Parameterized Complexity of Upper Edge Domination

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Abstract. In this paper we study a maximization version of the classical Edge Dominating Set (EDS) problem, namely, the Upper EDS problem, in the realm of Parameterized Complexity. In this problem, given an undirected graph $G$, a positive integer $k$, the question is to check whether $G$ has a minimal edge dominating set of size at least $k$. We obtain the following results for Upper EDS. We prove that Upper EDS admits a kernel with at most $4k^2 - 2$ vertices. We also design a fixed-parameter tractable (FPT) algorithm for Upper EDS running in time $2^{O(k)} \cdot n^{O(1)}$.

Keywords: Parameterized Complexity · FPT · treewidth · upper edge dominating set

1 Introduction

The dominating set problem and its variants have been extensively studied in the literature. Typically, researchers have considered this concept in terms of the minimisation problem MINIMUM DOMINATING SET, namely MIN DS: find a smallest set of vertices that dominate all vertices of the graph [14]. However, researchers have also considered the max-min variant, usually called UPPER DOMINATING SET, which we abbreviate to UPPER DS: A minimal dominating set is a dominating set in a graph that is not a proper subset of any other dominating set. Every minimum dominating set is a minimal dominating set, but the converse does not necessarily hold. Our goal here is to find an inclusion-wise minimal dominating set of largest size [9,10,12,14,15]. Both MIN DS and UPPER DS are NP-hard for general graphs; see [13], problem GT2] and [7], respectively. In 2021, Monnot, Fernau and Manlove [19] studied the edge variant of the (vertex) dominating set problem.

A set of edges $M$ of $G = (V, E)$ is called an edge dominating set if every edge of $E \setminus M$ is adjacent to some edge of $M$. Similarly, researchers have considered this concept in terms of the minimisation problem MINIMUM EDGE DOMINATING SET, namely MIN EDS: find a smallest set of edges that dominate all edges of the graph [15]. An edge dominating set $M$ of $G$ is said to be a minimal edge dominating set if no proper subset of $M$ is also an edge dominating set of $G$. The problem We consider in this paper is as follows:
Upper EDS

**Input:** A graph $G = (V, E)$ and an integer $k$.

**Question:** Does $G$ have a minimal edge dominating set $M \subseteq E$ of size at least $k$?

Whilst Min EDS has received considerable attention in the literature, the same is not true for Upper EDS. Min EDS is NP-hard in planar or bipartite graphs of maximum degree 3 [21] and in planar cubic graphs [15], whilst solvable in polynomial time in several graph classes (see [8] for a brief survey). On the other hand Upper EDS has been largely neglected: Upper EDS is NP-hard in bipartite graphs [18]. Monnot et al. [19] showed that this problem is not approximable within a ratio of $n^{\epsilon - \frac{1}{2}}$, for any $\epsilon \in (0, 1)$, assuming $P \neq NP$, where $n = |V|$. In this paper we enhance our understanding of the problem from the viewpoint of parameterized complexity. We refer to [10,11] for further details on parameterized complexity.

Our results are as follows:

- **Upper Edge Dominating Set** parameterized by the solution size $k$ admits a kernel of size $4k^2 - 2$.

- We prove that, given an $n$-vertex graph $G$ and its nice tree decomposition $T$ of width at most $\omega$, the size of a maximum Upper Edge Dominating Set of $G$ can be computed in time $45^\omega \cdot n^{O(1)}$. This gives a fixed-parameter tractable (FPT) algorithm for Upper EDS running in time $2^{O(k)} \cdot n^{O(1)}$

## 2 Preliminaries

Throughout this paper, we consider simple undirected graphs. A graph $G = (V, E)$ can be specified by the set $V$ of vertices and the set $E$ of edges. The *(open)* neighbourhood $N_G(v)$ of a vertex $v \in V$ is the set $\{u \mid (u, v) \in E\}$. The *(closed)* neighbourhood $N_G[v]$ of a vertex $v \in V$ is the set $\{v\} \cup N_G(v)$. The subgraph induced by $D \subseteq V$ is denoted by $G[D]$. Every edge has two endpoints and these two endpoints are called *adjacent*: if $v$ is an endpoint of $e$, we also say that $e$ and $v$ are *incident* and two edges $e$ and $e'$ are adjacent if they share a common endpoint. The *(closed)* neighbourhood $N[e]$ of an edge $e \in E(G)$ is the set $\{e' \mid e$ and $e'$ are adjacent$\}$.

An edge set $M \subseteq E$ is an *edge dominating set* if every edge $e \in E \setminus M$ is adjacent to some edge of $M$. Let $M \subseteq E$ be an edge dominating set. Define an edge $e \in E$ to be *private* if $e$ is dominated by exactly one edge of $M$. The following lemma demonstrates a connection between minimal edge dominating sets and private edges

**Lemma 1.** [19] Let $G = (V, E)$ be a graph and let $M \subseteq E$ be an edge dominating set. Then $M \subseteq E$ is a minimal edge dominating set if and only if every edge $e \in M$ has a private edge in $N[e]$.

The graph parameter that we explicitly use in this paper is treewidth. We review the concept of a tree decomposition, introduced by Robertson and Seymour in [20]. Treewidth is a measure of how “tree-like” the graph is.
Definition 1. [11] A tree decomposition of a graph \( G = (V, E) \) is a tree \( T \) together with a collection of subsets \( X_t \) (called bags) of \( V \) labeled by the vertices \( t \) of \( T \) such that \( \bigcup_{t \in T} X_t = V \) and (1) and (2) below hold:

1. For every edge \( uv \in E(G) \), there is some \( t \) such that \( \{u, v\} \subseteq X_t \).
2. (Interpolation Property) If \( t \) is a vertex on the unique path in \( T \) from \( t_1 \) to \( t_2 \), then \( X_{t_1} \cap X_{t_2} \subseteq X_t \).

Definition 2. [11] The width of a tree decomposition is the maximum value of \( |X_t| - 1 \) taken over all the vertices \( t \) of the tree \( T \) of the decomposition. The treewidth \( tw(G) \) of a graph \( G \) is the minimum width among all possible tree decompositions of \( G \).

A special type of tree decomposition, known as a nice tree decomposition, was introduced by Kloks [17]. The nodes in such a decomposition can be partitioned into four types:

Definition 3. [17] A tree decomposition is said to be a nice tree decomposition if the following conditions are satisfied:

1. All bags that correspond to leaves are empty. One of the leaves is considered as root node \( r \). Thus \( X_r = \emptyset \) and \( X_l = \emptyset \) for each leaf \( l \).
2. There are three types of non-leaf nodes:
   - Introduce node: a node \( t \) with exactly one child \( t' \) such that \( X_t = X_{t'} \cup \{v\} \) for some \( v \not\in X_{t'} \); we say that \( v \) is introduced at \( t \).
   - Forget node: a node \( t \) with exactly one child \( t' \) such that \( X_t = X_{t'} \setminus \{w\} \) for some \( w \in X_{t'} \); we say that \( w \) is forgotten at \( t \).
   - Join node: a node with two children \( t_1 \) and \( t_2 \) such that \( X_t = X_{t_1} = X_{t_2} \).

Note that, by the interpolation property of tree decomposition, a vertex \( v \in V(G) \) may be introduced several times, but each vertex is forgotten only once. To control the introduction of edges, sometimes one more type of node is considered in a nice tree decomposition called introduce edge node. An introduce edge node is a node \( t \), labeled with edge \( uv \in E(G) \), such that \( u, v \in X_t \) and \( X_t = X_{t'} \), where \( t' \) is the only child of \( t \). We say that node \( t \) introduces edge \( uv \). We additionally require that every edge of \( E(G) \) is introduced exactly once in the whole decomposition. It is known that if a graph \( G \) admits a tree decomposition of width at most \( tw \), then it also admits a nice tree decomposition of width at most \( tw \), that has at most \( O(n \cdot tw) \) nodes [10].

3 Kernelization algorithm for Upper EDS parameterized by solution size

In this section we give a kernelization algorithm for Upper EDS which matches the lower bound. We start with some simple reduction rules that clean up the
graph. The first reduction rule is based on the following trivial observation: If the graph \( G \) has an isolated vertex, the removal of this vertex does not change the solution, and this operation can be implemented in polynomial time. Thus, the following rule is safe.

**Reduction UEDS 1** If \( G \) contains an isolated vertex \( v \), remove \( v \) from \( G \). The resulting instance is \((G - v, k)\).

The second rule is also based on a simple observation. If \( G \) contains an isolated edge, it must be included in the solution.

**Reduction UEDS 2** If there is an isolated edge \((u, v)\) in \( G \), delete it and decrease \( k \) by 1. The new instance is \((G - \{u, v\}, k - 1)\).

In our kernelization algorithm, it is convenient to work with coloured graphs. We colour the vertices of \( G \) with four colours: blue, purple, red and green. The meaning of the colours is the following. The vertices of degree 1 are coloured blue; every vertex that is adjacent to a blue vertex is coloured purple; we colour a vertex red if all of its neighbours are coloured purple; and rest of the vertices are coloured green. We denote the set of blue, purple, red and green vertices by \( V_B, V_P, V_R \) and \( V_G \) respectively. We make a simple note that every green vertex has at least one green neighbour. Also it is easy to verify that this is a valid partition, that is, every vertex is coloured with exactly one colour. Based on the colouring, we give some simple reduction rules.

**Reduction UEDS 3** If there is a purple vertex \( p \) with more than one blue neighbour then reduce the number of blue neighbours to one.

Notice that the Reduction UEDS 3 does not influence the set of feasible solutions to the instance \((G, k)\). The fourth rule is based on the following observation.
Suppose $G$ has a green vertex $g$ of degree $\geq 2k$. Suppose $N(g) = \{v_1, \ldots, v_{2k}\}$. Note that each $v_i$ is either a green or purple vertex and hence of degree at least 2. We can easily construct an edge dominating set $M$ with $|M| \geq k$ of $G$ without including any edge incident to $g$.

**Reduction UEDS 4** If $G$ contains a green vertex $g$ of degree greater than or equal to $2k$, then conclude that we are dealing with a yes-instance.

![Illustration of Reduction Rule UEDS 4](image)

Note that $d(g) = 8$ and $d(v_i) \geq 2$ for all $1 \leq i \leq 8$. A minimal edge dominating set of size 6 that does not contain any edge incident to $g$, is shown here in orange colour.

The fifth rule is based on the observation that having $k$ blue vertices implies that there is a matching of size at least $k$. We can construct a minimal edge dominating set of $G$ which contains all the edges of the matching. This implies that we have a yes-instance.

**Reduction UEDS 5** If $G$ contains at least $k$ blue vertices, then conclude that we are dealing with a yes-instance.

Next, we present a reduction rule that applies when $G$ has some red vertices.

**Reduction UEDS 6** If $V_R$ is non-empty, then remove $V_R$ from $G$. The new instance is $(G - V_r, k)$.

We make some important observations before we prove the correctness of this reduction rule. Let $p$ be a purple vertex and for simplicity let $N_B(p) = \{b\}, N_R(p) = \{r_1, r_2\}$ and $N_G(p) = \{g_1, g_2\}$.

**Lemma 2.** For every purple vertex $p$, at least one edge incident to $p$ is included in the solution.

**Proof.** Every purple vertex $p$ is adjacent to a blue vertex $b$. The only way to dominate the edge $(p, b)$ is to include to the solution either the edge itself or another edge incident to $p$. \qed
Lemma 3. If a purple-red edge \((p, r)\) is included in the solution, then no other edges incident to \(p\) can be included in the solution.

Proof. Suppose \((p, r_1)\) is included in the solution, then we show that \((p, r_2)\) cannot be included in the solution. Assume, for the sake of contradiction, that both \((p, r_1)\) and \((p, r_2)\) are included in the solution \(M\). Notice that the edges incident to \(p\) are not private edges, as they are dominated by two edges \((p, r_1)\) and \((p, r_2)\) of \(M\). By Lemma 1, every edge of \(M\) has a private edge. Therefore the private edge of \((p, r_1)\) must be \((r_1, p')\) for some purple vertex \(p' \neq p\). This is impossible, as by Lemma 2, the purple vertex \(p'\) has an edge \(e'\) incident to \(p'\) in the solution. Therefore \((r_1, p')\) is not a private edge and hence \((p, r_1)\) does not have any private edge, a contradiction.

\(\square\)

Lemma 4. If a purple-blue \((p, b)\) edge is included in the solution, then no other edges incident to \(p\) can be included in the solution.

The proof of this lemma is essentially the same as the proof of Lemma 3.

Lemma 5. If a purple-green \((p, g)\) edge is included in the solution, then no other edge of the form purple-red or purple-blue incident to \(p\) can be included in the solution.

Based on the above observations, we will construct a solution \(S'\) from \(S\) such that no edge between a purple vertex and a red vertex is inside the set \(S\) and \(|S'| \geq |S|\). Let us assume that \(S\) contains edges of the form \((v_i, w_i)\) where \(v_i\) is coloured purple and \(w_i\) is coloured red. Due to observations 3 and 4, we know that there is a unique edge adjacent to \(v_i\) inside the solution and that is \((v_i, w_i)\). We replace every edge \((v_i, w_i)\) in \(S\) by an edge \((v_i, z_i)\) where \(z_i\) is a unique blue neighbour of \(v_i\).

Lemma 6. Reduction Rule \(\mathcal{R}\) is safe.

Proof. In one direction, we show that if \(M\) is a solution to \((G, k)\) then \(M'\) is a solution to \((G - V_r, k)\), where \(M'\) is obtained from \(M\) by replacing every purple-red edge \((p, r) \in M\) by the purple-blue edge \((p, b)\). First we prove that \(M'\) is an edge dominating set in \(G - V_r\). Assume, for the sake of contradiction, that \(M'\) is not an edge dominating set in \(G - V_r\), that is, an edge \(e\) in \(G - V_r\) is not dominated by \(M'\) but it was dominated by an edge \(e'\) in \(M\). Clearly, \(e'\) must
be a purple-red edge \((p, r)\). Therefore, one endpoint of \(e\) must be \(p\). Note that \(r\) cannot be an endpoint of \(e\) as it is an edge in \(G - V_r\). As \(e' = (p, r)\) is replaced by \((p, b)\) in \(M'\), and one endpoint of \(e\) is \(p\), \(M'\) dominates \(e\), a contradiction. Therefore, \(M'\) is an edge-dominating set in \(G - V_r\). Now, we claim that \(M'\) is a minimal edge dominating set in \(G - V_r\). Consider an edge \(e\) in \(M' \cap M\).

**Case 1.** Suppose \(e = (p, g)\) where \(p\) is a purple vertex and \(g\) is a green vertex. By Lemma 5, if \(e = (p, g)\) is in the solution, no purple-red \((p, r)\) or purple-blue \((p, b)\) edges can be included in the solution. Then, clearly \((p, b)\) is a private edge of \(e\) and it remains private edge of \(e\) in \(M'\) too.

**Case 2.** Suppose \(e = (p, b)\) where \(p\) is a purple vertex and \(b\) is a blue vertex. By Lemma 4, if the purple-blue \((p, b)\) edge is included in the solution, then no other edges incident to \(p\) can be included in the solution. Note that \(e = (p, b)\) is its own private edge in both \(M\) and \(M'\).

**Case 3.** Suppose \(e = (g_1, g_2) \in M\) or \(e = (p_1, p_2) \in M\). It can be verified that \(e\) will have a private edge in \(M'\) as well.

Next consider an edge \(e \in M' \setminus M\). By construction, \(e\) must be of the form \((p, b)\) which has replaced some purple-red edge \((p, r)\) in \(M\). By Lemma 4 if the purple-blue \((p, b)\) edge is included in the solution, then no other edges incident to \(p\) can be included in the solution. Therefore, no other edge in \(M'\) dominates the edge \((p, b)\) but itself.

For the other direction, let \(M\) be a minimal edge dominating set of \(G - V_R\). We claim that \(M\) is a minimal edge dominating set in \(G\). It is true because we get \(G\) from \(G - V_R\) by introducing some red vertices and some purple-red edges. By Lemma 2 for every purple vertex \(p\), at least one edge incident to \(p\) is included in the solution. Therefore, \(M\) dominates all edges of \(G\) including the newly introduced purple-red edges.

We claim the final reduction rule that explicitly bounds the size of the kernel.

**Reduction UEDS 7** Let \((G, k)\) be an input instance such that Reductions UEDS 1 to UEDS 6 are not applicable to \((G, k)\). If \(G\) has more than \(4k^2 - 2\) vertices, then conclude that we are dealing with a yes instance.

Since we cannot apply Reductions UEDS 1 to UEDS 6 anymore on graph \(G\), we have \(|V(G)| = |V_B| + |V_P| + |V_G|\). By Reduction Rule 5 if \(|V_B| \geq k\), then conclude that we are dealing with a yes-instance. Therefore, we can assume that there are less than \(k\) blue vertices. It also implies that there are less than \(k\) purple vertices.

Next, we note that if \(G\) has a minimal vertex cover of size \(2k\) then \((G, k)\) is a yes-instance. This is true because we can obtain a maximal matching of \(G\) of size greater than or equal to \(k\) by a greedy algorithm. Since the maximal matching is of size more than or equal to \(k\) then we are done as it is also a minimal edge dominating set. The endpoints of the maximal matching forms a vertex cover of \(G\). Now, consider the graph induced by the set of green vertices in \(G\). Since we computed a minimal vertex cover of size less than \(2k\), it implies that \(G[V_G]\) has
a vertex cover $S$ of size less than $2k$. Since we cannot apply Reduction UEDS \(1\) anymore on $G$, $G[V_G]$ has no isolated vertices. Thus every vertex of $G[V_G] - S$ should be adjacent to some vertex from $S$. By Reduction UEDS \(4\), every vertex of $G[V_G]$ has degree less than $2k$. It follows that $|V_G - S| < 2k|S|$ and hence $|V_G| < (2k + 1)|S|$. Since $|S| < 2k$, we have $|V_G| < (2k + 1)2k$. Therefore, we have $|V(G)| = |V_B| + |V_P| + |V_G| \leq (k - 1) + (k - 1) + (2k)(2k - 1) = 4k^2 - 2$, which concludes that the Reduction Rule UEDS \(7\) is safe.

Finally, we remark that all the reduction rules are trivially applicable in linear time. Thus we obtain the following theorem.

**Theorem 1.** Upper Edge Dominating Set parameterized by the solution size $k$ admits a kernel of size $4k^2 - 2$.

## 4 Single exponential time algorithm parameterized by solution size

In this section, we construct a $2^{O(k)}$ running time algorithm using dynamic programming parameterized by solution size. Assume we are given a Upper Edge Dominating Set instance $(G, k)$. We find a maximal matching $M$ in $G$. Note that a maximal matching in $G$ is also a minimal edge dominating set in $G$. If $|M| \geq k$, then we can clearly conclude that $(G, k)$ is a yes-instance, so assume otherwise, that is, $|M| < k$. The endpoints of the edges in $M$ constitute a vertex cover of $G$ of size at most $2k - 2$. As $G$ has a vertex cover of size bounded by $2k - 2$, we can construct a tree decomposition of $G$ with width at most $2k - 1$. Therefore, if Upper Edge Dominating Set can be solved in time $2^{O(\omega)}$ where $\omega$ is the treewidth of the input graph, then it can be solved in time $2^{O(k)}$, where $k$ is the solution size. We now prove the following theorem:

**Fig. 5.** The blue edges shows a minimal edge dominating set of above graph. Note that the edges incident to black vertices are not contained in the solution. Edge $(A, H)$ forms a $K_{1,1}$ and its endpoints are purple. Edges $(C, D)$ and $(D, F)$ forms a $K_{1,2}$ where $D$ is colored green and $C, D$ are colored $r_1$. 
Theorem 2. Given an \( n \)-vertex graph \( G \) and its nice tree decomposition \( T \) of width at most \( \omega \), the size of a maximum Upper Edge Dominating Set of \( G \) can be computed in time \( 45^\omega \cdot n^{O(1)} \).

Proof. Let \( (T, \{X_t\}_{t \in V(T)}) \) be a nice tree decomposition rooted at node \( s \) of the input graph \( G \). For a node \( t \) of \( T \), let \( V_t \) be the union of all bags present in the subtree of \( T \) rooted at \( t \), including \( X_t \). With each node \( t \) of the tree decomposition we associate a subgraph \( G_t = (V_t, E_t) \) where

\[
E_t = \{ e : e \text{ is introduced in the subtree rooted at } t \}.
\]

A coloring of bag \( X_t \) is a mapping \( f : X_t \to \{b, p, g, r_0, r_1\} \) assigning five different colors to vertices of the bag. We give intuition behind the five colours. Suppose \( A_t \subseteq E_t \) is a minimal edge dominating set of \( G_t \). It is easy to observe that subgraph \( (V_t, A_t) \) is the disjoint union of isolated vertices and stars, isomorphic to \( K_{1,r} \) for some \( r \geq 1 \). Isolated vertices are coloured black; endpoints of \( K_{1,1} \) are colored purple; the internal node of a star graph \( K_{1,r} \), with \( r \geq 2 \), is colored green and the \( r \) leaves are colored red. See Fig. ?, which provides an illustration of the colouring.

- **Black**, represented by \( b \). The meaning is that the edges incident to black vertices are not contained in the partial solution in \( G_t \).
- **Purple**, represented by \( p \). If two adjacent vertices \( u \) and \( v \) are coloured purple then \( e = (u, v) \) is contained in the partial solution in \( G_t \) but edges in \( N[e] \) are not contained in the partial solution.
- **Green**, represented by \( g \). For every green vertex \( u \), at least two edges incident to \( u \) are contained in the partial solution.
- **Red**, represented by \( r_0 \) and \( r_1 \). The meaning is that exactly one edge \( (u,v) \) incident to every red vertex \( u \) is contained in the partial solution, where \( v \) has to be a green vertex. Furthermore, vertices coloured \( r_1 \) must have at least one black neighbour where as vertices coloured \( r_0 \) have no black neighbours.

For each node \( t \) of \( T \), we construct a table \( dp_t(f, y, n_r, n_{r1}, n_c, \alpha, \beta) \in \{\text{true, false}\} \) where \( f \) is a colouring of the bag \( X_t \); \( y \) is a vector of length \( n \); \( n_r, n_{r1}, n_c \) are integers between 0 and \( n \); \( \alpha \) and \( \beta \) are integers between 0 and \( m \). The vector \( y \) is of length \( n \) and \( i \)th coordinate of vector \( y \) is

\[
y(i) = \begin{cases} 
0 & \text{if } v_i \text{ is incident to no edges in the partial solution in } G_t \\
1 & \text{if } v_i \text{ is incident to exactly one edge in the partial solution in } G_t \\
2 & \text{if } v_i \text{ is incident to at least two edges in the partial solution in } G_t.
\end{cases}
\]

We use \( n_r \) to denote the number of red vertices in \( G_t \); \( n_{r1} \) to denote the number of red vertices with a black neighbour in \( G_t \); \( n_c \) to denote the number of vertices in \( V_t \setminus X_t \) which satisfies the coloring condition. Note that a black vertex satisfies the coloring condition if the number of edges incident to it from the partial solution in \( G_t \) is zero. A purple or red vertex satisfies the coloring condition if the number of edges incident to it from the partial solution in \( G_t \) is exactly one.
green satisfies the coloring condition if the number of edges incident to it from the partial solution in \( G_t \) is greater than or equal to 2. Finally \( \alpha \) denotes the number of edges in the partial solution in \( G_t \) and \( \beta \) denotes the number of edges such that both the endpoints are colored black. We set \( dp_t(f, y, n_r, n_r, n_c, \alpha, \beta) = \text{true} \) if and only if there exists a subset \( A_t \subseteq E_t \) such that:

1. \( n_r = |\{ v \in V_t : f(v) \in \{ r_0, r_1 \} \}| \)
2. \( n_{r_1} = |\{ v \in V_t : f(v) = r_1 \}| \)
3. \( g(i) = |\{ e \in A_t : e \text{ is incident to } v_i \}| \) for all \( i \).
4. \( n_c \) is number of vertices in \( V_t \setminus X_t \) satisfying the coloring conditions stated above.
5. \( \alpha = |A_t| \)
6. \( \beta = \{(u, v) \in E_t : f(u) = f(v) = b \} \).

Note that the size of a minimal edge dominating set in \( G \) is \( \alpha \) for which we have \( dp_s(\emptyset, y, n_r, n_{r_1}, |V(G)|, \alpha, 0) = \text{true}, n_r = n_{r_1}, n_c = |V(G)| \) and \( \beta = 0 \). This is because we have \( G = G_s \), \( X_s = \emptyset \), which means that for \( X_s \) we have only one colouring, the empty function; \( n_r = n_{r_1} \) because we need every red vertex to have a black neighbour; \( n_c = |V(G)| \) because we want every vertex to satisfy the colouring condition; and finally \( \beta = 0 \) because we do not want to have any edge with both the endpoints coloured black in the final solution.

In the following, we compute all entries \( dp_t(f, y, n_r, n_{r_1}, n_c, \alpha, \beta) \) in a bottom-up manner. There are \( 5^\omega 3^\omega n^\omega m^2 = 15^\omega n^{O(1)} \) possible tuples \( (f, y, n_r, n_{r_1}, n_c, \alpha, \beta) \) at each node \( t \) of \( T \). Thus to prove Theorem 2 it suffices to show that each entry \( dp_t(f, y, n_r, n_{r_1}, n_c, \alpha, \beta) \) can be computed in time \( 3^\omega n^{O(1)} \) time assuming that the entries for the children of \( t \) are already computed.

Now we introduce some notations. Let \( X \subseteq V \) and consider a colouring \( f : X \mapsto \{ b, p, g, r_0, r_1 \} \). For \( \gamma \in \{ 1, 0, 1 \} \) and \( v \in V(G) \) a new colouring \( f_{v \mapsto \gamma} : X \cup \{ v \} \mapsto \{ b, p, g, r_0, r_1 \} \) is defined as follows:

\[
f_{v \mapsto \gamma}(x) = \begin{cases} 
  f(x) & \text{when } x \neq v \\
  \gamma & \text{when } x = v
\end{cases}
\]

Let \( f \) be a colouring of \( X \), then the notation \( f|_Y \) is used to denote the restriction of \( f \) to \( Y \), where \( Y \subseteq X \). We now proceed to present the recursive formulas for the values of \( dp_t(f, y, n_r, n_{r_1}, n_c, \alpha, \beta) \).

**Leaf node:** For a leaf node \( t \) we have that \( X_t = \emptyset \). Hence there is only one empty colouring. Observe that \( dp_t(f, y, n_r, n_{r_1}, n_c, \alpha, \beta) = \text{true} \) if and only if \( f = \emptyset, y(i) = 0 \) for all \( i \), \( n_r = n_{r_1} = n_c = 0 \), \( \alpha = 0 \), and \( \beta = 0 \). These conditions can be checked in \( O(1) \) time.

**Introduce vertex node:** Suppose \( t \) is an introduction node with a child \( t' \) such that \( X_t = X_{t'} \cup \{ v_i \} \) for some \( v_i \notin X_{t'} \). Recall that we have not introduced any edges adjacent to \( v_i \), so \( v_i \) is isolated in \( G_t \). For any colouring \( f \) of \( X_t \), we
consider the following cases:

**Case (i):** Let \( f(v_i) \in \{b, g, p\} \). Then, \( dp_t(f, y, n_r, n_{r_1}, n_c, \alpha, \beta) = \) true if and only if \( dp_t'(f|_{X_v}, y, n_r, n_{r_1}, n_c, \alpha, \beta) = \) true.

**Case (ii):** Let \( f(v_i) = r_0 \). Then, \( dp_t(f, y, n_r, n_{r_1}, n_c, \alpha, \beta) = \) true if and only if \( dp_t(f|_{X_v}, y, n_r - 1, n_{r_1}, n_c, \alpha, \beta) = \) true.

**Case (iii):** Let \( f(v_i) = r_1 \). Then, \( dp_t(f, y, n_r, n_{r_1}, n_c, \alpha, \beta) = \) false. This is because we need to be sure that we do not introduce an isolated vertex with color \( r_1 \); an isolated vertex cannot have a black neighbour.

Therefore, \( dp_t(f, y, n_r, n_{r_1}, n_c, \alpha, \beta) \) can be computed in \( O(1) \) time.

**Introduce edge node:** Suppose \( t \) is an introduction edge node labeled with edge \( v_i v_j \) and let \( t' \) be the child of \( t \). Let \( f \) be a coloring of \( X_t \). We consider two cases:

**Case (i):** Let \( (v_i, v_j) \in A_t \). Then, \( dp_t(f, y, n_r, n_{r_1}, n_c, \alpha, \beta) = \) true if and only if \( dp_t'(f', y', n_r, n'_{r_1}, n_c, \alpha', \beta') = \) true, where

1. \( f(v_i) = r_1, f'(v_i) = r_0 \) and \( f'(v_j) = b; f(v_k) = f'(v_k) \) for all \( k \neq i \).
2. the \( k \)th coordinate of vector \( y \) is
   \[
   y(k) = \begin{cases} 
   y'(k) + 1 & \text{for } k \in \{i, j\} \\
   y'(k) & \text{otherwise}
   \end{cases}
   \]
3. \( n_{r_1} = \begin{cases} 
   n'_{r_1} + 1 & \text{if } f(v_i) = r_0 \text{ and } f(v_j) = b \\
   n'_{r_1} & \text{otherwise}
   \end{cases} \)
4. \( \alpha = \alpha' + 1 \)
5. \( \beta = \begin{cases} 
   \beta' + 1 & \text{if } f(v_i) = b \text{ and } f(v_j) = b \\
   \beta' & \text{otherwise}
   \end{cases} \)

**Case (ii):** Let \( (v_i, v_j) \notin A_t \). In this case \( dp_t(f, y, n_r, n_{r_1}, n_c, \alpha, \beta) = \) true if and only if \( dp_t'(f', y, n_r, n'_{r_1}, n_c, \alpha, \beta) = \) true, where \( f, n_{r_1} \) and \( \beta \) satisfy the above recurrence relation.

Therefore, \( dp_t(f, y, n_r, n_{r_1}, n_c, \alpha, \beta) \) can be computed in \( O(k) \) time.

**Forget node:** Let \( t \) be a forget node with a child \( t' \) such that \( X_t = X_{t'} \setminus \{v_i\} \) for some \( v_i \in X_{t'} \). Then, \( dp_t(f, y, n_r, n_{r_1}, n_c, \alpha, \beta) = \) true if and only if
Since we assume that the number of nodes in a nice tree decomposition is maximum size of a minimal edge dominating set is the maximum upper EDS on graphs with bounded treewidth. We list some problems emerge from the solution size. This algorithm is obtained by constructing a dynamic programming when parameterized by treewidth which also provides a single exponential FPT algorithm parameterized by treewidth.

Therefore, \( dp_1(f, y, n_r, n_c, \alpha, \beta) \) can be computed in \( O(1) \) time.

**Join node:** Let \( t \) be a join node with children \( t_1 \) and \( t_2 \). Recall that \( X_t = X_{t_1} \cup X_{t_2} \). Then, \( dp_1(f, y, n_r, n_c, \alpha, \beta) \) is true if and only if there exist \( (f^1, y^1, n^1_r, n^1_c, \alpha^1, \beta^1) \) and \( (f^2, y^2, n^2_r, n^2_c, \alpha^2, \beta^2) \) such that

\[
dp_1(f^1, y^1, n^1_r, n^1_c, \alpha^1, \beta^1) = dp_2(f^2, y^2, n^2_r, n^2_c, \alpha^2, \beta^2) = \text{true},
\]

1. \( f(v_i) = f^1(v_i) = f^2(v_i) \) for all \( v_i \in X_t \)
2. \( y(i) = y^1(i) + y^2(i) \) for all \( 1 \leq i \leq n \)
3. \( n_r = n^1_r + n^2_r - |\{u \in X_t : f(u) \in \{r_0, r_1, r_2\}\}| \)
4. \( n_c = n^1_c + n^2_c - |\{u \in X_t : f(u) = r_1\}| \)
5. \( \alpha = \alpha^1 + \alpha^2 \)
6. \( \beta = \beta^1 + \beta^2 \)

There are at most \( 3^\omega \) possible pairs for \( (y^1, y^2) \) as \( y^2 \) is uniquely determined by \( y^1, n \) possible pairs for \( (n^1_r, n^2_r) \), \( (n^1_c, n^2_c) \); and \( m \) possible pairs for \( (\alpha^1, \alpha^2) \), and for \( (\beta^1, \beta^2) \). In total there are \( 3^\omega n^{O(1)} \) candidates. Each candidate can be checked in \( O(1) \) time. Therefore, \( dp_1(f, y, n_r, n_c, \alpha, \beta) \) can be computed in \( 3^\omega n^{O(1)} \) time.

Since we assume that the number of nodes in a nice tree decomposition is \( O(kn) \), the algorithm requires \( 45^\omega n^{O(1)} \) time. At the root node \( s \), we look at all records such that \( dp_s(\emptyset, y, n_r, n_c, \alpha, \beta) = \text{true} \), \( n_r = n_{r_1} \), \( n_c = |V(G)| \), \( \beta = 0 \). The maximum size of a minimal edge dominating set is the maximum \( \alpha \) satisfying

\[
dp_s(\emptyset, y, n_r, n_c, |V(G)|, \alpha, 0) = \text{true}.
\]

5 Conclusion

We proved that Upper EDS admits a kernel of size \( 4k^2 - 2 \) on general graphs. We have also provided a single exponential FPT algorithm when parameterized by treewidth which also provides a single exponential FPT algorithm parameterized by solution size. This algorithm is obtained by constructing a dynamic programming on graphs with bounded treewidth. We list some problems emerge from the results here: (1) It remains open whether Upper EDS on general graphs admits a linear kernel or a matching lower bound can be proved, (2) Can we improve the base of a single exponential FPT algorithm parameterized by treewidth?
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