The Mutual Information in Random Linear Estimation

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Abstract—We consider the estimation of a signal from the knowledge of its noisy linear random Gaussian projections, a problem relevant in compressed sensing, sparse superposition codes or code division multiple access just to cite few. There has been a number of works considering the mutual information for this problem using the heuristic replica method from statistical physics. Here we put these considerations on a firm rigorous basis. First, we show, using a Guerra-type interpolation, that the replica formula yields an upper bound to the exact mutual information. Secondly, for many relevant practical cases, we present a converse lower bound via a method that uses spatial evolution analysis and the I-MMSE theorem. This yields, in particular, a single letter formula for the mutual information and the minimal-mean-square error for random Gaussian linear estimation of all discrete bounded signals.

Random linear projections and random matrices are ubiquitous in computer science, playing an important role in machine learning [1], statistics [2] and communication [3]. In particular, the task of estimating a signal from its linear random projections has a myriad of applications such as compressed sensing (CS) [4], code division multiple access (CDMA) in communication [5], error correction via sparse superposition codes [6], or Boolean group testing [7]. It is particularly influential approach to this question has been through the use of the heuristic replica method of statistical physics. Here we put these considerations on a firm rigorous basis. First, we show, using a Guerra-type interpolation, that the replica formula yields an upper bound to the exact mutual information. Secondly, for many relevant practical cases, we present a converse lower bound via a method that uses spatial evolution analysis and the I-MMSE theorem. This yields, in particular, a single letter formula for the mutual information and the minimal-mean-square error for random Gaussian linear estimation of all discrete bounded signals.

In Gaussian random linear estimation, one is interested in reconstructing a signal \( s \in \mathbb{R}^N \) from few measurements \( y \in \mathbb{R}^M \) obtained from a random i.i.d Gaussian measurement matrix \( \phi \in \mathbb{R}^{M \times N} \).

\[
y = \phi s + z \sqrt{\Delta} \Leftrightarrow y_\mu = \sum_{i=1}^N \phi_{\mu i} s_i + z_\mu \sqrt{\Delta}, \tag{1}
\]

where the additive white Gaussian noise (AWGN) of variance \( \Delta \) is i.i.d with \( Z_\mu \sim N(0, 1) \), \( \mu \in \{1, \ldots, M\} \). The signal \( s \) to be reconstructed is made of \( L \) i.i.d sections \( s_i \in \mathbb{R}^L, l \in \{1, \ldots, L\} \) distributed according to a discrete prior \( P_0(s_i) = \sum_{l} \phi_{\mu i} \delta(l - a_i) \) with a finite number of terms and all \( a_i \)'s bounded. We will refer to such priors simply as discrete priors. Thus the total number of signal components is \( N = LB \). The case of priors that are mixtures of discrete and absolutely continuous parts can presumably be treated in the present framework but this leads to extra technical complications. The matrix \( \phi \) has i.i.d Gaussian entries \( \phi_{\mu i} \sim N(0, 1/L) \). The measurement rate is \( \alpha := M/N \). \( \mathrm{I} \) is referred as the CS model despite being more general than CS, and we borrow vocabulary of this field.

Define \( \bar{x} := x - s \), \( [\bar{x}]_\mu := \sum_{i=1}^M \phi_{\mu i} \bar{x}_i \). In the Bayesian setting, the posterior associated with the CS model is

\[
P^{\mathrm{cs}}(x|y) = \frac{\exp \left( -\frac{1}{2 \Delta} \sum_{\mu=1}^M ([\phi \bar{x}]_\mu - z_\mu \sqrt{\Delta})^2 \right)^L}{\mathcal{Z}^{\mathrm{cs}}(y)} \prod_{l=1}^L P_0(x_l),
\]

where \( y \) depends on the quenched random variables \( \phi, s, z \) through \( \mathrm{I} \). The denominator \( \mathcal{Z}^{\mathrm{cs}}(y) \) is the normalization,
or partition function, given by the integral of the numerator over all $x$ components. The Gibbs averages with respect to (w.r.t) this posterior are denoted by $\langle - \rangle$. For example the usual MMSE estimator is simply $E[X|Y] = \langle X \rangle$. The MI (per section) is then

$$i_{cs} := \frac{1}{L} \mathbb{E} \left[ \ln \left( \frac{P_{cs}(S, Y)}{P_0(S)P_{cs}(Y)} \right) \right] = \frac{\alpha B}{2} \mathbb{E} \left[ \ln Z_{cs}(Y) \right] - \frac{\alpha B}{2} \mathbb{E} \left[ \ln Z_{cs}(Y) \right],$$

where $\mathbb{E}$ is the expectation w.r.t all the quenched random variables, $P_{cs}(Y) = Z_{cs}(Y)$ and $P_{cs}(S, Y)$ is the joint distribution of the signal and the measurement. Note that $-E[\ln Z_{cs}(Y)]/L$ is referred as the free energy in the statistical physics literature.

The MMSE per section is $\text{mmse} := \mathbb{E}[\|S - \langle X \rangle\|^2]/L$. Unfortunately, this quantity is rather difficult to access rigorously from the MI. For this reason, it is more convenient to consider the measurement MMSE $\text{ymmse} := \mathbb{E}[\|\Phi(S - X)\|^2]/M$ which is directly related to the MI by an I-MMSE relation [15]

$$\frac{d\text{cs}}{d\Delta^{-1}} = \frac{\alpha B}{2} \text{ymmse}.$$  

Thus if we can compute the MI, we can compute the measurement MMSE and conversely. The measurement and usual MMSE’s are formally related by

$$\text{ymmse} = \frac{\text{mmse}}{1 + \text{mmse}/\Delta} + o_L(1),$$

where $\lim_{L \to \infty} o_L(1) = 0$. As we will see we can prove and use a slightly weaker form of such a relation for a “perturbed” model defined in [12].

The replica method yields the replica symmetric (RS) formula for the MI of model [12]. Let $v := \mathbb{E}[SY]/L = \sum_i p_i |a_i|^2 \psi(E; \Delta) := \alpha B \ln(1+E/\Delta) - E/(E+\Delta)/2$. The RS formula is $\lim_{L \to \infty} i_{cs} = \min_{E \in [0, v]} i_{RS}(E; \Delta)$ where

$$i_{RS}(E; \Delta) := \psi(E; \Delta) + i(S; S + \Sigma E; \Delta).$$

The second term on the r.h.s is the MI for a $B$-dimensional denoising model $\tilde{y} = S + \Sigma$ with $\Sigma(E; \Delta) = \alpha B/(E+\Delta)$,

$$i(S; \tilde{y}) := -\mathbb{E}_{S, \tilde{y}} \left[ \ln \left( e^{-\sum_i \langle \delta \tilde{y}_i, -\delta S_i, S_i \rangle^2 / 2 \Sigma} \right) \right] - \frac{B}{2}.$$

The main result of this paper is a complete proof of the RS formula for $B = 1$, $P_0$ discrete and s.t the RS potential [4] has at most three stationary points. As a consequence, we also get the large $L$ asymptotic formula for the measurement MMSE $\text{ymmse}$. For general $B$ and general $P_0$ we show that $i_{RS}(E; \Delta)$ is an upper bound to $\lim_{L \to \infty} i_{cs}$ (in the process we also prove the existence of the limit). We believe that with more work our method can be extended to prove the equality for this more general case.

B. Relation to previous works

Plenty of papers about structured linear problems make use of the replica formula. In statistical physics, these date back to the late 80’s with the study of the perceptron and neural networks [16–18]. Of particular influence has been the work of Tanaka on CDMA [12] which has opened the way to a large set of contributions in information theory [19, 20]. In particular, the MI (or the free energy) in CS has been considered in a number of publications, e.g. [10, 11, 21–26].

In a very interesting line of work, the replica formula has emerged following the study of AMP. Again, the story of this algorithm is deeply rooted in statistical physics, with the work of Thouless, Anderson and Palmer [27] (thus the name “TAP” sometimes given to this approach). The earlier version, to the best of our knowledge, appeared in the late 80’s in the context of the perceptron problem [18]. For linear estimation, it was again developed initially in the context of CDMA [28]. It is, however, only after the application of this approach to CS [9] that the method has gained its current popularity. Of particular importance has been the development of the rigorous proof of state evolution (SE) that allows to track the performance of AMP, using techniques developed by [29] and [30]. Such techniques are deeply connected to the analysis of iterative forms of the TAP equations by Bolthausen [31]. Interestingly, the SE fixed points correspond to the extrema of the RS formula, strongly hinting that AMP achieves the MMSE for many problems where it reaches the global minimum.

While our proof technique uses SE, it is based on two important additional ingredients. The first is Guerra’s interpolation method [32, 33], that allows in particular to show that the RS formula yields an upper bound to the MI. This was alreday done for the CDMA problem in [33] (for binary signals) and here we extend this work to any $B$ and discrete $P_0$. The converse requires more work and the use of spatial coupling and threshold saturation, that follows recent analysis of capacity-achieving spatially coupled codes [34–37]. Using SC in compressed sensing was proposed in [38], but it is only with the joint use of AMP that it was shown to be so powerful [10, 11, 13]. Similar ideas have been proposed for CDMA [39], group testing [40] and sparse superposition codes [41–43].

The authors have recently applied a similar strategy to the factorization of low rank matrices [44, 45]. This, we believe, shows that the developed techniques and results proved in this paper are not only relevant for random linear estimation, but also in a broader context, opening the way to prove many
results on estimation problems previously obtained with the heuristic replica method.

Finally we wish to point out that we have received a private communication from [46] who reached at the same time similar results using a very different approach.

C. Approximate message-passing and state evolution

AMP is deeply linked to [5]. Its asymptotic performance for the CS model can be rigorously tracked by SE in the scalar $B = 1$ case [13,29]. The vectorial $B \geq 2$ case requires extending the SE analysis rigorously, which at the moment has not been done to the best of our knowledge. Nevertheless, we conjecture that SE (see (7) below) tracks AMP for any $B$. This is numerically confirmed in [42] and proven for power allocated sparse superposition codes [47].

Denote $E(t) := \lim_{L \to \infty} E[\|S - \hat{S}(t)\|^2_2]/L$ the asymptotic average MSE obtained by AMP at iteration $t$, $\hat{S}(t)$ being the AMP estimate at $t$. Denote the MMSE associated with the denoising model (introduced in sec [A]) by $\text{mmse}(\Sigma^{-2}) := E[\|S - E[X|S + Z]\|^2_2]$. The SE recursion tracking AMP is

$$E^{(t+1)} = \text{mmse}(\Sigma(E^{(t)}; \Delta)^{-2}),$$

with the initialisation $E^{(0)} = v$. Monotonicity properties of $\text{mmse}(\Sigma^{-2})$ imply that $E^{(t)}$ is a decreasing sequence s.t $\lim_{t \to \infty} E^{(t)} = E^{(\infty)}$ exists. Let us give a natural definition for the AMP threshold.

**Definition 1.1 (AMP algorithmic threshold):** $\Delta_{\text{AMP}}$ is the supremum of all $\Delta$ s.t the fixed point equation associated with (7) has a unique solution for all noise values in $[0, \Delta]$.

**Remark 1.2 (SE and $i^{RS}$ link):** The extrema of (5) correspond to the fixed points of the SE recursion (7). Thus $\Delta_{\text{AMP}}$ is also the smallest solution of $\partial_i^{RS}/\partial E = \partial^2 i^{RS}/\partial E^2 = 0$; in other words it is the "first" horizontal inflexion point appearing in $i^{RS}(E; \Delta)$ when $\Delta$ increases.

D. Results: mutual information and measurement MMSE

Our first result states that the minimum of (5) upper bounds the asymptotic MI.

**Theorem 1.3 (Upper Bound):** Assume model (1) with any $B$ and discrete prior $P_0$. Then

$$\lim_{L \to \infty} i^{\text{cs}} \leq \min_{E \in [0,v]} i^{RS}(E; \Delta).$$

This result generalizes the one already obtained for CDMA in [48], and we note that a further generalization to more general priors that are mixtures of discrete and absolutely continuous parts (as long as the support is bounded) can also be achieved without any major change in our proof. The next result yields the equality in the scalar case.

**Theorem 1.4 (One letter formula for $i^{cs}$):** Take $B = 1$ and assume $P_0$ is a discrete prior such that $i^{RS}(E; \Delta)$ in (5) has at most three stationary points (as a function of $E$). Then for any $\Delta$

$$\lim_{L \to \infty} i^{\text{cs}} = \min_{E \in [0,v]} i^{RS}(E; \Delta).$$

(8)

It is conceptually useful to define the following threshold.

**Definition 1.5 (Information theoretic threshold):** Define $\Delta_{\text{Opt}} := \sup(\Delta \text{ s.t } \lim_{L \to \infty} i^{\text{cs}}$ is analytic in $[0, \Delta])$.

Theorem 1.4 gives us an explicit formula to compute the information theoretical threshold $\Delta_{\text{Opt}} = \Delta_{\text{RS}}$.

Using (3) and the theorem 1.4 we obtain the following.

**Corollary 1.6 (measurement MMSE):** Under the same assumptions than in in theorem 1.4 and for any $\Delta \neq \Delta_{\text{RS}}$ the measurement MMSE for model (1) satisfies

$$\lim_{L \to \infty} \frac{\text{ymmse}}{\text{ymmse}} = \frac{E}{1 + E/\Delta},$$

where $E$ is the unique global minimum of $i^{RS}(E; \Delta)$.

The proof of theorems 1.3 and 1.4 are discussed in sec. II and III. We conjecture that theorem 1.4 and corollary 1.6 hold for any $B$. Their proofs require a control of AMP by SE, a result that (to our knowledge) is currently available in the literature only for $B = 1$. Proving SE for all $B$ would imply these results for the vectorial case, and we believe that this is not out of reach. Moreover, despite our proof does not quite yield it, we conjecture that corollary 1.6 extends to the usual MSE instead of the measurement one. A second direction for generalizations is to consider $P_0$ a mixture of discrete and absolutely continuous parts.

A natural conjecture concerns the performance of AMP for estimation on the CS model (1): as $L \to \infty$, AMP initialized without any knowledge other than $P_0$ yields upon convergence the asymptotic measurement and usual MMSE if $\Delta < \Delta_{\text{AMP}}$ or $\Delta > \Delta_{\text{RS}}$. This conjecture would follow from our analysis (for $B = 1$) if one would show a relation of the type (4) is also valid for the corresponding AMP mean-square errors. This however requires an interesting extension of the theorems of [29].

E. The single first order phase transition scenario

In this contribution, we assume that $P_0$ is discrete and s.t (5) has at most three stationary points. Let us briefly discuss what this hypothesis entails.

Three scenarios are possible: $\Delta_{\text{AMP}} < \Delta_{\text{RS}}$ (one first order phase transition); $\Delta_{\text{AMP}} = \Delta_{\text{RS}} < \infty$ (one higher order phase transition); $\Delta_{\text{AMP}} = \Delta_{\text{RS}} = \infty$ (no phase transition). In the sequel we will consider the most interesting (and challenging) first order phase transition case where a gap between the algorithmic AMP and information theoretic performance appears. The cases of no or higher order phase transition, which present no algorithmic gap, follow as special cases from our proof. It should be noted that in these two cases spatial coupling is not really needed and the proof can be achieved by an "area theorem" as already showed in [49].

Recall the notation $E(\Delta) = \arg\min_{E \in [0,v]} i^{RS}(E; \Delta)$. At $\Delta_{\text{RS}}$, when the argmin is set with two elements, one can think of it as a discontinuous function.

The picture for the stationary points of (5) is as follows. For $\Delta < \Delta_{\text{AMP}}$ there is a unique stationary point which is a global minimum $E$ and we have $E = E(\infty)$. At $\Delta_{\text{AMP}}$, $i^{RS}$ develops a horizontal inflexion point, and for $\Delta_{\text{AMP}} < \Delta < \Delta_{\text{RS}}$ there are three stationary points: a local minimum corresponding to $E(\infty)$, a local maximum, and the global

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minimum $\bar{E}$. It is not difficult to argue that $\bar{E} < E^{(\infty)}$ in the interval $\Delta_{\text{AMP}} < \Delta < \Delta_{\text{RS}}$. At $\Delta_{\text{RS}}$ the local and global minima switch roles, so at this point the global minimum $\bar{E}$ has a jump discontinuity. For all $\Delta > \Delta_{\text{RS}}$ there is at least one stationary point which is the global minimum $\bar{E}$ and $\bar{E} = E^{(\infty)}$ (the other stationary points can merge and annihilate each other as $\Delta$ increases).

Finally we note that with the help of the implicit function theorem for real analytic functions we can show $\bar{E}(\Delta)$ is an analytic function of $\Delta$ except at $\Delta_{\text{RS}}$. Therefore $i^{\text{RS}}(\bar{E}, \Delta)$ is analytic in $\Delta$ except at $\Delta_{\text{RS}}$.

II. PROOF STRATEGY

A. A general interpolation

We have already seen that the RS formula (5) involves the MI of a denoising model (see above (6)). One of the main tools that we use is an interpolation between a simple denoising model and the original CS model (1). Consider a set of observations $[y, \bar{y}]$ from the following channels (here $Z \sim N(0, I_M), \bar{Z} \sim N(0, I_N)$)

$$\begin{cases} y = \phi s + z \frac{1}{\sqrt{\gamma(t)}}, \\ \bar{y} = s + \bar{z} \frac{1}{\sqrt{\gamma(t)}}, \end{cases}$$

where $t \in [0, 1]$ is the interpolating parameter and the signal-to-noise (SNR) functions $\gamma(t)$ and $\lambda(t)$ (let us call these SNR despite the signal power $\nu$ may be $\neq 1$) satisfy the constraint

$$\frac{\alpha B}{\gamma(t)} + \frac{\alpha B}{\Delta + E} = \Sigma(E; \Delta)^{-2}, \quad (9)$$

and $\gamma(0) = \lambda(1) = 0$, $\gamma(1) = 1/\Delta$, $\lambda(0) = \Sigma(E; \Delta)^{-2}$. We also require $\gamma(t)$ to be strictly increasing and $\lambda(t)$ strictly decreasing.

In order to prove concentration properties that are needed in our proofs, we will actually work with a more complicated perturbed interpolated model where we add a set of extra observations that come from another "side channel" denoising model $\bar{y} = s + \bar{z} / \sqrt{h}, Z \sim N(0, I_N)$. Here the SNR $h$ is "small" and one should keep in mind that it will be removed in the process of the proof, i.e., $h \to 0$ (from above).

Define $\tilde{y} := [\bar{y}, \bar{y}, \bar{y}]$ as the concatenation of all observations. Our central object of study is the posterior of this general perturbed interpolated model

$$P_{t,h}(x|\tilde{y}) = \exp \left( -\cal{H}_{t,h}(x|\tilde{y}) \right) \prod_{i=1}^{L} P_0(x_i), \quad (10)$$

where the Hamiltonian is

$$\cal{H}_{t,h}(x|\tilde{y}) := \frac{h}{2} \sum_{i=1}^{N} \left( \frac{\phi_x - \tilde{z}_i}{\sqrt{h}} \right)^2 + \frac{\gamma(t)}{2} \sum_{\mu=1}^{M} \left( \phi_{\mu} - \tilde{z}_i \right)^2 + \frac{\lambda(t)}{2} \sum_{i=1}^{N} \left( \tilde{x}_i - \frac{\bar{z}_i}{\sqrt{\lambda(t)}} \right)^2,$$

and $Z_{t,h}(\tilde{y})$ is the partition function (the integral of the numerator over all $x$ components). Note that the quenched random variables $\phi, z, \bar{z}$ and $\tilde{Z}$ are all independent. As before, expectations w.r.t the Gibbs measure (10) are denoted $\langle \cdot \rangle_{t,h}$, expectations w.r.t the quenched random variables by $\mathbb{E}$.

The MI $i_{t,h}$ for the perturbed interpolated model is defined similarly as (2). Note that $i_{t,0} = t^\gamma$.

**Remark 2.1 (SNR conservation):** Constraint (9), or SNR conservation, is essential. It expresses that as $t$ decreases from 1 to 0, we slowly decrease the SNR of the CS measurements and make up for it in the denoising model. When $t = 0$ the SNR vanishes for the CS model, and no information is available about $s$ from the compressed measurements, information comes only from the denoising model. Instead at $t = 1$ the noise is infinite in the denoising model and letting also $h \to 0$ we recover the CS model.

This constraint can be interpreted as follows. Given a CS model of SNR $\Delta^{-1}$, by remark 2.2 and (7), the global minimum of (5) is the MMSE of an "effective" denoising model of $\Sigma(E; \Delta)^{-2}$. Therefore, the interpolated model (11) (at $h = 0$) is asymptotically equivalent (in the sense that it has the same MMSE) as two independent denoising models: an "effective" one of SNR $\Sigma(E; \gamma(t)^{-1})^{-2}$ associated with the CS model, and another one with SNR $\lambda(t)$. Proving theorem 1.4 requires the interpolated model to be designed s.t its MMSE equals the MMSE of the CS model (1) for almost all $t$. Knowing that the estimation of $s$ in the interpolated model comes from independent channels, this MMSE constraint induces (9).

**Remark 2.2 (Nishimori identity):** We place ourselves in the Bayes optimal setting where $P_0, \Delta, \gamma(t), \lambda(t)$ and $h$ are known. The perturbed interpolated model is carefully designed, that is each of the three terms in (11) corresponds to a "physical" channel model, such that the Nishimori identity holds. This remarkable and general identity (from which many convenient "sub-identities" follow) plays an important role in our calculations. For any (integrable) function $g(x, s)$: if $s$ is the signal, then

$$\mathbb{E}[\langle g(X, S) \rangle_{t,h}] = \mathbb{E}[\langle g(X, X') \rangle_{t,h}],$$

where $X, X'$ are i.i.d vectors distributed according to the product measure of (10). We abuse notation here by denoting the posterior measure for $X$ and the product measure for $X, X'$ with the same bracket $\langle \cdot \rangle_{t,h}$.

B. Various MMSE’s

We will need the following I-MMSE lemma that straightforwardly extends to the perturbed interpolated model the usual I-MMSE theorem [50] for the vectorial denoising model. Let $\gamma \text{mmse}_{t,h} := \mathbb{E}[\|\Phi(S - (X \cdot t,h))\|_2^2]/M$. Then

**Lemma 2.1 (I-MMSE):** $d_{t,h}/d\gamma(t) = (\alpha B/2) \gamma \text{mmse}_{t,h}$.

Let us give a useful link between $\gamma \text{mmse}_{t,h}$ and the usual MMSE $E_{t,h} := \mathbb{E}[\|S - (X \cdot t,h)\|_2^2]/L$. For the perturbed interpolated model ($\phi$ i.i.d Gaussian), the following holds (proof sketch in sec [III-C]).

**Lemma 2.2 (MMSE relation):** For almost every (a.e) $h$, $\gamma \text{mmse}_{t,h} = E_{t,h}(1 + \gamma(t)/E_{t,h})^{-1} + O(1)$.

In this lemma $\lim_{L \to \infty} c_L(1) = 0$. However in our proof $c_L(1)$ is not uniform in $h$ and diverges as $h^{-1/2}$ as $h \to 0$. For this reason we cannot interchange the limits $L \to \infty$ and
\( h \to 0 \). This is not only a technicality, because in the presence of a first order phase transition one has to somehow deal with the discontinuity at \( \Delta_{RS} \).

C. The integration argument

We first remark that AMP is sub-optimal. Thus when used for inference over the CS model (1) (with \( t = 1, h = 0 \)) one gets \( \lim \inf_{L \to \infty} E_{1,h} \leq E(\infty) \). Adding new measurements can only improve optimal inference thus \( E_{1,h} \leq E_{1,0} \) and \( \lim \sup_{L \to \infty} E_{1,h} \leq E(\infty) \). Combining this with lemma 2.4 and using that \( E(1+E/\Delta)^{-1} \) is an increasing function of \( E \), one gets that for a.e \( h \)

\[
\lim \sup_{L \to \infty} \frac{\alpha B}{2} \text{MMSE}_{1,h} \leq \frac{\alpha B}{2} \frac{E(\infty)}{1 + E(\infty)/\Delta}.
\]

(12)

Now let us look at the case \( \Delta < \Delta_{AMP} \) first. In this noise regime \( E(\infty) = \tilde{E} \) the global minimum of \( i^{\text{RS}} \) (see remark 1.2) so we replace \( E(\infty) \) by \( \tilde{E} \) in the r.h.s of (12). Furthermore by a rather explicit differentiation one checks that \( di^{\text{RS}}(\tilde{E}; \Delta)/d\Delta^{-1} = (\alpha B/2E(1 + E/\Delta)^{-1}. \) Then, using also lemma 2.3 the inequality (12) becomes

\[
\lim \sup_{L \to \infty} \frac{di_{1,h}}{d\Delta^{-1}} \leq \frac{di^{\text{RS}}(\tilde{E}; \Delta)}{d\Delta} \quad \text{or} \quad \frac{di^{\text{RS}}(\tilde{E}; \Delta)}{d\Delta} \leq \lim \inf_{L \to \infty} \frac{di_{1,h}}{d\Delta}.
\]

Integrating the last inequality over \( [0, \Delta) \subset [0, \Delta_{AMP}] \) and using Fatou’s lemma we get

\[
i^{\text{RS}}(\tilde{E}; \Delta) - i^{\text{RS}}(\tilde{E}; 0) \leq \lim \inf_{L \to \infty}(i_{1,h}; |\Delta - i_{1,h}|_0).
\]

It is easy to see that \( i_{t,h} \) is concave, and thus continuous, in \( h \). Our interpolation proofs show superadditivity of this sequence so that the limit \( L \to \infty \) of \( i_{1,h} \) exists, and is concave and continuous in \( h \). As a consequence we can take the limit \( h \to 0 \) in the last inequality and permute the limits \( h \to 0 \) and \( L \to \infty \). Furthermore one can show that \( i^{\text{RS}}(\tilde{E}; 0) = \lim_{L \to \infty} i^{\text{CS}}_{|\Delta| = 0} = H(S) \) the Shannon entropy of \( S \sim P_0 \). So we obtain \( i^{\text{RS}}(\tilde{E}; \Delta) \leq \lim_{L \to \infty} i^{\text{CS}}_{|\Delta|} \) with which combined with theorem 1.3 yields theorem 1.4 for all \( \Delta \in [0, \Delta_{AMP}] \).

Notice that \( \Delta_{AMP} \leq \Delta_{Opt} \). While this might seem clear, it follows from \( \Delta_{RS} \geq \Delta_{AMP} \) (by their definitions) which together with \( \Delta_{AMP} > \Delta_{Opt} \) would imply from theorem 1.4 that \( \lim_{L \to \infty} i^{\text{CS}} \) is analytic at \( \Delta_{Opt} \), a contradiction.

Assume for a moment that \( \Delta_{Opt} = \Delta_{RS} \). Thus both \( \lim_{L \to \infty} i^{\text{CS}} \) and \( i^{\text{RS}}(\tilde{E}; \Delta) \) are analytic until \( \Delta_{Opt} \), since they are equal on \( [0, \Delta_{AMP}] \subset [0, \Delta_{RS}] \), implies by unicity of the analytic continuation that they are equal for all \( \Delta < \Delta_{RS} \). Concaity in \( \Delta \) implies continuity of \( \lim_{L \to \infty} i^{\text{CS}} \) which allows to conclude that theorem 1.4 holds at \( \Delta_{RS} \) too.

Now consider \( \Delta \geq \Delta_{RS} \). Then again \( E(\infty) = \tilde{E} \) the global minimum of \( i^{\text{RS}} \). We can start again from (12) with \( E(\infty) \) replaced by \( \tilde{E} \) and apply a similar integration argument with the integral now running from \( \Delta_{RS} \) to \( \Delta \). The validity of the replica formula at \( \Delta_{RS} \) that we just proved above is crucial to complete this argument.

It remains to show \( \Delta_{Opt} = \Delta_{RS} \). This is where SC and threshold saturation come as now crucial ingredients.

![Fig. 1. Spatially coupled measurement matrices \( \in \mathbb{R}^{M \times N} \) with a “band diagonal” structure. They are made of \( \Gamma \times \Gamma \) blocks indexed by \( (r, c) \) (here \( \Gamma = 9 \), each with \( N/\Gamma \) columns and \( M/\Gamma \) rows. The i.i.d entries inside block \( (r, c) \) are \( N(0, J_{r,c}/L) \). The coupling strength is controled by the variance matrix \( \Sigma \). We consider two slightly different constructions. The periodic matrix (left): it has \( w \) forward and \( w \) backward coupling blocks (here \( w = 2 \) with \( J_{r,c} = \Gamma/(2w+1) \) if \( |r-c| \leq w \) (mod \( \Gamma \), 0 else (white blocks with only zeros). The opened matrix (right): the coupling window \( w \) remains unchanged except at the boundaries where the periodicity is broken. Moreover the coupling strength is stronger at the boundaries (darker color).]

D. Proof of \( \Delta_{Opt} = \Delta_{RS} \) using spatial coupling

Spatial coupling: In order to show the equality of the thresholds, we need the introduction of two closely related spatially coupled CS models. Their construction is described by fig. 1 which shows two measurement matrices replacing the one of the CS model (1). On the left the matrix corresponds to taking periodic boundary conditions. This is called the periodic SC system. On the right the SC system is opened. This is called the seeded SC system because for this system, we assume that the signal components are known at the boundary blocks in \( B \), which size is of order \( w \) (see [43] for precise statements). The stronger variance at the boundaries of opened matrices help this information seed to trigger a reconstruction wave that propagates inward the signal. This phenomenon is what allows SC to reach such good results, namely reconstruction by AMP at low \( \alpha \).

Threshold saturation: AMP performance, when SC matrices are used, is tracked by an MSE profile \( E^{(t)} \): a vector in \([0, v]^\Gamma \) whose components are MSe’s describing the quality of the reconstructed signal, see [43] for details.

Consider the seeded SC system. The MSE profile \( E^{(t)} \) can be asymptotically computed by SE. The precence of the seed is reflected by \( E_r^{(t)} = 0 \) for all \( t \) if \( r \in B \), else

\[
E_r^{(t+1)} = \frac{1}{\Gamma} \sum_{c=1}^{\Gamma} J_{r,c} \text{MMSE}(\Sigma_{c}(E^{(t)}; \Delta)^{−2}) \quad \text{if} \quad r \not\in B,
\]

(13)

with initialization \( E_r^{(0)} = v \quad \forall \quad r \in \{1, \ldots, \Gamma\} \setminus B \), as required by AMP. Denote \( E^{(\infty)} \) the fixed point of this SE recursion. The algorithmic threshold of the seeded SC model is

\[
\Delta_{AMP}^{\epsilon} := \lim_{w \to \infty} \inf_{v \to \infty} \sup_{\Delta > 0 \quad \text{s.t.} \quad E^{(\infty)} \leq E_{\text{good}}(\Delta) \forall \Delta} \frac{\Delta}{\Delta + E_r^{(t)}},
\]

with three of us that when AMP is used for seeded SC systems, threshold saturation occurs, that is:

**Lemma 2.5 (Threshold saturation):** \( \Delta_{AMP}^{\epsilon} \geq \Delta_{RS} \)

Note that in fact the equality holds, but we shall not need it.

Invariant of the optimal threshold: Call the MI per section for the periodic and seeded SC systems, respectively,
We claim the following

\[ \Delta_{\text{AM}\text{P}} \leq \Delta_{\text{amp}} \leq \Delta_{\text{opt}} \leq \Delta_{\text{RS}}, \]

and therefore \( \Delta_{\text{opt}} \equiv \Delta_{\text{RS}} \).

The inequity chain: We claim the following

\[ \Delta_{\text{RS}} \leq \Delta_{\text{AM}\text{P}} \leq \Delta_{\text{opt}} \leq \Delta_{\text{RS}}, \]

and therefore \( \Delta_{\text{opt}} = \Delta_{\text{RS}} \).

The first inequality is lemma 2.5. The second follows from sub-optimality of AMP for the seeded SC system. The equality follows from lemma 2.6 (together with the discussion below it). The last inequality requires a final argument that we now explain.

Recall that \( \Delta_{\text{opt}} < \Delta_{\text{AMP}} \) is not possible. Let us show that \( \Delta_{\text{RS}} < \Delta_{\text{AMP}} \), \( \Delta_{\text{opt}} \) is also impossible. We proceed by contradiction so we suppose this is true. Then each side of (8) are analytic on \( \{0, \Delta_{\text{RS}}\} \) and since they are equal for \( \{0, \Delta_{\text{AMP}}\} \) \( \Delta_{\text{RS}} \), they must be equal on the whole range \( \{0, \Delta_{\text{RS}}\} \) and also at \( \Delta_{\text{AMP}} \) by continuity. For \( \Delta > \Delta_{\text{RS}} \) the fixed point of SE is \( E(\Delta) = \bar{E} \) the global minimum of \( i^{\text{RS}}(E; \Delta) \), hence, the integration argument can be used once more on an interval \( \Delta_{\text{RS}}, \Delta \) which implies that (8) holds for all \( \Delta \). But then \( i^{\text{RS}}(E; \Delta) \) is analytic at \( \Delta_{\text{RS}} \in ]\Delta_{\text{AMP}}, \Delta_{\text{opt}}[ \) which is a contradiction.

III. PROOFS

A. Upper bound using Guerra’s interpolation method

The goal of this section is to sketch the proof of theorem 1.3. First note that the denoising model has been designed specifically so that

\[ i_{0,0} = i^{\text{RS}}(E; \Delta) - \psi(E; \Delta), \]

see (5). By the fundamental theorem of calculus, we have \( i_{1,h} = i_{0,0} + \int_{0}^{1} dt \langle d t, \psi(t) / d t \rangle \). Using (14) and a bit of algebra this is equivalent to

\[ i_{1,h} = i^{\text{RS}}(E; \Delta) + (i_{0,0} - i_{1,0}) + \int_{0}^{1} d t R_{t,h}, \]

\[ R_{t,h} = \frac{d i_{t,h}}{d t} - \frac{\alpha B d \gamma(t)}{2} \frac{\gamma(t) E^{2}}{(1 + \gamma(t) E)^{2}}. \]

We derive a useful expression for the remainder \( R_{t,h} \) which shows that it is negative up to a negligible term. Straightforward differentiation gives \( d i_{t,h} / d t = (A + B) / (2L) \) where

\[ A := \frac{d \gamma(t)}{d t} \sum_{\mu=1}^{M} E \left( \langle [\phi X]^{2}_{\mu} - \gamma(t)^{-\frac{1}{2}} [\phi X]_{\mu}, Z_{\mu}, t, h \rangle \right), \]

\[ B := \frac{d \lambda(t)}{d t} \sum_{i=1}^{N} E \left( \langle X_{i}^{2} - \lambda(t)^{-\frac{1}{2}} X_{i}, Z_{i}, t, h \rangle \right). \]

These two quantities can be simplified using Gaussian integration by parts. For example, integrating by parts w.r.t. \( \gamma \),

\[ \gamma(t)^{-\frac{1}{2}} E \left( \langle [\phi X]^{2}_{\mu}, t, h, Z_{\mu} \rangle \right) = E \left( \langle [\phi X]^{2}_{\mu}, t, h, Z_{\mu} \rangle - \langle [\phi X]^{2}_{\mu}, t, h \rangle \right), \]

which allows to simplify \( A \). For \( B \) we proceed similarly with an integration by parts w.r.t. \( Z_{i} \), and find

\[ A = \frac{d \gamma(t)}{d t} \sum_{\mu=1}^{M} E \left( \langle [\Phi X]^{2}_{\mu}, t, h \rangle \right), \]

\[ B = \frac{d \lambda(t)}{d t} \sum_{i=1}^{N} E \left( \langle X_{i}^{2}, t, h \rangle \right). \]

Now, recalling the definitions of \( E_{1,h} \) and \( \text{ymmse}_{1,h} \), using lemma 2.6 and the snr conservation relation (9) we see that these two formulas are equivalent, respectively, to

\[ A = \frac{\alpha B d \gamma(t)}{2} \text{ymmse}_{1,h} \]

\[ B = \frac{d \lambda(t)}{d t} \frac{\gamma(t) E_{1,h}}{2(1 + \gamma(t) E_{1,h})}. \]

Notice that the first relation is true for a.e. \( h \). Finally, combining (16, 19) gives for a.e. \( h \)

\[ R_{t,h} \leq \frac{d \gamma(t)}{d t} \frac{\gamma(t)(E - E_{1,h})^{2}}{(1 + \gamma(t) E_{1,h})^{2} + \sigma_{L}(1)}. \]

Since \( \gamma(t) \) is an increasing function we see that, quite remarkably, \( R_{t,h} \) is negative up to a vanishing term. A similar interpolation technique ensures that the limit \( \lim_{h \to 0} \) exists. We therefore obtain from (16) that \( \lim_{h \to 0} \gamma_{1,h} \leq i^{\text{RS}}(E; \Delta) + \gamma_{0,0} - \gamma_{1,0} \) for a.e. \( h \). We can now take the limit \( h \to 0 \) along a suitable subsequence. It is easy to check \( \lim_{h \to 0} i_{0,0} = \gamma_{1,0} \). Also \( \lim_{h \to 0} \lim_{h \to -\infty} i_{1,h} = \lim_{h \to -\infty} i_{1,0} \), because concavity and continuity of \( i_{1,h} \) and existence of the \( L \to \infty \) limit imply that \( \lim_{h \to -\infty} i_{1,h} \) is also a concave and continuous function of \( h \). We conclude \( \lim_{h \to -\infty} i_{1,0} = i^{\text{RS}}(E; \Delta) \), that is equivalent to theorem 1.3.

B. Invariance of the mutual information

In this paragraph we sketch the proof of lemma 2.6, i.e. that the SC model has the same asymptotic MI per section than the original CS model (1). For that purpose, we compare three models: the decoupled \( w = 0 \) model, the SC 0 < \( w < (\Gamma - 1)/2 \) model and the homogeneous \( w = (\Gamma - 1)/2 \) model.

In all cases, a periodic matrix (fig. 1) is used. The RS MI associated with a generic SC model is

\[ t^{\text{per}}_{\Gamma,w}(E; \Delta) := \sum_{c=1}^{\Gamma} \left[ \psi(E_{c}; \Delta) + i \left( \bar{S}_{c} - \bar{S} + \frac{\bar{Z} \bar{S} \Sigma_{c}(E; \Delta)}{2} \right) \right], \]

where \( \Sigma_{c}(E; \Delta)^{-1} \) is given by (13) and the dependence on \( w \) is through the coupling variance matrix \( \bar{J} \). It will be convenient to restrict to a constant trial profile, i.e. \( E_{c} = E \forall c \in \{1, \ldots, \Gamma\} \) in what follows.

We first compare the mutual informations of the homogeneous \( w = (\Gamma - 1)/2 \) and SC (fixed \( w \)) models. A convenient way to compare the two MI is by re-writing \( t^{\text{per}}_{\Gamma,w} - t^{\text{per}}_{\Gamma,w} \) as

\[ (t^{\text{per}}_{\Gamma,w} - t^{\text{per}}_{\Gamma,w}) + (t^{\text{per}}_{\Gamma,w} - t^{\text{per}}_{\Gamma,w}) + (t^{\text{per}}_{\Gamma,w} - t^{\text{per}}_{\Gamma,w}). \]

Indeed each of these differences can be studied "easily". Note that, by construction, the homogeneous model matches the
This leads us to a new remainder difference $i_{\Gamma_{t, w}}^{\text{per}} \leq \lim_{L \to \infty} i_{\Gamma_{t, w}}^{\text{per}}$. Moreover, from (21), (13) and the fact that both the trial profile and coupling matrix are constant, one gets $i_{\Gamma_{t, w}}^{\text{per}} (E; \Delta) = i_{\Gamma_{t, w}}^{\text{RS}} (E; \Delta)$ given by (5). Thus the last difference in (22) is exactly what we studied in the previous section, i.e., the $t$-integral of (20) (up to a term that tends to 0 as $h \to 0$, see (15)).

The middle term in (22) is null due to the choice of a constant trial profile, making $i_{\Gamma_{t, w}}^{\text{RS}} (E; \Delta)$ independent of $w$.

It remains to study the first difference in (22). To do so, we follow the same strategy as in the previous section. Observations about “blocks” of the signal are coming from the periodic SC model with $\gamma(t)$ and from $\Gamma$ independent AWGN vectorial denoising models with $\gamma(t)$, s.t. (9) is verified and with the same boundary values and monotonicity properties as in sec. III-A. The (perturbed) Hamiltonian of this SC interpolated model is of the form (11), with an appropriate $\Gamma_{t, h}$ periodic matrix $\phi$. Calculations essentially similar to the ones of the previous sec. III-A show that the first difference in (22) equals the integral over $t \in [0, 1]$ of a new remainder (up to terms tending to 0 as $h \to 0$)

$$-rac{d \gamma}{dt} \left[ \sum_{r \in \mathbb{R}} \frac{\gamma(E - \sum_{c \in \mathbb{R}} E_{c, t, h})^2}{(1 + \gamma) E^2} + \delta_L(1), \right]$$

with $w := \{w, \ldots, w + r\}$ and the MMSE “per block” is $E_{c, t, h} := \mathbb{E} [\sum_{n \in \mathbb{R}} (X(t)_{c, h} - S_{c, h})^2]/\Gamma/L$ (where $\{c, h\}$ being the components part of the block $c$). Now notice that due to the periodicity of the SC construction used here, for any quantity indexed by the “spatial dimension”, one has $\sum_{n \in \mathbb{R}} U_r = \sum_{c \in \mathbb{R}} U_{c, h}/(2w + 1)$. This implies that $E_{c, t, h}$ is replaced by $E_{c, t, h}/(\Gamma(2w + 1))$ in (20). Thus (22) equals (up to $\delta_L(1)$ terms) the $t$-integral of the difference of remainders

$$\gamma \left[ \frac{(1 + \gamma) E^2}{1 + \gamma} \right] \int_0^1 dt \frac{d \gamma(t)}{dt} \left( \frac{E - \mathbb{E}(E_{c, t, h})^2}{1 + \gamma E_{c, t, h}} - \frac{E - \mathbb{E}(E_{c, t, h})^2}{1 + \gamma E_{c, t, h}} \right).$$

Now notice $(E - x)^2/(1 + \gamma x)$ is a convex function of $x$ which directly implies that the last expression is $\leq 0$. Thus we find

$$\lim_{L \to \infty} i_{\Gamma_{t, w}}^{\text{per}} \geq \lim_{L \to \infty} i_{\Gamma_{t, w}}^{\text{per}}.$$ 

Putting everything together, we obtain $\lim_{L \to \infty} i_{\Gamma_{t, w}}^{\text{per}} \geq \lim_{L \to \infty} i_{\Gamma_{t, w}}^{\text{per}} \geq \lim_{L \to \infty} i_{\Gamma_{t, w}}^{\text{per}}$. Since the two extreme limits are equal to $\lim_{L \to \infty} i_{\Gamma_{t, w}}^{\text{per}}$, we obtain the first part of lemma 2.6.

We conclude by the similar statement for the seeded system. It is not hard to show that the MI difference between those of the periodic and seeded SC systems is $O(w/\Gamma)$, thus vanishing as $\Gamma \to \infty$. As a consequence, for the seeded system and any fixed $w$ we have $\lim_{L \to \infty} \lim_{\Gamma \to \infty} i_{\Gamma_{t, w}}^{\text{seed}} = \lim_{L \to \infty} i_{\Gamma_{t, w}}^{\text{seed}}$ as well, proving the second part of lemma 2.6.

### C. Computing ymmse$_{t,h}$

Let us now prove lemma 2.4. A direct application of the Nishimori identity (remark 2.2) brings $2\mathbb{E}[(\Phi_X)^2_{t, h}] = \mathbb{E}[(\Phi_X^2)_{t, h}]$. Using this, (17) and the first equality of (18), we obtain that ymmse$_{t,h} = \sum_{\mu} \mathbb{E}[(\Phi_X)^2_{t, h}]/M$ is also equal to

$$\frac{1}{M} \sum_{\mu=1}^M \mathbb{E} \left[ (\Phi_X^2_{t, h}) - \mathbb{E} \left[ (\Phi_X^2_{t, h}) \right] \right].$$

Define $U_{\mu} := \sqrt{\gamma} (\Phi_X)_{t, h} - Z_{\mu}$. An integration by part w.r.t $\phi_{\mu i} \sim \mathcal{N}(0, 1/L)$ of (23) brings that

$$\text{ymmse}_{t,h} = \frac{1}{M} \sum_{\mu=1}^M \mathbb{E} \left[ (\Phi_X^2_{t, h}) \right].$$

The Nishimori identity allows to write that ymmse$_{t,h}$ equals

$$\frac{1}{M} \sum_{\mu=1}^M \left( - \sqrt{\gamma} \mathbb{E} \left[ (\Phi_X^2_{t, h}) \right] - \mathbb{E} \left[ (\Phi_X^2_{t, h}) \right] \right) \frac{1}{M} = \gamma_1 - \gamma_2,$$

together with $\gamma_1 := \mathbb{E} \left[ \sum_{\mu} Z_{\mu}^2 / M \right]$ and $\gamma_2 := \mathbb{E} \left[ \sum_{\mu} Z_{\mu}^2 / M \right]$. By the law of large numbers, $\sum_{\mu} Z_{\mu}^2 / M = 1 + o(1)$ almost surely as $L \to \infty$ so that using the Nishimori identity, we reach $\gamma_1 = E_{t,h} + o_{L}(1)$. Using similar concentration proofs as [48] (and this is actually the point where the perturbation of the interpolated model becomes fundamental), one can show for the second term that for a.e $h$

$$\gamma_2 = \sqrt{\gamma} \mathbb{E} \left[ \sum_{\mu=1}^M \frac{Z_{\mu}}{M} (\Phi_X^2_{t, h}) \right] \mathbb{E} \left[ \sum_{i=1}^N (X_i X_i^*)_{t, h} \right] + o_{L}(1),$$

where we used the Nishimori identity to identify $E_{t,h}$ in the second equality. From this and (23) we recognize $\gamma_2 = E_{t,h} \gamma(t) \text{ymmse}_{t,h} + o_{L}(1)$. Putting all pieces together, we get from (24) that ymmse$_{t,h} = E_{t,h} - E_{t,h} \gamma(t) \text{ymmse}_{t,h} + o_{L}(1)$ for a.e $h$, which leads lemma 2.4.
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