INVERSE SCATTERING METHOD AND VECTOR HIGHER ORDER NONLINEAR SCHRÖDINGER EQUATION

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Abstract

A generalized inverse scattering method has been developed for arbitrary n dimensional Lax equations. Subsequently, the method has been used to obtain N soliton solutions of a vector higher order nonlinear Schrödinger equation, proposed by us. It has been shown that under suitable reduction, vector higher order nonlinear Schrödinger equation reduces to higher order nonlinear Schrödinger equation. The infinite number of conserved quantities have been obtained by solving a set of coupled Riccati equations. A gauge equivalence is shown between the vector higher order nonlinear Schrödinger equation and the generalized Landau Lifshitz equation and the Lax pair for the latter equation has also been constructed in terms of the spin field, establishing direct integrability of the spin system.
1 Introduction

Integrable models, since its inception, occupy a distinguished position because of their striking predictability and the existence of a special class of extended solutions, called solitons. Solitons are characterised by non-dispersive localised wave pulses which produce as a result of the competition between linear dispersion and nonlinearity. Moreover, solitons retain their shapes even after collisions among themselves and consequently exhibit particle-like behaviour. Integrable models as well as solitons have applications in such diverse areas of physics as high energy physics, condensed matter physics, plasma physics, nonlinear optics and nuclear physics.

At present, a handful of nonlinear dynamical equations exist exhibiting soliton solutions. Historically, KdV equation \([1]\) is the most important one. It describes the dynamics of waves moving in shallow water. KdV and KP hierarchies also play an important role in describing non-critical strings \([2]\) through the construction of \(\tau\) functions. In KdV or KP hierarchy approach, evaluation of partition function and all correlation functions may be made directly through a set of integrable differential equations, bypassing the complicated technicalities involved in other approaches. Sine-Gordon equation \([3]\) is another well-known example of integrable equation, which first arose as a master equation for pseudo-spherical surfaces. Besides, it describes the propagation of magnetic flux in a Josephson junction transmission line, the propagation of dislocations in a crystal lattice and many other physical problems. Nonlinear Schrödinger
equation and its higher order generalizations [4,5] have also many applications in plasma physics and in nonlinear optics. In particular recently, a lot of excitement is veered around the soliton solutions of a higher order generalization of nonlinear Schrödinger equation, because of its potential application in high speed fibre communication systems. The key observation is that the dynamics of the higher order nonlinear Schrödinger equation (HNLS) not only takes care of dispersion loss, but also takes care of the propagation loss as the optical pulse propagates along the fibre. This is due to the fact that stimulated Raman effect, which compensates the propagation loss, already exists within the spectrum of HNLS equation. As a consequence, a very short pulse can be propagated over a long distance without distortion. Thus, it is optical solitons, which are responsible behind the successful propagation of optical pulses through the fibre over a long distance. Therefore, the importance of the study of integrable models and to find their soliton solutions is unquestionable.

There are several methods exist to obtain soliton solutions of a nonlinear evolution equation, like Hirota bilinear method, Painleve analysis, Bäcklund transformation and inverse scattering transform (IST). IST method, however, is the most elegant tool, which eventually proves the complete integrability of the evolution equation. IST method, on the other hand, is the most complicated one and is intimately connected with the existence of an auxiliary linear problem, known as Lax equations. Most of the known integrable evolution equations are, in fact, associated with $2 \times 2$ and $3 \times 3$ dimensional Lax opera-
tors. But there are some evolution equations whose associated linear equations, in general, cannot be cast in terms of $2 \times 2$ or $3 \times 3$ Lax operators. One such example is higher order nonlinear Schrödinger equation (HNLS) \[6\]:

$$i \partial_t q + \beta_1 \partial_{xx} q + \beta_2 |q|^2 \partial_x q + \beta_3 |q|^2 q + \beta_4 \partial_x (|q|^2) q + \beta_5 |q|^2 \partial_x q = 0$$ \hspace{1cm} (1)

where, $q$ is a complex field and $\beta_i$ for $i = 1, 2, \ldots, 5$ are the constant coefficients.

Notice that in absence of last three terms, (1) reduces to nonlinear Schrödinger equation. It is observed recently, for arbitrary values of the coefficients, (1) is associated with $n \times n$ Lax operators \[12\]. Interestingly, the parameters, $\beta$ are related to the dimension of the Lax operators. Unfortunately, IST method is not well understood for arbitrary dimensional Lax operators. Nonetheless, IST methods developed in the context of $2 \times 2$ \[11\] and $3 \times 3$ \[13\], \[14\] matrix Lax operators cannot be generalized straightforwardly to handle more general cases. Thus, in order to obtain soliton solutions of (1) by IST method, a generalized IST method, associated with $n$ dimensional Lax operator, is to be formulated. In this paper, we will, therefore, develop a generalized IST method, which ultimately leads to solving a set of coupled Gelfand Levitan Marchenko equations \[15\] for obtaining soliton solutions. We will use the generalized IST method to obtain soliton solutions for an evolution equation, whose dynamical field is an $(n - 1)$ component vector, $\vec{q}$. $\vec{q}$, in fact, is a vector generalization of the field $q$ given in (1). The evolution equation may, therefore, be called vector higher order nonlinear Schrödinger equation (VHNLS) and will be described in the next section. We will also show that under suitable reduction VHNLS equation...
reduces to (1).

The rigidity in the structures of the solitons reveals that the system has a huge underlying symmetry. This symmetry is, in general, manifested by the existence of an infinite number of conserved quantities. Existence of infinite number of conserved quantities is a key criterion for a field theoretical model to be called integrable in Liouville sense [16]. Construction of Lax pair of a dynamical system although itself is a good hint for integrability, many integrable models may not satisfy Liouville integrability criterion. In this paper we will obtain Riccati equations associated with an n dimensional Lax operator and indentify the generating function of the conserved charges. Consequently, we will find conserved charges explicitly in terms of the field variables and their derivatives.

Another interesting aspect, we will address, is to establish a connection between nonlinear field theories and spin systems. In this context, it is wellknown that nonlinear Schrödinger equation is gauge related to Landau Lifshitz equation [17]. Recently Sasa Satsuma equation [18], a particular version of HNLS equation is shown to be gauge related to a generalized Landau Lifshitz equation, where the spin field is associated with $SU(3)$ group [19, 20]. It is thus expected that a spin system may also exist corresponding to a vector generalisation of HNLS equation. We will establish a connection between the VHNLS equation, proposed by us and a generalized Landau Lifshitz equation and obtain a Lax pair for the spin system.
The organization of the paper is as follows. In section 2, the VHNLS equation is introduced and a Lax pair for the nonlinear equation is constructed. We also show a reduction procedure under which VHNLS equation reduces to a two parameter family of HNLS equation. In section 3, we show the existence of infinite number of conserved quantities, by solving a set of coupled Riccati equations. We develop the inverse scattering method for an \( n \) dimensional Lax operators in section 4 and obtain \( n \) Gelfand Levitan Marchenko equations. Section 5 deals with the solutions of Gelfand Levitan Marchenko equations and consequently find \( N \) soliton solutions. We show in section 6 the gauge equivalence of VHNLS equation and the generalised Landau Lifshitz equation and obtain a Lax pair for the generalised Landau Lifshitz equation in terms of the spin field. Section 7 is the concluding one.

2 Evolution Equation and Lax Pair

We propose a vector nonlinear evolution equation of the form

\[
\ddot{\mathbf{q}}_t + \epsilon \dddot{\mathbf{q}}_{xxx} + 3\epsilon (\mathbf{q}^2 \cdot \dot{\mathbf{q}}_x)\dot{\mathbf{q}} + 3\epsilon |\mathbf{q}|^2 \dot{\mathbf{q}}_x = 0
\]  

(2)

where, the dynamical field variable, \( \mathbf{q} = (q_1, q_2, \ldots, q_{n-1}) \), is an \( (n-1) \)-tuple vector. The equation (2) is an example of vector higher order nonlinear Schrödinger equation (VHNLS). The vector field \( \mathbf{q} \) may be interpreted as an \( (n-1) \) interacting optical modes describing the dynamics of a charge field with \( (n-1) \) colours. In fact, we will show under suitable reduction, (2) yields to the HNLS equation. 

[12]
In order to obtain Lax pair for (2), we first introduce the \( n \times n \) matrix linear eigenvalue problem as

\[
\frac{\partial \psi}{\partial x} = U(x, t, \lambda) \psi \tag{3a}
\]

\[
\frac{\partial \psi}{\partial t} = V(x, t, \lambda) \psi \tag{3b}
\]

where, \( \psi(x, t) \) is an \( n \)-tuple vector auxilliary field, \( \lambda \) is the spectral parameter and the Lax operators \( U(x, t) \) and \( V(x, t) \) are \( n \times n \) matrices. Let us now assume an explicit form of the Lax pair, \( U(x, t) \) and \( V(x, t) \), associated with the VHNLS equation, as

\[
U = -i\lambda \Sigma + A \tag{4a}
\]

\[
V = -\epsilon A_{xx} + \epsilon(A_x A - A A_x) + 2\epsilon A^3 - 2i\epsilon \lambda \Sigma (A^2 - A_x) + 4\epsilon \lambda^2 A - 4i\epsilon \lambda^3 \Sigma, \tag{4b}
\]

In the equation (4) \( \Sigma \) is a c-no. diagonal matrix and the matrix \( A(x, t) \) consists of dynamical fields, \( \vec{q}(x, t) \) and \( \vec{q}^\ast(x, t) \) only. It is interesting to note that the evolution equation for the matrix \( A(x, t) \) immediately follows from the compatibility condition, namely

\[
U_t(x, t) - V_x(x, t) + [U(x, t), V(x, t)] = 0
\]

provided \( A \) and \( \Sigma \) satisfy the conditions that

\[
\Sigma^2 = 1, \quad \Sigma A + A \Sigma = 0 \tag{5}
\]

and as a consequence, the nonlinear evolution equation for \( A(x, t) \) becomes

\[
A_t + \epsilon A_{xxx} - 3\epsilon (A^2 A_x + A_x A^2) = 0 \tag{6}
\]
It is now clear that various representations of the matrix $A$ in terms of the fields $\vec{q}$ and $\vec{q}^*$ yield to different nonlinear evolution equations. To associate (6) with the VHNLS equation (2), let us consider the explicit expressions of $\Sigma$ and $A(x, t)$ of the form satisfying the properties (5) as

$$\Sigma = \sum_{i=1}^{n-1} e_{ii} - e_{nn} \quad (7a)$$

$$A(x, t) = \sum_{i=1}^{n-1} q_i(x, t)e_{in} - \sum_{i=1}^{n-1} q_i^*(x, t)e_{ni} \quad (7b)$$

where, $e_{ij}$ is an $n \times n$ matrix whose only $(ij)$th element is unity, the rest elements being zero and $q_i(x, t)$ is the $i$th component of the dynamical field $\vec{q}$.

Substituting (7) in (6), the evolution equation becomes

$$q_{it} + \epsilon q_{ixx} + 3 \epsilon (\sum_{j=0}^{n-1} q_j^* q_j) q_i + 3 \epsilon (\sum_{j=0}^{n-1} q_j q_j^*) q_{ix} = 0 \quad (8)$$

which is nothing but VHNLS equation (2), written in the component form. Now in order to obtain HNLS equation (1), let us consider the following reduction.

Notice that all the dynamical fields $q_i$ in (8) are independent. However, instead of $(n-1)$ independent dynamical fields, if we restrict ourselves to only one dynamical field $q$ and its complex conjugate $q^*$ in (8), then all the $q_i$s are not independent. $q_i$s, in this case, may be chosen as either $q$ or $q^*$. If we, for example, choose $m$ number of $q_i$ as $q$ and the rest $(n-m-1)$ numbers as $q^*$, (8) reduces to HNLS equation

$$q_{it} + \epsilon q_{xxx} + 6 \epsilon |q|^2 q_x + 3(n-m-1) \epsilon (|q|^2) x q = 0. \quad (9)$$

The equation (9), to be precise, is a gauge equivalent version of HNLS equation.
Soliton solutions of (9) and those of HNLS equation are, in fact, related through a U(1) gauge transformation [12]. Two well known equations immediately follow from (9). If we choose \( m = n - 1 \), (9) gives rise to Hirota equation [21]

\[
q_t + \epsilon q_{xxx} + 6\epsilon |q|^2 q_x = 0
\]

(10)
after rescalling of \( q \) and \( q^* \) as \( q = (n - 1)^{-\frac{1}{2}} q \) and \( q^* = (n - 1)^{-\frac{1}{2}} q^* \) respectively.

On the other hand, if we choose \( m = (n - 1)/2 \), which is only possible for odd dimensional Lax pair, (9) reduces to Sasa Satsuma equation [18, 20, 22],

\[
q_t + \epsilon q_{xxx} + 6\epsilon |q|^2 q_x + 3\epsilon (|q|^2)_x q = 0,
\]

(11)
once again with appropriate scaling of the fields \( q \) and \( q^* \) respectively as \( q = (\frac{n-1}{2})^{-\frac{1}{2}} q \) and \( q^* = (\frac{n-1}{2})^{-\frac{1}{2}} q^* \).

### 3 Riccati Equation and Conserved Charges

In order to obtain Riccati equation, we first write the Lax equation (3a,4a,7) in the component form. For the first \((n-1)\) components of \( \Psi \), (3a) can be written in the form,

\[
\begin{align*}
\Psi_{1x} &= -i\lambda \Psi_1 + q_1 \Psi_n \\
\Psi_{2x} &= -i\lambda \Psi_2 + q_2 \Psi_n \\
&\vdots \\
\Psi_{n-1x} &= -i\lambda \Psi_{n-1} + q_{n-1} \Psi_n
\end{align*}
\]
\[ i.e. \]

\[
\Psi_{ix} = -i\lambda \Psi_i + q_i \Psi_n \tag{12a}
\]

where, \( i = 1, 2, \cdots, n - 1 \) and \( \Psi_i \) denotes the \( i \)th component of \( \Psi \). But for the \( n \)th component of \( \Psi \), the Lax equation (3a) has a different form as

\[
\Psi_{nx} = i\lambda \Psi_n - q_1^* \Psi_1 - q_2^* \Psi_2 - \cdots - q_{n-1}^* \Psi_{n-1}
\]

\[
= i\lambda \Psi_n - \sum_{j=1}^{n-1} q_j^* \Psi_j \tag{12b}
\]

Following now a similar procedure as in [20] we write,

\[
\Gamma_i = \frac{\Psi_i}{\Psi_n}, \tag{13}
\]

\( i = 1, 2, \cdots, n - 1 \), which are related to the conserved charges \( \alpha_{nn}(\lambda) \) in the following way,

\[
\ln \alpha_{nn}(\lambda) = \ln \Psi_n - i\lambda x |_{x \to \infty}
\]

\[
= -\int_{-\infty}^{\infty} dx \left( \sum_{i=1}^{n-1} q_i^* \Gamma_i \right) \tag{14}
\]

We will see in section 5 that \( \alpha_{nn}(\lambda) \) is, indeed, \((nn)\)th element of the scattering data matrix and more so it does not evolve with time. By using (12a,b) and (13), we may obtain a first order differential equation for each \( \Gamma_i \),

\[
\Gamma_{ix} + 2i\lambda \Gamma_i - \left( \sum_{j=1}^{n-1} q_j^* \Gamma_j \right) \Gamma_i - q_i = 0 \tag{15}
\]

The set of \((n-1)\) coupled nonlinear differential equations for \( \Gamma_i \) in (15) are called Ricatti equations. It is obvious from (14) that the solutions of Riccati
equations eventually determine the conserved quantities. Now in order to solve (15), we assume a series solution of $\Gamma_i$ as

$$\Gamma_i(x, \lambda) = \sum_{n=0}^{\infty} C^n_i(x) \lambda^{-n}$$

Substituting (16) into (15), the following recursion relations may be obtained.

$$C^i_0 = 0; \quad C^i_1 = \frac{q_i}{2t}$$

and

$$2t C^i_{k+2} + (C^i_{k+1})_x - \sum_{m=0}^{k-1} C^i_{k-1+m} \sum_{j=1}^{n-1} q_j^* C^j_m = 0$$

with $k = 0, 1, 2, \ldots$. The infinite number of Hamiltonians (conserved quantities) may explicitly be determined in terms of the dynamical field variables $q_i$ and their derivatives by expanding $\alpha_{nn}(\lambda)$ in the form,

$$\ln \alpha_{nn}(\lambda) = (n - 1) \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} H_l \lambda^{-l}$$

and thus comparing (18) with (14) and (16), $H_l$ becomes

$$H_l = \frac{(2t)^{2l+1}}{(-1)^l (n-1)} \int dx [\sum_{j=1}^{n-1} q_j^* C^j_l]$$

The explicit expressions of the first few low order Hamiltonians are given below.

$$H_1 = \frac{1}{n - 1} \int dx [\sum_{j=1}^{n-1} q_j^* q_j]$$

$$H_2 = \frac{1}{2(n - 1)} \int dx \sum_{j=1}^{n-1} [q_j^* q_{jx} - q_{jx} q_j]$$

$$H_3 = \frac{1}{n - 1} \int dx [\sum_{j=1}^{n-1} q_j^* q_j]^2 - \sum_{j=1}^{n-1} q_{jx}^2 q_{jx}$$
\[ H_4 = \frac{1}{2(n-1)} \int dx \sum_{j=1}^{n-1} [q_j^* q_{jxxx} - q_{jxxx}^* + 3(\sum_{k=1}^{n-1} |q_k|)^2 (q_j^* q_{jx} - q_{jx}^* q_j)] \]  

\[ H_5 = \frac{1}{n-1} \int dx \left[ \sum_{j=1}^{n-1} q_j^* q_{jxxx} + 2(\sum_{j=1}^{n-1} q_j^* q_j)^3 - (\sum_{j=1}^{n-1} (|q_j|^2)_x)^2 \right] 
\]

\[ - \frac{n-1}{4} \sum_{k=1}^{n-1} |q_j|^2 \sum_{j=1}^{n-1} q_j^* q_{jx} - 2 \sum_{j=1}^{n-1} q_j^* q_{jx} \sum_{k=1}^{n-1} q_k^* q_k \]  

We verify directly by using equations of motions that \( H_1, H_2, H_3, H_4 \) and \( H_5 \) are, indeed, constants of motions.

Let us now specialize to HNLS equation. If we assume that \( m \) number of \( q_i \)s are chosen as \( q \) and the rest \( n - m - 1 \) as \( q^* \), the Hamiltonians in (20) reduce to

\[ H_1 = \int dx |q|^2 \]  

\[ H_2 = (2m - n + 1) \int dx (q_x q^* - q q^*_x) \]  

\[ H_3 = \int dx ((n-1)|q|^4 - q^*_x q_x^*) \]  

\[ H_4 = (2m - n + 1) \int dx [3(n-1)|q|^2 (q_x q^* - q q^*_x) + \frac{1}{2} (q_{xxx} q^* - q q_{xxx}^*)] \]  

\[ H_5 = \int dx[q_x q_{xx}^* + 2(n-1)^2 |q|^6 + [(2m - n - 1)^2 - 4(n-1)]|q|^2 q_x q_{xx}^* \]

\[ - [(n-1) + \frac{2m(n-l-1)}{(n-1)}] (|q|^2)_x^2] \]  

Notice that \( H_1 \) in (21a) has a universal form, which implies that all solitons
have the same energy irrespective of their shapes. However, the momentum, $H_2$ crucially depends on the numbers of $q$, chosen in a given representation of the matrix $A$. For example, $H_2$ becomes zero for Sasa Satsuma case, i.e., for $m = (n - 1)/2$. In fact, all conserved quantities having even indices become trivial for Sasa Satsuma case. $H_3$ in (21c) also has a somewhat universal form, depending only on the dimensions of the Lax pairs. But the higher order conserved quantities starting from $H_5$ do not possess such universal forms. Their explicit forms depend both on the dimensionality of the matrix Lax pair and also on the representations of the matrix $A$.

4 Generalized Gelfand Levitan Marchenko Equations

We now generalize IST method suitable for studying $n$ dimensional Lax operators. The first step in this direction is to obtain a set of generalized Gelfand Levitan Marchenko equations. This generalisation is a nontrivial one and crucially depends on the properties of scattering data matrix. However, we will see that for three dimensional Lax operators generalized Gelfand Levitan Marchenko equations will reduce to Sasa Satsuma case as is expected. We broadly follow the treatment of Manakov, developed in the context of $3 \times 3$ Lax operator and for that assume Jost functions, for real $\lambda$, satisfy the boundary conditions,

$$
\Phi^{(i)} = e_{i} e^{-i\lambda x} \quad (22a)
$$
with \( i = 1, 2, \ldots, n - 1 \), but the \( n \)th one satisfies a different boundary condition like

\[
\Phi^{(n)} = e_n e^{i\lambda x} \tag{22b}
\]

as \( x \to -\infty \). Similarly, as \( x \to \infty \), other set of Jost functions satify the boundary conditions

\[
\Psi^{(i)} = e_i e^{-i\lambda x} \tag{23a}
\]

for \( i = 1, \ldots, n - 1 \) and for the \( n \)th one,

\[
\Psi^{(n)} = e_n e^{i\lambda x}. \tag{23b}
\]

In the equations (22) and (23), \( e_i \)'s are the basis vectors for an \( n \)-dimensional vector space. It follows from (7b) that \( A^\dagger = -A \) for real valued \( \lambda \) and thus we have

\[
\partial_x (\Psi^{(1)}\Psi^{(2)}) = 0 \tag{24}
\]

for any pair of solutions of equation (2a), \( \Psi^{(1)} \) and \( \Psi^{(2)} \), having the same eigenvalue. It is straightforward to show from (22) and (23) that

\[
\Phi^{(i)\dagger}\Phi^{(j)} = \Psi^{(i)\dagger}\Psi^{(j)} = \delta_{ij} \tag{25}
\]

for \( i, j = 1, 2, \ldots, n \). Since the set of Jost functions \( \Psi_i \) are linearly independent and the maximum number of independent Jost functions is \( n \), we may express the set of Jost functions \( \Phi_i \) as a linear combination of \( \Psi_i \) as

\[
\Phi^{(i)}(x, \lambda) = \sum_{j=1}^{n} a_{ij}(\lambda) \Psi^{(j)}(x, \lambda) \tag{26}
\]
where $\alpha_{ij}(\lambda)$ is the $(ij)$th element of scattering data matrix, which can be expressed by using (23) and (26) in the form

$$\alpha_{ij}(\lambda) = \Psi^{(j)\dagger}(x, \lambda)\Phi^{(i)}(x, \lambda)$$  \hspace{1cm} (27)

The orthogonality property of the scattering data matrix elements for real eigenvalues $\lambda$, subsequently follows from (22), (23) and (27) and thus we obtain

$$\sum_{k=1}^{n} \alpha_{ik}(\lambda)\alpha_{jk}(\lambda) = \delta_{ij}$$  \hspace{1cm} (28)

which finally gives

$$\Psi^{(i)}(x, \lambda) = \sum_{j=1}^{n} \alpha_{ji}^{\dagger}(\lambda)\Phi^{(j)}(x, \lambda)$$  \hspace{1cm} (29)

It is further interesting to see, by exploiting the properties of $[\alpha_{ij}]$ and $[\alpha_{ij}^{\dagger}]$ matrices, that we can write the element $\alpha_{ij}^{\dagger}$ as the cofactor of the elements of the matrix $[\alpha_{ij}]$. In particular, $\alpha_{ni}^{\dagger}$ element can be written as

$$\alpha_{ni}^{\dagger} = (-1)^{n+i}\text{det}[\tilde{\alpha}_{ni}]$$  \hspace{1cm} (30)

where $[\tilde{\alpha}_{ni}]$ is a $(n-1) \times (n-1)$ matrix, constructed from the $n \times n$ scattering matrix, $[\alpha_{ij}]$ with $n$th row and $i$th column being omitted, i.e. $\text{det}[\tilde{\alpha}_{ni}]$ is the minor of $\alpha_{ni}$ element of scattering matrix, $[\alpha_{ij}]$. Now by using (26) and (30), we obtain the following useful relations among the Jost functions $\Phi^{(i)}$ and $\Psi^{(i)}$.

The first $n-1$ Jost functions in (26) satisfy

$$\frac{1}{\alpha_{nn}^{*}(\lambda)} \sum_{j=1}^{n-1} (\text{Adj}[\tilde{\alpha}_{nn}])_{kj} \Phi^{(j)}e^{i\lambda x} = \Psi^{(k)}e^{i\lambda x} - \frac{\alpha_{nk}^{*}}{\alpha_{nn}^{*}} \Psi^{(n)}e^{i\lambda x}$$  \hspace{1cm} (31a)

with $k = 1, \ldots, n-1$, but the $n$th. Jost function, $\Phi^{(n)}$ obey the relation

$$\frac{1}{\alpha_{nn}} \Phi^{(n)}e^{-i\lambda x} = \Psi^{(n)}e^{-i\lambda x} + \frac{1}{\alpha_{nn}} \sum_{j=1}^{n-1} \alpha_{nj} \Phi^{(j)}e^{-i\lambda x}$$  \hspace{1cm} (31b)

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Notice that in deriving (31), we have used the following properties of the scattering data matrix.

\[ \alpha_{nk}^* \delta_{ij} = \sum_{l=1}^{n-1} [\tilde{\alpha}_{nk}]_{il} (\text{Adj} [\tilde{\alpha}_{nk}])_{lj} \]

It is important to mention that analyticity properties of the Jost functions and the elements of scattering matrix may be obtained from (31).

We are now going to derive the Gelfand, Levitan and Marchencho equation for an \( n \) dimensional Lax pair. Let us consider an integral representation of the Jost function \( \Psi^{(i)} \) for \( (i = 1, 2, \ldots, n) \). For the first \( n - 1 \) Jost functions we may choose the following integral representations

\[ \Psi^{(j)}(x, \lambda) = e_j e^{-i\lambda x} + \int_x^\infty dy K^{(j)}(x, y) e^{-i\lambda y} \]  (32a)

with \( j = 1, 2, \ldots, n - 1 \), while the \( n \)th Jost function may be written as

\[ \Psi^{(n)}(x, \lambda) = e_n e^{i\lambda x} + \int_x^\infty dy K^{(n)}(x, y) e^{i\lambda y}. \]  (32b)

where, \( e_i, \ i = 1, 2, \ldots, n \) are basis vectors for an \( n \) dimensional vector space and the kernels \( K^{(j)} \) and \( K^{(n)} \) are \( n \) dimensional column vectors, which may be written explicitly in the component form as

\[ K^{(j)}(x, y) = \sum_{m=1}^n K_{m}^{(j)}(x, y) e_m \]  (33a)

\[ K^{(n)}(x, y) = \sum_{m=1}^n K_{m}^{(n)}(x, y) e_m \]  (33b)
Substituting (32) in (31b), we obtain

\[ \frac{1}{\alpha_{nn}} \Phi^{(n)} = e_n e^{i\lambda x} + \int_x^\infty dy K^{(n)}(x, y)e^{i\lambda y} + \sum_{j=1}^{n-1} \frac{\alpha_{nj}}{\alpha_{nn}} e_j e^{-i\lambda x} + \sum_{j=1}^{n-1} \frac{\alpha_{nj}}{\alpha_{nn}} \int_x^\infty dy K^{(j)}(x, y)e^{-i\lambda y} \]  

Multiplying now both sides of (34) by \( \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda e^{-i\lambda z} \), with an assumption \( z > x \) and using the analyticity property of the Jost function, it follows that

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \frac{\Phi^{(n)}}{\alpha_{nn}(\lambda)} e^{-i\lambda z} = K^{(n)}(x, z) + \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \sum_{j=1}^{n-1} \frac{\alpha_{nj}(\lambda)}{\alpha_{nn}(\lambda)} e_j e^{-i\lambda (x+z)} + \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \int_x^\infty dy \frac{\alpha_{nj}(\lambda)}{\alpha_{nn}(\lambda)} K^{(j)}(x, y)e^{-i\lambda (y+z)} \]  

The L.H.S. of (35a) can be simplified further by taking into account that \( \frac{1}{\alpha_{nn}(\lambda)} \) is analytic in the lower half plane except at the points, say \( \lambda_j^* \) with \( Im\lambda_j^* > 0 \) and \( j = 1, 2, \ldots, N \), where \( \frac{1}{\alpha_{nn}(\lambda)} \) has \( N \) simple poles. Moreover, we assume that at the simple poles \( \lambda_j^* \), \( \Phi^{(n)} \) to be of the form

\[ \Phi^{(n)}(x, \lambda_j^*) = \sum_{p=1}^{n-1} C_{np}^{(j)} \Psi^{(p)}(x, \lambda_j^*) \]  

With these assumptions, L.H.S. of (35a) becomes

\[ -i \sum_{j=1}^{N} \frac{e^{-i\lambda_j^* z}}{\alpha'_{nn}(\lambda_j^*)} \sum_{p=1}^{n-1} C_{np}^{(j)} \Psi^{(p)}(x, \lambda_j^*) \]  

where \( \alpha'_{nn} \) denotes derivative with respect to \( \lambda \). By Substituting the integral representations of \( \Psi^{(p)}(x, \lambda_j^*) \) from (32a), (35c), i.e. the L.H.S. of (35a) finally reduces to

\[ -i \sum_{j=1}^{N} \frac{e^{-i\lambda_j^* z}}{\alpha'_{nn}(\lambda_j^*)} \sum_{p=1}^{n-1} C_{np}^{(j)} [e_p e^{-i\lambda_j^* x} + \int_x^\infty dy K^{(p)}(x, y)e^{-i\lambda_j^* y}] \]  

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Let us now introduce a function $F_p(x + y)$ as

$$F_p(x + y) = i \sum_{j=1}^{N} C^{(j)}_{np} e^{-i\lambda_j^*(x+y)} + \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \frac{\alpha_{np}(\lambda)}{\alpha_{nn}(\lambda)} e^{-i\lambda(x+y)}$$  \hspace{1cm} (36)

In terms of the function, $F_p(x + y)$ in (36), (35a) and (35d) together may be written in a compact form as

$$K^{(n)}(x, z) + \sum_{p=1}^{n-1} e_p F_p(x + z) + \sum_{p=1}^{n-1} \int_{x}^{\infty} K^{(p)}(x, y) F_p(z + y) = 0 \hspace{1cm} (37a)$$

The integral equation (37a) is one of the desired Gelfand Levitan Marchenko equations for the kernel $K^{(n)}$. Other integral equations for the kernels $K^{(p)}$ for $p = 1, 2, \cdots, n - 1$ may be obtained from (31a) and (32) in a similar way like that of (37a). The integral equations for the kernels $K^{(p)}$ thus turn out to be of the form

$$K^{(p)}(x, z) - e_n F^*_p(x + z) - \int_{x}^{\infty} dy K^{(n)}(x, z) F^*_p(z + y) = 0 \hspace{1cm} (37b)$$

provided $z > x$. In deriving (37b) we have used the following identity

$$C^*_m = \alpha^*_m(\lambda_j) = \sum_{i=1}^{n-1} [\overline{\alpha}_{np}(\lambda_j)]_{ki} (\text{Adj}[\overline{\alpha}_{np}(\lambda_j)])_{li} \delta_{kl}.$$  \hspace{1cm} (17)

The set of coupled equations (37) may be called as generalized Gelfand Levitan Marchenko equations. Substituting now (33b) and (37b) in (37a), to a first approximation, we find the Gelfand Levitan Marchenko equation for the $p$th component of $K^{(n)}$:

$$K^{(n)}_p(x, z) + F_p(x + z) + \sum_{m=1}^{n-1} \int_{x}^{\infty} ds K^{(n)}_m(x, s) \int_{x}^{\infty} dy F_m(y + z) F^*_m(y + s) = 0 \hspace{1cm} (38)$$

which will be used later to find the soliton solutions for VHNLS equation.
5 N Soliton Solutions

To obtain soliton solutions, let us associate dynamical fields $q_i(x,t)$ with the kernels, $K_i^{(n)}$ in (38). Substituting (32b) into Lax equation (2a) we find

$$q_i(x) = -2K_i^{(n)}(x,x)$$

and consequently, it is evident that the solution of (38) gives rise to soliton solutions in terms of scattering data elements. But before going to solve (38), we first compute time evolution of scattering data element. Notice that as $|x| \to \infty$, the Lax equation (3b,4b) leads to

$$\frac{\partial \Psi}{\partial t} = -4i\epsilon \lambda^3 \Sigma$$

which, in turn, determines time evolution of the scattering data elements. For example, scattering data elements, $\alpha_{nj}(\lambda, t)$ for $j = 1, 2, \cdots, n-1$ evolve with time as

$$\alpha_{nj}(\lambda, t) = \alpha_{nj}(\lambda, 0)e^{-8i\epsilon \lambda^3 t}$$

while the element $\alpha_{nn}(\lambda, t)$ is time invariant:

$$\alpha_{nn}(\lambda, t) = \alpha_{nn}(\lambda, 0),$$

which eventually justifies our conjecture in section 3 that $\alpha_{nn}$, indeed, can be associated with the conserved quantities. From (29) and (35b) it follows that the coefficients $C_{np}^{(j)}$ also satisfy the similar time dependence as $\alpha_{nj}$ in (41a) and thus

$$C_{np}^{(j)}(\lambda, t) = C_{np}^{(j)}(\lambda, 0)e^{-8i\epsilon \lambda^3 t}$$

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If we now restrict ourselves to soliton sector, $\alpha_{np}(\lambda)$ becomes trivial and the function $F_p(x + y)$, in this case, reduces to

$$F_p(x + y) = i \sum_{j=1}^{N} \frac{C^{(j)}_{np} e^{-i\lambda_j^*(x+y)}}{\alpha'_{nn}(\lambda_j^*)}.$$  

Time dependence of the function $F_p(x + y)$ may be obtained immediately from (42) as

$$F_p(x + y) = i N \sum_{j=1}^{N} C^{(j)}_{np}(\lambda, 0) e^{-8i\epsilon\lambda_j^3 t} e^{-i\lambda_j^*(x+y)}.$$  

(43)

To solve the integral equation (38) for the soliton solutions, the kernels $K_p^{(n)}(x + y)$ are assumed to be of the form

$$K_p^{(n)}(x + y) = \sum_{j=1}^{N} \omega_{pj}(x, t) e^{-i\lambda_j^* y}$$  

(44)

Substituting (43) and (44) in (38), we obtain a set of $n - 1$ algebraic equations:

$$K_p^{(n)}(x, x) + F_p(x + x) - \sum_{j,k,l=1}^{n} \sum_{r=1}^{n-1} \frac{\omega_{pl} e^{-i\lambda_j^* x}}{(\lambda_k - \lambda_j^*)(\lambda_k - \lambda_l^*)} \cdot \frac{C^{(j)}_{nr}(0) C^{(k)}_{nr}(0)}{\alpha'_{nn}(\lambda_j^*) \alpha'_{nn}(\lambda_k^*)} e^{i(2\lambda_k - \lambda_j^* - \lambda_l^*) x + 8i\epsilon(\lambda_k^3 - \lambda_j^3 - \lambda_l^3)t} = 0$$  

(45)

Solving (45) for $K_p(x, x)$ and subsequently using (39), the $N$ soliton solutions for each dynamical field $q_i(x, t)$ may be expressed as

$$q_i(x, t) = -2 \sum_{j=1}^{N} (BC^{-1})_{ij} e^{-i\lambda_j^* x}$$  

(46)

where, $B$ and $C$ are respectively $(n - 1) \times N$ and $N \times N$ matrices whose explicit forms are given by

$$(B)_{ij} = iC^{(j)}_{ni}(0) e^{-8i\epsilon\lambda_i^3 t - i\lambda_j^* x}$$
\[(C)_{ij} = \sum_{p=1}^{n-1} \sum_{k=1}^{N} \frac{C_{np}^{(j)}(0)C_{np}^{(k)}(0)e^{-i(\lambda_i^*+\lambda_j^*-2\lambda_k)x+8i\epsilon(\lambda_i^2-\lambda_j^2)t}}{\alpha_{nn}^{*}\alpha_{nn}(\lambda_i^*)(\lambda_k-\lambda_i^*)(\lambda_k-\lambda_j^*)} - \delta_{ij}\]

Let us now consider an explicit form of one soliton. One soliton solution, which follow from (46), may be written in the following form

\[q_i(x, t) = \frac{2F_i(x + x)}{1 + \sum_{p=1}^{n-1} |F_p(x + x)|^2 \frac{1}{(\lambda_i-\lambda_j)^2}} \quad (47)\]

where

\[F_i(x + y) = \frac{iC_{ni}^{(1)}(\lambda; 0)}{\alpha_{nn}^{*}(\lambda_1^*)} e^{-8i\epsilon\lambda_i^2t - i\lambda_1^2(x+y)}.\]

\[q_i(x, t)\] in (47) may be expressed in a more conventional way, by choosing the position of the pole at \(\lambda_1 = \frac{1}{2}(-\xi + i\eta)\) and by introducing \(e^{\gamma_i + i\delta_i} = \frac{C_{ni}^{(1)}(\lambda; 0)}{2\eta\alpha_{nn}(\lambda_1^*)}\) and thus

\[q_i(x, t) = 4\eta e^{\gamma_i - \eta x + \epsilon(\eta^2 - 3\xi^2)t + i(\delta_i + \xi + \epsilon\xi t - 3\epsilon\xi^2)} \frac{1}{1 + \sum_{p=1}^{n-1} e^{2\gamma_p} e^{-2\eta x + 2\epsilon(\eta^2 - 3\xi^2)t}} \quad (48)\]

which can further be simplified as

\[q_i(x, t) = \frac{2i\eta e^{\gamma_i - \Gamma_n + iQ_i}}{cosh P} \quad (49)\]

by introducing

\[e^{2\Gamma_n} = \sum_{p=1}^{n-1} e^{2\gamma_p}\]

\[P(x, t) = \eta x - \epsilon(\eta^2 - 3\xi^2)t - \Gamma_n\]

\[Q_i(x, t) = \xi x + \epsilon\xi(\xi^2 - 3\xi\eta)t + \delta_i\]

Once again, if we specialize to HNLS equation, each independent field \(q_i(x, t)\), depending on the models, reduces either to \(q\) or to \(q^*\) and as a consequence \(q_i\)
in [18] yields to
\[
q(x, t) = \frac{2i\eta e^{iB}}{\sqrt{(n-1)coshA}}
\] (50)
where, \(\tilde{A} = \eta x - \epsilon\eta(\eta^2 - 3\xi^2)t - \gamma - \frac{3}{2}\ln(n-1)\) and \(\tilde{B} = \xi x + \epsilon\xi(\xi^2 - 3\xi_2)t + \delta\).

It is interesting to note that we have obtained precisely the same expression for one soliton in [12].

6 Generalized Landau Lifshitz type equation as the Gauge equivalence system

We now show an interesting connection between the VHNLS equation and the generalized Landau Lifshitz type equation by exploiting the gauge equivalence of the Lax pairs of these two dynamical systems. The procedure is similar to that between the nonlinear Schrödinger equation and the standard Landau Lifshitz equation [17].

Under a local gauge transformation, the Jost function, \(\Psi(x, t, \lambda)\) changes to
\[
\tilde{\Psi} = g^{-1}(x, t)\Psi(x, t, \lambda)
\] (51)
where \(g(x, t) = \Psi(x, t, \lambda)|_{\lambda=0}\). We claim that \(g(x, t)\) is an element of \(SU(n)\) group. As a consequence of the gauge transformation (51), the Lax equations (52) become
\[
\begin{align*}
\tilde{\Psi}_x &= \tilde{U}(x, t, \lambda)\tilde{\Psi} \\
\tilde{\Psi}_t &= \tilde{V}(x, t, \lambda)\tilde{\Psi}
\end{align*}
\] (52a)
\[
\begin{align*}
\tilde{\Psi}_x &= \tilde{U}(x, t, \lambda)\tilde{\Psi} \\
\tilde{\Psi}_t &= \tilde{V}(x, t, \lambda)\tilde{\Psi}
\end{align*}
\] (52b)
where $\tilde{U}$ and $\tilde{V}$ are the new gauge transformed Lax pair, given by

$$
\tilde{U}(x, t, \lambda) = g^{-1}(U - U_0)g 
$$

$$
\tilde{V}(x, t, \lambda) = g^{-1}(V - V_0)g 
$$

with $U_0 = U|_{\lambda=0} = g_x(x,t)g^{-1}(x,t)$ and $V_0 = V|_{\lambda=0} = g_t(x,t)g^{-1}(x,t)$. This leads to

$$
A = g_x(x,t)g^{-1} 
$$

Since $A$ belongs to $su(n)$ algebra, $g(x,t)$ obviously belongs to $SU(n)$ group, which justifies our claim. We may now identify the spin field of the Landau Lifshitz type equation as

$$
S = g^{-1}(x,t)\Sigma g(x,t), \quad S^2 = 1 
$$

With this identification, the gauge transformed Lax pair (51) may be expressed in terms of the spin field $S$ (55) and its derivatives only, yielding

$$
\tilde{U} = -i\lambda S 
$$

$$
\tilde{V} = -4i\epsilon \lambda^3 S + 2\epsilon \lambda^2 SS_x + i\epsilon \lambda(S_{xx} + \frac{3}{2}SS_x^2) 
$$

In deriving (56), we have used the following important identities

$$
SS_x = 2g^{-1}\Sigma A g 
$$

$$
SS_x^2 = -4g^{-1}\Sigma A^2 g 
$$

$$
S_{xx} + SS_x^2 = 2g^{-1}\Sigma A_x g 
$$

The zero curvature condition of (56), namely

$$
\tilde{U}_t - \tilde{V}_x + [\tilde{U}, \tilde{V}] = 0 
$$
ultimately leads to the generalized Landau Lifshitz type equation

$$S_t + \epsilon S_{xxx} + \frac{3}{2} \epsilon (S_x^3 + SS_{xx}S_x + SSS_{xx}) = 0 \quad (57)$$

with $S \in SU(n)/(U(n-1))^n$.

7 Conclusion

We have formulated a generalized IST method for an $n$ dimensional Lax operator. Subsequently, the generalized IST method is used to obtain $N$ soliton solutions for a multi-component generalization of HNLS equation, proposed by us. We have, although, obtained soliton solutions for VHNLS equation, IST method developed here is quite general and is applicable for obtaining soliton solutions for all integrable nonlinear equations. The sech structure of one soliton solution of VHNLS equation is quite interesting, particularly in the context of nonlinear optics, since it can easily be produced from the output of a mode locked laser.

We have shown the integrability of VHNLS equation also in the Liouville sense. This has been achieved first by finding a set of coupled Riccati equations and subsequently by identifying the generating function for the conserved charges. It is found that the last diagonal element of the scattering data matrix may be identified as the generating function of the conserved charges. This is also confirmed from the time evolution of scattering data matrix elements. We have also established an intriguing relationship between the VHNLS
equation and a generalized Landau Lifshitz equation, where the spin field $S \in SU(n)/(U(n-1))^n$. Moreover, we have obtained a Lax pair for the spin system implying direct integrability of the generalised Landau Lifshitz equation.

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