Some Remarks on the Local Unitary Classification of Three-Qubit Pure States

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Abstract. Characterization of multipartite entanglement is in general a challenging task. The entanglement polytope is particularly useful to study three-qubit systems since it allows a qualitative analysis of the related non-local properties from a geometric perspective. In this work, using numerical approaches, we show that there exist a correspondence between some classes of entanglement and some specific regions of the polytope for three-qubit systems.

1. Introduction

Entanglement is a fundamental feature of multipartite quantum systems such that the properties of each one of the components cannot be described independently of the others [1,2]. Some efforts have been made in the study of production, identification and manipulation of entanglement [3]. Indeed, some practical applications of such a resource have been achieved over the years. Nevertheless, there are still some challenges to overcome. The more parties involved, the more complex the characterization of entanglement becomes [4]. For example, many inequivalent types of entanglement arise as the number of components increases, so discriminating between them constitutes a fundamental issue [3]. On the other hand, it is known that two quantum states with the same amount of entanglement are equivalent up to a local unitary transformation [5]. Thus, it is possible to accomplish a classification in terms of the corresponding orbits. For instance, in the case of three-qubit pure states the equivalence classes are labeled by five continuous parameters, so that the classification of entanglement involves a large amount of subjects [6,7]. Yet, the state vectors can be grouped according to their entanglement properties, which are determined by the five parameters. In contrast, there are only two entanglement types when a classification in terms of stochastic local operations and classical communication is considered [8].

A way of addressing the classification problem is through the entanglement polytope. For multiqubit pure states it is found that the minimum eigenvalues of the reduced single-qubit density matrices satisfy the so called polygon inequalities [9]. Such expressions, in the case of a three-qubit system, determine a polytope in a three-dimensional space. This scheme provides a criterion for witnessing and classification of multipartite entanglement only with the determination of the local spectra, since the information about non-local properties is encoded in such a convex body [5,10,11]. The aim of this contribution is to use this theoretical framework to deepen the understanding of the classification of three-qubit pure states under local unitary operations. For this purpose, a numerical identification of the correspondence between the local unitary types of entanglement with some regions of the polytope is accomplished.
This paper is organized as follows. In Section 2 we revisit the theory of the entanglement polytope. We present the defining inequalities and the geometrical image of some representative three-qubit pure states is considered. In Section 3 we review the local unitary entanglement classification and the numerical identification of their corresponding projections on the polytope is presented. Finally, some conclusions and perspectives are provided in Section 4.

2. Three-qubit entanglement polytope

We consider a system composed of three two-level systems $A$, $B$ and $C$. Such a system is described by a pure state $|\psi\rangle$ in the Hilbert space $\mathcal{H} = \mathcal{H}_2^\otimes 3$. In terms of the computational basis, the state reads

$$|\psi\rangle = \sum_{i,j,k=0}^{1} \psi_{ijk} |ijk\rangle, \quad \psi_{ijk} \in \mathbb{C}, \quad \sum_{i,j,k=0}^{1} |\psi_{ijk}|^2 = 1.$$  \hspace{1cm} (1)

Hereafter we shall adopt the notation $|i\rangle \otimes |j\rangle \otimes |k\rangle := |ijk\rangle$ for the basis product states.

Local information can be extracted from the reduced one-qubit density matrices $\{\rho_A, \rho_B, \rho_C\}$, obtained by tracing over the density matrix $\rho = |\psi\rangle \langle \psi|$. Let $\{\lambda_A, \lambda_B, \lambda_C\}$ be the set of minimum eigenvalues associated with the previous reduced density matrices, respectively. It is found that these eigenvalues satisfy the so called *polygon inequalities* \cite{9}:

$$\lambda_A \leq \lambda_B + \lambda_C, \quad \lambda_B \leq \lambda_A + \lambda_C, \quad \lambda_C \leq \lambda_A + \lambda_B,$$  \hspace{1cm} (2)

where each eigenvalue fulfills the condition $0 \leq \lambda_i \leq 1/2$, $i \in \{A, B, C\}$. Geometrically, working in the three-dimensional space $(\lambda_A, \lambda_B, \lambda_C)$, which we will refer to as *minimum eigenvalue space*, the polygon inequalities (2) determine a convex polytope, as it is shown in Figure 1. Consequently, the set of minimum eigenvalues corresponding to any $|\psi\rangle \in \mathcal{H}$ can be identified with a vector in $\mathbb{R}^3$, denoted $\vec{\lambda} = (\lambda_A, \lambda_B, \lambda_C)^T$, which we will call the *image* of $|\psi\rangle$ in the minimum eigenvalue space; this image is unique.

![Figure 1: Convex polytope determined by the polygon inequalities (2) in the three-dimensional space $(\lambda_A, \lambda_B, \lambda_C)$. The body is limited by the surfaces $\lambda_A = \lambda_B + \lambda_C$, $\lambda_B = \lambda_A + \lambda_C$, $\lambda_C = \lambda_A + \lambda_B$, $\lambda_A = 1/2$, $\lambda_B = 1/2$ and $\lambda_C = 1/2$. Consequently, five vertices are identified: $S = (0, 0, 0)^T$, $A = (0, 1/2, 1/2)^T$, $B = (1/2, 0, 1/2)^T$, $C = (1/2, 1/2, 0)^T$ and $G = (1/2, 1/2, 1/2)^T$. The image of the state in Eq. (3) is located at point $(1/3, 1/3, 1/3)^T$. The geometric center of $\triangle ABC$ (red dot). State (4) is situated at vertex $G$. Bi-separable states are located at edges $SA$, $SB$ and $SC$.](image-url)
As examples we consider the W and GHZ states

\[ |W\rangle = (|001\rangle + |010\rangle + |100\rangle) / \sqrt{3}, \]  
\[ |GHZ\rangle = (|000\rangle + |111\rangle) / \sqrt{2}. \]  

For these vectors the minimum triplet \((\lambda_A, \lambda_B, \lambda_C)^T\) is computed and it is found that the image of the first one is located at the point \(\hat{\lambda}_W = (1/3, 1/3, 1/3)^T\), while the GHZ state is placed at vertex \(G\) in Fig. 1. On the other hand, bi-separable states \(|\psi_A\rangle = |\phi_A\rangle |\phi_{BC}\rangle\), \(|\psi_B\rangle = |\phi_B\rangle |\phi_{AC}\rangle\) and \(|\psi_C\rangle = |\phi_C\rangle |\phi_{AB}\rangle\) are located at the edges \(SA, SB\) and \(SC\), respectively. The representative state vectors will be introduced in the next Section.

When we are interested in the entanglement properties of a pure state this framework constitute an elegant and powerful tool for detecting multipartite entanglement only with the determination of local spectra, that is, without measuring any correlation between the parties, which demands great experimental efforts [3, 11]. Certainly, in our case it is possible to detect the presence of genuine three-partite entanglement by locating the image of a pure state in the minimum eigenvalue space, given that some regions of the polytope are compatible only with certain type of entanglement [10, 11]. For example, when we consider states with equivalent entanglement under stochastic local operations and classical communication, states that can be converted into each other with a finite probability of success, there are two types of three-partite entanglement: W type and GHZ type [8]. In the former case the representative state vector is \(|W\rangle\), Eq. (3), and the states belonging to this type are projected to the tetrahedron \(SABC\). On the other hand, states in the GHZ type cover the entire polytope, and the representative state vector is \(|GHZ\rangle\), Eq. (4). This method has been implemented experimentally in the optical regime, turning out to be useful in identifying different entanglement types since fewer resources are required compared to a full tomographic reconstruction [3].

In the next section we consider a finer classification of the three-partite entanglement, obtained by considering local unitary operations. In this context infinite types of entanglement arise, nevertheless, the states can be grouped according to the values of some local unitary invariants, which determine their entanglement properties. We found that this very illustrative geometric framework turns out to be useful for gain insight into such a classification of states.

### 3. Geometric perspective of local unitary classes of entanglement

We say that two state vectors \(|\psi\rangle\) and \(|\varphi\rangle\) in the same Hilbert space are equivalent under local unitary operations if there exist local unitary matrices \(\{U_i\}\) such that

\[ |\psi\rangle = U_1 \otimes \cdots \otimes U_N |\varphi\rangle. \]  

The idea is to bring any state vector into a minimal form, so that two vectors are local-unitary-equivalent (LU-equivalent for short) if and only if their respective minimal forms coincide [1]. In the present case, any three-qubit pure state can be cast to a minimal form in which any vector is written in a unique form as a linear combination of five orthogonal basis product states. Indeed, by means of local unitary operations, the state in equation (1) can be written as [6, 7]:

\[ |\psi\rangle = c_0 |000\rangle + c_1 e^{i\phi} |100\rangle + c_2 |101\rangle + c_3 |110\rangle + c_4 |111\rangle, \]  

where \(c_i \geq 0\), \(\sum_i c_i^2 = 1\). Besides, in order to assure the uniqueness of the minimal form the parameter \(\phi\) must satisfy \(0 \leq \phi \leq \pi\) [7].

A classification defined by the minimal number of basis product states in the decomposition (6) has been accomplished and the corresponding entanglement classes were discussed in [6]. In the following we compute the image \((\lambda_A, \lambda_B, \lambda_C)^T\) for the representative state of each class.
The parameters defining each representative are discretized and sweeping their values one can identify the corresponding geometric image in the minimum eigenvalue space. The entanglement classes are:

- **Type 1 (Fully separable states).** For this kind of states the representative state vector is
  \[ |\psi_1\rangle = |000\rangle, \] (7)
  so that every fully separable state \( |\psi\rangle \in \mathcal{H} \) is LU-equivalent to \( |\psi_1\rangle \) and its image lies in the vertex \( S = (0, 0, 0) \) of the polytope.

- **Type 2a (Biseparable states).** We refer to as biseparable systems those for which one of its parts can be described independently of the other two. Any state \( |\psi\rangle \in \mathcal{H} \) for which party \( A \) is not entangled with parties \( B \) and \( C \) is LU-equivalent to the representative vector
  \[ |\psi_{2aA}\rangle = \cos \alpha |000\rangle + \sin \alpha |011\rangle, \] (8)
  for some \( 0 < \alpha \leq \pi/4 \). Equivalently, systems in which party \( B \) is not entangled with parties \( A \) and \( C \) have associated the state
  \[ |\psi_{2aB}\rangle = \cos \alpha |000\rangle + \sin \alpha |101\rangle. \] (9)
  Finally, the representative vector
  \[ |\psi_{2aC}\rangle = \cos \alpha |000\rangle + \sin \alpha |110\rangle, \] (10)
  corresponds to systems in which party \( C \) is not entangled with parties \( A \) and \( B \). The images in the minimum eigenvalue space of vectors (8), (9) and (10) are shown in Figure 2a, and correspond to edges \( SA, SB \) and \( SC \), respectively. In all cases \( \alpha \) takes values in the indicated interval and when \( \alpha = \pi/4 \) each state is mapped to the corresponding vertex \( A, B \) or \( C \). Edges \( SA, SB \) and \( SC \) constitute by themselves polytopes of two-qubit systems.

- **Type 2b.** The representative state vector is written as
  \[ |\psi_{2b}\rangle = \cos \alpha |000\rangle + \sin \alpha |111\rangle, \] (11)
  Figure 2b shows that the image of this type lies in the segment \( SG \). If \( \alpha = \pi/4 \) the GHZ state is obtained, see Eq. (4).

- **Type 3a.** This type is represented by the vector
  \[ |\psi_{3a}\rangle = \sin \alpha \sin \beta |000\rangle + \sin \alpha \cos \beta |101\rangle + \cos \alpha |110\rangle, \] (12)
  The image is mapped over the regions \( \triangle SAB, \triangle SAC, \triangle SBC \) and \( \triangle ABC \), the faces of the tetrahedron \( SABC \), as shown in Figure 2c. Note that when \( \cos \alpha = 1/\sqrt{3}, \cos \beta = 1/\sqrt{2} \), we obtain the state \( |W\rangle \), Eq. (3).

- **Type 3b.** In this case there are three representative states
  \[ |\psi_{3bA}\rangle = \sin \alpha \sin \beta |000\rangle + \sin \alpha \cos \beta |011\rangle + \cos \alpha |111\rangle, \] (13)
  \[ |\psi_{3bB}\rangle = \sin \alpha \sin \beta |000\rangle + \sin \alpha \cos \beta |101\rangle + \cos \alpha |111\rangle, \] (14)
  \[ |\psi_{3bC}\rangle = \sin \alpha \sin \beta |000\rangle + \sin \alpha \cos \beta |110\rangle + \cos \alpha |111\rangle, \] (15)
Figure 2: (a) Type 2a, biseparable states, Eqs. (8)-(10). Edge $SA$ (red) allows us to witness bipartite entanglement $BC$ (between parties $B$ and $C$). Equivalently, edges $SB$ (blue) and $SC$ (green) are the witnesses of bipartite entanglement $AC$ and $AB$, respectively. (b) Type 2b, Eq. (11). States are projected to the segment $SG$. (c) Type 3a, Eq. (12). The image of such a state lies on the faces of the tetrahedron $SABC$. (d) Type 3b, Eqs. (13)-(15). States are mapped to the regions $\triangle SAG$ (green), $\triangle SGB$ (blue) and $\triangle SCG$ (red).

where $0 < \alpha, \beta < \pi/2$. The equations (13), (14) and (15) have their images in the plane regions $\triangle SAG$, $\triangle SGB$ and $\triangle SCG$, respectively; see Figure 2d.

In the following, $c_0 = \sin \alpha \sin \beta \sin \gamma$, $c_1 = \sin \alpha \sin \beta \cos \gamma$, $c_2 = \sin \alpha \cos \beta$, $c_3 = \cos \alpha$, with $0 < \alpha, \beta, \gamma < \pi/2$.

- **Type 4a.** We write the representative state vector of such states as
  $$| \psi_{4a} \rangle = c_0 |000\rangle + c_1 |100\rangle + c_2 |101\rangle + c_3 |110\rangle.$$  

Figures 3a-3b show two perspectives of the image of the state (16), which corresponds to the tetrahedron $SABC$.

- **Type 4b.** There are two representative state vectors,
  $$| \psi_{4bA} \rangle = c_0 |000\rangle + c_1 |100\rangle + c_2 |110\rangle + c_3 |111\rangle,$$
  $$| \psi_{4bB} \rangle = c_0 |000\rangle + c_1 |100\rangle + c_2 |101\rangle + c_3 |111\rangle.$$

Two different perspectives of the corresponding images are shown in Figures 3c-3d. The state (17) is mapped to the polyhedron $SACG$ (region in red), while state (18) is mapped in a similar way to the polyhedron $SABG$ (region in blue).
Figure 3: (a)-(b) Type 4a, Eq. (16). States in this type are mapped to tetrahedron $SABC$. (c)-(d) Type 4b, Eqs. (17)-(18). The region in red (blue), in polyhedron $SACG$ ($SABG$), corresponds to the image of state (17) ((18)). It is noticeable that the images of both types exhibit a higher density of points around the segment $SG$.

- **Type 4c.** In this case the vector
  \[
  |\psi_{4c}\rangle = c_0 |000\rangle + c_1 |101\rangle + c_2 |110\rangle + c_3 |111\rangle
  \]  
  is the representative of such states. The image of this type is shown in Figures 4a-4b.

- **Type 4d.** We write the state vector representing this type as
  \[
  |\psi_{4d}\rangle = c_0 |001\rangle + c_1 |010\rangle + c_2 |100\rangle + c_3 |111\rangle.
  \]  
  Figures 4c-4d show that all regions of the polytope are mapped.

So far, the local unitary classification of three-qubit pure states and the visualization of the corresponding images in the minimum eigenvalue space have been presented. From our analysis it can be seen that there are some types which contain two or more representative vectors sharing similar entanglement properties. This is noticeable, for instance, in Figures 2a, 2d and 3d. Further, the polytope allows to get information about the spectral properties of the subsystems and also to know from what class it is possible to obtain bipartite entanglement. In the case of Type 1, the image is located at the origin, thus the minimum eigenvalue of each part is zero. For Type 2a, the separable subsystem always have associated the eigenvalue zero, and the other two parties have the same eigenvalue. Subsystems in Type 2b have associated the same minimum eigenvalue, so the image lies in the segment $SG$. Besides, from these kind
of states it is not possible to obtain bipartite entanglement, since if any of the parties perform a measure then the entanglement is destroyed. Geometrically, this is noticeable at a glance since we cannot approach arbitrarily to any of the edges in which biseparable states lie. This is not the case for Type 3a, from this class of states we can obtain bipartite entanglement between any pair of subsystems. For example, if party A finds that its subsystem is in the state $|1\rangle$, can be sure that parties B and C share an entangled state. On the other hand, if the subsystem turns out to be in the state $|0\rangle$, then party A can be sure that the other parties share no entanglement (this property has also been mentioned in [6]). Type 3b is an interesting one. Paying attention to state (13), with the same analysis as before we can notice that it is possible to obtain bipartite entanglement BC, but not AB nor AC. This can be noticed immediately from Figure 2d, since the image of this state lies in $\triangle SAG$, so we can approach arbitrarily to segment SA, but no to SB nor SC. Similarly, the same figure shows that from states (14) and (15) we can extract only bipartite entanglement AC and AB, respectively. From Type 4a we can extract any kind of bipartite entanglement. An interesting property of the image is that states in this type are more concentrated around segment SG. With respect to Type 4b, Figures 3c-3d show that from state (17) we can extract bipartite entanglement BC and AB, but no AC. On the other hand, state (18) only does not allow to extract bipartite entanglement AB. From types 4c and 4d all types of bipartite entanglement can be obtained, and again, states are more concentrated around segment SG.

Figure 4: (a)-(b) Type 4c, Eq. (19). States are mapped to tetrahedron $SABC$ and some region of tetrahedron $ABCG$ is also covered. (c)-(d) Type 4d, Eq. (20). States belonging to this type are projected to the entire polytope. Again, both types present a higher density of points around the segment $SG$. 

(a) 

(b) 

(c) 

(d)
around segment \(SG\). It is worth mentioning that, up to class 4a, the correspondence between the local unitary classification and the entanglement polytope has been reported in [10]. The present work also includes the study of classes 4b-4d.

4. Conclusions

The theory of the entanglement polytope and the local unitary classification of three-qubit pure states has been briefly revisited. The visualization of each type of entanglement in the so-called minimum eigenvalue space has been accomplish in a numerical fashion, thus recognizing a correspondence with some specific regions of the polytope, allowing to delve further into the understanding of the local unitary classification. More specifically, our results show that this framework proves to be a suitable tool for (i) to get information about the spectral properties of subsystems that share a common three-qubit pure state, and (ii) to discern information not only of global entanglement but also of bipartite entanglement available in a given state. Future work can be directed to the study of the rotational and reflectional symmetries of the polytope. On the other hand, quantitative information can be extracted defining measures based on euclidean distances in this three-dimensional space [11]. In addition, we can ask about the ways in which entanglement can be manipulated in this geometric scenario. Results in these directions will be reported elsewhere.

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