BLOCK FUSION SYSTEMS OF THE ALTERNATING GROUPS

MARTIN WEDEL JACOBSEN

Abstract. We describe a purely group-theoretic condition on an element $g$ of a finite group $G$ which implies that $g$ has coefficient zero in every central idempotent element of the group ring $RG$, provided that $R$ is a ring of prime characteristic. We use this condition to prove that the fusion system associated to a block of an alternating group is always isomorphic to the group fusion system of an alternating group.

Contents

1. Introduction 1
2. Preliminaries 2
3. Central idempotents of the group ring 5
4. The symmetric and alternating groups 8
References 17

1. Introduction

In the study of modular representations, the group ring does not factor into a product of matrix rings, as is the case with complex representations. Instead, one can assign to each irreducible factor a fusion system which provides some information about the structure of the factor. It is an open problem whether these block fusion systems always occur as group fusion systems as well. A few cases are known: the block fusion systems of symmetric groups are always group fusion systems of symmetric groups, and a similar result holds for the general linear groups over finite fields (see [2, Theorem 7.2 and Remarks 7.4]).

In this paper, we provide a new proof of the result on symmetric groups (Theorem 26), and we extend this proof to cover the alternating groups. We obtain that a block fusion system of an alternating group is always isomorphic to a group fusion system of an alternating group (Corollary 29).

In Section 2, we review the definition of block fusion systems and recall a few of their properties. We also prove a number of simple lemmas that will be needed later. In Section 3, we consider central idempotents of the group ring $RG$, where $R$ is any ring of prime characteristic $p$. We describe a group-theoretic condition on an element $g$ of $G$ that implies that $g$ has coefficient zero in all these central idempotents. In Section 4, we apply this condition to the symmetric and the alternating groups in order to derive restrictions on the possible defect groups. We

Date: May 10, 2014.
Supported by the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation.
then derive some properties of the centric subgroups of these defect groups and use these properties to determine the possible fusion systems.

Notation: Throughout this paper, we work in the group ring $R G$ where $R$ is a ring of characteristic $p$ and $G$ is a finite group. When $g$ is an element of $G$, $[g]$ is the conjugacy class of $g$, and when $C$ is a conjugacy class of $G$, $\Sigma C$ is the sum of the elements in $C$, considered as an element of $RG$.

An element of $G$ is called $p$-regular if it has order not divisible by $p$, and it is called a $p$-element if its order is a power of $p$. For any element $g$ of $G$, $g_p$ and $g_{p'}$ denote the $p$-part and the $p'$-part of $g$ (see Lemma 8). The notation $\prod_{i=a}^b f(i)$ always means the product $f(a) \cdot f(a+1) \cdots f(b)$.

For a finite set $M$, $S_M$ and $A_M$ are the symmetric and alternating groups on $M$, and for a natural number $n$, $S_n$ and $A_n$ are the symmetric and alternating groups on $\{1, \ldots, n\}$. $C_n$ is the cyclic group of order $n$. When $P$ is a subgroup of $G$, $\text{Aut}_G(P)$ is the group of automorphisms of $P$ that arise from conjugation with an element of $G$. Likewise, when $F$ is a fusion system on a group containing $P$, $\text{Aut}_F(P)$ is the automorphism group of $P$ in $F$. When $S$ is a Sylow $p$-subgroup of $G$, $F_S(G)$ is the fusion system on $S$ generated by $G$.

I would like to thank my advisor, Jesper Michael Møller, for introducing me to this problem and for providing invaluable guidance throughout my work on it.

2. Preliminaries

We review the definition and basic properties of blocks of a finite group, Brauer pairs associated to a block, and the fusion system associated to a block. For more detailed background, we refer to [1] and [3].

Let $k$ be an algebraically closed field of characteristic $p$ and let $G$ be a finite group. The group ring $kG$ may be decomposed as a direct product of $k$-algebras; each primitive factor is called a block of $G$. The unit element of a block is called the block idempotent. It is a primitive central idempotent of $kG$. We record a few basic facts about central idempotents:

**Proposition 1.** The product of any two distinct block idempotents of $kG$ is 0. Any central idempotent of $kG$ is the sum of distinct block idempotents. The sum of all the block idempotents of $kG$ is 1.

**Proof.** Apply [3, Corollary 4.2] to $Z(kG)$. \qed

**Lemma 2.** Let $A$ be a finite-dimensional algebra over $k$ and let $\varphi : A \to B$ be a surjective algebra homomorphism. The image of a primitive central idempotent of $A$ is a central idempotent of $B$ (possibly zero), and every central idempotent of $B$ that lies in $\varphi(Z(A))$ can be lifted to a central idempotent of $A$.

**Proof.** Since $\varphi$ is surjective, $\varphi(Z(A))$ is contained in $Z(B)$. This covers the first part. For the second part, apply [3, Theorem 3.2(b)] to $Z(A)$. \qed

We recall the definition of block fusion systems by means of Brauer pairs. More detailed information can be found in [1, Part IV].

**Definition 1.** A Brauer pair of $G$ is a pair $(P, e)$ where $P$ is a $p$-subgroup of $G$ and $e$ is a block idempotent of $kC_G(P)$. 
Definition 2. Let $Q \trianglelefteq P$ be $p$-subgroups of $G$. The Brauer homomorphism $\text{Br}_{P/Q} : (kC_G(Q))^P \to kC_G(P)$ is given by mapping $\sum_{g \in C_G(Q)} c_g g$ to $\sum_{g \in C_G(P)} c_g g$ (note the change in the range of the sum). It is a $k$-algebra homomorphism.

A Brauer pair whose first part is $P$ is also referred to as a Brauer pair at $P$. Note that there is an obvious one-to-one correspondence between Brauer pairs at the trivial group and block idempotents of $G$. When $Q$ is the trivial group, the Brauer homomorphism is denoted $\text{Br}_P$.

Definition 3. Inclusion of Brauer pairs is defined as follows. We write $(P,e) \trianglerighteq (Q,f)$ if $Q \trianglelefteq P$, $f \in (kC_G(Q))^P$, and $\text{Br}_{P/Q}(f)e = e$. We define the relation $\trianglerighteq$ between Brauer pairs to be the transitive closure of the relation $\trianglelefteq$. Conjugation of Brauer pairs is defined by $(P,e)^g = (P^g,e^g)$. A Brauer pair $(P,e)$ is said to be associated to a block idempotent $b$ of $G$ if $(1,b) \leq (P,e)$.

The condition $\text{Br}_{P/Q}(f)e = e$ deserves further explanation. Since $\text{Br}_{P/Q}$ is surjective, $\text{Br}_{P/Q}(f)$ is a central idempotent of $kC_G(Q)$; it therefore decomposes as a sum of distinct block idempotents. The condition $\text{Br}_{P/Q}(f)e = e$ then means that $e$ appears in this decomposition.

Proposition 3. Given a Brauer pair $(P,e)$ and a subgroup $Q$ of $G$ with $Q \trianglelefteq P$, there is a unique block idempotent $f$ of $C_G(Q)$ such that $(Q,f) \trianglelefteq (P,e)$. If in addition $Q \trianglelefteq P$, then $(Q,f) \leq (P,e)$. In particular, every Brauer pair of $G$ is associated to a unique block idempotent. Additionally, for a given block $b$, any two maximal Brauer pairs associated to $b$ are conjugate.

Definition 4. Let $b$ be a block of $G$. If $(P,e)$ is a maximal Brauer pair associated to $b$, then $P$ is called a defect group of $b$. The fusion system on $(P,e)$ associated to $b$ is a category defined as follows. The objects are the subpairs of $(P,e)$; the maps from $(Q,f)$ to $(Q',f')$ are the injective group homomorphisms $\varphi : Q \to Q'$ with the property that there is a $g \in G$ such that $\varphi(x) = x^g$ and $(Q,f)^g = (Q',f')$.

Proposition 4. The fusion system on $(P,e)$ associated to $b$ is a saturated fusion system on $P$. Because maximal Brauer pairs associated to $b$ are conjugate, the fusion system does not depend on the choice of maximal Brauer pair.

For the definition and properties of saturated fusion systems, we refer to [1] Part I. For this article, we will only need the following two facts.

Lemma 5. Let $P$ be a $p$-group, and let $\mathcal{F}$ be a saturated fusion system on $P$. Then $\text{Aut}_P(P)$ is a Sylow $p$-subgroup of $\text{Aut}_\mathcal{F}(P)$.

Proof. By [1] Definition I.2.2, $P$ is $\mathcal{F}$-conjugate to a fully automatized subgroup of $P$, which must be $P$ itself, since no proper subgroup of $P$ is isomorphic to $P$. The required property then follows from the definition of fully automatized.

Theorem 6 (Alperin’s fusion theorem, weak form). Let $P$ be a $p$-group, and let $\mathcal{F}$ be a saturated fusion system on $P$. Then $\mathcal{F}$ is uniquely determined by the groups $\text{Aut}_\mathcal{F}(Q)$ where $Q$ runs over the centric subgroups of $P$.

Proof. By [1] Theorem I.3.5, $\mathcal{F}$ is determined by the groups $\text{Aut}_\mathcal{F}(Q)$ where $Q$ runs over the $\mathcal{F}$-essential subgroups of $P$, together with the group $\text{Aut}_\mathcal{F}(P)$. By [1] Proposition I.3.3(a), an $\mathcal{F}$-essential subgroup of $P$ is $\mathcal{F}$-centric in $P$, so by definition it is also centric in $P$. Additionally, $P$ itself is also centric in $P$. □
We will make use of the following simple observation regarding defect groups.

**Lemma 7.** Let $e$ be a block idempotent of $G$, and let $P$ be a defect group of $e$. Then $e$ contains an element $a \in G$ such that $P$ is a Sylow $p$-subgroup of $C_G(a)$.

**Proof.** Since $P$ is a defect group of $e$, we have $\text{Br}_P(e) \neq 0$, so we may choose an $a \in G$ with nonzero coefficient in $\text{Br}_P(e)$. Then $a$ also has nonzero coefficient in $e$, and we have $a \in C_G(P)$. This obviously implies $P \subseteq C_G(a)$. Now let $S$ be a $p$-subgroup of $G$ that properly contains $P$. Since $P$ is a defect group of $e$ and all defect groups of $e$ are conjugate, there can be no Brauer pairs at $S$ associated to $e$. Then $\text{Br}_S(e) = 0$, so we have $a \notin C_G(S)$. This implies $S \not\subseteq C_G(a)$, so $P$ is a maximal $p$-subgroup of $C_G(a)$. It is then a Sylow $p$-subgroup of $C_G(a)$. □

We also collect a few group-theoretic lemmas we will need later.

**Lemma 8.** Let $G$ be a finite group and $p$ a prime. For any element $g$ of $G$, there are unique elements $g_p$ and $g_{p'}$ of $G$, such that $g_p$ is a $p$-element, $g_{p'}$ is $p$-regular, and $g = g_pg_{p'}$.

**Proof.** Let $g$ have order $p^km$ where $p \nmid m$, and let $x$ and $y$ be integers such that $xm + yp^k = 1$. Set $g_p = g^{xm}$ and $g_{p'} = g^{yp^k}$; then $g = g_pg_{p'} = g_{p'}g_p$. Further, $x$ and $p^k$ are coprime, so $g_p$ has order $p^k$; similarly, $g_{p'}$ has order $m$ since $m$ and $y$ are coprime. This proves the existence of $g_p$ and $g_{p'}$.

For uniqueness, suppose that we are given $g_p$ and $g_{p'}$ with the desired properties. Then we have $g^{xm} = (g_pg_{p'})^{xm} = g_p^{xm}g_{p'}^{xm} = g_p^{xm}$. Since $g_p$ and $g_{p'}$ commute and $g_{p'}$ has order $m$, $g_p$ has order $p^k$. As $g_p$ has order $p^k$, we get $g_p^{xm} = g_p^{xm+yp^k} = g_p$. Thus we must have $g_p = g^{xm}$. We similarly deduce $g_{p'} = g^{yp^k}$. □

The elements $g_p$ and $g_{p'}$ may be called the $p$-part and the $p'$-part of $g$, respectively.

**Lemma 9.** Let $G$ be a finite group, and let $p$ be a prime. Then there exists a number $m \in \mathbb{N}$ such that $g'^m = g'$ for any $g \in G$.

**Proof.** Write $|G| = p^k \cdot m$ with $m$ not divisible by $p$. Choose $n$ such that $p^n$ is congruent to $1$ modulo $m$ and $n \geq k$. Then if $g$ is a $p$-element, its order is a divisor of $p^k$, so $g_p^{p^n}$ is the identity. If $g$ is a $p$-regular element, its order is a divisor of $m$, so $g_p^{p^n} = g$. For an arbitrary $g \in G$, we then have

$$g^{pm} = (g_p \cdot g_{p'})^{pm} = g_p^{pm} \cdot g_{p'}^{pm} = g_p.$$ □

We call a number $q = p^n$ a $p$-regular exponent of $G$ if it satisfies the conditions of Lemma 9.

**Lemma 10.** Let $G$ be a finite group, $S$ a Sylow $p$-subgroup of $G$, and $a$ a $p$-element of $G$. Then the group $\langle S^{a^k} \mid k \in \mathbb{Z} \rangle$ contains $a$.

**Proof.** Let $H = \langle S^{a^k} \mid k \in \mathbb{Z} \rangle$. Since $S \subseteq H \subseteq G$ and $S$ is a Sylow $p$-subgroup of $G$, $S$ is also a Sylow $p$-subgroup of $H$. Then $\text{Syl}_p(H)$ is a subset of $\text{Syl}_p(G)$.

Since conjugation by $a$ maps $S^{a^k}$ into $S^{a^{k+1}}$, $a$ normalizes $H$. Then the group $\langle a \rangle$, whose order is a power of $p$, acts on $H$ by conjugation. In particular, it acts on $\text{Syl}_p(H)$ by conjugation. Any orbit of this action has size divisible by $p$ unless
it consists of a single element, and since $|\text{Syl}_p(H)|$ is congruent to 1 modulo $p$, the action then has a fixed point. That is, there is a group $T \in \text{Syl}_p(H)$ such that $T^a = T$. This implies that $\langle T, a \rangle$ is a $p$-subgroup of $G$ containing $T$. Because $T$ is a Sylow $p$-subgroup of $G$, we must then have $T = \langle T, a \rangle$, and then $a \in T \subseteq H$. □

**Lemma 11.** Let $a$ and $b$ be elements of the group $G$, and let $n \in \mathbb{N}$. Then $\prod_{i=0}^{n-1} a^{b^{-i}} = (ab)^n \cdot b^{-n}$.

**Proof.** By induction. The statement is clear for $n = 1$, and for higher $n$, we have

$$\prod_{i=0}^{n} a^{b^{-i}} = \left(\prod_{i=0}^{n-1} a^{b^{-i}}\right) \cdot a^{b^{-n}} = (ab)^n \cdot b^{-n} \cdot a^{b^{-n}} = (ab)^n \cdot a \cdot b^{-n}$$

$$= (ab)^n \cdot ab \cdot b^{-(n+1)} = (ab)^{n+1} \cdot b^{-(n+1)}$$

□

**Lemma 12.** Let $G$ be a finite group that acts on a set $X$, let $H$ be a subgroup of $G$, and let $O_1, \ldots, O_k$ be the orbits of the action of $H$ on $X$. Then the action of $CG(H)$ on $X$ induces an action on the set $\{O_1, \ldots, O_k\}$ by $g(O_i) = \{g(x) \mid x \in O_i\}$.

**Proof.** Let $O$ be an orbit of $X$ under $H$, let $x \in O$, and let $g \in CG(H)$. Then for any $h \in H$, we have $h(g(x)) = g(h(x))$, so the orbit of $X$ containing $g(x)$ is precisely $g(O)$. □

### 3. Central idempotents of the group ring

In this section, we analyze the central idempotents of $RG$, where $R$ is a ring of characteristic $p$. Note that we do not assume that $R$ is a field, or even a commutative ring. It is easily seen that $Z(RG)$ consists of all elements of the form $\sum_{i=1}^{k} r_i \cdot \Sigma C_i$ where $r_i \in Z(R)$ and each $C_i$ is a conjugacy class of $G$.

**Lemma 13.** Let $a \in G$, $n \in \mathbb{N}$, and suppose that for every conjugacy class $C$ in $G$, the coefficient of $a$ in $(\Sigma C)^p^n$ is zero. Then $a$ has coefficient zero in all central idempotents of $RG$.

**Proof.** Let $e$ be a central idempotent of $RG$, and write $e = \sum_{i=1}^{k} r_i \cdot \Sigma C_i$ for some elements $r_i \in Z(R)$ and conjugacy classes $C_i$ of $G$. Since $e$ is idempotent, we have $e^p = e$. As $Z(RG)$ is commutative, we may apply Freshman’s Dream, and we find

$$e = e^p = \left(\sum_{i=1}^{k} r_i \cdot \Sigma C_i\right)^p = \sum_{i=1}^{k} r_i^p \cdot (\Sigma C_i)^p$$

By assumption, $a$ has coefficient zero on the right hand side, so it also has coefficient zero in $e$. □

When $q$ is a $p$-regular exponent, it turns out to be fairly simple to describe $(\Sigma C)^q$:

**Theorem 14.** Let $G$ be a finite group, $C$ a conjugacy class of $G$, $a$ an element of $G$, and $q$ a $p$-regular exponent of $G$. If $a$ is not $p$-regular, then the coefficient of $a$ in $(\Sigma C)^q$ is zero. If $a$ is $p$-regular, let $S$ be a Sylow $p$-subgroup of $CG(a)$, and let $H$ be the set of all $p$-elements of $CG(S)$. Then the coefficient of $a$ in $(\Sigma C)^q$ is equal to the number of elements $h$ of $H$ such that $ah \in C$. 
Proof. Write $C^q$ for the set of $q$-tuples of elements of $C$, and define a map $\pi : C^q \to G$ by $\pi(x_1, \ldots, x_q) = \prod_{i=1}^q x_i$. Then clearly $(\Sigma C)^q = \sum_{\alpha \in C} \pi(\alpha)$.

$G$ acts on $C^q$ by $(x_1, \ldots, x_q)^g = (x_1^g, \ldots, x_q^g)$, and this action clearly satisfies $\pi(\alpha^g) = \pi(\alpha)^g$. We also define an action of $C_q$ on $C^q$ as follows. Fix a generator $\sigma$ of $C_q$ and define $(x_1, x_2, \ldots, x_q)^{\sigma} = (x_2, \ldots, x_q, x_1)$. Then for $\alpha = (x_1, x_2, \ldots, x_q)$, we find $\pi(\alpha^{\sigma}) = \prod_{i=2}^q x_i \cdot x_1 = x_1^{-1} \prod_{i=1}^q x_i = \pi(\alpha \cdot x_1) = \pi(\alpha)$, so $\pi(\alpha^{\sigma})$ and $\pi(\alpha)$ are conjugate in $G$. Additionally, this action commutes with the action of $G$, so we have an action of $G \times C_q$ on $C^q$.

Consider for a fixed $\alpha \in C^q$ the set $M = \{\pi(\alpha^g) \mid g \in G \times C_q\}$ where we count elements with multiplicity. We partition $M$ into $q$ sets $M_0, M_1, \ldots, M_{q-1}$ by defining $M_i = \{\pi(\alpha^{(g,\sigma^i)}) \mid g \in G\}$. Since $\pi(\alpha^{(g,\sigma^i)}) = \pi(\alpha^{\sigma^i} \cdot \alpha)$, $M_i$ consists of $|C_G(\pi(\alpha^{\sigma^i}))|$ copies of $[\pi(\alpha^{\sigma^i})]$. As $\pi(\alpha)$ and $\pi(\alpha^{\sigma^i})$ are conjugate in $G$, we have $|C_G(\pi(\alpha^{\sigma^i}))| = |C_G(\pi(\alpha))|$ and $[\pi(\alpha^{\sigma^i})] = [\pi(\alpha)]$, so $M$ consists of $q \cdot |C_G(\pi(\alpha))|$ copies of $[\pi(\alpha)]$. Now when $g$ runs over $G \times C_q$, $\alpha^g$ runs over $\alpha^{G \times C_q}$ exactly $|C_G \times C_q(\alpha)|$ times, so we find

$$\sum_{\beta \in \alpha^{G \times C_q}} \pi(\beta) = \frac{q \cdot |C_G(\pi(\alpha))|}{|C_G \times C_q(\alpha)|} \cdot \Sigma[\pi(\alpha)]$$

Now consider the map $\varphi : C_G \times C_q(\alpha) \to C_q$ defined as the restriction of the projection $G \times C_q \to C_q$. The kernel of $\varphi$ is clearly $C_G(\alpha)$, and the image is a subgroup of $C_q$ of order $p^k$, say. Then $|C_G \times C_q(\alpha)| = p^k \cdot |C_G(\alpha)|$. Since $\alpha^g = \alpha$ implies $\pi(\alpha^g) = \pi(\alpha^g) = \pi(\alpha)$, we also have $C_G(\alpha) \subseteq C_G(\pi(\alpha))$. Inserting this in the above equation, we get

$$\sum_{\beta \in \alpha^{G \times C_q}} \pi(\beta) = \frac{q}{p^k} \cdot |C_G (\pi(\alpha)) : C_G(\alpha)| \cdot \Sigma[\pi(\alpha)]$$

If $\varphi$ is not surjective, $p^k$ is a smaller power of $p$ than $q$, and then $\frac{q}{p^k}$ is divisible by $p$. Since we are working in characteristic $p$, this immediately implies $\sum_{\beta \in \alpha^{G \times C_q}} \pi(\beta) = 0$. Furthermore, the question of whether $\varphi$ is surjective depends only on $\alpha^{G \times C_q}$ since if $\alpha$ and $\alpha'$ lie in the same orbit, then $C_G \times C_q(\alpha)$ and $C_G \times C_q(\alpha')$ are conjugate in $G \times C_q$. We define $X \subseteq C^q$ to consist of those $\alpha \in C^q$ for which $\varphi$ is surjective; this is then a union of $G \times C_q$-orbits, and we have

$$(\Sigma C)^q = \sum_{\alpha \in C} \pi(\alpha) = \sum_{\alpha \in X} \pi(\alpha)$$

Now let $\alpha \in X$ and write $\alpha = (x_1, x_2, \ldots, x_q)$. Since the projection of $C_G \times C_q(\alpha)$ onto $C_q$ is surjective, there exists an $h \in G$ such that $(h, \sigma) \in C_G \times C_q(\alpha)$. We then have $(x_1, x_2, \ldots, x_q) = (x_1, x_2, \ldots, x_q)^{(h, \sigma)} = (x_2^h, \ldots, x_q^h, x_1^h)$. This shows that $x_i^{h^i} = x_i$ for $1 \leq i \leq q - 1$; by induction, we then obtain $x_i = x_1^{h^{i-1}}$ for $1 \leq i \leq q$. This shows that $\alpha$ is determined by $x_1$ and $h$ alone. Further, the equation $x_i^{h^{-1}} = x_q$ implies $x_i^{h^{-1}} = x_1$, so that $x_1$ and $h^i$ commute.

We say that $(x, h)$ is a defining pair for $\alpha$ if $x$ and $h^i$ commute and $\alpha = (x, x^{h^{-1}}, x^{h^{-2}}, \ldots, x^{h^{q-1}})$. The above then shows that every $\alpha \in X$ has a defining pair.

Let $\alpha \in X$ and let $(x, h)$ be a defining pair for $\alpha$. Since $q$ is a $p$-regular exponent of $G$, we have $h^q = h_p$. Then $h_p$ commutes with $x$, and for any $k \in \mathbb{Z}$, we have $x^{h^{-k}} = x^{h^{-k}} x_p^{-k} = x^{h^{-k}}$. Additionally, $h_p^k$ commutes with $x$ since it is the identity.
element, so \((x, h_p)\) is a defining pair for \(\alpha\). Then every \(\alpha \in X\) has a defining pair \((x, h)\) where \(h\) is a \(p\)-element; for these pairs, the condition that \(x\) and \(h^q\) commute is redundant since \(h^q\) is the identity. From now on, it will be assumed that if \((x, h)\) is a defining pair, then \(h\) is a \(p\)-element.

Let again \(\alpha \in X\) and let \((x, h)\) be a defining pair for \(\alpha\). Using Lemma 11, we find

\[
\pi(\alpha) = \prod_{i=0}^{q-1} x^{h^{-i}} = (xh)^q \cdot h^{-q} = (xh)^q = (xh)_p'
\]

In particular, \(\pi(\alpha)\) is a \(p\)-regular element. This proves the first part of the theorem.

For the second part, let \(a\) be a \(p\)-regular element, and let \(X_a\) be the set of \(\alpha\) in \(X\) satisfying \(\pi(\alpha) = a\). Then the coefficient of \(a\) in \((\Sigma C)^q\) is equal to \(|X_a|\). We have already seen that \(G\) acts on \(X\) by conjugation; since \(\pi(a^q) = \pi(a)^q\), \(C_G(a)\) maps \(X_a\) to itself under this action. Then \(C_G(a)\) acts on \(X_a\) by conjugation. Let \(S\) be a Sylow \(p\)-subgroup of \(C_G(a)\), and let \(X'_a\) be the set of fixed points of \(S\) in \(X_a\). Since \(S\) is a \(p\)-group, any orbit of \(S\) in \(X_a\) has size divisible by \(p\) unless it consists of a single element. Then \(|X_a|\) and \(|X'_a|\) are congruent modulo \(p\), so \(|X'_a|\) is equal to the coefficient of \(a\) in \((\Sigma C)^q\).

Now let \(\alpha \in X'_a\), and let \((x, h)\) be a defining pair for \(\alpha\); by the above, we have \((xh)_p' = a\). Let \(s \in S\); since \(\alpha^s = \alpha, s\) commutes with \(x^{h^k}\) for every \(k \in \mathbb{Z}\). Conjugating by \(h^k\), we get that \(s^{h^k}\) commutes with \(x\) for every \(k \in \mathbb{Z}\).

Define \(b = (xh)_p'\); then \(a\) and \(b\) commute and we have \(xh = (xh)_p' \cdot (xh)_p = ab\). Let \(s \in S\); we prove by induction that \(s^{h^k} = s^{h^k}\) for every \(k \in \mathbb{N}\) (and hence for every \(k \in \mathbb{Z}\)). This is clear for \(k = 0\), and for higher \(k\), we get

\[
\begin{align*}
  s^{h^{k+1}} &= (s^{h^k})^h = (s^{h^k})^{xh} = (s^{h^k})^{ab} = (s^{h^k})^b = s^{h^{k+1}}
\end{align*}
\]

where we have used the fact that \(x\) commutes with \(s^{h^k}\), the induction hypothesis, the equation \(xh = ab\), and the fact that \(a\) commutes with both \(s\) and \(b\).

We now see that \(s^{h^k}\) commutes with \(x\) for every \(k \in \mathbb{Z}\). Then \((S^{h^k} \mid k \in \mathbb{Z}) \subseteq C_G(x)\). Since \(S\) is a Sylow \(p\)-subgroup of \(C_G(a)\) and \(b \in C_G(a)\), Lemma 10 provides that \(b \in \langle S^{h^k} \mid k \in \mathbb{Z} \rangle\). Then \(b\) commutes with \(x\); since it also commutes with \(a\) and \(b\), the equation \(xh = ab\) implies that \(b\) commutes with \(h\). Since both \(b\) and \(h\) are \(p\)-elements, \(hb^{-1}\) is a \(p\)-element, and for any \(k \in \mathbb{Z}\), we have

\[
x^{(hb^{-1})^{-k}} = x^{h^{-k}b^{-k}} = x^{h^{-k}b^{-k}} = x^{h^{-k}}
\]

Then \((x, hb^{-1})\) is a defining pair for \(\alpha\), and we have \(xhb^{-1} = ab^{-1} = a\). Additionally, \(hb^{-1}\) centralizes \(S\), since \(s^h = s^b\) for every \(s \in S\).

We conclude that any \(\alpha \in X'_a\) has a defining pair \((x, h)\) such that \(xh = a\) and \(h\) centralizes \(S\). But \(x\) is determined by \(a\), since \(x\) is the first element of the tuple, and \(h\) is then determined by the equation \(xh = a\). Thus a defining pair of this form is unique. Conversely, it is easily seen that any pair \((x, h)\) such that \(x \in \mathcal{C}\), \(h\) is a \(p\)-element that centralizes \(S\), and \(xh = a\), is a defining pair of an element of \(X'_a\). Then these pairs are in one-to-one correspondence with the elements of \(X'_a\), so in order to determine \(|X'_a|\), we may simply count the pairs instead. Since \(xh = a\), these pairs are determined by \(h\) alone, and an element \(h \in G\) defines a valid pair if \(h\) is a \(p\)-element, \(h\) centralizes \(S\), and \(x = ah^{-1} \in \mathcal{C}\). Since the inverse of a \(p\)-element is a \(p\)-element, we may replace \(h\) by \(h^{-1}\). This is exactly the second part of the theorem. □
As a corollary to the first part of this theorem, we get the following well-known result:

**Corollary 15.** Let $a \in G$ be an element that is not $p$-regular. Then $a$ has coefficient zero in all block idempotents of $G$.

**Theorem 16.** Let $a$ be a $p$-regular element of $G$, let $S$ be a Sylow $p$-subgroup of $C_G(a)$, and suppose that $C_G(S)$ contains a normal abelian $p$-subgroup $P$ such that $a$ does not centralize $P$. Then for any conjugacy class $C$ of $G$ and any $p$-regular exponent $q$ of $G$, the coefficient of $a$ in $(\Sigma C)^q$ is zero. In particular, $a$ has coefficient zero in all central idempotents of $RG$.

**Proof.** Fix a $p$-regular exponent $q$ and a conjugacy class $C$, and let $M$ be the set of $p$-elements $h$ of $C_G(S)$ such that $ah \in C$. By Theorem 14, the coefficient of $a$ in $(\Sigma C)^q$ is equal to $|M|$, so we wish to prove that $|M|$ is divisible by $p$.

For each $h \in M$, let $M_h$ be the set of elements $h'$ of $M$ such that $g^{h'} = g^h$ for all $g \in P$. The sets $M_h$ then form a partition of $M$, so it is sufficient to show that each $M_h$ has size divisible by $p$. Fix an $M_h$, and define a map $\varphi : P \to P$ by $\varphi(g) = (g^{-1})^{ah} \cdot g$. Since both $a$ and $h$ lie in $C_G(S)$ and $P$ is normal in $C_G(S)$, $\varphi(g)$ is in fact an element of $P$, and because $P$ is abelian, $\varphi$ is a homomorphism. Furthermore, $\varphi$ only depends on the set $M_h$ and not on the element $h$; if we choose another $h' \in M_h$, we have $(g^{-1})^{ah'} \cdot g = (g^{-1})^{ah} \cdot g$ since $(g^{-1})^a \in P$.

If $\varphi(P)$ is the trivial group, then $g^{ah} = g$ for each $g \in P$; that is, the maps $g \mapsto g^a$ and $g \mapsto g^h$ are inverses on $P$. Since $a$ is $p$-regular and does not centralize $P$, the map $g \mapsto g^a$ is a nonidentity $p$-regular permutation of the elements of $P$. Then its inverse should also be a nonidentity $p$-regular permutation, but this is impossible, since $h$ is a $p$-element. Thus $\varphi(P)$ is not the trivial group.

Since $P$ is a normal $p$-subgroup of $C_G(S)$ and $h$ is a $p$-element, $(h, P)$ is a $p$-group. In particular, $h\varphi(g)$ is a $p$-element of $C_G(S)$ for every $g \in P$. Additionally, for any $g \in P$ we have

$$(ah)^g = g^{-1} \cdot (ah) \cdot g = (ah) \cdot (g^{-1})^{ah}g = ah\varphi(g)$$

Since $ah \in C$, it follows that $ah\varphi(g) \in C$. Then $h\varphi(g) \in M$, and since $P$ is abelian, we in fact have $h\varphi(g) \in M_h$. We may therefore define a right action of $\varphi(G)$ on $M_h$ by setting $h\varphi(g) = h\varphi(g)$. This is clearly a free action, and $\varphi(G)$ is a nontrivial $p$-group, so it follows that $M_h$ has size divisible by $p$.

The last statement now follows from Lemma 13. \qed

4. The symmetric and alternating groups

In this section, we apply Theorem 16 to the symmetric and alternating groups. We begin with some well-known results about Sylow subgroups, centralizers, and conjugation in these groups.

**Lemma 17.** Let $P$ be a Sylow $p$-subgroup of $S_n$, and write $n = \sum_{i=0}^{k} d_i p^i$ with $0 \leq d_i \leq p - 1$ for each $i$. Then $P$ is isomorphic to the direct product $\prod_{i=1}^{k} (W_i)^{d_i}$, where $W_i$ is isomorphic to a Sylow $p$-subgroup of $S_{p^i}$. The action on $\{1, \ldots, n\}$ of each factor $W_i$ in $S$ has a unique nontrivial orbit which has size $p^i$, on which $W_i$ acts using its action as a subgroup of $S_{p^i}$, and two distinct factors of $S$ have disjoint nontrivial orbits.
Lemma 18. Let $W$ be a Sylow 2-subgroup of $S_n$. Then $W' = W \cap A_n$ is a Sylow 2-subgroup of $A_n$, $W'$ has index 2 in $W$, and if $n \geq 4$, $W$ and $W'$ have the same orbits in $\{1, \ldots, n\}$.

Proof. The first two parts are well-known. For the last part, let $i$ and $j$ be two elements of $\{1, \ldots, n\}$ that are in the same orbit under $W$. Pick two distinct elements $k$ and $l$ of $\{1, \ldots, n\}$ that are both different from both $i$ and $j$; this is possible since $n \geq 4$. Pick a $\sigma \in W$ such that $\sigma(i) = j$, and let $\tau$ be the transposition interchanging $k$ and $l$; then either $\sigma$ or $\tau \sigma$ is even, and $\sigma \tau(i) = \sigma(i) = j$. Then $i$ and $j$ lie in the same orbit of $W'$.

Lemma 19. Let $a \in S_n$ be an element that has cycle type $c_1^{m_1}c_2^{m_2} \cdots c_r^{m_r}$. Then $C_{S_n}(a)$ is isomorphic to $\prod_{i=1}^r C_{c_i} \wr S_{m_i}$. The embedding of each factor into $G$ may be described thus: Fix an $l$ with $1 \leq l \leq r$, let $M \subseteq \{1, \ldots, n\}$ be the set of points that are moved by the $m_l c_l$-cycles of $a$, and relabel the points in $M$ by elements of $\{1, \ldots, m_l\} \times \mathbb{Z}/c_l$ in such a way that $a((x,y)) = (x,y+1)$. Then $C_{C_l} \wr S_{m_l}$ acts trivially on the points of $\{1, \ldots, n\}$ outside $M$, the $m_l$ copies of $C_{c_l}$ are generated by the elements $a_1, \ldots, a_{m_l}$ defined by $a_z((x,y)) = (x,y+1)$ and $a_y((z,y)) = (z,y)$ for $z \neq x$, and the subgroup $S_{m_l}$ acts on $M$ by $\sigma((x,y)) = (\sigma(x),y)$.

Lemma 20. Let $x$ and $y$ be elements of $S_m$, and regard $S_m$ as a subgroup of $S_n$ for some $n \geq m$. If $x$ and $y$ are conjugate in $S_n$, then they are conjugate in $S_m$. In particular, if $C$ is a conjugacy class of $S_n$, then $C \cap S_m$ is either empty or a conjugacy class in $S_m$.

Proof. Let $x$ and $y$ be elements of $S_m$, and suppose that there is an $s \in S_n$ such that $x^s = y$. Let $i \in \{1, \ldots, m\}$ be any element such that $s(i) \notin \{1, \ldots, m\}$; then $s(i)$ is a fixed point of $x$, and we have $y(i) = x^s(i) = s^{-1}x^s(i) = s^{-1}s(i) = i$. Then $i$ is a fixed point of $y$; conversely, if $i \in \{1, \ldots, m\}$ is not a fixed point of $y$, then $s(i) \in \{1, \ldots, m\}$. We may therefore pick an element $t \in S_m$ such that $t(i) = s(i)$ if $i$ is not a fixed point of $y$. Then any $i \in \{1, \ldots, n\}$ is a fixed point of either $y$ or $s^{-1}t$, since if it is not a fixed point of $y$, we have $s^{-1}t(i) = s^{-1}s(i) = i$. Then $y$ and $s^{-1}t$ commute, and we get $y = y^{s^{-1}t} = x^{s^{-1}t} = x^t$, so $x$ and $y$ are conjugate in $S_m$. The second part is immediate.

Lemma 21. Let $x$ and $y$ be elements of $A_m$, and regard $A_m$ as a subgroup of $A_n$ for some $n \geq m$. If $x$ and $y$ are conjugate in $A_n$, then either they are conjugate in $A_m$, or there exists an odd element $g \in S_m$ such that $x = y^g$. In particular, if $C$ is a conjugacy class of $A_n$, then $C \cap A_m$ is either empty, a conjugacy class of $A_m$, or a union of two conjugacy classes $C_1$ and $C_2$ of $A_m$ such that $C_1^g = C_2$ for every odd element $g \in S_m$.

Proof. Suppose $x$ and $y$ are elements of $A_m$ that are conjugate in $A_n$; then by Lemma 20, they are conjugate in $S_m$. This is the first part. For the second part, we note that $S_m$ acts transitively on $C \cap A_m$, and since $A_m$ is normal in $S_m$, $S_m$ acts on the set of $A_m$-orbits in $C \cap A_m$ (that is, the set of $A_m$-conjugacy classes contained in $C \cap A_m$). $A_m$ acts trivially on this set, while $S_m$ acts transitively, so $S_m/A_m$ acts transitively on this set. Since $S_m/A_m$ has order 2, there are at most two $A_m$-conjugacy classes in $C \cap A_m$, and if there are two, they are interchanged by the nontrivial element of $S_m/A_m$. \qed
Theorem 22. Let $G = S_n$ or $G = A_n$, and let $a$ be a $p$-regular element of $G$ containing at least $p$ $c$-cycles for some $c > 1$. In the case $G = A_n$ and $p = 2$, assume further that $a$ contains at least 4 $c$-cycles, or there exists a number $d \neq c$ such that $a$ contains at least $2d$ $d$-cycles. Then $a$ does not appear in any central idempotent of $RG$.

Proof. This will be proven by showing that $a$ satisfies the condition in Theorem. We first consider the case $G = S_n$. Let $a$ have cycle type $c_1^{m_1}c_2^{m_2} \cdots c_r^{m_r}$, and write $C_{S_n}(a)$ as $\prod_{i=1}^r C_{c_i} \wr S_{m_i}$, as in Lemma 19. Choose a $k$ such that $c_k > 1$ and $m_k \geq p$; this is possible by assumption. Since $a$ is $p$-regular, $c_k$ is not divisible by $p$, so the subgroup $S_{m_k}$ of $C_{c_k} \wr S_{m_k}$ contains a Sylow $p$-subgroup $T$ of $C_{c_k} / S_{m_k}$. Let $S$ be a Sylow $p$-subgroup of $C_{S_n}(a)$ containing $T$; since $C_{c_k} \wr S_{m_k}$ is a direct factor of $C_{S_n}(a)$, $T$ is then a direct factor of $S$. By the above, there is a complement of $C_{c_k} \wr S_{m_k}$ in $C_{S_n}(a)$ that acts trivially on $M$, so there is a complement of $T$ in $S$ that acts trivially on $M$.

Let $W$ be a direct factor of $T$ in the decomposition given in Lemma 17; such a factor exists because $m_k \geq p$. Under the usual action of $S_{m_k}$ on $\{1, \ldots, m_k\}$, there is then a subset $N$ of $\{1, \ldots, m_k\}$ with $|N|$ a nonzero power of $p$ such that $W$ acts transitively on $N$ and fixes all other points. Additionally, there is a complement of $W$ in $T$ that acts trivially on $N$. Transferring this to the action on $\{1, \ldots, n\}$ by using the relabeling given in Lemma 19 we see that $W$ acts on $N \times \mathbb{Z}/c_k$ with orbits $N \times \{1\}, N \times \{2\}, \ldots, N \times \{c_k\}$, and it acts trivially on all other points of $\{1, \ldots, n\}$. Combining this with the previous paragraph, we see that $W$ is a direct factor of $S$, and there is a complement of $W$ in $S$ that acts trivially on $N \times \mathbb{Z}/c_k$.

The group $W$ acts on $N$, and hence, so does $Z(W)$. We use this action to define $c_k$ different actions (labelled by the elements of $\mathbb{Z}/c_k$) of $Z(W)$ on $\{1, \ldots, n\}$. Still maintaining the above relabeling, we first specify that $Z(W)$ in all cases acts trivially on points outside $N \times \mathbb{Z}/c_k$. On $N \times \mathbb{Z}/c_k$, the $i$'th action is defined by $\sigma((x, i)) = (\sigma(x), i)$ and $\sigma((x, j)) = (x, j)$ for $j \neq i$. We now define the $c_k$ subgroups $Z_1, \ldots, Z_{c_k}$ of $G = S_n$ by letting $Z_i$ be the image of the natural map from $Z(W)$ into $S_n$ constructed from the $i$'th action. Then $Z_i$ acts transitively on $N \times \{i\}$ and trivially on all other points of $\{1, \ldots, n\}$. Since an element of $Z(W)$ is completely defined by its action on $N$, each $Z_i$ is isomorphic to $Z(W)$.

Since $N \times \{i\}$ and $N \times \{j\}$ are disjoint for $i \neq j$, $Z_i$ and $Z_j$ centralize each other and have trivial intersection. Define $P = \langle Z_i \mid 1 \leq i \leq c_k \rangle$; $P$ is then isomorphic to the direct product $\prod_{i=1}^{c_k} Z_i$. Furthermore, each $Z_i$ is an nontrivial abelian $p$-group, since it is isomorphic to $Z(W)$. Then $P$ is a nontrivial abelian $p$-group.

It is straightforward to verify that $W$, considered as a subgroup of $S_n$, centralizes the groups $Z_1, \ldots, Z_{c_k}$. Additionally, $W$ has a complement in $S$ that acts trivially on $N \times \mathbb{Z}/c_k$, and since the $Z_i$ act trivially outside $N \times \mathbb{Z}/c_k$, this complement also centralizes the $Z_i$. Then $S$ centralizes the $Z_i$, so we have $Z_i \subseteq C_{S_n}(S)$. This implies $P \subseteq C_{S_n}(S)$.

Now let $g \in C_{S_n}(S)$. Then $g$ centralizes $W$, so by Lemma 12 $g$ permutes the sets $N \times \{1\}, N \times \{2\}, \ldots, N \times \{c_k\}$. Let $i \in \mathbb{Z}/c_k$ be arbitrary, and choose $j \in \mathbb{Z}/c_k$ such that $g^{-1}(N \times \{i\}) = N \times \{j\}$; then $Z_i^g$ acts trivially on all points outside $N \times \{j\}$. Let $z \in Z_i$; because of the way we have defined $Z_i$, there exists an element $\tilde{z} \in Z(W)$ (now considered as a subgroup of $S_n$) such that $z((x, i)) = \tilde{z}((x, i))$ for all $x \in N$. Then we also have $z^g((x, j)) = \tilde{z}^g((x, j))$ for all $x \in N$, and since $g$ centralizes $W$, we have $\tilde{z}^g = \tilde{z}$. Then $z^g((x, j)) = \tilde{z}(x, j)$, and $z^g$ fixes all points outside $N \times \{j\}$.
Thus $z^9$ is an element of $Z_j$, so $Z_j^9 = Z_j$. Then $g$ permutes the groups $Z_1, \ldots, Z_{c_k}$, so $g$ normalizes $P$. This implies that $P$ is normal in $C_{S_n}(S)$. 

Finally, we note that since the action of $a$ on $N \times Z/c_k$ is given by $a((x, y)) = (x, y + 1)$, we have $Z_i^a = Z_{i-1}$. As $c_k > 1$, this implies that $a$ does not centralize $P$. Then $P$ satisfies all the conditions of Theorem 16.

This completes the proof for $G = S_n$. In the case $p > 2$, we see that every $p$-element of $S_n$ is contained in $A_n$; in particular, the groups $S$ and $P$ constructed above are contained in $A_n$. Further, $S$ is a Sylow $p$-subgroup of $C_{A_n}(a)$ because it is a Sylow $p$-subgroup of $C_{S_n}(a)$, and $P$ is normal in $C_{A_n}(S)$ because it is normal in $C_{S_n}(S)$. We may then apply Theorem 16 to prove the case $G = A_n$ and $p > 2$.

For the case $G = A_n$ and $p = 2$, we will need to modify the proof. Let again $a$ have cycle type $c_1^{m_1}c_2^{m_2}\cdots c_r^{m_r}$, and write $C_{S_n}(a) \cong \prod_{i=1}^{r}C_{c_i} \triangleleft S_{m_i}$. For each $i$, let $T_i$ be a Sylow 2-subgroup of $S_{m_i}$. Since $a$ is 2-regular, each $c_i$ is odd, so $T_i$ is also a Sylow 2-subgroup of $C_{c_i} \triangleleft S_{m_i}$, and $S = \prod_{i=1}^{r}T_i$ is a Sylow 2-subgroup of $C_{S_n}(a)$. The group $S' = S \cap A_n$ is then a Sylow 2-subgroup of $G(a)$.

Suppose that we can show that $C_{A_n}(S') = C_{A_n}(S)$. We can then take a $k$ with $m_k \geq 2$ and use it to construct the group $P \cong \prod_{i=1}^{r}Z_i$ as above, although it will only be a subgroup of $S_n$, not necessarily of $A_n$. If we set $P' = P \cap A_n$, we have $P' \triangleleft C_{A_n}(S)$, since $P \triangleleft C_{S_n}(S)$. As $C_{A_n}(S') = C_{A_n}(S)$, we get $P' \triangleleft C_{A_n}(S')$. Additionally, $Z_i \times Z_{i+1}$ is a subgroup of $P$ of order at most 4, and since $P'$ has index at most 2 in $P$, $P' \cap (Z_i \times Z_{i+1})$ is nontrivial. As $(P' \cap (Z_i \times Z_{i+1}))'' = P' \cap (Z_{i-1} \times Z_i)$ and $c_k \geq 3$, we find that $a$ does not centralize $P'$. We can then apply Theorem 16 completing the proof. This method can be used to cover most subcases, but a few will need to be treated separately.

We will need a small observation regarding the parity of elements in $T_i$: if $x \in T_i$, then $x$ is an odd element of $S_n$ if and only if it is an odd element of $S_{m_i}$. To see this, note that the action of $x$ on $\{1, \ldots, n\}$ is given by identifying $\{1, \ldots, m_i\}$ with $c_i$ pairwise disjoint subsets of $\{1, \ldots, n\}$ and letting $x$ act on each of these using its action on $\{1, \ldots, m_i\}$. Then $x$ in $S_n$ is the product of $c_i$ elements, each of which has the same parity as $x$ in $S_{m_i}$. As $c_i$ is odd, the result follows.

Assume first that $a$ contains 1 $c$-cycles for some $c > 1$, so there is a $k$ such that $m_k \geq 4$ and $c_k > 1$. By Lemma 17, we may write $T_k$ as a direct product $\prod_{j=1}^{r}W_j$ where each $W_j$ is a Sylow 2-subgroup of $S_{2n_j}$ for some $n_j$, and no two $n_j$ are equal. Assume without loss of generality that $W_1$ has the largest order among the $W_j$; in particular, the order of $W_1$ is at least 4, and the nontrivial orbits of the action of $W_1$ on $\{1, \ldots, n\}$ have size at least 4. We set $W = W_1$ and $W' = W \cap A_n$, by Lemma 16 the above observation regarding parity of elements, $W'$ is then a proper subgroup of $W$. Since $W$ has order at least 4, we further have that $W'$ has the same orbits as $W$ in $\{1, \ldots, n\}$.

Consider the case that there is an $l$ distinct from $k$ such that $m_l \geq 2$, so we may pick an element $u \in T_l$ that is a transposition when considered as an element of $S_{m_l}$. Then $u$ is odd when considered as an element of $S_n$, so $S'$ contains the elements $wu$ where $w \in W$ but $w \notin W'$. Since $u$ commutes with every element of $W$, we may define the subgroup $W_u$ of $S'$ to consist of these elements together with $W'$. Let $x \in C_{A_n}(S')$; then $x$ centralizes $W_u$, so it must permute the orbits of $W_u$ in $\{1, \ldots, n\}$. The orbits of $W_u$ consist of a number of orbits of size 2 from the action of $u$, and a number of orbits from the action of $W$, plus some fixed points. Since the orbits from $W$ have size at least 4, they are permuted separately from

\[\text{(Theorem 16)}\]
the orbits from \( u \). Then \( x \) permutes the nontrivial orbits of the action of \( u \) alone, so it follows that \( x \) actually commutes with \( u \). As \( x \) then centralizes both \( S' \) and \( u \), it must centralize \( S \). Then we have \( C_{A_n}(S') = C_{A_n}(S) \).

Now consider the case that \( m_i < 2 \) for \( i \neq k \) but \( t > 1 \). Then \( S' \) contains those elements \((w_1, w_2) \in W \times W_2\) for which \( w_1 \) and \( w_2 \) are both even or both odd. Similarly to the previous case, we note that the orbits of \( S' \cap (W \times W_2) \) consist of a number of orbits from \( W \) and a number of orbits from \( W_2 \), and the orbits from \( W \) do not have the same size as the orbits from \( W_2 \). As before, we then conclude that an element that centralizes this group in fact centralizes \( W \times W_2 \), from which it follows that \( C_{A_n}(S') = C_{A_n}(S) \).

There remains the case that \( m_i < 2 \) for \( i \neq k \) and \( t = 1 \). Here \( S = W \) and \( S' = W' \), and \( W' \) is nontrivial because \( W \) has order at least 4. Then \( Z(W') \) is nontrivial, and we use this group to construct a normal abelian \( p \)-subgroup of \( C_{A_n}(S') \) satisfying the conditions of Theorem \( 10 \) in the same manner as in the proof of the case \( G = S_n \).

Now consider the possibility that \( m_i < 4 \) for all \( i \), but there exists two distinct numbers \( k \) and \( l \) such that \( m_k \geq 2 \) and \( m_l \geq 2 \). In this case, each nontrivial \( T_i \) is cyclic of order 2, and the nontrivial element is a product of \( c_i \) disjoint transpositions. Suppose first that there exists a third number \( m \), distinct from both \( k \) and \( l \), such that \( T_m \) has order 2. Let \( s_k, s_l, \) and \( s_m \) be the nontrivial elements of \( T_k, T_l, \) and \( T_m \), respectively. These are all odd, being the product of an odd number of transpositions, so \( s_k s_l \) and \( s_k s_m \) are even and therefore elements of \( S' \). Now let \( x \in C_{A_n}(S') \). The orbits of \( s_k s_l \) consist of \( c_k \) orbits of size 2 from \( s_k \) and \( c_l \) orbits of size 2 from \( s_l \), plus fixed points, and \( x \) must permute the orbits of size 2. The orbits from \( s_k \) are also nontrivial orbits of \( s_k s_m \), while the orbits from \( s_l \) are fixed points of \( s_k s_m \). As \( x \) also centralizes \( s_k s_m \), \( x \) then cannot map an orbit from \( s_k \) into an orbit from \( s_l \). Then \( x \) permutes the orbits from \( s_k \), which implies that \( x \) centralizes \( s_k \). Then we have \( C_G(S') = C_G(S) \).

Finally we have the possibility that there is no such third number. Then \( S \) is isomorphic to \( T_k \times T_l \cong C_2 \times C_2 \), and \( S' \) is cyclic of order 2, with the nontrivial element being a product of \( c_k + c_l \) disjoint transpositions. Then we may write \( C_{S_n}(S') \cong S_n - e_i \times (C_2 \wr S_{c_k + c_l}) \), and \( C_{A_n}(S') = C_{S_n}(S') \cap A_n \). Clearly \( C_{S_n}(S') \) has the normal subgroup \( C_2 \wr (c_k + c_l) \), and setting \( P = C_2 \wr (2c_k + c_l) \cap A_n \), we obtain a normal abelian \( 2 \)-subgroup of \( C_{A_n}(S') \). As either \( c_k \geq 3 \) or \( c_l \geq 3 \), we see as in the first case above that \( a \) does not centralize \( P \). □

**Lemma 23.** Let \( G = S_n \) or \( G = A_n \), and let \( e \) be a block idempotent of \( G \). Then there exists a subset \( M \) of \( \{1, \ldots, n\} \) such that a Sylow \( p \)-subgroup of \( S_M \) (respectively \( A_M \)) is a defect group of \( e \).

**Proof.** By Lemma \( 6 \) we may choose an element \( a \) with nonzero coefficient in \( e \) such that the Sylow \( p \)-subgroups of \( C_G(a) \) are defect groups of \( e \); write the cycle type of \( a \) as \( 1^{m_1} c_1^{m_1} \cdots c_r^{m_r} \). By Theorem \( 22 \) we then have \( m_i < p \) for all \( i \) (except possibly in the case \( G = A_n \) and \( p = 2 \)). Write \( C_{S_n}(a) \cong S_m \times \prod_{i=1}^r C_{c_i} \times S_{m_i} \); then the factor \( \prod_{i=1}^r C_{c_i} \times S_{m_i} \) has order not divisible by \( p \), so the Sylow \( p \)-subgroups of \( C_{S_n}(a) \) are simply the Sylow \( p \)-subgroups of \( S_m \). It follows that a Sylow \( p \)-subgroup of \( S_m \) is a defect group of \( e \). Since \( S_m \) embeds into \( S_n \) as the subgroup consisting of permutations of the fixed points of \( a \), we choose \( M \) to be the set of fixed points
of α. This obviously works for $G = S_n$, and it also works for $G = A_n$ because the Sylow p-subgroups of $C_{A_n}(α)$ are the Sylow p-subgroups of $A_n ∩ S_M = A_M$.

In the exceptional case $G = A_n$ and $p = 2$, it is possible that there is a $k$ such that $m_k ≥ 2$. If this happens, we must have $m_i < 2$ for all $i ≠ k$, $m_k < 4$, and $m < 2$. Then $C_{S_n}(α)$ has order divisible by 2 but not by 4, and the nontrivial element of a Sylow 2-subgroup is a product of $c_k$ transpositions, which is odd because $c_k$ is odd. This implies that $C_{A_n}(α) = A_n ∩ C_{S_n}(α)$ contains no nontrivial elements of order 2, so its Sylow 2-subgroup is trivial. Then we may choose $M$ to be empty.

This is the restriction we need on the possible defect groups of blocks of symmetric groups. To determine the possible fusion systems, we will use Alperin’s fusion theorem. First, we will determine what the centric subgroups of the defect group look like.

**Lemma 24.** Let $P$ be a Sylow p-subgroup of $S_{pm}$ or $A_{pm}$ and let $Q$ be a centric subgroup of $P$. Then $Q$ has no fixed points in $\{1, \ldots, pm\}$ and $C_{S_{pm}}(Q)$ is a p-group.

**Proof.** We first consider the Sylow p-subgroups of $S_{pm}$. Since Sylow p-subgroups are conjugate, it is sufficient to prove the claim for a particular Sylow p-subgroup.

Let $z$ be an element of $S_{pm}$ consisting of $m$ p-cycles, and let $C = C_{S_{pm}}(z)$. By Lemma [19] $C$ is then isomorphic to $C_p \wr S_m \cong (C_p)^m ⋊ S_m$. We let $a_1, \ldots, a_m$ be the generators of the $m$ copies of $C_p$, so that $z = \prod_{i=1}^m a_i$, and we have the natural homomorphism $ϕ : (C_p)^m ⋊ S_m \to S_m$ whose kernel is $(C_p)^m$.

The order of $C$ is $p^m m! = \prod_{i=1}^m p_i$, which is equal to the product of those factors in $(pm)!$ that are divisible by $p$. Then $|C|$ is divisible by the same power of $p$ as $|S_{pm}|$, so a Sylow p-subgroup of $C$ is also a Sylow p-subgroup of $S_{pm}$. We pick one such group $P = (C_p)^m ⋊ T$, where $T$ is a Sylow p-subgroup of $S_m$. Clearly, $z$ is an element of $Z(P)$.

Let $Q$ be a centric subgroup of $P$. Then $Q$ contains $Z(P)$, so $z$ is an element of $Q$. Since $z$ has no fixed points, neither does $Q$.

Furthermore, $C_{S_{pm}}(Q)$ is contained in $C$. Suppose that $C_{S_{pm}}(Q)$ is not a p-group; we let $x$ be a nontrivial element of $C_{S_{pm}}(Q)$ of order not divisible by $p$. Since $x \in C$, we may write $x = bs$ with $b \in (C_p)^m$ and $s \in S_m$. Then $s$ is a nontrivial element of $S_m$ with order not divisible by $p$, since it is the image of $a$ under $ϕ$. Pick a $k \in \{1, \ldots, m\}$ that is not a fixed point of $s$, and let $K$ be the orbit of $k$ under $ϕ(Q)$.

If $s(k)$ is an element of $K$, there is a $t \in ϕ(Q)$ such that $t(k) = s(k)$. But since $a$ centralizes $Q$, $s = ϕ(x)$ must centralize $ϕ(Q)$, so $s$ and $t$ commute. By induction we then get $t^n(k) = t(t^{n-1}(k)) = t(s^{n-1}(k)) = s^{n-1}(t(k)) = s^n(k)$ for all $n \in N$. That is, the cycles in $s$ and $t$ containing $k$ are identical. But this is impossible since $s$ has order not divisible by $p$ and does not fix $k$, and $t$ is an element of a p-group. So $s(k) \not\in K$.

Now let $a = \prod_{i \in K} a_i$. By the definition of $K$, $a$ commutes with every element of $ϕ(Q)$; it also commutes with every element of $(C_p)^m$, since this group is abelian. Since $Q$ is contained in $(C_p)^m ⋊ ϕ(Q)$, $a$ then centralizes $Q$. As $a \in P$ and $Q$ is centric in $P$, it follows that $a \in Q$. This implies that $a$ commutes with $x = bs$; since it also commutes with $b$, it commutes with $s$. But this implies $s(K) = K$, which is a contradiction since $s(k) \not\in K$. 

□
This covers $S_{pm}$. When $p > 2$, $S_{pm}$ and $A_{pm}$ have the same Sylow $p$-subgroups, so the only case that remains is the Sylow 2-subgroups of $A_{2m}$.

Consider first the case that $m$ is even, so we may write $2m = 4l$ for some $l$. Let $z$ be the product of $2l$ disjoint transpositions; this is an even element, so it lies in $A_{4l}$. Set $C' = C_{S_{4l}}(z)$; then $C'$ is isomorphic to $(C_2 \wr S_{2l}) \cong (C_2)^{2l} \rtimes S_{2l}$, and if $T$ is a Sylow 2-subgroup of $S_{2l}$, then $P' = (C_2)^{2l} \rtimes T$ is a Sylow 2-subgroup of both $C'$ and $S_{4l}$. Set $C = C' \cap A_{4l}$ and $P = P' \cap A_{4l}$; then $C = C_{A_{4l}}(z)$ and $P$ is a Sylow 2-subgroup of $A_{4l}$ such that $z \in Z(P)$. We have the natural homomorphism $\varphi : (C_2)^{2l} \rtimes S_{2l} \to S_{2l}$.

Let $Q$ be a centric subgroup of $P$; then $Q$ contains $z$, so $C_{S_{4l}}(Q)$ is contained in $C'$. Suppose that $C_{S_{4l}}(Q)$ is not a 2-group, and let $x$ be a nontrivial element of odd order of $C_{S_{4l}}(Q)$. As in the case of $S_{pm}$, we write $x = bs$ with $b \in (C_2)^{2l}$ and $s \in S_{2l}$, and we pick a $k \in \{1, \ldots, 2l\}$ that is not a fixed point of $s$. Assume for the moment that we can pick $k$ such that it is not a fixed point of $\varphi(Q)$. We then let $K$ be the orbit of $k$ under $\varphi(Q)$, and let $a = \prod_{i \in K} a_i$ where $a_1, \ldots, a_{2l}$ are the generators of $(C_2)^{2l}$. The size of $K$ is a power of 2 different from 1, so it is even. Then $a$ is an element of $A_{2l}$, and by the same arguments as in the case of $S_{pm}$, we conclude both that $s(K) = K$ and $s(k) \not\in K$, which is a contradiction.

There remains the possibility that every element in $\{1, \ldots, 2l\}$ that is not a fixed point of $s$ is a fixed point of $\varphi(Q)$. We then let $K$ be the set of points that are not fixed points of $s$, and we write $(C_2)^{K}$ for the subgroup of $(C_2)^{2l}$ generated by the elements $a_i$ as $i$ runs over $K$. Then $R = (C_2)^K \cap A_{2l}$ is a nontrivial subgroup of $P$; by the definition of $K$, any element of $\varphi(Q)$ centralizes this group. Then every element of $R$ centralizes both $(C_2)^{4l}$ and $\varphi(Q)$, so they all centralize $Q$. As $Q$ is central in $P$, we get $R \subseteq Q$, so $x$ centralizes $R$. Since $b \in (C_2)^{3l}$, $b$ also centralizes $R$, and hence, so does $s$. But if we take any $k \in K$, $k$ is a part of a cycle in $s$ of length at least 3, so $a_k a_{s(k)}$ is an element of $R$ that does not commute with $s$. This is a contradiction.

We now consider the case that $m$ is odd, so we may write $2m = 4l + 2$. Let $z$ be a product of $2l$ transpositions, and set $C' = C_{S_{4l+2}}(z)$; then $C'$ is isomorphic to $(C_2 \wr S_{2l}) \times C_2$. We let $w$ be the nontrivial element of the $C_2$ factor, so that $w$ is the transposition interchanging the two fixed points of $z$. Let $P'$ be a Sylow 2-subgroup of $C'$; comparing orders, we find that $P'$ is also a Sylow 2-subgroup of $S_{4l+2}$. Set $C = C' \cap A_{4l+2}$ and $P = P' \cap A_{4l+2}$; then $P$ is a Sylow 2-subgroup of both $C$ and $A_{4l+2}$. Since $z$ is a central element of $C$, any centric subgroup of $P$ contains $z$, so $C_{A_{4l+2}}(Q)$ is contained in $C$. Since $C_{A_{4l+2}}(Q)$ has index at most 2 in $C_{S_{4l+2}}(Q)$, it is sufficient to prove that $C_{A_{4l+2}}(Q)$ is a 2-group.

Now consider the projection homomorphism $\pi : C' \to C_2 \wr S_{2l}$ with kernel $C_2 = \langle w \rangle$. For any element $x \in C'$, exactly one of $x$ and $xw$ is even, since $w$ is odd. This implies that when we restrict $\pi$ to $C$, we obtain an isomorphism $C \cong C_2 \wr S_{2l}$. Then $\pi(P)$ is a Sylow 2-subgroup of $C_2 \wr S_{2l}$, and it is enough to prove that the centralizer in $C_2 \wr S_{2l}$ of any centric subgroup of $\pi(P)$ is a 2-group. But this was shown as part of the case $G = S_{2l}$ and $p = 2$.

We can now determine the number of Brauer pairs at a centric subgroup of the defect group. This will allow us to determine the automorphism groups of these pairs, which is sufficient to determine the block fusion system.
Lemma 25. Let $e$ be a block idempotent of $S_n$, let $P$ be a defect group of $e$, and let $Q$ be a centric subgroup of $P$. Then there exists a unique Brauer pair at $Q$ associated to $e$.

Proof. Let $N \subseteq \{1, \ldots, n\}$ be the set of fixed points of $P$ and let $M$ be its complement. Then $P$ is a Sylow $p$-subgroup of $S_M$ and $|M|$ is divisible by $p$, so by Lemma 24, $C_{S_n}(Q)$ is isomorphic to $T \times S_N$ where $T$ is a $p$-group. Then every $p$-regular element of $C_{S_n}(Q)$ is contained in $S_N$, so we may identify the central idempotents of $kC_{S_n}(Q)$ with the central idempotents of $kS_N$.

Since the map $Br_Q : (kS_n)^Q \rightarrow kC_{S_n}(Q)$ is surjective, it restricts to a map $Br_Q : Z(kS_n) \rightarrow Z(kC_{S_n}(Q))$. By Lemma 26, any element of $Z(kS_n)$ lies in the image of this restricted map. This includes the primitive idempotents of $Z(kC_{S_n}(Q))$, so these can all be lifted to a primitive idempotent of $kS_n$. Conversely, $Br_Q$ then maps any primitive idempotent of $Z(kS_n)$ to either zero or a primitive idempotent. In particular, $Br_Q(e)$ is a primitive idempotent of $Z(kC_{S_n}(Q))$, so there is a unique Brauer pair at $Q$ associated to $e$.

Theorem 26. Let $e$ be a block idempotent of $S_n$, let $P$ be a defect group of $e$, and let $M$ be a subset of $\{1, \ldots, n\}$ such that $P$ is a Sylow $p$-subgroup of $S_M$. Then the block fusion system on $P$ is equal to $\mathcal{F}_P(S_M)$.

Proof. Let $\mathcal{F}$ be the block fusion system on $P$. By Alperin’s theorem, it is sufficient to prove that $\text{Aut}_P(Q) = \text{Aut}_{S_n}(Q)$ for each centric subgroup $Q$ of $P$. For each of these groups, there is a unique Brauer pair $(Q, e_Q)$ associated to $e$, so all elements of $S_n$ that normalize $Q$ also normalize $(Q, e_Q)$. Then we have $\text{Aut}_P(Q) = \text{Aut}_{S_n}(Q)$.

Since $S_M \subseteq S_n$, we have $\text{Aut}_{S_M}(Q) \subseteq \text{Aut}_{S_n}(Q)$, so we need only prove the reverse inclusion. Let $\varphi \in \text{Aut}_{S_n}(Q)$ be an automorphism represented by the element $g \in S_n$. Let $N$ be the set of fixed points of $Q$; then $M \cup N = \{1, \ldots, n\}$, although the two sets need not be disjoint. Since $g$ normalizes $Q$, $g$ maps $N$ to itself, so we may consider the element $h \in S_n$ defined by $h(x) = g(x)$ for $x \in N$ and $h(x) = x$ otherwise. Since $h$ only permutes the fixed points of $Q$, it must centralize $Q$. That is, $h$ represents the identity automorphism of $Q$. Then $gh^{-1}$ represents $\varphi$, and we have $gh^{-1}(x) = x$ for $x \in N$. This implies that $gh^{-1}$ is an element of $S_M$, so it follows that $\text{Aut}_{S_n}(Q) \subseteq \text{Aut}_{S_M}(Q)$.

Lemma 27. Let $e$ be a block idempotent of $A_n$, let $P$ be a defect group of $e$, and let $Q$ be a centric subgroup of $P$. Then there exist either one or two Brauer pairs at $Q$ associated to $e$, this number being the same for all $Q$. If there are two, then conjugation by $g \in N_{A_n}(Q)$ interchanges the two pairs if and only if the action of $g$ on the fixed points of $P$ is an odd permutation.

Proof. Let $N$ be the set of fixed points of $P$. By Lemma 24, $C_{S_n}(Q)$ is the direct product of a $p$-group with $S_N$, so every $p$-regular element of $C_{S_n}(Q)$ is contained in $S_N$. Then all $p$-regular elements of $C_{A_n}(Q)$ are contained in $A_n \cap S_N = A_N$, so we may identify the central idempotents of $kC_{A_n}(Q)$ with the central idempotents of $kA_N$. Since $N$ does not depend on the choice of $Q$, this shows that there is the same number of Brauer pairs at $Q$ associated to $e$ for all possible $Q$. We also note that since the map $Br_Q : (kA_n)^Q \rightarrow kC_{A_n}(Q)$ is surjective, it restricts to a map $Br_Q : Z(kA_n) \rightarrow Z(kC_{A_n}(Q))$.

Since $A_N$ is normal in $S_N$, $S_N$ acts on $kA_N$ by conjugation, and $A_N$ obviously acts trivially on $Z(kA_N)$ under this action. Hence $S_N/A_N$ acts on $Z(kA_N)$; let $\sigma$
be the nontrivial element of $S_N/A_N$. Let $x$ be any element of $Z(kA_N)$. Then $x$ is
also an element of $Z(kC_{A_n}(Q))$, and by Lemma 21 $x$ lies in $Br_Q(Z(kA_n))$ if and
only if $x^e = x$.

Now let $(Q, e_Q)$ be a Brauer pair at $Q$ associated to $e$; then $e_Q^e$ is again a central
idempotent of $kA_N$. Suppose first that $e_Q^e = e_Q$. Then $e_Q$ lies in $Br_Q(Z(kA_n))$, and
therefore lifts to a primitive central idempotent of $kA_n$. This primitive central
idempotent must be $e$, since $(Q, e_Q)$ is associated to $e$. That is, we have $Br_Q(e) = e_Q$,
so there is a unique Brauer pair at $Q$ associated to $e$.

Now suppose that $e_Q^e \neq e_Q$. Since $e_Q^e$ is trivial, $e$ then interchanges $e_Q$ and $e_Q^e$.
As $e_Q$ and $e_Q^e$ are distinct primitive central idempotents of $kA_N$, $e_Q + e_Q^e$ is a central
idempotent of $kA_N$. Since $(e_Q + e_Q^e) = e_Q^e + e_Q$, $e_Q + e_Q^e$ lies in $Br_Q(Z(kA_n))$.
Additionally, it must be a primitive idempotent in $Br_Q(Z(kA_n))$, since $e_Q$ does
not lie in $Br_Q(Z(kA_n))$ and both $e_Q$ and $e_Q^e$ are primitive in $Z(kC_{A_n}(Q))$. Then
$e_Q + e_Q^e$ lifts to a primitive idempotent of $Z(kA_n)$; since $(Q, e_Q)$ is associated to $e$,
this idempotent must be $e$. That is, we have $Br_Q(e) = e_Q + e_Q^e$, so there are two
Brauer pairs at $Q$ associated to $e$, namely $(Q, e_Q)$ and $(Q, e_Q^e)$.

For any $h \in N_{A_n}(Q)$, conjugation by $h$ must either fix both of these pairs or
interchange them, depending on whether $e_Q^h$ is equal to $e_Q$ or $e_Q^e$. This can only
depend on the action of $h$ on $N$, since $e_Q$ lies in $kA_N$, and by the above, we have
$e_Q^h = e_Q^e$ if and only if $h$ restricts to an odd element of $S_N$. \hfill \Box

**Theorem 28.** Let $e$ be a block idempotent of $A_n$, let $P$ be a defect group of $e$, and
let $M$ be the subset of $\{1, \ldots, n\}$ consisting of those elements that are not fixed
points of $P$. Then $P$ is a Sylow $p$-subgroup of $A_M$, and the block fusion system on
$P$ is equal to either $F_P(A_M)$ or $F_P(S_M)$. If $p = 2$, then the block fusion system on
$P$ is equal to $F_P(A_M)$.

**Proof.** By Lemma 23 there is a subset $M'$ of $\{1, \ldots, n\}$ such that $P$ is a Sylow $p$-subgroup of $A_M$. Then we clearly have $M \subseteq M'$, so $A_M \subseteq A_{M'}$, and since $P \subseteq A_M$, $P$ is a Sylow $p$-subgroup of $A_M$.

Let $F$ be the block fusion system on $P$. By Alperin’s theorem, it is enough to prove that either $Aut_F(Q) = Aut_{S_{M}}(Q)$ for every centric subgroup $Q$ of $P$, or
$Aut_F(Q) = Aut_{A_{M}}(Q)$ for all these groups.

Let $N$ be the set of fixed points of $P$, so that $\{1, \ldots, n\}$ is the disjoint union of $M$ and $N$. By Lemma 24 any centric subgroup of $Q$ has $N$ as its set of fixed
points. Then any element $x \in N_{A_n}(Q)$ satisfies $x(N) = N$, so we may write $x = ab$
with $a \in S_M$ and $b \in S_N$ with $a$ and $b$ either both even or both odd. Since $b$
centralizes $Q$, $x$ and $a$ represent the same automorphism of $Q$, so we always have
$Aut_F(Q) \subseteq Aut_{A_n}(Q) \subseteq Aut_{S_{M}}(Q)$.

Suppose that at every $Q$ there is a unique Brauer pair at $Q$ associated to $e$. Then
we have $Aut_F(Q) = Aut_{A_n}(Q)$. If $N$ contains at least two elements, there is an
element $b \in S_N$ of odd order. Then for any $a \in S_M$, either $a$ or $ab$ is an element of $G$, and $ab$ represents the same automorphism of $Q$ as $a$. We then have $Aut_{S_{M}}(Q) \subseteq
Aut_{A_n}(Q)$; combining this with the earlier results, we get $Aut_F(Q) = Aut_{S_{M}}(Q)$.
If $N$ consists of at most one element, $S_N$ is trivial, and we get $N_{A_n}(Q) = N_{A_M}(Q)$.
This implies $Aut_F(Q) = Aut_{A_{M}}(Q)$ for all $Q$.

Now suppose that at every $Q$ there are two Brauer pairs $(Q, e_Q)$ and $(Q, e_Q)$
associated to $e$. Let $x \in N_{A_n}(Q, e_Q)$; since $N_{A_n}(Q, e_Q) \subseteq N_{A_n}(Q)$, we may write
$x = ab$ above. Then by Lemma 27 $b$ is an even element of $S_N$, so $a$ is an even
element of $S_M$. This implies $\text{Aut}_F(Q) \subseteq \text{Aut}_{A_M}(Q)$. Conversely, conjugation by any element $a$ of $A_M$ fixes both Brauer pairs, since $a$ acts trivially on $N$, so we also have $\text{Aut}_{A_M}(Q) \subseteq \text{Aut}_F(Q)$. Hence we have $\text{Aut}_F(Q) = \text{Aut}_{A_M}(Q)$ for all $Q$.

Suppose now that $p = 2$ and we have found that $\text{Aut}_F(Q) = \text{Aut}_{S_M}(Q)$ for all $Q$. Let $P'$ be a Sylow 2-subgroup of $S_M$ containing $P$; then $P$ has index 2 in $P'$, so it is normal in $P'$. Then $\text{Aut}_{P'}(P)$ is a 2-subgroup of $\text{Aut}_F(P)$ containing $\text{Aut}_P(P)$, but since $F$ is a saturated fusion system, $\text{Aut}_{P'}(P)$ is a Sylow 2-subgroup of $\text{Aut}_F(P)$. Then we must have $\text{Aut}_{P'}(P) = \text{Aut}_P(P)$. Since $P'$ properly contains $P$, we find that $C_{P'}(P)$ must properly contain $C_P(P)$. That is, there is an odd element $x \in P'$ that centralizes $P$.

Now let $Q$ be a centric subgroup of $P$; then $x$ centralizes $Q$. Let $g \in S_M$ be an element representing an $F$-automorphism of $Q$; then $g$ and $gx$ are both elements of $S_M$ representing this automorphism, and one of them is even. That is, any $F$-automorphism can be represented by an element of $A_M$, so we have $\text{Aut}_F(Q) = \text{Aut}_{A_M}(Q)$.

**Corollary 29.** Let $e$ be a block idempotent of $A_n$, let $P$ be a defect group of $e$, and let $F$ be the block fusion system on $P$. Then there is a subset $L$ of $\{1, \ldots, n\}$ such that $P$ is contained in $A_L$ and $F = F_P(A_L)$.

**Proof.** Let $M$ be the subset of $\{1, \ldots, n\}$ consisting of those elements that are not fixed points of $P$. By Theorem 28, $P$ is then a Sylow $p$-subgroup of $A_M$, and $F$ is equal to either $F_P(A_M)$ or $F_P(S_M)$. If $F = F_P(A_M)$ we set $L = M$ and are done, so consider the case $F = F_P(S_M)$. In this case we have $p > 2$ and $\text{Aut}_F(P) = \text{Aut}_{S_M}(P)$. Then the complement of $M$ in $\{1, \ldots, n\}$ must contain at least two elements, since otherwise we would have $\text{Aut}_{A_n}(P) = \text{Aut}_{A_M}(P)$, which is impossible since $\text{Aut}_F(P) \subseteq \text{Aut}_{A_n}(P)$. Let $L$ consist of $M$ plus any two other elements of $\{1, \ldots, n\}$; since $p > 2$, $P$ is then a Sylow $p$-subgroup of $A_L$. By the same arguments as in the proof of Theorem 28, we find that $\text{Aut}_{A_L}(Q) = \text{Aut}_{S_M}(Q) = \text{Aut}_F(Q)$ for every centric subgroup $Q$ of $P$, so that $F = F_P(A_L)$.

**References**

[1] Michael Aschbacher, Radha Kessar, and Bob Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011. MR 2848834

[2] Radha Kessar, *Introduction to block theory*, Group representation theory, EPFL Press, Lausanne, 2007, pp. 47–77. MR 2336637 (2008f:20020)

[3] Jacques Thévenaz, *G-algebras and modular representation theory*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, Oxford Science Publications. MR 1365077 (96j:20017)

**Institut for Matematiske Fag, Universitetsparken 5, DK–2100 København**

**E-mail address:** martinw@math.ku.dk