Thermodynamics of black holes in \((n + 1)\)-dimensional Einstein-Born-Infeld-dilaton gravity

A. Sheykhi \(^1\),\(^2\)* and N. Riazi \(^1\)†

1. Physics Department and Biruni Observatory, Shiraz University, Shiraz 71454, Iran
2. Physics Department, Shahid Bahonar University, Kerman, Iran

We construct a new class of \((n + 1)\)-dimensional \((n \geq 3)\) black hole solutions in Einstein-Born-Infeld-dilaton gravity with Liouville-type potential for the dilaton field and investigate their properties. These solutions are neither asymptotically flat nor (anti)-de Sitter. We find that these solutions can represent black holes, with inner and outer event horizons, an extreme black hole or a naked singularity provided the parameters of the solutions are chosen suitably. We compute the thermodynamic quantities of the black hole solutions and find that these quantities satisfy the first law of thermodynamics. We also perform stability analysis and investigate the effect of dilaton on the stability of the solutions.

I. INTRODUCTION

Born-Infeld electrodynamics was first introduced in 1934 in order to obtain a classical theory of charged particles with finite self-energy [1]. In this regard the theory was successful. However, they did not remove all of the singularities associated with a point charge and nonlinear electrodynamics became less popular with the introduction of QED which provided much better agreement with experiment. In recent years, there has been great interest in the Born-Infeld type generalizations of Abelian and non-Abelian gauge theories. Such generalizations arise naturally in open superstrings and in D-branes [2]. The low energy effective action for an open superstring in loop calculations leads to Born-Infeld type actions [3]. It has also been observed that the Born-Infeld action arises as an effective action governing the dynamics of vector-fields on D-branes [4]. The nonlinearity of the electromagnetic field brings remarkable properties to avoid the black hole singularity problem which may contradict the strong version of the Penrose cosmic censorship conjecture in some cases. Actually a new nonlinear electromagnetism was proposed, which produces a nonsingular exact black hole solution satisfying the weak energy condition [5], and has distinct properties from Bardeen black holes [6]. Unfortunately, the Born-Infeld model is not of this type. The Born-Infeld action including a dilaton and an axion field, appears in the coupling of an open

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* email address: asheykhi@mail.uk.ac.ir
† email: riazi@physics.susc.ac.ir
superstring and an Abelian gauge field theory. This action, describing a Born-Infeld-dilaton-axion system coupled to Einstein gravity, can be considered as a non-linear extension in the Abelian field of Einstein-Maxwell-dilaton-axion gravity. Some efforts have been done to construct exact solutions of Einstein-Born-Infeld (EBI) gravity. Exact solutions of Born-Infeld systems with zero cosmological constant has been studied by many authors [7, 8]. Thermodynamics of Born-Infeld black holes have been discussed in [9]. The exact five-dimensional charged black hole solution in Lovelock gravity coupled to Born-Infeld electrodynamics was presented in [10]. Black hole solutions to the theory with derivative corrections to the Born-Infeld action were derived in [11]. A class of slowly rotating black hole solutions in Born-Infeld theory has been obtained in [12]. In the presence of cosmological constant, black hole solutions of Einstein-Born-Infeld (EBI) theory with positive [13, 14], zero or negative constant curvature horizons have also been constructed [15]. Recently, the thermodynamical properties of these black hole solutions in (A)dS spacetime have been studied [16]. Unfortunately, exact solutions to the Einstein-Born-Infeld equation coupled to matter fields are too complicated to find except in a limited number of cases. Indeed, exact solutions to the Einstein Born-Infeld dilaton (EBId) gravity are known only in three dimensions [17]. Numerical studies of the EBId system in four dimensional static and spherically symmetric spacetime have been done in [18]. Some limited classes (factorized solutions) of four dimensional black hole in EBId theory with magnetic or electric charge have also been constructed [19, 20]. Till now, exact black hole solutions of EBId gravity in more than four dimensions have not been constructed. Our aim in this paper is to construct a new class of exact, spherically symmetric solution in $(n+1)$-dimensional Einstein-Born-Infeld-dilaton gravity for an arbitrary value of coupling constant with one and two Liouville-type potentials and investigate their properties. Specially, we want to perform a stability analysis and investigate the effect of dilaton on the stability of the solutions.

The organization of this paper is as follows: In Sec. II we construct a class of $(n+1)$-dimensional black hole solutions in EBId theory with one and two liouville type potentials and general dilaton coupling and investigate their properties. In Sec. III we obtain the conserved and thermodynamical quantities of the $(n+1)$-dimensional black hole solutions and show that these quantities satisfy the first law of thermodynamics. In Sec. IV we perform a stability analysis and show that the dilaton creates an unstable phase for the solutions. We finish our paper with some concluding remarks.
II. FIELD EQUATIONS AND SOLUTIONS

We consider the \((n + 1)\)-dimensional \((n \geq 3)\) action in which gravity is coupled to dilaton and Born-Infeld fields

\[
S = \int d^{n+1}x \sqrt{-g} \left( \mathcal{R} - \frac{4}{n-1} (\nabla \Phi)^2 - V(\Phi) + L(F, \Phi) \right),
\]

(1)

where \(\mathcal{R}\) is the Ricci scalar curvature, \(\Phi\) is the dilaton field and \(V(\Phi)\) is a potential for \(\Phi\). The Born-Infeld \(L(F, \Phi)\) part of the action is given by

\[
L(F, \Phi) = 2 \gamma e^{\frac{4\alpha \Phi}{n-1}} \left( 1 - \sqrt{1 + e^{-\frac{8\alpha \Phi}{n-1} F^2}} \right),
\]

(2)

Here, \(\alpha\) is a constant determining the strength of coupling of the scalar and electromagnetic field and \(F^2 = F_{\mu\nu} F^{\mu\nu}\), where \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the electromagnetic field tensor and \(A_\mu\) is the electromagnetic vector potential. \(\gamma\) is called the Born-Infeld parameter with dimension of mass. In the limit \(\gamma \to \infty\), \(L(F, \Phi)\) reduces to the standard Maxwell field coupled to a dilaton field

\[
L(F, \Phi) = -e^{\frac{4\alpha \Phi}{n-1}} F_{\mu\nu} F^{\mu\nu}.
\]

(3)

On the other hand, \(L(F, \Phi) \to 0\) as \(\gamma \to 0\). It is convenient to set

\[
L(F, \Phi) = 2 \gamma e^{\frac{4\alpha \Phi}{n-1}} \mathcal{L}(Y),
\]

(4)

where

\[
\mathcal{L}(Y) = 1 - \sqrt{1 + Y},
\]

(5)

\[
Y = e^{-\frac{8\alpha \Phi}{n-1} F^2} \gamma.
\]

(6)

The equations of motion can be obtained by varying the action \(\mathcal{I}\) with respect to the gravitational field \(g_{\mu\nu}\), the dilaton field \(\Phi\) and the gauge field \(A_\mu\) which yields the following field equations

\[
\mathcal{R}_{\mu\nu} = \frac{4}{n-1} \left( \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu\nu} V(\Phi) \right) - 4 e^{-\frac{4\alpha \Phi}{n-1}} \partial_\nu \mathcal{L}(Y) F_{\eta\nu} F^{\eta\mu} + \frac{2\gamma}{n-1} e^{\frac{4\alpha \Phi}{n-1}} \left[ 2Y \partial_\nu \mathcal{L}(Y) - \mathcal{L}(Y) \right] g_{\mu\nu},
\]

(7)

\[
\nabla^2 \Phi = \frac{n-1}{8} \frac{\partial V}{\partial \Phi} + \gamma e^{\frac{4\alpha \Phi}{n-1}} \left[ 2Y \partial_\nu \mathcal{L}(Y) - \mathcal{L}(Y) \right],
\]

(8)

\[
\partial_\mu \left( \sqrt{-g} e^{\frac{4\alpha \Phi}{n-1}} \partial_\nu \mathcal{L}(Y) F^{\mu\nu} \right) = 0.
\]

(9)

In particular, in the case of the linear electrodynamics with \(\mathcal{L}(Y) = -\frac{1}{2} Y\), the system of equations \(\mathcal{I}\) reduce to the well-known equations of Einstein-Maxwell dilaton gravity \[21\].
We wish to find static and spherically symmetric solutions of the above field equations. The most general such metric can be written in the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2R^2(r)d\Omega^2_{n-1},$$

(10)

where, $d\Omega^2_{n-1}$ denotes the metric of an unit $(n - 1)$ sphere. The functions $f(r)$ and $R(r)$ should be determined. The electromagnetic fields equation can be integrated immediately, where all the components of $F_{\mu\nu}$ are zero except $F_{tr}$:

$$F_{tr} = \frac{qe^{\frac{4\Phi}{n-1}}}{(rR)^{n-1}} \sqrt{1 + \frac{2q^2\gamma}{\gamma(rR)^{2n-2}}},$$

(11)

where $q$ is an integration constant related to the electric charge of the black hole. It is interesting to consider three limits of (11). First, for large $\gamma$ (where the BI action reduces to Maxwell case) we have $F_{tr} = \frac{qe^{2\Phi}}{(rR)^{n-1}}$ as presented in [21]. On the other hand, if $\gamma \to 0$ we get $F_{tr} = 0$. Finally, in the absence of the dilaton field ($\alpha = 0$), it reduces to the case of $(n + 1)$-dimensional Einstein-Born-Infeld theory (see for example [14, 15]).

A. Black hole solutions with a Liouville type potential

First, we consider the action (1) with a Liouville type potential,

$$V(\Phi) = 2\Lambda e^{2\zeta\Phi},$$

(12)

where $\Lambda$ and $\zeta$ are constants. One may refer to $\Lambda$ as the cosmological constant, since in the absence of the dilaton field the action reduces to the action of EBI gravity with cosmological constant [14, 15]. Here, we have redefined the cosmological constant as $\Lambda = -n(n-1)/2l^2$. In order to solve the system of equations (7) and (8) for three unknown functions $f(r)$, $R(r)$ and $\Phi(r)$, we make the ansatz

$$R(r) = e^{2\alpha\Phi/(n-1)},$$

(13)

Using (13), the electromagnetic fields (11) and the metric (10), one can easily show that equations (7) and (8) have solutions of the form

$$f(r) = -\frac{(n-2)(\alpha^2 + 1)^2b^{-2\beta}\gamma^{2\beta}}{(\alpha^2 - 1)(n + \alpha^2 - 2)} - \frac{m}{r^{(n-1)(1-\beta)-1}} + \frac{2\gamma(\alpha^2 + 1)^2b^{2\beta}r^{2(1-\beta)}}{(n-1)(n - \alpha^2)} + \frac{2(\alpha^2 + 1)b^{3(n-1)}\gamma}{n-1}r^{(n-1)(\beta-1)+1}\int \Gamma r^{-2\beta} dr,$$

(14)
\[ \Phi(r) = \frac{(n - 1)\alpha}{2(1 + \alpha^2)} \ln\left( \frac{b}{r} \right), \quad (15) \]

where
\[ \Gamma \equiv \sqrt{\gamma \left( 2q^2 + \gamma b^{2\beta(n-1)}r^{2(n-1)(1-\beta)} \right)}. \quad (16) \]

The integral can be done in terms of hypergeometric functions and can be written in a compact form. The result is
\[ f(r) = \frac{2(n - 2)}{(\alpha^2 - 1)(n + \alpha^2 - 2)} \alpha^2 \beta \left( -2 \ln \left( \frac{2}{2\alpha^2 - n} \right) \right) \left( -2 \ln \left( \frac{2\alpha^2 + n - 2}{2n - 2} \right) \right) \times \]
\[ \left( 1 - 2F_1 \left( \frac{-1}{2}, \frac{\alpha^2 - n}{2\alpha^2 - n}, \frac{\alpha^2 + n - 2}{2n - 2}, \frac{-2q^2}{\gamma b^{2\beta(n-1)}r^{2(n-1)(1-\beta)}} \right) \right), \quad (17) \]

Here \( b \) is an arbitrary constant and \( \beta = \alpha^2 / (\alpha^2 + 1) \). In the above expression, \( m \) appears as an integration constant and is related to the ADM mass of the black hole. In order to fully satisfy the system of equations, we must have
\[ \zeta = \frac{2}{\alpha(n - 1)}, \quad \Lambda = \frac{(n - 1)(n - 2)\alpha^2}{2b^2(\alpha^2 - 1)}. \quad (18) \]

One may note that as \( \gamma \to \infty \) these solutions reduce to the \((n + 1)\)-dimensional charged dilaton black hole solutions given in Ref. [21]. In the absence of a non-trivial dilaton \((\alpha = \beta = 0)\), the above solutions reproduce correctly the asymptotically flat Born-Infeld black hole (see for example [9]). Using the fact that \( 2F_1(a, b, c, z) \) has a convergent series expansion for \(|z| < 1\), we can find the behavior of the metric for large \( r \). This is given by
\[ f(r) = \frac{-(n - 2)}{(\alpha^2 - 1)(n + \alpha^2 - 2)} \alpha^2 \beta \left( -2 \ln \left( \frac{2}{2\alpha^2 - n} \right) \right) \left( -2 \ln \left( \frac{2\alpha^2 + n - 2}{2n - 2} \right) \right) \times \]
\[ \left( 1 - 2F_1 \left( \frac{-1}{2}, \frac{\alpha^2 - n}{2\alpha^2 - n}, \frac{\alpha^2 + n - 2}{2n - 2}, \frac{-2q^2}{\gamma b^{2\beta(n-1)}r^{2(n-1)(1-\beta)}} \right) \right), \]
\[ \frac{(n - 2)}{(\alpha^2 - 1)(n + \alpha^2 - 2)} b^{-2\beta} \right) \frac{m}{r^{(n-1)(1-\beta)-1}} - \frac{2(\alpha^2 + 1)\alpha^2 b^{-2(n-2)\beta}q^2}{(n - 1)(\alpha^2 + n - 2)r^{2(n-2)(1-\beta)}} \left( \frac{\alpha^2}{2n^2} \right) \gamma \left( \frac{n(\alpha^2 + 3n - 4)}{2n^2} r^{2(n-2)(1-\beta)} \right). \quad (19) \]

Note that in the limit \( \gamma \to \infty \) and \( \alpha = \beta = 0 \), it has the form of Reissner-Nordstrom black hole. The last term in the right hand side of the above expression is the leading Born-Infeld correction to the RN black hole with dilaton field in the large \( \gamma \) limit.

**B. Black hole solutions with two Liouville type potential**

Second, we present exact, \((n + 1)\)-dimensional solutions to the EBId gravity equations with an arbitrary dilaton coupling parameter \( \alpha \) and dilaton potential
\[ V(\Phi) = 2\Lambda_0 e^{2\zeta_0 \Phi} + 2\Lambda e^{2\zeta \Phi}. \quad (20) \]
where $\Lambda_0$, $\Lambda$, $\zeta_0$ and $\zeta$ are constants. This kind of potential was previously investigated by a number of authors both in the context of $FRW$ scalar field cosmologies \cite{22} and EMd black holes \cite{21, 23, 24}. This generalizes further the potential \cite{12}. If $\zeta_0 = \zeta$, then (20) reduces to (12), so we will not repeat these solutions, thus we require $\zeta_0 \neq \zeta$. Again, using (13), the electromagnetic fields (14) and the metric (10), one can easily show that equations (7) and (8) have solutions of the form

$$f(r) = -\frac{(n-2) \left(\alpha^2 + 1\right)^2 b^{-2\beta} r^{2\beta}}{(\alpha^2 - 1)(n+\alpha^2 - 2)} - \frac{m}{r^{(n-1)(1-\beta)-1} \Gamma} + \frac{2\Lambda \left(\alpha^2 + 1\right)^2 b^{2\beta} r^{2(1-\beta)}}{(n-1)(\alpha^2 - n)}$$

$$+ \frac{2\gamma \left(\alpha^2 + 1\right)^2 b^{2\beta} r^{2(1-\beta)}}{(n-1)(n-\alpha^2)} - \frac{2(\alpha^2 + 1) b^{(3-n)\beta}}{n-1} r^{(n-1)(\beta-1)+1} \int \Gamma r^{-2\beta} dr,$$  \hspace{1cm} (21)

where $b$ is a arbitrary constant and $m$ the mass parameter. $\Phi(r)$ and $\Gamma$ are given by Eqs. (15)- (16). Setting $\Lambda = 0$ in (21) one recover (14). In order to fully satisfy the system of equations, we should have $\zeta = 2\alpha/(n-1)$, and a similar relation to Eq. (18) for $\zeta_0$ and $\Lambda_0$. We can also turn out the integration and express the solution in terms of hypergeometric function

$$f(r) = -\frac{(n-2) \left(\alpha^2 + 1\right)^2 b^{-2\beta}}{(\alpha^2 - 1)(n+\alpha^2 - 2)} r^{2\beta} - \frac{m}{r^{(n-1)(1-\beta)-1} \Gamma} + \frac{2\Lambda \left(\alpha^2 + 1\right)^2 b^{2\beta}}{(n-1)(\alpha^2 - n)} r^{2(1-\beta)} - \frac{2\gamma \left(\alpha^2 + 1\right)^2 b^{2\beta} r^{2(1-\beta)}}{(n-1)(\alpha^2 - n)}$$

$$\times \left(1 - \frac{1}{2} \left[\frac{\alpha^2 - n}{2n - 2}\right] \quad \frac{\left[\frac{\alpha^2 + n - 2}{2n - 2}\right]}{\gamma b^{2\beta(n-1)} r^{2(n-1)(1-\beta)}} \right) \hspace{1cm} (22)$$

One may note that as $\gamma \to \infty$ these solutions reduce to the $(n+1)$-dimensional charged dilaton black hole solutions given in Ref. \cite{21}. In the absence of a non-trivial dilaton ($\alpha = \beta = 0$), the above solutions reduce to the Born-Infeld black hole in (A)dS space presented in \cite{14, 15}. Again, we can find the behavior of the metric for large $r$. This is given by

$$f(r) = -\frac{(n-2) \left(\alpha^2 + 1\right)^2 b^{-2\beta}}{(\alpha^2 - 1)(n+\alpha^2 - 2)} r^{2\beta} - \frac{m}{r^{(n-1)(1-\beta)-1} \Gamma} + \frac{2\Lambda \left(\alpha^2 + 1\right)^2 b^{2\beta}}{(n-1)(\alpha^2 - n)} r^{2(1-\beta)}$$

$$+ \frac{2(\alpha^2 + 1) b^{-2(n-2)\beta}}{(n-1)(\alpha^2 + n - 2)} r^{2(n-2)(1-\beta)} - \frac{(1 + \alpha^2) b^{-2(2n-3)\beta} q^4}{\gamma(n-1)(\alpha^2 + 3n - 4)} r^{2(2n-3)(1-\beta)}$$  \hspace{1cm} (23)

Note that in the limit $\gamma \to \infty$ and $\alpha = \beta = 0$, it has the form of Reissner-Nordstrom (A)dS black hole \cite{25}. The last term in the right hand side of the above expression is the leading Born-Infeld correction to the RN(A)dS black hole with dilaton field in the large $\gamma$ limit \cite{21}.

**Properties of the solutions**

Since the solution (14) can be obtained from (21) by setting $\Lambda = 0$, in the remains part of paper we focus on (21). In order to study the general structure of these solutions, we first look for the
FIG. 1: The function $f(r)$ versus $r$ for $\alpha = 0.5$, $q = 1$ and $n = 4$. $m = 1$ (bold), $m = 1.6$ (continuous) and $m = 2$ (dashed).

FIG. 2: The function $f(r)$ versus $r$ for $q = 1$, $m = 2$ and $n = 4$. $\alpha = 0$ (bold), $\alpha = 0.5$ (continuous) and $\alpha = 0.7$ (dashed).

FIG. 3: The function $f(r)$ versus $r$ for $m = 1$ and $\alpha = 0.5$ and $n = 4$. $q = 0.55$ (bold), $q = 0.66$ (continuous) and $q = 1$ (dashed).
curvature singularities in the presence of dilaton gravity. It is easy to show that in the presence of dilaton field, the Kretschmann scalar $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ diverges at $r = 0$, it is finite for $r \neq 0$ and goes to zero as $r \to \infty$. Thus, there is an essential singularity located at $r = 0$. The spacetime is neither asymptotically flat nor (A)dS. It is notable to mention that these solutions do not exist for the string case where $\alpha = 1$. As one can see from Eq. (14) and (21), the solutions are ill-defined for $\alpha = \sqrt{n}$. The cases with $\alpha > \sqrt{n}$ and $\alpha < \sqrt{n}$ should be considered separately. In the first case where $\alpha > \sqrt{n}$, the spacetime has a cosmological horizon for positive values of the mass parameter, despite the sign of the cosmological constant $\Lambda$. In the second case where $\alpha < \sqrt{n}$, there exist a cosmological horizon for $\Lambda > 0$, while there is no cosmological horizons if $\Lambda < 0$. Indeed, in the latter case ($\alpha < \sqrt{n}$ and $\Lambda < 0$) the spacetimes associated with these solutions exhibit a variety of possible casual structures depending on the values of the metric parameters $\alpha$, $\gamma$, $m$, $q$, and
\( \Lambda \). One can obtain the casual structure by finding the roots of \( f(r) = 0 \). Unfortunately, because of the nature of the \( (21) \), it is not possible to find analytically the location of the horizons. To achieve understanding of the nature of the horizons, we first plot in figures \( (1) - (3) \), the function \( f(r) \) for some different values of mass parameter \( m \) and dilaton coupling \( \alpha \). For simplicity we have kept fixed the other parameters \( l = 1, b = 1, \gamma = 1 \). We find from these figures that our solutions can represent black hole, with inner and outer event horizons, an extreme black hole or a naked singularity provided the parameters of the solutions are chosen suitably. Then, we plot in figures \( (4) \) and \( (5) \), the mass parameter \( m \) as a function of the horizon radius \( r_h \) for some different values of Born-Infeld parameter \( \gamma \) and dilaton coupling \( \alpha \). Again we fixed \( l = 1, b = 1 \) and \( q = 1 \), for simplicity. It is easy to show that the mass parameter of the black hole can be expressed in terms of the horizon radius as

\[
m(r_h) = -\frac{(n - 2)}{(\alpha^2 - 1)} \frac{(\alpha^2 + 1) b^{2\beta} r_h^{n-2+\beta(3-n)}}{(n + \alpha^2 - 2)} + \frac{2\Lambda}{(n-1)(\alpha^2 - n)} - \frac{2\gamma}{(n-1)(\alpha^2 - n)} \times 1 - 2F_1 \left( \left[ -\frac{1}{2}, \frac{n + \alpha^2 - 2}{2n - 2} \right], \left[ \frac{\alpha^2 + n - 2}{2n - 2} \right], -\frac{2q^2}{\gamma b^{2\beta(n-1)} r_h^{2(n-1)(\beta-1)}} \right).
\]

Figures \( (4) \) and \( (5) \), show that for a given value of \( \alpha \) or \( \gamma \), the number of horizons depend on the choice of the value of the mass parameter \( m \). We see from these figures that up to certain value of the mass parameter \( m \), there are two horizons, and as we decrease the \( m \) further, the two horizons meet. In this case we get extremal black hole. Numerical calculations show that when this condition is satisfied, the temperature of the black hole vanishes. To see this better we obtain the temperature of the black hole on the horizon. The Hawking temperature of the black hole on the outer horizon \( r_+ \) can be calculated using the relation

\[
T_+ = \frac{\kappa}{2\pi} = \frac{f'(r_+)}{4\pi},
\]

where \( \kappa \) is the surface gravity. Then, one can easily show that

\[
T_+ = -\frac{(\alpha^2 + 1) b^{2\beta} r_+^{1-2\beta}}{2\pi (n-1)} \frac{(n - 2)(n - 1) b^{-2\beta} r_+^{n-4\beta-2}}{2(\alpha^2 - 1)} r_+^{4\beta-2} + (\Lambda - \gamma) + \Gamma b^{(1-n)\beta} r_+^{(1-n)(1-\beta)}
\]

\[
= -\frac{(n - \alpha^2) m}{4\pi (\alpha^2 + 1)} r_+^{(n-1)(\beta-1)} - \frac{(n - 2)(\alpha^2 + 1) b^{-2\beta} r_+^{2\beta-1}}{2\pi (\alpha^2 + n - 2)} - \frac{q^2 (\alpha^2 + 1) b^{2(n-1)\beta} r_+^{2(n-1)(1-\beta)-1}}{\pi (\alpha^2 + n - 2)}
\]

\[
\times 2F_1 \left( \left[ \frac{1}{2}, \frac{n + \alpha^2 - 2}{2n - 2} \right], \left[ \frac{3n + \alpha^2 - 4}{2n - 2} \right], \frac{2q^2 b^{2(1-n)\beta} r_+^{2(n-1)(\beta-1)}}{\gamma} \right).
\]

(26)
There is also an extreme value for the mass parameter in which the temperature of the black hole is zero. It is a matter of calculation to show that
\[ m_{\text{ext}} = \frac{2(n-2)(\alpha^2 + 1)^2 b^{-2\beta}}{(n-\alpha^2)(\alpha^2 + n - 2)} r_{\text{ext}}^{(2-n)(\beta-1)+\beta} + \frac{4q^2(\alpha^2 + 1)^2 b^{2(2-n)\beta}}{(n-\alpha^2)(\alpha^2 + n - 2)} r_{\text{ext}}^{(3-n)(1-\beta)-1} \times 2F_1 \left( \left[ \frac{1}{2}, \frac{n + \alpha^2 - 2}{2n - 2} \right], \left[ \frac{3n + \alpha^2 - 4}{2n - 2} \right], -\frac{2q^2 b^{2\beta(1-n)} r_{\text{ext}}^{2(n-1)(\beta-1)}}{\gamma} \right). \] (27)

Indeed the metric of Eqs. (10) and (21) has two inner and outer horizons located at \( r_- \) and \( r_+ \), provided the mass parameter \( m \) is greater than \( m_{\text{ext}} \), an extreme black hole in the case of \( m = m_{\text{ext}} \), and a naked singularity if \( m < m_{\text{ext}} \). Notice that as we increase \( \alpha \) or \( \gamma \), the extremal value of the mass, \( m_{\text{ext}} \), increases too (see figs. 4-5).

III. THERMODYNAMICS OF BLACK HOLE

In this section we want to compute the conserved and thermodynamics quantities of the EBId black hole. The metric presented in this paper is not be asymptotically flat, therefore we must use the quasilocal formalism to define the mass of the solutions (see [26, 27] for details). If we write the metric of spherically symmetric spacetime in the form
\[ ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2d\Omega_{n-1}^2, \] (28)
and the matter action contains no derivatives of the metric, then the quasilocal mass is given by
\[ M = \frac{n-1}{2} r^2 V(r)^{1/2} \left( V_0(r)^{1/2} - V(r)^{1/2} \right). \] (29)

Here \( V_0(r) \) is an arbitrary function which determines the zero of the energy for a background spacetime and \( r \) is the radius of the spacelike hypersurface boundary. When the spacetime is asymptotically flat, the ADM mass \( M \) is the \( M \) determined in (29) in the limit \( r \to \infty \). If no cosmological horizon is present, the large \( r \) limit of (29), is used to determine the mass. If a cosmological horizon is present one can not take the large \( r \) limit to identify the quasilocal mass. However, one can still identify the small mass parameter in the solution [26, 27]. For the solution under consideration, there is no cosmological horizon and if we transform the metric in the form [28], then we obtain the mass of the black hole
\[ M = \frac{b^{(n-1)\beta}(n-1)\omega_{n-1} m}{16\pi(\alpha^2 + 1)}, \] (30)
where \( \omega_{n-1} \) is the volume of the unit \( (n-1) \) sphere. Black hole entropy typically satisfies the so-called area law of the entropy [28]. This near universal law applies to almost all kinds of black
holes and black holes in Einstein gravity \cite{29}. It is a matter of calculation to show that the entropy of the black hole is

\[ S = b^{(n-1)\beta} \omega_{n-1} \frac{r_+^{(n-1)(1-\beta)}}{4}, \] (31)

which shows that the area law holds for the black hole solutions in dilaton gravity. Next, we calculate the electric charge of the solutions. To determine the electric field we should consider the projections of the electromagnetic field tensor on spacial hypersurfaces. The normal to such hypersurfaces is

\[ u^0 = \frac{1}{N}, \quad u^r = 0, \quad u^i = -\frac{V^i}{N}, \] (32)

where \( N \) and \( V^i \) are the lapse function and shift vector. Then the electric field is

\[ E_\mu = g^{\mu\rho} e^{\frac{4\omega}{n-1}} F_{\rho\nu} u^\nu, \]

and the electric charge can be found by calculating the flux of the electric field at infinity, yielding

\[ Q = \frac{q\omega_{n-1}}{4\pi}. \] (33)

The electric potential \( U \), measured at infinity with respect to the horizon, is defined by

\[ U = A_\mu \chi^\mu \big|_{r=\infty} - A_\mu \chi^\mu \big|_{r=r_+}, \] (34)

where \( \chi = \partial_t \) is the null generator of the horizon. One can easily show that the gauge potential \( A_t \) corresponding to the electromagnetic field \cite{30} can be written as

\[ A_t = \frac{qb^{(3-n)\beta}}{\Upsilon r_+} 2F_1\left(\left[\frac{1}{2}, \frac{n+\alpha^2-2}{2n-2}\right], \left[\frac{3n+\alpha^2-4}{2n-2}\right], -\frac{2q^2}{\gamma b^{2\beta(n-1)r_+2(n-1)(1-\beta)}}\right), \] (35)

where \( \Upsilon = (n-3)(1-\beta) + 1 \). Therefore, the electric potential may be obtained as

\[ U = \frac{qb^{(3-n)\beta}}{\Upsilon r_+} 2F_1\left(\left[\frac{1}{2}, \frac{n+\alpha^2-2}{2(n-1)}\right], \left[\frac{3n+\alpha^2-4}{2(n-1)}\right], -\frac{2q^2}{\gamma b^{2\beta(n-1)r_+2(n-1)(1-\beta)}}\right). \] (36)

Then, we consider the first law of thermodynamics for the black hole. In order to do this, we obtain the mass \( M \) as a function of extensive quantities \( S \) and \( Q \). Using the expression for the mass, the entropy and the charge given in Eqs. \cite{30}, \cite{31} and \cite{33} and the fact that \( f(r_+) = 0 \), one can obtain a Smarr-type formula as

\[ M(S, Q) = \frac{-(n-1)(n-2)(\alpha^2 + 1) b^{-\alpha^2} (4S)^{\frac{n-\alpha^2}{n-1}}}{16\pi(\alpha^2 - 1)(\alpha^2 + n - 2)} + \frac{(\alpha^2 + 1)(\Lambda - \gamma)b^{\alpha^2} (4S)^{\frac{n-\alpha^2}{n-1}}}{8\pi(\alpha^2 - n)} + \frac{(n-1)(\alpha^2 + 1) b^{\alpha^2} (4S)^{\frac{n-\alpha^2}{n-1}}}{8\pi(\alpha^2 - n)} 2F_1\left(\left[\frac{1}{2}, \frac{\alpha^2 - n}{2n-2}\right], \left[\frac{\alpha^2 + n - 2}{2n-2}\right], -\frac{2\pi^2 Q^2}{\gamma S^2}\right). \] (37)
FIG. 6: \((\partial^2 M/\partial S^2)_Q\) versus \(\alpha\) for \(q = 1\), \(\gamma = 1\) and \(n = 5\). \(q = 0.5\) (bold), \(q = 1\) (continuous) and \(q = 1.5\) (dashed).

FIG. 7: \((\partial^2 M/\partial S^2)_Q\) versus \(\alpha\) for \(q = 1\) and \(\gamma = 1\). \(n = 4\) (bold), \(n = 5\) (continuous) and \(n = 6\) (dashed).

One may then regard the parameters \(S\), and \(Q\) as a complete set of extensive parameters for the mass \(M(S,Q)\) and define the intensive parameters conjugate to \(S\) and \(Q\). These quantities are the temperature and the electric potential

\[
T = \left(\frac{\partial M}{\partial S}\right)_Q, \quad U = \left(\frac{\partial M}{\partial Q}\right)_S.
\]  

(38)

Numerical calculations show that the intensive quantities calculated by Eq. (38) coincide with Eqs. (26) and (36). Thus, these thermodynamics quantities satisfy the first law of thermodynamics

\[
dM = TdS + UdQ.
\]  

(39)

IV. STABILITY IN THE CANONICAL ENSEMBLE
Finally, we would like to study the stability of the EBId black hole we have just found. Specially, we want to investigate the effect of dilaton on the stability of the solutions. The stability of a thermodynamic system with respect to small variations of the thermodynamic coordinates is usually performed by analyzing the behavior of the entropy $S(M,Q)$ around the equilibrium. The local stability in any ensemble requires that $S(M,Q)$ be a convex function of the extensive variables or its Legendre transformation must be a concave function of the intensive variables. The stability can also be studied by the behavior of the energy $M(S,Q)$ which should be a convex function of its extensive variable. Thus, the local stability can in principle be carried out by finding the determinant of the Hessian matrix of $M(S,Q)$ with respect to its extensive variables $X_i$, $H^M_{X_iX_j} = \left[ \partial^2 M/\partial X_i\partial X_j \right]$ [30, 31]. In our case the mass $M$ is a function of entropy and charge. The number of thermodynamic variables depends on the ensemble that is used. In the canonical ensemble, the charge is a fixed parameter and therefore the positivity of the $(\partial^2 M/\partial S^2)_Q$
is sufficient to ensure local stability. Numerical calculations show that the black hole solutions are stable independent of the value of the charge and Born-Infeld parameters $q$ and $\gamma$, in any dimensions if $\alpha < \alpha_{\text{max}}$, while for $\alpha > \alpha_{\text{max}}$ the system has an unstable phase (see figs. 6-8). Notice that again we have kept fixed the other parameters $l = 1$, $b = 1$, $r_+ = .08$ in the figures 6-15. On the other hand, figures 9-11 show that there is always a low limit for electric charge $q_{\text{min}}$ for which the system is thermally stable provided $q > q_{\text{min}}$. It is worth noting that $\alpha_{\text{max}}$ and $q_{\text{min}}$ depend on the Born-Infeld parameter $\gamma$, and the dimensionality of space time. In figures 12-14 we plot $(\partial^2 M/\partial S^2)_Q$ versus Born-Infeld parameter $\gamma$ for different values of the parameters $q$, $\alpha$ and $n$. These figures show that, there is always a low limit for Born-Infeld parameter $\gamma_{\text{min}}$, for which $(\partial^2 M/\partial S^2)_Q$ is positive provided $\gamma > \gamma_{\text{min}}$. It is notable to mention that $\gamma_{\text{min}}$ decreases with increasing $q$ and increases with increasing $\alpha$ and $n$. Finally, in figure 15 we show more explicitly the thermal stability of the black hole solutions for small value of dilaton coupling $\alpha$, in any dimensions.
FIG. 12: \((\partial^2 M/\partial S^2)_Q\) versus \(\gamma\) for \(n = 5, q = 1\). \(\alpha = 0.5\) (bold), \(\alpha = 1.1\) (continuous) and \(\alpha = 1.4\) (dashed).

FIG. 13: \((\partial^2 M/\partial S^2)_Q\) versus \(\gamma\) for \(n = 5\) and \(\alpha = \sqrt{2}\). \(q = 0.8\) (bold), \(q = 1.2\) (continuous) and \(q = 1.5\) (dashed).

for a fixed value of the charge parameter \(q\) and arbitrary Born-Infeld parameter \(\gamma\).

V. CLOSING REMARKS

To sum up, we presented the \((n + 1)\)-dimensional Einstein-Born-Infeld action coupled to a dilaton field and obtained the equations of motion by varying this action with respect to the gravitational field \(g_{\mu\nu}\), the dilaton field \(\Phi\) and the gauge field \(A_\mu\). Then, we constructed a new class of charged, black hole solutions to \((n + 1)\)-dimensional \((n \geq 3)\) Einstein-Born-Infeld-dilaton theory with one and two Liouville-type potentials and investigated their properties. These solutions are neither asymptotically flat nor (anti)-de Sitter. These solutions do not exist for the string case where \(\alpha = 1\). In the presence of Liouville-type potential, we obtained exact solutions provided \(\alpha \neq \sqrt{n}\). In the particular case \(\gamma \rightarrow \infty\), these solutions reduce to the \((n + 1)\)-dimensional Einstein-Maxwell-dilaton black hole solutions given in Ref. [21], while in the absence of a non-
trivial dilaton ($\alpha = \beta = 0$), the above solutions reduce to the $(n + 1)$-dimensional Born-Infeld black hole in the presence of a cosmological constant presented in [14, 15]. We found that these solutions can represent black holes, with inner and outer event horizons, an extreme black hole or a naked singularity provided the parameters of the solutions are chosen suitably. We also computed temperature, mass, entropy, charge and electric potential of the black hole solutions and found that these quantities satisfy the first law of thermodynamics. We showed that these thermodynamic quantities are independent of the Born-Infeld parameter $\gamma$. We found a Smarr-type formula and performed a stability analysis in canonical ensemble by considering $(\partial^2 M/\partial S^2)_Q$ for the charged black hole solutions in $(n + 1)$ dimensions and showed that there is no Hawking-Page phase transition in spite of charge of the black hole provided $\alpha \leq \alpha_{\text{max}}$, independent of the values of the charge and Born-Infeld parameter, $q$, $\gamma$ and the dimensionality of the space time, while the solutions have an unstable phase for $\alpha \leq \alpha_{\text{max}}$. We found that there is always a low limit for electric charge $q_{\text{min}}$ for which $(\partial^2 M/\partial S^2)_Q$ is positive, provided $q > q_{\text{min}}$. It is worth to note that...
$\alpha_{\text{max}}$ and $q_{\text{min}}$ depend on the Born-Infeld parameter $\gamma$, and the dimensionality of space time. On the other hand, we found a lower limit for the Born-Infeld parameter $\gamma_{\text{min}}$, for which $\left(\partial^2 M/\partial S^2\right)_{Q}$ is positive provided $\gamma > \gamma_{\text{min}}$. It is notable to mention that $\gamma_{\text{min}}$ decreases with increasing $q$ and increases with increasing $\alpha$ and $n$. Note that the $(n+1)$-dimensional black hole solutions obtained here are static. Thus, it would be interesting if one could construct rotating black hole solutions in $(n+1)$ dimensions in the presence of dilaton and Born-Infeld fields.

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