STRATIFIED MORSE CRITICAL POINTS AND BRASSELET NUMBER ON
TORIC VARIETIES

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Abstract. The generalization of the Morse theory presented by Goresky and
MacPherson is a landmark that divided completely the topological and geomet-
rical study of singular spaces. In this work, we consider \((X,0)\) a germ at 0
of toric variety and non-degenerate function-germs \(f, g : (X,0) \to (\mathbb{C},0)\), and
we prove that the difference of the Brasselet numbers \(B_{f,X}(0)\) and \(B_{f,X \cap g^{-1}(0)}(0)\)
is related with the number of Morse critical points on the regular part of the
Milnor fiber of \(f\) appearing in a morsefication of \(g\), even in the case where \(g\)
has a critical locus with arbitrary dimension. This result connects topological
and geometric properties and allows us to determine many interesting formulae,
mainly in terms of the combinatorial information of the toric varieties.

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1. Introduction

The geometric and topological study of singular varieties or singular maps are
related to several classical problems in Mathematics. Singularity Theory, struc-
tured as a field of research, has risen from the works of Whitney, Mather and
Thom around 1960’s.

Whitney’s Embedding Theorem established the best linear bound on the small-
est dimensional Euclidean space, \(\mathbb{R}^{2m}\), in which an \(m\)-dimensional manifold \(M\)
embeds. An interesting question is to consider the types of singularities that occur

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when it is not possible to embed \( M \) in \( \mathbb{R}^n \) and to study their stability. Thom also played a major role by proving important stability results and with keen foresight predicted conjectures that directed the research in the field for years to come, see [39]. Mather, in a series of articles, advances the research in this area to new depths, establishing solid foundation for Singularity Theory [26, 27, 28, 29].

Suppose that \( X \) is a topological space, \( f \) is a smooth real valued function on \( X \), and \( c \) is a real number. The fundamental problem of Morse theory is to study the topological changes in the space \( X \) as the number \( c \) varies.

In classical Morse theory, the space \( X \) is taken to be a compact differentiable manifold. Let \( f \) be the projection onto the vertical coordinate axis. So, \( f(x) \) measures the height of the point \( x \). We imagine slowly increasing \( c \) and we watch how the topology changes.

In [18] the authors generalize Morse theory by extending the class of spaces to which it applies. This increase in generality allows us to apply Morse Theory to several new questions. This new theory had a great impact in the study of the topology and geometry of singular spaces, for instance.

Let us suppose that \( X \) is a compact Whitney stratified subspace of a manifold \( M \) and that \( f \) is the restriction to \( X \) of a smooth function on \( M \). We define a critical point of \( f \) to be a critical point of the restriction of \( f \) to any stratum. A critical value is, as before, the value of \( f \) at a critical point. Goresky and MacPherson in their book give the natural generalization of classical Morse theory on a manifold \( M \) to stratified spaces.

An important invariant associated to a germ of an analytic function \( f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) \) with an isolated critical point at the origin is denoted by \( \mu(f) \) and it is defined as

\[
\mu(f) := \dim_{\mathbb{C}} \mathcal{O}_n/J(f),
\]

where \( \mathcal{O}_n \) is the ring of germs of analytic functions at the origin, and \( J(f) \) is the Jacobian ideal of \( f \). This invariant, defined by Milnor in [32] and called the Milnor number of \( f \), provides information on the geometry of \( f \) and also information about the local topology of the hypersurface \( X = f^{-1}(0) \). For example, when \( f \) has an isolated critical point at the origin, the following invariants coincide up to sign:

(a) the Milnor number of \( f \) at the origin;
(b) the number of Morse critical points of a morseification of \( f \);
(c) the Poincaré-Hopf index of the complex conjugate of the gradient vector field of \( f \).

Let \( (X, 0) \) be the germ of an analytic singular space embedded in \( \mathbb{C}^n \) and \( f : (X, 0) \rightarrow (\mathbb{C}, 0) \) a germ of analytic function. Brasselet et al. introduced in [5] a generalization of (c), called Euler obstruction of \( f \), denoted by \( \text{Eu}_{f,X}(0) \). Roughly, it is the obstruction to extending a lifting of the conjugate of the gradient vector field of \( f \) as a section of the Nash bundle of \( (X, 0) \). It is then natural to compare \( \text{Eu}_{f,X}(0) \) to the Milnor number of \( f \) in the case of a singular germ \( (X, 0) \). This has been a question raised in [37]. The Euler obstruction of a function was investigated by
many authors as Ament et al. [1], Dalbelo and Pereira [11], Dutertre and Grulha [14], Massey [25], Seade, Tibăr and Verjovisky [37] to mention just a few. For an overview on the topic one can see [3] or the book [7].

The Euler obstruction of \( f \) is called such because of its relation with the local Euler obstruction of \( X \) defined by MacPherson [21] and denoted by \( \text{Eu}_X(0) \). Several authors have proved formulae for the local Euler obstruction. In particular, Brasselet et al. proved a Lefschetz-type formula in [4].

Inspired by the famous formula to compute the local Euler obstruction in terms of the polar multiplicities due to Lê and Teissier [20], Brasselet conjectured that it would be possible to compute the Euler obstruction of a function in terms of the relative polar varieties. Dutertre and Grulha [13] proved that, for a function germ \( f : (X,0) \to (\mathbb{C},0) \) with isolated singularity, the difference \( \text{Eu}_X(0) - \text{Eu}_{f,X}(0) \) can be computed in terms of the relative polar varieties. Then, the authors called this difference the Brasselet number, denoted by \( \text{B}_{f,X}(0) \) (see Definition 2.13). If \( f \) is linear and generic, then \( \text{B}_{f,X}(0) = \text{Eu}_X(0) \), hence it can be viewed as a generalization of the local Euler obstruction. Moreover, even if \( f \) has a non-isolated singularity it provides interesting results. For example, the Brasselet number satisfies a Lê-Greuel type formula (see [13, Theorem 4.4]), i.e., the difference of the Brasselet numbers \( \text{B}_{f,X}(0) \) and \( \text{B}_{f,X^g}(0) \) is measured by the number of Morse critical points on the regular part of the Milnor fiber of \( f \) appearing in a morsefication of \( g \), where \( g : (X,0) \to (\mathbb{C},0) \) is a prepolar function and \( X^g = X \cap g^{-1}(0) \).

A first generalization of this result was given by Santana [36], that considered the case where the function \( g \) has a stratified singular set of dimension 1 and proved that in this case the difference of the Brasselet numbers \( \text{B}_{f,X}(0) \) and \( \text{B}_{f,X^g}(0) \) is still related with the number of Morse critical points on the regular part of the Milnor fiber of \( f \) appearing in a morsefication of \( g \).

Although they are very interesting, the problems mentioned above are not easy in general. In Mathematics, an usual strategy is to prove some results in a controlled class of variety, but sufficiently wide to have a good felling if the result could hold in general. Even if it is not, if we find a wide and nice class in which we can deal with, it can be very useful.

A very important and wide class of algebraic varieties to study in Algebraic Geometry, Singularity Theory and Algebraic Topology are the toric varieties [16]. These are very interesting objects, with very useful properties and connexions. We can highlight its relation with elemental convex geometry, for example, which relates the combinatorial study of convex polytopes with algebraic torus actions.

Toric varieties can be characterized by the property that they have an action of the algebraic torus \( (\mathbb{C}^\ast)^d \), and with this action one decompose the variety in orbits all homeomorphic to torus. Due to this fact, and also the informations coming from the combinatorial residing in these varieties, many questions that was originally studies for functions defined on \( \mathbb{C}^d \) can be extended to functions defined on toric varieties. For instance, the famous result of Varchenko [40] which describes the topology of the Milnor fiber of a function \( f : (\mathbb{C}^d,0) \to (\mathbb{C},0) \) using the geometry of the Newton polygon of \( f \) are recently generalized by Matsui
and Takeuchi [31], presenting a formula for monodromy zeta functions of Milnor fibers over singular toric varities and also for Milnor fibers over non-degenerate complete intersection subvarieties of toric varieties.

Given \((Y,0) \subset (X,0)\) a germ of non-degenerate complete intersection in a toric variety \((X,0)\), Dalbelo and Hartmann [10] presented a relation between the Brasselet number of a function-germ \(f : (Y,0) \to (\mathbb{C},0)\) and the combinatorial that came from the Newton polygons associated to \(f\) and \(Y\). Such relation allowed the authors to derive sufficient conditions to obtain the invariance of the Euler obstruction for families of non-degenerate complete intersections which are contained in \((X,0)\) [10, Theorem 4.1].

In this work, we consider \((X,0)\) a germ of toric variety at the origin and \(f, g : (X,0) \to (\mathbb{C},0)\) non-degenerate function-germs and we prove that the difference of the Brasselet numbers \(B_{f,X}(0)\) and \(B_{f,X^g}(0)\) is related with the number of Morse critical points on the regular part of the Milnor fiber of \(f\) appearing in a morsefication of \(g\), even in the case where \(g\) has a critical locus with arbitrary dimension.

This paper is organized as follows. In Section 2 we present some background material concerning the Euler obstruction, Brasselet number and toric varieties, which will be used in the entire work. In Section 3, given \((X,0)\) the germ at the origin of a toric variety and \(g, f : (X,0) \to (\mathbb{C},0)\) non-degenerate polynomial function-germs we construct a good stratification \(V_f\) of the representative \(X\) relative to \(f\) and also a good stratification \(V'_g\) of \(X^g\) relative to \(f\). Moreover, using these stratifications we prove our main result, which stated that the difference of the Brasselet numbers \(B_{f,X}(0)\) and \(B_{f,X^g}(0)\) is related with the number of Morse critical points on the regular part of the Milnor fiber of \(f\) appearing in a morsefication of \(g\), even in the case where \(g\) has a \(d\)-dimensional critical locus, with arbitrary \(d\). In Section 4 we establish some results and corollaries that we obtain from the results of Section 3. In Section 5 we consider the case where \((X,0) = (\mathbb{C}^n,0)\) and we derive sufficient conditions to obtain the constancy of the number of Morse points, for families of non-degenerate hypersurfaces which are contained on \(X\).

2. Preliminaries notions and results

For convenience of the reader and to fix some notation we present some general facts in order to establish our results.

2.1. Euler obstruction. The local Euler obstruction is a singular invariant defined by MacPherson in [21] as a tool to prove the conjecture about the existence and unicity of the Chern classes in the singular case. Since then the Euler obstruction has been extensively investigated by many authors such as Brasselet and Schwartz [6], Sebastiani [38], Lê and Teissier [20], Sabbah [35], Dubson [12], Kashiwara [19] and others. For an overview of the Euler obstruction see [2]. Let us now introduce some objects in order to define the Euler obstruction.

Let \((X,0) \subset (\mathbb{C}^n,0)\) be an equidimensional reduced complex analytic germ of dimension \(d\) and \(X\) a sufficiently small representative of the germ in an open set \(U \subset \mathbb{C}^n\). We consider a complex analytic Whitney stratification \(\mathcal{V} = \{V_i\}\) of \(U\).
adapted to $X$ and we assume that $\{0\}$ is a stratum. We choose a small representative of $(X,0)$ such that 0 belongs to the closure of all the strata. We will denote it by $X$ and we will write $X = \bigcup_{i=0}^{d} V_i$ where $V_0 = \{0\}$ and $V_q = X_{\text{reg}}$ is the set of regular points of $X$. We will assume that the strata $V_0, \ldots, V_{q-1}$ are connected and that the analytic sets $\overline{V_0}, \ldots, \overline{V_{q-1}}$ are reduced.

Let $G(d,n)$ denote the Grassmanian of complex $d$-planes in $\mathbb{C}^n$. On the regular part $X_{\text{reg}}$ of $X$ the Gauss map $\phi : X_{\text{reg}} \to U \times G(d,n)$ is well defined by $\phi(x) = (x, T_x(X_{\text{reg}}))$.

**Definition 2.1.** The Nash transformation (or Nash blow up) $\tilde{X}$ of $X$ is the closure of the image $\text{Im}(\phi)$ in $U \times G(d,n)$. It is a (usually singular) complex analytic space endowed with an analytic projection map $\nu : \tilde{X} \to X$ which is a biholomorphism away from $\nu^{-1}(\text{Sing}(X))$.

The fiber of the tautological bundle $\mathcal{T}$ over $G(d,n)$, at point $P \in G(d,n)$, is the set of vectors $v$ in the $d$-plane $P$. We still denote by $\mathcal{T}$ the corresponding trivial extension bundle over $U \times G(d,n)$. Let $\tilde{\mathcal{T}}$ be the restriction of $\mathcal{T}$ to $\tilde{X}$, with projection map $\pi$. The bundle $\tilde{\mathcal{T}}$ on $\tilde{X}$ is called the Nash bundle of $X$.

An element of $\tilde{\mathcal{T}}$ is written $(x,P,v)$ where $x \in U$, $P$ is a $d$-plane in $\mathbb{C}^n$ based at $x$ and $v$ is a vector in $P$. The following diagram holds:

$$
\begin{array}{ccc}
\tilde{\mathcal{T}} & \hookrightarrow & \mathcal{T} \\
\pi \downarrow & & \downarrow \\
\tilde{X} & \hookrightarrow & U \times G(d,n) \\
\nu \downarrow & & \downarrow \\
X & \hookrightarrow & U.
\end{array}
$$

Adding the stratum $U \setminus X$ we obtain a Whitney stratification of $U$. Let us denote the restriction to $X$ of the tangent bundle of $U$ by $TU|_X$. We know that a stratified vector field $v$ on $X$ means a continuous section of $TU|_X$ such that if $x \in V_i \cap X$ then $v(x) \in T_x(V_i)$. From the Whitney condition (a), one has the following lemma.

**Lemma 2.2.** [6]. Every stratified vector field $v$ on a subset $A \subset X$ has a canonical lifting to a section $\tilde{v}$ of the Nash bundle $\tilde{\mathcal{T}}$ over $\nu^{-1}(A) \subset \tilde{X}$.

Now consider a stratified radial vector field $v(x)$ in a neighborhood of $\{0\}$ in $X$, i.e., there is $\varepsilon_0$ such that for every $0 < \varepsilon \leq \varepsilon_0$, $v(x)$ is pointing outwards the ball $B_\varepsilon$ over the boundary $S_\varepsilon := \partial B_\varepsilon$.

The following interpretation of the local Euler obstruction has been given by Brasselet and Schwartz in [6].

**Definition 2.3.** Let $v$ be a radial vector field on $X \cap S_\varepsilon$ and $\tilde{v}$ the lifting of $v$ on $\nu^{-1}(X \cap S_\varepsilon)$ to a section of the Nash bundle. The local Euler obstruction (or simply the Euler obstruction) $\text{Eu}_X(0)$ is defined to be the obstruction to extending $\tilde{v}$ as a nowhere zero section of $\tilde{\mathcal{T}}$ over $\nu^{-1}(X \cap B_\varepsilon)$.

More precisely, let $O(\tilde{v}) \in H^{2d}((\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon), \mathbb{Z})$ be the obstruction cocycle to extending $\tilde{v}$ as a nowhere zero section of $\tilde{\mathcal{T}}$ inside $\nu^{-1}(X \cap B_\varepsilon)$. The local
Euler obstruction $\text{Eu}_X(0)$ is defined as the evaluation of the cocycle $O(\nu)$ on the fundamental class of the pair $[(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon)]$ and therefore it is an integer.

In [4, Theorem 3.1], Brasselet, Lê and Seade proved a formula to make the calculation of the Euler obstruction easier.

**Theorem 2.4.** Let $(X,0)$ and $\mathcal{V}$ be given as before, then for each generic linear form $l$, there exists $\varepsilon_0$ such that for any $\varepsilon$ with $0 < \varepsilon < \varepsilon_0$ and $\delta \neq 0$ sufficiently small, the Euler obstruction of $(X,0)$ is equal to

$$\text{Eu}_X(0) = \sum_{i=1}^{q} \chi(V_i \cap B_\varepsilon \cap l^{-1}(\delta)) \cdot \text{Eu}_X(V_i),$$

where $\chi$ is the Euler characteristic, $\text{Eu}_X(V_i)$ is the Euler obstruction of $X$ at a point of $V_i$, $i = 1, \ldots, q$ and $0 < |\delta| \ll \varepsilon \ll 1$.

Let us give the definition of another invariant introduced by Brasselet, Massey, Parameswaran and Seade [5]. Let $f : X \to \mathbb{C}$ be a holomorphic function with isolated singularity at the origin given by the restriction of a holomorphic function $F : U \to \mathbb{C}$ and denote by $\nabla F(x)$ the conjugate of the gradient vector field of $F$ in $x \in U$,

$$\nabla F(x) := \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right).$$

Since $f$ has an isolated singularity at the origin, for all $x \in X \setminus \{0\}$, the projection $\tilde{\zeta}_i(x)$ of $\nabla F(x)$ over $T_x(V_i(x))$ is non-zero, where $V_i(x)$ is a stratum containing $x$. Using this projection, the authors constructed a stratified vector field over $X$, denoted by $\nabla f(x)$. Let $\tilde{\zeta}$ be the lifting of $\nabla f(x)$ as a section of the Nash bundle $\tilde{T}$ over $\tilde{X}$ without singularity over $\nu^{-1}(X \cap S_\varepsilon)$. Let $O(\tilde{\zeta}) \in \mathbb{H}^{2n}(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$ be the obstruction cocycle for extending $\tilde{\zeta}$ as a non-zero section of $\tilde{T}$ inside $\nu^{-1}(X \cap B_\varepsilon)$.

**Definition 2.5.** The local Euler obstruction of the function $f$, $\text{Eu}_{f,X}(0)$ is the evaluation of $O(\tilde{\zeta})$ on the fundamental class $[\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon)]$.

The next theorem compares the Euler obstruction of a space $X$ with the Euler obstruction of function defined over $X$.

**Theorem 2.6** ([5], Theorem 3.1). Let $(X,0)$ and $\mathcal{V}$ be given as before and let $f : (X,0) \to (\mathcal{C},0)$ be a function with an isolated singularity at $0$. For $0 < |\delta| \ll \varepsilon \ll 1$, we have

$$\text{Eu}_{f,X}(0) = \text{Eu}_X(0) - \sum_{i=1}^{q} \chi(V_i \cap B_\varepsilon \cap f^{-1}(\delta)) \cdot \text{Eu}_X(V_i).$$

In the stratified case, as we consider $\{0\}$ as a stratum, how could we “measure” the degeneracy of $f$ at this point? In order to have a good generalization of a morsefication in the singular case, we need to deal with the contribution of the variety at point in a $0$-dimensional strata. The idea is to characterize a kind of “Morse” point in this setting. Theses points are the generic points defined below.
Definition 2.7. Let $\mathcal{V} = \{V_\beta\}$ be a stratification of a reduced complex analytic space $X$ and $p$ be a point in a stratum $V_\beta$ of $\mathcal{V}$. A degenerate tangent plane of $\mathcal{V}$ at $p$ is an element $T$ of some Grassmanian manifold such that $T = \lim_{p_i \to p} T_{p_i} V_{\alpha_i}$, where $p_i \in V_{\alpha_i}$ and $V_{\alpha} \neq V_\beta$.

Definition 2.8. Let $(X, 0) \subset (U, 0)$ be a germ of complex analytic space in $\mathbb{C}^n$ equipped with a Whitney stratification and let $f : (X, 0) \to (\mathbb{C}, 0)$ be an analytic function, given by the restriction of an analytic function $F : (U, 0) \to (\mathbb{C}, 0)$. Then 0 is said to be a generic point of $f$ if the hyperplane $\text{Ker}(d_0F)$ is transverse in $\mathbb{C}^n$ to all degenerate tangent planes of the Whitney stratification at 0.

The definition of a morseification of a function is given as follows.

Definition 2.9. Let $\mathcal{W} = \{W_0, W_1, \ldots, W_q\}$, with $0 \in W_0$, a Whitney stratification of the complex analytic space $X$. A function $f : (X, 0) \to (\mathbb{C}, 0)$ is said to be Morse stratified if $\dim W_0 \geq 1$, $f|_{W_0} : W_0 \to \mathbb{C}$ has a Morse point at 0 and 0 is a generic point of $f$ with respect to $W_i$, for all $i \neq 0$. A stratified morseification of a germ of analytic function $f : (X, 0) \to (\mathbb{C}, 0)$ is a deformation $\tilde{f}$ of $f$ such that $\tilde{f}$ is Morse stratified.

Seade, Tibăr and Verjovsky [37, Proposition 2.3] proved that the Euler obstruction of a function $f$ is also related to the number of Morse critical points of a stratified morseification of $f$.

Proposition 2.10. Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a germ of analytic function with isolated singularity at the origin. Then,

$$\text{Eu}_{1,X}(0) = (-1)^{d-m},$$

where $m$ is the number of Morse points in $X_{\text{reg}}$ in a stratified morseification of $f$.

2.2. Stratifications and Brasselet number. Considering a complex analytic germ $(X, 0)$, a function-germ $f : (X, 0) \to (\mathbb{C}, 0)$, which is the restriction of a representative of an analytic function-germ $F : (U, 0) \to (\mathbb{C}, 0)$ and assuming that $X$ is equipped with a good stratification relative to $f$, Dutertre and Grulha [13] defined the Brasselet number, $B_{1,X}(0)$, and, under some conditions, obtained a Lé-Greuel formula for this invariant. When $f$ has an isolated singularity this number is given by the difference $B_{1,X}(0) = \text{Eu}_{X}(0) - \text{Eu}_{1,X}(0)$. When $f$ is linear and generic, it gives $\text{Eu}_{X}(0)$, hence $B_{1,X}(0)$ can be viewed as a generalization of the Euler obstruction.

In order to introduce the definition and properties of the Brasselet’s number, we need some notions concerning stratifications that we present below. For more details, we refer to Massey [23, 24]. For subsets $A \subset \mathbb{C}^n$, $B \subset \mathbb{C}^m$ and a function $f : A \to B$ we will fix the notation $A^f := A \cap f^{-1}(0)$.

Definition 2.11. A good stratification of $X$ relative to $\bar{f}$ is a stratification $\mathcal{V}$ of $X$ which is adapted to $X^f$, i.e., $X^f$ is a union of strata such that $\{V_\alpha \in \mathcal{V} \mid V_\alpha \not\subset X^f\}$. A Whitney stratification of $X \setminus X^f$ and such that for any pair of strata $(V_{\alpha}, V_{\beta})$ such that $V_{\alpha} \not\subset X^f$ and $V_{\beta} \subset X^f$, the $(\alpha_{i})$-Thom condition is satisfied. We call the strata included in $X^f$ the good strata.

By [18], given a stratification $\mathcal{S}$ of $X$, one can refine $\mathcal{S}$ to obtain a Whitney stratification $\mathcal{V}$ of $X$ which is adapted to $X^f$. Moreover, by [9, Theorem 4.3.2] (see
also [34]), the refinement \( \mathcal{V} \) satisfies the \( \alpha_f \)-Thom condition. This means that good stratifications always exist.

For instance, if \( \mathcal{V} \) is a Whitney stratification of \( X \) and \( f : X \to \mathbb{C} \) has a stratified isolated critical point, then
\[
\{ V_{\alpha} \setminus X^i, \ V_{\alpha} \cap X^i \setminus \{ 0 \}, \ {0} \mid \ V_{\alpha} \in \mathcal{V} \}
\]
is a good stratification for \( f \). We call it the good stratification induced by \( f \).

**Definition 2.12.** The critical locus of \( f \) relative to \( \mathcal{V} \), \( \Sigma_V f \), is defined by the union
\[
\Sigma_V f = \bigcup_{V_{\alpha} \in \mathcal{V}} \Sigma(f|_{V_{\alpha}}).
\]

Durterte and Grulha [13] defined the Brasselet number as follows.

**Definition 2.13.** Suppose that \( X \) is equidimensional. Let \( \mathcal{V} = \{ V_i \}_{i=0}^q \) be a good stratification of \( X \) relative to \( f \). The Brasselet number, \( B_{f,X}(0) \), is defined by
\[
B_{f,X}(0) = \sum_{i=1}^q \chi(V_i \cap B_{\varepsilon}(0) \cap f^{-1}(\delta)) \cdot \text{Eu}_X(V_i),
\]
where \( 0 < |\delta| \ll \varepsilon \ll 1 \).

In [13] the authors proved that the Brasselet number has a Lê-Greuel type formula, which relates this invariant with the number of Morse critical points. For present this property it is necessary the following definition.

**Definition 2.14.** Let \( \mathcal{V} \) be a good stratification of \( X \) relative to \( f \). We say that \( g : (X,0) \to (\mathbb{C},0) \) is prepolar with respect to \( \mathcal{V} \) at the origin if the origin is a stratified isolated critical point of \( g \).

Given \( f \) and \( g \) function-germs defined on \( (X,0) \), the \( (\alpha_f) \)-Thom condition in Definition 2.13 together with the hypothesis of \( g \) be prepolar guarantee that \( g : X \cap f^{-1}(\delta) \cap B_{\varepsilon} \to \mathbb{C} \) has no critical points on \( \{ g = 0 \} \) [23, Proposition 1.12] and so the number of stratified Morse critical points on the top stratum \( V_0 \cap f^{-1}(\delta) \cap B_{\varepsilon}(0) \) appearing in a morseification of \( g : X \cap f^{-1}(\delta) \cap B_{\varepsilon}(0) \to \mathbb{C} \) does not depend on the morsefication.

The following result shows that the Brasselet number satisfies a Lê-Greuel type formula [13, Theorem 4.4].

**Theorem 2.15.** Suppose that \( X \) is equidimensional and that \( g \) is prepolar with respect to \( \mathcal{V} \) at the origin. For \( 0 < |\delta| \ll \varepsilon \ll 1 \), we have
\[
B_{f,X}(0) - B_{f,X \circ g}(0) = (-1)^{d-1} m,
\]
where \( m \) is the number of stratified Morse critical points of a morseification of \( g : X \cap f^{-1}(\delta) \cap B_{\varepsilon} \to \mathbb{C} \) appearing on \( X_{\text{reg}} \cap f^{-1}(\delta) \cap \{ g \neq 0 \} \cap B_{\varepsilon} \). In particular, this number does not depend on the morsefication.

In [36, Theorem 3.2], Santana considered the case where the function \( g \) has a stratified singular set of dimension 1 and proved that in this case the difference of the Brasselet numbers \( B_{f,X}(0) \) and \( B_{f,X \circ g}(0) \) is still related with the number of
Morse critical points on the regular part of the Milnor fiber of $f$ appearing in a morseification of $g$. To prove this result the author considered that the function-germ $g$ is tractable.

The notion of tractability it be also important in our results, then in the sequence we present this concept. Aiming to define the notion of tractability, we need the following auxiliary definitions.

**Definition 2.16.** If $\mathcal{V} = \{V_\alpha\}$ is a stratification of $X$, the **symmetric relative polar variety** of $f$ and $g$ with respect to $\mathcal{V}$, $\tilde{\Gamma}_{f,g}(\mathcal{V})$, is the union $\bigcup_\lambda \tilde{\Gamma}_{f,g}(V_\lambda)$, where $\Gamma_{f,g}(V_\lambda)$ denotes the closure in $X$ of the critical locus of $(f,g)|_{V_\lambda \cap X'(\mathbb{C}^n)}$.

Using these varieties, we can introduce the notion of tractability.

**Definition 2.17.** A function $g : (X,0) \rightarrow (\mathbb{C},0)$ is **tractable at the origin with respect to a good stratification $\mathcal{V}$ of $X$ relative to $f : (X,0) \rightarrow (\mathbb{C},0)$** if the dimension of $\tilde{\Gamma}_{f,g}(\mathcal{V})$ is less or equal to 1 in a neighborhood of the origin and, for all strata $V_\alpha \subseteq X'$, $g|_{V_\alpha}$ has no critical points in a neighborhood of the origin except perhaps at the origin itself.

In the following we present Santana’s result.

**Theorem 2.18.** Suppose that $g$ is tractable at the origin with respect to $\mathcal{V}$ relative to $f$. Then, for $0 < |\delta| \ll \varepsilon \ll 1$,

$$B_{f,X}(0) - B_{f,X_0}(0) - \sum_{j=1}^{r} m_{f,b_j} \cdot (E_{u,X}(b_j) - E_{u,X_0}(b_j)) = (-1)^{d-1} m,$$

where $m$ is the number of stratified Morse critical points of a partial morseification of $g : X \cap f^{-1}(\delta) \cap B_\varepsilon \rightarrow \mathbb{C}$ appearing on $X_{\text{reg}} \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\varepsilon$, $\Sigma g = \{0\} \cup b_1 \cup \ldots \cup b_r$ and $m_{f,b_j}$ is the multiplicity of $f|_{b_j}$.

### 2.3. Toric Varieties

In what follow we present the definition of toric variety and the necessary background in order to state our results. As a reference for these definitions, we use [31].

Let $N \cong \mathbb{Z}^d$ be a $\mathbb{Z}$-lattice of rank $d$ and $\sigma$ a strongly convex rational polyhedral cone in $N_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} N$. We denote by $M$ the dual lattice of $N$ and the polar cone $\sigma^\circ$ of $\sigma$ in $M_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} M$ by

$$\sigma^\circ = \{ v \in M_\mathbb{R} \mid \langle u, v \rangle \geq 0 \text{ for any } u \in \sigma \},$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^d$. Then the dimension of $\sigma^\circ$ is $d$ and we obtain a semigroup $S_\sigma := \sigma^\circ \cap M$.

**Definition 2.19.** A $d$-dimensional affine Toric Variety $X_\sigma$ is defined by the spectrum of $\mathbb{C}[S_\sigma]$, i.e., $X = \text{Spec}(\mathbb{C}[S_\sigma])$.

There exists a natural action from the algebraic torus $T = \text{Spec}(\mathbb{C}[M]) \cong (\mathbb{C}^*)^d$ on $X_\sigma$, moreover the $T$-orbits of this action are in a 1-1 relation with the faces $\Delta$ of $\sigma (\Delta \lhd \sigma)$. Then, if $L(\Delta)$ is the smallest linear subspace of $M_\mathbb{R}$ containing $\Delta$, we denote by $T_\Delta$ the $T$-orbit in $\text{Spec}(\mathbb{C}[M \cap L(\Delta)])$ which corresponds to $\Delta \lhd \sigma$. The $d$-dimensional affine toric varieties are exactly those $d$-dimensional affine, normal
varieties admitting a $(\mathbb{C}^*)^d$-action with an open and dense orbit homeomorphic to $(\mathbb{C}^*)^d$. Besides, each $T$-orbit $T_\Delta$ is homeomorphic to $(\mathbb{C}^*)^r$, where $r$ is the dimension of $L(\Delta)$. Therefore, we obtain a decomposition $X_\sigma = \bigsqcup_{\Lambda \neq 0} T_\Lambda$ into $T$-orbits, which are homeomorphic to algebraic torus $(\mathbb{C}^*)^r$.

Since we study properties of the Brasselet number of function-germs on $X_\sigma$, let us introduce some concepts that will be necessary.

Consider $f : X_\sigma \to \mathbb{C}$ a polynomial function on $X_\sigma$, i.e., a function which corresponds to an element $f = \sum_{v \in S_\sigma} a_v \cdot v$ of $\mathbb{C}[S_\sigma]$, where $a_v \in \mathbb{C}$.

**Definition 2.20.** Let $f = \sum_{v \in S_\sigma} a_v \cdot v$ be a polynomial function on $X_\sigma$.

(a) The set $\{v \in S_\sigma; \ a_v \neq 0\} \subset S_\sigma$ is called the **support** of $f$ and we denote it by $\text{supp} f$.

(b) The **Newton polygon** $\Gamma_+(f)$ of $f$ is the convex hull of

$$\bigcup_{v \in \text{supp} f} (v + \sigma) \subset \sigma.$$

Now let us fix a function $f \in \mathbb{C}[S_\sigma]$ such that $0 \notin \text{supp} f$, i.e., $f : X_\sigma \to \mathbb{C}$ vanishes at the $T$-fixed point $0$. Considering $M(S_\sigma)$ the $\mathbb{Z}$-sublattice of rank $d$ in $M$ generated by $S_\sigma$, each element $v$ of $S_\sigma \subset M(S_\sigma)$ is identified with a $\mathbb{Z}$-vector $v = (v_1, \ldots, v_d)$ and to any $g = \sum_{v \in S_\sigma} b_v \cdot v \in \mathbb{C}[S_\sigma]$ we can associate a Laurent polynomial $L(g) = \sum_{v \in S_\sigma} b_v \cdot x^v$ on $T = (\mathbb{C}^*)^d$, where $x^v := x_1^{v_1} \cdot x_2^{v_2} \cdots x_d^{v_d}$.

**Definition 2.21.** A polynomial function $f = \sum_{v \in S_\sigma} a_v \cdot v \in \mathbb{C}[S_\sigma]$ is said to be **non-degenerate** if for any compact face $\nu$ of $\Gamma_+(f)$ the complex hypersurface

$$\{x = (x_1, \ldots, x_d) \in (\mathbb{C}^*)^d \mid L(f_\nu)(x) = 0\}$$

in $(\mathbb{C}^*)^d$ is smooth and reduced, where $f_\nu := \sum_{v \in \nu \cap S_\sigma} a_v \cdot v$.

It is possible to describe geometrical and topological properties of non-degenerate singularities by the combinatorics. This is done in [31] using mixed volume as follows.

For each face $\Delta \prec \sigma$ such that $\Gamma_+(f) \cap \Delta \neq \emptyset$, let $\beta_1^\Delta, \beta_2^\Delta, \ldots, \beta_{\dim(\Delta)}^\Delta$ be the compact faces of $\Gamma_+(f) \cap \Delta$ such that $\dim(\beta_i^\Delta) = \dim \Delta - 1$. Let $\Gamma_\Delta^\Delta$ be the convex hull of $\beta_i^\Delta \cup \{0\}$ in $L(\Delta)$ and consider the normalized $(\dim(\Delta))$-dimensional volume $\text{Vol}_{\mathbb{L}}(\Gamma_\Delta^\Delta) \in \mathbb{Z}$ of $\Gamma_\Delta^\Delta$ with respect to the lattice $M(S_\sigma \cap \Delta)$, where $M(S_\sigma \cap \Delta)$ denote the sublattice of $M(S_\sigma)$ generated by $S_\sigma \cap \Delta$.

Non-degeneracy can also be study in case of complete intersections defined on $X_\sigma$. Let $f_1, f_2, \ldots, f_k \in \mathbb{C}[S_\sigma]$ ($1 \leq k \leq d = \dim X_\sigma$) and consider the following subvarieties of $X_\sigma$:

$$V := \{f_1 = \cdots = f_{k-1} = f_k = 0\} \subset W := \{f_1 = \cdots = f_{k-1} = 0\}.$$

Assume that $0 \in V$. For each face $\Delta \prec \sigma$ such that $\Gamma_+(f_k) \cap \Delta \neq \emptyset$, we set

$$I(\Delta) = \{j = 1, 2, \ldots, k-1 \mid \Gamma_+(f_j) \cap \Delta \neq \emptyset\} \subset \{1, 2, \ldots, k-1\}$$

and $m(\Delta) = \sharp I(\Delta) + 1$.

Let $\mathbb{L}(\Delta)$ and $M(S_\sigma \cap \Delta)$ be as before and $\mathbb{L}(\Delta)^*$ the dual vector space of $\mathbb{L}(\Delta)$. Then $M(S_\sigma \cap \Delta)^*$ is naturally identified with a subset of $\mathbb{L}(\Delta)^*$ and the polar cone
\[ \tilde{\Delta} = \{ u \in L(\Delta)^* \mid \langle u, v \rangle \geq 0 \text{ for any } v \in \Delta \} \text{ of } \Delta \text{ in } L(\Delta)^* \text{ is a rational polyhedral convex cone with respect to the lattice } M(S_\sigma \cap \Delta)^* \text{ in } L(\Delta)^*. \]

**Definition 2.22.** (i) For a function \( f = \sum_{v \in \Gamma_+(f) \cap \Delta} a_v \cdot v \in \mathbb{C}[S_\sigma] \) on \( X_\sigma \) and \( u \in \tilde{\Delta} \), we set

\[
\Gamma(f); u = \{ v \in \Gamma_+(f) \cap \Delta \mid \langle u, v \rangle = \min \langle u, w \rangle, \text{ for } w \in \Gamma_+(f) \cap \Delta \}.
\]

We call \( \Gamma(f); u \) the supporting face of \( u \) in \( \Gamma_+(f) \cap \Delta \).

(ii) For \( j \in I(\Delta) \cup \{ k \} \) and \( u \in \tilde{\Delta} \), we define the \( u \)-part \( f^u_j \in \mathbb{C}[S_\sigma \cap \Delta] \) of \( f_j \) by

\[
f^u_j = \sum_{v \in \Gamma(f); u} a_v \cdot v \in \mathbb{C}[S_\sigma \cap \Delta],
\]

where \( f_j = \sum_{v \in \Gamma_+(f_j) \cap \Sigma} a_v \cdot v \in \mathbb{C}[S_\sigma] \).

By taking a \( \mathbb{Z} \)-basis of \( M(S_\sigma) \) and identifying the \( u \)-parts \( f^u_j \) with Laurent polynomials \( L(f^u_j) \) on \( T = (\mathbb{C}^*)^d \) as before, we have the following definition which does not depend on the choice of the \( \mathbb{Z} \)-basis of \( M(S_\sigma) \).

**Definition 2.23.** We say that \( (f_1, \ldots, f_k) \) is non-degenerate if for any face \( \Delta < \sigma \) such that \( \Gamma_+(f_k) \cap \Delta \neq \emptyset \) (including the case where \( \Delta = \sigma \)) and any \( u \in \text{Int}(\Delta) \cap M(S_\sigma \cap \Delta)^* \) the following two subvarieties of \( (\mathbb{C}^*)^d \) are non-degenerate complete intersections

\[
\left\{ x \in (\mathbb{C}^*)^d \mid L(f^u_j)(x) = 0, \forall j \in I(\Delta) \right\}; \left\{ x \in (\mathbb{C}^*)^d \mid L(f^u_j)(x) = 0, \forall j \in I(\Delta) \cup \{ k \} \right\}.
\]

For each face \( \Delta < \sigma \) of \( \sigma \) such that \( \Gamma_+(f_k) \cap \Delta \neq \emptyset \), let us set

\[
f_\Delta = \left( \prod_{j \in I(\Delta)} f_j \right) \cdot f_k \in \mathbb{C}[S_\sigma]
\]

and consider its Newton polygon \( \Gamma_+(f_\Delta) = \left\{ \sum_{j \in I(\Delta)} \Gamma_+(f_j) \right\} + \Gamma_+(f_k) \subset \sigma \). Let \( \gamma^1_\Delta, \ldots, \gamma^\ell_\Delta \) be the compact faces of \( \Gamma_+(f_\Delta) \cap \Delta(\neq \emptyset) \) such that \( \dim \gamma^1_\Delta = \dim \Delta - 1 \). Then for each \( 1 \leq i \leq \nu(\Delta) \) there exists an unique primitive vector \( u^1_i \in \text{Int}(\Delta) \cap M(S_\sigma \cap \Delta)^* \) which takes its minimal in \( \Gamma_+(f_k) \cap \Delta \) exactly on \( \gamma^1_i \).

For \( j \in I(\Delta) \cup \{ k \} \), set \( \gamma(f_j)^\Delta_i = \Gamma(f_j) \cap \Delta \) and \( d_i = \min_{w \in \Gamma_+(f_k) \cap \Delta} \langle u^1_i, w \rangle \). Note that we have

\[
\gamma(f_j)^\Delta_i = \sum_{j \in I(\Delta) \cup \{ k \}} \gamma(f_j)^\Delta_i
\]

for any face \( \Delta < \sigma \) such that \( \Gamma_+(f_k) \cap \Delta \neq \emptyset \) and \( 1 \leq i \leq \nu(\Delta) \). For each face \( \Delta < \sigma \) such that \( \Gamma_+(f_k) \cap \Delta \neq \emptyset \), \( \dim \Delta > m(\Delta) \), and \( 1 \leq i \leq \nu(\Delta) \), we set

\[
I(\Delta) \cup \{ k \} = \{ j_1, j_2, \ldots, j_{m(\Delta)-1}, k = j_{m(\Delta)} \}
\]

and

\[
K^\Delta_i := \sum_{a_{\Delta} \geq 0, a_{\Delta} \geq 1 \text{ for } a_{\Delta} \leq m(\Delta)-1} \text{Vol}_\Delta(\gamma(f_j)^\Delta_i, \ldots, \gamma(f_j)^\Delta_i, \gamma(f_j)^\Delta_i, \gamma(f_j)^\Delta_i),
\]

\[\alpha_1 \text{-times} \frac{\alpha_1 \text{-times}}{\alpha_m(\Delta) \text{-times}}\]
Here
\[ \text{Vol}_Z(\gamma(f_1)_{\alpha_1}^\Delta, \ldots, \gamma(f_j)_{\alpha_j}^\Delta, \ldots, \gamma(f_m(\Delta))_{\alpha_m(\Delta)}^\Delta) \]

is the normalized \((\dim \Delta - 1)\)-dimensional mixed volume with respect to the lattice
\( M(S_\alpha \cap \Delta) \cap L(\gamma^\Delta) \) (see Definition 2.6, pg 205 from [17]). For \( \Delta \) such that \( \dim \Delta - 1 = 0 \), we set
\[ K_\Delta^\Delta = \text{Vol}_Z(\gamma(f_k)_{\alpha_k}^\Delta, \ldots, \gamma(f_k)_{\alpha_k}^\Delta) := 1 \]

(in this case \( \gamma(f_k)_{\alpha_k}^\Delta \) is a point).

The class of non-degenerate singularity is open and dense when the Newton boundary (the union of the compact faces of the Newton polygon) is fixed.

3. **Good stratifications induced for non-degenerate functions**

From now on, let us denote by \( X_\sigma \) a sufficiently small representative of the germ of toric variety \((X_\sigma, 0)\), and by \( f \) a representative of the function-germ \( f : (X_\sigma, 0) \to (\mathbb{C}, 0) \).

In order to have a Santana’s type result we need to find suitable good stratifications of the representatives of \( X_\sigma \) and of \( X_\sigma^g \) near the origin without any hypothesis on the dimension of the singular set of \( f \) or \( g \). In the context of Toric Varieties, we present such suitable good stratifications in the next lemmas.

Let \( \mathcal{T} = \{ T_\Delta \}_{\Delta \in 0} \) be the Whitney stratification of \( X_\sigma \) induced by the torus action orbits. Given \( f : X_\sigma \to \mathbb{C} \) a non-degenerate function we denote by \( T_{\Delta_i} \) a stratum of \( \mathcal{T} \) contained in \( \Sigma_T f \).

**Lemma 3.1.** There exists a refinement for strata \( T_{\Delta_i} \subset \Sigma_T f \) which provides a refinement \( \mathcal{V}_i \) of \( \mathcal{T} \),
\[ \mathcal{V}_i = \left\{ T_\Delta \setminus \{ f = 0 \}, \quad T_\Delta \cap \{ f = 0 \} \setminus \Sigma_T f, \quad T_{\Delta_i} = \cup W_i \subset \Sigma_T f \right\} \]

which is a good stratification of \( X_\sigma \) relative to \( f \).

**Proof.** It is sufficient to prove that \( \mathcal{V}_i \) is a Whitney stratification of \( X_\sigma \).

Consider a pair of strata \((V_{\alpha}, V_{\beta})\) such that \( V_{\beta} \subset V_{\alpha} \). For a pair of strata of type \((T_{\Delta_1} \setminus \{ f = 0 \}, \ T_{\Delta_2} \setminus \{ f = 0 \})\) the Whitney’s condition (b) is valid, since \( T_{\Delta_1} \setminus \{ f = 0 \} \) is an open subset of \( T_{\Delta_1}, i = 1, 2 \), and \( (T_{\Delta_1}, T_{\Delta_2}) \) is Whitney regular. A pair of strata \((T_{\Delta_1} \setminus \{ f = 0 \}, \ T_{\Delta_2} \cap \{ f = 0 \} \setminus \Sigma_T f)\) is also Whitney regular if \( (T_{\Delta_1}, T_{\Delta_2}) \) is Whitney regular, since \( \{ f = 0 \} \setminus \Sigma_T f \) intersects \( T_{\Delta_2} \) transversely. For \( (T_{\Delta_1} \setminus \{ f = 0 \}, T_{\Delta_i}), (T_{\Delta_i} \subset \Sigma_T f, \) the Whitney’s condition (b) is also valid, since \( T_{\Delta_1} \setminus \{ f = 0 \} \) is an open subset of \( T_{\Delta_1} \) and \( (T_{\Delta_1}, T_{\Delta_2}) \) is Whitney regular. The pair \((T_{\Delta_1} \cap \{ f = 0 \} \setminus \Sigma_T f, T_{\Delta_2} \cap \{ f = 0 \} \setminus \Sigma_T f)\) is Whitney regular, since \( \{ f = 0 \} \setminus \Sigma_T f \) intersects \( T_{\Delta_1}, i = 1, 2 \), transversely and \( (T_{\Delta_1}, T_{\Delta_2}) \) is Whitney regular. The pair \((T_{\Delta_i} \subset \Sigma_T f, T_{\Delta_i} \subset \Sigma_T f)\) is, by hypothesis, Whitney regular. At last, the pair \((T_{\Delta} \cap \{ f = 0 \} \setminus \Sigma_T f, \ T_{\Delta_i} \subset \Sigma_T f)\) may not satisfied the Whitney’s condition (b). Since Whitney stratification always exist, one can refine \( T_{\Delta_i} \subset \Sigma_T f \) in such a way that each pair of type \((T_{\Delta} \cap \{ f = 0 \} \setminus \Sigma_T f, \ T_{\Delta_i} \subset \Sigma_T f)\) is Whitney regular and the result follows. \( \Box \)
Remark 3.2. If we consider \( f, g : X_\sigma \to \mathbb{C} \) non-degenerate functions such that \( \Sigma_T g \cap \{ f = 0 \} = \{ 0 \} \), then \( \Sigma_V, g = \cup \Delta \setminus \{ f = 0 \} \) where \( \Delta \) denotes faces of \( \bar{\sigma} \) satisfying \( \Gamma_{\sigma}(g) \cap \Delta = \emptyset \).

Let \( f, g : X_\sigma \to \mathbb{C} \) be non-degenerate functions. Applying Lemma 3.1 for \( g \), one obtains a good stratification of \( X_\sigma \) relative to \( g \)

\[
\mathcal{V}_g = \{ \Delta \setminus \{ g = 0 \}, \Delta \cap \{ g = 0 \} \setminus \Sigma_T g, \Delta_\sigma = \cup_i W_i^0 \subseteq \Sigma_T g \},
\]

where \( \cup_i W_i^0 \) is a refinement of \( \Delta_\sigma \) which guarantees the Whitney regularity of the pair of strata \( (\Delta | \{ g = 0 \} \setminus \Sigma_T g, W_i^0) \). But to state our main result we need more than that. The next lemma is a key-condition result for our main theorem.

Lemma 3.3. Suppose that \( g \) is tractable at the origin with respect to \( \mathcal{V}_i \) relative to \( f \) and that \( \Sigma_T g \cap \{ f = 0 \} = \{ 0 \} \). Then the refinement of \( \mathcal{V}_i \),

\[
\mathcal{V}' = \{ \mathcal{V}_i \cap \{ g = 0 \} \setminus \Sigma_T g, W_i^0 \cap \mathcal{V}_i, \Delta_\sigma = \cup_i W_i^0 \subseteq \Sigma_T g \mid \mathcal{V}_i \in \mathcal{V} \}
\]

is a good stratification of \( \mathcal{X}_{\sigma,g} \) relative to \( f|_{\mathcal{X}_g} \).\]

Proof. Since \( g \) is tractable at the origin with respect to \( \mathcal{V} \) relative to \( f \), for each strata \( \mathcal{X}_x \subseteq \mathcal{X}_\sigma \), \( g_{\mid \mathcal{X}_x} \) has an isolated singularity at the origin or is non-singular. Hence, \( W_i^0 \cap \{ f = 0 \} \setminus \Sigma_T f = \{ 0 \} \). Also, since \( \Sigma_T f \cap \{ f = 0 \} = \{ 0 \} \), we have \( \Sigma_T f \cap \Sigma_T g = \{ 0 \} \). So,

\[
\mathcal{V}' = \{ \Delta \cap \{ g = 0 \} \setminus \{ f = 0 \} \cup \Sigma_T g, \Delta \cap \{ f = 0 \} \cap \{ g = 0 \} \setminus \{ f = 0 \} \cup \Sigma_T g, W_i^0 \cap \{ f = 0 \}, W_i^0 \subseteq \Sigma_T g, W_i^0 \subseteq \Sigma_T f \mid \Delta \in \mathcal{T} \}.
\]

Notice that the sets in \( \mathcal{V}' \) are smooth varieties. First, \( \{ g = 0 \} \setminus \Sigma_W g \) intersects the open subset \( \Delta \setminus \{ f = 0 \} \subset \Delta \) transversely. On the other hand, since \( g \) is tractable at the origin with respect to \( \mathcal{V}_i \) relative to \( f \), \( \Delta \cap \{ f = 0 \} \cap \{ g = 0 \} \setminus \{ f = 0 \} \cup \Sigma_T f \) and \( W_i^0 \cap \{ g = 0 \} \setminus \{ f = 0 \} \subseteq \Sigma_T f \) are smooth subvarieties. Therefore, the sets in \( \mathcal{V}' \) are strata.

Since \( \Sigma_W g \cap \{ f = 0 \} = \{ 0 \} \), \( \{ X_0 \}^f \) is union of strata of type \( \Delta \cap \{ g = 0 \} \cap \{ f = 0 \} \setminus \Sigma_W f \cup \Sigma_T g \) and \( W_i^0 \cap \{ f = 0 \} \setminus \Sigma_W f \cup \Sigma_T g \).

Let us verify that the collection \( \{ \Delta \cap \{ g = 0 \} \setminus \{ f = 0 \} \cup \Sigma_W g, W_i^0 \setminus \{ f = 0 \} \} \) is a Whitney stratification of \( X_0^f \setminus \{ f = 0 \} \). The pair

\[
(\Delta_{\sigma, i} \cap \{ g = 0 \} \setminus \{ f = 0 \} \cup \Sigma_T g, \Delta_2 \cap \{ g = 0 \} \cap \{ f = 0 \} \cup \Sigma_T g)
\]

is Whitney regular, since \( \{ g = 0 \} \setminus \Sigma_T g \) intersects the open subset \( \Delta_{\sigma, i} \setminus \{ f = 0 \} \) transversely, \( i = 1, 2 \), and \( (\Delta_{\sigma, i}, \Delta_2) \) is Whitney regular. The pair \( \{ W_i^0 \setminus \{ f = 0 \} \}, W_i^0 \setminus \{ f = 0 \} \) is Whitney regular, since \( W_i^0 \setminus \{ f = 0 \} \) is an open subset of \( W_i^0 \), \( i = 1, 2 \), and \( (W_1^0, W_2^0) \) is Whitney regular. Since

\[
\Delta \cap \{ g = 0 \} \setminus \{ f = 0 \} \cup \Sigma_T g \not\subseteq W_i^0 \setminus \{ f = 0 \} \subseteq \Sigma_T g,
\]

we need to verify the pair

\[
(\Delta \cap \{ g = 0 \} \setminus \{ f = 0 \} \cup \Sigma_T g, W_i^0 \setminus \{ f = 0 \}) \quad \text{where} \quad W_i^0 \subseteq \Sigma_T g.
\]

This pair is Whitney regular, since \( \Delta \cap \{ g = 0 \} \setminus \{ f = 0 \} \cup \Sigma_T g \) and \( W_i^0 \setminus \{ f = 0 \} \) are open subsets of \( \Delta \cap \{ g = 0 \} \setminus \Sigma_T g \) and \( W_i^0 \), respectively, and \( (\Delta \cap \{ g = 0 \} \setminus \Sigma_T g, W_i^0) \) is Whitney regular.
Now we verify the Thom conditions. Let $\tilde{f}$ and $\tilde{g}$ be analytic extensions of $f$ and $g$, respectively, in a neighborhood of $p$.

1. Let $V_\gamma = \Delta \cap \{ f = 0 \} \setminus \Sigma_T f \cup \Sigma_T g$. Since $p \not\in \Sigma_T f$ and $p \not\in \Sigma_T g$,
   
   $T_p V_\gamma = T_p \Delta \cap T_p V(\tilde{g}) \cap T_p V(\tilde{f})$.

   (a) Let $V_\lambda = \Delta_{\lambda} \cap \{ f = 0 \} \setminus \Sigma_T g$. Since $p \not\in \{ f = 0 \}$,
   
   $T_p V(f|_{V_\lambda} - f|_{V_\lambda}) = T_p V(\tilde{f} - \tilde{f}(p_1)) \cap T_p (\Delta_{\lambda} \cap \{ g = 0 \} \setminus \{ f = 0 \} \cup \Sigma_T g)$
   
   $= T_p V(\tilde{f} - \tilde{f}(p_1)) \cap T_p (\Delta_{\lambda} \setminus \{ f = 0 \}) \cap T_p V(\tilde{g})$
   
   $= T_p V(\tilde{f} - \tilde{f}(p_1)) \cap T_p (\Delta_{\lambda}) \cap T_p V(\tilde{g})$.

   Since $p \not\in \Sigma_T f$, we have that $\lim_{p_1 \to p} T_p V(\tilde{f} - \tilde{f}(p_1))$ exists and it is equal to $T_p V(\tilde{f})$. Analogously, since $p \not\in \Sigma_T g$, we obtain that $\lim_{p_1 \to p} T_p V(\tilde{g}) = T_p V(\tilde{g})$. Also, since $(\Delta_{\lambda}, \Delta)$ is Whitney regular, we have the inclusion $T_p \Delta \subset \lim_{p_1 \to p} T_p \Delta_{\lambda}$. Hence
   
   $T_p V_\gamma = T_p \Delta \cap T_p V(\tilde{f}) \cap T_p V(\tilde{g}) \subseteq \lim_{p_1 \to p} T_p \Delta_{\lambda} \cap T_p V(\tilde{f}) \cap T_p V(\tilde{g})$
   
   $= \lim_{p_1 \to p} T_p V(f|_{V_\lambda} - f|_{V_\lambda}(p_1))$, where the last equality is justified by $g$ be tractable at the origin with respect to $V_\lambda$ and, therefore, $T_p V(\tilde{g})$ intersects $T_p (\Delta \cap \{ f = 0 \} \setminus \Sigma_T f) = T_p \Delta \cap T_p V(\tilde{f})$ transversely.

   (b) Let $V_\lambda = W_{\lambda}^g \setminus \{ f = 0 \}$, $W_{\lambda}^g \in \Sigma_T g$. Since $p \not\in \{ f = 0 \}$,
   
   $T_p V(f|_{V_\lambda} - f|_{V_\lambda}) = T_p V(\tilde{f} - \tilde{f}(p_1)) \cap T_p (W_{\lambda}^g \setminus \{ f = 0 \})$
   
   $= T_p V(\tilde{f} - \tilde{f}(p_1)) \cap T_p (W_{\lambda}^g)$. Since $p \not\in \Sigma_T f$, we have the equality $\lim_{p_1 \to p} T_p V(\tilde{f} - \tilde{f}(p_1)) = T_p V(\tilde{f})$. Moreover, since $\Delta \cap \{ g = 0 \} \setminus \Sigma_T g$, $W_{\lambda}^g$ is Whitney regular,
   
   $T_p (\Delta \cap \{ g = 0 \} \setminus \Sigma_T g) \subset \lim_{p_1 \to p} T_p W_{\lambda}^g$.

   Hence,
   
   $T_p V_\gamma = T_p \Delta \cap T_p V(\tilde{f}) \cap T_p V(\tilde{g}) \subseteq \lim_{p_1 \to p} T_p W_{\lambda}^g \cap \lim_{p_1 \to p} V(\tilde{f} - \tilde{f}(p_1))$
   
   $= \lim_{p_1 \to p} T_p V(f|_{V_\lambda} - f|_{V_\lambda}(p_1))$. Since $g$ is tractable at the origin with respect to $V_\lambda$, $T_p V(\tilde{g})$ intersects $T_p (\Delta \cap V(\tilde{f}))$ transversely. Then, $T_p V(\tilde{f}) = \lim_{p_1 \to p} V(\tilde{f} - \tilde{f}(p_1))$ intersects $\lim_{p_1 \to p} T_p W_{\lambda}^g$ transversely, which justifies the last equality.
(2) Let $V_f = W_f^i \cap \{g = 0\} \setminus \Sigma_T g$, where $W_f^i \in \Sigma_T f \subset \{f = 0\}$. Since $g$ is tractable at the origin with respect to $V_f$ relative to $f$, $\{g = 0\} \setminus \Sigma_T g$ intersects $\{f = 0\}$ transversely, so

$$T_p V_f = T_p (W_f^i \cap \{g = 0\} \setminus \Sigma_T g) = T_p W_f^i \cap T_p V(\tilde{g}).$$

(a) Let $V_\lambda = T_{\Delta_1} \cap \{g = 0\} \setminus \Sigma_T g \cup \{f = 0\}$. As we have seen,

$$\lim_{p_i \to p} T_{p_i} V(f|_{V_\lambda} - f|_{V_\lambda}(p_i)) = \lim_{p_i \to p} T_{p_i} V(\tilde{f} - \tilde{f}(p_i)) \cap T_{p_i}(T_{\Delta_1}) \cap T_{p_i} V(\tilde{g}).$$

Since $V_f$ is a good stratification of $X_\sigma$ relative to $f$,

$$T_p W_f^i \subseteq \lim_{p_i \to p} V(f|_{T_{\Delta_1} \setminus \{f = 0\}} - f|_{T_{\Delta_1} \setminus \{f = 0\}}(p_i)) = \lim_{p_i \to p} T_{p_i} V(\tilde{f} - \tilde{f}(p_i)) \cap T_{p_i}(T_{\Delta_1} \setminus \{f = 0\}) = \lim_{p_i \to p} T_{p_i} V(\tilde{f} - \tilde{f}(p_i)) \cap T_{p_i}(T_{\Delta_1}).$$

Nevertheless, since $p \notin \Sigma_T g$, $\lim_{p_i \to p} V(\tilde{g}) = T_p V(\tilde{g})$. Then,

$$T_p V_f = T_p W_f^i \cap T_p V(\tilde{g}) \subseteq \lim_{p_i \to p} T_{p_i} V(\tilde{f} - \tilde{f}(p_i)) \cap T_{p_i}(T_{\Delta_1}) \cap T_p V(\tilde{g}) = \lim_{p_i \to p} T_{p_i} V(\tilde{f} - \tilde{f}(p_i)) \cap T_{p_i}(T_{\Delta_1}) \cap T_{p_i} V(\tilde{g}) = \lim_{p_i \to p} T_{p_i} V(f|_{V_\lambda} - f|_{V_\lambda}(p_i)).$$

(b) Since $V_f \subset \nabla_\lambda$, the case where $V_\lambda = W_f^i \setminus \{f = 0\} \subset \Sigma_T g$ does not happen.

Using Remark 3.2 and Lemma 3.3 we compute the difference $B_{f, X_\sigma}(0) - B_{f, X^g_\sigma}(0)$ without the hypothesis about the dimension of $\Sigma_T f$ or $\Sigma_T g$.

**Theorem 3.4.** Let $X_\sigma \subset \mathbb{C}^n$ be a $d$-dimensional toric variety and $g, f : X_\sigma \to \mathbb{C}$ non-degenerate functions. Suppose that $g$ is tractable at the origin with respect to $V_f$ relative to $f$. Then, for $0 < |\delta| \ll \varepsilon \ll 1$,

$$B_{f, X_\sigma}(0) - B_{f, X^g_\sigma}(0) - \sum_{i=1}^r C_{f, W_i}(\text{Eu}_{X_\sigma}(w_i) - \text{Eu}_{X^g_\sigma}(w_i)) = (-1)^{d-1} m \quad (3.1)$$

where $m$ is the number of stratified Morse critical points of a partial morseification of $g : X_\sigma \cap f^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$ appearing on $X_{\text{reg}} \cap f^{-1}(\delta) \setminus \{g \neq 0\} \cap B_\varepsilon$ and $C_{f, W_i} = \chi(W_i^g \cap f^{-1}(\delta) \cap B_\varepsilon)$.

**Proof.** By [13, Corollary 4.3], for $0 < |\delta| \ll \varepsilon \ll 1$ we have

$$B_{f, X_\sigma}(0) - \sum_{V_i \subset V_f} \chi(V_i \cap X^g_\sigma \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot \text{Eu}_{X_\sigma}(V_i) = (-1)^{d-1} m.$$
If $V_i \not\subseteq \Sigma_{V_i} g$, $V_i$ intersects $\{g = 0\}$ transversely and $E_{X_\sigma}(V_i) = E_{X_\sigma}^g(V_i \cap g^{-1}(0))$, then
\[
\sum_{V_i \subset V_f} \chi(V_i \cap X^g_\sigma \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot E_{X_\sigma}(V_i) = \sum_{V_i \not\subseteq \Sigma_{V_i} g} \chi(V_i^g \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot E_{X_\sigma}^g(V_i^g) + \sum_{l=1}^q \chi(V_l^g \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot E_{X_\sigma}(V_l),
\]
where $V_i^g$ denotes the intersection $V_i \cap g^{-1}(0)$. From Remark 3.2, if $V_i \subset \Sigma_{V_i} g$ then $V_i = T^1_{\Delta_\sigma} \setminus \{f = 0\}$. Therefore $V_i$ can be decomposed as
\[
T^1_{\Delta_\sigma} \setminus \{f = 0\} = \bigcup_{t=1}^{k_{A_\sigma}} W^t_t.
\]
Then,
\[
\chi(T^1_{\Delta_\sigma} \cap g^{-1}(0) \cap f^{-1}(\delta) \cap B_\varepsilon) = \sum_{t=1}^{k_{A_\sigma}} \chi(W^t_t \cap f^{-1}(\delta) \cap B_\varepsilon) = \sum_{t=1}^{k_{A_\sigma}} C_{f, W_t},
\]
and we obtain
\[
\sum_{V_i \subset V_f} \chi(V_i \cap X^g_\sigma \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot E_{X_\sigma}(V_i) = \sum_{V_i \not\subseteq \Sigma_{V_i} g} \chi(V_i^g \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot E_{X_\sigma}^g(V_i^g) + \sum_{l=1}^q \sum_{t=1}^{k_{A_\sigma}} C_{f, W_t} \cdot E_{X_\sigma}(w_t).
\]
On the other hand,
\[
B_{f, X_\sigma^g}(0) = \sum_{V_i \not\subseteq \Sigma_{V_i} g} \chi(V_i^g \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot E_{X_\sigma}^g(V_i^g) + \sum_{l=1}^q \sum_{t=1}^{k_{A_\sigma}} C_{f, W_t} \cdot E_{X_\sigma}(w_t).
\]
Therefore,
\[
\sum_{V_i \subset V_f} \chi(V_i \cap X^g_\sigma \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot E_{X_\sigma}(V_i) = B_{f, X_\sigma^g}(0) - \sum_{l=1}^r C_{f, W_l}(E_{X_\sigma}(w_l) - E_{X_\sigma}(w_l)).
\]

4. Euler obstruction, Morse points and torus action

Using Theorem 3.4, we present formulas to compute the Brasselet number of a function defined on a non-degenerate hypersurface.

Let $X_\sigma \subset \mathbb{C}^n$ be a $d$-dimensional toric variety and $g, f : X_\sigma \to \mathbb{C}$ non-degenerate functions. Let $\mathcal{T}$ be the decomposition of $X_\sigma = \bigsqcup_{\Delta < \delta} T_\Delta$ into $T$-orbits, and $\mathcal{T}_g$ the decomposition of $X^g_\sigma = \bigsqcup_{\Delta < \delta} T_\Delta \cap X^g_\sigma$. Since $g$ is a non-degenerate function, $\mathcal{T}_g$ is a decomposition of $X^g_\sigma$ into smooth subvarieties [31, Lemma 4.1].
In the next results we work in the situation where $T_g$ is a Whitney stratification.

**Proposition 4.1.** Let $X_\sigma \subset \mathbb{C}^n$ be a $d$-dimensional toric variety and $g, f : X_\sigma \to \mathbb{C}$ non-degenerate functions. Suppose that $g$ is tractable at the origin with respect to $V_i$ relative to $f$. If $T_g$ is a Whitney stratification, then

$$B_{f, X_\sigma^g}(0) - B_{f, X_\sigma^g}(0) - \sum_{g_+ (g) \cap \Delta = \emptyset} C_{f, T_\Delta \cap X_\sigma^g} \cdot (\text{Eu}_{X_\sigma^g}(T_\Delta \cap X_\sigma^g) - \text{Eu}_{X_\sigma^g}(T_\Delta \cap X_\sigma^g)) = (-1)^{d-1} m,$$

where $m$ is the number of stratified Morse critical points of a partial morseification of $g : X_\sigma \cap f^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$ appearing on $X_{\sigma, reg} \cap f^{-1}(\delta) \cap \{g \neq 0\} \cap B_\varepsilon$ and

$$C_{f, T_\Delta \cap X_\sigma^g} = \chi(T_\Delta \cap f^{-1}(\delta) \cap B_\varepsilon) = (-1)^{\dim \Delta - 1} \sum_{i=1}^{\mu(\Delta)} \text{Vol}_\mathbb{Z}(\Gamma_i^\Delta).$$

**Proof.** As $T_g$ is a Whitney stratification, we do not need refine the strata $T_{\Delta_\sigma} \subset \Sigma_T g$ in the stratification $\mathbb{V}'$ that appear in Lemma 3.3. Therefore, denoting the intersections $V_i \cap g^{-1}(0)$ and $T_\Delta \cap g^{-1}(0)$ by $V_i^g$ and $T_\Delta^g$, respectively, we have

$$\sum_{V_i \subset V'_i} \chi(V_i^g \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot \text{Eu}_{X_\sigma^g}(V_i) = \sum_{V_i \not\subset V'_i} \chi(V_i^g \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot \text{Eu}_{X_\sigma^g}(V_i^g)$$

$$+ \sum_{g_+ (g) \cap \Delta = \emptyset} \chi(T_\Delta^g \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot \text{Eu}_{X_\sigma^g}(T_\Delta^g),$$

and

$$B_{f, X_\sigma^g}(0) = \sum_{V_i \not\subset V'_i} \chi(V_i^g \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot \text{Eu}_{X_\sigma^g}(V_i^g)$$

$$+ \sum_{g_+ (g) \cap \Delta = \emptyset} \chi(T_\Delta^g \cap f^{-1}(\delta) \cap B_\varepsilon) \cdot \text{Eu}_{X_\sigma^g}(T_\Delta^g).$$

Moreover, if $g_+ (g) \cap \Delta = \emptyset$ then $T_\Delta \subset X_\sigma^g$, since $g$ is non-degenerate. \qed

Let $l : \mathbb{C}^n \to \mathbb{C}$ be a generic linear form which is generic with respect to $X_\sigma$, then $V_l$ is a good stratification of $X_\sigma$ induced by $l$ (see [36, Lemma 3.3]). Therefore, Lemma 3.3 can be applied and we obtain a good stratification $V_l$ of $X_\sigma$ relative to $l$ such that $\mathbb{V}'$ is a good stratification of $X_\sigma^g$ relative to $l|_{X_\sigma^g}$.

**Remark 4.2.** Let $X_\sigma \subset \mathbb{C}^n$ be a $d$-dimensional toric variety. If $l$ is a generic linear form with respect to $X_\sigma$, $l$ does not have singularities in $X_\sigma$ out of the origin, in particular, $l$ does not have singularities on the dense orbit, which is diffeomorphic to $(\mathbb{C}^*)^d$. Hence $l$ is a non-degenerate function.

**Corollary 4.3.** Let $X_\sigma \subset \mathbb{C}^n$ be a $d$-dimensional toric variety, $g : (X_\sigma, 0) \to (\mathbb{C}, 0)$ a non-degenerate function and $l$ a generic linear form on $\mathbb{C}^n$. Suppose that $g$ is tractable at the origin with respect to $V_l$ relative to $l$. If $T_g$ is a Whitney stratification, then

$$\text{Eu}_{X_\sigma^g}(0) - \text{Eu}_{X_\sigma^g}(0) - \sum_{g_+ (g) \cap \Delta = \emptyset} C_{l, T_\Delta \cap X_\sigma^g} \cdot (\text{Eu}_{X_\sigma^g}(T_\Delta) - \text{Eu}_{X_\sigma^g}(T_\Delta)) = (-1)^{d-1} m,$$

where and $m$ is the number of stratified Morse critical points of a partial morseification of $g : X_\sigma \cap l^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$ appearing on $X_{\sigma, reg} \cap l^{-1}(\delta) \cap \{g \neq 0\} \cap B_\varepsilon$. and
Corollary 4.4. Let $X_\sigma = \mathbb{C}^n$ be a $n$-dimensional toric variety, $g : (X_\sigma, 0) \to (\mathbb{C}, 0)$ a non-degenerate function and $l$ a generic linear form on $\mathbb{C}^n$. Suppose that $g$ is tractable at the origin with respect to $\mathcal{V}_l$ relative to $l$. If $T_g$ is a Whitney stratification, then

$$\text{Eu}_{X_\sigma}^g(0) + \sum_{\gamma_{\pm}(g) \cap \Delta = \emptyset} (-1)^{\dim \Delta} \cdot \text{Eu}_{X_\sigma}^g(T_\Delta) = (-1)^n m + \sum_{\gamma_{\pm}(g) \cap \Delta = \emptyset} (-1)^{\dim \Delta} + 1,$$

where and $m$ is the number of stratified Morse critical points of a partial morsefication of $g : X_\sigma \cap l^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$ appearing on $X_{\sigma \text{reg}} \cap l^{-1}(\delta) \cap \{g \neq 0\} \cap B_\varepsilon$.

Proof. As $X_\sigma = \mathbb{C}^n$, the semigroup has exactly $n$ generators, more precisely $S_\sigma = \langle e_1^*, \ldots, e_n^* \rangle$. From this fact and the fact that $l$ is generic, for each face $\Delta$ such that $\gamma_{\pm}(g) \cap \Delta = \emptyset$ there exists exactly one compact face of $\gamma_{\pm}(1) \cap \Delta$. Then

$$C_{l,T_\Delta \cap X_\sigma^g} = \chi(T_\Delta \cap l^{-1}(\delta) \cap B_\varepsilon) = (-1)^{\dim \Delta - 1} \sum_{i=1}^{\mu(\Delta)} \text{Vol}_Z(\Gamma_i^\Delta),$$

to finish, remember that $l$ is linear, therefore $\text{Vol}_Z(\Gamma_i^\Delta) = 1$. □

As a consequence of Corollary 4.4, we have that if $X_\sigma = \mathbb{C}^n$ is a $n$-dimensional toric variety then the number of stratified Morse critical points is determined by the Euler obstruction of the hypersurface $X_\sigma^g$ and by the combinatorial of the torus action. However, this fact also happens in the case where $X_\sigma$ not necessarily is smooth. But, in this situation we need to consider the case where $(g, f) : (X_\sigma, 0) \to (\mathbb{C}^2, 0)$ is a non-degenerate complete intersection. More precisely, if $(g, f) : (X_\sigma, 0) \to (\mathbb{C}^2, 0)$ is a non-degenerate complete intersection and $T_g$ is a Whitney stratification, then by [10, Theorem 3.2]

$$B_{l,X_\sigma^g}(0) = \sum_{\gamma_{\pm}(f) \cap \Delta / \emptyset} (-1)^{\dim \Delta} \cdot \text{Eu}_{X_\sigma}^g(T_\Delta \cap X_\sigma^g).$$

By [10, Proposition 3.4],

$$B_{l,X_\sigma}(0) = \sum_{\gamma_{\pm}(f) \cap \Delta / \emptyset} (-1)^{\dim \Delta - 1} \left( \sum_{i=1}^{\mu(\Delta)} \text{Vol}_Z(\Gamma_i^\Delta) \right) \cdot \text{Eu}_{X_\sigma}(T_\Delta).$$

Moreover, since $(g, f)$ is a non-degenerate complete intersection, the functions $g$ and $f$ are non-degenerate, then by Proposition 4.1 the following equation holds

$$B_{l,X_\sigma}(0) - B_{l,X_\sigma^g}(0) - \sum_{\gamma_{\pm}(g) \cap \Delta / \emptyset} C_{l,T_\Delta \cap X_\sigma^g} \left( \text{Eu}_{X_\sigma}^g(T_\Delta \cap X_\sigma^g) - \text{Eu}_{X_\sigma^g}(T_\Delta \cap X_\sigma^g) \right) = (-1)^{d-1} m,$$

where

$$C_{l,T_\Delta \cap X_\sigma^g} = \chi(T_\Delta \cap f^{-1}(\delta) \cap B_\varepsilon) = (-1)^{\dim \Delta - 1} \sum_{i=1}^{\mu(\Delta)} \text{Vol}_Z(\Gamma_i^\Delta).$$
The Euler obstruction $\text{Eu}_{X_{\sigma}}(T_{\Delta})$ of the toric variety $X_{\sigma}$ in each stratum $T_{\Delta}$ is given by the combinatorial that came from the torus action (see [30]). Therefore, $m$ is completely determined by the Euler obstruction of the hypersurface $X_{\sigma}^0$, and the combinatorial of the torus action.

5. Constancy of Euler obstruction, Brasselet numbers and Morse points

In an unpublished lecture note [8], Briançon considered functions defined in $\mathbb{C}^n$ and observed that if $\{f_t\}$ is a family of isolated complex hypersurface singularities such that the Newton boundary of $f_t$ is independent of $t$ and $f_t$ is non-degenerate, then the corresponding family of hypersurfaces $\{f_t^{-1}(0)\}$ is Whitney equisingular (and hence topologically equisingular). A first generalization of this assertion to families with non-isolated singularities was given to the case of non-degenerate complete intersections by Oka [33] under a rather technical assumption.

In [15] Eyral and Oka restricted themselves to the case of hypersurfaces, and gave a new generalization under a simple condition. As in [33], to dealing with non-isolated singularities, the authors considered not only the compact faces of the Newton polygon but also “essential” non-compact faces. The union of the compact and essential non-compact faces forms the non-compact Newton boundary.

As application of Proposition 4.1 and also of [15, Theorem 3.6] we prove that the number of Morse critical points is invariant for a special family of hypersurfaces. This type of family is called admissible (see [15, Definition 3.5]). First, we need some new concepts and notations.

Definition 5.1. A deformation of a function germ $f : (X, 0) \to (\mathbb{C}, 0)$ is another function germ $F : (\mathbb{C} \times X, (0, 0)) \to (\mathbb{C}, 0)$ such that $F(0, x) = f(x)$, for all $x \in X$.

We assume that $F$ is origin preserving, that is, $F(t, 0) = 0$ for all $t \in \mathbb{C}$, so we have a 1-parameter family of function germs $f_t : (X, 0) \to (\mathbb{C}, 0)$ given by $f_t(x) = F(t, x)$. Moreover, associated to the family $f_t : (X, 0) \to (\mathbb{C}, 0)$ we have the family $X^{f_t} := X \cap f_t^{-1}(0)$ of subvarieties of $X$.

In the particular case of a polynomial function $f : (X, 0) \to (\mathbb{C}, 0)$, any polynomial deformation $f_t$ can be written as:

$$f_t(x) = f(x) + \sum_{i=1}^{r} \theta_i(t) \cdot h_i(x) \quad (5.1)$$

for some polynomials $h_i : (X, 0) \to (\mathbb{C}, 0)$ and $\theta_i : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, where $\theta_i(0) = 0$, for all $i = 1, \ldots, r$.

Suppose that $f : X_{\sigma} \to \mathbb{C}$ is a polynomial function defined on a toric variety $X_{\sigma}$. Consider a family as in Eq. (5.1) with $\Gamma_{+}(h_i) \subset \Gamma_{+}(f)$, for all $i = 1, \ldots, r$. Let $\gamma_{i_1}^{\Delta}, \gamma_{i_2}^{\Delta}, \ldots, \gamma_{i_{|\Delta|}}^{\Delta}$, and $\beta_{1}^{\Delta}, \beta_{2}^{\Delta}, \ldots, \beta_{\mu(\Delta)}^{\Delta}$ be the compact faces of $\Gamma_{+}(h_i) \cap \Delta$ and of $\Gamma_{+}(f) \cap \Delta$, respectively, such that $\dim \gamma_{i_1}^{\Delta} = \dim \beta_{j}^{\Delta} = \dim \Delta - 1$. If for each face $\Delta \prec \delta$, with $\Gamma_{+}(h_i) \cap \Delta \neq \emptyset$, we have that $\gamma_{i_l}^{\Delta} \cap \beta_{j}^{\Delta} = \emptyset$, for all $l = 1, \ldots, i_{\delta(\Delta)}$ and $j = 1, \ldots, \mu(\Delta)$, then

$$\Gamma_{+}(f_t) = \Gamma_{+}(f), \quad \text{for all } t \in \mathbb{C}.$$ 

That is, the Newton polygon $\Gamma_{1}(f_t)$ is independent of $t$. 

In the next definition and in Theorem 5.3 we consider \( X_\sigma = \mathbb{C}^n \).

**Definition 5.2.** We say that a family \( f_t^{-1}(0) \) of (possibly non-isolated) hypersurface singularities is **Whitney equisingular** if there exist a Whitney stratification of the hypersurface \( F^{-1}(0) \) in an open neighborhood \( U \) of the origin \( (0,0) \in \mathbb{C} \times \mathbb{C}^n \) such that the \( t \)-axis \( U \cap (D \times \{0\}) \) is a stratum, where \( D \) is an open disc centered at \( 0 \in \mathbb{C} \).

In [15] Eyral and Oka stratified \( \mathbb{C} \times \mathbb{C}^n \) in such a way that the hypersurface \( F^{-1}(0) \) is a union of strata. More precisely, they considered the following three types of strata:

- \( A_\Delta := U \cap (F^{-1}(0) \cap (C \times T_\Delta)) \) for \( \Delta \) such that \( \Gamma_+(f_0) \cap \Delta = \emptyset \);
- \( B_\Delta := U \cap (((C \times T_\Delta) \setminus (F^{-1}(0) \cap (C \times T_\Delta))) \) for \( \Delta \) such that \( \Gamma_+(f_0) \cap \Delta = \emptyset \);
- \( C_\Delta := U \cap (C \times T_\Delta) \) for \( \Delta \) such that \( \Gamma_+(f_0) \cap \Delta \neq \emptyset \).

The finite collection

\[
\mathcal{T}_t := \left\{ A_\Delta, B_\Delta \mid \Gamma_+(f_0) \cap \Delta = \emptyset \right\} \cup \left\{ C_\Delta \mid \Gamma_+(f_0) \cap \Delta \neq \emptyset \right\}
\]

is a stratification of the set \( U \cap (\mathbb{C} \times \mathbb{C}^n) \) for which \( U \cap F^{-1}(0) \) is a union of strata.

**Theorem 5.3 ([15], Theorem 3.6).** If the family of polynomial functions \( \{f_t\} \) is admissible, then the canonical stratification \( \mathcal{T}_t \) of \( U \cap (\mathbb{C} \times \mathbb{C}^n) \) described above is a Whitney stratification, and hence, the corresponding family of hypersurfaces \( \{f_t^{-1}(0)\} \) is Whitney equisingular.

If the family of polynomial functions \( \{f_t\} \) is admissible, then the stratification \( \mathcal{T}_t \) provides a Whitney stratification of each hypersurface \( f_t^{-1}(0) \), which we will denote by \( \mathcal{T}_t \). Moreover, for each \( t \) this stratification coincides with the stratification \( \mathcal{T}_t \) provided by the orbits of the torus action. Therefore, we have the following application of Proposition 4.1.

Let \( f_t : (X_\sigma,0) \rightarrow (\mathbb{C},0) \) and \( g_s : (X_\sigma,0) \rightarrow (\mathbb{C},0) \) be families of non-degenerate functions satisfying the following conditions:

1. The Newton polygon of \( f_t \) is independent of \( t \);
2. The family of polynomial functions \( \{g_s\} \) is admissible, accordingly to the Definition 3.5 of [15];
3. For each \( s, t \in \mathbb{C} \) the function \( g_s \) is tractable at the origin with respect to \( \mathcal{V}_t \), relative to \( f_t \).

**Theorem 5.4.** Let \( X_\sigma = \mathbb{C}^n \) be a \( n \)-dimensional toric variety and suppose that \( g_s, f_t : (X_\sigma,0) \rightarrow (\mathbb{C},0) \) are families of non-degenerate functions satisfying the conditions (1), (2) and (3) above. Then, the number of stratified Morse critical points, \( m_{st} \), of a partial morsecification of \( g_s : X_\sigma \cap f_t^{-1}(\delta) \cap B_\varepsilon \rightarrow \mathbb{C} \) appearing on \( X_{\sigma reg} \cap f_t^{-1}(\delta) \cap \{g_s \neq 0\} \cap B_\varepsilon \) is independent of \( t \) and \( s \).

**Proof.** From Proposition 4.1, for each \( s \) and \( t \), the equality below holds

\[
B_{f_t,X_\sigma}(0) - B_{f_t,X_\sigma}(0) - \sum_{\Gamma_+(g_s) \cap \Delta = \emptyset} C_{f_t,T_\Delta \cap X_\sigma^0}(1 - Eu_{X_\sigma^0}(T_\Delta \cap X_\sigma^0)) = (-1)^{d-1} m_{st}.
\]

Moreover, by Theorem 5.3 the family of hypersurfaces \( \{X_\sigma^0\} \) is Whitney equisingular. Then, there exists a linear function \( l : \mathbb{C}^n \rightarrow \mathbb{C} \) which is generic with respect
to $X^{g,s}_0$, for all $s$. From [10, Theorem 4.1], the number $E_{u,X^{g,s}_0}(T_\Delta \cap X^{g,s}_0)$ is constant for all $s$. Consequently the Brasselet numbers $B_{f_t,X^{g,s}_0}(0)$ and $B_{f_t,X^{s}_0}(0)$ are independent of $s$ and $t$ (see [10, Corollary 3.6] and the arguments used in its prove). Applying [31, Corollary 3.6], we also obtain the constancy of the number

$$C_{f_t,T_\Delta \cap X^{g,s}_0} = \chi(T_\Delta \cap f_t^{-1}(\delta) \cap B_\varepsilon) = (-1)^{\dim \Delta - 1} \sum_{i=1}^{\mu(\Delta)} \text{Vol}_\varepsilon(T^i_\Delta),$$

since the family $\{f_t\}$ satisfies (1) and $\{g_s\}$ satisfies (2). The independence of the sets

$$\{\Delta \prec \sigma | \Gamma_{g_s} \cap \Delta = \emptyset\}$$

and

$$\{\Delta \prec \sigma | \Gamma_{g_s} \cap \Delta \neq \emptyset\}$$

are also insured by condition (2). Therefore, the result follows.

□

Remark 5.5. For more details about the independence of $s$ on the family of polynomial functions $\{g_s\}$ we refer to [15, page 8].

As we said before, if the family of polynomial functions $\{f_t\}$ is admissible, the stratification $T_\sigma$ is Whitney. However this can may happen, even when the family is not admissible. As we see in the next example.

Example 5.6. Let $S_\sigma = \mathbb{Z}_3^3$ and let $X_\sigma = \mathbb{C}^3$ be the smooth 3-dimensional toric variety. Consider $g, l : X_\sigma \rightarrow \mathbb{C}$ non-degenerate functions, where

$$g(z_1, z_2, z_3) = z_2^2 - z_1^3 - z_1^2 z_3^2$$

and

$$l(z_1, z_2, z_3) = z_1 + z_2 + z_3.$$

The decomposition $V = \{V_0 = \{0\}, V_1 = \{\text{axis} - z_3\} \setminus \{0\}, V_2 = (X^{g,s}_0)_{\text{reg}}\}$ is a Whitney stratification of the non-degenerate hypersurface $X^{g,s}_0$, where $(X^{g,s}_0)_{\text{reg}}$ is the regular part of $X^{g,s}_0$ (see Figure 1). For a subset $I \subset \{1, 2, 3\}$ we denote the face $\sum_{i \in I} \mathbb{R}_+ e_i^* \sigma$ of $\sigma = (\mathbb{R}_+^3)^*$ by $\Delta_I$ and by $\Delta_0$ the face $\{0\}$, where $e_i^*$ denotes the elements of the canonical basis of $(\mathbb{R}^3)^*$. The Whitney stratification $T$ of $X_\sigma$ is given by the following $T$-orbits: $T_{\Delta_0} = \{0\}$, $T_{\Delta_1} = \mathbb{C}^* \times \{0\} \times \{0\}$, $T_{\Delta_2} = \{0\} \times \mathbb{C}^* \times \{0\}$, $T_{\Delta_3} = \{0\} \times \mathbb{C}^* \times \{0\}$.

Figure 1. Real illustration of $X^{g,s}_0$. 
The partition \[ T_{\Delta_1} = \{0\} \times \{0\} \times \mathbb{C}^*, \quad T_{\Delta_{12}} = \mathbb{C}^* \times \mathbb{C}^* \times \{0\}, \quad T_{\Delta_{13}} = \mathbb{C}^* \times \{0\} \times \mathbb{C}^*, \quad T_{\Delta_{23}} = \{0\} \times \mathbb{C}^* \times \mathbb{C}^*. \]

As \[ T_{\Delta_1} \cap X_0^g = T_{\Delta_3} = V_1 \]
the stratification \[ T_g = \{ T_{\Delta_1} \cap X_0^g, \Delta_0 \} \]
is a Whitney stratification of \( X_0^g \). The face \( \Delta_3 \) is the only non-zero dimensional face such that \( \Gamma_+(g) \cap \Delta_3 = \emptyset \). From Corollary 4.4,

\[ \text{Eu}_{X_0^g}(0) + (-1)^{\dim \Delta_3} \cdot \text{Eu}_{X_0^g}(T_{\Delta_3}) = (-1)^3 \cdot m + (-1)^{\dim \Delta_3} + 1. \]

Therefore,

\[ \text{Eu}_{X_0^g}(0) - \text{Eu}_{X_0^g}(T_{\Delta_3}) = -m - 1 + 1 = -m \]
since \( \dim \Delta_3 = 1 \).

We finish with an observation about good stratifications in toric varieties. Given \( X_\sigma \subset \mathbb{C}^n \) a d-dimensional toric variety and \( f : X_\sigma \to \mathbb{C} \) a non-degenerate function, the partition

\[ \{ X_\sigma^0 \cap T_\Delta, \quad T_\Delta \setminus X_\sigma^0, \quad \{0\} \quad T_\Delta \in \mathcal{T} \} \quad (5.2) \]
is not a Whitney stratification in general. However, when \( \dim \Sigma_\mathcal{T} f = 1 \) the stratification in Eq. (5.2) is a good stratification of \( X_\sigma \) relative to \( f \).

**Proposition 5.7.** Let \( f : X_\sigma \to \mathbb{C} \) be a non-degenerate function. If \( \dim \Sigma_\mathcal{T} f = 1 \), then

\[ \mathcal{T}_f = \{ X_\sigma^0 \cap T_\Delta, \quad T_\Delta \setminus X_\sigma^0, \quad \{0\} \quad T_\Delta \in \mathcal{T} \} \]
is a good stratification of \( X_\sigma \) relative to \( f \).

**Proof.** First we verify that \( X_\sigma^0 \) is a union of strata of \( \mathcal{T}_f \). Since \( \{f = 0\} \setminus \Sigma_\mathcal{T} f \) intersects \( T_\Delta \) transversely, \( T_\Delta \cap \{f = 0\} \setminus \Sigma_\mathcal{T} f \) is smooth. Let us describe \( \Sigma_\mathcal{T} f \). If \( T_\Delta \) contains a critical point of \( f|_{T_\Delta} \), then each point of \( T_\Delta \) is a critical point of \( f|_{T_\Delta} \), since \( f \) is non-degenerate. Therefore, \( \Sigma_\mathcal{T} f \) is a union of strata \( T_\Delta \). Hence, \( X_\sigma^0 \) is union of strata of type \( T_\Delta \cap \{f = 0\} \setminus \Sigma_\mathcal{T} f \) and \( T_\Delta \subset \Sigma_\mathcal{T} f \).

Nevertheless, \( \{ T_\Delta \setminus \{f = 0\} \mid T_\Delta \in \mathcal{T} \} \) is a Whitney stratification of \( X_\sigma \setminus X_\sigma^0 \), since \( T_\Delta \setminus \{f = 0\} \) is an open subset of \( T_\Delta \) and \( (T_\Delta \setminus \{f = 0\})_\Delta \) is a Whitney stratification of \( X_\sigma \).

Now we verify the Thom condition. Let \( V_\alpha = T_{\Delta_1} \setminus \{f = 0\} \) and \( V_\beta = T_{\Delta_2} \setminus \{f = 0\} \). Suppose that \( p \notin \Sigma_\mathcal{T} f \). Then

\[ T_p V_\beta = T_p (T_{\Delta_2} \setminus \{f = 0\}) = T_p T_{\Delta_2} \cap T_p V(\tilde{f}), \]
where \( \tilde{f} \) is an analytic extension of \( f \) in a neighborhood of \( p \). Therefore, since \( p_1 \notin \{f = 0\} \) we have that

\[ T_{p_1} V(f|_{V_\alpha} - f|_{V_\alpha}(p_1)) = T_{p_1} V(\tilde{f} - \tilde{f}(p_1)) \cap T_{p_1}(T_{\Delta_1} \setminus \{f = 0\}) = T_{p_1} V(\tilde{f} - \tilde{f}(p_1)) \cap T_{p_1}(T_{\Delta_1}) \]

and

\[ T = \lim_{i \to \infty} T_{p_1} V(f|_{V_\alpha} - f|_{V_\alpha}(p_1)) = \lim_{i \to \infty} T_{p_1} V(\tilde{f} - \tilde{f}(p_1)) \cap T_{p_1}(T_{\Delta_1}) \]

\[ \subseteq \lim_{i \to \infty} T_{p_1} V(\tilde{f} - \tilde{f}(p_1)) \cap \lim_{i \to \infty} T_{p_1}(T_{\Delta_1}). \]
Define $T_1 := \lim_{i \to \infty} T_{p_i}(Ta_1)$. Since Whitney’s condition (a) is valid over $(Ta_1, Ta_2)$, we have the inclusion $T_{p_i}Ta_2 \subseteq T_1$. If $p \notin \Sigma T f$, then $T_{p_i}V(f) \cap T_{p_i}Ta_2$ transversely. Hence, $T_{p_i}V(f) \cap T_{p_i}(T_{a_i})$ intersects $T_1$ transversely. As a consequence we obtain,

$$T = \lim_{i \to \infty} T_{p_i}V(fV_{a_i} - fV_{a_i}(p_i)) = \lim_{i \to \infty} T_{p_i}V(f(f(p_i)) \cap \lim_{i \to \infty} T_{p_i}(T_{a_i})$$

$$= T_{p_i}V(f) \cap T_1.$$

So, $T_{p_i}V_B = T_{p_i}Ta_2 \cap T_{p_i}V(f) \subseteq T_1 \cap T_{p_i}V(f) = T$. Suppose now that $p \in \Sigma T f$. Since Thom stratifications always exist, one may refine a stratum $Ta_2 \in \Sigma T f$ in such a way the Thom condition is satisfied. Since $\Sigma T f$ is one-dimensional, this refinement would be done by taking off a finite number of points. Then, in a small neighborhood of the origin, the Thom condition is verified over the strata of type $Ta_2 \subseteq \Sigma T f$.

When $X_g = \mathbb{C}^n$, the result above was also proved in [22, Remark 1.29], therefore Proposition 5.7 can be viewed as a generalization of Massey’s result.

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References

[1] D. A. H. Ament, J. J. Nuñio Ballesteros, B. Oréﬁce-Okamoto, and J. N. Tomazella, The Euler obstruction of a function on a determinantal variety and on a curve, Bull. Braz. Math. Soc. (N.S.) 47 (2016), no. 3, 955–970. MR 3549078
[2] J.-P. Brasselet, Local Euler obstruction, old and new, XI Brazilian Topology Meeting (Rio Claro, 1998), World Sci. Publ., River Edge, NJ, 2000, pp. 140–147. MR 1835967
[3] J.-P. Brasselet and N. G. Grulha Jr., Local Euler obstruction, old and new, II, Real and complex singularities, London Math. Soc. Lecture Note Ser., vol. 380, Cambridge Univ. Press, Cambridge, 2010, pp. 23–45. MR 2759085
[4] J.-P. Brasselet, D. T. Lê, and J. Seade, Euler obstruction and indices of vector ﬁelds, Topology 39 (2000), no. 6, 1193–1208. MR 1783853
[5] J.-P. Brasselet, D. B. Massey, A. J. Parameswaran, and J. Seade, Euler obstruction and defects of functions on singular varieties, J. London Math. Soc. (2) 70 (2004), no. 1, 59–76. MR 2064752
[6] J.-P. Brasselet and M.-H. Schwartz, Sur les classes de Chern d’un ensemble analytique complexe, The Euler-Poincaré characteristic (French), Astérisque, vol. 82, Soc. Math. France, Paris, 1981, pp. 93–147. MR 629125
[7] J.-P. Brasselet, J. Seade, and T. Suwa, Vector ﬁelds on singular varieties, Lecture Notes in Mathematics, vol. 1987, Springer-Verlag, Berlin, 2009. MR 2574165
[8] J. Briançon, “Le théorème de Kouchmirenko”, unpublished lecture notes.
[9] J. Briançon, P. Maisonobe, and M. Merle, Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom, Invent. Math. 117 (1994), no. 3, 531–550. MR 1283729
[10] T. M. Dalbelo and L. Hartmann, Brasseeet number and Newton polygons, Manuscripta Math. 162 (2020), no. 1-2, 241–269. MR 4079807
[11] T. M. Dalbelo and M. S. Pereira, Multitoric surfaces and Euler obstruction of a function, Internat. J. Math. 27 (2016), no. 10, 1650084, 21. MR 3554459
[12] A. Dubson, Classes caractéristiques des variétés singulières, C.R. Acad. Sc. Paris 287 (1978), no. 4, 237–240.
[13] N. Dutertre and N. G. Grulha, Jr., Lé-Greuel type formula for the Euler obstruction and applications, Adv. Math. 251 (2014), 127–146. MR 3130338
[14] ———, Some notes on the Euler obstruction of a function, J. Singul. 10 (2014), 82–91. MR 3300287
[15] C. Eyral and M. Oka, Non-compact Newton boundary and Whitney equisingularity for non-isolated singularities, Adv. Math. 316 (2017), 94–113. MR 3672903
[16] W. Fulton, Introduction to toric varieties, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. MR 1234037
[17] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, Discriminants, resultants and multidimensional determinants, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2008, Reprint of the 1994 edition. MR 2394437
[18] M. Goresky and R. MacPherson, Stratified Morse theory, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 14, Springer-Verlag, Berlin, 1988. MR 932724
[19] M. Kashiwara, Systems of microdifferential equations, Birkhäuser (1983).
[20] D. T. Lê and B. Teissier, Variétés polaires locales et classes de Chern des variétés singulières, Ann. of Math. (2) 114 (1981), no. 3, 457–491. MR 634426
[21] R. D. MacPherson, Chern classes for singular algebraic varieties, Ann. of Math. (2) 100 (1974), 423–432. MR 0361141
[22] D. B. Massey, Lé cycles and hypersurface singularities, Lecture Notes in Mathematics, vol. 1615, Springer-Verlag, Berlin, 1995. MR 1441075
[23] ———, Hypercohomology of Milnor fibres, Topology 35 (1996), no. 4, 969–1003. MR 1404920
[24] ———, Vanishing cycles and Thom’s a; condition, Bull. Lond. Math. Soc. 39 (2007), no. 4, 591–602. MR 2346940
[25] ———, Characteristic cycles and the relative local Euler obstruction, A panorama of singularities, Contemp. Math., vol. 742, Amer. Math. Soc., Providence, RI, [2020] ©2020, pp. 137–156. MR 4047795
[26] J. N. Mather, Stability of C∞ mappings. V. Transversality, Advances in Math. 4 (1970), 301–336 (1970). MR 275461
[27] ———, Stability of C∞ mappings. VI: The nice dimensions, Proceedings of Liverpool Singularities-Symposium, I (1969/70), 1971, pp. 207–253. Lecture Notes in Math., Vol. 192. MR 0293670
[28] ———, Stratifications and mappings, Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), 1973, pp. 195–232. MR 0368064
[29] ———, How to stratify mappings and jet spaces, Singularités d’applications différentiables (Sém., Plans-sur-Bex, 1975), 1976, pp. 128–176. Lecture Notes in Math., Vol. 535. MR 0455018
[30] Y. Matsui and K. Takeuchi, A geometric degree formula for A-discriminants and Euler obstructions of toric varieties, Adv. Math. 226 (2011), no. 2, 2040–2064. MR 2737807
[31] ———, Milnor fibers over singular toric varieties and nearby cycle sheaves, Tohoku Math. J. (2) 63 (2011), no. 1, 113–136. MR 2788778
[32] J. Milnor, Singular points of complex hypersurfaces, Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968. MR 0239612
[33] M. Oka, Canonical stratification of nondegenerate complete intersection varieties, J. Math. Soc. Japan 42 (1990), no. 3, 397–422. MR 1056828
[34] A. Parusiński, Limits of tangent spaces to fibres and the w; condition, Duke Math. J. 72 (1993), no. 1, 99–108. MR 1242881
[35] C. Sabbah, Quelques remarques sur la géométrie des espaces conormaux, Astérisque 330 (1985), 161–192.
[36] H. Santana, Brasselet number and function-germs with a one-dimensional critical set, 2019.
[37] J. Seade, M. Tibăr, and A. Verjovsky, Milnor numbers and Euler obstruction, Bull. Braz. Math. Soc. (N.S.) 36 (2005), no. 2, 275–283. MR 2152019
[38] M. Sebastiani, *Sur la formule de González-Verdier. (french)* [The González-Verdier formula], Bol. Soc. Brasil. Mat. 16 (1985), no. 1, 31–44.

[39] R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier (Grenoble) 6 (1955/56), 43–87. MR 87149

[40] A. N. Varchenko, *Zeta-function of monodromy and Newton’s diagram*, Invent. Math. 37 (1976), no. 3, 253–262. MR 0424806

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