LOOP MODELS, MARGINALLY ROUGH INTERFACES, 
AND THE COULOMB GAS *

JANÉ KONDEV

Department of Physics, Brown University
Providence, Rhode Island, 02912-1843, USA

Received (received date)
Revised (revised date)

We develop a coarse-graining procedure for two-dimensional models of fluctuating loops by mapping them to interface models. The result is an effective field theory for the scaling limit of loop models, which is found to be a Liouville theory with imaginary couplings. This field theory is completely specified by geometry and conformal invariance alone, and it leads to exact results for the critical exponents and the conformal charge of loop models. A physical interpretation of the Dotsenko-Fateev screening charge is found.

1. Introduction

Loop models are defined by drawing closed loops (which can come in one or more flavors) along the bonds of a two-dimensional lattice \( \mathcal{L} \), subject to the constraint that only loops of different flavor may cross. Boltzmann weights of different loop configurations are completely determined by specifying the weight (\( \rho \)) of all possible loop arrangements at a single vertex of \( \mathcal{L} \), and the fugacity (\( n \)) of every loop flavor.

The partition function can be written as:

\[
Z = \sum_{\mathcal{G}} \rho_1^{m_1} \rho_2^{m_2} \cdots \rho_V^{m_V} n_1^{l_1} n_2^{l_2} \cdots n_F^{l_F}
\]

where \( F \) is the number of loop flavors and \( V \) the number of allowed vertex configurations. The sum in Eq. (1) goes over all allowed loop configurations \( \mathcal{G} \); \( m_i \) \( (i = 1, \ldots, V) \) is the number of vertices of type \( i \), and \( l_j \) \( (j = 1, \ldots, F) \) the number of loops of flavor \( j \) in \( \mathcal{G} \).

The motivation for studying loop models comes from the observation that they are rather ubiquitous in the realm of two-dimensional statistical mechanics. Loop models appear very naturally when one considers domain boundaries in discrete spin models. For example, model \( A \) in Fig. 1 is equivalent to the \( Q = n^2 \) Potts model on the square lattice, where the loops run along the bonds of the medial lattice so as to encircle clusters of equal Potts spin.

In a somewhat different setting, loop models appear as space-time diagrams of certain one-dimensional quantum systems, where the quantum partition function can be written as a sum over loop configurations with appropriate weights. This loop representation of quantum models has

---

*Parts of this work were done in collaboration with C.L. Henley, J. deGier, and B. Nienhuis.
been successfully taken advantage of in recent Monte Carlo (MC) simulations of spin chains and ladders. The loop representation allows for non-local (loop) update moves in the MC algorithm, which lead to a considerable reduction in critical slowing down, and rather large system sizes can be simulated. On the analytic side, the loop representation of the Heisenberg spin chain has lead to a very nice geometrical interpretation of correlation functions. For instance, the spin-spin correlation function is associated with the probability that two points on the lattice are connected by a loop segment, in the appropriate loop model.

![Fig. 1. Allowed vertex configurations in fully packed loop models on the square and honeycomb lattice; every vertex is assigned the weight $\rho = 1$. The models A and B have a single loop flavor with fugacity $n$, while loops in model C come in two flavors (black and grey), with fugacities $n_1$ and $n_2$.]

For a specific choice of Boltzmann weights, loop models can be in a critical state characterized by power law correlations. Here we examine the critical properties of fully packed loop models. These loop models have non-vanishing and identical ($\rho_i = 1$) vertex weights only for configurations that satisfy the fully-packing constraint; this constraint ensures that every vertex of the lattice is covered by one, and only one loop of each flavor. We have studied three such models, shown in Fig. 1. Model A (No. of loop flavors, $F = 1$) is equivalent to the loop decomposition of the critical $Q = n^2$ Potts model, model $B$ ($F = 1$) is the zero-temperature phase of the $O(n)$ model, and model $C$ ($F = 2$) has recently been introduced as the loop generalization of the four-coloring model on the square lattice. The theoretical challenge we are faced with, is to find an effective description of a critical loop model in terms of a (conformal) field theory, from which critical exponents and other universal quantities can be calculated exactly.

The answer to the above stated theoretical challenge is provided by mapping a loop model to an interface model, where loops are identified with contour lines of the interface. Coarse-graining of the interface model leads to an effective field theory which describes the critical fluctuations of loops in terms of a fluctuating height field. This effective field theory is equivalent by a duality to a Coulomb gas of electric and magnetic charges, with an additional electric charge placed at infinity.

In the Coulomb gas approach to two-dimensional critical phenomena, the critical phase is identified with the vacuum phase of a gas of electric and magnetic charges.
the strength of the Coulomb interaction is parameterized by the coupling constant $g$. Critical exponents are readily calculated once the value of this coupling is known. In the interface representation of loop models, $g$ has the physical interpretation of the effective surface tension, or stiffness (see Eq. (1)). The effective stiffness is not a universal quantity and to calculate it in terms of the microscopic weights is typically as hard as solving the model exactly.$^a$ In this paper we show that the stiffness of the interface is completely specified by geometry and conformal invariance alone. More precisely, it is shown that demanding conformal invariance of the effective field theory of critical loop models completely fixes all the couplings in the theory.

2. Interface representations of loop models

The general procedure for constructing an effective field theory of a loop model is outlined here for the $A$ model. Details of this construction for models $B$ and $C$ can be found elsewhere.$^7,11$

Model $A$ is the well known polygon representation of the $Q = n^2$ Potts model, and it is critical for $0 \leq n \leq 2$. For $n > 2$ the model has a finite correlation length, roughly the size of the largest loop; in the limit $n \to \infty$ it orders in one of two states, which are related by a lattice translation, and in which all the loops are of length four.

The construction of an effective field theory for the scaling limit of a loop model can be broken up into three steps:

2.1. Orient the loops

Each loop is assigned an orientation by placing arrows along the bonds of the square lattice. In order to assign proper weights to the loops ($n \leq 2$) every left (right) turn is weighted by the phase factor $\exp(-i\epsilon_0 \pi/4)$ ($\exp(i\epsilon_0 \pi/4)$). For every closed loop on the square lattice the difference between the number of right, and the number of left turns is $\pm 4$. Therefore, by summing over the two possible orientations each loop is assigned the weight

$$n = 2 \cos(\pi \epsilon_0).$$  \hspace{1cm} (2)

2.2. Map to an interface model

Each loop is viewed as a contour line of an interface model. For the $A$ model this is the well known body-centered solid-on-solid (BCSOS) model$^12$; for the $B$ and $C$ models, interface models with two and three component height variables are obtained.$^13,8,10$ The microscopic heights $z$, of model $A$, are defined on the dual lattice, and $z$ increases (decreases) by $1/2$ every time an oriented loop is crossed from left to right (right to left). Coarse-graining of the microscopic height leads to an effective field theory of the loop model.

$^a$In the past, the value of the coupling constant has been determined for models that map to exactly solvable models by comparing the Coulomb-gas expression for a particular exponent, which is $g$ dependent, with the value of the same exponent found from the exact solution.$^9$
2.3. Coarse grain the interface model

The interface model is coarse-grained in terms of the ideal states. These states minimize the variance of the microscopic height, i.e., they are macroscopically flat. In the loop model these are states with the maximum number of loops on the lattice, or equivalently, the states which one finds in the limit $n \to \infty$. For finite $n$ the lattice breaks up into domains of ideal states, and each domain is assigned a coarse-grained height equal to the average microscopic height over the domain: $h = \langle z \rangle$. The continuum limit is taken by replacing the discrete heights by a height field $h(x)$. The height field $h(x)$ is compactified to a circle, i.e.,

$$ h \equiv h + \mathcal{R}, \quad \mathcal{R} = \{0, \pm 1, \pm 2, \ldots\} $$ (3)

where $\mathcal{R}$ is the repeat lattice; the vectors in this lattice are height differences between equivalent loop configurations.

The Euclidean action of the effective field theory, which describes the long-wavelength fluctuations of the microscopic height (or the critical fluctuations of the loops in the $A$ model), can be written in terms of the field $\phi(x) \equiv 2\pi h(x)$ as

$$ S[\phi] = S_g[\phi] + S_b[\phi] + S_p[\phi] $$

$$ S_g = \frac{g}{4\pi} \int d^2x \ [\nabla \phi]^2 $$

$$ S_b = \frac{i\varepsilon_0}{4\pi} \int d^2x \ R \phi $$

$$ S_p = \int d^2x \ w(\phi) $$ (4)

The three parts to the action have a simple geometrical interpretation:

(i) $S_g$ describes the Gaussian fluctuations of the height around the flat ideal states.

(ii) $S_b$ is the coupling of the height to the scalar curvature $R$. Namely, if a loop winds around a point of non-zero curvature, then the difference between the number of left and the number of right turns, for this loop, is no longer four; $S_b$ corrects for this. For the square lattice in the infinite plane this term corresponds to a background charge $2\varepsilon_0$ placed at the point at infinity.

(iii) $S_p$ is the potential term. Its origins are twofold: 1) it accounts for the discrete nature of the heights, and 2) it assigns proper weights ($n < 2$) to the loops, i.e., the operator $w(\phi)$ is conjugate to the loop weight. From Eq.(3) we conclude that $w(\phi)$ can be expanded into a Fourier series,

$$ w(\phi) = \sum_{E \in \mathcal{R}^*} w_E e^{iE\phi} $$ (5)

where $\mathcal{R}^*$ is the lattice dual to $\mathcal{R}$; here $\mathcal{R}^*$ is simply the set of integers. Only the most relevant vertex operator (i.e., exponential of the height field) appear-

---

Elements of $\mathcal{R}^*$ are the electric charges, while elements of $\mathcal{R}$ are the magnetic charges of the two-dimensional Coulomb gas, which is dual to the interface model.
ing in the above Fourier expansion is kept in the action. Upon examining the values which \( w(\phi) \) takes in the ideal states, we find this to be the vertex operator with electric charge

\[
E_w = 2 .
\]  
(6)

3. Calculation of the coupling

In order to calculate the value of the coupling \( g \), for model A in the critical phase \((0 \leq n \leq 2)\), we demand that the action \( S \) describe a conformally invariant field theory. This, as will be made clear below, is equivalent to the assumption that the loop fugacity does not change with the change of scale.

For \( S \) to describe a conformal field theory the potential term has to be marginal, i.e., the scaling dimension \( x_w \) of the operator \( w(\phi) \) is two. The dimension \( x_w \) can be expressed in terms of the coupling \( g \) and the background charge \( e_0 \) as:

\[
x_w = E_w (E_w - 2e_0) / 2g = 2 .
\]  
(7)

Using Eq.(6) we find the exact value of the coupling

\[
g = 1 - e_0 ;
\]  
(8)

both the background charge and \( g \) are continuous functions of the fugacity \( n \), Eq.(2). This value of the coupling agrees with the value found from the exact solution of the eight-vertex model. We emphasize that \( e_0(n) \) in Eq.(2) defines a whole family of critical models, whose conformal charges are given in Eq.(9).

4. Discussion

The field theory described by the Euclidean action in Eq. (4) is a Liouville theory with imaginary couplings, which has been suggested as the Lagrangean description of a two-dimensional Coulomb gas in the presence of a background charge. The potential term \( S_p \) is the screening charge, which was originally introduced into the Coulomb gas to ensure that the four-point correlation functions do not vanish. Here this operator appears quite naturally in the action as a result of the coarse graining, and it is the operator conjugate to the loop fugacity.

The term \( S_p \) also contains the locking potential, which enforces the condition that the heights are discrete. We find that the locking potential is marginal along the whole critical line \((n \leq 2)\) of the loop model. This leads to the somewhat surprising conclusion that the associated interface model is at the roughening transition for all \( n \leq 2 \), not just at the boundary at \( n = 2 \). In the case of model \( B \) Baxter arrived at the same conclusion from the exact solution of the related three-coloring model; he showed that the partition function has a line of essential singularities for \( 0 \leq n \leq 2 \).

For the \( B \) and \( C \) models the height field has more than one component and consequently the magnetic \((\textbf{M})\) and electric charges \((\textbf{E})\) are lattice vectors in \( \mathcal{R} \), and the
Loop models, Marginally Rough Interfaces, and the Coulomb Gas

dual $\mathcal{R}^*$. The proper identification of these lattices follows from the coarse-graining procedure. The fact that the height is compactified on $\mathcal{R}$, Eq. (3), leads to some interesting conclusions about the symmetry of these loop models in the continuum. Namely, for $n = 2$ ($B$ model) and $n_1 = n_2 = 2$ ($C$ model) the background charge vanishes, and these loop models are described, in the continuum, by the $SU(3)^{k=1}$ and the $SU(4)^{k=1}$ Wess-Zumino-Witten model respectively. The action written in terms of the height field is the free-field representation of these sigma models.

The conformal charge and all the critical exponents of the loop models can be expressed in terms of the coupling $g$ and the background charge $e_0$:

$$c = k - \frac{6e_0^2}{g}$$

$$x(E, M) = \frac{1}{2g} E \cdot (E - 2e_0) + \frac{g}{2} M^2,$$  \hspace{1cm} (9)

where $x(E, M)$ is the scaling dimension of the operator with electro-magnetic charge $(E, M)$, and $k = 1, 2,$ and $3$ for the $A, B,$ and $C$ model respectively. For example, the m-RSOS models which are described in the continuum by the minimal models of conformal field theory can be mapped to the the $A$ model with $e_0 = 1/(m + 1), (m > 2)$.

The conformal charge that we calculate from Eqs. (8) and (9) is $c = 1 - 6/m(m + 1)$, which is the well known expression.

Acknowledgments

It is a pleasure to acknowledge illuminating discussions with T. Spencer, J.B. Marston, and in particular J. Cardy, who pointed out to me the connection between marginal operators and screening charges in the Coulomb gas. I am indebted to C.L. Henley, J. deGier, and B. Nienhuis, my collaborators on related projects, who have helped shape my understanding of loop models in a significant way. This work was supported by the NSF through grant No. DMR 9357613.

References

1. S.O. Warnaar and B. Nienhuis, J. Phys. A 26, 2301 (1993).
2. R.J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, 1982)
3. H.G. Evertz, G. Lana, and M. Marcu, Phys. Rev. Lett. 70, 875 (1993); N. Kawashima and J.E. Gubernatis, Phys. Rev. Lett. 73, 1295 (1994); B. Frischmut, B. Ammon, and M. Troyer, preprint No. cond-mat/9601027.
4. M. Aizenman and B. Nachtergaele, Comm. Math. Phys. 164, 17 (1994).
5. H.W.J. Blöte and B. Nienhuis, Phys. Rev. Lett. 72, 1372 (1994).
6. M.T. Batchelor, J. Suzuki, and C.M. Yung, Phys. Rev. Lett. 73, 2646 (1994).
7. J. Kondev, J. deGier, and B. Nienhuis, submitted to J. Phys. A: Math. Gen.; preprint No. cond-mat/9603170.
8. J. Kondev and C.L. Henley, Phys. Rev. B52, 6628 (1995).
9. B. Nienhuis, in *Phase Transitions and Critical Phenomena*, Vol.11, ed. C. Domb and J.L. Lebowitz (Academic Press, New York, 1987)
10. J. Kondev and C.L. Henley, Nucl. Phys. B464, 540 (1996).
11. J. Kondev, in preparation.
12. H. van Beijeren, Phys. Rev. Lett. 38, 993 (1977).
13. D.A. Huse and A.D. Rutenberg, Phys. Rev. B45, 7536 (1992).
14. Vl.S. Dotsenko and V.A. Fateev, Nucl. Phys. B251, 691 (1985); S.D. Mathur, Nucl. Phys. B369, 433 (1992).
15. Vl.S. Dotsenko and V.A. Fateev, Nucl. Phys. B240, 312 (1984).
16. R.J. Baxter, J. Math. Phys. 11, 784 (1970).
17. N. Read, reported in Kagomé workshop (Jan. 1992), unpublished.
18. I. Affleck, Phys. Rev. Lett. 55, 1355 (1985).
19. O. Foda and B. Nienhuis, Nucl. Phys. B324, 643 (1989).