Hidden scale dependence in renormalon

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Abstract

In the framework of renormalon consideration, a role of the anomalous dependence of gluon propagator on the scale $\mu$ is shown for a large number of quark flavours as well as in the nonabelian theory, where the parameter of the "running" coupling constant evolution has a hidden dependence on the scale. A phenomenological matching between the modified renormalon and the static quark potential is described to fix the renormalon scales.

Introduction

In the QCD perturbation theory, an account for the next-to-leading order term over $\alpha_s$ is connected with the uncertainty, caused by a choice of the energy scale, determining the $\alpha_s(\mu^2)$ value, since the transition to the $\bar{\mu}$ scale leads to the substitution

$$\alpha_s(\mu^2) \approx \alpha_s(\bar{\mu}^2) \left( 1 - \alpha_s(\bar{\mu}^2) \frac{\beta_0}{4\pi} \ln \frac{\mu^2}{\bar{\mu}^2} \right),$$

where $\beta_0 = (11N - 2n_f)/3$, $N$ is the number of colours, $n_f$ is the number of quark flavours. So, the physical quantity, represented in the form

$$r = r_0 \alpha_s(\mu^2)\{1 + r_1 \alpha_s + O(\alpha_s^2)\}\]

with the given order of accuracy, will get the form

$$r = r_0 \alpha_s(\bar{\mu}^2)\left\{1 + \left( r_1 - \frac{\beta_0}{4\pi} \ln \frac{\mu^2}{\bar{\mu}^2} \right) \alpha_s + O(\alpha_s^2) \right\}$$

$$= r_0 \alpha_s(\bar{\mu}^2)\{1 + \bar{r}_1 \alpha_s + O(\alpha_s^2)\}. $$

Thus, the value of $r_1$ coefficient for the $\alpha_s$ correction depends on the scale of $\alpha_s(\mu^2)$ determination. In ref. [1], one offered to fix the $\mu$ scale so that the $r_1$ coefficient does
not contain terms, proportional to $\beta_0$. The latter procedure can be understood as one, leading to $r_1$ to be independent of the number of quark flavours $n_f \sim \beta_0$. As was shown in refs.\[2, 3\], such choice of scale has the strict sense in the framework of the $1/n_f$ expansion, where the $\alpha_s$ correction to the gluon propagator is determined by the fermion loop contribution into the vacuum polarization. So, the expression for such contribution depends on the regularization scheme, and in the next-to-leading order one has

$$\alpha_s(\mu^2) D(k^2, \mu) = \frac{\alpha_s(\mu^2)}{k^2} \left\{ 1 + \frac{\alpha_s n_f}{6\pi} \left( \ln \frac{k^2}{\mu^2} + C \right) \right\}, \quad (1)$$

where $D(k^2, \mu)$ determines the transversal part of the gluon propagator $D_{\mu\nu}(k^2, \mu) = \delta^{ab} D(k^2, \mu)(-g_{\mu\nu} + k_\mu k_\nu/k^2)$. The constant value $C$ is defined by the renormalization scheme, so $C_{\overline{\text{MS}}} = -\frac{5}{3}$ and $C_V = 0$ in the so-called $V$-scheme \[1\].

The summing of $(n_f \alpha_s)^n$ contributions into the gluon propagator leads to the expression

$$\alpha_s(\mu^2) D(k^2, \mu) = \frac{\alpha_s(e^C k^2)}{k^2}, \quad (2)$$

i.e., in fact, it results in the account for the "running" $\alpha_s$ value in respect to the gluon virtuality. Note, the $1/n_f$ consideration is exact for the abelian theory, where $\alpha_s D$ is the renormalization group invariant. Moreover, the $e^{-C \Lambda_{QCD}^2}$ value does not depend on the renormalization scheme, and therefore it results in the scheme-independence for the expression

$$\alpha_s(e^C k^2) = \frac{4\pi}{\beta_0 \ln(e^{C k^2} / \Lambda_{QCD}^2)}.$$

As one can see in the performed consideration, the transition to the nonabelian theory is done by the $2n_f/3 \to -\beta_0$ substitution, that is called as the procedure of "naive nonabelization", giving the correct "running" of the coupling constant.

Further, consider the physical quantity $r$, for which the first order $\alpha_s$-contribution is calculated as an integral over the gluon virtuality with the weight $F(k^2)

$$r = \int \frac{dk^2}{k^2} \alpha_s(\mu^2) F(k^2).$$

Then the account for the $(n_f \alpha_s)^n$ corrections to the gluon propagator leads to the substitution of "running" value $\alpha_s(e^C k^2)$ for $\alpha_s(\mu^2)$

$$\alpha_s(\mu^2) \to \alpha_s(\mu^2) \sum_{n=0}^\infty \left\{ -\frac{\alpha_s(\mu^2) \beta_0}{4\pi} \left( \ln \frac{k^2}{\mu^2} + C \right) \right\}^n, \quad (3)$$

so that in the offered procedure of the scale fixing \[4\], the introduction of the first order $n_f \alpha_s$-correction results in the substitution $\mu^2 \to \bar{\mu}^2 = \mu^2 \exp\{C + \langle \ln k^2 / \mu^2 \rangle\}$, where the average value is determined by the integral with the $F$ weight. However, in some cases \[2, 3\] the integration of the $n$-th order term in expansion \[3\] leads to the
$n!$ factorial growth of the coefficients for the expansion over $\alpha_s^n(\mu^2)$, so that this rising can be determined by the region of low virtualities as well as large ones of gluon. Then one says about the infrared and ultraviolet renormalons, respectively [2, 3, 4]. As was shown [2, 3], the renormalon leads to power-like uncertainties of the $r$ evaluation, so

$$\Delta r = \left( \frac{\Lambda_{QCD}}{\mu} \right)^k a_k,$$

where $k$ is positive for the infrared renormalon and it is negative for the ultraviolet one. Indeed, consider a quantity $F$, presented in the form

$$F(\alpha_s) = \sum_{n=0}^{\infty} f_n \left( \frac{\beta_0}{4\pi} \alpha_s(\mu^2) \right)^n,$$

and its Borel transformation, defined as

$$B[F](u) = f_0 \delta(u) + \sum_{n=0}^{\infty} \frac{f_{n+1}}{n!} u^n,$$

so that $F$ can be restored due to the integral operation

$$F(\alpha_s) = \int_0^{\infty} du B[F](u) \exp\left(-\frac{4\pi u}{\beta_0 \alpha_s(\mu^2)} \right).$$

In the case of the factorial divergence in the coefficients of expansion (5), the Borel transformation has singular points over $u$ [2, 3], so that after transform (6), a pole contribution at a point $u_k = k/2$

$$\Delta B[F](u) = \frac{a_k}{u - u_k},$$

depends on the rule of a way out the singularity, and

$$\Delta F \sim a_k \exp\left(-\frac{4\pi u_k}{\beta_0 \alpha_s(\mu^2)} \right) = a_k \left( \frac{\mu}{\Lambda_{QCD}} \right)^{-2u_k} = a_k \left( \frac{\Lambda_{QCD}}{\mu} \right)^k.$$

For the two-point correlator of heavy quark vector currents, for instance, one has $k = 4$ and the corresponding uncertainty can be, in fact, eliminated in the procedure of definition for the nonperturbative gluon condensate, having the same power over the infrared parameter [3]. However, for the renormalized mass of heavy quark the infrared renormalon with $k = 1$ does not correspond to some definite quark-gluon condensate, and the value $\Delta m(\mu) \sim \Lambda_{QCD}$ can not be adopted into a definition of a physical condensate.

In the present paper we modify the renormalon in the framework of QCD one-loop renormalization group and in the $1/n_f$ consideration by the account for the anomalous
dependence of the gluon propagator on the $\mu$ scale, so that under the ”naive non-abelization”, the modification appears in the determination of the evolution parameter for the ”running” coupling constant in different schemes of the renormalization. The standard renormalon corresponds to the specific choice of the fixed point of the gluon propagator dependence on the scale. The nonabelian evolution of the gluon propagator leads to a hidden scale dependence in the renormalon, because of the anomalous dependence on the $\mu$ scale. A phenomenological matching of the renormalization group invariant parameters of the modified renormalon $\Lambda_{QCD}$ and $\mu_g$, corresponding to the singularity points in the QCD coupling constant and in the gluon propagator, respectively, with the tension of string, giving the linear rise of the potential in the heavy quarkonium, and with the constant of the subleading coulomb interaction allows one to fix the physical parameters, which, in such way, point to a deviation from the standard renormalon on the $1\sigma$ confidence level. At asymptotically high virtualities of gluon, the presence of the second scale in addition to the evolution parameter of the QCD ”running” coupling constant is not essential. However, near the infrared region, the presence of the additional scale has to be included into the consideration, as it has for the potential in the heavy quarkonium or, for instance, for a construction of models for such nonperturbative quantities as the gluon condensate.

In Section 1 we use the one-loop renormalization group of QCD to find the anomalous scale dependence of the gluon propagator. In Section 2 we modify the renormalon in the framework of $1/n_f$ consideration and under the procedure of ”naive nonabelization” for the ”running” constant. In Section 3 we find the hidden scale dependence in the renormalon, because of the nonabelian evolution of the gluon propagator. In Section 4 we make the phenomenological matching of the modified renormalon parameters with the potential in the heavy quarkonium. In Section 5 we use a toy ansatz, motivated by the modified renormalon, to make a model estimate of the gluon condensate value. In Conclusion the obtained results are summarized.

1 Anomalous dependence of gluon propagator on scale

In the covariant gauge, the gluon propagator has the form

$$D_{\nu\lambda}^{ab}(k^2, \mu) = \frac{g^{ab}}{k^2\omega(\mu^2, k^2)} \left(-g_{\nu\lambda} + (1 - a_l(\mu^2)\omega(\mu^2, k^2))\frac{k_\nu k_\lambda}{k^2}\right),$$

where $a_l\omega$ is the renormalization group invariant, since the renormalization group transformations make ”bare” quantities, independent of the scale, in the form

$$a_l^B = Z_3 a_l,$$

$$\omega^B = Z_3^{-1}\omega,$$
where \( Z_3 \) is the renormalization constant of the gluon field \( A^B_{\mu} = Z_3^{1/2} A^B_{\mu} \). In the MS-scheme, one-loop calculations give

\[
Z_3^{\text{MS}} = 1 + \frac{1}{\epsilon} \frac{\alpha_s}{24\pi} (N(13 - 3a_l) - 4n_f) .
\]

\( Z_3^{\text{MS}} \) can be obtained from \( Z_3^{\text{MS}} \) by the substitution \( 1/\epsilon \to 1/\epsilon + \ln 4\pi - \gamma_E \), where \( \gamma_E = 0.5772 \ldots \) The differential equation for \( a_l \) has the following MS-scheme form

\[
\frac{da_l(\tau)}{d\tau} = \frac{\alpha_s}{12\pi} a_l(\tau) [(13N - 4n_f) - 3Na_l(\tau)] ,
\]

where \( \tau = \ln(\mu/\Lambda_{\text{QCD}}) \), and \( \alpha_s(\tau) = 2\pi/(\beta_0 \tau) \). Eq.(7) has the solution

\[
a_l(\tau) = \frac{(13N - 4n_f) C_a \tau^{n_a}}{1 + 3N C_a \tau^{n_a}} ,
\]

where

\[
n_a = \frac{13N - 4n_f}{6\beta_0} ,
\]

and \( C_a \) is the renormalization group invariant.

Then eq.(8) allows one to represent the gluon propagator in the form

\[
D^{ab}_{\nu\lambda}(k^2, \mu) = \frac{\delta^{ab}}{k^2} \frac{1}{1 - (1 - \omega(\mu_0^2))(\frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)})^{n_a}} \left( -g_{\nu\lambda} + (1 - b) \frac{k_\nu k_\lambda}{k^2} \right) ,
\]

where \( b \) is the arbitrary gauge parameter, being the renormalization group invariant. In eq.(9) we take into account the anomalous dependence for the gluon propagator on the scale, \( \omega = \omega(\mu^2) \), only.

The choice of fixed point \( \omega \equiv 1 \) corresponds to the standard renormalon. The consideration of the gluon propagator at \( \omega \neq 1 \) leads to the modified renormalon.

2 Modified renormalon in \( 1/n_f \) consideration

In the leading order over \( 1/n_f \) one has \( n_a = 1 \), and the cases with 1) \( \omega(\mu_0^2) > 1 \), 2) \( \omega(\mu_0^2) = 1 \) and 3) \( \omega(\mu_0^2) < 1 \) at some large \( \mu_0 \) values can be formally come to the different choices of the renormalization group invariant \( \mu_g \), such that \( \omega(\mu_g^2) = 0 \) and 1) \( \mu_g < \Lambda_{\text{QCD}} \), 2) \( \mu_g = \Lambda_{\text{QCD}} \) and 3) \( \mu_g > \Lambda_{\text{QCD}} \), respectively. The \( \alpha_s(\mu_g^2) \) value is considered as the formal one-loop expression

\[
\alpha_s(\mu^2) = \frac{4\pi}{\beta_0 \ln \frac{\mu^2}{\Lambda_{\text{QCD}}}} .
\]

\(^1\)In ref.[5] the solution with the fixed \( b = 3 \) value is considered and \( \omega \) is expressed through \( b/a_l \).
Thus, the $1/n_f$ consideration gives
\[
\frac{\alpha_s(\mu^2)}{\omega(\mu^2)} = \frac{\alpha_s(\mu^2)\alpha_s(\mu^2_g)}{\alpha_s(\mu^2_g) - \alpha_s(\mu^2)} = \alpha_s(e^d\mu^2) , \tag{10}
\]
where $d$ is the scheme-independent invariant of the one-loop renormalization group, and it is equal to
\[
d = \ln \frac{\Lambda^2_{QCD}}{\mu^2} = -\frac{4\pi}{\beta_0\alpha_s(\mu^2_g)}. \tag{11}
\]
Further, one can repeat the calculation of $(n_f\alpha_s)^n$ contributions into the gluon propagator with the substitution of value (10) for $\alpha_s(\mu^2)$, so that this replacement corresponds to the account for the anomalous dependence of the gluon propagator versus the $\mu$ scale. Then the scale fixing due to the next-to-leading order $\alpha_s$ correction results in the expression
\[
\bar{\mu}^2 = \mu^2 \exp\{C + \langle \ln k^2/\mu^2 \rangle + d\} ,
\]
and the summing of the corresponding contributions modifies eq.(2)
\[
\alpha_s(\mu^2)D(k^2,\mu) = \frac{\alpha_s(e^{C+d}k^2)}{k^2} = \frac{\alpha_s(e^{C}k^2)}{\omega(e^{C}k^2)k^2} . \tag{11}
\]
Next, one can define the $\nabla$-scheme, introducing $C_\nabla = -d$. Then
\[
\Lambda^\nabla_{QCD} = e^{5/6-d/2} \Lambda_{QCD}^{\overline{MS}} .
\]
In the $\nabla$-scheme the perturbative potential between the heavy quark and antiquark in the colour-singlet state will have the form
\[
V(q^2) = -\frac{4}{3} \frac{4\pi\alpha_s^\nabla(q^2)}{q^2} \tag{12}
\]
at $q^2 = k^2$. Potential (12) comes to the Richardson’s potential $[8]$, when one uses $\alpha_s^\nabla(q^2+\Lambda^2)$ instead of $\alpha_s^\nabla(q^2)$, and $\Lambda$ fixes the linearly rising part of potential, confining quarks with the distance increase,
\[
\Delta V_{lin}(x) = \sigma|x| = \frac{8\pi}{27} \Lambda_{lin}^2 |x| . \tag{13}
\]
The fitting of mass spectra for the charmonium and bottomonium in the Richardson’s potential gives
\[
\Lambda_{lin} = 398 \text{ MeV} . \tag{14}
\]
However, one usually takes $\sigma = 0.18 \pm 0.01 \text{ GeV}^2$ as the scale normalization for the lattice computations, so that this string tension is equal to that of the Cornell model $[7]$. Therefore, we will use
\[
\Lambda_{lin} = 425 \pm 15 \text{ MeV} . \tag{14}
\]
Value (14) has to be compared with the evolution scale \( \Lambda \) of the "running" coupling constant of QCD, where \( \alpha_{\text{MS}}(m_Z^2) = 0.117 \pm 0.005 \) corresponds to the one-loop value \( \Lambda_{\text{QCD}}^{\text{MS}} = 85 \pm 25 \) MeV [8]. If one assumes, that \( \Lambda \) does not depend on the number of flavours, then at the scale of "switching on (off)" the additional flavour of quarks \( \mu = m_{n_f+1} \), the QCD coupling has a discontinuity, related with the step-like change \( \beta_0(n_f) \rightarrow \beta_0(n_f+1) = \beta_0(n_f) - 2/3 \). One can avoid such discontinuities, if one supposes that \( \Lambda \) depends on the flavour number, so that \( \alpha_s(\mu^2 = m_{n_f+1}^2, \Lambda^{(n_f)}, n_f) = \alpha_s(\mu^2 = m_{n_f+1}^2, \Lambda^{(n_f+1)}, n_f+1) \). Then in the one-loop approximation one finds

\[
\Lambda^{(n_f)} = \Lambda^{(n_f+1)} \left( \frac{m_{n_f+1}}{\Lambda^{(n_f+1)}} \right)^{\omega_0(n_f)}.
\]

Setting \( \Lambda^{(5)} = 85 \pm 25 \) MeV, one gets

\[
e^{-d/2} = 1.42 \pm 0.40.
\]

Hence, the \( \alpha_s \) rescaling, accounting for eq.(13), results in the \( d \) value, that agrees with the standard renormalon \( (d \equiv 0) \) within the current accuracy up to 1\( \sigma \), corresponding to the confidence probability, equal to 30\%.

However, the quantity \( \alpha_s D \), obtained in the framework of \( 1/n_f \) consideration under the procedure of substitution \( 2n_f/3 \rightarrow -\beta_0(n_f) \), possesses the renormalization group properties, which disagree with the scale dependence of \( \alpha_s D \) in the nonabelian theory, where \( n_a \neq 1 \). The restoring of correct properties in the nonabelian renormalization group for the gluon propagator results in a hidden scale dependence of the renormalon.

### 3 Nonabelian generalization and hidden scale dependence

One has \( n_a \neq 1 \) in the nonabelian theory. Introduce the effective coupling constant \( \bar{\alpha}_s \)

\[
\bar{\alpha}_s(\mu^2, \Lambda(\mu)) = \frac{\alpha_s(\mu^2)}{\omega(\mu^2)} = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda^2(\mu))},
\]

where the evolution parameter \( \Lambda(\mu) \) has the following dependence on the \( \mu \) scale

\[
\Lambda(\mu) = \Lambda_{\text{QCD}} \left( \frac{\mu}{\Lambda_{\text{QCD}}} \right)^{\frac{\ln(\mu/\Lambda_{\text{QCD}})}{\ln(\mu/\Lambda_{\text{QCD}})}}. \quad (17)
\]

Thus, expansions in the perturbative theory of QCD built over the powers of \( \bar{\alpha}_s \), that has the following transformation properties, caused by the hidden scale dependence,

\[
\bar{\alpha}_s(\mu_0^2, \Lambda(\mu)) = \frac{\bar{\alpha}_s(\mu_1^2, \Lambda(\mu))}{1 + \frac{2n_a}{4\pi} \bar{\alpha}_s(\mu_1^2, \Lambda(\mu)) \ln \frac{\mu_0^2}{\mu_1^2}}, \quad (18)
\]

\( 7 \)
\[ \bar{\alpha}_s(\mu_2^2, \Lambda(\mu_2)) = \frac{\bar{\alpha}_s(\mu_1^2, \Lambda(\mu_1))}{1 + \frac{\beta_0}{4\pi} \bar{\alpha}_s(\mu_1^2, \Lambda(\mu_1)) \ln \frac{\mu_2^2}{\mu_1^2}} F(\mu_2, \mu_1), \]

where

\[ F(\mu_2, \mu_1) = \frac{1 - (1 - \omega(\mu_2^2))\left(\alpha_s(\mu_2^2)/\alpha_s(\mu_1^2)\right)}{1 - (1 - \omega(\mu_1^2))\left(\alpha_s(\mu_2^2)/\alpha_s(\mu_1^2)\right)^{n_a}}. \]

One finds \( F \equiv 1 \) at \( \omega \equiv 1 \) or \( n_a = 1 \), so that the choice of fixed point or the abelian case make us to return to the consideration, performed in the previous section. Then \( \Lambda(\mu) = \mu_g \), i.e. we come to the \( \nabla \)-scheme, and the standard consideration with \( \mu_g = \Lambda_{QCD} \) becomes valid for the choice of fixed point.

Further, it is evident, that the introduction of the fermion loop contribution into the gluon propagator results in the expression

\[ \alpha_s(\mu^2) D(k^2, \mu) = \frac{\bar{\alpha}_s(e^C k^2, \Lambda(\mu))}{k^2} = \frac{\alpha_s(e^{C+d(\mu)} k^2)}{k^2}, \]

where the hidden dependence on the \( \mu \) scale in \( \Lambda(\mu) \) or in \( d(\mu) \) restores the correct renormalization group properties of the \( \alpha_s D \) quantity in the nonabelian theory, so that

\[ d(\mu) = \ln \frac{\Lambda_{QCD}^2}{\Lambda^2(\mu)}. \]

Accounting for eq. (20), one can easily get the generalization of equations, giving the BLM-procedure of scale fixing in the "running" constant and the renormalon modification under \( d \rightarrow d(\mu) \).

However, physically measurable quantities have to be invariant under the renormalization group transformations, so that corresponding expressions can contain the \( \alpha_s D \) value, given at a physical scale, being invariant under the renormalization group and defining a normalization condition.

## 4 Phenomenology of scale fixing

Let us consider the momentum-space potential in the heavy quarkonium at \( q^2 \to 0 \) in the \( V \)-scheme under the Richardson’s prescription

\[ V(q^2) = -\frac{4}{3} \frac{4\pi}{q^2} \frac{\alpha_s(\mu^2(q^2))}{\omega(\mu^2(q^2))}, \]

where \( \mu^2(q^2) = q^2 + \mu_g^2 \), so that the expansion over the \( q^2/\mu^2_g \) powers gives

\[ V(q^2) = -\frac{4}{3} \frac{4\pi}{q^2} \left( \frac{8\pi}{\beta_0} \frac{\mu_g^2}{q^2} + \frac{4\pi}{\beta_0} \frac{\alpha_s(\mu_g^2)}{2} \right) + O(1), \]
so that at large $|x|$ values and at $\beta_0(n_f = 3) = 9$ one gets

$$V(x) = \frac{16\pi}{27} \mu_g^2 |x| - \frac{4}{3} \frac{1}{|x|} \left( \frac{4\pi}{9} - \frac{\alpha_s(\mu_g^2)}{2} \right) + o(1/|x|).$$  \hfill (22)

Note, that the coefficient at the scale, giving the $\sigma|x|$ term in the potential, is twice greater, than in eq.(13). If one accepts the given scheme of matching of the gluon propagator parameters with the parameters of potential, confining the quarks in the heavy quarkonium, then at $\mu^2 = \mu_g^2(1 + O(q^2/\mu_g^2))$, one will have

$$\Lambda(\mu) \simeq \Lambda(\mu_g) = \mu_g,$$

and we obtain, that the effective "running" coupling constant of QCD is given in the $\overline{V}$-scheme. Moreover, the $\Lambda_{\overline{\text{MS}}}$ value, determined in Section 2 in the $1/n_f$ consideration, is related with the evolution parameter of the coupling constant by the expression

$$\frac{\Lambda_{\overline{\text{MS}}}}{\mu_g} = \sqrt{2},$$

that is in a good agreement with the experimental value in eq.(16). Thus, the matching of the evolution scale $\mu_g$ with the gluon string tension in the heavy quarkonium results in the value

$$\mu_g^{\overline{\text{MS}}} = 136 \pm 4 \text{ MeV}$$

at $n_f = 3$.

Further, as was shown, the account for the fermion loop contribution into the gluon propagator gives the potential of the Richardson’s form, with $\Lambda(\mu) = \mu_g$. In such potential, the expansion over the $q^2/\mu_g^2$ powers gives

$$V^{\text{Rich}}(q^2) = - \frac{4}{3} \frac{4\pi}{q^2} \left( \frac{4\pi}{\beta_0} \frac{\mu_g^2}{q^2} + \frac{2\pi}{\beta_0} \right) + O(1).$$  \hfill (24)

Comparing eq.(21) and eq.(24), one notes, that the modified potential (22) at large distances can be obtained from the Richardson’s potential by the following

$$V(x) \approx \frac{8\pi}{27} \mu_g^2 |x| + V^{\text{Rich}}(x),$$  \hfill (25)

if one supposes

$$\alpha_s(\mu_g^2) = \frac{4\pi}{\beta_0},$$

i.e.

$$\ln \frac{\mu_g^2}{\Lambda^2_{QCD}} = 1.$$  \hfill (26)
Figure 1: The potentials, corresponding to the Cornell model (dots), the model with the "running" $\alpha_s/\omega$ value (solid circles) and with the "running" $\bar{\alpha}_s$ value plus the linear correction (empty circles).

The additional term in eq. (25) appears as the correction to the linear potential, confining the quarks, and its presence is due to the modification of the renormalon. Furthermore, the matching of the modified potential with the Richardson’s potential over the coulomb deviation from the linear rise at large $|x|$ allows one to fix the ratio $\mu_g/\Lambda_{QCD}$ or $\alpha_s(\mu^2_g)$ (see eq. (23)). As has to be expected, the presence of the additional scale in the $\alpha_sD$ quantity becomes unessential at low distances, i.e. at large momentum transfers, when the dependence on the scale is determined by the coupling constant evolution.

The comparison of the modified potential with eq. (25) and with the Cornell model potential is given on figure 1. As one can see, at large $|x| > 0.5$ fm, where the modification is valid, the Richardson’s potential with $\mu_g$, determined in eq. (23), and with the account for the additional linear term is the quite accurate approximation, that is also close to the potential of Cornell model. As was noted [9], the QCD-motivated potentials, fitting the spectra of charmonium and bottomonium, give one and the same form of the $V(|x|)$ dependence at $0.2 < |x| < 1$ fm with the accuracy up to an additive shift $\Delta V$, independent of $|x|$. This form is caused by the transition of the coulomb-like potential, following from the asymptotic freedom of QCD at low distances, to the linearly rising string-like potential, confining the quarks at large distances $|x| \geq 1$ fm.

In the present section we have found the connection between the scale of the QCD
"running" constant evolution and the string tension.

5 Model-dependent estimate of scales

A toy ansatz, based on the extraction of the pole singularity of $\omega$ in the gluon propagator in the euclidian space and in the $V$-scheme is given by the expression

$$D_{\text{mod}}(k^2) = \frac{4 \ln(\mu_g/\Lambda_{\text{QCD}})}{k^2 - \mu_g^2},$$

that is used for the one-loop calculation of the gluon condensate $\langle \frac{\alpha_s}{\pi} G^2 \rangle$ value. To get the renormalization group invariant expression in the dimensional regularization, one has to extend $D_{\text{mod}}$ into the $(4-2\epsilon)$-dimensional space, so that $D_{\text{mod}}^{(4-2\epsilon)} = D_{\text{mod}}(1+\epsilon z_1)$, where $z_1$ is a single-fold determined value, because of the requirement of the renormalization group invariance of the gluon condensate value. Straightforward calculations give

$$\langle \frac{\alpha_s}{\pi} G^2 \rangle = \frac{32}{\pi^2} \ln \frac{\mu_g}{\Lambda_{\text{QCD}}} \left( \ln \frac{\mu_g}{\Lambda_{\text{QCD}}} \right) \mu_g^4.$$  \hspace{1cm} (27)

Note, that in accordance with eq. (27), the gluon condensate tends to zero, if one has the purely perturbative-like expression for $D_{\text{mod}}$ at $\mu_g = 0$ or if one takes the fixed point $\mu_g = \Lambda_{\text{QCD}}$. The substitution of numbers for the scales, estimated in the previous section, results in

$$\langle \frac{\alpha_s}{\pi} G^2 \rangle \approx 0.011 \text{ GeV}^4,$$  \hspace{1cm} (28)

where $\mu_g = e^{5/6} \mu_g^{\text{MS}} = 316 \pm 12 \text{ MeV}$. Value (28) is in a good agreement with the gluon condensate estimate in the framework of QCD sum rules [10].

Conclusion

In the present paper we have shown, that in the $1/n_f$ consideration, the account for the anomalous scale dependence of the gluon propagator leads to the modification of the renormalon, because of the redefinition of the evolution scale for the "running" coupling constants of QCD. When one restores the correct renormalization group dependence of the gluon propagator on the scale in the nonabelian theory, the modified renormalon obtains the hidden scale dependence, that is expressed in the specific transformation properties of the effective "running" coupling constant. The phenomenological matching of the modified renormalon with the tension of gluon string between the static quarks and with the subleading coulomb interaction at large distances results in the fixing of the two renormalization group invariant scales of the modified renormalon, and it gives the normalization condition for the "running" coupling constant. The
modification of renormalon leads to the linear term, additional to the Richardson’s potential. Furthermore, we have made the model estimate of the gluon condensate value at the presence of two infrared parameters, determining the scales of the pole singularities in the ”running” QCD constant and in the gluon propagator up to one loop. At large virtualities of the gluon, the coupling constant is given by the expression

$$\bar{\alpha}_s(m^2, \mu_g) = \frac{2\pi}{\beta_0 \ln(m/\mu_g)} ,$$

to one loop, so that

$$\mu_g^{\overline{MS}}(n_f = 3) = 136 \pm 4 \text{ MeV},$$

that is in agreement with the experimental estimates of \(\alpha_s(m_Z^2)\). Moreover,

$$\alpha_s(\mu_g) = \frac{2\pi}{\beta_0 \ln(\mu_g/\Lambda_{QCD})} = \frac{4\pi}{9} .$$

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