Solitons in the duality-based matrix model

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Abstract: We analyze soliton solutions in the duality-based matrix model. There are two types of solution, a one soliton-antisoliton solution (with the constant boundary condition at infinity) and a periodic solution with an infinite number of solitons. It is shown that there is no finite number \((n > 1)\) of solitons at finite distances in the limit when the length of the box tends to infinity. Particularly, there is no finite number of \(\delta\) – function solitons in the singular limit.

Keywords: Multispecies Calogero model, Collective-field theory, Solitonic solutions.
1. Introduction

In [1], the “duality-based matrix model” was proposed in the collective-field formulation. It was conjectured to describe the hermitean matrix model. In the next paper [2], the same authors studied solitons and excitations in the duality-based matrix model. They claimed that there existed many soliton solutions with the constant boundary condition at infinity, i.e., topologically nontrivial BPS solitons. Furthermore, in the paper [3] they constructed solitons and giants in matrix models. The two-field model, associated with two types of giant graviton, was proposed as the duality-based matrix model. They claimed that they found the finite form of the \( n \)− soliton solution. The singular limit of this solution, particularly the finite number of \( \delta \)− function solutions was discussed.

On the other hand [4], we have studied BPS solitons in the Calogero model with distinguishable particles in the collective-field approach. This model includes the duality-based matrix model as a special case. We studied soliton solutions with the constant boundary condition at infinity. We only found a one-soliton-antisoliton solution and explicitly showed that there did not exist a finite two-soliton solution with the constant boundary condition. In the comment [5] we showed explicitly that there did not exist a singular limit of the form \( (1 - \lambda)[\delta(x - a) + \delta(x + a)] \), with \( a \) finite.

The purpose of this paper is to present an independent analysis of soliton solutions in order to confirm the correctness of our previous conclusions concerning the existence of only one soliton-antisoliton pair of solution in the \( L \to \infty \) limit. In this paper we extend a general construction [4] to periodic soliton solutions. We also find a periodic infinite number of solitons with an arbitrary finite period when the length of the box \( L \) tends to
infinity. However, in contrast to [3], we show that there is no finite number \( n > 1 \) of solitons at finite distances when \( L \to \infty \). Consequently, there is no finite number of \( \delta \) functions in the singular limit.

2. BPS equations and their solutions

The Hamiltonian [1], [2], [3], [4] for the duality-based matrix model in the collective-field formulation is

\[
H = \int dx \frac{\rho_1(x)}{2m_1} \left[ (\partial_x \pi_1(x))^2 + \left( \frac{\lambda_1 - 1}{2} \frac{\partial_x \rho_1}{\rho_1} + \int dy \frac{\lambda_1 \rho_1(y) + \rho_2(y)}{x-y} \right)^2 \right] \\
+ \int dx \frac{\rho_2(x)}{2m_2} \left[ (\partial_x \pi_2(x))^2 + \left( \frac{\lambda_2 - 1}{2} \frac{\partial_x \rho_2}{\rho_2} + \int dy \frac{\lambda_2 \rho_2(y) + \rho_1(y)}{x-y} \right)^2 \right],
\]

(1)

where

\[
\frac{\lambda_1}{m_1^2} = \frac{\lambda_2}{m_2^2} = \frac{1}{m_1 m_2}, \quad \lambda_1 \lambda_2 = 1, \quad 0 < \lambda_1 < 1.
\]

(2)

The corresponding BPS equations [3], [4], [5] are given by

\[
(\lambda_1 - 1) \partial_x \rho_1 = 2\pi \rho_1 (\lambda_1 \rho_1^H + \rho_2^H),
\]

\[
(\lambda_2 - 1) \partial_x \rho_2 = 2\pi \rho_2 (\lambda_2 \rho_2^H + \rho_1^H),
\]

(3)

where

\[
\rho^H(x) = \frac{1}{\pi} \int dy \frac{\rho(y)}{y-x}.
\]

(4)

Then it follows

\[
\rho_1 \rho_2 = c, \quad \lambda_i \neq 1.
\]

(5)

In this paper, we extend our construction of a one soliton-antisoliton solution [4] to periodic solutions of [3], [6] and obtain a general solution of two coupled BPS equations in the form presented in [3]. For the solutions with the constant boundary condition at infinity \((\rho_1, \rho_2 \to \text{const})\), for \(|x| \to \infty\), we construct a one soliton-antisoliton solution [4],

\[
\rho_1(x) = \alpha + r_1(x) > 0,
\]

and

\[
\rho_2(x) = \frac{c}{\alpha} + r_2(x) > 0,
\]

(6)

where \(r_1(x), r_2(x) \to 0\) when \(|x| \to \infty\). The functions \(r_1(x), r_2(x)\) satisfy

\[
(\lambda_1 - 1) \partial_x r_1 = 2\pi \lambda_1 r_1 r_1^H,
\]

\[
(\lambda_2 - 1) \partial_x r_2 = 2\pi \lambda_2 r_2 r_2^H,
\]

(7)
and are given by

\[ r_1(x) = \frac{\lambda_1 - 1}{\pi \lambda_1} \frac{b}{x^2 + b^2}, \]

(8)

\[ r_2(x) = \frac{\lambda_2 - 1}{\pi \lambda_2} \frac{a}{x^2 + a^2}, \]

(9)

where \( a, b > 0 \). From Eq. (5) and the normalization conditions

\[ \int dx \rho_1(x) = N_1 = L \alpha - \frac{1 - \lambda_1}{\lambda_1}, \]

(10)

\[ \int dx \rho_2(x) = N_2 = L \frac{c}{\alpha} + 1 - \lambda_1, \]

(11)

we find \((N_1, N_2 >> 1)\)

\[ a = \frac{N_1(\lambda_1 - 1)}{\alpha \pi N_2(1 - N_1^2 \lambda_1^2 / N_2^2)}, \]

\[ b = \frac{(1 - \lambda_1)}{\lambda_1 \alpha \pi (1 - N_2^2 / \lambda_1^2 N_1^2)}, \]

\[ c = \alpha^2 \frac{N_2}{N_1}. \]

(12)

We note in passing that for \( N_2 = \lambda_1 N_1 \), it follows \( a, b = \infty, \rho_1(x) = \alpha, \rho_2(x) = \frac{c}{\alpha} = \lambda_1 \alpha \).

We point out that our construction \([4]\) for \( \rho_1(x) = \alpha + r_1(x), \rho_2(x) = \frac{c}{\alpha} + r_2(x), \) (Eq. (3)), automatically satisfies the coupled BPS equations of a two-family system, (Eqs.(3)). To see that this is indeed the case, one has to use Eqs. (5) and (8) and rewrite \( \rho_1 \) and \( \rho_2 \) in terms of \( r_1 \) and \( r_2 \) as \( \rho_1 = \frac{\alpha + r_1}{\lambda_1} \) and \( \rho_2 = \frac{c}{\alpha} + \frac{r_2}{\lambda_1} \). By differentiating these relations with respect to \( x \) and using Eqs.(6), we end up with the coupled BPS equations (6). In this way, we have reduced the two-family Calogero model to two one-family Calogero decoupled systems (6). We note that the solutions of the BPS equation are simultaneously the solutions of the corresponding variational equation. The problem of finding solutions of the variational equation corresponding to a one-family Calogero model was solved in [6], [7] a long time ago. From these solutions and by using the identities

\[ \left( \frac{b}{x^2 + b^2} \right)^H = \frac{x}{x^2 + b^2} \]

and

\[ \left( \frac{\sinh u}{\cosh u - \cos kx} \right)^H = \frac{\sin kx}{\cosh u - \cos kx}, \]

(13)

one easily finds only two types of solutions of the BPS equations (7).
1. Aperiodic one-soliton solution

\[ r(x) = \frac{\lambda - 1}{\pi \lambda} \frac{b}{x^2 + b^2}, \quad b \in \mathbb{R}_+ \]  

(14)

with the property

\[ \int_{-\infty}^{\infty} dx r(x) = \frac{\lambda - 1}{\lambda}. \]  

(15)

2. Periodic solutions

\[ r(x) = \frac{\lambda - 1}{2\pi \lambda} k \frac{\sinh u}{\cosh u - \cos kx}, \quad \text{where} \quad u \geq 0, \quad k \in \mathbb{R}_+, \]  

(16)

with the property

\[ \int_{-\frac{\pi}{k}}^{\frac{\pi}{k}} dx r(x) = \frac{\lambda - 1}{\lambda}. \]  

(17)

The period \( \frac{2\pi}{k} \) is arbitrary. In the limit \( k \to 0 \), we find that the period \( \frac{2\pi}{k} \to \infty \) and, if \( u \to 0 \), one obtains the first “aperiodic solution” with finite \( b = \frac{2\sinh \frac{\pi}{k}}{k} \). Using our construction for the aperiodic one-soliton solution, we find a unique one-soliton-antisoliton solution and there is no other finite number \( (\lambda > 1) \) of soliton solutions (at mutually finite distances) of Eq. (3).

Let us now apply our general construction, Eqs. (16), to the periodic solutions \( r_1(x), r_2(x) \) of Eqs. (7) of the form (16) with the same period \( \frac{2\pi}{q} \). We obtain

\[ r_1(x) = \frac{\lambda_1 - 1}{2\pi \lambda_1} q \frac{\sinh u_1}{\cosh u_1 - \cos qx}, \]  

(18)

\[ r_2(x) = \frac{\lambda_2 - 1}{2\pi \lambda_2} q \frac{\sinh u_2}{\cosh u_2 - \cos qx}, \]  

(19)

where \( u_1 > u_2 \geq 0 \) and \( r_1(x), r_2(x) \) are related by the relation \( \frac{\lambda_1}{\lambda_2} = \frac{\sinh u_2}{\sinh u_1} \). The parameters \( u_1, u_2 \) can be expressed in terms of \( \rho_{10} = \frac{N_1}{L}, \quad \rho_{20} = \frac{N_2}{L} \) and \( q \). \( (\rho_{10}, \rho_{20}, q \text{ finite}) \). Note that \( q \) is a free arbitrary parameter which cannot be determined by the parameters of the model. From Eq. (16) we find

\[ \alpha = \frac{1 - \lambda_1}{2\pi \lambda_1} q \frac{\sinh u_1}{\cosh u_1 - \cosh u_2} > 0, \]  

(20)

\[ \frac{c}{\alpha} = \frac{1 - \lambda_1}{2\pi} q \frac{\sinh u_2}{\cosh u_1 - \cosh u_2} > 0, \]  

(21)

\[ \frac{c}{\lambda_1 \alpha^2} = \frac{\sinh u_2}{\sinh u_1}, \]  

(22)
whereas from Eqs. (6) it follows
\[ \alpha = \rho_{10} + \frac{1 - \lambda_1}{2\pi \lambda_1} q, \]
\[ c = \rho_{20} - \frac{1 - \lambda_1}{2\pi} q. \]  
(23)

Using equations (20 - 23) we obtain
\[ \coth \frac{u_1 + u_2}{2} = 2 + \frac{\lambda_1 \rho_{10} - \rho_{20}}{(1 - \lambda_1)q} 2\pi, \]
\[ \coth \frac{u_1 - u_2}{2} = \frac{\lambda_1 \rho_{10} + \rho_{20}}{(1 - \lambda_1)q} 2\pi. \]  
(24)

Note that for \( q \to 0 \), we have \( \alpha \to \rho_{10}, \; \frac{c}{\alpha} \to \rho_{20}, \; u_1, u_2 \to 0 \), (except for \( \lambda_1 \rho_{10} = \rho_{20} \) when \( u_1 = u_2 \), \( \coth u_1 = 2 \)). In this limit, we obtain the parameters \( a, b, c \), Eqs. (12), found in [4],

\[ a = \frac{2 \sinh \frac{u_2}{2}}{q}, \quad b = \frac{2 \sinh \frac{u_1}{2}}{q}, \quad \frac{c}{\alpha} = \frac{N_2}{N_1}. \]

For \( u_2 \to 0 \) and finite \( q \) and \( u_1 \), there exists an infinite number of \( \delta \)- function soliton solutions with the finite period \( \frac{2\pi}{q} \), with \( (1 - \lambda_1)q = 2\pi \rho_{20} \). Namely, the soliton solutions are
\[ \rho_1(x) = \alpha + \frac{\lambda_1 - 1}{2\pi \lambda_1} q \frac{\sinh u_1}{\cosh u_1 - \cos qx} = \alpha \frac{\sin^2 \frac{qx}{2}}{\sinh^2 \frac{u_1}{2} + \sin^2 \frac{qx}{2}}, \]  
(25)
\[ \rho_2(x) = (1 - \lambda_1) \sum_{i \in Z} \delta(x - \frac{2\pi}{q} i), \]  
(26)

where \( \rho_1(x)\rho_2(x) = 0 \) and \( \alpha = \frac{1 - \lambda_1}{2\pi \lambda_1} q \coth \frac{u_2}{2} \). Note that for \( u_1 = u_2 = 0 \) and \( q = 0 \), our solutions reduce to \( \rho_1 = \alpha \) and \( \rho_2 = \frac{c}{\alpha} \).

3. Discussion and conclusion

1. Our general construction of the solutions of the coupled BPS equations (3), \( \rho_1 = \alpha + r_1, \; \rho_2 = \frac{c}{\alpha} + r_2 \), (8), leads to the decoupled BPS equations (9) for one-family Calogero models for \( r_1(x), r_2(x) \), respectively, satisfying the relation \( \rho_1 \rho_2 = c \), (9).

2. There is only a one soliton-antisoliton solution (8), (9) with the constant boundary condition at \( |x| \to \infty \), (9) when \( q \to 0 \). There is also a periodic solution (18),(19) with a infinite number of solitons, with the finite period \( \frac{2\pi}{q} \) and \( L \to \infty \).
3. However, there is a general result that there are no solutions of the BPS Eq. (3) with a finite number \((n > 1)\) of solitons, mutually at finite distances, when \(L \to \infty\). Namely, for a finite number of solitons \(n\) in the box of length \(L\), \(q = \frac{2\pi n}{L} \to 0\) when \(L \to \infty\). Then the period \(2\pi q = \frac{L}{n} \to \infty\) and one obtains only a one soliton-antisoliton solution \([4]\), in contradiction with \([2], [3]\). Let us mention that we have explicitly shown that there does not exist a two-soliton solution at finite distance, with the constant boundary condition at infinity \([4], [5]\).

4. Consequently, there is no finite number \((n > 1)\) of \(\delta\) functions at mutually finite distances. Namely, \(q \to 0\) and \(u_2 \to 0\) imply \(\rho_{20} \to 0\), which is in contradiction with \(\rho_{10}, \rho_{20}\) finite.

5. Moreover, from the periodic \(\delta\)-function solution with finite \(q\)

\[
\rho_1(x) = \alpha \frac{\sin^2 \frac{2\pi x}{q}}{\sinh^2 \frac{\mu}{q} + \sin^2 \frac{2\pi x}{q}},
\]

\(\rho_2(x) = (1 - \lambda_1) \sum_{i \in \mathbb{Z}} \delta(x - \frac{2\pi i}{q}),\) (2)

where \(\alpha = \frac{1 - \lambda_1}{2\pi q} q \coth \frac{\mu}{2}\), we can easily see that when we take the limit \(q \to 0\), then, for finite \(u_1\), \(\alpha \to 0\) and \(\rho_1(x) \to 0\), leading to an unacceptable solution. In order that \(\alpha \neq 0\), one has to take the limit \(u_1 \to 0\). When \(\frac{u_1}{q} = b = \text{finite}\), we obtain only an aperiodic singular solution

\[
\rho_1(x) = \alpha \frac{x^2}{x^2 + b^2},
\]

\(\rho_2(x) = (1 - \lambda_1)\delta(x)\) (3)

found in \([4]\). Hence, there is no finite number \((n > 1)\) of \(\delta\) functions in the solution \(\rho_2\), when \(q \to 0\).

Particularly, we again point out \([4], [5]\) that \(\rho_2(x) = (1 - \lambda_1)\left(\delta(x - a) + \delta(x + a)\right)\), \(|a| < \infty\), and \(\rho_1(x) = \rho_0 \frac{(x^2 - a^2)^2}{(x^2 + a^2)(x^2 + b^2)}\), \(\rho_1 \rho_2 = 0\), is not a solution of Eqs. \([3], [5]\), in contradiction with \([2], [5]\), implying that their conclusion and conjecture (rational ansatz) \([2]\) are wrong.

6. In view of all afore-mentioned findings, we conclude that there is no finite \((n > 1)\) number of soliton solutions even in the genuine one-family Calogero model \((L \to \infty)\).

We note that the duality-based matrix model \([1], [2], [3]\) is not built on exact duality between two one-family Calogero models, \([8], [9], [10]\). The two-family (multispecies)
Calogero models with exact duality symmetry have been constructed in the collective-field formulation in a natural and unique way [10]. The analysis of the corresponding soliton solutions is in preparation [11].

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