CONVEX BODIES WITH FEW FACES

KEITH BALL AND ALAIN PAJOR

(Communicated by William J. Davis)

Abstract. It is proved that if \( u_1, \ldots, u_n \) are vectors in \( \mathbb{R}^k \), \( k \leq n \), \( 1 \leq p < \infty \) and

\[
 r = \left( \frac{1}{k} \sum_{i=1}^{n} |u_i|^p \right)^{1/p}
\]

then the volume of the symmetric convex body whose boundary functionals are \( \pm u_1, \ldots, \pm u_n \), is bounded from below as

\[
 |\{x \in \mathbb{R}^k : |(x, u_i)| \leq 1 \text{ for every } i\}|^{1/k} \geq 1/\sqrt{pr}.
\]

An application to number theory is stated.

0. Introduction

In [V], Vaaler proved that if \( Q_n = [-\frac{1}{2}, \frac{1}{2}]^n \) is the central unit cube in \( \mathbb{R}^n \) and \( U \) is a subspace of \( \mathbb{R}^n \) then the volume \( |U \cap Q_n| \), of the section of \( Q_n \) by \( U \) is at least 1. This result may be reformulated as follows: if \( u_1, \ldots, u_n \) are vectors in \( \mathbb{R}^k \), \( 1 \leq k \leq n \) whose Euclidean lengths satisfy \( \sum_{i=1}^{n} |u_i|^2 \leq k \) then

\[
 |\{x \in \mathbb{R}^k : |(x, u_i)| \leq 1 \text{ for every } i\}|^{1/k} \geq 2.
\]

A related theorem, (Theorem 1, below) in which the condition \( \sum |u_i|^2 \leq k \) is replaced by \( \max_{i} |u_i| \leq 1 \) was proved by Carl and Pajor [CP] and Gluskin [G]. Gluskin's methods enable him to obtain sharp results in limiting cases which in turn have applications in harmonic analysis. Results closely related to Theorem 1 were also obtained by Bárány and Füredi [BF] and Bourgain, Lindenstrauss and Milman [BLM].

Theorem 1. There is a constant \( \delta > 0 \) so that if \( u_1, \ldots, u_n \in \mathbb{R}^k \), \( 1 \leq k \leq n \) are vectors of length at most 1 then

\[
 |\{x \in \mathbb{R}^k : |(x, u_i)| \leq 1 \text{ for every } i\}|^{1/k} \geq \delta/\sqrt{1 + \log(n/k)}.
\]

The estimate is best possible if \( n \) is at most exponential in \( k \), apart from the value of the constant \( \delta \). This is demonstrated by an example which had
appeared some time earlier in a paper of Figiel and Johnson [FJ]. Theorem 1
gives a lower bound on the volume ratios of the unit balls of $k$-dimensional
subspaces of $l_\infty^n$ and hence on the distance of these subspaces from Euclidean
space.

Regarding Theorem 1 as a $p = \infty$ version of Vaaler's $p = 2$ result, Kashin
asked whether a similar result holds for $2 < p < \infty$. This question is answered
in the affirmative by the following theorem.

**Theorem 2.** Suppose $u_1, \ldots, u_n \in \mathbb{R}^k$ with $k \leq n$, $1 \leq p < \infty$ and let

$$
r = \left( \frac{1}{k} \sum_{i=1}^{n} |u_i|^p \right)^{1/p}.
$$

Then

$$
|\{x \in \mathbb{R}^k : |\langle x, u_i \rangle| \leq 1 \text{ for every } i\}|^{1/k} \geq \begin{cases} 
\frac{2\sqrt{2}}{\sqrt{p}r} & \text{if } p \geq 2 \\
1/r & \text{if } 1 \leq p \leq 2.
\end{cases}
$$

The lower bound is best possible (up to a constant) provided $e^p k \leq n \leq e^k$.

**Remark.** The slightly stronger result for $p \geq 2$ is isolated since for $p = 2$ it
gives back exactly Vaaler's result.

Theorem 1 follows immediately from Theorem 2 by a standard optimization
argument. If $(u_i)^n_{i=1}$ in $\mathbb{R}^k$ all have norm at most 1 then for any $p \in [1, \infty)$,

$$
\left( \frac{1}{k} \sum_{k} |u_i|^p \right)^{1/p} \leq \left( \frac{n}{k} \right)^{1/p}
$$

so that

$$
|\{x : |\langle x, u_i \rangle| \leq 1 \text{ for every } i\}|^{1/k} \geq 2\sqrt{2}/\sqrt{p} \left( \frac{n}{k} \right)^{1/p}
$$

(for $p \geq 2$) and the latter is at least $2/\sqrt{p} \sqrt{1 + \log(n/k)}$ when $p = 2(1 + \log(n/k))$.

With the careful use of well-known methods for estimating the entropy of
convex bodies it is possible to obtain more general (but less precise) estimates
than that provided by Theorem 2; (see [BP]). The purpose of this paper is to
provide a very short proof of Theorem 2, and a fortiori, Theorem 1.

Vaaler originally proved his theorem because of its applications to the geo-
metry of numbers. The last section of this paper includes a statement of the
generalization of Siegel's lemma which follows from Theorem 2.

1. **The lower bound**

The proof of Theorem 2 makes use of the following result from [MeP] which
was designed to extend Vaaler's theorem in a different direction: it estimates
the volumes of sections of the unit balls of the spaces $l_p^n$, $1 \leq p \leq \infty$. For $1 \leq p \leq \infty$, $n \in \mathbb{N}$ let

$$B_p^n = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \leq 1 \right\}$$

be the unit ball of $l_p^n$.

**Theorem 3.** Let $U$ be a $k$-dimensional subspace of $\mathbb{R}^n$; if $1 \leq p < q < \infty$ then

$$\frac{|B_p^n \cap U|}{|B_p^k|} \leq \frac{|B_q^n \cap U|}{|B_q^k|}. \quad \Box$$

**Remark.** The case $p = 2$, $q = \infty$ is Vaaler's theorem since then, the left side is 1 and the inequality states that

$$|B_\infty^n \cap U| \geq |B_\infty^k| \geq 2^k.$$

For notational convenience, the proof of Theorem 2 is divided into several short lemmas. The first is no more than a convenient form of Hölder’s inequality. For $k \in \mathbb{N}$, $S^{k-1}$ will denote the Euclidean sphere in $\mathbb{R}^k$ and $\sigma = \sigma_{k-1}$, the rotationally invariant probability measure on $S^{k-1}$. Also let $v_k$ be the volume of the Euclidean unit ball in $\mathbb{R}^k$.

**Lemma 4.** Let $C$ and $B$ be symmetric convex bodies in $\mathbb{R}^k$ with Minkowski gauges $\| \cdot \|_C$ and $\| \cdot \|_B$, respectively. Then for $p > 0$

$$\left( \frac{|C|}{|B|} \right)^{1/k} \geq \left( \frac{k + p}{k|B|} \right) \left( \int_B \|x\|_C^p \, dx \right)^{-1/p}.$$

**Proof.**

$$\left( \frac{|C|}{|B|} \right)^{1/k} = \left( \frac{v_k}{|B|} \int_{S^{k-1}} \|\theta\|_C^{-k} \, d\sigma(\theta) \right)^{1/k}$$

$$= \left( \frac{k v_k}{|B|} \int_{S^{k-1}} \left( \frac{\|\theta\|_B}{\|\theta\|_C} \right)^k \int_0^{\|\theta\|_B^{-1}} r^{k-1} \, dr \, d\sigma(\theta) \right)^{1/k}$$

$$= \left( \frac{1}{|B|} \int_B \left( \frac{\|x\|_B}{\|x\|_C} \right)^k \, dx \right)^{1/k}$$

$$\geq \left( \frac{1}{|B|} \int_B \left( \frac{\|x\|_B}{\|x\|_C} \right)^{-p} \, dx \right)^{-1/p}$$

$$= \left( \frac{k + p}{k|B|} \right) \left( \int_B \|x\|_C^p \, dx \right)^{-1/p}. \quad \Box$$

**(Lemma 4 appears in [MiP] as Corollary 2.2.)**
Lemma 5. Suppose \( u_1, \ldots, u_n \in \mathbb{R}^k \) with \( k \leq n \) and \( 1 \leq p < \infty \). Then
\[
\{ x \in \mathbb{R}^k : \langle x, u_i \rangle \leq 1 \text{ for every } i \}\]_{1/k}^{1/k} 
\geq 2 \left( \frac{k + p}{k} \sum_{i=1}^{n} \frac{1}{|B_p^k|} \int_{B_p^k} |\langle x, u_i \rangle|^p \, dx \right)^{-1/p}.
\]

Proof. Define \( T : \mathbb{R}^k \to \mathbb{R}^n \) by \((Tx)_i = \langle x, u_i \rangle\), \( 1 \leq i \leq n \) and let \( U = T(\mathbb{R}^k) \). The problem is to estimate from below
\[
|T^{-1}(B_\infty^n)|^{1/k} = |T^{-1}(U \cap B_\infty^n)|^{1/k}.
\]

By Theorem 3,
\[
|U \cap B_\infty^n|^{1/k} \geq 2 \left( \frac{|U \cap B_p^n|}{|B_p^k|} \right)^{1/k}
\]
and so
\[
|T^{-1}(B_\infty^n)|^{1/k} \geq 2 \left( \frac{|T^{-1}(B_p^n)|}{|B_p^k|} \right)^{1/k}.
\]

Regard \( T \) as an operator: \( l^k_p \to l^n_p \). Then by Lemma 4,
\[
2 \left( \frac{|T^{-1}(B_p^n)|}{|B_p^k|} \right)^{1/k} \geq 2 \left( \frac{k + p}{k|B_p^k|} \int_{B_p^k} \|Tx\|^p \, dx \right)^{-1/p}
\]
\[
= 2 \left( \frac{k + p}{k|B_p^k|} \int_{B_p^k} \sum_{i=1}^{n} |\langle x, u_i \rangle|^p \, dx \right)^{-1/p}.
\]

Proof of Theorem 2. Let \((u_i)_i^n\) and \( p \) be as above. For each \( i \) let \( v_i \) be the unit vector in the direction of \( u_i \). By Lemma 5,
\[
\{ x \in \mathbb{R}^k : \langle x, u_i \rangle \leq 1 \text{ for every } i \}\]_{1/k}^{1/k} 
\geq 2 \left( \frac{k + p}{k} \sum_{i=1}^{n} \frac{1}{|B_p^k|} \int_{B_p^k} |\langle x, u_i \rangle|^p \, dx \right)^{-1/p}
\]
\[
= 2 \left( \frac{k + p}{k} \sum_{i=1}^{n} |u_i|^p \cdot \frac{1}{|B_p^k|} \int_{B_p^k} |\langle x, v_i \rangle|^p \, dx \right)^{-1/p}
\]
\[
\geq 2 \left( \frac{1}{k} \sum_{i=1}^{n} |u_i|^p \right)^{-1/p} \min \left( \frac{k + p}{|B_p^k|} \int_{B_p^k} |\langle x, v \rangle|^p \, dx \right)^{-1/p}
\]
where the minimum is taken over all vectors \( v \) of Euclidean length 1. So to complete the proof it suffices to show that for such a vector \( v \),
\[
\left( \frac{k + p}{|B_p^k|} \int_{B_p^k} |\langle x, v \rangle|^p \, dx \right)^{1/p} \leq \begin{cases} \sqrt{p/2} & \text{if } p \geq 2 \\ 2 & \text{if } 1 \leq p < 2 \end{cases}
\]
Let \((x^{(j)})_1^k\) and \((v^{(j)})_1^k\) be the coordinates of the vectors \(x\) and \(v\) in \(\mathbb{R}^k\). For \(p \geq 2\), observe that the functions \((x^{(j)}v^{(j)})\) on \(B_p^k\) form a conditionally symmetric sequence, so by Khintchine’s inequality and Hölder’s inequality (for \(\sum_1^n v^{(j)2} = 1\)),

\[
\left( \frac{k + p}{|B_p^k|} \int_{B_p^k} \left| \sum_1^k x^{(j)}v^{(j)} \right|^p dx \right)^{1/p} \leq \sqrt{\frac{p}{2}} \left( \frac{k + p}{|B_p^k|} \int_{B_p^k} \left( \sum_1^k x^{(j)2}v^{(j)2} \right)^{p/2} dx \right)^{1/p} \leq \sqrt{\frac{p}{2}} \left( \frac{k + p}{|B_p^k|} \int_{B_p^k} \sum_1^n |x^{(j)}|^p v^{(j)2} dx \right)^{1/p} = \sqrt{\frac{p}{2}} \left( \frac{k + p}{|B_p^k|} \int_{B_p^k} |x^{(1)}|^p dx \right)^{1/p} = \sqrt{\frac{p}{2}}.
\]

For \(1 < p < 2\) it is easily checked that

\[
\left( \frac{k + p}{|B_p^k|} \int_{B_p^k} |(x, v)|^p dx \right)^{1/p} \leq (k + p)^{1/p} \left( \frac{1}{|B_p^k|} \int_{B_p^k} (x, v)^2 dx \right)^{1/2} = (k + p)^{1/p} \left( \frac{1}{|B_p^k|} \int_{B_p^k} (x^{(1)})^2 dx \right)^{1/2}
\]

and the last expression can be (rather roughly) estimated by 2 using standard inequalities involving logarithmically concave functions. □

Remark. The proof of Theorem 2 can be simplified even further if the integration over \(B_p^k\) is replaced by integration over \(S^{k-1}\) (and Hölder’s inequality applied here). The proof was presented as above because Lemma 5 has some intrinsic interest: for example it may be used to recover Gluskin’s precise estimate as follows. Suppose \(m \in \mathbb{N}\) and the vectors \((z_i)_1^m \in \mathbb{R}^k\) satisfy

\[|z_i| \leq (\log(1 + m/k))^{-1/2}, \quad 1 \leq i \leq m.\]

For \(\varepsilon > 0\), let \(W(\varepsilon)\) be the set

\[
\left\{ x \in \mathbb{R}^k : \max_j |x^{(j)}| \leq 1, \max_i |(x, z_i)| \leq \frac{1}{\varepsilon} \right\}.
\]
that is, $W(\varepsilon)$ is the intersection of the cube $B^k_\infty$ with $m$ "bands" of width at most $(2/\varepsilon)\sqrt{\log(1 + m/k)}$. Then $|W(\varepsilon)|^{1/k} \to 2$ as $\varepsilon \to 0$, uniformly in $k$ and $m$. To see this, apply Lemma 5 with $n = k + m$, the first $k$, $u_i$'s being the standard basis vectors $\mathbb{R}^k$ and the remaining $m$ being the vectors $(\varepsilon z_i)_1^m$. If $e_j$ is a standard basis vector,

$$\frac{k + p}{|B^k_p|} \int_{B^k_p} |\langle x, e_j \rangle|^p \, dx = 1$$

and so Lemma 5 (and the proof of Theorem 2) show that for each $p \geq 2$,

$$|W(\varepsilon)|^{1/k} \geq 2(1 + m/k)^{(p/2)^2} e^p (\log(1 + m/k))^{-1/p}$$

and the latter is at least $2/(1 + \sqrt{\varepsilon} \cdot \varepsilon)$ if $p = \max(2, 2 \log(1 + m/k))$. \[\square\]

As was briefly mentioned earlier, more general estimates than that of Theorem 2 are obtained in [BP] (for entropy numbers instead of volumes). It is worth noting however that even the argument of Theorem 2 can be used to give the following: there is a constant $c$ so that if $u_1, \ldots, u_n \in \mathbb{R}^k$, $k \leq n$ and $T: l^k_2 \rightarrow l^n_\infty$ is given by $(Tx)_i = \langle u_i, x \rangle$, $1 \leq i \leq n$, then the $k$th entropy number of $T$ satisfies

$$e_k(T) \leq c \sqrt{k} \left( \frac{1}{k} \sum_{i=1}^{n} |u_i|^p \right)^{1/p} \left( 1 + \log \frac{n}{k} \right)^{1/p}$$

and hence

$$e_k(T) \leq \frac{ec}{\sqrt{k}} \sqrt{1 + \log \frac{n}{k} \cdot \|T\|}$$

(taking $p = 2(1 + \log(n/k))$). To obtain this, one uses Schütz's estimates [S] for the entropy numbers of the formal identity from $l^n_p$ to $l^n_\infty$ in place of the result of Meyer and Pajor, an the dual Sudakov inequality of Pajor and Tomczak, [PT] in place of the application of Hölder's inequality.

2. AN APPLICATION TO LINEAR FORMS

As stated in the Introduction, Vaaler's original result has applications to the geometry of numbers. One such, a sharpened form of Siegel's lemma, is given in [BV]. Using the arguments of Bombieri and Vaaler and Theorem 2, one can obtain the generalization of their result, contained in Theorem 6, below. Some notation is needed. If $A$ is a $k \times n$ matrix of reals with independent rows ($1 \leq k \leq n$), denote by $v_j = v_j(A)$, $1 \leq j \leq k$, the rows of $A$. Let $(e_i)_1^n$ be the standard basis of $\mathbb{R}^n$ and denote by $c_i$, the distance (in the Euclidean norm) of $e_i$ from the span of the $v_j$'s in $\mathbb{R}^n$. (So if $A_i$ is the matrix with $k + 1$ rows, $v_1, \ldots, v_n, e_i$ then

$$c_i^2 = \frac{\det(A_i A_i^*)}{\det(A A^*)}$$

for $1 \leq i \leq n$.)

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Theorem 6. Let $A$ be a $k \times n$ matrix with rank $k$ and integral entries. With the notation above, the system $Ax = 0$ admits $n - k$ linearly independent solutions

$$z^{(r)} = (z_1^{(r)}, \ldots, z_n^{(r)}) \in \mathbb{Z}^n, \quad 1 \leq r \leq n - k$$

so that for every $p \geq 2$,

$$\prod_{1 \leq r \leq n-k} \max_i |z_i^{(r)}| \leq D^{-1} \sqrt{\frac{p}{2}} \left( \frac{1}{n-k} \sum_i c_i^p \right)^{1/p} \sqrt{\det AA^*}$$

where $D$ denotes the $G-C-D$ of all $k \times k$ determinants extracted from $A$.

Remark. The principal importance of such a generalization of Bombieri and Vaaler's result is that it takes into account, more strongly, the form of the matrix $A$. If the $c_i$'s are all about the same size, then for $p > 2$, the expression

$$\left( \frac{1}{n-k} \sum_i c_i^p \right)^{1/p}$$

is small compared with the corresponding expression in which $p$ is replaced by 2.

References

[BF] I. Bárány and Z. Füredi, Computing the volume is difficult, Discrete Comput. Geom. 2 (1987), 319–326.

[BLM] J. Bourgain, J. Lindenstrauss and V. D. Milman, Approximation of zonoids by zonotopes, Acta Math. 162 (1989), 73–141.

[BP] K. M. Ball and A. Pajor, On the entropy of convex bodies with "few" extreme points, in preparation.

[BV] E. Bombieri and J. Vaaler, On Siegel's lemma, Invent. Math. 73 (1983), 11–32.

[CP] B. Carl and A. Pajor, Gelfand numbers of operators with values in a Hilbert space, Invent. Math. 94 (1988), 479–504.

[FJ] T. Figiel and W. B. Johnson, Large subspaces of $l_\infty^n$ and estimates of the Gordon–Lewis constants, Israel J. Math. 37 (1980), 92–112.

[G] E. D. Gluskin, Extremal properties of rectangular parallepipeds and their applications to the geometry of Banach spaces, Mat. Sb. (N. S.) 136 (1988), 85–95.

[MeP] M. Meyer and A. Pajor, Sections of the unit ball of $l_p^n$, J. Funct. Anal. 80 (1988), 109–123.

[MiP] V. D. Milman and A. Pajor, Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed $n$-dimensional space, Israel seminar on G.A.F.A. (1987–88), Springer-Verlag, Lectures Notes in Math., vol. 1376, 1989.

[PT] A. Pajor and N. Tomczak-Jaegermann, Subspaces of small codimension of finite-dimensional Banach spaces, Proc. Amer. Math. Soc. 97 (1986), 637–642.

[S] C. Schütt, Entropy numbers of diagonal operators between symmetric Banach spaces, J. Approx. Theory 40 (1984), 121–128.

[V] J. D. Vaaler, A geometric inequality with applications to linear forms, Pacific J. Math. 83 (1979), 543–553.

Department of Mathematics, Texas A & M University, College Station, Texas 77843

U. E. R. de Mathématiques, Université de Paris VII, 2 Place Jussieu, 75251 Paris Cedex 05, France