An Adaptive High-Resolution and Low-Dissipation Hybrid Energy Consistent/WENOCU Scheme

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Abstract. We investigate the performance of the hybrid energy consistent/WENOCU4 scheme compared to the classical WENOCU4 scheme. It is borne out that the hybrid scheme preserves excellent shock-capturing ability without numerical oscillations. In the meantime, the hybrid scheme can achieve more accurate representations of smooth flow features which indicates its low-dissipation capacities. Moreover, hybrid scheme significantly outperforms WENOCU4 for its superior computational efficiency. Therefore, employing hybrid scheme is a much more cost-effective choice for the simulation of shock/turbulence interaction.

Keywords: Hybrid scheme; Energy consistent; WENOCU4; Low dissipation; High resolution.

1. Introduction
The applications of classical WENO schemes for the simulation of compressible turbulence are restrained due to the excessive numerical dissipation in smooth regions and the huge amount of computational cost incurred by the complex flux reconstruction process. Hybrid schemes have been widely applied due to the high-resolution, low-dissipation properties and superior computational efficiency. Hybrid schemes were firstly proposed by Adams [1] and further developed by Pirozzoli [2], which have been gradually obtained more extensive applications. The main idea of the hybrid scheme is that only the shock-capturing scheme activates near discontinuities while using linear discretization schemes elsewhere. The high-resolution shock sensor is the key ingredient for the stability and accuracy of schemes, especially the coupling and transition between the two separate sub-schemes. Traditional sensors such as Harten [3], Jameson [4] or Ren [5] strongly rely on first- or second-order discrete derivatives of the flow variables which exhibit poor computational efficiency and accuracy. Another class of sensors based on the smoothness indicators or WENO sub-stencil like Hill [6], Visbal [7] or Movahed [8] also exhibit misidentification to some extent. Our previous work proposed a new sensor [9] for the development of an efficient WENOCU scheme, and the sensor exerts excellent performance. In this work, the performances of the adaptive sensor are further evaluated by constructing a hybrid scheme.

The hybrid scheme developed in this paper is consisting of energy consistent scheme [10] and WENOCU4 scheme [11]. The energy consistent scheme is suitable for the simulation of smooth regions, and Pirozzoli [12] proved that the scheme based on the method of splitting the convective derivatives proposed by Kennedy et al. [13] holds generalized conservative approximations property. Energy consistent scheme is of great practical engineering application, since its nearly no dissipation and much less computational cost. WENOCU4 scheme is a typical improved WENO-based scheme.
The main objective of this paper is to investigate the resolution of the efficient shock sensor as well as the performance of the hybrid scheme. The paper is organized as follows: In Section 2 we briefly present the construction process of hybrid scheme, and in Section 3 we test the dissipation, robustness capabilities and computational efficiency of hybrid scheme. Conclusions are given in Section 4.

2. Numerical Methods

This section briefly describes the formulation of the hybrid energy consistent/WENOCU4 scheme in the context of the 1-D advection equation,

\[
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad u(x,0) = u_0(x), \quad -\infty < x < +\infty
\]

(1)

Where, \(u(x,t)\) means the transported quantity, and \(f(u)\) means the numerical flux, a conservation finite-difference semi-discretization form of Eq.1 as follows,

\[
L_j(v) = \frac{dv}{dt} = -\frac{1}{h} \left( \hat{f}_{j+1/2} - \hat{f}_{j-1/2} \right) \approx \frac{\partial f(u)}{\partial x}
\]

(2)

Where, \(v_j(t) \approx u(x_j,t)\), and \(\hat{f}_{j+1/2}\) is a suitable blending numerical flux. In all benchmark cases after shown, time integration is carried out by means of a classical third-order Runge-Kutta march. The hybrid numerical flux is defined as,

\[
\hat{f}_{j+1/2} = (1 - \varphi_{j+1/2}) \hat{f}_{j+1/2}^{\text{SMOOTH}} + \varphi_{j+1/2} \hat{f}_{j+1/2}^{\text{SHOCK}}
\]

(3)

Where, \(\hat{f}_{j+1/2}^{\text{SMOOTH}}\) is the numerical flux for WENOCU4 scheme and \(\hat{f}_{j+1/2}^{\text{SHOCK}}\) is the numerical flux for energy consistent scheme. It needs to emphasize that the parameter \(C\) introduced for WENOCU4 flux reconstruction is employed the constant \(C=1\) for guaranteeing numerical stability. \(\varphi_{j+1/2}\) is the adaptive switch, which is designed to be unity near shocks and zero in smooth regions. \(\varphi_{j+1/2} = \max(\varphi_j, \varphi_{j+1})\), \(\varphi_j\) is defined as,

\[
\varphi_j = \begin{cases} 
1, & \text{if } r_j \geq \bar{\varphi} \\
0, & \text{otherwise}
\end{cases}
\]

(4)

\(r_j\) is the nodal value of our shock sensor obtained by density to mark critical cells near shocks, and \(\varepsilon\) is the threshold value which is set 0.7. \(r_j\) is defined as follows,

\[
r_j = \frac{\sum_{i=1}^{N} \left( \frac{\omega_i}{d_i} - 1 \right)^2}{\min_{m=0,1,2} \left( \frac{1}{d_m} - 1 \right)^2 + 2}
\]

(5)

\(\omega_i\), \(d_i\) denote the nonlinear and optimal weights of WENOCU4 scheme, respectively.

In order to avoid the numerical oscillations caused by the coupling of sub-schemes, the buffer points method is employed by forcing \(\varphi_{j+1/2} = \varphi_{j+3/2} = 1\), whenever \(\varphi_{j+1/2} = 1\).

3. Results and Discussions

3.1. Applications to 1-D Euler Equations
3.1.1. Shock–Density Wave Interaction. We consider the Shu-Osher [14] problem which is generally preferred to test the shock-capturing and wave-disturbance propagation capabilities of schemes. The solution computed by WENO5 scheme with 4001 grid points as a reference solution. Fig. 1(a)-(b) show the calculated density and entropy profile as well as switch \( \phi \). No oscillations can be found for hybrid scheme and the main shock and multiple shocklets are detected accurately by the shock sensor. That is employing WENOCU4 near shocks while energy consistent scheme only activates in smooth regions. Moreover, for the hybrid scheme, WENOCU4 sub-scheme is also employed in partial entropy wave regions which is incurred by the limited length of sub-stencils of WENOCU4. However, hybrid scheme remains exhibit better resolution than WENOCU4.

3.1.2. Shock-Tube Problems. Lax [15] and Sod [16] problems are considered to study the capabilities of the schemes for capturing steady shocks. Both the reference exact solutions are obtained by WENO5 scheme with 10001 grid points. As illustrated in Fig.1(c)-(f), WENOCU4 scheme produces excessive smearing effect around the discontinuity. Hybrid scheme precisely marks the shock positions and no oscillations can be observed in the vicinity of shocks, demonstrating its excellent shock-resolving ability and numerical stability. In the meantime, hybrid scheme shows obviously lower numerical dissipation than WENOCU4.

3.2. Applications to 2-D Euler Equations

3.2.1. Double Mach Reflection. The inviscid Double Mach reflection case [17] is considered to investigate the shock-capturing, and contact discontinuity resolving capabilities of algorithm. Fig. 2(a)-(b) show the enlarged view for density profile, including both triple points and slip lines. The WENOCU4 is so dissipative that no vorticities can be observed. Hybrid scheme tends to develop more vorticities around the contact lines. However, the enhancement of dissipation performance is limited.

3.2.2. Riemann Problem. Another canonical test case, namely Riemann problem [18] is considered. Fig. 2(c)-(d) give the density profile of schemes. These schemes show obvious differences around the Mach stem region. The WENOCU4 scheme still contains excessive dissipation that only the main jet flow...
features can be captured, without any signatures of Kelvin-Helmholtz vortices. This is due to its excessive numerical dissipation. Hybrid scheme is able to capture abundant roll-up of the jet wake. It can be appreciated that the significant low dissipation properties of energy consistent sub-scheme, which is a noticeable result showing the superiority of the shock sensor and the high-resolution of the hybrid scheme.

### 3.2.3. Rayleigh–Taylor Instability

Fig. 2(e)-(f) show the density contours of schemes obtained from the inviscid Rayleigh–Taylor instability case [19]. WENOCU4 can not capture the detailed vortical structures, indicating its high-dissipation property. Apparently, hybrid scheme exhibits considerable enhancements in resolving the small-scale vorticities in the shear-layer regions. It indicates that the numerical dissipation of hybrid scheme is effectively confined in shock regions. Moreover, hybrid scheme induces a slight asymmetry structure, owing to small numerical perturbations.

Figure 2. Simulations of test cases for 2-D Euler equations, (a)-(b) Double Mach reflection problem: 38 density contours from 2 to 20.5 on 960×240 mesh; (c)-(d) Riemann problem: 32 density contours from 0.15 to 1.7 on 800×800 mesh; (e)-(f) Rayleigh–Taylor instability problem: 13 density contours from 0.95 to 2.15 on 240×960 mesh.

### 3.3. Applications to 3-D Euler Equations

Here, we consider the isotropic compressible turbulence case [20] to evaluate the turbulence simulation capabilities of schemes, which is computed on a 64³ grid without viscosity involved. The initial turbulence Mach number is $Ma_0 = 0.5$. $\tau$ denotes the large-eddy-turnover time, $t / \tau$ means the normalized time and $k$ is the wavenumber.

Fig. 3 shows the results of non-dimensional fluctuation, turbulence kinetic energy and energy spectra, respectively. Hybrid scheme performs better characteristics compared to WENOCU4, which indicate its superiority of turbulence simulation capabilities. Fig. 4 presents the coherent randomly vortical structures identified by $Q$ criteria and colored by vortices magnitude at $t = 3\tau$. As can be seen, different schemes resolve remarkable differences in the number of vortices, which is a visual outcome that hybrid scheme obviously outperforms WENOCU4 for its considerable low dissipation capacity.
3.4. Comparisons of Computational Efficiency

Table 1. Computational efficiency.

| Test Cases | Double Mach Reflection | Riemann | Rayleigh–Taylor Instability |
|------------|------------------------|---------|-----------------------------|
| WENOCU4    | 1                      | 1       | 1                           |
| hybrid     | 0.48                   | 0.85    | 0.72                        |

Table 1 displays the results of normalized computational cost computed based on the measured CPU time for the 2-D test cases. Although for specific benchmark case, the reduction of the computational cost for hybrid scheme with respect to WENOCU4 shows obvious difference, which is due to the complexity of different test cases. Generally, hybrid scheme behaves superior computational efficiency than WENOCU4 scheme.

4. Conclusions

In this paper, we analyze the performance of hybrid energy consistent/WENOCU4 scheme with an efficient adaptive shock sensor. With numbers of multi-dimensional benchmark cases of Euler equations, we confirm that hybrid scheme preserves excellent shock-capturing ability that is the locations of discontinuities can be detected accurately, and only WENOCU4 sub-scheme activates near shocks without numerical oscillations. In the meantime, hybrid scheme significantly outperforms WENOCU4 with regard to numerical resolution and dissipation capacities, especially for the 2-D and 3-D tests of Euler equations where much finer detailed vortical structures can be captured. The underlying reason is the extreme low dissipation property of energy consistent sub-scheme. Moreover, hybrid scheme exhibits obvious superior computational efficiency than WENOCU4 scheme. Therefore, employing hybrid scheme is a more cost-effective choice for the simulation of the interactions between turbulent flows and shock waves. The coupling methods of sub-schemes employed in the present work can be readily applied to other hybrid schemes.
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