Beyond Structured Programming

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Abstract

The correctness of a structured program is, at best, plausible. Though this is a step forward compared to what came before, it falls short of verified correctness. To verify a structured program according to Hoare’s method one is faced with the problem of finding assertions to fit existing code. In 1971 this mode of verification was declared by Dijkstra as too hard to be of practical use—he advised that proof and code were to grow together. A method for doing this was independently published by Reynolds in 1978 and by van Emden in 1979. The latter was further developed to attain the form of matrix code. This form of code not only obviates the need of fitting assertions to existing code, but helps in discovering an algorithm that reaches a given postcondition from a fixed precondition. In this paper a keyboard-editable version of matrix code is presented that uses E.W. Dijkstra’s guarded commands as starting point. The result is reached by using Floyd’s method rather than Hoare’s as starting point.

1 Introduction

Structured Programming is the name of a method introduced by E.W. Dijkstra in 1969 [6]. At the time the method was revolutionary: it advocated the elimination of the goto statement and the exclusive use of conditional, alternative, and repetitive clauses. In a few years structured programming evolved from controversy to orthodoxy. C.A.R. Hoare invented a logic-based verification method for structured programs [13].

In the late 1960’s Dijkstra developed the conviction that the looming “software crisis” could only be averted by formally verifying all code. At the time Hoare’s method was the only one available. It was found difficult to use in practice. Dijkstra maintained his conviction that code needed to be verified. He defended it in his 1971 paper “Concern for correctness as guiding principle in program composition” [7]. In this paper Dijkstra advocated developing the code and its correctness proof in parallel. He did not include any suggestions as to how this might be done. I accept concern for correctness as guiding principle in program composition. In this paper I review the first two stages of a published method that acts on this concern and present a recent, so far unpublished, third stage.

The method is based on Floyd’s verification method for flowcharts [12]. To verify such programs, assertions are attached to suitably selected labels in the code. These assertions have to be such that they are true of the computational state whenever execution reaches the associated label. This property is assured by the way the flowchart connects the assertions by tests or assignment statements. The connection is expressed by what Floyd calls “verification conditions”, which are triples written by Floyd as \( V_C(p; q) \) where \( C \) is a test or assignment statement, \( p \), the “precondition” is an assertion attached to the input node of \( C \), and the “postcondition” is an assertion attached to the output node of \( C \). Hoare wrote \( \{ p \} C \{ q \} \) instead of \( V_C(p; q) \). Most subsequent researchers wrote \( \{ p \} C \{ q \} \) and called it “Hoare triple”. In this paper I will use the latter notation and refer to the triple as “verification condition”.

1

5

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Floyd left some things unsaid concerning the connection between logic and code. With the missing connection in place, his idea deserves the name “Floyd Logic”. He gave only the application to flowcharts, although its scope is wider. This paper explains the wider scope and defines the nature of Floyd Logic.

Floyd’s method applies to flowcharts; Hoare’s method is formulated in terms of the control primitives favoured by the “structured programming” proposed by Dijkstra and widely embraced. This style of programming results in more readable code. Of course readable code is preferable to code that is unreadable without having any redeeming properties. But, without a formal Hoare-style proof, the correctness of a structured program is, at best, plausible. If we are more ambitious and want verified code, then a structured program presents us with complete code for which suitable assertions need to be found.

Such a situation was considered by Dijkstra in a 1971 paper \[7\] from which I quote

When correctness concerns come as an afterthought and correctness proofs have to be given once the program is already completed, the programmer can indeed expect severe troubles. If, however, he adheres to the discipline to produce the correctness proofs as he programs along, he will produce program and proof with less effort than programming alone would have taken.

Let us call this “Dijkstra’s Principle”. Here he implicitly enjoined his readers to find an alternative to Structured Programming. But about the form this alternative is to take, nothing more specific is to be found than that the programmer is to “produce the correctness proofs as he programs along”. In a later paper \[8\] he advocated

... challenging the choice of a posteriori verification of given programs as the most significant problem, the argument against that choice being, that programs are not “given” but must be designed. Instead of trying to find out which known proof-patterns are applicable when faced with a given program, the program designer can try to do it the other way round: by first choosing the proof-pattern that he wants to be applicable, he can then write his program in such a way as to satisfy the requirements of that proof.

The first alternative that might fit Dijkstra’s suggestions in \[7,8\] was proposed, according to R-J. Back \[3\], by J. Reynolds \[16\] and by M.H. van Emden \[19\]. Van Emden’s method, “programming with verifications”, was developed into the Matrix Code of 2014 \[20\].

To balance the generalities considered so far, let us look at a concrete example of Programming with Verification Conditions in the form of the program in Figure 1. The purpose of the function is to compute the greatest common divisor in time that is logarithmic in the greatest of its arguments. The coding style, or, rather, the absence of any, may well leave readers at a loss for words. I present the listing here to make the point that readability in the usual sense is not relevant, as we should be aiming at verifiability rather than mere readability.

And verified it is: the function in Figure 1 is verified in the sense that it is proved that executing a call $\text{gcd}(x, y)$ results in $\text{gcd}(x, y)$. The proof consists in the verification that whenever execution reaches a label, the assertion associated with that label (the one shown as a comment) holds. This follows from the truth of the verification conditions (in the sense of Floyd \[12\]) listed in Figure 2. It is verified so easily because that it was written with this goal in mind.

These verification conditions need to be read with some indulgence. An intuitive guideline is that both tests and assignment statements can be interpreted as binary relations over states and can be composed as binary relations with the symbol “;”. Moreover, assertions can be interpreted as sets of states. In the sequel these interpretations will be stated more precisely.

The validity of the verification conditions relies on equalities concerning the $\text{gcd}$ function shown in Figure 3. The method of obtaining a logarithmic-time algorithm based on these properties is due to Stein and brought to my attention by reading \[17\].

As truth is a property of each verification condition by itself, their ordering in Figure 2 does not matter. In Floyd’s verification method there is a close correspondence between the code and
int gcd(int x, int y) {
    S: // x=x0 & y=y0 & x0>0 & y0>0
        int z = 1; goto A;
    A: // gcd(x0,y0) = z*gcd(x,y)
        if (x == y) { z *= x; goto H; }
        if (x > y) { goto B; }
        if (x < y) { swap(&x, &y); goto B; }
        assert(0);
    B: // A & x>y
        if (y == 1) goto H;
        if (y > 1) goto E;
        assert(0);
    E: // B & y>1
        if (x%2 == 0) goto C;
        if (x%2 != 0) goto D;
        assert(0);
    C: // B & even(x)
        if (y%2 == 0) { x /= 2; y /= 2; z *= 2; goto B; }
        if (y%2 != 0) { x /= 2; goto A; }
        assert(0);
    D: // B & odd(x)
        if (y%2 == 0) { y /= 2; goto B; }
        if (y%2 != 0) { x = x/2-y/2; goto A; }
        assert(0);
    H: // z = gcd(x0, y0)
        return z;
}

Figure 1: Not your ordinary spaghetti code: every label is associated with an assertion. Each goto statement is to be read as a claim that the computational state satisfies the assertion associated with the label. The program can be read as a collection of verification conditions. Moreover, it can be seen that execution does not traverse a closed circuit through the code without decreasing |x−y|.
As the verification conditions are all true, it follows that, when label H is reached, z has the desired value.

{S} int z = 1 {A}
{A} x == y; z *= x {H}
{A} x > y {B}
{A} x < y; swap(&x, &y) {B}
{B} y == 1 {H}
{B} y > 1 {E}
{E} x%2 == 0 {C}
{E} x%2 != 0 {B}
{C} y%2 == 0; x /= 2; y /= 2; z *= 2 {B}
{C} y%2 != 0; x /= 2 {A}
{D} y%2 == 0; y /= 2 {B}
{D} y%2 != 0; x = x/2-y/2 {A}

Figure 2: Verification conditions for Figure 1
\[
gcd(x, x) = x \]
\[
gcd(x, 1) = 1 \]
\[
gcd(x, y) = \gcd(y, x) \]
\[
gcd(x, y) = 2 \cdot \gcd(x/2, y/2) \text{ if } \text{even}(x) \land \text{even}(y) \]
\[
gcd(x, y) = \gcd(x/2, y) \text{ if } \text{even}(x) \land \text{odd}(y) \]
\[
gcd(x, y) = \gcd(x, y/2) \text{ if } \text{odd}(x) \land \text{even}(y) \]
\[
gcd(x, y) = \gcd([x/2] - [y/2], y) \text{ if } \text{odd}(x) \land \text{odd}(y) \land x > y \]

Figure 3: Properties used in Stein’s algorithm of the \textit{gcd} function on positive integers.

the verification conditions. To make this correspondence easy to see, I have placed verification conditions with the same precondition next to each other. This makes it easy to transcribe the set of verification conditions into code. It matters little whether the code is in the form of a flowchart (as in Floyd’s paper) or in the form of C code (as in Figure 1). In both cases the correspondence between sets of verification conditions and code is close—Figure 1 can be compiled, yet in a sense is a theorem of mathematics. In case of trouble, one must realize that errors are not uncommon in theorems and their proofs. When some of these show by the code giving wrong results or no results, we are lucky, but we should not make the mistake of trying to debug the code: there must be an error in the theorem or in its transcription—tracking that down is a more productive activity than debugging code.

2 Structured Programming

“Structured programming” has been, and continues to be, widely influential and, arguably, widely misunderstood. It is therefore useful to examine how the term arose, its various meanings, and how it became influential.

As Dijkstra recounts in his retrospective memorandum EWD1308 [10], 1968 was a turning point in his career. At this point he realized that in the larger scheme of things his two biggest successes in software system implementation, the first Algol 60 compiler of 1960 and the recently completed THE operating system, should be regarded as mere “agility exercises”. He decided “to tackle the real problem of How to Do Difficult Things”. To get started he wrote a memorandum with title “Notes on structured programming”, completed August 1969. By the time it was published in 1971 [4], it had already become influential via the grapevine.

In 1968 Dijkstra had submitted a short note to the Communications of the ACM under the title “A case against the goto statement”. To expedite publication the editor turned it into a Letter to the Editor. Such letters do not carry a title and are supplied by the editor with a phrase summarizing the content. Thus it appeared under the heading “Goto Considered Harmful” [5]. Although Dijkstra did not discuss or even mention structured programming, the combined effect of the Letter to the Editor and “Notes on structured programming” was that structured programming was equated to abstaining from the use of the goto statement. In 2001 Dijkstra wrote [10] “IBM . . . stole the term ’Structured Programming’ and under its auspices Harlan D. Mills trivialized the original concept to the abolishment of the goto statement.” Indeed a far cry from the intention of structured programming, but who could have guessed it was meant as a step toward “learning How to do Difficult Things”.

I sense that Dijkstra’s sense of urgency was driven by two considerations: (1) the rapid advance
in computer hardware will present the opportunity to build systems requiring millions of lines of code and (2) without drastic methodological measures such systems will never be made to perform reliably. Dijkstra’s conclusion was that structured programming would not be enough; only a correctness proof would do. He realized that, for example, Hoare’s method of inserting assertions into a structured program is too hard to be practically usable and concluded [7] that correctness proof would only be feasible if the code was designed ab initio to be easily proved correct. The question of abstention from the goto statement did not come up. Correctly so: structured programming aims at readability; verifiability is a different, independent, and more important goal.

While Dijkstra was distressed by the trivialization of structured programming, C.A.R. Hoare reports [14] with equanimity the fate of the concept in the decades following:

The decisive breakthrough in the adoption of structured programming by IBM was the publication of a simple result in pure programming theory, the Bohm-Jacopini theorem. This showed that an arbitrary program with jumps could be executed by an interpreter written without any jumps at all; so in principle any task whatsoever can be carried out by purely structured code. This theorem was needed to convince senior managers of the company that no harm would come from adopting structured programming as a company policy; and project managers needed it to protect themselves from having to show their programmers how to do it by rewriting every piece of complex spaghetti code that might be submitted. Instead the programmers were just instructed to find a way, secure in the knowledge that they always could. And after a while, they always did.

One shudders to think of the contortions to which the code was subjected so that programmers could satisfy project managers: “Look! No goto’s!” But Hoare’s intention was to convey the impression that structured programming was responsible for more than cosmetic changes.

3 Floyd Logic

Both Floyd [12] and Hoare [13] use the same identifiers for variables in assertions as for variables in program code. Apparently they do not consider this double use worth even a passing remark, whereas it seems to some readers not obvious that there is any connection between the variables of logic (the things one quantifies over, for example) and the variables in a programming language (names for memory locations). Whatever connection Floyd [12] and Hoare [13] had in mind was obviated by Apt [1] where the need to make such connection is avoided by creating a new logic that included among its expressions some that can be recognized as programming language constructs. In this paper I follow Floyd and Hoare in leaving the language of assertions unchanged at first-order predicate logic. I therefore have to establish the connection between program variables and logic variables. I do this by defining the meaning of assertions and program fragments in a suitable way.

The meaning of an assertion is a set of states

Each assertion is a formula $F$ of first-order predicate logic. For each free variable in $F$ there is a program variable with the same identifier.

The collection $A$ of assertions of a program have a common vocabulary, which consists of function symbols, predicate symbols, and a common set $V$ of variables.

An interpretation for $A$ consists of a universe of discourse $D$, an assignment of relations over $D$ to the predicate symbols, and an assignment of functions over $D$ to the function symbols. The truth $M^I_1(F)$ of a formula $F$ depends not only on an interpretation $I$, but also on a state, which is a mapping of type $V \to D$ of each variable in $V$ to an element of $D$. Hence a state is a vector of elements of $D$ indexed by elements of $V$.

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3 In some logic texts a “valuation”.
The meaning \( M^I(F) \) of a formula \( F \) in an interpretation \( I \) is the set of states in which it is true under that interpretation: \( M^I(F) = \{ s \in (V \rightarrow D) \mid M^I_s(F) \} \). In other words, the meaning is a relation consisting of tuples of type \( V \rightarrow D \). In other words, the meaning is a subset of \( V \rightarrow D \).

Typically, the set \( V_F \) of free variables in \( F \) is a strict subset of \( V \). Then \( s \in M^I(F) \) implies that \( s' \in M^I(F) \) for any \( s' \) that only differs from \( s \) in variables not in \( V_F \).

Truth or falsity of a formula is determined by an interpretation and a state individually for this formula. In Floyd’s logic the unit of interpretation is extended to be the collection \( A \) of assertions of the same program. This is the only difference between Floyd’s logic and first-order predicate logic. It is a difference because \( A \), although consisting of formulas of first-order predicate logic, is not itself a sentence or formula of this logic.

The meaning of a program is a binary relation over states

By “program” I understand anything that can be characterized as a binary relation over the set of states; hence as a set of pairs of the form \( \langle \text{state before}, \text{state after} \rangle \).

Let us first establish what to include under “program”. We might as well make it general, observing that the structured programs of Hoare [13] are a special case of the flowcharts of [12], which are themselves a special case of programs defined as follows.

1. A formula \( F \) of first-order predicate logic is a program. The meaning of program \( F \) is the binary relation \( \{ \langle s, s \rangle \mid M^I_s(F) \} \), which is the set of pairs of equal states \( s \) in which the formula is true in \( I \).

2. \( "v := t" \) is a program, where \( v \) is a variable and \( t \) is a term, both “variable” and “term” in the sense of first-order predicate logic. Let \( M^I_t(t) \in D \) be the meaning of term \( t \) in state \( s \) according to Tarskian semantics. Then the meaning of \( "v := t" \) is the set \( \langle p, q \rangle \) of pairs of states such that \( q(v) = M^I_p(t) \) and \( q(x) = p(x) \) if \( x \in V \) is not the same variable as \( v \).

The meaning of a verification condition

\( \{ p \} C \{ q \} \) is true iff for all \( s \) and \( t \) in \( V \rightarrow D \), \( \langle s, t \rangle \in M^I(C) \land s \in M^I(p) \) implies \( t \in M^I(q) \).

4 Programming language

The listing of verification conditions in Figure 2 is satisfactory for the verification of the program in Figure 1. But we want to avoid having to find verification conditions for existing code. As we will show, it is easier to write verification conditions first and then write code that is verified by it. This is achieved by Matrix Code. However, its two-dimensional format makes it awkward to enter and edit. It helps to create an intermediate form in a text file that is edita ble by keyboard. This has to be so that it is easy to transcribe verification conditions to it and such that it is easy to write compilable code from it. This is a way of following Dijkstra’s Principle, and the only way I know of.

For this intermediate text the \( \text{if...fi} \) statement of Dijkstra’s guarded-command language [9] is a good starting point. We make two modifications. The first is to attach a label to each \( \text{if...fi} \) and to associate an assertion with this label. The second modification is to append \( \text{goto L} \) at the end of each guarded command to claim that the assertion labelled by \( L \) is true when execution reaches that point. In this way each guarded command closely mirrors a verification condition and is easy to transcribe into compilable code.

Guards are written as Algol boolean expressions, hence “=" for equality. The assignment operator is written as “:=”. Text between a label and an \( \text{if} \) is comment. It is used for the assertion associated with the label. The Pascal convention of \( \{ \ldots \} \) is used for comments.

As is the case in Dijkstra’s guarded-command language, \( \text{if...fi} \) where all guards are false is equivalent to \( \text{abort} \). This command can be added after every \( \text{fi} \) without altering the meaning, so is normally not written. It is useful when a replacement for \( \text{if...fi} \) is needed.
S: \( x=x_0 \& y=y_0 \)
  \[
  \begin{align*}
  &\text{if } \text{true} \rightarrow z := 1; \text{goto } A \\
  &\text{fi}
  
  
  A: \gcd(x_0,y_0) = z \gcd(x,y) \\
  &\text{if } x=y \rightarrow z := z \times x; \text{goto } H \\
  &| \ x>y \rightarrow \text{goto } B \\
  &| \ x<y \rightarrow \text{swap}(x, y); \text{goto } B \\
  &\text{fi}
  
  B: A \& x>y \\
  &\text{if } y=1 \rightarrow \text{goto } H \\
  &| \ y>1 \rightarrow \text{goto } E \\
  &\text{fi}
  
  E: B \& y>1 \\
  &\text{if } \text{even}(x) \rightarrow \text{goto } C \\
  &| \ \text{odd}(x) \rightarrow \text{goto } D \\
  &\text{fi}
  
  C: B \& \text{even}(x) \\
  &\text{if } \text{even}(y) \rightarrow x, y, z := x/2, y/2, 2 \times z; \text{goto } B \\
  &| \ \text{odd}(y) \rightarrow x := x/2; \text{goto } A \\
  &\text{fi}
  
  D: B \& \text{odd}(x) \\
  &\text{if } \text{even}(y) \rightarrow y := y/2; \text{goto } B \\
  &| \ \text{odd}(y) \rightarrow x := x/2 - y/2; \text{goto } A \\
  &\text{fi}
  
  H: \gcd(x_0,y_0) = z \\
  &\text{return } z
  
  \]

Figure 4: The Liffig program that is obtained by transcription from the verification conditions in Figure 2. From this Liffig program the compilable C code can be obtained by transcription. I regard neither the verification conditions nor the transcriptions to be in need of justification, but may be in need of scrutiny for possible errors. The name “Liffig” is derived from the \texttt{if...fi} construct of E.W. Dijkstra’s guarded-command language.

I call this language “Liffig” because it symbolizes an \texttt{if...if} preceded by a label, which is where the L comes from, and is followed by a goto, which is where the g comes from. The Liffig text to serve as intermediate between Figure 2 and Figure 1 is in Figure 4. I agree that it is presumptuous to dream up a name for such a meagre parody of a language. Yet, when one wants to talk about something it helps to have a name for that something. I can report that one gets used to “Liffig”.

5 Program construction by gathering snippets of truth

Dijkstra’s Principle is “to produce the correctness proofs as he programs along”. With a bit of good will, programming with verification conditions can be regarded as a realization of the principle. Here partial correctness is the key. Partial correctness means that the program cannot do anything wrong. It is short of total correctness when it does not enough. So one can always start by achieving partial correctness with a program having the desired precondition and postcondition that does nothing at all. Successively larger parts of the problem are solved by adding patently correct increments in the form of true verification conditions. The importance of programs and assertions as defined here is that these allow us to follow the latter approach. In Liffig, programs can grow by adding verification
conditions. By choosing verification conditions with sufficiently simple program components their truth is obvious. Because of this simplicity, the verification condition is a mere “snippet of truth”. Hence “program construction by gathering snippets of truth”, a useful slogan to characterize this realization of Dijkstra’s Principle.

5.1 Example: GCD

Let us follow Dijkstra’s Principle to obtain the program in Figure 4. The snippets of truth are obtained from properties of the gcd function (Figure 3), starting with gcd(x, x) = x.

S: x=x0 & y=y0
    if true -> z := 1; goto A
    fi
A: gcd(x0,y0) = z*gcd(x,y)
    if x=y -> z := z*x; goto H
    fi
B: A & x>y
    abort
H: gcd(x0,y0) = z
    return z

The partial correctness is very partial indeed: we only get a successful computation when the arguments are equal.

Using gcd(x, y) = gcd(y, x) we get

S: x=x0 & y=y0
    if true -> z := 1; goto A
    fi
A: gcd(x0,y0) = z*gcd(x,y)
    if x=y -> z := z*x; goto H
    | x>y -> goto B
    | x<y -> swap(x, y); goto B
    fi
B: A & x>y
    abort
H: gcd(x0,y0) = z
    return z

This does not solve a larger part of the problem, but opens the way to more effective program increments, such as the one that uses gcd(x, 1) = 1.

S: x=x0 & y=y0
    if true -> z := 1; goto A
    fi
A: gcd(x0,y0) = z*gcd(x,y)
    if x=y -> z := z*x; goto H
    | x>y -> goto B
    | x<y -> swap(x, y); goto B
    fi
B: A & x>y
    if y=1 -> goto H
    | y>1 -> goto E
    fi
Thoughts on the example  In what sense does Figure 1 specify an algorithm? With the method demonstrated here it is merely the result of the composition of two transcriptions of the equalities in Figure 3. Of course the equalities are not a random selection from among the infinity of equalities that are true of the gcd function. Rather than an algorithm, Stein’s contribution consists of this selection.

5.2 Example: Fast exponentiation

Exponentiation is often explained as iterated multiplication. This suggests an algorithm for computing $a^n$ that uses in the order of $n$ operations. This is improved by using $a^n = (a^2)^{n/2}$ at appropriate points in the algorithm. In this way the number of operations can be reduced to the order of log $n$.

This example, a staple of introductions to programming, has a wider significance as suggested by the fact that the same idea turns up in surprising places. Stepanov and McJones [17] and Stepanov and Rose [18] explain this wider significance by pointing out that $a^n$ can be interpreted as

\[
\underbrace{a + a + \cdots + a}_{n \text{ times}}
\]

in any algebra that has a binary associative $+$ and a neutral element. This explains why the same programming trick works to obtain multiplication from addition (“Egyptian multiplication” in [18]), the computation of high powers in modular arithmetic (used in cryptography), and computation of the $n$-th Fibonacci number in logarithmic time, to mention just a few examples.

For the purpose of this paper, fast exponentiation is a good example because of different attitudes of different authors. After the simplest version Stepanov and Rose dedicate a section, 2.2, to “Improving the algorithm”, where they address various places where needless tests or operations occur. Dijkstra and Feijen [11] point out such possibilities, but loftily dismiss them: “All such attempts probably make the program text less clear, and in any case much longer”. In this paper clarity of program text is not a valid criterion, verifiability is. Section 2.2 of [18] shows that improving the simplest, most elegant, version is an instructive exercise, although opinions apparently differ as to whether it is worthwhile.

In the remainder of this section we gather snippets of truth in a way that results in an algorithm that seems to be as efficient as the one developed in section 2.2 of [18].

We begin by documenting the precondition and the postcondition of the entire program.

S: \{\ n0 \geq 0 \ & \ a = a0 \ & \ n = n0 \ & \ z = 0 \ \}\n
H: \{ z = n0*a0 \}\n
We consider whether there is sufficiently simple C such that \{S\}C\{H\}. As there is none, we find an assertion that is a common generalization of the precondition and the postcondition. This is Dijkstra’s “proof pattern that he wants to be applicable” [8].

A: \ n \geq 0 \ & \ n0*a0 = z + n*a \n
We pick off the easy cases $n = 0$ and $n = 1$. 

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S: n0 >= 0 & a = a0 & n = n0 & z = 0
if true -> goto A
fi
A: n >= 0 & n0*a0 = z + n*a 
if n=0 -> goto H
| n!=0 -> { n > 0 } goto B
fi
B: A & n>0
if n = 1 -> z := z+a; goto H
| n != 1 -> { n > 1 } goto C
fi
C: A & n>1
"reduce n while maintaining A"
H: z = n0*a0

To reduce n we can decrement or divide by two, with priority given to the latter.

S: n0 >= 0 & a = a0 & n = n0 & z = 0
if true -> goto A
fi
A: n >= 0 & n0*a0 = z + n*a 
if n=0 -> goto H
| n!=0 -> { n > 0 } goto B
fi
B: A & n>0
if n = 1 -> z := z+a; goto H
| n != 1 -> { n > 1 } goto C
fi
C: A & n>1
if even(n) -> goto D
| odd(n) -> n := n-1; z := z+a; goto D
fi
D: A & n>1 & even(n)
  n := n/2; a := a+a; goto B
H: z = n0*a0

Thoughts on the example  An attractive feature of the if...fi construct is that it can contain any number of guarded commands. This gives separation of concerns in several directions: it allows correctness to be strictly partial for the time being; it allows non-determinism for the time being.

Yet in this example there are always two guarded commands without however being equivalent to if...then...else... The rigid adherence to the binary if...fi’s is a consequence of the goal to obtain code that does not duplicate tests.

5.3 Example: Table of primes

The most prominent example in Dijkstra’s chapter in [4] is to write a program that produces a table of the first thousand prime numbers. Dijkstra chose the method of trial division. Some readers reacted with “But everyone knows that the most efficient way to generate prime numbers is by using the Sieve of Eratosthenes” ([4], page 27).

Dijkstra proceeds with an excruciatingly detailed account of successive levels of refinement, until at the end (page 37) he writes “To give the algorithm an unexpected turn we shall assume the absence of a convenient remainder computation.” He shows a simple way of avoiding the remainder operation, which turns out to be the Sieve of Eratosthenes.

10
Abstinence from a built-in remainder operation comes at a cost, namely, the need to create an auxiliary array containing suitable multiples of as many of the smallest primes as this array will hold.

I work by successive refinement by developing two Trial Division and Sieve A.

Problem statement:

```plaintext
int p[1000]; int n := 1000;
S: int p[0..n-1] is allocated & n >= 2
H: p[0..n-1] contains the first n prime numbers in incr. order }
```

I can’t think of any verification condition with S as precondition and H as postcondition. Heuristic: if direct attack is not successful, then find related easier problem that can be solved. In this case, if not all of the n prime numbers can be produced, maybe k (k < n) of them can. Which k primes? The k largest? The k smallest? Last option is more plausible as it is likely that knowing smaller ones can help in finding larger ones. Specifically, we can find the next larger prime after the largest found so far.

```plaintext
int p[1000]; int n := 1000;
S: int p[0..n-1] is allocated & n >= 2
  p[0],p[1] := 2,3; k := 2; goto A
A: S & p[0..k-1] contains the first k prime numbers in incr. order
  if k = n -> goto H
  | k != n { 2 <= k < n } ->
    "create variable cand, compute candidate for next prime after p[k-1], and assign it to cand"
    fi
H: p[0..n-1] contains the first n prime numbers in incr. order
```

How to find the next prime? Find an easier problem, one that is easy to achieve initially and can be gradually nudged closer to cand being the next prime.

```plaintext
int p[1000]; int n := 1000;
S: int p[0..n-1] is allocated & n >= 2
  p[0],p[1] := 2,3; k := 2; goto A
A: S & p[0..k-1] contains the first k prime numbers in incr. order
  if k = n -> goto H
  | k != n { 2 <= k < n } ->
    cand := p[k-1]+2; j := 0; goto B
    fi
B: A & cand not divisible by any of p[0..j]
  if cand < p[j]^2 { cand is next prime after p[k-1] } ->
    p[k] := cand; k := k+1; goto A
    | cand >= p[j]^2
t    "set of possible values for cand needs to be shrunk"
    fi
H: p[0..n-1] contains the first n prime numbers in incr. order
```

This leaves us with an untreated case of uncertainty about the primality of cand.

```plaintext
int p[1000]; int n := 1000;
  int k, cand, j;
S: int p[0..n-1] is allocated & n >= 2
  p[0],p[1] := 2,3; k := 2; goto A
A: S & p[0..k-1] contains the first k prime numbers in incr. order
  if k = n -> goto H
```
k != n { 2 <= k < n } -> cand := p[k-1]+2; j := 0; goto B
fi
B: A & cand not divisible by any of p[0..j]
   if cand < p[j]^2 { cand is next prime after p[k-1] } ->
      p[k] := cand; k := k+1; goto A
   | cand >= p[j]^2 -> j := j+1; goto C
   fi
C: A & cand not divisible by p[0..j-1]
   if cand%p[j] = 0 { cand not a prime } ->
      cand := cand+2; j := 0; goto B
   | not cand%p[j] = 0 -> goto B
   fi
H: p[0..n-1] contains the first n prime numbers in incr. order

No sub-problem left unsolved. Ready for transcription to C. The transcription leaves the structure of the Liffig code intact, so that the verification conditions can be verified in the C code. This completes Trial Division.

To obtain Sieve we need modifications at the end of Trial Division. A modification occurs at the beginning as well: the declaration of the auxiliary array for the multiples of the first few primes. In Sieve A cand is tested for divisibility by p[j] by comparing it with mult[j]. If equal, then cand is not prime. Otherwise, and if mult[j] < cand, we increase mult[j] by p[j] and compare again. This leads to the following program.

int p[1000]; int n := 1000;
int mult[30]; // mult[i] for multiple of p[i]
in k, cand, j;
S: int p[0..n-1] is allocated & n >= 2
   p[0],p[1] := 2,3; k := 2; goto A
A: S & p[0..k-1] contains the first k prime numbers in incr. order
   k <= n
   if k = n -> goto H
   | k != n { 2 <= k < n } -> cand := p[k-1]+2; j := 0; goto B
   fi
B: A & cand not div by p[0..j]
   if cand < p[j]^2 { cand is next prime after p[k-1] } ->
      p[k] := cand; k := k+1; goto A
   | cand >= p[j]^2 -> j := j+1; mult[j] := p[j]; goto C
   fi
C: A & cand not div by p[0..j-1] & cand >= mult[j]
   if cand = mult[j] -> { cand is not a prime }
      cand += 2; j := 0; goto B
   | cand > mult[j] -> mult[j] += p[j]; goto D
   fi
D: B & cand not div by p[0..j-1]
   if cand < mult[j] -> { cand not div by p[j] } goto B
   | cand >= mult[j] -> goto C
   fi
H: p[0..n-1] contains the first primes in increasing order

Thoughts on the example  In Trial Division we used refinement by first posting informally the goal
"create variable cand, compute candidate
for next prime after $p[k-1]$, and assign it to cand"

and then resolving it. In the course of this resolution another refinement cropped up in the form of

"set of possible values for cand needs to be shrunk"

Distinct from the refinement technique is the strategy of developing the least ambitious version, Trial Division, first, and then finding a suitable modification to reach Sieve.

Turning Trial Division into a possibly novel version of the Sieve of Eratosthenes was a stroke of genius on the part of Dijkstra, which he passed by without remark. He noted the difficulty that a size for the array of multiples needed to be declared in advance. He wrote “number theory gives us 30 as a safe upper bound”. Presumably that means invoking the Prime Number Theorem to get an upper bound for the 1000-th prime, taking the square root of it and invoke the Prime Number Theorem again to conclude that the 30-th prime is a safe upper bound for that square root. I say, better make a guess at the size of the `mult` array and build an emergency exit into the code for the situation that this size turns out to be too small. Or switch to C++ and use the standard library’s `vector` container to store the multiples. Neither of these low-level maneuvers detract from Dijkstra’s brilliant idea.

### 6 Concluding remarks

**From Matrix Code to Liffig** Matrix code was born on a whiteboard. As a result it was little used. To be usable it is necessary to transform it to a format that is easy to store in a computer file and easy to edit with a keyboard. The key to such a format is the observation that code matrices are sparse, which suggests as alternative to the two-dimensional matrix format a list of triples consisting of column label, cell content, and row label. This is the list of verification conditions. The matrix format suggests ordering the triples by column label and using the `if...fi` of Dijkstra’s guarded command language. The desirability of verification by assertions suggests the use of labels to mark locations in the code where the assertions are intended to hold. It also suggests a formal way of claiming that a certain assertion holds that has already been labelled. For this the keyword `goto` is an obvious choice, as of course it has no other use. These observations amount to a description of the programming language Liffig.

**What counts as a proof?** In the 1930’s predicate logic stabilized into the currently conventional syntax and semantics. From the same decade stems a new standard of rigour in mathematics as exemplified by van der Waerden’s “Moderne Algebra” and the first fascicles produced by Bourbaki. Mathematicians have accepted as relaxed notion of rigour that in principle a formalisation in first-order predicate logic is possible, but that in practice an informal summary is preferable. I have adopted this relaxed notion of rigour.

When, for example in the verification conditions for the program for the table of primes I write

```plaintext
cand not div by p[0..j-1]
```

I mean the formula

$$\forall x. (0 \leq x \land x \leq j - 1 \Rightarrow \neg \text{div}(\text{cand}, p[x])).$$

which is still problematic from the point of view of predicate logic because of the “variable” $p[x]$.

**Need for an expanded version of predicate logic for assertions** In Section 3 Floyd’s logic was defined to include predicate logic as the language for the assertions. This is not adequate for most programs where variables have sorts (“types”) and are organized in arrays and other data structures. Proposals for such an extension are presented in Apt and Bezem [2].
What is imperative programming? Until recently I recognized two programming paradigms: imperative versus declarative. Imperative is characterized by being dominated by the concept of state, whereas state plays no role in declarative programming. Indeed, although Figure 1 is based on the equations in Figure 3, the relation is not close.

Lamport [15] has convinced me that the dichotomy of imperative versus declarative sketched above is not helpful. A program in Scheme or in Prolog, though declarative, still specifies a computation and according to Lamport all computation is performed by a state machine of some sort. This is obviously true of imperative program, but it is also true of programs in Scheme or Prolog, although typically less obviously so.

Still, structured programming is different from the kind of programming for which Scheme or Prolog is intended for. How can one characterize the difference? Not by whether a state machine is involved. The difference seems to lie in the structure of the state and in its visibility. In the kind of programming that Dijkstra is interested in, and that I loosely refer to as imperative programming, the structure of the state is simple: the state is a vector of primitive values indexed by variables. The state is visible: new values are created as values of expressions containing state components and decisions are taken on the basis of such expressions. In the kind of programming for which Scheme and Prolog are intended, the structure of the state is complex, containing stacks and pointers into them. These are not visible and are manipulated indirectly via the evaluation of expressions (Scheme) or via the elaboration of goal statements (Prolog).

“Goto Considered Harmful” Although the term “structured programming” does not occur in Dijkstra’s 1968 Letter to the Editor [5], the two have been strongly associated by the public. In the Letter Dijkstra advocates abolishing the goto statement in any programming language above the level of machine language and to use structured control exclusively. I end up using the goto statement exclusively. There must be something in the Letter that I disagree with. What is it?

Dijkstra opens the argument with

My first remark is that, although the programmer’s activity ends when he has constructed a correct program, the process taking place under control of his program is the true subject matter of his activity, for it is this process that has to accomplish the desired effect; it is this process that in its dynamic behaviour has to satisfy the desired specifications.

What I propose is to change “the true subject matter of his activity” from that process to the goal the process is to achieve. That goal is a static entity. It can be expressed as an assertion. The programmer’s activity should be to design achievable goals and to link their assertions with verification conditions. One can statically ascertain whether a set of verification conditions is sufficient for the task at hand.

To proceed thus is to work at a higher level than structured programming; if it needs a name, then “goal-directed programming” might do. From the point of view of Floyd, this is a step forward because it is assumed there that code has to be verified. Indeed goal-directed programming results in code that is verified in the sense of Floyd. But I see a more important advantage. Without knowing an algorithm to achieve the goal, the subgoals needed suggest themselves, together with verification conditions that link them to a complete set. As observed before, to get from a set of verification conditions to executable code is trivial mechanics. The examples in this paper demonstrate this miracle.

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