Entanglement and Dynamic Stability of Nash Equilibria in a Symmetric Quantum Game.

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March 31, 2022

Abstract

We study the evolutionary stability of Nash equilibria (NE) in a symmetric quantum game played by the recently proposed scheme of applying ‘identity’ and ‘Pauli spin flip’ operators on the initial state with classical probabilities. We show that in this symmetric game dynamic stability of a NE can be changed when the game changes its form, for example, from classical to quantum. It happens even when the NE remains intact in both forms.

1 Introduction

In a recent paper Eisert et al [1] have shown that when the players in a game have access to quantum strategies the game can give different Nash Equilibria (NE) that can give better payoffs. Marinatto and Weber [2] have proposed another interesting scheme to play a quantum game where players implement their ‘tactics’ on an initial strategy by a probabilistic choice between applying the ‘identity’ and ‘Pauli spin-flip’ operators. In a quantized game there can rise multiple NE similar to a classical game. For example classical version of the Battle of Sexes game gives three NE and its quantization by Marinatto and Weber’s scheme again gives three NE. In classical game theory dynamical stability is one of essential criteria to select among multiple NE that are solutions to a game and it is usually considered with respect to some adjustment process that underlies the game. Selten [3] introduced the idea of perturbations in normal form classical games to get rid of unreasonable NE as solutions to these games by using a static procedure based on intuitive equilibrium conditions of NE. Replicator dynamic [4][5] in evolutionary game theory is an adjustment process for which frequency of a particular strategy evolves according to its payoff relative to that of other strategies. For pure available classical strategies $S_1, ..., S_n$, let the proportion of players of $S_i$ at a particular time be $p_i, (i = 1, ..., n)$, so that
the average population strategy is the vector \( \mathbf{p} = (p_i) \), with the expected payoff of an \( S_i \) player being \( f_i(\mathbf{p}) \) and the overall expected payoff in the population being \( F(\mathbf{p}) = \sum p_i f_i(\mathbf{p}) \). Then the replicator dynamic is defined by the differential equation 

\[
\frac{dp_i}{dt} = p_i (f_i(\mathbf{p}) - F(\mathbf{p}))
\]

Thus the proportion of players which play the better strategies increase with time. The replicator dynamic is originally based on biological considerations but it has also found applications in Economics. Evolutionarily Stable Strategies (ESSs) of evolutionary game theory are attractors of replicator dynamic with an important property of stability. When a population plays an ESS it cannot be invaded by a small number of mutants.

Quantum games apart from finding applications in quantum information [4] have also been considered interesting from another perspective. Referring to Dawkins' "Selfish Gene" [5] Eisert et al [1] hinted that games of survival are being played already on molecular level where quantum mechanics dictates the rules. Self replication of DNA molecules and synthesis of proteins which govern all the processes of a living molecule are two essential ways [12] where quantum information exchange can be said to play an important role. A consideration of quantum games in such situations seems a good reason to make these games interesting as indicated by Eisert et al [1]. A decision by a DNA molecule, for example to replicate itself or to synthesize a protein, can be imagined as a set of instructions that can formulated as a quantum strategy. How can, then, a DNA molecule evolves and develops a procedure to prepare a particular protein, for example, and not some other; assuming the procedure constitutes a quantum strategy. Why the other strategies, leading to synthesis of a different protein for example, instead of the desired one are not successful? This brings to mind the notion of mutant strategy from ESS theory that tries to invade an ESS without success and ESS comes out as a winning or successful strategy having a property of stability against perturbations from mutants.

We assume life molecules in possession of many strategies out of which a few or just one is successful in the sense of its increased use with time. The notion of ESS derived from population biology problems but the concept remains valid even when, for example, there are just two players to play a bi-matrix game. In that case the ESS will be a strategy the players will choose to play almost all the time and the fraction of the total time will replace the usual term called ‘frequency’ that corresponds to fraction of the total population. The molecule deciding in favor of a particular set of steps out of many available options gave us a good reason to consider perturbations in quantum game like situation that we believe to exist in that domain of intermolecular interactions. Mutant strategies constituting perturbations in classical evolutionary game theory are not successful against the Evolutionarily Stable Strategy (ESS) and ESS appears as a stable strategy that cannot be invaded when mutants appear in small numbers.
Underlying process in the definition of ESS is replicator dynamic based on Darwinian idea of ‘survival of the fittest’ leading to increase with time the frequency of a strategy giving better pay off. Replicator dynamic is a simple and theoretically attractive process that can have, we believe, an intuitive appeal to be extended to quantum game like situations. Among the many possible quantum strategies that we suppose to be at the disposal of a life molecule the winning strategy, corresponding to synthesis of desired protein for example, can be imagined as a result of micro-level replicator dynamic that becomes responsible for the increased use of the winning stable strategy to such an extent that the molecule implements it almost all the time and mutant set of procedures resulting in undesirable synthesis cannot invade the stable solution called ESS coming out of replicator dynamic.

In this perspective the use of replicator dynamic to select among many NE a special one corresponding to a stable solution, in quantum game like situations that we believe to exist in molecules forming the basis of life, appears an interesting hypothesis related to the mechanism existing in molecules to formulate and evolve established and stable procedures to perform their fundamental functions. An adjustment process having fruitful results both in population biology problems and economics is simple and intuitively appealing to form a mechanism to provide stable solutions in the domain of molecular interactions.

We apply Marinatto and Weber’s scheme \[2\] to a symmetric game between two players. Our reason to select this scheme is that the idea of a mixed NE and mixed ESS can easily be given a meaning without some extra assumptions when the pure strategies of the quantized version of the game are taken as the implementation by the players of identity \(I\) and Pauli spin-flip operator \(\sigma_x\) respectively. In Marinatto and Weber’s scheme quantum ‘tactics’ are applied by selecting \(I\) and \(\sigma_x\) with classical probabilities similar to the case of mixed NE in classical games where pure strategies are chosen with classical probabilities. We then show how the ‘entanglement’ usually considered as a pure quantum phenomenon can be exploited to select and decide ESSs in quantum version of a game satisfying a few requirements. The game we consider is symmetric similar to corresponding examples in evolutionary game theory where ESS is usually defined for symmetric pairwise contests.

In an earlier letter \[13\] we showed that for certain asymmetric quantum games between two players ESSs can be made to appear or disappear via a control over the initial state even when the corresponding strategies remain NE for those initial states. Our motivation in this letter is to explore similar possibilities for symmetric quantum games. The ESS idea was originally defined for symmetric games and will be of more interest when entanglement and entangled states are exploited in those games as well. Even for possible molecular level games referred to earlier the symmetric contests seems more appropriate and interesting. It also brings the exciting subject of mathematical theory of ESSs in the domain of quantum games and also quantum mechanics in general. During the last thirty years the theory of ESSs has been developed by collaboration between mathematicians and evolutionary biologists. An extension of the ESS idea in quantum games is an opportunity for both quantum game theory and
classical theory of ESSs to broaden their domains and also look for topics of common interest.

2 Evolutionary Stability

An Evolutionarily Stable Strategy (ESS) was originally defined by Smith and Price [6] with the motivation that a population playing the ESS can withstand a small invading group. They considered a symmetric game where the players are anonymous. Let $P[x, y]$ be the payoff to a player playing $x$ against the player playing $y$. Strategy $x$ is an ESS if for any alternative strategy $y$, the following two requirements are satisfied:

\[ P[x, x] \geq P[y, x] \]  \hspace{1cm} (2)

and in the case were condition 2 is satisfied as an equality:

\[ P[x, y] > P[y, y] \]  \hspace{1cm} (3)

Requirement 2 is in fact the Nash requirement and requires that no single individual can gain by unilaterally changing strategy from $x$ to $y$. For a linear structure of the game the two requirements 2 and 3 are equivalent to that for any $y \neq x$ there is a critical positive value such that if the frequency of the $y$-mutant strategists is lower than this value (while the rest of the population sticks to the strategy $x$), it is better for everyone to stick to majority strategy $x$, better being defined in terms of individual maximization of the payoff function $P$. Thus, as ESS is a strategy which, if played by almost all members of a population, cannot be displaced by a small invading group playing any alternative strategy. An ESS will persist as the dominant strategy through time, so that strategies observed in the real world will tend to be ESSs [9]. ESS comes out dynamically stable with respect to replicator dynamic. In molecular level situations when replicator dynamic is taken as the underlying process the ESSs involving quantum strategies will similarly tend to be observed in real micro world.

3 Evolutionary stability in symmetric quantum games

In Marinatto and Weber’s scheme [2] an entangled ‘initial strategy’ is forwarded to two players on which they apply their ‘tactics’. To remain consistent with available literature on the theory of ESSs in classical games we preferred to call Marinatto and Weber’s ‘initial strategy’ an ‘initial state’ and call their ‘tactics’ as ‘strategies’. This change in terminology has no effect on the originally proposed scheme to play a quantum game [10]. We consider following symmetric
bi-matrix game between two players for two available classical strategies $S_1$ and $S_2$

\[
\begin{pmatrix}
(\alpha, \alpha) & (\beta, \gamma) \\
(\gamma, \beta) & (\sigma, \sigma)
\end{pmatrix}
\]  

with the constants of the matrix $\alpha, \beta, \gamma, \sigma$ satisfying the following conditions.

\[
\begin{align*}
\alpha, \beta, \gamma, \sigma & \geq 0 \\
(\sigma - \beta) & > 0 \\
(\gamma - \alpha) & \geq 0 \\
(\gamma - \alpha) & < (\sigma - \beta)
\end{align*}
\]  

We take the initial quantum state to play the above game to be 2:

\[
|\psi_{in}\rangle = a |S_1S_1\rangle + b |S_2S_2\rangle
\]  

where

\[
|a|^2 + |b|^2 = 1
\]  

The associated density matrix is 2:

\[
\rho_{ini} = |a|^2 |S_1S_1\rangle \langle S_1S_1| + ab^* |S_1S_1\rangle \langle S_2S_2| + a^* b |S_2S_2\rangle \langle S_1S_1| + |b|^2 |S_2S_2\rangle \langle S_2S_2|
\]  

Where $p$ and $q$ be the probabilities of two players to act with the operator $\hat{I}$. Payoff to a $p$ player against a $q$ player for the payoff matrix (4) are written as 2:

\[
P(p, q) = \alpha \left\{ pq |a|^2 + (1-p)(1-q) |b|^2 \right\} + \
\beta \left\{ p(1-q) |a|^2 + q(1-p) |b|^2 \right\} + \
\gamma \left\{ p(1-q) |b|^2 + q(1-p) |a|^2 \right\} + \
\sigma \left\{ pq |b|^2 + (1-p)(1-q) |a|^2 \right\}
\]  

The condition that makes $(p^*, p^*)$ a NE is given as:

\[
P(p^*, p^*) - P(p, p^*) = (p^* - p) \left[ - |a|^2 (\sigma - \beta) + |b|^2 (\gamma - \alpha) + p^* \{ (\sigma - \beta) - (\gamma - \alpha) \} \right] \geq 0
\]
From the relation (10) it is clear that there can be three NE i.e. the pure strategies $p^* = 0$, $p^* = 1$ and the mixed strategy $p^* = \frac{(\gamma - \alpha)|a|^2 - (\gamma - \alpha)|b|^2}{(\gamma - \alpha) + (\sigma - \beta)}$. In the earlier form of this letter we only considered the strategy $p^* = 0$ and showed that it is evolutionary stable in the classical game (4) with conditions (5) but not so in a quantum version of the same game. We are grateful to anonymous referee who hinted the need to study the problem in a more general and systematic way. We now consider the evolutionary stability of these three NE of the symmetric game (4) in three separate cases:

### 3.1 Case $p^* = 0$

For the strategy $p^* = 0$ to be a NE we should have:

$$P(0,0) - P(p,0) = \frac{p}{(\gamma - \alpha) + (\sigma - \beta)} \left[ |a|^2 - \frac{(\gamma - \alpha)}{(\gamma - \alpha) + (\sigma - \beta)} \right] \geq 0 \quad (11)$$

Now $[P(0,0) - P(p,0)] > 0$ when $1 \geq |a|^2 > \frac{(\gamma - \alpha)}{(\gamma - \alpha) + (\sigma - \beta)}$ and $p^* = 0$ is a pure ESS. However, at $|a|^2 = \frac{(\gamma - \alpha)}{(\gamma - \alpha) + (\sigma - \beta)}$ we have the difference (11) zero and $p^* = 0$ can be an ESS if

$$P(0,p) - P(p,p) = p \left\{ (\gamma - \alpha) + (\sigma - \beta) \right\} \left[ |a|^2 - \frac{(1-p)(\gamma - \alpha) + p(\sigma - \beta)}{(\gamma - \alpha) + (\sigma - \beta)} \right] > 0 \quad (12)$$

it can be written as:

$$P(0,p) - P(p,p) = p \left\{ (\gamma - \alpha) + (\sigma - \beta) \right\} \left\{ |a|^2 - F \right\} > 0 \quad (13)$$

Where $\frac{(\gamma - \alpha)}{(\gamma - \alpha) + (\sigma - \beta)} \leq F \leq \frac{(\sigma - \beta)}{(\gamma - \alpha) + (\sigma - \beta)}$ for the range $0 \leq p \leq 1$. In such a situation $p^* = 0$ can be an ESS only when $|a|^2 > \frac{(\sigma - \beta)}{(\gamma - \alpha) + (\sigma - \beta)}$ which is not possible because $|a|^2$ is fixed at $\frac{(\gamma - \alpha)}{(\gamma - \alpha) + (\sigma - \beta)}$. Therefore $p^* = 0$ is a stable NE or an ESS for $1 \geq |a|^2 > \frac{(\gamma - \alpha)}{(\gamma - \alpha) + (\sigma - \beta)}$ and at $|a|^2 = \frac{(\gamma - \alpha)}{(\gamma - \alpha) + (\sigma - \beta)}$ this NE becomes unstable. The classical game is obtained by fixing $|a|^2 = 1$ for which $p^* = 0$ is a stable NE. However, this NE does no remain stable when $|a|^2 = \frac{(\gamma - \alpha)}{(\gamma - \alpha) + (\sigma - \beta)}$ corresponding to an entangled initial state; though, the NE remains intact in both forms of the game.

### 3.2 Case $p^* = 1$

Similar to previous case we write the NE condition for the strategy $p^* = 1$ as:
\[ P(1, 1) - P(p, 1) = \frac{(1-p)}{(\gamma - \alpha) + (\sigma - \beta)} \left[ -|a|^2 + \frac{(\sigma - \beta)}{(\gamma - \alpha) + (\sigma - \beta)} \right] \geq 0 \quad (14) \]

Now \( p^* = 1 \) is a pure ESS for \( 0 \leq |a|^2 < \frac{(\sigma - \beta)}{(\gamma - \alpha) + (\sigma - \beta)} \). At \( |a|^2 = \frac{(\sigma - \beta)}{(\gamma - \alpha) + (\sigma - \beta)} \), we have the difference given in eq. (14) zero. The strategy \( p^* = 1 \), then, becomes an ESS when

\[ P(1, p) - P(p, p) = (1 - p) \left\{ (\gamma - \alpha) + (\sigma - \beta) \right\} \left[ -|a|^2 + \frac{(1-p)(\gamma - \alpha) + p(\sigma - \beta)}{(\gamma - \alpha) + (\sigma - \beta)} \right] > 0 \quad (15) \]

It is possible only when \( |a|^2 < \frac{(\gamma - \alpha)}{(\gamma - \alpha) + (\sigma - \beta)} \). Therefore the strategy \( p^* = 1 \) is a stable NE for \( 0 \leq |a|^2 < \frac{(\sigma - \beta)}{(\gamma - \alpha) + (\sigma - \beta)} \). Therefore, the strategy \( p^* = 1 \) not stable classically, i.e. for \( |a|^2 = 1 \), can be stable for an entangled initial state.

### 3.3 Case \( p^* = \frac{(\sigma - \beta)|a|^2 - (\gamma - \alpha)|b|^2}{(\sigma - \beta) - (\gamma - \alpha)} \)

In case of the mixed strategy

\[ p^* = \frac{(\sigma - \beta)|a|^2 - (\gamma - \alpha)|b|^2}{(\sigma - \beta) - (\gamma - \alpha)} \quad (16) \]

we have from the NE condition (10)

\[ P(p^*, p^*) - P(p, p^*) = 0 \quad (17) \]

The mixed strategy (16) becomes an ESS when

\[ P(p^*, p) - P(p, p) = (p^* - p) \left[ -|a|^2 (\sigma - \beta) + |b|^2 (\gamma - \alpha) + p \{(\sigma - \beta) - (\gamma - \alpha)\} \right] > 0 \quad (18) \]

for all \( p \neq p^* \). Write now the strategy \( p \) as \( p = p^* + \Delta \). For the mixed strategy of eq. (16) the payoff difference of the eq. (18) can be reduced to:

\[ P(p^*, p) - P(p, p) = -\Delta^2 \{ (\sigma - \beta) - (\gamma - \alpha) \} \quad (19) \]

So that, for the game defined in the conditions (16) the mixed strategy \( p^* = \frac{(\sigma - \beta)|a|^2 - (\gamma - \alpha)|b|^2}{(\sigma - \beta) - (\gamma - \alpha)} \) cannot be an ESS, though it can be a NE of the symmetric game.
4 Conclusion

Dynamic stability is a well known refinement concept in situations where multiple NE arise as solutions to a symmetric quantum game played between two players. ESS is a stable NE in a game when its dynamics puts a direct proportionality between frequency of a strategy and its relative payoff advantage usually termed as replicator dynamic. We suggested that this idea can be interesting not only in population biology but also in quantum game theory. We explored evolutionary stability of NE in a symmetric quantum game played between two players via a scheme proposed recently by Marinatto and Weber \[2\]. We showed that in this scheme the evolutionary stability of a pure strategy for a symmetric game can be changed by maneuvering the initial state even when the strategy remains a NE during such a maneuver. For example, in a symmetric game a pure strategy can be an ESS in the classical form of the game but it does not remain so in some quantized form of the same game. We also showed that in the proposed scheme \[2\] with its particular form of initial state the evolutionary stability of a mixed strategy, however, cannot be changed while retaining the corresponding NE. Our results show that entangled states have also potential roles to play in the stability properties of NE in symmetric games. It shows the possibility of achieving dynamic stability of NE in symmetric quantum games via the use of entangled states.

5 Acknowledgment

One of us (A.I) is grateful to Pakistan Institute of Lasers and Optics, Islamabad, for support.

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