Efficient Posterior Sampling for Bayesian Poisson Regression

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ABSTRACT

Poisson log-linear models are ubiquitous in many applications, and one of the most popular approaches for parametric count regression. In the Bayesian context, however, there are no sufficient specific computational tools for efficient sampling from the posterior distribution of parameters, and standard algorithms, such as random walk Metropolis-Hastings or Hamiltonian Monte Carlo algorithms, are typically used. Herein, we developed an efficient Metropolis-Hastings algorithm and importance sampler to simulate from the posterior distribution of the parameters of Poisson log-linear models under conditional Gaussian priors with superior performance with respect to the state-of-the-art alternatives. The key for both algorithms is the introduction of a proposal density based on a Gaussian approximation of the posterior distribution of parameters. Specifically, our result leverages the negative binomial approximation of the Poisson likelihood and the successful Pólya-gamma data augmentation scheme. Via simulation, we obtained that the time per independent sample of the proposed samplers is competitive with that obtained using the successful Hamiltonian Monte Carlo sampling, with the Metropolis-Hastings showing superior performance in all scenarios considered. Supplementary materials for this article are available online.

1. Introduction

Poisson log-linear models are common in statistics and represent one of the most popular choices to model how the distribution of count data varies with predictors. A typical assumption is that, under an independent sample of counts, \(y_1, \ldots, y_n\), the probability mass function of the generic \(y_i\) conditionally on a \(p\)-dimensional vector of covariates \(x_i\) is

\[
  f(y_i|\lambda_i) = \frac{\lambda_i^{y_i}}{y_i!} e^{-\lambda_i}, \quad \log(\lambda_i) = x_i^T \beta, \tag{1}
\]

where \(\beta\) is a \(p\)-dimensional vector of unknown coefficients. Linking the linear predictor \(x_i^T \beta\) and the parameter \(\lambda_i\), with the logarithm represents the most natural choice, as the logarithm is the canonical link for the Poisson family (Nelder and Wedderburn 1972). This model has broad application in several fields, including medicine and epidemiology (Frome 1983; Frome and Checkoway 1985; Hutchinson and Holtman 2005), manufacturing process control (Lambert 1992), analysis of accident rates (Joshua and Garber 1990; Miaou 1994), and crowd counting (Chan and Vasconcelos 2009), among others.

In the Bayesian context, model (1) does not enjoy any conjugacy property and, thus, regardless of the prior used, the posterior distribution of \(\beta\) is not available in close form. Consequently, inference is conducted using Markov chain Monte Carlo (MCMC) methods, which obtain a sample from the posterior distribution of the parameters. Several approaches have focused on how to easily obtain the posterior distribution of the coefficients of Poisson models without requiring complex tuning strategies or long computation times. In the context of count-valued time series, Frühwirth-Schnatter and Wagner (2006) proposed a formulation of the model based on two levels of data augmentation, to derive an efficient approximate Gibbs sampler. Frühwirth-Schnatter et al. (2009) exploited a data augmentation strategy to simplify the computation of hierarchical models for count and multinomial data. Data augmentation strategies have also been employed in the case of models for multivariate dependent count data (Karlis and Meligkotsidou 2005; Bradley, Holan, and Wikle 2018). However, the simplest Poisson regression in (1) still lacks a specific and efficient algorithm to sample from the posterior distribution of the parameters \(\beta\) for any prior choice, making the Metropolis-Hastings (Hastings 1970) or Hamiltonian Monte Carlo (HMC) (Neal 2011) algorithms the only available options.

On the other hand, several efficient computational strategies for binary regression models have been proposed in the literature. Using the probit link, Albert and Chib (1993) proposed an efficient data augmentation based on a latent Gaussian variable, while the more recent contribution by Polson, Scott, and Windle (2013) exploited the canonical logit link, introducing an efficient Pólya-gamma data augmentation scheme. Leveraging Polson, Scott, and Windle (2013) approach, we propose a novel approximation of the posterior distribution that can be exploited as proposal distribution of a Metropolis-Hastings algorithm or as importance density of an importance sampling for Poisson log-linear models with conditional Gaussian prior.
distributions on the regression parameters. With conditional Gaussian prior, we refer to a possibly hierarchical prior with conditional distribution \( \beta \sim N(b, B) \), with \( b \) and/or \( B \) random. Examples include straightforward Gaussian prior distributions with informative \((b, B)\) fixed using prior information, and scale mixtures of Gaussian where \( b \) is set to zero and the variance has a suitable hierarchical representation, such as the Bayesian lasso prior (Park and Casella 2008), the horseshoe prior, and its extensions (Carvalho, Polson, and Scott 2010; Piironen and Vehtari 2017).

More specifically, we introduce an approximation of the posterior density that exploits the negative binomial convergence to the Poisson distribution. Thanks to this result, we are able to leverage the Pólya-gamma data augmentation scheme of Polson, Scott, and Windle (2013) to derive an efficient sampling scheme. In the next section, we introduce and discuss the proposed algorithms, starting from the definition of an approximate posterior distribution whose sampling can be performed straightforwardly. Sampling from this approximate posterior is then used as proposal density for the Metropolis-Hastings or importance sampler. The performances of the proposed algorithms in terms of computational efficiency is compared with that of state-of-the-art methods in a simulation study. The article concludes with two illustrative applications.

2. Efficient Posterior Sampling Strategies

2.1. Approximate Posterior Distribution

Assume \( y_1, \ldots, y_n \) is an independent sample of counts from model (1). We introduce an approximation of the posterior density which exploits the negative binomial convergence to the Poisson distribution, that is, we approximate the contribution to the likelihood function \( f(y_i|\lambda_i) \) with \( \tilde{f}_r_i(y_i|\lambda_i) \) where

\[
\tilde{f}_r_i(y_i|\lambda_i) = \frac{(r_i + y_i - 1)}{(r_i - 1)} \frac{\lambda_i^{r_i}}{(r_i + \lambda_i)} \gamma(y_i|\lambda_i),
\]

which corresponds to the probability mass function of a negative binomial random variable with parameter \( r_i \), the number of failures until the experiment is stopped, and success probability \( \lambda_i/(r_i + \lambda_i) \). As \( r_i \) approaches infinity, this quantity converges to a Poisson likelihood.

Following Polson, Scott, and Windle (2013), we implement a data augmentation scheme based on the introduction of \( n \) Pólya-gamma random variables. A random variable \( X \) is said to follow a Pólya-gamma distribution with parameters \( \xi > 0 \) and \( \zeta \in \mathbb{R} \), in the following denoted as \( X \sim PG(\xi, \zeta) \), if

\[
X \overset{D}{=} \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{g_k}{(k - 1/2)^2 + \zeta^2/(4\pi^2)},
\]

where the \( g_k \sim \text{Gamma}(\xi, 1) \) are independent gamma random variables, and where \( \overset{D}{=} \) indicates equality in distribution. A key identity is that the binomial likelihood, parameterized by logits, can be represented as a scale mixture of Gaussians with respect to a Pólya-gamma distribution, that is

\[
(2^r \pi^-)^{-\xi} e^{r(-\xi/2)} \sum_{k=1}^{\infty} g_k e^{-\omega^2/2f_{\omega}(\omega; \xi, 0) d\omega},
\]

where \( f_{\omega}(\cdot; \xi, \zeta) \) denotes the density of a Pólya-gamma with parameters \( (\xi, \zeta) \).

This is also true for the negative binomial likelihood parameterized as in (2); indeed, we can rewrite each \( i \)th contribution to the approximate likelihood (2) by introducing augmented Pólya-gamma random variables \( \omega_i \sim PG(y_i + r_i, 0) \), that is,

\[
\tilde{f}_{r_i}(y_i|\beta) = \left( \frac{r_i + y_i - 1}{r_i + y_i} \right) \left( \frac{\lambda_i}{r_i + \lambda_i} \right) \gamma(y_i|\lambda_i),
\]

\[
\times \int_0^{+\infty} \exp \left( -\omega_i \left( \frac{Y_{r_i}^T \beta - \log r_i}{2} \right)^2 \right) f_{\omega_i}(\omega_i; y_i + r_i, 0) d\omega_i.
\]

More details on the derivation of this expression can be found in Appendix A of the supplementary materials.

In what follows, we assume that prior knowledge about the unknown \( \beta \) parameters is represented by a conditionally Gaussian prior, that is, \( \beta \sim N(b, B) \), with a possible hierarchical representation for the parameters \( b \) and \( B \). Examples include default informative Gaussian with fixed \((b, B)\) or scale mixtures of Gaussian where \( b \) is set to zero and the variance has a suitable hierarchical representation (Park and Casella 2008; Carvalho, Polson, and Scott 2010; Piironen and Vehtari 2017). The approximate posterior based on the conditionally Gaussian prior \( \beta \sim N(b, B) \) and approximate likelihood \( \prod_{i=1}^{n} \tilde{f}_{r_i}(y_i|\beta) \) is consistent with the successful Gibbs sampler of Polson, Scott, and Windle (2013); that is, sampling from the approximate posterior is equivalent to sampling iteratively from the following full conditionals

\[
\omega_i|\beta \sim PG(y_i + r_i, X_r^T \beta - \log r_i), \quad \beta|y, \omega \sim N_p(m_\omega, V_\omega),
\]

where \( V_\omega = (X_T \Omega X + B^{-1}) \) and \( m_\omega = V_\omega(X_T \kappa + B^{-1} b) \), with \( \Omega = \text{diag}(\omega_1, \ldots, \omega_n) \) and \( \kappa = (\omega_1 \log r_1 + (y_1 - r_1)/2, \ldots, \omega_n \log r_n + (y_n - r_n)/2) \). Derivation of this data augmentation strategy can be obtained in a straightforward manner starting from the model formulation in Equation (3) and following the original strategy of Polson, Scott, and Windle (2013).

The adherence of this approximate posterior to the true posterior highly depends on the values of \( r_i \), with larger values of \( r_i \) resulting in better approximations. However, when employing this result in posterior sampling, large values of \( r_i \) imply longer computation time due to the computational cost of sampling Pólya-gamma random variables with large parameters. Although the specific choice of \( r_i \) remains an open point—discussed later in Section 2.4—in the context of MCMC sampling, we propose to reduce the computational burden related to the sampling of \( n \) Pólya-gamma random variables marginalizing the Gaussian distribution in (4) with respect to the related Pólya-gamma density conditioned on \( \beta^{(t-1)} \), the last available \( \beta \) sampled. Since this marginalization is not in a closed form we introduce a second level of approximation of the true posterior. Specifically, we introduce \( q(\beta|\beta^{(t-1)}) \) a density that depends on \( \beta^{(t-1)} \), defined as the first-order Taylor expansion of the marginalized Gaussian distribution, that is,
\[
q(\beta | \beta^{(t-1)}) = (2\pi)^{-p/2} \text{det}(V_{E(\omega)})^{-1/2} \times \exp \left\{ -\frac{1}{2} (\beta - m_{E(\omega)})^T V_{E(\omega)}^{-1} (\beta - m_{E(\omega)}) \right\},
\]

where \( V_{E(\omega)} = (X^T \tilde{\Omega} X + B^{-1}) \), \( m_{E(\omega)} = V_{E(\omega)} (X^T \tilde{\kappa} + B^{-1} b) \), \( \tilde{\Omega} = \text{diag}(E(\omega_1), \ldots, E(\omega_n)) \), \( \tilde{\kappa} = (E(\omega_1) \log r_i + (y_i - r_i)/2) \), and for each \( i = 1, \ldots, n \) the conditional expectation of each \( \omega_i \) is simply

\[
E(\omega_i) = \frac{r_i + y_i}{2(x_i^T \beta^{(t-1)} - \log r_i)} \left( \frac{x_i^T \beta^{(t-1)} - r_i}{x_i^T \beta^{(t-1)} + r_i} \right),
\]

or equivalently

\[
\beta | \beta^{(t-1)} \sim N(m_{E(\omega)}, V_{E(\omega)}).
\]

The above construction is eventually used as the building block of efficient Metropolis-Hastings and importance sampling algorithms, as described in the following sections.

### 2.2. Metropolis-Hastings Sampler

We employ the above sampling mechanism as the proposal density in a Metropolis-Hastings algorithm. Consistent with this, at each iteration of the MCMC sampler, an additional step that accepts or rejects the proposed draw is introduced. Specifically, we assume that conditionally on the current state of the chain \( \beta^{(t-1)} \), a new value \( \beta^* \) is sampled from (6). Then, the acceptance probability

\[
\alpha(\beta^*, \beta^{(t-1)}) = \min \left\{ 1, \frac{\pi(\beta^*) | y) q(\beta^{(t-1)} | \beta^*)}{\pi(\beta^{(t-1)} | y) q(\beta^* | \beta^{(t-1)})} \right\},
\]

is evaluated to decide whether to accept or reject the proposed \( \beta^* \), where \( \pi(\beta^* | y) \) is the exact posterior distribution of \( \beta \) given the sample \( y_1, \ldots, y_n \).

To compute the acceptance probability in (7), the forward and backward transition densities \( q(\beta^* | \beta^{(t-1)}) \) and \( q(\beta^{(t-1)} | \beta^*) \) must be computed. Consistent with this, approximation (5) is particularly useful: without it, it would be necessary to compute the marginal density where the Pólya-gamma random variables are integrated out. However, the marginalization with respect to the Pólya-gamma density does not lead to a closed form expression; thus, the Metropolis-Hastings algorithm cannot be defined.

Clearly, for increasing \( r_i \) the proposal density (6) is closer to the true full conditional distribution; hence, the related acceptance rate will be higher, and the Metropolis-Hastings algorithm will be similar to a Gibbs sampler. On the other hand, setting this parameter to get a lower acceptance rate can result in smaller autocorrelation, and hence better mixing (Robert and Casella 2010). We discuss an approach to choose \( r_i \) balancing these two extremes in Section 2.4.

### 2.3. Adaptive Importance Sampler

The sampling mechanism (6) can also be exploited within the context of importance sampling, where the posterior expectation of a function of the parameter \( \beta \), \( E(h(\beta)) = \int h(\beta) \pi(\beta | y) d\beta \) is evaluated via Monte Carlo integration without direct sampling from \( \pi(\beta | y) \). To this end, the general approach is to define an importance density \( q(\beta) \) that is used to sample values \( \beta^{(1)}, \ldots, \beta^{(T)} \), which are eventually averaged to obtain an approximation of \( E(h(\beta)) \) through

\[
\hat{E}(h(\beta)) = \frac{1}{T} \sum_{t=1}^{T} \tilde{w}(\beta^{(t)}) h(\beta^{(t)}),
\]

with weights

\[
\tilde{w}(\beta^{(t)}) = \frac{\pi(\beta^{(t)} | y)}{q(\beta^{(t)})}.
\]

The efficiency of this algorithm is determined by the ability of the importance density to sample values relevant to the target distribution. To improve this aspect, we modify the original algorithm and, instead of using a fixed density \( q \), at each iteration we consider an adaptive proposal. Specifically, we make use of (5) as proposal density, but, unlike the Metropolis-Hastings algorithm, we update it only when the last sampled value moves toward a region with a higher posterior probability. Denoting with \( \beta^c \) the conditioning value, at each iteration we sample a new value from \( q(\beta^{(t)} | \beta^c) \), and if \( \pi(\beta^{(t)} | y) > \pi(\beta^c | y) \), we set \( \beta^c = \beta^{(t)} \). Thus, the importance density is adaptively updated and the weights become

\[
\hat{w}(\beta^{(t)}) = \frac{\pi(\beta^{(t)} | y)}{q(\beta^{(t)})}.
\]

Notice that if one already had an estimate of the mode of the posterior (e.g., resulting from a variational approach, as in Arridge et al. (2018), or a numerical optimization), such a value could be used as a fixed conditioning value. This approach would lead to the advantage of avoiding evaluation and update of \( \beta^c \) at each iteration; however, finding the model is not trivial in general, and the overall feasibility and convenience should be evaluated on a case-by-case basis.

### 2.4. Tuning Parameters \( r_i \)

The values of the parameters \( r_i, i = 1, \ldots, n \), have to be tuned to balance the tradeoff between acceptance rate and autocorrelation in the Metropolis-Hastings, and to control the mixing of the weights in the importance sampler. However, tuning \( n \) parameters is not practical, especially for moderate to large \( n \). The first simple solution sets all parameters equal to a single value \( r \), however, in our experience, this resulted in a low effective sample size for some of the sampled chains.

As an alternative strategy, we choose to tune instead the distance of the proposal density from the target posterior. As the expression of the posterior distribution is unknown, we control the distance between the Poisson and negative binomial likelihood. Based on Teerapobolarn (2012), we consider the upper bound of the relative error between the Poisson and negative binomial cumulative distribution functions. This result is particularly useful owing to its simplicity, which allows to analytically derive adequate parameters to bound the error to a specific value. Specifically, if \( Y \) is a Poisson random variable
3. Numerical Illustrations

3.1. Synthetic Data

We conducted a simulation study under various settings to compare the efficiency of the proposed Metropolis-Hastings and importance sampling methods. Specifically, for each combination of $n$ and $p$, we considered $B = 10$ independent dimensions of counts where $n = 25, 30, 100, 200$ and $p = 5, 10, 20$. For each combination of $n$ and $p$, we generated $B$ independent samples according to a Poisson distribution with mean $\lambda = \lambda_i$ and variance $\lambda_i$.

Hence, by setting an upper bound $d$ for the distance between the parameter vectors, we can automatically determine the values of $r_1, \ldots, r_p$ in the algorithm. The relative error in (8) may appear a smaller choice is driven by the need to have simple analytic expressions for the relationship between the relative error and for example the Kullback-Leibler divergence. However, the differences are less evident with increasing sample size.

For the horseshoe prior, we used the details of Makalic and Schmidt (2016) and fixed the global shrinkage parameter $\tau_0 = 1$. For the horseshoe prior, we used the details of Makalic and Schmidt (2016) and fixed the global shrinkage parameter $\tau_0 = 1$. The global shrinkage parameter $\tau_0$ is the default value of $\tau_0$ in Stan (Carpenter et al., 2017).

The experiments have been run on a Mac OS machine with 2 GB RAM and 2 GB CPU. The experiments have been run on a Mac OS machine with 2 GB RAM and 2 GB CPU. The experiments have been run on a Mac OS machine with 2 GB RAM and 2 GB CPU. The experiments have been run on a Mac OS machine with 2 GB RAM and 2 GB CPU. The experiments have been run on a Mac OS machine with 2 GB RAM and 2 GB CPU.
of covariates \( p \). The performance of the importance sampler remains competitive. As previously observed for the Gaussian prior, the differences are less evident for increasing sample size.

Our experience suggests that for increasing dimension, the performance of the methods gets worse but remains competitive for moderate \( p \) beyond the range considered here. Results for \( p = 50 \) are available in Appendix B of the supplementary materials.

3.2. Spike Train Data

Herein, we illustrate the proposed sampling method on spike train data, which describe the neurons’ activity in response to stimulation. This type of data is relatively new, and it arises from the observation of brain activity through the technique of calcium imaging. Thanks to this technique, it is possible to investigate the association between a set of external conditions and the neurons’ response. The dataset was generated using a small subset of the Allen Brain Observatory (Allen Institute for Brain Science 2016), which is a survey of the activity of neurons in mice in response to visual stimulation. Specifically, we applied the deconvolution method of Jewell et al. (2019) as described in de Vries et al. (2020) to count the activations of each neuron, to analyze how they are affected by the experimental conditions and other covariates available from the study.

In the context of neural studies, this approach is usually referred to as “encoding models,” as the interest is to predict the activity of a population of neurons in response to a given stimulus and other experimental conditions. Paninski, Pillow, and Lewi (2007) reviewed some methods commonly employed in encoding problems. Generalized linear models, which are flexible and yet interpretable, are one of the fundamental tools for investigating the response of neurons to external factors. In particular, the authors assert that assuming a Poisson distribution is a plausible assumption to model spike counts; hence, we regressed the estimated number of neurons’ activation on several continuous and categorical covariates available from the study.
The covariates are the depth of the neuron, the area of the visual cortex where the neuron is located (factor with 6 levels), the cre transgenic mouse line (factor with 13 levels), and the type of visual stimulation (factor with 4 levels). The depth of the neurons is discretized to 22 levels, ranging from 175 to 625 microns, thus, we could obtain a dataset having a full factorial design with 5 replications for each available covariate combination. Moreover, we included a quadratic term of the depth to improve the fitting. The obtained dataset is made of 920 observations on 23 variables.

We ran the proposed Metropolis-Hastings algorithm for 9000 iterations, discarding the first 5000 as burn-in. The computation time was 98 sec. We employed the Metropolis-Hastings sampler rather than the importance sampler as it was more stable. Our experience suggests that this is a general behavior and thus we recommend using the Metropolis-Hastings algorithm for high $p$. The posterior estimates of the coefficients of the dummies on three categorical variables are shown in Figure 3; and for the numeric covariate depth, the posterior mean and 95% credible intervals were equal to $-2.72 \times 10^{-3}$ ($-2.90 \times 10^{-3}, -2.42 \times 10^{-3}$) for the linear term, and $5.59 \times 10^{-6}$ ($5.11 \times 10^{-6}, 6.08 \times 10^{-6}$) for the quadratic term. Given these estimates of the coefficients, the number of spikes increased with the largest depths. Moreover, as shown in Figure 3, the response of neurons is heterogeneous across the cre-lines and, coherent with the results of de Vries et al. (2020), we obtained that the mean response is lower for the VISam, VISpm, and VISrl areas.

### 3.3. Betting Data

Poisson regression models have been widely adopted in sports analytics, where the response variable is the match score or total number of points. Modeling match scores in association football has recently gained considerable interest owing to the popularity of the betting market, and several modeling approaches have
been proposed. Classical methods use only the information of the teams playing (Dixon and Coles 1997; Petretta et al. 2021), while other authors have explored the possibility of introducing additional information, such as historical data and bookmakers’ odds (Egidi, Pauli, and Torelli 2018; Groll et al. 2019). In general, to describe the number of goals scored by each team in a match, the Poisson distribution is considered a valid assumption (Maher 1982; Lee 1997). Herein, we considered data of match scores from the Italian Serie A 2020–2021 season, which are publicly available at http://www.football-data.co.uk. We considered the number of goals as the variable of interest, and, as covariates, we included the fixed effects of the team, several betting odds (for different betting types and bookmakers), and an indicator of whether the team is playing home. The resulting dataset has 760 observations on 102 variables.

Following the same reasoning as in the previous section, we used the proposed Metropolis-Hastings algorithm. Moreover, we considered both a Gaussian prior centered at zero and a horseshoe prior distribution on the parameters, to compare the results and analyze the variables the two priors select. The results are depicted in Figure 4. For each explanatory variable, the posterior mean and credible interval were obtained using the two priors. The filled symbols indicate that the credible interval does not contain zero, showing that the shrinkage induced by the horseshoe prior selects only a few variables compared to the informative Gaussian prior. Moreover, the horseshoe prior induces a significant reduction of the amplitude of all credible intervals.

A more formal comparison between the two models is obtained using the conditional predictive ordinate (CPO) statistics (Gelfand, Dey, and Chang 1992; Geisser 1993; Gelfand and Dey 1994), which is defined for \( i = 1, \ldots, n \), as \( \text{CPO}_i = p(y_i | y_{-i}) \), where \( y_{-i} \) is the vector of observed data omitting the \( i \)th value. Figure 5 shows the boxplots of the resulting CPO statistics for the two models. The graph does not highlight any fundamental difference in the predictive capacity of the two models, implying that the horseshoe prior allows to obtain a more parsimonious model with a similar fit. This is also confirmed by the logarithm of the pseudo-marginal likelihood, which is commonly used as a summary of CPO’s (Ibrahim, Chen, and Sinha 2014). It is equal to \(-1172.055\) and to \(-1167.521\) for the models based on the Gaussian prior and the horseshoe, respectively.

4. Discussion

Motivated by the lack of specific computational tools for efficient sampling from the posterior distribution of regression parameters in Poisson log-linear models, we introduced an approximate posterior distribution used as the building block for the Metropolis-Hastings and importance sampling algorithms.
The introduced proposal distribution is based on two levels of approximation of the target density. The first level exploits the well-known convergence of the negative binomial likelihood to the Poisson likelihood; the second level introduces a Gaussian approximation of the data augmentation scheme of Polson, Scott, and Windle (2013) to sample from this negative binomial model. The accuracy of the overall approximation can be tuned by acting on the first of these two levels only: thanks to the availability of a simple expression of the relative error between the Poisson and negative binomial distribution, it is possible to fix an upper bound for the discrepancy between the two likelihoods. Although we do not have a handle on the second approximation level, the successful empirical results that we showed are reassuring.

The resulting density is particularly convenient for use as proposal distribution in Monte Carlo algorithms for two reasons: the possibility to tune its closeness to the target density through a single parameter, and the extremely simple form, which is multivariate normal.

The performances of the proposed solutions, in terms of mixing and computation time, were comparable or superior to those of the efficient Stan implementation of HMC in all scenarios considered and particularly when a hierarchical prior is assumed. The ease of application of our methods is further
enhanced by their availability via the R package bpr, which obtains the posterior distribution of several quantities of interest without the need for coding and with minimal tuning.

Supplementary Materials

Appendix A: Derivation of the data augmentation scheme for the negative binomial model.

Appendix B: Simulation results for \( p \) up to 50 covariates.

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