HIGHER GENTLE ALGEBRAS

JORDAN MCMAHON

Abstract. We introduce higher gentle algebras. Our definition allows us to
determine the singularity categories and subsequently show that higher gen-
tle algebras are Iwanaga-Gorenstein. Under extra assumptions, we show that
cluster-tilted algebras (in the sense of Oppermann-Thomas) of higher Auslander
algebras of type $A$ are higher gentle.

Contents

1. Introduction 1
2. Background 2
2.1. Tilting theory and $(d + 2)$-angulated categories 4
3. Higher gentle algebras 5
4. Tilted algebras of higher Auslander algebras of linearly oriented type $A$ 10
5. Examples 13
6. Acknowledgements 14
References 15

1. Introduction

Gentle algebras were introduced in [2], as a class of the special biserial alge-
bras introduced in [33]. Specifically, gentle algebras encompass the tilted algebras
of type $A_n$ and $\tilde{A}_n$. Since then, gentle algebras have appeared naturally in many
other contexts; prominent sources include triangulations of surfaces [1], [23], tilings
of surfaces [11] [32], as well as $m$-Calabi-Yau tilted algebras [12] and Brauer graph
algebras [31]. More general models have been attained recently [3] [26] [29]. Gen-
tle algebras are Gorenstein [14], and their singularities were described in [22].

As a natural generalisation of gentle algebras, we define higher gentle algebras
in Definition 2. Generalising the technique for calculating singularity categories
of Nakayama algebras [10] and higher Nakayama algebras [24], we arrive at the
following result.

Theorem 1.1 (Theorem 3.3). Let $A$ be a $d$-gentle algebra. There exists an idem-
potent $f$ such that $fAf$ is a gentle algebra and such that there is an equivalence of
categories

$$D_{sg}(A) \cong D_{sg}(fAf).$$
For an algebra $\Lambda$ of linearly-oriented type $A$, any rigid $\Lambda$ module $M$ gives rise to a gentle algebra $\operatorname{End}_{\Lambda}(M)^{\text{op}}$ \cite{2}. This may be generalised.

**Theorem 1.2** (Corollary \[4.3\]). Let $\Lambda$ be a $d$-Auslander algebra of linearly-oriented type $A$, $T$ a $d$-rigid $\Lambda$-module such that $\operatorname{add}(T) \subseteq C \subseteq \operatorname{mod}(\Lambda)$, where $C$ is the canonical $d$-cluster-tilting subcategory. Then $\operatorname{End}_{\Lambda}(T)^{\text{op}}$ is a $d$-gentle algebra.

The behaviour of higher cluster-tilting subcategories differs significantly from that of module categories. For example, the number of simple modules in the $d$-cluster-tilting subcategory of the module category of a $d$-Auslander algebra of linearly-oriented type $A$ is independent of the value of $d$. In general, the members of such a $d$-cluster-tilting subcategory do not, however, have a filtration by these simple modules. In this sense, the $d$-cluster-tilting subcategories a $d$-Auslander algebra of linearly orientated type $A$ behave like module categories only to some extent; to this same extent we are able to produce higher gentle algebras.

For the following result, let $A_{n}^{d}$ be the $d$- Auslander algebra of linearly-oriented type $A_{n}$, $C \subseteq \operatorname{mod}(A_{n}^{d})$ the canonical $d$-cluster-tilting subcategory and $\mathcal{O}_{A_{n}^{d}}$ the $(d+2)$-angulated cluster category of $A_{n}^{d}$.

**Proposition 1.3** (Corollary \[4.5\]). Let $S$ be a semisimple $A_{n}^{d}$-module in $C$ such that $\operatorname{Ext}_{A_{n}^{d}}^{d}(S, S) = 0$. Let $P$ be a basic projective $A_{n}^{d}$-module such that $\operatorname{Ext}_{A_{n}^{d}}^{d}(S, P) = 0$ and set $T := P \oplus \tau_{d}^{-1}(S)$. Then $\operatorname{End}_{\mathcal{O}_{A_{n}^{d}}}(T)^{\text{op}}$ is a $d$-gentle algebra.

If $T$ is tilting as an $A_{n}^{d}$-module, then the algebra $\operatorname{End}_{\mathcal{O}_{A_{n}^{d}}}(T)^{\text{op}}$ is a cluster-tilted algebra in the sense of Oppermann-Thomas. Unfortunately it is not always true that cluster-tilted algebras (in the sense of Oppermann-Thomas) of higher Auslander algebras of type $A$ are higher gentle, and we provide a counterexample in Section 5.

2. **Background**

Consider a finite-dimensional algebra $A$ over a field $k$, and fix a positive integer $d$. We will assume that $A$ is of the form $kQ/I$, where $kQ$ is the path algebra over some quiver $Q$ and $I$ is an admissible ideal of $kQ$. For two arrows in $Q \alpha : i \to j$ and $\beta : j \to k$, we denote their composition as $\beta \alpha : i \to k$. Let $A^{\text{op}}$ denote the opposite algebra of $A$. An $A$-module will mean a finitely-generated left $A$-module; by $\operatorname{mod}(A)$ we denote the category of $A$-modules. The functor $D = \operatorname{Hom}_{k}(-, k)$ defines a duality; let $\Omega$ be the syzygy functor and set $\tau_{d} = \tau \circ \Omega^{d-1}$ to be the $d$-Auslander-Reiten translation \[18\] Section 1.4]. For an $A$-module $M$, let $\operatorname{add}(M)$ be the full subcategory of $\operatorname{mod}(A)$ composed of all $A$-modules isomorphic to finite direct sums of copies of $M$. A subcategory $C$ of $\operatorname{mod}(A)$ is *precovering* if for any $M \in \operatorname{mod}(A)$ there is an object $C_{M} \in C$ and a morphism $f : C_{M} \to M$ such that for any morphism $X \to M$ with $X \in C$ factors through $f$;
that there is a commutative diagram:

\[
\begin{array}{c}
X \\
\downarrow \\
C_M \xrightarrow{f} M
\end{array}
\]

The object \(C_M\) is said to be the \textit{right} \(C\)-approximation of \(M\). The dual notion of precovering is \textit{preenveloping}. A subcategory \(C\) that is both precovering and preenveloping is called \textit{functorially finite}. For a finite-dimensional algebra \(A\), a functorially-finite subcategory \(C\) of \(\text{mod}(A)\) is a \textit{d-cluster-tilting subcategory} \cite{18} Definition 2.2 \cite{21} Definition 3.14 if it satisfies the following conditions:

\[
C = \{ M \in \text{mod}(A) \mid \text{Ext}_A^i(C, M) = 0 \quad \forall \ 0 < i < d \}.
\]

If there exists a \textit{d}-cluster-tilting subcategory \(C \subseteq \text{mod}(A)\) and \(\text{gl.dim}(A) \leq d\), then \(A\) is \textit{d-representation finite} in the sense of \cite{20}. The \textit{dominant dimension} of \(A\), \(\text{dom.dim}(A)\), is the number \(n\) such that for a minimal injective resolution of \(A\):

\[
0 \to A \to I_0 \to \cdots \to I_{n-1} \to I_n \to \cdots
\]

the modules \(I_0, \ldots, I_{n-1}\) are projective-injective and \(I_n\) is not projective. The class of \(d\)-representation-finite algebras were characterised by Iyama as follows.

\textbf{Theorem 2.1.} \cite{19} Proposition 1.3, Theorem 1.10] Let \(A\) be a finite-dimensional algebra with the property that \(\text{gl.dim}(A) \leq d\). Then there is a unique \(d\)-cluster-tilting subcategory \(C \subseteq \text{mod}(A)\) if and only if

\[
\text{dom.dim}(\text{End}_A(M)^{\text{op}}) \geq d + 1 \geq \text{gl.dim(End}_A(M)^{\text{op}})
\]

where \(M\) is an additive generator of the subcategory

\[
C = \text{add}(\{ \tau_i^d(DA) \mid i > 0 \}) \subseteq \text{mod}(A).
\]

Following Theorem 2.1, let \(\Gamma\) be a finite dimensional algebra satsifying

\[
\text{gl. dim.}(\Gamma) \leq d + 1 \leq \text{dom. dim.}(\Gamma)
\]

for some positive integer \(d \geq 1\). Then \(\Gamma\) is said to be a \(d\)-\textit{Auslander algebra}. An algebra \(A\) is called an \(n\)-\textit{Iwanaga-Gorenstein algebra} if it satisfies the following axioms:

\begin{enumerate}
\item \(\text{inj.dim}_A(A) \leq n\),
\item \(\text{proj.dim}_A(DA) \leq n\).
\end{enumerate}

An \(A\)-module \(M\) is said to be \textit{Gorenstein projective} (also referred to as \textit{maximal Cohen-Macaulay} in the literature, most notably in \cite{5}) if \(\text{Ext}_A^i(M, A) = 0\) for all \(i > 0\). The class of Gorenstein projective modules is denoted \(\text{GP}(A)\). Likewise, define a module \(M\) to be \textit{Gorenstein injective} if \(\text{Ext}_A^i(DA, M) = 0\) for all \(i > 0\), and denote by \(\text{GI}(A)\) the class of Gorenstein injective modules. For further information about Gorenstein homological algebra, we refer to \cite{7}.
Let $D^b(A)$ denote the bounded derived category of $\text{mod}(A)$. A complex of $A$-modules is said to be perfect if it is isomorphic in $D^b(A)$ to a finite complex of finitely generated projective $A$-modules. This gives a full subcategory of $D^b(A)$, denoted by $D^b_{\text{perf}}(A)$. The singularity category $D_{\text{sg}}(A)$ is defined as the Verdier quotient of $D^b(A)/D^b_{\text{perf}}(A)$ [5] [28]. The following theorem is a classical result of Buchweitz.

**Theorem 2.2.** [5, Theorem 4.4.1] Let $A$ be an $n$-Iwanaga-Gorenstein algebra. Then there is an equivalence of (triangulated) categories:

$$\mathbf{GP}(A) \cong D_{\text{sg}}(A).$$

2.1. **Tilting theory and $(d + 2)$-angulated categories.** Let $A$ be a finite-dimensional algebra. An $A$-module $T$ is a pre-$d$-tilting module [17], [25] if:

1. $\text{proj.dim}(T) \leq d$.
2. $\text{Ext}^i_A(T, T) = 0$ for all $0 < i \leq d$.

Then $T$ is in addition $d$-tilting if there exists an exact sequence

$$0 \rightarrow A \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_d \rightarrow 0$$

where $T_0, \ldots, T_d \in \text{add}(T)$. The importance of tilting modules is highlighted by the following theorem:

**Theorem 2.3 (Happel).** [17] Let $A$ be a finite-dimensional algebra, $T$ a $d$-tilting $A$-module and $B := \text{End}_A(T)^{\text{op}}$. Then the derived functor $R\text{Hom}_A(T, -)$ induces an equivalence of triangulated categories

$$D^b(A) \rightarrow D^b(\text{End}_A(T)^{\text{op}}).$$

The concept of a $(d + 2)$-angulated category was introduced by Geiss-Keller-Oppermann in [15]. We refer there, as well as to [4], for a definition.

**Theorem 2.4.** [15, Theorem 1] Let $\Lambda$ be a $d$-representation-finite algebra with $d$-cluster-tilting subcategory $\mathcal{C} \subseteq \text{mod}(\Lambda)$. Then there exists a $(d + 2)$-angulated category $\mathcal{U}_{\Lambda}$ with $d$-suspension functor $\Sigma^d$ and inverse $d$-suspension functor $\Sigma^{-d}$. Any $d$-exact sequence in $\mathcal{C}$

$$0 \rightarrow M_{d+1} \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow 0$$

induces a $(d + 2)$-angle in $\mathcal{U}_{\Lambda}$

$$M_{d+1} \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow \Sigma^d(M_{d+1}).$$

The functor $\tau_d$ acts on $\mathcal{C}$, and induces a functor $\mathcal{S}_d$ in $\mathcal{U}_{\Lambda}$. Oppermann-Thomas [27, Definition 5.22] defined the $(d + 2)$-angulated cluster category of $\Lambda$ to be the orbit category

$$\mathcal{O}_{\Lambda} := \mathcal{U}_{\Lambda}/\Sigma^{-d}(\mathcal{S}_d).$$
Theorem 2.5. [27, Theorem 5.2(1)] Let $\Lambda$ be a $d$-representation-finite algebra with $d$-cluster-tilting subcategory $C \subseteq \text{mod}(\Lambda)$. Then the $O_\Lambda$ is a $(d + 2)$-angulated category with $d$-suspension $[d]$. The isomorphism classes indecomposable objects of $O_\Lambda$ are in bijection with the indecomposable direct summands of $M \oplus \Lambda[d]$, where $M$ is an additive generator of $C$.

The concepts of rigid and cluster-tilting objects can be extended to the language of $(d + 2)$-angulated categories.

Definition 1. [27, Definition 5.3] Let $O_\Lambda$ be the $(d + 2)$-angulated cluster category of $\Lambda$ with $d$-suspension $[d]$. An object $T \in O_\Lambda$ is $d$-rigid if
\[ \text{Hom}_{O_\Lambda}(T, T[d]) = 0. \]

A $d$-rigid object $T \in O_\Lambda$ is a Oppermann-Thomas cluster-tilting object if any $X \in O_\Lambda$ occurs in a $(d + 2)$-angle
\[ X[-d] \to T_d \to \cdots \to T_1 \to T_0 \to X \]
with $T_i \in \text{add}(T)$ for all $0 \leq i \leq d$. The endomorphism algebra $\text{End}_{O_\Lambda}(T)^{\text{op}}$ is called a Oppermann-Thomas cluster-tilted algebra.

Tilting and Oppermann-Thomas cluster-tilting objects are related as follows.

Theorem 2.6. [27, Theorem 5.2, Theorem 5.5] Let $\Lambda$ be a $d$-representation-finite algebra with canonical $d$-cluster-tilting subcategory $C \subseteq \text{mod}(\Lambda)$. Let $T$ be a $d$-tilting $\Lambda$-module so that $\text{add}(T) \subseteq C$. Then

1. $T$ is an Oppermann-Thomas cluster-tilting object in $O_\Lambda$.
2. There is a $d$-cluster-tilting subcategory $D \subseteq \text{mod}(\text{End}_{O_\Lambda}(T)^{\text{op}})$.

3. Higher gentle algebras

An algebra $A = kQ/I$ is special biserial if

1. Each vertex of $Q$ has at most two arrows starting from it.
2. Each vertex of $Q$ has at most two arrows ending at it.
3. For each arrow $\alpha \in Q_1$, there is most one arrow $\beta$ such that $\alpha \beta \notin I$.
4. For each arrow $\gamma \in Q_1$, there is most one arrow $\beta$ such that $\beta \gamma \notin I$.

If moreover $A$ satisfies

1. The ideal $I$ is generated by paths of length at most two.
2. For each arrow $\alpha \in Q_1$, there is most one arrow $\beta$ such that $\alpha \beta \in I$.
3. For each arrow $\gamma \in Q_1$, there is most one arrow $\beta$ such that $\beta \gamma \in I$.

then $A$ is said to be gentle. In other words, on either side of each arrow in $Q$ there is at most one arrow such that the composition with this arrow is in $I$, and at most one such that the composition is not in $I$. Generalisations of special biserial algebras exist in the literature, such as special multiserial algebras [16], [34]. We are interested in generalising special biserial algebras in the following fashion.
An algebra $A = kQ/I$ contains an $m$-cube if there is a collection of paths between vertices $x$ and $y$ in the quiver of $A$ such that the underlying graph is an $m$-dimensional cube and any two paths defining a square face in this $m$-cube commute in $A$. For each arrow $\beta \in Q_1$, then an arrow $\alpha \in Q_1$ is a strong successor of $\beta$ in $A$ if $s(\alpha) = t(\beta)$, neither $\alpha\beta \in I$ nor are there are arrows $\alpha'$ and $\beta'$ and a relation $\alpha\beta - \alpha'\beta' \in I$. In this case $\beta$ is also a strong predecessor of $\alpha$ in $A$. Consider an algebra $A = kQ/I$ satisfying the following conditions, which we consider to some extent as a replacement the special biserial conditions.

(A1) Each vertex of $Q$ has at most $d$ arrows starting from it.
(A1') Each vertex of $Q$ has at most $d$ arrows ending at it.
(A2) For each arrow $\alpha \in Q_1$ there is at most one strong successor $\beta \in Q_1$ of $\alpha$ in $A$.
(A2') For each arrow $\beta \in Q_1$ there is at most one strong predecessor $\alpha \in Q_1$ of $\beta$ in $A$.
(A3) Let $\alpha \in Q_1$ be an arrow with strong successor $\beta$. Then for any $1 < m < d$ and any set of $m$ arrows $\beta_i$ indexed by $1 \leq i \leq m$ such that $s(\beta_i) = t(\alpha)$ and $\beta_i\alpha \notin I$ for all $1 \leq i \leq m$, there is a unique $(m+1)$-cube containing $\beta$ and all $\beta_i$.
(A3') Let $\alpha \in Q_1$ be an arrow with strong predecessor $\beta$. Then for any $1 < m < d$ and any set of $m$ arrows $\beta_i$ indexed by $1 \leq i \leq m$ such that $s(\alpha') = t(\beta_i)$ and $\alpha\beta_i \notin I$ for all $1 \leq i \leq m$, there is a unique $(m+1)$-cube containing $\beta$ and all $\beta_i$.
(A4) The ideal $I$ is generated by paths and commutativity relations of length two.

We say that two zero relations $\beta_i\alpha_i \in I$, $i \in \{1, 2\}$, sandwich a commutativity relation if there exist arrows $\gamma, \delta \in Q_1$ satisfying any of the diagrams in Figure 1.

**Definition 2.** An algebra $A = kQ/I$ satisfying axioms (A1)-(A4) is $d$-pre-gentle if it satisfies the following additional axioms:

(E1) For each arrow $\alpha \in Q_1$, there is at most one arrow $\beta$ such that $\alpha\beta \in I$.
(E2) For each arrow $\gamma \in Q_1$, there is at most one arrow $\beta$ such that $\beta\gamma \in I$.
(E3) There exists no commutativity relation that is sandwiched by zero relations.
(E4) For every idempotent $e$ of $A$, then $eAe$ satisfies axioms (E1)-(E3).

Finally, an algebra $B$ is $d$-gentle if:

1. There is a $d$-pre-gentle algebra $A$ and an idempotent $e$ of $A$ such that $B \cong eAe$.
2. For every idempotent $f$ of $B$, then the quiver of $fBf$ contains no $d$-cube.

The axiom forbidding any sandwiching by zero relations should be thought of as a replacement for having no zero relations of length greater than two. For any gentle algebra $A$ and any idempotent $e$ of $A$, then also $eAe$ is gentle. Once we introduce commutativity relations, we need some other means by which to control
the length of zero relations - this is achieved by forbidding the sandwiching of a commutativity relation by zero relations. Nevertheless, as we shall see in Example 1 this is the condition that would make the most sense to relax.

For studying $d$-gentle algebras, it is helpful to start with the notion of a localisable object. Localisable objects were introduced in [9]: the name stems from the localising subcategories studied by Geigle and Lenzing in [13, Section 2].

**Definition 3.** Let $A$ be a finite-dimensional algebra. Then an object $S \in \text{mod}(A)$ is a localisable if:

- the module $S$ is simple,
- $\text{proj.dim}_A(S) \leq 1$, and
- $\text{Ext}^1_A(S, S) = 0$.

Every localisable object $S$ can be expressed as $S \cong A/(f)$ for some idempotent $f$ in $A$, since $\text{Ext}^1_A(S, S) = 0$. This was generalised in the following sense in [24]:

**Definition 4.** Let $A$ be a finite-dimensional algebra and $e, f$ idempotents of $A$. Then $f$ is an fabric idempotent of $A$ with respect to $e$ (or simply $f$ is a fabric idempotent) if:

- the idempotent $f$ satisfies $\text{proj.dim}_A(A/(f)) \leq 1$. 

---

**Figure 1.** The four configurations whereby two zero relations sandwich a commutativity relation.
For every projective $A/\langle f \rangle$-module $P$, the module $\tau_A(P)$ is injective as an $A/\langle e \rangle$-module.

For every injective $A/\langle e \rangle$-module $I$, the module $\tau_A^{-1}(I)$ is projective as an $A/\langle f \rangle$-module.

**Theorem 3.1.** [6, Theorem 2.1] [8, Corollary 3.3] [30, Theorem 5.2] Let $A$ be a finite-dimensional algebra and $f$ an idempotent of $A$. Then there is an equivalence

$$D_{sg}(A) \cong D_{sg}(fAf)$$

if and only if the algebra $A$ satisfies $\operatorname{proj.dim}_{fAf}(fA) < \infty$ and $\operatorname{proj.dim}_A(M) < \infty$ for all modules $M \in \text{mod}(A/\langle f \rangle)$.

Fabric idempotents are useful because of the following result.

**Corollary 3.2.** [24, Corollary 3.7] Let $A$ be a finite-dimensional algebra with fabric idempotent $f$. If in addition $\operatorname{gl.dim}(A/\langle f \rangle) < \infty$, then there is an equivalence

$$D_{sg}(A) \cong D_{sg}(fAf).$$

By design, we obtain the following result.

**Theorem 3.3.** Let $A$ be a $d$-gentle algebra. Then there exists an idempotent $f$ such that $f$ is a product of fabric idempotents, $fAf$ is a gentle algebra and there is an equivalence of categories

$$D_{sg}(A) \cong D_{sg}(fAf).$$

**Proof.** We first prove the result for $d$-pre-gentle algebras. The proof is similar to the case for higher Nakayama algebras [24, Theorem 4.4]. Suppose there are four vertices $a, b, c, d \in Q_0$ and non-zero paths $w_1, w_2, w_3, w_4$ in $Q$ as follows

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\downarrow w_3 \\
\text{c} \\
\text{\downarrow w_4} \\
\text{d}
\end{array}
\begin{array}{c}
\text{b} \\
\begin{array}{c}
\downarrow w_2 \\
\text{x}
\end{array}
\end{array}
\end{array}
\]

Suppose that there does not exist a surjective morphism $I_d \twoheadrightarrow I_b$. Then either there is a vertex $x$ and a path $w_x : x \rightarrow b$ such that $w_2w_x \in I$

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\downarrow w_3 \\
\text{c} \\
\text{\downarrow w_4} \\
\text{d}
\end{array}
\begin{array}{c}
\text{b} \\
\begin{array}{c}
\downarrow w_2 \\
\text{x}
\end{array}
\end{array}
\end{array}
\]


There must be an inclusion $P_d \hookrightarrow P_c$, otherwise suppose there is a vertex $g$ and an path $w_g : d \to g$ such that $w_g w_4 \in I$:

\[
\begin{array}{c}
    a \\
    \downarrow w_3 \\
    c \\
\end{array} \quad \begin{array}{c}
    b \\
    \downarrow w_2 \\
    d \\
\end{array} \quad \begin{array}{c}
    x \\
    \downarrow w_g \\
    g \\
\end{array}
\]

Then we obtain the following forbidden diagram:

\[
\begin{array}{c}
    a \\
    \downarrow w_3 \\
    c \\
\end{array} \quad \begin{array}{c}
    b \\
    \downarrow w_2 \\
    x \\
\end{array} \quad \begin{array}{c}
    \quad \quad \\
    \quad \quad \\
    g \\
\end{array}
\]

Secondly, for any path from $c$ to some vertex $c'$, there is a path from $a$ to $c'$. Else there is a sandwiching by zero relations:

\[
\begin{array}{c}
    a \\
    \downarrow w_3 \\
    c \\
\end{array} \quad \begin{array}{c}
    b \\
    \downarrow w_2 \\
    x \\
\end{array} \quad \begin{array}{c}
    \quad \quad \\
    \quad \quad \\
    g \\
\end{array}
\]

So there must be injective morphisms $P_d \hookrightarrow P_c$ and $P_c \hookrightarrow P_a$. Choose the smallest idempotent $f$ of $A$ such that $\text{coker}(P_d \to P_c) \in \text{add}(A/(f))$. Then $f$ is a fabric idempotent by \cite{23} Proposition 3.2. The same argument on avoiding sandwiching by zero relations also implies that $\text{gl.dim}(A/(f)) \leq \infty$. Hence Corollary \ref{corollary} implies that $D_{sg}(A) \cong D_{sg}(fAf)$.

A similar argument can be made if there does not exist a surjective morphism $I_b \twoheadrightarrow I_a$, and a dual argument may be made if there exists neither an injective morphism $P_a \hookrightarrow P_c$ nor an injective morphism $P_c \hookrightarrow P_a$.

Since zero relations are of length two, we may remove all commutativity relations for $A$; reducing to a $d$-gentle algebra whose singularity category is equivalent to that of $A$, and that has no commutativity relations. In other words, a gentle algebra. Now suppose we have an arbitrary $d$-gentle algebra $\epsilon A \epsilon$ (where $A$ is a $d$-pre-gentle algebra), and a zero relation of length greater than two. It should be easily seen that any such relation may only occur if there were a series of commutativity relations in $A$ neighbouring a relation of length two. Precisely such a situation produces a diagram as above, and is compatible with producing fabric idempotents. So by applying the same reduction technique as above, up to removing vertices that do not appear in $\epsilon$, then we again arrive at a gentle algebra. \qed
Singularity categories for gentle algebras were described in [22], hence we may also describe the singularity category of any d-gentle algebra. Singularity categories are especially interesting for Iwanaga-Gorenstein algebras, because of Theorem 2.2.

For an algebra $A$ and idempotent $f$, let $B_f = \{ M \in \text{mod}(A) | fM = 0 \}$, then by [6, Lemma 2.2] the equivalence in Theorem 3.1 is induced by a functor $D^b(A) \to D^b(fAf)$ that has as a kernel the full subcategory of complexes with cohomology groups that are in $B_f$.

The following result generalises the main theorem of [14].

Corollary 3.4. Any d-gentle algebra is Iwanaga-Gorenstein.

Proof. Given a d-gentle algebra $A$, consider an infinite projective resolution (as a complex) of some injective $A$-module $I$

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to 0.$$ By Theorem 3.3, there is a product of fabric idempotents $f$, such that $fAf$ is gentle. Since $f$ is a product of fabric idempotents, $I \notin \text{mod}(A/\langle f \rangle)$. Applying the functor $\text{Hom}_A(Af, -)$ induces the sequence of projective $fAf$-modules

$$\cdots \to fP_n \to \cdots \to fP_1 \to fP_0 \to 0.$$ This is a projective resolution of the injective $fAf$-module $fI$, and so $fI$ has infinite projective dimension. Since $fAf$ is gentle and hence Iwanaga-Gorenstein, this is a contradiction. □

4. Tilted algebras of higher Auslander algebras of linearly oriented type $A$

Following the notation of Oppermann-Thomas [27, Definition 2.2], define the sets

$$I^d_m := \{(i_0, \ldots, i_d) \in \{1, \ldots, m\}^{d+1} | \forall x \in \{0, 1, \ldots, d-1\} : i_x + 2 \leq i_{x+1}\},$$

$$\overset{\circ}{I}^d_m := \{(i_0, \ldots, i_d) \in I^d_m | i_d + 2 \leq i_0 + m\}.$$ Given two increasing $(d+1)$-tuples of real numbers $X = \{x_0, x_1, \ldots, x_d\}$ and $Y = \{y_0, y_1, \ldots, y_d\}$, then $X$ intertwines $Y$ if $x_0 < y_0 < x_1 < y_1 < \cdots < x_d < y_d$. Denote by $X \vdash Y$ if $X$ intertwines $Y$. A collection of increasing $(d+1)$-tuples of real numbers is non-intertwining if no pair of elements intertwine (in either order).

In [19], Iyama describes an inductive construction of an $(i+1)$-Auslander algebra from an $i$-Auslander algebra. In particular, for the linearly oriented quiver $Q$ of type $A_n$, a $d$-Auslander algebra may be constructed, we call this algebra $A_n^d$. As part of this construction, the category $\text{mod}(A_n^d)$ has a canonical $d$-cluster-tilting subcategory, and it is unique by Theorem 2.1. By [27, Theorem 3.4], the indecomposable modules in this $d$-cluster-tilting subcategory, as well as the vertices of the quiver of $A_n^{d+1}$ may be labelled by $I^d_{n+2d}$. The quiver of $A_n^2$ is as follows:
The quiver of $A_3^4$ is as follows:

For each $I \in \mathbf{I}_{n+2d}^d$, denote by $M_I$ the object of the aforementioned $d$-cluster-tilting subcategory, and let $M$ be an additive generator of the subcategory. Then there is a combinatorial description of tilting $A_n^d$-modules.

**Theorem 4.1.** [27, Theorem 3.6(4), Theorem 4.4] Let $I, J \in \mathbf{I}_{n+2d}^d$. Then $\text{Ext}^d_{A_n^d}(M_I, M_J) \neq 0 \iff J \wr I$. Moreover, there are bijections between

- triangulations of the cyclic polytope $C(n+2d, 2d)$.
- non-intertwining collections of $\binom{n+2d-1}{d}$ $(d+1)$-tuples in $\mathbf{I}_{n+2d}^d$.
- isomorphism classes of summands of $A_n^d M$ which are tilting modules.

We refer to [27] for a definition of cyclic polytopes, as this is beyond the scope of this article.

**Theorem 4.2.** [27, Lemma 6.6, Proposition 6.1, Theorem 6.4] Consider the $d$-representation-finite algebra $A_n^d$, and $O_{A_n^d}$, the $(d+2)$-angulated cluster category of $A_n^d$. Then:

1. the indecomposable objects of the $(d+2)$-angulated category $O_{A_n^d}$ are indexed by $\mathbf{I}_{n+2d+1}^d$.
2. $\text{Hom}_{O_{A_n^d}}(M_I, M_J[d]) \neq 0 \iff I \wr J$ or $J \wr I$.
3. Triangulations of the cyclic polytope $C(n+2d+1, 2d)$ correspond bijectively to basic Oppermann-Thomas cluster-tilting objects in $O_{A_n^d}$.

The following result may now be obtained.

**Corollary 4.3.** Let $T$ be any $d$-rigid $A_n^d$-module in $C$, where $C \subseteq \text{mod}(A_n^d)$ is the canonical $d$-cluster-tilting subcategory. Let $B = \text{End}_{A_n^d}(T)^{op}$. Then $B$ is a $d$-gentle algebra.
Proof. Let $T$ be a $d$-rigid $A_n^d$-module in $C$. By Theorem 4.2, this corresponds to a set $I$ of non-intertwining subsets of $n+2d$. It is clear that $A_n^d$ is $d$-pre-gentle. Let $e$ be an idempotent of $A_n^d$ corresponding to $I$, then $eA_n^d e \cong B$. Finally, $eA_n^d e$ contains no $d$-cube, owing precisely to the $d$-rigid condition. \qed

We show that in some cases Oppermann-Thomas cluster-tilted algebras of type $A_n^d$ are $d$-gentle. Let $1 \leq i \leq n$, and let the subset given by $\{i, i+2, \ldots, i+2(d-1)\}$ (modulo $n$) be denoted by $I_i$.

**Proposition 4.4.** Let $I \subset \{1, 2, \ldots, n\}$ be a subset such that $i, j \in I$ implies $i \neq j + 1 (\text{mod } n)$. Let

$$T = \bigoplus_{x \in \{1, 2, \ldots, n\}} P_x \oplus \bigoplus_{i \in I} \tau_d^{-1} S_i.$$ 

Then $T$ is a $d$-tilting $A_n^d$-module, and $B := \text{End}_{O_{A_n^d}}(T)^{\text{op}}$ is a $d$-gentle algebra.

**Proof.** By [13, Theorem 2.3.1], for any algebra $\Lambda$ with global dimension $d$ and any two $\Lambda$-modules $M$ and $N$, there is an isomorphism

$$\text{Hom}_{\Lambda}(M, \tau_d(N)) \cong \text{Ext}_{\Lambda}^d(N, M).$$

So $\text{Ext}_{A_n^d}^d(T, T) \cong \text{Hom}_{A_n^d}(T, \oplus_{i \in I} S_i) = 0$. Observe that $\tau_d^{-1} S_i \cong S_{i-1}$. It is straightforward to see for any $i \in I$ that there is an exact sequence:

$$0 \rightarrow P_i \rightarrow P_{i-1, i+2, i+4, \ldots, i+2d-2} \rightarrow P_{i-1, i+1, i+4, \ldots, i+2d-2} \rightarrow \cdots \rightarrow P_{i-1} \rightarrow S_{i-1} \rightarrow 0$$

and hence $T$ is a $d$-tilting module.

Let $I$ be as above, and let $A$ be the algebra $\text{End}_{O_{A_n^d}}(A_n^d \oplus \oplus_{i \in I} \tau_d^{-1} S_i)^{\text{op}}$. By construction, for any zero relation $x \rightarrow y \rightarrow z$, one of $x, y, z$ is $I_i$ for some $1 \leq i \leq n$. It can now be seen that there can be no sandwiching by zero relations. Since every relation in $A$ is of length two, $A$ is a $d$-pre-gentle algebra. By definition, there is an idempotent $e$ such that $B = eAe$, and since $T$ is $d$-rigid there can be no $d$-cube in $B$. \qed

**Corollary 4.5.** Let $C \subseteq \text{mod}(A_n^d)$ be the canonical $d$-cluster-tilting subcategory and let $S$ be a semisimple $A_n^d$-module in $C$. Suppose that $\text{Ext}_{A_n^d}^d(S, S) = 0$ and let $P$ be a basic projective $A_n^d$-module such that $\text{Ext}_{A_n^d}^d(S, P) = 0$. For $T' := P \oplus \tau_d^{-1}(S)$, the algebra $\text{End}_{A_n^d}(T')^{\text{op}}$ is $d$-gentle.

**Proof.** Let $I \subset \{1, 2, \ldots, n\}$ be a subset such that $i, j \in I$ implies $i \neq j + 1 (\text{mod } n)$. There is a bijection between such subsets and semisimple $A_n^d$-modules in $C$ such that $\text{Ext}_{A_n^d}^d(S, S) = 0$. Let $T$ be defined as in the statement of Proposition 4.4, then $\text{End}_{A_n^d}(T)^{\text{op}}$ is a $d$-gentle algebra. So choose any projective module $P$ such that $\text{Ext}_{A_n^d}^d(S, P) = 0$. Then there must be an idempotent $e$ such that

$$\text{End}_{A_n^d}(T')^{\text{op}} = e\text{End}_{A_n^d}(T)^{\text{op}} e$$
and this must determine a \(d\)-gentle algebra.

5. Examples

Consider the collection

\[ I := \{135, 136, 137, 138, 139, 147, 148, 149, 157, 158, 159, 169, 179, 357, 579\}. \]

If \( T \) corresponds to the tilting \( A_d \)-module \( T \), then the algebra of \( B := \text{End}_{\mathcal{O}_{A_d}}(T)^{\text{op}} \) is as follows.

\[
\begin{array}{c}
\text{139} \\
\text{138 \rightarrow 149} \\
\text{137 \rightarrow 148 \rightarrow 159} \\
\text{136 \rightarrow 147 \rightarrow 158 \rightarrow 169} \\
\text{135 \rightarrow 357 \rightarrow 157 \rightarrow 579 \rightarrow 179}
\end{array}
\]

This is a 2-gentle algebra in the setting of Proposition 4.3. The proof of Theorem 3.3 shows that the singularity category of \( B \) is equivalent to that of the following gentle algebra:

\[
\begin{array}{c}
\text{136 \rightarrow 147} \\
\text{135 \rightarrow 357 \rightarrow 157 \rightarrow 579 \rightarrow 179} \\
\text{158 \rightarrow 169}
\end{array}
\]

It is unfortunately not true that for every tilting \( A_d \)-module \( T \), the algebra \( \text{End}_{\mathcal{O}_{A_d}}(T)^{\text{op}} \) is \(d\)-gentle. Nevertheless, this does not mean that singularity categories for such algebras are difficult to calculate. An example is the following algebra, which corresponds to the maximal non-intertwining collection

\[ I := \{135, 136, 137, 138, 148, 158, 168, 357, 358, 368\}. \]
Example 1.

This algebra is 2-Iwanaga-Gorenstein, but it is not 2-gentle; there is a sandwiching by zero relations:

This algebra has singularity category:

where the dotted lines denote the $\Omega$ orbit, and the composition of any two arrows is zero.

6. Acknowledgements

This paper was completed as part of my PhD studies, with the support of the Austrian Science Fund (FWF): W1230. I would like to thank my supervisor, Karin
Baur, for her continued help and support during my studies, as well as Ana Garcia Elsener for useful discussions.

REFERENCES

1. Ibrahim Assem, Thomas Brüstle, Gabrielle Charbonneau-Jodoin, and Pierre-Guy Plamondon, Gentle algebras arising from surface triangulations, Algebra & Number Theory 4 (2010), no. 2, 201–229.
2. Ibrahim Assem and Andrzej Skowroński, Iterated tilted algebras of type \(\tilde{A}_n\), Mathematische Zeitschrift 195 (1987), no. 2, 269–290.
3. Karin Baur and Raquel Coelho Simoes, A geometric model for the module category of a gentle algebra, arXiv preprint arXiv:1803.05802 (2018).
4. Petter Andreas Bergh and Marius Thaule, The axioms for \(n\)-angulated categories, Algebr. Geom. Topol. 13 (2013), no. 4, 2405–2428. MR 3073923
5. Ragnar-Olaf Buchweitz, Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings, (1987).
6. Xiao-Wu Chen, Singularity categories, Schur functors and triangular matrix rings, Algebr. Represent. Theory 12 (2009), no. 2-5, 181–191. MR 2501179
7. ______, Gorenstein homological algebra of Artin algebras, arXiv preprint arXiv:1712.04587 (2016).
8. ______, Unifying two results of Orlov on singularity categories, Abh. Math. Semin. Univ. Hambg. 80 (2010), no. 2, 207–212. MR 2734686
9. Xiao-Wu Chen and Henning Krause, Expansions of abelian categories, J. Pure Appl. Algebra 215 (2011), no. 12, 2873–2883. MR 2811570
10. Xiao-Wu Chen and Yu Ye, Retractions and Gorenstein homological properties, Algebras and Representation Theory 17 (2014), no. 3, 713–733.
11. Lucas David-Roesler and Ralf Schiffler, Algebras from surfaces without punctures, Journal of Algebra 350 (2012), no. 1, 218–244.
12. Ana Garcia Elsener, Gentle \(m\)-Calabi-Yau tilted algebras, arXiv preprint arXiv:1701.07968 (2017).
13. Werner Geigle and Helmut Lenzing, Perpendicular categories with applications to representations and sheaves, J. Algebra 144 (1991), no. 2, 273–343. MR 1140607
14. Ch. Geiß and I. Reiten, Gentle algebras are Gorenstein, Representations of algebras and related topics, Fields Inst. Commun., vol. 45, Amer. Math. Soc., Providence, RI, 2005, pp. 129–133. MR 2146244
15. Christoph Geiss, Bernhard Keller, and Steffen Oppermann, \(n\)-angulated categories, J. Reine Angew. Math. 675 (2013), 101–120. MR 3021448
16. Edward L. Green and Sibylle Schroll, Brauer configuration algebras: a generalization of Brauer graph algebras, Bull. Sci. Math. 141 (2017), no. 6, 539–572. MR 3698159
17. Dieter Happel, Triangulated categories in the representation theory of finite-dimensional algebras, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988. MR 935124
18. Osamu Iyama, Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories, Adv. Math. 210 (2007), no. 1, 22–50. MR 2298819
19. ______, Cluster tilting for higher Auslander algebras, Adv. Math. 226 (2011), no. 1, 1–61. MR 2735750
20. Osamu Iyama and Steffen Oppermann, \(n\)-representation-finite algebras and \(n\)-APR tilting, Trans. Amer. Math. Soc. 363 (2011), no. 12, 6575–6614. MR 2833569
21. Gustavo Jasso, \textit{n-abelian and n-exact categories}, Math. Z. \textbf{283} (2016), no. 3–4, 703–759. MR 3519980

22. Martin Kalck, \textit{Singularity categories of gentle algebras}, Bulletin of the London Mathematical Society \textbf{47} (2015), no. 1, 65–74.

23. Daniel Labardini-Fragoso, \textit{Quivers with potentials associated to triangulated surfaces}, Proceedings of the London Mathematical Society \textbf{98} (2009), no. 3, 797–839.

24. Jordan McMahon, \textit{Fabric idempotent ideals and homological dimensions}, arXiv preprint arXiv:1803.07186 (2018).

25. Yoichi Miyashita, \textit{Tilting modules of finite projective dimension}, Mathematische Zeitschrift \textbf{193} (1986), no. 1, 113–146.

26. Sebastian Opper, Pierre-Guy Plamondon, and Sibylle Schroll, \textit{A geometric model for the derived category of gentle algebras}, arXiv preprint arXiv:1801.09659 (2018).

27. Steffen Oppermann and Hugh Thomas, \textit{Higher-dimensional cluster combinatorics and representation theory}, J. Eur. Math. Soc. (JEMS) \textbf{14} (2012), no. 6, 1679–1737. MR 2984586

28. D. O. Orlov, \textit{Triangulated categories of singularities and D-branes in Landau-Ginzburg models}, Tr. Mat. Inst. Steklova \textbf{246} (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 240–262. MR 2101296

29. Yann Palu, Vincent Pilaud, and Pierre-Guy Plamondon, \textit{Non-kissing and non-crossing complexes for locally gentle algebras}, arXiv preprint arXiv:1807.04730 (2018).

30. Chrysostomos Psaroudakis, Øystein Skartsæterhagen, and Øyvind Solberg, \textit{Gorenstein categories, singular equivalences and finite generation of cohomology rings in recollements}, Trans. Amer. Math. Soc. Ser. B \textbf{1} (2014), 45–95. MR 3274657

31. Sibylle Schroll, \textit{Trivial extensions of gentle algebras and Brauer graph algebras}, Journal of Algebra \textbf{444} (2015), 183–200.

32. Raquel Coelho Simoes and Mark James Parsons, \textit{Endomorphism algebras for a class of negative Calabi-Yau categories}, Journal of Algebra \textbf{491} (2017), 32–57.

33. Andrzej Skowroński and J Waschbüscher, \textit{Representation-finite biserial algebras}, Journal für Mathematik-Didaktik \textbf{345} (1983), 23.

34. Hans-Joachim von Höhne and Josef Waschbüscher, \textit{Die Struktur n-reihiger Algebren}, Comm. Algebra \textbf{12} (1984), no. 9-10, 1187–1206. MR 738544