TWO BOUNDS ON THE NONCOMMUTING GRAPH

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Abstract. Erdős introduced the noncommuting graph, in order to study the number of commuting elements in a finite group. Despite the use of combinatorial ideas, his methods involved several techniques of classical analysis. The interest for this graph is becoming relevant in the last years for various reasons. Here we deal with a numerical aspect, showing for the first time an isoperimetric inequality and an analytic condition in terms of Sobolev inequalities. This last result holds in the more general context of weighted locally finite graphs.

1. Terminology and preliminary notions

If \( \Gamma \) denotes a locally finite graph (i.e.: each vertex of \( \Gamma \) has a finite number of neighbors) with vertex set \( V \) and edge set \( E \), two elements \( x, y \in V \) are in the relation \( x \sim y \) if \( x \) and \( y \) are adjacent and joined by an edge \( xy \). For a subset \( \Omega \subseteq V \),

\[ \partial \Omega = \{ xy \mid x \in \Omega \text{ and } y \in V - \Omega \}, \]

is the set of edges which join a vertex in \( \Omega \) with a vertex outside \( \Omega \). In presence of an orientation, each edge in \( \partial \Omega \) is oriented so that it points outwards from \( \Omega \). To \( \Gamma \), we associate the edge weight \( \sigma_{xy} > 0 \) for each \( xy \in E \), so for any \( S \subseteq E \) we define the measure

\[ \sigma(S) = \sum_{xy \in S} \sigma_{xy}. \]

Extending the function \( \sigma_{xy} \) by zero to those \( x, y \) which are not neighbors, we get a symmetric function from \( V \times V \) to \([0, +\infty[\). It will be also useful to introduce the vertex weight

\[ \mu_x : x \in V \longmapsto \mu_x = \sum_{y : y \sim x} \sigma_{xy} \in ]0, +\infty[. \]

In case \( \sigma_{xy} = 1 \) for each \( xy \in E \) (for instance, in unweighted graphs),

\[ \mu_x = \text{deg}(x) = |\{yx \mid y \sim x\}| \]

is the degree of \( x \), that is, the number of neighbors of the vertex \( x \). On the other hand, it is well defined the positive measure

\[ \mu : \Omega \subseteq V \longmapsto \mu(\Omega) = \sum_{x \in \Omega} \mu_x \in ]0, +\infty[. \]

If \( \Gamma \) is equipped with \( \sigma \) and \( \mu \) as above, we say that it is a weighted graph. In particular, if \( \Gamma_G \) is the noncommuting graph of a finite group \( G \) (i.e.: recall from [1] that \( \Gamma_G \) is defined by vertices \( x, y \in G - Z(G) = V \) joined by an edge \( xy \in E \) if \( x \) do not commute with \( y \)), there is neither weight nor orientation, so \( \mu(\Omega) = \sum_{x \in \Omega} \text{deg}(x) \) and \( \sigma(\partial \Omega) = |\partial \Omega| \). Important contributions on \( \Gamma_G \) can be found in [1, 8, 12], but the reader may refer to [16] for a recent

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survey\(^1\). \(\Gamma_G\) has interesting properties: it is always connected, of diameter 2 and hamiltonian (see [1, Propositions 2.1, 2.2]); moreover the planar and the regular cases are classified by [1, Propositions 2.3, 2.6]. To the best of our knowledge, there are no isoperimetric inequalities on its invariants and we are going to show one of these here for the first time.

Following [7, §5.2], it is possible to define the gradient operator

\[
\nabla : f \in \mathbb{R}^{|V \times V|} \mapsto \nabla f = \nabla_{xy} f = f(y) - f(x) \in \mathbb{R},
\]

where \(\mathbb{R}^{|V \times V|}\) is the set of all functions from \(V \times V\) to \(\mathbb{R}\), and \(\nabla_{xy}\) denotes the fact that there is a dependence from \(x, y \in V\) in the definition of \(\nabla\). Consequently,

\[
\Delta : f \in \mathbb{R}^{|V|} \mapsto \Delta f(x) = \frac{1}{\mu_x} \sum_{y, y \sim x} (\nabla_{xy} f) \sigma_{xy} \in \mathbb{R}
\]

is the Laplace operator. A natural variation of the Green’s Formula is

\[
\sum_{|\Omega| < \infty} \Delta f(x) \mu_x = \sum_{|\Omega| < \infty} (\nabla_{xy} f) \sigma_{xy} = \sum_{e \in \partial \Omega} (\nabla_e f) \sigma_e
\]

and, if \(f, g \in \mathbb{R}^{|V|}\) with either \(f\) or \(g\) of finite support, then

\[
\sum_{x \in V} \Delta f(x) g(x) \mu_x = -\frac{1}{2} \sum_{x, y \in V} (\nabla_{xy} f)(\nabla_{xy} g) \sigma_{xy} - \sum_{e \in E} (\nabla_e f)(\nabla_e g) \sigma_e.
\]

We will consider distance functions on \(V\), inspired by analogous contexts of riemannian geometry in [13][14]. The graph distance \(\rho_\xi(x)\) between \(x \in V\) and the fixed vertex \(\xi \in V\) is the number of edges in a shortest path (also called a graph geodesic) connecting them and, in particular,

\[
\rho : (\xi, x) \in V \times V \mapsto \rho(\xi, x) = \rho_\xi(x) \in \mathbb{N} \quad \text{and} \quad \rho_\xi : x \in V \mapsto \rho_\xi(x) \in \mathbb{N}.
\]

This is also known as the geodesic distance and we note that there are more than one shortest path between two vertices.\(^2\) In contrast with the case of undirected graphs, one may have \(\rho\) not symmetric a priori. But we only deal with graphs possessing a distance function as \(\rho\) and all we have said up to now is of course true for finite graphs (in particular, for \(\Gamma_G\)). This allows us to define, fixed \(r > 0\), a ball \(B_\xi(r) = \{x \in V \mid \rho_\xi(x) < r\}\). We show mainly two results in the present paper. One is a specialization to \(\Gamma_G\) of theorems in [7]. This provides an isoperimetric inequality for \(\Gamma_G\), which is unknown up to now. The second main result has more general interest and shows a characterization in terms of a Nash–type inequality of certain locally finite weighted graphs, which generalise the noncommuting graph.

### 2. First result

Following [6][7], we may restrict the investigations to graphs, whose geometric properties are analogous with some classical notions of the riemannian manifolds (see [5][9]). For a weighted graph \(\Gamma\) with a distance \(\rho\), the positive quantity

\[
\mu_\xi^\rho_x = \sum_{y : y \sim x, \rho_\xi(y) < \rho_\xi(x)} \sigma_{xy},
\]

\(^1\)This graph appears originally in certain combinatorial problems in group theory, related to conjectures of Erdős on the number of commuting elements in a group (see [15]). A probabilistic version of these ideas can be found in [10][11].

\(^2\) If there is no path connecting the two vertices, i.e., if they belong to different connected components, then conventionally the distance is defined as infinite. We also note that in case of a directed graph the distance \(\rho_\xi(x)\) is defined as the length of a shortest path from \(\xi\) to \(x\) consisting of arcs, provided at least one such path exists.
Theorem 2.2 (See [7], Theorem 6.3). If a weighted graph \( (\Gamma, \sigma) \) has \( P(\delta, \iota, R_0) \), then the following isoperimetric inequality is true:

\[
\sigma(\partial \Omega) \geq c \mu(\Omega)^{1 - \frac{1}{p}},
\]

where \( \Omega \subseteq V \) is finite, \( \omega = \inf \{ \mu_x \mid x \in V \} \), \( \omega' = \inf \{ \sigma_{xy} \mid x \sim y, x, y \in V \} \) and

\[
c = \frac{\omega' \omega^{\frac{1}{1-p}}}{4^{n+3} \nu R_0 + 1 \iota e^{2n}}.
\]

An inequality of Sobolev type (see [5]) is recalled below in our context. Note that the presence of an isoperimetric inequality is requested in the assumptions.

Theorem 2.3 (See [7], Theorem 7.6). If a weighted graph \( (\Gamma, \sigma) \) possess a finite subset \( \Omega \subseteq V \) of \( \mu(\Omega) < v_0 \) such that \( \sigma(\partial \Omega) \geq c \mu(\Omega)^{1 - \frac{1}{p}} \) for some \( c, v_0 > 0 \) and \( n > 1 \), then

\[
C(n, p) \left( \sum_{y \sim x \sim x', y \in V} |f(y) - f(x)|^p \sigma_{xy} \right)^{\frac{1}{p}} + cK(\Omega_0) \left( \sum_{x \in V} |f|^p \mu_x \right)^{\frac{1}{p}}
\]

\[
\geq \frac{c}{2^{1 + \frac{1}{p} - \frac{n}{p}}} \left( \sum_{x \in V} |f|^{\frac{np}{n-p}} \right)^{\frac{n-p}{np}}
\]

for any \( f \in \mathbb{R}^V \) of finite support, where (with the meaning of Theorem 2.2)

\[
c = \frac{\omega' \omega^{\frac{1}{1-p}}}{4^{n+3} \nu R_0 + 1 \iota e^{2n}},
\]

\( C(n, p) > 0 \) is a positive constant,

\( \Omega_0 = \{ x \in V \mid |f(x)| > 0 \} \)

and

\[
K(\Omega_0) = \begin{cases} 0 & \text{if } \mu(\Omega_0) \leq v_0, \\ v_0^{-1} \mu(\Omega_0)^{1 - \frac{1}{p}} & \text{if } \mu(\Omega_0) > v_0. \end{cases}
\]

What we said until now can be tested for the noncommuting graph.

Lemma 2.4. \( \Gamma_G \) satisfies (i)–(iii) of Definition 2.1.

Clearly satisfies \( \mu_\xi < \mu_x \) and allows us to introduce the ratio

\[
\nu_r = \inf \left\{ \frac{\mu_x}{\mu_\xi} \mid \xi \in V, x \in B_\xi(r) \right\},
\]

which correspond to the notion of relative isoperimetric dimension in [4, 5, 9]. In this spirit, Chung and others introduced the so called \( P(\delta, \iota, R_0) \) property.

Definition 2.1 (See [7]). We say that \( (\Gamma, \sigma) \) has \( P(\delta, \iota, R_0) \), when:

(i) \( |\nabla_{xy} \rho_\xi| \leq 1 \) for any \( \xi, x, y \in V \);

(ii) \( \exists \) a function \( q_\xi(x) \) and three constants \( \iota \geq 1, \delta > 0 \) and \( R_0 > 0 \) such that

(1) \( q_\xi(x) \geq 0 \) for all \( x \in V \), and \( q_\xi(x) = 0 \) if and only if \( x = \xi \);

(2) \( |\nabla_{xy} q_\xi| \leq \rho_\xi(x) + 1 \) for all \( \xi \in V \) and \( x, y \in B_\xi(R_0) \);

(3) \( \Delta q_\xi(x) \geq \delta \) for all \( \xi \in V \) and \( x \in B_\xi(R_0) \);

(iii) \( n = \delta \nu R_0 + 1 \geq 1 \).

The presence of an isoperimetric inequality can be deduced from \( P(\delta, \iota, R_0) \).
Proof. We begin to check (i). Recall that $\text{diam } \Gamma_G = 2$ and $|\nabla x \rho_\xi| = |\rho_\xi(y) - \rho_\xi(x)|$. Of course $|\nabla xy \rho_\xi| = 0$, whenever $x = y = \xi$. Assume $x \neq y$ and $x = \xi$. Since $\text{diam } \Gamma_G = 2$, we have 3 points in $V$ and two of them coincide, hence $\rho_\xi(y) = 1$ and $\rho_\xi(x) = 0$. The same argument applies when $x \neq y$ and $y = \xi$. Then in both cases $|\nabla xy \rho_\xi| = 1$. Assume now $x \neq y$, $x \neq \xi$ and $y \neq \xi$. Again the condition $\text{diam } \Gamma_G = 2$ implies $\rho_\xi(x) = \rho_\xi(y) = 1$ and so $|\nabla xy \rho_\xi| = 0$. This allows us to conclude that $|\nabla xy \rho_\xi| = 1$ for all $x, y, \xi \in V$. About (ii) of Definition 2.1 it is enough to put $q_\xi(x) = \frac{1}{2} \rho_\xi^2(x)$. In fact one can check easily (ii.1). About (ii.2)

$$\nabla xy q_\xi = q_\xi(y) - q_\xi(x) = \frac{1}{2} (\rho_\xi^2(y) - \rho_\xi^2(x)) = \frac{1}{2} (\rho_\xi(y) - \rho_\xi(x)) (\rho_\xi(y) + \rho_\xi(x)) \leq 1 \text{ from (i) above}$$

Finally, for any $\xi \in V$ and $x \in B_\xi(2) = B_\xi(R_0)$ we have

$$2 \deg(x) \Delta q_\xi(x) = \sum_{y, y \sim x} (\rho_\xi(y) - \rho_\xi(x)) (\rho_\xi(y) + \rho_\xi(x)) \geq 1$$

and so (ii.3) is realized with $\delta = 1$. (iii) is satisfied with $n = \nu_3$, but $\text{diam } \Gamma_G = 2$ implies $\nu_3 = \nu_2$ and so $n = \nu_2$ is better. □

The previous lemma provides information, which we summarize below.

Corollary 2.5. $\Gamma_G$ has $R_0 = \delta = \iota = 1$ and $n = \nu_2$ in Definition 2.1

Proof. See proof of Lemma 2.4 □

Now our first main result can be stated.

Theorem 2.6. $\Gamma_G$ satisfies the isoperimetric inequality

$$|\partial \Omega| \geq c \left( \sum_{x \in \Omega} \deg(x) \right)^{\frac{1}{2} - \frac{1}{2\nu_2}}$$

where $\Omega \subseteq G - Z(G)$, $\omega = \inf \{\deg(x) \mid x \in G - Z(G)\}$ and

$$c = \frac{\omega^{1 - \frac{1}{2}}}{4e\nu_2 - 3 \nu_2^2 \epsilon^{2\nu_2}}.$$

Proof. We specialize Theorem 2.2 by the use of Lemma 2.4 and Corollary 2.5 □

There are difficulties of computation for $\nu_2$ already for groups of order 8.

Example 2.7. Let $G = Q_8 = \{1, -1, i, -i, j, -j, k, -k \mid ij = k, jk = i, ki = j, i^2 = j^2 = k^2 = -1\}$ be the quaternion group of order 8. This presentation is not elegant in terms of generators and relations, but very useful for our aims. In fact we can see immediately that $\Gamma_G$ has $|V| = |Q_8 - Z(Q_8)| = |\{i, -i, j, -j, k, -k\}| = 6$, $|E| = 12$, $\deg(i) = \deg(j) = \deg(k) = 4$ and we confirm Propositions 2.3, 2.6 noting that $\Gamma_G$ is planar and regular. In order to compute $\nu_2$, fix $\xi = i$ and $x = j$. Here $\rho_i(y) = \rho_i(j) = 1$ for all $y \in V$ so that $\mu_i^j = 0$. But when $\xi = i$ and $x = -i$, $1 = \rho_i(y) < \rho_i(-i) = 2$ for all $y \in V - \{-i\}$ and so $\mu_i^{-i} = 1$. Since this argument may be repeated for $x = -j$ and $x = -k$, we conclude that $\nu_2 = 4$. This means that $c = \frac{4^{3/4}}{12 \pi^2}$. Here $\Omega = V$ confirms Theorem 2.6 by $12 \geq \left( \frac{4^{3/4}}{12 \pi^2} \right) \cdot 24^{3/4}$.

The following is the first example of Sobolev inequality for $\Gamma_G$.

Corollary 2.8. $\Gamma_G$ satisfies the thesis of Theorem 2.3 with $R_0 = \iota = \omega' = \sigma_{xy} = 1$, $\mu_x = \deg(x)$, $\nu_0 = 1 + \sum_{x \in \Omega} \deg(x)$, $n = \nu_2$. 

One of the most interesting problems is due to the optimality of the constants which appear in Theorem 2.3. This hasn’t been discussed properly in [7], but the same authors have produced a series of papers in the last ten years on the problem of weakening $P(t, \delta, R_0)$. Recently, some new metric spaces are considered in [2, 3] and they seem to be the natural contexts where the above property can be generalized. We don’t discuss this delicate aspect here.

3. Second result

The reader may observe that the condition $P(\delta, t, R_0)$ implies the isoperimetry, as explained in [7, Theorem 6.3], but, on the other hand, (\textcircled{\textbullet}) has the form of a Sobolev–Poincaré inequality when $K(\Omega_0) = 0$ (see [9] for details). This motivates us to characterize a special situation, by means of another well known inequality of Nash type. The following theorem illustrates such equivalence.

**Theorem 3.1.** If a weighted graph $(\Gamma, \sigma)$ has $P(\delta, t, R_0)$ with $\Omega \subseteq V$ of $\mu(\Omega) < \infty$ and $p = 2n/n - 2$, then the following conditions are equivalent for any $f \in \mathbb{R}^V$

$$(\dagger) \quad \left( \sum_{x \in V} |f(x)|^p \mu_x \right)^{\frac{2}{p}} \leq A(p) \sum_{y, y' \sim x} |f(y) - f(x)|^2 \sigma_{xy},$$

$$(\dagger\dagger) \quad \left( \sum_{x \in V} |f(x)|^2 \mu_x \right)^{1 + \frac{p}{2}} \leq B(p) \left( \sum_{y, y' \sim x} |f(y) - f(x)|^2 \sigma_{xy} \right) \left( \sum_{x \in V} |f(x)| \mu_x \right)^{\frac{p}{2}},$$

where $A(p)$ and $B(p)$ are (nonoptimal) constants depending only on $p$.

**Proof.** The property $P(\delta, t, R_0)$ is assumed, in order to be sure that there exists a graph satisfying an isoperimetric inequality (see Theorem 2.2), and, so, by Theorem 2.3 a Sobolev type inequality. In fact the proof of the equivalence among the conditions (\textcircled{\textbullet}) and (\textcircled{\textbullet\textbullet}), as we will see, doesn’t use the property $P(\delta, t, R_0)$. On the other hand, we put it in the assumptions of the theorem for this precise motivation.

$$(\dagger) \Rightarrow (\dagger\dagger).$$

We apply the Hölder inequality in the following form:

$$\sum_{x \in V} |f(x)|^2 \mu_x = \sum_{x \in V} |f(x)|^{\frac{2}{p}} \mu_x x \mu_x \mu_x$$

$$\leq \left( \sum_{x \in V} |f(x)|^{\frac{2}{p}} \mu_x \right)^{p-1} \left( \sum_{x \in V} |f(x)|^{\frac{2}{p}} \mu_x \right)^{\frac{p}{2}},$$

that is,

$$\sum_{x \in V} |f(x)|^2 \mu_x \leq \left( \sum_{x \in V} |f(x)|^p \mu_x \right)^{\frac{2}{p} - 1} \left( \sum_{x \in V} |f(x)|^2 \mu_x \right)^{\frac{p}{2}}$$

and by (\textcircled{\textbullet}) we upper bound the right side of the above inequality with

$$\leq \left( A(p) \sum_{y, y' \sim x} |f(y) - f(x)|^2 \sigma_{xy} \right)^{\frac{p}{2}} \left( \sum_{x \in V} |f(x)| \mu_x \right)^{\frac{p}{2}},$$
so that the \((1 + \frac{2}{n})\)th power implies

\[
\left( \sum_{x \in V} |f(x)|^2 \mu_x \right)^{1 + \frac{2}{n}} \leq \left( A(p) \sum_{y, y \sim x} \sum_{x, y \in V} |f(y) - f(x)|^2 \sigma_{xy} \right)^{1 + \frac{2}{n}} \left( \sum_{x \in V} |f(x)| \mu_x \right)^{1 + \frac{2}{n}}
\]

\[
\leq A(p) \left( \sum_{y, y \sim x} \sum_{x, y \in V} |f(y) - f(x)|^2 \sigma_{xy} \right)^{\left( 1 + \frac{2}{n} \right) \left( \frac{p}{p - 1} \right) \left( \frac{n}{n - 2} \right)} \left( \sum_{x \in V} |f(x)| \mu_x \right)^{\left( 1 + \frac{2}{n} \right) \left( \frac{p}{p - 1} \right) \left( \frac{n}{n - 2} \right)}
\]

since

\[
\left( 1 + \frac{2}{n} \right) \left( \frac{1}{p - 1} \right) \left( \frac{p}{2} \right) = \left( 1 + \frac{2}{n} \right) \left( \frac{1}{n - 2} \right) = \left( \frac{n + 2}{n} \right) \left( \frac{n - 2}{n + 2} \right) \left( \frac{n}{n - 2} \right) = 1
\]

and

\[
\left( 1 + \frac{2}{n} \right) \left( \frac{p - 2}{p - 1} \right) = \left( \frac{n + 2}{n} \right) \left( \frac{2n - 2}{n - 2} \right) = \left( \frac{n + 2}{n} \right) \left( \frac{4}{n - 2} \right) = \frac{4}{n}
\]

Therefore \((\dagger\dagger)\) follows with \(A(p) = B(p)\).

\[(\dagger\dagger) \Rightarrow (\dagger). \text{ Given } f \in \mathbb{R}^V \text{ and } k \in \mathbb{Z}, \text{ we define } U_k = \{ x \in V \mid |f(x)| < 2^k \}, \ V_k = \{ x \in V \mid 2^k \leq |f(x)| < 2^{k+1} \}, \ W_k = \{ x \in V \mid |f(x)| \geq 2^{k+1} \} \text{ and } f_k(x) = \begin{cases} 0, & \text{if } x \in U_k \\ |f_k(x)| - 2^k, & \text{if } x \in V_k \\ 2^k, & \text{if } x \in W_k. \end{cases}
\]

We note some useful properties of the way of writing \(f_k(x)\) as above. Firstly, \(V = U_k \cup V_k \cup W_k\), that is, \(V\) is the disjoint union of the sets \(U_k, V_k\) and \(W_k\). Secondly, \(f_k(x)\) is zero over \(U_k\), and this doesn’t give contribution in writing sums, while \(f_k(x)\) is constant over \(W_k\) once \(k\) is fixed. Thirdly, we have by construction that \(W_{k+1} \subseteq W_k\) for all \(k \in \mathbb{Z}\). From the first of these properties, we get easily that

\[
\sum_{y, y \sim x} \sum_{x, y \in V} |f_k(y) - f_k(x)|^2 \sigma_{xy} = \sum_{y, y \sim x} \sum_{x, y \in U_k} |f_k(y) - f_k(x)|^2 \sigma_{xy} + \sum_{y, y \sim x} \sum_{x, y \in V_k} |f_k(y) - f_k(x)|^2 \sigma_{xy} + \sum_{y, y \sim x} \sum_{x, y \in W_k} |f_k(y) - f_k(x)|^2 \sigma_{xy} \leq \sum_{y, y \sim x} \sum_{x, y \in V_k} |f_k(y) - f_k(x)|^2 \sigma_{xy}
\]

Now we apply \((\dagger)\) to each \(f_k(x)\) and, because of the above inequality, we find

\[(*) \quad \left( \sum_{x \in V} |f_k(x)|^2 \mu_x \right)^{1 + \frac{2}{n}} \leq B(p) \left( \sum_{y, y \sim x} \sum_{x, y \in V_k} |f_k(y) - f_k(x)|^2 \sigma_{xy} \right)^{\left( 1 + \frac{2}{n} \right) \left( \frac{p}{p - 1} \right) \left( \frac{n}{n - 2} \right)} \left( \sum_{x \in V} |f_k(x)| \mu_x \right)^{\left( 1 + \frac{2}{n} \right) \left( \frac{p}{p - 1} \right) \left( \frac{n}{n - 2} \right)}
\]
Estimating the \((1 + \frac{2}{n})^{th}\) root of the term on the left side of (*), we get
\[
2^{2k} \sum_{x \in V} |\chi_{W_k}(x)| \mu_x \leq \sum_{x \in W_k} |f_k(x)|^2 \mu_x \leq \sum_{x \in V} |f_k(x)|^2 \mu_x
\]
and so we have the following upper bound for this term:
\[
\left( 2^{2k} \sum_{x \in V} \chi_{W_k}(x) \mu_x \right)^{1 + \frac{2}{n}} \leq \left( \sum_{x \in V} |f_k(x)|^2 \mu_x \right)^{1 + \frac{2}{n}}.
\]

On the other hand, we estimate the \((4/n)\)th root of the following term in the right side of (*):
\[
\sum_{x \in V} |f_k(x)| \mu_x = \sum_{x \in V} |f_k(x)| \mu_x + \sum_{x \in W_k} |f_k(x)| \mu_x + \sum_{x \in W_k} |f_k(x)| \mu_x = 2^k \sum_{x \in V} \chi_{V_k}(x) \mu_x + 2^k \sum_{x \in V} \chi_{W_k}(x) \mu_x \leq 2^k \sum_{x \in V} \chi_{W_k}(x) \mu_x
\]
and so we have the following upper bound for this term:
\[
\left( \sum_{x \in V} |f_k(x)| \mu_x \right)^{\frac{4}{n}} \leq \left( 2^k \sum_{x \in V} \chi_{W_k}(x) \mu_x \right)^{\frac{4}{n}}.
\]
We conclude that (*) implies (**)
\[
\left( 2^{2k} \sum_{x \in V} \chi_{W_k}(x) \mu_x \right)^{1 + \frac{2}{n}} \leq B(p) \left( \sum_{y \sim x \in V_k} |f_k(y) - f_k(x)|^2 \sigma_{xy} \right) \left( 2^k \sum_{x \in V} \chi_{W_k}(x) \mu_x \right)^{\frac{4}{n}}.
\]

In order to manipulate the terms which appear in (**), we denote
\[
a_k = 2^{p k} \sum_{x \in V} \chi_{W_k}(x) \mu_x \quad \text{and} \quad b_k = \sum_{y \sim x \in V_k} |f_k(y) - f_k(x)|^2 \sigma_{xy},
\]
where \( p \) is always equal to \( 2n/(n - 2) \). Now
\[
a_{k+1} = 2^{p k + p} \sum_{x \in V} \chi_{W_k}(x) \mu_x
\]
and we rewrite (***) as
\[
2^{(2k-pk-p)\left(\frac{4}{n}\right)} \left( a_{k+1} \right)^{1 + \frac{2}{n}} \leq B(p) \left( \sum_{y \sim x \in V_k} |f_k(y) - f_k(x)|^2 \sigma_{xy} \right) \left( 2^{p k + p} \left( a_k \right)^{\frac{4}{n}} \right) 2^{\frac{4k}{n} - \frac{4pk}{n}} (a_k)^{\frac{4}{n}}.
\]
that is,
\[
\left( a_{k+1} \right)^{1 + \frac{2}{n}} \leq 2^{\frac{4k}{n} - \frac{4pk}{n} - (2k-pk+p)\left(1 + \frac{2}{n}\right)} B(p) \left( \sum_{y \sim x \in V_k} |f_k(y) - f_k(x)|^2 \sigma_{xy} \right) (a_k)^{\frac{4}{n}}.
\]
Since \((1 + \frac{2}{n}) \left(\frac{n}{n+2}\right) = 1\), we do the \(\left(\frac{n}{n+2}\right)^{th}\) power and get

\[
a_{k+1} \leq 2^{\frac{2k}{n+2}(1-p) - 2k + pk + p} \, B(p) \, \left(\sum_{y \sim x \in V_k} |f_k(y) - f_k(x)|^2 \sigma_{xy}\right) \, \left(\sum_{y \sim x \in V_k} |f_k(y) - f_k(x)|^2 \sigma_{xy}\right)^{\frac{n}{n+2}},
\]

but

\[
\frac{4k}{n+2}(1-p) - 2k + pk + p = \frac{4k}{p+2} (1-p) - 2k + pk + p.
\]

\[
\frac{4k}{p+2} (1-p) - 2k + pk + p = \left(\frac{p-2}{p-1}\right) (1-p)k - 2k + pk + p = -k(p-2) - 2k + pk + p = p
\]

and so

\[
(\sharp) \quad a_{k+1} \leq 2^p \, B(p) \, \left(\sum_{y \sim x \in V_k} |f_k(y) - f_k(x)|^2 \sigma_{xy}\right) \, \left(\sum_{y \sim x \in V_k} |f_k(y) - f_k(x)|^2 \sigma_{xy}\right)^{\frac{n}{n+2}}.
\]

Until now, we have shown that (\#) follows from (\#) via (**). But we may sum (\#) over \(k \in \mathbb{Z}\) and get

\[
\sum_{k \in \mathbb{Z}} a_k = \sum_{k \in \mathbb{Z}} a_{k+1} \leq 2^p \, B(p) \, \left(\sum_{k \in \mathbb{Z}} b_k\right)^{\frac{n}{n+2}} \, \left(\sum_{k \in \mathbb{Z}} a_k^2\right)^{\frac{n}{n+2}}
\]

and applying the Hölder inequality with conjugate exponents \(P = \frac{n}{n+2}\) and \(Q = 1 - \frac{n}{n+2} = \frac{2}{n+2}\), this quantity is upper bounded by

\[
\leq 2^p \, B(p) \, \left(\sum_{k \in \mathbb{Z}} b_k\right)^{\frac{n}{n+2}} \, \left(\sum_{k \in \mathbb{Z}} a_k\right)^{\frac{n}{n+2}}
\]

where we may even upper bound the last term a priori, getting

\[
\leq 2^p \, B(p) \, \left(\sum_{k \in \mathbb{Z}} b_k\right)^{\frac{n}{n+2}} \, \left(\sum_{k \in \mathbb{Z}} a_k\right)^{\frac{n}{n+2}}.
\]

This allows us to conclude

\[
\sum_{k \in \mathbb{Z}} a_k \leq 2^p \, B(p) \, \left(\sum_{k \in \mathbb{Z}} b_k\right)^{\frac{n}{n+2}} \, \left(\sum_{k \in \mathbb{Z}} a_k\right)^{\frac{n}{n+2}}
\]

hence

\[
(\sharp\sharp) \quad \sum_{k \in \mathbb{Z}} a_k \leq \left(2^p \, B(p) \, \left(\sum_{k \in \mathbb{Z}} b_k\right)^{\frac{n}{n+2}}\right)^{\frac{n}{n+2}} = 2^{\frac{p(n+2)}{n+2}} \, B(p) \, \left(\sum_{k \in \mathbb{Z}} b_k\right)^{\frac{n}{n+2}}.
\]

Now, on a hand \(\bigcup_{k \in \mathbb{Z}} V_k = V\) and so

\[
\sum_{k \in \mathbb{Z}} b_k = \sum_{k \in \mathbb{Z}} \left(\sum_{y \sim x \in V_k} |f_k(y) - f_k(x)|^2 \sigma_{xy}\right) \leq \sum_{y \sim x \in V} |f(y) - f(x)|^2 \sigma_{xy}.
\]
on another hand, we note that \( \bigcup_{k \in \mathbb{Z}} V_k = V \), that \( V_k = W_{k-1} - W_k \) and that the restriction \( |f|^p(V_k) \leq (2^{k+1})^p = 2^p(2^k) \), and so

\[
\sum_{x \in V} |f(x)|^p \mu_x = \sum_{k \in \mathbb{Z}} \left( \sum_{x \in V_k} |f(x)|^p \mu_x \right) \leq \sum_{k \in \mathbb{Z}} 2^p(2^k) \left( \sum_{x \in V_k} \chi_{W_k}(x) \mu_x \right) = 2^p \sum_{k \in \mathbb{Z}} 2^k \left( \sum_{x \in V} \chi_{W_{k+1}}(x) \mu_x \right) = 2^p \sum_{k \in \mathbb{Z}} \left( a_k - \frac{a_{k+1}}{2^p} \right) \sum_{x \in V} \chi_{W_k}(x) \mu_x
\]

Therefore we combine these last two inequalities with \((\dagger)\) and find that

\[
\left( \sum_{x \in V} |f(x)|^p \mu_x \right)^{2^p} \leq \left( 2^{p-1} \right)^{\frac{2^p}{p}} B(p) \sum_{x, y \sim x} |f(y) - f(x)|^2 \sigma_{xy},
\]

which gives exactly \((\ddagger)\) when \( A(p) = (2^p - 1)^{\frac{2^p}{p}} 2^{(p-1)} B(p) \).

The following corollary shows a Nash inequality for \( \Gamma_G \) for the first time.

**Corollary 3.2.** \( \Gamma_G \) satisfies the thesis of Theorem [24] with \( n = \nu_2 \), \( \mu_x = \deg(x) \), \( \sigma_{xy} = \nu = R_0 = \delta = 1 \).

**Proof.** Application of definitions, Lemma 2.4 and Theorem 3.1

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