Phase Structure of Compact Lattice $QED_3$

with Massless Fermions

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Abstract

In the framework of (2+1)-dimensional compact lattice QED with light fermions, we investigate the phase diagram in the $(\beta, N)$ plane. The approximations involved are related to an expansion of the effective fermionic action as a power series of the flavor number $N$. We also develop a new mechanism for understanding the $N$–critical phenomenon in the full theory. Our results for the specific heat indicate that only one phase does exist. We give strong evidences that this qualitative result should not be changed with the inclusion of higher order terms in the $N$ expansion.
There are several reasons which made it interesting the study of Quantum Field Theories in 2+1 dimensions and justify the increasing attention devoted to this field in recent time. Essentially, the motivation for such an analysis is two-fold:

i) There are many similarities between gauge theories in 2+1 dimensions and four dimensional QCD like quark confinement and spontaneous symmetry breaking, which suggest us to use three dimensional formulation of gauge theories as a simpler laboratory to study general features of QCD.

ii) Recently it has been shown that a very popular model in the area of high $T_c$ superconductivity, the Heisenberg model, can be reformulated exactly as a $SU(2)$ lattice gauge theory with massless dynamical staggered fermions in 2+1 dimensions at strong coupling ($\beta = 0$) [1]. Chiral-symmetry breaking in three-dimensional lattice gauge theories is the analog of the Neel phase of antiferromagnets. On the other hand the fact that we do not expect essential differences between the abelian and non abelian models, at least in the strong coupling region, suggests that the study of the abelian case should be relevant also in this context.

The first analytic studies of continuum $QED_3$ in the $1/N$ approximation [2,3] indicated that dynamical fermion mass generation takes place for any arbitrarily large number of flavors $N$. More recently, a reexamination of the $1/N$ expansion of the Dyson-Schwinger kernel opened the possibility for the existence of a critical $N_c \approx 32/\pi^2$ beyond which chiral symmetry is restored [4,5].

The results of the Dyson-Schwinger approach [4,5] seem to have been confirmed by the first numerical simulations of noncompact $QED_3$ [6,7]. However the small lattices and limited data for the fermion masses $m$ used in [6,7] to extrapolate to the chiral limit, makes this results not conclusive. Furthermore Pennington and Walsh [8] have recently found by numerical methods non perturbative solutions of the Dyson-Schwinger equation for the full fermion propagator, which exhibit chiral symmetry breaking for all values of $N$. The corresponding chiral condensate seems to decrease exponentially with $N$ and if it is true, this kind of behavior puts serious doubts on the feasibility of the $1/N$ approximation approach to $QED_3$.

Therefore, at present the situation is far from being clear and the aim of this letter is to increase our understanding of the $N$-critical phenomenon by studying numerically the three-dimensional compact model. We will pay special attention to the analysis of the phase diagram of this model in the
(β, N) plane and will develop a physical picture for understanding the origin of phase transitions in this parameter space. A similar study to that developed here but for the noncompact model is in progress and the results will be published in the next future.

The approach we have used in our calculations is based on the application of the microcanonical fermionic average method introduced in [9] and tested for the compact [10] and noncompact [11,12] formulations of the abelian model in four dimensions. We therefore refer to these references for technical details.

In order to explore the phase diagram in the (β, N) plane, we should search for non analyticities of thermodynamical quantities in this plane. A direct investigation of the zero mass limit of the chiral condensate, which would give us information on how chiral symmetry is realised in this model at different flavor numbers, has several technical difficulties. Mainly, an extrapolation of the numerical results for the chiral condensate to the massless limit is necessary and the final result may depend on the kind of fitting function used in this procedure, specially when the numerical value of the chiral condensate is very small.

This difficulty can be overcome if we look for other thermodynamical quantities like plaquette energy and specific heat since in such a case no symmetry forces them to be zero. This is indeed what was done in [11,12] to investigate the phase diagram and the nature of the fixed point in (3+1)-dimensional noncompact QED and the results were very satisfactory. Therefore in this letter we will concentrate on the investigation of the behavior of the plaquette energy and specific heat in the (β, N) plane whereas our results for the chiral condensate will be reported in a separate publication.

Our starting point is the definition of an effective fermionic action [9] which depends on the pure gauge plaquette energy E, fermion mass m and flavor number N and which is related to the gauge link variables by the expression

\[
e^{-S^F_{eff}(E,m,N)} = \frac{\int [dU_\mu(x)](det\Delta(m,U_\mu(x)))^{N/2}\delta(S_G(U_\mu(x)) - 3V E)}{\int [dU_\mu(x)]\delta(S_G(U_\mu(x)) - 3V E)}, \quad (1)
\]

where the integration is over all gauge link variables that are elements of the compact U(1) gauge group. V is the lattice volume, Δ is the fermionic matrix (we use staggered fermions) and the exponent 1/2 in the determinant
is due to the fermion doubling. \( S_G(U) \) is the full pure gauge plaquette energy defined as the sum over all lattice plaquettes of the cosine of the plaquette angle and the denominator in (1) is the density of states \( N(E) \) of fixed pure gauge normalized energy \( E \).

The partition function can be written now as a one-dimensional integral over the normalized pure gauge energy as

\[
Z = \int dE e^{-S_{\text{eff}}(E, m, N, \beta)}.
\]

where \( S_{\text{eff}}(E, m, N, \beta) \) is the full effective action related to the density of states \( N(E) \), the fermionic action \( S_{\text{eff}}^F(E, m, N) \) and the pure gauge energy \( E \) by

\[
S_{\text{eff}}(E, m, N, \beta) = -\ln N(E) - 3\beta V E + S_{\text{eff}}^F(E, m, N).
\]

This effective action diverges linearly with the lattice volume \( V \) in the infinite volume limit and in this limit, the thermodynamics of the system can be studied by means of the saddle point technique. The mean plaquette energy \( \langle E_p \rangle = E_0(m, \beta, N) \) will be given by the solution of the saddle point equation

\[
\frac{-1}{n(E)} \frac{dn(E)}{dE} + \frac{\partial S_{\text{eff}}^F(E, m, N)}{\partial E} - 3\beta = 0
\]

satisfying the minimum condition

\[
\left\{ \frac{1}{n(E)^2} \frac{d^2 n(E)}{dE^2} - \frac{1}{n(E)} \frac{dn(E)}{dE} \frac{\partial S_{\text{eff}}^F(E, m, N)}{\partial E^2} \right\}_{E_0(m, \beta, N)} > 0,
\]

where \( n(E) \) in (4), (5) is related to the density of states \( N(E) \) by the expression

\[
n(E) = N(E)^{1/V},
\]

and \( S_{\text{eff}}^F(E, m, N) \) is the effective fermionic action normalized by the lattice volume.

The specific heat \( C_\beta = \partial E_0 / \partial \beta \) can be obtained by derivating equation (4), and the final expression is
$$C_\beta = 3\{ \frac{1}{n(E)^2} \frac{d^2 n(E)}{dE^2} - \frac{1}{n(E)} \frac{d^2 n(E)}{dE^2} + \frac{\partial^2 S_{\text{eff}}^F(E, m, N)}{\partial E^2} \}^{-1}_{E_0(m, \beta, N)},$$  \hspace{1cm} (7)

Before discussing the results of our numerical simulations, let us do a qualitative analysis on the basis of expression (7) for the specific heat. The question now is: how a phase transition can be generated when the number of dynamical flavors is large enough? In the pure gauge model it has been shown that no phase transition occurs at any finite value of the inverse coupling constant $\beta$ [13,14]. In other words moving $\beta$ from zero to $\infty$, the plaquette energy takes continuously all the values from 0 to 1. Therefore the sum of the first two terms in the denominator of the specific heat (7) will be positive in all the [0,1) energy interval (convexity of the effective action!). Now let switch on dynamical fermions in the system and consider the $N$ parameter as a continuous parameter. It can be shown that the effective fermionic action $S_{\text{eff}}^F(E, m, N)$ diverges linearly with $N$ when $N$ goes to $\infty$. Then, if the sign of the second energy derivative of $S_{\text{eff}}^F(E, m, N)$ is negative in some energy interval and $N$ is large enough, the third contribution to the denominator of the specific heat (7) can compensate the pure gauge contribution and the global sign in the denominator of (7) could be changed in this way [15]. In such a case there will be some energy interval in which the minimum condition (5) of the saddle point equation (4) does not hold. This energy interval will not be accessible to the system and then a first order phase transition will appear. Decreasing now the value of the parameter $N$, we will find a critical value $N_c$, at which the energy interval where equation (5) does not hold will become a single point. This critical value $N_c$, where the specific heat diverges, will be the end point of a first order phase transition line in the $(\beta, N)$ plane.

Now let us analyse our numerical results in order to see how this general scheme is realized in compact $QED_3$. The feasibility of a direct computation of the effective fermionic action (1) is related to the form of the probability distribution function of the logarithm of the fermionic determinant at fixed pure gauge energy [16]. An alternative and more reliable way is to compute $S_{\text{eff}}^F$ by expanding it in a power series of $N$ (cumulant expansion) [12,16]
\[ -S^F_{\text{eff}}(E, m, N) = \frac{N}{2} \langle \ln \text{det} \Delta \rangle_E + \frac{N^2}{8} \{ \langle (\ln \text{det} \Delta)^2 \rangle_E - \langle \ln \text{det} \Delta \rangle^2_E \} + \ldots \]  

(8)

where \( \langle O \rangle_E \) means the mean value of the operator \( O(U_\mu(x)) \) computed with the probability distribution \( [dU_\mu(x)] \delta(S_G(U_\mu(x) - 3VE)) \).

We have done simulations on 8\(^3\), 10\(^3\), and 14\(^3\) lattices. On each lattice 100000 configurations at each fixed \( E \) were generated by an exact microcanonical algorithm. The fermionic matrix at zero mass was exactly diagonalized for 200 decorrelated configurations at each energy by means of a modified Lanczos algorithm [17]. From the eigenvalues obtained in this way we can compute the successive terms in (8) for any value of the fermion mass \( m \). Fig. 1 shows the \( \langle \ln \text{det} \Delta \rangle_E \) as a function of \( E \) on a 14\(^3\) lattice.

We have done canonical simulations of the effective theory described by the partition function (2). In these simulations we have approximated the effective fermionic action by its first contribution in the cumulant expansion (8) and measured the plaquette energy, specific heat, chiral condensate and Binder parameter. Details of these simulations will be reported in a longer publication. The most relevant thing which emerges from the analysis of the results of these simulations is the \((\beta, N)\) plane phase diagram for massless fermions plotted in Fig. 2. The striking feature of this phase diagram is the fact that the first order phase transition line ends at finite (negative) \( \beta \).

As a check of these results we have done the following simple calculation. We fitted the results for the first contribution to the effective fermionic action by a fifth degree polynomial (continuous line in Fig. 1). From the results of this fit and the computation of the specific heat in the pure gauge model, we found by means of equation (5) that the critical number of flavors at the end point of the first order phase transition line is \( N_c = 18.4 \) and the critical plaquette energy \( E_c = 0.48 \), in very good agreement with the results of the numerical simulation.

The fact that the first order phase transition line of Fig. 2 ends at a finite value of the inverse coupling constant \( \beta \) can be understood on the light of the energy dependence of the effective fermionic action plotted in Fig. 1. The second energy derivative of the effective fermionic action is negative in the small-intermediate energy region but it becomes positive in the large energy region. Therefore at large energy (or equivalently large \( \beta \) values) the specific heat will be finite and equation (5) will hold. In order to get a phase
transition line ending at $\beta = \infty$ an opposite convexity of $S_{eff}^F$ in the large energy region is necessary.

Might this qualitative result be changed with the inclusion of higher order terms in the $N$ expansion of the effective fermionic action? We think that there are several indications which make such a possibility highly unlikely. They are mainly:

i) The second energy derivative of the pure gauge effective action (sum of the first two terms in (5)) is, as stated before, positive for all $E > 0$. Even more, when the energy approaches the maximum value $E = 1$, this quantity diverges or equivalently the specific heat goes to zero when $\beta$ goes to $\infty$ in the pure gauge theory. Therefore in order to change the sign of (5) at $E$ near 1, we need not only a change in the convexity of the effective fermionic action at large energy but also a second energy derivative negatively divergent in this limit.

ii) The probability distribution of the logarithm of the fermionic determinant at large energy is very narrow (much more than in the small energy region) due to the fact that the density of states $n(E)$ goes to zero when $E$ goes to 1. In fact we have observed that the fluctuations of the logarithm of the determinant at fixed energy in the large energy region ($E > 0.85$) are independent of the lattice volume inside statistical errors, indicating that the first term in (8) is the only relevant one for $E$ near 1 in the infinite volume limit.

The physical picture which emerges from the phase diagram plotted in Fig. 2 is that only one phase does exist in the $(\beta,N)$ plane for massless compact $QED_3$. This suggests that chiral symmetry be spontaneously broken for any number of dynamical fermions. We can not give a definite answer at present to what extent this result will remain unchanged when all the higher order terms in the $N$ expansion of the effective fermionic action will be included. However we have presented strong arguments and numerical results which make hard to imagine that a qualitative change might occur.

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Figure Captions

Fig. 1. First contribution to the effective fermionic action (8) versus $E$ at several fermion masses and two flavors, obtained through microcanonical simulations. Statistical errors are invisible at this scale.

Fig. 2. Phase diagram in the $(\beta, N)$ plane. Statistical errors are invisible at this scale.