QUANTIZED FLAG MANIFOLDS AND NON-RESTRICTED MODULES OVER QUANTUM GROUPS AT ROOTS OF UNITY

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ABSTRACT. We give a proof of Lusztig’s conjectural multiplicity formula for non-restricted modules over the De Concini-Kac type quantized enveloping algebra at the \(\ell\)-th root of unity, where \(\ell\) is an odd prime power satisfying certain reasonable conditions.

1. INTRODUCTION

1.1. Let \(G\) be a connected, simply-connected, simple algebraic group over the complex number field \(\mathbb{C}\), and let \(\mathfrak{g}\) be its Lie algebra. The flag manifold \(\mathcal{B}\) of \(G\) plays a crucial role in the geometric representation theory. By the Borel-Weil theory one can construct a finite-dimensional rational \(G\)-module as the space of global sections of a \(G\)-equivariant line bundle on \(\mathcal{B}\). By the works of Beilinson-Bernstein [7] and Brylinsky-Kashiwara [10] around 1980 one can also construct infinite-dimensional \(\mathfrak{g}\)-modules using \(D\)-modules on \(\mathcal{B}\) instead of line bundles. More recently, this construction was generalized by Bezrukavnikov-Mirković-Rumynin [12, 13] and Bezrukavnikov-Mirković [14] to the situation when the base field \(k\) is of positive characteristic. In particular, it was proved in [14] as a consequence that Lusztig’s conjecture on non-restricted \(\mathfrak{g}\)-modules holds true, where \(\mathfrak{g}\) is the counterpart of \(\mathfrak{g}\) over \(k\).

1.2. The present paper arose out of the effort to give analogues for quantum groups of the above mentioned results for Lie algebras.

Let \(U_q(\mathfrak{g})\) be the quantized enveloping algebra of \(\mathfrak{g}\) over the rational function field \(\mathbb{Q}(q)\). It is a \(q\)-analogue of the enveloping algebra \(U(\mathfrak{g})\). The first task is to construct the quantized flag manifold \(\mathcal{B}_q\) for \(U_q(\mathfrak{g})\). Note that the ordinary flag manifold \(\mathcal{B}\) is a projective algebraic variety whose homogeneous coordinate algebra \(A\) is a certain subalgebra of the affine coordinate algebra \(\mathcal{O}(G)\) of \(G\). One can easily define a \(q\)-analogue \(A_q\) of \(A\) using \(U_q(\mathfrak{g})\); however, \(A_q\) is a non-commutative ring (except for the case \(G = SL_2\)). Hence in order to give a geometric meaning to \(\mathcal{B}_q\) we need the language of non-commutative algebraic geometry, developed in [3, 16, 39] following Manin’s idea [36]. Using \(A_q\) we can define as in [39, 31] (see also [26]) the abelian category \(\text{mod}(\mathcal{O}_{\mathcal{B}_q})\) which is regarded as the category of “coherent \(\mathcal{O}_{\mathcal{B}_q}\)-modules” on the virtual space \(\mathcal{B}_q = \text{Proj}(A_q)\).

Moreover, Lunts-Rosenberg [31] gave a definition of the category of “coherent \(D\)-modules” on the virtual space \(\mathcal{B}_q\), and formulated a Beilinson-Bernstein type correspondence for the quantized flag manifolds as a conjecture. This conjecture was

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settled in [40] by modifying the definition of the category of “coherent $D$-modules” on $B_q$.

1.3. Let us focus our attention on the quantum analogues of the results [12], [13], [14] on Lie algebras in positive characteristics. It is known that phenomena in positive characteristics in the ordinary world resemble those at roots of unity in the quantum world. Hence we consider the situation when the parameter $q$ is specialized to a complex number $\zeta$ which is an $\ell$-th root of unity. By the specialization $q \mapsto \zeta$ we obtain the De Concini-Kac type quantized enveloping algebra $U_{\zeta}(g)$ and the non-commutative graded ring $A_{\zeta}$. We can also construct an abelian category $\text{mod}(\mathcal{O}_{B_{\zeta}})$ which is regarded as the category of “coherent $\mathcal{O}_{B_{\zeta}}$-modules” on the virtual space $B_{\zeta} = \text{Proj}(A_{\zeta})$ (by [44] the virtual space $B_{\zeta}$ turns out to be a non-commutative scheme in the sense of [39]). The aim of the series [41], [42], [45] and the present paper is to give analogues of [12], [13], [14] using the quantized enveloping algebras and the quantized flag manifolds at roots of unity instead of the ordinary enveloping algebras and the ordinary flag manifolds in positive characteristics. Our arguments are basically parallel to those in [12], [13], [14] although some new ideas are needed. Sometimes an obvious geometric fact in positive characteristics requires additional algebraic arguments in the quantized situation (see for example Section 3.3 below).

For some reasons we assume

(a1) $\ell > 1$ is odd,

(a2) $\ell$ is prime to the order of the center of $G$,

(a3) $\ell$ is prime to 3 if $G$ is of type $G_2$

in the following. In general the situation becomes more complicated when the parameter $q$ is specialized to a root of unity. However, a good news in the case $\zeta$ is a root of unity is that we can relate the non-commutative scheme $B_{\zeta}$ with the ordinary scheme $B$ using Lusztig’s quantum Frobenius homomorphism. In fact, we have a morphism $\text{Fr} : B_{\zeta} \to B$ of non-commutative schemes, by which we obtain an actual sheaf of rings $\mathcal{D} = \text{Fr}_*\mathcal{O}_{B_{\zeta}}$ on the ordinary flag manifold $B$. The category $\text{mod}(\mathcal{O}_{B_{\zeta}})$ is naturally equivalent to the category $\text{mod}(\mathcal{D})$ of coherent $\mathcal{D}$-modules. In the case $G = SL_2$ we have $B_{\zeta} = B = \mathbb{P}^1$, and $\text{Fr} : \mathbb{P}^1 \to \mathbb{P}^1$ is given by $z \mapsto z^\ell$ for $z \in \mathbb{C} = \mathbb{A}^1 \subset \mathbb{P}^1$. In this case $\mathcal{D} = \text{Fr}_*\mathcal{O}_{\mathbb{P}^1}$ is commutative; however, in other cases where $G \neq SL_2$ the $\mathcal{O}_B$-algebra $\mathcal{D}$ turns out to be non-commutative.

1.4. Let us consider $D$-modules on $B_{\zeta}$. We have a category $\text{mod}(\mathcal{D}_{B_{\zeta}})$ of “coherent $\mathcal{D}_{B_{\zeta}}$-modules”, defined in the framework of non-commutative geometry; however, in our situation we can use $\mathcal{D} = \text{Fr}_*\mathcal{D}_{B_{\zeta}}$, which is an actual sheaf of rings on $B$ so that $\text{mod}(\mathcal{D}_{B_{\zeta}})$ is naturally equivalent to the category $\text{mod}(\mathcal{D})$ of coherent $\mathcal{D}$-modules on $B$. We have ring homomorphisms $\mathcal{D} \to \mathcal{D}, U_{\zeta}(g) \to \mathcal{D}, \mathcal{O}(H) \to \mathcal{D}$, where $\mathcal{O}(H)$ is the affine coordinate algebra of a maximal torus $H$ of $G$. Moreover, $\mathcal{O}(H) \to \mathcal{D}$ is a central embedding, and for $t \in H$ the specialization $\mathcal{D}_t = \mathbb{C} \otimes_{\mathcal{O}(H)} \mathcal{D}$ of $\mathcal{D}$ with respect to $\mathcal{O}(H) \ni t \mapsto \mathbb{C}$ is regarded as an analogue of the ring of twisted differential operators in the ordinary case.

Let $Z_{\text{Har}}(U_{\zeta}(g))$ be the Harish-Chandra center of $U_{\zeta}(g)$. It is a central subalgebra of $U_{\zeta}(g)$ isomorphic to $\mathcal{O}(H/W_0)$, where $W$ is the Weyl group and $H/W_0$ denotes the quotient of $H$ with respect to a twisted action $W \times H \ni (w, t) \mapsto w \circ t \in H$ of
W on $H$. For $t \in H$ we denote by $[t]$ its image in $H/W \circ$. For $[t] \in H/W \circ$ we denote by $\xi_{[t]} : Z_{Har}(U_{\xi}(g)) \to \mathbb{C}$ the algebra homomorphism given by $O(H/W \circ) \to \mathbb{C}$. We set $U_{\xi}(g)_{[t]} = \mathbb{C} \otimes_{Z_{Har}(U_{\xi}(g))} U_{\xi}(g)$ with respect to $\xi_{[t]}$. Then the ring homomorphism $U_{\xi}(g) \to \mathfrak{D}$ induces $U_{\xi}(g)_{[t]} \to \mathfrak{D}_t$ for $t \in H$.

In general, for a ring (resp. a sheaf of rings on a topological space) $R$ we denote by $\text{mod}_t(R)$ the category of finitely-generated (resp. coherent) $R$-modules. For $t \in H$ we denote by $\text{mod}_t(\mathfrak{D})$ (resp. $\text{mod}_t(U_{\xi}(g))$) the category of coherent $\mathfrak{D}$-modules (resp. finitely generated $U_{\xi}(g)$-modules) killed by some power of the image of $\text{Ker}(t : O(H) \to \mathbb{C})$ in $\mathfrak{D}$ (resp. Ker($\xi_{[t]}$) in $U_{\xi}(g)$). Then we have functors

\begin{align*}
(1.1) \quad R\Gamma : D^b(\text{mod}(\mathfrak{D}_t)) & \to D^b(\text{mod}(U_{\xi}(g)_{[t]})), \\
(1.2) \quad R\Gamma : D^b(\text{mod}(\mathfrak{D})) & \to D^b(\text{mod}_t(U_{\xi}(g))).
\end{align*}

Here, for an abelian category $\mathfrak{A}$ we denote its bounded derived category by $D^b(\mathfrak{A})$.

We conjecture that (1.1) (and hence (1.2)) give equivalences of triangulated categories if $t \in H$ is regular in the sense that $|W \circ t| = |W|$. In [12] and [13] we proved this conjecture in some cases. In particular, it holds if the following conditions are satisfied:

- (p) $\ell > 1$ is a power of a prime,
- (b) $t^\ell$ has finite order which is prime to $\ell$.

The condition (b) is harmless from the view point of the representation theory since we are reduced to the situation where (b) is satisfied using the parabolic induction (see [19]). But the assumption (p) should not be necessary. In fact in the case $G = SL_n$ we can show the conjecture without assuming (p). This will appear elsewhere.

1.5. Let us recall basics on the representations of $U_{\xi}(g)$. Take Borel subgroups $B^+$, $B^-$ of $G$ such that $B^+ \cap B^- = H$, and denote by $N^\pm$ the unipotent radical of $B^\pm$. Let $Z_{Fr}(U_{\xi}(g))$ be the Frobenius center of $U_{\xi}(g)$. It is a central subalgebra of $U_{\xi}(g)$ isomorphic to the coordinate algebra $O(K)$ of the affine algebraic group

$$K = \{ (g^+_h, g^- h^{-1}) \mid g^\pm \in N^\pm, h \in H \} \subset B^+ \times B^-$$

(see [18], [21]). For $k \in K$ we denote by $\xi^k : Z_{Fr}(U_{\xi}(g)) \to \mathbb{C}$ the algebra homomorphism given by $O(K) \to \mathbb{C}$. We set $U_{\xi}(g)^k = \mathbb{C} \otimes_{Z_{Fr}(U_{\xi}(g))} U_{\xi}(g)$ with respect to $\xi^k$, and consider $U_{\xi}(g)^k$-modules for each $k \in K$ (by a version of Schur’s lemma any irreducible $U_{\xi}(g)$-module is an irreducible $U_{\xi}(g)^k$-module for some $k \in K$). Define $\eta : K \to G$ by $\eta(x_1, x_2) = x_1 x_2^{-1}$. By [20], $U_{\xi}(g)^k$ is isomorphic to $U_{\xi}(g)^{k'}$ if $\eta(k)$ and $\eta(k')$ belong to the same conjugacy class of $G$. Hence for each conjugacy class $O$ of $G$ one can choose one $k \in K$ satisfying $\eta(k) \in O$. Set

$$\mathcal{X} = K \times_{H/W} (H/W \circ), \quad \mathcal{Y} = K \times_{H/W} H.$$

Here, $K \to H/W$ is the composite of $\eta : K \to G$ with the Steinberg map $G \to H/W$, and $H/W \circ \to H/W$ (resp. $H \to H/W$) is given by $[t] \mapsto [t^\ell]$ (resp. $t \mapsto [t^\ell]$). By [18], [21] (see also [43]) the total center $Z(U_{\xi}(g))$ of $U_{\xi}(g)$ is generated by $Z_{Har}(U_{\xi}(g))$ and $Z_{Fr}(U_{\xi}(g))$. Furthermore, the algebra homomorphisms $\xi_{[t]} : Z_{Har}(U_{\xi}(g)) \to \mathbb{C}$ ($[t] \in H/W \circ$) and $\xi^k : Z_{Fr}(U_{\xi}(g)) \to \mathbb{C}$ ($k \in K$) are compatible if and only if $(k, [t]) \in \mathcal{X}$. 


For \((k, [t]) \in \mathcal{X}\) we denote by \(\xi_{[t]}^k : Z(U_\mathcal{C}(\mathfrak{g})) \to \mathbb{C}\) the algebra homomorphism satisfying \(\xi_{[t]}^k|_{Z\mathcal{O}(U_\mathcal{C}(\mathfrak{g}))} = \xi_{[t]}^k\). We set \(U_\mathcal{C}(\mathfrak{g})_{[t]}^k = \mathcal{C} \otimes Z(U_\mathcal{C}(\mathfrak{g})) U_\mathcal{C}(\mathfrak{g})\) with respect to \(\xi_{[t]}^k\). We denote by \(\widehat{U}_k^k\) the completion of \(U_k\) at the maximal ideal \(\text{Ker}(\xi_{[t]}^k)\) of \(Z(U_\mathcal{C}(\mathfrak{g}))\).

1.6. We will use the identification \(\mathcal{B} = B\mathcal{G}\) in the following. Set

\[ \mathcal{V} = \{(B^{-g}, g, k, t) \in \mathcal{B} \times K \times H \mid gn(k)g^{-1} \in l\mathcal{N}^{-}\}. \]

Let \(p_1^\mathcal{V} : \mathcal{V} \to H\), \(p_2^\mathcal{V} : \mathcal{V} \to \mathcal{B}\) and \(p_3^\mathcal{V} : \mathcal{V} \to \mathcal{Y}\) be the projections. By \([41]\) \(\mathfrak{D}\) contains a central \(\mathcal{O}_\mathcal{B}\)-subalgebra isomorphic to \((p_3^\mathcal{V})_* \mathcal{O}_\mathcal{V}\). Since \(p_3^\mathcal{V}\) is an affine morphism, we can consider the localization \(\mathbb{D}\) of \(\mathfrak{D}\) on \(\mathcal{V}\). It is an \(\mathcal{O}_\mathcal{V}\)-algebra satisfying \((p_3^\mathcal{V})_* \mathfrak{D} = \mathfrak{D}\). For \((k, t) \in \mathcal{Y}\) we define a closed subvariety \(\mathcal{V}_{k,t}\) of \(\mathcal{V}\) by \(\mathcal{V}_{k,t} = (p_3^\mathcal{V})^{-1}(k, t)\), and denote by \(\mathbb{D}_{k,t}\) the formal neighbourhood of \(\mathcal{V}_{k,t}\) in \(\mathcal{V}\). We set

\[ \mathbb{D}_{k,t} = \mathcal{O}_{\mathcal{V}_{k,t}} \otimes_{\mathcal{O}_\mathcal{V}} \mathbb{D}, \quad \mathbb{D}_{k,t} = \mathcal{O}_{\mathcal{V}_{k,t}} \otimes_{\mathcal{O}_\mathcal{V}} \mathbb{D}. \]

They are an \(\mathcal{O}_{\mathcal{V}_{k,t}}\)-algebra and an \(\mathcal{O}_{\mathcal{V}_{k,t}}\)-algebra respectively. We denote by \(\text{mod}_{\mathcal{V}}^k(\mathbb{D})\) the category of coherent \(\mathbb{D}\)-modules supported on \(\mathcal{V}_{k,t}\). We also denote by \(\text{mod}_{\mathcal{V}_{k,t}}^k(U_k(\mathfrak{g}))\) the category of finitely-generated \(U_k(\mathfrak{g})\)-modules killed by some power of \(\text{Ker}(\xi_{[t]}^k)\). Then we have embeddings

\[
\begin{align*}
(1.3) & \quad \text{mod}(\mathbb{D}_{k,t}) \hookrightarrow \text{mod}_{\mathcal{V}}^k(\mathbb{D}) \hookrightarrow \text{mod}(\mathbb{D}_{k,t}), \\
(1.4) & \quad \text{mod}(U_k(\mathfrak{g})_{[t]}^k) \hookrightarrow \text{mod}_{\mathcal{V}_{k,t}}^k(U_k(\mathfrak{g})) \hookrightarrow \text{mod}(\widehat{U}_k^k) 
\end{align*}
\]

of abelian categories. Moreover, \((1.3)\) and \((1.4)\) induce isomorphisms

\[
\begin{align*}
(1.5) & \quad K(\text{mod}(\mathbb{D}_{k,t})) \cong K(\text{mod}_{\mathcal{V}}^k(\mathbb{D})), \\
(1.6) & \quad K(\text{mod}(U_k(\mathfrak{g})_{[t]}^k)) \cong K(\text{mod}_{\mathcal{V}_{k,t}}^k(U_k(\mathfrak{g})))
\end{align*}
\]

of the Grothendieck groups. Here, for an abelian category \(\mathfrak{A}\) we denote its Grothendieck group by \(K(\mathfrak{A})\). Note that \((1.1)\) and \((1.2)\) induce functors

\[
\begin{align*}
(1.7) & \quad R\Gamma : D^b(\text{mod}(\mathbb{D}_{k,t})) \to D^b(\text{mod}(U_k(\mathfrak{g})_{[t]}^k)), \\
(1.8) & \quad R\Gamma : D^b(\text{mod}_{\mathcal{V}_{k,t}}^k(\mathbb{D})) \to D^b(\text{mod}_{\mathcal{V}_{k,t}}^k(U_k(\mathfrak{g}))), \\
(1.9) & \quad R\Gamma : D^b(\text{mod}(\widehat{U}_k^k)) \to D^b(\text{mod}(\widehat{U}_k^k)).
\end{align*}
\]

The main result of \([41]\) is the split Azumaya property of \(\mathbb{D}_{k,t}\) and \(\widehat{U}_k^k\) for \((k, t) \in \mathcal{Y}\) under a certain condition on \(\ell\) depending on \((k, t) \in \mathcal{Y}\). The condition is void if \(\eta(k)\) is unipotent. In order that the property holds for any \((k, t) \in \mathcal{Y}\) we need to assume

\[
\begin{align*}
(1) & \quad \ell \text{ is prime to 3 if } G \text{ is of type } E_4, E_6, E_7, E_8; \\
(2) & \quad \ell \text{ is prime to 5 if } G \text{ is of type } E_8.
\end{align*}
\]

If \((c1), (c2)\) are satisfied, then there exist a locally free \(\mathcal{O}_{\mathcal{V}_{k,t}}\)-module \(\mathfrak{r}_{k,t}\) and a locally free \(\mathcal{O}_{\mathcal{V}_{k,t}}\)-module \(\widehat{r}_{k,t}\) of finite rank such that \(\mathbb{D}_{k,t} \cong \text{End}_{\mathcal{O}_{\mathcal{V}_{k,t}}}(\mathfrak{r}_{k,t})\) and \(\widehat{U}_k^k \cong \text{End}_{\mathcal{O}_{\mathcal{V}_{k,t}}}(\widehat{r}_{k,t})\). Hence we have equivalences

\[
\text{mod}(\mathbb{D}_{k,t}) \cong \text{mod}(\mathcal{O}_{\mathcal{V}_{k,t}}), \quad \text{mod}(\widehat{U}_k^k) \cong \text{mod}(\mathcal{O}_{\mathcal{V}_{k,t}}). \]
by the Morita theory. By the second equivalence we have also
\[
\text{mod}^k_B(D) \cong \text{mod}^k_B(O_Y),
\]
where \(\text{mod}^k_B(O_Y)\) denotes the category of coherent \(O_Y\)-modules supported on \(Y^k\).

We assume in the following that \(t\) satisfies (p) in addition to (a1), (a2), (a3). As noted above, \((1.1)\) is an equivalence if \(t\) is regular and satisfies (b). In this case we can also show that \((1.8)\) and \((1.9)\) give equivalences of triangulated categories. Therefore, we obtain the following equivalences of triangulated categories

\[
(1.10) \quad D^b(\text{mod}^k_B(O_Y)) \cong D^b(\text{mod}^k_B(U_c(g))),
\]
\[
(1.11) \quad D^b(\text{mod}(O_{\hat{V}^k})) \cong D^b(\text{mod}(\hat{U}_c(g)^k))
\]
if \(t\) is regular and the conditions (b), (c1), (c2) are satisfied.

1.7. The aim of the present paper is to give a more refined version of \((1.1)\) in terms of abelian categories. We fix \(\hat{k} \in K\) such that \(\hat{x} = \eta(\hat{k})\) is a unipotent element of \(G\), and consider the abelian category \(\text{mod}(U_c(g)^k)\) and its variants for \(t \in H\) such that \((\hat{k}, t) \in \mathcal{Y}\). We do not need to assume (c1), (c2) since \(\eta(\hat{k})\) is unipotent. Moreover, we have \((\hat{k}, t) \in \mathcal{Y}\) for \(t \in H\) if and only if \(t^\ell = 1\). Hence any \(t\) such that \((\hat{k}, t) \in \mathcal{Y}\) satisfies the condition (b). Denote by \(H_\ell\) the set of \(t \in H\) satisfying \(t^\ell = 1\). The set of regular elements in \(H_\ell\) is denoted by \(H_\ell^{\text{reg}}\).

Note that for \(t \in H_\ell\) the variety \(V^k_\ell\) is naturally isomorphic to the Springer fiber \(B^\hat{x} = \{B^{-} - g \in B \mid gxg^{-1} \in N^{-}\}\), and \(V^k_\ell\) is naturally isomorphic to the formal neighborhood \(\hat{B}^\hat{x}\) of \(B^\hat{x}\) in \(\hat{G} = \{(B^{-} - g, x) \in B \times G \mid gxg^{-1} \in B^{-}\}\). Hence we have an equivalence

\[
(1.12) \quad D^b(\text{mod}(\hat{U}_c(g)^k)) \cong D^b(\text{mod}(O_{\hat{B}^\hat{x}})) \quad (t \in H_\ell^{\text{reg}})
\]
of derived categories. Denote by \(\text{mod}_{\text{reg}}(O_\hat{G})\) the category of coherent \(O_\hat{G}\)-modules supported on \(B^\hat{x}\). We have \(\text{mod}^k_B(O_Y) \cong \text{mod}_{\text{reg}}(O_\hat{G})\), and hence

\[
(1.13) \quad D^b(\text{mod}^k_B(U_c(g))) \cong D^b(\text{mod}_{\text{reg}}(O_\hat{G})) \quad (t \in H_\ell^{\text{reg}}).
\]

Note that to ensure \(H_\ell^{\text{reg}} \neq \emptyset\) we need the condition

(a4) \(\ell \geq h_G\),

where \(h_G\) is the Coxeter number. We assume (a4) in the following.

Note that we have a non-standard \(t\)-structure of \(D^b(\text{mod}(O_{\hat{B}^\hat{x}}))\) called the exotic \(t\)-structure (see \([10], [14]\)). We denote its heart by \(\text{mod}^{\text{ex}}(O_{\hat{B}^\hat{x}})\). Our main result is the following.

**Theorem 1.1.** The equivalence \((1.12)\) of the triangulated categories induces an equivalence

\[
(1.14) \quad \text{mod}(\hat{U}_c(g)^k) \cong \text{mod}^{\text{ex}}(O_{\hat{B}^\hat{x}}) \quad (t \in H_\ell^{\text{reg}}).
\]
of abelian categories.

The exotic \(t\)-structure is defined using an affine braid group action on \(D^b(\text{mod}(O_{\hat{B}^\hat{x}}))\). Theorem \((1.1)\) is proved by showing that the corresponding affine braid group action on \(D^b(\text{mod}(\hat{U}_c(g)^k))\) is given by the wall-crossing functor.
1.8. To give an application to the representation theory we also need an equivariant version of (1.14). By replacing \( k \) with suitable \( k' \in K \) such \( \eta(k') \) is conjugate to \( \eta(k) \) we may assume from the beginning that there exists a maximal torus \( C \) of the centralizer \( Z_G(x) \) such that \( C \subset H \). Let \( \Lambda_C \) be the character group of \( C \). The natural grading of \( U_\xi(g) \) by weights induces a \( \Lambda_C \)-grading of \( \widehat{U}_\xi(g)_{[\xi]}^k \). We fix \( t \in H_{\text{reg}} \) in the following. Then (1.14) induces an equivalence

\[
\text{mod}(\widehat{U}_\xi(g)_{[\xi]}^k; C) \cong \text{mod}^{\text{ex}}(O_{\hat{B}^s}; C)
\]

of abelian categories. Here, \( \text{mod}(\widehat{U}_\xi(g)_{[\xi]}^k; C) \) denotes the category of finitely generated \( \Lambda_C \)-graded \( \widehat{U}_\xi(g)_{[\xi]}^k \)-modules satisfying a certain compatibility condition, and \( \text{mod}^{\text{ex}}(O_{\hat{B}^s}; C) \) denotes the category of \( C \)-equivariant exotic coherent \( O_{\hat{B}^s} \)-modules.

1.9. Let us describe application of (1.15) to the representation theory. We denote by \( \text{mod}_{[\xi]}(U_\xi(g)^k) \) the category of finitely generated \( U_\xi(g)^k \)-modules killed by some power of the image of \( \text{Ker}(\xi_{[\xi]}) \) in \( U_\xi(g)^k \). We also denote by \( \text{mod}_{[\xi]}(U_\xi(g)^k; C) \) (resp. \( \text{mod}_{[\xi]}(U_\xi(g); C) \)) the category of \( \Lambda_C \)-graded objects in \( \text{mod}_{[\xi]}(U_\xi(g)^k) \) (resp. \( \text{mod}_{[\xi]}(U_\xi(g)) \)) satisfying a certain compatibility condition. We have embeddings

\[
\text{mod}(U_\xi(g)^k_{[\xi]}; C) \subset \text{mod}_{[\xi]}(U_\xi(g)^k; C) \subset \text{mod}_{[\xi]}(U_\xi(g); C) \subset \text{mod}(\widehat{U}_\xi(g)_{[\xi]}^k; C)
\]

of abelian categories. Denote by \( \text{mod}(O_{B^s}; C) \) and \( \text{mod}_{B^s}(O_{\hat{G}}; C) \) the category of \( C \)-equivariant objects in \( \text{mod}(O_{B^s}) \) and \( \text{mod}_{B^s}(O_{\hat{G}}) \) respectively. Then we have embeddings

\[
\text{mod}(O_{B^s}; C) \subset \text{mod}_{B^s}(O_{\hat{G}}; C) \subset \text{mod}(O_{\hat{B}^s}; C)
\]

of abelian categories. Moreover, we have the following commutative diagram of functors:

\[
\begin{array}{ccc}
D^b(\text{mod}(O_{B^s}; C)) & \cong & D^b(\text{mod}_{B^s}(O_{\hat{G}}; C)) \\
\downarrow & & \downarrow \\
D^b(\text{mod}_{[\xi]}(U_\xi(g)^k; C)) & \cong & D^b(\text{mod}_{[\xi]}(U_\xi(g); C)) \\
& & \downarrow \\
D^b(\text{mod}(U_\xi(g)^k_{[\xi]}; C)) & \cong & D^b(\text{mod}(\widehat{U}_\xi(g)_{[\xi]}^k; C)).
\end{array}
\]

Let \( \{L_\sigma\}_{\sigma \in \Theta} \) be the set of irreducible objects of \( \text{mod}(U_\xi(g)^k_{[\xi]}; C) \). We denote by \( E_\sigma \) (resp. \( \hat{E}_\sigma \)) the projective cover of \( L_\sigma \) in \( \text{mod}_{[\xi]}(U_\xi(g)^k; C) \) (resp. \( \text{mod}(\widehat{U}_\xi(g)_{[\xi]}^k; C) \)). Lusztig’s conjecture gives a formula expressing \([E_\sigma]\) as a \( \mathbb{Z} \)-linear combination of the elements \([L_\tau]\) for \( \tau \in \Theta \) in the Grothendieck group

\[
K(\text{mod}(U_\xi(g)^k_{[\xi]}; C)) \cong K(\text{mod}_{[\xi]}(U_\xi(g)^k; C)) \cong K(\text{mod}_{[\xi]}(U_\xi(g); C)).
\]

For \( \sigma \in \Theta \) we denote by \( L_\sigma, E_\sigma, \hat{E}_\sigma \) the objects of \( \text{mod}^{\text{ex}}(O_{B^s}; C) \) corresponding to \( L_\sigma, E_\sigma, \hat{E}_\sigma \) respectively. By \( L_\sigma, E_\sigma \in \text{mod}_{[\xi]}(U_\xi(g); C) \) we have \( L_\sigma, E_\sigma \in D^b(\text{mod}_{B^s}(O_{\hat{G}}; C)) \). Our problem is to give a formula expressing \([E_\sigma]\) as a \( \mathbb{Z} \)-linear combination of the elements \([L_\tau]\) for \( \tau \in \Theta \) in the Grothendieck group

\[
K(\text{mod}(O_{B^s}; C)) \cong K(\text{mod}_{B^s}(O_{\hat{G}}; C)).
\]
Note that \( \{ L_\sigma \}_{\sigma \in \Theta} \) is the set of irreducible objects of \( \text{mod}^\text{ex}( \mathcal{O}_{B;G} ; C) \) contained in \( D^b(\text{mod}_{B_\xi}( \mathcal{O}_{G;C} )) \), and \( \hat{E}_\sigma \) is the projective cover of \( L_\sigma \) in \( \text{mod}^\text{ex}( \mathcal{O}_{B_\xi} ; C) \). Moreover, by

\[
E_\sigma = \mathbb{C} \otimes_{\mathbb{Z}[U_{\xi}(\mathfrak{g})]} \hat{E}_\sigma = \mathbb{C} \otimes_{\mathbb{Z}[U_{\xi}(\mathfrak{g})]} \hat{E}_\sigma
\]

with respect to \( \xi^k \), we have \( E_\sigma = \mathcal{O}_{\{ \xi \}} \otimes_{\hat{O}_G} \hat{E}_\sigma \), where \( \hat{G} \to G \) is induced by the projection \( \hat{G} \to G \). From these facts together with the result of [14] we obtain the desired formula expressing \( [E_\sigma] \) as a \( \mathbb{Z} \)-linear combination of the elements \( [L_\tau] \) for \( \tau \in \Theta \) in terms of a certain geometrically defined bilinear form and Lusztig’s conjectural canonical bases of certain equivariant \( K \)-groups (whose existence is proved in [14]). See Section 9 below for more details.

In the case \( \hat{x} \) is a regular unipotent element of the reductive part of a parabolic subgroup of \( G \), this formula implies a combinatorial character formula of \( L_\sigma \) in terms of baby Verma modules (see [34] and [35, Section 10]). We note that a recent work of Bezrukavnikov-Losev [11] should shed light to the character formula in the general situation.

In the special case \( \hat{x} = 1 \) we obtain a new proof of the character formula of the irreducible highest weight modules over the Lusztig type quantized enveloping algebra for \( \ell \) satisfying (a1), (a2), (a3), (a4), and (p), which was originally proved through the combination of [29], [33] and [28] (see also [2]).

We finally note that our result gives a proof of a conjecture of Bezrukavnikov-Mirković (see [14, 1.7.1]) for \( \ell \) satisfying (a1), (a2), (a3), (a4), (p).

1.10. The organization of this paper is as follows. In Section 2 we introduce basic notation on the quantized enveloping algebras and the quantized coordinate algebras. In Section 3 we recall fundamental results on the quantized flag manifold and \( D \)-modules on it in the situation where the parameter \( q \) is specialized to an arbitrary non-zero complex number. We assume that \( q \) is specialized to an \( \ell \)-th root of 1 starting from Section 4. In Section 4 we recall the structure theorem for the center of the quantized enveloping algebra. We also give a formula concerning the action the Harish-Chandra center on the tensor product of a \( U_{\xi}(\mathfrak{g}) \)-module and an integrable module over the Lusztig type quantized enveloping algebra. This is an analogue of Kostant’s result on the tensor product of a \( \mathfrak{g} \)-module and a finite-dimensional \( \mathfrak{g} \)-module. This plays essential role in the investigation of the translation functor. In Section 5 we use Frobenius morphism \( \text{Fr} : B_\zeta \to B \) to reformulate results on \( D \)-modules on \( B_\zeta \) using \( D \)-modules and \( ^e D \)-modules. In Section 6 we recall the Beilinson-Bernstein type correspondence given in [42], [45]. In Section 7 we formulate a torus equivariant version of the correspondence. In Section 8 after recalling the notion of the exotic sheaves we give a proof of the main theorem. An essential role is played by the wall crossing functor which is closely related to the translation functor. In Section 9 we explain how we can deduce Lusztig’s conjecture on non-restricted representations from the main theorem. In Appendix we give a brief account of certain results for quantized partial flag manifolds.
1.11. If $R$ is a ring, we denote its center by $Z(R)$. We also denote by $\text{Mod}(R)$ (resp. $\text{Mod}^r(R)$) the category of left (resp. right) $R$-modules. Its subcategory consisting of finitely generated left (resp. right) $R$-modules is denoted by $\text{mod}(R)$ (resp. $\text{mod}^r(R)$).

If $\Gamma$ is a free abelian group of finite rank and $R$ is a $\Gamma$-graded ring, we denote by $\text{Mod}^\Gamma(R)$ (resp. $\text{mod}^\Gamma(R)$) the category of $\Gamma$-graded left $R$-modules (resp. $\Gamma$-graded finitely generated left $R$-modules).

If $R$ is a sheaf of rings on a topological space, we denote by $\text{Mod}(R)$ the category of quasi-coherent left $R$-modules. Its subcategory consisting of coherent $R$-modules is denoted by $\text{mod}(R)$.

The structure sheaf of a scheme $X$ is denoted by $\mathcal{O}_X$. If $X$ is an affine algebraic variety, its coordinate algebra $\Gamma(X, \mathcal{O}_X)$ is denoted by $\mathcal{O}(X)$.

If $U$ is a Hopf algebra, we sometimes use Sweedler’s notation

$$\Delta^{(n)}(u) = \sum_{(u)} u(0) \otimes \cdots \otimes u(n) \quad (u \in U)$$

for the iterated comultiplication $\Delta^{(n)} : U \to U^{\otimes n+1}$.

2. Quantum Groups

2.1. Simple Lie algebra. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over the complex number field $\mathbb{C}$. We take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and denote by $\Delta \subset \mathfrak{h}^*$ the root system. Let $W \subset GL(\mathfrak{h}^*)$ be the Weyl group, and let

$$(\ , \ ) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$$

be the $W$-invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for short roots $\alpha$. For $\alpha \in \Delta$ the corresponding coroot is given by $\alpha^\vee = 2\alpha / (\alpha, \alpha) \in \mathfrak{h}^*$. We denote by $Q$ and $\Lambda$ the root lattice and the weight lattice respectively. We also set $Q^\vee = \sum_{\alpha \in \Delta^+} \mathbb{Z}\alpha^\vee$. Fix a set $\{\alpha_i\}_{i \in I}$ of simple roots, and denote by $\Delta^+$ the set of positive roots. We set

$$Q^+ = \sum_{\alpha \in \Delta^+} \mathbb{Z}\alpha, \quad \Lambda^+ = \{\lambda \in \Lambda \mid (\lambda, \alpha^\vee) \geq 0 \ (\alpha \in \Delta^+)\}.$$ 

For $i \in I$ we denote by $s_i \in W$ the simple reflection corresponding to $i$. We define subalgebras $\mathfrak{n}^+, \mathfrak{n}^-, \mathfrak{b}^+, \mathfrak{b}^-$ of $\mathfrak{g}$ by

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_{\pm \alpha}, \quad \mathfrak{b}^\pm = \mathfrak{h} \oplus \mathfrak{n}^\pm,$$

where $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$ denotes the root subspace.

Let $G$ be a connected, simply-connected, simple algebraic group with Lie algebra $\mathfrak{g}$. We define $H$, $N^\pm$, $B^\pm$ to be the connected closed subgroups of $G$ with Lie algebras $\mathfrak{h}$, $\mathfrak{n}^\pm$, $\mathfrak{b}^\pm$ respectively. For $\lambda \in \Lambda$ we denote the corresponding character of $H$ by

$$(2.2) \quad \theta_\lambda : H \to \mathbb{C}^\times.$$
2.2. Quantized enveloping algebra. For \( i \in I \) we set \( d_i = (\alpha_i, \alpha_i)/2 \in \mathbb{Z} \), and for \( i, j \in I \) we set \( a_{ij} = (\alpha_i^\vee, \alpha_j) \in \mathbb{Z} \). Let \( \mathbb{F} = \mathbb{Q}(q^{1/|\Lambda|/Q}) \) be the field of rational functions over \( \mathbb{Q} \) in the variable \( q^{1/|\Lambda|/Q} \). We define \( U_\mathbb{F}(\mathfrak{g}) \) to be the \( \mathbb{F} \)-algebra with 1 generated by the elements \( k_\lambda (\lambda \in \Lambda) \), \( e_i, f_i (i \in I) \) satisfying the fundamental relations

\[
\begin{align*}
(2.3) \quad & k_0 = 1, \quad k_\lambda k_\mu = k_{\lambda + \mu} \quad (\lambda, \mu \in \Lambda), \\
(2.4) \quad & k_\lambda e_i = q^{(\lambda, \alpha_i)} e_i k_\lambda, \quad k_\lambda f_i = q^{-(\lambda, \alpha_i)} f_i k_\lambda \quad (i \in I, \lambda \in \Lambda), \\
(2.5) \quad & e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - q_i^{-1}}{q_i - q_i^{-1}} k_i^{-1} \quad (i, j \in I), \\
(2.6) \quad & \sum_{r+s=1-a_{ij}} (-1)^s e_i^{(r)} e_j^{(s)} = \sum_{r+s=1-a_{ij}} (-1)^s f_i^{(r)} f_j^{(s)} = 0 \quad (i, j \in I, i \neq j),
\end{align*}
\]

where \( q_i = q^{a_i} \), \( k_i = k_{\alpha_i} \) for \( i \in I \), and \( e_i^{(r)} = e_i^r/[r]_q! \), \( f_i^{(r)} = f_i^r/[r]_q! \) for \( i \in I \), \( r \in \mathbb{Z}_{\geq 0} \). Here,

\[
[r]_q! = \prod_{n=1}^r \frac{t^n - t^{-n}}{t - t^{-1}} \in \mathbb{Z}[t, t^{-1}].
\]

Then \( U_\mathbb{F}(\mathfrak{g}) \) turns out to be a Hopf algebra by

\[
\begin{align*}
(2.7) \quad & \Delta(k_\lambda) = k_\lambda \otimes k_\lambda, \quad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \\
(2.8) \quad & \varepsilon(k_\lambda) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 1, \\
(2.9) \quad & S(k_\lambda) = k_{-\lambda}, \quad S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i
\end{align*}
\]

for \( \lambda \in \Lambda \), \( i \in I \). We define subalgebras \( U_\mathbb{F}(\mathfrak{h}) \), \( U_\mathbb{F}(\mathfrak{n}^\pm) \), \( U_\mathbb{F}(\mathfrak{b}^\pm) \) of \( U_\mathbb{F}(\mathfrak{g}) \) by

\[
U_\mathbb{F}(\mathfrak{h}) = \langle k_\lambda \mid \lambda \in \Lambda \rangle, \quad U_\mathbb{F}(\mathfrak{n}^+) = \langle e_i \mid i \in I \rangle, \quad U_\mathbb{F}(\mathfrak{n}^-) = \langle f_i \mid i \in I \rangle, \quad U_\mathbb{F}(\mathfrak{b}^+) = \langle k_\lambda, e_i \mid \lambda \in \Lambda, i \in I \rangle, \quad U_\mathbb{F}(\mathfrak{b}^-) = \langle k_\lambda, f_i \mid \lambda \in \Lambda, i \in I \rangle.
\]

We have \( U_\mathbb{F}(\mathfrak{h}) = \bigoplus_{\lambda \in \Lambda} \mathbb{F} k_\lambda \). For \( \lambda \in \Lambda \) we define a character

\[
(2.10) \quad \chi_\lambda : U_\mathbb{F}(\mathfrak{h}) \rightarrow \mathbb{F}
\]

by \( \chi_\lambda(k_\mu) = q^{(\lambda, \mu)} = (q^{1/|\Lambda|/Q})^{(\lambda, \mu)|\Lambda/Q} \) \( (\lambda, \mu \in \Lambda) \). Note \( (\Lambda, \Lambda) \subset \mathbb{Z} \setminus \mathbb{N} \). By our choice of \( q \).

Set \( A = \mathbb{Z}[q^{1/|\Lambda|/Q}, q^{-1/|\Lambda|/Q}] \). We define \( A \)-forms \( U_A(\mathfrak{g}) \) and \( U_L^A(\mathfrak{g}) \) of \( U_\mathbb{F}(\mathfrak{g}) \) as follows. The De Concini-Kac form \( U_A(\mathfrak{g}) \) is the \( A \)-subalgebra of \( U_\mathbb{F}(\mathfrak{g}) \) generated by \( k_\lambda (\lambda \in \Lambda) \), \( e_i, f_i (i \in I) \). The Lusztig form \( U_L^A(\mathfrak{g}) \) is the \( A \)-subalgebra of \( U_\mathbb{F}(\mathfrak{g}) \) generated by \( k_\lambda (\lambda \in \Lambda) \), \( e_i^{(r)}, f_i^{(r)} (i \in I, r \in \mathbb{Z}_{\geq 0}) \). Then \( U_A(\mathfrak{g}) \) and \( U_L^A(\mathfrak{g}) \) are Hopf algebras over \( A \).

Now assume that we are given \( \zeta \in \mathbb{C}^* \) together with a choice of \( \zeta^{1/|\Lambda|/Q} \in \mathbb{C}^* \) satisfying \( (\zeta^{1/|\Lambda|/Q})^{|\Lambda/Q} = \zeta \). We define Hopf algebras \( U_\zeta(\mathfrak{g}) \) and \( U_L^\zeta(\mathfrak{g}) \) over \( \mathbb{C} \) by

\[
(2.11) \quad U_\zeta(\mathfrak{g}) = \mathbb{C} \otimes_A U_A(\mathfrak{g}), \quad U_L^\zeta(\mathfrak{g}) = \mathbb{C} \otimes_A U_L^A(\mathfrak{g})
\]

with respect to \( A \rightarrow \mathbb{C} \) \( (q^{1/|\Lambda|/Q} \mapsto \zeta^{1/|\Lambda|/Q}) \).
2.3. **Harish-Chandra center.** Define subalgebras $U_\zeta(h)$, $U_\zeta(n^+)$, $U_\zeta(n^-)$ of $U_\zeta(g)$ by

$$U_\zeta(h) = \langle k_\lambda | \lambda \in \Lambda \rangle, \quad U_\zeta(n^+) = \langle e_i | i \in I \rangle, \quad U_\zeta(n^-) = \langle f_i | i \in I \rangle.$$ 

Then we have an isomorphism

$$U_\zeta(g) \cong U_\zeta(n^-) \otimes U_\zeta(h) \otimes U_\zeta(n^+) \quad (xyz \leftrightarrow x \otimes y \otimes z)$$

of $\mathbb{C}$-modules. Moreover, we have $U_\zeta(h) = \bigoplus_{\lambda \in \Lambda} \mathbb{C} k_\lambda$. We set

$$(2.12) \quad e U_\zeta(h) = \bigoplus_{\lambda \in \Lambda} \mathbb{C} k_2\lambda,$$

and define a twisted action of $W$ on $e U_\zeta(h)$ by

$$w \circ k_{2\lambda} = \zeta^{2(w\lambda - \lambda, \rho)} k_{2w\lambda} \quad (w \in W, \lambda \in \Lambda),$$

where $\rho \in \Lambda$ is defined by $(\rho, \alpha_i^\vee) = 1$ for any $i \in I$.

We define $Z_{Har}(U_\zeta(g))$ to be the $\mathbb{C}$-subalgebra of $Z(U_\zeta(g))$ generated by the image of $Z(U_\Lambda(g)) \rightarrow U_\zeta(g)$. We call $Z_{Har}(U_\zeta(g))$ the Harish-Chandra center of $U_\zeta(g)$.

Define

$$\Xi : Z_{Har}(U_\zeta(g)) \rightarrow U_\zeta(h)$$

as the composite of

$$Z_{Har}(U_\zeta(g)) \hookrightarrow U_\zeta(g) \cong U_\zeta(n^-) \otimes U_\zeta(h) \otimes U_\zeta(n^+) \xrightarrow{\varepsilon \otimes 1 \otimes \varepsilon} U_\zeta(h).$$

The following is well-known.

**Proposition 2.1.** The linear map $\Xi$ is an injective algebra homomorphism whose image is equal to

$$e U_\zeta(h)^{W_\circ} = \{ x \in e U_\zeta(h) | w \circ x = x \}.$$
2.4. Quantized coordinate algebra. We say that a left (resp. right) \( U_F(g) \)-module \( M \) is integrable if it is a direct sum of weight spaces

\[
M_\mu = \{ m \in M \mid am = \chi_\mu(a)m \text{ (resp. } ma = \chi_\mu(a)m) \text{ for } a \in U_F(h) \}
\]

for \( \mu \in \Lambda \), and if for any \( m \in M \) there exists \( n > 0 \) such that \( e_i^{(n)}m = f_i^{(n)}m = 0 \) (resp. \( me_i^{(n)} = mf_i^{(n)} = 0 \)) for any \( i \in I \). For \( \lambda \in \Lambda^+ \) we denote by \( \Delta_F(\lambda) \) the irreducible \( U_F(g) \)-module with highest weight \( \lambda \). It is known that integrable irreducible \( U_F(g) \)-modules are in one-to-one correspondence with \( \Lambda^+ \) via \( \Delta_F(\lambda) \leftrightarrow \lambda \). Define an \( \Lambda \)-form \( \Delta_F(\lambda) \) of \( \Delta_F(\lambda) \) by \( \Delta_F(\lambda) = U^F(\lambda) \). Set \( \Delta_F(\lambda) = \Lambda \) and if for any \( n > N \) such that \( \lambda \in \Lambda \) we have \( \chi_\lambda(U^F(\lambda)) \in \Lambda \), and \( \chi_\lambda(U^F(\lambda)) \) induces a character

\[
\chi_\lambda : U^F(\lambda) \to \mathbb{C}
\]

of \( U^F(\lambda) \). By abuse of notation we denote by

\[
\chi_\lambda : U^F(\lambda) \to \mathbb{C}
\]

the natural extension of \( (2.16) \) to \( U^F(\lambda) \) given by \( \chi_\lambda(f_i^{(n)}) = 0 \) for \( i \in I, n > 0 \).

We say that a left \( U^F(\lambda) \)-module (resp. \( U^F(\lambda) \)-module) \( M \) is integrable if it is a direct sum of weight spaces

\[
M_\mu = \{ m \in M \mid am = \chi_\mu(a)m \text{ for } a \in U^F(\lambda) \} \quad (\mu \in \Lambda),
\]

and if for any \( m \in M \) there exists \( N > 0 \) such that \( e_i^{(n)}m = f_i^{(n)}m = 0 \) (resp. \( f_i^{(n)}m = 0 \)) for any \( i \in I \) and \( n > N \). We define the notion of an integrable right \( U^F(\lambda) \)-module (resp. \( U^F(\lambda) \)-module) similarly.

Let \( a = g \) or \( b^- \). We denote by \( \text{Mod} \text{-int}(U^F(\lambda)) \) (resp. \( \text{Mod} \text{-int}(U^F(\lambda)) \)) the category of integrable left (resp. right) \( U^F(\lambda) \)-modules. Its full subcategory consisting of finite-dimensional objects is denoted by \( \text{mod} \text{-int}(U^F(\lambda)) \) (resp. \( \text{mod} \text{-int}(U^F(\lambda)) \)). Set \( U^F(\lambda)^* = \text{Hom}_C(U^F(\lambda), \mathbb{C}) \). It is a \( U^F(\lambda) \)-bimodule similarly to \( (2.15) \). Set \( A = G \) or \( B^- \) according to whether \( a = g \) or \( b^- \). We define \( \text{O}_F(A) \) to be the subspace of \( U^F(\lambda)^* \) consisting of \( \varphi \in U^F(\lambda)^* \) satisfying the following equivalent conditions:
(1) the left $U^L_\zeta(a)$-module $U^L_\zeta(a)\varphi$ is integrable,
(2) the right $U^L_\zeta(a)$-module $\varphi U^L_\zeta(a)$ is integrable.

Then $O_\zeta(A)$ is naturally a Hopf algebra and a $U^L_\zeta(\mathfrak{g})$-bimodule. It is known that $O_\zeta(G) \cong \mathbb{C} \otimes_\Lambda O_\Lambda(G)$ with respect to $q^{1/N} \mapsto \zeta^{1/N}$ (see e.g. [45, Proposition 3.2]).

For $\lambda \in \Lambda^+$ we set $\Delta_\zeta(\lambda) = \mathbb{C} \otimes_\Lambda \Delta_\Lambda(\lambda)$. It is a finite-dimensional integrable $U^L_\zeta(\mathfrak{g})$-module, called the Weyl module.

2.5. $R$-matrix. We denote by

$$\tau : U^L_\zeta(\mathfrak{b}^+) \times U^L_\zeta(\mathfrak{b}^-) \to \mathbb{F}$$

the Drinfeld pairing. It is a bilinear form characterized by the properties:

$$\tau(x, y_1 y_2) = (\tau \otimes \tau)(\Delta(x), y_1 \otimes y_2) \quad (x \in U^L_\zeta(\mathfrak{b}^+), \ y_1, y_2 \in U^L_\zeta(\mathfrak{b}^-)),
\tau(x_1 x_2, y) = (\tau \otimes \tau)(x_2 \otimes x_1, \Delta(y)) \quad (x_1, x_2 \in U^L_\zeta(\mathfrak{b}^+), \ y \in U^L_\zeta(\mathfrak{b}^-)),
\tau(k_\lambda, k_\mu) = q^{-(\lambda, \mu)} \quad (\lambda, \mu \in \Lambda),
\tau(e_i, f_j) = -\delta_{ij}(q_i - q_i^{-1})^{-1} \quad (i, j \in I),
\tau(e_i, k_\lambda) = \tau(k_\lambda, f_i) = 0 \quad (i \in I, \ \lambda \in \Lambda).
$$

For $\gamma \in Q^+$ we denote by $U^L_\zeta(\mathfrak{n}^+)_\gamma$ (resp. $U^L_\zeta(\mathfrak{n}^-)_\gamma$) the linear subspace of $U^L_\zeta(\mathfrak{n}^+)$ (resp. $U^L_\zeta(\mathfrak{n}^-)$) spanned by the elements of the form $e_{i_1} \cdots e_{i_r}$ (resp. $f_{i_1} \cdots f_{i_r}$) with $i_1, \ldots, i_r \in I, \sum_{s=1}^r \alpha_{i_s} = \gamma$. Then the restriction $\tau|_{U^L_\zeta(\mathfrak{n}^+)_\gamma \times U^L_\zeta(\mathfrak{n}^-)_\gamma}$ is non-degenerate for any $\gamma \in Q^+$. We denote by $R^\gamma \in U^L_\zeta(\mathfrak{n}^+)_\gamma \otimes_{\mathbb{F}} U^L_\zeta(\mathfrak{n}^-)_\gamma$ the corresponding canonical element. Set $U^L_\Lambda(\mathfrak{n}^\pm)_\pm \gamma = U^L_\Lambda(\mathfrak{n}^\pm) \cap U^L_\zeta(\mathfrak{n}^\pm)_\gamma$. Then we have

$$R^\gamma \in U^L_\Lambda(\mathfrak{n}^+)_\gamma \otimes_\Lambda U^L_\Lambda(\mathfrak{n}^-)_\gamma \subset U^L_\zeta(\mathfrak{n}^+)_\gamma \otimes_{\mathbb{F}} U^L_\zeta(\mathfrak{n}^-)_\gamma.
$$

Let $U^L_\zeta(\mathfrak{n}^\pm)_\pm \gamma$ be the subspace of $U^L_\zeta(\mathfrak{n}^\pm)$ generated by the image of $U^L_\Lambda(\mathfrak{n}^\pm)_\pm \gamma \to U^L_\zeta(\mathfrak{n}^\pm)$. We denote by $R^\gamma \in U^L_\zeta(\mathfrak{n}^+)_\gamma \otimes U^L_\zeta(\mathfrak{n}^-)_\gamma$ the specialization of $R^\gamma$.

Let $M, N \in \text{Mod}_{\text{int}}(U^L_\zeta(\mathfrak{g}))$. We regard $M \otimes N$ as an object of $\text{Mod}_{\text{int}}(U^L_\zeta(\mathfrak{g}))$ via the comultiplication $\Delta : U^L_\zeta(\mathfrak{g}) \to U^L_\zeta(\mathfrak{g}) \otimes U^L_\zeta(\mathfrak{g})$. We define a linear map

$$R^{M,N} : M \otimes N \to N \otimes M$$

by

$$R^{M,N}(m \otimes n) = \zeta^{-(\lambda, \mu)} \sum_{\gamma \in Q^+} P(R^\gamma(m \otimes n)) \quad (m \in M_\lambda, \ n \in N_\mu),$$

where $P : M \otimes N \to N \otimes M$ is the transposition.

The following fact is standard.

**Proposition 2.2.** For $M, N \in \text{Mod}_{\text{int}}(U^L_\zeta(\mathfrak{g}))$ the linear map (2.19) gives an isomorphism in $\text{Mod}_{\text{int}}(U^L_\zeta(\mathfrak{g}))$.

3. $D$-modules on the quantized flag manifolds

3.1. Quantized flag manifold. Define a subalgebra $A_{\mathfrak{g}}$ of $O_{\mathfrak{g}}(G)$ by

$$A_{\mathfrak{g}} = \{ \varphi \in O_{\mathfrak{g}}(G) \mid \varphi u = \varepsilon(u) \varphi \ (u \in \mathfrak{g}^-) \}.$$
The $U_F(g)$-bimodule structure of $O_F(G)$ given by (2.15) induces a $(U_F(g), U_F(h))$-bimodule structure of $A_F$. Setting

$$A_F(\lambda) = \{ \varphi \in A_F \mid \varphi a = \chi_\lambda(a) \varphi \ (a \in U_F(h)) \} \quad (\lambda \in \Lambda^+)$$

we have

$$A_F = \bigoplus_{\lambda \in \Lambda^+} A_F(\lambda).$$

Set $A_h = O_h(A) \cap A_F$. Then $A_h$ is an $A$-subalgebra of $O_h(A)$ satisfying

$$A_h = \bigoplus_{\lambda \in \Lambda^+} A_h(\lambda), \quad A_h(\lambda) = A_F(\lambda) \cap A_h.$$  

Moreover, it is a $(U_L(g), U_L(h))$-bimodule.

Setting $A_\zeta = C \otimes_A A_h$, $A_\zeta(\lambda) = C \otimes_A A_h(\lambda) \ (\lambda \in \Lambda^+)$ with respect to $A \to C \ (q^{1/N} \mapsto \zeta^{1/N})$ we have

$$A_\zeta = \bigoplus_{\lambda \in \Lambda^+} A_\zeta(\lambda).$$

Not that $A_\zeta$ is a $\Lambda$-graded $C$-algebra as well as a $(U_L^L(g), U_L^L(h))$-bimodule satisfying

$$A_\zeta(\lambda) = \{ \varphi \in A_\zeta \mid \varphi a = \chi_\lambda(a) \varphi \ (a \in U_L^L(h)) \} \quad (\lambda \in \Lambda^+).$$

We denote by $\text{Tor}_{\Lambda^+}(A_\zeta)$ the full subcategory of $\text{Mod}_A(A_\zeta)$ consisting of $M \in \text{Mod}_A(A_\zeta)$ satisfying

$$\forall m \in M, \exists N \text{ s.t. } \lambda \in \Lambda^+, \langle \lambda, \alpha_i \rangle > N \ (\forall i \in I) \implies A_\zeta(\lambda)m = 0,$$

and set

$$\text{(3.1)} \quad \text{Mod}(O_{B_\zeta}) = \text{Mod}_A(A_\zeta)/\text{Tor}_{\Lambda^+}(A_\zeta).$$

By definition $\text{Mod}(O_{B_\zeta})$ is the localization of the category $\text{Mod}_A(A_\zeta)$ with respect to the multiplicative system consisting of morphisms in $\text{Mod}_A(A_\zeta)$ whose kernel and cokernel belong to $\text{Tor}_{\Lambda^+}(A_\zeta)$.

**Remark 3.1.** The notations $B_\zeta$ and $O_{B_\zeta}$ do not have actual meanings; however, $\text{Mod}(O_{B_\zeta})$ is an actual abelian category.

By a general result the natural exact functor

$$\text{(3.2)} \quad \omega^* : \text{Mod}_A(A_\zeta) \to \text{Mod}(O_{B_\zeta})$$

admits a right adjoint

$$\text{(3.3)} \quad \omega_* : \text{Mod}(O_{B_\zeta}) \to \text{Mod}_A(A_\zeta).$$

Then $\omega_*$ is left exact and we have

$$\omega^* \circ \omega_* = \text{Id}$$

(see [37, Ch4]). We define a left exact functor

$$\text{(3.4)} \quad \Gamma : \text{Mod}(O_{B_\zeta}) \to \text{Mod}(C)$$

by $\Gamma(M) = (\omega_* M)(0)$. 

For $\mu \in \Lambda$ we define an exact functor
\begin{equation}
(\bullet)[\mu] : \text{Mod}(\mathcal{O}_{B_\zeta}) \to \text{Mod}(\mathcal{O}_{B_\zeta}) \quad (M \mapsto M[\mu])
\end{equation}
to be the functor induced by
\begin{equation}
(\bullet)[\mu] : \text{Mod}_\Lambda(A_\zeta) \to \text{Mod}_\Lambda(A_\zeta) \quad (M \mapsto M[\mu]),
\end{equation}
where
\begin{equation}
(M[\mu])(\lambda) = M(\lambda + \mu) \quad (\lambda \in \Lambda).
\end{equation}

Similarly we set
\begin{equation}
\text{mod}(\mathcal{O}_{B_\zeta}) = \text{mod}_\Lambda(A_\zeta) / (\text{Tor}_\Lambda + (A_\zeta) \cap \text{mod}_\Lambda(A_\zeta)).
\end{equation}
It is a full subcategory of \text{Mod}(\mathcal{O}_{B_\zeta}), and the functors (3.3), (3.4) induce
\begin{equation}
\omega^* : \text{mod}_\Lambda(A_\zeta) \to \text{mod}(\mathcal{O}_{B_\zeta}),
\end{equation}
\begin{equation}
\omega_* : \text{mod}(\mathcal{O}_{B_\zeta}) \to \text{mod}_\Lambda(A_\zeta).
\end{equation}

**Remark 3.2.** In the case $\zeta = 1$ the category \text{Mod}(\mathcal{O}_{B_1}) (resp. \text{mod}(\mathcal{O}_{B_1})) is naturally identified with the category \text{Mod}(\mathcal{O}_B) (resp. \text{mod}(\mathcal{O}_B)) consisting of quasi-coherent (resp. coherent) sheaves on the flag manifold $B = B^\times \setminus G$, and then (3.4) turns out to be the global section functor
\begin{equation}
\Gamma(B, \bullet) : \text{Mod}(\mathcal{O}_B) \to \text{Mod}(\mathcal{C}).
\end{equation}

Following [4] we give another description of the quantized flag manifolds.

**Definition 3.3.** We define an abelian category $\widetilde{\text{Mod}}(\mathcal{O}_{B_\zeta})$ as follows. An object of $\widetilde{\text{Mod}}(\mathcal{O}_{B_\zeta})$ is a vector space $M$ equipped with a left $\mathcal{O}_\zeta(G)$-module structure and a right integrable $U^L_\zeta(b^-)$-module structure such that
\begin{equation}
(fm)y = \sum_\langle y \rangle (fy(0))(my(1)) \quad (f \in \mathcal{O}_\zeta(G), \ m \in M, \ y \in U^L_\zeta(b^-)).
\end{equation}
A morphism in $\widetilde{\text{Mod}}(\mathcal{O}_{B_\zeta})$ is a linear map preserving the left $\mathcal{O}_\zeta(G)$-module structure and right $U^L_\zeta(b^-)$-module structure.

**Remark 3.4.** In the case $\zeta = 1$ the category $\widetilde{\text{Mod}}(\mathcal{O}_{B_1})$ is naturally identified with the category consisting of quasi-coherent sheaves on $G$ equivariant with respect to the action of $B^\times$ on $G$ given by the left multiplication. Hence it is also equivalent to \text{Mod}(\mathcal{O}_B).

We define a functor
\begin{equation}
\tilde{\omega}_* : \widetilde{\text{Mod}}(\mathcal{O}_{B_\zeta}) \to \text{Mod}_\Lambda(A_\zeta)
\end{equation}
by
\begin{equation}
\tilde{\omega}_* M = \sum_{\lambda \in \Lambda} (\tilde{\omega}_* M)(\lambda) = \bigoplus_{\lambda \in \Lambda} (\tilde{\omega}_* M)(\lambda) \subset M,
\end{equation}
\begin{equation}
(\tilde{\omega}_* M)(\lambda) = \{ m \in M \mid my = \chi_\lambda(y)m \ (y \in U^L_\zeta(b^-)) \}
\end{equation}
for $M \in \widetilde{\text{Mod}}(\mathcal{O}_{B_\zeta})$. 
For $\mu \in \Lambda$ we denote by $\mathcal{C}_\mu$ the one-dimensional right $U^L(\mathfrak{b}^-)$-module corresponding to $\chi_\mu : U^L(\mathfrak{b}^-) \to \mathbb{C}$, and define an exact functor

$$\text{(3.10)} \quad (\bullet)[\mu] : \text{Mod}(\mathcal{O}_{A_\xi}) \to \hat{\text{Mod}}(\mathcal{O}_{B_\xi})$$

by $M[\mu] = M \otimes \mathbb{C}_{-\mu}$. Here, the left $\mathcal{O}_\xi(G)$-module structure of $M[\mu]$ is given by that of the first factor $M$, and the right $U^L(\mathfrak{b}^-)$-module structure is given by the tensor product of right $U^L(\mathfrak{b}^-)$-modules $M$ and $\mathbb{C}_{-\mu}$.

We define a functor

$$\text{(3.11)} \quad \tilde{\omega}^* : \text{Mod}_A(A_\xi) \to \hat{\text{Mod}}(\mathcal{O}_{B_\xi})$$

by $\tilde{\omega}^* K = \mathcal{O}_\xi(G) \otimes_{A_\xi} K$. The left $\mathcal{O}_\xi(G)$-module structure of $\mathcal{O}_\xi(G) \otimes_{A_\xi} K$ is given by the left multiplication on the first factor. The right $U^L(\mathfrak{b}^-)$-module structure of $\mathcal{O}_\xi(G) \otimes_{A_\xi} K$ is given by the tensor product of two right $U^L(\mathfrak{b}^-)$-modules, where $K$ is regarded as a right $U^L(\mathfrak{b}^-)$-module by

$$k\gamma = \chi_\lambda(y)k \quad (k \in K(\lambda), \ y \in U^L(\mathfrak{b}^-)).$$

The following result is a consequence of the general theory due to Artin-Zhang [3].

**Proposition 3.5 ([4]).** The functor $\omega^* \circ \tilde{\omega}$ gives an equivalence

$$\text{(3.12)} \quad \hat{\text{Mod}}(\mathcal{O}_{B_\xi}) \cong \text{Mod}(\mathcal{O}_{B_\xi})$$

of abelian categories compatible with the shift functors $\text{(3.5)}, \text{(3.10)}$. Moreover, identifying $\hat{\text{Mod}}(\mathcal{O}_{B_\xi})$ with $\text{Mod}(\mathcal{O}_{B_\xi})$ via (3.12) we have

$$\text{(3.13)} \quad \tilde{\omega}^* = \omega^*, \quad \tilde{\omega} = \omega.$$  

**3.2. D-modules.** For $\varphi \in A_\Lambda$, $u \in U_\Lambda(\mathfrak{g})$, $\lambda \in \Lambda$ we define $\ell_\varphi, \partial_u, \sigma_\lambda \in \text{End}_A(A_\Lambda)$ by

$$\ell_\varphi(\psi) = \varphi\psi, \ \partial_u(\psi) = u\psi, \ \sigma_\lambda(\psi) = \psi\kappa_\lambda \quad (\psi \in A_\Lambda).$$

Here, $A_\Lambda$ is regarded as a $(U_\Lambda(\mathfrak{g}), U_\Lambda(\mathfrak{h}))$-bimodule via the embedding $U_\Lambda(\mathfrak{g}) \hookrightarrow U^L_\Lambda(\mathfrak{g}), U_\Lambda(\mathfrak{h}) \hookrightarrow U^L_\Lambda(\mathfrak{h})$. We define an $A$-subalgebra $D_\Lambda$ of $\text{End}_A(A_\Lambda)$ by

$$D_\Lambda = \langle \ell_\varphi, \partial_u, \sigma_\lambda, \delta_{2\lambda} \mid \varphi \in A_\Lambda, u \in U_\Lambda(\mathfrak{g}), \lambda \in \Lambda \rangle \subset \text{End}_A(A_\Lambda).$$

Then $D_\Lambda$ turns out to be a $\Lambda$-graded $A$-algebra by the $\Lambda$-grading $D_\Lambda = \bigoplus_{\lambda \in \Lambda} D_\Lambda(\lambda)$ given by

$$D_\Lambda(\lambda) = \{ P \in D_\Lambda \mid P(A_\Lambda(\mu)) \subset A_\Lambda(\lambda + \mu) \quad (\mu \in \Lambda) \}.$$  

For $\zeta \in \mathbb{C}^*$ we define a $\Lambda$-graded $\mathbb{C}$-algebra $D_\zeta$ by

$$D_\zeta = \mathbb{C} \otimes_A D_\Lambda,$$

where $A \to \mathbb{C}$ is given by $q^{1/N} \mapsto \zeta^{1/N}$. It is easily seen that in the $\mathbb{C}$-algebra $D_\zeta$ we have

$$\text{(3.14)} \quad z \in Z_{\text{Har}}(U_\zeta(\mathfrak{g})), \ \Xi(z) = \sum_{\lambda \in \Lambda} a_\lambda k_{2\lambda} \implies \partial_z = \sum_{\lambda \in \Lambda} a_\lambda \sigma_{2\lambda}.$$
Regarding $A_{\zeta}$ as a $\Lambda$-graded $\mathbb{C}$-subalgebra of $D_{\zeta}$ by $A_{\zeta} \rightarrow D_{\zeta}$ ($\varphi \mapsto \ell_{\varphi}$) we define a category $\text{Mod}(D_{B_{\zeta}})$ by

\[(3.15) \quad \text{Mod}(D_{B_{\zeta}}) = \text{Mod}_{\Lambda}(D_{\zeta})/(\text{Mod}_{\Lambda}(D_{\zeta}) \cap \text{Tor}_{A^+}(A_{\zeta})).\]

The natural exact functor

$$\omega^* : \text{Mod}_{\Lambda}(D_{\zeta}) \rightarrow \text{Mod}(D_{B_{\zeta}})$$

admits a right adjoint

$$\omega_* : \text{Mod}(D_{B_{\zeta}}) \rightarrow \text{Mod}_{\Lambda}(D_{\zeta}),$$

which is left exact. For $M \in \text{Mod}_{\Lambda}(D_{\zeta})$ the zero-th part $M(0)$ turns out to be a $U_{\zeta}$-module via $U_{\zeta} \rightarrow D_{\zeta}(0)$ ($u \mapsto \partial u$), and hence we obtain a left exact functor

\[(3.16) \quad \Gamma : \text{Mod}(D_{B_{\zeta}}) \rightarrow \text{Mod}(U_{\zeta}(g)) \quad (M \mapsto (\omega_* M)(0)).\]

For $t \in H$ we define full subcategories $\text{Mod}(D_{B_{\zeta},t})$ and $\text{Mod}_{t}(D_{B_{\zeta}})$ of $\text{Mod}(D_{B_{\zeta}})$ as follows. Let $\text{Mod}_{t}(D_{\zeta};t)$ be the full subcategory of $\text{Mod}_{\Lambda}(D_{\zeta})$ consisting of $M \in \text{Mod}_{\Lambda}(D_{\zeta})$ satisfying

$$\sigma_{2\lambda}|_{M(\mu)} = \theta_{\lambda}(t)\zeta^{2(\lambda,\mu)} \text{id} \quad (\mu \in \Lambda).$$

We also denote by $\text{Mod}_{t}(D_{\zeta})$ the full subcategory of $\text{Mod}_{\Lambda}(D_{\zeta})$ consisting of $M \in \text{Mod}_{\Lambda}(D_{\zeta})$ such that for any $m \in M(\mu)$ with $\mu \in \Lambda$ there exists some $n$ such that

$$(\sigma_{2\lambda} - \theta_{\lambda}(t)\zeta^{2(\lambda,\mu)})^n m = 0.$$ 

Then we define $\text{Mod}(D_{B_{\zeta},t})$ and $\text{Mod}_{t}(D_{B_{\zeta}})$ by

\[(3.17) \quad \text{Mod}(D_{B_{\zeta},t}) = \text{Mod}_{t}(D_{\zeta};t)/(\text{Mod}_{t}(D_{\zeta};t) \cap \text{Tor}_{A^+}(A_{\zeta})),\]

\[(3.18) \quad \text{Mod}_{t}(D_{B_{\zeta}}) = \text{Mod}_{t}(D_{\zeta})/(\text{Mod}_{t}(D_{\zeta}) \cap \text{Tor}_{A^+}(A_{\zeta})).\]

Note that $\text{Mod}(D_{B_{\zeta},t})$ is an analogue of the category of quasi-coherent $D_{B_{\nu}}$-modules, where $D_{B_{\nu}}$ is the ring of twisted differential operators on the ordinary flag manifold $B$ corresponding to the parameter $\nu \in h^*$. By restricting $\omega^*$ and $\omega_*$ we obtain natural exact functors

$$\omega^*_{t*} : \text{Mod}_{\Lambda}(D_{\zeta};t) \rightarrow \text{Mod}(D_{B_{\zeta},t}), \quad \omega^*_{t*} : \text{Mod}_{\Lambda}(D_{\zeta}) \rightarrow \text{Mod}_{t}(D_{B_{\zeta}}),$$

and their right adjoints

$$\omega_*^{t*} : \text{Mod}(D_{B_{\zeta},t}) \rightarrow \text{Mod}_{\Lambda}(D_{\zeta};t), \quad \omega_*^{t*} : \text{Mod}(D_{B_{\zeta}}) \rightarrow \text{Mod}_{\Lambda}(D_{\zeta}),$$

which are left exact. Moreover, $(3.16)$ induces

\[(3.19) \quad \Gamma : \text{Mod}(D_{B_{\zeta},t}) \rightarrow \text{Mod}(U_{\zeta}(g)_{|t}) \quad (M \mapsto (\omega_*^{t*} M)(0)),\]

\[(3.20) \quad \Gamma : \text{Mod}_{t}(D_{B_{\zeta}}) \rightarrow \text{Mod}_{|t}(U_{\zeta}(g)) \quad (M \mapsto (\omega_*^{t*} M)(0)).\]

by $(3.14)$. Here, $\text{Mod}_{|t}(U_{\zeta}(g))$ is the full subcategory of $\text{Mod}(U_{\zeta}(g))$ consisting of $M \in \text{Mod}(U_{\zeta}(g))$ such that for any $m \in M$ we have $\text{Ker}(\xi_{|t})^n m = 0$ for some $n$.

We define $\text{mod}(D_{B_{\zeta}})$, $\text{mod}(D_{B_{\zeta},t})$, $\text{mod}_{t}(D_{B_{\zeta}})$ similarly to $\text{Mod}(D_{B_{\zeta}})$, $\text{Mod}(D_{B_{\zeta},t})$, $\text{Mod}_{t}(D_{B_{\zeta}})$ respectively using $\text{mod}_{\Lambda}(D_{\zeta})$ instead of $\text{Mod}_{\Lambda}(D_{\zeta})$. They are full subcategories of $\text{Mod}(D_{B_{\zeta}})$, $\text{Mod}(D_{B_{\zeta},t})$, $\text{Mod}_{t}(D_{B_{\zeta}})$ respectively, and $(3.16), (3.19), (3.20)$. 

Remark 3.6. In our previous papers [41] and [42], we used, instead of the categories $\text{Mod}(D_{B_\zeta})$ and $\text{Mod}(D_{B_\zeta,t})$ ($t \in H$) described above, the categories $\text{Mod}(D^b_{B_\zeta})$, $\text{Mod}(D^b_{B_\zeta,t})$ defined starting from

$$D^b_{B_\zeta} = \langle \ell_v, \partial_u, \sigma_\lambda \mid \varphi \in A_\lambda, u \in U_\lambda(g), \lambda \in \Lambda \rangle \subset \text{End}_{A_\lambda}(A_\lambda)$$

which is slightly larger than $D_{B_\zeta}$. However, since the arguments in [41], [42] also work for our $\text{Mod}(D_{B_\zeta})$, $\text{Mod}(D_{B_\zeta,t})$ either, we use them in the present paper. We note that we are able to show $\text{Mod}(D^b_{B_\zeta,t}) \cong \text{Mod}(D_{B_\zeta,t^2})$.

For $\lambda \in \Lambda$ we define $t_\lambda \in H$ by

$$\theta_\mu(t_\lambda) = \zeta^{2(\lambda,\mu)} \quad (\mu \in \Lambda).$$

In [40] we proved in the case where $\zeta$ is transcendental that (3.22) gives an equivalence of abelian categories if $t = t_\lambda$ with $\lambda \in \Lambda^+$. This is an analogue of the Beilinson-Bernstein correspondence. For general $\zeta \in \mathbb{C}^\times$ we have the following conjecture.

Conjecture 3.7. The derived functors

$$R\Gamma : D^b(\text{mod}(D_{B_\zeta,t})) \to D^b(\text{mod}(U_\zeta(g)_{\{t\}}))$$

(3.25)

$$R\Gamma : D^b(\text{mod}(D_{B_\zeta})) \to D^b(\text{mod}_{\{t\}}(U_\zeta(g)))$$

(3.26)

of (3.22), (3.23) between bounded derived categories give equivalences of triangulated categories if $t \in H$ is regular in the sense that $|W \circ t| = |W|$.

In the case $\zeta$ is not a root of 1, Conjecture 3.7 is regarded as an analogue of a known result for twisted $D$-modules on the ordinary flag manifold $B$ over $\mathbb{C}$ (see [8], [28]). However, the proof in the ordinary case uses a certain integral transform, which is sometimes called the Radon transform, and is not directly applied to our case. It is an interesting problem to define the Radon transforms for quantized situation. Conjecture 3.7 in the case $\zeta$ is a root of 1 will be discussed in Section 6.

3.3. $U_{B_\zeta}$-modules. For $V \in \text{mod}_{\text{int}}(U^L_{\zeta}(g))$ we set $E_V = V \otimes A_\zeta$. We regard it as a right $A_\zeta$-module via the right multiplication on the second factor. We also regard it as a left $U^L_{\zeta}(g)$-module by

$$u(v \otimes \varphi) = \sum_{(u)} u(0)v \otimes \partial_{u(1)}\varphi \quad (u \in U^L_{\zeta}(g), v \in V, \varphi \in A_\zeta).$$

We will identify $E_V$ with $A_\zeta \otimes V$ through the isomorphism $R^{A_\zeta,V} : A_\zeta \otimes V \to V \otimes A_\zeta$ of $U^L_{\zeta}(g)$-modules (see (2.19)). We define a left $A_\zeta$-module structure of $E_V$ by the
left multiplication on the first factor of $E_V = A_\zeta \otimes V$. Then $E_V$ turns out to be an $A_\zeta$-bimodule. Moreover, by

$$R^{A_\zeta, V}(1 \otimes v) = v \otimes 1 \quad (v \in V)$$

we can regard $V$ as a $U_\zeta^L(g)$-submodule of $E_V$ by the embedding

$$V \hookrightarrow E_V \quad (v \mapsto v \otimes 1 \in V \otimes A_\zeta = E_V).$$

It is easily seen that the functor $E_V \otimes_{A_\zeta} (\bullet) : \text{Mod}_\Lambda(A_\zeta) \rightarrow \text{Mod}_\Lambda(A_\zeta)$ induces an exact functor

(3.27) $$V \otimes (\bullet) : \text{Mod}(O_{B_\zeta}) \rightarrow \text{Mod}(O_{B_\zeta}).$$

We refer the readers to [40, 4.3] for the properties of $E_V$ mentioned above.

**Proposition 3.8.** Assume that $V \in \text{mod}_{\text{int}}(U_\zeta^L(g))$ is equipped with a $U_\zeta^L(b^-)$-stable filtration

(3.28) $$V = V_1 \supset V_2 \supset \cdots \supset V_m \supset V_{m+1} = \{0\}$$

such that $V_j / V_{j+1}$ is a one-dimensional $U_\zeta^L(b^-)$-module with character $\xi_j \in \Lambda$. Then for $\mathcal{M} \in \text{Mod}(O_{B_\zeta})$ we have a functorial filtration

$$V \otimes \mathcal{M} = \mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots \supset \mathcal{M}_m \supset \mathcal{M}_{m+1} = \{0\}$$

of $V \otimes \mathcal{M}$ such that $\mathcal{M}_j / \mathcal{M}_{j+1} \cong \mathcal{M}[\xi_j]$ for $j = 1, \ldots, m$.

**Proof.** A proof of this result over $\mathbb{F}$ is given in [40, 4.4]. Here, we give a different proof using the equivalence (3.12).

It is easily seen that the functor

$$V \otimes (\bullet) : \text{Mod}(O_{B_\zeta}) \rightarrow \text{Mod}(O_{B_\zeta})$$

corresponding to (3.27) is given by $M \mapsto V \otimes M$. Here, the right $U_\zeta^L(b^-)$-module structure of $V \otimes M$ is given by the right $U_\zeta^L(b^-)$-module structure of $M$ (the action of $U_\zeta^L(b^-)$ on $V$ is trivial), and the left $O_\zeta(G)$-module structure of $V \otimes M$ is given by the composite of

$$O_\zeta(G) \otimes V \otimes M \xrightarrow{R^{O_\zeta(G), V}} V \otimes O_\zeta(G) \otimes M \xrightarrow{1 \otimes \alpha} V \otimes M,$$

where $\alpha : O_\zeta(G) \otimes M \rightarrow M$ is the left $O_\zeta(G)$-module structure of $M$.

Let us define $(V \otimes M)^\dagger \in \text{Mod}(O_{B_\zeta})$ as follows. As a vector space we have $(V \otimes M)^\dagger = V \otimes M$. The right $U_\zeta^L(b^-)$-module structure of $(V \otimes M)^\dagger$ is given by

$$(v \otimes m)y = \sum_{(y)} (Sy_{(0)}v \otimes my_{(1)}) \quad (v \in V, \ m \in M, \ y \in U_\zeta^L(b^-)),$$

and the left $O_\zeta(G)$-module structure of $(V \otimes M)^\dagger$ is given by the composite of

$$O_\zeta(G) \otimes V \otimes M \xrightarrow{\sum_{\zeta \in g} R^{O_\zeta(G)}_\zeta \otimes 1} V \otimes O_\zeta(G) \otimes M \xrightarrow{1 \otimes \alpha} V \otimes M,$$

where $R^{O_\zeta(G)}_\zeta : O_\zeta(G) \otimes V \rightarrow V \otimes O_\zeta(G)$ is given by

$$R^{O_\zeta(G)}_{\zeta, \gamma} = \sum_p r_p \otimes r'_p \implies R^{O_\zeta(G)}_{\zeta, \gamma}(\varphi \otimes v) = \zeta^{-(\lambda - \gamma, \mu)} \sum_p r'_p v \otimes \varphi r_p.$$
for \( v \in V \) and \( \varphi \in O(\mathcal{O}(G)) \) such that \( \varphi h = \chi(h)\varphi \) for any \( h \in U_{\xi}(h) \).

One can check that the linear map \( V \otimes M \to V \otimes M \) defined by
\[
v \otimes m \mapsto \sum_{i} v_i \otimes \varphi_i m \quad \text{if} \quad uv = \sum_{i} (\varphi_i, u)v_i \quad (u \in U_{\xi}(g), v \in V)
\]
gives an isomorphism \( V \otimes M \to (V \otimes M)^{1} \) in \( \operatorname{Mod}(O_{\mathcal{O}(G)}) \).

It remains to show that \( V_j \otimes M \) is a subobject of \( (V \otimes M)^{1} \in \operatorname{Mod}(O_{\mathcal{O}(G)}) \) and we have \( (V_j \otimes M)/(V_{j+1} \otimes M) \cong M[\xi_j] \). This is easily seen from the definition of \( (V \otimes M)^{1} \).

We define a \( \mathbb{C} \)-algebra structure of \( A_{\xi} \otimes U_{\xi}(g) \) by
\[
(\varphi \otimes u)(\varphi' \otimes u') = \sum_{(u)} \varphi \cdot \partial_{u(0)}(\varphi') \otimes u_{(1)}u' \quad (\varphi, \varphi' \in A_{\xi}, u, u' \in U_{\xi}(g)).
\]
We will identify \( A_{\xi} \) and \( U_{\xi}(g) \) as subalgebras of \( A_{\xi} \otimes U_{\xi}(g) \) by the embeddings \( \varphi \mapsto \varphi \otimes 1, u \mapsto 1 \otimes u \) for \( \varphi \in A_{\xi}, u \in U_{\xi}(g) \). We regard \( A_{\xi} \otimes U_{\xi}(g) \) as a \( \Lambda \)-graded \( \mathbb{C} \)-algebra by \( (A_{\xi} \otimes U_{\xi}(g))(\lambda) = A_{\xi}(\lambda) \otimes U_{\xi}(g) \). We have a canonical homomorphism
\[
A_{\xi} \otimes U_{\xi}(g) \to D_{\xi} \quad (\varphi \otimes u \mapsto \ell_{\varphi} \partial_u)
\]
of \( \Lambda \)-graded algebras. We set
\[
\operatorname{Mod}(U_{\mathcal{O}(G)}) = \operatorname{Mod}_{\Lambda}(A_{\xi} \otimes U_{\xi}(g))/\left(\operatorname{Mod}_{\Lambda}(A_{\xi} \otimes U_{\xi}(g)) \cap \operatorname{Tor}_{\Lambda^{+}}(A_{\xi})\right).
\]
Then \ref{3.27} induces an exact functor
\[
V \otimes (\bullet) : \operatorname{Mod}(U_{\mathcal{O}(G)}) \to \operatorname{Mod}(U_{\mathcal{O}(G)})
\]
for \( V \in \operatorname{mod}_{\mathbb{Z}}(U_{\xi}(g)) \). Here, for \( M \in \operatorname{Mod}_{\Lambda}(A_{\xi} \otimes U_{\xi}(g)) \) we regard \( V \otimes M \) as a \( U_{\xi}(g) \)-module by the tensor product of two \( U_{\xi}(g) \)-modules.

By Proposition \ref{3.8} we have the following.

**Proposition 3.9.** Let \( V \) as in Proposition \ref{3.8}. Let \( M \in \operatorname{Mod}(D_{\mathcal{O}(G)}^{\Lambda}) \), and regard it as an object of \( \operatorname{Mod}(U_{\mathcal{O}(G)}) \) via \ref{3.29}. Then \( V \otimes M \in \operatorname{Mod}(U_{\mathcal{O}(G)}) \) has a functorial filtration
\[
V \otimes M = M_1 \supset M_2 \supset \cdots \supset M_m \supset M_{m+1} = \{0\}
\]
such that \( M_j/M_{j+1} \cong M[\xi_j] \in \operatorname{Mod}(D_{\mathcal{O}(G)}^{\Lambda}(M)) \) for \( j = 1, \ldots, m \).

4. **Representation theory at roots of unity**

In the rest of this paper we assume that \( \zeta \in \mathbb{C}^{\times} \) is a primitive \( \ell \)-th root of unity, where \( \ell \) is an integer satisfying the following conditions (a1), (a2), (a3):

(a1) \( \ell > 1 \) is odd,
(a2) \( \ell \) is prime to \( |\Lambda/Q| \),
(a3) \( \ell \) is prime to 3 if \( \Delta \) is of type \( G_2 \).

We take \( \zeta^{1/|\Lambda/Q|} \) to be a primitive \( \ell \)-th root of unity (see (a2)).
4.1. **Center.** For $\alpha \in \Delta^+$ we define the positive and the negative root vectors $e_\alpha, f_\alpha \in U_\zeta(\mathfrak{g})$ using Lusztig’s braid group action (see [32]). Then the elements $k_{t\lambda}$ ($\lambda \in \Lambda$), $e'_\alpha, f'_\alpha$ ($\alpha \in \Delta^+$) belong to the center $Z(U_\zeta(\mathfrak{g}))$ of $U_\zeta(\mathfrak{g})$. We denote by $Z_{Fr}(U_\zeta(\mathfrak{g}))$ the subalgebra of $Z(U_\zeta(\mathfrak{g}))$ generated by them.

**Remark 4.1.** The elements $e_\alpha, f_\alpha$ actually depends on the choice of a reduced expression of the longest element of the Weyl group $W$; however, the subalgebra $Z_{Fr}(U_\zeta(\mathfrak{g}))$ is independent of the choice.

Define a closed subgroup $K$ of $G \times G$ by

$$K = \{(xs, ys^{-1}) \mid x \in N^+, y \in N^-, s \in H\}.$$ 

**Proposition 4.2** ([21], [22]).

(i) The algebra $Z_{Fr}(U_\zeta(\mathfrak{g}))$ is naturally isomorphic to the coordinate algebra $O(K)$ of $K$.

(ii) The algebra $Z_{Fr}(U_\zeta(\mathfrak{g})) \cap Z_{Har}(U_\zeta(\mathfrak{g}))$ is naturally isomorphic to $O(H/W)$.

(iii) We have an isomorphism

$$Z(U_\zeta(\mathfrak{g})) \cong Z_{Fr}(U_\zeta(\mathfrak{g})) \otimes_{Z_{Fr}(U_\zeta(\mathfrak{g})) \cap Z_{Har}(U_\zeta(\mathfrak{g}))} Z_{Har}(U_\zeta(\mathfrak{g})) \quad (z_1z_2 \leftrightarrow z_1 \otimes z_2)$$

of algebras.

Moreover, under the identification

$$Z_{Fr}(U_\zeta(\mathfrak{g})) \cong O(K), \quad Z_{Har}(U_\zeta(\mathfrak{g})) \cong O(H/W)$$

the inclusions

$$Z_{Fr}(U_\zeta(\mathfrak{g})) \cap Z_{Har}(U_\zeta(\mathfrak{g})) \to Z_{Har}(U_\zeta(\mathfrak{g}))$$

$$Z_{Fr}(U_\zeta(\mathfrak{g})) \cap Z_{Har}(U_\zeta(\mathfrak{g})) \to Z_{Fr}(U_\zeta(\mathfrak{g}))$$

correspond to $H/W \circ \to H/W$ ($[t] \mapsto [t^\ell]$), and

$$\kappa : K \to H/W$$

which is the composite of

$$\eta : K \to G \quad ((x_1, x_2) \mapsto x_1x_2^{-1})$$

with the Steinberg map $St : G \to H/W$ respectively. Hence $Z(U_\zeta(\mathfrak{g}))$ is isomorphic to the coordinate algebra $O(\mathcal{X})$ of the affine algebraic variety $\mathcal{X} = K \times_{H/W}(H/W \circ)$.

For $g \in G$ we denote its Jordan decomposition by $g = g_s g_u$. By definition we have

$$(4.1) \quad \mathcal{X} = \{(k, [t]) \in K \times (H/W \circ) \mid t^\ell \in \text{Ad}(G) (\eta(k)_s)\}.$$ 

For $k \in K$ we denote by

$$(4.2) \quad \xi^k : Z_{Fr}(U_\zeta(\mathfrak{g})) \to \mathbb{C}$$

the corresponding character of $Z_{Fr}(U_\zeta(\mathfrak{g}))$. For $(k, [t]) \in \mathcal{X}$ we have a character

$$(4.3) \quad \xi^k_{[t]} : Z(U_\zeta(\mathfrak{g})) \to \mathbb{C}$$
of the total center $Z(U_\zeta(\mathfrak{g}))$ such that $\xi^k_{[t]}|_{Z_{\text{Har}}(U_\zeta(\mathfrak{g}))} = \xi_{[t]}$, and $\xi^k_{[t]}|_{Z_{\text{Fr}}(U_\zeta(\mathfrak{g}))} = \xi^k$.

We set
\begin{align}
U_\zeta(\mathfrak{g})^k &= U_\zeta(\mathfrak{g}) \otimes_{Z_{\text{Fr}}(U_\zeta(\mathfrak{g}))} \mathbb{C} \quad (k \in K), \\
U_\zeta(\mathfrak{g})_{[t]} &= U_\zeta(\mathfrak{g}) \otimes_{Z(U_\zeta(\mathfrak{g}))} \mathbb{C} \quad ((k, [t]) \in \mathcal{X})
\end{align}
with respect to $\xi^k$, $\xi^k_{[t]}$ respectively. Recall also that we have
\begin{align}
U_\zeta(\mathfrak{g})_{[t]} &= U_\zeta(\mathfrak{g}) \otimes_{Z_{\text{Har}}(U_\zeta(\mathfrak{g}))} \mathbb{C} \quad ([t] \in H/W_\circ).
\end{align}

A version of Schur’s lemma tells us that for any irreducible $U_\zeta(\mathfrak{g})$-module $M$ there exists some $(k, [t]) \in \mathcal{X}$ such that $z|_M = \xi^k_{[t]}(z)\text{id}$ for any $z \in Z(U_\zeta(\mathfrak{g}))$. It is known by [20] that for $k, k' \in K$ we have $U_\zeta(\mathfrak{g})^k \cong U_\zeta(\mathfrak{g})^{k'}$ if $\eta(k)$ is conjugate to $\eta(k')$.

We say that $\hat{t} \in H$ is unramified if we have $\theta_{\alpha}(\hat{t}) = \zeta^{-2(\rho, \alpha)}$ for any $\alpha \in \Delta$ satisfying $\theta_{\alpha}(\hat{t}) = 1$. We denote by $H_\text{ur}$ the set of unramified elements of $H$.

For $k \in K$ we give a description of the set $\mathcal{X}^k$ consisting of the Harish-Chandra central characters $\xi_{[t]} : Z_{\text{Har}}(U_\zeta(\mathfrak{g})) \to \mathbb{C}$ which is compatible with the Frobenius central character $\xi^k : Z_{\text{Fr}}(U_\zeta(\mathfrak{g})) \to \mathbb{C}$. Fix $h \in H \cap \text{Ad}(G)(\eta(k)_s)$. Then we have $H \cap \text{Ad}(G)(\eta(k)_s) = W(h)$. By the above argument we have

$$\mathcal{X}^k \cong H(W(h))/W_\circ \cong H(h)/W_h \circ,$$

where
\begin{align*}
H(W(h)) &= \{ t \in H \mid t^\ell \in W(h) \}, \\
H(h) &= \{ t \in H \mid t^\ell = h \}, \\
W_h &= \{ w \in W \mid w(h) = h \}.
\end{align*}

It is known that $W_h$ is the Weyl group of the root system
\[ \Delta_h = \{ \alpha \in \Delta \mid \theta_{\alpha}(h) = 1 \}. \]

We denote by $H_\ell$ the set of $d \in H$ satisfying $d^\ell = 1$. By our assumption on $\ell$ we have
\begin{equation}
H_\ell = \{ t\lambda \mid \lambda \in \Lambda \} \cong \Lambda/\ell\Lambda
\end{equation}
(see [3, 22] for the notation). Set
\[ H_\text{ur}(h) = H(h) \cap H_\text{ur} = \{ \hat{t} \in H(h) \mid \theta_{\alpha}(\hat{t}) = \zeta^{-2(\rho, \alpha)} \ (\alpha \in \Delta_h) \}. \]

Then we have
\[ H(h) = \bigsqcup_{\hat{t} \in H_\text{ur}(h)} \hat{t}H_\ell. \]

It is easily seen that for $\hat{t} \in H_\text{ur}(h)$, $d \in H_\ell$, and $w \in W_h$ we have $w \circ (\hat{t}d) = \hat{t}(w(d))$.

Hence we obtain
\begin{align*}
\mathcal{X}^k &\cong H(h)/W_h \circ = \bigsqcup_{\hat{t} \in H_\text{ur}(h)} (\hat{t}H_\ell)/W_h \circ, \\
(\hat{t}H_\ell)/W_h \circ &\cong H_\ell/W_h \cong \Lambda/(W_h \times (\ell\Lambda)) \quad (\hat{t} \in H_\text{ur}(h)).
\end{align*}
4.2. Tensor product with integrable highest weight modules. Let $k \in K$, and take $h \in H$ such that $\text{Ad}(G)(\eta(k)_s) \cap H = W(h)$. We denote by $\text{mod}^k(U_\zeta(\mathfrak{g}))$ the category of finitely generated $U_\zeta(\mathfrak{g})$-modules on which $z - \xi^k(z)$ for $z \in Z_{Fr}(U_\zeta(\mathfrak{g}))$ acts locally nilpotently. For $[t] \in H(W(h))/W_\circ$ we set $\text{mod}^k(U_\zeta(\mathfrak{g})_{[t]}) = \text{mod}^k(U_\zeta(\mathfrak{g})) \cap \text{mod}(U_\zeta(\mathfrak{g})/[t])$. We also denote by $\text{mod}^k(U_\zeta(\mathfrak{g}))$ the category of finitely generated $U_\zeta(\mathfrak{g})$-modules on which $z - \xi^k(z)$ for $z \in Z(U_\zeta(\mathfrak{g}))$ acts locally nilpotently.

Assume $M \in \text{mod}^k(U_\zeta(\mathfrak{g}))$. Since $U_\zeta(\mathfrak{g})^k$ is finite-dimensional, $M$ is a finite-dimensional $U_\zeta(\mathfrak{g})$-module. In particular, it has a finite composition series. Let $N$ be a composition factor of $M$. By Schur’s lemma there exists some $[t] \in H(W(h))/W_\circ$ such that any $z \in Z_{\text{Har}}(U_\zeta(\mathfrak{g}))$ acts on $N$ by the scalar multiplication of $\xi_{[t]}(z)$. It follows that we have the direct sum decomposition

\[(4.8) \quad \text{mod}^k(U_\zeta(\mathfrak{g})) = \bigoplus_{[t] \in H(W(h))/W_\circ} \text{mod}^k(U_\zeta(\mathfrak{g})_{[t]}).\]

If $V$ is a $U_\zeta^k(\mathfrak{g})$-module, we will regard it as a $U_\zeta(\mathfrak{g})$-module via the natural homomorphism $U_\zeta(\mathfrak{g}) \to U_\zeta^k(\mathfrak{g})$ induced by $U_\zeta^k(\mathfrak{g}) \subset U_\zeta^k(\mathfrak{g})$. For a $U_\zeta(\mathfrak{g})$-module $M$ we regard $V \otimes M$ as a $U_\zeta(\mathfrak{g})$-module through the comultiplication of $U_\zeta(\mathfrak{g})$. We see easily the following.

**Lemma 4.3.** For $V \in \text{mod}_{\text{int}}(U_\zeta^k(\mathfrak{g}))$ and $M \in \text{mod}^k(U_\zeta(\mathfrak{g}))$ we have $V \otimes M \in \text{mod}^k(U_\zeta(\mathfrak{g}))$.

Let $V \in \text{mod}_{\text{int}}(U_\zeta^k(\mathfrak{g}))$. Fix $z \in Z_{\text{Har}}(U_\zeta(\mathfrak{g}))$. For $\nu \in \Lambda$ we define $a_\nu \in \mathfrak{g}^\vee U_\zeta(\mathfrak{h})$ by

$$\Xi(z) = \sum_{\mu \in \Lambda} c_\mu k_{2\mu} \in \mathfrak{g}^\vee U_\zeta(\mathfrak{h}) \implies a_\nu = \sum_{\mu \in \Lambda} c_\mu \xi^{2(\mu,\nu)} k_{2\mu} \in \mathfrak{g}^\vee U_\zeta(\mathfrak{h}),$$

and set

$$F_z(x) = \prod_{\nu \in \text{wt}(V)} (x - a_\nu) \in \mathfrak{g}^\vee U_\zeta(\mathfrak{h})[x].$$

Here, $\text{wt}(V)$ denotes the multi-set consisting of weights of $V$, where each weight $\nu$ of $V$ is counted $\dim V_\nu$-times. We see easily that $F_z(x) \in \mathfrak{g}^\vee U_\zeta(\mathfrak{h})[x]$, and hence we can write

\[(4.9) \quad F_z(x) = x^n + \sum_{k=0}^{n-1} \Xi(z_k) x^k \quad (z_k \in Z_{\text{Har}}(U_\zeta(\mathfrak{g}))),\]

where $n = \dim V$.

Define

$$\delta : Z_{\text{Har}}(U_\zeta(\mathfrak{g})) \to \text{End}(V) \otimes U_\zeta(\mathfrak{g})$$

to be the composite of

$$Z_{\text{Har}}(U_\zeta(\mathfrak{g})) \subset U_\zeta(\mathfrak{g}) \xrightarrow{\Delta} U_\zeta(\mathfrak{g}) \otimes U_\zeta(\mathfrak{g}) \xrightarrow{\sigma_\nu \otimes 1} \text{End}(V) \otimes U_\zeta(\mathfrak{g}),$$

where $\sigma_\nu : U_\zeta(\mathfrak{g}) \to \text{End}(V)$ is the corresponding representation.

We have the following analogue of Kostant’s result for the ordinary enveloping algebras.
Proposition 4.4. Let \( V \in \text{mod}_{\text{int}}(U_{\zeta}^L(g)) \). We assume that there exist a finite-dimensional integrable \( U_{\mathcal{F}}(g) \)-module \( V_{\mathcal{F}} \) and its \( U_{\mathcal{F}}(g) \)-stable \( \mathbb{A} \)-form \( V_{\mathcal{F}} \) such that \( V = \mathbb{C} \otimes_{\mathbb{A}} V_{\mathcal{F}} \) with respect to the specialization \( q \mapsto \zeta \). Let \( z \in Z_{\text{Har}}(U_{\zeta}(g)) \) and define \( z_k \in Z_{\text{Har}}(U_{\zeta}(g)) \) by (4.9). Then we have
\[
\delta(z)^n + \sum_{k=0}^{n-1} (1 \otimes z_k)\delta(z)^k = 0
\]
in \( \text{End}(V) \otimes U_{\zeta}(g) \).

Proof. Set \( \tilde{\mathbb{F}} = \mathbb{C}(q^{1/\Lambda/Q}), \tilde{\mathbb{A}} = \mathbb{C}[q^{1/\Lambda/Q}] \). The proof of the corresponding statement over \( \tilde{\mathbb{F}} \), where \( U_{\zeta}(g) \), \( V \), \( Z_{\text{Har}}(U_{\zeta}(g)) \) are replaced by \( \tilde{\mathbb{F}} \otimes_{\tilde{\mathbb{F}}} U_{\mathcal{F}}(g) \), \( \mathbb{F} \otimes_{\tilde{\mathbb{F}}} V_{\mathcal{F}} \), \( Z(\tilde{\mathbb{F}} \otimes_{\tilde{\mathbb{F}}} U_{\mathcal{F}}(g)) \) respectively, is similar to that for the ordinary enveloping algebras given in [30], [9]. Our assertion follows from this since \( U_{\zeta}(g) \) is a specialization of the \( \tilde{\mathbb{A}} \)-form \( \tilde{\mathbb{A}} \otimes_{\tilde{\mathbb{A}}} U_{\mathcal{F}}(g) \) of \( \tilde{\mathbb{F}} \otimes_{\tilde{\mathbb{F}}} U_{\mathcal{F}}(g) \) and since \( Z(\tilde{\mathbb{A}} \otimes_{\tilde{\mathbb{A}}} U_{\mathcal{F}}(g)) \to Z_{\text{Har}}(U_{\zeta}(g)) \) is surjective.

It follows easily from Proposition 4.4 the following.

Proposition 4.5. Let \( V \in \text{mod}_{\text{int}}(U_{\zeta}^L(g)) \) be as in Proposition 4.4. Assume \( M \in \text{mod}^k_{[t]}(U_{\zeta}(g)) \) for \([t] \in H(W(h))/W_o \). Then for \( z \in Z_{\text{Har}}(U_{\zeta}(g)) \) we have
\[
\prod_{\nu \in \text{wt}(V)} (z - \xi_{[tt]}(z))(V \otimes M) = 0.
\]

In general for \( V \in \text{mod}_{\text{int}}(U_{\zeta}^L(g)) \) we regard \( V^* = \text{Hom}_\mathbb{C}(V, \mathbb{C}) \) as an object of \( \text{mod}_{\text{int}}(U_{\zeta}^L(g)) \) by
\[
\langle uf, v \rangle = \langle f, (Su)v \rangle \quad (u \in U_{\zeta}^L(g), \ f \in V^*, \ v \in V).
\]
For \( \lambda \in \Lambda^+ \) we call \( \nabla_{\zeta}^* = \Delta_{\zeta}^*(\lambda)^* \in \text{mod}_{\text{int}}(U_{\zeta}^L(g)) \) the dual Weyl module with highest weight \( \lambda \). There exists a non-zero homomorphism \( \Delta_{\zeta}^*(\lambda) \to \nabla_{\zeta}^*(\lambda) \) of \( U_{\zeta}^L(g) \)-modules, which is unique up to constant multiple, and its image \( L_{\zeta}(\lambda) \) is an irreducible \( U_{\zeta}^L(g) \)-module. Moreover, any irreducible integrable \( U_{\zeta}^L(g) \)-module is isomorphic to \( L_{\zeta}(\lambda) \) for some \( \lambda \in \Lambda^+ \).

Corollary 4.6. Let \( V \in \text{mod}_{\text{int}}(U_{\zeta}^L(g)) \), and let \( L_{\zeta}(\lambda_1), \ldots, L_{\zeta}(\lambda_r) \) for \( \lambda_j \in \Lambda^+ \) be its composition factors. Let \( t \in H(W(h)) \), and define a subset \( \mathcal{A} \) of \( H/W_o \) (without multiplicity) by
\[
\mathcal{A} = \bigcup_{j=1}^{r} \{[tt]_\nu \mid \nu \in \text{wt}(\Delta_{\zeta}(\lambda_j))\}.
\]
Then for \( M \in \text{mod}^k_{[t]}(U_{\zeta}(g)) \) we have \( V \otimes M \in \bigoplus_{[t] \in \mathcal{A}} \text{mod}^k_{[t]}(U_{\zeta}(g)) \).

By (4.8) we have
\[
\text{mod}(\hat{U}_{\zeta}(g))^k = \bigoplus_{[t] \in H(W(h))/W_o} \text{mod}(\hat{U}_{\zeta}(g)|_{[t]}^k),
\]
and by Corollary 4.6 we easily obtain the following.
4.7. Let $V \in \text{mod}_{\text{int}}(U_\ell^L(\mathfrak{g}))$, $t \in H(W(h))$, $A \subset H/W$ be as in Corollary 4.6. Then for $M \in \text{mod}(\hat{U}_\ell(\mathfrak{g})^L_{\mathfrak{b}})$ we have $V \otimes M \in \bigoplus_{[\ell'] \in A} \text{mod}(\hat{U}_\ell(\mathfrak{g})^L_{\mathfrak{b}})$.

5. BACK TO THE ORDINARY flag manifold

The aim of this section is to relate the quantized flag manifold $B_\zeta$ with the ordinary flag manifold $B$ using the Frobenius morphism $\text{Fr} : B_\zeta \to B$, which is a morphism in the category of non-commutative schemes.

5.1. $\mathcal{O}$-modules. In the arguments below we identify $\text{Mod}(\mathcal{O}_{B_{\zeta}})$ (resp. $\text{mod}(\mathcal{O}_{B_{\zeta}})$) with the category $\text{Mod}(\mathcal{O}_B)$ (resp. $\text{mod}(\mathcal{O}_B)$) consisting of quasi-coherent (resp. coherent) $\mathcal{O}_B$-modules, where $B = B^-\backslash G$ is the ordinary flag manifold (see Remark 3.2). Note that for $\lambda \in \Lambda$

\[ \mathcal{O}_B[\lambda] := \omega^*(A_1[\lambda]) \in \text{mod}(\mathcal{O}_{B_{\zeta}}) = \text{mod}(\mathcal{O}_B) \]

is the invertible $\mathcal{O}_B$-module corresponding to $\lambda \in \Lambda$.

Let $\text{Fr} : U_\ell^L(\mathfrak{g}) \to U(\mathfrak{g})$ be Lusztig’s Frobenius homomorphism (see [32]). It induces an embedding $\mathcal{O}(G) \hookrightarrow \mathcal{O}_\zeta(G)$ of Hopf algebras. From this we obtain an embedding $A_1 \hookrightarrow A_\zeta$ of $\mathbb{C}$-algebras. Note that we have $A_1(\lambda) \hookrightarrow A_\zeta(\ell\lambda)$ for $\lambda \in \Lambda$. Define a subalgebra $A_\zeta^{(\ell)}$ of $A_\zeta$ by $A_\zeta^{(\ell)} = \bigoplus_{\lambda \in \Lambda^+} A_\zeta(\ell\lambda)$. We regard it as a $\Lambda$-graded algebra by $A_\zeta^{(\ell)}(\lambda) = A_\zeta(\ell\lambda)$ for $\lambda \in \Lambda$. It contains $A_1$ as a $\Lambda$-graded central subalgebra. Moreover, $A_\zeta^{(\ell)}$ is finitely generated as an $A_1$-module. Applying $\omega^* : \text{mod}_A(A_1) \to \text{mod}(\mathcal{O}_{B_{\zeta}})$ to $A_\zeta^{(\ell)}$ regarded as an object of $\text{mod}_A(A_1)$, we set

$\mathcal{D} = \text{Fr}_*\mathcal{O}_{B_{\zeta}} := \omega^* A_\zeta^{(\ell)} \in \text{mod}(\mathcal{O}_{B_{\zeta}}) = \text{mod}(\mathcal{O}_B)$.

Then $\mathcal{D}$ is naturally equipped with an $\mathcal{O}_B$-algebra structure through the multiplication of $A_\zeta^{(\ell)}$. As an $\mathcal{O}_B$-module it is coherent. Moreover, the category $\text{Mod}(\mathcal{D})$ (resp. $\text{mod}(\mathcal{D})$) of quasi-coherent (resp. coherent) $\mathcal{D}$-modules is identified with the category

$\text{Mod}_A(A_\zeta^{(\ell)})/(\text{mod}_A(A_\zeta^{(\ell)}) \cap \text{Tor}_A(A_1))$

(resp. $\text{mod}_A(A_\zeta^{(\ell)})/(\text{mod}_A(A_\zeta^{(\ell)}) \cap \text{Tor}_A(A_1))$).

Let

$\underline{\omega}^* : \text{Mod}_A(A_\zeta^{(\ell)}) \to \text{Mod}(\mathcal{D})$

be the canonical functor, and let

$\underline{\omega}_* : \text{Mod}(\mathcal{D}) \to \text{Mod}_A(A_\zeta^{(\ell)})$

be its right adjoint. Let

$\text{Fr}_* : \text{Mod}(\mathcal{O}_{B_{\zeta}}) \to \text{Mod}(\mathcal{D})$

be the functor induced by

$(\bullet)^{(\ell)} : \text{Mod}_A(A_\zeta) \to \text{Mod}_A(A_\zeta^{(\ell)})$,

$M \mapsto \bigoplus_{\lambda \in \Lambda} M(\ell\lambda)$.

By definition we have $\text{Fr}_* \circ \omega^* = \underline{\omega}^* \circ (\bullet)^{(\ell)}$ on $\text{Mod}_A(A_\zeta)$. By [31] Lemma 3.9 we have also the following.
**Lemma 5.1.**  (i) The functor (5.2) gives category equivalences:
\[
\text{Mod}(\mathcal{O}_B) \cong \text{Mod}(\mathcal{O}), \quad \text{mod}(\mathcal{O}_B) \cong \text{mod}(\mathcal{O}).
\]
(ii) We have \( \bar{\omega} \circ \text{Fr} = (\bullet)^{(l)} \circ \omega \) on \( \text{Mod}(\mathcal{O}_B) \).

For \( \lambda \in \Lambda \) we set
\[
O[\lambda] = \text{Fr}^*(\omega^*(A[\lambda])) = \mathcal{F}^t(A[\lambda])^{(l)} \in \text{Mod}(\mathcal{O}).
\]
Then \( \mathcal{O}[\lambda] \) is naturally an \( \mathcal{O} \)-bimodule. Moreover, the two \( \mathcal{O}_B \)-module structures of \( \mathcal{O}[\lambda] \) obtained by restricting the left and the right \( \mathcal{O} \)-module structures coincide.

Define
\[
(\bullet)[\lambda] : \text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}) \quad (\mathcal{M} \mapsto \mathcal{M}[\lambda])
\]
to be the functor induced by (5.3). Then we have
\[
(\mathcal{M}[\lambda] \otimes R \mathcal{O}[\mu] \cong \mathcal{O}[\lambda + \mu] \quad (\lambda, \mu \in \Lambda).
\]
Define
\[
O[\lambda] = \text{Fr}^*(\omega^*(A[\lambda])) = \omega^*(A[\lambda])^{(l)} \in \text{Mod}(\mathcal{O}).
\]
As a \( \mathcal{C} \)-algebra we have
\[
\mathcal{D} = \langle \partial_u, \ell, \sigma_2 \mid u \in U_{\zeta}(\mathfrak{g}), \varphi \in \mathcal{O}, \mu \in \Lambda \rangle
\]
locally on \( \mathcal{B} \). For \( \lambda \in \Lambda \) we have a natural \( \mathcal{D} \)-module structure of \( \mathcal{O}[\lambda] \). For \( \mu \in \Lambda \) we have \( \sigma_2 \mid \mathcal{O}[\lambda] = \zeta^{2(\mu, \lambda)} \text{id} \). For \( \varphi \in \mathcal{D} \) the action of \( \ell \varphi \) on \( \mathcal{D}[\lambda] \) is given by the left \( \mathcal{D} \)-module structure of \( \mathcal{O}[\lambda] \). Finally, for \( u \in U_{\zeta}(\mathfrak{g}) \) the action of \( \partial_u \) on \( \mathcal{D}[\lambda] \) is given by the composite of \( U_{\zeta}(\mathfrak{g}) \to U_{\zeta}(\mathfrak{g}) \) with (5.8).

In \( \mathcal{D} \) we have
\[
\partial_u \ell \varphi = \sum_{(u)} \ell u(0) \varphi \partial u(1) \quad (u \in U_{\zeta}(\mathfrak{g}), \varphi \in \mathcal{D}),
\]
\[
\ell \varphi \sigma_2 \mu = \sigma_2 \mu \ell \varphi \quad (\varphi \in \mathcal{D}, \mu \in \Lambda),
\]
\[
\partial_u \sigma_2 \mu = \sigma_2 \mu \partial_u \quad (u \in U, \mu \in \Lambda).
\]
In particular, $\sigma_{2\mu}$ belongs to the center of $\mathcal{D}$. It is also easily seen that $\partial_z$ for $z \in Z_{Fr}(U_\zeta(\mathfrak{g}))$ belongs to the center of $\mathcal{D}$ (see the proof of [41, Lemma 5.1]). For $t \in H$ we set

\begin{equation}
\mathcal{D}_t := \mathcal{D}/ \sum_{\mu \in \Lambda} \mathcal{D}(\sigma_{2\mu} - \theta_\mu(t)).
\end{equation}

We denote by mod$_t(\mathcal{D})$ the category of coherent $\mathcal{D}$-module $\mathcal{M}$ such that for any $\mu \in \Lambda$ and any section $m$ of $\mathcal{M}$ we have $(\sigma_{2\mu} - \theta_\mu(t))^n m = 0$ for some $n$. We have

\begin{align*}
\text{mod}(\mathcal{D}_{B_\zeta}) & \cong \text{mod}(\mathcal{D}), \\
\text{mod}(\mathcal{D}_{B_\zeta,t}) & \cong \text{mod}(\mathcal{D}_t), \\
\text{mod}_t(\mathcal{D}_{B_\zeta}) & \cong \text{mod}_t(\mathcal{D}).
\end{align*}

5.3. Localization on $\mathcal{V}$. We set

\begin{equation}
\mathcal{V} = \{(B^{-g}, k, t) \in B \times K \times H \ | \ g\eta(k)g^{-1} \in t^\ell N^{-}\};
\end{equation}

and consider the projections

\begin{align*}
p^Y_B : \mathcal{V} & \to B \quad ((B^{-g}, k, t) \mapsto B^{-g}), \\
p^Y_H : \mathcal{V} & \to H \quad ((B^{-g}, k, t) \mapsto t), \\
p^Y_Y : \mathcal{V} & \to \mathcal{Y} \quad ((B^{-g}, k, t) \mapsto (k, t)).
\end{align*}

We define a subalgebra $\mathfrak{Z}$ of $\mathcal{D}$ by

$\mathfrak{Z} = \langle \ell_\varphi, \partial_u, \sigma_{2\mu} \ | \ \varphi \in \mathcal{O}_B, \ u \in Z_{Fr}(U_\zeta(\mathfrak{g})), \ \mu \in \Lambda \rangle \subset \mathcal{D}$.

It is contained in the center of $\mathcal{D}$. By [41] we have

\begin{equation}
\mathfrak{Z} \cong (p^Y_B)_* \mathcal{O}_\mathcal{V}.
\end{equation}

Note that $p^Y_B$ is an affine morphism. Hence localizing $\mathcal{D}$ on $\mathcal{V}$ we obtain an $\mathcal{O}_\mathcal{V}$-algebra $^\sharp \mathcal{D}$ such that

\begin{equation}
\text{Mod}(\mathcal{D}) \cong \text{Mod}(^\sharp \mathcal{D}), \quad \text{mod}(\mathcal{D}) \cong \text{mod}(^\sharp \mathcal{D}).
\end{equation}

It is easily seen that $^\sharp \mathcal{D}$ is a coherent $\mathcal{O}_\mathcal{V}$-module. We have a natural $\mathbb{C}$-algebra homomorphism

\begin{equation}
U_\zeta(\mathfrak{g}) \to ^\sharp \mathcal{D} \quad (u \mapsto \partial_u).
\end{equation}

We define subvarieties $\mathcal{V}_t$ for $t \in H$, and $\mathcal{V}^k_t$ for $(k, t) \in \mathcal{Y}$ by

\begin{equation}
\mathcal{V}_t = (p^Y_H)^{-1}(t), \quad \mathcal{V}^k_t = (p^Y_Y)^{-1}(k, t).
\end{equation}

We denote by $\mathring{\mathcal{V}}_t$ (resp. $\mathring{\mathcal{V}}^k_t$) the formal neighborhood of $\mathcal{V}_t$ (resp. $\mathcal{V}^k_t$) in $\mathcal{V}$. 
Remark 5.2. Let $Y$ be a closed subvariety of a variety $X$. In this paper we consider the formal neighborhood $\tilde{Y}$ of $Y$ in $X$ only when $Y$ is a base change of an affine closed embedding $Z' \to Z$ for a projective morphism $X \to Z$. Hence following [14] we understand $\tilde{Y}$ to be the ordinary scheme $X \times_Z \text{Spec}(\mathcal{O}(Z))$, where $\mathcal{O}(Z)$ denotes the completion of $\mathcal{O}(Z)$ at the ideal defining $Z'$ (see [14, Section 0.1]).

For $t \in H$ we define an $\mathcal{O}_{V_t}$-algebra $\mathbb{D}_t$ and an $\mathcal{O}_{\tilde{V}_t}$-algebra $\mathbb{D}_t$ by

$$\mathbb{D}_t = \mathbb{D} \otimes_{\mathcal{O}_{V_t}} \mathcal{O}_{\tilde{V}_t}, \quad \mathbb{D}_t = \mathbb{D} \otimes_{\mathcal{O}_{\tilde{V}_t}} \mathcal{O}_{V_t}.$$ 

Then (5.29) induces $\mathbb{C}$-algebra homomorphisms

$$(5.29) \quad U_\zeta(g)_{|t} \to \mathbb{D}_t, \quad \tilde{U}_\zeta(g)_{|t} \to \mathbb{D}_t.$$ 

Since $\sigma_{2\mu} - \theta_\mu(t)$ for $\mu \in \Lambda$ generate the defining ideal of $V_t$ in $V$ we have

$$(5.30) \quad \text{mod}(\mathcal{D}_t) \cong \text{mod}(\mathbb{D}_t).$$

We denote by $\text{mod}_t(\mathbb{D})$ the category of coherent $\mathbb{D}$-modules supported on $V_t$. Then we have

$$(5.31) \quad \text{mod}_t(\mathbb{D}) \cong \text{mod}(\mathbb{D}).$$

For $(k, t) \in \mathcal{Y}$ we set

$$\mathbb{D}^k_t = \mathbb{D} \otimes_{\mathcal{O}_{V_t}} \mathcal{O}_{\tilde{V}_t}, \quad \tilde{\mathbb{D}}^k_t = \mathbb{D} \otimes_{\mathcal{O}_{\tilde{V}_t}} \mathcal{O}_{V_t}.$$ 

They are an $\mathcal{O}_{V_t}$-algebra and an $\mathcal{O}_{\tilde{V}_t}$-algebra respectively. We denote by $\text{mod}_t^k(\mathbb{D})$ the category of coherent $\mathbb{D}$-modules supported on $V_t^k$. Then we have

$$\text{mod}(\mathbb{D}^k_t) \subset \text{mod}_t^k(\mathbb{D}) \subset \text{mod}(\tilde{\mathbb{D}}^k_t).$$

Set

$$(5.32) \quad \mathcal{B}^k_t = \{ B^{-} g \in \mathcal{B} \mid g\eta(k)g^{-1} \in t^{f}N^{-} \} \quad ((k, t) \in \mathcal{Y}).$$

Note that we have a natural isomorphism $\mathcal{V}^k_t \cong \mathcal{B}^k_t$ for $(k, t) \in \mathcal{Y}$. Since the variety $\mathcal{B}^k_t$ depends only on $(k, t^f)$, we have a natural identification

$$(5.33) \quad \mathcal{V}^k_{td} \cong \mathcal{B}^k_t \quad ((k, t) \in \mathcal{Y}, \ d \in H_e).$$

Similarly we have

$$(5.34) \quad \tilde{\mathcal{V}}^k_{td} \cong \tilde{\mathcal{B}}^k_t \quad (d \in H_e).$$

Here, $\tilde{\mathcal{B}}^k_t$ is the formal neighborhood of $\mathcal{B}^k_t$ naturally embedded in the smooth subvariety

$$(5.35) \quad \tilde{\mathcal{G}}^t = \{(B^{-} g, k') \in \mathcal{B} \times K \mid g\eta(k')g^{-1} \in B^{-}\}$$

of $\mathcal{B} \times K$. We will sometimes regard $\mathbb{D}^k_t$ and $\tilde{\mathbb{D}}^k_t$ as an $\mathcal{O}_{\mathcal{B}^k_t}$-algebra and an $\mathcal{O}_{\tilde{\mathcal{B}}^k_t}$-algebra respectively.
5.4. Azumaya property. Let us recall the main results of [41]. In [41] it is proved using a result of [15] that \( \hat{\mathcal{D}} \) is an Azumaya algebra of rank \( \ell^{2|\Delta^+|} \) on \( \mathcal{V} \) if the following conditions are satisfied

(c1) \( \ell \) is prime to 3 if \( G \) is of type \( F_4, E_6, E_7, E_8 \);
(c2) \( \ell \) is prime to 5 if \( G \) is of type \( E_8 \).

Namely, \( \hat{\mathcal{D}} \) is a locally free \( \mathcal{O}_\mathcal{V} \)-module of rank \( \ell^{2|\Delta^+|} \), and for any \( v \in \mathcal{V} \) the fiber \( \hat{\mathcal{D}}(v) \) is isomorphic to the matrix algebra \( M_{\ell|\Delta^+|}(\mathbb{C}) \). Moreover, it is also shown that \( \hat{\mathcal{D}}^k \) and \( \hat{\mathcal{D}}^{k,\ell} \) are split Azumaya algebras for any \((k, t) \in \mathcal{V}\) under the same conditions (c1), (c2). Namely, there exists a locally free \( \mathcal{O}_{\mathcal{V}^k} \)-module \( \hat{\mathcal{R}} \) (resp. a locally free \( \mathcal{O}_{\mathcal{V}^{k,\ell}} \)-module \( \hat{\mathcal{R}} \)) of rank \( \ell^{|\Delta^+|} \) such that \( \hat{\mathcal{D}}^k \cong \mathcal{E}_{\text{nd}\mathcal{O}_{\mathcal{V}^k}}(\hat{\mathcal{R}}) \) (resp. \( \hat{\mathcal{D}}^{k,\ell} \cong \mathcal{E}_{\text{nd}\mathcal{O}_{\mathcal{V}^{k,\ell}}}(\hat{\mathcal{R}}) \)).

We now discuss the choice of the splitting bundles \( \hat{\mathcal{R}} \) and \( \hat{\mathcal{R}} \) of \( \hat{\mathcal{D}}^k \) and \( \hat{\mathcal{D}}^{k,\ell} \) for \((k, t) \in \mathcal{V} \) respectively. Note that we have a natural algebra homomorphism

\[
\mathcal{O}_\mathcal{V} \otimes_{Z(U_\mathcal{V}(g))} U_\mathcal{X}(g) \to \hat{\mathcal{D}}
\]

given by \( Z(U_\mathcal{V}(g)) \cong \mathcal{O}(\mathcal{X}) \to \mathcal{O}_\mathcal{V} \) corresponding to

\[
\mathcal{V} \to \mathcal{X} \quad ((B^{-g}, k, t) \mapsto (k, [t])).
\]

Let \((k, t) \in \mathcal{V}\). We first deal with the case where \( t \in \mathcal{H}_{ur} \). By using a result of [15] we proved in [41] that the restriction of (5.36) to the open subset \( (\mathcal{V}^\mathcal{H})^{-1}(\mathcal{H}_{ur}) \) of \( \mathcal{V} \) is an isomorphism. Hence we have

\[
\hat{\mathcal{D}}^k \cong \mathcal{O}_{\mathcal{V}^k} \otimes_{\mathbb{C}} U_\mathcal{X}(g)^k_{[t]}, \quad \hat{\mathcal{D}}^{k,\ell} \cong \mathcal{O}_{\mathcal{V}^{k,\ell}} \otimes_{\mathcal{O}(\mathcal{X}^k_{[t]})} U_\mathcal{X}(g)^k_{[t]},
\]

where \( \mathcal{O}(\mathcal{X})_{[t]} \) denotes the completion of \( \mathcal{O}(\mathcal{X}) \) at the maximal ideal corresponding to the point \((k, [t]) \in \mathcal{X}\). By [15] \( U_\mathcal{X}(g)^k_{[t]} \) is isomorphic to the matrix algebra \( M_{\ell|\Delta^+|}(\mathbb{C}) \) so that there exists a unique irreducible \( U_\mathcal{X}(g)^k_{[t]} \)-module \( K^k_{[t]} \). It follows that \( \hat{\mathcal{D}}^k \) is a split Azumaya algebra with splitting bundle

\[
\hat{\mathcal{R}}^k_t = \mathcal{O}_{\mathcal{V}^k} \otimes_{\mathbb{C}} K^k_{[t]},
\]

where we identify \( \mathcal{V}^k \) with \( \mathcal{B}^k_t \) via (5.33). Moreover, since \( \mathcal{O}(\mathcal{X})_{[t]} \) is a complete local ring, there exists a \( U_\mathcal{X}(g)^k_{[t]} \)-module \( \hat{K}^k_{[t]} \) which is free over \( \mathcal{O}(\mathcal{X})_{[t]} \) such that \( \mathbb{C} \otimes_{\mathcal{O}(\mathcal{X})_{[t]}} \hat{K}^k_{[t]} \cong K^k_{[t]} \) and \( \hat{U}_\mathcal{X}(g)^k_{[t]} \cong \text{End}_{U_\mathcal{X}(g)^k_{[t]}}(\hat{K}^k_{[t]}) \). Hence \( \hat{\mathcal{D}}^{k,\ell} \) is a split Azumaya algebra with splitting bundle

\[
\hat{\mathcal{R}}^{k,\ell}_{t} = \mathcal{O}_{\mathcal{V}^{k,\ell}} \otimes_{\mathcal{O}(\mathcal{X}^k_{[t]})} \hat{K}^k_{[t]}.
\]

Now we consider the case where \( t \in \mathcal{H} \) is not necessarily contained in \( \mathcal{H}_{ur} \). By [41] (see [41] Proposition 6.12 and its proof) we have the following.

**Theorem 5.3.** Let \( k \in K \) and \( \ell \in H_{ur} \) such that \((k, \ell) \in \mathcal{Y}\), and let \( \lambda \in \Lambda \). Then \( \hat{\mathcal{D}}^k_{tt,\lambda} \) and \( \hat{\mathcal{D}}^{k,\ell}_{tt,\lambda} \) are split Azumaya algebras with splitting bundles

\[
\hat{\mathcal{R}}^k_{t}[\lambda] = \mathcal{O}[\lambda] \otimes_{\mathcal{O}} \hat{\mathcal{R}}^k_{t}, \quad \hat{\mathcal{R}}^{k,\ell}_{t}[\lambda] = \mathcal{O}[\lambda] \otimes_{\mathcal{O}} \hat{\mathcal{R}}^{k,\ell}_{t}
\]

respectively. Here, the \( \mathcal{O} \)-module structures of \( \hat{\mathcal{R}}^k_t \) and \( \hat{\mathcal{R}}^{k,\ell}_t \) are given by \( \mathcal{O} \to \hat{\mathcal{D}} \).
Remark 5.5. The splitting bundle $R_k^b[\lambda]$ for $^t\mathcal{D}_t^k$ depends on the choice of $\lambda$. We have $t_\lambda = t_{\lambda + \ell\nu}$ for $\lambda, \nu \in \Lambda$ and

$$R_k^b[\lambda + \ell\nu] = \mathcal{D}[\ell\nu] \otimes O R_k^b[\lambda] \cong O_B[\nu] \otimes O G R_k^b[\lambda].$$

By the Morita equivalence we obtain the following.

Theorem 5.6. For $(k, \tilde{t}) \in K \times_{H/W} H_{ur}$ and $\lambda \in \Lambda$ we have

$$\text{mod}(^t\mathcal{D}_t^k) \cong \text{mod}(O_{B^k_t}) \quad (R_k^b[\lambda] \otimes O_{B^k_t} S \longleftrightarrow S),$$

$$\text{mod}(^t\mathcal{D}_t^k) \cong \text{mod}(O_{B^k_t}) \quad (\hat{R}_k^b[\lambda] \otimes O_{B^k_t} S \longleftrightarrow S).$$

Denote by $\text{mod}_{B^k_t}(O_{G'})$ the category of coherent $O_{G'}$-modules supported on $B^k_t$. By (5.39) we have also the following

$$\text{mod}_{B^k_t}(^t\mathcal{D}) \cong \text{mod}_{B^k_t}(O_{G'}).$$

6. Equivalence of derived categories

6.1. Choice of $\tilde{k} \in K$. As noted in Section 4, $U_\zeta(g)^k$ for $k \in K$ depends only on the conjugacy class of $G$ containing $\eta(k)$. Hence in the representation theory of $U_\zeta(g)$ we can select for each conjugacy class $O$ of $G$ any single $k$ satisfying $\eta(k) \in O$ and restrict ourselves to the investigation of $U_\zeta(g)^k$-modules. It is well-known that any conjugacy class of $G$ contains an element $g$ such that $g_s \in H$, $g_u \in N^+$ and there exists a maximal torus of the centralizer $Z_G(g)$ contained in $H$ (see [23, 22.3]). In the rest of this paper we fix $\tilde{k} \in K$, $\tilde{g} \in G$, $\tilde{h} \in H$, $\tilde{x} \in N^+$ such that

(k1) $\eta(\tilde{k}) = \tilde{g} = \tilde{h} \tilde{x} = \tilde{x} \tilde{h},$

(k2) there exists a maximal torus of $Z_G(\tilde{g})$ contained in $H$,

and restrict ourselves to the case where the Frobenius central character is given by $\xi^k$. Note that we can write $\tilde{k} = (\tilde{x} \tilde{h}^{1/2}, \tilde{h}^{-1/2})$, where $\tilde{h}^{1/2} \in H$ satisfies $(\tilde{h}^{1/2})^2 = \tilde{h}$. On the other hand by the theory of parabolic induction (see [19]) we are further reduced to the case where $\tilde{h}$ is exceptional. Here, a semisimple element $h$ of $G$ is called exceptional if the semisimple rank of $Z_G(h)$ is the same as that of $G$. Hence we further assume

(k3) $\tilde{h}$ is exceptional

in the following. We set $\tilde{G} = Z_G(\tilde{h})$. It is connected since $G$ is chosen to be simply-connected. Hence $\tilde{G}$ is a connected semisimple algebraic group with maximal torus $H$. We denote by $\tilde{\Delta} \subset \Delta$ the root system of $G$. We set $\Delta^+ = \Delta^+ \cap \Delta, Q = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha$. We denote by $W$ the Weyl group of $\tilde{\Delta}$.
6.2. Equivalences. In [15] we have proved Conjecture 3.7 in certain cases. In particular, it is true when the regular element \( t \in H \) satisfies \( t \in H(\hat{h}) \) for \( \hat{h} \) as above, and \( \ell \) satisfies the conditions:

\((k4)\) \( \ell \) is prime to \( |Q/\hat{Q}| \),
\((p)\) \( \ell \) is a power of a prime number

in addition to the conditions (a1), (a2), (a3). In the rest of this paper we also assume the conditions (k4), (p).

Recall that \( H(\hat{h}) \) consists of elements of the form \( \bar{t}d \) for \( \bar{t} \in H_{ur}(\hat{h}) \) and \( d \in H_\ell \). We fix \( \bar{t} \in H_{ur}(\hat{h}) \) in the following. Then \( td \) for \( d \in H_\ell \) is regular if and only if \( d \) is regular with respect to the action of \( \hat{W} \) on \( H_{\ell} \), i.e.

\[(6.1)\]

\[w \in \hat{W},\ w d = d \implies w = 1.\]

By (5.27), (5.29) we have left exact functors

\[(6.2)\]
\[\Gamma : \text{mod}(\mathcal{D}_{td}) \to \text{mod}(U_\zeta(g)_{[td]}),\]

\[(6.3)\]
\[\Gamma : \text{mod}_{td}(\mathcal{D}) \to \text{mod}_{[td]}(U_\zeta(g)),\]

\[(6.4)\]
\[\Gamma : \text{mod}(\widehat{\mathcal{D}}_{td}) \to \text{mod}(\widehat{U}_\zeta(g)_{[td]})\]

We have also right exact functors

\[(6.5)\]
\[\text{Loc}_{td} : \text{mod}(U_\zeta(g)_{[td]}) \to \text{mod}(\mathcal{D}_{td}),\]

\[(6.6)\]
\[\text{id}_d \text{Loc} : \text{mod}_{[td]}(U_\zeta(g)) \to \text{mod}_{td}(\mathcal{D}),\]

\[(6.7)\]
\[\text{Loc}_{td} : \text{mod}(\widehat{U}_\zeta(g)_{[td]}) \to \text{mod}(\widehat{\mathcal{D}}_{td})\]

in the opposite directions. Here, (6.5) and (6.7) are defined by

\[\text{Loc}_{td}(M) = \mathcal{D}_{td} \otimes_{U_\zeta(g)_{[td]}} M, \quad \text{Loc}_{td}(M) = \mathcal{D}_{td} \otimes_{U_\zeta(g)_{[td]}} M\]

respectively. For \( M \in \text{mod}_{[td]}(U_\zeta(g)) \) we have

\[\mathcal{D} \otimes_{U_\zeta(g)} M \in \bigoplus_{\ell' \in W_{\circ}(td)} \text{mod}_{\ell'}(\mathcal{D}),\]

and \( \text{id}_d \text{Loc}(M) \) is the component of \( \mathcal{D} \otimes_{U_\zeta(g)} M \) belonging to \( \text{mod}_{td}(\mathcal{D}).\)

**Theorem 6.1.** Assume that \( d \in H_\ell \) is regular in the sense of (6.1). Then the right derived functors of (6.2), (6.3), (6.4) and the left derived functors of (6.5), (6.6), (6.7) induce equivalences

\[(6.8)\]
\[D^b(\text{mod}(U_\zeta(g)_{[td]})) \simeq D^b(\text{mod}(\mathcal{D}_{td})),\]

\[(6.9)\]
\[D^b(\text{mod}_{[td]}(U_\zeta(g))) \simeq D^b(\text{mod}_{td}(\mathcal{D})),\]

\[(6.10)\]
\[D^b(\text{mod}(\widehat{U}_\zeta(g)_{[td]})) \simeq D^b(\text{mod}(\widehat{\mathcal{D}}_{td}))\]

of triangulated categories.

**Proof.** The first assertion (6.8) is already proved in [15] in view of (5.18) and (5.30). The main part in the argument of [15], which is used to establish (6.8), is to show \( R\Gamma(\mathcal{D}_t) \cong U_\zeta(g)_{[t]} \) for \( t \in H(\hat{h}) \). The proof of (6.10) is similar to that of
We need $R\Gamma(\hat{\mathcal{D}_t}) \cong \hat{U}_t(\mathfrak{g})|_{[t]}$ instead of $R\Gamma(\hat{\mathcal{D}_t}) \cong U_t(\mathfrak{g})|_{[t]}$. This also holds since we actually proved in [45] a stronger result
\[ R\Gamma(\mathcal{J}_t^*(\hat{\mathcal{D}})) \cong U_t(\mathfrak{g}) \otimes_{\mathcal{O}(H/W_\circ)} \mathcal{O}(H/W_\circ)|_{[t]} \]
Here $\mathcal{O}(H)_{[t]}$ and $\mathcal{O}(H/W_\circ)|_{[t]}$ denote the localizations of $\mathcal{O}(H)$ and $\mathcal{O}(H/W_\circ)$ at the maximal ideals corresponding to the points $t \in H$ and $[t] \in H/W_\circ$ respectively, and
\[ j_t : \mathcal{V} \times_H \text{Spec}(\mathcal{O}(H)_{[t]}) \to \mathcal{V} \]
is the natural morphism (see [45, Theorem 8.1]).

The assertion (6.9) follows easily from (6.10) since both sides of (6.9) are subcategories of those of (6.10). \qed

**Remark 6.2.** We suppose Theorem 6.1 holds without assuming the condition (p) (see Conjecture 3.7). If this is the case, then the arguments in the rest of this paper also work without assuming (p). We also note that we can show Theorem 6.1 without assuming (p) when $\mathfrak{g}$ is of type $A$.

For $d \in H_t$ we denote by $\text{mod}^k(\hat{\mathcal{D}}_{td})$ (resp. $\text{mod}^k_{td}(\hat{\mathcal{D}})$) the category consisting of coherent $\hat{\mathcal{D}}_{td}$-modules (resp. $\hat{\mathcal{D}}$-modules) supported on $\mathcal{V}_{td}^k$. Taking into account of the action of $Z_{Fr}(\hat{U}_t(\mathfrak{g}))$ we also obtain the following.

**Theorem 6.3.** Assume that $d \in H_t$ is regular in the sense of (6.1). Then we have equivalences
\begin{align}
(6.11) \quad & D^b(\text{mod}^k(U_t(\mathfrak{g})|_{[td]})) \cong D^b(\text{mod}^k(\hat{\mathcal{D}}_{td})), \\
(6.12) \quad & D^b(\text{mod}^k(U_t(\mathfrak{g}))) \cong D^b(\text{mod}^k_{td}(\hat{\mathcal{D}})), \\
(6.13) \quad & D^b(\text{mod}(\hat{U}_t(\mathfrak{g})_{[td]})) \cong D^b(\text{mod}(\hat{\mathcal{D}}_{td}^k))
\end{align}
of triangulated categories.

**7. Torus action**

**7.1. Torus $C$.** For $\lambda \in \Lambda$ we define $s_\lambda \in H$ by $\theta_\mu(s_\lambda) = \zeta(\lambda,\mu)$ for $\mu \in \Lambda$. Then we have $s_\lambda^2 = t_\lambda$. Let $\hat{s} \in H$ be the unique element satisfying $\hat{s}^2 = \hat{t}$, $\hat{s} = \hat{h}^{1/2}$. For $\lambda \in \hat{Q}$ we set $\hat{s}(\lambda) = \hat{s}s_\lambda$.

We set
\[ C = Z_H(\hat{\mathfrak{g}})^0 = Z_H(\hat{x})^0. \]
By our assumption (k2) $C$ is a maximal torus of $Z_G(\hat{\mathfrak{g}}) = Z_G(\hat{x})$. Define a subset $\Gamma$ of $\Delta^+$ by
\[ \hat{x} = \exp(\hat{a}), \quad \hat{a} = \sum_{\beta \in \Gamma} a_\beta, \quad a_\beta \in \mathfrak{g}_\beta \setminus \{0\}, \]
and set
\[ \Lambda'_C = \Lambda \cap \left( \sum_{\beta \in \Gamma} \mathbb{Q}_\beta \right), \quad \Lambda_C = \Lambda / \Lambda'_C. \]
We denote by \( \lambda \mapsto \overline{\lambda} \) the canonical map \( \Lambda \to \Lambda_C \). For \( \gamma = \overline{\lambda} \in \Lambda_C \) we set \( \theta_C^\gamma = \theta_\lambda|_C \). Then the character group of the torus \( C \) is identified with \( \Lambda_C \) by the correspondence \( \gamma \leftrightarrow \theta_C^\gamma \). We set

\[
(7.2) \quad C_\ell = \{ h \in C \mid h^\ell = 1 \},
\]

\[
(7.3) \quad Q_C = \{ \lambda \in Q \mid (\lambda, \Lambda_C^\prime) = \{0\} \}.
\]

We see easily that \( s_\lambda \in C_\ell \) for \( \lambda \in Q_C \).

**Lemma 7.1.** The group homomorphism \( Q_C \to C_\ell (\lambda \mapsto s_\lambda) \) induces an isomorphism

\( Q_C / \ell Q_C \cong C_\ell \)

of abelian groups.

**Proof.** Set \( Q_C^\prime = \{ \gamma \in Q^\prime \mid (\gamma, \Lambda_C^\prime) = 0 \} \). It is the lattice dual to \( \Lambda_C \) so that we have \( Q_C^\prime / \ell Q_C^\prime \cong C_\ell \). Hence it is sufficient to show that the natural map \( Q_C / \ell Q_C \to Q_C^\prime / \ell Q_C^\prime \) induced by \( Q_C \subset Q_C^\prime \) is bijective. This follows from our assumption on \( \ell \). \( \square \)

### 7.2. Graded modules.

Note that \( U_\zeta(\mathfrak{g}) \) is equipped with the natural weight space decomposition \( U_\zeta(\mathfrak{g}) = \bigoplus_{\lambda \in \Lambda} U_\zeta(\mathfrak{g})_{\lambda} \) such that \( U_\zeta(\mathfrak{h}) \subset U_\zeta(\mathfrak{g})_0 \), \( e_i \in U_\zeta(\mathfrak{g})_{\alpha_i} \), \( f_i \in U_\zeta(\mathfrak{g})_{-\alpha_i} \). We define a \( \Lambda_C \)-grading

\[
(7.4) \quad U_\zeta(\mathfrak{g}) = \bigoplus_{\gamma \in \Lambda_C} U_\zeta(\mathfrak{g})_{\gamma}
\]

of the algebra \( U_\zeta(\mathfrak{g}) \) by \( U_\zeta(\mathfrak{g})_{\gamma} = \bigoplus_{\lambda \in \Lambda, \overline{\lambda} = \gamma} U_\zeta(\mathfrak{g})_{\lambda} \) for \( \gamma \in \Lambda_C \), where the natural map \( Q \to \Lambda_C \) is denoted by \( \lambda \mapsto \overline{\lambda} \). We say that a \( U_\zeta(\mathfrak{g}) \)-module \( M \) is \( \Lambda_C \)-graded if we are given a grading

\[
(7.5) \quad M = \bigoplus_{\gamma \in \Lambda_C} M(\gamma)
\]

of \( M \) such that

\[
(7.6) \quad U_\zeta(\mathfrak{g})_{\gamma} M(\gamma') \subset M(\gamma + \gamma') \quad (\gamma, \gamma' \in \Lambda_C).
\]

**Lemma 7.2.** The \( \Lambda_C \)-grading \( (7.4) \) of \( U_\zeta(\mathfrak{g}) \) induces \( \Lambda_C \)-gradings of \( U_\zeta(\mathfrak{g})^k \) and \( U_\zeta(\mathfrak{g})^k \) for \( d \in H_\ell \).

**Proof.** Note \( U_\zeta(\mathfrak{g})^k = U_\zeta(\mathfrak{g}) / \mathcal{I} \), where \( \mathcal{I} \) is the ideal of \( U_\zeta(\mathfrak{g}) \) generated by \( e_\beta^k \), \( Sf_\beta - c_\beta \) for \( \beta \in \Delta^+ \), and \( k_{\lambda} - d_{\lambda} \) for \( \lambda \in \Lambda \). Here, \( c_\beta \in \mathbb{C}, \ d_{\lambda} \in \mathbb{C}^* \) are determined from \( k \). Hence in order to verify the assertion for \( U_\zeta(\mathfrak{g})^k \) it is sufficient to show that if \( c_\beta \neq 0 \) for \( \beta \in \Delta^+ \), we have \( \beta \in \text{Ker}(\Lambda \to \Lambda_C) \). This is easily seen from the definition of \( \xi^k \). The assertion for \( U_\zeta(\mathfrak{g})^k \) follows from \( Z_{\text{Harm}}(U_\zeta(\mathfrak{g})) \subset U_\zeta(\mathfrak{g})_0 \subset U_\zeta(\mathfrak{g})(0) \). \( \square \)

The \( \Lambda_C \)-gradings of \( U = U_\zeta(\mathfrak{g}), U_\zeta(\mathfrak{g})^k, U_\zeta(\mathfrak{g})^k \) determine left \( C \)-actions on them given by

\[
 s \cdot u = \theta_C^\gamma (s) u \quad (s \in C, \ \gamma \in \Lambda_C, \ u \in U(\gamma)).
\]

Correspondingly we have coactions \( U \to U \otimes \mathcal{O}(C) \).
Lemma 7.3. For \( M \in \text{mod}(U_{\zeta}(\mathfrak{g})^k) \) we have a simultaneous eigenspace decomposition

\[
M = \bigoplus_{\gamma \in \Lambda_C/\ell\Lambda_C} M(\langle \gamma \rangle),
\]

\[
M(\langle \gamma \rangle) = \{ m \in M \mid k_{\nu}m = \theta_{\nu}(\hat{s})\theta^C_{\gamma}(s_{\nu})m \ (\nu \in Q_C) \} \quad (\gamma \in \Lambda_C)
\]

with respect to the action of \( \{k_{\nu}\}_{\nu \in Q_C} \) on \( M \).

Proof. Since \( U_{\zeta}(\mathfrak{g})^k \) is finite-dimensional, \( M \) is finite-dimensional. By \( M \in \text{mod}(U_{\zeta}(\mathfrak{g})^k) \) we have \( k_{\nu}^2 - \theta_{\nu}(h^{1/2}) = 0 \) for any \( \nu \in \Lambda \) as an operator on \( M \). In particular, the action of \( k_{\nu} \) on \( M \) is diagonalizable. It follows that the action of \( \{k_{\nu}\}_{\nu \in \Lambda} \) on \( M \) is simultaneously diagonalizable since \( U_{\zeta}(\mathfrak{h}) \) is commutative. Hence we obtain

\[
M = \bigoplus_{h \in H} M(h), \quad M(h) = \{ m \in M \mid k_{\nu}m = \theta_{\nu}(h)m \ (\nu \in \Lambda) \}.
\]

For \( m \in M(h) \) we have

\[
\theta_{\nu}(h^\ell)m = \theta_{\nu}(h)k_{\nu}m = k_{\nu}(h^{1/2})m.
\]

for any \( \nu \in \Lambda \), and hence we have \( h^\ell = h^{1/2} \) if \( M(h) \neq 0 \). Therefore, by \( \{h \in H \mid h^\ell = h^{1/2}\} = \{\hat{s}s_{\lambda} \mid \lambda \in Q\} \) we obtain \( M = \bigoplus_{\lambda \in Q} M(\hat{s}s_{\lambda}) \), which gives the simultaneous eigenspace decomposition of \( M \) with respect to the action of \( \{k_{\nu}\}_{\nu \in \Lambda} \) on \( M \).

We restrict this to the action of \( \{k_{\nu}\}_{\nu \in Q_C} \). Assume \( \nu \in Q_C \). The we have

\[
k_{\nu}|_{M(\hat{s}s_{\lambda})} = \theta_{\nu}(\hat{s})\theta_{\nu}(s_{\lambda}) \text{id} = \theta_{\nu}(\hat{s})\theta_{\lambda}(s_{\nu}) \text{id} = \theta_{\nu}(\hat{s})\theta^C_{\gamma}(s_{\nu}) \text{id}
\]

for \( \lambda \in Q \). Moreover, for \( \lambda, \mu \in Q \) we have \( \theta_{\nu}(\hat{s})\theta^C_{\gamma}(s_{\nu}) = \theta_{\nu}(\hat{s})\theta^C_{\mu}(s_{\nu}) \) for any \( \nu \in Q_C \) if and only if \( \theta^C_{\lambda - \mu}(C) = 1 \). This condition is equivalent to \( \lambda - \mu \in \text{Ker}(Q \rightarrow \Lambda_C/\ell\Lambda_C) \).

It remains to show

\[
Q/\text{Ker}(Q \rightarrow \Lambda_C/\ell\Lambda_C) \cong \Lambda_C/\ell\Lambda_C.
\]

This follows from the surjectivity of the composite of \( Q \hookrightarrow \Lambda \rightarrow \Lambda_C \rightarrow \Lambda_C/\ell\Lambda_C \), which is a consequence of (a2). \( \square \)

We denote by \( \text{mod}(U_{\zeta}(\mathfrak{g})^k; C) \) the category of \( \Lambda_C \)-graded \( U_{\zeta}(\mathfrak{g}) \)-modules \( M \) contained in \( \text{mod}(U_{\zeta}(\mathfrak{g})^k) \) such that

\[
\nu \in Q_C, \ \gamma \in \Lambda_C, \ m \in M(\langle \gamma \rangle) \implies k_{\nu}m = \theta_{\nu}(\hat{s})\theta^C_{\gamma}(s_{\nu})m.
\]

We also denote by \( \text{mod}^k(U_{\zeta}(\mathfrak{g}); C) \) the category of \( \Lambda_C \)-graded \( U_{\zeta}(\mathfrak{g}) \)-modules \( M \) contained in \( \text{mod}^k(U_{\zeta}(\mathfrak{g})) \) such that

\[
\nu \in Q_C, \ \gamma \in \Lambda_C, \ m \in M(\langle \gamma \rangle) \implies \exists n \text{ s.t. } (k_{\nu} - \theta_{\nu}(\hat{s})\theta^C_{\gamma}(s_{\nu}))^n m = 0.
\]

Let \( d \in H_C \). By Lemma 7.2 the torus \( C \) acts on \( \hat{U}_{\zeta}(\mathfrak{g})^k \). Correspondingly we have a coaction

\[
\hat{U}_{\zeta}(\mathfrak{g})^k \rightarrow \hat{U}_{\zeta}(\mathfrak{g})_{[id]}^k \hat{\otimes} \mathcal{O}(C)
\]

where

\[
\hat{U}_{\zeta}(\mathfrak{g})^k_{[id]} \hat{\otimes} \mathcal{O}(C) = \lim_{n \to \infty} (U_{\zeta}(\mathfrak{g}) \otimes \mathcal{O}(C))/(\text{Ker}(\xi_{[id]}^k)^n U_{\zeta}(\mathfrak{g}) \otimes \mathcal{O}(C)).
\]
For a $\hat{U}_\zeta(\mathfrak{g})^k_{[id]}$-module $M$ we set
\[ M \otimes \mathcal{O}(C) = \lim_{\longrightarrow} (M \otimes \mathcal{O}(C))/((\text{Ker}(\xi^k_{[id]}))^n M \otimes \mathcal{O}(C)). \]

We denote by mod($\hat{U}_\zeta(\mathfrak{g})^k_{[id]}$; $C$) the category consisting of finitely generated $\hat{U}_\zeta(\mathfrak{g})^k_{[id]}$-module $M$ endowed with a right coaction $M \to M \otimes \mathcal{O}(C)$ of $\mathcal{O}(C)$ on $M$ such that for any $\nu \in Q_C$ and $n > 0$ the linear endomorphism $f_{\nu,n} \in \text{End}_\mathbb{C}(M/(\text{Ker}(\xi^k_{[id]}))^n M)$ induced by $f_{\nu} = \sigma(k_\nu) - \theta_\nu(s)\rho(s_\nu) \in \text{End}_\mathbb{C}(M)$ is nilpotent. Here, $\rho(s)$ for $s \in C$ denotes the action of $s$ on $M$ induced by the coaction of $\mathcal{O}(C)$ on $M$, and $\sigma(u)$ for $u \in U_\zeta(\mathfrak{g})$ denotes the action of $u$ on $M$ coming from the $\hat{U}_\zeta(\mathfrak{g})^k_{[id]}$-module structure.

7.3. Equivariant $\mathcal{O}$-modules. We have natural right actions of $C$ on $B$, $K$ and $\mathcal{V}$ given by

\begin{align*}
(7.9) & \quad B^{-g} \cdot h = B^{-gh} \quad (B^{-g} \in B), \\
(7.10) & \quad (x_1, x_2) \cdot h = (h^{-1}x_1h, h^{-1}x_2h) \quad ((x_1, x_2) \in K), \\
(7.11) & \quad (B^{-g}, k, t) \cdot h = (B^{-g} \cdot h, k \cdot h, t) \quad ((B^{-g}, k, t) \in \mathcal{V})
\end{align*}

for $h \in C$. We will also use the following right actions of $C$ on the same spaces given by the $\ell$-th powers:

\begin{align*}
(7.12) & \quad B^{-g} \star h = B^{-gh^\ell} \quad (B^{-g} \in B), \\
(7.13) & \quad (x_1, x_2) \star h = (h^{-\ell}x_1h^\ell, h^{-\ell}x_2h^\ell) \quad ((x_1, x_2) \in K), \\
(7.14) & \quad (B^{-g}, k, t) \star h = (B^{-g} \star h, k \star h, t) \quad ((B^{-g}, k, t) \in \mathcal{V})
\end{align*}

for $h \in C$.

Let $A_F = \bigoplus_{\lambda \in \Lambda} (A_F)_\lambda$ be the weight space decomposition of $A_F$ where $A_F$ is regarded as a $U_F(\mathfrak{h})$-module by restricting the left $U_F(\mathfrak{g})$-module structure. We define a $\Lambda_C$-grading $A_F = \bigoplus_{\gamma \in \Lambda_C} A_F(\gamma)$ of $A_F$ by $A_F(\gamma) = \bigoplus_{\lambda \in \Lambda, \gamma = \gamma} (A_F)_\lambda$. It induces a $\Lambda_C$-grading $A_\lambda = \bigoplus_{\gamma \in \Lambda_C} A_\lambda(\gamma)$ of $A_\lambda$ given by $A_\lambda(\gamma) = A_\lambda(\gamma) \cap A_\lambda$. Moreover, we have a $\Lambda_C$-grading $D_\lambda = \bigoplus_{\gamma \in \Lambda_C} D_\lambda(\gamma)$ of $D_\lambda$ given by $D_\lambda(\gamma) = \{ P \in D_\lambda \mid P(A_\lambda(\gamma')) \subset A_\lambda(\gamma + \gamma') \ (\gamma' \in \Lambda_C) \}$.

By the specialization $q \mapsto \zeta$ we obtain $\Lambda_C$-gradings

\begin{align*}
(7.15) & \quad A_\zeta = \bigoplus_{\gamma \in \Lambda_C} A_\zeta(\gamma), \quad D_\zeta = \bigoplus_{\gamma \in \Lambda_C} D_\zeta(\gamma)
\end{align*}

of $A_\zeta$, $D_\zeta$ respectively. We have $\partial_u, \ell_\varphi \in D_\zeta(\gamma)$ for $u \in U_\zeta(\mathfrak{g})(\gamma)$, $\varphi \in A_\zeta(\gamma)$, and $\sigma_2\lambda \in D_\zeta(0)$ for $\lambda \in \Lambda$. Note that the original $\Lambda$-gradings $A_\zeta = \bigoplus_{\lambda \in \Lambda} A_\zeta(\lambda)$, $D_\zeta = \bigoplus_{\lambda \in \Lambda} D_\zeta(\lambda)$ are compatible with the new $\Lambda_C$-gradings in the sense

\begin{align*}
A_\zeta = \bigoplus_{\lambda \in \Lambda_C, \gamma \in \Lambda_C} A_\zeta(\lambda) \cap A_\zeta(\gamma), \quad D_\zeta = \bigoplus_{\lambda \in \Lambda_C, \gamma \in \Lambda_C} D_\zeta(\lambda) \cap D_\zeta(\gamma).
\end{align*}

From (7.15) we obtain left actions of the torus $C$ on the $\mathbb{C}$-algebras $A_\zeta$, $D_\zeta$ given by

\begin{align*}
P & \in D_\zeta(\gamma) \quad \Longrightarrow \quad hP = \theta_\zeta^C(h)P \quad (h \in C), \\
\varphi & \in A_\zeta(\gamma) \quad \Longrightarrow \quad h\varphi = \theta_\zeta^C(h)\varphi \quad (h \in C).
\end{align*}
By restricting them to $A_\zeta^{(g)}$, $D_\zeta^{(g)}$ we obtain left actions of $C$ on $A_\zeta^{(g)}$, $D_\zeta^{(g)}$. By further restricting it to $A_1$ through the embedding $A_1 \hookrightarrow A_\zeta^{(g)}$ we obtain a left $C$-action on $A_1$. Identifying $A_1$ with the projective coordinate algebra of the ordinary flag manifold $B = B^- \backslash G$ we obtain the corresponding right $C$-action on $B$. This is given by (7.12). Define

$$\begin{align*}
p : B \times C &\rightarrow B, \\
a : B \times C &\rightarrow B
\end{align*}$$

by

$$\begin{align*}
p(x, h) &= x, \\
a(x, h) &= x \ast h
\end{align*}$$

respectively.

**Proposition 7.4.** The left action of $C$ on $A_\zeta^{(g)}$ (resp. $D_\zeta^{(g)}$) defined above gives a $C$-equivariant $O_B$-algebra structure of

$$\mathcal{D} = \omega^* A_\zeta^{(g)} \quad \text{(resp. } \mathcal{D} = \omega^* D_\zeta^{(g)})$$

with respect to the right $C$-action (7.12). Namely, we are given a homomorphism

$$F : \alpha^* \mathcal{D} \rightarrow p^* \mathcal{D} \quad \text{(resp. } F : \alpha^* \mathcal{D} \rightarrow p^* \mathcal{D})$$

of $O_{B \times C}$-algebras satisfying $F|_{B \times \{1\}} = \text{id}$ and the cocycle condition.

Similarly, we have the following.

**Proposition 7.5.** The sheaves $\mathcal{D}$, $\mathcal{D}_k$ and $\mathcal{D}_{id}^k$ ($d \in H_\zeta$) are $C$-equivariant $O_B$-algebras with respect to the right $C$-action (7.12).

We refer the reader to [27] for basics about equivariant coherent sheaves.

The right action (7.14) of $C$ on $V$ induces the right actions of $C$ on $V^k$ and $V_{id}^k$ for $d \in H_\zeta$;

$$\begin{align*}
(7.16) \quad (B^- g, \dot{k}, t) \ast h &= (B^- g h^{\dot{k}}, \dot{k}, t) \\
(7.17) \quad (B^- g, \dot{k}, id) \ast h &= (B^- g h^{\dot{k}}, \dot{k}, id)
\end{align*}$$

Moreover, the $C$-equivariant $O_B$-algebra structure of $\mathcal{D}$ (resp. $\mathcal{D}_k$, $\mathcal{D}_{id}^k$) is naturally lifted to the $C$-equivariant $O_V$-algebra (resp. $O_{V^k}$-algebra, $O_{V_{id}^k}$-algebra) structure of $\mathcal{D}$ (resp. $\mathcal{D}_k$, $\mathcal{D}_{id}^k$) with respect to (7.14) (resp. (7.16), (7.17)).

Assume that $\mathcal{M}$ is a $C$-equivariant $\mathcal{D}$-module. The $C$-algebra homomorphism $U_\zeta(g) \rightarrow \mathcal{D}$ induced by (5.10) gives an action

$$\begin{align*}
(7.18) \quad U_\zeta(g) &\rightarrow \text{End}_C(\mathcal{M}) \quad (u \mapsto \partial_u)
\end{align*}$$

of $U_\zeta(g)$ on $\mathcal{M}$. On the other hand since the action of the subgroup $C_\ell$ of $C$ on $V$ in (7.14) is trivial, we obtain an action

$$\begin{align*}
(7.19) \quad C_\ell &\rightarrow \text{End}_C(\mathcal{M}) \quad (h \mapsto \rho_h)
\end{align*}$$

of $C_\ell$ on $\mathcal{M}$.

We denote by mod($\mathcal{D}_{id}^k \ast C$) the category of $C$-equivariant coherent $\mathcal{D}_{id}^k \ast C$-module $\mathcal{M}$ such that for any $\nu \in Q_c$ and any section $m$ of $\mathcal{M}$ we have $\partial_{\nu} m = \theta_{\nu}(s) \rho_{s_\nu} m$ with respect to (7.18) and (7.19).
We also denote by \( \text{mod}^k_{\text{id}}(\mathcal{O}; \star C) \) the category of \( C \)-equivariant coherent \( \mathcal{O} \)-module \( \mathcal{M} \) contained in \( \text{mod}^k_{\text{id}}(\mathcal{O}) \) as a \( \mathcal{O} \)-module for any \( \nu \in Q_C \) and any section \( m \) of \( \mathcal{M} \) there exists \( n \) satisfying \( (\partial_{k_\nu} - \theta_\nu(s)\rho_{s\nu})^nm = 0 \) with respect to (7.18) and (7.19).

**Proposition 7.6.** Assume \( d \in H_\ell \) satisfies (6.1). Then (6.12) induces the equivalence

\[
D^k(\text{mod}^k_{\text{id}}(\mathcal{O}; \star C)) \simeq D^k(\text{mod}^k_{\text{id}}(U_{\zeta}(\mathfrak{g}); C))
\]

of triangulated categories.

**Proof.** We can easily show that the functors

\[
\Gamma : \text{mod}^k_{\text{id}}(\mathcal{O}) \to \text{mod}^k_{\text{id}}(U_{\zeta}(\mathfrak{g})),
\]

\[
\text{Loc}^k_{\text{id}} : \text{mod}^k_{\text{id}}(U_{\zeta}(\mathfrak{g})) \to \text{mod}^k_{\text{id}}(\mathcal{O})
\]

induce

\[
\Gamma : \text{mod}^k_{\text{id}}(\mathcal{O}; \star C) \to \text{mod}^k_{\text{id}}(U_{\zeta}(\mathfrak{g}); C),
\]

\[
\text{Loc}^k_{\text{id}} : \text{mod}^k_{\text{id}}(U_{\zeta}(\mathfrak{g}); C) \to \text{mod}^k_{\text{id}}(\mathcal{O}; \star C)
\]

respectively. Hence the desired result follows from Proposition 7.7 and Lemma 7.8 below. \( \square \)

Denote by \( \text{Mod}(\mathcal{O}; \star C) \) the category of \( C \)-equivariant quasi-coherent \( \mathcal{O} \)-modules.

**Proposition 7.7.** The abelian category \( \text{Mod}(\mathcal{O}; \star C) \) has enough injectives. Moreover, for any injective object \( \mathcal{I} \) of \( \text{Mod}(\mathcal{O}; \star C) \) we have

\[
R^k\Gamma(\mathcal{I}) = 0 \quad (k > 0).
\]

**Proof.** The first half is a consequence of the general theory of Grothendieck categories. Let us show the latter half. Since \( \mathcal{O} \) is a locally free \( \mathcal{O}_\mathcal{V} \)-module, we can show that \( \mathcal{I} \) is injective as an object of \( \text{Mod}(\mathcal{O}_\mathcal{V}; \star C) \) similarly to [27, Lemma 3.3.6]. Hence the assertion follows from [27, Lemma 3.3.9]. \( \square \)

Denote by \( \text{Mod}(U_{\zeta}(\mathfrak{g}); C) \) the category of \( \Lambda_C \)-graded \( U_{\zeta}(\mathfrak{g}) \)-modules. The following is a consequence of the standard arguments on the cohomologies of graded rings (see e.g. [24]).

**Lemma 7.8.** The abelian category \( \text{Mod}(U_{\zeta}(\mathfrak{g}); C) \) has enough projectives. Moreover, any projective object of \( \text{Mod}(U_{\zeta}(\mathfrak{g}); C) \) is projective as an object of \( \text{Mod}(U_{\zeta}(\mathfrak{g})) \).

We denote by \( (\mathcal{V}^k_{\text{id}} \times C) \) the formal neighborhood of \( \mathcal{V}^k_{\text{id}} \times C \) in \( \mathcal{V} \times C \). We define

\[
p : (\mathcal{V}^k_{\text{id}} \times C) \to \mathcal{V}^k_{\text{id}}, \quad a : (\mathcal{V}^k_{\text{id}} \times C) \to \mathcal{V}^k_{\text{id}}
\]

to be the morphisms induced by the projection \( \mathcal{V} \times C \to \mathcal{V} \) and the right \( C \)-action (7.17) respectively. Then \( \mathcal{O}^k_{\mathcal{V}^k_{\text{id}}} \) is a \( C \)-equivariant \( \mathcal{O}_{\mathcal{V}^k_{\text{id}}} \)-algebra in the sense that we are given a homomorphism \( F : a^*(\mathcal{O}^k_{\mathcal{V}^k_{\text{id}}}) \to p^*(\mathcal{O}^k_{\mathcal{V}^k_{\text{id}}}) \) satisfying \( F|_{\mathcal{V}^k_{\text{id}} \times \{1\}} = \text{id} \) and the cocycle condition.
We denote by mod($\mathcal{D}^k_{td}; \ast C$) the category of $C$-equivariant coherent $\mathcal{D}^k_{td}$-modules. Namely, its object is a coherent $\mathcal{D}^k_{td}$-module $\mathcal{M}$ equipped with an isomorphism $F_{3t}: a^*\mathcal{M} \to p^*\mathcal{M}$ of $p^*(\mathcal{D}^k_{td})$-modules satisfying $F_{3t}|_{\mathcal{V}^k_{td} \times \{1\}} = \text{id}$ and the cocycle condition.

Similarly to Proposition 7.4, we have the following

**Proposition 7.9.** Assume $d \in H_\ell$ satisfies (6.1). Then (6.13) induces the equivalence

$$D^b(\text{mod}(\mathcal{D}^k_{td}; \ast C)) \cong D^b(\text{mod}(\hat{U}_\zeta(\mathfrak{g})^k_{[id]; C}))$$

of triangulated categories.

### 7.4. Equivariant Morita equivalence

We next give a $C$-equivariant analogue of the Morita equivalence (5.38). We will use the identification

$$\mathcal{V}^k_{t;\lambda} \cong B^k_t \quad (\lambda \in \Lambda)$$

and regard $\mathcal{D}^k_{t;\lambda}$ with an $\mathcal{O}_{g^k_t}$-algebra in the following. We have the right $C$-action on $B^k_t$ given by

$$B^{-g} \cdot h = B^{-gh} \quad (B^{-g} \in B^k_t, \ h \in C).$$

Under the identification (7.20) this corresponds to the right $C$-action on $\mathcal{V}^k_{t;\lambda}$ induced by the right $C$-action (7.11) on $V$ (without $\ell$-th power). The right $C$-action on $B^k_t$ given by the $\ell$-th power is denoted by

$$B^k_t \times C \to B^k_t; \quad ((x, h) \mapsto x \ast h := x \cdot h^\ell).$$

Let us describe the action of $C$ on the splitting bundle $\mathcal{R}^k_{t;\lambda}$ of $\mathcal{D}^k_{t;\lambda}$ for $\lambda \in \Lambda$. Recall $\mathcal{R}^k_{t;\lambda} = \mathcal{O}_{g^k_t} \otimes C K^k_{[\lambda]}$, where $K^k_{[\lambda]}$ is the unique irreducible module over $U_\zeta(\mathfrak{g})^k_{[\lambda]}$. By [24] $K^k_{[\lambda]}$ has a unique (up to shift of grading) $\Lambda_C$-grading so that $K^k_{[\lambda]} \in \text{mod}(U_\zeta(\mathfrak{g})^k_{[\lambda]; C})$. Hence there exists a unique (up to tensoring with one-dimensional $C$-module) $C$-equivariant $\mathcal{D}^k_{t;\lambda}$-module structure of $\mathcal{R}^k_{t;\lambda}$ such that $\mathcal{R}^k_{t;\lambda} \in \text{mod}(\mathcal{D}^k_{t;\lambda}; \ast C)$. Not that for $\lambda \in \Lambda$ we have

$$\mathcal{D}^k_{t;\lambda} = \mathcal{O}[\lambda] \otimes \mathcal{D}^k_t \otimes \mathcal{D}[-\lambda], \quad \mathcal{R}^k_{t;\lambda} = \mathcal{O}[\lambda] \otimes \mathcal{R}^k_t.$$

Since $\mathcal{D}$ is a $C$-equivariant $\mathcal{O}_{g^k_t}$-algebra and since $\mathcal{O}[\lambda]$ is a $C$-equivariant $\mathcal{D}$-module, we have $\mathcal{R}^k_{t;\lambda} \in \text{mod}(\mathcal{D}^k_{t;\lambda}; \ast C)$.

We denote by mod($\mathcal{O}_{g^k_t}; C$) the category of $C$-equivariant coherent $\mathcal{O}_{g^k_t}$-modules with respect to the right action (7.21) of $C$ on $B^k_t$.

**Proposition 7.10.** The Morita equivalence (5.38) induces an equivalence

$$\text{mod}(\mathcal{D}^k_{t;\lambda}; \ast C) \cong \text{mod}(\mathcal{O}_{g^k_t}; C)$$

of categories.
PROOF. Note that (5.38) is given by
\[
H : \text{mod}(\mathcal{D}^k_{\ell \lambda}) \to \text{mod}(\mathcal{O}_{B_{\ell i}}) \quad (H(\mathcal{M}) = \mathcal{H}om_{\mathcal{D}^k_{\ell \lambda}} (\mathcal{O}_{B_{\ell i}}[\lambda], \mathcal{M})),
\]
\[
E : \text{mod}(\mathcal{O}_{B_{\ell i}}) \to \text{mod}(\mathcal{D}^k_{\ell \lambda}) \quad (E(S) = \mathcal{R}^k_{\ell i}[\lambda] \otimes_{\mathcal{O}_{B_{\ell i}}^k} S).
\]
Assume \( \mathcal{M} \in \text{mod}(\mathcal{D}^k_{\ell \lambda}; C) \). Then we have a natural \( C \)-equivariant \( \mathcal{O}_{B_{\ell i}} \)-module structure of \( H(\mathcal{M}) \) with respect to (7.22). However, since the action of \( C_{\ell} \) on \( H(\mathcal{M}) \) is trivial, \( H(\mathcal{M}) \) is in fact a \( C/C_{\ell} \)-equivariant \( \mathcal{O}_{B_{\ell i}} \)-module. Hence identifying \( C/C_{\ell} \) with \( C \) by \( hC_{\ell} \leftrightarrow h' \), we obtain \( H(\mathcal{M}) \in \text{mod}(\mathcal{O}_{B_{\ell i}}; C) \).

Assume \( \mathcal{S} \in \text{mod}(\mathcal{O}_{B_{\ell i}}; C) \). By lifting the \( C \)-equivariant \( \mathcal{O}_{B_{\ell i}} \)-module structure of \( \mathcal{S} \) via \( C \ni h \mapsto h^t \in C \) we obtain a new \( C \)-equivariant \( \mathcal{O}_{B_{\ell i}} \)-module structure of \( \mathcal{S} \) with respect to (7.22). By this we obtain \( E(\mathcal{S}) \in \text{mod}(\mathcal{D}^k_{\ell \lambda}; C) \).

We next give a \( C \)-equivariant analogue of the (5.39). We will use the identification (7.24)
\[
\mathcal{V}^k_{\ell \lambda} \cong \mathcal{B}^k_{\ell i} \quad (\lambda \in \Lambda)
\]
(see 5.34), and regard \( \hat{\mathcal{D}}^k_{\ell \lambda} \) as an \( \mathcal{O}_{\mathcal{B}_{\ell i}} \)-algebra in the following. The right \( C \)-action
\[
(B^{-} g, \hat{k}) \cdot h = (B^{-} gh, \hat{k} \cdot h) \quad ((B^{-} g, \hat{k}) \in \hat{G}', \ h \in C)
\]
on \( \hat{G}' \) induces a right \( C \)-action (7.25)
\[
(B^k_{\ell i} \times C)^{\wedge} \to \hat{B}^k_{\ell i}
\]
on \( \hat{B}^k_{\ell i} \), where \( (B^k_{\ell i} \times C)^{\wedge} \) is the formal neighborhood of \( B^k_{\ell i} \times C \) in \( \hat{G}' \times C \).

We denote by \( \text{mod}(\mathcal{O}_{\mathcal{B}_{\ell i}}; C) \) the category of \( C \)-equivariant coherent \( \mathcal{O}_{\mathcal{B}_{\ell i}} \)-modules with respect to the right \( C \)-action (7.25) on \( \hat{B}^k_{\ell i} \). Similarly to Proposition 7.10 we have the following.

**Proposition 7.11.** The Morita equivalence (5.39) induces an equivalence
\[
\text{mod}(\mathcal{D}^k_{\ell \lambda}; C) \cong \text{mod}(\mathcal{O}_{B_{\ell i}}; C)
\]
of categories.

Denote by \( \text{mod}_{\mathcal{B}_{\ell i}}(\mathcal{O}_{\mathcal{G}'}; C) \) the category of \( C \)-equivariant coherent \( \mathcal{O}_{\mathcal{G}'} \)-module supported on \( \mathcal{B}^k_{\ell i} \). By (5.40) and Proposition 7.11 we have the following.
\[
\text{mod}^k_{\ell \lambda}(\mathcal{D}^k; C) \cong \text{mod}_{\mathcal{B}^k_{\ell i}}(\mathcal{O}_{\mathcal{G}'}; C).
\]

8. **Equivalence of abelian categories**

In the rest of this paper we further pose the condition
\[
\text{(k4)} \quad \hat{h} = 1,
\]
which is stronger than (k3). In particular, we have \( \hat{\Delta} = \Delta, \hat{W} = W, \) and \( \hat{t} = t_{-\rho} \).

For \( \lambda \in \Lambda \) we set \( \hat{t}(\lambda) = t_{\lambda} = t_{\lambda - \rho} \).
8.1. Alcoves. Set 

\[ \Lambda_R = \mathbb{R} \otimes_{\mathbb{Z}} \Lambda. \]

For \( \lambda \in \Lambda \) we define \( \tau_\lambda : \Lambda_R \to \Lambda_R \) by \( \tau_\lambda(x) = x + \ell \lambda \) for \( x \in \Lambda_R \). We denote by \( W_a \) (resp. \( \tilde{W}_a \)) the group of affine transformations of \( \Lambda_R \) generated by \( W \) and \( \tau_\lambda \) for \( \lambda \in Q \) (resp. \( \Lambda \)). It is called the affine Weyl group (resp. extended affine Weyl group). The action of \( W_a \) is generated by the reflections \( s_{\alpha^\vee,n} (\alpha \in \Delta^+, n \in \mathbb{Z}) \) with respect to the hyperplanes \( \mathcal{H}_{\alpha^\vee,n} \) given by

\[ \mathcal{H}_{\alpha^\vee,n} = \{ \lambda \in \Lambda_R \mid \langle \alpha^\vee, \lambda \rangle = \ell n \}. \]

We set \( \Lambda^\text{reg}_R := \Lambda_R \setminus \bigcup_{\alpha \in \Delta^+, n \in \mathbb{Z}} \mathcal{H}_{\alpha^\vee,n} \). Note that \( d = t_\lambda \) for \( \lambda \in \Lambda \) is regular in the sense of (6.1) (for \( \tilde{W} = W \)) if and only if we have \( \lambda \in \Lambda^\text{reg} := \Lambda \cap \Lambda^\text{reg}_R \). The connected components of \( \Lambda^\text{reg}_R \) are called alcoves. The affine Weyl group \( W_a \) acts simply transitively on the set of alcoves. We call the alcove

\[ A_0 = \{ \mu \in \Lambda_R \mid \langle \alpha^\vee, \mu \rangle > 0 (\alpha \in \Delta^+), \ (\theta^\vee, \mu) < \ell \} \]

the fundamental alcove. Here, \( \theta^\vee \) is the highest coroot.

Set \( I_a = I \cup \{ 0 \} \), and define \( \mathcal{H}_i \) for \( i \in I \) by \( \mathcal{H}_i = \mathcal{H}_{\alpha_i^\vee,0} \) (\( i \in I \)), \( \mathcal{H}_0 = \mathcal{H}_{\theta^\vee,1} \). Then \( A_0 \) is surrounded by the hyperplanes \( \mathcal{H}_i \) (\( i \in I_a \)). For \( i \in I_a \) we denote by \( F_i \) the codimension one face of \( A_0 \) contained in \( \mathcal{H}_i \). We define \( s_i \) (\( i \in I_a \)) to be the reflection with respect to \( \mathcal{H}_i \). Then \( W \) and \( W_a \) are Coxeter groups with the canonical generator systems \( S = \{ s_i \mid i \in I \} \) and \( S_a = \{ s_i \mid i \in I_a \} \) respectively. We denote by \( \ell : W_a \to \mathbb{Z}_{\geq 0} \) the length function. Set \( \Omega = \{ w \in \tilde{W}_a \mid w(A_0) = A_0 \} \). It is a finite subgroup of \( \tilde{W}_a \) isomorphic to \( \tilde{W}_a/W_a \cong \Lambda/Q \). Note \( \tilde{W}_a \cong \Omega \rtimes W_a \). We extend the length function for \( W_a \) to \( \ell : \tilde{W}_a \to \mathbb{Z}_{\geq 0} \) by setting \( \ell(w \omega) = \ell(w) \) for \( w \in W_a, \omega \in \Omega \).

The following fact is crucially used in the theory of translation functors (see Jantzen [23, Part II, 7.7]).

**Lemma 8.1.** Assume that \( \lambda, \mu \in \Lambda \) are contained in the closure of the same alcove. Define \( \nu \in \Lambda^+ \) by \( \{ \nu \} = W(\mu - \lambda) \cap \Lambda^+ \). Assume that \( \xi \in \text{wt}(\Delta_<(\nu)) \) satisfies \( \lambda + \xi \in W_a \mu \). Then we have \( \xi \in W \nu \). Moreover, we have

\[ w_1 \lambda = \lambda, \quad w_1 \mu = \lambda + \xi, \]

for some \( w_1 \in W_a \).

We define the braid group \( \tilde{B}_a \) to be the group generated by the elements \( b_w \) for \( w \in \tilde{W}_a \) satisfying the fundamental relations

\[ b_w b_{w'} = b_{ww'}, \quad (w, w' \in \tilde{W}_a), \quad \ell(w w') = \ell(w) + \ell(w'). \]

We denote by \( B \) (resp. \( B_a \)) the subgroups of \( \tilde{B}_a \) generated by \( b_w \) for \( w \in W \) (resp. \( W_a \)). Then \( B \) and \( B_a \) are naturally isomorphic to the braid groups associated to the Coxeter groups \( W \) and \( W_a \) respectively. We denote by \( \tilde{B}_a^+ \) (resp. \( B_a^+ \)) the subsemigroup of \( \tilde{B}_a \) (resp. \( B_a \)) generated by \( b_w \) for \( w \in W_a \) (resp. \( \tilde{W}_a \)).
8.2. **Exotic sheaves.** We recall the definition of the exotic sheaves. Set
\[
\tilde{G} = \{(B^- g, x) \in B \times G \mid gxg^{-1} \in B^-\},
\]
and define a morphism
\[
\varrho_G : \tilde{G} \rightarrow G
\]
to be the obvious projection. We have
\[
\tilde{\varrho}_G^{-1}(\hat{x}) \cong B^\hat{x} := \{B^- g \in B \mid g\hat{x}g^{-1} \in B^-\}.
\]
We denote by \(\tilde{\mathcal{B}}^x\) the formal neighborhood of \(\tilde{\varrho}_G^{-1}(\hat{x})\) in \(\tilde{G}\). Note that we have
\[
B^x = B^\hat{x}_i, \quad \tilde{B}^x = \tilde{B}^\hat{x}_i
\]
in the notation of Section 5.

Set
\[
\mathfrak{g} = \{(B^- g, a) \in B \times \mathfrak{g} \mid \text{Ad}(g)(a) \in \mathfrak{b}^-\},
\]
and define a morphism
\[
\varrho_\mathfrak{g} : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}
\]
to be the obvious projection. The exponential map \(\exp : \mathfrak{g} \rightarrow G\) induces a morphism
\[
\exp : \tilde{\mathfrak{g}} \rightarrow \tilde{G} \quad ((B^- g, a) \mapsto (B^- g, \text{exp}(a)))
\]
of complex manifolds. Take \(\dot{a} \in \mathfrak{n}^+\) such that \(\exp(\dot{a}) = \hat{x}\). Then we have
\[
\varrho_\mathfrak{g}^{-1}(\dot{a}) \cong B^\hat{x} := \{B^- g \in B \mid \text{Ad}(g)(\dot{a}) \in \mathfrak{b}^-\}.
\]
It is easily seen that \(\exp\) induces isomorphisms
\[
B^\hat{x} \cong B^x, \quad \tilde{B}^\hat{x} \cong \tilde{B}^x
\]
of (algebraic) schemes, where \(\tilde{B}^\hat{x}\) is the formal neighborhood of \(B^\hat{x}\) in \(\tilde{\mathfrak{g}}\).

Let us transfer the definition of exotic sheaves from \(\tilde{B}^\hat{x}\) to \(\tilde{B}^x\). Note
\[
\tilde{G} \times_G \tilde{G} = \{(B^- g_1, B^- g_2, x) \in B \times B \times G \mid g_kxg_k^{-1} \in B^- (k = 1, 2)\},
\]
For \(i \in I\) we define a subvariety \(\Gamma_i\) of \(\tilde{G} \times_G \tilde{G}\) to be the closure of
\[
\{(B^- g, B^- \hat{s}_i(g), x) \in \tilde{G} \times_G \tilde{G} \mid gxg^{-1} \in B^- \cap \hat{s}_i^{-1}B^- \hat{s}_i\},
\]
where \(\hat{s}_i \in N_G(H) \subset G\) is a representative of \(s_i \in W \cong N_G(H)/H\). Let \(\varrho_{\mathfrak{g},i} : \Gamma_i \rightarrow G\) be the natural morphism, and set \(\Gamma^\hat{x}_i = \varrho_{\mathfrak{g},i}^{-1}(\hat{x})\). The projections \(p_{i,r} : \Gamma_i \rightarrow \tilde{G}\)
\((r = 1, 2)\) induce projective morphisms \(p^\hat{x}_i : \Gamma^\hat{x}_i \rightarrow \tilde{B}^\hat{x}\) of schemes, where \(\hat{\Gamma}^\hat{x}_i\) denotes the formal neighborhood of \(\Gamma^\hat{x}_i\) in \(\Gamma_i\).

By [14, 1.3.2] (see also [38]) we have a weak action
\[
J_b : D^b(\text{mod}(\mathcal{O}_{\tilde{B}^\hat{x}})) \xrightarrow{\cong} D^b(\text{mod}(\mathcal{O}_{\tilde{B}^x})) \quad (b \in \tilde{\mathfrak{a}})
\]
of \(\tilde{\mathfrak{a}}\) on \(D^b(\text{mod}(\mathcal{O}_{\tilde{B}^\hat{x}}))\) given by
\[
J_{b_{i\lambda}}(S) = R(p^\hat{x}_i,\lambda)_*((p^\hat{x}_i)_*S) \quad (i \in I),
\]
\[
J_{b_{\lambda\lambda}}(S) = \mathcal{O}_{\tilde{B}}[\lambda] \otimes_{\mathcal{O}_{\tilde{B}}} S \quad (\lambda \in \Lambda^+).
\]
Let $\varrho^x : \hat{B}^x \to \{\hat{x}\}$ be the natural morphisms induced by $g_G$, where $\{\hat{x}\}$ is the formal neighborhood of $\{\hat{x}\}$ in $G$. By [14, 1.5.1] there exists a $t$-structure $(D^{\leq 0}, D^{\geq 0})$ of the triangulated category $D^b(\text{mod}(\mathcal{O}_{\hat{B}_x}))$ given by

\[
D^{\leq 0} = \{ S \in D^b(\text{mod}(\mathcal{O}_{\hat{B}_x})) \mid R\varrho^x_*(\mathcal{J}_b S) \in D^{\leq 0}(\text{mod}(\mathcal{O}_{\{\hat{b}\}})) \ (b \in \hat{B}_a^+) \},
\]
\[
D^{\geq 0} = \{ S \in D^b(\text{mod}(\mathcal{O}_{\hat{B}_x})) \mid R\varrho^x_*(-1)S) \in D^{\geq 0}(\text{mod}(\mathcal{O}_{\{\hat{b}\}})) \ (b \in \hat{B}_a^+) \}.
\]

We denote by $\text{mod}^{ex}(\mathcal{O}_{\hat{B}_x})$ the heart of this $t$-structure. We call objects of $\text{mod}^{ex}(\mathcal{O}_{\hat{B}_x})$ exotic sheaves on $\hat{B}^x$. We define $\text{mod}^{ex}(\mathcal{O}_{\hat{B}_x}; C)$ to be the abelian category consisting of objects of $D^b(\text{mod}(\mathcal{O}_{\hat{B}_x}; C))$ which are contained in $\text{mod}^{ex}(\mathcal{O}_{\hat{B}_x})$ if we forget the $C$-action.

Recall that for $\lambda \in \Lambda^{\text{reg}}$ we have equivalences

\[
D^b(\text{mod}(\mathcal{U}_\zeta(\mathfrak{g})^{\hat{k}}_{\{\lambda\}})) \cong D^b(\text{mod}(\mathcal{O}_{\hat{B}_x})),
\]
\[
D^b(\text{mod}(\mathcal{O}_\zeta(\mathfrak{g})^{\hat{k}}_{\{\lambda\}}; C)) \cong D^b(\text{mod}(\mathcal{O}_{\hat{B}_x}; C))
\]

of triangulated categories by Theorem 6.3, Theorem 5.6, Proposition 7.9, Proposition 7.11.

The following is the main result of this paper.

**Theorem 8.2.** Let $\lambda_0 \in A_0 \cap \Lambda$. Then (8.7), (8.8) induce equivalences

\[
\text{mod}(\mathcal{U}_\zeta(\mathfrak{g})^{\hat{k}}_{\{\lambda_0\}}) \cong \text{mod}^{ex}(\mathcal{O}_{\hat{B}_x}),
\]
\[
\text{mod}(\mathcal{U}_\zeta(\mathfrak{g})^{\hat{k}}_{\{\lambda_0\}}; C) \cong \text{mod}^{ex}(\mathcal{O}_{\hat{B}_x}; C)
\]

of abelian categories.

Note that (8.10) easily follows from (8.9). The aim of this section is to give a proof of (8.9) following the argument of [14]. The point is to show that the wall-crossing functor for $\mathcal{U}_\zeta(\mathfrak{g})$-modules correspond to the action $\mathfrak{g}$ of $\hat{B}_a$ for coherent sheaves.

### 8.3. $\mathfrak{g}$-modules

We set

\[
\mathfrak{g} = \text{Fr}_\mathfrak{g} \mathcal{U}_{\hat{B}_x} := \mathfrak{g}_\lambda \left( (\mathcal{O}_\zeta \otimes \mathcal{U}_\zeta(\mathfrak{g}))^{\hat{k}} \right)
\]

(see (3.30) for the notation). It is a sheaf of $\mathbb{C}$-algebras on $\mathcal{B}$. It contains $\mathfrak{O}$ and $\mathcal{U}_\zeta(\mathfrak{g})$ as subalgebras, and

\[
\mathfrak{O} \otimes \mathcal{U}_\zeta(\mathfrak{g}) \to \mathfrak{g} \quad (\varphi \otimes u \mapsto \varphi u)
\]

is an isomorphism of sheaves. Moreover, we have

\[
u\varphi = \sum (\partial_{a(0)}(\varphi))u_{(1)} \quad (u \in \mathcal{U}_\zeta(\mathfrak{g}), \varphi \in \mathfrak{O})
\]

in $\mathfrak{g}$. We denote by $\mathfrak{g}^k$ the ideal of $\mathfrak{g}$ generated by $z - \xi^k(z)$ for $z \in \mathcal{Z}_{\text{Fr}}(\mathcal{U}_\zeta(\mathfrak{g}))$, and set

\[
\mathfrak{g}^k = \mathfrak{g}/\mathfrak{g}^k, \quad \hat{\mathfrak{g}}^k = \lim_n \frac{\mathfrak{g}}{(\mathfrak{g}^k)^n}.
\]

For $\lambda \in \Lambda$ we have natural algebra homomorphisms

\[
\mathfrak{g}^k \to \mathfrak{O}^k_{\{\lambda\}}, \quad \hat{\mathfrak{g}}^k \to \hat{\mathfrak{O}}^k_{\{\lambda\}}
\]
induced by \((8.29)\).

By \(\text{mod}(U_{L}(\xi)^{\pm}) \cong \text{mod}(\Omega)\) we obtain from \((8.31)\) an exact functor
\[
(8.12) \quad V \otimes (\bullet) : \text{mod}(\Omega) \to \text{mod}(\Omega)
\]
for \(V \in \text{mod}_{\text{int}}(U_{L}^{\pm}(\xi))\). This induces
\[
(8.13) \quad V \otimes (\bullet) : \text{mod}(\hat{\Omega}^{k}) \to \text{mod}(\hat{\Omega}^{k}), \quad V \otimes (\bullet) : \text{mod}(\hat{\Omega}^{k}) \to \text{mod}(\hat{\Omega}^{k}).
\]

By Proposition \(8.3\) we have the following.

**Proposition 8.3.** Let \(V \in \text{mod}_{\text{int}}(U_{L}^{\pm}(\xi))(\xi)\) and let
\[
(8.14) \quad V = V_{1} \supset V_{2} \supset \cdots \supset V_{m} \supset V_{m+1} = \{0\}
\]
be a \(U_{L}^{\pm}(\xi, b)-\text{stable filtration of } V\) such that \(V_{j}/V_{j+1}\) is a one-dimensional \(U_{L}^{\pm}(\xi, b)-\text{module with character } \xi_{j} \in \Lambda\).

Let \(\lambda \in \Lambda\). Let \(M \in \text{mod}(\hat{\Omega}^{k}_{(\xi_{j})})\) (resp. \(\text{mod}(\hat{\Omega}^{k}_{(\xi_{j})})\)), and regard it as an object of \(\text{mod}(\Omega^{k})\) (resp. \(\text{mod}(\hat{\Omega}^{k})\)) via \((8.11)\). Then \(V \otimes M \in \text{mod}(\Omega^{k})\) (resp. \(\text{mod}(\hat{\Omega}^{k})\)) has a functorial filtration
\[
V \otimes M = M_{1} \supset M_{2} \supset \cdots \supset M_{m} \supset M_{m+1} = \{0\}
\]
such that \(M_{j}/M_{j+1} = \Omega[\xi_{j}] \otimes \text{mod}(\hat{\Omega}^{k}_{(\xi_{j})})\) (resp. \(\text{mod}(\hat{\Omega}^{k}_{(\xi_{j})})\)) for \(j = 1, \ldots, m\).

### 8.4. Translation functor

Recall that we have a direct sum decomposition
\[
(8.15) \quad \text{mod}(\hat{U}_{\xi}(\xi^{\pm})) = \bigoplus_{\lambda \in \Lambda/W_{\xi}} \text{mod}(\hat{U}_{\xi}(\xi^{\pm}))
\]
of abelian categories (see \((4.10)\)). For \(\lambda \in \Lambda\) we denote by
\[
(8.16) \quad p_{\lambda} : \text{mod}(\hat{U}_{\xi}(\xi^{\pm})) \to \text{mod}(\hat{U}_{\xi}(\xi^{\pm}))
\]
the projection with respect to \((8.15)\).

Assume that \(\lambda, \mu \in \Lambda\) are contained in the closure of the same alcove. In this case we define an exact functor
\[
(8.17) \quad T_{\mu\lambda} : \text{mod}(\hat{U}_{\xi}(\xi^{\pm})) \to \text{mod}(\hat{U}_{\xi}(\xi^{\pm}))
\]
by
\[
T_{\mu\lambda}(M) = p_{\mu}(L_{\xi}(\nu) \otimes M),
\]
where \(\nu \in \Lambda^{+}\) is given by \(\{\nu\} = W(\mu - \lambda) \cap \Lambda^{+}\). This functor is called the translation functor.

**Remark 8.4.** We can show using Corollary \(4.6\) and Lemma \(8.1\) that
\[
T_{\mu\lambda}(M) \cong p_{\mu}(\Delta_{\xi}(\nu) \otimes M) \cong p_{\mu}(\nabla_{\xi}(\nu) \otimes M)
\]
for \(\lambda, \mu, \nu\) as above.

**Remark 8.5.** For \(\lambda, \mu \in \Lambda\) as above and \(w \in W_{\xi}\) we have \(T_{\mu\lambda} = T_{w\mu,w\lambda}\). Hence it is harmless to assume \(\lambda, \mu \in \Lambda_{0}\) as we do later.
Lemma 8.6. Assume that $\lambda, \mu \in \Lambda$ are contained in the closure of the same alcove. Then $T_{\lambda \mu}$ is left and right adjoint to $T_{\mu \lambda}$.

Proof. By symmetry it is sufficient to show that $T_{\lambda \mu}$ is right adjoint to $T_{\mu \lambda}$. Take $\nu_1, \nu_2 \in \Lambda^+$ satisfying $\{\nu_1\} = W(\mu - \lambda) \cap \Lambda^+$, $\{\nu_2\} = W(\lambda - \mu) \cap \Lambda^+$. Then we have $L(\nu_1) \cong L(\nu_2)^*$. Hence for $M \in \text{mod}(U_\zeta(g)^k_{[\ell(\lambda)]})$, $N \in \text{mod}(U_\zeta(g)^k_{[\ell(\mu)]})$ we have

$$\text{Hom}(T_{\lambda \mu}M, N) = \text{Hom}(p_\mu(L(\nu_1) \otimes M), N) \cong \text{Hom}(L(\nu_1) \otimes M, N) \cong \text{Hom}(L(\nu_2)^* \otimes M, N) \cong \text{Hom}(M, L(\nu_2) \otimes N) = \text{Hom}(M, T_{\lambda \mu}N).$$

$\square$

Proposition 8.7. Assume that $\mu \in \Lambda$ is contained in the closure of the facet containing $\lambda \in \Lambda$. Then for $\mathcal{M} \in D^b(\text{mod}(\mathfrak{D}^k_{i(\lambda)}))$ we have

$$T_{\mu \lambda}R\Gamma(\mathcal{M}) = R\Gamma(\mathfrak{D}[\mu - \lambda] \otimes_\mathcal{O} \mathcal{M}).$$

Proof. Let $\mathcal{M} \in D^b(\text{mod}(\mathfrak{D}^k_{i(\lambda)}))$. Take $\nu \in \Lambda^+$ such that $\{\nu\} = W(\mu - \lambda) \cap \Lambda^+$. Set $V = \Delta_\zeta(\nu)$, and choose its filtration as in (8.14). By Proposition 8.3 there exist $\mathcal{M}_j \in D^b(\text{mod}(\mathfrak{U}^k_{\text{mod}}))$ for $j = 1, \ldots, m + 1$ such that

$$\mathcal{M}_1 = V \otimes \mathcal{M}, \quad \mathcal{M}_{m+1} = 0,$$

and distinguished triangles

$$\mathcal{M}_{j+1} \to \mathcal{M}_j \to \mathfrak{D}[\xi_j] \otimes_\mathcal{O} \mathcal{M} \overset{+1}{\to}.$$

By applying $p_\mu \circ R\Gamma$ we obtain $M_j \in D^b(\text{mod}(\mathfrak{U}_\zeta(g)^k_{[\ell(\mu)]}))$ for $j = 1, \ldots, m + 1$ such that

$$M_1 = V \otimes R\Gamma(\mathcal{M}), \quad M_{m+1} = 0,$$

and distinguished triangles

$$M_{j+1} \to M_j \to p_\mu(R\Gamma(\mathfrak{D}[\xi_j] \otimes_\mathcal{O} \mathcal{M})) \overset{+1}{\to}.$$

Assume $p_\mu(R\Gamma(\mathfrak{D}[\xi_j] \otimes_\mathcal{O} \mathcal{M})) \neq \{0\}$. By $R\Gamma(\mathfrak{D}[\xi_j] \otimes_\mathcal{O} \mathcal{M}) \in D^b(\text{mod}(\mathfrak{U}_\zeta(g)^k_{[\ell(\mu)]}))$ we have $\lambda + \xi_j \in W_\alpha \mu$. By $\mu - \lambda - \xi_j \in Q$ and $\ell \lambda \cap Q = \ell Q$ we obtain $\lambda + \xi_j \in W_\alpha \mu$. By Lemma 8.1 this implies $\xi_j = \mu - \lambda$. Since $\mu - \lambda$ is an extremal weight of $V$, such $j$ is unique, and for this $j$ we have

$$T_{\mu \lambda}R\Gamma(\mathcal{M}) = R\Gamma(\mathfrak{D}[\xi_j] \otimes_\mathcal{O} \mathcal{M}) = R\Gamma(\mathfrak{D}[\mu - \lambda] \otimes_\mathcal{O} \mathcal{M}).$$

$\square$

8.5. Parabolic version. To proceed further we need (8.27) below which is a parabolic analogue of (8.7). Here, we only give a brief account of it. See Appendix for more details.

Let $J \subset I$. Let $P_J$ be the parabolic subgroup of $G$ corresponding to the parabolic subalgebra

$$p_J = b^- \oplus \bigoplus_{\alpha \in \Delta_J^+} g_\alpha$$
of $\mathfrak{g}$, where $\Delta^+_J = \sum_{j \in J} \mathbb{Z} \alpha_j \cap \Delta^+$, and let $\mathcal{P}_J = P^*_J \backslash G$ be the corresponding partial flag manifold.

Then we can consider the corresponding quantized partial flag manifolds $\mathcal{P}_{J,\zeta}$ at $\zeta$. More precisely, we have a category $\text{Mod}(\mathcal{O}_{\mathcal{P}_{J,\zeta}})$ (resp. $\text{mod}(\mathcal{O}_{\mathcal{P}_{J,\zeta}})$), which is an analogue of the the category $\text{Mod}(\mathcal{O}_{\mathcal{P}_J})$ (resp. $\text{mod}(\mathcal{O}_{\mathcal{P}_J})$) consisting of quasi-coherent (resp. coherent) sheaves on $\mathcal{P}_J$. Similarly to the case $J = \emptyset$ we can define an $\mathcal{O}_{\mathcal{P}_J}$-algebra $\mathcal{O}_J = \text{Fr}_s \mathcal{O}_{\mathcal{P}_{J,\zeta}}$ satisfying

$$\text{Mod}(\mathcal{O}_{\mathcal{P}_{J,\zeta}}) \cong \text{Mod}(\mathcal{O}_J), \quad \text{mod}(\mathcal{O}_{\mathcal{P}_{J,\zeta}}) \cong \text{mod}(\mathcal{O}_J).$$

Set

$$\Lambda^J = \{ \lambda \in \Lambda \mid \langle \lambda, \alpha_j^\vee \rangle = 0 \ (j \in J) \}.$$ 

For $\lambda \in \Lambda^J$ we have an $\mathcal{O}_J$-bimodule $\mathcal{O}_J[\lambda]$.

Let

$$(8.18) \quad \pi_J : \mathcal{B} \to \mathcal{P}_J$$

be the natural morphism. We set

$$(8.19) \quad \mathcal{D}_J = (\pi_J)_* \mathcal{B}, \quad \mathfrak{J}_J = (\pi_J)_* \mathfrak{J}.$$ 

Then $\mathcal{D}_J$ is a sheaf of $\mathbb{C}$-algebras on $\mathcal{P}_J$ containing a central subalgebra $\mathfrak{J}_J$. Set

$$(8.20) \quad \mathcal{V}_J = \{(P_J^* g, k, t) \in \mathcal{P}_J \times K \times H \mid g\eta(k)g^{-1} \in (\text{St}_{J}^{-1}(t^J))N_J^{-}\},$$

where $\text{St}_J : L_J \to H/W_J$ is the Steinberg map for $L_J$. Let

$$p_{\mathcal{P}_J}^{\mathcal{V}_J} : \mathcal{V}_J \to \mathcal{P}_J, \quad p_{\mathcal{Y}_J}^{\mathcal{V}_J} : \mathcal{V}_J \to \mathcal{Y}$$

be the projections. Then we have

$$(8.21) \quad \mathfrak{J}_J \cong (p_{\mathcal{P}_J}^{\mathcal{V}_J})_* \mathcal{O}_{\mathcal{V}_J}.$$ 

Localizing $\mathcal{D}_J$ on $\mathcal{V}_J$ we obtain an $\mathcal{O}_{\mathcal{V}_J}$-algebra $\sharp \mathcal{D}_J$. For $(k, t) \in \mathcal{Y}$ we set

$$\mathcal{V}_J^k = (p_{\mathcal{Y}_J}^{\mathcal{V}_J})^{-1}(k, t).$$

We denote by $\mathcal{V}^{k}_{J,t}$ the formal neighborhood of $\mathcal{V}^k_J$ in $\mathcal{V}_J$. Set

$$\sharp \mathcal{D}^{k}_{J,t} = \sharp \mathcal{D}_J \otimes_{\mathcal{O}_{\mathcal{V}_J}} \mathcal{O}_{\mathcal{V}^{k}_{J,t}}, \quad \mathcal{D}^{k}_{J,t} = \mathcal{D}_J \otimes_{\mathcal{O}_{\mathcal{V}_J}} \mathcal{O}_{\mathcal{V}^{k}_{J,t}}$$

respectively. We set

$$\mathcal{P}^k_J = \{ P^*_J g \in \mathcal{P}_J \mid g\hat{x}g^{-1} \in P^*_J \},$$

and denote by $\hat{P}^{k}_J$ the formal neighborhood of $\mathcal{P}^k_J$ in

$$\hat{G}_J = \{(P^*_J g, x) \in \mathcal{P}_J \times G \mid g\hat{x}g^{-1} \in P^*_J \}.$$ 

Let

$$\pi^k_J : \mathcal{B}^k \to \mathcal{P}^k_J, \quad \pi^k_J : \mathcal{B}^k \to \hat{P}^{k}_J$$

be the natural morphisms. We have canonical isomorphisms

$$\mathcal{V}^{k}_{J,t}(\lambda) \cong \mathcal{P}^k_J, \quad \mathcal{V}^{k}_{J,t}(\lambda) \cong \hat{P}^{k}_J \quad (\lambda \in \Lambda^J).$$

We identify $\sharp \mathcal{D}^{k}_{J,t}(\lambda)$ and $\sharp \mathcal{D}^{k}_{J,t}(\lambda)$ with an $\mathcal{O}_{\mathcal{P}_J}$-algebra and an $\mathcal{O}_{\hat{P}_J}$-algebra respectively in the following.
For $\lambda \in \Lambda^J$ we define a $z^{\mathcal{D}_J^k,\mathcal{D}_J^{k,\lambda}}$-module $R^k_{J,l}[\lambda]$ (resp. a $\hat{\mathcal{D}}^{k,\lambda}$-module $\hat{R}^k_{J,l}[\lambda]$) by

$$R^k_{J,l}[\lambda] = \mathcal{D}_J[\lambda] \otimes_{\mathcal{D}_J} (\mathcal{O}_{\mathcal{P}_J} \otimes_{\mathbb{C}} K^k_{[\lambda]}),$$

(resp. $\hat{R}^k_{J,l}[\lambda] = \mathcal{D}_J[\lambda] \otimes_{\mathcal{D}_J} (\mathcal{O}_{\mathcal{P}_J} \otimes_{\mathcal{D}_J[\lambda]} \hat{K}^k_{[\lambda]}))$.

By definition we have the following.

**Proposition 8.8.** For $\lambda \in \Lambda^J$ we have

$$(\pi^+_{J})^* R^k_{J,l}[\lambda] \cong \mathcal{R}^k_{Y}[\lambda], \quad (\pi^+_{J})^* \hat{R}^k_{J,l}[\lambda] \cong \hat{\mathcal{R}}^k_{Y}[\lambda].$$

Similarly to Theorem 5.3 we have the following result (see Theorem 8.19).

**Theorem 8.9.** Assume $\lambda \in \Lambda^J$. Then $z^{\mathcal{D}_J^k,\mathcal{D}_J^{k,\lambda}}$ (resp. $\hat{\mathcal{D}}^{k,\lambda}$) is a split Azumaya algebra with splitting bundle $R^k_{J,l}[\lambda]$ (resp. $\hat{R}^k_{J,l}[\lambda]$).

By the Morita equivalence we obtain the following.

**Theorem 8.10.** Assume $\lambda \in \Lambda^J$. The we have the following equivalences of abelian categories.

$$\text{mod}(z^{\mathcal{D}_J^k,\mathcal{D}_J^{k,\lambda}}) \cong \text{mod}(\mathcal{O}_{\mathcal{P}_J}),$$

$$\text{mod}(\hat{\mathcal{D}}^{k,\lambda}) \cong \text{mod}(\mathcal{O}_{\mathcal{P}_J})$$

Note that for $(k, t) \in \mathcal{Y}$ we have natural functors

$$\Gamma : \text{mod}(z^{\mathcal{D}_J^k}) \to \text{mod}(U_{\mathcal{C}}^k_{[\lambda]}),$$

$$\Gamma : \text{mod}(\hat{\mathcal{D}}^{k,\lambda}) \to \text{mod}(U_{\mathcal{C}}^k_{[\lambda]}),$$

where $\text{mod}(z^{\mathcal{D}_J^k})$ denotes the category of coherent $z^{\mathcal{D}_J^k}$-modules supported on $\mathcal{V}_{J,t}$.

Set $W_J = \{s_j \mid j \in J\} \subset W$, $Q_J = \sum_{j \in J} \mathbb{Z}a_j \subset Q$, and let $W_{J,a}$ be the subgroup of $W_J$ generated by $W_J$ and $\tau_\lambda$ for $\lambda \in Q_J$. We denote by $\Lambda^J_{\text{reg}}$ the set of $\lambda \in \Lambda^J$ satisfying the following equivalent conditions:

- if $w\tau = \tau$ for $w \in W$, then $w \in W_J$,
- if $y\lambda = \lambda$ for $y \in W_a$, then $y \in W_{J,a}$.

Similarly to Theorem 6.3 we have the following result (see Theorem 8.20).

**Theorem 8.11.** Assume $\lambda \in \Lambda^J_{\text{reg}}$. Then (8.25) and (8.26) induce equivalences

$$D^b(\text{mod}(z^{\mathcal{D}_J^k})) \cong D^b(\text{mod}(U_{\mathcal{C}}^k_{[\lambda]})),$$

$$D^b(\text{mod}(\hat{\mathcal{D}}^{k,\lambda})) \cong D^b(\text{mod}(\hat{U}_{\mathcal{C}}^k_{[\lambda]}))$$

of triangulated categories.

By Theorem 8.10 and Theorem 8.11 we obtain an equivalence

$$D^b(\text{mod}(\hat{U}_{\mathcal{C}}^k_{[\lambda]})) \cong D^b(\text{mod}(\mathcal{O}_{\mathcal{P}_J}))$$

of triangulated categories for $\lambda \in \Lambda^J_{\text{reg}}$. 
Proposition 8.12. Assume that $\lambda \in \Lambda$ is contained in the fundamental alcove $A_0$ and $\mu \in \Lambda$ lies on the facet $F \subset A_0$ corresponding to $J \subset I$. Then we have the following commutative diagram:

\[
\begin{array}{ccc}
D^b(\text{mod}(O_{B^s})) & \xrightarrow{R_{\pi_j}^*} & D^b(\text{mod}(O_{\tilde{B}^j})) \\
\cong \downarrow & & \cong \downarrow \\
D^b(\text{mod}(\tilde{U}_\zeta(g)^k_{[\ell(\lambda)]})) & \xrightarrow{T_{\mu\lambda}} & D^b(\text{mod}(\tilde{U}_\zeta(g)^k_{[\ell(\mu)]})).
\end{array}
\]

Proof. Let $S \in D^b(\text{mod}(O_{B^s}))$. By Proposition 8.7 and Proposition 8.8 we have

\[
T_{\mu\lambda} R\Gamma(\mathcal{R}_I^{\mu}[\lambda] \otimes O_{B^s} S) \cong R\Gamma(\mathcal{O}[\mu - \lambda] \otimes \mathcal{R}_I^{\mu}[\lambda] \otimes O_{B^s} S)
\]

\[
\cong R\Gamma(\mathcal{R}_I^{\mu}[\mu] \otimes O_{B^s} S)
\]

\[
\cong R\Gamma((\pi_j^* \mathcal{R}_I^{\mu}[\mu]) \otimes O_{B^s} S)
\]

\[
\cong R\Gamma(R_{\pi_j^*}((\pi_j^*)^*(\pi_j^* \mathcal{R}_I^{\mu}[\mu]) \otimes O_{B^s} S))
\]

\[
\cong R\Gamma(\mathcal{R}_I^{\mu}[\mu] \otimes O_{\tilde{B}^j} R_{\pi_j^*} S).
\]

Since $(\pi_j^*)^*$ is left adjoint to $R_{\pi_j^*}$, we obtain from Lemma 8.6 and Proposition 8.12 the following.

Proposition 8.13. Let $\lambda$, $\mu$, $J$ be as in Proposition 8.12. Then we have the following commutative diagram:

\[
\begin{array}{ccc}
D^b(\text{mod}(O_{\tilde{B}^j})) & \xrightarrow{(\pi_j^*)^*} & D^b(\text{mod}(O_{B^s})) \\
\cong \downarrow & & \cong \downarrow \\
D^b(\text{mod}(\tilde{U}_\zeta(g)^k_{[\ell(\mu)]})) & \xrightarrow{T_{\lambda\mu}} & D^b(\text{mod}(\tilde{U}_\zeta(g)^k_{[\ell(\lambda)]})).
\end{array}
\]

8.6. Wall crossing functor. Let $i \in I_a$. For $\lambda_0 \in \Lambda \cap A_0$, $\mu_0 \in \Lambda \cap F_0$ the exact functor

\[
R_{\lambda_0/\mu_0} = T_{\lambda_0\mu_0} \circ T_{\mu_0\lambda_0} : \text{mod}(\tilde{U}_\zeta(g)^k_{[\ell(\lambda_0)]}) \to \text{mod}(\tilde{U}_\zeta(g)^k_{[\ell(\mu_0)]})
\]

is called the wall crossing functor with respect to the wall $\mathfrak{H}_i$ of $A_0$. By Lemma 8.6 it is self adjoint and we have natural morphisms $\text{Id} \to R_{\lambda_0/\mu_0}$ and $R_{\lambda_0/\mu_0} \to \text{Id}$ of functors. We define functors

\[
\Gamma_{\lambda_0/\mu_0}^i, \Gamma_{\lambda_0/\mu_0}^* : D^b(\text{mod}(\tilde{U}_\zeta(g)^k_{[\ell(\lambda_0)]})) \to D^b(\text{mod}(\tilde{U}_\zeta(g)^k_{[\ell(\mu_0)]}))
\]

by

\[
\Gamma_{\lambda_0/\mu_0}^i = \text{cone}(R_{\lambda_0/\mu_0} \to \text{Id})[-1], \quad \Gamma_{\lambda_0/\mu_0}^* = \text{cone}(\text{Id} \to R_{\lambda_0/\mu_0}).
\]
Proposition 8.14. Let $i \in I_a$ and let $\lambda_0 \in \Lambda \cap A_0$, $\mu_0 \in \Lambda \cap F_0$. Assume $w \in \tilde{W}_a$ satisfies $ws_i \lambda_0 - \omega \lambda_0 \in Q^+$. Then we have the following commutative diagrams of functors:

$$
\begin{array}{ccc}
D^b(\text{mod}(\hat{\mathcal{D}}(w_{s_i} \lambda_0)_{[i(\lambda_0)]})) & \xrightarrow{\Delta[w_{s_i} \lambda_0 - \omega \lambda_0] \otimes_D \Delta} & D^b(\text{mod}(\hat{\mathcal{D}}(w_{s_i} \lambda_0)_{[i(\lambda_0)]})) \\
R \Gamma & & R \Gamma \\
D^b(\text{mod}(\hat{U}_c(\mathfrak{g})_{[i(\lambda_0)]})) & \xrightarrow{\Gamma_{\lambda_0/\mu_0}} & D^b(\text{mod}(\hat{U}_c(\mathfrak{g})_{[i(\lambda_0)]}))
\end{array}
$$

$$
\begin{array}{ccc}
D^b(\text{mod}(\hat{\mathcal{D}}(w_{s_i} \lambda_0)_{[i(\lambda_0)]})) & \xrightarrow{\Delta[w_{s_i} \lambda_0 - \omega \lambda_0] \otimes_D \Delta} & D^b(\text{mod}(\hat{\mathcal{D}}(w_{s_i} \lambda_0)_{[i(\lambda_0)]})) \\
R \Gamma & & R \Gamma \\
D^b(\text{mod}(\hat{U}_c(\mathfrak{g})_{[i(\lambda_0)]})) & \xrightarrow{\Gamma_{\lambda_0/\mu_0}} & D^b(\text{mod}(\hat{U}_c(\mathfrak{g})_{[i(\lambda_0)]}))
\end{array}
$$

Proof. For $\mathfrak{M} \in D^b(\text{mod}(\hat{\mathcal{D}}(w_{s_i} \lambda_0)_{[i(\lambda_0)]}))$ we have

$$
R_{\lambda_0/\mu_0}(R \Gamma(\mathfrak{M})) = T_{w_{s_i} \lambda_0, w_{s_i} \lambda_0} T_{w_{s_i} \lambda_0, \lambda_0} (R \Gamma(\mathfrak{M})) = T_{w_{s_i} \lambda_0, w_{s_i} \lambda_0} R \Gamma(D[w_{s_i} \lambda_0 - \omega \lambda_0] \otimes_D \mathfrak{M})
$$

by Proposition 8.11. Moreover, similarly to the proof of Proposition 8.11 we can show that for $\mathfrak{M} \in D^b(\text{mod}(\hat{\mathcal{D}}(w_{s_i} \lambda_0)_{[i(\lambda_0)]}))$ there exists a distinguished triangle

$$
R \Gamma(D[w_{s_i} \lambda_0 - \omega \lambda_0] \otimes_D \mathfrak{M}) \rightarrow T_{w_{s_i} \lambda_0, w_{s_i} \lambda_0} R \Gamma(\mathfrak{M}) \rightarrow R \Gamma(D[w_{s_i} \lambda_0 - \omega \lambda_0] \otimes_D \mathfrak{M}) \cong 1
$$

in $D^b(\text{mod}(\hat{U}_c(\mathfrak{g})_{[i(\lambda_0)]}))$. Hence we obtain a distinguished triangle

$$
R \Gamma(\mathfrak{M}) \rightarrow R_{\lambda_0/\mu_0}(R \Gamma(\mathfrak{M})) \rightarrow R \Gamma(D[w_{s_i} \lambda_0 - \omega \lambda_0] \otimes_D \mathfrak{M}) \cong 1
$$

The commutativity of the first diagram follows from this using the argument in the proof of [13, Lemma 2.2.3 (c)]. The proof of the second diagram is similar. □

Corollary 8.15. The functors $\mathcal{I}_{\lambda_0/\mu_0}$ and $\mathcal{I}_{\lambda_0/\mu_0}$ do not depend on the choice of $\mu_0$. Moreover, they are mutually inverse equivalences.

For $i \in I_a$ and $\lambda_0 \in \Lambda \cap A_0$ we define functors

$$
\mathcal{I}_{\lambda_0/\mu_0}^i, \mathcal{I}_{\lambda_0/\mu_0}^i : D^b(\text{mod}(\hat{U}_c(\mathfrak{g})_{[i(\lambda_0)]})) \rightarrow D^b(\text{mod}(\hat{U}_c(\mathfrak{g})_{[i(\lambda_0)]}))
$$

by $\mathcal{I}_{\lambda_0/\mu_0}^i = \mathcal{I}_{\lambda_0/\mu_0}^i = \mathcal{I}_{\lambda_0/\mu_0}^i$ for $\mu_0 \in \Lambda \cap F_0^i$.

Using Proposition 8.14 we can show the following as in [13].

Proposition 8.16. For $\lambda_0 \in \Lambda \cap A_0$ we have a weak action

$$
\begin{array}{c}
\mathcal{I}_b : D^b(\text{mod}(\hat{U}_c(\mathfrak{g})_{[i(\lambda_0)]})) \rightarrow D^b(\text{mod}(\hat{U}_c(\mathfrak{g})_{[i(\lambda_0)]}))
\end{array}
$$

of $\hat{W}_a$ on $D^b(\text{mod}(\hat{U}_c(\mathfrak{g})_{[i(\lambda_0)]}))$ satisfying $\mathcal{I}_b = \mathcal{I}_{\lambda_0/\mu_0}$ for $i \in I_a$ and $\mathcal{I}_b = T_{w_{s_i} \lambda_0}$ for $\omega \in \Omega$. Moreover, for any $\nu \in \Lambda^+$ we have the following commutative diagram of
functors:

\[ D^b(\text{mod}(\mathfrak{D}_k^{(\Lambda_0)})) \xrightarrow{\mathcal{O}_B[x] \otimes \mathcal{O}_B(y)} D^b(\text{mod}(\mathfrak{D}_k^{(\Lambda_0+\Theta)})) \]

\[ \xrightarrow{\text{RT}} \]

\[ D^b(\text{mod}(\mathcal{U}_k^{(\mathfrak{g})}_{[(\lambda_0)]})) \xrightarrow{1_b} D^b(\text{mod}(\mathcal{U}_k^{(\mathfrak{g})}_{[(\lambda_0)]})) \]

8.7. Proof of Theorem 8.2. Using Proposition 8.12 Proposition 8.13 for \( J = \{ i \} \) with \( i \in I \) we can show the following as in [38].

Proposition 8.17. For \( \lambda_0 \in \Lambda \cap A_0 \) we have the following commutative diagram of functors:

\[ D^b(\text{mod}(\mathcal{O}_B)) \xrightarrow{\mathcal{J}_b} D^b(\text{mod}(\mathcal{O}_B)) \]

\[ \cong \]

\[ D^b(\text{mod}(\mathcal{U}_k^{(\mathfrak{g})}_{[(\lambda_0)]})) \xrightarrow{1_b} D^b(\text{mod}(\mathcal{U}_k^{(\mathfrak{g})}_{[(\lambda_0)]})) \]

for any \( b \in \mathfrak{B}_a \)

Theorem 8.2 follows from Proposition 8.12 for \( J = I \) and Proposition 8.17 as in [13 1.6.5].

9. Lusztig’s conjecture

9.1. For \( \lambda \in \Lambda \) we denote by \( \text{mod}_{[(\lambda)]}(U_\mathfrak{c}(\mathfrak{g})) \) the category of finitely generated \( U_\mathfrak{c}(\mathfrak{g}) \)-modules on which \( z - \xi^k(z) \) for \( z \in Z_{\mathfrak{g}}(U_\mathfrak{c}(\mathfrak{g})) \) acts as zero and \( z - \xi^k_{[(\lambda)]}(z) \) for \( z \in Z_{\text{Hat}}(U_\mathfrak{c}(\mathfrak{g})) \) acts nilpotently. We also denote by \( \text{mod}_{[(\lambda)]}(U_\mathfrak{c}(\mathfrak{g}); C) \) the subcategory of \( \text{mod}(U_\mathfrak{c}(\mathfrak{g})^k; C) \) consisting of objects which belong to \( \text{mod}_{[(\lambda)]}(U_\mathfrak{c}(\mathfrak{g})^k) \) as a \( U_\mathfrak{c}(\mathfrak{g})^k \)-module. We have

\[ \text{mod}(U_\mathfrak{c}(\mathfrak{g})^k_{[(\lambda)]}) \subset \text{mod}_{[(\lambda)]}(U_\mathfrak{c}(\mathfrak{g})^k) \subset \text{mod}^k_{[(\lambda)]}(U_\mathfrak{c}(\mathfrak{g})) \subset \text{mod}(\mathcal{U}_k^{(\mathfrak{g})}_{[(\lambda)]}), \]

\[ \text{mod}(U_\mathfrak{c}(\mathfrak{g})^k_{[(\lambda)]}; C) \subset \text{mod}_{[(\lambda)]}(U_\mathfrak{c}(\mathfrak{g})^k; C) \subset \text{mod}^k_{[(\lambda)]}(U_\mathfrak{c}(\mathfrak{g}); C) \subset \text{mod}(\mathcal{U}_k^{(\mathfrak{g})}_{[(\lambda)]}; C). \]

It is known by Jantzen [21] that any irreducible \( U_\mathfrak{c}(\mathfrak{g})^k_{[(\lambda)]} \)-module \( L \) admits a unique (up to shift of grading) \( \Lambda_C \)-grading by which \( L \in \text{mod}^k_{[(\lambda)]}(U_\mathfrak{c}(\mathfrak{g})) \). We denote by \( \{ L_\sigma \}_{\sigma \in \Theta} \) the collection of irreducible \( U_\mathfrak{c}(\mathfrak{g})^k_{[(\lambda)]} \)-modules with \( \Lambda_C \)-gradings contained in \( \text{mod}^k_{[(\lambda)]}(U_\mathfrak{c}(\mathfrak{g})) \). The group \( \Lambda_C \) acts on \( \Theta \) freely by

\[ L_\sigma = L_\sigma[\gamma] \quad (\gamma \in \Lambda_C; \sigma \in \Theta), \]

where \( [\gamma] \) denotes the shift of grading, and the set of the irreducible \( U_\mathfrak{c}(\mathfrak{g})^k_{[(\lambda)]} \)-modules is parametrized by \( \Theta/\Lambda_C \). For \( \sigma \in \Theta \) we define \( E_\sigma \) (resp. \( F_\sigma \)) to be the
Recall that we have equivalences
\[ L \rightarrow \text{objects} \]
with respect to \( \xi^k : Z_{\mathfrak{U}}(U_\xi(g)) \rightarrow \mathbb{C} \).

If \( \mathfrak{A} \) is an abelian category or a triangulated category, we denote its Grothendieck
group by \( K(\mathfrak{A}) \).

By (9.2) we obtain
\[ K(\mod(U_\xi(g)^k; \mathcal{C})) \cong K(\mod(U_\xi(g); \mathcal{C})) \cong K(\mod(U_\xi(g)^k; \mathcal{C})) \]
\[ \subseteq K(\mod(U_\xi(g)^k; \mathcal{C})). \]

Note that \( K(\mod(U_\xi(g)^k; \mathcal{C})) \) is a free \( \mathbb{Z} \)-module with basis \( \{[L_\sigma]\}_{\sigma \in \Theta} \).

9.2. Recall that we have equivalences
\[ D^b(\mod(U_\xi(g)^k; \mathcal{C})) \cong D^b(\mod(O_\xi; \mathcal{C})) \quad (\lambda \in \Lambda_{\text{reg}}), \]
\[ D^b(\mod(U_\xi(g)^k; \mathcal{C})) \cong D^b(\mod(O_\xi; \mathcal{C})) \quad (\lambda \in \Lambda_{\text{reg}}) \]
of triangulated categories and an equivalence
\[ \mod(U_\xi(g)^k; \mathcal{C}) \cong \mod(\mathcal{O}_\xi; \mathcal{C}) \quad (\lambda \in \Lambda \cap A_0) \]
of abelian categories. We have the following identifications of the Grothendieck
groups:
\[ K(\mod(\mathcal{O}_\xi; \mathcal{C})) = K(\mod(\mathcal{O}_\xi; \mathcal{C})), \]
\[ K(\mod(\mathcal{O}_\xi; \mathcal{C})) = K(D^b(\mod(\mathcal{O}_\xi; \mathcal{C}))) \]
and the following commutative diagram:
\[ K(\mod(U_\xi(g)^k; \mathcal{C})) \xrightarrow{\cong} K(\mod(U_\xi(g)^k; \mathcal{C})) \]
\[ \cong K(\mod(\mathcal{O}_\xi; \mathcal{C})) \xrightarrow{\cong} K(\mod(\mathcal{O}_\xi; \mathcal{C})). \]

For \( \sigma \in \Theta \) we denote by \( \mathcal{L}_\sigma, \mathcal{E}_\sigma, \tilde{\mathcal{E}}_\sigma \) the objects of \( \mod(\mathcal{O}_\xi; \mathcal{C}) \) corresponding to the objects \( L_\sigma, E_\sigma, \tilde{E}_\sigma \) of \( \mod(U_\xi(g)^k; \mathcal{C}) \) respectively. We have the following description of \( \mathcal{L}_\sigma, \mathcal{E}_\sigma, \tilde{\mathcal{E}}_\sigma \).

**Proposition 9.1.**

(i) The collection \( \{\mathcal{L}_\sigma\}_{\sigma \in \Theta} \) is exactly the set of irreducible objects of \( \mod(\mathcal{O}_\xi; \mathcal{C}) \) contained in \( D^b(\mod(\mathcal{O}_\xi; \mathcal{C})) \).

(ii) For \( \sigma \in \Theta \) the exotic sheaf \( \tilde{E}_\sigma \) is a projective cover of \( \mathcal{L}_\sigma \) in \( \mod(\mathcal{O}_\xi; \mathcal{C}) \).

(iii) We have \( \mathcal{E}_\sigma = \mathcal{O}_{\{\xi\}} \otimes_{\mathcal{O}_\xi} \tilde{\mathcal{E}}_\sigma \) for \( \sigma \in \Theta \).

**Proof.** The assertions (i) and (ii) are obvious from the equivalence of categories. We can easily show (iii) from (9.3). \( \square \)

We have the following result due to Bezrukavnikov-Mirković (see [14, Proposition 5.1.4]).
Theorem 9.2 ([14]). For any σ ∈ Θ the exotic sheaf Ėσ is a locally free O_Eσ-module.

9.3. Let us identify B^{\tilde{x}} and ˜B^{\tilde{x}} with B^a and ˜B^a respectively via (8.5). Here, a ∈ g is the nilpotent element satisfying exp(\tilde{a}) = \tilde{x}. Take b, c ∈ g such that [\tilde{c}, \tilde{a}] = 2\tilde{a}, [\tilde{c}, \tilde{b}] = -2\tilde{b}, [\tilde{a}, \tilde{b}] = \tilde{c}, and set

\tilde{S}^a = \{(B^- g, a) ∈ \tilde{g} | a ∈ g_{nil} ∩ (\tilde{a} + J_g(\tilde{b}))\},

where g_{nil} denotes the set of nilpotent elements in g and J_g(\tilde{b}) is the centralizer of \tilde{b} in g. We may assume that C is a maximal torus of the simultaneous centralizer of \tilde{a}, \tilde{b}, \tilde{c}. The right action of C on \tilde{g} given by

(B^- g, a)h = (B^- gh, Ad(h)^{-1}(a)) \quad ((B^- g, a) ∈ \tilde{g}, h ∈ C)

preserves B^a, ˜B^a and S^a. We have the identifications

mod_{B^a}(O_{\tilde{g}}; C) = mod_{\tilde{B}^a}(O_{\tilde{g}}; C), \quad \text{mod}(O_{\tilde{B}^a}; C) = \text{mod}(O_{\tilde{B}^a}; C).

We define a right action of C' = C × C^\times on g and \tilde{g} by

a · (h, z) = z^2 Ad(h)^{-1}(\phi(z)^{-1})a \quad (a ∈ g, (h, z) ∈ C'),

(B^- g, a)(h, z) = (B^- g\phi(z)h, a · (h, z)) \quad ((B^- g, a) ∈ \tilde{g}, (h, z) ∈ C'),

where the homomorphism \phi : C^\times → G is given by \phi(exp(u)) = exp(uc) for u ∈ C. We have natural lifts L'_{σ}, E'_{σ}, ˜E'_{σ} ∈ D^b(mod(\tilde{O}_{\tilde{g}}; C'^)) of L_σ, E_σ, ˜E_σ ∈ D^b(mod(\tilde{O}_{\tilde{g}}; C)) (see [14, 5.3.2]). Moreover, for each σ ∈ Θ there exists uniquely a C'-equivariant locally free O_{S^a}-module ˜E'_{σ} whose restriction to the formal neighborhood of B^a in S^a coincides with the restriction of ˜E'_{σ} (see [14, 5.3.1]). By Proposition 9.4 and Theorem 9.2 we have

(9.10)

E'_{σ} = O(\tilde{x}) ⊗ B^a ˜E'_{σ}.

Let R_{C'} be the representation ring of C', and let \mathcal{R}_{C'} be its quotient field. The direct image with respect the closed embedding B^a ↪ S^a gives an embedding

(9.11) K(mod(O_{B^a}; C')) ⊂ K(mod(O_{S^a}; C'))

of free R_{C'}-modules of the same rank dim_Q H^*(B^a; Q) (see [35, Theorem 1.14]). This induces the identification

(9.12) \mathcal{R}_{C'} ⊗_{R_{C'}} K(mod(O_{B^a}; C')) = \mathcal{R}_{C'} ⊗_{R_{C'}} K(mod(O_{S^a}; C'))

of \mathcal{R}_{C'}-modules. We denote by f → f^* the automorphism of \mathcal{R}_{C'} induced by C' ∋ c → c^{-1} ∈ C'. Set

\nabla_a = \left(\sum_r (-1)^r [\bigwedge^r J_g(\tilde{b})]\right) \left(\sum_r (-1)^r [\bigwedge^r h]\right)^{-1} ∈ R_{C'}.

In [35, Conjecture 5.12, Conjecture 5.16] Lusztig conjectured the existence of certain canonical \mathbb{Z}[v, v^{-1}]-bases B_{S^a} and B_{S^a} of K(mod(O_{S^a}; C')) and K(mod(O_{S^a}; C')) respectively. Here \mathbb{Z}[v, v^{-1}] is identified with the representation ring of C^\times. This conjecture was proved by Bezrukavnikov-Mirković [14] as follows.

Theorem 9.3 ([14]). (i) The set \{[L'_{σ}]\}_σ ∈ Θ turns out to be Lusztig’s conjectural canonical base B_{S^a} of K(mod(O_{S^a}; C')).
(ii) The set \( \{ [\tilde{E}_\sigma'] \}_{\sigma \in \Theta} \) turns out to be Lusztig’s conjectural canonical base \( B_{\tilde{S}} \) of \( K(\text{mod}(\mathcal{O}_{\tilde{S}}; C')) \).

(iii) We have

\[
[\mathcal{E}_\sigma'] = \nabla_{\nu}^v [\tilde{E}_\sigma'] \quad (\sigma \in \Theta)
\]

in \( K(\text{mod}(\mathcal{O}_{B}^\sigma; C')) \subset K(\text{mod}(\mathcal{O}_{\tilde{S}}; C')) \).

We define \( n_{\tau,\sigma} \in \mathbb{Z}[v, v^{-1}] \) \( (\sigma, \tau \in \Theta) \) by

\[
(9.13) \quad [\mathcal{E}_\sigma'] = \sum_{\tau \in \Theta} n_{\tau,\sigma} [\mathcal{L}_\tau'] \quad (\sigma \in \Theta)
\]

in \( K(\text{mod}(\mathcal{O}_{B}^\sigma; C')) \). We note that there exists a description of \( n_{\tau,\sigma} \) in terms of \( B_{\tilde{S}} \) and a certain geometrically defined bilinear form on \( K(\text{mod}(\mathcal{O}_{\tilde{S}}; C')) \), which we omit here (see [35], [14] for more details). We obtain the following result conjectured by Lusztig [35, 17.2].

**Theorem 9.4.** For \( \sigma \in \Theta \) we have

\[
[E_\sigma] = \sum_{\tau \in \Theta} n_{\tau,\sigma}(1)[L_\tau] \quad (\sigma \in \Theta)
\]

in \( K(\text{mod}_{[i(\lambda)]}(U_\zeta(\mathfrak{g})^k; C')) \).

**Proof.** By (9.13) we have

\[
[\mathcal{E}_\sigma] = \sum_{\tau \in \Theta} n_{\tau,\sigma}(1)[\mathcal{L}_\tau] \quad (\sigma \in \Theta)
\]

in \( K(\text{mod}(\mathcal{O}_{B}^\sigma; C')) \). The desired formula follows from this by the equivalence of categories. \( \square \)

**Appendix A. Quantized Partial Flag Manifolds**

We give an account of certain results concerning the quantized partial flag manifolds, which are analogous to those in [31], [32], [35] for the quantized ordinary flag manifold (see also [6] for some related results). Some of them are shown by repeating the original arguments, and others are being deduced from the original results presented in [31], [32], [35].

**A.1. \( \mathcal{O} \)-modules.** Let \( J \subset I \). We set

\[
\Delta_J = \left( \sum_{j \in J} \mathbb{Z}\alpha_j \right) \cap \Delta,
\]

\[
\Lambda^J = \{ \lambda \in \Lambda \mid \langle \lambda, \alpha_j^\vee \rangle = 0 \ \ (j \in J) \} = \bigoplus_{i \notin J} \mathbb{Z}\omega_i,
\]

\[
\Lambda^{J+} = \Lambda^J \cap \Lambda^+ = \bigoplus_{i \notin J} \mathbb{Z}_{\geq 0}\omega_i.
\]

We define subalgebras \( I_J, n^-_J, p^-_J \) of \( \mathfrak{g} \) by

\[
I_J = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_J} \mathfrak{g}_\alpha, \quad n^-_J = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_J} \mathfrak{g}_{-\alpha}, \quad p^-_J = I_J \oplus n^-_J.
\]
The corresponding connected closed subgroups of $G$ are denoted as $L_J$, $N_J^-$, $P_J^-$ respectively.

Let $\zeta$ be a non-zero complex number equipped with a choice of $\zeta^{1/|\Lambda/Q|} \in \mathbb{C}^\times$ satisfying $(\zeta^{1/|\Lambda/Q|})^{|\Lambda/Q|} = \zeta$. We denote by $U^L_\zeta(p_J)$ the subalgebra of $U^L_\zeta(\mathfrak{g})$ generated by $U^L_\zeta(h)$, $e_j^{(n)}$, $f_i^{(n)}$ for $j \in J$, $i \in I$, $n \geq 0$. For $\lambda \in \Lambda^J$ the algebra homomorphism $\chi_\lambda : U^L_\zeta(h) \to \mathbb{C}$ is extended to

\[
\chi_{J,\lambda} : U^L_\zeta(p_J) \to \mathbb{C}
\]

by $\chi_{J,\lambda}(e_j^{(n)}) = \chi_{J,\lambda}(f_i^{(n)}) = 0$ for $j \in J$, $i \in I$, $n \in \mathbb{Z}_{\geq 0}$. We define a $\mathbb{C}$-subalgebra $A_{J,\zeta}$ of $A_\zeta$ by

\[
A_{J,\zeta} = \bigoplus_{\lambda \in \Lambda^J_{+}} A_\zeta(\lambda).
\]

It is a $\Lambda^J$-graded $\mathbb{C}$-algebra. We denote by $\text{Tor}_{\Lambda^J_+}(A_{J,\zeta})$ the full subcategory of $\text{Mod}_{\Lambda^J}(A_{J,\zeta})$ consisting of $M \in \text{Mod}_{\Lambda^J}(A_{J,\zeta})$ satisfying

\[
\forall m \in M, \exists N \text{ s.t. } \lambda \in \Lambda^J_{+}, \langle \lambda, \alpha_i^\vee \rangle > N (\forall i \in I \setminus J) \implies A_\zeta(\lambda)m = 0,
\]

and set

\[
\text{Mod}(\mathcal{O}_{P_{J,\zeta}}) = \text{Mod}_{\Lambda^J}(A_{J,\zeta})/\text{Tor}_{\Lambda^J_+}(A_{J,\zeta}).
\]

The natural exact functor

\[
\omega^*_J : \text{Mod}_{\Lambda^J}(A_{J,\zeta}) \to \text{Mod}(\mathcal{O}_{P_{J,\zeta}})
\]

admits a right adjoint

\[
\omega_{J*} : \text{Mod}(\mathcal{O}_{P_{J,\zeta}}) \to \text{Mod}_{\Lambda^J}(A_{J,\zeta}),
\]

which is left exact, and we have $\omega^*_J \circ \omega_{J*} = \text{Id}$. We define the global section functor

\[
\Gamma(\mathcal{P}_{J,\zeta}, \bullet) : \text{Mod}(\mathcal{O}_{P_{J,\zeta}}) \to \text{Mod}(\mathbb{C})
\]

by $\Gamma(\mathcal{P}_{J,\zeta}, N) = (\omega_{J*} N)(0)$.

We define a full subcategory $\text{mod}(\mathcal{O}_{P_{J,\zeta}})$ of $\text{Mod}(\mathcal{O}_{P_{J,\zeta}})$ by

\[
\text{mod}(\mathcal{O}_{P_{J,\zeta}}) = \text{mod}_{\Lambda^J}(A_{J,\zeta})/(\text{Tor}_{\Lambda^J_+}(A_{J,\zeta}) \cap \text{mod}_{\Lambda^J}(A_{J,\zeta})).
\]

The functors \[A.3\), \[A.4\)] induce

\[
\omega^*_J : \text{mod}_{\Lambda^J}(A_{J,\zeta}) \to \text{mod}(\mathcal{O}_{P_{J,\zeta}}),
\]

\[
\omega_{J*} : \text{mod}(\mathcal{O}_{P_{J,\zeta}}) \to \text{mod}_{\Lambda^J}(A_{J,\zeta}).
\]

For $\mu \in \Lambda^J$ we define and exact functor

\[
(\bullet)[\mu] : \text{Mod}(\mathcal{O}_{P_{J,\zeta}}) \to \text{Mod}(\mathcal{O}_{P_{J,\zeta}}) \quad (M \mapsto M[\mu])
\]

to be the functor induced by

\[
(\bullet)[\mu] : \text{Mod}_{\Lambda^J}(A_{J,\zeta}) \to \text{Mod}_{\Lambda^J}(A_{J,\zeta}) \quad (M \mapsto M[\mu]),
\]

where

\[
(M[\mu])(\lambda) = M(\lambda + \mu) \quad (\lambda \in \Lambda^J).
\]
Remark A.1. In the case $\zeta = 1$ the category $\text{Mod}(\mathcal{O}_{P_J^\lambda})$ (resp. $\text{mod}(\mathcal{O}_{P_J^\lambda})$) is naturally identified with the category $\text{Mod}(\mathcal{O}_{P_J})$ (resp. $\text{mod}(\mathcal{O}_{P_J})$) consisting of quasi-coherent (resp. coherent) sheaves on the partial flag manifold $P_J$ and if for any $U_J = P_J \setminus G$.

We say that a left $U_J^L(\mathfrak{p}_J)$-module $M$ is integrable if it is a direct sum of weight spaces

$$M_\mu = \{ m \in M \mid am = \chi_\mu(a)m \text{ for } a \in U_J^L(\mathfrak{h}) \} \quad (\mu \in \Lambda),$$

and if for any $m \in M$ there exists $N > 0$ such that $e_{i,j}^{(n)}m = f_{i,j}^{(n)}m = 0$ for $i \in I$, $j \in J$, $n > N$. The notion of an integrable right $U_J^L(\mathfrak{p}_J)$-module is defined similarly. We denote by $\text{Mod}^{\text{int}}(U_J^L(\mathfrak{p}_J))$ (resp. $\text{mod}^{\text{int}}(U_J^L(\mathfrak{p}_J))$) the category of left (resp. right) integrable $U_J^L(\mathfrak{p}_J)$-modules. We define $\mathcal{O}_\zeta(P_J)$ to be the subspace of $U_J^L(\mathfrak{p}_J)^*$ consisting of $\varphi \in U_J^L(\mathfrak{p}_J)^*$ such that the left $U_J^L(\mathfrak{p}_J)$-module $U_J^L(\mathfrak{p}_J)\varphi$ is integrable. Then $\mathcal{O}_\zeta(P_J)$ is naturally a Hopf algebra and a $U_J^L(\mathfrak{p}_J)$-bimodule.

We define an abelian category $\widetilde{\text{Mod}}(\mathcal{O}_{P_J^\zeta})$ similarly to the abelian category $\widetilde{\text{Mod}}(\mathcal{O}_{B_\zeta})$ using $U_J^L(\mathfrak{p}_J)$ instead of $U_J^L(b^-)$. Namely, an object of $\widetilde{\text{Mod}}(\mathcal{O}_{P_J^\zeta})$ is a vector space $M$ over $\mathbb{C}$ equipped with a left $\mathcal{O}_\zeta(G)$-module structure and a right integrable $U_J^L(\mathfrak{p}_J)$-module structure satisfying the natural compatibility condition.

Remark A.2. In the case $\zeta = 1$ the category $\widetilde{\text{Mod}}(\mathcal{O}_{P_J^\zeta})$ is naturally identified with the category consisting of quasi-coherent sheaves on $G$, which are equivariant with respect to the action of $P_J$ on $G$ given by the left multiplication. Hence it is also equivalent to $\text{Mod}(\mathcal{O}_{P_J})$.

Similarly to (3.9) we define a functor

$$(A.10) \quad \tilde{\omega}_J^* : \widetilde{\text{Mod}}(\mathcal{O}_{P_J^\zeta}) \to \text{Mod}_{\Lambda^J}(A_{J,\zeta})$$

by

$$\tilde{\omega}_J^* M = \sum_{\lambda \in \Lambda^J} (\tilde{\omega}_J^* M)(\lambda) = \bigoplus_{\lambda \in \Lambda^J} (\tilde{\omega}_J^* M)(\lambda) \subset M,$$

$$(\tilde{\omega}_J^* M)(\lambda) = \{ m \in M \mid \mu y = \chi_{J,\lambda}(y)m \ (y \in U_J^L(\mathfrak{p}_J)) \}$$

for $M \in \widetilde{\text{Mod}}(\mathcal{O}_{P_J^\zeta})$.

Similarly to (3.10) we define an exact functor

$$(A.11) \quad (\bullet)[\mu] : \widetilde{\text{Mod}}(\mathcal{O}_{P_J^\zeta}) \to \widetilde{\text{Mod}}(\mathcal{O}_{P_J^\zeta}) \quad (\mu \in \Lambda^J)$$

by $M[\mu] = M \otimes \mathbb{C}_{-\mu}$, where for $\mu \in \Lambda^J$ we denote by $\mathbb{C}_\mu$ the one-dimensional right $U_J^L(\mathfrak{p}_J)$-module corresponding to $\chi_{J,\mu} : U_J^L(\mathfrak{p}_J) \to \mathbb{C}$.

Similarly to (3.11) we define a functor

$$(A.12) \quad \tilde{\omega}_J^* : \text{Mod}_{\Lambda^J}(A_{J,\zeta}) \to \widetilde{\text{Mod}}(\mathcal{O}_{P_J^\zeta})$$

by $\tilde{\omega}_J^* K = \mathcal{O}_{J}(G) \otimes_{A_{J,\zeta}} K$.

The following generalization of Proposition 3.5 is also a consequence of the general theory due to Artin-Zhang [3]. We only give a sketch of the proof.
Theorem A.3. The functor $\omega_j^* \circ \tilde{\omega}_{J_*}$ gives an equivalence
\begin{equation}
\text{Mod}(\mathcal{O}_{P_{J,\zeta}}) \cong \text{Mod}(\mathcal{O}_{P_{J,\zeta}})
\end{equation}
of abelian categories compatible with the shift functors (A.9), (A.11). Moreover, identifying $\text{Mod}(\mathcal{O}_{P_{J,\zeta}})$ with $\text{Mod}(\mathcal{O}_{P_{J,\zeta}})$ via (A.13) we have
\begin{equation}
\tilde{\omega}_j^* = \omega_j^*, \quad \tilde{\omega}_{J_*} = \omega_{J_*}.
\end{equation}

**Proof.** By an obvious generalization of [3] Theorem 4.5 we can show that $\omega_j^* \circ \tilde{\omega}_{J_*}$ gives an equivalence which is compatible with the shift functors. Moreover, we have
\begin{equation}
\omega_{J_*} \circ \omega_j^* \circ \tilde{\omega}_{J_*} = \tilde{\omega}_{J_*}.
\end{equation}
Here, in order to show that the assumption of [3] Theorem 4.5 is satisfied, we need some properties of the derived induction functors for quantized enveloping algebras. More specifically, we use the Kempf type vanishing theorem [1, Corollary 5.7] and the vanishing of cohomologies in sufficiently large degrees [1, Theorem 5.8]. Then we obtain $\tilde{\omega}_{J_*} = \omega_{J_*}$ by (A.15). The remaining $\tilde{\omega}_j^* = \omega_j^*$ follows from the fact that $\tilde{\omega}_j^*, \omega_j^*$ are the left adjoint functors to $\tilde{\omega}_{J_*}, \omega_{J_*}$ respectively. \qed

**Lemma A.4.** We have
\begin{equation}
\omega_{J_*} \omega_j^* A_{J,\zeta} \cong A_{J,\zeta}.
\end{equation}

**Proof.** By Theorem A.3 it is sufficient to show $\tilde{\omega}_{J_*} \omega_j^* A_{J,\zeta} \cong A_{J,\zeta}$. This is obvious from definition. \qed

A.2. Equivariant objects. Similarly to [42 Section 4] we define $\text{Mod}^{eq}_\Lambda(A_{J,\zeta})$ to be the category consisting of $N \in \text{Mod}_\Lambda(A_{J,\zeta})$ equipped with a left integrable $U^L_\zeta(\mathfrak{g})$-module structure satisfying
\begin{equation}
U^L_\zeta(\mathfrak{g}) N(\lambda) \subset N(\lambda) \quad (\lambda \in \Lambda^J),
\end{equation}
\begin{equation}
\varphi n = \sum_{(u)} (u(0) \varphi) \cdot (u(1)n) \quad (u \in U^L_\zeta(\mathfrak{g}), \varphi \in A_{J,\zeta}, n \in N),
\end{equation}
and set
\begin{equation}
\text{Mod}^{eq}(\mathcal{O}_{P_{J,\zeta}}) = \text{Mod}^{eq}_\Lambda(A_{J,\zeta}) / \text{Mod}^{eq}_\Lambda(A_{J,\zeta}) \cap \text{Tor}_{\Lambda^J}(A_{J,\zeta}).
\end{equation}

**Remark A.5.** In the case $\zeta = 1$ the category $\text{Mod}^{eq}(\mathcal{O}_{P_{J,1}})$ is naturally identified with the category consisting of $G$-equivariant quasi-coherent $\mathcal{O}_{P_J}$-modules.

Similarly to [42 Section 4] we define $\tilde{\text{Mod}}^{eq}(\mathcal{O}_{P_{J,\zeta}})$ to be the category consisting of an object $M$ of $\text{Mod}(\mathcal{O}_{P_{J,\zeta}})$ equipped with a left integrable $U^L_\zeta(\mathfrak{g})$-module structure satisfying
\begin{equation}
\varphi m = \sum_{(u)} (u(0) \varphi) \cdot (u(1)m) \quad (u \in U^L_\zeta(\mathfrak{g}), \varphi \in \mathcal{O}_\zeta(G), m \in M).
\end{equation}

**Remark A.6.** In the case $\zeta = 1$ the category $\tilde{\text{Mod}}^{eq}(\mathcal{O}_{P_{J,1}})$ is naturally identified with the category consisting of quasi-coherent $\mathcal{O}_G$-modules equivariant under the left action of $G$ via the left multiplication and the right action of $P_J$ via the right multiplication.
For a left (resp. right) $U_L^\zeta(p_J)$-module $M$ we denote by $M^{[r]}$ (resp. $M^{[l]}$) the right (resp. left) $U_L^\zeta(p_J)$-module with the same underlying vector space as $M$ and the right (resp. left) action of $U_L^\zeta(p_J)$ given by

$$my = (Sy) \cdot m \quad (\text{resp. } gm = m \cdot (S^{-1}y)) \quad (m \in M, \quad y \in U_L^\zeta(p_J)).$$

We have $M^{[r][l]} = M$ and $M^{[l][r]} = M$.

Similarly to [2] Section 4 we have the following equivalences of abelian categories:

$$\text{Mod}_\text{int}(U_L^\zeta(p_J)) \cong \text{Mod}^{\text{eq}}(\mathcal{O}_{P_J^\zeta}) \cong \text{Mod}^{\text{eq}}(\mathcal{O}_{P_J^\zeta}).$$

Here, the first equivalence is given by associating $M \in \text{Mod}^{\text{eq}}(\mathcal{O}_{P_J^\zeta})$ to $(M^{U_L^\zeta(g)})^{[l]}$, where $M^{U_L^\zeta(g)}$ denotes the right $U_L^\zeta(p_J)$-module structure of $M$. The opposite correspondence is given by associating $M \in \text{Mod}^{\text{eq}}(\mathcal{O}_{P_J^\zeta})$ to $(M^{U_L^\zeta(g)})^{[r]}$ to $\mathcal{O}_{\zeta}(G) \otimes K \in \text{Mod}^{\text{eq}}(\mathcal{O}_{P_J^\zeta})$. Here, the left $\mathcal{O}_{\zeta}(G)$-module structure and the left $U_L^\zeta(g)$-module structure of $\mathcal{O}_{\zeta}(G) \otimes K$ are induced by those on the first factor $\mathcal{O}_{\zeta}(G)$, and the right $U_L^\zeta(p_J)$-module structure is given by

$$(\varphi \otimes k)y = \sum_{(y)} \varphi y(0) \otimes (Sy(1)) \cdot k \quad (\varphi \in \mathcal{O}_{\zeta}(G), \ k \in K, \ y \in U_L^\zeta(p_J)).$$

The second equivalence in (A.17) is induced by (A.13). We write

$$\mathcal{F}_J : \text{Mod}_\text{int}(U_L^\zeta(p_J)) \rightarrow \text{Mod}^{\text{eq}}(\mathcal{O}_{P_J^\zeta})$$

for the composite of (A.17).

A.3. **Induction functor.** Let $J_1 \subset J_2 \subset I$. We have the obvious restriction functor

$$\text{Res}^{J_1,J_2} : \text{Mod}_\text{int}(U_L^\zeta(p_{J_2})) \rightarrow \text{Mod}_\text{int}(U_L^\zeta(p_{J_1})).$$

The induction functor

$$\text{Ind}^{J_2,J_1} : \text{Mod}_\text{int}(U_L^\zeta(p_{J_1})) \rightarrow \text{Mod}_\text{int}(U_L^\zeta(p_{J_2})), \quad (A.20)$$

which is right adjoint to (A.19) is given as follows. For $M \in \text{Mod}_\text{int}(U_L^\zeta(p_{J_1}))$ we define a left $U_L^\zeta(p_{J_1})$-module structure of $\mathcal{O}_{\zeta}(P_{J_2}) \otimes M$ by

$$y \ast (\varphi \otimes m) = \sum_{(y)} \varphi \cdot S^{-1}y(0) \otimes y(1)m \quad (y \in U_L^\zeta(p_{J_1}), \ \varphi \in \mathcal{O}_{\zeta}(P_{J_2}), \ m \in M),$$

and set

$$\text{Ind}^{J_2,J_1}M = \{z \in \mathcal{O}_{\zeta}(P_{J_2}) \otimes M \mid y \ast z = \varepsilon(y)z \ (y \in U_L^\zeta(p_{J_1}))\}.$$ 

Then the left $U_L^\zeta(p_{J_2})$-module structure of $\mathcal{O}_{\zeta}(G) \otimes M$ given by

$$u(\varphi \otimes m) = u\varphi \otimes m \quad (u \in U_L^\zeta(p_{J_2}), \ \varphi \in \mathcal{O}_{\zeta}(P_{J_2}), \ m \in M)$$

induces a left integrable $U_L^\zeta(p_{J_2})$-module structure of $\text{Ind}^{J_2,J_1}M$. 

We have also the right module versions

\[(A.21) \quad \text{Res}^r_{J_1, J_2} : \text{Mod}^r_{\text{int}}(U^L_{\zeta}(p^{-1}_{J_2})) \to \text{Mod}^r_{\text{int}}(U^L_{\zeta}(p^{-1}_{J_1})),\]
\[(A.22) \quad \text{Ind}^r_{J_2, J_1} : \text{Mod}^r_{\text{int}}(U^L_{\zeta}(p^{-1}_{J_1})) \to \text{Mod}^r_{\text{int}}(U^L_{\zeta}(p^{-1}_{J_2})).\]

Here, \(\text{Res}^r_{J_1, J_2}\) is the obvious restriction functor, and \(\text{Ind}^r_{J_2, J_1}\) is given by \(\text{Ind}^r_{J_2, J_1}(M) = (\text{Ind}^r_{J_2, J_1}(M^{[l]}))^{[r]}\).

### A.4. Inverse and direct images.

For \(J_1 \subset J_2 \subset I\) we define functors

\[\begin{align*}
(\pi_{\zeta}^{J_2, J_1})^* &: \text{Mod}(O_{P_{J_2, \zeta}}) \to \text{Mod}(O_{P_{J_1, \zeta}}), \\
(\pi_{\zeta}^{J_2, J_1})_* &: \text{Mod}(O_{P_{J_1, \zeta}}) \to \text{Mod}(O_{P_{J_2, \zeta}}), \\
(\pi_{\zeta}^{J_2, J_1})^* &: \text{Mod}(O_{P_{J_2, \zeta}}) \to \text{Mod}(O_{P_{J_1, \zeta}}), \\
(\pi_{\zeta}^{J_2, J_1})_* &: \text{Mod}(O_{P_{J_1, \zeta}}) \to \text{Mod}(O_{P_{J_2, \zeta}})
\end{align*}\]

in the following.

The definition of

\[(A.23) \quad (\pi_{\zeta}^{J_2, J_1})_* : \text{Mod}(O_{P_{J_2, \zeta}}) \to \text{Mod}(O_{P_{J_1, \zeta}})\]

is simple. For \(M \in \text{Mod}(O_{P_{J_2, \zeta}})\) we have \((\pi_{\zeta}^{J_2, J_1})^* M = M\) as a left \(O_\zeta(G)\)-module and \((\pi_{\zeta}^{J_2, J_1})^* M = \text{Res}^r_{J_1, J_2} M\) as a right \(U^L_{\zeta}(p_{J_2}^{-1})\)-module. This is obviously an exact functor. By definition we have the following commutative diagram:

\[(A.24) \quad \begin{array}{ccc}
\text{Mod}_{A_{J_2, \zeta}}(A_{J_2, \zeta}) & \xrightarrow{A_{J_1, \zeta} \otimes A_{J_2, \zeta}(\bullet)} & \text{Mod}_{A_{J_1, \zeta}}(A_{J_1, \zeta}) \\
\tilde{\omega}_{J_2} & & \tilde{\omega}_{J_1} \\
\text{Mod}(O_{P_{J_2, \zeta}}) & \xrightarrow{(\pi_{\zeta}^{J_2, J_1})_*} & \text{Mod}(O_{P_{J_1, \zeta}})
\end{array}\]

We next give the definition of the functor

\[(A.25) \quad (\pi_{\zeta}^{J_2, J_1})_* : \text{Mod}(O_{P_{J_1, \zeta}}) \to \text{Mod}(O_{P_{J_2, \zeta}})\]

Let \(M \in \text{Mod}(O_{P_{J_1, \zeta}})\). As a right \(U^L_{\zeta}(p_{J_2}^{-1})\)-module we have \((\pi_{\zeta}^{J_2, J_1})_* M = \text{Ind}^r_{J_2, J_1} M\).

The left \(O_\zeta(G)\)-module structure of \(\text{Ind}^r_{J_2, J_1} M\) is given as follows. Let \(\alpha : O_\zeta(G) \otimes M \to M\) be the left \(O_\zeta(G)\)-module structure of \(M\). Since \(\alpha\) is a homomorphism of right \(U^L_{\zeta}(p_{J_2}^{-1})\)-module we have a homomorphism

\[\text{Ind}^r_{J_2, J_1} \alpha : \text{Ind}^r_{J_2, J_1}(O_\zeta(G) \otimes M) \to \text{Ind}^r_{J_2, J_1} M\]

of right \(U^L_{\zeta}(p_{J_2}^{-1})\)-modules. Note that \(O_\zeta(G)\) is an integrable right \(U^L_{\zeta}(p_{J_2}^{-1})\)-module. Hence by the standard property of the induction functor called the tensor identity we have \(\text{Ind}^r_{J_2, J_1}(O_\zeta(G) \otimes M) \cong O_\zeta(G) \otimes \text{Ind}^r_{J_2, J_1} M\). Hence we obtain a homomorphism

\[\tilde{\alpha} : O_\zeta(G) \otimes \text{Ind}^r_{J_2, J_1} M \to \text{Ind}^r_{J_2, J_1} M\]

of right \(U^L_{\zeta}(p_{J_2}^{-1})\)-modules. This gives the desired left \(O_\zeta(G)\)-module structure of \(\text{Ind}^r_{J_2, J_1} M\).
Define a functor

\[ F_{\zeta}^{J_2 J_1} : \text{Mod}_{A^{J_1}}(A_{J_1, \zeta}) \to \text{Mod}_{A^{J_2}}(A_{J_2, \zeta}) \]

by

\[ F_{\zeta}^{J_2 J_1}(M) = \bigoplus_{\lambda \in \Lambda^{J_2}} M(\lambda). \]

**Lemma A.7.** The following diagram is commutative:

(A.26) \[
\begin{array}{ccc}
\text{Mod}_{A^{J_1}}(A_{J_1, \zeta}) & \xrightarrow{\sim} & \text{Mod}_{A^{J_2}}(A_{J_2, \zeta}) \\
\sim \downarrow & & \sim \downarrow \\
\tilde{\text{Mod}}(\mathcal{O}_{P_{J_1, \zeta}}) & \xrightarrow{(\pi_{\zeta}^{J_2 J_1})_*} & \tilde{\text{Mod}}(\mathcal{O}_{P_{J_2, \zeta}}).
\end{array}
\]

**Proof.** For \( M \in \tilde{\text{Mod}}(\mathcal{O}_{P_{J_1, \zeta}}) \), \( \lambda \in \Lambda^{J_2} \) we have

\[
(\tilde{\omega}_{J_2*}(\pi_{\zeta}^{J_2 J_1})_* M)(\lambda) \cong \text{Hom}_{\text{Mod}^{\text{int}}_\mathcal{O}(U_{\zeta}^2(p_{J_2}))(\mathcal{O}_\zeta, (\pi_{\zeta}^{J_2 J_1})_* M)} \\
\cong \text{Hom}_{\text{Mod}^{\text{int}}_\mathcal{O}(U_{\zeta}^2(p_{J_2}))(\mathcal{O}_\zeta, \text{Ind}^{J_2 J_1}(M))} \\
\cong \text{Hom}_{\text{Mod}^{\text{int}}_\mathcal{O}(U_{\zeta}^2(p_{J_1}))(\mathcal{O}_\zeta, M)} \\
\cong (\tilde{\omega}_{J_1*} M)(\lambda)
\]

by the Frobenius reciprocity. \( \square \)

**Lemma A.8.** The functor \( (\pi_{\zeta}^{J_2 J_1})_* \) is right adjoint to \( (\pi_{\zeta}^{J_2 J_1})^* \).

**Proof.** For \( M \in \tilde{\text{Mod}}(\mathcal{O}_{P_{J_1, \zeta}}) \), \( N \in \tilde{\text{Mod}}(\mathcal{O}_{P_{J_2, \zeta}}) \) we have

\[
\text{Hom}( (\pi_{\zeta}^{J_2 J_1})^* N, M) = \text{Hom}_{\text{Mod}^{\text{int}}_\mathcal{O}(\mathcal{O}_{P_{J_1, \zeta}})}(N, M) \\
= \text{Hom}_{\text{Mod}^{\text{int}}_\mathcal{O}(U_{\zeta}^1(p_{J_1})))}(N, M) \cap \text{Hom}_{\text{Mod}^{\text{int}}_\mathcal{O}(\mathcal{O}_{\zeta(G)})}(N, M),
\]

\[
\text{Hom}(N, (\pi_{\zeta}^{J_2 J_1})_* M) = \text{Hom}_{\text{Mod}^{\text{int}}_\mathcal{O}(U_{\zeta}^1(p_{J_1})))}(N, \text{Ind}^{J_2 J_1} M) \cap \text{Hom}_{\text{Mod}^{\text{int}}_\mathcal{O}(\mathcal{O}_{\zeta(G)})}(N, \text{Ind}^{J_2 J_1} M) \\
\cong \text{Hom}_{\text{Mod}^{\text{int}}_\mathcal{O}(U_{\zeta}^1(p_{J_1})))}(N, M) \cap \text{Hom}_{\text{Mod}^{\text{int}}_\mathcal{O}(\mathcal{O}_{\zeta(G)})}(N, M).
\]

\( \square \)

We define functors

(A.27) \[
(\pi_{\zeta}^{J_2 J_1})^* : \text{Mod}(\mathcal{O}_{P_{J_2, \zeta}}) \to \text{Mod}(\mathcal{O}_{P_{J_1, \zeta}}),
\]

(A.28) \[
(\pi_{\zeta}^{J_2 J_1})_* : \text{Mod}(\mathcal{O}_{P_{J_1, \zeta}}) \to \text{Mod}(\mathcal{O}_{P_{J_2, \zeta}})
\]

as the functors corresponding to (A.23), (A.25) under the category equivalence in Theorem A.3. By (A.24) and Theorem A.3 we have the following.

**Lemma A.9.** The functor

\[ A_{J_1, \zeta} \otimes_{A_{J_2, \zeta}} (\bullet) : \text{Mod}_{A^{J_2}}(A_{J_2, \zeta}) \to \text{Mod}_{A^{J_1}}(A_{J_1, \zeta}) \quad (K \mapsto A_{J_1, \zeta} \otimes_{A_{J_2, \zeta}} K) \]
induces (A.27). Namely we have the following commutative diagram:

\[
\begin{array}{c}
\text{Mod}_{A^2}(A_{f_2, \zeta}) \quad \text{Mod}_{A^1}(A_{f_2, \zeta}) \\
\downarrow \omega_{f_2} \quad \downarrow \omega_{f_1} \\
\text{Mod}(\mathcal{O}_{f_2, \zeta}) \quad \text{Mod}(\mathcal{O}_{f_2, \zeta}).
\end{array}
\]

**Lemma A.10.** We have

\[(\pi^{f_2}_{\zeta})_* = \omega_{f_2} \circ F^{f_2}_{\zeta} \circ \omega_{f_1}.*\]

**Proof.** By (A.26) and Theorem A.3 we have

\[\omega_{f_2} \circ F^{f_2}_{\zeta} \circ \omega_{f_1} = \omega_{f_2} \circ \omega_{f_1} \circ (\pi^{f_2}_{\zeta})_* = (\pi^{f_2}_{\zeta})_*.\]

By Lemma A.8 we have the following.

**Lemma A.11.** The functor \((\pi^{f_2}_{\zeta})_*\) is right adjoint to \((\pi^{f_2}_{\zeta})^*\).

By Lemma A.9 and Lemma A.10 the functors (A.27), (A.28) induce

\[(A.29) \quad (\pi^{f_2}_{\zeta})^* : \text{mod}(\mathcal{O}_{f_2, \zeta}) \rightarrow \text{mod}(\mathcal{O}_{f_1, \zeta}),\]

\[(A.30) \quad (\pi^{f_2}_{\zeta})_* : \text{mod}(\mathcal{O}_{f_1, \zeta}) \rightarrow \text{mod}(\mathcal{O}_{f_2, \zeta}).\]

It is easily also seen that the functors (A.27), (A.28) induce

\[(A.31) \quad (\pi^{f_2}_{\zeta})^* : \text{Mod}^{eq}(\mathcal{O}_{f_2, \zeta}) \rightarrow \text{Mod}^{eq}(\mathcal{O}_{f_1, \zeta}),\]

\[(A.32) \quad (\pi^{f_2}_{\zeta})_* : \text{Mod}^{eq}(\mathcal{O}_{f_1, \zeta}) \rightarrow \text{Mod}^{eq}(\mathcal{O}_{f_2, \zeta}).\]

Similarly to [12] Lemma 4.9 we have the following.

**Lemma A.12.** We have the following commutative diagram of functors:

\[
\begin{array}{c}
\text{Mod}_{I}(U_{\zeta}(p_{f_2})) \quad \text{Mod}_{I}(U_{\zeta}(p_{f_2})) \\
\downarrow F_{f_1} \quad \downarrow F_{f_2} \\
\text{Mod}_{I}(\mathcal{O}_{f_1, \zeta}) \quad \text{Mod}_{I}(\mathcal{O}_{f_2, \zeta}).
\end{array}
\]

**A.5. Frobenius morphism.** We assume that \(\zeta \in \mathbb{C}^\times\) is a primitive \(\ell\)-th root of unity, where \(\ell\) is an integer satisfying the conditions (a1), (a2), (a3) in Section 4. We take \(\zeta^{1/[\Lambda/Q]}\) to be a primitive \(\ell\)-th root of unity (see (a2)).

Recall that Lusztig’s Frobenius homomorphism \(\text{Fr} : U_{\zeta}(\mathfrak{g}) \rightarrow U(\mathfrak{g})\) induces an embedding \(\mathcal{O}(G) \hookrightarrow \mathcal{O}_{\zeta}(G)\) of Hopf algebras. This gives an embedding \(A_1 \hookrightarrow A_{\zeta}\) of \(\mathbb{C}\)-algebras satisfying \(A_1(\lambda) \hookrightarrow A_{\zeta}(\lambda)\) for \(\lambda 

Define a \(\Lambda^f\)-graded algebra \(A_{f, \zeta}^{(f)} = \bigoplus_{\lambda \in \Lambda^f} A_{f, \zeta}^{(f)}(\lambda)\) by

\[A_{f, \zeta}^{(f)}(\lambda) = A_{f, \zeta}(\ell \lambda) \quad (\lambda \in \Lambda^f).\]
Then $A_{J,1} = \bigoplus_{\lambda \in \Lambda^J} A_1(\lambda)$ is a $A^J$-graded central subalgebra of $A^{(\ell)}_{J,\zeta}$. Moreover, $A^{(\ell)}_{J,\zeta}$ is finitely generated as an $A_{J,1}$-module. Applying $\omega_J^* : \mod_{A^J}(A_{J,1}) \to \mod(O_{P,\zeta})$ to $A^{(\ell)}_{J,\zeta}$ regarded as an object of $\mod_{A^J}(A_{J,1})$, we set

$$\mathcal{O}_J = \Fr_* O_{P,\zeta} := \omega_J^* A^{(\ell)}_{J,\zeta} \in \mod(O_{P,\zeta}) = \mod(O_J).$$

Then $\mathcal{O}_J$ is naturally equipped with an $O_{P,J}$-algebra structure through the multiplication of $A^{(\ell)}_{J,\zeta}$. Moreover, the category

$$\text{Mod}_{A^J}(A^{(\ell)}_{J,\zeta})/(\text{mod}_{A^J}(A^{(\ell)}_{J,\zeta}) \cap \text{Tor}_{A^J,+}(A_{J,1}))$$

(resp. $\mod_{A^J}(A^{(\ell)}_{J,\zeta})/(\mod_{A^J}(A^{(\ell)}_{J,\zeta}) \cap \text{Tor}_{A^J,+}(A_{J,1}))$)

is naturally identified with the category $\text{Mod}(\mathcal{O}_J)$ (resp. $\text{mod}(\mathcal{O}_J)$) of quasi-coherent (resp. coherent) $\mathcal{O}_J$-modules. Let

$$\mathfrak{w}_J^* : \text{Mod}_{A^J}(A^{(\ell)}_{J,\zeta}) \to \text{Mod}(\mathcal{O}_J)$$

be the canonical functor, and let

$$\mathfrak{w}_{J,*} : \text{Mod}(\mathcal{O}_J) \to \text{Mod}_{A^J}(A^{(\ell)}_{J,\zeta})$$

be its right adjoint. Let

$$(\bullet)^{(\ell)} : \text{Mod}_{A^J}(A^{(\ell)}_{J,\zeta}) \to \text{Mod}_{A^J}(A^{(\ell)}_{J,\zeta})$$

be the functor given by $M \mapsto \bigoplus_{\lambda \in \Lambda^J} M(\ell \lambda)$, and define a functor

(A.33) $\Fr_* : \text{Mod}(O_{P,\zeta}) \to \text{Mod}(\mathcal{O}_J)$

by the commutative diagram:

$$\begin{array}{ccc}
\text{Mod}_{A^J}(A^{(\ell)}_{J,\zeta}) & \xrightarrow{(\bullet)^{(\ell)}} & \text{Mod}_{A^J}(A^{(\ell)}_{J,\zeta}) \\
\omega_J^* \downarrow & & \downarrow \mathfrak{w}_J^* \\
\text{Mod}(O_{P,\zeta}) & \xrightarrow{\Fr_*} & \text{Mod}(\mathcal{O}_J).
\end{array}$$

Similarly to Lemma 5.1 we have the following.

**Lemma A.13.**  
(i) The functor (A.33) gives category equivalences:

$$\text{Mod}(O_{P,\zeta}) \cong \text{Mod}(\mathcal{O}_J), \quad \text{mod}(O_{P,\zeta}) \cong \text{mod}(\mathcal{O}_J).$$

(ii) We have the following commutative diagram:

$$\begin{array}{ccc}
\text{Mod}(O_{P,\zeta}) & \xrightarrow{\omega_J^*} & \text{Mod}_{A^J}(A^{(\ell)}_{J,\zeta}) \\
\Fr_* \downarrow & & \downarrow (\bullet)^{(\ell)} \\
\text{Mod}(\mathcal{O}_J) & \xrightarrow{\mathfrak{w}_{J,*}} & \text{Mod}_{A^J}(A^{(\ell)}_{J,\zeta}).
\end{array}$$
Now assume $J_1 \subset J_2 \subset I$. We denote by
\[
\pi^{J_2|J_1} : \mathcal{P}_{J_1} \to \mathcal{P}_{J_2}
\]
the natural morphism of ordinary algebraic varieties. By Lemma A.10 and Lemma A.13 we have the following.

**Lemma A.14.** The following diagram is commutative:

\[
\begin{array}{ccc}
\text{Mod}(\mathcal{O}_{\mathcal{P}_{J_1}}, \zeta) & \xrightarrow{(\pi^{J_2|J_1})_*} & \text{Mod}(\mathcal{O}_{\mathcal{P}_{J_2}}, \zeta) \\
\downarrow \text{Fr}_* & & \downarrow \text{Fr}_* \\
\text{Mod}(\mathcal{O}_{J_1}) & \xrightarrow{(\pi^{J_2|J_1})_*} & \text{Mod}(\mathcal{O}_{J_2}).
\end{array}
\]

**Lemma A.15.** $(\pi^{J_2|J_1})_* (\mathcal{O}_{J_1}) = \mathcal{O}_{J_2}$.

**Proof.** For $J \subset I$ we have
\[
\mathcal{O}_J = \mathcal{O}_J^{(\ell)} = \text{Fr}_*(\omega_J^* A_{J,\zeta}).
\]
Hence it is sufficient to show $(\pi^{J_2|J_1})_* (\omega_{J_1}^* A_{J_1,\zeta}) = \omega_{J_2}^* A_{J_2,\zeta}$ by Lemma A.14. By Lemma A.3 and Lemma A.10 we have
\[
(\pi^{J_2|J_1})_* (\omega_{J_1}^* A_{J_1,\zeta}) = \omega_{J_2}^* F_{\zeta}^{J_2|J_1} \omega_{J_1}^* A_{J_1,\zeta} = \omega_{J_2}^* F_{\zeta}^{J_2|J_1} A_{J_1,\zeta} = \omega_{J_2}^* A_{J_2,\zeta},
\]
\[
\square
\]
Hence by
\[
\text{Hom}(\mathcal{O}_{J_2}, \mathcal{O}_{J_2}) \cong \text{Hom}(\mathcal{O}_{J_2}, (\pi^{J_2|J_1})_* (\mathcal{O}_{J_1})) \\
\cong \text{Hom}((\pi^{J_2|J_1})^{-1} \mathcal{O}_{J_2}, \mathcal{O}_{J_1})
\]
we obtain a homomorphism
\[
(\pi^{J_2|J_1})^{-1} \mathcal{O}_{J_2} \to \mathcal{O}_{J_1}
\]
of sheaf of rings.

**Proof.** Recall that the vertical functors Fr$_*$ are equivalences. Hence it is sufficient to show that the functor $\mathcal{O}_{J_1} \otimes (\pi^{J_2|J_1})^{-1} \mathcal{O}_{J_2} (\pi^{J_2|J_1})^{-1} (\bullet)$ is left adjoint to
\[
(\pi^{J_2|J_1})_* : \text{Mod}(\mathcal{O}_{J_1}) \to \text{Mod}(\mathcal{O}_{J_2}).
\]
For $M \in \text{Mod}(\mathcal{O}_{J_1})$, $N \in \text{Mod}(\mathcal{O}_{J_2})$ we have
\[
\text{Hom}_{\mathcal{O}_{J_2}}(N, (\pi^{J_2|J_1})_* M) \\
\cong \text{Hom}_{(\pi^{J_2|J_1})^{-1} \mathcal{O}_{J_2}}((\pi^{J_2|J_1})^{-1} N, M) \\
\cong \text{Hom}_{\mathcal{O}_{J_1}}(\mathcal{O}_{J_1} \otimes (\pi^{J_2|J_1})^{-1} \mathcal{O}_{J_2} (\pi^{J_2|J_1})^{-1} N, M).
\]
\[
\square
A.6. The \( C \)-algebra \( \mathcal{D}_J \). Let \( J \subset I \). Recall that we have a sheaf \( \mathcal{D} = \text{Fr}_* \mathcal{D}_{\mathcal{B}_c} \) of \( C \)-algebras on \( \mathcal{B} \). Set

\begin{equation}
\mathcal{D}_J = (\pi^J)_{!*} \mathcal{D}.
\end{equation}

It is a sheaf of \( C \)-algebras on \( \mathcal{P}_J \) equipped with \( C \)-algebra homomorphisms

\begin{align}
U_\zeta(\mathfrak{g}) &\to \mathcal{D}_J \\
\mathcal{D}_J &\to \mathcal{D}_J \\
\mathbb{C}[\Lambda] &\to \mathcal{D}_J
\end{align}

satisfying the relations similar to (5.13), (5.14), (5.15). For \( t \in H \) we set

\begin{equation}
\mathcal{D}_{J,t} := \mathcal{D}_J / \sum_{\lambda \in \Lambda} \mathcal{D}_J(\sigma_{2\lambda} - \theta_{\lambda}(t)).
\end{equation}

**Theorem A.16.** Let \( t \in H \). Assume that \( \ell \) is a power of a prime number and that the order of \( t^\ell \) is finite and prime to \( \ell \). Then we have

\[ R\Gamma(\mathcal{P}_J, \mathcal{D}_{J,t}) \cong U_\zeta(\mathfrak{g})_{[t]} \].

We first recall the outline of the proof of Theorem A.16 in the case \( J = \emptyset \) given in [45].

The adjoint action of \( U_\mathbb{F}(\mathfrak{g}) \) on \( U_\mathbb{F}(\mathfrak{g}) \) given by

\[ \text{ad}(u)(v) = \sum_{(u)} u_{(0)} \cdot v \cdot S u_{(1)} \quad (u, v \in U_\mathbb{F}(\mathfrak{g})) \]

induces the adjoint action of \( U_\mathbb{F}(\mathfrak{g}) \) on \( U_\zeta(\mathfrak{g}) \). We set

\[ fU_\zeta(\mathfrak{g}) = \{ u \in U_\zeta(\mathfrak{g}) \mid \text{dim ad}(U_\mathbb{F}(\mathfrak{g}))(u) < \infty \}. \]

It is a subalgebra of \( U_\zeta(\mathfrak{g}) \). Define a subalgebra \( fD_\zeta \) of \( D_\zeta \) by

\[ fD_\zeta = \langle \ell_\varphi, \partial_u, \sigma_{2\lambda} \mid \varphi \in A_\zeta, u \in fU_\zeta(\mathfrak{g}), \lambda \in \Lambda \rangle. \]

We also set

\[ fD_{\zeta,t} = fD_\zeta / \sum_{\lambda \in \Lambda} fD_\zeta(\sigma_{2\lambda} - \theta_{\lambda}(t)). \]

Define objects \( f\mathcal{D}_{\mathcal{B}_c} \) and \( f\mathcal{D}_{\mathcal{B}_c,t} \) \( (t \in H) \) of \( \text{Mod}(\mathcal{O}_{\mathcal{B}_c}) \) by

\[ f\mathcal{D}_{\mathcal{B}_c} = \omega^*(fD_\zeta), \quad f\mathcal{D}_{\mathcal{B}_c,t} = \omega^*(fD_{\zeta,t}). \]

We also define objects \( f\mathcal{D} \) and \( f\mathcal{D}_t \) of \( \text{Mod}(\mathcal{D}) \) by

\[ f\mathcal{D} = \text{Fr}_* f\mathcal{D}_{\mathcal{B}_c}, \quad f\mathcal{D}_t = \text{Fr}_* f\mathcal{D}_{\mathcal{B}_c,t}. \]

Then \( f\mathcal{D} \) is the subalgebra of \( \mathcal{D} \) given by

\[ f\mathcal{D} = \langle \ell_\varphi, \partial_u, \sigma_{2\lambda} \mid \varphi \in \mathcal{D}, u \in fU_\zeta(\mathfrak{g}), \lambda \in \Lambda \rangle, \]

and we have

\[ f\mathcal{D}_t = f\mathcal{D} / \sum_{\lambda \in \Lambda} f\mathcal{D}(\sigma_{2\lambda} - \theta_{\lambda}(t)) \quad (t \in H). \]

Moreover, we have

\[ \mathcal{D} = f\mathcal{D} \otimes fU_\zeta(\mathfrak{g}) U_\zeta(\mathfrak{g}), \quad \mathcal{D}_t = f\mathcal{D}_t \otimes fU_\zeta(\mathfrak{g}) U_\zeta(\mathfrak{g}). \]
Since $U_\zeta(g)$ is a flat $I^f U_\zeta(g)$-module (both as a left module and a right module), we have

$$R\Gamma(\mathcal{B}, \mathcal{D}) = R\Gamma(\mathcal{B}, I^f \mathcal{D}) \otimes_{I^f U_\zeta(g)} U_\zeta(g) = R\Gamma(\mathcal{B}_\zeta, I^f \mathcal{D}_{\mathcal{B}_\zeta}) \otimes_{I^f U_\zeta(g)} U_\zeta(g),$$

$$R\Gamma(\mathcal{B}, \mathcal{D}_t) = R\Gamma(\mathcal{B}, I^f \mathcal{D}_t) \otimes_{I^f U_\zeta(g)} U_\zeta(g) = R\Gamma(\mathcal{B}_\zeta, I^f \mathcal{D}_{\mathcal{B}_\zeta, t}) \otimes_{I^f U_\zeta(g)} U_\zeta(g).$$

Here, $R\Gamma(\mathcal{B}_\zeta, \bullet)$ is the derived functor of the global section functor

$$\Gamma(\mathcal{B}_\zeta, \bullet) : \text{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \text{Mod}(\mathcal{C}) \quad (M \mapsto (\omega_\zeta M)(0)).$$

We denote by $e^* U_\zeta(g)$ the subalgebra of $U_\zeta(g)$ generated by $k_{2\lambda}$ for $\lambda \in \Lambda$ and $e_i$, $S f_i$ for $i \in I$. Set

$$(A.38) \quad V_\zeta = e^* U_\zeta(g)/ \sum_{i \in I} (S f_i) \cdot e^* U_\zeta(g).$$

The adjoint action of $U_\zeta^L(g)$ on $U_\zeta(g)$ induces a $U_\zeta^L(g^-)$-module structure of $V_\zeta$ so that $V_\zeta \in \text{Mod}_{\text{int}}(U_\zeta^L(g^-))$. We also regard $V_\zeta$ as an $\mathcal{O}(H)$-module by

$$\bar{\pi} \theta_\lambda = \frac{1}{u k_{2\lambda}} \quad (u \in e^* U_\zeta(g), \lambda \in \Lambda).$$

Then $V_\zeta$ is a $(U_\zeta^L(g^-), \mathcal{O}(H))$-bimodule. For $t \in H$ we set

$$V_{\zeta, t} = V_\zeta \otimes_{\mathcal{O}(H)} \mathbb{C}_t,$$

where $\mathbb{C}_t$ is the one-dimensional $\mathcal{O}(H)$-module given by $t : \mathcal{O}(H) \to \mathbb{C}$. By $I^f \mathcal{D}_{\mathcal{B}_\zeta}$ and $I^f \mathcal{D}_{\mathcal{B}_\zeta, t}$ are objects of $\text{Mod}^{\text{int}}(\mathcal{O}_{\mathcal{B}_\zeta})$, and the corresponding objects of $\text{Mod}_{\text{int}}(U_\zeta^L(g^-))$ are $V_\zeta$ and $V_{\zeta, t}$ respectively. We conjecture that the natural morphisms

$$(A.39) \quad I^f U_\zeta(g) \otimes_{Z_{\text{Har}}(U_\zeta(g))} \mathcal{O}(H) \to R\text{Ind}^H_\mathcal{O}(V_\zeta),$$

$$(A.40) \quad I^f U_\zeta(g) \otimes_{Z_{\text{Har}}(U_\zeta(g))} \mathbb{C}[t] \to R\text{Ind}^H_\mathcal{O}(V_{\zeta, t})$$

are isomorphisms. Here, $\mathbb{C}[t]$ is the one-dimensional $Z_{\text{Har}}(U_\zeta(g))$-module given by $\xi[t]$. We see by Lemma $A.12$ that if $(A.39)$ is an isomorphism, we have

$$(A.41) \quad R\Gamma(\mathcal{B}, \mathcal{D}) \cong U_\zeta(g) \otimes_{Z_{\text{Har}}(U_\zeta(g))} \mathcal{O}(H),$$

and if $(A.40)$ is an isomorphism, we have

$$(A.42) \quad R\Gamma(\mathcal{B}, \mathcal{D}_t) \cong U_\zeta(g)|_t.$$

The quantized coordinate algebra $\mathcal{O}_\zeta(G)$ is denoted as $\mathcal{O}_\zeta(G)_{\text{ad}}$ when it is regarded as a left $U_\zeta^L(g)$-module via the adjoint action

$$\text{ad}(u)(\varphi) = \sum_{(u)} u_{(0)} \cdot \varphi \cdot (S^{-1} u_{(1)}) \quad (u \in U_\zeta^L(g), \varphi \in \mathcal{O}_\zeta(G)).$$

A similar convention is also applied to $\mathcal{O}_\zeta(B^-)$ and $\mathcal{O}_\zeta(P^-_f)$. One can construct using the Drinfeld paring the natural isomorphisms

$$(A.43) \quad V_\zeta \cong \mathcal{O}_\zeta(B^-)_{\text{ad}}, \quad I^f U_\zeta(g) \cong \mathcal{O}_\zeta(G)_{\text{ad}}.$$
in \( \text{Mod}_{\text{int}}(U_L^J(b^-)) \) and \( \text{Mod}_{\text{int}}(U_L^J(g)) \) respectively (see [17, 45]). Then under the identification \( \text{A.43} \) the morphisms \( \text{A.39}, \text{A.40} \) become canonical morphisms

\[
\begin{align*}
\text{(A.44)} & \quad O_\zeta(G)_{\text{ad}} \otimes_{O(H/W)} O(H) \to R\text{Ind}^{J_0}(O_\zeta(B^-)_{\text{ad}}), \\
\text{(A.45)} & \quad O_\zeta(G)_{\text{ad}} \otimes_{O(H/W)} C_t \to R\text{Ind}^{J_0}(O_\zeta(B^-)_{\text{ad},t}),
\end{align*}
\]

where \( O_\zeta(B^-)_{\text{ad},t} = O_\zeta(B^-)_{\text{ad}} \otimes_{O(H)} C_t \).

Denote by \( O(H)_t \) the localization of \( O(H) \) at the maximal ideal \( \text{Ker}(t : O(H) \to \mathbb{C}) \). In [45] we established the isomorphism

\[
\text{(A.46)} \quad O_\zeta(G)_{\text{ad}} \otimes_{O(H/W)} O(H)_t \cong R\text{Ind}^{J_0}(O_\zeta(B^-)_{\text{ad}}) \otimes_{O(H)} O(H)_t
\]

using reduction to the case \( q = 1 \) (the assumption on \( \ell \) and \( t \in H \) in Theorem [A.16] is only used here). By [45], Proposition 4.5] this implies

\[
\text{(A.47)} \quad O_\zeta(G)_{\text{ad}} \otimes_{O(H/W)} C_t \cong R\text{Ind}^{J_0}(O_\zeta(B^-)_{\text{ad},t}).
\]

Hence Theorem [A.16] for \( J = \emptyset \) follows from the above argument.

Now let us give a proof of Theorem [A.16] for general \( J \). By Lemma A.14, Lemma A.12 we have

\[
R(\pi^{J_0})_f \mathcal{D} \cong \text{Fr}_*(R(\pi^{J_0})_f \mathcal{D}_{\zeta}) \cong \text{Fr}_*(R\text{Ind}^{J_0}(O_\zeta(B^-)_{\text{ad}})),
\]

and

\[
R(\pi^{J_0})_f \mathcal{D}_t \cong \text{Fr}_*(R(\pi^{J_0})_f \mathcal{D}_{\zeta,t}) \cong \text{Fr}_*(R\text{Ind}^{J_0}(O_\zeta(B^-)_{\text{ad},t})).
\]

Assume we could show

\[
\text{(A.48)} \quad R^i\text{Ind}^{J_0}(O_\zeta(B^-)_{\text{ad}}) \otimes_{O(H)} O(H)_t = 0 \quad (i \neq 0).
\]

Then by [45], Proposition 4.5] we have

\[
R\text{Ind}^{J_0}(O_\zeta(B^-)_{\text{ad},t}) \cong \text{Ind}^{J_0}(O_\zeta(B^-)_{\text{ad}}) \otimes_{O(H)} C_t,
\]

and hence

\[
R(\pi^{J_0})_f \mathcal{D}_t \cong \text{Fr}_*(R(\pi^{J_0})_f \mathcal{D}_{\zeta}) \otimes_{O(H)} C_t \cong ((\pi^{J_0})_* f \mathcal{D}) \otimes_{O(H)} C_t.
\]

Since \( U_\zeta(g) \) is a flat \( f U_\zeta(g) \)-module, we have

\[
R(\pi^{J_0})_* \mathcal{D}_t \cong R(\pi^{J_0})_* f \mathcal{D}_t \otimes_{f U_\zeta(g)} U_\zeta(g) \cong (((\pi^{J_0})_* f \mathcal{D}) \otimes_{O(H)} C_t) \otimes_{f U_\zeta(g)} U_\zeta(g)
\]

\[
\cong (\pi^{J_0})_* \mathcal{D} \otimes_{O(H)} C_t \cong \mathcal{D}_{\text{int}}.
\]

Hence we obtain

\[
R\Gamma(\mathcal{P}_{J}, \mathcal{D}_{\text{int}}) \cong R\Gamma(\mathcal{P}_{J}, R(\pi^{J_0})_* \mathcal{D}_t) \cong R\Gamma(\mathcal{B}, \mathcal{D}_t) \cong U_\zeta(g)[t]
\]

by Theorem [A.16] for \( J = \emptyset \). It remains to show (A.48). Set \( b^-_J = I_J \cap b^- \), and define \( U_\zeta^J(I_J) \) and \( U_\zeta^J(b^-_J) \) in a obvious way. We can similarly define the induction functor

\[
\text{Ind}' : \text{Mod}_{\text{int}}(U_L^J(b^-)) \to \text{Mod}_{\text{int}}(U_L^J(I_J)).
\]

By a standard fact concerning induction functors we have

\[
\text{For}(\text{Ind}^{J_0}(M)) \cong \text{Ind}'(\text{For}(M))
\]
for any $U^L_{\zeta}(b^-)$-module $M$. Here, $\text{For}$ denotes the forgetful functor. Define $\mathcal{O}_{\zeta}(B^-)$ in an obvious way. Then we can show that there exists an integrable $U^L_{\zeta}(p_J)$-module $N$ such that

$$\text{For}(\mathcal{O}_{\zeta}(B^-)_{\text{ad}}) \cong \mathcal{O}_{\zeta}(B^-)_{\text{ad}} \otimes N$$

in $\text{Mod}_{\text{int}}(U^L_{\zeta}(b^-))$. Hence we obtain

$$\text{For}(R^i\text{Ind}^H_\emptyset(\mathcal{O}_{\zeta}(B^-)_{\text{ad}})) \otimes_{\mathcal{O}(H)} \mathcal{O}(H)_t$$

$$\cong R^i\text{Ind}'(\text{For}(\mathcal{O}_{\zeta}(B^-)_{\text{ad}})) \otimes_{\mathcal{O}(H)} \mathcal{O}(H)_t$$

$$\cong R^i\text{Ind}'(\mathcal{O}_{\zeta}(B^-)_{\text{ad}} \otimes N) \otimes_{\mathcal{O}(H)} \mathcal{O}(H)_t$$

$$\cong (R^i\text{Ind}'(\mathcal{O}_{\zeta}(B^-)_{\text{ad}}) \otimes_{\mathcal{O}(H)} \mathcal{O}(H)_t) \otimes N.$$  

Here, the last isomorphism is a consequence of the tensor identity for the induction functor. Therefore, it is sufficient to show

$$R^i\text{Ind}'(\mathcal{O}_{\zeta}(B^-)_{\text{ad}}) \otimes_{\mathcal{O}(H)} \mathcal{O}(H)_t \quad (i \neq 0).$$

This is a consequence of (A.47) applied to the case $G = L_J$. The proof of Theorem A.16 is complete.

A.7. Localization on $\mathcal{V}_J$. Set

(A.49) \[ \mathcal{V}_J = \{(P^-_J g, k, t) \in P_J \times K \times H \mid \eta(k) g^{-1} \in (\text{St}_{J^{-1}}(\tau^t))N_J \}, \]

where $\text{St}_J : L_J \to H/W_J$ is the Steinberg map for $L_J$, and define a morphism $\pi^J : \mathcal{V} \to \mathcal{V}_J$ by $\pi^J((B^-_g, k, t)) = (P^-_J g, k, t)$.

**Lemma A.17.** We have $R(\pi^J)_* \mathcal{O}_\mathcal{V} = \mathcal{O}_{\mathcal{V}_J}$.

**Proof.** Define $a : B^- \to H$ and $a_J : P^-_J \to L_J$ using the natural isomorphisms $B^-/N^- \cong H$ and $P^-_J/N^- \cong L_J$ respectively. Recall

$$\tilde{G} = \{(B^-g, x) \in B \times G \mid gxg^{-1} \in B^-\}.$$  

We similarly set

$$\tilde{G}_J = \{(P^-_J g, x) \in P_J \times G \mid gxg^{-1} \in P^-_J\}.$$  

Then we have a natural morphism

$$g_J : \tilde{G} \to \tilde{G}_J \times_{H/W_J} H \quad ((B^-g, x) \mapsto (P^-_J g, x, a(gxg^{-1}))),$$

where $\tilde{G}_J \to H/W_J$ is given by $(P^-_J g, x) \mapsto \text{St}_J(a_J(gxg^{-1}))$, and $H \to H/W_J$ is given by $t \mapsto [t] = Wt$. We see easily that $\pi^J$ is the base change with respect to the smooth morphism

$$\mathcal{V} \to \tilde{G}_J \times_{H/W_J} H \quad ((P^-g, k, t) \mapsto (P^-g, \eta(k), t')).$$

Hence it is sufficient to show

(A.50) \[ Rg_J_* \mathcal{O}_{\tilde{G}} = \mathcal{O}_{\tilde{G}_J \times_{H/W_J} H}. \]

In the case $J = I$ $g_J$ is given by

$$\tilde{G} \to G \times_{H/W} H \quad ((B^-g, x) \mapsto (x, a(gxg^{-1}))),$$

and (A.50) is well-known (see, for example, [45 (8.13)]). In the general case (A.50) is shown using this special case for $G = L_J$. Details are omitted. \qed
Let 
\[ p^V_J : V_J \to P_J, \quad p^H_J : V_J \to H, \quad p^Y_J : V_J \to Y \]
be the projections.

We set 
\[ 3_J = (\pi^{J0})_* 3. \]

It is a central subalgebra of \( D_J = (\pi^{J0})_* D. \) We have
\[ (A.51) \quad 3_J \cong (p^V_J)_* O_{V_J}. \]
In fact
\[ (A.52) \quad 3_J = (\pi^{J0})_* 3 \cong (\pi^{J0})_* (p^V_J)_* (\pi^{J0})_* O_V \cong (p^V_J)_* O_{V_J} \]
by (5.25) and Lemma A.17. Hence localizing \( D_J \) on \( V_J \) we obtain an \( O_{V_J} \)-algebra \( \sharp \underline{D}_J. \) By definition we have
\[ \sharp \underline{D}_J = (\pi^{J0})_* (\sharp D). \]

We denote by \( \text{Mod}(\sharp \underline{D}_J) \) (resp. \( \text{mod}(\sharp \underline{D}_J) \)) the category of quasi-coherent (resp. coherent) \( \sharp \underline{D}_J \)-modules. Set
\[ V_{J,t} = (p^V_H)^{-1}(t) \quad (t \in H), \quad V_{J,t}^k = (p^V_Y)^{-1}(k,t) \quad ((k,t) \in Y), \]
and define an \( O_{V_{J,t}} \)-algebra \( \sharp \underline{D}_{J,t} \) and an \( O_{V_{J,t}}^k \)-algebra \( \sharp \underline{D}_{J,t}^k \) by
\[ \sharp \underline{D}_{J,t} = \sharp D \otimes O_{V_{J,t}}, \quad \sharp \underline{D}_{J,t}^k = \sharp D \otimes O_{V_{J,t}}^k \]
respectively. We have
\[ (A.52) \quad \text{mod}(\underline{D}_{J,t}) \cong \text{mod}(\sharp \underline{D}_{J,t}) \]
by definition.

Set
\[ V_{ur} = (p^V_H)^{-1}(H_{ur}), \quad V_{J,ur} = (p^V_Y)^{-1}(H_{ur}). \]

They are open subsets of \( V \) and \( V_J \) respectively. Let \( \pi^{J0}_{ur} : V_{ur} \to V_{J,ur} \) be the restriction of \( \pi^{J0}. \)

**Proposition A.18.**

(i) The restriction of \( \underline{D}_J \) to \( V_{J,ur} \) is an Azumaya algebra.

(ii) Assume \( (k,t) \in K \times_{H/W} H_{ur}. \) Then \( \sharp \underline{D}_{J,t}^k \) is a split Azumaya algebra with splitting bundle \( \underline{S}_{J,t}^k = K_t^k \otimes \mathcal{O}_{V_{J,t}} \) (see Section 5.2 for the notation).

**Proof.** We proved in [11] using the results of [15] that
\[ \sharp \underline{D}|_{V_{ur}} \cong O_{V_{ur}} \otimes Z_{\text{Has}}(U_\zeta(\mathfrak{g})) U_\zeta(\mathfrak{g}). \]

It follows that
\[ \sharp \underline{D}|_{V_{J,ur}} \cong (\pi^{J0}_{ur})_* (\sharp \underline{D}|_{V_{ur}}) \cong (\pi^{J0}_{ur})_* (O_{V_{ur}} \otimes Z_{\text{Has}}(U_\zeta(\mathfrak{g})) U_\zeta(\mathfrak{g})) \]
\[ \cong (\pi^{J0}_{ur})_* O_{V_{J,ur}} \otimes Z_{\text{Has}}(U_\zeta(\mathfrak{g})) U_\zeta(\mathfrak{g}) \]
\[ \cong O_{V_{J,ur}} \otimes Z_{\text{Has}}(U_\zeta(\mathfrak{g})) U_\zeta(\mathfrak{g}) \]
by Lemma A.17. Hence the desired results follow from [15].

Similarly to Theorem 5.3 we obtain from Proposition A.18 the following.
Theorem A.19. Assume \((k, \tilde{t}) \in K \times_{H/W} H_{ur}\) and \(\mu \in \Lambda^J\). Then \(\mathcal{D}^k_{J,\tilde{t},\mu}\) is a split Azumaya algebra with splitting bundle \(\mathcal{R}^k_{J,\tilde{t},\mu} = \mathcal{O}\) \(J[\mu] \otimes_{\mathcal{O}, J} \mathcal{R}^k_{J,\tilde{t}}\).

A.8. Equivalence of categories. We assume that \(\ell\) is a power of a prime number in the following. Let \(\lambda \in \Lambda^J\) and let \(\tilde{t} \in H\) such that the order of \(\tilde{t}\) is finite and prime to \(\ell\). Set \(t = \ell\lambda\). We also take \(k \in K\) such that \((k, \tilde{t}) \in \mathcal{Y}\).

Arguing as in [12, Sections 3.3, 3.4, 3.5] using Theorem A.16, Theorem A.19 we obtain the following.

Theorem A.20. Assume that \(t\) is \(J\)-regular in the sense that \(w \in W, w \circ t = t \implies w \in W\).

Then we have equivalences
\[
D^b\left(\text{mod}\left(U_{\mathcal{g}}[\ell]\right)\right) \cong D^b\left(\text{mod}^{\ell}(\mathcal{D}, t)\right),
\]
\[
D^b\left(\text{mod}^{k}_{\ell}(U_{\mathcal{g}})\right) \cong D^b\left(\text{mod}^{k}_{\ell}(\mathcal{D})\right)
\]
of triangulated categories.

By Theorem A.19 we obtain the following.

Theorem A.21. We have the equivalence
\[
\text{mod}^{\ell}(\mathcal{D}_{J, t}) \cong \text{mod}(\mathcal{O}_{V, J, t})
\]
of abelian categories.

We note that the arguments in A.6, A.7 also work for the formal version. In particular, we have
\[
\text{mod}^{\ell}(\mathcal{D}_{J, t}) \cong \text{mod}(\mathcal{O}_{V, J, t}),
\]
where \(\mathcal{V}_{J, t}\) is the formal neighborhood of \(V_{J, t}\) in \(V_J\), and \(\mathcal{D}_{J, t} = \mathcal{O}_{V, J} \otimes_{\mathcal{O}_{V_J}} \mathcal{D}_J\).

Moreover, we have
\[
D^b\left(\text{mod}\left(U_{\mathcal{g}}[\ell]\right)\right) \cong D^b\left(\text{mod}^{\ell}(\mathcal{D}_{J, t})\right)
\]
if \(t\) is \(J\)-regular. Therefore, if \(t\) is \(J\)-regular, then we have the equivalence
\[
D^b\left(\text{mod}\left(U_{\mathcal{g}}[\ell]\right)\right) \cong D^b\left(\text{mod}(\mathcal{O}_{V, J, t})\right)
\]
of triangulated categories.

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