In the context of Finsler-Randers theory we consider, for a first time, the cosmological scenario of the varying vacuum. In particular, we assume the existence of a cosmological fluid source described by an ideal fluid and the varying vacuum terms. We determine the cosmological history of this model by performing a detailed study on the dynamics of the field equations. We determine the limit of General Relativity, while we find new eras in the cosmological history provided by the geometrodynamical terms provided by the Finsler-Randers theory.

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1. INTRODUCTION

Since the pioneering discovery of the accelerating expansion of our Universe\cite{1,2} cosmology is now in the limelight of modern science. The physical mechanism able to explain this accelerating universe is one of the greatest challenges of modern physics. Within the realm of General Relativity (GR) this acceleration is easily accommodated by introducing a dark energy sector (DE)\cite{4} characterized by negative pressure. The simplest DE model arises with the inclusion of a positive and time-independent cosmological constant, namely $\Lambda$, in the gravitational equations of GR\cite{5,6}. The resulting cosmological scenario is widely known as the $\Lambda$-cosmology and this cosmological model is in agreement with a series of observational data, however it suffers from two major problems, for details see\cite{7}.

This naturally leads to think of several alternative $\Lambda$-cosmological models\cite{4} to investigate the same issue. One of the simplest and natural generalizations of the $\Lambda$-cosmology is to introduce time dependence in the $\Lambda$ term, which leads to varying vacuum cosmologies. On the other hand, apart from the concept of DE physics, an alternative route to mimic this accelerating phase appears either due to the direct modifications of GR leading directly to modified gravitational theories\cite{8,11,12} or by introducing new gravitational theories completely different from GR, such as the teleparallel equivalent of GR (TEGR)\cite{13}.

The models arising from this latter approach are usually known as the geometric dark energy (GDE) models. Although both DE and GDE models have been widely studied and acknowledged in the literature, research over the last several years has indicated that despite a large number of models, none of them can be considered to be a completely healthy and viable model able to portray the dynamical evolution of the universe. Most notably though, the physical nature and evolution of both DE and GDE are still unknown even after substantial cosmological research. Thus, the debates in search of a perfect cosmological theory have been the central theme of modern cosmology at present times. The studies so far clearly justify that there are definitely no reasons to favor any particular cosmological theory or model, at least in light of the recent cosmological observations.

An interesting gravitational theory in the context of the present accelerating expansion is based on the introduction of Finsler geometry, which gives rise to a wider geometrical picture of the universe extending the traditional Riemannian geometry. In other words, one can recover the Riemannian geometry as a special case of the Finslerian geometry. Thus Finslerian geometry is expected to provide more insights on the dynamics and evolution of the observed universe,
and as a consequence, the cosmology in Finslerian geometry gained significant attention in the scientific community (e.g. [14–21]). In particular, the Finsler-Randers (FR) metric [22] and the induced cosmological model [23, 24] is of special interest since the field equations include an extra geometrical term that acts as a DE fluid. As we pointed out the (FR) cosmological model contains in each point two metric structures, one Riemannian and one Finslerian so it can be considered as a direction-dependent \((-y)\) motion of the Riemannian /FRW model with osculating structure.

In the present article we consider a very general dynamical picture of the universe in which a time-dependent cosmological term is present within the context of Finslerian geometry. The presence of a time dependent cosmological term, \(\Lambda(t)\) actually inherits an interaction in the cosmic sector. These kind of models are widely accepted in the literature for their ability to describe various cosmological eras. The plan of the paper is as follows.

In Section 2 we briefly discuss the FR cosmology. The varying vacuum model is described in Section 3 where we present the field equations and the models of our analysis. Section 4 includes the main material of this work. In particular we present the dynamical analysis and we determine the cosmological evolution for the models of our consideration. Finally, in Section 5 we discuss our results.

2. FINSLER-RANDERS THEORY: AN OVERVIEW

The origin of the FR model is based on the Finslerian geometry [23, 24] which is a natural generalization of the traditional Riemannian geometry and it has gained considerable attention in the cosmological community, see for instance [23–29] for more details in this direction. In what follows we describe the basics of the Finslerian geometry.

Given a differentiable manifold \(M\), the Finsler space is generated from a generating differentiable function \(F(x, y)\) on the tangent bundle \(TM\) with \(F : TM \rightarrow R\), \(TM = T(M)\backslash \{0\}\). The function \(F\) is a one degree homogeneous function with respect to the variable \(y\) which is related to \(x\), as \(y = \frac{\partial F}{\partial x}\). here \(t\) is the time variable. In the FR space-time, we have

\[
F(x, y) = \sigma(x, y) + u_\mu(x) y^\mu, \quad \sigma(x, y) = \sqrt{a_{\mu\nu} y^\mu y^\nu},
\]

where \(a_{\mu\nu}\) is a Riemannian metric and \(u_\mu = (u_0, 0, 0, 0)\) is a weak primordial vector field with \(\|u_\mu\| \ll 1\). Let us note that the vector field \(u_\mu\) intrinsically contributes to the geometry of Finslerian space-time and this vector field introduces a preferred direction in the referred space time. The vector field \(u_\mu\) additionally causes a differentiation of geodesics from a Riemannian spacetime [30]. Although, there is a case where the geodesics of Riemannian and (FR) are identical. This happens when the covector \(u_\mu = \frac{\partial \sigma(x)}{\partial x^\mu}\). Now one can write down the Finslerian metric tensor using the Hessian of \(F\) as follows

\[
f_{\mu\nu} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^\mu \partial y^\nu}.
\]

One can now derive the Cartan tensor \(C_{\mu\nu\kappa} = \frac{1}{2} \frac{\partial f_{\mu\nu}}{\partial y^\kappa}\) using the Finslerian metric tensor given above. We also note that the component \(u_0\) can be given as \(u_0 = 2C_{000}\) [23].

Let us consider the gravitational equations in the FR cosmology in order to explore the dynamics of the universe within this context. The field equations in this context are

\[
L_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right), \tag{1}
\]

where \(L_{\mu\nu}\) denotes the Finslerian Ricci Tensor (for more details see [23]); \(g_{\mu\nu} = Fa_{\mu\nu}/\sigma; T_{\mu\nu}\) is the energy momentum tensor of the matter sector and \(T\) is the trace of \(T_{\mu\nu}\).

Now, consider the Finslerian perfect fluid with velocity 4-vector field \(u_\mu\) for which the energy momentum tensor takes the form \(T_{\mu\nu} = \text{diag} (\rho, -P f_{ij})\), where \(\{\mu, \nu\} \in \{0, 1, 2, 3\}\) and \(\{i, j\} \in \{1, 2, 3\}\); \(\rho\) and \(P\) respectively denote the total energy density and pressure of the underlying cosmic fluid [31].

For the above expression of the energy-momentum tensor, in a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric\(^1\),

\[
ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right),
\]

\(^1\) Let us note that the nonzero components of the Ricci tensor in the context are: \(L_{00} = 3\left( \frac{\dot{a}}{a} + 3\frac{\ddot{a}}{a^2} \dot{u}_0 \right)\) and \(L_{ii} = -(a\ddot{a} + 2\dot{a}^2 + \frac{11}{2} a^2 \dot{u}_0) / \Delta_{ii}\) where \((\Delta_{11}, \Delta_{22}, \Delta_{33}) = (1, r^2, r^2\sin^2\theta)\).
the gravitational field equations can be explicitly written as

\[ \dot{H} + H^2 + \frac{3}{4} H Z_t = -\frac{4\pi G}{3}(\rho + 3p), \]  

(2)

\[ \dot{H} + 3H^2 + \frac{11}{4} H Z_t = 4\pi G(\rho - p), \]  

(3)

where and the overdot represents the derivative with respect to the cosmic time and \( H \equiv \dot{a}/a \), is the Hubble rate and \( Z_t = \dot{u}_0(t) \). Now, combining Eqs. (2) and (3) one arrives at

\[ H^2 + H Z_t = \frac{8\pi G}{3}\rho \]  

(4)

Additionally, using the Bianchi identities one can have the conservation equation for the total fluid which goes as

\[ \dot{\rho} + 3H (\rho + p) + Z_t \left( \rho + \frac{3}{2}p \right) = 0. \]  

(5)

One can clearly understand that the modified dynamics in an FR universe are mainly affected by the the extra term \( H Z_t \). For \( Z_t = 0 \), which appears when \( u_0 \equiv 0 \), one can recover the usual Friedmann equation.

3. VARYING VACUUM IN A FINSLER RANDERS MODEL

In the framework of General Relativity the Running Vacuum Model (RVM) has been thoroughly studied at the background and perturbation levels respectively (see [32] and references therein). Here we want to extend the situation by including in the Finsler Randers geometry the concept of RVM. Notice that the time dependence of the vacuum energy density in the RVM is only through the Hubble rate, hence \( \dot{\rho}_\Lambda \neq 0 \).

Let us assume that we have a mixture of two fluids, namely, matter (labeled with the symbol \( m \)) and the varying vacuum (labeled with \( \Lambda \)), hence the total energy density and pressure of the total fluid are given by

\[ \rho = \rho_m + \rho_\Lambda, \quad p = p_m + p_\Lambda. \]  

(6)

The complete set of field equations reads

\[ 1 + \frac{Z_t}{H} = \frac{8\pi G}{3H^2}\rho_m + \frac{8\pi G}{3H^2}\rho_\Lambda \]  

(7)

\[ \frac{\dot{H}}{H^2} + 1 + \frac{3Z_t}{4H} = -\frac{4\pi G}{3H^2}(\rho_m + 3w_m\rho_m - 2\rho_\Lambda), \]  

(8)

where through the Bianchi equations the continuity equations become,

\[ \dot{\rho}_m + 3H(1 + w_m)\rho_m + Z_t \left( \rho + \frac{3}{2}p \right) = \dot{Q}, \]  

(9)

\[ \dot{\rho}_\Lambda = -\dot{Q}, \]  

(10)

where \( w_m = p_m/\rho_m \), is the equation-of-state parameter of the matter fluid and the term \( \dot{Q} \) appearing in (9) and (12) refers to the interaction rate between the matter and vacuum sectors. As one can quickly note that \( \dot{Q} = 0 \) actually recovers the usual non-interacting dynamics. It is easy to realize that the presence of interaction between these sectors certainly generalizes the cosmic dynamics and it is of utmost importance to address many cosmological puzzles. Due to the diverse characteristics, interacting models have gained a massive attention to the cosmological community because. The mechanism of an interaction in the dark sector of the universe is a potential route that may explain the cosmic coincidence problem [33–38] and provide a varying cosmological constant that could explain the tiny value of the cosmological constant leading to a possible solution to the cosmological constant problem [39]. In the past years, a cluster of interaction models have been studied by many researchers. Some of the interaction
models existing in the literature are [40–49] while some cosmological constraints on interacting models can be found in [50–69]. On the other hand, this model can also be seen as the particle creation model which has gained massive attention in the scientific society [70–78]. In this work we aim to study the generic evolution of the solution of the field equations for specific functional forms of the interaction term $\dot{Q}$. In the following we replace the interaction term $\dot{Q}$ with $Q = \dot{Q} - Z_t (\rho + \frac{4}{3} p)$ such that to remove the nonlinear term and rewrite the continuous equation in the friendly form

$$\dot{\rho}_m + 3H(1 + w_m)\rho_m = Q,$$

$$\rho_\Lambda = -Q,$$  (11, 12)

Following our previous works [79, 80] we study how the implementation of the Finsler geometry affects the varying vacuum scenarios studied in GR as well as how the implementation of the varying vacuum responds in a Finsler Randers scenario.

4. DYNAMICAL EVOLUTION

In this Section, we study the cosmological evolution of the different cosmological scenarios of varying vacuum in a FR geometrical background by using methods of the dynamical analysis [81, 82]. Specifically, we study the critical points of the field equations in order to identify the different cosmological eras that are accommodated by each scenario. The respective stability of these cosmological eras is determined by calculating the eigenvalues of the linearized system at the specific critical points.

In order to perform such an analysis we define proper dimensionless variables such that to rewrite the field equations as a set of algebraic-differential equations. The critical points of the system are considered to be the sets of variables for which all the ODE of the system are zero. These sets of variables correspond to a specific solution of the system and each to a different era of the cosmos that may be able to describe the observed universe. The eigenvalues of the above points are defining the stability of the critical points. Namely a critical point is stable/attractor when the corresponding eigenvalues have negative real parts. Thus, the eigenvalues are valuable tools that characterize the behavior of the dynamical system around the critical point [83].

Our approach is as follows. We consider a dynamical system of any dimension

$$\dot{x}^A = f^A (x^B),$$

and then a critical point of the system $P = P (x^B)$ which has to satisfy $f^A (P) = 0$. The linearized system around $P$ is written as

$$\delta \dot{x}^A = J^A_B \delta x^B, \quad J^A_B = \frac{\partial f^A (P)}{\partial x^B}.$$

where $J^A_B$ is the respective Jacobian matrix. We calculate the eigenvalues and eigenvectors and then express the general solution at the respective points as their expression. Since the linearized solutions are expressed in terms of the eigenvalues $\lambda_i$ and thus as functions of $e^{\lambda_i t}$, apparently when all these terms have their real part negative the respective solution of the critical point is stable and the point is an attractor, otherwise the point is a source.

Such an analysis is very useful in terms of defining viable theories that can describe the observable universe. Thus for a healthy theory to be viable, the critical point analysis should provide points where the universe will be accelerating and also these points to be stable. This analysis has been applied in various cosmological models, for instance see and references therein [79, 80, 84–96].

4.1. Dimensionless variables

We select to work in the $H$–normalization. Therefore, we define the dimensionless variables [81, 82]

$$\Omega_m = \frac{\rho_m}{3H^2}, \quad \Omega_z = \frac{Z_t}{H}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{3H^2}.$$  (13)

Thus, the first Friedmann equation gives the constraint equation

$$1 + \Omega_z = \Omega_m + \Omega_\Lambda$$  (14)
while the rest of the field equations are written as follows

\[
\frac{d\Omega_A}{d\ln a} = 2\Omega_A \left[ 1 + \frac{3}{4}(\Omega_m + \Omega_A - 1) + \frac{1}{2} \Omega_m(1 + 3w_m) - \Omega_A \right] - \frac{Q}{3H^3}, \tag{15}
\]

\[
\frac{d\Omega_m}{d\ln a} = 2\Omega_m \left[ 1 + \frac{3}{4}(\Omega_m + \Omega_A - 1) + \frac{1}{2} \Omega_m(1 + 3w_m) - \Omega_A \right] + \frac{Q}{3H^3} - 3\Omega_m(1 + w_m),
\]

where \( p_m = w_m \rho_m \). In the following we assume that \( w_m \in (-1, 1) \).

We proceed by determining the critical points of the dynamical system. Every point \( P \) has coordinates \( P = (\Omega_m, \Omega_A, \Omega_z) \), and describes a specific cosmological solution. For every point we determine the physical cosmological variables as well as the equation of the state parameter \( w_{tot} (P) \). In order to determine the stability of each critical point the eigenvalues of the linearized system around the critical point \( P \) are derived.

We remark that the second Friedmann equation with the use of the dimensionless variables reads

\[
\frac{\dot{H}}{H^2} = -1 - \frac{3}{4} \Omega_z - \frac{1}{2} \Omega_m(1 + 3w_m) + \Omega_A,
\]

from where we find that at a stationary point \( P \), the equation of state parameter for the effective fluid is

\[
w_{tot} (P) = -\frac{1}{3} + \frac{2}{3} \left( \frac{3}{4} \Omega_z + \frac{1}{2} \Omega_m(1 + 3w_m) - \Omega_A \right).
\]

In this work we study various functional forms for the interaction term \( Q \). In order to extend the results of \[79\], we assume that (A) the interaction term \( Q \) is proportional to the density of dark matter \[50\], that is, \( Q_A = 9nH\rho_m \) or equivalent \( Q_A \approx 9nH^3\Omega_m \), where the dimensionless parameter \( n \) is an indicator of the interaction strength; (B) \( Q \) is proportional to the density of the dark energy term, i.e. \( Q_B = 9nH^3\rho_A \) \[47, 51\]; (C) \( Q_C \) is proportional to the sum of the energy density of the dark sector of the universe, that gives \( Q_C = 9nH(\rho_A + \rho_m) \).

Motivated by the above functional forms \( Q \), which have given interesting results in General Relativity, \[79\], we propose some new interaction terms which are proportional to the energy density \( \Omega_z \). In particular we select the models (D) \( Q_D = 9nH^3\Omega_z \); (E) \( Q_E = 9nH^3\Omega_z + 9mH\rho_m \) and (D) \( Q_F = 9nH^3\Omega_z + 9mH\rho_m \). In these models \( m \) is a dimensionless parameter, an indicator of the interaction strength. Finally, in order to compare our results with the non-varying vacuum model we investigate the case where \( Q_G = -3\Omega_z \rho_m H^3 (1 + \frac{3}{2}w_m) \).

4.2. **Model A**: \( Q_A = 9nH\rho_m \)

For the first model that we consider \( Q_A = 9nH\rho_m \), the field equations are expressed as follows.

\[
\frac{d\Omega_A}{d\ln a} = 2\Omega_A \left[ 1 + \frac{3}{4}(\Omega_m + \Omega_A - 1) + \frac{1}{2} \Omega_m(1 + 3w_m) - \Omega_A \right] - 3n\Omega_m, \tag{18}
\]

\[
\frac{d\Omega_m}{d\ln a} = 2\Omega_m(1 + \frac{3}{4}(\Omega_m + \Omega_A - 1) + \frac{1}{2} \Omega_m(1 + 3w_m) - \Omega_A) + 3n\Omega_m - 3\Omega_m(1 + w_m) \tag{19}
\]

The dynamical system \[18\], \[19\] admits three critical points with coordinates \( \{\Omega_m, \Omega_A, \Omega_z\} \)

\[
A_1 = \{0, 0, -1\}, \quad A_2 = \{0, 1, 0\}, \quad A_3 = \{1 - \frac{n}{1 + \frac{n}{1 + w_m}}, \frac{n}{1 + w_m}, 0\}.
\]

Point \( A_1 \) always exists and describes an empty universe with equation of state parameter \( w_{tot} (A_1) = -\frac{3}{2} \). The universe accelerates with the contribution of the extra term introduced due to the Finsler-Randers Geometry. The eigenvalues of the linearized system near to point \( A_1 \) are \( \{ \frac{3}{2}, -\frac{3}{2} - 3(w_m - n) \} \), from where we can infer that the point is always a source, since one of the eigenvalues is always positive.

Point \( A_2 \) describes a de Sitter universe with equation of state parameter \( w_{tot} (A_2) = -1 \), where only the \( \Lambda \) term contributes in the evolution of the universe. The eigenvalues are derived to be \( \{-\frac{1}{2}, -3(w_m - n + 1)\} \), from where
FIG. 1: Phase space diagram for the dynamical system (18), (19). We consider $w_m = 0$, for $n < 1$. The unique attractor is the de Sitter point $A_2$.

TABLE I: Stationary points and physical parameters for the interaction model A.

| Point | $(\Omega_m, \Omega_\Lambda, \Omega_z)$ | Existence | $w_{tot}$ | Acceleration |
|-------|---------------------------------|-----------|------------|--------------|
| $A_1$ | $(0, 0, -1)$                    | Always    | $-\frac{1}{3}$ | Yes          |
| $A_2$ | $(0, 1, 0)$                     | Always    | $-1$       | Yes          |
| $A_3$ | $(1 - \frac{n}{1 + w_m}, \frac{n}{1 + w_m}, 0)$ | $w_m \neq -1, (n = 0, w_m > -1)$ or $(n > 0, w_m > -1 + n)$ | $w_m - n$ | $w_m \leq n - \frac{4}{3}$ |

we can infer that the point is an attractor when $w_m \geq n - 1$ or equivalently $n \leq 1 + w_m$. Because $n$ is the strength of the interaction of the varying vacuum and matter we assume this term to be close to zero (either positive or negative) and thus understand that the aforementioned condition is satisfied (we generally have that $w_m > -1$). Thus this point is of great physical interest.

Point $A_3$ exists for $n \geq 0$ (for $n < 0$ then $w_m < -1$ and it exists in the phantom region) and describes a universe dominated by the varying vacuum and the matter fluid; in the case where $w_m = 0$, point $A_3$ describes the $\Lambda$-CDM universe in the FR theory. The equation of state parameter is derived $w_{tot}(A_3) = w_m - n$, from where we conclude that the exact solution at the point describes an accelerated universe when $w_m \leq n + \frac{1}{3}$. The eigenvalues of the linearized system are $\{3(w_m - n + 1), 3(w_m - n + \frac{4}{3})\}$ and thus can be stable for $w_m < n - 1$.

The above results are summarized in Tables I and II. In addition in the Figs. 13-14 we present the evolution of the trajectories for the dynamical system of our study.

4.3. Model B: $Q_B = 9nH\rho_\Lambda$

For the second model of our analysis, where $Q_B = 9nH\rho_\Lambda$, the field equations become

TABLE II: Stationary points and stability conditions for the interaction model A.

| Point | Eigenvalues | Stability |
|-------|-------------|-----------|
| $A_1$ | $\{\frac{1}{3}, -\frac{1}{3} - 3(w_m - n)\}$ | Source |
| $A_2$ | $\{-\frac{1}{3}, -3(w_m - n + 1)\}$ | $w_m \geq n - 1$ |
| $A_3$ | $\{3(w_m - n + 1), 3(w_m - n + \frac{4}{3})\}$ | $w_m < n - 1$ |
FIG. 2: Evolution diagrams with time, for various energy densities of the dynamical system \[18], [19]. We consider the initial conditions (a) \( \Omega_m = 0.4, \Omega_\Lambda = 0.1 \) (b) \( \Omega_m = 0.7, \Omega_\Lambda = 0.1 \) (c) \( \Omega_m = 0.5, \Omega_\Lambda = 0.2 \) (d) \( \Omega_m = 0.2, \Omega_\Lambda = 0.3 \) (e) \( \Omega_m = 0.1, \Omega_\Lambda = 0.9 \) (f) \( \Omega_m = 0.2, \Omega_\Lambda = 0.3 \), for \( n < 1 \) and \( w_m = 0 \).

\[
\frac{d\Omega_\Lambda}{d\ln a} = 2\Omega_\Lambda \left[ 1 + \frac{3}{4}(\Omega_m + \Omega_\Lambda - 1) + \frac{1}{2}\Omega_m(1 + 3w_m) - \Omega_\Lambda \right] - 3n\Omega_\Lambda, \tag{20}
\]
FIG. 3: Phase space diagram for the dynamical system (20), (21). We consider \( w_m = 0 \), for \( n < 1 \). The unique attractor is the de Sitter point \( B_3 \).

### TABLE III: Stationary points and physical parameters for the interaction model B.

| Point | \((\Omega_m, \Omega_\Lambda, \Omega_z)\) | Existence | \(w_{tot}\) | Acceleration |
|-------|-----------------------------------|-----------|----------|-------------|
| \(B_1\) | \((0, 0, -1)\) | Always | \(-\frac{5}{6}\) | Yes |
| \(B_2\) | \((1, 0, 0)\) | Always | \(w_m\) | \(w_m \leq -\frac{1}{3}\) |
| \(B_3\) | \(\left(\frac{n}{1+w_m}, -\frac{n}{1+w_m}, 0\right)\) | \(w_m \neq -1, (1+w_m) \geq n \geq 0 -1+n\) | \(n \leq \frac{2}{3}\) |

\[
\frac{d\Omega_m}{d\ln a} = 2\Omega_m \left(1 + \frac{3}{4}(\Omega_m + \Omega_\Lambda - 1) + \frac{1}{2}\Omega_m(1+3w_m) - \Omega_\Lambda\right) + 3n\Omega_\Lambda - 3\Omega_m(1 + w_m),
\]

(21)

The dynamical system (20), (21) admits three critical points with coordinates

\[
B_1 = \{0, 0, -1\}, \ B_2 = \{1, 0, 0\}, \ B_3 = \left\{\frac{n}{1+w_m}, \frac{n}{1+w_m}, 0\right\}.
\]

Point \(B_1\) exists always and it corresponds to an empty universe with equation of state parameter \(w_{tot}(B_1) = -\frac{5}{6}\), that is accelerating due to the contribution of the extra term introduced by the Finsler-Randers geometrical background. The eigenvalues of the linearized system are \(\left\{\frac{1-6n}{2}, -\frac{5+6n}{2}\right\}\); hence the exact solution at the stationary point \(B_1\) it is stable when \(n > \frac{1}{3}\) and \(w_m > -\frac{1}{3}\). Thus this point is of great physical interest since it can describe a past or future acceleration phase.

Point \(B_2\) describes a universe dominated by matter, \(w_{tot}(B_2) = w_m\), and the exact solution at the point corresponds to an accelerated universe when \(w_m < -\frac{1}{3}\). The eigenvalues of the linearized system are derived to be \(\left\{3(w_m - n + 1), \frac{5+6n}{2}\right\}\), from where we observe that \(B_3\) is an attractor when \(n \leq \frac{1}{3}\) & \(w_m < n - 1\) or \(n > \frac{1}{3}\) & \(w_m < -\frac{5}{6}\).

Point \(B_3\) exists when \(w_m \neq -1, (1+w_m) \geq n \geq 0\) and it has the same physical properties with point \(A_3\). The eigenvalues of the linearized system near the stationary point are derived to be \(\left\{\frac{6n-1}{2}, -3(w_m - n + 1)\right\}\), from where we infer that the exact solution at \(B_3\) is stable for \(n < \frac{1}{3}\) & \(w_m > -1+n\).

The above results are summarized in Tables III and IV. In Figs. 3, 4 the evolution of trajectories for the dynamical system our study in phase space are presented.

4.4. **Model C:** \(Q_C = 9nH(\rho_\Lambda + \rho_m)\)

For the third model of our analysis, where \(Q_C = 9nH(\rho_\Lambda + \rho_m)\), the field equations read
FIG. 4: Evolution diagrams with time, for various energy densities of the dynamical system \((20), (21)\). We consider the initial conditions (a) \(\Omega_m = 0.4, \Omega_\Lambda = 0.1\) (b) \(\Omega_m = 0.7, \Omega_\Lambda = 0.1\) (c) \(\Omega_m = 0.5, \Omega_\Lambda = 0.2\) (d) \(\Omega_m = 0.2, \Omega_\Lambda = 0.3\) (e) \(\Omega_m = 0.1, \Omega_\Lambda = 0.9\) (f) \(\Omega_m = 0.2, \Omega_\Lambda = 0.3\), for \(n < 1\) and \(w_m = 0\).

\[ \frac{d\Omega_\Lambda}{d \ln a} = 2\Omega_\Lambda \left[ 1 + \frac{3}{4}(\Omega_m + \Omega_\Lambda - 1) + \frac{1}{2}\Omega_m(1 + 3w_m) - \Omega_\Lambda \right] - 3n\Omega_\Lambda - 3n\Omega_m, \]  

(22)
The dynamical system our study in phase space are presented.

With points linearized system around the critical points we determine the eigenvalues \( \{ \lambda_n \} \) where we considered (11).

\[
\begin{align*}
B_1 & \quad \{ \frac{1}{2}, -\frac{1}{2} \}, \quad n > \frac{1}{2} \text{ & } w_m > -\frac{5}{4} \\
B_2 & \quad \{ 3(w_m - n + 1), \frac{5+4w_m}{2} \}, \quad n \leq \frac{1}{2} \land w_m < n - 1 \text{ or } n > \frac{1}{2} \land w_m < -\frac{5}{4} \\
B_3 & \quad \{ \frac{6n-1}{2}, -3(w_m - n + 1) \}, \quad n < \frac{1}{2} \land w_m > -1 + n 
\end{align*}
\]

The above results are summarized in Tables V, VI and VII. Moreover, in Figs. 5, 6 the evolution of trajectories for the latter dynamical system admits the following critical points

\[
\begin{align*}
C_1 & \quad \{ 0, 0, -1 \}, \quad C_{2\pm} = \left\{ \frac{1}{2} \left( 1 \pm \sqrt{\frac{x}{1+w_m}} \right), \frac{1}{2} \left( 1 \mp \sqrt{\frac{x}{1+w_m}} \right), 0 \right\}, \\
& \quad \text{Existence } w_m \neq 0, n < 0 \land w_m \leq 4n - 1 \quad \frac{1}{2} \left( w_m - 1 \pm \sqrt{(1+w_m)x} \right), \\
& \quad \text{Acceleration } \text{Yes see [VI].}
\end{align*}
\]

\[
\frac{d\Omega_m}{d\ln a} = 2\Omega_m \left( 1 + \frac{3}{4}(\Omega_m + \Omega - 1) + \frac{1}{2}\Omega_m(1+3w_m) - \Omega \right) + 3n\Omega_A + 3n\Omega_m - 3\Omega_m(1+w_m), \quad (23)
\]

The latter dynamical system admits the following critical points

\[
C_1 = \{ 0, 0, -1 \}, \quad C_{2\pm} = \left\{ \frac{1}{2} \left( 1 \pm \sqrt{\frac{x}{1+w_m}} \right), \frac{1}{2} \left( 1 \mp \sqrt{\frac{x}{1+w_m}} \right), 0 \right\},
\]

where we considered \( x = 1 - 4n + w_m \).

The universe described by the exact solution at the stationary point \( C_1 \) has the same physical quantities with those of points \( A_1 \) and \( B_1 \). The eigenvalues of the linearized system are

\[
\begin{align*}
e_1 (C_1) & = -\frac{1}{2} \left( 2 + 3w_m + 3\sqrt{(1+w_m)x} \right) \\
e_2 (C_1) & = -\frac{1}{2} \left( 2 + 3w_m - 3\sqrt{(1+w_m)x} \right)
\end{align*}
\]

from where we can infer that the point is an attractor when \( \left\{ \frac{1}{12} < n < \frac{1}{2}, -\frac{3}{4} < w < -1 + 4n \right\}, \left\{ n > \frac{1}{2}, -\frac{2}{3} < w < 1 \right\}, \left\{ \frac{1}{12} < n < \frac{11}{12}, -1 + 4n < w < \frac{5-36\sqrt{3}}{36n-6} \right\}, \left\{ \frac{11}{12} < n < \frac{1}{2}, -1 + 4n < w < 1 \right\}. \]

Point \( C_{2\pm} \) exists for \( \{ n < 0 \land w_m \leq 4n - 1 \} \) or \( \{ n > 0 \land w_m \geq 4n - 1 \} \) and describes the same physical solutions with points \( A_3 \) and \( B_3 \). The equation of state parameter is \( w_m (C_{2\pm}) = \frac{1}{2} \left( -1 + w_m \pm \sqrt{(1+w_m)x} \right) \). From the linearized system around the critical points we determine the eigenvalues

\[
\begin{align*}
e_1 (C_{2\pm}) & = \frac{1}{4} \left( 2 + 3w_m \pm 9\sqrt{(1+w_m)x} \right) + \frac{1}{4} \sqrt{13 - 36n(1+w_m) \pm 12\sqrt{(1+w_m)x} + 6w_m(5+3w_m \mp 3\sqrt{(1+w_m)x})}, \\
e_2 (C_{2\pm}) & = \frac{1}{4} \left( 2 + 3w_m \pm 9\sqrt{(1+w_m)x} \right) - \frac{1}{4} \sqrt{13 - 36n(1+w_m) \pm 12\sqrt{(1+w_m)x} + 6w_m(5+3w_m \mp 3\sqrt{(1+w_m)x})}.
\end{align*}
\]

Therefore, point \( C_{2\pm} \) is always unstable while point \( C_{2\mp} \) is conditionally stable as shown in [VI].

The above results are summarized in Tables [VII] and [VII]. Moreover, in Figs. [VI] [VI] the evolution of trajectories for the dynamical system our study in phase space are presented.
FIG. 5: Phase space diagram for the dynamical systems (22), (23). We consider $w_m = 0$, for $n < 1$. The unique attractor is point $C_2$.

TABLE VI: Stationary points and stability conditions for the interaction model C.

| Point  | Stability                                                                 |
|--------|---------------------------------------------------------------------------|
| $C_1$  | $\left\{ \frac{1}{3} < n < \frac{1}{2}, -\frac{4}{3} < w < -1 + 4n \right\}$, $\left\{ n > \frac{1}{3}, -\frac{4}{3} < w < -1 \right\}$ |
| $C_2$  | Unstable                                                                 |
| $C_{2+}$ | $n < 0 : w_m < -1 + 4n$, $-1 < w_m < \frac{5 - 36n}{36n - 6}$, $\frac{5 - 36n}{36n - 6} < w_m < -\frac{2}{3}$ |
|        | $0 < n < \frac{1}{12} : w_m < -1$, $-1 + 4n < w_m < \frac{5 - 36n}{36n - 6}$, $\frac{5 - 36n}{36n - 6} < w_m < -\frac{2}{3}$ |
|        | $\frac{1}{12} < n < \frac{1}{2} : w_m < -1$                              |
|        | $n > \frac{1}{12} : w_m = \frac{5 - 36n}{36n - 6}$                       |
|        | $n < \frac{1}{12} : w_m = \frac{5 - 36n}{36n - 6}$                       |
|        | $-\frac{2}{3} \leq w_m < 1$, $n < \frac{5 - 36n}{40n + 4}$               |

4.5. Model D: $Q_D = 9nH^3\Omega_z$

In this scenario we shall consider an interaction of the form, $Q = Q(\Omega_z)$, that of course due to the constraint equation (14) means that

$$Q = Q(\Omega_m, \Omega_\Lambda)$$

So, if we consider the interaction term to be $Q = 9nH^3\Omega_z$, then it follows

$$Q = 9nH^3(\Omega_m + \Omega_\Lambda - 1).$$

and our system is now

TABLE VII: Acceleration conditions for the interaction model C for point $C_2$.

| Point  | Acceleration                                                                 |
|--------|-----------------------------------------------------------------------------|
| $C_{2\pm}$ | $n = 0$                                                                     |
|        | $n > 0 \& w_m \leq -1$ or $n < \frac{1}{3}$ and $4n - 1 \leq w_m$            |
|        | $\frac{2}{3} > n \geq \frac{1}{3}$ and $w_m > \frac{1}{3n - 1}$             |
|        | $n < 0$ and $[4n - 1 \geq w_m \text{ or } w_m \geq -1]$                     |
FIG. 6: Evolution diagrams with time, for the various densities for the dynamical system (22), (23). We consider the initial conditions (a) $\Omega_m = 0.4$, $\Omega_\Lambda = 0.1$ (b) $\Omega_m = 0.7$, $\Omega_\Lambda = 0.1$ (c) $\Omega_m = 0.5$, $\Omega_\Lambda = 0.2$ (d) $\Omega_m = 0.2$, $\Omega_\Lambda = 0.3$ (e) $\Omega_m = 0.1$, $\Omega_\Lambda = 0.9$ (f) $\Omega_m = 0.2$, $\Omega_\Lambda = 0.3$, for $n < 1$ and $w_m = 0$.

\[
\frac{d\Omega_\Lambda}{d \ln a} = 2\Omega_\Lambda \left[ 1 + \frac{3}{4}(\Omega_m + \Omega_\Lambda - 1) + \frac{1}{2}\Omega_m(1 + 3w_m) - \Omega_\Lambda \right] - 3n(\Omega_m + \Omega_\Lambda - 1),
\]  

(24)
FIG. 7: Phase space diagram for the dynamical system (24), (25). We consider $\Omega_m = 0$, $w_m = 0$, for $n < 1$. The unique attractor is the point $D_1$.

**TABLE VIII: Stationary points and physical parameters for the interaction model D.**

| Point | $(\Omega_m, \Omega_\Lambda, \Omega_z)$ | Existence | $w_{\text{tot}}$ | Acceleration |
|-------|-------------------------------------|-----------|-----------------|--------------|
| $D_1$ | $(0, 1, 0)$                         | Always    | $-1$            | Yes          |
| $D_2$ | $(1, 0, 0)$                         | Always    | $w_m$           | $w_m \leq -\frac{1}{3}$ |
| $D_3$ | $(6n\alpha, 6n\alpha(5 + 6w_m), (5 + 6w_m)\alpha)$ | $n < 0, w_m > -\frac{5}{6}$ or $n > \frac{5}{6}, 0 < w_m + \frac{5}{6} < n$ | $-\frac{5}{6}$ | Yes |

$$\frac{d\Omega_m}{d\ln a} = 2\Omega_m \left[1 + \frac{3}{4}(\Omega_m + \Omega_\Lambda - 1) + \frac{1}{2} \Omega_m(1 + 3w_m) - \Omega_\Lambda \right] + 3n(\Omega_m + \Omega_\Lambda - 1) - 3\Omega_m(1 + w_m), \quad (25)$$

The dynamical system (24), (25) admits three critical points with coordinates:

- $D_1 = (0, 1, 0)$, $D_2 = (1, 0, 0)$, $D_3 = \{6n\alpha, 6n\alpha(5 + 6w_m), (5 + 6w_m)\alpha\}$

where $\alpha = (36n(1 + w_m) - (5 + 6w_m))^{-1}$. Point $D_1$ describes a de Sitter universe with equation of state parameter $w_{\text{tot}}(D_1) = -1$, where only the varying vacuum term contributes in the evolution of the universe. The eigenvalues are derived to be $\{-\frac{1}{2}, -3(1 + w_m)\}$, so for $w_m \geq -1$ the point is always an attractor and this point is of great physical interest.

Point $D_2$ describes a universe dominated by matter, $w_{\text{tot}}(D_2) = w_m$, and the exact solution at the point corresponds to an accelerated universe when $w_m \leq -\frac{1}{3}$. The eigenvalues of the linearized system are $\left\{3(1 + w_m), \frac{(5 + 6w_m)}{2} \right\}$, from where we observe that this point is an attractor only when $w_m < -1$.

Point $D_3$ exists when $n < 0, w_m > -\frac{5}{6}$ or $n > \frac{5}{6}, 0 < w_m + \frac{5}{6} < n$ and it corresponds to a universe of two fluids and the contribution of the geometrical background of Finsler Randers that is always accelerating, that is, $w_{\text{tot}}(D_3) = -\frac{5}{6}$. Given that we consider the values of $n$ very small this solution describes a universe where matter decays in vacuum. The eigenvalues of the linearized system near the stationary point are $\left\{\frac{1}{2}, -\frac{(5 + 6w_m)}{2} \right\}$, so point $D_3$ is always a source, since one of the eigenvalues has always positive real part.

The above results are summarized in Tables VIII and IX. In Figs. 7, 8 the evolution of real trajectories for the dynamical system our study in phase space are presented.

### 4.6. Model E: $Q_E = 9nH^3\Omega_\Lambda + 9mH\rho_m$

Our system is now
FIG. 8: Evolution diagrams with time, for various energy densities of the dynamical system [24], [25]. We consider the initial conditions (a) \( \Omega_m = 0.4, \Omega_\Lambda = 0.1 \) (b) \( \Omega_m = 0.7, \Omega_\Lambda = 0.1 \) (c) \( \Omega_m = 0.5, \Omega_\Lambda = 0.2 \) (d) \( \Omega_m = 0.2, \Omega_\Lambda = 0.3 \) (e) \( \Omega_m = 0.1, \Omega_\Lambda = 0.9 \) (f) \( \Omega_m = 0.2, \Omega_\Lambda = 0.3 \), for \( n < 1 \) and \( w_m = 0 \).

\[
\frac{d\Omega_\Lambda}{d\ln a} = 2\Omega_\Lambda \left[ 1 + \frac{3}{4}(\Omega_m + \Omega_\Lambda - 1) + \frac{1}{2}\Omega_m (1 + 3w_m) - \Omega_\Lambda \right] - 3n(\Omega_m + \Omega_\Lambda - 1) - 3m\Omega_m,
\]  

(26)
TABLE IX: Stationary points and stability conditions for the interaction model D.

| Point | Eigenvalues                                      | Stability |
|-------|--------------------------------------------------|-----------|
| $D_1$ | $\{-\frac{1}{2}, -3(1 + w_m)\}$                | $w_m > -1$|
| $D_2$ | $\{3(1 + w_m), \frac{5 + 6w_m}{2}\}$            | $w_m < -1$|
| $D_3$ | $\left\{\frac{1}{2}, \frac{5 + 6w_m}{2}\right\}$ | Source    |

FIG. 9: Phase space diagram for the dynamical system (26), (27). We consider $w_m = 0$, for $n < 1$. The unique attractor is point $E_1$.

\[
\frac{d\Omega_m}{d\ln a} = 2\Omega_m \left[ 1 + \frac{3}{4} (\Omega_m + \Omega_\Lambda - 1) + \frac{1}{2} \Omega_m (1 + 3w_m) - \Omega_\Lambda \right] \\
+ 3n(\Omega_m + \Omega_\Lambda - 1) + 3m\Omega_m - 3\Omega_m (1 + w_m),
\]

(27)

The dynamical system (26), (27), admits three critical points with coordinates

$E_1 = \{0, 1, 0\}$, $E_2 = \{1, 0, 0\}$, $E_3 = \{6nb, 6nb(5 + 6w_m), (5 + 6w_m - 6b)b\}$.

where $b = (36n(1 + w_m) - (5 + 6w_m) + 6m)^{-1}$. Point $E_1$ describes a de Sitter universe with equation of state parameter $w_{\text{tot}}(E_1) = -1$, where only the varying vacuum term contributes in the evolution of the universe. The eigenvalues are derived to be $\{-\frac{1}{2}, -3(1 + w_m - m)\}$, so for $w_m > m - 1$ the point is always an attractor and thus this solution is of great physical interest.

Point $E_2$ describes a universe dominated by the varying vacuum and matter; when $w_m = 0$, point $E_2$ describes the Λ-CDM universe in the FR theory. The equation of state parameter is derived $w_{\text{tot}}(E_2) = w_m - m$, so this point describes an accelerated universe when $w_m \leq m - \frac{1}{2}$. The eigenvalues of the linearized system are \( \left\{3(1 + w_m - m), \frac{(5 + 6w_m - 6m)}{2}\right\} \) from where we can infer that the point is stable for $w_m < m - 1$.Given though the existence condition $m - 1 \leq w_m$ we consider the point to be unstable.

Point $E_3$ exists when $n$, $m$, and $w_m$ are constrained as presented in Table XII. Similarly with point $D_3$ this point corresponds to a universe of two fluids and the contribution of the geometrical background of Finsler Randers that is always accelerating($w_{\text{tot}}(E_3) = -\frac{5}{2}$). The eigenvalues of the linearized system near the stationary point are $\left\{\frac{1}{2}, -\frac{(5 + 6w_m)}{2} + 3m\right\}$, so point $E_3$ is always a source.

The above results are summarized in Tables X and XI. The trajectories of the dynamical system in the phase space are presented in Figs. 9, 10.
FIG. 10: Evolution diagrams with time, for various energy densities of the dynamical system \(26, 27\). We consider the initial conditions (a) \(\Omega_m = 0.4, \Omega_\Lambda = 0.1\) (b) \(\Omega_m = 0.7, \Omega_\Lambda = 0.1\) (c) \(\Omega_m = 0.5, \Omega_\Lambda = 0.2\) (d) \(\Omega_m = 0.2, \Omega_\Lambda = 0.3\) (e) \(\Omega_m = 0.1, \Omega_\Lambda = 0.9\) (f) \(\Omega_m = 0.2, \Omega_\Lambda = 0.3\), for \(n < 1\) and \(w_m = 0\).

| Point | \((\Omega_m, \Omega_\Lambda, \Omega_z)\) | Existence | \(w_{tot}\) | Acceleration |
|-------|--------------------------------|-----------|-------------|--------------|
| \(E_1\) | \((0, 1, 0)\) | Always | \(-1\) | Yes |
| \(E_2\) | \(1 - \frac{m}{1 + w_m}, 1 + \frac{m}{1 + w_m}, 0\) | \(w_m > -1, 0 \leq m \leq 1 + w_m\) | \(w_{tot} = m - \frac{1}{3}\) |
| \(E_3\) | \((6nb, 6nb(5 + 6w_m), (5 + 6w_m - 6m)b)\) | See Table XII | \(-\frac{3}{5}\) | Yes |
two fluids and the contribution of the geometrical background of Finsler Randers that is always accelerating.

The evolution of trajectories for the dynamical system our study in phase space are presented.

where we can infer that the point is stable for

\[ m < -\frac{5}{6} \]

an accelerated universe when

\[ \Lambda - \text{CDM universe in the FR theory.} \]

The equation of state parameter is derived

\[ \frac{\Omega_m + \alpha}{\Omega_m + \beta} = 0, \]

\[ \Omega_m + \beta = 0, \]

\[ \frac{\Omega_m + \alpha}{\Omega_m + \beta} = 0. \]

The existence conditions of point \( E_3 \) are

\[ 0 < m \leq \frac{1}{2} \]

and

\[ m > \frac{1}{2} \]

and

\[ n > 0 \]

and

\[ 0 < 5 + 6w_m \leq 6m + 6n. \]

The dynamical system (28), (29) admits three critical points with coordinates

\[ F_1 = \{0, 1, 0\}, \]

\[ F_2 = \{0, 1, 0\}, \]

\[ F_3 = \{0, 1, 0\}. \]

where

\[ c = (36n(1 + w_m) + (5 + 6w_m)(6m - 1))^{-1}. \]

Point \( F_1 \) describes a universe dominated by matter, \( w_{\text{tot}} (F_1) = w_m \), and the exact solution at the point corresponds to an accelerated universe for \( w_m \leq -\frac{1}{3} \). The eigenvalues of the linearized system are \( \{\frac{5 + 6w_m}{2}, 3(1 + w_m - m)\} \), from where we observe that this point is an attractor only when \( w_m < -\frac{5}{6} \) and \( 1 + w_m < m \). Thus this point provides a viable scenario of a matter dominated universe.

Point \( F_2 \) describes a universe dominated by the varying vacuum and matter; when \( w_m = 0 \), point \( F_3 \) describes the \( \Lambda - \text{CDM universe in the FR theory.} \)

The equation of state parameter is derived \( w_{\text{tot}} (F_2) = -m \), so this point describes an accelerated universe when \( m \leq \frac{1}{2} \). The eigenvalues of the linearized system are \( \{-3(1 + w_m - m), 3m - \frac{1}{2}\} \) from where we can infer that the point is stable for \( m < \frac{1}{2} \) and \( 1 + w_m > m \). We observe that for the theoretical values of \( m \) (very small) this is a stable point that describes an accelerated universe and thus it is extremely interesting from a physical point of view.

The existence conditions of point \( F_3 \) are given in Table \( \text{XXV} \). Similar with point \( E_3 \), it corresponds to a universe of two fluids and the contribution of the geometrical background of Finsler Randers that is always accelerating, that is, \( w_{\text{tot}} (F_3) = -\frac{5}{6} \). The eigenvalues of the linearized system near the stationary point are \( \{\frac{1}{2} - 3m, -\frac{5 + 6w_m}{2}\} \), so point \( F_3 \) is an attractor for \( m > \frac{1}{2} \) and \( w_m > -\frac{5}{6} \). The above results are summarized in Tables \( \text{XIII and XIV} \). In Figs. [13][12] the evolution of trajectories for the dynamical system our study in phase space are presented.
FIG. 11: Phase space diagram for the dynamical system (28), (29). We consider $w_m = 0$, for $n < 1$. The unique attractor is point $F_3$.

TABLE XIII: Stationary points and physical parameters for the interaction model F.

| Point | $(\Omega_m, \Omega_\Lambda, \Omega_z)$ | Existence | $w_{tot}$ | Acceleration |
|-------|--------------------------------------|-----------|-----------|--------------|
| $F_1$ | $(1, 0, 0)$                          | Always    | $w_m$     | $w_m \leq -\frac{4}{3}$ |
| $F_2$ | $(\frac{m}{1 + w_m}, \frac{1 - m}{1 + w_m}, 0)$ | $w_m > -1, 0 \leq m \leq 1 + w_m - 1 + m$ | $m \leq \frac{2}{3}$ |
| $F_3$ | $(6nc, 6nc(5 + 6w_m), (5 + 6w_m)(6m - 1)c)$ | See Table XIV | $-\frac{2}{6}$ | Yes |

4.8. Model G: $Q_G = -3 \left(1 + \frac{3}{2}w_m\right)\Omega_\Lambda \Omega_m H^3$.

For the last model that we consider the term that is intrinsically by the FR model, namely $Q_G = -3 \left(1 + \frac{3}{2}w_m\right)\Omega_\Lambda \Omega_m H^3$, and the field equations are expressed as follows.

$$
\frac{d\Omega_\Lambda}{d \ln a} = \frac{1}{2} [\Omega_\Lambda - \Omega_\Lambda^2 + \Omega_m (2 + 3w_m)(\Omega_m - 1) + \Omega_\Lambda \Omega_m (7 + 9w_m)]
$$

(30)

$$
\frac{d\Omega_m}{d \ln a} = -\frac{3}{2} \Omega_m (1 + 3w_m)(1 + \Omega_\Lambda - \Omega_m)
$$

(31)

The dynamical system (30), (31) admits four critical points with coordinates $\{\Omega_m, \Omega_\Lambda, \Omega_z\}$

$G_1 = \{0, 0, -1\}$, $G_2 = \{0, 1, 0\}$, $G_3 = \{1, 0, 0\}$, $G_4 = \{-\frac{1}{4} + \frac{6w_m}{4}, \frac{5 + 6w_m}{4 + 6w_m}, -2 - \frac{2}{4 + 6w_m}\}$

Point $G_1$ always exists and describes an empty universe with equation of state parameter $w_{tot}(G_1) = -\frac{5}{6}$. The universe accelerates with the contribution of the extra term introduced due to the Finsler-Randers Geometry. The eigenvalues of the linearized system near to point $G_1$ are $\{-\frac{2}{6} \Omega_m (1 + 3w_m)(1 + \Omega_\Lambda - \Omega_m)\}$, and thus the point is always a source.

TABLE XIV: Stationary points and stability conditions for the interaction model F.

| Point | Eigenvalues | Stability |
|-------|-------------|-----------|
| $F_1$ | $\{\Omega_m (\frac{m}{1 + w_m}), 3(1 + w_m - m)\}$ | $w_m < -\frac{2}{6}$ and $1 + w_m < m$ |
| $F_2$ | $\{-3(1 + w_m - m), 3m - \frac{3}{2}\}$ | Attractor for $m < \frac{1}{6}$ and $1 + w_m > m$ |
| $F_3$ | $\{\frac{1}{2} - 3m, -\frac{\Omega_\Lambda \Omega_m}{2}\}$ | Attractor for $m > \frac{1}{6}$ and $w_m > -\frac{2}{6}$ |
FIG. 12: Evolution diagrams with time, for various energy densities of the dynamical system \((28), (29)\). We consider the initial conditions (a) \(\Omega_m = 0.4, \Omega_\Lambda = 0.1\) (b) \(\Omega_m = 0.7, \Omega_\Lambda = 0.1\) (c) \(\Omega_m = 0.5, \Omega_\Lambda = 0.2\) (d) \(\Omega_m = 0.2, \Omega_\Lambda = 0.3\) (e) \(\Omega_m = 0.1, \Omega_\Lambda = 0.9\) (f) \(\Omega_m = 0.2, \Omega_\Lambda = 0.3\), for \(n < 1\) and \(w_m = 0\).

Point \(G_2\) describes a de Sitter universe with equation of state parameter \(w_{\text{tot}}(G_2) = -1\), where only the \(\Lambda\) term contributes in the evolution of the universe. The eigenvalues are derived to be \(\left\{-\frac{1}{2}, -\frac{3}{2}(1 + w_m)\right\}\), from where we can infer that the point is an attractor when \(w_m > -1\). Thus this point is of great physical interest.

Point \(G_3\) always exists and describes a matter dominated universe that is accelerating for \(w_m \leq -\frac{1}{3}\). The eigenvalues of the linearized system are \(\{3(1 + w_m), \frac{(5+6w_m)}{2}\}\) and thus can be stable only for \(w_m < -1\).
TABLE XV: Existence conditions for the stationary point $F_4$

| Point | Existence | Existence |
|-------|-----------|-----------|
| $F_3$ | $5 + 6w_m \geq 0$ | $m = \frac{1}{6}, n \neq 0$, for $m < \frac{1}{6}$, $n < 0$ or $m + n > \frac{1}{6}$ |
|       |           | $m > \frac{1}{6}$, $n > 0$ or $m + n < \frac{1}{6}$ |
|       | $5 + 6w_m \leq 0$ | $n > 0$ or $m + n < \frac{1}{6}$ |
|       |           | $n < 0$ or $m + n > \frac{1}{6}$ |
|       |           | $m \neq \frac{1}{6}$, $m + n = \frac{1}{6}$ or $n = 0$, $5 + 6w_m \neq 0$ |

FIG. 13: Phase space diagram for the dynamical system (30), (31). We consider $w_m = 0$, for $n < 1$. The unique attractor is the de Sitter point $G_2$.

Point $G_4$ exists only for $w_m = -\frac{5}{6}$ in which case it again describes a matter dominated universe, but in this case it is always accelerating. By studying its eigenvalues for $w_m = -\frac{5}{6}$ though we deduce that the point is always unstable.

The above results are summarized in Tables XVII and XVIII. In addition in the Figs. 13, 14 the evolution of trajectories for the dynamical system our study in phase space are presented.

5. DISCUSSION

We performed, for a first time, a detailed study on the dynamics of the varying vacuum model in a Finsler-Randers geometrical background. Specifically in the homogeneous and isotropic spatially flat FLRW spacetime we assumed the existence of an ideal gas fluid source which couples with the varying vacuum terms. That scenario follows from the interacting models where interaction in the dark sector has been proposed as a possible scenario to explain the cosmological observations. For the gravitational theory, we consider that of Finsler Randers from where a new geometrodynamical term is introduced and affects the dynamical evolution.

The functional form of varying vacuum model is in generally unknown but a dominating quadratic term in the

TABLE XVI: Stationary points and physical parameters for the interaction model G.

| Point | $(\Omega_m, \Omega_\Lambda, \Omega_z)$ | Existence $w_{tot}$ | Acceleration |
|-------|----------------------------------|---------------------|--------------|
| $G_1$ | $(0, 0, -1)$                      | Always $-\frac{5}{6}$ | Yes          |
| $G_2$ | $(0, 1, 0)$                       | Always $-1$         | Yes          |
| $G_3$ | $(1, 0, 0)$                       | Always $w_m$ $w_m \leq -\frac{1}{3}$ |
| $G_4$ | $\left(-\frac{1}{1+6w_m}, \frac{1+6w_m}{2+6w_m}, -2 - \frac{2}{2+6w_m}\right)$ | $w_m = -\frac{5}{6}, -\frac{2}{3}$ | Yes |
Hubble function has been found to be good a candidate. In this work we consider six different functional forms for the interaction between the components of the dark sector of the universe.

Models $Q_A$, $Q_B$ and $Q_C$ have been studied in a previous work in the case of GR [79]. In this work we recover the results of the previous study, that is, the limit of GR relativity is recovered, while there exists one possible era in the cosmological history which corresponds to the epoch where only the geometrodynamical term of the FR geometry contributes.
TABLE XVII: Stationary points and stability conditions for the interaction model G.

| Point | Eigenvalues | Stability |
|-------|-------------|-----------|
| $G_1$ | $\{-\frac{1}{2}, -\frac{3}{2}(1 + w_m)\}$ | Source |
| $G_2$ | $\{-\frac{1}{2}, -\frac{3}{2}(1 + w_m)\}$ | $w_m > -1$ |
| $G_3$ | $\{3(1 + w_m), \frac{(5+6w_m)}{2}\}$ | $w_m < -1$ |
| $G_4$ | $\{0, \frac{1}{2}\}$ | Unstable |

In addition, we considered three new interaction models, namely $Q_D$, $Q_E$ and $Q_F$ which depend also on the geometrodynamical term of FR. For these three models the limit of GR is recovered while now there is a new cosmological solution where the geometrodynamical term contributes along the terms of the dark sector. These new epochs describe accelerated universe. As far as the stability of these exact solutions are concerned, they can be stable or unstable, depending on the coupling constants of the models.

Finally $Q_G$ is the case without varying vacuum term. In this scenario we found four critical points which describe the matter dominated era, the de Sitter universe, the vacuum space and an exact solution which correspond to a point where all the fluid source contributes in the cosmological evolution.

From the results of this analysis we conclude that the varying vacuum cosmological scenario in the context of Finsler-Randers geometry can describe the basic epochs of cosmic history. In a future work we plan to test the performance of the current class of modified gravity models against the cosmological data.

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