A SINGULAR LIMIT PROBLEM FOR CONSERVATION LAWS RELATED TO THE ROSENAU EQUATION

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Abstract. We consider the Rosenau equation, which contains nonlinear dispersive effects. We prove that as the diffusion parameter tends to zero, the solutions of the dispersive equation converge to discontinuous weak solutions of the Burgers equation. The proof relies on deriving suitable a priori estimates together with an application of the compensated compactness method in the $L^p$ setting.

1. Introduction

Dynamics of shallow water waves that is observed along lake shores and beaches has been a research area for the past few decades in oceanography (see [1, 24]). There are several models proposed in this context: Korteweg-de Vries (KdV) equation, Boussinesq equation, Peregrine equation, regularized long wave (RLW) equation, Kawahara equation, Benjamin-Bona-Mahoney equation, Bona-Chen equation etc. These models were derived from first principles under various different hypothesis and approximations. They are all well studied and very well understood.

The dynamics of dispersive shallow water waves, on the other hand, is captured with slightly different models, like Rosenau-Kawahara equation, Rosenau-KdV equation, and Rosenau-KdV-RLW equation [2, 11, 12, 13, 17].

The Rosenau-KdV-RLW equation is

\begin{equation}
\partial_t u + a \partial_x u + k \partial_x u^n + b_1 \partial^3_{xxxx} u + b_2 \partial^3_{txxx} u + c \partial^5_{txxxx} u = 0, \quad a, k, b_1, b_2, c \in \mathbb{R},
\end{equation}

Here $u(t, x)$ is the nonlinear wave profile. The first term is the linear evolution one, while $a$ is the advection or drifting coefficient. $b_1$ and $b_2$ are the dispersion coefficients. The higher order dispersion coefficient is $c$, while the coefficient of nonlinearity is $k$ where $n$ is nonlinearity parameter. These are all known and given parameters.

In [17], the authors analyzed (1.1). They got solitary waves, shock waves and singular solitons along with conservation laws.

Considering the $n = 2, a = 0, k = 1, b_1 = 0, b_2 = -1, c = 1$:

\begin{equation}
\partial_t u + \partial_x u^2 + \partial^3_{xxxx} u - \partial^3_{txxx} u + \partial^5_{txxxx} u = 0.
\end{equation}

If $n = 2, a = 0, k = 1, b_1 = 0, b_2 = -1, c = 1$, (1.1) reads

\begin{equation}
\partial_t u + \partial_x u^2 - \partial^3_{txxx} u + \partial^5_{txxxx} u = 0,
\end{equation}

which is known as Rosenau-RLW equation.

Arguing in [7], we re-scale the equations as follows

\begin{equation}
\partial_t u + \partial_x u^2 + \beta \partial^3_{xxxx} u - \beta \partial^3_{txxx} u + \beta^2 \partial^5_{txxxx} u = 0,
\end{equation}

\begin{equation}
\partial_t u + \partial_x u^2 - \beta \partial^3_{txxx} u + \beta^2 \partial^5_{txxxx} u = 0,
\end{equation}

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where $\beta$ is the diffusion parameter.

In [3], the authors proved that the solutions of (1.4) and (1.5) converge to the unique entropy solution of the Burgers equation

$$\partial_t u + \partial_x u^2 = 0.$$  

Choosing $n = 2$, $a = 0$, $k = 1$, $b_2 = b_1 = 0$, $c = 1$, (1.1) reads

$$\partial_t u + \partial_x u^2 + \partial_{xxxx}^5 u = 0,$$

which is known as Rosenau equation (see [19, 20]). The existence and the uniqueness of the solution for (1.7) has been proved in [16].

Finally, if $n = 2$, $a = 0$, $k = 1$, $b_1 = 1$, $b_2 = 0$, $c = 1$, (1.1) reads

$$\partial_t u + \partial_x u^2 + \partial_{xxx}^3 u + \partial_{xxxx}^5 u = 0,$$

which is known as Rosenau-KdV equation.

In [23], the author discussed the solitary wave solutions and (1.8). In [12], a conservative linear finite difference scheme for the numerical solution for an initial-boundary value problem of the Rosenau-KdV equation is considered. In [10, 18], authors discussed the solitary solutions for (1.8) with solitary ansatz method. The authors also gave the two invariants for (1.8). In particular, in [18], the authors studied two types of soliton solutions: a solitary wave and a singular soliton. In [22], the authors proposed an average linear finite difference scheme for the numerical solution of the initial-boundary value problem for (1.8).

In this paper, we analyze (1.7). Arguing in [7], we re-scale the equations as follows

$$\partial_t u + \partial_x u^2 + \beta^2 \partial_{xxxx}^5 u = 0.$$  

We are interested in the no high frequency limit, we send $\beta \to 0$ in (1.9). In this way we pass from (1.9) to (1.6)

We prove that, as $\beta \to 0$, the solutions of converge (1.9) to the unique entropy solution of (1.6).

In order to do this, we can choose the initial datum and $\beta$ in two different ways.

Following [9, Theorem 7.1], the first choice is the following (see Theorem 2.1):

$$u_0 \in L^2(\mathbb{R}), \quad \beta = o(\varepsilon^4).$$

Since $\|\cdot\|_{L^4}$ is a conserved quantity for (1.9), the second choice is (see Theorem 3.1):

$$u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}), \quad \beta = O(\varepsilon^4).$$

It is interesting to observe that, while the summability on the initial datum in (1.11) is greater than the one in (1.10), the assumption on $\beta$ in (1.11) is weaker than the one in (1.10).

From the mathematical point of view, the two assumptions require two different arguments for the $L^\infty$–estimate (see Lemmas 2.2 and 3.1). Indeed, the proof of Lemma 2.2 under the assumption (1.10), is more technical than the one of Lemma 3.1.

The paper is organized in five sections. In Section 2, we prove the convergence of (1.9) to (1.6) in the $L^p$ setting, with $1 \leq p < 2$. In Section 3, we prove the convergence of (1.9) to (1.6) in the $L^p$ setting, with $1 \leq p < 4$. Sections A and B are two appendixes, where, choosing the initial datum in two different ways, we prove that the solutions of the Korteweg-de Vries equation converge to discontinuous weak solutions of (1.6) in the $L^p$ setting, with $1 \leq p < 2$. 
2. The Rosenau Equation: $u_0 \in L^2(\mathbb{R})$

In this section, we consider (1.9), and assume (1.10) on the initial datum.

We study the dispersion-diffusion limit for (1.9), namely we send $\beta \to 0$ and get (1.6). Therefore, we fix two small numbers $0 < \varepsilon, \beta < 1$ and consider the following fifth order problem

\[
\begin{align*}
(2.1) \quad & \frac{\partial u_{\varepsilon, \beta}}{\partial t} + \partial_x u_{\varepsilon, \beta}^2 + \beta^2 \partial_{xxxxx}^5 u_{\varepsilon, \beta} = \varepsilon \partial_{xx}^2 u_{\varepsilon, \beta}, & t > 0, x \in \mathbb{R}, \\
& u_{\varepsilon, \beta}(0, x) = u_{\varepsilon, \beta, 0}(x), & x \in \mathbb{R},
\end{align*}
\]

where $u_{\varepsilon, \beta, 0}$ is a $C^\infty$ approximation of $u_0$ such that

\[
(2.2) \quad \|u_{\varepsilon, \beta, 0}\|_{L^p_{\text{loc}}(\mathbb{R})}^2 + \left(\frac{\beta^2}{2} + \varepsilon^2\right) \|\partial_x u_{\varepsilon, \beta, 0}\|_{L^2(\mathbb{R})}^2 \leq C_0, \quad \varepsilon, \beta > 0
\]

and $C_0$ is a constant independent on $\varepsilon$ and $\beta$.

The main result of this section is the following theorem.

**Theorem 2.1.** Assume that (1.10) and (2.2) hold. Fix $T > 0$, if

\[
(2.3) \quad \beta = O\left(\varepsilon^4\right),
\]

then, there exist two sequences $\{\varepsilon_n\}_{n \in \mathbb{N}}$, $\{\beta_n\}_{n \in \mathbb{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function

\[
u_m \in L^\infty((0, T); L^2(\mathbb{R}))
\]

such that

i) $u_{\varepsilon_n, \beta_n} \to u$ strongly in $L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$, for each $1 \leq p < 2$,

ii) $u$ is a distributional solution of (1.6).

Moreover, if

\[
(2.4) \quad \beta = O\left(\varepsilon^4\right),
\]

iii) $u$ is the unique entropy solution of (1.6). 

Let us prove some a priori estimates on $u_{\varepsilon, \beta}$, denoting with $C_0$ the constants which depend only on the initial data.

Arguing as [3, Lemma 2.1], we have the following result.

**Lemma 2.1.** For each $t > 0$,

\[
(2.5) \quad \|u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|\partial_x u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C_0.
\]

**Lemma 2.2.** Fix $T > 0$. Assume (2.3) holds. There exists $C_0 > 0$, independent on $\varepsilon, \beta$ such that

\[
(2.6) \quad \|u_{\varepsilon, \beta}\|_{L^\infty((0, T) \times \mathbb{R})} \leq C_0 \beta^{-\frac{1}{4}}.
\]

Moreover,

i) the families $\{\beta^\frac{1}{2} \partial_x u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$, $\{\beta^\frac{1}{2} \partial_{xx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$, $\{\beta^\frac{3}{2} \partial_{xxx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$, $\{\beta^3 \partial_{xxxx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$, are bounded in $L^\infty((0, T); L^2(\mathbb{R}))$;

ii) the families $\{\beta^\frac{1}{2} \varepsilon \partial_x u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$, $\{\beta^\frac{1}{2} \varepsilon \partial_{xx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$, $\{\beta^\frac{3}{2} \varepsilon \partial_{xxx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$, $\{\beta^3 \varepsilon \partial_{xxxx} u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$ are bounded in $L^2((0, T) \times \mathbb{R})$. 

Proof. Let $0 < t < T$. Multiplying (2.1) by $-\beta^2 \partial_{xx}^2 u_{\varepsilon,\beta} - \beta \varepsilon \partial_{xxx}^2 u_{\varepsilon,\beta} + \varepsilon \partial_t u_{\varepsilon,\beta}$, we have

\[
\left(-\beta^2 \partial_{xx}^2 u_{\varepsilon,\beta} - \beta \varepsilon \partial_{xxx}^2 u_{\varepsilon,\beta} + \varepsilon \partial_t u_{\varepsilon,\beta}\right) \partial_t u_{\varepsilon,\beta} \\
+ 2 \left(-\beta^2 \partial_{xx}^2 u_{\varepsilon,\beta} - \beta \varepsilon \partial_{xxx}^2 u_{\varepsilon,\beta} + \varepsilon \partial_t u_{\varepsilon,\beta}\right) u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \\
+ \beta^2 \left(-\beta^2 \partial_{xx}^2 u_{\varepsilon,\beta} - \beta \varepsilon \partial_{xxx}^2 u_{\varepsilon,\beta} + \varepsilon \partial_t u_{\varepsilon,\beta}\right) \partial_{xxxx}^2 u_{\varepsilon,\beta} \\
= \varepsilon \left(-\beta^2 \partial_{xx}^2 u_{\varepsilon,\beta} - \beta \varepsilon \partial_{xxx}^2 u_{\varepsilon,\beta} + \varepsilon \partial_t u_{\varepsilon,\beta}\right) \partial_{xx}^2 u_{\varepsilon,\beta}.
\]

(2.7)

We observe

\[
\int_R \left(-\beta^2 \partial_{xx}^2 u_{\varepsilon,\beta} - \beta \varepsilon \partial_{xxx}^2 u_{\varepsilon,\beta} + \varepsilon \partial_t u_{\varepsilon,\beta}\right) \partial_t u_{\varepsilon,\beta} dx \\
= \frac{\beta^2}{2} \frac{d}{dt} \left\| \partial_x u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2 + \beta \varepsilon \left\| \partial_{xx}^2 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2 \\
+ \varepsilon \left\| \partial_t u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2.
\]

(2.8)

Since

\[
2 \int_R \left(-\beta^2 \partial_{xx}^2 u_{\varepsilon,\beta} - \beta \varepsilon \partial_{xxx}^2 u_{\varepsilon,\beta} + \varepsilon \partial_t u_{\varepsilon,\beta}\right) u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} dx \\
= -2 \beta^2 \int_R u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \partial_{xx}^2 u_{\varepsilon,\beta} dx - 2 \beta \varepsilon \int_R u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \partial_{xxx}^2 u_{\varepsilon,\beta} dx \\
+ 2 \varepsilon \int_R u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \partial_t u_{\varepsilon,\beta} dx,
\]

\[
\beta^2 \int_R \left(-\beta^2 \partial_{xx}^2 u_{\varepsilon,\beta} - \beta \varepsilon \partial_{xxx}^2 u_{\varepsilon,\beta} + \varepsilon \partial_t u_{\varepsilon,\beta}\right) \partial_{xxxx}^2 u_{\varepsilon,\beta} dx \\
= \frac{\beta^2}{2} \frac{d}{dt} \left\| \partial_{xx}^2 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2 + \beta \varepsilon \left\| \partial_{xxx}^2 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2 \\
+ \varepsilon \left\| \partial_t u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2,
\]

(2.7)

Integrating (2.7) on $R$ we have

\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \left\| \partial_x u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2 + \frac{\beta \varepsilon}{2} \left\| \partial_{xx}^2 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2 \right) \\
+ \frac{\beta^2}{2} \frac{d}{dt} \left\| \partial_{xx}^2 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2 + \beta \varepsilon \left\| \partial_{xx}^2 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2 \\
+ \beta \varepsilon \left\| \partial_{xxx}^2 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2 + \frac{\beta^2}{2} \frac{d}{dt} \left\| \partial_{xxx}^2 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2 \\
+ \beta^2 \left\| \partial_{xxxx}^2 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2 + \beta \varepsilon \left\| \partial_{xxxx}^2 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(R)}^2 \\
= 2 \beta^2 \int_R u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \partial_{xx}^2 u_{\varepsilon,\beta} dx + 2 \beta \varepsilon \int_R u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \partial_{xxx}^2 u_{\varepsilon,\beta} dx \\
- 2 \varepsilon \int_R u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \partial_t u_{\varepsilon,\beta} dx.
\]

(2.9)
Due to (2.3) and the Young inequality,

\[ 2\beta \int \frac{d}{dt} \left( \frac{1}{2} \| u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \| \partial_x u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 \right) \leq 2 \beta \int \frac{d}{dt} \left( \frac{1}{2} \| u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \| \partial_x u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 \right) + \frac{\beta^2}{2} \int \| \partial_{xx} u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 \right) \]

From (2.9) and (2.10) we gain

\[ 2\beta \int \left( \frac{1}{2} \| u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \| \partial_x u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 \right) \leq \int \left( \frac{1}{2} \| u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \| \partial_x u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 \right) \]

An integration on \((0,t)\), (2.2) and (2.5) give

\[ \frac{\beta^2}{2} \int_0^t \| \partial_{xx} u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 \right) \]

\[ \leq C_0 \int_0^t \left( \frac{1}{2} \| u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \| \partial_x u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 \right) \]

\[ \leq C_0 \left( 1 + \int \| u_{\varepsilon,\beta} \|_{L^2(\mathbb{R})}^2 \right) \]
We prove (2.10). Due to (2.5), (2.11) and the Hölder inequality,
\[
\begin{align*}
\quad u^2_{\varepsilon, \beta}(t, x) &= 2 \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \, dx \\
&\leq 2 \int_{\mathbb{R}} |u_{\varepsilon, \beta}| \left| \partial_x u_{\varepsilon, \beta} \right| \, dx \\
&\leq C_0 \sqrt{C_0 + C_0 \left\| u_{\varepsilon, \beta} \right\|^2_{L^\infty(0, T) \times \mathbb{R}}},
\end{align*}
\]
that is
\[
(2.12) \quad \left\| u_{\varepsilon, \beta} \right\|^4_{L^\infty(0, T) \times \mathbb{R}} \leq \frac{C_0}{\delta} \left( 1 + \left\| u_{\varepsilon, \beta} \right\|^2_{L^\infty(0, T) \times \mathbb{R}} \right).
\]
Introducing the notation
\[
(2.13) \quad y = \left\| u_{\varepsilon, \beta} \right\|_{L^\infty(0, T) \times \mathbb{R}}, \quad \delta = \frac{\beta}{2},
\]
(2.12) reads
\[
y^4 \leq \frac{C_0}{\delta} (1 + y^2).
\]
Arguing as [3] Lemma 2.3, we have
\[
(2.14) \quad y \leq C_0 \delta^{-\frac{1}{2}}.
\]
(2.10) follows from (2.13) and (2.14). (2.6) and (2.11) give
\[
\begin{align*}
\beta + \frac{\beta^2 \varepsilon^2}{2} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2 \varepsilon^2}{2} \left\| \partial_{xxx} u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
+ \frac{\beta \varepsilon^2}{2} \left\| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\beta \varepsilon}{2} \int_0^t \left\| \partial_x^2 u_{\varepsilon, \beta}(s, \cdot) \right\|^2_{L^2(\mathbb{R})} \, ds \\
+ \frac{\beta^3 \varepsilon}{2} \int_0^t \left\| \partial_t^3 u_{\varepsilon, \beta}(s, \cdot) \right\|^2_{L^2(\mathbb{R})} \, ds + \frac{\beta^3 \varepsilon}{2} \int_0^t \left\| \partial_{xxx} u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \, ds \\
+ \frac{\beta^3 \varepsilon}{2} \int_0^t \left\| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \, ds + \frac{\beta^3 \varepsilon}{2} \int_0^t \left\| \partial_{xxx} u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \, ds \leq C_0 \beta^{-\frac{1}{2}},
\end{align*}
\]
that is
\[
\begin{align*}
\beta + \frac{\beta^2 \varepsilon^2}{2} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\beta^3 \varepsilon}{2} \int_0^t \left\| \partial_t^3 u_{\varepsilon, \beta}(s, \cdot) \right\|^2_{L^2(\mathbb{R})} \, ds \\
+ \frac{\beta^3 \varepsilon}{2} \int_0^t \left\| \partial_{xxx} u_{\varepsilon, \beta}(s, \cdot) \right\|^2_{L^2(\mathbb{R})} \, ds + \frac{\beta^3 \varepsilon}{2} \int_0^t \left\| \partial_{xx} u_{\varepsilon, \beta}(s, \cdot) \right\|^2_{L^2(\mathbb{R})} \, ds \\
+ \frac{\beta^3 \varepsilon}{2} \int_0^t \left\| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \, ds + \frac{\beta^3 \varepsilon}{2} \int_0^t \left\| \partial_{xxx} u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \, ds \leq C_0.
\end{align*}
\]
Hence,
\[
\begin{align*}
\beta^\frac{1}{2} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})} &\leq C_0, \\
\beta^\frac{1}{2} \varepsilon \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})} &\leq C_0, \\
\beta^\frac{3}{2} \left\| \partial_{xxx} u_{\varepsilon, \beta}(t, \cdot) \right\|_{L^2(\mathbb{R})} &\leq C_0.
\end{align*}
\]
\[ \beta^3 \varepsilon \| \partial_x^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})} \leq C_0, \]

\[ \beta^3 \varepsilon \int_0^t \| \partial_x^4 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \leq C_0, \]

\[ \beta \varepsilon \int_0^t \| \partial_x^2 u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \leq C_0, \]

\[ \beta^5 \varepsilon \int_0^t \| \partial_x^5 u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \leq C_0, \]

\[ \beta^2 \varepsilon \int_0^t \| \partial_x^3 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \leq C_0, \]

for every \( 0 < t < T \).

To prove Theorem 2.1, the following technical lemma is needed [15].

**Lemma 2.3.** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^2 \). Suppose that the sequence \( \{ \mathcal{L}_n \}_{n \in \mathbb{N}} \) of distributions is bounded in \( W^{-1, \infty}(\Omega) \). Suppose also that

\[ \mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n}, \]

where \( \{ \mathcal{L}_{1,n} \}_{n \in \mathbb{N}} \) lies in a compact subset of \( H^{-1}_{\text{loc}}(\Omega) \) and \( \{ \mathcal{L}_{2,n} \}_{n \in \mathbb{N}} \) lies in a bounded subset of \( M_{\text{loc}}(\Omega) \). Then \( \{ \mathcal{L}_n \}_{n \in \mathbb{N}} \) lies in a compact subset of \( H^{-1}_{\text{loc}}(\Omega) \).

Moreover, we consider the following definition.

**Definition 2.1.** A pair of functions \((\eta, q)\) is called an entropy–entropy flux pair if \( \eta : \mathbb{R} \to \mathbb{R} \) is a \( C^2 \) function and \( q : \mathbb{R} \to \mathbb{R} \) is defined by

\[ q(u) = 2 \int_0^u \xi \eta'(\xi) d\xi. \]

An entropy-entropy flux pair \((\eta, q)\) is called convex/compactly supported if, in addition, \( \eta \) is convex/compactly supported.

We begin by proving the following result.

**Lemma 2.4.** Assume that (1.10), (2.2) and (2.3) hold. Then for any compactly supported entropy-entropy flux pair \((\eta, q)\), there exist two sequences \( \{ \varepsilon_n \}_{n \in \mathbb{N}}, \{ \beta_n \}_{n \in \mathbb{N}} \), with \( \varepsilon_n, \beta_n \to 0 \), and a limit function

\[ u \in L^\infty((0, T); L^2(\mathbb{R})), \]

such that

\[ u_{\varepsilon_n, \beta_n} \to u \quad \text{in} \quad L^p_{\text{loc}}((0, T) \times \mathbb{R}), \quad \text{for each} \quad 1 \leq p < 2, \]

\[ u \quad \text{is a distributional solution of (1.6).} \]

**Proof.** Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\). Multiplying (2.1) by \( \eta'(u_{\varepsilon, \beta}) \), we have

\[ \partial_t \eta(u_{\varepsilon, \beta}) + \partial_x q(u_{\varepsilon, \beta}) = \varepsilon \eta'(u_{\varepsilon, \beta}) \partial_x^2 u_{\varepsilon, \beta} + \beta^2 \eta'(u_{\varepsilon, \beta}) \partial_x^5 u_{\varepsilon, \beta} = I_{1, \varepsilon, \beta} + I_{2, \varepsilon, \beta} + I_{3, \varepsilon, \beta} + I_{4, \varepsilon, \beta}, \]
where
\[ I_{1,\varepsilon,\beta} = \partial_x (\varepsilon \eta'(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}), \]
\[ I_{2,\varepsilon,\beta} = -\varepsilon \eta''(u_{\varepsilon,\beta})(\partial_x u_{\varepsilon,\beta})^2, \]
\[ I_{3,\varepsilon,\beta} = \partial_x (\beta^2 \eta(u_{\varepsilon,\beta})\partial_{xxx}^4 u_{\varepsilon,\beta}), \]
\[ I_{4,\varepsilon,\beta} = -\beta \eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{xxx}^4 u_{\varepsilon,\beta}. \]

(2.17)

Fix \( T > 0 \). Arguing as [6, Lemma 3.2], we have that \( I_{1,\varepsilon,\beta} \to 0 \) in \( H^{-1}((0,T) \times \mathbb{R}) \), and \( \{I_{2,\varepsilon,\beta}\}_{\varepsilon,\beta > 0} \) is bounded in \( L^1((0,T) \times \mathbb{R}) \).

We claim that
\[ I_{3,\varepsilon,\beta} \to 0 \quad \text{in} \quad H^{-1}((0,T) \times \mathbb{R}), \quad T > 0, \quad \text{as} \quad \varepsilon \to 0. \]

By (2.3) and Lemma 2.2,
\[ \left\| \beta^2 \eta'(u_{\varepsilon,\beta})\partial_{xxx}^4 u_{\varepsilon,\beta} \right\|_{L^2((0,T) \times \mathbb{R})} \]
\[ \leq \beta^4 \left\| \eta' \right\|_{L^\infty(\mathbb{R})} \left\| \partial_{xxx}^4 u_{\varepsilon,\beta} \right\|_{L^2((0,T) \times \mathbb{R})} \]
\[ = \left\| \eta' \right\|_{L^\infty(\mathbb{R})} \frac{\beta^4}{\varepsilon} \left\| \partial_{xxx}^4 u_{\varepsilon,\beta} \right\|_{L^2((0,T) \times \mathbb{R})}^2 \]
\[ \leq C_0 \left\| \eta' \right\|_{L^\infty(\mathbb{R})} \varepsilon \to 0. \]

We have that \( \{I_{4,\varepsilon,\beta}\}_{\varepsilon,\beta > 0} \) is bounded in \( L^1((0,T) \times \mathbb{R}) \), \( T > 0 \).

Thanks to (2.3), Lemmas 2.1, 2.2 and the Hölder inequality,
\[ \left\| \beta^2 \eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial_{xxx}^4 u_{\varepsilon,\beta} \right\|_{L^1((0,T) \times \mathbb{R})} \]
\[ \leq \beta^2 \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \int_0^T \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\beta}\partial_{xxx}^4 u_{\varepsilon,\beta}| \, ds \, dx \]
\[ = \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \frac{\beta^2}{\varepsilon} \left\| \partial_x u_{\varepsilon,\beta} \right\|_{L^2((0,T) \times \mathbb{R})} \left\| \partial_{xxx}^4 u_{\varepsilon,\beta} \right\|_{L^2((0,T) \times \mathbb{R})} \]
\[ = \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \frac{\beta^4}{\varepsilon} \left\| \partial_x u_{\varepsilon,\beta} \right\|_{L^2((0,T) \times \mathbb{R})} \left\| \partial_{xxx}^4 u_{\varepsilon,\beta} \right\|_{L^2((0,T) \times \mathbb{R})} \leq C_0 \left\| \eta'' \right\|_{L^\infty(\mathbb{R})}. \]

Therefore, (2.15) follows from Lemmas 2.1, 2.2 and the \( L^p \) compensated compactness of [21].

Arguing as [3, Theorem 2.1], we have (2.16).

\[ \square \]

Following [14], we prove the following result.

**Lemma 2.5.** Assume that (1.10), (2.2) and (2.4) hold. Then for any compactly supported entropy-entropy flux pair \((\eta, q)\), there exist two sequences \(\{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}}\), with \(\varepsilon_n, \beta_n \to 0\), and a limit function
\[ u \in L^\infty((0,T); L^2(\mathbb{R})), \]
\[ (2.18) \]
such that (2.14) holds and
\[ \text{such that} \quad (2.14) \quad \text{holds and} \quad (2.18) \quad u \quad \text{is the unique entropy solution of} \quad (1.6). \]

**Proof.** Let us consider a compactly supported entropy-entropy flux pair \((\eta, q)\). Multiplying (2.1) by \(\eta'(u_{\varepsilon,\beta})\), we have
\[ \partial_t \eta(u_{\varepsilon,\beta}) + \partial_x q(u_{\varepsilon,\beta}) = \varepsilon \eta'(u_{\varepsilon,\beta})\partial_{xx}^2 u_{\varepsilon,\beta} + \beta^2 \eta(u_{\varepsilon,\beta})\partial_{xxx}^4 u_{\varepsilon,\beta}, \]
\[ = I_{1,\varepsilon,\beta} + I_{2,\varepsilon,\beta} + I_{3,\varepsilon,\beta} + I_{4,\varepsilon,\beta}. \]
where \(I_{1,\varepsilon,\beta}, I_{2,\varepsilon,\beta}, I_{3,\varepsilon,\beta}, I_{4,\varepsilon,\beta}\) are defined in (2.17).

As in Lemma 2.4, we obtain that \(I_{1,\varepsilon,\beta} \to 0\) in \(H^{-1}((0, T) \times \mathbb{R})\), \(\{I_{2,\varepsilon,\beta}\}_{\varepsilon,\beta>0}\) is bounded in \(L^1((0, T) \times \mathbb{R})\), \(I_{3,\varepsilon,\beta} \to 0\) in \(H^{-1}((0, T) \times \mathbb{R})\).

Let us show that
\[
I_{4,\varepsilon,\beta} \to 0 \quad \text{in} \quad L^1((0, T) \times \mathbb{R}), \quad T > 0.
\]
Thanks to (2.4), Lemmas 2.1, 2.2 and the H"older inequality,
\[
\|\beta \eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta} \partial_{xxx} u_{\varepsilon,\beta}\|_{L^1((0, T) \times \mathbb{R})}
\leq \beta^2 \|\eta''\|_{L^\infty(\mathbb{R})} \int_0^T \int_\mathbb{R} |\partial_x u_{\varepsilon,\beta} \partial_{xxx} u_{\varepsilon,\beta}| \, ds \, dx
\leq \|\eta''\|_{L^\infty(\mathbb{R})} \frac{\beta^2 \varepsilon}{\varepsilon} \|\partial_x u_{\varepsilon,\beta}\|_{L^2((0, T) \times \mathbb{R})} \|\partial_{xxx} u_{\varepsilon,\beta}\|_{L^2((0, T) \times \mathbb{R})}
\leq C_0 \|\eta''\|_{L^\infty(\mathbb{R})} \frac{\beta^2}{\varepsilon} \to 0.
\]
Arguing as [3, Theorem 2.1], we have (2.18). \(\square\)

**Proof of Theorem 2.4.** Theorem 2.1 follows from Lemmas 2.4 and 2.5. \(\square\)

### 3. The Rosenau Equation: \(u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})\).

In this section, we consider (1.9), and we assume (1.11) on the initial datum.

We consider the approximate problem (2.1), where \(u_{\varepsilon,\beta,0}\) is a \(C^\infty\) approximation of \(u_0\) such that
\[
u_{\varepsilon,\beta,0} \to u_0 \quad \text{in} \quad L^p_{\text{loc}}(\mathbb{R}), \quad 1 \leq p < 2, \quad \text{as} \quad \varepsilon, \beta \to 0,
\]
and
\[
\begin{align*}
\|u_{\varepsilon,\beta,0}\|_{L^4(\mathbb{R})}^4 + \|u_{\varepsilon,\beta,0}\|_{L^2(\mathbb{R})}^2 + \beta^2 + \varepsilon^2 \|\partial_x u_{\varepsilon,\beta,0}\|_{L^2(\mathbb{R})}^2 & \leq C_0, \quad \varepsilon, \beta > 0, \\
\beta \|\partial_x u_{\varepsilon,\beta}\|_{L^2(\mathbb{R})}^2 + \beta^2 \|\partial_{xxx} u_{\varepsilon,\beta}\|_{L^2(\mathbb{R})}^2 & \leq C_0, \quad \varepsilon, \beta > 0,
\end{align*}
\]
and \(C_0\) is a constant independent on \(\varepsilon\) and \(\beta\).

The main result of this section is the following theorem.

**Theorem 3.1.** Assume that (1.11) and (3.1) hold. Fix \(T > 0\), if (2.3) holds, there exist two sequences \(\{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}}\), with \(\varepsilon_n, \beta_n \to 0\), and a limit function
\[
u \in L^\infty((0, T); L^2(\mathbb{R}) \cap L^4(\mathbb{R})),
\]
such that
\[
\begin{align*}
i) & \quad u_{\varepsilon_n,\beta_n} \to u \quad \text{strongly in} \quad L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}), \quad \text{for each} \quad 1 \leq p < 4, \\
ii) & \quad u \text{ is the unique entropy solution of (1.6)}.
\end{align*}
\]

Let us prove some a priori estimates on \(u_{\varepsilon,\beta}\), denoting with \(C_0\) the constants which depend only on the initial data.

**Lemma 3.1.** Fix \(T > 0\). Assume (2.3) holds. There exists \(C_0 > 0\), independent on \(\varepsilon, \beta\) such that (2.6) holds. In particular, we have
\[
\beta \|\partial_x u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \beta^3 \|\partial_{xxx} u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2
+ \frac{3\beta\varepsilon}{2} \int_0^t \|\partial_{xxx} u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C_0,
\]
(3.2)
for every $0 < t < T$. Moreover,

\begin{equation}
\| \partial_x u_{\varepsilon, \beta} \|_{L^\infty((0,T) \times \mathbb{R})} \leq C_0 \beta^{-\frac{3}{2}}.
\end{equation}

**Remark 3.1.** Observe that the proof of Lemma 3.1 is simpler than the one of Lemma 2.2. Indeed, we only need to prove (2.6).

**Proof of Lemma 3.1.** Let $0 < t < T$. Multiplying (2.1) by $-\beta \frac{1}{2} \partial_{xx}^2 u_{\varepsilon, \beta}$, we have

\begin{equation}
-\beta \frac{1}{2} \partial_{xx}^2 u_{\varepsilon, \beta} \partial_t u_{\varepsilon, \beta} - 2\beta \frac{1}{2} u_{\varepsilon, \beta} \partial_t u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} - \beta \frac{3}{2} \partial_t \partial_{xxx} u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} = -\beta \frac{1}{2} \varepsilon (\partial_{xx}^2 u_{\varepsilon, \beta})^2.
\end{equation}

Since

\begin{align*}
-\beta \frac{1}{2} \int_{\mathbb{R}} \partial_{xx}^2 u_{\varepsilon, \beta} \partial_t u_{\varepsilon, \beta} \, dx &= \frac{\beta \frac{1}{2}}{2} \left. \frac{d}{dt} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})} \right|_0^t, \\
-\beta \frac{3}{2} \int_{\mathbb{R}} \partial_{xxx} u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} \, dx &= \frac{\beta \frac{3}{2}}{2} \left. \frac{d}{dt} \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})} \right|_0^t,
\end{align*}

integrating (3.1) on $\mathbb{R}$, we get

\begin{equation}
\frac{d}{dt} \left( \beta \frac{1}{2} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \beta \frac{3}{2} \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) + 2\beta \frac{1}{2} \varepsilon \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 = 4\beta \frac{1}{2} \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_t u_{\varepsilon, \beta} \partial_{xx}^2 u_{\varepsilon, \beta} \, dx.
\end{equation}

It follows from (2.10) and (3.5) that

\begin{equation}
\frac{d}{dt} \left( \beta \frac{1}{2} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \beta \frac{3}{2} \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) + \frac{3}{2} \beta \frac{1}{2} \varepsilon \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq C_0 \varepsilon \| u_{\varepsilon, \beta} \|_{L^\infty((0,T) \times \mathbb{R})}^2 \| \partial_t u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2.
\end{equation}

Integrating on $(0, t)$, from (2.5) and (3.1), we get

\begin{equation}
\begin{aligned}
\beta \frac{1}{2} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \beta \frac{3}{2} \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 &+ \frac{3}{2} \beta \frac{1}{2} \varepsilon \| \partial_{xx}^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \\
&\leq C_0 + C_0 \varepsilon \| u_{\varepsilon, \beta} \|_{L^\infty((0,T) \times \mathbb{R})}^2 \int_0^t \| \partial_x u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 \, ds \\
&\leq C_0 \left( 1 + \| u_{\varepsilon, \beta} \|_{L^\infty((0,T) \times \mathbb{R})}^2 \right).
\end{aligned}
\end{equation}

We prove (2.6). Due to (2.5), (3.6) and the Hölder inequality,

\begin{align*}
\| u_{\varepsilon, \beta} \|_{L^\infty((0,T) \times \mathbb{R})}^4 &\leq \frac{1}{2} \int_{-\infty}^\infty u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \, dx \\
&\leq \int_{\mathbb{R}} |u_{\varepsilon, \beta}| \left| \partial_x u_{\varepsilon, \beta} \right| \, dx \\
&\leq \| u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})} \\
&\leq \frac{C_0}{\beta \frac{1}{2}} \sqrt{1 + \| u_{\varepsilon, \beta} \|_{L^\infty((0,T) \times \mathbb{R})}^2},
\end{align*}

that is

\begin{equation}
\| u_{\varepsilon, \beta} \|_{L^\infty((0,T) \times \mathbb{R})}^4 \leq \frac{C_0}{\beta \frac{1}{2}} \left( 1 + \| u_{\varepsilon, \beta} \|_{L^\infty((0,T) \times \mathbb{R})}^2 \right).
\end{equation}

Arguing as Lemma 2.2 we have (2.6). (3.2) follows from (2.6) and (3.6).
Finally, we prove (3.3). Due to (2.5), (3.2) and the H"older inequality,
\[(\partial_x u_\varepsilon(t, x))^2 = 2 \int_{-\infty}^{x} \partial_x u_\varepsilon \partial_{x,x}^2 u_\varepsilon \, dx \leq 2 \iint_{\mathbb{R}} \partial_x u_\varepsilon \partial_{x,x}^2 u_\varepsilon \, dx \]
\[\leq \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \|\partial_{x,x}^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_0 \beta^{-\frac{3}{4}}.\]
Hence,
\[|\partial_x u_\varepsilon| \leq C_0 \beta^{-\frac{3}{4}},\]
which gives (3.3).

Following [4] Lemma 2.2], or [8] Lemma 4.2], we prove the following result.

**Lemma 3.2.** Fix $T > 0$. Assume (2.3) holds. Then:

i) the family $\{u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$ is bounded in $L^\infty((0, T); L^4(\mathbb{R}))$;

ii) the families $\{\varepsilon \partial_t u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$, $\{\beta^2 \varepsilon^2 \partial_{t, t}^2 u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$ are bounded in $L^\infty((0, T); L^2(\mathbb{R}))$;

iii) the families $\{\beta^2 \varepsilon^4 \partial_{t, t, t, t}^4 u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$, $\{\varepsilon^2 \partial_t u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$, $\{\beta^3 \varepsilon^4 \partial_{t, t, t, t}^4 u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$, $\{\beta^2 \varepsilon^2 \partial_{t, t}^2 u_{\varepsilon, \beta}\}_{\varepsilon, \beta}$ are bounded in $L^2((0, T) \times \mathbb{R})$.

**Proof.** Let $0 < t < T$. Let $A, B$ be some positive constants which will be specified later. Multiplying (2.11) by
\[u_{\varepsilon, \beta}^3 - A \beta \varepsilon \partial_{t, t}^3 u_{\varepsilon, \beta} - B \varepsilon \partial_t u_{\varepsilon, \beta},\]
we have
\[(u_{\varepsilon, \beta}^3 - A \beta \varepsilon \partial_{t, t}^3 u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta}) \partial_t u_{\varepsilon, \beta} + 2 \left(u_{\varepsilon, \beta}^3 - A \beta \varepsilon \partial_{t, t}^3 u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta}\right) u_{\varepsilon, \beta} \partial_t u_{\varepsilon, \beta} + \beta^2 \left(u_{\varepsilon, \beta}^3 - A \beta \varepsilon \partial_{t, t}^3 u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta}\right) \partial_{t, t, t, t}^5 u_{\varepsilon, \beta}

= \varepsilon \left(u_{\varepsilon, \beta}^3 - A \beta \varepsilon \partial_{t, t}^3 u_{\varepsilon, \beta} + B \varepsilon \partial_t u_{\varepsilon, \beta}\right) \partial_{t, t}^2 u_{\varepsilon, \beta}.
\]
An integration of (3.7) on \( \mathbb{R} \) gives

\[
\frac{d}{dt} \left( \frac{1}{4} \left\| u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^4(\mathbb{R})}^4 + \frac{B \varepsilon}{2} \left\| \partial_x u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{A \beta \varepsilon}{2} \left\| \partial^2_{xx} u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) + A \beta \varepsilon \left\| \partial^2_{xx} u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + B \varepsilon \left\| \partial_t u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
+ A \beta^2 \varepsilon \left\| \partial_{xxx}^4 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + B \beta^2 \varepsilon \left\| \partial_{xxx}^3 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right)
\]

\[
(3.8)
\]

Due to the Young inequality,

\[
2A \beta \varepsilon \int_{\mathbb{R}} \left| u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \right| \left| \partial^3_{xxx} u_{\varepsilon,\beta} \right| dx = \varepsilon \int_{\mathbb{R}} \left| 4A u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \right| \left| \sqrt{B} \beta \partial^3_{xxx} u_{\varepsilon,\beta} \right| dx \leq \frac{8A^2 \varepsilon}{B} \left\| u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{B \beta^2 \varepsilon}{2} \left\| \partial_{xxx}^3 u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]

\[
2B \varepsilon \int_{\mathbb{R}} \left| u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \right| \left| \partial_t u_{\varepsilon,\beta} \right| dx = \varepsilon \int_{\mathbb{R}} \left| \partial_t u_{\varepsilon,\beta} \right| \left| 2B \partial_t u_{\varepsilon,\beta} \right| dx \leq \frac{\varepsilon}{2} \left\| u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2B^2 \left\| \partial_t u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]

Therefore, from (3.8) we gain

\[
\frac{d}{dt} \left( \frac{1}{4} \left\| u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^4(\mathbb{R})}^4 + \frac{B \varepsilon}{2} \left\| \partial_x u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{A \beta \varepsilon}{2} \left\| \partial^2_{xx} u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) + A \beta \varepsilon \left\| \partial^2_{xx} u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + B \varepsilon \left( 1 - 2B \right) \left\| \partial_t u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq \frac{3 \beta^2}{2} \int_{\mathbb{R}} \left| u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \right| \left| \partial^4_{xxx} u_{\varepsilon,\beta} \right| dx.
\]

From (2.3), we have

\[
(3.10)
\]

\[
\beta \leq D^2 \varepsilon^4,
\]

where \( D \) is a positive constant which will be specified later. Due to (2.10), (3.10) and the Young inequality,

\[
3 \beta^2 \int_{\mathbb{R}} \left| u_{\varepsilon,\beta} \partial_x u_{\varepsilon,\beta} \right| \left| \partial^4_{xxx} u_{\varepsilon,\beta} \right| dx = \int_{\mathbb{R}} \left| \frac{3 \beta^2}{2} \frac{\partial^2_{xx} u_{\varepsilon,\beta}}{\varepsilon} \right| \left| \frac{\sqrt{A}}{\beta^2} \right| dx \leq \frac{9 \beta}{2 \varepsilon A} \int_{\mathbb{R}} \left| \partial^4_{xxx} u_{\varepsilon,\beta} \right|^2 dx + \frac{A \beta^3 \varepsilon}{2} \left\| \partial^4_{xxx} u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq \frac{9 \beta}{2 \varepsilon A} \left\| u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2((0,T) \times \mathbb{R})}^2 + \frac{A \beta^3 \varepsilon}{2} \left\| \partial^4_{xxx} u_{\varepsilon,\beta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]
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Then, (3.9) gives

\[ \frac{d}{dt} \left( \frac{1}{4} \| u_{\varepsilon, \beta}(t, \cdot) \|_{L^4(\mathbb{R})}^4 + \frac{B \varepsilon^2}{2} \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{A \beta \varepsilon}{2} \| \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) \]

\[ + A \beta \varepsilon \| \partial_x^2 u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \varepsilon B (1 - 2B) \| \partial_t u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \]

\[ + \varepsilon \left( \frac{5}{2} - \frac{8A^2}{B} - \frac{C_0D}{A} \right) \| u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})} \leq 0. \]

We search \( A, B \) such that

\[ \begin{cases} 1 - 2B > 0, \\ \frac{5}{2} - \frac{8A^2}{B} - \frac{C_0D}{A} > 0, \end{cases} \]

that is

\[ \begin{cases} B < \frac{1}{2}, \\ 16A^3 - 5BA + 2C_0BD < 0. \end{cases} \]

We choose

\[ B = \frac{1}{3}. \]

Therefore, the second equation of (3.12) reads

\[ 16A^3 - 5A + \frac{2}{3}C_0D < 0, \]

that is

\[ 48A^3 - 5A + 2C_0D < 0. \]

Let us consider the following function

\[ g(X) = 48X^3 - 5X + 2C_0D \]

We observe that

\[ \lim_{x \to -\infty} g(X) = -\infty, \quad g(0) = 2C_0D > 0, \quad \lim_{x \to \infty} g(X) = \infty. \]

Since \( g'(X) = 144X^2 - 5 \), we find that

\[ g \text{ is increasing in } \left( -\infty, -\frac{\sqrt{5}}{12} \right) \text{ and in } \left( \frac{\sqrt{5}}{12}, \infty \right). \]

Therefore,

\[ g \left( \frac{\sqrt{5}}{12} \right) = 16 \left( \frac{\sqrt{5}}{12} \right)^3 - \frac{5\sqrt{5}}{12} + 2C_0D. \]

Since we want that

\[ g \left( \frac{\sqrt{5}}{12} \right) < 0, \]
we choose
\begin{equation}
D < \frac{5\sqrt{5}}{27C_0}
\end{equation}

It follows from (3.16), (3.17), (3.18), (3.19), and (3.20) that the function \( g \) has three zeros \( A_1 < 0 < A_2 < A_3 \).

Therefore, (3.14) is verified when
\begin{equation}
A_2 < A < A_3.
\end{equation}

From (3.11), (3.13), and (3.21), we have
\[
\frac{d}{dt} \left( \frac{1}{4} \|u_{\varepsilon,\beta}(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{\varepsilon^2}{6} \|\partial_x u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{A \beta \varepsilon}{2} \|\partial_x^2 u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \\
+ A \beta \varepsilon \|\partial_x^2 u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon}{9} \|\partial_t u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
+ \frac{A \beta^3 \varepsilon}{2} \|\partial_{xxx} u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\beta^2 \varepsilon}{6} \|\partial_{xxx} u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
+ K_1 \varepsilon \|u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \|\partial_x u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq 0,
\]
where \( K_1 \) is a positive constant.

An integration on \((0,t)\) and (3.1) give
\[
\frac{1}{4} \|u_{\varepsilon,\beta}(t, \cdot)\|_{L^4(\mathbb{R})}^4 + \frac{\varepsilon^2}{6} \|\partial_x u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{A \beta \varepsilon}{2} \|\partial_x^2 u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
+ A \beta \varepsilon \int_0^t \|\partial_x^2 u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + \frac{\varepsilon}{9} \int_0^t \|\partial_t u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
+ \frac{A \beta^3 \varepsilon}{2} \int_0^t \|\partial_{xxx} u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + \frac{\beta^2 \varepsilon}{6} \int_0^t \|\partial_{xxx} u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
+ K_1 \varepsilon \int_0^t \|u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 \|\partial_x u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0.
\]

Hence,
\[
\|u_{\varepsilon,\beta}(t, \cdot)\|_{L^4(\mathbb{R})} \leq C_0, \\
\varepsilon \|\partial_x u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})} \leq C_0, \\
\beta^\frac{1}{2} \varepsilon \|\partial_x^2 u_{\varepsilon,\beta}(t, \cdot)\|_{L^2(\mathbb{R})} \leq C_0, \\
\beta \varepsilon \int_0^t \|\partial_x^2 u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0, \\
\varepsilon \int_0^t \|\partial_t u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0, \\
\beta^3 \varepsilon \int_0^t \|\partial_{xxx} u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0, \\
\beta^2 \varepsilon \int_0^t \|\partial_{xxx} u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0, \\
\varepsilon \int_0^t \|u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 \|\partial_x u_{\varepsilon,\beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0,
\]
for every \( 0 < t < T \).

We are ready for the proof of Theorem 3.1.
We claim that

By (2.3) and Lemma 3.2, observe that if \( \beta \)

Proof of Theorem (3.1). Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\). Multiplying (2.1) by \( \eta' (u_{\varepsilon, \beta}) \), we have

where \( I_1, \varepsilon, \beta, I_2, \varepsilon, \beta, I_3, \varepsilon, \beta, I_4, \varepsilon, \beta \) are defined in (2.17).

Fix \( T > 0 \). Arguing as [3, Theorem 2.1], we have that \( I_1, \varepsilon, \beta \rightarrow 0 \) in \( H^{-1}((0, T) \times \mathbb{R}) \), and \( \{I_2, \varepsilon, \beta\}_{\varepsilon, \beta > 0} \) is bounded in \( L^1((0, T) \times \mathbb{R}) \).

We claim that

By (2.3) and Lemma 3.2,

\[
\left\| \beta^2 \eta' (u_{\varepsilon, \beta}) \partial_x^4 u_{\varepsilon, \beta} \right\|^2_{L^2((0, T) \times \mathbb{R})} < \beta^4 \left\| \eta' \right\|_{L^\infty(\mathbb{R})} \left\| \partial_x^4 u_{\varepsilon, \beta} \right\|^2_{L^2((0, T) \times \mathbb{R})}
\]

\[
= \left\| \eta' \right\|_{L^\infty(\mathbb{R})} \beta^4 \varepsilon \left\| \partial_x^4 u_{\varepsilon, \beta} \right\|^2_{L^2((0, T) \times \mathbb{R})}
\]

\[
= \left\| \eta' \right\|_{L^\infty(\mathbb{R})} \beta^4 \varepsilon \left\| \partial_x^4 u_{\varepsilon, \beta} \right\|^2_{L^2((0, T) \times \mathbb{R})} \leq C_0 \left\| \eta' \right\|_{L^\infty(\mathbb{R})} \varepsilon^3 \rightarrow 0.
\]

Let us show that

\( I_4, \varepsilon, \beta \rightarrow 0 \) in \( L^1((0, T) \times \mathbb{R}) \), \( T > 0 \).

Thanks to (2.3), Lemmas 2.1 and 3.2 and the H"older inequality,

\[
\left\| \beta^2 \eta'' (u_{\varepsilon, \beta}) \partial_x u_{\varepsilon, \beta} \partial_x^4 u_{\varepsilon, \beta} \right\|_{L^1((0, T) \times \mathbb{R})} < \beta^2 \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \int_0^T \left\| \partial_x u_{\varepsilon, \beta} \partial_x^4 u_{\varepsilon, \beta} \right\|_{L^1(\mathbb{R})} ds dx
\]

\[
= \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \beta^2 \varepsilon \left\| \partial_x u_{\varepsilon, \beta} \right\|_{L^2((0, T) \times \mathbb{R})} \left\| \partial_x^4 u_{\varepsilon, \beta} \right\|_{L^2((0, T) \times \mathbb{R})}
\]

\[
= \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \beta^2 \varepsilon \left\| \partial_x u_{\varepsilon, \beta} \right\|_{L^2((0, T) \times \mathbb{R})} \left\| \partial_x^4 u_{\varepsilon, \beta} \right\|_{L^2((0, T) \times \mathbb{R})} \leq C_0 \left\| \eta'' \right\|_{L^\infty(\mathbb{R})} \varepsilon \rightarrow 0.
\]

Arguing as [3, Theorem 2.1], the proof is concluded.

Appendix A. The Korteweg-de Vries equation: the first case

In this appendix, we consider the Korteweg-de Vries equation

\[
\partial_t u + u \partial_x u + \beta \partial_x^3 u = 0.
\]

We augment (A.1) with the initial condition

\[
u(0, x) = u_0(x),
\]
on which we assume that

\[
0 \in L^2(\mathbb{R}), \quad -\infty < \int_\mathbb{R} u^3(x) dx < \infty.
\]

Observe that if \( \beta \rightarrow 0 \), we have (1.6).
We study the dispersion-diffusion limit for (A.1). Therefore, we fix two small numbers \( \varepsilon, \beta \) and consider the following third order approximation

\[
\begin{align*}
(A.4) & \quad \partial_t u_{\varepsilon, \beta} + u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} + \beta \partial_{xx}^3 u_{\varepsilon, \beta} = \varepsilon \partial_{xx}^2 u_{\varepsilon, \beta}, \quad t > 0, \ x \in \mathbb{R}, \\
u_{\varepsilon, \beta}(0, x) = u_{\varepsilon, \beta, 0}(x), \quad x \in \mathbb{R},
\end{align*}
\]

where \( u_{\varepsilon, \beta, 0} \) is a \( C^\infty \) approximation of \( u_0 \) such that

\[
(A.5) \quad u_{\varepsilon, \beta, 0} \to u_0 \quad \text{in} \quad L^p_{\text{loc}}(\mathbb{R}), \ 1 \leq p < 2, \ \text{as} \ \varepsilon, \ \beta \to 0,
\]

\[
\|u_{\varepsilon, \beta, 0}\|_{L^2(\mathbb{R})}^2 + \beta \|\partial_x u_{\varepsilon, \beta, 0}\|_{L^2(\mathbb{R})}^2 \leq C_0, \quad \varepsilon, \beta > 0,
\]

\[
- \infty < \int_{\mathbb{R}} u_{\varepsilon, \beta, 0}(x) \, dx < \infty, \quad \varepsilon, \beta > 0,
\]

and \( C_0 \) is a constant independent on \( \varepsilon \) and \( \beta \).

The main result of this section is the following theorem.

**Theorem A.1.** Assume that (A.2) and (A.5) hold. Fix \( T > 0 \), if

\[
(A.6) \quad \beta = O(\varepsilon^3),
\]

then, there exist two sequences \( \{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}} \), with \( \varepsilon_n, \beta_n \to 0 \), and a limit function

\[
u \in L^\infty((0, T); L^2(\mathbb{R})),
\]

such that

1. \( u_{\varepsilon_n, \beta_n} \to u \) strongly in \( L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}) \), for each \( 1 \leq p < 2 \),
2. \( u \) is a distributional solution of (1.6).

Moreover, if

\[
(A.7) \quad \beta = o(\varepsilon^3),
\]

\[
(iii) \quad u \text{ is the unique entropy solution of (1.6)}.
\]

Let us prove some a priori estimates on \( u_{\varepsilon, \beta} \), denoting with \( C_0 \) the constants which depend only on the initial data.

Arguing as [21], we have

\[
(A.8) \quad \|u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|\partial_x u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C_0,
\]

for every \( t > 0 \).

**Lemma A.1.** Fix \( T > 0 \). Assume that (A.6) holds. There exists \( C_0 > 0 \), independent on \( \varepsilon, \beta \) such that

\[
(A.9) \quad \|u_{\varepsilon, \beta}\|_{L^\infty((0, T) \times \mathbb{R})} \leq C_0\beta^{-\frac{1}{4}}.
\]

Moreover,

\[
(A.10) \quad \beta^2 \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \varepsilon \int_0^t \|\partial_{xx} u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 \, ds \leq C_0.
\]

**Proof.** Let \( 0 < t < T \). Multiplying (A.4) by \(-u_{\varepsilon, \beta}^2 - 2\beta \partial_{xx}^2 u_{\varepsilon, \beta}\), we have

\[
(A.11) \quad \begin{align*}
&(-u_{\varepsilon, \beta}^2 - 2\beta \partial_{xx}^2 u_{\varepsilon, \beta}) \partial_t u_{\varepsilon, \beta} + (-u_{\varepsilon, \beta}^2 - 2\beta \partial_{xx}^2 u_{\varepsilon, \beta}) u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \\
&\quad + \beta (-u_{\varepsilon, \beta}^2 - 2\beta \partial_{xx}^2 u_{\varepsilon, \beta}) \partial_{xx}^3 u_{\varepsilon, \beta} = \varepsilon (-u_{\varepsilon, \beta}^2 - 2\beta \partial_{xx}^2 u_{\varepsilon, \beta}) \partial_{xx}^2 u_{\varepsilon, \beta}.
\end{align*}
\]

Since

\[
\int_{\mathbb{R}} (-u_{\varepsilon, \beta}^2 - 2\beta \partial_{xx}^2 u_{\varepsilon, \beta}) \partial_t u_{\varepsilon, \beta} \, dx
\]
Due to (A.8), (A.12) and the Hölder inequality, integrating (A.11) on \( \mathbb{R} \), we get

\[
\frac{d}{dt} \left( -\frac{1}{3} \int_{\mathbb{R}} u_{\varepsilon, \beta}^3 dx + \beta \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \right) + 2\beta \varepsilon \int_{\mathbb{R}} \partial_{xx} u_{\varepsilon, \beta}(t, \cdot) \partial_{xx} u_{\varepsilon, \beta} dx \leq C_0 + 2\beta \varepsilon \int_{0}^{t} \| \partial_{xx} u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds.
\]

(A.5), (A.8) and an integration on \((0, t)\) give

\[
-\frac{1}{3} \int_{\mathbb{R}} u_{\varepsilon, \beta}^3 dx + \beta \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq C_0 + 2\varepsilon \int_{0}^{t} \| \partial_{xx} u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds.
\]

Again by (A.8), we have

\[
\beta \| \partial_x u_{\varepsilon, \beta}(t, \cdot) \|_{L^2(\mathbb{R})}^2 + 2\beta \varepsilon \int_{0}^{t} \| \partial_{xx} u_{\varepsilon, \beta}(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \leq C_0 + C_0 \| u_{\varepsilon, \beta} \|_{L^\infty((0, T) \times \mathbb{R})} + \frac{1}{3} \int_{\mathbb{R}} u_{\varepsilon, \beta}^3 dx
\]

(A.12)

Due to (A.8), (A.12) and the Hölder inequality,

\[
| \partial_{xx} u_{\varepsilon, \beta}(t, x) | = \left| 2 \int_{-\infty}^{x} u_{\varepsilon, \beta}(y) \partial_x u_{\varepsilon, \beta} dy \right| \leq \int_{\mathbb{R}} | u_{\varepsilon, \beta} | | \partial_x u_{\varepsilon, \beta} | dx \leq C_0 \frac{1}{\sqrt{\beta}} \left( 1 + \| u_{\varepsilon, \beta} \|_{L^\infty((0, T) \times \mathbb{R})} \right),
\]

that is

\[
\| u_{\varepsilon, \beta} \|_{L^\infty((0, T) \times \mathbb{R})}^4 \leq \frac{C_0}{\beta} \left( 1 + \| u_{\varepsilon, \beta} \|_{L^\infty((0, T) \times \mathbb{R})} \right).
\]
Arguing as [6] Lemma 2.5, we have (A.9).

Finally, (A.10) follows from (A.9) and (A.12). □

We begin by proving the following result.

**Lemma A.2.** Assume that (A.3), (A.5), and (A.6) hold. Then, for any compactly supported entropy-entropy flux pair \((\eta, q)\), there exist two sequences \(\{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}}\), with \(\varepsilon_n, \beta_n \to 0\), and a limit function \(u \in L^\infty((0, T); L^2(\mathbb{R}))\), such that (2.15) holds and

(A.13) \(u\) is a distributional solution of (1.6).

**Proof.** Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\). Multiplying (A.4) by \(\eta'(u, \beta)\), we have

\[
\partial_t \eta'(u, \beta) + \partial_x q(u, \beta) = \varepsilon \eta''(u, \beta) \partial_{xx}^2 u + \beta \eta'(u, \beta) \partial_{xx} u,
\]

where

\[
I_1, \varepsilon, \beta = \partial_x (\varepsilon \eta'(u, \beta) \partial_x u),
\]

\[
I_2, \varepsilon, \beta = -\varepsilon \eta''(u, \beta) (\partial_x u)^2,
\]

\[
I_3, \varepsilon, \beta = \partial_x (\beta \eta'(u, \beta) \partial_{xx} u),
\]

\[
I_4, \varepsilon, \beta = -\beta \eta''(u, \beta) \partial_x u \partial_{xx} u.
\]

Fix \(T > 0\). Arguing as [6] Lemma 3.2, we have that \(I_1, \varepsilon, \beta \to 0\) in \(H^{-1}((0, T) \times \mathbb{R})\), and \(\{I_2, \varepsilon, \beta\}_{\varepsilon, \beta > 0}\) is bounded in \(L^1((0, T) \times \mathbb{R})\).

We claim that

\[I_3, \varepsilon, \beta \to 0\quad \text{in} \quad H^{-1}((0, T) \times \mathbb{R}), \quad T > 0, \quad \text{as} \quad \varepsilon \to 0.\]

By (A.6) and Lemma A.1

\[
\|\beta \eta''(u, \beta) \partial_{xx} u\|_{L^2((0, T) \times \mathbb{R})}^2 \\
\leq \beta^2 \|\eta''\|_{L^\infty(\mathbb{R})} \|\partial_{xx}^2 u\|_{L^2((0, T) \times \mathbb{R})}^2 \\
= \|\eta''\|_{L^\infty(\mathbb{R})}^2 \beta \varepsilon \|\partial_{xx}^2 u\|_{L^2((0, T) \times \mathbb{R})}^2 \\
= \|\eta''\|_{L^\infty(\mathbb{R})} \frac{\beta \varepsilon}{\varepsilon} \|\partial_{xx}^2 u\|_{L^2((0, T) \times \mathbb{R})}^2 \leq C_0 \|\eta''\|_{L^\infty(\mathbb{R})} \varepsilon^2 \\ \to 0.
\]

Let us show that

\[\{I_4, \varepsilon, \beta\}\quad \text{is bounded in} \quad L^1((0, T) \times \mathbb{R}), \quad T > 0.\]

Thanks to (A.6), (A.8), Lemma A.1 and the H"older inequality,

\[
\|\beta \eta''(u, \beta) \partial_x u \partial_{xx} u\|_{L^1((0, T) \times \mathbb{R})} \\
\leq \beta \|\eta''\|_{L^\infty(\mathbb{R})} \int_0^T \int \partial_x u \partial_{xx} u \partial_{xx} u ds dx \\
= \|\eta''\|_{L^\infty(\mathbb{R})} \frac{\beta \varepsilon}{\varepsilon} \|\partial_x u \partial_{xx} u\|_{L^1(0, T) \times \mathbb{R})} \|\partial_{xx}^2 u\|_{L^2((0, T) \times \mathbb{R})} \\
\leq C_0 \|\eta''\|_{L^\infty(\mathbb{R})} \frac{\beta \varepsilon}{\varepsilon} \leq C_0 \|\eta''\|_{L^\infty(\mathbb{R})}.
\]
Arguing as in [21], we have (A.13). □

Lemma A.3. Assume (A.3), (A.4), and (A.6) hold. Then, for any compactly supported entropy-entropy flux pair \((\eta, q)\), there exist two sequences \(\{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}}\), with \(\varepsilon_n, \beta_n \to 0\), and a limit function \(u \in L^\infty((0, T); L^2(\mathbb{R}))\), such that (2.15) and (2.18) hold.

Proof. Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\). Multiplying (A.4) by \(\eta'(u_{\varepsilon, \beta})\), we have

\[
\partial_t \eta(u_{\varepsilon, \beta}) + \partial_x \eta(u_{\varepsilon, \beta}) = \varepsilon \eta'(u_{\varepsilon, \beta}) \partial_{xx} u_{\varepsilon, \beta} + \beta \eta'(u_{\varepsilon, \beta}) \partial_{xx} u_{\varepsilon, \beta} = I_1, \varepsilon, \beta + I_2, \varepsilon, \beta + I_3, \varepsilon, \beta + I_4, \varepsilon, \beta,
\]

where \(I_1, \varepsilon, \beta, I_2, \varepsilon, \beta, I_3, \varepsilon, \beta, I_4, \varepsilon, \beta\) are defined in (A.14). As in Lemma 2.3, we have that \(I_1, \varepsilon, \beta, I_3, \varepsilon, \beta \to 0\) in \(H^{-1}((0, T) \times \mathbb{R})\), \(\{I_2, \varepsilon, \beta\}_{\varepsilon, \beta > 0}\) is bounded in \(L^1((0, T) \times \mathbb{R})\), while \(I_4, \varepsilon, \beta \to 0\) in \(L^1((0, T) \times \mathbb{R})\).

Arguing as in [14], we have (2.18). □

Proof of Theorem A.1. Theorem A.1 follows from Lemmas A.2 and A.3. □

APPENDIX B. THE KORTWEG-DE VRIES EQUATION: THE SECOND CASE.

In this appendix, we argument (A.1) with the following initial datum

\[
(B.1) \quad u_0 \in L^2(\mathbb{R}).
\]

We consider the approximation (A.4), where \(u_{\varepsilon, \beta}\) is a \(C^\infty\) of \(u_0\) such that

\[
(B.2) \quad \|u_{\varepsilon, \beta, 0}\|_{L^2(\mathbb{R})} \leq 1, \quad 1 \leq p < 2, \quad \text{as} \quad \varepsilon, \beta \to 0,
\]

and \(C_0\) is a constant independent on \(\varepsilon\) and \(\beta\).

The main result of this section is the following theorem.

Theorem B.1. Assume that (B.1) and (B.2) hold. Fix \(T > 0\), if (2.3) holds, then, there exist two sequences \(\{\varepsilon_n\}_{n \in \mathbb{N}}, \{\beta_n\}_{n \in \mathbb{N}}\), with \(\varepsilon_n, \beta_n \to 0\), and a limit function \(u \in L^\infty((0, T); L^2(\mathbb{R}))\), such that

i) \(u_{\varepsilon_n, \beta_n} \to u\) strongly in \(L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})\), for each \(1 \leq p < 2\),

ii) \(u\) is the unique entropy solution of (1.6).

Let us prove some a priori estimates on \(u_{\varepsilon, \beta}\), denoting with \(C_0\) the constants which depend only on the initial data

Lemma B.1. Fix \(T > 0\). Assume that (2.3) holds. There exists \(C_0 > 0\), independent on \(\varepsilon, \beta\) such that (2.6) holds. Moreover,

\[
(B.3) \quad \beta \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{3\beta \varepsilon}{2} \int_0^t \|\partial_{xx} u_{\varepsilon, \beta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C_0.
\]

Proof. Let \(0 < t < T\). Multiplying (A.4) by \(-\beta \frac{d}{dt} \partial_x u_{\varepsilon, \beta}\), an integration on \(\mathbb{R}\) gives

\[
(B.4) \quad \beta \frac{d}{dt} \|\partial_x u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta \varepsilon \|\partial_{xx} u_{\varepsilon, \beta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 = 2\beta \frac{d}{dt} \int_{\mathbb{R}} u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta} \partial_{xx} u_{\varepsilon, \beta} dx.
\]
Due to \([2.3]\) and the Young inequality,
\[
2\beta^\frac{1}{2} \int_{\mathbb{R}} |u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta}| \partial^2_{xx} u_{\varepsilon, \beta} |dx = 2\beta^\frac{1}{2} \int_{\mathbb{R}} \frac{|u_{\varepsilon, \beta} \partial_x u_{\varepsilon, \beta}|}{\varepsilon^2} |\frac{1}{\varepsilon} \partial^2_{xx} u_{\varepsilon, \beta}| |dx|
\]
(B.5)
\[
\leq \frac{\beta^\frac{1}{2}}{\varepsilon} \int_{\mathbb{R}} u_{\varepsilon, \beta}^2 (\partial_x u_{\varepsilon, \beta})^2 |dx| + \frac{\beta^\frac{1}{2} \varepsilon}{2} \left\| \frac{1}{\varepsilon} \partial^2_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})}
\]
\[
\leq C_0 \varepsilon \left\| u_{\varepsilon, \beta} \right\|^2_{L^\infty((0, T) \times \mathbb{R})} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\beta^\frac{1}{2} \varepsilon}{2} \left\| \partial^2_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})}.
\]

It follows from (B.4) and (B.5) that
\[
\frac{\beta^\frac{1}{2} d}{dt} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{3\beta^\frac{1}{2} \varepsilon}{2} \left\| \partial^2_{xx} u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})}
\]
\[
\leq C_0 \varepsilon \left\| u_{\varepsilon, \beta} \right\|^2_{L^\infty((0, T) \times \mathbb{R})} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})}.
\]

Integrating on \((0, t)\), from (A.8) and (B.2), we have
\[
\frac{\beta^\frac{1}{2} d}{dt} \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{3\beta^\frac{1}{2} \varepsilon}{2} \int_0^t \left\| \partial^2_{xx} u_{\varepsilon, \beta}(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds
\]
(B.6)
\[
\leq C_0 + C_0 \varepsilon \left\| u_{\varepsilon, \beta} \right\|^2_{L^\infty((0, T) \times \mathbb{R})} \int_0^t \left\| \partial_x u_{\varepsilon, \beta}(t, \cdot) \right\|^2_{L^2(\mathbb{R})} ds
\]
\[
\leq C_0 \left( 1 + \left\| u_{\varepsilon, \beta} \right\|^2_{L^\infty((0, T) \times \mathbb{R})} \right).
\]

Arguing as Lemma 2.2, we have (2.6).
(B.3) follows from (2.6) and (B.6). □

We are ready for the proof of Theorem 3.1.

**Proof of Theorem 3.1.** Let us consider a compactly supported entropy–entropy flux pair \((\eta, q)\). Multiplying (A.4) by \(\eta'(u_{\varepsilon, \beta})\), we have
\[
\partial_t \eta(u_{\varepsilon, \beta}) + \partial_x q(u_{\varepsilon, \beta}) = \varepsilon \eta'(u_{\varepsilon, \beta}) \partial^2_{xx} u_{\varepsilon, \beta} + \beta \eta'(u_{\varepsilon, \beta}) \partial^2_{xx} u_{\varepsilon, \beta}
\]
\[
= I_{1, \varepsilon, \beta} + I_{2, \varepsilon, \beta} + I_{3, \varepsilon, \beta} + I_{4, \varepsilon, \beta},
\]
where \(I_{1, \varepsilon, \beta}, I_{2, \varepsilon, \beta}, I_{3, \varepsilon, \beta}, I_{4, \varepsilon, \beta}\) are defined in (A.14).

As in Lemma 2.4, we have that \(I_{1, \varepsilon, \beta}, I_{3, \varepsilon, \beta} \to 0 \) in \(H^{-1}((0, T) \times \mathbb{R})\), \(\{I_{2, \varepsilon, \beta}\}_{\varepsilon > 0}\) is bounded in \(L^1((0, T) \times \mathbb{R})\).

We claim that
\[
I_{3, \varepsilon, \beta} \to 0 \quad \text{in} \quad H^{-1}((0, T) \times \mathbb{R}), \quad T > 0, \quad \text{as} \quad \varepsilon \to 0.
\]

By (2.3) and (B.3),
\[
\left\| \beta \eta'(u_{\varepsilon, \beta}) \partial^2_{xx} u_{\varepsilon, \beta} \right\|^2_{L^2((0, T) \times \mathbb{R})}
\]
\[
\leq \beta^2 \left\| \eta' \right\|^2_{L^\infty(\mathbb{R})} \left\| \partial^2_{xx} u_{\varepsilon, \beta} \right\|^2_{L^2((0, T) \times \mathbb{R})}
\]
\[
= \left\| \eta' \right\|^2_{L^\infty(\mathbb{R})} R \left\| \partial^2_{xx} u_{\varepsilon, \beta} \right\|^2_{L^2((0, T) \times \mathbb{R})}
\]
\[
\leq C_0 \varepsilon^\frac{3}{2} \to 0.
\]

Let us show that
\[
I_{4, \varepsilon, \beta} \to 0 \quad \text{in} \quad L^1((0, T) \times \mathbb{R}), \quad T > 0.
\]
Thanks to (2.3), (A.8), and (B.3), and the Hölder inequality,

\[ \left| \beta \eta''(u_{x,\beta}) \partial_x u_{x,\beta} \right|_{L^1((0,T) \times \mathbb{R})} \leq \beta \| \eta'' \|_{L^\infty(\mathbb{R})} \int_0^T \int_{\mathbb{R}} |\partial_x u_{x,\beta} \partial_{xx}^2 u_{x,\beta}| \, ds \, dx \]

\[ = \beta \frac{\beta^2}{\varepsilon} \left\| \partial_x u_{x,\beta} \right\|_{L^2((0,T) \times \mathbb{R})} \left\| \partial_{xx}^2 u_{x,\beta} \right\|_{L^2((0,T) \times \mathbb{R})} \]

\[ \leq C_0 \| \eta'' \|_{L^\infty(\mathbb{R})} \varepsilon \rightarrow 0. \]

Arguing as in [14], the proof is concluded.

\[ \square \]

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