Abstract

We introduce Perelman’s $W$-entropy and prove the $W$-entropy formula along the geodesic flow on the $L^2$-Wasserstein space over compact Riemannian manifolds equipped with Otto’s Riemannian metric, which allows us to recapture a previous result due to Lott and Villani on the displacement convexity of $\text{sEnt} + n s \log s$ on $P^\infty_2(M)$ over Riemannian manifolds with non-negative Ricci curvature. To better understand the similarity between the $W$-entropy formula for the geodesic flow on the Wasserstein space and the $W$-entropy formula for the heat equation of the Witten Laplacian on the underlying manifolds, we introduce the Langevin deformation of flows on the Wasserstein space over Riemannian manifold, which interpolates the gradient flow and the geodesic flow on the Wasserstein space over Riemannian manifolds, and can be regarded as the potential flow of the compressible Euler equation with damping on manifolds. We prove the existence, uniqueness and regularity of the Langevin deformation on the Wasserstein space over the Euclidean space and compact Riemannian manifolds, and prove the convergence of the Langevin deformation for $c \to 0$ and $c \to \infty$ respectively. We prove an analogue of the Perelman type $W$-entropy formula along the Langevin deformation on the Wasserstein space on Riemannian manifolds. A rigidity theorem is proved for the $W$-entropy for the geodesic flow, and a rigidity model is also provided for the Langevin deformation on the Wasserstein space over complete Riemannian manifolds with the $CD(0,m)$-condition.

MSC2010 Classification: primary 53C44, 58J35, 58J65; secondary 60J60, 60H30.

Keywords: Langevin deformation of flows, Ricci curvature, Wasserstein space, $W$-entropy

*Research of S. Li has been supported by NSFC No. 11901569.
†Research of X.-D. Li has been supported by National Key RD Program of China (No. 2020YF0712700), NSFC No. 11771430 and No. 11688101, and Key Laboratory RCSDS, CAS, No. 2008DP173182.
1 Introduction

In his seminal paper [42], Perelman pointed out that the Ricci flow can be realized as the gradient flow of the $F$-functional and proved the monotonicity of its $W$-entropy, which plays a key role in his proof of the Poincaré conjecture. Meanwhile in the field of optimal transport, the infinite dimensional Riemannian geometry and the theory of the gradient flow on the Wasserstein space have been developed by Otto, Lott, McCann, Villani and Sturm [38, 40, 33, 52, 53, 12, 45, 46, 47] among others, which leads to the reveal of the deep relation between the gradient flow and the entropy formula. Moreover, the convexity of the Boltzmann-Shannon entropy or the Rényi entropy along geodesics on the Wasserstein space has been a key tool in Lott-Villani [33, 32, 52, 53] and Sturm [45, 46, 47] to develop a synthesis of comparison geometry on metric measure spaces with the extended notion of the curvature-dimension $\text{CD}(K, m)$-condition. See [1] for the further development on the theory of gradient flows on metric measure spaces.

We start with a brief description of the key concepts of the topic and then present our main results. Let $(M, g)$ be a complete Riemannian manifold equipped with a weighted volume measure $d\mu = e^{-f}d\nu$, where $f \in C^2(M)$ and $d\nu$ denotes the volume measure on $(M, g)$. The Witten Laplacian, which is self-adjoint and non-positively definite on $L^2(M, \mu)$, is defined as follows

$$L := \Delta - \nabla f \cdot \nabla.$$

Let $m > n$ be a constant. Following Bakry and Emery [3], we introduce the $m$-dimensional Bakry-Emery Ricci curvature associated with the Witten Laplacian $L$,

$$\text{Ric}_{m,n}(L) := \text{Ric} + \nabla^2 f - \frac{\nabla f \otimes \nabla f}{m-n}.$$

We also make the convention that when $m = n$, $\text{Ric}_{n,n}(L) = \text{Ric}$ and is only defined when $f$ is identically a constant. By [1][30], when $m > n$ is an integer, $\text{Ric}_{m,n}(L)$ is the horizontal projection of the Ricci curvature on the product manifold $\tilde{M} = M \times N$ with the warped product metric $\tilde{g} = g \otimes e^{-\frac{m}{n}f}g_N$, where $(N, g_N)$ is any $(m-n)$-dimensional complete Riemannian manifold. Following [3], we say that the weighted Riemannian manifold $(M, g, \mu)$ satisfies the $\text{CD}(K, m)$-condition for some constant $K \in \mathbb{R}$ and $m \in [n, \infty]$ if and only if

$$\text{Ric}_{m,n}(L) \geq K.$$

The Boltzmann-Shannon entropy of the probability measure $\rho d\mu$ with respect to the reference measure $\mu$ is defined by

$$\text{Ent}(\rho) := \int_M \rho \log \rho \, d\mu.$$

Let $P_2(M, \mu)$ (resp. $P_2^\infty(M, \mu)$) be the Wasserstein space (resp. the smooth Wasserstein space) of all probability measures $\rho(x) d\mu(x)$ with density function (resp. with
smooth density function \( \rho \) on \( M \) such that \( \int_M d^2(o, x)\rho(x)d\mu(x) < \infty \), where \( d(o, \cdot) \) denotes the distance function from a fixed point \( o \in M \).

By Otto \cite{Otto2001}, the tangent space \( TP_2^\infty(M, \mu) \) is identified as follows

\[
T_{pd\mu}P_2^\infty(M, \mu) = \{ s = \nabla^*_\mu(\rho\nabla \phi) : \phi \in C^\infty(M), \quad \int_M |\nabla \phi|^2 \rho d\mu < \infty \},
\]

where \( \nabla^*_\mu \) denotes the \( L^2 \)-adjoint of the Riemannian gradient \( \nabla \) with respect to the weighted volume measure \( d\mu \) on \((M, g)\). For \( s_i = \nabla^*_\mu(\rho\nabla \phi_i) \in T_{pd\mu}P_2^\infty(M, \mu), \ i \in \{1, 2\} \), Otto \cite{Otto2001} introduced the following infinite dimensional Riemannian metric on \( P_2^\infty(M, \mu) \)

\[
\langle \langle s_1, s_2 \rangle \rangle := \int_M \langle \nabla \phi_1, \nabla \phi_2 \rangle \rho d\mu,
\]

provided that

\[
\|s_i\|^2 := \int_M |\nabla \phi_i|^2 \rho d\mu < \infty, \quad i = 1, 2.
\]

Let \( TP_2(M, \mu) \) be the completion of \( T_{pd\mu}P_2^\infty(M, \mu) \) with Otto’s Riemannian metric. Then \( P_2(M, \mu) \) is a formal infinite dimensional Riemannian manifold.

By Benamou and Brenier \cite{Benamou2000}, for any given \( \mu_0 = \rho_0 d\mu \in P_2(M, \mu), \ i \in \{0, 1\} \), the \( L^2 \)-Wasserstein distance between \( \mu_0 \) and \( \mu_1 \) coincides with the geodesic distance between \( \mu_0 \) and \( \mu_1 \) on \( P_2(M, \mu) \) equipped with Otto’s infinite dimensional Riemannian metric, i.e.,

\[
W_2^2(\mu_0, \mu_1) = \inf \left\{ \frac{1}{2} \int_0^1 |\nabla \phi(x, t)|^2 \rho(x, t) d\mu(x) : \partial_t \rho = \nabla^*_\mu(\rho\nabla \phi), \ \rho(0) = \rho_0, \ \rho(1) = \rho_1 \right\}.
\]

Given \( \mu_0 = \rho(\cdot, 0) d\mu, \mu_1 = \rho(\cdot, 1) d\mu \in P_2^\infty(M, \mu) \), it is known that there is a unique minimizing Wasserstein geodesic \( \{\mu(t), t \in [0, 1]\} \) of the form \( \mu(t) = (F_t)_* \mu_0 \) joining \( \mu_0 \) and \( \mu_1 \) in \( P_2(M, \mu) \), where \( F_t \in \text{Diff}(M) \) is given by \( F_t(x) = \exp_x(-t\nabla \phi(\cdot, 0)) \) for an appropriate Lipschitz function \( \phi(\cdot, t) \) (see \cite{Benamou2000}). If the Wasserstein geodesic in \( P_2(M, \mu) \) belongs entirely to \( P_2^\infty(M, \mu) \), then the geodesic flow \( (\rho, \phi) \in TP_2^\infty(M, \mu) \) satisfies the transport equation and the Hamilton-Jacobi equation

\[
\begin{align*}
\partial_t \rho - \nabla^*_\mu(\rho \nabla \phi) &= 0, \\
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 &= 0,
\end{align*}
\]

with the boundary condition \( \rho(0) = \rho_0 \) and \( \rho(1) = \rho_1 \). When \( \rho_0 \in C^\infty(M, \mathbb{R}^+) \) and \( \phi_0 \in C^\infty(M) \), defining \( \phi(\cdot, t) \in C^\infty(M) \) by the Hopf-Lax solution

\[
\phi(x, t) = \inf_{y \in M} \left( \phi_0(y) + \frac{d^2(x, y)}{2t} \right),
\]

and solving the transport equation \((1.1)\) by the characteristic method, it is known that \((\rho, \phi)\) satisfies \((1.1)\) and \((1.2)\) with \( \rho(0) = \rho_0 \) and \( \phi(0) = \phi_0 \). See \cite{Villani2009} Sect. 5.4.7. See also
In view of this, the transport equation (1.1) and the Hamilton-Jacobi equation (1.2) describe the geodesic flow on the tangent bundle $TP_2^\infty(M,\mu)$ over the Wasserstein space $P_2(M,\mu)$.

Our first result of this paper is the following $W$-entropy formula for the geodesic flow on the Wasserstein space $P_2(M,\mu)$.

**Theorem 1.1.** Let $(M,g)$ be a compact Riemannian manifold or a complete Riemannian manifold with suitable bounded geometry condition, $f \in C^2(M)$, $d\mu = e^{-f}dv$. Let $(\rho,\phi): M \times [0,T] \to \mathbb{R}^+ \times \mathbb{R}$ be a geodesic in $P_2^\infty(M,\mu)$. For any $m \geq n$, define the $H_m$-entropy and $W_m$-entropy for the geodesic flow $(\rho,\phi)$ on $TP_2^\infty(M,\mu)$ as follows

$$H_m(\rho, t) = \text{Ent}(\rho(t)) + \frac{m}{2} \left( 1 + \log(4\pi t^2) \right),$$

and

$$W_m(\rho, t) = \frac{d}{dt}(tH_m(\rho, t)).$$

Then for all $t > 0$, we have

$$\frac{d}{dt}W_m(\rho, t) = t \int_M \left[ |Hess \phi - \frac{g}{t}|^2 + Ric_{m,n}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu + \frac{t}{m-n} \int_M |\nabla \phi \cdot \nabla f - \frac{m-n}{t}|^2 \rho d\mu. \tag{1.4}$$

In particular, if $Ric_{m,n}(L) \geq 0$, then $W_m(\rho, t)$ is increasing in time $t$ along the geodesic flow on $TP_2^\infty(M,\mu)$.

Note that

$$\frac{d}{dt}W_m(\rho, t) = \frac{d^2}{dt^2} \left( t\text{Ent}(\rho(t)) + mt \log t \right).$$

As a corollary of Theorem 1.1, we recapture the following result due to Lott-Villani.

**Corollary 1.2.** (33, 32) Let $M$ be a compact Riemannian manifold. Suppose that $Ric_{m,n}(L) \geq 0$. Then $t\text{Ent}(\rho(t)) + mt \log t$ is convex in time $t$ along the geodesic on $P_2(M,\mu)$.

Recall that, in our previous papers [21, 22, 33, 24, 26], inspired by the work of Perelman [12] and Ni [39], we have proved the following $W$-entropy formula for the heat equation associated with the Witten Laplacian on $(M,g,\mu)$. 

4
Theorem 1.3. (21, 22, 23, 24, 26) Let $M$ be a compact or a complete Riemannian manifold with suitable bounded geometry condition. Let $u$ be a positive solution of the heat equation
\[ \partial_t u = Lu. \] (1.5)
Define the $H_m$-entropy and the $W_m$-entropy as follows
\[ H_m(u, t) = \text{Ent}(u(t)) + \frac{m}{2} (1 + \log(4\pi t)), \]
and
\[ W_m(u, t) = \frac{d}{dt}(tH_m(u, t)). \]
Then
\[ \frac{d}{dt} W_m(u, t) = 2t \int_M \left[ \text{Hess} \log u + \frac{g}{2t} \right]^2 + \text{Ric}_{m,n}(L)(\nabla \log u, \nabla \log u) \] \[ + \frac{2t}{m-n} \int_M \left| \nabla \log u \cdot \nabla f - \frac{m-n}{t} \right|^2 \] \[ \cdot u d\mu. \] (1.6)
In particular, if $\text{Ric}_{m,n}(L) \geq 0$, then $W_m(u, t)$ is decreasing in time $t$ along the heat equation $\partial_t u = Lu$.

As a corollary of Theorem 1.3, we have the following

Corollary 1.4. Let $M$ be a compact or a complete Riemannian manifold with suitable bounded geometry condition. Suppose that $\text{Ric}_{m,n}(L) \geq 0$. Then $t\text{Ent}(u(t)) + \frac{m}{2} t \log t$ is convex in time $t$ along the heat equation $\partial_t u = Lu$.

The $W$-entropy formula (1.6) can be regarded as an analogue of Perelman’s $W$-entropy formula for the Ricci flow, and extends Ni’s $W$-entropy formula for the heat equation of the Laplace-Beltrami operator on Riemannian manifolds with non-negative Ricci curvature [39].

Theorem 1.1 and Theorem 1.3, Corollary 1.2 and Corollary 1.4, have very similar features. We first look at the heat equation case. When $m \in \mathbb{N}$, let $u_m(x, t) = \frac{1}{(4\pi t)^\frac{m}{2}} e^{-\frac{\|x\|^2}{4t}}$ be the heat kernel of the heat equation $\partial_t u = \Delta u$ on $\mathbb{R}^m$. The Boltzmann-Shannon entropy of the Gaussian heat kernel measure $u_m(x, t)dx$ is given by
\[ \text{Ent}(u_m(t)) = -\frac{m}{2} (1 + \log(4\pi t)). \]
Thus the $H_m$-entropy for the heat equation of the Witten Laplacian is given by
\[ H_m(u(t)) = \text{Ent}(u(t)) - \text{Ent}(u_m(t)), \]

\footnotemark{1}

Following Villani [52, 53], we call $H_m(u(t))$ the relative entropy even though it is slightly different from the classical definition of the relative entropy in probability theory.
and the $W_m$-entropy for the heat equation of the Witten Laplacian is given by the Boltzmann entropy formula in statistical mechanics

$$W_m(u,t) := \frac{d}{dt} \left(t[\text{Ent}(u(t)) - \text{Ent}(u_m(t))])\right).$$

(1.7)

This gives a natural probabilistic interpretation of the $W$-entropy for the heat equation of the Witten Laplacian on Riemannian manifolds. See also [22] for the probabilistic interpretation of the Perelman $W$-entropy for the Ricci flow. In [22, 23], Theorem 1.3 has been extended to complete Riemannian manifolds with bounded geometric condition and a rigidity theorem for the $W_m$-entropy has been proved on complete Riemannian manifolds with the $CD(0, m)$-condition. More precisely, we have the following

**Theorem 1.5.** Suppose that the assumption in Theorem 1.3 holds and $\text{Ric}_{m,n}(L) \geq 0$. Then $\frac{d}{dt}W_m(u, t) \geq 0$ on $[0, \infty)$. Moreover, $\frac{d}{dt}W_m(u, t) = 0$ holds at some $t = t_0 > 0$ if and only if $(M, g)$ is isometric to Euclidean space $\mathbb{R}^n$, $m = n$, and $u(x, t) = u_m(x, t)$ is the heat kernel of the heat equation $\partial_t u = \Delta u$ on $\mathbb{R}^m$.

For the geodesic flow on the Wasserstein space, the $W$-entropy formula can be formulated in the same way. When $m \in \mathbb{N}$, it is easy to check that

$$\rho_m(x, t) = \frac{1}{(4\pi t^2)^{m/2}}e^{-\frac{||x||^2}{4t}},$$

(1.8)

$$\phi_m(x, t) = \frac{||x||^2}{2t},$$

(1.9)

is a geodesic flow on $TP^\infty_2(\mathbb{R}^m)$, where $t > 0, x \in \mathbb{R}^m$. Moreover, it serves as the rigidity model. Indeed, the Boltzmann-Shannon entropy of the probability measure $\rho_m(t, x)dx$ is given by

$$\text{Ent}(\rho_m(t)) = -\frac{m}{2}(1 + \log(4\pi t^2)),$$

and the $H_m$-entropy for the geodesic flow on the Wasserstein space $P_2(M, \mu)$ are defined as

$$H_m(\rho(t)) = \text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t)).$$

(1.10)

The Boltzmann formula leads us to introduce

$$W_m(\rho, t) := \frac{d}{dt} \left(t[\text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t))]ight).$$

(1.11)

Similarly to the case of Theorem 1.3, we can extend the $W$-entropy formula (1.4) in Theorem 1.4 to complete Riemannian manifolds with bounded geometry condition. Moreover, a rigidity theorem can be also proved for the $W$-entropy for the geodesic flow on the Wasserstein space $P_2(M, \mu)$ over complete Riemannian manifolds with the $CD(0, m)$-condition. More precisely, we have the following (see also Theorem 3.7 in Section 3)
Theorem 1.6. Under the same condition as in Theorem 1.1, assume that \((\rho, \phi)\) is a smooth solution (with suitable growth condition) to the transport equation (1.1) and the Hamilton-Jacobi equation (1.2), and suppose that \(\text{Ric}_{m,n}(L) \geq 0\). Then \(\frac{d}{dt} W_m(\rho, t) \geq 0\) on \([0, \infty)\). Moreover, \(\frac{d}{dt} W_m(\rho, t) = 0\) holds at some \(t = t_0 > 0\) if and only if \((M, g)\) is isometric to \(\mathbb{R}^n\), \(n = m\), and \((\rho, \phi) = (\rho_n, \phi_m)\).

It is then natural to ask the following question: How to understand the similarity and the difference between the \(W\)-entropy formulas for the heat equation on a Riemannian manifold \(M\) and for the geodesic flow on the Wasserstein space over \(M\)? Can we pass through one of them to another one?

One of possible approaches to answer this question is to use the vanishing viscosity limit method from the heat equation to the Hamilton-Jacobi equation. However, it seems that this approach does not work in our situation.

In this paper, inspired by J.-M.Bismut’s work (see [5, 6]) on the deformation of hypoelliptic Laplacians on the tangent bundle over Riemannian manifolds, we introduce a deformation of geometric flows \((\rho, \phi) : [0, T] \to TP_2(M, \mu)\) by solving the following equations on \(TP_2(M, \mu)\) (the tangent bundle over the Wasserstein space \(P_2(M, \mu)\)),

\[
\begin{align*}
\partial_t \rho - \nabla^\mu \phi &= 0, \quad (1.12) \\
c^2 \left( \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) &= -\phi - \log \rho - 1, \quad (1.13)
\end{align*}
\]

where \(c \geq 0\). We call \((\rho, \phi)\) the Langevin deformation of flows and prove its analogue of \(W\)-entropy formula.

The Langevin deformation of flows interpolates the geodesic flow on the tangent bundle of the Wasserstein space \(P_2(M, \mu)\) and the heat equation of the Witten Laplacian on the underlying manifold \(M\), regarded as the gradient flow of the Boltzmann-Shannon entropy on the Wasserstein space \(P_2(M, \mu)\). Indeed, we can derive the heat equation and the geodesic flow as the limit of (1.12) and (1.13) in a proper sense by taking \(c \to 0\) and \(c \to \infty\) respectively. See the precise statement of this result in Section 5 and its proof in Section 7.

It turns out that the Langevin deformation of flows has a close connection with the compressible Euler equation with damping, such that its wellposedness and the rigorous proof of convergence can be obtained through the results on the corresponding compressible Euler equation with damping. In Section 6, using the Kato-Majda theory of the hyperbolic quasi-linear systems and the Hamilton-Jacobi theory, we prove that, for any given \(c > 0\), there exists \(T = T_c > 0\) such that the Cauchy problem of the system (1.12) and (1.13) has a unique smooth solution \((\rho, \phi) \in C^1([0, T], C^\infty(M, \mathbb{R}^+) \times C^\infty(M))\) with given initial data \((\rho_0, \phi_0) \in C^\infty(M, \mathbb{R}^+) \times C^\infty(M)\).

The following theorem provides us a dissipation formula for the Hamiltonian along the Langevin deformation of flows on \(TP_2(M, \mu)\).
Theorem 1.7. Let \((M,g)\) be a compact Riemannian manifold, \(f \in C^2(M)\) and \(d\mu = e^{-f}dv\). For any \(c \in (0, \infty)\), let \((\phi, \rho)\) be a smooth solution to (1.12) and (1.13). Define the Hamiltonian and the Lagrangian as follows

\[
H(\rho, \phi) = \frac{c^2}{2} \int_M |\nabla \phi|^2 \rho d\mu + \int_M \rho \log \rho d\mu,
\]

\[
L(\rho, \phi) = \frac{c^2}{2} \int_M |\nabla \phi|^2 \rho d\mu - \int_M \rho \log \rho d\mu.
\]

Then

\[
\frac{d}{dt} H(\rho, \phi) = -\int_M |\nabla \phi|^2 \rho d\mu, \tag{1.14}
\]

\[
\frac{d^2}{dt^2} L(\rho, \phi) = 2 \int_M \left[ c^{-2} |\nabla \phi + \nabla \log \rho|^2 + |\text{Hess}\phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu. \tag{1.15}
\]

In particular, \(H\) is always monotonically nonincreasing, and if the \(CD(0, \infty)\)-condition holds, i.e., \(\text{Ric}(L) = \text{Ric} + \nabla^2 f \geq 0\), then \(L(\rho, \phi)\) is convex along the Langevin deformation flow \((\rho, \phi)\) defined by (1.12) and (1.13).

To state the \(W\)-entropy type formula for the Langevin deformation of flows on \(TP_2(M, \mu)\), we need first to introduce the reference model in order to define the relative entropy functional as in (1.10). In Section 6, we prove that there is a special solution to the transport equation (1.12) and the deformed Hamilton-Jacobi equation (1.13) on Euclidean space \((\mathbb{R}^m, dx)\) when \(m \in \mathbb{N}\). More precisely, let \(u : (0, T) \to (0, \infty)\) be a smooth solution to the ODE

\[
 c^2 u'' + u' = \frac{1}{2u}, \tag{1.16}
\]

where \(T > 0\) is the lifetime of the solution \(u\). Let \(\alpha(t) = \frac{u'(t)}{u(t)}\), and let \(\beta(t) \in C((0, T), \mathbb{R})\) be the unique solution to the ODE

\[
c^2 \beta'(t) = -\beta(t) - m \log u(t) - \frac{m}{2} \log(4\pi) + 1,
\]

with any given initial data \(\beta(0) \in \mathbb{R}\). For \(x \in \mathbb{R}^m\) and \(t \in (0, T)\), define

\[
\rho_m(x, t) = \frac{1}{(4\pi u^2(t))^{m/2}} e^{-\frac{\|x\|^2}{4u^2(t)}},
\]

\[
\phi_m(x, t) = \frac{\alpha(t)}{2} \|x\|^2 + \beta(t).
\]

Then \((\rho_m, \phi_m)\) is a smooth solution of (1.12) and (1.13) on \((\mathbb{R}^m, dx)\). The above \((\rho_m, \phi_m)\) can be regarded as a reference model to (1.12) and (1.13).
Theorem 1.8. Let \((M, g)\) be a compact Riemannian manifold, \(f \in C^2(M)\), \(d\mu = e^{-f}dv\). For any \(c \geq 0\), let \(\alpha(t)\) be as above and \((\phi, \rho)\) be a smooth solution to (1.12) and (1.13). Then
\[
\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{1}{c^2} \int_M \frac{|\nabla \rho(t)|^2}{\rho(t)} d\mu + m \alpha^2(t)
\]
\[
= \int_M \left[|\text{Hess} \phi - \alpha(t)g|^2 + \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi)\right] \rho d\mu
\]
\[
+ (m - n) \int_M \left[\alpha(t) + \frac{\nabla \phi \cdot \nabla f}{m - n}\right]^2 \rho d\mu. \tag{1.17}
\]
In particular, if the \(CD(0, m)\)-condition holds, i.e., \(\text{Ric}_{m,n}(L) \geq 0\), we have
\[
\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{1}{c^2} \int_M \frac{|\nabla \rho(t)|^2}{\rho(t)} d\mu + m \alpha^2(t) \geq 0.
\]

Remark 1.9. The limiting cases \(c \to 0\) and \(c \to \infty\) can be specified as follows.

- When \(c = 0\), from (1.13) we have
\[
\phi = \log \rho + 1 = \frac{\delta \text{Ent}(\rho)}{\delta \rho}, \tag{1.18}
\]
which is the \(L^2\)-derivative of the Boltzmann-Shannon entropy \(\text{Ent}\) on \(P_2^\infty(M, \mu)\) equipped with Otto’s infinite dimensional Riemannian metric (38, 52, 53). In this case, \(\rho\) satisfies the heat equation
\[
\partial_t \rho = L\rho. \tag{1.19}
\]
Equivalently, when \(c = 0\), \((\rho, \phi)\) can be regarded as the gradient flow of the Boltzmann-Shannon entropy on \(P_2^\infty(M, \mu)\) equipped with Otto’s infinite dimensional Riemannian metric. In this case, we have \(u(t) = \sqrt{t}\), \(\alpha(t) = -\frac{1}{2t}\), and
\[
\phi_m(x, t) = -\frac{\|x\|^2}{4t} - \frac{m}{2} \log(4\pi t) + 1,
\]
\[
\rho_m(x, t) = \frac{1}{(4\pi t)^{m/2}} e^{-\frac{\|x\|^2}{4t}}.
\]
The second order entropy dissipation formula reads
\[
\frac{d^2}{dt^2} \text{Ent}(\rho(t)) = 2 \int_M \left[|\text{Hess} \log \rho|^2 + \text{Ric}(L)(\nabla \log \rho, \nabla \log \rho)\right] \rho d\mu,
\]
and we have

\[ \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{t^2} = 2 \int_M \left[ |\text{Hess} \log \rho - \frac{g}{2t}|^2 + \text{Ric}_{m,n}(L)(\nabla \log \rho, \nabla \log \rho) \right] \rho d\mu + \frac{1}{m-n} \int_M \left| \nabla \log \rho \cdot \nabla f + \frac{m-n}{2t} \right|^2 \rho d\mu. \] (1.20)

This is an equivalent form of the W-entropy formula in Theorem 1.3.

- When \( c = \infty \), to make the sense of the equation (1.13), \( \rho \) and \( \phi \) must satisfies the transport equation (1.1) and the Hamilton-Jacobi equation (1.2), i.e., \((\rho, \phi)\) is the geodesic flow on the tangent bundle over the Wasserstein space \( H^\infty_2(M, \mu) \). In this case, we have \( u(t) = t \), \( \alpha(t) = \frac{1}{t} \) and

\[ \phi_m(x, t) = \frac{|x|^2}{2t}, \]
\[ \rho_m(x, t) = \frac{1}{(4\pi t^2)^{m/2}} e^{-\frac{|x|^2}{4t}}. \]

The second order entropy dissipation formula reads

\[ \frac{d^2}{dt^2} \text{Ent}(\rho(t)) = \int_M \left[ |\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu, \]

and we have

\[ \frac{d^2}{dt^2} \text{Ent}(\rho(t)) = \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{t^2} = \int_M \left[ |\text{Hess} \phi - \frac{g}{t}|^2 + \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu + \frac{1}{m-n} \int_M \left| \nabla \phi \cdot \nabla f + \frac{m-n}{t} \right|^2 \rho d\mu. \] (1.21)

This is an equivalent form of the W-entropy formula in Theorem 1.7.

In general case \( 0 < c < \infty \), Theorem 1.8 suggests us to introduce a variant of the W-entropy as follows: for any \( 0 < t_0 < t < \infty \),

\[ W_c(\rho(t)) - W_c(\rho(t_0)) := \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{1}{c^2} \text{Ent}(\rho(t)) + 2 \int_{t_0}^t \alpha(s) \frac{d}{ds} \text{Ent}(\rho(s)) ds + \frac{1}{c^2} \int_{t_0}^t \int_M |\nabla \rho(s)|^2 \rho(s) d\mu ds. \] (1.22)
By direct calculation we can verify that
\[ \frac{d}{dt} W(\rho_m(t)) = -m \alpha^2(t). \]

In view of this, Theorem 1.8 can be reformulated as follows

**Theorem 1.10.** Let \( M \) be a compact Riemannian manifold. Under the above notations, we have
\[ \frac{d}{dt} (W_c(\rho(t)) - W_c(\rho_m(t))) = \int_M |\text{Hess} \phi - \alpha(t) g|^2 \rho d\mu + \int_M \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \rho d\mu + \frac{1}{m-n} \int_M |\nabla f \cdot \nabla \phi + (m-n) \alpha(t)|^2 \rho d\mu. \quad (1.23) \]

In particular, if \( \text{Ric}_{m,n}(L) \geq 0 \), then for all \( t > 0 \), we have the comparison inequality
\[ \frac{d}{dt} W_c(\rho(t)) \geq \frac{d}{dt} W_c(\rho_m(t)). \quad (1.24) \]

In particular, when \( m = n \) and \( \mu = v \), we have the following result on compact Riemannian manifolds with standard volume measure.

**Theorem 1.11.** Let \((M, g)\) be a compact Riemannian manifold. For any \( c \geq 0 \), let \( \alpha(t) \) be as above and \((\phi, \rho)\) be a smooth solution to the transport equation and the deformed Hamilton-Jacobi equation
\[ \partial_t \rho + \nabla \cdot (\rho \nabla \phi) = 0, \quad (1.25) \]
\[ c^2 \left( \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) = -\phi - \log \rho - 1. \quad (1.26) \]
\[ \frac{d}{dt} (W_c(\rho(t)) - W_c(\rho_n(t))) = \int_M |\text{Hess} \phi - \alpha(t) g|^2 \rho d\nu + \int_M \text{Ric}(\nabla \phi, \nabla \phi) \rho d\nu. \]

In particular, if \( \text{Ric} \geq 0 \), then for all \( t > 0 \), we have the comparison inequality
\[ \frac{d}{dt} W_c(\rho(t)) \geq \frac{d}{dt} W_c(\rho_n(t)). \]

**Remark 1.12.** The entropy formulas in Theorem 1.1, Theorem 1.3 and Theorem 1.10 suggest that the specific solution \((\rho_m, \phi_m)\) should play as the reference model for the rigidity theorem of the W-entropy on complete Riemannian manifolds with the \( CD(0, m) \)-condition. Similarly to Theorem 1.2 and Theorem 1.6, for any \( c \in (0, \infty) \), we can extend Theorem 1.3 to a class of smooth solutions \((\rho, \phi)\) on weighted complete Riemannian manifolds \((M, g, f)\) with natural bounded geometry condition. We can therefore expect that the following rigidity theorem holds: Let \( M \) be a complete Riemannian manifold with natural bounded geometry condition and with \( CD(0, m) \)-condition, i.e., \( \text{Ric}_{m,n}(L) \geq 0 \). Then \( \frac{d}{dt} W_c(\rho(t)) = \frac{d}{dt} W_c(\rho_n(t)) \) holds at some \( t = t_0 > 0 \), if and only if \( M \) is isometric to \( \mathbb{R}^n \), \( m = n \), \( f \) is a constant, \((\rho, \phi) = (\rho_m, \phi_m)\). To save the length of the paper, we omit the detail of the technical part of the proof of this result and will do it in a future work.
As we have already mentioned, the Langevin deformation of flows converges in a proper sense to the gradient flow of the Boltzmann-Shannon entropy on $TP_2^\infty(M, \mu)$ when $c \to 0$, and converges in a proper sense to the geodesic flow on $TP_2^\infty(M, \mu)$ when $c \to \infty$. In view of this, the $W$-entropy formula (6.105) indeed interpolates the $W$-entropy formula (1.4) for the geodesic flow on $TP_2^\infty(M, \mu)$ and the $W$-entropy formula (1.6) for the heat equation of the Witten Laplacian on $(M, \mu)$.

In [33, 45, 46, 47], Lott-Villani and Sturm proved that the Boltzmann entropy $\text{Ent}$ is $K$-convex along the geodesics on the Wasserstein space $P_2(M, \mu)$ if and only if the $CD(K, \infty)$-condition holds, i.e., $\text{Ric}(L) \geq K$, and the Rényi entropy $S_N(\rho \mu) = -\int_M \rho^{-1/N} d\mu$ is convex along the geodesic on $P_2(M, \mu)$ for all $N \geq m$ if and only if the $CD(0, m)$-condition holds. In view of this, we would like to raise the following conjecture for the characterization of the $CD(0, m)$-condition on complete Riemannian manifolds, which can be regarded as the converse of Corollary 1.2 and Corollary 1.4.

**Conjecture.** Let $(M, g)$ be a compact Riemannian manifold or a complete Riemannian manifold with bounded geometry condition, $f \in C_\infty(M)$ with $\nabla f \in C_\infty^b(M)$. Suppose that the $W_m$-entropy associated to the heat equation of the Witten Laplacian or the optimal transport problem is non-decreasing in $t$. Then the $CD(0, m)$-condition holds, i.e., $\text{Ric}_{m,n}(L) \geq 0$.

Now we mention some related work in the literature. In [36], McCann and Topping proved the contraction property of the $L^2$-Wasserstein distance between solutions of the backward heat equation on closed manifolds equipped with the Ricci flow, which extends a previous result for the Fokker-Planck equation on Euclidean space (see Otto [38]) and on complete Riemannian manifolds with suitable Bakry-Emery curvature condition (see Sturm and von Renesse [49]). See also [50, 51]. In [32], Lott further proved two convexity results of the Boltzmann-Shannon type entropy along the geodesics on the Wasserstein space over closed manifolds equipped with Ricci flow, which are closely related to Perelman’s results on the monotonicity of the $F$ and $W$-entropy functionals for Ricci flow. In [25], the authors extended Lott’s convexity results to the Wasserstein space on compact Riemannian manifolds equipped with Perelman’s Ricci flow.

The paper is organized as follows. In Section 2, we recall some elementary facts about the infinite dimensional Riemannian geometry on the Wasserstein space over Riemannian manifolds. In Section 3 we introduce the $W$-entropy for the geodesic flow on the Wasserstein space and prove the $W$-entropy formula (1.4) for the geodesic flow, i.e., Theorem 1.1. A rigidity theorem is proved. In Section 4 we introduce the Langevin deformation of geometric flows on the tangent bundle of a complete Riemannian manifold. In Section 5 we introduce the Langevin deformation of geometric flows on the Wasserstein space over Riemannian manifolds, and prove its corresponding $W$-entropy formula in Section 6. In Section 7, we provide the rigorous proof of the convergence of Langevin deformation for $c \to 0$ and $c \to \infty$ respectively.

To end this section, let us mention that the previous version [28] of the present paper has been posted on arXiv (arxiv1604.02596, 2016) but has not been submitted, while
Theorem 5.8 and its proof in Section 7 are new. Theorem 1.1 has been also stated (without giving the proof) and the Langevin deformation of flows (with the positive sign in front of $\nabla V$) has been introduced in our previous paper [27].

2 Otto’s calculus on Wassertsien space over weighted Riemannian manifolds

Let $(M, g)$ be a complete Riemannian manifold, $f \in C^2(M)$, and $d\mu = e^{-f}d\nu$, where $d\nu$ denotes the volume measure on $(M, g)$. Integration by parts formula shows that, for all $u \in C^\infty_0(M)$ and $X \in C^\infty_0(M, T M)$, we have

$$\int_M \langle X, \nabla u \rangle d\mu = \int_M \nabla^*_\mu X u d\mu,$$

where $\nabla^*_\mu$ denotes the $L^2$-adjoint of $\nabla$ with respect to $\mu$, and is given by

$$\nabla^*_\mu X = -\nabla \cdot X + \langle \nabla f, X \rangle.$$

The Witten Laplacian on $(M, g)$ with respect to $\mu$ is defined by

$$L = -\nabla^*_\mu \nabla.$$

More precisely, we have

$$L = \Delta - \nabla f \cdot \nabla.$$

By Bakry-Emery [3], the Bochner-Weitzenböck formula holds

$$L|\nabla u|^2 - 2\langle \nabla u, \nabla Lu \rangle = 2|\text{Hess} u|^2 + 2\text{Ric}(L)(\nabla u, \nabla u), \quad \forall u \in C^\infty(M),$$

where

$$\text{Ric}(L) := \text{Ric} + \nabla^2 f$$

is the infinite dimensional Bakry-Emery Ricci curvature associated with the Witten Laplacian $L$. Following [3], we say that $CD(K, \infty)$-condition holds if and only if $\text{Ric}(L) \geq K$.

Let

$$P^\infty(M, \mu) = \{\rho d\mu : \rho \in C^\infty(M), \rho \geq 0, \int_M \rho d\mu = 1\}.$$ 

For all $\rho d\mu \in P^\infty(M, \mu)$, the tangent space at $\rho d\mu$ is given by

$$T_{\rho d\mu}P^\infty(M, \mu) = \{s \in C^\infty(M) : \int_M s d\mu = 0\}.$$
By solving the Poisson equation $-\rho L\phi + \nabla \phi \cdot \nabla \rho = s$, there exists a unique function $\phi \in C^\infty(M)$ (up to a constant) such that

\[ s = V_\phi := \nabla_\mu^*(\rho \nabla \phi). \]

Following [32], $V_\phi$ can be identified as the vector field on $P_2^\infty(M, \mu)$ defined by

\[ (V_\phi F)(\rho d\mu) = \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} F(\rho d\mu + \varepsilon \nabla_\mu^*(\rho \nabla \phi) d\mu), \]

where $F \in C^\infty(P_2^\infty(M), \mathbb{R})$.

Let $P_2^\infty(M, \mu)$ be the Wasserstein space of probability measures $\rho d\mu \in P^\infty(M)$ with finite second moment

\[ \int_M d^2(o, \cdot) \rho d\mu < \infty, \]

where $o \in M$ is a fixed point. Similarly to Otto [35], we can introduce the infinite dimensional Riemannian metric on $T_{\rho d\mu}P_2^\infty(M, \mu)$ as follows

\[ \langle \langle s_1, s_2 \rangle \rangle = \int_M \langle \nabla \phi_1, \nabla \phi_2 \rangle \rho d\mu, \quad \forall s_i = V_{\phi_i} \in T_{\rho d\mu}P_2^\infty(M, \mu), \ i = 1, 2. \]

The tangent space of $P_2(M, \mu)$ at $\rho d\mu$, denoted by $T_{\rho d\mu}P_2(M, \mu)$, is defined as the completion of $T_{\rho d\mu}P_2^\infty(M, \mu)$ with respect to the norm $\|s\|^2 := \int_M |\nabla \phi|^2 \rho d\mu$ for $s = \nabla_\mu^* \cdot (\rho \nabla \phi)$.

The Wasserstein distance between $\mu_1$ and $\mu_2$ is defined by

\[ W_2^2(\mu_1, \mu_2) = \inf_\pi \int_{M \times M} d^2(x, y) d\pi(x, y), \]

where $\pi \in \prod(\mu_1, \mu_2)$, i.e., $\pi$ is a probability measure on $M \times M$ such that

\[ \int_M \pi(\cdot, d\mu_1) = \mu_1, \quad \int_M \pi(d\mu_2, \cdot) = \mu_2. \]

The following result extends Benamou and Brenier’s result [8] from Euclidean space to complete Riemannian manifolds.

**Theorem 2.1.** Let $(M, g)$ be a complete Riemannian manifold. Let $\mu_0 = \rho_0 d\mu$, $\mu_1 = \rho_1 d\mu$ be two probability measures in $P_2(M, \mu)$. Then

\[ W_2^2(\mu_0, \mu_1) = \inf \int_M \int_0^1 |\nabla \phi(x, t)|^2 \rho(x, t) d\mu(x) dt, \]

where $\rho : M \times [0, 1] \to [0, \infty)$ and $\phi : M \times [0, 1] \to \mathbb{R}$ satisfy

\[ \partial_t \rho - \nabla_\mu^*(\rho \nabla \phi) = 0, \quad (2.27) \]

\[ \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0, \quad (2.28) \]

\[ \rho(\cdot, 0) = \rho_0, \quad \rho(\cdot, 1) = \rho_1. \quad (2.29) \]
Proof. — The proof is analogue of the one in [8].

The function \( \phi \) in Theorem 2.1 is called the potential function, and \( v = \nabla \phi \) can be considered as the velocity of the curve \( \rho(\cdot, t) d\mu \) in \( P^\infty_2(M, \mu) \). The transport equation (2.27) and the Hamilton-Jacobi equation (2.28) describe the geodesic \( \rho_0 d\mu \) which links \( \rho_0 d\mu \) and \( \rho_1 d\mu \) in \( P^\infty_2(M, \mu) \) equipped with Otto’s infinite dimensional Riemannian metric.

Following Lott [32], we can prove the entropy dissipation formula along the geodesics on the Wasserstein space over a compact Riemannian manifold with weighted volume measure. When the potential function is constant, it is due to Lott [32]. To save length of the paper we omit the details of the proof.

Proposition 2.2. Let \((M, g)\) be a compact Riemannian manifold, \( f \in C^2(M) \). Let \( \rho : [0, 1] \times M \to \mathbb{R} \) be a positive solution of the transport equation
\[
\partial_t \rho - \nabla^*_\mu(\rho \nabla \phi) = 0.
\tag{2.30}
\]
Let \( \text{Ent}(\rho(t)) := \int_M \rho \log \rho d\mu \), then we have
\[
\frac{d}{dt} \text{Ent}(\rho(t)) = \int_M \langle \nabla \rho, \nabla \phi \rangle d\mu,
\tag{2.31}
\]
and
\[
\frac{d^2}{dt^2} \text{Ent}(\rho(t)) = -\int_M L \rho (\partial_t \phi + \frac{1}{2} |\nabla \phi|^2) d\mu + \int_M (|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho d\mu.
\tag{2.32}
\]

As a direct consequence of Proposition 2.2, we have the following

Theorem 2.3. For the geodesic \((\rho, \phi)\) on \( P^\infty_2(M, \mu) \), we have
\[
\frac{d^2}{dt^2} \text{Ent}(\rho(t)) = \int_M (|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho d\mu.
\]

In view of Theorem 2.3 the Hessian of the Boltzmann-Shannon entropy functional \( \text{Ent} \) on \( P^\infty_2(M, \mu) \) equipped with Otto’s infinite dimensional Riemannian metric is given by
\[
\text{Hess}_{P^\infty_2(M, \mu)} \text{Ent}(V_\phi, V_\phi) = \int_M (|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho d\mu.
\tag{2.33}
\]

As a corollary of Theorem 2.3, we have the following result due to Lott-Villani [33, 32], Sturm-von Renesse [16] and Sturm [47].

Corollary 2.4. If \( \text{Ric}(L) \geq 0 \), then \( \frac{d^2}{dt^2} \text{Ent}(\rho(t)) \geq 0 \), i.e., \( \text{Ent} \) is convex along geodesic in \( P^\infty_2(M, \mu) \).
3  The $W$-entropy formula for the geodesic flow on Wasserstein space

In this section, we introduce the $W$-entropy and prove its variational formula along the
gedesic flow on the Wasserstein space over compact Riemannian manifolds with weighted
volume measure, i.e., Theorem 1.1. We will also compare the $W$-entropy formula in
Theorem 1.1 with the $W$-entropy formula for the heat equation of the Witten Laplacian
on compact Riemannian manifolds (i.e., Theorem 1.3), and then introduce the $W$-entropy
for the optimal transport problem on compact or complete Riemannian manifolds with
weighted volume measure.

3.1  Proof of Theorem 1.1

Proof. — By Theorem 2.3, we have

$$
\frac{d}{dt} W_m(\rho, t) = t \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + 2 \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{t} \\
= t \int_M (|\text{Hess } \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho d\mu - 2 \int_M L \phi \rho d\mu + \frac{m}{t} \\
= t \int_M \left[ |\text{Hess } \phi - \frac{g}{t}|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu + 2 \int_M \nabla \phi \cdot \nabla f \rho d\mu + \frac{m - n}{t}. 
$$

Note that

$$
\text{Ric}(L)(\nabla \phi, \nabla \phi) + \frac{2}{t} \nabla \phi \cdot \nabla f + \frac{m - n}{t^2} \\
= \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) + \frac{1}{m - n} \left| \nabla \phi \cdot \nabla f + \frac{m - n}{t} \right|^2 .
$$

Thus

$$
\frac{d}{dt} W_m(\rho, t) \\
= t \int_M \left[ |\text{Hess } \phi - \frac{g}{t}|^2 + \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu + \frac{t}{m - n} \int_M \left| \nabla \phi \cdot \nabla f + \frac{m - n}{t} \right|^2 \rho d\mu.
$$

This proves the $W$-entropy formula (1.4) in Theorem 1.1.

As a corollary of Theorem 1.1, we can recapture the following result due to Lott-Villani [33]. See also Lott [32].

**Corollary 3.1.** (i.e., Corollary 1.2) If $\text{Ric}_{m,n}(L) \geq 0$, then $t \text{Ent} + mt \log t$ is convex in $t$
along the geodesic in $P_2^\infty(M, \mu)$.
3.2 The $W$-entropy for the geodesic flow on Wasserstein space

Let $\rho_n(t, x) = \frac{e^{-\|x\|^2/(4\pi t^2)}}{(4\pi t^2)^{n/2}}$ be the special solution of the transport equation (2.27) with the velocity $\phi_n(t, x) = \frac{|x|^2}{2t}$ on the Euclidean space $\mathbb{R}^n$. By calculus, the Boltzmann-Shannon entropy of $\rho_n(t, x)dx$ with respect to the Lebesgue measure on $\mathbb{R}^n$ is given by

$$\text{Ent}(\rho_n(t)) = -\frac{n}{2}(1 + \log(4\pi t^2)).$$

Let $(\rho(t), \phi(t))$ be smooth solution to (1.1) and (1.2). Define the Boltzmann-Shannon entropy

$$\text{Ent}(\rho(t)) = \int_M \rho \log \rho d\mu,$$  

(3.34)

By Theorem 2.3 we have

$$\frac{d}{dt}\text{Ent}(\rho(t)) = -\int_M \langle \nabla \rho, \nabla \phi \rangle d\mu = \int_M L\phi \rho d\mu,$$  

(3.35)

$$\frac{d^2}{dt^2}\text{Ent}(\rho(t)) = -\int_M \left[ |\nabla^2 \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu.$$  

(3.36)

Following Perelman [42], we introduce the $W$-entropy for the geodesic flow $(\rho, \phi)$ on $TP_2(M, \mu)$ as

$$W_m(\rho, t) := \frac{d}{dt} (tH_m(\rho, t)),$$  

(3.37)

where

$$H_m(\rho, t) = \text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t))$$  

(3.38)

is the difference between the Boltzmann-Shannon entropy of the probability measure $\rho d\mu$ with respect to the weighted volume measure $\mu$ on $(M, g)$ and the Boltzmann-Shannon entropy of the probability measure $\rho_m(t, x)dx$ with respect to the Lebesgue measure $dx$ on $\mathbb{R}^m$.

Substituting (3.35) into (3.38) and (3.37) we have

$$W_m(\rho, t) = \int_M [tL\phi - \log \rho - \text{Ent}(\rho_m(t))] \rho d\mu.$$  

(3.39)

Moreover, we can reformulate Theorem 1.1 as follows

$$\frac{d}{dt}W_m(\rho, t) = t \int_M \left[ \left| \text{Hess} \phi - \frac{g}{t} \right|^2 + \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu$$

$$+ \frac{t}{m-n} \int_M \left| \nabla \phi \cdot \nabla f + \frac{m-n}{t} \right|^2 \rho d\mu.$$  

In view of this, Theorem 1.1 can be reformulated as follows.
**Theorem 3.2.** Let $M$ be a compact Riemannian manifold. Let $(\rho, \phi)$ be a smooth geodesic flow on $TP_2^\infty(M, \mu)$, i.e., $\rho$ is a smooth solution to the transport equation (1.1) and $\phi$ is a smooth solution to the Hamilton-Jacobi equation (1.2). Let

$$W_m(\rho, t) = \frac{d}{dt} (t[\text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t))])$$

be the $W$-entropy associated to the optimal transport on $(M, g, \mu)$. Then

$$\frac{d}{dt}W_m(\rho, t) = t \int_M \left[ \text{Hess} \phi - \frac{g}{t} \right]^2 + \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \rho d\mu$$

$$+ \frac{t}{m-n} \int_M \left| \nabla \phi \cdot \nabla f - \frac{m-n}{t} \right|^2 \rho d\mu.$$

In particular, if $\text{Ric}_{m,n}(L) \geq 0$, then the Helmholtz free energy $S_m = t(\text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t)))$ associated with (1.1) and (1.2) is convex in time $t$.

**Corollary 3.3.** (i.e., Corollary [33, 32]) Let $M$ be a compact Riemannian manifold. Suppose that $\text{Ric}_{m,n}(L) \geq 0$. Then $t\text{Ent}(\rho(t)) + mt \log t$ is convex in time $t$ along the geodesic flow $(\rho(t), \phi(t))$ on $TP_2(M, \mu)$.

**Proof.** — By Theorem 1.1 if $\text{Ric}_{m,n}(L) \geq 0$, $t(\text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t))) = t\text{Ent}(\rho(t)) + \frac{mt}{2} \log (4\pi t^2) + 1$ is convex in $t$ along the geodesic $\rho(t)$ on $P_2^\infty(M, \mu)$. Note that

$$\frac{d^2}{dt^2} (t(\text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t)))) = \frac{d^2}{dt^2} (t\text{Ent}(\rho(t)) + mt \log t).$$

Hence $t\text{Ent} + mt \log t$ is convex in $t$ along the geodesic $\rho(t)$ on $P_2^\infty(M, \mu)$. For the general case of non smooth geodesic on $P_2(M, \mu)$, see Lott [32].

In particular, taking $f = 0$, $m = n$ and $L = \Delta$, we have the following

**Theorem 3.4.** Let $M$ be a compact Riemannian manifold. Let $\phi$ and $\rho$ be a smooth solution to the Hamilton-Jacobi equation and the transport equation

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0,$$

$$\partial_t \rho + \nabla \cdot (\rho \nabla \phi) = 0.$$  \hspace{1cm} (3.40) \hspace{1cm} (3.41)

Let

$$W_n(\rho, t) = \frac{d}{dt} (t[\text{Ent}(\rho(t)) - \text{Ent}(\rho_n(t))]).$$

Then

$$\frac{d}{dt}W_n(\rho, t) = t \int M \left[ \text{Hess} \phi - \frac{g}{t} \right]^2 + \text{Ric}(\nabla \phi, \nabla \phi) \rho d\mu.$$  \hspace{1cm} (3.42)

In particular, if $\text{Ric} \geq 0$, then the $W$-entropy $W_n$ associated with (3.40) and (3.41) is increasing in time $t$, and $t\text{Ent} + nt \log t$ is convex in $t$ along the geodesic flow $(\rho(t), \phi(t))$ on $TP_2(M, \nu)$. 

18
3.3 The case of complete noncompact Riemannian manifolds

To extend Theorem 1.1 to complete Riemannian manifolds with bounded geometry condition, we need the following

Theorem 3.5. Let \( M \) be a complete Riemannian manifold, and \( f \in C^2(M) \). Suppose that \( \text{Ric}(L) = \text{Ric} + \nabla^2 f \) is uniformly bounded on \( M \), i.e., there exists a constant \( C > 0 \) such that \( |\text{Ric}(L)| \leq C \). Let \( \rho \) and \( \phi \) be smooth solutions to the transport equation (1.1) and the Hamilton-Jacobi equation (1.2), and satisfying the following growth condition

\[
\int_M \left[ |\nabla \log \rho|^2 + |\nabla \phi|^2 + |\nabla^2 \phi|^2 + |L \phi|^2 + |\nabla L \phi|^2 \right] \rho d\mu < \infty,
\]

and there exist a point \( o \in M \), and some functions \( C_i \in C([0, T], \mathbb{R}^+) \) and \( \alpha_i \in C([0, T], \mathbb{R}^+) \) such that

\[
C_1(t)e^{-\alpha_1(t)d^2(x,o)} \leq \rho(x, t) \leq C_2(t)e^{\alpha_2(t)d^2(x,o)}, \quad \forall x \in M, t \in [0, T],
\]

and

\[
\int_M d^4(x,o)\rho(x,t)d\mu < \infty, \quad \forall t \in [0, T].
\]

Then the entropy dissipation formulas hold

\[
\partial_t \int_M \rho \log \rho \, d\mu = \int_M \nabla \phi \cdot \nabla \rho \, d\mu = -\int_M L \phi \rho \, d\mu,
\]

\[
\partial_t^2 \int_M \rho \log \rho \, d\mu = -\int_M \|\nabla^2 \phi^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)\| \rho \, d\mu.
\]

Proof. — Let \( \eta_k \) be an increasing sequence of functions in \( C_0^\infty(M) \) such that \( 0 \leq \eta_k \leq 1 \), \( \eta_k = 1 \) on \( B(o,k) \), \( \eta_k = 0 \) on \( M \setminus B(o,2k) \), and \( \|\nabla \eta_k\| \leq \frac{1}{k} \). By standard argument and integration by parts, we have

\[
\partial_t \int_M \rho \log \rho \eta_k \, d\mu = \int_M \nabla^* (\rho \nabla \phi)(\log \rho + 1) \eta_k \, d\mu
\]

\[
= \int_M \rho \nabla \phi \cdot \nabla \log \rho \eta_k \, d\mu + \int_M \rho \nabla \phi \cdot (\log \rho + 1) \nabla \eta_k \, d\mu
\]

\[
=: I_1 + I_2.
\]

Under the assumption of theorem, the Lebesgue dominated convergence theorem yields

\[
I_1 = \int_M \rho \nabla \phi \cdot \nabla \log \rho \eta_k \, d\mu \to \int_M \nabla \phi \cdot \nabla \rho \, d\mu.
\]

and

\[
I_1 = \int_M \nabla^* (\eta_k \nabla \phi) \rho \, d\mu = -\int_M \eta_k L \phi \rho \, d\mu + \int_M \nabla \eta_k \cdot \nabla \phi \rho \, d\mu \to -\int_M L \phi \rho.
\]
Thus
\[ \int_M \nabla \phi \cdot \nabla \rho d\mu = - \int_M L \rho d\mu. \]

On the other hand, since \( \int_M |\nabla \phi|^2 \rho d\mu < \infty \) and \( \int_M |\log \rho + 1|^2 \rho d\mu < \infty \), we have \( \int_M |\log \rho + 1| |\nabla \phi| \rho d\mu < \infty \). By Lebesgue dominated convergence theorem, we have
\[ I_2 = \int_M \rho \nabla \phi \cdot (\log \rho + 1) \nabla \eta d\mu \to 0. \]

This proves that
\[ \partial_t \int_M \rho \log \rho \mu = \int_M \nabla \phi \cdot \nabla \rho d\mu = - \int_M L \rho d\mu. \]

By standard argument, we have
\[
\partial_t \int_M L \rho \eta_k d\mu = \int_M L \left( \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) \rho \eta_k d\mu - \frac{1}{2} \int_M L |\nabla \phi|^2 \rho \eta_k d\mu + \int_M L \phi \partial_t \rho \eta_k d\mu
\]
\[ = -\frac{1}{2} \int_M L |\nabla \phi|^2 \rho \eta_k d\mu + \int_M L \phi \nabla_k^L (\rho \nabla \phi) \eta_k d\mu
\]
\[ = I_3 + I_4. \]

By the weighted Bochner formula and \( |\text{Ric}(L)| \leq C \), we have
\[ \int_M |L| |\nabla \phi|^2 |\rho d\mu = 2 \int_M |\nabla \phi \cdot \nabla L \phi + |\nabla^2 \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)| \rho d\mu
\]
\[ \leq 2 \int_M \left[ |\nabla \phi| |\nabla L \phi| + |\nabla^2 \phi|^2 + C \right] |\nabla \phi|^2 | \rho d\mu
\]
\[ < \infty \]
provided that
\[ \int_M [|\nabla \phi|^2 + |\nabla L \phi|^2 + |\nabla^2 \phi|^2] \rho d\mu < \infty. \]

Thus
\[ I_3 \to -\frac{1}{2} \int_M L |\nabla \phi|^2 \rho d\mu. \]
On the other hand, as $\int_M [ |L\phi|^2 + |\nabla\phi|^2 + |\nabla L\phi|^2 ] \rho d\mu < \infty$, we have

$$I_4 = \int_M L\phi \nabla^*_\mu (\rho \nabla \phi) \eta_k d\mu = \int_M \nabla (\eta_k L\phi) \cdot \rho \nabla \phi d\mu = \int_M [ L\phi \nabla \eta_k \cdot \nabla \phi + \eta_k \nabla L\phi \cdot \nabla \phi ] \rho d\mu \to \int_M \nabla L\phi \cdot \nabla \phi \rho d\mu.$$

In summary, under the assumption of theorem, we have

$$\partial_t \int_M L\phi \rho d\mu = - \frac{1}{2} \int_M L |\nabla \phi|^2 \rho d\mu + \int_M \nabla L\phi \cdot \nabla \phi \rho d\mu = - \int_M [ |\nabla^2 \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi) ] \rho d\mu.$$

This finishes the proof of Theorem 3.5.

Based on Theorem 3.5, Theorem 1.1 can be extended to complete Riemannian manifolds as follows.

**Theorem 3.6.** Let $M$ be a complete Riemannian manifold with bounded geometry condition. Under the same condition as in Theorem 3.5, the $W$-entropy formula in Theorem 1.1 remains true.

**Proof.** — The proof is similar to the one of Theorem 1.1.

### 3.4 The rigidity theorem for the $W$-entropy formula

Similarly to the monotonicity and rigidity theorem (i.e., Theorem 1.5) of the $W$-entropy for the heat equation associated with the Witten Laplacian over complete Riemannian manifolds with bounded geometry condition, which is the gradient flow of the Boltzmann-Shannon entropy on the Wasserstein space, we have the following monotonicity and rigidity theorem of the $W$-entropy for the geodesic flow on the Wasserstein space over complete Riemannian manifolds with bounded geometry condition.

**Theorem 3.7.** Let $M$ be a complete Riemannian manifold with bounded geometry condition and with $\text{Ric}_{m,n}(L) \geq 0$. Suppose that $(\rho, \phi)$ is a smooth geodesic flow on $TP^\infty_2 (M, \mu)$, i.e., $(\rho(t), \phi(t), t \in [0, \infty))$ is a smooth solution to the transport equation (1.1) and the Hamilton-Jacobi equation (1.2) satisfying the growth condition as required in Theorem 3.5. Then $W_m(\rho, t)$ is increasing in $t$, and $t \text{Ent}(\rho(t)) + mt \log t$ is convex in $t$. Moreover,
\[ \frac{d}{dt} W_m(\rho, t) = 0 \] at some \( t = \tau > 0 \) if and only if \( M \) is isometric to \( \mathbb{R}^n, n = m, \) and \((\rho(t), \phi(t)) = (\rho_n(t), \phi_n(t))\) for all \( t \geq 0. \) That is to say, the Euclidean space \( \mathbb{R}^n \) equipped with the Gaussian measure \( N(0, t^2) \) is the rigidity model for the \( W \)-entropy associated with the geodesic flow on \( TP_\infty^\infty(M, \mu) \) with natural growth condition at infinity over complete Riemannian manifold with bounded geometry condition and with \( \text{Ric}_{m,n}(L) \geq 0. \)

**Proof.** — The proof is as the same as the one of Theorem 2.5 in [22] (i.e., Theorem 1.6 as stated in Section 1). For the convenience of the reader, we give the detail here. Indeed, by the explicit expression of \( \frac{d}{dt} W_m(\rho, t) \) in Theorem 1.1, we see that \( \frac{d}{dt} W_m(\rho, t) = 0 \) holds at some \( t = \tau \) if and only if \( \nabla^2 \phi(x, \tau) = g_{\tau}, \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) = 0 \) and \( \nabla \phi \cdot \nabla f + \frac{m-n}{\tau} = 0. \)

Thus, \( \phi \) is a strict convex function on \( M, \) which implies that \( M \) is diffeomorphic to \( \mathbb{R}^n. \)

Integrating along the shortest geodesics between \( x_0 \) and \( x \) on \( M \) shows that

\[ 2\tau(\phi(x, \tau) - \phi(x_0, \tau)) = r^2(x_0, x), \quad \forall x \in M, \]

where \( x_0 \) is the minimum point of \( \phi(\cdot, \tau). \) This yields

\[ \Delta r^2(x_0, x) = 2n, \quad \forall x \in M, \]

which implies that \((M, g)\) is isometric to the Euclidean space \((\mathbb{R}^n, (\delta_{ij})).\) By the generalized Cheeger-Gromoll splitting theorem (see Theorem 1.3, p. 565, [6]), we can derive that \( f \) must be a constant and \( m = n. \)

Thus \( \phi(\cdot, \tau) \in C^\infty(\mathbb{R}^n) \) satisfies \( \nabla^2 \phi(x, \tau) = \frac{\delta_{ij}}{\tau}. \) This yields \( \nabla \phi(x, \tau) = \frac{\|x\|^2}{2\tau} \) up to an additional constant. By the Hopf-Lax formula for the solution to the Hamilton-Jacobi equation (2.28) with \( \phi(x, \tau) = \frac{\|x\|^2}{2\tau}, \)

we have

\[ \phi(t, x) = \inf_{y \in \mathbb{R}^n} \left\{ \frac{\|x\|^2}{2\tau} + \frac{\|x - y\|^2}{2(t - \tau)} \right\} = \frac{\|x\|^2}{2t}, \quad \forall t > \tau, \ x \in \mathbb{R}^n. \]

By the uniqueness of the smooth solution to the Hamilton-Jacobi equation (2.28), we see that \( \phi(x, t) = \frac{\|x\|^2}{2t} \) for all \( t > 0. \) Solving the transport equation (2.27) with the initial data \( \lim_{t \to 0} \rho(t, x) = \delta_0(x), \)

we have

\[ \rho(x, t) = \frac{e^{-\frac{\|x\|^2}{4t^2}}}{(4\pi t^2)^{n/2}}, \quad \forall t > 0, \ x \in \mathbb{R}^n. \]

This finishes the proof of Theorem 3.7.

**3.5 Comparison between Theorem 1.1 and Theorem 1.3**

Theorem 1.1 has the same feature as Theorem 1.3. Their proofs are also quite similar. Both of them are based on the dissipation formulas of the first order and the second order...
derivatives of the Boltzmann-Shannon entropy along the heat equation (i.e., the gradient flow of the Boltzmann-Shannon entropy) and the geodesic flow on the Wasserstein space $P_2^\infty(M, \mu)$.

Theorem 1.1 and Theorem 1.3 lead us to the following observation: On compact Riemannian manifolds $(M, g)$ with the weighted measure $d\mu = e^{-f}d\nu$, if the $\text{CD}(0, m)$-condition holds, then the relative Boltzmann-Shannon entropy $H_m(\rho, t) = \text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t))$ is convex along the geodesic flow on the Wasserstein space $P_2^\infty(M, \mu)$, and the relative Boltzmann-Shannon entropy $H_m(u, t) = \text{Ent}(u(t)) - \text{Ent}(u_m(t))$ is convex along the backward gradient flow of $\text{Ent}(u) = -\int_M u \log u d\mu$ on the Wasserstein space $P_2^\infty(M, \mu)$. This leads us to raise the question whether there is an essential reason for which the Boltzmann-Shannon entropy share the same convexity property along the geodesic flow and the gradient flow on $P_2^\infty(M, \mu)$.

On the other hand, there is a difference between the $W$-entropy formula for the heat equation of the Witten Laplacian on Riemannian manifold and the $W$-entropy formula for the geodesic flow on the Wasserstein space, i.e., $\frac{t}{\sqrt{2}}$ appears in (1.6), while $\frac{t}{\sqrt{2}}$ appears in (1.4), and their rigidity models are also different (see [23, 24, 26] and Theorem 3.7). An intuitive interpretation for this difference can be given as follows. The heat kernel of the Laplacian on $\mathbb{R}^m$ is the transition probability of Brownian motion starting from time 0 to time $t$. The mean square displacement (the variance of the distance that the “Brownian particle” moving along its trajectory) on $\mathbb{R}^m$ during the time interval $[0, t]$ is given by $E[|B_t|^2] = mt$. While for the geodesic flow (2.27) and (2.28), if we assume that the velocity of the “light particle” has the unit speed along each direction, the distance (denoted by $|X_t|$) of the “light particle” moving along the geodesic during time interval $[0, t]$ is indeed $|X_t| = t$. Hence, the mean square displacement of the “light particle” moving along the geodesic during time interval $[0, t]$ is $E[|X_t|^2] = t^2$. This explains intuitively why the rigidity model for the $W$-entropy for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the $\text{CD}(0, m)$-condition is the Gaussian space $(\mathbb{R}^m, g_0, N(0, tl^2))$, while the rigidity model for the $W$-entropy for the geodesic flow on the Wasserstein space over complete Riemannian manifolds with the $\text{CD}(0, m)$-condition is the Gaussian space $(\mathbb{R}^m, g_0, N(0, tl^2))$. Here $g_0$ denotes the Euclidean metric on $\mathbb{R}^m$, $\text{Id}$ is the unit matrix on $\mathbb{R}^m$, and $N(0, tl^2)$ denotes the Gaussian distribution on $\mathbb{R}^m$ with mean zero and variance $tl^2$.

### 4 Langevin deformation of flows on finite dimensional manifolds

In this section we introduce the Langevin deformation of geometric flows on the tangent bundle $TM$ over a complete Riemannian manifold $(M, g)$, which interpolates the geodesic flow on $TM$ and the gradient flow of a potential function on $M$. Our work has been inspired by Bismut [5, 6] who introduced the deformation of a family of hypoelliptic
Laplacians on $TM$, which is the infinitesimal generator of the Langevin diffusion process interpolating the geodesic flow on $TM$ and the Brownian motion on $M$. The ideas and results in this section will be extended in Section 6 to the infinite dimensional Wasserstein space over compact Riemannian manifolds.

4.1 The construction of the deformation of flows

We first describe J.-M. Bismut’s idea for the construction of a family of hypoelliptic Laplacians on the tangent bundle over Riemannian manifolds ([5, 6]). Let $c > 0$ be a parameter, let $(x_t, v_t)$ be the Langevin diffusion process on the tangent bundle $TM$ over a complete Riemannian manifold $M$ which solves the following stochastic differential equation

\begin{align*}
\dot{x} &= \frac{v}{c}, \quad (4.44) \\
\dot{v} &= -\frac{v}{c^2} dt + \frac{dw_t}{c}, \quad (4.45)
\end{align*}

where $dw_t$ denotes the Stratonovich differential of Brownian motion $w_t$ on $M$. This is the stochastic differential equation for the Langevin hypoelliptic diffusion process $(x_t, v_t)$ on the tangent bundle $TM$ over $M$. The position process $x_t$ satisfies the Langevin stochastic differential equation

\begin{equation}
\dot{x} = \dot{w}_t, \quad (4.46)
\end{equation}

where $\dot{w}_t$ denotes the Stratonovich derivation of Brownian path $w_t$ on $M$. As was pointed out by Bismut [5, 6], taking $c \to 0$, the limiting process $x_t$ is the Brownian motion on $M$, i.e.,

\begin{equation}
\dot{x} = \dot{w}_t,
\end{equation}

and when $c \to \infty$, to make sense the Langevin stochastic differential equation (4.46), the limiting process $x_t$ must satisfy the geodesic equation

\begin{equation}
\ddot{x} = 0.
\end{equation}

Thus the Langevin diffusion processes $(x_t, v_t)$ provide a deformation of geometric flows which interpolate the geodesic flows $\ddot{x} = 0$ on the tangent bundle $TM$ over $M$ and the Brownian motion $x_t = w_t$ on the underlying Riemannian manifold $M$.

Let $V$ be a smooth function on $M$. Instead of the above Langevin diffusion processes on $TM$, let us introduce the Langevin deformation of geometric flows on $TM$

\begin{align*}
\dot{x} &= v, \quad (4.47) \\
c^2 \dot{v} &= -v - \nabla V(x). \quad (4.48)
\end{align*}
Then \( x_t \) satisfies the second order ordinary differential equation

\[
c^2 \ddot{x} = -\dot{x} - \nabla V(x). \tag{4.49}
\]

The equation \((4.49)\) is indeed the Newton-Langevin equation which describes the motion of particles with mass \( c^2 \) moving in a fluid with friction coefficient 1 and with an external potential \( V \). In view of the classical ODE theory, for fixed \( c > 0 \) and under a suitable condition on \( V \)(for example \( \nabla V \) is Lipschitz), the above equations admit a unique local solution. It defines a family of geometric flows on \( TM \) which interpolates the geodesic flow \( \ddot{x} = 0 \) and the gradient flow \( \dot{x} = -\nabla V(x) \). Indeed, similarly to Bismut’s situation, we can rigorously prove that when \( c \to 0 \), the limiting flow \( x_t \) is the gradient flow of \( V \), i.e.,

\[
\dot{x}_t = -\nabla V(x_t),
\]

and when \( c \to \infty \), the limiting flow \( x_t \) must satisfy the geodesic equation

\[
\dot{x} = 0.
\]

In view of this, \((x_t, v_t)\) is a deformation of geometric flows on \( TM \) which interpolate the geodesic flows \( \ddot{x} = 0 \) on the tangent bundle \( TM \) over \( M \) and the gradient flow \( \dot{x}_t = -\nabla V(x_t) \) on the underlying Riemannian manifold \( M \). For the statement and proof of the convergence result, see Section 7.

Following Bismut \([5, 6]\) and Villani \([52]\), we use a Hamiltonian point of view to give an interpretation of the above deformation of flows. Let

\[
H(x, v) = \frac{c^2|v|^2}{2} + V(x) \tag{4.50}
\]

be the Hamiltonian energy of a particle moving in the tangent bundle, where \(-V\) is the external potential. Then

\[
\nabla H(x, v) = \left( \frac{\partial H}{\partial x}, \frac{\partial H}{\partial v} \right)^\tau = (\nabla V(x), c^2v)^\tau.
\]

Let

\[
A = \begin{pmatrix} 0 & c^{-2}Id \\ -c^{-2}Id & -c^{-4}Id \end{pmatrix}.
\]

Then

\[
A\nabla H(x, v) = (v, -c^{-2}\nabla V(x) - c^{-2}v)^\tau.
\]

Thus the deformation flow \((x_t, v_t)\) can be regarded as the “\(A\)-Hamiltonian flow”, given by

\[
\frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix} = A\nabla H \begin{pmatrix} x \\ v \end{pmatrix}. \tag{4.51}
\]

Therefore, \((x_t, v_t)\) admits the dissipative property as the gradient flow: the Hamiltonian is monotone along the Langevin deformation flow, while the Lagrangian is \(K\)-convex if \(-V\) is \(K\)-convex, i.e., \(\nabla^2 V \leq -K\). More precisely, we have
Proposition 4.1. Let \((x_t, v_t)\) be a smooth solution to (4.47) and (4.48) on \(T^*M\). Let

\[
H(x, v) = \frac{c^2|v|^2}{2} + V(x),
\]

\[
L(x, v) = \frac{c^2|v|^2}{2} - V(x).
\]

Then

\[
\frac{d}{dt}H(x_t, v_t) = -|v|^2 \leq 0.
\]

and

\[
\frac{d^2}{dt^2}L(x, v) = 2|\dot{v}|^2 - 2\nabla^2 V(v, v).
\]

In particular, if \(-V\) is \(K\)-convex, i.e., \(\nabla^2 V \leq -K\), we have

\[
\frac{d^2}{dt^2}L(x, v) \geq 2|\dot{v}|^2 + 2K|v|^2.
\]

Proof. — By direct calculation, we have

\[
\frac{d}{dt}H(x_t, v_t) = c^2 v \cdot \dot{v} + \nabla V \cdot \dot{x}
\]

\[
= v \cdot (-v - \nabla V(x)) + \nabla V \cdot v
\]

\[
= -|v|^2.
\]

On the other hand, we have

\[
\frac{d}{dt}L(x, v) = c^2 v \cdot \dot{v} - \nabla V \cdot \dot{x}
\]

\[
= v \cdot (-v - \nabla V(x)) - \nabla V \cdot v
\]

\[
= -|v|^2 + 2\nabla V \cdot v,
\]

and

\[
\frac{d^2}{dt^2}L(x, v) = -2v \cdot \dot{v} - 2\nabla^2 V(v, v) - 2\nabla V \cdot \dot{v}
\]

\[
= 2c^{-2} v \cdot (v + \nabla V(x)) - 2\nabla^2 V(v, v) + 2c^{-2} \nabla V \cdot (v + \nabla V(x))
\]

\[
= 2c^{-2}|v + \nabla V|^2 - 2\nabla^2 V(v, v)
\]

\[
= 2|\dot{v}|^2 - 2\nabla^2 V(v, v).
\]
4.2 The $W$-entropy formula for the deformation flow on $TM$

In this subsection, we introduce a variant of the $W$-entropy functional and to prove its monotonicity along the deformed flows $(x_t, v_t)$ on $TM$. Our first observation is the following

**Proposition 4.2.** For any $c > 0$, let $(x_t, v_t)$ be the Langevin deformation flow on $TM$ defined by (4.47) and (4.48). Then

$$\frac{d^2}{dt^2} V(x) + c^{-2} \frac{d}{dt} V(x) + c^{-2} |\nabla V|^2 = \nabla^2 V(v, v). \quad (4.53)$$

In particular, if $\nabla^2 V \geq K$, where $K \in \mathbb{R}$ is a constant, we have

$$\frac{d^2}{dt^2} V(x) + c^{-2} \frac{d}{dt} V(x) + c^{-2} |\nabla V|^2 \geq K|v|^2. \quad (4.54)$$

**Proof.** — Indeed, a simple calculation yields

$$\frac{d}{dt} V(x(t)) = \langle \nabla V, \dot{x} \rangle,$$
$$\frac{d^2}{dt^2} V(x(t)) = \nabla^2 V(\dot{x}, \dot{x}) + \langle \nabla V, \ddot{x} \rangle$$
$$= \nabla^2 V(v, v) - c^{-2} \langle \nabla V, v + \nabla V \rangle,$$

Hence

$$\frac{d^2}{dt^2} V(x) + c^{-2} \frac{d}{dt} V(x) + c^{-2} |\nabla V|^2 = \nabla^2 V(v, v) \geq K|v|^2.$$

This finishes the proof.

Thus from (4.53) we see that $\frac{d^2}{dt^2} V(x) + c^{-2} \frac{d}{dt} V(x) + c^{-2} |\nabla V|^2$ is a quantity that is invariant for $c \in [0, \infty]$. This leads us to introduce the $W$-entropy for the Langevin deformation flow on $TM$ such that its time derivative is given by

$$\frac{d}{dt} W_c(x, v) := \frac{d^2}{dt^2} V(x) + c^{-2} \frac{d}{dt} V(x) + c^{-2} |\nabla V|^2. \quad (4.55)$$

Thus if $\nabla^2 V \geq K$, by (4.53), we have for all $c \geq 0$,

$$\frac{d}{dt} W_c(x, v) = \nabla^2 V \geq K.$$

**Remark 4.3.** In fact, by the convergence result (Theorem 7.1) in Section 7, we see that the quantity $c^{-2} \frac{d^2}{dt^2} V(x) + c^{-2} |\nabla V|^2$ is well defined when $c$ approaches $0$ and $\infty$. More precisely, when $c = 0$, $\dot{x} = -\nabla V(x)$, we have $\ddot{x} = -\nabla^2 V \dot{x} = \nabla^2 V v$. Hence

$$\frac{d^2}{dt^2} V(x) = \nabla^2 V(\dot{x}, \dot{x}) + \nabla V \cdot \ddot{x} = 2 \nabla^2 V(v, v).$$
Moreover, when \( \nabla V \) are \( \nabla^2 V \) is \( K \)-Lipschitz and uniformly bounded, then there exists a constant \( T > 0 \) such that, as \( c \to 0 \), we have

\[
\ddot{x} = -c^{-2}(\dot{x} + \nabla V) \to -\nabla^2 V \cdot v \quad \text{(in the uniform convergence on } [0, T])
\]

we derive that, as \( c \to 0 \), then in the uniform convergence on \([0, T]\), we have

\[
c^{-2} \frac{d}{dt} V(x) + c^{-2} |\nabla V|^2 = \langle \nabla V, c^{-2}(\dot{x} + \nabla V) \rangle \to -\nabla^2 V(v, v),
\]

which implies

\[
\lim_{c \to 0} \frac{d}{dt} W_c(x, v) = \nabla^2 V(v, v) = \frac{d}{dt} W_0(x, v). \tag{4.56}
\]

On the other hand, when \( c \to \infty \), we have in the uniform convergence on \([0, T]\)

\[
\frac{d^2}{dt^2} V(x) = \nabla^2 V(\dot{x}, \ddot{x}) + \langle \nabla V, \ddot{x} \rangle = \nabla^2 V(v, v) - \langle \nabla V, c^{-2}(\dot{x} + \nabla V) \rangle \to \nabla^2 V(v, v),
\]

and

\[
c^{-2} \frac{d}{dt} V(x) + c^{-2} |\nabla V|^2 = \langle \nabla V, c^{-2}(\dot{x} + \nabla V) \rangle \to 0,
\]

which implies

\[
\lim_{c \to \infty} \frac{d}{dt} W_c(x, v) = \nabla^2 V(v, v) = \frac{d}{dt} W_\infty(x, v). \tag{4.57}
\]

The convergences (4.56) and (4.57) show that our definition of \( W_c \) is indeed an interpolation between the \( W \)-entropy for the gradient flow and that for the geodesic flow.

## 5 Langevin deformation of flows on Wasserstein space

In this section we extend the idea and results in Section 4 to the infinite dimensional Wasserstein space over compact Riemannian manifolds, and introduce the Langevin deformation of flows on the Wasserstein space \( P_\infty^2(M, \mu) \). Based on the connection between the Langevin deformation of flows on \( P_\infty^2(M, \mu) \) and the the compressible Euler equation with damping, we prove the well-posedness of the Langevin deformation of flows. Finally we show the convergence results of the Langevin deformation of flows when \( c \) approaches 0 and \( \infty \) respectively.
5.1 Langevin deformation of flows on $TP^\infty_2(M, \mu)$

Let $M$ be a compact Riemannian manifold, and $P^\infty_2(M, \mu)$ be the smooth Wasserstein space over $M$ equipped with the weighted volume measure $d\mu = e^{-f}dv$. Let $\rho : [0, T] \rightarrow P^\infty_2(M, \mu)$ be a smooth curve. Since $\dot{\rho} \in T_{\rho d\mu}P^\infty_2(M, \mu)$, there exists a function $\phi$ on $M$ such that

$$\dot{\rho} = \nabla^*_\mu (\rho \nabla \phi).$$

Equivalently, $\rho$ satisfies the transport equation with velocity $\phi$ on $P^\infty_2(M, \mu)$.

$$\partial_t \rho - \nabla^*_\rho (\rho \nabla \phi) = 0. \quad (5.58)$$

By Otto’s infinite dimensional Riemannian metric on $T_{\rho d\mu}P^\infty_2(M, \mu)$, we have

$$\|\dot{\rho}\|^2 = \int_M |\nabla \phi|^2 \rho d\mu.$$

Let $V \in C^1(\mathbb{R})$. Define the Hamiltonian and Lagrangian on $TP^\infty_2(M, \mu)$ by

$$H(\rho, \dot{\rho}) = \frac{\|\dot{\rho}\|^2}{2} + V(\rho), \quad (5.59)$$

$$L(\rho, \dot{\rho}) = \frac{\|\dot{\rho}\|^2}{2} - V(\rho). \quad (5.60)$$

Extending the idea in Section 4, let us introduce the following ODE on $TP^\infty_2(M, \mu)$

$$\partial_t \rho = v, \quad (5.61)$$

$$\partial_t v = -v - \nabla V(\rho) \quad (5.62)$$

The first equation is indeed the transport equation $(5.58)$, and the second equation can be written as

$$\nabla_\rho \dot{\rho} = -\dot{\rho} - \nabla V(\rho).$$

According to Otto [38], we have

$$\nabla V(\rho) = \nabla^*_\mu \left( \rho \nabla \frac{\delta V}{\delta \rho} \right),$$

where $\frac{\delta V}{\delta \rho}$ is the $L^2$-derivative of $V$ with respect to $\rho$, and the result in Lott [32] leads to

$$\nabla_\rho \dot{\rho} = \nabla^*_\mu \left( \rho \nabla \left( \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) \right), \quad (5.63)$$

Hence the second equation reads

$$\nabla^*_\mu \left( \rho \nabla \left( \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) \right) = -\nabla^*_\mu (\rho \nabla \phi) - \nabla^*_\mu \left( \rho \nabla \frac{\delta V}{\delta \rho} \right).$$
which is equivalent to (up to a additive constant)
\[
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = -\phi - \frac{\delta V}{\delta \rho}. \tag{5.64}
\]

Now for any parameter \(c > 0\), we can introduce the Langevin deformation of flows on \(TP^\infty_2(M, \mu)\) as follows
\[
\partial_t \rho - \nabla^\ast_\mu (\rho \nabla \phi) = 0, \tag{5.65}
\]
\[
c^2 \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) = -\phi - \frac{\delta V}{\delta \rho}. \tag{5.66}
\]

In Section 6.2, we will prove the well-posedness of the Langevin deformation of flows on \(TP^\infty_2(M, \mu)\). From its definition, we can see that, when \(c \to 0\), we may formally get
\[
\phi = -\frac{\delta V}{\delta \rho}, \tag{5.67}
\]
which yields that \(\rho\) is the gradient flow of \(V\) on the Wasserstein space \(P^\infty_2(M, \mu)\)
\[
\partial_t \rho = -\nabla^\ast_\mu \left( \rho \nabla \frac{\delta V}{\delta \rho} \right), \tag{5.68}
\]
and when \(c \to \infty\), we may formally get the geodesic flow on the tangent bundle over the Wasserstein space \(P^\infty_2(M, \mu)\), i.e.,
\[
\partial_t \rho - \nabla^\ast_\mu (\rho \nabla \phi) = 0, \tag{5.69}
\]
\[
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0. \tag{5.70}
\]
Indeed, the convergence in a precise sense will be rigorously proved. See Theorem 5.8 for the precise statement and see Section 7 for the proof.

### 5.2 The compressible Euler equation with damping

When \(\nu = 0\), the compressible Euler equations and the deformed Hamilton-Jacobi equation with the transport equation have been well studied in the literature at least in the Euclidean case. See e.g. Carles [9] and reference therein. The case of compact Riemannian manifolds is as the same as in the Euclidean case. On the other hand, the compressible Euler equation with damping on Euclidean space has been also well studied in the literature. See Wang and Yang [54]. See also Sideris, Thomases ad Wang [44] and reference therein. In this subsection, we develop a little bit more detail on the link between the deformed Hamilton-Jacobi equation and the compressible Euler equation with damping on compact Riemannian manifolds.
Let $u = \nabla \phi$, $\gamma = \frac{1}{c^2}$, and $p \in C^1(\mathbb{R})$ be such that $\nabla p(\rho) = \rho \frac{\partial V}{\partial \rho}(\rho)$, then the Langevin deformation of flows (5.65), (5.66) turns into

$$\partial_t \rho - \nabla^*_{\mu}(\rho u) = 0,$$

(5.71)

$$\partial_t u + u \cdot \nabla u = -\gamma u - \frac{1}{c^2} \nabla V'(\rho).$$

(5.72)

In the case $M = \mathbb{R}^n$ and $\mu = dx$, the above system is the compressible Euler equation with damping, where $\rho$ is the density of fluid, $u$ is the velocity of the fluid, and $\gamma = \frac{1}{c^2}$ is the friction constant, $p(\rho)$ is the pressure of the fluid. When $\gamma = 0$, the connections between the compressible Euler equations and the deformed Hamilton-Jacobi equation with the transport equation have been well studied in the literature, at least in the Euclidean case. See e.g. Carles [9] and reference therein. The case of compact Riemannian manifolds is the same as in the Euclidean case. On the other hand, there are extensive studies on the compressible Euler equation with damping on Euclidean space. See Wang and Yang [54], and also Sideris, Thomases and Wang [44] and reference therein.

In the case $M$ is a compact Riemannian manifold, we can regard (5.71) and (5.72) as the compressible Euler equation with damping on the compact Riemannian manifold $(M, g)$ equipped with the reference measure $\mu$. In this case, the compressible Euler equation with damping can be rewritten as follows

$$\partial_t \rho - \nabla^*_{\mu}(\rho u) = 0,$$

(5.73)

$$\partial_t u + \nabla u u = -\gamma u - \frac{1}{c^2} \nabla V'(\rho),$$

(5.74)

where $\nabla_u u$ denotes the Levi-Civita covariant derivative of the vector field $u$ along the velocity field $u$ of the trajectory of the fluid.

### 5.3 Well-posedness of the Langevin deformation of flows

In this subsection, we prove the existence and uniqueness of the Langevin deformation of flows (5.65) and (5.66) for any fixed $c \in (0, \infty)$.

Recall the Sobolev inequalities on compact Riemannian manifolds. We only consider the unweighted case $\mu = \nu$. The general case can be treated similarly. By [2], there exists a constant $C_{\text{Sob}} > 0$ such that

$$\|f\|_\frac{2n}{n-2} \leq C_{\text{Sob}}(\|\nabla f\|_2 + \|f\|_2), \quad \forall f \in C^\infty(M).$$

(5.75)

Moreover, for any $\alpha \in (0, 1)$, if $k > \alpha + \frac{n}{2}$, the Kondrakov embedding theorem holds

$$\|f\|_{C^{0,\alpha}} \leq C_{\alpha}\|f\|_{k,2}.$$}

(5.76)

In particular, we have

$$\|f\|_{\infty} \leq C_{\alpha}\|f\|_{k,2}.$$}

(5.77)
Let $H^s(M)$ denotes the Sobolev space equipped with the Sobolev norm
\[ \|f\|_{s,2} = \left( \sum_{|\alpha| \leq s} \|D^\alpha f\|_{L^2}^2 \right)^{1/2} \]
where $\alpha$ is a multi-index, and for any vector valued function $U = (u_1, \ldots, u_n)$ on $M$,
\[ \|U\|_{s,2} = \sum_{i=1}^n \|u_i\|_{s,2} \]
For any $s \in \mathbb{Z}$, we define the Banach space
\[ X_s = \{(f,g)| f \in H^2(M), g \in \oplus^n H^s(M)\} \]
equipped with the norm $\|(f,g)\|_{s,2} = \|f\|_{s,2} + \|g\|_{s,2}$.

The following result gives the local existence and uniqueness of the solution to the compressible Euler equation with damping on compact Riemannian manifolds.

**Theorem 5.1.** (Local existence and uniqueness of smooth solution) Let $M$ be $\mathbb{R}^n$ or a compact Riemannian manifold, $s = \left[\frac{n}{2}\right] + 1$. Let $V(\rho) = \int_M \rho \log \rho d\mu$ or $V(\rho) = \frac{1}{m-1} \int_M \rho^m d\mu$ for $m > 1$. Suppose that $(\rho_0, u_0) \in H^{s+1}(M)$ with $\rho_0 > 0$. Then, there exists a constant $T > 0$ such that the Cauchy problem of the compressible Euler with damping (5.71) and (5.72) has a unique smooth solution $(\rho, u)$ in $C([0, T], H^{s+1}(M)) \times H^s(M))$.

**Proof.** — This is a standard result which can be proved by the method in Kato [14] and Majda [34].

The following result gives the global existence and uniqueness of the solution to the compressible Euler equation with damping on compact Riemannian manifolds.

**Theorem 5.2.** (Global existence and uniqueness of smooth solution with small initial data) Let $M$ be $\mathbb{R}^n$ or a compact Riemannian manifold. Let $V(\rho) = \int_M \rho \log \rho d\mu$ or $V(\rho) = \frac{1}{m-1} \int_M \rho^m d\mu$ for $m > 1$. Let $s = \left[\frac{n}{2}\right] + 1$, $l \geq 2$. Then there exists $\delta_0 > 0$ such that if $(\rho_0, u_0) \in H^{s+l}(M)$ is a small smooth initial value in the sense that $\|\rho_0 - 1\|_{s+l,2} + \|u_0\|_{s+l,2} \leq \delta_0$ is sufficiently small. Then the Cauchy problem of the compressible Euler equation with damping (5.71) and (5.72) admits a unique global smooth solution $(\rho, u) \in C([0, \infty), H^{s+l}(M) \times H^{s+l-1}(M))$ with initial value $(\rho_0, u_0)$ and satisfying the following energy estimate
\[
\begin{align*}
\|\partial_t \rho(t)\|_{s+l-1,2}^2 + \|\rho(t) - 1\|_{s+l,2}^2 &+ \|u(t)\|_{s+l,2}^2 \\
+ \int_0^t \left( \|\partial_t \rho(r)\|_{s+l-1,2}^2 + \|
abla \rho(r)\|_{s+l-1,2}^2 + \|u(r)\|_{s+l,2}^2 \right) dr \\
&\leq C(\|\partial_t \rho(0)\|_{s+l-1,2} + \|\rho_0 - 1\|_{s+l,2} + \|u_0\|_{s+l,2}).
\end{align*}
\]
(5.78)
Proof. — In the case $M = \mathbb{R}^n$, $V(\rho) = \int_M \rho \log \rho d\mu$ or $V(\rho) = \frac{1}{m-1} \int_M \rho^m d\mu$ with $m > 1$, this is the well-established result due to Wang and Yang [54]. See also Sideris, Thomases ad Wang [44] for the case $M = \mathbb{R}^n$ and $V(\rho) = \frac{1}{m-1} \int_M \rho^m d\mu$ with $m > 1$. In the case $M$ is a compact Riemannian manifold and $V(\rho) = \frac{1}{m-1} \int_M \rho^m d\mu$ with $m > 1$, the proof of theorem is similar to the ones in [54, 44]. In the case $M$ is a compact Riemannian manifold and $V(\rho) = \int_M \rho \log \rho d\mu$, we can modify the proof of the main results in [54, 44]. The main point here is that on compact Riemannian manifold, the positivity of the initial data $\rho_0 > 0$ implies that there exists a constant $\varepsilon_0 > 0$ such that $\rho_0 \geq \varepsilon_0 > 0$, and the argument used in Wang and Yang [54] can be extended to the case $V(\rho) = \int_M \rho \log \rho d\mu$ on compact Riemannian manifolds. We can also modify the argument used in Sideris, Thomases ad Wang [44] by taking the sound speed to be $\sigma(\rho) = \log \rho$. To save the length of the paper, we omit the detail of the proof.

Now we turn back to the Langevin deformation of flows (5.65) and (5.66). The key point here is that we need to prove that if the initial value $u_0 = \nabla \phi_0$ for some function $\phi_0$, $u_t$ will keep the gradient form along $t > 0$. To see this, we show the following result.

Theorem 5.3. Let $M = \mathbb{R}^n$ or a compact Riemannian manifold, $(\rho, u)$ be a smooth solution to the compressible Euler equation with damping, i.e., (5.71) and (5.72). Let $\omega = du$. Suppose that $\frac{\nabla \rho}{\rho} = \nabla V'(\rho)$. Then

$$\partial_t \omega + u \cdot \nabla \omega + \omega \wedge \nabla u^* = -\gamma \omega. \quad (5.79)$$

Moreover, if $|\nabla u|_{L^\infty} \leq C$, then for all $t \in [0, T]$, we have\footnote{In the case $C = \max |\nabla u|_{L^\infty} < \gamma$, the above estimates holds for all $t \in [0, \infty)$.}

$$\|\omega(t)\|_{L^p} \leq \|\omega(0)\|_{L^p} e^{(C-\gamma)t}. \quad (5.80)$$

In particular, if $u_0$ is a closed form, so is $u(t, \cdot)$, i.e., $du_0 = 0$ implies $du(t, \cdot) = 0$ on $[0, T]$.

Proof. — We first prove that $\omega = du$ satisfies (5.79). By identifying $u$ with its dual $u^*$, and identifying $\nabla V'(\rho)$ with its dual $dV'(\rho)$, with respect to the Riemannian metric on $M$, we can rewrite (5.74) (or (5.72)) as follows

$$\partial_t u^* + \nabla_u u^* = -\gamma u^* - \frac{1}{c^2} dV'(\rho). \quad (5.81)$$

Taking exterior differentiation on the both sides of the compressible Euler equation (5.81), letting $\omega = du^* = \sum_{i=1}^n du_i \wedge e_i^* \in \Gamma(\Lambda^2 TM)$, and using $ddV'(\rho) = 0$, we have

$$\partial_t \omega + d\nabla_u u^* = -\gamma \omega. \quad (5.82)$$
Let \((e_i)\) be an ONB and normal at \(x \in M\), writing \(u = \sum_{i=1}^n u_i e_i\) and \(u^* = \sum_{i=1}^n u_i^* e_i^*\), we derive that

\[
d\nabla u u^* = \nabla_u (du^*) + \sum_{i=1}^n du_i \wedge \nabla_{e_i} u^* + \sum_{k=1}^n e_k^* \wedge R(e_k, u) u^*.
\]

(5.83)

Indeed, since \((e_i)\) is an ONB and normal at \(x \in M\), we have

\[
d\nabla u^* = \sum_{k=1}^n e_k^* \wedge \nabla_{e_k} (\nabla u u^*) = \sum_{i=1}^n du_i \wedge \nabla_{e_i} u^* + \sum_{k,i=1}^n u_i e_k^* \wedge \nabla_{e_k} \nabla_{e_i} u^*,
\]

and

\[
\nabla u du^* = \sum_{i=1}^n u_i \nabla_{e_i} (du^*) = \sum_{k,i=1}^n u_i e_k^* \wedge \nabla_{e_k} \nabla_{e_i} u^*.
\]

Hence

\[
d\nabla u u^* - \nabla_u (du^*) = \sum_{i=1}^n du_i \wedge \nabla_{e_i} u^* + \sum_{k,i=1}^n u_i e_k^* \wedge R(e_k, e_i) u^*.
\]

Moreover, we claim that for all \(u \in \Gamma(TM)\),

\[
\sum_{k=1}^n e_k^* \wedge R(e_k, u) u^* = 0.
\]

(5.84)

Indeed, for any \(i, j = 1, \ldots, n\), acting on \((e_i, e_j)\), we have

\[
\sum_{k=1}^n (e_k^* \wedge R(e_k, u) u^*) (e_i, e_j) = (R(e_i, u) u^*) (e_j) - (R(e_j, u) u^*) (e_i).
\]

Notice that

\[
(\nabla_{e_i} \nabla_u u^*) (e_j) = \nabla_{e_i} \nabla_u u_j + \nabla_{e_i} (u^* \nabla_u e_j),
\]

meanwhile we also have

\[
(\nabla_u \nabla_{e_i} u^*) (e_j) = \nabla_u \nabla_{e_i} u_j + \nabla_u (u^* \nabla_{e_i} e_j),
\]

such that

\[
(R(e_i, u) u^*) (e_j) = \nabla_{e_i} (u^* \nabla_u e_j) - \nabla_u (u^* \nabla_{e_i} e_j) = \langle u, R(e_i, u) e_j \rangle.
\]
Exchanging $e_i$ and $e_j$ leads to

$$(R(e_j, u^*)(e_i)) = \langle u, R(e_j, u) e_i \rangle.$$ 

Therefore we derive that

$$\sum_{k=1}^{n} (e_k^* \wedge R(e_k, u^*)(e_i, e_j)) = R(e_i, u, e_j, u) - R(e_j, u, e_i, u) = 0,$$

which finishes the proof of (5.84).

Then taking (5.83) and (5.84) into (5.82), we obtain

$$\partial_t \omega + \nabla_u \omega + \sum_{i=1}^{n} (\omega(e_i)) \wedge \nabla e_i u^* = -\gamma \omega,$$

which is written briefly as (5.79).

Now taking inner product with $|\omega|^{p-2} \omega$ in the both sides of (5.79), and integrating on $M$, we have

$$\int_M \langle D \partial_t \omega, |\omega|^{p-2} \omega \rangle d\mu + \int_M \langle \omega \wedge \nabla u, |\omega|^{p-2} \omega \rangle d\mu = -\gamma \|\omega\|_p^p$$

where $\frac{D}{dt} \omega = \partial_t \omega + u \cdot \nabla \omega$. Note that

$$\langle D \partial_t \omega, |\omega|^{p-2} \omega \rangle = \frac{1}{p} \frac{D}{dt} \|\omega\|^p.$$

Hence

$$\frac{1}{p} \frac{d}{dt} \|\omega(t)\|^p_p = - \int_M \langle \omega \wedge \nabla u, |\omega|^{p-2} \omega \rangle d\mu + \gamma \|\omega(t)\|^p_p$$

$$\leq \int_M \langle |\nabla u| |\omega(t)|^p d\mu - \gamma \|\omega(t)\|^p_p \rangle d\mu$$

$$\leq (\|\nabla u\|_{L^\infty} - \gamma) \|\omega(t)\|^p_p.$$ 

By Gronwall inequality, we have

$$\|\omega(t)\|^p_p \leq \|\omega(0)\|^p_p \exp \{p (\|\nabla u\|_{L^\infty} - \gamma) t \}.$$ 

Thus, if $\omega(0) = 0$, then for all $t \geq 0$, we have $\omega(t) = 0$. Hence, if $u^*(0, \cdot)$ is a closed one-form on $M$, then $u^*(t, \cdot)$ is also a closed one-form on $M$.

We now state the main result of this section, which gives the existence and uniqueness of the local solution to Langevin deformation of flows (5.65) and (5.66) for any fixed $c \in (0, \infty)$.
Theorem 5.4. Let $M = \mathbb{R}^n$ or be a compact Riemannian manifold, $c \in [0, \infty]$. Given $(\rho_0, \phi_0) \in TP_2^\infty(M, \mu)$ with $\rho_0, \phi_0 \in C^\infty(M)$, there exists $T = T_c > 0$ such that the Cauchy problem of the Langevin deformation of flows (5.65) and (5.66) has a unique solution $(\rho, \phi) \in C^1([0, T], C^\infty(M)^2)$.

Proof. — The cases $c = 0$ and $c = \infty$ are well known. For $c \in (0, \infty)$, consider the compressible Euler equation with damping on $M$

\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= -\frac{u}{c^2} - \frac{1}{c^2} \frac{\delta V}{\delta \rho}, \quad u|_{t=0} = \nabla \phi_0, \quad (5.85) \\
\partial_t \rho + \nabla^*_\mu(\rho u) &= 0, \quad \rho|_{t=0} = \rho_0. \quad (5.86)
\end{align*}
\]

By Theorem 5.1, if the initial data are in $H^s(M, \mu)$ for any $s > \frac{n}{2} + 1$, then there exists $T = T_c > 0$ such that above system has a unique solution $(\rho, u) \in C([0, T], H^s(M, \mu))^2$. Moreover, tame estimates show that the time of existence $T > 0$ can be chosen independent of $s > \frac{n}{2} + 1$.

By Theorem 5.3, if $u^*_0 = d\phi_0$, $u^*(t, \cdot)$ is closed on $M$. For all $(t, x) \in [0, T] \times M$, let

\[
\phi(t, x) = e^{-\gamma t} \phi_0(x) + e^{-\gamma t} \int_0^t e^{\gamma s} \left( f(\rho(s, x)) - \frac{1}{2} |u(s, x)|^2 \right) ds, \quad (5.87)
\]

where $\gamma = \frac{1}{c^2}$ and $f(\rho) = -\frac{1}{c^2} \frac{\delta V}{\delta \rho}$. We have

\[
\partial_t \phi = -\gamma \phi + f(\rho(t, x)) - \frac{1}{2} |u(t, x)|^2.
\]

Note that, as $u^*(t, \cdot)$ is a closed one-form on $M$, it holds that

\[
\nabla|u|^2 = 2u \cdot \nabla u.
\]

Hence we can check that

\[
\partial_t (\nabla \phi - u) = -\gamma (\nabla \phi - u).
\]

Note that at $t = 0$, $u(0) = \nabla \phi(0)$. Thus $u(t) = \nabla \phi(t)$ on $[0, T] \times M$. Substituting this into the compressible Euler equation with damping, we have

\[
\nabla \left( \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) = -\gamma \nabla \phi - \frac{1}{c^2} \nabla V'(\rho).
\]

This indicates that the Cauchy problem of (5.65) and (5.66) has a unique solution $(\rho, \phi) \in C([0, T], H^s(M) \times H^{s+1}(M))$ for all $s > \frac{n}{2} + 1$. The proof is completed. \qed

36
5.4 Entropy dissipation formulas along the deformation of flows

In this subsection we prove the entropy dissipation formulas along the Langevin deformation of flows on the Wasserstein space. In this subsection, we assume that $M$ is Euclidean space or a compact Riemannian manifold. By Theorem 5.4, for any $c \in [0, \infty]$, the Langevin deformation of flows on $TP^\infty_2(M, \mu)$ has a unique smooth solution (up to an additional constant) on $[0, T_c] \times P^\infty_2(M, \mu)$.

Similarly to the case of finite dimensional manifolds, we have the following variational formulas for the Hamiltonian and Lagrangian along the Langevin deformation of flows (5.65) and (5.66).

**Theorem 5.5.** Let $(\phi_t, \rho_t)$ be a smooth solution to Eq. (5.65) and Eq. (5.64) on $TP^\infty_2(M, \mu)$. Let

$$H(\rho, v) = \frac{c^2}{2} \int_M |\nabla \phi(x)|^2 \rho d\mu + V(\rho),$$

$$L(\rho, v) = \frac{c^2}{2} \int_M |\nabla \phi(x)|^2 \rho d\mu - V(\rho).$$

Then

$$\frac{d}{dt}H(\rho_t, \phi_t) = - \int_M |\nabla \phi_t|^2 \rho_t d\mu,$$

and

$$\frac{d^2}{dt^2}L(\rho_t, \phi_t) = 2c^{-2} \|\dot{\rho} + \nabla V(\rho)\|^2 - 2\nabla^2 V(\rho)(\dot{\rho}, \dot{\rho}) = c^2 \|\dot{\rho}\|^2 - 2\nabla^2 V(\dot{\rho}, \dot{\rho}).$$

In particular, if $-V$ is $K$-convex on $P^\infty_2(M, \mu)$, i.e., the Hessian of $-V$ on $P^\infty_2(M, \mu)$ satisfies

$$-\nabla^2 V(\rho) = -\text{Hess}_{P^\infty_2(M)} V(\rho) \geq K,$$

then

$$\frac{d^2}{dt^2}L(\rho_t, v_t) \geq 2K \int_M |\nabla \phi_t|^2 \rho_t d\mu + 2 \int_M \left[\nabla \left(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2\right)\right]^2 \rho d\mu.$$

**Proof.** — Notice that this result can be directly derived from Proposition 4.1 for that on $TP^\infty_2(M, \mu)$,

$$\frac{d}{dt}H(\rho, \phi) = -|v|^2 = -\int_M |\nabla \phi|^2 \rho d\mu.$$

Of course we could also prove it by direct computations.

$$\frac{d}{dt}H(\rho, \phi) = \frac{d}{dt} \left(\frac{c^2}{2} \int_M |\nabla \phi|^2 \rho d\mu + \langle \nabla V, \dot{\rho} \rangle\right).$$

37
Note that
\[
\frac{d}{dt} \left( \frac{c^2}{2} \int_M |\nabla \phi|^2 \rho \, d\mu \right) = c^2 \int_M \langle \nabla \phi, \nabla \partial_t \phi \rangle \rho \, d\mu + \frac{c^2}{2} \int_M |\nabla \phi|^2 \partial_t \rho \, d\mu
\]
\[
= c^2 \int_M \langle \nabla \phi, \nabla (\partial \phi + \frac{1}{2} |\nabla \phi|^2) \rangle \rho \, d\mu
\]
\[
= -\int_M \left( \nabla \phi, \nabla \left( \phi + \frac{\delta V}{\delta \rho} \right) \right) \rho \, d\mu.
\]

On the other hand
\[
\frac{d}{dt} V(\rho_t) = \langle \nabla V, \dot{\rho} \rangle = \int_M \langle \frac{\delta V}{\delta \rho}, \nabla \phi \rangle \rho \, d\mu.
\]

Hence
\[
\frac{d}{dt} H(\rho, \phi) = -\int_M \langle \nabla \phi, \nabla \left( \phi + \frac{\delta V}{\delta \rho} \right) \rangle \rho \, d\mu + \langle \nabla V, \dot{\rho} \rangle
\]
\[
= -\int_M |\nabla \phi|^2 \rho \, d\mu.
\]

The second dissipation formula in Theorem 5.5 can be proved by straightforwardly extending the argument used in the proof of Proposition 4.1 to the Wasserstein space \( P^\infty_2(M, \mu) \) equipped with Otto’s infinite dimensional Riemannian metric. To save the length of the paper, we omit the detail here.

5.5 Proof of Theorem 1.7

Applying Theorem 5.5 to \( V(\rho) = -\text{Ent}(\rho) = -\int_M \rho \log \rho \, d\mu \), we can derive Theorem 1.7.

Proof. — Indeed, applying Theorem 5.5 to \( V = -\text{Ent} \), we have
\[
\frac{d^2}{dt^2} L(\rho, \phi) = 2c^2 \|\dot{\rho}\|^2 + 2\text{Hess}_P^\infty(\text{Ent})(\dot{\rho}, \dot{\phi}).
\]

Here \( \text{Hess}_P^\infty(\text{Ent}) \) is the Hessian of \( \text{Ent} \) on the Wasserstein space \( P^\infty_2(M) \) equipped with Otto’s infinite dimensional Riemannian metric. By Theorem 2.3, we have
\[
\text{Hess}_P^\infty(\text{Ent})(\dot{\rho}, \dot{\phi}) = \int_M (|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho \, d\mu.
\]

On the other hand, we have
\[
c^2 \|\dot{\rho}\|^2 = c^2 \int_M \left| \nabla \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) \right|^2 \rho \, d\mu
\]
\[
= c^{-2} \int_M |\nabla \phi + \nabla \log \rho|^2 \rho \, d\mu.
\]
This finishes the proof of Theorem 1.7.

On compact manifolds with non-negative Bakry-Emery Ricci curvature, Theorem 1.7 implies that the Hamiltonian function $H$ is always monotonically nonincreasing and the Lagrangian function $L$ is always convex along the Langevin deformation flow $(\rho, \phi)$ which interpolates the geodesic flow and the backward gradient flow of $V$ on the Wasserstein space over a compact Riemannian manifold with weighted measure.

Note that, when $c = 0$ in Theorem 1.7, we have $\phi = -\log \rho - 1$, hence $\nabla \phi + \rho^{-1} \nabla \rho = 0$. This yields $\partial_t \rho = L \rho$, and

$$\frac{d^2}{dt^2} \operatorname{Ent}(\rho(t)) = 2 \int_M \left[ |\operatorname{Hess} \phi|^2 + \operatorname{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu.$$ 

If we formally take $c = \infty$ in Theorem 1.7, we have $\ddot{\rho} = 0$, and we obtain

$$\frac{d^2}{dt^2} H(\rho, \phi) = 2 \int_M \left[ |\operatorname{Hess} \phi|^2 + \operatorname{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu.$$ 

However, this formula is not correct. Indeed, when $c = \infty$, the kinetic energy term $\frac{1}{2} \int_M |\nabla \phi(t)|^2 \rho(t) d\mu$ is a constant along the geodesic flow $(\rho(t), \phi(t))$ on the Wasserstein space $P_2(M, \mu)$, thus $\frac{d^2}{dt^2} \int_M |\nabla \phi(t)|^2 \rho(t) d\mu = \infty$. In this case, we must replace the left hand side of (5.92) in Theorem 1.7 by the second order derivative of $\operatorname{Ent}(\rho(t))$, which is given by the entropy dissipation formula (Theorem 2.3).

$$\frac{d^2}{dt^2} \operatorname{Ent}(\rho(t)) = \int_M \left[ |\operatorname{Hess} \phi|^2 + \operatorname{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu.$$ 

In other words, we have the following

**Corollary 5.6.** Let $M$ be a compact Riemannian manifold, $f \in C^2(M)$. Then

(i) When $c = \infty$, we have $\ddot{\rho} = 0$, i.e., $(\rho, \phi)$ is a geodesic flow on $P_2(M, \mu)$, and satisfies the transport equation (5.69) and the Hamilton-Jacobi equation (5.70). Moreover

$$\frac{d^2}{dt^2} \operatorname{Ent}(\rho(t)) = \int_M \left[ |\operatorname{Hess} \phi|^2 + \operatorname{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu,$$

(ii) When $c = 0$, we have $\phi = -\log \rho - 1$, i.e., $\rho$ is a positive solution to the heat equation

$$\partial_t \rho = L \rho.$$

Moreover

$$\frac{d^2}{dt^2} \operatorname{Ent}(\rho(t)) = 2 \int_M \left[ |\operatorname{Hess} \phi|^2 + \operatorname{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu.$$
Remark 5.7. In [28], we introduced the Langevin deformation of flows as follows

\[ \partial_t \rho - \nabla^*_\mu (\rho \nabla \phi) = 0, \]
\[ c^2 \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) = -\phi + \frac{\delta V}{\delta \rho}. \]

(5.88)  
(5.89)

It is indeed the Langevin deformation between the backward gradient flow \( \partial_t \rho = \nabla^*_\mu \left( \rho \nabla \frac{\delta V}{\delta \rho} \right) \) on the Wasserstein space and the geodesic flow on the tangent bundle \( TP_2^{\infty}(M, \mu) \). Define the Hamiltonian and the Lagrangian on \( TP_2^{\infty}(M, \mu) \) by

\[ H(\rho, \phi) = \frac{c^2}{2} \int_M |\nabla \phi|^2 \rho d\mu + V(\rho), \]
\[ L(\rho, \phi) = \frac{c^2}{2} \int_M |\nabla \phi|^2 \rho d\mu - V(\rho). \]

Then

\[ \frac{d}{dt} L(\rho, \phi) = -\int_M |\nabla \phi|^2 \rho d\mu, \]
(5.90)

and

\[ \frac{d^2}{dt^2} H(\rho, \phi) = 2c^{-2} \| \dot{\rho} - \nabla V(\rho) \|^2 + 2 \nabla^2 V(\rho) \langle \dot{\rho}, \dot{\rho} \rangle \]
\[ = c^2 \| \dot{\rho} \|^2 + 2 \nabla^2 V(\dot{\rho}, \dot{\rho}). \]

(5.91)

In particular, if \( V \) is \( K \)-convex on \( P_2^{\infty}(M, \mu) \), i.e., the Hessian of \( V \) on \( P_2^{\infty}(M, \mu) \) satisfies \( \nabla^2 V(\rho) = \text{Hess}_{P_2^{\infty}(M)} V(\rho) \geq K \),

then

\[ \frac{d^2}{dt^2} H(\rho, \phi) \geq 2K \int_M |\nabla \phi|^2 \rho d\mu + 2 \int_M \left[ \nabla \left( \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) \right]^2 \rho d\mu. \]

(5.92)

In particular, taking \( V(\rho) = \text{Ent}(\rho) = \int_M \rho \log \rho d\mu \), we can derive that

\[ \frac{d}{dt} L(\rho, \phi) = -\int_M |\nabla \phi|^2 \rho d\mu, \]
(5.93)

\[ \frac{d^2}{dt^2} H(\rho, \phi) = 2 \int_M \left[ c^{-2} |\nabla \phi - \nabla \log \rho|^2 + |\text{Hess}\phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu. \]
(5.94)

Suppose that \( \text{Ric}(L) = \text{Ric} + \nabla^2 f \geq K \), then we have

\[ \frac{d^2}{dt^2} H(\rho, \phi) \geq 2K \int_M |\nabla \phi|^2 \rho d\mu + \frac{2}{n} \int_M |\Delta \phi|^2 \rho d\mu + 2c^{-2} \int_M |\nabla \phi - \nabla \log \rho|^2 \rho d\mu. \]
5.6 Convergence results

In this subsection we show the convergence results for the Langevin deformation of flows. Note that the $L^2$-adjoint of $\nabla$ with respect to the standard volume measure on $\mathbb{R}^d$ or a compact Riemannian manifold is $\nabla^* = -\nabla \cdot$. Taking $V(\rho) = \int \rho \log \rho$ in (5.65) and (5.66), and using $\frac{\delta V}{\delta \rho} = \log \rho + 1$, we get the Langevin deformation of flows as follows

$$\partial_t \rho + \nabla \cdot (\rho \nabla \phi) = 0,$$

(5.93)

$$c^2 (\partial_t \phi + \frac{1}{2} |\nabla \phi|^2) = -\phi - \log \rho - 1,$$

(5.94)

on $\mathbb{R}^d$ or compact manifold $M$, with initial value

$$\rho(0, x) = \rho_0(x), \quad \phi(0, x) = \phi_0(x).$$

Our aim is to prove that when $c \to 0$ and $c \to \infty$, the solution $(\rho, \phi)$ converges in a precise sense to that to the heat equation and to the geodesic flow respectively.

The key point of the proof still relies on the close connection between the Langevin deformation of flows and the compressible Euler equation with damping: we first prove the convergence for (5.71) and (5.72) with $V(\rho) = \int \rho \log \rho$, and then turn back to (7.113) and (7.114). We will first show the convergence in Euclidean space and then extend the results to compact manifolds. For simplicity, we consider the Laplace-Beltrami operator here instead of the Witten Laplacian. Under suitable assumptions on the potential function $f$, it is easy to prove that the convergence results also hold for the Witten Laplacian.

We now state the convergence theorem in a precise sense. To keep the paper to be more readable, we leave its proof to the Section 7.

**Theorem 5.8.** Let $M$ be $\mathbb{R}^n$ or a compact Riemannian manifold. Let $s \in \mathbb{N}$ with $s > \frac{d}{2} + 2$ and $c \in (0, \infty)$, $\bar{\rho} > 0$ be two constants. Let $(\rho^c, \phi^c)$ be the local solution to the initial value problem of Langevin deformation of flows (7.113), (7.114).

- Given the initial value $(\rho_0, \phi_0) \in TP^\infty_2(M, \mu)$ satisfying $\rho_0 \in H^s(\mathbb{R}^d)$, $\phi_0 \in H^{s+1}(M)$, there exists $T > 0$ which is independent of $c > 1$, we have as $c \to 0$

$$\sup_{t \in [0, T]} \|\rho^c - \rho^0\|_{L^1} \to 0,$$

where $\rho^0 \in C([0, T], H^s(\mathbb{R}^d))$ is the solution to the heat equation with the initial value $\rho_0$.

- Given the initial value $(\rho_0, \phi_0) \in TP^\infty_2(M, \mu)$ satisfying $\|\rho_0 - \bar{\rho}\|_{H^s(M)} + \|\phi_0\|_{H^{s+1}(M)} \leq \delta$ for some $\delta > 0$, there exists $T > 0$ which is independent of $c > 1$, such that as $c \to 0$, $\rho^c$ converges to the solution to the heat equation $\rho^0 \in C([0, T], \bar{\rho} + H^s(\mathbb{R}^d))$ in $C([0, T], H^s(B_R))$.
• Given initial value \((\rho_0, \phi_0) \in TP_2^\infty(M, \mu)\) with \(\log \rho_0 \in H^s(M)\), \(\phi_0 \in H^{s+1}(M)\), there exists \(T > 0\) which is independent of \(c > 1\), such that \(\rho^\infty dx\) weakly converges to \(\rho^\infty dx\) in \(C([0, T], P(\mathbb{R}^d))\) and \(\phi^\infty\) converges to \(\phi^\infty\) in \(C([0, T], C^1(B_R))\) as \(c \to \infty\), with \((\rho^\infty, \phi^\infty)\) satisfying

\[
\begin{align*}
\partial \rho^\infty + \nabla \cdot (\rho^\infty \nabla \phi^\infty) &= 0, \\
\partial_t \phi^\infty + \frac{1}{2} |\nabla \phi^\infty|^2 &= 0, \\
\rho^\infty(0, x) &= \rho_0(x), \quad \phi^\infty(0, x) = \phi_0(x).
\end{align*}
\] (5.95) (5.96)

Remark 5.9. Consider the initial value problem to the following Hamilton-Jacobi equation

\[
\begin{align*}
\partial \phi + \frac{1}{2} |\nabla \phi|^2 &= 0, \\
\phi(0, \cdot) &= \phi_0(\cdot).
\end{align*}
\]

When \(\phi_0 \in W^{2,\infty}\), by the method of characteristics, there exists \(T^* > 0\) such that the local solution satisfies \(\phi \in W^{2,\infty}([0, T^*], \mathbb{R}^d)\). Therefore, if the initial value in the above theorem satisfying \(c_0 \in H^{s+1}\) with \(s > \frac{d}{2} + 1\), we conclude that \(\phi^\infty \in W^{2,\infty} \cap C^1\), since \(H^{s+1}\) can be embedded into \(W^{2,\infty}\).

5.7 The model of deformation of flows on \(TP_2^\infty(\mathbb{R}^m, dx)\)

Let \(m \in \mathbb{N}, m \geq n\). In this subsection we introduce the model of Langevin deformation of flows \((5.63)\) and \((5.66)\) for \(V(\rho) = \int \rho \log \rho\) on \(TP_2^\infty(\mathbb{R}^m)\). As in the previous section, for simplicity we consider Laplace operator instead of Witten Laplacian and take \(\mu = \nu\).

Let \(u\) be a positive solution of the following ODE on \((0, T] \subset [0, \infty)\)

\[
c^2 u'' + u' = \frac{1}{2u},
\] (5.97)

with given initial datas \(u(0) > 0\) and \(u'(0) \in \mathbb{R}\). Note that, in the case \(c = 0\), \(u(t) = \sqrt{t}\) is a solution to \((5.97)\) on \((0, T]\) for any \(T > 0\), and in the case \(c = \infty\), \(u(t) = t\) is a solution to \((5.97)\) on \((0, T]\) for any \(T > 0\).

Theorem 5.10. Let \(u\) be a smooth solution to the ODE \((5.97)\). Let \(\alpha(t) = \frac{u'(t)}{u(t)}\), and \(\beta(t)\) be a smooth function such that

\[
c^2 \dot{\beta}(t) = -\beta(t) + m \log u(t) + \frac{m}{2} \log(4\pi) - 1,
\]

with a given initial data \(\beta(0) \in \mathbb{R}\). For \(x \in \mathbb{R}^m\) and \(t > 0\), let

\[
\rho(x, t) = \frac{1}{(4\pi u^2(t))^{m/2}} e^{-\frac{||x||^2}{4u^2(t)}},
\]

\[
\phi(x, t) = \frac{\alpha(t)}{2} ||x||^2 + \beta(t).
\]
Then \((\rho(x, t), \phi(x, t))\) satisfies the Langevin deformation of flows \((7.113)\) and \((7.114)\) on \(\mathcal{P}^2_\infty(\mathbb{R}^m)\).

**Proof.** Note that
\[
\nabla \phi(x, t) = \alpha(t)x.
\]
The transport equation \((7.113)\) has a special solution given by
\[
\rho(x, t) = \gamma^m(t)\rho_0(\gamma(t)x),
\]
where \(\rho_0(x)\) is any probability density on \(\mathbb{R}^m\) with respect to the Lebesgue measure, and \(\gamma\) is a smooth function in \(t\) which will be determined later. Indeed, we have
\[
\partial_t \rho = m\gamma^{m-1}\dot{\gamma}\rho_0(\gamma x) + \gamma^m\dot{\gamma}\langle \nabla \rho_0(\gamma x), x \rangle,
\]
\[
\nabla \cdot (\rho \nabla \phi) = m\gamma^m \alpha \rho_0(\gamma x) + \gamma^{m+1} \alpha \langle \nabla \rho_0(\gamma x), x \rangle.
\]
Thus, \((\rho, \phi)\) satisfies the transport equation if and only if
\[
\dot{\gamma} + \gamma \alpha = 0.
\]
Substituting \(\phi\) and \(\rho\) into \((7.114)\) leads to
\[
c^2 \left( \frac{\dot{\alpha}(t)||x||^2}{2} + \dot{\beta}(t) + \frac{\alpha^2(t)||x||^2}{2} \right) = -\frac{\alpha(t)||x||^2}{2} - \beta(t) - m \log \gamma(t) - \log \rho_0(\gamma(t)x) - 1.
\]
Changing the variable \(y = \gamma(t)x\), we have
\[
c^2 \left( \frac{\dot{\alpha}(t)||y||^2}{2\gamma^2(t)} + \dot{\beta}(t) + \frac{\alpha^2(t)||y||^2}{2\gamma^2(t)} \right) = -\frac{\alpha(t)||y||^2}{2\gamma^2(t)} - \beta(t) - m \log \gamma(t) - \log \rho_0(y) - 1.
\]
That is
\[
\left[ c^2 \left( \dot{\alpha}(t) + \alpha^2(t) \right) + \alpha(t) \right] \frac{||y||^2}{2\gamma^2(t)} + c^2 \dot{\beta}(t) = -\beta(t) - m \log \gamma(t) - \log \rho_0(y) - 1.
\]
In particular, taking
\[
\rho_0(y) = \frac{1}{(4\pi)^{\frac{m}{2}}} e^{-\frac{||y||^2}{4}},
\]
we can choose \(\alpha(t)\) and \(\beta(t)\) by solving the following ODEs
\[
c^2 \left( \dot{\alpha}(t) + \alpha^2(t) \right) + \alpha(t) = \frac{\gamma^2(t)}{2}, \quad (5.98)
\]
\[
c^2 \dot{\beta}(t) = -\beta(t) - m \log \gamma(t) + \frac{m}{2} \log(4\pi) - 1. \quad (5.99)
\]
Let \( u(t) = e^{\int_0^t \alpha(s) \, ds} \), and assume \( \gamma(0) = 1 \). Then \( \alpha = \frac{u'}{u} \), \( \dot{\alpha} = \frac{u''}{u} - \frac{u'^2}{u^2} \), \( \gamma(t) = \frac{1}{u(t)} \), and Eq. (5.98) is equivalent to

\[
c^2 \left( \frac{u''}{u} - \frac{u'^2}{u^2} + \frac{u'^2}{u^2} \right) + \frac{u'}{u} = \frac{1}{2u^2}.
\]

Thus \( u \) satisfies the following nonlinear ODE

\[
c^2 u'' + u' = \frac{1}{2u}.
\]

Note that, in the case \( c = 0 \), we can take \( \alpha(t) = \frac{1}{2t} \), \( u(t) = \sqrt{t} \) and \( \gamma(t) = \frac{1}{\sqrt{t}} \), \( t \in [0, T) \); and in the case \( c = \infty \), we can take \( \alpha(t) = \frac{1}{t} \), \( u(t) = t \) and \( \gamma(t) = \frac{1}{t} \), \( t \in [0, T) \).

\[\text{Remark 5.11. It is worth to point out that the equation (5.97) can be realized by finite dimensional Langevin deformation of flows (4.47) and (4.48) on } \mathbb{T} \mathbb{R} \setminus \{0\}. \text{ To this end, take } V(x) = -\frac{1}{2} \log |x| \text{ in (4.48), such that } V'(x) = -\frac{1}{2x}. \text{ Then } x \text{ satisfies the Langevin equation}
\]

\[
c^2 \ddot{x} + \dot{x} = \frac{1}{2x}.
\]

Given any initial position \( x(0) > 0 \) and initial velocity \( x'(0) \in \mathbb{R} \), there exists a unique solution \( x(t) \) to the above equations on a small interval \( [0, T) \subset [0, \infty) \) with the given initial datas \( x(0) \) and \( x'(0) \).

\[\text{Proposition 5.12. Let } (\rho_m, \phi_m) \text{ be defined as Theorem 5.10. Then}
\]

\[
\text{Ent}(\rho_m(t)) = -\frac{m}{2} (1 + \log(4\pi u^2(t))),
\]

\[
H(\rho_m(t), \phi_m(t)) = \frac{mc^2 u'(t)^2}{2} - \frac{m}{2} (1 + \log(4\pi u^2(t))).
\]

\[\text{Proof. — Indeed, } \rho_m(t)dx \text{ is the Gaussian distribution with variance } u^2(t).\]

\[\text{6 The } W \text{-entropy formula for the Langevin deformation of flows on Wasserstein space}
\]

In this section we prove Perelman’s type \( W \)-entropy formulas and monotonicity results for the Langevin deformation of flows on the Wasserstein space over compact Riemannian manifolds or on Euclidean spaces. Our results can be regarded as an interpolation between Langevin deformation of flowsthe \( W \)-entropy formula for the geodesic flow and the heat
flow on the Wasserstein space over compact Riemannian manifolds. We also provide the rigidity models for the W-entropy of the Langevin deformation of flows on the Wasserstein space over complete noncompact Riemannian manifolds. Langevin deformation of flows
In this section we only consider the case where the potential \( V \) is the Boltzmann-Shannon entropy \( V(\rho) = \text{Ent}(\rho) = \int_M \rho \log \rho d\mu \). We will study the case of the Rényi entropy \( V(\rho) = \frac{1}{m-1} \int_M \rho^m d\mu \) with \( m \neq 1 \) in future.

We first extend Proposition 4.2 to the Langevin deformation of flows \((\rho(t), \phi(t))\) on \( TP_2^\infty(M, \mu) \).

**Proposition 6.1.** Let \( c > 0 \). Let \( \rho, \phi \) be a smooth solution of the Langevin deformation of flows

\[
\begin{aligned}
\partial_t \rho - \nabla^* \mu(\rho \nabla \phi) &= 0, \\
c^2 \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) &= -\phi - \log \rho - 1.
\end{aligned}
\]

Then we have

\[
\frac{d^2}{dt^2} \text{Ent}(\rho) + \frac{1}{c^2} \frac{d}{dt} \text{Ent}(\rho) + \frac{1}{c^2} |\nabla \text{Ent}(\rho)|^2 = \int_M \left[ |\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu.
\]

(6.100)

In particular, if \( \text{Ric}(L) \geq K \), then for all \( c > 0 \), we have

\[
\frac{d^2}{dt^2} \text{Ent}(\rho) + \frac{1}{c^2} \frac{d}{dt} \text{Ent}(\rho) + \frac{1}{c^2} |\nabla \text{Ent}(\rho)|^2 \geq \frac{1}{n} \int_M |\Delta \phi|^2 \rho d\mu + K \int_M |\nabla \phi|^2 \rho d\mu.
\]

(6.101)

**Proof.** — By the same argument as used in the proof of Proposition 4.2 and (2.33), we can prove

\[
\frac{d^2}{dt^2} \text{Ent}(\rho) + \frac{1}{c^2} \frac{d}{dt} \text{Ent}(\rho) + \frac{1}{c^2} |\nabla \text{Ent}(\rho)|^2 = \text{Hess Ent}(\dot{\rho}, \dot{\rho}) = \int_M \left[ |\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu,
\]

which leads to (6.101) if \( \text{Ric}(L) \geq K \) by using the inequality \( |\text{Hess} \phi|^2 \geq \frac{1}{n} |\Delta \phi|^2 \).

### 6.1 Proof of Theorem 1.8

**Proof.** — The proof has the same spirit as those for Theorem 1.1 and Theorem 1.3. By Otto’s calculation on \( P_2(M, \mu) \), we have

\[
\frac{d}{dt} \text{Ent}(\rho(t)) = \nabla \text{Ent}(\rho(t)) \cdot \dot{\rho}(t) = -\int_M L\phi \rho d\mu.
\]
By Theorem 2.3 we have
\[
\frac{d^2}{dt^2} \text{Ent}(\rho(t)) = \nabla^2 \text{Ent}(\rho(t)) (\dot{\rho}(t), \ddot{\rho}(t)) + \nabla \text{Ent} \cdot \dot{\rho}(t)
\]
\[
= \int_M (|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho d\mu + \int_M \nabla (\log \rho + 1) \cdot \nabla \left( \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) \rho d\mu
\]
\[
= \int_M (|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho d\mu + \frac{1}{c^2} \int_M \nabla \rho \cdot \nabla (\phi - \log \rho - 1) d\mu
\]
\[
= \int_M (|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho d\mu - \frac{1}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} d\mu + \frac{1}{c^2} \int_M L \phi \rho d\mu.
\]
Then, similarly to the proof of Theorem 1.1 for the case \(m = n\) and \(L = \Delta\), we derive that
\[
\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + (2\alpha(t) + c^{-2}) \frac{d}{dt} \text{Ent}(\rho(t)) + c^{-2} \|\nabla \text{Ent}(\rho(t))\|^2 + m\alpha^2(t)
\]
\[
= \int_M [\|\text{Hess} \phi\|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)] \rho d\nu - 2\alpha(t) \int_M \Delta \phi \rho d\nu + m\alpha^2(t)
\]
\[
= \int_M [\|\text{Hess} \phi - \alpha(t) g\|^2 + \text{Ric}(\nabla \phi, \nabla \phi)] \rho d\nu.
\]
For the case \(m > n\), similar computations lead to
\[
\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + (2\alpha(t) + c^{-2}) \frac{d}{dt} \text{Ent}(\rho(t)) + c^{-2} \|\nabla \text{Ent}(\rho(t))\|^2 + m\alpha^2(t)
\]
\[
= \int_M [\|\text{Hess} \phi\|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)] \rho d\mu - 2\alpha(t) \int_M L \phi \rho d\mu + m\alpha^2(t)
\]
\[
= \int_M [\|\text{Hess} \phi - \alpha(t) g\|^2 + \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi)] \rho d\mu + (m - n) \int_M \left| \alpha(t) + \frac{\nabla \phi \cdot \nabla f}{m - n} \right|^2 \rho d\mu,
\]
where the last line is due to
\[
\text{Ric}(L)(\nabla \phi, \nabla \phi) + 2\alpha \nabla \phi \cdot \nabla f + (m - n) \alpha^2
\]
\[
= \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) + \frac{\|\nabla \phi \cdot \nabla f\|^2}{m - n} + 2\alpha \nabla \phi \cdot \nabla f + (m - n) \alpha^2
\]
\[
= \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) + (m - n) \left| \alpha + \frac{\nabla \phi \cdot \nabla f}{m - n} \right|^2.
\]
This completes the proof of Theorem 1.8.

6.2 The \(W\)-entropy for the Langevin deformation and rigidity model

Based on the formulas 1.22 and 1.17, we now introduce the \(W\)-entropy for the Langevin deformation of flows, as stated in Section 1. More precisely, we define the \(W\)-entropy for
the Langevin deformation of flows as follows: for all $0 < t_0 < t < \infty$,

$$W_c(\rho(t)) - W_c(\rho(t_0)) = \frac{d}{dt} \text{Ent}(\rho(t)) + \int_{t_0}^{t} \left( 2\alpha(s) + \frac{1}{c^2} \right) \frac{d}{ds} \text{Ent}(\rho(s)) ds$$

$$+ \frac{1}{c^2} \int_{t_0}^{t} \|\nabla \text{Ent}(\rho(s))\|^2 ds,$$

such that its time derivative is given by

$$\frac{d}{dt} W_c(\rho(t)) = \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left( 2\alpha(t) + \frac{1}{c^2} \right) \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{1}{c^2} \|\nabla \text{Ent}(\rho(t))\|^2. \quad (6.103)$$

**Proposition 6.2.** For the model $(\rho_m, \phi_m)$ on $(\mathbb{R}^m, dx)$, we have

$$\frac{d}{dt} W_c(\rho_m(t)) = -m \alpha^2(t). \quad (6.104)$$

**Proof.** — Indeed

$$\text{Hess} \phi_m = \alpha(t) g, \quad \text{Ric} = 0,$$

from which and Theorem 1.8 we derive (6.104).

### 6.3 Comparison theorem for the W-entropy for Langevin deformation

As a direct consequence of Theorem 1.8 and (6.104), we can derive the following comparison theorem for the $W$-entropy for Langevin deformation of flows.

**Theorem 6.3.** (i.e., Theorem 1.10)

$$\frac{d}{dt} (W_c(\rho(t)) - W_c(\rho_m(t))) = \int_M |\text{Hess} \phi - \alpha(t) g|^2 \rho d\mu + \int_M \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \rho d\mu$$

$$+ \frac{1}{m-n} \int_M |\nabla f \cdot \nabla \phi + (m-n)\alpha(t)|^2 \rho d\mu.$$

In particular, if $\text{Ric}_{m,n}(L) \geq 0$, then for all $t > 0$, we have the comparison theorem

$$\frac{d}{dt} W_c(\rho(t)) \geq \frac{d}{dt} W_c(\rho_m(t)). \quad (6.106)$$

We would like to point out that, similarly to the proof of Theorem 3.5, we can extend the above $W$-entropy formula to complete Riemannian manifolds with bounded geometry.
condition. To save the length of the paper, we omit the detail of this technical part. In view of this, the rigidity model of the \(W\)-entropy for the Langevin deformation of flows on complete Riemannian manifolds with non-negative Ricci curvature or complete and weighted Riemannian manifolds with \(\text{CD}(0, m)\)-condition should be \(M = \mathbb{R}^m\), \(m \in \mathbb{N}\), and \((\rho_m, \phi_m)\) given in Theorem 5.10.

To end this paper, let us give the following remark which illustrate that the \(W\)-entropy formula (1.17) is an interpolation between those of the geodesic flow and of the the heat flow on the Wasserstein space.

**Remark 6.4.** Indeed, when \(c \to \infty\), we have \(u(t) \to t\), \(\alpha(t) \to \frac{1}{t}\). This yields

\[
\lim_{c \to \infty} \frac{d}{dt} W_c(\rho(t)) = \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)).
\]

Hence the time derivative of the \(W\)-entropy for the geodesic flow should be given by

\[
\frac{d}{dt} W(\rho(t)) = \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)),
\]

and the \(W\)-entropy formula reads as (i.e., Theorem 1.1)

\[
\frac{d}{dt} (W(\rho(t)) - W(\rho_m(t))) = \int_M \left| \text{Hess} \phi - \frac{g}{t} \right|^2 \rho d\mu + \int_M \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \rho d\mu + \int_M \left| \nabla f \cdot \nabla \phi + \frac{m-n}{2t} \right|^2 \rho d\mu.
\]

On the other hand, when \(c \to 0\), we have \(u(t) \to \sqrt{t}\), \(\alpha(t) \to \frac{1}{2t}\). Moreover, the convergence results lead to

\[
\frac{d}{dt} \text{Ent}(\rho(t)) = \nabla \text{Ent}(\rho(t)) \cdot \dot{\rho}(t) = -\|\nabla \text{Ent}(\rho(t))\|^2.
\]

This yields

\[
\lim_{c \to 0} \frac{d}{dt} W_c(\rho(t)) = \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{1}{t} \frac{d}{dt} \text{Ent}(\rho(t)).
\]

Hence the time derivative of the \(W\)-entropy for the gradient flow should be given by

\[
\frac{d}{dt} W(\rho(t)) = \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{1}{t} \frac{d}{dt} \text{Ent}(\rho(t)),
\]

and the \(W\)-entropy formula reads as (i.e., Theorem 1.3)

\[
\frac{d}{dt} (W(\rho(t)) - W(\rho_m(t))) = \int_M \left| \text{Hess} \phi - \frac{g}{2t} \right|^2 \rho d\mu + \int_M \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \rho d\mu + \int_M \left| \nabla f \cdot \nabla \phi + \frac{m-n}{2t} \right|^2 \rho d\mu.
\]

Note that the factor in front of \(\frac{d}{dt} \text{Ent}(\rho(t))\) in the right hand side of the \(W\)-entropy formula is \(\frac{1}{t}\) for \(c = \infty\) but is \(\frac{2}{t}\) for \(c = 0\). For its reason, see Section 3.5.
7 Convergence of the Langevin deformation

7.1 The convergence of the deformation flow in finite dimension

Let \((x, v)\) be the smooth solution to the following Langevin equation on \(\mathbb{R}^d \times [0, T]\)

\[
\dot{x} = v, \quad \dot{v} = -v - \nabla V(x),
\]

with the initial data

\[
x(0) = x_0, \quad v(0) = v_0.
\]

The existence and uniqueness of solution to (7.107) and (7.108) follow from the Cauchy-Lipschitz theorem in ODE theory. The following convergence results can be proved by standard \(C^2\)-estimates. To save the length of this paper, we omit the details of proof.

**Theorem 7.1.** Let \(V \in C^2(\mathbb{R}^d)\) and assume that \(\nabla V\) is Lipschitz, i.e., there exists a constant \(K > 0\) such that

\[
|\nabla V(x_1) - \nabla V(x_2)| \leq K|x_1 - x_2|, \quad \forall \, x_1, x_2 \in \mathbb{R}^d.
\]

Then there exists \(T > 0\), such that the equations (7.107) and (7.108) have a unique solution \((x_t, v_t) \in C([0, T], C^2(\mathbb{R}^d)) \times C^1(\mathbb{R}^d))\). Moreover, when \(c \to 0\), the solution to (7.107), (7.108) uniformly converges on \([0, T]\) to the solution to the gradient flow of \(V\)

\[
\dot{y} = -\nabla V(y),
\]

\[
y(0) = x_0,
\]

and when \(c \to \infty\), the solution to (7.107), (7.108) uniformly converges on \([0, T]\) to the solution to the geodesic equation

\[
\ddot{z} = 0,
\]

\[
z(0) = x_0, \quad \dot{z}(0) = v_0.
\]

Indeed, letting \(y(t) \in C([0, T], C^2(\mathbb{R}^d))\) be the unique solution to (7.109), and \(z(t) \in C([0, T], C^2(\mathbb{R}^d))\) be the unique solution to (7.111), then

\[
\lim_{c \to 0} \sup_{t \in [0, T]} |x(t) - y(t)| = 0,
\]

\[
\lim_{c \to \infty} \sup_{t \in [0, T]} |x(t) - z(t)| = 0.
\]

Moreover, if \(\dot{x}(0) = \dot{y}(0) = \dot{z}(0), \ddot{x}(0) = \ddot{y}(0) = \ddot{z}(0)\), and if \(\nabla^2 V\) is \(K_2\)-Lipschitz and uniformly bounded, then
\[
\lim_{c \to 0} \sup_{t \in [0,T]} [|\dot{x}(t) - \dot{y}(t)| + |\ddot{x}(t) - \ddot{y}(t)|] = 0, \\
\lim_{c \to \infty} \sup_{t \in [0,T]} [|\dot{x}(t) - \dot{z}(t)| + |\ddot{x}(t)|] = 0.
\]

7.2 The convergence of the Langevin deformation of flows on Wasserstein space: \( c \to 0 \)

In this section we consider the Langevin deformation of flows on \( \mathbb{R}^d \) or compact manifold \( M \),

\[
\partial_t \rho + \nabla \cdot (\rho \nabla \phi) = 0, \tag{7.113}
\]

\[
c^2 (\partial_t \phi + \frac{1}{2} |\nabla \phi|^2) = -\phi - \log \rho - 1, \tag{7.114}
\]

with initial value

\[
\rho(0, x) = \rho_0(x), \quad \phi(0, x) = \phi_0(x).
\]

We will first discuss the convergence in Euclidean space when \( c \) approaches 0 and infinity respectively, and then extend the results to compact manifolds. For simplicity, we consider the Laplace-Beltrami operator here instead of the Witten Laplacian. Under suitable assumptions on the potential function \( f \), it is easy to prove that the convergence results also hold for the Witten Laplacian.

Consider the Cauchy problem of (7.113) and (7.114). In this section we start with the convergence results for the compressible Euler equation with damping. Let \( \nabla \phi = u \).

Taking differential on both sides of (7.114), we obtain

\[
\partial_t \rho^c + \nabla \cdot (\rho^c \nabla \phi^c) = 0, \tag{7.115}
\]

\[
c^2 (\partial_t \phi^c + \frac{1}{2} |\nabla \phi^c|^2) = -\phi^c - \frac{\nabla \rho^c}{\rho^c} \cdot \nabla u^c. \tag{7.116}
\]

In the following we use the notation \((\rho^c, u^c)\) instead of \((\rho, u)\) to emphasis its relevance with \( c \).

7.2.1 The relative entropy method for the convergence of local solution

Inspired by the works of Lattanzio-Tzavaras [19] and [20], we apply the relative entropy method to study the convergence of local solutions \((\rho^c, u^c)\). In [19], the method was employed to study the diffusive limit of entropy weak solutions to compressible isentropic Euler equation with damping only in one dimension. In this section we apply the relative entropy method to prove the convergence of entropy solutions for Langevin deformation of flows. It has its own interest to use the relative entropy method to prove the convergence,
since for the compressible Euler equation with damping, $L^\infty$ solution is more natural (because of the shocks) than local smooth solutions. As the well posedness of entropy solution to the compressible Euler equation with damping has not yet been established in high dimension, we will focus the case on one dimension.

For any \( \rho, \bar{\rho} \in \mathcal{P}(\mathbb{R}^d) \), define the density for the Boltzmann entropy as
\[
h(\rho) = \rho \log \rho.
\]
Then
\[
h'(\rho) = \frac{1}{\rho},
\]
and for the relative Boltzmann entropy \( h(\rho | \bar{\rho}) \) between \( \rho \) and \( \bar{\rho} \), we have
\[
h(\rho | \bar{\rho}) = h(\rho) - h(\bar{\rho}) - h'(\bar{\rho})(\rho - \bar{\rho}) = (\rho - \bar{\rho})^2 \int_0^1 \int_0^\tau h''(s\rho + (1-s)\bar{\rho}) dsd\tau \geq 0.
\]

Moreover, the density for the kinetic energy for \( (\rho, m) \) is given by
\[
k(\rho^c, m^c) = \frac{1}{2} \rho^c ||u^c||^2.
\]
We may also define the relative kinetic energy between \( (\rho, m) \) and \( (\bar{\rho}, \bar{m}) \) as
\[
k((\rho, m) | (\bar{\rho}, \bar{m})) = k(\rho, m) - k(\bar{\rho}, \bar{m}) - \langle \nabla k(\rho, m), (\rho - \bar{\rho}, m - \bar{m}) \rangle = \frac{1}{2} \rho \|u - \bar{u}\|^2.
\]

For (7.115) and (7.116), we define the entropy as
\[
\eta(\rho^c, m^c) = \rho^c \log \rho^c + \frac{c^2}{2} \rho^c ||u^c||^2,
\]
and the flux
\[
q(\rho^c, m^c) = c(1 + \log \rho^c)m^c + \frac{c^3}{2(\rho^c)^2} m^c,
\]
where \( m^c = \rho^c u^c \).

Let \( \rho^0 \) be the solution to the heat equation
\[
\frac{\partial \rho^0}{\partial t} = \Delta \rho^0,
\]
\[
\rho^0(0, x) = \rho_0(x),
\]
such that \( m^0 = \nabla \rho^0 \).

Thus the relative entropy between \( (\rho^c, m^c) \) and \( (\rho^0, m^0) \) is given by
\[
\eta(\rho^c, m^c | \rho^0, m^0) = \text{Ent}(\rho^c | \rho^0) + c^2 K(\rho^c, m^c | \rho^0, m^0)
\]
\[
= \int_{\mathbb{R}^d} \rho^c (\log \rho^c - \log \rho^0) + \frac{c^2}{2} \int_{\mathbb{R}^d} \rho^c ||u^c - u^0||^2.
\]
We have the following results for the relative entropy \( \eta(\rho^c, m^c | \rho^0, m^0) \).
Proposition 7.2. For smooth solution to the compressible Euler equation, we have

\[ \eta(\rho^c, m^c|\rho^0, m^0)(t) = \int_{\mathbb{R}^d} \rho(\log \rho - \log \rho^0) + \frac{c^2}{2} \int_{\mathbb{R}^d} \rho \| u^c - u^0 \|^2 \]

\[ \leq (\eta(\rho^c, m^c|\rho^0, m^0)(0) + c^4 \int_{\mathbb{R}^d} \rho^c |\Delta u^0 - u^0 \cdot \nabla u^0| e^{2\|\nabla u^0\|_{\infty}t}). \] (7.117)

Proof. — Notice that for smooth solutions, we have

\[ \partial_t \eta(\rho^c, m^c) + \frac{1}{c} \nabla \cdot q(\rho^c, m^c) = -\frac{|m^c|^2}{\rho^c}, \]

and \( u^c \) satisfies

\[ \partial_t u^c = -u^c \cdot \nabla u^c - c^{-2} u^c - c^{-2} \nabla \log \rho^c. \]

Thus we derive that

\[ \frac{d}{dt} \left( \int_{\mathbb{R}^d} \rho^c \log \rho^c + \frac{c^2}{2} \int_{\mathbb{R}^d} \rho^c |u^c|^2 \right) = - \int_{\mathbb{R}^d} \rho^c |u^c|^2. \] (7.118)

On the other hand, by the integration by parts formula, we have

\[ \frac{d}{dt} \left( -\int_{\mathbb{R}^d} \rho^c \log \rho^0 \right) = \int_{\mathbb{R}^d} \rho^c \langle u^c, \nabla \log \rho^0 \rangle - \int_{\mathbb{R}^d} (\nabla \rho^c, u_0) + \int_{\mathbb{R}^d} \rho^c \langle u^0, \nabla \log \rho^0 \rangle. \]

Moreover, since \( u^0 = -\nabla \log \rho^0 \) satisfies

\[ \partial_t u^0 = -2u^0 \cdot \nabla u^0 + \Delta u^0, \]

we derive that

\[ \frac{d}{dt} \int_{\mathbb{R}^d} \rho^c (|u^0|^2 - 2u^c \cdot u^0) \]

\[ = 2 \int_{\mathbb{R}^d} \rho \langle u^c, (u^0 - u^c) \cdot \nabla u^0 \rangle + 2 \int_{\mathbb{R}^d} \rho \langle (u^0 - u^c) \cdot (\Delta u^0 - 2\nabla u^0 \cdot u^0) + c^{-2}(u^c + \nabla \log \rho^c) \cdot u^0 \rangle. \]

Combining the above terms together leads to

\[ \frac{d}{dt} \eta(\rho^c, m^c|\rho^0, m^0) = \frac{d}{dt} \left( \int_{\mathbb{R}^d} \rho^c (\log \rho^c - \log \rho^0) + \frac{c^2}{2} \int_{\mathbb{R}^d} \rho^c \| u^c - u^0 \|^2 \right) \]

\[ \leq - \int_{\mathbb{R}^d} \rho^c \| u^c - u^0 \|^2 - c^2 \int_{\mathbb{R}^d} \rho^c \langle u^c - u^0, (u^c - u^0) \cdot \nabla u^0 \rangle \]

\[ + c^2 \int_{\mathbb{R}^d} \rho^c (u^0 - u^c) \cdot (\Delta u^0 - u^0 \cdot \nabla u^0). \]
By the Cauchy-Schwarz inequality, we have
\[
(u^0 - u^c) \cdot (\Delta u^0 - u^0 \cdot \nabla u^0) \leq \frac{1}{2}(\|u^0 - u^c\|^2 + \|\Delta u^0 - u^0 \cdot \nabla u^0\|^2).
\]
Together with the fact that \(\int_{\mathbb{R}^d} \rho^c (\log \rho^c - \log \rho^0) \geq 0\) we derive that
\[
\frac{d}{dt} \eta(\rho^c, m^c|\rho^0, m^0) 
\leq (4 + 2\|\nabla u^0\|_\infty) \left( \frac{c^2}{2} \int_{\mathbb{R}^d} \rho^c \|u^c - u^0\|^2 + \int_{\mathbb{R}^d} \rho^c (\log \rho^c - \log \rho^0) \right) + 2c^2 \|\Delta u^0 - u^0 \cdot \nabla u^0\|_\infty^2,
\]
i.e.
\[
\eta(\rho^c, m^c|\rho^0, m^0)(t) - \eta(\rho^c, m^c|\rho^0, m^0)(0) 
\leq \int_0^t (4 + 2\|\nabla u^0\|_\infty) \eta(\rho^c, m^c|\rho^0, m^0)(s) + 2c^2 \int_0^t \|\Delta u^0 - u^0 \cdot \nabla u^0\|_\infty^2(s) ds.
\]
Therefore by Gronwall’s inequality, we conclude that
\[
\eta(\rho^c, m^c|\rho^0, m^0)(t) 
\leq \left( \eta(\rho^c, m^c|\rho^0, m^0)(0) + 2c^2 \int_0^t \|\Delta u^0 - u^0 \cdot \nabla u^0\|_\infty^2(s) ds \right) e^{\int_0^t (4 + 2\|\nabla u^0\|_\infty) ds}.
\]

In our case, \(\eta(\rho^c, m^c|\rho^0, m^0)(0) = 0\), therefore (7.117) implies that when \(c \to 0\), \(h(\rho^c|\rho^0) \to 0\). Applying classical Csiszár-Kullback inequality, we obtain
\[
\int_{\mathbb{R}^d} |\rho^c - \rho_0| dx \leq h(\rho^c|\rho^0) \to 0.
\]
thus \(\rho^c\) converges to \(\rho_0\) in \(L^1([0, T] \times \mathbb{R}^d)\).

**Theorem 7.3.** Let \(\rho_0, u_0 \in H^s(\mathbb{R}^d)\) be the initial value, and there exists local solution \((\rho, u)\) in \(C([0, T], H^s)\) to (7.115) and (7.116). Then as \(c \to 0\), we have
\[
\sup_{t \in [0, T]} \|\rho - \rho_0\|_{L^1} \to 0.
\]

### 7.2.2 The convergence of strong solutions with small initial value

Inspired by the work of Coulombel-Goudon [11], we show the convergence of strong solutions in this part. Indeed, when \(c \to 0\), the convergence of (7.115), (7.116) is equivalent
to the strong relaxation limit of the isothermal Euler equations. Recall that Coulombel and Goudon [11] consider the isothermal Euler equations

\[ \partial_t \rho^c + \nabla \cdot (\rho^c \rho^c) = 0, \quad \partial_t \rho^c + \nabla \cdot (\rho^c \rho^c) + a^2 \nabla \nabla \nabla = \frac{1}{c^2} \rho^c, \]

with initial values

\[ \rho^c(0, x) = \rho_0(x), \quad \rho^c(0, x) = m_0(x), \]

where \( \rho^c : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow (0, +\infty) \) is the density, \( m^c : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) is the momentum, \( a > 0 \) is the speed of sound and \( 0 < c < 1 \) is the relaxation time. We may write \( m^c = \rho^c v^c \), where \( v^c \) is the velocity. According to the proof in [11], the value of \( a \) has no impact on the results, thus we take \( a = 1 \) in what follows.

Now define

\[ \rho^c(t, x) = \rho^c(t_c, x), \quad u^c(t, x) = \frac{1}{c} v^c(t_c, x), \]

then \( \rho^c \) and \( u^c \) satisfy

\[ \partial_t \rho^c + \nabla \cdot (\rho^c \rho^c) = 0, \quad \partial_t (\rho^c u^c) + \nabla \cdot (\rho^c u^c \otimes u^c) + \frac{1}{c^2} (\nabla \rho^c + \rho^c \nabla u^c) = 0, \]

which are exactly Equations (7.115), (7.116).

In [11], the authors proved that for small initial data, Equations (7.119), (7.120) admit a global solution, which satisfies a uniformly energy estimate. With the above scaling procedure, their uniform estimate leads to the following result for \( (\rho^c, u^c) \).

**Theorem 7.4.** Let \( \bar{\rho} > 0, \) and \( s \in \mathbb{N} \) with \( s > \frac{d}{2} + 1 \). Then there exists two constants \( \delta > 0 \) and \( C > 0 \) such that for any \( c \in (0, 1) \) and for initial data \( (\rho_0, u_0) \) satisfying

\[ \|\rho_0 - \bar{\rho}\|_{H^s(\mathbb{R}^d)} + \|\rho_0 u_0\|_{H^s(\mathbb{R}^d)} \leq \delta, \]

there exists a unique global solution \( (\rho^c, u^c) \) to (7.115), (7.116), such that \( (\rho^c - \bar{\rho}, u^c) \in C(\mathbb{R}^+ \times [0, T], H^s(\mathbb{R}^d)) \) satisfies

\[ \sup_{t \geq 0} (\|\rho^c - \bar{\rho}\|_{H^s(\mathbb{R}^d)}^2 + \|\rho^c u^c\|_{H^s(\mathbb{R}^d)}^2) + \int_0^\infty \|\rho^c u^c\|_{H^s(\mathbb{R}^d)}^2 ds \leq C(\|\rho_0 - \bar{\rho}\|_{H^s(\mathbb{R}^d)}^2 + \|\rho_0 u_0\|_{H^s(\mathbb{R}^d)}^2). \]

By the same argument as used in [11], we can prove the following convergence result for the strong solution with small initial value. To save the length of the paper, we omit the proof.

**Theorem 7.5.** Let \( (\rho^c, u^c) \) be the solution to Equations (7.115) and (7.116) with initial data \( (\rho_0, u_0) \). Let \( B_r \) be the ball in \( \mathbb{R}^d \) centered at the origin of radius \( r \). Under the same assumption as in Theorem 7.4, for any \( 0 < T, R < \infty, 0 < s' < s \), \( \rho^c \) converges in \( C([0, T], H^s(B_R)) \) towards \( \rho^0 \in C(\mathbb{R}^+, \bar{\rho} + H^s(\mathbb{R}^d)) \), which is the unique solution to the Cauchy problem of the heat equation

\[ \partial_t \rho^0 = \Delta \rho^0, \quad \rho^0(0, x) = \rho_0(x). \]
7.3 The convergence of the Langevin deformation of flows on Wasserstein space: \( c \to \infty \)

In this part we turn to the case when \( c \to \infty \). First rewrite (7.115), (7.116) into the following symmetric hyperbolic equations

\[
\begin{align*}
\partial_t \log \rho^c + u^c \cdot \nabla \log \rho^c + \nabla \cdot u^c &= 0, \\
c^2 \partial_t u^c + c^2 u^c \cdot \nabla u^c + \nabla \log \rho^c + u^c &= 0.
\end{align*}
\] (7.125) (7.126)

Let \( U^c = \begin{pmatrix} \log \rho^c \\ u^c \end{pmatrix} \), then the above equations turn into

\[
A_0(c) \partial_t U^c + \sum_{j=1}^n A_j(U^c, c) \partial_j U^c + B(0)U^c = 0,
\] (7.127)

where \( A_0(c) = \begin{pmatrix} 1 & 0 \\ 0 & c^2 \text{Id}_d \end{pmatrix}, B(0) = \begin{pmatrix} 0 & 0 \\ 0 & \text{Id}_d \end{pmatrix}, \) and

\[
A_j(U^c, c) = \begin{pmatrix} u^j & 0 & \cdots & 1 & \cdots & 0 \\ 0 & c^2 u^j & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 1 & 0 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & c^2 u^j \end{pmatrix}_{(d+1) \times (d+1)}.
\]

Following the proof in Klainerman-Majda [15, 16, 34], we obtain the convergence result.

**Theorem 7.6.** Let \( s \in \mathbb{N} \) with \( s > \frac{d}{2} + 1 \). Let \((\rho^c, u^c)\) be the solution to (7.115), (7.116), and there exists a constant \( M > 0 \), such that the initial data \((\rho_0, u_0)\) satisfies

\[
\| \log \rho_0 \|^2_{H^s} + \| u_0 \|^2_{H^s} \leq M.
\]

Then there exists \( T > 0 \), which is independent of \( c > 1 \), such that when \( c \to \infty \), \( \rho^c dx \) weakly converges to \( \rho^\infty dx \) in \( C([0, T], \mathcal{P}(\mathbb{R}^d)) \), and \( u^c \) converges to \( u^\infty(x, t) \in C([0, T], H^s) \cap C^1([0, T], H^{s-1}) \) in \( C([0, T], H^{s-1}(B_R)) \) for any \( R > 0, s' < s \). Moreover, \((\rho^\infty, u^\infty)\) satisfies

\[
\begin{align*}
\partial_t \rho^\infty + \nabla \cdot (\rho^\infty u^\infty) &= 0, \\
\partial_t u^\infty + u^\infty \cdot \nabla u^\infty &= 0,
\end{align*}
\]

\[
\rho^\infty(0, x) = \rho_0(x), \quad u^\infty(0, x) = u_0(x).
\] (7.128) (7.129) (7.129)
Remark 7.7. The limit equation (7.129) is a pressureless Euler equation, and the equations (7.128), (7.129) are called sticky particle system. There are a lot of research regarding this topic, see [7], [13] for the one-dimensional case, and [43] for the existence of measure solution in high dimension. As pointed out in [7], this system can formally be obtained through the compressible Euler equation with the pressure term approaching zero or the Boltzmann equation with the temperature going to zero. In [10] the authors rigorously proved that the entropy solution to the compressible Euler equation converges to that to the pressureless Euler equation when the pressure term goes to zero in one dimension.

Remark 7.8. As pointed out by Villani in [52], shocks do not arise in the case of optimal transportation. One could prove that for the geodesic flow, the velocity field $u_t$ is locally Lipschitz in both $t$ and $x$, see Theorem 5.51 in [52].

According to Klainerman-Majda [15, 16, 34], to prove Theorem 7.6, we need first to prove the following uniform a priori estimate.

Proposition 7.9. Assume that for initial value $(\log \rho_0, u_0)$, there exist two constants $M > 0$ and $s > \frac{d}{2} + 1$, such that

$$\| \log \rho_0 \|^2_{H^s} + \| u_0 \|^2_{H^s} \leq M. \quad (7.130)$$

Then there exists $T > 0$, which is independent of $c$, such that for any $c \geq 1$, the equation (7.127) admits classical solution $\log \rho^c, u^c \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$, satisfying

$$\sup_{[0,T]} c^{-2} \| \log \rho^c \|^2_{H^s} + \| u^c \|^2_{H^s} \leq R, \quad (7.131)$$

where $R$ is a constant independent of $c$.

To see this, we first prove the following a priori estimate as in [34].

Lemma 7.10. Assume that the conditions in Theorem 7.9 hold, and the equation (7.127) admits local classical solution $(\log \rho^c, u^c)$ on $[0, T^*_c]$, satisfying

$$\sup_{[0,T^*_c]} c^{-2} \| \log \rho^c \|^2_{H^s} + \| u^c \|^2_{H^s} \leq 2M, \quad (7.132)$$

for all $c \geq 1$. Then there exists a constant $C > 0$, which is independent of $c$, such that

$$\sup_{[0,T^*_c]} c^{-2} \| \log \rho^c \|^2_{H^s} + \| u^c \|^2_{H^s} \leq (c^{-2} \| \log \rho_0 \|^2_{H^s} + \| u_0 \|^2_{H^s}) e^{C(2M+1)T^*_c}. \quad (7.133)$$

Proof. — We apply the classical energy estimate method. First, multiply $U$ on both sides of the equation (7.127) and integrate on $[0, t]$, then we obtain

$$\langle A_0(c)U, U \rangle_0^t - \sum_j \int_0^t \langle \partial_j A_j U, U \rangle + 2 \int_0^t \langle B(0)U, U \rangle = 0.$$
Thus,
\[
\| \log \rho^c \|^2_{L^2(t)} + c^2 \| u^c \|^2_{L^2(t)} \\
\leq \| \log \rho_0 \|^2_{L^2} + c^2 \| u_0 \|^2_{L^2} + \int_0^t \| \partial_j u_j \|^2_{L^\infty}(\| \log \rho^c \|^2_{L^2}(s) + c^2 \| u^c \|^2_{L^2}(s))ds.
\]

By Sobolev inequality, for \( s > \frac{d}{2} + 1 \) we have
\[
\| \nabla u \|^2_{L^\infty} < C_s \| u \|^2_{H^s},
\]
which leads to
\[
c^{-2} \| \log \rho^c \|^2_{L^2(t)} + \| u^c \|^2_{L^2(t)} \leq c^{-2} \| \log \rho_0 \|^2_{L^2} + \| u_0 \|^2_{L^2} + C_s \int_0^t \| u^c \|^2_{H^s}(c^{-2} \| \log \rho^c \|^2_{L^2}(s) + \| u^c \|^2_{L^2}(s))ds.
\]

Now applying differential operator \( D^\alpha \) on both sides of the equation (7.127), where \( 1 \leq \alpha \leq s \), we obtain
\[
A_0(c)\partial_t D^\alpha U + \sum_{j=1}^n A_j(u, c)\partial_j D^\alpha U + B(0)D^\alpha U = -\sum_{\beta=0}^{\alpha-1} \sum_{j=1}^n D^{\alpha-\beta} (A_j(u, c)) D^\beta \partial_j U.
\]

Again using the energy estimate method, we derive that
\[
\langle A_0(c)D^\alpha U, D^\alpha U \rangle(t) \leq \langle A_0(c)D^\alpha U, D^\alpha U \rangle(0) + \int_0^t \left( \sum_{\beta=0}^{\alpha-1} \sum_{j=1}^n \| \partial_j A_j(u, c) \|^2_{L^\infty}(\| D^\alpha \log \rho^c \|^2_{L^2} + c^2 \| D^\alpha u^c \|^2_{L^2})ds \\
+ \sum_{\beta=0}^{\alpha-1} \sum_{j=1}^n \int_0^t (\| D^{\alpha-\beta} u_j^c \|^2_{L^2}(\| D^\beta \partial_j \log \rho^c \|^2_{L^2} + c^2 \| D^\beta \partial_j u^c \|^2_{L^2}) + (\| D^{\alpha} \log \rho^c \|^2_{L^2} + c^2 \| D^\alpha u^c \|^2_{L^2}))ds,
\]
where the second inequality is due to the Cauchy-Schwarz inequality
\[
\| ab \|_{L^1} \leq \| a \|_{L^2} \| b \|_{L^2},
\]

57
and also
\[
-2 \int_0^t \left( \sum_{\alpha=0}^{n-1} \sum_{j=1}^n D^{\alpha-\beta}(A_j(u,c)) D^\beta \partial_j U, D^\alpha U \right)(s) ds
\]
\[
= -2 \int_0^t \sum_{\beta=0}^{n-1} \sum_{j=1}^n \left( (D^{\alpha-\beta} u_j^c D^\beta \partial_j \log \rho^c, D^\alpha \log \rho^c) + c^2 (D^{\alpha-\beta} u_j^c D^\beta \partial_j u^c, D^\alpha u^c) \right) ds
\]
\[
\leq \int_0^t \sum_{\beta=0}^{n-1} \sum_{j=1}^n \left( \|D^{\alpha-\beta} u_j^c\|^2_{L^2} \|D^\beta \partial_j \log \rho^c\|^2_{L^2} + c^2 \|D^\beta \partial_j u^c\|^2_{L^2} + (\|D^\alpha \log \rho^c\|^2_{L^2} + c^2 \|D^\alpha u^c\|^2_{L^2}) \right) ds.
\]

Therefore we have
\[
c^{-2} \|D^\alpha \log \rho^c\|^2_{L^2(t)} + \|D^\alpha u^c\|^2_{L^2(t)} \leq c^{-2} \|D^\alpha \log \rho_0\|^2_{H^s} + \|D^\alpha u_0\|^2_{H^s} + \int_0^t \sum_{j=1}^n \|\partial_j u_j^c\|_{L^\infty} \left( c^{-2} \|D^\alpha \log \rho^c\|^2_{L^2} + \|D^\alpha u^c\|^2_{L^2} \right) ds
\]
\[
+ C \int_0^t \sum_{j=1}^n \left( \|u_j^c\|_{H^s} \left( c^{-2} \|D^\alpha \log \rho^c\|^2_{H^s} + \|u^c\|^2_{H^s} \right) + (c^{-2} \|D^\alpha \log \rho^c\|^2_{L^2} + \|D^\alpha u^c\|^2_{L^2}) \right) ds.
\]

Combining (7.135) and (7.137), then summing over $1 \leq \alpha \leq s$, we get
\[
(c^{-2} \|D^\alpha \log \rho^c\|^2_{H^s} + \|u^c\|^2_{H^s})(t)
\]
\[
\leq \left( c^{-2} \|D^\alpha \log \rho_0\|^2_{H^s} + \|u_0\|^2_{H^s} \right) + C_1 \int_0^t \|u^c\|_{H^s} \left( c^{-2} \|D^\alpha \log \rho^c\|^2_{H^s} + \|u^c\|^2_{H^s} \right) ds
\]
\[
+ C_2 \int_0^t (\|u^c\|^2_{H^s} + 1) (c^{-2} \|D^\alpha \log \rho^c\|^2_{H^s} + \|u^c\|^2_{H^s}) ds,
\]

where $C_1, C_2$ only depend on Sobolev constants. By Gronwall’s inequality and the assumption (7.132), we deduce that
\[
(c^{-2} \|D^\alpha \log \rho^c\|^2_{H^s} + \|u^c\|^2_{H^s})(t) \leq \left( c^{-2} \|D^\alpha \log \rho_0\|^2_{H^s} + \|u_0\|^2_{H^s} \right) e^{C(M+1)t},
\]
where the constant $C > 0$ is independent of $c$. This finishes the proof of the lemma.

Now we are ready to prove Proposition 7.9.

Proof. — We choose $T$ such that
\[
e^{C(2M+1)T} = 2.
\]
Notice that under the initial condition (7.130), (7.133) may directly lead to (7.132), where $T$ is independent of $c$. By taking $R = 2M$ we finish the proof.

Now we are in the position to prove Theorem 7.9. Since the uniform estimate (7.131) does not guarantee that the convergence of the equations, we need to prove uniform estimate on the time derivative.
Proof. — We first prove the uniform estimate of time derivative. Set $\partial_t U_c = V_c$, and take derivative on both sides of (7.127), then we obtain

$$A_0(c)\partial_t V_c + \sum_j A_j(U, c)\partial_j V_c + B(0)V_c = -\sum_j \partial_t A_j(U,c)\partial_j U_c.$$  

Thus

$$\frac{1}{2} \partial_t \langle A_0(c) V_c, V_c \rangle = \frac{1}{2} \sum_j \langle \partial_t A_j V_c, V_c \rangle - \langle B(0)V_c, V_c \rangle - \langle \partial_t A_j(U,c)\partial_j U_c, V_c \rangle.$$  

By Sobolev inequality, we derive that

$$\langle A_0(c) V_c, V_c \rangle(t) - \langle A_0(c) V_c, V_c \rangle(0) \leq C \int_0^t \left( \| u^c \|^2_{H^s} (\| \partial_t \log \rho^c \|^2_{L^2} + c^2 \| \partial_t u^c \|^2_{L^2}) + \| c^{-1} \nabla \log \rho^c \|^2_{L^\infty} (\| c\partial_t u^c \|^2_{L^2} + \| \partial_t \log \rho^c \|^2_{L^2}) \right) ds + C \int_0^t c^2 \| u^c \|^2_{L^2} \| \partial_t u^c \|^2_{L^2} ds.$$  

Hence

$$\left( \| \partial_t \log \rho^c \|^2_{L^2} + c^2 \| \partial_t u^c \|^2_{L^2} \right)(t) \leq \left( \| \partial_t \log \rho^c \|^2_{L^2} + c^2 \| \partial_t u^c \|^2_{L^2} \right)(0) + C \int_0^t \left( \| c^{-1} \nabla \log \rho^c \|^2_{H^s} + \| \nabla u^c \|^2_{H^s} \right) \left( \| \partial_t \log \rho^c \|^2_{L^2} + c^2 \| \partial_t u^c \|^2_{L^2} \right) ds.$$  

Note that $\| c^{-1} \nabla \log \rho^c \|^2_{H^s} + \| \nabla u^c \|^2_{H^s}$ is uniformly bounded by a constant $R > 0$ which is independent of $c$. Applying Gronwall’s inequality, we obtain

$$(c^{-2} \| \partial_t \log \rho^c \|^2_{L^2} + \| \partial_t u^c \|^2_{L^2})(t) \leq (c^{-2} \| \partial_t \log \rho^c \|^2_{L^2}(0) + \| \partial_t u^c \|^2_{L^2}(0)) e^{CT},$$

which implies that

$$\sup_{[0,T]} \| \partial_t u^c \|^2_{L^2} \leq C,$$  

(7.137)

where $C$ is a constant which is independent of $c$.

According to the uniform estimate (7.131), $u^c$ is uniformly bounded in $L^\infty([0,T], H^s)$, then there exists $u^\infty \in L^\infty([0,T], H^s)$ and a sequence $\{u^n\}$ in $L^\infty([0,T], H^s)$, such that $u^n$ weakly converges to $u^\infty$.

In view of Arzela-Ascoli theorem and (7.137), we get a subsequence $\{u^c\}$ of $u^n$(still denoted by $\{u^c\}$) which converges to $u^\infty$ in $C([0,T], L^2(B_R))$.

By the Nash inequality, for $s' < s$ and any function $v \in H^s(\mathbb{R}^d)$, we have

$$\|v\|_{H^{s'}} \leq C_s \|v\|_{L^2} \|v\|_{H^s}^{s'},$$

59
which ensures that \( u^c \) converges to \( u^\infty \) in \( C([0, T], H^{s'}(B_R)) \).

Since \( c^{-1}\nabla \log \rho^c \) and \( u^c \) are uniformly bounded in \( C([0, T], H^{s-1}) \) (Proposition 7.9), we deduce that \( \lim_{c \to \infty} c^{-2}(\nabla \log \rho^c + u^c) = 0 \) holds in \( C([0, T], H^s) \). By the fact \( u^c \in C([0, T], H^s) \), we have \( \nabla u^c \in C([0, T], H^{s-1}) \). This yields \( u^c \cdot \nabla u^c \in C([0, T], H^{s-1}) \) as

\[
\|u^c \cdot \nabla u^c\|_{H^{s-1}} \leq C_s \|u^c\|_{H^{s-1}} \|\nabla u^c\|_{H^{s-1}} \leq C_s \|u^c\|_{H^s} \|\nabla u^c\|_{H^{s-1}}.
\]

Moreover, Eq. (7.126) yields

\[
\partial_t u^c = -u^c \cdot \nabla u^c - c^{-2}(\nabla \log \rho^c + u^c) \in C([0, T], H^{s-1}) \text{ and } \lim_{c \to \infty}(\partial_t u^c + u^c \cdot \nabla u^c) = -\lim_{c \to \infty} c^{-2}(\nabla \log \rho^c + u^c) = 0 \in C([0, T], H^{s-1}).
\]

On the other hand, as a consequence of the Sobolev inequality: if \( u, v \in H^s \), \( s > \frac{d}{2} \), then \( uv \in H^s \), and

\[
\|uv\|_{H^s} \leq C_s \|u\|_{H^s} \|v\|_{H^s},
\]

we can derive that \( u^c \cdot \nabla u^c \) converges to \( u^\infty \cdot \nabla u^\infty \) in \( C([0, T], H^{s'-1}(B_R)) \).

Moreover, by (7.137) there exists \( v^\infty \in L^\infty([0, T], L^2) \) such that \( \partial_t u^\infty \) weakly converges to \( v^\infty \). Also by the fact that \( u^c \) converges to \( u^\infty \) in \( C([0, T], H^{s'}(B_R)) \), we have \( \partial_t u^c \) converges to \( \partial_t u^\infty \) in distribution, implying that \( \partial_t u^\infty = v^\infty \in L^\infty([0, T], L^2) \).

As a result \( \partial_t u^\infty + u^\infty \cdot \nabla u^\infty = 0 \) holds in the weak-* topology of \( L^\infty([0, T], L^2) \). Also by \( u^\infty \cdot \nabla u^\infty \in L^\infty([0, T], H^{s-1}) \cap C([0, T], H^{s'-1}(B_R)) \), we deduce that \( \partial_t u^\infty \in L^\infty([0, T], H^{s-1}) \cap C([0, T], H^{s'-1}(B_R)) \), and thus \( u^\infty \in Lip([0, T], H^{s-1}) \cap C^1([0, T], H^{s'-1}(B_R)) \).

Meanwhile, by Sobolev embedding theorem, for \( s > s' > \frac{d}{2} + 1 \) we have \( H^{s'-1}, H^{s-1} \subset C^0, H^{s'}, H^s \subset C^1 \), which implies that \( u^c, u^\infty \in C^1([0, T] \times \mathbb{R}^d) \), and hence \( u^c \in Lip([0, T] \times \mathbb{R}^d) \) converges to \( u^\infty \in Lip([0, T], H^{s-1}) \cap C^1([0, T] \times \mathbb{R}^d) \) in \( C([0, T], H^{s'}(B_R)) \) for any \( R > 0 \).

Now we claim that for the solution \( \rho^c \) to the continuity equation \( \partial_t \rho^c + \nabla \cdot (\rho^c u^c) = 0 \), \( \rho^c dx \) weakly converges to \( \rho^\infty dx \) in \( C([0, T], P(\mathbb{R}^d)) \), which satisfies (7.128).

Since \( u^c \) is Lipschitz in \( t \) and \( x \), according to Ambrosio-Gigli-Savaré 11 or Theorem 5.34 in Villani 52, we have \( \rho^c = X^c_t \rho_0 \), where \( X^c \) is the integral curve of \( u^c \),

\[
\frac{d}{dt} X^c(t, x) = u^c(t, X^c(t, x)),
\]

\[
X(0, x) = x.
\]

Now we prove that as \( c \to \infty \), \( X^c \) uniformly converges to \( X^\infty \) in \( C([0, T] \times B_R) \), for any \( R > 0 \) and \( X^\infty \) is the flow generated by the vector field \( u^\infty \) on \( \mathbb{R}^d \). Now take \( x \in B_R \), and we derive that

\[
|X^c(t, x) - X^\infty(t, x)| \\
\leq \int_0^t |u^c(s, X^c(s, x)) - u^\infty(s, X^c(s, x))|ds + \int_0^t |u^\infty(s, X^c(s, x)) - u^\infty(s, X^\infty(s, x))|ds \\
\leq \int_0^t |u^c(s, X^c(s, x)) - u^\infty(s, X^c(s, x))|ds + L \int_0^t |X^c(s, x) - X^\infty(s, x)|ds,
\]

60
where \( L \) is the Lipschitz constant of \( u^\infty \) in \( x \) direction on \( B_R \). Thus
\[
\sup_{t \in [0,T], x \in B_R} |X^c(t, x) - X^\infty(t, x)| \leq \sup_{t \in [0,T], x \in B_R} e^{LT} \int_0^t |u^c(s, X^c(s, x)) - u^\infty(s, X^c(s, x))| ds \\
\leq CT e^{LT} \sup_{s \in [0,T]} \|u^c - u^\infty\|_{H^s(B_R)} \to 0.
\]

Then the weak convergence of \( \rho^c(x) \) to \( \rho^\infty = X^\infty \rho_0 \) in \( C([0,T], \mathcal{P}(\mathbb{R}^d)) \) is a direct consequence of Lemma 5.2.1 in Ambrosio-Gigli-Savaré [1].

7.4 The convergence of the Langevin deformation of flows on compact manifolds

In this section we give the proof of Theorem 5.8.

**Proof.** — It suffices to prove that when \( u^\infty(0, x) = \nabla \phi_0(x) \), there exits a function \( \phi^\infty \), such that \( u^\infty = \nabla \phi^\infty \). Following the proof of Theorem 5.8, we construct \( \phi^c \) by \( u^c \). More precisely, define
\[
\phi^c(t, x) = \phi_0(t, x) - e^{-\frac{1}{2L}t} \int_0^t e^{\frac{1}{2L}s} \left( \frac{1}{C^2} \nabla \log \rho^c(s, x) + \frac{1}{2} |u^c(s, x)|^2 \right) ds.
\]

Following the proof of 7.6, we derive that \( c^{-1} \nabla \log \rho^c \) is uniformly bounded in \( C([0,T], H^{s-1}) \), thus \( e^{-\frac{1}{2L}t} \int_0^t e^{\frac{1}{2L}s} \nabla \log \rho^c(s, x) ds \) converges to 0 in \( C([0,T], H^{s-1}) \). Since \( u^c \) converges in \( C([0,T], H^s(B_R)) \) for any \( R > 0 \), \( s' < s \), we deduce that the convergence also holds in \( C([0,T], C^1(B_R)) \) according to Sobolev embedding theorem. Therefore we get the convergence of \( e^{-\frac{1}{2L}t} \int_0^t e^{\frac{1}{2L}s} |u^c(s, x)|^2 ds \) to \( \int_0^t |u^\infty(s, x)|^2 ds \) in \( C([0,T], C^1(B_R)) \). Now define
\[
\phi^\infty(t, x) = \phi_0(t, x) - \frac{1}{2} \int_0^t |u^\infty(s, x)|^2 ds.
\]

So we conclude that \( \phi^c \) converges to \( \phi^\infty \) in \( C([0,T], C^1(B_R)) \). The fact that \( u^\infty \in C^1([0,T] \times \mathbb{R}^d) \) implies that \( \phi^\infty \in C^1([0,T] \times \mathbb{R}^d) \). Also by Theorem 5.3, we know that when \( u^\infty(0, x) = \nabla \phi_0(x) \), \( u^\infty(s, x) = \nabla \phi^\infty(s, x) \) holds on \([0,T] \times \mathbb{R}^d\), i.e. \( \phi^\infty \) satisfies the equation (5.36), which ends our proof.

8 Further remarks

To end this paper, let us give some further remarks.
Remark 8.1. In [12], Erbar-Kuwada-Sturm introduced a new definition of the curvature-dimension condition on metric-measure spaces, called the entropic curvature-dimension condition. By [12], we say that the entropic curvature-dimension condition, denoted by $CD_{\text{Ent}}(K, m)$, holds if the Boltzmann entropy $\text{Ent}$ satisfies

$$\text{HessEnt} - \frac{1}{N} \nabla \text{Ent}^\otimes 2 \geq K,$$

where $K \in \mathbb{R}$, $N \geq n$ are two constants. As was pointed out in [12], when $M$ is a complete Riemannian manifold, the $CD_{\text{Ent}}(K, m)$ is equivalent to the $CD(K, m)$-condition.

Based on Erbar-Kuwada-Sturm’s new definition of the entropic curvature-dimension condition $CD_{\text{Ent}}(K, m)$, we can also prove the W-entropy inequalities for the geodesic flow, the gradient flow as well as the Langevin deformation of flows (of the Boltzmann-Shannon entropy) on the Wasserstein space over complete Riemannian manifolds. More precisely, we have the following

Theorem 8.2. Let $M$ be a complete Riemannian manifold of dimension $n$. Suppose that Erbar-Kawada-Sturm’s $CD_{\text{Ent}}(K, N)$ condition holds, i.e.,

$$\text{HessEnt} - \frac{1}{N} \nabla \text{Ent}^\otimes 2 \geq K,$$

where $K \in \mathbb{R}$, $N \geq n$ are two constants. Then

(i) for geodesic flow $(\rho(t), \phi(t))$ on $TP_2(M, \mu)$, we have

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{N}{t} \geq \frac{1}{N} \left| \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle \right| + \frac{N}{2} \| \dot{\rho}(t) \|^2.$$

(ii) for the gradient flow $\dot{\rho}(t) = -\nabla \text{Ent}(\rho(t))$ on $P_2(M, \mu)$, we have

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{1}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{N}{2} \left( K + \frac{1}{t} \right)^2 \geq \frac{2}{N} \left| \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle \right| + \frac{N}{2} \left( K + \frac{1}{t} \right) g.$$

Theorem 8.3. Let $c \in [0, \infty)$, and let $(\rho(t), \phi(t))$ be the Langevin deformation of flows on $TP_2(M, \mu)$. Suppose that Erbar-Kawada-Sturm’s $CD_{\text{Ent}}(K, N)$-condition holds for some constants $K \in \mathbb{R}$ and $N \in \mathbb{N}$ with $N \geq n$, i.e.,

$$\text{HessEnt} - \frac{1}{N} \nabla \text{Ent}^\otimes 2 \geq K.$$

Let $\alpha(t) = (\log u)'$ be as in Section 6 with $m = N$. Then

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left( 2\alpha(t) + \frac{1}{c^2} \right) \frac{d}{dt} \text{Ent}(\rho(t)) + N\alpha^2(t) + \frac{1}{c^2} \| \nabla \text{Ent}(\rho(t)) \|^2 \geq \frac{1}{N} \left| \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + N\alpha(t) \right|^2 + K\| \dot{\rho}(t) \|^2.$$
The above results might bring some new insights to the study of geometric analysis on non smooth metric measure spaces. To save the length of this paper, we omit the proof of the above results. See Section 9 of our previous preprint [28].

**Remark 8.4.** As we have mentioned in Remark 1.12 and Section 6, it is possible to extend the W-entropy formula to the Langevin deformation of flows on the Wasserstein space over complete Riemannian manifolds with bounded geometry condition. This will implies the rigidity theorem as mentioned in Section 6. To this end, we need only to verify some technical condition to exchange the differentiation and integration of the Boltzmann-Shannon entropy along the Langevin deformation of flows. To save the length of the paper, we omit the detail of the technical part. Moreover, as mentioned also in Section 6, this paper we only consider the Langevin deformation of flows for the Boltzmann-Shannon entropy $\text{Ent}(\rho) = \int_M \rho \log \rho \, d\mu$ on the Wasserstein space over Euclidean spaces and Riemannian manifolds. We can also consider the Langevin deformation of flows for the Rényi entropy $V(\rho) = \frac{1}{m-1} \int_M \rho^m \, d\mu$ with $m \neq 1$. This will be studied in future.

**Remark 8.5.** As we have mentioned in Section 1, McCann and Topping [36] proved the contraction property of the $L^2$-Wasserstein distance between solutions of the backward heat equation on closed manifolds equipped with the Ricci flow. See also [50, 51]. In [32], Lott proved the convexity of the Boltzmann-Shannon entropy along the geodesics on the Wasserstein space over closed manifolds equipped with Ricci flow, which is closely related to Perelman’s results on the monotonicity of the $F$ and $W$-entropy functionals for Ricci flow. In [25], the authors extended Lott’s convexity results to the Wasserstein space on compact Riemannian manifolds equipped with Perelman’s Ricci flow. Inspired by these works, we can introduce the Langevin deformation of flows on the Wasserstein space over manifolds equipped with time dependent metrics and potentials, in particular, to manifolds equipped with the Ricci flow and the $(K, m)$-Ricci flow as introduced in our previous papers [24, 26, 27]. This has been done and will be included in a forthcoming paper.

**Remark 8.6.** In [18], Kuwada and the second named author of this paper introduced the W-entropy for the heat flow on metric measure space and proved its monotonicity on metric measure space with $\text{RCD}(0,N)$ condition. It would be interesting to extend the main results of this paper to the W-entropy for the geodesic flow and Langevin deformation of flows on the Wasserstein space over metric measure spaces and prove their monotonicity on metric measure spaces with $\text{RCD}(0,N)$ condition. Moreover, we can raise the question how to extend the monotonicity and rigidity theorems to the W-entropy along the Langevin deformation of flows on the Wasserstein space over super Ricci flows on metric measure spaces as introduced by Sturm [48] and Kopfer-Sturm [17].

**Remark 8.7.** Finally, let us mention that one can also formally introduce the Langevin diffusions on the Wasserstein space over Euclidean space and Riemannian manifolds as Bismut [5, 6] did on tangent bundle over Riemannian manifolds. See Section 4. This gives an interpolation between the stochastic gradient flows (in particular, the Brownian motion) on the Wasserstein space and the geodesic flow (or more general the Hamiltonian flow) on the tangent bundle over the Wasserstein space. However, the question how to rigorously
define the infinite dimensional Brownian motion or more general infinite dimensional stochastic gradient flows on the Wasserstein space remains open.

Acknowledgement. The authors would like to thank Professors S. Aida, D. Bakry, J.-M. Bismut, G.-Q. Chen, F.-M. Huang, K. Kuwada, K. Kuwae, K.-T. Sturm, C. Villani and F.-Y. Wang for helpful discussions and their interests on this work.

References

[1] L. Ambrosio, N. Gigli, G. Savaré, Gradient flows in metric spaces and in the space of probability measures, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.

[2] T. Aubin, Nonlinear Analysis on Manifolds, Monge-Ampère Equations, Springer-Verlag, 1980

[3] D. Bakry, M. Emery, Diffusion hypercontractives, Sém. Prob. XIX, Lect. Notes in Maths. 1123 (1985) 177-206.

[4] A. L. Besse, Einstein manifolds, Ergebnisse der Mathematik (3) 10, Springer, Berlin, 1987.

[5] J.-M. Bismut, The hypoelliptic Laplacian on the cotangent bundle. J. Amer. Math. Soc. 18 (2005), no. 2, 379-476.

[6] J.-M. Bismut, Hypoelliptic Laplacian and orbital integrals. Annals of Mathematics Studies, 177. Princeton University Press, Princeton, NJ, 2011.

[7] Yann Brenier, Emmanuel Grenier, Sticky particles and scalar conservation laws, SIAM J. Numer. Anal. 35(1998), no. 6, 2317–2328.

[8] J.-D. Benamou, Y. Brenier, A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. Numer. Math. 84, 375-393 (2000)

[9] R. Carles, Semi-classical analysis for nonlinear Schrödinger equations, World Scientific Publishing, 2008.

[10] Gui-Qiang Chen, Hai-Liang Liu, Formation of δ-shocks and vacuum states in the vanishing pressure limit of solutions to the euler equations for isentropic fluids, SIAM J. Math. Anal. 34(2003), no. 4, 925–938.

[11] Jean-Francoise Coulombel, Thierry Goudon, The strong relaxation limit of the multidimensional isothermal euler equations, Transactions of the American Mathematical Society 359(2007), no. 2, 637–648.
[12] M. Erbar, K. Kuwada, K.-T. Sturm, On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces. Preprint. Available at: arXiv:1303.4382.

[13] Weinan E, Yu.G. Rykov, Ya.G. Sinai, Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics, Commun. Math. Phys 177(1996), 349–380.

[14] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems. Arch Rational Mech Anal. 58 (1975), 181-205.

[15] S. Klainerman, A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, Comm. Pure Appl. Math. 34(1981), 481–524.

[16] S. Klainerman, A. Majda, Compressible and incompressible fluids, Comm. Pure Appl. Math. 35(1982), 629–653.

[17] E. Kopfer, K.-T. Sturm, Heat flow on time-dependent metric measure spaces and super-Ricci flows. Comm. Pure Appl. Math. 71 (2018), no. 12, 2500-2608.

[18] K. Kuwada, X.-D. Li, Monotonicity and rigidity of the W-entropy on RCD(0,N) spaces, Manuscripta Math. 164 (2021), 119-149.

[19] Corrado Lattanzio, Athanasios E. Tzavaras, Relative entropy in diffusive relaxation, SIAM Journal on Mathematical Analysis 45(2013), no. 3, 1563–1584.

[20] Corrado Lattanzio, Athanasios E. Tzavaras, From gas dynamics with large friction to gradient flows describing diffusion theories, Communications in Partial Differential Equations 42(2017), no. 2, 261–290.

[21] X.-D. Li, On the W-entropy formula for the Witten Laplacian over weighted Riemannian manifolds, preprint 2006, and included in Thèse d’Habilitation à Diriger des Recherches, Université Paul Sabatier, 2007.

[22] X.-D. Li, Perelman’s entropy formula for the Witten Laplacian on Riemannian manifolds via Bakry-Emery Ricci curvature, Math. Ann. 353 (2012), 403-437.

[23] X.-D. Li, Hamilton’s Harnack inequality and the W-entropy formula on complete Riemannian manifolds, Stoch. Processes and Appl. 126 (2016) 1264-1283

[24] S. Li, X.-D. Li, W-entropy formula for the Witten Laplacian on manifolds with time dependent metrics and potentials, Pacific J. Math. Vol. 278, No. 1, 2015, 173-199.

[25] S. Li, X.-D. Li, On the convexity of Boltzmann type entropy on compact Riemannian manifolds equipped with Perelman’s Ricci flow, preprint, 2013.
[26] S. Li, X.-D. Li, Hamilton differential Harnack inequality and W-entropy for Witten Laplacian on Riemannian manifolds, J Funct Anal, 2018, 274: 3263-3290, https://doi.org/10.1016/j.jfa.2017.09.017.

[27] S. Li, X.-D. Li, W-entropy formulas on super Ricci flows and Langevin deformation on Wasserstein space over Riemannian manifolds, Science China Mathematics, 2018 Vol. 61 No. 8: 1385-1406, https://doi.org/10.1007/s11425-017-9227-7.

[28] S. Li, X.-D. Li, W-entropy formulas and Langevin deformation of flows on Wasserstein space over Riemannian manifolds, arXiv:1604.02596, 2016.

[29] S. Li, X.-D. Li, W-Entropy, Super Perelman Ricci Flows, and (K, m)-Ricci Solitons, J. Geom. Anal. (2020) 30, 3149-3180, https://doi.org/10.1007/s12220-019-00193-4

[30] J. Lott, Some geometric properties of the Bakry-Emery Ricci tensor, Comment. Math. Helv. 78 30 (2003) 865-883.

[31] J. Lott, Some geometric calculations on Wasserstein space, Comm. Math. Phys. 277 (2008), no. 2, 423-437.

[32] J. Lott, Optimal transport and Perelman’s reduced volume, Calc. Var. and Partial Differential Equations 36, 49-84 (2009).

[33] J. Lott, C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Annals of Math. 169, 903-991, 2009.

[34] A. Majda, Compressible fluid flow and systems of conservation laws in several space variables. Applied Mathematical Sciences. New York, Springer-Verlag, 1984.

[35] R. McCann, Polar factorization of maps on Riemannian manifolds, Geometric and Functional Analysis, 11, 3(2001), 589-608.

[36] R. McCann, P. Topping, Ricci flow, entropy and optimal transpotation, American Journal of Math., 132 (2010) 711-730.

[37] F. Nier, A semi-classical picture of quantum scattering, Ann. Sci. École Norm. Sup. (4) 29 (1996), 2, 149-183.

[38] F. Otto, The geometry of dissipative evolution equations: the porous medium equation, Commun. in Parial Differential Equations 26 (1 and 2), 101-174 (2001).

[39] L. Ni, The entropy formula for linear equation, J. Geom. Anal. 14 (1), 87-100, (2004).

[40] F. Otto, C. Villani, Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality, J. Funct. Anal. 173 (2000) 361-400.

[41] F. Otto, M. Westdickenberg, Eulerian calculus for the contraction in the Wasserstein distance. SIAM J. Math. Anal., 37(4):1227-1255 (electronic), 2005.
[42] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, http://arXiv.org/abs/maths0211159.

[43] Michael Sever, An existence theorem in the large for zero-pressure gas dynamics, Differential and Integral Equations 14(2001), no. 9, 1077–1092.

[44] T. Sideris, B. Thomases, D.H. Wang, Long time behavior of solutions to the 3D compressible Euler equations with damping, Comm. PDE. 28 (2003), no.3-4, 795-816.

[45] K.-T. Sturm, Convex functionals of probability measures and nonlinear diffusions on manifolds, J. Math. Pures Appl. (9) 84 (2005), no. 2, 149-168.

[46] K.-T. Sturm, On the geometry of metric measure spaces. I. Acta Math., 196(1): 65-131, 2006.

[47] K.-T. Sturm, On the geometry of metric measure spaces. II. Acta Math., 196(1), 133-177, 2006.

[48] K.-T. Sturm, Super-Ricci flows for metric measure spaces. J. Funct. Anal. 275 (2018), no. 12, 3504-3569.

[49] K.-T. Sturm, M.-K. von Renesse, Transport inequalities, gradient estimates, entropy, and Ricci curvature Comm. Pure Appl. Math., 58(7), 923–940, 2005.

[50] P. Topping, Lectures on the Ricci Flow, London Mathematical Society Lecture Note Series 325, Cambridge University Press, 2006.

[51] P. Topping, L-optimal transportation for Ricci flow, J. Reine Angew. Math., 636 (2009) 93–122.

[52] C. Villani, Topics in Mass Transportation, Grad. Stud. Math., Amer. Math. Soc., Providence, RI, 2003.

[53] C. Villani, Optimal Transport, Old and New, Springer, 2008.

[54] W. Wang, T. Yang, The pointwise estimates of solutions for Euler equations with damping in multi-dimensions, J. Diff. Equa. 173 (2001), 410-450.

Songzi Li, School of Mathematics, Renmin University of China, Beijing, 100872, China Email: sli@ruc.edu.cn

Xiang-Dong Li, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, No. 55, Zhongguancun East Road, Beijing, 100190, China E-mail: xdli@amt.ac.cn

and

School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, 100049, China