The integral representation of solutions of KZ equation and a modification by $\mathcal{K}$ operator insertion

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Abstract

A root of unity limit of the $q$-deformed Virasoro algebra is considered. The $\widehat{sl}(2)_k$ current algebra and the integral formulas of the solutions of the KZ equations can be realized by the $q$-deformed boson at the limit and an additional boson. We explicitly construct the integral representation of the four-point blocks with a $\mathcal{K}$-operator insertion.

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1 Introduction

The AGT relation [1] states that the instanton partition functions [2] of the four-dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theory are related to the correlation functions [3] of the two-dimensional conformal field theory with Virasoro symmetry. The extension to the similar correspondence between $SU(n)$ gauge theory and conformal field theory with $W_n$ symmetry has been constructed in [4, 5]. Since then, the both sides of the correspondence have been intensively studied by a number of people. For example, see [9–32].

There exists a natural generalization to the connection between the 2d theory with the $q$-deformed Virasoro/W symmetry and the five-dimensional gauge theory [33–39]. Recently, the elliptic Virasoro/6d correspondence is also proposed in [40, 41].

In our previous papers [42–44], we considered a $r$-th root of unity limit in $q$ and $t$ ($q \to \omega$, $t \to \omega$ with $\omega = e^{2\pi i}$) of the $q$-W/5d correspondence. In [42], we proposed a limiting procedure to get the Virasoro/W block in the 2d side from that in the $q$-deformed version. In [43], we have elaborated the limiting procedure and showed that the $\mathbb{Z}_r$-parafermionic CFT appears in the 2d side. $^1$

The another extension of the AGT relation by including various defects in the gauge theory have also been considered [46]. We are interested in the defects, the so-called surface operator, supported on two-dimensional submanifolds (for review for surface operator, see [47]). There are two kinds of the surface operators in a sense, i.e. two-dimensional defects brought either from 2d-defect of 4d-defects in the M5-brane construction of $\mathcal{N} = 2$ gauge theory. The instanton partition functions in the presence of a kind of surface operator are related with the conformal block with a degenerate field insertion [48–54]. In the present paper, we focus on the 4d-defects. It was conjectured that the instanton partition functions in this case are related to the affine $\widehat{sl}_k(n)$ current block with a mysterious operator, the so-called $K$ operator, insertion [55, 56]. More general cases of the surface operator insertion are discussed in [57, 58].

On the other hand, the defining relation of the $q$-Virasoro algebra is [59–61]

$$f(z'/z)\mathcal{T}(z)\mathcal{T}(z') - f(z/z')\mathcal{T}(z')\mathcal{T}(z) = \frac{(1 - q)(1 - t^{-1})}{(1 - p)}\left[\delta(pz/z') - \delta(p^{-1}z/z')\right], \quad (1.1)$$

where $p = q/t$ and

$$f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1 - q^n)(1 - t^{-n})}{(1 + p^n)} z^n\right), \quad \delta(z) = \sum_{n \in \mathbb{Z}} z^n. \quad (1.2)$$

$^1$The root of unity limit of the troidal algebra has also been considered in [45].
It is known that the $q$-deformed current $T(z)$ can be realized by the $q$-deformed Heisenberg algebra. In this paper, we consider the following root of unity limit in $q$ and $t$:

$$q \to 1, \quad t \to -1.$$  \hfill (1.3)

This limit has been considered in [63] and the $q$-Virasoro algebra (1.1) is reduced to the Lepowsky-Wilson’s $Z$-algebra [64, 65]. The $\widehat{sl}(2)_k$ current algebra can be realized by using the two types of bosons obtained from a $q$-deformed boson in this limit and an additional boson. In this formalism, we will first reconstruct the integral representation of the solution to the KZ equation [66] that the current blocks should satisfy. Then we consider the current block with the $K$ operator insertion and illustrate how to derive those as integral representation.

This paper is organized as follows: In section two, the root of unity limit of the $q$-deformed boson is considered. In section three, we see that the $\widehat{sl}(2)_k$ current algebra can be realized. In section four, integral formulas of the solutions to the KZ equation are constructed. In section five, we consider the four-point current block with the $K$ operator insertion and present its integral representation.

## 2 $q$-deformed boson at root of unity

As mentioned in Introduction, the $q$-deformed Virasoro current $T(z)$ can be realized by the $q$-deformed Heisenberg algebra [59],

$$[\alpha_0, \tilde{Q}] = 2,$$

$$[\alpha_n, \alpha_m] = -\frac{1}{n}(1 - q^{-n})(1 - t^n)(1 + p^n)\delta_{n+m,0}. \hfill (2.1)$$

The $q$-deformed boson is defined by

$$\tilde{\varphi}(z) = \tilde{\varphi}_{\text{even}}(z) + \tilde{\varphi}_{\text{odd}}(z), \hfill (2.2)$$

where

$$\tilde{\varphi}_{\text{even}}(z) = \beta \frac{1}{2} \tilde{Q} + \beta \frac{1}{2} \alpha_0 \log z - \sum_{n \neq 0} \frac{1}{q^n - q^{-n}} \alpha_{2n} z^{-2n}, \hfill (2.3)$$

$$\tilde{\varphi}_{\text{odd}}(z) = -\sum_{n \in \mathbb{Z}} \frac{1}{q^{(2n+1)/2} - q^{-(2n+1)/2}} \alpha_{2n+1} z^{-(2n+1)}. \hfill (2.4)$$

Let us consider the following limit of the $q$-deformed boson,

$$q = t^\alpha = e^{-\frac{h}{\sqrt{\beta}}}, \quad t = (-1)^{\tilde{k}} e^{-\sqrt{\beta}h}, \quad p = q/t = (-1)^{-\tilde{k}} e^{Q \sqrt{\beta}h}, \quad h \to 0. \hfill (2.5)$$

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where \( k \) is an odd number and \( Q_E = \sqrt{\beta - \frac{1}{\sqrt{\beta}}} \) and \( \alpha = \frac{1}{\beta} \). In order to take this limit, the following condition is demanded:

\[
\alpha \left( i\pi k - \frac{h}{\sqrt{\alpha}} + 2\pi m_+ \right) = -\sqrt{\alpha h} + 2\pi m_-. \tag{2.6}
\]

where \( m_\pm \) are positive integers. Thus the parameter \( \alpha \) has to satisfy

\[
\alpha = \frac{1}{\beta} = -\frac{2m_-}{2m_+ + k} \equiv \frac{2}{2 + k}, \tag{2.7}
\]

where

\[
k = \frac{2(m_+ - m_-) + \tilde{k}}{m_-}. \tag{2.8}
\]

In the section 3, we will see that the parameter \( k \) serves as the level of the \( \hat{sl}(2)_k \) algebra. In the limit (2.5), the even and odd part of the \( q \)-boson are expanded in powers of \( h \) as

\[
\bar{\varphi}_{\text{even}}(z) \equiv \phi_1(z) + \mathcal{O}(h), \tag{2.9}
\]

\[
\bar{\varphi}_{\text{odd}}(z) \equiv \phi_2(z) + \mathcal{O}(h), \tag{2.10}
\]

where

\[
\phi_1(z) = Q + \phi^{(1)}_0 \log z - \sum_{n \neq 0} \frac{\phi^{(1)}_{2n}}{2n} z^{-2n}, \tag{2.11}
\]

\[
\phi_2(z) = -\sum_{n \in \mathbb{Z}} \frac{\phi^{(2)}_{2n+1}}{2n+1} z^{-(2n+1)}. \tag{2.12}
\]

with the commutation relations,

\[
[\phi^{(1)}_{2n}, \phi^{(1)}_{2m}] = (k + 2)(2n)\delta_{n+m,0}; \quad [\phi^{(1)}_0, Q] = k + 2, \tag{2.13}
\]

\[
[\phi^{(2)}_{2n+1}, \phi^{(2)}_{-2m-1}] = -k(2n + 1)\delta_{m,n}. \tag{2.14}
\]

### 3 Affine \( \hat{sl}_k(2) \) algebra

#### 3.1 Free field realization

There exists the well-known free field realization in terms of a free boson and \( \beta-\gamma \) system i.e. the Wakimoto representation [62]. However, we introduce another realization of the \( \hat{sl}_k(2) \) current algebra in terms of the bosons obtained in the previous section because it is useful in order to consider the insertion of the \( K \) operator which we will see in the section 5.
The following additional bosons are required:

\[ \phi_0(z) = - \sum_{n \in \mathbb{Z}} \frac{\phi_{2n+1}^{(0)}}{2n+1} z^{-(2n+1)}, \quad (3.1) \]

with

\[ [\phi_{2n+1}^{(0)}, \phi_{-2m-1}^{(0)}] = k(2n + 1)\delta_{n,m}. \quad (3.2) \]

This has the same algebraic structure as \( \phi^{(2)}(z) \) (times i). Following [63], let us introduce

\[ \beta(z) = \partial \phi_0(z), \quad (3.3) \]
\[ x(z) = : (\partial \phi_1(z) + \partial \phi_2(z)) e^{k \frac{\varphi(z)}{2}} :, \quad (3.4) \]

where \( \varphi(z) = \phi_0(z) + \phi_2(z) \) and the symbol :: stands for the normal ordering defined in the standard way. Using \( \beta(z) \) and \( x(z) \), we can explicitly construct the \( \hat{sl}(2) \) currents of the level \( k \) as follows:

\[ E(w) = \frac{1}{2} \left[ (\beta(z) - x_e(z)) \right]_{w=z^2}, \quad (3.5) \]
\[ F(w) = \frac{1}{2z^2} \left[ (\beta(z) + x_e(z)) \right]_{w=z^2}, \quad (3.6) \]
\[ H(w) = \left( \frac{x_o(z)}{z} + \frac{k}{2z^2} \right) \left|_{w=z^2} \right., \quad (3.7) \]

where

\[ x_e(z) = \frac{1}{2} (x(z) + x(-z)), \quad x_o(z) = \frac{1}{2} (x(z) - x(-z)). \quad (3.8) \]

Note that the currents are defined on \( w \)-plane. In fact, one can easily check that \( E(w), F(w) \) and \( H(w) \) serve precisely as the affine \( \hat{sl}(2)_k \) currents,

\[ H(w_1)H(w_2) \sim \frac{2k}{(w_1 - w_2)^2}, \]
\[ H(w_1)E(w_2) \sim \frac{2}{w_1 - w_2} E(w_2), \quad (3.9) \]
\[ H(w_1)F(w_2) \sim \frac{-2}{w_1 - w_2} F(w_2), \]
\[ E(w_1)F(w_2) \sim \frac{k}{(w_1 - w_2)^2} + \frac{1}{w_1 - w_2} H(w_2). \]

The stress tensor with the central charge \( c = \frac{3k}{k+2} \) can also be constructed by the Sugawara construction,

\[ T(w) = \frac{1}{4z^2} \left\{ \frac{1}{k} (\partial \phi_0)^2 + \frac{1}{k} (\partial \phi_1)^2 - \frac{1}{k} (\partial \phi_2)^2 - \frac{1}{k} \left( \frac{\partial}{z} + \frac{1}{z} \right) \partial \phi_1 \right. \]
\[ \left. + \frac{1}{z} \left( \partial \phi_1 \cosh \frac{k}{2} \varphi + \partial \phi_2 \sinh \frac{k}{2} \varphi \right) + \frac{k(k+4)}{4kz^2} \right\} \left|_{w=z^2} \right., \quad (3.10) \]
where \( \kappa = k + 2 \) as usual.

### 3.2 spin \( j/2 \) representation

The operators corresponding to the highest (respectively, lowest) weight state of the spin 1/2 representation are given by

\[
V_{1/2,1/2}(w) = \frac{1}{2} : \left( e^{\alpha_1 \phi_1(z)} + e^{\alpha_2 \phi_2(-z)} \right) :_{w = z^2} = \frac{k}{2\pi} : e^{\frac{1}{k} \phi(z)} \cosh \left( \frac{1}{k} \varphi(z) \right) :_{w = z^2},
\]

\[
V_{1/2,1/2}(w) = \frac{1}{2} : \left( e^{\alpha_1 \phi_1(z)} - e^{\alpha_2 \phi_2(-z)} \right) :_{w = z^2} = \frac{k}{2\pi} : e^{\frac{1}{k} \phi(z)} \sinh \left( \frac{1}{k} \varphi(z) \right) :_{w = z^2},
\]

where the repeated indices \( i \) are summed over for \( i = 0, 1, 2 \) and

\[
(\alpha^0, \alpha^1, \alpha^2) = \left( \frac{1}{k}, \frac{1}{\kappa}, \frac{1}{k} \right).
\]

In general, the operators corresponding to the states belonging to the spin \( j/2 \) representation \( (j \in \mathbb{Z}_{\geq 0}) \) are given by

\[
V_{j/2,j/2-2m}(w) = z^{\frac{k}{2\pi} - m} : e^{j \alpha_1 \phi_1} \cosh^{2-m} \left( \frac{1}{k} \varphi(z) \right) \sinh^m \left( \frac{1}{k} \varphi(z) \right) :_{w = z^2}, \quad 0 \leq m \leq j.
\]

In particular, the operator corresponding to the highest weight state is

\[
V_{j/2}(w) \equiv V_{j/2, j/2}(w) = z^{\frac{k}{2\pi}} : e^{j \alpha_1 \phi_1} \cosh^{j} \left( \frac{1}{k} \varphi(z) \right) :_{w = z^2}.
\]

The vertex operator \( V_{j/2,j/2-2m}(w) \) has the expected behavior,

\[
H(w_1) V_{j/2,j/2-2m}(w_2) \sim \frac{2 \left( \frac{j}{2} - m \right)}{w_1 - w_2} V_{j/2,j/2-2m}(w_2),
\]

\[
E(w_1) V_{j/2,j/2-2m}(w_2) \sim \frac{m}{w_1 - w_2} V_{j/2,j/2-2m+1}(w_2),
\]

\[
F(w_1) V_{j/2,j/2-2m}(w_2) \sim \frac{j - m}{w_1 - w_2} V_{j/2,j/2-2m-1}(w_2),
\]

On the other hand, it is easy to check that \( V_{j/2,j/2-2m}(w) \) is also primary operator with the scaling dimension \( \Delta_j = \frac{j(j+1)}{4\kappa} \).

### 4 Solutions of KZ equation

Let \( V = V_{j_1} \otimes V_{j_2} \otimes \cdots \otimes V_{j_N} \). Here \( V_{j_n}, 0 \leq n \leq N \) is the \( (j_n + 1) \)-dimensional lowest weight module for \( \mathfrak{sl}_2 \). The standard Chevalley basis of \( \mathfrak{sl}_2 \) is denoted by \( \{e, f, h\} \) with \([e, f] =\)
The lowest weight state \( v_j \in V_j \) is defined by

\[
f \cdot v_j = 0, \quad h \cdot v_j = -j v_j, \quad e^{j+1} \cdot v_j = 0.
\]

(4.1)

The KZ equation [66] (for review, see [67]) is written as

\[
\kappa \frac{\partial}{\partial w_n} \Psi(w) = \left( \sum_{m=1, m \neq n}^{N} \frac{\Omega_{mn}}{w_m - w_n} \right) \Psi(w), \quad n = 1, \ldots, N.
\]

(4.2)

where \( \Psi(w) \) is the function which takes value in \( V \) and

\[
\Omega_{mn} = e_m f_n + f_m e_n + \frac{1}{2} h_m h_n.
\]

(4.3)

Here \( x_n \) stands for the action of \( x \in \mathfrak{sl}_2 \) on \( V_j \), i.e.

\[
x_n = 1 \otimes 1 \otimes \cdots \otimes \underbrace{x}^{\mathsf{v}} \otimes \cdots \otimes 1,
\]

(4.4)

\[
x_m y_n = 1 \otimes 1 \otimes \cdots \otimes \underbrace{x}^{\mathsf{v}} \otimes \cdots \otimes \underbrace{y}^{\mathsf{v}} \otimes \cdots \otimes 1
\]

(4.5)

In this section, we want to get the integral formulas of the solution \( s \) of (4.2) in the free field realization constructed in the previous section.

### 4.1 A simple solution

Let us define

\[
X(z) =: \cosh \left( \frac{1}{k} (\phi_0 + \phi_2)(z) \right) :, \quad Y(z) = \frac{1}{z} : \sinh \left( \frac{1}{k} (\phi_0 + \phi_2)(z) \right) :, \quad Z_j(z) = z^{\frac{k j}{2k}} : e^{-k \phi_1(z)} .
\]

(4.6)

Then the operator (3.14) is expressed by

\[
V_{j/2, j/2-m}(w) = Z_j(z) X(z)^{j-2m} Y(z)^m \bigg|_{w = z^2}.
\]

(4.8)

Note that

\[
X(z_1) X(z_2) =: X(z_1) X(z_2) :, \quad Y(z_1) Y(z_2) =: Y(z_1) Y(z_2) :, \quad X(z_1) Y(z_2) =: X(z_1) Y(z_2) :, \quad (4.9)
\]

\[
\prod_{i=1}^{N} Z_{j_i}(z_i) = \prod_{m<n}^{N} \left( z_m^2 - z_n^2 \right)^{\frac{j_m j_n}{2k}} : \prod_{n=1}^{N} Z_{j_n}(z_n) :. \quad (4.10)
\]
Let us choose the Fock vacuum $|\Omega\rangle$ and the conjugate $\langle \Omega |$ as
\[
\phi_{2n+1}^{(0)}|\Omega\rangle = 0, \quad \phi_{2n+1}^{(1)}|\Omega\rangle = 0, \quad n \geq 0, \\
\langle \Omega |\phi_{2n+1}^{(0)} = 0, \quad \langle \Omega |\phi_{2n+1}^{(1)} = 0, \quad n < 0.
\]

The highest weight state of the spin $j/2$ representation is given by
\[
|j\rangle = e^{\frac{j}{2}(j-\frac{1}{2})Q}|\Omega\rangle, \quad \langle j | = \langle \Omega |e^{-\frac{j}{2}(j-\frac{1}{2})Q}.
\]

We denote by $\mathcal{H}_j$ the highest weight module over $\hat{\mathfrak{sl}}(2)_k$ generated from $|j\rangle$. Then the operator $\mathcal{V}_{j/2,j/2-m}(w)$ plays a role to map $\mathcal{H}_{j_0}$ onto $\mathcal{H}_{j_0+j}$ for any $j_0 \in \mathbb{Z}_{\geq 0}$. From the product of $N$ $\mathcal{V}_{j/2}$’s we obtain
\[
\langle j | \prod_{n=1}^{N} \mathcal{V}_{j_{n}/2}(w_n) |0\rangle = \prod_{m<n}^N (w_m - w_n)^{\frac{j_m-j_n}{2}} \equiv \psi_0(w), \quad j = \sum_{n=1}^{N} j_n. \tag{4.13}
\]

Consequently the simple solution of (4.2) is given by
\[
\Psi_0(w) = \psi_0(w)v = \prod_{m<n}^N (w_m - w_n)^{\frac{j_m-j_n}{2}}v = \langle j | \prod_{n=1}^{N} \mathcal{V}_{j_{n}/2}(z_n) |0\rangle v, \quad j = \sum_{n=1}^{N} j_n. \tag{4.14}
\]

where $v = v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_N} \in V$.

### 4.2 screening charge

The screening current is defined by \(^2\)
\[
S(t) = t^{-\kappa \phi_2(t)}e^{\frac{-2}{\kappa} \phi_1(t)} \bigg|_{t=t^2}, \tag{4.15}
\]
which satisfies
\[
\beta(z)S(t) \sim 0, \\
x_e(z)S(t) \sim \frac{\partial}{\partial \tau} A_e(z, \tau), \tag{4.16}
\]
\[
x_o(z)S(t) \sim \frac{\partial}{\partial \tau} A_o(z, \tau),
\]

\(^2\)The screening current (4.15) can be obtained by taking the root of unity limit of the $q$-deformed screening current $\tilde{S}(z) = e^{\tilde{\varphi}(z)}$ up to the overall factor.
where
\[
A_e(z, \tau) = \kappa \frac{t^{k+4}}{z^2 - t^2} \left( e^{\frac{2}{k}(\phi_0 + \phi_2)(t)} + e^{-\frac{2}{k}(\phi_0 + \phi_2)(t)} \right) e^{-\frac{2}{k+2}\phi_1(t)} \bigg|_{\tau = t^2},
\]
(4.17)
\[
A_o(z, \tau) = \kappa \frac{t^{k+4}}{z^2 - t^2} \left( e^{\frac{2}{k}(\phi_0 + \phi_2)(t)} - e^{-\frac{2}{k}(\phi_0 + \phi_2)(t)} \right) e^{-\frac{2}{k+2}\phi_1(t)} \bigg|_{\tau = t^2}.
\]
(4.18)

The screening charge defined by
\[
U = \int_C d\tau S(\tau),
\]
(4.19)
commutes with the \(\hat{sl}(2)_k\) currents \(E(w), H(w), F(w)\). Here we postulate the cycle \(C\) on \(w\)-plane is chosen appropriately.

### 4.3 intertwining operator

In this section, we consider the intertwining operator \(\Phi_j^m(z) : \mathcal{H}_{j_0} \to \mathcal{H}_{j_0 + j - 2m} \otimes V_j, \forall j_0 \in \mathbb{Z}_{\geq 0}\). Let us introduce the following formal operator:
\[
\gamma(z) = X(z)^{-1}Y(z).
\]
(4.20)
The intertwining operator of level \(m = 0\) can be constructed in terms of \(Z_j(z), X(z)\) and \(Y(z)\) as
\[
\Phi_j^0(z)u = Z_j(z)X^j(z)e^{-\gamma(z) \otimes e}(u \otimes v_j),
\]
(4.21)
where \(u \in \mathcal{H}_{j_0}\) and \(v_j \in V_j\) is the lowest weight state. In fact, \(\Phi_j^0(z)\) satisfies the intertwining relation,
\[
\Phi_j^0(z)J^A_n = (J^A_n \otimes 1 + (z^2)^n \cdot 1 \otimes A)\Phi_j^0(z) \quad A = e, f, h.
\]
(4.22)
By using the intertwining operators, \(\Psi_0(w)\) is given by
\[
\Psi_0(w) = \langle j | \prod_{n=1}^N \Phi_{j_n}^0(z_n)|0\rangle, \quad j = \sum_{n=1}^N j_n.
\]
(4.23)
The general \(m\) intertwining operator can be obtained by multiplying \(m\) screening charges to \(\Phi_j^0(z)\),
\[
\Phi_j^m(z)u = \Phi_j^0(z)U^m,
\]
(4.24)
and the solutions of the KZ equation are given by
\[
\Psi_m(w) = \langle j - 2m | \prod_{i=1}^N \Phi_{j_i}^{m_i}(z_i)|0\rangle, \quad j = \sum_{i=1}^N j_i, \quad m = \sum_{i=1}^N m_i.
\]
(4.25)
which reproduces the well-known integral formulas. For example, we obtain, in the case of \( m = 1 \),

\[
\Psi_1(w) = \langle j - 2 \mid \prod_{n=1}^{N-1} \Phi_{j_n}(z_n) \Phi_{j_N}(z_N) \mid 0 \rangle
\]

\[
= \prod_{m<n} (w_m - w_n)^{-\frac{im}{2\pi}} \int_C \prod_{n=1}^{N} (w_n - \tau)^{-\frac{im}{2\pi}} \left( \sum_{n=1}^{N} \frac{-e_n}{w_n - \tau} \right) v,
\]

(4.26)

and in the case of \( m = 2 \),

\[
\Psi_2(w) = \langle j - 4 \mid \prod_{n=1}^{N-1} \Phi_{j_n}(z_n) \Phi_{j_N}(z_N) \mid 0 \rangle
\]

\[
= \prod_{m<n} (w_m - w_n)^{-\frac{im}{2\pi}} \int_C \prod_{n=1}^{N} \prod_{p=1}^{2} (w_n - \tau_p)^{-\frac{im}{2\pi}} \left( \tau_1 - \tau_2 \right)^2 \times \left( \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{e_m e_n}{(w_m - \tau_1)(w_n - \tau_2)} \right) v,
\]

(4.27)

where \( \tau = (\tau_1, \tau_2) \) and \( C = C_1 \times C_2 \). Here we have used

\[
\Phi_j^0(z) S(\tau) = (w - \tau)^{-\frac{m}{2\pi}} \left\{ : \Phi_j^0(z) S(\tau) : - \frac{\tau - \frac{\phi_1(t)}{2}}{w - \tau} \left( (1 \otimes e) - (w \otimes f) \right) : e^{-\frac{i}{2\pi} (2\phi_1(t) + \phi_1(t'))} : \right\},
\]

(4.28)

\[
S(\tau) S(\tau') = (\tau - \tau')^{\frac{m}{2\pi}} \left\{ : S(\tau) S(\tau') : - \frac{k(\tau \tau') - \frac{i}{2\pi} (\tau + \tau')}{(\tau - \tau')^2} : e^{-\frac{2}{\pi} (\phi_1(t) + \phi_1(t'))} : \right\}.
\]

(4.29)

5 Insertion of \( \mathcal{K} \) operator

In the paper [55], the authors proposed the insertion of the \( \mathcal{K} \) operator to the \( \hat{sl}(2)_k \) current block in order to establish the AGT relation in the presence of a full surface operator. The generalization to the \( \hat{sl}(n)_k \) current block is also discussed in [56]. The \( \mathcal{K} \) operator is defined by \(^3\)

\[
\mathcal{K}^\dagger = \exp \left( \sum_{n=1}^{\infty} \frac{1}{2n - 1} [J_{n-1}^+ + J_n^-] \right).
\]

(5.1)

In our formalism, the \( \mathcal{K} \) operator has the following simple expression in terms of \( \phi_0(z) \):

\[
\mathcal{K}^\dagger = e^{-\phi_0^{(1)}(1)},
\]

(5.2)

\(^3\)Since the \( \mathcal{K} \) operator we will use below is equal to \( \mathcal{K}^\dagger(1, 1) \) presented in [56], we denote it by \( \mathcal{K}^\dagger \).
where \( \phi_0^{(+)}(z) \) is the positive modes of \( \phi_0(z) \).

Therefore it is meaningful to consider the four-point correlation function with a \( \mathcal{K} \) operator insertion,

\[
\tilde{\Psi}_{m_1,m_2}(w_2) \equiv \langle j - 2m | \Phi_{j_1}^{m_1}(1)e^{-\phi_0^{(+)}(1)}\Phi_{j_2}^{m_2}(z_2)|j_3 \rangle.
\] (5.3)

Since the screening current \( S(z) \) does not include \( \phi_0(z) \), it is trivial that the \( \mathcal{K} \) operator commutes with \( U \). However, the action of \( e^{-\phi_0^{(+)}(1)} \) on \( \Phi_0^{j_1}(z) \) yields nontrivial result and we obtain

\[
\tilde{\Psi}_{m_1,m_2}(w_2) = \langle j - 2m | \Phi_{j_1}^{m_1}(1)\tilde{\Phi}_{j_2}^{m_2}(z_2)|j_3 \rangle,
\] (5.4)

where

\[
\tilde{\Phi}_{j}^{m}(z)u = \tilde{\Phi}_{j}^{0}(z)U^m u
= Z_j(z)\tilde{X}(z)e^{-\tilde{\gamma}(z)\otimes e}U^m u \otimes v_j,
\] (5.5)

with

\[
\tilde{X}(z) = X(z) + z^2 Y(z),
\]
\[
\tilde{Y}(z) = Y(z) + X(z),
\]
\[
\tilde{\gamma}(z) = \tilde{X}^{-1}(z)\tilde{Y}(z).
\] (5.6)

It is easy to get the explicit integral formulas for the four point current block with the \( \mathcal{K} \) operator insertion. For example,

\[
\tilde{\Psi}_{0,0}(w_2) = (1 - w_2)^{\frac{1}{24} - \frac{j_2}{2} - \frac{1}{2}w_2^2}j_1) (v_{j_1} \otimes e^{-v_{j_2}}),
\] (5.7)
\[
\tilde{\Psi}_{0,1}(w_2) = (1 - w_2)^{\frac{1}{24} - \frac{j_2}{2} - \frac{1}{2}w_2^2} \int_C d\tau (1 - \tau)^{-\frac{j_1}{2}}(w_2 - \tau)^{-\frac{j_2}{2}} (w_2 - \tau)^{-\frac{j_4}{2}}
\]
\[
\left\{ \frac{-e_1}{1 - \tau} + \frac{-e_2 + w_2 f_2}{w_2 - \tau} \right\} (v_{j_1} \otimes e^{-v_{j_2}}).
\] (5.8)

### 6 Summary

To summarize, we have reproduced the free field realization of the \( sl(2)_k \) current algebra and the integral formulas of the KZ equation by using three types of chiral bosons which are obtained from the \( q \)-deformed boson in the root of unity limit. In addition, we have derived the integral formulas for the modified four point current blocks.

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\(^4\)The original \( \mathcal{K} \) operator in [55] can be realized by the negative modes of \( \phi_0 \).
Finally, we give a comment on the modification of KZ equation. The insertion of the $\mathcal{K}$ operator would modify the original KZ equation. Now, let us examine the OPE with the stress-energy tensor (3.10),

$$T(w_1)\tilde{\Phi}_j^0(w_2) \sim \frac{1}{(w_1 - w_2)^2} \frac{j(j+2)}{4\kappa} \tilde{\Phi}_j^0(w_2) + \frac{1}{w_1 - w_2} \partial_{w_2} \tilde{\Phi}_j^0(w_2)$$

$$- \frac{1}{w_1 - w_2} \left( j \tilde{X}^{-1} Y \otimes 1 + \tilde{X}^{-2} \tilde{Y} Y \otimes e \right) \tilde{\Phi}_j^0(w_2).$$

The last term is the additional one. The OPEs with $H(w)$, $F(w)$ and $E(w)$ have also the extra terms. The modification of the KZ equation is provided by these unusual behaviors in $\tilde{\Phi}_j^0(z)$ with respect to the Virasoro algebra and the current algebra. Such an equation may have the solution displayed in this paper, for example, (5.7) and (5.8) and may be related to the quantum isomonodromy equation proposed in [68].

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