ON THE DIFFUSIVE STRESS RELAXATION FOR MULTIDIMENSIONAL VISCOELASTICITY

DONATELLA DONATELLI AND CORRADO LATTANZIO

Abstract. This paper deals with the rigorous study of the diffusive stress relaxation in the multidimensional system arising in the mathematical modeling of viscoelastic materials. The control of an appropriate high order energy shall lead to the proof of that limit in Sobolev space. It is shown also as the same result can be obtained in terms of relative modulate energies.

1. Introduction

The aim of this paper is the study of the diffusive relaxation limit for a model in multidimensional viscoelasticity. To this end, we shall consider the following semilinear hyperbolic system

\[
\begin{align*}
\partial_t \tilde{F}_{i\alpha} - \partial_{\alpha} \tilde{v}_i &= 0 \\
\partial_t \tilde{v}_i - \partial_{\alpha} \tilde{S}_{i\alpha} &= 0 \\
\partial_t \tilde{S}_{i\alpha} - \frac{\mu}{\epsilon} \partial_{\alpha} \tilde{v}_i &= -\frac{1}{\epsilon} \tilde{S}_{i\alpha} + \frac{1}{\epsilon} T_{i\alpha}(\tilde{F}).
\end{align*}
\]

(1.1)

In (1.1), \(i, \alpha = 1, 2, 3\); \(\tilde{F}\) and \(\tilde{v}\) are respectively the deformation gradient and the velocity, while \(\epsilon > 0\) stands for the relaxation parameter. Moreover, we assume the stress tensor \(T\) to be smooth and, from now on, we sum on repeated indices. Putting formally \(\epsilon = 0\) in (1.1), we recover the equilibrium relation

\[
\tilde{S} = T(\tilde{F}) + \nabla_x \tilde{v}
\]

(1.2)

and the equilibrium dynamic is hence given by the incomplete parabolic system for viscoelasticity

\[
\begin{align*}
\partial_t \tilde{F}_{i\alpha} - \partial_{\alpha} \tilde{v}_i &= 0 \\
\partial_t \tilde{v}_i - \partial_{\alpha} T_{i\alpha}(\tilde{F}) &= \mu \partial_{\alpha} \partial_{\alpha} \tilde{v}.
\end{align*}
\]

(1.3)

The rigorous proof of this diffusive relaxation limit will be showed in the framework of Sobolev spaces, as long as the reduced system (1.3) admits (regular) solutions (Section 2). To perform this task, we follow the approach already used in [15] to analyze a singular Euler–Poisson approximation for the incompressible Navier–Stokes equations and previously in [8] for singular perturbations of general first order hyperbolic pseudo–differential equations. More precisely, via standard energy estimates, we shall obtain a suitable
donatella donatelli and corrado lattanzio

bound in $H^3$ for the differences $\bar{v} - \hat{v}$, $\bar{F} - \hat{F}$ and $\bar{S} - \hat{S}$, for well-prepared initial data, namely when they approach equilibrium in that space. This result will imply in particular the existence of (classical) solutions of (1.1) in an $\epsilon$–independent time interval $[0, T]$ and their convergence toward the solutions of (1.3) in that interval. Moreover, we shall recast this convergence result also in terms of relative modulated energy techniques (Section 2.1).

The results contained in this paper are part of a more general project which intend to study connections between (semilinear) system of conservation laws with diffusive source terms and (possible degenerate) parabolic systems. The mathematical study of the these connections starts from the papers of Kurtz [9] and McKean [14], where the authors introduced for the first time a parabolic scaling for hyperbolic systems, to put into evidence their diffusive behavior. Afterwards, this scaling has been extensively used in the analysis of hyperbolic–parabolic relaxation limits, both for weak solutions, by means of compensated compactness techniques (among others, see [10, 7] and the references therein), and for classical solutions (see, for instance, [6, 12]). It is worth to observe that, as for the one–dimensional model treated in [5], in the present case the equilibrium system turns out to be incompletely parabolic, which forces ourselves to study the relaxation limit in the context of regular solutions, because standard compensated compactness techniques do not apply. In this framework, in addition to the already mentioned paper [5], further results dealing with diffusive relaxation toward a degenerate parabolic limit are confined to the case of a class of BGK–type approximations to a Cauchy problem for multidimensional degenerate scalar parabolic equations [1].

Finally, we point out that, as in [5], the relaxation limit in the present case has also a physical interpretation in terms of mathematical models in the study of viscoelastic materials [10, 3]. More precisely, it can be viewed as the passage from the viscosity of the memory type to the viscosity of the rate type. Indeed, at the equilibrium, the stress–strain response $\bar{S}$ is given by (1.2), while in (1.1) $\bar{S}$ can be recast as follows

$$\bar{S} = \frac{\mu}{\epsilon} \bar{F} - \int_{-\infty}^{t} \frac{1}{\epsilon} e^{-\frac{t-\tau}{\epsilon}} \left( \frac{\mu}{\epsilon} \bar{F} - T(\bar{F}) \right) (\tau) d\tau,$$

that is, the viscous effect comes from a memory term.

2. Energy estimates and trend to equilibrium

In this section we study the relaxation limit $\epsilon \downarrow 0$ for system (1.1) in the framework of Sobolev spaces, showing the convergence of its solutions toward the solutions of (1.3), as long as the latter exist.

Remark 2.1. The local well–posedness in Sobolev spaces for large data of the model (1.3) is guaranteed by the results on general weakly parabolic systems [4 Theorem 1 and Remark 2]. To recover the lack of parabolicity of the model, the authors proposed in the paper appropriate conditions on
the first order terms, which must be controlled by the diffusion term. These sufficient conditions leading to the local existence result reduce in our case to
\[
3 \sum_{\alpha=1}^{3} |(R_{\alpha} - A_{\alpha}(F)^T)v_{\alpha}|^2 \leq C\mu \sum_{\alpha=1}^{3} |v_{\alpha}|^2, \tag{2.1}
\]
for any \( F \in \text{Mat}_{3 \times 3} \simeq \mathbb{R}^9 \) and for any \( v_{\alpha} \in \mathbb{R}^3, \alpha = 1, 2, 3 \). In (2.1), the matrices \( R_{\alpha}, A_{\alpha}(F)^T \in \text{Mat}_{9 \times 3} \) are defined by
\[
R_{\alpha} = \begin{pmatrix}
\delta_{1\alpha} I_{3 \times 3} \\
\delta_{2\alpha} I_{3 \times 3} \\
\delta_{3\alpha} I_{3 \times 3}
\end{pmatrix}, \quad A_{\alpha}(F) = \nabla_F \begin{pmatrix} T_{1\alpha}(F) \\
T_{2\alpha}(F) \\
T_{3\alpha}(F)
\end{pmatrix} = \left( \frac{\partial T_{i\alpha}(F)}{\partial F_{j\beta}} \right)_{ij\beta},
\]
for \( \alpha = 1, 2, 3 \). Hence condition (2.1) is fulfilled if and only if
\[
\nabla_F T(F) \leq \Gamma I, \tag{2.2}
\]
for any \( F \) under consideration. Condition (2.2) has been considered already in [5] in the one-dimensional case to prove global existence and convergence of relaxation limit in Sobolev norms for large solutions. Moreover, it stands for the subcharacteristic condition assumed in [11] to control the relaxation limit in the hyperbolic–hyperbolic regime. In the multidimensional case, (2.2) gives local existence of smooth solutions to (1.3), thanks to [4], while global smooth solutions can be constructed for small perturbation-type initial data.

Let \( (\widehat{F}, \widehat{v})^T \) be the solution of (1.3), belonging to \( C^1([0, T]; H^j(\mathbb{R}^3)) \) for any \( j > \frac{3}{2} + 2 \) and define \( \widehat{S} \) by (1.2). Then the differences \( v = \bar{v} - \widehat{v}, F = \bar{F} - \widehat{F} \) and \( S = \bar{S} - \widehat{S} \) verify the following semilinear system
\[
\begin{cases}
\partial_t F_{i\alpha} - \partial_\alpha v_i = 0 \\
\partial_t v_i - \partial_\alpha S_{i\alpha} = 0 \\
\partial_t S_{i\alpha} - \frac{\mu}{\epsilon} \partial_\alpha v_i = - \frac{1}{\epsilon} S_{i\alpha} + \frac{1}{\epsilon} \left( T_{i\alpha}(F + \widehat{F}) - T_{i\alpha}(\widehat{F}) \right) - \partial_t \widehat{S}_{i\alpha}.
\end{cases} \tag{2.3}
\]
We rewrite (2.3) in vectorial notations as follows
\[
W_t + A_{\alpha}^\epsilon \partial_\alpha W = R(W; \widehat{W}), \tag{2.4}
\]
where
\[
W = (F, v, S)^T, \quad \widehat{W} = (\widehat{F}, \widehat{v}, \widehat{S})^T,
\]
\[
A_{\alpha}^\epsilon = \begin{pmatrix}
0_{9 \times 9} & -R_{\alpha} & 0_{9 \times 9} \\
0_{3 \times 3} & 0_{3 \times 3} & -R_{\alpha}^T \\
0_{9 \times 9} & -\frac{\mu}{\epsilon} R_{\alpha} & 0_{9 \times 9}
\end{pmatrix} \in \text{Mat}_{21 \times 21},
\]
\[
R(W; \widehat{W}) = \begin{pmatrix}
0 \\
-\frac{1}{\epsilon} S_{i\alpha} + \frac{1}{\epsilon} \left( T_{i\alpha}(F + \widehat{F}) - T_{i\alpha}(\widehat{F}) \right) - \partial_t \widehat{S}_{i\alpha}
\end{pmatrix}.
\]
and system (2.4) is coupled with initial data

\[ W_0(x) = (\tilde{F}_0(x) - \tilde{F}(x, 0), \tilde{v}_0(x) - \tilde{v}(x, 0), \tilde{S}_0(x) - \tilde{S}(x, 0)) \]

In the next lemma, we show system (2.4) is symmetrizable and hence it is equipped with a positive definite energy, at least for \( \epsilon \ll 1 \) (see [5] for the same analysis in 1-D).

**Lemma 2.2.** For \( \epsilon \ll 1 \), the matrix

\[
B' = \begin{pmatrix}
\frac{\mu}{\epsilon} I_{9 \times 9} & 0_{9 \times 3} & -I_{9 \times 9} \\
0_{3 \times 9} & (\frac{\mu}{\epsilon} - 1) I_{3 \times 3} & 0_{3 \times 9} \\
-I_{9 \times 9} & 0_{9 \times 3} & I_{9 \times 9}
\end{pmatrix}
\]

defines a positive definite symmetrizer of (2.4) and

\[ E'(W) = \langle B'W, W \rangle_{L^2(\mathbb{R}^3)} \]

is a positive definite energy for solutions of (2.4).

**Proof.** A direct computation shows

\[
B'A'_0 = \begin{pmatrix}
0_{9 \times 9} & 0_{9 \times 3} & 0_{9 \times 9} \\
0_{3 \times 9} & 0_{3 \times 3} & -(\frac{\mu}{\epsilon} - 1) R_{\alpha} \\
0_{9 \times 9} & -(\frac{\mu}{\epsilon} - 1) R_{\alpha} & 0_{9 \times 9}
\end{pmatrix},
\]

that is, \( B' \) symmetrizes \( A'_0 \). Moreover,

\[
E'(W) = \int_{\mathbb{R}^3} \left( \frac{\mu}{\epsilon} |F|^2 - 2F_{\alpha}S_{\alpha} + \left( \frac{\mu}{\epsilon} - 1 \right) |v|^2 + |S|^2 \right) dx
\]

\[
\geq \frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{\mu}{\epsilon} |F|^2 + \frac{\mu}{\epsilon} |v|^2 + |S|^2 \right) dx,
\]

for \( \epsilon \ll 1 \), say \( \epsilon < \frac{\mu}{4} \). \( \Box \)

The above result implies the energy \( \epsilon E'(W) \) is equivalent to the (square of the) \( L^2 \) norm of \( (F(\cdot, t), v(\cdot, t), \sqrt{\epsilon}S(\cdot, t))^T \), for any \( t > 0 \) and for \( \epsilon \ll 1 \). Thus we define the following high–order energy:

\[
E'(t) = \sum_{|\gamma| \leq 3} \epsilon E'(\partial_{\gamma} W),
\]

where \( \gamma \) is a multi–index. Hence we shall establish our convergence result in terms of the norm defined in (2.5), which is equivalent for \( \epsilon \) sufficiently small to the (square of the) \( H^3 \) norm of the vector \( (F(\cdot, t), v(\cdot, t), \sqrt{\epsilon}S(\cdot, t))^T \).

**Theorem 2.3.** Let \( (\tilde{F}, \tilde{v})^T \) be a solution of (1.3) belonging to the space \( C^1([0, T]; H^4(\mathbb{R}^3)) \) and define \( \tilde{S} \) by (1.2). Let us assume the initial data \( (\tilde{F}_0, \tilde{v}_0, \tilde{S}_0)^T \in H^3(\mathbb{R}^3) \) for (1.1) verify

\[
E'(0) = \sum_{|\gamma| \leq 3} \epsilon E'(\partial_{\gamma} W_0^\epsilon) \to 0,
\]
DIFFUSIVE RELAXATION FOR VISCOELASTICITY

as $\epsilon \downarrow 0$. Then there exist $\epsilon_0 > 0$ and a constant $C_T > 0$ such that for any $0 < \epsilon \leq \epsilon_0$ and any $t \in [0, T]$, there exist a (unique strong) solution $(\tilde{\nabla}^2, \tilde{\nabla}(\cdot, t), \sqrt{\tilde{\tau}}(\cdot, t)) \in H^3(\mathbb{R}^3)$ of (2.7) verifying

$$
E^\epsilon(t) = \sum_{|\alpha| \leq 3} \epsilon E^\epsilon(\partial_\gamma W^\epsilon) \leq C_T(\epsilon^2 + E^\epsilon(0)).
$$

(2.6)

**Proof.** From classical results of local well-posedness of Sobolev solutions for symmetrically hyperbolic systems (see, for instance, [13, 18, 17]), we know there exists a maximal time $T^\epsilon > 0$ such that, for any $t \in [0, T^\epsilon)$,

$$
E^\epsilon(t) \leq M^\epsilon,
$$

(2.7)

where the value $M^\epsilon$, decaying to zero as $\epsilon \downarrow 0$, will be chosen later. The result of the theorem is then equivalent to prove $T^\epsilon \geq T$, which shall be obtained by showing equality in (2.7) cannot be achieved for $T^\epsilon < T$, provided $M^\epsilon$ is properly chosen [15].

To perform the energy estimate needed to prove (2.6), we apply the symmetrizer $B^\epsilon$ to the partial derivative of the nonhomogeneous term $R(W; \tilde{W})$, and we obtain

$$
B^\epsilon \partial_\gamma R(W; \tilde{W}) = \left( \begin{array}{c}
\frac{1}{\epsilon} \partial_\gamma S_{i\alpha} - \frac{1}{\epsilon} \partial_\gamma \left( T_{i\alpha}(F + \tilde{F}) - T_{i\alpha}(\tilde{F}) \right) + \partial_\gamma \partial_\gamma S_{i\alpha} \\
0 \\
- \frac{1}{\epsilon} \partial_\gamma S_{i\alpha} + \frac{1}{\epsilon} \partial_\gamma \left( T_{i\alpha}(F + \tilde{F}) - T_{i\alpha}(\tilde{F}) \right) - \partial_\gamma \partial_\gamma S_{i\alpha}
\end{array} \right).
$$

(2.8)

We start by computing the time derivative of the energy $E^\epsilon(\partial_\gamma W)$. By integrating by parts and by taking into account Lemma 2.2 we get

$$
\frac{d}{dt}E^\epsilon(\partial_\gamma W) = 2(B^\epsilon \partial_\gamma R(W; \tilde{W}), \partial_\gamma W)_{L^2(\mathbb{R}^3)}
$$

$$
= \frac{2}{\epsilon} \langle \partial_\gamma S_{i\alpha}, \partial_\gamma F_{i\alpha} \rangle_{L^2(\mathbb{R}^3)}
$$

$$
+ \frac{2}{\epsilon} \langle \partial_\gamma \left( T_{i\alpha}(F + \tilde{F}) - T_{i\alpha}(\tilde{F}) \right), \partial_\gamma S_{i\alpha} - \partial_\gamma S_{i\alpha} \rangle_{L^2(\mathbb{R}^3)}
$$

$$
- \frac{2}{\epsilon} \| \partial_\gamma S \|^2_{L^2(\mathbb{R}^3)} + \langle \partial_\gamma \partial_\gamma S_{i\alpha}, \partial_\gamma F_{i\alpha} - \partial_\gamma S_{i\alpha} \rangle_{L^2(\mathbb{R}^3)}
$$

$$
= I_1 + I_2 + I_3 + I_4
$$

(2.9)

Now we estimate separately each one of the terms $I_1$, $I_2$, $I_4$. For $I_1$ we have

$$
I_1 = \frac{2}{\epsilon} \langle \partial_\gamma S_{i\alpha}, \partial_\gamma F_{i\alpha} \rangle_{L^2(\mathbb{R}^3)} \leq \frac{1}{2\epsilon} \| \partial_\gamma S \|^2_{L^2(\mathbb{R}^3)} + \frac{2}{\epsilon} \| \partial_\gamma F \|^2.
$$

The term $I_2$ can be controlled as follows

$$
I_2 = \frac{2}{\epsilon} \langle \partial_\gamma \left( T_{i\alpha}(F + \tilde{F}) - T_{i\alpha}(\tilde{F}) \right), \partial_\gamma S_{i\alpha} - \partial_\gamma S_{i\alpha} \rangle_{L^2(\mathbb{R}^3)}
$$

$$
= \frac{2}{\epsilon} \langle \partial_\gamma \left( \nabla_{k\eta}T_{i\alpha}(\tilde{F})F_{k\eta} \right), \partial_\gamma S_{i\alpha} - \partial_\gamma S_{i\alpha} \rangle_{L^2(\mathbb{R}^3)}
$$

$$
+ \frac{2}{\epsilon} \langle \partial_\gamma \left( T_{i\alpha}(F + \tilde{F}) - T_{i\alpha}(\tilde{F}) - \nabla_{k\eta}T_{i\alpha}(\tilde{F})F_{k\eta} \right), \partial_\gamma S_{i\alpha} - \partial_\gamma S_{i\alpha} \rangle_{L^2(\mathbb{R}^3)}
$$
there exists a constant $\hat{\gamma}$. Then, from (2.13) it follows $M_k \leq \frac{1}{4\epsilon} \|\partial_\gamma S\|^2_{L^2(\mathbb{R}^3)} + \frac{2}{\epsilon} \|\partial_\gamma F\|^2_{L^2(\mathbb{R}^3)} + \frac{16}{\epsilon} \|\partial_\gamma (\nabla_{k\eta} T_{ia}(\hat{F}) F_k\eta)\|^2_{L^2(\mathbb{R}^3)}$ + $\frac{16}{\epsilon} \|\partial_\gamma \left( \int_0^1 d\theta \int_0^1 d\omega \nabla^2_{k\eta\xi} T_{ia}(\omega \theta F + \hat{F}) \theta F_k\eta F_l\xi \right)\|^2_{L^2(\mathbb{R}^3)}, \quad (2.10)$

where we used the notations

$\nabla_{k\eta} T_{ia}(F) = \frac{\partial T_{ia}(F)}{\partial F_k\eta}, \quad \nabla^2_{k\eta\xi} T_{ia}(F) = \frac{\partial^2 T_{ia}(F)}{\partial F_k\eta \partial F_l\xi}.$

Finally we estimate $I_4$,

$I_4 = 2\langle \partial_t \partial_\gamma \hat{S}_{ia}, \partial_\gamma F_{ia} - \partial_\gamma S_{ia} \rangle_{L^2(\mathbb{R}^3)}$

$\leq 2\epsilon \|\partial_t \partial_\gamma \hat{S}\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{\epsilon} \|\partial_\gamma S\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{\epsilon} \|\partial_\gamma F\|^2_{L^2(\mathbb{R}^3)}.$ \quad (2.11)

By using (2.4), (2.10) and (2.11) in (2.8) we get

$d\frac{dt}{\epsilon} E^\epsilon(\partial_\gamma W) = 2\epsilon \|\partial_t \partial_\gamma \hat{S}\|^2_{L^2(\mathbb{R}^3)} - \frac{1}{4\epsilon} \|\partial_\gamma S\|^2_{L^2(\mathbb{R}^3)} + \frac{5}{\epsilon} \|\partial_\gamma F\|^2_{L^2(\mathbb{R}^3)}$

$+ \frac{16}{\epsilon} \|\partial_\gamma \left( \int_0^1 d\theta \int_0^1 d\omega \nabla^2_{k\eta\xi} T_{ia}(\omega \theta F + \hat{F}) \theta F_k\eta F_l\xi \right)\|^2_{L^2(\mathbb{R}^3)}$

$+ \frac{16}{\epsilon} \|\partial_\gamma \left( \nabla_{k\eta} T_{ia}(\hat{F}) F_k\eta \right)\|^2_{L^2(\mathbb{R}^3)} \quad (2.12)$

Now we multiply (2.12) by $\epsilon$ we sum in $\gamma, |\gamma| \leq 3$ and by taking into account Sobolev inequalities we end up with

$d\frac{dt}{\epsilon} \mathcal{E}(t) + \frac{1}{4} \|S\|^2_{H^3(\mathbb{R}^3)} \leq C(\|\hat{F}\|_{L^\infty}) \mathcal{E}(t) + c(\|F\|_{L^\infty}, \|\hat{F}\|_{L^\infty}) \mathcal{E}(t)^2$

$+ \mathcal{E}(t)^3 + 2\epsilon^2 \|\partial_t \hat{S}\|^2_{H^3(\mathbb{R}^3)} \quad (2.13)$

To simplify notations, we denote with $\hat{}$ all constants depending on $\hat{W}$ and its derivatives. Fixing $M_\epsilon \leq 1$, in particular we have $\|F\|_{L^\infty(\mathbb{R}^3)} \leq 1$ and hence there exists a constant $\hat{\epsilon}_2 > 0$ such that $c(\|\hat{F}\|_{L^\infty}) + c(\|F\|_{L^\infty}, \|\hat{F}\|_{L^\infty}) \leq \hat{\epsilon}_2$

Then, from (2.13) it follows

$d\frac{dt}{\epsilon} \mathcal{E}(t) \leq \hat{\epsilon}_1 \epsilon^2 + \hat{\epsilon}_2 \mathcal{E}(t), \quad \text{for any } t \in [0, T^\epsilon]$

and the Gronwall Lemma implies

$\mathcal{E}(t) \leq \left( \mathcal{E}(0) + \hat{\epsilon}_1 \epsilon^2 t \right) e^{\hat{\epsilon}_2 t} \quad \text{for any } t \in [0, T^\epsilon]. \quad (2.14)$

If we choose $M_\epsilon = \left( \mathcal{E}(0) + \hat{\epsilon}_1 \epsilon^2 T \right)^{1/2}$, we see that, for $\epsilon$ sufficiently small, we cannot reach inequality in (2.10) for $T^\epsilon < T$. This proves that $T^\epsilon \geq T$ and thus that (2.14) is valid on $[0, T]$, which conclude the proof. \qed
2.1. Relative modulated energy approach. The control of the relaxation limit contained in the above theorem can be recast in terms of relative modulated energy techniques, already used in [2] for semilinear relaxation approximation of incompressible Navier–Stokes equations and in [19, 11] in the hyperbolic–hyperbolic stress relaxation for elasticity with memory. To illustrate this method, we shall obtain here the \( L^2 \) control of the difference between the equilibrium \((\tilde{F}, \tilde{v})\), solution of (1.3), and its relaxation approximation \((F, \hat{v})\). To this end, let us observe that we can eliminate in system (1.1) the off-equilibrium variable \( \bar{S} \) to obtain

\[
\begin{aligned}
\partial_t \bar{F}_{ia} - \partial_{ia} \bar{v}_i &= 0 \\
\partial_t \bar{v}_i - \partial_\alpha \partial_\alpha \bar{v}_i &= 0, \quad (2.15)
\end{aligned}
\]

Then, let \((\bar{F}, \bar{v})\) be a smooth, uniformly bounded solution of (2.15) for any \( t \in [0, T] \) and let us denote with \( \Gamma \) the corresponding bound on the characteristic speed of the relaxation approximation, that is

\[
\nabla \rho T(\bar{F}) \leq \Gamma \rho,
\]

for any \( \bar{F} \) under consideration. For \( \lambda > 1 \) arbitrary, we define our relative modulated energy as follows:

\[
\mathcal{H}_{rm} = \frac{1}{2} (|\bar{v} - \tilde{v}|^2 + |\bar{F} - \tilde{F}|^2) + \epsilon (\bar{v}_i - \tilde{v}_i) \partial_t (\bar{v}_i - \tilde{v}_i) + \frac{1}{2} \epsilon^2 \lambda |\partial_t (\bar{v} - \tilde{v})|^2 + \frac{1}{2} \epsilon \lambda \mu |\nabla_\alpha (\bar{v} - \tilde{v})|^2 + \epsilon \lambda \partial_\alpha (\bar{v}_i - \tilde{v}_i) (T_{ia}(\bar{F}) - T_{ia}(\tilde{F})),
\]

with associated flux given by

\[
\mathcal{Q}_{\alpha, rm} = (\bar{v}_i - \tilde{v}_i) (T_{ia}(\bar{F}) - T_{ia}(\tilde{F})) + \mu (\bar{v}_i - \tilde{v}_i) \partial_\alpha (\bar{v}_i - \tilde{v}_i) + \epsilon \lambda \mu \partial_\alpha (\bar{v}_i - \tilde{v}_i) (T_{ia}(\bar{F}) - T_{ia}(\tilde{F})).
\]

Such an energy is obtained modulating the standard energy estimates of (1.3) by higher order contributions of acoustic waves, to take advantage of the dissipation coming from the relaxation term. The above energy verifies

\[
\begin{aligned}
\partial_t \mathcal{H}_{rm} - \partial_\alpha \mathcal{Q}_{\alpha, rm} + (\mu |\nabla_\alpha (\bar{v} - \tilde{v})|^2 - \epsilon \lambda \partial_\alpha (\bar{v}_i - \tilde{v}_i) \nabla j_\beta T_{ia}(\bar{F}) \partial_\beta (\bar{v}_j - \tilde{v}_j)) \\
+ \epsilon (\lambda - 1) |\partial_t (\bar{v} - \tilde{v})|^2
\end{aligned}
\]

\[= \partial_\alpha (\bar{v}_i - \tilde{v}_i) (\bar{F}_{ia} - \tilde{F}_{ia} - (T_{ia}(\bar{F}) - T_{ia}(\tilde{F})))
\]

\[- \epsilon \partial_\alpha (\bar{v}_i - \tilde{v}_i) - \epsilon^2 \lambda \partial_\alpha (\bar{v}_i - \tilde{v}_i)
\]

\[+ \epsilon \lambda \partial_\alpha (\bar{v}_i - \tilde{v}_i) (\nabla j_\beta T_{ia}(\bar{F}) - \nabla j_\beta T_{ia}(\tilde{F})) \partial_\beta (\bar{F})
\]

\[:= \mathcal{R}^{\epsilon}. \quad (2.16)
\]

Before proving (2.16), let us emphasize that, for \( \epsilon \ll 1 \), say \( \epsilon < \min\{\frac{1}{\rho}, \frac{1}{\rho_{01}}, 1\} \), and for \( \lambda > 1 \) properly chosen,

\[
(\mu |\nabla_\alpha (\bar{v} - \tilde{v})|^2 - \epsilon \lambda \partial_\alpha (\bar{v}_i - \tilde{v}_i) \nabla j_\beta T_{ia}(\bar{F}) \partial_\beta (\bar{v}_j - \tilde{v}_j)) \geq C_{11} |\nabla_\alpha (\bar{v} - \tilde{v})|^2,
\]
\[ C_2 \varphi^\varepsilon(t) \geq \int_{\mathbb{R}^3} \mathcal{H}_{rm} \, dx \geq C_3 \varphi^\varepsilon(t), \]

where
\[ \varphi^\varepsilon(t) := \int_{\mathbb{R}^3} \left( |\bar{v} - \tilde{v}|^2 + |\bar{F} - \tilde{F}|^2 + \epsilon^2 |\partial_t (\bar{v} - \tilde{v})|^2 + \epsilon |\nabla_{\alpha} (\bar{v} - \tilde{v})|^2 \right) \, dx \]
and \( C_1, C_2, C_3 > 0 \) depend on \( \mu \) and \( \Gamma \) and not on \( \epsilon \). Moreover, using again Schwartz inequality, we control the error terms in (2.16) to obtain
\[ |\mathcal{R}^\varepsilon| \leq (\epsilon^2 K_1 + K_2 \mathcal{H}_{rm}) + \frac{C_1}{2} |\nabla_{\alpha} (\bar{v} - \tilde{v})|^2, \]
with \( K_1, K_2 > 0 \) depending only on \( \mu, \Gamma \) and the equilibrium solution \((\tilde{F}, \bar{v})\) and its derivatives and not on \( \epsilon \). Hence, we integrate (2.10) with respect to \( x \) and we use Gronwall Lemma to get for any \( t \in [0, T]\)
\[ \varphi^\varepsilon(t) \leq C_T (\epsilon^2 + \varphi^\varepsilon(0)), \]
that is, the same kind of control of the relaxation limit obtained in Theorem 2.3 for well-prepared initial data, that is for \( \varphi^\varepsilon(0) \to 0 \) as \( \epsilon \downarrow 0 \).

To prove (2.16), we first observe that the differences \( \bar{F} - \tilde{F} \) and \( \bar{v} - \tilde{v} \) verify
\begin{align*}
\begin{cases}
\partial_t (\bar{F}_{ia} - \tilde{F}_{ia}) = \partial_{\alpha} (\bar{v}_i - \tilde{v}_i) \\
\partial_t (\bar{v}_i - \tilde{v}_i) = \partial_{\alpha} (T_{ia}(\bar{F}) - T_{ia}(\tilde{F})) + \mu \partial_{\alpha} \partial_{\alpha} (\bar{v}_i - \tilde{v}_i) - \epsilon \partial_t^2 (\bar{v}_i - \tilde{v}_i) - \epsilon \partial_{\alpha}^2 \tilde{v}_i.
\end{cases}
\end{align*}

(2.17)

Then we multiply (2.17)1 by \( \bar{F}_{ia} - \tilde{F}_{ia} \) and (2.17)2 by \( \bar{v}_i - \tilde{v}_i \) and we sum over all indices to obtain
\begin{align*}
\partial_t \left( \frac{1}{2} |\bar{v} - \tilde{v}|^2 + |\bar{F} - \tilde{F}|^2 \right) + \epsilon (\bar{v}_i - \tilde{v}_i) \partial_t (\bar{v}_i - \tilde{v}_i) \\
- \partial_{\alpha} \left( (\bar{v}_i - \tilde{v}_i) (T_{ia}(\bar{F}) - T_{ia}(\tilde{F})) + \mu (\bar{v}_i - \tilde{v}_i) \partial_{\alpha} (\bar{v}_i - \tilde{v}_i) \right) \\
+ \mu |\nabla_{\alpha} (\bar{v} - \tilde{v})|^2 - \epsilon |\partial_t (\bar{v} - \tilde{v})|^2 \\
= \partial_{\alpha} (\bar{v}_i - \tilde{v}_i) (\bar{F}_{ia} - \tilde{F}_{ia} - (T_{ia}(\bar{F}) - T_{ia}(\tilde{F}))) - \epsilon \partial_t^2 \tilde{v}_i (\bar{v}_i - \tilde{v}_i).
\end{align*}

(2.18)

To take advantage of the dissipation coming from the relaxation, we must modulate the above relation with an higher order energy, coming from the damped wave equation in (2.17)2. To this end, we fix \( \lambda > 1 \) and we multiply this relation by \( \epsilon \lambda \partial_t (\bar{v}_i - \tilde{v}_i) \) and sum over \( i \) to obtain
\begin{align*}
\partial_t \left( \frac{1}{2} \epsilon^2 \lambda |\partial_t (\bar{v} - \tilde{v})|^2 + \frac{1}{2} \epsilon \lambda \mu |\nabla_{\alpha} (\bar{v} - \tilde{v})|^2 \right) \\
- \partial_{\alpha} \left( \epsilon \lambda \mu \partial_t (\bar{v}_i - \tilde{v}_i) \partial_{\alpha} (\bar{v}_i - \tilde{v}_i) \right) \\
+ \epsilon^2 \lambda |\partial_t (\bar{v} - \tilde{v})|^2 - \epsilon \lambda \partial_t (\bar{v}_i - \tilde{v}_i) \partial_{\alpha} (T_{ia}(\bar{F}) - T_{ia}(\tilde{F})) \\
= -\epsilon^2 \lambda \partial_t \tilde{v}_i \partial_t (\bar{v}_i - \tilde{v}_i).
\end{align*}

(2.19)
We interchange $x$ and $t$ derivatives in the last term in the left of (2.19) as follows
\begin{align*}
- \partial_t (\bar{v}_i - \hat{v}_i) & \partial_\alpha (T_{i\alpha}(\bar{F}) - T_{i\alpha}(\hat{F})) = - \partial_\alpha (\bar{v}_i - \hat{v}_i) \partial_t (T_{i\alpha}(\bar{F}) - T_{i\alpha}(\hat{F})) \\
+ \partial_t \left( \partial_\alpha (\bar{v}_i - \hat{v}_i) (T_{i\alpha}(\bar{F}) - T_{i\alpha}(\hat{F})) \right) & - \partial_\alpha \left( \partial_t (\bar{v}_i - \hat{v}_i) (T_{i\alpha}(\bar{F}) - T_{i\alpha}(\hat{F})) \right) \\
= & - \partial_\alpha (\bar{v}_i - \hat{v}_i) \nabla_{j\beta} T(\bar{F}) \partial_\beta (\bar{v}_j - \hat{v}_j) \\
& - \partial_\alpha (\bar{v}_i - \hat{v}_i) \left( \nabla_{j\beta} T_{i\alpha}(\bar{F}) - \nabla_{j\beta} T_{i\alpha}(\hat{F}) \right) \partial_t \hat{F}_{j\beta} \\
+ & \partial_t \left( \partial_\alpha (\bar{v}_i - \hat{v}_i) (T_{i\alpha}(\bar{F}) - T_{i\alpha}(\hat{F})) \right) - \partial_\alpha \left( \partial_t (\bar{v}_i - \hat{v}_i) (T_{i\alpha}(\bar{F}) - T_{i\alpha}(\hat{F})) \right).
\end{align*}

Finally, using this identity, (2.19) and (2.18) give (2.16).

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Donatella Donatelli — Dipartimento di Matematica Pura ed Applicata, Università di L’Aquila, Via Vetoio, 67010 Coppito (AQ), Italy
E-mail address: donatell@univaq.it

Corrado Lattanzio — Sezione di Matematica per l’Ingegneria, Dipartimento di Matematica Pura ed Applicata, Università di L’Aquila, Piazzale E. Pontieri, 2, Monteluco di Roio, 67040 L’Aquila, Italy
E-mail address: corrado@univaq.it