Density operators that extremize Tsallis entropy
and thermal stability effects

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Abstract
Quite general, analytical (both exact and approximate) forms for discrete probability distributions (PD’s) that maximize Tsallis entropy for a fixed variance are here investigated. They apply, for instance, in a wide variety of scenarios in which the system is characterized by a series of discrete eigenstates of the Hamiltonian. Using these discrete PD’s as “weights” leads to density operators of a rather general character. The present study allows one to vividly exhibit the effects of non-extensivity. Varying Tsallis’ non-extensivity index $q$ one is seen to pass from unstable to stable systems and even to unphysical situations of infinite energy.

1 Introduction
Tsallis’ thermostatistics is today a new paradigm for statistics, with applications to several scientific disciplines [1, 2, 3, 4, 5]. Notwithstanding its manifold applications, some details of the basic thermostatistical formalism remain unexplored. This is why analytical results are to be welcome, specially if they are, as the ones to be here investigated, of a very general nature. We will provide this type of results for discrete probability distributions of fixed variance that maximize Tsallis’ entropy.

Given a discrete probability distribution (DPD) $p = \{p_k\}$, its Tsallis’ information measure (or entropic form) is defined as

$$H_q(p) = \frac{1}{q - 1} \left(1 - \sum_{k=-\infty}^{+\infty} p_k^q\right).$$  \hspace{1cm} (1)
It is a classical result that as \( q \to 1 \), Tsallis entropy reduces to Shannon entropy

\[
H_1(p) = - \sum_{k=\infty}^{\infty} p_k \log p_k.
\]

Without loss of generality, we will here consider only centered random variables of fixed variance.

The aim of this paper is to provide accurate estimates of i) the parameters of the DPD’s and ii) their behavior. The maximizers of Tsallis’ information measure under variance constraint in the continuous, multivariate case have been discussed in [6].

Consider a quantum system whose eigenstates are characterized by a set of quantum numbers that we collectively denote with an integer, running index \( k \) (Cf. Eq. (1)), that will of course also label the eigensolutions \( |\psi_k\rangle, \epsilon_k \) of the pertinent time-independent Schrödinger equation

\[
\mathcal{H} |\psi_k\rangle = \epsilon_k |\psi_k\rangle, \tag{2}
\]

with \( \mathcal{H} \) the Hamiltonian. Let \( p_k \equiv |\psi_k|^2 \) be the probability of finding our system in the state \( |\psi_k\rangle \). The mixed state

\[
\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|, \tag{3}
\]

commutes with the Hamiltonian by construction and represents thus a bona fide possible stationary state of the system. If we now find a physical quantity \( Z \) whose mean value is proportional to the variance, we can interpret \( \rho \) as the mixed state that maximizes Tsallis measure subject to the a-priori known expectation value of such a physical quantity. We will show below that, in these circumstances, universal expressions can be given for the \( p_k \)'s, and thus for \( \rho \).

We discuss possible applications in the forthcoming section. Afterwards, after having introduced some definitions and notations, we characterize the discrete Tsallis maximizers for fixed variance in both the \( q < 1 \) and the \( q > 1 \) cases, and discuss thermal stability questions. We pass then to analyze extensions to the multivariate cases. For the sake of completeness, some of the proofs of assertions referenced to in what follows are given in Annex.

## 2 Possible physical applications

Several physical models can be adapted to the scenario described above (Cf. Eqs. (4) and (6)). The weights \( p_k \) in (6) will be of the Tsallis-power law form. With these weights, \( \rho \) maximizes Tsallis’ entropy subject to the constraint of a constant variance, which introduces a Lagrange multiplier that we will call \( \beta \).

Given a physical quantity \( Z \) whose mean value is proportional to the variance, \( \rho \) is that mixed state which maximizes Tsallis measure subject to the a-priori
known $\langle Z \rangle$—value. In the examples below $Z$ is the system’s energy $E$, but many other possibilities can be imagined. Thus, $\rho$ will be the state maximizing Tsallis’ $H_q$ for a fixed value of the expectation value $U = \langle E \rangle = \text{Tr} [H \rho]$ of the Hamiltonian. As a consequence, the multiplier $\beta$ can be thought of as an “inverse temperature” $T$, that is, set $\beta = 1/(k_B T)$, with $k_B$ the Boltzmann constant. This is so because we are at liberty of imagining that $U$ is kept constant because it is in contact with a heat reservoir \[7\]. Of course, this is not necessarily the case. $\rho$ exists by itself and is a legitimate stationary mixed state of our system. But we can think of $\beta$ as either a “real” or an “equivalent” inverse temperature.

Consider, for example, a system for which the energy spectrum consists of a denumerable set of $N$ ($N$ possibly infinite) energy levels labeled by a quantum number $k$ with $p_{-k} = p_k$, so that all levels exhibit a degeneracy

$$g_k = 2 \text{ for } k \neq 0; \; k > 0; \; \text{and } g_0 = 1,$$

i.e., the sums in (1) run from 0 up to $\infty$ and each summand is multiplied by $g_k$. Within the present framework, $U$ \[1\]

$$U = \langle E \rangle = \sum_{k=0}^{+\infty} g_k p_k E_k,$$

becomes numerically equal to the DPD’s variance, which by definition is fixed and assumedly known a priori. As just stated, we may think, if we wish, that our system is in contact with a heat reservoir, which fixes the mean energy, and that the associated Lagrange multiplier $\beta$ can be assimilated to an inverse temperature $T$. Among many examples of such a scenario we mention here:

- the planar rotor \[8\], where $k$ is the magnetic quantum number corresponding to the azimuthal angle usually denoted by $\phi$. The level-energies $E_k$ are proportional to $k^2$ and

$$E_k = C E k^2; \; C_E \text{ has dimension of energy.} \quad (6)$$

We have $C_E = \hbar^2/2M_I$, with $M_I$ the system’s moment of inertia \[9\]. For simplicity’s sake we take here $C_E = 1$, but retain, of course, its energy-units.

- the three-dimensional rigid rotator \[8\], although $k$ now means the quantum number $L$ associated to orbital angular momentum (for large $k$, the spectrum of \[6\] looks like that of the 3D rigid rotor) and, for all $k \geq 0$, $g_k = 1$.

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\[1\]Mean values linear in the PD are employed here. They are quite legitimate and were used by Tsallis in his seminal 1988 paper \[2\]. The Legendre structure of thermodynamics is definitely respected using them in conjunction with Tsallis’ entropy \[3\]. Recently, Ferri, Martinez and Plastino have shown \[4\] that PD’s obtained in this manner can be easily translated into PD’s constructed via MaxEnt using $q$—expectation values evaluated à la Tsallis-Mendes-Plastino \[5\], showing thereby that there is a one-to-one correspondence between the two types of PD.
• a vibrating string of length $2l$ with fixed ends $[10]$, whose energy eigenvalues are $E_k \propto [k\pi/2l]^2$.

• the case of a particle (of mass $m$) in a box subject to periodic boundary conditions (at $(-L, +L)$) whose energy values are $\epsilon_k = [2\pi^2 \hbar^2 k^2/(mL^2)]$ $[7]$.

3 Definitions and Notations

In what follows, $m$ is a positive real number and $n$ a positive integer. Often, $m$ will take the special form of an odd integer $m = 2n + 1$. The solutions to the problem of maximization of Tsallis entropy “under variance constraint” (here equivalent to “for fixed mean energy”) will be called discrete Tsallis-probability laws (DTPL’s) in this manuscript. Since (see preceding Section) fixed mean energy can be thought of as implying contact with a heat reservoir at the fixed temperature $T = 1/(k_B \beta)$ we will call the pertinent Lagrange multipliers $\beta$ inverse temperatures in what follows. DTPL’s are given by the following two theorems.

**Theorem 1** If $\frac{1}{3} < q < 1$, for $k_B T = 1/\beta$, with $k_B$ the Boltzmann constant, the discrete probability law $p$ with zero mean and mean energy $U$ (equivalently, variance $\sigma^2$) that maximizes Tsallis’ entropy is defined as

$$p_k = \Pr\{X = k\} = f^{-1}_q(T)(1 + \frac{k^2}{k_B T})^{\frac{1}{1-q}} \quad \forall k \in \mathbb{Z}^+, \quad (7)$$

where $f_q(T)$ is the partition function, given by

$$f_q(T) = \sum_{k=0}^{+\infty} g_k (1 + \frac{k^2}{k_B T})^{\frac{1}{1-q}}, \quad (8)$$

and $\beta = 1/k_B T$ is a real positive Lagrangian multiplier such that

$$U \equiv \sigma^2 = \sum_{k=0}^{+\infty} g_k k^2 p_k.$$

In the limit $T \to \infty$, our probability distribution converges to the classical discrete Gaussian distribution $[11]$.

Notice the existence of a lower bound $q = 1/3$. Smaller $q$ values are unphysical because for them the mean energy of the model (for the PD $[7]$) becomes infinite. Additionally (see below) the system turns out to be thermodynamically unstable because the specific heat becomes negative. In this case where $q < 1$, we define the real positive parameter $m$ that we need below by

$$m = \frac{1 + q}{1 - q}. \quad (9)$$

$^2$the dependence of $p$ on $\beta = 1/k_B T$ is omitted for notational simplicity
Theorem 2 If $q > 1$, the discrete probability distribution $p$ with zero mean and mean energy $U$ (equivalently, variance $\sigma^2$) that maximizes Tsallis entropy is defined as

$$p_k = \Pr\{X = k\} = \begin{cases} f_q^{-1}(T)(1 - \frac{k^2}{k_BT})^{\frac{1}{q-1}} & \forall k \in [0, \lfloor k_BT \rfloor - 1] \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

where $f_q(T)$ is the partition function

$$f_q(T) = \sum_{k=0}^{|\beta|-1} g_k (1 - \frac{k^2}{k_BT})^{\frac{1}{q-1}}, \quad (11)$$

$\lfloor k_BT \rfloor$ denotes the integer part of $k_BT$ whose inverse, namely, $\beta$, is a real positive Lagrangian multiplier such that

$$U = \sigma^2 = \sum_{k=0}^{\infty} g_k k^2 p_k.$$

In the present case where $q > 1$, we define $m$ as a real positive parameter related to $q$ by

$$m = \frac{q + 1}{q - 1}, \quad (12)$$
to be of use below. Note that this definition differs from definition (9) in the case $q < 1$.

We do not give the proofs of theorems (1) and (2) here. That laws defined by (8) and (11) are the solutions of the Tsallis entropy maximization problem can be obtained by extending the results of the continuous case as presented in [6]. The cases i) $q < 1$ and ii) $q > 1$ differ essentially by the fact that the latter has a finite support. As far as we know, this is a novel way, in the $q$–literature, of dealing with the celebrated Tsallis-cut-off [2].

We remark on the fact that the forms (7) and (10) coincide with the distribution-form given by Tsallis in his pioneer 1988 paper [2] (see also [5]), namely,

$$p_k = Z_q^{-1}[1 - (q - 1)\beta^* k^2]^{1/(q-1)}, \quad (13)$$

that, for $q \to 1$, tends to the discrete Gaussian [2]

$$Z_1^{-1} \exp [-\beta^* k^2]. \quad (14)$$

In the former case (Eq. (7)) we have $q < 1$, while in the later (Eq. (10)) $q > 1$. Thus, in Eq. (7) we have $\beta = (1-q)\beta^*$, while for (10) $\beta = (q-1)\beta^*$. Of course, according to (7), (10), and (14), for $q \to 1$ we have $\beta^* \to \beta$. 

5
4 Approximate treatments for $q < 1$

4.1 The two regimes

Let us recall that, for $q < 1$, $m = (1 + q)/(1 - q)$. Thus, $m$ is large if $q \to 1$ (Boltzmann limit (BL)), while a small positive value of $m$ entails $q \to -1$. Careful inspection of the partition function $f_q(T)$ in the cases $m = 3 \leftrightarrow q = 1/2$, $m = 5 \leftrightarrow q = 2/3$, and $m = 7 \leftrightarrow q = 3/4$ indicates that two regimes should be distinguished:

- the first regime, called ”power law” regime, corresponds to the case $T \ll 1$ (low temperatures)
- the second regime called the ”Student-t” regime, corresponds to the case $T \gg 1$ (high temperatures)

Anticipating the results provided by the following theorems, we note that the cases ”small $T$”/”large $T$” can be translated as joint range values of parameters $U$ and $m \equiv (1 + q)/(1 - q)$ : typically, the large $T$ case corresponds to jointly large values of $U$ and $m$, as expected, while the small case corresponds to jointly small values of these parameters. This is illustrated on the curve below (Fig. 1), where the large $T$ property is translated as $T \geq 100$, and the small $T$ property as $T \leq 0.01$. Note that the left bound $n = 1/2$ corresponds to $m = 2n + 1 = 2$, i.e., $q = 1/3$, and thus to an (unphysical) infinite $U$. For economy’s sake we set herefrom

$$a = \sqrt{k_B T}.$$  \hspace{1cm} (15)

4.2 The power law regime

In the power-law regime ($T \ll 1$), a detailed characterization of the distribution can be obtained for all real values of $m$, as shown in the following theorem.

**Theorem 3** Assuming that $T \ll 1 \; (a \ll 1)$ and with $m$ defined by (14), the following approximations hold:

1. for the partition function

$$f_q(T) \simeq 1 + 2a^{m+1}\zeta(m+1),$$

2. for the probability law

$$p_k \simeq \begin{cases} \frac{a^{m+1}}{1+2a^{m+1}\zeta(m+1)}k^{-m-1} & \text{if } k \neq 0 \\ \frac{1}{1+2a^{m+1}\zeta(m+1)} & \text{if } k = 0 \end{cases},$$  \hspace{1cm} (16)

and corresponds thus to a discrete power-law. It tends to a Gaussian for $q \to 1$. 


3. for the mean energy

\[ U \simeq 2a^{m+1}\zeta(m-1). \]  

(17)

4. If, moreover, \( m = 2n + 1 \) with \( n \in \mathbb{N}, n \geq 1 \), the characteristic function can be approximated as

\[ \phi_m(u) \simeq \frac{1}{(1 + 2a^{2n+2}\zeta(2n+2))} \left(1 + 2a^{2n+2}(-1)^n(2\pi)^{2n+2}(2n+2)!B_{2n+2}(\frac{u}{2\pi})\right), \]

where \( B_n(u) \) denotes the Bernoulli polynomial of order \( n \).

**Proof.** The proof is given in Annex 1.

The specific heat \( C = \frac{dU}{dT} \) is proportional to \( (m+1) \). This is positive, and thus the system stable, for all \( 1 < q < 1 \).

Notice that, according to (16), only the ground state is populated at \( T = 0 \), as it should.

The probability \( p_k \) decreases exponentially with \( k \), being thus concentrated around its mean value \( EX = 0 \). As an example, if \( a = 0.1 \), the following table shows, for several values of \( q > \frac{1}{3} \), the first values of \( p_k \).

|        | \( k = 0 \)    | \( k = \pm 1 \) | \( k = \pm 2 \) | \( k = \pm 3 \) |
|--------|----------------|-----------------|-----------------|-----------------|
| \( q = \frac{1}{4} \) | 0.99978        | 9.9978 \times 10^{-3} | 6.2486 \times 10^{-6} | 1.2343 \times 10^{-6} |
| \( q = \frac{1}{5} \) | 0.99998        | 9.9998 \times 10^{-6} | 3.1249 \times 10^{-7} | 4.1151 \times 10^{-8} |
| \( q = \frac{2}{3} \) | \simeq 1.00000 | 1.00000 \times 10^{-6} | 1.5625 \times 10^{-8} | 1.3717 \times 10^{-9} |
| \( q = \frac{3}{4} \) | \simeq 1.00000 | 1.00000 \times 10^{-7} | 7.8125 \times 10^{-10} | 4.5725 \times 10^{-11} |

Table 1: first values of \( p_k \) for several values of \( q \) and \( a = 0.1 \)

4.3 The Student-t regime

In the case of the Student-t regime (high temperatures), our main result writes as follows:

**Theorem 4** \( \forall \varepsilon > 0, \exists a_0 > 0 \) and \( \exists m_0 > 0 \) such that if \( a \geq a_0 \) and \( 2 < m \leq m_0 \) then

\[ \sum_{k=0}^{\infty} g_k \left(1 + \frac{k^2}{a^2}\right)^{-\frac{m+1}{2}} - a\sqrt{\pi} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)} \leq \varepsilon. \]  

(18)

The proof of this result is given in Annex 2. The values of \( a_0 \) and \( m_0 \) such that (18) holds are given in Fig. 2 for \( \epsilon = 10^{-9} \). As a direct consequence of this result, we have the following explicit approximations, valid if, simultaneously, i) \( a \gg 1 \) (\( T \gg 1 \)) and ii) \( q \to 1 \).

**Proposition 5** If \( a \gg 1 \), \( m \gg 1 \), and \( \frac{a^2}{m} \gg 1 \), then the following approximations hold with high accuracy:
• for the partition function
\[ f_q(a) \simeq a^{\sqrt{\frac{m}{2}}/a} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(m/2\right)} \times \frac{\Gamma\left(m/2\right)}{\Gamma\left(m+1/2\right)}, \]

• for the probability law (remember that \( U = \sigma^2 \))
\[ p_k \simeq \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(m/2\right)\Gamma\left(\frac{1}{2}\right)} \sigma \sqrt{m-2} \left(1 + \frac{k^2}{\sigma^2(m-2)}\right)^{-\frac{m+1}{2}}, \quad (19) \]

• while for the mean energy
\[ U \simeq \frac{a^2}{m-2}. \quad (20) \]

Note that here
\[(m - 2) U = k_B T = \beta^{-1}, \quad (21)\]
a result that will be needed below.

\textbf{Proof}, the expression of the partition function results from theorem (4). Using a by-product of proof of theorem (3), the mean energy \( U \) reads
\[ U = a^2 \frac{\Gamma\left(m+1/2\right)}{\Gamma\left(m/2\right)\Gamma\left(1/2\right)} \frac{\Gamma\left(m+1/2\right)}{\Gamma\left(m+1/2\right)} - 1 = \frac{a^2}{m-2}. \]

As a consequence, \( a \simeq \sqrt{U(m-2)} \) and the law writes
\[ p_k \simeq \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(m/2\right)\Gamma\left(\frac{1}{2}\right)} \sqrt{U(m-2)} \left(1 + \frac{k^2}{U(m-2)}\right)^{-\frac{m+1}{2}} \]

Notice that in this case the specific heat is proportional to \((m - 2)^{-1}\). This is negative, and thus the system unstable, unless \( q > 1/3 \). In other words, the system is stable in the interval \((1/3, 1]\). We remark also that, as a consequence of (17) and (20), the mean energy \( U \) is an increasing function of \( a \) (and thus of \( T \)), as it should.

\textbf{Remark 6} A remarkable result here is that expression (19) is exactly the sampled version - with the same partition function - of the continuous maximizer of the Tsallis entropy whose expression is recalled here [12]:
\[ f(x) = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(m/2\right)\Gamma\left(\frac{1}{2}\right)} \sqrt{U(m-2)} \left(1 + \frac{x^2}{U(m-2)}\right)^{-\frac{m+1}{2}}, \quad \forall x \in \mathbb{R} \]

This entails that, at very high \( T \), one can approximate sums over discrete energy levels by an integral. In order to understand the situation, remember that the results holds for \( m \gg 1 \), which entails \( q \sim 1 \), i.e., the Boltzmann limit. In this limit, replacing sums by integrals is a commonplace text-book procedure.
5 The $q > 1$ instance

5.1 A general result and its consequences

Recall that in the case $q > 1$, $m = (q+1)/(q-1)$. The equivalent of theorem 4 writes as follows:

Theorem 7 $\forall \varepsilon > 0$, $\exists a_0 > 0$ and $\exists m_0 > 0$ such that if $a \geq a_0$ and $m \geq m_0$ then

$$|\sum_{k=-a-1}^{a-1} (1 - \frac{k^2}{a^2})^{m-1} - a\frac{\sqrt{\pi}}{\Gamma(m/2 + 1)}| \leq \varepsilon. \quad (22)$$

Proof. The proof of this result is given in Annex 3. It essentially follows the steps of the proof of theorem 4.

We depict in Fig. 3 the area (north-east), delimited in the plane $(a,m)$, that contains values of $a$ and $m$ for which approximation (22) holds, with $\varepsilon = 10^{-10}$.

Corollary 8 For $a \gg 1$ ($T \gg 1$) the mean energy $U$ verifies

$$U \simeq \frac{a^2}{m + 2} \quad (23)$$

Proof. Denoting $n = \frac{m-1}{2}$ and $f_n(a)$ for $f_q(a)$, the normalization constant verifies the following recurrence

$$f_{n+1}(a) = \sum_{k=-a}^{a} (1 - \frac{k^2}{a^2})^{n+1} = \sum_{k=-a}^{a} (1 - \frac{k^2}{a^2})^{n} - \frac{1}{a^2} \sum_{k=-a}^{a} k^2(1 - \frac{k^2}{a^2})^{n} = f_n(a) - \frac{U}{a^2} f_{n+1}(a)$$

so that

$$U = a^2(1 - \frac{f_{n+1}(a)}{f_n(a)}).$$

Moreover, using the result of theorem 7, we have

$$f_n(a) \simeq \frac{\Gamma(\frac{m}{2} + 1)}{a\sqrt{\pi} \Gamma(\frac{m+1}{2})}$$

we obtain after some algebra

$$U \simeq \frac{a^2}{m + 2}. \quad (24)$$

We see from the form of (24) that the specific heat $C$ is proportional to $1/(m+2)$. Thus, the is system stable for all $q > 1$. Also, since $a^2 = k_BT$, we see that the mean energy $U$ is an increasing function of $T$, as it should.
5.2 Convergence to a sampled Student-r law

As a consequence of theorem (7), the Tsallis maximizers can be approximated as follows, for \( a \) large enough (\( T \) large enough).

**Theorem 9** If \( a \gg 1 \) then the following approximation holds

\[
p_k \simeq \frac{\Gamma\left(\frac{m}{2} + 1\right)}{\sqrt{\pi} \Gamma\left(\frac{m+1}{2}\right) \sqrt{U} \sqrt{m + 2}} \left(1 - \frac{k^2}{U(m + 2)}\right)^{\frac{m-1}{2}}.
\]

**Proof.** From (23), we have

\[a^2 \simeq (m + 2)U.\]

and the result follows from Theorem (4). \(\blacksquare\)

Note that here

\[(m + 2)U = k_B T = \beta^{-1},\]

a result that will be needed below. We remark the similarity to the corresponding \( q < 1 \) probability law.

Eq. (25) corresponds to the sampled version of a continuous Student-r distribution - which maximizes Tsallis entropy for fixed mean energy - with the same partition function.

5.3 Detailed results

In the case where \( m \) is an odd integer, more detailed results can be obtained concerning the behavior of this probability law.

**Theorem 10** For \( a \gg 1 \), if \( m = 2n + 1 \) is an odd integer with \( n \geq 2 \), then

\[
f_m(a) = \frac{2^{2n+1}(n!)^2}{(2n+1)!} a + o(a^{-2})
\]

and \( \forall a \geq 1 \), and if \( m = 3 \)

\[
f_3(a) = a \frac{4}{3} - \frac{a^{-1}}{3}.
\]

Moreover, the mean energy \( U \) verifies

\[U = \frac{a^2}{m + 2} + o(a^{-1}).\]

**Proof.** the proof is given in Annex 4. \(\blacksquare\)

This quasi linearity of \( f_m(a) \) in \( a \) for high temperatures is illustrated in Fig. 4, while the quadratic behavior of \( U \) vs. \( a \) is depicted in Fig. 5.
6 Convergence to the discrete normal distribution

We show now that, in the case where parameter $m$ grows to infinity, both discrete Tsallis distributions corresponding to either $q < 1$ or $q > 1$ converge to the discrete normal distribution.

**Proposition 11** If $a \gg 1$ and if the mean energy $U$ is fixed to a constant value, then, as $m$ grows to infinity (i.e. $q \to 1^\pm$), both discrete Tsallis probability distributions (7) and (10) converge to the maximizer of Shannon’s discrete entropy, i.e., the discrete normal distribution.

**Proof.** in the case $q < 1$, the result follows by taking $m \to +\infty$ in (19) with

$$\lim_{m \to +\infty} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{2U\pi}}$$

and (Cf. (21))

$$\lim_{m \to +\infty} (1 + \frac{k^2}{U(m-2)})^{-\frac{m+1}{2}} = \exp\left(-\frac{k^2}{2U}\right).$$

In the case $q > 1$, the result follows similarly by taking $m \to +\infty$ in (25) with

$$\lim_{m \to +\infty} \frac{\Gamma\left(\frac{m}{2}\right) + 1}{\sqrt{\pi} \Gamma\left(\frac{m+1}{2}\right)\sqrt{m+2}} = \frac{1}{\sqrt{2\pi}}$$

and (Cf. (26))

$$\lim_{m \to +\infty} (1 - \frac{k^2}{U(m+2)})^{-\frac{m+1}{2}} = \exp\left(-\frac{k^2}{2U}\right).$$

This result shows that, for high enough temperatures, and if $m \to \infty$ (or equivalently $q \to \pm 1$) the distribution that maximizes Tsallis’ entropy converges to the Boltzmann one, both for $q < 1$ and $q > 1$. Of course, for fixed $T \gg 1$ and $q \to +\infty$, we obtain the uniform distribution, whereas for $T \ll 1$ and $q \to +\infty$ we obtain the deterministic case $p_k = \delta_k$.

7 Extension to the multivariate case

In the last part of this communication we give a heuristic discussion of the multivariate discrete Tsallis laws with $q < 1$.

**Theorem 12** if $\frac{d}{d+2} < q < 1$ then the probability $p_{k_1,\ldots,k_d} = Pr\{X_1 = k_1, \ldots, X_d = k_d\}$ defined by
\[ p_{k_1,\ldots,k_d} = f_{m,d}^{-1}(a_1,\ldots,a_d)(1 + \sum_{i=1}^{d} \frac{k_i^2}{a_i^2})^{-\frac{m+d}{2}}; \quad k_i \in \mathbb{Z}^+ , \]

maximizes Tsallis \( d \)-variate entropy

\[ H_q(p) = \frac{1}{q-1}(1 - \sum_{k_1,\ldots,k_d} p_k^q) \]

for fixed mean energy

\[ \sum_{k_i} g_{k_i} k_i^2 p_{k_i} = u_i^2 . \]

The partition function \( f_{m,d}(a_1,\ldots,a_d) \) is defined as follows:

\[ f_{m,d}(a_1,\ldots,a_d) = \sum_{k_1,\ldots,k_d} (1 + \sum_{i=1}^{d} \frac{k_i^2}{a_i^2})^{-\frac{m+d}{2}} \]

In the case where \( \forall i, a_i \gg 1 \), this function can be approximated as

\[ f_{m,d}(a_1,\ldots,a_d) \simeq \pi^{\frac{d}{2}} \frac{\Gamma(m)}{\Gamma(m+\frac{d}{2})} \prod_{i=1}^{d} a_i . \]

**Proof.** The proof can be derived from that given for the continuous case in [6]. It is detailed in Annex 5.

We remark here that property (29) is nothing but renormalizability: any subsystem of a system distributed according to a discrete Tsallis law is itself distributed according to a discrete Tsallis law. We note, too, that renormalization does not change the value of parameter \( m \).

### 8 Conclusions

In this report, we have derived in analytic fashion some elementary properties of the discrete univariate and multivariate Tsallis laws (probability distributions...
$p_k$ running over a discrete index $k$) for both $q < 1$ and $q > 1$. Most of the ensuing results are given by approximations, but numerical simulations show that they can be regarded as very accurate ones.

The discrete probability distributions $p_k$ define, for any Hamiltonian $\mathcal{H}$ whose eigenvalue equation reads $\mathcal{H}|\psi_k\rangle = \epsilon_k|\psi_k\rangle$ mixed states of the form

$$\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|,$$

that commute with the Hamiltonian by construction and represent thus bona fide stationary states of the system defined by $\mathcal{H}$. These weights $p_k$ are of the Tsallis-power law form. With these weights, $\rho$ maximizes Tsallis’ entropy subject to variance constraint, which introduces a Lagrange multiplier $\beta$. In this work we have considered situations for which the system’s mean energy is proportional to the variance.

Our present consideration allows one to nitidly appreciate the effects of non-extensivity, as, varying $q$, one passes from unstable to stable systems and even to unphysical situations of infinite energy.

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FIGURE CAPTIONS

Fig. 1: Plot of $\log_{10}(\sqrt{U})$ vs. $[1/(q - 1)]$ for $T = 100$ (upper curve) and $T = 0.01$ (lower curve) that illustrates the regions of applicability of the Student-t and power law regimes.

Fig. 2: Student-t regime for $q < 1$: the values of $a_0$ as a function of $m_0$ (see text).

Fig. 3: The plane $(a, m)$: values of $a$ and $m$ for which approximation (22) holds, with $\varepsilon = 10^{-10}$ (see text).

Fig. 4: $q > 1$ regime: quasi linearity of $f_m(a)$ in $a$ for high temperatures. The four different curves correspond, respectively, to $m=3.5, 7, 9, 11$, from top to bottom. Dots indicate the approximate value as given by Eq. (17). The continuous line represent exact values.

Fig. 5: $q > 1$ regime: quadratic behavior of $U$ vs. $a$. The three curves correspond, respectively, to $m=3, 5, 7$, from top to bottom. Dots indicate the approximate value as given by Eq. (18). The continuous line represent exact values.
Annex: proofs

For the sake of completeness we give here some of the proofs of assertions referenced to in the text.

8.1 Annex 1: proof of theorem (3)

1. the normalizing factor writes

\[ f_q(a) = \sum_{k=-\infty}^{+\infty} \left( 1 + \frac{k^2}{a^2} \right)^{-\frac{m+1}{2}} = 1 + 2 \sum_{k=1}^{+\infty} \left( 1 + \frac{k^2}{a^2} \right)^{-\frac{m+1}{2}} \]

\[ \simeq 1 + 2 \sum_{k=1}^{+\infty} \left( \frac{k^2}{a^2} \right)^{-\frac{m+1}{2}} = 1 + 2a^{m+1} \zeta(m+1) \]

2. the approximated expression of the distribution as given by (16) is a direct consequence of the preceding result

3. We denote indifferently \( f_m \) as \( f_q \). As a consequence of the preceding result and remarking that

\[ U = f_m^{-1}(a)^2 \sum_{k=-\infty}^{+\infty} \frac{k^2}{a^2} \left( 1 + \frac{k^2}{a^2} \right)^{-\frac{m+1}{2}} \]

\[ = f_m^{-1}(a)^2 \sum_{k=-\infty}^{+\infty} \left( 1 + \frac{k^2}{a^2} \right)^{-\frac{m+1}{2}} - f_m^{-1}(a)^2 \sum_{k=-\infty}^{+\infty} \frac{1}{(1 + \frac{k^2}{a^2})^{\frac{m+1}{2}}} \]

\[ = a^2 \left( \frac{f_m-2}{f_m} - 1 \right) \]

we obtain

\[ U = a^2 \left( \frac{f_{2n-1}}{f_{2n+1}} - 1 \right) \simeq a^2 \left( \frac{1 + 2a^{2n} \zeta(2n)}{1 + 2a^{2n+2} \zeta(2n+2)} - 1 \right) = 2a^{2n+2} \frac{(\zeta(2n) - a^2 \zeta(2n+2))}{1 + 2a^{2n+2} \zeta(2n+2)} \]

\[ \simeq 2a^{2n+2} (\zeta(2n) - a^2 \zeta(2n+2))(1 - 2a^{2n+2} \zeta(2n+2)) \simeq 2a^{m+1} \zeta(m-1) \]

4. if \( m = 2n + 1 \) with \( n \in \mathbb{N} \), the characteristic function writes

\[ \phi_X(u) = f_m^{-1}(a) \sum_{k=-\infty}^{+\infty} \cos(ku) \left( 1 + \frac{k^2}{a^2} \right)^{-\frac{m+1}{2}} = f_m^{-1}(a)(1 + 2 \sum_{k=1}^{+\infty} \cos(ku) \left( 1 + \frac{k^2}{a^2} \right)^{-\frac{m+1}{2}}) \]

\[ \simeq f_m^{-1}(a)(1 + 2a^{m+1} \sum_{k=1}^{+\infty} \cos(ku) \frac{k^{m+1}}{k^{m+1}}) \]

But [13, 1.443] writes

\[ \sum_{k=1}^{+\infty} \frac{\cos(k \pi x)}{k^{2n}} = \frac{(-1)^{n-1} (2\pi)^{2n}}{2n!} B_{2n} \left( \frac{x}{2} \right) \quad (0 \leq x \leq 2) \]

(31)
so that

\[ \phi_X(u) \simeq f_m^{-1}(a)(1 + 2a^{m+1} \sum_{k=1}^{\infty} \frac{\cos(ku)}{k^{m+1}}) \]

\[ \simeq \frac{1}{1 + 2a^{m+1}} \zeta(m + 1) \]

\[ (1 + 2a^{m+1} \frac{(-1)^n (2\pi)^{2n+2}}{2 (2n + 2)!} B_{2n+2} \frac{u}{2\pi}) \quad (0 \leq u \leq 2\pi) \]

### 8.2 Annex 2: proof of theorem (4)

Denote the Gamma function as

\[ \gamma_{\mu}(v) = \frac{1}{\Gamma(\mu)} e^{-v} v^{\mu-1}, \quad v \geq 0. \]

Using the Gamma integral, we have, with \( \mu = \frac{m+1}{2} \),

\[ \sum_{k=-\infty}^{+\infty} (1 + \frac{k^2}{a^2})^{-\mu} = \sum_{k=-\infty}^{+\infty} \int_{0}^{+\infty} e^{-\frac{k^2}{a^2}v} \gamma_{\mu}(v) dv. \]

Consider an arbitrary value \( a^2 \) and divide the integral into two parts:

\[ \int_{0}^{+\infty} e^{-\frac{k^2}{a^2}v} \gamma_{\mu}(v) dv = \int_{0}^{a^2} e^{-\frac{k^2}{a^2}v} \gamma_{\mu}(v) dv + \int_{a^2}^{+\infty} e^{-\frac{k^2}{a^2}v} \gamma_{\mu}(v) dv. \]

We will show that

\[ I_1 = \sum_{k=-\infty}^{+\infty} \int_{a^2}^{+\infty} e^{-\frac{k^2}{a^2}v} \gamma_{\mu}(v) dv \]

can be made arbitrary small provided \( a \) is large enough, and for that choice of \( a \), the absolute value of \( I_2 = \sum_{k=-\infty}^{+\infty} \int_{0}^{a^2} e^{-\frac{k^2}{a^2}v} \gamma_{\mu}(v) dv - a\sqrt{\pi} \frac{\Gamma(\mu-\frac{1}{2})}{\Gamma(\mu)} \) can be made arbitrary small by choosing \( \mu \) small enough.

In term \( I_1 \), \( v \geq a^2 \) so that \( \exp(-\frac{k^2}{a^2}v) \leq \exp(-k^2) \) so that

\[ I_1 = \sum_{k=-\infty}^{+\infty} \int_{a^2}^{+\infty} e^{-\frac{k^2}{a^2}v} \gamma_{\mu}(v) dv \leq \sum_{k=-\infty}^{+\infty} e^{-k^2} \int_{a^2}^{+\infty} \gamma_{\mu}(v) dv \leq 2 \int_{a^2}^{+\infty} \gamma_{\mu}(v) dv \]

since \( \sum_{k=-\infty}^{+\infty} e^{-k^2} \simeq \sqrt{\pi} = 1.7726 \). As \( \int_{a^2}^{+\infty} \gamma_{\mu}(v) dv \) is the residue of the converging integral of a positive function, a proper choice of \( a_0 \) yields

\[ a \geq a_0 \implies I_1 \leq \varepsilon / 2. \]
The second term writes

\[ I_2 = \sum_{k=\infty}^{+\infty} \int_{0}^{\infty} e^{-\frac{k^2}{a^2}} \gamma_\mu(v) dv - a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} \]

\[ = \int_{0}^{a^2} \left( \sum_{k=\infty}^{+\infty} e^{-\frac{k^2}{a^2}} \gamma_\mu(v) dv - a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} \right) \]

\[ = \left( \int_{0}^{a^2} \sum_{k=\infty}^{+\infty} e^{-\frac{k^2}{a^2}} \gamma_\mu(v) dv - \int_{0}^{a^2} a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} \gamma_\mu(v) dv \right) \]

\[ + \left( \int_{0}^{a^2} a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} \gamma_\mu(v) dv - a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} \right). \]

For the first difference, we use the fact that, for \( a \geq a_1 \),

\[ \left| \sum_{k=\infty}^{+\infty} e^{-\frac{k^2}{a^2}} - a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} \right| \leq \varepsilon \]

so that

\[ \int_{0}^{a^2} \left( \sum_{k=\infty}^{+\infty} e^{-\frac{k^2}{a^2}} \gamma_\mu(v) dv - \int_{0}^{a^2} a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} \gamma_\mu(v) dv \right) \leq \varepsilon \int_{0}^{a^2} \gamma_\mu(v) dv \]

This r.h.s. integral can be \( \leq \frac{1}{4} \) as soon as \( a \geq a_1 \). Choosing \( a \geq \max(a_1, a_2) \) yields

\[ \left| \int_{0}^{a^2} \sum_{k=\infty}^{+\infty} e^{-\frac{k^2}{a^2}} \gamma_\mu(v) dv - \int_{0}^{a^2} a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} \gamma_\mu(v) dv \right| \leq \varepsilon/4. \]

The second difference writes

\[ \int_{0}^{a^2} a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} \gamma_\mu(v) dv - a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} = a\sqrt{\pi} \int_{0}^{a^2} \frac{1}{\sqrt{v}} \gamma_\mu(v) dv - \int_{0}^{+\infty} a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} \gamma_\mu(v) dv \]

\[ = a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} \left( \int_{0}^{a^2} \gamma_{\mu - \frac{1}{2}}(v) dv - 1 \right). \]

Now, for \( a \geq \max(a_0, a_1) \) fixed, function \( f(\mu) = a\sqrt{\pi} \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(\mu)} (\int_{0}^{a^2} \gamma_{\mu - \frac{1}{2}}(v) dv - 1) \) is continuous in 0 and such that

\[ \lim_{\mu \to 0^+} f(\mu) = 0 \]

so that \( \exists \mu_0 \) such that

\[ \mu \leq \mu_0 \implies |f(\mu)| \leq \varepsilon/4. \]
8.3 Annex 3: proof of theorem (7)

Using the complex version of the Gamma integral as given in [13, 3.382.6], we have

\[ p^{\nu - 1} = \frac{\Gamma(\nu)}{2\pi} e^{\beta p} \int_{-\infty}^{+\infty} e^{ipx} (\beta + ix)^{\nu} dx \]

if \( p > 0, \text{Re}(\beta) > 0, \text{Re} (\nu) > 0 \). Choosing \( \nu - 1 = \frac{m-1}{2} \) and \( \beta = 1 \), we deduce that

\[
(1 - k^2 a^2)^{\frac{m-1}{2}} = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\pi} e^{1 - \frac{k^2}{a^2}} \int_{-\infty}^{+\infty} e^{i(1 - \frac{k^2}{a^2})x} (1 + ix)^{\frac{m+1}{2}} dx
\]

so that

\[
\sum_{k=-(a-1)}^{a-1} (1 - k^2 a^2)^{\frac{m-1}{2}} = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\pi} \int_{-\infty}^{+\infty} \sum_{k=-(a-1)}^{a-1} e^{-i(1+ix)} \frac{k^2}{a^2} e^{1+ix} (1 + ix)^{\frac{m+1}{2}} dx.
\]

When \( a \gg 1 \),

\[
\sum_{k=-(a-1)}^{a-1} e^{-i(1+ix)} \frac{k^2}{a^2} \approx \frac{a \sqrt{\pi}}{\sqrt{1+ix}}
\]

so that

\[
f_m(a) \approx \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\pi} a \sqrt{\pi} \int_{-\infty}^{+\infty} e^{1+ix} (1 + ix)^{\frac{m}{2}+1} dx = \frac{\Gamma\left(\frac{m+1}{2}\right) a \sqrt{\pi}}{2\pi} \int_{-\infty}^{+\infty} e^{1+ix} (1 + ix)^{\frac{m}{2}+1} dx
\]

\[
= \frac{\Gamma\left(\frac{m+1}{2}\right) a \sqrt{\pi}}{2\pi} \frac{2\pi}{\Gamma\left(\frac{m}{2}+1\right)} = a \sqrt{\pi} \Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{m}{2}+1\right).
\]

8.4 Annex 4: proof of theorem (10)

Denote \( n = \frac{m}{2} \) and \( f_n(a) = \sum_{k=-(a-1)}^{a-1} (1 - \frac{k^2}{a^2})^n \). If \( m \) is an odd integer then \( n \) is integer and \( \tilde{f}(n, a) \) can be expanded as follows

\[
\tilde{f}(n, a) = \sum_{k=0}^{a-1} \left(1 - \frac{k^2}{a^2}\right)^n = \sum_{k=0}^{a-1} a^2 \sum_{p=0}^{n} \binom{n}{p} \left(\frac{k^2}{a^2}\right)^p = \frac{a^2}{2p+1} \sum_{p=0}^{n} \binom{n}{p} (-1)^p a^{-2p} \sum_{k=0}^{a-1} k^{2p}.
\]

But it is well-known [13, 0.121] that, if \( p \neq 0 \),

\[
\sum_{k=0}^{a-1} k^{2p} = \frac{1}{2p+1} \sum_{l=0}^{2p} \binom{2p+1}{l} B_l a^{2p+1-l} = \frac{1}{2p+1} a^{2p+1} - \frac{1}{2} a^{2p} + \frac{p}{6} a^{2p-1} + o(a^{2p-2})
\]
since $B_3 = 0$, and if $p = 0$, $\sum_{k=0}^{a-1} k^{2p} = a$, so that

$$\hat{f}_n(a) = \sum_{p=0}^{n} \binom{n}{p} (-1)^p a^{-2p} \sum_{k=0}^{a-1} k^{2p}$$

$$= a + \sum_{p=1}^{n} \binom{n}{p} (-1)^p a^{-2p} \left( \frac{1}{2p+1} a^{2p+1} - \frac{1}{2} a^{2p} + \frac{p}{6} a^{2p-1} + o(a^{2p-2}) \right)$$

$$= a \left( \sum_{p=0}^{n} \binom{n}{p} (-1)^p \right) - \frac{1}{2} \sum_{p=1}^{n} \binom{n}{p} (-1)^p + \frac{a^{-1}}{6} \sum_{p=1}^{n} \binom{n}{p} (-1)^p p + o(a^{-2})$$

Moreover, since

$$\sum_{p=0}^{n} \binom{n}{p} \frac{(-1)^p}{2p+1} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(n+1)}{\Gamma(\frac{3}{2} + n)} = \frac{(2^n n!)^2}{(2n+1)!}$$

by Euler’s duplication formula, and as

$$n > 0 \implies \sum_{p=1}^{n} \binom{n}{p} (-1)^p = (1 - 1)^n - 1 = -1$$

and

$$\sum_{p=1}^{n} \binom{n}{p} (-1)^p p = -n \sum_{p=0}^{n} \binom{n-1}{p} (-1)^p = -n \delta_{n,1} = \begin{cases} -1 & \text{if } n = 1 \\ 0 & \text{else} \end{cases}$$

we get finally, for $n \geq 2$,

$$\hat{f}_n(a) = a \frac{(2^n n!)^2}{(2n+1)!} + \frac{1}{2} + o(a^{-2}) , \ f_n(a) = a \frac{2(2^n n!)^2}{(2n+1)!} + o(a^{-2}).$$

and in the case $n = 1$,

$$\hat{f}_1(a) = a \frac{2}{3} + \frac{1}{2} - \frac{a^{-1}}{6} , \ f_1(a) = a \frac{4}{3} - \frac{a^{-1}}{3}$$

The partition function verifies the following recurrence

$$f_{n+1}(a) = \sum_{k=-a}^{+a} (1 - \frac{k^2}{a^2})^{n+1} = \sum_{k=-a}^{+a} (1 - \frac{k^2}{a^2})^n \frac{1}{a^2} \sum_{k=-a}^{+a} k^2 \left( 1 - \frac{k^2}{a^2} \right)^n$$

$$= f_n(a) - \frac{U}{a^2} f_n(a)$$

so that

$$U = a^2 \left( 1 - \frac{f_{n+1}(a)}{f_n(a)} \right).$$

Since $U \geq 0$, this is an alternate proof that $f_n(a)$ is decreasing in $n$. Moreover, using (27), we obtain after some algebra

$$U = \frac{a^2}{2n+3} + o(a^{-1}).$$

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9 Annex 5: proof of theorems (12) and (13)

We apply the same technique as in the univariate case, and represent the partition function as a mixture of discrete Gaussian using the Gamma integral as follows:

\[ f_{m,d}(a_1, \ldots, a_d) = \sum_{k_1, \ldots, k_d} (1 + \sum_{i=1}^{d} \frac{k_i^2}{a_i^2})^{-\frac{m+d}{2}} \]

\[
= \sum_{k_1, \ldots, k_d} \int_{0}^{+\infty} e^{-\sum_{i=1}^{d} \frac{k_i^2}{a_i^2} \gamma_{m+d}(v)} dv = \int_{0}^{+\infty} \sum_{k_1, \ldots, k_d} e^{-\sum_{i=1}^{d} \frac{k_i^2}{a_i^2} \gamma_{m+d}(v)} dv \\
= \int_{0}^{+\infty} \prod_{i=1}^{d} \gamma_{m+d}(v) dv \simeq \int_{0}^{+\infty} \left( \prod_{i=1}^{d} a_i \sqrt{\pi} \right) \gamma_{m+d}(v) dv \\
= \pi^\frac{d}{2} \left( \prod_{i=1}^{d} a_i \right) \int_{0}^{+\infty} v^{-\frac{d}{2}} \gamma_{m+d}(v) dv = \pi^\frac{d}{2} \left( \prod_{i=1}^{d} a_i \right) \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m+d}{2})}.
\]

Using the same approach, the \(d'\)-variate marginal law writes

\[ p_{k_1, \ldots, k_{d'}} = \sum_{k_{d'+1}, \ldots, k_d} p_{k_1, \ldots, k_{d'}} = f_{m,d}^{-1}(a_1, \ldots, a_d) \sum_{k_{d'+1}, \ldots, k_d} (1 + \sum_{i=1}^{d} \frac{k_i^2}{a_i^2})^{-\frac{m+d}{2}} \]

\[
= f_{m,d}^{-1}(a_1, \ldots, a_d) \sum_{k_{d'+1}, \ldots, k_d} \left( (1 + \sum_{i=1}^{d} \frac{k_i^2}{a_i^2}) + \sum_{i=d'+1}^{d} \frac{k_i^2}{a_i^2} \right)^{-\frac{m+d}{2}} \\
= f_{m,d}^{-1}(a_1, \ldots, a_d)(1 + \sum_{i=1}^{d'} \frac{k_i^2}{a_i^2})^{-\frac{m+d}{2}} \sum_{k_{d'+1}, \ldots, k_d} (1 + \sum_{i=d'+1}^{d} \frac{k_i^2}{a_i^2})^{-\frac{m+d}{2}}
\]

with

\[ \tilde{a}_i^2 = a_i^2 (1 + \sum_{i=1}^{d'} \frac{k_i^2}{a_i^2}) \]

But, with \(m' = m + d'\),

\[
\sum_{k_{d'+1}, \ldots, k_d} \left(1 + \sum_{i=d'+1}^{d} \frac{k_i^2}{a_i^2}\right)^{-\frac{m-d}{2}} \simeq \pi^\frac{d-d'}{2} \left( \prod_{i=d'+1}^{d} a_i \right) \frac{\Gamma(\frac{m'}{2})}{\Gamma(\frac{m'+d-d'}{2})} \\
= \pi^\frac{d-d'}{2} \left( \prod_{i=d'+1}^{d} a_i \right) \frac{\Gamma(\frac{m'}{2})}{\Gamma(\frac{m'+d-d'}{2})} \left( 1 + \sum_{i=1}^{d'} \frac{k_i^2}{a_i^2} \right)^{\frac{d-d'}{2}}
\]

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so that

\[
p_{k_1,\ldots,k_{d'}} \simeq \frac{\pi^{-d} \Gamma\left(\frac{m+d}{2}\right) \pi^{-d'} \left(\prod_{i=1}^{d'} a_i\right) \Gamma\left(\frac{m'}{2}\right)}{(\prod_{i=1}^{d} a_i^i)^{m+d/2}} \left(1 + \sum_{i=1}^{d'} \frac{k_i^2}{d_i^2}\right)^{-\frac{m+d'}{2}}
\]

\[
= \frac{\Gamma\left(\frac{m+d'}{2}\right)}{\pi^{d'} (\prod_{i=1}^{d'} a_i) \Gamma\left(\frac{m}{2}\right)} \left(1 + \sum_{i=1}^{d'} \frac{k_i^2}{d_i^2}\right)^{-\frac{m+d'}{2}}.
\]