A BASIS OF THE BASIC $\mathfrak{sl}(3,\mathbb{C})^\sim$-MODULE

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Abstract. J. Lepowsky and R. L. Wilson initiated the approach to combinatorial Rogers-Ramanujan type identities via the vertex operator constructions of representations of affine Lie algebras. In this approach the first new combinatorial identities were discovered by S. Capparelli through the construction of the level 3 standard $A_2^{(2)}$-modules. We obtained several infinite series of new combinatorial identities through the construction of all standard $A_1^{(1)}$-modules; the identities associated to the fundamental modules coincide with the two Capparelli identities. In this paper we extend our construction to the basic $A_1^{(1)}$-module and, by using the principal specialization of the Weyl-Kac character formula, we obtain a Rogers-Ramanujan type combinatorial identity for colored partitions. The new combinatorial identity indicates the next level of complexity which one should expect in Lepowsky-Wilson’s approach for affine Lie algebras of higher ranks, say for $A_n^{(1)}$, $n \geq 2$, in a way parallel to the next level of complexity seen when passing from the Rogers-Ramanujan identities (for modulus 5) to the Gordon identities for odd moduli $\geq 7$.

Introduction

J. Lepowsky and R. L. Wilson gave in [LW] a Lie-theoretic interpretation and proof of the classical Rogers-Ramanujan identities in terms of representations of the affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{C})^\sim$. The identities are obtained by expressing in two ways the principal characters of vacuum spaces for the principal Heisenberg subalgebra of $\hat{\mathfrak{g}}$. The product sides follow from the principally specialized Weyl-Kac character formula; the sum sides follow from the vertex operator construction of bases parametrized by partitions satisfying difference 2 conditions. Very roughly speaking, for a level 3 standard $\hat{\mathfrak{g}}$-module $L$ with a highest weight vector $v_0$, Lepowsky and Wilson construct $\mathbb{Z}$-operators $\{Z(j) \mid j \in \mathbb{Z}\}$ which commute with the action of the principal Heisenberg subalgebra, and show that

$$Z(j_1)Z(j_2)\cdots Z(j_s)v_0, \quad j_1 \leq j_2 \leq \cdots \leq j_s < 0$$

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is a spanning set of the vacuum space $\Omega_L$ of the principal Heisenberg subalgebra. This spanning set is reduced to a basis by using the vertex operator formula

$$\sum_{i \geq 0} a_i (Z(m-i)Z(n+i) + Z(n-i)Z(m+i))$$

or, equivalently, which satisfy the difference 2 conditions $Z(j)Z(j)$, $Z(j-1)Z(j)$, $j < 0$.

Finally, Lepowsky and Wilson prove the linear independence of this set of vectors, and this gives a Lie-theoretic proof of the Rogers-Ramanujan identities.

Lepowsky-Wilson’s approach is also possible for other affine Lie algebras and for other constructions of vertex operators, as in [C], [LP], [Ma] and [Mi], for example. In this approach the first new combinatorial identities were discovered by S. Capparelli through the construction of level 3 standard $A_2^{(2)}$-modules in the principal picture. We obtained in [MP2] several infinite series of new combinatorial identities through the homogeneous construction of all standard $A_1^{(1)}$-modules; for the fundamental modules and the $(1,2)$-specialization the identities coincide with the two Capparelli identities. In this paper we follow the ideas developed in [MP1] and [MP2] and construct a basis of the basic $A_2^{(1)}$-module parametrized by colored partitions.

In order to describe our main result, let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and let $e_i$, $h_i$, $f_i$ be the Chevalley generators of $\mathfrak{g}$. Set $\mathfrak{h} = \mathbb{C}$-span $\{h_1, h_2\}$ and let $B$ be the ordered basis

$$X_1 = [e_1, e_2], X_2 = e_1, X_3 = e_2, X_4 = h_1, X_5 = h_2, X_6 = f_2, X_7 = f_1, X_8 = [f_2, f_1],$$

$$X_1 \succ X_2 \succ X_3 \succ X_4 \succ X_5 \succ X_6 \succ X_7 \succ X_8.$$ 

Let $\tilde{\mathfrak{g}}$ be the affine Lie algebra associated with $\mathfrak{g}$ (cf. [K]), spanned by elements $x(n), x \in \mathfrak{g}, n \in \mathbb{Z}$, the canonical central element $c$ and a derivation $\tilde{d}$. Then we have a Poincaré-Birkhoff-Witt spanning set

$$X_{i_1}(j_1)X_{i_2}(j_2)\ldots X_{i_s}(j_s)v_0, \quad j_1 \leq j_2 \leq \cdots \leq j_s < 0, \quad X_{i_r} \preceq X_{i_{r+1}} \text{ if } j_r = j_{r+1},$$

of the basic $\tilde{\mathfrak{g}}$-module $L(\Lambda_0)$ with a highest weight vector $v_0$. We may refer to monomials $X_{i_1}(j_1)X_{i_2}(j_2)\ldots X_{i_s}(j_s)$ of the above form as ordered monomials in the universal enveloping algebra $U(\tilde{\mathfrak{g}})$.

The above Poincaré-Birkhoff-Witt spanning set can be reduced to a basis in the following way: Let $R$ be the set of ordered monomials of the form

$$X_{i_1}(j_1)X_{i_2}(j), \text{ with "colors" } i_1i_2 : 11, 21, 22, 31, 32, 33, 41, 42, 43, 44, 51, 52, 53, 54, 55,$$

$$62, 64, 65, 66, 73, 74, 75, 76, 77, 85, 86, 87, 88,$$

$$X_{i_1}(j-1)X_{i_2}(j), \text{ with "colors" } i_1i_2 : 11, 12, 13, 14, 15, 16, 17, 18, 22, 24, 26, 27, 28,$$

$$33, 35, 36, 37, 38, 47, 48, 56, 58, 66, 68, 77, 78, 88,$$

$$X_3(j-1)X_4(j)X_1(j), \quad X_8(j-1)X_4(j-1)X_6(j); \text{ for all } j < 0.$$
We shall say that an ordered monomial \( u = X_{i_1}(j_1)X_{i_2}(j_2) \ldots X_{i_s}(j_s) \) satisfies the difference \( \mathcal{R} \) conditions if for any given \( d \in \mathcal{R} \) the monomial \( u \) does not contain as factors all the simple factors of \( d \), or, by abuse of language, if \( u \) does not contain \( d \) as a factor. For example, \( u = X_1(-2)X_5(-1)X_3(-1) \) does not satisfy the difference \( \mathcal{R} \) conditions since it contains both of the simple factors of \( d = X_1(-2)X_3(-1) \in \mathcal{R} \). On the other hand, \( X_1(-6)X_5(-3)X_3(-1) \) obviously satisfies the difference \( \mathcal{R} \) conditions since the “differences” between the “parts” \( X_1(-6), X_5(-3) \) and \( X_3(-1) \) are “big enough”, and the monomial \( X_1(-6)X_5(-3)X_3(-1) \) cannot contain any \( d \) from \( \mathcal{R} \).

Now we can state our main result:

**Theorem A.** The set of vectors

\[
X_{i_1}(j_1)X_{i_2}(j_2) \ldots X_{i_s}(j_s)v_0 \in L(\Lambda_0),
\]

where the monomials \( X_{i_1}(j_1)X_{i_2}(j_2) \ldots X_{i_s}(j_s) \) are ordered and satisfy the difference \( \mathcal{R} \) conditions, is a basis of the basic \( \mathfrak{g} \)-module \( L(\Lambda_0) \).

By analogy with the Rogers-Ramanujan case, we start with the vertex operator formula \( X_1(z)^2L(\Lambda_0) = 0 \) (cf. [LP]), and extract the coefficients of the powers of \( z \):

\[
\sum_{i+j=n} X_1(i)X_1(j) = 0 \quad \text{on} \quad L(\Lambda_0).
\]

Using the adjoint action of \( \mathfrak{g} \) on \( X_1(z)^2 \) we get a 27-dimensional space of relations which annihilate \( L(\Lambda_0) \). The set of quadratic monomials listed above is the set of leading terms of the components of these relations (Lemma 1). We also construct two relations with the leading terms \( X_3(j - 1)X_4(j)X_1(j) \) and \( X_8(j - 1)X_4(j - 1)X_6(j) \) (Lemma 8). By using these relations we can reduce the Poincaré-Birkhoff-Witt spanning set to a basis, the proof is given at the end of the next section.

The proof of linear independence is based on the same ideas that we used in the \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \) case, the main result being Theorem 11 which is analogous to Theorem 9.1 in [MP2]. The main ingredient of the proof is a construction of \( (64 + 35 + 35 + 27) \)-dimensional space of relations among relations (Proposition 3), which is obtained from the vertex operator algebra structure on \( L(\Lambda_0) \) (cf. [FLM] and [MP2, Section 8]). These relations among relations are first reformulated in terms of embeddings (Proposition 10, in a way the analogue of Lemma 9.2 in [MP2]), the general statement of Theorem 11 is obtained by using Lemma 9.4 in [MP2]. In the next section we introduce all necessary notions and facts needed in the proof, except that we for a few details refer to [MP2].

As a consequence of Theorem A, and the principally specialized Weyl-Kac character formula, we obtain a combinatorial identity for colored partitions of Rogers-Ramanujan type (Theorem B).

The new combinatorial identity indicates the next level of complexity which one should expect in Lepowsky-Wilson’s approach for affine Lie algebras of higher ranks, say for \( A_1^{(1)}, n \geq 2 \), in a way parallel to the next level of complexity seen when passing from the Rogers-Ramanujan identities (for modulus 5) to the Gordon identities for odd moduli \( \geq 7 \). This is due to the appearance of two additional cubic terms in the set \( \mathcal{R} \) of “mainly difference 2 conditions” — a new combinatorial phenomenon not seen in the \( A_1^{(1)} \) case. These cubic terms make the proof of linear independence
“as complicated as” the proof in the case of level 2 standard $A_1^{(1)}$-modules, suggesting, at least as far as the complexity of partitions is concerned, some kind of duality between “level 1 rank 2 identities” and “level 2 rank 1 identities”.

It is clear that Lepowsky-Wilson’s approach for higher rank affine Lie algebras should lead to combinatorial Rogers-Ramanujan type identities for colored partitions. It should be noted that similar combinatorial identities also appear as a consequence of the classical $q$-series approach, see, for example, [AAG] and the references therein.

A BASIS OF THE BASIC MODULE

As was already stated, let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{F})$ and let $e_i, h_i, f_i$ be the Chevalley generators of $\mathfrak{g}$ and $\mathfrak{h} = \mathbb{F}\text{-span}\{h_1, h_2\}$, where $\mathbb{F} = \mathbb{C}$ or any other field of characteristic 0. Let $B$ be the ordered basis

$$X_1 = [e_1, e_2], X_2 = e_1, X_3 = e_2, X_4 = h_1, X_5 = h_2, X_6 = f_2, X_7 = f_1, X_8 = [f_2, f_1],$$

$$X_1 \succ X_2 \succ X_3 \succ X_4 \succ X_5 \succ X_6 \succ X_7 \succ X_8.$$ 

Let $\tilde{\mathfrak{g}}$ be the affine Lie algebra associated with $\mathfrak{g}$ spanned by the elements $x(n)$, $x \in \mathfrak{g}$, $n \in \mathbb{Z}$, the canonical central element $c$ and a derivation $d$. Set

$$\tilde{B} = \{b(n) \mid b \in B, n \in \mathbb{Z}\},$$

$$\tilde{B}_- = \tilde{B}_{\leq 0} \cup \{X_i(0) \mid i = 6, 7, 8\},$$

$$\tilde{B}_{< 0} = \{b(n) \mid b \in B, n \in \mathbb{Z}_{< 0}\},$$

so that $\tilde{B}, \tilde{B}_-, \tilde{B}_{\leq 0}$ parametrize bases of the Lie algebras $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathbb{F}c$, $\tilde{\mathfrak{n}}_-$ and $\tilde{\mathfrak{g}}_{< 0}$ (respectively). We choose the order $\preceq$ on $\tilde{B}$ defined by

$$b_1(j_1) \prec b_2(j_2) \iff j_1 < j_2 \text{ or } j_1 = j_2, \ b_1 < b_2.$$ 

The set of colored partitions $\mathcal{P}(\tilde{B})$ is defined as the set of all maps $\pi : \tilde{B} \to \mathbb{N}$, where $\pi(a)$ equals zero for all but finitely many $a \in \tilde{B}$. Clearly $\pi$ is determined by its values $(\pi(a) \mid a \in \tilde{B})$ and we shall write $\pi$ as the monomial $\pi = \prod_{a \in \tilde{B}} a^{\pi(a)}$. We may also think of the colored partition $\pi = \prod b_i(j_i)$ as

$$\pi = (b_1(j_1), \ldots, b_s(j_s)), \ b_1(j_1) \preceq \cdots \preceq b_s(j_s),$$

where $b_i(j_i) \in \tilde{B}$ are called the parts of $\pi$, $|\pi| = j_1 + \cdots + j_s$ the degree of $\pi$ and $\ell(\pi) = s \geq 0$ the length of $\pi$. Since the basis elements $X_i$ are weight vectors for the adjoint action of $\mathfrak{g}$, we define the $\mathfrak{h}$-weight $\text{wt}(\pi)$ as the sum of the $\mathfrak{h}$-weights of the $b_i$’s. We shall think of $\pi$ as the “plain” partition $\text{sh} \pi$ of the form $j_1 \leq j_2 \leq \cdots \leq j_s$ “colored” with “colors” $b_1, b_2, \ldots, b_s$ from the set of colors $B = \{X_1, \ldots, X_8\}$. Sometimes we shall shortly say that $X_{i_1}(j_1)X_{i_2}(j_2)$ is colored with colors $i_1i_2$.

Let $\pi, \pi' \in \mathcal{P}(\tilde{B})$, $\pi = (a_1, \ldots, a_s)$, $\pi' = (a_1', \ldots, a_s')$. We extend the order $\preceq$ on $\tilde{B} \subset \mathcal{P}(\tilde{B})$ to the order on $\mathcal{P}(\tilde{B})$ defined by

$$\pi \prec \pi'$$

if $\pi \neq \pi'$ and one of the following statements holds:

(i) $\ell(\pi) > \ell(\pi'),$
(ii) \( \ell(\pi) = \ell(\pi'), |\pi| < |\pi'| \),
(iii) \( \ell(\pi) = \ell(\pi'), |\pi| = |\pi'| \) and there is \( i, \ell(\pi) \geq i \geq 1 \), such that \( |a_j| = |a'_j| \) for \( \ell(\pi) \geq j > i \) and \( |a_i| < |a'_i| \),
(iv) \( \text{sh} \pi = \text{sh} \pi' \) and there is \( i, \ell(\pi) \geq i \geq 1 \), such that \( a_j = a'_j \) for \( \ell(\pi) \geq j > i \) and \( a_i < a'_i \).

We also order the plain partitions by the requirements (i)–(iii). So if (i), (ii) or (iii) holds, we have that \( \text{sh} \pi < \text{sh} \pi' \) and (hence) \( \pi < \pi' \).

We denote by \( U_1(\mathfrak{g}) \) the quotient of the universal enveloping algebra of \( \mathfrak{g} \) by the ideal generated by \( c - 1 \), and we denote by \( \tilde{U}_1(\mathfrak{g}) \) the completed enveloping algebra, where \( x_i \to x \) if for each \( \mathfrak{g} \)-module \( V \) of level 1 in the category \( \mathcal{O} \) and each vector \( v \) in \( V \) there is \( i_0 \) such that \( i \geq i_0 \) implies \( x_i v = x_{i_0} v = x v \) [MP1], [MP2, 6.4]. For \( \pi \in \mathcal{P}(B) \) set

\[
\begin{align*}
u(\pi) &= b_1(j_1) \ldots b_s(j_s) \in U_1(\mathfrak{g}), \\
U[\pi] &= \mathbb{F}\text{-span}\{\nu(\pi') \mid \pi' \succ \pi\}, \\
U(\pi) &= \mathbb{F}\text{-span}\{\nu(\pi') \mid \pi' \succ \pi\},
\end{align*}
\]

the closure taken in \( \tilde{U}_1(\mathfrak{g}) \). For \( u \in U[\pi] \), \( u \notin U(\pi) \), we say that \( \pi \) is the leading term of \( u \) and we write \( \bar{\ell}(u) = \pi \).

Recall that the generalized Verma \( \mathfrak{g} \)-module \( N(A_0) \cong U(\mathfrak{g}_{<0}) \) has the structure of vertex operator algebra (cf. [MP2, Section 3]). Set \( r_{2\theta} = X_1(-1)X_1(-1)1 \) and \( R = U(\mathfrak{g})r_{2\theta} \). Then \( R \) is an irreducible \( \mathfrak{g} \)-module with highest weight vector \( r_{2\theta} \).

Set \( \bar{R} = \bigoplus_{n \in \mathbb{Z}} R(n), \quad R(n) = \{r(n) \mid r \in R\} \),

where \( Y(r, z) = \sum_{n \geq 0} r_n z^{-n-1} \) denotes the vertex operator associated with the vector \( r \) and \( r(n) = r_{n+1} \) for \( r \in R \). We think of \( \bar{R} \) as a subset of the completed enveloping algebra \( \tilde{U}_1(\mathfrak{g}) \). Each nonzero element in \( \bar{R} \) has a leading term. Moreover, since \( \bar{R} \) is generated by the adjoint action of \( \mathfrak{g} \) on the elements of the form

\[
(1) \quad r_{2\theta}(n) = (r_{2\theta})_{n+1} = \sum_{i+j=n} X_1(i)X_1(j), \quad n \in \mathbb{Z},
\]

we can describe the set \( \bar{\ell}(\bar{R}) = \{\bar{\ell}(r) \mid r \in \bar{R} \setminus \{0\}\} \) explicitly:

**Lemma 1.** The set \( \bar{\ell}(\bar{R}) \) consists of the elements of the form

\[
X_{i_1}(j)X_{i_2}(j), \quad \text{with colors } i_1i_2 : 11, 21, 22, 31, 32, 33, 42, 43, 44, 51, 52, 53, 54, 55, 62, 64, 65, 66, 73, 74, 75, 76, 77, 85, 86, 87, 88, 93, 35, 36, 37, 38, 47, 48, 56, 58, 66, 68, 77, 78, 88,
\]

\[
X_{i_1}(j-1)X_{i_2}(j), \quad \text{with colors } i_1i_2 : 11, 12, 13, 14, 15, 16, 17, 18, 22, 24, 26, 27, 28, 33, 35, 36, 37, 38, 47, 48, 56, 58, 66, 68, 77, 78, 88,
\]

where \( j \in \mathbb{Z} \).

**Proof.** For \( n = 2j - 1 \) the element \( r_{2\theta}(2j - 1) = 2 \cdot X_1(j-1)X_1(j) + \ldots \) has the leading term \( X_1((j-1)X_1(j) \). The adjoint action of \( f_1 \) on \( r_{2\theta}(2j - 1) \) gives the element \( 2X_3(j-1)X_1(j) + 2X_1(j-1)X_3(j) + \ldots \) with the leading term \( X_1(j-1)X_3(j) \). In
this way we can construct a basis of the \( g \)-module \( R(n) \subset \tilde{R} \) of dimension 27. The case when \( n = 2j \) is treated similarly. \( \square \)

Clearly we can fix a map \( \theta (\tilde{R}) \to \tilde{R}, \rho \mapsto r(\rho) \), such that \( \theta (r(\rho)) = \rho \). Moreover, we will assume that this map is such that \( r(\rho) \) is homogeneous of \( h \)-weight \( \text{wt}(\rho) \) and degree \( |\rho| \), and that the coefficient \( c \) of “the leading term” \( X_i(n)X_j(m) \) in “the expansion” of \( r(X_i(n)X_j(m)) = cX_i(n)X_j(m) + \ldots \) is chosen to be \( c = 1 \). Note that \( \{ r(\rho) \mid \rho \in \theta (\tilde{R}) \} \) is a basis of \( \tilde{R} \) described explicitly by Lemma 1.

Let

\[
\tilde{g} \otimes \tilde{R} = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{i+j=n} g(i) \otimes R(j)
\]

and use infinite sum notation to denote the elements in \( \tilde{g} \otimes \tilde{R} \) by e.g.

\[
\sum_{i+j=n} x^{(i)}(i) \otimes r^{(j)}(j),
\]

\( x^{(i)} \in g, r^{(j)} \in R \). Since \( \tilde{g} \) and \( \tilde{R} \) are loop modules, \( \tilde{g} \otimes \tilde{R} \) becomes a \( \tilde{g} \)-module in the natural way. Define a linear map \( \Psi : \tilde{g} \otimes \tilde{R} \to \overline{U_1(\tilde{g})} \) by linear extension of

\[
\Psi : \sum_{i+j=n} x^{(i)}(i) \otimes r^{(j)}(j) \mapsto \sum_{i<0, i+j=n} x^{(i)}(i)r^{(j)}(j) + \sum_{i \geq 0, i+j=n} r^{(j)}(j)x^{(i)}(i).
\]

Then \( \Psi \) is not a \( \tilde{g} \)-module map, but it is a \( g \)-module map.

**Lemma 2.** The following elements in \( \tilde{g} \otimes \tilde{R} \) are for all \( n \in \mathbb{Z} \) highest weight vectors of \( g \)-submodules of dimensions 64, 35, 35 and 27 (respectively):

\[
q_{64}(n) = \sum_{j \in \mathbb{Z}} (3j - n)X_1(j) \otimes r_{2n}(n - j),
\]

\[
q_{35}(n) = (f_1(-1)h_1(0) - f_1(0)h_1(-1)) \cdot q_{64}(n + 1),
\]

\[
q_{35}(n) = (f_2(-1)h_2(0) - f_2(0)h_2(-1)) \cdot q_{64}(n + 1),
\]

\[
q_{27}(n) = (f_2(-1)h_2(0) - f_2(0)h_2(-1)) \cdot q_{35}(n + 1).
\]

**Proof.** Note first that with

\[
q' = (f_1(-1)h_1(0) - f_1(0)h_1(-1)) \cdot q,
\]

(a) \( e_1 q = e_1(-1)q = 0 \) implies \( e_1q' = 0 \) and \( e_2q = e_2(-1)q = 0 \) implies \( e_2q' = 0 \), and

(b) \( e_1(-1)q = e_1(-2)q = (h_1(-2)h_1(0) - h_1(-1)h_1(-1))q = 0 \) implies \( e_1(-1)q' = 0 \)

and \( e_2(-1)q = e_2(-2)q = 0 \) implies \( e_2(-1)q' = 0 \).

It is clear that \( q_{64}(n) \) is a highest weight vector. Moreover, for \( i = 1, 2 \) and \( j \in \mathbb{Z} \) we have \( e_i(j)q_{64}(n) = 0 \) and \( (h_i(-2)h_i(0) - h_i(-1)h_i(-1))q_{64}(n) = 0 \). Now by applying (a) and (b) we see that the listed vectors are highest weight vectors (of course, provided they are not zero, see Lemma 5). Since the generators \( f_1 \) and \( f_2 \) act nilpotently, these vectors generate finite dimensional \( g \)-modules with dimensions given by the Weyl formula. \( \square \)
Proposition 3. For all $n \in \mathbb{Z}$ we have
\[
\Psi(q_{64}(n)) = \Psi(q_{35}(n)) = \Psi(q_{35}(n)) = 0, \\
\Psi(q_{27}(n)) = c(n)r_{29}(n) \text{ for some } c(n) \in \mathbb{F}.
\]

Proof. Since we have $L_{-1}X_1(-1)^31 = 3X_1(-2)X_1(-1)^21$, where $L_{-1}$ is a Virasoro algebra element, we can write the vertex operator associated with this vector in two different ways:
\[
d \frac{d}{dz} \left( Y(X_1(-1)1, z)Y(r_{29}, z) \right) = 3 \left( d \frac{d}{dz} Y(X_1(-1)1, z) \right) Y(r_{29}, z).
\]
The coefficients of this relation give $\Psi(q_{64}(n)) = 0$ for all $n \in \mathbb{Z}$.

For $i = 1, 2$ set $A_i = f_i(-1)h_i(0) - f_i(0)h_i(-1)$. Since $x(i) \otimes r(j) \mapsto x(i)r(j)$ and $x(i) \otimes r(j) \mapsto r(j)x(i)$ are $\mathfrak{g}$-module maps, it is clear that
\[
\Psi(A_1 \sum_{i<0,i+j=n} x(i) \otimes r(j)) = A_1 \Psi(\sum_{i<0,i+j=n} x(i) \otimes r(j)), \\
\Psi(A_1 \sum_{i>1,i+j=n} x(i) \otimes r(j)) = A_1 \Psi(\sum_{i>1,i+j=n} x(i) \otimes r(j)).
\]
This implies that
\[
\Psi(q_{35}(n - 1)) = \Psi(A_1q_{64}(n)) = A_1 \Psi(q_{64}(n)) \\
+ \Psi \left( A_1 \sum_{j=0}^{0} (3j - n)X_1(j) \otimes r_{29}(n - j) \right) \\
- A_1 \left( \sum_{j=0}^{0} (3j - n)r_{29}(n - j)X_1(j) \right).
\]

Now note that the last two terms consist of products of $X_1(i)$ or $f_1 \cdot X_1(i)$ with $r_{29}(i)$ or $f_1 \cdot r_{29}(i)$, $i \in \mathbb{Z}$, differing only in the order they are written. Since these elements commute (cf. (1)), the last two terms cancel. This together with $\Psi(q_{64}(n)) = 0$ gives $\Psi(q_{35}(n - 1)) = 0$. The relation $\Psi(q_{35}(n)) = 0$ is proved in a similar way.

Arguing as before we get
\[
\Psi(q_{27}(n - 2)) = \Psi(A_2A_1q_{64}(n)) = A_2A_1 \Psi(q_{64}(n)) \\
+ \Psi \left( A_2A_1 \sum_{j=0}^{1} (3j - n)X_1(j) \otimes r_{29}(n - j) \right) \\
- A_2A_1 \left( \sum_{j=0}^{1} (3j - n)r_{29}(n - j)X_1(j) \right).
\]
Now note that the last two terms consist of products of $X_1(i)$, $f_1 \cdot X_1(i)$, $f_2 \cdot X_1(i)$ or $f_2f_1 \cdot X_1(i)$ with $r_{29}(i)$, $f_1 \cdot r_{29}(i)$, $f_2 \cdot r_{29}(i)$ or $f_2f_1 \cdot r_{29}(i)$, $i \in \mathbb{Z}$, differing only in the order they are written. Since $R$ is a loop module, that is $[x(i), r(j)] = (x \cdot r)(i + j)$,
we can commute these elements and the last two terms cancel except for an element in $R$ of h-weight 20 and degree $n - 2$. This together with $\Psi(q_{27}(n)) = 0$ gives that $\Psi(q_{27}(n - 2))$ is proportional to $r_{27}(n - 2)$. □

Recall that $N^1(\Lambda_0) = \bar{R}N(\Lambda_0)$ is the maximal $\mathfrak{g}$-submodule of $N(\Lambda_0)$. The elements $r(n) = r_{n+1} \in \bar{R}$ annihilate the basic $\mathfrak{g}$-module $L(\Lambda_0) = N(\Lambda_0)/N^1(\Lambda_0)$, so we call them relations. On the other hand $\Psi(q_{64}(n)) = 0$ reads

$$\sum_{j<0}(3j - n)X_1(j)r_{2\theta}(n - j) + \sum_{j\geq0}(3j - n)r_{2\theta}(n - j)X_1(j) = 0 \text{ on } N(\Lambda_0),$$

so we shall sometimes say that $\Psi(q_{64}(n)) = 0$, or $q_{64}(n) \in \bar{g} \otimes \bar{R}$, are relations among relations. If we extend $\Psi : \bar{g} \otimes \bar{R} + \bar{F} \otimes \bar{R} \rightarrow \bar{U}_1(\mathfrak{g})$ by $\Psi(1 \otimes r(n)) = r(n)$, then we can write $\Psi(q_{27}(n)) = c(n)r_{2\theta}(n)$ as $\Psi(q_{27}(n) - c(n) \otimes r_{2\theta}(n)) = 0$. By abuse of notation we shall sometimes write or think $q_{27} \in \ker \Psi$.

For colored partitions $\kappa$, $\rho$ and $\pi = \kappa \rho$ we shall write $\kappa = \pi/\rho$ and $\rho \subset \pi$. We shall say that $\rho \subset \pi$ is an embedding (of $\rho$ in $\pi$). For an embedding $\rho \subset \pi$, where $\rho \in \mathfrak{\ell}(\bar{R})$, we define the element $u(\rho \subset \pi)$ in $\bar{U}_1(\mathfrak{g})$ by

$$u(\rho \subset \pi) = \begin{cases} u(\pi/\rho)r(\rho) & \text{if } |\rho| > |\pi/\rho|, \\ r(\rho)u(\pi/\rho) & \text{if } |\rho| \leq |\pi/\rho|. \end{cases}$$

It is clear that each nonzero $q \in \bar{g} \otimes \bar{R}$ (or $q \in \bar{g} \otimes \bar{R} + \bar{F} \otimes \bar{R}$) can be written in the form

$$q = \sum_{\rho' \subset \pi} C_{\rho',\pi} u(\pi/\rho') \otimes r(\rho') + \sum_{\rho' \subset \pi'} C_{\rho',\pi'} u(\pi'/\rho') \otimes r(\rho')$$

for some $\pi$ of length 3 (or $\leq 3$) and some $C_{\rho',\pi}, C_{\rho',\pi'} \in \bar{F}$, where at least one coefficient $C_{\rho',\pi}$ is nonzero. We shall say that $\pi$ is the leading term of $q$ and we write $\pi = \mathfrak{\ell}(q)$. The assumption $q \in \ker \Psi$, i.e.

$$\sum_{\rho' \subset \pi} C_{\rho',\pi} u(\rho' \subset \pi) + \sum_{\rho' \subset \pi'} D_{\rho',\pi'} u(\rho' \subset \pi') = 0,$$

implies that $\sum_{\rho' \subset \pi} C_{\rho',\pi} = 0$ (cf. [MP2, Lemma 6.4.1]). Hence for such $q$ there must be at least two (different) embeddings $\rho' \subset \pi$, $\rho'' \subset \pi$ with the corresponding coefficients being nonzero. For a colored partition $\pi$ of length 3 set

$$N(\pi) = \max\{\#(\mathcal{E}(\pi)) - 1, 0\}, \quad \mathcal{E}(\pi) = \{\rho \in \mathfrak{\ell}(\bar{R}) : \rho \subset \pi\}.$$

It is clear that $0 \leq N(\mathfrak{\ell}(q)) \leq 2$ for $q \in \bar{g} \otimes \bar{R}$. The argument above shows that $N(\mathfrak{\ell}(q)) \geq 1$ for $q \in \ker \Psi$.

**Lemma 4.** Let $Q \subset \ker \Psi$ be a subspace of dimension $n$. Then

$$\sum_{\pi \in \mathfrak{\ell}(Q)} N(\pi) \geq n,$$

where $\mathfrak{\ell}(Q) = \{\mathfrak{\ell}(q) : q \in Q \setminus \{0\}\}$.

**Proof.** Let $Q_{[\pi]} = \{q \in Q : \mathfrak{\ell}(q) \geq \pi\}$ and $Q_{(\pi)} = \{q \in Q : \mathfrak{\ell}(q) > \pi\}$. Let $\dim Q_{[\pi]}/Q_{(\pi)} = m(\pi) = m \leq n$, $m \geq 1$ and let $\rho_1 \subset \pi, \ldots, \rho_s \subset \pi$ (where
s = s(π) be all possible embeddings in π. Let π* be such that $Q(π) = Q[π*]$. Then we can write a basis of $Q[π]/Q(π)$ in the form
\[
\begin{align*}
c_{11}u(π/ρ_1) \otimes r(ρ_1) + \cdots + c_{1s}u(π/ρ_s) \otimes r(ρ_s) + v_1 + Q(π), \\
c_{21}u(π/ρ_1) \otimes r(ρ_1) + \cdots + c_{2s}u(π/ρ_s) \otimes r(ρ_s) + v_2 + Q(π), \\
\cdots \cdots \\
c_{m1}u(π/ρ_1) \otimes r(ρ_1) + \cdots + c_{ms}u(π/ρ_s) \otimes r(ρ_s) + v_m + Q(π),
\end{align*}
\]
where the vectors $v_i$ are of the form
\[
v_i = \sum_{π < π' < π*} \sum_{ρ \in E(π')} d_{i, ρ, π'} u(π'/ρ) \otimes r(ρ) + \sum_{π* < π' < π*} \sum_{ρ \in E(π')} e_{i, ρ, π'} u(π'/ρ) \otimes r(ρ).
\]
Assume that rank $(c_{ij}) < m$. Then the rows are linearly dependent. By taking a nontrivial linear combination of the basis elements we get a vector in $Q$ of the form
\[
v = \sum_{π < π' < π*} \sum_{ρ \in E(π')} d_{ρ, π'} u(π'/ρ) \otimes r(ρ) + \sum_{π* < π' < π*} \sum_{ρ \in E(π')} e_{ρ, π'} u(π'/ρ) \otimes r(ρ).
\]
The coefficients $d_{ρ, π'}$, $ρ \in E(π')$, $π < π' < π*$, must be zero since otherwise $Q(π) \neq Q[π*]$. But then $v \in Q(π) = Q[π*]$ and our nontrivial linear combination of basis elements is zero in $Q[π]/Q(π)$, a contradiction.

Hence the rank of the matrix $(c_{ij})$ is $m$ and we have $s \geq m$. Assume that $s = m$. Then the matrix $(c_{ij})$ is regular and the set of vectors of the form $u(π/ρ_1) \otimes r(ρ_1) + v_1 + Q(π), \ldots, u(π/ρ_s) \otimes r(ρ_s) + v_m + Q(π)$ (vectors $v_i$ as above) is a basis of $Q[π]/Q(π)$. In particular, we have a vector $u \in Q$ of the form $u = u(π/ρ_1) \otimes r(ρ_1) + v_1$ such that $Ψ(u) \neq 0$, a contradiction.

Hence $s > m$, i.e. $N(π) = s(π) - 1 \geq m(π)$. Since $\dim Q = \sum m(π)$, the lemma follows. □

Remark. Let us keep the notation from the proof of Lemma 4 and let us assume that $N(π) = m(π)$. Then there are altogether $s = m + 1$ possible embeddings $ρ_1 \subset π, \ldots, ρ_s \subset π$ and the rank of $(c_{ij})$ is $s = m - 1$. Let us assume that the first $s - 1$ columns of $(c_{ij})$ are linearly independent. Then for each $1 \leq i \leq s - 1$ there is a vector in $Q$ of the form
\[
u(π/ρ_i) \otimes r(ρ_i) + d_i u(π/ρ_s) \otimes r(ρ_s) + \sum_{π < π' < π*} \sum_{ρ \in E(π')} d_{i, ρ, π'} u(π'/ρ) \otimes r(ρ)
\]
for some $d_i, d_{i, ρ, π'} \in F$. As was shown above, the assumption $Q \subset kerΨ$ implies $d_i \neq 0$. But then for each $i, j \in \{1, \ldots, s\}$ there are $d_{ij}, d_{i, j, ρ, π'} \in F$ such that
\[
(3) \ u(π/ρ_i) \otimes r(ρ_i) + d_{ij} u(π/ρ_j) \otimes r(ρ_j) + \sum_{π < π' < π*} \sum_{ρ \in E(π')} d_{i, j, ρ, π'} u(π'/ρ) \otimes r(ρ) \in kerΨ.
\]

As above we denote with by dot the adjoint action of $U(g)$ and set
\[
Q_A(n) = U(g) \cdot q_A(n) \quad \text{for} \quad n \in Z \quad \text{and} \quad A = 27, 35, 35, 64.
\]
Lemma 5. For \( j \in \mathbb{Z} \) we have:

\[
\text{sh}(\ell(q)) = \begin{cases} 
  j^3 & \text{for } q \in Q_{27}(3j) \oplus Q_{35}(3j) \oplus Q_{35}(3j), \\
  (j-1)j(j+1) & \text{for } q \in Q_{64}(3j), \\
  (j-1)^2j & \text{for } q \in Q_{27}(3j-1) \oplus Q_{35}(3j-1) \oplus Q_{35}(3j-1) \oplus Q_{64}(3j-1), \\
  (j-1)^2j & \text{for } q \in Q_{27}(3j-2) \oplus Q_{35}(3j-2) \oplus Q_{35}(3j-2) \oplus Q_{64}(3j-2).
\end{cases}
\]

Proof. Note first that each \( q_A(n) \), \( A = 27, 35, 35, 64 \), is a sum of elements of the form \( C_{\rho',\pi'} u(\pi'/\rho') \otimes r(\rho') \) with \( |\pi'| = n \) and \( \ell(\pi') = 3 \) (and additional terms with \( \ell(\pi') = 2 \) appearing in \( q_{27} \)), and that for such colored partitions \( \text{sh}(\pi') = abc \), where \( a \leq b \leq c, a + b + c = |\pi'| \). The smallest possible shapes are

\[
\begin{align*}
&j^3 < (j-1)j(j+1) < \ldots \quad \text{when } |\pi'| = 3j, \\
&(j-1)^2j \prec \ldots \quad \text{when } |\pi'| = 3j - 1, \\
&(j-1)^2j \prec \ldots \quad \text{when } |\pi'| = 3j - 2.
\end{align*}
\]

In the case of \( q_{64}(3j) \) we see that for \( \text{sh}(\pi') = j^3 \) the coefficient \( C_{\rho',\pi'} \) equals \( 3j - 3j = 0 \), and that for \( \text{sh}(\pi') = (j-1)j(j+1) \) the corresponding coefficients are not zero. Hence \( \text{sh}(\ell(q_{64})) = (j-1)j(j+1). \) Since the projection

\[
\phi: Q_{64}(3j) \to g(j-1) \otimes R(2j+1) + g(j+1) \otimes R(2j-1), \\
\phi: U(g)q_{64}(3j) \to U(g)(-3X_1(j-1) \otimes r_{2\theta}(2j+1) + 3X_1(j+1) \otimes r_{2\theta}(2j-1))
\]

is a \( g \)-module map, for each \( q \in U(g)q_{64}(3j), q \neq 0 \), we have \( \phi(q) \neq 0. \) Hence \( \text{sh}(\ell(q_{64})) = (j-1)j(j+1). \)

In the case of \( q_{64}(3j-1) \) we see that for \( \text{sh}(\pi') = (j-1)^2j \) the coefficients \( C_{\rho',\pi'} \) equal \( 3j - 3 - 3j + 1 = -2 \neq 0 \) and \( 3j - 3j + 1 = 1 \neq 0 \), and hence \( \text{sh}(\ell(q_{64})) = (j-1)^2j. \) Since the projection

\[
\phi: U(g)q_{64}(3j-1) \to U(g)(-2X_1(j-1) \otimes r_{2\theta}(2j) + X_1(j) \otimes r_{2\theta}(2j-1))
\]

is a \( g \)-module map, for each \( q \in U(g)q_{64}(3j-1), q \neq 0 \), we have \( \phi(q) \neq 0. \) Hence \( \text{sh}(\ell(q)) = (j-1)^2j. \)

The other cases \( q_A(n), A = 27, 35, 35, \) are similar (except for more complicated expressions for \( q_A \)): we have to show that at least one coefficient \( C_{\rho',\pi'} \neq 0 \) for each of the shapes \( j^3, (j-1)j^2, \) and \( (j-1)^2j. \)

Note that Lemma 4 and Lemma 5 imply that:

\[
\sum_{\text{sh}(\pi) = j^3} N(\pi) \geq 27 + 35 + 35 = 97,
\]
\[
\sum_{\text{sh}(\pi) = (j-1)j(j+1)} N(\pi) \geq 64,
\]
\[
\sum_{\text{sh}(\pi) = (j-1)^2j} N(\pi) \geq 27 + 35 + 35 + 64 = 161,
\]
\[
\sum_{\text{sh}(\pi) = (j-1)^2j} N(\pi) \geq 27 + 35 + 35 + 64 = 161.
\]

By direct counting we see the following:
Lemma 6.
\[ \sum_{\text{sh}(\pi)=j^3} N(\pi) = 97, \]
\[ \sum_{\text{sh}(\pi)=(j-1)j(j+1)} N(\pi) = 64, \]
\[ \sum_{\text{sh}(\pi)=(j-1)j^2} N(\pi) = 162, \]
\[ \sum_{\text{sh}(\pi)=(j-1)^2j} N(\pi) = 162. \]

Lemmas 4, 5 and 6 imply that for all but two h-weight spaces \( Q(n)_\mu \) we have
\[ \sum_{\ell(\pi)=3,|\pi|=n,\text{wt}(\pi)=\mu} N(\pi) = \dim Q(n)_\mu, \]
where \( n = 3j, 3j-1, 3j-2 \) and \( Q(n) = Q_{27}(n) \oplus Q_{35}(n) \oplus Q_{35}(n) \oplus Q_{64}(n) \). The next lemma describes precisely these two exceptions. There we shall use the notation
\[
\begin{array}{cccccc}
\text{o} & \text{o} & 3 & 3 & \text{o} \\
\text{o} & \text{o} & 5 & 4 & \text{o} & 3 \\
\text{e} & \text{e} & 1 & 1 & \text{e} & 1 \\
\end{array}
\]
where the Young diagram represents the plain partition \((-2)(-1)^2\), we shall think of \((j-1)^2j\) in general, the numbers 351, 341 and 531 represent colorings. So we have listed three colored partitions (from left to right): \( X_3(j-1)X_5(j)X_1(j) \prec X_3(j-1)X_4(j)X_1(j) \prec X_3(j-1)X_3(3)X_1(j) \). Moreover, with circes and bullets we denote all possible embeddings \( X_5(j)X_1(j) \subset X_3(j-1)X_5(j)X_1(j) \), \( X_3(j-1)X_5(j) \subset X_3(j-1)X_5(j)X_1(j) \) and \( X_3(j)X_1(j) \subset X_3(j-1)X_3(3)X_1(j) \) for the listed colored partitions (cf. Lemma 1).

Lemma 7. (a) For \( \mu = \alpha_1 + 2\alpha_2 \) and \( j \in \mathbb{Z} \) there are the following 10 partitions \( \pi \) such that \( \text{sh}(\pi) = (j-1)j^2 \) and \( \text{wt}(\pi) = \mu \):
\[
\begin{array}{cccccc}
\text{o} & \text{o} & 1 & \text{ • } & 1 & \text{ • } 2 & \text{ • } 3 & \text{ • } 1 & \text{ • } 3 & 3 & \text{ • } 4 & \text{ • } 7 \\
\text{ • } 5 & \text{ • } 4 & 3 & \text{ • } 3 & \text{ • } 7 & \text{ • } 5 & 4 & 3 & 3 & 1 \\
\text{ • } 3 & \text{ • } 3 & \text{ • } 3 & 3 & \text{ • } 2 & \text{ • } 1 & \text{ • } 1 & 1 & 1 & 1 & 1. \\
\end{array}
\]
Moreover, for such partitions \( \sum N(\pi) = 7 = \dim Q(3j-1)_\mu + 1 \).

(b) For \( \mu = -\alpha_1 - 2\alpha_2 \) and \( j \in \mathbb{Z} \) there are the following 10 partitions \( \pi \) such that \( \text{sh}(\pi) = (j-1)^2j \) and \( \text{wt}(\pi) = \mu \):
\[
\begin{array}{cccccc}
\text{ • } 6 & \text{ • } 6 & \text{ • } 8 & 6 & 7 & 8 & 8 & 8 & 8 \\
\text{ • } 5 & \text{ • } 4 & \text{ • } 2 & 6 & 6 & 5 & 4 & 6 & 6 & 8 \\
\text{ • } 8 & \text{ • } 8 & \text{ • } 8 & 8 & 7 & 6 & 6 & 5 & 4 & 2. \\
\end{array}
\]
Moreover, for such partitions \( \sum N(\pi) = 7 = \dim Q(3j-2)_\mu + 1 \).

Let \( \rho_1 \subset \pi, \rho_2 \subset \pi \) be two embeddings, \( \rho_1, \rho_2 \in \mathfrak{gl}(\widetilde{R}) \). We would like to construct a relation among relations of the form
\[ u(\rho_1) \subset \pi \in \mathbb{R}^* u(\rho_2) \subset \pi + \mathbb{R}^{\text{span}}\{u(\rho \subset \pi') \mid \rho \in \mathfrak{gl}(\widetilde{R}), \rho \subset \pi', \pi \prec \pi'\} \]
(the closure taken in \( U_1(\mathfrak{g}) \)). If the colored partition \( \pi \) is such that \( \ell(\pi) = 3 \) and that for \( \mu = \text{wt}(\pi) \) the relation (4) holds, then the proof of Lemma 4 shows that \( s(\pi) = m(\pi) + 1 \) and that there is an element \( q \in \ker \Psi \) of the form (see (3))
\[ q = u(\pi/\rho_1) \otimes r(\rho_1) + c u(\pi/\rho_2) \otimes r(\rho_2) + \sum_{\pi \prec \pi'} \sum_{\rho \in \mathcal{F}(\pi')} c_{\rho,\pi'} u(\pi' / \rho) \otimes r(\rho), \]
and this implies (5). However, the equality (4) does not hold for all weights \( \mu \). In the next two lemmas we identify embeddings which do not appear in relations among relations of the form (5):
Lemma 8. Let \( j \in \mathbb{Z} \). Then
\[
\ell (X_3(j-1)r(X_5(j)X_1(j)) - r(X_3(j-1)X_5(j))X_1(j)) = X_3(j-1)X_4(j)X_1(j),
\]
\[
\ell (r(X_5(j)X_1(j)) - r(X_5(j)X_6(j+1)) = X_8(j)X_4(j)X_6(j+1).
\]

Proof. By calculating the first few terms of \( r(X_5(j)X_1(j)) \) and \( r(X_3(j-1)X_5(j)) \) we get
\[
X_3(j-1)r(X_5(j)X_1(j)) - r(X_3(j-1)X_5(j))X_1(j)
\]
\[
= X_3(j-1)X_5(j)X_1(j) + X_3(j-1)X_4(j)X_1(j) + \ldots
\]
\[
- (X_3(j-1)X_5(j)X_1(j) + X_5(j-1)X_3(j)X_1(j) + \ldots)
\]
\[
= X_3(j-1)X_4(j)X_1(j) - X_5(j-1)X_3(j)X_1(j) + \ldots,
\]
where the dots represent sums of terms of higher shape. The other equality is proved similarly. \( \Box \)

Lemma 9. Let \( j \in \mathbb{Z} \). Then there is no element \( q \) in \( \ker \Psi \) such that the leading term \( \ell(q) \) is either \( X_3(j-1)X_5(j)X_1(j) \) or \( X_8(j-1)X_5(j-1)X_6(j) \).

Proof. Set \( \pi = X_3(j-1)X_5(j)X_1(j) \) and let \( q \in \ker \Psi \) be of the form
\[
q = X_3(j-1)X_5(j)X_1(j) - X_1(j)X_3(j-1)X_5(j)
\]
\[
+ \sum_{\pi \prec \pi', \rho \in \mathcal{E}(\pi')} c_{\rho, \pi'} u(\pi'/\rho) \otimes r(\rho).
\]

From Lemma 7(a) we see that \( X_3(j-1)X_5(j)X_1(j) \prec X_3(j-1)X_4(j)X_1(j) \prec X_5(j-1)X_3(j)X_1(j) \) and that \( \mathcal{E}(X_3(j-1)X_4(j)X_1(j)) = \emptyset \). Hence
\[
\Psi(q) = X_3(j-1)X_5(j)X_1(j) - X_1(j)X_3(j-1)X_5(j)
\]
\[
+ \sum_{X_3(j-1)X_4(j)X_1(j) < \pi', \rho \in \mathcal{E}(\pi')} c_{\rho, \pi'} u(\pi'/\rho) r(\rho).
\]

Now Lemma 8 implies that the leading term \( \ell(\Psi(q)) = X_3(j-1)X_4(j)X_1(j) \), and in particular this implies that \( \Psi(q) \neq 0 \), a contradiction.

The case of \( X_8(j-1)X_5(j-1)X_6(j) \) is proved similarly. \( \Box \)

We may conclude our discussion with the following:

Proposition 10. Let \( \pi \) be a colored partition of length 3 and let \( \rho_1 \subset \pi \), \( \rho_2 \subset \pi \) be two embeddings, \( \rho_1, \rho_2 \in \ell(R) \). Let
\[
\pi \neq X_3(j-1)X_5(j)X_1(j), \quad \pi \neq X_8(j-1)X_5(j-1)X_6(j).
\]

Then the relation (5) holds:
\[
u(\rho_1 \subset \pi) \in \mathbb{F}^\pi u(\rho_2 \subset \pi) + \mathbb{F}^\pi \text{span}\{u(\rho \subset \pi') \mid \rho \in \ell(R), \rho \subset \pi', \pi \prec \pi'\}.
\]

Proof. We have seen that the relation (5) holds for all cases except when \( \text{wt}(\pi) = \alpha_1 + 2\alpha_2 \) or \( \text{wt}(\pi) = -\alpha_1 - 2\alpha_2 \). So consider the first case: all possible \( \pi \)'s are listed
in Lemma 7(a), altogether five of them allow two or more than two embeddings, say
\[ \pi_1 < \pi_2 < \pi_3 < \pi_4 < \pi_5 = X_3(j - 1)X_5(j)X_1(j), \]
and
\[ \sum_{i=1}^{4} N(\pi_i) = 6. \]

Set \( Q = Q(3j - 1)_{\alpha + \lambda} + 2\alpha_2 \). Since \( Q \subset \ker \Psi \), Lemma 9 implies that \( Q_{[\pi_5]} = 0 \). Since \( Q = Q_{[\pi_i]} \supset \cdots \supset Q_{[\pi_5]} = Q_{(\pi_i)} = 0 \), we have
\[ \dim Q = \sum_{i=1}^{4} \dim (Q_{[\pi_i]}/Q_{(\pi_i)}) = 4 \sum_{i=1}^{4} m(\pi_i) = 6. \]

From the proof of Lemma 4 we see that \( N(\pi_i) = s(\pi_i) - 1 \geq m(\pi_i) \). Hence \( \sum N(\pi_i) = \sum m(\pi_i) \) implies that \( N(\pi_i) = m(\pi_i) \) for \( i = 1, 2, 3, 4 \), which, as remarked, implies the relation (5) for \( \pi = \pi_1, \ldots, \pi_4 \).

The case when \( \omega(\pi) = -\alpha_1 - 2\alpha_2 \) is proved similarly. \( \square \)

Set
\[ \mathcal{R} = \ell(\tilde{R}) \cup \{X_3(j - 1)X_4(j)X_1(j), X_5(j - 1)X_4(j - 1)X_6(j) | j \in \mathbb{Z}\}. \]

We extend the map \( \ell(\tilde{R}) \to \tilde{R}, \rho \to r(\rho) \), to the map \( \mathcal{R} \to \tilde{R}U(\mathfrak{g}) \) by
\[ r(X_3(j - 1)X_4(j)X_1(j)) = X_3(j - 1)r(X_5(j)X_1(j)) - r(X_3(j - 1)X_5(j))X_1(j), \]
\[ r(X_5(j)X_4(j)X_6(j + 1)) = r(X_5(j)X_5(j))X_6(j + 1) - X_8(j)r(X_5(j)X_6(j + 1)), \]
so that \( \ell(r(\rho)) = \rho \). We still have that \( r(\rho) \) is homogeneous of \( h \)-weight \( \omega(\rho) \) and degree \( |\rho| \), and that the coefficient \( c \) of “the leading term” \( u(\rho) \) in “the expansion” of \( r(\rho) = c u(\rho) + \ldots \) is chosen to be \( c = 1 \). For an embedding \( \rho \subset \pi \), where \( \rho \in \mathcal{R} \), we define the element \( u(\rho \subset \pi) \) in \( \overline{U_1(\mathfrak{g})} \) the same way as before (cf. (2)). For \( \pi \in \mathcal{P}(\tilde{B}) \) set
\[ \mathcal{R}_{(\pi)} = \overline{\mathbb{F}-\text{span}}\{u(\rho \subset \pi') | \rho \in \mathcal{R}, \rho \subset \pi', \pi' \prec \pi'\}, \]
the closure taken in \( \overline{U_1(\mathfrak{g})} \).

Note that for \( \pi = X_3(j - 1)X_5(j)X_1(j), \rho_1 = X_5(j)X_1(j), \rho_2 = X_3(j - 1)X_5(j), \) we have
\[ u(\rho_1 \subset \pi) \in u(\rho_2 \subset \pi) + \mathcal{R}_{(\pi)} \]
and \( u(\rho_1 \subset \pi) = r(X_3(j - 1)X_4(j)X_1(j)) + u(\rho_2 \subset \pi) + \text{shorter terms} \) (cf. (2)).

So our construction gives
\[ u(\rho_1 \subset \pi) \in \mathbb{F}^* u(\rho_2 \subset \pi) + \mathcal{R}_{(\pi)} \]
for any two embeddings \( \rho_1 \subset \pi, \rho_2 \subset \pi \) when \( \rho_1, \rho_2 \in \ell(\tilde{R}) \) and \( \ell(\pi) = 3 \).

Our main result about relations among relations is the following:
Theorem 11. Let $\rho_1 \subset \pi$, $\rho_2 \subset \pi$ be two embeddings, $\rho_1, \rho_2 \in \mathcal{R}$. Then

\[
(6) \quad u(\rho_1 \subset \pi) \in \mathbb{F}^\times u(\rho_2 \subset \pi) + \mathcal{R}(\pi).
\]

We prove the theorem in several steps, basically following the ideas used in the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{F})$:

The case when $\rho_1 \rho_2 \subset \pi$ is easy: We “expand” the product $r(\rho_1)r(\rho_2)$ in two different ways and get (6). For example, for $\pi = X_3(j - 1)X_5(j)X_5(j)X_1(j)$ we have

\[
\begin{align*}
    r(\rho_1) &= r(X_3(j - 1)X_5(j)) = X_3(j - 1)X_5(j) + X_5(j - 1)X_3(j) + \ldots, \\
    r(\rho_2) &= r(X_5(j)X_1(j)) = X_5(j)X_1(j) + X_4(j)X_1(j) + \ldots,
\end{align*}
\]

where the dots represent sums of terms of higher shape. So we can “expand” the product $r(\rho_1)r(\rho_2)$ in two ways and get

\[
\begin{align*}
    r(\rho_1)X_5(j)X_1(j) - X_3(j - 1)X_5(j)r(\rho_2) &= -r(\rho_1)X_4(j)X_1(j) + X_5(j - 1)X_3(j)r(\rho_2) + \ldots \\
    &\in \mathcal{R}(X_3(j - 1)X_5(j)X_5(j)X_1(j)).
\end{align*}
\]

where dots represent a sum of terms of higher shape. For later purposes note that

\[
\begin{align*}
    \theta(X_3(j - 1)X_5(j)r(\rho_2)) &= X_5(j - 1)X_5(j)X_3(j)X_1(j) \geq X_3(j - 1)X_5(j)X_4(j)X_1(j)
\end{align*}
\]

and hence

\[
(7) \quad r(\rho_1)X_5(j)X_1(j) - X_3(j - 1)X_5(j)r(\rho_2) \in -r(\rho_1)X_4(j)X_1(j) + \mathcal{R}(X_3(j - 1)X_5(j)X_4(j)X_1(j)),
\]

The case when $\rho_1, \rho_2 \in \theta(\tilde{R})$ follows from the discussion above. In particular, relations among relations (5) hold for all $\pi$ of length 3 except $X_3(j - 1)X_5(j)X_1(j)$ or $X_5(j - 1)X_5(j - 1)X_6(j)$.

The remaining case is when $\rho_1 \in \mathcal{R}$ “intersects” $\rho_2 \in \mathcal{R}\setminus \theta(\tilde{R})$: Define $\pi \cup \nu$ and $\pi \cap \nu$ by

\[
(\pi \cup \nu)(a) = \max\{\pi(a), \nu(a)\}, \quad (\pi \cap \nu)(a) = \min\{\pi(a), \nu(a)\},
\]

and denote the partition $1 = \prod_{a \in \tilde{B}} a^0$ with no parts and length 0 as $\varnothing$. By using Lemma 1 we see the following:

Lemma 12. Let $\pi$ be a colored partition, $\rho_1 \in \mathcal{R}$ and $\rho_2$ either $X_3(j - 1)X_4(j)X_1(j)$ or $X_5(j - 1)X_4(j - 1)X_6(j)$, $j \in \mathbb{Z}$. Assume that $\rho_1, \rho_2 \subset \pi$, $\pi = \rho_1 \cup \rho_2$, $\rho_1 \cap \rho_2 \neq \varnothing$ and $\ell(\pi) \geq 4$. Then $\pi$ is one of the following colored partitions:

\[
\begin{align*}
    &X_a^\bullet(j - 2)X_a^\bullet(j - 1)X_4(j)X_1(j), \quad a = 3, 1, \\
    &X_a^\bullet(j - 1)X_3^\bullet(j - 1)X_4(j)X_1(j), \quad a = 7, 5, 4, 3, 2, 1, \\
    &X_a^\bullet(j - 1)X_3^\bullet(j - 1)X_3^\bullet(j)X_1(j), \quad a = 2, 1, \\
    &X_a^\bullet(j - 1)X_3^\bullet(j - 1)X_4(j)X_3^\bullet(j), \quad a = 1, \\
    &X_3^\bullet(j - 1)X_4(j)X_1(j)X_3^\bullet(j), \quad a = 8, 7, 6, 5, 3, \\
    &X_3(j - 1)X_4^\bullet(j)X_4(j)X_3^\bullet(j), \quad a = 7, 6, 5, 4, 3, 2,
\end{align*}
\]
Here we used the multiplicative notation and the embedding \( \rho_1 \subset \pi \) is denoted with bullets.

**Lemma 13.** Let \( \pi \) be one of the partitions of length 4 listed in Lemma 12. Set
\[
\pi_0 = \begin{cases} 
X_a(i)X_3(j-1)X_5(j)X_1(j) & \text{if } \pi = X_a(i)X_3(j-1)X_4(j)X_1(j), \\
X_a(i)X_5(j-1)X_5(j-1)X_6(j) & \text{if } \pi = X_a(i)X_8(j-1)X_4(j-1)X_6(j).
\end{cases}
\]
Let \( \pi' \) be such that \( \text{wt}(\pi') = \text{wt}(\pi) \) and \( \pi_0 < \pi' \prec \pi \). Then there is no embedding \( \rho \subset \pi' \) such that \( \rho \) is of the form \( X_3(j'-1)X_5(j')X_1(j') \) or \( X_8(j'-1)X_5(j'-1)X_6(j') \).

**Proof.** First note that \( \text{sh}(\pi') = \text{sh}(\pi) \), that \( \text{wt}(X_a) \) is a root or zero, and that
\[
\text{wt}(X_3(j'-1)X_4(j')X_1(j')) = 2a_2 + a_1, \quad \text{wt}(X_8(j'-1)X_3(j'-1)X_6(j')) = -2a_2 - a_1.
\]
Let \( \pi = X_a(i)X_3(j-1)X_4(j)X_1(j) \). Then there is no root \( \beta \) such that \( \text{wt}(X_a) + 2a_2 + a_1 = \beta - 2a_2 - a_1 \), and hence \( \pi' \) cannot contain a partition of the form \( X_8(j'-1)X_3(j'-1)X_6(j') \).

Assume that \( \pi' = X_{a'}(i')X_3(j'-1)X_5(j')X_1(j') \). If \( a \neq 4 \) and \( a \neq 5 \), then \( \text{wt}(\pi') = \text{wt}(\pi) \) implies that \( a' = a \), and it is easy to see that this implies that \( \pi' = \pi_0 \). If \( a = 4 \) or \( a = 5 \), then \( \text{wt}(\pi') = \text{wt}(\pi) \) gives \( a' = 4 \) or \( a' = 5 \) and it is easy to see that then \( \pi_0 < \pi' \prec \pi \) does not hold.

The case when \( \pi = X_a(i)X_8(j-1)X_4(j-1)X_6(j) \) is similar. \( \Box \)

**Lemma 14.** Let \( \pi \) be one of the partitions listed in Lemma 12 and let \( \rho_1, \rho_2 \subset \pi \).

Then
\[
(8) \quad u(\rho_1 \subset \pi) \in \mathbb{F}^* u(\rho_2 \subset \pi) + \mathcal{R}(\pi).
\]

**Proof.** Let \( \pi = X_3(j-1)X_5(j)X_4(j)X_1(j) \). From (7) we get
\[
\begin{align*}
    r(X_3(j-1)X_4(j)X_1(j))X_5(j) &= X_3(j-1)X_5(j)r(X_5(j)X_1(j)) - r(X_3(j-1)X_5(j))X_5(j)X_1(j) + \mathcal{R}(\pi) \\
    &= r(X_3(j-1)X_5(j))X_4(j)X_1(j) + \mathcal{R}(\pi) ,
\end{align*}
\]
and this implies (8) for any \( \rho_1, \rho_2 \subset \pi \).

Let \( \pi = X_a(i)X_3(j - 1)X_4(j)X_1(j) \) and assume that there is no embedding \( \rho \subset \pi \) such that \( \rho \) is of the form \( X_3(j' - 1)X_5(j') \) or \( X_5(j')X_1(j') \), that is \( \pi \neq X_3(j - 1)X_5(j)X_4(j)X_1(j) \). Clearly there is no embedding \( \rho \subset \pi \) such that \( \rho \) is of the form \( X_8(j' - 1)X_5(j') \) or \( X_5(j')X_6(j') \). Hence for every two embeddings \( \rho_1, \rho_2 \subset \pi, \rho_1, \rho_2 \in \varepsilon(R) \), the relation (5) holds.

From the definition (2) we have

\[
u = u(X_3(j - 1)X_4(j)X_1(j) \subset X_a(i)X_3(j - 1)X_4(j)X_1(j))
\]

\[= X_a(i)r(X_3(j - 1)X_4(j)X_1(j)) + v\]

\[= X_a(i)X_3(j - 1)r(X_5(j)X_1(j)) - X_a(i)r(X_3(j - 1)X_5(j))X_1(j) + v,\]

where \( v \in R(\pi) \) may arise from commuting elements to the indicated positions. Note that the leading term of \( u \) is \( \pi \).

If \( i = j - 2 \), then \( a = 3, 1 \), and by using the relation (5) for two embeddings in \( X_a(j - 2)X_3(j - 1)X_5(j) \) we get from (9)

\[u \in X_a(i)X_3(j - 1)r(X_5(j)X_1(j)) - r(X_a(i)X_3(j - 1)X_5(j)X_1(j)) + R(\pi_0),\]

where \( \pi_0 \) is defined in Lemma 13. By using two different expansions, we get \( u \in R(\pi_0) \), i.e.

\[u = \sum_{\pi_0 < \pi' \leq \pi} \sum_{\rho \in E(\pi')} c_{\rho, \pi'} u(\pi'/\rho) + R(\pi)\]

for some coefficients \( c_{\rho, \pi'} \in \mathbb{F} \). By Proposition 10 and Lemma 13 for all embeddings \( \rho \subset \pi', \pi_0 < \pi' \leq \pi, \) we can apply (5), so for each \( \pi' \) we choose a particular \( \rho_{\pi'} \subset \pi', \rho_{\pi'} \in \varepsilon(R) \), and (by using (5) if necessary) we get

\[u = \sum_{\pi_0 < \pi' \leq \pi} c_{\pi'} u(\pi'/\rho_{\pi'}) + R(\pi)\]

for some coefficients \( c_{\pi'} \in \mathbb{F} \). Since \( \varepsilon(u) = \pi \), we get by induction (cf. [MP2, Lemma 9.4]) that \( c_{\pi'} = 0 \) for \( \pi_0 < \pi' < \pi \), and hence

\[X_a(j - 2)r(X_3(j - 1)X_4(j)X_1(j)) \in c_{\pi} u(\pi/\rho_{\pi})r(\rho_{\pi}) + R(\pi)\]

for some particular embedding \( \rho_{\pi} \subset \pi \), \( \rho_{\pi} \in \varepsilon(R) \). Now (11) and (5) imply (8) for any \( \rho_1, \rho_2 \subset \pi \).

If \( i = j - 1 \), then \( a = 7, 5, 4, 3, 2, 1 \), and by using the relation (5) for two embeddings in \( X_a(j - 1)X_3(j - 1)X_5(j) \) we get from (9)

\[u \in X_a(i)X_3(j - 1)r(X_5(j)X_1(j)) - r(X_a(i)X_3(j - 1)X_5(j)X_1(j)) + R(\pi_0)\]

for all \( a = 7, 5, 4, 3, 2, 1 \).

If \( i = j \), then \( a \in \{8, \ldots, 1\} \setminus \{5\} \). By using the relation (5) for two embeddings in \( X_a(j - 1)X_3(j - 1)X_5(j) \) we get from (9)

\[u \in X_a(i)X_3(j - 1)r(X_5(j)X_1(j)) - r(X_a(i)X_3(j - 1))X_5(j)X_1(j) + R(\pi_0)\]

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for \( a = 8, 7, 6, 3 \). By using (5) for two embeddings in \( X_5(j)X_a(j)X_1(j) \) we get from (9)

\[
u \in X_3(j-1)X_5(j)r(X_a(i)X_1(j)) - r(X_3(j-1)X_5(j))X_a(i)X_1(j) + \mathcal{R}(\pi_0)\]

for \( a = 2, 1 \). For \( a = 4 \), by using (5), we get from (9) that

\[
u = CX_3(j-1)r(X_5(j)X_a(i))X_1(j) + \mathcal{R}(\pi_0)\]

Since \( \pi_0 \prec \pi = \theta(u) \), this implies that \( C = 0 \) and \( u \in \mathcal{R}(\pi_0) \).

If \( i = j + 1 \), then \( a \in \{8, \ldots, 1\} \), and by using (5) for two embeddings in \( X_5(j)X_1(j)X_a(j+1) \) we get from (9)

\[
u \in X_3(j-1)X_5(j)r(X_1(j)X_a(i)) - r(X_3(j-1)X_5(j))X_1(j)X_a(i) + \mathcal{R}(\pi_0)\]

for all \( a \in \{8, \ldots, 1\} \).

In either of these cases we get \( u \in \mathcal{R}(\pi_0) \), i.e. (10), and we argue as before that (8) holds for any \( \rho_1, \rho_2 \subseteq \pi \).

The case when \( \pi = X_a(i)X_5(j-1)X_4(j-1)X_6(j) \) is similar.

Finally, let us consider the case \( \pi = X_5(j-2)X_8(j-1)X_4(j-1)X_1(j-1)X_0(j) \), \( \rho_1 = X_8(j-1)X_4(j-1)X_6(j) \). By applying relations for the embeddings in \( X_3(j-2)X_s(j-1)X_4(j-1)X_1(j-1)X_6(j) \) and \( X_3(j-2)X_s(j-1)X_4(j-1)X_6(j) \) that we have already proved we get

\[
r(X_3(j-2)X_4(j-1)X_1(j-1))X_8(j-1)X_6(j)\]

\[
= r(X_3(j-2)X_8(j-1)X_4(j-1)X_1(j-1)X_6(j)) + \mathcal{R}(\pi)\]

\[
= X_3(j-2)X_1(j-1)r(X_8(j-1)X_4(j-1)X_6(j)) + \mathcal{R}(\pi),\]

so (8) holds in this case as well. \( \square \)

Lemma 14 is the last step in the proof of Theorem 11.

Proof of Theorem A. Note first that the order \( \asymp \) on \( \mathcal{P}(\bar{B}_{<0}) \) is a (reverse) well order which behaves well with respect to multiplications (cf. Lemmas 6.2.1 and 6.2.2 in [MP2]). Since the relations \( r(\rho), \rho \in \mathcal{R} \), vanish on \( L(\Lambda_0) \), we see by induction that the set of vectors

\[
u(\pi)v_0, \quad \pi \in \mathcal{P}(\bar{B}_{<0}) \setminus (\mathcal{P}(\bar{B})\mathcal{R})\]

form a spanning set of \( L(\Lambda_0) \). Of course, here \( \pi \in \mathcal{P}(\bar{B}_{<0}) \setminus (\mathcal{P}(\bar{B})\mathcal{R}) \) means that \( u(\pi) \) satisfies the difference \( \mathcal{R} \) condition defined previously.

Since \( L(\Lambda_0) = N(\Lambda_0)/N^1(\Lambda_0) \), linear independence of this spanning set will follow from Proposition 6.3.2 in [MP2], provided we can construct a basis of \( N^1(\Lambda_0) \) parametrized by \( \mathcal{P}(\bar{B}_{<0}) \cap (\mathcal{P}(\bar{B})\mathcal{R}) \).

Since \( N^1(\Lambda_0) = U(\mathfrak{g}, 0)\bar{R}1 \) (see Section 5 in [MP2]), the set of vectors

\[
u(\rho \subseteq \pi)1, \quad \rho \in \mathcal{R}, \pi \in \mathcal{P}(\bar{B}_{<0}) \cap (\mathcal{P}(\bar{B})\mathcal{R}), \rho \subseteq \pi\]

is a spanning set of \( N^1(\Lambda_0) \). Now for each \( \pi \) we choose precisely one \( \rho = \rho(\pi) \) such that \( \rho \subseteq \pi \). By using Theorem 11, and induction on \( \asymp \), we see that

\[
u(\rho(\pi) \subseteq \pi)1, \quad \pi \in \mathcal{P}(\bar{B}_{<0}) \cap (\mathcal{P}(\bar{B})\mathcal{R})\]

is a spanning set as well. Since \( \theta_{N(\Lambda_0)}(\nu(\rho(\pi) \subseteq \pi)1) = \nu(\pi)1 \), this spanning set is obviously linearly independent, and hence the desired basis of \( N^1(\Lambda_0) \). \( \square \)
Let $A$ be a nonempty set and denote by $\mathcal{P}(A)$ the set of all maps $f : A \to \mathbb{N}$, where $f_a = f(a)$ equals zero for all but finitely many $a \in A$. We say that $f$ is a partition and for $f_a > 0$ we say that $a$ is a part of $f$.

For nonempty subsets $A_1, \ldots, A_s \subset \mathbb{Z}_{>0}$, $s \geq 1$, let $A = A_1 \bigcup \cdots \bigcup A_s$ be a disjoint union of sets. We call the elements of $\mathcal{P}(A)$ colored partitions with parts in $A$, where for $c \in \{1, \ldots, s\}$ and $j \in A_c$ we say that $j$ is of color $c$ and of degree $|j| = j \in \mathbb{Z}_{>0}$. We define the degree $|f|$ of $f \in \mathcal{P}(A)$ as $|f| = \sum_{a \in A} |a| f_a$ and we say that $f$ is a colored partition in $s$ colors of the nonnegative integer $|f|$.

The explicit construction of the basic module for the affine Lie algebra $\mathfrak{sl}(3, \mathbb{C})^\sim$, given by Theorem A, implies the following combinatorial identity of Rogers-Ramanujan type:

**Theorem B.** The number of partitions $f$ in one color (say $A = \mathbb{Z}_{>0}$) of a nonnegative integer $n$ such that each part appears at most twice (i.e. $f_a \leq 2$) equals the number of partitions $f$ in three colors (say $A = \{r, r, r \mid r \in \mathbb{Z}_{>0}\}$) of $n$ such that each part appears at most once (i.e. $f_a \leq 1$), but subject to the conditions

\begin{align*}
(1) & \quad f_a > 0 \text{ implies } |r| \equiv \pm 1 \text{ mod } 3, \\
(2) & \quad |r| > 1, \ |r| > 1, \ |r| \neq 2, \\
(3) & \quad f_a + f_b \leq 1 \text{ for } ||a| - |b|| \leq 1,
\end{align*}

and the conditions

\begin{align*}
(4) & \quad f_{3i+2} + f_{3i+3} + f_{3i+4} + f_{3i+5} \leq 1, \\
& \quad f_{3i+2} + f_{3i+3} + f_{3i+4} + f_{3i+5} \leq 1, \\
& \quad f_{3i+2} + f_{3i+3} + f_{3i+4} + f_{3i+5} \leq 1, \\
& \quad f_{3i+1} + f_{3i+2} + f_{3i+3} + f_{3i+4} \leq 1, \\
& \quad f_{3i+1} + f_{3i+2} + f_{3i+3} + f_{3i+4} \leq 1, \\
& \quad f_{3i+1} + f_{3i+2} + f_{3i+3} + f_{3i+4} \leq 1, \\
& \quad f_{3i+1} + f_{3i+2} + f_{3i+3} + f_{3i+4} \leq 1,
\end{align*}

\begin{align*}
(5) & \quad f_{3i+1} + f_{3i+3} + f_{3i+5} \leq 2, \\
& \quad f_{3i+1} + f_{3i+3} + f_{3i+5} \leq 2
\end{align*}

for all $i$ in $\mathbb{N}$.

**Proof.** By using the Lepowsky numerator formula for the Weyl-Kac character formula (cf. [K], [L]), we can write the principally specialized character of the basic $\mathfrak{sl}(3, \mathbb{C})^\sim$-module as the infinite product

$$
\prod_{r \not\equiv 0 \mod 3} (1 - q^r)^{-1} = \prod_{r \geq 1} (1 + q^r + q^{2r}).
$$

This product can be interpreted as the generating function $\sum_{n \geq 0} c_n q^n$ of the partition function $c_n$ counting the number of partitions $f$ of a nonnegative integer $n$ such that each part $r$ appears at most twice.
On the other hand, using Theorem A we can describe the principally specialized character of the basic module $L(\Lambda_0)$: Consider the isomorphism of monoids $\varphi: \mathcal{P}(\bar{B}_{<0}) \to \mathcal{P}(A)$ defined by the bijection

$$
\varphi: \bar{B}_{<0} \to A = \{j \mid j \geq 2\} \bigsqcup \{i \mid j \geq 2\} \bigsqcup \{3i \pm 2 \mid i \geq 1\},
$$

$$
X_1(-i) \mapsto 3i - 2, \quad X_8(-i) \mapsto 3i + 2,
X_2(-i) \mapsto 3i - 1, \quad X_7(-i) \mapsto 3i + 1,
X_3(-i) \mapsto 3i - 1, \quad X_6(-i) \mapsto 3i + 1,
X_4(-i) \mapsto 3i, \quad X_5(-i) \mapsto 3i.
$$

In particular, we (would) have $f_1 = X_7(0) \mapsto 1$, $f_2 = X_6(0) \mapsto 1$, $f_0 = X_1(-1) \mapsto 1$, and by setting $|n| = |\bar{n}| = |\underline{n}| = n \in \mathbb{Z}_{>0}$ the map

$$
X_s(-i) \mapsto \deg X_s(-i) = |\varphi(X_s(-i))|
$$

induces the principal specialization of the character of the basic module: the degree of the vector $X_{i_1}(j_1)X_{i_2}(j_2) \ldots X_{i_s}(j_s)v_0$ is the sum $\sum \deg X_{i_r}(j_r)$.

Note that $\mathcal{P}(\bar{B}_{<0}) \setminus (\mathcal{P}(\bar{B}) \mathcal{R})$ is a partition ideal (cf. [A], [MP2, 11.2]) defined by the difference $\mathcal{R}$ conditions. The condition that a colored partition $\pi = (\pi_a \mid a \in \bar{B}_{<0})$ does not contain $X_s(j)X_s(j)$ for $j < 0$ and $s = 1, \ldots, 8$ can be written as $\pi X_s(j) \leq 1$, meaning that the part $X_s(j)$ may appear in $\pi$ at most once. Since $\varphi$ is an isomorphism, for the corresponding colored partition $f = \varphi(\pi) = \pi \circ \varphi^{-1}$, $f = (f_a \mid a \in A)$, this condition reads that the part $a$ may appear in $f$ at most once. Now the condition that $\pi$ does not contain $X_1(-i - 1)X_1(-i)$ can be written as $\pi X_1(-i - 1) + \pi X_1(-i) \leq 1$, and the corresponding condition for $f$ as $f_{3i+1}f_{3i-2} + f_{3i+1}f_{3i-2} \leq 1$.

If we write down all difference $\mathcal{R}$ conditions for $\pi$ and the corresponding conditions for $f$, we can arrange them to take the form (3)–(5), as stated in the theorem. The conditions (5) come from the two cubic terms in $\mathcal{R}$, the conditions (2) represent the “initial” conditions $f_1 \cdot v_0 = 0$ and $f_2 \cdot v_0 = 0$ for the basic module. \[\square\]

Remark. Let us remark that the conditions (3) and (4) stated in Theorem B represent over fifty conditions of the form $f_a + f_b \leq 1$. We tried to write them in a compact and comprehensive form, but still without understanding in what form this identity might be generalized to higher levels or higher ranks.
References

[AAG] K. Alladi, G. E. Andrews and B. Gordon, *Refinements and generalizations of Capparelli’s conjecture on partitions*, J. Algebra 174 (1995), 636-658.

[A] G. E. Andrews, *The theory of partitions*, Encyclopedia of math. and appl., Addison-Wesley, Amsterdam, 1976.

[C] S. Capparelli, *A construction of the level 3 modules for the affine Lie algebra $A_2^{(2)}$ and a new combinatorial identity of the Rogers-Ramanujan type*, Trans. AMS 348 (1996), 481–501.

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex operator algebras and the Monster*, Pure and Applied Math., Academic Press, San Diego, 1988.

[K] V. G. Kac, *Infinite-dimensional Lie algebras* 3rd ed., Cambridge Univ. Press, Cambridge, 1990.

[L] J. Lepowsky, *Application of the numerator formula to k-rowed plane partitions*, Advances in Math. 35 (1980), 179–194.

[LP] J. Lepowsky and M. Primc, *Structure of the standard modules for the affine Lie algebra $A_1^{(1)}$*, Contemporary Math. 46 (1985).

[LW] J. Lepowsky, R. L. Wilson, *The structure of standard modules, I: Universal algebras and the Rogers-Ramanujan identities*, Invent. Math. 77 (1984), 199–290; *II: The case $A_1^{(1)}$, principal gradation*, Invent. Math. 79 (1985), 417–442.

[Ma] M. Mandia, *Structure of the level one standard modules for the affine Lie algebras $B_1^{(1)}$, $F_4^{(1)}$ and $G_2^{(1)}$*, Memoirs American Math. Soc. 362 (1987).

[MP1] A. Meurman, M. Primc, *Annihilating ideals of standard modules of $\mathfrak{sl}(2,\mathbb{C})^\sim$ and combinatorial identities*, Advances in Math. 64 (1987), 177–240.

[MP2] A. Meurman and M. Primc, *Annihilating fields of standard modules of $\mathfrak{sl}(2,\mathbb{C})^\sim$ and combinatorial identities*, Memoirs American Math. Soc. 652 (1999).

[Mi] K. C. Misra, *Level one standard modules for affine symplectic Lie algebras*, Math. Ann. 287 (1990), 287–302.

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