More on the Bernoulli*-Taylor formula for extended umbral calculus

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Abstract

One presents here the $\psi$-Bernoulli-Taylor* formula of a new sort with the rest term of the Cauchy type recently derived by the author in the case of the so called $\psi$-difference calculus which constitutes the representative for the purpose case of extended umbral calculus. The central importance of such a type formulas is beyond any doubt - and recent publications do confirm this historically established experience. Its links via umbrality to combinatorics are known at least since Rota and Mullin-Rota source papers then up to recently extended by many authors to be indicated in the sequel.

KEY WORDS: umbral calculus, Bernoulli formula, Graves-Heisenberg-Weyl algebra(**)

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* see below : a historical remark based on Academician N.Y.Sonin article published in Petersburg in 19-th century. We owe this information and the article to Professor O.V.Viscov from Moscow.

** see: C. Graves, On the principles which regulate the interchange of symbols in certain symbolic equations, Proc. Royal Irish Academy 6(1853–1857), 144-152
1 One Historical Remark

Here are the famous examples of expansion
\[ \partial_0 = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \frac{d^n}{dx^n} \]

or
\[ \epsilon_0 = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \frac{d^n}{dx^n} \]

where \( \partial_0 \) is the divided difference operator while \( \epsilon_0 \) is at the zero point evaluation functional. If one compares these with "series universalissima" of J.Bernoulli from Acta Erudicorum (1694) (see commentaries in [12]) and with
\[ \exp \{ yD \} = \sum_{k=0}^{\infty} \frac{y^k D^k}{k!}, \quad D = \frac{d}{dx}, \]

then confrontation with B.Taylor’s "Methodus incrementorum directa et inversa" (1715), London; entitles one to call the expansion formulas considered in this note "Bernoulli - Taylor formulas" or (for \( n \to \infty \)) "Bernoulli - Taylor series" [1].

Information: Johann Bernoulli was elected a fellow of the academy of St Petersburg. Johann Bernoulli - the Discoverer of Series Universalissima was "Archimedes of his age" and this is indeed inscribed on his tombstone.

2 Introduction

A. From here now \( \psi \) denotes an extension of \( \langle \frac{1}{n} \rangle_{n \geq 0} \) sequence to quite arbitrary one (the so called "admissible"- see: Markowsky) and the specific choices are for example: Fibonomialy -extended (\( \langle F_n \rangle \) - Fibonacci sequence) \( \langle \frac{1}{F_n} \rangle_{n \geq 0} \) or just ”the usual” \( \langle \frac{1}{n} \rangle_{n \geq 0} \) or Gauss \( q \)-extended \( \langle \frac{1}{n_q} \rangle_{n \geq 0} \) admissible sequences of extended umbral operator calculus - see more in [16, 13, 14, 15].

The simplicity of the first steps to be done while identifying general properties of such [6,7,8,16,13-15] \( \psi \)-extensions consists in natural notation i.e. here - in writing objects of these extensions in mnemonic convenient upside down notation [15], [13]

\[ \frac{\psi(n-1)}{\psi_n} = n\psi, n\psi! = n\psi(n - 1)\psi!, n > 0, x\psi = \frac{\psi(x - 1)}{\psi(x)}, \quad (1) \]


\[ x^k_\psi = x_\psi(x-1)_\psi(x-2)_\psi... (x-k+1)_\psi \]  \hspace{1cm} (2)

\[ x_\psi(x-1)_\psi... (x-k+1)_\psi = \frac{\psi(x-1)_\psi(x-2)_\psi... \psi(x-k)}{\psi(x)_\psi(x-1)... \psi(x-k+1)} \]  \hspace{1cm} (3)

If one writes the above in the form \( x_\psi \equiv \frac{\psi(x-1)}{\psi(x)} \equiv \Phi(x) \equiv \Phi_x \equiv x_\Phi \), one sees that the name upside down notation is legitimate.

You may consult \[10\], \[9\], \[14,15\] for further development and use of this notation.

With such an extension we frequently though not always may “\( \psi \)-mnemonic” repeat with exactly the same simplicity and beauty most of what was done by Rota \[10\], \[15\]. Accordingly the extension of notions and formulas with its elementary essential content and context to general case of \( \psi \)-umbral instead of umbral or \( q \)-umbral calculi case only - is sometimes automatic \[10\], \[14,15\] (see corresponding earlier references there and necessary definitions).

**B.** While deriving the Bernoulli-Taylor \( \psi \)-formula one is tempted to adapt the ingenious Viskov’s method \[2\] of arriving to formulas of such type for various pairs of operations. In our case these would be \( \psi \)-differentiation and \( \psi \)-integration (see: Appendix). However straightforward application of Viscov methods in \( \psi \)-extensions of umbral calculus leads to sequences which are not normal (Ward) hence a new invention is needed. This expected and verified here invention is the new specific \( \ast_\psi \) product of analytic functions or formal series. This note is based on \[3\] where the derivation of this new form of Bernoulli-Taylor \( \ast_\psi \)-formula was delivered due to the use of a specific \( \ast_\psi \)-product of formal series.

### 3 Classical Bernoulli-Taylor formulas with the rest term of the Cauchy type by Viskov method

Let us consider the obvious identity

\[ \sum_{k=0}^{n} (\alpha_k - \alpha_{k+1}) = \alpha_0 - \alpha_{n+1} \]  \hspace{1cm} (4)

in which \(4\) we now put \( \alpha_k = a^kb^k; a, b \in \mathcal{A} \). \( \mathcal{A} \) is an associative algebra with unity over the field F=R,C. Then we get

\[ \sum_{k=0}^{n} a^k(1-ab)b^k = 1 - a^{n+1}b^{n+1}; a, b \in \mathcal{A} \]  \hspace{1cm} (5)
Numerous choices of \( a, b \in A \) result in many important specifications of (5).

**Example 1.** Let \( \mathcal{F} \) denotes the linear space of sufficiently smooth functions \( f : F \rightarrow F \). Let

\[
a : \mathcal{F} \rightarrow \mathcal{F}; \quad (af)(x) = \int_a^b f(t)dt,
\]
\[
b : \mathcal{F} \rightarrow \mathcal{F}; \quad (bf)(x) = \left( \frac{d}{dx} f \right)(x);
\]
\[
l : \mathcal{F} \rightarrow \mathcal{F}; \quad (lf)(x) = f(x).
\]

Then \([b,a]=1-ab=\varepsilon_\alpha\) where \( \varepsilon_\alpha \) is evaluation functional on \( \mathcal{F} \) i.e.

\[
\varepsilon_\alpha(f) = f(\alpha)
\]

Using now the text-book integral Cauchy formula \( (k > 0) \)

\[
(a^k f)(x) = \int_a^x \frac{(x-t)^{k-1}}{(k-1)!} f(t)dt,
\]

and under the choice \( (6) \) one gets from (5) the well-known Bernoulli-Taylor formula

\[
f(x) = \sum_{k=0}^{n} \frac{(x-\alpha)^k}{k!} f^{(k)}(\alpha) + R_{n+1}(x)
\]

with the rest term \( R_{n+1}(x) \) in the Cauchy form

\[
R_{n+1}(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt
\]

**Example 2.** Let \( \mathcal{F} \) denotes the linear space of functions \( f : Z_+ \rightarrow F; Z_+ = N \cup \{0\} \). Let

\[
a : Z_+ \rightarrow \mathcal{F}; \quad (af)(x) = \sum_{k=0}^{x-1} f(k),
\]
\[
b : Z_+ \rightarrow \mathcal{F}; \quad (bf)(x) = f(x+1) - f(x),
\]
\[
l : Z_+ \rightarrow \mathcal{F}; \quad (lf)(x) = f(x).
\]

It is easy to see that \([b,a]=1-ab=\varepsilon_0\) where \( \varepsilon_0 \) is evaluation functional i.e. \( \varepsilon_0(f) = f(0) \). \( b = \Delta \) is the standard difference operator with its left inverse.
definite summation operator a. The corresponding $\Delta$ - calculus Cauchy formula is also known (see formula (31 p.310 in [5]);

$$(a^k f)(x) = \sum_{r=0}^{x-1} \frac{(x - r - 1)^{k-1}}{(k-1)!} f(r); k > 0$$  \hspace{1cm} (12)

where $x^n = x(x-1)(x-2)...(x-n+1)$.

Under the choice (11) one gets from (5) the $\Delta$ - calculus Bernoulli - Taylor formula [1]

$$f(x) = \sum_{k=0}^{n} \frac{x^k}{k!} (\Delta^k f)(0) + R_{n+1}(x)$$  \hspace{1cm} (13)

with the rest term $R_{n+1}(x)$ in the Cauchy $\Delta$ form

$$R_{n+1}(x) = \sum_{r=0}^{x-1} \frac{(x - r - 1)^n}{n!} (\Delta^{n+1} f)(r);$$  \hspace{1cm} (14)

4 "*$_\psi$ realization" of Bernoulli identity.

Now a specifically new form of the Bernoulli-Taylor formula with the rest term of the Cauchy type as well as Bernoulli-Taylor series is to be supplied in the case of $\psi$-difference umbral calculus (see [5-8] and [9,10] and references therein). For that to do we use natural $\psi$-umbral representation [13,14] of Graves-Heisenberg-Weyl (GHW) algebra [11,12] generators $\hat{p}$ and $\hat{q}$ and then we use Bernoulli identity (15)

$$\hat{p} \sum_{k=0}^{n} \frac{(-\hat{q})^k \hat{p}^k}{k!} = \frac{(-\hat{q})^n \hat{p}^{n+1}}{n!}$$  \hspace{1cm} (15)

derived by Viskov from (11) under the substitution (see (28) in [2])

$$\alpha_0 = 0, \ \alpha_k = (-1)^k (\hat{q})^{k-1} \hat{p}^k (k - 1)!, \ k = 1, 2,...$$
due to $\hat{p} \hat{q}^n = \hat{q}^n \hat{p} + n \hat{q}^{n-1}$ (n=1,2,...) resulting by induction from

$$[\hat{p}, \hat{q}] = 1$$  \hspace{1cm} (16)

**Example 1.** The choice $\hat{p} = D \equiv \frac{d}{dx}$ and $\hat{q} = \dot{x} - y, y \in F; \dot{x} f(x) = x f(x)$ after substitution into Bernoulli identity (15) and integration $\int_{\alpha}^{\dot{x}} dt$ gives the Bernoulli - Taylor formula (9).
Example 2. The choice \( \hat{p} = \Delta \) and \( \hat{q} = \hat{x} \circ E^{-1} \) where \( E^\alpha f(x) = f(x + \alpha) \) after substitution into Bernoulli identity (15) and "\( \Delta \) - integration" \( \sum_{r=0}^{\alpha-1} r \frac{n}{n!} (\nabla^{n+1} f)(r + 1) \) gives the Bernoulli - Mac laurin formula of the following form (\( \alpha, x \in \mathbb{Z}, \nabla = 1 - E^{-1} \)) with the rest term \( R_{n+1}(x) \)

\[
 f(0) = \sum_{k=0}^{n} \frac{\alpha^k}{k!} (-1)^{k+1} (\nabla^k f)(\alpha) + R_{n+1}(\alpha); \tag{17}
\]

\[
 R_{n+1}(\alpha) = (-1)^n \sum_{r=0}^{\alpha-1} r \frac{n}{n!} (\nabla^{n+1} f)(r + 1). \tag{18}
\]

Example 3. Here \( f^{(k)} \equiv \partial_{\psi}^k f \) and \( f(x) \ast_{\psi} g(x) \equiv f(\hat{x}_{\psi})g(x) \) - see Appendix. The choice \( \hat{p} = \partial_{\psi} \) and \( \hat{q} = \hat{x}_{\psi} \) (\( z = x - y \)) where \( \hat{x}_{\psi} x^n = \frac{n+1}{(n+1)_{\psi}} x^{n+1} \) after substitution into Bernoulli identity (15) and "\( \partial_{\psi} \) - integration" \( \int_{\alpha}^{x} d_{\psi} t \) (see: Appendix) gives another Bernoulli - Taylor \( \psi \)-formula of the form:

\[
 f(x) = \sum_{k=0}^{n} \frac{1}{k!} (x - \alpha)^{k\ast_{\psi}} f^{(k)}(\alpha) + R_{n+1}(x) \tag{19}
\]

with the rest term \( R_{n+1}(x) \) in the Cauchy-form

\[
 R_{n+1}(x) = \frac{1}{n!} \int_{\alpha}^{x} d_{q} t (x - t)^{n\ast_{q}} f^{(n+1)}(t) dt \tag{20}
\]

In the above notation \( x^{0\ast_{\psi}} = 1, \ x^{n\ast_{\psi}} \equiv x \ast_{\psi} (x^{(n-1)\ast_{\psi}}) = x \ast_{\psi} \ldots \ast_{\psi} x = \frac{n!}{n_{\psi}!} x^n; \ n \geq 0. \)

Naturally \( \partial_{\psi} x^{n\ast_{\psi}} = n x^{(n-1)\ast_{\psi}} \) and in general \( f, g \) - may be formal series for which

\[
 \partial_{\psi} (f \ast_{\psi} g) = (Df) \ast_{\psi} g + f \ast_{\psi} (\partial_{\psi} g) \tag{21}
\]

i.e. Leibniz \( \ast_{\psi} \) rule holds \[13, 14, 15\].

Summary: These another forms of both the Bernoulli - Taylor formula with the rest term of the Cauchy type \[3\] as well as Bernoulli - Taylor series are quite easily handy due to the technique developed in \[13, 14\] where one may find more on \( \ast_{\psi} \) product devised perfectly suitable for the Ward’s "calculus of sequences" \[6\] or more exactly \( \ast_{\psi} \) is devised perfectly suitable for the so-called \( \psi \) - extension on Finite Operator Calculus of Rota (see \[9, 10, 14, 15\] and references therein)
5 Appendix

*ψ product

Let $n - \psi \equiv \psi_n; \psi_n \neq 0$: $n > 0$. Let $\partial_\psi$ be a linear operator acting on formal series and defined accordingly by $\partial_\psi x^n = n\psi x^{n-1}$.

We introduce now a intuition appealing $\partial_\psi$-difference-ization rules for a specific new $*\psi$ product of functions or formal series. This $*\psi$ product is what we call: the $\psi$-multiplication of functions or formal series as specified below.

**Notation A.1.**

$x * \psi x^n = \hat{x}_\psi(x^n) = \frac{(n+1)}{(n+1)_\psi} x^{n+1}; \quad n \geq 0$ hence $x * \psi 1 = (1_\psi)^{-1} x \neq x$

therefore $x * \psi \alpha 1 = \alpha 1 * \psi x = x * \psi \alpha = \alpha * \psi x = \alpha (1_\psi)^{-1} x$ and $\forall x, \alpha \in F$; $f(x) * \psi x^n = f(\hat{x}_\psi)x^n$.

For $k \neq n$ $x^n * \psi x^k \neq x^k * \psi x^n$ as well as $x^n * \psi x^k \neq x^{n+k}$ - in general.

In order to facilitate the formulation of observations accounted for on the basis of $\psi$-calculus representation of GHW algebra we shall use what follows.

**Definition A.1.** With Notation A.1. adopted define the $*\psi$ powers of $x$ according to $x^n*\psi \equiv x * \psi x^{(n-1)*_\psi} = \hat{x}_\psi(x^{(n-1)*_\psi}) = x * \psi x * \psi ... * \psi x$

$= \frac{n!}{n_\psi!} x^n; \quad n \geq 0$. Note that $x^n*\psi x^k*\psi = \frac{n!}{n_\psi!} x^{(n+k)*_\psi} \neq x^k * \psi x^n * \psi = \frac{k!}{k_\psi!} x^{(n+k)*_\psi}$ for $k \neq n$ and $x^0 * \psi = 1$.

This noncommutative $\psi$-product $*\psi$ is devised so as to ensure the following observations.

**Observation A.1**

a) $\partial_\psi x^n*\psi = n x^{(n-1)*_\psi}; \quad n \geq 0$

b) $\exp\{\alpha x\} = \exp\{\alpha \hat{x}_\psi\} 1$

c) $\exp\{\alpha x\} * \psi (\exp\{\beta \hat{x}_\psi\} 1) = (\exp\{\alpha + \beta \hat{x}_\psi\}) 1$

d) $\partial_\psi(x^k * \psi x^n*\psi) = (Dx^k) * \psi x^n*\psi + x^k * \psi (\partial_\psi x^n*\psi)$

e) $\partial_\psi(f * \psi g) = (Df) * \psi g + f * \psi (\partial_\psi g); \ f, g$ - formal series

f) $f(\hat{x}_\psi)g(\hat{x}_\psi) 1 = f(x) * \psi \tilde{g}(x); \ \tilde{g}(x) = g(\hat{x}_\psi) 1$.

**Umbral ”~” Note:** $\tilde{g}(x) = g(\hat{x}_\psi) 1$ defines the map $\sim$: $g \mapsto \tilde{g}$ i.e. $\sim$: $P \mapsto P$ which is an umbral operator.

**ψ-Integration** Let: $\partial_\psi x^n = x^{n-1}$. The linear operator $\partial_\psi$ is identical with divided difference operator. Let $\hat{Q} f(x) f(qx)$. Recall also that to the
"$\partial_q$ difference-ization" there corresponds the $q$-integration which is a right inverse operation to "$q$-difference-ization". Namely

$$F(z) \equiv \left( \int_q \varphi \right)(z) := (1 - q) z \sum_{k=0}^{\infty} \varphi(q^k z) q^k$$

(22)

i.e.

$$F(z) \equiv \left( \int_q \varphi \right)(z) = (1 - q) z \sum_{k=0}^{\infty} q^k \hat{Q}^k \varphi(z) = \left( (1 - q) z \frac{1}{1 - q \hat{Q}} \right)(z).$$

(23)

Of course

$$\partial_q \circ \int_q = \text{id}$$

(24)

as

$$\frac{1 - q \hat{Q}}{(1 - q)} \partial_0 \left( (1 - q) \hat{z} \frac{1}{1 - q \hat{Q}} \right) = \text{id}.$$ 

(25)

Naturally (25) might serve to define a right inverse operation to "$q$-difference-ization" $(\partial_q \varphi)(x) = \frac{1 - q \hat{Q}}{(1 - q)} \partial_0 \varphi(x)$ and consequently the "$q$-integration" as represented by (22) and (23). As it is well known the definite $q$-integral is a numerical approximation of the definite integral obtained in the $q \to 1$ limit.

Finally we introduce the analogous representation for $\partial_\psi$ difference-ization

$$\partial_\psi = \hat{n}_\psi \partial_\psi; \quad \hat{n}_\psi x^{n-1} = n_\psi x^{n-1}; \quad n \geq 1$$

(26)

Then

$$\int_\psi x^n = \left( \hat{x} \frac{1}{n_\psi} \right) x^n = \frac{1}{(n + 1)_\psi} x^{n+1}; \quad n \geq 0$$

(27)

and of course $(\int_\psi f \equiv \int d_\psi)$

$$\partial_\psi \circ \int_\psi = \text{id}$$

(28)

Naturally

$$\partial_\psi \circ \int_\psi \int_{-\infty}^{\infty} f(t) d_\psi t = f(x)$$
The formula of "per partes" $\psi$-integration is easily obtainable from (Observation A.1 e) and it reads:

$$\int_a^b (f \ast_\psi \partial_\psi g)(t)d_\psi t = [(f \ast_\psi g)(t)]_a^b - \int_a^b (Df \ast_\psi g)(t)d_\psi t$$ (29)

Closing Remarks:

I. All these above may be quite easily extended [15] to the case of any $Q \in \text{End}(P)$ linear operator that reduces by one the degree of each polynomial [16]. Namely one introduces [15]:

**Definition A.2.**

$$\hat{x}_Q \in \text{End}(P), \hat{x}_Q : F[x] \to F[x]$$

such that $(x^n) = \frac{(n+1)}{(n+1)_\psi} q_{n+1}; n \geq 0; \text{where } Qq_n = nq_{n-1}$. Then $\ast_Q$ product of formal series and $Q$-integration are defined almost mnemonic analogously.

II. In 1937 Jean Delsarte [17] had derived the general Bernoulli-Taylor formula for a class of linear operators $\delta$ including linear operators that reduce by one the degree of each polynomial. The rest term of the Cauchy-like type in his Taylor formula (I) is given in terms of the unique solution of a first order partial differential equation in two real variables. This first order partial differential equation is determined by the choice of the linear operator $\delta$ and the function $f$ under expansion. In our Bernoulli-Taylor formula (16)–(17) or in its straightforward $\ast_Q$ product of formal series and $Q$-integration generalization - there is no need to solve any partial differential equation.

III. In [18] (1941) Professor J. F. Steffensen - the Master of polynomials application to actuarial problems

(see: http://www.math.ku.dk/arkivet/jfsteff/stfarkiv.htm)

supplied a remarkable derivation of another Bernoulli-Taylor formula with the rest of "$Q$-Cauchy type" in the example presenting the "Abel poweroids"

IV. The recent paper [19] (2003) by Mourad E. H. Ismail, Denis Stanton may serve as a kind of indication for pursuing further investigation. There the authors have established two new q-analogues of a Taylor series expansion for polynomials using special Askey-Wilson polynomial bases. As "byproducts" their important paper includes also new summation theorems, quadratic transformations for q-series and new results on a q-exponential function.

V. Let us also draw an attention to two more different publication on the subject which are the ones referred to as [20,21]. The $q$-Bernoulli theorems
are named here and there above as \( q \)-Taylor theorems. The corresponding \((q,h)\)-Bernoulli theorem for the \( \partial_{q,h} \)-difference calculus of Hahn \[22\] might be also obtained as the the one \((q,0)\)-Bernoulli i.e. \( q \)-Bernoulli theorem constituting here the special case the Viskov method \[2\] application. This is so because the \( \partial_{q,h} \)-difference calculus of Hahn \[22\] may be reduced to \( q \)-calculus of Thomae-Jackson \[5,23\] due to the following observation. Let

\[
h \in R, (E_{q,h}\varphi)(x) = \varphi(qx + h)
\]

and let

\[
(\partial_{q,h}\varphi)(x) = \frac{\varphi(x) - \varphi(qx + h)}{(1-q)x - h}
\] (30)

Then (see Hann \[5\]and \[22\])

\[
\partial_{q,h} = E_1 - \frac{h}{1-q} \partial_q E_1, \frac{h}{1-q}.
\] (31)

Due to (30) it is easy now to derive corresponding formulas including Bernoulli-Taylor \( \partial_{q,h} \)-formula obtained in \[24\] by the Viskov method \[2\] which for

\[
q \to 1, h \to 0
\]

recovers the content of one of the examples in \[2\] , while for

\[
q \to 1, h \to 1
\]

one recovers the content of the another example in \[2\]. The case \( h \to 0 \) is included in the formulas of \( q \)-calculus of Thomae-Jackson easy to be specified : see \[22\] (see also thousands of up-date references there). For Bernoulli-Taylor Formula (presented during PTM - Convention Lodz - 2002) : contact \[23\] for its recent version.

The comparison of the all above quoted ways to arrive at extended Bernoulli formulae we leave for another exhibition of similar investigations.

VI. As indicated right after Observation A.1.e) the rule \( \tilde{g}(x) = g(\tilde{x}) \cdot 1 \) defines the map \( \sim: g \mapsto \tilde{g} \) which is an umbral operator \( \sim: P \mapsto P \). It is mnemonic extension of the corresponding \( q \) - definition by Kirschenhofer \[24\] and Cigler \[25\]. This umbral operator (without reference to to \[24,25\]) had been already used in theoretical physics aiming at Quantum Mechanics on the lattice \[26\]. The similar aim is represented by \[27,28\] (see further references there) where incidence algebras are being prepared for that purpose (Dirac notation included). As it is well known - the classical umbral \[29,30\] and extended \[10\] finite operator calculi may be formulated in the reduced
incidence algebra language. Hence both applications of related tools to the same goal are expected to meet at the arena of GHW algebra description of both [14,13].

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