Localization counteracts decoherence in noisy Floquet topological chains

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The topological phases of periodically-driven, or Floquet systems, rely on a perfectly periodic modulation of system parameters in time. Even the smallest deviation from periodicity leads to decoherence, causing the boundary (end) states to leak into the system’s bulk. Here, we show that in one dimension this decay of topologically protected end states depends fundamentally on the nature of the bulk states: a dispersive bulk results in an exponential decay, while a localized bulk slows the decay down to a diffusive process. The localization can be due to disorder, which remarkably counteracts decoherence even when it breaks the symmetry responsible for the topological protection. We derive this result analytically, using a novel, discrete-time Floquet-Lindblad formalism and confirm our findings with the help of numerical simulations. Our results are particularly relevant for experiments, where disorder can be tailored to protect Floquet topological phases from decoherence.

Introduction.—In recent years, Floquet systems have been established as a novel paradigm of physics hosting a plethora of unique and fascinating phenomena. Periodic modulation of a quantum system’s Hamiltonian is not only a powerful tool to engineer exotic, effectively static models, but even more intriguingly it is a pathway to realize novel phases of matter that do not have time-independent counterparts. Experimental studies of such phases raise a fundamental question: How robust are the new phenomena against unavoidable imperfections in the implementation of the periodic driving? Here, we address this question for the specific case of Floquet topological phases [1–21]. The unique feature of these phases is that they have topologically protected boundary states even if the bulk bands have vanishing topological invariants [4, 7] — which is impossible in a static system. We ask, then, what the fate of these boundary states is in the presence of imperfect or noisy driving.

Indeed, in any realistic experimental scenario, it is impossible to achieve exact invariance under discrete time translations and thus, to conserve quasienegy. A case in point are realizations of anomalous Floquet topological insulators with photonic waveguides [20, 21], where the propagation of light along the waveguide direction emulates time evolution [22, 23], and deviations from periodicity appear due to fabrication defects. The latter cause the simultaneous presence of both static (or quenched) disorder as well as temporal randomness, or noise. However, their combined effect on the topological boundary states of Floquet systems has to our knowledge never been addressed.

We study this problem for a Floquet topological system defined on a 1D ladder and subject to a piecewise-constant drive (see Fig. 1). The drive consists of a number of steps during which the Hamiltonian is held constant. We use a simple but generic model for deviations from periodicity, namely a random and uncorrelated-in-time duration of each driving step. Formally, this amounts to timing noise in the piecewise-constant Hamiltonian of the system. We focus on how imperfections in the periodic driving lead to the decay of the topologically protected end state. While a system initialized in a
Floquet eigenstate and evolved with such a noisy Hamiltonian will always lose its memory of the initial state, we show that the rate of decay crucially depends on the nature of the bulk. If the bulk is dispersive the end state decays exponentially in time. For localized bulk states it fades out diffusively. We prove this by deriving a discrete-time Floquet-Lindblad equation that describes the noisy time evolution of the system density matrix. We first apply it to a clean system that can exhibit both a dispersive and localized bulk, and then compare our results to numerical simulations of a system in which the bulk is localized by disorder.

Model.—We consider a model of spinless fermions hopping on a 1D ladder with sites labeled by \((j, s)\), where \(j\) denotes a doublet of sites and \(s\) is the sublattice index (see Fig. 1). The system is subject to a periodic drive with frequency \(\omega\) and period \(T = 2\pi/\omega\), which consists of four steps of equal duration \(T/4\). In each step, only hopping across certain bonds is allowed, as shown in Fig. 1. The piecewise-constant Hamiltonian reads

\[
H(t) = -\sum_{\mu, \nu} J_{\mu \nu}(t) \left( c_{\mu}^\dagger c_{\nu} + \text{h.c.} \right),
\]

where \(c_{\mu}^{(t)}\) are fermionic annihilation (creation) operators and \(\mu = (j, s)\). The hopping amplitudes are constant during each step \(i = 1, \ldots, 4\)

\[
J_{\mu \nu}(t) = J_{\mu \nu}^i \quad \text{for} \quad (i - 1)T/4 \leq t < iT/4,
\]

where \(J_{\mu \nu}^i = J\) for the active bonds, and zero otherwise. The active bonds in steps 1 and 3 are the rungs of the ladder, connecting sites \((j, s)\) to \((j - s, -s)\). In step 2, sites \((j, s)\) link to \((j - 2s, -s)\) (across doublets) and in step 4 to \((j, -s)\) (within doublets). This driving protocol is illustrated in Fig. 1. Later we will introduce timing noise to this model by varying the duration of each step.

We treat the system within the framework of Floquet theory. The time evolution over one full driving cycle is described by the Floquet operator, \(U_F = T \exp \left( -i \int_0^T H(t) dt \right)\), where \(T\) denotes time-ordering and we set \(\hbar = 1\) throughout this work. Repeated application of \(U_F\) yields the stroboscopic evolution at multiples of the period \(T\). As such, by diagonalizing \(U_F |\psi_\alpha\rangle = e^{-i\epsilon_\alpha T} |\psi_\alpha\rangle\) we can describe the system in terms of its Floquet eigenstates \(|\psi_\alpha\rangle\) and their associated quasienergies \(\epsilon_\alpha\). For the piecewise-constant Hamiltonian in Eq. (1), the Floquet operator factorizes as \(U_F = U_4 U_3 U_2 U_1\), where [24]

\[
U_i = P_i (\cos(\phi) + i \sin(\phi) H_i/J) P_i + Q_i.
\]

Here, the phase \(\phi =JT/4\) and \(H_i\) denotes the Hamiltonian in step \(i\), i.e., \(H_i = -\sum_{\mu \nu} J_{\mu \nu}^i \left( c_{\mu}^\dagger c_{\nu} + \text{h.c.} \right)\). The operators \(P_i\) are projection operators onto all lattice sites appearing in \(H_i\) and \(Q_i = 1 - P_i\). Note that for \(i = 1, 3\) all lattice sites are involved and thus \(P_i = 1\), whereas in step 2 the lattice sites \((1, +)\) and \((L - 1, -)\) are excluded and, similarly, in step 4 the sites \((0, -), (L, +)\) are excluded.

The properties of Floquet eigenstates depend crucially on the value of the phase \(\phi\). A special case occurs at the resonant driving point \(\phi = \pi/2\), where the otherwise dispersive bulk is fine-tuned to form a localized flat band: in every step of the driving protocol, a particle is fully transferred from one lattice site to another, acquiring a phase of \(\pi/2\). As such, particles initialized in the Floquet eigenstates \(|j, \pm\rangle = c_{j, \pm}^\dagger |0\rangle\) encircle the \(j\)-th plaquette in opposite directions for \(s = \pm\) during each period. In one period, each state inside the chain accumulates a phase of 2\(\pi\) resulting in a flat bulk-band at quasienergy \(\varepsilon = 0\). At the ends of the chain, however, there are two states which have no doublet partners, \(|0, -\rangle\) and \(|L, +\rangle\), and therefore skip two steps, resulting in an accumulated phase of \(\pi\) and a corresponding quasienergy of \(\varepsilon = \omega/2\).

Away from resonant driving, i.e., \(\phi = \pi/2 + \delta\phi\), the bulk band becomes dispersive with a bandwidth \(\alpha \delta\phi\) and the bulk states \(|\beta\rangle\) are delocalized over the entire chain. The left and right end states \(|\epsilon_{l/r}\rangle\) at quasienergy \(\omega/2\) are topologically protected and remain exponentially localized on the boundaries, with the ladder’s sublattice symmetry responsible for their protection. If it is broken, for instance by an onsite potential, the end states can shift away from \(\omega/2\) and be pushed into the system’s bulk [24]. Experimentally fine-tuning the system to resonant driving is unrealistic [20, 21], suggesting the dispersive bulk as the generic case. The point of resonant driving still provides a natural example of a localized bulk in our model and a good starting point to study the effect of noise in the periodic drive.

Timing noise and Floquet-Lindblad equation.—We introduce deviations from the perfectly-periodic time dependence of \(H(t)\), Eq. (1), by including timing noise into the driving protocol. The duration of step \(i\) in the \(n\)-th cycle is thus not exactly \(T/4\), but rather \(T/4 + \tau_{ni}\). We take \(\tau_{ni}\) to be random numbers from a Gaussian distribution with zero mean, \(\tau_{ni} = 0\), and fluctuations \(\tau_{ni+1} \sim \tau^2 \delta_{ni} \delta_{n'i'}\), i.e., the deviations are uncorrelated between different steps and driving cycles. Further, we consider the noise strength \(\tau\) to be weak, \(\tau \ll T\).

The noise-averaged effective description of the system’s dynamics is captured in a Lindblad-like equation. To derive the latter, we first note that the expectation value of any observable \(O\) can be written as \(\langle O \rangle_n = \langle \psi_n | O | \psi_n \rangle = \text{tr} (O \rho_n)\), with the density matrix \(\rho_n = |\psi_n\rangle \langle \psi_n|\) at time \(t = nT\), an initial condition \(\rho_0 = |\psi_0\rangle \langle \psi_0|\), and \(\text{tr}\) denoting the noise average. As the timing noise is uncorrelated for different cycles \(n\), we can derive a stroboscopic evolution equation for the density matrix considering the evolution over one cycle \(\rho_{n+1} = U_{F,n+1} \rho_n U_{F,n+1}^\dagger\). To this end we introduce a “noisy Floquet operator” \(U_{F,n} = U_{4,n} U_{3,n} U_{2,n} U_{1,n}\), with the \(U_{i,n}\) as defined by Eq. (3) with \(\phi \rightarrow \phi + J \tau_{ni}\). To second order in the noise
strength $\tau$, the evolution equation takes the form of a discrete-time Floquet-Lindblad equation [24]:
\[ \rho_{n+1} = U_F \left( \rho_n + \tau \sum_i \mathcal{D}[L_i] \rho_n \right) U_F^\dagger. \] (4)

$\mathcal{D}[L] \rho = L \rho L - \frac{1}{2} \{ L^2, \rho \}$ is the so-called dissipator, and the self-adjoint quantum jump operators are given by
\[ L_1 = H_I, \quad L_2 = U_1^\dagger H_2 U_1, \quad L_3 = U_1^\dagger U_2^\dagger H_3 U_2 U_1, \quad \text{and} \quad L_4 = U_1^\dagger U_2^\dagger U_3^\dagger H_4 U_3 U_2 U_1. \]

As in the usual continuous-time Lindblad equation (see, e.g., [25]), dissipation originates from perturbatively eliminating fluctuations that are uncoupled on the characteristic time scale of the system dynamics — the main difference being that usually these fluctuations correspond to additional quantum degrees of freedom, while here they are induced by noise. The coherent evolution due to $U_F$ conserves the occupations of Floquet eigenstates, while dissipation leads to transitions. Here, we study the noise-induced decay of the occupation of, say, the end state $|e\rangle$, $\rho_n^e = \langle e | \rho_n | e \rangle$, starting from the pure state $\rho_0 = |e\rangle \langle e|$. We study the system’s dynamics close to resonant driving $\phi = \pi/2 + \delta\phi$ with $\delta\phi \ll \pi/2$. There, the matrix elements of the operators $L_i$ on the end states take the particularly simple form $\langle e | L_i | e \rangle = 0$ and $\langle e | L_i^2 | e \rangle = J^2$ for $i = 1, 3$ and 0 for $i = 2, 4$ [24]. We thus find
\[ \rho_{n+1}^e = (1 - 2J^2\tau^2) \rho_n^e + \tau^2 \sum_i \sum_{bb'} \langle e | L_i | b \rangle \langle b' | L_i | e \rangle \rho_n^{bb'}, \] (5)

where $\rho_n^{bb'} = \langle b | \rho_n | b' \rangle$. In the following, we discuss separately the limit of approaching resonant driving $\delta\phi \to 0$ and resonant driving $\delta\phi = 0$ itself, which exhibit fundamentally different behaviors.

**Exponential Decay.**—In the more generic case of $\delta\phi \to 0$ the off-diagonal terms in the density matrix oscillate strongly, thus reducing the dynamics to an exponential decay. To see this, we note that the bulk states are dispersive with quasienergies that are much smaller than the driving frequency $\epsilon_b \ll \omega$, but, crucially, nonzero. The end state is exponentially localized, while the bulk states are spread out over its whole length. Thus, the matrix elements of the quasilocal operators $L_i$ connecting end state and bulk decay for larger system sizes: $\langle e | L_i | b \rangle \sim J/\sqrt{L}$. Further, due to the noise the off-diagonal terms of the bulk’s density matrix $\rho_n^{bb'}$ are populated with a rate $\sim J^2\tau^2/L$ per period, but acquire phases from the coherent time evolution at a rate proportional to the energy differences $(\epsilon_b - \epsilon_e) \sim \delta\phi$. For system sizes $L \gg J^2\tau^2/\delta\phi$ these off-diagonal terms dephase faster than they are populated, and we can approximate the density matrix to be diagonal at long times. Assuming that the bulk states are populated evenly from excitations out of the end state, the diagonals are $\rho_n^{bb} = O(1/L)$. The sum over bulk states in Eq. (5) thus scales with the inverse of the system size and can be neglected for large enough systems, resulting in an exponential decay of the end state with a rate $2J^2\tau^2$:
\[ \rho_{n+1}^e = (1 - 2J^2\tau^2) \rho_n^e + O(1/L). \] (6)

A qualitatively similar result has been found in a work considering decoherence in quantum walks [26].

Heuristically, the finite bandwidth of the bulk band allows excitations from the end state into the bulk to move away from the edge fast enough to not lead to blocking or even backflow. This is the origin of the fast (exponential) decay. Conversely, for a localized bulk, we expect a qualitatively different behavior. This can be achieved by tuning close enough to resonant driving, i.e., $J^2\tau^2/\delta\phi \gg L$, which we give as a bound on tolerable deviations from resonant driving in experiments. Alternatively and more realistically, the bulk can be localized by disorder. We show in the following how both scenarios lead to diffusive decay of the end-state occupation.

**Diffusive Decay.**—At resonant driving, the Floquet-Lindblad equation can be analyzed exactly and simplifies into a diffusion equation at long times. We start by noting that due to the full degeneracy of the bulk band, the bulk basis is not fixed. In particular, we can choose basis states that are spread over the system, for which no phase factors are accumulated for the off-diagonal terms $\langle b | \rho_n | b \rangle$ and $\rho_n^{bb} = O(1/L)$ for all of the $O(L^2)$ elements at any time. Thus, the sum in Eq. (5) is of order one and has to be taken into account more carefully.

Choosing the real-space basis $|b\rangle = |j, s\rangle$, the calculation simplifies significantly and we can treat the evolution equation exactly for any state in the ladder. Now, the off-diagonal elements of $\rho_n$ are not populated from the diagonal [24] and we disregard them throughout the calculation since we are interested in a diagonal initial state $\rho_0 = |e\rangle \langle e|$. With this we find the diagonal elements $\rho_n^{j,s} = \langle j, s | \rho_n | j, s \rangle$ to evolve under the noisy periodic driving as
\[ \rho_{n+1}^{j,s} = (1 - 4J^2\tau^2) \rho_n^{j,s} + J^2\tau^2 \left( \rho_n^{j-s,-s} + 2\rho_n^{j,-s} + \rho_n^{j+s,-s} \right). \] (7)

The total occupation of the doublet $j$ is given by the sum $\rho_n^j = \rho_n^{j,+} + \rho_n^{j,-}$. It obeys a discrete diffusion equation, while the relative occupation of the two sublattice sites, $\sigma_n^j = \rho_n^{j,+} - \rho_n^{j,-}$, is exponentially suppressed:
\[ D_t \rho_n^j = \Delta D^2 z \rho_n^j, \] (8)
\[ D_t \sigma_n^j = \left( -8J^2\tau^2 - \frac{J^2\tau^2}{T} D_z^2 z \right) \sigma_n^j. \] (9)

The diffusion coefficient is $\Delta = J^2\tau^2a^2/T$ and we introduced the discrete time derivative $D_t f_n = (f_{n+1} - f_n)/T$ and second order spatial derivative $D_z^2 f = (f_{j+1} - 2f_j + f_{j-1})/a^2$ with lattice constant $a$. While this diffusion
equation describes the dynamics of an arbitrary bulk state $|j, s\rangle$, we emphasize that in this case bulk and end states can be treated on an equal footing using the lattice-site basis. Thus, the wave function of a particle initialized in an end state $|e\rangle = |0, -\rangle$ spreads diffusively into the bulk, assuming a Gaussian profile of width $\sim \sqrt{n}$. 

Disorder — While the drastic slow down of the end state decay seems out of reach at a fine-tuned point in parameter space, its cause is ultimately a localized bulk band. This suggests that adding disorder should have a similar effect and in the following we argue that it indeed results in a diffusive process, effectively protecting the end state against decay. We consider disorder in either the hopping matrix elements or the on-site potential. To add hopping disorder we allow for site-dependent fluctuations on the hopping terms, $J_{\mu\nu}^l + \delta J_{\mu\nu}$. For the onsite disorder we add a random chemical potential with strength $v_\mu$ to the Hamiltonian. We take $\delta J_{\mu\nu}$ and $v_\mu$ to be uncorrelated and uniformly distributed in the interval $[-V, V]$, with the disorder strength $V$. In the presence of disorder, the Floquet eigenstates $|l\rangle$ are localized on a length scale $\xi$. They are labeled according to their center of mass along the 1D chain, such that $|l\rangle$ can be thought of as being the left nearest neighbor of $|l+1\rangle$. As observed above, the off-diagonal density-matrix elements play typically no role in the long-time behavior of the system. Therefore, Eq. (4) can again be reduced to a classical master equation [24],

$$\rho_{n+1}^l = \rho_n^l + \sum_{l' \neq l} W_{l'\rightarrow l} \left( \rho_n^{l'} - \rho_n^l \right),$$

(10)

where $\rho_n^l = \langle l | \rho_n | l \rangle$ and $W_{l'\rightarrow l} = \tau^2 \sum_i |\langle l' | L_i | l \rangle|^2$. The operators $L_i$ are defined as in Eq. (4). Since these operators are quasilocal, the transition probabilities $W_{l'\rightarrow l}$ only connect nearby states. Thus, the time evolution of the density matrix is reduced to a random walk with disordered transition probabilities, and as such must be diffusive [27]. Note that this behavior only requires that the bulk states are localized, independently of the type of disorder. As such, on-site disorder will also lead to an increased robustness of the end state, even though it breaks the sublattice symmetry required for its protection. For small strengths of the chemical potential disorder $V$, the end modes will shift from the quasiequilibrium $\omega/2$ by an amount $\sim V$, but will still decay diffusively.

Numerics — To corroborate our analytical results, we numerically integrated the Schrödinger equation with the driven and noisy Hamiltonian (1) for a ladder with 200 rungs. Figure 2 shows the survival probability of the end state $s = |\langle e | \psi_n \rangle|^2$ (averaged over noise realizations) for a single particle initialized at the edge, $|\psi_0\rangle = |e\rangle$. As expected, we find an exponential decay of the end state amplitude at a rate $2J^2\tau^2$ [cf. Eq. (6)] for a dispersive bulk, while localized bulk states lead to a much slower, diffusive decay — irrespective of whether localization is due to fine-tuning to resonant driving in a clean system or disorder. Note that numerically, we have studied both hopping and on-site disorder [24].

Conclusions — We have studied the combined effect of quenched disorder and noise in a 1D Floquet topological phase. We have shown analytically and numerically that a dispersive bulk causes an exponential decay of the end mode due to noise, while a localized bulk is associated with a qualitatively slower, diffusive decay. The formalism we have developed provides a comprehensive framework to address timing noise in Floquet systems with piecewise constant driving, including periodically “kicked” (non-topological) systems which have been the focus of recent experimental [28–30] and numerical [31] studies. For future work it will serve as a ground to extend the Lindblad-Floquet formalism to smooth drivings with correlated noise, or study the combined effect of space and time randomness in higher-dimensional topological phases.

Relying only on localization and not on systems details, our results hold for many 1D topological phases, such as the SSH [32] or the Kitaev [33] chain, and are highly relevant to realizations of Floquet topological phases [1, 3, 18–21]. In fact, we have shown that purposely including quenched disorder can increase the robustness of a topological boundary against unavoidable losses in an experimental setup. This increase can be by orders of magnitude at long times, and occurs even when the disorder breaks the symmetries required for topological pro-
tection. Our results can be readily tested in experiments, since the 1D ladder we consider is a simplified version of the already experimentally realized 2D anomalous Floquet topological phases [20, 21].

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[1] Takuya Kitagawa, Erez Berg, Mark Rudner, and Eugene Demler, “Topological characterization of periodically driven quantum systems,” Phys. Rev. B 82, 235114 (2010).
[2] Liang Jiang, Takuya Kitagawa, Jason Alicea, A. R. Akhmerov, David Pekker, Gil Refael, J. Ignacio Cirac, Eugene Demler, Mikhail D. Lukin, and Peter Zoller, “Majorana fermions in equilibrium and in driven cold-atom quantum wires,” Phys. Rev. Lett. 106, 220402 (2011).
[3] Takuya Kitagawa, Matthew A. Broome, Alessandro Fedrizzi, Mark S. Rudner, Erez Berg, Ivan Kassal, Alán Aspuru-Guzik, Eugene Demler, and Andrew G. White, “Observation of topologically protected bound states in photonic quantum walks,” Nature Communications 3, 882 EP – (2012).
[4] Mark S. Rudner, Netanel H. Lindner, Erez Berg, and Michael Levin, “Anomalous edge states and the bulk-edge correspondence for periodically driven two-dimensional systems,” Phys. Rev. X 3, 031005 (2013).
[5] Matthew D. Reichl and Erich J. Mueller, “Floquet edge states with ultracold atoms,” Phys. Rev. A 89, 063628 (2014).
[6] David Carpentier, Pierre Delplace, Michel Fruchart, and Krzysztof Gawedzki, “Topological index for periodically driven time-reversal invariant 2d systems,” Phys. Rev. Lett. 114, 106806 (2015).
[7] Frederik Nathan and Mark S Rudner, “Topological singularities and the general classification of Floquet bloch systems,” New Journal of Physics 17, 125014 (2015).
[8] I. C. Fulga and M. Maksymenko, “Scattering matrix invariants of Floquet topological insulators,” Phys. Rev. B 93, 075405 (2016).
[9] Daniel Leykam, M. C. Rechtsman, and Y. D. Chong, “Anomalous topological phases and unpaired dirac cones in photonic Floquet topological insulators,” Phys. Rev. Lett. 117, 013902 (2016).
[10] Hoi Chun Po, Lukasz Fidkowski, Takahiro Morimoto, Andrew C. Potter, and Ashvin Vishwanath, “Chiral Floquet phases of many-body localized bosons,” Phys. Rev. X 6, 041070 (2016).
[11] M. Maksymenko, I. C. Fulga, M.T. Rieder, N. Lindner, and E. Berg, “Exotic phases in the periodically driven kitaev model,” (2017), in preparation.
[12] Takahiro Morimoto, Hoi Chun Po, and Ashvin Vishwanath, “Floquet topological phases protected by time glide symmetry,” Phys. Rev. B 95, 195155 (2017).
[13] A. Quelle, C. Weitenberg, K. Sengstock, and C. Morais Smith, “Driving protocol for a floquet topological phase without static counterpart,” (2017), arxiv, arXiv:1704.00306 [cond-mat.quant-gas].
[14] Rahul Roy and Fenner Harper, “Floquet topological phases with symmetry in all dimensions,” Phys. Rev. B 95, 195128 (2017).
[15] Fenner Harper and Rahul Roy, “Floquet topological order in interacting systems of bosons and fermions,” Phys. Rev. Lett. 118, 115301 (2017).
[16] Paraj Titum, Erez Berg, Mark S. Rudner, Gil Refael, and Netanel H. Lindner, “Anomalous floquet-anderson insulator as a nonadiabatic quantized charge pump,” Phys. Rev. X 6, 021013 (2016).
[17] Dong E. Liu, Alex Levchenko, and Harold U. Baranger, “Floquet majorana fermions for topological qubits in superconducting devices and cold-atom systems,” Phys. Rev. Lett. 111, 047002 (2013).
[18] Wenchao Hu, Jason C. Pllllay, Kan Wu, Michael Pasek, Perry Ping Shum, and Y. D. Chong, “Measurement of a topological edge invariant in a microwave network,” Phys. Rev. X 5, 011012 (2015).
[19] Fei Gao, Zhen Gao, Xihang Shi, Zhaoju Yang, Xiao Lin, Hongyi Xu, John D. Joannopoulos, Marin Soljačić, Hong-sheng Chen, Ling Lu, Yidong Chong, and Baile Zhang, “Probing topological protection using a designer surface plasmon structure,” Nature Communications 7, 11619 EP – (2016).
[20] Sebabrata Mukherjee, Alexander Spracklen, Manuel Valiente, Erik Andersson, Patrik Öhberg, Nathan Goldman, and Robert R. Thomson, “Experimental observation of anomalous topological edge modes in a slowly driven photonic lattice,” Nature Communications 8, 13918 EP – (2017).
[21] Lukas J. Maczewsky, Julia M. Zeuner, Stefan Nolte, and Alexander Szameit, “Observation of photonic anomalous floquet topological insulators,” Nature Communications 8, 13756 EP – (2017).
[22] S. Longhi, “Quantum-optical analogies using photonic structures,” Laser & Photonics Reviews 3, 243-261 (2009).
[23] Alexander Szameit and Stefan Nolte, “Discrete optics in femtosecond-laser-written photonic structures,” J. Phys. B 43, 163001 (2010).
[24] See Supplemental Material for a detailed discussion of the topological properties of the model considered in the main text, the effect of adding disorder to this model, and details of the derivation of the Floquet-Lindblad equation, both for stroboscopically driven systems with timing noise in general and the specific model we consider.
[25] Crispin W Gardiner and Peter Zoller, Quantum Noise, 2nd ed., Springer series in synergetics, Vol. 56 (Springer, Berlin Heidelberg, 2000).
[26] Thorsten Groh, Stefan Brakhane, Wolfgang Alt, Dieter Meschede, Janos K. Asbőth, and Andrea Alberti, “Robustness of topologically protected edge states in quantum walk experiments with neutral atoms,” Phys. Rev. A 94, 013620 (2016).
[27] B.D. Hughes, Random Walks and Random Environments, Volume 1: Random Walks (Oxford University Press, Oxford, 1995).
[28] Windell H. Oskay, Daniel A. Steck, and Mark G. Raizen, “Timing noise effects on dynamical localization,”
in *Chaos, Solitons and Fractals*, Vol. 16 (2003) pp. 409–416.

[29] M. Bitter and V. Milner, “Experimental Observation of Dynamical Localization in Laser-Kicked Molecular Rotors,” *Phys. Rev. Lett.* **117**, 144104 (2016), arXiv:1603.06918.

[30] M. Bitter and V. Milner, “Control of quantum localization and classical diffusion in laser-kicked molecular rotors,” *Phys. Rev. A* **95**, 013401 (2017).

[31] T. Cadez, R. Mondaini, and P. D. Sacramento, “Dynamical localization and the effects of aperiodicity in Floquet systems,” (2017), arXiv:1707.07420.

[32] W. P. Su, J. R. Schrieffer, and A. J. Heeger, “Solitons in polyacetylene,” *Phys. Rev. Lett.* **42**, 1698–1701 (1979).

[33] A Yu Kitaev, “Unpaired majorana fermions in quantum wires,” *Physics-Uspekhi* **44**, 131 (2001).
Supplemental Material: Localization counteracts decoherence in noisy Floquet topological chains

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In this Supplemental Material, we provide a detailed discussion of the topological properties of the model considered in the main text, the effect of adding disorder to this model, and details of the derivation of the Floquet-Lindblad equation, both for stroboscopically driven systems with timing noise in general and the specific model we consider.

I. FLOQUET TOPOLOGICAL PHASE OF THE 1D LADDER

The 1D ladder model introduced in the main text is an example of an anomalous Floquet topological phase, one in which topologically protected end modes appear even though the bulk bands have vanishing topological invariants. To see this, we write the Hamiltonian and the Floquet operator in momentum space by “straightening out” the ladder, as shown in Fig. 1.

Figure 1. For an infinite 1D ladder (top), we can determine the momentum-space Hamiltonian by rearranging the site doublets denoted by $j$ such that they become the unit cells of an infinite chain (bottom). The hoppings on the rungs of the ladder (blue) then connect neighboring unit cells. Half of the hoppings on the legs of the ladder (green) connect sites within a unit cell, while the other half (orange) become next-nearest-neighbor hoppings.

The resulting $2 \times 2$ momentum-space Hamiltonian reads

$$H(k) = \begin{pmatrix} 0 & q(k) \\ q^*(k) & 0 \end{pmatrix},$$

where $q(k) = J^4 + J^1 e^{i k} + J^2 e^{2i k}$.

As in the main text, we denote by $J^i$ the hopping active during the $i$th step of the time evolution, and by $H_i(k)$ the corresponding Hamiltonian. Since we consider real hopping amplitudes $J^i$, the Hamiltonian Eq. (1) exhibits time-reversal, particle-hole, as well as sublattice symmetries

$$H(k) = T H^*(-k) T^{-1}$$
$$H(k) = -P H^*(-k) P^{-1}$$
$$H(k) = \Gamma H(-k) \Gamma^{-1}$$

Introducing Pauli matrices $\sigma_{x,y,z}$ acting on the sublattice degree of freedom, the symmetry operators are $\Gamma = \sigma_z$, $T = K$, and $P = \sigma_y K$, with $K$ complex conjugation. Notice that $T^2 = P^2 = +1$, such that $H(k)$ belongs to class BDI in the Altland-Zirnbauer classification [1].

Owing to the simple form of the Hamiltonian and of the driving protocol, the Floquet operator

$$U_F(k) = e^{-i \frac{T}{4} H_4(k)} e^{-i \frac{K}{4} H_3(k)} e^{-i \frac{K}{4} H_2(k)} e^{-i \frac{K}{4} H_1(k)}$$

(3)
can be computed exactly, although its form is involved for generic parameter values. At the resonant driving point however, when the strength of the active hopping \( J \) is such that \( JT/4 = \pi/2 \), the bulk Floquet operator becomes the identity operator \( U^F_{\text{F}}(k) = 1 \). As such, the bulk Floquet bands are flat and positioned at zero quasienergy. In addition, this form of \( U^F_{\text{F}} \) explains the anomalous nature of the topological phase: since Floquet eigenstates become momentum-independent at the resonant driving point, they must be topologically trivial. Away from the fine-tuned value \( JT/4 = \pi/2 \), bulk bands are dispersive and show a parabolic band touching point at \( k = \pi \), as shown in Fig. 2.

Figure 2. Bandstructure of the Floquet operator Eq. (3) for \( JT/4 = \pi/2 \) (dashed) and \( JT/4 = \pi/3 \) (solid). At the resonant driving point the bulk bands are flat, whereas they disperse otherwise and form a parabolic touching point at \( k = \pi \) and \( \varepsilon = 0 \).

As discussed in the main text, the 1D ladder hosts a topological Floquet phase, such that end states appear in a finite system despite the topologically trivial bands. The presence of end states at quasienergies \( \varepsilon = \pm \omega/2 \) can be deduced directly from the real-space Floquet system at the resonant driving point, but remain pinned to these quasienergies also away from it. This behavior is linked to the symmetries of the Floquet operator Eq. (3). While the stroboscopic driving protocol explicitly breaks time-reversal symmetry, the sublattice symmmetry of the Hamiltonian Eq. (1) implies that

\[
U^F_{\text{F}}(k) = \sigma_z U^*_{\text{F}}(-k)\sigma_z. \tag{4}
\]

As such, the sublattice symmetry on the level of the instantaneous Hamiltonian translates into an effective particle-hole symmetry on the level of \( U^F_{\text{F}} \). Due to this symmetry, momentum eigenstates must come in pairs, related by \( \varepsilon \to -\varepsilon \) and \( k \to -k \), as seen in Fig. 2. In real space, particle-hole symmetry relates states at the same position but opposite values of quasienergy, explaining the robustness of the end modes. Even if the system is perturbed away from resonant driving, the end modes cannot couple due to the bulk gap so they cannot shift in quasienergy away from the particle-hole symmetric \( \varepsilon = \omega/2 = -\omega/2 \).

To determine the topological invariant responsible for the presence of end states, we use the method of Ref. [2], which is based on evaluating the time-evolution operator at all times during the driving cycle. Writing

\[
U_t(k) = \mathcal{T} \exp \left(-i \int_0^t dt' H(k,t') \right), \tag{5}
\]

the \( \mathbb{Z}_2 \) topological invariant can be determined as the parity of the total number of times the eigenphases of both \( U_t(k = 0) \) and \( U_t(k = \pi) \) cross \( \pi \) in the interval \( t \in [0,T] \). Due to the simple form of the piecewise constant Hamiltonian Eq. (1), these eigenphases can be evaluated exactly, since \( H(k = 0) = (J^4 + J^{1,3} + J^2)\sigma_x \) and \( H(k = \pi) = (J^4 - J^{1,3} + J^2)\sigma_x \). Therefore, the two eigenphases of \( U_t(k = 0,\pi) \) are opposite and increase or decrease linearly in each driving step, depending on whether \( J^i \) enters the Hamiltonian with a positive or negative sign. At resonant driving, for \( k = 0 \) the eigenphases are monotonic as a function of \( t \) and show a single crossing at \( \pi \), when \( t = T/2 \). For \( k = \pi \), the eigenphases reverse direction in steps 1 and 3 as compared to steps 2 and 4, such that they never cross \( \pi \). As such, the total number of \( \pi \)-crossings is odd, as shown in Fig. 3, implying a topologically non-trivial phase.
II. TIMING NOISE IN FLOQUET SYSTEMS WITH PIECEWISE CONSTANT DRIVING

Here we derive the discrete-time Floquet-Lindblad equation (FLE) that describes the time evolution of periodically driven systems with timing noise. First we consider the simplest case of binary driving (or, equivalently, a system subject to periodic kicks), and then generalize our results to driving cycles comprising an arbitrary number of steps. We discuss the conditions under which the FLE can be reduced to a classical master equation. Then, we apply the general formalism to the model system considered in the main text.

A. Binary piecewise constant driving

Let us begin with the simplest case, which is a binary and piecewise constant driving protocol. We first discuss perfectly periodic driving, and then modifications due to timing noise. Without noise, a driving cycle consists of the following two steps: first, the Hamiltonian $H_1$ is applied for a time $T_1$, and then another Hamiltonian $H_2$ is applied for a time $T_2$. The full duration of the driving cycle is thus $T = T_1 + T_2$, and the Floquet operator $U_F$, which describes the evolution of the system during one period, is given by

$$U_F = Te^{-i\int_0^T dt H(t)} = U_2U_1 = e^{-iT_2H_2}e^{-iT_1H_1}. \tag{6}$$

Here, $T$ denotes time ordering, and $H(t)$ is the time-dependent Hamiltonian

$$H(t) = \begin{cases} H_1, & nT \leq t < nT + T_1, \\ H_2, & nT + T_1 \leq t < (n+1)T. \end{cases} \tag{7}$$

The integer $n$ counts the number of driving cycles. A Floquet operator with the same form as in Eq. (6) arises in periodically “kicked” systems, in which $H_2$ is applied as an instantaneous pulse at multiples of the period $T$. Such a scenario is described by the following time-dependent Hamiltonian:

$$H(t) = H_1 + \lambda \sum_{n\in\mathbb{N}} \delta(t - nT)H_2. \tag{8}$$

For this type of driving, the noiseless evolution is given by the Floquet operator $U_F = e^{-i\lambda H_2}e^{-iT H_1}$. In both of these driving schemes, the state of the system at multiples of the driving period, $|\psi_n\rangle = |\psi(nT)\rangle$, can be obtained by repeated application of $U_F$, i.e., $|\psi_n\rangle = U^n_F |\psi_0\rangle$, where $|\psi_0\rangle$ is the initial state of the system.

We now proceed to discuss how the above driving protocols are modified in the presence of timing noise. For simplicity, we assume that there is noise only in the first step of the driving cycle. The generalization of our considerations to include noise in the second step or to a driving protocol that comprises more than two steps is straightforward and summarized in the next section. If there is timing noise in the first step, the Hamiltonian $H_1$ is not applied exactly
for a time \( T_1 \). Instead, in the \( n \)-th cycle, it is applied for \( T_1 + \tau_n \), where \( \tau_n \) is a random number with zero mean, \( \overline{\tau_n^2} = \tau^2 \). Strictly speaking, causality requires that \( \tau_n \geq -T_1 \), but formally we can relax this constraint, and we simply require that noise is weak in the sense that \( \tau \ll T_1 \). (In particular, this allows us to take the distribution of the \( \tau_n \) to be Gaussian as in the numerical results presented in the main text.) The precise form of the probability distribution is not important in the following. Moreover, we assume that the time shifts in different cycles are uncorrelated, i.e., \( \overline{\tau_n \tau_m} = \tau^2 \delta_{nm} \). Under these conditions, after \( n \) driving cycles the state of the system is given by

\[
|\psi_n\rangle = e^{-iT_2 H_2}e^{-i(T_1 + \tau_n)H_1} \cdots e^{-iT_2 H_2}e^{-i(T_1 + \tau_1)H_1} |\psi_0\rangle = U_{F,n} \cdots U_{F,1} |\psi_0\rangle.
\]

(9)

In the last equality, we introduced the noisy Floquet operator \( U_{F,n} \), which describes the evolution of the system during the \( n \)-th driving cycle. It can be written as

\[
U_{F,n} = T e^{-i \int_{t_{n-1}}^{t_n} dt H(t)} = e^{-iT_2 H_2}e^{-i(T_1 + \tau_n)H_1},
\]

(10)

where \( t_n = n T + \sum_{n'=1}^{n} \tau_n \) depends on all prior time shifts. The time-dependent Hamiltonian is now given by

\[
H(t) = \begin{cases} 
H_1, & t_n \leq t < t_n + T_1 + \tau_n, \\
H_2, & t_n + T_1 + \tau_n \leq t < t_n + T_1 + \tau_n + T_2.
\end{cases}
\]

(11)

For the kicking protocol defined by Eq. (8), the noisy Hamiltonian assumes the form

\[
H(t) = H_1 + \lambda \sum_n \delta(t - t_n)H_2,
\]

(12)

where \( t_n \) is defined as above, i.e., the duration between two kicks is \( t_{n+1} - t_n = T + \tau_n \). The corresponding Floquet operator reads as in Eq. (10), only with \( U_2 = e^{-iT_2 H_2} \) replaced by \( e^{-i\lambda H_2} \). In both cases, the evolution of the state \( |\psi_n\rangle = |\psi(t_n)\rangle \) during one driving cycle is given by

\[
|\psi_{n+1}\rangle = U_{F,n+1} |\psi_n\rangle = U_2 e^{-i(T_1 + \tau_{n+1})H_1} |\psi_n\rangle.
\]

(13)

Evidently, the state of the system at time \( t_n \) depends on the particular noise realization, i.e., on the sequence of all prior time shifts \( \tau_1, \ldots, \tau_n \). Taking the average over noise realizations, the expectation value of an observable \( O \) can be written as

\[
\langle O_n \rangle = \langle \psi_n | O | \psi_n \rangle = \text{tr}\left( O |\psi_n\rangle \langle \psi_n| \right) = \text{tr}(O \rho_n),
\]

(14)

where \( \rho_n = |\psi_n\rangle \langle \psi_n| \) is the density matrix that describes the noise-averaged state of the system at time \( t_n \). Thus, to evaluate the expectation values of observables, it is sufficient to track the evolution of \( \rho_n \), and in the following we derive an evolution equation for this quantity in the weak-noise limit. A key point that facilitates this derivation is that the noise average can be performed for each driving cycle individually because the \( \tau_1, \ldots, \tau_n \) are statistically independent. Thus, rewriting Eq. (13) for the density matrix, we obtain

\[
\rho_{n+1} = \frac{U_{F,n+1} \rho_n U_{F,n+1}^\dagger}{U_{F,n+1}^\dagger U_{F,n+1}}.
\]

(15)

For weak noise, we can expand the noisy floquet operator (10) in the time shift. Keeping terms up to second order in \( \tau_{n+1} \), we find

\[
U_{F,n+1} = U_F \left( 1 - i\tau_{n+1} H_1 - \frac{\tau_{n+1}^2}{2} H_1^2 \right),
\]

(16)

where \( U_F = U_2 U_1 \) is the Floquet operator for perfectly periodic driving. Inserting this form in Eq. (15), the average over noise becomes straightforward, and we obtain (dropping terms of cubic and higher order in \( \tau_{n+1} \))

\[
\rho_{n+1} = U_F \left( 1 - i\tau_{n+1} H_1 - \frac{\tau_{n+1}^2}{2} H_1^2 \right) \rho_n \left( 1 + i\tau_{n+1} H_1 - \frac{\tau_{n+1}^2}{2} H_1^2 \right) U_F^\dagger
\]

\[
= U_F \left[ \rho_n - i\tau_{n+1} [H_1, \rho_n] + \frac{\tau_{n+1}^2}{2} \left( H_1 \rho_n H_1 - \frac{1}{2} \{ H_1^2, \rho_n \} \right) \right] U_F^\dagger
\]

\[
= U_F \left[ \rho_n - i\tau_{n+1} [H_1, \rho_n] + \frac{\tau_{n+1}^2}{2} \left( H_1 \rho_n H_1 - \frac{1}{2} \{ H_1^2, \rho_n \} \right) \right] U_F^\dagger
\]

\[
= U_F \left( \rho_n + \frac{\tau^2}{2} D[L_1] \rho_n \right) U_F^\dagger.
\]

(17)
This discrete-time evolution equation combines the usual noise-free coherent stroboscopic Floquet evolution, described by the Floquet operator $U_F$, with dissipative dynamics as familiar from quantum master equations in Lindblad form. In particular, in analogy to the usual continuous-time Lindblad equation, we identify the quantum jump operator $L_1 = H_1$ and the dissipator

$$\mathcal{D}[L] \rho = \frac{\mathcal{L} \rho}{\mathcal{L}} \frac{1}{2} \{ L^2, \rho \} = \frac{1}{2} [[L, \rho], L].$$

(18)

In the Floquet-Lindblad equation (FLE) (17), these two elements of the evolution — coherent evolution and dissipation — are applied in a staggered fashion, i.e., the map $\rho_n \mapsto \rho_{n+1}$ is a composition of $\rho \mapsto \rho + \tau^2 \mathcal{D}[L_1] \rho$ and $\rho \mapsto \mathcal{L}\rho U_F^\dagger$. For comparison, the continuous-time form of the Lindblad equation reads

$$\frac{d\rho}{dt} = -i[H, \rho] + \gamma \mathcal{D}[L] \rho.$$

(19)

We note that while in the present context of noise in Floquet systems the jump operator $L_1 = H_1$ is always Hermitian, this is not the case in general. Then, the dissipator should be modified to $\mathcal{D}[L] \rho = \mathcal{L}\rho \frac{1}{2} \{ L^1, \rho \}$. In the usual (continuous-time) quantum master equation in Lindblad form, the dissipator describes the effect of an environment or bath on the system dynamics, and assumes the above time-local form if (i) the system-bath coupling is weak and (ii) the bath correlation time is much shorter than the time scales of the system dynamics (see, e.g., [3]). Conditions (i) and (ii) justify the Born and Markov approximations, respectively, which are made in derivations of the Lindblad equation. In the derivation of the FLE, the expansion in the noise strength is analogous to the Born approximation, and the noise we consider is Markovian (i.e., uncorrelated on the intrinsic time scale $T$ of the evolution of system) by assumption. Just like the usual master equation, the FLE can immediately be seen to be trace-preserving and completely positive.

**B. Multi-step piecewise constant driving**

The above derivation can be generalized straightforwardly to extended driving protocols, defined in terms of a sequence of Hamiltonians $H_i$ with $i = 1, 2, \ldots, M$ which are applied for times $T_i$ so that the duration of a full driving cycle is $T = \sum_{i=1}^M T_i$. Assuming — as in the main text — that there is timing noise in each step of the driving cycle, the FLE that generalizes Eq. (17) takes the form

$$\rho_{n+1} = U_F \left( \rho_n + \tau^2 \sum_{i=1}^M \mathcal{D}[L_i] \rho_n \right) U_F^\dagger,$$

(20)

where $U_F = U_M \cdots U_1$ with $U_i = e^{-iT_i H_i}$. Here we take the shifts $\tau_{ni}$ in different steps and driving cycles to be uncorrelated and identically distributed, $\sum_{n'\neq n''} \tau_{ni} \tau_{n'i'} = \tau^2 \delta_{nn'} \delta_{ii'}$. The jump operators are given by

$$L_i = U_F^\dagger U_M \cdots U_i H_i U_{i-1} \cdots U_1 = U_i^\dagger \cdots U_{i-1}^\dagger H_i U_{i-1} \cdots U_1.$$

(21)

As above, the jump operators are Hermitian, $L_i^\dagger = L_i$.

**C. Reduction to classical master equation**

In many cases of practical interest, the discrete Lindblad equation for the density matrix can be reduced to a classical master equation for the diagonal elements of the density matrix written in the basis of Floquet eigenstates $|\alpha\rangle$. The latter are the right eigenvectors of the Floquet operator, $U_F |\alpha\rangle = e^{-iT \epsilon_\alpha} |\alpha\rangle$, and $\epsilon_\alpha$ is the respective quasienergy. In this basis, the density matrix can be written as

$$\rho = \sum_{\alpha, \beta} \rho^{\alpha \beta} |\alpha\rangle \langle \beta|.$$

(22)

The condition for obtaining a classical master equation is that upon repeated application of $U_F$ in Eq. (20), the off-diagonal elements of the density matrix dephase more rapidly than they are repopulated from the diagonal. Then, for the diagonal elements $\rho^\alpha = \rho^{\alpha \alpha}$ we obtain the discrete master equation

$$\rho_{n+1}^{\alpha} = \sum_{\beta} W_{\beta \rightarrow \alpha} \rho_{n}^{\beta},$$

(23)
where the transition probabilities are given by

\[ W_{\beta \rightarrow \alpha} = \delta_{\alpha \beta} + \tau^2 \sum_i \left( |\langle \alpha | L_i | \beta \rangle|^2 - \langle \alpha | L_i^2 | \alpha \rangle \right) \delta_{\alpha \beta}. \]  

(24)

The transition probabilities are symmetric, \( W_{\alpha \rightarrow \beta} = W_{\beta \rightarrow \alpha} \), and conservation of the trace of the density matrix — in other words, conservation of probability — is valid in steps where \( \sum_\beta W_{\beta \rightarrow \alpha} = 1 \). Thus, the probability to remain in state \( \alpha \) can be written as \( W_{\alpha \rightarrow \alpha} = 1 - \sum_{\beta \neq \alpha} W_{\beta \rightarrow \alpha} \), and Eq. (23) can be recast as

\[ \rho_{\alpha \alpha}^{\alpha+1} = \rho_{\alpha \alpha}^\alpha + \sum_{\beta \neq \alpha} W_{\beta \rightarrow \alpha} (\rho_{\alpha \beta}^\beta - \rho_{\alpha \alpha}^\alpha), \]  

(25)

which is the form quoted in the main text.

III. DECAY OF AN END-STATE IN A NOISY FLOQUET TOPOLOGICAL CHAIN

Here, we apply the formalism derived above to derive the time evolution of an imperfectly driven ladder as introduced in the main text. We show that timing noise leads to diffusion exactly on resonance and to exponential decay away from resonance. The microscopic parameter entering the discussion is the hopping parameter \( J = 2\pi/T + \delta J \), with \( \delta J \) being a measure for the bulk’s bandwidth. We assume that \( \delta J \) is small, enabling us to make analytical progress. In the following, we will distinguish the point of resonant driving \( \delta J = 0 \) from the limit \( \delta J \rightarrow 0 \) and show how they capture the diffusive and exponential decay, respectively. Working in the eigenbasis of the Floquet operator, we find that transitions between Floquet states are induced by the noise at order \( \delta J^2 \). For small deviations from resonance, in the calculation of the transition probabilities we can set \( \delta J = 0 \) (thus, ignoring terms \( O(\delta J^2) \)). The crucial point is that we have to be careful in taking \( \delta J \rightarrow 0 \): starting from finite \( \delta J \) and sending it to zero, the bulk states remain delocalized with an arbitrarily small but finite quasienergy and bandwidth; on the other hand, working exactly on resonance, we should take the bulk states to be localized with an exact degeneracy of all states at quasienergy 0.

Employing the formalism of the Floquet-Lindblad equation derived above, we note that the periodic drive considered in the main text comprises \( M = 4 \) individual steps. Thus, the Floquet operator is \( U_F = U_3U_2U_2U_1 \), with the individual steps \( U_i = e^{-iTH_i}/4 \) where \( H_i = -\sum_{\mu \nu} J_{\mu \nu} (c_\mu^\dagger c_\nu + \text{h.c.}) \), as introduced in the main text. The time evolution during each step can thus be evaluated assuming single particle states and using the fact that the sum over bonds in \( H_i \) is over mutually disconnected pairs of neighboring lattice sites. Using the identities

\[ H_i^{2n} = J^{2n} \sum_{(\mu \nu)_i} (n_\mu + n_\nu) \quad \text{and} \quad H_i^{2n+1} = J^{2n+1}(-H_i/J), \]  

(26)

with the sum running over all bonds affected by the hopping in step \( i \), we find

\[ U_i = 1 + (\cos(\phi) - 1) \sum_{(\mu \nu)_i} (n_\mu + n_\nu) + i \sin(\phi) H_i/J, \]  

(27)

with \( \phi = JT/4 \). Equivalently, we can represent the \( U_i \) in the single particle basis with the projector on all involved lattice sites \( P_i = \sum_{(\mu \nu)_i} (|\mu\rangle \langle \mu| + |\nu\rangle \langle \nu|) \) and its complement \( Q_i = 1 - P_i \):

\[ U_i = P_i (\cos(\phi) + i \sin(\phi) H_i/J) \]  

(28)

\[ P_i + Q_i, \]

as used in the main text. In steps 1 and 3 all lattice sites are involved and \( Q_1 = Q_3 = 0 \), while in steps 2 and 4 one site at each end drops out of the respective Hamiltonian and thus \( Q_2 = |1, +\rangle \langle 1, +| + |L - 1, -\rangle \langle L - 1, -| \) and \( Q_4 = |0, -\rangle \langle 0, -| + |L, +\rangle \langle L, +| \). The jump operators \( L_i \) introduced above are \( L_1 = H_1, \) \( L_2 = U_1^\dagger H_2 U_1, \) \( L_3 = U_1^\dagger U_2^\dagger H_3 U_3 U_2 U_1, \) and \( L_4 = U_1^\dagger U_2^\dagger U_3^\dagger H_4 U_4 U_3 U_2 U_1. \)

The form of the jump operators simplifies further exactly on resonance, when \( \phi = \pi/2 \) and during each step of the driving protocol a particle is fully transferred from one lattice site to another. Then, we find \( L_1 = H_1, \) \( L_2 = H_1 H_2, \) \( L_3 = H_1 (H_2 + iQ_2) H_3 (H_3 - iQ_3) H_1, \) and \( L_4 = H_1 (H_2 + iQ_2) H_3 H_4 H_3 (H_2 - iQ_3) H_1. \) Moreover, on resonance the Floquet eigenstates take a particularly simple form: The end states are localized to the outermost lattice sites \( |e_1\rangle = |0, -\rangle \) and \( |e_r\rangle = |L, +\rangle \), while for the bulk states we can choose the lattice site basis \( |b\rangle = |j, s\rangle \) with \( j = 1, \ldots, L - 1 \) and \( s = \pm \). In the lattice basis, the jump operators are represented as \( L_1 = J \sum_j (|j, +\rangle \langle j, -| + \text{h.c.}) \), \( L_2 = J \sum_j (|j, +\rangle \langle j - 1, -| + \text{h.c.}) \), \( L_3 = J \sum_j (|j, +\rangle \langle j + 1, -| + \text{h.c.}) \), and \( L_4 = J \sum_j (|j, +\rangle \langle j - 1, +| + \text{h.c.}) \). Focusing on
the left end state $|e\rangle \equiv |e_1\rangle$, we immediately see that the diagonal matrix elements of the jump operators vanish, $\langle e|L_i|e\rangle = 0$, and that the left end state is an eigenstate of the square of the jump operators:

$$L_i^2 |e\rangle = \begin{cases} J^2 |e\rangle, & i = 1, 3, \\ 0, & i = 2, 4. \end{cases}$$

(29)

This, in turn, implies that $\langle e|L_i^2|e\rangle = J^2$ for $i = 1, 3$ and $\langle e|L_i|e\rangle = 0$ otherwise, while $\langle b|L_i^2|e\rangle = 0$. Below, we investigate the stability of the chain’s end state very close to as well as exactly on resonance. In both cases, we can use the above expressions for matrix elements of the jump operators which we derived for resonant driving. The crucial difference comes from the matrix elements $\langle b|L_i|e\rangle$, which we discuss in detail below.

In the following, we restrict our attention to the left end of the chain which we denote by $|e\rangle$, and we disregard the other one, assuming a semi-infinite system. Then, the system’s density matrix can be represented in the Floquet eigenbasis as

$$\rho_n = \rho_n^e |e\rangle \langle e| + \sum_b \left( \delta \rho_n^{eb} |e\rangle \langle b| + \rho_n^{be} |b\rangle \langle e| \right) + \sum_{bb'} \rho_n^{bb'} |b\rangle \langle b'|.$$

(30)

Evolving the density matrix by one driving period with the FLE (20), we find for the occupation of the chain’s left end state:

$$\rho_{n+1}^e = \left( 1 + \tau^2 \sum_i \left( |\langle e|L_i|e\rangle|^2 - |\langle e|L_i^2|e\rangle|^2 \right) \rho_n^e + \tau^2 \sum_i \sum_b \left[ \left( \langle e|L_i|e\rangle \langle b|L_i|e\rangle - \frac{1}{2} \langle b|L_i^2|e\rangle \right) \rho_n^{eb} + \left( \langle e|L_i|b\rangle \langle b|L_i|e\rangle - \frac{1}{2} \langle e|L_i^2|b\rangle \right) \rho_n^{be} \right] + \tau^2 \sum_i \sum_{bb'} \langle e|L_i|b\rangle \langle b'|L_i|e\rangle \rho_n^{bb'} \right).$$

(31)

This can be simplified using the matrix elements evaluated above, and we obtain (cf. Eq. (5) of the main text)

$$\rho_{n+1}^e = \left( 1 - 2\tau^2 J^2 \right) \rho_n^e + \tau^2 \sum_i \sum_{bb'} \langle e|L_i|b\rangle \langle b'|L_i|e\rangle \rho_n^{bb'},$$

(32)

where, in particular, coherences between the end state and bulk states drop out.

### A. Exponential Decay for a Dispersive Bulk

We first analyze the noisy stroboscopic time evolution of the end states in Eq. (32) for a dispersive bulk, i.e., $\delta \phi \neq 0$, for which the bulk states acquire a finite bandwidth $\propto \delta \phi$ and are delocalized out over the full length of the system. An analytical treatment is possible only close to resonant driving, and hence we focus on the limit $\delta \phi \to 0$. This allows us to approximate the end state $|e\rangle$ and the jump operators with their forms at resonant driving. Importantly, the bulk states have finite albeit arbitrarily small quasienergies.

For the matrix elements connecting the end to the bulk states, $\langle b|L_i|e\rangle$, we need to keep in mind that the jump operators transfer electrons only between lattice sites within a short range of each other. Assuming that the bulk states are evenly spread out over the entire chain, they carry a normalization factor of $\sim 1/\sqrt{L}$, leading to the following scaling of the matrix elements with system size:

$$\langle b|L_i|e\rangle \sim \frac{1}{\sqrt{L}}.$$  

(33)

Moreover, it is important to note that in the delocalized basis the off-diagonal part of the bulk density matrix $\rho_n^{bb'}$ for $b \neq b'$ is non-zero. These matrix elements are getting populated by excitations from the end state at a rate $\tau^2 \langle b|L_i|e\rangle \langle e|L_i|b'\rangle \sim \tau^2/L$ and, to leading order, evolve coherently with $U_F$ per period. The off-diagonal elements therefore pick up phase factors of $e^{i\delta \phi}$ for each full period. No matter how small the quasienergies, these phases accumulate and average to zero for long enough times. Only the diagonal elements are static and contribute to the sum over bulk states in Eq. (32), which then decays with the system size as can be seen from a simple dimensional analysis

$$\sum_b \rho_n^{bb} |\langle e|L_i|b\rangle|^2 \sim \frac{1}{L}.$$  

(34)
For large enough systems and long enough times, the contribution from this sum can be neglected and leaves us with an exponential decay of the end state (cf. Eq. (6) of the main text)

$$\rho_{n+1}^e = (1 - \tau^2 J^2) \rho_n^e + O(1/L),$$

confirming the intuitive picture that a dispersive bulk would carry away any excitation out of the edge. This contrasts the case of the localized bulk at resonant driving in which excitations get stuck close to the end state and have a finite return probability, as we discuss in the next section.

## B. Diffusive Decay at Resonant Driving

We now focus on the dynamical evolution of the density matrix for a flat bulk band at resonant driving. As explained above, in this case we can work in the basis of lattice sites, in which the end states are $|e_i\rangle = |0, -\rangle \equiv |e\rangle$ and $|e_r\rangle = |L, +\rangle$, while the bulk states are given by $|b\rangle = |j, s\rangle$ with $j = 1, \ldots, L - 1$ and $s = \pm$. Note that the lattice site basis for the bulk is an arbitrary choice since the bulk band is fully degenerate.

In the equation for the end state population (32), only $L_1$ and $L_3$ connect the end state to the bulk states, such that $\langle e|L_i|b_j\rangle = J$ for $j = 1, 3$ and $|b_1\rangle = |1, +\rangle$ and $|b_3\rangle = |1, -\rangle$. All other elements $\langle e|L|b\rangle$ are zero. Then, the dynamical equation reduces to

$$\rho_{n+1}^{0,-} = (1 - 2\tau^2 J^2) \rho_n^{0,-} + \tau^2 J^2 (\rho_n^{1,+} + \rho_n^{0,-}),$$

where we introduced a notation for the density matrix diagonal labeled by lattice sites $\rho_n^{bb} = \rho_n^{i,s}$. We note that all off-diagonal elements of the density matrix drop out of Eq. (36) — a property that is also true for the evolution equation of the population of a generic bulk state. Thus, starting from an initially diagonal density matrix, $\rho_n = |e\rangle \langle e|$, the density matrix remains diagonal at all times. Importantly, the time evolution for the end state is strongly coupled to its neighboring bulk states. To understand its behavior in the long-time limit we can alternatively look at the dynamical equation for a generic bulk state,

$$\rho_{n+1}^{0,s} = (1 - \tau^2 \sum_i \sum_{l,s'} |l,s'| L_i |j,s\rangle \langle j,s'|^2) \rho_n^{0,s} + \tau^2 \sum_i \sum_{l,s'} |l,s'| L_i |j,s\rangle \langle j,s'|^2 \rho_n^{l,s'}$$

$$= (1 - 4\tau^2 J^2) \rho_n^{0,s} + \tau^2 J^2 (2\rho_n^{1,-s} + \rho_n^{1,+s,-} + \rho_n^{0,-s,-s}).$$

The last equation is Eq. (7), and can straightforwardly be reduced to the diffusion equation (8) for the total occupation of doublets $\rho_n = \rho_n^{1,+} + \rho_n^{0,-}$. As bulk and end state can be treated in the same way at resonant driving, being localized to one lattice site each, the behavior of the end state for long times can be inferred immediately. It obeys a diffusion equation against a hard wall boundary for a particle initialized right next to this boundary.

Note that also in the resonant driving case we can choose to describe the bulk in a delocalized basis, rendering the off-diagonal elements $\rho_n^{bb'}$ non-zero. However, in contrast to the dispersive case, these elements are static in the stroboscopic time-evolution due to the flat bulk band and can thus not be neglected.

## IV. NUMERICAL SIMULATIONS AND DISORDER EFFECTS

In this part of the Supplemental Material, we show numerically that hopping disorder also slows the end mode decay down to a diffusive process. This is indicated in Fig. 4, and is a consequence of the fact that hopping disorder in this model leads to a localization of all bulk states. While localization is expected for on-site disorder, which breaks the sublattice symmetry, this is not immediately obvious in the case of random hoppings, since the particle-hole symmetry of the Floquet operator, Eq. (4), is preserved for every disorder realization. One-dimensional particle-hole symmetric chains may enter a so-called critical phase, characterized by the presence of delocalized states [4–7]. In the 1D ladder model, however, Floquet bulk bands form a parabolic (as opposed to linear) band touching point at the particle-hole symmetric quasienergy $\varepsilon = 0$, meaning that the bulk state velocity vanishes already in the clean limit. In this way, the 1D critical phase is avoided, all bulk states become localized, and the decay is diffusive as in the case of on-site disorder.

[1] Alexander Altland and Martin R. Zirnbauer, “Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures,” Phys. Rev. B 55, 1142 (1997).
Figure 4. As in the main text, we plot the survival probability of an initial end mode, \( s = (\langle \psi(0)|\psi(t) \rangle)^2 \) as a function of the number of noisy driving cycles, \( n \). All system parameters, as well as the blue and yellow curves are the same as in the main text Fig. 2. The orange curve however is obtained by introducing disorder in the hopping amplitudes, using \( V_{\text{hop}}T/4 = 0.75 \), and averaging over 4000 independent realizations of disorder and noise. Since hopping disorder also localizes all bulk states, the decay is again diffusive, as indicated by the dashed line.

[2] Liang Jiang, Takuya Kitagawa, Jason Alicea, A. R. Akhmerov, David Pekker, Gil Refael, J. Ignacio Cirac, Eugene Demler, Mikhail D. Lukin, and Peter Zoller, “Majorana fermions in equilibrium and in driven cold-atom quantum wires,” Phys. Rev. Lett. 106, 220402 (2011).

[3] Crispin W Gardiner and Peter Zoller, Quantum Noise, 2nd ed., Springer series in synergetics, Vol. 56 (Springer, Berlin Heidelberg, 2000).

[4] P. W. Brouwer, A. Furusaki, I. A. Gruzberg, and C. Mudry, “Localization and Delocalization in Dirty Superconducting Wires,” Phys. Rev. Lett. 85, 1064–1067 (2000).

[5] Olexei Motrunich, Kedar Damle, and David A. Huse, “Griffiths effects and quantum critical points in dirty superconductors without spin-rotation invariance: One-dimensional examples,” Phys. Rev. B 63, 224204 (2001).

[6] P. W. Brouwer, A. Furusaki, and C. Mudry, “Universality of delocalization in unconventional dirty superconducting wires with broken spin-rotation symmetry,” Phys. Rev. B 67, 014530 (2003).

[7] Ilya A. Gruzberg, N. Read, and Smitha Vishveshwara, “Localization in disordered superconducting wires with broken spin-rotation symmetry,” Phys. Rev. B 71, 245124 (2005).