Hermann type actions on a pseudo-Riemannian symmetric space

Naoyuki Koike

Abstract

In this paper, we first investigate the geometry of the orbits of the isotropy action of a semi-simple pseudo-Riemannian symmetric space by investigating the complexified action. Next we investigate the geometry of the orbits of a Hermann type action on a semi-simple pseudo-Riemannian symmetric space. By considering two special Hermann type actions on a semi-simple pseudo-Riemannian symmetric space, we recognize an interesting structure of the space. As a special case, we recognize an interesting structure of the complexification of a semi-simple pseudo-Riemannian symmetric space. Also, we investigate a homogeneous submanifold with flat section in a pseudo-Riemannian symmetric space under certain conditions.

1 Introduction

In Riemannian symmetric spaces, the notion of an equifocal submanifold was introduced by Terng-Thorbergsson in [36]. This notion is defined as a compact submanifold with flat section such that the normal holonomy group is trivial and that the focal radius functions for each parallel normal vector field are constant. However, the condition of the equifocality is rather weak in the case where the Riemannian symmetric spaces are of non-compact type and the submanifold is non-compact. So we [17, 18] have recently introduced the notion of a complex equifocal submanifold in a Riemannian symmetric space $G/K$ of non-compact type. This notion is defined by imposing the constancy of the complex focal radius functions in more general. Here we note that the complex focal radii are the quantities indicating the positions of the focal points of the extrinsic complexification of the submanifold, where the submanifold needs to be assumed to be complete and of class $C^\omega$ (i.e., real analytic). On the other hand, Heintze-Liu-Olmos [13] has recently defined the notion of an isoparametric submanifold with flat section in a general Riemannian manifold as a submanifold such that the normal holonomy group is trivial, its sufficiently close parallel submanifolds are of constant mean curvature with respect to the radial direction and that the image of the normal space at each point by the normal exponential map is flat and totally geodesic. We [18] showed the following fact:

All isoparametric submanifolds with flat section in a Riemannian symmetric space $G/K$ of non-compact type are complex equifocal and that conversely, all curvature-adapted and complex equifocal submanifolds are isoparametric ones with flat section.

Here the curvature-adaptedness means that, for each normal vector $v$ of the submanifold, the Jacobi operator $R(\cdot, v)v$ preserves the tangent space of the submanifold invariantly
and the restriction of $R(\cdot, v)v$ to the tangent space commutes with the shape operator $A_v$, where $R$ is the curvature tensor of $G/K$. Note that curvature-adapted hypersurfaces in a complex hyperbolic space (and a complex projective space) mean so-called Hopf hypersurfaces and that curvature-adapted complex equifocal hypersurfaces in the space mean Hopf hypersurfaces with constant principal curvatures, which are classified by J. Berndt [2]. Also, he [3] classified curvature-adapted hypersurfaces with constant principal curvatures (i.e., curvature-adapted complex equifocal hypersurfaces) in the quaternionic hyperbolic space. As a subclass of the class of complex equifocal submanifolds, we [19] defined the notion of a proper complex equifocal submanifold in $G/K$ as a complex equifocal submanifold whose lifted submanifold to $H^0([0, 1], g)$ ($g := \text{Lie}G$) through some pseudo-Riemannian submersion of $H^0([0, 1], g)$ onto $G/K$ is proper complex isoparametric in the sense of [17], where we note that $H^0([0, 1], g)$ is a pseudo-Hilbert space consisting of certain kind of paths in the Lie algebra $g$ of $G$. For a $C^\omega$-submanifold $M$, we [18] showed that $M$ is proper complex equifocal if and only if the lift of the complexification $M^c$ (which is a submanifold in the anti-Kaehlerian symmetric space $G^c/K^c$) of $M$ to $H^0([0, 1], g^c)$ ($g^c := \text{Lie}G^c$) by some anti-Kaehlerian submersion of $H^0([0, 1], g^c)$ onto $G^c/K^c$ is proper anti-Kaehlerian isoparametric in the sense of [18]. This fact implies that a proper complex equifocal submanifold is a complex equifocal submanifold whose complexification has regular focal structure. Let $G/K$ be a Riemannian symmetric space of non-compact type and $H$ be a closed subgroup of $G$. If the $H$-action is proper and there exists a complete embedded flat submanifold meeting all $H$-orbits orthogonally, then it is called a complex hyperpolar action. Principal orbits of a complex hyperpolar action are complex equifocal. If $H$ is a symmetric subgroup of $G$ (i.e., $(\text{Fix}\sigma)_0 \subset H \subset \text{Fix}\sigma$ for some involution $\sigma$ of $G$), then the $H$-action is called a Hermann type action, where $\text{Fix}\sigma$ is the fixed point group of $\sigma$ and $(\text{Fix}\sigma)_0$ is the identity component of the group. Hermann type actions are complex hyperpolar. We ([18,19]) showed the following fact:

**Principal orbits of a Hermann type action are curvature-adapted and proper complex equifocal.**

Similarly, we can define the notions of a complex equifocal submanifold, proper complex equifocal one and a curvature-adapted one in a pseudo-Riemannian symmetric space (see Section 2). Also, we can define the notions of a complex hyperpolar action and a Hermann type action on a pseudo-Riemannian symmetric space. We [23] showed the following fact:

**All isoparametric submanifolds with flat section in a pseudo-Riemannian symmetric space $G/K$ are complex equifocal. Conversely all curvature-adapted complex equifocal submanifolds such that $A$ and $R(\cdot, v)v$ are semi-simple for any normal vector $v$ are isoparametric ones with flat section, where $A_v$ is the shape operator and $R$ is the curvature tensor of $G/K$ and the semi-simplenesses of $A_v$ and $R(\cdot, v)v$ mean that their complexifications are diagonalizable.**

L. Geatti and C. Gorodski [9] has recently showed that a polar representation of a real reductive algebraic group on a pseudo-Euclidean space has the same closed orbits as the isotropy representation (i.e., the linear isotropy action) of a pseudo-Riemannian symmetric space (see Theorem 1 of [9]). Also, they showed that the principal orbits of the polar representation through a semi-simple element (i.e., the orbit through a regular element (in the sense of [9])) is an isoparametric submanifold by investigating the com-
plexified representation (see Theorem 11 (also Example 12) of [9]), where an isoparametric submanifold means a pseudo-Riemannian submanifold such that the (restricted) normal holonomy group is trivial and that the shape operator for each (local) parallel normal vector field is semi-simple and has constant complex principal curvature. All isoparametric submanifolds in this sense are isoparametric ones (with flat section) in the sense of [13]. Let $G/H$ be a (semi-simple) pseudo-Riemannian symmetric space (equipped with the metric $\langle , \rangle$ induced from the Killing form of the Lie algebra $\mathfrak{g}$ of $G$). In this paper, we first investigate the complexified shape operators of the orbits of the isotropy action of $G/H$ (i.e., the $\mathfrak{g}$-invariant semisimple normal field is semi-simple and has constant complex principal curvature. All isoparametric submanifolds mean a pseudo-Riemannian submanifold such that the (restricted) normal field is semi-simple and has constant complex principal curvature. All isoparametric submanifolds in this sense are isoparametric ones (with flat section) in the sense of [13].

**Theorem A.** Let $G/H$ be a (semi-simple) pseudo-Riemannian symmetric space, $H'$ be a symmetric subgroup of $G$, $\sigma$ (resp. $\sigma'$) be an involution of $G$ with $(\text{Fix} \sigma)_0 \subset H \subset \text{Fix} \sigma$ (resp. $(\text{Fix} \sigma')_0 \subset H' \subset \text{Fix} \sigma'$), $L := (\text{Fix} (\sigma \circ \sigma'))_0$ and $\mathfrak{l} := \text{Lie} L$. Assume that $G$ is not compact and $\sigma \circ \sigma' = \sigma' \circ \sigma$. Then the following statements (i) and (ii) hold:

(i) The orbit $H'(eH)$ of the $H'$-action on $G/H$ is a reflective pseudo-Riemannian submanifold and it is homothetic to the semi-simple pseudo-Riemannian symmetric space $H'/H \cap H'$. For each $x \in H'(eH)$, the section $\Sigma_x$ of $H'(eH)$ through $x$ is homothetic to the semi-simple pseudo-Riemannian symmetric space $L/H \cap H'$.

(ii) Let $M$ be a principal orbit of the $H'$-action through a point $\exp_G(w)H$ ($w \in \mathfrak{q} \cap \mathfrak{q}'$ s.t. $\text{ad}(w)|_1$ : semi-simple) of $\Sigma_x \setminus F$, where $\mathfrak{q} := \text{Ker}(\sigma + id) = T_{eH}(G/H)$, $\mathfrak{q}' := \text{Ker}(\sigma' + id)$ and $F$ is a focal set of $H'(eH)$. Then $M$ is curvature-adapted and proper complex equifocal, for any normal vector $v$ of $M$, $R(\cdot, v)v$ and the shape operator $A_v$ are semi-simple. Hence it is an isoparametric submanifold with flat section.

**Remark 1.1.** (i) Since $\bigcup_{w \in \mathfrak{q} \cap \mathfrak{q}' \text{ s.t. } \text{ad}(w)|_1 \text{ semi-simple}} (H' \cap H)(\exp_G(w)H)$ is an open dense subset of $L(eH)$, it is shown that $\bigcup_{w \in \mathfrak{q} \cap \mathfrak{q}' \text{ s.t. } \text{ad}(w)|_1 \text{ semi-simple}} H'(\exp_G(w)H)$ is an open dense subset of $G/H$.

(ii) It is shown that, if $M$ is a curvature-adapted complex equifocal submanifold and, for any normal vector $v$ of $M$, $R(\cdot, v)v$ and $A_v$ are semi-simple, then it is an isoparametric submanifold with flat section (see Proposition 9.1 of [23]).

(iii) When we take a Riemannian symmetric space of non-compact type as $G/H$ in this theorem, we have $\bigcup_{x \in H'(eH)} \Sigma_x = G/H$ and $F = \emptyset$.

L. Geatti [8] has recently defined a pseudo-Kaehlerian structure on some $G$-invariant domain of the complexification $G^c/H^c$ of a semi-simple pseudo-Riemannian symmetric space $G/H$. On the other hand, we [23] have recently defined an anti-Kaehlerian structure on the whole of the complexification $G^c/H^c$. By applying Theorem A to the complexification $G^c/H^c$ (equipped with the natural anti-Kaehlerian structure) of a semi-simple pseudo-Riemannian symmetric space $G/H$ and a symmetric subgroup $G$ of $G^c$, we recognize an interesting structure of $G^c/H^c$. Here we note that an involution $\sigma$ of $G^c$ with
(Fix σ)_0 ⊂ H^c ⊂ Fix σ and the conjugation τ of G^c with respect to G are commutative. In this case, the group corresponding to L in the statement of Theorem A is the dual G^{*^μ} of G with respect to H. Hence we have the following fact.

\[ H' \cap (eH) \]
\[ \Sigma_{x_1} \]
\[ \Sigma_{eH} \]
\[ \Sigma_{x_2} \]

\[ H'(eH) \]
\[ H'y \]
\[ \Sigma_{eH} \]
\[ \Sigma_{x_1} \]
\[ \Sigma_{x_2} \]

Fig. 1.

Corollary B. Let G^c/H^c and G^{*μ} be as above. Then the following statements (i) and (ii) hold:

(i) The orbit G(eH^c) is a reflective pseudo-Riemannian submanifold and it is homothetic to the pseudo-Riemannian symmetric space G/H. For each \( x \in G(eH^c) \), the section \( \Sigma_x \) of \( G(eH^c) \) through \( x \) is homothetic to the pseudo-Riemannian symmetric space G^{*μ}/H.

(ii) For principal orbits of the G-action on G^c/H^c, the same fact as the statement (ii) of Theorem A holds.

By considering two special Hermann type actions on a semi-simple pseudo-Riemannian symmetric space, we obtain the following interesting fact for the structure of the semi-simple pseudo-Riemannian symmetric space.

Theorem C. Let G/H and σ be as in Theorem A, θ the Cartan involution of G with \( \theta \circ \sigma = \sigma \circ \theta \), \( K := (\text{Fix } \theta)_0 \) and \( L := (\text{Fix}(\sigma \circ \theta))_0 \). Then the following statements (i) and (ii) hold:

(i) The orbits \( K(eH) \) and \( L(eH) \) are reflective submanifolds satisfying \( T_{eH}(G/H) = T_{eH}(K(eH)) \oplus T_{eH}(L(eH)) \) (orthogonal direct sum), \( K(eH) \) is anti-homothetic to the
Riemannian symmetric space $K/H \cap K$ of compact type and $L(eH)$ is homothetic to the Riemannian symmetric space $L/H \cap K$ of non-compact type. Also, the orbit $K(eH)$ has no focal point.

(ii) Let $M_1$ be a principal orbit of the $K$-action and $M_2$ be a principal orbit of the $L$-action through a point of $K(eH) \setminus F$, where $F$ is the focal set of $L(eH)$. Then $M_i$ ($i = 1, 2$) are curvature-adapted and proper complex equifocal, for any normal vector $v$ of $M_i$, $R(\cdot, v)v|_{T_x M_i}$ ($x$ : the base point of $v$) and the shape operator $A_v$ are diagonalizable. Hence they are isoparametric submanifolds with flat section.

**Remark 1.2.** For any involution $\sigma$ of $G$, the existence of a Cartan involution $\theta$ of $G$ with $\theta \circ \sigma = \sigma \circ \theta$ is assured by Lemma 10.2 in [1].

By applying Theorem C to the complexification $G^c/H^c$ (equipped with the natural anti-Kaehlerian structure) of a semi-simple pseudo-Riemannian symmetric space $G/H$, we recognize the interesting structure of $G^c/H^c$. In this case, the groups corresponding to $K, L$ and $H \cap K$ in the statement of Theorem C are as follows. Let $\sigma$ be an involution of $G$ with $(\text{Fix} \sigma)_0 \subset H \subset \text{Fix} \sigma$, $\theta$ be a Cartan involution of $G$ commuting with $\sigma$ and set $K_\theta := \text{Fix} \theta$. Let $G^*$ be the compact dual of $G$ with respect to $K_\theta$, $H^*$ be the compact dual of $H$ with respect to $H \cap K_\theta$ and $(G^d, H^d)$ be the dual of semi-simple symmetric pair $(G, H)$ in the sense of [26]. Then $G^*, G^d$ and $H^*$ correspond to $K, L$ and $H \cap K$ in the statement of Theorem C, respectively. Hence we have the following fact.

**Corollary D.** Let $G^c/H^c$ be the complexification (equipped with the natural anti-Kaehlerian structure) of a semi-simple pseudo-Riemannian symmetric space $G/H$, $G^*$ (resp. $H^*$) be the compact dual of $G$ (resp. $H$) and $(G^d, H^d)$ be the dual of $(G, H)$. Then the following statements (i) and (ii) hold:

(i) The orbits $G^*(eH^c)$ and $G^d(eH^c)$ are reflective submanifolds of $G^c/H^c$ satisfying $T_{eH^c}(G^c/H^c) = T_{eH^c}(G^*(eH^c)) \oplus T_{eH^c}(G^d(eH^c))$ (orthogonal direct sum), $G^*(eH^c)$ is anti-homothetic to the Riemannian symmetric space $G^*/H^*$ of compact type and $G^d(eH^c)$ is homothetic to the Riemannian symmetric space $G^d/H^*$ of non-compact type. Also, the orbit $G^*(eH^c)$ has no focal point.
(ii) For principal orbits of the $G^*$-action and $G^d$-action on $G^e/H^e$, the same fact as the statement (ii) of Theorem C holds.

Remark 1.3. In the case where $G/H$ in the statement of Corollary D is a Riemannian symmetric space of non-compact type, we have $G^d = G$.

Homogeneous submanifolds with flat section in a pseudo-Riemannian symmetric space are complex equifocal. We obtain the following fact for a homogeneous submanifold with flat section in a semi-simple pseudo-Riemannian symmetric space which admits a reflective focal submanifold, where a reflective submanifold means a totally geodesic pseudo-Riemannian submanifold with section.

Theorem E. Let $M$ be a homogeneous submanifold with flat section in a semi-simple pseudo-Riemannian symmetric space $G/H$. Assume that $M$ admits a reflective focal submanifold $F$ such that $\mathfrak{n}_h(g^{-1}_s T g H F)$ is a non-degenerate subspace of $\mathfrak{h}$, where $gH$ is an arbitrary point of $F$ and $\mathfrak{n}_h(g^{-1}_s T g H F)$ is the normalizer of $g^{-1}_s T g H F$ in $\mathfrak{h}$. Then $M$ is a principal orbit of a Hermann type action.

Remark 1.4. (i) For the $H'$-action in Theorem A, we have $\mathfrak{n}_h(T_{eH}(H'(eH))) = \mathfrak{n}_h(q \cap h') = \mathfrak{h} \cap h' + 3_{\mathfrak{h} \cap q'}(q \cap h')$, where $3_{\mathfrak{h} \cap q'}(q \cap h')$ is the centralizer of $q \cap h'$ in $\mathfrak{h} \cap q'$. Hence, if $3_{\mathfrak{h} \cap q'}(q \cap h') = \{0\}$, then $\mathfrak{n}_h(T_{eH}(H'(eH)))$ is a non-degenerate subspace of $\mathfrak{h}$. Thus almost all principal orbits of the $H'$-action have $H'(eH)$ as a reflective focal submanifold as in the statement of Theorem E.

(ii) For the $K$-action in Theorem C, we have $\mathfrak{n}_h(T_{eH}(K(eH))) = \mathfrak{n}_h(q \cap \mathfrak{f}) = \mathfrak{h} \cap \mathfrak{f} + 3_{\mathfrak{h} \cap \mathfrak{f}'}(q \cap \mathfrak{f})$. Hence, $\mathfrak{n}_h(T_{eH}(K(eH)))$ is a non-degenerate subspace of $\mathfrak{h}$. Similarly, for the $L$-action in Theorem C, it is shown that $\mathfrak{n}_h(T_{eH}(L(eH)))$ is a non-degenerate subspace of $\mathfrak{h}$. Thus almost all principal orbits of the $K$-action (resp. the $L$-action) have $K(eH)$ (resp. $L(eH)$) as a reflective focal submanifold as in the statement of Theorem E.

2 New notions in a pseudo-Riemannian symmetric space

In this section, we shall define new notions in a (semi-simple) pseudo-Riemannian symmetric space, which are analogies of notions in a Riemannian symmetric space of non-compact type defined in [18]. Let $M$ be an immersed pseudo-Riemannian submanifold with flat section (that is, $g^{-1}_s T_x^\perp M$ is abelian for any $x = gH \in M$) in a (semi-simple) pseudo-Riemannian symmetric space $N = G/H$ (equipped with the metric induced from the Killing form of $\mathfrak{g} := \text{Lie} G$), where $T_x^\perp M$ is the normal space of $M$ at $x$. Denote by $A$ the shape tensor of $M$. Let $v \in T_x^\perp M$ and $X \in T_x M$ ($x = gK$), where $T_x M$ is the tangent space of $M$ at $x$. Denote by $\gamma_v$ the geodesic in $N$ with $\gamma_v(0) = v$, where $\dot{\gamma}_v(0)$ is the velocity vector of $\gamma_v$ at 0. The strongly $M$-Jacobi field $Y$ along $\gamma_v$ with $Y(0) = X$ (hence $Y'(0) = -A_v X$) is given by

$$Y(s) = (P_{\gamma_v|[0,s]} \circ (D^c_{\gamma_v} - s D^s_{\gamma_v} \circ A_v))(X),$$

(2.1)
where $Y'(0) = \tilde{\nabla}_t Y$ ($\tilde{\nabla}$ : the Levi-Civita connection of $N$), $P_{\gamma_t|_0}$ is the parallel translation along $\gamma_t|_0$ and $D^0_{sv}$ (resp. $D^i_{sv}$) is given by

$$D^0_{sv} = g_s \circ \cos(\sqrt{-1}\text{ad}(sg^{-1}_s v)) \circ g^{-1}_s$$

(resp. $D^i_{sv} = g_s \circ \frac{\sin(\sqrt{-1}\text{ad}(sg^{-1}_s v))}{\sqrt{-1}\text{ad}(sg^{-1}_s v)} \circ g^{-1}_s$).

Here ad is the adjoint representation of the Lie algebra $\mathfrak{g}$. All focal radii of $M$ along $\gamma_t$ are obtained as real numbers $s_0$ with Ker($D^0_{sv} - s_0 D^i_{sv} \circ A_v$) $\neq \{0\}$. So, we call a complex number $z_0$ with Ker($D^0_{sv} - z_0 D^i_{sv} \circ A_v$) $\neq \{0\}$ a complex focal radius of $M$ along $\gamma_t$ and call dim Ker($D^0_{sv} - z_0 D^i_{sv} \circ A_v$) the multiplicity of the complex focal radius $z_0$, where $A^c_v$ is the complexification of $A_v$ and $D^0_{sv}$ (resp. $D^i_{sv}$) is a $\mathbb{C}$-linear transformation of $(T_y N)^c$ defined by

$$D^0_{sv} = g_s \circ \cos(\sqrt{-1}\text{ad}(z_0 g^{-1}_s v)) \circ (g^c_s)^{-1}$$

(resp. $D^i_{sv} = g_s \circ \frac{\sin(\sqrt{-1}\text{ad}(z_0 g^{-1}_s v))}{\sqrt{-1}\text{ad}(z_0 g^{-1}_s v)} \circ (g^c_s)^{-1}$),

where $g^c_s$ (resp. ad$^c$) is the complexification of $g_s$ (resp. ad). Here we note that, in the case where $M$ is of class $C^i$, complex focal radii along $\gamma_t$ indicate the positions of focal points of the (extrinsic) complexification $M^c(\to G^c/H^c)$ of $M$ along the complexified geodesic $\gamma^c_{t,v}$. Here $G^c/H^c$ is the pseudo-Riemannian symmetric space equipped with the metric induced from the Killing form of $\mathfrak{g}^c$ regarded as a real Lie algebra (which is called the anti-Kaehlerian symmetric space associated with $G/H$), $M^c$ and the complexified immersion of $M^c$ into $G^c/H^c$ are defined as in [23] and $\iota$ is the natural embedding of $G/H$ into $G^c/H^c$. Furthermore, assume that the normal holonomy group of $M$ is trivial. Let $\tilde{v}$ be a parallel unit normal vector field of $M$. Assume that the number (which may be 0 and $\infty$) of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$. Furthermore assume that the number is not equal to 0. Let $\{r_{i,x} \mid i = 1, 2, \cdots \}$ be the set of all complex focal radii along $\gamma_{\tilde{v}_x}$, where $|r_{i,x}| < |r_{i+1,x}|$ or $|r_{i,x}| = |r_{i+1,x}|$ & Re $r_{i,x} > \Re r_{i+1,x}$ or $|r_{i,x}| = |r_{i+1,x}|$ & Re $r_{i,x} = \Re r_{i+1,x}$ & Im $r_{i,x} = -\Im r_{i+1,x} < 0$. Let $r_i$ ($i = 1, 2, \cdots \)$ be complex valued functions on $M$ defined by assigning $r_{i,x}$ to each $x \in M$. We call these functions $r_i$ ($i = 1, 2, \cdots \)$ complex focal radius functions for $\tilde{v}$. We call $r_i \tilde{v}$ a complex focal normal vector field for $\tilde{v}$. If, for each parallel unit normal vector field $\tilde{v}$ of $M$, the number of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$, each complex focal radius function for $\tilde{v}$ is constant on $M$ and it has constant multiplicity, then we call $M$ a complex equifocal submanifold. Also, if parallel submanifolds sufficiently close to $M$ has constant mean curvature with respect to the radial direction, then we call $M$ an isoparametric submanifold with flat section. It is shown that all isoparametric submanifolds with flat section are complex equifocal and that, conversely, all curvature-adapted complex equifocal submanifold with complex diagonalizable shape operators and Jacobi operators are isoparametric submanifolds with flat section (see Theorem 9.1 of [23]).

Let $N = G/H$ be a (semi-simple) pseudo-Riemannian symmetric space and $\pi$ be the natural projection of $G$ onto $G/H$. Let $\sigma$ be an involution of $G$ with $\text{Fix} \sigma_0 \subset H \subset \text{Fix} \sigma$ and denote by the same symbol $\sigma$ the involution of $\mathfrak{g} := \text{Lie} \ G$. Let $\mathfrak{h} := \{X \in \mathfrak{g} \mid \sigma(X) = X\}$ and $\mathfrak{q} := \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$, which is identified with the tangent space $T_{eH} N$. Let $\langle , \rangle$ be the Killing form of $G$. Denote by the same symbol $\langle , \rangle$ both the bi-invariant
pseudo-Riemannian metric of $G$ induced from $\langle \cdot, \cdot \rangle$ and the pseudo-Riemannian metric of $N$ induced from $\langle \cdot, \cdot \rangle$. Let $\theta$ be a Cartan involution of $G$ with $\theta \circ \sigma = \sigma \circ \theta$. Denote by the same symbol $\theta$ the involution of $\mathfrak{g}$ induced from $\theta$. Let $\mathfrak{f} := \{X \in \mathfrak{g} \mid \theta(X) = X\}$ and $\mathfrak{p} := \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. From $\theta \circ \sigma = \sigma \circ \theta$, it follows that $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{h} \cup \mathfrak{p}$ and $\mathfrak{q} = \mathfrak{q} \cap \mathfrak{h} \cup \mathfrak{q} \cap \mathfrak{p}$. Set $\mathfrak{g}_\pm := \mathfrak{p}$, $\mathfrak{g}_- := \mathfrak{f}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{g}_\pm} := -\pi^*_{\mathfrak{g}_\pm} \langle \cdot, \cdot \rangle + \pi^*_{\mathfrak{g}_\pm} \langle \cdot, \cdot \rangle$, where $\pi_{\mathfrak{g}_-}$ (resp. $\pi_{\mathfrak{g}_+}$) is the projection of $\mathfrak{g}$ onto $\mathfrak{g}_-$ (resp. $\mathfrak{g}_+$). Let $H^0([0,1], \mathfrak{g})$ be the space of all $L^2$-integrable paths $u : [0,1] \to \mathfrak{g}$ (with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}_0}$). It is shown that $(H^0([0,1], \mathfrak{g}), \langle \cdot, \cdot \rangle_{\mathfrak{g}_0})$ is a pseudo-Hilbert space. Let $H^1([0,1], G)$ be the Hilbert Lie group of all absolutely continuous paths $g : [0,1] \to G$ such that the weak derivative $g'$ of $g$ is squared integrable (with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}_0}$), that is, $g^{-1}_* g' \in H^0([0,1], \mathfrak{g})$. Define a map $\phi : H^0([0,1], \mathfrak{g}) \to G$ by $\phi(u) = g_u(1)$ ($u \in H^0([0,1], \mathfrak{g})$), where $g_u$ is the element of $H^1([0,1], G)$ satisfying $g_u(0) = e$ and $g_u^{-1} g'_u = u$. We call this map the parallel transport map (from 0 to 1). This submersion $\phi$ is a pseudo-Riemannian submersion of $(H^0([0,1], \mathfrak{g}), \langle \cdot, \cdot \rangle_{\mathfrak{g}_0})$ onto $(G, \langle \cdot, \cdot \rangle)$. Denote by $\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C}, \mathfrak{q}^\mathbb{C}, \mathfrak{f}^\mathbb{C}, \mathfrak{p}^\mathbb{C}$ and $\langle \cdot, \cdot \rangle^\mathbb{C}$ the complexifications of $\mathfrak{g}, \mathfrak{h}, \mathfrak{q}, \mathfrak{f}, \mathfrak{p}$ and $\langle \cdot, \cdot \rangle$. Set $\mathfrak{g}^\mathbb{C}_\pm := \sqrt{-1} \mathfrak{f}^\mathbb{C} + \mathfrak{p}$ and $\mathfrak{g}^\mathbb{C} := \mathfrak{f} + \sqrt{-1} \mathfrak{p}$. Set $\langle \cdot, \cdot \rangle^\mathbb{C} := 2\text{Re}(\cdot, \cdot)^\mathbb{C}$ and $\langle \cdot, \cdot \rangle^\mathbb{C}_{\mathfrak{g}_\pm} := -\pi^*_{\mathfrak{g}_\pm} \langle \cdot, \cdot \rangle^\mathbb{C} + \pi^*_{\mathfrak{g}_\pm} \langle \cdot, \cdot \rangle^\mathbb{C}$, where $\pi^*_{\mathfrak{g}_-}$ (resp. $\pi^*_{\mathfrak{g}_+}$) is the projection of $\mathfrak{g}^\mathbb{C}$ onto $\mathfrak{g}^\mathbb{C}_-$. Let $H^0([0,1], \mathfrak{g}^\mathbb{C})$ be the space of all $L^2$-integrable paths $u : [0,1] \to \mathfrak{g}^\mathbb{C}$ (with respect to $\langle \cdot, \cdot \rangle^\mathbb{C}_{\mathfrak{g}_0}$). Define a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle^\mathbb{C}_0$ of $H^0([0,1], \mathfrak{g}^\mathbb{C})$ by $\langle u, v \rangle^\mathbb{C}_0 := \int_0^1 \langle u(t), v(t) \rangle^\mathbb{C} dt$. It is shown that $(H^0([0,1], \mathfrak{g}^\mathbb{C}), \langle \cdot, \cdot \rangle^\mathbb{C}_0)$ is an infinite dimensional anti-Kaehlerian space. See [18] about the definition of an infinite dimensional anti-Kaehlerian space. In similar to $\phi$, the parallel transport map $\phi^\mathbb{C} : H^0([0,1], \mathfrak{g}^\mathbb{C}) \to G^\mathbb{C}$ for $G^\mathbb{C}$ is defined. This submersion $\phi^\mathbb{C}$ is an anti-Kaehlerian submersion. Let $\pi : G \to G/H$ and $\pi^\mathbb{C} : G^\mathbb{C} \to G^\mathbb{C}/H^\mathbb{C}$ be the natural projections. By imitating the proof of Theorem 1 of [18], we can show that, in the case where $M$ is of class $C^\omega$, the following statements (i) $\sim$ (iii) are equivalent:

(i) $M$ is complex equifocal,
(ii) each component of $(\pi \circ \phi)^{-1}(M)$ is complex isoparametric,
(iii) each component of $(\pi^\mathbb{C} \circ \phi^\mathbb{C})^{-1}(M^\mathbb{C})$ is anti-Kaehlerian isoparametric.

See [18] about the definitions of a complex isoparametric submanifold and an anti-Kaehlerian isoparametric submanifold. In particular, if each component of $(\pi \circ \phi)^{-1}(M)$ is proper complex isoparametric in the sense of [17], that is, for each normal vector $v$ of $(\pi \circ \phi)^{-1}(M)$, there exists a pseudo-orthonormal base of the complexified tangent space consisting of the eigenvectors of the complexified shape operator for $v$, then we call $M$ a proper complex equifocal submanifold. For $C^\omega$-submanifold $M$ in $G/H$, it is shown that $M$ is proper complex equifocal if and only if $(\pi^\mathbb{C} \circ \phi^\mathbb{C})^{-1}(M^\mathbb{C})$ is proper anti-Kaehlerian isoparametric in the sense of [18], that is, for each normal vector $v$ of $(\pi^\mathbb{C} \circ \phi^\mathbb{C})^{-1}(M^\mathbb{C})$, there exists a $J$-orthonormal base of the tangent space consisting $J$-eigenvectors of the shape operator for $v$, where $J$ is the complex structure of $(\pi^\mathbb{C} \circ \phi^\mathbb{C})^{-1}(M^\mathbb{C})$. See [18] the definitions of $J$-orthonormal base and $J$-eigenvalue. Proper anti-Kaehlerian isoparametric submanifolds are interpreted as ones having regular focal structure among anti-Kaehlerian isoparametric submanifolds. From this fact, proper complex equifocal submanifolds are interpreted as ones whose complexification has regular focal structure among complex equifocal submanifolds.

Next we shall recall the notions of a complex Jacobi field and the parallel translation along a holomorphic curve, which are introduced in [23], and we state some facts related
to these notions. These notions and facts will be used in the next section. Let \((M, J, g)\) be an anti-Kaehlerian manifold, \(\nabla\) (resp. \(R\)) be the Levi-Civita connection (resp. the curvature tensor) of \(g\) and \(\nabla^c\) (resp. \(R^c\)) be the complexification of \(\nabla\) (resp. \(R\)). Let \((TM)^{(1,0)}\) be the holomorphic vector bundle consisting of complex vectors of \(M\) of type \((1,0)\). Note that the restriction of \(\nabla^c\) to \(TM^{(1,0)}\) is a holomorphic connection of \(TM^{(1,0)}\) (see Theorem 2.2 of [6]). For simplicity, assume that \((M, J, g)\) is complete even if the discussion of this section is valid without the assumption of the completeness of \((M, J, g)\).

Let \(\gamma : C \to M\) be a complex geodesic, that is, \(\gamma(z) = \exp_{\gamma(0)}((\text{Re}\, z)\gamma_\ast((\frac{d}{dz})0) + (\text{Im}\, z)\gamma_\ast((\frac{d}{dz})0))\), where \((z)\) is the complex coordinate of \(C\) and \(s := \text{Re}\, z\). Let \(Y : C \to (TM)^{(1,0)}\) be a holomorphic vector field along \(\gamma\). That is, \(Y\) assigns \(Y_z \in (T\gamma(z)M)^{(1,0)}\) to each \(z \in C\) and, for each holomorphic local coordinate \((U, (z_1, \cdots, z_n))\) of \(M\) with \(U \cap \gamma(C) \neq \emptyset\), \(Y_i : \gamma^{-1}(U) \to C\) \((i = 1, \cdots, n)\) defined by \(\gamma_z = \sum_{i=1}^n Y_i(z)((\frac{d}{dz})\gamma_z)\) are holomorphic. If \(Y\) satisfies
\[
\nabla^c\gamma_\ast((\frac{d}{dz})0) + \nabla^c\gamma_\ast((\frac{d}{dz})0) + (\gamma_\ast((\frac{d}{dz})0)) = 0,
\]
then we call \(Y\) a complex Jacobi field along \(\gamma\). It is shown that, for a complex geodesic \(\gamma\), \(\gamma\) is a complex geodesic for each \(\gamma_\ast((\frac{d}{dz})0)\) is a complex Jacobi field along \(\gamma : \delta(\cdot, 0)\). A vector field \(X\) on \(M\) is said to be real holomorphic if the Lie derivation \(L_X J\) of \(J\) with respect to \(X\) vanishes. It is known that \(X\) is a real holomorphic vector field if and only if the complex vector field \(X - \sqrt{-1}JX\) is holomorphic. Let \(\gamma : C \to M\) be a complex geodesic and \(Y\) be a holomorphic vector field along \(\gamma\). Denote by \(Y\) the real part of \(Y\). Then it is shown that \(Y\) is a complex Jacobi field along \(\gamma\) if and only if, for any \(z_0 \in C\), \(s \mapsto (Y_\Re)_{s:=z_0}\) is a Jacobi field along the geodesic \(\gamma_{z_0}(s) := \gamma(s_{z_0})\). Next we shall recall the notion of the parallel translation along a holomorphic curve. Let \(\alpha : D \to (M, J, g)\) be a holomorphic curve, where \(D\) is an open set of \(C\). Let \(Y\) be a holomorphic vector field along \(\alpha\). If \(\nabla^c\gamma_\ast((\frac{d}{dz})0) Y = 0\), then we say that \(Y\) is parallel. Let \(\alpha : D \to (M, J, g)\) be a holomorphic curve. For \(z_0 \in D\) and \(v \in (T\alpha(z_0)M)^{(1,0)}\), there uniquely exists a parallel holomorphic vector field \(Y\) along \(\alpha\) with \(Y_{z_0} = v\). We denote \(Y_1\) by \((P_\alpha)_{z_0}(v)\). It is clear that \((P_\alpha)_{z_0}(v)\) is a \(C\)-linear isomorphism of \((T\alpha(z_0)M)^{(1,0)}\) onto \((T\alpha(z_1)M)^{(1,0)}\). We call \((P_\alpha)_{z_0}(v)\) the parallel translation along \(\alpha\) from \(z_0\) to \(z_1\). We consider the case where \((M, J, g)\) is an anti-Kaehlerian symmetric space \(G^c/H^c\). For \(v \in (T_{g^0}H^c(G^c/H^c)^c\), we define \(C\)-linear transformations \(D^c\) of \((T_{g^0}H^c(G^c/H^c)^c\) by \(D^c \circ (g^0_{-1}) - 1\) and \(D^c \circ (g^0_{-1}) - 1\), respectively, where \(ad_{g^0}^c\) is the complexification of the adjoint representation of \(g^0\). Let \(Y\) be a holomorphic vector field along \(\gamma^c\). Define \(\tilde{\gamma} : D \to (T_{g^0}K^c(G^c/K^c)^{(1,0)}\) by \(\tilde{\gamma}_z = (P^c_{g^0})_{z_0}(Y_z)\) \((z \in D)\), where \(D\) is the domain of \(\gamma^c\). Then we have
\[
Y_z = (P^c_{g^0})_{0}, z (\tilde{D}^c_{D^c(1,0)}(Y_0) + z\tilde{D}^c_{D^c(1,0)}(\frac{d\tilde{\gamma}}{dz} | z=0))\]
3 The isotropy action of a pseudo-Riemannian symmetric space

In this section, we investigate the complexified shape operators of the orbits of the isotropy action of a semi-simple pseudo-Riemannian symmetric space by investigating the complexified action. Let $G/H$ be a (semi-simple) pseudo-Riemannian symmetric space (equipped with the metric $\langle \ , \ \rangle$ induced from the Killing form $B$ of $\mathfrak{g}$) and $\sigma$ be an involution of $G$ with $(\text{Fix}\sigma)_{0} \subset H \subset \text{Fix}\sigma$. Denote by the same symbol $\sigma$ the differential of $\sigma$ at $e$. Let $\mathfrak{h} := \text{Lie}\sigma$ and $\mathfrak{q} := \text{Ker}(\sigma + \text{id})$, which is identified with $T_{eH}(G/H)$. Let $\theta$ be a Cartan involution of $G$ with $\theta \circ \sigma = \sigma \circ \theta$, $J := \text{Ker}(\theta - \text{id})$ and $\mathfrak{p} := \text{Ker}(\theta + \text{id})$. Let $\mathfrak{g}^{c}, \mathfrak{h}^{c}, \mathfrak{q}^{c}, \mathfrak{f}^{c}, \mathfrak{p}^{c}$ and $\langle \ , \ \rangle^{c}$ be the complexifications of $\mathfrak{g}, \mathfrak{h}, \mathfrak{q}, \mathfrak{f}, \mathfrak{p}$ and $\langle \ , \ \rangle$, respectively. The complexification $\mathfrak{q}^{c}$ is identified with $T_{eHc}(G^{c}/H^{c})$. Under this identification, $\sqrt{-1}X \in \mathfrak{q}^{c}$ corresponds to $J_{c}eX \in T_{eHc}(G^{c}/H^{c})$, where $J$ is the complex structure of $G^{c}/H^{c}$. Give $G^{c}/H^{c}$ the metric (which also is denoted by $\langle \ , \ \rangle^{c}$) induced from the Killing form $B_{A}$ of $\mathfrak{g}^{c}$ regarded as a real Lie algebra. Note that $B_{A}$ coincides with $2\text{Re}B^{c}$ and $(J, \langle \ , \ \rangle^{c})$ is an anti-Kahlerian structure of $G^{c}/H^{c}$, where $B^{c}$ is the complexification of $B$. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{q}$ (that is, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$ and each element of $\mathfrak{a}$ is semi-simple). The dimension of $\mathfrak{a}$ is called the rank of $G/H$. Without loss of generality, we may assume that $\mathfrak{a} = \mathfrak{a} \cap J + \mathfrak{a} \cap \mathfrak{p}$. Let $\mathfrak{a}^{c} := \{X \in \mathfrak{a}^{c} | \text{ad}(a)^{2}X = \alpha(a)^{2}X \text{ for all } a \in \mathfrak{a}^{c}\}$ and $\mathfrak{h}^{c} := \{X \in \mathfrak{h}^{c} | \text{ad}(a)^{2}X = \alpha(a)^{2}X \text{ for all } a \in \mathfrak{a}^{c}\}$ for each $\alpha \in (\mathfrak{a}^{c})^{*}$ and $(\mathfrak{a}^{c})^{*}$ the (C-)dual space of $\mathfrak{a}^{c}$) and $\Delta := \{\alpha \in (\mathfrak{a}^{c})^{*} | \mathfrak{a}^{c} \neq \{0\}\}$. Then we have

$$
\begin{align*}
\mathfrak{q}^{c} = \mathfrak{a}^{c} + \sum_{\alpha \in \Delta_{+}} \mathfrak{a}^{c}_{\alpha} \quad \text{and} \quad \mathfrak{b}^{c} = \mathfrak{h}^{c}(\mathfrak{a}^{c}) + \sum_{\alpha \in \Delta_{+}} \mathfrak{h}^{c}_{\alpha},
\end{align*}
$$

where $\Delta_{+}(\subset \Delta)$ is the positive root system under some lexicographical ordering and $\mathfrak{h}^{c}(\mathfrak{a}^{c})$ is the centralizer of $\mathfrak{a}^{c}$ in $\mathfrak{h}^{c}$. Let $\tilde{\mathfrak{a}}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$ and $\mathfrak{g}_{\tilde{\mathfrak{a}}}^{c} := \{X \in \mathfrak{g}^{c} | \text{ad}(a)X = \alpha(a)X \text{ for all } a \in \mathfrak{a}^{c}\}$ for each $\tilde{\alpha} \in (\mathfrak{a}^{c})^{*}$ and $\tilde{\Delta} := \{\tilde{\alpha} \in (\mathfrak{a}^{c})^{*} | \mathfrak{g}_{\tilde{\alpha}}^{c} \neq \{0\}\}$. Then we have $\mathfrak{g}^{c} = \tilde{\mathfrak{a}}^{c} + \sum_{\tilde{\alpha} \in \tilde{\Delta}} \mathfrak{g}_{\tilde{\alpha}}^{c}$ and $\dim_{\mathbb{C}}\mathfrak{g}_{\tilde{\alpha}}^{c} = 1$ for each $\tilde{\alpha} \in \tilde{\Delta}$. Also, we have $\Delta = \delta_{\tilde{\mathfrak{a}}}^{\mathfrak{a}} | \tilde{\alpha} \in \tilde{\Delta} \} \setminus \{0\}$, $\mathfrak{q}^{c} = (\sum_{\tilde{\alpha} \in \tilde{\Delta} \text{ s.t. } \tilde{\alpha}_{\mathfrak{a}} = \pm 0} ) \cap \mathfrak{q}^{c} (\alpha \in \Delta)$ and $\mathfrak{h}^{c} = (\sum_{\tilde{\alpha} \in \tilde{\Delta} \text{ s.t. } \tilde{\alpha}_{\mathfrak{a}} = \pm 0} ) \cap \mathfrak{h}^{c} (\alpha \in \Delta)$. The following fact is well-known.

**Lemma 3.1.** For each $\alpha \in \Delta$, $\alpha(\mathfrak{a} \cap \mathfrak{p}) \subset \mathbb{R}$ and $\alpha(\mathfrak{a} \cap J) \subset \sqrt{-1}\mathbb{R}$.

**Remark 3.1.** Each element of $\mathfrak{a} \cap \mathfrak{p}$ (resp. $\mathfrak{a} \cap J$) is called a hyperbolic (resp. elliptic) element.

Take $E_{\tilde{\alpha}}(\neq 0) \in \mathfrak{g}_{\tilde{\alpha}}^{c}$ for each $\tilde{\alpha} \in \tilde{\Delta}$ and set $Z_{\tilde{\alpha}} := c_{\tilde{\alpha}}(E_{\tilde{\alpha}} + \sigma E_{\tilde{\alpha}})$ and $Y_{\tilde{\alpha}} := c_{\tilde{\alpha}}(E_{\tilde{\alpha}} - \sigma E_{\tilde{\alpha}})$, where $c_{\tilde{\alpha}}$ is one of two solutions of the complex equation

$$
z^{2} = \frac{\alpha(a_{\alpha})}{B^{c}(E_{\tilde{\alpha}} - \sigma E_{\tilde{\alpha}}, E_{\tilde{\alpha}} - \sigma E_{\tilde{\alpha}})}.
$$

Then we have $\text{ad}(a)Z_{\tilde{\alpha}} = \tilde{\alpha}(a)Y_{\tilde{\alpha}}$ and $\text{ad}(a)Y_{\tilde{\alpha}} = \tilde{\alpha}(a)Z_{\tilde{\alpha}}$ for any $a \in \mathfrak{a}^{c}$. Hence we have $Z_{\tilde{\alpha}} \in \mathfrak{h}_{\tilde{\alpha}}^{c} \cap \mathfrak{a}$ and $Y_{\tilde{\alpha}} \in \mathfrak{q}_{\tilde{\alpha}}^{c} \cap \mathfrak{a}$. Furthermore, for $\alpha \in \mathfrak{a}^{c}$, it is shown that $\mathfrak{h}_{\alpha}^{c}$ (resp. $\mathfrak{g}_{\alpha}^{c}$) contains $\mathfrak{h}_{\tilde{\alpha}}^{c}$ (resp. $\mathfrak{g}_{\tilde{\alpha}}^{c}$) for any $\tilde{\alpha} \in \tilde{\Delta}$.
For simplicity, set $G/H$ is shown. L. Verhoczki [38] investigated the shape operator $s$ of orbits of the isotropy complexification of $M \in \Delta$. First we shall show the statement (i) by imitating the proof of Proposition 3 in [38]. Let $a$ be a Cartan subspace of $q$ containing $w$ and $q^c = a^c + \sum_{\alpha \in \Delta_+} q^c_{\alpha}$ be the root space decomposition with respect to $a^c$. Then the following statements (i) and (ii) hold:

(i) $g^{-1}_s(T_x M)^c = \sum_{\alpha \in \Delta_+ s.t. \alpha(w) \notin \sqrt{-1}\pi Z} q^c_{\alpha}$ and $g^{-1}_s(T_x^1 M)^c = a^c + \sum_{\alpha \in \Delta_+ s.t. \alpha(w) \notin \sqrt{-1}\pi Z} q^c_{\alpha}$ hold. In particular, if $M$ is a principal orbit, then we have $g^{-1}_s(T_x M)^c = \sum_{\alpha \in \Delta_+} q^c_{\alpha}$ and $g^{-1}_s(T_x^1 M)^c = a^c$.

(ii) Let $H_x$ be the isotropy group of $H$ at $x$ and set $H_x(g_s a) = \{h \cdot g_s a \mid a \in a, h \in H_x\}$. Then $H_x(g_s a)$ is open in $T_x^1 M$ and, for any $v := h \cdot g_s a \in H_x(g_s a)$, we have $A_{\alpha}^s|_{h \cdot g_s a} = -\frac{\sqrt{-1}\alpha(w)}{\tan(\sqrt{-1}\alpha(w))} \text{id}$ (for $\alpha \in \Delta_+ s.t. \alpha(w) \notin \sqrt{-1}\pi Z$), where $A^c$ is the complexification of $A$.

Proof. First we shall show the statement (i) by imitating the proof of Proposition 3 in [38]. Let $M^c$ be the extrinsic complexification of $M$, that is, $M^c := H^c \cdot x$ (or $G^c/H^c$), where $G/H$ is identified with $G(eH^c)$. We shall investigate $T_x(M^c)$ instead of $(T_x M)^c$ because $(T_x M)^c$ is identified with $T_x(M^c)$. Let $a\alpha$ ($\alpha \in \Delta$), $\tilde{\Delta}$, $Z_{\tilde{\alpha}}$ and $Z_{\tilde{\alpha}}$ ($\tilde{\alpha} \in \tilde{\Delta}$) be the above quantities defined for $a$ and a Cartan subalgebra $\tilde{\alpha}$ of $q$ containing $a$. Let $\tilde{\alpha} \in \tilde{\Delta}$ and $\alpha := \tilde{\alpha}|_{a^c}$. Since $[Z_{\tilde{\alpha}}, w] = -\alpha(w)Y_{\tilde{\alpha}}$ and $[Z_{\tilde{\alpha}}, Y_{\tilde{\alpha}}] = \alpha(a\alpha)a\alpha$, we have

$$\frac{d}{dt}|_{t=0}\text{Ad}_{G^c}(\exp t Z_{\tilde{\alpha}})w = -\alpha(w)Y_{\tilde{\alpha}},$$

where $\text{Ad}_{G^c}$ is the adjoint representation of $G^c$. Hence we have

$$T_w \text{Ad}_{G^c}(H^c)w = \sum_{\alpha \in \Delta_+ s.t. \alpha(w) \neq 0} q^c_{\alpha}.$$
On the other hand, we have $W_1 = (d\text{Exp})_w(\sqrt{\langle w, w \rangle_{Y_{\tilde{a}}}} Y_{\tilde{a}})$. Hence we have

\begin{equation}
(d\text{Exp})_w(Y_{\tilde{a}}) = \frac{\sin(\sqrt{-1} \alpha(w))}{\sqrt{-1} \alpha(w)} g_{\tilde{a}} Y_{\tilde{a}}.
\end{equation}

Since $M^c = \text{Exp}(\text{Ad}_{G^c}(H^c)w)$, we have $T_x(M^c) = (d\text{Exp})_w(T_w(\text{Ad}_{G^c}(H^c)w))$. Hence the relations in the statement (i) follow from (3.1).

Next we shall show the statement (ii). The $H_x$-action on $T_x(G/H)$ preserves $T_xM$ and $T_x^\perp M$ invariantly, respectively. The $H_x$-action on $T_x^\perp M$ is so-called slice representation and it is equivalent to an $s$-representation (the linear isotropy representation of a pseudo-Riemannian symmetric space). Therefore $H_x(g, a)$ is open in $T_x^\perp M$. In the sequel, we shall show the remaining part of the statement (ii) by imitating the proof of Theorem 1 in [38] for the isotropy action of a Riemannian symmetric space of compact type. Denote by $\hat{A}$ the shape tensor of $M^c$. Under the identification of $(T_x M^c)$ with $(T_x M)^c$, the complexified shape operator $A^c_w$ is identified with $\hat{A}_w$. Hence we suffice to investigate $\hat{A}_w$ instead of $A^c_w$. Let $\alpha$ be an element of $\triangle \subset \alpha|_H = \alpha$. Also, in case of $2\alpha \notin \triangle$, it is clear that $\text{Ad}(w \alpha) \in \triangle \subset \alpha|_H = \alpha$. Also, in case of $2\alpha \notin \triangle$, it is clear that $\text{Ad}(w \alpha) \in \triangle \subset \alpha|_H = \alpha$. Hence $\hat{H}_x^c := \text{Exp}(\alpha \alpha)$. Easily we can show

$$\text{Ad}_{G^c}(\text{Exp} \ z \alpha) = \cos(k^2 z \alpha(a)) a_{\alpha} - \frac{1}{k} \sin(k^2 z \alpha(a)) Y_{\alpha \tilde{a}} \quad (k = 1, 2).$$

From the relation, it follows that $\text{Ad}(\hat{H}_x^c)(w)$ is a complex hypersurface in $\tilde{a} := Ca_{\alpha} + \hat{q}_{2\alpha}$. In the other hand, it is clear that $\text{Ad}(\hat{H}_x^c)(w)$ is contained in the complex hypersphere $\text{Exp}(\alpha \alpha)$. Hence $\text{Ad}(\hat{H}_x^c)(w)$ coincides with this complex hypersphere. The vector $w$ is expressed as $w = \frac{\alpha(w)}{\alpha(a)} a_{\alpha} + b$ for some $b \in \alpha^{-1}(0)$. Then we have

$$\text{Ad}_{G^c}(\text{Exp} \ z \alpha) = \frac{\alpha(w)}{\alpha(a)} a_{\alpha} - \frac{1}{k} \sin(k^2 z \alpha(a)) Y_{\alpha \tilde{a}} \quad (k = 1, 2).$$

From this relation, it follows that $\text{Ad}(\hat{H}_x^c)(w)$ coincides with the complex hypersphere $\text{Exp}(\alpha \alpha)$. Set $\hat{Q}^c_{\alpha} := \text{Exp}(\alpha \alpha)$ and $\hat{Q}^c_{\alpha} := \text{Exp}(\alpha \alpha)$. It is easy to show that $\hat{Q}^c_{\alpha}$ is a totally geodesic complex rank one anti-Kaehlerian symmetric space in $G^c/H^c$. Furthermore, by imitating the proof of Proposition 4 in [38], it is shown that $\hat{Q}^c_{\alpha}(b)$ is a totally geodesic complex rank one anti-Kaehlerian symmetric space and it is isometric to $\hat{Q}^c_{\alpha}$. In fact, a map $\phi : \hat{Q}^c_{\alpha} \to \hat{Q}^c_{\alpha}(\alpha \alpha)$ of $b \to \alpha \alpha$, $\hat{H}_x \cdot x$ is a complex geodesic hypersphere of complex radius $\sqrt{\alpha(w)}$ in $\hat{Q}^c_{\alpha}(b)$. Set $\hat{Q}^c := \text{Exp}(\alpha \alpha + \hat{q}_{2\alpha})$, which is isometric to the anti-Kaehlerian product $\hat{Q}^c_{\alpha}(b) \times C^r$. Since $\text{Ad}(\hat{H}_x^c)(w)$ is equal to the complex hypersphere of complex radius $\sqrt{\alpha(w)}$ in $\hat{Q}^c_{\alpha}(b)$. Set $\hat{Q}^c := \text{Exp}(\alpha \alpha + \hat{q}_{2\alpha})$, which is isometric to the anti-Kaehlerian product $\hat{Q}^c_{\alpha}(b) \times C^r$. We have $\hat{Q}^c := \text{Exp}(\alpha \alpha + \hat{q}_{2\alpha})$, which is isometric to the anti-Kaehlerian product $\hat{Q}^c_{\alpha}(b) \times C^r$. Therefore $\hat{H}_x \cdot x$ is a component of $M^c \cap \hat{Q}^c_{\alpha}$.
shape tensor of $\tilde{H}_a^c \cdot x \leftrightarrow \tilde{Q}_a^c$. Since $\tilde{Q}_a^c$ is totally geodesic in $G^c/H^c$ and $T_x^c(M^c)$ con-
tains the normal space of $H^c_\alpha \cdot x$ in $Q^c_\alpha$, it follows from pseudo-Riemannian version of
Lemma 6 of [38] that $\tilde{A}_{g,a} \alpha$ preserves $T_x \tilde{H}_a^c \cdot x$ invariantly and that $\tilde{A}_{g,a} \alpha$ on
$T_x(\tilde{H}_a^c \cdot x)$. Let $\phi$ be the above isometry of $\tilde{Q}_a^c$ onto $\tilde{Q}_a^c(b)$. Set $r_0 := \frac{\alpha(w)}{\alpha(c_0)}$ and denote
by $\tilde{A}$ the shape tensor of $\tilde{H}_a^c \cdot (r_0 a_\alpha) \leftrightarrow \tilde{Q}_a^c$. Clearly we have $\phi(\tilde{H}_a^c \cdot (r_0 a_\alpha)) = \tilde{H}_a^c \cdot x$
and $\phi_*(\text{exp}_{G^c}(r_0 a_\alpha)) \cdot (a_\alpha) = g_s a_\alpha$. Hence we have $\tilde{A}_{g,a} \alpha = \phi_* \circ \tilde{A}_{g,a} \alpha \circ \phi_*^{-1}$.
For simplicity, set $\tilde{Y} := \text{exp}_{G^c}(r_0 a_\alpha)$. Now we shall investigate $\tilde{A}_{g,a} \alpha$. Define a complex
geodesic variation $\delta : C^2 \to G^c/H^c$ by

$$
\delta(z, u) := \text{Exp}(z_0 \cos u \cdot a_\alpha + \sqrt{-1} \alpha(r_0 a_\alpha) \sin u \cdot Y_\alpha) \quad ((z, u) \in C^2).
$$

Set $W := \frac{\partial}{\partial u}|_{u=0}$. Since $W$ is a complex Jacobi field along $\gamma_{r_0 a_\alpha}^c$, it follows from (2.2) that

$$
W = \frac{\sin(\sqrt{-1} \alpha(r_0 a_\alpha) \cdot \sqrt{-1} \alpha(r_0 a_\alpha))}{\sqrt{-1} \alpha(r_0 a_\alpha)} \left( P_{\gamma_{r_0 a_\alpha}^c} \right)_{0,z} (Y_\alpha).
$$

We have

$$
\frac{\partial\delta}{\partial \alpha} \bigg|_{u=0} \frac{\partial\delta}{\partial z} = \frac{\partial\delta}{\partial \alpha} \bigg|_{u=0} \frac{\partial\delta}{\partial z} = W' = W
$$

$$
= \cos(\sqrt{-1} \alpha(r_0 a_\alpha)) \left( P_{\gamma_{r_0 a_\alpha}^c} \right)_{0,z} Y_\alpha \in T_{\text{Exp}(r_0 a_\alpha) \tilde{H}_a^c} \cdot (r_0 a_\alpha)
$$

and hence

$$
\tilde{A}_{g,a} \alpha W_1 = -\cos(\sqrt{-1} \alpha(r_0 a_\alpha)) \left( P_{\gamma_{r_0 a_\alpha}^c} \right)_{0,z} Y_\alpha,
$$

which together with (3.2) and $\alpha(b) = 0$ deduces

$$
\tilde{A}_{g,a} \alpha \tilde{g}_* Y_\alpha = -\frac{\sqrt{-1} \alpha(a_\alpha)}{\tan(\sqrt{-1} \alpha(w))} \tilde{g}_* Y_\alpha.
$$

Therefore we have

$$
\tilde{A}_{g,a} \alpha g_s Y_\alpha = -\frac{\sqrt{-1} \alpha(a_\alpha)}{\tan(\sqrt{-1} \alpha(w))} g_s Y_\alpha.
$$

Similarly we have

$$
\tilde{A}_{g,a} \alpha g_s Y_\alpha = -\frac{2 \sqrt{-1} \alpha(a_\alpha)}{\tan(2 \sqrt{-1} \alpha(w))} g_s Y_\alpha.
$$

Take $\tilde{b} \in \alpha^{-1}(0)$. Since $\tilde{Q}_a^c(b)$ is totally geodesic and $T_x^c(\tilde{Q}_a^c(b) \cap M^c \cap T_x^c \tilde{H}_a^c \cdot x)$ is parallel
along $\tilde{H}_a^c \cdot x$ with respect to the normal connection of $\tilde{Q}_a^c(b) \leftrightarrow G^c/H^c$, we have

$$
\tilde{A}_{g,a} \alpha g_s Y_\alpha = \tilde{A}_{g,a} \alpha g_s Y_\alpha = 0.
$$

Take an arbitrary $a \in a$. We can express as $a = \frac{\alpha(a)}{\alpha(c_0)} a_\alpha + \tilde{b}$ for some $\tilde{b} \in \alpha^{-1}(0)$. Thus,
for each $a \in a$, we have

$$
\tilde{A}_{g,a} \alpha g_s Y_\alpha = -\frac{\sqrt{-1} \beta(a)}{\tan(\sqrt{-1} \beta(w))} \text{id} \quad (\beta \in \Delta_+ \text{ s.t. } \beta(w) \notin \sqrt{-1} \pi Z).
$$
Take an arbitrary \( h_{xx}g_\ast a \in H_x(g_\ast a) \) \((a \in a, \ h \in H_x)\). Since \( h \) is an isometry of \( G^c/H^c \), we have \( \hat{A}_{h_{xx}g_\ast a} = h_{xx} \circ \hat{A}_{g_\ast a} \circ h_{xx}^{-1} \). Hence we have

\[
\hat{A}_{h_{xx}g_\ast a}|_{h_{xx}g_\ast q^0} = -\frac{\sqrt{-1}}{\tan(\sqrt{-1})} \id \ (\beta \in \Delta_+ \ s.t. \ \beta(w) \notin \sqrt{-1}\pi \mathbb{Z}).
\]

Therefore, we obtain the relation in the statement (ii). q.e.d.

4 Shape operators of partial tubes

In this section, we investigate the shape operators of partial tubes over a pseudo-Riemannian submanifold with section in a (semi-simple) pseudo-Riemannian symmetric space \( G/H \) equipped with the metric induced from the Killing form of \( g := \text{Lie} \ G \). Let \( M \) be a pseudo-Riemannian submanifold with section in \( G/H \), that is, for each \( x = gH \) of \( M \), \( g_{\ast}^{-1}T^\perp_xM \) is a Lie triple system. Let \( t(M) \) be a connected submanifold in the normal bundle \( T^\perp M \) of \( M \) such that, for any curve \( c : [0,1] \to M \), \( P_c^\perp (t(M) \cap T^\perp_{c(0)}M) = t(M) \cap T^\perp_{c(1)}M \) holds, where \( P^\perp \) is the parallel transport along \( c \) with respect to the normal connection. Denote by \( F \) the set of all critical points of the normal exponential map \( \exp^\perp \) of \( M \). Assume that \( t(M) \cap F = \emptyset \). Then the restriction \( \exp^\perp|_{t(M)} \) of \( \exp^\perp \) to \( t(M) \) is an immersion of \( t(M) \) into \( G/H \). Assume that \( \exp^\perp|_{t(M)} : t(M) \hookrightarrow G/H \) is a pseudo-Riemannian submanifold. Then we call \( t(M) \) a partial tube over \( M \). Define a distribution \( D^V \) on \( t(M) \) by \( D^V_v = T_v(t(M) \cap T^\perp_{\pi(v)}M) \ (v \in t(M)) \), where \( \pi \) is the bundle projection of \( T^\perp M \). We call this distribution a vertical distribution on \( t(M) \). Let \( X \in T_{\pi(v)}M \). Take a curve \( c \) in \( M \) with \( \bar{c}(0) = X \). Let \( \widetilde{\gamma} \) be a parallel normal vector field along \( c \) with \( \widetilde{\gamma}(0) = v \). Denote by \( \bar{X} \) the velocity vector \( \bar{\gamma}(0) \) of the curve \( \bar{\gamma} \) in \( T^\perp M \) at 0. We call \( \bar{X} \) the horizontal lift of \( X \) to \( v \). Define a distribution \( D^H \) on \( t(M) \) by \( D^H_v = \{ \bar{X} \in T_{\pi(v)}M \} \ (v \in t(M)) \). We call this distribution a horizontal distribution on \( t(M) \). From (2.1), we have

\[
\exp^\perp(\bar{X}) = P_{\gamma_v} (D^H_v X - D^H_v(\gamma_vX)).
\]

Assume that \( t(M) \) is contained in the \( \varepsilon \)-tube \( t_{\varepsilon}(M) := \{ v \in T^\perp M \mid \sqrt{\langle v,v \rangle} = \varepsilon \} \) \((\varepsilon \neq 0)\).

Define a subbundle \( D^\perp \) of the normal bundle \( T^\perp t(M) \) of \( t(M) \) by \( D^\perp_v := T^\perp_v t(M) \cap T_v(t_{\varepsilon}(M)) \ (v \in t(M)) \). Clearly we have \( T_v t(M) = D^H_v \oplus D^V_v \) (orthogonal direct sum) and \( T^\perp_v t(M) = D^\perp_v \oplus \text{Span}\{\bar{\gamma}_v(1)\} \) (orthogonal direct sum), where \( \bar{\gamma}_v \) is defined by \( \bar{\gamma}_v(t) := tv \). Denote by \( A \) (resp. \( A^\perp \)) the shape tensor of \( M \) (resp. \( t(M) \)). Also, denote by \( A^\varepsilon \) that of a submanifold \( t(M) \cap T^\perp_xM \) in \( \exp^\perp(T^\perp_xM) \) immersed by \( \exp^\perp|_{t(M) \cap T^\perp_xM} \). In the sequel, we omit \( \exp^\perp \). For a real analytic function \( F \) and \( v \in T_{gH}(G/H) \), we denote the operator \( g_\ast \circ F(\text{ad}(g_\ast^{-1}v)) \circ g_\ast^{-1} \) by \( F(\text{ad}(v)) \) for simplicity. Then, by imitating the proof of Proposition 3.1 in [19], we can show the the following relations.

**Proposition 4.1.** Let \( v \in t(M) \) and \( w \in D^\perp_v \). Also, let \( \pi(v) = g_1H \), \( g_2 := \exp_G(g_1^{-1}v) \) and \( g := g_1g_2g_1^{-1} \), where \( \exp_G \) is the exponential map of the Lie group \( G \).

(i) For \( Y \in D^\perp_v \), we have

\[
A^\perp_{g_\ast v} Y = A^\pi(v)_{g_\ast v} Y, \quad A^\perp_w Y = A^\pi(v)_w Y.
\]
(ii) Assume that $\text{Span}\{g_*^{-1}v, (g_1g_2)_*^{-1}w\}$ is abelian. Then, for $X \in T_{\pi(v)}M$, we have
\begin{align}
A^t_u\tilde{v} = \sqrt{-1}\text{ad}(g_1^{-1}w)\frac{\text{sin}(\sqrt{-1}\text{ad}(v))}{\sqrt{-1}\text{sin}(\sqrt{-1}\text{ad}(v))}(A_{g^{-1}w}X) \\
+ \left(\frac{\text{ad}(v)}{\text{ad}(v)} - \text{id} + \frac{\sqrt{-1}\text{sin}(\sqrt{-1}\text{ad}(v)) + \text{ad}(v)}{\text{ad}(v)^2}\right)\text{x ad}(g_*^{-1}w)(A_{v}X) .
\end{align}

Remark 4.1. The parallel translation $P_{\gamma_v}$ along $\gamma_v$ is equal to $g_*$.  

5 Proper complex equifocality

In this section, we investigate the proper complex equifocality of a complex equifocal submanifold in a pseudo-Riemannian symmetric space. Let $G/H$ be a (semi-simple) pseudo-Riemannian symmetric space and $R$ be the curvature tensor of $G/H$. First we prepare the following lemma for a curvature-adapted submanifold with flat section such that the normal holonomy group is trivial.

Lemma 5.1. Let $M$ be a curvature-adapted submanifold in $G/H$ with flat section such that the normal holonomy group is trivial. Assume that, for any normal vector $v$ of $M$, $A_v$ and $\text{ad}(g_*^{-1}v)$ are semi-simple, where $A$ is the shape tensor of $M$ and $g$ is an element of $G$ such that $gH$ is the base point of $v$. Then, for any $x \in M$, $\{A_v \mid v \in T_x^\perp M\} \cup \{R(\cdot, v)v|_{T_xM} \mid v \in T_x^\perp M\}$ is a commuting family of linear transformations of $T_xM$.

Proof. Let $v_i \in T_x^\perp M$ ($i = 1, 2$). Since $M$ has flat section, $R(\cdot, v_1)v|_{T_xM}$ and $R(\cdot, v_2)v|_{T_xM}$ commute with each other. Since $M$ has flat section and the normal holonomy group is trivial, $A_{v_1}$ and $A_{v_2}$ commute with each other. In the sequel, we shall show that $R(\cdot, v_1)v|_{T_xM}$ and $A_{v_2}$ commute with each other. Let $x = gH$. Since $g_*^{-1}T_x^\perp M$ is abelian and, for any $v \in T_x^\perp M$, $\text{ad}(g_*^{-1}v)$ is semi-simple, there exists a Cartan subspace $\mathfrak{a}$ of $\mathfrak{q} (= T_eH(G/H))$ containing $\mathfrak{b} := g_*^{-1}(T_x^\perp M)$. Let $\triangle$ be the root system with respect to $\mathfrak{a}$ and set $\overline{\triangle} := \{\alpha|_{\mathfrak{b}} \mid \alpha \in \triangle \text{ s.t. } \alpha|_{\mathfrak{b}} \neq 0\}$. For each $\beta \in \overline{\triangle}$, we set $\mathfrak{q}_\beta \triangleq \{X \in \mathfrak{q} \mid \text{ad}(b)^2(X) = \beta(b)^2X \ (\forall b \in \mathfrak{b})\}$. Then we have $\mathfrak{q}^c = \mathfrak{g}^c(\mathfrak{b}^c) + \sum_{\beta \in \overline{\triangle}_+} \mathfrak{q}_\beta^c$, where $\overline{\triangle}_+$ is the positive root system under some lexicographical ordering and $\mathfrak{g}^c(\mathfrak{b}^c)$ is the centralizer of $\mathfrak{b}^c$ in $\mathfrak{q}^c$. Consider
\begin{align}
D := \{v \in (T_x^\perp M)^c \mid \text{Span}\{g_*^{-1}v\} \cap \bigcup_{(\beta_1, \beta_2) \in \overline{\triangle}_+ \times \overline{\triangle}_+ \ s.t. \beta_1 \neq \beta_2} (l_{\beta_1} \cap l_{\beta_2}) = \emptyset\},
\end{align}
where $l_{\beta} := \beta_*^{-1}(1)$ ($i = 1, 2$). It is clear that $D$ is open and dense in $(T_x^\perp M)^c$. Take $v \in D$. Since $\beta(g_*^{-1}v)$’s ($\beta \in \overline{\triangle}_+$) are mutually distinct, the decomposition $(T_xM)^c = g_*($$\mathfrak{g}^c(\mathfrak{b}^c) \oplus \mathfrak{b}^c$) + $\sum_{\beta \in \overline{\triangle}_+} g_*\mathfrak{q}_\beta^c$ is the eigenspace decomposition of $R^c(\cdot, v)v|_{(T_xM)^c}$. Since $M$
is curvature-adapted and hence \( R^{c}(\cdot, v)v|_{(T_xM)^{c}}^{c}, A_{v}^{c} = 0 \), we have

\[
(T_xM)^{c} = \sum_{\lambda \in \text{Spec } A_{v}^{c}} \left( g_{*}(\mathfrak{T}^{c}(b^{c}) \oplus b^{c}) \cap \text{Ker}(A_{v}^{c} - \lambda \text{id}) \right) \\
+ \sum_{\beta \in \Delta_{+}} g_{*}q_{\beta} \cap \text{Ker}(A_{v}^{c} - \lambda \text{id})).
\]

(5.1)

Suppose that (5.1) does not hold for some \( v_0 \in (T_x^\perp M)^{c} \setminus D \). Then it is easy to show that there exists a neighborhood \( U \) of \( v_0 \) in \((T_x^\perp M)^{c}\) such that (5.1) does not hold for any \( v \in U \). Clearly we have \( U \cap D = \emptyset \). This contradicts the fact that \( D \) is dense in \((T_x^\perp M)^{c}\). Hence (5.1) holds for any \( v \in (T_x^\perp M)^{c} \setminus D \). Therefore, (5.1) holds for any \( v \in (T_x^\perp M)^{c} \). In particular, (5.1) holds for \( v_2 \). On the other hand, the decomposition \((T_xM)^{c} = g_{*}(\mathfrak{T}^{c}(b^{c}) \oplus b^{c}) + \sum_{\beta \in \Delta_{+}} g_{*}q_{\beta}\) is the common eigenspace decomposition of \( R^{c}(\cdot, v)v|_{(T_xM)^{c}}^{c} \)'s \((v \in (T_x^\perp M)^{c})\). From these facts, we have

\[
(T_xM)^{c} = \sum_{\lambda \in \text{Spec } A_{v}^{c}} \sum_{\mu \in \text{Spec } R^{c}(\cdot, v_1|_{(T_xM)^{c}})} \left( \text{Ker}(R^{c}(\cdot, v_1|_{(T_xM)^{c}} - \mu \text{id}) \cap \text{Ker}(A_{v_2}^{c} - \lambda \text{id}) \right),
\]

which implies that \( R^{c}(\cdot, v_1|_{(T_xM)^{c}}) \) and \( A_{v_2}^{c} \) commute with each other. This completes the proof.

q.e.d.

By this lemma, Lemma 5.3, Propositions 5.6 and 5.7 of [17] (these lemmas are valid even if the ambient space is a pseudo-Riemannian symmetric space), we can show the following fact.

**Proposition 5.2.** Let \( M \) be a curvature-adapted complex equifocal submanifold in \( G/H \). Assume that, for any normal vector \( v \) of \( M \), \( A_{v} \) and \( \text{ad}(g_{v}^{-1}v) \) are semi-simple and that \( \pm \beta(g_{v}^{-1}v) \notin \text{Spec } A_{v}^{c}|_{g_{v}q_{\beta}} \) (\( \beta \in \Delta_{+} \)), where \( g \) is an element of \( G \) such that \( gH \) is the base point of \( v \). Then \( M \) is proper complex equifocal.

**Proof.** Let \( \widetilde{M} := (\pi \circ \phi)^{-1}(M) \) and denote by \( \widetilde{A} \) the shape tensor of \( \widetilde{M} \). Fix \( u \in \widetilde{M} \) and \( v \in T_{u}^\perp \widetilde{M} \). For simplicity, set \( x(= gH) = (\pi \circ \phi)(u) \) and \( \tau := (\pi \circ \phi)^{*}(v) \). According to Lemma 5.1, it follows from the assumptions that \( A_{v}^{c} \) commutes with \( R^{c}(\cdot, w|_{(T_xM)^{c}}) \)'s \((w \in (T_x^\perp M)^{c})\). Also, it follows from the assumptions that \( A_{v}^{c} \) and \( R^{c}(\cdot, w|_{T_xM}) \)'s \((w \in (T_x^\perp M)^{c})\) are diagonalizable. Hence they are simultaneously diagonalizable, that is, we have

\[
(T_x^\perp M)^{c} = \sum_{\lambda \in \text{Spec } A_{v}^{c}} \sum_{\beta \in \Delta_{+}} (g_{*}q_{\beta} \cap \text{Ker}(A_{v}^{c} - \lambda \text{id})).
\]

On the other hand, by the assumption, we have \( \pm \beta(g_{v}^{-1}\tau) \notin \text{Spec } A_{v}^{c}|_{g_{*}q_{\beta}} \) for each \( \beta \in \Delta_{+} \). Therefore, it follows from Lemma 5.3, Proposition 5.6 and 5.7 of [17] that there exists a pseudo-orthonormal base of \((T_v\widetilde{M})^{c}\) consisting of eigenvectors of \( \widetilde{A}_{v}^{c} \). Therefore \( \widetilde{M} \) is proper complex isoparametric, that is, \( M \) is proper complex equifocal.

q.e.d.
Proof of Theorem A. Since \( T_{eH}(H'(eH)) = q \cap h' \) and \( q \cap h' \) is a non-degenerate subspace of \( q \), we see that \( H'(eH) \) is a pseudo-Riemannian submanifold. Since \( \sigma \circ \sigma' = \sigma' \circ \sigma \), we can show that \( H'(eH) \) is a reflective submanifold by imitating the first-half part of the proof of Lemma 4.2 in [19]. Thus the first-half part of the statement (i) is shown. Furthermore, by imitating the second-half part of the proof of Lemma 4.2 in [19], we can show the second-half part of the statement (i). In the sequel, we shall show the statement (ii). Let \( M \) be a principal orbit of the \( H' \)-action as in the statement (ii). For simplicity, set \( x := \exp_G(w)H \) and \( g := \exp_G(w) \), where \( w \) is as in the statement (ii). By imitating the second-half part of the proof of Lemma 4.2 in [17], it is shown that \( M \) is a partial tube over \( H'(eH) \) and \( M \cap \Sigma_{eH} \) is an orbit of the isotropy action of the symmetric space \( \Sigma_{eH} (\cong L/H \cap H') \). Since \( M \) is a principal orbit, \( M \cap \Sigma_{eH} \) is a principal orbit of the isotropy action. Hence, since \( \text{ad}(w)|_k \) is semi-simple, \( b := g_s^{-1}T_x 1 \) is a Cartan subspace of \( q \cap q' \) by (i) of Proposition 3.2. Take a Cartan subspace \( a \) of \( q \) containing \( b \). Let \( q^c = a^c + \sum_{\alpha \in \Delta_+} q^c_{\alpha} \) be the root space decomposition with respect to \( a^c \). Set \( \Delta_{\text{aff}} := \{ \alpha|_{\text{aff}} \mid \alpha \in \Delta, \alpha|_{\text{aff}} \neq 0 \} \) and \( q^c_{\beta} := \sum_{\alpha \in \Delta, \alpha|_{\text{aff}} = \beta} q^c_{\alpha} (\beta \in \Delta_{\text{aff}}) \). Then we have \( q^c = 3q^c(b^c) + \sum_{\beta \in (\Delta_{\text{aff}})^+} q^c_{\beta}, \) where \( (\Delta_{\text{aff}})^+ \) is the positive root system under some lexicographical ordering. Also, since \( q^c \cap h'^c \) and \( q^c \cap q'^c \) are \( \text{ad}(b)^2 \)-invariant for any \( b \in b^c \), we have \( q^c \cap h'^c = 3q^c(b^c) \cap h'^c + \sum_{\beta \in (\Delta_{\text{aff}})^+} (q^c_{\beta} \cap h'^c) \) and \( q^c \cap q'^c = b^c + \sum_{\beta \in (\Delta_{\text{aff}})^+} (q^c_{\beta} \cap q'^c) \). Hence we have

\[
(T_x M)^c = g_x^c(3q^c(b^c) \cap h'^c) + \sum_{\beta \in (\Delta_{\text{aff}})^+} (g_x^c(q^c_{\beta} \cap h'^c) + g_x^c(q^c_{\beta} \cap q'^c)),
\]

\[
(T_{eH}(H'(eH)))^c = 3q^c(b^c) \cap h'^c + \sum_{\beta \in (\Delta_{\text{aff}})^+} (q^c_{\beta} \cap h'^c)
\]

and

\[
(T_x (M \cap \Sigma_{eH}))^c = \sum_{\beta \in (\Delta_{\text{aff}})^+} g_x^c(q^c_{\beta} \cap q'^c).
\]

Also we have \( T_{x}^\perp = g_x b \). Take \( v \in T_x^\perp \) \( M = g_x b \). It is clear that \( R(\cdot, v)w \) is semi-simple. Since \( H'(eH) \) is totally geodesic, it follows from (ii) of Proposition 4.1 and (4.1) that \( A_v^c \tilde{X}_w = 0 \) (\( X \in 3q^c(b^c) \cap h'^c) \) and

\[
A_v^c \tilde{X}_w = \sqrt{-1} \beta(g_s^{-1}v) \tan(\sqrt{-1} \beta(w)) \tilde{X}_w \quad (X \in q^c_{\beta} \cap h'^c (\beta \in (\Delta_{\text{aff}})^+)).
\]

Also, since \( M \cap \Sigma_{eH} \) is a principal orbit of the isotropy action of \( \Sigma_{eH} (\cong L/H \cap K) \), it follows from Proposition 3.2 and (i) of Proposition 4.1 that

\[
A_v^c Y = \frac{\sqrt{-1} \beta(g_s^{-1}v)}{\tan(\sqrt{-1} \beta(w))} Y \quad (Y \in g_s(q^c_{\beta} \cap q'^c))
\]

up to constant-multiple, where we note that the induced metric on \( \Sigma_{eH} (\cong L/H \cap K) \) is homothetic to the metric induced from the Killing form of \( f \). Thus \( A_v^c \) is diagonalizable, that is, it is semi-simple and we have
adapted. Next we shall show that $M$ is proper complex equifocal. Since $g^{-1}_x T^x M$ is a Cartan subspace of $q \cap \gamma'$ for each $x = qH \in M$, $M$ has flat section. Since $M$ is a principal orbit of the $H'$-action, each normal vector of $M$ extend to an $H'$-equivariant normal vector field, which is parallel with respect to the normal connection of $M$ has flat section. From this fact, it follows that the normal holonomy group of $M$ is trivial. Furthermore, it follows from the homogeneity of $M$ that $M$ is complex equifocal. From (6.1) and (6.2), we have $\Spec(A^c_{\gamma} | g \in q \{0\}) \subset \{ \sqrt{-1} \beta (g^{-1}_x v) \tan(\sqrt{-1} \beta (w)), -\sqrt{-1} \beta (g^{-1}_x v) \tan(\sqrt{-1} \beta (w)) \} (\beta \in (\Delta q)^+)$, that is, $\pm \beta (g^{-1}_x v) \notin \Spec(A^c_{\gamma} | g \in q \{0\})$. Therefore, it follows from Proposition 5.2 that $M$ is proper complex equifocal. Furthermore it follows from the result of [23] stated in Introduction that $M$ is an isoparametric submanifold with flat section. This completes the proof.

q.e.d.

Next we prove Theorem C.

**Proof of Theorem C.** According to Theorem A, we have only to show that $K(eH)$ has no focal point and that, for any normal vector $v$ of $M_1$, $R(\cdot, v)|_{T^1_1 M_1}$ and $A_v$ are diagonalizable. Let $q = f + p$ be the Cartan decomposition of $q$ associated with $\theta$. Take an arbitrary normal vector $v$ of $K(eH)$ at $eH$. Take a maximal abelian subspace $b$ of $q \cap p$ containing $v$ and a Cartan subspace $a$ of $q$ containing $b$. Let $q^c = a^c + \sum_{a \in \Delta_a} q_a^c$ be the root space decomposition of $q^c$ with respect to $a^c$. Let $\Delta_b := \{a|_b | a \in \Delta \text{ s.t. } a|_b \neq 0 \}$ and $q_\beta := (\sum_{a \in \Delta \text{ s.t. } a|_b = \beta} q_a^c) \cap q (\beta \in \Delta_b)$. Since $b \subset p$, we have $\beta(b) \subset \R (\beta \in \Delta_b)$ (see Lemma 3.1) and hence $q = 3q(b) + \sum_{\beta \in (\Delta_b)^+} q_\beta$. Furthermore, since $\text{ad}(b)^2(q \cap f) \subset q \cap f$ for any $b \in b$, we have $q \cap f = 3q(b) \cap f + \sum_{\beta \in (\Delta_b)^+} (q_\beta \cap f)$. Let $X \in q_\beta \cap f (\beta \in (\Delta_b)^+)$, $Y$ be the strongly $K(eH)$-Jacobi field along $\gamma_w$ with $Y(0) = X$. Since $K(eH)$ is totally geodesic, we have $Y(s) = \cos(h(s) \beta(v)) P_{\gamma_v|q, a}(X)$. Since $\beta(v)$ is a real number, $Y$ has no zero point. Also any strongly $K(eH)$-Jacobi field $\tilde{Y}$ along $\gamma_w$ with $\tilde{Y}(0) \in 3q(b) \cap f$ is expressed as $\tilde{Y}(s) = P_{\gamma_v|q, a}(\tilde{Y}(0))$ and hence it has no zero point. On the other hand, since $K(eH)$ is reflective and hence it has section, any non-strongly $K(eH)$-Jacobi field along $\gamma_w$ has no zero point. After all there exists no focal point of $K(eH)$ along $\gamma_w$. From the arbitrariness of $v$, it follows that $K(eH)$ has no focal point. For convenience, set $H_1 := K, H_2 := L, b_1 := f, b_2 := L, q_1 := p$ and $q_2 := f \cap q + p \cap f$. Let $M_1$ (resp. $M_2$) be a principal orbit of the $H_1$-action (resp. the $H_2$-action) through $x_1 = \exp_{G}(w_1)H \in H_1(eH)$ ($w_1 \in q \cap q_1$) (resp. $x_2 = \exp_{G}(w_2)H \in H_1(eH) \setminus F (w_2 \in q \cap q_2)$). Set $g_i := \exp_{G}(w_i)$ ($i = 1, 2$). Since $b_1 := g_1^{-1}(T_{\gamma_1}^1 M_1)$ and $b_2 := g_2^{-1}(T_{\gamma_2}^2 M_2)$ are maximal abelian subspaces of $q \cap p$ and $q \cap f$, respectively, they are maximal split abelian subspaces of $q$. Hence we have the root space decomposition $q = 3q(b_i) + \sum_{\beta \in \Delta_+^i} q_\beta$ of $q$ with respect to $b_i$ ($i = 1, 2$), where $q_\beta := \{X \in q | \text{ad}(b)^2(X) = (-1)^{i-1} \beta(b) X \ (\forall b \in b_i) \}$ by Lemma 3.1 and $\Delta_+^i$ is the positive root system of $\Delta^i := \{\beta \in b_1^* | q_\beta \neq \{0\} \}$ with respect to a lexicographical ordering. Also, it is easy to show that $q \cap q_1 = 3q(b_i) \cap q_1 + \sum_{\beta \in \Delta_+^i} (q_\beta \cap q_1)$.
and $q \cap q_i = b_i + \sum_{\beta \in \Delta^i_+} (q_\beta \cap q_i)$, where $i = 1, 2$. Hence we have

$$T_{x_i} M_i = g_{i*}(\mathfrak{z}_q(b_i) \cap \mathfrak{h}_i) + \sum_{\beta \in \Delta^i_+} (g_{i*}(q_\beta \cap \mathfrak{h}_i) + g_{i*}(q_\beta \cap q_i)),$$

and

$$T_{eH}(H_i(eH)) = \mathfrak{z}_q(b_i) \cap \mathfrak{h}_i + \sum_{\beta \in \Delta^i_+} (q_\beta \cap \mathfrak{h}_i)$$

where $\Sigma_{eH}$ is the section of $H_i(eH)$ through $eH$. Take $v_i \in T_{x_i} M_i = g_{i*} b_i$. It is clear that $R(\cdot, v_i) v_i$ is diagonalizable. Denote by $A^i$ the shape tensor of $M_i$. By using Propositions 3.2, 4.1 and (4.1), we can show $A^i v_i \tilde{X}_{w_i} = 0$ $(X \in \mathfrak{z}_q(b_i) \cap \mathfrak{h}_i)$,

$$A^i v_i \tilde{X}_{w_i} = \sqrt{-1} i \beta (g_{i*}^{-1} v_i) \tan(\sqrt{-1} i \beta (w_i)) \tilde{X}_{w_i} \quad (X \in q_\beta \cap \mathfrak{h}_i \ (\beta \in \Delta^i_+))$$

and

$$A^i_{v_i} Y = -\frac{\sqrt{-1} i \beta (g_{i*}^{-1} v_i)}{\tan(\sqrt{-1} i \beta (w_i))} Y \quad (Y \in g_{i*}(q_\beta \cap q_i) \ (\beta \in \Delta^i_+)).$$

Thus $A^i_{v_i}$ is diagonalizable. This completes the proof. q.e.d.

Next we shall prove Theorem E. By imitating the proof of Lemma 2.1 of [21], we can show the following fact.

**Lemma 6.1.** Let $G = (G \times G)/\Delta G$ be a semi-simple Lie group equipped with the bi-invariant pseudo-Riemannian metric induced from the Killing form of $\mathfrak{g} + \mathfrak{g}$, $H'$ be a closed subgroup of $G \times G$ and $\mathfrak{a}$ be an abelian subspace of the normal space $T_{eH}^+ (H' \cdot e)$ of $H' \cdot e$. Set $\Sigma := \exp_G(\mathfrak{a})$. Then all $H'$-orbits through $\Sigma$ meet $\Sigma$ orthogonally.

By using this lemma and imitating the proof of Lemma 2.2 of [21], we can show the following fact.

**Lemma 6.2.** Let $G/H$ be a semi-simple pseudo-Riemannian symmetric space, $H'$ be a closed subgroup of $G$ and $\mathfrak{a}$ be an abelian subspace of the normal space $T_{eH}^+ (H' \cdot eH)$ of $H'(eH)$. Set $\Sigma := \text{Exp}(\mathfrak{a})$. Then all $H'$-orbits through $\Sigma$ meet $\Sigma$ orthogonally.

By using this lemma, we prove Theorem E.

**Proof of Theorem E.** Let $M$, $F$ and $G/H$ be as in the statement of Theorem E. Without loss of generality, we may assume that $G$ is simply connected. Since $M$ is homogeneous, there exists a closed subgroup $H_1$ of $G$ having $M$ as an orbit. Without loss of generality, we may assume that $H_1(eH) = M$. Set $\Sigma := \text{Exp}(T_{eH}^+M)$. Since $M$ has flat section, that is, $T_{eH}^+M$ is abelian, it follows from Lemma 6.2 that all $H_1$-orbits through $\Sigma$ meet $\Sigma$ orthogonally. Hence their dimensions are lower than $\dim M + 1$. This fact together
with \( \dim M + \dim \Sigma = \dim G/H \) implies that all \( H_1 \)-orbits through \( W \) are of the same dimension as \( \dim M \) for some neighborhood \( W \) of \( eH \) in \( \Sigma \) and they are principal orbits. Set \( U := H_1 \cdot W \), which is an open set of \( G/H \). Fix \( g_0 H \in F \). Set \( H_2 := g_0^{-1} H_1 g_0, t := T \circ H g_0^{-1} F \) and \( t^+ := T \circ H g_0^{-1} F \). Furthermore set \( \mathfrak{h}' := n_0(t) + t \) and \( \mathfrak{q}' := (\mathfrak{h} \oplus n_0(t)) + t^+ \). Since \( n_0(t) \) is a non-degenerate subspace of \( \mathfrak{h} \) by the assumption, we have \( \mathfrak{g} = \mathfrak{h}' \oplus \mathfrak{q}' \) (orthogonal direct sum). Since \( F \) is a reflective by the assumption, \( t \) and \( t^+ \) are Lie triple systems. By using this fact, we can show \( [\mathfrak{h}', \mathfrak{h}] \subset \mathfrak{h}', [\mathfrak{h}', \mathfrak{q}'] \subset \mathfrak{q}' \) and \( [\mathfrak{q}', \mathfrak{q}'] \subset \mathfrak{h}' \). Thus the connected subgroup \( H' \) of \( G \) having \( \mathfrak{h}' \) as its Lie algebra is symmetric, where we use the simply connectedness of \( G \). That is, the \( H' \)-action on \( G/H \) is a Hermann type action. Easily we can show \( T(e((H_2 \times H) \cdot e)) = \text{pr}_q(b_2) + \mathfrak{h} \) and \( T(e((H' \times H) \cdot e)) = \text{pr}_q(h') + \mathfrak{h} = t + \mathfrak{h} \), where \( \text{pr}_q \) is the orthogonal projection of \( \mathfrak{g} \) onto \( \mathfrak{q} \) and \( b_2 := \text{Lie} H_2 \). Since \( \pi^{-1}(H_2(eH)) = (H_2 \times H) \cdot e \), we have \( T(e(H_2(eH))) = \text{pr}_q(T(e((H_2 \times H) \cdot e))) = \text{pr}_q(b_2) \), that is, \( \text{pr}_q(b_2) = t \). Hence we have \( T(e((H' \times H) \cdot e)) = T(e((H_2 \times H) \cdot e)) \), which implies \( (H' \times H) \cdot e = (H_2 \times H) \cdot e \). Therefore we have \( H'(eH) = H_2(eH) \). Set \( \Sigma' := \text{Exp}(T_{g_0^{-1}}(g_0^{-1} M)) \), which passes through \( eH \). Set \( \mathfrak{a}' := T_{eH} \Sigma' \), which is abelian. Since \( T_{eH}(H'(eH)) = T_{eH}(H_2(eH)) \) includes \( \mathfrak{a}' \), it follows from Lemma 6.2 that all \( H' \)-orbits and all \( H_2 \)-orbits through \( \Sigma' \) meet \( \Sigma' \) orthogonally. Since all \( H_2 \)-orbits through \( g_0^{-1} W(\subset \Sigma') \) are principal and hence \( T_{g_0^{-1}}(H_2(gH)) = g_0 H \cdot \Sigma' \) for all \( gH \in g_0^{-1} W \), we have \( T_{gH}(H'(gH)) \subset T_{gH}(H_2(gH)) \) for all \( gH \in g_0^{-1} W \). On the other hand, we have \( \text{pr}_H(b_2) \subset \text{pr}_H(T_{eH}(H_2 \times H) \cdot e)) = T_{eH}(H_2(eH)) = t \), that is, \( \text{pr}_H(b_2) \subset n_0(t) \), where \( \text{pr}_H \) is the orthogonal projection of \( \mathfrak{g} \) onto \( \mathfrak{h} \). Hence we have \( b_2 \subset T_{eH}(H_2(eH)) \subset \mathfrak{h} \), that is, \( H_2 \subset H' \). Therefore we see that \( H'(gH) = H_2(gH) \) for all \( gH \in g_0^{-1} W \). In particular, \( g_0^{-1} M \) is a principal orbit of the \( H' \)-action. Hence \( M \) is a principal orbit of the Hermann type action \( g_0 H' g_0^{-1} \). This completes the proof.

q.e.d.

### 7 Cohomogeneities of special Hermann type actions

In this section, we shall list up the cohomogeneities of the \( K \)-action and the \( L \)-action as in Theorem C on irreducible (semi-simple) pseudo-Riemannian symmetric spaces \( G/H \) in terms of the fact that the cohomogeneity of the \( K \)-action (resp. \( L \)-action) is equal to the rank of \( L/H \cap K \) (resp. \( K/H \cap K \)). In Tables 1 \~ 5, \( A \cdot B \) denotes \( A \times B/\Pi \), where \( \Pi \) is the discrete center of \( A \times B \). The symbol \( SO_0(1,8) \) in Table 6 denotes the universal covering of \( SO_0(1,8) \) and the symbol \( \alpha \) in Table 6 denotes an outer automorphism of \( G^2_2 \).
| $G/H$ | $K$ | $L$ |
|---|---|---|
| $SL(n, \mathbb{R})/SO_0(p, n-p)$ | $SO(n)$ | $(SL(p, \mathbb{R}) \times SL(n-p, \mathbb{R})) \cdot \mathbb{R}$ |
| | $n-1$ | $p$ |
| $SL(n, \mathbb{R})/SL(n, \mathbb{C}) \cdot U(1)$ | $SO(n)$ | $SO_0(p, n-p)$ |
| | $p$ | $p$ |
| $SU^*(2n)/SO^*(2n)$ | $Sp(n)$ | $SL(n, \mathbb{C}) \cdot U(1)$ |
| | $n-1$ | $n$ |
| $SU^*(2n)/SU^*(2n) \cdot U(1)$ | $Sp(n)$ | $SO^*(2n)$ |
| | $p$ | $p$ |
| $SU^*(2n)/(SU^*(2p) \times SU^*(2n-2p) \times U(1))$ | $Sp(n)$ | $SU^*(2p) \times SU^*(2n-2p) \times U(1)$ |
| | $n-1$ | $p$ |
| $SU(p, q)/SO_0(p, q)$ | $S(U(p) \times U(q))$ | $SO_0(p, q)$ |
| | $p$ | $n-1$ |
| $SU(p, p)/SO^*(2p)$ | $S(U(p) \times U(p))$ | $Sp(p, \mathbb{R})$ |
| | $p$ | $p-1$ |
| $SU(p, p)/Sp(p, \mathbb{R})$ | $S(U(p) \times U(p))$ | $SO^*(2p)$ |
| | $p$ | $p-1$ |
| $SU(p, p)/SL(p, \mathbb{C}) \cdot U(1)$ | $S(U(p) \times U(p))$ | $SL(p, \mathbb{C}) \cdot U(1)$ |
| | $p$ | $p-1$ |
| $SU(2p, 2q)/Sp(p, q)$ | $S(U(2p) \times U(2q))$ | $Sp(p, q)$ |
| | $p$ | $n-1$ |
| $SU(p, q)/SU(q) \times U(p-i, q-j)$ | $S(U(p) \times U(q))$ | $SU(p-i, q) \times U(q-j)$ |
| | $\min\{p-i, j\}$ | $\min\{i, p-i\} + \min\{j, q-j\}$ |

Table 1.
| $G/H$ | $K$ | $L$ |
|------|-----|-----|
| $SL(n, C)/SO(n, C)$ | $SU(n)$ | $SL(n, R)$ |
| $n - 1$ | $n - 1$ | |
| $SL(n, C)/SL(n, R)$ | $SU(n)$ | $SO(n, C)$ |
| $[\frac{n}{2}]$ | $n - 1$ | |
| $SL(n, C)/(SL(p, C) \times SL(n - p, C) \times U(1))$ | $SU(n)$ | $SU(p, n - p)$ |
| $(p \leq \frac{n}{2})$ | $p$ | $p$ |
| $SL(n, C)/SU(p, n - p) \ (p \leq \frac{n}{2})$ | $SU(n)$ | $SL(p, C) \times SL(n - p, C) \times U(1)$ |
| $n - 2$ | $p$ | |
| $SL(2n, C)/Sp(n, C)$ | $SU(2n)$ | $SU^*(2n)$ |
| $n - 1$ | $n - 1$ | |
| $SL(2n, C)/SU^*(2n)$ | $SU(2n)$ | $Sp(n, C)$ |
| $n$ | $n - 1$ | |
| $SO_o(p, q)/SO_o(i, j) \times SO_o(p - i, q - j)$ | $SO_o(p) \times SO(q)$ | $SO_o(p - i, j) \times SO_o(i, q - j)$ |
| $\min\{p - i, j\}$ | $\min\{i, p - i\}$ | $\min\{i, q - j\}$ | $\min\{j, q - j\}$ |
| $SO_o(p, p)/SO_o(p, C)$ | $SO_o(p) \times SO(p)$ | $SL(p, R) \cdot U(1)$ |
| $p$ | $[\frac{n}{2}]$ | |
| $SO_o(p, p)/SL(p, R) \cdot U(1)$ | $SO_o(p) \times SO(p)$ | $SO_o(p, C)$ |
| $[\frac{n}{2}]$ | $[\frac{n}{2}]$ | |
| $SO_o(2p, 2q)/SU(p, q) \cdot U(1) \ (p \leq q)$ | $SO_o(2p) \times SO(2q)$ | $SU(p, q) \cdot U(1)$ |
| $p$ | $[\frac{n}{2}] + [\frac{n}{2}]$ | |
| $SO^*(2n)/SO^*(2p) \times SO^*(2n - 2p)$ | $U(n)$ | $SU(p, n - p) \cdot U(1)$ |
| $(p \leq \frac{n}{2})$ | $p$ | $p$ |
| $SO^*(2n)/SU^*(2p) \cdot U(1)$ | $U(n)$ | $SO^*(2p) \times SO^*(2n - 2p)$ |
| $(p \leq \frac{n}{2})$ | $[\frac{n}{2}] + [\frac{n - p}{2}]$ | $p$ |
| $SO^*(2n)/SO(n, C)$ | $U(n)$ | $SO(n, C)$ |
| $[\frac{n}{2}]$ | $n$ | |
| $SO^*(4n)/SU^*(2n) \cdot U(1)$ | $U(2n)$ | $SU^*(2n) \cdot U(1)$ |
| $n - 1$ | $n - 1$ | |
| $SO(n, C)/SO(p, C) \times SO(n - p, C)$ | $SO(n)$ | $SO_o(p, n - p)$ |
| $(p \leq \frac{n}{2})$ | $p$ | $p$ |
| $SO(n, C)/SO_o(p, n - p)$ | $SO(n)$ | $SO(p, C) \times SO(n - p, C)$ |
| $(p \leq \frac{n}{2})$ | $[\frac{n}{2}] + [\frac{n - p}{2}]$ | $p$ |
| $SO(2n, C)/SL(n, C) \cdot SO(2, C)$ | $SO(2n)$ | $SO^*(2n)$ |
| $[\frac{n}{2}]$ | $[\frac{n}{2}]$ | |
| $SO(2n, C)/SO^*(2n)$ | $SO(2n)$ | $SL(n, C) \cdot SO(2, C)$ |
| $n$ | $[\frac{n}{2}]$ | |

Table 2.
| $G/H$ | $K$ | $L$ |
|---|---|---|
| $Sp(n, R)/SU(p, n - p) \cdot U(1)$ ($p \leq \frac{n}{2}$) | $U(n)$ | $Sp(p, R)$ $\times Sp(n - p, R)$ |
| | $n$ | $p$ |
| $Sp(n, R)/Sp(p, R) \times Sp(n - p, R)$ ($p \leq \frac{n}{2}$) | $U(n)$ | $SU(p, n - p) \cdot U(1)$ |
| | $p$ | $p$ |
| $Sp(n, R)/SL(n, R) \cdot U(1)$ | $U(n)$ | $SL(n, R) \cdot U(1)$ |
| | $n - 1$ | $n - 1$ |
| $Sp(2n, R)/Sp(n, C)$ | $U(2n)$ | $Sp(n, C)$ |
| | $n$ | $n$ |
| $Sp(p, q)/SU(p, q) \cdot U(1)$ | $Sp(p) \times Sp(q)$ | $SU(p, q) \cdot U(1)$ |
| | $p$ | $p + q$ |
| $Sp(p, p)/SU^*(2p) \cdot U(1)$ | $Sp(p) \times Sp(p)$ | $Sp(p, C)$ |
| | $p$ | $p$ |
| $Sp(p, p)/Sp(p, C)$ | $Sp(p) \times Sp(p)$ | $SU^*(2p) \cdot U(1)$ |
| | $p - 1$ | $p$ |
| $Sp(p, q)/Sp(i, j) \times Sp(p - i, q - j)$ | $Sp(p) \times Sp(q)$ | $Sp(p - i, j)$ $\times Sp(i, q - j)$ |
| | $\min\{p - i, j\}$ | $\min\{i, p - i\}$ |
| | $+\min\{i, q - j\}$ | $+\min\{j, q - j\}$ |
| $Sp(n, C)/SL(n, C) \cdot SO(2, C)$ | $Sp(n)$ | $Sp(n, R)$ |
| | $n$ | $n$ |
| $Sp(n, C)/Sp(n, R)$ | $Sp(n)$ | $SL(n, C) \cdot SO(2, C)$ |
| | $n$ | $n$ |
| $Sp(n, C)/Sp(p, C) \times Sp(n - p, C)$ ($p \leq \frac{n}{2}$) | $Sp(n)$ | $Sp(p, n - p)$ |
| | $p$ | $p$ |
| $Sp(n, C)/Sp(p, n - p)$ ($p \leq \frac{n}{2}$) | $Sp(n)$ | $Sp(p, C)$ $\times Sp(n - p, C)$ |
| | $n$ | $p$ |

Table 3.
| $G/H$               | $K$                    | $L$                           | $\text{cohom}_K$ | $\text{cohom}_L$ |
|--------------------|------------------------|-------------------------------|-------------------|-------------------|
| $E_6^0/Sp(4,\mathbb{R})$ | $Sp(4)/\{\pm 1\}$     | $SL(6,\mathbb{R}) \times SL(2,\mathbb{R})$ | 6                 | 4                 |
| $E_6^0/SL(6,\mathbb{R}) \times SL(2,\mathbb{R})$ | $Sp(4)/\{\pm 1\}$     | $Sp(4,\mathbb{R})$           | 4                 | 4                 |
| $E_6^0/Sp(2,2)$     | $Sp(4)/\{\pm 1\}$     | $SO_0(5,5) \cdot \mathbb{R}$ | 6                 | 2                 |
| $E_6^0/Sp(5,5) \cdot \mathbb{R}$ | $Sp(4)/\{\pm 1\}$     | $Sp(2,2)$                     | 2                 | 2                 |
| $E_6^0/SU^*(6) \cdot SU(2)$ | $Sp(4)/\{\pm 1\}$     | $F_4^4$                       | 4                 | 1                 |
| $E_6^0/F_4^4$       | $Sp(4)/\{\pm 1\}$     | $SU^*(6) \cdot SU(2)$        | 2                 | 1                 |
| $E_6^0/Sp(1,3)$     | $SU(6) \cdot SU(2)$   | $F_4^4$                       | 4                 | 2                 |
| $E_6^0/Sp(4,\mathbb{R})$ | $SU(6) \cdot SU(2)$   | $Sp(1,3)$                     | 1                 | 2                 |
| $E_6^0/SU(2,4) \cdot SU(2)$ | $SU(6) \cdot SU(2)$   | $SO_0(4,6) \cdot U(1)$       | 4                 | 2                 |
| $E_6^0/SO(4,6) \cdot U(1)$ | $SU(6) \cdot SU(2)$   | $SU(2,4) \cdot SU(2)$        | 2                 | 2                 |
| $E_6^0/SU(3,3) \cdot SL(2,\mathbb{R})$ | $SU(6) \cdot SU(2)$   | $SU(3,3) \cdot SL(2,\mathbb{R})$ | 4                 | 4                 |
| $E_6^0/SO^*(10) \cdot U(1)$ | $SU(6) \cdot SU(2)$   | $SO^*(10) \cdot U(1)$        | 2                 | 2                 |
| $E_6^0/Sp(2,2)$     | $Spin(10) \cdot U(1)$ | $Sp(2,2)$                     | 2                 | 6                 |
| $E_6^{14}/SU(2,4) \cdot SU(2)$ | $Spin(10) \cdot U(1)$ | $SU(2,4) \cdot SU(2)$        | 2                 | 4                 |
| $E_6^{14}/SU(1,5) \cdot SL(2,\mathbb{R})$ | $Spin(10) \cdot U(1)$ | $SO^*(10) \cdot U(1)$        | 2                 | 2                 |
| $E_6^{14}/SO^*(10) \cdot U(1)$ | $Spin(10) \cdot U(1)$ | $SU(1,5) \cdot SL(2,\mathbb{R})$ | 2                 | 2                 |
| $E_6^{14}/SO_0(2,8) \cdot U(1)$ | $Spin(10) \cdot U(1)$ | $SO_0(2,8) \cdot U(1)$       | 2                 | 2                 |
| $E_6^{14}/F_4^{-20}$ | $Spin(10) \cdot U(1)$ | $F_4^{-20}$                   | 1                 | 2                 |
| $E_6^{26}/Sp(1,3)$  | $F_4$                  | $SU^*(6) \cdot SU(2)$        | 2                 | 4                 |
| $E_6^{26}/SU^*(6) \cdot SU(2)$ | $F_4$                  | $Sp(1,3)$                     | 1                 | 4                 |
| $E_6^{26}/SO_0(1,9) \cdot U(1)$ | $F_4$                  | $F_4^{-20}$                   | 1                 | 1                 |
| $E_6^{26}/F_4^{-20}$ | $F_4$                  | $SO_0(1,9) \cdot U(1)$       | 2                 | 1                 |
| $E_6^0/E_6^0$       | $E_6$                  | $Sp(4,\mathbb{C})$           | 4                 | 6                 |
| $E_6^0/Sp(4,\mathbb{C})$ | $E_6$                  | $E_6^0$                       | 6                 | 6                 |
| $E_6^0/E_6^0$       | $E_6$                  | $SL(6,\mathbb{C}) \cdot SL(2,\mathbb{C})$ | 6                 | 4                 |
| $E_6^0/SL(6,\mathbb{C}) \cdot SL(2,\mathbb{C})$ | $E_6$                  | $E_6^2$                       | 4                 | 4                 |
| $E_6^0/E_6^{14}$    | $E_6$                  | $SO(10,\mathbb{C}) \cdot Sp(1)$ | 6                 | 2                 |
| $E_6^0/SO(10,\mathbb{C}) \cdot Sp(1)$ | $E_6$                  | $E_6^{14}$                    | 2                 | 2                 |
| $E_6^0/F_4^{-20}$   | $E_6$                  | $F_4^{26}$                    | 2                 | 2                 |
| $E_6^0/E_6^{26}$    | $E_6$                  | $F_4^{2}$                     | 4                 | 2                 |

Table 4.
| $G/H$ | $K$ | $L$ | cohom$K$ | cohom$L$ |
|-------|-----|-----|----------|----------|
| $E_7^5 / SU(8, \mathbb{R})$ | $SU(8)/{\pm 1}$ | $SU(8, \mathbb{R})$ | 7 | 7 |
| $E_7^5 / SU^*(8)$ | $SU(8)/{\pm 1}$ | $E_8^6 - U(1)$ | 7 | 3 |
| $E_7^5 / E_6^5 - U(1)$ | $SU(8)/{\pm 1}$ | $SU^*(8)$ | 3 | 3 |
| $E_7^5 / SU(4, 4)$ | $SU(8)/{\pm 1}$ | $SO_0(6, 6) \cdot SL(2, \mathbb{R})$ | 7 | 4 |
| $E_7^5 / SO_0(6, 6) \cdot SL(2, \mathbb{R})$ | $SU(8)/{\pm 1}$ | $SU(4, 4)$ | 4 | 4 |
| $E_7^5 / SO^*(12) - SU(2)$ | $SU(8)/{\pm 1}$ | $E_8^6 - U(1)$ | 4 | 2 |
| $E_7^5 / E_0^6 - U(1)$ | $SU(8)/{\pm 1}$ | $SO^*(12) - SU(2)$ | 3 | 2 |
| $E_7^5 / SU(4, 4)$ | $SO^*(12) - SU(2)$ | $SU(4, 4)$ | 4 | 7 |
| $E_7^5 / SU(2, 6)$ | $SO^*(12) - SU(2)$ | $E_8^6 - U(1)$ | 4 | 3 |
| $E_7^5 / E_0^6 - U(1)$ | $SO^*(12) - SU(2)$ | $SU(2, 6)$ | 2 | 3 |
| $E_7^5 / SO^*(12) - SL(2, \mathbb{R})$ | $SO^*(12) - SU(2)$ | $SO^*(12) - SL(2, \mathbb{R})$ | 4 | 4 |
| $E_7^5 / SO_0(4, 8) - SU(2)$ | $SO^*(12) - SU(2)$ | $SO_0(4, 8) - SU(2)$ | 4 | 4 |
| $E_7^5 / E_6^{14} - U(1)$ | $SO^*(12) - SU(2)$ | $E_6^{14} - U(1)$ | 2 | 3 |
| $E_7^5 / SU^*(8)$ | $E_6 - U(1)$ | $SU^*(8)$ | 3 | 7 |
| $E_7^5 / SU(2, 6)$ | $E_6 - U(1)$ | $SO^*(12) - SU(2)$ | 3 | 5 |
| $E_7^5 / SO^*(12) - SU(2)$ | $E_6 - U(1)$ | $SU(2, 6)$ | 2 | 5 |
| $E_7^5 / SO_0(2, 10) - SL(2, \mathbb{R})$ | $E_6 - U(1)$ | $E_6^{14} - U(1)$ | 2 | 2 |
| $E_7^5 / E_6^{14} - U(1)$ | $E_6 - U(1)$ | $SO_0(2, 10) \cdot SL(2, \mathbb{R})$ | 3 | 2 |
| $E_7^5 / E_6^{26} - U(1)$ | $E_6 - U(1)$ | $E_6^{26} - U(1)$ | 2 | 3 |
| $E_7^5 / E_7^7$ | $E_7$ | $SL(8, \mathbb{C})$ | 7 | 7 |
| $E_7^5 / SL(8, \mathbb{C})$ | $E_7$ | $E_7^7$ | 7 | 7 |
| $E_7^5 / E_7^5$ | $E_7$ | $SO(12, \mathbb{C}) \cdot SL(2, \mathbb{C})$ | 7 | 4 |
| $E_7^5 / SO(12, \mathbb{C}) - SL(2, \mathbb{C})$ | $E_7$ | $E_7^5$ | 4 | 4 |
| $E_7^5 / E_7^5$ | $E_7$ | $E_7^5$ | 7 | 3 |
| $E_7^5 / E_6^5 \cdot C^*$ | $E_7$ | $E_7^5$ | 3 | 3 |
| $E_7^5 / SO^*(16)$ | $SO^*(16)$ | $E_7^5 - SL(2, \mathbb{R})$ | 4 | 4 |
| $E_7^5 / E_7^5 - SL(2, \mathbb{R})$ | $SO^*(16)$ | $SO^*(16)$ | 4 | 4 |
| $E_7^5 / SO_0(8, 8)$ | $SO^*(16)$ | $SO_0(8, 8)$ | 8 | 8 |
| $E_7^5 / E_7^5 - Sp(1)$ | $SO^*(16)$ | $E_7^5 - Sp(1)$ | 4 | 4 |
| $E_7^5 / SO^*(16)$ | $E_7 \cdot Sp(1)$ | $SO^*(16)$ | 4 | 8 |
| $E_7^5 / SO_0(4, 12)$ | $E_7 \cdot Sp(1)$ | $E_7^5 - Sp(1)$ | 4 | 4 |
| $E_7^5 / E_7^5 - Sp(1)$ | $E_7 \cdot Sp(1)$ | $SO_0(4, 12)$ | 4 | 4 |
| $E_7^5 / E_7^5 - SL(2, \mathbb{R})$ | $E_7 \cdot Sp(1)$ | $E_7^5 - SL(2, \mathbb{R})$ | 4 | 4 |
| $E_7^5 / E_7^5$ | $E_8$ | $SO(16, \mathbb{C})$ | 8 | 8 |
| $E_7^5 / SO(16, \mathbb{C})$ | $E_8$ | $E_8$ | 8 | 8 |
| $E_7^5 / E_7^{24}$ | $E_8$ | $E_7^5 \times SL(2, \mathbb{C})$ | 8 | 4 |
| $E_7^5 / E_7^{24}$ | $E_8$ | $E_7^{24}$ | 4 | 4 |

Table 5.
Table 6.

| $G/H$ | $K$ | $L$ | $\text{cohom}_K$ | $\text{cohom}_L$ |
|-------|-----|-----|-----------------|-----------------|
| $F_4^1/Sp(1,2) \cdot Sp(1)$ | $Sp(3) \cdot Sp(1)$ | $SO_0(4,5)$ | 4 | 1 |
| $F_4^1/Sp(3,4,5)$ | $Sp(3) \cdot Sp(1)$ | $Sp(1,2) \cdot Sp(1)$ | 1 | 1 |
| $F_4^1/Sp(3,\mathbb{R}) \cdot SL(2,\mathbb{R})$ | $Sp(3) \cdot Sp(1)$ | $Sp(3,\mathbb{R}) \cdot SL(2,\mathbb{R})$ | 4 | 4 |
| $F_4^{-20}/Sp(1,2) \cdot Sp(1)$ | $Spin(9)$ | $SO_0(1,8)$ | 1 | 1 |
| $F_4^{-20}/SO_0(1,8)$ | $Spin(9)$ | $Sp(1,2) \cdot Sp(1)$ | 1 | 1 |
| $G_2^C/F_4^1$ | $F_4$ | $Sp(3,\mathbb{C}) \cdot SL(2,\mathbb{C})$ | 4 | 4 |
| $F_4^1/Sp(3,\mathbb{C}) \cdot SL(2,\mathbb{C})$ | $F_4$ | $F_4$ | 4 | 4 |
| $F_4^C/F_4^{20}$ | $F_4$ | $SO(9,\mathbb{C})$ | 4 | 1 |
| $G_2^C/Sp(9,\mathbb{C})$ | $F_4$ | $F_4^{20}$ | 1 | 1 |
| $G_2^2/SL(2,\mathbb{C}) \times SL(2,\mathbb{R})$ | $SO(4)$ | $\alpha(SO(4))$ | 2 | 2 |
| $G_2^2/\alpha(SO(4))$ | $SO(4)$ | $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ | 2 | 2 |
| $G_2^2/G_2^2$ | $G_2$ | $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ | 2 | 2 |
| $G_2^2/SL(2,\mathbb{C}) \times SL(2,\mathbb{R})$ | $G_2$ | $G_2^2$ | 2 | 2 |

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Department of Mathematics, Faculty of Science, Tokyo University of Science 26 Wakamiya Shinjuku-ku, Tokyo 162-8601, Japan (e-mail: koike@ma.kagu.tus.ac.jp)