LOCAL RIGIDITY OF PARTIALLY HYPERBOLIC ACTIONS: SOLUTION OF THE GENERAL PROBLEM VIA KAM METHOD 1)

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ABSTRACT. We consider a broad class of partially hyperbolic algebraic actions of higher-rank abelian groups. Those actions appear as restrictions of full Cartan actions on homogeneous spaces of Lie groups and their factors by compact subgroups of the centralizer. The common property of those actions is that hyperbolic directions generate the whole tangent space. For these actions we prove differentiable rigidity for perturbations of sufficiently high regularity. The method of proof is KAM type iteration scheme. The principal difference with previous work that used similar methods is very general nature of our proofs: the only tool from analysis on groups is exponential decay of matrix coefficients and no specific information about unitary representations is required.

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1) Based on research supported by NSF grant DMS 1002554.
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Comment on the title. This is the fourth paper after [3], [6] and [31] with a title that begins with “Local rigidity of partially hyperbolic actions”. These papers can be thought of as parts of a series that carries out major steps in realization of a program. The first two papers appeared with Roman numerals I and II after this generic title, followed by a specific description. Unfortunately the title of [31] has neither a numeral, that should have been “III”, nor a specific description, that could have been “Weyl Chamber flows on non-split groups and algebraic K-theory”. Thus it would be ambiguous to use a numeral in the title of the present paper: it should be “IV” but then “III” would be missing, and with “III” we would put [31] out of the series to which it belongs.

Acknowledgements. We would like to thank Gregory Margulis for referring us to Lojasiewicz inequality and Roger Howe for discussion of matrix coefficients decay on twisted symmetric spaces. Livio Flaminio suggested a method of obtaining tame estimates in the centralizer direction in a different setting that inspired our arguments on that topic. Discussions with Ralf Spatzier helped to clarify the essential preliminary step in our construction: reduction to a perturbation in the neutral direction.

1. The setting

We consider actions of higher rank abelian groups $\mathbb{Z}^k \times \mathbb{R}^\ell$, $k + \ell \geq 2$ that come from the following general algebraic construction:

Let $G$ be a connected Lie group, $A \subseteq G$ a closed Abelian subgroup which is isomorphic to $\mathbb{Z}^k \times \mathbb{R}^\ell$, $L$ a compact subgroup of the centralizer $Z(A)$ of $A$, and $\Gamma$ a cocompact lattice in $G$. Then $A$ acts by left translation on the compact space $M = L \backslash G/\Gamma$. 

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Throughout this paper $G$ will always denote a semisimple connected Lie group of $\mathbb{R}$-rank $\geq 2$ without compact factors and with finite center.

1.1. **Symmetric space examples.** Let $D$ be the connected component of a split Cartan subgroup of $G$. $D$ is isomorphic to $\mathbb{R}^{\text{rank}_G}$. Suppose $\Gamma$ is an irreducible torsion-free cocompact lattice in $G$. The centralizer of $Z(D)$ of $D$ splits as a product $Z(D) = KD$ where $K$ is compact and commutes with $D$. Let $D_+$ be a closed subgroup of $D$ isomorphic to $\mathbb{R}^\ell \times \mathbb{Z}^k$, $k + \ell \geq 2$.

Let $\Phi$ denote the restricted root system of $G$. Then the Lie algebra $\mathfrak{g}$ of $G$ decomposes

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{d} + \sum_{\phi \in \Phi} \mathfrak{g}_\phi$$

where $\mathfrak{g}_\phi$ is the root space of $\phi$ and $\mathfrak{k}$ and $\mathfrak{d}$ are the Lie algebras of $K$ and $D$ respectively. Elements of $\mathfrak{d} \setminus \bigcup_{\phi \in \Phi} \ker(\phi)$ are regular elements. Connected components of the set of regular elements are Weyl chambers.

We will consider actions of higher rank subgroups of $D$ by left translations on double coset spaces $L \setminus G/\Gamma$ where $L \subset K$ is a connected subgroup.

The action of $D$ on $G/\Gamma$ is sometimes referred to as full Cartan action. The action of on $K \setminus G/\Gamma$ is called Weyl chamber flow. It is Anosov (normally hyperbolic with respect to the orbit foliation). Regular elements of the Weyl chamber flow are normally hyperbolic. Full Cartan action coincides with the Weyl chamber flow if and only if the group $G$ is $\mathbb{R}$-split.

1.2. **Twisted symmetric space examples.** Let $\rho : \Gamma \to SL(m, \mathbb{Z})$ be a representation of $\Gamma$ which admits no invariant subspace with eigenvalue 1. Then $\Gamma$ acts on the $N$-torus $\mathbb{T}^N$ via $\rho$. By Margulis’ superrigidity theorem [23], semisimplicity of the algebraic hull $H$ of $\rho(\Gamma)$ and the non-compactness of $\rho(\Gamma)$ the representation $\rho$ of $\Gamma$ extends to a rational homomorphism $\mathfrak{g} \to H_{ad}$ over $\mathbb{R}$ where $H_{ad}$ is the adjoint group of $H$. Note that $\rho(\Gamma)$ has finite center $Z$ as follows for example from Margulis’ finiteness theorem [23], then $G$ acts on orbifold $\mathbb{R}^N/Z$ via $\rho$, which can be lifted to a representation of $G$ on $\mathbb{R}^N$. We still denote by $\rho$.

Now assume notations of the previous section. Then we have semi-direct product Lie group $G \rtimes \mathbb{R}^N$ twisted by $\rho$. The multiplication of elements in $G \rtimes \mathbb{R}^N$ is defined by

$$(g_1, r_1) \cdot (g_2, r_2) = (g_1 g_2, \rho(g_2^{-1}) r_1 + r_2).$$

Then $\Gamma \rtimes \mathbb{Z}^N$ is a lattice in $G \rtimes \mathbb{R}^N$. We can view $G \rtimes \mathbb{R}^N/\Gamma \rtimes \mathbb{Z}^N$ as a torus bundle over $G/\Gamma$. So we may assume without loss of generality that $\rho$ is irreducible over $\mathbb{R}$ on $\mathbb{R}^N$.

Let $\Phi_1$ be the set of weights of the representation $\rho$. Then the Lie algebra $\mathfrak{g} + \mathbb{R}^N$ of $G \rtimes \mathbb{R}^N$ decomposes

$$\mathfrak{g} + \mathbb{R}^N = \mathfrak{k} + \mathfrak{d} + \sum_{\phi \in \Phi} \mathfrak{g}_\phi + \sum_{\mu \in \Phi_1} \mathfrak{d}_\mu$$
where $v^\mu$ is the weight space of $\mu$. We call the representation $\rho$ on $\mathbb{R}^N$ Anosov or hyperbolic, if there is no 0 weight in $\Phi_1$ and genuinely partially hyperbolic on $\mathbb{R}^N$ if $0 \in \Phi_1$. Elements of $\mathcal{D}\setminus \bigcup_{\phi \in \Phi \cup \Phi_1 \setminus \{0\}} \ker(\phi)$ are regular elements. Connected components of the set of regular elements are Weyl chambers.

Similarly to the symmetric space setting we will consider actions of higher rank subgroups of $D$ by left translations on double coset spaces $L\backslash G \ltimes \mathbb{R}^N / \Gamma \ltimes \mathbb{Z}^N$ where $L \subset K$ is a connected subgroup.

The action of the whole group $D$ on $K\backslash G \ltimes \mathbb{R}^N / \Gamma \ltimes \mathbb{Z}^N$ is called twisted Weyl chamber flow. It is Anosov if and only if the representation $\rho$ is Anosov.

**Remark 1.1.** The requirement “$\Gamma$ which admits no invariant subspace with eigenvalue 1” is necessary for rigidity. Otherwise, there is a factor $\mathbb{R}^{N_i}$ on which the $G$ acts trivially.

### 1.3. Regular restrictions and coarse Lyapunov distributions.

Let $X$ be a double coset space $L\backslash G / \Gamma$ as in symmetric space examples or $L\backslash G \ltimes \mathbb{R}^N / \Gamma \ltimes \mathbb{Z}^N$ as in twisted symmetric space examples.

**Definition 1.1.** A two-dimensional plane in $\mathbb{P} \subset D$, the Lie algebra of $D$, is in regular position if it contains at least one regular element.

Let $D_+ \subset D$ be a closed subgroup which contains a lattice $L$ in a plane in regular position and let $D_+ = \exp D_+$.

**Definition 1.2.** The action $\alpha_{D_+}$ of $D_+$ by left translations on $X$ will be referred to as a higher-rank regular restriction of split Cartan actions or just a regular restriction for short.

Throughout this paper terms “symmetric space examples” and “twisted symmetric space examples” are used as synonyms for “higher rank regular restrictions of split Cartan actions” in the corresponding cases.

If a plane contains a lattice in regular position, then there exists a linearly independent basis of the plane consisting of two regular elements in the lattice which we denote by $a$ and $b$. $a$ and $b$ will be referred to as regular generators.

**Definition 1.3.** Coarse Lyapunov distributions are defined as minimal non-trivial intersections of stable distributions of various action elements.

In the setting of present paper those are homogeneous distributions or their perturbations, that integrate to homogeneous foliations called coarse Lyapunov foliations; see [4, Section 2] and [12] for detailed discussion in greater generality.

The standard root system comes from the decomposition of $\mathfrak{g}$, the Lie algebra of $G$, into the eigenspaces of adjoint representation of $\mathcal{D}$ whose elements are simultaneously diagonalizable. For any connected subgroup $P$ of $D$ with Lie algebra $\mathfrak{p}$, we can also consider the decomposition of $\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{p}$ and the resulting root system
is called the root system with respect to $P$. Then the coarse Lyapunov distributions of regular restrictions both these classes of examples are the positively proportional root spaces obtained from root systems with respect to $D_+$. For the symmetric space examples let $r$ be the smallest integer such that the coarse Lyapunov distributions of full Cartan actions as well as their commutators of length $r$ span the tangent space at any $x \in M$, then $r$ also has the same property for any regular restriction. For the twisted symmetric space examples, notice that in $G + \mathbb{R}^N \cdot (X, 0), (0, t) = d\rho(X)t$ where $X, \in G$ and $d\rho$ is the induced Lie algebra representation on $\mathbb{R}^N$ from $\rho$. By complete irreducibility of semisimple groups, there is a decomposition of $\mathbb{R}^N = \bigoplus \mathbb{R}^{N_i}$ such that $\rho$ is irreducible over $\mathbb{R}$ on each $\mathbb{R}^{N_i}$ and hence $\rho$ is irreducible over $\mathbb{R}$ on each $\mathbb{R}^{N_i}$. It follows that there exists an integer $r_1$ such that the coarse Lyapunov distributions of full Cartan actions as well as their commutators of length $r_1$ also span the tangent space at any $x \in M$. Similarly $r_1$ has the same property for any regular restriction.

Let $L$ be the Lie algebra of the group $L$. Regular restrictions are partially hyperbolic:

- For the symmetric space examples the neutral distribution is $L \setminus N$ where $N = \mathbb{R} + D$;
- for the Anosov twisted symmetric space examples the neutral distribution is $L \setminus N$ where $N = \mathbb{R} + D$;
- for the genuinely partially hyperbolic twisted symmetric space examples the neutral distribution is $L \setminus N$ where $N = \mathbb{R} + D + \sum v^0$ and $v^0$ are weight spaces of 0 weight.

Fix positive definite inner products $\langle \cdot \rangle_1$ on $\mathbb{R} + D$ and $\langle \cdot \rangle_2$ on $v^0$ that are invariant under $\text{Ad}_K$. Then $\text{Ad}_K \subset O(N, \langle \cdot \rangle)$. Let $L^\perp$ be the orthogonal complement of $L$. Then $L^\perp = L \setminus N$ and both $L$ and $L^\perp$ are invariant under $\text{Ad}_{Z(L)}$.

Remark 1.2. Notice that the neutral distribution for any regular restriction coincides with the homogeneous distribution into cosets of the centralizer of
the full Cartan action, or its factor by $L$ in the case of actions on double coset spaces.

We call the following simple fact the uniqueness decomposition property which will be often used in the future: there exist bounded, open, connected neighborhoods $U_L$ and $U_{L^\perp}$ of 0 in $L$ and $L^\perp$ respectively, such that the mapping \( T : (U_L, U_{L^\perp}) \to \exp(U_L) \exp(U_{L^\perp}) \) is a diffeomorphism of $U_L \times U_{L^\perp}$ onto an open neighborhood of $e$ in $Z(D)$.

1.4. Rigidity. One can naturally think of $D_+$ as the image of an embedding $i_0 : \mathbb{Z}^k \times \mathbb{R}^\ell \to D_+$. Then one can consider the action $\alpha_{i_0}$ of $A = \mathbb{Z}^k \times \mathbb{R}^\ell$ on $M$ given by

\[
\alpha_{i_0}(a, x) = i_0(a) \cdot x
\]

Thus $\alpha_{i_0}$ is $\alpha_{D_+}$ with a fixed system of generators. We will say that $A$ action $\alpha_{i_0}$ generates $D_+$ action $\alpha_{D_+}$. Of course, $D_+$ can be obtained as the image of different embeddings; corresponding actions of $A$ differ by a time change.

A standard perturbation of the action $\alpha_{i_0}$ is an action $\alpha_i$ where $i : A \to D$ is a homomorphism close to $i_0$. Since a small perturbation of an embedding is still an embedding a standard perturbation also generates a regular restriction $\alpha_{D'_+}$ where $D'_+ \subset D$ is the image of $i$.

Remark 1.3. In the hyperbolic situation, i.e. for $D_+ = D$ any standard perturbation is simply a time change corresponding to an automorphism of the acting group but in the partially hyperbolic cases standard perturbations are usually essentially different from each other.

A proper notion of rigidity for algebraic actions, i.e homogeneous actions, their factors, as well as affine actions, states that any perturbation of the action in a properly defined regularity class is conjugate to an algebraic action obtained by perturbing acting subgroup. In our setting this translates into the following definition.

Definition 1.4. An action $\alpha_{i_0}$ of $A = \mathbb{Z}^k \times \mathbb{R}^\ell$ on $M$ is $C^{k,r,\ell}$ locally rigid if any $C^k$ perturbation $\tilde{\alpha}$ which is sufficiently $C^r$ close to $\alpha_{i_0}$ on a compact generating set is $C^\ell$ conjugate to a standard perturbation $\alpha_i$.

An action $\alpha_{D_+}$ is $C^{k,r,\ell}$ locally rigid if there exists a homomorphism $i_0 : \mathbb{Z}^k \times \mathbb{R}^\ell \to D$ whose image equals $D_+$ such that $\alpha_{i_0}$ is $C^{k,r,\ell}$ locally rigid.

Remark 1.4. It is immediately obvious that if $\alpha_{i_0}$ is locally rigid then the same is true for any time change obtained by an automorphism of $A$; hence the notion of local rigidity for $\alpha_{D_+}$ depends only of the subgroup $D_+$.

2. History of the rigidity problem

2.1. Rigidity of hyperbolic actions. Differentiable rigidity of higher rank algebraic Anosov actions including Weyl chamber flows and twisted Weyl chamber flows was proved in the mid-1990s [19]. The proof consists of two major parts:
(i) An a priori regularity argument that shows smoothness of the Hirsch-Pugh-Shub orbit equivalence [10]. The key part of the argument is the theory of non-statinary normal forms developed in [9].

(ii) Cocycle rigidity used to “straighten out” a time change; it is proved by a harmonic analysis method in [17].

Both these ingredients also appear in the present paper.

2.2. Difference between hyperbolic and partially hyperbolic actions. The next step is to consider algebraic partially hyperbolic actions. Unlike the hyperbolic case, the a priori regularity method is not directly applicable here since individual elements of such actions are not even structurally stable. In addition to their work on hyperbolic rigidity, the first author and R. Spatzier also considered cocycle problem for certain partially hyperbolic actions in [18] and proved essential cocycle rigidity results.

Before proceeding with the chronological account let us explain an essential point that also plays a role in the present work.

In the hyperbolic case smooth orbit rigidity reduces the local differentiable rigidity problem to rigidity of vector valued cocycles. For, the expression of the old time, i.e. that of the unperturbed action through the new time, i.e. that of the perturbed action is a cocycle over the unperturbed action with values in the acting group that is a vector space or its subgroup. This is the scheme of [19]. In other words, smooth orbit rigidity reduces the differentiable conjugacy problem to a time change problem.

In the partially hyperbolic cases instead of smooth orbit rigidity one may hope at best to have smooth rigidity of neutral foliation when the scheme of [19] is applicable. For that one needs the following property:

(\(B^\prime\)) The stable directions of various action elements or, equivalently, coarse Lyapunov distributions, (see above Section 1.3), together with the orbit direction, generate the tangent space as a Lie algebra, i.e. those distributions and their brackets of all orders generate the Lie algebra linearly.

In this paper we deal with cases, namely regular restrictions of split Cartan actions, where an even stronger property (\(B\)) holds:

(\(B\)) The stable directions of various action elements generate the tangent space as a Lie algebra.

Still even after smooth rigidity of the orbit foliation has been established, the problem of differentiable conjugacy is not reduced to a cocycle problem over the unperturbed action; rather it reduces to the cocycle problem over the perturbed action; furthermore, the values of the cocycle may be in a non-abelian group: it is actually the exponential of the central (neutral) distribution.

2.3. Previous work on partially hyperbolic rigidity.
2.3.1. \textit{KAM method}. The work on the differentiable rigidity of partially hyperbolic but not hyperbolic actions started in earnest in 2004 \cite{3} (complete proofs appeared in print in \cite{5}). The situation considered in that work, actions by commuting partially hyperbolic automorphisms of a torus, is algebraically more amenable than the one of \cite{18} or the present paper, but is geometrically more subtle because stable directions for different action elements commute as any homogeneous distributions on the torus do. To handle this problem a new method was introduced: a KAM type iteration scheme formally similar to that employed by J.Moser \cite{28} in the higher-rank version of a conventional setting for application of KAM scheme – diffeomorphisms of the circle.

While causing geometric difficulties, abelian nature of the torus situation helps both with algebra and with analysis involved in carrying out the iterative scheme. It allows for an explicit calculation of solutions of linearized conjugacy equation, as well as the splitting for “almost cocycles” that transforms producing needed tame estimates into manageable problems of Fourier analysis. At the time applications of the KAM scheme even to the most basic semisimple situations looked very problematic. It seemed that specific information from representation theory would be needed that may be available for some semisiple Lie groups and not for others. Only in the present work new algebraic and analytic insights and tools are developed that made possible applications of the KAM type iterative scheme to all semisimple and various other cases.

2.3.2. \textit{Geometric/Algebraic K-theory method}. In \cite{4, 6} a different method has been developed that bypasses subtleties of analysis and representation theory altogether. It builds upon the already mentioned observation that the problem of differentiable conjugacy reduces to the cocycle problem over the perturbed action. Solution of coboundary equations for actions satisfying condition (B) are built along broken paths consisting of pieces of stable foliations for different elements of the action. It was first observed in \cite{4} that consistency of such a construction follows in certain cases from description of generators and relations in the ambient Lie group. In \cite{6} another key ingredient was introduced: under certain circumstances the web of Lyapunov foliations is so robust that the construction of solutions of the coboundary equation carries out to the \textit{perturbed action} thus providing a solution of the conjugacy problem in the Hölder category. After that smoothness of the conjugacy is established by a priori regularity method as in \cite{19}.

A great strength of this method is that it requires only $C^2$ closeness for the perturbation (possibly even less), unlike the KAM scheme where number of derivatives is large and dependent on the data. But this comes with a price. In order to carry out this scheme:

(i) very detailed information about specific generators and relations in the ambient group is required, and
(ii) coarse Lyapunov distributions for the partially hyperbolic restriction should be the same as for the ambient Cartan action.

The latter requirement leads to an extra general position (generic) assumption on the acting subgroup of $D$ that, while open and dense in the spaces of actions, (i.e. of embeddings $i_0 : \mathbb{Z}^k \times \mathbb{R}^l \to D$) is more restrictive than (28) and is clearly far from necessary for rigidity.

In [4, 6] the special case $G = SL(n, \mathbb{R})(n \geq 3)$ is considered. In this case necessary algebraic information can be extracted from the classical work of Steinberg, Matsumoto and Milnor, see [29, 25, 27]. The approach of [4, 6] was further employed in [2], [31] and [32] for extending cocycle rigidity and differentiable rigidity from $SL(n, \mathbb{R})/\Gamma$ and $SL(n, \mathbb{C})/\Gamma$ to compact homogeneous spaces obtained from simple split Lie groups and some non-split Lie groups. All actions considered in [4, 5, 2, 31] are “generic restrictions” and in [32] only a special example of non-generic restrictions is considered.

There is no way to deal with more general situations like regular restrictions of Weyl chamber flows treated in the present paper. Unlike the case of generic restrictions, in both symmetric space examples and twisted symmetric space examples coarse Lyapunov distributions of regular restrictions may be different from those of full Cartan actions.

Another problem with the geometric method is that it requires consideration of simple Lie groups case-by-case. While in [2, 32] algebraic results needed for the treatment of actions on homogeneous spaces of split Lie groups are still deduced from the literature with a moderate effort, treatment of non-split series $SO^\rho(m, n), |m - n| \geq 2$ and $SU(m, n)$ in [31] requires an algebraic tour de force. The remaining classical cases that are quaternionic groups, and non-split exceptional groups, would require even heavier algebraic calculations although results of the present work indicate that rigidity is likely to hold with lower regularity requirements on the closeness of perturbations. It is unlikely that this method can be adapted to the case of semisimple but not simple groups.

The geometric method can be applied to some hyperbolic twisted symmetric spaces examples, which will appear in another paper by the second author, but it can’t deal with genuinely partially hyperbolic cases. For these cases there are no good invariant layers isomorphic to simple groups under perturbations as in the hyperbolic twisted symmetric spaces examples and due to the abelian nature of torus bundles geometric structures are not robust under perturbations.

2.3.3. Comparison. To summarize, two methods developed to prove rigidity for partially hyperbolic actions can be viewed as somewhat complementary: the method of the present paper requires less specific information and hence applies in a greater variety of settings to a broader class of actions, while the geometric/algebraic $K$-theory method, whose applicability is more limited,
produces stronger conclusions by requiring lower regularity for perturbations.

3. Statement of results

In this paper we develop an approach for proving local differentiable rigidity of higher rank abelian groups, i.e. $\mathbb{Z}^k \times \mathbb{R}^\ell$, $k+\ell \geq 2$, based on KAM-type iteration scheme, that was first introduced in [3,5] where local differentiable rigidity was proved for $\mathbb{Z}^k$, $k \geq 2$ actions by partially hyperbolic automorphisms of a torus. Here we deal with actions of $\mathbb{Z}^k \times \mathbb{R}^\ell$, $k+\ell \geq 2$ on $L\backslash G/\Gamma$ and $L\backslash G \ltimes \mathbb{R}^n/\Gamma \ltimes \mathbb{Z}^n$ that are regular restrictions of split Cartan actions.

The work on commuting toral automorphisms together with the present work covers all significant instances of higher rank partially hyperbolic algebraic actions with an exception of actions by automorphisms of nilmanifolds that is the subject of a work in progress by Damjanovic and collaborators.

3.1. Some Lie groups preliminaries. Lojasiewicz inequality [22, Theorem 4.1] implies the following statement: for any subalgebra $\mathfrak{S} \subseteq \mathfrak{N}$ there exist constants $d, q, \delta > 0$ such that $n_1, n_2 \in \mathfrak{L}$ any $1 \geq \gamma > 0$, if $||[n_1, n_2]|| \leq \gamma$ then there exist $n_1', n_2' \in \mathfrak{S}$ such that

$$(3.1) \quad ||[n_1, n_2]|| = 0, ||n_1 - n_1'|| \leq d\gamma^\delta, ||n_2 - n_2'|| \leq d\gamma^\delta, \text{ if } ||n_1|| + ||n_2|| \leq q.$$ 

Definition 3.1. Let $\delta(q)$ be the maximum of all $\delta$ satisfying (3.1) and let $\delta_0(\mathfrak{S}) = \max\{\delta(t), 0 < t \leq q\}$.

As we will see now, for the symmetric space examples and Anosov twisted symmetric space examples for any subalgebra $\mathfrak{S} \subseteq \mathfrak{N}$ we have $\delta_0(\mathfrak{S}) \geq \frac{1}{2}$.

For the genuinely partially hyperbolic examples, if $G$ is split over $\mathbb{R}$, then $\mathfrak{N}$ is abelian and hence $\delta_0(\mathfrak{S}) = \infty$ for any $\mathfrak{S} \subseteq \mathfrak{N}$. If $G$ is non-split over $\mathbb{R}$, that is the compact part $\mathfrak{K}$ is nontrivial, $\delta_0$ depends on the the representation $\rho$. Even if $G$ is quasi-split over $\mathbb{R}$, that is $\mathfrak{K}$ is nontrivial but abelian, there is no definite answer about $\delta_0$.

Lemma 3.1. For the symmetric space examples and the Anosov twisted symmetric space examples, for any subalgebra $\mathfrak{S} \subseteq \mathfrak{N}$, $\delta_0(\mathfrak{S}) \geq \frac{1}{2}$.

Proof. For these cases $\mathfrak{S} = \mathfrak{R} \cap \mathfrak{G} + \mathfrak{D} \cap \mathfrak{G}$. Suppose $n_1, n_2 \in \mathfrak{S}$ and $||[n_1, n_2]|| \leq \gamma$. Since $\mathfrak{R} \cap \mathfrak{S}$ is compact it can be written as a direct sum $\mathfrak{R} \cap \mathfrak{S} = \hat{\mathfrak{R}}_0 + \hat{\mathfrak{R}}_s$ where the ideals $\hat{\mathfrak{R}}_s$ and $\hat{\mathfrak{R}}_0$ are semisimple and abelian, respectively. Let $p$ be the projection from $\mathfrak{S}$ to $\hat{\mathfrak{R}}_s$. Let $K'$ be the connected subgroup in $G$ with Lie algebra $\hat{\mathfrak{R}}_s$. Then $K'$ is compact. Let $U$ be the maximal torus in $K'$ containing $p(n_1)$ with Lie algebra $\mathfrak{U}$. Let $(\hat{\mathfrak{R}}_s)_C$ be the complexification of $\hat{\mathfrak{R}}_s$ and let $\mathfrak{U}'$ denote the subalgebra of $(\hat{\mathfrak{R}}_s)_C$ generated by $\mathfrak{U}$. Let $\Delta^*$ denote the set of nonzero roots of $(\hat{\mathfrak{R}}_s)_C$ with respect to $\mathfrak{U}'$. Then there exists $E_\psi, F_\psi, (\psi \in (\Delta^*)_+)$ a base of $\hat{\mathfrak{R}}_s$ (mod $\mathfrak{U}$) and

$$[H, E_\psi] = -i\psi(H)F_\psi$$
$$[H, F_\psi] = i\psi(H)E_\psi$$
for all \( H \in \mathcal{U} \).

Let \( p(n_2) = \sum_{\psi \in (\Delta^*)_+} (x_{\psi}E_{\psi} + y_{\psi}F_{\psi}) + u \) where \( u \in \mathcal{U} \). Let
\[
\mathcal{O} = \{ \psi \in (\Delta^*)_+ | \text{either} \ |x_{\psi}| > \sqrt{\gamma} \text{ or } |y_{\psi}| > \sqrt{\gamma} \}
\]

and let \( p'(n_2) = \sum_{\psi \in \mathcal{O}} (x_{\psi}E_{\psi} + y_{\psi}F_{\psi}) + u \), then \( \|p'(n_2) - p(n_2)\| < C\sqrt{\gamma} \). By assumption \( \psi(p(n_1)) < \sqrt{\gamma} \) if \( \psi \in \mathcal{O} \). Since the maximum cardinality of the set of linearly independent elements in \( \mathcal{O} \) is not greater than \( \dim \mathcal{U} \) then there exists \( p'(n_1) \in \mathcal{U} \) such that \( \|p'(n_1) - p(n_1)\| < C\sqrt{\gamma} \) and \( \psi(p'(n_1)) = 0 \) for any \( \psi \in \mathcal{O} \). Then we also have \( [p'(n_1), p'(n_2)] = 0 \). Hence we let \( n'_1 = n_1 - p(n_1) + p'(n_1) \) and \( n'_2 = n_2 - p(n_2) + p'(n_2) \). Then \( n'_1, n'_2 \) satisfy (3.1) for \( \delta = \frac{1}{2} \).

3.2. Main theorems. In what follows \( G \) always denotes a connected semisimple Lie group of real rank greater than one.

In the symmetric space case we prove rigidity for regular restrictions.

**Theorem 1.** Let \( \alpha_{D_+} \) be a restriction of the split Cartan action \( \alpha \) on a factor \( L \backslash G/\Gamma \) of \( M = G/\Gamma \) to a \( D_+ \subset D \) in regular position. Then there exists a constant \( \ell = \ell(\alpha_{D+}, M) \) such that \( \alpha_{D+} \) is \( C^{\infty, \ell, \infty} \)-locally rigid.

Recall that for the twisted symmetric space examples, if \( \rho \) is irreducible on \( \mathbb{R}^N \) and the weights do not contain 0, then we call the resulted twisted symmetric space hyperbolic and if they contain 0 we call them irreducible genuinely partially hyperbolic.

**Theorem 2.** Let \( \alpha_{D_+} \) be the restriction of the split Cartan action on a factor of a hyperbolic twisted symmetric space \( M = G \times \mathbb{R}^N/\Gamma \times \mathbb{Z}^N \) to \( D_+ \subset D \) in regular position. Then there exists a constant \( \ell = \ell(\alpha_{D+}, M) \) such that \( \alpha_{D+} \) is \( C^{\infty, \ell, \infty} \)-locally rigid.

**Theorem 3.** Let \( \alpha_{D_+} \) be the restriction of the split Cartan action of the irreducible genuinely partially hyperbolic twisted symmetric space on \( X = L \backslash G \times \mathbb{R}^N/\Gamma \times \mathbb{Z}^N \) to \( D_+ \subset D \) in regular position. Let \( \mathfrak{z} \) be the Lie algebra of \( Z(D) \cap Z(L) \). If \( \delta_0(\mathfrak{z}) > \frac{1}{4} \) then there exists a constant \( \ell = \ell(\alpha_{D+}, M) \) such that \( \alpha_{D+} \) is \( C^{\infty, \ell, \infty} \)-locally rigid.

**Remark 3.1.** The condition \( \delta_0(\mathfrak{z}) > \frac{1}{4} \) can not be weakened in the framework of our approach. This condition will be used in finding two commuting regular generators in small neighborhoods of two in general not commuting elements in the iterative step. The norm of the commutator of these two not commuting elements can at best be estimated by the fourth power of the error, so \( \delta_0 > \frac{1}{4} \) is necessary. In fact, this condition also applies to Theorem 1 and 2. We omit it since in these cases \( \delta_0 \geq \frac{1}{2} \) by Lemma 3.1.

If \( G \) is split over \( \mathbb{R} \), then \( \delta_0 = \infty \), hence for all irreducible genuinely partially hyperbolic representations, we have rigidity of higher rank regular restrictions.
Thus what is not fully covered by our theorems is rigidity of general higher rank irreducible genuinely partially hyperbolic symmetric space examples for non-split groups, as well as higher rank restrictions that contain no regular elements. The following result demonstrates the existence of irreducible genuinely partially hyperbolic symmetric space examples for non-split groups for which \( \delta_0(3) > \frac{1}{4} \) for any \( L \), and hence Theorem 3 applies.

**Theorem 4.** For \( G \) and \( \Gamma \) with assumed notations irreducible genuinely partially hyperbolic twisted symmetric space examples exist. For these particular examples \( \delta_0(3) \geq \frac{1}{3} \) if \( G \) is quasi-split or the semisimple part of \( K \) is \( SO(3) \).

As for higher rank non-regular restriction of Cartan actions the there three issues here.

1. First thing is property (B). It is necessary for our method and it is satisfied in many cases.
2. Next issue is that the neutral foliation is not any more a combination of an abelian and compact parts. This changes somewhat the character of linearized conjugacy equations since adjoint action is not any more by isometries. In many situations this problem is not particularly serious.
3. And finally there is a control over \( \delta_0 \) that even in the symmetric space cases need not be greater than 1/4 any more. However, again in many situations a proper estimate can be carried out.

To summarize, one can define a reasonably broad class of non-regular restrictions to which our results extend with a moderate amount of technical changes.

To put our results into perspective let us mention that a scheme similar to that of the present paper applies to certain parabolic cases, i.e. homogeneous actions of unipotent abelian groups. In those situations there are no stable manifolds altogether and hence geometric considerations cannot even get started. While the first paper on the subject [7] uses lots of specific information about the actions at hand and specific constructions of splitting, insights and constructions introduced in the present paper allow to treat parabolic situations of considerably greater generality [8].

### 3.3. Structure of the proof of Theorems 1, 2 and 3

All three theorems are proved simultaneously; along the way we sometimes have to consider three cases separately. Typically distinctions between the symmetric space and twisted hyperbolic symmetric space cases are minimal, and appear at the level of notations. However, the genuinely partially hyperbolic twisted symmetric space case sometimes requires a separate argument, see proofs of Lemmas 5.1, 5.2 and 5.3.

In the next section we formulate without proofs essential ingredients used in the construction of solutions of conjugacy equations. Those include:
• Decay estimates for matrix coefficients for irreducible unitary representations in the form presented in [20]. Those are essential in the first reduction step and construction of distribution solutions for coboundary equations.

• Elliptic regularity results used previously in the work on cocycle rigidity in the form presented in [18] that allow to deduce global regularity of solutions.

In Sections 4.1 and 4.2 we carry out two essential reductions: to the special class of perturbations along the neutral foliation of the original action, and to the conjugacy problem for one generator of the action. After that we derive the linearized conjugacy equation and consider linearized commutativity condition in Section 4.3.

In Section 5 we solve the linearized conjugacy equation in appropriately chosen Sobolev spaces and produce tame estimates for the solutions. This is done in two steps: we first solve linearized equation for a single generator with tame estimates assuming vanishing of naturally defined obstructions (Lemma 5.1) and then show vanishing of the obstructions in the case of higher rank actions due to the linearized commutativity condition thus producing a common solution for the linearized conjugacy equation (Lemma 5.3).

The next and more delicate step is producing a tame splitting for “almost cocycles” in Section 6. We first do that in auxiliary “incomplete” or “partial” Sobolev spaces where the norms include differentiation only along hyperbolic directions and hence no loss of regularity appears in the solution of linearized conjugacy equations (Section 6.2), and then use that to obtain tame estimates for the splitting in the “real” Sobolev spaces (Section 6.3). There are several essential new ingredients in constructions of the solutions of linearized equations and the splitting that are explained at the beginning of Section 6.1 and in Section 6.1.

Finally, in Section 7 the iteration process is carried out. It consists of two parts: The first part (Sections 7.3–7.5) that follows the general scheme developed in [5], and the parameter adjustment (Section 7.6) that is specific for our situation since we prove the conjugacy not with the original action but with its algebraic perturbation. It is only in the second part that condition $\delta_0 > 1/4$ appears, see derivation of inequality (7.37) and subsequent calculation leading to (7.38).

3.4. Elliptic regularity and decay of matrix coefficients. Now we formulate two important facts that form the analysis that are used in the proofs. The first one summarizes elliptic regularity results that we need, and the second contains essential information on the decay of matrix coefficients.

Let $X_1, \cdots, X_\ell$ be smooth vector fields on $M$ whose commutators of length at most $r$ span the tangent space at each point on $M$ and satisfy the following technical condition:
that the vector $v$

**Theorem 6.** [20, appendix] There exist constants $C_i$.

**Corollary 3.1.**

**Theorem 5.** [18] Suppose $f$ is in $L^2(M)$ or a distribution on $M$ and $m$ is a positive integer greater than $r$. Denote by $H_s$ the $s$'th Sobolev space of $M$ with Sobolev norm $\| \cdot \|_s$. If the $m$'th partial derivative $X^m_i (f)$ exists as a continuous or a $L^2(M)$ function, then $f \in H^{\ell-1}$ with estimate

$$
\|f\|_m \leq C_m (\sum_{j=1}^{\ell} \sum_{i=1}^{m} \|X^j_i f\|_0 + \|f\|_0)
$$

where $\|\cdot\|_0$ denotes the $L^2$-norm.

Recall that $G$ is a semisimple Lie group of real rank $\geq 2$.

Let us fix a Riemannian metric $\text{dist}(\cdot, \cdot)$ on $G$ bi-invariant with respect to $K$. Let $\pi$ be a unitary representation of $G$ on a Hilbert space $H$. Say that a vector $v \in H$ is $\delta$-Lipschitz if

$$
\delta = \sup_{g \in G - \{e\}} \frac{\|\pi(g)v - v\|}{\text{dist}(e, g)} < \infty;
$$

we will refer to the number $\delta$ as the $\delta$-Lipschitz coefficient of $v$, and say that the vector $v$ is $\delta$-Lipschitz.

**Theorem 6.** [20, appendix] Let $\pi$ be an irreducible unitary representation of $G$ with discrete kernel. There exist constants $\gamma, E > 0$, dependent only on $G$ such that if $v_i, i = 1, 2,$ be $\delta_i$-Lipschitz vectors in the representation space of $\pi$ then for any $g \in G$

$$
|\langle \pi(g) v_1, v_2 \rangle| \leq (E\|v_1\|_1 \|v_2\|_1 + \delta_1 \|v_2\|_1 + \delta_2 \|v_1\|_1 + \delta_1 \delta_2) e^{-\gamma \text{dist}(e, g)}.
$$

**Corollary 3.1.** There exist constants $\gamma, E > 0$, dependent only on $G$ such that if $f_i, i = 1, 2,$ be $C^1$-functions on $M$ where $M = G/\Gamma$ or $M = G \ltimes \mathbb{R}^N/\Gamma \ltimes \mathbb{Z}^N$ orthogonal to the constants then for any $g \in G$

$$
|\langle f_1(g), f_2 \rangle| \leq E(\|f_1\|_0 \|f_2\|_0 + \|f_1\|_{C^1} \|f_2\|_0 + \|f_2\|_{C^1} \|f_1\|_0 + \|f_1\|_{C^1} \|f_2\|_{C^1}) e^{-\gamma \text{dist}(e, g)}
$$

where $\langle \cdot, \cdot \rangle$ the inner product in $L^2(M)$ with respect to the Haar measure.

**Proof.** Denote by $\rho_0$ the regular representation of $G$ on the space $L^2(M)$ and let $V$ be a non-trivial irreducible component of $L^2(M)$. If we can show that the kernel of $\rho_0$ is discrete, then Theorem 6 applies since every $C^1$ function $f$ on $M$ is $\delta$-Lipschitz with $\delta \leq C\|f\|_{C^1}$ where $C$ is a constant only dependent on $M$.

If $g_0$ is in the kernel of $\rho_0$ then we have $f(g_0g^{-1}) = f$ for every $f \in V$ and $g \in G$. Let $H$ be the subgroup generated by $\{g_0g^{-1}, g \in G\}$. Obviously $H$ is a normal subgroup of $G$. $H$ is discrete if and only if $g_0$ is in the center of $G$. If $H$ is not discrete then the non-compactness follows from the fact...
there is no nontrivial compact normal subgroups in $G$. Then there exists a non-compact one parameter subgroup $\{h_t\}$ in $H$.

For the symmetric space examples, since $\Gamma$ is irreducible, there are no $L^2$-functions on $G/\Gamma$ orthogonal to the constants which are invariant under a non-compact element in $G$ by Moore’s ergodicity theorem \[33\]. Hence every non-trivial irreducible component of $L^2(G/\Gamma)$ has discrete kernel.

For the twisted symmetric space examples we want to show ergodicity of noncompact one parameter subgroups of $G$ on $G \ltimes \mathbb{R}^N/\Gamma \ltimes \mathbb{Z}^N$. Brezin and Moore \[1\] show that a one parameter subgroup of $G$ acts ergodically on $G \ltimes \mathbb{R}^N/\Gamma \ltimes \mathbb{Z}^N$ if and only if the quotient flows on the maximal Euclidean quotient and the maximal semisimple quotient are ergodic. The Euclidean quotient is $G \ltimes \mathbb{R}^N/E$ where $[G \ltimes \mathbb{R}^N, G \ltimes \mathbb{R}^N] \subseteq E$. Notice that the derived group is $G \ltimes \mathbb{R}^N$, as follows from \textcolor{red}{(3.1)}. Hence the maximal Euclidean quotient is a point. The latter quotient is obtained by factoring $G \ltimes \mathbb{R}^N$ by the closure of $\Gamma \ltimes \mathbb{R}^N$ which is isomorphic to $G/\Gamma$. Since $\Gamma$ is irreducible, all noncompact one parameter subgroups of $G$ are ergodic on $G/\Gamma$. Then we get ergodicity of noncompact one parameter subgroups of $G$ on $G \ltimes \mathbb{R}^N/\Gamma \ltimes \mathbb{Z}^N$, especially for $\{h_t\}$. Hence every non-trivial irreducible component of $L^2(G \ltimes \mathbb{R}^N/\Gamma \ltimes \mathbb{Z}^N)$ has discrete kernel. □

4. Preparatory steps and notations

We begin by describing two important steps that reduce the proof of our theorems to a more specialized situation.

4.1. Smooth conjugacy for neutral foliations. Let $\tilde{\alpha}$ be a $C^\infty$ action that is $C^r$ close to $\alpha_{D_+}$. The first key step in the proof of our results is finding a $C^\infty$ diffeomorphism $H_1$ (neutral foliations conjugacy) that maps the neutral foliation of the perturbed action $\tilde{\alpha}$ onto that of the unperturbed action $\alpha_{D_+}$. This is done by the application of \textcolor{red}{[19, Theorem 1]}. A short comment is in order. The quoted theorem has words “Anosov action” in its assumptions and its conclusion is existence of smooth orbit equivalence for such an action satisfying some technical assumptions, and its perturbation. However, the proof applies verbatim to our situation replacing the orbit foliation by the neutral foliation. This is transparent already from the outline at the beginning of Section 2.2 in \textcolor{red}{[19]} and from the fact any one parameter subgroup in $D_+$ acts ergodically (see Corollary \textcolor{red}{3.1}). Furthermore, the foliation conjugacy $H_1$ is $C^r$ close to identity where $r$ is the least integer such that the coarse Lyapunov distributions of $\alpha_{D_+}$ as well as their commutators of length $r$ span the tangent space at any $x \in M$. Let $\tilde{\alpha} = H_1^{-1} \circ \alpha' \circ H_1$. The proof consists of showing that the Hirsch-Pugh-Shub neutral foliation conjugacy that is transversally unique can be chosen to be $C^\infty$ along coarse Lyapunov foliations without loss of regularity. The principal tool here is the theory of non-stationary normal forms, see \textcolor{red}{[9]}. After that global smoothness follows from general elliptic regularity results, i.e. Theorem \textcolor{red}{5} as in \textcolor{red}{[19]}, \textcolor{red}{[6]} or \textcolor{red}{[24]}. 
Conjugating \( \tilde{\alpha} \) by this diffeomorphism \( H_1 \) produces an action whose neutral foliation is the same as for \( \alpha_{D_+} \). Thus general local conjugacy problem for perturbations of \( \alpha_{D_+} \) is reduced to considering a special class of perturbations along the neutral foliation. As we will see later the analytic machinery we use to construct the conjugacy inductively depends crucially on the special form of the linearized conjugacy equation for such perturbations. At this step there is a crucial difference with [5]. In the torus situation considered there the Hirsch-Pugh-Shub neutral foliation conjugacy can \textit{a priori} only be shown to be smooth along coarse Lyapunov directions. But because of the lack of condition (B) elliptic regularity cannot be used and global smoothness does not follow. However, on the torus this is not too high price to pay for considering general linearized conjugacy equation: only stronger requirement for the regularity of the perturbation. This equation still is solved with tame estimates and tame splitting is produced. This is due to the fact that smooth functions on the torus have super-polynomial decay of Fourier coefficients that result to super-exponential decay of correlations for such functions under the action of any irreducible partially hyperbolic automorphism. By contrast, in the semisimple and other cases at hand there is a particular speed of exponential decay of matrix coefficients, however smooth the functions are, and it is not sufficient to construct distribution solutions for the twisted coboundary equations that appear in the linearization of the conjugacy problem for general perturbations. We comment on this point more specifically later, see Remark [4.2].

4.2. Reduction to a conjugacy for a single generator. We will say that a homeomorphism \( \theta : X \to X \) preserves a foliation \( F \) everywhere if \( \theta(F(x)) \subseteq F(x) \) for any \( x \in X \).

The following Lemma shows that obtaining a \( C^\infty \)-conjugacy preserving the neutral foliation everywhere for one regular generator suffices for the proofs of Theorems 1, 2 and 3:

**Lemma 4.1.** Let \( a \in D_+ \) be a regular element for \( \alpha_{D_+} \). Suppose \( \tilde{\alpha} \) is a sufficiently small \( C^1 \) perturbation of \( \alpha_{D_+} \) and \( H \) is a \( C^1 \) map of \( X = L\setminus M \) that is \( C^1 \) close to identity and preserves the neutral foliation everywhere. If \( H \) satisfies:

\[
\tilde{\alpha}(a) \circ H = H \circ \alpha(a') 
\]

for some element \( a' \in Z(D) \cap Z(L) \), then \( H \) conjugates the corresponding maps for all the elements of the action i.e. there exists a homomorphism \( i_0 : D_+ \to Z(D) \cap Z(L) \) such that for all \( d \in D_+ \) we have

\[
\tilde{\alpha}(d) \circ H = H \circ \alpha(i_0(d) \cdot d). 
\]

**Proof.** Let \( d \) be any element \( D_+ \), other than \( a \). It follows from (4.1) and commutativity that

\[
\tilde{\alpha}(a) \circ F = F \circ \alpha(a') 
\]
where

\[ F = \tilde{\alpha}(d)^{-1} \circ H \circ \alpha(d). \]

Therefore

\[ (4.2) \quad \alpha(a') \circ F^{-1} \circ H = F^{-1} \circ H \circ \alpha(a'). \]

We can lift \( \theta = F^{-1} \circ H \) to a map \( \tilde{\theta} \) on \( M \). Since \( \theta \) is \( C^1 \) close to identity and preserves the neutral foliation everywhere, there exist a \( C^1 \) small map \( R : M \to \mathcal{L}^\perp \) such that \( \tilde{\theta} = \exp(R) \cdot I \). Since \( \tilde{\theta} \) is \( L \)-fiber preserving \( R \) satisfies:

\[ l' \cdot \exp(\text{Ad}_l(l^{-1}x)) = \exp(R(x)) \quad \text{for any } l \in L. \]

where \( l' \in L \) is dependent on \( l \). If \( \|R\|_{C^0} \) is small enough by unique decomposition property near identity then it follows

\[ (4.3) \quad \text{Ad}_l(l^{-1}x) = R(x) \quad \text{for any } l \in L. \]

Let \( p \) be the natural projection \( p : M \to M \setminus \mathbb{M} \). Then

\[ \theta(\bar{x}) = p(\exp(R(x)) \cdot p(x)) \]

for any \( x \in M \) satisfying \( p(x) = \bar{x} \). We write \( \theta = \exp(R) \cdot I \) without confusion. By \( (4.2) \) we have:

\[ l(x) \cdot a' \cdot \exp(R(x)) \cdot x = \exp(R(a'x)) \cdot a' \cdot x \quad \text{for any } x \in M \]

where \( l : M \to L \) is a \( C^1 \) map, or equivalently

\[ l \cdot \exp(\text{Ad}_a R) = \exp(R \circ \alpha(a')) \]

By assumption \( a' \in Z(L) \) \( \text{Ad}_a R \) takes values in \( \mathcal{L}^\perp \), then use unique decomposition property again the above equation is reduced to

\[ (4.4) \quad \text{Ad}_a R = R \circ \alpha(a'). \]

If \( \text{Ad}_a \) is diagonalizable, the equation \( (4.3) \) reduces to several equations of the following type:

\[ (4.5) \quad \lambda \omega = \omega \circ \alpha(a') \]

where \( \omega \) is a \( C^1 \) function and \( \lambda \) is an eigenvalue of \( \text{Ad}_a \) with \( |\lambda| = 1 \).

Consider integrals on \( M \) with respect to the Haar measure \( \mu \):  

\[ \int_M \lambda \omega d\mu = \int_M \omega \circ \alpha(a') d\mu = \int_M \omega d\mu. \]

Hence \( \int_M \omega d\mu = 0 \) or \( \lambda = 1 \).

If \( \int_M \omega d\mu = 0 \) by Corollary \( (3.1) \) for any \( f \in C^1(M) \) there exist \( C, \gamma > 0 \) such that

\[ (4.6) \quad |\langle \omega((a')^n), f \rangle| \leq Ce^{-|n|\gamma} \quad \text{for all } n \in \mathbb{Z}. \]

While on the other hand

\[ (4.7) \quad |\langle \omega, f \rangle| = |\langle \lambda^n \omega, f \rangle| = |\langle \omega((a')^n), f \rangle|, \quad \forall n \in \mathbb{Z}, \]

where

\[ F = \tilde{\alpha}(d)^{-1} \circ H \circ \alpha(d). \]
so \( \omega \) is a 0 distribution. Since \( \omega \) is also \( C^1 \), then \( \omega = 0 \).

If \( \lambda = 1 \), let \( \omega' = \omega - \int_M \omega \, d\mu \), then \( \omega' = \omega' \circ \alpha (a') \). Similarly to the above argument \( \omega' = 0 \), hence \( \omega = \int_M \omega \, d\mu \). This implies \( R \) is constant.

If there are Jordan blocks for \( \text{Ad}_{a'} \), the argument above is still sufficient to deduce that \( R \) has to be constant. Namely, if there is, say, a 3-Jordan block, then equation (4.4) reduces to

\[
\begin{pmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix} =
\begin{pmatrix}
\omega_1 \circ \alpha (a') \\
\omega_2 \circ \alpha (a') \\
\omega_3 \circ \alpha (a')
\end{pmatrix}
\]

This implies

\[
\begin{align*}
\lambda \omega_1 + \omega_2 &= \omega_1 \circ \alpha (a') \\
\lambda \omega_2 + \omega_3 &= \omega_2 \circ \alpha (a') \\
\lambda \omega_3 &= \omega_3 \circ \alpha (a')
\end{align*}
\]

From the third equation above, if \( \lambda \neq 1 \), using deday of matrix coefficients as in the case of a simple eigenvalue, we deduce that \( \omega_3 = 0 \). Substituting into the second equation, we obtain \( \omega_2 = 0 \) and finally using this fact in the first equation above, we obtain \( \omega_1 = 0 \). If \( \lambda = 1 \) then by the same argument as in the case of a simple eigenvalue \( \omega_3 = c_3 \) for some \( c_3 \in \mathbb{C} \). Substituting into the second equation, we get \( c_3 = 0 \) by integrating each side and then it follows \( \omega_2 = c_2 \) for some \( c_2 \in \mathbb{C} \). Using a similar argument again by substituting into the first equation we obtain \( c_2 = 0 \) and then \( \omega_1 = c_1 \) for some \( c_1 \in \mathbb{C} \). One can obviously by induction obtain that \( R \) is constant for Jordan blocks of any dimension; hence it follows \( R \) is constant. Since \( R \) satisfies (4.3) \( \exp (R) \in \mathbb{Z}(L) \) and therefore \( F = H \circ \alpha (d') \) for some constant \( d' \in \mathbb{Z}(L) \) and

\[
H = \hat{\alpha} (d)^{-1} \circ H \circ \alpha (i_0 (d) \cdot d)
\]

for an arbitrary \( d \in D_+ \), where \( i_0 (d) = d^{\prime -1} \).

As has been explained in the previous section, the smoothness of the diffeomorphism \( H \) follows from the non-stationary normal forms theory and Theorem 5.

\[\square\]

**Remark 4.1.** Alternatively Lemma 4.1 can be proved using the “geometric” construction of solutions for cohomological equations first introduced in [14] and developed in the setting of abelian group actions in [4]. We prefer the more analytic proof to demonstrate that the machinery of periodic cycle functionals that is central for the geometric approach can be avoided in the present setting.

4.3. **Conjugacy problem and linearization.** Now we proceed to the main part of the proof of our theorem. Since it is carried out via a KAM-type iteration scheme we first need to deduce linearized conjugacy equation in a convenient form on \( X = L \setminus M \).
Due to Lemma 4.11 it is sufficient to prove the existence of a $C^1$ conjugacy $H$ preserving neutral foliation $\mathfrak{L}^\perp$ everywhere such that
\begin{equation}
\tilde{\alpha}(d) \circ H = H \circ \alpha(i_0(d)) \quad \text{for } \forall d \in D_+ \tag{4.8}
\end{equation}
for some homomorphism $i_0 : D_+ \to \exp(\mathfrak{N}) \cap Z(L)$.

Any map on $X$ can be lifted to a map on $M$. For any $d \in D_+$ the lifts of $\tilde{\alpha}(d)$ and $H$ are $\exp(R(d)) \cdot \alpha(i_0(d))$ and $\exp(\Omega) \cdot I$ respectively. Since we are considering only perturbations along the neutral foliation and correspondingly looking for a conjugacy that preserve leaves of that foliation, this means that $\exp(R)$ and $\exp(\Omega)$ take values in $\exp(\mathfrak{L}^\perp)$ and hence
\[
R, \Omega : M \to \mathfrak{L}^\perp \text{ are } C^\infty \text{ maps close to identity.}
\]

Let $\bar{e}$ be the image of the identity $e$ of $G$ or $G \ltimes \mathbb{R}^N$ on $M$. For a small neighborhood $V$ of the tangent bundle $T_eG$ or $T_eG \ltimes \mathbb{R}^N$, we can identify the metrics on $V$ and $\exp V \cdot x$ for any $x \in M$.

Since $\tilde{\alpha}$ is $L$-fiber preserving $R : M \to \mathfrak{L}^\perp$ satisfies the adjoint $L$-invariant condition
\begin{equation}
\text{Ad}_l R(l^{-1} x) = R(x) \quad \text{for any } l \in L. \tag{4.9}
\end{equation}

Then $\Omega : M \to \mathfrak{L}^\perp$ is also adjoint $L$-invariant and for any $\bar{x} \in X$ and any $x \in M$ such that $p(x) = \bar{x}$, where $p$ is the natural projection $M \to L \setminus M$
\[
H(\bar{x}) = p(\exp(\Omega(x)) \cdot p(x))
\]
\[
\tilde{\alpha}(d)(\bar{x}) = p(\exp(R(x)) \cdot \alpha(i_0(d)) \cdot p(x)) \quad \forall d \in D_+
\]

We write without confusion $H = \exp(\Omega) \cdot I$ and $\tilde{\alpha}(d) = \exp(R(d)) \cdot \alpha(i_0(d))$ for any $d \in D_+$.

One can consider the problem of finding a conjugacy as a problem of solving the following non-linear functional equation:
\begin{equation}
\exp(R \circ \tilde{H}) \cdot (i_0(d) \exp(\Omega)i_0(d)^{-1}) = l \cdot \exp(\Omega \circ \alpha \circ i_0(d)). \tag{4.10}
\end{equation}

where $l : M \to L$ and $\tilde{H}$ is the lift of $H$. Let $S(R, \Omega) = R \circ \tilde{H} + \text{Ad}_{i_0(d)} \Omega - \Omega \circ \alpha(i_0(d))$, then
\[
\exp(S(R, \Omega)) \exp(\text{Res}_1(R, \Omega)) = l
\]
where $\text{Res}_1(R, \Omega)$ is quadratically small with respect to $R$ and $\Omega$. By assumption the image of the homomorphism $i_0$ is in $Z(L)$; hence $S(R, \Omega)$ takes values in $\mathfrak{L}^\perp$. By the unique decomposition property
\begin{equation}
\exp(S(R, \Omega)) = \exp(\text{Res}(R, \Omega)) \tag{4.11}
\end{equation}
where $\text{Res}(R, \Omega)$ is quadratically small with respect to $R$ and $\Omega$.

Thus the operator $S$ has the following form:
\[
S(R, \Omega) = S(0, 0) + D_1 S(R, 0) R + D_2 S(0, \Omega) \Omega + \text{Res}_2(R, \Omega) = R + \text{Ad}_{i_0(d)} \Omega - \Omega \circ \alpha(i_0(d)) + \text{Res}_2(R, \Omega) \tag{4.12}
\]
where $\text{Res}_2(R, \Omega)$ is quadratically small with respect to $R$ and $\Omega$. Here $D_1S(R, 0)$ and $D_2S(0, \Omega)$ denote Frechet derivatives of the map $S$ in the first and second variable, respectively, at the point $(0, 0)$ so that the linearization of $S$ at $(0, 0)$ is equal to $R + \text{Ad}_{i_0(d)} \Omega - \Omega \circ \alpha(i_0(d))$.

Combining (4.11) and (4.12) we obtain the linearized equation of (4.10):

\[
\Omega \circ \alpha(i_0(d)) - \text{Ad}_{i_0(d)} \Omega = R. 
\]

(4.13)

Remark 4.2. In deriving the algebraic form of the linearized equation we have not used specific form of our perturbation. For a general perturbation $R$ takes values in the whole Lie algebra and hence $\Omega$ also must take values there but the linearized equation has the same form (4.13). There is however a crucial difference: for our special perturbations $\text{Ad}$ have eigenvalues of absolute value one and, as we have seen in Lemma 4.1 and will soon see, weak exponential estimates on the decay of matrix coefficients given by Theorem 5 are sufficient to show vanishing of functions, see (4.6) and to construct special distribution solutions $\Omega_{(+)}$, see (5.5). In general however the eigenvalues of the $\text{Ad}$ beat the exponential decay of matrix coefficients and the right-hand part in (4.7) or (5.5) diverges too fast to produce even a distribution.

The equation (4.13) actually consists of infinitely many equations corresponding to different elements $d \in D_+$ of the action and we need to find a common approximate solution $\Omega$ to all those equations.

Lemma 4.1 shows that it is enough to produce a conjugacy for one regular element of the action. It is clear however that, in general, it is not possible to produce a $C^\infty$ conjugacy for a perturbation of a single element of the action, since a single element of a genuinely partially hyperbolic action is not even structurally stable. Indeed, as we will soon see in Lemma 5.1 that there are infinitely many obstructions to solving linearized equation for one generator. Therefore, we consider two regular elements of the action, and reduce the problem of solving the linearized equation (4.13) to solving simultaneously the following system:

\[
\begin{align*}
\Omega \circ \alpha(a_1) - \text{Ad}_{a_1} \Omega &= R_a, \\
\Omega \circ \alpha(a_2) - \text{Ad}_{a_2} \Omega &= R_b
\end{align*}
\]

(4.14)

where $a_1 = i_0(a)$ and $a_2 = i_0(b)$ are regular commuting elements of $Z(D)$ close to $a$ and $b$, respectively and

\[
R_a := R(a) \quad R_b := R(b). 
\]

(4.15)

If the solution of the system (4.14) on $M$ has an adjoint $L$-invariant solution then it implies that the solution can descend to a map on $X$ and then we obtain a conjugacy on $X$. Hence throughout the paper, we only solve the system (4.14) on $M$ and show that if $R$ satisfies adjoint $L$-invariant condition then the solution also does.

Notice that since $\Omega$ has values in the Lie algebra $\mathfrak{N}$ of the centralizer of the unperturbed action $\text{Ad}_a \Omega = \text{Ad}_b \Omega = \text{Id}$. Moreover, the centralizer acts
on itself by isometries. Hence \( \text{Ad}_{a_1} \) and \( \text{Ad}_{a_2} \) act on \( \Omega \) as isometries close to identity.

Assume now that \( \exp(R) \cdot (\alpha \circ i_0) \) is a commutative action. Linearized form of commutativity relation gives the following twisted cocycle condition:

\[
R_b \circ a_1 - \text{Ad}_{a_1} R_b - R_a \circ a_2 + \text{Ad}_{a_2} R_a = 0.
\]

We will now justify this formal linearization by showing that \( \exp(a) \) is in \( (\mathfrak{L}, \mathfrak{R}) \). By the Baker-Campbell-Hausdorff formula there exists \( C \in \mathfrak{L} \) such that

\[
\det(1 - \text{Ad}_{a_1} \circ \text{Ad}_{a_2}) = \exp(C).
\]

Then by \([21, \text{Appendix II}]\) it follows that there exists \( C_m > 0 \) only dependent on \( m, \eta \), where\( C_m \) is a constant only dependent on \( m, \eta \).

**Lemma 4.2.** Let \( \tilde{a} = \exp(R) \cdot (\alpha \circ i_0) \) be a commutative \( C^\infty \) action on \( X = L \setminus M \) where \( R \) has values in \( \mathfrak{L}^\perp \) and the image of the homomorphism \( i_0 \) is in \( Z(L) \cap Z(D) \). If \( d(i_0(a), a) + d(i_0(b), b) \leq \eta \) and \( \| R_a, R_b \|_{C^\infty} \leq 1 \), then for \( 0 \leq m \leq s - 1 \) the following inequalities hold

\[
\| L(R_b, R_a) \|_{C^m} \leq C_{m, \eta} \| R_a \|_{C^m} \| R_b \|_{C^m} + 1
\]

where \( C_{m, \eta} \) is a constant only dependent on \( m, \eta \).

**Proof.** Re-writing the commutativity condition \( \tilde{a} \circ \tilde{b} = \tilde{b} \circ \tilde{a} \) in the Lie algebra terms we obtain

\[
\exp(R_a \cdot (\exp(R_b(x)) \cdot a_2 \cdot x)) = \exp(R_a \cdot (\exp(R_b(x)) \cdot a_2 \cdot x)) \cdot a_1 \cdot \exp(R_b(x)) \cdot a_2 \cdot x
\]

where \( l : M \to L \). Equivalently

\[
\exp(R_a \cdot (\exp(R_b \cdot a_2)) \cdot \exp(\text{Ad}_{a_1} R_b) = l \cdot \exp(R_b \cdot (\exp(R_a) \cdot a_1)) \cdot \exp(\text{Ad}_{a_2} R_a).
\]

Let \( S = R_a \cdot (\exp(R_b) \cdot a_2) + \text{Ad}_{a_1} R_b - R_b \cdot (\exp(R_a) \cdot a_1) - \text{Ad}_{a_2} R_a \), then

\[
(4.17) \quad \exp(S + \text{Res}(R_a, R_b)) = l
\]

By the Baker-Campbell-Hausdorff formula there exists \( C_{m, \eta} > 0 \) only dependent on \( m, \eta \) such that

\[
\| \text{Res}(R_a, R_b) \|_{C^m} \leq C_{m, \eta} \| R_a \|_{C^m} \| R_b \|_{C^m}.
\]

By assumption \( i_0 \) valued in \( Z(L) \) then \( S \) takes values in \( \mathfrak{L}^\perp \). By unique decomposition property

\[
(4.18) \quad \| S \|_{C^m} \leq C_{m, \eta} \| R_a \|_{C^m} \| R_b \|_{C^m}.
\]

Furthermore

\[
L(R_b, R_a) \cdot (a_1, a_2) = R_b \circ a_1 - \text{Ad}_{a_1} R_b - R_a \circ a_2 + \text{Ad}_{a_2} R_a
\]

\[
= (R_a \cdot (\exp(R_b) \cdot a_2) - R_a \circ a_2) - (R_b \cdot (\exp(R_a) \cdot a_1) - R_b \circ a_1)
\]

\[
- (R_a \cdot (\exp(R_b) \cdot a_2) + \text{Ad}_{a_1} R_b - R_b \cdot (\exp(R_a) \cdot a_1) - \text{Ad}_{a_2} R_a),
\]

then by \([21, \text{Appendix II}]\) it follows that there exists \( C_m > 0 \) only dependent on \( m \) such that

\[
\| R_a \cdot (\exp(R_b) \cdot a_2) \|_{C^m} \leq C_m \| R_a \|_{C^m} \| R_b \|_{C^m} \| R_a \|_{C^m} \| R_b \|_{C^m} \| R_a \|_{C^m} \| R_b \|_{C^m}.
\]

\[
(4.19) \quad \| R_b \cdot (\exp(R_a) \cdot a_1) - R_b \circ a_1 \|_{C^m} \leq C_m \| R_b \|_{C^m} \| R_b \|_{C^m}.
\]
Then combining (4.18) and (4.19) we obtain desired estimate
\[
\|L(R_b, R_a)^{\phi_1,\phi_2}\|_{C^m} \leq 2C_m\|R_b\|_{C^m}\|R_a\|_{C^{m+1}} + C_{n,m}\|R_a\|_{C^m}\|R_b\|_{C^m} \\
\leq (2C_m + C_{m,n})\|R_a, R_b\|_{C^m}\|R_b, R_a\|_{C^{m+1}}.
\]

\[\square\]

4.4. Some notation. We try as much as possible to develop a unified system of notations for symmetric space examples and twisted symmetric space examples. We will use notations from this section throughout subsequent sections. So the reader should consult this section if an unfamiliar symbol appears.

(1) Let \( U = Z(D) \times S^1_b \), where \( S^1_b \) is the set of complex numbers of absolute value one. For any \( x \in U \), \( B_\eta(x) \) denotes \( \{ y \in U | d(x, y) \leq \eta \} \); for any \( y \in Z(D) \), \( B_\eta(y) \) denotes \( \{ x \in Z(D) | d(x, y) \leq \eta \} \).

(2) Fix positive definite inner products \( \langle \cdot, \cdot \rangle_1 \) on \( \mathfrak{g} \) and \( \langle \cdot, \cdot \rangle_2 \) on \( \mathbb{R}^N \) that are invariant under \( \text{Ad}_K \) and \( \rho(K) \) respectively. Let \( \langle \cdot \rangle = \langle \cdot \rangle_1 + \langle \cdot \rangle_2 \) and \( \mathfrak{L}^\perp \) be the orthogonal complement of \( \mathfrak{L} \) in \( \mathfrak{N} \).

In the setting of twisted symmetric space examples for \( X + v \in \mathfrak{G} + \mathbb{R}^N \) where \( X \in \mathfrak{G} \) and \( v \in \mathbb{R}^N \), let \( \|X + v\|_2 \) denote \( \max\{\|X\|_1, \|v\|_2\} \), where \( \|\cdot\|_1 \) and \( \|\cdot\|_2 \) are induced by \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \) respectively.

(3) For symmetric space examples, let \( X_1, \cdots, X_p \) be a base of \( \{g^\phi\}_{\phi \in \Phi} \), the root spaces of the Cartan action of \( D \), such that their commutators of length at most \( r \) span \( \mathfrak{R} + \mathfrak{D} \). Let \( Y_1, \cdots, Y_q \) be a base of \( \mathfrak{N} \) generated by these commutators.

For twisted symmetric space examples let \( v_1, \cdots, v_{N-N_0} \) be a base of \( \{v^\mu\}_{\mu \in \Phi_1\setminus0} \) the non-zero weight spaces of the Cartan action of \( D \), such that together with \( X_1, \cdots, X_p \) their commutators of length at most \( r \) span \( \mathfrak{N} = \mathfrak{R} + \mathfrak{D} + \mathbb{R}^{N_0}. \) Let \( u_1, \cdots, u_{N_0}, Y_1, \cdots, Y_q \) be a base of \( \mathfrak{N} \) generated by these commutators.

(4) For any function \( f \) on \( M \), \( v \in \mathfrak{G} \) (correspondingly \( v \in \mathfrak{G} + \mathbb{R}^N \)), \( v^m(f) \) denotes the \( m \)th partial derivative along direction \( v \) if it exists.

(5) Let \( m > r \) be an integer. For symmetric space examples, let \( \mathcal{L}^m \) to be the subspace of \( L^2(M) \) such that \( f \) and \( X^k_j(f) \) exist as \( L^2 \) functions for \( 1 \leq j \leq p, k \leq m \); for twisted symmetric space examples, let \( \mathcal{L}^m \) be the subspace of \( L^2(M) \) such that \( f \) \( X^k_j(f) \) and \( v^k_i(f) \) exist as \( L^2 \) functions for \( 1 \leq j \leq p, 1 \leq i \leq N - N_0, k \leq m \).
By Theorem 5, $L^m$ for $m > r$ can be made into a Hilbert space with the norm
\begin{equation}
\|f\|_m \overset{\text{def}}{=} \left( \sum_{j=1}^p \sum_{i=1}^m \|X_j^i f\|_0^2 + \|f\|_0^2 \right)^{1/2}
\end{equation}
for symmetric space examples and
\begin{equation}
\|f\|'_m \overset{\text{def}}{=} \left( \sum_{n=1}^{N-N_0} \sum_{j=1}^p \sum_{i=1}^m \|X_j^i f\|_0^2 + \|f\|_0^2 \right)^{1/2}
\end{equation}
for twisted symmetric space examples.

Denote by $H^s$ the $s$’th Sobolev space of $M$ with Sobolev norm $\|\cdot\|_s$ and let $\|\cdot\|_{C^r}$ stand for $C^r$ norm for functions on $M$. Let $L^r_0 = \{ f \in L^r \mid \int_M f = 0 \}$ and $H^r_0 = \{ f \in H^r \mid \int_M f = 0 \}$.

Assume $f \in H^{s+\sigma}$ where $\sigma > \frac{\dim M}{2} + 1$ and $s \in \mathbb{N}$. The following relations hold by Sobolev embedding theorem
\begin{equation}
\|f\|_s \leq C \|f\|_{C^s} \quad \text{and} \quad \|f\|_{C^s} \leq C_s \|f\|_{s+\sigma}
\end{equation}
In particular, one may take $\sigma = \frac{\dim M}{2} + 1 + \delta$ with small $\delta > 0$.

(6) For a map $F$ with coordinate functions $f_i$, $1 \leq i \leq n_0$ define: $\|F\|_r = \max_{1 \leq i \leq n_0} \|f_i\|_r$ and $\int_M F = (\int_M f_1 d\mu, \ldots, \int_M f_{n_0} d\mu)$ where $\mu$ is the Haar measure. For two maps $F, G$ define $\|F, G\|_r = \max\{\|F\|_r, \|G\|_r\}$. $\|F\|_{C^r}$, $\|F\|'_r$, and $\|F, G\|_{C^r}$ are defined similarly. We write $F \in C^r$ if $f_i \in C^r$ for $1 \leq i \leq n_0$. $F \in H^s$, $F \in H^s_0$, $F \in L^s$ and $F \in L^s_0$ are defined similarly. We say that $F$ is a distribution if coordinates $f_i$ are distribution, $1 \leq i \leq n_0$.

For any connected subgroup $L \subseteq K$, let $S_L$ be the set of maps from $M$ to $L^\perp$.
\begin{equation}
L^r_{0,L} \overset{\text{def}}{=} \{ F \in L^r \cap S_L \mid Ad_l^{-1} F(lx) = F(x) \}, \forall l \in L
\end{equation}
and
\begin{equation}
H^r_{0,L} \overset{\text{def}}{=} \{ F \in H^r_0 \cap S_L \mid Ad_l^{-1} F(lx) = F(x) \}, \forall l \in L.
\end{equation}

(7) For the symmetric space examples, any $z \in Z(D)$ can be decomposed as $z = dk$ where $\log d \in D$ and $\log k \in \mathfrak{k}$; for the twisted symmetric space examples, any $z \in Z(D)$ can be decomposed as $z = dkv$ where $\log d \in D$, $\log k \in \mathfrak{k}$ and $v \in \mathfrak{v}^0$. In either case we call $d$ the split part of $z$ and $k$ the compact part of $z$. We call $a_1, a_2 \in Z(D)$ linearly independent if their splits are linearly independent over $\mathbb{R}$. If $a_1, a_2 \in Z(D)$ are linearly independent with split parts $d_1$ and $d_2$ respectively then $\sum_{\phi \in \Phi} |\phi(j_1 \log d_1 + j_2 \log d_2)| > 0$ for any $j_1, j_2 \in \mathbb{R}$.
with $j_1^2 + j_2^2 \neq 0$.

(8) For $(z, \lambda) \in U$ and a function $f$ on $M$, define the twisted coboundary operators:

$$(z, \lambda)^{\tau} f = f(z) - \lambda f$$

In what follows $\lambda$ will be either equal to 1 or to an eigenvalue of $\text{Ad}_k$ or $\rho(k)$, where $k$ is the compact part of $z$. Recall that $|\lambda| = 1$. For any $s > 0$, $(z, \lambda)^{\tau}$ is a bounded linear operator on $L^s_0$. Denote the operator norm by $\|(z, \lambda)^{\tau}\|_s$.

(9) Let $f$ be a function or a distribution. We introduce notations for the following formal sums:

$$\sum_{j=0}^{+\infty} f \overset{\text{def}}{=} -\sum_{j \geq 0} \lambda^{-(j+1)} f \circ z^j$$

$$\sum_{j \leq -1} f \overset{\text{def}}{=} \sum_{j \leq -1} \lambda^{-(j+1)} f \circ z^j$$

$$\sum_{j=-\infty}^{+\infty} f \overset{\text{def}}{=} \sum_{j=-\infty}^{+\infty} \lambda^{-(j+1)} f \circ z^j$$

(10) For $z \in Z(D)$ and a map $F : M \to \mathfrak{g}$, define the twisted coboundary operator:

$$\mathcal{T}_z F = F(z) - \text{Ad}_z F.$$

Similarly to the scalar case we introduce notations for the following formal sums:

$$\sum_z F \overset{\text{def}}{=} \sum_{j=-\infty}^{+\infty} \text{Ad}_z^{-j+1} F \circ z^j.$$

(11) For two functions $\theta, \varphi$ and maps $F, G : M \to \mathfrak{g}$, for $(a_1, \lambda_1), (a_2, \lambda_2) \in U$ define the following operators:

$$\mathcal{L}(\theta, \varphi)_{(\lambda_1, \lambda_2)}^{(a_1, a_2)} \overset{\text{def}}{=} (a_2, \lambda_2)^{\tau} \theta - (a_1, \lambda_1)^{\tau} \varphi$$

$$\mathcal{L}(F, G)_{(a_1, a_2)}^{(a_2, a_1)} \overset{\text{def}}{=} F \circ a_2 - \text{Ad}_{a_2} F - G \circ a_1 + \text{Ad}_{a_1} G = \mathcal{T}_{a_2} F - \mathcal{T}_{a_1} G.$$

We will sometimes use generic notation $\mathcal{L}(F, G)$ when the generators are either not specified or clear from the context.

(12) In what follows, $C$ will denote any constant that depends only on the given action $\alpha_D$ with two linearly independent regular generators and on the dimension of group $G$. $C_{x,y,z,...}$ will denote any constant.
that in addition to the above depends also on parameters $x, y, z, \ldots$.

(13) Smoothing operators and some norm inequalities

The space $M$ is compact, then there exists a collection of smoothing operators $S_t : C^\infty(M) \to C^\infty(M)$, $t > 0$, such that the following holds:

$$
\|S_t f\|_{C^{s+s'}} \leq C_{s,s'} t^{s'} \|f\|_{C^{s}}
$$

$$
\|(I - S_t) f\|_{C^{s-s'}} \leq C_{s,s'} t^{-s'} \|f\|_{C^{s}}
$$

5. Solution of the linearized equation

We consider scalar equations that appear as projections of the linearized conjugacy equations on common eigenspaces for two commuting $\text{Ad}$ operators.

5.1. Cohomological stability. First we define obstructions to solvability of the linearized conjugacy equation, i.e. twisted coboundary equation, for a single element and show that vanishing of those obstructions implies solvability of the equation with tame estimates wrt. Sobolev norms. The latter property is an instance of cohomological stability, the notion first defined in [13].

Scheme of the proof is as follows:

1. Decay estimates for matrix coefficients imply existence of two distribution solutions obtained by iteration in positive and negative directions: one of those solutions is differentiable along stable directions and the other along unstable directions.

2. Vanishing of the obstructions implies that those distribution solutions coincide.

3. Condition (B) allows to apply elliptic regularity and deduce that solution is really a smooth function. At this stage however there is a large loss of regularity, roughly from $m$th norm to $\frac{m}{2}$th norm.

4. Since solution along the stable and unstable directions is given by explicit exponentially converging “telescoping sums” they can be differentiated without loss of regularity. Up to this point the proof follows the same general scheme as in [17] although we obtain more elaborate information about estimates in different norms.

5. Remaining directions are those of the centralizer of the acting element; in particular, the adjoint representation acts by isometries and hence derivatives of all orders in those direction are bounded. Tame estimates follow form that and from the fact that those vector-fields can be expressed as polynomial of hyperbolic ones, i.e. from condition (B).
Lemma 5.1. Let \( a \) and \( b \) be two regular generators for the unperturbed action \( \alpha_D \). Let \((z, \lambda) \in B_\eta(a, 1) \cup B_\eta(b, 1)\) with \( \eta \) small enough and \( \theta \in \mathcal{H}_0^m \), \( m \in \mathbb{N} \) and \( m \geq m_i \), where

\[
m_1 \overset{\text{def}}{=} \max\{r \dim M/2 + r^2 + 4r, r \dim M/2 + rr_0 + 4r, r_0 r + 4r\}
\]

where \( r_0 \) is a positive constant defined in (5.13) and (5.14) below respectively, depending on the eigenvalues of \( \text{Ad}_a \) and \( \text{Ad}_b \). If

\[
\sum_{j=0}^{+\infty} \lambda^{-(j+1)} \theta \circ z^j = 0
\]

as a distribution, then the equation

\[
\Omega \circ z - \lambda \Omega = \theta
\]

have a solution \( \Omega \in \mathcal{H}_0^{m-r-1} \) and the following estimate holds

\[
\|\Omega\|_{m-r-1} \leq C_{m, \eta} \|\theta\|_m.
\]

We present a detailed proof for the symmetric space examples showing in particular how the above scheme is implemented, and then describe additional ingredients that appear in the twisted cases.

Proof for symmetric space examples. Distribution solution. To find the solution \( \Omega \) let us first show that the formal solutions

\[
\Omega_{(+)} = \left[\sum_{j \geq 0} \lambda^{-(j+1)} \theta \circ z^j\right]_{j \geq 0}
\]

are distributions.

Denote \( z = dk \) where \( d \) is the split part and \( k \) is the compact part. Since \( m > \frac{\dim M}{2} + 2 \) then \( \theta \in C^1(M) \).

Let \( g \in C^\infty(G/\Gamma) \). By Corollary 3.11 there exist constants \( \gamma, E > 0 \) only dependent on \( G \) such that

\[
|\langle \lambda^{-(j+1)} \theta (z^j), g \rangle| \leq E \|\theta\|_{C^1} \|g\|_{C^1} e^{-\gamma j l(z)}
\]

where \( l(z) = \frac{1}{2} \sum_{\phi \in \Phi} |\phi(d)| \). Hence \( \sum_{j=0}^{+\infty} \langle \lambda^{-(j+1)} \theta (z^j), g \rangle \) converges absolutely, and there is a constant \( C > 0 \) such that \( \sum_{j=0}^{+\infty} \langle \lambda^{-(j+1)} \theta (z^j), g \rangle \leq C \|g\|_{C^1} \). Thus \( \Omega_+ \) and similarly \( \Omega_- \) are distributions. By assumption \( \sum (z, \lambda) \theta = 0 \) hence \( \Omega_+ \overset{\text{def}}{=} \Omega_- = \Omega_+ \). This gives a formal solution \( \Omega \).

Smoothness in stable and unstable directions. Next we will show differentiability of \( \Omega \) along the stable and unstable foliations of \( z \) by using both of its forms.

Let \( \phi \) be the root corresponding to \( X_i \). If \( \phi(\log d) < 0 \), we may use the \( \Omega_+ \) form for the solution to obtain the following bound on \( s \)th derivative

\[
\sum_{j=0}^{\infty} X_i^s (\lambda^{-(j+1)} \theta \circ z^j) = \sum_{j=0}^{\infty} e^{s \phi(\log d)} \lambda^{-(j+1)} Z_j^s (\theta) \circ z^j
\]
where \( Z_j = \text{Ad}^j_k(X_i) \) for all \( 1 \leq s \leq m \). Notice that \( \|X_i\| = \|\text{Ad}_k(X_i)\| \) hence the left-hand side of (5.11) converges absolutely in \( L^2 \) norm on \( M \) and we get

\[
(5.8) \quad \|X_i^s(\Omega)\|_0 \leq C_{s,\eta}\|\theta\|_s.
\]

Similarly, if \( \phi(\log d) > 0 \) using the form \( \Omega = \Omega_\eta \), the estimate \( \|X_i^s(\Omega)\|_0 \leq C_{s,\eta}\|\theta\|_s \) holds if \( 1 \leq s \leq m \).

**Smoothness of the solution.** By Theorem 3, \( \Omega \in \mathcal{H}_{\eta}^{m-1} \) and it follows that

\[
(5.9) \quad \|\Omega\|_{m-1} \leq C_{m,\eta}(\|\theta\|_m + \|\Omega\|_0).
\]

By assumption \( \frac{m}{2} - 1 - \frac{\dim M}{2} - 2 \geq 1 \), then \( \Omega \in C^1(M) \). Using Sobolev embedding theorem together with (5.9) we have

\[
(5.10) \quad \|\Omega\|_{C^1} \leq C\|\Omega\|_{\frac{m}{2}+3} \leq C\|\Omega\|_{m-1} \leq C_{m,\eta}(\|\theta\|_m + \|\Omega\|_0).
\]

By Corollary 3.1 we have

\[
\|\Omega\|^2_0 = |\sum_{j=0}^{\infty} \lambda^{-(j+1)}\langle z^j, \Omega \rangle| \leq \sum_{j=0}^{\infty} |\langle \theta(z^j), \Omega \rangle| \leq A_0(\|\Omega\|_0 + \|\Omega\|_{C^1})
\]

where \( 0 < A_0 \leq C_{\eta}\|\theta\|_{C^1} \). Together with (5.10) we have

\[
(5.11) \quad \|\Omega\|_0 \leq C_{m,\eta}\|\theta\|_m.
\]

Substituting (5.11) into (5.9) it follows

\[
(5.12) \quad \|\Omega\|_{m-1} \leq C_{m,\eta}\|\theta\|_m.
\]

**Tame estimates in stable and unstable directions.** Let \( x \) stand for either \( a \) or \( b \). Define \( r_0 \) to be the minimal positive integer satisfying:

\[
\chi(\log x) + r_0\phi(\log x) > 0 \quad \text{if } \phi(\log x) > 0
\]

\[
(5.13) \quad \chi(\log x) + r_0\phi(\log x) < 0 \quad \text{if } \phi(\log x) < 0
\]

for any \( \chi, \phi \in \Phi \).

Let \( \eta \) be small enough such that

\[
\chi(\log g) + r_0\phi(\log g) > 0 \quad \text{if } \phi(\log g) > 0
\]

\[
(5.14) \quad \chi(\log g) + r_0\phi(\log g) < 0 \quad \text{if } \phi(\log g) < 0
\]

for any \( \chi, \phi \in \Phi \) and any \( g \in Z(D) \) with either \( d(g, a) \leq \eta \) or \( d(g, b) \leq \eta \).

By assumption \( \frac{m}{2} - 1 > \frac{\dim M}{2} + r_0 + 2 \), then \( \Omega \in C^{r_0+1}(M) \) with bound

\[
(5.15) \quad \|\Omega\|_{C^{r_0+1}} \leq C\|\Omega\|_{m-1} \leq C_{m,\eta}\|\theta\|_m.
\]

Fix \( X_i, 1 \leq i \leq p \) with corresponding root \( \phi \). If \( \phi(\log d) < 0 \) then

\[
X_i\Omega = -\sum_{j \geq 0} e^{j\phi(\log d)}\lambda^{-(j+1)}(Z_j\theta) \circ z^j
\]

where \( Z_j = \text{Ad}^j_k(X_i) \).
Note that \( \|X_j\| = \|Z_j\| \) for all \( j \), then for any \( 1 \leq j \leq q \) we have
\[
(5.16) \quad \|Y_j^s(X_i\Omega)\|_0 \leq C_{m,\eta}\theta_m \quad \text{if } s \leq m - 1.
\]
For any \( X_j, 1 \leq j \leq p \) with the corresponding root \( \chi \), if \( \chi(\log d) < 0 \) we have
\[
(5.17) \quad \|X_j^s(X_i\Omega)\|_0 \leq C_{m,\eta}\theta_m \quad \text{if } s \leq m - 1.
\]
For any \( X_l, 1 \leq l \leq p \) with the corresponding root \( \nu \), if \( \nu(\log d) > 0 \), let
\[
P_j = \text{Ad}_k^l(X_l)
\]
and let \( P_j^{\nu}Z_j \) act on each side of equation (5.3) we get
\[
e^{\phi(\log d) + r\nu(\log d)}(P_j^{\nu}Z_{j+1}\Omega) = z - \lambda P_j^{\nu}Z_j\Omega = P_j^{\nu}Z_j\theta
\]
where each side is continuous function for any \( j \in \mathbb{Z} \) by (5.15).

By assumption of \( r_0 \), \( e^{\phi(\log d) + r\nu(\log d)} > 1 \), then
\[
X_j^{r_0}(X_i\Omega) = P_j^{r_0}Z_0(X_i\Omega)
\]

(5.19)
\[
e^\sum_{j \leq -1} e^{j\phi(\log d) + jr\nu(\log d)}\lambda^{-j-1}(P_j^{\nu}Z_j\theta) \circ z^j.
\]
Then if \( 0 \leq s \leq m - 1 - r_0 \) we have
\[
(5.20) \quad \|X_j^{r_0+s}(X_i\Omega)\|_0 \leq C_{m,\eta}\theta_m.
\]
By assumption \( \frac{m}{r} - 1 > r_0 + 2 \) and (5.12) we also have
\[
(5.21) \quad \|X_i^s(X_i\Omega)\|_0 \leq C_{m,\eta}\theta_m \quad \text{if } s \leq r_0.
\]
Combine (5.16), (5.17), (5.20) and (5.21) by Theorem 3 \( X_i\Omega \in \mathcal{H}_0^{m-2} \) with estimate
\[
(5.22) \quad \|X_i\Omega\|_{m-2} \leq C_{m,\eta}\theta_m.
\]
If \( \phi(d) > 0 \), the arguments follows in a similar way. Similarly we get
\[
(5.23) \quad \|X_j\Omega\|_{m-2} \leq C_{m,\eta}\theta_m
\]
for all \( 1 \leq j \leq p \).

**Tame estimates in neutral directions.** In the universal enveloping algebra \( U(\mathfrak{g}) \) let \( p_j \) be a polynomial with degree no greater than \( r \) such that
\[
p_j(X_j^{(1)}, \cdots, X_j^{(q)}) = Y_j, \quad 1 \leq j \leq q.
\]
Such a polynomial exists due to the condition (23). Note that
\[
(5.24) \quad Y_j^s(\Omega) = Y_j^{s-1}p_j(X_j^{(1)}, \cdots, X_j^{(i)})\Omega
\]
Recall that \( C^0 \) norms of all powers of \( Y_j \) are uniformly bounded since \( Y_j \) is the generates action by isometries. By (5.23) if \( s - 1 + r - 1 \leq m - 2 \) we have
\[
\|Y_j^s(\Omega)\|_0 \leq C\sum_{j(i)} \|X_j^{(i)}\Omega\|_{r-1+s-1} \leq C_{m,\eta}\theta_m
\]
combine with (5.8) by Theorem 5 \( \Omega \in \mathcal{H}_0^{m-r-1} \) with estimate
\[
(5.25) \quad \|\Omega\|_{m-r-1} \leq C_{m,\eta}\theta_m.
\]
It follows that there exists a solution $\Omega$ to equation (5.3) stratifying estimates (5.4) providing the condition (5.2) is satisfied. \hfill $\square$

**Proof for twisted symmetric space examples.** We assume notations in the previous part if there is no confusion.

(i) If $\rho$ is Anosov then the neutral distribution is still $\mathcal{O} + \mathcal{R}$. Then $z = (dk, 0)$ where $d$ is the split part and $k$ is the compact part. Notice for any $(g, t) \in G \ltimes \mathbb{R}^N$, $z \cdot (g, t) = (dkg, t)$, then the statement follows essentially verbatim as in the symmetric space examples.

(ii) If $\rho$ is genuinely partially hyperbolic then the neutral distribution is $\mathcal{O} + \mathcal{R} + \mathcal{V}^0$. Let $z = (dk, t_0)$ where $d$ is the split part, $k$ is the compact part and $t_0 \in \mathcal{V}^0$ then $z^j = (d^j k^j, t_j)$ where $t_j = \sum_{i=0}^{j-1} \rho(k)^{-i} t_0$ for any $j$.

First let us show that two formal solutions $\Omega$ (5.4) providing the condition (5.2) is satisfied.

(i) If $\phi$ is small enough then $\Omega$ is Anosov then the neutral distribution is still $\mathcal{O} + \mathcal{R} + \mathcal{V}^0$. Let $z = (dk, t_0)$ where $d$ is the split part, $k$ is the compact part and $t_0 \in \mathcal{V}^0$ then $z^j = (d^j k^j, t_j)$ where $t_j = \sum_{i=0}^{j-1} \rho(k)^{-i} t_0$ for any $j$.

By Corollary 3.1 there exist constants $\gamma, E > 0$ only dependent on $G$ such that for any $f \in C^\infty(G \ltimes \mathbb{R}^N / \Gamma \ltimes \mathbb{Z}^N), j \in \mathbb{Z}$ we have

$$
|\langle \lambda^{-j+1} \theta(z^j x), f(x) \rangle |
= |\langle \theta((dk)^j, 0)x), f((0, -t_j)x) \rangle |
\leq E \|\theta\| c_1 \|f((0, -t_j)x)\| e^{-\gamma j \|l(z)\|}
\leq CE\|t_0\| \|\theta\| c_1 \|f\| c_1 \|j\| e^{-\gamma j \|l(z)\|}
$$

(5.26)

where $l(z) = \frac{1}{2} \sum_{\phi \in \Phi} |\phi(d)|$. Hence $\sum_{j=0}^{+\infty} \langle \lambda^{-j+1} \theta(z^j), f \rangle$ converges absolutely, and thus $\Omega_+$ and $\Omega_-$ are distributions. By the assumption $\sum_{j=0}^{+\infty} \langle z, \lambda \theta \rangle = 0$ we have $\Omega = \Omega_+ = \Omega_- = 0$.

Note that for any $X_i, 1 \leq i \leq p$ with corresponding root $\phi$ we have

$$
\text{Ad}_z(X_i) = e^{\phi \langle \log d \rangle} \text{Ad}_{(k, t_0)}(X_i)
= e^{\phi \langle \log d \rangle} \text{Ad}_{k}(X_i) - e^{\phi \langle \log d \rangle} \rho(k) d \rho(X_i) \rho(t_0)
$$

(5.27)

and for any $v_i, 1 \leq i \leq N - N_0$ with corresponding weight $\mu$ we have

$$
\text{Ad}_z(v_i) = e^{\mu \langle \log d \rangle} \rho(k) v_i = e^{\mu \langle \log d \rangle} \text{Ad}_{(k, t_0)}(v_i),
$$

(5.28)

the remaining part follows almost the same way as in the symmetric space examples if we can show $\|\text{Ad}_{(k, t_0)}(X_i)\|$ and $\|\text{Ad}_{(k, t_0)}(v_i)\|$ are bounded for any $j \in \mathbb{Z}$. It is obvious for the latter one since $\|\text{Ad}_{(k, t_0)}(v_i)\| = \|\rho(k) v_i\| = \|v_i\|$ by assumption.

For the first one if $\eta$ is small enough then

$$
\|\rho(k) d \rho(X_i) t_0\| = \|d \rho(X_i) t_0\| \leq \|X_i\| = \text{Ad}_k(X_i)
$$

then we have

$$
\|\text{Ad}_{(k, t_0)}(X_i)\| = \|\text{Ad}_k(X_i)\|.
$$

$\square$
Corollary 5.1. If the equation
\[ \Omega \circ z - \lambda \Omega = \theta \]  
has a solution in \( C^1(M) \cap \mathcal{H}_0^0 \) then it is unique.

Proof. If \( \Omega' \in C^1(M) \cap \mathcal{H}_0^0 \) is also a solution of equation (5.29), then \( \lambda^{-j}(\Omega - \Omega') \circ z^j = \Omega - \Omega' \) for any \( j \in \mathbb{Z} \). Using Corollary 3.1 as in (5.6) and (5.26) the left side is a 0 distribution. Since \( \Omega - \Omega' \in C^1(M) \) then \( \Omega = \Omega' \). \( \Box \)

Lemma 5.2. Let \( \eta \) be sufficiently small and \( m \in \mathbb{N} \) with \( m \geq m_1 + r + 1 \) where \( m_1 \) is defined as in Lemma 5.1. For any \( z \in B_\eta(a) \cup B_\eta(b) \) and any \( \mathcal{H}_0^m \) map \( \mathcal{F} : M \to \mathfrak{r} \). If
\[ \sum_{j=-\infty}^{+\infty} \text{Ad}_z^{-j}(\mathcal{F} \circ z^j) = 0 \]  

as a distribution, then the equation
\[ \Lambda \circ z - \text{Ad}_z \Lambda = \mathcal{F} \]  
have a solution \( \Lambda \in \mathcal{H}_0^{m-2r-2} \) and the following estimate holds
\[ \| \Lambda \|_{m-2r-2} \leq C_{m,\eta} \| \theta \|_m. \]

Furthermore if \( z \in Z(L) \) and \( \mathcal{F} \in \mathcal{H}_0^{m,L} \), then \( \Lambda \in \mathcal{H}_0^{m-2r-2,L} \).

Proof for symmetric space examples and hyperbolic twisted examples. Let \( \mathfrak{N}_C \) be the complexification of the subalgebra \( \mathfrak{N} = \mathfrak{k} + \mathfrak{d} \). There exists an orthonormal basis that diagonalizes \( \text{Ad}_z \). As usual, this basis may be chosen to consists of several real vectors and several pairs of complex conjugate vectors. Equations (5.30) and (5.31) split into finitely many equations of the form
\[ \sum_{j=-\infty}^{+\infty} \lambda^{-j}(\varphi \circ z^j) = 0 \]  
and
\[ \omega \circ z - \lambda \omega = \varphi \]
where \( \varphi \) is a \( \mathcal{H}_0^m \) function and \( \lambda \) in \( S^1_C \) is corresponding eigenvalue of \( \text{Ad}_z \). Notice that since the coefficients and the right-hand part of the equation (5.31) are real-valued, the unique solution in \( C^1(M) \cap \mathcal{H}_0^0 \) (see Corollary 5.1) is real-valued as well. Then the conclusion follows directly from Lemma 5.1. \( \Box \)

Proof for genuinely partially hyperbolic twisted examples. Now the neutral distribution is \( \mathfrak{N} = \mathfrak{d} + \mathfrak{k} + \mathfrak{v}^0 \). Let \( z = (dk,t) \) where \( d \) is the split part, \( k \) is compact part and \( t \in \mathfrak{v}^0 \). Consider as before the complexification \( \mathfrak{N}_C \) of
\( \mathfrak{H} \). By (5.27) and (5.28) there exists an orthonormal basis in \( \mathfrak{H}_C \) such that \( \text{Ad}_z \) and \( \text{Ad}_{z^{-1}} \) are represented in that basis by matrices \( J_1 \) and \( J_2 \) where

\[
J_i = \begin{pmatrix}
A_i & 0 & 0 \\
B_i & D_i & 0 \\
0 & 0 & E_i
\end{pmatrix}.
\]

Here \( E_i = I_{\text{dim}(D_i)} \), \( A_i \) are \( \text{dim} K \times \text{dim} K \) diagonal matrices, \( D_i \) are \( \text{dim} v_0 \times \text{dim} v_0 \) diagonal matrices all of whose eigenvalues are of absolute value 1 all elements of \( B_i \) have absolute value smaller than 1 if \( \eta \) is small enough. This basis can be chosen as in the symmetric space case to include real vectors and pairs of complex conjugate vectors.

Then equations (5.31) have the form:

\[
\Lambda \circ z - J_1 \Lambda = \Theta
\]

and condition (5.30) can be written as

\[
\sum_{j = -\infty}^{+\infty} J_1^{-(j+1)} \Theta \circ z^j = 0.
\]

We will show that the formal solutions

\[
\Lambda(\pm) = \begin{pmatrix} - \\ + \end{pmatrix} \sum_{\substack{j \geq 0 \\
(j \leq -1)}} J_1^{-(j+1)} \Theta \circ z^j
\]

are in fact \( \mathcal{H}^{m-2r-2} \) solutions. Denote by \( \Lambda_\pm^i \) and \( \Lambda_\pm^i \) the \( i \)-th coordinates of \( \Lambda_- \) and \( \Lambda_+ \) respectively.

Denote entries of the matrices \( J_1 \) and \( J_2 \) by \( q_1^{ij} \) and \( q_2^{ij} \) correspondingly and let \( q_1^{ij} = \lambda_i \) for \( 1 \leq i, j \leq t_0 \) where \( t_0 = \text{dim} \mathcal{D} + \text{dim} v_0 + \text{dim} \mathfrak{H} \).

By simply comparing coefficients, it is easy to obtain the following relation between the coefficients of \( \text{Ad}_z \) and \( \text{Ad}_{z^{-1}} \):

\[
q_1^{ij} + \lambda_i \lambda_j q_2^{ij} = 0
\]

for \( 1 \leq j \leq \dim(\mathfrak{H}) \), \( \dim(\mathfrak{H}) + 1 \leq i \leq \dim(\mathfrak{H}) + \dim v_0 + 1 \).

Let the coordinate functions of \( \Theta \) be \( \vartheta_i \), \( 1 \leq i \leq t_0 \). For \( 1 \leq i \leq \dim(\mathfrak{H}) \) or \( \dim(\mathfrak{H}) + \dim v_0 + 1 \leq i \leq t_0 \), the \( i \)-th equation in (5.35) becomes:

\[
\omega_i \circ z - \lambda_i \omega_i = \vartheta_i
\]

and the condition (5.36) splits as

\[
\sum_{j = -\infty}^{+\infty} \lambda_i^{-(j+1)} \vartheta_i \circ z^j = 0.
\]

Then the existence of a solution follows the same way as in Lemma 5.1. Moreover, the estimate:

\[
\|\omega_i\|_{m-r-1} \leq C_{m,\eta} \|\vartheta_i\|_m \leq C_{m,\eta} \|\Theta\|_m
\]
holds for \(1 \leq i \leq \dim(\frak{r})\) or \(\dim(\frak{r}) + \dim\frak{v}^0 + 1 \leq i \leq t_0\). Equality \(\omega_i = \Lambda_i^i = -\Lambda_i^{i-1}\) follows directly from Lemma 5.1.

For \(n_0 \geq i \geq \dim(\frak{r}) + 1\) the \(i\)-th equation in (5.35) is:

\[
(5.42) \quad \omega_i \circ a_1 - \lambda_i \omega_i = \vartheta_i + \sum_{j=1}^{k_0} q_1^{ij} \omega_j
\]

where \(n_0 = \dim(\frak{r}) + \dim\frak{v}^0\) and \(k_0 = \dim(\frak{r})\). By [5.38] the \(i\)-th coordinate function of \(J_1^{-(j+1)}\) \(\omega \circ z^j\) is

\[
(5.43) \quad \lambda_i^{-(j+1)} \vartheta_i \circ z^j + \sum_{k=1}^{k_0} \sum_{n=0}^{-j-2} q_1^{ik} \lambda_k^n \lambda_i^{-j-2-n} \vartheta_k \circ z^j \quad \text{for any } j \leq -2
\]

\[
(5.44) \quad \lambda_i^{-(j+1)} \vartheta_i \circ z^j - \sum_{k=1}^{k_0} \sum_{n=-1}^{-j-1} q_1^{ik} \lambda_k^n \lambda_i^{-j-n-2} \vartheta_k \circ z^j \quad \text{for any } j \geq 0
\]

and it follows:

\[
\Lambda_i^- = \vartheta_i \circ z^{-1} + \sum_{j \leq -2} (\lambda_i^{-(j+1)} \vartheta_i \circ z^j + \sum_{k=1}^{k_0} \sum_{n=0}^{-j-2} q_1^{ik} \lambda_k^n \lambda_i^{-j-2-n} \vartheta_k \circ z^j)
\]

\[
\Lambda_i^+ = -\sum_{j \geq 0} (\lambda_i^{-(j+1)} \vartheta_i \circ z^j + \sum_{k=1}^{k_0} \sum_{n=-1}^{-j-1} q_1^{ik} \lambda_k^n \lambda_i^{-j-n-2} \vartheta_k \circ z^j).
\]

Using Corollary 3.1 and the fact that \(e^{-\delta |j|}\) decreases faster than any negative power of \(|j|\) for any \(\delta > 0\) and \(j \in \mathbb{Z}\), similar to (5.26) we can show that both \(\Lambda_i^-\) and \(\Lambda_i^+\) are distributions. Now we use the fact that all the subsequent equations are solved i.e. we substitute all \(\vartheta_j\) for all \(1 \leq j \leq k_0\) into above functions using their expression as in (5.39). This implies:

\[
\Lambda_i^- = \sum_{j \leq -1} \lambda_i^{-(j+1)}(\vartheta_i(z^j)) + \sum_{j=1}^{k_0} q_1^{ik} \omega_j(z^j) + \lim_{j \to -\infty} \sum_{k=1}^{-j-1} \sum_{n=1}^{k_0} q_1^{ik} \lambda_k^n \lambda_i^{-n-j-1} \omega_k(z^j)
\]

\[
\Lambda_i^+ = -\sum_{j \geq 0} \lambda_i^{-(j+1)}(\vartheta_i(z^j)) + \sum_{j=1}^{k_0} q_1^{ik} \omega_j(z^j) + \lim_{j \to +\infty} \sum_{k=1}^{j+1} \sum_{n=1}^{k_0} q_1^{ik} \lambda_i^{-j-n-2} \lambda_k^{-n} \omega_k(z^{j+1}).
\]

By Corollary 3.1 and similar to (5.26) we can show without difficulty that

\[
\lim_{j \to -\infty} \sum_{k=1}^{-j-1} \sum_{n=1}^{k_0} q_1^{ik} \lambda_k^n \lambda_i^{-n-j-1} \omega_k(z^j) \quad \text{and} \quad \lim_{j \to +\infty} \sum_{k=1}^{j+1} \sum_{n=1}^{k_0} q_1^{ik} \lambda_i^{-j-n-2} \lambda_k^{-n} \omega_k(z^{j+1})
\]

are 0 distributions. Hence it follows

\[
(5.45) \quad \Lambda_i^- - \Lambda_i^+ = \sum_{j=-\infty}^{+\infty} \lambda_i^{-(j+1)}(\vartheta_i(z^j)) + \sum_{j=1}^{k_0} q_1^{ik} \omega_j(z^j))
\]

moreover by assumption (5.36) the right side is 0 distribution.
Thus the equation (5.42) satisfies the solvability condition (5.2) and notice that its right-hand side is a $\mathcal{H}_0^{m-r-1}$ function by (5.41) therefore we may use Lemma 5.1 again to conclude that the equation (5.42) has a solution $\omega_i \in \mathcal{H}_0^{m-2r-2}$ and

$$\omega_i = \Lambda^-_i = -\Lambda^+_i$$

(5.46)

for any $n_0 \geq i \geq \dim(\mathfrak{g}) + 1$.

As a consequence of assumption (5.41) this solution satisfies the estimate

$$\|\omega_i\|_{m-2r-2} \leq C_{m,\eta}\|\vartheta_i\| + \sum_{j=1}^{k_0} q_{ik}^j \omega_j \|_{m-r-1} \leq C_{m,\eta}\|\Theta\|_m$$

(5.47)

for $n_0 \geq i \geq \dim(\mathfrak{g}) + 1$ if $m-r-1 \geq m_1$.

Combine (5.41) and (5.47) we obtain a $\mathcal{H}_0^{m-2r-2}$ solution $\Lambda$ of the equation (5.31) with estimate:

$$\|\Lambda\|_{m-2r-2} \leq C_{m,\eta}\|\mathcal{F}\|_m.$$ 

(5.51)

Now we derive a similar result for “partial” Sobolev norms $\|\cdot\|_m'$ defined in Section 4.4 (5). The argument is actually simpler since it only involves differentiability along stable and unstable direction and no loss of regularity appears.

**Corollary 5.2.** For any $z \in B_\eta(a) \cup B_\eta(b)$ with $\eta$ sufficiently small and for any $L^m_0$ map $\mathcal{F} : M \to \mathfrak{N}$, $m \in \mathbb{N}$ with $m > \frac{\dim M}{2} + 3r$ if

$$\sum_{j=-\infty}^{+\infty} \text{Ad}_z^{-(j+1)}(\mathcal{F} \circ z^j) = 0$$

(5.49)

as a distribution, then the equation

$$\Omega \circ z - \text{Ad}_z \Omega = \mathcal{F}$$

(5.50)

has a solution $\Omega \in L^m_0$ such that

$$\|\Omega\|_{m'} \leq C_{m,\eta}\|\Theta\|_m'.$$

(5.51)

Furthermore if $z \in Z(L)$ and $\mathcal{F} \in L^m_{0,L}$, then $\Omega \in L^m_{0,L}$.
Proof. If \( \frac{n}{r} - 1 - \frac{\dim M}{2} - 1 > 1 \) then \( F \in C^1(\mathcal{M}) \). Using the same method as in Lemma 5.1 we show the two formal solutions \( \Omega^{(+)} \) for any coordinate of \( F \) in (5.5) are distributions. The differentiability of \( \Omega \) along all stable and unstable foliations and the estimates follows from (5.7).

Using the same method as in Lemma 5.2 we show the two formal solutions \( \Lambda^{(+)} \) in (5.37) are also distributions. The estimates of \( \Lambda \) follows similarly from (5.41) and (5.47) just by substituting norm \( \| \cdot \| \) with \( \| \cdot \|' \) and by noticing that there is no loss of regularity by above discussion in the first part. Finally the invariance of \( \Omega \) under \( L \) is the same as (5.48).

\( \square \)

Similarly to Corollary 5.1 we also have the following “uniqueness” property:

**Corollary 5.3.** If the equation

\[
\Omega \circ z - \text{Ad}_z \Omega = F
\]

has a solution in \( C^1(\mathcal{M}) \cap \mathcal{H}_0^0 \) then it is unique.

**Proof for symmetric space examples and hyperbolic twisted examples.** For both these cases \( \text{Ad}_z \) has diagonal form in \( \mathfrak{h}_C \) and then the equation (5.52) splits into finitely many equations of the form

\[
\omega \circ z - \lambda \omega = \theta
\]

where \( \lambda \) is an eigenvalue of \( \text{Ad}_z \) and \( \theta \in C^1(\mathcal{M}) \cap \mathcal{H}_0^0 \) is the coordinate function of \( F \) under new basis. By Corollary 5.1 the solution of the above equation is unique hence the solution of (5.54) is unique. \( \square \)

**Proof for genuinely partially hyperbolic twisted examples.** We assume notations in Lemma 5.2 if there is no confusion. If \( \Omega' \in C^1(\mathcal{M}) \cap \mathcal{H}_0^0 \) is also a solution of equation (5.52), then

\[
J_1^{-j}(\Omega - \Omega') \circ z^j = \Omega - \Omega'
\]

for any \( j \in \mathbb{Z} \).

For \( 1 \leq i \leq \dim(\mathfrak{a}) \) or \( \dim(\mathfrak{a}) + \dim v + 1 \leq i \leq t_0 \), the trivialization of the \( i \)-th equation in (5.54) follows the same ways as in the previous part.

For \( n_0 \geq i \geq \dim(\mathfrak{a}) + 1 \) the left side of \( i \)-th equation in (5.54) has the same form either as (5.43) or as (5.44) depending on \( j \) by substituting \( \theta_i \) with \( \theta_i \in C^1(\mathcal{M}) \cap \mathcal{H}_0^0 \) are coordinate functions of \( \Omega - \Omega' \) under new basis. Using Corollary 5.1 as in (5.26) we can show without difficulty that both (5.43) and (5.44) are 0 distributions as \( j \to \infty \). By above argument \( \theta_i = 0 \) for \( n_0 \geq i \geq \dim(\mathfrak{a}) + 1 \). Hence \( \Omega = \Omega' \).

\( \square \)

5.2. Higher rank trick and trivialization of cohomology. Now we will show that in the higher rank case obstructions to the solution of the linearized conjugacy equation vanish. The reason for that is the linearized form of the commutation relation (4.16) \( L(\mathcal{F}, \mathcal{G})^{(a_1, a_2)} = 0 \) that mean that the pair \( \mathcal{F}, \mathcal{G} \) form a twisted cocycle over the homogeneous action generated by \( a_1 \) and \( a_2 \). Joint solvability of the linearized conjugacy equations for
commuting elements means that this cocycles is a coboundary, hence corresponding twisted first cohomology is trivial. The proof consists of three parts:

1. Reduction of the vector values linearized conjugacy equation (4.14) and linearized commutativity condition 4.16 to scalar equations. This is straightforward for the symmetric space and Anosov twisted symmetric space cases but for genuinely partially hyperbolic twisted symmetric space examples requires certain algebraic manipulations somewhat similar to those that appear in the case of commuting toral automorphisms in the presence of Jordan blocks, see [5, Section 3.2].

2. The “higher rank trick” that proves vanishing of the obstructions (5.2). It appears in virtually identical form in all proofs of cocycle and differentiable rigidity for actions of higher rank abelian groups that use some form of dual, i.e. harmonic analysis arguments. For its earliest appearance see Lemmas 4.3, 4.6 and 4.7 in [17].

3. Application of Lemma 5.1 in the case of Sobolev norms and Corollary 5.2 for partial Sobolev norms $\|\cdot\|_{m}^\prime$.

**Lemma 5.3.** Let $\eta$ be sufficiently small and $m \in \mathbb{N}$ with $m \geq m_1 + r + 1$ where $m_1$ is defined as in Lemma 5.1. For any $a_1 \in B_\eta(a)$ and $a_2 \in B_\eta(b)$ where $a_1, a_2$ commute and any two $\mathcal{H}_0^m$ maps $F, G : M \rightarrow \mathfrak{N}$ satisfying $L(F, G)^{(a_1, a_2)} = 0$, or, equivalently, $\mathcal{T}_{a_2} F = \mathcal{T}_{a_1} G$ then the equations

\[
\begin{align*}
\Omega \circ a_1 - \text{Ad}_{a_1} \Omega &= F \\
\Omega \circ a_2 - \text{Ad}_{a_2} \Omega &= G
\end{align*}
\]

or equivalently

\[
F = \mathcal{T}_{a_1} \Omega, \quad G = \mathcal{T}_{a_2} \Omega
\]

have a common solution $\Omega \in \mathcal{H}_0^{m-2r-2}$ with the following estimate

\[
\|\Omega\|_{m-2r-2} \leq C_{m, \eta} \|F, G\|_m.
\]

Furthermore if $a_1, a_2 \in Z(L)$ and $F, G \in \mathcal{H}_{0,L}^m$, then $\Omega \in \mathcal{H}_{0,L}^{m-2r-2}$.

**Proof for symmetric space examples.** We consider symmetric space examples at first. Let $a_i = d_i k_i$, $i = 1, 2$ where $d_i$ is the split part and $k_i$ is compact part. Consider the complexification $\mathfrak{N}_C$ of the subalgebra $\mathfrak{N} = \mathfrak{S} + \mathfrak{D}$. There exists an orthonormal basis that diagonalizes $\text{Ad}_{a_1}$ and $\text{Ad}_{a_2}$. As usual, this basis may be chosen to consists of several real vectors and several pairs of complex conjugate vectors. The equations $L(F, G)^{(a_1, a_2)} = 0$ and (5.55) split into finitely many equations of the form

\[
L(\varphi, \vartheta)^{(a_1, a_2)}_{(\lambda_1, \lambda_2)} = (a_1, \lambda_1)^\tau \varphi - (a_2, \lambda_2)^\tau \vartheta = 0
\]

1 Somewhat similar arguments already appeared in the proof of Lemma 5.2.
and
\[ \omega \circ a_1 - \lambda_1 \omega = \vartheta \]
\[ \omega \circ a_2 - \lambda_2 \omega = \varphi \]  
(5.58)

where \( \vartheta, \varphi \) are \( H^0 \) functions and \( \lambda_1 \) and \( \lambda_2 \) in \( S_1 \) are corresponding eigenvalues of \( \text{Ad}_{a_1} \) and \( \text{Ad}_{a_2} \), respectively. Notice that since the coefficients and right-hand parts of the equations (5.55) are real-valued, the unique solution in \( H^0 \) (see Corollary 5.1) is real-valued as well.

By the assumption (5.57) we get
\[ \sum_{j=-n}^{n} \lambda_1^{-j} \vartheta(a_2 a_1^j) = \lambda_1^{-n} \vartheta(a_2 a_1^{-n}) = \lambda_1^{-n} \varphi(a_2 a_1^{-n}). \]

By Corollary 3.1 the right-hand converges to 0 as a distribution when \( n \to \infty \). Hence using notation from Section 4.4 (9)
\[ \sum_{j=0}^{n} \lambda_2^{-j} \vartheta(a_2) = \sum_{j=0}^{n} \vartheta \]  
(5.59)

as distributions.

Let \( \phi \in \Phi \). For any \( j, n \in \mathbb{Z} \) write
\[ \sum_{\phi \in \Phi} |\phi(j \log d_1 + n \log d_2)| = (|j| + |n|) \sum_{\phi \in \Phi} |\phi(j \log d_1 + j_2 \log d_2)| \]

where \( j_1 = \frac{j}{|j| + |n|} \) and \( j_2 = \frac{n}{|j| + |n|} \). If \( \eta \) is small enough then \( a_1 \) and \( a_2 \) are also linearly independent elements. Hence
\[ c_0 = \min_{(r_1, r_2) \in \mathbb{R}^2} \frac{1}{2} (\sum_{\phi \in \Phi} |\phi(r_1 \log d_1 + r_2 \log d_2)|) > 0. \]

For any \( f \in C^\infty(G/\Gamma) \) by Corollary 3.1 there exist constants \( \gamma, E > 0 \) only dependent on \( G \) such that
\[ \sum_{n=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\langle \lambda_1^{-(j+1)} \lambda_2^{-(n+1)} \vartheta(a_2 a_1^j), f \rangle| \leq \sum_{n=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\vartheta(a_2 a_1^j), f \rangle| \leq E \| \vartheta \|_{C^1} \| f \|_{C^1} e^{-\gamma c_0 (|n| + |j|)} < \infty. \]

Hence the sum \( \sum_{n=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\langle \lambda_1^{-(j+1)} \lambda_2^{-(n+1)} \vartheta(a_2 a_1^j), f \rangle| \) converges absolutely and thus \( \sum_{(a_2, \lambda_2)} \sum_{(a_1, \lambda_1)} \vartheta \) is a distribution. On the other hand
by iterating equation \( (5.59) \) we obtain:

\[
\sum_{\lambda} \lambda^2 \vartheta(a_k^2) = \sum_{\lambda} \vartheta
\]

as distributions for any \( j \in \mathbb{Z} \). Therefore

\[
\sum_{\lambda}^{(a_2, \lambda_2)} \sum_{\lambda}^{(a_1, \lambda_1)} \vartheta = \sum_{k} \lambda_2^{-(k+1)} \vartheta(a_k^2) = \sum_{k} \lambda_2^{-(k+1)} \sum_{\lambda} \vartheta
\]

The series in the left hand side of \( (5.60) \) is not a distribution unless \( \sum_{\lambda}^{(a_1, \lambda_1)} \vartheta \) is a 0 distribution. Similarly \( \sum_{\lambda}^{(a_2, \lambda_2)} \varphi \) is also a 0 distribution.

This is the “higher rank trick”!

By Lemma 5.1 each equation of \( (5.58) \) has a \( \mathcal{H}^{m-r-1} \) solution. Moreover, \( \omega \) solves the first equation, i.e. \( (a_1, \lambda_1)^T \omega = \vartheta \) then by \( (5.57) \) we have

\[
(a_2, \lambda_2)^T(a_1, \lambda_1)^T \omega = (a_2, \lambda_2)^T \vartheta = (a_1, \lambda_1)^T \varphi
\]

Since operators \( (a_2, \lambda_2)^T \) and \( (a_2, \lambda_2)^T \) commute this implies

\[
(a_1, \lambda_1)^T((a_2, \lambda_2)^T \omega - \varphi) = 0
\]

By Corollary 5.1 \( (a_1, \lambda_1)^T \) is an injective operator if \( \omega, \varphi \in C^1(M) \), that is \( m - r - 1 > \dim \mathcal{M} + 2 \). Therefore \( (a_2, \lambda_2)^T \omega - \varphi = 0 \) i.e. \( \omega \) solves the second equation as well.

**Proof for twisted symmetric space examples.** We assume notations from the previous part if there is no confusion.

1. If \( \rho \) is Anosov then the neutral distribution is still \( \mathcal{D} + \mathcal{R} \). Let \( a_i = d_i k_i \), \( i = 1, 2 \) where \( d_i \) is the split part and \( k_i \) is compact part. Notice for any \( (g, t) \in G \times \mathbb{R}^N \), \( a_i \cdot (g, t) = (d_i k_i g, t) \), \( i = 1, 2 \) then the statement follows essentially verbatim as in the case of symmetric space examples.

2. If \( \rho \) is genuinely partially hyperbolic, then the neutral distribution is \( \mathcal{R} = \mathcal{D} + \mathcal{R} + \mathcal{V}^0 \). Let \( a_i = (d_i k_i, t_i) \), \( i = 1, 2 \) where \( d_i \) is the split part, \( k_i \) is compact part and \( t_i \in \mathcal{V}^0 \). We repeat with appropriate modifications construction from the proof of Lemma 5.2. Consider as before the complexification \( \mathcal{M}_C \) of \( \mathcal{M} \). By \( (5.27) \) and \( (5.28) \) there exists an orthonormal basis in \( \mathcal{M}_C \) such that \( \text{Ad}_{a_i}, i = 1, 2 \) has the following form in this basis

\[
J_i = \begin{pmatrix}
A_i & 0 & 0 \\
B_i & D_i & 0 \\
0 & 0 & E_i
\end{pmatrix}
\]

where \( E_i = I_{\dim(\mathcal{D})} \), \( A_i \) are \( \dim \mathcal{R} \times \dim \mathcal{R} \) diagonal matrices and \( D_i \) are \( \dim \mathcal{V}^0 \times \dim \mathcal{V}^0 \) diagonal matrices all of whose eigenvalues are of absolute value 1 and every element of \( B_i \) with absolute value smaller than 1 if \( \eta \) is small enough. This basis can be chosen as in the symmetric space case to include real vectors and pairs of complex conjugate vectors.
Then equations (5.55) have the form:

\[
\Omega \circ a_1 - J_1\Omega = \Theta \\
\Omega \circ a_2 - J_2\Omega = \Psi
\]

(5.61)

and the condition \(L(\Psi, \Theta)^{(a_1,a_2)} = 0\) can be written as

\[
J_2\Theta - \Theta \circ a_2 = J_1\Psi - \Psi \circ a_1.
\]

(5.62)

Denote \(J_1 = (q_1^{ij})\) and \(J_2 = (q_2^{ij})\) and let \(q_1^{ij} = \lambda_i\) and \(q_2^{ij} = \mu_i\) for \(1 \leq i, j \leq t\), where \(t_0 = \dim \mathcal{O} + \dim \mathcal{v}^0 + \dim \mathcal{r}\). Since \(\text{Ad}_{a_1}\) and \(\text{Ad}_{a_2}\) commute, by comparing coefficients, one obtains the following relation between the coefficients of \(\text{Ad}_{a_1}\) and \(\text{Ad}_{a_2}\):

\[
\lambda_i q_2^{ij} + \mu_j q_1^{ij} = \mu_i q_1^{ij} + \lambda_j q_2^{ij}
\]

(5.63)

for \(1 \leq j \leq \dim(\mathcal{r}),\) \(1 \leq i \leq \dim(\mathcal{r}) + \dim(\mathcal{v}^0)\).

Let the coordinate functions of \(\Theta\) and \(\Psi\) be \(\vartheta_i\) and \(\varphi_i\), \(1 \leq i \leq t\), respectively. For \(1 \leq i \leq \dim(\mathcal{r})\) or \(\dim(\mathcal{r}) + \dim(\mathcal{v}^0) + 1 \leq i \leq t\), the \(i\)-th pair of equations in (5.61) is:

\[
\omega_i \circ a_1 - \lambda_i \omega_i = \vartheta_i \\
\omega_i \circ a_2 - \mu_i \omega_i = \varphi_i
\]

(5.64)

and the condition \(L(\Psi, \Theta)^{(a_1,a_2)} = 0\) splits as

\[
L(\varphi_i, \vartheta_i)^{(a_1,a_2)} = (a_1, \lambda_i)^\tau \varphi_i - (a_2, \mu_i)^\tau \vartheta_i = 0
\]

(5.65)

If we can show that \(\sum_{(a_2,\mu_i)}^{(a_2,\mu_i)} \sum_{(a_1,\lambda_i)}^{(a_1,\lambda_i)} \vartheta_i\) is a distribution then \(\sum_{(a_1,\lambda_i)}^{(a_1,\lambda_i)} \varphi_i = 0\) and hence the existence of a common solution follows the same way as in the previous part.

As before for integers \(j, n\) we define \(j_1 = \frac{j}{|j|+|n|}\) and \(j_2 = \frac{n}{|j|+|n|}\). If \(\eta\) is small enough then \(a_1\) and \(a_2\) are also linearly independent elements hence

\[
c_0 = \min_{(r_1, r_2) \in \mathbb{R}^2, |r_1|+|r_2|=1} \frac{1}{2} \left( \sum_{\phi \in \Phi} |\phi(r_1 \log d_1 + r_2 \log d_2)| \right) > 0.
\]

For any \(f \in C^\infty(M)\), let \(c_i^j = d_i^j k_i^j, t_{n,j} = \sum_{l=0}^{n-1} \rho(k_1)^{-j} \rho(k_2)^{-lt_2} + \sum_{l=0}^{j-1} \rho(k_1)^{-lt_1}, i = 1, 2\) for any \(n, j \in \mathbb{Z}\), by Corollary \[5.1\] there exist constants \(\gamma, E > 0\)
only dependent on $G$ satisfying
\[
\sum_{n=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\langle \mu_i^{-(n+1)} \lambda_i^{(j+1)} \partial_i (a_2^n a_1^j), f \rangle|
\leq \sum_{n=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\langle \partial_i (a_2^n a_1^j), f \rangle|
\leq \sum_{n=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\langle \partial_i ((c_2^n e_1^j, 0)x), f((0, -t_{n,j})x) \rangle|
\leq \sum_{n=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} E \| \partial \|_{C^1} \| f \|_{C^1} (\| t_1 \| + \| t_2 \|)(|j| + |n|)e^{-\gamma c_0 (|n| + |j|)} < \infty
\]
then the sum $\sum_{n=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\langle \mu_i^{-(n+1)} \lambda_i^{(j+1)} \partial (a_2^n a_1^j), f \rangle|$ converges absolutely. Hence $\sum_{(a_2, \mu_i)} \sum_{(a_1, \lambda_i)} \partial_i$ is a distribution.

Therefore, using (5.65) the same way as in the previous part, we deduce that there exist $\omega_i \in H^{m-r-1}_0$ which solve simultaneously the equations in (5.64). Moreover, for $1 \leq i \leq \dim(\mathcal{R})$ or $\dim(\mathcal{R}) + \dim \mathcal{V} + 1 \leq i \leq t_0$, the estimate:
\[
(5.66) \quad \| \omega_i \|_{m-r-1} \leq C_{m,\eta} \| \partial_i, \varphi_i \|_m \leq C_{m,\eta} \| \Theta, \Phi \|_m
\]
follows from Lemma 5.1.

For $\dim(\mathcal{R}) + \dim \mathcal{V} \geq i \geq \dim(\mathcal{R}) + 1$ the $i$-th pair of equations in (5.61) is:
\[
\omega_i \circ a_1 - \lambda_i \omega_i - \sum_{j=1}^{k_0} q_1^{ij} \omega_j = \vartheta_i
\]
\[
(5.67) \quad \omega_i \circ a_2 - \mu_i \omega_i - \sum_{j=1}^{k_0} q_2^{ij} \omega_j = \varphi_i
\]
where $n_0 = \dim(\mathcal{R}) + \dim \mathcal{V}$ and $k_0 = \dim(\mathcal{R})$ and the assumption (5.62) for $\omega_i$ and $\varphi_i$ splits as:
\[
(5.68) \quad \mu_i \vartheta_i + \sum_{j=1}^{k_0} q_2^{ij} \vartheta_j - \vartheta_i \circ a_2 = \lambda_i \varphi_i + \sum_{j=1}^{k_0} q_1^{ij} \varphi_j - \varphi_i \circ a_1
\]
Now we use the fact that all the subsequent pairs of equations are solved i.e. we substitute all $\vartheta_j$ and $\varphi_j$ for all $1 \leq j \leq k_0$ into (5.68) using their expression as in (5.64). This implies:
\[
\sum_{j=1}^{k_0} q_2^{ij} (a_1, \lambda_j)^{\tau} \omega_j - (a_2, \mu_i)^{\tau} \vartheta_i = \sum_{j=1}^{k_0} q_1^{ij} (a_2, \mu_j)^{\tau} \omega_j - (a_1, \lambda_i)^{\tau} \varphi_i
\]
Since \( \text{Ad}_{a_1} \) and \( \text{Ad}_{a_2} \) commute, we can use the equation (5.63) for the coefficients and the linearity of operators \((a_2, \mu_j)^{\tau}\) and \((a_1, \lambda_j)^{\tau}\), to simplify the above expression to:

\[
(a_2, \mu_i)^{\tau}(\vartheta_i + \sum_{j=1}^{k_0} q_{1}^{ij} \omega_j) = (a_1, \lambda_i)^{\tau}(\varphi_i + \sum_{j=1}^{k_0} q_{2}^{ij} \omega_j)
\]

Thus the functions \( \vartheta_i + \sum_{j=1}^{k_0} q_{1}^{ij} \omega_j \) and \( \varphi_i + \sum_{j=1}^{k_0} q_{2}^{ij} \omega_j \) satisfy the solvability condition

\[
L(\vartheta_i + \sum_{j=1}^{k_0} q_{1}^{ij} \omega_j, \varphi_i + \sum_{j=1}^{k_0} q_{2}^{ij} \omega_j) = 0 \quad (5.69)
\]

they are in \( \mathcal{H}^{m-r-1}_0 \) by (5.66) therefore we may use previous part again to conclude that the pair of equations (5.67) has a common solution \( \omega \in \mathcal{H}^{m-2r-2}_0 \). As a consequence of assumptions (5.66) this solution satisfies the estimate

\[
\| \omega_i \|_{m-2r-2} \leq C_{m,\eta}\|\vartheta_i, \varphi_i\|_{m} \leq C_{m,\eta}\|\Theta, \Phi\|_{m} 
\]

for \( n_0 \geq i \geq \dim(\mathcal{K}) + 1 \) if \( m - r - 1 \geq m_1 \).

We obtain the following estimate for the norm of the \( \mathcal{H}^{m-2r-2}_0 \) solution \( \Omega \) of the system (5.55):

\[
\| \Omega \|_{m-2r-2} \leq C_{m,\eta}\|\mathcal{F}, \mathcal{G}\|_{m}. 
\]

Both for the symmetric space examples and hyperbolic twisted symmetric space examples it is obvious that

\[
\Omega = \Lambda (_{-}^{+}) = \left( ^{+}_{-} \right) \sum_{j \geq 0 \atop j \leq -1} J^{-(j+1)}_i \Theta \circ a^{\pm}_1. 
\]

For the genuinely partially hyperbolic twisted symmetric space examples, similar to the former cases for the diagonal blocks \( A \) and \( E \) of \( \text{Ad}_{a_1} \) and \( \text{Ad}_{a_2} \), that is for \( 1 \leq i \leq \dim(\mathcal{K}) \) or \( \dim(\mathcal{K}) + \dim \mathcal{V} + 1 \leq i \leq t_0 \), the \( i \)-th coordinate \( \Omega_i \) of \( \Omega \) is the \( i \)-th coordinate of \( \Lambda (_{-}^{+}) \).

For \( n_0 \geq i \geq \dim(\mathcal{K}) + 1 \), by (5.69)

\[
\sum_{j=1}^{k_0} (\vartheta_i + \sum_{j=1}^{k_0} q_{1}^{ij} \omega_j) = 0 \quad \text{as a distribution}
\]

combined with (5.45) the \( i \)-th coordinate of \( \Lambda_- - \Lambda_+ \) is also 0. Then we get \( \Lambda_- = \Lambda_+ \). By Lemma 5.2 and Corollary 5.3 \( \Omega = \Lambda_- = \Lambda_+ \). Invariance of \( \Omega \) under \( \text{Ad}_L \) follows the same way as in (5.48).  

Now we consider the case of partial Sobolev norms \( \| \cdot \|_m' \).
Corollary 5.4. Let $\eta$ be sufficiently small and $m \in \mathbb{N}$ with $m > \frac{r \dim M + 3r}{r}$. For any $a_1 \in B_{\eta}(a)$ and $a_2 \in B_{\eta}(b)$ where $a_1, a_2$ commute and for any two $F, G : M \to \mathfrak{M}$ satisfying $L(F, G)_{(a_2, a_1)} = 0$, the equations
\[
\begin{align*}
\Omega \circ a_1 - \text{Ad}_{a_1} \Omega &= F \\
\Omega \circ a_2 - \text{Ad}_{a_2} \Omega &= G
\end{align*}
\] or, using our compact notation, $\mathcal{T}_{a_1} \Omega = F$, $\mathcal{T}_{a_2} \Omega = G$ have a common solution $\omega \in L^m_0$ with the following estimate
\[
\|\Omega\|_m' \leq C_{m, \eta} \|F, G\|_m'.
\]
Furthermore if $a_1, a_2 \in Z(L)$ and $F, G \in L^m_{0,L}$, then $\Omega \in L^m_{0,L}$.

Proof. If $m - 1 - \frac{\dim M}{2} - 1 > 1$ then $F, G \in C^1(M)$. Using the same method as in Lemma 5.3 we show that both $\sum_{j=-\infty}^{+\infty} \text{Ad}_{a_1}^{(j+1)} (F \circ a_1^j)$ and $\sum_{j=-\infty}^{+\infty} \text{Ad}_{a_2}^{(j+1)} (G \circ a_2^j)$ are 0 distributions. Then by Corollary 5.2 each equation of (5.73) has a $L^m_0$ solution. Moreover, they coincide. If $\Omega$ solves the first equation, i.e. $\mathcal{T}_{a_1} \Omega = F$, then by assumption $L(F, G)_{(a_1, a_2)} = 0$ we have $\mathcal{T}_{a_2} F = \mathcal{T}_{a_1} G$ and thus
\[
\mathcal{T}_{a_2} \mathcal{T}_{a_1} \Omega = \mathcal{T}_{a_1} \Omega = F = \mathcal{T}_{a_1} G.
\]
Since operators $\mathcal{T}_{a_1}$ and $\mathcal{T}_{a_2}$ commute this implies
\[
\mathcal{T}_{a_1} (\mathcal{T}_{a_2} \Omega - G) = 0
\]
By Corollary 5.3 $\mathcal{T}_{a_1}$ is an injective operator if $\Omega, G \in C^1(M)$ which is satisfied by our assumption. Therefore $\mathcal{T}_{a_2} \Omega - G = 0$ i.e. $\Omega$ solves $\Omega \circ a_2 - \text{Ad}_{a_2} \Omega = G$ as well. Then estimate (5.74) follows by Corollary 5.2. 

\[\square\]

6. Approximate solution of linearized equation

6.1. The splitting problem. As we mentioned in Section 2.2 conjugacy problem cannot be reduced to a cohomology problem for the unperturbed action. In other words, perturbations do not satisfy cocycle equations exactly. However they satisfy those equations approximately. The method of proof of our main theorems is based on the iteration procedure. At each step we have an almost cocycle and show that it is an almost coboundary. To achieve that we need to show that an almost cocycle $F, G$ can be split into a real cocycle and an error term that can be estimated tamely through the values of the coboundary operator $L(F, G)$.

From the general functional analysis point of view the problem does not look very hopeful. We have a bounded operator with infinite-dimensional co-kernel and without a spectral gap. What may help of course is that we are content with a finite loss of regularity but still in general tame splitting is not likely. So one needs to use special features of the operators to construct desired splittings.
The first thing that comes to mind is to use the fact that our action is a part of the actions of the whole group $G$ (we ignore additional factorization for the sake of this discussion). One can split the unitary representation of $G$ in $L^2(M)$ into irreducibles that are orthogonal not only with respect to the $L^2$ norm itself but also with respect to Sobolev norms, and try to construct splitting in each irreducible representation space. A similar approach works for the actions of $Z^k$ by automorphisms of the torus [5]. In the semisimple case it has been successfully applied to the unipotent action on homogeneous spaces of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ in [7]. There are some other cases where this approach may (and should) work, such as actions by automorphisms of nilmanifolds, partially hyperbolic actions on factors of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, or unipotent actions on homogeneous spaces of $SL(2,\mathbb{C})$. In all these cases one uses specific explicit constructions of the splittings that do not depend on regularity of functions involved that can be loosely described as “pushing the obstructions to the lowest possible level”. However, even for simple Lie groups of rank $\geq 2$ the structure of irreducible representations is too complicated to carry out similar specific constructions.

We solve the problem in a fairly general way by using the algebraic structure of the coboundary operators that allows to reduce the problem of splitting to orthogonal projections in sufficiently high Sobolev spaces. While algebra is transparent, the analysis part is involved and subtle. In order to carry out our method for a parametric family of operators one needs to work with operators in a fixed Hilbert space; in other words not to allow any loss of regularity. This is the reason we consider Sobolev spaces $L^m_{0,L}$ where derivatives are only considered in the hyperbolic directions and no loss of regularity appears in the solution of the coboundary equations, see Corollaries [5.2] and [5.4]. Splittings for those spaces are constructed in the next section.

However spaces $L^m_{0,L}$ play an auxiliary role: they cannot be used in the iteration scheme. We use this splitting in Section 6.3 to get tame estimates uniform in parameters for the splitting in the “real” Sobolev spaces $H^m_{0,L}$ (see (6.28) and (6.30) in the proof of Lemma 6.6). This is the only place in the main line of arguments where auxiliary norms $\|\cdot\|_m$ appears but it seems to be crucial.

6.2. Construction of splittings in $L^m_{0,L}$. We begin with a preparatory lemma about properties of coboundaries. The main part here is the last statement that asserts that on the intersection of images of two commuting coboundary operators the product of those operators can be inverted in the space $L^m_{0,L}$.

\textbf{Lemma 6.1.} If $m \in \mathbb{N}$ with $m > \frac{\ell \dim M}{2} + 3r$ and if $\eta$ is small enough the following properties hold for any $z \in B_\eta(a) \cup B_\eta(b)$ with $z \in Z(L)$:}

\footnote{This can also be used instead of specific constructions in the situations considered in [5, 7].}
(1) if \( f = T_z \Omega \) and \( \Omega \in \mathcal{L}^0_{0,L} \) then \( \sum_z f = 0 \) as a distribution.
(2) if \( T_z \Omega = 0 \) with \( \Omega \in \mathcal{L}^n_{0,L} \) then \( \Omega = 0 \).
(3) if \( T_z \Omega = f \in \mathcal{L}^m_{0,L} \) and \( \Omega \in C^1 \cap \mathcal{L}^0_{0,L} \) then \( \Omega \in \mathcal{L}^m_{0,L} \) and
\[
|\Omega(m)| \leq C(m,\eta) ||f||_m.
\]
(4) if \( z_1 \in B_3(0) \) and \( z_2 \in B_3(0) \) and \( z_1, z_2 \in Z(L) \) and \( f \in \mathcal{L}^m_{0,L} \) satisfying \( \sum z_1 f = 0 \) and \( \sum z_2 f = 0 \) as distributions then there exists \( \Omega \in \mathcal{L}^m_{0,L} \) satisfying \( T_{z_1} T_{z_2} \Omega = f \) and
\[
|\Omega(m)| \leq C(m,\eta) ||f||_m.
\]

**Proof.** We assume notations of Lemma 5.2 if there is no confusion.

(1) By assumption we get
\[
\sum_{j=-n}^{n} \text{Ad}_{z}^{-1}(j+1) f(z^j) = \text{Ad}_{z}^{-1}(k+1) \Omega(z^{k+1}) - \text{Ad}_{z}^{0} \Omega(z^{-n}).
\]

First consider symmetric space examples and hyperbolic twisted symmetric space examples. For both these cases \( \text{Ad}_{z} \) has diagonal form in \( \mathfrak{m}_C \) and then any coordinate function of \( (6.1) \) has the form
\[
\lambda^{-(k+1)} \vartheta(z^{k+1}) - \lambda^{n} \vartheta(z^{-n})
\]
where \( \lambda \) is an eigenvalue of \( \text{Ad}_{z} \) and \( \theta \in \mathcal{L}^0_{0,L} \).

By (5.6) and (5.26) \( \lambda^{-(k+1)} \vartheta(z^{k+1}) \) and \( \lambda^{n} \vartheta(z^{-n}) \) converge to 0 as distributions as \( k \to \infty \) and \( n \to \infty \) respectively, then \( \sum (z,\lambda) f \) is a 0 distribution.

Now consider genuinely partially hyperbolic twisted symmetric space examples.

For \( 1 \leq i \leq \dim(\mathfrak{r}) \) or \( \dim(\mathfrak{r}) + \dim \mathfrak{u}^0 + 1 \leq i \leq t_0 \), the \( i \)-th coordinate in \( (6.1) \) has the form:
\[
\lambda_i^{-(k+1)} \vartheta_i(z^{k+1}) - \lambda_i^{n} \vartheta_i(z^{-n})
\]
where \( \theta_i \in \mathcal{L}^m_{0,L} \). Arguing as above we see that expression \( (6.2) \) converges to a distribution as \( k \to \infty \) and \( n \to \infty \).

For \( n_0 \geq i \geq \dim(\mathfrak{r}) + 1 \) by (5.43) and (5.44) the \( i \)-th coordinate of \( \text{Ad}_{z}^{(k+1)} \Omega(z^{k+1}) \) and \( \text{Ad}_{z}^{n} \Omega(z^{-n}) \) in \( (6.1) \) are equal to
\[
-\lambda_i^{-(k+1)} \vartheta_i(z^{k+1}) + \sum_{j=1}^{k_0} \sum_{n=-1}^{-k-2} q_{ij}^{n} \lambda_i^{n-k-n-2} \vartheta_j(z^{k+1})
\]
and
\[
+\lambda_i^{n} \vartheta_i(z^{-n}) + \sum_{k=1}^{n-2} \sum_{j=0}^{k_0} q_{ik}^{j} \lambda_k^{j} \lambda_i^{n-1-j} \vartheta_k(z^{-n})
\]
where \( \vartheta_i \in \mathcal{L}^m_{0,L} \).

Using Corollary 3.1 as in (5.26) we see that those expressions converge to 0 distributions as \( k \to \infty \) and \( n \to \infty \) respectively.

Combining both these cases we deduce that \( \sum (z,\lambda) f \) is a 0 distribution.
(2) A direct consequence of Corollary 5.3 since if \( \Omega \in \mathcal{L}^m_{0,L} \), then \( \Omega \in C^1 \).

(3) By (1), we get \( \sum_z f = 0 \) as a distribution. Then by Corollary 5.2 there exist \( \Omega_1 \in \mathcal{L}^m_{0,L} \) satisfying: \( \mathcal{T}_z \Omega_1 = f \) such that \( \| \Omega_1 \|_m' \leq C_m \eta \| f \|_m' \).

By (2) \( \Omega \) and \( \Omega_1 \) coincide.

(4) By assumption \( \sum_z^1 f = 0 \) and \( \sum_z^2 f = 0 \) it follows from Corollary 5.4 that there exist \( \Omega_1, \Omega_2 \in \mathcal{L}^m_{0,L} \) satisfying: \( \mathcal{T}_z \Omega_1 = f \) and \( \mathcal{T}_z \Omega_2 = f \) with estimate \( \| \Omega_1, \Omega_2 \|_m' \leq C_m \eta \| f \|_m' \). Since \( \mathcal{T}_z \Omega_1 = \mathcal{T}_z \Omega_2, \) then by Corollary 5.4 there exists \( \Omega \in \mathcal{L}^m_{0,L} \) satisfying: \( \mathcal{T}_z \Omega = \Omega_1 \) and \( \mathcal{T}_z \Omega = \Omega_2 \) with estimate \( \| \Omega \|_m' \leq C_m \eta \| \Omega_1, \Omega_2 \|_m' \leq C_m \eta \| f \|_m' \) and \( \mathcal{T}_z \mathcal{T}_z \Omega = \mathcal{T}_z \Omega_1 = f \).

For any \( z \in B_\eta(a) \cup B_\eta(b) \) with \( z \in Z(L) \) let

\[ U_z^m = \{ f \in \mathcal{L}^m_{0,L} \mid \sum_z f \text{ is a 0 distribution} \} \]

By Lemma 6.1 if \( m > \frac{\dim M}{2} + 3r \), for \( \eta \) sufficiently small \( U_z^m \) is a closed subspace of \( \mathcal{L}^m_{0,L} \).

For any pair \( z_1, z_2 \) where \( z_1 \in B_\eta(a) \) and \( z_2 \in B_\eta(b) \) let the orthogonal complement of \( U_{z_1,z_2}^{(3,m)} = U_{z_1}^m \cap U_{z_2}^m \) in \( U_{z_1}^m \) be \( U_{z_1,z_2}^{(1,m)} \) and in \( U_{z_2}^m \) be \( U_{z_2}^{(2,m)} \). Then we have a decomposition

\[ U_{z_1}^m + U_{z_2}^m = U_{z_1,z_2}^{(1,m)} \bigoplus U_{z_1,z_2}^{(2,m)} \bigoplus U_{z_1,z_2}^{(3,m)} \]

For any \( f \in U_{z_1}^m + U_{z_2}^m \), write \( f = \sum_{i=1}^3 f_i \) where \( f_i \in U_{z_i}^{(i,m)} \). By the Open Mapping Theorem there exists \( C > 0 \) such that

\[ \| f \|_m' \leq \sum_{i=1}^3 \| f_i \|_m' \leq C_m \eta \| f \|_m' \]

where \( C_m \eta \) is dependent on \( m, z_1, z_2 \).

Now we are ready to construct the advertised splitting. It can be described as follows: given a pair of functions that represent an “almost cocycle”, i.e. \( \psi, \theta \in \mathcal{L}^m_{0,L} \), \( a_1, a_2 \in Z(L) \) and \( L(\psi, \theta)^{(a_1,a_2)} = \omega \), project the coboundary \( \mathcal{T}_a \theta \) orthogonally to the space \( U_{z_1}^m \cap U_{z_2}^m \) of joint coboundaries. By Lemma 6.1 (4) this projection is in the image of the product of the coboundary operators that can be inverted in the space \( \mathcal{L}^m_{0,L} \) producing a function \( \Omega \in \mathcal{L}^m_{0,L} \). Coboundaries \( \mathcal{T}_a \Omega \) and \( \mathcal{T}_{a_2} \Omega \) form a cocycle that approximates our pair \( \psi, \theta \) with an error of the order of \( \omega \).

**Lemma 6.2.** Let \( \eta \) be sufficiently small and \( m \in \mathbb{N} \) with \( m > \frac{\dim M}{2} + 3r \), \( a_1 \in B_\eta(a) \) and \( a_2 \in B_\eta(b) \) where \( a_1, a_2 \) commute and \( a_1, a_2 \in Z(L) \).

Suppose that \( \psi, \theta \in \mathcal{L}^m_{0,L} \) and \( L(\psi, \theta)^{(a_1,a_2)} = \omega \).

Then there exists \( \Omega \in \mathcal{L}^m_{0,L} \) such that

\[ \| \theta - \mathcal{T}_{a_1} \Omega \|_m' \leq C_m \eta \| \theta \|_m' \]
\[ \| \psi - \mathcal{T}_{a_2} \Omega \|_m' \leq C_m \eta \| \psi \|_m' \]
\[ \| \Omega \|_m \leq C_m \eta \| \theta, \psi \|_m \]
Proof. Let $p_t$ be the projection from $U_{a_1}^m + U_{a_2}^m$ to $U_{(a_1,a_2)}^{(i,m)}$, $1 \leq i \leq 3$. By (1) of Lemma 6.1, $T_{a_2} \theta \in U_{a_2}^m$ and $T_{a_1} \psi \in U_{a_1}^m$ then we write using the decomposition (6.3):

(6.8) \hspace{1cm} T_{a_2} \theta = p_2(T_{a_2} \theta) + p_3(T_{a_2} \theta),

(6.9) \hspace{1cm} T_{a_1} \psi = p_1(T_{a_1} \psi) + p_3(T_{a_1} \psi).

Since $p_3(T_{a_2} \theta) \in U_{a_1}^m \cap U_{a_2}^m$ and $p_3(T_{a_1} \psi) \in U_{a_1}^m \cap U_{a_2}^m$ then by (4) of Lemma 6.1 there exist $\Omega, \Omega' \in \mathcal{L}^m_{0,L}$ such that

(6.10) \hspace{1cm} T_{a_1} T_{a_2} \Omega = p_3(T_{a_2} \theta),

(6.11) \hspace{1cm} T_{a_1} T_{a_2} \Omega' = p_3(T_{a_1} \psi)

satisfying

$$\|\Omega\|_m' \leq C_{m,\eta}\|p_3(T_{a_2} \theta)\|_m' \leq C_{m,\eta}\|T_{a_2} \theta\|_m \leq C_{m,\eta}\|\theta\|_m.$$

Hence (6.7) holds. Now we are going to show that $\Omega$ satisfies (6.5) and (6.6). (Naturally, by symmetry all three inequalities will also hold for $\Omega'$).

Substituting (6.10) into (6.8) we get

(6.12) \hspace{1cm} T_{a_2} (\theta - T_{a_2} \Omega) = p_2(T_{a_2} \theta).

Since $L(\psi, \theta)_{(a_1,a_2)} = T_{a_1} \psi - T_{a_2} \theta = \omega$ and $T_{a_1} \psi \in U_{(a_1,a_2)}^{(1,m)} \oplus U_{(a_1,a_2)}^{(3,m)}$, we have $-p_2(T_{a_2} \theta) = p_2 \omega$. Then by (6.4)

(6.13) \hspace{1cm} \|p_2(T_{a_2} \theta)\|_m' = \|p_2 \omega\|_m' \leq C_{m,a_1,a_2} \|\omega\|_m'.

Combining (6.12) and (6.13) and using (3) of Lemma 6.1 we have

$$\|\theta - T_{a_2} \Omega\|_m' \leq C_{m,\eta}\|p_2(T_{a_2} \theta)\|_m' \leq C_{m,\eta}C_{m,a_1,a_2} \|\omega\|_m'.$$

This gives (6.5).

Substituting (6.10) and (6.11) into $p_3(T_{a_1} \psi) - p_3(T_{a_2} \theta) = p_3 \omega$ gives

(6.14) \hspace{1cm} T_{a_1} T_{a_2} \Omega' - T_{a_1} T_{a_2} \Omega_2 = p_3 \omega.

Since $T_{a_2} \theta \in U_{(a_1,a_2)}^{(2,m)} \oplus U_{(a_1,a_2)}^{(3,m)}$, $p_1(T_{a_1} \psi) = p_1 \omega$. Then by (6.4) we have

(6.15) \hspace{1cm} \|p_1(T_{a_1} \psi)\|_m' = \|p_1 \omega\|_m' \leq C_{m,a_1,a_2} \|\omega\|_m'.

Substituting (6.11) into (6.9) gives

(6.16) \hspace{1cm} T_{a_1} (\psi - T_{a_2} \Omega') = p_1(T_{a_1} \psi).

Combining (6.14) and (6.16) gives

$$T_{a_1} (\psi - T_{a_2} \Omega) = T_{a_1} (\psi - T_{a_2} \Omega') + (T_{a_1} T_{a_2} \Omega' - T_{a_1} T_{a_2} \Omega) = p_1(T_{a_1} \psi) + p_3 \omega.$$
then use (3) of Lemma 6.1 again and combine with (6.15):
\[
\| \psi - \mathcal{T}_{a_2} \Omega \|_m' \\
\leq C_{m,\eta} \| p_1(\mathcal{T}_{a_1} \psi) + p_3 \omega \|_m' \\
\leq C_{m,\eta} \| p_1(\mathcal{T}_{a_1} \psi) \|_m' + C_{m,\eta} \| p_3 \omega \|_m' \\
\leq C_{m,\eta} C_{m,a_1,a_2} \| \omega \|_m + C_{m,\eta} \| \omega \|_m',
\]
i.e. (6.6) also holds. □

Next we will show there exists a upper bound for \( C_{m,a_1,a_2} \) in \( B_\delta(a) \cup B_\delta(b) \) for sufficiently small \( \eta \) if \( a_1 \) and \( a_2 \) commute and \( a_1, a_2 \in Z(L) \). The proof uses a fairly straightforward compactness argument that works because we operate in a fixed Hilbert space.

**Lemma 6.3.** In the notation of the previous lemma the constant \( C_{m,a_1,a_2} \) can be chosen to depend only on \( m \) and \( \eta \).

**Proof.** Let \( U' = \{(a_1, a_2) \in B_\delta(a) \times B_\delta(b) | a_1 \text{ and } a_2 \text{ commute and } a_1, a_2 \in Z(L) \} \). Since \( U' \) is a compact set we just need to show: for any pair \( a_1, a_2 \in U' \) there exists \( \delta, C' > 0 \) such that for any pair \( z_1, z_2 \in U' \) satisfying \( z_1 \in \mathcal{B}_\delta(a_1) \) and \( z_2 \in \mathcal{B}_\delta(a_2) \), \( C_{m,z_1,z_2} < C' \).

If not, for any \( n \in \mathbb{N} \) there exist \( z_n, b_n \in \mathcal{B}_\frac{1}{n}(a_1), b_n \in \mathcal{B}_\frac{1}{n}(a_2) \) where \( z_n, b_n \in U' \) and \( \theta_n \in U_{(z_n,b_n)}^{(1,n)} \), \( \psi_n \in U_{(z_n,b_n)}^{(2,n)} \) with \( \| \theta_n \|_m' + \| \psi_n \|_m' = 1 \) while \( \| \omega_n \|_m < \frac{1}{n} \) where \( \omega_n = \theta_n - \psi_n \). By assumption there exists a sequence \( c_n \to 0 \) as \( n \to \infty \) such that
\[
(6.17) \quad \| \mathcal{T}_{z_n} - \mathcal{T}_{a_1} \|_m' + \| \mathcal{T}_{b_n} - \mathcal{T}_{a_2} \|_m' \leq c_n.
\]
Since \( \theta_n \in U_{z_n}^m \) and \( \psi_n \in U_{b_n}^m \) by Corollary 5.2 there exists \( \Omega_n, \Omega'_n \in \mathcal{L}_{0,L}^m \) such that
\[
(6.18) \quad \mathcal{T}_{z_n} \Omega_n = \theta_n \quad \mathcal{T}_{b_n} \Omega'_n = \psi_n
\]
with estimate
\[
(6.19) \quad \| \Omega_n, \Omega'_n \|_m' \leq C_{m,\eta} \| \theta_n, \psi_n \|_m'.
\]
Let \( \mathcal{T}_{a_1} \Omega_n - \mathcal{T}_{a_2} \Omega'_n = \omega'_n \) then combining (6.17) and (6.18) we obtain
\[
\| \omega'_n \|_m' \leq \| \omega_n \|_m + \| \mathcal{T}_{z_n} \Omega_n - \mathcal{T}_{a_1} \Omega_n \|_m' + \| \mathcal{T}_{b_n} \Omega'_n - \mathcal{T}_{a_2} \Omega'_n \|_m'
\]
\[
\leq \frac{1}{n} + c_n \| \theta_n \|_m' + \| \psi_n \|_m' = \frac{1}{n} + c_n.
\]
By Lemma 5.2 there exists \( \phi_n \in \mathcal{L}_{0}^m \) satisfying
\[
\| \phi_n - \mathcal{T}_{a_2} \phi_n \|_m' \leq C_{m,a_1,a_2} \| \omega'_n \|_m'
\]
\[
\| \phi_n - \mathcal{T}_{a_1} \phi_n \|_m' \leq C_{m,a_1,a_2} \| \omega'_n \|_m'
\]
\[
(6.21) \quad \| \phi_n \|_m' \leq C_{m,\eta} \| \Omega_n, \Omega'_n \|_m'.
\]
Let $c = \max \{ \|T_{zn}\|, \forall z \in U' \}$ then by (6.17), (6.19), (6.20) and (6.21) we have
\[
\|T_{zn}(\Omega_n - T_{bn}\phi_n)\|_m' + \|T_{bn}(\Omega'_n - T_{zn}\phi_n)\|_m' \\
\leq c\|\Omega_n - T_{bn}\phi_n\|_m' + c\|\Omega'_n - T_{zn}\phi_n\|_m'
\]
\[
\leq c\|\Omega_n - T_{a2}\phi_n\|_m' + c\|\Omega'_n - T_{a2}\phi_n\|_m' + c\|\Omega'_n - T_{a1}\phi_n\|_m'
\]
\[
\leq 2cC_m,a_1,a_2\|\phi_n\|_m' + 2cc_n\|\phi_n\|_m'
\]
\[
\leq 2cC_m,a_1,a_2(\frac{1}{n} + c_n) + 2cc_nC_m,\eta.
\]
Hence
(6.22) \[\|T_{zn}(\Omega_n - T_{bn}\phi_n)\|_m' + \|T_{bn}(\Omega'_n - T_{zn}\phi_n)\|_m' \to 0\]
as $n \to \infty$.

On the other hand notice
\[T_{zn}T_{bn}\phi_n = T_{bn}T_{zn}\phi_n \in U_{(zn,bn)}^{(3, m)}\]
then by (6.18) for any $n \in \mathbb{N}$ we have
\[
\|T_{zn}(\Omega_n - T_{bn}\phi_n)\|_m' + \|T_{bn}(\Omega'_n - T_{zn}\phi_n)\|_m' = \|\theta_n - T_{zn}T_{bn}\phi_n\|_m' + \|\psi_n - T_{zn}T_{bn}\phi_n\|_m'
\]
\[
\geq \|\theta_n\|_m' + \|\psi_n\|_m'
\]
\[
= 1
\]
which contradicts (6.22).

6.3. Construction of splittings in $H_{0,L}^m$. The following lemma is proved exactly the same way as Lemma 6.1 using Lemma 5.2 instead of Corollary 5.2 and Lemma 5.3 instead of Corollary 5.3

Lemma 6.4. Let $m_1$ be defined as in Lemma 6.1. If $\eta$ is small enough the following properties hold for any $z \in B_{\eta}(a) \cup B_{\eta}(b)$ and $z \in Z(L)$:

1. if $f = T_z\Omega$ and $\Omega \in H^m_{0,L}$ then $\sum f = 0$ if $m \geq r \dim M + 3r$.
2. if $T_z\Omega = 0$ with $\Omega \in H^m_{0,L}$ then $\Omega = 0$ if $m \geq r \dim M + 3r$.
3. if $T_z\Omega = f \in H^m_{0,L}$ and $\Omega \in C^1 \cap H^0_{0,L}$ then $\Omega \in H^m_{0,L}$ satisfying $\sum f = 0$ and $\sum f = 0$ as distributions then there exists $\Omega \in H^m_{0,L}$ satisfying $T_{z_1}T_{z_2}\Omega = f$ and $\|\Omega\|_{m-2r-2} \leq C_{m,\eta}\|f\|_m$ if $m \geq m_1 + r + 1$.
4. if $z_1 \in B_{\eta}(a)$ and $z_2 \in B_{\eta}(b)$ and $z_1, z_2$ commute with $z_1, z_2 \in Z(L)$ and $f \in H^m_{0,L}$ satisfying $\sum f = 0$ and $\sum f = 0$ as distributions then there exists $\Omega \in H^m_{0,L}$ satisfying $T_{z_1}T_{z_2}\Omega = f$ and $\|\Omega\|_{m-2r-4} \leq C_{m,\eta}\|f\|_m$ if $m \geq m_1 + 2r + 2$.

Notice finite loss of regularity in statements (3) and (4). Let
\[V^m_z = \{ f \in H^m_{0,L} \sum f = 0 \text{ is a 0 distribution} \}.\]
Lemma 6.5. If \( m \geq m_1 + r + 1 \) then \( V^m_z \) is a closed subspace of \( \mathcal{H}^m_{0, L} \).

Proof. If \( f_n \in V^m_z \) and \( f_n \to f \) in \( \mathcal{H}^m_{0, L} \), by (3) of Lemma 6.4, there exist \( \Omega_n \in \mathcal{H}^m_{0, L} \) such that \( \mathcal{T}_z \Omega_n = f_n \) with estimates \( \| \Omega_n \|_{m-2r-2} \leq C_m, \eta \| f_n \|_m \).

Then there exists a subsequence \( \Omega_{k_n} \to \Omega \) for some \( \Omega \in \mathcal{H}^m_{0, L} \) by Rellich’s Lemma where \( \epsilon \) is sufficiently small. By continuity of the operator \( \mathcal{T}_z \) we get \( \mathcal{T}_z \Omega = f \). Then by (1) of Lemma 6.4 \( f \in V^m_z \).

Hence \( V^m_z \) is a closed subspace. \( \square \)

Now we proceed similarly to the previous section. Let \( a_1, a_2 \) be commuting elements in \( Z(L) \) and \( a_1 \in B_\eta(a), a_2 \in B_\eta(b) \). Denote the orthogonal complement of \( V^{(3, m)}_{(a_1, a_2)} = V^m_{a_1} \cap V^m_{a_2} \) in \( V^m_{a_1} \) by \( V^{(1, m)}_{(a_1, a_2)} \) and in \( V^m_{a_2} \) by \( V^{(2, m)}_{(a_1, a_2)} \).

Then we have a decomposition

\[ V^m_{a_1} + V^m_{a_2} = V^{(1, m)}_{(a_1, a_2)} \bigoplus V^{(2, m)}_{(a_1, a_2)} \bigoplus V^{(3, m)}_{(a_1, a_2)}. \]

For any \( f \in V^m_{a_1} + V^m_{a_2} \), write \( f = \sum_{i=1}^3 f_i \) where \( f_i \in V^{(i, m)}_{(a_1, a_2)} \).

Next lemma is the central part of the splitting argument. Its conclusion is similar to that Lemma 6.3 but since we deal with ordinary Sobolev norms estimates, they are weaker in three respects: (i) initial data have high regularity \( s > r(m+1) \) compared to the regularity \( m \) appearing in the estimates; (ii) there is fixed \( (2r+2) \) loss of regularity in the estimates; (iii) quality of approximation is not linear in the norm error but is estimated by a certain power of that norm.

Lemma 6.6. Let \( \eta \) be sufficiently small, \( a_1, a_2 \) as before, and \( m \in \mathbb{N} \) such that \( m \geq m_1 + 2r + 2, s > r(m+1) \), Suppose that \( \theta, \psi \in \mathcal{H}^m_{0, L} \), and \( L(\psi, \theta)^{(a_1, a_2)} = \omega \) with \( \| \omega \|_s = R_0 \leq 1 \). Then there exists \( \Omega \in \mathcal{H}^m_{0, L} \) such that

\[
\begin{align*}
(6.23) \quad & \| \theta - \mathcal{T}_{a_1} \Omega \|_{m-4r-4} \leq C_{m, \eta, s} \| \omega \|_m^{\frac{s-2m+1-m-2r}{s-2m+1-m-2r}}, \\
(6.24) \quad & \| \psi - \mathcal{T}_{a_2} \Omega \|_{m-4r-4} \leq C_{m, \eta, s} \| \omega \|_m^{\frac{s-2m+1-m-2r}{s-2m+1-m-2r}}, \\
(6.25) \quad & \| \Omega \|_{m-4r-4} \leq C_{m, \eta} \| \theta, \psi \|_m.
\end{align*}
\]

Proof. The structure of the proof is similar to that of Lemma 6.2.

Let \( \mathcal{P}_i, 1 \leq i \leq 3 \) be the orthogonal projection from \( V^m_{a_1} + V^m_{a_2} \) to \( V^{(i, m)}_{(a_1, a_2)} \). By (1) of Lemma 6.4 \( \mathcal{T}_{a_2} \theta \in V^m_{a_2} \) and \( \mathcal{T}_{a_1} \psi \in V^m_{a_1} \) so that

\[
\begin{align*}
(6.26) \quad & \mathcal{T}_{a_2} \theta = \mathcal{P}_2(\mathcal{T}_{a_2} \theta) + \mathcal{P}_3(\mathcal{T}_{a_2} \theta), \\
(6.27) \quad & \mathcal{T}_{a_1} \psi = \mathcal{P}_1(\mathcal{T}_{a_1} \psi) + \mathcal{P}_3(\mathcal{T}_{a_1} \psi).
\end{align*}
\]
Notice that since $s > r(m + 1)$ then $p_3(T_{a_2} \theta) \in \mathcal{H}_0^m$. Using Theorem 5.3, Lemma 6.3 and properties of smoothing operators, we have for any $t > 0$:

(6.28)

$$\|P_2(T_{a_2} \theta)\|_m = \|T_{a_2} \theta - \mathcal{P}_3(T_{a_2} \theta)\|_m \leq \|T_{a_2} \theta - p_3(T_{a_2} \theta)\|_m$$

$$\leq C_{r(m+1)}\|T_{a_2} \theta - p_3(T_{a_2} \theta)\|_{r(m+1)}^r$$

$$\leq C_{r(m+1)}C_{r(m+1)}\|\omega\|_{r(m+1)}^r \leq C_{r(m+1)}\|\omega\|_{r(m+1)}^r$$

$$\leq C_{r,m,\eta}(I - S_t)\|\omega\|_{r(m+1)}^r + C_{r,m,\eta}S_t\|\omega\|_{r(m+1)}^r$$

$$\leq C_{r,m,\eta}C_{s,s-r(m+1)}t^{r(m+1)-s}\|\omega\|_s + C_{r,m,\eta}C_{r(m+1)-r,m}t^{r(m+1)-m}\|\omega\|_m.$$  

If we let $t = R_0^{\frac{1}{2s-4r^2}}$, then we have

$$\|P_2(T_{a_2} \theta)\|_m \leq C_{r,m,\eta}C_{s,s-r(m+1)}\|\omega\|_s + C_{r,m,\eta}C_{r(m+1)-r,m}R_0^{\frac{1}{2s-4r^2}}\|\omega\|_m.$$

(6.29)

Similarly we have

(6.30)  $$\|P_1(T_{a_1} \psi)\|_m \leq C_{r,m,\eta}\|\omega\|_m^{\frac{1}{2s-4r^2}}.$$ 

**Remark 6.1.** As we mentioned before, derivation of (6.29) and (6.30) (more precisely, the second inequality in (6.28) and similar inequality for $\psi'$) is the only but crucial place where the auxiliary norms $\|\cdot\|'$ appear in the main line of proof of our main theorems. Those norms can be used due to condition (B). \[3\]

Now we repeat the arguments from the proof of Lemma 6.2 with appropriate modifications for the norms. Algebra is identical.

By (1) of Lemma 6.4 there exist $\Omega, \Omega' \in \mathcal{H}_0^{m-4r^2}$ such that

(6.31)  $$T_{a_1}T_{a_2} \Omega = \mathcal{P}_3(T_{a_2} \theta),$$

(6.32)  $$T_{a_1}T_{a_2} \Omega' = \mathcal{P}_3(T_{a_1} \psi)$$

satisfying

$$\|\Omega\|_{m-4r^2} \leq C_{r,m,\eta}\|\mathcal{P}_3(T_{a_2} \theta)\|_m \leq C_{r,m,\eta}\|T_{a_2} \theta\|_m$$

(6.33)  $$\leq C_{r,m,\eta}C_{r,m,\eta}\|\theta\|_m.$$ 

This gives (6.25). Similar inequality holds for $\Omega'$ although we do not use it.

\[3\]Variations of our method work in other situations where that condition does not hold or even where there are no stable directions altogether. [5]. However in those situations the “target” action is unique. Hence there is no problem of uniformity of estimates in the parametric family of standard perturbations that we handle in Lemma 6.3.
Substituting (6.31) into (6.26) we obtain

\[ \mathcal{T}_{a_2}(\theta - \mathcal{T}_{a_1}\Omega) = \mathcal{P}_2(\mathcal{T}_{a_2}\theta). \]

By (6.29) and (3) of Lemma 6.4 the following inequalities hold:

\[
\|\theta - \mathcal{T}_{a_1}\Omega\|_{m-4r-4} \\
\leq C_{m,\eta}\|\mathcal{P}_2(\mathcal{T}_{a_2}\theta)\|_m \\
\leq C_{m,\eta}C_{m,\eta,s}\|\mathcal{O}\|_m^{\frac{2m+3-m-r}{m}}.
\]

(6.35)

Thus (6.23) holds. Substituting (6.31) and (6.32) into the identity

\[ \mathcal{P}_3(\mathcal{T}_{a_1}\psi) - \mathcal{P}_3(\mathcal{T}_{a_2}\theta) = \mathcal{P}_3\omega \]

we have

\[ \mathcal{T}_{a_1}\mathcal{T}_{a_2}\Omega' - \mathcal{T}_{a_1}\mathcal{T}_{a_2}\Omega = \mathcal{P}_3\omega. \]

(6.36)

Substituting (6.32) into (6.27) we have

\[ \mathcal{T}_{a_1}(\psi - \mathcal{T}_{a_2}\Omega') = \mathcal{P}_1(\mathcal{T}_{a_1}\psi). \]

(6.37)

Combine (6.36) and (6.37) it follows

\[ \mathcal{T}_{a_1}(\psi - \mathcal{T}_{a_2}\Omega) = \mathcal{T}_{a_1}(\psi - \mathcal{T}_{a_2}\Omega') + (\mathcal{T}_{a_1}\mathcal{T}_{a_2}\Omega' - \mathcal{T}_{a_1}\mathcal{T}_{a_2}\Omega) = \mathcal{P}_1(\mathcal{T}_{a_1}\psi) + \mathcal{P}_3\omega, \]

then use (3) of Lemma 6.4 again and combine (6.30) we have

\[
\|\psi - \mathcal{T}_{a_2}\Omega\|_{m-4r-4} \\
\leq C_{m,\eta}\|\mathcal{P}_1(\mathcal{T}_{a_1}\psi) + \mathcal{P}_3\omega\|_m \\
\leq C_{m,\eta}\|\mathcal{P}_1(\mathcal{T}_{a_1}\psi)\|_m + C_{m,\eta}\|\mathcal{P}_3\omega\|_m \\
\leq C_{m,\eta}C_{m,\eta,s}\|\mathcal{O}\|_m^{\frac{2m+3-m-r}{m}} + C_{m,\eta}\|\mathcal{O}\|_m \\
\leq C_{m,\eta,s}\|\mathcal{O}\|_m^{\frac{2m+3-m-r}{m}}.
\]

This gives (6.24) and completes the proof.

Using the Sobolev embedding theorem we translate estimates for the previous lemma to those in $C^m$ norms that are used in our iteration process. Recall that regularity threshold $m_1$ has been defined in (5.1), (5.13) and (5.14). Thus we obtain

**Corollary 6.1 (Main Estimate).** Let $\eta$ be sufficiently small and $m \in \mathbb{N}$ with $m \geq m_0 = m_1 - \frac{\dim M}{2} - 2r - 3$. For any $a_1 \in B_\eta(a)$ and $a_2 \in B_\eta(b)$ where $a_1, a_2$ commute and $a_1, a_2 \in Z(L)$ and any two $C^s$ $L$-invariant
maps $\mathcal{F}, \mathcal{G} : M \to \mathfrak{N}$ with $L(\mathcal{G}, \mathcal{F})(a_1, a_2) = \Psi$ satisfying $\int_M \mathcal{F}(x)dx = \int_M \mathcal{G}(x)dx = 0$ and $\|\Psi\|_{C^s} \leq 1$, then there exists $\Omega \in C^m$ such that

\[
\|\mathcal{F} - (\Omega \circ a_1 - \text{Ad}_{a_1})\|_{C^m} \leq C_{m, \eta, s} \|\Psi\|_{C^{m+\sigma}}^\gamma(m+\sigma, s)
\]

\[
\|\mathcal{G} - (\Omega \circ a_2 - \text{Ad}_{a_2})\|_{C^m} \leq C_{m, \eta, s} \|\Psi\|_{C^{m+\sigma}}^\gamma(m+\sigma, s)
\]

\[
\|\Omega\|_{C^m} \leq C_{m, \eta} \|\mathcal{F}, \mathcal{G}\|_{C^{m+\sigma}}.
\]

where $s \geq s(m)$ and

\[
s(m) = r(m + \dim M/2 + 4r + 6),
\]

$\sigma = \lfloor \frac{\dim M}{2} + 4r + 6 \rfloor$ and $\gamma(m, s) = \frac{s - 2rm + m - 2r}{s + rm - r}$.

7. Iteration procedure and completion of proof

7.1. Scheme of proof. Let $\bar{e}$ be the image of the identity $e$ of $G$ (corr. $G \ltimes \mathbb{R}^N$) on $M$ and $d(\cdot, \cdot)$ a right invariant metric on $G$ (corr. $G \ltimes \mathbb{R}^N$). For a small neighborhood $V$ of the tangent bundle $T_eG$ or $T_eG \ltimes \mathbb{R}^N$, we can identify the metrics on $V$, exp $V$ and exp $V \cdot \bar{e}$, that is, for any $x \in V \cdot \bar{e}$, by writing $x = \exp(v) \cdot \bar{e}$ where $v \in V$ then $d(x, \bar{e}) = d(\exp v, e) = d(v, 0)$. For simplicity we denote $\|x\| = \|\exp(v)\| = \|v\| = d(x, \bar{e})$.

For any continuous map $f$ valued on $\exp V$ (corr. $\exp V \cdot \bar{e}$), we can define $\|f\|_{C^0} = \max_{y \in D(f)} d(f(y), e)$ (corr. $\|f\|_{C^0} \overset{\text{def}}{=} \max_{y \in D(f)} d(f(y), \bar{e})$) where $D(f)$ is the domain of $f$.

Assuming $\alpha'$ is a $C^\infty$ action that is $C^\ell_r$ close to $\alpha_{D_1}$ (where $\ell$ is fixed and will be determined in the proof, see (7.28)), there is a $C^\infty$ orbit conjugacy $H_1$ between $\alpha'$ and $\alpha_{D_1}$ which is $C^\ell$ close to identity, see Section 4.1. We define a linear $L$-averaging operator “$-$” on the set of smooth maps from $M$ to $\mathfrak{N}$:

\[
\tilde{f}(x) \overset{\text{def}}{=} \int_L \text{Ad}_{l^{-1}} f(l \cdot x)dl.
\]

Let $\tilde{\alpha} = H_1^{-1} \circ \alpha' \circ H_1$. We can represent $\tilde{\alpha} = \exp(R) \cdot \alpha_{D_1}$ where $R$ is close to 0 in $C^\ell$ norm and we will show that $\tilde{\alpha}$ is smoothly conjugate to $\alpha$. The conjugacy is produced for two regular generators and by Lemma 4.4 it works for all elements of the action. This proof is similar to the iterative proof in [4] with essential additions of the balancing of the norms explained below and the parameter adjustment argument in Section 7.3.

Following the scheme described in Section 4.3 $R$ is an $L$-invariant map on $M$ (i.e, $R = R$) valued on $L^\perp$, then at each step of the iterative procedure we solve the linearized equation (4.14):

\[
\Omega \circ \alpha(a_1) - \text{Ad}_{a_1} \Omega = R_a,
\]

\[
\Omega \circ \alpha(a_2) - \text{Ad}_{a_2} \Omega = R_b
\]

approximately where $a_1 \in B_y(a) \cap Z(L)$, $a_2 \in B_y(b) \cap Z(L)$ and $a_1$ and $a_2$ commute. By the Main Estimate (Corollary 6.1) this linearized equation has an approximate solution $\Omega$ which is $C^\infty$ although we can only compare
its $C^m$ norm to the in $C^{m+\sigma}$ of $R$ where $\sigma$ is large but fixed. The norm of the error by construction in the Main Estimate is comparable to a certain power of the norm of $L(R_b,R_a)^{(a_1,a_2)}$. Thus it is small with respect to $R$ by Lemma 4.2, but comparison again comes with fixed loss of derivatives.

Very essential part of the argument is the proper balancing of various norms. Notice that in the Main Estimate the estimate with fixed loss of regularity is obtained for the $C^m$ norm for an $m$ above the threshold $m_0$ determined by the data (see (5.1), (5.13), and (5.14) under the additional assumption that much higher $C^s$ norm is bounded, where $s = s(m)$ (see (6.38)), in particular $s > rm$. Thus we cannot simply make closeness assumption for a fixed norm and offset the finite loss of regularity by straightforward application of smoothing. Instead we assume closeness of the perturbation to the original action in a high $C^\ell$ norm where $\ell$ is defined in (7.25). Essential requirement is $\ell \geq 3\ell_0 > 3s(m_0)$, see (7.25). In particular $l > 3rm_0$. In the iterative step we use smoothing both for the data and for the solution. By making very strong requirements on the decrease of the $C^0$ norms of the successive errors that are allowed by the quadratic convergence and controlling the growth of $C^\ell$ norms (see (7.24)), we guarantee via interpolation inequalities fast decrease of still high $C^{\ell_0}$ norms. Since $\ell_0 > s(m_0)$ this allows for successive applications of the Main Estimate.

At the end we guarantee convergence of conjugacies in $C^1$ and appeal to the a priori regularity to conclude that is is $C^\infty$. Of course our argument also gives convergence in $C^{\ell_0}$ but since we only work in finite regularity we do not produce $C^\infty$ conjugacy directly.

### 7.2. Smoothing operators and some norm inequalities.

To overcome this fixed loss of derivatives at each step of the iteration process, it is standard (see for example [30]) to introduce the family of smoothing operators: $\{S_t, t \in \mathbb{R}\}$. Since $S_tR$ is not necessarily $L$-invariant we combine smoothing with $L$-averaging and will solve approximately the following system:

\begin{align}
\Omega \circ \alpha(a_1) - \text{Ad}_{a_1}\Omega &= S_tR_a, \\
\Omega \circ \alpha(a_2) - \text{Ad}_{a_2}\Omega &= S_tR_b.
\end{align}

(7.1)

Denote the averages $\int_{\mathcal{M}} S_tR_ad\mu = A(S_tR_a)$ and $\int_{\mathcal{M}} S_tR_b d\mu = A(S_tR_b)$. Recall that $L(\mathcal{F}, \mathcal{G})^{(a_1,a_2)} = \mathcal{F} \circ a_2 - \text{Ad}_{a_2}\mathcal{F} - \mathcal{G} \circ a_1 + \text{Ad}_{a_1}\mathcal{G} = \mathcal{T}_{a_2}\mathcal{F} - \mathcal{T}_{a_1}\mathcal{G}$. 


By Lemma 4.2 and \( L \)-invariance of the data i.e. \( \mathcal{R}_b = R_b \) and \( \mathcal{R}_a = R_a \), we have

\[
\| L (S_t \mathcal{R}_b - \mathcal{A}(S_t \mathcal{R}_b), S_t \mathcal{R}_a - \mathcal{A}(S_t \mathcal{R}_a))^{(a_1, a_2)} \|_{C_m}
\]

\[
= \| L (S_t \mathcal{R}_b, S_t \mathcal{R}_a)^{(a_1, a_2)} - \int_M L (S_t \mathcal{R}_b, S_t \mathcal{R}_a)^{(a_1, a_2)} d\mu \|_{C_m}
\]

\[
\leq C \| L (S_t \mathcal{R}_b, S_t \mathcal{R}_a)^{(a_1, a_2)} \|_{C_m}
\]

\[
\leq C \| L (R_b, R_a)^{(a_1, a_2)} \|_{C_m} + C \| L ((I - S_t)R_b, (I - S_t)R_a)^{(a_1, a_2)} \|_{C_m}
\]

\[
\leq C \| L (R_b, R_a)^{(a_1, a_2)} \|_{C_m} + C_{m, j} \| (I - S_t)R_a, (I - S_t)R_b \|_{C_m}
\]

\[
\leq C \| L (R_b, R_a)^{(a_1, a_2)} \|_{C_m} + C_{m, j, \eta} t^{-j} \| R_a, R_b \|_{C_{j+1}}
\]

\[
\leq C_{m, j} \| R_a, R_b \|_{C_m} \| R_b, R_a \|_{C_{m+1}} + C_{m, j, \eta} t^{-j} \| R_a, R_b \|_{C_{j+1}}
\]

for any \( j \in \mathbb{N} \).

### 7.3. Iterative step and the error estimate

At each step of the iterative scheme we first choose a smoothing operator \( S_t \) with an appropriately chosen \( t \). In order to solve approximately

\[
\Omega \circ a_1 - \text{Ad}_{a_1} \Omega = S_t \mathcal{R}_a - \mathcal{A}(S_t \mathcal{R}_a)
\]

\[
\Omega \circ a_2 - \text{Ad}_{a_2} \Omega = S_t \mathcal{R}_b - \mathcal{A}(S_t \mathcal{R}_b)
\]

we use the Main Estimate (Corollary 6.1) to obtain an approximate solution \( \Omega \in C^{m_0} \):

\[
\Omega \circ a_1 - \text{Ad}_{a_1} \Omega = S_t \mathcal{R}_a - \mathcal{A}(S_t \mathcal{R}_a) - \mathcal{N}(S_t \mathcal{R}_a - \mathcal{A}(S_t \mathcal{R}_a))
\]

\[
\Omega \circ a_2 - \text{Ad}_{a_2} \Omega = S_t \mathcal{R}_b - \mathcal{A}(S_t \mathcal{R}_b) - \mathcal{N}(S_t \mathcal{R}_b - \mathcal{A}(S_t \mathcal{R}_b))
\]

such that for any \( m \leq m_0 \)

\[
\| \Omega \|_{C_m} \leq C_{m_0, \eta} \| S_t \mathcal{R}_a, S_t \mathcal{R}_b - \mathcal{A}(S_t \mathcal{R}_a, S_t \mathcal{R}_b) \|_{C^{m_0 + \eta}}
\]

\[
\leq C C_{m_0, \eta} \| S_t \mathcal{R}_a, S_t \mathcal{R}_b \|_{C^{m_0 + \eta}}
\]

\[
\leq C_{m_0, \eta} \| R_a, R_b \|_{C^{m_0 + \eta}}
\]

and also

\[
\| \Omega \|_{C_m} \leq C_{m_0, \eta} \| S_t \mathcal{R}_a, S_t \mathcal{R}_b - \mathcal{A}(S_t \mathcal{R}_a, S_t \mathcal{R}_b) \|_{C^{m_0 + \eta}}
\]

\[
\leq C C_{m_0, \eta} \| S_t \mathcal{R}_a, S_t \mathcal{R}_b \|_{C^{m_0 + \eta}}
\]

\[
\leq C_{m_0, \eta} t^{-j} \| R_a, R_b \|_{C^{m_0 + \eta}}
\]

for any \( j \in \mathbb{N} \). Here we used the properties of smoothing operators. As was explained above, applicability of the Main Estimate will be guaranteed by uniform boundedness of the data in \( C^{\ell_0} \) for a sufficiently large \( \ell_0 \). The error terms \( \mathcal{N}(S_t \mathcal{R}_a - \mathcal{A}(S_t \mathcal{R}_a)) \) and \( \mathcal{N}(S_t \mathcal{R}_b - \mathcal{A}(S_t \mathcal{R}_b)) \) will be estimated by the Main Estimate.

Then we define \( H = \exp(S_s \Omega) \cdot I \) for a certain \( s > 0 \). Notice that by properties of smoothing operators \( \| S_s \Omega \|_{C^1} \leq C \| \Omega \|_{C^1} \), and since \( \Omega \) is small in \( C^1 \) throughout the iteration, then \( H \) is invertible and \( H \) projects into
an invertible map \( h \) on \( X \). This is the conjugacy at the iterative step. Let \( \tilde{\alpha}^{(1)} = h^{-1} \circ \tilde{\alpha} \circ h \). This is the new action and we need to estimate is distance form a certain standard algebraic perturbation of the unperturbed action \( \alpha_{D+} \).

For any \( d \in D_+ \), \( \tilde{\alpha}^{(1)}(d) \) can be lifted to a map on \( M \) which we still denote by \( \tilde{\alpha}^{(1)}(d) \) without confusion. Hence there exists \( l_d : M \to L \) such that

\[
H \circ \tilde{\alpha}^{(1)}(d) = l_d \cdot (\tilde{\alpha}(d) \circ H)
\]

so that

\[
\tilde{\alpha}^{(1)} = H^{-1}(l \cdot (\tilde{\alpha} \circ H)) = \exp(-S_s\Omega \circ \tilde{\alpha}^{(1)}) \cdot l \cdot \exp(R \circ H) \cdot \exp(\text{Ad}_{\alpha_{D+}} S_s\Omega) \cdot \alpha_{D+}
\]

\[
= l \cdot \exp(-S_s\Omega \circ (l^{-1} \cdot \tilde{\alpha}^{(1)})) \cdot \exp(S_s\Omega(\exp(AR) \cdot \alpha_{D+}))
\]

\[
\cdot \exp(-S_s\Omega(\exp(AR) \cdot \alpha_{D+})) \cdot \exp(S_s\Omega \circ \alpha_{D+})
\]

\[
\cdot \exp(-S_s\Omega \circ \alpha_{D+}) \cdot \exp(R \circ H) \cdot \exp(-R) \cdot \exp(S_s\Omega \circ \alpha_{D+})
\]

\[
\cdot \exp(-S_s\Omega \circ \alpha_{D+} \cdot \exp(\text{Ad}_{\alpha_{D+}} S_s\Omega) \cdot \exp(-AR)
\]

\[
\cdot \exp(S_s\Omega \circ \alpha_{D+} \cdot -R - \text{Ad}_{\alpha_{D+}} S_s\Omega + AR)
\]

\[
\cdot \exp(-S_s\Omega \circ \alpha_{D+} \cdot R + \text{Ad}_{\alpha_{D+}} S_s\Omega - AR)
\]

\[
\cdot \exp(AR) \cdot \alpha_{D+}.
\]

The new error is:

\[
\exp(R'_{(1)}) = l^{-1} \cdot \tilde{\alpha}^{(1)} \cdot (\exp(AR) \cdot \alpha_{D+})^{-1}
\]

and it can be decomposed as \( \exp(R'_{(1)}) = E_2 \cdot \exp(E_1) \) where:

1. \( \exp(E_1) \) is the error coming from solving the linearized equation only approximately:

\[
E_1 = R(S_i \overline{R} - A(S_i \overline{R})) + (I - S_i)\overline{R} + A((I - S_i)\overline{R})
\]

\[
+ (I - \text{Ad}_{\alpha_{D+}}) \circ (I - S_s)\Omega
\]

2. \( E_2 \) is the standard error coming from the linearization:

\[
E_2 = \exp(-S_s\Omega \circ (l^{-1} \cdot \tilde{\alpha}^{(1)})) \cdot \exp(S_s\Omega(\exp(AR) \cdot \alpha_{D+}))
\]

\[
\cdot \exp(-S_s\Omega(\exp(AR) \cdot \alpha_{D+})) \cdot \exp(S_s\Omega \circ \alpha_{D+})
\]

\[
\cdot \exp(-S_s\Omega \circ \alpha_{D+}) \cdot \exp(R \circ H) \cdot \exp(-R) \cdot \exp(S_s\Omega \circ \alpha_{D+})
\]

\[
\cdot \exp(-S_s\Omega \circ \alpha_{D+}) \cdot \exp(R) \cdot \exp(\text{Ad}_{\alpha_{D+}} S_s\Omega) \cdot \exp(-AR)
\]

\[
\cdot \exp(S_s\Omega \circ \alpha_{D+} \cdot -R - \text{Ad}_{\alpha_{D+}} S_s\Omega + AR)
\]
**Estimate of $E_1$ in $C^0$.** Using Main Estimate, properties of smoothing operators and inequality (7.2) for $j = \ell - m_0 - \sigma$ we have for any $m \leq m_0$

\begin{equation}
\|R(\overline{S_t R} - A(S_t R))\|_{C^m} \\
\leq C_{m_0, \eta, \ell_0} \|L(\overline{S_t R_B} - A(S_t R_B), \overline{S_t R_a} - A(S_t R_a))^{(a_1, a_2)}\|_{C^{m_0 + \sigma}} \\
\leq C_{m_0, \eta, \ell_0} \|R_B, R_a\|_{C^{m_0 + \sigma}} \|R_B, R_a\|_{C^{m_0 + \sigma + 1}} \\
+ C_{m_0, \eta, \ell, \ell_0} t^{-\gamma(\ell - m_0 - \sigma)} \|R_B, R_a\|_{C^\ell}
\end{equation}

where $\gamma = \gamma(m_0 + \sigma, \ell_0)$ providing $\|L(\overline{S_t R_B} - A(S_t R_B), \overline{S_t R_a} - A(S_t R_a))^{(a_1, a_2)}\|_{C^{t_0}}$ is bounded throughout the procedure for well chosen $\ell_0$ and $t$ where $a_1 = i_0(a)$, $a_2 = i_0(b)$ and $\gamma$ is as in Main Estimate determined by $m_0, \sigma, \ell_0$.

Also, using properties of smoothing operators:

\begin{equation}
\|(I - S_t) R\|_{C^0} \leq C\|\overline{(I - S_t) R}\|_{C^0} \\
\leq C t^{-\ell} \|R\|_{C^\ell}
\end{equation}

and thus

\begin{equation}
\|A((I - S_t) R)\|_{C^0} \leq C t^{-\ell} \|R\|_{C^\ell}
\end{equation}

and by (7.6)

\begin{equation}
\|(I - \text{Ad}_{\alpha_{D_+}}) \circ (I - S_s) \Omega\|_{C^0} \\
\leq C \|\overline{(I - S_s) \Omega}\|_{C^0} \leq C_{m_0, \eta, \ell_0} \|\Omega\|_{C^{m_0}} \\
\leq C_{m_0, \eta, \ell_0} t^{-\gamma \ell - (\ell - m_0 - \sigma)(\ell - m_0 - \sigma)} \|R_B, R_a\|_{C^\ell}
\end{equation}

Thus, combining (7.7), (7.8), (7.9) and (7.10) we have for $\gamma = \gamma(m_0 + \sigma, \ell_0)$ (since $R$ is kept bounded in $C^{t_0}$ throughout the iteration):

\begin{equation}
\|E_1\|_{C^0} \leq C_{m_0, \eta, \ell_0} \|R\|_{C^m} \|R\|_{C^{m_0 + \sigma}} \|R\|_{C^{m_0 + \sigma + 1}} + C t^{-\ell} \|R\|_{C^\ell} \\
+ C_{m_0, \eta, \ell_0} t^{-\gamma(\ell - m_0 - \sigma)} \|R\|_{C^\ell} + C_{m_0, \eta, \ell_0} s^{-\gamma \ell - (\ell - m_0 - \sigma)} \|R\|_{C^\ell}
\end{equation}

\begin{align*}
E_1 &= \exp(-S_s \Omega \circ (\alpha_s^{-1} \cdot \alpha(1))) \cdot \exp(S_s \Omega(\exp(AR) \cdot \alpha_{D_+})) \\
E_2 &= \exp(-S_s \Omega(\exp(AR) \cdot \alpha_{D_+})) \cdot \exp(S_s \Omega \circ \alpha_{D_+}) \\
E_3 &= \exp(-S_s \Omega \circ \alpha_{D_+}) \cdot \exp(R \circ H \cdot \exp(-R \cdot \exp(S_s \Omega \circ \alpha_{D_+})) \\
E_4 &= \exp(-S_s \Omega \circ \alpha_{D_+}) \cdot \exp(R) \cdot \exp(\text{Ad}_{\alpha_{D_+}} S_s \Omega) \cdot \exp(-AR) \\
&\quad \cdot \exp(S_s \Omega \circ \alpha_{D_+} - R - \text{Ad}_{\alpha_{D_+}} S_s \Omega + AR)
\end{align*}

then we have

\[\|E_2\|_{C^0} \leq \|E_1\|_{C^0} + \|E_2\|_{C^0} + \|E_3\|_{C^0} + \|E_4\|_{C^0}\]
which follows from the fact that for any small enough \( x_1, x_2, x_3, x_4 \in \exp(V) \) by right invariance of metric \( d \) we have
\[
d(x_1x_2x_3x_4, e) \\
\leq d(x_1x_2x_3x_4, x_2x_3x_4) + d(x_2x_3x_4, x_3x_4) + d(x_3x_4, x_4) + d(x_4, e) \\
= d(x_1, e) + d(x_2, e) + d(x_3, e) + d(x_4, e).
\]
Thus there are four terms that we estimate as follows:

**First term.** Using (7.5) we obtain
\[
\|\exp(-S_\delta \Omega \circ (l_d^{-1} \cdot \tilde{\alpha}^{(1)})) \cdot \exp(S_\delta \Omega (\exp(AR) \cdot \alpha_{D_+}))\|_{C^0} \\
\leq \|S_\delta \Omega (\exp(AR) \cdot \alpha_{D_+}) - S_\delta \Omega \circ (l_d^{-1} \cdot \tilde{\alpha}^{(1)})\|_{C^0} + C\|S_\delta \Omega\|^2_{C^0} \\
\leq C\|S_\delta \Omega\|_{C^1} \|R'(1)\|_{C^0} + C\|\Omega\|^2_{C^0} \\
\leq C\|\Omega\|_{C^1} \|R'(1)\|_{C^0} + C\|\Omega\|^2_{C^0} \\
(7.12) \\
\leq \frac{1}{4} \|R'(1)\|_{C^0} + C_{m_0, \eta} \|R\|_{C_{m_0+\sigma}^0}
\]
thus this term is absorbed into \( \|R'(1)\|_{C^0} \) providing \( \|\Omega\|_{C^1} \) remains sufficiently small throughout the procedure.

**Second term.** We estimate similarly to \( E_1 \):
\[
\|\exp(-S_\delta \Omega (\exp(AR) \cdot \alpha_{D_+})) \cdot \exp(S_\delta \Omega \circ \alpha_{D_+})\|_{C^0} \\
\leq \|S_\delta \Omega (\exp(AR) \cdot \alpha_{D_+}) - S_\delta \Omega \circ \alpha_{D_+}\|_{C^0} + C\|S_\delta \Omega\|^2_{C^0} \\
\leq C\|S_\delta \Omega\|_{C^1} \|R\|_{C^0} + C\|S_\delta \Omega\|^2_{C^0} \\
\leq C\|\Omega\|_{C^1} \|R\|_{C^0} + C\|\Omega\|^2_{C^0} \\
(7.13) \\
\leq C_{m_0, \eta} \|R\|_{C^0} \|R\|_{C_{m_0+\sigma}^0} + C_{m_0, \eta} \|R\|_{C_{m_0+\sigma}^0}^2.
\]

**Third term.** We first notice:
\[
\|\exp(R \circ H) \exp(-R)\|_{C^0} \\
\leq C\|R \circ H - R\|_{C^0} + C\|R\|_{C^0}^2 \\
\leq C\|R\|_{C^1} \|S_\delta \Omega\|_{C^0} + C\|R\|_{C^0}^2 \\
(7.14) \\
\leq C\|R\|_{C^1} \|\Omega\|_{C^0} + C\|R\|_{C^0}^2,
\]
then use (7.5), the fact that \( \|\Omega\|_{C^1} \) and \( \|R\|_{C^0} \) are bounded throughout the procedure and (7.14) we get
\[
\|\exp(-S_\delta \Omega \circ \alpha_{D_+}) \cdot \exp(R \circ H) \cdot \exp(-R) \cdot \exp(S_\delta \Omega \circ \alpha_{D_+})\|_{C^0} \\
\leq \|\text{Ad}_{S_\delta \Omega \circ \alpha_{D_+}}\|_{C^0} \|\exp(R \circ H) \exp(-R)\|_{C^0} \\
\leq C(\|S_\delta \Omega \circ \alpha_{D_+}\|_{C^0} + 1) \|\exp(R \circ H) \exp(-R)\|_{C^0} \\
\leq C(\|\Omega\|_{C^0} + 1) \|\exp(R \circ H) \exp(-R)\|_{C^0} \\
\leq C\|R\|_{C^1} \|\Omega\|_{C^0} + C\|R\|_{C^0}^2 \\
(7.15) \\
\leq C_{m_0, \eta} \|R\|_{C_{m_0+\sigma}^0} \|R\|_{C^1} + C\|R\|_{C^0}^2.
Fourth term. Use (7.12) and the fact $C^0$-norm of Lie brackets between $S_s \Omega$, $R$, $AR$ and $\text{Ad}_{\alpha_{D_+}} S_s \Omega$ are uniformly bounded by $C_{m_0, \eta} \|R\|_{C^{m_0 + \sigma}}^2$, hence we have

$$\|\exp(-S_s \Omega \circ \alpha_{D_+}) \cdot \exp(R) \cdot \exp(\text{Ad}_{\alpha_{D_+}} S_s \Omega) \cdot \exp(-AR) \cdot \exp(S_s \Omega \circ \alpha_{D_+} - R - \text{Ad}_{\alpha_{D_+}} S_s \Omega + AR)\|_{C^0} \leq C_{m_0, \eta} \|R\|_{C^{m_0 + \sigma}}^2.$$  \hspace{1cm} (7.16)

By combining (7.12), (7.13), (7.15), (7.16) we obtain the following estimate for $E_2$:

$$\|E_2\|_{C^0} \leq C_{m_0, \eta} \|R\|_{C^{m_0 + \sigma}}^2.$$ \hspace{1cm} (7.17)

By combining (7.11) and (7.17) we obtain an estimate for the new error for $\gamma = \gamma(m_0 + \sigma, \ell_0)$:

$$\|R'(1)\|_{C^0} \leq \|E_1\|_{C^0} + \|E_2\|_{C^0} \leq C_{m_0, \eta, s} \|R\|_{C^{m_0 + \sigma}}^\gamma \|R\|_{C^{m_0 + \sigma + 1}}^\gamma + C_\ell t^{-\ell} \|R\|_{C^\ell}^\gamma + C_{m_0, \eta, \ell_0} \|R\|_{C^\ell}^\gamma + C_{m_0, \eta, \ell} \|R\|_{C^\ell}^\gamma \leq C_{m_0, \eta} \|R\|_{C^{m_0 + \sigma}}^2.$$  \hspace{1cm} (7.18)

This completes the $C^0$ estimate.

Now we need to prepare for the coordinate change that will make the constant term quadratically small with respect to $\|R'(1)\|_{C^0}$.

Since $R$ is $L$-invariant then $AR \in Z(D) \cap Z(L)$, then $\exp(AR) \cdot \alpha_{D_+}$ can descend to $X$, then there exists smooth map $R' : M \to \mathfrak{L}^\perp$ such that

$$\tilde{\alpha}^{(1)}(p(x)) = p(\exp(R'(x)) \cdot \exp(AR) \cdot \alpha_{D_+}(x)) \quad \forall x \in M$$  \hspace{1cm} (7.19)

where $p$ is the natural projection from $M$ to $X = L \setminus M$. Notice $l^{-1} \cdot \tilde{\alpha}^{(1)}$ descends to the same map on $X$ as $\tilde{\alpha}^{(1)}$ does then combine (7.18) we have

$$\|R'\|_{C^n} \leq C \|R'(1)\|_{C^n} \quad \forall n \in \mathbb{N}.$$  \hspace{1cm} (7.19)

Since $R'$ is $L$-invariant then $AR' \in Z(D) \cap Z(L)$ and $\exp(AR') \cdot \exp(AR) \cdot \alpha_{D_+}$ can descend to $X$, hence there exists a smooth map $R(1) : M \to \mathfrak{L}^\perp$ such that

$$\tilde{\alpha}^{(1)}(p(x)) = p(\exp(R(1)(x)) \cdot \exp(AR') \cdot \exp(AR) \cdot \alpha_{D_+}(x)) \quad \forall x \in M$$  \hspace{1cm} (7.20)

and hence

$$\|R(1)\|_{C^n} \leq C \|R'\|_{C^n} \quad \forall n \in \mathbb{N}.$$  \hspace{1cm} (7.21)

Combine (7.19) and (7.20) we have

$$l \cdot \exp(R') = \exp(R(1)) \cdot \exp(AR').$$
where \( l : M \to L \). Notice \( \log(AR') \in \mathcal{L} \) and thus it follows
\[
R' = R(1) + AR' + \text{Res}
\]
where \( \|\text{Res}\|_{C^0} \leq C\|R', R(1)\|_{C^0} \leq C\|R'(1)\|_{C^0}^2 \). Then integrate each side of above equation we have
\[
\left\| \int_M R(1)d\mu \right\| = \left\| \int_M \text{Resd}\mu \right\| \leq C\|R'(1)\|_{C^0}^2
\]
and
\[
\left\| R(1) \right\|_{C^0} \leq C\|R'(1)\|_{C^0} \leq C_{m_0, \eta, s}\|R\|_{C^{m_0+\sigma}} + C_{\ell} t^{-\ell} \|R\|_{C^\ell} + C_{m_0, \eta, \ell} t^{-m_0} \|R\|_{C^\ell} + C_{m_0, \eta, \ell} t^{-m_0} \|R\|_{C^\ell} + C_{m_0, \eta} \|R\|_{C^\ell}
\]
\[
R(1) \leq C_{\ell, m_0, \eta} (S_t \Omega)_{C^\ell} + 1 + \|R\|_{C^\ell} \leq C_{\ell, m_0, \eta} (S_{t/m_0} \Omega)_{C^\ell} + 1 + \|R\|_{C^\ell}
\]
(7.23)

7.4. The iteration scheme. To set up the iterative process we first let:
\[
R^{(0)} = R; \quad \tilde{\alpha}^{(0)} = \tilde{\alpha}; \quad \alpha^{(0)} = \alpha_D; \quad H^{(0)} = I
\]
Now construct \( R^{(n)} \) inductively for every \( n \): for \( R^{(n)} \) choose an appropriate number \( t_n \) to obtain \( S_{t_n} R^{(n)} \) which produces, after solving approximately the linearized equation, new \( \Omega^{(n)} \). Then we construct new abelian action \( \alpha^{(n+1)} \) as follows: At first define
\[
H^{(n)} = \exp(\Omega^{(n)}) \cdot I
\]
\[
\tilde{\alpha}^{(n+1)} = (H^{(n)})^{-1} \circ \tilde{\alpha}^{(n)} \circ H^{(n)}
\]
\[
\exp(R'_{(n+1)}) = \tilde{\alpha}^{(n+1)} \cdot \left( \exp(AR^{(n)}) \cdot \alpha^{(n)} \right)^{-1}
\]
Now let
\[
\alpha^{(n+1)} = \exp(AR_{(n+1)}') \cdot \exp(AR^{(n)}) \cdot \alpha^{(n)}
\]
\[
\exp(R_{(n+1)}) = \tilde{\alpha}^{(n+1)} \cdot (\alpha^{(n+1)})^{-1}
\]
Notice \( \alpha^{(n+1)} \) is not necessarily an abelian action, so we need to do some coordinate change: find a new \( \exp(AR^{(n)})' \) close enough to \( \exp(AR'_{(n+1)}) \cdot \exp(AR^{(n)}) \) such that \( \alpha^{(n+1)} = \exp(AR^{(n)})' \cdot \alpha^{(n)} \) is abelian and thus define
\(R^{(n+1)} = \tilde{\alpha}^{(n+1)} \circ (\alpha^{(n+1)})^{-1}\) (see Section 7.6). Consequently:

\[\tilde{\alpha}^{(n+1)} = (H^{(n)})^{-1} \circ (H^{(n-1)})^{-1} \circ \cdots \circ (H^{(0)})^{-1} \circ \tilde{\alpha} \circ H^{(0)} \circ \cdots \circ H^{(n)} = H_n^{-1} \circ \tilde{\alpha} \circ H_n\]

where \(H_n = H^{(0)} \circ \cdots \circ H^{(n)}\).

Let \(\delta_1 = \frac{\delta_0}{2} + \frac{1}{5}\) where \(\delta_0\) is defined in Definition 3.1. To ensure \(C^1\) convergence of the process set:

\[
\|R^{(n)}\|_{C^0} \leq \varepsilon_n = \varepsilon^{(k_n)} \\
\|R^{(n)}\|_{C^1} \leq \varepsilon_n^{-1} \\
s_n = t_n = \varepsilon_n^{-\frac{1}{k_0+2}}
\]

where \(k = \frac{k_0+1}{k_0}\) and \(k_0 > \max\left\{\frac{7k_0 + 1}{4}, 4\delta_1, \frac{1}{1-\frac{1}{3+\delta_0}}\right\}\). Let \(\ell_0 \in \mathbb{N}\) large enough such that

\[
\ell_0 > \frac{r(m_0 + 2r)}{2} + \frac{\dim M}{2} + 6,
\]

(7.25) \[1 > \gamma(m_0 + \sigma, \ell_0) > \frac{1}{2} + \frac{1}{8\delta_1}\]

where \(\tau = \max\left\{\frac{k_0+1}{k_0}, \frac{1}{2\delta_1}\right\}\). Notice that \(\tau > 1\).

From interpolation inequalities it follows that:

\[
\|R^{(n)}\|_{C^0} \leq C_1\|R^{(n)}\|_{C^0}^{1-\frac{t_0}{2}} \leq C_1\varepsilon_n^{-1} < \varepsilon_n^{-\frac{1}{2}} \\
\|R^{(n)}\|_{C^0+1} \leq C_1\|R^{(n)}\|_{C^0}^{1-\frac{t_0+1}{2}} \leq C_1\varepsilon_n^{-1} < \varepsilon_n^{-\frac{1}{2}}
\]

(7.26) then by (7.2)

\[
\|L(S_{t_n}\tilde{R}_b^{(n)} - A(S_{t_n}\tilde{R}_b^{(n)}), S_{t_n}\tilde{R}_a^{(n)} - A(S_{t_n}\tilde{R}_a^{(n)}))^{(a_n, a_n^2)}\|_{C^0} < \varepsilon_n^{-\frac{1}{2}}
\]

(7.27) \[\leq C_1\varepsilon_n^{-\frac{1}{2}} + C_1\varepsilon_n^{-2\delta_0}\|R^{(n)}\|_{C^0} < \varepsilon_n^{-\frac{1}{2}} < 1.
\]

At this point fix \(\ell\):

\[
\ell \geq \max\{3\ell_0, \frac{2m_0 + 2\sigma}{1 - \frac{1}{k_0}}, \frac{2m_0 + 2\sigma + 2}{2 - \frac{r}{\frac{1}{2} + \frac{1}{3+\delta_0}}}, \frac{2\tau}{\tau - \frac{1}{\delta_1+\delta_0}}, (\tau + 2 + m_0)(k_0\sigma + 2)\}
\]

This seemingly cumbersome condition will be needed to estimate \(\|R^{(n)}\|_{C^0}\).
7.5. Convergence. By induction it is proved that all the bounds (7.24) hold for every \( n \in \mathbb{N} \). By (7.23) and by the inductive assumption we obtain
\[
\| R_{(n+1)} \|_{C^{\ell}} \leq C_{\ell,n} (s^{\ell-m_0 \ell^{-(\ell-m_0 - \sigma)}} \| R \|_{C^{\ell}} + 1 + \| R \|_{C^{\ell}})
\]
\[
\leq C_{\ell,n} \left( \varepsilon_n^{\frac{\ell-m_0}{\kappa_0 \sigma + 2}} \varepsilon_n^{\frac{-m_0 - \sigma}{\kappa_0 \sigma + 2} - 1} + 1 + \varepsilon_n^{-1} \right)
\]
\[
\leq C_{\ell,n} \left( \varepsilon_n^{\frac{1}{\kappa_0 \sigma + 2}} - 1 + 1 + \varepsilon_n^{-1} \right) \leq 2C_{\ell,n} \varepsilon_n^{\frac{-k_0 + 2}{\kappa_0 \sigma + 2} - 1}
\]
\[
< \varepsilon_n^{-1} = \varepsilon_n^{-1} \leq \varepsilon_n^{1-n+1}.
\]

Now by (7.26) and (7.27) Main Estimate applies to get (7.7). From interpolation inequalities it follows that:
\[
\| R^{(n)} \|_{C^{m_0 + \sigma}} \leq C_{\ell} \| R^{(n)} \|_{C^{0}}^{1-\frac{m_0 + \sigma}{\ell}} \| R^{(n)} \|_{C^{\ell}}^{\frac{m_0 + \sigma}{\ell}}.
\]
\[
\| R^{(n)} \|_{C^{m_0 + \sigma + 1}} \leq C_{\ell} \| R^{(n)} \|_{C^{0}}^{1-\frac{m_0 + \sigma + 1}{\ell}} \| R^{(n)} \|_{C^{\ell}}^{\frac{m_0 + \sigma + 1}{\ell}}.
\]
\[
(7.29)
\]
\[
\| R^{(n)} \|_{C^{1}} \leq C_{\ell} \| R^{(n)} \|_{C^{0}}^{1-\frac{1}{\ell}} \| R^{(n)} \|_{C^{\ell}}^{\frac{1}{\ell}}
\]

Along with (7.22) and this implies for \( \gamma = \gamma(m_0 + \sigma, \ell_0) \):
\[
\| R_{(n+1)} \|_{C^{0}} \leq C_{m_0, \eta, \delta} \| R \|_{C^{0}}^{\gamma} \| R \|_{C^{m_0 + \sigma + 1}}^{\gamma} + C_{\ell, \ell_0} \| R \|_{C^{\ell}}^{\gamma} + C_{m_0, \eta, \ell_0} \| \ell^{-(\ell-m_0 - \sigma)} \| \| R \|_{C^{\ell}}
\]
\[
+ C_{m_0, \eta, \ell_0} \| R \|_{C^{m_0 + \sigma}}^{2}.
\]
\[
\leq C_{m_0, \eta, \ell_0} \varepsilon_n^{\gamma(2-\frac{2m_0 + 2\sigma + 2}{\ell})} + C_{\ell} \varepsilon_n^{1-\frac{1}{\kappa_0 \sigma + 2}}
\]
\[
+ C_{m_0, \eta, \ell_0} \varepsilon_n^{\gamma} + C_{m_0, \eta, \ell_0} \varepsilon_n^{1-\frac{m_0 - \sigma}{\kappa_0 \sigma + 2}} + C_{m_0, \eta, \ell_0} \varepsilon_n^{\gamma} + C_{m_0, \eta, \ell_0} \varepsilon_n^{\frac{-m_0 - \sigma}{\kappa_0 \sigma + 2} - 1}.
\]
\[
= C(\varepsilon_n^{2} + \varepsilon_n^{2} + \varepsilon_n^{2} + \varepsilon_n^{2} + \varepsilon_n^{2}) \leq \varepsilon_n^{7}
\]

where \( \tau = \max\left\{ \frac{k_0 + 1}{k_0}, \frac{1}{2m_0} \right\} \) providing
\[
\gamma(2-\frac{2m_0 + 2\sigma + 2}{\ell}) > \tau, \quad \frac{\ell}{k_0 \sigma + 2} - 1 > \tau
\]
\[
\gamma(\ell - \frac{m_0 - \sigma}{k_0 \sigma + 2}) - \gamma > \tau, \quad \frac{\ell - m_0 - \sigma}{k_0 \sigma + 2} + \frac{m_0}{k_0 \sigma + 2} - 1 > \tau
\]
\[
2 - \frac{2m_0 + 2\sigma}{\ell} > \tau
\]

all inequalities above are satisfied for
\[
\ell > \max\left\{ (\tau + 2 + m_0)(k_0 \sigma + 2), \frac{2m_0 + 2\sigma + 2}{2 - \frac{1}{\frac{m_0}{k_0 \sigma + 2}}} \right\}, \quad k_0 > 4\delta_1.
\]

Using interpolation inequalities we can get the \( C^{1} \) bound for \( R_{(n+1)} \):
\[ \| R_{(n+1)} \|_{C^1} \leq C_\ell \| R_{(n+1)} \|_{C^0}^{1 - \frac{1}{\ell}} \| R_{(n+1)} \|_{C^\ell}^{1 - \frac{1}{\ell}} \leq C_\ell \varepsilon_n \tau^{(1 - \frac{1}{\ell}) - \frac{1}{\ell}} \leq \frac{1}{\delta_1 + \delta_0} \]

providing \( \tau (1 - \frac{1}{\ell}) - \frac{1}{\ell} > \frac{1}{\delta_1 + \delta_0} \) which is satisfied for \( \ell > \frac{2\tau}{\tau - \frac{1}{\delta_1 + \delta_0}} \) and \( k_0 > \frac{1}{\delta_1 + \delta_0} \).

### 7.6. Construction of coordinate changes.

By induction we just formed:

\[ \tilde{a}^{(n+1)} = \exp(R_{(n+1)}^a) \cdot \alpha(a_n) \]
\[ \tilde{b}^{(n+1)} = \exp(R_{(n+1)}^b) \cdot \alpha(b_n) \]

where \( \alpha(a_n) = \alpha(n+1)(a) \) and \( \alpha(b_n) = \alpha(n+1)(a) \) with \( a_n, b_n \in Z(L) \) and

\[ \| R_{(n+1)}^a, R_{(n+1)}^b \|_{C^0} \leq \varepsilon_n^{\frac{1}{2}} \]
\[ \| R_{(n+1)}^a, R_{(n+1)}^b \|_{C^\ell} \leq \varepsilon_n^{\frac{1}{2} + \frac{k_0}{q_0 + 1}}. \]

Using (7.22) we have

\[ \| A R_{(n+1)}^a, A R_{(n+1)}^b \| \leq \varepsilon_n^{\frac{1}{2}} \]

Since \( \tilde{a}^{(n+1)} \) and \( \tilde{b}^{(n+1)} \) commute, we can argue similarly to the proof of Lemma 4.2. Let \( \exp(X) = \log(a_n b_n (a_n b_n)^{-1}) \), then

\[ X = -R_{(n+1)}^a (\exp(R_{(n+1)}^b) \cdot b_n) - Ad_{a_n} R_{(n+1)}^b \]
\[ + R_{(n+1)}^b (\exp(R_{(n+1)}^a) \cdot a_n) + Ad_{b_n} R_{(n+1)}^a + \text{Res} \]
\[ = R_{(n+1)}^b \circ a_n - Ad_{a_n} R_{(n+1)}^b - R_{(n+1)}^a \circ b_n + Ad_{b_n} R_{(n+1)}^a \]
\[ + R_{(n+1)}^b (\exp(R_{(n+1)}^a) \cdot a_n) - R_{(n+1)}^b \circ a_n \]
\[ + R_{(n+1)}^a \circ b_n - R_{(n+1)}^a (\exp(R_{(n+1)}^b) \cdot b_n) + \text{Res} \]

where

\[ \| \text{Res} \|_{C^0} \leq C \| R_{(n+1)}^a, R_{(n+1)}^b \|_{C^0}^2 \leq C \varepsilon_n^{\frac{1}{2}} < \varepsilon_n^{\frac{1}{2} + \frac{k_0}{q_0 + 1}}. \]

By (7.31) and along proof line in Lemma 4.2 we have

\[ \| R_{(n+1)}^b (\exp(R_{(n+1)}^a) \cdot a_n) - R_{(n+1)}^b \circ a_n \|_{C^0} \leq C \varepsilon_n^{\frac{1}{2}} < \varepsilon_n^{\frac{1}{2} + \frac{k_0}{q_0 + 1}}, \]

similarly we also have

\[ \| R_{(n+1)}^a (\exp(R_{(n+1)}^b) \cdot b_n) - R_{(n+1)}^a \circ b_n \|_{C^0} < \varepsilon_n^{\frac{1}{2} + \frac{k_0}{q_0 + 1}}. \]

And thus if we integrate each side of (7.33), since the left side is constant and norm of integral of

\[ R_{(n+1)}^b \circ a_n - Ad_{a_n} R_{(n+1)}^b - R_{(n+1)}^a \circ b_n + Ad_{b_n} R_{(n+1)}^a \]
is bounded by $\frac{1}{k_0}$ by $(7.32)$, combine $(7.31)$ and $(7.33)$ it follows that

$$
\|X\| \leq \frac{1}{k_0} + \frac{1}{k_0} + \frac{1}{k_0} + \frac{1}{k_0} < 4\varepsilon_n \frac{1}{k_0}
$$

hence we have

$$
\|(\log(a_n), \log(b_n))\| \leq C\|X\| \leq 4C\varepsilon_n \frac{1}{k_0} < \varepsilon_n \frac{2}{k_0}.
$$

(7.36)

By assumption of $\delta_0$ and using $(7.36)$ there exist $n_1', n_2' \in \mathbb{Z}$ where $3$ is the Lie algebra of $Z(D) \cap Z(L)$ such that $[n_1', n_2'] = 0$ and

$$
\|n_1' - \log(a_n), n_2' - \log(b_n)\| \leq \|CX\|^{\delta_0 - \epsilon} \leq \varepsilon_n \frac{2}{k_0}.
$$

(7.37)

In order for the process to converge we need the power in the right-hand part of $(7.37)$ to be greater than $\frac{k_0 + 1}{k_0}$ for some $\epsilon > 0$. Since $\delta_1 = \frac{\delta_0}{2} + \frac{1}{8}$ this is equivalent to

$$
\frac{2\delta_0}{2} + \frac{\delta_0}{2} > \frac{k_0 + 1}{k_0}
$$

or

$$
\delta_0 > \frac{1}{16} \frac{2k_0}{k_0 + 1} - \frac{7}{4}
$$

which is satisfied if $k_0 > \frac{7}{4} + \frac{1}{16}$. Notice that the minimum of the right-hand side of above inequality is $\frac{1}{4}$. That is the key assumption needed to carry out the proof.

Let $\tilde{a}_n = \exp(n_1')$ and $\tilde{b}_n = \exp(n_2')$ and also let

$$
R_a^{(n+1)} = \log(\exp(R_a^{(n)})a_n(\tilde{a}_n)^{-1}),
$$

$$
R_b^{(n+1)} = \log(\exp(R_b^{(n)})b_n(\tilde{b}_n)^{-1})
$$

(7.39)

combine $(7.31)$ and $(7.37)$ it follows

$$
\|R_a^{(n+1)}, R_b^{(n+1)}\|_{C^0} \leq C\|R_a^{(n+1)}, n_1' - \log(a_n), n_2' - \log(b_n)\|_{C^0} \leq \varepsilon_n \frac{k_0 + 1}{k_0}
$$

$$
\|R_a^{(n+1)}, R_b^{(n+1)}\|_{C^1} \leq C\|R_a^{(n+1)}, R_b^{(n+1)}\|_{C^0} \leq C\varepsilon_n \frac{k_0 + 1}{k_0 + 1} \leq \varepsilon_n^{-1}.
$$

Then we get

$$
\tilde{a}^{(n+1)} = \exp(R_a^{(n+1)}) \cdot a^{(n+1)}(a)
$$

$$
\tilde{a}^{(n+1)} = \exp(R_b^{(n+1)}) \cdot a^{(n+1)}(b)
$$

(7.40)

where $a^{(n+1)}(a) = a(\tilde{a}_n)$ and $a^{(n+1)}(b) = a(\tilde{b}_n)$.

Using $(7.3)$ and $(7.29)$ we may check the $C^1$ bound for $\Omega$:

$$
\|\Omega^{(n+1)}\|_{C^1} \leq C\|R_a^{(n)}, R_b^{(n)}\|_{C^0}^{\alpha} \leq C\varepsilon_n \frac{1}{k_0} \frac{2a_0^m n^{2+2\sigma}}{\ell} < C\varepsilon_n \frac{1}{k_0} \frac{2a_0^m n^{2+2\sigma}}{\ell}
$$

providing $1 - \frac{2a_0^m n^{2+2\sigma}}{\ell} > \frac{1}{k_0}$ which is satisfied for $\ell > \frac{2a_0^m n^{2+2\sigma}}{1 - \frac{1}{k_0}}$. 

Thus for sufficiently small \( \|R\|_{C^0} \) and \( \|R\|_{C^\ell} \) the process converges to a solution \( \Omega \in C^1 \) with \( \|\Omega\|_{C^1} < \frac{1}{4} \). Hence \( \exp \Omega \) conjugates \( \tilde{\alpha} \) with a standard perturbation of \( \sigma_{D^+} \).

Since the \( C^1 \) conjugacy thus constructed is also an orbit conjugacy between the neutral foliations, it is \( C^\infty \) by [19, Theorem 1] it is \( C^\infty \). This completes the proof of our theorems.

**Remark 7.1.** The last argument is a shortcut that is available due to condition (\( B \)). However the general method of the present paper is applicable to certain cases where this condition either does not hold as in the setting of [5] or where there is no hyperbolicity to begin with as in [7, 8]. A direct application of our scheme with orthogonal projection in a fixed Sobolev space and iteration procedure described above would only produce conjugacy of finite regularity and a priori estimates are not available in those situations. However, the iterative scheme may be modified by increasing \( \ell \) along the way. The idea is to achieve closeness of conjugacy constructed on an iteration step to identity in a high norm that will allow to raise the threshold \( m_0 \) in the Main Estimate.

8. Rigid genuinely partially hyperbolic twisted examples

8.1. Preliminaries on arithmetic groups. To prove the statement of Theorem 4 we first recall some algebraic notations and theorems.

Let \( H \) be an algebraic group defined over \( \mathbb{Q} \), and we can specify an embedding \( H \hookrightarrow GL(n, \mathbb{C}) \). Then we can define \( H(\mathbb{Z}) = H(\mathbb{C}) \cap GL(n, \mathbb{Z}) \) and \( H(\mathbb{Q}) = H(\mathbb{C}) \cap GL(n, \mathbb{Q}) \). Obviously \( H(\mathbb{Z}) \) and \( H(\mathbb{Q}) \) depend on the embedding \( H \hookrightarrow GL(n, \mathbb{C}) \).

**Definition 8.1.** A subgroup \( S \) of \( H(\mathbb{Q}) \) is called an arithmetic subgroup if it is commensurable with \( H(\mathbb{Z}) \), i.e., the intersection \( S \cap H(\mathbb{Z}) \) has finite index in both \( S \) and \( H(\mathbb{Z}) \). In particular, \( H(\mathbb{Z}) \) and its subgroups of finite index are arithmetic groups.

Now we give the definition of arithmetic subgroups of Lie groups. Since a Lie group may not be equal to the real locus of an algebraic group, we need a more general definition of arithmetic subgroups.

**Definition 8.2.** Let \( L \) be a Lie group, \( \Lambda \subseteq L \) be a discrete subgroup. \( \Lambda \) is called an arithmetic subgroup of \( L \) if there exists an algebraic group \( H \) defined over \( \mathbb{Q} \) and a Lie group homomorphism \( \varphi : L \rightarrow H(\mathbb{R}) \) whose kernel is compact such that the image \( \varphi(\Lambda) \) is an arithmetic subgroup of \( H(\mathbb{Q}) \).

One has the following fundamental theorem due to Margulis:

**Theorem 7.** (Margulis’ arithmeticity theorem [23]) Suppose \( L \) is a connected semisimple Lie group without compact factors. If the rank of \( L \) is at least 2, then every irreducible lattice in \( L \) is arithmetic.
8.2. Proof of Theorem [4]. By Margulis arithmeticity theorem \( \Gamma \) is arithmetic. Hence there exists \( \varphi : G \to H(\mathbb{R}) \) whose kernel is compact such that the image \( \varphi(\Gamma) \) is an arithmetic subgroup of \( H(\mathbb{Q}) \). Since \( G \) has no compact factor, the kernel of \( \varphi \) is finite. Consider the adjoint representation of \( \varphi(\Gamma) \) on the Lie algebra of \( \varphi(G) \) which is isomorphic to \( \mathfrak{g} \). Since \( \text{Ad}_{\varphi(\Gamma)} \in SL(m, \mathbb{Q}) \) \( (m = \dim \mathfrak{g}) \) with bounded denominators, there exists a base of \( \mathfrak{g} \) such that \( \text{Ad}_{\varphi(\Gamma)} \in SL(m, \mathbb{Z}) \). Also notice the adjoint representation admits no invariant subspace of eigenvalue 1 hence \( \text{Ad} \circ \varphi \) is an genuinely partially hyperbolic representation of \( G \) on \( \mathbb{R}^m \).

If \( G \) is quasi-split then obviously \( \mathfrak{g} \) is abelian hence \( \delta_0 = \infty \). If \( \mathfrak{g} = \mathfrak{g}_0 + \mathfrak{so}(3) \) where the ideal \( \mathfrak{g}_0 \) is abelian, next we will show \( \delta_0 \geq \frac{1}{3} \) in \( \mathfrak{g} \). In fact we just need to show for any \( n_1, n_2, t_1, t_2 \in \mathfrak{so}(3) \), if \( \|[(n_1, t_1), (n_2, t_2)]\| < \gamma \) then there exist \( n'_1, n'_2, t'_1, t'_2 \in \mathfrak{so}(3) \) such that \( \|[(n'_1, t'_1), (n'_2, t'_2)]\| = 0 \) with \( d(n'_1, n_1) < C\gamma^\frac{1}{3} \) and \( d(t'_1, t_1) < C\gamma^\frac{1}{4} \) for \( i = 1, 2 \).

There exist bases of \( \mathfrak{so}(3) \) such that \( \text{ad}_{n_1} \) and \( \text{ad}_{n_2} \) have the forms \( \text{ad}_{n_i} = \begin{pmatrix} 0 & \theta_i & 0 \\ -\theta_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \) \( i = 1, 2 \) respectively. We can assume either \( |\theta_1| > \gamma^\frac{1}{3} \) or \( |\theta_2| > \gamma^\frac{1}{4} \). Otherwise there exists \( C > 0 \) such that \( \|n_i\| < C\gamma^\frac{1}{3}, i = 1, 2 \). Then let \( n'_i = 0 \) and \( t'_i = t_i, i = 1, 2 \).

Assume that \( |\theta_1| > \gamma^\frac{1}{3} \). Denote \( \text{ad}_{t_1} = \begin{pmatrix} 0 & q & b \\ -q & 0 & c \\ -b & -c & 0 \end{pmatrix}, \) \( n_2 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \) and \( t_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \) under the basis. Then by assumption we have

\[
|\theta_1 x_1| < \gamma, \quad |\theta_1 y_1| < \gamma, \quad |\theta_1 y_2 - qy_1 - bz_1| < \gamma \\
-|\theta_1 x_2 + qx_1 - cz_1| < \gamma, \quad |bx_1 + cy_1| < \gamma.
\]

Since \( |\theta_1| > \gamma^\frac{1}{3} \) then \( |x_1| < \gamma^\frac{2}{3} \) and \( |y_1| < \gamma^\frac{2}{3} \). Then it follows:

\[|d_1 = \theta_1 y_2 - bz_1| < C\gamma^\frac{2}{3} \text{ and } |d_2 = \theta_1 x_2 + cz_1| < C\gamma^\frac{2}{3} \]

Then let \( n'_1 = n_1, n'_2 = (0, 0, z_1), t'_1 = t_1 \) and \( t'_2 = (x_2 - d_2 \theta_1^{-1}, y_2 - d_1 \theta_1^{-1}, z_2) \). Notice \( ||d_2 \theta_1^{-1}, d_2 \theta_1^{-1}|| \leq C\gamma^\frac{1}{3} \) then it is easy to check they satisfy the requirement.

Hence \( \delta_0 > \frac{1}{3} \) works for all the following examples: \( G = SO(m + 3, m), SU(m+2, m) \) or \( Sp(m+1, m) \) or it’s product with any quasi-split semisimple Lie groups.

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