1-stable fluctuations in branching Brownian motion at critical
temperature II: general functionals

Pascal Maillard and Michel Pain

March 19, 2021

Abstract

Let $\mu_t$ denote the critical derivative Gibbs measure of branching Brownian motion at time $t$. It has been proved by Madaule [16] and Maillard and Zeitouni [18] that $\mu_t$ converges weakly to the random measure $Z_\infty \frac{\sqrt{2/\pi x^2} e^{-x^2/2}}{x^2} dx$, where $Z_\infty$ is the limit of the derivative martingale. In this paper, we are interested in the fluctuations that occur in this convergence and prove for a large class of functions $F$ that

$$\sqrt{t} \left( \int F d\mu_t - Z_\infty \int_0^\infty F(x) \frac{\sqrt{2/\pi x^2} e^{-x^2/2}}{x^2} dx - \frac{c(F) \log t}{\sqrt{t}} Z_\infty \right) \to S_F^{\frac{1}{2}}, \quad \text{in law},$$

where $c(F)$ is a constant depending on $F$ and $(S_r^{\frac{1}{2}})_{r \geq 0}$ is a 1-stable Lévy process independent of $Z_\infty$. Moreover, we extend this result to a functional convergence, and we identify precisely the particles responsible for the fluctuations. In particular, this proves the following result for the critical additive martingale $(W_t)_{t \geq 0}$:

$$\sqrt{t} \left( \frac{1}{\sqrt{t}} W_t - \frac{1}{\sqrt{2}} Z_\infty \right) \to S_Z, \quad \text{in law},$$

where here $(S_r)_{r \geq 0}$ is a Cauchy process independent of $Z_\infty$, confirming a conjecture by Mueller and Munier [19] in the physics literature.

1 Introduction

1.1 Definitions and result

Branching Brownian motion (BBM) is a branching Markov process defined as follows. Initially, there is a single particle at the origin. Each particle moves according to a Brownian motion with variance $\sigma^2 > 0$ and drift $\rho \in \mathbb{R}$, during an exponentially distributed time of parameter $\lambda > 0$ and then splits into a random number of new particles, chosen accordingly to a reproduction law $\mu$. These new particles start the same process from their place of birth, behaving independently of the others. The system goes on indefinitely, unless there is no particle at some time. This paper is the sequel of [17] by the authors, where more motivations for the study of BBM are given.

Let $L$ denote a random variable on $\mathbb{N} := \{0, 1, \ldots\}$ with law $\mu$. Our assumptions in this paper concerning the reproduction law are

$$E[L] > 1 \quad \text{and} \quad E[L^2] < \infty. \quad (1.1)$$

The first inequality implies that the underlying Galton-Watson tree $T$ is supercritical and the event $S$ of survival of the population has positive probability. Let $\mathbb{P}^* := \mathbb{P}(\cdot|S)$ be the conditional probability given the survival.
Let $N(t)$ be the set of particles alive at time $t$ and $X_u(t)$ the position of particle $u$ at time $t$. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration associated with the BBM. As in [17] and in the branching random walk literature [1, 3], we take $\sigma = 1 = \rho = 1/2$ and $\lambda = 1/(2\mathbb{E}[L-1])$ in order to have, for any $t > 0$,

$$
\mathbb{E} \left[ \sum_{u \in N(t)} e^{-X_u(t)} \right] = 1, \quad \mathbb{E} \left[ \sum_{u \in N(t)} X_u(t) e^{-X_u(t)} \right] = 0 \quad \text{and} \quad \mathbb{E} \left[ \sum_{u \in N(t)} X_u(t)^2 e^{-X_u(t)} \right] = t,
$$

and to simplify calculations. Moreover, we define the \textit{critical additive martingale}

$$W_t := \sum_{u \in N(t)} e^{-X_u(t)}, \quad t \geq 0,$$

which converges a.s. to 0 [21], and the \textit{derivative martingale}

$$Z_t := \sum_{u \in N(t)} X_u(t) e^{-X_u(t)}, \quad t \geq 0.$$

It has been proved by Lalley and Sellke [15] for binary branching and then by Yang and Ren [23] under the optimal assumption $\mathbb{E}[L \log^2_2 L] < \infty$ that the derivative martingale converges $\mathbb{P}^*$-a.s. to a positive limit $Z_{\infty}$. Moreover, it has been proved for the branching random walk by Aïdékon and Shi [3] that after a proper renormalization, the critical additive martingale converges also to the same limit:

$$\sqrt{t}W_t \underset{t \to \infty}{\longrightarrow} \sqrt{\frac{2}{\pi}}Z_{\infty}, \quad \text{in probability}, \quad (1.2)$$

and this result extends to the case of BBM.

The study of BBM has been mainly focused on the behavior of the extremal particles (those which are at a distance of order 1 from the minimum of BBM): Bramson [6, 5] and Lalley and Sellke [15] proved the following convergence for the position of the lowest particle of BBM

$$\mathbb{P} \left( \min_{u \in N(t)} X_u(t) \geq \frac{3}{2} \log t + x \right) \underset{t \to \infty}{\longrightarrow} \mathbb{E} \left[ e^{-c^*e^x Z_{\infty}} \right], \quad (1.3)$$

for some positive constant $c^*$, and Aïdékon, Berestycki, Brunet and Shi [2], Arguin, Bovier and Kistler [4] described the limit of the whole extremal process. One can note that the derivative martingale plays a role in the behavior of the extremal particles. Indeed, for any $t \leq s$ such that $t$ and $s-t$ are large, extremal particles at time $s$ descend of the particles at time $t$ that mainly contribute to $Z_t$: these are the particles with a position of order $\sqrt{t}$ at time $t$ and we call them the \textit{front} of the BBM.

A way to pick a typical particle of the front is to consider the \textit{critical derivative measure} defined by

$$\sum_{u \in N(t)} X_u(t) e^{-X_u(t)} \delta_{X_u(t)/\sqrt{t}}.$$ 

This measure converges weakly as $t \to \infty$: more precisely, Madaule [16] (for the branching random walk) and Maillard and Zeitouni [18] proved that, for any $F: \mathbb{R} \to \mathbb{R}$ continuous and bounded, we have

$$Z_t(F) := \sum_{u \in N(t)} X_u(t) e^{-X_u(t)} F \left( \frac{X_u(t)}{\sqrt{t}} \right) \underset{t \to \infty}{\longrightarrow} \mathbb{E}[F(R_1)]Z_{\infty}, \quad \text{in probability}, \quad (1.4)$$

where $(R_s)_{s \geq 0}$ denotes a 3-dimensional Bessel process. Note that this convergence generalizes (1.2) for the critical additive martingale. It shows that the derivative martingale is the key quantity to describe the front of BBM.
In a previous paper [17], the authors studied the fluctuations of the derivative martingale around its limit and proved that, under $\mathbb{P}^*$,

$$\sqrt{t} \left( Z_t - Z_\infty - \frac{\log t}{\sqrt{2\pi t}} Z_\infty \right) \overset{\text{in law}}{\longrightarrow} S_Z, \quad t \to \infty,$$

where $S_Z$ is a random mixture of totally asymmetric 1-stable laws. More precisely, for any $\lambda \in \mathbb{R}$, they show that

$$\mathbb{E} \left[ \exp \left( i\lambda \sqrt{t} \left( Z_t - Z_\infty - \frac{\log t}{\sqrt{2\pi t}} Z_\infty \right) \right) \right]_{\mathcal{F}_t} \overset{\text{as } t \to \infty}{\longrightarrow} \exp \left( -Z_\infty \left[ \frac{\pi}{2} |\lambda| - i\lambda \left( \sqrt{\frac{2}{\pi}} \log |\lambda| + \sqrt{\frac{2}{\pi}} \mu_Z \right) \right] \right), \quad \text{in probability},$$

with also an extension to a functional convergence. Here, the constant $\mu_Z$ is defined as follows\(^1\):

$$\mu_Z = \lim_{x \to \infty} \mathbb{E}[Z_\infty 1_{Z_\infty \leq x}] - \log x - \gamma + 1,$$

where $\gamma$ is the Euler-Mascheroni constant. Moreover, they prove the following explicit control\(^2\) that will turn out to be useful in the sequel: For every $K > 0$ there exists $C > 0$ such that, for any $\delta > 0$ and $t \geq 2$, we have

$$\mathbb{P}(|Z_\infty - Z_t| \geq \delta) \leq t^{-K} + C \frac{(\log t)^2}{\delta \sqrt{t}}. \quad (1.7)$$

In this paper, we want to generalize the convergence (1.5) to all the $Z_t(F)$ and, in particular, study the fluctuations of the additive martingale $W_t$ around its limit, which corresponds to the case $F : x \mapsto 1/x$.

We now state the main result of the paper. Note that we consider functions such that $F(x)$ to diverge as $x^{-\alpha}$ at 0 for some $\alpha \in [0, 2)$. This allows us to apply our result to the additive martingale.

**Theorem 1.1.** Let $F : \mathbb{R} \to \mathbb{R}$. Assume $F$ is twice differentiable on $(0, \infty)$ and, for any $x > 0$, $|F''(x)| \leq C x^{-\alpha - 2} e^{C x}$ for some $\alpha \in [0, 2)$ and $C > 0$. Let $H(u) := \mathbb{E}[F(R_{1 u} 1_{u < 1}) - F(R_1)]$ for $u \geq 0$. Then, conditionally on $\mathcal{F}_{\varepsilon t}$, as $t \to \infty$ and then $\varepsilon \to 0$, the finite-dimensional distributions of

$$\sqrt{t} \cdot \left( Z_{at}(F) - \mathbb{E}[F(R_1)] Z_\infty + \frac{\log t}{2\sqrt{at}} Z_\infty \sqrt{\frac{2}{\pi}} \int_0^\infty H(u) d\left(-\frac{1}{\sqrt{u}}\right)\right)_{a \in (0, \infty)}$$

converge weakly in probability to

$$\left( \int_0^\infty H(u) dL_{-Z_\infty/\sqrt{\alpha u}} \right)_{\alpha \in (0, \infty)},$$

where $(L_s)_{s \in \mathbb{R}}$ is a spectrally positive 1-stable Lévy process indexed by $\mathbb{R}$ independent of $Z_\infty$ and with characteristic function given by

$$\mathbb{E} \left[ e^{i\lambda (L_s - L_1)} \right] = \exp \left( -|\lambda| + i\lambda \frac{2}{\pi} (\log |\lambda| - \mu_Z) \right).$$

\(^1\)In [17], the +1 in the definition of $\mu_Z$ was mistakenly missing, as pointed out to us by the authors of [7].

\(^2\)In fact, the statement is slightly weaker in [17], but the proof can be readily adapted so as to yield the current statement, by following the proof idea of Proposition 4.1.
Remarks. Our proof identifies precisely the particles responsible for the fluctuations. These are the particles coming below the level $\frac{1}{2}\log t + K$ after time $\varepsilon t$ for $K$ a large constant. See the end of Subsection 1.2 for more details.

Remark 1.3. This result can be directly extended to the joint convergence of the same process but for different functions $F_1, \ldots, F_n: \mathbb{R} \to \mathbb{R}$. Indeed, to prove this it is enough to apply the previous theorem to the function $F = \sum_{i=1}^n \lambda_i F_i$ for any $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

Remark 1.4. In [17], the fluctuations of the derivative martingale (see (1.6)) have been proved under the assumption $\mathbb{E}[L(\log_++ L)^3] < \infty$, instead of $\mathbb{E}[L^2] < \infty$. We believe that Theorem 1.1 also holds under this weaker assumption and that it is optimal. Nevertheless, the proof of fluctuations of the derivative Gibbs measure in this paper involves several additional technicalities and, therefore, we chose to avoid the truncation arguments needed to work under the assumption $\mathbb{E}[L(\log_++ L)^3] < \infty$.

Applying this to the function $F: x \mapsto 1/x$, we obtain the fluctuations in the convergence of $\sqrt{t}W_t$ towards $\sqrt{2/\pi}Z_\infty$ in (1.2). This confirms a conjecture by Mueller and Munier [19] in the physics literature, who also noticed that, in this case, the $(\log t)/\sqrt{t}$ term cancels. Furthermore, the limiting Lévy process in this case is Cauchy. We state it here only as a one-dimensional convergence.

Corollary 1.5. We have the following convergence in distribution

$$\sqrt{t} \left( \sqrt{t}W_t - \sqrt{\frac{2}{\pi}}Z_\infty \right) \xrightarrow{t \to \infty} S_{Z_\infty},$$

where $(S_r)_{r \geq 0}$ is a Cauchy Lévy process independent of $Z_\infty$ with $\mathbb{E}[e^{iAS_1}] = \exp(-2|\lambda| - \frac{2\log 2}{\pi}i\lambda)$.

1.2 Motivations and comments

The last decade has seen tremendous activity on the extremes of branching Brownian motion and other systems belonging to the so-called $F$-KPP universality class, see the references in [17]. The fluctuations of the extremes of branching Brownian motion have arisen recent interest [19, 17, 20] because of the manifestation of a certain universality of the fluctuations, witnessed by the ubiquity of 1-stable distributions. This contrasts with the $\alpha$-stable distributions with $\alpha > 1$ appearing in the fluctuations of the subcritical additive martingales of the branching random walk, which have been studied by Rösler, Topchii and Vatutin [22], Iksanov and Kabluchko [11], Iksanov, Kolesko and Meiners [12, 13] and Hartung and Klímovsky [10], also in the case of complex BBM.

This article is a continuation of the article [17], in which we studied the fluctuations of the derivative martingale $Z_t$. This martingale, which plays a crucial role in the description of the extremal particles, can be seen as the partition function of the derivative Gibbs measure defined above. This Gibbs measure is supported by particles at distance of order $\sqrt{t}$ from the extremes. It is natural to consider the more general functionals $Z_t(F)$ of the Gibbs measure, supported by the same particles. A specific example is the critical additive martingale $W_t$, for which Mueller and Munier [19] conjectured the result proven in Corollary 1.5. In this example, the 1-stable distribution arising in the description of the fluctuations is actually a (symmetric) Cauchy distribution. The underlying reason for this is not entirely clear to us, see the end of Section 2 for more details. Moreover, it is fascinating that all possible values of the asymmetry parameter can be obtained through varying $F$, which is a new feature compared to previous results.

Another motivation for studying the whole family $Z_t(F)$ allows to explore what happens when the function $F$ becomes more and more singular at the origin. In this article, if $F(x) \sim x^{-\alpha}$ as $x \to 0$, for some $\alpha < 2$, we show that the fluctuations are still of order $1/\sqrt{t}$. We believe that this
value is sharp, since the constant \( \int_0^\infty H(u) \frac{du}{u} \) from Theorem 1.1 becomes infinite or ill-defined at \( \alpha = 2 \), as one can easily check. In contrast, when \( \alpha > 3 \), we expect the functional \( Z_t(F) \) to be supported by the extremal particles, since the function then is not integrable anymore against the law of the Bessel-3 process started at zero. Hence, we expect the fluctuations to be of order 1. This motivates the following conjecture:

**Conjecture 1.6.** If \( 2 < \alpha < 3 \), the fluctuations of \( Z_t(F) \) are of order \( t^\beta \) with \( -\frac{1}{2} < \beta < 0 \).

In fact, Conjecture 1.6 should be related to recent work by Mytnik-Roquejoffre-Ryzhik [20]. In that work, the authors study the fluctuations of the number of particles in the limiting extremal process. They show that the number of particles below a high level \( x \) has fluctuations of order \( 1/x \). We expect that the limiting process describes well the particles at time \( t \) as long as \( x \ll \sqrt{t} \). Indeed, the density of the limiting process grows as \( xe^x \) [9], which is also the asymptotic behaviour of the density predicted by (1.4). Hence, it seems reasonable to assume that the number of particles in branching Brownian motion at distance \( t^\beta \) from the minimum at time \( t \), \( 0 < \beta < 1/2 \), also has fluctuations of order \( O(t^{-\beta}) \), and that it is these particles that contribute to the fluctuations of \( Z_t(F) \) when \( 2 < \alpha < 3 \).

In contrast, the current article treats the particles at distance of order \( \sqrt{t} \) from the minimum. We expect this to be the regime where the comparison with the extremal process breaks down, i.e. where the fluctuations predicted by [20] should cease to hold.

**Proof ideas.** Let us briefly describe the main ideas of the proof. A more detailed overview is provided in Section 2. We start by recalling the idea from [17] to study the fluctuations of the derivative martingale \( Z_t \): we introduce a “barrier” from time \( t \) on at the point \( \frac{1}{2} \log(t + \beta_t) \), where \( \beta_t \to \infty \) slowly enough. The goal is to treat specially the particles that hit the barrier, which are exactly those contributing to the fluctuations of \( Z_t \). Indeed, the contribution to \( Z_\infty \) of the descendants of such a particle is equal in law to \( e^{-\beta_t} Z_\infty / \sqrt{t} \), and the number of such particles is of the order \( e^{\beta_t} \). Since \( Z_\infty \) is in the domain of attraction of a 1-stable distribution, this leads to the 1-stable Lévy process.

The above-mentioned proof idea relied on the fact that a suitably shifted version of \( Z_t \) is a martingale when the particles are killed at the barrier. Indeed, when restricting to the particles that do not hit the barrier, the conditional expectation of \( Z_\infty \) conditioned on \( \mathcal{F}_t \) is equal to this shifted version of \( Z_t \), allowing the use of Chebychev’s inequality in order to bound the difference between \( Z_t \) and \( Z_\infty \), restricted to these particles. For general functionals \( Z_t(F) \), this strategy breaks down and we have to compare \( Z_t(F) \) with \( Z_s \) at a time \( s \) before the time \( t \). However, when adapting the “barrier strategy” mentioned above, it turns out that one cannot get optimal estimates in one step. We therefore implement a *multiscale*, or *bootstrap strategy*: We introduce a barrier between times \( t^a \) and \( t \), estimate the particles not touching the barrier by first an second moments, and the particles hitting the barrier by first moments. This yields a first estimate on the fluctuations of \( Z_t(F) \) of order \( t^{-b} \), for some \( b \) depending on \( a \). Then we iterate: we take another \( a' > a \) and introduce again a barrier between times \( t^{a'} \) and \( t \). However, we now make use of our previous estimate on fluctuations at two places: first, to get a good estimate on the conditional expectation of \( Z_t(F) \) conditioned on \( \mathcal{F}_{t^{a'}} \), and second, in order to estimate the contribution to \( Z_t(F) \) of the descendants of particles hitting the barrier. With this multiscale argument, after a finite number of steps (in fact, only two), the errors become smaller that \( 1/\sqrt{t} \), and we are able to conclude.

This idea makes it clear that the fluctuations come again from the contributions to \( Z_t(F) \) of the killed particles and helps to read the definition of the limit in Theorem 1.1. Indeed, for \( u > 0 \), the number of particles hitting the barrier between times \( ut \) and \( (u+du)t \) is approximately \((2\pi)^{-1}Z_\infty e^{\beta_t}u^{-3/2}du\). Moreover, if \( u < 1 \), each of them contributes by (1.4) approximately as an independent copy of \( e^{-\beta_t} \mathbb{E}[F(\sqrt{1 - ut}) - F(R_1)]Z_\infty \) to the difference \( Z_t(F) - \mathbb{E}[F(R_1)]Z_\infty \).
On the other hand, if $u > 1$, the killed particles contributes only to $-\mathbb{E}[F(R_1)]Z_\infty$, so its contribution is an independent copy of $e^{-\beta t}\mathbb{E}[-F(R_1)]Z_\infty$. Note that the fluctuations of the front are due to the particles that come extremely low, around $\frac{1}{2}\log t$, which is far below the usual minimum of BBM at a time of order $t$.

### 1.3 Organization of the paper and notation

The paper is organized as follows. In Section 2, we present the strategy for the proof of Theorem 1.1, which is obtained through several steps in Sections 4 to 7. Section 3 is devoted to preliminary results concerning branching Brownian motion, in particular when particles are killed at 0. Other technical results concerning the 3-dimensional Bessel process are shown in Section B.

In the sequel, we denote by $C$ a positive constant that can change from line to line and depend implicitly on some parameters indicated in the statement of each result. Moreover, for $x \in \mathbb{R}$, we set $x_+ = \max(x, 0)$. For $\varphi_1: \mathbb{R}_+ \to \mathbb{R}$ and $\varphi_2: \mathbb{R}_+ \to \mathbb{R}_+$, we say that $\varphi_1(t) = O(\varphi_2(t))$ as $t \to \infty$ if $\lim_{t \to \infty} \frac{\varphi_1(t)}{\varphi_2(t)} = 0$ and that $\varphi_1(t) = O(\varphi_2(t))$ as $t \to \infty$ if $\limsup_{t \to \infty} |\varphi_1(t)|/\varphi_2(t) < \infty$.

Let $(B_t)_{t \geq 0}$ denote a standard Brownian motion and $(R_t)_{t \geq 0}$ a 3-dimensional Bessel process. For $u, v \in \mathbb{T}$, we say that $u \leq v$ if $v$ is a descendant of $u$. If $n \geq 1$ is an integer, we set $[1, n] := \{1, 2, \ldots, n\}$.

### 2 Strategy of the proof

A consequent part of this paper will be dedicated to the proof of Proposition 2.1 below, which gives the fluctuations of a translated version of $Z_t(F)$ defined by

$$Z_t(F, \gamma_t) := \sum_{u \in N(t)} (X_u(t) - \gamma_t) + e^{-X_u(t)}F\left(\frac{X_u(t) - \gamma_t}{\sqrt{t}}\right),$$

where we take $\gamma_t := \frac{1}{2}\log t + \beta_t$ with $(\beta_t)_{t \geq 0}$ a family of positive real numbers such that

$$\beta_t \xrightarrow{t \to \infty} \infty \quad \text{and} \quad \frac{\beta_t}{\log t} \xrightarrow{t \to \infty} 0. \quad (2.1)$$

For the study of the fluctuations of the derivative martingale in [17], $Z_t(1, \gamma_t)$ was also the most appropriate quantity to look at.

Moreover, we will work at first under the following stronger assumptions for the function $F: \mathbb{R} \to \mathbb{R}$: for some $\kappa \geq 1$, we will assume

(A1$_\kappa$) For any $x > 0$, $|F(x)| \leq e^{\kappa x}$;

(A2$_\kappa$) For any $0 < y \leq x$, $|F(x) - F(y)| \leq (x - y)e^{\kappa x}$.

**Proposition 2.1.** Let $\kappa \geq 1$. For any function $F: \mathbb{R} \to \mathbb{R}$ satisfying (A1$_\kappa$)-(A2$_\kappa$), we have

$$\sqrt{t}\left(Z_t(F, \gamma_t) - \mathbb{E}[F(R_1)]Z_\infty \right) \overset{(law)}{\longrightarrow} S(F),$$

where $S(F)$ is a r.v. such that $\mathbb{E}[e^{i\lambda S(F)}]Z_\infty = \exp(-Z_\infty[c_2(F)|\lambda| + i\lambda(c_1(F)\log|\lambda| + c_3(F))])$ and, recalling $H(u) := \mathbb{E}[F(\sqrt{1-u}R_1)1_{u<1}-F(R_1)]$ for $u \geq 0$,

$$c_1(F) := \frac{1}{\sqrt{2\pi}} \int_0^1 H(u) \frac{du}{u^{3/2}},$$

$$c_2(F) := \frac{1}{2} \sqrt{\frac{\pi}{2}} \int_0^1 |H(u)| \frac{du}{u^{3/2}},$$

$$c_3(F) := \frac{1}{\sqrt{2\pi}} \int_0^1 H(u)(-\mu_Z + \log H(u)) \frac{du}{u^{5/2}}.$$
We indicate here the major steps of the proof of Proposition 2.1. We consider some $a \in (0, 1)$ and set
\[
\tilde{Z}_t^{\kappa, a}(F; \gamma_t) := \sum_{u \in \mathcal{N}(t)} (X_u(t) - \gamma_t) + e^{-X_u(t)} \mathbb{1}_{r \in [t^a, T]} \mathbb{1}_{X_u(r) > \gamma_t} F\left(\frac{X_u(t) - \gamma_t}{\sqrt{t}}\right),
\]
where particles are killed when coming below $\gamma_t$ during $[t^a, t]$. Let $\mathcal{L}^{\kappa, a}$ denote the stopping line of the killed particles and, for $u \in \mathcal{L}^{\kappa, a}$, $T_u$ the time of the death of $u$

\[
\mathcal{L}^{\kappa, a} := \{u \in T : \text{there exists } s \in [t^a, t] \text{ s.t. } u \in \mathcal{N}(s), X_u(s) \leq \gamma_t \text{ and } \forall r \in [t^a, s), X_u(r) > \gamma_t\};
\]

\[
T_u := \inf\{s \geq t^a : u \in \mathcal{N}(s) \text{ and } X_u(s) \leq \gamma_t\},
\]

see Figure 1 for an illustration. Let $\mathcal{F}_{\mathcal{L}^{\kappa, a}}$ be the $\sigma$-algebra associated with the stopping line $\mathcal{L}^{\kappa, a}$ (see Chauvin [8] for formal definitions). Then, we have the following decomposition
\[
Z_t(F; \gamma_t) - \mathbb{E}[F(R_1)]Z_t(1, \gamma_t) = \tilde{Z}_t^{\kappa, a}(F - \mathbb{E}[F(R_1)]), \gamma_t) + \sum_{u \in \mathcal{L}^{\kappa, a}} \Omega_t^{(u)},
\]
where $\Omega_t^{(u)}$ is the contribution of the progeny of the killed particle $u$ in $Z_t(F; \gamma_t)$, defined by
\[
\Omega_t^{(u)} := \sum_{u \in \mathcal{N}(t) \text{ s.t. } u \leq v} (X_u(t) - \gamma_t) + e^{-X_u(t)} \left(F\left(\frac{X_u(t) - \gamma_t}{\sqrt{t}}\right) - \mathbb{E}[F(R_1)]\right).
\]

Then, if $a \in (2/3, 3/4)$, Proposition 2.1 is a consequence of the two following propositions, that show that the fluctuations of $Z_t(F)$ come from the particles hitting $\gamma_t$ between times $t^a$ and $t$.

**Proposition 2.2.** Let $F : \mathbb{R} \to \mathbb{R}$ satisfying Assumptions (A1$_a$)-(A2$_a$) If $a \in (1/2, 3/4)$, then we have the following convergence in probability
\[
\sqrt{t} \tilde{Z}_t^{\kappa, a}(F - \mathbb{E}[F(R_1)]), \gamma_t) \xrightarrow{(P) t \to \infty} 0.
\]

**Proposition 2.3.** Let $F : \mathbb{R} \to \mathbb{R}$ satisfying Assumptions (A1$_a$)-(A2$_a$) and $S(F)$ as defined in Proposition 2.1. If $a \in (2/3, 1)$, then we have the following convergence in distribution
\[
\sqrt{t} \left(\sum_{u \in \mathcal{L}^{\kappa, a}} \Omega_t^{(u)} - c_1(F) \frac{\beta F}{\sqrt{t}} Z_\infty\right) \xrightarrow{(law) t \to \infty} S(F).
\]
The main tool to prove these propositions is a first concentration result for $Z_{\ell}(F,\gamma_{\ell}) - \mathbb{E}[F(R_{1})]Z_{\ell}(1,\gamma_{\ell})$ at the scale $t^{-1/3}$, see Proposition 4.1. It is proved using the same decomposition (2.2), with the choice $a = 2/3$ and $\gamma_{\ell} = \frac{1}{2} \log t$, but with cruder bounds: the control of $\tilde{Z}_{\ell}^{a,\gamma_{\ell}}(F - \mathbb{E}[F(R_{1})],\gamma_{\ell})$ is obtained via a first and second moment argument given $\mathcal{F}_{\ell}$ and the contributions of killed particles are controlled using only a first moment bound with an additional barrier at 0. This is done in Section 4.

In Section 5, we study more carefully the terms in decomposition (2.2) to prove Proposition 2.2 and Proposition 2.3. We control $\tilde{Z}_{\ell}^{a,\gamma_{\ell}}(F - \mathbb{E}[F(R_{1})],\gamma_{\ell})$ using again a first and second moment calculation given $\mathcal{F}_{\ell}$. The main difference is that we can now use Proposition 4.1 to improve our estimate of the conditional first moment. This is done in Subsection 5.1.

For the contributions of killed particles, note that, conditionally on $\mathcal{F}_{L^a,\gamma_{\ell}}$, the $\Omega_{\ell}^{(a)}$ for $u \in L^{a,\gamma_{\ell}}$ are independent random variables with the same law as $e^{-\gamma_{\ell}}Z_{t-T_{u}}(F - \mathbb{E}[F(R_{1})])$ (but still with a $\sqrt{t}$ scaling in the function $F$). Hence, we can use Proposition 4.1 to replace each $\Omega_{\ell}^{(a)}$ by an independent copy of $e^{-\gamma_{\ell}}H(T_{u,t}/t)Z_{\infty}$, see Subsection 5.2. Then, we have to estimate the sum of these random variables along the stopping line to prove Proposition 2.3 in Subsection 5.4.

Remaining steps. We now explain the strategy to obtain Theorem 1.1 from Proposition 2.1. In Section 6, we analyze precisely the difference $Z_{\ell}(F,\gamma_{\ell}) - Z_{\ell}(F)$, even for functions $F$ diverging at 0. This difference is of order $\gamma_{\ell}/\sqrt{t}$ and is responsible for $(\log t)/\sqrt{t}$ term appearing in Theorem 1.1. This explains in particular why there is no such term for the fluctuations of the additive martingale in Corollary 1.5: indeed $W_{t}$ is invariant by shifts (that is if $F(x) = 1/x$, $Z_{\ell}(F,\gamma_{\ell}) = Z_{\ell}(F)$, up to the contribution of particles below $\gamma_{\ell}$ at time $t$).

Theorem 1.1 is proved in Section 7. We first prove in Proposition 7.1 a version of Theorem 1.1 for functions $F$ satisfying Assumptions (A1a)-(A2a) and with $Z_{at}(F)\gamma_{\ell}$ instead of $Z_{at}(F)$. For this, we introduce again a barrier at level $\gamma_{\ell}$, but between times $ct$ and $\infty$ and proceed in a similar way as described above. The main difference is that the conditional first moment of $Z_{at}(F,\gamma_{\ell}) - \mathbb{E}[F(R_{1})]$ without the killed particles is not negligible: instead we estimate it precisely using Proposition 2.1.

Then the main remaining task is to generalize the result to functions diverging at 0. For this, we compare $Z_{at}(F,\log t)$ with its conditional expectation given $\mathcal{F}_{(1-\varepsilon)at}$, once particles have been killed at level $c\log t$ between times $(1-\varepsilon)at$ and $at$, where $c$ has to be taken closer and closer to $\frac{3}{4}$ when $\alpha$ approaches 2. Then, this conditional expectation can be written as $Z_{(1-\varepsilon)t}(F^{1-\varepsilon},\log t)$ where $F_{1-\varepsilon}$ is now bounded close to zero, so we can apply the first version of the theorem to this function to conclude.

3 Preliminary results

3.1 Some results concerning branching Brownian motion

In the sequel, we allow the BBM to start at an arbitrary point $x \in \mathbb{R}$, in which case we write $\mathbb{P}_{x}$ and $\mathbb{E}_{x}$ instead of $\mathbb{P}$ and $\mathbb{E}$. For a particle $u \in \mathcal{N}(t)$ and $s \leq t$, we write $X_{u}(s)$ for the position of the ancestor of $u$ alive at time $s$. Under our settings, the many-to-one formula is stated as follows: for any $x \in \mathbb{R}$, $t \geq 0$ and any measurable function $F: \mathcal{C}([0,t]) \to \mathbb{R}_{+}$, where $\mathcal{C}([0,t])$ denotes the set of continuous function from $[0,t] \to \mathbb{R}$, we have

$$
\mathbb{E}_{x}\left[\sum_{u \in \mathcal{N}(t)}e^{-X_{u}(t)}F(X_{u}(s), s \in [0,t])\right] = e^{-x}\mathbb{E}_{x}[F(B_{s}, s \in [0,t])],
$$

where $(B_{s})_{s \geq 0}$ denotes a standard Brownian motion, starting from $x$ under $\mathbb{P}_{x}$. We will use repetitively the following consequence, using the link (B.3) between the Brownian motion staying
positive and the 3-dimensional Bessel process: for any $x > 0$, $t \geq 0$ and any measurable function $\varphi : \mathbb{R} \to \mathbb{R}_+$, we have

$$E_x \left[ \sum_{u \in \mathcal{N}(t)} e^{-X_u(t)} \varphi(X_u(t)) 1_{\forall s \in [0,t], X_u(s) > 0} \right] = x e^{-x} E_x \left[ \frac{\varphi(R_t)}{R_t} \right], \quad (3.2)$$

where $(R_s)_{s \geq 0}$ denotes a 3-dimensional Bessel process, starting from $x$ under $P_x$.

Since we assumed that $E[L^2] < \infty$, the many-to-two formula holds (see [6, Lemma 10]). We only state it here in a special case: for any $x > 0$, $t \geq 0$ and any measurable function $\varphi : \mathbb{R} \to \mathbb{R}_+$, we have

$$E_x \left[ \left( \sum_{u \in \mathcal{N}(t)} e^{-X_u(t)} \varphi(X_u(t)) 1_{\forall s \in [0,t], X_u(s) > 0} \right)^2 \right] = Ke^{-x} \int_0^t dr \int_0^\infty E_y \left[ \sum_{u \in \mathcal{N}(t-r)} e^{-X_u(t-r)} \varphi(X_u(t-r)) 1_{\forall s \in [0,t-r], X_u(s) > 0} \right]^2 e^y q_r(x, y) dy + e^{-x} \int_0^\infty e^{-y} \varphi(y)^2 q_r(x, y) dy, \quad (3.3)$$

where $q_r(x, y) := (2\pi r)^{-1/2} (e^{-(x-y)^2/2r} - e^{-(x+y)^2/2r})$ for any $x, y > 0$ and $K$ is a positive constant. Note that $q_r(x, \cdot)$ is the density of a Brownian motion at time $r$ starting from $x$ and killed at 0.

We conclude this subsection by stating two useful bounds concerning the minimum of the BBM, see Equations (C.1) and (C.2) of [17]). The first one concerns the global minimum of the BBM: for any $M > 0$, we have

$$P \left( \exists s \geq 0, \min_{u \in \mathcal{N}(s)} X_u(s) \leq -M \right) \leq e^{-M} . \quad (3.4)$$

The second one deals with the minimum at a given time: there exists $C > 0$ such that, for any $s \geq 2$ and $x \in (-\infty, \sqrt{s}]$,

$$P \left( \min_{u \in \mathcal{N}(s)} X_u(s) \leq \frac{3}{2} \log s - x \right) \leq C(1 + x^2)e^{-x}. \quad (3.5)$$

3.2 Branching Brownian motion killed at 0

In this subsection, we prove preliminary results controlling first and second moments for the following quantity of BBM killed at 0, defined for $\Delta \in \mathbb{R}$ and $t \geq r \geq 0$ by

$$\tilde{Z}_r(F, \Delta) = \tilde{Z}_r(F, \Delta, t) := \sum_{u \in \mathcal{N}(r)} (X_u(r) - \Delta) 1_{X_u(r) > 0} e^{-X_u(r)} 1_{\forall q \in [0, r], X_u(q) > 0} F \left( \frac{X_u(r) - \Delta}{\sqrt{t}} \right). \quad (3.6)$$

Note that the $\sqrt{t}$ scaling in the function $F$ is not necessarily corresponding to the time $r$ where particles are considered. Moreover, since the scaling in the function $F$ is very frequently $\sqrt{t}$ in this paper, we usually skip the dependence in $t$ in the notation $\tilde{Z}_r(F, \Delta)$ and use the whole notation only when the scaling is not $\sqrt{t}$. Most results will be stated in the case $\Delta \geq 0$ to avoid some annoying error terms.

**Lemma 3.1.** For any $x > 0$, $\Delta \in \mathbb{R}$ and $t \geq r \geq 0$, we have the two following bounds

$$E_x \left[ \tilde{Z}_r(1, \Delta) \right] = e^{-x} \left( 1 \wedge \frac{x}{\sqrt{t}} \right) \left( 1 \wedge \frac{\Delta}{\sqrt{r}} \right),$$

$$E_x \left[ \tilde{Z}_r(1, \Delta) \right] \leq Ce^{-x} \left( x + |\Delta| \left( 1 \wedge \frac{x}{\sqrt{r}} \right) \right).$$

9
Proof. Using the many-to-one formula (3.2), we have
\[
\mathbb{E}_x[Z_r(1, \Delta)] = x e^{-x} \mathbb{E}_x \left[ \frac{(R_r - \Delta)_+}{R_r} \right]
\]
and, therefore, we get
\[
\left| \mathbb{E}_x[Z_r(1, \Delta)] - xe^{-x} \right| \leq xe^{-x} \mathbb{E}_x \left[ \frac{(R_r - \Delta)_+ - R_r}{R_r} \right] \leq xe^{-x} \mathbb{E}_x \left[ \frac{\Delta}{R_r} \right] = xe^{-x} \frac{\Delta}{\sqrt{r}} \mathbb{E}_{x/\sqrt{r}} \left[ \frac{1}{R_1} \right]
\]
using Lemma B.1, and the first inequality is proved. The second one follows. \( \square \)

Lemma 3.2. Let \( \kappa \geq 1 \). There exists a constant \( C = C(\kappa) > 0 \), such that for any function \( F : \mathbb{R} \to \mathbb{R} \) satisfying Assumptions (A1\( \kappa \))-(A2\( \kappa \)) and any \( x > 0, \ t \geq r > 0 \) and \( \Delta \geq 0 \), we have
\[
\mathbb{E}_x\left[ Z_r(F - \mathbb{E}[F(R_1)], \Delta) \right] \leq C xe^{-x} \left( \frac{x^2}{r} \left( 1 + \frac{x^4}{r^2} \right)e^{\kappa x/\sqrt{r}} + \frac{\Delta}{\sqrt{r}} + \frac{t - r}{\sqrt{tr}} \right),
\]
\[
\mathbb{E}_x\left[ \left| Z_r(F - \mathbb{E}[F(R_1)], \Delta) \right| \right] \leq C xe^{-x} \left( e^{\kappa x/\sqrt{r}} + \frac{\Delta}{\sqrt{r}} \left( 1 + \frac{r}{\sqrt{r}} \right) \right).
\]

Proof. For the first bound, using the many-to-one formula (3.2), we have
\[
\mathbb{E}_x\left[ Z_r(F, \Delta) \right] = xe^{-x} \mathbb{E}_x \left[ \frac{1}{R_1}(R_r - \Delta)_+ F \left( \frac{R_r - \Delta}{\sqrt{t}} \right) \right] = xe^{-x} \mathbb{E}_{x/\sqrt{r}} \left[ \frac{(R_1 - \Delta)_+}{R_1} F \left( \frac{1}{\sqrt{t}} \left( R_1 - \frac{\Delta}{\sqrt{t}} \right) \right) \right],
\]
using the scaling property of the Bessel process. Then, we use the following decomposition:
\( Z_r(F - \mathbb{E}[F(R_1)], \Delta) = (Z_r(F, \Delta) - \mathbb{E}[F(R_1)]xe^{-x}) - \mathbb{E}[F(R_1)](Z_r(1, \Delta) - xe^{-x}) \). On the other hand, applying (3.7), we have
\[
\left| \mathbb{E}_x[Z_r(F, \Delta)] - \mathbb{E}[F(R_1)]xe^{-x} \right| = xe^{-x} \mathbb{E}_{x/\sqrt{r}} \left[ \frac{(R_1 - \Delta)_+}{R_1} F \left( \frac{1}{\sqrt{t}} \left( R_1 - \frac{\Delta}{\sqrt{t}} \right) \right) \right] - \mathbb{E}[F(R_1)] \]
\[
\leq C xe^{-x} \left( \frac{x^2}{r} \left( 1 + \frac{x^4}{r^2} \right)e^{\kappa x/\sqrt{r}} + \frac{\Delta}{\sqrt{r}} + \frac{t - r}{\sqrt{tr}} \right),
\]
using Lemma B.3 with \( \alpha = \sqrt{r}/t, \ y = x/\sqrt{r}, \ \eta = \Delta/\sqrt{r} \) and the fact \( 1 - \sqrt{r}/t \leq (t - r)/\sqrt{tr} \).

On the other hand, using Lemma 3.1 and that \( \mathbb{E}[F(R_1)] \leq C \) by Assumption (A1\( \kappa \)), we get
\[
\left| \mathbb{E}[F(R_1)](\mathbb{E}_x[Z_r(1, \Delta)] - xe^{-x}) \right| \leq C xe^{-x} \frac{\Delta}{\sqrt{r}}.
\]
Combining (3.8) and (3.9), it proves the first bound.

We now deal with the second bound. Proceeding as in (3.7), we get
\[
\mathbb{E}_x\left[ \left| Z_r(F, \Delta) \right| \right] \leq xe^{-x} \mathbb{E}_{x/\sqrt{r}} \left[ \frac{(R_1 - \Delta)_+}{R_1} F \left( \frac{1}{\sqrt{t}} \left( R_1 - \frac{\Delta}{\sqrt{t}} \right) \right) \right] \]
\[
\leq C xe^{-x} e^{\kappa x/\sqrt{r}},
\]
(3.10)
using Lemma B.5. Furthermore, using Lemma 3.1, we have

\[ E_x \left[ \left| E[F(R_1)] \tilde{Z}_r(1, \Delta) \right| \right] \leq C e^{-x} \Delta \left( 1 \wedge \frac{x}{\sqrt{r}} \right) = C \sqrt{\frac{t}{r}} e^{-x} \Delta \left( 1 \wedge \frac{\sqrt{r}}{x} \right), \]

and the second bound follows.

**Remark 3.3.** Let \( \kappa \geq 1 \). Let \( F: \mathbb{R} \to \mathbb{R} \) be such that \( |F(x)| \leq \kappa (1 + x^{-1}) e^{\kappa x} \) for any \( x > 0 \). Then, Equation (3.10) still holds (because we can still apply Lemma B.5 in this case). It follows that there exists \( C = C(\kappa) > 0 \) such that, for \( r \) large enough (depending only on \( \kappa \)), \( M \geq 1 \) and \( \gamma \geq 0 \) satisfying \( \gamma + M \leq \sqrt{r} \), we have

\[ E \left[ |Z_r(F, \gamma, r)| \mathbb{1}_{\forall q \in [0, r], \min_{u \in \mathcal{N}(q)} X_u(q) > -M} \right] \leq e^{M} E_M \left[ |\tilde{Z}_r(F, \gamma + M, r)| \right] \leq CM. \quad (3.11) \]

Moreover, it follows from (3.4) that \( P(\exists q \in [0, r], \min_{u \in \mathcal{N}(q)} X_u(q) \leq -M) \leq e^{-M} \). Therefore, for any \( L > 0 \),

\[ P(|Z_r(F, \gamma, r)| \geq L) \leq e^{-M} + CML^{-1}. \quad (3.12) \]

It follows that the process \( \{|Z_r(F, \gamma, r)|\}_{r > 0} \) is tight. Note that here the scaling in function \( F \) is \( \sqrt{r} \), because this equation also be applied in the sequel with a scaling different from \( \sqrt{r} \).

**Lemma 3.4.** Let \( \kappa \geq 1 \). There exist constants \( C = C(\kappa) > 0 \), \( t_0 = t_0(\kappa) > 0 \), such that for any function \( F: \mathbb{R} \to \mathbb{R} \) satisfying Assumptions (A1\( \kappa \))- (A2\( \kappa \)) and any \( x > 0 \), \( t \geq t_0 \) and \( 2 \leq r \leq t \), we have

\[ E_x \left[ \left( \tilde{Z}_r(F - E[F(R_1)], 0) \right)^2 \right] \leq C e^{-x} \left( 1 + \frac{t - r}{t r}^2 + \frac{x}{2} \right) \]

*Proof.* For brevity, we write \( \overline{F} := F - E[F(R_1)] \). It follows from (3.3) that, with \( q_v(x, y) := (2\pi)^{-1/2} (e^{-((x-y)^2/2v)} - e^{-(x+y)^2/2v}) \),

\[ E_x \left[ \left( \tilde{Z}_r(\overline{F}, 0) \right)^2 \right] = K e^{-x} \int_0^r dv \int_0^\infty E_y \left[ \tilde{Z}_{r-v}(\overline{F}, 0) \right]^2 e^{y} q_v(x, y) \, dy \]

\[ + e^{-x} \int_0^\infty e^{-y} \left( y \overline{F} \left( \frac{y}{\sqrt{t}} \right) \right)^2 q_r(x, y) \, dy. \quad (3.13) \]

We first deal with the second term in the right-hand side of (3.13). Using Assumption (A1\( \kappa \)) and that \( q_r(x, y) \leq Cxy/\sqrt{r}^3/2 \), we get

\[ \int_0^\infty e^{-y} \left( y \overline{F} \left( \frac{y}{\sqrt{t}} \right) \right)^2 q_r(x, y) \, dy \leq C \int_0^\infty e^{-y} \left( ye^{y/\sqrt{r}} \right)^2 \frac{xy}{\sqrt{r}^3/2} \, dy \leq \frac{C x}{\sqrt{r}^3/2}, \]

choosing \( t_0 \) large enough such that \( 2\kappa / \sqrt{t_0} \leq 1/2 \).

Now, we deal with the first term in the right-hand side of (3.13): for this, we split the integral on \([0, r]\) in two pieces. Let start with the part \( v \in [0, r/2] \). Using the first bound of Lemma 3.2 and that \( r - v \geq r/2 \) we have

\[ \left| E_y \left[ \tilde{Z}_{r-v}(\overline{F}, 0) \right] \right| \leq C y e^{-y} \left( \frac{y^2}{r} \left( 1 + \frac{y^4}{r^2} \right) e^{y/\sqrt{r}} + \frac{t - r + v}{\sqrt{t r}} \right) \]

\[ \leq C y e^{-y} \left( \frac{y^2}{r} \left( 1 + \frac{y^4}{r^2} \right) e^{y/5} + \frac{t - r + v}{\sqrt{t r}} \right), \]

11
using that \( r \geq 1 \) and choosing \( t_0 \) large enough such that \( \kappa / \sqrt{t_0} \leq 1 / 5 \). Therefore, noting that \( y^2 (1 + y^3) e^{y/5} \leq C e^{y/4} \), we get

\[
\int_0^{r/2} dv \int_0^\infty \mathbb{E}_y \left[ \tilde{Z}_{r-v}(F,0) \right]^2 e^y q_v(x,y) \, dy
\leq C \int_0^{r/2} dv \int_0^\infty y^2 e^{-y} \left( \frac{e^{y/2}}{r^2} + \left( \frac{t-r}{\sqrt{tr}} \right)^2 + \left( \frac{v}{\sqrt{tr}} \right)^2 \right) q_v(x,y) \, dy
\leq C \int_0^\infty y^2 e^{-y} \left( \frac{e^{y/2}}{r^2} + \left( \frac{t-r}{\sqrt{tr}} \right)^2 + \left( \frac{v}{\sqrt{tr}} \right)^2 \right) q_v(x,y) \, dy + \frac{1}{tr} \int_0^{r/2} v^2 q_v(x,y) \, dv. \quad (3.14)
\]

Then, on the one hand, we have, using that \( q_v(x,y) \leq \sqrt{2/\pi} e^{-v^3/2xy} \),

\[
\int_0^{r/2} v^2 q_v(x,y) \, dv \leq \int_0^{r/2} v^2 \sqrt{\frac{2}{\pi x y}} \, dv \leq C x y ^{3/2},
\]

and, on the other hand, \( \int_0^\infty q_v(x,y) \, dv = 2(x \wedge y) \leq 2y \). Thus, it follows that (3.14) is at most

\[
C \int_0^\infty y^3 e^{-y} \left( \frac{e^{y/2}}{r^2} + \left( \frac{t-r}{\sqrt{tr}} \right)^2 + \left( \frac{v}{\sqrt{tr}} \right)^2 + \left( \frac{v}{\sqrt{t}} \right) \right) \, dy \leq C \left( \frac{1}{r^2} + \left( \frac{t-r}{tr} \right)^2 + \frac{x}{\sqrt{t}} \right),
\]

and it concludes our work with the part \( v \in [0,r/2] \).

Now, we deal with the part \( v \in [r/2,r] \) and have, using the second bound of Lemma 3.2,

\[
\left| \mathbb{E}_y \left[ \tilde{Z}_{r-v}(F,0) \right] \right| \leq C y e^{-y} e^{y/\sqrt{t}} \leq C y e^{-3y/4},
\]

using that \( C/\sqrt{t} \leq 1/4 \). Noting also that \( q_v(x,y) \leq C x y / r^{3/2} \) because \( v \geq r/2 \), we get

\[
\int_0^r dv \int_0^\infty \mathbb{E}_y \left[ \tilde{Z}_{r-v}(F,0) \right]^2 e^y q_v(x,y) \, dy \leq C \int_0^\infty e^{-y/2} \int_0^r \frac{2 x y}{v^{3/2}} \, dv \, dy \leq C \frac{x}{\sqrt{r}}.
\]

This concludes the proof. \( \square \)

**Lemma 3.5.** Let \( \alpha \in (0,2) \) and define \( f_\alpha(x) = x^{-\alpha} \). Then, there exists \( C = C(\alpha) > 0 \), such that for all \( t \geq 2 \),

\[
\mathbb{E}_x [\tilde{Z}_t(f_\alpha,0)^2] \leq C e^{-x} \times \begin{cases} 1 + \frac{x}{\sqrt{t}}, & \alpha < 1 \\ 1 + \frac{x}{\sqrt{t}} \log t, & \alpha = 1 \\ 1 + \frac{x}{\sqrt{t}} t^{\alpha-1}, & \alpha \in (1,2) \end{cases}
\]

**Proof.** We start as in the proof of (3.4). Thus,

\[
\mathbb{E}_x \left[ \left( \tilde{Z}_t(f_\alpha,0)^2 \right) \right] = K e^{-x} \int_0^t dr \int_0^\infty \mathbb{E}_y \left[ \tilde{Z}_{t-r}(f_\alpha,0)^2 \right] e^y q_r(x,y) \, dy + e^{-x} \int_0^\infty e^{-y} \left( y f_\alpha \left( \frac{y}{\sqrt{t}} \right) \right)^2 q_t(x,y) \, dy \quad (3.16)
\]

\[
=: T_1 + T_2.
\]

We start again with the second term. Using the bound \( q_t(x,y) \leq C x y / t^{3/2} \) and the definition of \( f_\alpha \), we have

\[
T_2 \leq C e^{-x} \frac{x}{\sqrt{t}} t^{\alpha-1} \int_0^\infty y^{1+2(1-\alpha)} e^{-y} \, dy, \quad (3.17)
\]

and the last integral is finite since \( \alpha < 2 \).
As for the first term, we first note that by the many-to-one lemma, we have
\[ e^{y}E_{y}[\bar{Z}_{t-r}(f_{\alpha},0)] = yE_{y}\left[ f_{\alpha}\left(\frac{R_{t-r}}{\sqrt{t}}\right)\right] = yt^{\alpha/2}E_{y}\left[ (R_{t-r})^{-\alpha}\right] \]  
(3.18)

We now split the integral over \( r \) in the definition of \( T_{t} \) into three parts, according to whether \( r \in [0,t/2], \ r \in [t/2, t-1] \) or \( r \in [t-1,t] \). For the first part, note that by Bessel scaling, we have
\[ E_{y}\left[ (R_{t-r})^{-\alpha}\right] = (t-r)^{-\alpha/2}E_{y}/\sqrt{r}[R_{1}^{-\alpha}] \leq E_{0}[R_{1}^{-\alpha}](t-r)^{-\alpha/2} \]  
(3.19)

where the last inequality comes from standard comparison results for one-dimensional diffusions, using that \( x^{-\alpha} \) is decreasing in \( x \). Using that \( E_{0}[(R_{1})^{-\alpha}] \) is finite since \( \alpha < 3 \), (3.18) and (3.19) give, for every \( r \in [0,t/2] \),
\[ e^{y}E_{y}[\bar{Z}_{t-r}(f_{\alpha},0)] \leq Cy. \]

It follows that
\[ \int_{0}^{t/2} dr \int_{0}^{\infty} E_{y}\left[ \bar{Z}_{t-r}(f_{\alpha},0)\right]^{2} e^{y}q_{r}(x,y) dy \leq C \int_{0}^{t/2} dr \int_{0}^{\infty} y^{2}e^{-y}q_{r}(x,y) dy \]  
(3.20)

Integrating first over \( r \) and using the Green’s function identity \( \int_{0}^{\infty} q_{r}(x,y) dr = 2(x \wedge y) \), we readily get
\[ \int_{0}^{t/2} dr \int_{0}^{\infty} y^{2}e^{-y}q_{r}(x,y) dy \leq C. \]  
(3.21)

The second part is bounded similarly, but using the estimate \( q_{r}(x,y) \leq Cy/y^{3/2} \) instead of the Green function identity. Integrating over \( y \), this gives,
\[ \int_{t/2}^{t-1} dr \int_{0}^{\infty} E_{y}\left[ \bar{Z}_{t-r}(f_{\alpha},0)\right]^{2} e^{y}q_{r}(x,y) dy \leq Cxt^{-3/2+\alpha} \int_{1}^{t/2} u^{-\alpha} du \]
\[ = C' \frac{x}{\sqrt{t}} \times \begin{cases} 1, & \alpha < 1 \\ \log t, & \alpha = 1 \\ t^{\alpha-1}, & \alpha \in (1,2) \end{cases} \]  
(3.22)

Finally, for the third part, we use again the estimate \( q_{r}(x,y) \leq Cy/y^{3/2} \), as well as the bound
\[ E_{y}[(R_{t-r})^{-\alpha}] \leq Cy^{-\alpha}, \quad r \geq t-1, \]  
(3.23)

which is easily obtained. Together with (3.18), this gives
\[ \int_{t-1}^{t} dr \int_{0}^{\infty} E_{y}\left[ \bar{Z}_{t-r}(f_{\alpha},0)\right]^{2} e^{y}q_{r}(x,y) dy \leq Cxt^{-3/2+\alpha} \int_{0}^{\infty} y^{3-2\alpha}e^{-y} dy \]
\[ = C' \frac{x}{\sqrt{t}} t^{\alpha-1}, \]  
(3.24)

using again that \( \alpha < 2 \). Gathering (3.16), (3.17), (3.20), (3.21), (3.22) and (3.24), the statement follows. □

3.3 Number of killed particles

In this subsection, we study BBM starting with a single particle at \( x > 0 \) and where particles are killed by hitting \( 0 \), and focus on the stopping line \( \mathcal{L} \) of the killed particles defined by
\[ \mathcal{L} := \{ u \in \mathbb{T} : \text{there exists } s \geq 0 \text{ s.t. } u \in \mathcal{N}(s), X_{u}(s) \leq 0 \text{ and } \forall r \in [0,s), X_{u}(r) > 0 \}. \]
Moreover, for $u \in \mathcal{L}$, let $T_u := \inf\{s \geq 0 : u \in \mathcal{N}(s) \text{ and } X_u(s) \leq 0\}$ be the killing time of $u$. Lastly, for $I \subset \mathbb{R}_+$ an interval, let $\mathcal{L}_I := \{u \in \mathcal{L} : T_u \in I\}$ denote the subset of particles that are killed at 0 during the time interval $I$.

Firstly, the following explicit formula follows from Lemma 4.5 of [17]: for any measurable function $\varphi : \mathbb{R} \to \mathbb{R}_+$ and any $x > 0$, we have

$$
\mathbb{E}_x \left[ \sum_{u \in \mathcal{L}} \varphi(T_u) \right] = xe^{-x} \int_0^\infty \varphi(r) \frac{e^{-x^2/2r}}{\sqrt{2\pi r^{3/2}}} \, dr.
$$

(3.25)

Now, we prove two lemmas bounding the first and second moment of the number of particles killed during the time interval $[s, \infty)$.

**Lemma 3.6.** There exists $C > 0$ such that, for any $s, x > 0$, we have

$$
\mathbb{E}_x \left[ \#\mathcal{L}_{[s, \infty)} \right] \leq C e^{-x} \left( 1 \wedge \frac{x}{\sqrt{s}} \right).
$$

*Proof.* It has already been shown in Lemma 4.5 of [17] that $\mathbb{E}_x[\#\mathcal{L}] = e^{-x}$. Moreover,

$$
\mathbb{E}_x \left[ \#\mathcal{L}_{[s, \infty)} \right] = xe^{-x} \int_s^\infty \frac{e^{-x^2/2r}}{\sqrt{2\pi r^{3/2}}} \, dr \leq C xe^{-x} \int_s^\infty \frac{1}{r^{3/2}} \, dr \leq C \frac{x}{\sqrt{s}} e^{-x}.
$$

Combining both results, it proves the result. \qed

**Lemma 3.7.** There exists $C > 0$ such that, for any $s, x > 0$, we have

$$
\mathbb{E}_x \left[ \left( \#\mathcal{L}_{[s, \infty)} \right)^2 \right] \leq C \frac{xe^{-x}}{\sqrt{s}}.
$$

*Proof.* For this proof, we use the change of measure introduced in Subsection 4.3 of [17]. We follow the proof of Lemma 4.8 in [17], but note that here, since we assumed that $\mathbb{E}[L^2] < \infty$, we do not need to use a truncation of the number of offspring of the spine. Firstly, note that

$$
\mathbb{E}_x \left[ \left( \#\mathcal{L}_{[s, \infty)} \right)^2 \right] = \mathbb{E}_x \left[ \#\mathcal{L}_{[s, \infty)} \left( \#\mathcal{L}_{[s, \infty)} - 1 \right) \right] + \mathbb{E}_x \left[ \#\mathcal{L}_{[s, \infty)} \right]
$$

and $\mathbb{E}_x[\#\mathcal{L}_{[s, \infty)}] \leq C xe^{-x}/\sqrt{s}$ by Lemma 3.6, so we only have to deal with the first summand.

Following the proof of Lemma 4.8 in [17], we introduce $\tau_u := \inf\{r \geq 0 : X_u(r) = 0\}$ with $\inf\emptyset = \infty$ and have, for any $t \geq s$,

$$
\mathbb{E}_x \left[ \#\mathcal{L}_{[s, t)} \left( \#\mathcal{L}_{[s, t)} - 1 \right) \right] = \mathbb{E}_x \left[ \sum_{u \in \mathcal{N}(t)} 1_{\tau_u \in [s, t)} (\#\mathcal{L}_{[s, t)} - 1) \right] = e^{-x} \mathbb{E}_{Q_x} \left[ \sum_{r \in \Pi_t} (O_{w_r} - 1) \mathbb{E}_{X_{w_r}(r)} \left[ \#\mathcal{L}_{[s-r \vee 0, t-r]} \right] \right].
$$

Then, denoting $O_{w_r}$ the number of offspring of $w_r$ and $\Pi_t$ the set of branching times of the spine before $t$ and using the spinal decomposition description, we get

$$
\mathbb{E}_x \left[ \#\mathcal{L}_{[s, t)} \left( \#\mathcal{L}_{[s, t)} - 1 \right) \right] = e^{-x} \mathbb{E}_{Q_x} \left[ \sum_{r \in \Pi_t} (O_{w_r} - 1) \mathbb{E}_{X_{w_r}(r)} \left[ \#\mathcal{L}_{[s-r \vee 0, t-r]} \right] \right]
$$

$$
\leq C e^{-x} \mathbb{E}_{Q_x} \left[ \sum_{r \in \Pi_t} e^{-X_{w_r}(r)} \left( 1 \wedge \frac{X_{w_r}(r)}{\sqrt{(s-r) \vee 0}} \right) \right],
$$

using Lemma 3.6 and that $\mathbb{E}[\hat{L}] < \infty$, where $\hat{L}$ has the size-biased distribution of $L$. Denoting by $(B_r)_{r \geq 0}$ a Brownian motion started at $x$ under $\mathbb{P}_x$ and setting $\tau := \inf\{r \geq 0 : B_r = 0\}$,
it follows from the spinal decomposition description and from the formula for expectations of additive functionals of Poisson point processes that

\[
\mathbb{E}_x \left[ \# \mathcal{L}_{[s,t]} \left( \# \mathcal{L}_{[s,t]} - 1 \right) \right] \leq C e^{-x} \mathbb{E}_x \left[ \mathbb{1}_{\tau \in [s,t]} \int_0^t e^{-B_r} \left( 1 \wedge \frac{B_r}{\sqrt{(s-r) \vee 0}} \right) dr \right] \\
\leq C e^{-x} \mathbb{E}_x \left[ \mathbb{1}_{\tau \geq s} \int_0^{s/2} e^{-B_r} \frac{B_r}{\sqrt{s}} dr \right] + C e^{-x} \mathbb{E}_x \left[ \mathbb{1}_{\tau \geq s} \int_s^t e^{-B_r} dr \right] \\
\leq C e^{-x} \sqrt{s} \mathbb{E}_x \left[ \int_0^t B_r e^{-B_r} dr \right] + C e^{-x} \int_{s/2}^\infty \mathbb{E}_x \left[ e^{-B_r} \mathbb{1}_{\tau \geq t} \right] dr
\]

(3.26)

On the one hand, the first expectation in the right-hand side of (3.26) is smaller than \(C x\) using (B.1). On the other hand, it follows from (B.2) that, for any \(r, x > 0\), \(\mathbb{E}_x[e^{-B_r} \mathbb{1}_{\tau \geq r}] \leq C x r^{-3/2}\), and therefore the second term in the right-hand side of (3.26) is smaller than \(C x e^{-x}/\sqrt{s}\) and it concludes the proof. \(\square\)

4 A first control on the rate of convergence

We fix \(a \in (0,1)\) and kill particles between times \(t^a\) and \(t\) at some level \(\gamma\). We set, for any \(r, t \geq 0\) and \(\gamma, \Delta \in \mathbb{R}\),

\[
Z_t(F, \Delta, t) = Z_t(F, \Delta) := \sum_{u \in \mathcal{N}(r)} (X_u(r) - \Delta)_+ e^{-X_u(r)} F\left(\frac{X_u(r) - \Delta}{\sqrt{t}}\right),
\]

\[
\tilde{Z}_t^{v, \gamma}(F, \Delta, t) = \tilde{Z}_t^{v, \gamma}(F, \Delta) := \sum_{u \in \mathcal{N}(r)} (X_u(r) - \Delta)_+ e^{-X_u(r)} \mathbb{1}_{\forall q \in [v, r], X_u(q) > \gamma} F\left(\frac{X_u(r) - \Delta}{\sqrt{t}}\right).
\]

(4.1)

As before, we omit the dependence in \(t\) in the notation \(Z_t(F, \Delta)\) or \(\tilde{Z}_t^{v, \gamma}(F, \Delta)\) except when the scaling in the function \(F\) is not \(\sqrt{t}\). In this section, the main goal is to prove the following proposition, which is finally proved in Subsection 4.4, but some of the established lemmas will be reused afterwards.

**Proposition 4.1.** Let \(\kappa \geq 1\) and \(K > 0\). There exist \(C = C(\kappa, K) > 0\) and \(t_0 = t_0(\kappa, K) > 0\), such that for any function \(F: \mathbb{R} \to \mathbb{R}\) satisfying Assumptions (A1\(\kappa\))- (A2\(\kappa\)), \(t \geq t_0, \Delta \in [0, K \log t]\) and \(\delta > 0\), we have

\[
P(|Z_t(F, \Delta) - \mathbb{E}[F(R_1)]Z_t(1, \Delta)| \geq \delta) \leq t^{-K} + \frac{C(\log t)^2}{\delta t^{1/3}}
\]

In order to prove this, we use a similar strategy as the one for Proposition 2.1 presented in Section 2, by introducing a barrier at \(\gamma\) from time \(t^a\) to time \(t\). We will eventually choose \(\gamma = \frac{1}{2} \log t\), but we keep some generality for \(\gamma\) throughout the section. Moreover, we will first prove this concentration result for \(Z_t(F, \gamma) - \mathbb{E}[F(R_1)]Z_t(1, \gamma)\), and only afterwards prove that we can replace \(\gamma\) by any other shift \(\Delta\).

Let \(\mathcal{L}_{v, \gamma}\) denote the stopping line of the killed particles, \(T_u\) the time of the death of a killed particle \(u\) and \(\mathcal{F}_{\mathcal{L}_{v, \gamma}}\) be the \(\sigma\)-algebra associated with the stopping line. Then, we have the following decomposition

\[
Z_t(F, \gamma) - \mathbb{E}[F(R_1)]Z_t(1, \gamma) = \tilde{Z}_t^{v, \gamma}(F - \mathbb{E}[F(R_1)], \gamma) + \sum_{u \in \mathcal{L}_{v, \gamma}} \Omega_t^{(u)},
\]

(4.2)
where $\Omega^{(u)}_t$ is the contribution of the progeny of the killed particle $u$ in $Z_t(F - \mathbb{E}[F(R_1)], \gamma)$ and is defined by

$$
\Omega^{(u)}_t := \sum_{v \in \mathcal{N}(t) \text{ s.t. } u \leq v} (X_u(t) - \gamma) + e^{-X_u(t)} \left( F\left( \frac{X_u(t) - \gamma}{\sqrt{t}} \right) - \mathbb{E}[F(R_1)] \right).
$$

Note that conditionally on $\mathcal{F}_{t-a}$, the $\Omega^{(u)}_t$ for $u \in \mathcal{E}^{t,a}$ are independent with respectively the same law as $Z_{t-T_a}(F - \mathbb{E}[F(R_1)], \gamma)$ under $\mathbb{P}_{X_u(T_a)}$.

### 4.1 The branching Brownian motion with barrier between times $t^a$ and $t$

In this subsection, our aim is to deal with the first part in the decomposition (4.2): we prove that $\tilde{Z}_t^{t^a,\gamma}(F - \mathbb{E}[F(R_1)], \gamma)$ is small, by controlling its first moment and its variance conditionally on $\mathcal{F}_{t^a}$. These results will be reused for the proof of Proposition 2.2 in Subsection 5.1.

#### Lemma 4.2

Let $\kappa \geq 1$. There exists a constant $C = C(\kappa) > 0$, such that for any function $F: \mathbb{R} \to \mathbb{R}$ satisfying Assumptions (A1$_\kappa$)-(A2$_\kappa$) and any $a \in (0,1)$, $\gamma \in \mathbb{R}$ and $t$ large enough (depending only on $\kappa$ and $a$), we have

$$
\mathbb{E}\left[ \tilde{Z}_t^{t^a,\gamma}(F - \mathbb{E}[F(R_1)], \gamma) \right] \leq \frac{C}{t^{1-a}} Z_{t^a}(x \to e^{x}, \gamma, t^a),
$$

$$
\text{Var}\left( \tilde{Z}_t^{t^a,\gamma}(F - \mathbb{E}[F(R_1)], \gamma) \right) \leq \frac{Ce^{-\gamma}}{\sqrt{t}} Z_{t^a}(x \to 1 + x^{-1}, \gamma, t^a).
$$

**Proof.** For brevity, we write $\mathcal{F} := F - \mathbb{E}[F(R_1)]$. Let start with the first moment. Using the branching property at time $t^a$, we get

$$
\mathbb{E}\left[ \tilde{Z}_t^{t^a,\gamma}(\mathcal{F}, \gamma) \right] \leq \sum_{v \in \mathcal{N}(t^a)} e^{-\gamma} \mathbb{1}_{X_u(t^a) > \gamma} \mathbb{E}_{X_u(t^a)}[\tilde{Z}_{t-t^a}(\mathcal{F}, 0)].
$$

Then, applying the first bound of Lemma 3.2 with $r = t$ and noting that $t - t^a \geq t/2$ for $t$ large enough, we have, for any $y > 0$,

$$
\mathbb{E}_y[\tilde{Z}_{t-t^a}(\mathcal{F}, 0)] \leq Cye^{-y} \left( \frac{y^2}{t} + \frac{t^a}{y^2} \right) e^{\sqrt{y} + t^a} \leq Cye^{-y} \frac{t^a}{t} e^{\sqrt{y} + \sqrt{t^a}},
$$

for $t$ large enough. The result follows.

We can now deal with the second moment. Using the branching property at time $t^a - t^a$, we get

$$
\text{Var}\left( \tilde{Z}_t^{t^a,\gamma}(\mathcal{F}, \gamma) \right) \leq \sum_{v \in \mathcal{N}(t^a)} e^{-2\gamma} \mathbb{1}_{X_u(t^a) > \gamma} \mathbb{E}_{X_u(t^a)}\left[ \left( \tilde{Z}_{t-t^a}(\mathcal{F}, 0) \right)^2 \right].
$$

(4.3)

Then, applying Lemma 3.4 with $r = t - t^a$, we have, for any $y > 0$,

$$
\mathbb{E}_y\left[ \tilde{Z}_{t-t^a}(\mathcal{F}, 0)^2 \right] \leq C e^{-y} \left( \frac{1}{t^2} + \frac{t^a}{t^2} + \frac{y}{\sqrt{t}} \right) \leq \frac{Ce^{-y}}{\sqrt{t}} \left( \frac{t^a}{y} + 1 \right),
$$

(4.4)

and the result follows.

In the following lemma, we combine the first and second moments calculation to get a concentration result on $\tilde{Z}_t^{t^a,\gamma}(F - \mathbb{E}[F(R_1)], \gamma)$ around 0. Note that the bound depends on an additional parameter $M$, that will be chosen afterwards either as $\log t$ or as a fixed large constant.
Lemma 4.3. Let $\kappa \geq 1$ and $K > 0$. There exists a constant $C = C(\kappa, K) > 0$, such that for any function $F: \mathbb{R} \to \mathbb{R}$ satisfying Assumptions (A1a)-(A2a) and any $a \in (0, 1)$, $t$ large enough (depending only on $\kappa, K$ and $a$), $|\gamma| \leq K \log t$, $M \in [1, K \log t]$ and $\delta > 0$, we have

$$
P\left(\left|\tilde{Z}^a_{t^a, \gamma}(F - E[F(R_1)], \gamma)\right| \geq \delta \right) \leq e^{-M} + \frac{CM}{\delta t^{a-1}} + \frac{Ce^{-\gamma}}{\delta^2 \sqrt{t}} M.
$$

Proof. For brevity, we write $\mathcal{P} := F - E[F(R_1)]$. We introduce a barrier at $-M$ between times $0$ and $t^a$ by considering the event $A := \{\forall r \in [0, t^a], \min_{u \in \mathcal{N}(r)} X_u(r) > -M\}$. Using (3.4), we have $\mathbb{P}(A^c) \leq e^{-M}$ and, therefore,

$$
P\left(\left|\tilde{Z}^a_{t^a, \gamma}(\mathcal{P}, \gamma)\right| \geq \delta \right) \leq e^{-M} + \mathbb{P}\left(A, \left|\tilde{Z}^a_{t^a, \gamma}(\mathcal{P}, \gamma)\right|_{\mathcal{F}_{t^a}} \geq \frac{\delta}{2}\right) + \mathbb{P}\left(A, \left|\tilde{Z}^a_{t^a, \gamma}(\mathcal{P}, \gamma) - E\left[\tilde{Z}^a_{t^a, \gamma}(\mathcal{P}, \gamma)\right]_{\mathcal{F}_{t^a}}\right| \geq \frac{\delta}{2}\right).
$$

(4.5)

Then, using Markov’s inequality and the first part of Lemma 4.2, the second term in the right-hand side of (4.5) is smaller than

$$
\frac{2}{\delta} E\left[1_A \left|\tilde{Z}^a_{t^a, \gamma}(\mathcal{P}, \gamma)\right|_{\mathcal{F}_{t^a}}\right] \leq \frac{C}{\delta t^{a-1}} E\left[1_A Z_{t^a}(x \mapsto e^{C_x \gamma}, t^a)\right]
$$

and it follows from (3.11) that $E[1_A Z_{t^a}(x \mapsto e^{C_x \gamma}, t^a)] \leq CM$ for $t$ large enough. On the other hand, using Chebyshev’s inequality and the second part of Lemma 4.2, the third term in the right-hand side of (4.5) is smaller than

$$
\frac{4}{\delta^4} \mathbb{E}\left[1_A \operatorname{Var}\left(\tilde{Z}^a_{t^a, \gamma}(\mathcal{P}, \gamma)\right)_{\mathcal{F}_{t^a}}\right] \leq \frac{Ce^{-\gamma}}{\delta^2 \sqrt{t}} E\left[1_A Z_{t^a}(x \mapsto 1 + x^{-1}, \gamma, t^a)\right] \leq \frac{Ce^{-\gamma}}{\delta^2 \sqrt{t}} M,
$$

using again (3.11). It concludes the proof. \hfill \Box

4.2 Number of killed particles at level $\gamma$

In this subsection, we state some straightforward consequences of the results in Subsection 3.3, concerning the number of particles killed by the barrier at $\gamma$. This results will be used repetitively throughout the paper. Before stating them, we define, for any interval $I \subset [t^a, t]$,

$$
\mathcal{L}^a_{I^a, \gamma} := \{u \in \mathcal{L}^a_{\gamma} : T_u \in I\}
$$

the subset of $\mathcal{L}^a_{\gamma}$ consisting of particles killed during the time interval $I$. Note that $\mathcal{L}^a_{(t^a, t)}$ plays a special role, because particles killed at time $t^a$ can be strictly below $\gamma$. We are not considering these particles in this subsection.

Firstly, it follows from (3.25) that, for any measurable function $\varphi: (t^a, t) \to \mathbb{R}_+$ and any $x > 0$,

$$
\mathbb{E}\left[\sum_{u \in \mathcal{L}^a_{(t^a, t)}} \varphi(T_u) \mid \mathcal{F}_{t^a}\right] = \sum_{v \in \mathcal{N}(t^a)} (X_v(t^a) - \gamma) e^{-\gamma} \int_{t^a}^t \varphi(r) e^{- \frac{(X_v(r^a) - \gamma)^2}{2(r - t^a)}} \frac{2}{\sqrt{2\pi(r - t^a)^{3/2}}} dr \leq Ce^{\gamma} Z_{t^a}(1, \gamma) \int_{t^a}^t \varphi(r) \frac{2}{\sqrt{2\pi(r - t^a)^{3/2}}} dr.
$$

(4.6)

Now, we consider some $s_1 \in [t^a, t)$ and state some different controls on $\#\mathcal{L}_{(s_1, t)}$.

Lemma 4.4. For any $t > 0$, $\gamma \in \mathbb{R}$ and $s_1 \in [t^a, t)$, we have

$$
\mathbb{E}\left[\#\mathcal{L}^a_{(s_1, t)} \mid \mathcal{F}_{t^a}\right] \leq Ce^{\gamma} \left(W_{t^a} \wedge \frac{Z_{t^a}(1, \gamma)}{\sqrt{s_1 - t^a}}\right).
$$
Proof. This inequality follows from branching property at time $t^a$ and Lemma 3.6.

**Lemma 4.5.** If $\gamma$ and $s_1$ are depending on $t$ such that $|\gamma| = O(\log t)$ and $s_1 \in [t^a, t)$, then, for any function $\varepsilon: \mathbb{R}_+ \to \mathbb{R}$ such that $\varepsilon(t) \to 0$ as $t \to \infty$, we have the following convergence in probability

$$
\varepsilon(t)\sqrt{s_1e^{-\gamma}\#L^{s_1,t}_{[s_1,t]} \to 0.}
$$

**Proof.** Using Lemma 4.4, we get

$$
\mathbb{E}[\varepsilon(t)\sqrt{s_1e^{-\gamma}\#L^{s_1,t}_{[s_1,t]}|F_{t^a}] \leq C\varepsilon(t)\sqrt{s_1} \left(W_{t^a} \wedge \frac{Z_{t^a}(1,\gamma)}{\sqrt{s_1 - t^a}}\right) \leq C\varepsilon(t)\left(\sqrt{2t^a}W_{t^a} + \sqrt{2Z_{t^a}(1,\gamma)}\right),
$$

by distinguishing cases $s_1 \leq 2t^a$ and $s_1 > 2t^a$. Using that $Z_{t^a}(1,\gamma) \to Z_{\infty}$ and $\sqrt{2t^a}W_{t^a} \to \sqrt{2/\pi}Z_{\infty}$ in probability, we get that $\mathbb{E}[\varepsilon(t)\sqrt{s_1e^{-\gamma}\#L^{s_1,t}_{[s_1,t]}|F_{t^a}] \to 0$ in probability and the result follows.

**4.3 Contributions of killed particles**

In this subsection, we deal with the contributions of the particles killed by the barrier at level $\gamma$ between times $t^a$ and $t$. This relies on the following lemma, proved by a crude triangle inequality and first moment bound on the contributions $\Omega^{(u)}_t$.

**Lemma 4.6.** Let $\kappa \geq 1$ and $K > 0$. There exists a constant $C = C(\kappa,K) > 0$, such that for any function $F: \mathbb{R} \to \mathbb{R}$ satisfying Assumptions (A1$\kappa$)-(A2$\kappa$) and any $a \in (0,1)$, $t$ large enough (depending only on $\kappa$, $K$ and $a$), $0 \leq \gamma \leq K \log t$, $M \in [1,K \log t]$ and $\delta > 0$, we have

$$
\mathbb{P}\left(\sum_{u \in L^{t^a}} \left|\Omega^{(u)}_t\right| \geq \delta\right) \leq e^{-M} + \frac{C(\log t)^2}{\delta t^{\gamma/2}}.
$$

**Proof.** For brevity, we write $\overline{F} := F - \mathbb{E}[F(R_t)]$. We introduce a barrier at $-M$: we set $A_s := \{\forall r \in [0,s], \min_{u \in N(r)} X_u(r) > -M\}$ for any $s \geq 0$. We will bound the first moment of the quantity of interest on the event $A_s$. Conditionally on $F_{L^{t^a},\gamma}$ and on the event $A_t$, for each $u \in L^{t^a,\gamma}$, the contribution $|\Omega^{(u)}_t|$ is stochastically dominated by $e^M\tilde{Z}_{t-t_u}(\overline{F},\gamma + M)$ under $\mathbb{P}_{X_u(t^a)_{t_u}}\mathbb{1}_{A_s}$. Therefore, distinguishing between particles killed at time $t^a$ or after, we get

$$
\mathbb{E}\left[\mathbb{1}_{A_t} \sum_{u \in L^{t^a,\gamma}} \left|\Omega^{(u)}_t\right| \geq \delta \right| F_{L^{t^a,\gamma}}
$$

\begin{align*}
&\leq \mathbb{1}_{A_a}\left(\sum_{u \in L^{t^a,\gamma}} e^M\mathbb{E}_{X_u(t^a)_{t_u}}\left[\tilde{Z}_{t-t_u}(\overline{F},\gamma + M)\right] + \sum_{u \in L^{t^a,\gamma}} e^M\mathbb{E}_{\gamma + M}\left[\tilde{Z}_{t-t_u}(\overline{F},\gamma + M)\right]\right) \\
&\leq \mathbb{1}_{A_a}\left(\sum_{u \in L^{t^a,\gamma}} (X_u(t^a) + M)e^{-X_u(t^a)}\left(e^{c(\gamma(M + M))} + \frac{M}{\sqrt{t^a}}\right) \\
&\quad + \left(\#L^{t^a,\gamma}\right) \cdot C(\gamma + M)e^{-\gamma}\left(e^{c(\gamma + M)\sqrt{t}} + 1\right)\right),
\end{align*}

(4.7)

using the second bound of Lemma 3.2. Since $X_u(t^a) \leq \gamma$ for each $u \in L^{t^a,\gamma}$ and $M \leq \sqrt{t^a}$ for $t$ large enough, the first moment of the first term on the right-hand side of 4.7 is smaller than

$$
\mathbb{E}\left[\mathbb{1}_{A_a} \sum_{u \in N(t^a)} (X_u(t^a) + M)e^{-X_u(t^a)}\mathbb{1}_{X_u(t^a) \leq \gamma}\right] \leq MP_{M}(R_{t^a} \leq \gamma + M)
$$

18
using the many-to-one lemma (3.2). Then, we have
\[ \mathbb{P}_M(R_1 \leq \gamma + M) \leq \mathbb{P}(R_1 \leq (\gamma + M)/\sqrt{t^2}) \leq C(\gamma + M)^3 \leq C \frac{(\log t)^3}{t^{3\alpha/2}} \]

using the density of $R_1$. On the other hand, applying Lemma 4.4, the first moment of the second term on the right-hand side of 4.7 is smaller than
\[ C(\gamma + M)e^{-\gamma}E \left[ 1_{A_{\alpha}} E \left[ \#L_{t^2}^{\gamma, \gamma} | \mathcal{F}_{t^2} \right] \right] \leq C(\gamma + M)E[1_{A_{\alpha}}W_{t^2}] \leq C \frac{(\log t)^2}{\sqrt{t^2}}, \]

where the last inequality follows from (3.11). Therefore the first moment of the right-hand side of 4.7 is smaller than $(\log t)^2/\sqrt{t^2}$. Combining this with $\mathbb{P}(A^c) \leq e^{-M}$ and Markov's inequality, this concludes the proof. \qed

4.4 Rate of convergence of $Z_t(F, \Delta)$

Proof of Proposition 4.1. Let $a \in (0,1)$ and $\gamma \in [0, K \log t]$ to be chosen later. Using (4.2), Lemma 4.3 and Lemma 4.6 with $M = (K + 1) \log t$, we get

\[ \mathbb{P}(|Z_t(F, \gamma) - E[F(R_1)|Z_t(1, \gamma)| \geq \delta) \leq t^{-K-1} + C(\log t)^2 \left( \frac{t^{-3/2}}{\delta} + \frac{e^{-\gamma}}{\delta^2} + t^{-\alpha/2} \gamma \right). \]

The optimal choice of $a$ is $a = 2/3$. We also choose $\gamma = \frac{1}{2} \log t$. Then the previous bound becomes $t^{-K-1} + C(\log t)^2(t^{-1/3} \delta^{-1} + t^{-1/3} \delta^{-2})$ and, noting that $t^{-1/3} \delta^{-2}$ is either smaller than $t^{-1/3} \delta^{-1}$ or greater than 1, we get
\[ \mathbb{P}(|Z_t(F, \gamma) - E[F(R_1)|Z_t(1, \gamma)| \geq \delta) \leq t^{-K-1} + C(\log t)^2 t^{-1/3} \delta^{-1}. \]

We now remove the shift $\gamma$. It follows from Assumption (A2_{\alpha}) (see (B.9)) that
\[ |Z_t(F, \gamma) - Z_t(F, \Delta)| \leq C \frac{|\Delta - \gamma|}{\sqrt{t}} \sum_{u \in N(t)} (X_u(t)+1)e^{-X_u(t)}e^{xX_u(t)/\sqrt{t}} \]
\[ \leq C \frac{(\log t)}{\sqrt{t}} Z_t(x \mapsto (1 + x^{-1})e^{x}, 0) \]

Then, using (3.12) with $M = K \log t$, we get
\[ \mathbb{P}(|Z_t(F, \gamma) - Z_t(F, \Delta)| \geq \delta) \leq t^{-K-1} + C(\log t)^2 t^{-1/2} \delta^{-1}. \tag{4.8} \]

This can be applied to $F = 1$ to bound $|Z_t(1, \gamma) - Z_t(1, \Delta)|$ as well. The result follows. \qed

5 Precise behavior of $Z_t(F, \gamma_t)$

In this section, we prove Propositions 2.2 and 2.3, which imply together Proposition 2.1.

5.1 Proof of Proposition 2.2

Note that Lemma 4.3 with $\delta = \epsilon/\sqrt{t}$ and $M$ large would imply directly Proposition 2.2 in the case $a < 1/2$. However, we want to cover the case $1/2 < a < 3/4$ in order to avoid having too many particles killed by the barrier. For this, we have to improve the estimate of the conditional first moment using Proposition 4.1.
Proof of Proposition 2.2. For brevity, we write \( \overline{F} := F - \mathbb{E}[F(R_1)] \). We first estimate the conditional first moment of \( \tilde{Z}^{\alpha, \gamma_t}_t (\overline{F}, \gamma_t) \) given \( \mathcal{F}_{t^a} \). It follows from the branching property at time \( t^a \) and from (3.7) that

\[
\mathbb{E} \left[ \tilde{Z}^{\alpha, \gamma_t}_t (\overline{F}, \gamma_t) \mid \mathcal{F}_{t^a} \right] = \sum_{v \in N(t^a)} e^{-\gamma_t} \mathbb{1}_{X_v(t^a) > \gamma_t} \mathbb{E} X_v(t^a) - \gamma_t \left[ \tilde{Z}^{\alpha, \gamma_t}_t (\overline{F}, 0) \right] = \sum_{v \in N(t^a)} (X_v(t^a) - \gamma_t) + e^{-X_v(t^a)} \mathbb{E} (X_v(t^a) - \gamma_t) / \sqrt{t - t^a} \left[ \overline{F}(R_{1-t^a-1}) \right].
\]

(5.1)

Applying Lemma B.4 with \( \varepsilon = t^a - 1 \) and \( x = (X_v(t^a) - \gamma_t) \sqrt{t - t^a} / (t - t^a) \) for \( t \) large enough, we get (recall \( \overline{F} := F - \mathbb{E}[F(R_1)] \))

\[
\left| \mathbb{E} (X_v(t^a) - \gamma_t) / \sqrt{t - t^a} \right| \leq C t^{2(a-1)} \left( 1 + \left( \frac{X_v(t^a) - \gamma_t}{\sqrt{t^a}} \right)^4 \right) e^{3 \kappa (X_v(t^a) - \gamma_t) / \sqrt{t}}.
\]

Moreover, recalling the definition of \( G \) in (B.10), we have

\[
G \left( \frac{X_v(t^a) - \gamma_t}{\sqrt{t^a}} \right) - G \left( \frac{X_v(t^a) - \gamma_t}{\sqrt{t^a}} \right) \leq C t^{a-1} \left( \frac{X_v(t^a) - \gamma_t}{\sqrt{t^a}} \right)^2.
\]

Coming back to (5.1), we get

\[
\mathbb{E} \left[ \tilde{Z}^{\alpha, \gamma_t}_t (\overline{F}, \gamma_t) \mid \mathcal{F}_{t^a} \right] \leq t^a - 1 |Z^{\alpha}_v(G, \gamma_t, t^a)| + C t^{2(a-1)} Z^{\alpha}_v(x \mapsto e^{4\kappa x}, \gamma_t, t^a).
\]

On the one hand, it follows from (3.12) with \( M = 10 \log t \) that, for any \( \delta > 0 \),

\[
\mathbb{P} \left( t^{2(a-1)} Z^{\alpha}_v(x \mapsto e^{4\kappa x}, \gamma_t, t^a) \geq \delta \right) \leq t^{-10} + C(\log t) t^{2(a-1)} \delta^{-1}.
\]

On the other hand, note that \( \mathbb{E}[G(R_1)] = 0 \) since \( \mathbb{E}[R_1^2] = 3 \), so we can apply Proposition 4.1 to \( Z^{\alpha}_v(G, \gamma_t, t^a) \) to get

\[
\mathbb{P} \left( t^{a-1} |Z^{\alpha}_v(G, \gamma_t, t^a)| \geq \delta \right) \leq t^{-10} + C(\log t)^2 t^{-a/3} (\delta t^{1-a})^{-1} \leq t^{-10} + C(\log t)^2 t^{3-1-a} \delta^{-1}.
\]

Finally, this proves that for any \( \delta > 0 \)

\[
\mathbb{P} \left( \mathbb{E} \left[ \tilde{Z}^{\alpha, \gamma_t}_t (\overline{F}, \gamma_t) \mid \mathcal{F}_{t^a} \right] \geq \delta \right) \leq 2 t^{-10} + C(\log t)^2 \left( t^{3-1-a} + t^{2(a-1)} \right) \delta^{-1}.
\]

Let \( \varepsilon \in (0, 1) \) and \( M \geq 1 \). We use this new input in the proof of Lemma 4.3 (see (4.5)) to get, with \( \delta = \varepsilon / \sqrt{t} \) and recalling \( \gamma_t = \frac{\varepsilon}{t} + \beta_t \),

\[
\mathbb{P} \left( \left| \tilde{Z}^{\alpha, \gamma_t}_t (F - \mathbb{E}[F(R_1)], \gamma_t) \right| \geq \frac{\varepsilon}{\sqrt{t}} \right) \leq e^{-M} + C(\log t)^2 \left( t^{3-1-a} + t^{2(a-1)} \right) + \frac{C e^{-\beta_t}}{\varepsilon^2} M \to \infty e^{-M},
\]

using \( a < 3/4 \) and \( \beta_t \to \infty \). Letting \( M \to \infty \) proves the result.

\[ \square \]

5.2 Simplifying the contribution of killed particles

This subsection contains some preliminary work for the proof of Proposition 2.3. We recall some notation introduced in Subsection 4.2. For any interval \( I \subset [t^a, t] \), let \( \mathcal{L}^{\alpha, \gamma_t}_I := \{ u \in \mathcal{L}^{\alpha, \gamma_t} : T_u \in \]

...
$I$} be the subset of $\mathcal{L}^{t_1,\gamma_t}$ consisting of particles killed during the time interval $I$. Recall that for $u \in \mathcal{L}^{t_1,\gamma_t}$, the contribution of the killed particle $u$ in $Z_{t}(F, \gamma_t)$ is

$$\Omega_{t}^{(u)} := \sum_{v \in N(t) \text{ s.t. } u \leq v} (X_v(t) - \gamma_t) + e^{-X_v(t)} \left( F \left( \frac{X_v(t) - \gamma_t}{\sqrt{t}} \right) \right) - E[F(R_1)].$$

Given $\mathcal{F}_{\mathcal{L}^{t_1,\gamma_t}}$, the $\Omega_{t}^{(u)}$ for $u \in \mathcal{L}^{t_1,\gamma_t}$ are independent with respectively the same distribution as $e^{-\gamma_t}Z_{t-T_a}(F - E[F(R_1)], 0)$ under $\mathbb{P}$. We set

$$H(u) := E[F(R_{1-u})] - E[F(R_1)], \quad u \in [0, 1]. \quad (5.2)$$

Then the following lemma shows that the contributions $\Omega_{t}^{(u)}$ can be replaced by $e^{-\gamma_t}H(T_a/t)Z_{\infty}^{(u)}$, where $Z_{\infty}^{(u)}$ is the limit of the derivative martingale of the BBM rooted at $u$, defined as

$$Z_{\infty}^{(u)} := \lim_{t \to \infty} \sum_{v \in N(s) \text{ s.t. } u \leq v} (X_v(s) - X_u(T_a)) e^{-(X_v(s) - X_u(T_a))}.$$ 

The proof essentially relies on Proposition 4.1 applied to each $\Omega_{t}^{(u)}$.

**Lemma 5.1.** Let $F: \mathbb{R} \to \mathbb{R}$ satisfying Assumptions (A1), (A2). If $a \in (2/3, 1)$, then we have the following convergence in probability,

$$\sqrt{t} \left( \sum_{u \in \mathcal{L}^{t_1,\gamma_t}} \Omega_{t}^{(u)} - \sum_{u \in \mathcal{L}^{t_1,\gamma_t}} H(T_a/t) e^{-\gamma_t} Z_{\infty}^{(u)} \right) \xrightarrow{t \to \infty} 0.$$ 

**Proof.** Note that, as a consequence of (3.5), we have

$$\mathbb{P}(\mathcal{L}^{t_1,\gamma_t} \neq \emptyset) \leq C(\log t) e^{\gamma t} t^{-3a/2} \xrightarrow{t \to \infty} 0,$$

since $a > 1/3$. Moreover, setting $t_2 = t - te^{-2\delta t}$, it follows from (4.6) that

$$E \left[ \# \mathcal{L}^{t_2,\gamma_t} \big| \mathcal{F}_{t_2} \right] \leq Ce^{\gamma t} Z_{\infty}(1, \gamma_t) \int_{t_2-t_1}^{t_1} e^{r} \frac{dr}{r^{3/2}} \leq Ce^{\gamma t} Z_{\infty}(1, \gamma_t) \frac{t-t_2}{t_2} - \frac{t}{t_1} \leq CZ_{\infty}(1, \gamma_t) e^{-\gamma t} \xrightarrow{t \to \infty} 0,$$

in probability, since $Z_{\infty}(1, \gamma_t)$ is tight by Remark 3.3. Therefore, it is now sufficient to prove that, for any $\varepsilon \in (0, 1)$

$$\mathbb{P} \left( \sum_{u \in \mathcal{L}^{t_1,\gamma_t}} \left| \Omega_{t}^{(u)} - H(T_a/t) e^{-\gamma_t} Z_{\infty}^{(u)} \right| \geq \frac{\varepsilon}{\sqrt{t}} \right) \xrightarrow{t \to \infty} 0. \quad (5.3)$$

We set $N := \# \mathcal{L}^{t_1,\gamma_t}$. Given $\mathcal{F}_{\mathcal{L}^{t_1,\gamma_t}}$, the $\Omega_{t}^{(u)} - H(T_a/t) e^{-\gamma_t} Z_{\infty}^{(u)}$ for $u \in \mathcal{L}^{t_1,\gamma_t}$ are independent with the same law as $e^{-\gamma_t}(Z_{t-T_a}(F - E[F(R_1)], 0) - H(T_a/t)Z_{\infty})$ under $\mathbb{P}$, so we get

$$\mathbb{P} \left( \sum_{u \in \mathcal{L}^{t_1,\gamma_t}} \left| \Omega_{t}^{(u)} - H(T_a/t) e^{-\gamma_t} Z_{\infty}^{(u)} \right| \geq \frac{\varepsilon}{\sqrt{t}} \right| \mathcal{F}_{\mathcal{L}^{t_1,\gamma_t}} \right) \leq \sum_{u \in \mathcal{L}^{t_1,\gamma_t}} \chi \left( T_a, \frac{\varepsilon e^{\gamma t}}{N\sqrt{t}} \right), \quad (5.4)$$

where we set $\chi(s, \delta) := \mathbb{P}(\left| Z_{t-s}(F - E[F(R_1)], 0) - H(s/t)Z_{\infty} \right| \geq \delta)$. Then, we write

$$\chi(s, \delta) \leq \mathbb{P}(\left| Z_{t-s}(F - H(s/t), 0) \right| \geq \frac{\delta}{2}) + \mathbb{P}(\left| H(s/t)(Z_{t-s} - Z_{\infty}) \right| \geq \frac{\delta}{2}). \quad (5.5)$$
Note that $Z_{t-s}(F - E[F(R_l)]) - H(s/t, 0) = Z_{t-s}(F_{s/t} - E[F_{s/t}(R_l)], 0, t - s)$ with $F_u(x) := F(x\sqrt{1-u})$ for $u \in [0, 1]$ and $x \in \mathbb{R}$. Note that the function $F_u$ satisfies Assumptions (A1u)-(A2u) (with the same $\kappa$ as $F$), so we can apply Proposition 4.1 to the first term on the right-hand side of (5.5), uniformly in $s$. For the second term on the right-hand side of (5.5), we use that $H(s/t) \leq C$ and (1.7). This gives

$$\chi(s, \delta) \leq t^{-10} + \frac{C(\log(t-s))^2}{\delta(t-s)^{1/3}} + \frac{C(\log(t-s))^2}{\delta \sqrt{t-s}} \leq t^{-10} + \frac{C(\log t)^2 \epsilon^{2\beta t/3}}{\delta^{1/3}},$$

using that $t \geq t - s \geq t - t_2 = t e^{-2\beta t}$. Therefore, (5.4) is smaller than

$$N \cdot \left( t^{-10} + \frac{C(\log t)^2 \epsilon^{2\beta t/3}}{\delta^{1/3}} \right) N \frac{\sqrt{t}}{\epsilon \alpha^{1/2}} = t^{-10} N + C \frac{\epsilon^{5\beta t/3}}{\epsilon t^{1/4}} \left( \sqrt{t} e^{-\gamma t} N \right)^2 \xrightarrow{t \to \infty} 0,$$

in probability, using Lemma 4.5, $\beta_t = o(\log t)$ by (2.1) and $a > 2/3$. It proves (5.3).

### 5.3 Convergence of sums along the stopping line

In this section, we prove the following lemma that will be useful in the proof of Proposition 2.3.

**Lemma 5.2.** Let $Y : [0, 1] \to \mathbb{C}$ be a measurable function such that there exists $C > 0$ such that, for any $u \in [0, 1]$, $|Y(u)| \leq C u^{3/4}$. If $a \in (2/3, 1)$, then we have the following convergence in probability,

$$\beta_t \left( e^{-\beta t} \sum_{u \in \mathbb{C}^{[t^a, t^{-a}]}_t} Y(T_u/t) - \frac{Z_\infty}{\sqrt{2\pi}} \int_0^1 Y(u) \frac{du}{u^{3/2}} \right) \xrightarrow{t \to \infty} 0.$$

**Proof.** The idea of the proof is to use a first and second moment calculation given $F_{s/t}$ to prove that the sum converges to the integral faster than $1/\beta_t$. We set $t_1 := t \beta_t^{-6}$ and

$$\Xi_t := e^{-\beta t} \sum_{u \in \mathbb{C}^{[t^a, t^{-a}]}_t} Y(T_u/t) - \frac{Z_\infty}{\sqrt{2\pi}} \int_{t_1}^1 Y(u) \frac{du}{u^{3/2}}.$$

The first step consists in showing that $\beta_t \Xi_t \to 0$ in probability, by the means of a first and second moment calculation. We start with the first moment: by the first part of (4.6), we have

$$E \left[ \sum_{u \in \mathbb{C}^{[t^a, t^{-a}]}_t} Y(T_u/t) \bigg| F_{s/t} \right] = \sum_{v \in \mathbb{N}^{(t^a)}} (X_v(t^a) - \gamma_t) + e^{-(X_v(t^a) - \gamma_t)} \int_{t_1}^t \frac{e^{-(X_v(t^a) - \gamma_t)^2/2(ut-t^a)}}{\sqrt{2\pi}(r-t^a)^{3/2}} Y(r/t) \, dr.$$

Changing variables with $u = r/t$, we get

$$E \left[ \sum_{u \in \mathbb{C}^{[t^a, t^{-a}]}_t} Y(T_u/t) \bigg| F_{s/t} \right] \leq \sum_{v \in \mathbb{N}^{(t^a)}} (X_v(t^a) - \gamma_t) + e^{-(X_v(t^a) - \gamma_t)} \int_{t_1}^t \frac{e^{-(X_v(t^a) - \gamma_t)^2/2(ut-t^a)}}{\sqrt{2\pi}(1-t^a/u)^{3/2}} \left( \frac{1}{1-t^a/u} \right)^{3/2} Y(u) \, du.$$

Now, for $y > 0$ and $u \in [t_1/t, 1]$, we have

$$\frac{e^{-y^2/2(ut-t^a)}}{(1-t^a/u)^{3/2}} - 1 \leq e^{-y^2/2(ut-t^a)} \left( \frac{1}{1-t^a/u} \right)^{3/2} - 1 + \left| e^{-y^2/2(ut-t^a)} - 1 \right| \leq \left| e^{-y^2/2(ut-t^a)} - 1 \right| + \frac{y^2}{2(ut-t^a)} \leq C \left( \frac{t^a}{t_1} + \frac{y^2}{t_1} \right) \leq C \rho_t^{6t^a} \left( 1 + \frac{y^2}{t^a} \right),$$
for \( t \) large enough, recalling that \( t_1 = t\beta_t^{-6} \), and it leads to

\[
|\mathbb{E}[\Xi_t|\mathcal{F}_t]| \leq \sum_{v \in \mathcal{N}(t^a)} (X_v(t^a) - \gamma_t) e^{-X_v(t^a)} C\beta_t^{6a/t} \left(1 + \frac{(X_v(t^a) - \gamma_t)^2}{t^a}\right) \int_0^1 \frac{|Y(u)|}{u^{3/2}} \, du
\]

\[
\leq C\beta_t^{6a/t} t^{a-1} Z_{t^a}(x \mapsto 1 + x^2, \gamma_t, t^a),
\]  \hspace{1cm} (5.6)

because the integral is finite. We now deal with \( \Xi_t \), using notation of Subsection 3.3. Applying \((1.2)\) and \((1.3)\), and from \( (\ref{eq:2}) \), \( \gamma_t \) to bound the number of such particles. For \( \Xi_t \) in probability, and from \((\ref{eq:2})\) in probability. But, we have

\[
\text{For } \Xi_t \text{ in probability, and from } (\ref{eq:2}), \text{ we can now apply Markov’s and Chebyshev’s inequalities given } \mathcal{F}_t \text{ to get}
\]

\[
\mathbb{P}\left( |\Xi_t| \geq \frac{\varepsilon}{\beta_t} \bigg| \mathcal{F}_t \right) \leq \frac{2\beta_t}{\varepsilon} \mathbb{E}[|\Xi_t|] + \left(\frac{2\beta_t}{\varepsilon}\right)^2 \text{Var}(\Xi_t) \mathcal{F}_t
\]

\[
\leq \left(\frac{C}{\varepsilon}\right) \frac{\beta_t t^{a-1} + \frac{C}{\varepsilon}\beta_t^2 e^{-\beta_t}}{Z_{t^a}(x \mapsto 1 + x^2, \gamma_t, t^a)},
\]  \hspace{1cm} (5.7)

(5.6) and (5.7). But, it follows from Remark 3.3 that \( (Z_{t^a}(x \mapsto 1 + x^2, \gamma_t, t^a))_{t>0} \) is tight, and therefore, it proves that \( \beta_t \Xi_t \to 0 \) in probability and concludes the first step.

The second step consists of dealing with the remaining terms, that are

\[
\Xi_t^{(1)} := \sum_{u \in \mathcal{C}(\mathcal{F}_t, \gamma_t)} e^{-\beta_t} T_u(T_u/t),
\]

\[
\Xi_t^{(2)} := \frac{Z_{t^a}(1, \gamma_t)}{\sqrt{2\pi}} \int_0^{t_1/t} \frac{Y(u)}{u^{3/2}} \, du,
\]

\[
\Xi_t^{(3)} := \frac{Z_{t^a} - Z_{t^a}(1, \gamma_t)}{\sqrt{2\pi}} \int_0^1 \frac{Y(u)}{u^{3/2}} \, du.
\]

so that the convergence of Lemma 5.2 can be written \( \beta_t (\Xi_t + \Xi_t^{(1)} + \Xi_t^{(2)} - \Xi_t^{(3)}) \to 0 \) in probability. For \( \Xi_t^{(3)} \), recalling that the integral is finite, we have to prove that \( \beta_t (Z_{t^a}(1, \gamma_t) - Z_{t^a}) \to 0 \) in probability. But, we have \( Z_{t^a}(1, \gamma_t) = Z_{t^a} - \gamma_t W_{t^a} \) and it follows from \((\ref{eq:1.5})\) that \( \beta_t (Z_{t^a} - Z_{t^a}) \to 0 \) in probability, and from \((1.2)\) that \( \beta_t \gamma_t W_{t^a} \to 0 \). Thus, it proves that \( \beta_t \Xi_t \to 0 \) in probability. We now deal with \( \Xi_t^{(2)} \). For this, we bound the following integral, using \( |Y(u)| \leq C u^{3/4} \),

\[
\left| \int_0^{t_1/t} \frac{Y(u)}{u^{3/2}} \, du \right| \leq C \int_0^{t_1/t} \frac{du}{u^{3/4}} \leq C \left(\frac{t_1}{t}\right)^{1/4} = C \beta_t^{-3/2},
\]  \hspace{1cm} (5.8)

Hence, \( |\Xi_t^{(2)}| \leq C \beta_t^{-1/2} Z_{t^a}(1, \gamma_t) \to 0 \), because \( Z_{t^a}(1, \gamma_t) \to Z_{t^a} \). Finally, we deal with \( \Xi_t^{(1)} \), by bounding its first moment given \( \mathcal{F}_t \). For particles \( u \) with \( T_u \in (t^a, 2t^a] \), we use \( |Y(T_u/t)| \leq C t^{3(a-1)/4} \) and Lemma 4.4 to bound the number of such particles. For particles \( u \) with \( T_u \in (2t^a, t_1] \), we apply \((4.6)\). This yields

\[
\mathbb{E}[|\Xi_t^{(1)}|] \mathcal{F}_t \leq C e^{-\beta_t} \left(e^{\gamma} W_{t^a} t^{3(a-1)/4} + e^{\gamma} Z_{t^a}(1, \gamma_t) \int_{2t^a}^{t_1} \frac{|Y(r/t)|}{r^{3/2}} \, dr\right)
\]

\[
\leq CZ_{t^a}(1 + x^{-1}, \gamma_t, t^a) \left(\frac{t^{(a-1)/4} + \int_0^{t_1/t} |Y(u)|}{u^{3/2}} \, du\right)
\]

\[
\leq CZ_{t^a}(1 + x^{-1}, \gamma_t, t^a) \beta_t^{-3/2},
\]
applying again (5.8). It follows that \( \beta_t \Xi^{(1)}_t \to 0 \) in probability (using Remark 3.3). Therefore, we showed that \( \beta_t \Xi^{(i)}_t \to 0 \) in probability for each \( i \in \{1, 2, 3\} \) and this concludes the proof. \( \square \)

### 5.4 Convergence of the contributions of the killed particles

In this section, we prove Proposition 2.3, that is the convergence in law of the sum of contributions \( \Omega^{(a)}_t \) of the particles killed by the barrier.

**Proof of Proposition 2.3.** Using Lemma 5.1, the convergence we have to prove is equivalent to

\[ \sum_{u \in L^{(a)}_{t[0, 1]}} e^{-\beta_t} H(T_u/t)Z^{(a)}_\infty \xrightarrow{\text{law}} c_1(F)\beta_t Z_\infty \xrightarrow{t \to \infty} S(F). \]  

(5.9)

On the other hand, applying Lemma 5.2 to the function \( \Upsilon = H \) (which satisfies \( |H(u)| \leq Cu \) for \( u \in [0, 1] \) as a consequence of Assumption (A2a)), we have

\[ \beta_t \left( \sum_{u \in L^{(a)}_{t[0, 1]}} e^{-\beta_t} H(T_u/t) - c_1(F)Z_\infty \right) \xrightarrow{\text{law}} 0, \]

recalling that \( c_1(F) = \frac{1}{\sqrt{2\pi}} \int_0^1 H(u) \frac{du}{\omega u^\gamma}. \) Therefore, the result follows from Lemma 5.3 below. \( \square \)

**Lemma 5.3.** Let \( F : \mathbb{R} \to \mathbb{R} \) satisfying Assumptions (A1a)-(A2a) and \( S(F) \) as defined in Proposition 2.1. If \( a \in (2/3, 1) \), then we have the following convergence in distribution

\[ \sum_{u \in L^{(a)}_{t[0, 1]}} e^{-\beta_t} H(T_u/t)\left(Z^{(a)}_\infty - \beta_t\right) \xrightarrow{\text{law}} S(F). \]  

(5.10)

**Proof of Lemma 5.3.** For brevity, we denote by \( \Gamma_t \) the sum on the left-hand side of (5.10). Recalling the definition of \( S(F) \) in Proposition 2.1, first note that it is sufficient to prove that, for any \( \lambda \in \mathbb{R} \), the following convergence holds in probability

\[ \mathbb{E}\left[e^{i\lambda \Gamma_t} | \mathcal{F}_{L^{(a)}_{t[0, 1]}} \right] \xrightarrow{\text{law}} \exp(-Z_\infty[c_2(F)|\lambda| + i\lambda(c_1(F)\log|\lambda| + c_3(F))]). \]  

(5.11)

By the branching property, we have

\[ \mathbb{E}\left[e^{i\lambda \Gamma_t} | \mathcal{F}_{L^{(a)}_{t[0, 1]}} \right] = \prod_{u \in L^{(a)}_{t[0, 1]}} \Psi_{Z_\infty}(\lambda e^{-\beta_t} H(T_u/t)) \exp(-i\lambda \beta_t e^{-\beta_t} H(T_u/t)), \]  

(5.12)

where \( \Psi_{Z_\infty} \) denotes the characteristic function of \( Z_\infty \). It has been shown in [17, Equation (1.11)] that there exists a continuous function \( \psi : \mathbb{R} \to \mathbb{C} \), with \( \psi(0) = 0 \), such that for any \( \lambda \in \mathbb{R} \),

\[ \Psi_{Z_\infty}(\lambda) = \exp\left(-\frac{\pi}{2}|\lambda| + i\lambda(-\log|\lambda| + \mu_Z) + \lambda\psi(\lambda)\right). \]  

(5.13)

Using this development of \( \Psi_{Z_\infty} \) around 0, we get, for any \( x \in [0, 1] \),

\[ \Psi_{Z_\infty}(\lambda e^{-\beta_t} H(x)) \exp(-i\lambda \beta_t e^{-\beta_t} H(x)) = \exp\left(e^{-\beta_t} \Upsilon_\lambda(x) + \lambda e^{-\beta_t} H(x)\psi(\lambda e^{-\beta_t} H(x))\right), \]

with \( \Upsilon_\lambda(x) := -\frac{\pi}{2}|\lambda H(x)| + i\lambda H(x)(-\log|\lambda H(x)| + \mu_Z) \). Therefore, (5.12) is equal to

\[ \exp\left(e^{-\beta_t} \sum_{u \in L^{(a)}_{t[0, 1]}} \left[ \Upsilon_\lambda(T_u/t) + \lambda H(T_u/t)\psi(\lambda e^{-\beta_t} H(T_u/t))\right]\right). \]
Then, using that, for any $z_1, z_2 \in \{z \in \mathbb{C} : \text{Re}(z) \leq 0\}$, $|e^{z_1} - e^{z_2}| \leq 2 \land |z_2 - z_1|.$

$$\begin{align*}
\left| \mathbb{E}[e^{\alpha T_f} | \mathcal{F}_{L^{a, \gamma_1}}] - \exp \left( e^{-\beta_{\gamma_1}} \sum_{u \in L^{a, \gamma_1}} \Upsilon_{\lambda}(T_u/t) \right) \right| &\leq e^{-\beta_{\gamma_1}} \sum_{u \in L^{a, \gamma_1}} \left| \lambda H(T_u/t) \psi \left( \lambda e^{-\beta_{\gamma_1}} H(T_u/t) \right) \right| \\
&\leq \varepsilon_{\lambda}(t) e^{-\beta_{\gamma_1}} \sum_{u \in L^{a, \gamma_1}} |H(T_u/t)|, \tag{5.14}
\end{align*}$$

with $\varepsilon_{\lambda}(t) \to 0$ as $t \to \infty$ depending only on $\lambda$, using that $|H(T_u/t)| \leq C$. Then it follows from Lemma 5.2 applied to $Y = |H|$ that the right-hand side of (5.14) tends to 0 in probability. On the other hand, since $|H(x)| \leq Cx$ for $x \in [0, 1]$ by Assumption (A2), we have $|\Upsilon_{\lambda}(x)| \leq Cx^{3/4}$. Therefore, we can apply Lemma 5.2 again but with the function $\Upsilon_{\lambda}$ to get

$$\exp \left( e^{-\beta_{\gamma_1}} \sum_{u \in L^{a, \gamma_1}} \Upsilon_{\lambda}(H(T_u/t)) \right) \xrightarrow{t \to \infty} \exp \left( \frac{Z_{\infty}}{\sqrt{2\pi}} \int_0^1 \Upsilon_{\lambda}(H(u)) \frac{du}{u^{3/2}} \right). \tag{5.15}$$

The right-hand side of (5.15) coincides with the right-hand side of (5.11), recalling the definition of the $\tilde{c}_i(F)$’s and of $\Upsilon_{\lambda}$. This concludes the proof. \qed

6 Dealing precisely with shifts

The goal of this section is to prove precise estimates for $Z_t(F) - Z_t(F, \Delta)$, holding for function $F$ satisfying the assumptions of Theorem 1.1, that is which can be diverging at 0 as $x^{-\alpha}$ for some $\alpha, \kappa \geq 0$:

(H1) For any $x > 0$, $|F(x)| \leq x^{-\alpha}e^{e^x}$;

(H2) $F$ is differentiable on $(0, \infty)$ and, for any $x > 0$, $|F'(x)| \leq x^{-\alpha-1}e^{e^x}$;

(H3) $F$ is twice differentiable on $(0, \infty)$ and, for any $x > 0$, $|F''(x)| \leq x^{-\alpha-2}e^{e^x}$.

6.1 Rate of convergence for functions with a divergence at 0

In this subsection, we prove a first estimate, that generalizes Proposition 4.1 to functions $F$ which can be diverging at 0 as $x^{-\alpha}$ for $\alpha \in (0, 2)$ (note that we have to go up to 3 here).

**Proposition 6.1.** Let $\alpha \in (0, 3), \kappa \geq 1$ and $K > 0$. Let $F$ be a function satisfying Assumptions (H1)- (H2). There exists $C = C(\kappa, \alpha, K) > 0$ and $t_0 = t_0(\kappa, \alpha, K) > 0$, such that for any $t \geq t_0$, $\Delta \in [0, K \log t]$ and $\delta > 0$, we have

$$\mathbb{P}(\|Z_t(F - \mathbb{E}[F(R_1)], \Delta)\| \geq \delta) \leq t^{-K} + C(\log t)^{2\delta} t^{(\alpha-3)/12} + C(\log t)^2 e^{\Delta t^{-3/2+K-1}} 1_{n \geq 2}.$$

The main idea of the proof is to apply Proposition 4.1 to a modified version of $F$ that satisfies Assumptions (A1)-(A2). We introduce the function

$$\tilde{F}_t(x) := F(x)1_{x > x_t} + F(x_t)1_{x \leq x_t},$$

where $x_t \to 0$ as $t \to \infty$ and will be chosen explicitly later but we assume for now that $(\log t)^2/\sqrt{t} \leq x_t \leq 1$. We first prove the following lemma showing we can replace $F$ by $\tilde{F}_t$. \[25\]
Lemma 6.2. Let $\alpha \in (0, 3)$, $\kappa \geq 1$ and $K > 0$. Let $F$ be a function satisfying Assumption (H1_{a,\kappa}). There exists $C = C(\kappa, \alpha, K) > 0$ and $t_0 = t_0(\kappa, \alpha, K) > 0$, such that for any $t \geq t_0$, $\Delta \in [0, K \log t]$ and $\delta > 0$, we have
\[
\mathbb{P}\left(\left|Z_t(\tilde{F} - \mathbb{E}[F(R_1)], \Delta) - Z_t(\tilde{F}_t - \mathbb{E}[\tilde{F}_t(R_1)], \Delta)\right| \geq \delta\right) 
\leq 2t^{-K} + C(\log t)x_t^{3-\alpha}\delta^{-1} + C(\log t)^2e^{Kt^{-3/2+K^{-1}}\mathbb{1}_{\alpha \geq 2}}.
\]

Proof. We set $M = K \log t$. We first work in the case $\alpha < 2$ and bound the first moment of $Z_t(F - \tilde{F}_t, \Delta)$ on the event $A := \{\forall r \in [0, t], \min_{u \in \mathbb{N}(r)} X_u(r) > -M\}$. By Assumption (H1_{a,\kappa}), we have $|F - \tilde{F}_t|(x) \leq Cx^{-\alpha}\mathbb{1}_{x \leq x_t}$. By the many-to-one lemma (3.2), we have
\[
\mathbb{E}\left[\left|Z_t(F - \tilde{F}_t, \Delta)\right| \mathbb{1}_A\right] \leq e^M\mathbb{E}_M\left[\tilde{Z}_t(\tilde{F} - \mathbb{E}[\tilde{F}_t], \Delta + M)\right] 
= M\mathbb{E}_M\left[\frac{(R_t - \Delta - M)}{R_t} \cdot |F - \tilde{F}_t|\left(\frac{R_t - \Delta - M}{\sqrt{t}}\right)\right] 
\leq C\mathbb{E}_M/\sqrt{t}\left[\frac{1}{R_t(R_t - \eta)^{\alpha-1}}\mathbb{1}_{0 < R_t - \eta \leq x_t}\right],
\]
where we set $\eta = (\Delta + M)/\sqrt{t}$. We apply (B.5) bounding $e^{-(z-y)^2/2}(1 - e^{-2zy}) \leq 2zy$, to get that the last expectation is smaller than
\[
\int_{\eta}^{x_t+\eta} \frac{Cz \, dz}{(z - \eta)^{\alpha-1}} \leq C\left(\int_{\eta}^{2\eta} \frac{\eta \, dz}{(z - \eta)^{\alpha-1}} + \int_{\eta}^{x_t+\eta} \frac{dz}{z^{\alpha-2}}\right) \leq C\left(\eta^{3-\alpha} + (x_t + \eta)^{3-\alpha}\right),
\]
where for the first integral we used that $\alpha < 2$. Finally, noting that $\eta \leq x_t$ for $t$ large enough, we get
\[
\mathbb{E}\left[\left|Z_t(F - \tilde{F}_t, \Delta)\right| \mathbb{1}_A\right] \leq C(\log t)x_t^{3-\alpha}.
\]
In the case $\alpha \in [2, 3)$, we have to remove some particles before taking the first moment, in order to get rid of the integral on $[\eta, 2\eta]$ in (6.2) which is infinite when $\alpha \geq 2$. For this, let $B := \{\min_{u \in \mathbb{N}(t)} X_u(t) - \Delta > (\log t)/K\}$, and the same calculation shows
\[
\mathbb{E}\left[\left|Z_t(F - \tilde{F}_t, \Delta)\right| \mathbb{1}_{A \cap B}\right] \leq C\mathbb{E}_M/\sqrt{t}\left[\frac{1}{R_t(R_t - \eta)^{\alpha-1}}\mathbb{1}_{(\log t)/(K\sqrt{t}) < R_t - \eta \leq x_t}\right] 
\leq CM \int_{\eta/(\log t)/(K\sqrt{t})}^{x_t+\eta} \frac{z \, dz}{(z - \eta)^{\alpha-1}} \leq C(\log t)x_t^{3-\alpha},
\]
where we use that for $z \geq \eta + (\log t)/(K\sqrt{t})$, we have $z - \eta \geq c\eta$, where the constant $c$ depends on $K$. Using that $\mathbb{P}(A^c) \leq e^{-M}$ by (3.4) and $\mathbb{P}(B^c) \leq C(\log t)^2e^{Kt^{-3/2+K^{-1}}}$ by (3.5), we proved that
\[
\mathbb{P}\left(\left|Z_t(F - \tilde{F}_t, \Delta)\right| \geq \delta\right) \leq e^{-M} + C(\log t)x_t^{3-\alpha}\delta^{-1} + C(\log t)^2e^{Kt^{-3/2+K^{-1}}\mathbb{1}_{\alpha \geq 2}}.
\]
On the other hand, using again $|F - \tilde{F}_t|(x) \leq Cx^{-\alpha}\mathbb{1}_{x \leq x_t}$ and the density of $R_1$ in (B.4), we get
\[
\left|\mathbb{E}[F(R_1)] - \mathbb{E}[\tilde{F}_t(R_1)]\right| \leq \int_0^{x_t} \frac{Cz^2 \, dz}{z^{\alpha}} \leq Cx_t^{3-\alpha},
\]
so a bound similar to (6.4) holds for $(\mathbb{E}[F(R_1)] - \mathbb{E}[\tilde{F}_t(R_1)])Z_t(1, \Delta)$ by (3.12). \qed

We can now prove Proposition 6.1 by applying Proposition 4.1 to a rescaled version of $\tilde{F}_t$.
Proof of Proposition 6.1. Note that by Assumption \((H1_{α,κ})\), we have \(|\tilde{F}_t(x)| \leq x^{-α} e^{cx} \cdot e^{cx} \) for any \(x > 0\). Moreover, by Assumption \((H2_{α,κ})\) and distinguishing cases, we get that, for any \(0 < y \leq x\), \(|\tilde{F}_t(x) - \tilde{F}_t(y)| \leq x^{-α-1}(x-y)e^{cx}\). This proves that the function \(x^3 \tilde{F}_t\) satisfies Assumptions \((A1_κ)\)-(\(A2_κ\)), hence we can apply Proposition 4.1 to this function to get
\[
P\left(\left|Z_t(\tilde{F}_t - \mathbb{E}[\tilde{F}_t(R_1)], \Delta)\right| \geq \delta \right) \leq t^{-K} + C(\log t)^2 t^{-1/3} x_t^{-1-α^{-1}}.
\]
We combine this with Lemma 6.2, choosing \(x_t = t^{-1/12}\), which is the optimal choice given by the equation \(x_t^3 = t^{-1/3} x_t^{-1-α}\). This proves the result. \(\square\)

6.2 Precise effect of a shift

Proposition 6.3. Let \(α \in (0,2)\) and \(κ \geq 1\). Let \(F\) be a function satisfying Assumptions \((H1_{α,κ})-(H2_{α,κ})-(H3_{α,κ})\). Let \(ε > 0\). Uniformly in \(\Delta \in [0, (\frac{2}{3} - ε) \log t]\), we have the following convergence in probability
\[
\sqrt{t} \cdot \left( Z_t(F) - Z_t(F, \Delta) + \left( \frac{1}{\sqrt{2π}} \int_0^1 H(u) \frac{du}{u^{3/2}} - \sqrt{\frac{1}{2π}} \mathbb{E}[\tilde{F}_t(R_1)] \right) Z_\infty \frac{\Delta}{\sqrt{t}} \right) \xrightarrow{t \to ∞} 0.
\]

Proof. We first note that
\[
\frac{1}{\sqrt{2π}} \int_0^1 H(u) \frac{du}{u^{3/2}} = \sqrt{\frac{1}{2π}} \mathbb{E}[\tilde{F}_t(R_1)] - \mathbb{E} \left[ \frac{F(R_1) + F'(R_1)}{R_1} \right],
\]
which is proved by direct calculation in Appendix \(A\). We introduce the event \(E_t := \{\min_{u \in \mathcal{N}(t)} X_u(t) > \Delta + 1\}\). Using \((3.5)\), we have \(P(E_t') \leq C(\log t)^2 t^{-ε} \to 0\) as \(t \to ∞\) and, therefore it is now sufficient to prove that
\[
\mathbbm{1}_{E_t} \sqrt{t} \cdot \left( Z_t(F) - Z_t(F, \Delta) - \mathbb{E} \left[ \frac{F(R_1) + F'(R_1)}{R_1} \right] Z_∞ \frac{\Delta}{\sqrt{t}} \right) \xrightarrow{t \to ∞} 0. \tag{6.6}
\]
Firstly, on event \(E_t\), we have \(Z_t(F) - Z_t(F, \Delta) = Z_t(\Phi_t, \Delta)\), where we set, for \(x \in \mathbb{R}\),
\[
\Phi_t(x) := \frac{1}{x} [(x + δ) F(x + δ) - x F(x)] \mathbbm{1}_{x > 0}, \quad \text{with} \quad δ := \frac{\Delta}{\sqrt{t}}.
\]
Moreover, let \(\Phi(x) := \frac{1}{x} (F(x) + x F'(x)) \mathbbm{1}_{x > 0}\). Note that, at the first order, \(\Phi_t\) behaves as \(δ\Phi\). With this in mind, we split the proof of \((6.6)\) into the two following convergences in probability:
\[
\mathbbm{1}_{E_t} \sqrt{t} \cdot Z_t(\Phi_t - δ\Phi, \Delta) \xrightarrow{t \to ∞} 0, \tag{6.7}
\]
\[
\Delta \cdot (Z_t(\Phi, Δ) - \mathbb{E}[\Phi(R_1)] Z_∞) \xrightarrow{t \to ∞} 0. \tag{6.8}
\]
For \((6.8)\), we simply note that it follows from Assumptions \((H1_{α,κ})-(H2_{α,κ})-(H3_{α,κ})\) for \(F\), that \(\Phi/3\) satisfies Assumptions \((H1_{α+1,κ})-(H2_{α+1,κ})\) and therefore we can apply Proposition 6.1 to \(\Phi/3\), since \(α + 1 < 3\). Combining this with \((1.7)\), this proves \((6.8)\).

We now prove \((6.7)\). For \(x > 0\), we have
\[
x(\Phi_t(x) - δ\Phi(x)) = \int_0^δ [F(x + u) + (x + u) F'(x + u)] - [F(x) + x F'(x)] \, du.
\]
Using Assumptions \((H2_{α,κ})-(H3_{α,κ})\) for \(F\), we can bound the integrand by \(u 3x^{-α-1} e^{cx}\) and therefore we get \(|\Phi_t(x) - δ\Phi(x)| \leq Cδ^2 x^{-α-2} e^{cx}\) for any \(x > 0\). Note that this is a strong divergence at 0, which is not integrable against the density of \(R_1\) if \(α \geq 1\), which implies that \(Z_t(x \mapsto x^{-α-2} e^{cx}, \Delta)\) is not tight. However, we will bound its first moment, on the event
\[ A := \{ \forall r \in [0, t], \min_{u \in \mathcal{N}(r)} X_u(r) > -M \} \text{ intersected with } E_t. \] Using the many-to-one formula (3.2), we get

\[
E[|Z_t(\Phi_t - \delta \Phi, \Delta)| 1_{A \cap E_t}] \leq M E_M \left[ \frac{(R_t - \Delta - M)z}{R_t} \cdot |\Phi_t - \delta \Phi| \left( \frac{R_t - \Delta - M}{\sqrt{t}} \right) 1_{R_t > \Delta + 1 + M} \right]
\]

\[
\leq C M \delta^2 E_M \left[ \frac{e^{c(R_t - \eta)}}{R_t(R_t - \eta)^{\alpha + 1}} 1_{R_t > \eta + t^{-1/2}} \right].
\]

using \(|\Phi_t(x) - \delta \Phi(x)| \leq C \delta^2 x^{-\alpha - 2} e^{\alpha x}
\]

and setting \(\eta := (\Delta + M)/\sqrt{t}\). We can assume that \(t\) is large enough so that \(\eta < 1/2\). We bound the density of \(R_t\) under \(P_M/\sqrt{t}\), given in (B.5), by \(z^2\)

if \(z \leq 1\) and by \(z^2 e^{-(z-M/\sqrt{7})^2/2}\) if \(z > 1\). Thus, the last expectation is smaller than

\[
\int_1^{\infty} C z \frac{dz}{(z-\eta)^{\alpha + 1}} + \int_1^{\infty} C e^{\alpha z} e^{-(z-M/\sqrt{7})^2/2} dz \leq \int_1^{2\eta} C \eta \frac{dz}{(z-\eta)^{\alpha + 1}} + \int_1^{2\eta} C \frac{dz}{2z^\alpha} + C
\]

\[
\leq C \eta^{\alpha/2} + \eta^{\alpha - 1} + C \leq C (log t)^{\alpha/2},
\]

using \(\eta \leq C (log t)/\sqrt{7}\). Hence, since \(\delta \leq C (log t)/\sqrt{7}\), we get

\[
\sqrt{7} \cdot E[|Z_t(\Phi_t - \delta \Phi, \Delta)| 1_{A \cap E_t}] \leq C (log t)^4 t^{\alpha/2 - 1} \xrightarrow{t \to \infty} 0,
\]

which proves (6.7) and concludes the proof. \(\square\)

7 Proof of the theorem

In this section, our aim is to prove Theorem 1.1. In a first subsection, we prove a weaker version of the theorem (Proposition 7.1), in which we consider \(Z_t(F, \gamma_t)\) instead of \(Z_t(F)\) and we work in the case of a function \(F\) without a divergence at 0. Then, we have already seen in Section 6 how to replace \(Z_t(F, \gamma_t)\) by \(Z_t(F)\), see Proposition 6.3. Hence our remaining task will be to show how to deal with functions that can diverge as \(x^{-\alpha}\) at 0, with \(\alpha \in [0, 2]\), and this is the goal of the second subsection. Finally, in a third subsection, we prove Theorem 1.1.

7.1 A first version of the theorem

In this section, we prove the following result, close to Theorem 1.1: the two differences are that the function \(F\) is assumed to satisfy Assumptions (A1\(a\))-(A2\(a\)) and we deal with \(Z_{at}(F, \gamma_t, at)\) instead of \(Z_{at}(F)\). As in (2.1), we consider \(\gamma_t := 1/2 \log t + \beta_t\) with \((\beta_t)_{t \geq 0}\) a family of positive real numbers such that

\[
\beta_t \xrightarrow{t \to \infty} \infty \quad \text{and} \quad \frac{\beta_t}{\log t} \xrightarrow{t \to \infty} 0.
\]

**Proposition 7.1.** Let \(F\) be a function satisfying Assumptions (A1\(a\))-(A2\(a\)). Let \(\varepsilon > 0\). Then, conditionally on \(\mathcal{F}_t\), as \(t \to \infty\) and then \(\varepsilon \to 0\), the finite-dimensional distributions of

\[
\sqrt{t} \cdot \left( Z_{at}(F, \gamma_t, at) - E[F(R_t)] Z_{\infty} - c_3(F) \frac{\beta_t}{\sqrt{at}} Z_{\infty} \right)_{a \in (0, \infty)}
\]

converge in probability to

\[
\left( \int_0^\infty H(u) dL_{-Z_{\infty}/\sqrt{M}} \right)_{a \in (0, \infty)}
\]

where \((L_s)_{s \in \mathbb{R}}\) is a spectrally positive \(1\)-stable Lévy process indexed by \(\mathbb{R}\) with characteristic function given by

\[
E[e^{i\lambda(L_a - L_{-1})}] = \exp \left( -|\lambda| + i\lambda \frac{2}{\pi} (\log |\lambda| - \mu Z) \right),
\]

with \(H(u) := E[F(R_{t-u}) 1_{u < 1} - F(R_t)]\) for \(u \geq 0\), and where \(c_3(F) = \frac{1}{\sqrt{2\pi}} \int_0^1 H(u) \frac{du}{u^{3/2}}\).

28
Proof. We fix some small \( \varepsilon > 0 \). We add an absorbing barrier at \( \gamma_t \) between time \( \varepsilon t \) and \( \infty \). Let \( Z_{\varepsilon t, \gamma_t} \) be the particles absorbed during the time interval \( I \). For some \( a > 0 \), assuming that \( \varepsilon < a \), we have

\[
Z_{at}(F, \gamma_t, at) = \tilde{Z}_{\varepsilon t, \gamma_t}(F, \gamma_t, at) + \sum_{u \in L_{\varepsilon t, \gamma_t}} \Omega_{at}^{(u)}
\]

where we define \( \Omega_{at}^{(u)} \) as the contribution of particle \( u \) in \( Z_{at}(F, \gamma_t, at) \) (note that before it was the contribution in \( Z_{at}(F - E[F(R_1)], \gamma_t, at) \) by

\[
\Omega_{at}^{(u)} := \sum_{v \in N(at), u \leq v} (X_v(at) - \gamma_t) e^{-X_v(at)} F\left(\frac{X_v(at) - \gamma_t}{\sqrt{at}}\right).
\]

We decompose \( Z_{\infty} \) as follows, with \( \tilde{Z}_{\varepsilon t, \gamma_t} \) the a.s. limit of the non-negative martingale \( (\tilde{Z}_{s, \gamma_s})_{s \geq \varepsilon t} \),

\[
Z_{\infty} = \tilde{Z}_{\varepsilon t, \gamma_t} + \sum_{u \in L_{\varepsilon t, \infty}} e^{-\gamma_t} Z_{\infty}^{(u)}
\]

which was the decomposition used in [17] to obtain the fluctuations of the derivative martingale.

We first prove (as \( t \to \infty \) and then \( \varepsilon \to \infty \))

\[
\sqrt{t}\left(\tilde{Z}_{\varepsilon t, \gamma_t}(F, \gamma_t) - E[F(R_1)]\tilde{Z}_{\varepsilon t, \gamma_t} - \frac{1}{\sqrt{2\pi}} \int_0^{\varepsilon/a} H(u) \frac{du}{u^{3/2}} \frac{\beta_t}{\sqrt{at}} Z_{\infty}\right) \xrightarrow{t \to \infty, \varepsilon \to 0} 0.
\]

By [17, Lemma 5.1], we know that \( \sqrt{t}(\tilde{Z}_{\varepsilon t, \gamma_t} - Z_{\varepsilon t}(1, \gamma_t)) \to 0 \) in probability. Therefore, (7.4) will follow from \( \sqrt{t}\tilde{Z}_{\varepsilon t, \gamma_t}(F, \gamma_t) \to 0 \) in probability, where we set \( F = F - E[F(R_1)] \). Proceeding as in (5.1), we have

\[
E[\tilde{Z}_{\varepsilon t, \gamma_t}(F, \gamma_t)] = Z_{\varepsilon t}(F^{\varepsilon/a} - E[F(R_1)], \gamma_t),
\]

where \( F^h(x) := E[\sqrt{h} F(R_1)] \). Moreover, proceeding as in (4.3) and (4.4) (using here Lemma 3.4 with \( r = (1 - \varepsilon)t \)), we get

\[
\text{Var}\left(\tilde{Z}_{\varepsilon t, \gamma_t}(F, \gamma_t) | \mathcal{F}_t\right) \leq e^{-\gamma_t} \sum_{v \in N(\varepsilon t)} 1_{X_v(\varepsilon t) > \gamma_t} e^{-X_v(\varepsilon t)} \left(\varepsilon^2 + \frac{X_v(\varepsilon t) - \gamma_t}{\sqrt{t}}\right) \leq \frac{e^{-\beta_t}}{t} Z_{\varepsilon t}(x \mapsto x^{-1} + 1, \gamma_t, \varepsilon t).
\]

Combining (7.5) and (7.6) proves that \( \sqrt{t}(\tilde{Z}_{\varepsilon t, \gamma_t}(F, \gamma_t) - Z_{\varepsilon t}(F^{\varepsilon/a} - E[F(R_1)], \gamma_t)) \to 0 \) in probability. On the other hand, we know from Lemma 7.2 below (taking \( t \) and \( h \) to be \( \varepsilon t \) and \( \varepsilon/a \)) that

\[
\sqrt{at}\left(Z_{\varepsilon t}(F^{\varepsilon/a}, \gamma_t) - E[F(R_1)]Z_{\varepsilon t}(1, \gamma_t) - \frac{1}{\sqrt{2\pi}} \int_0^{\varepsilon/a} H(u) \frac{du}{u^{3/2}} \frac{\beta_t}{\sqrt{at}} Z_{\infty}\right) \xrightarrow{t \to \infty, \varepsilon \to 0} 0.
\]

This proves (7.4).

We now deal with the contributions of killed particles. First note that it follows from Lemma 5.1 (more precisely from its proof, see (5.3)), that

\[
\sqrt{t}\sum_{u \in L_{\varepsilon t, \gamma_t}} \left| \Omega_{at}^{(u)} - E[F(R_1 - T_u)/(at)]\right| e^{-\gamma_t} Z_{\infty}^{(u)} \xrightarrow{t \to \infty} 0.
\]
Therefore, combining this with (7.4) and the decompositions (7.2) and (7.3), it is now sufficient to prove that, conditionally on \( \mathcal{F}_{at} \), as \( t \to \infty \) and then \( \varepsilon \to 0 \), the finite-dimensional distributions of

\[
\sqrt{t} \left( \sum_{u \in \mathcal{E}_{[\varepsilon, \infty)}} H(T_u/(at)) e^{-\gamma_t} Z_\infty^{(u)} - \frac{1}{\sqrt{2\pi}} \int_\varepsilon^\infty H(u) \frac{du}{u^{3/2}} \beta_t Z_\infty \right)_{a \in (0, \infty)}
\]

(7.8)

converge in probability to (7.1). Firstly, it follows from the proof of Lemma 5.2 that

\[
\beta_t \left( \sum_{u \in \mathcal{E}_{[\varepsilon, \infty)}} H(T_u/(at)) - \frac{Z_\infty}{\sqrt{2\pi}} \int_\varepsilon^\infty H(u) \frac{du}{u^{3/2}} \right) \to 0 \text{ in probability as } t \to \infty.
\]

Therefore, we can replace (7.8) by

\[
\left( \sum_{u \in \mathcal{E}_{[\varepsilon, \infty)}} H(T_u/(at)) e^{-\beta_t} (Z_\infty^{(u)} - \beta_t) \right)_{a \in (0, \infty)}
\]

(7.9)

The convergence of this quantity follows from a characteristic function calculation similar to the proof of Lemma 5.3.

**Lemma 7.2.** Let \( F \) be a function satisfying Assumptions (A1c)–(A2c). For \( h > 0 \), define \( F^h(x) = \mathbb{E}_{\sqrt{h} x} [F(R_{1-h})] \) and \( \gamma_t^h = \gamma_t + O(\log h) \), where the \( O(\ldots) \) term holds as \( h \to 0 \) uniformly in \( t \). As \( t \to \infty \) and then \( h \to 0 \), we have

\[
\sqrt{\frac{h}{t}} \left( Z_t(F^h, \gamma_t^h) - \mathbb{E}[F(R_1)] Z_t(1, \gamma_t) - \sqrt{\frac{1}{2\pi}} \int_0^h H(u) \frac{du}{u^{3/2}} \beta_t Z_\infty \right) \to 0 \text{ in probability as } t \to \infty.
\]

**Proof.** We first fix \( h > 0 \). By Proposition 2.1, since \( \gamma_t^h = \frac{1}{2} \log t \to \infty \) as \( t \to \infty \), we get

\[
\sqrt{t} \left( Z_t(F^h, \gamma_t^h) - \mathbb{E}[F(R_1)] Z_t(1, \gamma_t) - c_1(F^h) \frac{1}{\sqrt{t}} \log t Z_\infty \right) \overset{\text{law}}{\to} \tilde{S}(F^h),
\]

where \( c_1(F) \) and \( \tilde{S}(F^h) \) are defined in Proposition 2.1, here in terms of the function

\[
H^h(u) = \mathbb{E} \left[ F(R_{1-u}) - F(R_1) \right] = \mathbb{E}[F(R_{1-u}) - F(R_1)] = H(uh), \quad u \in [0, 1]
\]

where the second equality follows from the Markov property of the Bessel process. Recall also that, by Assumption (A2c), we have \( |H(u)| \leq Cu \). Therefore, with a change of variable, we get

\[
c_2(F^h) = \frac{1}{2} \sqrt{\frac{h\pi}{2}} \int_0^h |H(u)| \frac{du}{u^{3/2}} = O(h),
\]

\[
c_1(F^h) = \sqrt{\frac{h}{2\pi}} \int_0^h H(u) \frac{du}{u^{3/2}} = O(h),
\]

\[
c_3(F^h) = \frac{1}{2} \sqrt{\frac{h}{2\pi}} \int_0^h H(u)(-\mu_Z + \log H(u)) \frac{du}{u^{4/2}} = O(h \log h),
\]

where the \( O(\ldots) \) terms are meant to hold as \( h \to 0 \). It follows that

\[
\frac{\gamma_t^h}{\sqrt{h}} c_1(F^h) \to 0 \quad \text{and} \quad \frac{1}{\sqrt{h}} \tilde{S}(F^h) \to 0 \text{ in probability as } h \to 0.
\]

Altogether, this shows the result.\[\Box\]
7.2 Regularizing the function at 0

We now want to deal with functions diverging as \( x^{-\alpha} \) at 0, for some \( \alpha \in [0, 2) \). The following result will allow us to regularize such a function at 0: note that the function \( F^{1-\varepsilon} \) is smooth and locally bounded at 0.

**Lemma 7.3.** Let \( F \) be a function satisfying Assumptions \((H1_{\alpha,\kappa})-(H2_{\alpha,\kappa})-(H3_{\alpha,\kappa})\). For \( h > 0 \), define \( F^h(x) = \mathbb{E}_{\sqrt{t}x}[F(R_{1-h})] \). Let \( c \in (\frac{1}{2}, \sqrt{2}, \frac{3}{2}) \). As \( t \to \infty \) and then \( \varepsilon \to 0 \), we have

\[
\sqrt{t} \left( Z_t(F, c \log t) - Z_{(1-\varepsilon)t}(F^{1-\varepsilon}, c \log t) - \left( \frac{c-\frac{1}{2}}{\sqrt{2\pi}} \log t \right) Z_{\infty} \int_{1-\varepsilon}^1 \mathbb{E}[F(R_{u-t})] \frac{du}{u^{3/2}} \right) \overset{P}{\to} 0.
\]

**Proof.** We introduce a barrier at \( \gamma = c \log t \) between times \((1-\varepsilon)t\) and \( t \). We decompose again

\[
Z_t(F, \gamma) = \tilde{Z}_t^{(1-\varepsilon)t, \gamma}(F, \gamma) + \sum_{u \in \mathcal{L}^{(1-\varepsilon)t, \gamma}} \Omega_t^{(u)}.
\]

Proceeding as in (5.1), we have

\[
\mathbb{E} \left[ \tilde{Z}_t^{(1-\varepsilon)t, \gamma}(F, \gamma) \bigg| \mathcal{F}_{(1-\varepsilon)t} \right] = Z_{(1-\varepsilon)t}(F^{1-\varepsilon}, \gamma).
\]

Moreover, using the branching property at time \((1-\varepsilon)t\) as in (4.3) and applying Lemmas 3.4 and 3.5, we get

\[
\text{Var} \left( \tilde{Z}_t^{(1-\varepsilon)t, \gamma}(F, \gamma) \bigg| \mathcal{F}_{(1-\varepsilon)t} \right) \leq e^{-\gamma} \sum_{v \in \mathcal{N}^{(1-\varepsilon)t}} e^{-X_v((1-\varepsilon)t)} \left( 1 + \frac{X_v((1-\varepsilon)t)}{\sqrt{\varepsilon t}} g_v(t) \right),
\]

where \( g_v(t) \) equals 1 if \( \alpha < 1 \), \( \log t \) if \( \alpha = 1 \) and \( t^{\alpha-1} \) if \( \alpha > 1 \). For fixed \( \varepsilon > 0 \), this is a \( O(t^{-\varepsilon}g_v(t)t^{-1/2}) \) in probability as \( t \to \infty \). Since \( c > \alpha - \frac{1}{2} \), this shows that the conditional variance is a \( o(t^{-1}) \) in probability and we conclude that

\[
\sqrt{t} \left( \tilde{Z}_t^{(1-\varepsilon)t, \gamma}(F, \gamma) - Z_{(1-\varepsilon)t}(F^{1-\varepsilon}, \gamma) \right) \overset{P}{\to} 0.
\]

We now have to estimate the contributions of killed particles. First note that with high probability, since \( c < 3/2 \), no particle is killed exactly at time \((1-\varepsilon)t\). Moreover, fixing some \( \delta \in (0, \frac{3}{2} - \varepsilon) \), it follows from (4.6) that, with high probability, no particle is killed in the time interval \([t-\delta, t]\). Therefore, we can deal only with the particles killed in the time interval \((1-\varepsilon)t, t-\delta t\).

Recall that, given \( \mathcal{F}_{(1-\varepsilon)t, \gamma} \), the \( \Omega_t^{(u)} \) for \( u \in \mathcal{L}^{(1-\varepsilon)t, \gamma} \) are independent with respectively the same distribution as \( t^{-\varepsilon}Z_{t-t^u}(F, 0, t) \) under \( \mathbb{P} \). But \( Z_{t-t^u}(F, 0, t) = Z_{t-th}(F, h, 0, t-th) \), with \( F_h(x) = F(\sqrt{1-h}x) \). Since \((1-h)^{\alpha/2}F_h\) satisfies Assumptions \((H1_{\alpha,\kappa})-(H2_{\alpha,\kappa})\), we can apply Lemma 7.4 below to obtain, with \( G(h) := \mathbb{E}[F(R_{1-h})] \),

\[
\mathbb{E} \left[ \sum_{u \in \mathcal{L}^{(1-\varepsilon)t, \gamma}} \left( \Omega_t^{(u)} - t^{-\varepsilon}G(T_u/t)Z_{\infty}^{(u)} \right) \bigg| \mathcal{F}_{(1-\varepsilon)t, \gamma} \right] \leq C t^{-c-\frac{1}{13}} \sum_{u \in \mathcal{L}^{(1-\varepsilon)t, \gamma}} (1-T_u/t)^{-\alpha/2}.
\]

Applying Lemma 4.4 if \( T_u \leq (1-\varepsilon/2)t \) and (4.6) if \( T_u > (1-\varepsilon/2)t \), the conditional expectation of the left-hand side given \( \mathcal{F}_{(1-\varepsilon)t} \) is smaller than

\[
C t^{-c-1/13} \left( e^{\frac{3}{2}W_{(1-\varepsilon)t}(\varepsilon/2)-(\alpha/2)} + t^{\varepsilon}Z_{(1-\varepsilon)t}(1, \gamma) \int_{(1-\varepsilon/2)t}^t \frac{(1-r/\varepsilon)^{\alpha/2}}{(r - (1-\varepsilon)t)^{3/2}} dr \right)
\leq C t^{-1/13} \left( W_{(1-\varepsilon)t} + Z_{(1-\varepsilon)t}(1, \gamma) t^{-1/2} \int_{1-\varepsilon/2}^1 \frac{(1-u)^{-\alpha/2}}{u^{3/2}} du \right),
\]

31
where \( C_\varepsilon \) can depend on \( \varepsilon \). Since the integral is finite and \( W_{(1-\varepsilon)t} = O(t^{-1/2}) \), this last quantity is a \( o(t^{-1/2}) \) in probability. This means that it is now sufficient to prove that

\[
\sqrt{t} \left( \sum_{u \in \mathcal{L}^{(1-\varepsilon),\gamma}_{(1-\varepsilon)t,t-\varepsilon t}} t^{-\varepsilon} G(T_u/t)Z_{\infty}^{(u)} - \frac{Z_{\infty}}{\sqrt{2\pi}} \int_{1-\varepsilon}^{1} G(u) \frac{du}{u^{3/2}} \right) \xrightarrow{P} t^{-\infty,\varepsilon \to 0} 0. \tag{7.10}
\]

with \( \beta = \gamma - \frac{1}{2} \log t = \left( c - \frac{1}{2} \right) \log t \). On the one hand, from a characteristic function calculation similar to the proof of Lemma 5.3, we get

\[
\sum_{u \in \mathcal{L}^{(1-\varepsilon),\gamma}_{(1-\varepsilon)t,t-\varepsilon t}} e^{-\beta G(T_u/t)} \left( Z_{\infty}^{(u)} - \beta \right) \xrightarrow{P} t^{-\infty,\varepsilon \to 0} 0.
\]

On the other hand, it follows from the proof of Lemma 5.2 that

\[
\beta \left( \sum_{u \in \mathcal{L}^{(1-\varepsilon),\gamma}_{(1-\varepsilon)t,t-\varepsilon t}} e^{-\beta G(T_u/t)} - \frac{Z_{\infty}}{\sqrt{2\pi}} \int_{1-\varepsilon}^{1} G(u) \frac{du}{u^{3/2}} \right) \xrightarrow{P} t^{-\infty,\varepsilon \to 0} 0.
\]

Therefore, (7.10) is proved and this concludes the proof.

**Lemma 7.4.** Let \( \alpha \in (0, 2) \), \( \kappa \geq 1 \) and \( K > 0 \). Let \( F \) be a function satisfying Assumptions (H1\( \alpha,\kappa \))-(H2\( \alpha,\kappa \)). There exists \( C = C(\kappa, \alpha, K) > 0 \) and \( t_0 = t_0(\kappa, \alpha, K) > 0 \), such that for any \( t \geq t_0 \) and \( d \in [0, K] \), we have

\[
\mathbb{E} \left[ t^{-d} | Z_{t}(F) - \mathbb{E}[F(R_1)]Z_{\infty} | \wedge 1 \right] \leq t^{-d-1/3}. \tag{7.11}
\]

**Proof.** We write the expectation as an integral

\[
\mathbb{E} \left[ t^{-d} | Z_{t}(F) - \mathbb{E}[F(R_1)]Z_{\infty} | \wedge 1 \right] = \int_{0}^{1} P \left( | Z_{t}(F) - \mathbb{E}[F(R_1)]Z_{\infty} | \geq xt^{d} \right) \, dx
\]

For \( x \leq t^{-d}(\alpha-3)/12 \), we bound the probability by 1. Otherwise, we apply Proposition 6.1 combined with (1.7) to get

\[
P \left( | Z_{t}(F) - \mathbb{E}[F(R_1)]Z_{\infty} | \geq xt^{d} \right) \leq t^{-K-1} + C(\log t)^2 x^{-1} t^{-d} t^{\alpha-3)/12}.
\]

The result follows using that \( \alpha < 2 \). \( \square \)

### 7.3 Proof of the theorem

**Proof of Theorem 1.1.** The proof is essentially a combination of previous results. We first apply Proposition 6.3 to replace \( Z_{at}(F) \) by \( Z_{at}(F, c \log t) \) for some \( c \in (\alpha - \frac{1}{2}, \frac{3}{2}) \). Then, for some small fixed \( \varepsilon > 0 \), we replace \( Z_{at}(F, c \log t) \) by \( Z_{(1-\varepsilon)at}(F^{1-\varepsilon}, c \log t) \) using Proposition 2.3. Applying again Proposition 6.3, we replace \( Z_{(1-\varepsilon)at}(F^{1-\varepsilon}, c \log t) \) by \( Z_{(1-\varepsilon)at}(F^{1-\varepsilon}, (\gamma t) \). The function \( F_{1-\varepsilon} \) satisfies Assumptions (A1\( \alpha \))-(A2\( \alpha \)) up to a rescaling by a constant depending on \( \varepsilon \), so we can apply Proposition 7.1 to get the multidimensional convergence of \( Z_{(1-\varepsilon)at}(F^{1-\varepsilon}, (\gamma t) \). Finally, we let \( \varepsilon \to 0 \) so that the distribution of the limit converges to the desired distribution and the error term coming from the regularization step tends to 0. Keeping track of the constants in the \((\log t)/\sqrt{t} \) term, this proves Theorem 1.1. \( \square \)
A Two constants depending of $F$ are identical

The relation (6.5) could actually be deduced from our proof, since if it was wrong, the final result in Theorem 1.1 would depend on the choice of $\beta_t$. However, we provide a direct proof here.

Proof of Equation (6.5). Recall $H(u) = \mathbb{E}[F(R_{1-u}) - F(R_1)] = \mathbb{E}[F(\sqrt{1-u}R_1) - F(R_1)]$ for $u \in (0, 1)$. We first introduce artificially a factor $\sqrt{1-u}$ and integrate by part

$$
\int_0^1 H(u) \frac{du}{u^{3/2}} = \int_0^1 \sqrt{1-u} H(u) \frac{du}{u^{3/2}} \sqrt{1-u} = \left[ \sqrt{1-u} H(u) \left( \frac{-2\sqrt{1-u}}{\sqrt{u}} \right) \right]_0^1 + \int_0^1 \left( \frac{-H(u)}{2\sqrt{1-u}} + \sqrt{1-u} H'(u) \right) 2\sqrt{1-u} \, du.
$$

We have $(1-u)H(u) \to 0$ as $u \to 1$ and $H(u)/\sqrt{u} \to 0$ as $u \to 0$ as consequences of Assumptions (H1α,κ)-(H2α,κ), so the boundary terms are zero and we get

$$
\int_0^1 H(u) \frac{du}{u^{3/2}} = -\int_0^1 \mathbb{E}[F(\sqrt{1-u}R_1) + \sqrt{1-u}R_1 F'(\sqrt{1-u}R_1)] \frac{du}{\sqrt{u}} + \int_0^1 \mathbb{E}[F(R_1)] \frac{du}{\sqrt{u}}.
$$

The second term equals $2\mathbb{E}[F(R_1)]$, so we focus on the first one. By (B.4), it equals

$$
-\int_0^1 \int_0^\infty \left( F(\sqrt{1-u}z) + \sqrt{1-u}z F'(\sqrt{1-u}z) \right) \sqrt{\frac{2}{\pi}} z^{-3/2} e^{-z^2/2} \, dz \frac{du}{\sqrt{u}} = -\int_0^\infty \sqrt{\frac{2}{\pi}} y^2 e^{-y^2/2} (F(y) + y F'(y)) \left( \int_0^1 e^{-uy^2/2(1-u)} \frac{du}{(1-u)^{3/2} \sqrt{u}} \right) dy,
$$

with the change of variables $y = \sqrt{1-u}z$. Then, with $v = y\sqrt{u}/1-u \Leftrightarrow u = v^2/(y^2 + v^2)$, we have

$$
\int_0^1 e^{-uy^2/2(1-u)} \frac{du}{(1-u)^{3/2} \sqrt{u}} = \int_0^1 \frac{e^{-v^2/2}}{y} \left( \frac{y^2}{y^2 + v^2} \right)^{3/2} \left( \frac{v^2}{y^2 + v^2} \right)^{-1/2} \frac{2v(y^2 + v^2) - v^2 2v}{(y^2 + v^2)^2} \, dv.
$$

Thus, we finally get

$$
\int_0^1 H(u) \frac{du}{u^{3/2}} = 2\mathbb{E}[F(R_1)] - \int_0^\infty \sqrt{\frac{2}{\pi}} y^2 e^{-y^2/2} (F(y) + y F'(y)) \frac{\sqrt{2\pi}}{y} \, dy
$$

$$
= 2\mathbb{E}[F(R_1)] - \sqrt{2\pi} \mathbb{E} \left[ F'(R_1) + \frac{F(R_1)}{R_1} \right]
$$

and the result is proved.

B Technical results on the 3-dimensional Bessel process

B.1 Brownian motion killed at 0

Let $(B_s)_{s \geq 0}$ denotes a standard Brownian motion starting from $x$ under $\mathbb{P}_x$ and $\tau := \inf\{ r \geq 0 : B_r = 0 \}$. Using that the Green function of the Brownian motion killed at 0 is $G(x, y) = 2(x \wedge y) \leq 2x$ for $x, y > 0$, one can get the following bound, for any $x > 0$,

$$
\mathbb{E}_x \left[ \int_0^\tau B_r e^{-B_r} \, dr \right] \leq Cx,
$$

(B.1)

33
obtained by a direct computation. Furthermore, we set \( q_r(x, y) := (2\pi r)^{-1/2}(e^{-(x-y)^2/2r} - e^{-(x+y)^2/2r}) \) for any \( x, y, r > 0 \). Then, \( q_r(x, \cdot) \) is the density of a Brownian motion at time \( r \) starting from \( x \) and killed at 0. Using that \( q_r(x, y) \leq \sqrt{2/\pi} r^{-3/2} xy \), it follows that, for any \( x, t > 0 \),

\[
\mathbb{E}_x\left[e^{-R_t \mathbb{1}_{\tau \geq t}} \right] \leq C \frac{x}{t^{3/2}},
\]

where \( \tau := \inf\{ r \geq 0 : B_r = 0 \} \).

### B.2 Some calculations on the law of \( R_1 \)

Let \( (R_s)_{s \geq 0} \) denotes a 3-dimensional Bessel process starting from \( x \) under \( \mathbb{P}_x \). Recall that, for \( x > 0 \), one has the following link between the 3-dimensional Bessel process and the Brownian motion (see Imhof [14]):

for any \( y, z > 0 \), \( x, y, r > 0 \), and the density of \( R_0 \) under \( \mathbb{P} \), the density of \( R_0 \) is \( z \mapsto \sqrt{2/\pi} z^2 r^{-3/2} e^{-z^2/2r} 1_{z > 0} \).

Moreover, under \( \mathbb{P} = \mathbb{P}_0 \), the density of \( R_0 \) is \( z \mapsto \sqrt{2/\pi} z^2 r^{-3/2} e^{-z^2/2r} 1_{z > 0} \).

In this subsection, we estimate the expectations of some functions of \( R_1 \). First recall that the density of \( R_1 \) under \( \mathbb{P} = \mathbb{P}_0 \) is

\[
z \mapsto \sqrt{2/\pi} z^2 e^{-z^2/2} 1_{z > 0},
\]

and the density of \( R_1 \) under \( \mathbb{P}_y \), for \( y > 0 \), is

\[
z \mapsto \frac{z}{y} e^{-(y-z)^2/2} (1 - e^{-2zy}) 1_{z > 0} = \sqrt{\frac{2}{\pi y}} z e^{-z^2/2} e^{-y^2/2} \sinh(zy) 1_{z > 0}.
\]

Note that, for any \( \lambda > 0 \) and \( a \in [0, 3] \), \( \mathbb{E}[(R_1)^{-a} e^{\lambda R_1}] \) is finite. We establish now several lemmas, that will be used in the next subsection.

**Lemma B.1.** For any \( y > 0 \) and \( a \in [0, 3] \), we have

\[
\mathbb{E}_y[(R_1)^{-a}] \leq C \left(1 + \frac{1}{y^a}\right).
\]

**Proof.** Using (B.5) and that, for any \( y, z > 0 \), \( 1 - e^{-2zy} \leq 2zy \) and \( 1 - e^{-2zy} \leq 1 \), we have

\[
\mathbb{E}_y[(R_1)^{-a}] = \int_{0}^{\infty} \frac{1}{z^a} \frac{z}{\sqrt{2\pi} y} e^{-(y-z)^2/2} (1 - e^{-2zy}) \, dz \\
\leq \int_{0}^{y/2} \frac{1}{\sqrt{2\pi} z^{a-1}} e^{-(y-z)^2/2} 2zy \, dz + \int_{y/2}^{\infty} \frac{1}{\sqrt{2\pi} \sqrt{z^{a-1}}} e^{-(y-z)^2/2} \, dz \\
\leq \frac{2}{\sqrt{2\pi}} \int_{0}^{y/2} \frac{1}{y^a} z^{-a} e^{-y^2/8} \, dz + \frac{1}{\sqrt{2\pi}} \frac{2^{a-1}}{y^a} \int_{y/2}^{\infty} e^{-(y-z)^2/2} \, dz \\
\leq C \left(y^{3-a} e^{-y^2/8} + \frac{1}{y^a}\right).
\]

Moreover, we have \( \mathbb{E}_y[(R_1)^{-a}] \leq \mathbb{E}[(R_1)^{-a}] < \infty \), and therefore it proves the result.

**Lemma B.2.** There exists \( C > 0 \) such that, for any \( y > 0 \) and \( \lambda > 0 \), we have

\[
\mathbb{E}_y\left[\left(\frac{1}{R_1} + 1\right) e^{\lambda R_1}\right] \leq C(1 + \lambda^2) e^{\lambda^2 y}.
\]

34
Proof. Using that, for any \( z > 0, 1 - e^{-2yz} \leq 1 \), we have,

\[
E_y \left[ \left( \frac{1}{R_1} + 1 \right) e^{\lambda R_1} \right] \leq \frac{1}{y} \int_0^\infty (1 + z) e^{\lambda z} e^{-(z-y)^2/2} \frac{dz}{\sqrt{2\pi}} = \frac{1}{y} e^{\lambda y} e^{\lambda^2/2} \int_{-y-\lambda}^\infty (1 + z + y + \lambda) e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} \leq C e^{\lambda y} e^{\lambda^2/2} (1 + y + \lambda),
\]

and, on the other hand, using that, for any \( y, z > 0, 1 - e^{-2yz} \leq 2yz \),

\[
E_y \left[ \left( \frac{1}{R_1} + 1 \right) e^{\lambda R_1} \right] \leq 2 \int_0^\infty (z + 2) e^{\lambda z} e^{-(z-y)^2/2} \frac{dz}{\sqrt{2\pi}} \leq C e^{\lambda y} e^{\lambda^2/2} (1 + y^2 + \lambda^2),
\]

by proceeding as before. The result follows by choosing the first bound if \( y \geq 1 \) and the second one if \( y < 1 \). \( \square \)

### B.3 Some consequences of Assumptions \((A_{1,\kappa})-(A_{2,\kappa})\)

We establish here some technical lemmas concerning the function \( F \), relying on Assumptions \((A_{1,\kappa})-(A_{2,\kappa})\).

**Lemma B.3.** Let \( \kappa \geq 1 \). There is a constant \( C = C(\kappa) > 0 \) such that for any function \( F : \mathbb{R} \to \mathbb{R} \) satisfying \((A_{1,\kappa})-(A_{2,\kappa})\) and any \( y \geq 0, \alpha \in (0,1] \) and \( \eta \geq 0 \), we have

\[
\left| E_y \left[ \left( \frac{1 - (R_1 \eta)}{R_1} \right) F(\alpha(R_1 - \eta)) \right] - E[F(R_1)] \right| \leq C y^2 (1 + y^4) e^{\kappa y} + \eta (1 - \alpha).
\]

**Proof.** We prove the inequality in three steps. The first step is to prove that

\[
\left| E_y \left[ \left( \frac{1 - (R_1 \eta)}{R_1} \right) F(\alpha(R_1 - \eta)) \right] - E \left[ \left( \frac{1 - (R_1 \eta)}{R_1} \right) F(\alpha(R_1 - \eta)) \right] \right| \leq C y^2 (1 + y^4) e^{\kappa y}. \tag{B.6}
\]

The left-hand side of (B.6) is equal to

\[
\left| \int_0^\infty \frac{(z - \eta) z}{z} F(\alpha(z - \eta)) \sqrt{\frac{2}{\pi y}} e^{-z^2/2} \left( e^{-y^2/2} \sinh(zy) - zy \right) \frac{dz}{\sqrt{2\pi}} \right| \tag{B.7}
\]

But, for \( y, z > 0 \), we have

\[
\left| e^{-y^2/2} \sinh(zy) - zy \right| \leq e^{-y^2/2} |\sinh(zy) - zy| + e^{-y^2/2} - 1 |zy| \leq e^{-y^2/2} \sum_{k \geq 1} \frac{(zy)^{2k+1}}{(2k+1)!} + \frac{zy^3}{2} \leq e^{-y^2/2} (zy)^3 \sum_{k \geq 0} \frac{(zy)^{2k}}{(2k+3)!} + zy^3 \leq e^{-y^2/2} (zy)^3 e^{zy} + zy^3.
\]

Then, using Assumption \((A_{1,\kappa})\) (because of the factor \((z - \eta)_+ \), \( F \) is only evaluated at positive points) and that \( \eta \geq 0 \), (B.7) is smaller than

\[
\int_0^\infty e^{\kappa y} \sqrt{\frac{2}{\pi}} e^{-z^2/2} y^3 \left( e^{-y^2/2} z^3 e^{zy} + z \right) \frac{dz}{\sqrt{2\pi}} \leq C y^2 \int_0^\infty (z^4 + 1) e^{\kappa y} e^{-z^2/2} \left( 1 + e^{zy} e^{-y^2/2} \right) \frac{dz}{\sqrt{2\pi}} \leq C y^2 \left( 1 + \int_0^\infty (z^4 + 1) e^{\kappa y} e^{(z-y)^2/2} \frac{dz}{\sqrt{2\pi}} \right) \leq C y^2 \left( 1 + \int_0^\infty ((z+y)^4 + 1) e^{\kappa y} e^{-(z-y)^2/2} \frac{dz}{\sqrt{2\pi}} \right) \leq C y^2 \left( 1 + (1 + y^4)e^{\kappa y} \right).
\]
and this proves (B.6). The second step consists in proving that
\[ E \left[ \left( \frac{R_1 - \eta}{R_1} \right) F(\alpha(R_1 - \eta)) - F(\alpha R_1) \right] \leq C \eta. \] (B.8)

This follows from the fact that for any \( x > 0 \), by Assumption (A2_\kappa), we have
\[
|(x - \eta) F(\alpha(x - \eta)) - x F(\alpha x)| \leq (x - \eta) |F(\alpha(x - \eta)) - F(\alpha x)| + |F(\alpha x)| \cdot |(x - \eta)| - x |
\leq x \alpha e^{\alpha x} + e^{\alpha x} \eta. \] (B.9)

Since \( E[(1 + R_1^{-1}) e^{\kappa R_1}] < \infty \), this proves (B.8). Finally, the third step consists in the bound
\[ |E[F(\alpha R_1)] - E[F(R_1)]| \leq C(1 - \alpha), \] which follows easily from Assumption (A2_\kappa). The result follows from this combined with (B.6) and (B.8).

We now establish the following lemma. It controls a quantity similar to the one in the previous lemma but with \( \eta = 0 \). However, here we keep the first order error term explicit.

Lemma B.4. Let \( \kappa \geq 1 \). There is a constant \( C = C(\kappa) > 0 \) such that for any function \( F: \mathbb{R} \to \mathbb{R} \) satisfying (A1_\kappa) and any \( \varepsilon < 1/2 \) and \( x \geq 0 \), we have
\[
E_x \sqrt{\varepsilon} [F(R_{1-\varepsilon})] - E_x [F(R_1)] + \varepsilon G(x) \leq C \varepsilon^2 (1 + x^4) e^{2 \varepsilon \sqrt{x}},
\]
where
\[
G(x) := E_x [F(R_1)] \left( \frac{3}{2} - \frac{x^2}{2} \right) + E_x [R_1^2 F(R_1)] \left( \frac{x^2}{6} - \frac{1}{2} \right). \] (B.10)

Proof. Write \( \sinhc(z) = \sinh(z)/z \) and note that for \( z \geq 0 \),
\[
\sinhc(z) = 1 + \frac{z^2}{6} + O(z^4 e^z). \] (B.11)

We have
\[
E_x \sqrt{\varepsilon} [F(R_{1-\varepsilon})] = \frac{1}{(1 - \varepsilon)^{3/2}} \int_0^\infty \frac{2}{\pi} z^2 e^{-\frac{z^2}{2(1-\varepsilon)}} \sinhc \left( \frac{z \sqrt{\varepsilon}}{1 - \varepsilon} \right) e^{-\frac{z^2}{2(1-\varepsilon)}} F(z) \, dz.
\]

We expand each term in \( \varepsilon \):
\[
\frac{1}{(1 - \varepsilon)^{3/2}} = 1 + \frac{3}{2} \varepsilon + O(\varepsilon^2)
\]
\[
e^{-\frac{z^2}{2(1-\varepsilon)}} = e^{-\frac{z^2}{2}} \left( 1 - \frac{z^2}{2} \varepsilon + O((z^4 + z^2) \varepsilon^2) \right)
\]
\[
e^{-\frac{z^2}{2(1-\varepsilon)}} = 1 - \frac{x^2}{2} \varepsilon + O((x^4 + x^2) \varepsilon^2).
\]

Using these expansions, as well as (B.11), it is straightforward to check that the lemma holds. We only detail the term arising from the \( O(\cdot) \) term in (B.11). This term equals
\[
O \left( x^4 \varepsilon^2 \int_0^\infty \frac{2}{\pi} z^6 e^{-\frac{(z-\sqrt{\varepsilon})^2}{2(1-\varepsilon)}} |F(z)| \right).
\]

Using that \( e^{-\frac{(z-\sqrt{\varepsilon})^2}{2(1-\varepsilon)}} \leq e^{-\frac{(z-\sqrt{\varepsilon})^2}{2}} \), as well as \( z^6 |F(z)| \leq C e^{Cz} \) for some \( C \), by the assumption on \( F \), the above integral can be bounded using a standard normal random variable \( Z \) by
\[
2C E[e^{CZ + C\sqrt{\varepsilon z}}] = O(e^{C\sqrt{\varepsilon}}).
\]

The remaining details are left to the reader. \( \square \)
The last lemma holds for a function satisfying an assumption weaker than \((A_{1, \kappa})\).

**Lemma B.5.** Let \(\kappa \geq 1\). Let \(F: \mathbb{R} \to \mathbb{R}\) be such that \(|F(x)| \leq (1 + x^{-1})e^{\kappa x}\) for any \(x > 0\). There is a constant \(C = C(\kappa) > 0\) such that for any \(y \geq 0\), \(\alpha \in (0,1]\) and \(\eta \geq 0\), we have

\[
\mathbb{E}_y \left[ \frac{(R_1 - \eta)_+}{R_1} |F(\alpha(R_1 - \eta))| \right] \leq Ce^{\kappa \alpha y}.
\]

**Proof.** Using that \((R_1 - \eta)_+/R_1 \leq 1\) and the assumption on \(F\), we get

\[
\mathbb{E}_y \left[ \frac{(R_1 - \eta)_+}{R_1} |F(\alpha(R_1 - \eta))| \right] \leq \mathbb{E}_y \left[ \left(1 + \frac{1}{R_1}\right)e^{\kappa \alpha R_1} \right] \leq Ce^{\kappa \alpha y},
\]

using Lemma B.2.

**References**

[1] E. Aïdékon. Convergence in law of the minimum of a branching random walk. *Ann. Probab.*, 41(3A):1362–1426, 2013.
[2] E. Aïdékon, J. Berestycki, É. Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probab. Theory Related Fields*, 157(1-2):405–451, 2013.
[3] E. Aïdékon and Z. Shi. The Seneta-Heyde scaling for the branching random walk. *Ann. Probab.*, 42(3):959–993, 2014.
[4] L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probab. Theory Related Fields*, 157(3-4):535–574, 2013.
[5] M. Bramson. Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem. Amer. Math. Soc.*, 44(285):iv+190, 1983.
[6] M. D. Bramson. Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, 31(5):531–581, 1978.
[7] D. Buraczewski, A. Iksanov, and B. Mallein. On the derivative martingale in a branching random walk. *arXiv:2002.05215*, 2020.
[8] B. Chauvin. Product martingales and stopping lines for branching Brownian motion. *Ann. Probab.*, 19(3):1195–1205, 1991.
[9] A. Cortines, L. Hartung, and O. Louidor. The structure of extreme level sets in branching Brownian motion. *The Annals of Probability*, 47(4):2257–2302, 2019.
[10] L. Hartung and A. Klimovsky. The phase diagram of the complex branching Brownian motion energy model. *Electron. J. Probab.*, 23:Paper No. 127, 27, 2018.
[11] A. Iksanov and Z. Kabluchko. A central limit theorem and a law of the iterated logarithm for the Biggins martingale of the supercritical branching random walk. *J. Appl. Probab.*, 53(4):1178–1192, 2016.
[12] A. Iksanov, K. Kolesko, and M. Meiners. Stable-like fluctuations of Biggins’ martingales. *Stochastic Process. Appl.*, 129(11):4480–4499, 2019.
[13] A. Iksanov, K. Kolesko, and M. Meiners. Fluctuations of Biggins’ martingales at complex parameters. *Ann. Inst. Henri Poincaré Probab. Stat.*, 56(4):2445–2479, 2020.
[14] J.-P. Imhof. Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *J. Appl. Probab.*, 21(3):500–510, 1984.
[15] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.*, 15(3):1052–1061, 1987.
[16] T. Madaule. First order transition for the branching random walk at the critical parameter. *Stochastic Process. Appl.*, 126(2):470–502, 2016.

[17] P. Maillard and M. Pain. 1-stable fluctuations in branching Brownian motion at critical temperature I: The derivative martingale. *Ann. Probab.*, 47(5):2953–3002, 2019.

[18] P. Maillard and O. Zeitouni. Slowdown in branching Brownian motion with inhomogeneous variance. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(3):1144–1160, 2016.

[19] A. H. Mueller and S. Munier. Phenomenological picture of fluctuations in branching random walks. *Phys. Rev. E*, 90:042143, Oct 2014.

[20] L. Mytnik, J.-M. Roquejoffre, and L. Ryzhik. Fisher-KPP equation with small data and the extremal process of branching Brownian motion. *arXiv:2009.02042*, 2020.

[21] J. Neveu. Multiplicative martingales for spatial branching processes. In *Seminar on Stochastic Processes, 1987 (Princeton, NJ, 1987)*, volume 15 of *Progr. Probab. Statist.*, pages 223–242. Birkhäuser Boston, Boston, MA, 1988.

[22] U. Rösler, V. A. Topchii, and V. A. Vatutin. The rate of convergence for weighted branching processes [translation of Mat. Tr. 5 (2002), no. 1, 18–45; MR1918893 (2003g:60146)]. *Siberian Adv. Math.*, 12(4):57–82 (2003), 2002.

[23] T. Yang and Y.-X. Ren. Limit theorem for derivative martingale at criticality w.r.t. branching Brownian motion. *Statist. Probab. Lett.*, 81(2):195–200, 2011.