REGULAR HOMOTOPY CLASSES OF IMMERSIONS
OF 3-MANIFOLDS INTO 5-SPACE

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Abstract. We give geometric formulae which enable us to detect (completely in some cases) the regular homotopy class of an immersion with trivial normal bundle of a closed oriented 3-manifold into 5-space. These are analogues of the geometric formulae for the Smale invariants due to Ekholm and the second author. As a corollary, we show that two embeddings into 5-space of a closed oriented 3-manifold with no 2-torsion in the second cohomology are regularly homotopic if and only if they have Seifert surfaces with the same signature. We also show that there exist two embeddings $F_0$ and $F_8: T^3 \to \mathbb{R}^5$ of the 3-torus $T^3$ with the following properties: (1) $F_0 \circ h$ is regularly homotopic to $F_8$ for some immersion $h: S^3 \to \mathbb{R}^5$, and (2) the immersion $h$ as above cannot be chosen from a regular homotopy class containing an embedding.

1. Introduction

The Smale-Hirsch theory [5, 12] reduces the problem of enumerating regular homotopy classes of immersions to homotopy theory. However, the problem of determining the regular homotopy class of a given immersion remains nontrivial. Here we address this problem in the case of immersions of 3-manifolds into $\mathbb{R}^5$ having trivial normal bundles. We concentrate on immersions with trivial normal bundles, since we want to apply our results to the following question: Which regular homotopy classes of immersions of an oriented 3-manifold into $\mathbb{R}^5$ contain embeddings? Note that embeddings of oriented 3-manifolds into $\mathbb{R}^5$ have trivial normal bundles.

History.

The problem of reading off the regular homotopy class of an immersion from its picture was posed by Smale [12]. For the particular case of immersions of the $n$-sphere $S^n$ into $\mathbb{R}^{2n}$, it had already been shown by Whitney [17] that the double points of an immersion $S^n \hookrightarrow \mathbb{R}^{2n}$ determine its regular homotopy class. This result has been extended by Ekholm [4, 3] to immersions $S^n \hookrightarrow \mathbb{R}^{2n-1} (n \geq 4)$ and $S^n \hookrightarrow \mathbb{R}^{2n-2}$.

On the other hand, a result by Hughes and Melvin [7] suggested that no further extension was possible. Namely they have shown that there are embeddings $S^{4k-1} \hookrightarrow \mathbb{R}^{4k+1}$ not regularly homotopic to the standard one. Therefore the multiple points do not determine the regular homotopy class of an immersion.

Still the original problem posed by Smale has been answered by Ekholm and Szűcs [4] for immersions $S^{4k-1} \hookrightarrow \mathbb{R}^{4k+1}$ by considering “singular Seifert surfaces” bounded by the immersions.

Immersions of closed oriented 3-manifolds into $\mathbb{R}^5$ were investigated by Wu [18]. Using the homotopy theoretical reduction provided by the Smale-Hirsch theory he has shown that the regular homotopy classes of immersions of a closed oriented 3-manifold $M^3$ into $\mathbb{R}^5$ can be characterised by two invariants as follows.

(1) The first one is a second cohomology class of $M^3$: “half of the normal Euler class”.
(2) The second invariant is an integer (for immersions with trivial normal bundles).
The geometric meaning of these invariants was not very clear from Wu’s work.

Now we can list our results:

i) We shall give a geometric description of the first invariant in the first part of Section 3. We call it “the Wu invariant”. Note that for an embedding $M^3 \hookrightarrow \mathbb{R}^5$ the Wu invariant is either zero or an element of order two in the second cohomology group $H^2(M^3; \mathbb{Z})$ of $M^3$.

ii) We show that given any second cohomology class in $H^2(M^3; \mathbb{Z})$, which is zero or has order two, there exists an embedding $M^3 \hookrightarrow \mathbb{R}^5$ of the oriented 3-manifold $M$ with Wu invariant equal to the given cohomology class (Theorem 3.8).

iii) We define an action of the group Imm$[S^3, \mathbb{R}^5]$ of regular homotopy classes of immersions of $S^3$ into $\mathbb{R}^5$ on the set Imm$[M^3, \mathbb{R}^5]_0$ of regular homotopy classes of immersions of the 3-manifold $M^3$ into $\mathbb{R}^5$ with trivial normal bundles by taking connected sums of immersions. In Section 4, we show that this action is effective and its orbits coincide with the regular homotopy classes of immersions having the same Wu invariant.

iv) In Section 5, we express the second invariant (which is an integer) through singularities of any singular Seifert surface. Actually we prove two formulae, analogous to those proved by Ekholm and Szűcs [4] for immersions of $S^3$ into $\mathbb{R}^5$.

v) In Section 6 we prove a few rather surprising corollaries concerning the question mentioned above about the regular homotopy classes containing embeddings. Although these corollaries hold more generally, we formulate them for the simplest manifold for which these surprising phenomena occur, namely for the 3-dimensional torus $T^3$:

**Corollary 6.2.** There exist embeddings $F_0$ and $F_8 : T^3 \hookrightarrow \mathbb{R}^5$ of the 3-torus $T^3$ with the following properties: (1) There is an immersion $h : S^3 \hookrightarrow \mathbb{R}^5$ such that $F_8$ is regularly homotopic to the connected sum $F_0\#h$, but (2) this immersion $h$ cannot be chosen from the regular homotopy class of an embedding.

**Corollary 6.3.** There exists an immersion $h : S^3 \hookrightarrow \mathbb{R}^5$ with the following properties: (1) $h$ is not regularly homotopic to an embedding $S^3 \hookrightarrow \mathbb{R}^5$, but (2) for any embedding $E : T^3 \hookrightarrow \mathbb{R}^5$, the connected sum $E\#h : T^3 \hookrightarrow \mathbb{R}^5$ is again regularly homotopic to an embedding $T^3 \hookrightarrow \mathbb{R}^5$.

In other words, the double points of the immersion $h$ cannot be eliminated by regular homotopy, but they can be eliminated after being taken connected sum with any embedding of the 3-torus.

vi) In Section 7, we give some corollaries concerning the number of cusps a singular Seifert surface can have.

We will use the following notation throughout the paper. By Imm$[M, N]$ we denote the set of regular homotopy classes of immersions of a manifold $M$ into a manifold $N$ and by Emb$[M, N]$ the subset of Imm$[M, N]$ consisting of all regular homotopy classes containing an embedding. By Imm$[M, N]_x$ we denote the subset of Imm$[M, N]$ consisting of all regular homotopy classes of immersions with normal Euler class $\chi \in H^*(M; \mathbb{Z})$. In particular for an oriented 3-manifold $M^3$ by Imm$[M^3, \mathbb{R}^5]_0$ we denote the set of regular homotopy classes of immersions $M^3 \hookrightarrow \mathbb{R}^5$ with trivial normal bundles. By $M^3$ we will always denote a smooth, closed, oriented 3-manifold.

We denote by $\Gamma_2(M^3)$ the finite subgroup of $H^2(M^3; \mathbb{Z})$ formed by zero and all elements of order two.

The punctured manifold of $M^3$ will be denoted by $M^3_0$; i.e. $M^3_0 := M^3 \setminus \text{int } D^3$ ($D^3$ is a 3-disk) and the $i$-skeleton of a manifold $M$ by $\text{sk}_iM$.

Throughout the paper, manifolds and maps are of class $C^\infty$. The symbol “$\simeq$” denotes an appropriate isomorphism between algebraic objects; “$\sim$” and “$\sim_r$” mean “homotopic” and “regularly homotopic” respectively.
2. Geometric formulae for the Smale invariants
given by Hughes-Melvin and by Ekhom-Sz"ucs

Hughes [3] has shown that \( \text{Imm}[S^n, \mathbb{R}^N] \) has a group structure under connected sum and the Smale invariant

\[
\Omega : \text{Imm}[S^n, \mathbb{R}^N] \to \pi_n(V_{N,n})
\]

actually gives a group isomorphism, where \( V_{N,n} \) denotes the Stiefel manifold of \( n \) frames in the \( N \)-space.

Hughes and Melvin [4] investigated the problem: Which regular homotopy classes contain embeddings in the set \( \text{Imm}[S^{4k-1}, \mathbb{R}^{4k+1}] \)? Their result for the case of embeddings of \( S^3 \) into \( \mathbb{R}^5 \) can be summarised by the following diagram:

\[
\begin{array}{ccc}
\Omega : & \text{Imm}[S^3, \mathbb{R}^5] \cup \text{Emb}[S^3, \mathbb{R}^5] & \to \pi_3(V^3, \mathbb{R}) \\
(\pi) & f \mapsto & 24\mathbb{Z} \\
& \sigma(f) & 3\sigma(V_f)/2,
\end{array}
\]

where \( \Omega \) is the Smale invariant and \( \sigma(V_f) \) is the signature of an oriented Seifert surface \( V_f \) for \( f \). Here, \( \sim \) denotes a group isomorphism.

Their result has been extended by the third author to \( \mathbb{Z}_2 \)-homology \( 3 \)-spheres [14]. Trying to extend it further was the starting point of the present paper.

As we mentioned in the introduction,

1. this result shows the impossibility of determining the regular homotopy class of an immersion by its multiple points, and

2. it has been shown in [4] that Hughes-Melvin’s formula stating that the Smale invariant of an embedding is equal to \( 3/2 \) times the signature of any Seifert surface bounded by the embedding can be extended to immersions if we allow “singular Seifert surfaces”.

Since we are going to generalise the result (2) to immersions of arbitrary oriented 3-manifolds with trivial normal bundles, we recall it in detail.

Let \( f : S^3 \hookrightarrow \mathbb{R}^5 \) be an immersion and \( V^4 \) an arbitrary compact oriented 4-manifold with \( \partial V^4 = S^3 \). The map \( f \) extends to a generic map \( \tilde{f} : V^4 \to \mathbb{R}^5 \) which has no singular points near the boundary. This map \( \tilde{f} \) has isolated cusps, each of which has a sign. Let us denote by \( \#\Sigma^{-1}(\tilde{f}) \) their algebraic number. Let us denote by \( \Omega(f) \) the Smale invariant of \( f \). The first formula for the Smale invariant given by [3] is the following:

**Theorem 2.1** (Theorem 1 (a) in [3]).

\[
\Omega(f) = \frac{1}{2}(3\sigma(V^4) + \#\Sigma^{-1,1}(\tilde{f})).
\]

Before giving the second formula for the Smale invariant we recall some definitions from [3]. Let \( V^4 \) be as above and let \( \tilde{f} : V^4 \to \mathbb{R}^6_+ \) be a generic map nonsingular near the boundary, such that \( \tilde{f}^{-1}(\mathbb{R}^5) = \partial V^4 = S^3 \) and \( \tilde{f}|_{\partial V^4} \sim f \) in \( \mathbb{R}^5 \). Here \( \mathbb{R}^6_+ \) is the upper half space in \( \mathbb{R}^6 \) and \( \mathbb{R}^5 = \partial \mathbb{R}^6_+ \). The double point set \( \Delta(\tilde{f}) \subset V^4 \) of the map \( \tilde{f} \) is 2-dimensional, and on the boundary of \( \tilde{f}(\Delta(\tilde{f})) \) lies the image \( \Sigma \) of the singular points of \( \tilde{f} \), which is 1-dimensional. Furthermore, triple points appear as isolated points, each of which has a sign. We denote the algebraic number of triple points of \( \tilde{f} \) by \( t(\tilde{f}) \).

**Definition 2.2** (Definition 4 in [3]). Let \( \Delta = \Delta(f) \) be the set of double points of \( f \) and \( f(\Delta) \) its image. Then \( f(\Delta) \) is a closed 1-manifold in \( \mathbb{R}^5 \). Since \( S^3 \) is 2-connected, \( f \) has a unique (up to homotopy) normal framing \( (n_1, n_2) \). Let \( f(\Delta)' \) be the manifold obtained from \( f(\Delta) \) by shifting slightly along the vector field, \( q \mapsto n_1(p_1) + n_1(p_2) \) for \( q = f(p_1) = f(p_2) \in f(\Delta) \). Define \( L(f) \) to be the linking number in \( \mathbb{R}^5 \) of \( f(S^3) \) and \( f(\Delta)' \).
**Definition 2.3** (Definition 5 in [4]). Let Σ′ be a copy of Σ shifted slightly along the outward normal vector field of Σ in (the closure of) $\hat{f}(\Delta(\hat{f}))$. Define the linking number $l(\hat{f}) \in \mathbb{Z}$ of the map $\hat{f}$ to be the linking number of Σ′ and $\hat{f}(V^4)$ in $\mathbb{R}^6_+$.

**Theorem 2.4** (Theorem 1 (b) in [4], see also [4] Remark 3)). The Smale invariant of $f$ is given by the formula:

$$\Omega(f) = \frac{1}{2}(3\sigma(V^4) + 3l(\hat{f}) - 3l(\hat{f}) + L(f)).$$

As is stated in [4], Remark 5), the above definition of $l$ makes sense for immersions of arbitrary closed oriented 3-manifolds. Furthermore, the definition of $L$ can be extended to the case of framed immersions — immersions with trivial normal bundles and endowed with normal framings — of closed oriented 3-manifolds, which are not necessarily $S^3$. We review this extension here, since it will be necessary later.

**Definition 2.5** (see [4], Remark 5)). Let $F$ be a framed immersion with a normal framing $\nu$, of a closed oriented 3-manifold $M^3$ into $\mathbb{R}^5$. Let $\Delta = \Delta(F)$ be the set of double points of $F$ and $F(\Delta)$ its image. Let $n_1$ and $n_2$ be the two linearly independent normal vectors of $F$ determined by $\nu$. Define the vector field $w$ along $F(\Delta)$ by $w(q) = n_1(p_1) + n_2(p_2)$, where $F(p_1) = F(p_2) = q$. Then, define $L_\nu(F)$ to be the linking number in $\mathbb{R}^3$ of $F(M^3)$ and $F(\Delta)$ pushed off along $w$ out of $F(M^3)$.

**Remark 2.6.** We will show later in the proof of Theorem 5.8 that $L_\nu(F)$ actually does not depend on the choice of the normal framing $\nu$ for an immersion $F : M^3 \looparrowright \mathbb{R}^5$ with trivial normal bundle of any closed oriented 3-manifold $M^3$ (see Remark 5.10).

3. IMMERSIONS OF 3-MANIFOLDS INTO $\mathbb{R}^5$ — WU’S RESULT AND THE GEOMETRY BEHIND IT

In this section, we first recall the result of Wu [15] on the enumeration of regular homotopy classes of immersions of an oriented 3-manifold $M^3$ into $\mathbb{R}^5$. Then we study the structure of $\text{Imm}[M^3, \mathbb{R}^5]|_0$ — the set of regular homotopy classes of immersions with trivial normal bundles. We look especially into a certain second cohomology class which characterises (the regular homotopy class of) an immersion on (the regular neighbourhood of) the 2-skeleton of $M^3$.

We fix a trivialisation $\tau$ of $TM^3$ once and for all. Then the Smale–Hirsch theory provides a bijection $\text{Imm}[M^3, \mathbb{R}^5]^\tau \cong [M^3, V_{5,3}]$. Based on this bijection, Wu [15] has shown the following.

**Theorem 3.1** (Theorem 2 in [15], see also [4], Theorem 3)). The normal Euler class $\chi_F$, for an immersion $F : M^3 \looparrowright \mathbb{R}^5$ is of the form $2C$ for some $C \in H^2(M^3; \mathbb{Z})$, and for any $C \in H^2(M^3; \mathbb{Z})$, there is an $F$ such that $\chi_F = 2C$. Furthermore,$$
\text{Imm}[M^3, \mathbb{R}^5]|_\chi \cong \bigcap_{C \in H^2(M^3; \mathbb{Z}) \text{ with } 2C = \chi} H^3(M^3; \mathbb{Z})/(4C \sim H^1(M^3; \mathbb{Z})),
$$
where $\text{Imm}[M^3, \mathbb{R}^5]|_\chi$ is the set of regular homotopy classes of immersions with normal Euler class $\chi \in H^2(M^3; \mathbb{Z})$, “$\sim$” represents the cup product and “$\cong$” denotes a bijection.

**Remark 3.2.** Let us consider the $S^1$-bundle $SO(5) \to V_{5,3}$ and choose the generator $\Sigma^2 \in H^2(V_{5,3}; \mathbb{Z})$ for which the Euler class of this bundle coincides with $2\Sigma^2$. Let $\xi_F : M^3 \to V_{5,3}$ be the map associated with an immersion $F : M^3 \looparrowright \mathbb{R}^5$ (and the fixed trivialisation $TM^3 = M^3 \times \mathbb{R}^3$). The first part of the above theorem is based on the fact that the normal Euler class $\chi_F$ is the pullback $\xi_F^*(2\Sigma^2)$ for the generator $\Sigma^2$ for $H^2(V_{5,3}; \mathbb{Z}) \cong \mathbb{Z}$. Hence, the class $C \in H^2(M^3; \mathbb{Z})$ which appears in the second part of the theorem is nothing but the pullback $\xi_F^*(\Sigma^2)$ of the generator $\Sigma^2$ of $H^2(V_{5,3}; \mathbb{Z}) \cong \mathbb{Z}$.
Recall that $\Gamma_2(M^3)$ is the finite set $\{C \in H^2(M^3; \mathbb{Z}) \mid 2C = 0\}$. Hereafter we shall concentrate on immersions with trivial normal bundles, i.e. on $\text{Imm}[M^3, \mathbb{R}^5]_0$. By Theorem 3.1 this set can be identified with $\Gamma_2(M^3) \times \mathbb{Z}$ (this identification depends on the trivialisation of $T M^3$). Note that Theorem 3.1 can also be applied to the nonclosed 3-manifold $M^3$ and gives the following bijection:

$$\text{Imm}[M^3, \mathbb{R}^5]_0 \approx \Gamma_2(M^3).$$

Note that elements of order two in $H^2(M^3; \mathbb{Z})$ can be naturally identified with those in $\mathcal{H}^2(M^3, \mathbb{Z})$.

**Definition 3.3.** The projection $c : \text{Imm}[M^3, \mathbb{R}^5]_0 \to \Gamma_2(M^3)$ is called the Wu invariant of the immersion of the parallelised 3-manifold with trivial normal bundle. It can be described in two equivalent ways:

1. $c(F) = \xi_F^2(\Sigma^2)$ or
2. $c(F) = F|_{F^3}$ if we use the identification $\text{Imm}[M^3, \mathbb{R}^5]_0 \approx \Gamma_2(M^3)$.

Next we shall give a more geometric description of the Wu invariant $c(F) \in \Gamma_2(M^3)$.

A normal trivialisation $\nu$ of $F$ (together with the tangent trivialisation and the differential of $F$) defines a map $\pi_1(M^3) \to \pi_1(SO(5))$, i.e. an element $\tilde{c}_F$ in $H^1(M^3; \mathbb{Z}_2)$. If we change $\nu$ by an element $\zeta \in [M^3, SO(2)] = H^1(M^3; \mathbb{Z}_2)$, then the class $\tilde{c}_F$ changes by $[\rho(\zeta)]$, where $\rho$ is the mod 2 reduction map $H^1(M^3; \mathbb{Z}) \to H^1(M^3; \mathbb{Z}_2)$. Hence the coset of $\tilde{c}_F$ in $H^1(M^3; \mathbb{Z}_2)/[\rho(H^1(M^3; \mathbb{Z}))]$ does not depend on $\nu$. The cokernel of $\rho$ can be identified with $\Gamma_2(M^3)$ by the canonical map induced by the Bockstein homomorphism (see below). Under this identification, the coset of $\tilde{c}_F$ corresponds to the Wu invariant $c(F) \in \Gamma_2(M^3) = \{C \in H^2(M; \mathbb{Z}) \mid 2C = 0\}$.

Thus the Wu invariant describes the immersion on the 1-skeleton (and then actually also on the 2-skeleton, since $\pi_2(SO(5)) = 0$), if a trivialisation of the tangent bundle is fixed. On the other hand for any immersion $M^3 \looparrowright \mathbb{R}^5$ there is a trivialisation of the tangent bundle such that the Wu invariant of the immersion under the given tangent bundle trivialisation is zero.

This geometric description will follow from the lemmas below.

Let $F : M^3 \looparrowright \mathbb{R}^5$ be an immersion with trivial normal bundle. Choosing a normal framing $\nu$ for $F$, we naturally obtain a map $\varphi_{\nu,F} : M^3 \to SO(5)$.

**Lemma 3.4.** Let $x$ be the generator of $H^2(SO(5); \mathbb{Z}) \approx \mathbb{Z}_2$ and $\varphi_{\nu,F}^* : H^2(SO(5); \mathbb{Z}) \to H^2(M^3; \mathbb{Z})$. Then $\varphi_{\nu,F}^*(x)$ does not depend on the choice of the normal framing $\nu$, and is equal to $c(F)$.

**Proof.** From the Gysin exact sequence of the $SO(2)$-bundle $p : SO(5) \to V_{5,3}$, it is easy to see that $p^*(\Sigma^2) = x$. Hence $\varphi_{\nu,F}^*(x)$ is equal to $c(F)$ and does not depend on the choice of the normal framing $\nu$. \qed

Consider the following commutative diagram:

$$
\begin{array}{ccc}
H^1(SO(5); \mathbb{Z}) & \xrightarrow{\varphi_{\nu,F}^*} & H^1(M^3; \mathbb{Z}_2) \\
\downarrow{\varphi_{\nu,F}^*} & & \downarrow{\varphi_{\nu,F}^*} \\
H^1(M^3; \mathbb{Z}) & \xrightarrow{\beta} & H^2(M^3; \mathbb{Z}) & \xrightarrow{\varphi_{\nu,F}^*} & H^2(SO(5); \mathbb{Z}) \\
\downarrow{\varphi_{\nu,F}^*} & & \downarrow{\varphi_{\nu,F}^*} & & \downarrow{\varphi_{\nu,F}^*} \\
0 & \rightarrow & \mathbb{Z} & \xrightarrow{x_2} & \mathbb{Z} & \rightarrow & \mathbb{Z}_2 & \rightarrow & 0,
\end{array}
$$

where each horizontal line is part of the cohomology exact sequence associated with the coefficient exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{x_2} \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$, each vertical line is the homomorphism induced by $\varphi_{\nu,F}$, and each $\beta$ is the Bockstein homomorphism.

**Lemma 3.5.** Let $y$ be the generator of $H^1(SO(5); \mathbb{Z}_2) \approx \mathbb{Z}_2$. Then $\beta(\varphi_{\nu,F}^*(y))$ does not depend on the choice of the normal framing $\nu$, and is equal to $c(F)$. \qed
Proof. Since $H^1(SO(5); \mathbb{Z}) = 0$ and $H^2(SO(5); \mathbb{Z}) \cong \mathbb{Z}_2$ in the diagram above, we see that $\beta(y) = x$. Hence $\varphi_{\nu,F}(x) = \beta(\varphi_{\nu,F}(y))$. Then the result follows directly from Lemma 3.4. \qed

Remark 3.6. Lemma 3.4 implies that the coset of the class $\varphi_{\nu,F}(y) \in H^1(M^3; \mathbb{Z}_2)$ modulo $\rho(H^1(M^3; \mathbb{Z}))$ does not depend on the choice of the normal framing $\nu$.

Note also that
\[
\Gamma_2(M^3) = \text{Ker}(H^1(M^3; \mathbb{Z}) \xrightarrow{\times 2} H^2(M^3; \mathbb{Z}))
\]
is isomorphic to $H^3(M^3; \mathbb{Z}_2)/\rho(H^1(M^3; \mathbb{Z}))$ by the exact sequence above. Thus, under this isomorphism, we have $c(F) = \varphi_{\nu,F}(x) = [\varphi_{\nu,F}(y)]$, where the bracket means the coset modulo the image of $\rho$.

Since the class $\overline{\nu}_F \in H^1(M^3; \mathbb{Z}_2)$ mentioned above is nothing but the homomorphism $\rho(\nu,F) : H_1(M^3; \mathbb{Z}) \to \text{Hom}(\nu,F)$, it is clear that
\[
\varphi_{\nu,F}(y) = y \circ (\varphi_{\nu,F}) = \overline{\nu}_F : H_1(M^3; \mathbb{Z}) \to \mathbb{Z}_2,
\]
where we consider the second y as the identity homomorphism $H_1(SO(5); \mathbb{Z}) \to \mathbb{Z}_2$. Thus, we have also the description $c(F) = [\overline{\nu}_F]$.

Henceforth we often identify $\Gamma_2(M^3)$ with $H^1(M^3; \mathbb{Z}_2)/\rho(H^1(M^3; \mathbb{Z}))$ and consider $c(F)$ as an element of $H^1(M^3; \mathbb{Z}_2)/\rho(H^1(M^3; \mathbb{Z}))$.

Remark 3.7. Since we have fixed a trivialisation of $TM^3$, the set $H^1(M^3; \mathbb{Z}_2)$ can be identified with the set of homotopy classes of trivialisations of $TM^3|_{sk_2 M^3}$, where $sk_2 M^3$ denotes the 2-skeleton of $M^3$, i.e. with the set of spin structures of $M^3$. The above observations show that an immersion of $M^3$ into $\mathbb{R}^5$ with trivial normal bundle determines a spin structure of $M^3$ up to elements of $\rho(H^1(M^3; \mathbb{Z}))$.

Now we can state the main result of this section, which claims that each $\mathbb{Z}$-component in $\text{Imm}(M^3, \mathbb{R}^5)|_0 \approx \mathbb{Z} \amalg \mathbb{Z} \amalg \cdots \amalg \mathbb{Z} = \Gamma_2(M^3) \times \mathbb{Z}$ contains an embedding.

Theorem 3.8. For every element $C \in H^1(M^3; \mathbb{Z}_2)/\rho(H^1(M^3; \mathbb{Z}))$, there exists an embedding $F : M^3 \to \mathbb{R}^5$ with the Wu invariant $c(F) = C$.

Proof. (A) First assume that $C = [0]$, where $0 \in H^1(M^3; \mathbb{Z}_2)$. Consider the spin structure of $M^3$ corresponding to $0 \in H^1(M^3; \mathbb{Z}_2)$ under the identification in Remark 3.7. Then by a result of Kaplan, there exists a framed 4-manifold $W$ such that

1. $\partial W = M^3$,
2. the framing of $W$ restricted to the 2-skeleton $sk_2 M^3$ of $M^3$ coincides with the given spin structure of $M^3$,
3. $W$ has a special handlebody decomposition, consisting of one 0-handle and some 2-handles attached to the 0-handle simultaneously.

Since $W$ is a nonclosed spin 4-manifold, it is parallelisable and so it can be immersed into $\mathbb{R}^5$. Furthermore, since $W$ has a 2-complex as its spine, its immersion into $\mathbb{R}^5$ can be deformed into an embedding. Take an embedding $F : W \to \mathbb{R}^5$ and set $F = F|_{M^3} : M^3 \to \mathbb{R}^5$.

Now it suffices to show that $c(F) = [0]$. Taking the normal framing $\nu$ for $F$ corresponding to the normal vector field of $M^3 \subset W$, we see that the map $\varphi_{\nu,F} : M^3 \to SO(5)$ restricted to the 2-skeleton can be written as the following composition:
\[
\varphi_{\nu,F}|_{sk_2 M^3} : sk_2 M^3 \to W \to SO(5),
\]
where $W \to SO(5)$ is the map determined by the differential of $F$ using the given framing on $W$. Thus $\varphi_{\nu,F}(y) = 0 \in H^1(M^3; \mathbb{Z}_2) \approx H^1(sk_2 M^3; \mathbb{Z}_2)$, since $H^1(W; \mathbb{Z}_2) = 0$.

(B) In the case when $C$ cannot be written as $[0]$ with $0 \in H^1(M^3; \mathbb{Z}_2)$, put $C = [\gamma]$ with $\gamma \neq 0 \in H^1(M^3; \mathbb{Z}_2)$.

As in (A), consider the spin structure of $M^3$ corresponding to $\gamma \in H^1(M^3; \mathbb{Z}_2)$, and take an embedding $\tilde{G} : W \to \mathbb{R}^5$ of a 4-manifold $W$ with $\partial W = M^3$ such that $W$ has the same properties as in (A). Set $G = \tilde{G}|_{M^3}$. 
In order to show that \( C(G) = \{ \gamma \} \), recall that we have fixed a trivialisation \( \tau \) of \( TM^3 \) to obtain the map \( c : Imm[M^3, R^5]_0 \to \Gamma_2(M^3) \).

If we take another trivialisation \( \tau' \) of \( TM^3 \), then we obtain another map
\[
c' : Imm[M^3, R^5]_0 \to \Gamma_2(M^3)
\]
and with respect to this new trivialisation \( \tau' \), we have a new map
\[
\varphi'_{\nu', G} : M^3 \to SO(5),
\]
and a new induced homomorphism
\[
\varphi'_{\nu', G}^* : \text{H}^1(SO(5); Z_2) \to \text{H}^1(M^3; Z_2)
\]
which satisfies Lemma 3.3. Furthermore, if we choose \( \tau' \) so that \( \tau'|_{sk_2M^3} \) corresponds to \( -\gamma = \gamma \in \text{H}^1(M^3; Z_2) \) as a spin structure, then we have
\[
\varphi'_{\nu', G}^*(y) = \varphi_{\nu', G}^*(y) - \gamma \quad \text{in} \quad \text{H}^1(M^3; Z_2),
\]
and hence
\[
c'(G) = c(G) - C \quad \text{in} \quad \text{H}^1(M^3; Z_2)/\rho(\text{H}^1(M^3; Z)).
\]
Since the restriction of \( \varphi'_{\nu', G} : M^3 \to SO(5) \) to the 2-skeleton \( sk_2M^3 \) can be decomposed into the same composition
\[
\varphi'_{\nu', G}|_{sk_2M^3} : sk_2M^3 \to W \to SO(5),
\]
as in (A), by the same reason as in (A) we have \( c'(G) = c(G) - C = 0 \) and hence \( c(G) = C \).

Thus \( G \) is a required embedding. \( \square \)

4. Modifying immersions on a disk

In this section we define an effective action of the group \( Imm[S^3, R^5] \approx Z \) on the set \( Imm[M^3, R^5]_0 \) and show that its orbits coincide with the sets of regular homotopy classes having the same Wu invariant \( c \).

Let \( M^3 \) be a closed oriented 3-manifold. Let \( D^3 \) be a 3-disk, which from now on we will often identify with the northern hemisphere of the 3-sphere \( S^3 \). Fix an inclusion \( D^3 \subset M^3 \), and put \( M^3 = M^3 \setminus \text{int} D^3 \). Suppose \( F_0 : M^3 \to R^5 \) is an immersion with trivial normal bundle such that \( F_0|_{D^3} \) coincides with the standard embedding \( S^3 \to R^5 \) restricted to the northern hemisphere. For an immersion \( f : S^3 \to R^5 \), we may assume that \( f \) restricted to the southern hemisphere is standard. Then consider the map
\[
\sharp_{F_0} : \text{Imm}[S^3, R^5] \to \text{Imm}[M^3, R^5] \quad f \mapsto F_0\sharp f,
\]
where \( (F_0\sharp f)|_{M^3} = F_0|_{M^3} \) and \( (F_0\sharp f)|_{D^3} = f|_{D^3} \). The normal bundle of \( F_0 \) is trivial and if \( F_0 \) is modified on \( D^3 \), then its normal bundle does not change. Furthermore, \( c(F_0) \) also does not change under this operation, since the invariant \( c \) is determined by the immersion restricted to a neighbourhood of the 2-skeleton of \( M^3 \). Therefore, if we define, for \( C \in H^2(M^3; Z) \) with \( 2C = 0 \),
\[
\text{Imm}[M^3, R^5]_0^{C} := \{ F \in \text{Imm}[M^3, R^5]_0 \mid c(F) = C \},
\]
then we can in fact define the map
\[
\sharp_{F_0} : \text{Imm}[S^3, R^5] \to \text{Imm}[M^3, R^5]^{c(F_0)}.
\]

The following proposition is an analogue of Proposition 3.1 in [14].

**Proposition 4.1.** The map
\[
\sharp_{F_0} : \text{Imm}[S^3, R^5] \to \text{Imm}[M^3, R^5]^{c(F_0)}
\]
is a bijection.
Proof. As it has been mentioned in the paragraph just after Remark 3.2, we have
\[ \text{Imm}[M^3, R^5]_0^{c(F_0)} \approx H^3(M^3; Z) = 0. \]
This means that \( \sharp f_0 \) is surjective.

Let us prove the injectivity. For two immersions \( f \) and \( g : S^3 \rightarrow R^5 \), we have
\[ F_0 \sharp f \sim_r F_0 \sharp g \iff \xi_{F_0 \sharp f} \simeq \xi_{F_0 \sharp g} : M^3 \rightarrow V_{5,3}. \]
Note that \( \xi_{F_0 \sharp f} \) and \( \xi_{F_0 \sharp g} \) are homotopic on the 2-skeleton of \( M^3 \).

Let \( \Delta^3_{\xi_{F_0 \sharp f}, \xi_{F_0 \sharp g}} \) be the difference 3-cochain between \( \xi_{F_0 \sharp f} \) and \( \xi_{F_0 \sharp g} \), that is, the 3-cochain which assigns \( \Omega(f) - \Omega(g) \in \pi_3(V_{5,3}) \) to \( D^3 \) considered as a 3-cell, and \( 0 \in \pi_3(V_{5,3}) \) to the other 3-cells. Then we have
\[ \xi_{F_0 \sharp f} \simeq \xi_{F_0 \sharp g} : M^3 \rightarrow V_{5,3} \iff \Delta^3_{\xi_{F_0 \sharp f}, \xi_{F_0 \sharp g}} \text{ is a coboundary,} \]
using the fact that \( 2c(F_0) = 0 \) and the following lemma.

**Lemma 4.2** (Theorem 8A in [17], see also [14, Lemma 2.1]). Two maps \( \xi \) and \( \eta : M^3 \rightarrow V_{5,3} \) are homotopic if and only if
\[
\begin{align*}
(a) & \quad \xi^*(\Sigma^2) = \eta^*(\Sigma^2) \in H^2(M^3; Z), \\
(b) & \quad \text{there exist a 1-cocycle } X^1 \text{ and a 2-cochain } Y^2 \text{ such that} \end{align*}
\]
\[ \Delta^3_{\xi, \eta} = 4X^1 - \xi^*(\Sigma^2) + \delta Y^2. \]

Note that \( \Sigma^2 \) is the generator of \( H^2(V_{5,3}; Z) \approx Z \) and that
\[ \xi_{F_0 \sharp f}^*(\Sigma^2) = \xi_{F_0 \sharp g}^*(\Sigma^2) = \xi_{F_0}^*(\Sigma^2) = c(F_0) \]
(see Remark 3.2 and Definition 3.3).

Now if \( \Omega(f) = \Omega(g) \), then \( \Delta^3_{\xi_{F_0 \sharp f}, \xi_{F_0 \sharp g}} \) (which we abbreviate here to \( \Delta^3 \)) is trivial, hence a coboundary. Conversely, if \( \Delta^3 \) is a coboundary, then there exists a 2-cochain \( Y^2 \) such that \( \delta Y^2 = \Delta^3 \). Thus, we have
\[ Y^2(\partial D^3) = \delta Y^2(D^3) = \Delta^3(D^3) = \Omega(f) - \Omega(g). \]

Furthermore,
\[ Y^2(\partial D^3) = -Y^2(\partial(M^3 \setminus \text{int } D^3)) = -\Delta^3(M^3 \setminus \text{int } D^3) = 0, \]
since \( \Delta^3 \) takes 0 on all 3-cells of \( M^3 \setminus \text{int } D^3 \). Therefore, we have
\[ \Delta^3_{\xi_{F_0 \sharp f}, \xi_{F_0 \sharp g}} \text{ is a coboundary} \]
\[ \iff \Omega(f) = \Omega(g) \in \pi_3(V_{5,3}) \]
\[ \iff f \sim_r g : S^3 \rightarrow R^5. \]

This completes the proof of Proposition 4.1 (Note that this last argument also shows that the map \( \sharp f_0 \) is indeed well-defined.) \( \square \)

**Remark 4.3.** The proposition above implies that for each \( C \in \Gamma_2(M^3) \), the action of \( \text{Imm}[S^3, R^5] \) on \( \text{Imm}[M^3, R^5]_0^{c(F_0)} \)
\[ \text{Imm}[M^3, R^5]_0^C \times \text{Imm}[S^3, R^5] \rightarrow \text{Imm}[M^3, R^5]_0^C \]
\[ (F, f) \rightarrow F \sharp f \]
is effective and transitive.
5. Geometric Formulae for \( \text{Imm}[M^3, \mathbb{R}^5]_0 \)

In this section, we shall define an integer invariant 
\[
i : \text{Imm}[M^3, \mathbb{R}^5]_0 \to \mathbb{Z},
\]
which together with the Wu invariant \( c \) will give a bijection 
\[
(c, i) : \text{Imm}[M^3, \mathbb{R}^5]_0 \to \Gamma_2(M^3) \times \mathbb{Z}.
\]

Actually we shall give two different geometric expressions for \( i \) — denoted by \( i_a \) and \( i_b \) — which are analogues of the formulae in [4]. Then we show that they coincide.

**Definition 5.1.** Let \( M^3 \) be a closed oriented 3-manifold. We denote by \( \alpha = \alpha(M^3) \) the dimension of the \( \mathbb{Z}_2 \) vector space \( \tau H_1(M^3; \mathbb{Z}) \otimes \mathbb{Z}_2 \), where \( \tau H_1(M^3; \mathbb{Z}) \) is the torsion subgroup of \( H_1(M^3; \mathbb{Z}) \). Note that the number of elements in the set \( \Gamma_2(M^3) \) is equal to \( 2^\alpha \).

The following lemma can be found in [10, Theorem 2.6]. For reader’s convenience we give a short proof here.

**Lemma 5.2.** Let \( M^3 \) be a closed oriented 3-manifold and \( W \) a compact oriented spin 4-manifold with boundary \( M^3 \). Then the signature \( \sigma(W) \) of \( W \) has the same parity as \( \alpha(M^3) \).

**Proof.** By [8], there exists a 1-connected compact oriented spin 4-manifold \( V \) with \( \partial V = M \) such that the spin structures on \( M^3 \) induced by \( V \) and \( W \) coincide with each other. Then by Rohlin’s theorem, \( \sigma(V) - \sigma(W) = \sigma(V \cup -W) \equiv 0 \) (mod 16). Thus we have only to show the assertion for \( V \) instead of \( W \).

Since \( V \) is 1-connected, we have the exact sequence
\[
0 \to H_2(M^3; \mathbb{Z}) \to H_2(V; \mathbb{Z}) \to H_2(\partial V; \mathbb{Z}) \to H_1(M^3; \mathbb{Z}) \to 0,
\]
where the map \( Q \) is identified with the intersection form of \( V \) through the Poincaré-Lefschetz duality \( H_2(V; \mathbb{Z}) \cong H^2(\partial V; \mathbb{Z}) \). Taking an appropriate basis for \( H_2(V; \mathbb{Z}) \), we may assume that the matrix representative of \( Q \) is of the form \( Q_0 \oplus Q_1 \), where \( Q_0 \) is the zero form and \( Q_1 \) is nonsingular. Then \( Q_1 \) can be regarded as a presentation matrix of \( \tau H_1(M^3; \mathbb{Z}) \) and its size has the same parity as \( \sigma(V) \). Now, since \( V \) is spin, its intersection form is of even type. Thus all the diagonal entries of \( (Q_1)_2 \) are zero, where \( (Q_1)_2 \) denotes the reduction modulo two. Now, it is an easy exercise to show that the size of such a matrix has the same parity as the dimension of the \( \mathbb{Z}_2 \) vector space \( \tau H_1(M^3; \mathbb{Z}) \otimes \mathbb{Z}_2 \) which it presents. Thus the result follows. \( \square \)

**Definition 5.3.** Let \( F : M^3 \cong \mathbb{R}^5 \) be an immersion with trivial normal bundle. Let \( W^4 \) be any compact oriented 4-manifold with \( \partial W^4 = M^3 \) and \( \hat{F} : W^4 \to \mathbb{R}^5 \) a generic map nonsingular near the boundary such that \( \hat{F}|_{\partial W^4} = F \). (We can choose such a generic map \( \hat{F} \), since \( F \) is an immersion with trivial normal bundle.) Denote the algebraic number of cusps of \( \hat{F} \) by \( \# \Sigma^{1,1}(\hat{F}) \). Then define
\[
i_a(F) = \frac{3}{2}(\sigma(W^4) - \alpha(M^3)) + \frac{1}{2}\# \Sigma^{1,1}(\hat{F}).
\]
We will see later that this is always an integer.

**Definition 5.4.** Let \( F : M^3 \cong \mathbb{R}^5 \) be an immersion with trivial normal bundle. Let \( W^4 \) be any compact oriented 4-manifold with \( \partial W^4 = M^3 \) and \( \hat{F} : W^4 \to \mathbb{R}^5 \) a generic map nonsingular near the boundary such that \( \hat{F}^{-1}(\mathbb{R}^5) = \partial W^4 \) and \( \hat{F}|_{\partial W^4} = F \). Then define
\[
i_b(F) = \frac{3}{2}(\sigma(W^4) - \alpha(M^3)) + \frac{1}{2}(3t(\hat{F}) - 3l(\hat{F}) + L_\nu(F)),
\]
where \( t(\hat{F}) \) is the algebraic number of triple points of \( \hat{F} \), \( l(\hat{F}) \) measures the linking of the singularity set of \( \hat{F} \) with the rest of the image \( \hat{F}(W^4) \) (see Definition 3), and \( L_\nu(F) \) is the linking in \( \mathbb{R}^5 \) of the image \( F(M^3) \) and the double point set of \( F \) pushed out of \( F(M^3) \) with
respect to a normal framing $\nu$ for $F$ (see Definition 2.3). It will be shown that $i_b$ also takes only integer values.

**Lemma 5.5.** The map $i_a$ is well-defined. That is, for an immersion $F : M^3 \hookrightarrow \mathbb{R}^5$ with trivial normal bundle, $i_a(F)$ does not depend on the 4-manifold $W^4$ or on the choice of the generic map $\tilde{F} : W^4 \to \mathbb{R}^5$.

**Proof.** Let $\tilde{F} : W \to \mathbb{R}^5$ and $\tilde{F}' : W' \to \mathbb{R}^5$ be generic maps of the compact oriented 4-manifolds $W$ and $W'$ respectively as in Definition 5.3. By Lemma 3 in [13], we obtain that $H^\text{collars}$. Hence — by the relative version of Hirsch’s theorem — there is an immersion $\tilde{W}$ in a neighbourhood of $M^3$ in $W \cup -W'$. Now we obtain

$$\# \Sigma^{1,1}(\tilde{F} \cup -\tilde{F}') + 3\sigma(W \cup -W') = 0$$

by Lemma 3 in [13], which claims that for any generic map $g : X^4 \to \mathbb{R}^5$ of a closed oriented 4-manifold $X^4$, the equality $\# \Sigma^{1,1}(g) + 3\sigma(X^4) = 0$ holds. Thus the result follows.

Suppose that the normal vector fields along $F(M^3)$ in $\tilde{F}(W)$ and in $\tilde{F}'(W')$ are homotopic. Then after a suitable deformation near $F(M^3)$, we may regard $\tilde{F}(W) \cup \tilde{F}'(W')$ as the image of a generic map of the closed 4-manifold $W \cup -W'$ which is an immersion on a neighbourhood of $M^3$ in $W \cup -W'$. Now we may assume that both normal vector fields are of equal constant length. Let $M_+$ and $M_+'$ be formed by the endpoints of vectors of these two normal vector fields. Then we may assume that $M_+$ and $M_+'$ intersect each other along a surface $S_+$ which is the image of a surface $S \subset M^3$ by the map associating to $x \in M^3$ the endpoint of the normal vector at $F(x)$. Thus, $\tilde{F}(W) \cup \tilde{F}'(W')$ can be deformed, by a small modification near $F(M^3)$, into the image of a smooth generic map which has only Whitney umbrella singular points on $S$ and no other singularities nearby. Now we can apply Lemma 3 in [13] again. This completes the proof. \qed

**Theorem 5.6.** Let $F$ and $G : M^3 \hookrightarrow \mathbb{R}^5$ be two immersions with trivial normal bundles such that $c(F) = c(G)$. Then $F$ and $G$ are regularly homotopic if and only if $i_a(F) = i_a(G)$.

**Proof.** Let us first prove that $i_a(F) = i_a(G)$ if $F$ and $G$ are regularly homotopic.

Suppose that $F$ and $G$ are regularly homotopic. Let $H : M^3 \times [0, 1] \hookrightarrow \mathbb{R}^5 \times [0, 1]$ be the track of a generic regular homotopy $H_t : M^3 \hookrightarrow \mathbb{R}^5$ between $F$ and $G$. Then we can take a normal vector field $\nu_t$ of $H_t$, which determines normal vector fields $\nu_F$ and $\nu_G$ of $F : M^3 \hookrightarrow \mathbb{R}^5$ and $G : M^3 \hookrightarrow \mathbb{R}^5$ for $t = 0$ and $t = 1$ respectively.

(The argument below follows that of Wells in [13, p. 288].) There is a small positive $\varepsilon$ such that on the collars $M^3 \times (0, \varepsilon)$ and $M^3 \times (1 - \varepsilon, 1)$ the maps $(x, t) \mapsto F(x) + t \cdot \nu_F(x)$ and $(x, t) \mapsto G(x) + (t - 1) \cdot \nu_G(x)$, respectively, are immersions. Let us denote these immersions by $h_{(0, \varepsilon)}$ and $h_{(1-\varepsilon, 1)}$ respectively.

The tangent bundle of the cylinder $M^3 \times I$ is $TM^3 \times I = \pi^*(TM^3) \oplus \varepsilon^1$, where $I = [0, 1]$, $\pi : M^3 \times I \to M^3$ is the projection, and $\varepsilon^1$ is the trivial line bundle. We may assume that the regular homotopy $H_t$ is such that $H_t = F$ for $t < 2\varepsilon$ and $H_t = G$ for $t > 1 - 2\varepsilon$. Now we define the bundle monomorphism $\Phi : TM^3 \times I \to \mathbb{R}^5$ as $dH_t \oplus \nu_t$, i.e. for a tangent vector $(\pi^*(v), s) \in T(x, t)M^3 \times I$, where $v \in TM^3$ and $s \in \varepsilon^1$, put $\Phi(\pi^*(v), s) = dH_t(v) + s \cdot \nu_t$.

This map is monomorphic on each fibre indeed, since $\nu_t$ is normal to $dH_t(TM^3)$. Furthermore, it coincides with the differentials of the immersions $h_{(0, \varepsilon)}$ and $h_{(1-\varepsilon, 1)}$ over the collars. Hence — by the relative version of Hirsch’s theorem — there is an immersion $H' : M^3 \times I \hookrightarrow \mathbb{R}^5$ which coincides with the given immersions on the collars.

Let $F : W_F \to \mathbb{R}^5$ and $G : W_G \to \mathbb{R}^5$ be generic maps as in Definition 5.3. Now via the smoothing process as in the proof of Lemma 5.3, we obtain a generic map $F \cup H' \cup G : W_F \cup (M^3 \times [0, 1]) \cup -W_G \to \mathbb{R}^5$ of the closed oriented 4-manifold $W_F \cup (M^3 \times [0, 1]) \cup -W_G$. By Lemma 3 in [13] we obtain that $i_a(F) = i_a(G)$.

Now let us show that $i_a$ is injective on the set $\text{Imm}(M^3, \mathbb{R}^5)_{i_0}$ of regular homotopy classes of immersions having the same Wu invariant $C$. Notice that the analogous map $i_a^S :$
Imm[$S^3, R^5] \to Z$ has been shown to be an isomorphism in $[4]$ (see Theorem 2.1 in the present paper).

Let us consider the effective and transitive group action described in the previous section restricted to just one orbit:

\[ \text{Imm}[M^3, R^5]_0 \times \text{Imm}[S^3, R^5] \to \text{Imm}[M^3, R^5]_0. \]

Choose an embedding $F_0$ in $\text{Imm}[M^3, R^5]_0$ (by Theorem 3.8, such an embedding exists). Let us consider the map

\[ \text{imm}[S^3, R^5] \to \text{Imm}[M^3, R^5]_0, \]

\[ f \mapsto F_0 \# f. \]

This map is bijective by Proposition 1.1. Compose it with the map

\[ i_a - i_a(F_0) : \text{imm}[M^3, R^5]_0 \to Q. \]

The resulting composition $\text{Imm}[S^3, R^5] \to \text{Imm}[M^3, R^5]_0 \to Q$ coincides with the map $i_a$. Indeed, for any embedded Seifert surface $W_{F_0}$ for $F_0$ we have

\[ i_a(F_0) = \frac{3}{2}(\sigma(W_{F_0}) - \alpha(M^4)), \]

and for a singular Seifert surface of $F_0 \# f$ one can take the boundary connected sum of the Seifert surface $W_{F_0}$ for $F_0$ with any singular Seifert surface for $f$.

Hence the map $i_a$ is injective. 

\[ \text{Lemma 5.7.} \text{ For an immersion } F : M^3 \hookrightarrow R^5 \text{ with trivial normal bundle and a normal framing } \nu, i_b(F) \text{ does not depend on the 4-manifold } W^4 \text{ or on the choice of the generic map } F : W^4 \to R^6. \]

\[ \text{Proof.} \text{ Let } \hat{F} : W \to R^6 \text{ and } \hat{F}' : W' \to R^6 \text{ be generic maps of the compact oriented 4-manifolds } W \text{ and } W' \text{ respectively as in Definition 5.4. Then, we obtain a generic map } \hat{F} \cup \hat{F}' : W \cup -W' \to R^6. \text{ From the fact that for any generic map } g : X^4 \to R^6 \text{ of a closed oriented 4-manifold } X^4, \text{ the equality } \sigma(X^4) - l(g) + t(g) = 0 \text{ holds (see [4, Lemma 4]), the result follows.} \]

The following theorem states that the invariant $i_b$ is well-defined, and it is a regular homotopy invariant. In particular it implies that $i_b(F)$ does not depend on the choice of the normal framing $\nu$ for $F$, either.

\[ \text{Theorem 5.8.} \text{ Let } F \text{ and } G : M^3 \hookrightarrow R^5 \text{ be two immersions with trivial normal bundles such that } c(F) = c(G). \text{ Then } F \text{ and } G \text{ are regularly homotopic if and only if } i_b(F) = i_b(G). \]

\[ \text{Proof.} \text{ Let } F : M^3 \hookrightarrow R^5 \text{ be an immersion with trivial normal bundle. By Theorem 3.8, there exists an embedding } F_0 : M^3 \hookrightarrow R^5 \text{ with } c(F_0) = c(F). \text{ Furthermore, by Proposition 1.1, there exists an immersion } f : S^3 \hookrightarrow R^5 \text{ — unique up to regular homotopy — such that } F \sim_r F_0 \# f. \text{ Let } H : M^3 \times I \hookrightarrow R^5 \times I \text{ be the track of a generic regular homotopy between } F \text{ and } F_0 \# f. \text{ Let } t(H) \text{ be the algebraic number of triple points of } H. \]

Take a compact 4-manifold $W_F$ and a generic map $\hat{F} : W_F \to R^6$ as in Definition 5.4 for $F$, and let $\hat{F}_0 : W_{F_0} \to R^5$ be a Seifert surface for $F_0$. Then, we can deform $\hat{F}(W_F) \cup H(M^3 \times I) \cup \hat{F}_0(W_{F_0})$ into the image of a generic map of the manifold $W_F \cup (M^3 \times I) \cup -W_{F_0}$ into $R^5 \times I \subset R^6$ bounded by $f : S^3 \hookrightarrow R^5$ (see Figure ??). Therefore, by Theorem 2.4, we have

\[ \Omega(f) = \frac{1}{2}(3\sigma(W_F \cup (M^3 \times I) \cup -W_{F_0}) - 3t(H) + 3t(\hat{F}) - 3l(\hat{F}) + L(f)) \]

\[ = \frac{3}{2}(\sigma(W_F) - \sigma(W_{F_0}) + t(\hat{F}) - l(\hat{F})) + \frac{1}{2}(L(f) - 3t(H)). \]

Here we need the following lemma.
Lemma 5.9. We have
\[ L(f) = L_\nu(F) + 3t(H) \]
for any normal framing \( \nu \) for \( F \).

Proof. We can extend \( \nu \) to the whole cylinder \( M^3 \times I \) using the regular homotopy \( H \). We shall denote this extended normal field also by \( \nu \). Since on \( D^3 \times \{0\} \subset M^3 \times \{0\} \), the normal field \( \nu \) coincides with the homotopically unique normal framing of \( (F_0 \sharp f)|_{D^3} \), we have
\[ L_\nu(F_0 \sharp f) = L(f). \]
(Indeed, recall that for an immersion the invariant \( L_\nu \) was defined as the linking number of the image of the immersion with the curves formed by the double points of the immersion pushed off out of the image of the immersion into the direction defined by \( \nu \). Since \( F_0 \) is an embedding the equality follows.)

Furthermore, one can show that
\[ L_\nu(F_0 \sharp f) - L_\nu(F) = 3t(H) \]
in the same way as in [4]. This completes the proof of Lemma 5.9. \( \square \)
Thus, \( L_\nu(F) \) does not depend on the choice of \( \nu \) and we obtain
\[
\Omega(f) = \frac{3}{2}(\sigma(W_F) - \sigma(W_{F_0}) + t(\hat{F}) - l(\hat{F})) + \frac{1}{2}L_\nu(f)
\]
\[
= \frac{1}{2}(3\sigma(W_F) + 3t(\hat{F}) - 3l(\hat{F}) + L_\nu(F)) - \frac{3}{2}\sigma(W_{F_0}).
\]
Since the regular homotopy class of \( F|M^3 \) is determined by \( c(F) \), again by Proposition 1.1, the regular homotopy class of \( F \) is completely determined by \( \Omega(f) \), hence by \( i_b(F) = \Omega(f) + 3(\sigma(W_{F_0}) - \alpha(M^3))/2 \). This completes the proof of Theorem 5.8. 

\[ \square \]

Remark 5.10. Theorem 5.8 implies that \( i_b(F) \) does not depend on the choice of \( \hat{F} \) or on the normal framing \( \nu \) for \( F \). Hence \( L_\nu(F) \) does not depend on the choice of the normal framing \( \nu \) for \( F \), either. Thus, we can use the notation \( L(F) \) for an immersion \( F \) with trivial normal bundle of a closed oriented 3-manifold from now on; then, the formula displayed in Definition 5.4 can be written as
\[
i_b(F) = \frac{3}{2}(\sigma(W^4) - \alpha(M^3)) + \frac{1}{2}(3t(\hat{F}) - 3l(\hat{F}) + L(F)).
\]

Remark 5.11. We see easily that for an embedding \( F_0 : M^3 \hookrightarrow \mathbb{R}^5 \) we have \( i_a(F_0) = i_b(F_0) \); furthermore, this is always an integer by Lemma 5.2. Therefore, by Theorems 2.1, 5.8, and 2.4, \( i_a \) and \( i_b \) take integer values for general immersions (with trivial normal bundles). Furthermore, considering the action of \( \text{Imm}[S^3, \mathbb{R}^5] \) on \( \text{Imm}[M^3, \mathbb{R}^5]_0 \), we see that in fact \( i_a = i_b \) (using Theorems 2.1 and 2.4). Thus, we obtain the surjective map
\[
i : \text{Imm}[M^3, \mathbb{R}^5]_0 \longrightarrow \mathbb{Z}
\]
defined in two ways as \( i_a \) and as \( i_b \), so that the map
\[
(c, i) : \text{Imm}[M^3, \mathbb{R}^5]_0 \longrightarrow \Gamma_2(M^3) \times \mathbb{Z}
\]
gives a bijection.

Corollary 5.12. Let \( M^3 \) be a closed oriented 3-manifold such that \( H^2(M^3; \mathbb{Z}) \) has no 2-torsion. Then two embeddings \( M^3 \hookrightarrow \mathbb{R}^5 \) are regularly homotopic if and only if they have Seifert surfaces with the same signature.

Proof. Since \( H^2(M^3; \mathbb{Z}) \) has no 2-torsion, \( c(F) \) is equal to 0 \( \in H^2(M^3; \mathbb{Z}) \) for every embedding \( F : M^3 \hookrightarrow \mathbb{R}^5 \). If we take a Seifert surface \( W^4 \hookrightarrow \mathbb{R}^5 \) for an embedding \( F \), then clearly \( i_a(F) = i_b(F) = 3(\sigma(W^4) - \alpha(M^3))/2 \). Thus the result follows directly from Theorems 5.6 and 5.8. 

\[ \square \]

Remark 5.13. As a consequence of Theorems 5.6 and 5.8, we have that if two embeddings \( F \) and \( G : M^3 \hookrightarrow \mathbb{R}^5 \) of an arbitrary closed oriented 3-manifold \( M^3 \) into \( \mathbb{R}^5 \) are regularly homotopic, then their Seifert surfaces have the same signature. In fact, we can also prove this fact by using a simpler argument as follows.

Let \( \hat{F} : W_F \hookrightarrow \mathbb{R}^5 \) and \( \hat{G} : W_G \hookrightarrow \mathbb{R}^5 \) be Seifert surfaces for \( F \) and \( G \) respectively. Let \( H : M^3 \times I \hookrightarrow \mathbb{R}^5 \times I \) be the track of a generic regular homotopy between \( F \) and \( G \). Then, using \( \hat{F}, \hat{G}, \) and \( H \), we obtain a generic immersion \( h : W_F \cup (M^3 \times I) \cup -W_G \hookrightarrow \mathbb{R}^5 \) after an appropriate smoothing process. Put \( X^4 := W_F \cup (M^3 \times I) \cup -W_G \). Let \( \chi \in H^2(X^4; \mathbb{Z}) \) be the normal Euler class of the immersion \( h \) and \( e \in H_2(X^4; \mathbb{Z}) \) its Poincaré dual. Then, \(-p_1(X^4) = \chi \searrow \chi \) and \(-e \) is represented by the set of double points \( \Delta := \{ x \in X^4 \mid h(x) = h(y) \text{ for some } y \neq x \} \) of \( h \) (see, p. 44, Lemmas 1 and 4). Since \( \Delta \subset M^3 \times I \subset X^4 \), we have
Thus, we have \( \sigma(X^4) = -p_1(X^4)/3 = 0 \) and hence \( \sigma(W_F) = \sigma(W_G) \).

6. Embeddings of \( T^3 \) into \( \mathbb{R}^5 \)

In this section, we prove the two surprising corollaries formulated in the introduction. Recall that these corollaries claim roughly that for the 3-dimensional torus \( T^3 \) “more regular homotopy classes of immersions into \( \mathbb{R}^5 \) contain embeddings than for the 3-sphere”.

(The proofs will show that there are twice as many regular homotopy classes containing embeddings for the torus as for the sphere.)

Since \( \Gamma_2(T^3) = 0 \), the set \( \text{Imm}(T^3, \mathbb{R}^5)_0 \) can be identified with the set of integers. Therefore, by Proposition 4.1, we see that

\[
\sharp_{F_0} : \text{Imm}[S^3, \mathbb{R}^5] \to \text{Imm}[T^3, \mathbb{R}^5]_0
\]

is a bijection for any immersion \( F_0 : T^3 \hookrightarrow \mathbb{R}^5 \) with trivial normal bundle. Furthermore, \( i : \text{Imm}(T^3, \mathbb{R}^5)_0 \to \mathbb{Z} \) in Section 3 gives a complete invariant of regular homotopy.

Obviously there exists an embedding \( F_0 : T^3 \hookrightarrow \mathbb{R}^5 \) which bounds a 4-manifold of signature 0 (e.g. \( D^2 \times S^1 \times S^1 \)) in \( \mathbb{R}^5 \). Furthermore, we show the following.

**Proposition 6.1.** (a) There exists an embedding \( F_8 : T^3 \hookrightarrow \mathbb{R}^5 \) having a Seifert surface \( W_{F_8} \hookrightarrow \mathbb{R}^5 \) with signature 8.

(b) The signature of any Seifert surface of any embedding \( T^3 \hookrightarrow \mathbb{R}^5 \) is divisible by 8.

**Proof.** (a) It is known that there exists a spin 4-manifold \( W = W_{F_8} \) of signature 8 with \( \partial W = T^3 \), see [8] for example. Furthermore, \( W \) can be chosen to have a special handlebody decomposition with one 0-handle and some 2-handles with even framings (see [8]). As we have seen in the proof of Theorem 3.8, such a 4-manifold \( W \) embeds into \( \mathbb{R}^5 \). Such an embedding, restricted to the boundary \( \partial W \), gives a required embedding \( F_8 : T^3 \hookrightarrow \mathbb{R}^5 \).

(b) It is shown also in [2] that \( T^3 \) with any spin structure spin-bounds either the solid torus \( D^2 \times S^1 \times S^1 \) or the above mentioned \( W_{F_8} \). Now let \( V \) be any Seifert surface of any embedding \( T^3 \hookrightarrow \mathbb{R}^5 \). Then \( V \) induces a spin structure on \( T^3 \), which is induced also from a spin structure either on the solid torus or on \( W_{F_8} \).

Therefore, either \( V \cup (D^2 \times S^1 \times S^1) \) or \( V \cup W_{F_8} \) carries a spin structure, and so by Rohlin’s theorem its signature is divisible by 16. By Novikov’s additivity these signatures are \( \sigma(V) - \sigma(D^2 \times S^1 \times S^1) = \sigma(V) \) and \( \sigma(V) - \sigma(W_{F_8}) = \sigma(V) - 8 \) respectively. In both cases \( \sigma(V) \) is divisible by 8.

**Corollary 6.2.** There exist two embeddings \( F_0 \) and \( F_8 : T^3 \hookrightarrow \mathbb{R}^5 \) such that there is an immersion \( h : S^3 \supseteq \mathbb{R}^5 \) with \( F_0 \sharp h \sim_{r} F_8 \), but \( h \) cannot be chosen from a regular homotopy class containing an embedding.

**Proof.** Let \( F_0 \) and \( F_8 \) be the embeddings described above in Proposition 6.1, i.e. they have Seifert surfaces \( W_{F_0} \) and \( W_{F_8} \) of signatures \( \sigma(W_{F_0}) = 0 \) and \( \sigma(W_{F_8}) = 8 \) respectively.

From the fact that

\[
\sharp_{F_0} : \text{Imm}[S^3, \mathbb{R}^5] \to \text{Imm}[T^3, \mathbb{R}^5]_0
\]

is a bijection, there is an immersion \( h : S^3 \supseteq \mathbb{R}^5 \) such that \( F_0 \sharp h \sim_{r} F_8 \). Let us assume that \( h \) is regularly homotopic to an embedding. Then the following sequence of equalities hold:

\[
12 = \frac{3}{2} \sigma(W_{F_0}) = i(F_8) = i(F_0 \sharp h) = i(F_0) + \Omega(h) = 0 + 24k
\]
for an integer \( k \in \mathbb{Z} \), which is a contradiction. Here the fourth equality \( i(F_0 \# h) = i(F_0) + \Omega(h) \) follows from the obvious remark that for a Seifert surface of the connected sum \( F_0 \# h \) one can choose the boundary connected sum of the Seifert surfaces of the embeddings \( F_0 \) and \( h \). The Smale invariant \( \Omega(h) \) is a multiple of 24 by the result of Hughes and Melvin \( \square \) recalled in Section \( \square \).

**Corollary 6.3.** There exists an immersion \( h : S^3 \hookrightarrow \mathbb{R}^5 \) not regularly homotopic to any embedding \( S^3 \hookrightarrow \mathbb{R}^5 \) such that for any embedding \( E : T^3 \hookrightarrow \mathbb{R}^5 \), the connected sum \( E \# h \) of \( F \) and \( h \) is again regularly homotopic to an embedding \( T^3 \hookrightarrow \mathbb{R}^5 \).

**Proof.** Let \( h : S^3 \hookrightarrow \mathbb{R}^5 \) be any immersion with Smale invariant \( \Omega(h) = 12 \). Then \( h \) is not regularly homotopic to an embedding again by the result of Hughes-Melvin \( \square \). We show that for any embedding \( E : T^3 \hookrightarrow \mathbb{R}^5 \) the connected sum \( E \# h \) is regularly homotopic to an embedding. For this purpose we first compute the \( i \)-invariant of \( E \# h \) and then produce an embedding with the same \( i \)-invariant. Since the \( i \)-invariant determines the regular homotopy class completely, we obtain that \( E \# h \) is regularly homotopic to the embedding.

The following sequence of equalities hold:

\[
i(E \# h) = i(E) + \Omega(h) = \frac{3}{2} \sigma(W_E) + 12 = \frac{3}{2} \cdot 8k + 12 = 12(k + 1).
\]

Here the first equality holds, since we can form the boundary connected sum of the Seifert surface \( W_E \) of \( E \) with that of \( h \) even if the latter is a “singular” one (i.e. it is a generic map with boundary \( h \), not necessarily an embedding). The number \( k = \sigma(W_E)/8 \) is an integer by part (b) of Proposition \( \square \). Let \( e_n : S^3 \hookrightarrow \mathbb{R}^5 \) be an embedding with Smale invariant \( \Omega(e_n) = 24n \), where \( n = k/2 \) if \( k \) is even and \( n = (k + 1)/2 \) if \( k \) is odd.

Now we see that if \( k \) is even, then \( i(F_0 \# e_n) = i(E \# h) \). If \( k \) is odd, then \( i(F_0 \# e_n) = i(E \# h) \). \( \square \)

### 7. Number of cusps of singular Seifert surfaces

The present section is not along the main line of the paper. Here as a byproduct of our formulae for \( i_a \) and \( i_b \), their correctness, integrality and coincidence, we prove a few simple corollaries on the number of cusps a singular Seifert surface can have.

**Corollary 7.1.** In the notation of Definition \( \square \), the residue class of the algebraic number \( \# \Sigma^{1,1}(\bar{F}) \) of cusps of \( \bar{F} \) modulo 3 does not depend on the 4-manifold \( W^4 \) or on the choice of the map \( F \), and it depends only on the immersion \( F \) of the 3-manifold \( M^3 \) into \( \mathbb{R}^5 \). \( \square \)

Naturally arises the problem: how to read off this modulo 3 residue class from \( F \)? It turns out that \( \# \Sigma^{1,1}(\bar{F}) \equiv L_\nu(F) \) (mod 3) (see Definition \( \square \)). This follows from the definitions of the invariants \( i_a \) and \( i_b \) (Definitions \( \square \) and \( \square \)) and their coincidence (see Remark \( \square \)).

**Corollary 7.2.** Let \( W^4 \) be a closed connected spin 4-manifold, and \( f : W^4 \rightarrow \mathbb{R}^5 \) a generic map. It is well-known that the singularity set is a surface \( \Sigma(f) \) with isolated cusps on it. Then the number of cusps on each component of \( \Sigma(f) \) is even.

**Proof.** Otherwise, restricting \( f \) to the closure of an appropriate neighbourhood of a component of \( \Sigma(f) \), we would get a contradiction to the integrality of the invariant \( i_a \) and Lemma \( \square \).

**Corollary 7.3.** In the conditions of the previous corollary, let \( M^3 \) be a null-homologous oriented submanifold of \( W^4 \) disjoint from (the closure of) the double point set of \( f \). Then in both parts of the set \( W \smallsetminus M \) the algebraic number of cusps is divisible by 6. \( \square \)
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