Phase transitions for products of characteristic polynomials under Dyson Brownian motion

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Abstract

We study the averaged products of characteristic polynomials for the Gaussian and Laguerre $\beta$-ensembles with external source, and prove Pearcey-type phase transitions for particular full rank perturbations of source. The phases are characterised by determining the explicit functional forms of the scaled limits of the averaged products of characteristic polynomials, which are given as certain multidimensional integrals, with dimension equal to the number of products.

1 Introduction

1.1 Matrix-valued Brownian motion

Let $Y$ be an $N \times N$ standard Gaussian matrix with real, complex or quaternion entries, the latter represented as particular $2 \times 2$ complex matrices. Forming $G = \frac{1}{2}(Y + Y^\ast)$ then gives a real symmetric, complex Hermitian or self-dual quaternion random matrix. This is the construction of matrices drawn from the Gaussian orthogonal, Gaussian unitary and Gaussian symplectic ensembles respectively (GOE, GUE and GSE) — the naming relates to the subset of unitary matrices which diagonalise $G$. Associated with each ensemble is a Dyson index $\beta$, which takes on the value 1, 2, or 4 depending on the number of independent real and imaginary parts in a single entry of $G$.

The dynamical description of the eigenvalues of GOE, GUE and GSE random matrices as diffusion processes was first stated by Dyson [28] in 1962. Replacing the real and imaginary parts of each entry of $Y$ by independent Brownian motions, Dyson constructed a matrix-valued random process and further observed that the eigenvalues satisfy a system of stochastic differential equations (SDEs), specifying what is known as Dyson Brownian motion. In subsequent years, Dyson Brownian motion has been both an important research topic in its own right, and an effective tool in random matrix theory; see e.g. [2,23,32,39,45,48].

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In particular, see Erdős and Yau’s book [33] for a comprehensive survey on its application to the universality problem for Wigner matrices.

To give some more detail, let \( \{ B_{j,k}(t), \tilde{B}_{j,k}(t) : 1 \leq j \leq k \leq N \} \) be a set of i.i.d. real-valued standard Brownian motions. The symmetric (\( \beta = 1 \)) and complex Hermitian (\( \beta = 2 \)) matrix-valued Brownian motion \( H(t) \), is a random process on matrices with entries equal to

\[
H_{j,k} = \begin{cases} 
\frac{1}{\sqrt{2}} (B_{j,k}(t) + i(\beta - 1) \tilde{B}_{j,k}(t)), & j < k, \\
\frac{1}{\sqrt{2}} B_{j,j}, & j = k.
\end{cases}
\]

(1.1)

For \( t \geq 0 \), let \( x_1(t), \ldots, x_N(t) \) denote the eigenvalues of the Hermitian matrix

\[
X(t) = H(t) + X(0)
\]

(1.2)

where \( X(0) \) is a fixed symmetric (\( \beta = 1 \)) or complex Hermitian (\( \beta = 2 \)) matrix. Let \( B_1(t), \ldots, B_N(t) \) denote standard Brownian motions. A fundamental observation of Dyson [28] (see [5] and [41] for textbook treatments) is that the eigenvalues satisfy a system of SDEs

\[
dx_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{j \neq i} \frac{1}{x_i(t) - x_j(t)} dt, \quad i = 1, \ldots, N,
\]

(1.3)

with \( \{ x_i(0) \}_{i=1}^N \) the eigenvalues of \( X(0) \).

Also of interest is the extension of (1.2) to chiral matrices specified by

\[
X(t) = \begin{bmatrix} 0_{p \times p} & Z(t) \\ Z^*(t) & 0_{N \times N} \end{bmatrix} + \begin{bmatrix} A & 0 \\ A^* & 0_{N \times N} \end{bmatrix}.
\]

(1.4)

Here \( Z(t) \) is a \( p \times N \) random matrix with entries i.i.d. real (\( \beta = 1 \)) or complex (\( \beta = 2 \)) Brownian motions, while \( A \) is a fixed \( p \times N \) matrix with real (\( \beta = 1 \)) or complex (\( \beta = 2 \)) entries. For \( p \geq N \), this matrix structure implies \( X(t) \) has \( p - N \) zero eigenvalues. The remaining \( 2N \) eigenvalues occur in \( \pm \) pairs, with the positive eigenvalues equal to the eigenvalues of the matrix

\[
(Z(t) + A)^* (Z(t) + A).
\]

(1.5)

It was shown by Bru [16] in the real case, and by Konig and O’Connell [44] in the complex case that the eigenvalues of (1.5) evolve as the system of SDEs

\[
dx_i(t) = 2\sqrt{x_i(t)} dB_i(t) + \beta \left( p + \sum_{j \neq i} \frac{x_i(t) + x_j(t)}{x_i(t) - x_j(t)} \right) dt, \quad i = 1, \ldots, N,
\]

(1.6)

with \( \{ x_i(0) \}_{i=1}^N \) the eigenvalues of \( A^* A \).

Dyson, in his selected papers [30], commented on the matrix-valued Brownian motion: “the physical motivation for introducing it is that it represents a system whose Hamiltonian is a sum of two parts, one known and one unknown”. For the known part, introduce the notation \( x_i(0) = f_i \) \( (i = 1, \ldots, N) \) for the eigenvalues of \( X(0) \) in (1.2) and of \( A^* A \) in (1.5), whereas for the eigenvalues for general \( t \) in these matrices introduce the notation \( x_i(t) = x_i \) \( (i = 1, \ldots, N) \). Let \( \Delta_N(x) := \prod_{1 \leq j < k \leq N} (x_k - x_j) \) denote the Vandermonde product. The
eigenvalue probability density for the matrices \( \text{E} \) can be calculated as [35] Eqs. (11.101) and (13.146) with \( t \to 1, \tilde{\beta} \to 1/2T \) then \( T \to \sqrt{T} \)

\[
P^{(G)}_{t, N}(x) = \frac{1}{\Gamma_{\beta,N}} t^{-\frac{N(2+\beta(N-1))}{2}} 
\prod_{i=1}^{N} e^{-\frac{1}{2t}(x_i^2+f_i^2)} |\Delta_N(x)|^\beta \quad \text{and} \quad 0F_0^{(2/\beta)} \left( \frac{1}{\sqrt{t}} x, \frac{1}{\sqrt{t}} \right), \quad (1.7)
\]

while for the functional form of the eigenvalue probability density for the matrices \( \text{E} \) we have [35] Eqs. (11.105) and (13.147) with \( t \to 1, \tilde{\beta} \to 1/T \) then \( T \to t \) and the change of variables \( x_i^2 \to x_i, (x_{i(0)})^2 \to f_i \)

\[
P^{(L)}_{t, N}(x) = \frac{1}{Z_{a,\beta,N}} t^{-\frac{1}{2} - \frac{N}{2} - \beta N(1)} 
\prod_{i=1}^{N} x_i a^{-1} e^{-\frac{1}{2t}(x_i+f_i)} \times |\Delta_N(x)|^\beta \quad 0F_1^{(2/\beta)} (a + \frac{1}{2} \beta(N-1); \frac{1}{t} x; \frac{1}{t} f), \quad (1.8)
\]

where \( a = \frac{\tilde{\beta}}{2}(p - N + 1) \). In the present setting the multivariate functions \( 0F_0^{(2/\beta)} \) and \( 0F_1^{(2/\beta)} \) are most naturally defined as the matrix integrals

\[
0F_0^{(2/\beta)} (x; f) = \int_{Q(N)} e^{\text{tr}(Q\Delta,tQ^{-1}A)} dQ,
\]

\[
1F_0^{(2/\beta)} (x; f) = \int_{Q(N)} dQ \int_{Q(p)} dR e^{\text{tr}(Q\Delta,tR^{-1}A)} dR.
\]

with \( \Delta = [\sqrt{\Delta}, 0_{N \times (p-N)}] \). Here \( Q(N) \) denotes the classical groups \( O(N) \) \( (\beta = 1) \), \( U(N) \) \( (\beta = 2) \) and \( Sp(2N) \) \( (\beta = 4) \), and for \( Q \in Q(N) \), \( dQ \) denotes the corresponding normalized Haar measure. The normalization constants \( \Gamma_{\beta,N} \) and \( Z_{a,\beta,N} \) are given in Appendix [11] and more detail about hypergeometric functions is given in Appendix [12].

The SDEs [13] and [14] have meaning for general \( \beta > 0 \), and can be shown to correspond to the Fokker-Planck dynamics of certain particle systems on the line and half line respectively, which interact via a logarithmic potential in the presence of a heat bath at inverse temperature \( \beta \); see e.g. [35] Ch. 11. The particle probability density functions are again given by [17] and [18], but now with \( 0F_0^{(2/\beta)} \) and \( 0F_1^{(2/\beta)} \) defined as multivariate hypergeometric functions based on Jack polynomials; see [35] Ch. 13 and Appendix [11] for a brief summary. It is furthermore the case that \( 0F_0^{(2/\beta)} \) and \( 0F_1^{(2/\beta)} \) for general \( \beta > 0 \) correspond to the eigenvalue PDF of recursively defined random matrices [36], which in turn for \( \beta = 1, 2 \) and 4 correspond to the Gaussian and Laguerre ensembles with an external source; see [24][35][37]. In keeping with this, we will denote the ensembles corresponding to [17] and [18] as \( \text{GE}_{\beta, t, N}(x; f) \) and \( \text{LE}_{a,\beta,t,N}(x; f) \) — in words Gaussian \( \beta \)-ensemble with a source, and Laguerre \( \beta \)-ensemble with a source — respectively.
1.2 Products of characteristic polynomials — Riemann zeros

The celebrated Riemann hypothesis in prime number theory asserts that the complex zeros of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{primes}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1,$$

when analytically continued in the whole complex plane, are all of the form $s = \frac{1}{2} \pm iE$, $E > 0$. These are termed the Riemann zeros. There is a conjecture attributed to Hilbert and Pólya [52] asserting that the Riemann zeros correspond to the eigenvalues of an as yet unknown unbounded self adjoint operator. From the topic of quantum chaos there is a prediction [15] that the highly excited energy levels of a generic Schrödinger operator have the same statistics as the bulk eigenvalues of large real symmetric (assuming a time reversal symmetry), or complex Hermitian (no time reversal symmetry). In keeping with these points, and based on analytic evidence from the work of Montgomery [46], and large scale, high precision numerical work of Odlyzko [47] of the $10^{20}$-th Riemann zero and over 70 million of its neighbours, the Montgomery–Odlyzko law asserts that the large Riemann zeros have the same statistical properties as the bulk eigenvalues of large complex Hermitian matrices.

Keating and Snaith [42] extended the Montgomery–Odlyzko law by proposing the use of the characteristic polynomial of Haar distributed random unitary matrices $Z(U, \theta) := \det(I - U e^{-i\theta})$ to model the statistical properties of $\zeta(s)$ for $s = \frac{1}{2} + iT$, $T \gg 1$. It has been known since the work of Dyson [29] that the statistical properties of the eigenvalues of large Haar distributed unitary matrices coincide with the statistical properties of the bulk eigenvalues of large GUE matrices. Specifically, with $N = \log T$ (the average spacing between eigenvalues and Riemann zeros then agree to leading order) it was hypothesised that the statistical properties of $Z(U, \theta)$ correctly give the corresponding statistical properties of $\zeta(0.5 + iT)$ up to known arithmetic factors. A celebrated example exhibited in [42] applied the moments formula for $Z(U, \theta)$ [8, 42]

$$\langle |Z(U, \theta)|^{2k} \rangle = \prod_{j=1}^{N} \frac{\Gamma(j) \Gamma(j + 2k)}{\Gamma(j + k)^2} \sim \frac{G^2(k + 1)}{G(2k + 1)} N^{k^2}, \quad (1.9)$$

where $G(z)$ denotes the Barnes $G$-function, and satisfies $G(z + 1) = \Gamma(z)G(z)$, to predict the corresponding asymptotic formula for the moments of the Riemann zeta function

$$\frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt \sim \frac{G^2(k + 1)}{G(2k + 1)} a(k) \left( \log \frac{t}{2\pi} \right)^{k^2},$$

where $a(k)$ is a known number theoretic constant.

1.3 Products of characteristic polynomials — phase transition

The general topic of averages of products and ratios of characteristic polynomials in random matrix ensembles has attracted an enormous amount of research;
papers relevant to the present study, where we focus attention on

\[ K_{t,N}^{(G)}(s; f) := \left\langle \prod_{j=1}^{n} \prod_{k=1}^{N} (s_j - \sqrt{\frac{2}{\beta}} x_k) \right\rangle_{x \in \text{GE}_{\beta,t,N}(x; f)}, \]  

(1.10)

and

\[ K_{t,N}^{(L)}(s; f) := \left\langle \prod_{j=1}^{n} \prod_{k=1}^{N} (s_j - \sqrt{\frac{2}{\beta}} x_k) \right\rangle_{x \in \text{LE}_{\beta,t,N}(x; f)}, \]  

(1.11)

include \[3, 7, 17, 19, 20, 22, 26, 27, 38\]. In the case \( n = 1 \) and with \( \beta = 2 \) these averages are well known to relate to multiple orthogonal polynomials associated with the corresponding ensembles; see \[11, 25\].

We seek the asymptotic form of \(1.10\) and \(1.11\) in settings corresponding to what for \( \beta = 2 \) has been termed the Pearcey universality class; see \[1, 12, 18, 43, 50\]. In the Brownian motion picture, this corresponds to the circumstance when in the ensemble \( \text{GE}_{\beta,t,N}(x; f) \) the eigenvalues take on one of two values chosen symmetrically about the origin. As time increases, the eigenvalues spread, first in the neighbourhood of the two values, then eventually colliding in the neighbourhood of the origin. It is literally a “collision” with the origin that is the physical mechanism of the transition in the ensemble \( \text{LE}_{\beta,t,N}(x; f) \), when the initial condition has all but a finite number of eigenvalues concentrated at a point some distance away on the positive half axis.

Earlier studies for general \( \beta > 0 \) have considered phase transitions in the finite rank case, say \( f_{r+1} = \cdots = f_N = 0 \), specifying the scaling limits of the averages \(1.10\) and \(1.11\); see \[36\] \((n = 1 \text{ and finite } r)\), \[29\] \((r = 0 \text{ and finite } n)\) and \[27\] \((\text{finite } n, r)\). This follows works on the scaling limits of the distribution function of the largest eigenvalue in the cases \( \beta = 1, 2 \) and 4 corresponding to a variant of the Laguerre \( \beta \)-ensemble with a source known as the general variance Wishart ensemble \[31, 37\]. The analogue of finite rank is then a spiking of the covariance matrix; see \[6\] for spiked complex Wishart matrices and \[13, 14, 37, 40, 51\] for spiked real and quaternion Wishart matrices.

The starting point is to implement duality formulas for the averaged products of characteristic polynomials in the two ensembles \( \text{GE}_{\beta,t,N}(x; f) \) and \( \text{LE}_{\beta,t,N}(x; f) \), due to Desrosiers \[24\] Proposition 8 \((\text{see also } 36)\), which read

\[ \left\langle \prod_{j=1}^{n} \prod_{k=1}^{N} (s_j - i \sqrt{\frac{2}{\beta}} x_k) \right\rangle_{x \in \text{GE}_{\beta,t,N}(x; f)} = \left\langle \prod_{k=1}^{N} \prod_{j=1}^{n} (x_j - i \sqrt{\frac{2}{\beta}} f_k) \right\rangle_{x \in \text{GE}_{4/\beta,1,n}(x; s)}, \]  

(1.12)

and

\[ \left\langle \prod_{j=1}^{n} \prod_{k=1}^{N} (s_j + \sqrt{\frac{2}{\beta}} x_k) \right\rangle_{x \in \text{LE}_{\beta,t,N}(x; f)} = \left\langle \prod_{k=1}^{N} \prod_{j=1}^{n} (x_j + \sqrt{\frac{2}{\beta}} f_k) \right\rangle_{x \in \text{LE}_{2n/\beta,4/\beta,1,n}(x; s)}, \]  

(1.13)

where the variables \( s_j \) denote the argument of characteristic polynomials. We stress that the duality relation has transformed the \( N \)-dimensional integral on
the LHS into an \( n \)-dimensional integral on the RHS, which is suitable for asymptotic analysis as \( N \to \infty \) whenever \( n \) is fixed, at least in principle; see [26,27,34] for some earlier relevant results.

The rest of the paper is organized as follows. Asymptotic analysis relating to the Pearcey universality class for the Gaussian \( \beta \)-ensemble with a source is undertaken in Section 2, and that for the Laguerre \( \beta \)-ensemble with a source in Section 3. Appendix 4.1 lists some constants, and Appendix 4.2 gives a brief account of the hypergeometric functions in (1.7) and (1.8).

### 2 Transition for the Gaussian ensemble

In this section we study scaled limits of the averaged product of characteristic polynomials for the Gaussian \( \beta \)-ensemble with a source, under the assumption that the vector \( f \) of initial eigenvalues takes on just two values chosen symmetrically about the origin, and in (close to) equal proportion. Rescaling the time as \( t = \hat{t}/N \), we will see that as \( \hat{t} \) changes a critical value \( t_c \) is reached, and so distinguishing three different regimes: (i) subcritical regime of \( \hat{t} > t_c \); (ii) critical regime of \( \hat{t} = t_c \); and (iii) supercritical regime of \( \hat{t} < t_c \).

Two families of multivariate functions of extended Selberg type are required. One is the Pearcey weighted function

\[
P^{(\alpha)}_{n,m}(\tau; y; \sigma) := \frac{1}{\Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}^n} e^{-\sum_{j=1}^{n} \left( \frac{1}{4} u_j^4 + \tau^2 u_j^2 \right)} \prod_{j=1}^{n} \prod_{k=1}^{m} (iu_j + \sigma_k) |\Delta_n(u)|^{\frac{4}{\alpha}} F_{\alpha}^{(\alpha)}(iu; y) d^n u,
\]

where \( \alpha \in \mathbb{R}_+, \tau \in \mathbb{R} \) and \( y \in \mathbb{R}^n, \sigma \in \mathbb{R}^m \). The other is the Gaussian weighted function

\[
G^{(\alpha)}_{n,m}(y; \sigma) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}^n} e^{-\sum_{j=1}^{n} \frac{1}{2} (u_j^2 - y_j^2)} \prod_{j=1}^{n} \prod_{k=1}^{m} (iu_j + \sigma_k) |\Delta_n(u)|^{\frac{1}{\alpha}} F_{\alpha}^{(\alpha)}(iu; y) d^n u.
\]

The latter has been defined in [27] except that the trivial factor \( e^{\sum_{j=1}^{n} y_j^2/2} \) was absent there. We immediately see from the duality relation (1.12) that

\[
G^{(\alpha)}_{n,m}(y; \sigma) = (i\sqrt{\alpha})^{mn} G^{(1/\alpha)}_{m,n}(i\sqrt{\alpha} \sigma; i\sqrt{\alpha} y).
\]

**Theorem 2.1.** For a fixed integer \( r \geq 0 \), suppose that \( N \) is a positive integer such that \( N + r \) is even and moreover that

\[
f_{r+1} = \cdots = f_{(N+r)/2} = -f_{1+(N+r)/2} = \cdots = -f_N = \sqrt{\beta/2} b, \ b > 0.
\]

With (1.10), as \( N \to \infty \) the following hold uniformly for any \( y_1, \ldots, y_n \) in a compact subset of \( \mathbb{R} \).

(i) When \( t > b^2 \), let \( f_1, \ldots, f_r \) be in a compact subset of \( \mathbb{R} \) and set

\[
s_j = \frac{t}{\sqrt{t - b^2}} y_j, \ j = 1, \ldots, n.
\]
For even $n$ we have
\[
\lim_{N \to \infty} \frac{1}{\Psi_{\text{sub}}} K_{t/N,N}^{(G)}(s; f) = \gamma_{n/2}(4/\beta) e^{-t \sum_{j=1}^n y_j} e_1 F_1(\beta/2)(n/\beta; 2n/\beta; 2y),
\]
where the function $e_1 F_1$ is defined in (4.13) of Appendix 4.2.

(ii) When $t = b^2(1 + \tau/\sqrt{N})^{-1}$, let $f_{m+1}, \ldots, f_r$ be in a compact subset of $R \setminus \{0\}$ with $0 \leq m \leq r$, and set
\[
s_j = N^{-\frac{1}{4}} y_j, \quad j = 1, \ldots, n \quad \text{and} \quad f_k = \sqrt{\beta} N^{-\frac{1}{2}} \sigma_k, \quad k = 1, \ldots, m
\]
with $\tau, \sigma_1, \ldots, \sigma_m \in R$. We have
\[
\lim_{N \to \infty} \frac{1}{\Psi_{\text{crit}}} K_{t/N,N}^{(G)}(s; f) = P_{n,m}^{(\beta/2)}(\tau; y; \sigma).
\]

(iii) When $t < b$, let $f_{m+1}, \ldots, f_r$ be in a compact subset of $R \setminus \{0\}$ with $1 \leq m \leq r$, and set
\[
s_j = \sqrt{\frac{(b^2-t)}{b^2}} y_j, \quad j = 1, \ldots, n \quad \text{and} \quad f_k = \sqrt{\frac{\beta b^2}{2(b^2-t)}} \sigma_k, \quad k = 1, \ldots, m
\]
with $\sigma_1, \ldots, \sigma_m \in R$. We have
\[
\lim_{N \to \infty} \frac{1}{\Psi_{\text{sup}}} K_{t/N,N}^{(G)}(s; f) e^{\frac{t}{2} \sum_{j=1}^n y_j^2} = G_{n,m}^{(\beta/2)}(y; \sigma).
\]

Here $\gamma_{n/2}(4/\beta)$, $\Psi_{\text{sub}}$, $\Psi_{\text{crit}}$ and $\Psi_{\text{sup}}$ are constants given in Appendix 4.1.

The proofs of Theorem 2.1 and Theorem 3.1 below build on asymptotic results for integrals of Selberg type; see Corollary 3.11 and Corollary 3.12 in [26]. Our new finding is to establish two families of phase transitions at the origin by adjusting external parameters properly and then by doing very subtle calculations. In the unitary case of $\beta = 2$, one can exactly solve the Gaussian and Laguerre ensembles with external source by using the famous HCIZ formula (see e.g. [35]), and so gets more delicate results such as correlation functions and the distribution of the largest eigenvalue, see [18, 50] and [43]. However, to our knowledge, for $\beta \neq 2$, even in the orthogonal case of $\beta = 1$, the Pearcey-type results have previously been investigated only for a single characteristic polynomial.

**Proof of Theorem 2.1.** Introduce scaled variables
\[
s_j = v + \frac{1}{\rho N} y_j, \quad j = 1, \ldots, n,
\]
where $v$ and $\rho$ are yet to be determined. With the assumption (2.4) in mind, application of the duality formula (1.12) shows
\[
K_{t/N,N}^{(G)}(s; f) = \frac{1}{\Gamma_{4/\beta,n}} (-t)^n N (N/t)^{\frac{\beta}{4}} e^{\frac{n(n-1)}{2} \sum_{j=1}^n (v + \frac{1}{\rho N} y_j)^2} I_N,
\]
(2.12)
where
\[
I_N = \int_{\mathbb{R}^n} e^{-N\sum_{j=1}^n p(x_j)} \prod_{j=1}^n \prod_{k=1}^n (x_j - i\sqrt{2/\beta} f_k) \prod_{j=1}^n (x_j^2 + b^2)^{-r/2} \times |\Delta_0(x)^{4/\beta} \mathcal{F}_0(\beta/2)(x/t; iy/\rho) d^n x.
\]
(2.13)

Here the exponent is given by
\[
p(z) = \frac{1}{2t}(z^2 - 2ivz - t \log(z^2 + b^2)).
\]
(2.14)

We apply now the method of steepest descent to analyse the large \( N \) form of the integral. From (2.14), the saddle point equation reads
\[
p'(z) = \frac{z - iv}{t} - \frac{z}{z^2 + b^2} = 0,
\]
(2.15)

which further reduces to the cubic equation
\[
\xi^3 - v\xi^2 - (b^2 - t)\xi + vb^2 = 0,
\]
(2.16)

where \( \xi = -iz \). The above equation has appeared in [4, 10] and the detailed analysis about its solutions plays a central role in establishing the local eigenvalue statistics in the bulk and at the soft edge in the case \( \beta = 2 \), where there is a determinantal structure.

To proceed, we take \( v = 0 \). Then, as the time \( t \) changes, there is a critical value \( t_c = b^2 \) at which (2.15) has a double root, so distinguishing the three different regimes: (i) subcritical for \( t > t_c \); critical for \( t = t_c \); and (iii) supercritical for \( t < t_c \).

**Subcritical case: \( t > b^2 \).** In this case the two solutions of (2.15) are \( z_\pm = \pm \sqrt{t - b^2} \), and moreover \( p(z) \) attains the same minimum over the real axis exactly at \( z = z_\pm \). Obviously, we have
\[
p(z_\pm) = \frac{1}{2} - \frac{b^2}{2t} = \frac{1}{2} \log t, \quad p''(z_\pm) = \frac{2}{t^2}(t - b^2) > 0.
\]
(2.17)

Recall the definition of the scaled variables (2.5), by [26, Corollary 3.12] we conclude that as \( N \to \infty \) the major contribution to the integral \( I_N \) comes from the case that \( n/2 \) variables lie in the neighborhood of \( z_+ \), and the other \( n/2 \) in that of \( z_- \). Taking \( \rho = \sqrt{t - b^2}/t \), simple calculations then show
\[
I_N \sim \left( \frac{n}{n/2} \right) \left( \Gamma_{4/\beta, n/2} \right)^2 e^{-1/2nN(1 - 2s)} t^{-1/2} \prod_{k=1}^n (- - (t - b^2) - \frac{2}{\beta} f_k) \times \left( t/\sqrt{2N(t - b^2)} \right)^{n+\beta(n-2)} \mathcal{F}_0(\beta/2)(1(\xi), (-1)(\xi); iy).
\]
(2.18)

Together with (2.12), upon noting the following identity (cf. [26, Corollary 2.3])
\[
\mathcal{F}_0(\beta/2)(1(\xi), (-1)(\xi); iy) = e^{-i\sum_{j=1}^n y_j} \mathcal{F}_1(\beta/2)(\beta/n/2; 2n/\beta; 2iy),
\]
(2.19)

where the function \( \mathcal{F}_1 \) is defined in [4, 13] of Appendix 4.2, we thus complete the subcritical case (i).
Critical case: } t = b^2(1 + \tau/\sqrt{N})^{-1}. We recall the definition of the scaled variables (2.7) and let \rho = N^{-1/4}. Changing variables x_j = N^{-1/4}u_j in (2.13), and noting the power series expansion

\[ p(x_j) = -\log b + \frac{1}{2N}(\tau u_j^2 + \frac{1}{2}u_j^4) + O(N^{-\frac{1}{2}}), \]

(2.20)
a simple calculation shows

\[ I_N \sim b^{(N-r)}N^{-\frac{(N-1)n-(m+1)n}{n}(i)} \prod_{j=m+1}^{N} \left( \sqrt{\frac{2}{\beta}}f_j \right)^{n} \]

\[ \int_{\mathbb{R}^n} \prod_{j=1}^{n} \left( e^{-\frac{2\pi}{b^2}u_j^2 + \frac{i}{b}j} \prod_{k=1}^{m} (iu_j + \sigma_k) \right) |\Delta_n(u)|^{4/\beta} e^{\beta/2}(iu; y) d^n u. \]

(2.21)

Combining (2.12) and the definition (2.1), one obtains

\[ K_{t/\sqrt{N}, N}^{(G)}(s, f) \sim \Psi_{\text{cri}} P_{n,m}^{(\beta/2)}(\tau; y; \sigma), \]

(2.22)

where \Psi_{\text{cri}} is given in Appendix 4.1.

Supercritical case: } t < b^2. In this case, p(z) attains its global minimum over the real axis exactly at z = 0. Let \rho = b/\sqrt{Nt(b^2 - t)} and change variables x_j = b\sqrt{t}u_j/\sqrt{N(b^2 - t)} in (2.13). Noting the Taylor expansion

\[ p(x_j) = -\log b + \frac{1}{2N}u_j^2 + O(N^{-\frac{1}{2}}), \]

(2.23)

and recalling the scaled variables in (2.10), we get the leading contribution

\[ I_N \sim (-i)^{mn}b^{(N-r)} \left( b\sqrt{t}/\sqrt{N(b^2 - t)} \right)^{(n-1)n+(m+1)n} \prod_{j=m+1}^{n} \left( \sqrt{\frac{2}{\beta}}f_j \right)^{n} \]

\[ \int_{\mathbb{R}^n} \prod_{j=1}^{n} \left( e^{-\frac{2\pi}{b^2}u_j^2 + \frac{i}{b}j} \prod_{k=1}^{m} (iu_j + \sigma_k) \right) |\Delta_n(u)|^{4/\beta} e^{\beta/2}(iu; y) d^n u. \]

(2.24)

Substituting the above into (2.12), we thus get

\[ K_{t/\sqrt{N}, N}^{(G)}(s, f) \sim \Psi_{\text{sup}} e^{-\frac{2\pi}{b^2} \sum_{j=1}^{n} y_j^2} G_{n,m}^{(\beta/2)}(y; \sigma). \]

(2.25)

The proof of the theorem is thus completed. \qed

We conclude this section with a few remarks about the assumption (2.4):

(i) This (or a similar) initial condition has been used to investigate the Peccey phenomenon in random matrices with source and non-intersecting Brownian motions, see e.g. [11,12,13,20,50]. (ii) The techniques introduced in the proof of Theorem 2.1 are applicable to a wider class of initial conditions, say, \( f_{r+1} = \cdots = f_{r+N_1} = b_1 \) and \( f_{r+N_1} = \cdots = f_N = b_2 \) with \( N_1/N \to c \in (0, 1) \). In this case the key equation satisfied by saddle points, like (2.14), will become more complicated and it is believed that there is no new interesting phenomenon, so we don’t proceed further.
3 Transition for the Laguerre ensemble

In this section we also assume that all but finitely many source eigenvalues, say \( r \), are the same. A hard edge phase transition can be described as follows as the time \( t \) changes. We first need to define two families of multivariate functions of Selberg type, for \( a > 0 \) one is defined to be

\[
P_{n,m}^{(a,a)}(y;\tau) := \frac{1}{Z_{a,2/a,n}} \int_{\mathbb{R}_+^n} \prod_{i=1}^n u_i^{-1} e^{-\tau u_i} \prod_{j=1}^m (u_i + \sigma_j) \times |\Delta_n(u)|^{\frac{1}{2}} \, {}_0F_1^{(a)}(a; n-1; u; -y) \, d^n u,
\]  

(3.1)

while the other reads

\[
W_{n,m}^{(a,a)}(y;\tau) := \frac{1}{Z_{a,2/a,n}} \int_{\mathbb{R}_+^n} \prod_{i=1}^n u_i^{-1} e^{-\tau u_i+y} \prod_{j=1}^m (u_i + \frac{1}{a} \sigma_j) \times |\Delta_n(u)|^{\frac{1}{2}} \, {}_0F_1^{(a)}(a+1; n-1; u; -y) \, d^n u.
\]  

(3.2)

It immediately follows from the duality relation (1.13) that the latter satisfies

\[
W_{n,m}^{(a,a)}(y;\tau) = \alpha^{-mn} W_{m,n}^{(a,a,1/a)}(-\sigma; -y).
\]  

(3.3)

Theorem 3.1 (Hard edge phase transition). Suppose that for a fixed integer \( r \geq 0 \),

\[
f_{r+1} = \cdots = f_N = \beta b/2, \quad b > 0.
\]  

(3.4)

With (1.11), as \( N \to \infty \) the following hold for any \( y_1, \ldots, y_n \) in a compact subset of \([0,\infty)\).

(i) For \( t > b \), let \( f_1, \ldots, f_r \) be in a compact subset of \([0,\infty)\) and set

\[
s_j = \frac{t^2 - b}{t - b} \frac{y_j}{N^2}, \quad j = 1, \ldots, n.
\]  

(3.5)

We have

\[
\lim_{N \to \infty} \Phi_{\text{sub}} \frac{1}{\Phi_{\text{cri}}} K_{t/N,N}^{(L)}(s;f) = {}_0F_1^{(\beta/2)}(2(a+n-1)/\beta;-y).
\]  

(3.6)

(ii) For \( t = b(1 - \tau/\sqrt{N}) \), let \( f_{m+1}, \ldots, f_r > 0 \) with \( 0 \leq m \leq r \) and set

\[
s_j = \frac{b y_j}{N^{3/2}}, \quad j = 1, \ldots, n \quad \text{and} \quad f_k = \frac{\beta \sigma_k}{2 \sqrt{N}}, \quad k = 1, \ldots, m
\]  

(3.7)

with \( \tau \in \mathbb{R} \) and \( \sigma_1, \ldots, \sigma_m \geq 0 \). We have

\[
\lim_{N \to \infty} \frac{1}{\Phi_{\text{cri}}} K_{t/N,N}^{(L)}(s;f) = B_{n,m}^{(2a/\beta,\beta/2)}(\tau;y;\sigma).
\]  

(3.8)

(iii) For \( t < b \), let \( f_{m+1}, \ldots, f_r > 0 \) with \( 1 \leq m \leq r \) and set

\[
s_j = \frac{t(b - t)}{b} \frac{y_j}{N}, \quad j = 1, \ldots, n \quad \text{and} \quad f_k = \frac{bt}{b - t} \sigma_j, \quad k = 1, \ldots, m
\]  

(3.9)

with \( \sigma_1, \ldots, \sigma_m \geq 0 \). We have

\[
\lim_{N \to \infty} \frac{1}{\Phi_{\text{sup}}} K_{t/N,N}^{(L)}(s;f) e^{(t/b) \sum_{i=1}^n y_i} = W_{n,m}^{(2a/\beta,\beta/2)}(y;\sigma).
\]  

(3.10)
Here $\Phi_{\text{sub}}, \Phi_{\text{cri}}$ and $\Phi_{\text{sup}}$ are constants given in Appendix 4.4.

**Proof.** Recalling the assumption (3.4), by the duality formula (1.13) we find

$$K_{t/N,N}^{(L)}(s,f) = \frac{1}{Z_{2a/\beta,4/\beta,n}}(-1)^{nN}(N/t)^{2n(a+n-1)/\beta}e^{(N/t)\sum_{i=1}^{n} s_i}I_N,$$  \hspace{1cm} (3.11)

where

$$I_N = \int_{R^n} e^{-N \sum_{i=1}^{n} p(x_i)} \prod_{i=1}^{n} \left( x_i^{\frac{a}{\beta}} - 1 (x_i + b)^{-r} \prod_{j=1}^{r} (x_i + \frac{2}{\beta} f_j) \right) \times |\Delta_n(x)|^{4/\beta} _{0}F_{1}^{(\beta/2)}(c; N x/t; -N s/t) \, dx.$$  \hspace{1cm} (3.12)

Here $c := 2(a + n - 1)/\beta$ and

$$p(z) = \frac{z}{t} - \log(z + b).$$  \hspace{1cm} (3.13)

Applying now the method of steepest descent, we see from (3.13) that the saddle point equation reads

$$p'(z) = \frac{1}{t} - \frac{1}{z + b} = 0,$$  \hspace{1cm} (3.14)

and further implies the saddle point $z_0 = t - b$. Next, we establish the hard-edge limits in three different cases: (i) subcritical regime of $t > b > 0$, (ii) critical regime of $t = b > 0$ and (iii) supercritical regime of $0 < t < b$.

**Subcritical case:** $z_0 > 0$. In this case $p(z_0) = 1 - bt^{-1} - \log t$ and $p''(z_0) = t^{-2} > 0$. Moreover, $p(z)$ attains the minimum over the positive real axis at $z = z_0$. Recalling the scaled variables in (3.5), by [26, Corollary 3.11] we conclude that as $N \to \infty$ the leading term of the integral $I_N$ comes from the neighborhood of $z_0$. Noting the change of variables (3.5), we have

$$I_N \sim \int_{R^n} e^{-Np(z_0) - N e^{\alpha z_0}} \sum_{i=1}^{n} (x_i - z_0)^2 \left( x_0^{\frac{a}{\beta}} - 1 (z_0 + b)^{-r} \right)^n \times \prod_{j=1}^{r} (z_0 + \frac{2}{\beta} f_j)^n |\Delta_n(x)|^{4/\beta} _{0}F_{1}^{(\beta/2)}(c; \frac{z_0}{t - b}; y) \, dx.$$  \hspace{1cm} (3.15)

$$= e^{-N(1 - \frac{1}{t} - \log t)(t - b)(\frac{2}{\beta} - 1)nL^{-rn}} \prod_{j=1}^{r} (t - b + \frac{2}{\beta} f_j)^n \times (t/\sqrt{N})^{2n(n-1) + n \Gamma_{4/\beta,n} \, _{0}F_{1}^{(\beta/2)}(c; -y)},$$  \hspace{1cm} (3.16)

where the constant $\Gamma_{4/\beta,n}$ is given in (4.11).

Together with (3.11) we thus get

$$K_{t/N,N}^{(L)}(s,f) \sim \Phi_{\text{sub}} \, _{0}F_{1}^{(\beta/2)}(c; -y).$$  \hspace{1cm} (3.17)

**Critical case:** $z_0 = 0$. Recall the double scaling $t = b(1 - \tau/\sqrt{N})$ and the scaled variables in (3.7). After the change of variables $x_i = u_i/\sqrt{N}$ in (3.12), noting the Taylor expansion

$$p(x_i) = -\log b + \frac{1}{N}(\tau u_i + \frac{1}{2} u_i^2) + O(N^{-\frac{3}{2}}),$$  \hspace{1cm} (3.18)

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a simple calculation shows

\[ I_N \sim b^n(N-r) \left( \frac{1}{\sqrt{N}} \right)^{\frac{\beta}{2}(a+n-1)n+mn} \prod_{j=m+1}^{r} \left( \frac{2}{\beta f_j} \right)^n \int_{\mathbb{R}_+^n} \phi_1^{(\beta/2)}(c; u; -y) \]

\[ \times \prod_{i=1}^{n} \left( u_i^{2\beta-1} e^{-\tau u_i - \frac{1}{2} \sigma_i^2} \prod_{j=1}^{m} (u_i + \sigma_j) \right) |\Delta_n(u)|^{4/\beta} d^n u. \]  

Combining (3.11) and (3.1), we obtain

\[ K_{t/N,N}^{(L)}(s, f) \sim \Phi_{\text{cri}} B^{(2a/\beta, \beta/2)}(r; y; \sigma). \]  

**Supercritical case:** \( z_0 < 0 \). In this case \( p(z) \) attains its minimum over the positive real axis at \( z = 0 \). Change variables \( x_i = btu_i/(N(b-t)) \) in (3.12) and note the Taylor expansion

\[ p(x_i) = -\log b + N^{-1}u_i + O(N^{-2}). \]  

Recalling now the scaled variables in (3.9), we have

\[ I_N \sim b^n(N-r) \left( \frac{bt}{N(b-t)} \right)^{\frac{\beta}{2}(a+n-1)n+mn} \prod_{j=m+1}^{r} \left( \frac{2}{\beta f_j} \right)^n \int_{\mathbb{R}_+^n} \phi_2^{(\beta/2)}(c; u; -y) \]

\[ \times e^{-\sum_{i=1}^{n} u_i \prod_{j=1}^{m} (u_i + \frac{2}{\beta} \sigma_j)} |\Delta_n(u)|^{4/\beta} d^n u. \]  

Substituting the above into (3.11), we thus get

\[ K_{t/N,N}^{(L)}(s, f; t/N) \sim (-1)^n N^n b^n(N-r) \left( \frac{b}{b-t} \right)^{\frac{\beta}{2}(a+n-1)n+mn} \left( \frac{t}{N} \right)^{mn} \prod_{j=m+1}^{r} \left( \frac{2}{\beta f_j} \right)^n \]

\[ \times \int_{\mathbb{R}_+^n} e^{(1-\frac{\beta}{2}) \sum_{i=1}^{n} y_i} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \prod_{i=1}^{n} \left( u_i^{2\beta-1} e^{-\tau u_i - \frac{1}{2} \sigma_i^2} \prod_{j=1}^{m} (u_i + \frac{2}{\beta} \sigma_j) \right) \]

\[ \times |\Delta_n(u)|^{4/\beta} d^n u_1 \cdots d^n u_n. \]  

From this, together with the duality formula (1.13) and the definition (3.2), we have the desired result

\[ K_{t/N,N}^{(L)}(s, f) \sim \Phi_{\text{sup}} e^{-t/(b)} \sum_{i=1}^{n} y_i W^{(2a/\beta, \beta/2)}_{n,m}(y; \sigma). \]  

**Remark 3.2.** Theorem 3.1 (i) also holds even when \( b = 0 \), which can be proved by following almost the same procedure.

### 4 Appendices

#### 4.1 Constants

The normalization constants in the Gaussian and Laguerre \( \beta \)-ensembles with a source, being independent of the source and \( t \), are the normalizations corresponding to the case \( f = (0, \ldots, 0) \) and \( t = 1 \). The generalized hypergeometric
functions are then equal to unity, and we can read off from [35, Eq. (1.160)] and [35, eqn (3.128)] that

\[ \Gamma_{\beta,n} = (2\pi)^{n/2} \prod_{j=1}^{n} \frac{\Gamma(1+j\beta/2)}{\Gamma(1+\beta/2)}. \]  

(4.1)

and

\[ Z_{a,\beta,N} = \prod_{j=0}^{N-1} \frac{\Gamma(1+(1+j)\beta/2)\Gamma(a+j\beta/2)}{\Gamma(1+\beta/2)}. \]  

(4.2)

A systematic way to obtain these evaluations is to use Selberg integral theory; see [35, §4.7].

The constants associated with the Gaussian \( \beta \)-ensemble with a source appearing in Theorem 2.1 are

\[ \Psi_{\text{sub}} = (-i)^{n(N-r)}2(t-b^2)^{-\frac{1}{2}+\frac{2}{\beta}} \prod_{j=1}^{r} \left( t - b^2 + \frac{2}{\beta} \right)^{n}, \]  

(4.3)

\[ \gamma_m(\beta') = \left( \frac{2m}{m} \right) \prod_{j=1}^{m} \frac{\Gamma(1+\beta'j/2)}{\Gamma(1+\beta'(m+j)/2)} \]  

(4.4)

\[ \Psi_{\text{crt}} = (-i)^{n(r+N)}b^{n(N-r)}N^{-\frac{1}{2}}(1-m)n + \frac{1}{2}r(n-1)n \prod_{j=m+1}^{r} \left( \sqrt{\frac{2}{\beta}} f_j \right)^n, \]  

(4.5)

and

\[ \Psi_{\text{sup}} = (-i)^{n(N+r)}b^{n(N-r)}(b/\sqrt{b^2-t})^{\frac{r}{2}}(n-1)n \prod_{j=m+1}^{r} \left( \sqrt{\frac{2}{\beta}} f_j \right)^n. \]  

(4.6)

The constants associated with the Laguerre \( \beta \)-ensemble with a source appearing in Theorem 3.1 are

\[ \Phi_{\text{sub}} = \frac{\Gamma_{4/\beta,n}}{Z_{2a/\beta,4/\beta,n}} \left( 1 - \frac{b}{t} \right)^{\frac{n}{2}} \prod_{j=1}^{r} \left( 1 - \frac{b}{t} + \frac{2}{\beta f_j} \right)^n \times (-1)^{nN}e^{-nN(1-\frac{1}{\beta} - \log t)N^{\frac{n-1}{2}}+(\frac{n}{2} + \frac{1}{2})n}, \]  

(4.7)

\[ \Phi_{\text{cri}} = (-1)^{aN}b^{n(N-r)}b^{-\frac{n}{2}(a+n-1)}N^{-\frac{(a+n-1)N}{n}} \prod_{j=m+1}^{r} \left( \frac{2}{\beta} f_j \right)^n, \]  

(4.8)

and

\[ \Phi_{\text{sup}} = (-1)^{nN}b^{n(N-r)} \left( \frac{b}{b-t} \right)^{\frac{n}{2}(a+n-1)+mn} \left( \frac{t}{N} \right)^{mn} \prod_{j=m+1}^{r} \left( \frac{2}{\beta} f_j \right)^n. \]  

(4.9)
4.2 Jack polynomials and hypergeometric functions

This appendix provides a brief review of Jack polynomials and hypergeometric functions [49, 53]; see also [35, Chapter 12].

A partition $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_i, \ldots)$ is a sequence of non-negative integers $\kappa_i$ such that

$$\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_i \geq \cdots$$

and only a finite number of the terms $\kappa_i$ are non-zero. The number of non-zero terms is referred to as the length of $\kappa$, and is denoted $\ell(\kappa)$. We shall not distinguish between two partitions that differ only by a string of zeros. The weight of a partition $\kappa$ is the sum

$$|\kappa| := \kappa_1 + \kappa_2 + \cdots$$

of its parts, and its diagram is the set of points $(i, j) \in \mathbb{N}^2$ such that $1 \leq j \leq \kappa_i$. Reflection in the diagonal produces the conjugate partition $\kappa' = (\kappa'_1, \kappa'_2, \ldots)$. The set of all partitions of a given weight is partially ordered by the dominance order: $\kappa \leq \sigma$ if and only if $\sum_{i=1}^{\kappa_i} \kappa_i \leq \sum_{i=1}^{\sigma_i} \sigma_i$ for all $\kappa$.

Let $\Lambda_n(x)$ be the algebra of symmetric polynomials in $n$ variables $x_1, \ldots, x_n$ with coefficients in the field $\mathbb{F} = \mathbb{Q}(\alpha)$, which is the field of rational functions in the parameter $\alpha > 0$. It is invariant under the action of homogeneous differential operators related to the Calogero-Sutherland models [9]:

$$E_k = \sum_{i=1}^{n} x_i^k \frac{\partial}{\partial x_i}, \quad D_k = \sum_{i=1}^{n} x_i^k \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{1 \leq i \neq j \leq n} \frac{x_i^k}{x_i - x_j} \frac{\partial}{\partial x_i}, \quad k = 0, 1, 2, \ldots.$$

The operators $E_1$ and $D_2$ can be used to define the Jack polynomials. Indeed, for each partition $\kappa$, there exists a unique symmetric polynomial $P^{(\alpha)}_\kappa(x)$ that satisfies the following two conditions [10]:

1. $P^{(\alpha)}_\kappa(x) = m_\kappa(x) + \sum_{\mu < \kappa} c_{\kappa \mu} m_\mu(x)$ \hspace{1cm} (triangularity) \hspace{1cm} (4.10)
2. $\left(D_2 - \frac{2}{\alpha}(n-1)E_1\right) P^{(\alpha)}_\kappa(x) = \epsilon_\kappa P^{(\alpha)}_\kappa(x)$ \hspace{1cm} (eigenfunction) \hspace{1cm} (4.11)

where $\epsilon_\kappa, c_{\kappa \mu} \in \mathbb{F}$. Because of the triangularity condition, $\Lambda_n(x)$ is also equal to the span over $\mathbb{F}$ of all Jack polynomials $P^{(\alpha)}_\kappa(x)$, with $|\kappa| \leq n$.

For $(i, j) \in \kappa$, let $a_\kappa(i, j) = \kappa_i - j$ and $l_\kappa(i, j) = \kappa_j - i$. Introduce the hook-length of $\kappa$ defined by

$$h^{(\alpha)}_\kappa = \prod_{(i, j) \in \kappa} \left(1 + a_\kappa(i, j) + \frac{1}{\alpha} l_\kappa(i, j)\right),$$

and the $\alpha$-deformation of the Pochhammer symbol by

$$[x]^{(\alpha)}_\kappa = \prod_{1 \leq i \leq \ell(\kappa)} \left(x - \frac{i-1}{\alpha}\right)_{\kappa_i}.$$

Here $(x)_j \equiv x(x+1) \cdots (x+j-1)$. We now turn to the precise definition of the hypergeometric series associated with Jack polynomials, see e.g. [53]. Given $p, q \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, let $a_1, \ldots, a_p, b_1, \ldots, b_q$ be complex numbers such that
(i − 1)/α − b_j \notin \mathbb{N}_0 \text{ for all } i \in \mathbb{N}_0. \text{ The } (p, q)\text{-type hypergeometric series in two }\text{ sets of } n \text{ variables } x = (x_1, \ldots, x_n) \text{ and } y = (y_1, \ldots, y_n) \text{ is defined as}

\begin{equation}
\begin{split}
pFq_{\alpha}(a_1, \ldots, a_p; b_1, \ldots, b_q; x; y) &= \sum_{k=0}^{\infty} \sum_{|\kappa| = k} \frac{1}{h_\kappa} \frac{[a_1]_\kappa^{(\alpha)} \cdots [a_p]_\kappa^{(\alpha)}}{[b_1]_\kappa^{(\alpha)} \cdots [b_q]_\kappa^{(\alpha)}} P_\kappa^{(\alpha)}(x) P_\kappa^{(\alpha)}(y) .
\end{split}
\end{equation}

(4.12)

where the shorthand notation 1^{(n)} stands for 1, \ldots, 1 with n times. In particular, when \(y_1 = \cdots = y_n = 1\), it reduces to the hypergeometric series

\begin{equation}
\begin{split}
pFq_{\alpha}(a_1, \ldots, a_p; b_1, \ldots, b_q; x) &= \sum_{k=0}^{\infty} \sum_{|\kappa| = k} \frac{1}{h_\kappa} \frac{[a_1]_\kappa^{(\alpha)} \cdots [a_p]_\kappa^{(\alpha)}}{[b_1]_\kappa^{(\alpha)} \cdots [b_q]_\kappa^{(\alpha)}} P_\kappa^{(\alpha)}(x).
\end{split}
\end{equation}

(4.13)

Note that when \(p \leq q\), the series (4.13) converges absolutely for all \(x \in \mathbb{C}^n\).

Setting \(x = 0^{(n)}\) in either (4.12) we see that the only term contributing is \(\kappa\) having all parts equal to zero, which shows

\begin{equation}
\begin{split}
pFq_{\alpha}(a_1, \ldots, a_p; b_1, \ldots, b_q; 0^{(n)}; y) = 1.
\end{split}
\end{equation}

(4.14)

Less immediate but also of interest is the simplification of (4.12) in the case \(y = y^{(n)}\). Thus (see e.g. [35, Eq. (13.63)])

\begin{equation}
\begin{split}
0Fq^{(\alpha)}(x; y^{(n)}) = 0Fq^{(\alpha)}(yx_1, \ldots, yx_n) = \exp\left(\sum_{i=1}^{n} x_i\right).
\end{split}
\end{equation}

(4.15)

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