Flux-free conductance modulation in a helical Aharonov–Bohm interferometer

Hisao Taira\textsuperscript{1,3,4} and Hiroyuki Shima\textsuperscript{2,3}

\textsuperscript{1} Division of Applied Physics, Graduate School of Engineering, Hokkaido University, Sapporo, Hokkaido, 060-8628, Japan
\textsuperscript{2} Division of Applied Physics, Faculty of Engineering, Hokkaido University, Sapporo, Hokkaido, 060-8628, Japan
\textsuperscript{3} Department of Applied Mathematics 3, LaCaN, Universitat Politècnica de Catalunya (UPC), Barcelona 08034, Spain
\textsuperscript{4} Department of Physics, The Chinese University of Hong Kong (CUHK), Shatin, New Territories, Hong Kong

E-mail: taira@eng.hokudai.ac.jp

Received 12 February 2010, in final form 10 May 2010
Published 1 June 2010
Online at stacks.iop.org/JPhysCM/22/245302

Abstract
A novel conductance oscillation in a twisted quantum ring composed of a helical atomic configuration is theoretically predicted. The internal torsion of the ring is found to cause a quantum phase shift in the wavefunction that describes the electron’s motion along the ring. The resulting conductance oscillation is free from magnetic flux penetrating inside the ring, which is in complete contrast with the case for the ordinary Aharonov–Bohm effect observed in untwisted quantum rings.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
The Aharonov–Bohm (AB) effect is a pivotal manifestation of geometric phase governing quantum dynamics [1]. It occurs when a charged particle travels along a coherent loop threaded by external magnetic flux; the particle’s wavefunction acquires an additional quantum phase that influences an interference pattern. Experimental efforts to confirm its topological nature [2, 3] as well as possible applications toward quantum computation [4, 5] have made it a topic of broader interest than ever [6, 7].

The AB effect was originally predicted for charged particles moving around magnetic flux. Since then, it has been generalized to neutral particles having magnetic [8–10] or electric [11, 12] dipole momenta that travel around a line of electric or magnetic charges, respectively. A unified picture for the three phenomena was established in the framework of electromagnetic duality, which further predicted another interference phenomenon called the dual AB effect [13]. It is noteworthy that the AB effect has many analogs; in fact, light penetrating through an optical medium [14], quasiparticles moving in Bose–Einstein condensates [15–18], and particles in a gravitational background [19–21] have been suggested to exhibit AB-like phenomena. In addition, recent attempts to reveal the Dirac fermion dynamics in the AB ring [22] and to unveil the ponderomotive AB effect driven by laser pulses [23] are quite intriguing. The series of studies evidences the relevance of the effect to diverse fields in physics.

In the present paper, we propose a distinct class of the AB-like interference effects for non-interacting charged particles that move along a helical circuit, i.e., a quantum ring consisting of helical atomic structure. Surprisingly, the effect requires no magnetic flux threading inside the ring, which is in complete contrast to the case for the ordinary Aharonov–Bohm effect. Such a flux-free interference effect originates from a torsion-induced vector potential $\mathbf{A}_{\text{eff}}$ that appears in the effective Hamiltonian describing the particle’s motion along the helical circuit [24–26]; a similar torsion-induced effect was found in twisted optical waveguides [27]. We demonstrate that an additional phase associated with $\mathbf{A}_{\text{eff}}$ results in a conductance oscillation whose pattern is determined by the helicity of the atomic configuration.

2. The basic equation and electronic eigenstates
We consider a twisted quantum ring that has the ring radius $R_1$ and a uniform circular cross-section with the tube radius $R_2$.
such as [35, 36] or theoretically suggested [30–34].

The vectors (appropriate reference frame equation (5) have the form

\[ g_{ab} = \frac{1}{\sqrt{8}} g_{0b} \sqrt{g} g^{ab} \phi + V \phi = E \phi, \]

(1)

where \( \mu \equiv -\hbar^2/(2m^*) \) with an effective mass \( m^* \) and

\[ g = \text{det}[g_{ab}], \quad g_{ab} = \partial_a \mathbf{R} \cdot \partial_b \mathbf{R}, \]

(2)

\[ g^{ab} = g^{-1}_{ab} \quad [a, b = 0, 1, 2]. \]

In equation (1), \( V \) represents a potential that confines the transverse motion of electrons within the cross-section. The components of the tensor \( g_{ab} \) in equations (2) are given by

\[ g_{00} = \gamma^4 + \tau^2 (q_1^2 + q_2^2), \quad g_{01} = g_{10} = -\tau q_2, \]

\[ g_{02} = g_{20} = \tau q_1, \]

\[ g_{ij} = \delta_{ij}, \quad [i, j = 1, 2], \]

where \( \gamma = (1 - \sum_{i=1}^{2} \kappa_i q_i) \) and \( \kappa_i = e_0 \cdot \partial_0 e_i \). The elements \( g^{ab} \) are those of the 3 \times 3 matrix \( [g^{ab}] \) inverse to \( [g_{ab}] \), and thus they can be read as (see reference [37])

\[ g^{00} = \gamma^{-4}, \quad g^{01} = g^{10} = \gamma^{-4} \tau q_2, \]

\[ g^{02} = g^{20} = -\gamma^{-4} \tau q_1, \]

\[ g^{ij} = \delta_{ij} + \gamma^{-4} \tau^2 [q_i^2 + q_j^2] \delta_{ij} - q_i q_j. \]

(4)

For simplicity, we assume that the geometric deviation of the twisted frame from the orthogonal one is sufficiently smooth and small that \( (\kappa_1^2 + \kappa_2^2)^{1/2} R_2 \ll 1 \) and \( \tau R_2 \ll 1 \). These assumptions allow us to obtain [24, 38]

\[ \mu \left[ \partial_0^2 + \partial_2^2 + \left( \partial_0 - \frac{i \tau L}{\hbar} \right)^2 + \frac{1}{4 R_1^2} \right] \phi + V \phi = E \phi, \]

(5)

where \( L \equiv -i\hbar (q_1 \partial_2 - q_2 \partial_1) \) is the angular momentum operator in the cross-section. The eigenfunctions of equation (5) have the form

\[ \phi(q_0, q_1, q_2) = \psi(q_0) \sum_{n=1}^{N} c_n u_n(q_1, q_2). \]

(6)

Figure 1. Two-terminal electron interferometer based on a twisted quantum ring encircling external current flow \( I_{\text{ext}} \). Each wavefunction \( \psi_s(x) \) (s = 1, 2, 3, 4) describes the electron’s motion lying at the lead (s = 1, 4) or the arc (s = 2, 3) as indicated in the figure.

Here, \( u_j(q_1, q_2) \) are \( N \)-fold eigenfunctions in the cross-section and \( \psi(q_0) \) describes the axial motion of electrons along the twisted wire. It follows from equations (5) and (6) that \( \psi(q_0) \) obeys an effective one-dimensional Schrödinger equation such as

\[ \mu \left[ \left( \partial_0 - \frac{i \tau \langle L \rangle}{\hbar} \right)^2 + \frac{1}{4 R_1^2} - \frac{\tau^2}{\hbar^2} \left( \langle L^2 \rangle - \langle L \rangle^2 \right) \right] \psi(q_0) = \epsilon \psi(q_0). \]

(7)

The angular brackets \( \langle \cdots \rangle \) in equation (7) indicate taking an expectation value with respect to the two-dimensional ground-state function in the cross-section. We see from equation (7) that the term \( \tau \langle L \rangle/\hbar \) plays the role of the effective vector potential \( A_{\text{eff}} \) that we mentioned earlier. Hence, nonzero values of \( \tau \) and \( \langle L \rangle \) are expected to yield a quantum phase shift in the wavefunction \( \psi(q_0) \), which originates from the helical atomic configuration in the ring. If the ring has non-uniform cross-section, equation (7) contains a spatially dependent scalar potential that stems from the geometric curvature of the cylindrical surface of the ring [39–42]. This potential, nevertheless, requires no qualitative revision in the conclusion of this paper.

To obtain a finite \( \langle L \rangle \), we suppose an external current \( I_{\text{ext}} \) that penetrates through the center of the ring as shown in figure 1. The operator \( L \) is then given by \( L = -i\hbar \partial_0 \partial_\theta - eB (q_1^2 + q_2^2)/2 \), where \( \theta \) is the angular coordinate in the cross-section, \( B = \mu_0 I_{\text{ext}}/\ell \), \( \ell = 2\pi R_1 \) and \( \mu_0 \) is the permeability of vacuum. We also assume a constant \( \tau \) throughout the ring and a parabolic potential well \( V(q_1, q_2) = m^* \omega_0^2 (q_1^2 + q_2^2)/2 \) that strongly confines the transverse motions of electrons within the cross-section of radius \( R_2 \ll R_1 \); the parameter \( \omega_0 \) determines the steepness of the potential. These assumptions allow us to write the eigenfunctions of equation (7) as [38]

\[ \psi(q_0) = \psi_{\text{am}}(q_0) \exp \left( - \frac{i \tau \hbar}{\langle L \rangle} \int_{q_0}^{q_{00}} (L) dq_0 \right). \]

(8)

where \( \psi_{\text{am}} \) is an eigenfunction for an untwisted quantum ring (i.e., \( \tau = 0 \)). The ground-state function \( u(q_1, q_2) \) in the cross-section 5 The value of \( N \) depends on the eigenenergy \( E \) of equation (5) to which \( \phi \) belongs, and also on the shape of the cross-section in general.
section reads \[ u_s(q_1, q_2) = \frac{1}{\sqrt{2\pi \hbar \Omega}} \exp\left(-\frac{q_1^2 + q_2^2}{2\hbar \Omega}\right), \tag{9} \]

where \( \ell = \sqrt{\hbar/(m^* \Omega)} \), \( \Omega = \sqrt{\omega^2_c + (\omega_c/2)^2} \) and \( \omega_c = eB/m^* \) is the cyclotron frequency. Then, we can prove that

\[ \langle L \rangle = -\frac{I_{ext}}{I_0} \frac{1}{\sqrt{4 + \left(\frac{2\hbar}{m^* \omega_c \ell}\right)^2}}, \quad I_0 = \frac{m^* \omega_c \ell}{e \mu_0}. \tag{10} \]

Equations (8) and (10) state that nonzero \( I_{ext} \) gives rise to a quantum phase shift by \( \tau(L) \hbar / (\ell \hbar) \) in the eigenstate \( \psi(q_0) \), in which the magnitude of the shift is determined by the formula (10).

Figure 2 shows how \( \langle L \rangle \) depends on the external current \( I_{ext} \) normalized by \( I_0 \). \( \langle L \rangle \) decreases monotonically with increasing \( I_{ext} \), while it converges to \(-h\) (or \(+h\)) in the limit of \( I_{ext} \to +\infty \) (or \(-\infty\)). Furthermore, the decay in \( \langle L \rangle \) is steep only within the region \(-5 < I_{ext}/I_0 < 5\). These features of \( \langle L \rangle \) imply that the torsion-induced phase shift \( \tau(L) \hbar / (\ell \hbar) \) shows a significant response to the change in \( I_{ext} \) when \( I_{ext}/I_0 \) lies within the region above. The flux-free interference effect in a helical circuit is a direct consequence of the torsion-induced phase shift, as demonstrated later.

3. An electron interferometer with a twisted quantum ring

Let us consider the conductance of a twisted-ring-based interferometer measured at two terminals, as illustrated in figure 1. Two different paths connecting the two points \( P \) and \( Q \) have the same length of \( \pi R_1 \). The electron’s path along two semi-infinite leads and the two semicircular arcs is parametrized by \( x \) such that \( x = 0 \) at \( P \) and \( x = \pi R_1 \) at \( Q \); electron flow inserted from \( x = -\infty \) bifurcates at \( P \), passing through either of the two branches (i.e., semicircles of the ring) until becoming confluent at \( Q \), and then flowing away toward \( x = +\infty \).

The wavefunctions \( \psi_s \) \((s = 1, 2, 3, 4)\) corresponding to the four different regions depicted in figure 1 are given by \[ \psi_1(x) = A_1 e^{ikx} + B_1 e^{-ikx}, \]
\[ \psi_2(x) = A_2 e^{-i(k+\alpha)x} + B_2 e^{-i(k-\alpha)x}, \]
\[ \psi_3(x) = A_3 e^{i(k-\alpha)x} + B_3 e^{-i(k+\alpha)x}, \quad \psi_4(x) = A_4 e^{ikx}. \tag{11} \]

Here, \( k \) is the wavenumber of the incident electron, and \( a \equiv \tau(L)/\hbar \) is a wavenumber shift caused by \( I_{ext} \).

The wavefunctions \( \psi_s \) satisfy the connection conditions:

\[ \psi_1 = \psi_2 = \psi_3 \text{ and } \theta_1 = \theta_2 = \theta_3 \text{ at } P \text{ and } \psi_4 = \psi_2 = \psi_3 \text{ and } \theta_4 = \theta_2 = \theta_3 \text{ at } Q. \]

Applying the conditions to equations (11), we obtain the conductance \( G \) of the system by using the two-terminal Landauer formula [46],

\[ G = \frac{2e^2}{h} |A_4|^2 = \frac{2e^2}{h} |(1 + \theta_1)A_2 + 2\theta_1(A_3 - 1)|^2, \tag{12} \]

where

\[ A_2 = \alpha \gamma - \beta \gamma', \quad A_3 = -\beta' A_2 + \gamma', \tag{13} \]

and

\[ \alpha(k, a) = k[2(2k + a) - (k + a)(\theta_2/\theta_1) + (k - a)\theta_2], \]
\[ \beta(k, a) = k[2(k + a) + (\theta/a)\theta_1 + (k - a)\theta_2], \]
\[ \gamma(k, a) = 2k[2k + a + (k - a)\theta_2], \tag{14} \]

with the notation \( \theta_1 = \exp(-i\kappa \ell), \theta_2 = \exp(ia \ell) \) and \( \kappa(k, a) = \xi(k, -a) \) for \( \xi = \alpha, \beta, \gamma \). Once we have determined the dimensionless parameters \( k \ell \) and \( a \ell \) (or equivalently, \( k \ell, \alpha \ell, \tau \ell \) and \( (\kappa \ell, h) \)), we can evaluate \( G \) by using equations (12)–(14). If we impose \( a = 0 \) in equation (14), the expression for \( G \) reduces to that for an ordinary untwisted interferometer [45]:

\[ G = \frac{2e^2}{h} \frac{32}{41 - 9 \cos(2k \ell)}. \tag{15} \]

To make concise arguments, we omit the electron’s spin-dependent transport [47] and impurities/structural disorder [48] in the system, though each of these is expected to yield interesting consequences, similarly to the case for untwisted systems.

4. Results

Figures 3 and 4 show the plots of the dimensionless conductance \( \tilde{G} = G/(2e^2/h) \) as a function of the dimensionless current \( \tilde{I}_{ext} = I_{ext}/I_0 \) for various values of \( \tau \ell \) and \( k \ell \). In figure 3 (or figure 4), we fix \( k \ell = 1.0 \) (or \( \tau \ell = 1.0 \)) and choose several values of \( \tau \ell \) (or \( k \ell \)) as indicated. In both figures, the curves of \( \tilde{G} \) exhibit waveforms within the region \( |\tilde{I}_{ext}| < 5 \) but smooth (or almost constant) behaviors outside the region: the two contrasting features of \( \tilde{G} \) in the two regions are attributed to the nonlinear response in \( \langle L \rangle \) to \( I_{ext} \), which will be discussed shortly.
The most important observation in figure 3 is an oscillation in $\bar{G}$ for large $t\ell$, which does not take place in figure 4. This conductance oscillation is what we call the flux-free interference effect, the peculiar phenomenon to the twisted-ring based AB interferometer. The magnitude of the oscillation can be enhanced (i.e., the oscillation period shortened) by increasing $t\ell$, since a larger $t\ell$ results in a larger torsion-induced phase shift $\tau(L)/h$. What value of $t\ell$ should be required for observing the flux-free conductance oscillation strongly depends on the condition of $k\ell$ that we choose. When $k\ell = 1.0$, for instance, the oscillation disappears for $\tau \ll 10.0$ as shown in figure 3. An exhaustive study covering wide ranges of the parameter space makes clear the required values of $t\ell$ and $k\ell$, which will be given elsewhere.

It is also noteworthy that the conductance oscillation in the present system is not periodic against the variation in $I_{\text{ext}}$, which differs from the case for the ordinary AB effect driven by penetrating magnetic flux. The non-periodic character is a consequence of the nonlinear dependence of $\langle L \rangle$ on $I_{\text{ext}}$ (see equation (10) and figure 2). We have seen from figure 2 that $\langle L \rangle$ decreases steeply with increasing $I_{\text{ext}}$ around $I_{\text{ext}} \sim 0$ (for $|I_{\text{ext}}| \gg 0$) and it decreases gently in the regions of $|I_{\text{ext}}| \gg 0$. This means that the phase shift $\tau(L)/h$ shows sensitive (insensitive) response to a change in $I_{\text{ext}}$ at $I_{\text{ext}} \sim 0$ ($|I_{\text{ext}}| \gg 0$), and thus it reaches $2\pi$ for only a small (very large) increase in $I_{\text{ext}}$ at $I_{\text{ext}} \sim 0$ ($|I_{\text{ext}}| \gg 0$). As a result, the conductance oscillates densely (sparsely) at $I_{\text{ext}} \sim 0$ ($|I_{\text{ext}}| \gg 0$) since the oscillation stems from the quantum interference caused by the phase shift $\tau(L)/h$.

As a by-product, we show in figure 4 the $I_{\text{ext}}$ dependence of $\bar{G}$ for a fixed $t\ell$ ($\tau \ell = 1.0$) and various $k\ell$. At $k\ell \ll 1.0$, a sharp peak arises at $I_{\text{ext}} = 0$ whose peak width broadens gradually with increasing $k\ell$. The peak height at $I_{\text{ext}} = 0$ is given by equation (15); thus it oscillates with increasing $k\ell$. When $k\ell$ exceeds 1.0, then $G$ becomes almost constant over the range of $I_{\text{ext}}$ that we have considered, and it shows a slight hollow at $I_{\text{ext}} = 0$. The almost constant behavior of $G(I_{\text{ext}})$ indicates that the torsion-induced phase shift gives little contribution to the motion of electrons whose energies are large enough to satisfy $k\ell \gg 1.0$.

5. Discussion

We remark that the numerical results in figures 3 and 4 are based on the presence of $I_{\text{ext}}$ that threads the center of the ring. An experimental realization of such a setup may not be feasible in a straightforward manner, since the ring radius $R_1$ should be small enough to keep the quantum coherence of mobile electrons. Still, we can build a setup equivalent to the above by applying an external magnetic field $B_{\text{ext}}$, instead of $I_{\text{ext}}$, to a portion of the ring in a tangential direction. This is because the tangential field $B_{\text{ext}}$ engenders nonzero $(L)$ in the cross-section and thus a phase shift in $\psi(q_0)$. A field strength of $B_0 \sim 10 \text{T}$ is required to observe the flux-free interference effect, provided the ring of the cross-sectional radius $R_2 \sim 10 \text{nm}$; we can estimate this from the relations of $B_{\text{ext}}$ and $L$ to $B_0 \sim m\omega_0^2 R_2^2/2$ (see section 2). This field strength is accessible in the existing nanometric measurements, thus supporting the experimental feasibility of our results.

It is noted that our attention has been limited to non-interacting electrons. When taking into account a Coulomb interaction between electrons, the amplitude of the conductance oscillation is expected to decrease to a degree. In fact, such the amplitude reduction caused by the Coulomb interaction was experimentally observed [49, 50] and theoretically considered [51] for untwisted AB interferometers, where sizable interference patterns still remain to be obtained. A further interesting issue would be transport properties of the Tomonaga–Luttinger liquid (TLL) state in an AB interferometer. In one-dimensional systems, the slightest correlation between electrons can lead to TLL states, a highly collective state of matter [52]. Recent studies have revealed that the conductance of TLL based AB interferometers shows anomalous interference patterns that are substantially different from those of the ordinary (non-interacting) AB system [53, 54]. These features pose a question as to what happens in the helical AB interferometer composed of TLL states, which requires newly establishing bosonization theory for a twisted quantum wire.
6. Conclusion

We theoretically predicted a non-trivial conductance oscillation in a twisted quantum ring that is free from threading magnetic flux. The helical atomic configuration inside the ring gives rise to a phase shift in the electron’s eigenstates by $\tau(L)\ell/h$, with $\tau$ the internal torsion of the ring, $L$ the angular momentum expectation value in the cross-section, and $\ell$ the ring perimeter. The phase shift induces a flux-free conductance oscillation in response to a change in the external current $I_{\text{ext}}$ or a tangential magnetic field $B_{\text{ext}}$, in which neither $I_{\text{ext}}$ nor $B_{\text{ext}}$ yields magnetic flux threading the ring. Our results suggest an untouched quantum nature in actual low-dimensional nanostructures composed of helical atomic configurations.

Acknowledgments

We would like to express our thanks to K Yakubo for illuminating discussions. One of the authors (HT) acknowledges K W Yu and M Arroyo for their comments and hospitality during stays in CUHK and UPC. HT also thanks K W Y u and M Arroyo for their comments and hospitality during stays in CUHK and UPC. HT also thanks K W Y U and M Arroyo for their comments and hospitality during stays in CUHK and UPC. HT also thanks K W Y U and M Arroyo for their comments and hospitality during stays in CUHK and UPC.

References

[1] Aharonov Y and Bohm D 1959 Phys. Rev. 115 485
[2] Tonomura A, Osakabe N, Matsuda T, Kawasaki T, Endo J, Yano S and Yamada H 1986 Phys. Rev. Lett. 56 792
[3] Caprez A, Barwick B and Batelaan H 2007 Phys. Rev. Lett. 99 210401
[4] Ionicioiu R 2003 Phys. Rev. A 68 034305
[5] Fischer A M, Campo V L Jr, Portnoi M E and Römer R A 2009 Phys. Rev. Lett. 102 096405
[6] Schwinger Schrögl U and Schuster C 2008 J. Phys.: Condens. Matter 20 383201
[7] Cano A and Paul I 2009 Phys. Rev. B 80 153401
[8] Aharonov Y and Casher A 1984 Phys. Rev. Lett. 53 319
[9] Cimmino A, Opat G I, Klein A G, Kaiser H, Werner S A, Arif M and Clothier R 1989 Phys. Rev. Lett. 63 380
[10] Sangster K, Hinds E A, Barnett S M and Riis E 1993 Phys. Rev. Lett. 71 3641
[11] He X-G and McKellar B H J 1993 Phys. Rev. A 47 3424
[12] Wilkens M 1994 Phys. Rev. Lett. 72 5
[13] Dowling J P, Williams C P and Franson J D 1999 Phys. Rev. Lett. 83 2486
[14] Cook R J, Feam H and Milonni P W 1995 Am. J. Phys. 63 705
[15] Haldane F D M and Wu Y-S 1985 Phys. Rev. Lett. 55 2887
[16] Kivelson S A and Spivak B Z 1992 Phys. Rev. B 45 10490
[17] Sonin E B 1997 Phys. Rev. B 55 458
[18] Mel’nikov A S 2001 Phys. Rev. Lett. 86 4108
[19] Ford L H and Vilenkin A 1981 J. Phys. A: Math. Gen. 14 2353
[20] Reznik B 1995 Phys. Rev. D 51 3108
[21] Bakke F and Kurtado C 2009 Phys. Rev. D 80 024033
[22] Coatescu I I and Papp E 2007 J. Phys.: Condens. Matter 19 242206
[23] Barwick B and Batelaan H 2008 New J. Phys. 10 083036
[24] Takagi S and Tanazawa T 1992 Prog. Theor. Phys. 87 561
[25] Mitchell K A 2001 Phys. Rev. A 63 042112
[26] Ńin M V and Magarill L I 2002 Phys. Rev. B 66 205308
[27] Longhi S 2009 Laser Photon. Rev. 3 243
[28] Cohen-Karuni T, Segev L, Srur-Lavi O, Cohen S R and Jeleselevich E 2006 Nat. Nanotechnol. 1 36
[29] Nagapriya K S, Goldbart O, Kaplan-Ashti I, Seiftig G, Tennre R and Jeleselevich E 2008 Phys. Rev. Lett. 101 195501
[30] Balakrishnan R and Dandoloff R 2007 Nonlinearity 21 1
[31] da Fonseca A F and Galvão D S 2008 Phys. Rev. Lett. 92 175502
[32] Arias I and Arroyo M 2008 Phys. Rev. Lett. 100 085503
[33] Wang Z, Xu X, Gao F and Weber W J 2008 Phys. Rev. B 77 224413
[34] Vassilev V M, Djondjorov P A and Mladenov I M 2008 J. Phys. A: Math. Gen. 41 435201
[35] Jensen H and Koppe H 1971 Ann. Phys. 63 586
[36] da Costa R C T 1998 Phys. Rev. A 23 1982
[37] Shima H and Nakayama T 2010 Higher Mathematics for Physics and Engineering (Berlin: Springer-Verlag)
[38] Taira H and Shima H 2010 J. Phys.: Condens. Matter 22 075301
[39] Taira H and Shima H 2007 Surf. Sci. 601 5270
[40] Shima H, Yoshoi R and Onno J 2009 Phys. Rev. B 79 201401(R)
[41] Ono S and Shima H 2009 Phys. Rev. B 79 235407
[42] Atanasov V, Dandoloff R and Saxena A 2009 Phys. Rev. B 79 033404
[43] Fock V 1928 Z. Phys. 47 446
[44] Darwin C G 1930 Proc. Camb. Phil. Soc. 27 86
[45] Yaku K and Ohe J 2000 J. Phys. Soc. Japan 69 2170
[46] Hod O, Baez R and Rabani E 2008 J. Phys.: Condens. Matter 20 383201
[47] Ying Y, Jin G and Ma Y 2009 J. Phys.: Condens. Matter 21 275801
[48] Heinrichs J 2009 J. Phys.: Condens. Matter 21 295701
[49] Taruchu S, Honda T and Saku T 1995 Solid State Commun. 94 413
[50] Ihnatsenka S and Zozoulenko I V 2008 Phys. Rev. B 77 235304
[51] Kottmäki V and Räsänen E 2010 arXiv:1003.1330v1
[52] Voit I 1995 Rep. Prog. Phys. 58 977
[53] Aristov D N and Wölfle P 2009 Phys. Rev. B 80 045109
[54] Krive I V, Palevski A, Shekhter R I and Jonson M 2010 Low Temp. Phys. 36 119