GLOBAL UNIQUENESS FOR SEMILINEAR EQUATIONS INVOLVING THE FRACTIONAL LAPLACIAN

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ABSTRACT. We investigate inverse boundary value problems for nonlocal semilinear heat and wave equations. We recover nonlinear terms in the semilinear equations from the knowledge of corresponding Dirichlet-to-Neumann maps. Our proofs are based on the Runge approximation and the unique continuation property for fractional Laplacian.

1. Introduction and main results

In this article we investigate inverse problems of heat and wave equations involving fractional Laplacian operator with zeroth order nonlinear perturbations. The study of inverse problems involving fractional Laplace began with the fundamental work [GSU20] by Ghosh, Salo and Uhlmann. In [GSU20], they proposed and proved a Calderón type inverse problem for a linear fractional Laplace operator. One of the key tools for studying this type of problems is the Runge approximation property, which is a consequence of the fractional unique continuation property (fUCP), i.e. if $u = (-\Delta)^s u = 0$ in certain open set, then $u = 0$ everywhere. We refer readers to [Lin20, LL20, GRSU20, Li21] for some recent works involving inverse problems for fractional Laplacian. Comparing to the study of inverse problems involving fractional order operator, the study involving nonlinear terms goes back to Isakov [Isa01] and has been under extensive study in the literature. In [Isa01] Isakov studied nonlinear inverse problems for elliptic and parabolic equations using first order linearization techniques. In [LLLS21] the authors successfully implemented higher order linearization techniques to solve inverse problem for elliptic equations involving power type nonlinearity. The method is also used to solve several nonlinear inverse problems; see for instance [FKOU21, CFK+21, KU20, HL22] and references therein.

The study of inverse problems related to semilinear wave equations started with fundamental work [KLU18] by Kurylev, Lassas and Uhlmann. For more results in this direction we refer readers to [FO20, LLPMT20] and references therein. We study both heat and wave equations, and we start with the heat equation first.

Let $n \geq 1$ be a non-negative integer and $0 < s < 1$. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $\Omega^c := \mathbb{R}^n \setminus \overline{\Omega}$. Let $W$ be any bounded Lipschitz domain in $\Omega^c$. Let $u = u(t,x)$ satisfies the following fractional diffusion equation with nonlinear term $q = q(t,x,z)$:

\[
\begin{align*}
\partial_t u(t,x) + (-\Delta)^s u(t,x) + q(t,x,u(t,x)) &= 0 & \text{in } & \Omega_T \equiv (0,T) \times \Omega, \\
u(t,x) &= f(t,x) & \text{in } & \Omega_T^c \equiv (0,T) \times \Omega^c, \\
u(0,x) &= 0 & \forall & x \in \Omega,
\end{align*}
\]
for certain appropriate exterior data $f = f(t, x) \in C_c^\infty(W_T)$, where $W_T := (0, T) \times W$. Here, the fractional Laplacian $(-\Delta)^s$ is defined via the Fourier transform:

$$\mathcal{F}((-\Delta)^s v)(\xi) := |\xi|^{2s} \hat{v}(\xi) \quad \forall \xi \in \mathbb{R}^n,$$

where $\hat{v} = \mathcal{F}v$ is the Fourier transform of distribution $v$. Given any open sets $V$ and $W$ in $\Omega^s$, we define the DN-map corresponding to (1.1) as follows:

$$\Lambda_q^{\text{heat}}(f) := (-\Delta)^s u{|_{W_T}} \quad \text{for all “sufficiently small” } f \in C_c^\infty(W_T),$$

where $u$ is the unique solution of (1.1), see Proposition 2.10. The main result of this article is the following.

**Theorem 1.1.** Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, let and $W, V \subset \Omega^s$ be any open sets, both with Lipschitz boundary, satisfying $\nabla \cap \Omega = \emptyset$ and $\mathcal{W} \cap \overline{\Omega} = \emptyset$. Fix an integer $m \geq 2$ and a positive number $\delta > 0$. Assume that $q (j = 1, 2)$ satisfies following assumptions:

1. $f_q(t, x) \in C_{-1}^m((-\delta, \delta))$.
2. $\forall (t, x) \in \Omega_T$.
3. There exists a non-decreasing function $\Phi : (-\delta, \delta) \rightarrow \mathbb{R}_+$ such that

$$\sup_{(t, x) \in \Omega_T, |z| \leq \epsilon} |\partial_z q_j(t, x, z)| \leq \Phi(\epsilon)$$

for all $0 < \epsilon < \delta$ and $\lim_{\epsilon \to 0} \Phi(\epsilon) = 0$.

4. Given any $k = 2, 3, \ldots, m + 1$, there exists $M_k$ (depending on $k$) such that

$$\sup_{(t, x) \in \Omega_T, |z| < \delta} |\partial^k q_j(t, x, z)| \leq M_k.$$

Then there exists a constant $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(n, s, \Omega, T, \delta)$ such that, if $\Lambda_q^{\text{heat}}(f) = \Lambda_{q_2}^{\text{heat}}(f)$ for all $f \in C_c^\infty(W_T)$ satisfying

$$\|f\|^2_{L^2(0, T; H^s(\mathbb{R}^n)))} + \|(-\Delta)^s f\|^2_{L^2(\Omega_T)} \leq \tilde{\epsilon}_0,$$

then we have

$$\partial^k q_1(t, x, 0) = \partial^k q_2(t, x, 0) \quad \forall (t, x) \in \Omega_T, \quad k = 0, 1, 2, \ldots, m.$$  

In other words, we can recover the m-jet $\{\partial^k q(t, x, 0)\}_{k=0}^m$ of $q(t, x, z)$ in $\Omega_T$. Additionally, if we assume $z : = q(t, x, z)$ is analytic for $(t, x) \in \Omega_T$, then we have

$$q_1(t, x, z) = q_2(t, x, z) \quad \forall (t, x) \in \Omega_T, \forall z \in I.$$

**Remark 1.1.** The condition (Q.3) in Theorem 1.1 implies $\partial_z q(\cdot, 0) = 0$ in $\Omega_T$. By (Q.2) we also have $q(\cdot, 0) = 0$ in $\Omega_T$. Therefore it is enough to show (1.5) for $k = 2, 3, \ldots, m$.

Using similar tools in this article we also consider a nonlinear inverse problem for fractional wave equations in one spatial dimension. Let $u = u(t, x)$ satisfies

$$\begin{align*}
\partial_t^2 u(t, x) + (-\Delta)^s u(t, x) + q(t, x, u(t, x)) &= 0 \quad \text{in } \Omega_T, \\
u(t, x) &= f(t, x) \quad \text{in } \Omega_T, \\
u(0, x) &= \partial_t u(0, x) = 0 \quad \text{for all } x \in \Omega,
\end{align*}$$

for certain appropriate exterior data. We can define the following hyperbolic DN-map corresponding to (1.7) as follows:

$$\Lambda_q^{\text{wave}}(f) := (-\Delta)^s u{|_{W_T}} \quad \text{for all “sufficiently small” } f \in C_c^\infty(W_T),$$
where $u$ is the unique solution of (1.7), see Proposition 5.4 for the well-posedness. Using similar ideas, we also can prove the following result:

**Theorem 1.2.** Let $n = 1$ and $1/2 < s < 1$. Let $\Omega \subset \mathbb{R}^1$ be a bounded open set in $\mathbb{R}^1$, let and $W, V \subset \Omega^e$ be any open sets satisfying $\overline{V} \cap \overline{\Omega} = \emptyset$ and $\overline{W} \cap \overline{\Omega} = \emptyset$. Fixing any integer $m \geq 2$ and a positive number $\delta > 0$. Assume that $q_j$ $(j = 1, 2)$ satisfies (Q.1)–(Q.4). Then there exists a constant $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(s, \Omega, T, \delta)$ such that, if $\Lambda_{q_1}^\text{wave}(f) = \Lambda_{q_2}^\text{wave}(f)$ for all $f \in C^\infty_c(W_T)$ satisfying (1.4), then we have (1.5). Additionally, if we assume $z : \rightarrow q(t, x, z)$ is analytic for $(t, x) \in \Omega_T$ then we have (1.6).

The main difficulty in proving Theorem 1.2 is the regularity issue of the solution; we give a detailed explanation in Remark 1.2. This difficulty restricts us to prove Theorem 1.2 only for $n = 1$ for the time being. The method we used requires $L^\infty(\Omega_T)$-regularity for the linear fractional wave equation. The $L^\infty(\Omega_T)$-regularity is required to guarantee the well-posedness of the nonlinear equation, and it is essential to prove that the linearization (of the nonlinear equation) is well-defined. However, we are only able to obtain this regularity in the case when $n = 1$ and $1/2 < s < 1$. If one can prove the well-posedness of (1.7) for general $n \in \mathbb{N}$ and $0 < s < 1$, then Theorem 1.2 immediately extend for general $n \in \mathbb{N}$ and $0 < s < 1$.

There are only a few works available in the literature related to inverse problems for fractional wave equations. However, to motivate our work, we mention some closely related works. In [KLIW21], the authors studied an inverse problem involving fractional wave equation, while in [LLL21], they solved an inverse problem for hyperbolic systems. In [LL19] the authors recovered the nonlinear term by using Cauchy data set instead of the DN-map.

**Remark 1.2.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $w \in H^1_0(\Omega)$ be a solution of

$$
\begin{aligned}
-\Delta w &= F \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

If $F \in L^p(\Omega)$ for $2 \leq p < \infty$, then the standard elliptic regularity result gives $w \in W^{2,p}(\Omega)$ (see [GT01] or [JLS17, Proposition A.1] in terms of other norms). In particular, when $p = 2$, we know $w \in H^2(\Omega)$. However, there is no such result for the fraction setting. Using [GSU20, Lemma 2.3], given any $F \in H^{-s}(\Omega)$, there exists a unique solution $w \in \tilde{H}^s(\Omega)$ of

$$
\begin{aligned}
(-\Delta)^s w &= F \quad \text{in } \Omega, \\
w &= 0 \quad \text{in } \Omega^e
\end{aligned}
$$

However, we cannot guarantee $w \in H^{2s}(\mathbb{R}^n)$ in general even when we assume $F \in C^\infty(\Omega)$. From [RO16, Lemma 5.4], we know that the function $w_0(x) = (1 - |x|^2)^*_+ \in C^s(\mathbb{R}^n)$ satisfies (1.8) with $\Omega$ replaced by the unit disk and $F = c$ for some positive constant $c > 0$. The optimal (global) regularity of (1.8) we know is only the Hölder regularity $C^s(\mathbb{R}^n)$, see [RO16, Proposition 7.2]. Therefore, for the nonlinear fractional wave equation, we shall use the embedding $H^s(\mathbb{R}^n) \subset L^\infty(\Omega)$ to obtain the $L^\infty$-regularity, which only holds for $n = 1$ and $1/2 < s < 1$, see Lemma 5.2.

The rest of the paper is organized as follows. We discuss the forward problems for the fractional diffusion equation in §2. Following, we prove a Runge approximation for the fractional diffusion equation in §3. With these tools at hand, we prove Theorem 1.1 in §4. Finally, we also sketch the proof of Theorem 1.2 in §5. To make our paper self-contained,
we also present the proof of the well-posedness of the linear fractional diffusion equation (Proposition 2.2) in Appendix A.

2. The forward problem for the fractional diffusion equation

In this section, we prove some preliminaries that we need in this work.

2.1. Fractional Sobolev spaces. We use notations for fractional Sobolev spaces as in [KLW21]. To make the paper self-contained, we give brief introductions to them. For \( \alpha \in \mathbb{R} \), denote as \( H^\alpha(\mathbb{R}^n) \) the standard \( L^2 \)-based fractional Sobolev space, which is defined via Fourier transform [DNPV12, Kwa17, Ste16]. For \( s \in (0,1) \), in fact

\[
H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \mid \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{2} + s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\}
\]

(ass set) with equivalent norm:

\[
\|u\|_{H^s(\mathbb{R}^n)}^2 = \|u\|_{L^2(\mathbb{R}^n)}^2 + [u]_{H^s(\mathbb{R}^n)}^2,
\]

where

\[
[u]_{H^s(\mathbb{R}^n)}^2 = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy.
\]

Here, (2.1) is called the Aronszajn-Gagliardo-Slobodeckij seminorm, see [DNPV12, equation (2.2)] for reference.

Let \( \mathcal{O} \) be any open set in \( \mathbb{R}^n \), and let \( \alpha \in \mathbb{R} \). We define the following Sobolev spaces:

\[
H^\alpha(\mathcal{O}) = \{ u|_{\mathcal{O}} \mid u \in H^\alpha(\mathbb{R}^n) \}
\]

\[
\tilde{H}^\alpha(\mathcal{O}) := \text{closure of } C^\infty_c(\mathcal{O}) \text{ in } H^\alpha(\mathbb{R}^n)
\]

\[
H^\alpha_0(\mathcal{O}) := \text{closure of } C^\infty_c(\mathcal{O}) \text{ in } H^\alpha(\mathcal{O})
\]

\[
\tilde{H}^\alpha_0(\mathcal{O}) = \{ u \in H^\alpha(\mathbb{R}^n) \mid \text{supp}(u) \subset \mathcal{O} \}.
\]

The Sobolev space \( H^\alpha(\mathcal{O}) \) is complete under the quotient norm

\[
\|u\|_{H^\alpha(\mathcal{O})} := \inf \left\{ \|v\|_{H^\alpha(\mathbb{R}^n)} \mid v \in H^\alpha(\mathbb{R}^n) \text{ and } v|_{\mathcal{O}} = u \right\}.
\]

It is easy to see that \( \tilde{H}^\alpha(\mathcal{O}) \subset H^\alpha_0(\mathcal{O}) \), and that \( H^\alpha_0(\mathcal{O}) \) is a closed subspace of \( H^\alpha(\mathbb{R}^n) \). If \( \Omega \) is a bounded Lipschitz domain, then we also have following identifications (with equivalent norms):

\[
\begin{aligned}
\tilde{H}^\alpha(\Omega) &= H^\alpha_{\Omega} \\
(H^\alpha_{\Omega})' &= H^{-\alpha}(\Omega) \text{ and } (H^\alpha(\Omega))' = H^{-\alpha}_{\Omega} \\
H^\alpha(\Omega) &= H^\alpha_\Omega = H^\alpha_0(\Omega)
\end{aligned}
\]

\( \forall \alpha \in \mathbb{R} \), \( \forall \alpha \in \mathbb{R} \), \( \forall -1/2 < s < 1/2 \), see e.g. [GSU20, Section 2A], [McL00, Chapter 3], and [Tri02]. Moreover, for any measurable set \( A \subset \mathbb{R}^n \) we use the following notations:

\[
(f,g)_{L^2(A)} := \int_A f \, g \, dx, \quad (F,G)_{L^2(AT)} := \int_0^T \int_A F \, G \, dx \, dt.
\]
2.2. Well-posedness for the linear equation. We state the well-posedness of the linear fractional diffusion equation. Let $T > 0$, $s \in (0,1)$, and $a = a(t,x) \in L^\infty(\Omega_T)$, and we consider the following initial-exterior value problem:
\[
\begin{cases}
(\partial_t + (-\Delta)^s + a)u = F & \text{in } \Omega_T, \\
u = f & \text{in } \Omega_T, \\
u = \varphi & \text{in } \{0\} \times \mathbb{R}^n,
\end{cases}
\]
(2.2)
where $f \in C_c^\infty(W_T)$ for some open set with Lipschitz boundary $W \subset \Omega$ satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$, and $\varphi \in \tilde{H}^0(\Omega) = \{ \varphi \in L^2(\mathbb{R}^n) \mid \text{supp } \varphi \subset \overline{\Omega} \}$. Setting $v := u - f$, we then consider the following linear equation with zero exterior data:
\[
\begin{cases}
(\partial_t + (-\Delta)^s + a)v = \tilde{F} & \text{in } \Omega_T, \\
v = 0 & \text{in } \Omega_T, \\
v = \varphi & \text{in } \{0\} \times \mathbb{R}^n,
\end{cases}
\]
(2.3)
where $\tilde{F} = F - (-\Delta)^sf$. Now it suffices to study the well-posedness of (2.3).

Define functions $v : [0, T] \rightarrow \tilde{H}^s(\Omega)$ and $\tilde{F} : [0, T] \rightarrow L^2(\Omega)$ by
\[
[v(t)](x) := v(t,x) \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^n,
\]
\[
[\tilde{F}(t)](x) := \tilde{F}(t,x) \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^n.
\]
(2.4)
Let $\langle \cdot, \cdot \rangle$ be the $H^{-s}(\Omega) \oplus \tilde{H}^s(\Omega)$ duality pair. Multiplying (2.3) by any $\phi \in \tilde{H}^s(\Omega)$ gives
\[
\langle v'(t), \phi \rangle + \mathcal{B}[v, \phi; t] = (\tilde{F}(t), \phi)_{L^2(\Omega)} \quad \text{for } 0 \leq t \leq T,
\]
where $\mathcal{B}[v, \phi; t]$ is the bilinear form given by
\[
\mathcal{B}[v, \phi; t] := \int_{\mathbb{R}^n} (-\Delta)^{s/2}v(t)(-\Delta)^{s/2}\phi \, dx + \int_{\Omega} a(t, \cdot) v(t) \phi \, dx.
\]

**Definition 2.1 (Weak solutions).** We said that $v$ is a weak solution of (2.3), if
(a) $v \in L^2(0, T; \tilde{H}^s(\Omega))$ and $v' \in L^2(0, T; H^{-s}(\Omega))$;
(b) $\langle v'(t), \phi \rangle + \mathcal{B}[v, \phi; t] = (\tilde{F}(t), \phi)_{L^2(\Omega)}$ for all $\phi \in \tilde{H}^s(\Omega)$ for (almost) all $0 \leq t \leq T$;
(c) $v(0) = \varphi$,

where $\mathcal{B}$ and $\tilde{F}$ are defined according to (2.4).

**Proposition 2.2 (Well-posedness).** Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $a \in L^\infty(\Omega_T)$. For any $\tilde{F} \in L^2(\Omega_T)$ and $\varphi \in \tilde{H}^0(\Omega)$, there exists a unique weak solution $v$ of (2.3). This unique weak solution satisfies the following estimate:
\[
\|v\|_{L^\infty(0,T;L^2(\Omega))} + \|v\|_{L^2(0,T;\tilde{H}^s(\Omega))} + \|\partial_t v\|_{L^2(0,T;H^{-s}(\Omega))} \leq C(\|\varphi\|_{L^2(\Omega)} + \|\tilde{F}\|_{L^2(\Omega_T)})
\]
(2.5)
for some constant $C = C(n, s, T, \|a\|_{L^\infty(\Omega_T)})$. If we further assume $\varphi \in \tilde{H}^s(\Omega)$, then $v \in L^\infty(0,T;\tilde{H}^s(\Omega))$ and $\partial_t v \in L^2(\Omega_T)$. In this case, the unique weak solution also satisfies the following estimate:
\[
\|v\|_{L^\infty(0,T;\tilde{H}^s(\Omega))} + \|\partial_t v\|_{L^2(\Omega_T)} \leq C(\|\varphi\|_{\tilde{H}^s(\Omega)}^2 + \|\tilde{F}\|_{L^2(\Omega_T)}^2)
\]
(2.6)
for some constant $C = C(n, s, T, \|a\|_{L^\infty(\Omega_T)})$. 
The proof of Proposition 2.2 is similar to the standard well-posedness proof of the classical diffusion equation. Although, we present the proof in Appendix A in order to make the paper self-contained.

Corollary 2.3. Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, let and $W \subset \Omega^e$ be any open set with Lipschitz boundary satisfying $W \cap \Omega = \emptyset$. Let $a \in L^\infty(\Omega_T)$. Then for any $F \in L^2(\Omega_T)$, $\varphi \in \tilde{H}^0(\Omega)$, and $f \in C_c^\infty(W_T)$, there exists a unique weak solution $u = v + f$ of (2.2) satisfying
\[
\|u - f\|_{L^\infty(0,T;L^2(\Omega))} + \|u - f\|^2_{L^2(0,T;\tilde{H}^s(\Omega))} + \|\partial_t(u - f)\|^2_{L^2(0,T;H^{-s}(\Omega))} \\
\leq C(\|\varphi\|^2_{L^2(\Omega)} + \|F - (-\Delta)^s f\|^2_{L^2(\Omega_T)})
\]
for some constant $C = C(n,s,T,\|a\|_{L^\infty(\Omega_T)})$. If we further assume $\varphi \in \tilde{H}^s(\Omega)$, then the unique weak solution $u$ also satisfies the following estimate:
\[
\|u - f\|^2_{L^\infty(0,T;H^s(\mathbb{R}^n))} + \|\partial_t u\|^2_{L^2(\Omega_T)} \leq C(\|\varphi\|^2_{H^s(\Omega)} + \|F - (-\Delta)^s f\|^2_{L^2(\Omega_T)})
\]
for some constant $C = C(n,s,T,\|a\|_{L^\infty(\Omega_T)})$.

We skip the proof of Corollary 2.3 as it is a straightforward consequence of Proposition 2.2.

2.3. Maximum principle for the linear equation. Modifying the ideas in [LL19, Proposition 3.1] or [RO16, Proposition 4.1], we can obtain the following proposition:

Proposition 2.4 (Maximum principle). Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and let $a \in L^\infty(\Omega_T)$. Suppose that $u \in L^2(0,T;H^s(\mathbb{R}^n))$ is a weak solution of
\[
(\partial_t + (-\Delta)^s + a)u = F \quad \text{in} \quad \Omega_T, \\
u = f \quad \text{in} \quad \Omega_T^c, \\
u = \varphi \quad \text{on} \quad \{0\} \times \mathbb{R}^n,
\]
If
\[
F \geq 0 \text{ in } \Omega_T, \quad f \geq 0 \text{ in } \Omega_T^c, \quad \varphi \geq 0 \text{ in } \mathbb{R}^n,
\]
then $u \geq 0$ in $\Omega_T$.

Proof. Let $M$ be a real number which shall be determined later. We define
\[
u_M(t,x) := e^{-Mt}u(t,x) \quad \text{in} \quad \Omega_T, \\
a_M(t,x) := a(t,x) + M \quad \text{in} \quad \Omega_T, \\
F_M(t,x) := e^{-Mt}F(t,x) \quad \text{in} \quad \Omega_T, \\
f_M(t,x) := e^{-Mt}f(t,x) \quad \text{in} \quad \Omega_T^c.
\]
We see that
\[
(\partial_t + (-\Delta)^s + a_M)u_M = F_M \quad \text{in} \quad \Omega_T, \\
u_M = f_M \quad \text{in} \quad \Omega_T^c, \\
u_M = \varphi \quad \text{on} \quad \{0\} \times \mathbb{R}^n.
\]
We choose $M = \|a\|_{L^\infty(\Omega_T)}$, then $a_M \geq 0$ in $\Omega_T$. 


Write \( u_M = u_+^M - u_0^M \), where \( u_+^M = \max\{u_M, 0\} \) and \( u_0^M = \max\{-u_M, 0\} \). Since \( u_M \in L^2(0, T; H^s(\mathbb{R}^n)) \), then \( u_+^M \in L^2(0, T; H^s(\mathbb{R}^n)) \). Since \( u_M = f_M \geq 0 \) in \( \Omega_T^c \), hence \( u_0^M = 0 \) in \( \Omega_T \), which implies
\[
 u_0^M \in L^2(0, T; \tilde{H}^s(\Omega)).
\]

Testing the first equation of (2.10) by \( u_0^M \), we have
\[
0 \leq (F_M(t), u_0^M(t))_{L^2(\Omega)} \quad \text{(because \( F_M \geq 0 \) and \( u_0^M \geq 0 \) in \( \Omega_T \))}
\]
\[
= \int_\Omega (\partial_t u_M(t)) u_0^M(t) \, dx + \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u_M(t)(-\Delta)^{\frac{s}{2}} u_0^M(t) \, dx + \int_{\Omega} a_M(t, \cdot) u_M u_0^M \, dx
\]
\[
\text{(2.11)} \quad = -\frac{d}{dt} \left( \frac{1}{2} \int_\Omega |u_0^M(t)|^2 \, dx \right) + \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u_M(t)(-\Delta)^{\frac{s}{2}} u_0^M(t) \, dx - \int_{\Omega} a_M(t, \cdot) |u_0^M|^2 \, dx
\]
for all \( 0 < t < T \). In [LL19, Proposition 3.1] or [RO16, Proposition 4.1], they have showed that
\[
\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u_M(t)(-\Delta)^{\frac{s}{2}} u_0^M(t) \, dx \leq 0 \quad \text{for all } 0 < t < T.
\]
Combining (2.11) and (2.12), we obtain
\[
\frac{d}{dt} \left( \int_\Omega |u_0^M(t)|^2 \, dx \right) \leq 0 \quad \text{for all } 0 < t < T.
\]
Since \( u_0^M = 0 \) on \( \mathbb{R}^n \times \{0\} \) (because \( \varphi \geq 0 \) in \( \mathbb{R}^n \)), then we conclude
\[
\int_\Omega |u_0^M(t)|^2 \, dx = 0 \quad \text{for all } 0 < t < T,
\]
which completes our proof. 

\[
\square
\]

\begin{corollary}[Comparison principle] \label{cor2.5}
Given any \( n \in \mathbb{N} \) and \( 0 < s < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), and let \( a \in L^\infty(\Omega_T) \). Let \( u_1 \) and \( u_2 \) be weak solutions of
\[
\begin{cases}
(\partial_t + (-\Delta)^{s} + a)u_j = F_j & \text{in } \Omega_T, \\
u_j = f_j & \text{in } \Omega_T^c, \\
u_j = \varphi_j & \text{on } \{0\} \times \mathbb{R}^n,
\end{cases}
\]
for \( j = 1, 2 \). If
\[
F_1 \geq F_2 \text{ in } \Omega_T, \quad f_1 \geq f_2 \text{ in } \Omega_T^c, \quad \varphi_1 \geq \varphi_2 \text{ in } \mathbb{R}^n,
\]
then \( u_1 \geq u_2 \) in \( \Omega_T \).
\end{corollary}

\begin{proof}
By applying Proposition \ref{prop2.4} with \( u = u_1 - u_2 \), this can be proved immediately. \qedhere
\end{proof}

\begin{remark}[Proposition \ref{prop2.4} as well as Corollary \ref{cor2.5} also imply the uniqueness part of Proposition \ref{prop2.2} and Corollary \ref{cor2.3}] \end{remark}

2.4. \textit{\( L^\infty \)-bounds of solutions of the linear equation.} The following lemma can be found in [LL19, Lemma 3.4] (with \( a \equiv 0 \)) or [RO16, Lemma 5.1].

\begin{lemma}[Elliptic barrier] \label{lem2.6}
Given any \( n \in \mathbb{N} \) and \( 0 < s < 1 \). Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \). There exists a function \( \phi = \phi(x) \in C_0^\infty(\mathbb{R}^n) \) such that
\[
\begin{cases}
(-\Delta)^s \phi \geq 1 & \text{in } \Omega, \\
\phi \geq 0 & \text{in } \mathbb{R}^n, \\
\phi \leq C & \text{in } \Omega,
\end{cases}
\]
\end{lemma}
for some constant $C = C(n, s, \Omega)$.

If we define $\Phi(t, x) = e^{t}\phi(x)$, we immediately obtain the following corollary:

**Corollary 2.7** (Parabolic barrier). Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. There exists a function $\Phi \in C^{\infty}(\mathbb{R} \times \mathbb{R}^n)$ such that

\[
\begin{cases}
(\partial_t + (-\Delta)^s)\Phi \geq 1 & \text{in } \Omega_T, \\
\Phi \geq 0 & \text{in } [0, T) \times \mathbb{R}^n, \\
\Phi \leq C & \text{in } \Omega_T,
\end{cases}
\]

for some constant $C = C(n, s, T, \Omega)$.

Using the barrier in Corollary 2.7, we now can obtain the following $L^\infty$-bound for the solution of (2.2).

**Proposition 2.8.** Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and let $a \in L^\infty(\Omega_T)$. Suppose that $u \in L^2(0,T;H^s(\mathbb{R}^n))$ is a weak solution of

\[
\begin{cases}
(\partial_t + (-\Delta)^s + a)u = F & \text{in } \Omega_T, \\
u = f & \text{in } \Omega^*_T, \\
u = 0 & \text{on } \{0\} \times \mathbb{R}^n,
\end{cases}
\]

with $F \in L^\infty(\Omega_T)$ and $f \in L^\infty(\Omega^*_T)$. Then

\[
\|u\|_{L^\infty(\Omega_T)} \leq C(\|f\|_{L^\infty(\Omega^*_T)} + \|F\|_{L^\infty(\Omega_T)})
\]

for some constant $C = C(n, s, T, \Omega, \|a\|_{L^\infty(\Omega_T)})$.

**Proof.** Using the functions given in (2.9) with $M = \|a\|_{L^\infty(\Omega_T)}$, we know that

\[
\begin{cases}
(\partial_t + (-\Delta)^s + a_M)u_M = F_M & \text{in } \Omega_T, \\
u_M = f_M & \text{in } \Omega^*_T, \\
u_M = 0 & \text{on } \{0\} \times \mathbb{R}^n,
\end{cases}
\]

with $a_M \geq 0$. Let

\[
v(t, x) := \|F_M\|_{L^\infty(\Omega^*_T)} + \|F_M\|_{L^\infty(\Omega_T)}\Phi(t, x) \geq 0 \quad \text{in } [0, T) \times \Omega,
\]

where $\Phi$ is the barrier given in Corollary 2.7. We see that

\[
(\partial_t + (-\Delta)^s + a_M)v \geq (\partial_t + (-\Delta)^s)\Phi \geq \|F_M\|_{L^\infty(\Omega_T)}(\partial_t + (-\Delta)^s)\Phi \geq \|F_M\|_{L^\infty(\Omega_T)}
\]

and hence

\[
(\partial_t + (-\Delta)^s + a_M)(v \pm u_M) \geq 0 \quad \text{in } \Omega_T.
\]

On the other hand, we see that

\[
v \pm u_M = v \pm f_M \geq \|f_M\|_{L^\infty(\Omega^*_T)} \pm f_M \geq 0 \quad \text{in } \Omega^*_T,
\]

as well as

\[
v \pm u_M = v \geq 0 \quad \text{on } \{0\} \times \mathbb{R}^n.
\]

From (2.13a), (2.13b), (2.13c) and Proposition 2.4, we know that

\[
v \geq \pm u_M \quad \text{in } \Omega_T,
\]
which implies that
\[
\|u_M\|_{L^\infty(\Omega_T)} \leq \|v\|_{L^\infty(\Omega_T)} \leq \|f_M\|_{L^\infty(\Omega_T^\delta)} + C\|F_M\|_{L^\infty(\Omega_T)},
\]
where \(C = C(n, s, T, \Omega)\) is the constant given in Corollary 2.7. Finally, since
\[
|u(t, x)| = e^{M|t|}u_M(t, x)| \leq e^{T\|a\|_{L^\infty(\Omega_T)}}\|u_M\|_{L^\infty(\Omega_T)} \quad \text{in } \Omega_T,
\]
\[
|F_M(t, x)| = e^{-M|t|}|F(t, x)| \leq \|F\|_{L^\infty(\Omega_T)} \quad \text{in } \Omega_T,
\]
\[
|f_M(t, x)| = e^{-M|t|}|f(t, x)| \leq \|f\|_{L^\infty(\Omega_T^\delta)} \quad \text{in } \Omega_T^\delta,
\]
we arrive at the conclude. \(\square\)

Combining Corollary 2.3 and Proposition 2.8, we obtain the following well-posedness result, and we omit the proof.

**Proposition 2.9.** Given any \(n \in \mathbb{N}\) and \(0 < s < 1\). Let \(\Omega \subset \mathbb{R}^n\) be a bounded Lipschitz domain in \(\mathbb{R}^n\), let and \(W \subset \Omega^c\) be any open set with Lipschitz boundary satisfying \(W \cap \Omega = \emptyset\). Then for any \(F \in L^\infty(\Omega_T)\) and \(f \in C_c^\infty(W_T)\), there exists a unique weak solution \(u\) of

\[
\begin{cases}
(\partial_t + (-\Delta)^s + a)u = F & \text{in } \Omega_T, \\
u = f & \text{in } \Omega_T^c, \\
u = 0 & \text{in } \{0\} \times \mathbb{R}^n,
\end{cases}
\]

satisfying
\[
\|u\|_{L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)}^2 + \|\partial_t u\|_{L^2(\Omega_T)}^2 \\
\leq C\left(\|F\|_{L^\infty(\Omega_T)}^2 + \|f\|_{L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)}^2 + \|(-\Delta)^s f\|_{L^2(\Omega_T)}^2\right)
\]
for some constant \(C = C(n, s, T, \|a\|_{L^\infty(\Omega_T)}, \Omega)\).

### 2.5. Well-posedness for the nonlinear equation

We now state the well-posedness of (1.1) for small exterior data:

**Proposition 2.10.** Given any \(n \in \mathbb{N}\) and \(0 < s < 1\). Let \(\Omega \subset \mathbb{R}^n\) be a bounded Lipschitz domain in \(\mathbb{R}^n\), let and \(W \subset \Omega^c\) be any open set with Lipschitz boundary satisfying \(W \cap \Omega = \emptyset\). Fixing any parameter \(\delta > 0\). Assume that \(q\) satisfies the following assumptions:

1. For each \((t, x) \in \Omega_T\), the mapping \(z \mapsto q(t, x, z)\) is in \(C^{n+1}((-\delta, \delta))\).
2. \(q(t, x, 0) = 0\) for all \((t, x) \in \Omega_T\).
3. There exists a non-decreasing function \(\Phi : (-\delta, \delta) \to \mathbb{R}_+\) such that

\[
\sup_{(t, x) \in \Omega, |z| \leq \epsilon} |\partial_z q(t, x, z)| \leq \Phi(\epsilon)
\]

for all \(0 < \epsilon < \delta\) and \(\lim_{\epsilon \to 0} \Phi(\epsilon) = 0\).

Then there exists a sufficiently small parameter \(\bar{\epsilon}_0 = \bar{\epsilon}_0(n, s, \Omega, T, \delta) > 0\) such that the following statement holds: Given any \(f \in C_c^\infty(W_T)\) with

\[
\|f\|_{\text{ext}}^2 = \|f\|_{L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)}^2 + \|(-\Delta)^s f\|_{L^2(\Omega_T)}^2 \leq \bar{\epsilon}_0,
\]

there exists a unique solution \(u \in L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)\) of (1.1) with

\[
\|u\|_{L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} \leq C\|f\|_{\text{ext}}
\]
for certain constant \(C = C(n, s, T, \Omega)\).
Remark 2.2. In order to prove Proposition \ref{prop:existence}, we only need \(q\) to be \(C^1\)-smooth in \(z\) variable. However to recover \(m\)-th jet of \(q\) we need the assumption (Q.i).

Remark 2.3. In \cite[Theorem 11.2]{MBRS16}, they showed that there exist infinitely many solutions \(w_j\) to
\[
\begin{cases}
(-\Delta)^sw_j + q(x, w_j) + h(x) = 0 & \text{in } \Omega, \\
w_j = 0 & \text{in } \Omega^c,
\end{cases}
\]
such that \(\|w_j\|_{H^s(\mathbb{R}^n)} \to \infty\) as \(j \to \infty\). Therefore, the smallness assumption on \(f\) seems to be necessary to ensure the uniqueness of the solution of (1.1).

Proof of Proposition \ref{prop:existence}. Step 1: Initialization. Given any \(f \in C_c^\infty(\Omega_T)\), from Proposition \ref{prop:existence1}, there exists a unique solution \(u_0 = u_0(t, x)\) of
\[
\begin{cases}
(\partial_t + (-\Delta)^s)u = 0 & \text{in } \Omega_T, \\
u = f & \text{in } \Omega_T^c, \\
u = 0 & \text{in } \{0\} \times \mathbb{R}^n,
\end{cases}
\]
with
\[
\|u_0\|_{L^\infty(0,T;H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n_T))} \leq C\|f\|_{\text{ext}}
\]
for some constant \(C = C(n, s, T, \Omega)\). If \(u\) is a solution of (1.1), then the reminder function \(v \equiv u - u_0\) satisfies
\[
\begin{cases}
(\partial_t + (-\Delta)^s)v = F(v) \equiv -q(t, x, (v + u_0)(t, x)) & \text{in } \Omega_T, \\
v = 0 & \text{in } \Omega_T^c, \\
v = 0 & \text{in } \{0\} \times \mathbb{R}^n.
\end{cases}
\]

Again, using Proposition \ref{prop:existence1}, given any \(F = F(t, x) \in L^\infty(\Omega_T)\), there exists a unique solution \(SF \in L^\infty(0,T;H^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)\) of
\[
\begin{cases}
(\partial_t + (-\Delta)^s)SF = F & \text{in } \Omega_T, \\
SF = 0 & \text{in } \Omega_T^c, \\
SF = 0 & \text{in } \{0\} \times \mathbb{R}^n,
\end{cases}
\]
with
\[
\|SF\|_{L^\infty(0,T;H^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)} \leq C\|F\|_{L^\infty(\Omega_T)}
\]
for some constant \(C = C(n, s, T, \Omega)\). In other words, the solution operator
\[
S : L^\infty(\Omega_T) \to L^\infty(0,T;H^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)
\]
of (2.17) is a bounded linear operator.

Step 2: Contraction. Let \(\epsilon = \|f\|_{\text{ext}}, \) and we define
\[
X_\epsilon := \left\{ v \in L^\infty(0,T;H^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T) \mid \|v\|_{L^\infty(0,T;H^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)} \leq \epsilon \right\}.
\]

We first show that
\[
S \circ F(v) \in X_\epsilon \quad \text{for all } v \in X_\epsilon.
\]

Given any \(v \in X_\epsilon\), using (2.15), we know
\[
\|u_0 + v\|_{L^\infty(0,T;H^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)} \leq C\epsilon.
\]
By choosing $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(n, s, \Omega, T, \delta) > 0$ to be sufficiently small, we can guarantee that $2C\epsilon \leq 2C\tilde{\epsilon}_0 < \delta$. From (Q.i) and (Q.ii), by using the mean value theorem, we can find a function $0 \leq \zeta(t, x) \leq 1$ such that
\begin{equation}
F(v)(t, x) = q(t, x, (u_0 + v)(t, x)) - q(t, x, 0) = \partial_x q(t, x, (\zeta(u_0 + v))(t, x))(u_0 + v)(t, x)
\end{equation}
for all $x \in \Omega$.

Therefore, using (Q.iii), combining (2.20) and (2.21), we obtain
\[\|F(v)\|_{L^\infty(\Omega_T)} \leq \Phi(C\epsilon)\|u_0 + v\|_{L^\infty(\mathbb{R}^d)} \leq C\Phi(C\epsilon)\epsilon.\]

Using (2.18), we then obtain
\[\|S \circ F(v)\|_{L^\infty(0, T; \tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^d)} \leq \tilde{C}\Phi(C\epsilon)\epsilon.\]

Since $\Phi$ is non-decreasing, using the assumption (Q.iii), possibly choosing a smaller $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(n, s, \Omega, T, \delta) > 0$, we can assure that $\Phi(C\epsilon) \leq \Phi(C\tilde{\epsilon}_0) \leq \tilde{C}^{-1}$, and we obtain
\begin{equation}
\|S \circ F(v)\|_{L^\infty(0, T; \tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^d)} \leq \epsilon,
\end{equation}
which concludes (2.19).

We next show that
\begin{equation}
S \circ F \text{ is a contraction on } X_\epsilon.
\end{equation}
Let $v_1, v_2 \in X_\epsilon$, similar to (2.20), we have
\begin{equation}
\|u_0 + v_j\|_{L^\infty(0, T; \tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^d)} \leq C\epsilon \quad \text{for all } j = 1, 2.
\end{equation}

From (Q.i), by using the mean value theorem, we can find a function $0 \leq \zeta(t, x) \leq 1$ such that
\begin{equation}
F(v_1)(t, x) - F(v_2)(t, x) = q(t, x, (u_0 + v_1)(t, x)) - q(t, x, (u_0 + v_2)(t, x)) = \partial_x q(t, x, (\zeta(u_0 + v_1) + (1 - \zeta)(u_0 + v_2))(t, x))(v_1 - v_2)(t, x).
\end{equation}

Using (2.24), we know that
\begin{equation}
\|\zeta(u_0 + v_1) + (1 - \zeta)(u_0 + v_2)\|_{L^\infty(\mathbb{R}^d)} \leq C\epsilon \leq C\tilde{\epsilon}_0.
\end{equation}

Since $C\tilde{\epsilon}_0 < \delta$, using (Q.iii), combining (2.25), we have
\[\|F(v_1) - F(v_2)\|_{L^\infty(\Omega_T)} \leq \Phi(C\tilde{\epsilon}_0)\|v_1 - v_2\|_{L^\infty(\Omega_T)} \leq \Phi(C\tilde{\epsilon}_0)\|v_1 - v_2\|_{L^\infty(0, T; \tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^d)}.\]

Using (2.18), we then obtain
\[\|S \circ F(v_1) - S \circ F(v_2)\|_{L^\infty(0, T; \tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^d)} \leq \tilde{C}\Phi(C\tilde{\epsilon}_0)\|v_1 - v_2\|_{L^\infty(0, T; \tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^d)}.\]

Since $\Phi$ is non-decreasing, using (Q.iii), possibly choosing a smaller $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(n, s, \Omega, T, \delta) > 0$, we can assure that $\Phi(C\tilde{\epsilon}_0) \leq \frac{1}{2}\tilde{C}^{-1}$, and we obtain
\[\|S \circ F(v_1) - S \circ F(v_2)\|_{L^\infty(0, T; \tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^d)} \leq \frac{1}{2}\|v_1 - v_2\|_{L^\infty(0, T; \tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^d)},\]
which concludes (2.23).

**Step 3: Conclusion.** From (2.19) and (2.23), by using the Banach fixed point theorem, there exists a unique $v \in X_\epsilon$ such that $v = S \circ F(v)$, that is, there exists a unique $v \in X_\epsilon$.
satisfies (2.16). Hence we know that \( u \equiv v + u_0 \in L^\infty(0, T; H^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T) \) is the unique solution of (1.1). Moreover, from (2.15) and (2.22), we can conclude (2.14). \( \square \)

3. THE RUNGE APPROXIMATION FOR THE FRACTIONAL DIFFUSION EQUATION

The following unique continuation property for \((-\Delta)^s\) (see [GSU20]) is crucial for our work.

**Lemma 3.1** (Antilocality). Suppose \( u = (-\Delta)^s u = 0 \) in \( \Omega_T \), for some open set \( \Omega \subset \mathbb{R}^n \), then \( u \equiv 0 \) in \( \mathbb{R}^n_T \).

We now prove the Runge approximation property for the diffusion equation:

**Proposition 3.2.** Given any \( n \in \mathbb{N} \) and \( 0 < s < 1 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), let \( W \subset \Omega^c \) be any open set with Lipschitz boundary satisfying \( \overline{W} \cap \Omega = \emptyset \). Fixing any \( a \in L^\infty(\Omega_T) \). For each \( f \in C_c^\infty(W_T) \), let \( \mathcal{P}_a f \in L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T) \) be the unique solution (see Proposition 2.9) of

\[
\begin{cases}
(\partial_t + (-\Delta)^s + a(t, x))\mathcal{P}_a f = 0 & \text{in } \Omega_T, \\
\mathcal{P}_a f = f & \text{in } \Omega_T^c, \\
\mathcal{P}_a f = 0 & \text{on } \{0\} \times \mathbb{R}^n.
\end{cases}
\]

Then the set
\[
\mathcal{D} := \{ \mathcal{P}_a f|_{\Omega_T} \mid f \in C_c^\infty(W_T) \}
\]
is dense in \( L^2(\Omega_T) \).

**Proof.** Using the Hahn-Banach theorem (see e.g. [Bre11, Corollary 1.8]), we only need to show the following: if \( v \in L^2(\Omega_T) \) satisfies

\[
(\mathcal{P}_a f, v)_{L^2(\Omega_T)} = 0 \quad \text{for all } f \in C_c^\infty(W_T),
\]

then \( v \equiv 0 \) in \( \Omega_T \). By Proposition 2.2, there exists a unique \( \tilde{w} \in L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T) \) such that

\[
\begin{cases}
(\partial_t + (-\Delta)^s + a(T - t, x))\tilde{w} = v(T - t, x) & \text{in } \Omega_T, \\
\tilde{w} = 0 & \text{in } \Omega_T^c, \\
\tilde{w} = 0 & \text{on } \{0\} \times \mathbb{R}^n.
\end{cases}
\]

Define \( w(t, x) := \tilde{w}(T - t, x) \), then

\[
\begin{cases}
(-\partial_t + (-\Delta)^s + a(t, x))w = v(t, x) & \text{in } \Omega_T, \\
w = 0 & \text{in } \Omega_T^c, \\
w = 0 & \text{on } \{T\} \times \mathbb{R}^n.
\end{cases}
\]

We note that

\[
(\mathcal{P}_a f, v)_{L^2(\Omega_T)} = (\mathcal{P}_a f - f, v)_{L^2(\Omega_T)} \quad \text{(because supp } (f) \cap \overline{\Omega_T} = \emptyset) \\
= (\mathcal{P}_a f - f, (-\partial_t + (-\Delta)^s + a(t, x))w)_{L^2(\Omega_T)} \\
= -((\mathcal{P}_a f, \partial_t w)_{L^2(\Omega_T)} + (\mathcal{P}_a f, aw)_{L^2(\Omega_T)}) \quad \text{(because supp } (f) \cap \overline{\Omega_T} = \emptyset) \\
+ ((\partial_t + (-\Delta)^s)w)_{L^2(\mathbb{R}^n)} \quad \text{(because supp } (\mathcal{P}_a f - f) \subset \overline{\Omega_T}) \\
= ((\partial_t + (-\Delta)^s + a)\mathcal{P}_a f, w)_{L^2(\Omega_T)} - (f, (-\Delta)^s w)_{L^2(\mathbb{R}^n)} \\
= -(f, (-\Delta)^s w)_{L^2(W_T)}.
\]
Combining this equality with (3.1), we obtain
\[(f, (-\Delta)^s w)_{L^2(W_T)} = 0 \quad \text{for all } f \in C_c^\infty(W_T),\]
which implies \((-\Delta)^s w = 0\) in \(W_T\). Since \(w = 0\) in \(W_T\), using Lemma 3.1, we conclude \(w \equiv 0\) in \(\mathbb{R}^n_T\), and hence from (3.2), we conclude that \(v = 0\) in \(\Omega_T\).

\[\square\]

4. The inverse problems for the fractional diffusion equation

In this section we perform higher order linearization to the nonlinear fractional diffusion equation (1.1) as well as the DN map (1.2), which is also nonlinear. For each linearization step we derive certain identities and combine them with the Runge approximation to recover partial derivatives of \(q\). We start with the zeroth order linearization.

4.1. Zeroth order linearization. Let \(u_j^\varepsilon\) be the unique solution of

\[
\begin{aligned}
\partial_t u_j^\varepsilon + (-\Delta)^s u_j^\varepsilon + q_j(\cdot, u_j^\varepsilon) &= 0 \quad \text{in } \Omega_T, \\
u_j^\varepsilon &= \varepsilon \cdot g = \varepsilon_1 g_1 + \cdots + \varepsilon_m g_m \quad \text{in } \Omega_T^\varepsilon, \\
u_j^\varepsilon &= 0 \quad \text{on } \{0\} \times \mathbb{R},
\end{aligned}
\]

(4.1)

where \(g = (g_1, \cdots, g_m) \in (C_c^\infty(W_T))^m\). Since \(q_j (j = 1, 2)\) satisfies (Q.i)–(Q.iii) in Proposition 2.10, there exists a constant \(\varepsilon_0 = \varepsilon_0(n, s, \Omega, T, \delta, g) > 0\) with \(\varepsilon_0 \leq \bar{\varepsilon}_0\), where \(\bar{\varepsilon}_0\) is the constant given in Proposition 2.10, such that the following statement holds: Given any \(\varepsilon\) with

\[|\varepsilon| = \max_{1 \leq k \leq m} |\varepsilon_k| \leq \varepsilon_0,\]

there exists a unique solution \(u_j^\varepsilon \in L^\infty(0, T; \tilde{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)\) of (4.1) with

\[(4.2) \quad \|u_j^\varepsilon\|_{L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} \leq C(n, s, \Omega, T, g, m)|\varepsilon|.
\]

Therefore, the corresponding DN-map is

\[
\Lambda_{q_j}(\varepsilon \cdot g) = (-\Delta)^s u_j^\varepsilon |_{W_T} \quad \text{for all } 0 \leq |\varepsilon| < \varepsilon_0.
\]

We now show that \(\varepsilon \to u_j^\varepsilon\) is continuous in the following sense:

**Lemma 4.1.** The mapping \(\varepsilon \to u_j^\varepsilon\) is continuous in \(L^\infty(0, T; \tilde{H}^s(\Omega))\), that is,

\[
\lim_{|\theta| \to 0} \|u_j^{\varepsilon+\theta} - u_j^\varepsilon\|_{L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} = 0 \quad \text{for each } |\varepsilon| < \varepsilon_0.
\]

(4.3)

**Proof.** Let \(\theta = (\theta_1, \cdots, \theta_m) \in \mathbb{R}^m\) with \(|\theta| \leq |\varepsilon|\) and \(|\varepsilon| + |\theta| < \varepsilon_0\). We define \(\delta_\theta u_j^\varepsilon = u_j^{\varepsilon+\theta} - u_j^\varepsilon\), and observe that

\[
\begin{aligned}
\partial_t + (-\Delta)^s \delta_\theta u_j^\varepsilon &= G \quad \text{in } \Omega_T, \\
\delta_\theta u_j^\varepsilon &= \theta \cdot g \quad \text{in } \Omega_T^\varepsilon, \\
\delta_\theta u_j^\varepsilon &= 0 \quad \text{on } \{0\} \times \mathbb{R}^n,
\end{aligned}
\]

where

\[G = -q_j(\cdot, u_j^{\varepsilon+\theta}) + q_j(\cdot, u_j^\varepsilon).
\]

From Proposition 2.9, we know that

\[
\|\delta_\theta u_j^\varepsilon\|_{L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} \leq C(\|G\|_{L^\infty(\Omega_T)} + |\theta|).
\]

(4.4)

Using mean value theorem on the \(z\) variable of \(q\), there exists \(0 \leq \zeta(t, x) \leq 1\) such that

\[G = -\partial_z q_j(\cdot, \zeta u_j^{\varepsilon+\theta} + (1-\zeta) u_j^\varepsilon) \delta_\theta u_j^\varepsilon \quad \text{in } \Omega_T.
\]
From (4.2), we know that
\[ \| \zeta u_j^{\epsilon+\theta} + (1 - \zeta)u_j^\epsilon \|_{L^\infty(\Omega_T; H^s(\Omega))} \leq C|\epsilon| \] (because \(|\theta| \leq |\epsilon|\)).

Using (Q.3), we know that
\[ \| \mathcal{G} \|_{L^\infty(\Omega_T)} \leq \Phi(C|\epsilon|)\| \delta \theta u_j^\epsilon \|_{L^\infty(\Omega_T)} \leq \Phi(C|\epsilon|)\| \delta \theta u_j^\epsilon \|_{L^\infty(0, T; \dot{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_t)}. \]

Substituting this inequality into (4.4), we obtain
\[
\| \delta \theta u_j^\epsilon \|_{L^\infty(0, T; \dot{H}^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} \leq C\Phi(C|\epsilon|)\| \delta \theta u_j^\epsilon \|_{L^\infty(0, T; \dot{H}^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} + C|\theta|.
\]

Since \( \Phi \) is non-decreasing, using the assumption (Q.3), and possibly by choosing a smaller constant \( \epsilon_0 = \epsilon_0(n, s, \Omega, T, \delta, g, m) > 0 \), we can assure that \( \Phi(C|\epsilon|) \leq \Phi(C\epsilon_0) \leq \frac{1}{2}C^{-1} \), and hence we have
\[
\| \delta \theta u_j^\epsilon \|_{L^\infty(0, T; \dot{H}^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} \leq 2C|\theta|,
\]
which implies (4.3). The proof is done. \( \square \)

Taking \( \epsilon = 0 \) in (4.1), we obtain
\[
\begin{aligned}
\begin{cases}
\partial_t u_j^0 + (-\Delta)^s u_j^0 + q_j(\cdot, u_j^0) = 0 & \text{in } \Omega_T, \\
u_j^0 = 0 & \text{in } \Omega_T^e, \\
u_j^0 = 0 & \text{on } \{0\} \times \mathbb{R}.
\end{cases}
\end{aligned}
\]

From (4.2), we know that
\[ u_j^0 \equiv 0 \text{ in } \mathbb{R}^n_T. \]

### 4.2. First order linearization.
Acting a formal derivative operator \( \partial_{\epsilon_1} \) to (4.1), we obtain
\[
\begin{aligned}
\begin{cases}
(\partial_t + (-\Delta)^s + \partial_x q_j(\cdot, u_j^\epsilon))(\partial_{\epsilon_1} u_j^\epsilon) = 0 & \text{in } \Omega_T, \\
\partial_{\epsilon_1} u_j^\epsilon = g_1 & \text{in } \Omega_T^e, \\
\partial_{\epsilon_1} u_j^\epsilon = 0 & \text{on } \{0\} \times \mathbb{R}.
\end{cases}
\end{aligned}
\]

Using (Q.3), we know that
\[ \| \partial_x q_j(\cdot, u_j^\epsilon) \|_{L^\infty(\Omega_T)} \leq \Phi(C\epsilon_0) \leq 1. \]

Therefore, using Proposition 2.9, given any \( \epsilon \) with \(|\epsilon| < \epsilon_0 \), there exists a unique solution \( v_j^\epsilon \in L^\infty(0, T; \dot{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T) \) to (4.5) with
\[
\| v_j^\epsilon \|_{L^\infty(0, T; \dot{H}^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} \leq C\| g_1 \|_{\text{ext}}.
\]

Here, \( v_j^\epsilon \) is just an intermediate function which will be dropped after showing that \( \partial_{\epsilon_1} u_j^\epsilon \) is well-defined.

**Lemma 4.2.** There exists a constant \( \epsilon_0 = \epsilon_0(n, s, \Omega, T, \delta, g, m) > 0 \) with \( 0 < \epsilon_0 < \overline{\epsilon}_0 \), where \( \overline{\epsilon}_0 \) is given in Proposition 2.10, such that for each \( \epsilon \) with \(|\epsilon| < \epsilon_0 \), we have
\[
\lim_{\epsilon_1 \to 0} \| v_j^\epsilon - \delta_{\epsilon_1} u_j^\epsilon \|_{L^\infty(0, T; \dot{H}^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} = 0,
\]
where
\[ \delta_{\epsilon_1} u_j^\epsilon = \frac{u_j^{\epsilon+\epsilon_1} - u_j^\epsilon}{\epsilon_1} \text{ for all } (t, x) \in \Omega_T, \]
provided that \(|\epsilon| + |\epsilon_1| < \epsilon_0 \).
Proof. Let \( \epsilon_1 \) satisfies \( |\epsilon_1| \leq |\epsilon| \) and \( |\epsilon| + |\epsilon_1| < \epsilon_0 \). Note that
\[
\begin{cases}
(\partial_t + (-\Delta)^s)(v^\epsilon_j - \delta_\epsilon u^\epsilon_j) = G_1 & \text{in } \Omega_T, \\
(v^\epsilon_j - \delta_\epsilon u^\epsilon_j) = 0 & \text{in } \Omega^c_T, \\
(v^\epsilon_j - \delta_\epsilon u^\epsilon_j) = 0 & \text{on } \{0\} \times \mathbb{R}^n,
\end{cases}
\]
with

\[-G_1 = \partial_z q(\cdot, u^\epsilon_j) v^\epsilon_j - \frac{q_j(\cdot, u^{\epsilon + \epsilon_1 \epsilon_1}) - q_j(\cdot, u^\epsilon_j)}{\epsilon_1}.
\]

From Proposition 2.9, we know that

\[(4.8) \quad \|v^\epsilon_j - \delta_\epsilon u^\epsilon_j\|_{L^\infty(0,T;\mathbb{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)} \leq C\|G_1\|_{L^\infty(\Omega_T)}.
\]

Using the mean value theorem on the \( z \) variable of \( q \), there exists \( 0 \leq \zeta(t, x) \leq 1 \) such that
\[
-G_1 = [\partial_z q(\cdot, u^\epsilon_j) - \partial_z q_j(\cdot, \zeta u^{\epsilon + \epsilon_1 \epsilon_1} + (1 - \zeta)u^\epsilon_j)] v^\epsilon_j + \partial_z q_j(\cdot, \zeta u^{\epsilon + \epsilon_1 \epsilon_1} + (1 - \zeta)u^\epsilon_j)] [v^\epsilon_j - \delta_\epsilon u^\epsilon_j]
\]

Using mean value theorem on the \( z \) variable of \( \partial_z q \), there exists \( 0 \leq \eta(t, x) \leq 1 \) such that
\[
-G_1 = -\zeta \partial^2_z q(\eta u^\epsilon_j - (1 - \eta)(\zeta u^{\epsilon + \epsilon_1 \epsilon_1} + (1 - \zeta)u^\epsilon_j)) (u^{\epsilon + \epsilon_1 \epsilon_1} - u^\epsilon_j) v^\epsilon_j + \partial \partial_z q_j(\cdot, \zeta u^{\epsilon + \epsilon_1 \epsilon_1} + (1 - \zeta)u^\epsilon_j)] [v^\epsilon_j - \delta_\epsilon u^\epsilon_j]
\]

From (4.2) and \( |\epsilon_1| \leq |\epsilon| \), we have
\[
\begin{align*}
\|\eta u^\epsilon_j - (1 - \eta)(\zeta u^{\epsilon + \epsilon_1 \epsilon_1} + (1 - \zeta)u^\epsilon_j)\|_{L^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} & \leq C|\epsilon|, \\
\|\zeta u^{\epsilon + \epsilon_1 \epsilon_1} + (1 - \zeta)u^\epsilon_j\|_{L^\infty(0,T;\mathbb{H}^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} & \leq C|\epsilon|.
\end{align*}
\]

Hence, by (Q.3) and (Q.4), we know that
\[
\begin{align*}
\|\zeta \partial^2_z q(\eta u^\epsilon_j - (1 - \eta)(\zeta u^{\epsilon + \epsilon_1 \epsilon_1} + (1 - \zeta)u^\epsilon_j))\|_{L^\infty(\Omega_T)} & \leq M_2, \\
\|\partial \partial_z q_j(\cdot, \zeta u^{\epsilon + \epsilon_1 \epsilon_1} + (1 - \zeta)u^\epsilon_j)\|_{L^\infty(\Omega_T)} & \leq \Phi(|\epsilon|).
\end{align*}
\]

Hence, by using (4.6) we know that
\[
\|G_1\|_{L^\infty(\Omega_T)} \leq CM_2\|g_1\|_{\text{ext}}\|u^{\epsilon + \epsilon_1 \epsilon_1} - u^\epsilon_j\|_{L^\infty(\Omega_T)} + \Phi(|\epsilon|)\|v^\epsilon_j - \delta_\epsilon u^\epsilon_j\|_{L^\infty(\Omega_T)}.
\]

Combining this with (4.8), we have
\[
\begin{align*}
\|v^\epsilon_j - \delta_\epsilon u^\epsilon_j\|_{L^\infty(0,T;\mathbb{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)} & \leq \tilde{C}M_2\|g_1\|_{\text{ext}}\|u^{\epsilon + \epsilon_1 \epsilon_1} - u^\epsilon_j\|_{L^\infty(\Omega_T)} + \Phi(|\epsilon|)\|v^\epsilon_j - \delta_\epsilon u^\epsilon_j\|_{L^\infty(0,T;\mathbb{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)}.
\end{align*}
\]

Since \( \Phi \) is non-decreasing, using the limiting assumption of \( \Phi \) in (Q.3), possibly choosing a smaller \( \epsilon_0 = \epsilon_0(n, s, \Omega, T, \delta, g, m) > 0 \), we can assure that \( \Phi(|\epsilon|) \leq \frac{1}{2} \), and hence
\[
\|v^\epsilon_j - \delta_\epsilon u^\epsilon_j\|_{L^\infty(0,T;\mathbb{H}^s(\Omega)) \cap L^\infty(\mathbb{R}^n_T)} \leq \tilde{C}M_2\|g_1\|_{\text{ext}}\|u^{\epsilon + \epsilon_1 \epsilon_1} - u^\epsilon_j\|_{L^\infty(\Omega_T)}.
\]

Finally, using Lemma 4.1, we conclude (4.7). \( \square \)

**Remark 4.1.** Lemma 4.2 shows that \( \partial_z u^\epsilon_j := v^\epsilon_j = \lim_{\epsilon_1 \to 0} \delta_\epsilon u^\epsilon_j \) is rigorously defined for \( |\epsilon| < \epsilon_0 \). As promised, we drop \( v^\epsilon_j \) from now on.
Using (Q.4), we also see that \( \partial_{\epsilon_1} u_j^\epsilon \big|_{\epsilon=0} \) satisfies
\[
\begin{aligned}
(\partial_t + (-\Delta)^s)(\partial_{\epsilon_1} u_j^\epsilon) &= 0 \quad \text{in } \Omega_T, \\
\partial_{\epsilon_1} u_j^\epsilon \big|_{\epsilon=0} &= g_1 \quad \text{in } \Omega_T^c, \\
\partial_{\epsilon_1} u_j^\epsilon \big|_{\epsilon=0} &= 0 \quad \text{on } \{0\} \times \mathbb{R}^n.
\end{aligned}
\]
(4.9)

By uniqueness of solution (see Proposition 2.9), we know that
\[
\partial_{\epsilon_1} u_1^\epsilon \big|_{\epsilon=0} = \partial_{\epsilon_1} u_2^\epsilon \big|_{\epsilon=0} = 0 \quad \text{in } \Omega_T.
\]
For later convenience, we simply denote
\[
\partial_{\epsilon_1} u_j^\epsilon \big|_{\epsilon=0} = \partial_{\epsilon_1} u_j^\epsilon \big|_{\epsilon=0} = 0 \quad \text{in } \Omega_T.
\]

In the next lemma we show that the information from DN-map can pass to first-order linearized DN-map:

**Lemma 4.3.** If \( \Lambda_{q_1}(f) = \Lambda_{q_1}(f) \) for all \( f \in C^\infty(W_T) \) with \( \|f\|_{\text{ext}} \leq \bar{\epsilon}_0 \), where \( \bar{\epsilon}_0 \) is the constant given in Proposition 2.10, then there exists a constant \( \epsilon_0 = \epsilon_0(n, s, \Omega, T, \delta, g, m) > 0 \) with \( 0 < \epsilon_0 < \bar{\epsilon}_0 \) such that
\[
(\Delta)^s \partial_{\epsilon_1} u_j^\epsilon \big|_{\Omega_T} = (\Delta)^s \partial_{\epsilon_1} u_j^\epsilon \big|_{\Omega_T}
\]
for all \( \epsilon \) with \( |\epsilon| \leq \epsilon_0 \).

**Proof.** We have
\[
\begin{aligned}
\| (\Delta)^s \partial_{\epsilon_1} u_j^\epsilon \big|_{\Omega_T} &= \frac{\Lambda_{q_1}((\epsilon + \epsilon_1 e_1) \cdot g) - \Lambda_{q_1}(\epsilon \cdot g)}{\epsilon_1} \\
&= \| (\Delta)^s (\partial_{\epsilon_1} u_j^\epsilon - \frac{u_j^\epsilon + \epsilon_1 e_1 - u_j^\epsilon}{\epsilon_1}) \|_{L^\infty(0,T;H^{-s}(V))} \\
&\leq \| (\Delta)^s (\partial_{\epsilon_1} u_j^\epsilon - \frac{u_j^\epsilon + \epsilon_1 e_1 - u_j^\epsilon}{\epsilon_1}) \|_{L^\infty(0,T;H^{-s}(\mathbb{R}^n))} \\
&\leq C \| (\partial_{\epsilon_1} u_j^\epsilon - \frac{u_j^\epsilon + \epsilon_1 e_1 - u_j^\epsilon}{\epsilon_1}) \|_{L^\infty(0,T;H^{s}(\mathbb{R}^n))}.
\end{aligned}
\]
From Lemma 4.2, we have
\[
\lim_{\epsilon_1 \to 0} \| (\Delta)^s \partial_{\epsilon_1} u_j^\epsilon \big|_{\Omega_T} = \frac{\Lambda_{q_1}((\epsilon + \epsilon_1 e_1) \cdot g) - \Lambda_{q_1}(\epsilon \cdot g)}{\epsilon_1} = 0.
\]
Combining this equality with the assumption \( \Lambda_{q_1} = \Lambda_{q_1} \), we conclude (4.11). \( \square \)

4.3. **Second order linearization.** First of all, we recall (4.6):
\[
\begin{aligned}
\| \partial_{\epsilon_1} u_j^\epsilon \|_{L^\infty(0,T;H^{s}(\mathbb{R}^n))} &\leq C \| g_1 \|_{\text{ext}}, \\
| \| \partial_{\epsilon_1} u_j^\epsilon \|_{L^\infty(0,T;H^{s}(\mathbb{R}^n))} &\leq C \| g_2 \|_{\text{ext}},
\end{aligned}
\]
see Lemma 4.2.

Acting a formal derivative operator \( \partial_{\epsilon_2} \) on (4.5), we obtain
\[
\begin{aligned}
(\partial_t + (-\Delta)^s + \partial_{\epsilon_2} q_j(\cdot, u_j^\epsilon))(\partial_{\epsilon_1 \epsilon_2} u_j^\epsilon) + \partial_{\epsilon_2}^2 q_j(\cdot, u_j^\epsilon)(\partial_{\epsilon_1} u_j^\epsilon)(\partial_{\epsilon_2} u_j^\epsilon) &= 0 \quad \text{in } \Omega_T, \\
\partial_{\epsilon_1 \epsilon_2} u_j^\epsilon &= 0 \quad \text{in } \Omega_T^c, \\
\partial_{\epsilon_1 \epsilon_2} u_j^\epsilon &= 0 \quad \text{on } \{0\} \times \mathbb{R}^n.
\end{aligned}
\]
(4.13)
Since the term $\partial_{x}^{2}g_{j}(\cdot, u_{j}^{\varepsilon})(\partial_{t_{1}}u_{j}^{\varepsilon})(\partial_{t_{2}}u_{j}^{\varepsilon})$ is bounded in $\Omega_{T}$, using Proposition 2.9, there exists a unique solution
\[ v_{j}^{\varepsilon} \in L^{\infty}(0, T; \tilde{H}^{s}(\Omega)) \cap L^{\infty}(\mathbb{R}^{n}) \]
to (4.13) with
\[ \|v_{j}^{\varepsilon}\|_{L^{\infty}(0, T; \tilde{H}^{s}(\Omega)) \cap L^{\infty}(\mathbb{R}^{n})} \leq C\|\partial_{x}^{2}g_{j}(\cdot, u_{j}^{\varepsilon})(\partial_{t_{1}}u_{j}^{\varepsilon})(\partial_{t_{2}}u_{j}^{\varepsilon})\|_{L^{\infty}(\Omega_{T})} \leq CM_{2}\|g_{1}\|_{\text{ext}}\|g_{2}\|_{\text{ext}}. \quad \text{(using (Q.4) and (4.12))} \]
Again, $v_{j}^{\varepsilon}$ is temporary notation, which will be dropped after showing $\partial_{t_{1}, t_{2}}u_{j}^{\varepsilon}$ is well-defined. We emphasize that we have already dropped $v_{j}^{\varepsilon}$, so this will not conflict with the one used in Section 4.2, see Remark 4.1.

**Lemma 4.4.** There exists a constant $\varepsilon_{0} = \varepsilon_{0}(n, s, \Omega, T, \delta, g, m) > 0$ with $0 < \varepsilon_{0} < \varepsilon_{0}$, where $\varepsilon_{0}$ is given in Proposition 2.10, such that for each $\varepsilon$ with $|\varepsilon| < \varepsilon_{0}$, we have
\[ \lim_{\varepsilon_{2} \to 0} \|v_{j}^{\varepsilon} - \varepsilon_{2}\partial_{t_{1}}u_{j}^{\varepsilon}\|_{L^{\infty}(0, T; H^{s}(\mathbb{R}^{n}) \cap L^{\infty}(\mathbb{R}^{n}))} = 0, \]
where
\[ \varepsilon_{2}\partial_{t_{1}}u_{j}^{\varepsilon} = \frac{\partial_{t_{1}}u_{j}^{\varepsilon} + \varepsilon_{2}\varepsilon_{2}}{\varepsilon_{2}} \quad \text{in} \quad \Omega_{T}, \]
provided $|\varepsilon| + |\varepsilon_{2}| < \varepsilon_{0}$.

**Proof.** Let $\varepsilon_{2}$ satisfies $|\varepsilon_{2}| \leq |\varepsilon|$ and $|\varepsilon| + |\varepsilon_{2}| < \varepsilon_{0}$. Note that
\[ \begin{cases} (\partial_{t} + (-\Delta)^{s})(v_{j}^{\varepsilon} - \varepsilon_{2}\partial_{t_{1}}u_{j}^{\varepsilon}) = G_{2} & \text{in} \quad \Omega_{T}, \\ v_{j}^{\varepsilon} - \varepsilon_{2}\partial_{t_{1}}u_{j}^{\varepsilon} = 0 & \text{in} \quad \Omega_{T}, \\ v_{j}^{\varepsilon} - \varepsilon_{2}\partial_{t_{1}}u_{j}^{\varepsilon} = 0 & \text{on} \quad \{0\} \times \mathbb{R}^{n}, \end{cases} \]
where
\[ -G_{2} = \partial_{x}q_{j}(\cdot, u_{j}^{\varepsilon})v_{j}^{\varepsilon} + \partial_{x}^{2}q_{j}(\cdot, u_{j}^{\varepsilon})(\partial_{t_{1}}u_{j}^{\varepsilon})(\partial_{t_{2}}u_{j}^{\varepsilon}) - \frac{\partial_{x}q_{j}(\cdot, u_{j}^{\varepsilon} + \varepsilon_{2}\varepsilon_{2})\partial_{t_{1}}u_{j}^{\varepsilon} + \varepsilon_{2}\varepsilon_{2}}{\varepsilon_{2}} \partial_{t_{1}}u_{j}^{\varepsilon} - \partial_{x}q_{j}(\cdot, u_{j}^{\varepsilon})\partial_{t_{1}}u_{j}^{\varepsilon}. \]
After some computation we can write
\[ -G_{2} = G_{21} + G_{22} + G_{23}, \]
where
\[ G_{21} = \partial_{x}q_{j}(\cdot, u_{j}^{\varepsilon})[v_{j}^{\varepsilon} - \varepsilon_{2}\partial_{t_{1}}u_{j}^{\varepsilon}], \]
\[ G_{22} = [\partial_{x}^{2}q_{j}(\cdot, u_{j}^{\varepsilon})(\partial_{t_{2}}u_{j}^{\varepsilon}) - \frac{\partial_{x}q_{j}(\cdot, u_{j}^{\varepsilon} + \varepsilon_{2}\varepsilon_{2})\partial_{t_{1}}u_{j}^{\varepsilon} + \varepsilon_{2}\varepsilon_{2}}{\varepsilon_{2}}](\partial_{t_{1}}u_{j}^{\varepsilon} + \varepsilon_{2}\varepsilon_{2}), \]
\[ G_{23} = \partial_{x}^{2}q_{j}(\cdot, u_{j}^{\varepsilon})\partial_{t_{2}}u_{j}^{\varepsilon}[\partial_{t_{1}}u_{j}^{\varepsilon} - \partial_{t_{1}}u_{j}^{\varepsilon} + \varepsilon_{2}]. \]
Note that
\[ \|v_{j}^{\varepsilon} - \delta_{t_{1}}\partial_{t_{1}}u_{j}^{\varepsilon}\|_{L^{\infty}(0, T; H^{s}(\mathbb{R}^{n}) \cap L^{\infty}(\mathbb{R}^{n}))} \leq C\|G_{2}\|_{L^{\infty}(\Omega_{T})}. \]
Possibly choosing a smaller $\varepsilon_{0}$, we have
\[ \|G_{21}\|_{L^{\infty}(\Omega_{T})} \leq \frac{1}{2}\|v_{j}^{\varepsilon} - \delta_{t_{1}}\partial_{t_{1}}u_{j}^{\varepsilon}\|_{L^{\infty}(0, T; H^{s}(\mathbb{R}^{n}) \cap L^{\infty}(\mathbb{R}^{n}))}. \]
Using the mean value theorem and Lemma 4.2, we know that
\[
\lim_{\epsilon_2 \to 0} \| G_{22} \|_{L^\infty(\Omega_T)} + \lim_{\epsilon_2 \to 0} \| G_{23} \|_{L^\infty(\Omega_T)} = 0.
\]
Therefore, using arguments similar to Lemma 4.2, we can conclude (4.15).

**Remark 4.2.** Lemma 4.4 shows that \( \partial_{\epsilon_1 \epsilon_2} u^\epsilon_j := v^\epsilon_j = \lim_{\epsilon_2 \to 0} \delta_{\epsilon_2} \partial_{\epsilon_1} u^\epsilon_j \) is rigorously defined for \( |\epsilon| < \epsilon_0 \).

Similar to Lemma 4.3, we can obtain the following lemma.

**Lemma 4.5.** If \( \Lambda_q(f) = \Lambda_{q_1}(f) \) for all \( f \in C^\infty_c(W_T) \) with \( \|f\|_{\text{ext}} \leq \tilde{\epsilon}_0 \), where \( \tilde{\epsilon}_0 \) is the constant given in Proposition 2.10, then there exists a constant \( \epsilon_0 = \epsilon_0(n, s, \Omega, T, \delta, g, m) > 0 \) with \( 0 < \epsilon_0 < \tilde{\epsilon}_0 \) such that
\[
(\Delta)^s \partial_{\epsilon_1 \epsilon_2} u^\epsilon_1|_{V_T} = (\Delta)^s \partial_{\epsilon_1 \epsilon_2} u^\epsilon_2|_{V_T}
\]
for all \( \epsilon \) with \( |\epsilon| \leq \epsilon_0 \).

**Proof.** Using similar arguments as in Lemma 4.3 (with Lemma 4.4), we can show that (4.11) implies (4.16). Then we conclude the lemma by Lemma 4.3.

Now we are ready to prove Theorem 1.1 for \( m = 2 \).

**Proof of Theorem 1.1 for \( m = 2 \).** Using (Q.3) and (4.10), we know that \( \partial_{\epsilon_1 \epsilon_2} u^\epsilon_j|_{\epsilon=0} \) satisfies
\[
\begin{aligned}
&\left\{ 
\begin{array}{ll}
(\partial_t + (\Delta)^s)(\partial_{\epsilon_1 \epsilon_2} u^\epsilon_j|_{\epsilon=0}) \\
+ \partial^2 q_j(\cdot, 0) (\partial_{\epsilon_1} u^\epsilon|_{\epsilon=0})(\partial_{\epsilon_2} u^\epsilon|_{\epsilon=0}) = 0 & \text{in } \Omega_T, \\
\partial_{\epsilon_1 \epsilon_2} u^\epsilon_j|_{\epsilon=0} = 0 & \text{in } \Omega^c_T, \\
\partial_{\epsilon_1 \epsilon_2} u^\epsilon_j|_{\epsilon=0} = 0 & \text{on } \{0\} \times \mathbb{R}^n.
\end{array}
\right.
\end{aligned}
\]
Hence, we know that \( v := \partial_{\epsilon_1 \epsilon_2} u^\epsilon_1|_{\epsilon=0} - \partial_{\epsilon_1 \epsilon_2} u^\epsilon_2|_{\epsilon=0} \) satisfies
\[
\begin{aligned}
&\left\{ 
\begin{array}{ll}
(\partial_t + (\Delta)^s) v + (\partial^2 q_1(\cdot, 0) - \partial^2 q_2(\cdot, 0)) (\partial_{\epsilon_1} u^\epsilon|_{\epsilon=0})(\partial_{\epsilon_2} u^\epsilon|_{\epsilon=0}) = 0 & \text{in } \Omega_T, \\
v = 0 & \text{in } \Omega^c_T, \\
v = 0 & \text{on } \{0\} \times \mathbb{R}^n.
\end{array}
\right.
\end{aligned}
\]
From Lemma 4.5, we know that \( (\Delta)^s v|_{V_T} = 0 \). Since \( v = 0 \) in \( V_T \), using the unique continuation property of fractional Laplacian in Lemma 3.1, we conclude that \( v \equiv 0 \). Therefore, we know that
\[
(\partial^2 q_1(\cdot, 0) - \partial^2 q_2(\cdot, 0)) (\partial_{\epsilon_1} u^\epsilon|_{\epsilon=0})(\partial_{\epsilon_2} u^\epsilon|_{\epsilon=0}) = 0.
\]
Since \( g_1, g_2 \in C^\infty_c(W_T) \) are arbitrary, using (4.9) and the Runge approximation for fractional diffusion equation in Proposition 3.2, we conclude
\[
\partial^2 q_1(\cdot, 0) - \partial^2 q_2(\cdot, 0) = 0 \quad \text{in } \Omega_T,
\]
which proves Theorem 1.1 for \( m = 2 \).
4.4. Higher order linearization. For each $2 \leq p \leq m$, we denote $\partial_{(p)} = \partial_{\epsilon_1} \cdots \partial_{\epsilon_p}$. By repeating formal differentiations to the equation (4.13), we obtain the following $p$-th order linearization

$$
\begin{align*}
\begin{cases} 
(\partial_t + (-\Delta)^s)\partial_{(p)} u_j^\epsilon + \partial_{(p)} q_j(\cdot, u_j^\epsilon) = 0 & \text{in } \Omega_T, \\
\partial_{(p)} u_j^\epsilon = 0 & \text{in } \Omega_T^c, \\
\partial_{(p)} u_j^\epsilon = 0 & \text{on } \{0\} \times \mathbb{R}^n,
\end{cases}
\end{align*}
$$

where we simply denote $\partial_{(p)} = \partial_{\epsilon_1} \cdots \partial_{\epsilon_p}$. By induction, we can verify

$$
\partial_{(p)} q_j(\cdot, u_j^\epsilon) = \partial_{z} q_j(\cdot, u_j^\epsilon) \partial_{(p)} u_j^\epsilon + \sum_{\ell=2}^{p-1} \partial_{z}^\ell q_j(\cdot, u_j^\epsilon) T_{\ell}^p(u_j^\epsilon) + \partial_{p}^\ell q_j(\cdot, u_j^\epsilon) \prod_{\ell=1}^{p} \partial_{\epsilon_\ell} u_j^\epsilon,
$$

where $T_{\ell}^p(u_j^\epsilon)$ is a generic notation (in order $p$ linearization) signifying a combination of the terms $\partial_{p}^\ell u_j^\epsilon$ with multi-index $\alpha$ satisfying $1 \leq |\alpha| \leq p - 1$. The following facts can be proved using strong induction on $m$:

1. Functions $\partial_{(p)} u_j^\epsilon \in L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)$ are well-defined for each $2 \leq p \leq m$.
2. There exists $\epsilon_0 = \epsilon_0(n, s, \Omega, T, \delta, g, m) > 0$ with $0 < \epsilon_0 < \bar{\epsilon}_0$, where $\bar{\epsilon}_0$ is the constant given in Proposition 2.10, such that

$$
\lim_{\epsilon_p \to 0} \|\partial_{(p)} u_j^\epsilon - \delta_{\epsilon_p} \partial_{(p-1)} u_j^\epsilon\|_{L^\infty(0, T; H^s(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n_T)} = 0 \quad \text{for all } 2 \leq p \leq m,
$$

where

$$
\delta_{\epsilon_p} \partial_{(p-1)} u_j^\epsilon = \frac{\partial_{(p-1)} u_j^{\epsilon + \epsilon_p \epsilon_p} - \partial_{(p-1)} u_j^\epsilon}{\epsilon_p} \quad \text{in } \Omega_T,
$$

provided $|\epsilon| + |\epsilon_p| < \epsilon_0$.
3. Moreover, if $\Lambda_{q_j}(f) = \Lambda_{q_j}(f)$ for all $f \in C_1(\mathbb{R}^n)$ with $\|f\|_{\text{ext}} \leq \bar{\epsilon}_0$, then we have

$$
(-\Delta)^s \partial_{(p)} u_j^\epsilon|_{V_T} = (-\Delta)^s \partial_{(p)} u_j^\epsilon|_{V_T} \quad \text{for all } 2 \leq p \leq m.
$$

Using the observations above, we are now ready to prove our main result.

**Proof of Theorem 1.1.** We prove by induction on $m$. We assume the following hypothesis:

$$
\partial_{p}^\ell q_1(\cdot, 0) = \partial_{z}^\ell q_1(\cdot, 0) \quad \text{for all } 2 \leq p \leq m - 1.
$$

Using (4.10), we see that

$$
\partial_{(m)} q_j(\cdot, u_j^\epsilon)|_{\epsilon=0} = \partial_{z} q_j(\cdot, 0) \partial_{(m)} u_j^\epsilon|_{\epsilon=0} + \sum_{\ell=2}^{m-1} \partial_{z}^\ell q_j(\cdot, 0) T_{\ell}^m(u_j^\epsilon)|_{\epsilon=0} + \partial_{z}^m q_j(\cdot, 0) \prod_{\ell=1}^{m} \partial_{\epsilon_\ell} u_j^\epsilon|_{\epsilon=0}.
$$

Using (4.18), we know that

$$
\sum_{\ell=2}^{m-1} \partial_{z}^\ell q_j(\cdot, 0) T_{\ell}^m(u_j^\epsilon)|_{\epsilon=0} = \sum_{\ell=2}^{m-1} \partial_{z}^\ell q_j(\cdot, 0) T_{\ell}^m(u_j^\epsilon)|_{\epsilon=0}.
$$
Therefore, we know that $v := \partial_{\kappa(n)}u^\kappa|_{\kappa=0} - \partial_{\kappa(n)}u_{\kappa}|_{\kappa=0}$ satisfies
\[
\begin{cases}
(\partial_t + (-\Delta)^s)v + (\partial_{\kappa}^m q_1(\cdot, 0) - \partial_{\kappa}^m q_2(\cdot, 0)) \prod_{\ell=1}^m \partial_{\ell\kappa}u_{\kappa}|_{\kappa=0} = 0 & \text{in } \Omega_T, \\
v = 0 & \text{in } \Omega_1, \\
v = 0 & \text{on } \{0\} \times \mathbb{R}^n.
\end{cases}
\]
Using (4.17), we know that
\[(-\Delta)^s v|_{V_\kappa} = 0.\]
Since $v = 0$ in $V_\kappa$, by using the unique continuation principle of the fractional Laplacian (see Lemma 3.1), we conclude that $v \equiv 0$. Therefore, we know that
\[
(\partial_{\kappa}^m q_1(\cdot, 0) - \partial_{\kappa}^m q_2(\cdot, 0)) \prod_{\ell=1}^m \partial_{\ell\kappa}u_{\kappa}|_{\kappa=0} = 0 \quad \text{in } \Omega_T.
\]
Since $g_1, \ldots, g_m \in C^\infty_c(W_T)$ are arbitrary, using (4.9) and the Runge approximation for the fractional diffusion equation proved in Proposition 3.2, we conclude
\[
\partial_{\kappa}^m q_1(\cdot, 0) - \partial_{\kappa}^m q_2(\cdot, 0) = 0 \quad \text{in } \Omega_T,
\]
which completes the proof of Theorem 1.1. □

5. Analogous result for the fractional wave equation

We now recall following results from [KLW21, Corollary 2.2], which we use later to prove the well-posedness of (1.7) with small exterior data and to solve inverse problem as well.

**Lemma 5.1 ([KLW21]).** Given any $n \in \mathbb{N}$ and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $\mathbb{R}^n$, let and $W \subset \Omega^c$ be any open set with Lipschitz boundary satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$. Let $a \in L^\infty(\Omega_T)$. Then for any $F \in L^2(\Omega_T)$, $f \in C^\infty_c(W_T)$, $\psi \in H^0(\Omega)$, $\varphi \in H^s(\Omega)$, there exists a unique solution $u$ of

\[
\begin{cases}
(\partial_t^2 + (-\Delta)^s + a)u = F & \text{in } \Omega_T, \\
u = f & \text{in } \Omega_T, \\
u = \varphi, \quad \partial_t u = \psi & \text{on } \{0\} \times \mathbb{R}^n.
\end{cases}
\]

satisfying

\[
\|u - f\|_{L^\infty(0,T;L^2(\Omega^c))} + \|\partial_t u\|_{L^\infty(0,T;L^2(\Omega))} \leq C(\|\varphi\|_{H^s(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 + \|F - (-\Delta)^s f\|_{L^2(\Omega_T)}^2)
\]

for some constant $C = C(n, s, T, \|a\|_{L^\infty(\Omega_T)})$.

**Remark 5.1.** It is interesting to compare (5.2) with (2.7): both solutions (wave and diffusion) have regularity $L^\infty(0,T;H^s(\mathbb{R}^n))$.

We need the following Sobolev embedding to obtain $L^\infty(\Omega_T)$-regularity of the solution, which is a special case of [DNPV12, Theorem 8.2]:

**Lemma 5.2 ([DNPV12]).** Let $n = 1$ and $1/2 < s < 1$. There exists a constant $C = C(s, \Omega)$ such that

\[
\|f\|_{C^{\alpha}(\Omega)} \leq C(\|f\|_{L^2(\Omega)}^2 + [f]_{H^s(\Omega)}^2) \leq C\|f\|_{H^s(\mathbb{R}^1)}^2
\]

for any $f \in L^2(\Omega)$ with $\alpha = (2s - 1)/2$.

Therefore, Lemma 5.1 implies the following result.
**Proposition 5.3.** Let $n = 1$ and $1/2 < s < 1$. Let $\Omega \subset \mathbb{R}^1$ be a bounded open set in $\mathbb{R}^n$, let and $W \subset \Omega^c$ be any open set satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$. Then for any $\tilde{F} \in L^\infty(\Omega_T)$ and $f \in C_c^\infty(W_T)$, there exists a unique weak solution $u$ of (5.1) satisfying

$$
\|u\|_{L^\infty(0,T;H^s(\mathbb{R}^1)) \cap L^\infty(\mathbb{R}^1_+)} + \|\partial_t u\|_{L^\infty(0,T;L^2(\Omega))} \\
\leq C(\|\varphi\|_{H^s(\Omega)}^2 + \|\psi\|_{L^2(\Omega)}^2 + \|F\|_{L^2(\Omega_T)}) \\
+ \|f\|_{L^\infty(0,T;H^s(\mathbb{R}^1)) \cap L^\infty(\mathbb{R}^1_+)}^2 + \|(-\Delta)^s f\|_{L^2(\Omega_T)}^2)
$$

for some constant $C = C(s, T, \|a\|_{L^\infty(\Omega_T)}, \Omega)$.

Using the same argument as in Proposition 2.10, we also can prove the well-posedness of (1.7) for some exterior data:

**Proposition 5.4.** Let $n = 1$ and $1/2 < s < 1$. Let $\Omega \subset \mathbb{R}^1$ be a bounded open set in $\mathbb{R}^n$, let and $W \subset \Omega^c$ be any open set satisfying $\overline{W} \cap \overline{\Omega} = \emptyset$. Fixing any parameter $\delta > 0$. Assume that $q$ satisfies (Q.i)–(Q.iii). There exists a sufficiently small parameter $\bar{\varepsilon}_0 = \varepsilon_0(s, \Omega, T, \delta) > 0$ such that the following statement holds: Given any $f \in C_c^\infty(W_T)$ with

$$
\|f\|_{\text{ext}}^2 \equiv \|f\|_{L^\infty(0,T;H^s(\mathbb{R}^1) \cap L^\infty(\mathbb{R}^1_+))} + \|(-\Delta)^s f\|_{L^2(\Omega_T)}^2 \leq \bar{\varepsilon}_0,
$$

there exists a unique solution $u \in L^\infty(0,T;H^s(\mathbb{R}^1)) \cap L^\infty(\mathbb{R}^1_T)$ of (1.7) with

$$
\|u\|_{L^\infty(0,T;H^s(\mathbb{R}^1)) \cap L^\infty(\mathbb{R}^1_+)} \leq C\|f\|_{\text{ext}}
$$

for some constant $C = C(s, T, \Omega)$.

Finally, the inverse problem for the nonlinear fractional wave equation (1.7), i.e. Theorem 1.2, can be proved using exactly the same idea as in Theorem 1.1, see Section 4.

**Appendix A. Well-posedness of the linear fractional diffusion equation.** We first prove the uniqueness of weak solution of (2.2) as well as (2.3). It suffices to show the following statement: If $u$ a weak solution of (A.1)

$$
\begin{cases}
(\partial_t + (-\Delta)^s + a)v = 0 & \text{in } \Omega_T, \\
v = 0 & \text{in } \Omega^c, \\
v = 0 & \text{in } \{0\} \times \mathbb{R}^n,
\end{cases}
$$

then $u \equiv 0$. Multiplying the first equation of (A.1) by $v$, we obtain

$$
0 = \langle \varphi', \varphi \rangle + B[\varphi, \varphi; t] = \frac{d}{dt} \left( \frac{1}{2} \|v(t)\|_{L^2(\Omega)}^2 \right) + B[\varphi, \varphi; t] \\
\geq \frac{d}{dt} \left( \frac{1}{2} \|v(t)\|_{L^2(\Omega)}^2 \right) - \|a\|_{L^\infty(\Omega_T)} \|v(t)\|_{L^2(\Omega)}^2,
$$

that is,

$$
\frac{d}{dt} \left( \|v(t)\|_{L^2(\Omega)}^2 \right) \leq 2\|a\|_{L^\infty(\Omega_T)} \|v(t)\|_{L^2(\Omega)}^2.
$$

Using the Grönwall’s inequality in [Eva10, Section B.2], we conclude that $\|v(t)\|_{L^2(\Omega)}^2 = 0$ for all $0 \leq t \leq T$, and hence $u \equiv 0$. The uniqueness is proved.
A.2. Existence of weak solution. Now it suffices prove that there exists a weak solution of (2.3).

Step 1: Galerkin approximation. We now set up the Galerkin approximation for (2.3). Similar to [KLIW21, Appendix A], we consider an eigenbasis \( \{w_k\}_{k \in \mathbb{N}} \) associated with the Dirichlet fractional Laplacian in a bounded domain \( \Omega \). We normalize these eigenfunctions so that

\[
\{w_k\}_{k \in \mathbb{N}} \text{ be an orthogonal basis in } \tilde{H}^s(\Omega), \\
\{w_k\}_{k \in \mathbb{N}} \text{ be an orthonormal basis in } L^2(\Omega).
\]

Given any fixed integer \( m \in \mathbb{N} \), we consider the following anzats:

(A.2) \[ v_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k. \]

Plugging the anzats (A.2) into Definition 2.1(b), we obtain

(A.3) \[
\begin{aligned}
&\left\{ \begin{array}{ll}
(v_m^t(t), w_k)_{L^2(\Omega)} + B[v_m, w_k; t] = (\tilde{F}(t), w_k)_{L^2(\Omega)} & \text{for all } 0 \leq t \leq T, \\
d_m^k(0) = (\tilde{\varphi}, w_k)_{L^2(\Omega)}.
\end{array} \right.
\end{aligned}
\]

Note that

\[
(v_m^t, w_k)_{L^2(\Omega)} = (d_m^k)'(t)
\]

\[
B[v_m, w_k] = \sum_{k=1}^{m} e^{kt} d_m^k(t) \quad \text{with} \quad e^{kt}(t) := B[w_t, w_k; t].
\]

This shows that \( d_m^k(t) \) satisfies the following linear system of ordinary differential equation (ODE):

(A.4) \[
\begin{aligned}
&\left\{ \begin{array}{ll}
(d_m^k)'(t) + \sum_{k=1}^{m} e^{kt} d_m^k(t) = (\tilde{F}(t), w_k)_{L^2(\Omega)} & \text{for all } 0 \leq t \leq T, \\
d_m^k(0) = (\tilde{\varphi}, w_k)_{L^2(\Omega)}.
\end{array} \right.
\end{aligned}
\]

Therefore, the standard ODE theory guarantees the existence and uniqueness of such \( d_m^k(t) \), and thus (A.2) is a valid discretization of (2.3).

Step 2: Energy estimate. Multiplying (A.3) by \( d_m^k(t) \), and summing over \( k = 1, \ldots, m \), we have

(A.5) \[ (v_m^t, v_m)_{L^2(\Omega)} + B[v_m, v_m; t] = (\tilde{F}, v_m)_{L^2(\Omega)}. \]

The following Hardy-Littlewood-Sobolev inequality can be found in [KLIW21, equation (A.11)] or [Pon16, Proposition 15.5]:

(A.6) \[
\|v_m\|_{L^2(\mathbb{R}^1)} \leq C(n, s) \|\phi\|_{\tilde{H}^s(\mathbb{R}^n)} \leq C(n, s) \|(-\Delta)^{s/2} \phi\|_{L^2(\mathbb{R}^1)} \quad \text{for all } \phi \in \tilde{H}^s(\Omega).
\]

On the other hand, we observe that

\[
(v_m^t, v_m)_{L^2(\Omega)} = \frac{d}{dt} \left( \frac{1}{2} \|v_m\|_{L^2(\Omega)}^2 \right).
\]

Hence, from (A.4) we have

(A.7) \[
\frac{d}{dt} \left( \frac{1}{2} \|v_m\|_{L^2(\Omega)}^2 \right) + \|v_m\|_{\tilde{H}^s(\Omega)}^2 \leq C(n, s, \|a\|_{L^\infty(\Omega)}) \left( \|v_m\|_{L^2(\Omega)}^2 + \|\tilde{F}\|_{L^2(\Omega)}^2 \right)
\]
for all $0 \leq t \leq T$. Using the Grönwall’s inequality in [Eva10, Section B.2], we have
\[
\|v_m(t)\|^2_{L^2(\Omega)} \leq e^{Ct} \left( \|v_m(0)\|^2_{L^2(\Omega)} + C \int_0^t \|\tilde{F}(s)\|^2_{L^2(\Omega)} \, ds \right) \quad \text{for all } 0 \leq t \leq T.
\]
Since
\[
\|v_m(0)\|^2_{L^2(\Omega)} = \sum_{k=1}^m \| (\tilde{\varphi}, w_k)_{L^2(\Omega)} \|^2 \leq \sum_{k=1}^\infty \| (\tilde{\varphi}, w_k)_{L^2(\Omega)} \|^2 = \| \varphi \|^2_{L^2(\Omega)},
\]
then we have
\[
(A.7) \quad \sup_{0 \leq t \leq T} \|v_m(t)\|^2_{L^2(\Omega)} \leq C_{s, T, \| a \|_{\infty}} \left( \| \varphi \|^2_{L^2(\Omega)} + \| \tilde{F} \|^2_{L^2(\Omega_T)} \right).
\]
Integrating (A.6) on $t \in [0, T]$, we have
\[
(A.8) \quad \|v_m\|^2_{L^2(0, T; \tilde{H}^s(\Omega))} \leq C_{s, \| a \|_{\infty}} \left( \|v_m\|^2_{L^2(\Omega_T)} + \| \tilde{F} \|^2_{L^2(\Omega_T)} \right).
\]
Combining (A.7) and (A.8), we obtain the following energy estimate:
\[
(A.9) \quad \sup_{0 \leq t \leq T} \|v_m(t)\|^2_{L^2(\Omega)} + \|v_m\|^2_{L^2(0, T; \tilde{H}^s(\Omega))} \leq C(n, s, T, \| a \|_{\infty(T)}) \left( \| \varphi \|^2_{L^2(\Omega)} + \| \tilde{F} \|^2_{L^2(\Omega_T)} \right).
\]
Fixing any $\phi \in \tilde{H}^s(\Omega)$ with $\| \phi \|_{\tilde{H}^s(\Omega)} \leq 1$, we write $\phi = \phi_1 + \phi_2$, where $\phi_1 \in \text{span} \{w_k\}_{k=1}^m$ and $(\phi_2, w_k)_{L^2(\Omega)} = 0$ for $k = 1, \ldots, m$. Using (A.3), we see that
\[
(v_m'(t), \phi)_{L^2(\Omega)} = (v_m'(t), \phi_1)_{L^2(\Omega)} = (\tilde{F}, \phi_1) - \mathcal{B}[v_m, \phi_1; t].
\]
Since $\|\phi_1\|_{\tilde{H}^s(\Omega)} \leq 1$, then
\[
|(v_m'(t), \phi)_{L^2(\Omega)}| \leq C(\| \tilde{F}(t) \|^2_{L^2(\Omega)} + \|v_m\|^2_{\tilde{H}^s(\Omega)}).
\]
Hence we know that
\[
\|v_m'(t)\|^2_{\tilde{H}^{-s}(\Omega)} = \sup_{\|\phi\|_{\tilde{H}^s(\Omega)} \leq 1} |(v_m'(t), \phi)_{L^2(\Omega)}| \leq C_{\| a \|_{\infty}} \left( \| \tilde{F}(t) \|^2_{L^2(\Omega)} + \|v_m\|^2_{\tilde{H}^s(\Omega)} \right).
\]
Integrating the inequality above on $t \in [0, T]$, and combining the result with (A.9), we obtain
\[
\begin{aligned}
\sup_{0 \leq t \leq T} \|v_m(t)\|^2_{L^2(\Omega)} + \|v_m\|^2_{L^2(0, T; \tilde{H}^s(\Omega))} + \|v_m'(t)\|^2_{L^2(0, T; \tilde{H}^{-s}(\Omega))} \\
\leq C_{s, T, \| a \|_{\infty}} \left( \| \varphi \|^2_{L^2(\Omega)} + \| \tilde{F} \|^2_{L^2(\Omega_T)} \right).
\end{aligned}
\]
(A.10)

**Step 3: Passing to the limit.** By (A.10), we can extract a subsequence of $\{v_m\}_{m \in \mathbb{N}}$, here we still denote this subsequence as $\{v_m\}_{m \in \mathbb{N}}$, such that
\[
\begin{cases}
 v_m \rightharpoonup v \quad \text{weakly in } L^2(0, T; \tilde{H}^s(\Omega)), \\
v_m' \rightharpoonup v' \quad \text{weakly in } L^2(0, T; H^{-s}(\Omega)).
\end{cases}
\]
Given any fixed integer $N$, we write
\[
\tilde{v}(t) := \sum_{k=1}^N d^k(t)w_k,
\]
where $d^k(t)$ $(k = 1, \ldots, N)$ are arbitrary smooth functions (not the one in (A.2)). Choosing $m \geq N$, multiplying (A.3) by $d^k(t)$, and summing over $k = 1, \ldots, N$, we obtain

(A.12) \[ \int_0^T \left( (v'_m(t), \tilde{v}(t))_{L^2(\Omega)} + B[v_m, \tilde{v}; t] \right) dt = \int_0^T (\tilde{F}(t), \tilde{v}(t))_{L^2(\Omega)} dt. \]

Taking $m \to +\infty$ in (A.12), and from (A.11), we know that

(A.13) \[ \int_0^T \left( (v'(t), \tilde{v}(t)) + B[v, \tilde{v}; t] \right) dt = \int_0^T (\tilde{F}(t), \tilde{v}(t))_{L^2(\Omega)} dt. \]

Due to the arbitrariness of $N$ and \( \{d^k\}_{k=1}^N \), we have

\[ (v', \phi) + B[v, \phi; t] = (\tilde{F}(t), \phi)_{L^2(\Omega)} \quad \text{for all } \phi \in \tilde{H}^s(\Omega). \]

This together with (A.10) verify Definition 2.1(a)(b).

It remains to show $v$ verifies Definition 2.1(c). To that end, let us choose any $\tilde{v} \in C^1(0, T; \tilde{H}^s(\Omega))$ with $\tilde{v}(T) = 0$. From (A.13), we have

(A.14) \[ \int_0^T \left( (\tilde{v}'(t), v(t))_{L^2(\Omega)} + B[v, \tilde{v}; t] \right) dt = \int_0^T (\tilde{F}(t), \tilde{v}(t))_{L^2(\Omega)} dt + (\tilde{v}'(0), v(0))_{L^2(\Omega)}. \]

Similarly, from (A.12), we have

(A.15) \[ \int_0^T \left( (\tilde{v}'(t), v_m(t))_{L^2(\Omega)} + B[v_m, \tilde{v}; t] \right) dt = \int_0^T (\tilde{F}(t), \tilde{v}(t))_{L^2(\Omega)} dt + (\tilde{v}'(0), v_m(0))_{L^2(\Omega)}. \]

Combining (A.11) and (A.15), we obtain

\[ \int_0^T \left( (\tilde{v}'(t), v(t))_{L^2(\Omega)} + B[v, \tilde{v}; t] \right) dt = \int_0^T (\tilde{F}(t), \tilde{v}(t))_{L^2(\Omega)} dt + (\tilde{v}'(0), v(0))_{L^2(\Omega)}. \]

Comparing this with (A.14), we see that

\[ (\tilde{v}'(0), v(0))_{L^2(\Omega)} = (\tilde{v}'(0), \varphi)_{L^2(\Omega)}. \]

Due to the arbitrariness of $\tilde{v}$, we conclude that $v$ verifies Definition 2.1(c).

**Step 4. Higher regularity.** We now further assume $\varphi \in \tilde{H}^s(\Omega)$. Multiplying (A.3) by $(d^k_m)'(t)$, and summing over $k = 1, \ldots, m$, we have

(A.16) \[ (v'_m, v'_m)_{L^2(\Omega)} + B[v_m, v'_m] = (\tilde{F}, v'_m)_{L^2(\Omega)}. \]

Note that we have

\[ B[v_m, v'_m] = \int_{\mathbb{R}^n} (-\Delta)^{\frac{a}{2}} v_m(t)(-\Delta)^{\frac{a}{2}} v'_m(t) dx + \int_{\Omega} a(t, x)v_m(t, x)v'_m(t, x) dx \]

\[ = \frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{a}{2}} v_m(t)|^2 dx \right) + \int_{\Omega} a(t, x)v_m(t, x)v'_m(t, x) dx, \]

and

\[ \int_{\Omega} a(t, x)v_m(t)v'_m(t) dx \leq \epsilon \|v'_m(t)\|^2_{L^2(\Omega)} + C\epsilon^{-1}\|v_m(t)\|^2_{L^2(\Omega)} \]

and

\[ |(\tilde{F}, v'_m)_{L^2(\Omega)}| \leq \epsilon \|v'_m(t)\|^2_{L^2(\Omega)} + C\epsilon^{-1}\|\tilde{F}(t)\|^2_{L^2(\Omega)}. \]
so (A.16) implies
\[
\|v'_m(t)\|_{L^2(\Omega)}^2 + \frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)\hat{v}_m(t)|^2 \, dx \right) \\
\leq 2\epsilon \|v'_m(t)\|_{L^2(\Omega)}^2 + C\epsilon^{-1} \|v_m(t)\|_{L^2(\Omega)}^2 + C\epsilon^{-1} \|\hat{F}(t)\|_{L^2(\Omega)}^2.
\]
Choosing \(\epsilon = 1/4\), we obtain
\[
(A.17) \quad \|v'_m(t)\|_{L^2(\Omega)}^2 + \frac{d}{dt} \left( \int_{\mathbb{R}^n} |(-\Delta)\hat{v}_m(t)|^2 \, dx \right) \leq C(\|v_m(t)\|_{L^2(\Omega)}^2 + \|\hat{F}(t)\|_{L^2(\Omega)}^2).
\]
Given any \(0 \leq \tilde{t} \leq T\), we integrate (A.17) on \(t \in [0, \tilde{t}]\), we obtain
\[
\int_0^\tilde{t} \|v'_m(t)\|_{L^2(\Omega)}^2 \, dt + \int_{\mathbb{R}^n} |(-\Delta)\hat{v}_m(\tilde{t})|^2 \, dx - \int_{\mathbb{R}^n} |(-\Delta)\hat{v}_m(0)|^2 \, dx \\
\leq C\left( \int_0^\tilde{t} \|v_m(t)\|_{L^2(\Omega)}^2 \, dt + \int_0^\tilde{t} \|\hat{F}(t)\|_{L^2(\Omega)}^2 \right) \\
\leq C\(\|v_m\|_{L^2(\Omega_T)}^2 + \|\hat{F}\|_{L^2(\Omega_T)}^2\),
\]
combining this inequality with (A.5), we obtain
\[
\|v'_m\|_{L^2(\Omega_T)}^2 + \|v_m\|_{L^\infty(0,T;\dot{H}^s(\Omega))}^2 \\
\leq C(\|v'_m\|_{L^2(\Omega_T)}^2 + \sup_{0 \leq \tilde{t} \leq T} \int_{\mathbb{R}^n} |(-\Delta)\hat{v}_m(\tilde{t})|^2 \, dx) \\
\leq C\left( \int_{\mathbb{R}^n} |(-\Delta)\hat{v}_m(0)|^2 \, dx + \|v_m\|_{L^2(\Omega_T)}^2 + \|\hat{F}\|_{L^2(\Omega_T)}^2 \right) \\
\leq C(\|v_m(0)\|_{\dot{H}^s(\Omega)}^2 + \|v_m\|_{L^2(\Omega_T)}^2 + \|\hat{F}\|_{L^2(\Omega_T)}^2).
\]
(A.18)

Since
\[
\|v_m(0)\|_{\dot{H}^s(\Omega)}^2 = \sum_{k=1}^{m} |(g, w_k)_{L^2(\Omega)}|^2 \|w_k\|_{\dot{H}^s(\Omega)}^2 \leq \sum_{k=1}^{\infty} |(g, w_k)_{L^2(\Omega)}|^2 \|w_k\|_{\dot{H}^s(\Omega)}^2 = \|\varphi\|_{\dot{H}^s(\Omega)},
\]
then (A.18) implies
\[
\|v'_m\|_{L^2(\Omega_T)}^2 + \|v_m\|_{L^\infty(0,T;\dot{H}^s(\Omega))}^2 \leq C(\|\varphi\|_{\dot{H}^s(\Omega)}^2 + \|v_m\|_{L^2(\Omega_T)}^2 + \|\hat{F}\|_{L^2(\Omega_T)}^2).
\]
Therefore, combining this inequality with (A.10), we obtain
\[
\|v'_m\|_{L^2(\Omega_T)}^2 + \|v_m\|_{L^\infty(0,T;\dot{H}^s(\Omega))}^2 \leq C(\|\varphi\|_{\dot{H}^s(\Omega)}^2 + \|\hat{F}\|_{L^2(\Omega_T)}^2).
\]
Finally, taking the limit \(m \to \infty\), we complete our proof.

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REFERENCES

[Bre11] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011. MR2759829, doi:10.1007/978-0-387-70914-7.

[CFK+21] C. I. Carstea, A. Feizmohammadi, Y. Kian, K. Krupchyk, and G. Uhlmann. The Calderón inverse problem for isotropic quasilinear conductivities. *Adv. Math.*, 391:Paper No. 107956, 31, 2021. MR4300916, doi:10.1016/j.aim.2021.107956, arXiv:2103.05917.

[DNPV12] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012. MR2944369, doi:10.1016/j.bulsci.2011.12.004, arXiv:1104.4345.

[Eva10] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010. MR2597943, doi:10.1090/gsm/019.

[FKOU21] A. Feizmohammadi, K. Krupchyk, L. Oksanen, and G. Uhlmann. Reconstruction in the Calderón problem on conformally transversally anisotropic manifolds. *J. Funct. Anal.*, 281(9):Paper No. 109191, 25, 2021. MR4291510, doi:10.1016/j.jfa.2021.109191, arXiv:2009.10280.

[GRSU20] T. Ghosh, A. Rüland, M. Salo, and G. Uhlmann. Unique uniqueness and reconstruction for the fractional Calderón problem with a single measurement. *J. Funct. Anal.*, 279(1):108505, 42, 2020. MR4083776, doi:10.1016/j.jfa.2020.108505, arXiv:1801.04449.

[GSU20] T. Ghosh, M. Salo, and G. Uhlmann. The Calderón problem for the fractional Schrödinger equation. *Analysis & PDE*, 13(2):455–475, 2020. MR4078233, doi:10.2140/apde.2020.13.455, arXiv:1609.09248.

[GT01] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order (reprint of the 1998 edition)*, volume 224 of *Classics in Mathematics*. Springer-Verlag Berlin Heidelberg, 2001. MR1814364, doi:10.1007/978-3-642-61798-0.

[HL22] B. Harrach and Y.-H. Lin. Simultaneous recovery of piecewise analytic coefficients in a semilinear elliptic equation. *arXiv preprint*, 2022. arXiv:2201.04594.

[Isa01] V. Isakov. Uniqueness of recovery of some quasilinear partial differential equations. *Comm. Partial Differential Equations*, 26(11-12):1947–1973, 2001. MR1876409, doi:10.1081/PDE-100107813.

[JLS17] V. Julin, T. Liimatainen, and M. Salo. $p$-harmonic coordinates for Hölder metrics and applications. *Comm. Anal. Geom.*, 25(2):395–430, 2017. MR3690246, doi:10.4310/CAG.2017.v25.n2.a5, arXiv:1507.03874.

[KLU18] Y. Kurylev, M. Lassas, and G. Uhlmann. Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations. *Invent. Math.*, 212(3):781–857, 2018. MR3802298, doi:10.1007/s00222-017-0780-y, arXiv:1405.3386.

[KLW21] P.-Z. Kow, Y.-H. Lin, and J.-N. Wang. The Calderón problem for the fractional wave equation: Uniqueness and optimal stability. *arXiv preprint*, 2021. arXiv:2105.11324.

[KU20] K. Krupchyk and G. Uhlmann. Partial data inverse problems for semilinear elliptic equations with gradient nonlinearities. *Math. Res. Lett.*, 27(6):1801–1824, 2020. MR4216606, doi:10.4310/MRL.2020.v27.n6.a10, arXiv:1909.08122.

[Kwa17] M. Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fractional Calculus and Applied Analysis*, 20(1):7–51, 2017. MR3613319, doi:10.1515/fca-2017-0002, arXiv:1507.03874.

[Li21] L. Li. An inverse problem for a fractional diffusion equation with fractional power type nonlinearities. *arXiv preprint*, 2021. arXiv:2104.00132.

[Lin20] Y.-H. Lin. Monotonicity-based inversion of fractional semilinear elliptic equations with power type nonlinearities. *arXiv preprint*, 2020. arXiv:2005.07163.

[LL19] R.-Y. Lai and Y.-H. Lin. Global uniqueness for the fractional semilinear Schrödinger equation. *Proc. Amer. Math. Soc.*, 147(3):1189–1199, 2019. MR3896066, doi:10.1090/proc/14319, arXiv:1710.07404.
[LL20] R.-Y. Lai and Y.-H. Lin. Inverse problems for fractional semilinear elliptic equations. *arXiv preprint*, 2020. arXiv:2004.00549.

[LLL21] Y.-H. Lin, H. Liu, and X. Liu. Determining a nonlinear hyperbolic system with unknown sources and nonlinearity. *arXiv preprint*, 2021. arXiv:2107.10219.

[LLLS21] M. Lassas, T. Liimatainen, Y. Lin, and M. Salo. Inverse problems for elliptic equations with power type nonlinearities. *J. Math. Pures Appl. (9)*, 145:44–82, 2021. MR4188325, doi:10.1016/j.matpur.2020.11.006, arXiv:1903.12562.

[LLPMT20] M. Lassas, T. Liimatainen, L. Potenciano-Machado, and T. Tyni. Uniqueness and stability of an inverse problem for a semi-linear wave equation. *arXiv preprint*, 2020. arXiv:2006.13193.

[MBRS16] G. Molica Bisci, V. D. Radulescu, and R. Servadei. *Variational methods for nonlocal fractional problems*, volume 162 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2016. MR3445279, doi:10.1017/CBO9781316282397.

[McL00] W. McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, 2000. MR1742312.

[Pon16] A. C. Ponce. *Elliptic PDEs, measures and capacities. From the Poisson equations to nonlinear Thomas-Fermi problems*. EMS Tracts in Mathematics, volume 23. European Mathematical Society (EMS), Zürich, 2016. MR3675703, doi:10.4171/140.

[RO16] X. Ros-Oton. Nonlocal elliptic equations in bounded domains: a survey. *Publ. Mat.*, 60(1):3–26, 2016. MR3447732, doi:10.5565/PUBLMAT_60116_01, arXiv:1504.04099.

[Ste16] E. M. Stein. *Singular integrals and differentiability properties of functions (PMS-30)*, volume 30. Princeton university press, 2016. MR0290095, doi:10.1515/9781408883882.

[Tri02] H. Triebel. Function spaces in Lipschitz domains and on Lipschitz manifolds. characteristic functions as pointwise multipliers. *Revista Matemática Complutense*, 15(2):475–524, 2002. MR1951822, doi:10.5209/rev_REMA.2002.v15.n2.16910.