On randomized confidence intervals for the binomial probability

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Abstract

Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically Bernoulli($\theta$) distributed. Also suppose that our aim is to find an exact confidence interval for $\theta$ that is the intersection of a $1 - \alpha/2$ upper confidence interval and a $1 - \alpha/2$ lower confidence interval. The Clopper-Pearson interval is the standard such confidence interval for $\theta$, which is widely used in practice. We consider the randomized confidence interval of Stevens, 1950 and present some extensions, including pseudorandomized confidence intervals. We also consider the “data-randomized” confidence interval of Korn, 1987 and point out some additional attractive features of this interval. We also contribute to the discussion about the practical use of such confidence intervals.

Keywords: Binomial confidence interval; data-randomized confidence interval; randomized confidence interval

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1. Introduction

Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed (iid), each with a $\text{Bernoulli}(\theta)$ distribution ($\theta \in [0, 1]$). Let $X = (X_1, \ldots, X_n)$. Our objective is to find a confidence interval for $\theta$ of the form $[\ell(X, V), u(X, V)]$, where the interval endpoints may depend on an auxiliary random variable $V$, such that both of the following conditions are satisfied:

1. $P_{\theta}(\theta > u(X, V)) \leq \frac{\alpha}{2}$ for all $\theta$ \hfill (1)
2. $P_{\theta}(\theta < \ell(X, V)) \leq \frac{\alpha}{2}$ for all $\theta$. \hfill (2)

Of course, such a confidence interval has infimum coverage probability that is greater than or equal to $1 - \alpha$. The conditions (1) and (2) make the endpoints of the confidence interval, $\ell(X, V)$ and $u(X, V)$, easy to interpret. Confidence intervals that satisfy these conditions are the discrete-data analogue of an equi-tailed confidence interval based on continuous data. The solution favoured by statistical practitioners is to find a non-randomized confidence interval for $\theta$ based on $Y = X_1 + X_2 + \cdots + X_n$, which has a $\text{Binomial}(n, \theta)$ distribution. The resulting Clopper-Pearson interval (Clopper and Pearson, 1934) is widely used in practice. Of course, if the conditions (1) and (2) were to be replaced by the less stringent requirement that $P_{\theta}(\ell(X, V) \leq \theta \leq u(X, V)) \geq 1 - \alpha$ for all $\theta$ then other non-randomized confidence intervals such as that of Blaker (2000) would come into consideration. Nonetheless, there is still a lively interest in randomized and related confidence intervals, as evidenced by e.g. Geyer and Meeden (2005) and the resulting published comments. In the present paper, we will compare various randomized, “pseudorandomized” and “data-randomized” confidence intervals that satisfy (1) and (2) with the Clopper-Pearson confidence interval (described, for the reader’s convenience, in Section 2).

A randomized confidence interval can be found by considering the artificial data $Z = Y + V$, where $V$ and $Y$ are independent random variables and $V$ has a uniform distribution on $(0, 1)$ (Stevens, 1950). Equi-tailed $1 - \alpha$ confidence intervals based on $Z$
dominate the $1 - \alpha$ Clopper-Pearson confidence intervals. In Section 3, we review this randomized confidence interval and introduce some extensions to the idea of a randomized confidence interval, including pseudorandomized confidence intervals. In Section 4, we review the usual objections to the use of randomized confidence intervals in practice and note a new objection based on the need to condition on an ancillary statistic.

Korn (1987) introduced a “data-randomized” confidence interval for $\theta$ that uses the data itself to generate the randomization. This confidence interval does not require the use of an auxiliary variable $V$ and overcomes some of the objections to the use of randomized confidence intervals in practice. In Section 5, we review this confidence interval and point out some additional attractive features of this interval. In Section 6, we note the objections of Senn (2007ab) to the use of such intervals in practice and note a further objection based on an invariance argument.

In Section 7, we consider the properties of an unusual confidence interval for $\theta$ that turns out to be a “data-randomized” confidence interval. We also explain why we expect this confidence interval to have properties that are inferior to the “data-randomized” confidence interval of Korn (1987).

2. Clopper-Pearson confidence interval for the binomial probability

Let $f_\theta(y) = P_\theta(Y = y)$ and $F_\theta(y) = P_\theta(Y \leq y)$. For observed value $y$ of $Y$, the Clopper-Pearson $1 - \alpha$ confidence interval for $\theta$ is found as follows. The p-value for testing the null hypothesis $H_0 : \theta = \tilde{\theta}$ against the alternative hypothesis $H_A : \theta < \tilde{\theta}$ is $P_\theta(Y \leq y)$. A $1 - \alpha$ upper confidence interval for $\theta$ is $\{\theta : P_\theta(Y \leq y) > \alpha/2\}$. We replace all upper confidence intervals of the form $[b, c)$ by $[b, c]$. This does not decrease the coverage probability of this confidence interval and leaves its length unchanged. The upper endpoint of the Clopper-Pearson $1 - \alpha$ confidence interval is 1 for $y = n$; otherwise it is the solution for $\theta$ of

$$P_\theta(Y \leq y) = F_\theta(y) = \frac{\alpha}{2}. \quad (3)$$
The lower endpoint of this interval is 0 for \( y = 0 \); otherwise it is the solution for \( \theta \) of

\[
P_\theta(Y \geq y) = 1 - F_\theta(y - 1) = \frac{\alpha}{2}.
\]

Convenient expressions for the solutions of these equations are described e.g. by Casella and Berger, (2002, p.454). We denote the Clopper-Pearson interval by \([\ell_{CP}(Y), u_{CP}(Y)]\).

3. Randomized confidence interval for the binomial probability

The randomized confidence interval of Stevens (1950) can be found by considering the artificial data \( Z = Y + V \), where \( V \) and \( Y \) are independent random variables and \( V \sim U(0,1) \). Also assume that either \( V \in [0,1) \) or \( V \in (0,1] \). For observed values \( y, v \) and \( z \) of \( Y, V \) and \( Z \), respectively, this confidence interval for \( \theta \) is found as follows. The p-value for testing \( H_0 : \theta = \bar{\theta} \) against \( H_A : \theta < \bar{\theta} \) is \( P_\theta(Z \leq z) \). A \( 1 - \alpha \) upper confidence interval for \( \theta \) is \( \{ \theta : P_\theta(Z \leq z) > \alpha/2 \} \). We replace all upper confidence intervals of the form \([b, c)\) by \([b, c]\). This does not decrease the coverage probability of this confidence interval and leaves its length unchanged. The upper endpoint of the randomized interval is 1 for \( y = n \) and \( v > \alpha/2 \); otherwise it is the solution for \( \theta \) of

\[
P_\theta(Z \leq z) = vf_\theta(y) + F_\theta(y - 1) = (1 - v)F_\theta(y - 1) + vF_\theta(y) = \frac{\alpha}{2}.
\]

The lower endpoint of this interval is 0 for \( y = 0 \) and \( v < 1 - \alpha/2 \); otherwise it is the solution for \( \theta \) of

\[
P_\theta(Z \geq z) = (1 - v)f_\theta(y) + 1 - F_\theta(y) = (1 - v)(1 - F_\theta(y - 1)) + v(1 - F_\theta(y)) = \frac{\alpha}{2}.
\]

Denote the resulting confidence interval by \([\ell_R(y, v), u_R(y, v)]\). This interval satisfies both of the following conditions:

\[
P_\theta(\theta > u_R(Y, V)) = \frac{\alpha}{2} \text{ for all } \theta
\]

\[
P_\theta(\theta < \ell_R(Y, V)) = \frac{\alpha}{2} \text{ for all } \theta.
\]

By comparing the left-hand sides of (5) with (3) and (6) with (4), we find that for every \( y \) and \( v \in [0,1] \), this confidence interval is contained strictly within the Clopper-Pearson \( 1 - \alpha \) confidence interval. The confidence interval lower endpoint \( \ell_R(y, v) \) is a
nondecreasing function of \( v \) that is strictly increasing for (a) \( y = 0 \) and \( v \geq 1 - \alpha/2 \) and (b) all \( y \geq 1 \). The confidence interval upper endpoint \( u_R(y, v) \) is nondecreasing function of \( v \) that is strictly increasing for (a) \( y = n \) and \( v \leq \alpha/2 \) and (b) all \( y \leq n - 1 \).

The confidence interval \([\ell_R(Y, V), u_R(Y, V)]\) dominates the \( 1 - \alpha \) Clopper-Pearson confidence interval. This domination is possible because “losses for interval estimation and hypothesis testing are not usually convex” (Casella and Berger, 1999, p.484). For the \( 1 - \alpha \) Clopper-Pearson confidence interval \([\ell_{CP}(Y), u_{CP}(Y)]\), \( P_{\theta}(\theta > u_{CP}(Y)) \) and \( P_{\theta}(\theta < \ell_{CP}(Y)) \) are discontinuous functions that typically take values well above \( \alpha/2 \) for some values of \( \theta \). By contrast, the confidence interval \([\ell_R(Y, V), u_R(Y, V)]\) has ideal coverage properties. The excellent theoretical properties of this randomized interval can be traced to the fact that the addition of \( V \) to \( Y \) has “split” each observation \( y \) into a continuous set of values, where the values that \( y \) is split into are less than all of the values that \( y + 1 \) is split into for each \( y = 0, \ldots, n - 1 \). In the language of Kabaila and Lloyd (2006), \( Z \) is a “refinement” of \( Y \).

As described in Appendix A, the confidence interval \([\ell_R(Y, V), u_R(Y, V)]\) may be generalized by allowing the lower and upper endpoints to depend on different random variables \( V_\ell \) and \( V_u \), respectively, where each of these random variables is uniformly distributed on \((0, 1)\). However, as explained in Appendix A, there seems to be no advantage to be gained from this generalization.

As described in Appendix B, we may also construct a randomized confidence interval for \( \theta \) using an auxiliary discrete random variable \( W \). This random variable may be viewed as an approximation to \( V \), which has a uniform distribution on \((0, 1)\).

The usual interpretation of the coverage probability of a confidence interval is that, in a sequence of independent repetitions of the statistical experiment that gave rise to this confidence interval, the long-run proportion of confidence intervals that includes the parameter is equal to the coverage probability. As described in Appendix C, this interpretation allows us to consider confidence intervals for \( \theta \) that are influenced by an appropriately-chosen auxiliary deterministic sequence, instead of the observed value of
an auxiliary random variable such as $V$. These deterministic sequences may be pseudo-random, quasi-random or possess a very obvious pattern. What we do in this appendix is to replace expectations by the corresponding long-run averages.

4. Objections to the use of randomized confidence intervals in practice

Cox and Hinkley (1974, p.100) view randomization of this type as “a mathematical artifice” that is “of no direct practical importance”. Two very cogent objections to the use of randomized confidence intervals in practice are the following:

(1) Two scientists using the same procedure to construct a randomized $1 - \alpha$ confidence interval for $\theta$ based on the same observed value $y$ will, with probability 1, produce different confidence intervals.

(2) The randomized interval is influenced by an auxiliary random variable $V$ that has no relation to the problem under consideration.

These two reasons are presented, for example, by Kiefer (1987, p.50) and Korn (1987, p.707). Would the first of these objections be reduced if the following procedure were adopted? A website maintained by a reputable organisation would, upon the provision of the name of the user and the title of a project, provide an observation $v$ of $V \sim U(0, 1)$ derived from a genuinely random source, such as electronic thermal noise. Together with this observation, this website would provide a identification number. The user would then use this observation $v$ to construct his/her realisation of a randomized confidence interval and report this interval, together with $v$ and this identification number. The website would permanently list the names of all users, projects, identification numbers and values of $v$.

In Appendix C, we show how confidence intervals depending on an appropriately-chosen auxiliary deterministic sequence have the desired long-run properties. Such a sequence may be pseudorandom, quasi-random or may be a sequence with a very obvious pattern. However, it would seem that the alarm experienced by practitioners in response to having their confidence interval being influenced by an auxiliary variable increases as
we move from random variable to pseudorandom variable to quasi-random variable to a variable showing a very obvious pattern.

We now add a third reason for rejecting the use of randomized confidence intervals in practice. Statisticians who believe that inference should be carried out conditional on an appropriate ancillary statistic (see e.g. Cox and Hinkley, 1974) would have the following objection to the use of such a randomized confidence interval in practice. The random variables $Y$ and $V$ can be recovered from the random variable $Z$. The statistic $V$ has a distribution that does not depend on $\theta$ i.e. it is an ancillary statistic. Carrying out inference conditional on $V = v$ is equivalent to carrying out inference based solely on $Y$, leading to a non-randomized confidence interval.

5. “Data-randomized” confidence interval for the binomial probability

An apparent solution to the first of the objections described in the previous section and a mitigation of the second and third objections has been proposed by Korn (1987). This author defines $W$ to be the one-sided p-value from the Wilcoxon rank-sum test for testing the null hypothesis that the ones in the sequence $X_1, \ldots, X_n$ are randomly distributed in this sequence against the alternative hypothesis that they come near the beginning of this sequence. The distribution of $W$, conditional on $Y = y$, is uniform on \( \left\{ \frac{1}{\binom{y}{y}}, \frac{2}{\binom{y}{y}}, \ldots, \frac{\binom{n}{y}}{\binom{y}{y}} \right\} \), so that it does not depend on $\theta$. This conditional distribution stochastically dominates the distribution of $V \sim U(0, 1)$. Korn (1987) uses this to prove that (7), stated in Appendix B, holds true.

Korn (1987) does not describe how the lower endpoint of his randomized confidence interval should be found. Based on the work presented in Appendix B, it is clear that the lower endpoint of this interval should be found as follows. Define the discrete random variable $\tilde{W}$ by the requirement that, conditional on $Y = y$, $\tilde{W} = W - 1/\binom{y}{y}$. Thus, conditional on $Y = y$, $\tilde{W}$ is uniformly distributed on \( \left\{ 0, \frac{1}{\binom{y}{y}}, \ldots, \frac{\binom{n}{y} - 1}{\binom{y}{y}} \right\} \). Hence the data-randomized confidence interval for $\theta$ is $[\ell_R(Y, \tilde{W}), u_R(Y, W)]$. This interval satisfies (7) and (8) (stated in Appendix B). If $n$ is not too small then (9) and (10) (stated in Appendix B) are also satisfied and the expected length functions of the confidence
intervals \([\ell_R(Y, V), u_R(Y, V)]\) and \([\ell_R(Y, \tilde{W}), u_R(Y, W)]\) are approximately equal. It is straightforward to show that \([\ell_R(Y, \tilde{W}), u_R(Y, W)]\) dominates the \(1 - \alpha\) Clopper-Pearson confidence interval. This confidence interval eliminates the first of the objections raised in Section 4 since the data determines the randomization. Consequently, Korn (1987) calls these “data-randomized” confidence intervals.

The excellent theoretical properties of this data-randomized interval can be traced to the fact that the addition of \(W\) to \(Y\) has “split” each observation \(y\) into \(\binom{n}{y}\) values, where the values that \(y\) is split into are less than all of the values that \(y + 1\) is split into for each \(y = 0, \ldots, n - 1\). In the language of Kabaila and Lloyd (2006), \(Y + W\) is a “refinement” of \(Y\). The upper endpoints of the data-randomized confidence intervals are based on \(Y + W\), which can take

\[
\sum_{y=0}^{n} \binom{n}{y} = \sum_{y=0}^{n} \binom{n}{y} 1^{n-y}1^y = (1 + 1)^n = 2^n
\]

possible values. The excellent theoretical properties of this data-randomized confidence interval suggest that it can be chosen as a standard against which other data-randomized confidence intervals can be judged.

Of course, conditional on \(Y = y\), there are \(\binom{n}{y}\) equally-likely distinct locations of the \(y\) ones. Any one-to-one correspondence between these distinct locations and the integers \(1, 2, \ldots, \binom{n}{y}\) can be used, in the obvious way, to generate a random variable with the same conditional distribution as \(W\). This random variable could be used as an alternative to \(W\) to construct a data-randomized confidence interval with the same coverage and expected length properties as \([\ell_R(Y, \tilde{W}), u_R(Y, W)]\).

6. Objections to the use of data-randomized confidence intervals in practice

Senn (2007ab) has objected to data-randomized inference procedures on two general grounds that specialise in the present circumstance to the following:

(1) The “split” of each observation \(y\) leads to quite an arbitrary ranking of the values into which \(y\) is split. Senn (2007a) says that any such split should be based only on
some meaningful comparison of the values that arise from a given observation $y$. In the present circumstance, there does not appear to be any meaningful comparison that could be used as the basis for this split.

(2) The confidence interval described by Korn (1987) is only one of many possible data-randomized confidence intervals with the same theoretical properties. If $W$ is replaced by $W^*$, which is obtained by calculating $W$ after the observations have undergone a given permutation then the resulting data-randomized confidence intervals have the same theoretical properties. Thus users of data-randomized confidence intervals will only be able to find a unique $1 - \alpha$ confidence interval for given data $x_1, \ldots, x_n$ if a convention can be established that the interval is based only on the auxiliary random variable $W$ proposed by Korn (1987) (and not some alternative auxiliary random variable $W^*$ with similar properties). However, establishing such a convention does not seem realistic.

Now it might be argued that the improvement in the properties of the confidence interval for $\theta$ justifies the “splitting” of each observation $y$ and that such a split does not require any meaningful comparison of the values that make up this split. However, even if a convention could be enforced that the data-randomized confidence intervals for $\theta$ are based only on Korn’s auxiliary random variable $W$, these confidence intervals would still not satisfy the invariance property described in Example 2.35 on Cox and Hinkley (1974).

7. Confidence intervals for $\theta$ based on splitting the Bernoulli data into two groups of approximately equal relatively prime size

As before, suppose that $X_1, X_2, \ldots, X_n$ are independent and identically Bernoulli($\theta$) distributed and that our aim is to find a confidence interval for $\theta$ that satisfies (1) and (2). In this section, we consider the properties of a confidence interval for $\theta$ that is obtained as follows. Suppose that $n = n_1 + n_2$, where $n_1$ and $n_2$ are relatively prime and as close as possible. Form the following estimator of $\theta$:

$$\hat{\theta} = \frac{1}{2} \left( \frac{Y_1}{n_1} + \frac{Y_2}{n_2} \right),$$
where \( Y_1 = X_1 + \cdots + X_{n_1} \) and \( Y_2 = X_{n_1+1} + \cdots + X_n \). This is an unbiased estimator of \( \theta \). Consider the following procedure for finding a \( 1 - \alpha \) confidence interval for \( \theta \). The Clopper-Pearson interval \([\ell_{CP}(Y), u_{CP}(Y)]\) is the intersection of upper and lower \( 1 - \alpha/2 \) confidence intervals for \( \theta \) that are based on inverting the family of hypothesis tests using the test statistic \( Y \) (or, equivalently, the test statistic \( Y/n \)). We can construct an analogous confidence interval that is the intersection of upper and lower \( 1 - \alpha/2 \) confidence intervals for \( \theta \) that are based on the test statistic \( \hat{\Theta} \). Let us denote this confidence interval by \([\ell^\dagger(\mathbf{X}), u^\dagger(\mathbf{X})]\). This confidence interval is obtained by deterministically splitting the data into two parts. A random splitting of the data into two parts is considered by Decrouez and Hall (2013b).

For concreteness, consider the particular case that \( n = 47 \). Form the following estimator of \( \theta \):

\[
\hat{\Theta} = \frac{1}{2} \left( \frac{Y_1}{23} + \frac{Y_2}{24} \right),
\]

where \( Y_1 = X_1 + \cdots + X_{23} \) and \( Y_2 = X_{24} + \cdots + X_{47} \). We have obtained \( Y_1 \) and \( Y_2 \) by splitting the 47 Bernoulli trials into two groups of approximately equal relatively prime size. How do the confidence intervals \([\ell_{CP}(Y), u_{CP}(Y)]\) and \([\ell^\dagger(\mathbf{X}), u^\dagger(\mathbf{X})]\) compare? We expect the estimator \( \hat{\Theta} \) to be a somewhat less efficient estimator of \( \theta \) than the maximum likelihood estimator \( Y/n \). This is because we give the same weight to the estimators \( Y_{1/23} \) and \( Y_{2/24} \), when the more accurate estimator \( Y_{2/24} \) should have been given a larger weight. On the other hand, the estimator \( \hat{\Theta} \) has \( 24 \times 25 - 1 = 599 \) possible values, whereas \( Y/n \) has only 48 possible values. We therefore expect that the \( P_\theta(\theta < \ell^\dagger(\mathbf{X})) \) and \( P_\theta(\theta > u^\dagger(\mathbf{X})) \) will tend to be closer to \( \alpha/2 \) than \( P_\theta(\theta < \ell_{CP}(Y)) \) and \( P_\theta(\theta > u_{CP}(Y)) \), respectively (cf. Decrouez and Hall, 2013a). The fact that the estimator \( \hat{\Theta} \) has many more possible values than the estimator \( Y/n \) can also be expected to lead to a shortening of the confidence intervals that will, to some extent, compensate or even overcome the inefficiency of the estimator \( \hat{\Theta} \) by comparison with the estimator \( Y/n \).

Observe, however, that the confidence interval \([\ell^\dagger(\mathbf{X}), u^\dagger(\mathbf{X})]\) may be viewed as
a data-randomized confidence interval for $\theta$. As suggested in Section 5, we use the data-randomized confidence interval of Korn (1987) as the standard against which we judge $[\ell(X), u(X)]$. We expect $[\ell(X), u(X)]$ to have coverage and expected length properties that are inferior to the data-randomized confidence interval of Korn (1987). For a start, the upper endpoints of the confidence intervals of Korn (1987) are based on a statistic that can take $2^{47} \approx 1.407 \times 10^{14}$ values. This is much larger than the 599 possible values of the statistic $\hat{\Theta}$, on which $[\ell(X), u(X)]$ is based. In addition, the statistic $\hat{\Theta}$ orders the data in the wrong way. The statistic on which the confidence interval is based should always take a larger value for observed value $y = t + 1$ than for $y = t$. However, for $y_1 = t$ and $y_2 = 0$ the observed value is $y = t$ and $\hat{\theta} = t/46$, which exceeds $\hat{\theta} = (t+1)/48$ when $y_1 = 0$ and $y_2 = t+1$ (so that the observed value is $y = t+1$) when $t > 23$. Also, $\hat{\theta}$ takes the same value, $1/2$, for $(y_1, y_2) = (0, n_2)$ (so that $y = n_2$) and $(y_1, y_2) = (n_1, 0)$ (so that $y = n_1$). In the language of Kabaila and Lloyd (2006), $\hat{\Theta}$ is not a “refinement” of $Y$. The fact that $\hat{\Theta}$ orders the data in the wrong way may be interpreted as just another manifestation of the inefficiency of this estimator. This means that if we are prepared to consider data-randomized confidence intervals then we should be using the data-randomized confidence interval of Korn (1987) instead of the confidence interval $[\ell(X), u(X)]$ based on $\hat{\Theta}$.

8. Discussion

Various kinds of randomized, pseudorandomized and data-randomized “equi-tailed” confidence intervals for the binomial probability, based on iid Bernoulli observations, have been reviewed. Of course, randomization, pseudorandomization and data-randomization can be combined in various ways. For example, we could combine randomization with data-randomization. Undoubtedly, such confidence intervals will continue to be of theoretical interest.

The standard confidence interval that satisfies the “equi-tailed” coverage constraints described in the paper is the Clopper-Pearson interval, which is not randomized (or pseudorandomized or data-randomized). Broadening the class of allowable interval es-
timators to include either randomization, pseudorandomization or data-randomization (or a combination of some of these) may be viewed as allowing one to use an additional resource. The theoretical question is: How well is this additional resource being used? We have asked and answered this question in the case of the unusual confidence interval for the binomial probability described in Section 7. Of course, whether randomized, pseudorandomized or data-randomized confidence intervals will ever be used in practice is open to question.

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Appendix A: A generalization of the randomized confidence intervals

A generalization of the randomized confidence interval \([\ell_R(Y, V), u_R(Y, V)]\) is \([\ell_R(Y, V_\ell), u_R(Y, V_u)]\), where \(Y\) and \((V_\ell, V_u)\) are independent, \(V_\ell \sim U(0, 1)\) and \(V_u \sim U(0, 1)\). One could, for example, choose \(V_\ell = 1 - V_u\). This interval satisfies the following conditions:

\[
P_\theta(\theta > u_R(Y, V_u)) = \frac{\alpha}{2} \quad \text{for all } \theta
\]

\[
P_\theta(\theta < \ell_R(Y, V_\ell)) = \frac{\alpha}{2} \quad \text{for all } \theta.
\]

Also, the confidence intervals \([\ell_R(Y, V), u_R(Y, V)]\) and \([\ell_R(Y, V_\ell), u_R(Y, V_u)]\) have the same expected length functions. There seems to be no advantage to be gained from this generalization. For example, suppose that \(V_\ell = 1 - V_u\). In this case, the confidence interval lower endpoint \(\ell_R(y, v_\ell) = \ell_R(y, 1 - v_u)\) is a decreasing function of \(v_u\) for (a) \(y = 0\) and \(1 - v_u \geq 1 - \alpha/2\) and (b) all \(y \geq 1\). This means that, for given observed value \(y\), the main effect of increasing \(v_u\) is to widen the confidence interval. In statistical practice, confidence interval width is interpreted as a measure of the accuracy of the estimation of \(\theta\). It does not seem helpful to report (according to this interpretation)
varying apparent accuracies of estimation of $\theta$ (depending on the value of $v_n$), for the same observed value $y$.

Appendix B: Randomized confidence intervals that depend on an auxiliary discrete random variable

Suppose that the random variable $W$ is such that, conditional on $Y = y$, $W$ is uniformly distributed on $\{1/M(y), 2/M(y), \ldots, M(y)/M(y)\}$, where $M(y)$ is an integer greater than 1 for each $y = 0, \ldots, n$. A particular case is that $M(y) = M$ for $y = 0, \ldots, n$. Define the discrete random variable $\tilde{W}$ by the requirement that, conditional on $Y = y$, $\tilde{W} = W - 1/M(y)$. Thus, conditional on $Y = y$, $\tilde{W}$ is uniformly distributed on $\{0, 1/M(y), \ldots, (M(y) - 1)/M(y)\}$. Let $F_V$, $F_W$ and $F_{\tilde{W}}$ denote the cumulative distribution functions of $V \sim U(0,1)$, $W$ and $\tilde{W}$, respectively. Using the facts that $F_W$ is stochastically larger than $F_V$ and $F_V$ is stochastically larger than $F_{\tilde{W}}$, it may be shown that the confidence interval $[\ell_R(Y, W), u_R(Y, W)]$ satisfies the following conditions:

$$P_\theta(\theta > u_R(Y, W)) \leq \frac{\alpha}{2} \text{ for all } \theta$$

(7)

$$P_\theta(\theta < \ell_R(Y, W)) \leq \frac{\alpha}{2} \text{ for all } \theta.$$  

(8)

If the smallest of the $M(y)$'s is not too small then, in addition,

$$P_\theta(\theta > u_R(Y, W)) \approx \frac{\alpha}{2} \text{ for all } \theta$$

(9)

$$P_\theta(\theta < \ell_R(Y, W)) \approx \frac{\alpha}{2} \text{ for all } \theta$$

(10)

and the expected length functions of the confidence intervals $[\ell_R(Y, V), u_R(Y, V)]$ and $[\ell_R(Y, \tilde{W}), u_R(Y, W)]$ are approximately equal. These results may be interpreted as resulting from the fact that $W$ may be viewed as an approximation to $V$, which has a uniform distribution on $(0,1)$. 

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Appendix C: Confidence intervals that depend on an auxiliary deterministic sequence

Suppose that \( Y_1, Y_2, \ldots \) are independent and identically Binomial\((n, \theta)\) distributed. In other words, suppose that we carry out independent repetitions of the statistical experiment that gives rise to \( Y \). Let \( v_1, v_2, \ldots \) be a deterministic sequence of real numbers such that either \( v_k \in [0, 1) \) for every \( k = 1, 2, \ldots \) or \( v_k \in (0, 1] \) for every \( k = 1, 2, \ldots \). Now suppose that \( v_1, v_2, \ldots \) is uniformly distributed modulo 1, as defined by Kuipers and Niederreiter (1974). For any given irrational number \( \lambda \), the sequence \( v_k = \{n\lambda\} \), where \( \{a\} \) denotes the fractional part of \( a \), possesses these properties and may be viewed as a pseudorandom sequence. The van der Corput sequence (defined e.g. on page 127 of Kuipers and Niederreiter, 1974) possesses these properties and may be viewed as a quasi-random sequence. Suppose that, given the observation \( y_k \) of \( Y_k \), we compute the confidence interval \( [\ell_R(y_k, v_k), u_R(y_k, v_k)] \). We use the notation

\[
\mathcal{I}(A) = \begin{cases} 
1 & \text{if } A \text{ is true} \\
0 & \text{if } A \text{ is false}
\end{cases}
\]

where \( A \) is an arbitrary statement. It may be shown that, for each \( \theta \),

\[
\frac{1}{m} \sum_{k=1}^{m} \mathcal{I}(\theta > u_R(Y_k, v_k)) \quad \text{converges almost surely to } \frac{\alpha}{2}
\]

and

\[
\frac{1}{m} \sum_{k=1}^{m} \mathcal{I}(\theta < \ell_R(Y_k, v_k)) \quad \text{converges almost surely to } \frac{\alpha}{2}
\]

as \( m \to \infty \). It may also be shown that, similarly, the long-run average lengths of the confidence intervals \([\ell_R(Y, V), u_R(Y, V)]\) and \([\ell_R(y_k, v_k), u_R(y_k, v_k)]\) are the same.

Alternatively, we may suppose that \( w_1, w_2, \ldots \) is a deterministic sequence of real numbers such that \( w_1, w_2, \ldots \) is a periodic sequence with period \( N \), where \((w_1, w_2, \ldots, w_N)\) is a permutation of \((1/N, 2/N, \ldots, N/N)\). There are both pseudorandom sequences (found using e.g. mixed congruential generators with maximal possible cycle length \( N \)) and sequences with very obvious pattern (e.g. \( 1/N, 2/N, \ldots, N/N, 1/N, 2/N, \ldots \)) that satisfy these conditions. Define the sequence \( \tilde{w}_1, \tilde{w}_2, \ldots \) by \( \tilde{w}_k = w_k - 1/N \) for
$k = 1, 2, \ldots$. Suppose that, given the observation $y_k$ of $Y_k$, we compute the confidence interval $[\ell_R(y_k, \hat{w}_k), u_R(y_k, w_k)]$. It may be shown that, for each $\theta$,

$$
\frac{1}{m} \sum_{k=1}^{m} I(\theta > u_R(Y_k, v_k)) \text{ converges almost surely to a number } \leq \frac{\alpha}{2} \text{ and }
\frac{1}{m} \sum_{k=1}^{m} I(\theta < \ell_R(Y_k, v_k)) \text{ converges almost surely to a number } \leq \frac{\alpha}{2}
$$
as $m \to \infty$.

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