Investigation of Temperature Statistics in Turbulent Rayleigh-Bénard Convection using PDF Methods

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Abstract. Rayleigh-Bénard convection in the turbulent regime is studied using statistical methods. Exact evolution equations for the probability density function of temperature and velocity are derived from first principles within the framework of the Lundgren-Monin-Novikov hierarchy. The unclosed terms arising in the form of conditional averages are estimated from direct numerical simulations. Focussing on the statistics of temperature, the theoretical framework allows to interpret the statistical results in an illustrative manner, giving deeper insight into the connection between dynamics and statistics of Rayleigh-Bénard convection.

1. Introduction

Rayleigh-Bénard convection is a paradigm of a pattern forming system far from equilibrium. Convective fluid motion in a vessel is induced by a vertical temperature gradient between the bottom and top boundaries due to buoyancy forces. In dependence on this temperature gradient, the geometry of the experiment and the fluid properties, a whole zoo of instabilities has been observed ranging from laminar, spatially coherent convective motion over spatially ordered but temporally chaotic up to highly turbulent fluid motion. We refer the reader to reviews available on the topic (Busse, 1978; Bodenschatz et al., 2000).

Recently, much effort has been devoted to the analysis of turbulent Rayleigh-Bénard (RB) convection both by experimental as well as theoretical means (Siggia, 1994; Ahlers et al., 2009). Direct numerical simulations allow one to consider the dynamical and statistical properties of RB turbulence and the transitions between different types of flows in fine detail.

It is obvious that the analysis of turbulent convective fluid motion has to be based on a combination of tools from dynamical systems theory, statistical physics, and the theory of stochastic processes. A necessary step is the statistical formulation of the underlying basic fluid dynamic equations, which for the most simple case are the Oberbeck-Boussinesq equations for the velocity field \( u(x, t) \), the temperature field \( T(x, t) \), and the pressure field \( p(x, t) \):

\[
\begin{align*}
\frac{\partial}{\partial t} u(x, t) + u(x, t) \cdot \nabla u(x, t) &= -\nabla p(x, t) + \Pr \Delta u(x, t) + \Pr Ra T(x, t) e_z \\
\frac{\partial}{\partial t} T(x, t) + u(x, t) \cdot \nabla T(x, t) &= \Delta T(x, t) \\
\nabla \cdot u(x, t) &= 0
\end{align*}
\]
Figure 1: Volume renderings of instantaneous snapshots of two corresponding fields. Both volume renderings have been done with the open source renderer Voreen (Meyer-Spradow et al. 2009). (a) Temperature field $T(x)$. Blue marks cold, red hot temperatures. (b) Vertical velocity field $u_z(x)$. Blue marks negative, red/yellow positive velocities.

The equations have been non-dimensionalized using the Rayleigh number $Ra = \frac{\alpha g \delta T h^3}{\nu \kappa}$, which is a dimensionless measure of the temperature gradient across the fluid layer (with thermal expansion coefficient $\alpha$, gravitational acceleration $g$, outer temperature difference $\delta T$, and distance of top and bottom plate $h$), as well as the Prandtl number $Pr = \frac{\nu}{\kappa}$ as the ratio of kinematic viscosity $\nu$ to heat conductivity $\kappa$ of the fluid. Thus, the vertical spatial coordinate obeys $z \in [0, 1]$, and the boundary conditions of the temperature at the bottom and top plate are $T(z = 0) = \frac{1}{2}$ and $T(z = 1) = -\frac{1}{2}$. For the velocity, no-slip boundary conditions $u(z=0) = u(z=1) = 0$ are assumed. These equations are solved numerically by a suitably designed penalization approach described in section 3. Snapshots of two different fields are exhibited in figure 1.

The statistical analysis is based on joint probability density functions (PDFs) for the temperature and the velocity at a single point in space and time. The basic fluid dynamic equations require the validity of certain relations among these PDFs. From these relations, corresponding expressions relating the various moments of the fields can be derived. For the case of incompressible turbulence, these relations have been formulated by Lundgren (1967), Monin (1967) and Novikov (1968), and Ulinich & Lyubimov (1969) and are sometimes known as the Lundgren-Monin-Novikov (LMN) hierarchy. They are directly related to Hopf’s functional equation, which can be viewed as the basic statistical formulation of the Navier-Stokes equation in the Eulerian framework (Monin & Yaglom, 2007). Similar relations can be derived for the corresponding Lagrangian quantities (Friedrich, 2003). It is evident that an analogous treatment is feasible for the Oberbeck-Boussinesq equations.

In the present article we will use this approach in order to analyze the single-point temperature probability density function for stationary turbulent RB convection. Our analysis combines direct numerical simulations with the relation of the LMN hierarchy for the single-point PDF. The result is a partial differential equation for the temperature PDF. This equation is unclosed due to the fact that it contains unclosed expressions which can be related to fluid pressure, viscous dissipation and heat diffusion. However, these expressions can be treated by introducing conditional averages, which can be extracted from direct numerical simulations. This leads to a partial differential equation for the joint temperature-velocity PDF. The derivation of this relation is outlined in section 2.

A similar approach has been performed by Novikov (1993), and more recently by Wilczek et al. for the PDFs of vorticity (Wilczek & Friedrich, 2009) and velocity (Wilczek et al., 2011) for stationary, isotropic turbulence. On the other hand, modeling of unclosed terms is also a possible method, as performed by e.g. Pope (1981, 2000). As we shall indicate, the analysis of the
evolution equation for the temperature PDF yields a comprehensive description of the dynamical processes in RB convection.

This proceedings article is a short version of a paper published elsewhere [Lülff et al., 2011] and is structured as follows: In section 2, we will derive the evolution equation for the temperature-velocity joint PDF. We then reduce the joint PDF to the temperature PDF and make use of statistical symmetries to cut down the complexity of the evolution equation and present a descriptive way to deal with this equation involving the method of characteristics. These theoretical results are complemented by results from direct numerical simulations, which will be discussed in section 3 followed by a summary in section 4.

2. Derivation of the Evolution Equation for the Probability Density Function

The statistical analysis is based on the single point joint probability density function for the temperature and the velocity, defined as

$$ f(\tau, v; x, t) = \langle \delta(\tau - T(x, t)) \delta(v - u(x, t)) \rangle \quad . $$  (2)

The brackets $\langle \cdot \rangle$ denote an ensemble average, in contrast to spatial averages $\langle \cdot \rangle_V$ and $\langle \cdot \rangle_A$ over the whole fluid volume, or a horizontal plane at height $z$, respectively, that are used later on. Also, it is important to distinguish between the sample space variables $\tau$, $v$ and the corresponding realizations of the temperature and velocity fields $T(x, t)$, $u(x, t)$.

The aim is to formulate an equation which characterizes this PDF. Starting from the Oberbeck-Boussinesq equations, we derive an evolution equation for the single-point joint probability density function of velocity and temperature along the lines of the LMN hierarchy (Lundgren, 1967; Monin, 1967; Novikov, 1968) for the case of incompressible turbulence. Calculating spatial and temporal derivatives of (2) and inserting the basic equations (1), we end up with the following evolution equation for the joint PDF:

$$ \frac{\partial}{\partial t} f + v \cdot \nabla f = - \frac{\partial}{\partial \tau} \left[ \langle \Delta T | \tau, v, x, t \rangle f \right] - \nabla_v \cdot \left[ \langle (\vec{\nabla} p | v, x, t) + \text{Pr}(\Delta u | v, x, t) + \text{Pr Ra} \tau e_z \rangle f \right] $$  (3)

Here, our ansatz to treat the unclosed term that appear in the derivation was to express them as conditional averages $\langle \cdot | \tau, v, x, t \rangle$ that can be estimated from DNS data (cf. Wilczek & Friedrich, 2009; Lülff et al., 2011; Wilczek et al., 2011).

In the following we shall restrict our attention to the reduced temperature PDF $h(\tau; x, t) = \int d^3 v f(\tau; v; x, t)$ and its evolution equation, because already this equation gives insight into the connection between Rayleigh-Bénard dynamics and temperature statistics. Projecting (3) onto the temperature part by performing the aforementioned integration and employing homogeneity in horizontal planes and stationarity in time, the resulting evolution equation for the temperature PDF reads

$$ \frac{\partial}{\partial z} \langle u_z | \tau, z \rangle h(\tau; z) + \frac{\partial}{\partial \tau} \left[ \langle \Delta T | \tau, z \rangle h(\tau; z) \right] = 0 \quad . $$  (4)

This first-order partial differential equation (PDE) can be analyzed with the help of the method of characteristics. Applying this method, one can find curves $(\tau(s), z(s))$ in the $\tau,z$-phase space parametrized by $s$ along which the PDE (4) transforms into an ordinary differential equation for $h(s) = \langle \tau(s); z(s) \rangle$ which can be integrated.

The characteristic curves may be identified as solutions of

$$ \frac{d}{ds} \tau(s) = \left. \langle \Delta T | \tau, z \rangle \right|_{\tau=\tau(s), z=z(s)} \quad , \quad \frac{d}{ds} z(s) = \left. \langle u_z | \tau, z \rangle \right|_{\tau=\tau(s), z=z(s)} \quad . $$  (5)
Along these curves, the PDE (1) becomes an ODE, which can be integrated to yield

$$h(s) = h(s_0) \exp \left[ - \int_{s_0}^{s} ds \left( \frac{\partial}{\partial \tau} \langle \Delta T | \tau, z \rangle + \frac{\partial}{\partial z} \langle u_z | \tau, z \rangle \right)_{\tau = \tau(s), z = z(s)} \right]. \quad (6)$$

This equation describes the evolution of the PDF along a trajectory \((\tau(s), z(s))\) starting at point \((\tau(s_0), z(s_0))\) in phase space. A particularly appealing property of this formalism is that it allows to interpret the statistical results in an illustrative manner, because the characteristics, i.e. trajectories in \(\tau, z\)-phase space, show the evolution of the “averaged” physical process.

It is tempting to interpret the characteristics as a kind of Lagrangian dynamics of a tracer particle inside the RB cell. However, the dynamics of a tracer particle is stochastic, whereas the characteristics defined by (5) describe purely deterministic trajectories and, thus, take the stochastic properties into account only in an averaged way. In a sense, the characteristics describe the averaged evolution of an ensemble of fluid particles that are defined by their initial condition in the \(\tau, z\)-plane.

Thinking of turbulent RB convection with some physical intuition, one can expect certain features from the statistical quantities introduced in this section. The conditionally averaged vertical velocity \(\langle u_z | \tau, z \rangle\) should show positive correlation with the temperature, i.e. it should mirror the well-known fact that hot fluid rises up and cold fluid sinks down. Also the no-slip boundaries should be recognizable for \(z \gtrless 0\) and \(z \lesssim 1\), respectively. The absolute value of the heat diffusive term \(\langle \Delta T | \tau, z \rangle\) should be highest near the boundaries because of the sharp change of the temperature profile. As the characteristics in a way describe the average path a fluid particle takes through \(\tau, z\)-phase space, the typical Rayleigh-Bénard cycle of fluid heating up at the bottom, rising up, cooling down at the top and sinking down again should find its correspondence in the statistical quantities describing the evolution of the PDF. Actual numerical studies of these quantities will be discussed in detail in section [3]

3. Numerical Results

The benefit of our theoretical approach is that we can easily provide it with measurements and data in the form of numerical results. To this end, we solve the basic Oberbeck-Boussinesq equations (1) with a standard dealiased pseudospectral code on a three-dimensional equidistant Cartesian grid with periodic boundary conditions. For an introduction into this topic the reader is referred to [Boyd (2001)] and [Canuto et al. (1987)].

Periodic boundaries are required in horizontal direction, but in vertical direction Dirichlet conditions for velocity and temperature (i.e. no-slip boundaries of constant temperature) are needed. They are enforced by a volume penalization ansatz [Angot et al. (1999), Schneider 2005, Keetels et al. (2007)]: The fluid domain \(\Omega = [0, L_x] \times [0, L_y] \times [0, 1] \subset \mathbb{R}^3\) is embedded in a computational domain that is extended by a layer of thickness \(d\) in z-direction, \(\Omega_c = [0, L_x] \times [0, L_y] \times [-d, 1 + d]\). Inside the fluid domain \(\Omega \subset \Omega_c\) the unaltered Oberbeck-Boussinesq equations are solved, while in the appended extra regions \(\Omega_c \setminus \Omega\) a strong exponential damping \((-\frac{1}{\eta} u\) and \(-\frac{1}{\eta} \theta\), respectively, with \(\eta \ll 1\)) is added to the evolution equations (1) of velocity and temperature that damps the fields to zero. By simulating the deviation from the linear temperature profile, \(\theta(x, t) := T(x, t) + (z - 1/2)\), instead of the temperature itself, the desired boundary conditions read \(u = 0\) and \(\theta = 0\) for \(z = 0\) and \(z = 1\). This change of variables \(T \to \theta\) allows us to make use of the volume penalization approach in a straightforward manner.

The reason we choose this numerical scheme instead of often used Chebyshev-based codes, described for example by [Boyd (2001)] and used by e.g. [Clercx & Bruneau (2006) and Schumacher (2009)], is because it allows for almost arbitrary shaped boundaries and sidewalls. Although this feature is not used in the present paper due to the required horizontal homogeneity, it even allows...
Figure 2: (a) The mean temperature $\langle T(z) \rangle_A$ and a color plot of the logarithm of the temperature PDF $h(\tau; z)$ as a function of $z$. The additional solid gray lines mark the contour line for $\langle T(z) \rangle_A \pm \sqrt{\langle \left( T - \langle T(z) \rangle_A \right)^2 \rangle_A / 2}$ and indicate the standard deviation of temperature at height $z$. The horizontal dashed lines indicate the positions of slices in $\tau$-direction at fixed height and are located at $z \in \{ 1/2, 1/4, 4\delta_T, 2\delta_T, \delta_T \}$, where $\delta_T = 1/2\text{Nu}$ is the thermal boundary layer thickness. (b) The logarithm of the temperature PDF $h(\tau; z)$ for different values of $z$. The upper abscissa is scaled in units of the globally taken standard deviation of temperature, $T_{\text{rms}} = \sqrt{\langle T^2 \rangle_V}$. 

Our theoretical derivation relies on the concept of ensemble averages. Of course, through our numerics we can only access a finite subset of all possible ensemble members. So due to the statistical symmetries and by assuming ergodicity, the ensemble average is substituted for the sharp change of the temperature PDF from a $\delta$-function at the boundaries across the boundary layer to a shape exhibiting larger tails in the bulk. In addition to these tails, another feature of the PDF is the hump close to the $\tau=0$-line. This hump corresponds to the most probable value of the temperature. One expects two different dynamical features to be responsible for this special shape; a tempting explanation would be to attribute the hump to the background temperature field of mean temperature, and account the wings for large $|\tau|$ for plumes that carry fluid that is much colder or hotter than the surrounding fluid. An evidence for this is the shape of the PDF close to the bottom or top boundaries (but still outside the boundary layers): At $z = 4\delta_T$, though
Due to the non-dimensionalization, the velocity is given in units of the heat diffusion velocity, i.e. the velocity with which heat would be transported from plate to plate by pure heat diffusion.

The most probable temperature value is moved slightly towards lower temperatures, the PDF exhibits a large tail at high temperatures. The interpretation is that mostly cold fluid gathers in the lower regions of the bulk, being almost at rest (compare the region of \( \langle u_z | \tau, z = 4\delta T \rangle \) in figure 3(b) corresponding to the hump), while very hot fluid is a more rare event, because hot fluid is convected away quickly due to plume dynamics. The reason why the hot fluid takes greater temperature values than the cold fluid (in terms of absolute value) is that very cold fluid detaching from the top plate already heats up on its way down.

PDFs of the same shape for Rayleigh numbers of the same order are reported by Emran & Schumacher [2008], where also the dependence on the vertical coordinate is taken into account. The experimental data of Castaing et al. [1989] and Ching [1993] show a more pronounced exponential shape of the temperature PDF, which can be attributed to the difference in the Rayleigh numbers which are several orders of magnitude above ours; the numerical data of Emran & Schumacher [2008] suggests that the PDFs become more exponential with increasing Rayleigh number.

Figure 3(a) and 4 exhibit the conditional averages introduced in section 2. One can clearly observe the features that were suggested in the aforementioned section. The conditional vertical velocity \( \langle u_z | \tau, z \rangle \) is high (low) for hot (cold) fluid respectively, and the no-slip boundary conditions manifest in the fact that \( \langle u_z | \tau, z \rangle \) is close to zero for \( z \geq 0 \) and \( z \leq 1 \). Additionally, one observes a stripe close to the \( \tau = 0 \)-line of almost vanishing vertical velocity which coincides with the reddish core (the hump, i.e. the most probable value) of the temperature PDF in figure 2(a). The interpretation is that fluid that is as hot as the mean temperature is neutrally buoyant and neither moves up nor down. Another striking feature is the sudden increase of the vertical velocity for high \( \tau \) near the boundary layer, i.e. for \( z = \delta T \), which we attribute to rising plumes that detach from the hot bottom plate. Again, it must be stressed that these interpretations hold only in an averaged sense.

Figure 4 shows that the conditional heat diffusion term \( \langle \Delta T | \tau, z \rangle \) is (in terms of absolute value) highest at the boundaries, with the term being positive (negative) at the hot bottom (cold top) plate. On the contrary, in the bulk the absolute value is high (low) for very cold (hot) fluid, i.e. in the wings of the temperature PDF. Additionally, the \( \tau \)-slice near the boundary in figure 4(b) shows an under- and overshoot. The connection of these unique features to the RB
dynamics has yet to be understood.

By combining the two aforementioned conditional averages to the vector field \( \bar{V} \) that defines the characteristics as suggested in section 2, one arrives at the vector field depicted in figure 5(a) – one of our central results. It is easy to interpret this graph by tracing the vector field; one can qualitatively reconstruct the typical RB cycle of fluid heating up at the bottom, rising up while starting to cool down, cooling down drastically at the top plate, falling down towards the bottom plate while warming up a bit and heating up again at the bottom. It is especially illustrative to see that the main contribution of cooling and heating (i.e. biggest movement in \( \tau \)-direction of phase space) takes place near the boundaries, highlighting the importance of the boundary layers, while obviously the biggest movement in \( z \)-direction occurs in the bulk.

Yet one has to consider, for example, that although hot fluid rises up very quickly (referring to the vectors pointing upwards at the right side in figure 5(a)), this does not contribute much to heat transport because these events occur rarely, as indicated by the temperature PDF shown along with the vector field governing the characteristics. To take this into account, figure 5(b) shows the vector field of the characteristics multiplied by the temperature PDF, which defines a probability current.

4. Summary

In the present work, we have analyzed the single-point temperature PDF on the basis of the Lundgren-Monin-Novikov hierarchy by truncating the hierarchy on the first level via the introduction of conditional averages.

We have first derived the evolution equation of the full joint PDF of temperature and velocity. Then we focused on the temperature PDF only, which is the central point of our paper, and obtained an evolution equation for it by reducing the joint PDF equation. We assumed rather weak symmetry conditions of statistical stationarity in time and homogeneity in lateral spatial directions; these conditions should be fulfilled at reasonably high aspect ratios even for closed vessels, i.e. are a good approximation of experimental setups in the bulk of the flow. Under these symmetry considerations, the evolution equation of the temperature PDF becomes fairly simple. The arising conditional averages of vertical velocity \( \langle u_z \rangle \) and temperature diffusion \( \langle \Delta T \rangle \) are estimated by direct numerical simulations using a suitably designed penalization approach,
and features of them are discussed. It shows that expected features such as properties of the temperature and velocity boundary layers, correlation of temperature and velocity and so on are related to the form of these conditional averages that naturally come up in our derivations.

The evolution equation of the temperature PDF is treated by the method of characteristics. Due to the applied symmetry conditions, the phase space which describes our system becomes two-dimensional, spanned by temperature $\tau$ and vertical coordinate $z$. Because of this reduced dimensionality of the system, the method of characteristics yields a descriptive view of the RB dynamics, resulting in the vector field describing the evolution in $\tau, z$-phase space. The characteristics, i.e. trajectories in $\tau, z$-phase space, are found to reproduce the typical cycle of a fluid parcel. The regions of the main transport in $\tau$-direction have been identified as the boundary layers, while the major movement in $z$-direction takes place in the bulk. This highlights the importance of the boundary layers to the heat transport. It would be very interesting to obtain the statistical quantities describing the evolution of the PDF directly from experiments, e.g. from measurements of instrumented particles as described by Gasteuil et al. (2007).

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