Kinetic–MHD simulations of gyroresonance instability driven by CR pressure anisotropy

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ABSTRACT
The transport of cosmic rays (CRs) is crucial for the understanding of almost all high-energy phenomena. Both pre-existing large-scale magnetohydrodynamic (MHD) turbulence and locally generated turbulence through plasma instabilities are important for the CR propagation in astrophysical media. The potential role of the resonant instability triggered by CR pressure anisotropy to regulate the parallel spatial diffusion of low-energy CRs (\( \lesssim 100 \text{ GeV} \)) in the interstellar and intracluster medium of galaxies has been shown in previous theoretical works. This work aims to study the gyroresonance instability via direct numerical simulations, in order to access quantitatively the wave–particle scattering rates. For this, we employ a 1D PIC–MHD code to follow the growth and saturation of the gyroresonance instability. We extract from the simulations the pitch-angle diffusion coefficient \( D_{\mu \nu} \) produced by the instability during the linear and saturation phases, and a very good agreement (within a factor of 3) is found with the values predicted by the quasi-linear theory (QLT). Our results support the applicability of the QLT for modelling the scattering of low-energy CRs by the gyroresonance instability in the complex interplay between this instability and the large-scale MHD turbulence.

Key words: MHD – plasmas – turbulence – cosmic rays.

1 INTRODUCTION
Basic processes of transport of cosmic rays (CRs) are crucial for understanding most of high-energy phenomena, ranging from solar flares (Yan, Lazarian & Petrosian 2008), \( \gamma \)-ray emission of molecular clouds associated with supernova remnants (Nava & Gabici 2013), to remote cosmological objects such as \( \gamma \)-ray bursts (Zhang et al. 2013). The interaction of CRs with magnetohydrodynamic (MHD) turbulence is thought to be the principal mechanism to scatter and isotropize CRs (e.g. Ginzburg 1966; Jokipii 1966; Wentzel 1969; Schlücker 2002; Yan 2015 and references therein). In addition to the large-scale MHD turbulence, small-scale instability generated perturbations also play crucial roles. In the context of acceleration in supernova shocks, studies of instabilities have been one of the major efforts in the field since the acceleration efficiency is essentially determined by the confinement at the shock region and magnetic field amplification (e.g. Bell & Lucek 2001; Bell 2004; Yan, Lazarian & Schlücker 2012; Caprioli & Spitkovsky 2014; Brose, Telezhinskys & Pohl 2016).

The diffusive propagation of CRs in the Galaxy, far from sources, is considered to be regulated by the interactions with the interstellar medium (ISM) background turbulence. However, due to the damping of the background turbulence (particularly the fast modes), CRs with energies below \( \sim 100 \text{ GeV} \) are expected to be mainly influenced by the self-generated instabilities. In fact, the small-scale instabilities and large-scale turbulence are not independent of each other. First, the instability generated waves can be damped through the interaction with the large-scale turbulence (Yan & Lazarian 2002; Farmer & Goldreich 2004; Lazarian 2016). Secondly, the large-scale compressible turbulence also generates small-scale waves through thermal plasma (e.g. Schekochihin et al. 2005; Santos-Lima et al. 2014, 2016; Sironi & Narayan 2015) and CR resonant instabilities (Lazarian & Beresnyak 2006; Yan & Lazarian 2011). The compression/expansion and shear from the large-scale ISM turbulence produce deformations in the local particle pitch-angle distribution due to the conservation of the first adiabatic invariant. Such anisotropic distributions are subjected to various instabilities. Waves generated through instabilities enhance the scattering rates of the particles, and their distribution then relaxed to the state of marginal state of instability even in the collisionless environment (Schekochihin & Cowley 2006; Kunz, Schekochihin & Stone 2014; Riquelme, Quataert & Verscharen 2015; Sironi & Narayan 2015; Santos-Lima et al. 2016). In particular, the anisotropy in the CR pressure induces a (gyro)resonant instability. Unlike the streaming instability (e.g. Skilling 1970, 1971, 1975; Amato & Blasi 2009), this gyroresonance instability does not require the bulk motion of...
CRs. The wave grows at the expense of the free energy from CRs’ anisotropy induced by the large-scale turbulent motions. In the case that the energy growth rate reaches the turbulence energy cascading rate, turbulence is damped (Yan & Lazarian 2011). This is one of the feedbacks from CRs on turbulence.

Yan & Lazarian (2011, YL11 hereafter) proposed an analytical equilibrium model of the CR diffusion resulting from their scattering by the gyroresonance instability, based on the quasi-linear theory (QLT). They derived the dependence of the diffusion coefficients with the parameters of the medium turbulence, taking into account self-consistently the damping effect of the instability on the large-scale compressible cascade. They found this mechanism to be important for the CR propagation in collisionless medium, such as the halo and hot ionized medium of the Galaxy, and also in the intracluster medium (ICM) of galaxies.

These previous findings motivate more detailed studies on this scattering mechanism and its role in the CR propagation in both the ISM and ICM. The saturation state of the instability as well as the accuracy of the CR scattering rates based on the QLT is addressed in this work. For this aim, we use direct numerical simulations to follow the evolution of an initially unstable CR distribution propagating in a thermal background plasma. These simulations are based on a hybrid technique that combines particle-in-cell (PIC) and MHD (e.g. Lucek & Bell 2000; Bai et al. 2015). This first study is restricted to unidirectional propagating modes, parallel to the mean field (one-dimensional geometry).

This paper is organized in the following way: in Section 2, we present the basic equations that describe the relevant CR interactions with the background thermal plasma, the dispersion relation for the transverse modes propagating parallel to the magnetic field for a distribution of CRs with pressure anisotropy, and the QLT prediction for the CR pitch-angle diffusion coefficient. In Section 3, we describe the numerical methods employed in our numerical simulations, and the results are presented in Section 4. Finally, in Section 5, we summarize our findings and offer our conclusions.

2 CR PRESSURE ANISOTROPY

GYRORESONANCE INSTABILITY

2.1 CR + thermal plasma description

The relevant physical phenomena for the CR propagation (considered only as ions here) in the ISM or ICM take place on scales larger than or of the order of the CR kinetic scales, many orders of magnitude above the kinetic scales of the thermal ions in the medium. Hence, for studying the effects of the interactions between CRs and the thermal plasma, it is natural to retain the kinetic description only for the CRs, employing the Vlasov equation for the evolution of their distribution function \( f(r, p, t) \):

\[
\frac{\partial f}{\partial t} + v \cdot \nabla f + q \left( E + \frac{1}{c} v \times B \right) \cdot \nabla f = 0,
\]

(1)

where \( v = p/\gamma m_\text{cr} \) is the CR velocity, \( \gamma = (1 - v^2/c^2)^{-1/2} \) is the light speed, \( m_\text{cr} \) is the CR rest mass, \( q \) is the CR charge (elementary charge \( e \) in our case), \( E \) and \( B \) are the electric and magnetic fields. The mass-domminating thermal plasma is described by the MHD approximation, modified to account for the CR presence:

\[
\frac{\partial p}{\partial t} + \nabla \cdot (pu) = 0,
\]

(2)

\( \rho \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right] + \nabla P_\text{th} - \frac{1}{4\pi} (\nabla \times B) \times B \]

\[-= -\frac{q}{c} J \times B - qn_e E, \]

(3)

\[
\frac{\partial B}{\partial t} = -c \nabla \times E,
\]

(4)

\[
\nabla \times B = \frac{4\pi}{c} \left( J + J_\text{cr} \right), \quad \nabla \cdot B = 0,
\]

(5)

where \( \rho, u, P_\text{th}, J_\text{cr} \) are the density, velocity, thermal pressure, and current density fields of the thermal plasma, \( n_e \) and \( J_\text{cr} \) are the number and current density of the CRs:

\[
n_c(r, t) = \int dp f(r, v, t),
\]

(6)

\[
J_\text{cr}(r, t) = q \int dp v f(r, v, t).
\]

(7)

In the ideal case when the resistivity can be neglected, the electric field is given by

\[
E = -\frac{1}{c} u \times B.
\]

(8)

The above equations (Bykov et al. 2013) assume (i) the thermal plasma is non-relativistic, and (ii) quasi-neutrality of the full plasma: \( qn_i + qn_e = en_i \), where \( n_i, e \) is the number density of the thermal ions/electrons, \( q_i \) is the ions charge, and \( e \) is the elementary charge. For simplicity, we consider the thermal plasma composed only of protons and electrons, and the CRs as protons (\( q = q_i = e, m_c = m_i = m_p \)), where \( m_p \) is the proton mass. The system must be closed by an equation for the evolution of \( \rho_\text{th} \).

2.2 Development of CR pressure anisotropy

The momentum distribution of CRs propagating in a region where the local magnetic field intensity \( B \) is changing slowly in time (compared to the CR gyroperiod) becomes distorted as the perpendicular component of the particles’ momentum \( p_\perp \) modifies according to the conservation of the first adiabatic invariant \( \propto p_\perp^2 / B \) (e.g. Longair 2011). In this way, an initially isotropic distribution function \( f_0(p)dp \propto p^2 \mathrm{d}p \) assumes an elliptic shape in the momentum space:

\[
f(p)dp \propto (\xi p_\perp^2 + p_\parallel^2)^{-(1+\delta)/2} \mathrm{d}p,
\]

(9)

where \( \parallel, \perp \) refer to the directions parallel/perpendicular to the local magnetic field \( B \). In general, we can parametrize small deviations from the isotropy by using an expansion in the distribution of the cosine of the pitch angle \( \mu = p_\parallel / |p| \):

\[
f(p)dp = N(p) p^2 dp g(\mu) \mathrm{d}\mu \mathrm{d}\psi,
\]

(10)

where \( \psi \) is the gyrophase and \( |\beta|, |\chi| \ll 1 \). The above distribution is obviously restricted to the case of uniform anisotropy over all \( p \). In this study, we neglect the dipole component \( \beta \) (no CR bulk velocity). In this case, the above distribution function translates in an anisotropy \( A \) of the CR pressure components \( P_\perp, P_\parallel \) defined by

\[
A \equiv P_\perp / P_\parallel - 1,
\]

(11)

where

\[
P_\perp \equiv \frac{1}{2} \int dp f(p) p_\perp^2, \quad P_\parallel \equiv \int dp f(p) p_\parallel.
\]

(12)
The correspondence between $\chi$ and $A$ is

$$\chi = \frac{5A}{2A + 3} = \frac{5}{3} + \mathcal{O}(A^2).$$ (13)

The analytical model in YL11 provides the following estimates for $A: \gtrsim 10^{-3}$ for the galactic halo and $\gtrsim 10^{-4}$ for the hot ISM and ICM. Although this work is dedicated to study the isolated effects of CR pressure anisotropy, it should be observed that we expect a large-scale drift of CRs flowing from the galactic sources to outside the Galaxy, with this dipole anisotropy (of the order of $B$) observed to be $10^{-4} - 10^{-3}$ at the Earth (Skilling 1970; Di Sciascio & Iuppa 2014). A systematic study on the combined effect of both kinds of anisotropy is beyond the scope of the present work, and will be addressed elsewhere.

### 2.3 Parallel-propagating transverse modes

The dispersion relation of the ordinary linear MHD waves can be modified depending on the CR distribution function. In fact, the free energy provided by the anisotropic pressure of CRs can turn the MHD waves unstable (e.g. Schlickeiser 2002). We focus here only on the parallel-propagating modes, for simplicity and also because they have the fastest growth rates, and therefore should be more important for CR scattering. Bykov et al. (2013) present the linear dispersion relation for the general case of a distribution function described by equation (10). In the absence of CR bulk velocity ($\beta = 0$), the linear dispersion relation for $k > 0$ is

$$\omega^2 = v_A^2 \left\{ k^2 \mp k \frac{4\pi q_n A}{B_0} \int_{p_{\min}}^{p_{\max}} \sigma(p, k) N(p) p^2 dp \right\},$$ (14)

where we assume $|\omega| \ll |\Omega|$, $\Omega = \Omega_0/\gamma$ is the CR cyclotron frequency with $\Omega_0 = qB_0/m_e c$, $v_A = B_0/\sqrt{\gamma\rho}$ is the Alfvén speed, $p_{\min}$ and $p_{\max}$ are the minimum and maximum values of the CR momentum distribution $N(p)$, and

$$\sigma(p, k) = \frac{3}{4} \int_{-1}^{1} (1 - \mu^2) \mu \frac{d\mu}{\Omega},$$ (15)

where $\varpi$ correspond to the left and right circular polarization, respectively. We adopt the definition of polarization used in Stix (1962) and Gary (1993): the waves with left polarization rotate in the same sense as protons, irrespective of the wave propagation direction, while the waves with right polarization rotate with the same sense as electrons.

According to the dispersion relation given by equation (14), for small anisotropies ($|\chi| \ll 1$), the real ($\omega_r$) and imaginary ($\Gamma$) parts of the frequency are

$$\omega_r(k) = v_A k + \mathcal{O}(\chi),$$ (16)

$$\Gamma^{L,R}(k) = \mp v_A \frac{2\pi q_n A}{B_0} \int_{p_{\min}}^{p_{\max}} \sigma(p, k) N(p) p^2 dp \right\} + \mathcal{O}(\chi^2).$$ (17)

Using the analytical solution in Bykov et al. (2013), the imaginary part in $\sigma(p, k)$ (equation 15) comes from the pole (resonant) contribution and is given by

$$\Im \{\sigma(p, k)\} = -\frac{3\pi}{4} \left[ \left( \frac{\Omega_0 m_e}{k p} \right)^2 - \left( \frac{\Omega m_e}{k p} \right)^2 \right] \times \mathcal{H}(p - m_e\Omega_0/k),$$ (18)

where $\mathcal{H}(x)$ is the Heaviside step function. Assuming the following power law for the momentum distribution:

$$N(p) = \frac{(1 - \alpha)}{(p_{\max}^p - p_{\min}^p)^{p - 2 - \alpha}}$$ (19)

for $p_{\min} < p < p_{\max}$ and zero otherwise, the integration in equation (17) gives

$$\Im \left\{ \int_{p_{\min}}^{p_{\max}} \sigma(p, k) N(p) p^2 dp \right\} (k) \sim -\frac{3\pi}{4} \left( \frac{\alpha - 1}{\alpha + 1} \right) \frac{c}{\gamma k r_p} \Omega_0 \left( \frac{n_e}{n_i} \right) A \left( kr_p \right)^{\alpha - 1},$$ (20)

where $r_p = p_{\min} / m_e \Omega_0$ is the Larmor radius for the lower energy particles (in the limit of zero pitch angle), and assuming $\alpha > 2$, $p_{\max} / p_{\min} \gg 1$. Now we can rewrite equation (17) as

$$\Gamma^{L,R}(k) \sim \pm \frac{\pi}{2} \frac{(\alpha - 1)}{(\alpha + 1)} \frac{c}{\gamma v_A} \Omega_0 \left( \frac{n_e}{n_i} \right) A \left( kr_p \right)^{\alpha - 1},$$ (21)

where $\Gamma^{L,R}(k) \sim \pm \frac{\pi}{2} \frac{(\alpha - 1)}{(\alpha + 1)} \frac{c}{\gamma v_A} \Omega_0 \left( \frac{n_e}{n_i} \right) A \left( kr_p \right)^{\alpha - 1}$. $\chi$ is the total helicity of the wave component of $B(k)$. $\sigma_{\parallel} = -1$ for waves with $L$ polarization and forward propagation or $R$ polarization and backward propagation; $\sigma_{\perp} = +1$ for $L$ polarization and backward propagation or $R$ polarization and forward propagation. The time interval $\delta t$ is understood as the time interval under

\[1\] The solution in equation (14) is rigorously derived for $\Im(\omega) > 0$. The validity of this expression for $\Im(\omega) < 0$ comes from the analytical continuation of the solution.
which the diffusion process is considered, and it is much shorter than the time-scale of change of the distribution function due to the scattering itself (see Appendix). Therefore, the above definition of the diffusion coefficient is valid for describing the evolution of distribution functions averaged in a time interval \( \sim \delta t \). Because the system we are focusing in this study is statistically homogeneous and the theoretical \( D_{\nu}^{\mu} \) does not depend on the gyrophase, we can consider the average of the Fokker–Planck equation in space and gyrophase.

The effect of \( D_{\nu}^{\mu} \) is to reduce gradually the anisotropies in \( \mu \) of the distribution function: the source of free energy of the instability. Therefore, after the linear phase, the initial distribution function of the CRs will evolve reducing the \( \mu \) anisotropy, then reducing the instability growth rate. During this last saturation phase, the distribution function: the source of free energy of the instability. This means that the anisotropy distribution over \( p \) does not evolve uniformly, i.e. \( A = A(p) \) on the linear phase.

The CR mean free path along the field lines (from the scattering of particles along this direction) is related to \( D_{\nu}^{\mu} \) via

\[
\lambda_{\nu} = \frac{3}{4} \int_{\mu=1}^{1} d\mu \left( \frac{1 - \mu^2}{v_{\text{scatt}}} \right) = \frac{3}{8} \int_{\mu=1}^{1} d\mu \left( \frac{1 - \mu^2}{D_{\nu}^{\mu}} \right),
\]

where \( v_{\text{scatt}} = 2D_{\nu}^{\mu}/(1 - \mu^2) \) is the CR scattering rate.

### 3 NUMERICAL METHODS

In order to study the evolution of the CR pressure anisotropy instability described by the MHD + CR kinetic equations (see Section 2.1), we use a hybrid MHD–particle code that represents the MHD fields \( (\rho, u, P, B) \) in cells defined by a grid over the simulation domain, while the CR distribution function is sampled by a collection of macroparticles whose orbits are directly solved. The CR macroscopic fields needed for the evolution of the MHD variables \((n\text{cr}, J\text{cr}, u\text{cr})\) are calculated by a process of deposition of the macroparticles on the grid cells (PIC method). This MHD–PIC coupling is described in detail in Bai et al. (2015).

The grid cell variables are evolved by the following set of equations:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0,
\]

\[
\frac{\partial}{\partial t} (\rho u) + \nabla \cdot \left\{ \rho uu + \left[ P_{th} + \frac{B^2}{8\pi} \right] I - \frac{BB}{4\pi} \right\} = -F_{\text{cr}},
\]

\[
\frac{\partial e}{\partial t} + \nabla \cdot \left\{ \left[ \epsilon + P_{th} + \frac{B^2}{8\pi} \right] u - \frac{(u \cdot B) B}{4\pi} + \frac{c}{4\pi} (E - E_0) \times B \right\} = -u_{\text{cr}} \cdot F_{\text{cr}},
\]

\[
\frac{\partial B}{\partial t} + c \nabla \times E = 0,
\]

where \( I \) is the unitary dyadic tensor, \( \epsilon = \rho u^2/2 + B^2/8\pi + P_{th}/(y_{th} - 1) \) is the total energy density with \( y_{th} = 5/3 \) the adiabatic index of the thermal gas, \( E_0 \) and \( E \) are the electric fields given by

\[
E_0 = -\frac{1}{c} u \times B,
\]

\[
E = E_0 - \frac{1}{c} \frac{n_{\text{cr}}}{n_e} (u_{\text{cr}} - u) \times B,
\]

where \( n_{\text{cr}} = n_{\text{cr}} + n_i \) is as before the electron number density and \( F_{\text{cr}} \) is the force density felt by the CRs:

\[
F_{\text{cr}} = qn_{\text{cr}}E + \frac{1}{c} J_{\text{cr}} \times B.
\]

The orbits of the macroparticles are evolved using the Lorentz force with the electromagnetic fields interpolated from the grid values:

\[
\frac{dp_j}{dt} = qE + \frac{q}{c} v_j \times B,
\]

where \( p_j \) and \( v_j \) are the momentum and velocity of the particle \( j \).

It should be pointed out that the electric field \( E \) (equation 30) differs from \( E_0 \) by the inclusion of the CR Hall effect, which is not taken into account in equation (8). However, the effects of this term (of the order of \( n_{\text{cr}}/n_i \) for \( n_{\text{cr}} \ll n_i \)) can be considered negligible if \( u_{\text{cr}} \sim u \), which is the case for the transverse Alfvén modes we are interested in.

The above equations are solved in a periodic, Cartesian, one-dimensional domain. The MHD equations are discretized using a conservative formulation; the fluxes are calculated using the Harten-Lax-van Leer Discontinuity solver (adapted from the Athena code; Stone et al. 2008) and linear interpolation. The particles are evolved using the relativistic Boris pusher (adapted from the Skeleton PIC codes; Decyk 1995, 2007) with a leap-frog scheme, and first-order weighting for the particles deposited on the grid. The full time integration (MHD + particles) is performed using Runge–Kutta of second order. We verified the second-order convergence of our code implementation against several linear test problems, as MHD waves with cold and not-cold CR distributions (with different relative densities \( n_{\text{cr}}/n_i \)) and the non-resonant Bell instability.

### 4 RESULTS

#### 4.1 Simulation parameters

The initial conditions of the performed simulations consist of homogeneous MHD fields with null velocity and mean magnetic field parallel to the axis of the simulation grid, and a distribution of CRs with anisotropy \( \theta_0 \) [distribution \( g(\mu) \) given by equation (10)], superimposed by a flat spectrum of circularly polarized waves with magnetic field amplitude \( |B_0| \).

The momentum distribution of CRs has power-law index \( \alpha = 2.8 \), with \( \gamma_{\text{min}} = 2 \) and \( \gamma_{\text{max}} = 10 \) [distribution \( N(p) \) given by equation (19)]. We fix the ratio \( v_{\text{th}}/c = 10^{-2} \). We vary the parameters \( A_0 \) and \( A_{\text{cr}}/n_i \).

Tables 1 and 2 summarize the parameters used for the simulations: \( n_{\text{cr}}/n_i \), \( v_{\text{th}}/c \), ratio between CR kinetic and magnetic energy density \( \beta_{\text{cr}} \equiv W_{\text{cr}}/W_{\text{th}} \), ratio between thermal and magnetic pressures \( P_{\text{th}}/P_{\text{cr}} \), domain size \( L \), grid resolution \( NX \), number of particles per grid cell \( NP/NX \), \( A_0 \) \((|B_0|/B_0)^2 \), polarization \( P \), and total helicity \( \sigma_H \).

The choice of parameters \( (v_{\text{th}}/c, n_{\text{cr}}/n_i) \) is motivated by the conditions in the Galactic halo and hot ionized medium, estimated in

| \( \gamma_{\text{min}} \) | \( P_{\text{th}}/P_{\text{cr}} \) | \( \gamma_{\text{max}}/\gamma_{\text{min}} \) | \( P_{\text{max}}/P_{\text{min}} \) | \( \alpha \) | \( L \Omega_0/c \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| 10^{-2}         | 2              | 2              | 10             | 2.8            | 500            |

\( E = E_0 - \frac{1}{c} \frac{n_{\text{cr}}}{n_e} (u_{\text{cr}} - u) \times B, \)

where \( n_{\text{cr}} = n_{\text{cr}} + n_i \) is as before the electron number density and \( F_{\text{cr}} \) is the force density felt by the CRs:
YLI1. The high values of $\beta_{cr}$ and $A_0$ (compared to the estimates by YLI1) are chosen in order to maximize the growth rate of the fluctuations in the resonant energy interval, so that the waves amplified by the instability achieve higher amplitudes than the magnetic fluctuations (noise) caused by the limited number of macroparticles that sample the CR distribution function.

4.2 Linear phase of instability

Figs 1 and 2 show the dispersion relation extracted from the runs $d1$ and $d2$ ($A_0 = -0.3$, waves with polarization $R$ and $L$, respectively). Each point in the figures is calculated by fitting the time-series of one Fourier component of the magnetic field, between $r = 0$ and $t_{end}$ (all the points of the same colour/style in each plot of Figs 1 and 2 are extracted from one single run). In this way, we determine for each wavenumber $k$ the phase, the phase speed, and the growth/damping rate. Fig. 1 shows the growth/damping rate $\Gamma(k)$, while Fig. 2 shows the real frequency $\omega(k)$ of the wave spectrum. The analytical solution from the dispersion relation (equation 14) is shown for comparison. The fitted values agree quite well with the theoretical values, except for large wavenumbers (inside the grey area), for which the numerical dissipation $\propto k^3$ dominates $\Gamma(k)$. Such agreement is observed also in the other runs, with the agreement in $\Gamma(k)$ better for the simulations with larger values of $|\Gamma_{max}|$. For anisotropies smaller than $|A_0| = 0.1$ (and $v_x/c = 10^{-2}$, $n_c/n_i = 10^{-4}$), for the same resolution and number of particles, the quality of the fitted values for $\Gamma(k)$ decays faster, due to the noise caused by the limited number of particles.

We present in the left column of Fig. 3 the magnetic field power spectrum of models with different initial anisotropies $A_0 = +0.1$, +0.2, +0.3 (models $l$–$l$) at different times during the linear phase of the instability, when the distribution of particles is still almost identical to the initial one and the magnetic energy in the waves increases exponentially. The blue shaded region shows the wavenumbers for which the growth rate (for waves with polarization $L$) is a power law with index related to the CR momentum distribution power-law index. The power spectrum in this region increases in amplitude but keeping the slope during the measured times constant, this increase is faster for the higher absolute initial anisotropy. For wavenumbers smaller than those in the blue shaded region, the amplitude remains nearly constant, as expected for the zero growth rate in this region.

Using this magnetic energy spectrum, we estimate the lower limit of the pitch-angle diffusion coefficient $\tilde{D}_{\mu\mu}(\delta t)$ provided by the QLT [equation (A32) in Appendix]:

$$
\tilde{D}_{\mu\mu}(\mu, p, \delta t) = \frac{\Omega^2(1 - \mu^2)}{2} \int_0^\infty dk \frac{B(k, t)^2}{B_0^2}
\times \left\{ \left[ 1 + \frac{\sigma_{\mu}(k, t)}{\sigma_{\nu}(k, t)} \right] \frac{\sin[(\nu \mu k + \Omega) \delta t]}{(\nu \mu k + \Omega)} + \left[ 1 - \frac{\sigma_{\mu}(k, t)}{\sigma_{\nu}(k, t)} \right] \frac{\sin[(\nu \mu k - \Omega) \delta t]}{(\nu \mu k - \Omega)} \right\}. \quad (33)
$$

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Table 2. Model parameters.

| Run | $A_0$ | $n_c/n_i$ | $\beta_{cr} | | \Gamma_{max}/\Omega_0 | | | $|B_{\mu\mu}|^2/B_0^2$ | $P$ | $\sigma_{\mu}$ | $t_{end}\Omega_0$ | NX | NP/NX |
|-----|--------|-----------|-------------|----------------|-------------|--------|-------------|--------|----------------|----------------|--------|--------|
| $d1$ | -0.3   | $10^{-4}$ | 5.2         | $1.4 \times 10^{-3}$ | $5 \times 10^{-4}$ | $R(+1)$ | +1         | 10$^3$   | 4096             | 4096             |        |        |
| $d2$ | -0.3   | $10^{-4}$ | 5.2         | $1.4 \times 10^{-3}$ | $5 \times 10^{-4}$ | $L(-1)$ | -1         | 10$^3$   | 4096             | 4096             |        |        |
| $l1$ | +0.1   | $10^{-4}$ | 5.2         | $3 \times 10^{-4}$   | $10^{-12}$    | $L(-1)$ | 0          | 2 $\times 10^3$| 4096             | 4096             |        |        |
| $l2$ | +0.2   | $10^{-4}$ | 5.2         | $6 \times 10^{-4}$   | $10^{-12}$    | $L(-1)$ | 0          | 2 $\times 10^3$| 4096             | 4096             |        |        |
| $l3$ | +0.3   | $10^{-4}$ | 5.2         | $9 \times 10^{-4}$   | $10^{-12}$    | $L(-1)$ | 0          | 2 $\times 10^3$| 4096             | 4096             |        |        |
| $l4$ | +0.2   | $2.5 \times 10^{-5}$ | 1.3 | $2 \times 10^{-4}$   | $10^{-12}$    | $L(-1)$ | 0          | 2 $\times 10^3$| 4096             | 4096             |        |        |
| $l5$ | +0.2   | $4 \times 10^{-4}$ | 20.8 | $2.4 \times 10^{-3}$ | $10^{-12}$    | $L(-1)$ | 0          | 2 $\times 10^3$| 4096             | 4096             |        |        |
| $s1$ | +0.1   | $10^{-4}$ | 5.2         | $3 \times 10^{-4}$   | $2 \times 10^{-12}$ | R, L   | 0          | 2 $\times 10^5$| 4096             | 1024             |        |        |
| $s2$ | 0       | $10^{-4}$ | 5.2         | 0                     | $2 \times 10^{-12}$ | R, L   | 0          | 2 $\times 10^5$| 4096             | 1024             |        |        |
| $s3$ | -0.1   | $10^{-4}$ | 5.2         | $4 \times 10^{-4}$   | $2 \times 10^{-12}$ | R, L   | 0          | 2 $\times 10^5$| 4096             | 1024             |        |        |
| $s4$ | -0.2   | $10^{-4}$ | 5.2         | $8 \times 10^{-4}$   | $2 \times 10^{-12}$ | R, L   | 0          | 2 $\times 10^5$| 4096             | 1024             |        |        |
| $s5$ | -0.3   | $10^{-4}$ | 5.2         | $1.4 \times 10^{-3}$ | $2 \times 10^{-12}$ | R, L   | 0          | 2 $\times 10^5$| 4096             | 1024             |        |        |
| $s6$ | 0       | $10^{-4}$ | 5.2         | 0                     | $2 \times 10^{-12}$ | R, L   | 0          | 10$^5$   | 4096             | 512              |        |        |
| $s7$ | 0       | $10^{-4}$ | 5.2         | 0                     | $2 \times 10^{-12}$ | R, L   | 0          | 10$^5$   | 4096             | 2048             |        |        |
Figure 2. Real frequency of waves for models with initial anisotropy $A_0 = -0.3$ (models $d_1$ and $d_2$ in Table 2). Red points: waves with polarization $L$; blue crosses: waves with polarization $R$. The theoretical value given by equation (14) is shown for comparison (solid black line). The grey area comprehends the wavelengths $\leq 32$ grid cells, where the growth/damping rate is dominated by the numerical dissipation.

\[
\sigma_{H}(k, t) = -2 \frac{k}{|k|} \text{Im} \left\{ \frac{E'(k, t)E^{*}(k, t)}{|E(k, t)|^2} \right\}
\]

where the overbar means an average in time between $t - \delta t$ and $t$; here we neglect the real frequency of the waves $\omega_r(k) \ll \Omega$ in the resonance function. The helicity spectrum $\sigma_{H}(k, t)$ is calculated from the transverse electric field spectrum (Gary 1993):

\[
\sigma_{H}(k, t) = -2 \frac{k}{|k|} \text{Im} \left\{ \frac{E'(k, t)E^{*}(k, t)}{|E(k, t)|^2} \right\}.
\]

We compare the quasi-linear estimative with the directly measured diffusion coefficient

\[
D_{\mu\mu}(t, \mu, p, \delta t) = \left\langle \left[ \mu(t) - \mu(t - \delta t) \right]^2 \right\rangle_{\mu},
\]

where $\mu(t - \delta t)$ and $\mu(t)$ are cosine of the pitch angle for the same particle at two consecutive times separated by $\delta t$ (see e.g. Xu & Yan 2013; Weidl et al. 2015; Cohet & Marcowith 2016). The average $\langle \cdot \rangle$ is taken over all the particles in the simulation with momentum and pitch-angle cosine in a small interval $[p, p + \Delta p], [\mu, \mu + \Delta \mu]$.

The distributions in momentum of the ratio $\langle D_{\mu\mu}(\delta t) / \tilde{D}_{Q_{L}}(\delta t) \rangle_{\mu}$ (averaged over all the values of $\mu$, for $|\mu| \leq 0.95$) are shown in the right column of Fig. 3, for the same models and times as for...
the energy spectra shown in the left column. We employ a time interval \( \delta \Omega_0 = 10^2 \). For the fastest growing mode between these runs (model IS in Table 2), we have \( \delta \Gamma_{\text{max}} \approx 2.4 \). For all models, the ratio \( \langle \tilde{D}_{\mu \mu}(\delta t) / \tilde{D}_{\mu \mu}(\delta t) \rangle_\mu \) is very close to one.

Fig. 4 is similar to Fig. 3, but it shows a comparison between models with fixed anisotropy \( A_0 \) and different values of \( n_{\text{cr}} / n_i \) (models l2, l4, and l5 in Table 2). While the models with lower growth rate of the instability \( \Gamma_{\text{max}} \) (models l2 and l4) present the better agreement between \( D_{\mu \mu}(\delta t) \) and \( \tilde{D}_{\mu \mu}(\delta t) \), the model with higher \( n_{\text{cr}} / n_i \) (model l5) shows a diffusion rate \( D_{\mu \mu}(\delta t) \) about two times larger than \( \tilde{D}_{\mu \mu}(\delta t) \) for the final time.

4.3 Instability saturation

Fig. 5 shows the time evolution of the CR pressure anisotropy and of the total magnetic energy of the waves (normalized by the energy of the mean field) for models s1–s5 starting with different anisotropies \( A_0 \) (see Table 2). It shows clearly the reduction of the absolute total anisotropy, although the values do not achieve zero by the end of our simulations. One control simulation with initial zero anisotropy (model s2) is shown to be stable and to continue isotropic until the end of the simulation. After \( t \Omega_0 \approx 10^4 \), the total magnetic energy in the fluctuations begins to saturate, at higher values for the models with higher \( |A_0| \). The vertical lines in Fig. 5 indicate the times in different regimes of the instability: \( t \Omega_0 = 2 \times 10^3 \) (linear phase), \( t \Omega_0 = 2 \times 10^4 \) (beginning of the saturation phase), and \( t \Omega_0 = 10^5 \) and \( 2 \times 10^5 \) (late times during the saturation phase). We observe that the energy of the waves in the isotropic case also increases initially and saturates, caused by the numerical noise due to the limited number of particles. This value is smaller for higher number of particles (see Fig. 6 for a comparison between simulations with different NP/NX). It indicates the minimum limit in the instability growth rate that we can simulate (with fixed resolution), below which the properties are dominated by the numerical noise.

We show in Fig. 7 the magnetic power spectrum, the ratio \( \langle D_{\mu \mu}(\delta t) / \tilde{D}_{\mu \mu}(\delta t) \rangle_\mu \), and the anisotropy distribution in the momentum \( A(p) \), for two different models \( (A_0 = +0.1 \text{ and } -0.3) \). We plot each quantity at the four times (indicated by different colours) of the different regimes described above (see also Fig. 5). In the beginning of the saturation phase, the magnetic energy in the blue shaded region is already saturated for the highest wavenumbers, but it is still growing for the smallest wavenumbers.
Saturation times (from smaller to larger $t^\ast$) for different stages of the instability evolution: linear, early saturation, and late saturation. The vertical lines mark four different stages of the instability evolution: linear, early saturation, and late saturation times (from smaller to larger $t$).

In Fig. 9, we show the scattering rate $\nu_{\text{scatter}} \equiv \langle 2D\mu_t/(1 - \mu^2) \rangle_{\text{scatt}}$ during the saturated phase (at the final time of the simulation) for three models $A_0 = -0.1, -0.2, -0.3$. The distribution as a function of the CR momentum is almost flat for the momentum range that is isotropized by this time, but decaying for larger energies. This is expected as the high-energy particles are resonant with waves that are still growing.

5 SUMMARY AND CONCLUSIONS

Using one-dimensional hybrid PIC–MHD simulations, we study numerically the evolution of the CR gyroresonance instability, triggered by a distribution of CR protons with initial anisotropy (with respect to the local mean magnetic field direction) in pressure ($P_{\perp} \neq P_{\parallel}$). We restricted our analysis to parallel-propagating modes, which are the fastest growing modes. During the linear phase of the instability, the growth rate and phase speed of the modes with right and left circular polarization show excellent agreement with the theoretical dispersion relation, for both initial setups with $P_{\perp} > P_{\parallel}$ and $P_{\perp} < P_{\parallel}$.

In all our simulations, the non-linear wave–particle effects are important. After a short initial period of exponential growth of the waves, the scattering and consequent isotropization of the CR momentum distribution is the mechanism that gradually saturates the instability growth. The low-energy CRs are isotropized faster than those of the higher energies. The amplitude of the waves and the scattering rate of particles during the saturation phase are larger for initially larger maximum instability growth rate $\Gamma_{\text{max}}$.

We extracted from the simulations the pitch-angle diffusion coefficient $D_{\mu_t}$ for the evolution of the CR distribution function averaged over a time-scale $\delta t \lesssim \Gamma_{\text{max}}^{-1}$, and we find the empirical values in good agreement with the QLT estimates for static waves (within a factor of 3). This agreement is shown to be better for our simulations with smaller $\Gamma_{\text{max}}$. Indeed, due to limitations imposed by the noise caused by the low sample of macroparticles in the PIC technique, all our simulations have parameters that produce a maximum growth rate of the instability much higher than expected in realistic situations. None the less, this direct confirmation of the applicability of the QLT to estimate the CR scattering by the gyroresonance instability is a valuable support for theoretical models connecting the large-scale turbulence cascade with the 'microphysics' of the
CR gyroresonance instability

Figure 7. Normalized power spectrum of magnetic field $|B(k)|^2/B_0^2$ (upper row), $\langle D_{\mu\nu}(\delta t)/\bar{D}_{\mu\nu}^{CL}(\delta t)\rangle_{\mu}$ ratio (middle row), and distribution of CR pressure anisotropy $A$ in momentum (bottom row) for models s1 and s5 (see Table 2) with different anisotropies $A_0$: $+0.1$ (left column) and $-0.3$ (right column). The lines with different colours indicate different times (shown in Fig. 5): $t\Omega_0 = 2 \times 10^3$ (grey solid line), $t\Omega_0 = 2 \times 10^4$ (red dashed line), $t\Omega_0 = 10^5$ (blue dot–dashed line), $t\Omega_0 = 2 \times 10^5$ (green long-dashed line). In the power spectrum plots (upper row), the grey area indicates the wavenumber interval where numerical dissipation dominates (wavelengths $\leq 32$ grid cells); the blue area represents the wavenumber interval where $k_{cr}/p_{max} < k < k_{cr}/p_{min}$.

CR instabilities (YL11), and for subgrid models in large-scale simulations involving CR transport (e.g. Everett & Zweibel 2011; Evoli & Yan 2014; del Valle, Romero & Santos-Lima 2015; Pfrommer et al. 2017), as direct numerical simulations cannot cover the huge range of scales involved.

In conclusion, the outcome of this work provides a solid foundation for developing further investigations on the role of the CR gyroresonance instability on CR propagation. This includes for example the use of a setup where the pressure anisotropy is generated naturally by continuous compression or shear (e.g. Kunz et al. 2014; Riquelme et al. 2015; Sironi & Narayan 2015), the combined effect with the streaming instability generated by the presence of CR drift, and the use of three-dimensional simulations that would better represent the turbulence cascade from the instability generated waves, necessary for understanding the decaying of the instability and the effect of successive large-scale random compressions/expansions provided by the large-scale turbulence (Melville, Schekochihin & Kunz 2016). As already stressed before, the precise quantitative knowledge of the scattering provided by microscopic instabilities is an indispensable ingredient for building more realistic models for the CR propagation in the Galaxy and in the ICM, needed for our correct understanding and interpretation of almost all high-energy phenomena.

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Figure 8. Distribution of the pressure anisotropy in momentum $A(p)$ for the modified models $s1$ (upper panel) and $s5$ (bottom panel). The modification consists in the extension of the CR distribution, originally given by a single power law ($\propto p^{-2.8}$) in the compact momentum interval $[p_{\text{min}}, p_{\text{max}}]$, to include the low-momentum interval $[0.1 p_{\text{min}}, p_{\text{min}}]$ where it assumes a steep, growing power law ($\propto p^{5.6}$). The lines with different colours indicate different times: $t/\Omega_1 = 2 \times 10^3$ (grey solid line), $t/\Omega_1 = 2 \times 10^4$ (red dashed line), and $t/\Omega_1 = 10^5$ (blue dot–dashed line).

Figure 9. Distribution of the scattering rate $\nu_{\text{scatt}} = (2 D_{\mu\mu}(\delta \tau)(1 - \mu^2))_{\mu}$ (normalized by $\Omega_0$) at time $\tau_{\Omega_0} = 2 \times 10^3$ (later time during the saturation phase, see the last vertical line in Fig. 5) for models with different initial anisotropies $A_0 = -0.1$ (blue dashed line), $-0.2$ (green dot–dashed line), $-0.3$ (black solid line), corresponding to models $s3$–$s5$ in Table 2.

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REFERENCES

Achatz U., Steinacker J., Schlickeiser R., 1991, A&A, 250, 266
Amato E., Blasi P., 2009, MNRAS, 392, 1591
Bai X.-N., Caprioli D., Sironi L., Spitkovsky A., 2015, ApJ, 809, 55
Bell A. R., 2004, MNRAS, 353, 550
Bell A. R., Lucek S. G., 2001, MNRAS, 321, 433
Brose R., Telezhinsky I., Pohl M., 2016, A&A, 593, A20
Bykov A. M., Brandenburg A., Malkov M. A., Osipov S. M., 2013, Space Sci. Rev., 178, 201
Caprioli D., Spitkovsky A., 2014, ApJ, 783, 91
Cohet R., Marcowith A., 2016, A&A, 588, A73
Decyk V. K., 1995, Comput. Phys. Commun., 87, 87
Decyk V. K., 2007, Comput. Phys. Commun., 177, 95
del Valle M. V., Romero G. E., Santos-Lima R., 2015, MNRAS, 448, 207
Di Sciascio G., Iuppa R., 2014, preprint (arXiv:1407.2144)
Everett J. E., Zweibel E. G., 2011, ApJ, 739, 60
Evoli C., Yan H., 2014, ApJ, 782, 36
Farmer A. J., Goldreich P., 2004, ApJ, 604, 671
Gary S. P., 1993, Theory of Space Plasma Microinstabilities. Univ. Press, Cambridge
Ginzburg V. L., 1966, Sov. Astron., 9, 877
Jokipii J. R., 1966, ApJ, 146, 480
Kunz M. W., Schekochihin A. A., Stone J. M., 2014, Phys. Rev. Lett., 112, 205003
Lazarian A., 2016, ApJ, 833, 131
Lazarian A., Beresnyak A., 2006, MNRAS, 373, 1195
Longair M. S., 2011, High Energy Astrophysics. Cambridge Univ. Press, Cambridge
Lucek S. G., Bell A. R., 2000, MNRAS, 314, 65
Melville S., Schekochihin A. A., Kunz M. W., 2016, MNRAS, 459, 2701
Nava L., Gabici S., 2013, MNRAS, 429, 1643
Pfrommer C., Pakmor R., Schaal K., Simpson C. M., Springel V., 2017, MNRAS, 465, 4500
Riquelme M. A., Quataert E., Verscharen D., 2015, ApJ, 800, 27
Santos-Lima R., de Gouveia Dal Pino E. M., Pfrommer C., Palla F., Falceta-Gonçalves D., Lazarian A., Nakwacki M. S., 2014, ApJ, 781, 84
Santos-Lima R., Yan H., de Gouveia Dal Pino E. M., Lazarian A., 2016, MNRAS, 460, 2492
Schekochihin A. A., Cowley S. C., 2006, Phys. Plasmas, 13, 056501
Schekochihin A. A., Cowley S. C., Kulsrud R. M., Hammett G. W., Sharma P., 2005, ApJ, 629, 139
Schlickeiser R., 2002, Cosmic Ray Astrophysics. Springer-Verlag, Berlin
Sironi L., Narayan R., 2015, ApJ, 800, 88
Skilling J., 1970, MNRAS, 147, 1
Skilling J., 1971, ApJ, 170, 265
Skilling J., 1975, MNRAS, 172, 557
Sist T. H., 1962, The Theory of Plasma Waves. McGraw-Hill, New York
Stone J. M., Gardiner T. A., Teuben P., Hawley J. F., Simon J. B., 2008, ApJS, 178, 137
Weidl M. S., Jenko F., Teaca B., Schlickeiser R., 2015, ApJ, 811, 8
Wentzel D. G., 1969, ApJ, 156, 303
Xu S., Yan H., 2013, ApJ, 779, 140
Yan H., 2015, in Lazarian A., de Gouveia Dal Pino E. M., Melioli C., eds, Astrophysics and Space Science Library, Vol. 407, Magnetic Fields in Diffuse Media. Springer-Verlag, Berlin, p. 253
Yan H., Lazarian A., 2002, Phys. Rev. Lett., 89, 281102
Yan H., Lazarian A., 2011, ApJ, 731, 35 (YL11)
Yan H., Lazarian A., Petrosian V., 2008, ApJ, 684, 1461
Yan H., Lazarian A., Schlickeiser R., 2012, ApJ, 745, 140
Zhang B., Yan H., 2011, ApJ, 726, 90
**APPENDIX: QUASI-LINEAR FOKKER–PLANCK PITCH-ANGLE DIFFUSION COEFFICIENT**

Consider the operational definition of the pitch-angle diffusion coefficient convenient to analyse the particle simulation:

\[
D_{\mu\mu}(X(t_0), t_0) = \frac{\langle \mu(t) - \mu(0) \rangle^2}{2(t - t_0)},
\]  

(A1)

where the angle brackets \( \langle \cdot \rangle \) represent an ensemble over all the particles with coordinates \( X = (X, Y, Z, \mu, p, \psi) \) at time \( t_0 \), where \( (X, Y, Z) \) are the coordinates of the guiding centre, and \( (p, \mu, \psi) \) are the momentum, cosine of pitch angle, and gyrophase, respectively. In what follows, we provide the derivation of \( D_{\mu\mu} \) for particles travelling in a homogeneous plasma superimposed by a spectrum of small-amplitude transverse electromagnetic waves, with circular polarization and propagating parallel to the mean magnetic field. We also show that the above definition coincides – using similar hypotheses – with the \( D_{\mu\mu} \) coefficient in the Fokker–Planck equation for the evolution of the (ensemble) averaged distribution function derived through the QLT (see e.g. Achatz, Steinacker & Schlickeiser 1991; Schlickeiser 2002).

Without loss of generality, we take \( t_0 = 0 \) from here:

\[
D_{\mu\mu} = \frac{\langle \mu(t) - \mu(0) \rangle^2}{2t},
\]

where \( t \) is the prime means the integration is to be performed along the unperturbed particles orbit coordinates \( X' \). Adapting a Cartesian coordinate frame with the \( x \)-axis pointing in the direction of the mean magnetic field \( B_0 \),

\[
X' = X + v(t)X, \\
Y' = Y, \\
Z' = Z, \\
p' = p, \\
\mu' = \mu, \\
\psi' = \psi - \epsilon \Omega s,
\]

(A3)

with \( v = p/ym \) is the particle speed, \( \gamma = (1 - v^2/c^2)^{-1/2} \) is the light speed, \( m \) is the rest mass of the particles, \( e = q/|q| \) is the signal of the particle charge, and \( \Omega = |q|B_0/ymc \) is the particle cyclotron frequency (the unprimed phase-space coordinates are to be understood at time \( t = 0 \)).

Assuming the existence of a correlation time \( t_c \) so that the correlation \( \langle \mu(s)\mu(s + t) \rangle \) decays fast for \( |t| > t_c \), then the main contribution to the integral (A2) comes from the interval \(-t_c \leq \tau \leq t_c \). For \( t \gg t_c \), the integrals of the second integral can be replaced by \(-t \) to \( t \): 

\[
D_{\mu\mu} \approx \frac{1}{2t} \int_0^t ds \int_0^t dt \langle \mu(s)\mu(s + t) \rangle, \\
\]

(A4)

where in the last approximation we assumed the dependence of the correlation in \( \tau \) to be approximately symmetric around zero.

Additionally, if the correlation \( \langle \mu(t)\mu(t + \tau) \rangle \) does not depend on \( \tau \), or if it changes on a time-scale \( \tau' \gg t \),

\[
D_{\mu\mu} \approx \int_0^t dt \langle \mu(0)\mu'(\tau) \rangle,
\]

(A5)

which coincides with the expression obtained from the QLT. In the remaining of this appendix, we work out one expression for \( D_{\mu\mu} \) in terms of the wave spectrum description.

Writing the transverse magnetic fluctuations in wave components with left (L) or right (R) circular polarization:

\[
\delta B_{\mu}(x, t) = \sum_n \int_{-\infty}^{\infty} dk \delta B_{\mu,n}(k, x, t)
\]

\[
= \sum_n \int_{-\infty}^{\infty} dk \left[ \frac{1}{2} \delta A_{\mu}^L(k, x, t) + \delta A_{\mu}^R(k, x, t) \right]
\]

(A6)

\[
\delta B_{\mu}(x, t) = \sum_n \int_{-\infty}^{\infty} dk \delta B_{\epsilon,n}(k, x, t)
\]

\[
= \sum_n \int_{-\infty}^{\infty} dk \frac{1}{2} \sign(l_n) \left[ -\delta A_{\mu}^L(k, x, t) + \delta A_{\mu}^R(k, x, t) \right],
\]

(A7)

where \( l_n \) is the direction of propagation of the wave, \( n = b, f \) are waves propagating backward \( (n = b, l_n = -1) \) and forward \( (n = f, l_n = 1) \) to the mean magnetic field direction. The components \( \delta A_{\mu}^n \) \((\alpha = L, R)\) are monochromatic waves given by

\[
\delta A_{\mu}^n(k, x, t) = A_{\mu}^n(k, t) \exp \left\{ i \left[ k x - \alpha \omega_{\mu}^n(k) t \right] \right\}
\]

\[
= |A_{\mu}^n(k, t)| \exp \left\{ i \left[ k x - \alpha \omega_{\mu}^n(k) t \right] \right\},
\]

(A8)

with

\[
|A_{\mu}^n(k, t)| = |A_{\mu}^n(k, 0)| \exp \left[ \Gamma_{\mu}^n(k) t \right]
\]

(A9)

the absolute amplitude of the wave, and \( \omega_{\mu}^n, \Gamma_{\mu}^n \) are the real frequency and growth/damping rate:

\[
\omega_{\mu}^n(k) = \omega_{\mu}^n(k) + i \Gamma_{\mu}^n(k).
\]

(A10)

with \( \omega_{\mu}^n(-k) = -\omega_{\mu}^n(k) \) and \( \Gamma_{\mu}^n(-k) = \Gamma_{\mu}^n(k) \). Because the magnetic field is real, \( \phi_{\mu}^n(-k) = -\phi_{\mu}^n(k) \) and \( \delta A_{\mu}^n(-k, x, t) = \delta A_{\mu}^n(k, x, t) \).

Assuming that the waves have random phases,

\[
\langle A_{\mu}^n(k)\delta A_m^{\alpha^*}(k') \rangle = |A_{\mu}^n(k)|^2 \delta_{\mu m} \delta(k - k'),
\]

(A11)

where the brackets \( \langle \cdot \rangle \) indicate average on an ensemble of realizations. From the Maxwell–Faraday equation relating the electromagnetic fields of the transverse waves

\[
\frac{\partial B}{\partial t} = -c \nabla \times E,
\]

(A12)

we can express the electric field in terms of the waves \( A_{\mu}^n \):

\[
\delta E_{\mu}(x, t) = \sum_n \int_{-\infty}^{\infty} dk \delta E_{\mu,n}(k, x, t)
\]

\[
= \sum_n \int_{-\infty}^{\infty} \frac{dk}{kc} \frac{i}{k} \delta B_{\epsilon,n}(k, x, t)
\]

\[
= \sum_n \int_{-\infty}^{\infty} \frac{dk}{kc} \frac{i}{2} \sign(l_n) \left[ -\frac{\omega_{\mu}^n(k)}{kc} \delta A_{\mu}^L(k, x, t) + \frac{\omega_{\mu}^n(k)}{kc} \delta A_{\mu}^R(k, x, t) \right],
\]

(A13)

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\[ \delta E_z(x,t) = \sum_n \int_{-\infty}^{\infty} dk \delta E_{n}(k,x,t) \]

\[ = \sum_n \int_{-\infty}^{\infty} \frac{1}{kc} \frac{\partial}{\partial t} \delta B_{y,n}(k,x,t) \]

\[ = \sum_n \int_{-\infty}^{\infty} \frac{1}{2k^2} \left[ \frac{\omega_{n}^2(k)}{kc} \delta A_{n}^y(k,x,t) \right. \]

\[ \left. + \frac{\omega_{n}^2(k)}{kc} \delta A_{n}^z(k,x,t) \right]. \] (A14)

Now from the Lorentz force on the particle

\[ \dot{p} = e e \left[ \delta E + \frac{1}{c} v \times (B_0 + \delta B) \right], \] (A15)

it is straightforward to show that

\[ \dot{\mu} = \frac{\Omega}{B_0} \sqrt{1 - \mu^2} \times \left[ \delta B_y \left[ \exp(\psi) + \exp(-\psi) \right] \right. \]

\[ + \left. \delta B_z \left[ \exp(\psi) - \exp(-\psi) \right] \right]. \] (A16)

In the environments we are interested in, the phase speed of the waves \(\omega_n^2/k \approx \nu_\lambda < c\). Therefore, we neglect the contribution of the electric field in equation (A16):

\[ \dot{\mu} = \frac{\Omega}{B_0} \sqrt{1 - \mu^2} \left\{ \delta B_y \left[ \exp(\psi) + \exp(-\psi) \right] \right. \]

\[ + \left. \delta B_z \left[ \exp(\psi) - \exp(-\psi) \right] \right\}. \] (A17)

Using equation (A17) in combination with equations (A3), (A6), and (A7)

\[ \langle \dot{\mu}(s)\dot{\mu}(s + \tau) \rangle = \frac{\Omega^2}{B_0^2} \left( 1 - \mu^2 \right) \sum_n \int_{0}^{\infty} dk \]

\[ \times \left\{ |A_n^y(k,s)|^2 \exp \left\{ -i \left[ k \nu \mu - \omega_n^2(k) + \ln(i\omega_\lambda)\omega, \right] \tau \right\} \right. \]

\[ + \left. |A_n^z(k,s)|^2 \exp \left\{ -i \left[ k \nu \mu - \omega_n^2(k) - \ln(i\omega_\lambda)\omega, \right] \tau \right\} \right\}. \] (A18)

Then integrating equation (A18),

\[ \int_{0}^{\tau} \langle \dot{\mu}(s)\dot{\mu}(s + \tau) \rangle = \frac{\Omega^2}{B_0^2} \left( 1 - \mu^2 \right) \sum_n \int_{0}^{\infty} dk \]

\[ \times \left\{ |A_n^y(k,s)|^2 \mathcal{R}(n, L, k, t) + |A_n^z(k,s)|^2 \mathcal{R}(n, R, k, t) \right\}, \] (A19)

where the resonance function \( \mathcal{R} \) (Schlickeiser 2002; Weidl et al. 2015) is defined by

\[ \mathcal{R}(n, \alpha, k, t) \equiv \left\{ \frac{1 - \exp(\Gamma_n^\alpha(k) t)}{\Gamma_n^\alpha(k) + \nu_n^2(k) t} \right\} \exp \left\{ -i \left( \frac{\nu_n^2(k) t}{\Gamma_n^\alpha(k)} \right) \right\}, \]

\[ = -\Gamma_n^\alpha(k) \left[ 1 - \exp(\Gamma_n^\alpha(k) t) \right] \frac{\cos(\theta_n^\alpha(k) t)}{\left( \Gamma_n^\alpha(k) + \theta_n^\alpha(k) t \right)^2}, \] (A20)

where

\[ \theta_n^\alpha(k) \equiv k \nu_\mu - \omega_n^\alpha(k) - P(\alpha)\ln(i\omega_\lambda)\omega, \] (A21)

and \( P(\alpha) \) is the polarization \((-1\text{ for }\alpha = L \text{ and } +1\text{ for }\alpha = R)\).

For constant-amplitude waves \((\Gamma_n^\alpha(k) \to 0)\),

\[ \lim_{\alpha \to 0} \mathcal{R}(n, \alpha, k, t) = \frac{\sin(\theta_n^\alpha(k) t)}{\theta_n^\alpha(k)} \] (A22)

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waves ($\Gamma_n > 0$),

$$R(n, \alpha, k, t) \geq \frac{\sin [\vartheta_n(k)t]}{\vartheta_n(k)}.$$  \hspace{1cm} (A30)

Now returning to equation (A19) and using the notation

$$|A^n_{\alpha}(k, t)|^2 \equiv \frac{1}{t} \int_0^t ds |A^n_{\alpha}(k, s)|^2,$$  \hspace{1cm} (A31)

and

$$\tilde{D}_{\mu \mu}(t) = \frac{\Omega^2}{2 R_0^2} (1 - \mu^2) \sum_{n, \alpha} \int_0^\infty dk |A^n_{\alpha}(k, t)|^2 \sin \left(\frac{\vartheta_n(k)t}{\vartheta_n(k)}\right).$$  \hspace{1cm} (A32)

we obtain (for growing waves)

$$D_{\mu \mu}(t) \geq \tilde{D}_{\mu \mu}(t).$$  \hspace{1cm} (A33)