Grade free product formulæ from Graßmann Hopf gebras

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ABSTRACT In the traditional approaches to Clifford algebras, the Clifford product is evaluated by recursive application of the product of a one-vector (span of the generators) on homogeneous i.e. sums of decomposable (Graßmann), multi-vectors and later extended by bilinearity. The Hestenesian 'dot' product, extending the one-vector scalar product, is even worse having exceptions for scalars and the need for applying grade operators at various times. Moreover, the multivector grade is not a generic Clifford algebra concept. The situation becomes even worse in geometric applications if a meet, join or contractions have to be calculated.

Starting from a naturally graded Graßmann Hopf gebra, we derive general formulæ for the products: meet and join, comeet and cojoin, left/right contraction, left/right cocontraction, Clifford and co-Clifford products. All these product formulæ are valid for any grade and any inhomogeneous multivector factors in Clifford algebras of any bilinear form, including non-symmetric and degenerated forms. We derive the three well known Chevalley formulæ as a specialization of our approach and will display co-Chevalley formulæ. The Rota–Stein cliffordization is shown to be the generalization of Chevalley deformation. Our product formulæ are based on invariant theory and are not tied to representations/matrices and are highly computationally effective. The method is applicable to symplectic Clifford algebras too.

Keywords: Graßmann Hopf gebra, contraction, cocontraction, Chevalley deformation, Rota–Stein cliffordization, Clifford product, Clifford coproduct, meet, join, comeet, cojoin, contractions, cocontractions, linear duality, categorial duality, Graßmann-Cayley algebra

1 Introduction

1.1 Preliminary note

Beside some rumour during the conference, we continue to use algebra, cogebra and Hopf gebra as technical terms. In our eyes these names fit into mathematical nomenclature having also a linguistic background. The most striking argument is, however, that it is misleading to call a cogebra a coalgebra making use of and pointing to the term algebra. By duality one sees that cogebras contain in principle the same amount of information as algebras. One could (should?) come up with a linear cogebra theory not making use of any algebraic structure or knowledge. It

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seems necessary to us to put the finger into the wound of the missed opportunity to develop algebra and coalgebra on the same footing and beg for pardon to those who feel linguistically offended by our naming.

### 1.2 Synopsis

The present paper will gather *grade free* product formulæ for almost all algebra and cogebras related to Graßmann-, Graßmann-Cayley, and Clifford algebras. This does not mean that we abandon the multivector structures of these algebras but that we come up with formulæ which are valid for general multivector polynomials, i.e. for general elements \( x \) from the algebra \( A \) or cogebr \( C \). In present literature important product formulæ are given only on generators or homogeneous elements of certain grades, and have to be expanded by iteration and linearity to the general case. Among these the most important Clifford product has to be calculated this way!

In \[17\] we find formulæ (1.21a-c), (1.22a,c), (1.23a,b), (1.25b,c) etc. where even restrictions like that the grade of one homogeneous algebra element has to be less or equal to the grade of another such element, e.g. (1.23a) and (1.25b,c), have to be assumed. The situation even goes worse if dot and inner products are considered. It is not our aim to criticise but to overcome this deficiencies. During this course we will gain lots of insights into the (almost) perfectly dual structure of algebras and cogebras.

To reach our goal we will see that we have to employ algebra and cogebras. Furthermore we will take as our point of departure the well behaved Graßmann Hopf algebra. Firstly we will show that the Graßmann-Cayley algebra is related to the Graßmann Hopf algebra by dualizing the coproduct. Then by deformation we will reach contractions and Clifford algebras. It will turn out that the Chevalley deformation having a grade restriction is a particular case of the general Rota-Stein *cliffordization* obeying no grade restriction.

Using categorial duality, we can write down immediately dualized versions of all algebraic well know structures coming up with a self dual Graßmann-Cayley double algebra, cocontractions, Chevalley codeformation, and with Clifford cogebras etc.

Categorial duality employs a most powerful and beautiful symmetry. To reach our results cogebras are inevitable. We strongly believe that only a fully dual treatment of (projective) geometry, linear coalgebra, invariant and deformation theory will prove powerful enough to overcome recent problems in mathematics and physics.

### 1.3 The grading

Since in various discussions, which took place during the ICCA 6, it became clear to me that the concept of grading and filtration seems not to be common ground, we will first settle down this issue here.

A *graded* \( \mathbb{k} \)-module \( A \) (graded \( \mathbb{k} \)-vector space, or simply *linear space*) is
a (finite) family of \( \mathbb{k} \)-modules \( \{ A_n \} \) where \( n \) runs through the non-negative integers. \( n \) is called degree or step, grade etc. The degree of an element \( a \) is denoted in various ways, \( \partial a = |a| = \text{length}(a) = \deg(a) = \ldots \). Let \( A, B \) be graded \( \mathbb{k} \)-modules. A graded morphism \( f \) is a family of morphisms \( \{ f_i \} \) such that the \( f_i : A_i \rightarrow B_i \) are morphisms of \( \mathbb{k} \)-modules. An element \( a \in A \) is called homogeneous of degree \( r \) iff one has \( a \in A_i \) and \( a \notin A_j, i \neq j \). Any module can be trivially graded by declaring its degree to be zero.

A grading can be introduced also by the action of an abelian group \( G \) such that the modules \( A_i \) are invariant subspaces of the group labelled by a representation index (character): \( G \cdot A_i \subset A_j, \chi_j(A_i) = i\delta_{i,j} \).

A filtration is defined in analogous way demanding the weaker obstruction to modules, morphisms etc. that they consists of or map into the spaces of same or lower degree

\[
f_i : A_i \rightarrow \bigoplus_{j \leq i} B_j.
\] (1.1)

We will later note that products emerging from cliffordization will be in general not graded morphisms but only obey a filtration.

Example: 2.1.1

Consider a polynomial \( p(x) = a_0 + a_1 x + a_2 x^2 + \ldots \in \mathbb{k}[[x]] \) in one variable \( x \) over the ring (field) \( \mathbb{k} \). The 1-dimensional spaces spanned by \( x^q \) are \( \mathbb{k} \)-modules \( A_i \). Monoms \( \alpha x^q \) are homogeneous elements of degree \( q \). Polynomials in several commuting complex variables \( \mathbb{C}[[z, w]] \) can be graded by their total degree: \( p(z, w) = a_0 + a_{1,0} z + a_{0,1} w + a_{2,0} z^2 + a_{1,1} z w + a_{0,2} w^2 + \ldots \). The \( \mathbb{C} \)-spaces of degree \( q \) have dimensions \( q + 1 \). Observe that one could introduce a finer grading by specifying a multidegree composed from the degree in \( z \) and \( w \), e.g. degree \( \partial(z^4 w^3) = \partial(z^4) + \partial(w^3) = 4 + 3 = 7 \), multidegree \( \partial(z^4 w^3) = (4,3) \).

A binary product \( m \) (a binary multiplication) is a morphism from the space \( B \simeq A \otimes A \) into the space \( A \). If \( A \) is a graded space we can define the grading of \( A \otimes A \) to be the sum of the grades of the homogeneous factors, i.e. \( \partial(A_i \otimes A_j) = \partial A_i + \partial A_j \), turning \( B \) into a graded module. A product \( m : B \rightarrow A \) is graded if it is a graded morphism. In other terms

\[
m : A_i \otimes A_j \subset A_{i+j}
\] (1.2)

Note that this definition of a product implies bilinearity but not associativity. We denote the product by \( m(a \otimes b) = m(a, b) \), or in infix notation or even by juxtaposition \( ab \). Let \( \alpha, \beta \) be ring elements we have right and left distributive laws (linearity)

\[
m(\alpha a + \beta b, c) = \alpha m(a, c) + \beta m(b, c)
\]
\[
m(a, \alpha b + \beta c) = \alpha m(a, b) + \beta m(a, c)
\] (1.3)

If a product acts on two adjacent slots of a higher tensor space (two out of a larger number of arguments) it is easily proven to be multilinear. Since we deal with
associative products mainly, we assume \( m \) to be associative from now on. In this case we can define \( m(a \otimes \ldots \otimes b) = m(a, \ldots, b) = m(a, m(\ldots)) \) where the order of binary multiplications is irrelevant.

**Example: 2.1.2**

 Canonical examples of graded binary products are tensor products, Grassmann and symmetric products. Let \( A = a \ldots b, \ B = c \ldots d \) be two words in a tensor algebra \( T(V) \) generated by the letters \( a, b, \ldots \in V \) linear over some ring \( k \). A grading is defined by the length of the words \( \text{length}(A) = r, \ \text{length}(B) = s \) the tensor product is concatenation. We find

\[
m(A \otimes B) = AB \quad \text{length}(m(A \otimes B)) = \text{length}(A) + \text{length}(B) \quad (1.4)
\]

This allows to decompose the tensor algebra, viewed as module, into a sum of disjoint submodules containing homogeneous elements \( T(V) = k \oplus V \oplus V \otimes 2^1 \oplus \ldots \) The elements of \( V \otimes^r \) need not to be decomposable (products of generators) but may be sums of products of generators.

Let \( e_1, e_2, \ldots \) be *generators* of a Grassmann algebra \( V^\wedge = k \oplus V \oplus V^{\wedge^2} + \ldots \). A grading can be defined by the number of generators in a monomial. If we define 0 to have any grade, we find that the Grassmann wedge product is a graded product

\[
V^{\wedge^r} \wedge V^{\wedge^s} \subset V^{\wedge^{r+s}} \quad \text{length}(V^{\wedge^r} \wedge V^{\wedge^s}) = \text{length}(V^{\wedge^r}) + \text{length}(V^{\wedge^s}) = r + s. \quad (1.5)
\]

A symmetric product in \( k[[a, b, \ldots]] \) defined as the usual point wise product of polynomials is graded.

\[
m(a^3b^2 \otimes (c^2d + d^3)) = a^3b^2c^2d + a^3b^2d^3 \quad (1.6)
\]

\[
\text{length}(m(a^3b^2 \otimes (c^2d + d^3)) = \text{length}(a^3b^2) + \text{length}(c^2d + d^3) = 5 + 3.
\]

It is possible to derive algebras from the tensor algebra by factoring out bilateral ideals. These ideals are generated by elements fulfilling some relations. In the case of the Grassmann and symmetric algebras they read for \( x, y \in V \)

\[
 I_{Gr} = \text{gen}\{ x \otimes y + y \otimes x \} \\
 I_{sym} = \text{gen}\{ x \otimes y - y \otimes x \} \quad (1.7)
\]

Since these ideals are graded, the factored algebras remain to be graded by the \( \mathbb{Z} \)-grading inherited from the tensor algebra. Without going into detail of this construction, we find immediately that the ideal of a Clifford algebra generated as follows

\[
 I_{Cl} = \text{gen}\{ x \otimes y + y \otimes x - 2g(x, y)\text{Id} \}, \quad (1.8)
\]

where \( g(x, y) \) is the symmetric polar bilinear form of a quadratic form \( Q \) on \( V \), is no longer \( \mathbb{Z} \)-graded since tensors of different degree are identified.
As a good example to this claim and a counter example to the widely accepted assumption that a Clifford algebra comes up with generic ‘multivectors’ i.e. a \( \mathbb{Z} \)-grading may serve the quaternions.

**Example: 2.1.3**

Let \( 1, i, j, k \) be the standard basis of the quaternions, obeying the relations

\[
\begin{align*}
  k &= ij \\
  ij &= -ji \\
  jk &= -kj \\
  ki &= -ik \\
  ii &= jj = kk = ijk = -1
\end{align*}
\]

We obtain a grading using the following length function. Assume that \( 1, j \) are generators and define

\[
\text{length}(1) = 0 \quad \text{length}(i) = 1 \quad \text{length}(j) = 1 \quad \text{length}(k) = 2.
\]

However, the roles of \( i, j, k \) are fully symmetric and we could have chosen that \( j, k \) are generators so that \( i = jk \) which would have lead us to a second different \( \mathbb{Z} \)-grading

\[
\text{length}(1) = 0 \quad \text{length}(j) = 1 \quad \text{length}(k) = 1 \quad \text{length}(i) = 2.
\]

Hence there is no unique such grading present in the quaternions. The argument above using the tensor algebra and factorization shows that such a grading cannot uniquely be established in any Clifford algebra. Only the \( \mathbb{Z}_2 \)-grading or *parity grading* defined by the length function modulo 2 is generic.

Adding a multivector structure to a Clifford algebra depends on additional choices, e.g. by the choice of particular elements being generators. In fact one has to choose in which way a Graßmann algebra having multivectors is embedded in a Clifford algebra. We are consequently using such an identification in the present work and all gradings we refer to are derived from the grading of the tensor and Graßmann algebras.

### 1.4 Algebra and cogebra

We will informally introduce the notion of a cogebra by dualizing the algebra structure. In category theory one uses *commutative diagrams* (CD) for this purpose, however, we also make frequent use of *tangles*, see discussion and references in [12]. The difference between both pictures is that they are dual in the sense that arrows and objects change their graphical representation. A product \( m \) may be seen as a morphisms (arrow) acting on objects (source and target points) in a CD. In the tangle analog we represent morphisms by points and objects by lines (arrows, implicitly red downwards unless otherwise specified). Categorial duality is the operation which reverses all arrows or mirrors all tangles at a horizontal line. The therefrom generated dualized morphisms are named using the
prefix ‘co’, e.g. a product changes into a coproduct. In graphical notation we get

\[
\begin{array}{c}
A \otimes A \\
\downarrow m \approx \begin{array}{c}
\Delta \Rightarrow \dual \Leftarrow \duality \Rightarrow C \\
A \otimes C
\end{array}
\end{array}
\] (1.12)

Tangles can be red like processes in physics, e.g. think of Feynman diagrams, or flow diagrams in computer science. Elements or spaces enter at the top flow down and suffer at the vertices, representing morphisms, some action. A binary product combines two inputs into one output, while a binary coproduct has one input and two outputs. Such a representation is called graphical calculus. Some details and references may be found in [12]. If one calculates with tangles an equality is sometimes called a move. The coproduct is a \(1 \rightarrow 2\) map algebraically given as

\[
\Delta : C \rightarrow C \otimes C.
\] (1.13)

The coproduct is in general an indecomposable tensor. It is very convenient to introduce the Sweedler notation [26]

\[
\Delta(x) = \sum_r a_r \otimes b_r = \sum_{(x)} x_{(1)} \otimes x_{(2)} = x_{(1)} \otimes x_{(2)}
\]

\[
\Delta(x^i) = \sum_i \Delta_{ik} x^i = \sum_{(i)} a^i_{(r)} \otimes b^k_{(r)} \text{ w.r.t. an arb. basis} (1.14)
\]

The \(\Delta_{ik}\) are called section coefficients, these constitute a sort of comultiplication table. Associativity dualizes to coassociativity and its axiom reads as CD or tangle:

\[
\begin{array}{c}
C \xrightarrow{\Delta} C \otimes C \\
\Delta \downarrow \Delta \otimes \Id \quad \Id \otimes \Delta \\
C \otimes C \xrightarrow{\Delta} C \otimes C \otimes C
\end{array} \quad = \quad \begin{array}{c}
\Delta \\
\Delta \otimes \Id \quad \Id \otimes \Delta \\
\Delta \otimes \Id \otimes \Delta \otimes \Id
\end{array} \] (1.15)

A coproduct may have a counit which is defined once more by dualizing the axioms of the unit. We find \((\epsilon \otimes \Id)\Delta = \Id = (\Id \otimes \epsilon)\Delta\) or graphically

\[
\begin{array}{c}
\begin{array}{c}
k \otimes C \\
\Delta \approx \epsilon \otimes \Id
\end{array}
\end{array} 
\begin{array}{c}
\begin{array}{c}
C \otimes C \\
\Id \otimes \epsilon
\end{array}
\end{array} 
\begin{array}{c}
\begin{array}{c}
C \otimes k \quad \Id \otimes \epsilon \quad \epsilon \otimes \Id \quad \Delta \\
\Delta \approx \Delta
\end{array}
\end{array} \] (1.16)

The pair \(\mathcal{A} = (A, m)\) is called an (associative possibly unital) algebra and the dualized structure \(\mathcal{C} = (C, \Delta)\) is called a (coassociative possibly counital) cogebra.
1.5 Linear duality

Since we have already used categorial duality we have a need to introduce the technical term linear duality for the conventional dual. Any possibly graded finite dimensional $k$-module $A$ comes naturally, i.e. functorially, with a linear dual $A^* \simeq \text{lin-hom}(A, k)$. Elements $\omega$ of $A^*$ are called linear forms. We will freely use the notations

$$\omega(x) = \langle \omega | x \rangle = \text{eval}(\omega \otimes x) \quad (1.17)$$

Arrows are used to indicate the type of the space. Downwards oriented lines represent the space $A$ while upward oriented lines depict the dual space $A^*$. The action of a linear dual on a space is called evaluation map, denoted as eval, due to symmetry we can define the action the opposite way around also, thereby identifying $A$ with the double dual $A^{**}$. In terms of tangles we write:

$$\Downarrow \text{eval} \quad \Uparrow \text{eval} \quad (1.18)$$

1.6 Product co-product duality (by evaluation)

The evaluation map provides a natural (functorial) connection of products and co-products on $A$ and $A^*$. The action of a linear form $\omega$ on a product $m(a \otimes b)$ shall be rewritten as the sum of scalar products of actions of some tensor $\omega(1) \otimes \omega(2)$ on the argument $a \otimes b$ of $m$. In the tangle picture this means that one pulls the product from the two right down-strands to the single left up-strand. During this process the product tangle gets mirrored (rotated by $\pi$) and turns into a coproduct tangle acting on the dual space. The right equation dualizes a product on $A^* \otimes A^*$.

$$\Downarrow \text{eval} \quad \Uparrow \text{eval} \quad (1.19)$$

In terms of algebraic formulæ we can write this as

$$\text{eval}(m(\omega \otimes \omega') \otimes x) = (\text{eval} \otimes \text{eval})(\omega \otimes \omega' \otimes \Delta(x)) = \omega(x(2))\omega'(x(1))$$

$$\text{eval}(\omega \otimes m(x \otimes y)) = (\text{eval} \otimes \text{eval})(\Delta(\omega) \otimes x \otimes y) = \omega_{(1)}(y)\omega_{(2)}(x) \quad (1.20)$$

Using the evaluation map any product induces a coproduct on the dual space and vice versa [20].

$$\Delta : A \rightarrow A \otimes A \quad \Leftrightarrow \quad m^* : A^* \otimes A^* \rightarrow A^*$$

$$m : A \otimes A \rightarrow A \quad \Leftrightarrow \quad \Delta^* : A^* \rightarrow A^* \otimes A^* \quad (1.21)$$

Working with a space $A$ and a dual space $A^*$ we are still free to choose i) a product and co-product on $A$ or on $A^*$, ii) a product $m$ on $A$ and $m^*$ on $A^*$, i.e. Graßmann-Cayley case or iii) a coproduct $\Delta$ on $A$ and $\Delta^*$ on $A^*$. 
2 Graßmann Hopf algebra

We will define the Graßmann Hopf algebra using the notion of letters and words, i.e. choosing a basis. Of course one could reformulate the following results basis free also. However, it will become important that the structure is unique up to isomorphy only. The notion of a Graßmann Hopf algebra is standard and may be found in \cite{26}, however we need to introduce some subtleties which will be used later on and are explained at length in \cite{12}. The terms Hopf algebra and Hopf algebra denote in general different structures but coincide in the Graßmann Hopf case, however these terms are distinct e.g. for Clifford Hopf algebras.

An euclidian coproduct of a Graßmann exterior product $\bigvee$ (meet of hyperplanes) on $A^*$ of an element $x \in A$ is defined as the sum over all those tensors $x(1) \otimes x(2)$ which multiply back to the element $x$. Our terminology reflects the usage of the euclidian dual isomorphism $\delta : V \rightarrow V^*$. Hence we consider the splits

$$I_x \equiv (x) := \left\{(a, b) \mid m(a \otimes b) = x\right\}$$

$$I_x(1) \equiv x(1) = a \quad I_x(2) \equiv x(2) = b$$

and obtain

$$\Delta(x) := \sum_{I_x} I_x(1) \otimes I_x(2) = \sum_{(x)} x(1) \otimes x(2) = x(1) \otimes x(2)$$

$$m \circ \Delta(x) = \sum_{(x)} m(x(1) \otimes x(2)) = \left|I_x\right| x$$

The Graßmann exterior algebra over a vector space $V^\wedge$ having a wedge product $\wedge$ can be obtained by factoring the tensor product modulo antisymmetrization. The single transposition needed for antisymmetrization is called Graßmann crossing or graded switch and is defined as

$$\otimes \xrightarrow{\pi_2} \wedge$$

$$\hat{\tau}(A \otimes B) = (-1)^{[A][B]} B \otimes A \quad \text{on homogeneous elements.} \quad (2.3)$$

We obtain

$$\Delta_{\wedge}(\text{Id}) = \text{Id} \otimes \text{Id}$$

$$\Delta_{\wedge}(a) = a \otimes \text{Id} + \text{Id} \otimes a$$

$$\Delta_{\wedge}(a \wedge b) = a \wedge b \otimes \text{Id} + a \otimes b - b \otimes a + \text{Id} \otimes a \wedge b$$

$$\vdots$$

$$\Delta_{\wedge}(x) = x(1) \otimes x(2). \quad (2.4)$$

The sign stemming from the permutations is included in Sweedler notation. To establish a basis in a Graßmann algebra we need a termordering on the elements.
2. Grade free product formulæ from Graßmann Hopf gebras

Let \( a, b, c \ldots \in V \) extended to \( V^\wedge \) to be able to decide if we should solve for \( ab \) or \( ba = -ab \). The splits of a word \( A = ab \ldots d \) into two blocks \( B = a \ldots c, C = b \ldots d \) is such that in every block \( B, C \) the termordering remains valid. In the Graßmann case we find that a word of length \( r \) obeys \( 2^r \) such splits. The euclidian dualized wedge coproduct is found to be: i) co-unital with counit \( \epsilon : V^\wedge \to k \), ii) co-associative, iii) (linear) dual to the exterior product (denoted as ‘vee’ \( \vee \)) on the dual space of linear forms \( V^\star \wedge : \Delta^\star = \vee \) and iv) can be obtained in a combinatorial way by a sum of all ‘splits’ of the exterior products into 2 blocks.

The pair \((V^\wedge, \wedge)\) is called Graßmann algebra while the pair \((V^\wedge, \Delta_\wedge)\) is called Graßmann cogebra. If the coproduct is dualized from the vee \( \vee \) we denote it as \( \Delta_\vee \).

If the following compatibility laws are valid, and if one can proof that an antipode exists we can establish a Graßmann Hopf algebra.

In Hopf algebras one demands as compatibility laws that product and unit are cogebra morphisms and that coproduct and counit are algebra morphisms.

Finally we give the axioms for the antipode, an anti-homomorphism, and a generalization of the inverse

\[
S(x^{(1)}) \wedge x^{(2)} = \epsilon(x) \mathrm{Id} = x^{(1)} \wedge S(x^{(2)}) \quad \forall x \in V^\wedge
\]  

\[
S \circ \mathrm{Id} = \mathrm{Id} \circ \epsilon = \mathrm{Id}_{\mathrm{conv}}
\]

A Graßmann Hopf algebra is defined as the following septuple \( H^\wedge = (V^\wedge, \wedge, \mathrm{Id}, \Delta_\wedge, \epsilon; \hat{\tau}, S) \) fulfilling the above axioms. A classification of convolution algebras obeying a product and a coproduct can be found in [12]. There it was demonstrated that if a convolutive unit \( \mathrm{Id}_{\mathrm{conv}} \) and an antipode exits then the product and coproduct induce all other structure tensors in a Hopf algebra. This idea goes back to Oziewicz [22, 24].

The rest of the paper is devoted to the task of showing that almost all algebraic structures needed in geometry and physics can be derived in a plain and natural way from the common generic root of Graßmann Hopf algebra. In this way we follow Oziewicz [21] from Graßmann to Graßmann-Cayley, Clifford, etc. adding in the same time the dual structures:
We will have no space to discuss the last point here, see [4, 12, 5].

3 Graßmann-Cayley double algebra

3.1 Integrals and the bracket

A left (right) integral is an element $\mu_L$ ($\mu_R$) $\in A^*$, i.e. a comultivector of the unital cogebral $A^*$ fulfilling:

\[
\mu_R = \mu_R \quad \mu_L = \mu_L
\]  

(3.1)

\[
(\text{Id} \otimes \mu_R)\Delta(x) = \mu_R(x)\text{Id} \quad (\mu_L \otimes \text{Id})\Delta(x) = \mu_L(x)\text{Id}
\]  

(3.2)

Graßmann Hopf gebras are bi-augmented, bi-connected, see [20, 12] and possses a unique left/right integral $\mu$. Integrals in general do not exist in Clifford Hopf gebras [12].

The bracket $[\ldots]$ of invariant theory is defined to be a multilinear alternating normalized map of $s$ multivector arguments having total degree $n$, i.e. $\partial A_0 + \partial A_1 + \ldots + \partial A_s = n = \dim V$ and otherwise zero.

\[
[A_0, \ldots, A_s] : \otimes^s V^{*s} \longrightarrow k \\
[A_0, \ldots, A_s] \equiv (\mu \circ \wedge^s)(A_0 \otimes \ldots \otimes A_s)
\]  

(3.3)

In fact this is a determinantal map. The unique integral $\mu$ of a Graßmann Hopf algebra turns out to be the projection onto the coefficient of the highest grade element. This allows to define the bracket in Graßmann Hopf algebraic terms:

\[
[A_0, \ldots, A_s]_\mu \cong \sum_{\sigma} \left[ A_{\sigma(0)} \otimes \cdots \otimes A_{\sigma(s)} \right]_\mu
\]  

(3.4)

In what follows it is important to realize, that the bracket is a sort of cup-tangle on $n$-strands or equivalently an $n \to 0$ map. While the evaluation map in (1.18)
was a pairing of a space and dual space, the bracket, using two arguments, constitute a self pairing \([\ldots] : V^\wedge \otimes V^\wedge \rightarrow k\). In terms of tangles we can, however, easily transfer notions from one to the other case. This will be used in the next subsection.

3.2 Meet and join, linear logic

Let \(A\) be an extensor, i.e. a homogenous decomposable multivector which can be written as \(A = a_0 \wedge \ldots \wedge a_r\). The linear space \(\overline{A} = \text{span}\{a_0, \ldots, a_r\}\) is called support of \(A\). The join \((A \wedge B)\) is defined as the disjoint union of the supports \(\overline{A}, \overline{B}\), i.e. \(A \wedge B = \overline{A} \cap \overline{B}\), and zero otherwise \([7, 3]\). In logical terms this is an exclusive or (XOR) on linear spaces.

Geometrically spoken the join connects disjoint geometric elements. Two points are joint to span a line, a point and a line may span a plane etc. It was already clear to Graßmann that one needs a second operation called meet (his regressive product, a section) which allows to compute common subspaces thereby lowering the degree of the algebraic objects. We will show that this notion is natural to a Graßmann Hopf algebra.

Historical note: The meet or \(\lor\)-product was introduced by H. Graßmann as 'eingewandtes Produkt' in \([5]\) using what later was called the rule of the common factor. He weakened this concept and renamed the operation to the regressive product in the second Ausdehnungslehre \([A2,1862]\) \([14]\) using there the unary operation of 'Ergänzung'. This is the notion of an orthogonal complement and was denoted by a vertical line \(a \rightarrow |a|\) such that \(a \wedge |a| = I\) where \(I\) is an element of maximal grade. In logical terms this operation is a negation on a linear space.

The Ergänzung makes explicite use of the total dimension of the underlying space \(V\) via the element \(I\), as it is also well known in logic that negation is based on a maximal element in an orthomodular lattice. This Ergänzungs operation of taking the orthogonal complement needs, spoken in geometrical terms, necessarily a symmetric polarity which leads necessarily to a symmetric polar bilinear form! It is this place where a restriction enters. Hence we can address the Ergänzung as linear NOT in linear logic.

We call the following rule the de Morgan law for linear spaces. It can be found in Graßmann’s A2, \([14]\) and was reinvented several times, see second line.

\[
\begin{align*}
|\overline{(A \lor B)}| &= (|\overline{A}|) \wedge (|\overline{B}|) \\
A \lor B &= I^{-1} \cdot ((I \cdot A) \wedge (I \cdot B))
\end{align*}
\]

\([1862, A2]\) needs a 'dot' product \(3.5\)

It should be remarked that the usage of a dot- or scalar product is still more restrictive than the assumption of orthomodularity which fixes only a class of polarities having the same determinant.

A universal or master formula for the meet of \(r\) factors not using any symmetric polarity was given by Alfred Lotze in 1955 \([18]\). Lotze showed in a note 1

---

1 We make use of the definition given by Doubilet, Rota & Stein \([7]\) which is for two factors, but
added in proof of the above cited paper, that the meet product turns out to be an
exterior product also. Moreover, Lotze showed that the 'double meet' (meet w.r.t.
the meet) is again probably up to a sign the original wedge product. This is a
remarkable and beautiful duality. Furthermore, it shows that we can safely reject
the idea of Rota to switch the notion of wedge and vee products to come up with
an direct analogy to set theory since duality spoils a fixed relation. Finally this
duality shows that it is irrelevant what is a point and what a hyperplane, but these
notions can be interchanged provided one interchanges also the meaning of meet
and join. This is the celebrated duality of projective geometry.

We are ready to define the meet now entirely in terms of the Graßmann Hopf
algebra as (the signs are due to a reordering of factors):

\[
A \lor B := (a_1 \land \ldots \land a_r) \lor (b_1 \land \ldots \land b_s)
= [B^{(1)}, A] B^{(2)} = A^{(1)} [B, A^{(2)}] = \pm [A, B^{(1)}] B^{(2)} = \pm A^{(1)} [A^{(2)}, B]
\]

The tangle definition of the meet reads:

\[
\lor := \pm \begin{array}{c}
\Delta \\
\land
\end{array}
\]

This definition still needs the notion of a maximal grade to exist, but works out
properly for arbitrary not necessarily symmetric non-degenerate bilinear forms
too. The meet is a sort of contraction w.r.t. the self pairing induced by the bracket,
see below.

### 3.3 Comeet and cojoin

Having the tangle definition it is simply a matter of dualizing to come up with
the notion of a cojoin and comeet. The cojoin turns out to be just the Graßmann
coproduct \( \Delta \land \). The co-meet \( \Delta \lor \) is given by categorial duality and involves the
obvious notion of a cointegral.

\[
\Delta \lor := \pm \begin{array}{c}
\Delta \\
\land
\end{array}
\]

The comeet is a coproduct, i.e. a \( 1 \to 2 \) map, it may be called cocontraction w.r.t
the cobracket.

### 3.4 Graßmann-Cayley and fourfold algebra

The Graßmann-Cayley algebra is defined to be the di-algebra \( GC(\lor, \land) \) hav-
ing two associative unital binary products. The various duality relations allow

---

uses a more compact notation.
us to identify the Graßmann-Cayley algebra with the Graßmann Hopf algebras \( H_\wedge \) or \( H_\vee \) over \( V^\wedge \) or \( V^{*\vee} \) and to introduce a Graßmann-Cayley cogebera \( GC(\Delta_\wedge, \Delta_\vee) \). In a CD this dualities read as:

\[
\begin{align*}
GC(\wedge, \wedge) &\quad \leftrightarrow \quad GC(\wedge, \vee) \\
H_\wedge(\wedge, \vee) &\quad \leftrightarrow \quad H_\vee(\wedge, \vee)
\end{align*}
\]

(3.9)

Note that in Graßmann Hopf algebras the exterior product and the exterior coproduct are independent. This has some subtle consequences and was the motivation to use wedge and vee for the exterior products on \( V^\wedge \) and \( V^{*\vee} \), see [13]. This independence makes it useful to introduce the fourfold algebra:

\[
H_\wedge \oplus H_\vee \simeq GC(\wedge, \vee, \wedge, \vee).
\]

(3.10)

It would be interesting to investigate in which way this is a Graßmann-Cayley Hopf di-algebra. A reasonable assumption is to relate the wedge \( \wedge \) and \( \vee \) vee product using an analogy of a co-(quasi) triangular structure (which might be trivial), see [5].

### 4 Bilnear forms and contractions

#### 4.1 Scalar and coscalar products

A **scalar product** \( B \) on \( V \otimes V \) is a map in the set \( \text{lin-hom}(V \otimes V, k) \) or similarly on the dual space \( D \in \text{lin-hom}(V^* \otimes V^*, k) \). A **coscalar product** \( C \) is an element of the set \( \text{lin-hom}(k, V \otimes V) \) or from \( \text{lin-hom}(k, V^* \otimes V^*) \).

\[
\begin{align*}
V &\quad B \quad V^* \\
D &\quad C \quad E \\
V \otimes V &\quad B \quad [k \quad D \quad V^* \otimes V^*]
\end{align*}
\]

(4.1)

Scalar products are \( 2 \rightarrow 0 \) maps, i.e. cup-tangles while coscalar products are \( 0 \rightarrow 2 \) maps, i.e. cap-tangles. However, on the linear spaces \( V^\wedge \) and \( V^{*\vee} \) we have to give a meaning to a scalar product \( B^\wedge \), resp. \( D^\vee \) in a *canonical* way. Later on we will investigate Clifford algebras where the scalar product is the polar bilinear form of a quadratic form on \( V \) and the algebra structure allows to define a unique generalization.

If we demand that the scalar product is extended by an exponential map one can check that this is related to co-(quasi) triangular structures. Furthermore can show that only exponentially generated scalar products \( B^\wedge \) on \( V^\wedge \otimes V^\wedge \) come up with associative algebraic structures during a deformation process [4, 12, 5].

The cup-tangles for scalar an coscalar products can be looked at in two ways, either as scalar products or as duality in \( \text{lin-hom}(V^\wedge, V^{*\vee}) \) resp. \( \text{lin-hom}(V^{*\vee}, V^\wedge) \).
This reads:

\[
\begin{align*}
\mathcal{B} & \sim \mathcal{B}^\text{eval} \\
\text{dualized: } C & \quad (4.2)
\end{align*}
\]

Hence we define the *canonically induced scalar product* \( B^\wedge \), which fulfils the axioms of a co-(quasi) triangular structure, as:

\[
B^\wedge = \exp_\wedge(B) = \epsilon \otimes \epsilon + B_{ij} \epsilon^i \otimes \epsilon^j + B_{\{i_1i_2\},|j_1j_2\}} \epsilon^{i_1} \wedge \epsilon^{i_2} \otimes \epsilon^{j_1} \wedge \epsilon^{j_2} + \ldots
\]

\[
B^\wedge = \bigoplus_{k=1} B \bigoplus \frac{1}{2!} B^2 \bigoplus \frac{1}{3!} B^3 \ldots
\]

\[
(4.3)
\]

The coscalar product \( C^\Delta \) is obtained in the same way by categorial duality, i.e. mirroring the tangle horizontally (rotating by \( \pi \)).

### 4.2 Contraction

Using the scalar product \( B^\wedge \) as cup-tangle, we can once more exploit product coproduct duality. This time all input spaces are of the same type and we get

\[
\wedge = \bigoplus \Delta^\wedge
\]

\[
(4.4)
\]

This motivates the definition of the *right contraction* in terms of a tangle equation as

\[
L_B := \Delta^\wedge
\]

\[
(4.5)
\]

Moving the product from left to right in the product coproduct duality \([1.19]\) gives

\[
\bigoplus \Bigcup_B := \Bigcup_B
\]

\[
(4.6)
\]
which motivates the definition of the left contraction as:

\[ \mathcal{B} := \begin{array}{c}
\bullet \\
\Delta^\wedge \\
\bullet
\end{array} \]

(4.7)

These two definitions are valid for arbitrary inhomogeneous elements of any grade. While in textbooks one finds such a definition using the pairing e.g. [16], there is no direct constructive rule for their evaluation. Since we can directly compute coproducts, our formulæ

\[ \mathcal{B}(A \otimes B) = B^\wedge(A, B_{(1)}) B_{(2)} \]

\[ \mathcal{B}(A \otimes B) = A_{(1)} B^\wedge(A_{(2)}, B) \]

(4.8)

are constructive and free of any grade, homogeneity or decomposability restrictions.

4.3 Chevalley formulæ for any grade

In Ref. [18] Chevalley introduced a recursive method to compute the contraction. From the properties of the pairing he derived the following well known rules for the left contraction. Of course analogous formulæ hold for right contractions. Let \( x, y \in V \) and \( u, v, w \in V^\wedge \) the left contraction obeys

i) \[ x \mathcal{B} y = B(x, y) \text{Id} = \epsilon(x \circ y) \text{Id} \]

ii) \[ x \mathcal{B} (u \wedge v) = (x \mathcal{B} u) \wedge v + \hat{u} \wedge (x \mathcal{B} v) \]

iii) \[ u \mathcal{B} (v \mathcal{B} w) = (u \wedge v) \mathcal{B} w, \]

(4.9)

where \( \hat{u} = (-1)^{\partial u} u \) is the grade involution which turns out to be the antipode of the Graßmann Hopf algebra [12].

To show that the above tangle definition of the left contraction is a generalization of Chevalley deformation we have to show that the three rules (4.9i–iii) follow from the tangle definition. But our main aim is to generalize the Chevalley relations to arbitrary inhomogeneous algebra elements of any grade.

**Theorem: 2.4.1**

The left contraction as defined in (4.7) for arbitrary algebra elements generalizes the Chevalley formulæ (4.3–iii) and reduces to them for the one-vector specialization. The graded crossing \( \hat{\tau} \) induces the grade involution (antipode) in the graded Leibnitz rule (4.9i).

**Proof:** of i): We compute the defining tangle of the left contraction on two grade one elements \( a, b \in V \):

\[ \mathcal{B}(a \otimes b) = (B^\wedge \otimes \text{Id})((\text{Id} \otimes \Delta)(a \otimes b)) \]

\[ = (B^\wedge \otimes \text{Id})(a \otimes b \otimes \text{Id} + a \otimes \text{Id} \otimes b) \]

\[ = B(a, b) \text{Id}. \]

recall that \( \Delta(b) = b \otimes \text{Id} + \text{Id} \otimes b \)

(4.10)
But the tangle definition is now valid for arbitrary elements

\[ \mathcal{J}_B(u \otimes v) = \left( B^\wedge \otimes \text{Id} \right) \left( \left( \text{Id} \otimes \Delta \right)(u \otimes v) \right) = \left( B^\wedge (u, v_1) \right) v_2 \]  

(4.11)

where only those terms survive having \( \partial u = \partial v_1 \).

**Proof:** of iii): We compute using tangles the following equation

\[ \mathcal{J}_B \]

\[ a) \quad \mathcal{J}_B \]

\[ b) \quad \mathcal{J}_B \]

\[ c) \quad \mathcal{J}_B \]

\[ \mathcal{J}_B \]

(4.12)

We have used the definition of the contraction (4.7) in a), product coproduct duality (1.19) in b), coassociativity (1.15) in c). This formula was already valid for any grade.

**Proof:** of ii): The most complicated case is relation (4.9). We compute firstly the tangle equation for the general case and prove that the restriction to a one-vector argument yields the well known graded Leibnitz rule.

\[ \mathcal{J}_B \]

\[ a) \quad \mathcal{J}_B \]

\[ b) \quad \mathcal{J}_B \]

\[ c) \quad \mathcal{J}_B \]

\[ \mathcal{J}_B \]

(4.13)

We have used the definition of the contraction (4.7) in a), the compatibility of algebra and cogebera structure (2.5) in b), product coproduct duality (1.19) in c),
and the following property of the crossing (4.14) in d).

\[ B^{cs}_{\tau_{sd}} = \tau^{ca}_{ds} B^{sb}. \]  \hfill (4.14)

In algebraic terms the above given tangle equation (4.13) reads:

\[ w \bigtriangleup_B (u \wedge v) = (-1)^{|w_1|} (w_2 \bigtriangleup_B u) \wedge (w_1 \bigtriangleup_B v), \]  \hfill (4.15)

To finish the proof we reduce the general formula to the case of a one vector contraction, i.e. we let \( w \rightarrow a \in V \)

\[ a \bigtriangleup_B (u \wedge v) = (a \bigtriangleup_B u) \wedge v + \hat{u} \wedge (a \bigtriangleup_B v), \]  \hfill (4.16)

remembering the definition of the graded switch (2.3) and specializing also to a one vector argument in the first tensor slot

\[ \hat{\tau} (a \otimes u) = (-1) \partial a \partial u (u \otimes a) = ((-1) \partial u u) \otimes a \]

\[ = \hat{u} \otimes a. \]  \hfill (4.17)

we obtain the well known graded Leibnitz rule (4.9ii).

Our calculation shows that the grade involution \( \hat{u} \) originates in the graded switch \( \hat{\tau} \). The crossing is thus related to the derivation property. This observation has tremendous impact on commutation relations. Let

\[ a_i \Leftrightarrow a_i \wedge \]

\[ a_i \Leftrightarrow a_i \bigtriangleup \]  \hfill (4.18)

an note that equation (4.9ii) defines then the commutation relations of such creation and annihilation operations, i.e. a CAR algebra.

\[ a_i a_j^\dagger | \phi \rangle = \langle a_i \big| a_j^\dagger \rangle_{\delta} | \phi \rangle - a_j^\dagger a_i \big| \phi \rangle \]  \hfill (4.19)

we have thus shown that all Chevalley deformation formul\ae{} follow from the Graßmann Hopf gebra generically. Note that our formul\ae{} allow to compute expressions having arbitrary grade or even being inhomogeneous. This will eventually be explored but see [9, 11].

4.4 Left/right cocontractions

Recalling that we had defined cap-tangles from co-scalar products we can employ dualized product coproduct duality to define left and right cocontractions. We write the tensor of the coscalar product as \( C^{(1)} \otimes C^{(2)} \) and define the left cocontraction via:

\[ = \Rightarrow \quad \Delta \quad := \]  \hfill (4.20)
The right cocontraction follows from

\[ \Delta (\hat{C} \cdot) = C^\wedge_{(1)} \otimes (C^\wedge_{(2)} \wedge x) \]

\[ \Delta (\hat{C} \wedge) = (x \wedge C^\wedge_{(1)}) \otimes C^\wedge_{(2)} \]

(4.22)

4.5 Co-Chevalley formulæ

Having an exterior coproduct \( \Delta \) and a cocontraction we can write down immediately the formulæ of co-Chevalley deformation. Let \( x \in V \), \( u \in V^\wedge \) we find

\[ \Delta_{(\hat{C} \cdot)} (\text{id}) = C^\wedge_{(1)} \otimes C^\wedge_{(2)} \]

\[ \Delta_{(\hat{C} \wedge)} (u) = C^\wedge_{(1)} \otimes C^\wedge_{(2)} \wedge u \]

\[ (\text{id} \otimes \Delta) \Delta_{(\hat{C} \cdot)} (u) = C^\wedge_{(1)} \otimes \Delta (C^\wedge_{(2)} \wedge u) \]

\[ = (-1)^{\delta u_{(1)} \delta C^\wedge_{(22)}} C^\wedge_{(1)} \otimes C^\wedge_{(21)} \wedge u_{(1)} \otimes C^\wedge_{(22)} \wedge u_{(2)} \]

\[ (\text{id} \otimes \Delta) \Delta_{(\hat{C} \wedge)} (x) = C^\wedge_{(1)} \otimes C^\wedge_{(21)} \otimes C^\wedge_{(22)} \wedge x \]

\[ + (-1)^{\delta C^\wedge_{(22)}} C^\wedge_{(1)} \otimes C^\wedge_{(21)} \wedge x \otimes C^\wedge_{(22)} \]

\[ (\Delta \otimes \text{id}) \Delta_{(\hat{C} \cdot)} (u) = C^\wedge_{(11)} \otimes C^\wedge_{(12)} \otimes C^\wedge_{(22)} \wedge u \]

(4.23)

where \( i \) is equivalent to the coscalar product, \( i' \) is the general left cocontraction, \( ii \) is the general cocontraction on a coproduct and \( ii' \) the corresponding co-Leibnitz rule for a one vector argument, while \( iii \) dualizes the general formula (4.9 iii). These formulæ are new according to our knowledge.

5 Deformation and cliffordization

5.1 Chevalley deformation, cliffordization

Composing the contraction and the exterior multiplication Chevalley [6] defined the Clifford product, denoted here as \( \& c \), as an element \( \gamma_x \) of the endomorphism
algebra $\text{End}(V^\wedge)$. Let $x \in V$ and $u \in V^\wedge$ he defined

$$
\gamma_x \in \text{End}(V^\wedge) \quad \quad \gamma : V \otimes V^\wedge \to V^\wedge \\
\gamma_x u := x \mathcal{J}_B u + x \wedge u = x \& c u
$$

(5.1)

Having now general expressions for the contraction and the wedge product at hand, its an easy task to write down a grade free Clifford product. This formula, i.e. the leftmost tangle in (5.2), was obtained by Rota and Stein [25] using Laplace Hopf algebras and has been called ‘Rota-sausage’ for obvious reasons by Ozievicz [23, 13]. This process is a very general deformation of an algebra and not tied to the Clifford case only. Rota and Stein coined the term cliffordization but it may also be addressed as a Drinfeld twist in certain circumstances. The cup-tangle in the deformation is a co-(quasi) triangular structure. However, only our approach makes it explicite that cliffordization is nothing but the generalized Chevalley deformation and is directly composed from left or right contraction and the exterior product, examine therefor middle and right tangle.

$$
\phantom{\text{Example: 2.5.1}}
\text{Example: 2.5.2}
\phantom{\text{Example: 2.5.2}}
$$

(5.2)

Note that this product is no longer graded, since we find

$$
\& c : V^{\wedge^r} \otimes V^{\wedge^s} \to V^{\wedge^{r+s}} \oplus \ldots \oplus V^{\wedge^{|r-s|}}
$$

(5.3)

but obeys only a filtration. This filtration depends on the chosen generators i.e. basis. Since the proof that this deformation comes up with the Clifford product is given in [25] we give only a few examples:

**Example: 2.5.1**

Product of two one-vectors using (5.2) middle and right tangle yields

$$
a \& c b = (a_{1(1)} \wedge (a_{2(2)} \mathcal{J}_B b))
\quad = a \wedge (\text{Id} \mathcal{J}_B b) + \text{Id} \wedge (a \mathcal{J}_B b)
\quad = a \wedge b + a \mathcal{J}_B b = \gamma_a b
\quad a \& c b = (a \mathcal{L}_b b_{(1)}) \wedge b_{(2)}
\quad = (a \mathcal{L}_b \text{Id}) \wedge b + (a \mathcal{L}_b) \wedge \text{Id}
\quad = a \wedge b + b \mathcal{L}_b a = \gamma_a b.
$$

(5.4)

**Example: 2.5.2**
Product of two bivectors using (5.2) middle tangle

\[(a \wedge b) \& c \,(x \wedge y) = (a \wedge b) \wedge (x \wedge y) + a \wedge (b \mathcal{J}_B((x \wedge y)))
\]
\[- b \wedge (a \mathcal{J}_B((x \wedge y))) + \text{Id} \wedge ((a \wedge b) \mathcal{J}_B(x \wedge y))
\]
\[= \ a \wedge (b \wedge (x \wedge y) + b \mathcal{J}_B(x \wedge y))
\]
\[+ a \mathcal{J}_B(b \wedge (x \wedge y) + b \mathcal{J}_B(x \wedge y)) - (a \mathcal{J}_B b)(x \wedge y)
\]
\[= \gamma_a (\gamma_b(x \wedge y)) - (a \mathcal{J}_B b)(x \wedge y)
\]
\[= (\gamma_a \wedge \gamma_b)(x \wedge y) = \gamma_{a\wedge b}(x \wedge y) . \quad (5.5)
\]

One can prove that \[ [12] \]:

- The Grassmann Hopf algebra unit \( \text{Id} \) remains to be the unit, also denoted as \( \text{Id} \), of the Clifford product if \( B^\wedge \) is exponentially generated.
- The Clifford product is associative if and only if \( B^\wedge \) is exponentially generated.
- The counit projects products onto the bilinear form \( \epsilon(u \& v) = B^\wedge(u, v) \)
  \(( = \langle 0 | u \& v | 0 \rangle \) . This can be used as vacuum expectation value in quantum field theory.
- If \( F^\wedge \) is exponentially generated from an antisymmetric bilinear form \( F = -F^T \) then is the deformed product \& \( c \wedge \) again an exterior product.
- The deformation w.r.t such an \( F \) encodes the Wick transformation of (fermionic) quantum field theory in Hopf algebraic terms \[11\].
- Exponentially generated bilinear forms fulfill the axioms of a co-(quasi)-triangular structure.
- Clifford Hopf algebras are biaugmented but neither connected nor coconnected, see \[21\] for definitions.

These properties result from considering an arbitrary, not exponentially generated bilinear form \( BF \) on \( V^\wedge \otimes V^\wedge \) in the cliffordization

\[ a \circ b := BF(a_{(2)}, b_{(1)})a_{(1)} \wedge b_{(2)} \quad (5.6) \]

Examining unit, associativity, etc. yields the above claims, see \[12\].

5.2 Co-cliffordization, co-Chevalley deformation

From the Rota sausage tangle (5.2), that is from the tangle definition of the Clifford product, we derive by categorial duality the co-cliffordization and the Clifford
2. Grade free product formulæ from Graßmann Hopf gebras

A **coproduct** denoted as $\Delta_c$.

$$\Delta_c := \Delta m_A$$

(5.7)

Obviously duality tells us that this coproduct is derived from co-Chevalley deformation also

$$= \Delta_c = \Delta_c$$

(5.8)

and depends thus on the coscalar product in the same manner as the Clifford product depends on the scalar product.

### 5.3 Deformation from cochains

It was shown in [11] that the Wick transformation of normalordered operator products into (non renormalized) time ordered operator products can be given by a cliffordization w.r.t. an antisymmetric scalar product $F$ exponentially generalized to $F^\wedge$. This is important since renormalization can then be introduced using the Epstein-Glaser formalism. There is a hope that this can also be achieved by a product deformation [4, 5]. We will not go into this difficult case, but try to show that the normal ordering transformation is topologically trivial. Therefore we show that the antisymmetric exponentially generated bilinear form $F^\wedge$ can be derived from a cocycle. For precise definitions see [19].

An $r$-cochain is defined to be a map $p : \otimes^r V^\wedge \to k$. A cochain may act in a convolution product, defined as $f * g = m(f \otimes g)\Delta$, like an endomorphism $P = p \star \text{Id} = \text{Id} \star p$ from $V^\wedge \overset{p}{\to} V^\wedge$ where $\Delta$ is the Graßmann coproduct and $m$ the product in $k$. Let furthermore $\partial$ be a (group like) co-boundary operator, so that $\partial p$ is a cocycle.

Tailoring a 1-cochain to obtain $F^\wedge$ as a particular 2-cocycle for Wick reordering leads to the following requirements for the special cochain $p$. Let $p(\text{Id}) = 1$, $p(a) = 0 \ \forall a \in V$, $p(a \wedge b) = p_{ab} \in k$ and expand the cochain via the Laplace like property $p(u \wedge v \wedge w) = \pm p(u_1, v)p(u_2, w)$ to $V^\wedge$. Then define operators $P$ and $P^{-1} : V^\wedge \to V^\wedge$

$$P(x) = p(x_1)x_2 \quad P^{-1}(x) = p^{-1}(x_1)x_2$$

(5.9)
which are assumed to be commutative under convolution, i.e.

\[ P = p \ast \text{Id} = \text{Id} \ast p \]

\[ P^{-1} = p^{-1} \ast \text{Id} = \text{Id} \ast p^{-1} \]

The circle product \( \circ \) is defined as a product homomorphic to the exterior wedge product under \( P \).

\[ P(x \circ y) = P(x) \wedge P(y), \quad P^{-1}(x \wedge y) = P^{-1}(x) \circ P^{-1}(y) \]

The 2-cocycle derived from the cochain \( p \) is a bilinear form denoted as \( \partial P \). It is formally invertible and reads explicit

\[ \partial P(u, v) = p(u(1))p(v(2))p^{-1}(u(2) \wedge v(1)) \]

\[ \partial P^{-1}(u, v) = p^{-1}(u(1))p^{-1}(v(2))p(u(2) \wedge v(1)) \]

One needs to show that this bilinear form is i) antisymmetric, ii) exponentially generated and iii) that the above homomorphism can be rewritten as a cliffordization w.r.t. this bilinear form. In terms of tangles one has to prove the '＝' in the following tangle equation

\[ \circ :\begin{array}{c}
\text{\scriptsize o} \\
p
\text{\scriptsize p}
\end{array} = \begin{array}{c}
p \\
p
\text{\scriptsize p}
\end{array} \quad \text{\scriptsize p}^{-1}
\]

where the rightmost tangle is called \textit{owl tangle}. This was done in [12]. However, it is well known from deformation quantization that not every deformation can be written as a homomorphism of products. Furthermore, since the bilinear form \( \partial P \) is equivalent to a 2-cocycle we see that both products, wedge and \( \circ \equiv \wedge \), are topologically equivalent. However, the related Hopf algebras are quite different [11]. While the Graßmann Hopf algebra w.r.t. the wedge is biconnected, that w.r.t. the dotted wedge \( \hat{\wedge} \) is not.

6 Outlook

For lack of place we will not give a summary but want to recall shortly the main idea and its further eventual development. Indeed the most striking feature of our approach is its complete duality between algebra and cogebras. Moreover we might have convinced the reader that cogebras structures are implicitly used e.g. in determinants, combinatorial identities, more explicit in the Graßmann-Cayley di-algebra having two associative products one related to a coproduct on the dual space, and most strikingly in the Clifford product and the
very general procedure of cliffordization. It has become clear during the course of our work that the Graßmann Hopf algebra is the core and starting point to develop systematically almost all algebraic structures and less known costructures. We might remark at this point that it is possible on a formal level to perform the same reasoning starting with the symmetric Hopf algebra and deforming it into Weyl or synonymously symplectic Clifford algebras. We have no time to show that cliffordization is also computationally very efficient but see [1, 2].

The most intriguing questions for further research are among others the following:

i) Can a linear cogebra theory be developed including geometrical meaning without making recourse to the algebra side of the world?

ii) Is there a set of axioms which directly characterizes Clifford Hopf algebras?

iii) What is the link of this bigebraic mathematics to geometry and physics? We know already, that the deformation has to do with quantization and the propagator of quantum field theory [8, 10], but this relation should be deepened.

iv) Since we deal with alternating multivector fields this structure is very close to string and M-theory, what is the concrete relation?

Lots more questions could be added, but we will insist in a final statement, probably of morally nature. Regarding the present development one cannot go for the algebra only approach any longer. We hope that this chapter will push forward this idea.

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