Spontaneous decay of artificial atoms in a three-qubit system

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Abstract. We study the evolution of qubits amplitudes in a one-dimensional chain consisting of three equidistantly spaced noninteracting qubits embedded in an open waveguide. The study is performed in the frame of single-excitation subspace, where the only qubit in the chain is initially excited. We show that the dynamics of qubits amplitudes crucially depend on the value of $kd$, where $k$ is the wave vector and $d$ is a distance between neighbor qubits. If $kd$ is equal to an integer multiple of $\pi$, then the qubits are excited to a stationary level. In this case, it is the dark states which prevent qubits from decaying to zero, even though they do not contribute to the output spectrum of photon emission. For other values of $kd$, the excitations of qubits exhibit the damping oscillations which represent the vacuum Rabi oscillations in a three-qubit system. In this case, the output spectrum of photon radiation is determined by a subradiant state which has the lowest decay rate. We also investigated the case with the frequency of a central qubit being different from that of the edge qubits. In this case, the qubits’ decay rates can be controlled by the frequency detuning between the central and the edge qubits.

1 Introduction

Quantum bits can be implemented with a variety of quantum systems, such as trapped ions [1], superconducting qubits [2,3], photons [4–6], and quantum dots [7,8]. In particular, superconducting qubits have emerged as one of the leading candidates for scalable quantum processor architecture.

Superconducting qubits coupled to photons propagating in an open waveguide [9–11] allow for the investigation of the fascinating world of quantum light–matter interactions in one dimension [12–14]. Even though the properties of multi-qubit 1D systems have been extensively studied both theoretically [15–19] and experimentally [20–22], less attention has been paid to a detail investigation of dynamic properties of the few-qubit systems which are the building blocks for quantum gates [23,24]. Moreover, the scaling laws for decay rates that have been found for multi-qubit systems [25] cannot obviously be applied for systems containing few qubits.

As is known, the superconducting qubits can be technologically addressed and controlled individually [14]. Therefore, it is important to know the evolution of the probability amplitude of any qubit in a superconducting circuit.

Here, we explore the dynamic properties of a quantum circuit consisting of a three-qubit linear chain, which is strongly coupled to a common waveguide. The motivation for this choice is that it is a simplest system with non-trivial properties for which a full analytical treatment can be obtained.

Our treatment is based on two approximations. First, we restrict the Hilbert space to a single-excitation subspace. The motivation for this is that throughout the paper, we consider a system where initially the only qubit is excited and there are no photons in a system. Our main assumption is that any qubit during its evolution emits only a single photon at the moment. Second, we apply the Wigner–Weisskopf approximation which allows us to obtain a set of linear differential equations for the qubits’ amplitudes. This approximation is justified if the decay rate of a qubit is much less than its transition frequency. From mathematical point of view, the Wigner–Weisskopf approximation is obtained by continuing the frequency to negative axis, thus, neglecting small deviations from exponential decay.

We investigate a dynamic behavior of the qubits amplitudes with the only qubit in the chain being initially excited. For three-qubit system, we find the analytic expressions for the complex energies and for the collective states which define a temporal behavior of qubits amplitudes. We show that the dynamics of qubits amplitudes crucially depend on the value of $kd$, where $k$ is the wave vector and $d$ is a distance between neighbor qubits. If $kd$ is equal to integer multiple of $\pi$, the
qubits are excited to a stationary level. In this case, these are the dark states which prevent qubits from decaying to zero, even though they do not contribute to the output photon spectrum. For other values of $kd$, the excitations of qubits have oscillatory behavior and are gradually damped out to zero. We also investigate the case with the frequency of a central qubit being different from that of the edge qubits. In this case, the qubits’ decay rates can be controlled by the frequency detuning between the central and the edge qubits. As the frequency detuning between central and edge qubits increases, the decay rates of qubits also increase. This property is very important for the implementation of the efficient control and readout protocols where a fast reset of the excited qubits to their ground state is essential \[26,27\].

The paper is structured as follows.

In Sect. 2, we begin by introducing a Jaynes–Cummings Hamiltonian for atom–light interactions. We truncate the Hilbert space to a single-excitation subspace and obtain a set of linear integro-differential equations for the qubits amplitudes.

In Sect. 3, in the frame of Wigner–Weisskopf approximation, we derive a set of linear differential equations for the qubits amplitudes $\beta_n(t)$, which allow for a direct numerical simulations.

A comprehensive analysis of the dynamics of the qubits amplitudes is given in Sect. 4. We show that for $kd = n\pi$, the qubits amplitudes become “frozen” at the constant level. The reason for this is the dark states which prevent qubits from decaying to zero. For the values of $kd$ which are not integer multiple of $\pi$, the qubits’ amplitudes gradually damped out to zero. In this section, we also calculated the probability amplitude of the photon emission. We find that for $kd = (2n + 1)\pi$, the evolution of the photon amplitude consists of clearly seen steps. These steps can be attributed to the interrelation between temporal behaviors of the different qubits amplitudes.

In Sect. 5, we calculate a spectral density of photon radiation from a three-qubit chain. We show that for $kd = n\pi$, the dark states do not contribute to output radiation near resonance. In this case, the output spectral density has a Lorentzian lineshape. For $kd = (2n + 1)\pi/2$, a spectral density exhibits two peaks which are a signature of the vacuum Rabi oscillations of qubits amplitudes. In general, if $kd$ is not equal to an integer multiple of $\pi$, the width of a spectral line is defined by the deep subradiant states.

In Sect. 6, we formulate the dynamics of a three-qubit chain with the aid of a non-Hermitian Hamiltonian which is obtained after the elimination of the photon variables. We find the collective states which are eigenvectors of non-Hermitian Hamiltonian and show how the qubits amplitudes $\beta_n(t)$ can be expressed in terms of these collective states.

In Sect. 7, we consider a three-qubit system in which the frequency $\Omega_0$ of the central qubit is different from that of the edge qubits. We show that using the detuning $\delta \Omega = \Omega - \Omega_0$ as external parameter, we can control the decay rates of the qubits’ amplitudes. As the frequency detuning between central and edge qubits increases, the decay rates of qubits also increase. For this case, a spectral density of photon radiation is also calculated for different values of $\delta \Omega$.

The main results of the paper are summarized in Sect. 8.

### 2 Formulation of the problem

We consider a linear chain of three equally spaced qubits which are coupled to photon field in an open waveguide (see Fig. 1).

A distance between neighbor qubits is equal to $d$. The Hilbert space of every qubit consists of the excited state $|e\rangle$ and the ground state $|g\rangle$. The Hamiltonian which accounts for the interaction between qubits and the electromagnetic field is as follows (we use $\hbar = 1$ throughout the paper):

$$H = H_0 + \sum_k \omega_k a_k^+ a_k + H_{\text{int}},$$  \hspace{1cm} (1)

where $H_0$ is Hamiltonian of bare qubits

$$H_0 = \frac{1}{2} \sum_{n=1}^{3} \left(1 + \sigma_z^{(n)}\right) \Omega_n,$$

$$H_{\text{int}} = \sum_{n=1}^{3} \sum_k g_k^{(n)} e^{-ikx_n} \sigma_-^{(n)} a_k^+ + h.c.$$ \hspace{1cm} (2)

The quantity $g_k^{(n)}$ in (2) is the coupling between $n$th qubit and the photon field in a waveguide. Below, we consider a single-excitation subspace with either a single photon is in a waveguide and all qubits are in the ground state, Fig. 1b, or there are no photons in a waveguide with the only $n$th qubit in the chain being excited, Fig. 1a. Therefore, we truncate Hilbert space...
In fact, the quantity

\[ |\Psi, 1_k \rangle = |g_1, g_2, g_3 \rangle \otimes |1_k \rangle ; \]

\[ |1, 0_k \rangle = \xi_1, g_2, g_3 \rangle \otimes |0_k \rangle ; \]

\[ |2, 0_k \rangle = |g_1, e_2, g_3 \rangle \otimes |0_k \rangle ; \]

\[ |3, 0_k \rangle = |g_1, g_2, e_3 \rangle \otimes |0_k \rangle . \]

The Hamiltonian (2) preserves the number of excitations (number of excited qubits + number of photons). In our case, the number of excitations is equal to one (see Fig. 1). Therefore, at any instant of time, the system will remain within a single-excitation subspace. The wave-function of an arbitrary single-excitation state can then be written in the form

\[ |\Psi \rangle = \sum_{n=1}^{3} \beta_n(t) e^{-i\Omega_n t} |n, 0_k \rangle + \sum_k \gamma_k(t) e^{-i\omega_k t} |G, 1_k \rangle , \tag{3} \]

where \( \beta_n(t) \) is the amplitude of \( n \)-th qubit, and \( \gamma_k(t) \) is a single-photon amplitude which is related to a spectral density of spontaneous emission

\[ S(\omega_k, t) = |\gamma_k(t)|^2. \tag{4} \]

The function (3) is normalized to unity

\[ 3 \sum_{n=1}^{3} \beta_n(t)^2 + \sum_{k} \gamma_k(t)^2 = 1. \tag{5} \]

From (5), we can find the full probability of photon emission from the three-qubit system

\[ P_{ph}(t) = \sum_k |\gamma_k(t)|^2 = 1 - \sum_{n=1}^{3} \beta_n(t)^2. \tag{6} \]

In fact, the quantity \( P_{ph}(t) \) is the probability to find the emitted photon at the moment \( t \).

The equations for qubits amplitudes \( \beta_n(t) \) in (3) can be found from time-dependent Schrodinger equation \( i\hbar d|\Psi \rangle/dt = H|\Psi \rangle \). For the amplitudes \( \gamma_k(t), \beta_n(t) \), we obtain

\[
\frac{d\beta_n}{dt} = -\sum_{m \neq n}^{3} g_k^{*}(m) g_k^{(n)} e^{-i(kx_{m}-x_{n})t} \int_{0}^{t} \beta_m(t') e^{-i(\omega_{k}-\Omega_{n})t'} dt',
\]

\[
-\sum_{k}^{3} g_k^{(n)} e^{-ikx_n} \int_{0}^{t} \beta_m(t') e^{i(\omega_{k}-\Omega_{n})t'} dt',
\]

\[ \gamma_k(t) = -i \sum_{n=1}^{3} g_k^{(n)} e^{-ikx_n} \int_{0}^{t} \beta_m(t') e^{i(\omega_{k}-\Omega_{n})t'} dt'. \tag{8} \]

According to (8), there are no photons in the system at \( t = 0 \). Our goal is to find the evolution of the amplitudes \( \beta_n(t) \) for any qubit in the chain when the only \( n_0 \)-th qubit \( (n_0 = 1, 2, \text{ or } 3) \) is initially excited

\[
\begin{cases}
\beta_{n_0}(0) = 1, \\
\beta_n(0) = 0, \ n \neq n_0.
\end{cases}
\]

### 3 Equations for the qubits’ amplitudes

We assume that the first and the third qubits are identical \((\Omega_1 = \Omega_3 = \Omega, g_k^{(1)} = g_k^{(3)} = g_k)\). The frequency and the coupling of the second qubit are different \((\Omega_2 \equiv \Omega_0, g_k^{(2)} \equiv g_k^{(0)})\). A distance between central qubit and the edge qubits is equal to \( d \). We take the origin in the location of the second qubit: \( x_1 = -d, x_2 = 0, x_3 = +d \). In the frame of Wigner-Weisskopf approximation, Eq. (7) for the qubits amplitudes can be reduced to the following set of linear differential equations (see Appendix A for the derivation):

\[
\frac{d\bar{\beta}_1}{dt} = -\frac{\Gamma}{2} \bar{\beta}_1 - i\frac{\Omega - \Omega_0}{2} \bar{\beta}_1(t)
\]

\[-\bar{\beta}_2(t) \frac{1}{2} \left( \frac{\Omega_0}{\Omega} \Gamma \right)^{1/2} \sqrt{\Gamma \Omega} e^{-2ikd} - \Gamma \bar{\beta}_3(t) e^{2ikd},
\]

\[
\frac{d\bar{\beta}_3}{dt} = -\frac{\Gamma}{2} \bar{\beta}_3(t) - i\frac{\Omega - \Omega_0}{2} \bar{\beta}_3(t)
\]

\[-\bar{\beta}_2(t) \frac{1}{2} \left( \frac{\Omega_0}{\Omega} \Gamma \right)^{1/2} \sqrt{\Gamma \Omega} e^{-2ikd},
\]

where

\[
\begin{cases}
\bar{\beta}_1(t) = e^{-i(\Omega - \Omega_0)t/2} \beta_1(t), \\
\bar{\beta}_2(t) = e^{i(\Omega - \Omega_0)t/2} \beta_2(t).
\end{cases}
\]

\[
\begin{align*}
&k = \Omega/v_g, \ k_0 = \Omega_0/v_g, \ \Gamma, \ \Gamma_0 \text{ are the rates of spontaneous emission into the waveguide mode from edge qubits and from the central qubit, respectively.} \\
&\text{Within a single-excitation subspace and the Wigner-Weisskopf approximation, Eqs. (9), (10), and (11) are exact. They are used for the calculations throughout the paper without any additional assumptions.}
\end{align*}
\]

### 4 The dynamics of three identical qubits

For three identical qubits, we obtain from (9)–(11) the following equations:
\[
\frac{d\beta_n}{dt} = -\frac{\Gamma}{2} \sum_{m=1}^{3} \beta_m(t)e^{ikd|m-n|}, \quad (n = 1, 2, 3) \quad (13)
\]

which can be expanded explicitly as:

\[
\begin{align*}
\frac{d\beta_1}{dt} &= -\frac{\Gamma}{2}\beta_1(t) - \frac{\Gamma}{2}\beta_2(t)e^{ikd} - \frac{\Gamma}{2}\beta_3(t)e^{2ikd} \\
\frac{d\beta_2}{dt} &= -\frac{\Gamma}{2}\beta_2(t) - \frac{\Gamma}{2}\beta_1(t) + \beta_3(t) \\
\frac{d\beta_3}{dt} &= -\frac{\Gamma}{2}\beta_3(t) - \frac{\Gamma}{2}\beta_1(t)e^{2ikd} - \frac{\Gamma}{2}\beta_2(t)e^{ikd},
\end{align*}
\]

where \( k = \Omega/v_y \), and for simplicity, we remove the bar over \( \beta_n(t) \).

The general solution of Eq. (14) can be written as follows:

\[
\beta_i(t) = \sum_{m=1}^{3} a_{m}^{(i)}e^{\lambda_m t}, \quad i = 1, 2, 3. \quad (15)
\]

The quantities \( \lambda_m \) are characteristic roots, which can be found by equating to zero the determinant of Eq. (14)

\[
\left( \lambda_i + \frac{\Gamma}{2} \right) \delta_{mn} + \frac{\Gamma}{2} e^{ikd|m-n|} (1 - \delta_{mn}). \quad (16)
\]

The quantities \( a_{m}^{(i)} \) in (15) are defined by initial conditions

\[
\begin{align*}
\sum_{m=1}^{3} a_{m}^{(n)} &= 1, \\
\sum_{m=1}^{3} a_{m}^{(i)} &= 0, \quad i \neq n_0,
\end{align*}
\]

where \( n_0 \) is the number of initially excited qubit.

From determinant of (16), we obtain the exact expressions for \( \lambda_m \)

\[
\begin{align*}
\lambda_1 &= -\frac{\Gamma}{4}e^{2ikd} - \frac{\Gamma}{4}e^{ikd}\sqrt{e^{2ikd} + 8} - \frac{\Gamma}{2} \mathbf{1}
\lambda_2 &= -\frac{\Gamma}{4}e^{2ikd} + \frac{\Gamma}{4}e^{ikd}\sqrt{e^{2ikd} + 8} - \frac{\Gamma}{2} \mathbf{1}
\lambda_3 &= \frac{\Gamma}{2}e^{ikd} - \frac{\Gamma}{2}
\end{align*}
\]

From (17), we see that a sum

\[
\sum_{i=1}^{3} \lambda_i = -\frac{3}{2}\Gamma \quad (18)
\]

does not depend on \( kd \). The expression (18) is a special case of the more general sum rule for \( N \) qubit system

\[
\sum_{i=1}^{N} \Gamma_i = NT/2, \quad (19)
\]

where \( \Gamma_i = -\text{Re}(\lambda_i) \). The sum rule (19) states that there are no other losses in the system other than the coherent spontaneous emission into a waveguide. In general, as is seen from (15), all \( \Gamma_i \)'s contribute to the decay rate of a concrete qubit.

From (17), we see that the decay rates \( \Gamma_i \) depend on \( kd \). This dependence is shown in Fig. 2.

From Fig. 2, we see that the dependence of \( \text{Re}(\lambda_i) \) on \( kd \) is periodic, and all the values of \( \text{Re}(\lambda_i) \) are negative. The latter means that as the time proceeds, the system evolves to its stationary state. In addition, as is seen from Fig. 2, the sum rule (19) for \( N = 3 \) holds for every value of \( kd \).

Consider now the solution of Eq. (14). By subtracting the third equation in (14) from the first one, we obtain

\[
\frac{d}{dt}(\beta_1 - \beta_3) = -\frac{\Gamma}{2} (1 - e^{2ikd})(\beta_1 - \beta_3). \quad (20)
\]

It follows from (20) that if the edge qubits are initially not excited, \( \beta_1(0) = \beta_3(0) = 0 \), then this difference remains zero for all times, \( \beta_1(t) = \beta_3(t) \equiv \beta(t) \). This is quite reasonable from symmetry consideration: if a central qubit is initially excited, the temporal behavior of the amplitudes of the edge qubits must be the same. Therefore, for this case, three Eq. (14) can be reduced to two equations

\[
\begin{align*}
\frac{d\beta_2}{dt} &= -\frac{\Gamma}{2}\beta_2(t) - \Gamma \beta(t)e^{ikd} \\
\frac{d\beta}{dt} &= -\frac{\Gamma}{2}\beta(t) (1 + e^{2ikd}) - \frac{\Gamma}{2}\beta_2(t)e^{ikd},
\end{align*}
\]

where \( \beta_2(0) = 1, \quad \beta(0) = 0 \).

The characteristic roots of (21) are equal to \( \lambda_1 \) and \( \lambda_2 \) which are given in (17). Therefore, the solution of Eq. (21) reads

\[
\begin{align*}
\beta_2(t) &= b_1e^{\lambda_1 t} + b_2e^{\lambda_2 t}, \\
\beta(t) &= a_1e^{\lambda_1 t} + a_2e^{\lambda_2 t},
\end{align*}
\]

\[
\begin{align*}
\beta_2(t) &= -\frac{\Gamma}{2}\beta_2(t) - \Gamma \beta(t)e^{ikd} \\
\frac{d\beta}{dt} &= -\frac{\Gamma}{2}\beta(t) (1 + e^{2ikd}) - \frac{\Gamma}{2}\beta_2(t)e^{ikd}, \quad (21)
\end{align*}
\]
where from initial conditions
\[
\begin{align*}
    b_1 + b_2 &= 1, \\
    a_1 + a_2 &= 0.
\end{align*}
\]  
(23)

Another two conditions follow from (21) for time derivatives at \( t = 0 \):
\[
\begin{align*}
    b_1 \lambda_1 + b_2 \lambda_2 &= -\frac{\Gamma}{2}, \\
    a_1 \lambda_1 + a_2 \lambda_2 &= -\frac{\Gamma}{2} e^{ikd}. \\
\end{align*}
\]  
(24)

From (23) and (24), we obtain
\[
\begin{align*}
    b_1 &= -\frac{\Gamma + \lambda_2}{\lambda_1 - \lambda_2}; \\
    b_2 &= \frac{\Gamma + \lambda_1}{\lambda_1 - \lambda_2}; \\
    a_1 &= -\frac{\Gamma}{2} e^{ikd}; \\
    a_2 &= \frac{\Gamma}{2} e^{ikd}.
\end{align*}
\]  
(25)

Using the explicit expressions (17), we obtain
\[
\begin{align*}
    a_1 &= \frac{1}{R}; \\
    a_2 &= -\frac{1}{R}; \\
    b_1 &= \frac{R - e^{ikd}}{2R}; \\
    b_2 &= \frac{R + e^{ikd}}{2R},
\end{align*}
\]  
(27)

where \( R = \sqrt{e^{2kd} + 8} \).

Now, we assume that the first qubit in the chain is initially excited
\[
\beta_1(0) = 1, \quad \beta_2(0) = 0, \quad \beta_3(0) = 0.
\]

In this case, all amplitudes behave differently, so that Eq. (14) should be used from which the solution can be straightforwardly obtained
\[
\begin{align*}
    \beta_1(t) &= \frac{b_1}{2} e^{\lambda_1 t} + \frac{b_2}{2} e^{\lambda_2 t} + \frac{1}{2} e^{\lambda_3 t} \\
    \beta_2(t) &= \frac{1}{R} \left( e^{\lambda_1 t} - e^{\lambda_2 t} \right) \\
    \beta_3(t) &= \frac{b_2}{2} e^{\lambda_1 t} + \frac{b_2}{2} e^{\lambda_2 t} - \frac{1}{2} e^{\lambda_3 t},
\end{align*}
\]  
(28)

where \( b_1 \) and \( b_2 \) are given in (27). For special cases when \( kd \) is integer multiple of \( \pi \), we may obtain from (22) and (28) very simple forms for the temporal behavior of the qubits' amplitudes.

4.1 \( kd = \pi n \)

For this case, \( \lambda_1 = -\frac{3}{2} \Gamma, \lambda_2 = \lambda_3 = 0 \) if \( n \) is even number, and \( \lambda_1 = 0, \lambda_2 = -\frac{3}{2} \Gamma, \lambda_3 = 0 \) if \( n \) is odd number. The calculations show that no matter which qubit in the chain is initially excited (central or edge qubit), its evolution is the same.

\[
\beta_{\text{exc}}(t) = \frac{1}{3} e^{-3\Gamma t/2} + \frac{2}{3}.
\]  
(29)

The expression (30) is valid for any \( n \) if the central qubit is excited. If the first qubit is excited and \( n \) is the odd number, the amplitudes of unexcited qubits evolve with the opposite phases: \( \beta_2(t) = \beta_{\text{unexc}}(t), \beta_3(t) = -\beta_{\text{unexc}}(t) \).

It worth noting the equality of the amplitudes of the second and the third qubit (if the first qubit is initially excited), even though they are located at different distance from the excited qubit. This is the consequence of Wigner–Weisskopf (or Markov) approximation: in this special case, there is no interference between qubits, so that all unexcited qubits feel the photon field simultaneously no matter how far they are from the excited qubit. From (29) and (30), we find a full probability of photon emission (6).

\[
P_{\text{ph}}(t) = 1 - 2|\beta_{\text{unexc}}(t)|^2 - |\beta_{\text{exc}}(t)|^2 = \frac{1}{3} \left( 1 - e^{-3\Gamma t} \right).
\]  
(31)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{The probability amplitudes for \( kd = n\pi \) of excited, \( |\beta_{\text{exc}}(t)|^2 \) (the upper line) and unexcited \( |\beta_{\text{unexc}}(t)|^2 \) (the lower line) qubits. The probability of a full photon emission is shown between the qubit lines.}
\end{figure}

The probability of qubits amplitudes and a full probability of photon emission are shown in Fig. 3.

As is seen from Fig. 3, as the time proceeds, the qubits amplitudes become “frozen” at some level. We see from (29) and (30) that this frozenness is solely due to the dark states \( \lambda_n = 0 \), which prevent the qubits' amplitudes from decaying to zero. The frozenness can be lifted if \( kd \) is not equal to an integer multiple of \( \pi \). Then, all qubits are damped to zero with the rate being determined by the root of \( \lambda_n \) which has the lowest real part.
If $n$ is even number, then $\lambda = \gamma d \equiv (2n + 1) \pi/2$. The oscillatory behavior of the qubits amplitudes is a signature of the vacuum Rabi oscillations initially excited. The evolution of the qubits amplitudes for half-integer multiple of $\pi$ with the central qubit being initially excited is shown in Fig. 4.

The evolution of qubits amplitudes for $kd$ equals the half-integer multiple of $\pi$ with the central qubit being initially excited. The oscillatory behavior of the qubits amplitudes is a signature of the vacuum Rabi oscillations.

4.2 $kd = \pi(2n + 1)/2$

If $n$ is even number, then $\lambda_1 = -\frac{\gamma}{2} (1 + i \sqrt{7})$, $\lambda_2 = -\frac{\gamma}{2} (1 - i \sqrt{7})$, $\lambda_3 = -\Gamma$, and if $n$ is odd number, then $\lambda_1 = -\frac{\gamma}{2} (1 - i \sqrt{7})$, $\lambda_2 = -\frac{\gamma}{2} (1 + i \sqrt{7})$, $\lambda_3 = -\Gamma$.

If the central qubit is initially excited, we obtain

$$\beta_1(t) = \beta_3(t) = (-1)^{n+1} i \frac{2}{\sqrt{7}} e^{-\frac{\gamma}{2} t} \sin \left( \frac{\sqrt{7}}{4} \Gamma t \right),$$
$$\beta_2(t) = e^{-\frac{\gamma}{2} t} \left( \cos \left( \frac{\sqrt{7}}{4} \Gamma t \right) + (-1)^{n} \frac{1}{\sqrt{7}} \sin \left( \frac{\sqrt{7}}{4} \Gamma t \right) \right).$$

(32)

For initially excited edge qubit, we obtain

$$\beta_1(t) = \frac{e^{-\frac{\gamma}{2} t}}{2} \left( \cos \left( \frac{\sqrt{7}}{4} \Gamma t \right) + \frac{1}{\sqrt{7}} \sin \left( \frac{\sqrt{7}}{4} \Gamma t \right) \right) + \frac{1}{2} e^{-\Gamma t},$$
$$\beta_2(t) = (-1)^{n+1} i \frac{2}{\sqrt{7}} e^{-\frac{\gamma}{2} t} \sin \left( \frac{\sqrt{7}}{4} \Gamma t \right),$$
$$\beta_3(t) = e^{-\frac{\gamma}{2} t} \left( \cos \left( \frac{\sqrt{7}}{4} \Gamma t \right) + \frac{1}{\sqrt{7}} \sin \left( \frac{\sqrt{7}}{4} \Gamma t \right) \right) - \frac{1}{2} e^{-\Gamma t}.$$  

(33)

The evolution of qubits amplitudes for half-integer multiple of $\pi$ with the central qubit being initially excited is shown in Fig. 4.

The evolution of $P_{ph}(t)$ reveals two clear visible steps where $dP_{ph}(t)/dt = 0$. These steps can be attributed to the interrelation between the temporal dynamics of the different amplitudes. As is seen from this figure, the steps on $P_{ph}(t)$ curve are in the vicinity of extremum points of qubits amplitudes. Physically, these steps are the signature of the trapping of the photon radiation.

As is seen in Fig. 3, near the first step, the radiation emitted by excited qubit is absorbed by unexcited qubits, so that the rate of the output radiation is not changed. The evolution of qubits’ amplitudes for half-integer multiple of $\pi$ with the first qubit being initially excited is shown in Fig. 5.

Here, we also see the vacuum Rabi oscillations of the qubits’ amplitudes. In addition, the evolution of qubit’s amplitude depends on its distance from the excited qubit. The greater is the distance of a qubit from the excited one, the less it is affected by the excitation.

5 Spectral density of photon radiation

The quantity $\gamma_k(t)$ in (3) allows for the calculation of a spectral density (4) of spontaneous emission into a waveguide, $S(\omega, t)$. Equation (8) for identical qubits reads

$$\gamma(\omega, t) = -igk \sum_{n=1}^{3} e^{-ikx_n} \int_0^t \beta_n(t') e^{i(\omega - \Omega)t'} dt'.$$  

(34)

Using the expression (15) for $\beta_n(t)$, we obtain

$$\gamma(\omega, t) = gk \sum_{n,m=1}^{3} a_j^{(n)} e^{-ikx_n} \frac{1 - e^{i(\omega - \Omega - i\lambda_m)t}}{\omega - \Omega - i\lambda_m}.$$  

(35)

As $g_k$ in (34) and (35) depends on the waveguide length, $L$ and the light velocity $v_g$ (see (A16)), we factorize $\gamma_k(t)$ as follows:

$$\gamma_k(t) \equiv \left( \frac{v_g}{2L\Omega} \right)^{1/2} \left( \frac{\Gamma}{\Omega} \right)^{1/2} f_k(t).$$

\[ \text{Image 4 and 5:} \text{Graphs showing the evolution of qubits amplitudes for } kd \text{ equals the half-integer multiple of } \pi \text{ with the central qubit being initially excited.} \]

\[ \text{Image 6:} \text{Graph showing the spectral density of photon radiation.} \]
Therefore, we define spectral density in the following form:

\[
S(\omega, t) = \frac{|\gamma_k(t)|^2}{\Gamma \Omega} = \frac{\Gamma}{\Omega} |f_k(t)|^2. \tag{36}
\]

Equation (35) is a rather general expression. As the time proceeds, only the term with the lowest real part of \(\lambda_m\) survives. The concise analytical results can be obtained only for several simple cases. Below, we calculate the spectral density for \(kd = n\pi\), using the amplitudes (29) and (30) in (34).

When a central qubit is initially excited, we obtain for \(\gamma_k(t)\) the following result:

\[
\gamma_k(t) = -g_k \frac{2}{3} \left( 1 - (-1)^n \cos \left( \frac{n\pi}{\Gamma} \right) \right) \frac{e^{(\omega-\Omega)t}}{\omega-\Omega} - 
- g_k \frac{1}{3} \left( 1 - (-1)^n \cos \left( \frac{n\pi}{\Gamma} \right) \right) \frac{e^{(\omega-\Omega+i\frac{3\pi}{2})t}}{\omega-\Omega+i\frac{3\pi}{2}}.
\tag{37}
\]

If the first qubit is initially excited, we obtain

\[
\gamma_k(t) = -g_k \frac{1}{3} \left( 2 e^{i(\pi n \pi)} - e^{-i(\pi n \pi)} - (-1)^n \right) \times 
\times \frac{e^{(\omega-\Omega)t}}{\omega-\Omega} -
- g_k \frac{1}{3} \left( 1 - (-1)^n \cos \left( \frac{n\pi}{\Gamma} \right) \right) \frac{e^{(\omega-\Omega+i\frac{3\pi}{2})t}}{\omega-\Omega+i\frac{3\pi}{2}}.
\tag{38}
\]

It worth noting that in expressions (37), (38), a wave vector \(k\) is related to a running frequency \(\omega\): \(k = \omega/v_q\). The first lines in (37) and (38) are related to the dark states \((\Lambda_n = 0)\). They do not lead to a singularity near resonance \(\omega \approx \Omega\), since \(kd\)-dependent prefactors in this term tend to zero more rapidly than the denominator does. At the point of a resonance, these terms exactly equal to zero. A physical reason for this is that the dark states do not interact with a photon field and, therefore, cannot contribute to the photon emission. Therefore, near the resonance, a spectral density of spontaneous emission can be approximated by a Lorentzian form with a full width at half the height of the resonance line being equal to \(3\Gamma\):

\[
\gamma_k(t) \approx g_k \frac{e^{i(\omega-\Omega+i\frac{3\pi}{2})t} - 1}{(\omega-\Omega+i\frac{3\pi}{2})^2} \quad \lim_{t \to \infty}.
\tag{39}
\]

\[
S(\omega, t \to \infty) = \frac{\Gamma \Omega}{(\omega-\Omega)^2 + (\frac{3\pi}{2})^2}.
\tag{40}
\]

For \(kd = (2n+1)\pi/2\), the qubits amplitudes are given in Eqs. (32) and (33). The calculation of (34) with these amplitudes results in the following expressions for \(\gamma_k(t)\)

with the central qubit being initially excited:

\[
\gamma_k(t) = -g_k \left[ \frac{1}{2} \left( 1 - (-1)^n \right) \right] \cos kd -
- g_k \frac{1}{2} \left( 1 + (-1)^n \right) \cos kd.
\tag{41}
\]

If the edge qubit is initially excited, we obtain

\[
\gamma_k(t) = -i g_k \left[ \frac{1}{2} \left( 1 - \frac{i}{\sqrt{7}} \right) \right] \cos kd -
- i g_k \frac{1}{2} \left( 1 + \frac{i}{\sqrt{7}} \right) \cos kd.
\tag{42}
\]

In expressions (41), (42), \(kd = \frac{\pi}{2} (2n + 1)\pi/2\).

The spectral density of photon radiation (36) calculated from (41) and (42) for \(n = 1, kd = 1.5\pi, t = 20/\Gamma, \Gamma/\Omega = 10^{-3}\) is shown in Fig. 6.
Two peaks at these plots are a clear signature of vacuum Rabi oscillations.

This spectrum is similar to the transmission spectrum of the high-Q resonator in the presence of a superconducting qubit interacting with the fundamental mode of the resonator [28], where the distance between the peaks is the measure of the interaction between qubit and the resonator mode (see Fig. 4 in [28]). It is not unexpectedly, since in our case, two edge qubits act as the quantum mirrors in an open cavity [19]. The distance between two peaks in Fig. 6a is the measure of the photon-mediated interaction between a central and the edge qubits. As is seen from (41), this interaction is equal to $\sqrt{2}T/2$ which is close to the calculated value shown in Fig. 6a.

6 Effective non-Hermitian Hamiltonian and collective states

The elimination of photon variables allows us to express the photon-mediated interaction between identical qubits in terms of a non-Hermitian effective Hamiltonian, which in the Markovian approximation reads [19,29]

$$H_{\text{eff}} = -i \frac{\Gamma}{2} \sum_{m,n=1}^{3} e^{i k|x_m - x_n|} \sigma^+_m \sigma_n,$$  (43)

where $k = \Omega/v_g$, $v_g$ is the velocity of electromagnetic wave in a waveguide, $x_n$ is the position of $n$th qubit, and $\sigma^+_n, \sigma_n$ are raising and lowering spin operators for $n$th qubit.

The rate of spontaneous emission $\Gamma$ of an individual qubit is defined by the Fermi golden rule

$$\Gamma = 2\pi \sum_k |g_k|^2 \delta(\omega_k - \Omega).$$  (44)

It follows from (43) that the photon-mediated interaction between qubits in such a system results in the coherent $J_{mn} = \Gamma \sin (k |x_m - x_n|)/2$ and dissipative $\Gamma_{mn} = \Gamma \cos (k |x_m - x_n|)$ rates. The coherent rate shifts the positions of the qubits resonances, while the dissipative rate gives rise to the additional spontaneous emission into the waveguide mode. Unlike the real atoms with short-range dipole–dipole interaction; here, a coherent interaction $J_{mn}$ is a long-range one: every qubit is sensitive to its distant neighbor.

The wave-function for Hamiltonian (43) can be expressed in terms of a superposition of the single excited states

$$\Psi(t) = \sum_{n=1}^{3} \beta_n(t) |n\rangle.$$  (45)

Even though the wave-functions (3) and (45) are different, the qubits amplitudes $\beta_n(t)$ in these expressions are the same quantities. Indeed, the equations for $\beta_n(t)$ in (45) can be derived from the time-dependent Schrodinger equation $i d\Psi/dt = H_{\text{eff}} \Psi$

$$\frac{d\beta_n}{dt} = -\frac{\Gamma}{2} \sum_{m=1}^{3} \beta_m(t) e^{i k|x_m - x_n|}, \quad (n = 1, 2, 3).$$  (46)

Equation (46) are nothing but a set of Eq. (14). Therefore, within a single-excitation subspace, two Hamiltonians (1) and (43) are equivalent in that they provide the same equations for the qubits amplitudes.

In a single-excitation subspace, the Hamiltonian (43) has three collective eigenfunctions

$$|\Psi_i(t)\rangle = e^{-i \tilde{E}_i t} \sum_{n=1}^{3} \alpha_n^{(i)} |n\rangle; \quad (i = 1, 2, 3),$$  (47)

where $\tilde{E}_i$ is a complex energy

$$\tilde{E}_i = E_i - i \frac{\Gamma_i}{2}.$$  (48)

The quantities $E_i$ and $\Gamma_i$ depend on the system parameters $\Gamma, k, x_n$. The wave vectors with $\Gamma_i < \Gamma$ are called superradiant states; those with $\Gamma_i > \Gamma$ are called superradiant states. The coefficients in (47) can be obtained from the Schrodinger equation $H_{\text{eff}} \Psi = E \Psi$

$$E_j \sum_{n=1}^{3} \alpha_n^{(j)} |n\rangle = -i \frac{\Gamma}{2} \sum_{m,n=1}^{3} e^{i k d|m-n|} \alpha_m^{(j)} |n\rangle; \quad (j = 1, 2, 3).$$  (49)

Complex energies can be found by equating to zero the determinant of the matrix

$$\left( E + i \frac{\Gamma}{2} \right) \delta_{mn} + i \frac{\Gamma}{2} e^{i k d|m-n|} (1 - \delta_{mn}); \quad (m, n = 1, 2, 3).$$  (50)

A comparison between the determinants of (16) and (50) shows that the quantities $\lambda_n$ are related to those of complex energies, $\tilde{E}_n = \tilde{E}_n = i \lambda_n$ where $\lambda_n$ are given in (17).

From (49), we obtain three equations for the coefficients

$$\left( E_j + i \frac{\Gamma}{2} \right) \alpha_1^{(j)} + i \frac{\Gamma}{2} e^{i k d} \alpha_2^{(j)} + i \frac{\Gamma}{2} e^{2i k d} \alpha_3^{(j)} = 0,$$

$$\left( E_j + i \frac{\Gamma}{2} \right) \alpha_2^{(j)} + i \frac{\Gamma}{2} e^{i k d} \alpha_1^{(j)} + i \frac{\Gamma}{2} e^{2i k d} \alpha_3^{(j)} = 0,$$

$$\left( E_j + i \frac{\Gamma}{2} \right) \alpha_3^{(j)} + i \frac{\Gamma}{2} e^{i k d} \alpha_2^{(j)} + i \frac{\Gamma}{2} e^{2i k d} \alpha_1^{(j)} = 0,$$  (51)

where for simplicity, we remove the bar over $E_i$. 

```
Because Hamiltonian (43) is non-Hermitian, the eigenfunctions (47) are neither normalized nor orthogonal. It is known that a correct calculation of the coefficients \( \alpha_n^{(i)} \) in (47) requires a bi-orthogonal set of eigenfunctions \( |\Psi_i(t)\rangle \), which are a solution of the Schrödinger equation for \( H_{\text{eff}}^\dagger \). In our case \( H_{\text{eff}}^\dagger = H_{\text{eff}}^* \) with the consequence that the complex conjugate of \( H_{\text{eff}} \) is an eigenstate of \( H_{\text{eff}}^\dagger \). Therefore, the conditions for normalization and orthonormality between eigenfunctions of these two sets lead to the following equations for the coefficients [30, 31]:

\[
\sum_{n=1}^{3} (\alpha_n^{(i)})^2 = 1, \quad i = 1, 2, 3, \quad (52)
\]

\[
\sum_{n=1}^{3} \alpha_n^{(i)} \alpha_n^{(j)} = 0, \quad i \neq j, \quad i, j = 1, 2, 3. \quad (53)
\]

It is worth noting that the coefficients \( \alpha_n^{(i)} \) in (52) and (53) are in general complex quantities. In what follows, we use Eq. (51) and the conditions (52) and (53) for the calculation of the coefficients \( \alpha_n^{(i)} \).

As the energies \( E_n \) are obtained from the determinant of Eq. (51), these equations are not independent. Therefore, for the calculations of the coefficients \( \alpha_n^{(i)} \), we may take any two of them. For subsequent calculations, we take first and second equations in (51). First, we calculate the coefficients \( \alpha_n^{(1)} \). With \( E_1 = i\lambda_1 \) determined in (17), two first equations in (51) read as follows:

\[
(e^{ikd} + R) \alpha_1^{(1)} - 2\alpha_2^{(1)} - 2e^{ikd}\alpha_3^{(1)} = 0
\]

\[
(e^{ikd} + R) \alpha_2^{(1)} - 2\alpha_1^{(1)} - 2\alpha_3^{(1)} = 0, \quad (54)
\]

where \( R \) is given in (27).

Additional third equation is given by the normalizing condition (52). Therefore, for the wave-function \( |\Psi_1(t)\rangle \), we find the following set of the coefficients \( \alpha_n^{(1)} \):

\[
\alpha_1^{(1)} = \alpha_3^{(1)} = \pm \frac{e^{ikd}(e^{ikd} + R) + 2}{D(R)}, \quad (55)
\]

\[
\alpha_2^{(1)} = \pm \frac{2e^{ikd} + R}{D(R)}, \quad (56)
\]

where

\[
D(R) = (4e^{4ikd} + 34e^{2ikd} + 14e^{ikd}R + 4e^{3ikd}R + 16)^{1/2}.
\]

Similar calculations for \( E_2 = i\lambda_2 \) provide the result that differs from (55), (56) only by the sign of \( R \). Therefore, for the wave-function \( |\Psi_2(t)\rangle \), we find the following set of the coefficients \( \alpha_n^{(2)} \):

\[
\alpha_1^{(2)} = \alpha_3^{(2)} = \pm \frac{e^{ikd}(e^{ikd} - R) + 2}{D(-R)}, \quad (57)
\]

\[
\alpha_2^{(2)} = \pm \frac{3e^{ikd} - R}{D(-R)}. \quad (58)
\]

The signs in right-hand side of Eqs. (55), (56), and (57), (58) must be the same (plus or minus) for all four expressions. This requirement follows from the orthonormality condition (53):

\[
\alpha_1^{(1)}\alpha_1^{(2)} + \alpha_2^{(1)}\alpha_2^{(2)} + \alpha_3^{(1)}\alpha_3^{(2)} = 0. \quad (59)
\]

Finally, for \( E_3 = i\lambda_3 \), we obtain two equations

\[
\alpha_1^{(3)} + e^{-ikd}\alpha_2^{(3)} + \alpha_3^{(3)} = 0,
\]

\[
\alpha_1^{(3)} + e^{ikd}\alpha_2^{(3)} + \alpha_3^{(3)} = 0, \quad (60)
\]

which provide with account for the normalizing condition (52) the following result:

\[
\alpha_1^{(3)} = \pm \frac{1}{\sqrt{2}}; \quad \alpha_2^{(3)} = \mp \frac{1}{\sqrt{2}}; \quad \alpha_3^{(3)} = 0. \quad (61)
\]

Therefore, the coefficients of collective states \( |\Psi_1(t)\rangle \) and \( |\Psi_2(t)\rangle \) depend on \( kd \), while those of \( |\Psi_3(t)\rangle \) is \( kd \)-independent.

Below, we consider special cases. For \( kd = 2\pi n \), where \( n \) is integer, we obtain from (55), (56) \( \alpha_1^{(1)} = \alpha_2^{(1)} = \alpha_3^{(1)} = \pm \frac{1}{\sqrt{3}} \). However, the calculation of the coefficients \( \alpha_n^{(2)} \) from (57), (58) is not straightforward: at this point, both the numerator and the denominator in these equations are equal to zero. To resolve this uncertainty, we put in these equations \( kd = 2\pi n + \epsilon \), where \( \epsilon \) tends to zero. In this case, both the numerator and the denominator tend to zero as \( \epsilon \), and their ratio is finite. The calculations show that as \( kd \) tends to \( 2\pi n \), the coefficients \( \alpha_n^{(2)} \) tend to their finite values:

\[
\alpha_1^{(2)} = \alpha_3^{(2)} = \frac{1}{\sqrt{6}}; \quad \alpha_2^{(2)} = -\frac{2}{\sqrt{6}}, \quad (62)
\]

For \( kd = (2n + 1)\pi \), the picture is vice versa: \( \alpha_1^{(1)} = \alpha_3^{(1)} = \frac{1}{\sqrt{6}}; \quad \alpha_2^{(1)} = \frac{2}{\sqrt{6}}, \) while \( \alpha_1^{(2)} = \alpha_2^{(2)} = \alpha_3^{(2)} = \pm \frac{1}{\sqrt{3}} \).

For \( kd = (2n + 1)\pi/2 \), we obtain from (55), (56), and (57), (58)

\[
\alpha_1^{(1)} = \alpha_3^{(1)} = \frac{1}{\sqrt{2}} (7 + i(-1)^n5\sqrt{7})^{1/2}, \quad (63)
\]

\[
\alpha_2^{(2)} = \frac{(-1)^n3 + i\sqrt{7}}{\sqrt{2} (7 + i(-1)^n5\sqrt{7})^{1/2}},
\]

\[
\alpha_3^{(2)} = \frac{(-1)^n3 + i\sqrt{7}}{\sqrt{2} (7 + i(-1)^n5\sqrt{7})^{1/2}}.
\]
Below, we summarize these results in the explicit forms of collective wave-functions for above special cases. If \( kd \) is equal to integer multiple of \( \pi \), we obtain

\[
|\Psi_1(t)\rangle = e^{-\frac{i}{2} t} \frac{1}{\sqrt{3}} (|1\rangle + |2\rangle + |3\rangle)
\]

\[
|\Psi_2(t)\rangle = \frac{1}{\sqrt{6}} (|1\rangle - 2|2\rangle + |3\rangle)
\]

\[
|\Psi_3(t)\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |3\rangle).
\]

Therefore, two of these collective states \(|\Psi_2(t)\rangle\) and \(|\Psi_3(t)\rangle\) are dark as their decay widths are zero.

For \( kd = (2n + 1)\pi/2 \), the collective wave-functions are as follows:

\[
|\Psi_1(t)\rangle = e^{-\frac{i}{2} t} e^{-i \frac{\pi}{2} t} (-i)^n \frac{1}{\sqrt{2}} (7 - i(-1)^n 5\sqrt{7})^{1/2}
\]

\[
\times \left((-i + (-1)^n \sqrt{7}) (|1\rangle + |3\rangle) + ((-1)^n 3 - i \sqrt{7}) |2\rangle\right)
\]

\[
|\Psi_2(t)\rangle = -\frac{e^{-\frac{i}{2} t} e^{-i \frac{\pi}{2} t} (-i)^n \frac{1}{\sqrt{2}}}{\sqrt{2} (7 + i(-1)^n 5\sqrt{7})^{1/2}}
\]

\[
\times \left((i + (-1)^n \sqrt{7}) (|1\rangle + |3\rangle) - ((-1)^n 3 + i \sqrt{7}) |2\rangle\right)
\]

\[
|\Psi_3(t)\rangle = e^{-\frac{i}{2} t} \frac{1}{\sqrt{2}} (|1\rangle - |3\rangle),
\]

where \( n \) is any integer.

It is instructive to rewrite (65) in terms of the dark, \(|D\rangle\) and bright, \(|B\rangle\) states for a two qubits (the first qubit and the third one) separated by \( \lambda/2 \). As was shown in [20], for this configuration, the dark state is decoupled from the waveguide and it can be revealed only via its interaction with a probe (central) qubit. This property follows directly from (65). For \( n = 0 \), we obtain from (65)

\[
|\Psi_1(t)\rangle = e^{-\frac{i}{2} t} e^{-i \frac{\pi}{2} t} \sqrt{2} (7 - i5\sqrt{7})^{1/2}
\]

\[
\times \left((-i + \sqrt{7}) \sqrt{2} |D\rangle \otimes |g_2\rangle + (3 - i \sqrt{7}) |G\rangle \otimes |e_2\rangle\right)
\]

\[
|\Psi_2(t)\rangle = -\frac{e^{-\frac{i}{2} t} e^{-i \frac{\pi}{2} t} \sqrt{2}}{\sqrt{2} (7 + i5\sqrt{7})^{1/2}}
\]

\[
\times \left((i + \sqrt{7}) \sqrt{2} |D\rangle \otimes |g_2\rangle - (3 + i \sqrt{7}) |G\rangle \otimes |e_2\rangle\right)
\]

\[
|\Psi_3(t)\rangle = e^{-\frac{i}{2} t} |B\rangle \otimes |g_2\rangle,
\]

where \(|G\rangle = |g_1 g_3\rangle\) and

\[
|D\rangle = \frac{|e_1 g_3\rangle + |g_1 e_3\rangle}{\sqrt{2}}; \quad |B\rangle = \frac{|e_1 g_3\rangle - |g_1 e_3\rangle}{\sqrt{2}}.
\]

As is seen from the third equation in (66), the bright state of the two-qubit system decays independently on the presence of the second (central) qubit. However, the decay of the dark state can be revealed only through its entanglement with the second qubit [20]. In fact, the interaction between second qubit and the dark state formed by two edge qubits gives rise to the vacuum Rabi oscillations which are shown in (32), (33), and in Figs. 4 and 5. A qualitative correspondence between vacuum Rabi oscillations in Fig. 4 and in Fig. 3 in [20] can be easily seen.

6.1 Relation between qubits amplitudes and collective states

Here, we show how the qubits amplitudes \( \beta_n(t) \) are related to collective wave-functions \(|\Psi_i(t)\rangle\) (47). We write the dynamic wave-function \(|\Psi(t)\rangle\) (45) as a decomposition over the collective states (47)

\[
|\Psi(t)\rangle = \sum_{i=1}^{3} A_i |\Psi_i(t)\rangle = \sum_{i,n=1}^{3} A_i e^{-i E_n t} \alpha_n^{(i)} |n\rangle.
\]

From (45), we obtain

\[
\beta_n(t) = \sum_{i=1}^{3} e^{-i E_n t} A_i \alpha_n^{(i)}
\]

with the initial conditions

\[
\beta_{n_0}(0) = \sum_{i=1}^{3} A_i \alpha_{n_0}^{(i)} = 1
\]

\[
\beta_n(0) = \sum_{i=1}^{3} A_i \alpha_n^{(i)} = 0; \quad n \neq n_0,
\]

where \( n_0 \) is the sequence number of excited qubit.

The probability amplitude \( |\beta_n(t)|^2 \) can directly be expressed in terms of collective state wave-functions \(|\Psi_i(t)\rangle\)

\[
|\beta_n(t)|^2 = \langle n | \sum_{i,j=1}^{3} A_i A_j^\ast \langle \Psi_i | \langle \Psi_j | \rangle |n\rangle.
\]

From linear algebraic equation (69), we can find coefficients \( A_i \) and, therefore, restore the qubits’ amplitudes \( \beta_n(t) \). However, if the number of qubits is large, this procedure is not convenient for computer simulations. The main reason is that it requires: first, the calculation of \( N \) complex energies from determinant of \( N \times N \) matrix analogous to (50); second, the calculations of \( \alpha_n^{(i)} \) from non-linear conditions (52) and (53); and third, the solution of a system of \( N \) linear algebraic equations analogous to (69). Every of these three steps is not simple from a mathematical point of view. It is
more convenient to directly compute the qubits amplitudes \( \beta_i(t) \) from a set of the linear differential equation (13), which allow us to completely avoid all three steps we mentioned above.

Nevertheless, we should like to mention some interesting consequences that follow from Eqs. (68), (69), and (70). First, from (68), we see that the dark states \( \langle ImE_i = 0 \rangle \) contribute to \( \beta_i(t) \), even though they do not contribute to the spectrum of the photon emission. Second, it follows from (70) that if specific collective state \( \langle |\Psi_i(t)\rangle \rangle \) does not contain qubit state \( |n\rangle \), then this collective state does not take part in the formation of the dynamics of the qubits amplitudes \( \beta_i(t) \). As was shown above [see (61)], the state \( |\Psi_3(t)\rangle \) does not contain the qubit state \( |2\rangle \). Therefore, independently on the value of \( kd \), only two collective states \( |\Psi_1(t)\rangle \) and \( |\Psi_2(t)\rangle \) take part in the formation of the dynamics of the qubits amplitudes \( \beta_i(t) \).

7 Three non-identical qubits

Here, we consider a system in which all three qubits are identical except for the frequency of a second qubit which has a different value \( \Omega_0 \). For this case, we rewrite Eqs. (9), (10), and (11)

\[
\frac{d\beta_1}{dt} = -\frac{\Gamma}{2} \beta_1(t) - i\frac{\delta\Omega}{2} \beta_1(t) - \beta_2(t) \frac{1}{2} \left( \frac{\Omega_0}{\Omega} \right)^{1/2} e^{ik_0d} - \frac{\Gamma}{2} \beta_3(t)e^{2ikd}
\]

\[
\frac{d\beta_2}{dt} = -\frac{\Gamma}{2} \beta_2(t) + i\frac{\delta\Omega}{2} \beta_2(t) - \frac{1}{2} \left( \frac{\Omega_0}{\Omega} \right)^{1/2} e^{ik_0d} \left( \beta_1(t) + \beta_3(t) \right)
\]

\[
\frac{d\beta_3}{dt} = -\frac{\Gamma}{2} \beta_3(t) - i\frac{\delta\Omega}{2} \beta_3(t) - \beta_1(t) \frac{\Gamma}{2} e^{2ikd} - \beta_2(t) \frac{1}{2} \left( \frac{\Omega_0}{\Omega} \right)^{1/2} e^{ik_0d}
\]

where \( \delta\Omega = \Omega - \Omega_0 \), \( k = \frac{\Omega}{\omega} \), \( k_0 = \frac{\Omega_0}{\omega} \).

Characteristic roots can be found from the determinant of Eqs. (71), (72), and (73)

\[\lambda_1 = -\frac{\Gamma}{2} \left( 1 + \frac{1}{2} e^{2ikd} \right) - \frac{\Gamma}{4} e^{ikd} \left( e^{2ikd} + 8e^{i(k_0-k)d} + 4i\frac{\delta\Omega}{\Gamma} - 4 \left( \frac{\delta\Omega}{\Gamma} \right)^2 \right) e^{-2ikd}
\]

\[\lambda_2 = -\frac{\Gamma}{2} \left( 1 + \frac{1}{2} e^{2ikd} \right) + \frac{\Gamma}{4} e^{ikd} \left( e^{2ikd} + 8e^{i(k_0-k)d} + 4i\frac{\delta\Omega}{\Gamma} - 4 \left( \frac{\delta\Omega}{\Gamma} \right)^2 \right) e^{-2ikd}
\]

\[
\lambda_3 = \frac{\Gamma}{2} e^{i2kd} - \frac{\Gamma}{2} - i\frac{\delta\Omega}{2}
\]

The calculations show that Eqs. (74) and (75) provide no dark states \( \langle \lambda_{1,2} = 0 \rangle \) if \( \delta\Omega \) is not equal to zero. The quantities \( \text{Re}(\lambda_{1,2}) \) are always negative at any \( kd \). The exception is the third root \( \lambda_3 \), with real part being equal to zero for \( kd = n\pi \). Therefore, if the root \( \lambda_3 \) does not contribute to the dynamics of qubits, then at any \( kd \), the qubits' amplitudes will decay to zero with the rate being dependent on \( \delta\Omega \).

The dependence of real parts of \( \lambda_n \) on \( kd \) is shown in Fig. 7 for \( \delta\Omega = \Gamma \).

If the second (central) qubit is excited, the amplitudes of the first and the third qubit are the same \( \beta_1(t) = \beta_3(t) \equiv \beta(t) \). Then, three equations (71), (72), and (73) reduce to two equations, for \( \beta_2 \) and \( \beta \)

\[
\frac{d\beta}{dt} = -\frac{\Gamma}{2} \left( 1 + e^{2ikd} + i\frac{\delta\Omega}{\Gamma} \right) \beta(t) - \beta_2(t) \frac{\Gamma}{2} \left( \frac{\Omega_0}{\Omega} \right)^{1/2} e^{i2kd},
\]

\[
\frac{d\beta}{dt} = -\frac{\Gamma}{2} \left( 1 - i\frac{\delta\Omega}{\Gamma} \right) \beta(t) - \beta_2(t) \frac{\Gamma}{2} \left( \frac{\Omega_0}{\Omega} \right)^{1/2} e^{i2kd}.
\]

The characteristic roots of these equations are equal to \( \lambda_1 \) and \( \lambda_2 \) given above in Eqs. (74) and (75).

The solution of Eqs. (77) and (78), with account for initial conditions for the amplitudes, \( \beta_2(0) = 1 \), \( \beta(0) = 0 \) and their time derivatives

\[
\left. \frac{d\beta}{dt} \right|_{t=0} = -\frac{\Gamma}{2} \left( \frac{\Omega_0}{\Omega} \right)^{1/2} e^{ik_0d},
\]

\[
\left. \frac{d\beta_2}{dt} \right|_{t=0} = -\frac{\Gamma}{2} \left( 1 - i\frac{\delta\Omega}{\Gamma} \right),
\]
is similar to (22)

\[
\begin{align*}
\beta_2(t) &= b_1 e^{\lambda_1 t} + b_2 e^{\lambda_2 t} \\
\beta(t) &= a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}
\end{align*}
\]

with

\[
\begin{align*}
b_1 &= -\frac{\Gamma}{2} \left( 1 - i\delta\Omega \right) + \lambda_2 \\
b_2 &= \frac{\Gamma}{2} \left( 1 - i\delta\Omega \right) + \lambda_1 \\
a_1 &= -\frac{\Gamma}{2} \left( \frac{\Omega_0}{\Omega} \right)^{1/2} e^{ikad} \\
a_2 &= \frac{\Gamma}{2} \left( \frac{\Omega_0}{\Omega} \right)^{1/2} e^{ikad}
\end{align*}
\] (83)

Using the detuning \(\delta\Omega\) as the external parameter, we can control the decay rates of the qubits amplitudes. This is shown in Fig. 8 for several values of the detuning \(\delta\Omega\) with the second qubit being initially excited. In the case shown in Fig. 8, the decay rates are governed by real parts of two roots, \(\lambda_1\) and \(\lambda_2\) with \(|Re\lambda_1| < |Re\lambda_2|\). As \(\delta\Omega/\Gamma\) increases from 0 to 1, \(|Re\lambda_1|\) decreases from 1.5 to 1.346, while \(|Re\lambda_2|\) increases from 0 to 0.154. Even though the first root tends to slow down the decay rate as \(\delta\Omega/\Gamma\) increases, its influence for \(t \gg \Gamma\) becomes negligible, so that the main contribution to the decay rate for large times comes from the second root which speeds up the decay rate as \(\delta\Omega/\Gamma\) increases.

Therefore, the greater is \(\delta\Omega\), the more is its influence on the qubit decay rates.

If the first qubit is initially excited, the solution is similar to (28)

\[
\begin{align*}
\beta_1(t) &= \frac{b_2}{2} e^{\lambda_1 t} + \frac{b_1}{2} e^{\lambda_3 t} + \frac{1}{2} e^{\lambda_2 t} \\
\beta_2(t) &= a_1 \left( e^{\lambda_1 t} - e^{\lambda_2 t} \right) \\
\beta_3(t) &= \frac{b_2}{2} e^{\lambda_1 t} + \frac{b_1}{2} e^{\lambda_3 t} - \frac{1}{2} e^{\lambda_2 t},
\end{align*}
\] (84)

where \(\lambda_1, \lambda_2, \lambda_3,\) and \(a_1, b_1, b_2\) are given in (74), (75), (76), (82), (83).

For this case, we show in Fig. 9 the temporal behavior of qubits’ amplitudes for several values of the detuning \(\delta\Omega\). In the absence of detuning (the panel \(a\)), we obtain the result shown in Fig. 3. However, for non-zero detuning, the amplitudes of the first and the third qubits decay to 0.25, while the second qubit decays to zero. This difference can be explained by the influence of dark states [the last terms in the expressions for \(\beta_1(t)\) and \(\beta_3(t)\) in (84)]. In general, as the detuning increases, the decay rates increase as well.

The subsystem consisting of a central qubit and the symmetric qubit array can be described as an analogue to a cavity QED system. In this description, the central qubit acts as a two-level atom and the symmetric qubit array mimics a high-finesse cavity, with the qubits array acting as an atomic mirror [19]. It seems that this analogy is supported by Fig. 3 where two dark states (64) of the whole system prevent the qubit amplitudes from the damping. If this analogy worked, we would expect that the detuning of a central qubit would not lead to any damping at all. In real cavity, the qubit which is large detuned from the cavity resonance cannot exchange its energy with the cavity via real photons. However, as is seen from Fig. 8, a full analogy between resonance cavity and the qubit array does not exist. Even a small detuning of the central qubit from the frequency of the qubit array results in the disruption of the dark states that, in turn, leads to the damping of the qubits amplitudes.

### 7.1 Spectral density of spontaneous photon emission

For the calculation of the spectral density of photon emission, we use Eq. (8) together with the substitution of \(\beta_n(t)\) from Eq. (12).

\[
\begin{align*}
\gamma_k(t) &= -igk e^{ikd} \int_0^t \bar{\beta}_1(t') e^{i(\Omega-\Omega_0)t'/2} e^{i(\omega-\Omega)t'} dt' \\
&- igk \int_0^t \bar{\beta}_2(t') e^{-i(\Omega-\Omega_0)t'/2} e^{i(\omega-\Omega)t'} dt' \\
&- igk e^{-ikd} \int_0^t \bar{\beta}_3(t') e^{i(\Omega-\Omega_0)t'/2} e^{i(\omega-\Omega)t'} dt',
\end{align*}
\] (85)
If peak (Fig. 10a) with the width equal to $3\Gamma$ (see also approximately equal to ($\omega$ narrow peaks. The distance between the peaks $\Delta$ detuning are shown in Fig. 10 for $qubit was initially excited. $kd = 3\pi/2$, respectively. In both cases, the second qubit was initially excited.

If $kd = 0$ and $kd = 2\pi$, we see a single Lorentzian peak (Fig. 10a) with the width equal to $3\Gamma$ (see also (40). If $\delta \Omega \neq 0$, a Lorentzian peak splits into two narrow peaks. The distance between the peaks $\Delta_{\omega}$ is approximately equal to $|Im\lambda_1| + |Im\lambda_2|$. The width of the peaks is determined by the root with a lowest real part. In the case shown in Fig. 10b–f, the width of the peaks and their height are approximately equal to $|Re\lambda_2|$ and $\Omega/|Re\lambda_2|$, respectively. To estimate the order of these quantities, we have found numerically the range of variation for $Re\lambda_2$, $Im\lambda_1$, and $Im\lambda_2$. As $\delta \Omega/\Gamma$ increases from 0.2 to 1, the quantities $Re\lambda_2$, $Im\lambda_1$, and $Im\lambda_2$ vary from $-0.006\Gamma$ to $-0.154\Gamma$, from $+0.028\Gamma$ to $+0.202\Gamma$, and from $-0.032\Gamma$ to $-0.205\Gamma$, respectively.

If $\delta \Omega = 0$ and $kd = 3\pi/2$, there are two similar peaks shown in Fig. 11a (see also Fig. 6a). The distance between the peaks and their widths are determined by the roots $\lambda_1$ and $\lambda_2$: $Im\lambda_1=-Im\lambda_2=(\sqrt{7}/4)\Gamma$, $Re\lambda_1=Re\lambda_2=-0.25\Gamma$ (see Eq. (17)). As $\delta \Omega/\Gamma$ increases from 0.2 to 1, the quantities $Re\lambda_1$ and $Re\lambda_2$ are of the same order of magnitude: $Re\lambda_1$ varies from $-0.287\Gamma$ to $-0.399\Gamma$, and $Re\lambda_2$ varies from $-0.213\Gamma$ to $-0.100\Gamma$. This behavior is confirmed in Fig. 11b–f where the width of the left peak increases, while the width of the right peak decreases. The distance between the peaks are determined by the quantities $Im\lambda_1$ and $Im\lambda_2$ which vary from $-0.708\Gamma$ to $-0.890\Gamma$, and from $+0.638\Gamma$ to $+0.821\Gamma$, respectively. Similar to the spectra shown in Fig. 6, two-peak spectra in Fig. 11 are due to the photon-mediated interaction between central qubit and the two edge qubits which act as the atomic mirrors [19]. As the detuning is increased, the spectrum becomes more asymmetric: the feature that is not observed in a qubit-resonator cavity.

We also calculated the temporal behavior of the full probability of the photon emission, $P_{ph}(t)$ as a function of the detuning. This dependence is shown in Fig. 12. It
is evident from the definition (6) that $P_{ph}(0) = 0$, and as the time increases, it approaches either 1 if all qubits amplitudes damp out to zero, or a constant value if not all qubits decay to zero. Because the detuning influences the rate of the qubits damping, the output rate of the photon emission, $P_{ph}/dt$, also depends on the detuning. The greater is the detuning, the greater is the rate of the photon emission.

### 8 Conclusion

In this paper, we have thoroughly investigated the dynamics behavior of qubits amplitudes in 1D chain consisting of three qubits embedded in an open waveguide. Within a single-excitation subspace, we have found the evolution of qubits amplitudes if one of the qubits was initially excited. We have shown that even though the dark states do not contribute to the output photon emission, they influence the evolution of qubits amplitudes in that they prevent the qubits amplitudes from decaying to zero. We have found the collective eigenstates of a three-qubit system and have shown how the qubits’ amplitudes can be expressed in terms of the amplitudes of the collective states. We also calculated the spectral density of the output photon emission from three-qubit system and studied its dependence on the $kd$ value. We studied the case when the frequency of the second qubit was different from that of the edge qubits. In this case, the rates of qubits decay crucially depend on the frequency detuning between central and edge qubits. The greater is the detuning, the greater is the rate of the qubits damping. This opportunity to control the qubits damping rates seems to be very important for the optimization of the measurement procedures and probably can be revealed in multi-qubit systems.

We hope that this research will prove useful for the development of the efficient control and readout protocols for a few-qubit gates and quantum processors.

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### Author contributions

YSG wrote the manuscript and contributed to its theoretical interpretation. AAS and AGM performed computer simulations. All authors discussed the results and commented on the manuscript. The authors declare that they have no competing interests.
Data Availability Statement The manuscript has associated data in a data repository.

Appendix A: Equations for qubits amplitudes in the Wigner–Weisskopf approximation

We assume that the first and the third qubits are identical ($\Omega_1 = \Omega_3 \equiv \Omega$, $g_k^{(1)} = g_k^{(3)} \equiv g_k$). The frequency and the coupling of the second qubit are different ($\Omega_2 \equiv \Omega_0$, $g_k^{(2)} \equiv g_k^{(0)}$). A distance between central qubit and the edge qubits is equal to $d$. We take the origin in the location of the second qubit: $x_1 = -d, x_2 = 0, x_3 = +d$. For this case, we expand the Eq. (7) as a set of three equations

$$\frac{d\beta_1}{dt} = -\sum_k g_k^2 \int_0^t \beta_1(t')e^{i(\omega_k - \Omega)(t-t')}dt' - \sum_k g_k g_k^{(0)} e^{-ikd} e^{i(\Omega_0 - \Omega)t} \int_0^t \beta_2(t')e^{-i(\omega_k - \Omega_0)(t-t')}dt' - \sum_k g_k^2 e^{-2kd} \int_0^t \beta_3(t')e^{-i(\omega_k - \Omega)(t-t')}dt'$$

$$\frac{d\beta_2}{dt} = -\sum_k (g_k^{(0)})^2 \int_0^t \beta_2(t')e^{-i(\omega_k - \Omega_0)(t-t')}dt' - \sum_k g_k g_k^{(0)} e^{ikd} e^{i(\Omega_0 - \Omega)t} \int_0^t \beta_1(t')e^{-i(\omega_k - \Omega)(t-t')}dt' - \sum_k g_k^2 e^{2kd} \int_0^t \beta_3(t')e^{-i(\omega_k - \Omega)(t-t')}dt'$$

$$\frac{d\beta_3}{dt} = -\sum_k g_k^2 \int_0^t \beta_3(t')e^{-i(\omega_k - \Omega)(t-t')}dt' - \sum_k 2g_k^2 e^{2kd} \int_0^t \beta_1(t')e^{-i(\omega_k - \Omega)(t-t')}dt' - \sum_k g_k g_k^{(0)} e^{-ikd} e^{i(\Omega_0 - \Omega)t} \int_0^t \beta_2(t')e^{-i(\omega_k - \Omega_0)(t-t')}dt'$$

According to Wigner–Weisskopf approach, we replace $\beta_1(t'), \beta_2(t'), \beta_3(t')$ in the integrands with $\beta_1(t), \beta_2(t), \beta_3(t)$ and take them out of the integrals

$$\frac{d\beta_1}{dt} = -\beta_1(t) \sum_k g_k^2 I_k(\Omega, t)$$

$$\frac{d\beta_2}{dt} = -\beta_2(t) \sum_k g_k g_k^{(0)} e^{-ikd} e^{i(\Omega - \Omega_0)t} I_k(\Omega_0, t)$$

$$\frac{d\beta_3}{dt} = -\beta_3(t) \sum_k g_k^2 e^{-2kd} I_k(\Omega, t),$$

where

$$I_k(\Omega, t) = \int_0^t e^{-i(\omega_k - \Omega)(t-t')}dt' = \int_0^t e^{-i(\omega_k - \Omega)\tau}d\tau$$

$$\approx \int_0^\infty e^{-i(\omega_k - \Omega)\tau}d\tau = \pi\delta(\omega_k - \Omega) - iP.v. \left(\frac{1}{\omega_k - \Omega}\right)$$

where $P.v.$ is a Cauchy principal value integral.

We can remove oscillating exponents in (A4)–(A6) with the aid of the substitution

$$\beta_1(t) = e^{i(\Omega_0 - \Omega)t/2} \beta_1(t)$$

$$\beta_2(t) = e^{-i(\Omega_0 - \Omega)t/2} \beta_2(t)$$

$$\beta_3(t) = e^{i(\Omega_0 - \Omega)t/2} \beta_3(t).$$

In addition, we assume that the coupling constants $g_k$ are even functions of $k$ ($g_k = g_{-k}$). Then, from (A4)–(A6), we obtain

$$\frac{d\beta_1}{dt} = -\beta_1(t) \sum_k g_k^2 I_k(\Omega, t) - i\frac{\Omega - \Omega_0}{2} \beta_1(t),$$

$$\frac{d\beta_2}{dt} = -\beta_2(t) \sum_k g_k g_k^{(0)} \cos(\omega_k) I_k(\Omega_0, t)$$

$$\frac{d\beta_3}{dt} = -\beta_3(t) \sum_k g_k^2 \cos(2\omega_k) I_k(\Omega_0, t),$$

$$\frac{d\beta_1}{dt} = -\beta_1(t) \sum_k g_k^2 I_k(\Omega, t)$$

$$\frac{d\beta_2}{dt} = -\beta_2(t) \sum_k (g_k^{(0)})^2 I_k(\Omega_0, t) + i\frac{\Omega - \Omega_0}{2} \beta_2(t),$$

$$\frac{d\beta_3}{dt} = -\beta_3(t) \sum_k g_k g_k^{(0)} \cos(\omega_k) I_k(\Omega, t)$$

$$\frac{d\beta_3}{dt} = -\beta_3(t) \sum_k g_k^2 \cos(2\omega_k) I_k(\Omega, t).$$
$$\frac{d\tilde{\beta}_0}{dt} = -\tilde{\beta}_0(t) \sum_k g_k^2 I_k(\Omega, t) - \frac{i}{2} \Omega \tilde{\beta}_0(t)$$

$$-\tilde{\beta}_1(t) 2 \sum_{k>0} g_k^2 \cos(2kd) I_k(\Omega, t)$$

$$-\tilde{\beta}_2(t) 2 \sum_{k>0} g_k g_k^{(0)} \cos(kd) I_k(\Omega_0, t). \quad (A11)$$

The next step is to relate the coupling constants $g_k$ to the qubit decay rate of spontaneous emission into waveguide mode. In accordance with Fermi golden rule, we define the qubit decay rates by the following expressions:

$$\Gamma = 2\pi \sum_k g_k^2 \delta(\omega_k - \Omega) \quad (A12)$$

$$\Gamma_0 = 2\pi \sum_k \left(g_k^{(0)}\right)^2 \delta(\omega_k - \Omega_0), \quad (A13)$$

where

$$g_k = \sqrt{\frac{\omega_k D^2}{2\hbar\varepsilon_0 V}}.$$  
(A14)

$D$ is the matrix element of the qubit’s dipole moment operator, and $V$ is the effective volume where the interaction between qubit and electromagnetic field takes place. For 1D case, a summation over $k$ is replaced by the integration over $\omega$ in accordance with the prescription

$$\sum_k \Rightarrow \frac{L}{2\pi} \int_{-\infty}^{\infty} dk = \frac{L}{2\pi} \int_0^\infty dk \text{Re} \Omega_0 / \pi v_g \int_0^\infty \omega \omega_k = \int_0^\infty \omega \omega_k \left( g_k(0) \right)^2 \delta(\omega_k - \omega), \quad (A15)$$

where $L$ is a length of the waveguide, and we assumed a linear dispersion law, $\omega_k = v_g k$. The application of (A15) to, for example, (A12), allows the relation between a coupling constant $g_k$ and the decay rate $\Gamma$

$$g_k = \sqrt{\frac{\Omega D^2}{2\hbar\varepsilon_0 V}} \quad (A16)$$

Therefore, we may relate the coupling constants $g_k^{(0)}$, $g_{\Omega_0}$, with their respective decay rates

$$g_k^{(0)} = \sqrt{\frac{\Omega D^2}{2\hbar\varepsilon_0 V}} = \left( \frac{\Omega}{\Omega_0} \right)^{1/2} \left( \frac{v_g \Gamma_0}{2L} \right)^{1/2}, \quad (A17)$$

$$g_{\Omega_0} = \sqrt{\frac{\Omega_0 D^2}{2\hbar\varepsilon_0 V}} = \left( \frac{\Omega_0}{\Omega} \right)^{1/2} \left( \frac{v_g \Gamma}{2L} \right)^{1/2}, \quad (A18)$$

$$g_{\Omega_0} = \sqrt{\frac{\Omega_0 D^2}{2\hbar\varepsilon_0 V}} = \left( \frac{v_g \Gamma}{2L} \right)^{1/2}. \quad (A19)$$

Now, we can calculate the different terms in (A9)–(A11). We begin with the sum in the first line in (A9)

$$\sum_k g_k^2 I_k(\Omega, t) = \sum_k \frac{g_k^2}{\omega_k - \Omega} = \frac{\Gamma}{2} - iP.v. \sum_k \left( \frac{g_k^2}{\omega_k - \Omega} \right) \approx \frac{\Gamma}{2},$$

(A20)

where $\Gamma$ is given in (A12).

The second term in (A20) gives rise to the shift of the qubit frequency. Therefore, we incorporate it in the renormalized qubit frequency and will not write it explicitly any more. The sum in the second line in (A9) is calculated as follows:

$$2 \sum_{k>0} g_k g_k^{(0)} \cos(kd) I_k(\Omega_0, t)$$

$$= \frac{L}{v_g} \int_0^\infty g_k(0) g_k \cos(\omega_k - \Omega_0) d\omega_k$$

$$- 2 i P.v. \sum_{k>0} \left( \frac{g_k g_k^{(0)} \cos(kd)}{\omega_k - \Omega_0} \right)$$

$$= \frac{L}{v_g} \sum_{k>0} g_k^{(0)} \cos(kd) - i \frac{L}{v_g \pi} g_{\Omega_0} g_{\Omega_0}^{(0)} \int_0^\infty \frac{d\omega}{\omega - \Omega_0} \cos \left( \frac{\Omega - \Omega_0}{v_g} \right).$$

(A21)

For principal value integral in (A21), we obtain with a good accuracy (see Appendix B)

$$P.v. \int_0^\infty \frac{d\omega}{\omega - \Omega_0} \approx -\pi \sin \left( \frac{\Omega_0}{v_g} \right).$$

As is shown in Appendix B, the result (A22) is obtained by a continuation of the frequency to negative axis, the trick which lies at the mathematical background of the Wigner–Weisskopf approximation.

Therefore, we finally obtain

$$2 \sum_{k>0} g_k g_k^{(0)} \cos(kd) I_k(\Omega_0, t) = \frac{L}{v_g} g_{\Omega_0} g_{\Omega_0} e^{i k d}$$

$$= \frac{1}{2} \left( \frac{\Omega_0}{\Omega} \right)^{1/2} \sqrt{\Gamma_0} e^{i k d}. \quad (A23)$$

Similar calculations give for the last sum in (A9)

$$2 \sum_{k>0} g_k^2 \cos(2kd) I_k(\Omega, t) = \frac{\Gamma}{2} e^{2ikd}. \quad (A24)$$
Collecting together (A20), (A23), and (A24), we write the final form of Eq. (A9)

\[
\frac{d\tilde{\beta}_1}{dt} = -\frac{\Gamma_0}{2} \tilde{\beta}_1(t) - i \frac{\Omega - \Omega_0}{2} \tilde{\beta}_1(t) - \frac{1}{2} \left( \Omega_0 \right)^{1/2} \sqrt{\Gamma T_0 e^{ikd}} - \frac{\Gamma}{2} \tilde{\beta}_2(t) e^{2ikd}.
\]

(A25)

Similar calculations for Eqs. (A10) and (A11) yield the following result:

\[
\frac{d\tilde{\beta}_2}{dt} = -\frac{\Gamma_0}{2} \tilde{\beta}_2(t) + i \frac{\Omega - \Omega_0}{2} \tilde{\beta}_2(t) - \frac{1}{2} \left( \Omega_0 \right)^{1/2} \sqrt{\Gamma T_0 e^{ikd}} (\tilde{\beta}_1(t) + \tilde{\beta}_3(t))
\]

(A26)

\[
\frac{d\tilde{\beta}_3}{dt} = -\frac{\Gamma_0}{2} \tilde{\beta}_3(t) - i \frac{\Omega - \Omega_0}{2} \tilde{\beta}_3(t) - \frac{1}{2} \left( \Omega_0 \right)^{1/2} \sqrt{\Gamma T_0 e^{ikd}}.
\]

(A27)

**Appendix B: Proof of Eq. (A22)**

The integral

\[
\int_0^\infty d\omega \frac{\cos \left( \frac{\omega}{a} \right)}{\omega - \Omega} = \int_0^\infty dx \frac{\cos (a x)}{x - 1},
\]

where \( a = k_0 d \), can be expressed in terms of sine and cosine integrals, \( ci \) and \( si \) [32]

\[
\int_0^\infty dx \frac{\cos (a x)}{x - 1} = -\cos(a)Ci(a) - \sin(a)[si(a) + \pi] = -\cos(a)Ci(a) - \sin(a) \left[ Si(a) + \frac{\pi}{2} \right].
\]

(B1)

where

\[
Ci(a) = -\int_a^\infty \frac{\cos t}{t} dt; \quad Si(a) = \int_0^a \frac{\sin t}{t} dt.
\]

(B2)

The integrand in left-hand side in (B1) has a singular point at \( x = 1 \) which manifests itself as a singularity of \( Ci(a) \) at \( a \to 0 \) in right-hand side in (B1). Therefore, we calculate integral (B1) as Cauchy principal value integral

\[
P.v. \int_0^\infty \frac{\cos a x}{x - 1} dx \approx P.v. \int_0^\infty \frac{\cos a x}{x - 1} dx = P.v. \int_\infty^{\infty} \frac{\cos a (t + 1)}{t} dt
\]

Fig. 13 Comparison of principle value \( F_1 \) (right-hand side of (B8)) with the “exact” expression \( F_2 \) (right-hand side of (B1))
Therefore, we finally obtain
\[ P.v. \int_{0}^{\infty} \frac{\cos ax}{x-1} dx \approx -\pi \sin a. \] (B8)

Below, in Fig. 13, we compare the \( kd \)-dependence of (B1) with that of (B8). We see a noticeable discrepancy for \( kd < \pi/4 \). For \( kd > \pi/2 \), two curves are almost identical.

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