Search by a Metamorphic Robotic System in a Finite 3D Cubic Grid

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Abstract
We consider search in a finite 3D cubic grid by a metamorphic robotic system (MRS), that consists of anonymous modules. A module can perform a sliding and rotation while the whole modules keep connectivity. As the number of modules increases, the variety of actions that the MRS can perform increases. The search problem requires the MRS to find a target in a given finite field. Doi et al. (SSS 2018) demonstrate a necessary and sufficient number of modules for search in a finite 2D square grid. We consider search in a finite 3D cubic grid and investigate the effect of common knowledge. We consider three different settings. First, we show that three modules are necessary and sufficient when all modules are equipped with a common compass, i.e., they agree on the direction and orientation of the x, y, and z axes. Second, we show that four modules are necessary and sufficient when all modules agree on the direction and orientation of the vertical axis. Finally, we show that five modules are necessary and sufficient when all modules are not equipped with a common compass. Our results show that the shapes of the MRS in the 3D cubic grid have richer structure than those in the 2D square grid.

Keywords. Distributed system, metamorphic robotic system, search, and 3D cubic grid

1 Introduction
Swarm intelligence has shed light to collective behavior of autonomous entities with simple rules, such as ant, boid, and particles. The notion is applied to a collection of robots and swarm robotics has attracted much attention in the past two decades. Each autonomous element of the system is called a robot, module, agent, process, and sensor, and a variety of swarm robot systems have been investigated such as the autonomous mobile robot system [15], the population protocol model [3], the programmable particles [4], Kilobot [14], and 3D Catoms [16]. Dumitrescu et al. considered the metamorphic robotic system (MRS), that consists of a collection of modules in the infinite 2D square grid [9, 10]. The modules are anonymous, i.e., they are indistinguishable. They are autonomous and uniform, i.e., each module autonomously decides its movement by a common algorithm. They are oblivious, i.e., each module has no memory of past. Thus, each module decides its behavior by observing other modules in nearby cells. Each module can perform a sliding to a side-adjacent cell and a rotation by 90 degrees around a cell. The modules must keep connectivity, which is defined by side-adjacency of cells occupied by modules. The authors considered reconfiguration of the MRS, that requires the MRS to change the initial shape to a specified final shape [9]. They showed that any horizontally convex connected initial shape of an MRS can be transformed to any convex final shape via a straight chain shape. Later Dumitrescu and Pach showed that

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any connected initial shape can be transformed to any connected final shape via a straight chain shape \cite{8}. In other words, the MRS has the ability of “universal reconfiguration.”

Reconfiguration can generate dynamic behavior of the MRS. Dumitrescu et al. demonstrated that the MRS can move forward by repeating a reconfiguration \cite{10}. They showed a reconfiguration that realizes the fastest \textit{locomotion}. Doi et al. pointed out that the oblivious modules can use the shape of the MRS as global memory and the MRS can solve more complicated problem as the number of modules increases. They investigated \textit{search} in a finite 2D square grid, that requires the MRS to find a target cell in a finite rectangle \textit{field} \cite{7}. Each module does not know the position of the target cell or the initial configuration of the MRS. They showed that the MRS can find the target cell by sweeping each row of the field without such a priori knowledge. Specifically, if the modules agree on the cardinal directions (i.e., north, south, east and west), three modules are necessary and sufficient, otherwise five modules are necessary and sufficient. Nakamura et al. considered \textit{evacuation} of the MRS from a finite rectangular field in the 2D square grid \cite{13}. There is a hole (i.e., two side-adjacent cells) on the wall of the field, and the MRS is required to exit from it from an arbitrary initial position and arbitrary initial shape. They showed that two modules are necessary and sufficient when the modules agree on the cardinal directions, otherwise four modules are necessary and sufficient.

In this paper, we investigate the effect of common knowledge on search by the MRS in the 3D cubic grid. We consider the following three cases: (i) modules equipped with a common \textit{compass} (i.e., they agree on the direction and orientation of \(x\), \(y\), and \(z\) axes). (ii) modules equipped with a common vertical axes (i.e., they agree on the direction and orientation of \(z\) axes), and (iii) modules not equipped with a common compass (i.e., they have no agreement on directions and orientations). We demonstrate that three modules are necessary and sufficient when the modules are equipped with a common compass and five modules are necessary and sufficient when the modules are not equipped with a common compass. The numbers of sufficient modules in the 3D cubic grid are the same as those in the 2D square grid \cite{7} because the MRS has more states in the 3D cubic grid than in the 2D square grid. For the intermediate case with a common vertical axis, we demonstrate that four modules are necessary and sufficient. Thus, our results in the 3D cubic grid show a smooth trade-off between the computational power of the MRS and common knowledge among modules, that the previous results in the 2D square grid could not find. We present search algorithms for these three settings and show the necessity by examining the state transition graph of the MRS.

\textbf{Related work.} Reconfiguration of swarm robot systems have been discussed for the MRS \cite{8,9}, autonomous mobile robots \cite{11,15} and the programmable particles \cite{5,6}. Michail et al. considered the \textit{programmable matter system}, that is similar to the MRS and investigated reconfiguration by rotations only and that by rotations and slidings \cite{12}. They showed that the combination of rotations and slidings guarantees universal reconfiguration, while rotations only cannot. They also presented \(O(n^2)\)-time reconfiguration algorithm by rotations and slidings. Almethen et al. considered reconfiguration by \textit{line-pushing}, where each module is equipped with the ability of pushing a line of modules \cite{1}. They presented \(O(n\log n)\)-time universal reconfiguration algorithm that does not promise connectivity of intermediate shapes and \(O(n\sqrt{n})\)-time reconfiguration algorithm that transforms a diagonal line into a straight chain with preserving connectivity. The same authors later showed that their programmable matter system has the ability of universal reconfiguration and \(O(n\sqrt{n})\)-time reconfiguration algorithm together with \(\Omega(n\log n)\) lower bound \cite{2}.
2 Preliminary

We consider a metamorphic robotic system (MRS) in a finite 3D cubic grid. A metamorphic robotic system consists of a collection of anonymous (i.e., indistinguishable) modules. A module can observe the positions of other modules in nearby cells, computes its next movement with a common algorithm, and performs the movement.

Each cell of the 3D cubic grid can adopt at most one module at a time. Cell \((x, y, z)\) is the cell surrounded by grid points \((x, y, z), (x+1, y, z), (x, y+1, z), (x, y, z+1), (x+1, y+1, z), (x+1, y, z+1), (x, y+1, z+1), (x+1, y+1, z+1)\). Cells \((x+1, y, z), (x, y+1, z), (x, y, z+1), (x+1, y+1, z), (x+1, y, z+1), (x, y+1, z+1), (x+1, y+1, z+1)\) are side-adjacent to cell \((x, y, z)\). We consider the positive \(x\) direction as East, the positive \(y\) direction as North, and the positive \(z\) direction is Up.

The MRS moves in a finite field, which is a cuboid of width \(w\), depth \(d\) and height \(h\) with its two diagonal cells being \((0, 0, 0)\) and \((w-1, d-1, h-1)\). We consider two types of planes; the first type is a set of cells forming a plane perpendicular to one of the \(x\), \(y\), and \(z\) axes. The second type is a set of cells parallel to one of the \(x\), \(y\), and \(z\) axes and diagonal to the remaining two axes. For example, \(\{(x, y, z)|y = s\}\) for some \(s \in \mathbb{Z}\) is a plane of the first type and \(\{(x, y, z)|x+y = s'\}\) for some \(s' \in \mathbb{Z}\) is a plane of the second type. A line of cells is a set of cells forming a horizontal or vertical line on a plane. For example, \(\{(x, y, z)|y = u, z = v\}\) for some \(u, v \in \mathbb{Z}\) is a line and \(\{(x, y, z)|x + y = u', z = v'\}\) for some \(u', v' \in \mathbb{Z}\) is a line.

The field is surrounded by six planes, which we call walls. More precisely, the walls are \(\{(x, y, z)|x = -1\}\) (the West wall), \(\{(x, y, z)|y = -1\}\) (the South wall), \(\{(x, y, z)|z = -1\}\) (the Bottom wall), \(\{(x, y, z)|x = w\}\) (the East wall), \(\{(x, y, z)|y = d\}\) (the North wall), and \(\{(x, y, z)|z = h\}\) (the Top wall).

All modules synchronously perform observation, computation, and movement in each discrete time \(t = 0, 1, 2, \ldots\). A configuration of the MRS is the set of cells occupied by the modules. We say two modules are side-adjacent if they are in the two side-adjacent cells. We also say that a module \(m\) is side-adjacent to cell \(c\) if the cell occupied by \(m\) is side-adjacent to \(c\). Given a configuration of the MRS, consider a graph where each vertex corresponds to a module and there is an edge between two vertices if the corresponding modules are side-adjacent. If this graph is connected, we say the MRS is connected.

A module can perform two types of movements, sliding and rotation.

1. Sliding: When two modules \(m_i\) and \(m_j\) are side-adjacent, another module \(m_k\) can move from a cell side-adjacent to \(m_i\) to an empty cell side-adjacent to \(m_k\) and \(m_j\) along \(m_i\) and \(m_j\). During the movement, \(m_i\) and \(m_j\) cannot perform any movement. See Figure 1 as an example.

2. Rotation: When two modules \(m_i\) and \(m_j\) are side-adjacent, \(m_i\) can move to a cell side-adjacent to \(m_j\) by rotating \(\pi/2\) in some direction. There are six cells side-adjacent to \(m_j\) and \(m_i\) can move to four of them by rotation. During the movement, \(m_j\) cannot move and the cells that \(m_i\) passes must be empty. See Figure 2 as an example.

Note that several modules can move at the same time as long as their moving tracks do not overlap. The modules must keep two types of connectivity at each time step.

1. At the beginning of each time step, the modules must be connected.
2. At each time step, the modules that do not move must be connected.

We assume that each module obtains the result of an observation and moves to the next cell in its local \(x\)-\(y\)-\(z\) coordinate system. We assume that the origin of the local coordinate system of a module is its current cell and all local coordinate systems are right-handed. In this paper, we consider three types of MRSs with different degree of agreement on
the coordinate system. When all modules agree on the directions and orientations of $x$, $y$, and $z$ axes, we say the MRS is equipped with a common compass. When all modules do not agree on the directions or orientations of $x$, $y$, and $z$ axes, we say the MRS is not equipped with a common compass. Hence, local coordinate systems are not consistent among the modules. As an intermediate model, we consider modules that agree on the direction and orientation of the vertical axis. In this case, we say the MRS is equipped with a common vertical axis. The state of the MRS is its local shape. If the modules are equipped with a common compass or a common vertical axis, the state of the MRS contains common directions and orientations. Otherwise the state of the MRS does not contain any directions and orientations.

The search problem requires the MRS to find a target placed at one cell in the field from a given initial configuration. We call the cell containing the target the target cell. The MRS finds a target when one of its modules enters the target cell.

When a module executes the common algorithm, the input is the observation of cells in a cube of size $(2k + 1) \times (2k + 1) \times (2k + 1)$ centered at the module (i.e., its $k$-neighborhood). It detects whether each cell in its $k$-neighborhood is a wall cell or not and whether it is occupied by a module or not. Let $C_m$ be the set of cells occupied by some modules, $C_w$ be the set of wall cells, and $C_e$ be the set of the remaining (i.e., empty) cells of an observation. More precisely, each set of cells is a set of coordinates of the corresponding cells observed in the local coordinate system of the module. For example, in Figure 2, the result of an observation at the red module is $C_m = \{(0, -1, 0), (1, -1, 0)\}$, $C_w = \emptyset$, and $C_e$ is the remaining cells. When a common algorithm outputs coordinate $(a, b, c)$ at a module, the module moves to $(a, b, c)$ in its local coordinate system.

When we describe an algorithm, the elements of $C_m$, $C_w$, and $C_e$ are specified in a "canonical coordinate system," i.e., the global coordinate system. When the modules are equipped with a common compass, without loss of generality, we assume that the common compass is identical to the global coordinate system. Thus, each module obtains its movement by checking $C_m$, $C_w$, and $C_e$. When the modules agree on a common vertical axis, without loss of generality, we assume that the vertical axis is identical to the $z$ axis of the global coordinate system. Each module computes its movement by rotating the current observation by $\pi/2$, $\pi$, $3\pi/2$, and $2\pi$ around the common vertical axis and comparing the results with $C_m$, $C_w$, and $C_e$. It selects the output with movement and if there are multiple outputs with movement it nondeterministically selects one of them. When the modules are not equipped with a common compass, a module checks 24 rotations of the current observation and selects an output in the same way as the above case.
3 Search algorithms for MRSs in a finite 3D cubic grid

In this section, we present search algorithms with small number of modules. Our proposed algorithms are based on a common strategy. Since the MRS does not know the position of the target cell, we make the MRS visit all cells of the field. The proposed algorithms slice the field into planes and the MRS visits the cells of each plane by sweeping each row or column of the plane. Thus, the proposed algorithms are extensions of the search algorithms by Doi et al. in a finite 2D cubic grid [7].

3.1 Search with a common compass

We show the following theorem by a search algorithm for the MRS of three modules equipped with a common compass.

**Theorem 1** The MRS of three modules equipped with a common compass can solve the search problem in a finite 3D cubic grid from any initial configuration.

The proposed algorithm considers each plane \( \{(x, y, z) | x + y = s\} \) for \( s = 1, 2, \ldots \) and the MRS moves along each line \( \{(x, y, z) | x + y = s, z = t\} \) for \( t = 0, 1, 2, \ldots \). Figure 3 shows a moving track of the proposed algorithm. The MRS continues to search each plane until it reaches the northeasternmost plane. Then, it moves along the edges of the field so that it returns to the southwesternmost plane. It starts searching each plane again to visit all cells of the field.

The MRS moves forward or turns by repeating a sequence of movements, that we call a move sequence. The proposed algorithm consists of the following move sequences.

- **Move sequence** \( M_{NW} \) (Figure 4). The blue module is in cell \((x, y, z)\) at the first. By this move sequence, the green module reaches cell \((x - 1, y + 1, z)\). By repeating \( M_{NW} \) \( n \) times, any of the modules visit the cells \((x - k, y + k, z) | 0 \leq k \leq n\). That is, it visits all the cells of the horizontal line \( \{(x, y, z) | x + y = s, z = t\} \).

- **Move sequence** \( M_{TurnNW} \) (Figure 5). By this move sequence, the MRS changes its move sequence from \( M_{NW} \) to \( M_{SE} \).

- **Move sequence** \( M_{SE} \) (Figure 6). The blue module is in cell \((x, y, z)\) at the first. By this move sequence, the green module reaches cell \((x + 1, y - 1, z)\). By repeating \( M_{SE} \) \( n \) times, any of the modules visit the cells \((x + k, y - k, z) | 0 \leq k \leq n\). That is, it visits all the cells of the horizontal line \( \{(x, y, z) | x + y = s, z = t\} \).

- **Move sequence** \( M_{TurnSE} \) (Figure 7). By this move sequence, the MRS changes its move sequence from \( M_{SE} \) to \( M_{NW} \).

- **Move sequence** \( M_{T} \) (Figure 8). By this move sequence, the MRS changes its move sequence from \( M_{SE} \) to \( M_{D} \).

- **Move sequence** \( M_{D} \) (Figure 9). The blue module is in cell \((x, y, z)\) at the first. By this move sequence, the green module reaches cell \((x, y, z - 1)\). By repeating \( M_{D} \) \( n \) times, any of the modules visit the cells \((x, y, z - k) | 0 \leq k \leq n\). That is, it visits all the cells of the line \( \{(x, y, z) | x = s, y = t\} \).

- **Move sequence** \( M_{B} \) (Figure 10). By this move sequence, the MRS changes its move sequence from \( M_{D} \) to \( M_{NW} \).

- **Move sequence** \( M_{NECorner} \) (Figure 11). By this move sequence, the MRS changes its move sequence from \( M_{D} \) to \( M_{WallBottom} \).
• Move sequence $M_{WallBottom}$ (Figure 12). The blue module is in cell $(x, y, 0)$ at the first. By this move sequence, the green module reaches cell $(x, y - 1, 0)$ along the edge. By repeating $M_{WallBottom}$ $n$ times, any of the modules visit the cells $(x, y - k, 0)(0 \leq k \leq n)$. That is, it visits all the cells of the line $\{(x, y, z)|x = s, z = 0\}$.

• Move sequence $M_{SWCorner}$ (Figure 13). By this move sequence, the MRS changes its move sequence from $M_{WallBottom}$ to $M_{Up}$.

• Move sequence $M_{Up}$. (Figure 14) The blue module is in cell $(0, 0, z)$ at the first. By this move sequence, the green module reaches cell $(0, 0, z + 1)$. By repeating $M_{Up}$ $n$ times, any of the modules visit the cells $(0, 0, k)(0 \leq k \leq n)$. That is, it visits all the cells of the line $\{(x, y, z)|x = 0, y = 0\}$.

Figure 3: Search by three modules equipped with common compass

The proposed algorithm consists of the following seven steps.

**Step 1** The MRS repeats $M_{NW}$, that makes it move in the northwest direction along a horizontal line on a plane $\{(x, y, z)|x + y = s\}$ for some $s$.

**Step 2** When the MRS reaches the north or west wall, it changes the moving direction to southeast by $M_{TurnNW}$.

**Step 3** The MRS repeats $M_{SE}$, that makes it move in the southeast direction along a horizontal line on $\{(x, y, z)|x + y = s\}$. This movement makes the MRS move along the same horizontal line as Step 1.

**Step 4** If the MRS is adjacent to the top wall it moves to the plane $\{(x, y, z)|x + y = s + 1\}$ by $M_T$. Then, it repeats $M_D$ shown in Figure 9 that makes it move down along the south wall or east wall until it reaches the bottom wall. Then, it leaves the wall by $M_B$. It starts searching the new plane by repeating Steps 1, 2, 3, and 4. Otherwise, it proceeds to Step 5.

**Step 5** When the MRS reaches the south or east wall, it moves to the row above by $M_{TurnSE}$. Then, it repeats Steps 1, 2, and 3 so that it visits all cells on the new horizontal line.

**Step 6** When the MRS reaches the northeast corner of the top wall, the algorithm sends the MRS back to the southwest corner, where the MRS starts searching by repeating Steps 1 to 5. It moves along the northeast edge until it reaches the northeast corner of the bottom wall by $M_D$. Then, it moves along the east edge of the bottom wall until it reaches the south east corner of the bottom wall by $M_{NECorner}$ and repeating $M_{WallBottom}$. It moves along the south edge of the bottom wall until it reaches the southwest corner of the bottom wall by repeating $M_{WallBottom}$. Finally, it moves along the southeast edge until it reaches the southwest corner of the top wall by $M_{SWCorner}$ and repeating $M_{Up}$. Then, the MRS returns to Step 4.
Figure 4: Move to northwest. In each figure, the red module moves. When the blue module is in cell \((x, y, z)\) at the start, after this move sequence, the green model reaches cell \((x - 1, y + 1, z)\).

Figure 5: Turn on the north or west wall. In the first figure, the red module moves.

Figure 6: Move to southeast. In each figure, the red module moves. When the blue module is in cell \((x, y, z)\) at the start, after this move sequence, the green model reaches \((x + 1, y - 1, z)\).

Figure 7: Turn on the south or east wall. In each figure, the red module moves.

Figure 8: Move around the top of the north or east wall. In each figure, the red module moves.
Figure 9: Move down on the north or east wall. In each figure, the red module moves. When
the blue module is in cell \((x, y, z)\) at the start, after this move sequence, the green model
reaches cell \((x, y, z - 1)\).

Figure 10: Leaving the bottom of the north or east wall. In each figure, the red module
moves.

Figure 11: Move on the northeast corner. In the first figure, the red module moves.

Figure 12: Move along the bottom of the wall. In each figure, the red module moves. When
the blue module is in cell \((x, y, 0)\) at the start, after this move sequence, the green model
reaches cell \((x, y - 1, 0)\).

Figure 13: Move on the southwest corner. In the first figure, the red module moves.
Table 3.1 shows the input and the output of the proposed algorithm. Each element specifies a part of the input (especially, \( C_w \) and \( C_e \)), and the MRS does not care whether other cells than those specified are walls or not.

When the MRS is on a plane \( \{(x, y, z) | x + y = s \} \) for some \( s \), it visits all cells in the horizontal line \( \{(x, y, z) | x + y = s, z = t \} \) for some \( t \) by repeating Steps 1, 2, and 3. Then, it proceeds to the horizontal line \( \{(x, y, z) | x + y = s, z = t + 1 \} \) by Step 5. By repeating Steps 1, 2, 3, and 5, it eventually reaches the top wall. Then, it starts searching for cells in \( \{(x, y, z) | x + y = s + 1 \} \) by Step 4 and 6.

Repeating the above movement, the MRS eventually reaches the northeast corner of the top wall. At this point, it may have not yet visited the cells near the south west corner. Steps 6 enables the MRS visit these cells by moving it to the southwest corner of the top wall and starting Steps 1 again.

There exist initial configurations that satisfies no condition of Table 3.1. We add exceptional transformation rules from such initial configurations. Figure 15 shows all states of three modules equipped with a common compass. Observe that any configuration can be transformed to another one in one time step. (Note that more than one module can move in one time.) Hence, even if the initial state of the MRS does not match any entry of Table 3.1, the MRS can be transformed to one of the entries and the MRS can start search from any initial configuration.

Table 1: Search algorithm for the MRS of three modules equipped with a common compass

| \( M_{SE} \) | \( C_m \) | \( C_w \) | \( C_e \) | Output |
|------------|---------|---------|---------|--------|
| (0,0,1),(1,0,1) | (2,0,0) | (1,0,0) |
| (1,0,0),(1,0,−1) | (0,0,1),(0,−1,1) | (0,−2,0) | (0,−1,0) |
| (0,−1,0),(0,−1,−1) | (0,0,1),(1,0,1) | (2,0,0) | (0,−1,0) |

| \( M_{TurnSE} \) | \( (0,0,1),(1,0,1) \) | \( (2,0,0) \) | \( (0,0,−1), (0,0,1) \) | \( (1,0,1) \) |
|------------|---------|---------|-------------|--------|
| (0,0,1),(0,−1,1) | (0,−2,0) | (0,1,1) |
| (0,−1,0),(0,−2,0) | (0,0,−1) | (0,−1,1) |

| \( M_{NW} \) | \( (−1,0,0),(−1,0,1) \) | \( (0,0,−1),(0,1,−1) \) | \( (0,1,0),(0,1,1) \) | \( (−2,0,0),(−1,0,−1) \) |
|------------|---------|---------|-------------|--------|
| (0,0,1),(0,−1,1) | (0,−2,0) | (0,1,0) |
| (0,−1,0),(0,−2,0) | (0,0,−1) | (0,−1,1) |

| \( M_{TurnNW} \) | \( (0,−1,0),(0,−1,1) \) | \( (0,1,0) \) | \( (0,0,1) \) |
|------------|---------|---------|--------|
| (1,0,0),(1,0,1) | (−1,0,0) | (0,0,1) |

| \( M_T \) | \( (0,0,1),(1,0,1) \) | \( (2,0,0),(0,0,2) \) | \( (1,0,1) \) |
|------------|---------|---------|--------|
| (1,0,0),(1,0,−1) | (2,0,0),(0,0,1) | (1,0,1) |
| (0,0,1),(0,1,1) | (1,0,0),(0,0,2) | (0,1,0) |
| (0,0,1),(0,−1,1) | (0,0,2),(0,0,−2) | (0,−1,0) |

| \( M_D \) | \( (0,0,1),(0,1,−1) \) | \( (1,0,0) \) | \( (0,0,−1) \) |
|------------|---------|---------|--------|
| (0,0,1),(0,1,1) | (1,0,0) | (0,1,−1) |
| (0,0,−1),(0,0,−2) | (1,0,0) | (0,−1,−1) |
| (1,0,0),(1,0,−1) | (0,−1,0) | (0,0,−1) |

| \( M_B \) | \( (0,1,0),(0,1,1) \) | \( (1,0,0),(0,0,−1) \) | \( (0,−1,0) \) |
|------------|---------|---------|--------|
| (0,0,−1),(0,0,−2) | (0,−1,0) | (−1,0,−1) |
Otherwise

\( x = \text{lem} \) in a finite 3D grid if no pair of modules have identical observation in an initial configuration. The MRS of four modules with a common vertical axis can solve a search problem on a plane with a common compass.

### 3.2 Search with a common vertical axis

We show the following theorem by a search algorithm for the MRS of four modules equipped with a common vertical axis.

**Theorem 2** The MRS of four modules with a common vertical axis can solve a search problem in a finite 3D grid if no pair of modules have an identical observation in an initial configuration.

We prove Theorem 2 by a search algorithm. The proposed algorithm considers each plane \( x = s \) for \( s = 0, 1, 2, \ldots \) and \( y = s' \) for \( s' = 0, 1, 2, \ldots \). The MRS moves along each vertical line \( x = s, y = t \) when it is on the plane \( x = s \) for \( t = 0, 1, 2, \ldots \) and \( x = t', y = s' \) when it is on the plane \( y = s' \) for \( t' = 0, 1, 2, \ldots \). Figure 16 shows an execution of the proposed algorithm.

The MRS continues to search each plane perpendicular to the \( x \)-axis until it reaches the east wall. Then, it starts to search each plane perpendicular to the \( y \)-axis. The algorithm description is given in Table 3.2.

Table 1: Search algorithm for the MRS of three modules equipped with a common compass.

| \( M_{NECorner} \) | \( (0, 0, -1), (0, -1, -1) \) | \( (0, 0, 0), (0, 0, -2) \) | \( (0, 2, 0) \) |
| \( M_{WallBottom} \) | \( (0, 0, 0), (1, -1, 0) \) | \( (0, 0, -1) \) | \( (1, 1, 0) \) |
| \( M_{SWCorner} \) | \( (0, 0, -1), (0, 0, -2) \) | \( (0, 0, 0), (0, 0, -2) \) | \( (0, 0, 0) \) |
| \( M_{Up} \) | \( (0, -1, 0), (0, 0, -1) \) | \( (0, 0, 0), (0, 0, -2) \) | \( (0, 0, 0) \) |
| \( M_{Down} \) | \( (0, 0, 0), (0, 0, -1) \) | \( (0, 0, 0), (0, 0, -2) \) | \( (0, 0, 0) \) |
| \( M_{TurnUp} \) | \( (0, 0, 0), (0, 0, 2) \) | \( (0, 0, 0), (0, -1, 0) \) | \( (0, 0, 0) \) |

The proposed algorithm consists of the following move sequences.

- Move sequence \( M_{Down} \) (Figure 18). The blue module is in cell \( (x, y, z) \) at the start. By this move sequence, the green module reaches cell \( (x, y, z - 1) \). By repeating \( M_{Down} \) \( n \) times, any of the modules visit the cells \( (x, y, z - k) (0 \leq k \leq n) \). That is, it visits all the cells of the horizontal line \( \{(x, y, z) | x = s, y = t \} \).

- Move sequence \( M_{Up} \) (Figure 19). The blue module is in cell \( (x, y, z) \) at the start. By this move sequence, the green module reaches cell \( (x, y, z + 1) \). By repeating \( M_{Up} \) \( n \) times, any of the modules visit the cells \( (x, y, z + k) (0 \leq k \leq n) \). That is, it visits all the cells of the horizontal line \( \{(x, y, z) | x = s, y = t \} \).

- Move sequence \( M_{TurnUp} \) (Figure 20). By this move sequence, the MRS changes its move sequence from \( M_{Up} \) to \( M_{Down} \).

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Figure 14: Move up on the southwest corner. In each figure, the red module moves. When the blue module is in cell $(0, 0, z)$ at the start, after this move sequence, the green module reaches cell $(0, 0, z + 1)$.

Figure 15: States of the MRS of three modules equipped with a common compass

Figure 16: Example of a search by four modules
• Move sequence $M_{TurnD}$ (Figure 21). By this move sequence, the MRS changes its move sequence from $M_{Down}$ to $M_{Up}$.

• Move sequence $M_{B1}$ (Figure 22). By this move sequence, the MRS changes its move sequence from $M_{Down}$ to $M_{B2}$.

• Move sequence $M_{B2}$ (Figure 23). The blue module is in cell $(x, y, 0)$ at the start. By this move sequence, the green module reaches cell $(x + 1, y, 0)$. By repeating $M_{B2}$ $n$ times, any of the modules visit the cells $(x + k, y, 0) (0 \leq k \leq n)$. That is, it visits all the cells of the horizontal line $\{(x, y, z)|y = s, z = 0\}$.

• Move sequence $M_{B3}$ (Figure 24). By this move sequence, the MRS changes its move sequence from $M_{B2}$ to $M_{Up}$.

• Move sequence $M_{Corner}$ (Figure 25). By this move sequence, the MRS changes its move sequence from $M_{Down}$ to $M_{B1}$.

The proposed algorithm consists of the following six steps. We use north, south, east, and west for explanation, however each module does not need to know the directions.

**Step 1** The MRS repeats $M_{Down}$, that makes it move in the down direction along a vertical line on a plane $\{(x, y, z)|y = s\}$ for some $s$.

**Step 2** When the MRS reaches the bottom wall, it changes the direction to up by $M_{TurnD}$.

**Step 3** The MRS repeats $M_{Up}$, that makes it move in the up direction. This movement makes the MRS move along the same vertical line as Step 1.

**Step 4** If the MRS is adjacent to the bottom of the west wall, it moves to the next plane $\{(x, y, z)|y = s - 1\}$ by $M_{B1}$. Then it moves along the bottom wall in the east direction by repeating $M_{B2}$. When the MRS reaches the east wall, it performs $M_{B3}$. Then it starts searching the next plane by Step 1. Otherwise, it proceeds to Step 5.

**Step 5** When the MRS reaches the top wall, it moves west by one row by $M_{TurnUp}$. Then it returns to Step 1.

**Step 6** When the MRS reaches the southwest corner, it performs $M_{Corner}$, and performs Step 4, that makes it start searching the plane obtained by rotation the current search plane by 90 degrees. Thus, the search planes are first perpendicular to the $x$ axis and the MRS moves to east. Then, the search planes are perpendicular to the $y$ axis and the MRS moves to north. Third the search planes are perpendicular to the $x$ axis and the MRS moves to west. Finally, the search planes are perpendicular to the $y$ axis and the MRS moves to south.

Depending on its initial state, MRS may start from the middle of the above track. When the MRS is on a plane $\{(x, y, z)|y = s\}$ for some $s$, it visits all cells on a vertical line $\{(x, y, z)|x = t, y = s\}$ by Steps 1, 2, and 3. Then, it proceeds to a vertical line $\{(x, y, z)|x = t - 1, y = s\}$ by Step 5. By repeating Steps 1, 2, 3, and 5, it visits all cells on the plane $\{(x, y, z)|y = s\}$. By Step 4, it moves to the bottom of east wall in the next plane $y = s - 1$, then it starts searching the plane. By repeating Steps 1 to 5, it eventually reaches the southwest corner of the bottom wall. It starts searching the vertical line $\{(x, y, z)|x = 1, y = d - 2\}$ by Step 6.
Figure 17: Example of a search by four modules

Figure 18: Move to down. In each figure, the red module moves. When the blue module is in cell \((x, y, z)\) at the start, after this move sequence, the green model reaches \((x, y, z - 1)\).

Figure 19: Move to up. In each figure, the red module moves. When the blue module is in cell \((x, y, z)\) at the start, after this move sequence, the green model reaches \((x, y, z + 1)\).

Figure 20: Turn on the up wall. In each figure, the red modules move.
Figure 21: Turn on the down wall. In the first figure, the red module moves.

Figure 22: Move on the bottom of the wall. In each figure, the red modules move.

Figure 23: Move along the down wall. In each figure, the red module moves. When the blue module is in cell \((x, y, 0)\) at the start, after this move sequence, the green model reaches \((x + 1, y, 0)\).

Figure 24: Move on the bottom of the wall to rise. In each figure, the red modules move.

Figure 25: Move on the bottom corner. In each figure, the red module moves.
Figure 26: Configurations of the MRS of four modules equipped with a common vertical axis

Figure 27: Transition graph of the MRS of four modules equipped with a common vertical axis
Depending on the initial configuration, the search may not be possible, and even if it is possible, the MRS cannot start the search directly because some initial configurations do not satisfy the conditions in Table 3.2.

Figure 26 shows all possible configurations for four modules not equipped with a common vertical axis. The configurations containing the coloured modules are symmetric, i.e., in each figure the modules painted by the same color may have an identical observation and move symmetrically. The MRS cannot start the search because it cannot move from their positions. Therefore, if there are pairs of modules with the same observation in an initial configuration, the search is not possible.

The proposed algorithm guarantees that the MRS finds the target when all modules have different observations in an initial configuration. Figure 27 shows these initial configurations can be transformed to $S_{4}^{4}L_{3}^{3}$ by a sequence of transformations. Even if some walls prevent these transformations, the MRS can leave the wall because its state is asymmetric. If the MRS satisfies the condition of the fourth movement of Table 3.2, the MRS can start the search directly. Otherwise, the MRS cannot move because it touches a wall. In this case, as shown in Figure 28, the MRS in $S_{4}^{4}L_{3}^{3}$ on a wall can change its direction by a single rotation and in this new configuration it can perform one of these movements. By adding the above rules to the algorithm, the MRS can start the search.

Table 2: Search algorithm for the MRS of four modules equipped with a common vertical axis

| Movement | $C_{m}$                      | $C_{w}$ | $C_{e}$ | Output |
|----------|------------------------------|---------|---------|--------|
| $M_{Down}$ | $(0, 0, 1), (0, 1, 1), (0, 1, -1)$ | $(0, 0, -2)$ | $(0, 0, -1)$ |        |
|          | $(0, 0, 1), (0, 1, 1), (0, 1, 2)$ | $(0, 0, -1)$ | $(0, 1, -1)$ |        |
|          | $(0, 0, 1), (0, 0, -2), (0, -1, -2)$ | $(0, 0, -3)$ | $(0, -1, -1)$ |        |
|          | $(0, 1, 0), (0, 1, -1), (0, 1, -2)$ | $(1, 0, 0)$ | $(0, 0, -1)$ |        |
| $M_{TurnD}$ | $(0, 0, -1), (0, 0, -2)$ | $(0, 0, -3)$ |         | $(0, -1, -1)$ |
| $M_{Up}$   | $(-1, 0, 1), (0, -1, 1), (0, 0, 1)$ |         | $1, 0, 1$ |        |
|           | $(-1, 0, 0), (-2, 0, 0), (0, -1, 0)$ | $(0, 0, 2)$ | $(-1, 0, 1)$ |        |
|           | $(1, 0, 0), (1, -1, 0), (1, 0, 1)$ | $(0, 0, -1)$ | $(0, 0, 1)$ |        |
|           | $(0, 0, 0), (0, 1, 1), (0, 1, -1)$ | $(0, 0, -1)$ | $(0, 0, 1)$ |        |
| $M_{TurnU}$ | $(0, 1, 0), (1, 1, 0), (1, 1, 1)$ | $(0, 0, 2)$ | $1, 0, 0$ |        |
|           | $(-1, 0, 0), (2, 0, 0), (1, -1, 0)$ | $(0, 0, 2)$ | $1, 0, 1$ |        |
|           | $(1, -1, 0)$ | $(0, 0, 2)$ |        |        |
### Table 2: Search algorithm for the MRS of four modules equipped with a common vertical axis

| Start | Move 1 | Move 2 | Move 3 | Move 4 | Move 5 |
|-------|--------|--------|--------|--------|--------|
| (0, 0, −1), (1, 0, −1), (1, −1, −1) | (0, 0, 1) | (0, 0, 0), (0, 0, −2) | (0, 0, −3), (−1, 0, 0) | (−1, 0, 1), (0, 1, 0) |
| (0, 1, 0), (−1, 0, 0), (−1, 1, 0) | (0, 0, 2) | (0, 0, −3) | (−1, 0, 0) | (−1, 0, 0) |
| M_{B1} | (0, 1, 0), (0, 1, 1), (0, 1, −1) | (0, 1, 0), (0, 0, −2) | (0, 0, 0) | (0, 0, 0) |
| (0, 0, −1), (0, 1, 0), (−1, 1, −1) | (0, 1, 0), (0, 0, −3) | (−1, 0, 0) | (−1, 0, 0) |
| M_{B2} | (0, 1, 0), (−1, 1, 0), (−2, 1, 0) | (0, 0, 0) | (−2, 0, 0) | (−1, 0, 0) |
| (0, 1, 0), (1, 1, 0), (1, −1, 0) | (0, 0, −3) | (−1, 0, 0) | (−1, 0, 0) |
| M_{B3} | (0, 1, 0), (1, 1, 0), (1, 0, 1) | (0, 0, −1), (−2, 0, 0) | (0, 1, 1) |
| (−1, 0, 0), (−2, 0, 0), (−2, −1, 0) | (0, 0, −1) | (−1, −1, 0) |
| M_{Corner} | (0, 0, −1), (0, 0, −2), (−1, 0, −2) | (0, 1, 0), (1, 0, 0), (0, 0, −3) | (−1, 0, 0) |
| (1, 0, 0), (0, 0, −1), (1, 0, −1) | (0, 1, 0), (2, 0, 0), (0, 0, −2) | (0, −1, −1) |
| (0, 0, −1), (−1, 0, −1), (−1, −1, −1) | (0, 1, 0), (0, 1, 0), (0, 0, −2) | (0, 0, −2) |
| Otherwise | Otherwise | Otherwise | (0, 0, 0) |

### 3.3 Search without a common compass

We show the following theorem by a search algorithm for the MRS of five modules not equipped with a common compass.

**Theorem 3** The MRS of five modules not equipped with a common compass can solve a search problem in a finite 3D grid if no pair of modules have an identical observation in an.
We prove Theorem 3 by a search algorithm for the MRS of three modules not equipped with a common compass. The proposed algorithm considers each plane perpendicular to one of the $x$, $y$, and $z$ axis. The choice of the axis depends on the initial configuration of the MRS, and the modules do not need to know the global coordinate system. In the following, without loss of generality, we assume that the MRS considers planes perpendicular to the $x$ axis, i.e., $x = s, y = s, z = s$, respectively, for $s = 0, 1, 2, \ldots$. It moves along each vertical line $\{(x, y, z) | x = s, y = t, z = u\}$ or horizontal line $\{(x, y, z) | x = s, y = u, z = t\}$ for $u = 0, 1, 2, \ldots$ on the plane. Figure 29 shows an execution of the algorithm.

The MRS continues to search each plane perpendicular to the $x$-axis until it reaches the east wall. Then, it changes the search direction from $x^+$ to $x^-$ direction and it starts to search each plane perpendicular to the $x$-axis until it reaches the west wall.

The algorithm description is given in Table 3.3.

**Figure 29: Example of a search with five modular robots**

The proposed algorithm consists of the following move sequences.

- **Move sequence $M_{Forward}$** (Figure 31). The blue module is in cell $(x, y, z)$ at the start. By this move sequence, the green module reaches cell $(x, y, z - 1)$. By repeating $M_{Forward}$ $n$ times, any of the modules visit the cells $(x, y, z - k)(0 \leq k \leq n)$. That is, it visits all the cells of the horizontal line $\{(x, y, z) | x = s, y = t\}$.

- **Move sequence $M_{Back}$** (Figure 33). The blue module is in cell $(x, y, z)$ at the start. By this move sequence, the green module reaches cell $(x, y, z + 1)$. By repeating $M_{Back}$ $n$ times, any of the modules visit the cells $(x, y, z + k)(0 \leq k \leq n)$. That is, it visits all the cells of the horizontal line $\{(x, y, z) | x = s, y = t\}$.

- **Move sequence $M_{TurnB}$** (Figure 32). By this move sequence, the MRS changes its move sequence from $M_{Forward}$ to $M_{Back}$.

- **Move sequence $M_{TurnF}$** (Figure 34). By this move sequence, the MRS changes its move sequence from $M_{Back}$ to $M_{Forward}$.
- Move sequence $M_{Edge}$ (Figure 35). By this move sequence, the MRS changes its move sequence from $M_{Forward}$ to $M_{TurnB}$.

- Move sequence $M_{Corner}$ (Figure 36). By this move sequence, the MRS changes its move sequence from $M_{Forward}$ to $M_{TurnB}$.

The proposed algorithm consists of the following six steps. We use down direction for explanation, but each module does not need to know the down direction.

**Step 1** The MRS repeats $M_{Forward}$, that makes it move to the down direction along a vertical line on a plane \( \{(x,y,z) | x = s\} \) for some \( s \).

**Step 2** When the MRS reaches the bottom wall, it changes the direction to up by $M_{TurnB}$.

**Step 3** The MRS repeats $M_{Back}$, that makes it move in the up direction along a vertical line followed in Step 1.

**Step 4** If the MRS is adjacent to the bottom of the west wall, it moves to plane \( \{(x,y,z) | x = s + 1\} \) by $M_{TurnF}$, and starts searching the new plane by repeating Steps 1, 2, and 3.

**Step 5** When the MRS reaches the top wall or the bottom wall, it moves to the southern row by $M_{Edge}$. Then it repeats Steps 1, 2, and 3 again.

**Step 6** When the MRS reaches a corner of the field, perform $M_{Corner}$ that makes it changes the search direction from the positive \( x \) direction to the negative \( x \) direction.

When the MRS is on a plane \( \{(x,y,z) | x = s\} \) for some \( s \), it visits all the cells on line \( \{(x,y,z) | x = s, y = t\} \) by Steps 1, 2, and 3. Then, it proceeds to a new line \( \{(x,y,z) | x = s, y = t - 1, z = u\} \) by Step 5. By repeating Steps 1, 2, 3, and 5, it visits all cells on plane \( x = s \). By Step 4, it starts searching a new plane \( x = s + 1 \). By repeating Step 1 to 5, it eventually reaches the corner adjacent to the south wall and the east wall. By Step 6, it starts searching west wall. By repeating Step 1 to 6 twice, the MRS visits all cells of the field.

![Figure 30: Example of a search with five modular robots](image)

Depending on the initial configuration, the search may not be possible, and even if it is possible, the MRS cannot start the search directly because some initial configurations does
Figure 31: Move to forward. In each figure, the red modules move. When the blue module is in cell \((x, y, z)\) at the start, after this move sequence, the green model reaches \((x, y, z - 1)\).

Figure 32: Turn to back. In each figure, the red modules move.

Figure 33: Move to back. In each figure, the red modules move. When the blue module is in cell \((x, y, z)\) at the start, after this move sequence, the green model reaches \((x, y, z + 1)\).

Figure 34: Turn to forward. In each figure, the red modules move.
Figure 35: Move on the edge. In each figure, the red modules move.

Figure 36: Move on the corner. In each figure, the red modules move.

Figure 37: Configurations of the MRS of five modules not equipped with a common compass
not satisfy the conditions in Table 3.3. Figure 37 shows all possible configurations for five modules not equipped with a common compass. The configurations containing the coloured modules are symmetric, i.e., in each figure the modules painted by the same color may have an identical observation and move symmetrically. The MRS cannot start the search because it cannot move from their positions. Therefore, if there are pairs of modules with the same observation in an initial configuration, the search is not possible.

We show that the search is possible when all modules have different observations in the initial configuration. Figure 38 shows that the configurations $S^5_L, S^5_S, S^5_U$ can transform into $S^5_P$, $S^5_W, S^5_X, S^5_Z$ can transform into $S^5_F$, $S^5_M$ can transform into $S^5_B$, and the other configurations can transform into $S^5_Q$. Even if some walls prevents these transformations, the MRS can leave the wall because its state is asymmetric. Then, any configuration can be transformed to $S^5_Q$. If the MRS satisfies the condition of the fourth or fifth movement of Table 3.3 the MRS can start the search directly. Otherwise, the MRS cannot move because it touches a wall. In this case, as shown in Figure 39 the MRS in $S^5_Q$ on a wall can change its direction by a single sliding and in this new configuration it can perform one of these movements. By adding the above rules to the algorithm, the MRS can start the search.

![Figure 38: Transition graph of the MRS of five modules not equipped with a common compass](image1)

![Figure 39: Transformation of $S^5_Q$](image2)

Table 3: Search algorithm for the MRS of five modules not equipped with a common compass

| $M_{Forward}$ | $C_{in}$ | $C_{w}$ | $C_{c}$ | Output |
|---------------|----------|---------|---------|--------|
|               | $(0, 0, 1), (0, -1, 1)$, $(0, 0, 2), (0, 0, 3)$ | $(0, 0, 4)$ | $(-1, 0, 1)$ |
|               | $(0, 1, 0), (0, 1, -1)$, $(0, 1, 1), (0, 1, 2)$ | $(0, 0, 3)$ | $(0, 1)$ |
|               | $(0, 0, -1), (0, 0, -2)$, $(0, -1, -2), (0, 0, -3)$ | $(0, 0, 1)$ | $(-1, 0, -1)$ |
|               | $(0, 1, 0), (-1, 1, 0)$, $(0, 1, -1), (-1, 1, -1)$ | $(0, 0, 1)$ | $(0, 1, 1)$ |
|               | $(1, 0, 0), (1, -1, 0)$, $(0, 0, -1), (1, 0, -1)$ | $(0, 0, 2)$ | $(0, -1, 1)$ |
|               | $(0, 0, 0), (1, 1, 0)$, $(1, 1, 1), (1, 1, -1)$ | $(0, 0, 2)$ | $(0, 1, 1)$ |
|               | $(1, 0, 0), (0, 0, -1)$, $(0, 0, 0), (1, 1, 0)$ | $(0, 0, 1)$ | $(1, 0, 1)$ |
Table 3: Search algorithm for the MRS of five modules not equipped with a common compass

| $M_{TurnB}$          | (1, 0, −1), (1, 0, −2) | (1, 0, 0), (0, 0, 1), (1, 0, −1), (0, 0, 2), (1, −1, 0) |
|----------------------|-------------------------|----------------------------------------------------------|
|                      | (0, 1, 0), (0, 1, 1), (0, 1, −1), (0, 1, 2)             | (0, 0, 3), (1, 0, 0), (−1, 1, 0)                           |
|                      | (0, 0, −1), (0, 0, −2) | (0, 0, 1), (1, 0, 0), (0, −1, −1)                         |
| $M_{Back}$           | (0, 0, −1), (−1, 0, −1), (0, 0, −2), (1, 0, −2)        | (0, 0, 1), (0, 0, −3), (0, −1, −1)                        |
|                      | (1, 0, 0), (1, 0, 1), (1, 0, −1), (2, 0, −1)           | (0, 0, 2), (0, 0, −1)                                    |
|                      | (−1, 0, 0), (−1, 0, 1), (−2, 0, 1), (−1, 0, 2)        | (0, 0, 3), (0, 0, −1), (−1, 0, −1)                        |
|                      | (0, 1, 0), (0, 1, −1), (−1, 1, −1), (0, 1, −2)        | (0, 0, −3), (0, 0, −1)                                   |
|                      | (0, 0, 1), (−1, 0, 1), (0, 0, 2), (0, −1, 2)          | (0, 0, −1), (−1, 0, 0)                                   |
|                      | (0, 0, 1), (1, 0, 0), (1, −1, 1), (1, 0, 2)           | (0, 0, −1), (2, 0, 0), (1, 0, 0)                          |
|                      | (0, 0, −1), (−1, 0, −1), (0, −1, −1), (−1, 0, −2)    | (−1, 0, 0)                                               |
|                      | (0, 0, −1), (1, 0, −1), (1, −1, −1), (1, 0, −2)      | (1, 0, 0)                                               |
|                      | (0, 1, 0), (−1, 1, 0), (−1, 1, 1), (0, 1, −1)         | (0, 0, −2), (0, 0, −1)                                  |
|                      | (0, 1, 0), (0, 1, 1), (−1, 1, 1), (0, 1, 2)           | (0, 0, −1), (0, 0, 3), (1, 1, 0)                          |
| $M_{TurnF}$          | (0, 1, 0), (0, 1, −1), (−1, 1, −1), (0, 1, −2)        | (0, 0, −3), (−1, 1, 0)                                  |
|                      | (1, 0, 0), (1, 0, 1), (1, −1, 1), (1, 0, −1)          | (0, 0, −2), (0, 0, −1)                                  |
|                      | (1, 0, 0), (1, 0, 1), (1, 0, 2), (0, 0, 2)            | (0, 0, −1), (1, −1, 0)                                  |
|                      | (0, 1, 0), (0, 1, 1), (0, 1, 2), (−1, 1, 2)           | (0, 0, −1), (1, 1, 0)                                   |
|                      | (−1, 0, 0), (−1, 0, 1), (−1, 0, 2), (−2, 0, 2)        | (0, 0, −1), (0, 0, 1)                                   |
|                      | (0, 0, 1), (1, 0, 1), (0, 0, 2), (−1, 0, 2)           | (0, 0, −1), (1, 0, 0)                                   |
|                      | (1, 0, 0), (1, 0, −1), (2, 0, −1), (1, 0, −2)        | (0, 0, −3), (1, −1, 0)                                  |
|                      | (0, 1, 0), (0, 1, −1), (1, 1, −1), (1, 1, −2)        | (0, 0, −3), (1, 1, 0)                                   |
| $M_{Edge}$           | (0, 0, 1), (0, −1, 1), (0, 0, 2), (0, 0, 3)           | (0, 0, 4), (1, 0, 0), (−1, 0, 1)                          |
|                      | (0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 1, −1)           | (0, 0, 3), (1, 0, 0), (0, 0, 1)                           |
|                      | (0, 0, −1), (0, 0, −2), (0, −1, −2), (0, 0, −3)      | (0, 0, 1), (1, 0, 0), (−1, 0, −1)                        |
Table 3: Search algorithm for the MRS of five modules not equipped with a common compass

| \( (0, 1, 0), (1, 1, 0), (1, 1, 1), (1, 1, -1) \) | \( (2, 0, 0), (0, 0, 2) \) | \( (0, 2, 0) \) | \( (0, 1, 1) \) |
| \( (1, 0, 0), (0, 0, -1), (1, 0, -1), (1, 0, -2) \) | \( (0, 0, 1), (2, 0, 0) \) | \( (0, 1, -1) \) |
| \( (0, 0, 1), (-1, 0, 1), (0, 0, 2), (-1, 0, 2) \) | \( (0, 0, 3), (1, 0, 0) \) | \( (0, 1, 1) \) |
| \( (0, 0, -1), (0, 1, -1), (-1, 0, -1), (-1, 1, -1) \) | \( (0, 0, 1), (1, 0, 0) \) | \( (0, 1, 0) \) |
| \( (1, 0, 0), (0, 1, 0), (1, 1, 0), (1, 0, 1) \) | \( (0, 0, 2), (2, 0, 0) \) | \( (-1, 1, 0) \) |
| \( (0, 1, 0), (-1, 1, 0), (-2, 1, 0), (0, 1, 1) \) | \( (0, 0, 2), (1, 0, 0) \) | \( (-1, 0, 0) \) |
| \( (0, 1, 0), (1, 1, 0), (-1, 1, 0), (1, 1, 1) \) | \( (0, 0, 2), (2, 0, 0) \) | \( (-1, 0, 0) \) |
| \( (0, 0, -1), (-1, 0, -1), (-2, 0, -1), (-1, -1, -1) \) | \( (0, 0, 1), (1, 0, 0) \) | \( (-1, 0, 0) \) |
| \( (0, 0, -1), (1, 0, -1), (-1, 0, -1), (-1, -1, -1) \) | \( (0, 0, 1), (2, 0, 0) \) | \( (-1, 0, 0) \) |
| \( (0, 0, -1), (1, 0, -1), (2, 0, -1), (-1, -1, -1) \) | \( (0, 0, 1), (3, 0, 0) \) | \( (-1, 0, -1) \) |

\( M_{\text{Corner}} \)

| \( (0, 0, -1), (-1, 0, -1), (-1, -1, -1), (0, -2) \) | \( (0, 0, 1), (1, 0, 0), (0, 1, 0) \) | \( (-1, 0, 0) \) |
| \( (0, 0, 1), (-1, 0, 1), (-1, -1, 0), (0, 0, 2) \) | \( (0, 0, 3), (1, 0, 0), (0, 1, 0) \) | \( (-1, 0, 0) \) |
| \( (0, 0, 1), (1, 0, 1), (0, -1, 0), (0, 0, 2) \) | \( (0, 0, 3), (2, 0, 0), (0, 1, 0) \) | \( (0, -1, 0) \) |
| \( (-1, 0, 0), (-1, -1, 0), (-1, 0, -1), (0, -1, 0) \) | \( (0, 0, 2), (1, 0, 0), (0, 1, 0) \) | \( (0, -1, 0) \) |
| \( (0, 0, -1), (1, 0, -1), (0, -1, 0), (0, 0, 2) \) | \( (0, 0, 1), (2, 0, 0), (0, 1, 0) \) | \( (0, -1, 0) \) |
| \( (-1, 0, 0), (-1, -1, 0), (-1, 0, -1), (0, -2, 0) \) | \( (0, 0, 2), (1, 0, 0), (0, -2, 0) \) | \( (0, 0, 1) \) |
| \( (-1, 0, 0), (-1, 1, 0), (-1, 0, 1), (0, 2, 0) \) | \( (0, 0, 2), (1, 0, 0), (0, 2, 0) \) | \( (0, 0, 1) \) |

Otherwise Otherwise Otherwise \( (0, 0, 0) \)

4 Necessary number of modules

We show that the three algorithms presented in Section 3.1 uses the minimum number of modules for each settings.

**Theorem 4** The MRS of less than three modules equipped with a common compass cannot solve the search problem in a finite 3D cubic grid.

**Proof.** In the case of one module, the MRS cannot move because there is no other static module during sliding or rotation.
In the case of two modules, we show that there is a cell which cannot be visited by the MRS. Doi et al. showed that in the 2D square grid two modules equipped with a common compass can move straight to one of the eight directions (north, south, east, west, northeast, northwest, southeast, and southwest) when they observe no wall [7]. By the same discussion, the MRS of two modules equipped with a common compass in 3D cubic grid can move straight to one direction, and the possible moving directions are eight diagonal directions in addition to those eight directions on planes perpendicular to $x$, $y$, or $z$ axis when they can observe no wall. Figure 40 shows an example of a 3D diagonal move.

Without loss of generality, we assume that the MRS continues to move in positive $x$ direction in the area where the two modules observe no wall. If it enters the area from other walls than the west wall, it cannot visit the cells with smaller $x$ coordinates on the straight moving track. Hence, the MRS must start the straight move when some modules can observe the west wall.

Assume that the MRS can visit all cells of the field. There exists a subsequence of configurations where the two modules observes only the west wall. We focus on the sequence of states of the MRS. Let $T_W = S_1, S_2, \ldots, S_\ell$ be a sequence of states of the MRS in the area where the MRS observes only west wall. $T_W$ consists of different states, otherwise the MRS cannot leave the west wall. Additionally, in the transformation from $S_\ell$ to the next state, say $S_{\ell+1}$, the MRS enters the area where it observes no wall. As shown in Figure 41 when we fix the bottom westmost module, there are three states of the MRS. The number of states where the modules observe only the west wall is at most $3k$. Thus, the length of $T_W$ is at most $3k$. In addition, the maximum distance that the MRS can move by $T_W$ in the $y$ direction or the $z$ direction is $3k$, since the maximum number of coordinates that it can travel by one transition is one in each direction.

Since the MRS cannot move in the negative $x$ direction from the area where the modules cannot observe the west wall, the only way for the MRS to enter the area where the modules can observe the west wall is to move from the area where the modules observe two or more walls. Thus, the transformation sequence $T_W$ always starts around the edge of the field that touches the west wall. However, since the maximum distance that the MRS can travel by $T_W$ is $3k$, it cannot reach the coordinates $(a, b, c)$ such that $(0 \leq a \leq k, 3k + k < b < d - (3k + k), 3k + k < c < h - (3k + k))$. Hence, it is impossible for the two modules to visit all cells.

We next show the necessary number of modules equipped with a common vertical axis.

**Theorem 5** The MRS of less than four modules equipped with a common vertical axis cannot...
solve the search problem in a finite 3D cubic grid.

Proof.

In the case of one module, the MRS cannot move because it cannot perform any sliding or rotation.

In the case of two modules, there are two possible states of the MRS. Let $S_A$ be the state, where the two modules form a vertical line, and $S_B$ be the state, where the two modules form a horizontal line. There exists only one horizontal line state because the modules do not agree on $x$ axis or $y$ axis. In $S_A$, one of the two modules can perform a rotation because they agree on a common vertical axis. Any rotation in $S_A$ results in $S_B$. In $S_B$, both modules obtain the same observation if their local coordinate systems are symmetric against their midpoint, and if one of them moves then the other also moves. Thus, the two modules cannot perform any movement. Consequently, the two modules cannot move to any direction.

In the case of three modules, we check possible movements of the MRS by the state transition graph shown in Figure 42. State $S_3^a$ cannot be transformed to any other configuration because both endpoint modules get the same observation. Therefore, it is necessary to move only in the $S_3^b, S_3^c, S_3^d,$ and $S_3^e$. However, no matter which transformation of the $S_3^b, S_3^c, S_3^d,$ and $S_3^e$, the MRS cannot move in the east, west, south or south direction. Therefore, when there is no wall in the vicinity, it becomes impossible to move, and it is impossible to search the whole field.

We finally show the necessary number of modules not equipped with a common compass.

**Theorem 6** The MRS of less than five modules not equipped with a common compass cannot solve the search problem in a finite 3D cubic grid.

Proof.

In the case of one module, the MRS cannot move because it cannot perform any sliding or rotation.

In the case of two modules, the MRS can take one state, where the two modules obtain the same observation if their local coordinate systems are symmetric against their midpoint. Thus, if one of them moves then the other also moves and the two modules cannot perform any movement.

In the case of three modules, there are two possible states of the MRS, i.e., the L-shape shown in the middle of the Figure 42 and the I-shape, i.e., a line. In the L-shape, the two endpoint modules obtain the same observation if their local coordinate systems are symmetric against the center module. The center module cannot perform any movement due to connectivity. The two endpoint modules cannot perform a sliding because they move into the same cell and there is no static modules during a sliding. If the two modules perform a rotation, the resulting state is the L-shape or the I-shape. In the I-shape, the two endpoint modules obtain the same observation in the worst case. They can only perform a rotation,
Figure 43: State transition graph for the MRS of 4 modules not equipped with a common compass

and a resulting configuration is the L-shape or the I-shape. In any movement in the L-shape and the I-shape, the center module does not move and the MRS does not move to any direction. Consequently, the MRS cannot move to any direction.

In the case of four modules, the state transition graph is shown in Figure 43. In each state, modules that may have the same observation is painted in red, and each arrow shows that its starting point state can be transformed to its end point in one time step. The MRS cannot change its state in $S_4^O$ and $S_4^K$. In $S_4^O$, the four modules have the same observation, and if one of them moves, the other also move. Thus, they cannot keep a static module during movement. Also in $S_4^K$, the three red module have the same observation, and if one of them moves, the others also move. It cause a conflict of their movement paths or transformation into $S_4^K$. Thus, they can transform only into $S_4^K$.

Next, we can find $S_4^S$, $S_4^Z$, and $S_4^I$ form a loop. However, the MRS cannot move to any direction by repeating these transitions. The MRS enters this loop from $S_4^N$, thus it cannot move to any direction from $S_4^N$. We also find $S_4^T$ and $S_4^L$ forms a loop and $S_4^L$ has arrows to $S_4^O$, $S_4^K$, $S_4^T$, and $S_4^N$. Thus, the MRS cannot move to any direction from $S_4^T$ and $S_4^L$. Therefore, there is no way for MRS to move.

5 Conclusion and future work

In this paper, we considered search by the single MRS in the finite 3D cubic grid. We demonstrated a trade-off between the common knowledge and the necessary and sufficient number of modules for search. When the modules are equipped with a common compass, three modules are necessary and sufficient. When the modules are not equipped with a common compass, five modules are necessary and sufficient. As an intermediate case, when the modules are equipped with a common vertical axis, four modules are necessary and sufficient. We finally note that the proposed algorithms depend on parallel movements, i.e., they are not designed for the centralized scheduler.

Our future goal is a distributed coordination theory for the MRS. First, reconfiguration and locomotion of a single MRS in the 3D cubic grid have not been discussed yet. Second, it is important to consider interaction among multiple MRSs such as rendezvous, collision avoidance, and collective search. Finally, the MRS is expected to solve more complicated tasks by interaction with the environment.
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