COUNTING ALGEBRAIC TORI OVER $\mathbb{Q}$ BY ARTIN CONDUCTOR

JUNGIN LEE
School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu,
Seoul 02455, Korea;
email: jilee.math@gmail.com

Abstract

In this paper we count the number $N_n^{\text{tor}}(X)$ of $n$-dimensional algebraic tori over $\mathbb{Q}$ whose Artin conductor of the associated character is bounded by $X$. This can be understood as a generalization of counting number fields of given degree by discriminant. We suggest a conjecture on the asymptotics of $N_n^{\text{tor}}(X)$ and prove that this conjecture follows from Malle’s conjecture for tori over $\mathbb{Q}$. We also prove that $N_2^{\text{tor}}(X) \ll X^{1+\varepsilon}$, and this upper bound can be improved to $N_2^{\text{tor}}(X) \ll X(\log X)^{1+\varepsilon}$ under the assumption of the Cohen-Lenstra heuristics for $p = 3$.

Contents

1 Introduction ............................................. 1
  1.1 Counting number fields by discriminant .................. 1
  1.2 Counting algebraic tori over $\mathbb{Q}$ by Artin conductor .... 2

  1.3 Main results and the structure of the paper ............. 3

2 Preliminaries ........................................... 4
  2.1 Classification of tori over $\mathbb{Q}$ ..................... 4

  2.2 Computation of $C(T)$ .................................. 5

  2.3 Counting number fields by conductor ................... 7

  2.4 Discriminant of compositum of number fields .......... 8

3 Malle’s conjecture for tori over $\mathbb{Q}$ .................. 8

4 Counting algebraic tori over $\mathbb{Q}$ of dimension 2 .... 11
  4.1 Classification of 2-dimensional tori over $\mathbb{Q}$ ....... 11

  4.2 Asymptotics of $N_2^{\text{tor}}(X; H_{12,A})$ ............... 13

  4.3 Upper bound of $g(X)$ .................................. 15

  4.4 Main theorems ....................................... 18

1 Introduction

1.1 Counting number fields by discriminant

Counting number fields by discriminant is one of the most important topics in arithmetic statistics. Let $L/K$ be an extension of number fields. (Throughout the paper, all number fields are taken to be subfields of a fixed algebraic closure of $\mathbb{Q}$.) Denote the absolute value of the discriminant of $L$ by $D_L$. Also take $\delta_{L/K}$ to be the relative discriminant ideal of $L/K$, $N_{K/\mathbb{Q}}$ to be the absolute norm and denote $D_{L/K} := N_{K/\mathbb{Q}}(\delta_{L/K})$. 

1
For a number field \( K \) and an integer \( n \geq 2 \), denote by \( N_{K,n}(X) \) the number of degree \( n \) extensions \( L \) of \( K \) such that \( D_{L/K} \leq X \) and denote \( N_a(X) := N_{Q,n}(X) \) for simplicity. A folk conjecture (sometimes attributed to Linnik) states that \( N_{K,n}(X) \sim c_{K,n}X \) as \( X \to \infty \) for some constant \( c_{K,n} > 0 \). This conjecture has been proved for \( n \leq 5 \) by the work of Davenport-Heilbronn [11], Datskovsky-Wright [10], Bhargava [3, 4] and Bhargava-Shankar-Wang [5].

Now let \( G \neq 1 \) be a transitive subgroup of the symmetric group \( S_n \). Denote by \( N_{K,n}(X; G) \) the number of degree \( n \) extensions \( L \) of \( K \) such that \( D_{L/K} \leq X \) and \( \mathrm{Gal}(L/K) \cong G \). Here \( L^c \) denotes the Galois closure of \( L/K \) and the Galois group \( \mathrm{Gal}(L^c/K) \) is permutation-isomorphic to \( G \). Denote \( N_n(X; G) := N_{Q,n}(X; G) \) for simplicity. Malle’s conjecture [22, 23] states that

\[
N_{K,n}(X; G) \sim c_{K,G}X^{\frac{1}{\mu(G)}}(\log X)^{b(K,G)-1}
\]

for some positive integers \( a(G), b(K,G) \) and a constant \( c_{K,G} > 0 \). There is also Malle’s weak conjecture which states that \( X^{\frac{1}{\mu(G)}} \ll N_{K,n}(X; G) \ll X^{\frac{1}{\mu(G)}+\varepsilon} \) for any \( \varepsilon > 0 \).

The numbers \( a(G) \) and \( b(K,G) \) are defined as follows. For any \( g \in G \leq S_n \), define the index of \( g \) by

\[
\text{ind}(g) := n - \text{the number of orbits of } g \text{ on } \{1, 2, \ldots, n\}
\]

and let \( a(G) := \min_{g \in G \setminus \{1\}} \text{ind}(g) \). For any field \( k \), denote its algebraic closure by \( \overline{k} \) and its absolute Galois group by \( G_k \). Define the \( K \)-conjugacy classes of \( G \) to be the orbits of the action of \( G_K \) on the conjugacy classes of \( G \) via the cyclotomic character. Since all elements of a \( K \)-conjugacy class \( C \) have the same index, its index can be defined by the index of any element of \( C \). The number \( b(K,G) \) is defined to be the number of \( K \)-conjugacy classes of \( G \) whose index is \( a(G) \). Since \( a(G) = 1 \) implies that \( b(K,G) = 1 \) [23, Lemma 2.2], Malle’s conjecture implies the asymptotics \( N_{K,n}(X) \sim c_{K,n}X \).

Malle’s conjecture was proved for the abelian extensions by Mäki [20] for \( K = \mathbb{Q} \) and Wright [36] for general \( K \). The case \( G = S_n \) for \( 3 \leq n \leq 5 \) was proved by Davenport-Heilbronn [11], Datskovsky-Wright [10], Bhargava [3, 4] and Bhargava-Shankar-Wang [5]. The product of these two cases, i.e. \( G = S_d \times A \) and \( n = d | A | \) for \( 3 \leq d \leq 5 \) and an abelian group \( A \), was recently proved by Wang [33] and Masri-Thorne-Tsai-Wang [24]. See also [6, 7, 15, 19, 25, 28] for more cases where the conjecture has been settled.

In the opposite direction, Klüners [18] found a counterexample \( G = C_3 \wr C_2 \cong S_3 \times C_3 \leq S_6 \) for Malle’s conjecture. Törmä [30] proposed a modified version of Malle’s conjecture (with the same \( a(G) \) and the different \( b(K,G) \)) which takes into account Klüners’ counterexample and agrees with the heuristics for function field case. Alberts [1] provided more evidences that Törmä’s modification is correct.

### 1.2 Counting algebraic tori over \( \mathbb{Q} \) by Artin conductor

In this paragraph, we explain how counting number fields can be regarded as a special case of counting (algebraic) tori over \( \mathbb{Q} \). The functor from the category of tori over \( \mathbb{Q} \) to the category of \( \mathbb{Q} \)-lattices which maps a torus \( T \) to its character group \( X^*(T) := \text{Hom}(T, \mathbb{Q}^\times) \) is an anti-equivalence of categories. Let \( T \) be an \( n \)-dimensional torus over \( \mathbb{Q} \) whose splitting field is \( L \). The \( \mathbb{Q} \)-action on \( X^*(T) \) gives the representation

\[
\rho_T : G_\mathbb{Q} \to \text{Aut}(X^*(T)) \cong \text{GL}_n(\mathbb{Z})
\]

such that \( \ker(\rho_T) = G_L \) and \( G_T := \text{im}(\rho_T) \) is isomorphic to \( \text{Gal}(L/\mathbb{Q}) \). Note that \( \rho_T \) and \( G_T \) are well-defined only up to conjugation since the isomorphism \( \text{Aut}(X^*(T)) \cong \text{GL}_n(\mathbb{Z}) \) depends on the choice of basis. Now let \( C(T) \) denote the Artin conductor of the character associated to the representation

\[
\rho : \text{Gal}(L/\mathbb{Q}) \cong G_\mathbb{Q}/G_L \to \text{Aut}(X^*(T)_\mathbb{Q}) \cong \text{GL}_n(\mathbb{Q})
\]
induced by $\rho_T$. It is always a positive integer.

Let $K$ be a number field of degree $n$ with Galois closure $K^c$, $G = \text{Gal}(K^c/Q)$ and $G' = \text{Gal}(K^c/K) \leq G$. Consider an $n$-dimensional torus $T = R_{K/Q} G_m$ (Weil restriction of $G_m$) over $Q$. The splitting field of $T$ is $K^c$ and its character group is given by

$$X^*(T) \cong \text{Ind}_{G_H}^{G} X^*(G_m) = \text{Ind}_{G_H}^{G} Z$$

(cf. [29, Proposition 11.4.22]).

Since $T$ splits over $K^c$, the $G_Q$-action on $X^*(T)$ factors through $G$ and $X^*(T) \cong \text{Ind}_{G_Q}^{G} Z$ as $G$-modules. Therefore the character of the representation $\rho : G \to \text{Aut}(X^*(T)_Q) \cong \text{GL}_n(Q)$ is $\text{Ind}_{G_Q}^{G} 1_{G'}$ (induced character of the trivial character $1_{G'}$), whose Artin conductor is $C(R_{K/Q} G_m) = D_K$ by [27, Corollary VII.11.8]. This shows that counting tori over $Q$ of dimension $n$ by Artin conductor can be understood as a generalization of counting number fields of degree $n$ by discriminant.

Denote by $N_n^{\text{tor}}(X)$ the number of the isomorphism classes of tori over $Q$ of dimension $n$ such that $C(T) \leq X$. For a finite subgroup $H \neq 1$ of $\text{GL}_n(Z)$, denote by $N_n^{\text{tor}}(X; H)$ the number of such tori $T$ over $Q$ such that $G_T$ is conjugate to $H$ in $\text{GL}_m(Z)$.

1.3 Main results and the structure of the paper

We start with some preliminaries on counting algebraic tori. First we explain how to classify tori $T$ over $Q$ such that $G_T$ is conjugate to given finite subgroup $H \neq 1$ of $\text{GL}_n(Z)$ in Section 2.1. We discuss the computation of the Artin conductor $C(T)$ in Section 2.2. After that, we review some backgrounds on counting number fields in Section 2.3 and 2.4.

In Section 3 we provide conjectures on the number of the isomorphism classes of tori over $Q$. First we suggest an asymptotics of the number $N_n^{\text{tor}}(X)$.

**Conjecture 1.1.** (Conjecture 3.1) For every $n \geq 1$, there exists a constant $c_n > 0$ satisfying

$$N_n^{\text{tor}}(X) \sim c_n X(\log X)^{n-1}.$$  \hspace{1cm} (2)

Next we provide an analogue of Malle’s conjecture for tori over $Q$. This conjecture is not new; it is a direct consequence of a more general conjecture of Ellenberg and Venkatesh [13, Question 4.3]. There is a further generalization of the conjecture by Ellenberg, Satriano and Zuerick-Brown [12, Conjecture 4.14] which specializes both to Batyrev-Manin conjecture and to Malle’s conjecture.

**Conjecture 1.2.** (Conjecture 3.2) For every $n \geq 1$ and a finite subgroup $1 \neq H \leq \text{GL}_n(Z)$,

$$N_n^{\text{tor}}(X; H) \sim c_H X^{\frac{1}{\prod b(h)}}(\log X)^{b(H)-1}$$  \hspace{1cm} (3)

where the positive integers $a(H), b(H)$ and a constant $c_H > 0$ depend only on $H$. For the identity matrix $I_n \in \text{GL}_n(Z)$, the number $a(H)$ is given by

$$a(H) := \min_{h \in H \backslash \{I_n\}} \text{rank}(h - I_n)$$  \hspace{1cm} (4)

and the number $b(H)$ is given by the number of the orbits $C$ of the action of $G_Q$ on the conjugacy classes of $H$ via the cyclotomic character such that $\text{rank}(h - I_n) = a(H)$ for some (equivalently, all) $h \in C$.

We present three remarks on this conjecture. First, the above conjecture is a generalization of Malle’s conjecture (Remark 3.3). Secondly, the conjecture is compatible with the direct product of $H$ (Remark 3.4).
Finally and most importantly, if Conjecture 1.2 holds for every finite nontrivial subgroup of GL_n(Z), then Conjecture 1.1 is true for n (Corollary 3.6).

In Section 4 we concentrate on the 2-dimensional case. Based on the results in Section 2.1 and 2.2, we classify 2-dimensional tori over \( \mathbb{Q} \) and compute their Artin conductors in Section 4.1. Section 4.2 to 4.4 is devoted to the asymptotics of the number of 2-dimensional tori over \( \mathbb{Q} \). Since Conjecture 1.2 can be easily proved when \( n = 2 \) and \( H \neq H_{12,A} \), the essential new result is the estimation of \( N_2^{tor}(X; H_{12,A}) \). Here \( H_{12,A} \) is a finite subgroup of \( GL_2(\mathbb{Z}) \) isomorphic to the dihedral group \( D_6 \), which appears in Section 4.1.

The following theorem summarizes the main results of Section 4. The first part of the theorem is unconditional, and the second part is under the assumption of the Cohen-Lenstra heuristics for \( p = 3 \). See Conjecture 4.5 for a precise version of the Cohen-Lenstra heuristics used in this paper.

**Theorem 1.3.** (1) (Theorem 4.9)
\[
X \ll N_2^{tor}(X; H_{12,A}) \ll \varepsilon X^{1+\frac{2+log 2}{log log X}}.
\]  
\[
X \log X \ll N_2^{tor}(X) \ll \varepsilon X^{1+\frac{2+log 2}{log log X}}.
\]

(2) (Theorem 4.10) Assume that Cohen-Lenstra heuristics (Conjecture 4.5) holds for \( p = 3 \) and every \( \alpha > 0 \). Then we have
\[
N_2^{tor}(X; H_{12,A}) \leq N_2^{tor}(X) \ll \varepsilon X(\log X)^{1+\varepsilon}.
\]

Here we briefly explain why the estimation of the asymptotics of \( N_2^{tor}(X; H_{12,A}) \) is difficult compared to the case \( N_2^{tor}(X; H_{8,A}) \). The asymptotics of \( N_2^{tor}(X; H_{8,A}) \) follows from the work of Altug, Shankar, Varma and Wilson [2], where both analytic techniques and geometry-of-numbers methods were used. In particular, the parametrization of \( D_4 \)-quartic fields via certain pairs of ternary quadratic forms following Bhargava [3] and Wood [35] was essential. However, such parametrization is not yet known for \( D_6 \)-sextic fields. In the sequel of the paper, we estimate \( N_2^{tor}(X; H_{12,A}) \) by understanding a \( D_6 \)-sextic field as a compositum of an \( S_3 \)-cubic field and a quadratic field.

2 Preliminaries

2.1 Classification of tori over \( \mathbb{Q} \)

For an \( n \)-dimensional torus \( T \) over \( \mathbb{Q} \), we have defined a finite subgroup \( G_T \leq GL_n(\mathbb{Z}) \) (well-defined up to conjugation) which is isomorphic to the Galois group of the splitting field of \( T \). In this section, we explain how to compute \( n \)-dimensional tori \( T \) over \( \mathbb{Q} \) such that \( G_T \) is conjugate to the given finite subgroup \( H \) of \( GL_n(\mathbb{Z}) \).

We explain the general method and provide an explicit computation for one example. A complete classification of 2-dimensional tori over \( \mathbb{Q} \) will be provided in Section 4.

Recall that the functor \( T \mapsto X^*(T) \) from the category of tori over \( \mathbb{Q} \) to the category of \( G_\mathbb{Q} \)-lattices is an anti-equivalence of categories. Its inverse functor maps a \( G_\mathbb{Q} \)-lattice \( M \) to a torus \( T_M \) defined by \( T_M(R) := Hom_{G_\mathbb{Q}}(M, R^*_\mathbb{Q}) \) for any \( \mathbb{Q} \)-algebra \( R \). Suppose that an \( n \)-dimensional torus \( T \) over \( \mathbb{Q} \) corresponds to a \( G_\mathbb{Q} \)-lattice \( M \) of rank \( n \). If the splitting field of \( T \) is \( L \), then we have \( \ker(\overline{G_\mathbb{Q}} \rightarrow Aut(M)) = G_L \) and
\[
T(R) \cong Hom_{Gal(L/\mathbb{Q})}(M, R^*_L).
\]

Fix a \( \mathbb{Z} \)-basis \( x_1, \ldots, x_n \) of \( M \). An element \( \phi \in Hom_{Gal(L/\mathbb{Q})}(M, R^*_L) \) is determined by \( n \) values
\[
v_1 := \phi(x_1), \ldots, v_n := \phi(x_n) \in R^*_L = (R_L/\mathbb{Q} \mathbb{G}_m)(R),
\]
whose relations are given by
\[ g \cdot v_j = \phi(g \cdot x_j) = \phi(\sum_{i=1}^{n} A_{ij} x_i) = \prod_{i=1}^{n} v_i^{A_{ij}} \] (7)
for each \( g \in \text{Gal}(L/\mathbb{Q}) \) corresponds to \( A \in G_T \). Note that it is enough to check the relations (7) for generators of \( \text{Gal}(L/\mathbb{Q}) \). Now we provide an example to illustrate this.

**Example 2.1.** Let \( H \) be the finite subgroup of \( \text{GL}_2(\mathbb{Z}) \) generated by two elements \( g = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \) and \( h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). It is same as the group \( H_{12, A} \) in Section 4. By the relations \( g^6 = h^2 = (gh)^2 = 1 \), \( H \) is isomorphic to the dihedral group \( D_6 \) of order 12. Let \( T \) be a 2-dimensional torus over \( \mathbb{Q} \). By the relations \( g^6 = h^2 = (gh)^2 = 1 \), \( H \) is isomorphic to the dihedral group \( D_6 \) of order 12. Let \( T \) be a 2-dimensional torus over \( \mathbb{Q} \). By the equation (7), \( T \) is determined by \( v_1, v_2 \in \mathbb{R}_{L/\mathbb{Q}} \mathbb{G}_m \) such that
\[
\begin{align*}
gv_1 &= v_1^{\varphi_{11}} v_2^{\varphi_{21}} = v_1 v_2, \quad gv_2 = v_1^{\varphi_{12}} v_2^{\varphi_{22}} = v_1^{-1} \\
hv_1 &= v_1^{\varphi_{11}} v_2^{\varphi_{21}} = v_2, \quad hv_2 = v_1^{\varphi_{12}} v_2^{\varphi_{22}} = v_1.
\end{align*}
\]
A simple computation shows that
\[
T = \{ (v_1, v_2) \in (\mathbb{R}_{L/\mathbb{Q}} \mathbb{G}_m)^2 : gv_1 = v_1 v_2, \; hv_1 = v_2, \; hv_2 = v_1 \} = \{ v_1 \in \mathbb{R}_{La/\mathbb{Q}} \mathbb{G}_m : g\varphi_{11}v_1 = v_1 \} = \{ v_1 \in \mathbb{R}_{La/\mathbb{Q}} \mathbb{G}_m : v_1 \cdot g^2v_1 = 1 \} = \{ v_1 \in \mathbb{R}_{La/\mathbb{Q}} \mathbb{G}_m : v_1 \cdot g^3v_1 = 1, v_1 \cdot g^4v_1 \} = \{ v_1 \in \mathbb{R}_{La/\mathbb{Q}} \mathbb{G}_m : v_1 \cdot g^3v_1 = 1 \} = \{ v_1 \in \mathbb{R}_{La/\mathbb{Q}} \mathbb{G}_m : v_1 \cdot g^3v_1 = 1 \} = \{ v_1 \in \mathbb{R}_{La/\mathbb{Q}} \mathbb{G}_m : N_{La/L_3}(v_1) = 1 \} \subset \mathbb{R}_{La/\mathbb{Q}} \mathbb{G}_m.
\]

**2.2 Computation of \( C(T) \)**

After the classification of tori over \( \mathbb{Q} \), we need to compute the Artin conductor \( C(T) \) for each torus \( T \) over \( \mathbb{Q} \). This can be done by combining the following two basic facts.

**Proposition 2.2.**

1. For a number field \( K \), \( C(\mathbb{R}_{K/\mathbb{Q}} \mathbb{G}_m) = D_K \). (See Section 1.2).

2. If \( 1 \rightarrow T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow 1 \) is an exact sequence of tori over \( \mathbb{Q} \), then \( C(T_2) = C(T_1)C(T_3) \). (The exact sequence gives an isogeny \( T_2 \sim T_1 \times T_3 \) so
\[
X^*(T_2)^{\mathbb{Q}} \cong X^*(T_1 \times T_3)^{\mathbb{Q}} \cong X^*(T_1)^{\mathbb{Q}} \times X^*(T_3)^{\mathbb{Q}}
\]
and \( C(T_2) = C(T_1)C(T_3) \).)

We introduce some notations for tori over \( \mathbb{Q} \). For an extension \( L/K \) of number fields,
\[
R^{(1)}_{L/K} \mathbb{G}_m := \ker(\mathbb{R}_{L/K} \mathbb{G}_m \rightarrow \mathbb{R}_{K/\mathbb{Q}} \mathbb{G}_m)
\]
is a torus over \( K \) of dimension \( [L : K] - 1 \) and
\[
T_{L/K} := R_{K/\mathbb{Q}}(R^{(1)}_{L/K} \mathbb{G}_m).
\]
is a torus over $\mathbb{Q}$ of dimension $[L : \mathbb{Q}] - [K : \mathbb{Q}]$. Note that $T_{L/\mathbb{Q}} = R_{L/\mathbb{Q}}^{(1)} \mathbb{G}_m$ is the norm-one torus. Taking $R_{K/\mathbb{Q}}$ on the exact sequence

$$1 \to R_{L/K}^{(1)} \mathbb{G}_m \to R_{L/K} \mathbb{G}_m \xrightarrow{N_{L/K}} \mathbb{G}_m \to 1$$

of tori over $K$, we obtain an exact sequence

$$1 \to T_{L/K} \to R_{L/\mathbb{Q}} \mathbb{G}_m \xrightarrow{R_{K/\mathbb{Q}}(N_{L/K})} R_{K/\mathbb{Q}} \mathbb{G}_m \to 1$$

of tori over $\mathbb{Q}$ (cf. [32, Section 3.12]). Proposition 2.2 implies that

$$C(T_{L/K}) = \frac{D_L}{D_K}. \quad (8)$$

Now let $K_1$ and $K_2$ be linearly disjoint number fields and $L = K_1K_2$. Recall that $K_1$ and $K_2$ are linearly disjoint if and only if $[L : \mathbb{Q}] = [K_1 : \mathbb{Q}][K_2 : \mathbb{Q}]$. The following lemma enables us to compute the number $C(T)$ for a torus $T = T_{L/K_1} \cap T_{L/K_2}$.

**Lemma 2.3.** Let $K_1$, $K_2$ and $L$ as above. Then $T_{L/K_1} \cap T_{L/K_2}$ is a torus over $\mathbb{Q}$ and the sequence

$$1 \to T_{L/K_1} \cap T_{L/K_2} \to T_{L/K_1} \xrightarrow{N_{K_2/Q}(N_{L/K_2})} T_{K_2/\mathbb{Q}} \to 1$$

is exact.

**Proof.** Denote $\alpha := R_{K_2/\mathbb{Q}}(N_{L/K_2})$. We have $N_{K_2/\mathbb{Q}}(N_{L/K_2}(x)) = N_{K_1/\mathbb{Q}}(N_{L/K_1}(x)) = 1$ for $x \in T_{L/K_1}$ so the map $\alpha$ is well-defined. Also it is trivial that $\ker(\alpha) = T_{L/K_1} \cap T_{L/K_2}$. From the equivalence of categories between the category of tori over $\mathbb{Q}$ and the category of $G_{\mathbb{Q}}$-lattices, it is enough to show that

$$\beta := X^*(\alpha) : X^*(T_{K_2/\mathbb{Q}}) \to X^*(T_{L/K_1})$$

is injective and $\text{coker}(\beta)$ is torsion-free.

Let $\sigma_1, \ldots, \sigma_{n_1}$ be the embeddings of $K_1$ into $\mathbb{C}$ and $\sigma'_1, \ldots, \sigma'_{n_2}$ be the embeddings of $K_2$ into $\mathbb{C}$. Since $K_1$ and $K_2$ are linearly disjoint, the embeddings of $L$ into $\mathbb{C}$ are $\tau_{ij}$ (1 ≤ $i$ ≤ $n_1$, 1 ≤ $j$ ≤ $n_2$) such that $\tau_{ij} | K_1 = \sigma_i$ and $\tau_{ij} | K_2 = \sigma'_j$. We have

$$X^*(T_{K_2/\mathbb{Q}}) \cong \text{coker}(X^*(\mathbb{G}_m) \to X^*(R_{K_2/\mathbb{Q}} \mathbb{G}_m)) \cong \text{coker}(\mathbb{Z} \xrightarrow{x \mapsto (x, \ldots, x)} \prod_{j=1}^{n_2} \mathbb{Z}_{ij})$$

and

$$X^*(T_{L/K_1}) \cong \text{coker}(X^*(R_{K_1/\mathbb{Q}}) \to X^*(R_{L/\mathbb{Q}} \mathbb{G}_m)) \cong \prod_{i=1}^{n_1} \text{coker}(\mathbb{Z}_i \xrightarrow{x_i \mapsto (x_i, \ldots, x_i)} \prod_{j=1}^{n_2} \mathbb{Z}_{ij})$$

as $\mathbb{Z}$-modules. Here $\mathbb{Z}_i$, $\mathbb{Z}_j$, $\mathbb{Z}_{ij}$ are isomorphic to $\mathbb{Z}$ and the subscripts are indices. Since the diagram

$$\begin{array}{ccc}
T_{L/K_1} & \longrightarrow & R_{L/\mathbb{Q}} \mathbb{G}_m \\
\downarrow S_{L/K_2} & & \downarrow S_{L/K_2} \\
T_{K_2/\mathbb{Q}} & \longrightarrow & R_{K_2/\mathbb{Q}} \mathbb{G}_m
\end{array}$$

commutes, the map $\beta$ is induced by the map

$$\prod_{j=1}^{n_2} (\mathbb{Z}_j \xrightarrow{x_j \mapsto (x_j, \ldots, x_j)} \prod_{i=1}^{n_1} \mathbb{Z}_{ij}).$$

For $A := \mathbb{Z}^{n_2}/(1, \ldots, 1) \mathbb{Z} \cong \mathbb{Z}^{n_2-1}$, the map $\beta$ is given by

$$\beta : A \to A^{n_1} (a \mapsto (a, \ldots, a)).$$

It is obviously injective and has a torsion-free cokernel. \qed
Now the exact sequence (9) gives
\[
\dim(T_{L/K_1} \cap T_{L/K_2}) = \dim T_{L/K_1} - \dim T_{K_2/Q} = ([L : Q] - [K_1 : Q]) - ([K_2 : Q] - 1) = ([K_1 : Q] - 1)([K_2 : Q] - 1).
\]
Also the equation (8) and Lemma 2.3 imply that
\[
C(T_{L/K_1} \cap T_{L/K_2}) = \frac{C(T_{L/K_1})}{C(T_{K_2/Q})} = \frac{D_L}{D_{K_1}D_{K_2}}.
\]

2.3 Counting number fields by conductor

In this section, we summarize the known results on counting number fields by conductor. For an integer \(n \geq 2\) and a subgroup \(G \leq S_n\), denote by \(\text{NF}_n(G)\) the set of the isomorphism classes of degree \(n\) number fields whose Galois closure has a Galois group permutation-isomorphic to \(G\). Denote \(\text{NF}_2(C_2)\) by \(\text{NF}_2\). For an abelian number field \(M\), denote its conductor by \(\text{Cond}(M)\). First we provide an asymptotics of the number of abelian number fields ordered by conductor.

**Proposition 2.4.** ([21, Theorem 4 and 5]) For a finite abelian group \(G\), the number \(N^{\text{con}}(X; G)\) of abelian number fields \(L\) such that \(\text{Gal}(L/Q) \cong G\) and the conductor of \(L\) is at most \(X\) is given by
\[
N^{\text{con}}(X; G) \sim c_G X (\log X)^{d(G)}
\]
for some constant \(c_G > 0\) and a nonnegative integer \(d(G)\). Here the integer \(d(G)\) is given by
\[
d(G) = \prod_{i=1}^r (e_i + 1) - 2
\]
when \(G \cong \prod_{i=1}^r \mathbb{Z}/p_i^{e_i}\mathbb{Z}\). In particular, \(d(C_4) = 1\) and \(d(C_6) = 2\).

Let \(L_4 \in \text{NF}_4(C_4)\) and denote its quadratic subfield by \(L_2\). By the conductor-discriminant formula [27, VII.11.9], the conductor of \(L_4\) is given by \(\text{Cond}(L_4) = \left(\frac{D_{L_4}}{D_{L_2}}\right)^{\frac{1}{2}}\). Similarly, if \(L_6 \in \text{NF}_6(C_6)\) and \(L_i (i = 2, 3)\) is a subfield of \(L_6\) of degree \(i\) then the conductor of \(L_6\) is given by \(\text{Cond}(L_6) = \left(\frac{D_{L_6}}{D_{L_2}D_{L_3}}\right)^{\frac{1}{2}}\). By the above proposition, we have
\[
\#\left\{L_4 \in \text{NF}_4(C_4) : \frac{D_{L_4}}{D_{L_2}} \leq X\right\} \sim c_1 X^{\frac{1}{2}} \log X
\]  
\[\text{(11)}\]
and
\[
\#\left\{L_6 \in \text{NF}_6(C_6) : \frac{D_{L_6}}{D_{L_2}D_{L_3}} \leq X\right\} \sim c_2 X^{\frac{1}{2}} \log^2 X
\]  
\[\text{(12)}\]
for some constants \(c_1, c_2 > 0\).

We have a similar result for some non-abelian number fields. Altuğ, Shankar, Varma and Wilson [2] counted the number of \(D_4\)-quartic fields ordered by conductor using both analytic techniques and geometry-of-numbers methods.

**Proposition 2.5.** ([2, Theorem 1]) For \(L \in \text{NF}_4(D_4)\), denote its unique quadratic subfield by \(K\). Then
\[
\#\left\{L \in \text{NF}_4(D_4) : \frac{D_L}{D_K} \leq X\right\} = c(D_4) X \log X + O(X \log \log X)
\]
for a constant \(c(D_4) := \frac{3}{4} \prod_p \left(1 - \frac{1}{p^2} - \frac{2}{p^3} + \frac{2}{p^4}\right) > 0\) (product over all primes).
2.4 Discriminant of compositum of number fields

Following the exposition of [33, Section 2], we give a description of the discriminant of compositum of two number fields. Let \( M \) be a degree \( n \) number field and \( p \) be a prime. Denote the Galois closure of \( M \) by \( M^c \) and the inertia group of \( M \) at \( p \) by \( I_{M,p} \). For a nonzero integer \( m \), denote the exponent of \( p \) in \( m \) by \( v_p(m) \).

Assume that \( M \) is tamely ramified at \( p \). Then the inertia group \( I_{M,p} \) is cyclic so we can choose its generator \( g_{M,p} \). Since \( g_{M,p} \in I_{M,p} \subset \text{Gal}(M^c/Q) \subset S_n \), we can define the index

\[
\text{ind}(g_{M,p}) := n - \text{the number of orbits of } g_{M,p} \text{ on } \{1, 2, \ldots, n\}
\]

which satisfies the formula \( v_p(D_{M,p}) = \text{ind}(g_{M,p}) \).

**Proposition 2.6.** ([33, Theorem 2.2 and 2.3]) Let \( K_1 \) and \( K_2 \) be number fields such that \( K_1^c \cap K_2^c = Q \) and \( p \) be a prime such that both of \( K_1 \) and \( K_2 \) are tamely ramified at \( p \). Suppose that \( g_{K_1,p} = \prod_k c_k \) (product of disjoint cycles) and \( g_{K_2,p} = \prod_i d_i \). Then

\[
v_p(D_{K_1 K_2}) = m_1 m_2 - \sum_{k,l} \gcd(|c_k|, |d_l|),
\]

where \( m_i \) is the degree of \( K_i \) and \( |c| \) denotes the length of the cycle \( c \). If the least common multiple of \( |c_k| \) and the least common multiple of \( |d_l| \) are coprime, then we have

\[
v_p(D_{K_1 K_2}) = v_p(D_{K_1}) \cdot m_2 + v_p(D_{K_2}) \cdot m_1 - v_p(D_{K_1} v_p(D_{K_2}).
\]

We also introduce a lemma which concerns the product distribution appears in counting number fields. It is useful when we consider the compositum of two linearly disjoint number fields.

**Proposition 2.7.** Let \( F_i(X) = \# \{ s \in S_i : s \leq X \} \) (\( i = 1, 2 \)) be the asymptotic distribution of some multi-set \( S_i \) consists of a sequence of elements of \( \mathbb{R}_{\geq 1} \). Suppose that \( F_i(X) \sim A_i X^{r_i} (\log X)^{r_1} \) for \( n_i > 0 \), \( r_i \geq 0 \) and define the product distribution

\[
\dot{P}(X) := \# \{ (s_1, s_2) \in S_1 \times S_2 : s_1 s_2 \leq X \}.
\]

1. ([33, Lemma 3.1]) If \( n_1 = n_2 = n \), then

\[
\dot{P}(X) \sim A_1 A_2 \frac{r_1! r_2!}{(r_1 + r_2 + 1)!} n X^n (\log X)^{r_1 + r_2 + 1}.
\]

2. ([33, Lemma 3.2]) If \( n_1 > n_2 \), then there exists a constant \( C > 0 \) such that

\[
\dot{P}(X) \sim CX^{n_1} (\log X)^{r_1}.
\]

3 Malle’s conjecture for tori over \( Q \)

In this section, we provide analogues of Linnik’s and Malle’s conjectures for tori over \( Q \) and study the relation between them. First we give a conjecture on the number of the isomorphism classes of \( n \)-dimensional tori over \( Q \) counted by Artin conductor.

**Conjecture 3.1.** For every \( n \geq 1 \), there exists a constant \( c_n > 0 \) satisfying

\[
N_n^{\text{tor}}(X) \sim c_n X (\log X)^{n-1}.
\]

We have the following simple comments on this conjecture.
(a) The conjecture is true for \( n = 1 \). Every one-dimensional torus over \( \mathbb{Q} \) is \( \mathbb{G}_m \) or \( T_{L/\mathbb{Q}} \) for a quadratic field \( L \). Since \( C(T_{L/\mathbb{Q}}) = D_L \) by the equation (8), we have

\[
N^{\text{tor}}_1(X) = N_2(X) + O(1) = \frac{6}{\pi^2} X + O(X^{\frac{1}{2}}).
\]

The case \( n = 2 \) will be discussed in Section 4.

(b) It is easy to prove that \( N^{\text{tor}}_n(X) \gg X(\log X)^{n-1} \) for each \( n \geq 1 \). Denote

\[
S_1 := \left\{ (L_1, \cdots, L_n) \in \text{NF}_n^2 : \prod_{i=1}^n D_{L_i} \leq X \right\}
\]

and let \( S_2 \) be the set of the isomorphism classes of tori \( T \) over \( \mathbb{Q} \) which are isomorphic to \( \prod_{i=1}^n T_{L_i/\mathbb{Q}} \) for some quadratic fields \( L_1, \cdots, L_n \) and \( C(T) = \prod_{i=1}^n D_{L_i} \leq X \). One can prove that

\[
|S_1| \sim \frac{1}{(n-1)!} \left( \frac{6}{\pi^2} \right)^n X(\log X)^{n-1}
\]

by induction on \( n \) using Proposition 2.7. The map \( S_1 \to S_2 \) defined by

\[
(L_1, \cdots, L_n) \mapsto \prod_{i=1}^n T_{L_i/\mathbb{Q}}
\]

is surjective and the size of each fiber of the map is at most \( n! \). Therefore

\[
N^{\text{tor}}_n(X) \geq |S_2| \geq \frac{|S_1|}{n!} \gg X(\log X)^{n-1}.
\]

Next we provide an analogue of Malle’s conjecture for tori over \( \mathbb{Q} \). Since the following conjecture is a direct consequence of Malle’s conjecture with modified weights suggested by Ellenberg and Venkatesh [13, Example 4.4], it is not new. The important point is that this conjecture implies the above conjecture on the asymptotics of \( N^{\text{tor}}_n(X) \) (see Corollary 3.6).

**Conjecture 3.2.** For every \( n \geq 1 \) and a finite subgroup \( 1 \neq H \leq \text{GL}_n(\mathbb{Z}) \),

\[
N^{\text{tor}}_n(X; H) \sim c_H X^{\pi(E_H)} (\log X)^{b(H)-1}
\]

where the positive integers \( a(H), b(H) \) and a constant \( c_H > 0 \) depend only on \( H \). For the identity matrix \( I_n \in \text{GL}_n(\mathbb{Z}) \), the number \( a(H) \) is given by

\[
a(H) := \min_{h \in H \setminus \{I_n\}} \text{rank}(h - I_n)
\]

and the number \( b(H) \) is given by the number of the orbits \( \mathcal{C} \) of the action of \( G_\mathbb{Q} \) on the conjugacy classes of \( H \) via the cyclotomic character such that \( \text{rank}(h - I_n) = a(H) \) for some (equivalently, all) \( h \in \mathcal{C} \).

It is easy to show that the number \( b(H) \) is well-defined. If \( h, h' \in H \) are conjugate, then \( h - I_n \) and \( h' - I_n \) are also conjugate so they have the same rank. If \( h \) and \( h' \) are in the same \( G_\mathbb{Q} \)-orbit, then \( h \) and \( h' \) are powers of one another so \( \text{rank}(h - I_n) = \text{rank}(h' - I_n) \).

Now we provide two remarks on Malle’s conjecture for tori over \( \mathbb{Q} \).
Remark 3.3. Let $\mathcal{P}_n$ be the group of the permutation matrices in $GL_n(\mathbb{Z})$. There is a canonical isomorphism $\mathcal{P}_n \cong S_n$ which maps $h \in \mathcal{P}_n$ to $\sigma \in S_n$ such that $h_{ij} = 1$ if and only if $i = \sigma(j)$. Let $H$ be the subgroup of $\mathcal{P}_n$ which corresponds to a transitive subgroup $G \leq S_n$. Also let $T$ be a torus over $\mathbb{Q}$ whose splitting field $L$ and $G_T$ is conjugate to $H$.

Following the argument of Section 2.1, $T$ is determined by $v_1, \cdots, v_n \in R_{L/\mathbb{Q}} \mathbb{G}_m$ such that

$$h \cdot v_i = \prod_{j=1}^n v_j^{h_{ji}} = v_{\sigma(i)}$$

for all $h \in H$ corresponds to $\sigma \in G$. By the transitivity of $G$, $v_2, \cdots, v_n$ are determined by $v_1$ and $v_1$ is fixed by every element of $H_1 := \{h \in H : h_{11} = 1\}$. Since $H_1$ is an index $n$ subgroup of $H \cong \text{Gal}(L/\mathbb{Q})$ and $H_1$ has no nontrivial subgroup which is normal in $H$, it corresponds to a Galois group $\text{Gal}(L/K)$ for a degree $n$ number field $K$ such that $K^c = L$. Therefore $T = R_{K/\mathbb{Q}} \mathbb{G}_m$ for a degree $n$ number field $K$ so

$$N_n^{\text{tor}}(X; H) = N_n(X; G).$$

For every $h \in H$ corresponds to $\sigma \in G$, we have

$$\text{rank}(h-I_n) = n - \text{the number of orbits of } \sigma \text{ on } \{1, 2, \cdots, n\} = \text{ind}(\sigma)$$

so $a(H) = a(G)$. The equality $b(H) = b(G)$ is trivial. This shows that Conjecture 3.2 is a generalization of Malle’s conjecture for number fields.

The next remark shows that the conjecture is compatible with the direct product of $H$.

Remark 3.4. Suppose that the conjecture is true for $1 \neq H_i \leq GL_{n_i}(\mathbb{Z})$ ($i = 1, 2$) and define

$$H := \left\{ d(h_1, h_2) := \begin{pmatrix} h_1 & O \\ O & h_2 \end{pmatrix} \in GL_{n_1+n_2}(\mathbb{Z}) : h_i \in H_i \right\}.$$ 

The formula $\text{rank}(d(h_1, h_2) - I_{n_1+n_2}) = \text{rank}(h_1 - I_{n_1}) + \text{rank}(h_2 - I_{n_2})$ implies that

$$a(H) = \min_{(h_1, h_2) \neq (I_{n_1}, I_{n_2})} \text{rank}(h_1 - I_{n_1}) + \text{rank}(h_2 - I_{n_2}) = \min(a(H_1), a(H_2)).$$

Denote by $C_i$ a conjugacy class of $H_i$ and $\tilde{C}_i$ a $G_\mathbb{Q}$-orbit on conjugacy classes of $H_i$ (via the cyclotomic character) containing $C_i$. The conjugacy classes of $H$ are of the form

$$C = d(C_1, C_2) := \{d(h_1, h_2) : h_i \in C_i\}$$

and a $G_\mathbb{Q}$-orbit on $H$ containing $d(C_1, I_{n_2})$ (resp. $d(I_{n_1}, C_2)$) are $d(\tilde{C}_1, I_{n_2})$ (resp. $d(I_{n_1}, \tilde{C}_2)$). Therefore the number $b(H)$ is given by

$$b(H) = \begin{cases} b(H_1) + b(H_2) & (a(H_1) = a(H_2)) \\ b(H_2) & (a(H_1) > a(H_2)) \\ b(H_1) & (a(H_1) < a(H_2)) \end{cases}.$$ 

If $T$ is a torus over $\mathbb{Q}$ such that $G_T$ is conjugate to $H$ in $GL_{n_1+n_2}(\mathbb{Z})$, then $T \cong T_1 \times T_2$ where $T_i$ is an $n_i$-dimensional torus such that $G_{T_i}$ is conjugate to $H_i$ in $GL_{n_i}(\mathbb{Z})$. We also have $C(T) = C(T_1)C(T_2)$. Therefore $N_{n_1+n_2}^{\text{tor}}(X; H)$ is bounded above by the product distribution $P(X)$ of $N_{n_1}^{\text{tor}}(X; H_1)$ and $N_{n_2}^{\text{tor}}(X; H_2)$.

By Proposition 2.7, we have

$$P(X) \sim C X^{\frac{1}{\text{rank}(H)}} (\log X)^{b(H)-1}$$

for some $C > 0$ depends only on $H_1$ and $H_2$. This implies that

$$N_{n_1+n_2}^{\text{tor}}(X; H) \ll H X^{\frac{1}{\text{rank}(H)}} (\log X)^{b(H)-1}.$$
The next proposition is an analogue of [23, Lemma 2.2]. Our proof is elementary, but it is not as simple as the proof of [23, Lemma 2.2].

**Proposition 3.5.** Let $H \neq 1$ be a finite subgroup of $GL_n(\mathbb{Z})$ with $a(H) = 1$. Then $b(H) \leq n$.

**Proof.** Assume that $n \geq 2$ and denote $I_n$ by $I$ for simplicity. Let $h \neq I$ be an element of $H$ so $h^n = I$ for some $m > 1$. If $\text{rank}(h - I) = 1$, then $h = I + vwT$ for some $v, w \in \mathbb{Z}^n$. This implies that $(h - I)^2 = (vwT)^2 = c(h - I)$ for $c = v \cdot w$ so the minimal polynomial of $h$ is $(x - 1)(x - t)$ for $t = c + 1$. The product of the eigenvalues of $h$ (counted with multiplicity) should be $\pm 1$ so $t = \pm 1$. Also $x - t | \frac{x^m - 1}{x - 1}$ implies that $t = -1$.

Assume that $a(H) = 1$ and $b(H) \geq n + 1$. Then there are $h_1, \ldots, h_{n+1} \in H$ which are not conjugate in $H$ and $\text{rank}(h_i - I) = 1$ for each $i$. Denote an order of $h \in H$ by $\text{ord}(h)$.

- Let $\{x, y\} \subset \{h_1, \ldots, h_{n+1}\}$ and assume that $\text{ord}(xy) = 2k + 1$ for $k \in \mathbb{Z}_{>0}$. Then $y = (xy)^k x(xy)^{-k}$ so $x$ and $y$ are conjugate, which is impossible. Therefore the order of $xy$ is even.

- Let $\{x, y, z\} \subset \{h_1, \ldots, h_{n+1}\}$ and assume that $\text{ord}(xy), \text{ord}(xz) \geq 4$. Then $(x, y, z)$ is an infinite group by the classification of finite Coxeter groups, which contradicts to the fact that $H$ is finite. Therefore at least one of $xy$ or $xz$ has order 2.

For every $i$, there are at most one $j \neq i$ such that $\text{ord}(h_i, h_j) > 2$. Note that if $\text{ord}(h_i, h_j) = 2$, then $h_i$ and $h_j$ commute. We may assume that $\text{ord}(h_i, h_j) > 2$ if and only if $\{i, j\} = \{2t - 1, 2t\}$ for some $1 \leq t \leq m$, where $m$ is an integer such that $0 \leq m \leq \frac{n + 1}{2}$. Now let $\text{ord}(h_{2t-1} h_{2t}) = 2\alpha_t$ ($1 \leq t \leq m$) for positive integers $\alpha_1, \ldots, \alpha_m > 1$ and consider the set

$$S := \{h_1, h_1(h_2)^{\alpha_1}, \ldots, h_{2m-1}(h_{2m-1}(h_{2m})^{\alpha_m}, h_{2m+1}, h_{2m+2}, \ldots, h_{n+1}\}.$$

Since $u_t := h_{2t-1}(h_{2t-1} h_{2t})^{\alpha_t}$ is conjugate to one of $h_{2t-1}$ or $h_{2t}$ and $u_t \neq h_{2t-1}$, we have $|S| = n + 1$ and $\text{rank}(h - I) = 1$ for every $h \in S$. Also every element of $S$ has order 2 and the elements of $S$ are pairwise commute so they are simultaneously diagonalizable, i.e. there exists $g \in GL_n(\mathbb{C})$ such that $g S g^{-1} \subset D_n$ for the group $D_n$ of diagonal matrices in $GL_n(\mathbb{Z})$. Now we have

$$\# \{h \in D_n : \text{rank}(h - I) = 1\} = n < |g S g^{-1}|,$$

which is a contradiction. \qed

**Corollary 3.6.** Conjecture 3.2 implies Conjecture 3.1.

**Proof.** We have

$$N_n^{\text{tor}}(X) = \sum_H N_n^{\text{tor}}(X; H),$$

where $H$ runs through the conjugacy classes of finite nontrivial subgroups of $GL_n(\mathbb{Z})$. It is a finite sum by the classical result of Minkowski [26]. Now the corollary follows from Proposition 3.5 and the fact that $a(D_n) = 1$ and $b(D_n) = n$ for the group $D_n$ of diagonal matrices in $GL_n(\mathbb{Z})$. \qed

4 Counting algebraic tori over $\mathbb{Q}$ of dimension 2

4.1 Classification of 2-dimensional tori over $\mathbb{Q}$

In this section, we give a classification of 2-dimensional tori over $\mathbb{Q}$ and compute their Artin conductors. The classification can be done as in Example 2.1 and it can be also found in the paper of Voskresenskii [31]. In the
course of the computation of the Artin conductor $C(T)$, Proposition 2.2 and the formulas (8) and (10) will be repeatedly used.

In each case, denote the splitting field of a torus $T$ by $L$ and identify $\text{Gal}(L/Q)$ with $G_T$. Let $C_m$ be the cyclic group of order $m$, $D_m$ be the dihedral group of order $2m$ and $S_m$ be the symmetric group of degree $m$. For simplicity, denote $D_i := D_{L_i}$, $D'_i := D_{L'_i}$, and so on.

The following list gives the classification of 2-dimensional tori over $Q$ (except for the trivial case $G_2^n$), together with their Artin conductors. Since there are 12 conjugacy classes of finite nontrivial subgroups of $GL_2(Z)$, the classification gives 12 types of 2-dimensional tori over $Q$.

(i) $G_T \cong C_2 : T$ is one of the following types.

(a) $G_T = H_{2,A} := \langle \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \rangle : T = T_{L_2/Q}$ and $C(T) = D_2^2$.
(b) $G_T = H_{2,B} := \langle \left( \begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix} \right) \rangle : T = G_m \times T_{L'/Q}$ and $C(T) = D_L$.
(c) $G_T = H_{2,C} := \langle \left( \begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix} \right) \rangle : T = R_{L'/Q} \times G_m$ and $C(T) = D_L$.

Note that $G_m \times T_{L'/Q}$ and $R_{L'/Q} \times G_m$ are isogenous, but not isomorphic. This corresponds to the fact that $H_{2,B}$ and $H_{2,C}$ are conjugate in $GL_2(Q)$, but not conjugate in $GL_2(Z)$.

(ii) $G_T \cong C_3 : T$ is the following type.

(a) $G_T = H_{3,A} := \langle \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \rangle : T = T_{L_3/Q}$ and $C(T) = D_3$.

(iii) $G_T = (g) \cong C_4 : L_4 = L$ has a unique quadratic subfield $L_2 = L^{g^2}$. $T$ is the following type.

(a) $G_T = H_{4,A} := \langle \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \rangle : T = T_{L_4/L_2}$ and $C(T) = \frac{D_4}{D_2}$.

(iv) $G_T = (g,h) \cong C_2 \times C_2 : L$ has 3 quadratic subfields $L_1 = L^g$, $L_2 = L^h$ and $L_3 = L^{gh}$. By the conductor-discriminant formula [27, VII.11.9], we have $D_L = D_1 D_2 D_3$. $T$ is one of the following types.

(b) $G_T = H_{4,B} := \langle \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right), \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \rangle : T = T_{L/L_1}$ and $C(T) = \frac{D_L}{D_1} = D_2 D_3$.
(c) $G_T = H_{4,C} := \langle \left( \begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix} \right), \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \rangle : T = T_{L_2/L_1} \times T_{L_3/L_2}$ and $C(T) = D_1 D_2$.

(v) $G_T = (g) \cong C_6 : L$ has a unique cubic subfield $L_3 = L^g$ and a unique quadratic subfield $L_2 = L^{g^2}$. $T$ is the following type.

(a) $G_T = H_{6,A} := \langle \left( \begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix} \right) \rangle : T = T_{L_6/L_2} \cap T_{L_6/L_3}$ and $C(T) = \frac{D_6}{D_2 D_3}$.

(vi) $G_T = (g,h : g^3 = h^2 = (gh)^2 = 1) \cong S_3 : L_6 = L$ has 3 isomorphic cubic subfields $L^h$, $L^{gh}$ and $L^{g^2 h}$, denoted by $L_3$ and a unique quadratic subfield $L_2 = L^g$. By the work of Hasse [17], we have $D_6 = D_2^3 D_2$. (See [14, Theorem 4] for its generalization to Frobenius groups.) $T$ is one of the following types.

(b) $G_T = H_{6,B} := \langle \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right), \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \rangle : T = T_{L_6/L_3}$ and $C(T) = D_3$.
(c) $G_T = H_{6,C} := \langle \left( \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right), \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \rangle : T = T_{L_6/L_2} \cap T_{L_6/L_3}$ and $C(T) = \frac{D_6}{D_2 D_3} = D_3$.

(vii) $G_T = (g,h : g^4 = h^2 = (gh)^2 = 1) \cong D_4 : L_4 = L^{gh}$ is a quartic $D_4$-field with a unique quadratic subfield $L_2 = L^{(g^2 gh)}$. $T$ is the following type.

(a) $G_T = H_{8,A} := \langle \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right), \left( \begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix} \right) \rangle : T = T_{L_4/L_2}$ and $C(T) = \frac{D_4}{D_2}$. 

12
In this section, we estimate the number of such \( L \).

\[ L = L(\langle g, h : g^6 = h^2 = (gh)^2 = 1 \rangle) \cong D_6 : \ L_6 = L^{gh} \in N_{F_6}(D_6) \] has a unique cubic subfield \( L_3 = L(\langle g^3, gh \rangle) \in N_{F_3}(S_3) \) and a unique quadratic subfield \( L_2 = L(\langle g^2, h \rangle) \). \( T \) is the following type.

\[ (a) \ G_T = H_{12,A} := \langle \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \rangle : \ T = T_{L_6/L_3} \cap T_{L_6/L_2} \text{ and } C(T) = \frac{D_6}{D_3 D_2} \]

The asymptotics of \( N_{2,tor}(X; H) \) for \( H \neq H_{12,A} \) can be easily computed. Therefore the essential new result of Section 4 is the estimation of \( N_{2,tor}(X; H_{12,A}) \).

**Proposition 4.1.** Conjecture 3.2 holds for every finite nontrivial subgroup of GL_2(\mathbb{Z}) which is not conjugate to \( H_{12,A} \).

**Proof.** Since Malle’s conjecture is true for abelian number fields and \( S_3 \)-cubic fields, the conjecture is true if \( H \) is one of

\[ H_{2,A}, H_{2,B}, H_{2,C}, H_{3,A}, H_{6,B} \text{ and } H_{6,C}. \]

By Proposition 2.7, the conjecture is true if \( H \) is \( H_{4,B} \) or \( H_{4,C} \). By the equations (11) and (12), the conjecture is true if \( H \) is \( H_{4,A} \) or \( H_{6,A} \). By Proposition 2.5, the conjecture is true for \( H = H_{8,A} \). \( \square \)

### 4.2 Asymptotics of \( N_{2,tor}(X; H_{12,A}) \)

Let \( L \) be a \( D_6 \)-sextic field with a unique cubic subfield \( F \in NF_3(S_3) \) and a unique quadratic subfield \( K \). By (viii) in Section 4, the number \( N_{2,tor}(X; H_{12,A}) \) is equal to the number of \( L \in NF_6(D_6) \) such that

\[ C(L) := \frac{D_L}{D_F D_K} \leq X. \]

In this section, we estimate the number of such \( L \) based on the strategy of the paper [24], where the authors proved Malle’s conjecture for \( D_6 \)-sextic fields. First we express \( v_p(D_L) \) (the \( p \)-adic valuation of \( D_L \)) in terms of \( v_p(D_F) \) and \( v_p(D_K) \).

**Proposition 4.2.**

\[ C(L) = \frac{D_F D_K^2}{C m^2} \] (16)

where \( C = 2^a 3^b \) for \( 0 \leq a \leq 9, 0 \leq b \leq 3 \) and \( m \) is the product of the primes \( p > 3 \) which divides both \( D_F \) and \( D_K \).

**Proof.** Since \( L = FK \) and the number fields \( F \) and \( K \) are linearly disjoint, we have

\[ \text{lcm}(D_F^2, D_K^4) \mid D_L \mid D_F^2 D_K^4 \]

so \( N := \frac{D_F^2 D_K^4}{D_L} \) is a positive integer which divides \( \text{gcd}(D_F^2, D_K^4) \). We need to show that \( N = C m^2. \)

- If \( p > 3 \) and \( p \nmid \text{gcd}(D_F, D_K) \), then \( v_p(N) = 0 = v_p(C m^2). \)
- If \( p > 3 \) and \( p \mid \text{gcd}(D_F, D_K) \), then \( v_p(N) = 2 = v_p(C m^2) \) by Proposition 2.6 (cf. [24, Table 1]).
- If \( p \in \{2,3\} \), then \( 0 \leq v_p(N) \leq v_p(D_K^4) \) so \( v_2(N) \leq 9 \) and \( v_3(N) \leq 3. \) \( \square \)
Denote the set of positive squarefree integers by Sqf. For a prime \( p \), denote by \( \text{Sqf}_p \) the set of positive integers which are squarefree outside \( p \). By the above proposition,

\[
N_{tor}^2(X; H_{12,A}) = \# \{ L \in \text{NF}_6(D_6) : C(L) \leq X \} \\
\leq \# \left\{ (F, K) \in \text{NF}_3(S_3) \times \text{NF}_2 : \frac{D_F D_K^2}{m^2} \leq \beta X \right\} \\
=: A_1(\beta X)
\]

for \( \beta := 2^9 3^3 \). Since the number of quadratic fields \( K \) satisfying \( m | D_K \) and \( D_K m \leq \left( \frac{X}{D_F} \right)^{\frac{1}{2}} \) for given \( F \) and \( m \) is at most \( 2^{\left( \frac{X}{D_F} \right)^{\frac{1}{2}}} \), we have

\[
A_1(X) \leq \sum_{m \in \text{Sqf}} \sum_{\substack{F \in \text{NF}_3(S_3) \\text{such that} \\frac{D_F}{m} \leq X \\text{and} \\gcd(m, 6) = 1 \\text{and} \ m \text{ is squarefree}}} 2 \left( \frac{X}{D_F} \right)^{\frac{1}{2}}.
\]

The Galois closure \( F^c \) of \( F \) has a unique quadratic subfield \( E \). Then \( D_F = D_E f^2 \) for some \( f \in \text{Sqf}_3 \). By [9, Theorem 2.5 and Proposition 3.7], the number of \( F \in \text{NF}_3(S_3) \) such that \( D_F = D_E f^2 \) for given \( E \) and \( f \) is \( O(h_3(E) \cdot 2^{w(f)}) \) where \( h_3(E) \) is the size of the 3-torsion subgroup of the class group of \( E \), \( w(f) \) is the number of prime divisors of \( f \) and the implied constant is absolute. Now let

\[
S_1 := \{ p \mid m : F \text{ is not totally ramified at } p \} \\
S_2 := \{ p \mid m : F \text{ is totally ramified at } p \} \\
m_i := \prod_{p \in S_i} p \ (i = 1, 2).
\]

Then \( m_1 \) and \( m_2 \) are coprime, squarefree integers such that

\[
m_1 | D_E, m_2 | f, m_1 m_2 = m \text{ and } \gcd(m_1 m_2, 6) = 1.
\]

Now the inequality (18) transforms into

\[
A_1(X) \ll \sum_{m_1, m_2 \in \text{Sqf} \\gcd(m_1 m_2, 6) = 1} \sum_{m_1 \mid D_E} \sum_{m_2 \mid f} \sum_{\substack{E \in \text{NF}_2 \\text{such that} \\frac{D_E}{m_1 f} \leq X \\text{and} \ D_E f^2 \leq X \\text{and} \ \gcd(m_1 m_2, 6) = 1 \\text{and} \ m_1 \mid D_E \\text{and} \ m_2 \mid f}} \left( \frac{X}{D_E f^2} \right)^{\frac{1}{2}} h_3(E) \cdot 2^{w(f)}.
\]

We estimate the right-hand side of the above inequality by summing over the intervals

\[
D_E f^2 \in [B, 2B)
\]

for \( B = 2^i \ (0 \leq i \leq \log_2 X) \). The inequality (19) implies that

\[
A_1(X) \ll \sum_{i=0}^{\lfloor \log_2 X \rfloor} A_2(X; 2^i)
\]

14
for
\[ A_2(X; B) := \sum_{m_1, m_2 \in \text{Sqf}} \sum_{l = 1}^{\lfloor \log_2 X \rfloor} \sum_{f \in \text{Sqf}} \left( \frac{X}{D_E f^2} \right)^{\frac{1}{2}} h_3(E) \cdot 2^{w(f)} \]
\[ \leq \frac{X^{\frac{1}{2}}}{B^2} \sum_{f \in \text{Sqf}} \sum_{m_2 \mid f} 2^{w(f)} \sum_{m_1 \mid D_E} h_3(E) \sum_{m_1 \mid D_E} 1 \]
\[ \leq \frac{X^{\frac{1}{2}}}{B^2} \sum_{f \in \text{Sqf}} \sum_{m_2 \mid f} 2^{w(f)} \tau(f) \sum_{E \in \text{NF}_2} h_3(E) \tau(D_E). \tag{21} \]

Here \( \tau(n) \) denotes the number of divisors of \( n \). The inequalities (20) and (21) show that it is essential to give an upper bound of the function
\[ g(X) := \sum_{E \in \text{NF}_2} h_3(E) \tau(D_E), \]
which will be done in the next section.

4.3 Upper bound of \( g(X) \)

The estimation of the function \( g(X) \) is the key part of Section 4. In this section, we provide both conditional and unconditional results on the upper bound of \( g(X) \).

**Proposition 4.3.**
\[ g(X) \ll \varepsilon X^{1 + \frac{\log_{\log X} X}{\log \log X}} \ll \varepsilon X^{1+\varepsilon}. \tag{22} \]

**Proof.** A classical result of Wigert [34] states that for any \( \varepsilon > 0 \),
\[ \max_{n < X} \tau(n) < X \frac{\log_{\log X} X}{\log \log X} \]
for sufficiently large \( X \). Therefore
\[ g(X) \ll \varepsilon X^{\frac{\log_{\log X} X}{\log \log X}} \sum_{E \in \text{NF}_2} h_3(E) \ll \varepsilon X^{1 + \frac{\log_{\log X} X}{\log \log X}} \ll \varepsilon X^{1+\varepsilon} \]
by the theorem of Davenport and Heilbronn [11, Theorem 3]. \( \Box \)

The following corollary will be useful for the estimation of \( A_2(X; B) \); see Theorem 4.9.

**Corollary 4.4.** For any \( Y \in [1, 2X] \), \( g(Y) \ll \varepsilon Y X^{\frac{\log_{\log X} X}{\log \log X}} \) as \( X \to \infty \).

Now we consider the following version of the Cohen-Lenstra heuristics. Denote the set of the isomorphism classes of real (resp. imaginary) quadratic fields by \( \text{NF}_2^+ \) (resp. \( \text{NF}_2^- \)). The elements of \( \text{NF}_2^+ \) and \( \text{NF}_2^- \) are ordered by the absolute values of their discriminants. For a number field \( K \) and a prime \( p \), denote the size of the \( p \)-torsion subgroup of the class group of \( K \) by \( h_p(K) \).

**Conjecture 4.5.** (Cohen-Lenstra) Let \( p \) be an odd prime and \( \alpha \) be a positive integer.

1. ([8, (C10)]) The average of \( \prod_{0 \leq i < \alpha} (h_p(K) - p^i) \) for \( K \in \text{NF}_2^+ \) is \( p^{-\alpha} \).
The average of \( \prod_{0 \leq i < \alpha} (h_p(K) - p^i) \) for \( K \in \text{NF}_2^* \) is 1.

By [11, Theorem 3], the above conjecture is true when \( p = 3 \) and \( \alpha = 1 \). It is the only known case of the conjecture.

**Remark 4.6.** If the conjecture is true for a fixed prime \( p \) and each of \( 1 \leq \alpha \leq m \), then the \( m \)-th moment of the number \( h_p(K) \) for quadratic fields \( K \) is given by

\[
\sum_{K \in \text{NF}_2^*} h_p(K)^m \sim cX
\]

for some explicit constant \( c > 0 \) depends only on \( p \) and \( m \).

The above conjecture enables us to obtain the following upper bound of \( g(X) \), which is much stronger than the upper bound given in Proposition 4.3.

**Proposition 4.7.** Let \( m \geq 2 \) be an integer. Under the assumption of Conjecture 4.5 for \( p = 3 \) and each of \( 1 \leq \alpha \leq m \), we have

\[
g(X) \ll_m X (\log X)^{2\frac{m}{m+1}-1}.
\]  

(23)

In particular, if Conjecture 4.5 holds for \( p = 3 \) and every \( \alpha > 0 \), then

\[
g(X) \ll X (\log X)^{1+\varepsilon}.
\]  

(24)

**Proof.** Assume that Conjecture 4.5 holds for \( p = 3 \) and \( 1 \leq \alpha \leq m \). Let

\[
a_r(X) := \# \{ E \in \text{NF}_2 : D_E \leq X \text{ and } h_3(E) = 3^r \}
\]

and denote by \( w(k) \) the number of prime divisors of an integer \( k \). By the theorem of Hardy and Ramanujan [16], there are constants \( c_1, c_2 > 0 \) such that

\[
\# \{ E \in \text{NF}_2 : D_E \leq X \text{ and } w(D_E) = k \} < \frac{c_1 X}{\log X} \cdot \frac{(\log \log X + c_2)^k - 1}{(k - 1)!}
\]  

(25)

for every \( X \geq 2 \) and \( k \geq 1 \).

For a quadratic field \( E \), \( v_2(D_E) \leq 3 \) so \( \tau(D_E) \leq 2^{v_2(D_E)+1} \). Therefore

\[
g(X) \leq \sum_{r=0}^{\infty} \sum_{k=1}^{\infty} 3^r 2^{k+1} \# \{ E \in \text{NF}_2 : D_E \leq X, h_3(E) = 3^r \text{ and } w(D_E) = k \}
\]

\[
\leq \sum_{r=0}^{\infty} \sum_{k=1}^{\infty} 3^r 2^{k+1} \min \left\{ a_r(X), \frac{c_1 X}{\log X} \cdot \frac{(\log \log X + c_2)^k - 1}{(k - 1)!} \right\}
\]  

(26)

by the inequality (25). Suppose that \( X \geq 2 \) and Denote

\[
S(r, k) := \min \left\{ a_r(X), \frac{c_1 X}{\log X} \cdot \frac{(\log \log X + c_2)^k - 1}{(k - 1)!} \right\}.
\]

To bound the sum \( \sum_{(r,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}} 3^r 2^{k+1} S(r, k) \), we divide the set \( \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1} \) into two parts:

\[
R_{1,m} := \left\{ (r,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1} : 2^{k-1} < 3^{r(m-1)} \right\}
\]

\[
R_{2,m} := \left\{ (r,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1} : 2^{k-1} \geq 3^{r(m-1)} \right\}.
\]

Now we estimate the sum \( \sum_{(r,k) \in R_{i,m}} 3^r 2^{k+1} S(r, k) \) for \( i = 1, 2 \).
(i) Since $3^r 2^k + 1 < 4 \cdot (3^r)^m$ for any $(r, k) \in R_{1,m}$, we have
\[
\sum_{(r,k) \in R_{1,m}} 3^r 2^k + 1 < 8 \cdot (3^r)^m
\]
for given $r \geq 0$. Therefore
\[
\sum_{(r,k) \in R_{1,m}} 3^r 2^k + 1 S(r,k) \leq \sum_{r=0}^{\infty} \sum_{k \geq 1} 3^r 2^k + 1 a_r(X) \leq 8 \sum_{r=0}^{\infty} (3^r)^m a_r(X) \tag{27}
\]
and
\[
\sum_{r=0}^{\infty} (3^r)^m a_r(X) = \sum_{r=0}^{\infty} \sum_{E \in NF_2 \atop D_E \leq X} \sum_{h_3(E) = 3^r} h_3(E)^m \tag{28}
\]
by Remark 4.6. The inequalities (27) and (28) imply that
\[
\sum_{(r,k) \in R_{1,m}} 3^r 2^k + 1 S(r,k) \ll_m X. \tag{29}
\]

(ii) Since $3^r 2^k + 1 \leq 4 \cdot (2^k-1)^{\frac{m}{\log X}}$ for any $(r, k) \in R_{2,m}$, we have
\[
\sum_{r \geq 0} \sum_{(r,k) \in R_{2,m}} 3^r 2^k + 1 \leq 6 \cdot (2^k-1)^{\frac{m}{\log X}}
\]
for given $k \geq 1$. Therefore
\[
\sum_{(r,k) \in R_{2,m}} 3^r 2^k + 1 S(r,k) \leq \sum_{k=1}^{\infty} \sum_{r \geq 0} \sum_{(r,k) \in R_{2,m}} 3^r 2^k + 1 S(r,k)
\]
\[
\leq 6 \sum_{k=1}^{\infty} (2^k-1)^{\frac{m}{\log X}} c_1 \frac{X (\log X + c_2)^k}{k!}
\]
\[
\leq 6c_1 \frac{X}{\log X} \sum_{k=1}^{\infty} (2^k-1)^{\frac{m}{\log X}} (\log X + c_2)^k
\]
\[
= 6c_1 \frac{X}{\log X} e^{2 \frac{m}{\log X}} (\log X + c_2)
\]
\[
\leq (6c_1 e^{4c_2}) \frac{X (\log X)^2}{\log X}
\]
\[
\ll (6c_1 e^{4c_2}) \frac{X (\log X)^2}{\log X} ^{\frac{m}{\log X}}.
\]
so
\[
\sum_{(r,k) \in R_{2,m}} 3^r 2^k + 1 S(r,k) \ll X (\log X)^2 ^{\frac{m}{\log X}} - 1. \tag{30}
\]
Now the proposition follows from the inequalities (26), (29) and (30). \qed

The following corollary follows from Proposition 4.7 and the fact that $g(X) = 0$ for every $X < 3$.

**Corollary 4.8.** Assume that Conjecture 4.5 holds for $p = 3$ and every $\alpha > 0$. Then for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that $g(X) \leq c_\varepsilon X (\log X)^{1+\varepsilon}$ for every $X \geq 1$.
4.4 Main theorems

In this section, we give asymptotic upper and lower bounds for $N_2^{\text{tor}}(X; H_{12,A})$ and $N_2^{\text{tor}}(X)$.

Theorem 4.9.

\[ X \ll N_2^{\text{tor}}(X; H_{12,A}) \ll \varepsilon X^{1+\frac{\log 2+\varepsilon}{\log \log X}} \]  
\[ X \log X \ll N_2^{\text{tor}}(X) \ll \varepsilon X^{1+\frac{\log 2+\varepsilon}{\log \log X}}. \]  

Proof. By Proposition 4.1, the upper and lower bounds of $N_2^{\text{tor}}(X)$ follows from the upper and lower bounds of $N_2^{\text{tor}}(X; H_{12,A})$. (Note that $a(H) = 1$ and $b(H) = 2$ for $H \in \{H_{4,B}, H_{4,C}, H_{8,A}\}$.) By the inequality (21) and Corollary 4.4, we have

\[
A_2(X; B) \leq \frac{X^{\frac{1}{2}}}{B^{\pi}} \sum_{f \in \text{Sqf}_f} 2^w(f) \tau(f) g\left(\frac{2B}{f^2}\right) \ll \varepsilon \frac{X^{\frac{1}{2}}}{B^{\pi}} \sum_{f \in \text{Sqf}_f} 2^w(f) \tau(f) \cdot \frac{2B}{f^2} X^{\frac{\log 2+\varepsilon}{\log \log X}} \ll \varepsilon \frac{X^{\frac{1}{2} + \frac{\log 2+\varepsilon}{\log \log X}}}{B^{\pi}} \cdot \sum_{f} 2^w(f) \tau(f) \frac{f}{f^2} \ll \varepsilon X^{\frac{1}{2} + \frac{\log 2+\varepsilon}{\log \log X}} B^\beta \]

for $B \leq X$. The last inequality is due to the fact that $2^w(f) \tau(f) \ll \varepsilon f^\varepsilon$, which implies the convergence of the sum $\sum_{f} 2^w(f) \tau(f) \frac{f}{f^2}$. Now the inequalities (17), (20) and (32) imply that

\[
N_2^{\text{tor}}(X; H_{12,A}) \leq A_1(\beta X) \ll \sum_{i=0}^{\left\lfloor \log_2(\beta X) \right\rfloor} A_2(\beta X; 2^i) \ll \varepsilon (\beta X)^{\frac{1}{2} + \frac{\log 2+\varepsilon}{\log \log X}} \sum_{i=0}^{\left\lfloor \log_2(\beta X) \right\rfloor} 2^i \ll \varepsilon X^{1 + \frac{\log 2+\varepsilon}{\log \log X}}.
\]

A trivial upper bound $C(L) \leq D_F D_K^2$ implies that

\[
N_2^{\text{tor}}(X; H_{12,A}) = \# \{ L \in \text{NF}_0(D_0) : C(L) \leq X \} \geq \# \{ (F, K) \in \text{NF}_3(S_3) \times \text{NF}_2 : D_F D_K^2 \leq X \text{ and } K \not\in F^c \} \geq \# \{ F \in \text{NF}_3(S_3) : 16D_F \leq X \text{ and } \mathbb{Q}(\sqrt{-3}) \not\in F^c \} + \# \{ F \in \text{NF}_3(S_3) : 9D_F \leq X \text{ and } \mathbb{Q}(\sqrt{3}) \not\in F^c \} \geq \# \{ F \in \text{NF}_3(S_3) : 16D_F \leq X \} \gg X.
\]

We have $X(\log X)^N \ll \varepsilon X^{1 + \frac{\log 2+\varepsilon}{\log \log X}} \ll \varepsilon X^{1+\varepsilon}$ for every positive integer $N$. Under the assumption of the Cohen-Lenstra heuristics for $p = 3$, we obtain a much better upper bound.
Theorem 4.10. Assume that Conjecture 4.5 holds for \( p = 3 \) and every \( \alpha > 0 \). Then we have

\[
N^\text{tor}_2(X; H_{12,A}) \leq N^\text{tor}_2(X) \ll X (\log X)^{1+\varepsilon}.
\] (33)

Proof. The inequality (21) and Corollary 4.8 imply that

\[
A_2(X; B) \ll \frac{X^{\frac{5}{2}}}{B^{\frac{3}{2}}} \sum_{f \in \text{Sqf}_3} \frac{2^{w(f)} \tau(f) g \left(\frac{2B}{f^2}\right)}{f^{< (2B)^{\frac{3}{2}}}} 
\]

\[
\ll \varepsilon \frac{X^{\frac{5}{2}}}{B^{\frac{3}{2}}} \sum_{f \in \text{Sqf}_3} \frac{2^{w(f)} \tau(f) \cdot 2B \log \left(\frac{2B}{f^2}\right)^{1+\varepsilon}}{f^{2}} 
\]

\[
\ll \varepsilon X^{\frac{5}{2}} (\log 2B)^{1+\varepsilon} B^{\frac{3}{2}} \sum_{f} \frac{2^{w(f)} \tau(f)}{f^{2}} 
\]

\[
\ll \varepsilon X^{\frac{5}{2}} (\log X)^{1+\varepsilon} B^{\frac{3}{2}}
\]

for \( B \leq X \). The proof can be completed as in the previous theorem. \( \square \)

Acknowledgments

The author is supported by a KIAS Individual Grant (MG079601) at Korea Institute for Advanced Study. We thank Jordan Ellenberg for pointing out that Conjecture 3.2 is a consequence of a more general conjecture in [13], and Joachim König for his corrections to this paper. We also thank Sungmun Cho, Frank Thorne, Jacob Tsimerman and Melanie Matchett Wood for their helpful comments.

References

[1] B. Alberts, Statistics of the first Galois cohomology group: A refinement of Malle’s conjecture, arXiv:1907.06289, to appear in Algebra Number Theory.

[2] S. A. Altuğ, A. Shankar, I. Varma and K. H. Wilson, The number of \( D_4 \)-fields ordered by conductor, J. Eur. Math. Soc. 23 (2021), no. 8, 2733-2785.

[3] M. Bhargava, The density of discriminants of quartic rings and fields, Ann. of Math. (2) 162 (2005), no. 2, 1031-1063.

[4] M. Bhargava, The density of discriminants of quintic rings and fields, Ann. of Math. (2) 172 (2010), no. 3, 1559-1591.

[5] M. Bhargava, A. Shankar and X. Wang, Geometry-of-numbers methods over global fields I: Prehomogeneous vector spaces, arXiv:1512.03035.

[6] M. Bhargava and M. M. Wood, The density of discriminants of \( S_3 \)-sextic number fields, Proc. Amer. Math. Soc. 136 (2008), no. 5, 1581-1587.

[7] H. Cohen, F. Diaz y Diaz and M. Olivier, Enumerating quartic dihedral extensions of \( \mathbb{Q} \), Compos. Math. 133 (2002), no. 1, 65-93.

[8] H. Cohen and H. W. Lenstra Jr., Heuristics on class groups of number fields, Number Theory, Noordwijkerhout 1983, Lecture Notes in Math. 1068, 33-62, Springer, Berlin, 1984.

[9] H. Cohen and F. Thorne, Dirichlet series associated to cubic fields with given quadratic resolvent, Michigan Math. J. 63 (2014), no. 2, 253-273.

[10] B. Datskovsky and D. J. Wright, Density of discriminants of cubic extensions, J. Reine Angew. Math. 386 (1988), 116-138.
COUNTING ALGEBRAIC TORI OVER $\mathbb{Q}$ BY ARTIN CONDUCTOR

[11] H. Davenport and H. Heilbronn, On the density of discriminants of cubic fields. II, Proc. Roy. Soc. London Ser. A 322 (1971), no. 1551, 405-420.

[12] J. S. Ellenberg, M. Satriano and D. Zureick-Brown, Heights on stacks and a generalized Batyrev-Manin-Malle conjecture, arXiv:2106.11340.

[13] J. S. Ellenberg and A. Venkatesh, Counting extensions of function fields with bounded discriminant and specified Galois group, in Geometric Methods in Algebra and Number Theory, Progr. Math. 235, 151-168, Birkhäuser Boston, Boston, MA, 2005.

[14] C. Fieker and J. Klüners, Minimal discriminants for fields with small Frobenius groups as Galois groups, J. Number Theory 99 (2003), no. 2, 318-337.

[15] É. Fouvry and P. Koymans, Malle’s conjecture for nonic Heisenberg extensions, arXiv:2102.09465.

[16] G. H. Hardy and S. Ramanujan, The normal number of prime factors of a number $n$, Q. J. Math. 48 (1917), 76-92.

[17] H. Hasse, Arithmetische Theorie der kubischen Zahlkörper auf klassenkörpertheoretischer Grundlage, Math. Z. 31 (1930), no. 1, 565-582.

[18] J. Klüners, A counter example to Malle’s conjecture on the asymptotics of discriminants, C. R. Math. Acad. Sci. Paris 340 (2005), no. 6, 411-414.

[19] J. Klüners, The distribution of number fields with wreath products as Galois groups, Int. J. Number Theory 8 (2012), no. 3, 845-858.

[20] S. Mäki, On the density of abelian number fields, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 54 (1985).

[21] S. Mäki, The conductor density of abelian number fields, J. Lond. Math. Soc. (2) 47 (1993), no. 1, 18-30.

[22] G. Malle, On the distribution of Galois groups, J. Number Theory 92 (2002), no. 2, 315-329.

[23] G. Malle, On the distribution of Galois groups II, Exp. Math. 13 (2004), 129-135.

[24] R. Masri, W. Tsai and J. Wang, Malle’s conjecture for $G \times A$ with $G=S_3,S_4,S_5$, arXiv:2004.04651.

[25] H. Mehta, Counting number fields by discriminant, PhD Thesis, University of South Carolina, 2020.

[26] H. Minkowski, Zur Theorie der positiven quadratischen Formen, J. Reine Angew. Math. 101 (1887), 196-202.

[27] J. Neukirch, Algebraic number theory, Grundlehren der Mathematischen Wissenschaften 322, Springer, Berlin, 1999.

[28] A. Shankar and I. Varma, Malle’s Conjecture for Galois octic fields over $\mathbb{Q}$, in preparation.

[29] T. A. Springer, Linear algebraic groups, 2nd ed., Progr. Math. 9, Birkhäuser, Boston, 1998.

[30] S. Türkelli, Connected components of Hurwitz schemes and Malle’s conjecture, J. Number Theory 155 (2015), 163-201.

[31] V. E. Voskresenskiǐ, On two-dimensional algebraic tori, Izv. Akad. Nauk SSSR Ser. Mat. 29 (1965), no. 1, 239-244.

[32] V. E. Voskresenskiǐ, Algebraic groups and their birational invariants, Transl. Math. Monogr. 179, Amer. Math. Soc., Providence, RI, 1998.

[33] J. Wang, Malle’s conjecture for $S_n \times A$ for $n = 3,4,5$, Compos. Math. 157 (2021), no. 1, 83-121.

[34] S. Wigert, Sur l’ordre de grandeur du nombre des diviseurs d’un entier, Ark. Mat. 3 (1907), no. 18, 1-9.

[35] M. M. Wood, Moduli spaces for rings and ideals, PhD Thesis, Princeton University, 2009.

[36] D. J. Wright, Distribution of discriminants of abelian extensions, Proc. Lond. Math. Soc. (3) 58 (1989), no. 1, 17-50.