On the Computation of the Third Order Terms of the Series Defining the Center Manifold for Systems of Delay Differential Equation

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Abstract

When studying a general system of delay differential equation with a single constant delay, we encounter a certain lack of uniqueness in determining the coefficient of one of the third order terms of the series defining the center manifold. We solve this problem by considering a perturbation of the considered problem, perturbation that allows us to remove the singularity. The result generalizes a similar result obtained for scalar differential equations (J. Dyn. Diff. Eqns., 24/2012).

Keywords: delay differential systems of equations, dynamical systems, center manifold, degenerate Hopf bifurcation, Bautin bifurcation.

AMS MSC 2010: 34K19, 34K18.

1 Introduction

We consider an equation of the form

\[ \dot{x}(t) = f(x(t), x(t - r)), \]

where \( x : [-r, T) \to \mathbb{R}^n, f : D \subset \mathbb{R}^{2n} \to \mathbb{R}^n, (0, 0) \in D \subset \mathbb{R}^{2n}, D \) open, \( f(0, 0) = 0, r > 0 \). We assume \( f \in C^k(D), k > 3 \). We attach to this equation an initial condition of the form

\[ x(s) = \phi(s), s \in [-r, 0], \]

with \( \phi : [-r, 0] \to \mathbb{R}^n \) a continuous function.

Since \( x(t) = 0 \) is a solution (an equilibrium) of (1), we consider the linearized of the equation around this solution

\[ \dot{x}(t) = Ax(t) + Bx(t - r), \]
where $A, B \in M_{n \times n}(\mathbb{R})$. Thus, eq. (1) may be written as

$$\dot{x}(t) = Ax(t) + Bx(t-r) + \hat{f}(x(t), x(t-r)), \quad (4)$$

$\hat{f}$ being the nonlinear part of $f$.

The characteristic equation attached to (3) is

$$\det(\lambda I - A - e^{-\lambda r}B) = 0. \quad (5)$$

**Hypothesis H.** We assume that (5) has a pair of pure imaginary complex conjugate solutions $\lambda_{1,2} = \pm \omega i$, with $\omega > 0$, and all other eigenvalues have negative real part. We also assume that the eigenvalues $\lambda_{1,2}$ are simple, (i.e. each of them has an one-dimensional corresponding eigenspace).

Under this hypothesis, it is known that an invariant manifold, called *center manifold*, exists in a certain function space that will be defined below. The dimension of the invariant manifold is two, it is the graph of a function depending on two complex conjugated scalar variables. The reduction of the problem to this manifold allows some bifurcations studies to be performed, when the problem depends also upon some parameters.

In order to find an approximation of the reduced to the center manifold problem, an approximation of the center manifold is required. This is obtained by considering the series of powers of the function defining the center manifold, and by computing (using a method that will be recalled in Section 2), the first coefficients of this series.

When one arrives to the third order terms, one sees that a singularity occurs, i.e. the system of algebraical equations determining a certain value of the coefficient of $z^2\tau$ has singular matrix. In this situation either the problem has an infinity of solutions, or it has no solution at all.

We show that our problem has an infinity of solutions (Section 3) and we present a method to select a significant solution from among them (Section 4). A similar problem was solved for the case of a single differential delay equation ($n = 1$) in [5].

We must add that the knowledge of the third order terms of the center manifold series is necessary for the study of degenerate Hopf bifurcation (when the problem presents one varying parameter) or Bautin bifurcation (when the problem presents two varying parameters) - see, for a scalar equation, [6].

2 Preliminaries

In this section the center manifold is defined and the method of approximating it is recalled. These are basic well-known ideas, but we recall them for the completeness of the paper.
2.1 The center manifold

This subsection relies on [2], [3], [4]. In order to define the center manifold, we need the space

\[ B = \{ \psi : [-r, 0] \mapsto \mathbb{R}^n \mid \psi \text{ continuous on } [-r, 0] \}, \]

and its complexification \( B_C = B + iB \), both with sup norm.

The eigenfunctions corresponding to \( \lambda_{1,2} = \pm \omega i \) (see hypothesis H) are the elements of \( B_C \) given by \( \varphi_{1,2}(s) = \varphi_{1,2}(0)e^{\pm \omega is} \), where \( \varphi_{1,2}(0) \) are solutions of

\[ (\pm \omega i I - e^{\mp \omega ir} B)\varphi_{1,2}(0) = 0. \]  

(6)

We consider the subspace of \( B_C \) defined by

\[ M = \text{Span}\{ \varphi_1, \varphi_2 \} \] (\( \varphi_{1,2} \) are column vectors). In [2], [3] a projector \( P : B_C \to M \) is defined, and by setting \( N = (I - P)B_C \), we have \( B_C = M \oplus N \). We review here the construction of this projector. The adjoint equation (associated to the linear equation (3)) is defined as

\[ \dot{y}(s) = -Ay(s) - By(s + r). \]  

(7)

The corresponding characteristic equation is

\[ \det(\lambda I + A + Be^{\lambda r}) = 0, \]

that, obviously has, together with eq. (5), the solutions \( \pm \omega i \).

The corresponding eigenfunctions are \( \psi_1(\zeta) = \psi_1(0)e^{-\omega i\zeta} \), \( \psi_2(\zeta) = \psi_2(0)e^{\omega i\zeta} \), \( \zeta \in [0, r] \) where \( \psi_j(0) \) are row vectors, solutions of the vectorial equations

\[ \psi_{1,2}(0)(\mp \omega i I + A + Be^{\mp \omega ir}) = 0. \]  

(8)

The following bilinear form is defined ([2], [3]) on \( C([0, r], \mathbb{C}^n) \times B_C \)

\[ \langle \psi, \varphi \rangle = \psi(0)\varphi(0) + \int_0^r \psi(\zeta + r)B\varphi(\zeta)d\zeta, \]  

(9)

where \( \psi \) is a row vector and \( \varphi \) is a column vector.

Linear combinations of the functions \( \psi_j, j = 1, 2 \), denoted by \( \Psi_i, i = 1, 2 \) are then constructed, such that \( \langle \Psi_i, \varphi_j \rangle = \delta_{ij} \). For this, the matrix \( E = (e_{ij})_{1 \leq i,j \leq 2} \), with \( e_{ij} = \langle \psi_i, \varphi_j \rangle \), is computed (see the Appendix). We obtain \( \Psi_1(\zeta) = \Psi_1(0)e^{-\omega i\zeta} \), with

\[ \Psi_1(0) = \frac{1}{\psi_1(0)\varphi_1(0) + \psi_1(0)B\varphi_1(0)e^{-\omega ir}}\psi_1(0). \]

The projector defined ([2], [3]) on \( B_C \), with values in \( M \) is given by

\[ P(\phi) = \langle \Psi_1, \phi \rangle \varphi_1 + \langle \Psi_2, \phi \rangle \varphi_2, \phi \in B_C. \]
In the hypothesis $H$ above, it is proved [3] that a local invariant manifold, called the center manifold, exists, that is a smooth manifold, tangent to the space $\mathcal{M}$ at the point $x = 0$ and being the graph of a function $w(\cdot)$ defined on a neighborhood of zero in $\mathcal{M}$ with values in $\mathcal{N}$. Since all eigenvalues different of $\lambda_{1,2}$ have strictly negative real part, the trajectory of any point that does not lie on the center manifold approaches, when $t \to \infty$, the center manifold.

In order to approximate the center manifold, the function $w(u, \overline{\nu}) := w(u \varphi_1 + \overline{\nu} \varphi_2)$ is written as a series of powers $u$ and $\overline{\nu}$,

$$w(u, \overline{\nu}) = \sum_{i+j \geq 2} \frac{1}{i!j!} w_{i,j} u^i \overline{\nu}^j,$$

where for each $i,j$, $w_{i,j} \in \mathcal{B}_{\mathcal{C}}$, hence it is a vectorial function on $[-r,0]$.

### 2.2 Computation of the coefficients $w_{i,j}$

The vectorial functions $w_{i,j}$ of (10) are determined by using the following relation [8], [7], [4] that is a consequence of the invariance of the center manifold.

$$\frac{\partial}{\partial s} \sum_{j+k \geq 2} \frac{1}{j!k!} w_{j,k}(s) u^j \overline{\nu}^k = \sum_{j+k \geq 2} \frac{1}{j!k!} g_{j,k} u^j \overline{\nu}^k \varphi_1(s) +$$

$$+ \sum_{j+k \geq 2} \frac{1}{j!k!} \overline{\nu}^j \overline{\nu}^k \varphi_2(s) + \frac{\partial}{\partial t} \sum_{j+k \geq 2} \frac{1}{j!k!} w_{j,k}(s) u^j \overline{\nu}^k.$$

From this, differential equations for each $w_{j,k}$ are obtained, by matching the terms of degree $j$ in $u$ and of degree $k$ in $\overline{\nu}$. To determine the integration constants, the following relation is used:

$$\frac{d}{dt} \sum_{j+k \geq 2} \frac{1}{j!k!} w_{j,k}(0) u^j \overline{\nu}^k + \sum_{j+k \geq 2} \frac{1}{j!k!} g_{j,k} u^j \overline{\nu}^k \varphi_1(0) +$$

$$+ \sum_{j+k \geq 2} \frac{1}{j!k!} \overline{\nu}^j \overline{\nu}^k \varphi_2(0) =$$

$$= A \sum_{j+k \geq 2} \frac{1}{j!k!} w_{j,k}(0) u^j \overline{\nu}^k + B \sum_{j+k \geq 2} \frac{1}{j!k!} w_{j,k}(-r) u^j \overline{\nu}^k +$$

$$+ \sum_{j+k \geq 2} \frac{1}{j!k!} f_{j,k} u^j \overline{\nu}^k.$$

### 3 The system for $w_{2,1}(-r)$, $w_{2,1}(0)$

By matching, in (11) and (12), the terms that contain $u^2 \overline{\nu}$, we get the system of differential equations and the set of conditions for $w_{2,1}$. These are
\[
\frac{d}{ds} w_{2,1}(s) = \omega i w_{2,1}(s) + g_{2,1} \varphi_1(0) e^{\omega i s} + \overline{g}_{1,2} \varphi_2(0) e^{-\omega i s} + 2w_{2,0}(s)g_{1,1} + w_{1,1}(s)g_{2,0} + 2w_{0,2}(s)\overline{g}_{0,2},
\]

(that is a system of \(n\) equations) and the \(n\) conditions:

\[
\omega i w_{2,1}(0) + 2w_{2,0}(0)g_{1,1} + w_{1,1}(0)g_{2,0} + 2w_{1,1}(0)\overline{g}_{1,1} + w_{0,2}(0)\overline{g}_{0,2} + g_{2,1} \varphi_1(0) + \overline{g}_{1,2} \varphi_2(0) = A w_{2,1}(0) + B w_{2,1}(-r) + f_2,1.
\]

From these we obtain the system of equations for \(w_{21}(0)\) and \(w_{21}(-r)\) (vectorial equations):

\[
-e^{-\omega i r} w_{2,1}(0) + w_{2,1}(-r) = -g_{2,1} r e^{-\omega i r} \varphi_1(0) + \frac{i}{2\omega} \overline{g}_{1,2}(e^{\omega i r} - e^{-\omega i r})\varphi_2(0) - 2g_{1,1} e^{-\omega i r} \int_{-r}^{0} w_{2,0}(\theta) e^{-\omega i \theta} d\theta - (g_{2,0} + 2g_{1,1}) e^{-\omega i r} \int_{-r}^{0} w_{1,1}(\theta) e^{-\omega i \theta} d\theta - \overline{g}_{0,2} e^{-\omega i r} \int_{-r}^{0} w_{0,2}(\theta) e^{-\omega i \theta} d\theta,
\]

\[
-(\omega I - A)w_{2,1}(0) + B w_{2,1}(-r) = g_{2,1} \varphi_1(0) + \overline{g}_{1,2} \varphi_2(0) - f_{2,1} + 2g_{1,1} w_{2,0}(0) + (g_{2,0} + 2g_{1,1}) w_{1,1}(0) + \overline{g}_{0,2} w_{0,2}(0).
\]

The matrix of this system is the \(2n \times 2n\) matrix (below, “\(I\)” is the \(n \times n\) identity matrix)

\[
\begin{pmatrix}
-e^{-\omega i r}I & I \\
-(\omega I - A)B
\end{pmatrix}.
\]

We denote the right hand side of (13) by \(R_1\) and the right hand side of (14) by \(R_2\). We multiply the vectorial equation (13) to the left by B and, by subtracting the second equation from the first we get the system

\[
(\omega I - A - Be^{-\omega i r})w_{21}(0) = BR_1 - R_2.
\]

Since \(\omega i\) is an eigenvalue for our problem, the determinant of the matrix of the above system is equal to zero. Hence, at this moment, we don’t know whether the system has solutions or not. The following result solves this problem.

**Proposition 3.1.** The system of equations (13) has solutions.

**Proof.** Since \(-\omega i\) is a simple eigenvalue, the matrix

\[
M = \omega i I - A - Be^{-\omega i r}
\]
has an one-dimensional kernel. The same is true for the matrix 
\((\omega i - A - Be^{-\omega ir})^T\). The kernel of this latter matrix is spanned by \(\Psi_1(0)^T\), since we have 
\[
\Psi_1(0)(\omega i - A - Be^{-\omega ir}) = 0.
\]

For a \(n \times n\) matrix \(M\) the equality \(\mathcal{R}(M) = (\mathcal{N}(M^T))^\perp\) holds (where \(\mathcal{R}\) is the range and \(\mathcal{N}\) is the null space of the matrices in brackets). Hence, in order to prove that system (15) has solutions, we show that \(BR_1 - R_2\) is orthogonal on \(\mathcal{N}((\omega i - A - Be^{-\omega ir})^T)\), that is 
\[
\Psi_1(0)(BR_1 - R_2) = 0.
\] (16)

We prove this by showing that the following relations hold:

\[\Psi_1(0)\left(-g_{2,1}re^{-\omega ir}B\varphi_1(0) - g_{2,1}\varphi_1(0) + f_{2,1}\right) = 0; \quad (R1a)\]
\[
\Psi_1(0)\left(\frac{i}{2\omega}g_{1,2}B\varphi_1(0)(e^{\omega i r} - e^{-\omega ir}) - g_{1,2}\varphi_1(0)\right) = 0; \quad (R1b)\]
\[
\Psi_1(0)\left(-2g_{1,1}e^{-\omega ir}B\int_{-r}^{0}w_{2,0}(\theta)e^{-\omega i\theta}d\theta - 2g_{1,1}w_{2,0}(0)\right) = 0; \quad (R2)\]
\[
\Psi_1(0)\left(-(g_{2,0} + 2\overline{g}_{1,1})e^{-\omega ir}B\int_{-r}^{0}w_{1,1}(\theta)e^{-\omega i\theta}d\theta - (g_{2,0} + 2\overline{g}_{1,1})w_{1,1}(0)\right) = 0; \quad (R3)\]
\[
\Psi_1(0)\left(\overline{g}_{0,2}e^{-\omega ir} - B\int_{-r}^{0}w_{0,2}(\theta)e^{-\omega i\theta}d\theta - \overline{g}_{0,2}w_{0,2}(0)\right) = 0. \quad (R4)\]

We prove these relations one by one.

\((R1a)\). This can be written as 
\[-g_{2,1}re^{-\omega ir}\Psi_1(0)B\varphi_1(0) - g_{2,1}\Psi_1(0)\varphi_1(0) + g_{2,1} = 0.\]

If \(g_{2,1} = 0\), then the relation is satisfied, if \(g_{2,1} \neq 0\), we divide by \(g_{2,1}\) and obtain 
\[-re^{-\omega ir}\Psi_1(0)B\varphi_1(0) - \Psi_1(0)\varphi_1(0) + 1 = 0,\]

that we rewrite as 
\[
\Psi_1(0)\varphi_1(0) + \Psi_1(0)B\varphi_1(0)\int_{-r}^{0}e^{-\omega i(\zeta + r)}e^{\omega i\xi}d\xi = 1.
\]

But this is the true relation 
\[\langle \Psi_1, \varphi_1 \rangle = 1.\]
Relation \((R1b)\) leads, after dividing it by \(\overline{g}_{1,2}\) (assumed non-null) to
\[
\Psi_1(0)\overline{\varphi}_1(0) + \frac{i}{2\omega}\Psi_1(0)B\overline{\varphi}_1(0)(e^{-\omega ir} - e^{\omega ir}) = 0
\]
that can be written as
\[
\Psi_1(0)\overline{\varphi}_1(0) + \Psi_1(0)B\overline{\varphi}_1(0) \int_{-r}^{0} (e^{-\omega i(\zeta + r)}e^{-\omega i\zeta})d\zeta = 0
\]
this last being equivalent to
\[
\langle \Psi_1, \overline{\varphi}_1 \rangle = 0,
\]
that holds true.

Relations \((R2)-(R4)\). If \(g_{1,1} = 0\), then relation \((R2)\) holds. We assume \(g_{1,1} \neq 0\). We also assume \(g_{2,0} + 2\overline{g}_{1,1} \neq 0\) and \(\overline{g}_{0,2} \neq 0\).

After dividing with the assumed non-zero coefficients, relations \((R2)-(R4)\) have the form:
\[
\Psi_1(0) \left(-e^{-\omega ir}B \int_{-r}^{0} w_{jk}(\theta)e^{-\omega i\theta}d\theta - w_{jk}(0) \right) = 0;
\]
that can be written as
\[
\Psi_1(0) w_{jk}(0) + \int_{-r}^{0} \Psi_1(0)e^{-\omega i(\theta + r)}Bw_{jk}(\theta)d\theta = 0
\]
\((j, k \geq 0, j + k = 2)\) that is
\[
\langle \Psi_1, w_{j,k} \rangle = 0,
\]
that holds true, since each of the functions \(w_{j,k}\) belongs to the complementary of \(\mathcal{M}\). Hence all relations \((R2)\) - \((R4)\) are proved.

By adding the five relations proved we obtain \((16)\) and the conclusion of our Proposition follows. \(\square\)

4 How to compute \(w_{2,1}(0)\)

Now, in order to solve the system for \(w_{2,1}(0)\), we use the following ideas. We take the vector \(v_1 = \Psi_1(0)^T/|\Psi_1(0)|\), and we complete it to a orthonormal basis in \(\mathbb{C}^n\), \(\mathcal{B} = \{v_1, v_2, ..., v_n\}\) (column vectors).

We write the system \(Mw_{2,1}(0) = BR_1 - R_2\) in the basis \(\mathcal{B}\).

Thus, we set \(x_j = w_{2,1}(0) \cdot v_j\) (the coordinates of \(w_{2,1}(0)\) with respect to \(\mathcal{B}\)), and we obtain the system in the new basis (the “\(\cdot\)” represents the scalar product in the space of \(n\) dimensional column vectors):
\[
\begin{cases}
(v_1 \cdot Mv_1)x_1 + (v_1 \cdot Mv_2)x_2 + ... + (v_1 \cdot Mv_n)x_n = v_1 \cdot (BR_1 - R_2), \\
(v_2 \cdot Mv_1)x_1 + (v_2 \cdot Mv_2)x_2 + ... + (v_2 \cdot Mv_n)x_n = v_2 \cdot (BR_1 - R_2), \\
\vdots \\
(v_n \cdot Mv_1)x_1 + (v_n \cdot Mv_2)x_2 + ... + (v_n \cdot Mv_n)x_n = v_n \cdot (BR_1 - R_2).
\end{cases}
\]

Since \(\Psi_1(0)M = 0\) (here 0 is a row vector), we have \(v_1 \cdot Mv_j = 0\) for any \(j = 1, ..., n\) and since \(v_1 \cdot (BR_1 - R_2) = 0\), the first equation is identically satisfied, hence we can not use it.

We assumed that the eigenvalues \(\pm \omega_i\) are simple, hence the rank of the matrix \(M\) is \(n - 1\), and the system of \(n - 1\) equations obtained by removing the first equation can be solved with respect to \(n - 1\) variables. Thus only the first equation is responsible for the singularity of the problem.

In order to remove this singularity, as in the scalar case, we consider a perturbation of our problem, of the form

\[
\dot{x}(t) = A_\epsilon x(t) + B_\epsilon x(t - r) + \tilde{f}(x(t), x(t - r)),
\]

where \(A_\epsilon, B_\epsilon\) depend smoothly enough on \(\epsilon > 0\), \(\lim_{\epsilon \to 0} A_\epsilon = A, \lim_{\epsilon \to 0} B_\epsilon = B\), and they are chosen such that, for small enough \(\epsilon\), the linearized problem attached to (17) admits the eigenvalues \(\lambda_{1,2} = \mu_\epsilon \pm \omega_\epsilon i\), with \(\mu_\epsilon > 0\), while all other eigenvalues have negative real part.

It follows that \(\lim_{\epsilon \to 0} \mu_\epsilon = 0, \lim_{\epsilon \to 0} \omega_\epsilon = \omega\).

Since \(\mu_\epsilon > 0\), the problem (17) admits an unstable manifold, tangent to the space \(M_\epsilon\) spanned by the two eigenfunctions corresponding to the two eigenvalues \(\lambda_{1,2}\), i.e. \(\varphi_{1,2}(s) = \varphi_{1,2}(0)e^{(\mu_\epsilon \pm i\omega_\epsilon)s}\).

This unstable manifold is the graph of a function \(w_\epsilon\) defined on \(M_\epsilon\) and with values in a subspace \(N_\epsilon\) of \(B_C\), such that \(B_C = M_\epsilon \oplus N_\epsilon\).

Now, as in Section 2 for the non-perturbed problem, the following mathematical objects are constructed (1):

- the adjoint of the linear equation and its eigenfunctions i.e. the functions \(\psi_1(s) = \psi_{11}(0)e^{-\lambda_1 s}, \psi_2(s) = \psi_{12}(0)e^{-\lambda_2 s}, s \in [0, r]\);

- the corresponding bilinear form, denoted also by \(\langle \cdot, \cdot \rangle\) given by

\[
\langle \psi, \varphi \rangle = \psi(0)\varphi(0) + \int_{-r}^{0} \psi(\zeta + r)B_\epsilon \varphi(\zeta)d\zeta.
\]

We find the functions \(\Psi_{\epsilon 1}, \Psi_{\epsilon 2}\), such that \(\langle \Psi_{\epsilon 1}, \varphi_{\epsilon j} \rangle = \delta_{i,j}\) (see the Appendix). The projector \(P_\epsilon : B_C \to M_\epsilon\), is defined by

\[
P_\epsilon(\phi) = \langle \Psi_{\epsilon 1}, \phi \rangle \varphi_1 + \langle \Psi_{\epsilon 2}, \phi \rangle \varphi_2.
\]

We set \(N_\epsilon = (I - P_\epsilon)(B_C)\). Note that

\[
\langle \Psi_{\epsilon j}, w_\epsilon \rangle = 0, \; j = 1, 2.
\]
since \( w_\epsilon \) takes values in \( \mathcal{N}_\epsilon \).

The problem reduced to the unstable manifold is

\[
\frac{dv}{dt} = \lambda v + \Psi_{\epsilon 1}(0) j(v\varphi_{\epsilon 1} + \overline{\varphi}_{\epsilon 2} + w_\epsilon (v\varphi_{\epsilon 1} + \overline{\varphi}_{\epsilon 2})).
\]

(18)

We consider the function

\[
w_\epsilon (v, \overline{\varphi}) := w_\epsilon (v\varphi_{\epsilon 1} + \overline{\varphi}_{\epsilon 2})
\]

and we write:

\[
w_\epsilon (v, \overline{\varphi}) = \sum_{i+j \geq 2} \frac{1}{i! j!} w_{\epsilon, i, j} v^i \overline{\varphi}^j.
\]

As in the non-perturbed case, the coefficients \( w_{\epsilon, i, j} \) are found by solving differential equations coming from relations similar to (11), (12) with \( w_{j, k}, g_{j, k}, \ldots \) replaced by \( w_{\epsilon, k}, g_{\epsilon, k}, \ldots \).

The equation for \( w_{\epsilon, 2,1} \) is

\[
\frac{dw_{\epsilon, 2,1}(s)}{ds} = (2\lambda_\epsilon + \overline{\lambda}_\epsilon) w_{\epsilon, 2,1}(s) + g_{\epsilon, 2,1}\varphi_1(0) e^{\lambda_\epsilon s} + \overline{g}_{\epsilon, 1, 2,1} \overline{\varphi}_{\epsilon 1}(0) e^{\overline{\lambda}_\epsilon s} + 2w_{\epsilon, 2,0}(s) g_{\epsilon, 1, 1} + w_{\epsilon, 1, 1}(s) g_{\epsilon, 2, 0} + 2w_{\epsilon, 1, 1}(s) \overline{g}_{\epsilon, 1, 1} + w_{\epsilon, 0, 2}(s) \overline{g}_{\epsilon, 0, 2},
\]

while the condition to determine the integration constant is

\[
(2\lambda_\epsilon + \overline{\lambda}_\epsilon) w_{\epsilon, 2,1}(0) + 2w_{\epsilon, 2,0}(0) g_{11} + w_{\epsilon, 1, 1}(0) g_{\epsilon, 2, 0} + 2w_{\epsilon, 1, 1}(0) \overline{g}_{\epsilon, 1, 1} + w_{\epsilon, 0, 2}(0) \overline{g}_{\epsilon, 0, 2} + g_{\epsilon, 2, 1}\varphi_1(0) + \overline{g}_{\epsilon, 1, 2,1} \overline{\varphi}_{\epsilon 1}(0) = A_\epsilon w_{\epsilon, 2,1}(0) + B_\epsilon w_{\epsilon, 2,1}(-r) + f_{\epsilon, 2,1}.
\]

By integrating the differential equation above between \(-r\) and 0, we find the system of equations for \( w_{\epsilon, 2,1}(-r), w_{\epsilon, 2,1}(0) \):

\[
- e^{-(2\lambda_\epsilon + \overline{\lambda}_\epsilon)r} w_{\epsilon, 2,1}(0) + w_{\epsilon, 2,1}(-r) = - \frac{1}{2\lambda_\epsilon + \overline{\lambda}_\epsilon} g_{\epsilon, 2,1}\varphi_1(0) (e^{-\lambda_\epsilon r} - e^{-(2\lambda_\epsilon + \overline{\lambda}_\epsilon)r} - \overline{g}_{\epsilon, 1, 2,1} \overline{\varphi}_{\epsilon 1}(0) (e^{-\overline{\lambda}_\epsilon r} - e^{-(2\lambda_\epsilon + \overline{\lambda}_\epsilon)r} - 2g_{\epsilon, 1, 1} e^{-(2\lambda_\epsilon + \overline{\lambda}_\epsilon)r} \int_{-r}^{0} w_{\epsilon, 2, 0}(\theta) e^{-(2\lambda_\epsilon + \overline{\lambda}_\epsilon)\theta} d\theta -
\]

\[
- (g_{\epsilon, 2, 0} + 2g_{\epsilon, 1, 1}) e^{-(2\lambda_\epsilon + \overline{\lambda}_\epsilon)r} \int_{-r}^{0} w_{\epsilon, 1, 1}(\theta) e^{-(2\lambda_\epsilon + \overline{\lambda}_\epsilon)\theta} d\theta -
\]

\[
- \overline{g}_{\epsilon, 0, 2} e^{-(2\lambda_\epsilon + \overline{\lambda}_\epsilon)r} \int_{-r}^{0} w_{\epsilon, 0, 2}(\theta) e^{-(2\lambda_\epsilon + \overline{\lambda}_\epsilon)\theta} d\theta,
\]

\[
(A_\epsilon - 2\lambda_\epsilon - \overline{\lambda}_\epsilon I) w_{\epsilon, 2,1}(0) + B_\epsilon w_{\epsilon, 2,1}(-r) = g_{\epsilon, 2,1}\varphi_1(0) + \overline{g}_{\epsilon, 1, 2,1} \overline{\varphi}_{\epsilon 1}(0) - f_{\epsilon, 2,1} +
\]

(20)
+2w_{e2,2}(0)g_{11} + w_{e1,1}(0)g_{e2,0} + 2w_{e1,1}(0)\overline{g}_{e1,1} + w_{e0,2}(0)\overline{g}_{e0,2}.

As in the scalar case, we can easily prove the following result.

**Proposition 4.1.** When \( \epsilon \to 0 \), the coefficients of system (19)-(20) tend to the coefficients of system (13)-(14).

The proof repeats the ideas of the proof of the similar assertion from the scalar problem, in [5], hence we skip it.

We now denote the right-hand side of the two equations by \( R_{\epsilon 1}, R_{\epsilon 2} \), respectively. As in the non-perturbed case, we multiply the first vectorial equation at the left by \( B_{\epsilon} \) and we subtract the second vectorial equation from the such modified first equation.

The matrix of the obtained \( n \times n \) system is

\[
M_{\epsilon} = -B_{\epsilon} e^{-(2\lambda_{\epsilon} + \overline{\lambda}_{\epsilon})t} - A_{\epsilon} + 2\lambda_{\epsilon} I + \overline{\lambda}_{\epsilon} I,
\]

and the system has the form:

\[
M_{\epsilon} w_{\epsilon 2,1}(0) = B_{\epsilon} R_{\epsilon 1} - R_{\epsilon 2}.
\]

Its matrix, \( M_{\epsilon} \), has determinant different of zero because otherwise the number \( 2\lambda_{\epsilon} + \overline{\lambda}_{\epsilon} \) would be an eigenvalue, with real part equal to \( 3\mu_{\epsilon} \), that contradicts the fact that all eigenvalues have real part \( \leq \mu_{\epsilon} \).

Now we construct a basis, \( B_{\epsilon} \), of orthonormal vectors in \( \mathbb{C}^n \), containing the vector \( v_{\epsilon 1} = \Psi_{\epsilon 1}(0)^T / |\Psi_{\epsilon 1}(0)| \) (column vectors).

The vectors of this basis, \( v_{\epsilon j}, j = 2, ..., n, \) should be chosen such that \( \lim_{\epsilon \to 0} v_{\epsilon j} = v_j \), where \( v_j \) are the vectors of the basis \( B \) from the non-perturbed case. We express the system of equations (22) as

\[
\begin{align*}
(v_{\epsilon 1} \cdot M_{\epsilon} v_{\epsilon 1})x_1 + (v_{\epsilon 1} \cdot M_{\epsilon} v_{\epsilon 2})x_2 + ... + (v_{\epsilon 1} \cdot M_{\epsilon} v_{\epsilon n})x_n &= v_{\epsilon 1} \cdot (B_{\epsilon} R_{\epsilon 1} - R_{\epsilon 2}), \\
(v_{\epsilon 2} \cdot M_{\epsilon} v_{\epsilon 1})x_1 + (v_{\epsilon 2} \cdot M_{\epsilon} v_{\epsilon 2})x_2 + ... + (v_{\epsilon 2} \cdot M_{\epsilon} v_{\epsilon n})x_n &= v_{\epsilon 2} \cdot (B_{\epsilon} R_{\epsilon 1} - R_{\epsilon 2}), \\
&\vdots \\
(v_{\epsilon n} \cdot M_{\epsilon} v_{\epsilon 1})x_1 + (v_{\epsilon n} \cdot M_{\epsilon} v_{\epsilon 2})x_2 + ... + (v_{\epsilon n} \cdot M_{\epsilon} v_{\epsilon n})x_n &= v_{\epsilon n} \cdot (B_{\epsilon} R_{\epsilon 1} - R_{\epsilon 2}).
\end{align*}
\]

The way we chose the vectors ensures us that, when \( \epsilon \to 0 \), the above system tends to the one obtained for \( \epsilon = 0 \). Now we shall treat the first equation of this system. First we prove

**Proposition 4.2.** For any \( \epsilon > 0 \), we have

\[
\Psi_{\epsilon 1}(0) M_{\epsilon} = \mu_{\epsilon} h_1(\epsilon),
\]

\[
\Psi_{\epsilon 1}(0) (B_{\epsilon} R_{\epsilon 1} - R_{\epsilon 2}) = \mu_{\epsilon} h_2(\epsilon),
\]

where the functions \( h_j, j = 1, 2 \) have finite limit for \( \epsilon \to 0 \).
Proof. By using the fact that $\Psi_{\epsilon_1}(0)$ is an eigenvalue for the perturbed problem, i.e.

$$\Psi_{\epsilon_1}(0)(\lambda_\epsilon I - A_\epsilon - B_\epsilon e^{-\lambda_\epsilon r}) = 0,$$

we have

$$\Psi_{\epsilon_1}(0)M_\epsilon = \Psi_{\epsilon_1}(0)\left(-B_\epsilon e^{-(2\lambda_\epsilon + \overline{\lambda}_R)r} - A_\epsilon + 2\lambda_\epsilon I + \overline{\lambda}_R I\right) =$$

$$= \Psi_{\epsilon_1}(0)\left(-B_\epsilon e^{-(2\lambda_\epsilon + \overline{\lambda}_R)r} + B_\epsilon e^{-\lambda_\epsilon r} + \lambda_\epsilon I + \overline{\lambda}_R I\right) =$$

$$= \Psi_{\epsilon_1}(0)\left(B_\epsilon e^{-\lambda_\epsilon r}(1 - e^{-2\mu_\epsilon r}) + 2\mu_\epsilon I\right) =$$

$$= \Psi_{\epsilon_1}(0)\left[2\mu_\epsilon \Psi_{\epsilon_1}(0)\left(B_\epsilon e^{-\lambda_\epsilon r}(1 - \frac{2\mu_\epsilon r}{2!} + \frac{(2\mu_\epsilon r)^2}{3!} + ...) + 2\mu_\epsilon I\right)\right].$$

By denoting

$$h_1(\epsilon) = 2\Psi_{\epsilon_1}(0)\left[B_\epsilon e^{-\lambda_\epsilon r}(1 - \frac{2\mu_\epsilon r}{2!} + \frac{(2\mu_\epsilon r)^2}{3!} + ...) + 1\right]$$

and remarking that

$$\lim_{\epsilon \to 0} h_1(\epsilon) = 2\Psi_{\epsilon_1}(0)\left(B e^{-\omega_\epsilon r} + 1\right),$$

we obtain the first assertion of the Proposition.

Now, in order to treat the term $B_\epsilon R_{\epsilon_1} - R_{\epsilon_2}$, inspired by the proof of Proposition 3.1, we write

$$\Psi_{\epsilon_1}(0)(B_\epsilon R_{\epsilon_1} - R_{\epsilon_2}) = E_{\epsilon_{1a}} + E_{\epsilon_{1b}} + E_{\epsilon_2} + E_{\epsilon_3} + E_{\epsilon_4},$$

where

$$E_{\epsilon_{1a}} = \frac{-g_{\epsilon_{21}}}{\lambda_\epsilon + \lambda_\epsilon} \Psi_{\epsilon_1}(0)B_\epsilon \varphi_{\epsilon_1}(0)(e^{-\lambda_\epsilon r} - e^{-(2\lambda_\epsilon + \overline{\lambda}_R)r}) - g_{\epsilon_{21}} \Psi_{\epsilon_1}(0)\varphi_{\epsilon_1}(0) + g_{\epsilon_{21}},$$

$$E_{\epsilon_{1b}} = \frac{-\overline{g}_{\epsilon_{12}}}{2\lambda_\epsilon} \Psi_{\epsilon_1}(0)B_\epsilon \overline{\varphi}_{\epsilon_1}(0)(e^{-\lambda_\epsilon r} - e^{-(2\lambda_\epsilon + \overline{\lambda}_R)r}) - \overline{g}_{\epsilon_{12}} \Psi_{\epsilon_1}(0)\overline{\varphi}_{\epsilon_1}(0),$$

$$E_{\epsilon_2} = -2\tilde{g}_{\epsilon_{11}} e^{-(2\lambda_\epsilon + \overline{\lambda}_R)r} \Psi_{\epsilon_1}(0)B_\epsilon \int_{-r}^{0} w_{\epsilon_2,0}(\theta) e^{-(2\lambda_\epsilon + \overline{\lambda}_R)\theta} d\theta - 2\theta_{\epsilon_{11}} \Psi_{\epsilon_1}(0) w_{\epsilon_2,0}(0),$$

$$E_{\epsilon_3} = -(g_{\epsilon_{20}} + 2\overline{g}_{\epsilon_{11}}) e^{-(2\lambda_\epsilon + \overline{\lambda}_R)r} \Psi_{\epsilon_1}(0)B_\epsilon \int_{-r}^{0} w_{\epsilon_1,1}(\theta) e^{-(2\lambda_\epsilon + \overline{\lambda}_R)\theta} d\theta -$$

$$-(g_{\epsilon_{20}} + 2\overline{g}_{\epsilon_{11}}) \Psi_{\epsilon_1}(0) w_{\epsilon_1,1}(0),$$

$$E_{\epsilon_4} = -\overline{g}_{\epsilon_{02}} e^{-(2\lambda_\epsilon + \overline{\lambda}_R)r} \Psi_{\epsilon_1}(0)B_\epsilon \int_{-r}^{0} w_{\epsilon_0,2}(\theta) e^{-(2\lambda_\epsilon + \overline{\lambda}_R)\theta} d\theta - \overline{g}_{\epsilon_{02}} \Psi_{\epsilon_1}(0) w_{\epsilon_0,2}(0).$$
$E_{c1a}$. We have

\[ E_{c1a} = -g_{c12} \Psi_{c1}(0) B_c \varphi_{c1}(0) \int_{-\rho}^{0} e^{-\lambda_r(s+r)} e^{(2 \lambda_r + \lambda_c) s} ds - g_{c21} \Psi_{c1}(0) \varphi_{c1}(0) + g_{c21}. \]

We consider the function $\rho_c(s) = \varphi_{c1}(0) e^{(2 \lambda_r + \lambda_c) s}$, $s \in [-\rho, 0]$ and remark that

\[ E_{c1a} = g_{c21} (1 - \langle \Psi_{c1}, \rho_c \rangle). \]

But we know that $1 = \langle \Psi_{c1}, \varphi_{c1} \rangle$, and, substituting this above, we find

\[ E_{c1a} = g_{c21} (\langle \Psi_{c1}, \varphi_{c1} \rangle - \langle \Psi_{c1}, \rho_c \rangle) = g_{c21} (\langle \Psi_{c1}, \varphi_{c1} - \rho_c \rangle), \]

and

\[ \varphi_{c1}(s) - \rho_c(s) = \varphi_{c1}(0) \left( e^{\lambda_r s} - e^{(2 \lambda_r + \lambda_c) s} \right) = e^{\lambda_r s} (1 - e^{2 \mu s}) \varphi_{c1}(0) = \]

\[ = -2 \mu s e^{\lambda_r s} \left( 1 + \frac{2 \mu s}{2!} + \frac{(2 \mu s)^2}{3!} + \ldots \right) \varphi_{c1}(0). \quad (23) \]

The series in the parenthesis converges uniformly and its sum is a bounded function on $[-\rho, 0]$. We see that

\[ \lim_{\epsilon \to 0} E_{c1a} / \mu_c = g_{c21} \langle \Psi_1, \rho \rangle, \]

where $\rho(s) = -2 s e^{\lambda_r s} \varphi_{c1}(0)$.

Now we pass to $E_{c1b}$.

\[ E_{c1b} = \frac{-\overline{g}_{c1,2}}{2 \lambda_c} \Psi_{c1}(0) B_c \varphi_{c1}(0) \left( e^{-\lambda_c r} - e^{-(2 \lambda_c + \lambda_r) \rho} \right) - \overline{g}_{c1,2} \Psi_{c1}(0) \varphi_{c1}(0) = \]

\[ = -\overline{g}_{c1,2} \left[ \Psi_{c1}(0) \overline{\varphi}_{c1}(0) + \Psi_{c1}(0) B_c \overline{\varphi}_{c1}(0) e^{-(2 \lambda_c + \lambda_r) \rho} \frac{1}{-2 \lambda_c} (1 - e^{2 \lambda_c r}) \right] = \]

\[ = -\overline{g}_{c1,2} \left[ \Psi_{c1}(0) \overline{\varphi}_{c1}(0) + \Psi_{c1}(0) B_c \overline{\varphi}_{c1}(0) \int_{-\rho}^{0} e^{-(2 \lambda_c + \lambda_r) (s+r)} e^{\lambda_r s} ds \right]. \]

We define the function $\eta_c(\zeta) = \Psi_{c1}(0) e^{-(2 \lambda_c + \lambda_r) \zeta}$, $\zeta \in [0, r]$ and get

\[ E_{c1b} = -\overline{g}_{c1,2} \langle \eta_c, \varphi_{c2} \rangle. \]

Since $\langle \Psi_{c1}, \varphi_{c2} \rangle = 0$, we have

\[ -\langle \eta_c, \varphi_{c2} \rangle = \langle \Psi_{c1}, \varphi_{c2} \rangle - \langle \eta_c, \varphi_{c2} \rangle = \langle \Psi_{c1} - \eta_c, \varphi_{c2} \rangle, \]

and

\[ \Psi_{c1}(\zeta) - \eta_c(\zeta) = \Psi_{c1}(0) \left( e^{-\lambda_c \zeta} - e^{-(2 \lambda_c + \lambda_r) \zeta} \right) = \Psi_{c1}(0) e^{-\lambda_c \zeta} (1 - e^{-(\lambda_c + \lambda_r) \zeta}). \]
= \Psi_{\epsilon_1}(0)e^{-\lambda_\epsilon\zeta}2\mu_\epsilon\zeta(1-\frac{2\mu_\epsilon\zeta}{2!}+\frac{(2\mu_\epsilon\zeta)^2}{3!}+...).

Hence
\lim_{\epsilon \to 0} \frac{E_{\epsilon_1 b}}{\mu_\epsilon} = \Psi_{\epsilon_1}(0), \quad \zeta \in [0, r].

We define
\alpha_{2,0} := -2g_{\epsilon,11}, \quad \alpha_{1,1} := -g_{\epsilon,20} - 2\Psi_{\epsilon_1,11}, \quad \alpha_{0,2} := -\Psi_{\epsilon_0,2}.

The expressions \( E_i \), \( i = 2, 3, 4 \), can be written as
\[ \alpha_{j,k} \left( \Psi_{\epsilon_1}(0) \cdot w_{\epsilon, j,k}(0) + \Psi_{\epsilon_1}(0) \cdot B_\epsilon \int_{-r}^{0} e^{-\lambda_\epsilon \epsilon + \lambda_\epsilon \epsilon} w_{\epsilon, j,k}(s) ds \right) = \alpha_{j,k}(\eta, w_{\epsilon, j,k}), \]
where \( j, k > 0 \), \( j + k = 2 \).

We know that
\[ \langle \Psi_{\epsilon_1}, w_{\epsilon, j,k} \rangle = 0, \]
and we may write
\[ \langle \eta, w_{\epsilon, j,k} \rangle = \langle \eta, w_{\epsilon, j,k} \rangle - \langle \Psi_{\epsilon_1}, w_{\epsilon, j,k} \rangle = \langle \eta - \Psi_{\epsilon_1}, w_{\epsilon, j,k} \rangle, \]
and
\[ \eta(\zeta) - \Psi_{\epsilon_1}(\zeta) = \Psi_{\epsilon_1}(0)(e^{-(2\lambda_\epsilon + \lambda_\epsilon)\zeta} - e^{-\lambda_\epsilon\zeta}) = \Psi_{\epsilon_1}(0)e^{-\lambda_\epsilon\zeta} \left[ e^{-2\mu_\epsilon\zeta} - 1 \right] = \]
\[ = -2\mu_\epsilon e^{-\lambda_\epsilon\zeta} \left( 1 - \frac{2\mu_\epsilon\zeta}{2!} + \frac{(2\mu_\epsilon\zeta)^2}{3!} - ... \right) \Psi_{\epsilon_1}(0). \]
Hence
\[ \lim_{\epsilon \to 0} \langle \eta, w_{\epsilon, j,k} \rangle / \mu_\epsilon = -\langle \eta, w_{j,k} \rangle. \]

From the above equalities we see that
\[ \lim_{\epsilon \to 0} (E_{\epsilon_2} + E_{\epsilon_3} + E_{\epsilon_4}) / \mu_\epsilon = 2g_{1,1} \langle \eta, w_{2,0} \rangle + (g_{2,0} + 2\Psi_{1,1}) \langle \eta, w_{1,1} \rangle + \Psi_{0,2} \langle \eta, w_{0,2} \rangle. \]

Finally, we have, by setting \( h_2(\epsilon) = \Psi_{\epsilon_1}(0) \cdot (B_\epsilon R_{\epsilon 1} - R_{\epsilon 2}) / \mu_\epsilon \),
\[ \lim_{\epsilon \to 0} h_2(\epsilon) = g_{21} \langle \Psi_1, \rho \rangle + \Psi_{1,2} \langle \eta, \varphi_2 \rangle + 2g_{1,1} \langle \eta, w_{2,0} \rangle + (g_{2,0} + 2\Psi_{1,1}) \langle \eta, w_{1,1} \rangle + \Psi_{0,2} \langle \eta, w_{0,2} \rangle, \]
where
\[ \rho(s) = -2se^{\omega_\epsilon s} \varphi_1(0), \quad s \in [-r, 0], \quad (24) \]
\[ \eta(\zeta) = 2\zeta e^{-\omega_\epsilon \epsilon} \Psi_1(0), \quad \zeta \in [0, r]. \quad (25) \]

This proves the second affirmation of the Proposition. □
The proof of Proposition 4.2 allows us to assert

**Theorem 4.1** The value in zero of the function \( w_{2,1} \) is given by

\[
w_{2,1}(0) = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n,
\]

where \( v_1 = \Psi(0)^T/|\Psi(0)| \), the set of vectors \( v_1, \ldots, v_n \) is an orthonormal basis of \( \mathbb{C}^n \) and \( x_1, \ldots, x_n \) are solutions of the system

\[
\begin{aligned}
(\Psi(0)\tilde{M}v_1)x_1 + (\Psi(0)\tilde{M}v_2)x_2 + \ldots + (\Psi(0)\tilde{M}v_n)x_n &= \tilde{R}, \\
(v_2 \cdot Mv_1)x_1 + (v_2 \cdot Mv_2)x_2 + \ldots + (v_2 \cdot Mv_n)x_n &= v_2 \cdot (BR_1 - R_2), \\
& \vdots \nonumber \\
(v_n \cdot Mv_1)x_1 + (v_n \cdot Mv_2)x_2 + \ldots + (v_n \cdot Mv_n)x_n &= v_n \cdot (BR_1 - R_2),
\end{aligned}
\]

with \( \tilde{M} = 2 (Be^{-\omega_ir} + 1) \), \( M = \omega I - A - Be^{-\omega_ir} \),

\[
\tilde{R} = g_{21}(\Psi, \rho) + \frac{g_{11}(\eta, \varphi_2) + 2g_{11}(\eta, \varphi_{2,0}) + (g_{2,0} + 2g_{1,1})(\eta, \varphi_{1,1}) + g_{0,2}(\eta, \varphi_{2,0})}{1 - e^{-2\omega_ir}}
\]

\[
\Psi(\zeta) = [\psi(0)\varphi_1(0) + \psi(0)B\varphi_1(0)e^{-\omega_ir} - \psi(0)Be^{-\omega_ir}\varphi_1(0)]^{-1}e^{-2\omega_ir}\psi(0),
\]

\[
\rho(s) = -2se^{\omega_is}\varphi(0), \quad s \in [-r, 0], \quad \text{and} \quad \eta(\zeta) = 2\zeta e^{-\omega_is}\Psi(0), \quad \zeta \in [0, r].
\]

Once \( w_{2,1}(0) \) known, \( w_{2,1}(-r) \) is computed from (13).

## 5 Appendix

### I. We compute here the matrix \( E \) with elements \( e_{ij} = \langle \psi_i, \varphi_j \rangle \).

\[
e_{11} = \langle \psi_1, \varphi_1 \rangle = \psi_1(0)\varphi_1(0) + \psi(0)B\varphi_1(0) \int_{-r}^{0} e^{-r(\omega_i)\zeta}e^{\omega_i}d\zeta = \psi_1(0)\varphi_1(0) + \psi(0)B\varphi_1(0)e^{-\omega_ir}.
\]

\[
e_{21} = \langle \psi_2, \varphi_1 \rangle = \psi_2(0)\varphi_1(0) + \psi(0)B\varphi_1(0) \int_{-r}^{0} e^{-r(\omega_i)\zeta}e^{\omega_i}d\zeta = \psi_2(0)\varphi_1(0) + \psi(0)B\varphi_1(0)e^{-\omega_ir} - \psi(0)Be^{-\omega_ir}\varphi_1(0).
\]

In the expression above, we use (8) to eliminate \( \psi_2(0)Be^{-\omega_ir}\varphi_1(0) \) and (6) to eliminate \( \psi_2(0)Be^{-\omega_ir}\varphi_1(0) \). We get

\[
e_{21} = \psi_2(0)\varphi_1(0) + \frac{1}{2\omega_i} (\psi_2(0)\omega_i\varphi_1(0) - \psi_2(0)^2A\varphi_1(0) - \psi_2(0)\omega_i\varphi_1(0) + \psi_2(0)A\varphi_1(0)) =
\]

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\[ \psi_2(0)\varphi_1(0) - \psi_2(0)\varphi_1(0) = 0. \]

For \( e_{12} \), as for \( e_{21} \), we find
\[ e_{12} = 0. \]

Then
\[ e_{22} = \langle \psi_2, \varphi_2 \rangle = \psi_2(0)\varphi_2(0) + \psi_2(0)B_e\varphi_2(0) \int_{-r}^{0} e^{(\zeta + r)\omega_i}e^{-\zeta\omega_i}d\zeta = \psi_2(0)\varphi_2(0) + \psi_2(0)B_e\varphi_2(0)e^{\omega_ir}, \]

Now, by using
\[ \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = E^{-1} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\det E} \begin{pmatrix} e_{22} & 0 \\ 0 & e_{11} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \]

we will get \( \Psi_1(\zeta) = \Psi_1(0)e^{-\omega_\zeta} \), with
\[ \Psi_1(0) = \frac{e_{22}}{\det E} \psi_1(0) = \frac{1}{e_{11} e_{22}} \psi_1(0) = \frac{1}{\psi_1(0)\varphi_1(0) + \psi_1(0)B_e\varphi_1(0)e^{-\omega_ir}} \psi_1(0). \]

**II.** We compute the matrix \( E_e \) with elements \( \tilde{e}_{ij} = \langle \psi_{e_i}, \varphi_{e_j} \rangle \):

\[ \tilde{e}_{11} = \langle \psi_{e_1}, \varphi_{e_1} \rangle = \psi_{e_1}(0)\varphi_{e_1}(0) + \psi_{e_1}(0)B_e\varphi_{e_1}(0) \int_{-r}^{0} e^{-\lambda_e(s+r)}e^{\lambda_e s} ds = \psi_{e_1}(0)\varphi_{e_1}(0) + \psi_{e_1}(0)B_e\varphi_{e_1}(0)e^{-\lambda_e r}. \]

\[ \tilde{e}_{12} = \langle \psi_{e_1}, \varphi_{e_2} \rangle = \psi_{e_1}(0)\varphi_{e_2}(0) + \psi_{e_1}(0)B_e\varphi_{e_2}(0) \int_{-r}^{0} e^{-\lambda_e(s+r)}e^{\lambda_e s} ds = \psi_{e_1}(0)\varphi_{e_2}(0) + \frac{1}{-\lambda_e + \lambda_e} \psi_{e_1}(0)(B_e\varphi_{e_2}(0)e^{-\lambda_e r} - B_e\varphi_{e_2}(0)e^{-\lambda_e r}) = \psi_{e_1}(0)\varphi_{e_2}(0) + \frac{1}{-\lambda_e + \lambda_e} \left[ (\lambda_e \psi_{e_1}(0) - \psi_{e_1}(0)A_e)\varphi_{e_2}(0) - \psi_{e_1}(0)(\bar{\lambda}_e \varphi_{e_2}(0) - A_e\varphi_{e_2}(0)) \right] = \psi_{e_1}(0)\varphi_{e_2}(0) + \frac{1}{-\lambda_e + \lambda_e} (\lambda_e - \bar{\lambda}_e)\psi_{e_1}(0)\varphi_{e_2}(0) = 0. \]

Here we used the relations
\[ \psi_{e_1}(0)(-\lambda_e I + A_e + B_e e^{-\lambda_e r}) = 0, \quad (\bar{\lambda}_e I - A_e - B_e e^{-\lambda_e r})\varphi_{e_2}(0) = 0. \]

Now, by taking the complex conjugate we obtain
\[ \tilde{e}_{21} = \langle \psi_{e_2}, \varphi_{e_1}(0) \rangle = 0, \]

\[ \tilde{e}_{22} = \langle \psi_{e_2}, \varphi_{e_2} \rangle = \psi_{e_2}(0)\varphi_{e_2}(0) + \psi_{e_2}(0)B_e\varphi_{e_2}(0) \int_{-r}^{0} e^{-\bar{\lambda}_e(s+r)}e^{\bar{\lambda}_e s} ds = \psi_{e_2}(0)\varphi_{e_2}(0) + \psi_{e_2}(0)B_e\varphi_{e_2}(0)e^{-\bar{\lambda}_e r}. \]

We have \( \det E = \tilde{e}_{11}\tilde{e}_{22} \) and
\[ \Psi_{e_1}(\zeta) = \frac{\tilde{e}_{22}}{\tilde{e}_{11}\tilde{e}_{22}} \psi_{e_1}(\zeta) = \frac{1}{\tilde{e}_{11}} e^{-\lambda_e \zeta} \psi_{e_1}(0). \]
References

[1] Faria, T. (2002). Normal forms for RFDE in finite dimensional spaces -section 8.3 of J. Hale, L.T. Magalhaes, W. Oliva, Dynamics in infinite dimensions, Applied Mathematical Sciences, 47, Springer.

[2] Hale, J. (1977). Theory of functional differential equations, Springer, New York.

[3] Hale, J., and Verduyn Lunel, S. M. (2003). Introduction to functional differential equation, Springer, Berlin.

[4] Ion, A. V. (2004). On the Bautin bifurcation for systems of delay differential equations, Acta Univ. Apulensis, 8, 235-246, arXiv:1111.1559v1.

[5] Ion, A. V. (2012). On the computation of the third order terms of the series defining the center manifold for a scalar delay differential equation, J. Dyn. Diff. Eqns, 24, 2, 325-340.

[6] Ion, A. V., and Georgescu, R. M. (2013). Bautin bifurcation in a delay differential equation modelling leukemia, Nonlinear Anal.: Theory, Methods and Appl., 82, 142-157.

[7] Mircea, G., Neamțu, M., and Opris, D. (2003). Dynamical systems from economy, mechanics, biology, described by delay equations, Mirton, Timişoara (in Romanian).

[8] Wischert, W., Wunderlin, A., Pelster, A., Olivier, M., and Grosambert J. (1994). Delay induced instabilities in nonlinear feedback systems, Physical Review E, 49, 1, 203-219.