The Renormalization Group Method and Quantum Groups: the postman always rings twice

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Abstract

We review some of our recent results concerning the relationship between the Real-Space Renormalization Group method and Quantum Groups. We show this relation by applying real-space RG methods to study two quantum group invariant Hamiltonians, that of the XXZ model and the Ising model in a transverse field (ITF) defined in an open chain with appropriate boundary terms. The quantum group symmetry is preserved under the RG transformation except for the appearance of a quantum group anomalous term which vanishes in the classical case. This is called the quantum group anomaly. We derive the new qRG equations for the XXZ model and show that the RG-flow diagram obtained in this fashion exhibits the correct line of critical points that the exact model has. In the ITF model the qRG-flow equations coincide with the tensor product decomposition of cyclic irreps of $SU_q(2)$ with $q^4 = 1$.

TO HONOR JERZY IN THIS CELEBRATED DATE

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1 Introduction

The Renormalization Group method has become one of the basic concepts in Physics, ranging from areas such as Quantum Field Theory and Statistical Mechanics to Condensed Matter Physics. The many interesting and relevant models encountered in these fields are usually not exactly solvable except for some privileged cases in one dimension. It is then when we resort to the RG method to retrieve the essential features of those systems in order to have a qualitative understanding of what the physics of the model is all about. This understanding is usually recast in the form of a RG-flow diagram were the different possible behaviours of the model leap to the eyes.

Many authors in the past have contributed significantly to the idea of renormalization and it is out of the scope of this paper to give a detailed account on this issue here. We shall be dealing with the the version of the RG as introduced by Wilson [1] and Anderson [2] in their treatment of the Kondo problem, and subsequent developments of these ideas carried out by Drell et al. at the SLAC group [3] and Pfeuty et al. [4].

It was Wilson in the late sixties and early seventies who set up the framework of the method in its more thorough and complete version. According to his own words, he did so in his search for a better understanding of what a quantum field theory is. To this end he also introduced another tool, the Operator Product Expansion (OPE) for the field operators. Both the RG method and the OPE have become two cornerstones in modern Quantum Field Theory.

It has been long known that the physics of (1+ 1)-dimensional quantum many-body systems and 2D statistical field theories were very special in many aspects when compared to their higher dimensional generalizations. It was after the seminal work by Belavin, Polyakov and Zamolodchikov (BPZ) [5] when the special role of conformal symmetry in two dimensions was brought about in connection to those many special properties exhibited by 2D systems such as, criticality, integrability, etc... The basic tool employed by BPZ to develop their conformal program was precisely one of the tools introduced by Wilson, the OPE, which with the help of two-dimensional conformal symmetry is powerful enough so as to classify the local fields forming the local algebra according to the irreducible representations of the Virasoro algebra and to determine the correlation functions of those fields.

Among all the Conformal Field Theories studied by BPZ they singled out what they called Minimal Models as those for which the conformal program is more successful in determining their properties the best. The minimal models, also called Rational Conformal Field Theories (RCFT), are those CFT which contain finite number of primary fields in the operator algebra. For these models the anomalous dimensions of the operators, or equivalently the critical exponents, are known exactly and moreover, their correlations functions can be computed as solutions of special systems of linear differential equations. The underlying phenomenon responsible for such remarkable features is the truncation of the operator algebra, that is, the primary fields form a closed operator algebra.

After the introduction of Quantum Groups by Drinfeld [6] in the mid-eighties, some CFT theorists realized the relationship between these new algebraic structures and those appearing in the RCFT. In reference [7] it was shown that the representation theory of $SL(2, q)$ with $q$ a root of unity provides solutions to the polynomial equations of RCFT, as well as a rather efficient way to compute the duality matrices. The rationality condition is met by requiring $q$ to be a root of unity. In this case the representation theory of the quantum group is significantly different from the classical one. The first surprise is that when $q^N = 1$ there are only $N - 1$ distinct finite dimensional irreps with spins $j = 0, 1/2, 1, \ldots, (N - 1)/2$, this last one irrep being singular. This truncation of the representation theory of quantum groups was put in correspondence [3] with the previous truncation of the operator algebra found by BPZ in the minimal models.

With this historical perspective in mind, what we have suggested by introducing what we call the Renormalization Quantum Group Method (qRG for short) [8] is to establish the connection of the truncation of states characteristic of the Real-Space RG methods (the other tool introduced by Wilson)
with the special features of the (1+1)-dimensional systems exemplified by the RCFT. The connection we have found can be casted into the following squematical form,

\[ q \text{RG-truncation} \leftrightarrow \text{RCFT} \]

whose precise content will be explained in the following sections with several examples.

Physicists working in field theory and condensed matter generalized the Real Space Renormalization Group methods introduced by Wilson \[1\] to other problems by using the Kadanoff’s concept of block \[2, 3\]. The Block method (BRG) has the advantage of being conceptually and technically simple, but it lacks of numerical accuracy or may even produce wrong results. For this reason the analytical BRG methods were largely abandoned in the 80’s in favor of numerical methods such as the Quantum Monte Carlo approaches. In the last few years there has been new developments in the numerical RG methods motivated by a better understanding of the errors introduced by the splitting of the lattice into disconnected blocks. A first step was put forward in \[3\] where a combination of different periodic boundary conditions applied to every block lead to the correct energy levels of a simple tight-binding model. This method however has not been generalized to models describing interactions. Recently, we have clarified the role played by the boundary conditions in the real-space renormalization group method \[10\] by constructing a new analytical BRG-method which is able to give the exact ground state of the model and the correct \(1/N^2\)-law for the energy of the first excited state in the large \(N\) (size)-limit.

A further step in the generalization to interacting models was undertaken by White in \[11\] where a Density Matrix algorithm (DMRG) is developed. The main idea is to take into account the connection of every block with the rest of the system when choosing the states which survive the truncation procedure. The standard prescription is to choose the lowest energy states of the block Hamiltonian. Instead, in the DMRG method one replaces the block Hamiltonian by a block density matrix and chooses the eigenstates of this matrix with the highest eigenvalues. The density matrix is constructed out of the ground state of a superblock which contains the desired block.

In these notes we want to explain another RG method, the qRG method, which uses the concept of quantum groups. This mathematical notion emerged in the study of integrable systems and it has been applied to conformal field theory, invariants of knots and manifolds, etc. \[6, 12\]. The new application of quantum groups that we envisage has been partially motivated by the aforementioned work of White, Noack and collaborators and it is probably related to it. This relation is suggested by the fact that quantum groups describe symmetries in the presence of non-trivial boundary conditions. The typical example to understand this property of quantum groups is given by the 1D Heisenberg-Ising model with anisotropic parameter \(\Delta\). The isotropic model \(\Delta = \pm 1\) is invariant under the rotation group \(SU(2)\), but as long as \(|\Delta| \neq 1\) this symmetry is broken down to the rotation group \(U(1)\) around the z-axis. One can “restore” this full rotation symmetry by adding appropriate boundary operators to the Hamiltonian of the open chain. The classical group \(SU(2)\) becomes then the quantum group \(SU_q(2)\), where the quantum parameter is related to the anisotropy by \(\Delta = \frac{q+q^{-1}}{2}\) \[13, 14\]. The “restoration” of a classical symmetry into a \(q\)-symmetry is achieved at the price of deforming the algebra and the corresponding addition rule of angular momentum. The \(q\)-sum rule, which is called the comultiplication, becomes non local and violates parity. The total raising (lowering) operators acting on the whole chain are a sum of the raising (lowering) operators acting at every single site times a non-local term involving all the remaining sites which appear in an asymmetric way: sites located to the left or to the right at a given site contribute differently.

These features of \(q\)-groups made them specially well-suited to implement a RG method which takes into account the correlation between neighboring blocks. In the forthcoming sections we show how this can be done explicitly in two examples in 1D: the Heisenberg-Ising model and the Ising model in a transverse field (ITF).

This paper is organized as follows. In Sect.2 we present a brief introduction to the Block Renormalization Group methods based upon the concept of the intertwiner operator \(T\). This will allow us to see
the truncation procedure inherent to the BRG method as tensoring representations of the Hamiltonian symmetry algebra, the intertwiner operator $T$ being a Clebsch-Gordan operator. In Sect.3 we use the Heisenberg-Ising model to show how the truncation process pertaining to the real-space RG is nothing but a tensor product decomposition of irreps of the symmetry algebra. The properties of the model are qualitative and quantitative well described by the BRG equations in the massive region $\Delta > 1$, while in the massless region one predicts the massless spectrum but not criticality at each value of $\Delta > 1$. This latter fact is rather subtle and elusive. In Sect. 4 we set up the foundations of the qRG method using the Heisenberg-Ising model as an example. We derive the new qRG equations and show that the RG-flow diagram obtained in this fashion exhibits the correct line of critical points that the exact model has. Moreover, the qRG equations for the renormalized spin operators show the appearence of a novel feature: the quantum group anomaly. In Sect. 5 we apply the qRG method to another model, the Ising model in a transverse field (ITF model). Here, the qRG-flow equations coincide with the tensor product decomposition of cyclic irreps of $SU_q(2)$ with $q^4 = 1$. Sect.6 is devoted to conclusions and prospective.

2 A Brief Review of Block Renormalization Group Methods (BRG)

In this section the block renormalization group method is revisited and we present a new and unified reformulation of it based on the idea of the intertwiner operator $T$ to be discussed below. For a more extensive account on this method we refer to [15] and chapter 11 of reference [16] and references therein.

The block RG-method is a real-space RG-method introduced and developed by the SLAC group [3]. Let us recall that Wilson developed his numerical real-space renormalization group procedure to solve the Kondo problem [1]. It was clear from the beginning that one could not hope to achieve the accuracy Wilson obtained for the Kondo problem when dealing with more complicated many-body quantum Hamiltonians such as Heisenberg, Hubbard, etc ... The key difference is that in the Kondo model there exists a recursion relation for Hamiltonians at each step of the RG-elimination of degrees of freedom. The existence of such recursion relation facilitates enormously the work, but as it happens it is specific of impurity problems.

From the numerical point of view, the Block Renormalization Group procedure proved to be not fully reliable in the past particularly in comparison with other numerical approaches, such as the Quantum MonteCarlo method which were being developed at the same time. This was one of the reasons why the BRG methods remained undeveloped during the ’80’s until the begining of the ’90’s when they are making a comeback as one of the most powerful numerical tools when dealing with zero temperature properties of many-body systems.

Let us first summarize the main features of the real-space RG. The problem that one faces generically is that of diagonalizing a quantum lattice Hamiltonian $H$, i.e.,

$$H|\psi > = E|\psi >$$  \hspace{1cm} (2.1)

where $|\psi >$ is a state in the Hilbert space $\mathcal{H}$. If the lattice has $N$ sites and there are $k$ possible states per site then the dimension of $\mathcal{H}$ is simply

$$\text{dim}\mathcal{H} = k^N$$ \hspace{1cm} (2.2)

As a matter of illustration we cite the following examples: $k = 4$ (Hubbard model), $k = 3$ (t-J model), $k = 2$ (Heisenberg model) etc.

When $N$ is large enough the eigenvalue problem (2.1) is out of the capability of any human or computer means unless the model turns out to be integrable which only happens in some instances in $d = 1$.

These facts open the door to a variety of approximate methods among which the RG-approach is one of the most relevant. The main idea of the RG-method is the mode elimination or thinning of the
degrees of freedom followed by an iteration which reduces the number of variables step by step until a more manageable situation is reached. These intuitive ideas give rise to a well defined mathematical description of the RG-approach to the low lying spectrum of quantum lattice hamiltonians.

To carry out the RG-program it will be useful to introduce the following objects:

- \( \mathcal{H} \): Hilbert space of the original problem.
- \( \mathcal{H}' \): Hilbert space of the effective degrees of freedom.
- \( H \): Hamiltonian acting in \( \mathcal{H} \).
- \( H' \): Hamiltonian acting in \( \mathcal{H}' \) (effective Hamiltonian).
- \( T \): embedding operator : \( \mathcal{H}' \rightarrow \mathcal{H} \)
- \( T^\dagger \): truncation operator : \( \mathcal{H} \rightarrow \mathcal{H}' \)

The problem now is to relate \( H, H' \) and \( T \). The criterium to accomplish this task is that \( H \) and \( H' \) have in common their low lying spectrum. An exact implementation of this is given by the following equation:

\[
HT = TH'
\]  

which imply that if \( \Psi'_{E'} \) is an eigenstate of \( H' \) then \( T\Psi'_{E'} \) is an eigenstate of \( H \) with the same eigenvalue (unless it belongs to the kernel of \( T \): \( T\Psi'_{E'} = 0 \)), indeed,

\[
HT\Psi'_{E'} = TH'\Psi'_{E'} = E'T\Psi'_{E'}
\]

To avoid the possibility that \( T\Psi' = 0 \) with \( \Psi' \neq 0 \), we shall impose on \( T \) the condition,

\[
T^\dagger T = 1_{\mathcal{H}'}
\]

such that

\[
\Psi = T\Psi' \Rightarrow \Psi' = T^\dagger \Psi
\]

Condition (2.5) thus establishes a one to one relation between \( \mathcal{H}' \) and \( \text{Im}(T) \) in \( \mathcal{H} \).

Observe that Eq. (2.3) is nothing but the commutativity of the following diagram:

\[
\begin{array}{ccc}
\mathcal{H}' & \xrightarrow{T} & \mathcal{H} \\
\downarrow & & \downarrow \\
\mathcal{H}' & \xrightarrow{T} & \mathcal{H}
\end{array}
\]

Eqs. (2.3) and (2.5) characterize what may be called exact renormalization group method (ERG) in the sense that the whole spectrum of \( H' \) is mapped onto a part (usually the bottom part) of the spectrum of \( H \). In practical cases though the exact solution of Eqs. (2.3) and (2.5) is not possible so that one has to resort to approximations (see later on). Considering Eqs. (2.3) and (2.5) we can set up the effective Hamiltonian \( H' \) as:

\[
H' = T^\dagger HT
\]  

This equation does not imply that the eigenvectors of \( H' \) are mapped onto eigenvectors of \( H \). Notice that Eq.(2.7) together with (2.5) does not imply Eq. (2.3). This happens because the converse of
Eq. (2.5), namely $TT^\dagger \neq 1_H$ is not true, since otherwise this equation together with (2.4) would imply that the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic while on the other hand the truncation inherent to the RG method assumes that $\text{dim}\mathcal{H}' < \text{dim}\mathcal{H}$.

The fact that $T^\dagger T \neq 1_H$ reflects nothing but the irreversibility of the RG-transformation. Indeed, we go from $\mathcal{H}$ to $\mathcal{H}'$ as prescribed by eq. (2.7) but we cannot reverse that equation.

What Eq. (2.7) really implies is that the mean energy of $H'$ for the states $\Psi'$ of $\mathcal{H}'$ coincides with the mean energy of $H$ for those states of $\mathcal{H}$ obtained through the embedding $T$, namely,

$$<\Psi'|H'|\Psi'> = <T\Psi'|HT\Psi'>$$

In other words $T\Psi'$ is used as a variational state for the eigenstates of the Hamiltonian $H$. In particular $T$ should be chosen in such a way that the states truncated in $\mathcal{H}$, which go down to $\mathcal{H}'$, are the ones expected to contribute the most to the ground state of $H$. Thus Eq. (2.7) is the basis of the so called variational renormalization group method (VRG). As a matter of fact, the VRG method was the first one to be proposed. The ERG came afterwards as a perturbative extension of the former (see later on).

More generally, any operator $O$ acting in $\mathcal{H}$ can be “pushed down” or renormalized to a new operator $O'$ which acts in $\mathcal{H}'$ defined by the formula,

$$O' = T^\dagger OT$$

Notice that Eq. (2.7) is a particular case of this equation if choose $O$ to be the Hamiltonian $H$.

In so far we have not made use of the all important concept of the block, but a practical implementation of the VRG or ERG methods does require it. The central role played by this concept makes all the real-space RG-methods to be block methods.

Once we have established the main features of the RG-program, there is quite freedom to implement specifically these fundamentals. We may classify this freedom in two aspects:

- The choice of how to reduce the size of the lattice.
- The choice of how many states to be retained in the truncation procedure.

We shall address the first aspect now. There are mainly two procedures to reduce the size of the lattice:

- by dividing the lattice into blocks with $n_s$ sites each. This is the blocking method introduced by Kadanoff to treat spin lattice systems.
- by retrieving site by site of the lattice at each step of the RG-program. This is the procedure used by Wilson in his RG-treatment of the Kondo problem. This method is clearly more suitable when the lattice is one-dimensional.

We shall be dealing with the Kadanoff block methods mainly because they are well suited to perform analytical computations and because they are conceptually easy to be extended to higher dimensions. On the contrary, the DMRG method introduced by White [11] works with the Wilsonian numerical RG-procedure what makes it intrinsically one-dimensional and difficult to be generalized to more dimensions.

The first step of the BRG method consists in assembling the set of lattice points into disconnected blocks of $n_s$ sites each.

In this fashion there are a total of $N' = N/n_s$ blocks in the whole chain. This partition of the lattice into blocks induces a decomposition of the Hamiltonian (2.1) into an intrablock Hamiltonian $H_B$ and a interblock Hamiltonian $H_{BB}$:
\[ H = H_B + \lambda H_{BB} \]  

where \( \lambda \) is a coupling constant which is already present in \( H \) or else it can be introduced as a parameter characterizing the interblock coupling and in this latter case one can set it to one at the end of the discussion.

Observe that the block Hamiltonian \( H_B \) is a sum of commuting Hamiltonians each acting on every block. The diagonalization of \( H_s \) can thus be achieved for small \( n_s \) either analytically or numerically. Eq. (2.10) suggests that we should search for solutions of the intertwiner equation (2.3) in the form of a perturbative expansion in the interblock coupling constant parameter \( \lambda \), namely,

\[ T = T_0 + \lambda T_1 + \lambda^2 T_2 + \ldots \]  

\[ H' = H'_0 + \lambda H'_1 + \lambda^2 H'_2 + \ldots \]  

To zeroth order in \( \lambda \) Eq. (2.3) becomes

\[ H_B T_0 = T_0 H'_0 \]  

Since \( H_B \) is a sum of disconnected block Hamiltonians \( h_{j'}^{(B)} \), \( j' = 1, \ldots, N' \) implicitly defined through the relation

\[ H_B = \sum_{j'=1}^{N'} h_{j'}^{(B)} \]  

one can search for a solution of \( T_0 \) in a factorized form

\[ T_0 = \prod_{j'=1}^{N'} T_{0,j'} \]  

and an effective Hamiltonian \( H'_0 \) which acts only at the site \( j' \) of the new chain,

\[ H'_0 = \sum_{j'=1}^{N'} h_{j'}^{(s')} = H'_{s'} \]  

Observe that \( H'_{s'} \) is nothing but a site-Hamiltonian for the new chain. Eq. (2.13) becomes for each block:

\[ h_{j'}^{(B)} T_{0,j'} = T_{0,j'} h_{j'}^{(s')} \]  

The diagonalization of \( h_{j'}^{(B)} \) for \( j' = 1, \ldots, N' \) will allow us to write

\[ h_{j'}^{(B)} = \sum_{i=1}^{k'} |i\rangle_j \langle j'| + \sum_{\alpha=1}^{k^{ns}-k'} |\alpha\rangle_j \langle j'| \]  

where \( |i\rangle_j > |\alpha\rangle_j \) for \( j = 1, \ldots, k' \) are the \( k' \)-lowest energy states of \( h_{j'}^{(B)} \). Moreover, we suppose that \( h_{j'}^{(B)} \) is the same Hamiltonian for each block so that \( \epsilon_i \) does not depend on the block.

The truncated Hamiltonian \( h_{j'}^{(s')} \) and the intertwiner operator \( T_{0,j'} \) are then given by:

\[ h_{j'}^{(s')} = \sum_{i=1}^{k'} |i\rangle_j \langle j'| \]  

\[ h_{j'}^{(s')} = \sum_{i=1}^{k'} |i\rangle_j \langle j'| \]
\[ T_{0,j'} = \sum_{i=1}^{k'} \langle i | j' \rangle \langle j' | i \rangle' \] (2.20)

Later on we shall show examples of these relations.

The final outcome of this analysis is that the effective Hamiltonian \( H' \) has a similar structure to the one we started with, namely \( H \). The operators involved in \( H'_s \) and \( H'_s' \) may by all means differ from those of \( H_s \) and \( H_{s'} \), but in some cases the only difference shows up as a change in the coupling constants. This is known as the renormalization of the bare coupling constants. When this is the case, one may easily iterate the RG-transformation and study the RG-flows.

### 3 Block RG-Approach to the Heisenberg-Ising Model

To exemplify the standard BRG-method we shall study a 1d-lattice Hamiltonian, the Heisenberg-Ising model. Prior to considering in detail the Antiferromagnetic (AF) Heisenberg model we shall make some general considerations concerning Hamiltonians which commute with a symmetry group \( G \). Notice that for the AF Heisenberg model \( G \) is nothing but the rotation group \( SU(2) \). Let us call \( g \) an element of the group \( G \) and \( \Pi_H(g) \) a representation of \( g \) acting on the Hilbert space \( \mathcal{H} \). We say that \( G \) is a symmetry group of the Hamiltonian \( H \) if

\[ [H, \Pi_H(g)] = 0 \quad \forall g \in G \] (3.1)

Similarly we want the effective Hamiltonian \( H' \) to be invariant under the action of \( G \) acting now on the Hilbert space \( \mathcal{H}' \),

\[ [H', \Pi_{H'}(g)] = 0 \quad \forall g \in G \] (3.2)

For this to be the case, the RG-transformation must preserve the symmetry of the original hamiltonian. This can be simply achieved if one choses \( T \) as the intertwiner or Clebsch-Gordan operator. Indeed we may recall that if a representation say \( \Pi_{V_3} \) is contained in the tensor product \( \Pi_{V_1} \otimes \Pi_{V_2} \) then one can define the CG-operator as,

\[ C_3^{12} : V_3 \rightarrow V_1 \otimes V_2 \] (3.3)

which in fact satisfies the intertwiner condition:

\[ (\Pi_{V_1} \otimes \Pi_{V_2})(g) C_3^{12} = C_3^{12} \Pi_{V_3}(g) \] (3.4)

This equation expresses the commutativity of the following diagram,

\[ \begin{array}{ccc}
V_3 & \xrightarrow{C_3^{12}} & V_1 \otimes V_2 \\
\Pi_3 \downarrow & & \downarrow \Pi_1 \otimes \Pi_2 \\
V_3 & \xrightarrow{C_3^{12}} & V_1 \otimes V_2
\end{array} \] (3.5)

Thus from a theoretical point of view, the truncation process pertaining to the real-space RG is nothing but a tensor product decomposition of representations of the group \( G \). To make this point more explicit let us suppose that \( \mathcal{H} \) and \( \mathcal{H}' \) are given as tensor products as:

\[ \mathcal{H} = \otimes^N V \] (3.6)

\[ \mathcal{H}' = \otimes^N V' \] (3.7)
where $V$ and $V'$ are irreducible representation spaces of $G$. Then the block method of the previous section applied to this case is equivalent to the tensor product decomposition:

$$V \otimes n_s \otimes V \rightarrow V'$$ (3.8)

In Eq. (3.8) one is establishing that the irrep $\Pi_{V'}$ is contained in the tensor product of $N$ copies of the irrep $\Pi_V$.

Obviously, the tensor product decomposition usually contains different irreps. The criterion to choose a particular irrep, or a collection of irreps is the one of minimum energy. All the states of a given irrep $V'$ will have the same energy.

The summary of the discussion so far is that the intertwiner operator $T_0$ can be identified with the Clebsch-Gordan operator,

$$T_0 = C_{V'}^{V \otimes n_s \otimes V} : V' \rightarrow V \otimes n_s \otimes V$$ (3.9)

Let us illustrate this ideas with the AF Heisenberg-Ising model whose Hamiltonian is given by:

$$H_N = J \sum_{j=1}^{N-1} (S^x_j S^x_{j+1} + S^y_j S^y_{j+1} + \Delta S^z_j S^z_{j+1})$$ (3.10)

where $\Delta \geq 0$ is the anisotropic parameter and $J > 0$ for the antiferromagnetic case. If $\Delta = 1$ one has the AF-Heisenberg model which was solved by Bethe in 1931. If $\Delta = 0$ one has the XX-model which can be trivially solved using a Jordan-Wigner transformation which maps it onto a free fermion model. For the remaining values of $\Delta$ the model is also solvable by Bethe ansatz and it is the 1D relative of the 2D statistical mechanical model known as the 6-vertex or XXZ-model.

The region $\Delta > 1$ is massive with a doubly degenerate ground state in the thermodynamic limit $N \rightarrow \infty$ characterized by the non-zero value of the staggered magnetization,

$$m_{st} = \left\langle \frac{1}{N} \sum_j S^z_j (-1)^j \right\rangle$$ (3.11)

The region $0 \leq \Delta \leq 1$ is massless and the ground state is non-degenerate with a zero staggered magnetization. The phase transition between the two phases has an essential singularity.

We would like next to show which of these features are captured by a real-space RG-analysis. The rule of thumb for the RG-approach to half-integer spin model or fermion model is to consider blocks with an odd number of sites. This allows in principle, although not necessarily, to obtain effective Hamiltonians with the same form as the original ones. Choosing for (3.10) blocks of 3 sites we obtain the block Hamiltonian:

$$\frac{1}{J} H = \vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \epsilon (S^z_1 S^z_2 + S^z_2 S^z_3)$$

$$= \frac{1}{2} \left\{ [\vec{S}_1 + \vec{S}_2 + \vec{S}_3]^2 - (\vec{S}_1 + \vec{S}_3)^2 - 3/4 \right\} + \epsilon (S^z_1 S^z_2 + S^z_2 S^z_3)$$ (3.12)

$\epsilon \equiv \Delta - 1$.

If $\epsilon = 0$ the block Hamiltonian $H_B$ is invariant under the $SU(2)$ group and according to the introduction to this section, we should consider the tensor product decomposition:

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$$ (3.13)

The particular way of writing $H_B$ given in Eq. (3.12) suggests to compose first $\vec{S}_1$ and $\vec{S}_3$ and then, the resulting spin with $\vec{S}_2$. The result of this compositions is given as follows:
\[ |\frac{3}{2}, \frac{3}{2}\rangle = |↑↑↑\rangle, \quad E_B = J/2 \]  \hspace{1cm} (3.14)

\[ |\frac{3}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{3}}(|↑↓↑⟩ + |↓↑↑⟩ + |↑↑↓⟩), \quad E_B = J/2 \]  \hspace{1cm} (3.15)

\[ |\frac{1}{2}, \frac{1}{2}\rangle_1 = \frac{1}{\sqrt{2}}(|↑↑↑⟩ - |↓↑↑⟩), \quad E_B = 0 \]  \hspace{1cm} (3.16)

\[ |\frac{1}{2}, \frac{1}{2}\rangle_0 = \frac{1}{\sqrt{6}}(2|↑↓↑⟩ - |↓↑↑⟩ - |↑↑↓⟩), \quad E_B = -J \]  \hspace{1cm} (3.17)

Hence for \( \epsilon = 0 \) we could choose the spin 1/2 irrep. with basis vectors \( |\frac{1}{2}, \frac{1}{2}\rangle_0 \) and \( |\frac{1}{2}, -\frac{1}{2}\rangle_0 \) in order to define the intertwiner operator \( T_0 \).

However, if \( \epsilon \neq 0 \) the states (3.14) - (3.17) are not eigenstates of (3.12). The full rotation group is broken down to the rotation around the z-axis. The states \( |\frac{3}{2}, \frac{1}{2}\rangle \) and \( |\frac{1}{2}, \frac{1}{2}\rangle_0 \) are mixed in the new ground state which is given by:

\[ |+\frac{1}{2}\rangle = \frac{1}{\sqrt{1 + 2x^2}}(2|\frac{1}{2}, \frac{1}{2}\rangle_1 + \sqrt{2x}|\frac{3}{2}, \frac{1}{2}\rangle) \]  \hspace{1cm} (3.18)

where

\[ x = \frac{2(\Delta - 1)}{8 + \Delta + 3\sqrt{\Delta^2 + 8}} \]  \hspace{1cm} (3.19)

and its energy is,

\[ E_B = -\frac{J}{4}[\Delta + \sqrt{\Delta^2 + 8}] \]  \hspace{1cm} (3.20)

along with its \( |-\frac{1}{2}\rangle \) partner. This are now the two states retained in the RG method. To be more explicit, we have

\[ |+\frac{1}{2}\rangle = \frac{1}{\sqrt{6(1 + 2x^2)}}[(2x + 2)|↑↓↑⟩ + (2x - 1)|↑↑↓⟩ + (2x - 1)|↓↑↑⟩] \]  \hspace{1cm} (3.21)

\[ |-\frac{1}{2}\rangle = -\frac{1}{\sqrt{6(1 + 2x^2)}}[(2x + 2)|↓↑↓⟩ + (2x - 1)|↓↓↑⟩ + (2x - 1)|↑↓↓⟩] \]  \hspace{1cm} (3.22)

The intertwiner operator \( T_0 \) reads then,

\[ T_0 = |+\frac{1}{2}\rangle\langle↑'⟩ + |-\frac{1}{2}\rangle\langle↓'⟩ \]  \hspace{1cm} (3.23)

where \( |↑'⟩ \) and \( |↓'⟩ \) form a basis for the space \( V' = C^2 \). The RG-equations for the spin operators \( \vec{S}_i \) \((i = 1, 3)\) are then given by

\[ T_0^\dagger \vec{S}_x T_0 = \xi_x \vec{S}_x \quad i = 1, 3. \]  \hspace{1cm} (3.24)

\[ T_0^\dagger \vec{S}_y T_0 = \xi_y \vec{S}_y \quad i = 1, 3. \]  \hspace{1cm} (3.25)

\[ T_0^\dagger \vec{S}_z T_0 = \xi_z \vec{S}_z \quad i = 1, 3. \]  \hspace{1cm} (3.26)

where \( \xi_x \), etc are the renormalization factors which depend upon the anisotropy parameter by,
\[ \xi^x = \xi^y \equiv \frac{2(1+x)(1-2x)}{3(1+2x^2)} \quad (3.27) \]

\[ \xi^z \equiv \frac{2(1+x)^2}{3(1+2x^2)} \quad (3.28) \]

Observe the symmetry between the sites \( i = 1 \) and 3 which is a consequence of the even parity of the states \((3.21) - (3.22)\).

The renormalized Hamiltonian can be easily obtained using Eqs.\((3.24)-(3.28)\) and \((3.10)\), and apart from an additive constant it has the same form as \(H\), namely [17],

\[ T_0^\dagger H_N(J, \Delta)T_0 = \frac{N}{3} e_B(J, \Delta) + H_{N/3}(J', \Delta') \quad (3.29) \]

where

\[ J' = (\xi^x)^2 J \quad (3.30) \]

\[ \Delta' = (\frac{\xi^z}{\xi^x})^2 \Delta \quad (3.31) \]

Iterating these equations we generate a family of Hamiltonians \(H^{(m)}_{N/3}(J^{(m)}, \Delta^{(m)})\). The energy density of the ground state of \(H_N\) in the limit \(N \to \infty\) is then given by,

\[ \lim_{N \to \infty} \frac{E_0}{N} = e_{BRG} = \frac{1}{3} \sum_{m=0}^{\infty} e_B(J^{(m)}, \Delta^{(m)}) \quad (3.32) \]

where initially \(J^{(0)} = J\), \(\Delta^{(0)} = \Delta\) and Eqs.\((3.30)-(3.31)\) provide the flow of the coupling constants.

The analysis of Eq.\((3.31)\) shows that there are 3 fixed points corresponding to the values \(\Delta = 0\) (isotropic XX-model), \(\Delta = 1\) (isotropic Heisenberg model) and \(\Delta = \infty\) (Ising model).

The computation of \(e_{BRG}^{\infty}\) in this case is facilitated by the fact that \((3.32)\) becomes a geometric series at the fixed point. The exact results concerning the models \(\Delta = 0\) and \(\Delta = 1\) are extracted from references [18] and [19]. The case with \(\Delta \to \infty\) is exact because the states \(|\pm \frac{1}{2}\rangle\) given in \((3.21) - (3.22)\) tend in that limit to the exact ground state \(|\uparrow\downarrow\uparrow\rangle\) and \(|\downarrow\uparrow\downarrow\rangle\) of the Ising model. As a matter of fact,

\[ |\pm \frac{1}{2} \rangle \approx_{\Delta \to \infty} \frac{1}{\Delta} \left( |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\downarrow\rangle \right) \]

The region \(0 < \Delta < 1\) which flows under the RG-transformation to the XX-model is massless since both \(J^{(m)}\) and \(\Delta^{(m)}\) go to zero. We showed at the beginning of this section that all this region is critical (a line of fixed points) and therefore massless. The RG-equations \((3.30)-(3.31)\) are not able to detect this criticality except at the point \(\Delta = 0\). Only the masslessness property is detected.

The region \(\Delta > 1\) which flows to the Ising model is massive and this follows from the fact that the product \(J^{(m)} \Delta^{(m)}\) goes in the limit \(m \to \infty\) to a constant quantity \(J^{(\infty)} \Delta^{(\infty)}\) which can be computed from Eqs. \((3.30)-(3.31)\) and \((3.23)\),

\[ J^{(\infty)} \Delta^{(\infty)} = \prod_{m=0}^{\infty} \frac{4}{9} \frac{(1+x_m)^4}{(1+2x_m^2)^2} \quad (3.33) \]
where $x_m$ is given by (3.19) with $\Delta$ replaced $\Delta^{(m)}$. This quantity gives essentially the mass gap above the ground state and also the end-to-end or LRO order (Long Range Order) given by the expectation value $|\langle \vec{S}(1) \cdot \vec{S}(N) \rangle|$ in the limit $N \to \infty$.

In summary, the properties of the Heisenberg-Ising model are qualitatively and quantitatively well described in the massive region $\Delta > 1$ while in the massless region $0 < \Delta < 1$ one predicts the massless spectrum but no criticality at each value of $\Delta$. This latter fact is rather subtle and elusive. One would like to construct a RG-formalism such that the Hamiltonian $H_N(\Delta)$ would be a fixed point Hamiltonian for every value of $\Delta$ in the range from -1 to 1. Hence we postpone this discussion to next section.

The phase transition between the two regimes is correctly predicted to happen at the value $\Delta = 1$. This is a consequence of the rotational symmetry, namely at $\Delta = 1$ the system is $SU(2)$ invariant and the RG transformation has been defined as to preserve this symmetry. When $\Delta \neq 1$ the $SU(2)$ symmetry is broken and this is reflected later on in the RG-flow of the coupling constant $\Delta$. The region $0 < \Delta < 1$ corresponds to a central charge $c = 1$, namely, it is realize by a boson compactified in a circle which radius depends on $\Delta$. We may wonder whether the criticality of the region $|\Delta| \leq 1$ is due to some non-trivial symmetry underlying the anisotropic Hamiltonian $H$.

4 Quantum Groups and the Block Renormalization Group Method for the Heisenberg-Ising Model

We present in this section a novel treatment of the Block Renormalization Group method for one dimensional quantum Hamiltonians based on the introduction of a quantum group. Our aim is to address the important questions left open in the previous section and to clarify the peculiar role played by the Renormalization group in the one dimensional physics [8].

Let us consider the following open spin chain Hamiltonian,

$$H_N(q,J) = \frac{J}{4} \left\{ \sum_{j=1}^{N-1} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \frac{q + q^{-1}}{2} \sigma_j^z \sigma_{j+1}^z - \frac{q - q^{-1}}{2} (\sigma_1^z - \sigma_N^z) \right\}$$

(4.1)

where $q$ is an arbitrary quantum parameter. This Hamiltonian is known to be integrable [20], [23], [13], [24]. In an interesting paper Pasquier and Saleur [14] established the $q$-group invariance of (4.1) which served to get a better understanding of the interplay between $q$-groups and CFT at a discrete or lattice level. We have already talked about the relation between $q$-groups and CFT in the introduction when reference [6] was mentioned, but this concerned the models in the continuum. What we would like to show now is that the real-space RG method applied to (4.1) may be perhaps the way to link both the discrete and continuous approaches between the relation of $q$-groups and CFT.

As a matter of fact, (4.1) is invariant under the following quantum group generators [14]: $S^+, S^-$ and $S^z$,

$$S^z = \frac{1}{2} \sum_{j=1}^{N} \sigma_j^z$$

(4.2)

$$S^\pm = \sum_{j=1}^{N} q^{-\frac{1}{2}(\sigma_{j}^z + \cdots \sigma_{j-1}^z)} \sigma_j^\pm q^{\frac{1}{2}(\sigma_{j+1}^z + \cdots \sigma_N^z)}$$

(4.3)

which satisfies the quantum group algebra:

$$[S^+, S^-] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}}$$

(4.4)

$$[S^z, S^\pm] = \pm S^\pm$$

(4.5)
In the limit \(q \rightarrow 1\) one recovers from Eqs. \((4.2)-(4.5)\) the usual algebra and addition rules of \(su(2)\). For \(q \neq 1\) this algebra is the quantum universal enveloping algebra \(U_q(su(2))\), or simply the quantum \(su(2)\) group denoted by \(SU_q(2)\). The important property is that the generators \((4.2)-(4.3)\) commute with the Hamiltonian \((4.1)\):

\[
[H_N(q), S^\pm] = [H_N(q), S^z] = 0 \quad (4.6)
\]

If we set

\[
\Delta = \frac{q + q^{-1}}{2} \quad (4.7)
\]

we observe that the bulk terms of Eq. \((4.1)\) and the one in Eq. \((3.10)\) coincide. The only difference appears in the boundary term of Eq. \((4.1)\) which is essential for the existence of the quantum group symmetry.

There are two important cases which we can consider:

\[
q : \text{real and positive} \quad \Rightarrow \quad \Delta \geq 1 \quad (4.8)
\]

\[
|q| = 1 \quad \Rightarrow \quad |\Delta| \leq 1 \quad (4.9)
\]

If \(q\) is real then \(H_N(q)\) is Hermitian while if \(q\) is a phase then \(H_N(q)\) is not Hermitian but nevertheless its spectrum is real for \(H_N(q)\) and \(H_N(q^{-1})\) are related by a similarity transformation. This case is the most interesting one. In particular if we write \(q\) as a root of unity \(q = e^{i\pi \mu} \), then the Hamiltonian \((4.1)\) is a critical Hamiltonian with a Virasoro central algebra given by,

\[
q = e^{i\pi \mu} \quad \Rightarrow \quad c = 1 - \frac{6}{\mu(\mu + 1)} \quad (4.10)
\]

However for the time being let us keep \(q\) as an arbitrary parameter characterizing the anisotropy of the model \((4.7)\).

Let us try to apply the Block Renormalization Group Method to the analysis of the Hamiltonian \((4.1)\). To this end we have to write \((4.1)\) in the following form:

\[
H_N(q) = \sum_{j=1}^{N-1} h_{j,j+1}(q, J) \quad (4.11)
\]

\[
h_{j,j+1}(q, J) = \frac{J}{4} \left[ \sum_{j=1}^{N-1} \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \frac{q + q^{-1}}{2} \sigma_j^z \sigma_{j+1}^z - \frac{q - q^{-1}}{2} (\sigma_j^z - \sigma_{j+1}^z) \right] \quad (4.12)
\]

Observe that each boundary term in \(h_{j,j+1}\) cancels one another leaving only those at the end of open chain (i.e. \(j = 1 \) and \(N\)). The nice feature of the site-site Hamiltonians \((4.12)\) is that all of them commute independently with the \(q\)-group generators \(S^\pm\) and \(S^z\):

\[
[h_{j,j+1}, S^\pm] = [h_{j,j+1}, S^z] = 0 \quad \forall j = 1, \ldots, N - 1 \quad (4.13)
\]

Using \(h_{j,j+1}\) we can construct \(q\)-group invariant block Hamiltonians. If the block has for example 3 sites then we will have,

\[
H_B = h_{1,2} + h_{2,3} \quad (4.14)
\]

In the isotropic case (i.e. \(q = 1\)) we employ the Clebsch-Gordan decomposition \((3.13)\) in order to find the eigenstates of the block Hamiltonian \(H_B\) \((\epsilon = 0)\) given in \((3.12)\). For quantum groups, we can also
perform $q$-CG decompositions. The new feature is that now the $q$-CG coefficients depend on the value of $q$. For generic values of $q$ the analogue of Eqs. (3.14) - (3.17) are given by:

$$\begin{align*}
|\frac{3}{2}, \frac{3}{2}\rangle &= |\uparrow\uparrow\uparrow\rangle \\
E_B &= \frac{Jq + q^{-1}}{4}
\end{align*}$$

(4.15)

$$\begin{align*}
|\frac{3}{2}, \frac{1}{2}\rangle &= \frac{1}{\sqrt{[3]_q}}((|\uparrow\downarrow\uparrow\rangle + q|\downarrow\uparrow\uparrow\rangle + q^{-1}|\uparrow\uparrow\downarrow\rangle)
\end{align*}$$

(4.16)

$$\begin{align*}
|\frac{1}{2}, \frac{1}{2}\rangle_1 &= \frac{1}{\sqrt{2(q + q^{-1} - 1)}}((q^{1/2}|\uparrow\uparrow\downarrow\rangle + (q^{1/2} - q^{-1/2})|\uparrow\downarrow\uparrow\rangle - q^{-1/2}|\downarrow\uparrow\uparrow\rangle) \\
E_B &= \frac{J^2 - q - q^{-1}}{4}
\end{align*}$$

(4.17)

$$\begin{align*}
|\frac{1}{2}, \frac{1}{2}\rangle_0 &= \frac{1}{\sqrt{2(q + q^{-1} + 1)}}((q^{1/2} + q^{-1/2})|\uparrow\downarrow\downarrow\rangle - q^{-1/2}|\downarrow\uparrow\uparrow\rangle - q^{1/2}|\uparrow\uparrow\downarrow\rangle) \\
E_B &= \frac{J^2 + q + q^{-1}}{4}
\end{align*}$$

(4.18)

$$\begin{align*}
|\frac{1}{2}, -\frac{1}{2}\rangle_0 &= \frac{1}{\sqrt{2(q + q^{-1} + 1)}}((-q^{1/2} + q^{-1/2})|\downarrow\uparrow\downarrow\rangle - q^{1/2}|\downarrow\downarrow\uparrow\rangle - q^{1/2}|\uparrow\downarrow\uparrow\rangle) \\
E_B &= \frac{J^2 + q + q^{-1}}{4}
\end{align*}$$

(4.19)

where $[3]_q$ denotes the $q$-number with the usual definition

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{N}$$

(4.20)

The normalization of the states in Eqs. (4.15) - (4.19) is as if $q$ were always a real number, a bra vector $\langle |$ means transposing a ket vector.

If $q$ goes to 1 the vectors in Eqs. (4.15) - (4.19) go over the vectors in Eqs. (3.14) - (3.17). On the other hand, for $q \neq 1$ there is a certain similarity between the states $|\frac{1}{2}, \pm \frac{1}{2}\rangle_0$ and the states $|\pm \frac{1}{2}\rangle$ given in Eqs. (3.21) - (3.22). However the main difference is that $|\frac{1}{2}, \pm \frac{1}{2}\rangle_0$ are not parity invariant ($1 \leftrightarrow 3$), $(2 \leftrightarrow 2)$, while states (3.21) - (3.22) are.

If we let $q$ go to zero then $\Delta$ goes to $+\infty$. Let us recall that in this limit the states $|\pm \frac{1}{2}\rangle$ (3.21) - (3.22) go to the exact ground state of the Hamiltonian (3.12). However in the case of the states $|\frac{1}{2}, \pm \frac{1}{2}\rangle_0$ one does not recover the exact eigenstates in this limit. This shows that the states (4.15) - (4.19) are not appropriate for a discussion of the AF region $\Delta > 1$ (or $q$ real). Hence we shall confine ourselves to the critical region $|\Delta| < 1$ where $q$ is a pure phase. As we did already for the isotropic case...
we shall truncate the basis (4.15) - (4.19) to the states $|\frac{1}{2}, \pm \frac{1}{2}\rangle_0$ and therefore the intertwiner operator $T_0$ is given by,

$$T_0 = \frac{1}{2}, \frac{1}{2}\rangle_0 \langle \uparrow |' + \frac{1}{2}, -\frac{1}{2}\rangle_0 \langle \downarrow |'$$  \hspace{1cm} (4.21)

which satisfies the normalization condition,

$$T_0^t T_0 = 1$$  \hspace{1cm} (4.22)

where the superscript $t$ stands for the transpose of the operator (instead of the adjoint). This means that we can get the effective Hamiltonian $H'$ through the formula:

$$H' = T_0^t H T_0$$  \hspace{1cm} (4.23)

The RG-equations for the spin operators $\vec{S}_i$ ($i = 1, 3$) are then given by

$$T_0^t \vec{S}_i^x T_0 = \xi \vec{S}_i^x \ i = 1, 3. \hspace{1cm} (4.24)$$

$$T_0^t \vec{S}_i^y T_0 = \xi \vec{S}_i^y \ i = 1, 3. \hspace{1cm} (4.25)$$

$$T_0^t \vec{S}_i^z T_0 = \xi \vec{S}_i^z + \eta_1 \ i = 1, 3. \hspace{1cm} (4.26)$$

where $\xi$ is the renormalization factor which depends upon the anisotropy parameter through the $q$-parameter in the following fashion,

$$\xi = \frac{q + q^{-1} + 2}{2(q + q^{-1} + 1)} \hspace{1cm} (4.27)$$

and

$$\eta_1 = -\eta_3 := \eta = \frac{q - q^{-1}}{4(q + q^{-1} + 1)} \hspace{1cm} (4.28)$$

Observe that there are quite a few remarkable differences between this “quantum” renormalization prescription Eqs. (4.24) - (4.27), with respect to the ordinary renormalization expressed in Eqs. (3.24) - (3.28). To begin with, the renormalization constant $\xi$ is common to all the spin operators regardless of its spatial component. This is a reflection of the $SU_q(2)$ preservation of the RG method adopted. And last but not least, observe the presence in Eq. (4.26) of an extra term proportional to the identity operator. We may call this term a quantum group anomaly.

Using these equations we can compute the renormalization of the block-block Hamiltonian $h_{BB}$ which turns out to be of the same form as the original site-site Hamiltonian (4.12) with the same value of $q$, namely

$$T_0^t h_{3k,3k+1}(q,J) T_0 = \frac{J}{4} \xi^2 [\sigma_{k+1}^{tx} \sigma_k^{tx} + \sigma_{k+1}^{ty} \sigma_k^{ty} + \frac{q + q^{-1}}{2} \sigma_k^{tz} \sigma_{k+1}^{tz} - \frac{q - q^{-1}}{2} (\sigma_k^{tz} - \sigma_{k+1}^{tz}) + e_{BB}(q,J)]$$  \hspace{1cm} (4.29)

with

$$e_{BB}(q,J) = J \frac{(q - q^{-1})^2 (3q + 3q^{-1} + 4)}{32(q + q^{-1} + 1)^2} \hspace{1cm} (4.30)$$

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Combining Eqs.(4), (4.30) and (4.15) - (4.19) we finally arrive at quantum group RG-equations,

\[ T_0^t H_N(q, J) T_0 = H_{N/3}(q', J') + \frac{N}{3} e_B(q, J) + (\frac{N}{3} - 1) e_{BB}(q, J) \]  

(4.31)

with

\[ e_B(q, J) = -J \frac{2 + q + q^{-1}}{4} \]  

(4.32)

\[ q' = q \]  

(4.33)

\[ J' = \xi^2 J \]  

(4.34)

Hence we obtain a quite remarkable result we were searching of, namely, that the coupling constant \( \Delta \) or alternatively \( q \) does not flow under the RG-transformation, while \( J^{(m)} \) goes to zero in the limit when \( m \to \infty \), which in turn implies that theory is massless.

The computation of the ground state energy of \( H_N \) for any value of \( N \) is very simple since one has only to compute a geometrical series. The result is

\[ E_0(N) = N \frac{1 - (\frac{\xi^2}{3})^M}{3 - \xi^2} (e_B + e_{BB}) - \frac{1 - \xi^{2M}}{1 - \xi^2} e_{BB} \]  

(4.35)

where \( M \) is the number of RG-steps we have to make in order to resolve the chain of \( N = 3^M \) sites.

As a check of the validity of this expression and because of its own interest as well, we shall consider the case \( q = e^{i\pi/3} \) in (4.35) which yields,

\[ E_0(N, q = e^{i\pi/3}) = -\frac{3}{8} N + \frac{3}{8} \]  

(4.36)

This expression coincides with the exact result obtained through Bethe ansatz in reference [20]. Recall that the finite-size corrections to the free-energy for conformally invariant two-dimensional systems behaves as [21], [22],

\[ E = eN + e_s - \frac{\pi c}{24 N} \]  

(4.37)

where \( e \) is the bulk energy per unit length and \( e_s \) is the surface energy (which vanishes for periodic boundary conditions). Since the term proportional to \( 1/N \) is absent in (4.36) one observes that this value of \( q \) corresponds to a central extension \( c = 0 \) of the Virasoro algebra, in agreement with equation (4.10) (\( \mu = 2 \)).

At first sight it looks surprising that an approximation method such as the Block Renormalization Group yields the exact result at least in the case \( q = e^{i\pi/3} \). The peculiarity of this value of \( q \), and in general when \( q \) is a root of unity, has been noticed in various contexts [12]: conformal field theory, quantum groups and in fact they are intimately related.

The first thing to be noticed is that at \( q = e^{i\pi/3} \) the two denominators of the states \( |\frac{3}{2}, m \rangle \) and \( |\frac{1}{2}, \frac{1}{2} \rangle \) vanishes reflecting the fact that the “norm” of these states is zero, i.e., they are null states and therefore they must be dropped out in a consistent theory. It has been shown in [14] that because of \( (S^\pm)^3 = 0 \) the 6 states \( |\frac{3}{2}, m \rangle \) and \( |\frac{1}{2}, m \rangle \) do not form two irreps of dimensions 4 and 2 but rather a simple indecomposable but not irreducible representation of \( SU_q(2) \). All these facts motivates that the tensor product decomposition (3.13) for the case \( q = e^{i\pi/3} \) should be really be taken as

\[ \left( \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \right)_{q = e^{i\pi/3}} = \frac{1}{2} \]  

(4.38)
where the irrep 1/2 on the right hand side denotes the one generated by |1/2, m\rangle_0.

This truncation is mathematically consistent and coincides precisely with the truncation we have adopted in our Block Renormalization Group approach to the q-group invariant Hamiltonian (4.1).

From a physical point of view, the truncation (4.38) means that the states |1/2, 1/2⟩ and |1/2, -1/2⟩ are not "good excited states" above the "local ground state" given by |1/2, m⟩_0. In other words, above the ground state there are not well behaved excited states. This is why the central extension is c = 0 which means that the unique state in the theory is actually the ground state. What the BRG method does is to pick up that piece of the ground state which projects onto a given block! In the case of q = e^{iπ/3} we have therefore construct for chains with N = 3^M sites the exact ground state of the model through the BRG method. It is worthwhile to point out that this derivation is independent of the Bethe ansatz construction and relies completely on the quantum group symmetry.

Another interesting example is provided by q = e^{iπ/4} which corresponds to the critical Ising model (c = 1/2). The RHS for this q in eq. (4.38) contains two spin-1/2 irreps. According to the qRG method the truncation of the spin-3/2 irrep should be a legitimate operation involving no approximation at all. In references [7] the representation theory of q-groups was put in one-to-one correspondence with that of Rational Conformal Field Theories (RCFT). There it was observed that the truncation inherent in the construction of the RCFT’s has a parallel in the truncation of the representation theory of q-groups with q a root of unity. The result we have obtained in this letter suggests that q-group truncations can be carried over a RG analysis of q-group invariant chains. In other words, using q-groups we can safely truncate states in the block RG method. We may summarize this discussion schematically as we have mentioned in the introduction.

5 qRG Treatment of the ITF Model

This simple model has been widely used to test the validity of BRG methods [3], [1]. The Hamiltonian of an open chain is given by \( H = \sum_{j=1}^{N-1} h_{j,j+1} \) where

\[
h_{j,j+1} = -(J\sigma_j^x \sigma_{j+1}^x + p \sigma_j^z + p' \sigma_{j+1}^z) \tag{5.1}
\]

The standard choice is \( p = p' = \Gamma/2 \), in which case (5.1) has 4 different eigenvalues. The BRG method with a block with two sites chooses just the 2 lowest ones. However if \( (p,p') = (0,0) \) or \( (0,\Gamma) \) the Hamiltonian (5.1) has two doubly degenerate eigenvalues ±ε_B (ε_B = \( \sqrt{J^2 + T^2} \)). This choice is not parity invariant but it implements the self-duality property of the ITF model, yielding the exact value of the critical fixed point of the ITF which appears at \( (\Gamma/J)_c = 1 \) [25]. In the following we shall make the choice \( (p,p') = (\Gamma,0) \). This degeneracy of the spectrum of (5.1) has a q-group origin. The relevant quantum group is again SU_q(2) with \( q^4 = 1 \). However the representations involved are not a q-deformation of the spin 1/2 irrep. as in the previous example, but rather a new class of irreps. which only exist when \( q \) is a root of unity. They are called cyclic irreps. and neither are highest weight nor lowest weight representations as the more familiar regular irreps. If we call E, F and K the generators of \( SU_q(2) \), which correspond essentially to \( S^+ \), \( S^- \) and \( q^{2S_z} \) in the notation of the previous example, then a cyclic irrep. acting at a single site of the chain is given by:

\[
E_j = a\sigma_j^x, \quad F_j = b\sigma_j^y, \quad K_j = \lambda\sigma_j^z \tag{5.2}
\]

where \( a = \frac{1}{2}\sqrt{\lambda^2 - 1}, \quad b = -\frac{1}{2}\sqrt{1 - \lambda^{-2}} \). The parameter \( \lambda \) is the label of the cyclic irrep. Strictly speaking, we have a particular kind of cyclic irreps. Indeed, the ones which allow the existence of an intertwiner for their tensor product. Using (5.2) and the addition rule of \( SU_q(2) \) we can get the representation of E, F and K acting on the whole chain:
\[ E = a \sum_{j=1}^{N} \lambda_j^{-1} \sigma_j^z \cdots \sigma_{j-1}^z \sigma_j^x \]  

(5.3)

\[ F = b \sum_{j=1}^{N} \lambda_j^{-N} \sigma_j^y \sigma_{j+1}^z \cdots \sigma_N^z \]  

(5.4)

\[ K = \lambda_N \prod_{j=1}^{N} \sigma_j^z \]  

(5.5)

Now it is a simple exercise to check that these operators commute with (5.1),

\[ [h_{j,j+1}, E] = [h_{j,j+1}, F] = [h_{j,j+1}, K] = 0, \ \forall j \]  

assuming that we choose

\[ \lambda = \Gamma / J \]  

(5.6)

The last of the equalities in (5.6) expresses the well-known \( Z_2 \)-symmetry of the ITF-model which allows one to split the spectrum of the Hamiltonian into an even and odd subsectors. *The other two symmetries are new* and explain the degeneracy of the spectrum of \( h_{j,j+1} \). By all means the whole Hamiltonian \( H = \sum_j h_{j,j+1} \) is also invariant under (5.3). Notice that \( H \) differs from the standard ITF simply in a term at one of the ends of the chain. This is the same mechanism as for the XXZ Hamiltonian: one needs properly chosen operators at the boundary in order to achieve quantum group invariance. Similarly as for the XXZ model the RG-analysis of the ITF becomes a problem in representation of quantum groups: blocking is equivalent to tensoring representations. What is the tensor product of cyclic irreps.? Here it is important to realize that all cyclic irreps. of \( SU_q(2) \) have dimension 2, what distinguishes them is the value of \( \lambda \). The tensor product decomposition of two cyclic irrep. \( \lambda_1 \) and \( \lambda_2 \) of the type given in (5.2) is given by:

\[ [\lambda_1] \otimes [\lambda_2] = 2 [\lambda_1 \lambda_2] \]  

(5.8)

where the 2 means that \( \lambda_1 \lambda_2 \) appears twice in the tensor product. If we perform a blocking of two sites we will get two cyclic irreps. corresponding to \( \lambda^2 \). Then we expect from \( q \)-group representation theory that the new effective Hamiltonian \( h'_{j,j+1} \) will have the same form as (5.1) but with new renormalized coupling constants \( J' \) and \( \Gamma' \) satisfying:

\[ \lambda' = \Gamma'/J' = (\Gamma/J)^2 = \lambda^2 \]  

(5.9)

This is indeed the result obtained in [27]. We arrive therefore at the conclusion that *the RG-flow of the ITF Hamiltonian (5.1) is equivalent to the tensor product decomposition of cyclic irreps of \( SU_q(2) \). This \( q \)-group interpretation of the RG-flow is independent of the size of the blocks: for a \( n \)-site block the RG-flow would be \( \lambda \rightarrow \lambda^n \). The fixed point \( \lambda = 1 \) of (5.3) describes the critical regime of the ITF Hamiltonian and it corresponds to a singular point in the manifold of cyclic irreps. [27]. At \( \lambda = 1 \) the operators (5.3) are still symmetries of the Hamiltonian (\( a, b \) taking any non-zero value) and they recall the Jordan-Wigner map between Pauli matrices and 1d-lattice fermions. Of the two equivalent irreps \( \lambda^2 \) appearing in the tensor product \( \lambda \otimes \lambda \) we only pick up one of them, which is the lowest energy. Observe that the loss of information implied by the RG method is in the ITF case somehow redundant information from the \( q \)-group point of view. The RG-flow in \( \lambda \) is in some sense exact and not affected by the RG-procedure.
The results we have obtained in the ITF model may perhaps be realized in more complicated models. Namely, that the space of coupling constants or equivalently, the space of Hamiltonians of a given theory which is where the RG takes place, is the Spec manifold of an underlying quantum algebra, so that the RG-flow is given by the tensor product decomposition of the algebra. For this to work we have to consider quantum groups with a rich and “exotic” representation theory, which is indeed the case when \( q \) is a root of unity. In these cases we know from references [26] and [27] that the Spec of these quantum groups is indeed very rich. We have used here the simplest situation. As we said above, it would be interesting to know whether there are more complicated realizations of these ideas. The chiral Potts model is a potential candidate for this realization due to its well known connection to \( SU_q(2) \) with \( q^N = 1 \).

6 Conclusions

In this paper we have presented a brief description of the Renormalization Quantum Group Method (qRG) which is specially well-suited to treat 1D quantum lattice Hamiltonians. We have applied real-space RG methods to study two quantum group invariant Hamiltonians, namely, the Heisenberg-Ising model and the Ising model in a transverse field (ITF model). They are defined in an open chain with appropriate boundary terms. The defining feature of this qRG method is that the quantum group symmetry is preserved under the RG transformations except for the appearance of a quantum group anomalous term which vanishes in the classical case. We have called it the quantum group anomaly.

As for the Heisenberg-Ising model, with the aide of the qRG equations we have shown that the \( q \) parameter describing the anisotropy coupling constant \( \Delta \) does not flow under the RG-transformation when \( q \) is of modulus one, while the coupling constant \( J \) goes to zero implying that the theory is massless. In this fashion, the RG-flow diagram obtained with the qRG method gives the correct line of critical points exhibited by the exact model.

In the ITF model, we have shown that the qRG-flow coincides with the tensor product decomposition of cyclic irreps of \( SU_q(2) \) with \( q^4 = 1 \). Cyclic irreps. were used in [28] to derive the Boltzmann weights of the \( \mathbb{Z}_N \)-chiral Potts model [29]. In [28] the labels of the cyclic irreps. have the meaning of rapidities rather than coupling constants as in our realization.

The \( \mathbb{Z}_2 \) CP-model is nothing but the ITF model. We may wonder whether the general \( \mathbb{Z}_N \)-chiral Potts model admits a \( q \)-group RG treatment along the lines of this work. This problem will be considered elsewhere. A model that admits a qRG analysis is the XY model with a magnetic field \( h \) (XYh). The results will be presented in [30]. It suffices to say here that the \( q \)-group underlying the model is \( SU_q(2) \) with \( q^4 = 1 \) and the representation used are the so called nilpotent irreps. [12], which are also described by a parameter \( \lambda \) analoge to that in (5.2) and related to the magnetic field \( h \). The XYh model is equivalent to a free fermion with chemical potential. The results we are obtaining can be translated into a \( q \)-group symmetry between fermions, either free as in the XY or ITF models or interacting as the XXZ model. Another interesting model of interacting fermions is the Hubbard model, which has been studied using RG-methods in [31]. The integrability of the 1D Hubbard model [32] suggests that it might be studied using our qRG techniques.

All the Hamiltonians analysed in this letter are one dimensional, so the quantum groups are of the type that we know. Despite the fact that the Yang-Baxter equation (the precursor of \( q \)-groups) has a higher dimensional analogue called the Zamolodchikov or tetrahedron equation [33], the corresponding high dimensional analogue of quantum groups is not known. This fact represents a barrier to a qRG analysis of Hamiltonians defined in dimensions higher than one.

Another possibility, which is suggested by our results, would be to define quantum groups as those which contain symmetries which are anomalous under RG transformations. This definition is independent of the space dimensionality. The quantum anomalous term in equation (4.26), and an analogous term also present in our qRG treatment of the ITF model, gives a discrete realization of this idea. A
continuum analogue of this anomaly is given by the Feigin-Fuchs current, which has an anomalous operator product expansion with the energy-momentum tensor $[34]$. At this point it may be worth recalling the continuous version of quantum groups in CFT of reference $[35]$, which uses the Feigin-Fuchs or free field realization of the latter. Putting all these arguments together, we arrive at the conclusion that quantum groups are indeed defined by symmetries anomalous under RG transformations. This point of view about quantum groups may set up the pathway to new developments in the field.

Finally, it is somewhat amusing the way in which the word group enters the title of these notes. It refers both to the Renormalization Group method and to Quantum Groups, but neither of them are really groups!

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