Regularity results for a class of widely degenerate parabolic equations

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Abstract

Motivated by applications to gas filtration problems, we study the regularity of weak solutions to the strongly degenerate parabolic PDE

$$u_t - \text{div} \left( (|Du| - \nu)^{p-1} \frac{Du}{|Du|} \right) = f \quad \text{in } \Omega_T = \Omega \times (0, T),$$

where $$\Omega$$ is a bounded domain in $$\mathbb{R}^n$$ for $$n \geq 2$$, $$p \geq 2$$, $$\nu$$ is a positive constant and $$(\cdot)_+$$ stands for the positive part. Assuming that the datum $$f$$ belongs to a suitable Lebesgue-Sobolev parabolic space, we establish the Sobolev spatial regularity of a nonlinear function of the spatial gradient of the weak solutions, which in turn implies the existence of the weak time derivative $$u_t$$. The main novelty here is that the structure function of the above equation satisfies standard growth and ellipticity conditions only outside a ball with radius $$\nu$$ centered at the origin. We would like to point out that the first result obtained here can be considered, on the one hand, as the parabolic counterpart of an elliptic result established in [5], and on the other hand as the extension to a strongly degenerate context of some known results for less degenerate parabolic equations.

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1 Introduction and statement of the results

In this paper, we study the local regularity properties of weak solutions $$u : \Omega_T \to \mathbb{R}$$ to strongly degenerate parabolic equations of the type

$$u_t - \text{div} \left( (|Du| - \nu)^{p-1} \frac{Du}{|Du|} \right) = f \quad \text{in } \Omega_T = \Omega \times (0, T),$$

(1.1)

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for exponents $p \geq 2$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$), $T > 0$, $\nu$ is a positive constant and $(\cdot)_+$ stands for the positive part.

The main feature of this PDE is that the structure function satisfies standard growth and ellipticity conditions for a growth rate $p \geq 2$, but only outside a ball with radius $\nu$ centered at the origin.

The elliptic version of the above equation naturally arises in optimal transport problems with congestion effects, and the regularity properties of its weak solutions have been widely investigated: see, for example, [2, 3, 4] and [5]. As far as we know, no parabolic counterpart of such works is available in the literature. On the other hand, we would like to point out that a motivation for studying equations of the type (1.1) can be found in Section 1.1 below.

Here, we establish the Sobolev spatial regularity of a nonlinear function of the spatial gradient $Du$ of the weak solutions to equation (1.1) (see Theorem 1.1 below), which in turn implies the Sobolev time regularity of the solutions (cf. Theorem 1.2), by assuming that the datum $f$ belongs to a suitable Lebesgue-Sobolev parabolic space. These results are obtained by adapting the techniques for the evolutionary $p$-Laplacian to this more degenerate context. In fact, for less degenerate parabolic problems, these issues have been widely investigated, as one can see, for example, in [9, 10] (where $f = 0$) and in [20]. Moreover, establishing the Sobolev regularity of the solutions with respect to time, once the higher differentiability in space has been obtained, is a quite usual fact in these problems: see, for instance, [15, 16, 17].

The distinguishing feature of equation (1.1) is that the principal part behaves like a $p$-Laplace operator only at infinity. Before giving the main results of this paper, let us summarize a few previous results on this topic: the regularity of solutions to parabolic problems with asymptotic structure of $p$-Laplacian type has been explored in [12], where a BMO regularity has been proved for solutions to asymptotically parabolic systems in the case $p = 2$ and $f = 0$ (see also [13], where the local Lipschitz continuity of weak solutions with respect to the spatial variable is established). In addition, we want to mention the results contained in [6], where nonhomogeneous parabolic problems involving a discontinuous nonlinearity and an asymptotic regularity in divergence form of $p$-Laplacian type are considered. There, the authors establish a global Calderón-Zygmund estimate by converting a given asymptotically regular problem to a suitable regular problem.

One of the main novelties of this work is an observation of interpolative nature, which allows us to suitably weaken the assumptions on the datum $f$: this comes from an idea that has already been exploited in the recent paper [7], in the elliptic setting. In fact, our assumption on the regularity of $f$ is weaker than those considered in the mentioned works.

The first result we prove in this paper is the following theorem, which can be considered as the parabolic counterpart of Theorem 4.2 in [5]. We refer to Section 2 for notation and definitions.

**Theorem 1.1.** Let $n \geq 2$, $p \geq 2$, $\frac{np + 4}{np + 4 - n} \leq \vartheta < \infty$ and $f \in L^\vartheta (0, T; W^{1,\vartheta}(\Omega))$. Moreover, assume that

$u \in C^0 \left( (0, T); L^2(\Omega) \right) \cap L^p (0, T; W^{1,p}(\Omega))$

is a weak solution of equation (1.1). Then the solution satisfies

$H^p_\vartheta(Du) \in L^2_{\text{loc}} \left( 0, T; W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^n) \right)$,

where

$H^p_\vartheta(Du) := (|Du| - \nu)^{p/2} \frac{Du}{|Du|}$. 

Furthermore, the following estimate

\[ \int_{Q_{\varepsilon/2}(z_0)} |DH^\varepsilon(Du)|^2 \, dz \leq c \left( \nu \|Df\|_{L^\varepsilon(Q_{R_0})} + \|Df\|_{L^\varepsilon(Q_{R_0})}^{\frac{np+4}{np+2-n}} \right) + \frac{c}{R^2} \left( \|Du\|_{L^p(Q_{R_0})}^2 + \|Du\|_{L^p(Q_{R_0})}^2 + \nu^p + \nu^2 \right) \]  

(1.2)

holds true for any parabolic cylinder \( Q_\varepsilon(z_0) \subset Q_R(z_0) \subset Q_{R_0}(z_0) \Subset \Omega_T \) and a positive constant \( c \) depending at most on \( n, p, \vartheta \) and \( R_0 \).

One of the main tools in the proof of Theorem 1.1 is the difference quotients technique used in the spatial directions. Here we will argue as in [9, Lemma 5.1] and [10, Theorem 4.1], but we need to take into account the strong degeneracy of equation (1.1). This is why we obtain the Sobolev spatial regularity not for the usual function \( V_\varrho(Du) := |Du|^\frac{2}{p-2} Du \), but for the vector field \( H^\varrho(Du) \), which vanishes in the set where equation (1.1) becomes degenerate.

The proof of Theorem 1.1 is also based on a comparison argument with the solutions of a family of much less degenerate parabolic problems having a smooth inhomogeneity. To the solutions of these problems we can apply the \( a \ priori \) estimates that we establish in Section 3 whose constants are independent of the less degenerate principal parts. Thereafter, we show that the \( L^p \)-norms of the spatial gradients of such solutions are uniformly bounded, and this allows us to transfer the higher differentiability in space of the comparison maps to the solution of our equation (see Section 4 below).

As we anticipated earlier, from the previous result we can easily deduce that \( u \) admits a weak time derivative \( u_t \), which belongs to the local Lebesgue space \( L^\min\{\vartheta, \varrho'\}(\Omega_T) \), where \( \varrho' = p/(p-1) \) is the conjugate exponent of \( p \). The idea is roughly as follows. Consider equation (1.1); since the above theorem tells us that in a certain pointwise sense the second spatial derivatives of \( u \) exist, then we may develop the expression under the divergence symbol; this will give us an expression that equals \( u_t \), from which we get the desired summability of the time derivative. Such an argument must be made more rigorous. Furthermore, we also need to make explicit \( a \ priori \) local estimates. These are provided in the following

**Theorem 1.2.** Under the assumptions of Theorem 1.1 the time derivative of the solution exists in the weak sense and satisfies

\[ \partial_t u \in L^\min\{\vartheta, \varrho'\}(\Omega_T). \]

Furthermore, the following estimate

\[ \left( \int_{Q_{\varepsilon/2}(z_0)} |\partial_t u|^\min\{\vartheta, \varrho'\} \, dz \right)^{\frac{1}{\min\{\vartheta, \varrho'\}}} \leq c \|f\|_{L^\varrho(Q_{R_0})} + c \|Du\|_{L^p(Q_{R_0})}^{\frac{p-2}{p}} \left( \nu \|Df\|_{L^\varrho(Q_{R_0})} + \|Df\|_{L^\varepsilon(Q_{R_0})}^{\frac{np+4}{np+2-n}} \right)^{\frac{1}{2}} + \frac{c}{R} \left( \|Du\|_{L^p(Q_{R_0})}^2 + \|Du\|_{L^p(Q_{R_0})}^2 + \nu^p + \nu^2 \right) \|Du\|_{L^p(Q_{R_0})}^{\frac{2}{p}} \]  

(1.3)

holds true for any parabolic cylinder \( Q_\varepsilon(z_0) \subset Q_R(z_0) \subset Q_{R_0}(z_0) \Subset \Omega_T \) and a positive constant \( c \) depending on \( n, p, \vartheta \) and \( R_0 \).
To conclude this introduction, it is worth pointing out that, starting from the weaker assumption

\[ f \in L^{np+4-n}(0, T; W^{1, np+4-n}(\Omega)) \],

Sobolev regularity results such as those of Theorems 1.1 and 1.2 seem not to have been established yet for weak solutions to parabolic PDEs that are far less degenerate than equation (1.1). In particular, the results contained in this paper can be easily extended to the case \( \nu = 0 \), i.e. to the evolutionary \( p \)-Poisson equation

\[ ut - \text{div} \left( |Du|^{p-2} Du \right) = f \quad \text{in} \ \Omega_T, \quad (1.4) \]

with \( f \in L^{np+4-n}(0, T; W^{1, np+4-n}(\Omega)) \). Therefore, our results permit to improve the existing literature, already for equations of the form (1.4), which exhibit a milder degeneracy.

1.1 Motivation

Before describing the structure of this paper, we wish to motivate our study by stressing that, in the case \( n \leq 3 \), degenerate equations of the form (1.1) may arise in gas filtration problems taking into account the initial pressure gradient.

The existence of significant deviations from the linear Darcy filtration law has been established for many systems consisting of a fluid and a porous medium (e.g., the filtration of a gas in argillous rocks). One of the manifestations of this nonlinearity is the existence of a limiting (initial) pressure gradient, i.e. the minimum value of the pressure gradient for which fluid motion occurs. In general, fluid motion still takes place for subcritical values of the pressure gradient, but very slowly; on reaching the limiting value of the pressure gradient, there is a marked acceleration of the filtration. Therefore, the limiting-gradient concept provides a good approximation for velocities which are not too low.

In accordance with some experimental results (see [1]), under certain physical conditions one can take the gas filtration law in the very simple form

\[
\begin{cases}
  \mathbf{v} = -\frac{k}{\mu} D\mathbf{P} \left[ 1 - \frac{G}{|D\mathbf{P}|^2} \right] & \text{if } |D\mathbf{P}| \geq G, \\
  \mathbf{v} = 0 & \text{if } |D\mathbf{P}| < G,
\end{cases}
\]

where \( \mathbf{v} = \mathbf{v}(x, t) \) is the filtration velocity, \( k \) is the rock permeability, \( \mu \) is the gas viscosity, \( \mathbf{P} = \mathbf{P}(x, t) \) is the pressure and \( G \) is a positive constant. Under this assumption we obtain a particularly simple expression for the gas mass velocity (flux) \( j \), which contains only the gradient of the pressure squared, just as in the usual gas filtration problems:

\[
\begin{cases}
  j = \rho\mathbf{v} = -\frac{k}{2\mu C} D\mathbf{P}^2 - G \frac{D\mathbf{P}^2}{|D\mathbf{P}|^2} & \text{if } |D\mathbf{P}| > G, \\
  j = 0 & \text{if } |D\mathbf{P}| \leq G,
\end{cases}
\]

(1.5)

where \( \rho \) is the gas density and \( C \) is a positive constant. Substituting expression (1.5) into the gas mass-conservation equation, we obtain the basic equation for the pressure:

\[
\begin{cases}
  \frac{\partial \mathbf{P}}{\partial t} = \frac{k}{2\mu m} \text{div} \left( D\mathbf{P}^2 - G \frac{D\mathbf{P}^2}{|D\mathbf{P}|^2} \right) & \text{if } |D\mathbf{P}| > G, \\
  \frac{\partial \mathbf{P}}{\partial t} = 0 & \text{if } |D\mathbf{P}| \leq G,
\end{cases}
\]

(1.6)

where \( m \) is a positive constant. Equation (1.6) implies, first of all, that the steady gas motion is...
described by the same relations as in the steady motion of an incompressible fluid if we replace
the pressure of the incompressible fluid with the square of the gas pressure. Moreover, if the
gas pressure differs very little from some constant pressure \( P_0 \), or if the gas pressure differs
considerably from a constant value only in regions where the gas motion is nearly steady, then
the equation for the gas filtration in the region of motion can be “linearized” following L. S.
Leibenson, thus obtaining (see (1) again):

\[
\begin{cases}
\frac{\partial P^2}{\partial t} = \frac{kP_0}{\mu} \text{div} \left[ D P^2 - G \frac{P^2}{|D P^2|} \right] & \text{if } |D P^2| > G, \\
\frac{\partial P^2}{\partial t} = 0 & \text{if } |D P^2| \leq G.
\end{cases}
\]  

(1.7)

Now, setting \( u = P^2 \) and performing an appropriate scaling, equation (1.7) turns into

\[
\frac{\partial u}{\partial t} - \text{div} \left[ (|Du| - 1) + \frac{Du}{|Du|} \right] = 0,
\]

which is nothing but equation (1.1) in the case \( p = 2, \nu = 1 \) and \( f = 0 \). This is why (1.1) is
sometimes called the Leibenson equation in the literature.

The paper is organized as follows. Section 2 is devoted to the preliminaries: after a list
of some classic notations and some essentials estimates, we recall the basic properties of the
difference quotients of Sobolev functions. In Section 3 we establish some \textit{a priori} estimates
that will be needed to demonstrate Theorem 1.1, whose proof is contained in Section 4. Using
the existence of second spatial derivatives, we then infer the existence of the weak time derivative of the solutions. The corresponding arguments, which imply Theorem 1.2 are given in
Section 5.

2 Notations and preliminaries

In this paper we shall denote by \( C \) or \( c \) a general positive constant that may vary on different
occasions. Relevant dependencies on parameters and special constants will be suitably empha-
sized using parentheses or subscripts. The norm we use on \( \mathbb{R}^n \) will be the standard Euclidean
one and it will be denoted by \( | \cdot | \). In particular, for the vectors \( \xi, \eta \in \mathbb{R}^n \), we write \( \langle \xi, \eta \rangle \) for
the usual inner product and \( |\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}} \) for the corresponding Euclidean norm.

For points in space-time, we will frequently use abbreviations like \( z = (x, t) \) or \( z_0 = (x_0, t_0) \),
for spatial variables \( x, x_0 \in \mathbb{R}^n \) and times \( t, t_0 \in \mathbb{R} \). We also denote by \( B(x_0, \rho) = B_\rho(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| < \rho \} \) the open ball with radius \( \rho > 0 \) and center \( x_0 \in \mathbb{R}^n \); when not impor-
tant, or clear from the context, we shall omit to denote the center as follows: \( B_\rho \equiv B(x_0, \rho) \).
Unless otherwise stated, different balls in the same context will have the same center. Moreover,
we use the notation

\[
Q_\rho(z_0) := B_\rho(x_0) \times (t_0 - \rho^2, t_0), \quad z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}, \quad \rho > 0,
\]

for the backward parabolic cylinder with vertex \((x_0, t_0)\) and width \( \rho \). We shall sometimes omit
the dependence on the vertex when all the cylinders occurring in a proof share the same vertex.
Finally, for a general cylinder \( Q = A \times (t_1, t_2) \), where \( A \subset \mathbb{R}^n \) and \( t_1 < t_2 \), we denote by

\[
\partial_{\text{par}} Q := (\bar{A} \times \{ t_1 \}) \cup (\partial A \times (t_1, t_2))
\]

the usual \textit{parabolic boundary} of \( Q \).
We now recall some tools that will be useful to prove our results. For the auxiliary function $H_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$H_\lambda (\xi) := \begin{cases} (|\xi| - \nu)^+ \frac{\lambda}{|\xi|} & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } \xi = 0, \end{cases}$$

where $\lambda > 0$ is a parameter, we record the following estimates, which can be obtained by suitably modifying the proof of Lemma 4.1 in [5].

**Lemma 2.1.** If $2 \leq p < \infty$, then for every $\xi, \eta \in \mathbb{R}^n$ we get

$$\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle \geq \frac{4}{p^2} \left| H_\frac{p}{2}(\xi) - H_\frac{p}{2}(\eta) \right|^2,$$

$$|H_{p-1}(\xi) - H_{p-1}(\eta)| \leq (p-1) \left( \left| H_\frac{p}{2}(\xi) \right|^\frac{p}{p-2} + \left| H_\frac{p}{2}(\eta) \right|^\frac{p}{p-2} \right) \left| H_\frac{p}{2}(\xi) - H_\frac{p}{2}(\eta) \right|.$$

In the following, we shall also use the auxiliary function $V_p : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$V_p (\xi) := |\xi|^{\frac{p-2}{2}} \xi,$$

where $p \geq 2$. For the above function, we recall the following estimates:

**Lemma 2.2.** If $2 \leq p < \infty$, then for every $\xi, \eta \in \mathbb{R}^n$ we get

$$|V_p (\xi) - V_p (\eta)|^2 \leq \frac{p^2}{4} \langle |\xi|^{p-2} \xi - |\eta|^{p-2} \eta, \xi - \eta \rangle,$$

$$||\xi|^{p-2} \xi - |\eta|^{p-2} \eta| \leq (p-1) \left( |\xi|^{\frac{p-2}{2}} + |\eta|^{\frac{p-2}{2}} \right) |V_p (\xi) - V_p (\eta)|.$$
for some exponents $1 \leq p, q < \infty$. Then the following estimate
\[
\int_{Q_r(z_0)} |v|^{p+pq/n} \, dz \leq c \left( \sup_{s \in (t_0-r^2,t_0)} \int_{B_r(x_0)} |v(x,s)|^q \, dx \right)^{p/n} \int_{Q_r(z_0)} |Dv|^p \, dz
\]
holds true for a positive constant $c$ depending at most on $n$, $p$ and $q$.

We conclude by recalling the following

**Definition 2.5.** A function $u \in C^0((0,T); L^2(\Omega)) \cap L^p(0,T; W^{1,p}(\Omega))$ is a weak solution of equation (1.1) if and only if for any test function $\varphi \in C_0^\infty(\Omega_T)$ the following integral identity holds:
\[
\int_{\Omega_T} (u \cdot \partial_t \varphi - \langle H_{p-1}(Du), D\varphi \rangle) \, dz = -\int_{\Omega_T} f \varphi \, dz. \tag{2.1}
\]

### 2.1 Difference quotients

We recall here the definition and some elementary properties of the difference quotients that will be useful in the following (see, for example, [11]).

**Definition 2.6.** For every vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^N$ the finite difference operator in the direction $x_s$ is defined by
\[
\tau_{s,h}F(x) = F(x + he_s) - F(x),
\]
where $h \in \mathbb{R}$, $e_s$ is the unit vector in the direction $x_s$ and $s \in \{1, \ldots, n\}$.

The difference quotient of $F$ with respect to $x_s$ is defined for $h \in \mathbb{R} \setminus \{0\}$ as
\[
\Delta_{s,h}F(x) = \frac{\tau_{s,h}F(x)}{h}.
\]

When no confusion can arise, we shall omit the index $s$ and simply write $\tau_h$ or $\Delta_h$ instead of $\tau_{s,h}$ or $\Delta_{s,h}$, respectively.

**Proposition 2.7.** Let $F$ be a function such that $F \in W^{1,p}(\Omega)$, with $p \geq 1$, and let us consider the set
\[
\Omega_{|h|} := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > |h| \}.
\]

Then:

(a) $\Delta_hF \in W^{1,p}(\Omega_{|h|})$ and $D_i(\Delta_hF) = \Delta_h(D_iF)$ for every $i \in \{1, \ldots, n\}$.

(b) If at least one of the functions $F$ or $G$ has support contained in $\Omega_{|h|}$, then
\[
\int_{\Omega} F \Delta_hG \, dx = -\int_{\Omega} G \Delta_{-h}F \, dx.
\]

(c) We have
\[
\Delta_h(FG)(x) = F(x + he_s)\Delta_hG(x) + G(x)\Delta_hF(x).
\]
The next result about the finite difference operator is a kind of integral version of Lagrange Theorem and can be obtained by combining Lemma 8.1 in [11] with the theorem on page 3 of [19].

**Lemma 2.8.** If \(0 < \rho < R\), \(|h| < \frac{R-\rho}{2}\), \(1 < q < +\infty\) and \(F \in L^1_{\text{loc}}(B_R, \mathbb{R}^N)\) is such that \(DF \in L^q(B_R, \mathbb{R}^{N \times n})\), then

\[
\int_{B_\rho} |\tau_h F(x)|^q \, dx \leq c^q(n) |h|^q \int_{B_R} |DF(x)|^q \, dx.
\]

Moreover, if \(F \in L^q(B_R, \mathbb{R}^N)\) then we have

\[
\int_{B_\rho} |F(x + he_s)|^q \, dx \leq \int_{B_R} |F(x)|^q \, dx.
\]

Finally, we recall the following fundamental result, whose proof can be found in [11, Lemma 8.2]:

**Lemma 2.9.** Let \(F : \mathbb{R}^n \rightarrow \mathbb{R}^N\), \(F \in L^q(B_R, \mathbb{R}^N)\) with \(1 < q < +\infty\). Suppose that there exist \(\rho \in (0, R)\) and a constant \(M > 0\) such that

\[
\sum_{s=1}^n \int_{B_\rho} |\tau_{s,h} F(x)|^q \, dx \leq M^q |h|^q
\]

for every \(h\) with \(|h| < \frac{R-\rho}{2}\). Then \(F \in W^{1,q}(B_R, \mathbb{R}^N)\). Moreover

\[
\|DF\|_{L^q(B_R)} \leq M
\]

and

\[
\Delta_{s,h} F \rightarrow D_s F \quad \text{in } L^q_{\text{loc}}(B_R), \text{ as } h \rightarrow 0,
\]

for each \(s \in \{1, \ldots, n\}\).

### 3 A priori estimates

In this section, we shall derive some a priori estimates for the solutions \(u_\varepsilon\) of equation (3.2) below, which is much less degenerate than equation (1.1). Such estimates will play a key role in the proof of Theorem 1.1 (see Section 4).

Let \(n \geq 2, p \geq 2, \frac{np+1}{np+4-n} \leq \vartheta < \infty\) and \(f \in L^{\vartheta}(0, T; W^{1,\vartheta} (\Omega))\). For \(\varepsilon \in [0, 1]\) and a couple of standard, non-negative, radially symmetric mollifiers \(\phi_1 \in C^\infty_0(B_1(0))\) and \(\phi_2 \in C^\infty_0((-1, 1))\) we write

\[
f_\varepsilon(x, t) := \int_{-1}^1 \int_{B_1(0)} f(x - \varepsilon y, t - \varepsilon s) \phi_1(y) \phi_2(s) \, dy \, ds,
\]

where \(f\) is meant to be extended by zero outside \(\Omega_T\). Let us observe that \(f_0 = f\) and \(f_\varepsilon \in C^\infty(\Omega_T)\) for every \(\varepsilon \in (0, 1]\).

Now we consider a domain in space-time denoted by \(\Omega'_{1,2} := \Omega' \times (T_1, T_2)\), where \(\Omega' \subseteq \Omega\) is...
a bounded domain with smooth boundary and \((T_1, T_2) \subseteq (0, T)\). For our purposes, in the following we will need the definitions below.

**Definition 3.1.** Let \(\varepsilon \in (0, 1]\). A function \(u_\varepsilon \in C^0((T_1, T_2); L^2(\Omega')) \cap L^p(T_1, T_2; W^{1,p}(\Omega'))\) is a weak solution of the equation

\[
\partial_t u_\varepsilon - \text{div} \left( H_{p-1}(Du_\varepsilon) + \varepsilon |Du_\varepsilon|^{p-2}Du_\varepsilon \right) = f_\varepsilon \quad \text{in } \Omega'_{1,2}
\]

if and only if for any test function \(\varphi \in C^\infty_0(\Omega'_{1,2})\) the following integral identity holds:

\[
\int_{\Omega'_{1,2}} \left( u_\varepsilon \cdot \partial_t \varphi - \langle H_{p-1}(Du_\varepsilon) + \varepsilon |Du_\varepsilon|^{p-2}Du_\varepsilon, D\varphi \rangle \right) \, dz = -\int_{\Omega'_{1,2}} f_\varepsilon \varphi \, dz. \tag{3.3}
\]

**Definition 3.2.** Let \(\varepsilon \in (0, 1]\) and \(g \in C^0([T_1, T_2]; L^2(\Omega')) \cap L^p(T_1, T_2; W^{1,p}(\Omega'))\). In this framework, we identify a function

\(u_\varepsilon \in C^0([T_1, T_2]; L^2(\Omega')) \cap L^p(T_1, T_2; W^{1,p}(\Omega'))\)

as a weak solution of the Cauchy-Dirichlet problem

\[
\left\{
\begin{array}{ll}
\partial_t u_\varepsilon - \text{div} \left( H_{p-1}(Du_\varepsilon) + \varepsilon |Du_\varepsilon|^{p-2}Du_\varepsilon \right) = f_\varepsilon & \text{in } \Omega'_{1,2}, \\
u_\varepsilon = g & \text{on } \partial_{\text{par}} \Omega'_{1,2},
\end{array}
\right. \tag{3.4}
\]

if and only if \(3.2\) holds and moreover, \(u_\varepsilon \in g + L^p(T_1, T_2; W^{1,p}_0(\Omega'))\) and \(u_\varepsilon(\cdot, T_1) = g(\cdot, T_1)\) in the \(L^2\)-sense, that is

\[
\lim_{t \to (T_1)^+} \|u_\varepsilon(\cdot, t) - g(\cdot, T_1)\|_{L^2(\Omega')} = 0. \tag{3.5}
\]

Therefore, the initial condition \(u_\varepsilon = g\) on \(\Omega' \times \{T_1\}\) has to be understood in the usual \(L^2\)-sense \(3.5\), while the condition \(u_\varepsilon = g\) on the lateral boundary \(\partial \Omega' \times (T_1, T_2)\) has to be meant in the sense of traces, i.e. \((u_\varepsilon - g)(\cdot, t) \in W^{1,p}_0(\Omega')\) for almost every \(t \in (T_1, T_2)\).

**Remark 3.3.** The regularized parabolic equation \(3.2\) fulfills standard \(p\)-growth conditions. The advantage of considering the associated Cauchy-Dirichlet problem \(3.4\) arises from the fact that the existence of a unique solution \(u_\varepsilon \in C^0((T_1, T_2); L^2(\Omega')) \cap L^p(T_1, T_2; W^{1,p}(\Omega'))\) satisfying the requirements of Definition \(3.2\) can be ensured by the classic existence theory for parabolic equations (see [18] Chapter 2, Theorem 1.2 and Remark 1.2).

Now we shall use the well-known difference quotients method in the spatial directions together with the properties of the functions \(H_\lambda\) and \(V_p\) to establish the following result.

**Proposition 3.4.** Let \(\varepsilon \in (0, 1]\), \(n \geq 2\), \(p \geq 2\), \(\frac{np+4}{np+4-n} \leq \vartheta < \infty\) and \(f \in L^\vartheta(0, T; W^{1,\vartheta}(\Omega))\). Moreover, assume that

\(u_\varepsilon \in C^0((T_1, T_2); L^2(\Omega')) \cap L^p(T_1, T_2; W^{1,p}(\Omega'))\)

is a weak solution of equation \(3.3\). Then the solution satisfies

\[
H_\varphi(Du_\varepsilon) \in L^2_{\text{loc}}(T_1, T_2; W^{1,2}_{\text{loc}}(\Omega', \mathbb{R}^n)) \quad \text{and} \quad Du_\varepsilon \in L^\infty_{\text{loc}}(T_1, T_2; L^2_{\text{loc}}(\Omega', \mathbb{R}^n)). \tag{3.6}
\]
Furthermore, the following estimate

\[
\sup_{t \in (\tau_0 - (\rho/2)^2, \tau_0)} \| Du_\varepsilon(\cdot, t) \|_{L^2(B_{\rho/2}(x_0))} + \int_{Q_{\rho/2}(x_0)} | DH_\varepsilon (Du_\varepsilon) |^2 \, dz 
\leq c \left[ \| Df_\varepsilon \|_{L^p(Q_{\rho R_0})} \left( \int_{Q_{\rho R_0}} | Du_\varepsilon |^p \, dz \right)^{\frac{1}{p}} + \rho^{-2} \int_{Q_{\rho R_0}} (| Du_\varepsilon |^p + 1) \, dz \right]
\]

(3.7)

holds true for any parabolic cylinder \( Q_\rho (z_0) \subset Q_{R_0} (z_0) \subset \Omega_{1,2} \) and a positive constant \( c \) depending only on \( n \) and \( p \).

**Proof.** By a slight abuse of notation, for \( w \in L^1_{\text{loc}}(\Omega_{1,2}, \mathbb{R}^N) \) and \( s \in \{1, \ldots, n\} \), \( h \neq 0 \), we set (when \( x + he_s \in \Omega' \))

\[
\tau_h w(x, t) \equiv \tau_{s,h} w(x, t) := w(x + he_s, t) - w(x, t),
\]

\[
\Delta_h w(x, t) \equiv \Delta_{s,h} w(x, t) := \frac{w(x + he_s, t) - w(x, t)}{h},
\]

where \( e_s \) is the unit vector in the direction \( x_s \).

Since \( u_\varepsilon \) is a weak solution of equation (3.2), we have

\[
\int_{\Omega_{1,2}} \left( u_\varepsilon \cdot \partial_t \varphi - \langle H_{p-1}(Du_\varepsilon) + \varepsilon | Du_\varepsilon |^{p-2} Du_\varepsilon, D\varphi \rangle \right) \, dz = -\int_{\Omega_{1,2}} f_\varepsilon \varphi \, dz,
\]

for every test function \( \varphi \in C^\infty_0(\Omega_{1,2}') \). Replacing \( \varphi \) with \( \tau_{s,h} \varphi \), where \( 0 < |h| < \text{dist}(\text{supp} \varphi, \partial \Omega_{1,2}') \), by virtue of the properties of the finite difference operator, we get

\[
\int_{\Omega_{1,2}} \left( \tau_h u_\varepsilon \cdot \partial_t \varphi - \langle \tau_h H_{p-1}(Du_\varepsilon) + \varepsilon \tau_h | Du_\varepsilon |^{p-2} Du_\varepsilon, D\varphi \rangle \right) \, dz = -\int_{\Omega_{1,2}} \tau_h f_\varepsilon \cdot \varphi \, dz.
\]

We now replace \( \varphi \) by \( \varphi_{\sigma} \equiv \phi_{\sigma} * \varphi \) in the previous equation, where \( \{ \phi_{\sigma} \}, \sigma > 0 \), denotes the family of standard, non-negative, radially symmetric mollifiers in \( \mathbb{R}^{n+1} \). This yields, for \( 0 < \sigma \ll 1 \)

\[
\int_{\Omega_{1,2}} \left( (\tau_h u_\varepsilon)_{\sigma} \cdot \partial_t \varphi - \langle (\tau_h H_{p-1}(Du_\varepsilon))_{\sigma} + \varepsilon (\tau_h | Du_\varepsilon |^{p-2} Du_\varepsilon)_{\sigma}, D\varphi \rangle \right) \, dz = -\int_{\Omega_{1,2}} (\tau_h f_\varepsilon)_{\sigma} \varphi \, dz.
\]

Now, in the last equation we choose the test function \( \varphi \equiv \Phi(\tau_h u_\varepsilon)_{\sigma} \), where \( \Phi \in C^\infty_0(\Omega_{1,2}') \) is a smooth function which will be specified later. After an integration by parts and then letting \( \sigma \to 0 \), we obtain

\[
-\frac{1}{2} \int_{\Omega_{1,2}} | \tau_h u_\varepsilon |^2 \partial_t \Phi \, dz + \int_{\Omega_{1,2}} \Phi \langle \tau_h H_{p-1}(Du_\varepsilon) + \varepsilon \tau_h | Du_\varepsilon |^{p-2} Du_\varepsilon, D\tau_h u_\varepsilon \rangle \, dz
\]

\[
= -\int_{\Omega_{1,2}} \langle \tau_h H_{p-1}(Du_\varepsilon), D\Phi \rangle \tau_h u_\varepsilon \, dz - \varepsilon \int_{\Omega_{1,2}} \langle \tau_h | Du_\varepsilon |^{p-2} Du_\varepsilon, D\Phi \rangle \tau_h u_\varepsilon \, dz
\]

\[
+ \int_{\Omega_{1,2}} \tau_h f_\varepsilon \cdot \Phi \cdot \tau_h u_\varepsilon \, dz,
\]

(3.8)

for any smooth function \( \Phi \in C^\infty_0(\Omega_{1,2}') \), provided that \( |h| \) is small enough. Note that an
approximation argument yields the same identity for any \( \Phi \in W^{1,\infty}(\Omega'_{1,2}) \) with compact support in \( \Omega'_{1,2} \) and any sufficiently small \( h \in \mathbb{R} \setminus \{0\} \). In what follows, we will denote by \( c_k \) and \( c \) some positive constants which do not depend either on \( h \) or \( \varepsilon \).

Now, let us consider a parabolic cylinder \( Q_{\rho}(z_0) \subset Q_{R_0}(z_0) \in \Omega'_{1,2} \). For a fixed time \( t_1 \in (t_0 - \rho^2, t_0) \) and \( \delta \in (0, t_0 - t_1) \), we choose \( \Phi(x, t) = \tilde{\chi}(t)\chi(t)\eta^2(x) \) with \( \chi \in W^{1,\infty}((T_1, T_2), [0, 1]) \), \( \chi \equiv 1 \) on \( (T_1, t_0 - \rho^2) \) and \( \partial_t \chi \geq 0 \), \( \eta \in C_0^{\infty}(B_{\rho}(x_0), [0, 1]) \), and with the Lipschitz continuous function \( \tilde{\chi} : (T_1, T_2) \to \mathbb{R} \) defined by

\[
\tilde{\chi}(t) = \begin{cases} 
1 & \text{if } t \leq t_1, \\
\text{affine} & \text{if } t_1 < t < t_1 + \delta, \\
0 & \text{if } t \geq t_1 + \delta.
\end{cases}
\]

With such a choice of \( \Phi \), equation (3.8) becomes

\[
-\frac{1}{2} \int_{\Omega'_{1,2}} |\tau_h u_e|^2 \eta^2(x) \chi(t) \partial_t \tilde{\chi}(t) \, dz - \frac{1}{2} \int_{\Omega'_{1,2}} |\tau_h u_e|^2 \eta^2(x) \tilde{\chi}(t) \partial_t \chi(t) \, dz \\
+ \int_{\Omega'_{1,2}} \tilde{\chi}(t) \chi(t) \eta^2(x) \langle \tau_h H_{p-1}(Du_e), D\tau_h u_e \rangle \, dz + \varepsilon \int_{\Omega'_{1,2}} \tilde{\chi}(t) \chi(t) \eta^2(x) \langle \tau_h [[Du_e]]^{p-2}Du_e, D\tau_h u_e \rangle \, dz \\
= -\frac{1}{2} \int_{\Omega'_{1,2}} \tilde{\chi}(t) \chi(t) \eta(x) \langle \tau_h H_{p-1}(Du_e), D\eta \rangle \tau_h u_e \, dz \\
- 2 \varepsilon \int_{\Omega'_{1,2}} \tilde{\chi}(t) \chi(t) \eta(x) \langle \tau_h [[Du_e]]^{p-2}Du_e, D\eta \rangle \tau_h u_e \, dz + \int_{\Omega'_{1,2}} (\tau_h f_\varepsilon)(\tau_h u_e) \tilde{\chi}(t) \chi(t) \eta^2(x) \, dz.
\]

Setting \( Q'_{1,2} := B_{\rho}(x_0) \times (t_0 - \rho^2, t_1) \) and letting \( \delta \to 0 \) in the previous equality, for every \( t_1 \in (t_0 - \rho^2, t_0) \) we get

\[
\frac{1}{2} \int_{B_{\rho}(x_0)} \chi(t_1) \eta^2(x) \left| \tau_h u_e(x, t_1) \right|^2 \, dx + \int_{Q'_{1,2}} \chi(t) \eta^2(x) \langle \tau_h H_{p-1}(Du_e), D\tau_h u_e \rangle \, dz \\
+ \varepsilon \int_{Q'_{1,2}} \chi(t) \eta^2(x) \langle \tau_h [[Du_e]]^{p-2}Du_e, D\tau_h u_e \rangle \, dz \\
= -\frac{1}{2} \int_{Q'_{1,2}} \chi(t) \eta(x) \langle \tau_h H_{p-1}(Du_e), D\eta \rangle \tau_h u_e \, dz + \int_{Q'_{1,2}} (\tau_h f_\varepsilon)(\tau_h u_e) \chi(t) \eta^2(x) \, dz \\
- 2 \varepsilon \int_{Q'_{1,2}} \chi(t) \eta(x) \langle \tau_h [[Du_e]]^{p-2}Du_e, D\eta \rangle \tau_h u_e \, dz + \frac{1}{2} \int_{Q'_{1,2}} (\partial_t \chi) \eta^2(x) \left| \tau_h u_e \right|^2 \, dz
\]

where we have used that \( \partial_t \tilde{\chi} \) converges to a Dirac delta distribution as \( \delta \to 0 \), together with the \( L^2(\Omega') \)-valued continuity of \( u_\varepsilon \). In the following, we estimate \( I_1, I_2 \) and \( I_3 \) separately. Let us first consider \( I_1 \). Using Lemma 2.1 together with Young’s inequality, we obtain

\[
|I_1| \leq \frac{2(\rho - 1)^2}{\beta} \int_{Q'_{1,2}} \chi(t) |D\eta|^2 \left( \left| H_\beta(Du_e(x + he_\varepsilon, t)) \right|^{\frac{p}{p-2}} + \left| H_\beta(Du_e) \right|^{\frac{p}{p-2}} \right)^2 |\tau_h u_e|^2 \, dz \\
+ \frac{\beta}{2} \int_{Q'_{1,2}} \chi(t) \eta^2(x) \left| \tau_h H_\beta(Du_e) \right|^2 \, dz,
\]

(3.10)
where $\beta > 0$ will be chosen later. As for $I_3$, by Lemma 2.2 we similarly have
\[
|I_3| \leq \frac{2\varepsilon (p - 1)^2}{\beta} \int_{Q^t_1} \chi(t) |D\eta|^2 \left( |Df_c(x + he_s, t)| \frac{p-2}{2} + |Df_c| \right)^2 |\tau_h u_c|^2 \, dz
+ \frac{\varepsilon \beta}{2} \int_{Q^t_1} \chi(t) |\eta|^2 (x) |\tau_h V_p(Df_c)|^2 \, dz.
\] (3.11)

We now turn our attention to $I_2$. Taking advantage of the properties of $f_c$, $u_c$, $\chi$ and $\eta$, and using Hölder’s inequality with exponents $p$ and $p'$ together with Lemma 2.8 we obtain
\[
|I_2| \leq c_1(n) \beta \frac{|h|^2}{\varepsilon} \|Df_c\|_{L^{p'}(Q_{R_0})} \|Du_c\|_{L^p(Q_{R_0})},
\] (3.12)

provided that $|h|$ is suitably small. Now, by virtue of Lemma 2.1 we also have
\[
\frac{4}{p^2} \int_{Q^t_1} \chi(t) |\eta|^2 (x) |\tau_h H_{\beta}^2(Du_c)|^2 \, dz \leq \int_{Q^t_1} \chi(t) |\eta|^2 (x) (\tau_h H_{p-1}(Du_c), D\tau_h u_c) \, dz.
\] (3.13)

Similarly, by Lemma 2.2 we obtain
\[
\frac{4\varepsilon}{p^2} \int_{Q^t_1} \chi(t) |\eta|^2 (x) |\tau_h V_p(Du_c)|^2 \, dz \leq \varepsilon \int_{Q^t_1} \chi(t) |\eta|^2 (x) (\tau_h |Du_c|^{p-2}Du_c), D\tau_h u_c) \, dz.
\] (3.14)

Collecting estimates (3.9), (3.10), (3.11), (3.12), (3.13) and (3.14), and choosing $\beta = 4/p^2$, we arrive at
\[
\int_{B_{\rho}(x_0)} \chi(t_1) |\eta|^2 (x) |\tau_h u_c(x, t_1)|^2 \, dx + \int_{Q^t_1} \chi(t) |\eta|^2 (x) |\tau_h H_{\beta}^2(Du_c)|^2 \, dz
+ \varepsilon \int_{Q^t_1} \chi(t) |\eta|^2 (x) |\tau_h V_p(Du_c)|^2 \, dz
\leq c_2(n, p) \int_{Q^t_1} \chi |D\eta|^2 \left( \left| H_{\beta}^2(Du_c(x + he_s, t)) \right| \frac{p-2}{2} + \left| H_{\beta}^2(Du_c) \right| \right)^2 (\partial_t \chi)^2 \, dz
+ c_2(n, p) \int_{Q^t_1} \varepsilon \chi |D\eta|^2 \left( \left| Du_c(x + he_s, t) \right| \frac{p-2}{2} + \left| Du_c \right| \right)^2 (\partial_t \chi)^2 \, dz
+ c_2(n, p) \frac{|h|^2}{\varepsilon} \|Df_c\|_{L^{p'}(Q_{R_0})} \|Du_c\|_{L^p(Q_{R_0})},
\]

which holds for every $t_1 \in (t_0 - \rho^2, t_0)$ and every sufficiently small $h \in \mathbb{R} \setminus \{0\}$. Recalling that $0 < \varepsilon \leq 1$ and using the definition of the function $H_{\beta}^2$, from the above estimate we get
\[
\int_{B_{\rho}(x_0)} \chi(t_1) |\eta|^2 (x) |\tau_h u_c(x, t_1)|^2 \, dx + \int_{Q^t_1} \chi(t) |\eta|^2 (x) |\tau_h H_{\beta}^2(Du_c)|^2 \, dz
\leq c_3(n, p) \int_{Q^t_1} \chi |D\eta|^2 \left( \left| Du_c(x + he_s, t) \right| \frac{p-2}{2} + \left| Du_c \right| \right)^2 (\partial_t \chi)^2 \, dz
+ c_3(n, p) \frac{|h|^2}{\varepsilon} \|Df_c\|_{L^{p'}(Q_{R_0})} \|Du_c\|_{L^p(Q_{R_0})},
\]

which holds for every $t_1 \in (t_0 - \rho^2, t_0)$ and every suitably small $h \neq 0$.

We now choose a cut-off function $\eta \in C_0^\infty(B_\rho(x_0))$ with $\eta \equiv 1$ on $B_{\rho/2}(x_0)$ such that $0 \leq \eta \leq 1$.
and $|D\eta| \leq C/\rho$. For the cut-off function in time, we choose the piecewise affine function $\chi : (T_1, T_2) \to [0, 1]$ with $\chi \equiv 0$ on $(T_1, t_0 - \rho^2)$, $\chi \equiv 1$ on $(t_0 - \rho^2, T_2)$ and $\partial_t \chi \equiv \frac{\rho}{\rho^2}$ on $(t_0 - \rho^2, t_0 - (\rho/2)^2)$.

Dividing both sides of the previous estimate by $|h|^2$ and using the properties of $\chi$ and $\eta$, we obtain

\[
\sup_{t_0 - (\rho/2)^2 < t < t_0} \int_{B_{\rho/2}(x_0)} |\Delta_h u_\varepsilon(x, t)|^2 \, dx + \int_{Q_{\rho/2}(z_0)} \left| \Delta_h H_{\varepsilon}^2(Du_\varepsilon) \right|^2 \, dz \\
\leq c_4(n, p) \rho^{-2} \int_{Q_{\rho}(z_0)} \left[ \left( |Du_\varepsilon(x + he, t)|^{2 - \frac{2}{p}} + |Du_\varepsilon|^{\frac{2 - 2}{p}} \right)^2 + 1 \right] |\Delta_h u_\varepsilon|^2 \, dz \\
+ c_4(n, p) \|Df_\varepsilon\|_{L^p(Q_{R_0})} \|Du_\varepsilon\|_{L^p(Q_{R_0})}. 
\]

(3.15)

Now we set

\[
I_5 := \int_{Q_{\rho}(z_0)} \left[ \left( |Du_\varepsilon(x + he, t)|^{2 - \frac{2}{p}} + |Du_\varepsilon|^{\frac{2 - 2}{p}} \right)^2 + 1 \right] |\Delta_h u_\varepsilon|^2 \, dz.
\]

If $p > 2$, using Hölder’s inequality with exponents $\left( \frac{p}{2}, \frac{p}{p-2} \right)$ and the properties of the difference quotients, we can control $I_5$ as follows

\[
I_5 \leq c_5(n) \left( \int_{Q_{R_0}} |Du_\varepsilon|^p \, dz \right)^{\frac{2}{p}} \left( \int_{Q_{R_0}} \left[ |Du_\varepsilon|^{p-2} + 1 \right]^{\frac{2}{p-2}} \, dz \right)^{\frac{p-2}{p}} \\
\leq c_6(n, p) \left( \int_{Q_{R_0}} |Du_\varepsilon|^p \, dz \right)^{\frac{2}{p}} \left( \int_{Q_{R_0}} \left[ |Du_\varepsilon|^{p} + 1 \right] \, dz \right)^{\frac{p-2}{p}} \\
\leq c_6(n, p) \int_{Q_{R_0}} (|Du_\varepsilon|^p + 1) \, dz,
\]

provided that $|h|$ is sufficiently small. Joining estimates (3.15) and (3.16), for $p > 2$ we then have

\[
\sup_{t_0 - (\rho/2)^2 < t < t_0} \int_{B_{\rho/2}(x_0)} |\Delta_s h u_\varepsilon(x, t)|^2 \, dx + \int_{Q_{\rho/2}(z_0)} \left| \Delta_s h H_{\varepsilon}^2(Du_\varepsilon) \right|^2 \, dz \\
\leq c \left( \|Df_\varepsilon\|_{L^p(Q_{R_0})} \left( \int_{Q_{R_0}} |Du_\varepsilon|^p \, dz \right)^{\frac{2}{p}} + \rho^{-2} \int_{Q_{R_0}} (|Du_\varepsilon|^p + 1) \, dz \right),
\]

with $c \equiv c(n, p) > 0$. Since the above estimate holds for every $s \in \{1, \ldots, n\}$ and every sufficiently small $h \in \mathbb{R} \setminus \{0\}$, for $p > 2$, by Lemma 2.9 we may conclude that

$H_{\varepsilon}^2(Du_\varepsilon) \in L^2_{\text{loc}}(T_1, T_2; W^{1,2}_{\text{loc}}(\Omega', \mathbb{R}^n))$.

Finally, if $p = 2$, arguing in a similar fashion we reach the same conclusions. Moreover, letting $h \to 0$ in the above estimate, we obtain the Caccioppoli-type inequality (3.7), which in turn implies the validity of (3.6).
As a consequence of Proposition 3.4, we are able to establish the following higher integrability result for the spatial gradient $Du_\varepsilon$, whose proof is based on Sobolev embedding theorem in space applied slicewise:

**Proposition 3.5.** Under the assumptions of Proposition 3.4, we obtain that

$$Du_\varepsilon \in L^{p+\frac{4}{n}}_{loc}(\Omega'_{1,2}, \mathbb{R}^n).$$

Moreover, there exists a positive constant $C_1 = C_1(n,p)$ such that for any cylinder $Q_\gamma(z_0) \subset Q_\rho(z_0) \in \Omega'_{1,2}$ there holds

$$\int_{Q_\gamma(z_0)} (|Du_\varepsilon| - \nu)^{p+\frac{4}{n}} \, dz \leq C_1 \left[ \sup_{t_0 - \rho^2 < t < t_0} \int_{B_\rho(z_0)} |Du_\varepsilon(x,t)|^2 \, dx \right]^{\frac{2}{n}} \cdot \int_{Q_\rho(z_0)} \left( |DH_{\varepsilon}^p(Du_\varepsilon)|^2 + \frac{1}{(\rho - \gamma)^2} |Du_\varepsilon|^p \right) \, dz.$$

In addition, the following estimate

$$\int_{Q_{\rho/2}(z_0)} (|Du_\varepsilon| - \nu)^{p+\frac{4}{n}} \, dz \leq C_2 \left[ \|Df\|_{L^p(Q_R_0)} \left( \int_{Q_{R_0}} |Du_\varepsilon|^p \, dz \right)^{\frac{1}{p}} + \rho^{-2} \int_{Q_{R_0}} (|Du_\varepsilon|^p + 1) \, dz \right]^{1+\frac{4}{n}}$$

(3.17)

holds true for any parabolic cylinder $Q_\rho(z_0) \subset Q_{2\rho}(z_0) \subset Q_{R_0}(z_0) \in \Omega'_{1,2}$ and a positive constant $C_2 = C_2(n,p)$.

**Proof.** Let us introduce the following function

$$G_\varepsilon := \frac{2n}{np+4} \left| H_{\varepsilon}^p(Du_\varepsilon) \right|^\frac{1}{np+1}.$$

In what follows, we will denote by $c_k$ some positive constants which do not depend on $\varepsilon$. From the definition of $G_\varepsilon$, we obtain

$$|DG_\varepsilon| = \frac{2}{p} \left| H_{\varepsilon}^p(Du_\varepsilon) \right|^\frac{1}{np} \left| D \left[ H_{\varepsilon}^p(Du_\varepsilon) \right] \right| \leq c_1 \left| H_{\varepsilon}^p(Du_\varepsilon) \right|^\frac{1}{np} \left| D \left[ H_{\varepsilon}^p(Du_\varepsilon) \right] \right|,$$

(3.18)

where $c_1 \equiv c_1(n,p) > 0$. For $B_\rho(x_0) \in \Omega'$, let $\varphi \in C_0^\infty(B_\rho(x_0))$ and $\chi \in W^{1,\infty}((T_1, T_2))$ be two non-negative cut-off functions with $\chi(T_1) = 0$ and $\partial_t \chi \geq 0$. Now, fix a time $t_0 \in (T_1, T_2)$ and apply the Sobolev embedding theorem on the time slices $\Sigma_t := B_\rho(x_0) \times \{t\}$ for almost every $t \in (T_1, t_0)$, to infer that
after the other. In the following, we estimate \( J \) where, in the second to last line, we have applied Minkowski’s and Young’s inequalities one after the other. In the following, we estimate \( J_1(t) \) and \( J_2(t) \) separately. Let us first consider \( J_1(t) \). Using (3.18) and Hölder’s inequality, we deduce

\[
J_1(t) \leq c_4(n, p) \left( \int_{\Sigma_t} \varphi^{2/n} \left( |Du_e| - \nu \right)^{2/n} \left| \frac{1}{n+2} D\varphi \left( Du_e \right) \right|^{2/n+2} dx \right)^{n+2/(n+4)}
\]

\[
\leq c_4(n, p) \int_{\Sigma_t} \varphi^2 \left| \frac{1}{n+2} D\varphi \left( Du_e \right) \right|^2 dx \left( \int_{\text{supp}(\varphi)} \left| Du_e \right|^2 dx \right)^{2/n}
\]

We now turn our attention to \( J_2(t) \). Using the definition of \( G_e \) and Hölder’s inequality, we can conclude

\[
J_2(t) = \frac{4n^2}{(np + 4)^2} \left( \int_{\Sigma_t} \left| Du_e \right|^2 \left| \frac{1}{n+2} D\varphi \left( Du_e \right) \right|^{2/n+2} dx \right)^{(n+4)/(n+2)}
\]

\[
\leq \frac{4n^2}{(np + 4)^2} \left( \int_{\Sigma_t} \left| Du_e \right|^2 \left| Du_e \right|^{2/n+2} dx \right)^{n+2/(n+4)}
\]

Putting together the last three estimates and integrating with respect to time, we obtain

\[
\int_{Q_{T_1, t_0}} \chi \varphi^2 \left( |Du_e| - \nu \right)^{p+4/n} dz \leq c_5(n, p) \int_{T_1}^{t_0} \chi \left( \int_{\text{supp}(\varphi)} |Du_e(x, t)|^2 dx \right)^{2/n} \cdot \left[ \int_{\Sigma_t} \left( \varphi^2 \left| \frac{1}{n+2} D\varphi \left( Du_e \right) \right|^2 + \left| D\varphi \right|^2 \left| Du_e \right|^p \right) dx \right] dt \leq c_5(n, p) \left[ \sup_{T_1 < t < t_0, \chi(t) \neq 0} \int_{\text{supp}(\varphi)} |Du_e(x, t)|^2 dx \right]^{2/n} \cdot \int_{Q_{T_1, t_0}} \chi \left( \varphi^2 \left| \frac{1}{n+2} D\varphi \left( Du_e \right) \right|^2 + \left| D\varphi \right|^2 \left| Du_e \right|^p \right) dz,
\]

(3.19)
where we have used the abbreviation $Q_{T_1,t_0} := B_{\rho}(x_0) \times (T_1, t_0)$. Now we perform a particular choice of the cut-off functions $\chi$ and $\varphi$ involved above. For a parabolic cylinder $Q_{\rho}(z_0) \Subset \Omega_{1,2}$ we choose $\chi \in W^{1,\infty}((T_1, T_2))$ such that

$$\chi \equiv 0 \text{ on } (T_1, t_0 - \rho^2], \quad \chi \equiv 1 \text{ on } [t_0 - \gamma^2, T_2) \quad \text{and} \quad \partial_t \chi \geq 0,$$

with $0 < \gamma < \rho$. As for $\varphi \in C^\infty_0(B_{\rho}(x_0))$, we assume that $\varphi \equiv 1$ on $B_{\gamma}(x_0)$, $0 \leq \varphi \leq 1$ and $|D\varphi| \leq \frac{C}{\rho - \gamma}$. With these choices (3.19) turns into

$$\int_{Q_{\rho}(z_0)} (|Du_\varepsilon| - \nu)^{p+\frac{4}{n}} \, dz \leq c_0(n, p) \left[ \sup_{t_0 - \rho^2 < t < t_0} \int_{B_{\rho}(x_0)} |Du_\varepsilon(x, t)|^2 \, dx \right]^{\frac{2}{p}} \cdot \int_{Q_{\rho}(z_0)} \left( |DH^{\frac{2}{p}}(Du_\varepsilon)|^2 + \frac{1}{(\rho - \gamma)^2} |Du_\varepsilon|^p \right) \, dz. \tag{3.20}$$

We now choose $\gamma = \rho/2$ and use (3.7) with $Q_{\rho}(z_0)$ replaced by $Q_{2\rho}(z_0)$, in order to estimate the first and second integral on the right-hand side of (3.20). After changing notation about the cylinders involved, we finally obtain the inequality (3.17), which ensures that $Du_\varepsilon \in L^{p+\frac{4}{n}}_{\text{loc}}(\Omega_{1,2}; \mathbb{R}^n)$.

\section{Proof of Theorem 1.1}

We now prove Theorem 1.1 by dividing the proof into three steps. The first step consists in constructing a family of Cauchy-Dirichlet problems, for which we are allowed to use the \textit{a priori} estimates from Propositions 3.4 and 3.5. At this stage, Lemma 2.3 will play a key role in deriving an \textit{a priori} estimate for the solutions to these problems (i.e. the comparison maps): we specifically refer to estimate (3.13) below.

In the second step, we will show that the $L^p$-norms of the spatial gradients of the comparison maps are actually uniformly bounded (see estimate (4.23)), and this is where Lemma 2.4 will come into play.

Finally, in the third step, we shall use a standard comparison argument, as well as the results obtained in the previous steps, to reach the desired conclusion.

\textbf{Proof of Theorem 1.1.} \textbf{Step 1: a priori estimate for the comparison maps.}

We shall keep the notation introduced for the proof of Proposition 3.4, starting from the case $p > 2$. For $\varepsilon \in (0, 1]$ and a couple of standard, non-negative, radially symmetric mollifiers $\phi_1 \in C^\infty_0(B_1(0))$ and $\phi_2 \in C^\infty_0((-1, 1))$, we define the function $f_\varepsilon$ as in (3.1), where $f$ is meant to be extended by zero outside $\Omega_T$. Therefore, we have that $f_\varepsilon \in C^\infty(\Omega_T)$.

Now, for any fixed $\varepsilon \in (0, 1]$ let us define the comparison map

$$u_\varepsilon \in C^0([t_0 - R_0^2, t_0]; L^2(B_{R_0}(x_0))) \cap L^p((t_0 - R_0^2, t_0; W^{1,p}(B_{R_0}(x_0))))$$

as the unique energy solution of the Cauchy-Dirichlet problem
\[
\begin{align*}
\partial_t u_\varepsilon - \text{div} (H_{p-1}(Du_\varepsilon) + \varepsilon |Du_\varepsilon|^{p-2}Du_\varepsilon) &= f_\varepsilon & \text{in } Q_{R_0}(z_0) \\
u_\varepsilon &= u & \text{on } \partial_{\text{par}}Q_{R_0}(z_0),
\end{align*}
\]

where \(Q_{R_0}(z_0) \subset \Omega_T\) and the initial-boundary condition is meant in the sense of Definition 3.2 (see [18, Chapter 2] or [8, Chapter 9] for the existence). Moreover, let us fix a positive number \(R < R_0\) and arbitrary radii \(R/2 \leq r < \ell < \rho < \gamma < \lambda r < R\), with \(1 < \lambda < 2\). In what follows, we will denote by \(c_k\) some positive constants which do not depend either on \(h\) or \(\varepsilon\).

For a fixed time \(t_1 \in (t_0 - \rho^2, t_0)\) and \(\delta \in (0, t_0 - t_1)\), we choose \(\Phi(x, t) = \tilde{\chi}(t)\chi(t)\eta^2(x)\) with \(\chi \in W^{1,\infty}((t_0 - R_0^2, t_0), [0, 1])\), \(\chi \equiv 0\) on \((t_0 - R_0^2, t_0 - \rho^2)\) and \(\partial_t \chi \geq 0\), \(\eta \in C_\infty^0(B_\rho(x_0), [0, 1])\), and with the Lipschitz continuous function \(\tilde{\chi} : (t_0 - R_0^2, t_0) \to \mathbb{R}\) defined by

\[
\tilde{\chi}(t) = \begin{cases} 
1 & \text{if } t \leq t_1, \\
\text{affine} & \text{if } t_1 < t < t_1 + \delta, \\
0 & \text{if } t \geq t_1 + \delta.
\end{cases}
\]

Setting \(Q^t_1 := B_\rho(x_0) \times (t_0 - \rho^2, t_1)\) and arguing as in the first part of the proof of Proposition 3.4 from [11] we get

\[
\begin{align*}
\frac{1}{2} \int_{B_\rho(x_0)} \chi(t_1)\eta^2(x) |\tau_h u_\varepsilon(x, t_1)|^2 dx + \int_{Q^t_1} \chi(t)\eta^2(x) \langle \tau_h H_{p-1}(Du_\varepsilon), D\tau_h u_\varepsilon \rangle dz \\
+ \varepsilon \int_{Q^t_1} \chi(t)\eta^2(x) \langle \partial_t \chi \rangle \langle \tau_h |Du_\varepsilon|^{p-2}Du_\varepsilon \rangle, D\tau_h u_\varepsilon \rangle dz \\
= -2 \int_{Q^t_1} \chi(t)\eta(x) \langle \tau_h H_{p-1}(Du_\varepsilon), D\eta \rangle \tau_h u_\varepsilon dz + \int_{Q^t_1} \langle \tau_h f_\varepsilon \rangle \langle \tau_h u_\varepsilon \rangle \chi(t)\eta^2(x) dz \\
- 2 \varepsilon \int_{Q^t_1} \chi(t)\eta(x) \langle \tau_h |Du_\varepsilon|^{p-2}Du_\varepsilon \rangle, D\eta \rangle \tau_h u_\varepsilon dz + \frac{1}{2} \int_{Q^t_1} \langle \partial_t \chi \rangle \eta^2(x) |\tau_h u_\varepsilon|^2 dz \\
=: A_1 + A_2 + A_3 + A_4,
\end{align*}
\]

for every sufficiently small \(h \in \mathbb{R} \setminus \{0\}\) and every \(t_1 \in (t_0 - \rho^2, t_0)\). In the following, we estimate \(A_1, A_2\) and \(A_3\) separately. Let us first consider \(A_1\). Using Lemma 2.1 together with Young’s inequality with \(\sigma > 0\) and exponents \((2, 2)\), we obtain

\[
|A_1| \leq \frac{2(p-1)^2}{\sigma} \int_{Q^t_1} \chi(t) |D\eta|^2 \left( \left| H_\varepsilon(Du_\varepsilon(x + he, t)) \right|^{\frac{p-2}{p}} + \left| H_\varepsilon(Du_\varepsilon) \right|^{\frac{p-2}{p}} \right)^2 |\tau_h u_\varepsilon|^2 dz \\
+ \frac{\sigma}{2} \int_{Q^t_1} \chi(t)\eta^2(x) \left| \tau_h H_\varepsilon(Du_\varepsilon) \right|^2 dz.
\]
As for \( A_3 \), by Lemma 2.2 we similarly have

\[
|A_3| \leq \frac{2\varepsilon (p - 1)^2}{\sigma} \int_{Q^1} \chi(t) |D\eta| |Du_e(x + he_x, t)|^{\frac{p-2}{2}} + |Du_e|^{\frac{p-2}{2}} |\tau_h u_e|^2 \, dz \\
+ \frac{\varepsilon \sigma}{2} \int_{Q^1} \chi(t) \eta^2(x) |\tau_h V_p(Du_e)|^2 \, dz. \tag{4.4}
\]

We now turn our attention to \( A_2 \). Thanks to Propositions 3.4 and 3.5, we know that

\[
H_2(Du_e) \in L^2_{loc}(t_0 - R^2_0, t_0; W^{1,2}_{loc}(B_{R_0}(x_0), \mathbb{R}^n)), \tag{4.5}
\]

\[
Du_e \in L^\infty_{loc}(t_0 - R^2_0, t_0; L^2_{loc}(B_{R_0}(x_0), \mathbb{R}^n)) \cap L^{p+\frac{4}{n}}_{loc}(Q_{R_0}(x_0), \mathbb{R}^n)
\]

and

\[
\int_{Q_{\gamma}(x_0)} (|Du_e| - \nu)^{p+\frac{4}{n}} \leq c_1(n, p) \left[ \sup_{t_0 - (\lambda r)^2 < t < t_0} \int_{B_{\lambda r}(x_0)} |Du_e(x, t)|^2 \, dx \right]^{\frac{p}{2}} \cdot \int_{Q_{\lambda r}(x_0)} \left( |DH_2(Du_e)|^2 + \frac{1}{(\lambda r - \gamma)^2} |Du_e|^p \right) \, dz. \tag{4.5}
\]

Therefore, taking advantage of the properties of \( f, u, \chi \) and \( \eta \), and using Hölder’s inequality with exponents \( \left( \frac{np+4}{np+4-n}, p + \frac{4}{n} \right) \) together with Lemma 2.8 we obtain

\[
A_2 \leq c_2(n, p) |h|^2 \left( \int_{Q_{\gamma}(x_0)} |Df_\gamma|^{\frac{np+4}{np+4-n}} \, dz \right)^{\frac{np+4-n}{np+4}} \left( \int_{Q_{\gamma}(x_0)} |Du_e|^{p+\frac{4}{n}} \, dz \right) \tag{4.6}
\]

provided that \(|h|\) is suitably small. Now, by virtue of Lemma 2.1 we have

\[
\frac{4}{p^2} \int_{Q^1} \chi(t) \eta^2(x) |\tau_h H_2(Du_e)|^2 \, dz \leq \int_{Q^1} \chi(t) \eta^2(x) \langle \tau_h H_{p-1}(Du_e), D\tau_h u_e \rangle \, dz. \tag{4.7}
\]

Similarly, by Lemma 2.2 we obtain

\[
\frac{4\varepsilon}{p^2} \int_{Q^1} \chi(t) \eta^2(x) |\tau_h V_p(Du_e)|^2 \, dz \leq \varepsilon \int_{Q^1} \chi(t) \eta^2(x) \langle \tau_h [||Du_e||^{p-2}Du_e], D\tau_h u_e \rangle \, dz. \tag{4.8}
\]

Collecting estimates (4.2), (4.3), (4.4), (4.6), (4.7) and (4.8), we arrive at
\[
\int_{B_{\rho}(x_0)} \chi(t_1) \eta^2(x) \left| \tau_{h} u_{c}(x, t_1) \right|^2 \, dx + \int_{Q_{t_1}} \chi(t) \eta^2(x) \left| \tau_{h} H_{\xi}^{\alpha}(Du_{c}) \right|^2 \, dz \\
+ \varepsilon \int_{Q_{t_1}} \chi(t) \eta^2(x) \left| \tau_{h} V_{p}(Du_{c}) \right|^2 \, dz \\
\leq \sigma \, c_{3}(p) \int_{Q_{t_1}} \chi(t) \eta^2(x) \left| \tau_{h} H_{\xi}^{\alpha}(Du_{c}) \right|^2 \, dz + \varepsilon \sigma \, c_{3}(p) \int_{Q_{t_1}} \chi(t) \eta^2(x) \left| \tau_{h} V_{p}(Du_{c}) \right|^2 \, dz \\
+ \frac{c_{3}(p)}{\sigma} \int_{Q_{t_1}} \chi(t) |D\eta|^2 \left( \left| H_{\xi}^{\alpha}(Du_{c}(x + h e_{s}, t)) \right|^\frac{p-2}{p} + \left| H_{\xi}^{\alpha}(Du_{c}) \right|^\frac{p-2}{p} \right)^2 \left| \tau_{h} u_{c} \right|^2 \, dz \\
+ \frac{\varepsilon \, c_{3}(p)}{\sigma} \int_{Q_{t_1}} \chi(t) |D\eta|^2 \left( \left| Du_{c}(x + h e_{s}, t) \right|^\frac{p-2}{p} + \left| Du_{c} \right|^\frac{p-2}{p} \right)^2 \left| \tau_{h} u_{c} \right|^2 \, dz \\
+ c_{4}(n, p) |h|^2 \left( \int_{Q_{t_1}} |Df_{e}| \frac{np+4-n}{np+4} \, dz \right) \left( \int_{Q_{t_1}} |Du_{c}|^{p+\frac{4}{n}} \, dz \right) \frac{n}{np+4} \\
+ c_{3}(p) \int_{Q_{t_1}} (\partial_{t}\chi) \eta^2(x) \left| \tau_{h} u_{c} \right|^2 \, dz.
\]

Choosing \( \sigma = (2 \, c_{3}(p))^{-1} \) and reabsorbing the first two integrals in the right-hand side of (4.9) by the left-hand side, we get

\[
\int_{B_{\rho}(x_0)} \chi(t_1) \eta^2(x) \left| \tau_{h} u_{c}(x, t_1) \right|^2 \, dx + \int_{Q_{t_1}} \chi(t) \eta^2(x) \left| \tau_{h} H_{\xi}^{\alpha}(Du_{c}) \right|^2 \, dz \\
+ \varepsilon \int_{Q_{t_1}} \chi(t) \eta^2(x) \left| \tau_{h} V_{p}(Du_{c}) \right|^2 \, dz \\
\leq c_{5}(p) \int_{Q_{t_1}} \chi(t) |D\eta|^2 \left( \left| H_{\xi}^{\alpha}(Du_{c}(x + h e_{s}, t)) \right|^\frac{p-2}{p} + \left| H_{\xi}^{\alpha}(Du_{c}) \right|^\frac{p-2}{p} \right)^2 \left| \tau_{h} u_{c} \right|^2 \, dz \\
+ \varepsilon \, c_{5}(p) \int_{Q_{t_1}} \chi(t) |D\eta|^2 \left( \left| Du_{c}(x + h e_{s}, t) \right|^\frac{p-2}{p} + \left| Du_{c} \right|^\frac{p-2}{p} \right)^2 \left| \tau_{h} u_{c} \right|^2 \, dz \\
+ c_{6}(n, p) |h|^2 \left( \int_{Q_{t_1}} |Df_{e}| \frac{np+4-n}{np+4} \, dz \right) \left( \int_{Q_{t_1}} |Du_{c}|^{p+\frac{4}{n}} \, dz \right) \frac{n}{np+4} \\
+ c_{5}(p) \int_{Q_{t_1}} (\partial_{t}\chi) \eta^2(x) \left| \tau_{h} u_{c} \right|^2 \, dz,
\]

which holds for every \( t_1 \in (t_0 - \rho^2, t_0) \) and every sufficiently small \( h \in \mathbb{R} \setminus \{0\} \).

We now choose a cut-off function \( \eta \in C_{0}^{\infty}(B_{\rho}(x_0)) \) with \( \eta \equiv 1 \) on \( B_{\ell}(x_0) \) such that \( 0 \leq \eta \leq 1 \) and \( |D\eta| \leq C/(\rho - \ell) \). For the cut-off function in time, we choose \( \chi \in W^{1,\infty}((t_0 - R_{0}^2, t_0), [0, 1]) \) such that

\[
\chi \equiv 0 \quad \text{on} \quad (t_0 - R_{0}^2, t_0 - \rho^2),
\]

\[
\chi \equiv 1 \quad \text{on} \quad [t_0 - \ell^2, t_0)
\]

and

\[
\partial_{t}\chi \leq \frac{C}{(\rho - \ell)^2} \quad \text{on} \quad (t_0 - \rho^2, t_0 - \ell^2).
\]
Dividing both sides of the previous estimate by \( |h|^2 \), using the properties of \( \chi \), \( \eta \) and \( u_\varepsilon \), and applying Young’s inequality with exponents \( \left( \frac{p}{p-2}, \frac{p}{2} \right) \), we get

\[
\sup_{t \in (t_0 - \varepsilon^2, t_0)} \int_{B_t(x_0)} \left| \Delta_h u_\varepsilon (x, t) \right|^2 \, dx + \int_{Q_t(x_0)} \left| \Delta_h H_{\frac{p}{2}} (D u_\varepsilon) \right|^2 \, dz + \varepsilon \int_{Q_t(x_0)} \left| \Delta_h V_{p} (D u_\varepsilon) \right|^2 \, dz 
\leq \frac{c_7(p)}{(\rho - \ell)^2} \left( \int_{Q_\rho(x_0)} \left[ \left( \left| H_{\frac{p}{2}} (D u_\varepsilon(x + h e_\varepsilon, t)) \right| \right] \right|^\frac{p}{2} + \left| H_{\frac{p}{2}} (D u_\varepsilon) \right|^\frac{p}{2} \right)^2 \left| \Delta_h u_\varepsilon \right|^2 \, dz 
+ \frac{\varepsilon c_7(p)}{(\rho - \ell)^2} \int_{Q_\rho(x_0)} \left( \left| D u_\varepsilon(x + h e_\varepsilon, t) \right|^\frac{p}{2} + \left| D u_\varepsilon \right|^\frac{p}{2} \right)^2 \left| \Delta_h u_\varepsilon \right|^2 \, dz 
+ c_6(n, p) \left( \int_{Q_{\gamma}(x_0)} \left| D f_{\varepsilon} \right|^\frac{np + 4}{n p + 4} \, dz \right)^{\frac{np + 4 - n}{np + 4}} \left( \int_{Q_{\gamma}(x_0)} \left| D u_\varepsilon \right|^\frac{p + \frac{4}{n}}{n} \, dz \right)^{\frac{n}{np + 4}} 
\leq \frac{c_8(p)}{(\rho - \ell)^2} \int_{Q_{\gamma}(x_0)} \left| D u_\varepsilon \right|^p \, dz + \frac{c_9(p)}{(\rho - \ell)^2} \int_{Q_\rho(x_0)} \left| \Delta_h u_\varepsilon \right|^p \, dz + \frac{c_{10}(p)}{(\rho - \ell)^2} \int_{Q_\rho(x_0)} \left| \Delta_h u_\varepsilon \right|^p \, dz 
+ c_6(n, p) \left( \int_{Q_{\gamma}(x_0)} \left| D f_{\varepsilon} \right|^\frac{np + 4}{n p + 4} \, dz \right)^{\frac{np + 4 - n}{np + 4}} \left( \int_{Q_{\gamma}(x_0)} \left| D u_\varepsilon \right|^\frac{p + \frac{4}{n}}{n} \, dz \right)^{\frac{n}{np + 4}} 
\leq \frac{c_{11}(n, p)}{(\rho - \ell)^2} \left( \int_{Q_{\gamma}(x_0)} \left| D u_\varepsilon \right|^p \, dz + \int_{Q_{\gamma}(x_0)} \left| D u_\varepsilon \right|^2 \, dz \right) 
+ c_6(n, p) \left( \int_{Q_{\gamma}(x_0)} \left| D f_{\varepsilon} \right|^\frac{np + 4}{n p + 4} \, dz \right)^{\frac{np + 4 - n}{np + 4}} \left( \int_{Q_{\gamma}(x_0)} \left| D u_\varepsilon \right|^\frac{p + \frac{4}{n}}{n} \, dz \right)^{\frac{n}{np + 4}} ,
\]

where we have used Lemma 2.8. Therefore, letting \( h \to 0 \) in (4.10), we obtain

\[
\sup_{t \in (t_0 - \varepsilon^2, t_0)} \left\| D u_\varepsilon (\cdot, t) \right\|^2_{L_2(B_t(x_0))} + \int_{Q_t(x_0)} \left| D H_{\frac{p}{2}} (D u_\varepsilon) \right|^2 \, dz + \varepsilon \int_{Q_t(x_0)} \left| D V_{p} (D u_\varepsilon) \right|^2 \, dz 
\leq \frac{c_{11}(n, p)}{(\rho - \ell)^2} \left( \int_{Q_{\gamma}(x_0)} \left| D u_\varepsilon \right|^p \, dz + \int_{Q_{\gamma}(x_0)} \left| D u_\varepsilon \right|^2 \, dz \right) 
+ c_6(n, p) \left( \int_{Q_{\gamma}(x_0)} \left| D f_{\varepsilon} \right|^\frac{np + 4}{n p + 4} \, dz \right)^{\frac{np + 4 - n}{np + 4}} \left( \int_{Q_{\gamma}(x_0)} \left| D u_\varepsilon \right|^\frac{p + \frac{4}{n}}{n} \, dz \right)^{\frac{n}{np + 4}} .
\]

Now we use (4.5) in order to estimate the last integral as follows
\[
\int_{Q_{\gamma}(z_0)} |Du_\varepsilon|^p + \frac{4}{n} \, dz \\
= \int_{Q_{\gamma} \cap \{|Du_\varepsilon| \geq \nu\}} (|Du_\varepsilon| - \nu + \nu)^{p + \frac{4}{n}} \, dz + \int_{Q_{\gamma} \cap \{|Du_\varepsilon| < \nu\}} |Du_\varepsilon|^{p + \frac{4}{n}} \, dz \\
\leq c_{13}(n, p) \int_{Q_{\gamma}} (|Du_\varepsilon| - \nu)^{p + \frac{4}{n}} \, dz + c_{13}(n, p) \nu^{p + \frac{4}{n}} |Q_{R_0}| \\
\leq c_{14}(n, p) \left[ \sup_{t \in (t_0 - (\lambda \varepsilon)^2, t_0)} \int_{B_{\lambda \varepsilon}(x_0)} |Du_\varepsilon(x, t)|^2 \, dx \right]^{\frac{4}{n}} \cdot \int_{Q_{\lambda \varepsilon}(z_0)} \left( DH^2_\varepsilon(Du_\varepsilon) + \frac{|Du_\varepsilon|^p}{(\lambda r - \gamma)^2} \right) \, dz \\
+ c_{14}(n, p) \nu^{p + \frac{4}{n}} |Q_{R_0}|,
\]
where $|Q_{R_0}|$ denotes the $(n+1)$-dimensional Lebesgue measure of the parabolic cylinder $Q_{R_0}(z_0)$. Plugging the above estimate into (4.11) and applying Young’s inequality with $\theta \in (0, 1)$ and exponents $\left( \frac{np + 4 - n}{np + 4 - n}, \frac{np + 4 - n}{np + 4 - n} \right)$, we get

\[
\sup_{t \in (t_0 - \varepsilon^2, t_0)} \|Du_\varepsilon(\cdot, t)\|_{L^2(B_{\varepsilon}(x_0))}^2 + \int_{Q_{\varepsilon}(z_0)} \left| DH^2_\varepsilon(Du_\varepsilon) \right|^2 \, dz \\
\leq c_{12}(n, p) \left( \|Du_\varepsilon\|_{L^{\rho}(Q_R)}^p + \|Du_\varepsilon\|_{L^2(Q_R)}^p \right) + c_{15}(n, p) \nu \|Df_\varepsilon\|_{L^{np + 4 - n}(Q_R)}^n |Q_{R_0}|^{\frac{n}{np + 4}} \\
+ c_{15}(n, p) \|Df_\varepsilon\|_{L^{np + 4 - n}(Q_R)} \left[ \sup_{t \in (t_0 - (\lambda \varepsilon)^2, t_0)} \|Du_\varepsilon(\cdot, t)\|_{L^2(B_{\lambda \varepsilon}(x_0))}^2 \right]^{\frac{2}{np + 4}} \cdot \int_{Q_{\lambda \varepsilon}(z_0)} \left( DH^2_\varepsilon(Du_\varepsilon) + \frac{|Du_\varepsilon|^p}{(\lambda r - \gamma)^2} \right) \, dz \\
\leq c_{12}(n, p) \left( \|Du_\varepsilon\|_{L^{\rho}(Q_R)}^p + \|Du_\varepsilon\|_{L^2(Q_R)}^p \right) + c_{16}(n, p, R_0) \nu \|Df_\varepsilon\|_{L^{np + 4 - n}(Q_R)}^n \\
+ c_{17}(n, p) \theta^{\frac{n + 4 - np}{np + 4 - n}} \|Df_\varepsilon\|_{L^{np + 4 - n}(Q_R)} \left[ \sup_{t \in (t_0 - (\lambda \varepsilon)^2, t_0)} \|Du_\varepsilon(\cdot, t)\|_{L^2(B_{\lambda \varepsilon}(x_0))}^2 \right] \\
+ \theta \int_{Q_{\lambda \varepsilon}(z_0)} \left| DH^2_\varepsilon(Du_\varepsilon) \right|^2 \, dz + \frac{\theta}{(\lambda r - \gamma)^2} \|Du_\varepsilon\|_{L^p(Q_R)}^p.
\]

Now, if we choose $\theta = \frac{1}{2}$ and set

\[
\Psi(\xi) := \sup_{t \in (t_0 - \xi^2, t_0)} \|Du_\varepsilon(\cdot, t)\|_{L^2(B_{\xi}(x_0))}^2 + \int_{Q_{\xi}(z_0)} \left| DH^2_\varepsilon(Du_\varepsilon) \right|^2 \, dz,
\]

then the previous estimate turns into

\[
\Psi(\xi) \leq \Psi(\ell) \leq \frac{1}{2} \Psi(\lambda r) + c_{12}(n, p) \left( \|Du_\varepsilon\|_{L^{\rho}(Q_R)}^p + \|Du_\varepsilon\|_{L^2(Q_R)}^p \right) \\
+ \frac{1}{2} \|Du_\varepsilon\|_{L^{p}(Q_R)}^p + c_{18}(n, p, R_0) \left( \nu \|Df_\varepsilon\|_{L^{np + 4 - n}(Q_R)}^n + \|Df_\varepsilon\|_{L^{np + 4 - n}(Q_R)}^n \right). \\
(4.12)
\]
Since (4.12) holds for all \( \frac{R}{2} \leq r < \ell < \rho < \gamma < \lambda r < R \), with \( 1 < \lambda < 2 \), we can now choose

\[
\ell = r + \frac{1}{4} (\lambda r - r), \quad \rho = r + \frac{1}{2} (\lambda r - r) \quad \text{and} \quad \gamma = r + \frac{3}{4} (\lambda r - r),
\]
thus obtaining

\[
\Psi(r) \leq \frac{1}{2} \Psi(\lambda r) + \frac{c_{19}(n, p)}{r^2(\lambda - 1)^2} \left( \|Du_\varepsilon\|_{L^p(Q_R)}^p + \|Du_\varepsilon\|_{L^2(Q_R)}^2 \right) + c_{18}(n, p, R_0) \left( \left\| Df_\varepsilon \right\|_{L^\infty L^{n+p+4}(Q_R)}^{n+p+4} + \left\| Df_\varepsilon \right\|_{L^\infty L^{n+p+4}(Q_R)}^{n+p+4} \right),
\]

which holds for all \( \frac{R}{2} \leq r < \lambda r < R \), with \( 1 < \lambda < 2 \). Therefore, the use of Lemma 2.3 with \( r_0 = \frac{R}{2} \) and \( r_1 = R \) yields

\[
\Psi(R/2) \leq \frac{c_{20}(n, p)}{R^2} \left( \|Du_\varepsilon\|_{L^p(Q_R)}^p + \|Du_\varepsilon\|_{L^2(Q_R)}^2 \right) + c_{21}(n, p, R_0) \left( \left\| Df_\varepsilon \right\|_{L^\infty L^{n+p+4}(Q_R)}^{n+p+4} + \left\| Df_\varepsilon \right\|_{L^\infty L^{n+p+4}(Q_R)}^{n+p+4} \right),
\]
that is,

\[
\sup_{t \in (t_0 - R^2/4, t_0)} \|Du_\varepsilon(\cdot, t)\|_{L^2(B_{R/2}(x_0))}^2 + \int_{Q_{R/2}(x_0)} \left| DH_\varepsilon(Du_\varepsilon) \right|^2 dz \leq \frac{c_{20}(n, p)}{R^2} \left( \|Du_\varepsilon\|_{L^p(Q_R)}^p + \|Du_\varepsilon\|_{L^2(Q_R)}^2 \right) + c_{21}(n, p, R_0) \left( \left\| Df_\varepsilon \right\|_{L^\infty L^{n+p+4}(Q_R)}^{n+p+4} + \left\| Df_\varepsilon \right\|_{L^\infty L^{n+p+4}(Q_R)}^{n+p+4} \right),
\]
which is the a priori estimate we were looking for.

Our next aim is to prove that the norms \( \|Du_\varepsilon\|_{L^p(Q_{R_0})} \) are all bounded by a constant independent of \( \varepsilon \) (and therefore the same will be true for the \( L^2 \)-norms, since \( p > 2 \)).

**Step 2: the uniform boundedness of the norms \( \|Du_\varepsilon\|_{L^p(Q_{R_0})} \).**

In order to have an (uniform in \( \varepsilon \)) energy estimate for \( |Du_\varepsilon|^p \), we now proceed by testing equations (1.1) and (1.1) for \( u \) and \( u_\varepsilon \), respectively, with the map \( \varphi = g(t)(u_\varepsilon - u) \), where \( g \in W^{1, \infty} (\mathbb{R}) \) is chosen such that

\[
g(t) = \begin{cases} 
1 & \text{if } t \leq t_2, \\
-\frac{1}{\omega}(t - t_2 - \omega) & \text{if } t_2 < t < t_2 + \omega, \\
0 & \text{if } t \geq t_2 + \omega,
\end{cases}
\]
with \( t_0 - R_0^2 < t_2 < t_2 + \omega < t_0 \), and then letting \( \omega \to 0 \). We observe that at this stage it is important that \( u_\varepsilon \) and \( u \) agree on the parabolic boundary \( \partial_{par}Q_{R_0}(x_0) \). We also note that the following computations are somewhat formal concerning the use of the time derivative, but they
can easily be made rigorous, for example by the use of Steklov averages. We skip this, since it is a standard procedure. With the previous choice of $\varphi$, for almost every $t_2 \in (t_0 - R_0^2, t_0)$ we find

\[
\frac{1}{2} \int_{B_{R_0}(x_0)} |u_\varepsilon(x, t_2) - u(x, t_2)|^2 \, dx + \int_{Q_{R_0, t_2}} \langle H_{p-1}(Du_\varepsilon) - H_{p-1}(Du), Du_\varepsilon - Du \rangle \, dz \\
+ \varepsilon \int_{Q_{R_0, t_2}} |Du_\varepsilon|^p \, dz = \int_{Q_{R_0, t_2}} (f - f_\varepsilon)(u_\varepsilon - u) \, dz,
\]

(4.14)

where we have used the abbreviation $Q_{R_0, t_2} = B_{R_0}(x_0) \times (t_0 - R_0^2, t_2)$. Using Lemma 2.1 the Cauchy-Schwarz inequality as well as Young’s inequality with $\mu > 0$ and exponents $(p, p')$, from (4.11) we infer

\[
\frac{1}{2} \sup_{t \in (t_0 - R_0^2, t_0)} \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2(B_{R_0}(x_0))}^2 + \frac{4}{p^2} \int_{Q_{R_0}(x_0)} \left| H_{p'}(Du_\varepsilon) - H_{p'}(Du) \right|^2 \, dz \\
+ \varepsilon \int_{Q_{R_0}(x_0)} |Du_\varepsilon|^p \, dz \\
\leq \int_{Q_{R_0}(x_0)} |f - f_\varepsilon| \, dz + \varepsilon \int_{Q_{R_0}(x_0)} |Du_\varepsilon|^{p-1} |Du| \, dz \\
\leq \int_{Q_{R_0}} |f - f_\varepsilon| \, dz + \frac{\varepsilon}{p^2 \mu^2} \int_{Q_{R_0}} |Du|^p \, dz + \frac{\varepsilon}{p^2 \mu^2} \int_{Q_{R_0}} |Du_\varepsilon|^p \, dz.
\]

Choosing $\mu = \left( \frac{p'}{2} \right)^{\frac{p'}{2}}$ and reabsorbing the last integral in the right-hand side of (4.15) by the left-hand side, we arrive at

\[
\sup_{t \in (t_0 - R_0^2, t_0)} \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2(B_{R_0}(x_0))}^2 + \int_{Q_{R_0}} \left| H_{p'}(Du_\varepsilon) - H_{p'}(Du) \right|^2 \, dz + \varepsilon \int_{Q_{R_0}} |Du_\varepsilon|^p \, dz \\
\leq \varepsilon c_1(p) \int_{Q_{R_0}(x_0)} |Du|^p \, dz + c_1(p) \int_{Q_{R_0}(x_0)} |f - f_\varepsilon| \, dz.
\]

(4.16)

Now we set

\[
A_5 := \int_{Q_{R_0}(x_0)} |f - f_\varepsilon| \, u_\varepsilon - u \, dz
\]

and observe that

\[
\left( p + \frac{4}{n} \right)' = \frac{np + 4}{np + 4 - n} \geq \left( p + \frac{2p}{n} \right)' = \frac{np + 2p}{np + 2p - n} \quad \text{for every } p \geq 2.
\]

(4.17)

Therefore, we can apply Hölder’s inequality with exponents $\left( \frac{np + 2p}{np + 2p - n}, p + \frac{2p}{n} \right)$ and Lemma 2.1 with $v = u_\varepsilon - u$, $r = R_0$ and $q = 2$ to estimate $A_5$ as follows:
\[ A_5 \leq \left( \int_{Q_{R_0}(z_0)} |f - f_\varepsilon|^{\frac{np + 2p}{np + 2p - n}} \, dz \right) \frac{\int_{Q_{R_0}(z_0)} |u_\varepsilon - u|^{\frac{p + 2p}{n}} \, dz}{\frac{np + 2p}{np + 2p - n}} \]

\[ \leq c_2(n, p) \| f - f_\varepsilon \|_{L^{\frac{np + 2p}{np + 2p - n}}(Q_{R_0})} \left( \sup_{t \in (t_0 - R_0^2, t_0)} \| u_\varepsilon(\cdot, t) - u(\cdot, t) \|_{L^2(B_{R_0}(x_0))}^2 \right)^\frac{1}{n + 2} \cdot \left( \int_{Q_{R_0}(z_0)} |Du_\varepsilon - Du|^p \, dz \right)^\frac{n}{np + 2p}. \]  

(4.18)

Combining estimates (4.16) and (4.18), recalling that \( 0 < \varepsilon \leq 1 \) and applying Young’s inequality with \( \beta > 0 \) and exponents \( \left( \frac{np + 2p}{np + p - n}, n + 2, \frac{np + 2p}{n} \right) \), we obtain

\[ \sup_{t \in (t_0 - R_0^2, t_0)} \| u_\varepsilon(\cdot, t) - u(\cdot, t) \|_{L^2(B_{R_0}(x_0))}^2 + \int_{Q_{R_0}} \left| H_{x}^2(Du_\varepsilon) - H_{x}^2(Du) \right|^2 \, dz + \varepsilon \int_{Q_{R_0}} |Du_\varepsilon|^p \, dz \]

\[ \leq c_1(p) \int_{Q_{R_0}} |Du|^p \, dz + c_3(n, p) \beta^{\frac{n + p}{n - np - p}} \| f - f_\varepsilon \|_{L^{\frac{np + 2p}{np + 2p - n}}(Q_{R_0})} \]

\[ + \beta \left[ \sup_{t \in (t_0 - R_0^2, t_0)} \| u_\varepsilon(\cdot, t) - u(\cdot, t) \|_{L^2(B_{R_0}(x_0))} \right] + \beta \int_{Q_{R_0}} |Du_\varepsilon - Du|^p \, dz \]

\[ \leq (c_1(p) + 2^{p-1} \beta) \int_{Q_{R_0}} |Du|^p \, dz + c_3(n, p) \beta^{\frac{n + p}{n - np - p}} \| f - f_\varepsilon \|_{L^{\frac{np + 2p}{np + 2p - n}}(Q_{R_0})} \]

\[ + \beta \left[ \sup_{t \in (t_0 - R_0^2, t_0)} \| u_\varepsilon(\cdot, t) - u(\cdot, t) \|_{L^2(B_{R_0}(x_0))} \right] + 2^{p-1} \beta \int_{Q_{R_0}} |Du_\varepsilon|^p \, dz. \]  

(4.19)

Now, let us notice that

\[ \int_{Q_{R_0}(z_0)} |Du_\varepsilon|^p \, dz \]

\[ = \int_{Q_{R_0} \cap \{ |Du_\varepsilon| \geq \nu \}} (|Du_\varepsilon| - \nu + \nu)^p \, dz + \int_{Q_{R_0} \cap \{ |Du_\varepsilon| < \nu \}} |Du_\varepsilon|^p \, dz \]

\[ \leq 2^{p-1} \int_{Q_{R_0}} [ (|Du_\varepsilon| - \nu)^p + \nu^p ] \, dz + \nu^p |Q_{R_0}| \]  

(4.20)

\[ = 2^{p-1} \int_{Q_{R_0}} \left( \left| H_{x}^2(Du_\varepsilon) - H_{x}^2(Du) + H_{x}^2(Du) \right|^2 + \nu^p \right) \, dz + \nu^p |Q_{R_0}| \]

\[ \leq 2^p \int_{Q_{R_0}} \left( \left| H_{x}^2(Du_\varepsilon) - H_{x}^2(Du) \right|^2 + H_{x}^2(Du) \right)^2 \, dz + \nu^p |Q_{R_0}| \]

\[ \leq 2^p \int_{Q_{R_0}} \left| H_{x}^2(Du_\varepsilon) - H_{x}^2(Du) \right|^2 \, dz + 2^{p+1} \int_{Q_{R_0}} (|Du|^p + \nu^p) \, dz. \]
\[(1 + 2^p \varepsilon) \int_{Q_{R_0}} |Du_\varepsilon|^p \, dz + 2^p \sup_{t \in (t_0 - R_0^2, t_0)} \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2\left(B_{R_0}(x_0)\right)}^2 \]
\[\leq \left(c_4(p) + 2^{2p-1}\beta\right) \int_{Q_{R_0}} ([Du]^p + \nu^p) \, dz + c_5(n, p) \beta^{\frac{n+p}{np+2p-n}} \|f - f_\varepsilon\|_{L^{np+2p-\nu}Q_{R_0}^{np+2p-n}(Q_{R_0})} \tag{4.21} \]
\[+ 2^p \beta \sup_{t \in (t_0 - R_0^2, t_0)} \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2\left(B_{R_0}(x_0)\right)}^2 \]
\[+ 2^{2p-1}\beta \int_{Q_{R_0}} |Du_\varepsilon|^p \, dz. \]

Now, choosing \(\beta = 1/2^p\) and reabsorbing the last two terms in the right-hand side of (4.21) by the left-hand side, we obtain
\[
\int_{Q_{R_0}} |Du_\varepsilon|^p \, dz + \sup_{t \in (t_0 - R_0^2, t_0)} \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2\left(B_{R_0}(x_0)\right)}^2 \]
\[\leq c_6(p) \int_{Q_{R_0}} ([Du]^p + \nu^p) \, dz + c_7(n, p) \|f - f_\varepsilon\|_{L^{np+2p-\nu}Q_{R_0}^{np+2p-n}(Q_{R_0})}. \]

Moreover, since \(f \in L^{\frac{np+4}{np+4-n}}(\Omega_T)\), recalling (3.1), by virtue of (4.17) we have
\[
f_\varepsilon \to f \quad \text{strongly in} \quad L^{\frac{np+2p}{np+2p-n}}(Q_{R_0}), \quad \text{as} \ \varepsilon \to 0. \tag{4.22} \]

Hence, there exists a finite positive number \(M\), independent of \(\varepsilon\), such that
\[
\|f - f_\varepsilon\|_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})} \leq M \quad \text{for all} \ \varepsilon \in (0, 1],
\]
from which we can finally deduce
\[
\|Du_\varepsilon\|_{L^p(Q_{R_0})}^p + \sup_{t \in (t_0 - R_0^2, t_0)} \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2\left(B_{R_0}(x_0)\right)}^2 \]
\[\leq c_8(n, p, R_0) \left( \|Du\|_{L^p(Q_{R_0})}^p + \nu^p + \|f - f_\varepsilon\|_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})} \right) \leq K \quad \text{for every} \ \varepsilon \in (0, 1], \tag{4.23} \]
for some positive constant \(K\) depending on \(n, p, \nu, M, R_0\) and \(\|Du\|_{L^p(Q_{R_0})}\), but not on \(\varepsilon\).

Joining (4.16), (4.18) and (4.23), and applying Minkowski’s inequality, we get the following \textit{comparison estimate}
\[
\sup_{t \in (t_0 - R_0^2, t_0)} \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2\left(B_{R_0}(x_0)\right)}^2 \leq \varepsilon c_1(p) \|Du\|_{L^p(Q_{R_0})}^p \]
\[+ c_9(n, p) \|f - f_\varepsilon\|_{L^{\frac{np+2p}{np+2p-n}}(Q_{R_0})} \left( \sup_{t \in (t_0 - R_0^2, t_0)} \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2\left(B_{R_0}(x_0)\right)}^2 \right)^{\frac{n+2}{n+2}} \]
\[\cdot \left( \int_{Q_{R_0}} |Du_\varepsilon - Du|^p \, dz \right)^{\frac{n}{np+2p}}. \]
In order to obtain an estimate for the finite difference \( \tau_{s,h} \), for increments \( h \in \mathbb{R} \setminus \{0\} \) such that \(|h|\) is suitably small.

Now we shall fix \( \varrho \in (0,R) \) and consider the finite difference operator \( \tau_{s,h} \), defined in Section 2.1 for increments \( h \in \mathbb{R} \setminus \{0\} \) such that \(|h|\) is suitably small.

In order to obtain an estimate for the finite difference \( \tau_{s,h} H^2_\varphi(Du) \), we use the following comparison argument:

\[
\int_{Q_{\varrho/2}(z_0)} \left| \tau_{s,h} H^2_\varphi(Du) \right|^2 \, dz \leq 4 \int_{Q_{\varrho/2}(z_0)} \left| \tau_{s,h} H^2_\varphi(Du) \right|^2 \, dx \, dt \\
+ 4 \int_{Q_{\varrho/2}(z_0)} \left| H^2_\varphi(Du(x + he_s, t)) - H^2_\varphi(Du(x + he_s, t)) \right|^2 \, dx \, dt \\
+ 4 \int_{Q_{\varrho/2}(z_0)} \left| H^2_\varphi(Du) - H^2_\varphi(Du) \right|^2 \, dx \, dt.
\]

Combining the above inequality with estimates (4.13), (4.23) and (4.24), for every \( s \in \{1, \ldots, n\} \) we get

\[
\leq \varepsilon c_1(p) \left\| Du \right\|^p_{L^p(Q_{R_0})} \\
+ c_{10}(n, p, R_0) \left\| f - f_\varepsilon \right\|_{L^{np+2p+\frac{np}{n+1}p}(Q_{R_0})} \left( \left\| Du \right\|^p_{L^p(Q_{R_0})} + \left\| f - f_\varepsilon \right\|_{L^{np+2p+\frac{np}{n+1}p}(Q_{R_0})} \right) \\
+ \varepsilon c_2(p) \left\| Du \right\|^p_{L^p(Q_{R_0})} + \varepsilon c_2(p) \left\| Du \right\|^p_{L^p(Q_{R_0})} + \varepsilon c_2(p) \left\| Du \right\|^p_{L^p(Q_{R_0})} + \varepsilon c_2(p) \left\| Du \right\|^p_{L^p(Q_{R_0})}
\]

\[
\leq \varepsilon c_1(p) \left\| Du \right\|^p_{L^p(Q_{R_0})} \\
+ c_{11}(n, p, R_0) \varepsilon c_2(p) \left\| Du \right\|^p_{L^p(Q_{R_0})} + \varepsilon c_2(p) \left\| Du \right\|^p_{L^p(Q_{R_0})} + \varepsilon c_2(p) \left\| Du \right\|^p_{L^p(Q_{R_0})} + \varepsilon c_2(p) \left\| Du \right\|^p_{L^p(Q_{R_0})}
\]

where, in the last line, we have used estimate (4.23) again.

**Step 3: the conclusion.**

Now we shall fix \( \varrho \in (0,R) \) and consider the finite difference operator \( \tau_{s,h} \), defined in Section 2.1 for increments \( h \in \mathbb{R} \setminus \{0\} \) such that \(|h|\) is suitably small.
the proof of Theorem 1.1. Let us first show that $p > 5$ follows from Theorem 1.1 in this case. Therefore, from now on we will assume that $p > 5$. This section is devoted to the study of the existence and regularity of the time derivative of the weak solutions to equation (1.1), under the assumptions of Theorem 1.1. Indeed, we are now in position to give the estimate (1.2).

\[ \varepsilon c_2(p) \| Du \|_{L^p(Q_{R_0})} + c_4(n, p, R_0) |h|^2 \left( \| \nu \|_{L^{np+4}}(Q_{R_0}) + \| Df\|_{L^{np+4}}(Q_{R_0}) \right) \]

\[ + c_4(n, p, R_0) \| f - f_\varepsilon \|_{L^{np+2p}}(Q_{R_0}) \left( \| Du \|_{L^p(Q_{R_0})} + \| f - f_\varepsilon \|_{L^{np+2p}}(Q_{R_0}) \right) \]

\[ \cdot \left( \| Du \|_{L^p(Q_{R_0})} + \| f - f_\varepsilon \|_{L^{np+2p}}(Q_{R_0}) \right) \]

which holds for every $\varepsilon \in (0, 1]$ and every sufficiently small $h \in \mathbb{R} \setminus \{0\}$. Therefore, recalling (3.1) and letting $\varepsilon \to 0$ in (4.25), by virtue of (4.22) we obtain

\[ \int_{Q_{R/2}(\zeta_0)} |\Delta_{sl} H_2^x(Du) |^2 \, dz \leq c_4(n, p, R_0) \left( \| Df \|_{L^{np+4}}(Q_{R_0}) + \| Df \|_{L^{np+4}}(Q_{R_0}) \right) \]

\[ + c_4(n, p, R_0) \frac{R_0^2}{R^2} \left( \| Du \|_{L^p(Q_{R_0})} + \| Du \|_{L^p(Q_{R_0})} + \nu^p + \nu^2 \right), \]

which holds for every $s \in \{1, \ldots, n\}$ and every sufficiently small $h \in \mathbb{R} \setminus \{0\}$. This proves the desired result for $p > 2$. Moreover, letting $h \to 0$ in the above inequality, we also obtain estimate (4.2).

Finally, when $p = 2$, arguing in a similar fashion we can reach the same conclusions. \qed

5 The time derivative: proof of Theorem 1.2

This section is devoted to the study of the existence and regularity of the time derivative of the weak solutions to equation (1.1), under the assumptions of Theorem 1.1. Indeed, we are now in position to give the

**Proof of Theorem 1.2** We shall keep both the notation and the parabolic cylinders used for the proof of Theorem 1.1. Let us first show that

\[ H_{p-1}(Du) \in L^{p'}_{loc} \left( 0, T; W^{1,p'}_{loc}(\Omega, \mathbb{R}^n) \right). \]

Observe that for $p = 2$ we have $H_{p-1}(Du) = H_2^x(Du)$, so that assertion (5.2) immediately follows from Theorem 1.1 in this case. Therefore, from now on we will assume that $p > 2$.

Let us notice that for every $\xi \in \mathbb{R}^n$ we have

\[ H_{p-1}(\xi) = \mathcal{F}(H_2^x(\xi)), \]

where $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ is the function defined by

\[ \mathcal{F}(\eta) := |\eta|^\frac{p-2}{p} \eta, \]
which is locally Lipschitz continuous for \( p > 2 \). Thus, the function \( H_{p-1}(Du) \) is weakly differentiable with respect to the \( x \)-variable by virtue of the chain rule in Sobolev spaces. From the definitions of \( \mathcal{F} \) and \( H_{p}^{-1} \), it follows that

\[
|D_{\eta} \mathcal{F}(H_{p}^{-1}(Du))| \leq c_{1} |H_{p}^{-1}(Du)|^{\frac{p-2}{p}} \leq c_{1} |Du|^{\frac{p-2}{p}}, \tag{5.2}
\]

for some positive constant \( c_{1} \equiv c(n, p) \). Now, applying the chain rule, the Cauchy-Schwarz inequality and estimate (5.2), we obtain

\[
|D_{x} H_{p-1}(Du)|^{p'} \leq c_{2}(n, p) |D_{\eta} \mathcal{F}(H_{p}^{-1}(Du))|^{p'} |D_{x} H_{p}^{-1}(Du)|^{p'} \leq c_{3} |Du|^{\frac{(p-2)p'}{2}} |D_{x} H_{p}^{-1}(Du)|^{p'}, \tag{5.3}
\]

where \( c_{3} \equiv c_{3}(n, p) > 0 \). Using (5.3), Hölder’s inequality with exponents \( \left( \frac{2(p-1)}{p-2}, \frac{p}{p'} \right) \) and estimate (1.2), we get

\[
\left( \int_{Q_{\theta/2}(z_{0})} |D H_{p-1}(Du)|^{p'} dz \right)^{\frac{1}{p'}} \leq c_{4}(n, p) \|Du\|^{\frac{p-2}{L^{p}(Q_{\theta/2})}} \left( \int_{Q_{\theta/2}(z_{0})} |DH_{p}^{-1}(Du)|^{2} dz \right)^{\frac{1}{2}} \leq c_{5} \|Du\|^{\frac{p-2}{L^{p}(Q_{\theta/2})}} \left[ \nu \|Df\|_{L^{p}(Q_{\theta/2})} + \|Df\|_{L^{p}(Q_{\theta/2})}^{\nu} \right]^{\frac{1}{2}} + \frac{c_{5}}{R} \left[ \|Du\|_{L^{p}(Q_{\theta/2})}^{2p-2} + \|Du\|_{L^{p}(Q_{\theta/2})}^{p} + (\nu + \nu^{2}) \|Du\|_{L^{p}(Q_{\theta/2})}^{p-2} \right]^{\frac{1}{2}}, \tag{5.4}
\]

where \( c_{5} \) is a positive constant depending on \( n, p, \theta \) and \( R_{0} \). Note that the right-hand side of (5.4) is finite, and this implies (5.1).

Now, let \( \kappa = \min \{ \theta, p' \} \). Going back to the weak formulation (2.1), thanks to (5.1) we can perform a partial integration in the second term on the left-hand side with respect to the spatial variables. We thus obtain

\[
\int_{Q_{\theta/2}(z_{0})} u \cdot \varphi_{t} \, dz = -\int_{Q_{\theta/2}(z_{0})} \left( \sum_{\alpha=1}^{n} D_{\alpha} [(H_{p-1}(Du)]_{\alpha} + f \right) \cdot \varphi \, dz,
\]

for any \( \varphi \in C_{0}^{\infty}(Q_{\theta/2}(z_{0})) \), and the desired conclusion immediately follows from (1.2) if \( p = 2 \), and from (5.4) if \( p > 2 \), since \( f \in L^{q}(0, T; W^{1,q}(\Omega)) \) with \( \frac{np + 4}{np + 4 - n} \leq \theta < \infty \) and

\[
\frac{np + 4}{np + 4 - n} = \left( p + \frac{4}{n} \right) < p' \quad \text{for every } p \geq 2.
\]

Furthermore, we can now observe that

\[
\partial_{t} u = \sum_{\alpha=1}^{n} D_{\alpha} [(H_{p-1}(Du)]_{\alpha} + f \quad \text{in } Q_{\theta/2}(z_{0}),
\]
from which we can infer
\[
\left(\int_{Q_{\rho/2}(z_0)} |\partial_t u|^p \, dz\right)^{\frac{1}{p}} \leq n \|DH_{p-1}(Du)\|_{L^p(Q_{\rho/2}(z_0))} + \|f\|_{L^p(Q_{\rho/2}(z_0))} \leq c_6 \|DH_{p-1}(Du)\|_{L^p(Q_{\rho/2}(z_0))} + c_6 \|f\|_{L^p(Q_{\rho/2}(z_0))},
\]
where \(c_6\) is a positive constant depending on \(n, p, \vartheta\) and \(R_0\). Thus, we can now deduce estimate (1.3) from (5.5) and (1.2) if \(p = 2\), and by combining (5.4) and (5.5) when \(p > 2\).

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