The $x_i$-eigenvalue problem on some new fuzzy spheres

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Abstract
We study the eigenvalue equation for the ‘Cartesian coordinates’ observables $x_i$ on the fully $O(2)$-covariant fuzzy circle $\{S^1_\Lambda\}_{\Lambda \in \mathbb{N}}$ ($i = 1, 2$) and on the fully $O(3)$-covariant fuzzy 2-sphere $\{S^2_\Lambda\}_{\Lambda \in \mathbb{N}}$ ($i = 1, 2, 3$) introduced in (Fiore and Pisacane 2018 J. Geom. Phys. 132 423–51). We show that the spectrum and eigenvectors of $x_i$ fulfill a number of properties which are expected for $x_i$ to approximate well the corresponding coordinate operator of a quantum particle forced to stay on the unit sphere.

Keywords: fuzzy spaces, $O(3)$-covariance, spectra of space coordinates, localized states

1. Introduction and preliminaries

Since their introduction fuzzy spaces have raised a lively interest as a non-perturbative technique in quantum field or string theory based on a finite-discretization of space(time) alternative to lattices. A fuzzy space is a particular type of noncommutative deformation of a space, more precisely a sequence $\{A_n\}_{n \in \mathbb{N}}$ of finite-dimensional (noncommutative) algebras such that as $n$ diverges $A_n$ goes to the commutative algebra $A$ of regular functions (with pointwise product) on an ordinary manifold $M$ (in particular, the dimension of $A_n$ diverges). Its main advantage with respect to a lattice discretization is that the algebras $A_n$ can carry representations of a Lie group (not only of a discrete one). The first and seminal fuzzy space is the Fuzzy 2-Sphere (FS) $S^2_\Lambda$ of Madore and Hoppe [1, 2]: $A_n \simeq M_n(\mathbb{C})$ (the algebra of complex $n \times n$ matrices) is generated by coordinate operators $\{x_i\}_{i=1}^3$ fulfilling
\[
[x_i, x_j] = \frac{2i}{\sqrt{n^2 - 1}} e^{i\Delta x_k}, \quad x_k x_k = 1
\]  
(1)

(here \( n \in \mathbb{N} \setminus \{1\} \), and sum over repeated indices is understood); in fact these are obtained by the rescaling

\[
x_i = \frac{2L_i}{\sqrt{n^2 - 1}}, \quad i = 1, 2, 3
\]  
(2)

of the elements \( L_i \) of the standard basis of \( so(3) \) in the irreducible representation \( (\pi_l, V_l) \) characterized by \( L^2 := L_i L_i = l(l + 1) \), or equivalently of dimension \( n = 2l + 1 \). In a quantum field theory (QFT) on a fuzzy space the ‘cutoff’ \( n \) works as a regularizing parameter of ultraviolet divergences, because integration over fields amounts to integration over matrices of a ‘field theory’ (QFT) on a ‘fuzzy space’ the cutoff and the sharpness of the potential well are parametrized by (and diverge with) the natural number \( \Lambda \) —it reduces the number of massive Kaluza–Klein modes of a field theory on \( M' \) to a finite value \( [9] \); finally, it has been recently proposed \([10]\) that \( n \) may also parametrize the large (but finite) amount of information hidden in a black hole. In the matrix model formulations of \( M \)-theory \([11, 12]\) and string theory \([13]\) fuzzy spaces may arise as subalgebras giving the leading contribution to the path-integrals over larger matrix algebras; they respectively lead to quantized branes (in particular, the 5-brane) in a 11- or 10-dimensional spacetime.

Relations (1) are covariant under \( SO(3) \), but not under the whole \( O(3) \), in particular not under inversion of axes \( x_i \to -x_i \). This is to be contrasted with the \( O(3) \)-covariance of the ordinary sphere \( S^2 \), where the right-hand side of (1) is zero. Moreover, while the Hilbert space \( V_l \) of the FS carries an irreducible representation of \( SO(3) \), that \( L^2(S^2) \) of a quantum particle on \( S^2 \) decomposes as the direct sum of all the irreducible representations of \( SO(3) \).

To overcome these shortcomings, in \([14, 15]\) we have built new fuzzy spheres \( \{S^2_\Lambda \}_{\Lambda \in \mathbb{N}} \) and \( \{S^2_\Lambda \}_{\Lambda \in \mathbb{N}} \), which are a fully \( O(2) \)-covariant fuzzy circle and a fully \( O(3) \)-covariant fuzzy 2-sphere, respectively; the right-hand side of (1) depends on the angular momentum components and therefore is parity invariant as in Snyder commutation relations \([16]\), see equation \( (27) \) below. The Hilbert space on which the algebra \( \mathcal{A}_\Lambda \) of \( S^2_\Lambda \) acts decomposes as the direct sum \( \mathcal{H}_\Lambda = \bigoplus_{l=0}^\infty V_l \) of all the irreducible representations of \( SO(3) \) up to the cutoff, and therefore also in this aspect \( S^2_\Lambda \) better approximates the configuration space \( S^2 \) in the limit \( \Lambda \to \infty \). We have constructed these fuzzy spheres imposing a suitable energy cutoff on a quantum particle subject to a confining potential well \( V(r) \) with a very sharp minimum on the sphere of radius \( r = 1 \) in the Euclidean spaces \( \mathbb{R}^2, \mathbb{R}^3 \), respectively; the cutoff and the sharpness of the potential well are parametrized by (and diverge with) the natural number \( \Lambda \). We think that these new fuzzy spheres might have applications not only in QFT or string theory, but also in some condensed matter physics problem as models with an effective one- or two-dimensional configuration space in the form of a circle, a cylinder or a sphere (like nanotubes, quantum waveguides, cylindrical or spherical sheets of graphene, etc). In fact, in these cases parity is respected, and the restriction to the circle, cylinder or sphere is an effective one obtained ‘\( a \ posteriori \)’ from the exact dynamics in higher dimension.
In the present work we start to study the localizability on $S^d_\Lambda$, $d = 1, 2$, more precisely the eigenvalue equation for the coordinates $x_i$. This is preparatory to a number of purposes. As we shall explain, eigenvectors of one of the coordinates are close to the most localized states. Localized states, especially when arranged in systems of coherent states [17], will be an extremely useful tool for studying path integrals (partition and correlation functions) in QFT over the $S^d_\Lambda$ (as over other fuzzy spaces, see e.g. [18, 19]), as well as the quantum metric aspects of the $S^d_\Lambda$, in particular the "distance" (either the spectral distance of Connes [20–22], or alternative ones, see e.g. [23, 24]) between two localized states. For $x_i$ ($i = 1, \ldots, D \equiv d + 1$) to approximate well and in an $O(D)$-equivariant way the corresponding coordinate of a quantum particle forced to stay on the unit sphere $S^d$, its spectrum $\Sigma_{x_i}$ should fulfill at least the following properties, which are fulfilled also by the Madore FS:

1. The spectrum $\Sigma_{x_i}$ of each $x_i$, for all choices of the orthogonal axes, is the same.
2. If $\alpha$ is an eigenvalue of $x_i$, then also $-\alpha$ is.
3. In the commutative limit the spectrum $\Sigma_{x_i}$ becomes uniformly dense in $[-1, 1]$, in particular the maximal and the minimal eigenvalues converge to 1 and $-1$, respectively.

We are going to show that $\Sigma_{x_i}$ on $S^d_\Lambda$ fulfills these and other properties. Among the latter, one, not shared by the FS, justifies why in our opinion (see section 5) $S^2_\Lambda$ can be interpreted as a fuzzy configuration space, while the FS should be interpreted only as a fuzzy spin phase space: namely that the eigenstate of $x_3$ with maximal eigenvalue, which is very localized around the North pole of $S^2$, is an eigenstate of $L_3$ with zero eigenvalue. We adopt [17] as a measure of the localization of a state the square space-uncertainty (dispersion) in the ambient Euclidean space $\mathbb{R}^D$ ($D = 2, 3$), i.e. the expectation value on the state (variance)

$$\Delta x^2 := \left< (x - \langle x \rangle)^2 \right> = \langle x^2 \rangle - \langle x \rangle^2 = \sum_{i=1}^{D} \langle x_i^2 \rangle - \sum_{i=1}^{D} \langle x_i \rangle^2,$$

(4)

which is manifestly $O(D)$-invariant. This symmetry means $(\Delta x)^2_\psi = (\Delta R x)^2_\psi$ for every state $\psi \in \mathcal{H}_\Lambda$ and $O(D)$-transformation $R$, and implies that minimizing (4) or $\langle x^2 \rangle - \langle x \rangle^2$ with a fixed $i \in \{1, \ldots, D\}$ is equivalent. On the other hand, since on our fuzzy spheres $x^2 \simeq 1$, this approximately amounts to maximizing $\langle x_i \rangle^2$, what occurs on the $x_i$-eigenstate with highest eigenvalue. Therefore the eigenstate $\chi$ with maximal eigenvalue of any coordinate is very localized (almost an optimally localized state; the latter are the closest to 'classical' states). Evaluating (4) on the approximation of $\chi$ determined in the present work is sufficient to prove [17] the bound

$$\Delta x_{\min}^2 \leq \frac{C}{(\Lambda + 1)^2},$$

where $C = 3.5$ for $S^1_\Lambda$ and $C = 11$ for $S^2_\Lambda$, which is lower than in the Madore–Hoppe FS.

The plan of the paper is as follows. In section 2 we briefly recall the construction procedure [14, 15] of these fuzzy spaces and how to diagonalize Toeplitz tridiagonal matrices; in sections 3 and 4 we study the $x_i$-eigenvalue equation on $S^1_\Lambda, S^2_\Lambda$ respectively; in section 5 we compare results on $S^2_\Lambda$ and FS; in the appendix we have concentrated lengthy calculations and complex proofs.
2. Preliminaries

2.1. Construction procedure of the fuzzy spheres \( S^d \) in brief

Here we recall how quantum mechanics on the \( O(D) \)-covariant fuzzy spheres \( S^d \) \( (D = d + 1, d = 1, 2) \) has been introduced in [14]. We start with a zero-spin quantum particle in \( \mathbb{R}^D \) configuration space with Hamiltonian

\[
H = -\frac{1}{2} \Delta + V(r) = -\frac{1}{2} \left[ \partial^2_r + (D - 1) \frac{1}{r^2} \partial_r - \frac{1}{r^2} L^2 \right] + V(r).
\]

Here \( \Delta := \partial_i \partial_i, \partial_i := \partial / \partial x_i (i = 1, ..., D), \quad r^2 := x^2 = x_i x_i, \quad \partial_r := \partial / \partial r. \) We use dimensionless coordinates \( x_i \), momentum components \( p_i := -i \partial_i \) and Hamiltonian \( H; \quad x_i, p_i \) generate the Heisenberg algebra \( \mathcal{O} \) of observables. Moreover \( L_{ij} := x_ip_j - x_j p_i \) are the angular momentum components, and \( L^2 := L_{ij}L^{ij} \) is the square angular momentum (in normalized units), i.e. the Laplacian on the sphere \( S^d \). The canonical commutation relations as well as \( H \) are invariant under all orthogonal transformations, including parity. We choose \( V(r) \) as a confining potential with a very sharp minimum at \( r = 1 \), i.e. with \( V'(1) = 0 \) and very large \( k := V''(1)/4 > 0 \), and fix \( V_0 := V(1) \) so that the ground state has zero energy, \( E_0 = 0 \). We choose an energy cutoff \( \mathcal{E} \) satisfying first of all the condition

\[
V(r) \simeq V_0 + 2k(r - 1)^2 \quad \text{if} \ r \ \text{fulfils} \quad V(r) \leq \mathcal{E},
\]

so that \( V(r) \) is approximately harmonic in the classical region \( \mathcal{V} \) compatible with the energy cutoff \( V(r) \leq \mathcal{E} \). Then we project the theory onto the finite-dimensional Hilbert subspace \( \mathcal{H}_\mathcal{E} \subset \mathcal{H} \equiv L^2(\mathbb{R}^D) \) spanned by \( \psi \) fulfilling the eigenvalue equation

\[
H \psi = E \psi, \quad \psi \in L^2(\mathbb{R}^D), \quad E \leq \mathcal{E}.
\]

This entails replacing every observable \( A \) by \( \overline{A} \):

\[
A \mapsto \overline{A} := P_\mathcal{E} A P_\mathcal{E},
\]

where \( P_\mathcal{E} \) is the projection on \( \mathcal{H}_\mathcal{E} \). In particular we thus construct the fuzzy Cartesian coordinates \( \overline{x}_i \) and angular momentum components \( \overline{L}_{ij} = \overline{L}_{ij} = \overline{L}_{ij} \). Replacing the Ansatz \( \psi = f(r)Y(\varphi, ..., \theta) \) (\( Y \) are eigenfunctions of \( L^2 \) and of the elements of a Cartan subalgebra of \( so(D) \); \( r, \varphi, ..., \theta \) are polar coordinates) transforms the PDE \( H \psi = E \psi \) into an ODE in the unknown \( f(r) \). At leading order in \( 1/k \) the latter is the eigenvalue equation of a \( 1 \)-dimensional harmonic oscillator, and the lowest eigenvalues are \( E_{j,n} = j(j + d - 1) + 2n \left( \sqrt{2k} + d - 2 \right), \quad j, n \in \mathbb{N}_0 \). Choosing a \( \Lambda \in \mathbb{N} \) fulfilling \( \Lambda(\Lambda + d - 1) < \Lambda \left( \sqrt{2k} + d - 2 \right) \), and setting \( \mathcal{E} = \mathcal{E}_\Lambda = \Lambda(\Lambda + d - 1) \), we ’freeze’ all radial excitations and make the spectrum consist only of the eigenvalues \( E_{j,n} \equiv E_{j,0} = j(j + d - 1) \) of \( \overline{H} \), which make up the lower part of the spectrum of \( L^2 \)-the Laplacian on \( S^d \), as wished. Correspondingly, we re-denote \( \mathcal{H}_\mathcal{E}, P_\mathcal{E} \) as \( \mathcal{H}_\Lambda, P_\Lambda \). Consistency with (6) requires \( k \) to be a function of \( \Lambda \) growing sufficiently fast with \( \Lambda \), e.g. \( k(\Lambda) \geq \Lambda^2(\Lambda + 1)^2 \). Finally, we denote as \( A_\Lambda \) the algebra \( \text{End}(\mathcal{H}_\Lambda) \) of observables on \( \mathcal{H}_\Lambda \). Below we shall remove the bar and denote the generic operator \( \overline{A} \in A_\Lambda \) as \( A \).

The manifest \( O(D) \)-equivariance of \( S^d \) has a number of welcome consequences, in particular implies that the spectra \( \Sigma_{\Lambda} \) fulfill properties 1,2 mentioned in the introduction.
2.2. Diagonalization of Toeplitz tridiagonal matrices

A real Toeplitz tri-diagonal matrix is a \( n \times n \) matrix
\[
P_n(a, b, c) := \begin{pmatrix}
a & b & 0 & 0 & 0 & 0 \\
c & a & b & 0 & 0 & 0 \\
0 & c & a & b & 0 & 0 \\
& & & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & a & b \\
0 & 0 & 0 & \cdots & c & a \\
0 & 0 & 0 & \cdots & 0 & c
\end{pmatrix}
\]
where \( a, b, c \in \mathbb{R} \). (8)

Its eigenvalues are (see e.g. [25] pp 2–3)
\[
\lambda_h = a + 2\sqrt{bc} \cos \left( \frac{h\pi}{n+1} \right), \quad h = 1, \ldots, n
\]
and the corresponding eigenvectors \( \chi^h \) are columns with the following components
\[
\chi^h_k = \left( \frac{c}{b} \right)^{\frac{1}{2}} \sin \left( \frac{hk\pi}{n+1} \right), \quad h, k = 1, 2, \ldots, n,
\]
up to normalization. In the symmetric case \( b = c \) all eigenvalues are real and the highest one is clearly \( \lambda_1 \); the norm of \( \chi^1 \) is easily computed:
\[
\|\chi^1\|^2 = \sum_{k=1}^n \sin^2 \left( \frac{k\pi}{n+1} \right) = \frac{n + 1}{2}.
\]

3. The fuzzy circle \( S_1^\Lambda \)

3.1. Preliminaries

The case \( D = 2 \) leads to the \( O(2) \)-covariant fuzzy circle \( S_1^\Lambda \). A suitable orthonormal basis \( B := \{ \psi_\Lambda, \psi_{\Lambda-1}, \ldots, \psi_{-1} \} \) of the Hilbert space \( \mathcal{H}_\Lambda \) consists of eigenvectors of the angular momentum \( L \equiv L_{12} \),
\[
L\psi_n = n\psi_n. \quad (12)
\]

Beside the hermitean cartesian coordinates \( x_1, x_2 \) we use the hermitean conjugate ones\(^1\)
\[
x_{\pm} := x_1 \pm ix_2;
\]
they act as follows:
\[
x_+\psi_n = b_{n+1}\psi_{n+1}, \quad x_-\psi_n = b_n\psi_{n-1}, \quad b_n := \begin{cases} 
1 + \frac{n(n-1)}{k} & \text{if } 1 - \Lambda \leq n \leq \Lambda, \\
n/k & \text{otherwise}.
\end{cases}
\]

It is easy to see that \( b_{-\Lambda} = b_{\Lambda+1} = 0 \), \( b_n = b_{1-n} \) if \( \Lambda + 1 \geq n \geq 0 \). The operator \( x^2 := x_1^2 + x_2^2 = (x_1^2 + x_2^2 + x_-x_+) / 2 \) represents the square distance from the origin. We denote as \( P_m \) the projection over the 1-dim subspace spanned by \( \psi_m \). The above formulae lead to the following \( O(2) \)-equivariant algebraic relations:

\(^1\)We have changed conventions with respect to [14]: the \( x_i \) (\( i = 1, 2 \)) as defined here equal the \( \xi_i = \pi/a \) of [14]
where \( a = 1 + \frac{1}{2\sqrt{2}} + O(\frac{1}{a}) \) is just a normalization factor; the \( x_{\pm} \) as defined here equal \( \sqrt{2} \xi_{\pm} = \sqrt{2}\xi_{\pm} / a \) of [14].
\[ [L, x_\pm] = \pm x_\pm, \quad x_+^\dagger = x_-, \quad L^\dagger = L, \]  
(14)

\[ [x_+, x_-] = -\frac{2L}{k} + \left[ 1 + \frac{\Lambda(\Lambda + 1)}{k} \right] \left( \tilde{P}_\Lambda - \tilde{P}_{-\Lambda} \right), \]  
(15)

\[ x^2 = 1 + \frac{L^2}{k} - \left[ 1 + \frac{\Lambda(\Lambda + 1)}{k} \right] \frac{\tilde{P}_\Lambda + \tilde{P}_{-\Lambda}}{2}. \]  
(16)

Formula (16) shows that \( x^2 \) is not the identity, but a function of \( L^2 \), hence the \( \psi_m \) are its eigenvectors; its eigenvalues (except on \( \psi_{\pm\Lambda} \)) are close to 1, slightly grow with |\( m \)| and collapse to 1 as \( \Lambda \to \infty \).

Finally, we showed that there is a sequence of \( O(2) \)-covariant \( * \)-algebra isomorphisms \( A_\Lambda \simeq M_N(\mathbb{C}) \simeq \pi_\Lambda[Uso(3)], \) where \( \pi_\Lambda \) is the \( (2\Lambda + 1) \)-dimensional unitary irreducible representation of \( Uso(3) \).

3.2. Spectrum of \( x_1 \) in the \( O(2) \)-equivariant fuzzy circle

In this subsection we analyze the spectrum of \( x_1 \). This is not a restriction because the algebraic relations (14)–(17) are covariant under \( O(2) \) transformations \( x \mapsto x' = Rx \), \( L \) is covariant under 2-dimensional rotations, \( L \to -L \) under \( x_1 \)-inversion and the same applies under \( x_2 \)-inversion; this implies that the spectra \( \Sigma_{x_i}(\Lambda) \) of all coordinate operators \( x_i \) are equal, and for this reason we can focus our attention to \( x_1 \). The spectrum \( \Sigma_{x_1} \) for \( \Lambda = 1, 2 \) is presented in formulae (B.1) and (B.2) of the appendix.

More generally, on the basis \( B \) of \( \mathcal{H}_\Lambda \) the operator \( x_1 \) is represented by the \((2\Lambda + 1) \times (2\Lambda + 1)\) symmetric tri-diagonal matrix (see (8))

\[
X^\Lambda = \frac{1}{2} \begin{pmatrix}
0 & b_\Lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
b_\Lambda & 0 & b_{\Lambda-1} & 0 & 0 & 0 & 0 & 0 \\
0 & b_{\Lambda-1} & 0 & b_{\Lambda-2} & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & b_{2-\Lambda} & 0 & b_{1-\Lambda} & 0 \\
0 & 0 & 0 & \cdots & 0 & b_{1-\Lambda} & 0 & 0 \\
\end{pmatrix} = X^\Lambda_0 + O \left( \frac{1}{\Lambda^2} \right),
\]

where \( X^\Lambda_0 := \frac{1}{2} P(0, 1, 1) \), and it is obvious that all the eigenvalues of \( X^\Lambda \) are real.

Let \( \Sigma^\Lambda_0 := \{ \tilde{\alpha}_h(\Lambda) \}_{h=1}^{2\Lambda+1} \) be the set of the eigenvalues of \( X^\Lambda_0 \) arranged in descending order; according to (9) one has

\[
\tilde{\alpha}_h(\Lambda) = \cos \left( \frac{h\pi}{2\Lambda + 2} \right), \quad h = 1, 2, \cdots, 2\Lambda + 1. \]  
(18)

It is easy to see that \( \alpha \in \Sigma^\Lambda_0 \Rightarrow -\alpha \in \Sigma^\Lambda_0 \), all the eigenvalues of \( \Sigma^\Lambda_0 \) are simple, \( \tilde{\alpha}_1(\Lambda + 1) \geq \tilde{\alpha}_1(\Lambda) \) and \( \Sigma^\Lambda_0 \) becomes uniformly dense in \([-1, 1]\) as \( \Lambda \to \infty \).

In appendix B we show that the same holds true also for the spectrum \( \Sigma^\Lambda \) of \( X^\Lambda \), in particular we prove...
Theorem 3.1.

(A) If $\alpha$ is an eigenvalue of $X_{\Lambda}$, then also $-\alpha$ is.
(B) For all $\Lambda$, all eigenvalues of $X_{\Lambda}$ are simple; we denote them as $\alpha_1(\Lambda), \alpha_2(\Lambda), \ldots, \alpha_{2\Lambda+1}(\Lambda)$, in decreasing order.
(C) Let $k(\Lambda) \geq \Lambda(\Lambda - 1)(2\Lambda + 3)^2(2\Lambda + 4)^4/4\pi^4$, then
$$\alpha_1(\Lambda + 1) > \alpha_1(\Lambda) \quad \forall \Lambda \in \mathbb{N}. \quad (19)$$
(D) $\Sigma_{\Lambda}$ becomes uniformly dense in $[-1,1]$ as $\Lambda \to \infty$, in particular
$$\lim_{\Lambda \to \infty} \alpha_1(\Lambda) = 1 \quad \text{and} \quad \alpha_1(\Lambda) \geq 1 - \frac{\pi^2}{8(\Lambda + 1)^2} \quad \forall \Lambda \in \mathbb{N}. \quad (20)$$

Let $\chi := \sum_{n=-\Lambda}^{\Lambda} \chi_n \psi_n$, the eigenvalue equation $x_1 \chi = \alpha \chi$ amounts to
$$b_{\Lambda} \chi_{n}(\Lambda) = \alpha \chi_{n+\Lambda}, \quad b_{n} \chi_{n-1} + b_{n+1} \chi_{n+1} = \alpha \chi_{n} \quad \text{if} \quad |n| < \Lambda; \quad (21)$$
on the other hand, $b_{n} \to 1$ in the commutative limit and in appendix B.4 we show that $\alpha_{n}(\Lambda) \simeq \cos \left( \frac{\hbar \pi}{2 \Lambda + 2} \right)$ in the limit $\Lambda \to +\infty$, so (10) and (11) imply
$$x_1 \chi_{n}(\Lambda) = \alpha_{n}(\Lambda) \chi_{n}(\Lambda) \implies \chi_{n}(\Lambda) \simeq \sqrt{\frac{2}{2\Lambda + 2}} \sin \left( \frac{\hbar \pi n}{2\Lambda + 2} \right). \quad (22)$$

4. The fuzzy sphere $S^2_{\Lambda}$

4.1. Preliminaries

The case $D = 3$ leads to the $O(3)$-covariant fuzzy sphere $S^2_{\Lambda}$. A suitable orthonormal basis $B := \{ \psi_{l}^m : l, m \in \mathbb{Z}, |m| \leq l \leq \Lambda \}$ of the Hilbert space $H_{\Lambda}$ consists of common eigenvectors of the angular momentum component $L_3 \equiv L_{12}$ and of the square angular momentum operator $L_2$.
$$L_3 \psi_{l}^m = m \psi_{l}^m, \quad L_2^2 \psi_{l}^m = l(l+1) \psi_{l}^m. \quad (23)$$

We define $x_0 := x_3, L_0 := L_3$ and beside the hermitean cartesian coordinates $x_1, x_2$ and angular momentum components $L_1, L_2$ we use the hermitean conjugate ones
$$x_{\pm} := \frac{x_1 \pm ix_2}{\sqrt{2}}, \quad L_{\pm} := \frac{L_1 \pm iL_2}{\sqrt{2}}; \quad (24)$$
they act as follows (here $a \in \{0, +, -\}$):
$$L_{\pm} \psi_{l}^m = \sqrt{(l+m)(l+m+1)} \psi_{l}^{m \pm a}, \quad x_{0} \psi_{l}^m = \begin{cases} c_{l} A_{l}^{m,a} \psi_{l-1}^{m+a} + c_{l+1} B_{l}^{m,a} \psi_{l+1}^{m+a} & \text{if} \ l < \Lambda, \\ c_{l} A_{l}^{m,a} \psi_{l-1}^{m+a} & \text{if} \ l = \Lambda, \\ 0 & \text{otherwise}, \end{cases} \quad (25)$$
where
$$A_{l}^{m,a} = \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}, \quad A_{l}^{m,a} = \pm \frac{1}{\sqrt{2}} \sqrt{\frac{(l+m)(l-m-1)}{(2l-1)(2l+1)}}, \quad A_{l}^{m,a} = B_{l}^{m,a}. \quad (26)$$
\[ c_l := \sqrt{1 + \frac{\ell^2}{k}} \quad 1 \leq \ell \leq \Lambda, \quad c_0 = c_{\Lambda+1} = 0, \quad \text{with} \quad k = k(\Lambda) \geq \Lambda^2 (\Lambda + 1)^2. \]  

(25)

The choice (25) is compatible with all \( V(r) \) having the same \( V(1), V'(1) = 0 \) and \( V''(1) = 4k \), up to \( O \left( \frac{1}{\Lambda^2} \right) \).

The operator \( x^2 := x_1^2 + x_2^2 + x_3^2 = x_+ x_- + x_+ x_- + x_3^2 \) represents the square distance from the origin. We denote as \( P_l \) the projection over the \( L^2 = l(l+1) \) eigenspace. The above formulæ lead to the following \( O(3) \)-equivariant algebraic relations:

\[
\begin{align*}
[ L_i, x_j ] &= i \epsilon_{ijk} x_k, \quad [ L_i, L_j ] = i \epsilon_{ijk} L_k, \quad x_j^\dagger = x_j, \quad L_j^\dagger = L_j, \\
x_i L_i &= 0, \quad [ x_i, x_j ] = i \epsilon_{ijk} \left( -\frac{l}{k} + k \tilde{P}_\Lambda \right) L_k & \quad i, j, h \in \{1, 2, 3\}, \\
 x^2 &= 1 + \frac{L^2 + 1}{k} - \left[ 1 + \frac{(\Lambda + 1)^2}{k} \right] \frac{\Lambda + 1}{2\Lambda + 1} \tilde{P}_\Lambda, \\
\prod_{l=0}^\Lambda [ L^2 - l(l+1) ] &= 0, \quad \prod_{m=-l}^l ( L_3 - ml ) \tilde{P}_l = 0, \quad (x_3)^{2\Lambda+1} = 0.
\end{align*}
\]

(26-28)

Formula (28) shows that \( x^2 \) is not the identity, but a function of \( L^2 \), hence the \( \psi^m_\Lambda \) are its eigenvectors; its eigenvalues (except on \( \psi^\Lambda_\Lambda \)) are close to 1, slightly grow with \( l \) and collapse to 1 as \( \Lambda \to \infty \).

Finally, we showed that there is a sequence of \( O(3) \)-covariant \( * \)-algebra isomorphisms \( A_\Lambda \cong MN(\mathbb{C}) \cong \pi_\Lambda (Uso(3)), \) where \( \pi_\Lambda \) is the \( (\Lambda + 1)^2 \)-dimensional unitary irreducible representation of \( Uso(4) \).

### 4.2. Spectrum of \( x_0 \) in the \( O(3) \)-equivariant fuzzy sphere

In this subsection we do the analysis of the spectrum of \( x_0 \), this is not a restriction since the covariance of the algebra under \( O(3) \) transformations \( x \mapsto x' = Rx, \; L \mapsto L' = RL \) implies that the spectra \( \Sigma_\Lambda (\Lambda) \) of all coordinate operators \( x_i \) of our fuzzy space are equal; on the other hand, because of \( [x_0, L_0] = 0 \), we can simultaneously diagonalize \( x_0 \) and \( L_0 \).

Equation (22) and

\[
\begin{align*}
\{ L_0 \chi^\beta_\alpha = \beta \chi^\beta_\alpha, \\
\chi^\alpha_\alpha = \chi^\beta_\beta,
\end{align*}
\]

(30)

imply

\[
\beta = m \in \{-\Lambda, -\Lambda + 1, \cdots, \Lambda - 1, \Lambda\}, \quad \text{and} \quad \chi^m_\alpha = \sum_{L=|m|}^\Lambda \chi^m_{\alpha L} \psi^m_L.
\]

(31)
so \( x_0 \chi^m = \alpha \chi^m \) can be re-written as

\[
\begin{cases}
\chi^m_{\alpha,|m|+1} = \chi^m_{\alpha,|m|+1} + 1A_{|m|+1}^0 \\
\chi^m_{\alpha,|m|+1} = \chi^m_{\alpha,|m|+1} + 2A_{|m|+2}^1 \\
\chi^m_{\alpha,|m|+1} = \chi^m_{\alpha,|m|+1} + 3A_{|m|+3}^2 \\
\vdots \\
\chi^m_{\alpha,|m|-2} = \chi^m_{\alpha,|m|-2} + 1B_{|m|-2}^0 \\
\chi^m_{\alpha,|m|-1} = \chi^m_{\alpha,|m|-1} + c_{|m|-1}A_{|m|-1}^0 \\
\chi^m_{\alpha,|m|-1} = \chi^m_{\alpha,|m|-1} + c_{|m|-1}A_{|m|-1}^0 \\
\end{cases}
\]

which in turn can be rewritten in the matrix form \( B_m(\Lambda) \chi = \alpha \chi \), where

\[
\chi = \begin{pmatrix} \chi^m_{\alpha,|m|}, \chi^m_{\alpha,|m|+1}, \ldots, \chi^m_{\alpha,|m|} \end{pmatrix}^T
\]

and \( B_m(\Lambda) \) is the following \( n(\Lambda; m) \times n(\Lambda; m) \) symmetric tridiagonal matrix

\[
B_m(\Lambda) =
\begin{pmatrix}
0 & c_{|m|+1}A_{|m|+1}^0 & 0 & 0 & 0 & 0 \\
c_{|m|+1}A_{|m|+1}^0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{|m|+2}A_{|m|+2}^1 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{|m|+3}A_{|m|+3}^2 & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & c_{|m|-1}A_{|m|-1}^0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

or equivalently \( M_m(\Lambda; \alpha) \chi = 0 \), where \( 0 \) here is the null vector, and we have abbreviated

\[
n = n(\Lambda; m) := \Lambda - |m| + 1, \quad M_m(\Lambda; \alpha) := B_m(\Lambda) - \alpha I_{n(\Lambda; m)}.
\]

It is well known that the problem of determining analytically the eigenvalues of a square matrix of large rank is absolutely not trivial, but the \( B_m(\Lambda) \) have several good properties (for example they are symmetric and tri-diagonal) which will help us in studying their spectra. We start with the following

**Remark 1.** All the eigenvalues of \( B_m(\Lambda) \) are real, and \( B_m(\Lambda) \equiv B_{-m}(\Lambda) \) implies that we can restrict our attention to the cases \( \beta = m \in \{0, 1, \ldots, \Lambda\} \).

As for the fuzzy circle, we prove

**Theorem 4.1.**

(A) If \( \alpha \) is an eigenvalue of \( B_m(\Lambda) \), then also \( -\alpha \) is.

(B) For all \( \Lambda, m \), all eigenvalues of \( B_m(\Lambda) \) are simple; we denote them as \( \alpha_1(\Lambda; m), \alpha_2(\Lambda; m), \ldots, \alpha_{n(\Lambda; m)}(\Lambda; m) \), in decreasing order.

(C) Let \( \alpha_1(\Lambda; m) \) be the highest eigenvalue of \( B_m(\Lambda) \), then

\[
\alpha_1(\Lambda; 0) > \alpha_1(\Lambda; 1) > \cdots > \alpha_1(\Lambda; \Lambda),
\]

and

\[
\alpha_1(\Lambda + 1; 0) > \alpha_1(\Lambda; 0) \quad \text{definitively, if } k(\Lambda) \geq \Lambda^6.
\]

(D) \( \Sigma_{\alpha_1(\Lambda)} \) becomes uniformly dense in \([-1, 1]\) as \( \Lambda \to \infty \), in particular

\[
\lim_{\Lambda \to +\infty} \alpha_1(\Lambda; 0) = 1 \quad \text{and} \quad \alpha_1(\Lambda; 0) \geq 1 - \frac{\pi^2}{2(\Lambda + 2)^2} \quad \forall \Lambda \geq 2.
\]
Item (C) of last theorem allows us to make a connection between our localized states and the classical ones because the $\alpha_1 (\Lambda; 0)$-eigenstate approximates a quantum particle on $S^2$ concentrated (because of the above equivalence between the $\alpha_1 (\Lambda; 0)$-eigenstate and the most localized state of our fuzzy space [17]) on the North pole and rotating around the $x_3$-axis; on the other hand, if we consider a classical particle forced to stay on $S^2$ and in the position $(0, 0, 1)$, then it must be

$$L_3 = (L_1 \times L_2)_3 = 0,$$

as for our case.

Note that, the spectrum $\Sigma_{\beta_0(\Lambda)}$ contains exactly $\Lambda + 1$ eigenvalues and the highest one fulfills (33), for this reason we focus our attention only on that matrix.

It is important to point out that the proof of item (D) can be trivially re-arranged in order to prove that it holds for $\Sigma_{\beta_0(\Lambda)}$ and $\alpha_1 (\Lambda; m)$ also if $m > 0$ is any other fixed integer.

Let $m \in \mathbb{N}_0$ and assume that $\chi^m_\alpha := \sum_{l=m}^{\Lambda} \chi^m_{\alpha l} \Phi^m_l$ is a common eigenstate of $x_0$ and $L_0$; let $\{\tilde{\alpha}_h (\Lambda; m)\}_{h=1}^{\Lambda - m + 1}$ be the set of the eigenvalues of $P_{\Lambda-m+1} (0, \frac{1}{2}, \frac{1}{2})$ arranged in descending order; according to (9) one has

$$\tilde{\alpha}_h (\Lambda, m) = \cos \left( \frac{h \pi}{\Lambda - m + 2} \right), \quad h = 1, 2, \ldots, \Lambda - m + 1.$$

We can prove (as for appendix B.4) that $\alpha_h (\Lambda; m) \simeq \cos \left( \frac{h \pi}{\Lambda - m + 2} \right)$ in the limit $\Lambda \to +\infty$, although in this case $c\alpha_1^{0, m} \to \frac{1}{2}$. On the other hand, when $|m| \ll 1$, we can approximate well $c\alpha_1^{0, m} \simeq \frac{1}{2}$ in the commutative limit, for this reason we believe that

$$\chi^m_{\alpha_0(\Lambda, m), l} \simeq \sqrt{\frac{2}{\Lambda - m + 2}} \sin \left( \frac{h l \pi}{\Lambda - m + 2} \right),$$

as for the $D = 2$ case.

5. Conclusions and comparison with the Madore fuzzy sphere

In the analysis of the spectra $\Sigma_{\alpha}(\Lambda)$ of our fuzzy spaces we have proved the following:

1. $O(D)$-equivariance: the spectrum $\Sigma_{\alpha}$ of each $\alpha_i$, for all choices of the orthogonal axes, is the same.

2. Parity property:

   $$\alpha \in \Sigma_{\alpha}(\Lambda) \Rightarrow -\alpha \in \Sigma_{\alpha}(\Lambda).$$

3. Monotonicity of the maximal eigenvalue with respect to $\Lambda$:

   $$\max \Sigma_{\alpha}(\Lambda) < \max \Sigma_{\alpha}(\Lambda + 1) \quad \text{and} \quad \lim_{\Lambda \to +\infty} \left[ \max \Sigma_{\alpha}(\Lambda) \right] = 1.$$  

4. Density property

   $$\Sigma_{\alpha}(\Lambda) \text{ becomes uniformly dense in } [-1, 1] \text{when } \Lambda \to +\infty.$$

5. On our fuzzy sphere $S^2_\Lambda$ the state $\chi$ most localized around the North pole fulfills the property $L_3 \chi = 0$ (item (C) of theorem 4.1), as the generalized quantum state (distribution) $2\delta(\theta)/\sin \theta \simeq \delta(\chi_1) \delta(\chi_2)$ on $S^2$ concentrated on the North pole (here $\theta$ is the colatitude); the classical counterpart of this property is that the classical particle on $S^2$ in the position $(0, 0, 1)$ has zero $L_3$ (z-component of the angular momentum).
It is important to underline that these are welcome properties for a $x_i$-operator which is required to approximate well, in the commutative limit, the $x_i$-coordinate of a quantum particle forced to stay on the unit sphere $S^d$.

Moreover, the spectrum of $L_i$ is $\Sigma_{L_i}(\Lambda) = \{-\Lambda, 1-\Lambda, \ldots, \Lambda\}$ for all $i = 1, 2, 3$, by the $SO(3)$-covariance, and fulfills properties 1, 2 (the multiplicity of the eigenvalue $m$ is $\Lambda - |m| + 1$).

In the Madore fuzzy sphere, since the $x_i$ are obtained by the rescaling (2) of angular momentum operators acting in an irreducible representation, then all $x_i$ have again the same spectrum as $x_3$, by $SO(3)$-covariance, and this is obtained by the rescaling of the spectrum of $L_3$; this leads to the eigenvalues (all simple) and eigenvectors
\[
x_3\varphi_m = \frac{m}{\sqrt{\Lambda^2 + \Lambda}} \varphi_m \quad \text{with } m \in \{-\Lambda, \cdots, \Lambda\},
\]
where we have set $\Lambda \equiv (n - 1)/2$. Hence also in this case properties 1–4 are fulfilled.

However, for this reason there is no longer room for independent observables playing the role of angular momentum operators on the carrier Hilbert space $V_\Lambda$, and property 5 is lost.

For this reason, and the other ones mentioned in the introduction, from our point of view it is more natural to interpret the $x_i$ simply as the spin phase space algebra, not as a fuzzyfication of the algebra of configuration space observables on $S^d$.

Appendix A. A very useful proposition

In the next proofs we often use the following

**Proposition A.1.** Let $A = (a_{ij})_{i,j=1}^n$ be a square matrix such that $a_{ij} \geq 0 \ \forall i,j$, then there exist a vector $\tilde{\chi} \in \mathbb{R}_+^n$ fulfilling
\[
\|\tilde{\chi}\|_2 = 1 \quad \text{and} \quad \|A[\tilde{\chi}]\|_2 = \|A\|_2.
\]

**Proof.** By definition
\[
\|A\|_2 = \sup_{\|\chi\|_2=1} \|A[\chi]\|_2,
\]
the Weierstrass theorem implies that
\[
\sup_{\|\chi\|_2=1} \|A[\chi]\|_2 = \max_{\|\chi\|_2=1} \|A[\chi]\|_2 = \|A\|_2 \quad \text{(A.1)},
\]
so we can consider a vector $\tilde{\chi} \in \mathbb{R}_+^n$ fulfilling (A.1) and $\|\tilde{\chi}\|_2 = 1$. We need to prove that $\tilde{\chi}_i \geq 0$ for all $i$. If we suppose that $\tilde{\chi}_j < 0$ for some $j$ in $\{1, 2, \cdots, n\}$, then we can define the vector $\hat{\chi} := (|\tilde{\chi}_1|, |\tilde{\chi}_2|, \cdots, |\tilde{\chi}_n|)^T$. It is such that $\|\hat{\chi}\|_2 = \|\tilde{\chi}\|_2 = 1$ and
\[
\|A[\hat{\chi}]\|_2 = \sqrt{\sum_{i=1}^n \left( \sum_{j=1}^n a_{ij} \hat{\chi}_j \right)^2} \leq \sqrt{\sum_{i=1}^n \left( \sum_{j=1}^n a_{ij} \tilde{\chi}_j \right)^2} = \sqrt{\sum_{i=1}^n \left( \sum_{j=1}^n a_{ij} \tilde{\chi}_j \right)^2} = \|A[\tilde{\chi}]\|_2.
\]
This last inequality proves that we can consider in the realization of the maximum the ‘positive’ vector $\tilde{\chi}$, instead of $\tilde{\chi}$, so the proof is finished. $\square$

**Proposition A.2.** Let $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$ be square matrices fulfilling $0 \leq a_{ij} \leq b_{ij} \; \forall i,j$, then

$$\|A\|_2 \leq \|B\|_2.$$ 

**Proof.** According to proposition A.1 we can consider a vector $\tilde{\chi} \in \mathbb{R}_+^n$ with $\|\tilde{\chi}\|_2 = 1$ fulfilling $\|A\|_2 = \|A[\tilde{\chi}]\|_2$; so

$$\|A\|_2 = \sqrt{\sum_{i=1}^n \left( \sum_{j=1}^n a_{ij} \tilde{\chi}_j \right)^2} \leq \sqrt{\sum_{i=1}^n \left( \sum_{j=1}^n b_{ij} \tilde{\chi}_j \right)^2} = \|B\|_2. \square$$

**Appendix B. The proofs of theorems of section 3.2**

**B.1. Proof of item (A) in theorem 3.1**

Consider the unitary and involutive operator $U_1 = U_1^\dagger = U_1^{-1}$ corresponding to the inversion operator of the $x_1$-axis (this exists by the $O(2)$-covariance of our model\(^2\): $U_1 x_1 U_1 = -x_1$, $U_1 x_2 U_1 = x_2$. Then $x_1 \chi = \alpha \chi$ implies $x_1 (U_1 \chi) = -\alpha (U_1 \chi)$, i.e. $U_1 \chi$ is an eigenvector of $x_1$ with the opposite eigenvalue.

**B.2. Proof of item (B) in theorem 3.1**

According to the last proof, if $I_n$ is the $n \times n$ identity matrix and $M_{\lambda} (\alpha) := X_\lambda + \alpha I_{2\lambda+1}$, then the eigenvalue problem for $X_\lambda$ is equivalent to solve $\det [M_{\lambda} (\alpha)] = 0$. In order to do this we define $M_{\lambda}^\alpha$ as the $n \times n$ submatrix of $M_{\lambda}$ formed by the first $n$ rows and columns, then

$$p_{\lambda} (\alpha) := \det [M_{\lambda} (\alpha)] \quad \text{and} \quad p_{\lambda}^\alpha (\alpha) := \det \{M_{\lambda}^\alpha (\alpha)\}.$$ 

It is not difficult to see that

- when $\Lambda = 1$, then

$$\begin{vmatrix}
\alpha & \beta_1 & 0 \\
\beta_1 & \alpha & \beta_2 \\
0 & \beta_2 & \alpha
\end{vmatrix} = \alpha \left[ \alpha^2 - \frac{(b_0)^2}{4} - \frac{(b_1)^2}{4} \right] =: p_1 (\alpha) \quad \Rightarrow \quad \begin{cases}
\alpha_1(1) = \sqrt{\frac{(b_0)^2 + (b_1)^2}{4}} = \sqrt{\frac{2}{\gamma}}, \\
\alpha_2(1) = 0, \\
\alpha_3(1) = -\sqrt{\frac{(b_0)^2 + (b_1)^2}{4}} = -\sqrt{\frac{2}{\gamma}},
\end{cases} \quad \text{(B.1)}$$

\(^2\) $U_1$ is obtained by projection on $\mathcal{H}_\Lambda$ of the original unitary operator $\hat{U}_1$ acting on $\mathcal{L}^1 (\mathbb{R}^2)$ as follows:

$\hat{U} : \psi(x_1, x_2) \rightarrow \psi(-x_1, x_2).$
• when \( \Lambda = 2 \), then

\[
p_2(\alpha) := \begin{vmatrix}
\alpha & \frac{b_2}{2} & 0 & 0 & 0 \\
\frac{b_2}{2} & \alpha & \frac{b_2}{2} & 0 & 0 \\
0 & \frac{b_2}{2} & \alpha & \frac{b_2}{2} & 0 \\
0 & 0 & \frac{b_2}{2} & \alpha & \frac{b_2}{2} \\
0 & 0 & 0 & \frac{b_2}{2} & \alpha
\end{vmatrix}
\]

\[
= \alpha \left\{ \alpha^4 - \alpha^2 \left[ \frac{(b_2)^2 + (b_1)^2 + (b_0)^2 + (b_{-1})^2}{4} + \frac{(b_1b_{-1})^2 + (b_2b_0)^2 + (b_2b_{-1})^2}{16} \right] \right\}
\]

\[
\Rightarrow \begin{align*}
\alpha_1(2) &= \sqrt{\frac{1}{2} \sqrt{A_2} + \sqrt{B_2}} = \frac{1}{2} \sqrt{3 + \frac{1}{3}}, \\
\alpha_2(2) &= \sqrt{\frac{1}{2} \sqrt{A_2} - \sqrt{B_2}} = \frac{1}{2} \sqrt{1 + \frac{1}{3}}, \\
\alpha_3(2) &= 0, \\
\alpha_4(2) &= -\sqrt{\frac{1}{2} \sqrt{A_2} - \sqrt{B_2}} = -\frac{1}{2} \sqrt{1 + \frac{1}{3}}, \\
\alpha_5(2) &= -\sqrt{\frac{1}{2} \sqrt{A_2} + \sqrt{B_2}} = -\frac{1}{2} \sqrt{3 + \frac{1}{3}},
\end{align*}
\]

(B.2)

because \( A_2 := (b_2)^2 + (b_1)^2 + (b_0)^2 + (b_{-1})^2 = 4 \left( 1 + \frac{1}{3} \right) \) and

\[
B_2 := 2 \left[ (b_1b_0)^2 + (b_2b_0)^2 + (b_{-1}b_0)^2 + (b_2b_{-1})^2 - (b_1b_{-1})^2 - (b_2b_0)^2 \right]
\]

\[
+ (b_1)^4 + (b_1)^4 + (b_0)^4 + (b_{-1})^4 = 4.
\]

• in general, when \( \Lambda > 2 \), one can calculate \( p_{\Lambda} (\alpha) \) through the use of this recursion formula:

\[
\begin{align*}
p_{\Lambda}^2 (\alpha) &:= \det \{ M_{\Lambda}^2 (\alpha) \} = \alpha^2 - \left( \frac{b_\Lambda}{\alpha} \right)^2, \\
p_{\Lambda}^3 (\alpha) &:= \det \{ M_{\Lambda}^3 (\alpha) \} = \alpha \left[ \alpha^2 - \frac{(b_\Lambda)^2 + (b_{\Lambda-1})^2}{4} \right], \\
p_{\Lambda}^4 (\alpha) &:= \alpha \left[ p_{\Lambda}^3 (\alpha) \right] - \left( \frac{b_{\Lambda-1}}{2} \right)^2 p_{\Lambda}^2 (\alpha), \\
p_{\Lambda}^5 (\alpha) &:= \alpha \left[ p_{\Lambda}^4 (\alpha) \right] - \left( \frac{b_{\Lambda-2}}{2} \right)^2 p_{\Lambda}^3 (\alpha),
\end{align*}
\]

(B.3)

So the claim is true because of (B.3) and the following

**Theorem B.1 (The Favard theorem, [26] (p. 60)).** Let \( \{ p_n(x) = x^n + \cdots \} (n = 0, 1, \cdots) \) be a sequence of polynomials with real coefficients, satisfying a recursion formula

\[
p_n(x) = (x - \beta_n) p_{n-1}(x) - \sum_0^{n-2} p_n(x)
\]

with positive \( \Sigma_0 \) and real \( \beta_n \); then there exists a distribution \( d\alpha \) such that

\[
\int_{-\infty}^{+\infty} p_n(x) p_m(x) d\alpha(x) = 0 \quad (m \neq n).
\]
**Theorem B.2 ([27] (p 44)).** The zeros of the orthogonal polynomials \( p_n(x) \), associated with distribution \( d\alpha(x) \) on the interval \([a, b]\) are distinct and are located in the interior of the interval \([a, b]\).

**B.3. Proof of item (C) in theorem 3.1**

First of all, we have to recall that \( \rho(A) = \|A\|_2 \) for every symmetric matrix \( A \), where \( \rho(A) \) is the spectral radius, i.e.

\[
\rho(A) := \max \{ |\lambda_j| : \lambda_j \in \Sigma_A \}.
\]

From \( 1 \leq b_n < \sqrt{1 + \frac{\Lambda(\Lambda - 1)}{k(\Lambda)}} \) and proposition A.2 we can infer

\[
\alpha_1(\Lambda) = \|X^A\|_2 \leq \sqrt{1 + \frac{\Lambda(\Lambda - 1)}{k(\Lambda)}} \left\| P_{2\Lambda + 1} \left( 0, 1, \frac{1}{2} \right) \right\|_2 = \sqrt{1 + \frac{\Lambda(\Lambda - 1)}{k(\Lambda)}} \cos \left( \frac{\pi}{2\Lambda + 2} \right)
\]

and

\[
\alpha_1(\Lambda + 1) = \|X^{A+1}\|_2 \geq \left\| P_{2\Lambda + 3} \left( 0, 1, \frac{1}{2} \right) \right\|_2 = \cos \left( \frac{\pi}{2\Lambda + 4} \right).
\]

On the other hand, by algebraic calculations, one can easily see that

\[
\sqrt{1 + \frac{\Lambda(\Lambda - 1)}{k(\Lambda)}} \cos \left( \frac{\pi}{2\Lambda + 2} \right) \leq \cos \left( \frac{\pi}{2\Lambda + 4} \right)
\]

is equivalent to

\[
k(\Lambda) \geq \frac{\Lambda(\Lambda - 1)}{\cos^2 \left( \frac{\pi}{2\Lambda + 2} \right) - \cos^2 \left( \frac{\pi}{2\Lambda + 4} \right)}
\]

\[
= \frac{\Lambda(\Lambda - 1)}{2 \sin \left( \frac{\pi(2\Lambda + 3)}{(2\Lambda + 2)(2\Lambda + 4)} \right) \sin \left( \frac{\pi}{(2\Lambda + 2)(2\Lambda + 4)} \right) \left[ \cos \left( \frac{\pi}{2\Lambda + 4} \right) + \cos \left( \frac{\pi}{2\Lambda + 2} \right) \right]}.
\]

And using

\[
a + 1 \geq \frac{1 + a}{a(a + 2)} \cos^2 \left( \frac{\pi}{2\Lambda + 2} \right) + \cos \left( \frac{\pi}{2\Lambda + 2} \right) \leq \frac{1}{2} \quad \forall \Lambda \in \mathbb{N} \quad \text{and} \quad \sin x \geq x^2 \quad \forall x \in \left[ 0, \frac{1}{2} \right],
\]

we obtain

\[
\frac{\Lambda(\Lambda - 1)}{2 \sin \left( \frac{\pi(2\Lambda + 3)}{(2\Lambda + 2)(2\Lambda + 4)} \right) \sin \left( \frac{\pi}{(2\Lambda + 2)(2\Lambda + 4)} \right) \left[ \cos \left( \frac{\pi}{2\Lambda + 4} \right) + \cos \left( \frac{\pi}{2\Lambda + 2} \right) \right]} < \frac{\Lambda(\Lambda - 1)}{4 \left( \frac{\pi}{2\Lambda + 2} \left( \frac{\pi}{2\Lambda + 4} \right) \right)^2}
\]

\[
< \frac{1}{4\pi^2 \Lambda(\Lambda + 1)(2\Lambda + 2)^2(2\Lambda + 3)^2(2\Lambda + 4)^2}.
\]
According to this,
\[ k(\Lambda) \geq \frac{1}{4\pi^2} \Lambda(\Lambda - 1)(2\Lambda + 2)^2(2\Lambda + 3)^2(2\Lambda + 4)^4 \Rightarrow \alpha_1(\Lambda) < \alpha_1(\Lambda + 1) \quad \forall \Lambda \in \mathbb{N}. \]

**B.4. Proof of item (D) in theorem 3.1**

The scheme of the proof is the following:

- We firstly prove
  \[ \lim_{\Lambda \to +\infty} \alpha_1(\Lambda) = 1. \] (B.5)
- Then we note that, in the limit \( \Lambda \to +\infty \), \( X^\Lambda \) can be approximated by \( P_\Lambda(0, \frac{1}{2}, \frac{1}{2}) \); so we consider the spectra of both matrices.
- For every \( \Lambda \in \mathbb{N} \) we define a continuous, odd and increasing (with respect to \( x \)) function \( G_\Lambda(x) \) mapping one spectrum into the other.
- Through lemma B.1 and B.2 we prove theorem B.3, which tells us that \( \lim_{\Lambda \to +\infty} G_\Lambda(x) = x \forall x \in [-1, 1] \).
- Finally, in theorem B.4, we prove that \( G_\Lambda \to I \) uniformly, and this trivially implies the claim of (D).

As for the previous proof, from
\[ \frac{1}{2} \leq b_n^2 \leq \frac{1}{2} \sqrt{1 + \frac{\Lambda(\Lambda - 1)}{k}} \quad \forall n \in \{ \Lambda, \Lambda - 1, \cdots, 2 - \Lambda, 1 - \Lambda \} \]
and proposition A.2 we obtain
\[ \left\| P_{2\Lambda+1}(0, \frac{1}{2}, \frac{1}{2}) \right\|_2 \leq \left\| X^\Lambda \right\|_2 \leq \left\| P_{2\Lambda+1}(0, \sqrt{1 + \frac{\Lambda(\Lambda - 1)}{k}}, \sqrt{1 + \frac{\Lambda(\Lambda - 1)}{k}}) \right\|_2, \]
which is equivalent to
\[ \cos \left( \frac{\pi}{2\Lambda + 2} \right) \leq \alpha_1(\Lambda) \leq \sqrt{1 + \frac{\Lambda(\Lambda - 1)}{k}} \cos \left( \frac{\pi}{2\Lambda + 2} \right). \] (B.6)

This and \( k = k(\Lambda) \geq \Lambda^2 (\Lambda + 1)^2 \) concludes the proof of (B.5).

The inequality (20)_2 follows trivially from (B.6), \( \cos x \geq 1 - \frac{x^2}{2} \forall x \in [0, 1] \) and \( \frac{\pi}{2\Lambda+2} \leq 1 \forall \Lambda \in \mathbb{N} \).

Corollary 6.3.8 in [28] p. 370 states that (here \( M_\text{n} \) is the space of \( n \times n \) complex matrices)

*Let \( A, E \in M_\text{n} \) assume that \( A \) is Hermitian and that \( A + E \) is normal, let \( \{\lambda_1, \cdots, \lambda_n\} \) be the eigenvalues of \( A \) arranged in increasing order \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) and let \( \{\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n\} \) be the eigenvalues of \( A + E \), ordered so that \( \text{Re} (\tilde{\lambda}_1) \leq \text{Re} (\tilde{\lambda}_2) \leq \cdots \leq \text{Re} (\tilde{\lambda}_n) \). Then
\[ \left[ \sum_{i=1}^{n} |\tilde{\lambda}_i - \lambda_i|^2 \right]^\frac{1}{2} \leq \|E\|_2. \] (B.7)*

According to this, setting \( A := P_{2\Lambda+1}(0, \frac{1}{2}, \frac{1}{2}), E := X^\Lambda - P_{2\Lambda+1}(0, \frac{1}{2}, \frac{1}{2}) \), then \( A \) and \( A + E \) are both symmetric, so (B.7) becomes
\[
\left[ \sum_{i=1}^{2\Lambda+1} |\alpha_i(\Lambda) - \tilde{\alpha}_i(\Lambda)|^2 \right]^{\frac{1}{2}} \leq \|E\|_2.
\]
From \(\sqrt{1+x} \leq 1 + \frac{x}{2}\), \(k = k(\Lambda) \geq \Lambda^2 (\Lambda + 1)^2\) and \(|n| \leq \Lambda\) we obtain
\[
\frac{1}{2} \left[ \sqrt{1 + \frac{n(n-1)}{k}} - 1 \right] \leq \frac{n(n-1)}{4k} \leq \frac{1}{4(\Lambda+1)^2},
\]
so proposition A.2 implies
\[
\|E\|_2 \leq \left\| P_{2\Lambda+1} \left( 0, \frac{1}{4(\Lambda+1)^2}, \frac{1}{4(\Lambda+1)^2} \right) \right\|_2 = \frac{1}{2(\Lambda+1)^2} \cos \left( \frac{\pi}{2\Lambda+2} \right) < \frac{1}{2(\Lambda+1)^2}
\]
and then
\[
\sum_{i=1}^{2\Lambda+1} |\alpha_i(\Lambda) - \tilde{\alpha}_i(\Lambda)|^2 < \frac{1}{2(\Lambda+1)^2} \forall \Lambda.
\]  
(B.8)
For every \(\Lambda \in \mathbb{N}\) we can define a continuous function \(G_{\Lambda} : [-1,1] \rightarrow [-\alpha_1(\Lambda), \alpha_1(\Lambda)]\) such that \(G_{\Lambda} \left( \tilde{\alpha}_n(\Lambda) \right) = \alpha_n(\Lambda), G_{\Lambda}(-x) = -G_{\Lambda}(x), G_{\Lambda}(x) = \alpha_1(\Lambda) \forall x \in [\tilde{\alpha}_1(\Lambda), 1]\), for instance we can join two ‘consecutive’ points \((\tilde{\alpha}_i(\Lambda), \alpha_i(\Lambda))\) and \((\tilde{\alpha}_{i+1}(\Lambda), \alpha_{i+1}(\Lambda))\) by a straight line; furthermore, because of
\[
G_{\Lambda} \left( \tilde{\alpha}_n(\Lambda) \right) = \alpha_n(\Lambda) < G_{\Lambda} \left( \tilde{\alpha}_{n-1}(\Lambda) \right) = \alpha_{n-1}(\Lambda),
\]
we can assume that every function \(G_{\Lambda}(x)\) is also increasing with respect to \(x\).

The \(G_{\Lambda}(x)\) are all odd functions so we can restrict our attention to the \(x \in [0,1]\), but it is also true that the continuity and the monotonicity of every \(G_{\Lambda}\) imply that
\[
\forall \varepsilon > 0, \forall x \in [0,1] \exists \delta = \delta(\varepsilon, \Lambda, x) s.t. y \in [0,1],\begin{cases} |x-y| < \delta \Rightarrow |G_{\Lambda}(x) - G_{\Lambda}(y)| < \varepsilon, \\
|x-y| > \delta \Rightarrow |G_{\Lambda}(x) - G_{\Lambda}(y)| > \varepsilon. \end{cases}
\]
At this point we need to prove the following

**Lemma B.1.** Let \(\varepsilon > 0\) and \(\pi \in [0,1]\) such that
\[
\limsup_{\Lambda \rightarrow +\infty} |\pi - G_{\Lambda}(\pi)| = 0,
\]
then
\[
\liminf_{\Lambda \rightarrow +\infty} \delta(\varepsilon, \Lambda, \pi) = \delta(\varepsilon, \pi) > 0.
\]  
(B.9)

**Proof.** Let \(\varepsilon > 0\) and assume, per absurdum, that
\[
\liminf_{\Lambda \rightarrow +\infty} \delta(\varepsilon, \Lambda, \pi) = 0,
\]
then we can find a sequence \(\{\Lambda_n\}_{n \in \mathbb{N}}\) such that
\[
\lim_{n} \delta(\varepsilon, \Lambda_n, \pi) = 0
\]  
(B.10)
and, correspondingly, because of (B.10) we can assume that \(n\) is sufficiently large so that we can find \(x \in [0,1]\) with \(\frac{\varepsilon}{2} > |\pi - x| > \delta(\varepsilon, \Lambda_n, \pi)\), \(|\pi - G_{\Lambda_n}(\pi)| < \frac{\varepsilon}{2}\) and \(|G_{\Lambda_n}(\pi) - G_{\Lambda_n}(x)| > \varepsilon\);
then
\[ |x - G_{\tilde{\Lambda}}(x)| = |x - \tilde{x} - G_{\tilde{\Lambda}}(\tilde{x}) + G_{\tilde{\Lambda}}(\tilde{x}) - G_{\tilde{\Lambda}}(x)| \]
\[ \geq \left| G_{\tilde{\Lambda}}(\tilde{x}) - G_{\tilde{\Lambda}}(x) \right| - |\tilde{x} - x| - |\tilde{x} - G_{\tilde{\Lambda}}(\tilde{x})| \]
\[ \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \]

This last inequality and (18) imply that there exist a finite set of indices \( I \) with \( |I| = m(n) \) such that the corresponding eigenvalues of \( P_{\tilde{\Lambda}+1} \), in symbols \( \{ \tilde{\alpha}_i(\tilde{\Lambda}_n) \}_{i \in I} \), fulfill
\[ \frac{\varepsilon}{4} > |\tilde{x} - \tilde{\alpha}_i(\tilde{\Lambda}_n)| > \delta(\varepsilon, \tilde{\Lambda}_n, \tilde{x}) \quad \forall i \in I \implies |\tilde{\alpha}_i(\tilde{\Lambda}_n) - G_{\tilde{\Lambda}}[\tilde{\alpha}_i(\tilde{\Lambda}_n)]| > \frac{\varepsilon}{2} \quad \forall i \in I \]
and of course (B.10) implies that \( m(n) \xrightarrow{n \to +\infty} +\infty \), so
\[ \lim_{n} \left[ \sum_{i \in I} |\tilde{\alpha}_i(\tilde{\Lambda}_n) - G_{\tilde{\Lambda}}[\tilde{\alpha}_i(\tilde{\Lambda}_n)]|^2 \right] = +\infty, \]
which disagrees with (B.8), so the proof is finished. \( \square \)

Let
\[ A := \{ x \in [0, 1] : \limsup_{n} |x - G_{\Lambda}(x)| = 0 \} \]
we have that \( 0 \in A \) and also that \( 1 \in A \) because
\[ \lim_{\Lambda \to +\infty} \alpha_1(\Lambda) = \lim_{\Lambda \to +\infty} \tilde{\alpha}_1(\Lambda) = \lim_{\Lambda \to +\infty} G_{\Lambda}[\tilde{\alpha}_1(\Lambda)] = 1. \]

In order to prove item (D) in theorem 3.1 we need the following

**Lemma B.2.** If \( 0 \leq \pi \leq 1, \pi \in A \), then \( \exists \sigma > 0 \) such that \( x \in \max \{ \pi - \sigma, 0 \}, \min \{ \pi + \sigma, 1 \} \implies x \in A \).

**Proof.** Let \( \varepsilon > 0 \), then lemma B.1 implies
\[ \liminf_{\Lambda \to +\infty} \delta(\varepsilon, \Lambda, \pi) = \tilde{\delta}(\varepsilon, \tilde{x}) > 0; \]
so, if we set \( \sigma := \min \left\{ \frac{\delta(\varepsilon, \pi)}{2}, \varepsilon \right\} \) and we take \( x \in \max \{ \pi - \sigma, 0 \}, \min \{ \pi + \sigma, 1 \} \), then
\[ \limsup_{\Lambda \to +\infty} |x - G_{\Lambda}(x)| = \limsup_{\Lambda \to +\infty} |x - G_{\Lambda}(x) - \tilde{x} + \tilde{x} - G_{\tilde{\Lambda}}(\tilde{x}) + G_{\tilde{\Lambda}}(\tilde{x})| \leq \limsup_{\Lambda \to +\infty} |x - \tilde{x}| + |\tilde{x} - G_{\tilde{\Lambda}}(\tilde{x})| + |G_{\Lambda}(x) - G_{\tilde{\Lambda}}(\tilde{x})| \leq 2\varepsilon, \]
of course \( \varepsilon \) can be chosen arbitrary small, so the proof is finished. \( \square \)

According to this, we can trivially infer that

**Corollary B.1.**

\[ A = [0, 1] \]

or
\[ A = [0, x_1] \cup [x_2, x_3] \cup \cdots \cup [x_n, 1] \quad \text{and} \quad B := [0, 1] \setminus A = [x_1, x_2] \cup [x_3, x_4] \cup \cdots, \]

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where \( x_1 < x_2 < x_3 < x_4 \cdots \) are suitable points of \([0, 1] \).

We are now ready to prove the following

**Theorem B.3.**

\[ A = [0, 1] \]

**Proof.** Let us assume, per absurdum, that \( A \neq [0, 1] \), then corollary B.1 implies

\[ B := [0, 1] \setminus A = [x_1, x_2] \cup [x_3, x_4] \cup \cdots, \]

(B.11)

so if \( x \in A, \delta > 0, x_1 - \delta < x < x_1 \) and \( \limsup_{\Lambda \to +\infty} |x_1 - G_\Lambda (x_1)| = k > 0 \), then

\[ \limsup_{\Lambda \to +\infty} |G_\Lambda (x) - G_\Lambda (x_1)| = \limsup_{\Lambda \to +\infty} |G_\Lambda (x) - x_1 + x_1 - G_\Lambda (x_1)| \]

\[ \leq \limsup_{\Lambda \to +\infty} |G_\Lambda (x) - x_1| + |x_1 - G_\Lambda (x_1)| \]

\[ \leq \delta + k, \]

(B.12)

because \( x \in A \).

On the other hand

\[ \limsup_{\Lambda \to +\infty} |G_\Lambda (x) - G_\Lambda (x_1)| = \limsup_{\Lambda \to +\infty} |G_\Lambda (x) - x_1 + x_1 - G_\Lambda (x_1)| \]

\[ \geq \limsup_{\Lambda \to +\infty} |x_1 - G_\Lambda (x_1)| - |G_\Lambda (x) - x_1| \]

\[ \geq k - \delta. \]

(B.13)

According to this, we obtain

\[ \lim_{\delta \to 0} \left[ \limsup_{\Lambda \to +\infty} |G_\Lambda (x) - G_\Lambda (x_1)| - k \right] = 0 \]

so we can infer that \( \limsup_{\Lambda \to +\infty} G_\Lambda (x_1) = k + x_1 \) and we can find a sequence \( \{ \tilde{\Lambda}_n \}_{n \in \mathbb{N}} \) such that

\[ \lim_{n \to +\infty} G_{\tilde{\Lambda}_n} (x_1) = k + x_1, \]

but we also know that \( G_\Lambda (x) \) is increasing with respect to \( x \), so

\[ \liminf_{n \to +\infty} G_{\tilde{\Lambda}_n} (x) \geq k + x_1 \quad \forall x \in \left[ x_1, x_1 + \frac{k}{2} \right]. \]

This implies

\[ \liminf_{n \to +\infty} \left| x - G_{\tilde{\Lambda}_n} (x) \right| \geq k + x_1 - \left( x_1 + \frac{k}{2} \right) = \frac{k}{2} \quad \forall x \in \left[ x_1, x_1 + \frac{k}{2} \right]. \]

This last inequality and (18) imply that there exist a finite set of indices \( I \) with \(|I| = m(n)\) such that the correspondings eigenvalues of \( P_{\Sigma \tilde{\Lambda}_n} (0, \frac{1}{2}, \frac{1}{2}) \), in symbols \( \{ \tilde{\alpha}_i (\tilde{\Lambda}_n) \}_{i \in I} \), fulfill
\[ \tilde{\alpha}_i \left( \tilde{\Lambda}_n \right) \in \left[ x_1, x_1 + \frac{k}{2} \right] \quad \forall i \in I \implies |\tilde{\alpha}_i \left( \tilde{\Lambda}_n \right) - G_{\tilde{\Lambda}_n} \left( \tilde{\alpha}_i \left( \tilde{\Lambda}_n \right) \right)| > \frac{k}{4} \quad \forall i \in I \]

and of course \( m(n) \xrightarrow{n \to +\infty} +\infty \), so

\[
\lim_{n \to +\infty} \sum_{i \in I} |\tilde{\alpha}_i \left( \tilde{\Lambda}_n \right) - G_{\tilde{\Lambda}_n} \left( \tilde{\alpha}_i \left( \tilde{\Lambda}_n \right) \right)|^2 = +\infty,
\]

which disagrees with (B.8), so the proof is finished. \( \square \)

According to this, we have that

\[
\lim_{\Lambda \to +\infty} G_{\Lambda}(x) = x \quad \forall x \in [0, 1],
\]

in the next theorem we will always denote the sequence \( \{\Lambda\}_{\Lambda \in \mathbb{N}} \) and its subsequences with the same notation.

**Theorem B.4.**

\[
\lim_{\Lambda \to +\infty} \left( \max_{x \in [0,1]} \{|x - G_{\Lambda}(x)|\} \right) = 0.
\]

**Proof.** Let us assume, per absurdum, that

\[
\lim_{\Lambda \to \infty} \max_{x \in [0,1]} \{|x - G_{\Lambda}(x)|\} = M > 0,
\]

and set

\[ x_{\Lambda} := \max_{x \in [0,1]} \{|x - G_{\Lambda}(x)|\} ; \]

we have that (up to a suitable subsequence)

\[
\lim_{\Lambda \to \infty} |x_{\Lambda} - G_{\Lambda}(x_{\Lambda})| = M.
\]

The sequence \( \{x_{\Lambda}\}_{\Lambda \in \mathbb{N}} \) is bounded, so we have that (up to a further suitable subsequence)

\[
\lim_{\Lambda \to +\infty} x_{\Lambda} = \bar{x} \in [0, 1] = A,
\]

at this point, let us choose \( \varepsilon, x \) so that

\[
0 < \varepsilon < \frac{M}{8} , \quad \sigma := \min \left\{ \frac{\delta(\varepsilon, \bar{x})}{2}, \frac{M}{8} \right\} > 0 , \quad x \in \left[ \bar{x} - \frac{\sigma}{2}, \bar{x} + \frac{\sigma}{2} \right]
\]

and \( \Lambda \) such that

\[
|x_{\Lambda} - G_{\Lambda}(x_{\Lambda})| > \frac{M}{2} , \quad |x - x_{\Lambda}| < \sigma , \quad |\bar{x} - x_{\Lambda}| < \sigma,
\]

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then we obtain (if \( \Lambda \) is sufficiently large)
\[
|\mathbf{x} - G_{\Lambda}(\mathbf{x})| \geq |\mathbf{x}_L - G_{\Lambda}(\mathbf{x}_L)| - \frac{M}{2} - |\mathbf{x}_R - G_{\Lambda}(\mathbf{x}_R)| - \left| G_{\Lambda}(\mathbf{x}) - G_{\Lambda}(\mathbf{x}) \right|
\]
This last inequality implies that there exist a finite set of indices \( I \) with \( |I| = m(\Lambda) \) such that the corresponding eigenvalues of \( P_{2\tilde{\Lambda} + 1}(0, \frac{1}{2}, \frac{1}{2}) \), in symbols \( \{\tilde{\alpha}_i(\Lambda)\}_{i \in I} \), fulfill
\[
\tilde{\alpha}_i(\Lambda) \in \left[ \frac{\pi}{2} - \frac{\sigma}{2}, \frac{\pi}{2} + \frac{\sigma}{2} \right] \quad \forall i \in I \quad \Rightarrow \quad |\tilde{\alpha}_i(\Lambda) - G_{\Lambda}(\tilde{\alpha}_i(\Lambda))| > \frac{M}{8} \quad \forall i \in I
\]
and of course \( m(\Lambda) \xrightarrow{\Lambda \to +\infty} +\infty \), so
\[
\lim_{\Lambda \to +\infty} \left[ \sum_{i \in I} \left| \tilde{\alpha}_i(\Lambda) - G_{\Lambda}(\tilde{\alpha}_i(\Lambda)) \right|^2 \right] = +\infty,
\]
which disagrees with (B.8), so the proof is finished. \( \square \)

We are now ready to complete the proof of item (D) of theorem 3.1, because if \( \varepsilon > 0 \) the last theorem implies that there exists a \( \tilde{\Lambda} = \tilde{\Lambda}(\varepsilon) \) such that \( |\mathbf{x} - G_{\Lambda}(\mathbf{x})| < \varepsilon \) \( \forall \Lambda > \tilde{\Lambda} \) and \( \forall \mathbf{x} \in [0, 1] \), while (18) implies
\[
|\tilde{\alpha}_{i+1}(\Lambda) - \tilde{\alpha}_i(\Lambda)| = \left| \cos \left( \frac{(n+1)\pi}{\Lambda + 1} \right) - \cos \left( \frac{n\pi}{\Lambda + 1} \right) \right| = 2 \sin \left( \frac{2n+1}{\Lambda + 1} \right) \sin \left( \frac{\pi}{\Lambda + 1} \right) \leq 2 \sin \left( \frac{2\pi}{\Lambda + 1} \right),
\]
this means that there exists a \( \tilde{\Lambda} = \tilde{\Lambda}(\varepsilon) \) such that \( |\tilde{\alpha}_i(\Lambda) - \tilde{\alpha}_{i+1}(\Lambda)| < \varepsilon \) \( \forall \Lambda > \tilde{\Lambda}, \forall i \).

Finally, if we set \( \tilde{\Lambda}(\varepsilon) = \max \left\{ \tilde{\Lambda}(\varepsilon), \tilde{\Lambda}(\varepsilon) \right\} \), then \( \forall \Lambda > \tilde{\Lambda} \) we obtain
\[
|\alpha_i(\Lambda) - \alpha_{i+1}(\Lambda)| \leq |\tilde{\alpha}_i(\Lambda) - \tilde{\alpha}_{i+1}(\Lambda)| + |\alpha_{i+1}(\Lambda) - \tilde{\alpha}_{i+1}(\Lambda)| + |\tilde{\alpha}_i(\Lambda) - \tilde{\alpha}_{i+1}(\Lambda)| + |\tilde{\alpha}_i(\Lambda) - \tilde{\alpha}_{i+1}(\Lambda)|
\]
\[
< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,
\]
so the proof is completed.

Appendix C. The proofs of theorems of section 4.2

C.1. Proof of item (A) in theorem 4.1

Consider the unitary and involutive operator \( U_0 = U_0^* = U_0^{-1} \) corresponding to the inversion operator of the \( x_3 \)-axis (this exists by the \( O(3) \)-covariance of our model): \( U_0 x_0 U_0 = -x_0 \).
\( U_0 x_\pm U_0 = x_\pm \). Then \( x_0 \chi = \alpha \chi \) implies \( x_0 (U_0 \chi) = -\alpha (U_0 \chi) \), i.e. \( U_0 \chi \) is an eigenvector of \( x_0 \) with the opposite eigenvalue.

C.2. Proof of item (B) in theorem 4.1

According to the last proof, we can equivalently set \( M_m(\Lambda; \alpha) := B_m(\Lambda) + \alpha I_{\Lambda - m + 1} \), then the eigenvalue problem for \( B_m(\Lambda) \) is equivalent to solve \( \det [M_m(\Lambda; \alpha)] = 0 \); in order to do this we define \( M_m^h \) as the \( h \times h \) submatrix of \( M_m \) formed by the first \( h \) rows and columns, then
\[ p_n(\Lambda; m) (\alpha) := \det [M_m(\Lambda; \alpha)] \quad \text{and} \quad p_n^h(\Lambda; m) (\alpha) := \det \{ M_m^h (\Lambda; \alpha) \}, \]

where \( n(\Lambda; m') := \Lambda - |m'| + 1 \) is the degree of the polynomial \( p_n(\Lambda; m') \).

It is not difficult to see that

- when \( n = 1 \leftrightarrow |m| = \Lambda \), then \( \alpha = 0 \);
- when \( n = 2 \), then
  \[
  \begin{vmatrix}
  \alpha & c_{\Lambda A}^{0,\Lambda-1} \\
  c_{\Lambda A}^{0,\Lambda-1} & \alpha
  \end{vmatrix} = \alpha^2 - \left( c_{\Lambda A}^{0,\Lambda-1} \right)^2 =: p_2(\alpha) \quad \Rightarrow \quad \alpha_{1,2} = \pm c_{\Lambda A}^{0,\Lambda-1},
  \]
- when \( n = 3 \), then
  \[
  \begin{vmatrix}
  \alpha & c_{\Lambda A}^{-1,\Lambda-2} & 0 \\
  c_{\Lambda A}^{-1,\Lambda-2} & \alpha & c_{\Lambda A}^{0,\Lambda-2} \\
  0 & c_{\Lambda A}^{0,\Lambda-2} & \alpha
  \end{vmatrix} = \alpha \left[ \alpha^2 - \left( c_{\Lambda A}^{0,\Lambda-2} \right)^2 \right] - \alpha \left( c_{\Lambda A}^{-1,\Lambda-2} \right)^2
  \]
  \[=: p_3(\alpha);\]
- in general, let \( \bar{n} = n(\Lambda; \bar{m}) \), then one can calculate \( p_n(\alpha) \) through the use of this recursion formula

\[
\begin{aligned}
p_1^2(\alpha) &:= \det \{ M_m^2 (\Lambda; \alpha) \}, \\
p_1^3(\alpha) &:= \det \{ M_m^3 (\Lambda; \alpha) \}, \\
p_1^4(\alpha) &:= \alpha \left[ p_1^3(\alpha) \right] - \left( c_{m+2}^{0,m} \right)^2 p_1^2(\alpha), \\
p_1^5(\alpha) &:= \alpha \left[ p_1^4(\alpha) \right] - \left( c_{m+4}^{0,m} \right)^2 p_1^3(\alpha), \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
p_{\bar{n}}^\bar{n}(\alpha) &:= \alpha \left[ p_{\bar{n}-1}^{\bar{n}-1}(\alpha) \right] - \left( c_{\bar{m}}^{0,\bar{m}} \right)^2 p_{\bar{n}-2}^{\bar{n}-2}(\alpha).
\end{aligned}
\]  

Then the proof of item (B) follows trivially from (C.1), theorem B.1 and theorem B.2, as for appendix B.2.

**C.3. Proof of (33) in theorem 4.1**

In this proof we will use the following theorem (here \( \{ p_n \}_{n \in \mathbb{N}} \) is a sequence of orthogonal polynomials):

**Theorem C.1.** [27] (p. 46) Let \( x_1 < x_2 < \cdots < x_2 \) be the zeros of \( p_n(x) \). Then each interval \([x_{i-1}, x_{i+1}]\) contains exactly one zero of \( p_{n+i}(x) \).

Anyway, this is the scheme of the proof:

- We firstly use theorem B.1 to prove that there exist a \( \mathbb{R} \)-measure such that the polynomials \( \{ p_m^{h}(n; \Lambda; m) \}_{m=1}^{m(\Lambda; m)} \) are orthogonal with respect to that measure; this implies that we can apply theorem B.2 getting that all the roots of every polynomial \( p_m^{h}(n; \Lambda; m) \) are real and simple.
• Then we use lemma C.2 and theorem C.1 to prove also that

$$\rho \left( B_m \right) = \| B_m \|_2 < \| B_{m-1}^{n(\Lambda,m)} \|_2 = \rho \left( B_{m-1}^{n(\Lambda,m)} \right).$$

where $\rho$ is the spectral radius.

• This last inequality involving the spectral radii trivially implies (33).

According to this, let us start with the first point of this scheme.

**Lemma C.1.** The roots of $p_n^{h(\Lambda,m)}(\alpha)$ are real and simple, and if $\alpha_{i}^{(\Lambda,m)}(\alpha) > \alpha_{i-1}^{(\Lambda,m)}(\alpha) > \cdots > \alpha_{0}^{(\Lambda,m)}(\alpha)$ are the zeros of $p_n^{h(\Lambda,m)}(\alpha)$, then every interval $[\alpha_{i+1}^{(\Lambda,m)}(\alpha), \alpha_{i}^{(\Lambda,m)}(\alpha)]$ contains exactly one zero of $p_n^{h(\Lambda,m)}(\alpha)$.

**Proof.** The matrices $B_{m-1}^{h(\Lambda,m)}$ are all symmetric, so the roots of $p_n^{h(\Lambda,m)}(\alpha)$ are real; while the sequence of polynomials $\{p_n^{h(\Lambda,m)}(\alpha)\}_{h=1}^n$ fulfill the recurrence relation (C.1) and because of theorem B.1 we can infer that there exists a distribution $d\Theta(\alpha)$ such that

$$\int_{-\infty}^{+\infty} p_n^{h(\Lambda,m)}(\alpha) p_n^{h(\Lambda,m)}(\alpha) d\Theta(\alpha) = 0 \quad (j \neq h).$$

Finally, we can apply theorem B.2 and theorem C.1 to the set $\{p_n^{h(\Lambda,m)}(\alpha)\}_{h=1}^n$ of polynomials, so the proof is finished. □

We firstly prove an inequality involving the $B_{m}$-matrix elements, which implies the aforementioned inequality between the spectral radii.

**Lemma C.2.** Let

$$1 \leq m \leq \Lambda, \quad j \in \mathbb{N}_0, \quad 1 \leq l := m + j \leq \Lambda; \quad (C.2)$$

then

$$c_l A_{l+1}^{0,m-1} > c_{l+1} A_{l}^{0,m}. \quad (C.3)$$

**Proof.** Because of (24) and (25), we obtain

$$c_l A_{l+1}^{0,m-1} = \sqrt{1 + \frac{l^2}{k}} \sqrt{\frac{(l + m - 1)(l - m + 1)}{4l^2 - 1}}$$

and

$$c_{l+1} A_{l}^{0,m} = \sqrt{1 + \frac{(l + 1)^2}{k}} \sqrt{\frac{(l + m + 1)(l - m + 1)}{4(l + 1)^2 - 1}},$$

then (C.3) becomes

$$\left( 1 + \frac{l^2}{k} \right) \left( l + m - 1 \right) \left( 4l^2 - 1 \right) - \left( 1 + \frac{(l + 1)^2}{k} \right) \left( l + m + 1 \right) \left( 4(l + 1)^2 - 1 \right) > 0$$

$\forall 1 \leq m \leq \Lambda$ and $1 \leq l \leq \Lambda$; by algebraic calculations, one can prove that the last inequality is equivalent to the following one:
\[
\left[ \frac{k + \hat{l}}{A} \right] (l + m - 1)(2l + 3) - \left[ \frac{k + (l + 1)^2}{B} \right] (l + m + 1)(2l - 1) > 0 \quad (C.4)
\]

\(1 \leq m \leq \Lambda\) and \(1 \leq l \leq \Lambda\).

Furthermore, one has
\[
A = \quad 2k\hat{l}^2 + 2k\hat{l}m + k\hat{l} + 3km - 3k + 2\hat{l}^3 + 2\hat{l}^2m + l^3 + 3l^2m - 3l^2,
\]
\[
B = \quad 2k\hat{l}^2 + 2k\hat{l}m + k\hat{l} - km - k + 2\hat{l}^3 + 2\hat{l}^3m + 5l^3 + 3l^2m + 3l^2 - l - m - 1;
\]
finally, (C.4) becomes
\[
A - B = 4km - 2k - 4\hat{l}^3 - 6\hat{l}^2 + l + m + 1 > 0
\]

\(1 \leq m \leq \Lambda\) and \(1 \leq l \leq \Lambda\).

From \(k(\Lambda) \geq \Lambda^2 (\Lambda + 1)^2\) we obtain
\[
4km - 2k - 4\hat{l}^3 - 6\hat{l}^2 + l + m + 1 \geq 2\Lambda^2(\Lambda + 1)^2 - 4\Lambda^3 - 6\Lambda^2 = 2\Lambda^2(\Lambda^2 - 2) \geq 0 \quad \forall \Lambda \geq 2,
\]
while when \(\Lambda = 1\)
\[
4km - 2k - 4\hat{l}^3 - 6\hat{l}^2 + l + m + 1 \geq 2[1^2(2)] - 4 - 6 + 3 = 1,
\]
so the proof is finished. \(\square\)

**Lemma C.3.** Let \(m \geq 1\), then
\[
\|B_m\|_2 < \|B_{m-1}^{(\Lambda,m)}\|_2.
\]

**Proof.** The matrices \(B_m\) and \(B_{m-1}^{(\Lambda,m)}\) have the same dimensions, they are explicitly
\[
B_m = \begin{pmatrix}
0 & c_m & 0 & 0 & \cdots & 0 & 0 \\
c_m & c_{m+1} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & c_{\Lambda A_{\Lambda}^{\text{aw}}}
\end{pmatrix}
\]
and
\[
B_{m-1}^{(\Lambda,m)} = \begin{pmatrix}
0 & c_m & 0 & 0 & \cdots & 0 & 0 \\
c_m & c_{m+1} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & c_{\Lambda - 1 A_{\Lambda - 1}^{\text{aw}}}
\end{pmatrix}
\]
Lemma C.2, together with proposition A.2, (C.5) and (C.6), imply
\[ \| B_m \|_2 < \| B_{m-1}^{(\Lambda,m)} \|_2, \]
so the proof is finished. \( \square \)

At this point, let \( \alpha_1 (\Lambda) := \max \{ \alpha_1 (\Lambda; 0) ; \alpha_1 (\Lambda; 1) ; \cdots ; \alpha_1 (\Lambda; \Lambda) \} \) and assume, per absurdum, that \( \alpha_1 (\Lambda) = \alpha_1 (\Lambda; m) \) with \( m > 0 \). We can take the matrix \( B_{m-1} \) and its elements; from lemma C.3 we can infer
\[ \| B_m \|_2 < \| B_{m-1}^{(\Lambda,m)} \|_2; \] (C.7)
and from lemma C.1 we know that the eigenvalues of \( B_{m-1}^{(\Lambda,m)} \) ‘separate’ the ones of \( B_m \), then
\[ \rho \left( B_{m-1}^{(\Lambda,m)} \right) < \rho \left( B_{m-1} \right). \] (C.8)

The inequalities (C.7) and (C.8) lead us to \( \alpha_1 (\Lambda) < \alpha_1 (\Lambda; m-1) \), but this is not possible. We can then conclude that \( \alpha_1 (\Lambda) = \alpha_1 (\Lambda; 0) \) and with the same procedure we can prove the other inequalities in (33).

C.4. Proof of (34) in theorem 4.1

Let
\[ \hat{B}_0 (\Lambda) := \begin{pmatrix} 0 & A_1^{0,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1^{0,0} & 0 & A_2^{0,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2^{0,0} & 0 & A_3^{0,0} & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & A_{\Lambda-1}^{0,0} & 0 & A_{\Lambda}^{0,0} \\ 0 & 0 & 0 & 0 & 0 & A_{\Lambda}^{0,0} & 0 \end{pmatrix} \]
and its spectrum \( \{ \hat{\alpha}_i (\Lambda; 0) \}_{i=1}^{\Lambda+1} \), where the eigenvalues are arranged in descending order.

First of all, from \( 1 \leq c_1 \leq \sqrt{1 + \frac{\Lambda^2}{k(\Lambda)}} \quad \forall 1 \leq l \leq \Lambda \) and proposition A.2 we obtain
\[ \alpha_1 (\Lambda; 0) = \| B_0 (\Lambda) \|_2 \leq \sqrt{1 + \frac{\Lambda^2}{k(\Lambda)}} \| \hat{B}_0 (\Lambda) \|_2 = \sqrt{1 + \frac{\Lambda^2}{k(\Lambda)}} \hat{\alpha}_1 (\Lambda; 0) \]
and \( \alpha_1 (\Lambda + 1; 0) = \| B_0 (\Lambda + 1) \|_2 \geq \| \hat{B}_0 (\Lambda + 1) \|_2 = \hat{\alpha}_1 (\Lambda + 1; 0) \); then, by algebraic calculations, one has
\[ \sqrt{1 + \frac{\Lambda^2}{k(\Lambda)}} \hat{\alpha}_1 (\Lambda; 0) \leq \hat{\alpha}_1 (\Lambda + 1; 0) \Leftrightarrow k(\Lambda) \geq \frac{\Lambda^2 [\hat{\alpha}_1 (\Lambda; 0)]^2}{[\hat{\alpha}_1 (\Lambda + 1; 0)]^2 - [\hat{\alpha}_1 (\Lambda; 0)]^2}, \] (C.9)

As done for appendix B.2, one can use theorems B.1 and B.2 to prove that \( \hat{\alpha}_1 (\Lambda + 1; 0) > \hat{\alpha}_1 (\Lambda; 0) \forall \Lambda \in \mathbb{N} \), while it is obvious that
\[ \sqrt{\frac{\rho}{4\rho - 1}} > \frac{1}{2} \quad \forall \Lambda \in \mathbb{N} \quad \Rightarrow \quad \| \hat{B}_0 (\Lambda) \|_2 = \hat{\alpha}_1 (\Lambda; 0) > \cos \left( \frac{\pi}{\Lambda + 2} \right) \quad \forall \Lambda \in \mathbb{N}; \]
finally, in appendix B.3.5 we prove $\alpha_1(\Lambda;0) \to 1$ when $\Lambda \to +\infty$.

According to this, one has $\hat{\alpha}_1(\Lambda;0) \uparrow 1$, $\hat{\alpha}_1(\Lambda;0) = \cos\left(\frac{\pi}{\sqrt{\Lambda+2}}\right) + \varepsilon(\Lambda)$ with $\varepsilon(\Lambda) \geq 0$ and $\varepsilon(\Lambda) \to 0$.

It is well known that $\cos x = 1 - \frac{x^2}{2} + o\left(x^3\right)$, then it is obvious that $\varepsilon(\Lambda) = \frac{1}{\Lambda} + o\left(\frac{1}{\Lambda}\right)$ when $\Lambda \to +\infty$ is not possible, because it is in constrast with $\hat{\alpha}_1(\Lambda;0) = \cos\left(\frac{\pi}{\sqrt{\Lambda+2}}\right) + \varepsilon(\Lambda) \leq 1 \forall \Lambda$; for the same reason, it must be

$$\varepsilon(\Lambda) < \frac{\pi^2}{2(\Lambda+2)^2} \quad \text{when} \quad \Lambda \to +\infty.$$ 

Finally, this and

$$\cos\left(\frac{\pi}{\Lambda+3}\right) - \cos\left(\frac{\pi}{\Lambda+2}\right) = \frac{\pi}{\Lambda^2} + o\left(\frac{1}{\Lambda^2}\right)$$

imply

$$\hat{\alpha}_1(\Lambda+1;0) - \hat{\alpha}_1(\Lambda;0) = \frac{C}{\Lambda^2} + o\left(\frac{1}{\Lambda^2}\right) \quad \text{when} \quad \Lambda \to +\infty,$$

for a suitable constant $\tilde{C} > 0$.

Coming back to (C.9), from $\sqrt{\frac{1}{3}} \leq \hat{\alpha}_1(\Lambda;0) < 1 \forall \Lambda \in \mathbb{N}$ we can infer

$$\frac{\Lambda^2[\hat{\alpha}_1(\Lambda;0)]^2 - [\hat{\alpha}_1(\Lambda+1;0)]^2}{[\hat{\alpha}_1(\Lambda+1;0)]^2} \leq 1$$

when $\Lambda \to +\infty$.

Then

$$k(\Lambda) \geq \Lambda^6 \Rightarrow \alpha_1(\Lambda+1;0) > \alpha_1(\Lambda;0) \quad \text{definitively.}$$

C.5. Proof of item (D) in theorem 4.1

First of all, from $\sqrt{\frac{\pi}{\sqrt{\Lambda+1}}} > \frac{1}{2} \forall \Lambda \in \mathbb{N}$ and proposition A.2 we obtain

$$\left\|P_{\Lambda+1}\left(0,\frac{1}{2},\frac{1}{2}\right)\right\|_2 = \cos\left(\frac{\pi}{\Lambda+1}\right) < \alpha_1(\Lambda;0) = \left\|B_0(\Lambda)\right\|_2, \quad (C.10)$$

then the inequality (35.2) follows trivially from (C.10), $\cos x \geq 1 - \frac{x^2}{2} \forall x \in [0, 1]$ and $\frac{\pi}{\sqrt{\Lambda+2}} \leq 1 \forall \Lambda \geq 2$.

On the other hand, if $x_1$ is the $x_0$-eigenvector having $\alpha_1(\Lambda,0)$ eigenvalue, then $L_0x_1 = 0$, which implies

$$\langle x_1, x_+x_1 \rangle = 0, \quad \langle x_1, x-x_1 \rangle = 0 \Rightarrow \langle x_1, x_1 \rangle = 0, \quad \langle x_1, x_2x_1 \rangle = 0;$$

so, from

$$(\Delta x)_{x_1}^2 := \langle x_1, x^2x_1 \rangle - \sum_{i=1}^3 \langle x_1, x_ix_1 \rangle^2 \geq 0,$$
we obtain
\[ [\alpha_1(\Lambda, 0)]^2 = \langle \chi_1, x_0 \chi_1 \rangle \leq \langle \chi_1, x^2 \chi_1 \rangle \leq 1 + \frac{\Lambda(\Lambda + 1)}{k(\Lambda)}. \] (C.11)

It is obvious that (C.10) and (C.11) trivially imply
\[ \lim_{\Lambda \to +\infty} \alpha_1(\Lambda, 0) = 1. \]

Once proved this, then the proof of (D) is essentially the same of appendix B.4, the only difference is that here \( A = P_{\Lambda+1}(0, 1/2, 1/2), A + E = B_0(\Lambda) \) and \( \|E\|_2 \leq 2 \left\{ \sqrt{1 + \frac{1}{12}} \sqrt{\frac{1}{4} - \frac{1}{2}} \right\} \), which follows from proposition A.2, (9) and

\[ c(A)^{0,m} = \sqrt{1 + \frac{\rho^2}{k(\Lambda)}} \sqrt{\frac{\rho^2 - m^2}{4 \rho^2 - 1}} \leq \sqrt{1 + \frac{1}{(\Lambda + 1)^2}} \left[ \frac{1}{2} + \left( \frac{1}{\sqrt{4 \rho^2 - 1}} - \frac{1}{2} \right) \right] \]
\[ = \sqrt{1 + \frac{1}{(\Lambda + 1)^2}} \left[ \frac{1}{2} + \left( \frac{1}{\sqrt{4 \rho^2 - 1}} + \frac{1}{2} \right) \right] \]
\[ \leq \sqrt{1 + \frac{1}{(\Lambda + 1)^2}} \left[ \frac{1}{2} + \frac{1}{12} \right]. \]

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