Norm estimates for selfadjoint Toeplitz operators on the Fock space

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Abstract
An estimate for the norm of selfadjoint Toeplitz operators with a radial, bounded and integrable symbol is obtained. This emphasizes the fact that the norm of such operator is strictly less than the supremum norm of the symbol. Consequences for time-frequency localization operators are also given.

1 Introduction
The Bargmann-Fock space $\mathcal{F}^2(\mathbb{C})$ is the Hilbert space consisting of those analytic functions $f \in H(\mathbb{C})$ such that

$$\|f\|_{\mathcal{F}^2}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-\pi|z|^2} \, dA(z) < +\infty,$$

where $dA(z)$ denotes the Lebesgue measure. $\mathcal{F}^2(\mathbb{C})$ admits a reproducing kernel $K_w(z) = e^{\pi w z}$, which means that

$$f(w) = \langle f, K_w \rangle, \quad f \in \mathcal{F}^2(\mathbb{C}).$$

The normalized monomials

$$e_n(z) = \left( \frac{\pi^n}{n!} \right)^{\frac{1}{2}} z^n, \quad n \geq 0,$$

form an orthonormal basis. For a fixed $a \in \mathbb{C}$ the translation operator

$$T_a : \mathcal{F}^2(\mathbb{C}) \to \mathcal{F}^2(\mathbb{C}), \quad (T_a f)(z) = f(z - a)e^{-\pi|a|^2 + \pi \overline{a} z},$$

is an isometry (see [13]). We denote $d\lambda(z) = e^{-\pi|z|^2} \, dA(z)$, so $\mathcal{F}^2(\mathbb{C})$ is a closed subspace of $L^2(\mathbb{C}, d\lambda)$. The orthogonal projection

$$P : L^2(\mathbb{C}, d\lambda) \to \mathcal{F}^2(\mathbb{C})$$
is the integral operator
\[(Pf)(z) = \int_C f(w)K_w(z) \, d\lambda(w).\]

For a measurable and bounded function \(F\) on \(C\) the Toeplitz operator with symbol \(F\) is defined as
\[T_F(f)(z) = P(Ff)(z) = \int_C F(w)f(w)K_w(z) \, d\lambda(w).\]

The systematic study of Toeplitz operators on the Fock space started in [3, 4]. Since then it has been a very active research area. We refer to [14, Chapter 6], where boundedness and membership in the Schatten classes is discussed.

It is obvious that
\[T_F : \mathcal{F}^2(C) \to \mathcal{F}^2(C)\]
is a bounded operator and
\[
\|T_F(f)\| \leq \|Ff\|_{L^2(C, d\lambda)} \leq \|F\|_{\infty} \cdot \|f\|. 
\]
In particular, \(\|T_F\| \leq 1\) whenever \(\|F\|_{\infty} \leq 1\). If moreover \(T_F\) is compact, which happens for instance when \(F \in L^1(C)\), then \(\|T_F\|\) is strictly less than 1 but, as far as we know, no precise estimate for the norm is known. The main result of the paper gives a bound for \(\|T_F\|\) in the case that the symbol \(F\) is radial, real-valued, and satisfies some integrability condition. For Toeplitz operators with radial symbols we refer to [11].

Besides Toeplitz operators on the Fock space we consider time-frequency localization operators with Gaussian window, also known as anti-Wick operators. They where introduced by Daubechies [7] as filters in signal analysis and can be obtained from Toeplitz operators on the Fock space after applying Bargmann transform.

## 2 Toeplitz operators on the Fock space

The Toeplitz operator defined by a real valued symbol \(F\) is self-adjoint. This is immediate from the identity
\[
\langle T_F(f), g \rangle = \int_C F(z)f(z)\overline{g(z)}d\lambda(z)
\]
for all \(f, g \in \mathcal{F}^2(C)\). In this case we have
\[
\|T_F\| = \sup_{\|f\|=1} |\langle T_F(f), f \rangle| \leq \sup_{\|f\|=1} \int_C |F(z)| \cdot |f(z)|^2 d\lambda(z).
\]
A symbol $F$ is said to be radial with respect to $a \in \mathbb{C}$ if $F(z) = g(|z - a|)$ for some bounded and measurable function $g$ on $[0, +\infty)$. The main result of the paper is as follows.

**Theorem 1.** Let $F \in L^1(\mathbb{C}) \cap L^\infty(\mathbb{C})$ be a real-valued and radial symbol with respect to $a \in \mathbb{C}$. Then

$$\|TF\| \leq \|F\|_\infty \left(1 - \exp \left( - \frac{\|F\|_1}{\|F\|_\infty} \right) \right).$$

For the proof we will need some auxiliary results. First we observe that for $|F(z)| = g(|z|)$ and $f = \sum_{n=0}^{\infty} b_n e_n$ we have, after changing to polar coordinates,

$$\int_{\mathbb{C}} |F(z)| \cdot |f(z)|^2 d\lambda(z) = \sum_{n=0}^{\infty} |b_n|^2 \int_{\mathbb{C}} |g(|z|)| e_n(z)|^2 \cdot d\lambda(z)$$

$$= \sum_{n=0}^{\infty} |b_n|^2 2\pi \int_0^\infty g(r) \frac{r^{2n+1}}{n!} e^{-\pi r^2} dr$$

$$= \sum_{n=0}^{\infty} |b_n|^2 \int_0^\infty g\left(\sqrt{\frac{t}{\pi}}\right) \frac{t^n}{n!} e^{-t} dt.$$  

The $d$-dimensional Lebesgue measure of a set $\Omega \subset \mathbb{R}^d$ is denoted $|\Omega|$ both for $d = 1$ and $d = 2$.

**Lemma 2.** Let $I \subset [0, +\infty)$ be a measurable set with finite Lebesgue measure. Then

$$\frac{1}{n!} \int_I s^n e^{-s} \, ds \leq 1 - e^{-|I|}.$$  

**Proof:** (a) We first assume that $I$ is a finite union of bounded intervals. Let $t_n > 0$ the absolute maximum of $h(s) = \frac{e^s}{n!} e^{-s}$. Then $h$ increases on $[0, t_n]$ and decreases on $[t_n, +\infty)$. We consider $a \leq t_n \leq b$ such that

$$t_n - a = |I \cap [0, t_n]| , \quad b - t_n = |I \cap [t_n, +\infty)|.$$
Then
\[
\frac{1}{n!} \int_I s^n e^{-s} \, ds \leq \int_a^b h(s) \, ds = \frac{e^{-a}}{n!} \int_0^{b-a} (t+a)^n e^{-t} \, dt
\]
\[
= \sum_{k=0}^n \binom{n}{k} \frac{a^{n-k}}{n!} e^{-a} \int_0^{|I|} t^k e^{-t} \, dt
\]
\[
= \sum_{k=0}^n \frac{a^{n-k}}{(n-k)!} \frac{1}{k!} e^{-a} \int_0^{|I|} t^k e^{-t} \, dt
\]
\[
\leq \sup_{0 \leq k \leq n} \frac{1}{k!} \int_0^{|I|} t^k e^{-t} \, dt = \int_0^{|I|} e^{-t} \, dt.
\]
For the last identity observe that
\[
\frac{1}{k!} \int_0^s t^k e^{-t} \, dt = 1 - e^{-s} \sum_{j=0}^k \frac{s^j}{j!}.
\]

(b) For a general measurable set $I$ with finite measure the conclusion follows from part (a) and the fact that for every $\varepsilon > 0$ there is a set $J$, finite union of bounded intervals, with the property that
\[
|J \setminus I| + |I \setminus J| \leq \varepsilon.
\]

\[\square\]

**Lemma 3.** Let $(I_k)_{k=1}^N$ be disjoint sets with finite measure and $0 \leq \varepsilon_k \leq 1$ for every $1 \leq k \leq N$. Then, for every $p \in \mathbb{N}_0$ we have
\[
\sum_{k=1}^N \varepsilon_k \int_{I_k} \frac{t^p}{p!} e^{-t} \, dt \leq 1 - \exp \left( - \sum_{k=1}^N \varepsilon_k |I_k| \right).
\]

**Proof:** We denote $n$ the number of indexes $k$ such that $0 < \varepsilon_k < 1$ and we proceed by induction on $n$. For $n = 0$ this is the content of lemma 2. Let us now assume $n = 1$. Let $1 \leq j \leq N$ be the coordinate with the property that $0 < \varepsilon_j < 1$ and check that
\[
\psi(\varepsilon) := \sum_{k \neq j} \int_{I_k} \frac{t^p}{p!} e^{-t} \, dt + \varepsilon \int_{I_j} \frac{t^p}{p!} e^{-t} \, dt + \exp \left( - \sum_{k \neq j} |I_k| - \varepsilon |I_j| \right) \leq 1
\]
for every $0 \leq \varepsilon \leq 1$. In fact, $\psi(0) \leq 1$ and $\psi(1) \leq 1$ follow from Lemma 2. Moreover, the critical point $\varepsilon_0$ of $\psi$ satisfies
\[
\int_{I_j} \frac{t^p}{p!} e^{-t} \, dt = |I_j| \exp \left( - \sum_{k \neq j} |I_k| - \varepsilon_0 |I_j| \right).
\]
Hence
\[
\psi(\varepsilon_0) = \sum_{k \neq j} \int_{I_k} \frac{t^p}{p!} e^{-t} \, dt + \varepsilon_0 |I_j| \exp \left( - \sum_{k \neq j} |I_k| - \varepsilon_0 |I_j| \right) \\
+ \exp \left( - \sum_{k \neq j} |I_k| - \varepsilon |I_j| \right)
\]
\[
= \sum_{k \neq j} \int_{I_k} \frac{t^p}{p!} e^{-t} \, dt + (1 + \varepsilon_0 |I_j|) \exp \left( - \sum_{k \neq j} |I_k| - \varepsilon |I_j| \right)
\]

Since
\[
1 + \varepsilon_0 |I_j| \leq \exp \left( \varepsilon_0 |I_j| \right)
\]
we conclude
\[
\psi(\varepsilon_0) \leq \sum_{k \neq j} \int_{I_k} \frac{t^p}{p!} e^{-t} \, dt + \exp \left( - \sum_{k \neq j} |I_k| \right) \leq 1.
\]

Let us assume that the Lemma holds for \( n = \ell \) (0 \( \leq \ell < N \)) and let \( n = \ell + 1 \). We consider the function \( \psi : [0,1]^{\ell+1} \rightarrow \mathbb{R} \) defined by
\[
\psi(\varepsilon) := \sum_{k=1}^{\ell+1} \varepsilon_k \int_{I_k} \frac{t^p}{p!} e^{-t} \, dt + \sum_{j} \int_{I_j} \frac{t^p}{p!} e^{-t} \, dt + \exp \left( - \sum_{k} \varepsilon_k |I_k| - \sum_{j} |I_j| \right)
\]
for \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_{\ell+1}) \). The induction hypothesis means that \( \psi(\varepsilon) \leq 1 \) whenever \( \varepsilon \) is in the boundary of \([0,1]^{\ell+1}\). The lemma is proved after checking that \( \psi(\varepsilon_0) \leq 1 \), where \( \varepsilon_0 \) is a critical point of \( \psi \). Proceeding as before,

\[
\psi(\varepsilon_0) = \left( \sum_{k=1}^{\ell+1} \varepsilon_k |I_k| + 1 \right) e^{-\sum_{k} \varepsilon_k |I_k|} e^{-\sum_{j} |I_j|} + \sum_{j} \int_{I_j} \frac{t^p}{p!} e^{-t} \, dt \leq \exp \left( - \sum_{j} |I_j| \right) + \sum_{j} \int_{I_j} \frac{t^p}{p!} e^{-t} \, dt \leq 1.
\]

\( \square \)

**Proof of Theorem 7**: We first assume \( a = 0 \), that is, \( F \) is radial. After replacing \( F \) by \( G = \frac{F}{\|F\|_\infty} \), if necessary we can assume that \( \|F\|_\infty = 1 \). Since \( F \) is radial we have \( F(z) = g(|z|) \). We aim to prove that
\[
\int_C |g(|z|)| \cdot |f(z)|^2 e^{-\pi|z|^2} \, dA(z) \leq 1 - \exp \left( - 2\pi \int_0^\infty r |g(r)| \, dr \right)
\]
for every entire function $f(z) = \sum_{n=0}^{\infty} b_n e_p$ such that $\sum_{p=0}^{\infty} |b_p|^2 = 1$. We have

$$\int_{\mathbb{C}} |g(|z|)| \cdot |F(z)|^2 e^{-\pi|z|^2} \, dA(z) = \sum_{p=0}^{\infty} |b_p|^2 \int_0^{\infty} \left| g\left(\sqrt{\frac{t}{\pi}}\right)\right| \cdot \frac{tp}{p!} e^{-t} \, dt.$$  

Let us first assume

$$g = \sum_{k=1}^{N} \varepsilon_k \chi_{I_k}, \ |\varepsilon_k| \leq 1, \quad (1)$$

where $(I_k)_{k=1}^{N}$ are disjoint intervals. Then, Lemma 2 gives

$$\sum_{p=0}^{\infty} |b_p|^2 \int_0^{\infty} \left| g\left(\sqrt{\frac{t}{\pi}}\right)\right| \cdot \frac{tp}{p!} e^{-t} \, dt \leq 1 - \exp\left(-\sum_{k=1}^{N} |\varepsilon_k||J_k|\right)$$

$$= 1 - \exp\left(-2\pi \int_{J_k} \! \! \! r |g(r)| \, dr\right)$$

$$= 1 - \exp\left(-\|F\|_1\right).$$

We used $J_k = \left\{ t : \sqrt{\frac{t}{\pi}} \in I_k \right\}$ and $|J_k| = 2\pi \int_{I_k} \! \! \! r \, dr$. Theorem 1 is proved for $g$ as in (1). Let us now assume that $\|g\|_\infty \leq 1$ and $g \in L^1(\mathbb{R}^+, r\,dr) \cap L^\infty(\mathbb{R}^+)$. Then there is sequence $(g_n)_n$ of step functions as in (1) such that

$$\lim_{n \to \infty} \int_0^{\infty} |g_n(r) - g(r)| \, r \, dr = 0.$$  

We put $F_n(z) := g_n(|z|)$. Since

$$\lim_{n \to \infty} \|T_F - T_{F_n}\| \leq \lim_{n \to \infty} \|F_n - F\|_1 = 0,$$

we finally conclude

$$\|T_F\| \leq 1 - \exp\left(-\|F\|_1\right).$$

In the case $a \neq 0$, the identity

$$\int_{\mathbb{C}} g(|z - a|)|f(z)|^2 d\lambda(z) = \int_{\mathbb{C}} g(|u|) \left(T_{-a} \hat{f}\right)(u) d\lambda(u)$$

and the fact that $T_{-a}$ is an isometry gives the conclusion. We can also argue from the fact that $T_{-a} \circ T_F = T_{G} \circ T_{-a}$, where $G(z) = g(|z|)$. □
In particular, if $\Omega \subset \mathbb{C}$ presents radial symmetry with respect to some point then
\[
\int_{\Omega} |f(z)|^2 \, d\lambda(z) \leq \left(1 - e^{-|\Omega|}\right) \cdot \int_{\mathbb{C}} |f(z)|^2 \, d\lambda(z)
\]  
for every $f \in \mathcal{F}^2(\mathbb{C})$.

The question arises whether inequality (2) holds for every subset $\Omega$. This is related to a conjecture by Abreu and Speckbacher in [1] (see the next section). We do not have an answer to this question except for monomials or its translates.

**Example 4.** Let $k_w = e^{-\pi |w|^2}K_w$ be the normalized reproducing kernel of $\mathcal{F}^2(\mathbb{C})$. Then, for every set $\Omega \subset \mathbb{C}$ with finite measure we have
\[
\int_{\Omega} |k_w(z)|^2 \, d\lambda(z) \leq 1 - e^{-|\Omega|}.
\]

**Proof.** In fact, $k_w = T_w(e_0)$. Hence
\[
\int_{\Omega} |k_w(z)|^2 \, d\lambda(z) = \int_{\Omega-w} d\lambda(z)
\]
and the conclusion follows from the fact that the last integral attains its maximum when $\Omega$ is a disc centered at $w$ (see [1, Proposition 9]). \(\square\)

It is easy to check that when $\Omega$ is a disc centered at point $\omega$ the inequality in Example 4 is an identity.

**Proposition 5.** Let $\Omega \subset \mathbb{R}^2$ be a set with finite measure. Then, for every $n \in \mathbb{N}$ and $a \in \mathbb{C}$,
\[
\int_{\Omega} |T_a(e_n)(z)|^2 \, d\lambda(z) \leq 1 - e^{-|\Omega|}.
\]

**Proof.** Since
\[
\int_{\Omega} |T_a(e_n)(z)|^2 \, d\lambda(z) = \int_{\Omega-a} |e_n(z)|^2 \, d\lambda(z)
\]
we can assume that $a = 0$. For every $\theta \in [0, 2\pi]$ we denote
\[
\Omega_\theta = \left\{ r \geq 0 : re^{i\theta} \in \Omega \right\}.
\]
Then
\[
\int_{\Omega} |e_n(z)|^2 \, d\lambda(z) = \frac{\pi^n}{n!} \int_{\Omega} |z^n| \, \frac{e^{-|z|^2}}{2\pi} \, dA(z)
\]
\[
= \frac{\pi^n}{n!} \int_0^{2\pi} \left( \int_{\Omega} r^{2n} e^{-\pi r^2} \, dr \right) \frac{d\theta}{2\pi}
\]
\[
= \int_0^{2\pi} \left( \int_{I_\theta} t^n \, e^{-t} \, dt \right) \frac{d\theta}{2\pi},
\]
where
\[
I_\theta = \{ t = \pi r^2 : r \in \Omega_\theta \}.
\]
Since $|\Omega| < \infty$ then a.e. $\theta \in [0, 2\pi]$ we have
\[
|I_\theta| = 2\pi \int_{\Omega_\theta} r \, dr < +\infty.
\]
Moreover, by Lemma 2
\[
\int_0^{2\pi} \left( \int_{I_\theta} t^n \, e^{-t} \, dt \right) \frac{d\theta}{2\pi} \leq \int_{I_\theta} \left( 1 - e^{-|I_\theta|} \right) \frac{d\theta}{2\pi}.
\]
Finally we consider the convex function $f(t) = e^{-t} - 1$ and the probability measure $\frac{d\theta}{2\pi}$ and put $h(\theta) = |I_\theta|$. Jensen’s inequality gives
\[
f \left( \int_0^{2\pi} h(\theta) \frac{d\theta}{2\pi} \right) \leq \int_0^{2\pi} f(h(\theta)) \frac{d\theta}{2\pi},
\]
which means
\[
\int_0^{2\pi} \left( 1 - e^{-|I_\theta|} \right) \frac{d\theta}{2\pi} \leq 1 - \exp \left( - \int_0^{2\pi} |I_\theta| \frac{d\theta}{2\pi} \right)
\]
\[
= 1 - \exp \left( - \int_0^{2\pi} \left( \int_{\Omega_\theta} r \, dr \right) \, d\theta \right)
\]
\[
= 1 - e^{-|\Omega|}.
\]
We finish the section with some examples of sets $\Omega$ with infinite Lebesgue measure for which the Toeplitz operator with symbol $F = \chi_\Omega$ has norm as small as we want.
Proposition 6. For every \( \varepsilon > 0 \) there exists \( \Omega \) with infinite Lebesgue measure such that
\[
\int_{\Omega} |f(z)|^2 d\lambda(z) \leq \varepsilon \int_{\mathbb{C}} |f(z)|^2 d\lambda(z)
\]
for every \( f \in \mathcal{F}^2 \).

Proof. Let us consider arbitrary \( f \in \mathcal{F}^2, R > 0 \) and \( \Omega \subset \mathbb{C} \). We denote \( C_R = \int_0^R 2\pi r e^{-\pi r^2} \, dr \). From
\[
|f(0)|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \quad \forall r > 0,
\]
we get
\[
|f(0)|^2 \cdot C_R \leq \int_0^R \left( \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right) re^{-\pi r^2} \, dr
= \int_{D(0,R)} |f(w)|^2 e^{-\pi |w|^2} \, dA(w).
\]
Then, for every \( z \in \mathbb{C} \),
\[
|f(z)|^2 e^{-\pi |z|^2} = |(T_{-z} f)(0)|^2 \leq C_R^{-1} \cdot \int_{D(0,R)} |(T_{-z} f)(w)|^2 e^{-\pi |w|^2} \, dA(w)
= C_R^{-1} \cdot \int_{D(z,R)} |f(w)|^2 e^{-\pi |w|^2} \, dA(w).
\]
Finally
\[
\int_{\Omega} |f(z)|^2 e^{-\pi |z|^2} dA(z) \leq C_R^{-1} \cdot \int_{\Omega} \left( \int_{\mathbb{C}} \chi_{D(w,R)}(z) |f(w)|^2 e^{-\pi |w|^2} \, dA(w) \right) \, dA(z)
\leq C_R^{-1} \cdot \int_{\mathbb{C}} |f(w)|^2 e^{-\pi |w|^2} \varphi(w) \, dA(w),
\]
where
\[
\varphi(w) = \int_{\Omega} \chi_{D(w,R)}(z) \, dA(z) = |\Omega \cap D(w,R)|.
\]
Now, take \( \Omega \) the union of infinitely many disks of Lebesgue measure \( \delta = \varepsilon \cdot C_R \), sufficiently separated so that each disk \( D(w,R) \) can only cut one of them. Then \( \varphi(w) \leq \varepsilon \) for every \( w \in \mathbb{C} \) and the conclusion follows. \( \square \)
3 Time-frequency localization operators

For $F \in L^1(\mathbb{C})$ we denote by $H_F : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ the localization operator

$$H_F f = \int_\mathbb{C} F(z) \langle f, \pi(z)h_0 \rangle \pi(z)h_0 \, dA(z).$$

Here $h_0(t) = 2^{1/4}e^{-\pi t^2}$ is the Gaussian and $\pi(z)$ is the time-frequency shift, defined for $z = x + i\omega$ as

$$\pi(z)f(t) = e^{2\pi i\omega t}f(t-x), \; f \in L^2(\mathbb{R}).$$

In case $F$ is the characteristic function of a set $\Omega$ we write $H_\Omega$ instead of $H_{\chi_\Omega}$. We refer to [5] or [6, Chapter 4] for general facts concerning localization operators.

For $f, g \in L^2(\mathbb{R})$, the expression

$$\langle V_g f, \rangle(z) := \langle f, \pi(z)g \rangle$$

is the short time Fourier transform of $f$ with window $g$, known as Gabor transform in the case where the window $g = h_0$ is the Gaussian.

If $F$ is real-valued then $H_F$ is a selfadjoint operator on $L^2(\mathbb{R})$, hence

$$\|H_F\| = \sup_{\|f\|_2 = 1} |\langle H_F f, f \rangle| \leq \sup_{\|f\|_2 = 1} \int_\mathbb{C} |F(z)| \cdot |(V_{h_0} f)(z)|^2 \, dA(z).$$

There is a connection between localization operators and Toeplitz operators on the Fock space via de Bargmann transform.

The Bargmann transform is the surjective and unitary operator

$$\mathcal{B} : L^2(\mathbb{R}) \rightarrow \mathcal{F}^2(\mathbb{C})$$

defined as

$$(\mathcal{B}f)(z) = 2^{1/4} \int_\mathbb{R} f(t)e^{2\pi i z t^2 - \pi t^2} \, dt.$$ 

It was introduced in [2] and has the important property that the Hermite functions are mapped into normalized analytic monomials. More precisely, $\mathcal{B}(h_n) = e_n$, where $h_n$ is defined via the so called Rodrigues formula as

$$h_n(t) = \frac{2^{1/4}}{\sqrt{n!}} \left( \frac{-1}{2\sqrt{\pi}} \right)^n e^{\pi t^2} \frac{d^n}{dt^n} \left( e^{-2\pi t^2} \right), \; n \geq 0.$$
Then \((h_n)_{n \geq 0}\) forms an orthonormal basis for \(L^2(\mathbb{R})\). The Gabor transform of Hermite functions is well-known (see for instance [9, Chapter 1.9]). In fact, for \(z = x + i\xi\),

\[
\langle h_n, \pi(z)h_0 \rangle = e^{-i\pi x \xi - \frac{\pi |x|^2}{2}} \cdot \frac{\pi^n}{n!} z^n.
\]

Since for \(z = x + i\xi\) we have ([10, 3.4.1])

\[
(V_{h_0} f)(x, -\xi) = e^{i\pi x \xi} \cdot (Bf)(z) \cdot e^{-\frac{\pi |z|^2}{2}}
\]

then, for every \(f \in L^2(\mathbb{R})\) and \(F \in L^1(\mathbb{C}) \cap L^\infty(\mathbb{C})\) we obtain

\[
\int_{\mathbb{C}} |F(z)| \cdot |(V_{h_0} f)(z)|^2 \ dA(z) = \int_{\mathbb{C}} |F(z)| \cdot |(Bf)(z)|^2 \ d\lambda(z).
\]

Consequently, all the estimates in the previous section can be translated into estimates concerning localization operators.

Abreu, Speckbacher conjecture in [1] that, among all the sets with a given measure, \(\|H_\Omega\|\) attains its maximum when \(\Omega\) is a disc, up to perturbations of Lebesgue measure zero. This turns out to be equivalent to the validity of inequality (2) for every function in the Fock space or, equivalently, to the fact that

\[
\|f\|_2^2 \leq e^{|\Omega|} \int_{\mathbb{C}\setminus\Omega} |(V_{h_0} f)(z)|^2 \ dA(z) \ \forall f \in L^2(\mathbb{R}).
\]

This should be compared with [3, Theorem 4.1]. In this regard it is worth noting that Nazarov [12] proved the existence of two absolute constants \(A, B\) such that

\[
\|f\|_2^2 \leq Ae^{B|S|\cdot|\Sigma|}
\]

for every \(f \in L^2(\mathbb{R})\) and for any pair \((S, \Sigma)\) of sets with finite measure.

From Theorem [1] and Proposition [3] we get the following.

**Corollary 7.** Let \(F \in L^1(\mathbb{C}) \cap L^\infty(\mathbb{C})\) be a real-valued and radial symbol with respect to \(a \in \mathbb{C}\). Then

\[
\|H_F\| \leq \|F\|_\infty \left(1 - \exp\left(-\frac{\|F\|_1}{\|F\|_\infty}\right)\right).
\]

**Corollary 8.** Let \(\Omega \subset \mathbb{R}^2\) be a set with finite measure. Then, for every \(n \in \mathbb{N}\),

\[
|\langle H_\Omega h_n, h_n \rangle| \leq 1 - e^{-|\Omega|}.
\]
We fix a non-zero window \( g \in L^2(\mathbb{R}) \). The modulation space \( M^1(\mathbb{R}) \), also known as Feichtinger algebra, is the set of tempered distributions \( f \in S'(\mathbb{R}) \) such that

\[
\| f \|_{M^1} := \int_{\mathbb{C}} |\langle f, \pi(z)g \rangle| dA(z) < +\infty.
\]

The use of different windows \( g \) in the definition of \( M^1(\mathbb{R}) \) yields the same spaces with equivalent norms. It is well known that \( M^1(\mathbb{R}) \) is continuously included in \( L^2(\mathbb{R}) \) and

\[
\| f \|_2 = \| V_g f \|_2 \leq \| V_g f \|_1
\]

whenever \( f \in M^1(\mathbb{R}) \) and \( \| g \|_2 = 1 \). See for instance [10, 3.2.1] for the first identity.

**Proposition 9.** Let \( \Omega \subset \mathbb{R}^2 \) be a set with finite measure. Then, for every \( f \in M^1(\mathbb{R}) \) and \( n \in \mathbb{N}_0 \) we have

\[
\int_{\Omega} |(V_{h_0} f)(z)|^2 dA(z) \leq \| V_{h_n} f \|_1^2 \cdot (1 - e^{-|\Omega|}).
\]

**Proof.** It suffices to prove the proposition under the additional assumption that \( \| V_{h_n} f \|_1 = 1 \). Fixed \( n \in \mathbb{N}_0 \) we consider the set

\[
B := \{ \pi(z) h_n : z \in \mathbb{C} \} \subset L^2(\mathbb{R}).
\]

Then

\[
B^\circ := \{ g \in L^2(\mathbb{R}) : |\langle g, \pi(z) h_n \rangle| \leq 1 \} = \{ g \in L^2(\mathbb{R}) : \| V_{h_n} g \|_\infty \leq 1 \}.
\]

We have

\[
|\langle f, g \rangle| = |\langle V_{h_n} f, V_{h_n} g \rangle| \leq \| V_{h_n} f \|_1 \cdot \| V_{h_n} g \|_\infty \leq 1
\]

for every \( g \in B^\circ \), which means that \( f \in B^{\circ\circ} \). According to the bipolar theorem,

\[
f = L^2 - \lim_{k \to \infty} f_k
\]

where each \( f_k \) is in the absolutely convex hull of \( B \). For each \( k \in \mathbb{N} \) we can find scalars \((\alpha_j)_{j=1}^N\) and points \((z_j)_{j=1}^N\) such that \( f_k = \).
\[ \sum_{j=1}^{N} \alpha_j \pi(z_j) h_n \quad \text{and} \quad \sum_{j=1}^{N} |\alpha_j| \leq 1. \]

Then
\[
\left( \int_{\Omega} |(V_{h_0} f_k)(z)|^2 \, dA(z) \right)^{\frac{1}{2}} = \left( \int_{\Omega} |\langle f_k, \pi(z) \varphi \rangle|^2 \, dA(z) \right)^{\frac{1}{2}} \\
\leq \sum_{j=1}^{N} |\alpha_j| \left( \int_{\Omega} |\langle \pi(z) h_n, \pi(z) \varphi \rangle|^2 \, dA(z) \right)^{\frac{1}{2}} \\
= \sum_{j=1}^{N} |\alpha_j| \left( \int_{\Omega} |\langle h_n, \pi(z - z_j) \varphi \rangle|^2 \, dA(z) \right)^{\frac{1}{2}} \\
= \sum_{j=1}^{N} |\alpha_j| \left( \int_{\Omega - z_j} |\langle h_n, \pi(z) \varphi \rangle|^2 \, dA(z) \right)^{\frac{1}{2}} \\
= \sum_{j=1}^{N} |\alpha_j| \left| \langle H_{\Omega - z_j} h_n, h_n \rangle \right|^2 \leq (1 - e^{-|\Omega|})^{\frac{1}{2}}.
\]

Finally,
\[
\int_{\Omega} |(V_{h_0} f)(z)|^2 \, dA(z) = \lim_{k \to \infty} \int_{\Omega} |(V_{h_0} f_k)(z)|^2 \, dA(z) \leq 1 - e^{-|\Omega|}.
\]

The next result is a direct consequence of Proposition 6 and should be compared with [1, Proposition 8].

**Corollary 10.** For every \( \varepsilon > 0 \) there exists \( \Omega \) with infinite Lebesgue measure such that
\[ \|H_\Omega\| \leq \varepsilon. \]

**References**

[1] L.D. Abreu, M. Speckbacher; *Donoho-Logan Large Sieve Principles for Modulation and Polyanalytic Fock Spaces.* [arXiv:1808.02258](https://arxiv.org/abs/1808.02258)

[2] V. Bargmann; *On a Hilbert space of analytic functions and an associated integral transform.* Comm. Pure Appl. Math. **14** (1961), 187–214.

[3] C.A. Berger, L.A. Coburn; *Toeplitz operators and quantum mechanics.* J. Funct. Anal. **68** (1986), 273–299.
[4] C.A. Berger, L.A. Coburn; *Toeplitz operators on the Segal-Bargmann space.* Trans. Amer. Math. Soc. 301 (1987), 813–829.

[5] E. Cordero, K. Gröchenig; *Time-frequency analysis of localization operators.* J. Funct. Anal. 205 (2003), 107–131.

[6] E. Cordero, L. Rodino; *Time-Frequency Analysis of Operators.* De Gruyter Studies in Mathematics, 75 (2020).

[7] I. Daubechies; *Time-frequency Localization Operators: a Geometric Phase Space Approach,* IEEE Trans. Inform. Theory, 34(4) (1988), 605-612.

[8] C. Fernández, A. Galbis; *Annihilating sets for the short time Fourier transform.* Adv. Math. 224 (2010), 1904–1926.

[9] G. B. Folland; *Harmonic Analysis in Phase Space.* Ann. Math. Stud., Vol.122, *Princeton Univ. Press,* Princeton, N.J. (1989).

[10] K. Gröchenig; *Foundations of Time-Frequency Analysis,* *Birkhäuser* (2001).

[11] S.M. Grudsky, N.L. Vasilevski; *Toeplitz operators on the Fock space: radial component effects.* Integral Equations Operator Theory 44 (2002), 10–37.

[12] F.L. Nazarov; *Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type.* (Russian) Algebra i Analiz 5 (1993), 3–66; translation in St. Petersburg Math. J. 5 (1994), 663–717.

[13] K. Zhu; *Invariance of Fock spaces under the action of the Heisenberg group.* Bull. Sci. Math. 135 (2011), 467–474.

[14] K. Zhu; *Analysis on Fock spaces.* *Graduate Texts in Mathematics,* 263. Springer, New York, 2012. x+344 pp.