Equilibration on average of temporally non-local observables in quantum systems

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Temporally non-local observables can evade equilibration on average in quantum systems. Equilibration on average refers to the evolution of a system’s time-dependent properties towards a constant value and remaining close to it for most times. We generalize this notion with the stronger requirement that equilibration holds for correlations between observations of a subsystem at multiple times as it evolves unitarily along with its environment with respect to a general global Hamiltonian. We address the attainability and robustness of this strong equilibration and show that for measurements that are made coherently, using a quantum memory, it is possible that the corresponding temporally non-local observables fail to equilibrate, violating previous bounds on the equilibration of quantum systems. For temporally uncorrelated observables, our results are a direct generalisation of this seminal earlier work. Finally, we support our analytical results with numerical calculations for a subsystem of a closed XX spin chain.

A fundamental question at the core of statistical mechanics is that of how thermal equilibrium arises from purely quantum mechanical laws in closed systems. There are three main approaches to resolving this conundrum: typicality [1–6], argues that small subsystems of a composite are in thermal equilibrium for almost all pure states of the whole; dynamical equilibration on average [7–12], which demonstrates that time-dependent quantities of quantum systems evolve towards fixed values and stay close to them for most times, even if they eventually deviate greatly from it; and the eigenstate thermalization hypothesis [13–18], which argues that the expectation values of a ‘physical observable’ at long times are indistinguishable for an isolated system from a thermal one.

What these approaches have in common is that they look at the statistical properties of the state of the system at long times, however, finding a system in or close to an equilibrium state does not necessarily imply all observable properties of the system have equilibrated. In particular, when only a subsystem is measured, it may be that correlations in time, manifesting in the statistics of sequential observations, maintain information about the initial perturbation. It is unclear whether these temporally non-local quantities also equilibrate in general; that is, whether they are most often found close to some average value.

In this Letter we focus on dynamical equilibration of these quantities, asking how, and how quickly, do temporally non-local properties of a system relax to be close to their equilibrium values, and present the necessary conditions for a stronger notion of equilibration to hold, characterised by the relaxation of joint observables at multiple times to equilibrium values. Moreover, we show that when the observables themselves are temporally correlated, corresponding to coherent measurements making use of an external memory, strong equilibration cannot be guaranteed; in general, it is possible that at any finite time following a perturbation, a subsystem could be distinguished from one in equilibrium given sufficient quantum memory. We back up our analytical bounds with a numerical example that shows this effect. We provide generalisations of the landmark work of Ref. [11], which we now briefly recapitulate before presenting our main results.

Equilibration — In the past decade, the programme of equilibration has focused on bounding the fluctuations of observable expectation values from above as a function of time around equilibrium, from which conclusions about equilibration of the state of the system itself have been drawn [5, 10]. The basic mechanism behind equilibration is that of dephasing [8, 19], and equilibration will occur as long as the initial state, following a perturbation, a subsystem could be distinguished from one in equilibrium. The only further assumption is that there are not too many degenerate energy levels [20], which ensures that the majority of the system plays a dynamical role [9].

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Specifically, quantum equilibration of subsystems refers to the phenomenon where a subpart of a quantum system relaxes towards some steady state, while the whole evolves unitarily. As in Ref. [11], we consider the dynamics of a $d_S$-dimensional subpart of a $d_E d_S$-dimensional system, where the whole (environment $E$ and subsystem $S$) is evolved by a unitary operator $U = \exp(-iHt)$. This approach is entirely general, with results depending on energetic properties of the Hamiltonian $H$, such as the number of distinct energy levels $\mathcal{D} \leq d$ and the maximum number of energy gaps $N(\epsilon)$ in an energy window of width $\epsilon > 0$.

In this minimal setting, the authors of Ref. [11] prove two important results on equilibration by showing the closeness between the subpart state $\rho_S(t)$ and the infinite time average state $\omega_S := \text{tr}_E \omega$ with $\omega = \lim_{T \rightarrow \infty} \rho^T$, where we denote by $\mathcal{X}^T = T^{-1} \int_0^T X(t) \, dt$ the time-average over a finite time window $[0, T]$. Specifically, they show that the time-averaged trace distance $D(\alpha, \beta) := \frac{1}{2} \text{tr} |\alpha - \beta|$ between the aforementioned states is bounded from above as

$$D(\rho_S(t), \omega_S)^2 \leq \sqrt{\frac{d_S^2 N(\epsilon) f(cT)}{4 d_{\text{eff}}}},$$

with $f(cT) = 1 + \frac{8 \log_2 \mathcal{D}}{cT}$, and so-called effective dimension $d_{\text{eff}} := 1/\sum_{\mu} \text{tr} (\pi_n \rho)^2$, where $\pi_n$ is a projector onto the $n$th eigenspace of $H$. The latter quantifies the number of energy levels contributing significantly to the dynamics of the initial state $\rho$. This result relies on a preliminary result for fluctuations of the expectation value of a closed system observable $A := \sum_{\mu} a_\mu |a_\mu\rangle \langle a_\mu|$ around equilibrium,

$$|\text{tr}[A(\rho(t) - \omega)]|^2 \leq ||A||^2 \frac{N(\epsilon) f(cT)}{d_{\text{eff}}},$$

where $||A||$ denotes the largest singular value of $A$. This translates directly to the case of subsystems with $A$ acting solely on subsystem $S$.

In general, we have the hierarchy $1 \leq d_{\text{eff}} \leq \mathcal{D} \leq d$, and Eqs. (1) and (2) imply that equilibration is attained for large $d_{\text{eff}}$, i.e., for systems whose energy eigenstates have a large enough overlap with the initial state. It has been argued, on physical grounds, that the effective dimension takes a large value in realistic situations [5, 8], increasing exponentially in the number of constituents of generic many-body systems [12], and it has been proven that it takes a large value for local Hamiltonian systems whenever correlations in the initial state decay rapidly [21]. As for the distinguishability measures in Eqs. (1) and (2) themselves, both constitute meaningful quantifiers of equilibration: for the observable expectation value $\text{tr}[A \rho(t)]$, a small variance relates to its concentration around its mean [22], whilst for trace distance, where by definition $D(\rho_S(t), \omega_S) = \max_A |\text{tr}((\rho_S(t) - \omega_S)A)|$, it implies that even with an optimal observable, the time-evolved state stays close to the steady state for most times, which can be justified through Markov’s inequality, $P[X \geq \epsilon] \leq \epsilon / f$. We will now show that similar results can be generalised to sets of observables across multiple times.

**Strong Equilibration** — To derive our main results, we consider the same minimal scenario as above, except that the subsystem $S$ can be interrogated multiple times, possibly jointly with an ancillary system, as depicted in Fig. 2. That is, an initial state $\rho_0$, an observation $A_0$ is made on $S$ along with an ancilla $\Gamma$, which is initially uncorrelated in state $\omega$. After the first observation, the environment and system evolve unitarily again for a time $\Delta t_1$ until another observation $A_1$ is made on $\Gamma$, and so on for $k$ steps. The joint expectation value of the series of observations is given by $\langle A_k, \ldots, A_0 \rangle := \text{tr}[A_k U_k \cdots A_0 U_0 (\rho \otimes \omega)]$, where $U_t(\cdot) = e^{-iH \Delta t(\cdot)}$ acts on $SE$, while $A_t(\cdot) := \sum_{\mu} a_\mu |a_\mu\rangle \langle a_\mu|$ acts on $S$. The ancillary space $\Gamma$ can be interpreted as a quantum memory device, and carries information about previous interactions with the system.

The information about the intrinsic dynamical process, i.e., the initial $SE$ state and the joint unitary evolutions with their respective timescales at each step, can be encoded in a positive semi-definite tensor $\mathcal{T}$; similarly, the sequence of observations can be encoded in a tensor $\Lambda$. This simplifies the joint expectation value $\langle A_k, \ldots, A_0 \rangle$ as the inner product $\langle \Lambda \mathcal{T} \rangle = \text{tr}[\Lambda \mathcal{T}]$, which can be seen as a generalisation of the Born rule to multi-time step quan-
tum processes [24]. Here, Υ becomes an unnormalized many-body density operator, and Λ an observable. Formally, Υ is the Choi state [25] of a quantum process, and contains all accessible information about a quantum process including that of the temporal correlations [26, 27]. It plays a central role in the process tensor formalism [28–30] and it is the quantum generalisation of a stochastic process [31].

We define a process on subsystem $S$ as having strongly equilibrated when the statistics of any multi-time observation $\Lambda$ are indistinguishable, for most starting times $t_0$, from that when the $SE$ system was initially in the equilibrium steady state $\omega$. This can be phrased in terms of the process $\Upsilon$, which is a function of $t_0$ and the set of fixed times between observations $\Delta t := (\Delta t_1, \ldots, \Delta t_k)$. In analogy with $\omega_S$, we define $\Omega$ as the infinite time-average of $\Upsilon$ over the first time evolution $t_0$, but we keep the subsequent process identical:

$$\Omega := \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} \Upsilon \; dt_0.$$  

Comparing $\Omega$ with $\Upsilon$, time-averaged over the first time of evolution $t_0$, tells us how well equilibration holds for correlations between sequential observations.

Before presenting our main results we note that when the process ends at the first measurement (of $A_0$) we have $\Upsilon(t_0) = \rho_S(t_0)$ and $\Upsilon(t_\infty) = \omega_S$, i.e., the processes become the corresponding quantum states and we recover Eq. (1) and Eq. (2). This is detailed in Appendix A.

Trace Distance Distinguishability — The natural extension of the central quantity in Eq. (1) is the trace distance $D(\Upsilon, \Omega)$ (normalized to have a maximum value of one), which here has an operational interpretation as the optimal distinguishability between the two processes, i.e., under the optimal set of observations. As before, we average over a finite time window on $t_0$, which yields the following generalisation of Eq. (1):

**Result 1.** For any $k$-step process $\Upsilon$ with initial $SE$ state $\rho$ and an evolution generated at each step by a time-independent Hamiltonian,

$$D(\Upsilon, \Omega) \leq \sqrt{\frac{\mathcal{R}}{d_S}} \left[ d_S^2 N(\epsilon) f(c_{T_0}) \right] \quad d_{\text{eff}}$$

where $\mathcal{R} := \max_{t_0} \text{rank}[\Upsilon - \Omega]$ for $t_0 \in [0,T_0]$ and all other quantities are defined as in Eq. (1).

Full details of the proof can be found in Appendix B. There, we bound the trace distance between the processes with the Schatten 2-norm, which we then time-average over the first evolution and further bound through standard matrix inequalities, ultimately employing the one for observable strong equilibration derived below. The main difference we find with standard equilibration is in the quantity $\mathcal{R}$, with Eq. (1) being recovered for $k = 0$ with $\mathcal{R} = d_S$.

The quantity $\mathcal{R}$ is hard to compute in full generality; it will be small (at least to a good approximation) when $\Upsilon$ and $\Omega$ are similar, but its maximum possible value grows exponentially with the number of timesteps $k$. However, even when $\mathcal{R}$ is large and the bound in Eq. (4) is not close to zero, deviations from strong equilibration may not be easy to observe. The trace distance between the processes $\Upsilon$ and $\Omega$ is equivalently defined by $D(\Upsilon, \Omega) := \max_{\mathcal{A}} \text{tr} | (\Upsilon - \Omega) \mathcal{A} |$ with the optimization being over operators $\mathcal{A}$ on the full space spanning multiple time steps, so in general actually distinguishing the two processes will demand a highly temporally non-local measurement across all time steps of the process. We can incorporate temporal non-locality between observations by introducing a limited ancillary memory, as suggested above, thus obtaining a more meaningful and refined upper bound on observable distinguishability.

**Observable Strong Equilibration** — We now look at the degree to which deviations from strong equilibration can be observed, by upper-bounding the fluctuations of the multi-time expectation value of $\Lambda$, depicted in Fig. 2, around its equilibrium value, time-averaged over an initial time window $[0,T_0]$, $||\Lambda(\Upsilon - \Omega)||^2_{T_0} = \left[ \text{tr} | \Lambda(\Upsilon - \Omega) \mathcal{A} |^2 \right]_{T_0}$. This generalises the standard case in Eq. (2). Our result quantifies how well a multi-time observable can tell the process with initial state $\rho$ from the one initially in equilibrium within a finite-time window for the first evolution. The case that is of particular interest is the multi-time step scenario with correlated observations.

Temporal correlations in observables are carried through the ancillary space $\Gamma$, which can be thought of as a memory carrier; any $\Lambda$ can be represented as a sequence of correlated observables on a joint $\Gamma T$ system, as depicted in Fig. 2. Both classically correlated observations, where the measurement basis is conditioned on past outcomes, and coherent quantum correlated measurements can be represented in this way [27]. The case of infinite memory and the case of completely uncorrelated observables are then extreme limits of this general setting.

**Result 2.** For any $k$-step process $\Upsilon$ with initial $SE$ state $\rho$ and an evolution generated by a time-independent Hamiltonian, and for any temporally non-local observable $\Lambda$, corresponding to a sequence of temporally local observables $\{A_i\}_{i=0}^{k}$ acting on a joint $\Gamma T$ system,

$$\frac{||\Lambda(\Upsilon - \Omega)||^2_{T_0}}{||A_0||^2 \cdots ||A_k||^2} \leq d_S^2 N(\epsilon) f(c_{T_0}) \quad d_{\text{eff}}$$

with $\| \cdot \|$ denoting operator norm (largest singular value), and the remaining quantities defined as in Eq. (1).
The proof is given in Appendix C, with the standard result for observable equilibration, Eq. (2), recovered with $k = 0$. As in the result given by Short and Farrelly [11], equilibration will depend on the overall scale set by the observables as determined by their operator norm. The difference in this case is that $d_{1}$, the dimension of the ancillary spaces carrying the correlations in $\Lambda$, might allow for observation of deviations from strong equilibration; this has a direct intuitive interpretation in terms of memory size, as large memory blocks will potentially allow one to access more information and tell more easily the difference between both processes. In order to highlight the significance of our bound, we will now consider three important cases and demonstrate numerically that it is still satisfied, even when the standard bound in Eq. (2) is violated.

Case 1. Temporally local observables — In the simplest case when there are no correlations between observables on $S$ and $\Lambda$ is simply the tensor-product of the individual Choi states of these observables, i.e., $d_{1} = 1$, the bound in Eq. (5) becomes equivalent to that in Eq. (2). This implies that, if a subsystem equilibrates in the usual sense, deviations from equilibration cannot be observed in the correlations between outcomes of sequential measurements for which the basis is independent of past outcomes (equivalent to there being no correlations in $\Lambda$).

Case 2. The infinite memory limit — As the full ancillary space dimension is increased, the upper bound in Eq. (2) becomes infinite. While the tightness of our bound is not guaranteed, we now argue that, in this case, it is always possible to violate the principle of strong equilibration for a class of systems with $d_{S}d_{E}$ and $T_{0}$ finite. Suppose we prepare our memory in state $|0\rangle \otimes n_{F}$, where $n_{F} \gg 1$. Now choose each operator at the $i$th step, $A_{i}$, to be a swap between the state in $S$ and the $i$th memory $|0\rangle$ state. By feeding such a fixed state at every step into subsystem $S$ for a great many steps, and as long as $d_{E}$ is finite, the process in the environment can only converge to a fixed state, i.e., we say that the process in the environment is forgetful about the initial state, and the rate at which this happens will depend on how information scrambling the unitary dynamics is at each step.

The no-hiding theorem [32] then tells us that after such large number of steps the information of the initial state $\rho$ cannot reside elsewhere but within the memory $\Gamma$. Our result implies then that in the limit of an infinite dimensional memory we are able to fully distinguish between $\rho(t_{0})$ and $\omega$ on average for $t_{0}$ within any finite time-window when the environment is forgetful.

Case 3. Finite memory — In the general scenario where observations carry memory, which could be quantum or classical[33], we now demonstrate with a numerical example that increasing $d_{1}$ leads to increasing observed deviations from equilibrium. We investigate a closed $XX$ spin chain with $n$-sites, given by the Hamiltonian $H = J \sum_{i=1}^{n} (\sigma_{x}^{(i)} \sigma_{x}^{(i+1)} + \sigma_{y}^{(i)} \sigma_{y}^{(i+1)})$, where $\sigma_{x}^{(i)}$ denotes a $\sigma_{x}$ Pauli matrix acting on the $i$-th site and similarly for $\sigma_{y}^{(i)}$ (identifying $n+1$ with 1); the coupling $J$ is specified for each case. This is a similar model to the one studied in Ref. [12]. Here we take a subset of the $n$ sites to be the system and the rest to be the environment.

As initial state we take an infinite temperature thermal (maximally mixed) state in $E$ together with a $|0\rangle$ state in $S$, i.e., $\rho = \frac{1}{d_{E}} \otimes |0\rangle |0\rangle$. With this, we obtain numerically the right hand side of Eq. (5) for $k = 0$, i.e., the upper bound in the standard observable equilibration case given by Short and Farrelly in Ref. [11], either for observables with operator norm one or with the difference of expectation values normalised by the operator norm. Full details of the setup are given in Appendix D.

We show in Fig. 3 that the standard observable equilibration bound is violated in this model for the case of $d_{S} = 2$ and $d_{E} = 2^{2}$ for a particular choice of $ST$ observable $h_{ST}$ measured at two subsequent time steps with $d_{1}$ sufficiently large. Furthermore, the average observable distinguishability keeps on increasing with the dimension of the ancillary space.

Discussion and Conclusions — We have introduced a stronger notion of equilibration that pertains to observations made across multiple times. In a similar way to usual notion of equilibration for observables at a single time, we have put bounds on the degree to which it holds that depend on the Hamiltonian driving the evolution. In particular, we have shown that subsystems of a closed time-independent Hamiltonian system will display equilibration on average for independent sequential observations, but that when the observables themselves are sufficiently temporally correlated, deviations from strong equilibration can be observed generically. As a proof of
concept, we have shown this explicitly for the particular case of two-time correlated observables on part of an Ising spin chain. The fact that this example involves a nearest neighbour interaction suggests that observation of deviations from strong equilibration should be feasible with currently available experimental systems [34, 35]. In fact, our results may help to interpret the lack of equilibrium behaviour of trapped ion spin chains [36].

Strong equilibration over temporally local observations is expected intuitively through decoherence arguments [37], however, when the observables carry memory, our result can be interpreted as having an enhanced access to the information stored within the initial state, which usually gets scrambled across system and environment between each observation. Furthermore, our argument in the case of a large memory by means of the no-hiding theorem [32] supports this intuition. It is as yet unclear under which circumstances strong equilibration can occur without the dynamics being Markovian, i.e., memoryless. We have previously shown that most processes are close to Markovian, and hence strongly equilibrate, in the strong coupling limit [38], but outside this regime the relationship between the two properties is less transparent.

A critical related open question pertains to the timescale over which strong equilibration occurs (to the degree that it does). Recent results for standard equilibration timescales in isolated systems relate these to the degree of locality of interaction in the Hamiltonian and in observables [39, 40], finding that highly local Hamiltonians and observables take longer to equilibrate. While in this Letter we do not approach this important problem in depth, it is worth emphasizing that our results hold generally for any choice of fixed time-intervals \(\Delta t_i\) and parameter \(T_0\) in the initial time-average interval, so that the aforementioned results can potentially be extended formally to our notion of strong equilibration to obtain further insights into the relationship between spatial and temporal locality.

Finally, equilibrium as a principle requires not only closeness of a system to a steady state but also that the system counteracts any external perturbations. This is already a well-known notion in chemistry, going under the name of Le Châtelier’s principle, and a similar notion has been explored in quantum theory, known as return to equilibrium [41, 42]. This usually refers to finite systems, initially in equilibrium, undergoing equilibration after having a quench applied [21]. The strong equilibration framework we have laid out here suggests itself as a platform for further studying the robustness of concepts such as return to equilibrium in a more general scenario.

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are the superoperators corresponding to the outcomes at single instances of uncorrelated controls.

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Appendix A: The process tensor

The process tensor is defined as a linear, completely positive (CP) and trace non-increasing map \( \mathcal{T} \) from a set of CP maps \( \{ A_i \} \) referred to as control operations, e.g. measurements, to a quantum state, and its action can be described as a multi-time open system evolution, e.g. for joint unitary evolution of an environment \( E \) plus system \( S \), with \( \dim(\mathcal{H}_E \otimes \mathcal{H}_S) = d_E d_S \), a k-step process is determined by

\[
\mathcal{T}_{k:0}[\{ A_i \}_{i=0}^{k-1}] = \text{tr}_E[U_k A_{k-1} \cdots A_0 U_0(\rho)]
\]

(A1)

where \( \rho \) is an initial joint \( SE \) state, \( U \) are unitary maps acting on \( SE \), and the maps \( A \) act solely on subsystem \( S \).

The associated Choi state of a time-evolved process tensor with initial state \( \rho \) is then given by

\[
\Upsilon_{k:0} = \text{tr}_E[U_{k:0}(\rho \otimes \psi^{\otimes k}) U_{k:0}^\dagger]
\]

(A2)

where \( \psi = \sum |ii\rangle\langle jj| \) is maximally entangled and unnormalized, and where here

\[
U_{k:0} := (U_k \otimes 1) S_k \cdots (U_1 \otimes 1) S_1 (U_0 \otimes 1),
\]

(A3)

with all identity operators \( 1 \) in the total ancillary system and with the \( U_i \) being \( SE \) unitary operators at step \( i \), and

\[
S_i := \sum_{\alpha, \beta} \mathcal{S}_{\alpha \beta} \otimes 1_{A_1 B_1 \cdots A_i-1 B_i-1} \otimes |\beta\rangle\langle\alpha| \otimes 1_{B_i A_i+1 B_i+1 \cdots A_k B_k},
\]

(A4)

with \( \mathcal{S}_{\alpha \beta} = 1_E \otimes |\alpha\rangle\langle\beta| \). This can be visualised as the quantum circuit depicted in Figure 4 when the unitary evolution is determined by a time-independent Hamiltonian, as we detail below, and we highlight that \( \Upsilon \) is defined directly with the first input being the initial state \( \rho \) (as opposed to half of a maximally entangled state in \( S \)). Explicitly, it can be written as

\[
\Upsilon_{k:0} = \sum \text{tr}_E \left[ U_k \mathcal{S}_{\alpha_k, \beta_k} \cdots U_1 \mathcal{S}_{\alpha_1, \beta_1} U_0 \rho U_0^\dagger \mathcal{S}_{\gamma_1, \delta_1}^\dagger U_1^\dagger \cdots \mathcal{S}_{\gamma_k, \delta_k}^\dagger U_k^\dagger \right] \otimes |\beta_1 \alpha_1 \cdots \beta_k \alpha_k\rangle \langle \delta_1 \gamma_1 \cdots \delta_k \gamma_k|.
\]

(A5)

Notice that \( \mathcal{S}_{\alpha \beta} = \mathcal{S}_{\alpha \beta}^\dagger, \mathcal{S}_{ab} \mathcal{S}_{cd}^\dagger = \delta_{ad} \mathcal{S}_{ac} \) and \( \text{tr}(\mathcal{S}_{ab}) = d_E \delta_{ab} \). Also notice that the resulting Choi process tensor state belongs to the whole \( S \) plus ancillary system, which has dimension \( d_S^{2k+1} \).
The equilibrium state is the time-averaged state
\[ \omega := \bar{\rho}^\infty := \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} \rho(t_0) \, dt_0, \]
which becomes \( \omega = \sum_m \rho_{mn} |n⟩⟨n| \) in a given Hamiltonian eigenbasis \( \{|n⟩\} \). We denote by \( \overline{X^T_0} := \frac{1}{T_0} \int_0^{T_0} X(t_0) \, dt_0 \) the time average of \( X \) over \( t_0 \) on an interval \( [0, T_0] \); all the averages are taken over this first evolution time-parameter \( t_0 \).

The process tensor Choi state corresponding to a full process with this initial state then takes the form
\[ \Omega_{\Delta t_k \cdots \Delta t_1} := \overline{\Omega_{\Delta t_k \cdots \Delta t_1}^\infty} = tr_E \left[ U_{k:1} S_1(\omega \otimes \psi^{\otimes k}) S_1^\dagger U_{k:1}^\dagger \right], \]
and so we are interested in quantifying the difference with the corresponding generic Choi state for \( k \)-steps,
\[ \Upsilon_{\Delta t_k \cdots \Delta t_1}^T_{t_0} - \Omega_{\Delta t_k \cdots \Delta t_1} = tr_E \left\{ U_{k:1} S_1 ([U_0 \rho U_0^\dagger - \omega] \otimes \psi^{\otimes k}) S_1^\dagger U_{k:1}^\dagger \right\}. \]

We omit the labels for the fixed time intervals, \( \Delta t_i \), and unless stated otherwise, we refer to \( k \)-step processes.

II. Reduction to the usual subsystem equilibration

We highlight that thus defined, for \( k = 0 \) this Choi state corresponds to the system’s state difference
\[ \Upsilon_{t_0} - \Upsilon_{t_0}^\infty = tr_E \left( U_0 \rho U_0^\dagger - \omega \right) = \rho_S(t_0) - \omega_S, \]
so that our results must contain those in Ref. [11] for \( k = 0 \). This occurs because we define the Choi state of the whole process in terms of the initial state \( \rho \), as opposed to the case for a quantum channel with no fixed initial input.
Appendix B: Trace distance distinguishability multi-time equilibration

Similar to standard subsystem equilibration, we look for a bound on the distinguishability of a general quantum process with the corresponding initially on-equilibrium one. Since we’re dealing with unnormalized Choi states, $\Upsilon$ and $\Omega$, with $\text{tr}(\Upsilon) = \text{tr}(\Omega) = d_S^2$, we look at the trace distance $D(\Upsilon, \Omega) := n\|\Upsilon - \Omega\|_1$ with normalization $n = 1/2d_S^k$. We first bound the one-norm by $\|X\|_1 \leq \sqrt{\text{rank}(X)}\|X\|_2 = \sqrt{\text{rank}(X)}\sqrt{\text{tr}(XX^\dagger)}$. For our purposes, $X$ will depend on $t_0$, so we further take the upper-bound with the maximum rank of $X$ in time $t_0 \in [0, T_0]$ to avoid an extra time-average involving this term.

Let us consider first the case of a one-step process, $k = 1$, with initial $SE$ state $\rho$ (assumed pure for now, but that can be extended to any mixed state as detailed below in Appendix C) so that

$$(\Upsilon_{\Delta t_{t_0}} - \Omega_{\Delta t_{t_0}})^2 = \sum \text{tr}_E \left[ U_1 \mathcal{S}_{ab}(U_0 \rho U_0^\dagger - \omega) \mathcal{S}_{dc} U_1^\dagger \right] \text{tr}_E \left[ U_1 \mathcal{S}_{ab}(U_0 \rho U_0^\dagger - \omega) \mathcal{S}_{dc} U_1^\dagger \right] \otimes |\beta\alpha\rangle\langle \gamma|\langle \delta|\langle dc|ba|$$

and so

$$\text{tr}[(\Upsilon_{\Delta t_{t_0}} - \Omega_{\Delta t_{t_0}})^2] = \sum \text{tr}[\left[ U_1 \mathcal{S}_{ab}(U_0 \rho U_0^\dagger - \omega) \mathcal{S}_{dc} U_1^\dagger \right]^2]$$

$$\leq \text{tr}_E \left[ \left( \sum U_1 \mathcal{S}_{ab} \omega \mathcal{S}_{ab} U_1^\dagger \right)^2 \right],$$

where the sums only include those terms where $m \neq n$ and $\mu \neq \nu$, and where again we are taking disjoint intervals for the respective time parameters over the first evolution and the rest. Looking first at the infinite time-average over $t_0$,

$$\text{tr}[(\Upsilon_{\Delta t_{t_0}} - \Omega_{\Delta t_{t_0}})^2] = \sum \rho_{nm} \rho_{nm} \text{tr}_S \left\{ \text{tr}_E \left[ U_1 \mathcal{S}_{ab} |m\rangle\langle n| \mathcal{S}_{dc} U_1^\dagger \right] \text{tr}_E \left[ U_1 \mathcal{S}_{cd} |n\rangle\langle m| \mathcal{S}_{ba} U_1^\dagger \right] \right\}$$

and so

$$\text{tr}[(\Upsilon_{\Delta t_{t_0}} - \Omega_{\Delta t_{t_0}})^2] = \sum \rho_{nm} \rho_{nm} \text{tr}_E \left( \sum U_1 \mathcal{S}_{ab} \omega \mathcal{S}_{ab} U_1^\dagger \right)^2 \leq \text{tr}_E \left( \sum U_1 \mathcal{S}_{ab} \omega \mathcal{S}_{ab} U_1^\dagger \right)^2$$

and it can be seen from the weak-subadditivity of the Renyi entropy (right hand side of Lemma 4.3 in [13]) that $\text{tr}(X_A^2) \leq \text{rank}(X_B) \text{tr}(X_A^2) \leq \text{dim}(X_B) \text{tr}(X_A^2)$, so

$$\text{tr}[(\Upsilon_{\Delta t_{t_0}} - \Omega_{\Delta t_{t_0}})^2] \leq d_S \text{tr}(\sum U_1 \mathcal{S}_{ab} \omega \mathcal{S}_{ab} U_1^\dagger)^2] = d^2_S \text{tr}(\omega^2) \leq d^2_S \text{tr}(\omega^2).$$

For a given number of time-steps $k$ then, it follows that

$$\text{tr}[(\Upsilon_{t_0} - \Omega)^2] \leq d^2_S k \text{tr}(\omega^2) = d^2_S \frac{k+1}{d_S},$$

and thus also

$$D(\Upsilon_{t_0}, \Omega) \leq \frac{1}{2} \sqrt{\frac{\mathcal{R}' d_S}{d_{\text{eff}}}},$$

where here

$$\mathcal{R}' := \lim_{T_0 \to \infty} \mathcal{R} := \lim_{T_0 \to \infty} \max_{0 \leq t_0 \leq T_0} \text{rank}(\Upsilon_{t_0} - \Omega).$$

The result in Ref. [11] for $k = 0$ is recovered with $\mathcal{R} = d_S$.

A generalisation to a finite-time average can be given through a Fourier decomposition

$$\Upsilon_{t_0} - \Omega = \sum_{n_1, \ldots, n_{2k+1}} \alpha_{n_1, \ldots, n_{2k+1}} (\Delta t_k, \ldots, \Delta t_1, t_0) F_{n_1, \ldots, n_{2k+1}}.$$
where \( \alpha_i \) are complex numbers carrying all time dependence and \( \mathbf{F}_{1, \ldots, 2k+1} := \mathcal{F}_1 \otimes (\mathcal{F}_2 \otimes \mathcal{F}_3) \psi \otimes \cdots \otimes (\mathcal{F}_{2k} \otimes \mathcal{F}_{2k+1}) \psi \),
where each \( \sqrt{d_S} \mathcal{F}_i \) is unitary. From \([11]\) it then follows that

\[
\frac{D(\Upsilon_{t_{0}}, \Omega)}{T_0} \leq \frac{1}{2} \sqrt{\mathcal{R} \left( \sum_{n} n^2 \left| \text{tr}(\Upsilon_{t_{0}} - \Omega) \mathbf{F}_{n}^\dagger \right|^2 \right)} T_0,
\]

and so, as \( \prod_{t=1}^{2k+1} \left( \sum_{n=1}^{d_S^3} \| \mathbf{F}_{n} \|^2 \right) = \sigma_S^{2k+1} \), also from Eq. \((C10)\),

\[
\frac{D(\Upsilon_{t_{0}}, \Omega)}{T_0} \leq \frac{1}{2} \sqrt{\frac{\mathcal{R} \ d_S N(c)}{d_{\text{eff}}} \left( 1 + \frac{8 \log_2 d_{\text{eff}}}{c \ T_0} \right)}.
\]

**Appendix C: Observable multi-time equilibration**

We consider a set of observables \( \{ A_i \}_{i=0}^{k} \) acting locally at each time-step, i.e., between each step of evolution, of a quantum stochastic process. Explicitly, by observables we mean Hermitian operators \( A_i = A_i^\dagger \) with a spectral decomposition \( A_i = \sum \lambda_n^{(i)} \Pi_n^{(i)} \) where \( \{ \Pi_n^{(i)} \} \) is a set of orthogonal operators, or measurement operators in system \( S \), with outcomes \( \lambda_n^{(i)} \).

We first look at the case of two intervention with temporally local observables \( A_0 \) and \( A_1 \). The Choi state of the two observables entering the process can be written as \( \Lambda_{1:0} := \text{tr}_\mathcal{A} [ (A_1 \otimes \mathcal{T}_\mathcal{C}) \psi] \otimes (A_0 \otimes \mathcal{T}) \psi = A_1(\mathbb{1}) \otimes (A_0 \otimes \mathcal{T}) \psi \), where by \( \mathcal{A} \) denote the map associated to the observable \( A_i \), i.e., \( A(\cdot) := \sum \lambda_n \Pi_n(\cdot) \Pi_n \). Here we first compute the square of the difference between expectation values \( \langle A_{1:0} \rangle_{\Upsilon - \Omega} \) averaged on \( t_0 \) over an infinite time-interval. We have

\[
(\Upsilon_{\Delta t_{1}, t_0} - \Omega_{\Delta t_{1}})A_{1:0} = \sum \text{tr}_E \left[ U_1 \ \mathcal{S}_{\alpha \beta} (U_0 \rho U_0^\dagger - \omega) \mathcal{S}_{\delta \gamma} U_1^\dagger \right] A_1 \otimes |\beta \rangle \langle \beta | \langle \delta | A_0 | \gamma \rangle,
\]

so

\[
\text{tr}[(\Upsilon_{\Delta t_{1}, t_0} - \Omega_{\Delta t_{1}})A_{1:0}] = \sum \text{tr}_S \left( \text{tr}_E \left[ U_1 (U_0 \rho U_0^\dagger - \omega) \mathcal{S}_{\delta \gamma} U_1^\dagger \right] A_1 \right) \langle \delta | A_0 | \gamma \rangle,
\]

and

\[
| \text{tr}[(\Upsilon_{\Delta t_{1}, t_0} - \Omega_{\Delta t_{1}})A_{1:0}]^2 | = \sum \text{tr}_S \left( \text{tr}_E \left[ U_1 (U_0 \rho U_0^\dagger - \omega) \mathcal{S}_{ab} U_1^\dagger \right] A_1 \right) \langle a | A_0 | b \rangle \langle c | A_0 | d \rangle,
\]

where \( \rho_{mn} = \langle m | \rho | n \rangle = \langle m | \phi | \phi | n \rangle \); by choosing this energy eigenbasis it is implied that \( m \neq n \) and \( \mu \neq \nu \). Here we take the first time-interval over the parameter \( t_0 \) as disjoint from the rest of time-intervals; this can be thought of as having a different clock on the remaining steps of the process. Taking the time-average over \( t_0 \),

\[
\frac{\left| \text{tr}[(\Upsilon_{\Delta t_{1}, t_0} - \Omega_{\Delta t_{1}})A_{1:0}]^2 \right|}{T_0} \leq \sum \rho_{mn}^2 \text{tr}_S \left( \text{tr}_E \left[ U_1 | m \rangle \langle n | \mathcal{S}_{ab} U_1^\dagger \right] A_1 \right) \langle a | A_0 | b \rangle \langle c | A_0 | d \rangle
\]

\[
\leq \text{tr} \left[ U_1 \mathcal{I} \otimes (A_0) U_1^\dagger (\mathcal{I} \otimes A_1) U_1 \right] \langle a | A_0 | b \rangle \langle c | A_0 | d \rangle
\]

\[
\leq \left( \text{tr} \left[ (A_0)^\dagger \right] \right) \left( \text{tr} \left[ (A_0^2) \right] \right) \leq \left( \| A_0 \| \right) \left( \| A_0 \| \right) \leq \| A_0 \|^2 \| A_1 \|^2 \text{tr}(\omega^2),
\]

where in the fourth line we used \( \text{tr}(AB^\dagger) \leq \sqrt{\text{tr}(A^AA^\dagger) \text{tr}(B^BB^\dagger)} \) (Cauchy-Schwarz inequality), while in the fifth line we used \( \text{tr}(XY) \leq \| X \| \text{tr}(Y) \) for positive \( X \) and \( Y \), together with \( \| P \otimes Q \| \leq \| P \| \| Q \| \) with \( \| \cdot \| \) the operator norm (the largest singular value).
The extension to an arbitrary number of time-steps \( k \) is straightforward then as

\[
|\text{tr}[(Y_{t_0} - \Omega)A_{k:0}]|^2 \leq \text{tr}(\omega^2) \prod_{i=0}^{k} \|A_i\|^2 = \frac{1}{\Delta_{\text{eff}}} \prod_{i=0}^{k} \|A_i\|^2,
\]

where

\[
A_{k:0} = A_k(1) \otimes (A_i \otimes 1) \psi
\]

with the standard result in Ref. [11] for quantum states recovered with \( k = 0 \). The extension to initial mixed states follows in the same way as in [11] via purification taking care that under the purified Hamiltonian \( H' = H \otimes 1 \), the purity of the time-averaged purified state \( \omega' \) is not equal to that of \( \omega \), even when \( \Delta'_{\text{eff}} = \Delta_{\text{eff}} \).

We can then generalize to the average over a finite time-interval \([0, T_0]\) with the same argument given by [11]. Starting from Eq. (C3), let \( \alpha = (m, n) \) and \( \beta = (\nu, \mu) \) such that e.g. \( G_\alpha = E_m - E_n \), and

\[
M_{\alpha\beta} = \exp[-i t_0(G_\alpha - G_\beta)]^{T_0},
\]

and also let \( v_\alpha = \sum \rho_{mn} \text{tr}_S \left\{ \text{tr}_E \left[ U_1|m\rangle\langle n| G_{ab} U_1^\dagger \right] A_1 \right\} \langle a|A_0|b \rangle \) (sum on \( a, b \), so that

\[
|\text{tr}[(Y_{\Delta t:0} - \Omega_{\Delta t})A_{1:0}]|^2 \leq \sum_{\alpha} M_{\alpha\beta} v_\beta^* \leq \|M\| \sum |v_\alpha|^2 \leq \frac{1}{\Delta_{\text{eff}}} \|M\| \|A_0\|^2 \|A_1\|^2,
\]

where on the third line we made use of Eq. (C5). Now, from [11] we know that

\[
\|M\| \leq N(\epsilon) \left( 1 + \frac{8 \log_2 D}{\epsilon T_0} \right),
\]

for all \( \epsilon > 0 \), where \( N(\epsilon) \) is the maximum number of energy gaps in an interval of size \( \epsilon \) and where here \( D \) is the number of energy levels. It follows then that for any number of time-steps \( k \),

\[
|\text{tr}[(Y_{t_0} - \Omega)A_{k:0}]|^2 \leq \frac{N(\epsilon)}{\Delta_{\text{eff}}} \left( 1 + \frac{8 \log_2 D}{\epsilon T_0} \right) \prod_{i=0}^{k} \|A_i\|^2,
\]

which can be rearranged and expressed as in Result 1.

### I. Correlated control operations

To allow for general (quantum and classical) correlations on a set of control operations (observables), and ancillary space \( \Gamma \) can be introduced such that each observable will act on \( ST \), thus turning the space \( \Gamma \) into a memory space of the interactions of the observable with \( S \). A circuit diagram for this process, with all operations being correlated, would look like those in Fig. 2. The corresponding Choi state of the full control would take the form

\[
\Lambda^{\text{corr}} = \text{tr}_\Gamma[(A_k \otimes 1)S_k \cdots S_1(A_0(1) \otimes \psi^{\otimes k})S_1^\dagger \cdots S_k^\dagger],
\]

with the swaps being between system \( S \) and half of each maximally entangled state. \( \Gamma \) is equivalent to an environment for the process tensor of the observable control \( \Lambda^{\text{corr}}, \) so it satisfies \( \text{dim}(\Gamma) = d_1 \leq d_S \) for a minimal dilation, i.e., the dimension above which an increase of dimensionality is redundant to describe the properties of the control \( \Lambda^{\text{corr}} \). The dimension \( d_1 \) could be thought of as the size of the memory (regardless of how much of it is actually used). Let us first consider a one-step process with

\[
\Lambda^{\text{corr}}_{1:0} = \text{tr}_\Gamma[(B \otimes 1)S(A \otimes \psi)S^\dagger].
\]
Decomposing the observables in a basis for $\Gamma$ and $S$, ordered $\{ |g \rangle \otimes |a \rangle \}$, as in

$$A = \sum A_{ghab}|ga\rangle\langle hb|,$$

we can recast Eq. (C12) as

$$A^{\text{corr}}_{1:0} = \sum_{g,h=1}^{dc} A_{gh}^{(S)} \otimes (B^{(S)}_{gh} \otimes \mathbb{1})\psi,$$

where $X_{gh}^{(S)} = \sum X_{ghab}|a\rangle\langle b|$ is a state in subsystem $S$ for each $g, h = 1, \ldots, d\Gamma$. We may then generalize the bound on Eq. (C4) as

$$\left| \text{tr}(\tilde{\Omega}_{\Delta t, t_0} - \Omega_{\Delta t}) A^{\text{corr}}_{1:0} \right|^2 \leq \text{tr} \left[ \sum U_1 \omega(\mathbb{1} \otimes A^{(S)}_{gh}) U_1^\dagger (\mathbb{1} \otimes B^{(S)}_{gh}) U_1 \omega(\mathbb{1} \otimes A^{(S)}_{gh}) U_1^\dagger (\mathbb{1} \otimes B^{(S)}_{gh}) \right]$$

$$\leq \sqrt{\text{tr} \left[ \sum \omega^2 (\mathbb{1} \otimes A^{(S)}_{gh}) U_1^\dagger (\mathbb{1} \otimes B^{(S)}_{gh}) U_1 (\mathbb{1} \otimes A^{(S)}_{gh}) \right]}^2$$

$$\leq |\text{tr}_{\Gamma}(\mathbb{1} \otimes A)(U_{1}^\dagger \otimes \mathbb{1})(\mathbb{1} \otimes B)| \text{tr}_{\Gamma}(\mathbb{1} \otimes A)(\mathbb{1} \otimes B)$$

$$\leq \frac{d^2}{\text{d}_{\text{eff}}} \| A \|^2 \| B \|^2,$$

where in the fifth line we used $\| \text{tr}_{\Gamma}(X_{AB}) \| \leq d_B \| X_{AB} \|$ (which holds in a more general way for any $p$-norm [44]). For a minimal dilation then this is upper bounded by $d_S^2/\text{d}_{\text{eff}} \| A \|^2 \| B \|^2$.

This can then be written for a $k$-step process as

$$\left| \text{tr}(\tilde{\Omega}_{t_0} - \Omega) A^{\text{corr}} \right|^2 \leq \frac{d^2}{\text{d}_{\text{eff}}} \prod_{i=0}^{k} \| A_i \|^2,$$

which is then upper bounded for the minimal case through $d\Gamma \leq d_S^{2k+1}$.

More generally, one can consider the difference of expectations in processes whose controls are correlated in blocks of a given order, $(\ell_j)_{j=1}^{n_k}$ with $\sum n_k = k + 1$, as each of these will contribute a factor $d_{\ell_j}^2 \leq d_S^{2(\ell_j-1)}$ to the upper bound, then

$$\left| \text{tr}(\tilde{\Omega}_{t_0} - \Omega) A^{(1,\ldots,\ell_k)} \right|^2 \leq \frac{d^2_{\ell_1} \cdots d^2_{\ell_k}}{\text{d}_{\text{eff}}} \prod_{i=0}^{k} \| A_i \|^2.$$  

Similar to the previous cases, Eq. (5) follows for a finite time-average over $t_0$ as well. A fully correlated control is then one such that $A^{\text{corr}}_{k:0} = A^{k+1}_{k:0}$. The two extreme cases of a fully correlated control and local observables is recovered, correspondingly, when there is only one $\ell = k + 1$ and when all $(\ell_j = 1)_{j=1}^{k+1}$. The purely local non-Markovian control case would saturate the bound with all $d\Gamma_j = 1$.

**Appendix D: Numerical example**

We consider a simple one-dimensional Ising model specified by the Hamiltonian

$$H = J \sum_{\ell=1}^{n} (\sigma_x^{(\ell)} \sigma_x^{(\ell+1)} + \sigma_y^{(\ell)} \sigma_y^{(\ell+1)}),$$

where the label $n + 1$ is identified with 1, and where $\sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|$ and $\sigma_y = i(|1\rangle\langle 0| - |0\rangle\langle 1|)$, for a single qubit in system $S$ and three qubits in $E$, that is, we take $n = 4$. 

The Hamiltonian has five distinct eigenvalues, $\pm 4\sqrt{2}|J|$, $\pm 4|J|$ and 0, so considering the biggest gap among them we may choose $\epsilon = 8\sqrt{2}|J|$ as our energy interval width, where there are $N(\epsilon) = \frac{5}{2}$ = 10 energy gaps. As initial state we take a maximally mixed state in $E$ together with a $|0\rangle$ state in $S$, i.e., $\rho = \frac{1}{d_E} \otimes |0\rangle\langle 0|$ with which we then compute $d_{\text{eff}} = 64/5$. This finally gives

$$B_{\text{std}} := \frac{N(\epsilon)f(\epsilon T_0)}{d_{\text{eff}}} = \frac{25}{32} \left( 1 + \frac{\log_2(5)}{\sqrt{2}|J| T_0} \right),$$

which is the upper-bound on standard equilibration.

Now we are mainly interested in looking at the average fluctuations of the joint expectation values around initial equilibrium in a one-step process through two correlated observables for different dimensions of the memory space $\Gamma$. To do this consistently and at the same time without picking a particular choice of observables, we sample a random Hermitian $2 \times 2$ matrix, $\gamma$, from the Gaussian Unitary Ensemble, i.e., with normally distributed entries with mean zero and variance one, with which we defined

$$h_{ST} := \frac{1}{2} \left\{ \exp[-i\gamma \otimes (1+n_\Gamma)] + \exp[i\gamma \otimes (1+n_\Gamma)] \right\},$$

where $n_\Gamma$ is related to the dimension of the ancillary space by $d_\Gamma = 2^n_\Gamma$. Then $h_{ST}$ is a $d_S d_\Gamma = 2^{1+n_\Gamma}$ dimensional Hermitian matrix with $\|h_{ST}\| \leq 1$.

In particular our results can be reproduced with

$$\gamma = \begin{pmatrix} 0.481032 & 0.499149 - 0.00935478 i \\ 0.499149 + 0.00935478 i & 0.705875 \end{pmatrix},$$

and the finite-time averages shown in Fig. 3 can be analytically computed as functions of $T_0$: this allows to verify their respective infinite-time limits; let us denote

$$\Lambda_{ST} := \text{tr}_\Gamma \left[ (h_{ST} \otimes 1) S(h_{ST} \otimes \psi) S^\dagger \right],$$

then for such setup, i.e., with $J = 1/16$ and $\Delta t_1 = 16\pi$, we obtain

| $\Gamma$ | $\langle \Lambda_{ST} \rangle_{T_0 \to 0}$ | $\langle \Lambda_{ST} \rangle_{T_0 \to \infty}$ |
|---|---|---|
| 2 | 0.23 | 0.15 |
| $2^2$ | 0.73 | 0.95 |
| $2^3$ | 3.81 | 3.41 |
| $2^4$ | 15.46 | 14.62 |
| $2^5$ | 62.51 | 60.85 |

whereas $\lim_{T_0 \to 0} B_{\text{std}} = \infty$ and $\lim_{T_0 \to \infty} B_{\text{std}} = 0.78$. 