Topological Structure of Single-Valued Neutrosophic Hesitant Fuzzy Sets and Data Analysis for Uncertain Supply Chains

Muhammad Riaz 1,*, Yahya Almalki 2, Sania Batool 1 and Shaista Tanveer 1

1 Department of Mathematics, University of the Punjab, Lahore 54590, Pakistan; saniabatool04@gmail.com (S.B.); shaistarandhawa04@gmail.com (S.T.)
2 Department of Mathematics, College of Sciences, King Khalid University, Abha 61413, Saudi Arabia; yalmalki@kku.edu.sa
* Correspondence: mriaz.math@pu.edu.pk

Abstract: From production to retail, the food supply chain (FSC) encompasses all stages of food production. Food is now transmitted across continents over long ranges. People depend on supply chains for basic necessities such as food, water, drinks, etc. Any disruption in these shipment pipelines poses a serious threat to human life. Supplier selection (SS) has been identified as a crucial component of FSC, which has been contemplated as a multi-criteria decision-making (MCDM) problem in many studies. The failure of some specific MCDM problems is due to failure in contemplating the relationships between alternatives under uncertain circumstances. To address such challenges, we present a contemporary method for designating green suppliers based on single-valued neutrosophic hesitant fuzzy (SVNHF) information, in which the input assessment is taken into account using single-valued neutrosophic hesitant fuzzy numbers (SVNHFNs). The foremost purpose of this analysis is to construct a topological structure on single-valued neutrosophic hesitant fuzzy sets (SVNHFs) as well as to validate several key properties with examples. We discuss certain properties of SVNHF topology such as the SVNHF closure, SVNHF interior, SVNHF exterior, and SVNHF frontier. We also examine the conceptualization of the SVNHF dense set and SVNHF base in SVNHF topology using comprehensive examples. Eventually, to demonstrate and validate the superiority and inferiority ranking (SIR) method and choice value (CV) method in terms of their rationality and scientific basis, a real-world example of supplier selection in a food supply chain is provided. A comparative analysis is also performed to discuss the symmetry, validity and advantage of the proposed techniques.

Keywords: SVNHFS; SVNHF topology; uncertain supply chain; symmetry; SIR method; CV method

1. Introduction

Data analysis and information aggregation techniques have been an increasing focus in various fields such as engineering, healthcare, economics, environmental concerns, and decision making. Due to uncertain information and limitations in data analysis, we cannot seek accurate and ideal evaluation in MCDM problems. To resolve such circumstances, Zadeh [1] introduced “fuzzy set theory” initially with the concept of membership function on behalf of an exact real number in [0, 1] to express the degree of belonging of objects under a criterion. The components of membership (MG) and non-membership (NMG) of objects were addressed by Atanassov [2] in terms of an “Intuitionistic fuzzy set (IFS)”. “Pythagorean fuzzy set” (PFS), a new method for coping with vagueness when considering membership degree $\mu$ and non-membership degree $\nu$ was proposed by Yager [3]. It can characterize uncertain information more adequately and accurately than IFSs. Although IFSs and PFSs can effectively report the attribute values in MCDM in the vast majority of instances, there are a few instances where they are deficient. In accordance with the constraints imposed by IFSs and PFSs, the attribute value cannot
be represented by both IFSs and PFSs when the square sum of $\omega^T$ and $\omega^F$ degrees exceeds unity. In order to cope with this scenario, Yager [4] established the notion of the q-rung orthopair fuzzy set (q-ROFS), which can be considered a generalization of IFS and PFS. The “picture fuzzy set” (PiFS) was developed by Cuong [5], and the “spherical fuzzy set” (SFS) initiated by Mahmood et al. [6], Ashraf et al. [7], and Gündogdu and Kahraman [8]. The idea of a neutrosophic set was suggested by Smarandache [9]. Some extensions of fuzzy sets and their constraints are expressed in Table 1.

Table 1. Representations of some fuzzy sets and fuzzy numbers as well as their constraints.

| Fuzzy Sets | Fuzzy Numbers | Constraints |
|------------|---------------|-------------|
| IFS [2]    | $(\omega^T, \omega^F)$ | $\omega^T, \omega^F \in [0, 1]$, $0 \leq \omega^T + \omega^F \leq 1$ |
| PFS [3]    | $(\omega^T, \omega^F)$ | $\omega^T, \omega^F \in [0, 1]$, $0 \leq (\omega^T)^2 + (\omega^F)^2 \leq 1$ |
| q-ROFS [4] | $(\omega^T, \omega^F)$ | $\omega^T, \omega^F \in [0, 1]$, $0 \leq (\omega^T)^q + (\omega^F)^q \leq 1$, $q \geq 1$ |
| PiFS [5]   | $(\omega^T, \omega^I, \omega^F)$ | $\omega^T, \omega^I, \omega^F \in [0, 1]$, $0 \leq \omega^T + \omega^I + \omega^F \leq 1$ |
| SFS [6–8]  | $(\omega^T, \omega^I, \omega^F)$ | $\omega^T, \omega^I, \omega^F \in [0, 1]$, $0 \leq (\omega^T)^2 + (\omega^I)^2 + (\omega^F)^2 \leq 1$ |
| NS [9]     | $(T, I, F)$    | $T, I, F \in [0, 1]$, $T + I + F \in [0, 3]$ |
| SVNS [10]  | $(T, I, F)$    | $T, I, F \in [0, 1]$, $T + I + F \in [0, 3]$ |

Smarandache [9] developed the “neutrosophic set” (NS) and neutrosophic logic. The neutrosophic logic is a conventional framework for determining truth, indeterminacy, and falsification. In the NS, indeterminacy is the main focus, although truth and falsity are basic components. This presumption is critical in a variety of situations, as in information fusion, which involves combining data from multiple sensors. Furthermore, because it can handle vague and imperfect information, neutrosophy appears to be preferable for modeling uncertainties, as it is rare to have complete information at one’s disposal when making decisions. The NS derives its component values from a subset of $[0, 1]$ that is either real or non-standard. However, in real-world engineering and scientific problems, using an NS with values from a real standard or non-standard subset of $[0, 1]$ is difficult. However, a single-valued neutrosophic set (SVNS) [10] derives its components $T, I, F \in [0, 1]$ with $T + I + F \in [0, 3]$. An SVNS [10] is a type of neutrosophic set that adds a new way to represent the undetermined, vague, imperfect, and inconsistent statistics that subsist in the real world. It is better suited for dealing with unspecified and inconsistent data.

Molodtsov [11] proposed “soft set” (SS) theory to cope with uncertainty of parameters and their approximate elements. Hashmi et al. [12] introduced the m-polar neutrosophic set and extended it to m-polar neutrosophic topological structure with clustering analysis and healthcare. It can be difficult to regulate an element’s membership in a fixed set at times, which could be due to a misunderstanding of a collection of different values. Torra [13] suggested the notion of “hesitant fuzzy sets” (HFSs) as a generalization of FSs to better describe this situation, which permits membership degree to aid in the collection of practicable values in the closed interval $[0, 1]$. Experts have used HFSs to choose a variety of possible MGs to evaluate objects under suitable criteria. The theory of HFSs has a diverse set of applications in a variety of disciplines, such as for computational intelligence, clustering, healthcare, and MCDM problems.

In an HFS, because of the doubts of decision makers, there is only one truth-membership hesitant function, and it is impossible to manifest this problem using only a few different values assigned by truth, indeterminacy, and falsity membership degrees. As a result, it can only represent one type of hesitancy statistics in this situation and cannot manifest three types of hesitancy statistics. To handle uncertain problems, the idea of a single-valued neutrosophic hesitant fuzzy set (SVNHS) was introduced by Jun [14]. The truth-membership hesitancy function (TMFF), the indeterminacy-membership hesitancy function (IMHF),
and the falsity-membership hesitancy function (FMHF) are three parts of the SVNHFS that can exhibit three types of hesitancy information in this state. Tanuwijaya et al. [15] proposed a novel SVNHF time series model and applied it to stock index forecasting in Indonesia and Argentina. Aggregation operators of SVNHFS were introduced by Liu et al. [16]. Wang and Li [17] proposed generalized single-valued neutrosophic hesitant fuzzy prioritized aggregation operators. The TOPSIS method for neutrosophic hesitant fuzzy multi-attribute decision making was extended by Giri et al. [18].

The CV method is a renowned and widely used MCDM basis for evaluating a set of choices using a set of criteria. Each choice is contrasted with the others by calculating a number of ratios, one per choice criterion. Every ratio is multiplied by the proportional weight of the criterion in consideration. The selection of one or more options from the set of alternatives based on the number of criteria is a fundamental task in MCDM problems.

Xu [19] proposed the SIR method, which is an extension of the PROMETHEE method. The SIR method is an important MCDM approach which can grasp real data and supply the system user with six different preference structures. According to the two ranking lists, the SIR method ranks the alternatives more accurately. This method ranks alternatives using a superiority ranking list and an inferiority ranking list. The great feature of using the SIR method is that it incorporates the possessions of other MCDM techniques such as TOPSIS, SAW, and PROMETHEE. Some applications of the SIR method are given in Table 2.

| Researchers          | Benchmarks         | Applications                              |
|----------------------|--------------------|-------------------------------------------|
| Tam et al. [20]      | SIR method         | Concrete pump selection                   |
| Tom and Tong [21]    | SIR method         | Developments in the project concerning the location of the large-scale harbor |
| Liu [22]             | IF SIR method      | Supply chain management                   |
| Ma et al. [23]       | HF SIR method      | Selection of outstanding teachers from overseas |
| Peng and Yang [24]   | PF SIR method      | Investment in internet stocks             |
| Rouhani [25]         | Fuzzy SIR method   | Software selection in IT field           |
| Chen [26]            | PF PROMETHEE method with superiority and inferiority PFNs | Bridge construction |
| Tavana et al. [27]   | IFG SIR method     | Solution of third-party reverse logistics problem |
| Zhao et al. [28]     | SIR method with HFL prioritized value | Sustainable energy technology evaluation |
| Geetha and Narayanamoorthy [29] | PF SIR method | For investment selection of the internet Stock marketing companies |
| Jie et al. [30]      | IVIF SIR method    | Engineering investment selection          |

Certain novel concepts of neutrosophic sets, neutrosophic logic, and neutrosophic probability were explored in [31]. Seikh and Dutta [32] developed a matrix games model based on SVNSSs. Saha and Paul [33] proposed generalized weighted exponential similarity measures for SVNSSs. Alcantud et al. [34] developed generalized OWA aggregation operators and multi-agent decision making with N-soft sets. Sitara et al. [35] proposed the notion
of q-rung picture fuzzy graph structures for decision analysis. Riaz et al. [36] studied recent trends in pharmaceutical logistics and supply chain management based on distance and similarity measures for bipolar fuzzy soft sets. Farid and Riaz [37] investigated the properties of Einstein interactive geometric information aggregation with q-ROFSs. Zararsiz and Riaz [38] introduced the idea of bipolar fuzzy metric spaces with applications in decision making. Riaz et al. [39] proposed topological data analysis with spherical fuzzy soft AHP-TOPSIS for an environmental mitigation system. Riaz et al. [40] proposed the idea of interval-valued linear Diophantine fuzzy Frank aggregation operators for computational intelligence and MCDM problems.

The foremost purpose of the paper is to construct the topological structure on a single-valued neutrosophic hesitant fuzzy set and to derive significant results. These results are explained with the help of examples. We define certain concepts of SVHF topology such as the interior of SVNHFS, the closure of SVNHFS, the exterior of SVNHFS, dense sets and the base of SVNHF topology. We establish an extension of the SIR technique towards SVNHF topology to deal with uncertain MCDM problems. Moreover, to demonstrate and validate the SIR method and the CV method, a practical example of supplier selection in a food supply chain is provided. A comparative analysis is also given to discuss the validity and advantage of the proposed techniques.

The organization of the rest of the paper is as follows. In Section 2, we examine some elementary conceptions such as NS, SVNS, HFS, SVNHFS, SF, and AF of SVNHFNs and operations on SVNHFSs. In Section 3, we introduce the topological structure of SVNHFSs. Section 4 introduces the SIR method, and the CV method is developed in Section 5 for SVNHF information. In Section 6, an application of the SIR method and the CV method for SVNHF information is illustrated for data analysis in uncertain supply chains. Section 5 concludes the article and discusses future directions.

2. Preliminaries

In this section, we review the notions of NS [9], SVNS [10], HFS [13], and SVNHFS [14].

Definition 1 ([9]). Let \( \mathcal{K} \) be a set. A NS \( \mathcal{A} \) in \( \mathcal{K} \) is specified by the three components defined by
\[
\mathcal{A} = \{ \langle \xi, (T_A, I_A, N_A) \rangle : \xi \in \mathcal{K} \}
\]
where \( T_A(\xi), I_A(\xi), N_A(\xi) \subseteq [0^-, 1^+] \) such that \( 0^- \leq \sup T_A(\xi) + \sup I_A(\xi) + \sup N_A(\xi) \leq 3^+ \).

Definition 2 ([10]). Let \( \mathcal{K} \) be a set. A SVNS \( \mathcal{S} \) in \( \mathcal{K} \) can be expressed as
\[
\mathcal{S} = \{ \langle \xi, (T_S, I_S, F_S) \rangle : \xi \in \mathcal{K} \}
\]
where the components are described by mappings \( T_S, I_S, F_S : \mathcal{S} \rightarrow [0, 1] \). The components also satisfy the condition \( 0 \leq T_S(\xi) + I_S(\xi) + F_S(\xi) \leq 3 \).

Example 1. Let \( \mathcal{K} = \{ \xi_1, \xi_2, \xi_3, \xi_4 \} \) be a set. Then, an SVNS in \( \mathcal{K} \) is of the form
\[
\mathcal{S} = \{ \langle \xi_1, (0.235, 0.476, 0.667) \rangle, \langle \xi_2, (0.133, 0.345, 0.198) \rangle, \langle \xi_3, (0.421, 0.782, 0.385) \rangle, \langle \xi_4, (0.347, 0.847, 0.667) \rangle \}.
\]

Definition 3 ([13]). Assume that \( \mathcal{K} \) is a set. An HFS set can be defined as follows:
\[
\mathfrak{h} = \{ \langle \varphi, (h(\varphi)) \rangle : \varphi \in \mathcal{K} \}
\]
where \( h(\varphi) \) yields a finite subset of \([0, 1]\). Then \( h(\varphi) \) denotes an HF-element that involve some values in \([0, 1]\) indicating the MGs of element \( \varphi \in \mathcal{K} \) to the set \( \mathfrak{h} \).
Example 2. Let $\mathcal{K} = \{ \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \}$ be a set. Then, the HFS can be expressed as

$$h = \{ \langle \varphi_1, \{0.645, 0.125, 0.667\} \rangle, \langle \varphi_2, \{0.133, 0.345, 0.988, 0.259\} \rangle, \langle \varphi_3, \{0.511, 0.178\} \rangle, \langle \varphi_4, \{0.481, 0.872\} \rangle, \langle \varphi_5, \{0.666, 0.879\} \rangle \}$$

where, $h(\varphi_1) = \{0.645, 0.125, 0.667\}$, $h(\varphi_2) = \{0.133, 0.345, 0.988, 0.259\}$, $h(\varphi_3) = \{0.511, 0.178\}$, $h(\varphi_4) = \{0.481, 0.872\}$, and $h(\varphi_5) = \{0.666, 0.879\}$ are HF-elements corresponding to $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ and $\varphi_5$, respectively.

Definition 4 ([14]). Let $\mathcal{K}$ be a set; a SVNHFS in $\mathcal{K}$ is defined as

$$\mathcal{N}^Y = \{ \langle \omega^Y, \omega^H(\omega^Y), \omega^I(\omega^Y), \omega^V(\omega^Y) \rangle | \omega^Y \in \mathcal{K} \}$$

in which $\omega^H(\omega^Y), \omega^I(\omega^Y)$ and $\omega^V(\omega^Y)$ are three sets of some values in $[0, 1]$, denoting the possible truth hesitant degrees (THD), indeterminacy hesitant degrees (IHD), and falsity hesitant degrees (FHD) of the element $\omega^Y \in \mathcal{K}$ to the set $\mathcal{N}^Y$, respectively, with the conditions $0 \leq \omega^H(\omega^Y), \omega^I(\omega^Y) \leq 1$ and $0 \leq \omega^H(\omega^Y) + \omega^I(\omega^Y) + \omega^V(\omega^Y) \leq 3$, where $\omega^H(\omega^Y), \omega^I(\omega^Y), \omega^V(\omega^Y) \in \omega^H(\omega^Y), \omega^I(\omega^Y), \omega^V(\omega^Y)$ respectively. The set and the certainty function $\mathcal{N}^Y$ is called a single-valued neutrosophic hesitant fuzzy element (SVNFHE). It can be expressed as $N = \{ \omega^H, \omega^I, \omega^V \}$.

Example 3. Let $\mathcal{K} = \{ \omega^1_Y, \omega^2_Y, \omega^3_Y, \omega^4_Y \}$ be a set. Then, an SVNFHE $\mathcal{N}^Y$ in $\mathcal{K}$ can be expressed as follows:

$$\mathcal{N}^Y = \{ \langle \omega^1_Y, \{0.645, 0.125, 0.667\}, \{0.211\}, \{0.111, 0.200\} \rangle, \langle \omega^2_Y, \{0.133, 0.345\}, \{0.988, 0.259\}, \{0.335, 0.121\} \rangle, \langle \omega^3_Y, \{0.511\}, \{0.124, 0.654\}, \{0.652, 0.783, 0.284\} \rangle, \langle \omega^4_Y, \{0.481, 0.872\}, \{0.100, 0.321\}, \{0.865\} \rangle \}$$

Definition 5 ([14]). A SVNFHE $\mathcal{N}^Y$ is said to be null SVNFHE if $\omega^H(\omega^Y) = 0, \omega^I(\omega^Y) = 0, \omega^V(\omega^Y) = 0$, where $i$ varies according to alternatives, and the null SVNFHS is denoted as $0\mathcal{N}^Y$. A SVNFHE $\mathcal{N}^Y$ is said to be an absolute SVNFHE if $\omega^H(\omega^Y) = 1, \omega^I(\omega^Y) = 0, \omega^V(\omega^Y) = 0$, where $i$ varies according to alternatives, and the absolute SVNFHS is denoted as $1\mathcal{N}^Y$.

Definition 6 ([41]). Let $N_i = \langle \omega^H_i, \omega^I_i, \omega^V_i \rangle (i = 1, 2, \ldots, n)$ be the collection of SVNFHEs. Then, the score function (SF) $S(N_i)$, the accuracy function (AF) $A(N_i)$, and the certainty function (CF) $C(N_i)$ of $N_i (i = 1, 2, \ldots, n)$ can be defined as follows:

$$S(N_i) = \frac{1}{3} \left[ 2 + \frac{1}{\lambda^H} \sum_{\omega^H \in \omega^H(\omega^Y)} \omega^H - \frac{1}{\lambda^I} \sum_{\omega^I \in \omega^I(\omega^Y)} \omega^I - \frac{1}{\lambda^V} \sum_{\omega^V \in \omega^V(\omega^Y)} \omega^V \right]$$ (1)

$$A(N_i) = \frac{1}{\lambda^H} \sum_{\omega^H \in \omega^H(\omega^Y)} \omega^H - \frac{1}{\lambda^I} \sum_{\omega^I \in \omega^I(\omega^Y)} \omega^I + \frac{1}{\lambda^V} \sum_{\omega^V \in \omega^V(\omega^Y)} \omega^V$$ (2)

$$C(N_i) = \frac{1}{\lambda^H} \sum_{\omega^H \in \omega^H(\omega^Y)} \omega^H$$ (3)
SVNHFSs are used to express the method of ranking alternatives as follows, based on the concepts of SF, AF, and CF on SVNHFSs. Let $N_1$ and $N_2$ be two SVNHFEs; the ranking method is:

1. If $S(N_1) > S(N_2)$, then $N_1$ is superior to $N_2$, designated by $N_1 > N_2$.
2. If $S(N_1) = S(N_2)$ and $A(N_1) > A(N_2)$, then $N_1$ is superior to $N_2$, designated by $N_1 > N_2$.
3. If $S(N_1) = S(N_2)$, $A(N_1) = A(N_2)$ and $C(N_1) > C(N_2)$, then $N_1$ is superior to $N_2$, designated by $N_1 > N_2$.
4. If $S(N_1) = S(N_2)$, $A(N_1) = A(N_2)$ and $C(N_1) = C(N_2)$, then $N_1$ is equal to $N_2$, designated by $N_1 \sim N_2$.

**Definition 7** ([14]). Let $N_i (i = 1, 2, \ldots, k)$ be a cluster of SVNHFEs; the SVNHFWA operator is also an SVNHFE defined by

$$SVNHFWA(N_1, N_2, \ldots, N_k) = \sum_{i=1}^{k} \theta_i N_i$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ is the weight vector of $N_i (i = 1, 2, \ldots, k)$ with $\theta_i > 0, \sum_{i=1}^{k} \theta_i = 1$.

The weight of each component is considered by the SVNHFWA operators, and the aggregation is also an SVNHFE.

**Definition 8** ([41]). Let $N_1 = \langle \omega^\mu_1, \omega^\gamma_1, \omega^\nu_1 \rangle$ and $N_2 = \langle \omega^\mu_2, \omega^\gamma_2, \omega^\nu_2 \rangle$ be two SVNHFEs. Then, the normalized Hamming distance between $N_1$ and $N_2$ is defined as follows:

$$D(N_1, N_2) = \frac{1}{3} \left( \left| \frac{1}{\omega^\mu_1} \sum_{\omega^\mu_1 \in \omega^\mu_1(\omega^\mu)} \omega^\mu_1 \right| - \frac{1}{\omega^\mu_2} \sum_{\omega^\mu_2 \in \omega^\mu_2(\omega^\mu)} \omega^\mu_2 \right) + \left| \frac{1}{\omega^\gamma_1} \sum_{\omega^\gamma_1 \in \omega^\gamma_1(\omega^\gamma)} \omega^\gamma_1 \right| - \frac{1}{\omega^\gamma_2} \sum_{\omega^\gamma_2 \in \omega^\gamma_2(\omega^\gamma)} \omega^\gamma_2 \right) + \left| \frac{1}{\omega^\nu_1} \sum_{\omega^\nu_1 \in \omega^\nu_1(\omega^\nu)} \omega^\nu_1 \right| - \frac{1}{\omega^\nu_2} \sum_{\omega^\nu_2 \in \omega^\nu_2(\omega^\nu)} \omega^\nu_2 \right)$$

where $\omega^\mu_1, \omega^\gamma_1$ and $\omega^\nu_1$ are numbers of feasible membership values in $N_k$ for $k = 1, 2$.

**Definition 9.** Let us consider two SNVHFEs $N_1 = \{ \omega^\mu_1(\omega^\mu), \omega^\gamma_1(\omega^\gamma), \omega^\nu_1(\omega^\nu) \}$ and $N_2 = \{ \omega^\mu_2(\omega^\mu), \omega^\gamma_2(\omega^\gamma), \omega^\nu_2(\omega^\nu) \}$. Some operations on SNVHFEs are defined in [14], which are the following:

1. Complement: The complements of SVNHFEs $N_1$ and $N_1$ can be expressed as follows:
   - $N_1^c = \{ \omega^\nu_1(\omega^\nu), 1 - \omega^\gamma_1(\omega^\gamma), \omega^\mu_1(\omega^\mu) \}$,
   - $N_2^c = \{ \omega^\gamma_2(\omega^\gamma), 1 - \omega^\nu_2(\omega^\nu), \omega^\mu_2(\omega^\mu) \}$.
2. Inclusion: $N_1 \subseteq N_2 \iff \omega^\mu_1(\omega^\mu) \leq \omega^\mu_2(\omega^\mu), \omega^\gamma_1(\omega^\gamma) \geq \omega^\gamma_2(\omega^\gamma)$ and $\omega^\nu_1(\omega^\nu) \geq \omega^\nu_2(\omega^\nu)$ for each $\omega^\mu \in N^\mu$.
3. The union of two SVNHFEs is defined as follows:
   - $N_1 \cup N_2 = \{ \omega^\mu \in (\omega^\mu_1 \cup \omega^\mu_2) | \omega^\mu \geq \max(\omega^\mu_1^-, \omega^\mu_2^-), \omega^\gamma \in (\omega^\gamma_1 \cap \omega^\gamma_2) | \omega^\gamma \leq \min(\omega^\gamma_1^+, \omega^\gamma_2^+) \}$.
4. The intersection of two SVNHFEs is defined as follows: \( N_1 \cap N_2 = \{ \omega^\mu \in (\omega_1^\mu \cap \omega_2^\mu) | \omega^\mu \leq \min(\omega_1^\mu, \omega_2^\mu), \omega^I \leq (\omega_1^I \cup \omega_2^I), \omega^\nu \geq \max(\omega_1^\nu, \omega_2^\nu) \} \).

5. \( N_1 \oplus N_2 = \{ \omega^\mu_1 \oplus \omega^\mu_2, \omega^I_1 \oplus \omega^I_2, \omega^\nu_1 \oplus \omega^\nu_2 \}
\begin{align*}
&= \omega^\mu_1 \omega^\mu_2, \omega^I_1 \omega^I_2, \omega^\nu_1 \omega^\nu_2 \in \{ \omega^\mu_1, \omega^\mu_2 \}, \{ \omega^I_1, \omega^I_2 \}, \{ \omega^\nu_1, \omega^\nu_2 \} \}.
\end{align*}

6. \( N_1 \otimes N_2 = \{ \omega^\mu_1 \otimes \omega^\mu_2, \omega^I_1 \otimes \omega^I_2, \omega^\nu_1 \otimes \omega^\nu_2 \}
\begin{align*}
&= \omega^\mu_1 \omega^\mu_2, \omega^I_1 \omega^I_2, \omega^\nu_1 \omega^\nu_2 \in \{ \omega^\mu_1 \omega^\mu_2 \}, \{ \omega^I_1 + \omega^I_2 - \omega^I_1 \omega^I_2 \},
\end{align*}
\begin{align*}
&\{ \omega^\nu_1 + \omega^\nu_2 - \omega^\nu_1 \omega^\nu_2 \} \}.
\end{align*}

7. \( \theta N_1 = \{ \Omega^\theta \} \cup \{ (1 - (1 - \omega^\mu_1)^\theta), \{ \omega^I_1 \theta \}, \{ \omega^\nu_1 \theta \} \}, \theta > 0.
\end{align*}

8. \( N_1^\theta = \{ \Omega^\theta \} \cup \{ (1 - (1 - \omega^I_1)^\theta), \{ 1 - (1 - \omega^I_1)^\theta \}, \{ 1 - (1 - \omega^I_1)^\theta \} \}, \theta > 0.
\end{align*}

3. SVNHF Topology

Ye [14] proposed the idea of SVNHFS as an efficient model for modeling uncertainties. Biswas et al. [41] suggested the notions of SF, AF, and CF for SVNHFEs. In this section, the notion of SVNHF topology is introduced using fundamental characteristics of SVNHFSs.

Definition 10. Let \( K \) be a set and \( \tau \) be the collection of SVNHFSs in \( K \). Then, \( \tau \) is called an SVNHF topology if it satisfies following properties:

1. \( 0^NY, 1^NY \in \tau. \)
2. For each \( N^Y \in \tau, i \in \Omega, \cup_{i \in \Omega} N^Y_i \in \tau. \)
3. For any \( N^Y_1, N^Y_2 \in \tau, N^Y_1 \cap N^Y_2 \in \tau. \)

Then \( (K, \tau) \) is called SVNHF topological space.

Example 4. Let \( K = \{ \omega_1^0, \omega_2^0, \omega_3^0, \omega_4^0 \} \) be a set. Let us consider

\[
N^Y_1 = \{ (\omega_1^0, 0.321, 0.567, 0.411), \{ 0.102 \}, \{ 0.536, 0.844, 0.689 \}), \]
\[
(\omega_2^0, 0.213, 0.469, 0.328), \{ 0.650, 0.679 \}, \{ 0.998, 0.450 \}), \]
\[
(\omega_3^0, 0.404, 0.500), \{ 0.308 \}, \{ 0.792, 0.670, 0.666 \}), \]
\[
(\omega_4^0, 0.210, 0.410, 0.589), \{ 0.752, 0.890, 0.786 \}, \{ 0.450 \}), \]
\[
N^Y_2 = \{ (\omega_1^0, 0.600, 0.580, 0.893), \{ 0.100 \}, \{ 0.520, 0.440, 0.250 \}), \]
\[
(\omega_2^0, 0.469, 0.480, 0.850), \{ 0.210, 0.650 \}, \{ 0.450, 0.100 \}), \]
\[
(\omega_3^0, 0.540, 0.600), \{ 0.300 \}, \{ 0.150, 0.150, 0.655 \}), \]
\[
(\omega_4^0, 0.589, 0.650, 0.890), \{ 0.095, 0.350, 0.750 \}, \{ 0.400 \}) \}
\]

any two SVNHFSs in \( K \). Tables 3 and 4 show the union and intersection, respectively, of the SVNHFSs \( N^Y_1 \) and \( N^Y_2 \).

Table 3. Union of SVNHFSs.

| Union | \( 0^NY \) | \( N^Y_1 \) | \( N^Y_2 \) | \( 1^NY \) |
|-------|------------|------------|------------|------------|
| \( 0^NY \) | \( 0^NY \) | \( N^Y_1 \) | \( N^Y_2 \) | \( 1^NY \) |
| \( N^Y_1 \) | \( N^Y_1 \) | \( N^Y_1 \) | \( N^Y_1 \) | \( 1^NY \) |
| \( N^Y_2 \) | \( N^Y_1 \) | \( N^Y_1 \) | \( N^Y_1 \) | \( 1^NY \) |
| \( 1^NY \) | \( 1^NY \) | \( 1^NY \) | \( 1^NY \) | \( 1^NY \) |
Table 4. Intersection of SVNHF topologies.

| Intersection | $^0\mathcal{N}^Y$ | $^1\mathcal{N}^Y$ | $^2\mathcal{N}^Y$ | $^1\mathcal{N}^Y$ |
|--------------|------------------|------------------|------------------|------------------|
| $^0\mathcal{N}^Y$ | $^0\mathcal{N}^Y$ | $^0\mathcal{N}^Y$ | $^0\mathcal{N}^Y$ | $^0\mathcal{N}^Y$ |
| $^1\mathcal{N}^Y$ | $^0\mathcal{N}^Y$ | $^1\mathcal{N}^Y$ | $^0\mathcal{N}^Y$ | $^1\mathcal{N}^Y$ |
| $^2\mathcal{N}^Y$ | $^0\mathcal{N}^Y$ | $^2\mathcal{N}^Y$ | $^0\mathcal{N}^Y$ | $^2\mathcal{N}^Y$ |
| $^1\mathcal{N}^Y$ | $^0\mathcal{N}^Y$ | $^1\mathcal{N}^Y$ | $^0\mathcal{N}^Y$ | $^1\mathcal{N}^Y$ |

We see that

\[ \tau_1 = \{ ^0\mathcal{N}^Y, ^1\mathcal{N}^Y \} \]
\[ \tau_2 = \{ ^0\mathcal{N}^Y, ^1\mathcal{N}^Y, ^2\mathcal{N}^Y \} \]
\[ \tau_3 = \{ ^0\mathcal{N}^Y, ^2\mathcal{N}^Y, ^1\mathcal{N}^Y \} \]
\[ \tau_4 = \{ ^0\mathcal{N}^Y, ^1\mathcal{N}^Y, ^2\mathcal{N}^Y, ^1\mathcal{N}^Y \} \]

are SVNHF topologies on $\mathcal{K}$, where

\[ ^0\mathcal{N}^Y = \{ \langle \omega_1^Y, \{0.000, 0.000, 0.000\}, \{1.000\}, \{1.000, 1.000, 1.000\} \}, \langle \omega_2^Y, \{0.000, 0.000, 0.000\}, \{1.000, 1.000\}, \{1.000, 1.000, 1.000\} \}, \langle \omega_3^Y, \{0.000, 0.000, 0.000\}, \{1.000\}, \{1.000, 1.000, 1.000\} \}, \langle \omega_4^Y, \{0.000, 0.000, 0.000\}, \{1.000, 1.000, 1.000\}, \{1.000\} \} \} \]

is a null SVNHF in $\mathcal{K}$.

\[ ^1\mathcal{N}^Y = \{ \langle \omega_1^Y, \{1.000, 1.000, 1.000\}, \{0.000\}, \{0.000, 0.000, 0.000\} \}, \langle \omega_2^Y, \{1.000, 1.000, 1.000\}, \{0.000, 0.000, 0.000\} \}, \langle \omega_3^Y, \{1.000, 1.000\}, \{0.000\}, \{0.000, 0.000, 0.000\} \}, \langle \omega_4^Y, \{1.000, 1.000, 1.000\}, \{0.000, 0.000, 0.000\} \} \} \]

is an absolute SVNHF in $\mathcal{K}$.

**Definition 11.** If ($\mathcal{K}, \tau$) is an SVNHF topological space over $\mathcal{K}$, then the members of SVNHF topology $\tau$ are called SVNHF open sets. That is, if $^\mathcal{N}^Y \in \tau$, then $^\mathcal{N}^Y$ is called an SVNHF open set.

**Theorem 1.** If ($\mathcal{K}, \tau$) is any SVNHF topological space, then
1. $^0\mathcal{N}^Y$ and $^1\mathcal{N}^Y$ are SVNHF open sets;
2. $\bigcup_{\omega \in \Omega} ^\mathcal{N}^Y_\omega$ is an SVNHF open set, where each $^\mathcal{N}^Y_\omega$ is an SVNHF open set;
3. $\bigcap_{i=1}^n ^\mathcal{N}^Y_i$ is an SVNHF open set, where each $^\mathcal{N}^Y_i$ is an SVNHF open set.

**Proof.** 1. From the definition of SVNHF topology $\tau$, $^0\mathcal{N}^Y, ^1\mathcal{N}^Y \in \tau$. Hence, $^0\mathcal{N}^Y$ and $^1\mathcal{N}^Y$ are SVNHF open sets.
2. Let $\{ (^\mathcal{N}^Y_\omega)_{\omega \in \Omega} \}$ be SVNHF open sets. Then, $^\mathcal{N}^Y_\omega \in \tau$. By the definition of $\tau$

\[ \bigcup_{\omega \in \Omega} ^\mathcal{N}^Y_\omega \in \tau \]

Hence, $\bigcup_{\omega \in \Omega} ^\mathcal{N}^Y_\omega$ is SVNHF open set.
3. Let $^\mathcal{N}^Y_1, ^\mathcal{N}^Y_2, \ldots, ^\mathcal{N}^Y_n$ be SVNHF open sets. Then, by the definition of $\tau$,

\[ \bigcap_{i=1}^n ^\mathcal{N}^Y_i \in \tau \]
Hence, $\cap_{i=1}^{n} N^Y_i$ is an SVNHF open set.

\[\square\]

**Definition 12.** Let $(K, \tau)$ be an SVNHF topological space over $K$. If the complement of an SVNHF is SVNHF open, then it is called an SVNHF closed set in $K$. That is, $N^Y$ is called an SVNHF closed set if, and only if, $(N^Y)^c \in \tau$.

**Theorem 2.** Assume that $(K, \tau)$ is an SVNHF topological space. Then,

1. $0 N^Y, 1 N^Y$ are SVNHF closed sets over $K$;
2. $(\cap N^Y_\alpha)_{\alpha \in \Omega}$ is an SVNHF closed set over $K$, where each $N^Y_\alpha$ is an SVNHF closed set;
3. $(\cup N^Y_\alpha)$ is an SVNHF closed set over $K$, for any SVNHF closed sets $N^Y_1$ and $N^Y_2$.

**Proof.** The proof is obvious. \[\square\]

**Definition 13.** Let $(K, \tau_1)$ and $(K, \tau_2)$ be two SVNHF topological spaces over same set $K$. If $\tau_1 \subseteq \tau_2$, then $\tau_1$ is said to be SVNHF coarser or SVNHF weaker than $\tau_2$, and $\tau_2$ is said to be SVNHF finer or SVNHF stronger than $\tau_1$.

*If $\tau_1 \not\subseteq \tau_2$ or $\tau_2 \not\subseteq \tau_1$, then these SVNHF topologies are not comparable.*

**Example 5.** From Example 4, let us consider $\tau_2 = \{0 N^Y, 1 N^Y, N^Y\}$ and $\tau_4 = \{0 N^Y, N^Y 1 N^Y, N^Y\}$, two SVNHF topologies on $K$. It is comprehensible that $\tau_2 \subseteq \tau_4$. Thus, $\tau_2$ is SVNHF coarser than $\tau_4$ and $\tau_4$ is SVNHF finer than $\tau_2$.

**Definition 14.** Assume a universal set $K$, and the assemblage of all SVNHF$S$s $\tau$ is defined over $K$. Then, $\tau$ is an SVNHF discrete topology on $K$, and $(K, \tau)$ is known as SVNHF discrete topological space over $K$.

**Definition 15.** Suppose that $K$ is a universal set and $\tau = \{0 N^Y, 1 N^Y\}$. Then, $\tau$ is an SVNHF non-discrete topology on $K$ and $(K, \tau)$ is an SVNHF non-discrete topological space over $K$.

**Theorem 3.** Suppose that $(K, \tau_1)$ and $(K, \tau_2)$ are two SVNHF topological spaces over identical universes of discourse $K$; then, $(K, \tau_1 \cap \tau_2)$ is an SVNHF topological space over $K$.

**Proof.** The proof is obvious. \[\square\]

**Definition 16.** Let us consider an SVNHF topological space $(K, \tau)$. For any SVNHF $N^Y$ of $K$, the SVNHF interior $(N^Y)^{\circ}$ is interpreted as the union of all SVNHF open subsets of $N^Y$. $(N^Y)^{\circ}$ is the largest SVNHF open set contained in $N^Y$.

**Example 6.** From Example 4, we see that $\tau = \{0 N^Y, N^Y, N^Y 1 N^Y, N^Y\}$ is an SVNHF topology on $K$. Consider an SVNHF defined on $K = \{\omega_1, \omega_2, \omega_3, \omega_4\}$

\[N^Y = \{(\omega_1, 0.400, 0.575, 0.589), (0.101), (0.530, 0.540, 0.570), (\omega_2, 0.369, 0.470, 0.638), (0.400, 0.660), (0.700, 0.200), (\omega_3, 0.500, 0.550, 0.305), (0.400, 0.380, 0.660), (\omega_4, 0.330, 0.500, 0.700), (0.400, 0.611, 0.770), (0.440)\}\]

Then, $(N^Y)^{\circ} = 0 N^Y \cup N^Y_1 = N^Y_1$.

**Theorem 4.** Assume that $(K, \tau)$ is an SVNHF topological space over $K$, $N^Y_1$ and $N^Y_2$ are SVNHF$S$s over $K$. Then,

1. $(N^Y_1)^{\circ} \subseteq N^Y_1$
2. $(N^Y_2)^{\circ} = ((N^Y_1)^{\circ})^{\circ}$
3. $N^Y_1$ is an SVNHF open set $\iff (N^Y_1)^{\circ} = N^Y_1$
4. \((N^Y_1)^o \subseteq (N^Y_1)^o\) if \(N^Y_1 \subseteq N^Y_2\)
5. \((N^Y_1)^o \cap (N^Y_2)^o = (N^Y_1 \cap N^Y_2)^o\)
6. \((N^Y_1)^o \cup (N^Y_2)^o \subseteq (N^Y_1 \cup N^Y_2)^o\)

**Proof.**
1. This is obvious from the definition of the SVNHF interior.
2. Since \((N^Y_1)^o\) is an SVNHF open set and it is also the biggest SVNHF open subset of itself, \((N^Y_1)^o = ((N^Y_1)^o)^o\).
3. If \(N^Y_1\) is an SVNHF open subset, then \(N^Y_1\) will be an SVNHF interior of itself since it is the largest SVNHF open subset. Conversely, if \((N^Y_1)^o = N^Y_1\), then \(N^Y_1\) is an SVNHF open set because \((N^Y_1)^o\) is SVNHF open.
4. Since \(N^Y_1 \subseteq N^Y_2\), from part (1), \((N^Y_1)^o \subseteq N^Y_1 \subseteq N^Y_2\). \((N^Y_1)^o\) is an SVNHF open subset of \(N^Y_2\) and so, by the definition of \((N^Y_2)^o\), we have \((N^Y_1)^o \subseteq (N^Y_2)^o\).
5. From part (4),
\[
\begin{align*}
(N^Y_1 \cap N^Y_2)^o &\subseteq (N^Y_1)^o \cap (N^Y_2)^o \\
\Rightarrow (N^Y_1 \cap N^Y_2)^o &\subseteq (N^Y_1)^o \text{ and } (N^Y_1 \cap N^Y_2)^o \subseteq (N^Y_2)^o
\end{align*}
\]
so that
\[
(N^Y_1 \cap N^Y_2)^o \subseteq (N^Y_1)^o \cap (N^Y_2)^o
\]
Furthermore, since \((N^Y_1)^o \subseteq N^Y_1\), \((N^Y_2)^o \subseteq N^Y_2\), \((N^Y_1)^o \cap (N^Y_2)^o \subseteq N^Y_1 \cap N^Y_2\), so that \((N^Y_1)^o \cap (N^Y_2)^o\) is an SVNHF open subset of \(N^Y_1 \cap N^Y_2\). Hence,
\[
(N^Y_1)^o \cap (N^Y_2)^o \subseteq (N^Y_1 \cap N^Y_2)^o
\]
Thus,
\[
(N^Y_1 \cap N^Y_2)^o = (N^Y_1)^o \cap (N^Y_2)^o
\]
6. From \(N^Y_1 \subseteq N^Y_1 \cup N^Y_2\), \(N^Y_2 \subseteq N^Y_1 \cup N^Y_2\)
we have
\[
(N^Y_1)^o \subseteq (N^Y_1 \cup N^Y_2)^o, (N^Y_2)^o \subseteq (N^Y_1 \cup N^Y_2)^o
\]
so that, because \((N^Y_1)^o \cup (N^Y_2)^o\) is SVNHF open,
\[
(N^Y_1)^o \cup (N^Y_2)^o \subseteq (N^Y_1 \cup N^Y_2)^o.
\]

**Definition 17.** Let us consider an SVNHF topological space \((K, \tau)\). For any SVNHF \(N^Y\) of \(K\), the SVNHF closure \(\overline{N^Y}\) is the intersection of all SVNHF closed super sets of \(N^Y\).

**Example 7.** Consider an SVNHF topology \(\tau = \{0, N^Y, N^Y_1, N^Y_2, N^Y_3, N^Y_4\}\) on \(K\) defined in Example 4 and an SVNHF space defined in Example 6. For SVNHF closure, we have to find the complements of \(N^Y_1, N^Y_2, N^Y_3, N^Y_4\) and \(N^Y_5\).

\[
\begin{align*}
(N^Y_1)^c &= \{(0.536, 0.844, 0.689), (0.321, 0.567, 0.411), (0.989, 0.450), (0.350, 0.321), (0.213, 0.469, 0.328), (0.792, 0.670, 0.666), (0.692), (0.404, 0.500)\} \\
(N^Y_2)^c &= \{(0.520, 0.440, 0.250), (0.900), (0.600, 0.580, 0.893), (0.450, 0.100), (0.790, 0.350), (0.469, 0.480, 0.850), (0.150, 0.150, 0.655), (0.700), (0.540, 0.600)\} \\
(N^Y_3)^c &= \{(0.400), (0.905, 0.650, 0.250), (0.589, 0.650, 0.890)\} \\
(0)^c &= \{(0.000), (0.000, 0.000, 0.000), (0.100, 0.000, 0.000), (0.000, 0.000, 0.000), (0.000, 0.000, 0.000)\}
\end{align*}
\]
Then, \( (\overline{\text{SVNHFS}})^c = 1^Y \).

**Theorem 5.** Suppose that \( K \) is a universal set, that \((K, \tau)\) is an SVNHF topological space over \( K \), and that \( \mathcal{N}_1^Y \) and \( \mathcal{N}_2^Y \) are SVNHF sets over \( K \). Then,

1. \( \overline{\mathcal{N}_1^Y} = 0^Y \) and \( \overline{\mathcal{N}_2^Y} = 1^Y \)
2. \( \mathcal{N}_1^Y \subseteq \mathcal{N}_1^Y \)
3. \( \mathcal{N}_1^Y \) is an SVNHF closed set \( \iff \mathcal{N}_1^Y = \overline{\mathcal{N}_1^Y} \)
4. \( \overline{\mathcal{N}_1^Y} = \overline{\mathcal{N}_1^Y} \)
5. \( \mathcal{N}_1^Y \subseteq \mathcal{N}_2^Y \) if \( \mathcal{N}_1^Y \subseteq \mathcal{N}_2^Y \)
6. \( \mathcal{N}_1^Y \cup \mathcal{N}_2^Y = \mathcal{N}_1^Y \cup \mathcal{N}_2^Y \)
7. \( \mathcal{N}_1^Y \cap \mathcal{N}_2^Y \subseteq \mathcal{N}_1^Y \cap \mathcal{N}_2^Y \)

**Proof.** 1. By definition, \( 0^Y \subseteq \overline{\mathcal{N}_1^Y} \). Since \( 0^Y \) is an SVNHF closed superset of itself, \( \overline{\mathcal{N}_1^Y} \subseteq 0^Y \). Thus, \( \overline{\mathcal{N}_1^Y} = 0^Y \). Similarly, \( \overline{\mathcal{N}_2^Y} = 1^Y \).
2. By definition, \( \mathcal{N}_1^Y \subseteq \mathcal{N}_2^Y \), because \( \mathcal{N}_1^Y \) is the intersection of all SVNHF closed superset of \( \mathcal{N}_1^Y \).
3. The proof is obvious.
4. Since \( \mathcal{N}_1^Y \) is an SVNHF closed set, by (3) we have \( \overline{\mathcal{N}_1^Y} = \overline{\mathcal{N}_1^Y} \).
5. Suppose \( \mathcal{N}_1^Y \subseteq \mathcal{N}_2^Y \) as \( \mathcal{N}_2^Y \subseteq \mathcal{N}_2^Y \). Therefore, \( \mathcal{N}_1^Y \subseteq \mathcal{N}_2^Y \). This means that \( \mathcal{N}_2^Y \) is an SVNHF closed superset of \( \mathcal{N}_1^Y \). Thus, \( \mathcal{N}_1^Y \subseteq \mathcal{N}_2^Y \).
6. As we know that \( \mathcal{N}_1^Y \subseteq \mathcal{N}_1^Y \cup \mathcal{N}_2^Y \) and \( \mathcal{N}_2^Y \subseteq \mathcal{N}_1^Y \cup \mathcal{N}_2^Y \), by using part (5), \( \overline{\mathcal{N}_1^Y} \subseteq \overline{\mathcal{N}_1^Y} \cup \overline{\mathcal{N}_2^Y} \)
and \( \overline{\mathcal{N}_2^Y} \subseteq \overline{\mathcal{N}_1^Y} \cup \overline{\mathcal{N}_2^Y} \).
\[
\iff \overline{\mathcal{N}_1^Y} \cup \overline{\mathcal{N}_2^Y} \subseteq \overline{\mathcal{N}_1^Y} \cup \overline{\mathcal{N}_2^Y}.
\]
Conversely, suppose that \( \mathcal{N}_1^Y \subseteq \mathcal{N}_2^Y \) and \( \mathcal{N}_2^Y \subseteq \mathcal{N}_2^Y \). Thus, \( \mathcal{N}_1^Y \cup \mathcal{N}_2^Y \subseteq \mathcal{N}_1^Y \cup \mathcal{N}_2^Y \).
Since \( \mathcal{N}_1^Y \cup \mathcal{N}_2^Y \) is an SVNHF closed superset of \( \mathcal{N}_1^Y \cup \mathcal{N}_2^Y \), therefore, \( \overline{\mathcal{N}_1^Y} \cup \overline{\mathcal{N}_2^Y} \subseteq \overline{\mathcal{N}_1^Y} \cup \overline{\mathcal{N}_2^Y} \).
Thus, \( \overline{\mathcal{N}_1^Y} \cup \overline{\mathcal{N}_2^Y} = \overline{\mathcal{N}_1^Y} \cup \overline{\mathcal{N}_2^Y} \).
7. If \( \mathcal{N}_1^Y \cap \mathcal{N}_2^Y \subseteq \mathcal{N}_1^Y \cap \mathcal{N}_2^Y \), then \( \overline{\mathcal{N}_1^Y} \cap \overline{\mathcal{N}_2^Y} \subseteq \overline{\mathcal{N}_1^Y} \) and \( \overline{\mathcal{N}_1^Y} \cap \overline{\mathcal{N}_2^Y} \subseteq \overline{\mathcal{N}_2^Y} \). Thus, \( \overline{\mathcal{N}_1^Y} \cap \overline{\mathcal{N}_2^Y} \subseteq \overline{\mathcal{N}_2^Y} \).

**Definition 18.** Let us consider an SVNHF topological space \((K, \tau)\). For any SVNHF set \( \mathcal{N}^Y \) of \( K \), the SVNHF exterior \( \text{Ext}(\mathcal{N}^Y) \) is interpreted as the interior of the complement of SVNHF \( \mathcal{N}^Y \).

**Example 8.** From Example 4, consider an SVNHF topology \( \tau = \{0^Y, \mathcal{N}_1^Y, \mathcal{N}_2^Y, 1^Y\} \) on \( K \) and an SVNHF defined in Example 6. For SVNHF exterior, we have to find the complement of \( \mathcal{N}^Y \):

\[
(\mathcal{N}^Y)^c = \{(0.500, 0.500, 0.500, 0.500), (0.899), (0.400, 0.575, 0.589), (0.700, 0.200), (0.600, 0.340), (0.369, 0.470, 0.638), (0.400, 0.380, 0.660), (0.695), (0.500, 0.550), (0.440), (0.600, 0.389, 0.230), (0.330, 0.500, 0.700)\}
\]

Then, \( \text{Ext}(\mathcal{N}^Y) = ((\mathcal{N}^Y)^c)^c = 0^Y \).
Theorem 6. Suppose that \((K, \tau)\) is an SVNHF topological space over \(K\), and that \(N^*_Y\) and \(N^*_{Y'}\) are SVNHF\(S\) over \(K\). Then,
1. \((N^*_Y)^c\) contains the largest SVNHF open set \(\text{Ext}(N^*_Y)\).
2. \((N^*_Y)^c\) is SVNHF open if and only if \(\text{Ext}(N^*_Y) = (N^*_Y)^c\).
3. \(N^*_Y \subseteq N^*_{Y'} \implies \text{Ext}(N^*_Y) \subseteq \text{Ext}(N^*_{Y'})\).

Proof. Straightforward. \(\square\)

Definition 19. Let us consider an SVNHF topological space \((K, \tau)\). For any SVNHF\(S\) \(N^Y\) of \(K\), the SVNHF frontier \(\text{Fr}(N^Y)\) is interpreted as the intersection of \(N^Y\) and \((N^Y)^c\).

Example 9. From Example 4, consider an SVNHF topology \(\tau = \{0, N^Y_1, N^Y_2, N^Y\} \) on \(K\) and an SVNHF\(S\) defined in Example 6. The SVNHF frontier is \(\text{Fr}(N^Y) = \overline{N^Y} \cap (N^Y)^c = N^Y\).

Theorem 7. Consider an SVNHF topological space \((K, \tau)\) and \(N^Y\) as an SVNHF; then,
1. \(\text{Fr}(N^Y) = \text{Fr}((N^Y)^c)\).
2. \((\text{Fr}(N^Y))^c = (N^Y)^c \cup ((N^Y)^c)^c\).
3. If \(N^Y\) is SVNHF open, then \(\text{Fr}(N^Y) \subseteq (N^Y)^c\).

Proof. 1. By the definition of the SVNHF frontier, \(\text{Fr}(N^Y) = \overline{N^Y} \cap (N^Y)^c = (N^Y)^c \cap \overline{N^Y} = \left(\text{Fr}(N^Y)^c\right)^c\).
2. Since \(\text{Fr}(N^Y)^c = \overline{N^Y} \cap (N^Y)^c\), by taking the SVNHF complement on both sides, we obtain \(\left(\text{Fr}(N^Y)^c\right)^c = (\overline{N^Y} \cap (N^Y)^c)^c = (N^Y)^c\).
3. Let \(N^Y\) be an SVNHF open set; this yields that \((N^Y)^c\) is SVNHF closed. Utilizing (2), \(\text{Fr}(N^Y)^c \subseteq (N^Y)^c\), and by (1), we obtain \(\text{Fr}(N^Y) \subseteq (N^Y)^c\).

\(\square\)

Theorem 8. Suppose that \((K, \tau)\) is an SVNHF topological space over \(K\) and that \(N^Y\) is an SVNHF; then,
1. \((N^Y)^c = ((N^Y)^c)^c\)
2. \(\overline{N^Y} = N^Y \cup \text{Fr}(N^Y)\).
3. \((N^Y)^o = N^Y \setminus \text{Fr}(N^Y)\).
4. For any subset \(N^Y\) in \((K, \tau)\), \(N^Y\) is open if, and only if, \(N^Y \cap \text{Fr}(N^Y)\) is a null SVNHF\(S\).
5. For any subset \(N^Y\) in \((K, \tau)\), \(N^Y\) is closed if, and only if, \(N^Y \subseteq \text{Fr}(N^Y)\).
6. For any subset \(N^Y\) in \((K, \tau)\), \(N^Y\) is both open and closed if, and only if, \(\text{Fr}(N^Y)\) is a null SVNHF\(S\).

Proof. The proof is obvious. \(\square\)

Definition 20. Consider an SVNHF topological space \((K, \tau)\). A SVNHF\(S\) \(N^Y\) is termed dense in \(K\) if \(\overline{N^Y} = 1\) \(N^Y\).

Example 10. Let us consider SVNHF topological space given in Example 4 an SVNHF\(S\) \(N^Y\) defined in Example 6. We see that \(\overline{N^Y} = 1\) \(N^Y\). This shows that \(N^Y\) is dense in \(K\).

Definition 21. Consider an SVNHF topological space \((K, \tau)\). A sub-collection \(B\) of \(\tau\) is called an SVNHF base for \(\tau\) if every SVNHF open set in \(\tau\) is a union of members of \(B\).

Example 11. From Example 4, consider SVNHF topology \(\tau = \{0, N^Y_1, N^Y_2, N^Y\}\) over \(K\). Then, \(B = \{0, N^Y, N^Y_1, N^Y_2, N^Y\}\) is an SVNHF base for \(\tau\).
4. Extension of SIR Method for SVNHF Information

In this section, a new MCDM method is developed based on Algorithm 1, which is an extension of the SIR method to SVNHFSs.

**Algorithm 1 (SIR method for SVNHFSs)**

Let $\mathcal{K} = \{\omega_i^k : i = 1, 2, \ldots, l\}$ be an assemblage of substitutes/alternatives and $\mathcal{C} = \{\omega_j^c : j = 1, 2, \ldots, m\}$ is the collection of attributes. Assume that $\mathcal{E} = \{\omega_k^e : k = 1, 2, \ldots, n\}$ be the collection of experts with weight vectors $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$. Suppose $\mathcal{V}(k) = (p_{ij}^{(k)})_{l \times m}$ ($i = 1, 2, \ldots, l; j = 1, 2, \ldots, m; k = 1, 2, \ldots, n$) is the SVNHF decision matrix, where $p_{ij}^{(k)}$ designates the accredits value that substitutes $\omega_i^k$ and persuades the accredits $\omega_j^c$ designated by expert $\omega_k^e$. The accredits weighted decision matrix is $\mathbf{w} = (\mathbf{w}_j^{(k)})_{n \times m}$, where $\mathbf{w}_j^{(k)}$ designates the weight value of the accredits $\omega_j^c$ designated by expert $\omega_k^e$.

A novel approach based on SVNHF-SIR is addressed below:

**Step 1:** Calculate the discrete/individual measure degree $\rho_k$ ($k = 1, 2, \ldots, n$) via the weights of experts, which take the form of SVNHFSs. The relative closeness coefficient is procured as follows:

$$\rho_k = \frac{d(\mathcal{M}_k, \mathcal{M}^-)}{d(\mathcal{M}_k, \mathcal{M}^+) + d(\mathcal{M}_k, \mathcal{M}^-)}.$$  \hspace{1cm} (6)

where $\mathcal{M}^+ = (\max\{\omega_k^l\}, \min\{\omega_i^l\}, \min\{\omega_j^k\})$, $\mathcal{M}^- = (\min\{\omega_k^l\}, \max\{\omega_i^l\}, \max\{\omega_j^k\})$. It is easily obtained that $0 \leq \mathcal{M}_k \leq 1$ and if $\rho_k \to 1$, then $\mathcal{M}_k \to \mathcal{M}^+$; if $\rho_k \to 0$, then $\mathcal{M}_k \to \mathcal{M}^-$.

**Step 2:** To make the sum into a unit, normalize the $\rho_k$ ($k = 1, 2, \ldots, n$) and obtain as follows:

$$\omega_k = \frac{\rho_k}{\sum_{k=1}^{n} \rho_k} = \frac{\rho_k}{\rho_1 + \rho_2 + \cdots + \rho_n}.$$  \hspace{1cm} (7)

We obtain the vector of real numbers that have been normalized $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^\mu$ as discrete/individual measure degrees.

**Step 3:** Employ the SVNHF-WA operator to aggregate individual perspectives into group perspectives as follows:

1. **Discrete/individual attributes’ weights integration:**

$$\mathbf{w}_j = \text{SVNHF-WA}_{\omega_k} (\mathbf{w}_j^{(1)}, \mathbf{w}_j^{(2)}, \ldots, \mathbf{w}_j^{(n)}) = \left\{ \prod_{i=1}^{n} \left( 1 - \omega_i^{(k)} \right)^{\omega_k} \right\} \left\{ \prod_{i=1}^{n} \left( \omega_i^{(k)} \right)^{\omega_k} \right\}$$ \hspace{1cm} (8)

2. **Discrete/individual decision matrix integration:**

$$p_{ij} = \text{SVNHF-WA}_{\omega_k} (p_{ij}^{(1)}, p_{ij}^{(2)}, \ldots, p_{ij}^{(n)}) = \left\{ \prod_{k=1}^{n} \left( 1 - \omega_k^{(ij)} \right)^{\omega_k} \right\} \left\{ \prod_{k=1}^{n} \left( \omega_k^{(ij)} \right)^{\omega_k} \right\}.$$  \hspace{1cm} (9)

From this step, the group-integrated decision matrix $p = (p_{ij})_{l \times m}$ and the attribute weight vector $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ are acquired.
Algorithm 1 Cont.

Step 4: Acquire the SVNHF superiority/inferiority matrix

1. Attain the performance function $f$:

   $$ f_{ij} = j(p_{ij}) = \frac{d(p_{ij}, \bar{p}^-)}{d(p_{ij}, \bar{p}^-) + d(p_{ij}, \bar{p}^+)} $$

   (10)

   where $\bar{p}^- = \{a_{ij}^c, (\min\{a_{ij}^d\}, \max\{a_{ij}^d\}, \max\{a_{ij}^d\})\}$,

   $\bar{p}^+ = \{a_{ij}^c, (\max\{a_{ij}^d\}, \min\{a_{ij}^d\}, \min\{a_{ij}^d\})\}$. It is easily obtained that $0 \leq f_{ij} \leq 1$

   and if $f_{ij} \to 1$, then $\bar{p}_{ij} \to \bar{p}^+$; if $f_{ij} \to 0$, then $\bar{p}_{ij} \to \bar{p}^-$.

2. Acquire the preference intensity $\mathcal{P}I_i(\omega_i^t, \omega_i^l)$:

   We define $\mathcal{P}I_i(\omega_i^t, \omega_i^l)(i, t = 1, 2, \cdots, i \neq t; j = 1, 2, \cdots, m)$ as the preference intensity of alternatives $\omega_i^t$ to the corresponding attribute $\omega_j^c$, which is given as follows:

   $$ \mathcal{P}I_i(\omega_i^t, \omega_i^l) = \phi_i(f_{ij} - f_{ij}) = \phi_i(d) $$

   (11)

   where $\phi_i(d)$ is a non-decreasing function from the real number to $[0, 1]$. Normally, $\phi_i(d)$ is from a set of six generalized threshold functions [42], or interpreted by the experts themselves.

3. Acquire superiority matrix and inferiority matrix:

   Superiority index (S-index): $\mathcal{S} = (S_{ij})_{1 \times m}$

   $$ S_{ij} = \sum_{t=1}^{m} \mathcal{P}I_i(\omega_i^t, \omega_i^l) = \sum_{t=1}^{m} \phi_i(f_{ij} - f_{ij}) $$

   (12)

   Inferiority index (I-index): $\mathcal{I} = (I_{ij})_{1 \times m}$

   $$ I_{ij} = \sum_{t=1}^{m} \mathcal{P}I_i(\omega_i^t, \omega_i^l) = \sum_{t=1}^{m} \phi_i(f_{ij} - f_{ij}) $$

   (13)

Step 5: Calculate the superiority flow and inferiority flow as follows:

S-flow

$$ \Psi^>(\omega_i^t) = SVNHFWA_{S_{ij}}(\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_m) $$

(14)

$$ = \{(1 - \prod_{j=1}^{m}(1 - \overline{w}_j)^{S_{ij}}), \{ \prod_{j=1}^{m}(\overline{w}_j)^{S_{ij}} \}, \{ \prod_{j=1}^{m}(\overline{w}_j)^{S_{ij}} \} \} $$

I-flow

$$ \Psi^<\omega_i^t) = SVNHFWA_{I_{ij}}(\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_m) $$

(15)

$$ = \{(1 - \prod_{j=1}^{m}(1 - \overline{w}_j)^{I_{ij}}), \{ \prod_{j=1}^{m}(\overline{w}_j)^{I_{ij}} \}, \{ \prod_{j=1}^{m}(\overline{w}_j)^{I_{ij}} \} \} $$

By using Equation (1), we calculate the score function of the corresponding S-flow $\Psi^>(\omega_i^t)$ and I-flow $\Psi^<\omega_i^t)$, respectively. Hence, we obtain the S-flow and I-flow of alternative $\omega_i^t$ as $\omega_i^t(\Psi^>(\omega_i^t), \Psi^<\omega_i^t))$. It seems that if the S-flow $\Psi^>(\omega_i^t)$ is larger and the I-flow $\Psi^<\omega_i^t)$ is smaller, the alternative $\omega_i^t$ is preferable.

Step 6: Superiority ranking rule (SRR-Rule):

SRR - Rule1. If $\Psi^>(\omega_i^t) > \Psi^>(\omega_i^t)$ and $\Psi^<\omega_i^t) < \Psi^<\omega_i^t)$, then $\omega_i^t > \omega_i^l$;

SRR - Rule2. If $\Psi^>(\omega_i^t) > \Psi^>(\omega_i^t)$ and $\Psi^<\omega_i^t) = \Psi^<\omega_i^t)$, then $\omega_i^t > \omega_i^l$;

SRR - Rule3. If $\Psi^>(\omega_i^t) = \Psi^>(\omega_i^t)$ and $\Psi^<\omega_i^t) < \Psi^<\omega_i^t)$, then $\omega_i^t > \omega_i^l$. 


Algorithm 1 Cont.

Inferiority ranking rule (IR-Rule):
IR - Rule1. If $\Psi^>(\omega_i^t) < \Psi^>(\omega_j^t)$ and $\Psi^<(\omega_i^t) > \Psi^<(\omega_j^t)$, then $\omega_i^t < \omega_j^t$;
IR - Rule2. If $\Psi^>(\omega_i^t) < \Psi^>(\omega_j^t)$ and $\Psi^<(\omega_i^t) = \Psi^<(\omega_j^t)$, then $\omega_i^t < \omega_j^t$;
IR - Rule3. If $\Psi^>(\omega_i^t) = \Psi^>(\omega_j^t)$ and $\Psi^<(\omega_i^t) > \Psi^<(\omega_j^t)$, then $\omega_i^t < \omega_j^t$.

Step 7: By incorporating the SR-Rule and the IR-Rule, we can achieve the best alternative $\omega_i^l (i = 1, 2, \cdots, l)$.

A flow chart of the SIR method for supplier selection is shown in Figure 1.

---

**Figure 1.** Flow chart of SIR method for supplier selection.

5. Extension of CV Method for SVNHF Information

In this section, a new MCDM method is developed in Algorithm 2, which is extension of the CV method to SVNHFSSs. Table 5 includes various types of cases where different systems were used to help with operations or decision-making in the food supply chain.

Algorithm 2 (CV method for SVNHFSSs)

**Step 1:** Obtain the decision matrices from the decision makers, with alternative $\omega_j^C$ evaluated on the basis of criterion $\omega_i^C$, given in Table 6. The aggregated decision matrix $\overline{p}_{ij}$ is obtained using step 1, step 2 and step 3(2).

**Step 2:** Decision makers also give weight $w_i$ to the criteria with the condition that the sum of weights must be equal to 1. Then, the multiplication of decision matrix is computed with criteria weights, to obtain the matrix $\overline{q}_{ij}$.

**Step 3:** Find the score function of each SVNHF.

**Step 4:** Compute the ranking of the alternatives according to their score function values.
Case Study

The food industry is critical for delivering the essentials for a variety of uses and tendencies [43]. Food must be stored, supplied, and marketed as soon as it is cultivated or produced so that it can reach the ultimate customers on time. Every year, worldwide food loss would supply more than enough to nourish the world’s nearly 1 billion starving people. In Pakistan, it is anticipated that 40% of food is wasted. Food is produced in sufficient quantities to feed the overall population of Pakistan, but due to food waste, an expected 6 out of 10 inhabitants go to bed without dinner. Pakistan stands 107th out of 118 developing countries on the International Poverty Index. Approximately one-third of all produced food is destroyed or wasted each year (approximately 1.3 billion tonnes) [44]. Two-thirds of all waste in food (roughly 1 billion tonnes) occurs at the supply chain stage, which encompasses cultivation, shipments, and storage [45].

The term FSCM has been utilized to depict the activities or procedures occurring during the yield, dispersion, and the use of various foods in order to preserve their quality and safety in effective and efficient ways [46,47]. The relevance of factors such as safety, food quality, and freshness within a specified time frame distinguishes FSCM from many other supply chains including furniture logistics and supply chain management, making the underpinning supply chain more convoluted and unmanageable [48]. As the challenges of global coordination have increased, the attention turned from a single echelon, such as food production, to the effectiveness and efficiency of the comprehensive supply chain. That is, the food supply chain resources such as vehicles, storage areas, transport services, and laborers will be used proficiently to ensure the quality and safety of food through effective efforts including optimization decisions [49]. The relevance of value chains in FSCM is that they benefit both consumers and producers. The traceability of food has become increasingly popular in recent decades, with a wide range of applications. Because of the emergence of food, FSCM is becoming more dynamic and complex in order to meet the diversifying and globalized industries.

FSCM IT Systems

There is no doubt that IT systems are crucial in FSCM because so much can go wrong, such as vehicles, food suppliers, data entry, and so on. The decision-making systems and traceability for FSCM are used as examples of current state-of-the-art situations that professionals can use when instituting IT-based solutions. A food’s traceability consists in a data trail that follows the physiological trail of the food through different phases [50]. Some systems track food all the way from the retailer to the farm, while others only examine key stages of the supply chain. Some traceability systems gather information only for tracking food products to the minute of manufacturing or logistics path, while others track only superficial data, such as in a vast geographical area [51]. Aside from FSCM’s traceability systems, other decision-making processes in the food industry include implementation, strategy, vehicle maintenance, and WMS.

The internationalization of food production, logistics, and utilization has resulted in an integrated world for FSCM, whose models are critical in ensuring consistently high standards and security in food products [52]. Quality of food, high delivery performance, and food security appear to be more important in these models. Multi-objective considerations are also common; for example, food quality assurance is incorporated into decision models. Recently, supply chain effectiveness and value chain evaluation have received special attention, since the international FSCM is becoming increasingly important.
### Table 5. Cases involving IT systems in FSCM that have been reported.

| Instance | Firm | Network | Enhancement |
|----------|------|---------|-------------|
| Aramyan et al. [53] | Tomato firm | Performance measurement system | Efficiency and flexibility have both improved, food quality has improved, and there is a faster response time. |
| Bevilacqua et al. [54] | Tronto Valley | ARIS | Three types of costs are being reduced; improved traceability. |
| Pagell and Wu [55] | Pizza restaurants | TQM Lean/JIT | Enhanced information sharing, superior quality, enhanced logistical efficiency. |
| Tuncel and Alpan [56] | A medium size | Risk management | The percentage of orders completed on time has increased to 90.6%, with risk reduction rising by 9.9%. |
| Zhu et al. [57] | A food manufacturer | Customer cooperation system | Customer cooperation has improved; internal environment management has been improved. |
| Jacxsens et al. [58] | A fresh producer | Food safety management system | Food of higher quality; improved risk management ability. |
| Friel et al. [59] | Agri-food supply chain | H&S food decision-making system | A more nutritious diet, with improved environmental sustainability. |
| Savino et al. [60] | A chestnut company | Value chain management system | Increased long-term viability, CO₂ reduced emissions, enhanced value chain. |
| Banasik et al. [61] | A mushroom manufacturer | Supply chain management system | Overall profitability increased by 11%, with improved environmental performance |
| Sgarbossa and Russo [62] | 6 Firms | FSCM system | Conserving energy, costs of disposal avoided, enhanced productivity. |

Choosing a supplier is a critical element of any business’s operations. Reputation, reliability, service, cost, and value for money are all important considerations. The aim of supplier selection is to identify the best supplier who delivers the best value for money in terms of product or service. Suitable supplier selection yields good profit and quality in the end. The supplier is treated as an integral part of the organization in this strategic alliance. All purchasers must choose a supplier, and it is a critical step in the acquisition process. Purchasers should go through several stages of decision making and develop their own selection criteria for selecting appropriate suppliers.

Some interesting studies for supply chain can be seen in [63–65].

### 6. MCDM Process

The RH Flour Mills in Lahore wants to find the best supplier for one of its key components in the manufacturing process. Four suppliers were left as alternatives. The four criteria considered were: quality and safety, delivery, social responsibility, and service. The suppliers are evaluated using the recommended methodology by a group of decision makers. In multi-criteria decision making with a fuzzy environment, four decision makers were chosen, consisting of supplier experts and expert academics. The steps in the procedure for selecting the best green supplier are as follows.
The decision-making process using Algorithm 1 is illustrated as follows:

Let $K = \{\omega_1^k, \omega_2^k, \omega_3^k, \omega_4^k\}$ be the set of alternatives and $C = \{\omega_1^c, \omega_2^c, \omega_3^c\}$ be the set of attributes. Assume that $\mathcal{E} = \{\omega_1^e, \omega_2^e, \omega_3^e\}$ is the set of experts. Then, the single-valued neutrosophic hesitant fuzzy decision matrices are expressed in Table 6, the weights of the experts are given in Table 7, and the weights of the attributes are shown in Table 8.

Table 6. Single-valued neutrosophic hesitant fuzzy decision matrices.

| $\omega_1^k$ | $\omega_1^c$ | $\omega_2^c$ | $\omega_3^c$ | $\omega_4^c$ |
|--------------|-------------|--------------|--------------|--------------|
| $\omega_1$  | {0.125, 0.326}, {0.444}, {0.236} | {0.230}, {0.200}, {0.912} | {0.200}, {0.111, 0.253}, {0.751} |
| $\omega_2$  | {0.111}, {0.750, 0.905}, {0.216} | {0.250}, {0.112}, {0.445} | {0.523}, {0.521}, {0.423}, {0.624} |
| $\omega_3$  | {0.149}, {0.242}, {0.889} | {0.246, 0.925}, {0.851}, {0.124} | {0.259, 0.590}, {0.125}, {0.300} |
| $\omega_4$  | {0.195}, {0.860}, {0.120, 0.120} | {0.315}, {0.866}, {0.606} | {0.400, 0.580}, {0.430}, {0.118} |

Table 7. The weights of the experts.

| Experts | SVNHFes                  |
|---------|--------------------------|
| $\omega_1$ | {0.831, 0.120}, {0.310}, {0.456, 0.511} |
| $\omega_2$ | {0.802}, {0.408, 0.300}, {0.472} |
| $\omega_3$ | {0.711, 0.177}, {0.234, 0.500}, {0.500, 0.250} |

Table 8. The weights of the attributes.

| $\omega_1^c$ | $\omega_2^c$ | $\omega_3^c$ |
|--------------|--------------|--------------|
| $\omega_1$  | {0.311}, {0.200}, {0.500} | {0.111, 0.210}, {0.100}, {0.805} | {0.200, 0.666}, {0.555}, {0.213} |
| $\omega_2$  | {0.207}, {0.300, 0.804}, {0.274} | {0.750}, {0.455}, {0.123} | {0.100}, {0.578}, {0.920} |
| $\omega_3$  | {0.170, 0.701}, {0.230}, {0.240} | {0.380}, {0.335, 0.510}, {0.200} | {0.310}, {0.456, 0.687}, {0.836} |

Step 1: Compute the individual measure degree $\rho_k (k = 1, 2, 3)$ using Equation (6), given by

$$\rho = (0.432, 0.466, 0.573)^\mu$$

Step 2: Acquire the normalized vector using Equation (7), given as

$$\omega = (0.294, 0.317, 0.390)^\mu$$

Step 3: The attribute weights can be obtained using Equation (8), which are expressed as follows:

- $\mathbf{w}_1 = \{(0.226, 0.480), (0.240, 0.328), (0.310)\}$
- $\mathbf{w}_2 = \{(0.483, 0.501), (0.258, 0.304), (0.258)\}$
- $\mathbf{w}_3 = \{(0.216, 0.394), (0.520, 0.611), (0.576)\}$
The aggregated decision matrices can be obtained using Equation (9), and can be written as follows:

\[
(\mathbf{p}_{ij})_{4 \times 3} = \begin{bmatrix}
\{0.232, 0.283, 0.389, 0.336\}, \{0.425\}, \{0.223\} \\
\{0.319\}, \{0.313, 0.349, 0.331, 0.369\}, \{0.294\} \\
\{0.279\}, \{0.253, 0.261\}, \{0.877\} \\
\{0.605\}, \{0.254\}, \{0.120, 0.171, 0.359, 0.215\}
\end{bmatrix}
\]

**Step 4:** Acquire the performance function \(f_{ij}\) using Equation (10):

\[
(f_{ij})_{4 \times 3} = \begin{bmatrix}
0.586 & 0.540 & 0.464 \\
0.625 & 0.644 & 0.599 \\
0.179 & 0.767 & 0.688 \\
1.000 & 0.198 & 0.601
\end{bmatrix}
\]

The threshold attribute function was set to

\[
\phi_k(d) = \begin{cases} 
0.01 & \text{if } d > 0 \\
0.00 & \text{if } d \leq 0 
\end{cases}
\]

Acquire the superiority matrix (\(S\)-matrix) using Equation (12):

\[
S = \begin{bmatrix}
0.01 & 0.01 & 0 \\
0.02 & 0.02 & 0.01 \\
0 & 0.03 & 0.02 \\
0.03 & 0.01 & 0
\end{bmatrix}
\]

Acquire the inferiority matrix (\(I\)-matrix) using Equation (13):

\[
I = \begin{bmatrix}
0.02 & 0.02 & 0.03 \\
0.01 & 0.01 & 0.02 \\
0.03 & 0 & 0 \\
0 & 0.03 & 0.01
\end{bmatrix}
\]

**Step 5:** Measure the flow of superiority and inferiority using Equations (14) and (15), which are exhibited in Tables 9 and 10.

**Table 9. SVNHF superiority flow.**

| \(\Psi > (\omega_i^f)\) | \(S(\Psi > (\omega_i^f))\) |
|-------------------|------------------|
| \(\omega_1^f\) | \{0.00709, 0.00981\}, \{0.00498, 0.00632\}, \{0.00568\} | 0.6657 |
| \(\omega_2^f\) | \{0.01634, 0.02356\}, \{0.01516, 0.01875\}, \{0.01712\} | 0.6619 |
| \(\omega_3^f\) | \{0.02097, 0.02685\}, \{0.02334, 0.02745\}, \{0.02502\} | 0.6597 |
| \(\omega_4^f\) | \{0.0111, 0.02228\}, \{0.0176, 0.02206\}, \{0.02082\} | 0.6587 |

**Table 10. SVNHF inferiority flow.**

| \(\Psi < (\omega_i^f)\) | \(S(\Psi < (\omega_i^f))\) |
|-------------------|------------------|
| \(\omega_1^f\) | \{0.02066, 0.03144\}, \{0.02556, 0.03097\}, \{0.02864\} | 0.6564 |
| \(\omega_2^f\) | \{0.01175, 0.01769\}, \{0.01538, 0.01854\}, \{0.0172\} | 0.6602 |
| \(\omega_3^f\) | \{0.00678, 0.01444\}, \{0.0072, 0.00984\}, \{0.0093\} | 0.6643 |
| \(\omega_4^f\) | \{0.01665, 0.01897\}, \{0.01294, 0.01523\}, \{0.0135\} | 0.6634 |

**Step 6:** Integrate Table 9 with the \(SR\)-Rule, and the following seems to be accessible:

\(\omega_1^f \succ \omega_2^f \succ \omega_3^f \succ \omega_4^f\)
Combine Table 10 with the \( IR \)-Rule, and the following seems to be accessible:

\[
\omega^1_x \succ \omega^2_x \succ \omega^4_x \succ \omega^3_x
\]

**Step 7:** The most desirable alternative, according to the results of the \( SR \)-Rule and the \( IR \)-Rule, is \( \omega^1_x \). A representation of SVNHF superiority and inferiority flow is shown in Figure 2.

![SVNHF superiority and inferiority flow](image)

**Figure 2.** Representation of SVNHF superiority and inferiority flow.

The decision-making process using Algorithm 2 is illustrated as follows: **Step 1:** Consider SVNHF decision matrices given in Table 6. Obtain aggregated decision matrix \((\overline{p}_{ij})_{4 \times 3}\) using Step 1, Step 2, and Step 3. **Step 2:** The decision makers provide weights to three criteria as \( w_1 = 0.235, w_2 = 0.312, \) and \( w_3 = 0.453, \) with \( \sum w_i = 1 \).)

\[
(\overline{p}_{ij})_{4 \times 3} = \begin{pmatrix}
0.232, 0.283, 0.289, 0.336, & 0.425, & 0.223 \\
0.319, & 0.313, & 0.349, & 0.331, & 0.369, & 0.294 \\
0.279, & 0.253, & 0.261, & 0.677 \\
0.605, & 0.254, & 0.120, & 0.171, & 0.359, & 0.215
\end{pmatrix}
\]

\[
(\overline{p}_{ij})_{4 \times 1} = \begin{pmatrix}
0.484, 0.495, 0.496, 0.507, & 0.0005, & 0.0007, & 0.0021, & 0.0022 \\
0.446, 0.455, & 0.0006, & 0.0006, & 0.0005, & 0.0011, & 0.0017, & 0.0013 \\
0.429, & 0.529, & 0.0003, & 0.0006, & 0.0026 \\
0.461, & 0.485, & 0.0022, & 0.0024, & 0.0005, & 0.0008, & 0.0016, & 0.0010
\end{pmatrix}
\]

**Step 3:** Compute the score values of each alternative. The score values of the alternatives are given in Table 11.

| Alternatives | Score Values |
|--------------|--------------|
| \( \omega^1_x \) | 0.8309 |
| \( \omega^2_x \) | 0.8158 |
| \( \omega^3_x \) | 0.8253 |
| \( \omega^4_x \) | 0.8233 |

**Step 4:** Rank the alternatives according to their score values.

\[
\omega^1_x \succ \omega^3_x \succ \omega^4_x \succ \omega^2_x
\]

As a result, \( \omega^1_x \) is the best supplier among the four alternatives according to the qualities of all criteria.
Comparative Analysis

This paper develops new techniques for modeling uncertainties using SVNHF information. We compare the ranking of alternatives using proposed SIR method and the CV method for SVNHFSSs. If we use the SVNHF SIR approach to assemble the alternatives, they are ranked for superiority flow as

\[ \omega_1^x \succ \omega_2^x \succ \omega_3^x \succ \omega_4^x \]

and for inferiority flow as

\[ \omega_1^x \succ \omega_2^x \succ \omega_4^x \succ \omega_3^x \]

On the other hand, when we use the technique of the CV method, the ranking of the alternatives becomes

\[ \omega_1^x \succ \omega_3^x \succ \omega_4^x \succ \omega_2^x \]

Based on these findings, it is clear that the ranking of the alternatives is not same. However, the optimal alternative \( \omega_1^x \) remains identical in both MCDM methods.

The ranking of alternatives using the SIR method and the CV method is also shown in Figure 3.

![Figure 3. Ranking of alternatives using the SIR method and the CV method.](image)

7. Conclusions

This paper was designed to introduce the concept of single-valued neutrosophic hesitant fuzzy (SVNHF) topology and its applications in data analysis for uncertain supply chains. An SVNHFSS is a hybrid structure of a hesitant fuzzy set (HFS) and a single-valued neutrosophic set (SVNS), which is a novel concept for modeling uncertainties in real-life circumstances with key features of three membership functions: truth-hesitancy membership function, indeterminacy-hesitancy membership function and falsity-hesitancy membership function. Using the characteristics of SVNHFSSs, we defined the notion of SVNHF topology. We investigated the fundamental properties of SVNHF topology, such as the SVNHF closure, the SVNHF interior, the SVNHF exterior, and the SVNHF frontier, as well as the SVNHF dense set and the SVNHF base, with the help illustrative examples. Eventually, to demonstrate and validate the SIR method and the CV method in terms of rationality and scientific basis, a real-life example of supplier selection in a food supply chain was provided. A comparative analysis was also given to discuss the validity and advantage of proposed techniques. The proposed methods can be further extended to investigate the dynamics of human decision analysis, humanized computing, data analysis, computational intelligence, and healthcare.
Author Contributions: M.R.: conceptualization, formal analysis, and supervision. Y.A.: investigation, supervision, and funding acquisition. S.B.: methodology and writing—review and editing. S.T.: investigation and writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University, Abha, Saudi Arabia, for funding this work through a research groups program under grant number RGP2/211/43.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are thankful to the Editor-in-Chief and reviewers for their valuable suggestions to improve the quality of this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy sets. Inf. Control. 1965, 8, 338–353. [CrossRef]
2. Atanassov, T.K. Intuitionistic fuzzy sets. Fuzzy Sets Syst. 1986, 20, 87–96. [CrossRef]
3. Yager, R.R. Pythagorean fuzzy subsets. In Proceedings of the Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, AB, Canada, 24–28 June 2013; pp. 57–61.
4. Yager, R.R. Generalized orthopair fuzzy sets. IEEE Trans. Fuzzy Syst. 2017, 25, 1222–1230. [CrossRef]
5. Cuen, B.C. Picture fuzzy sets. J. Comput. Sci. Cybern. 2014, 30, 409–420.
6. Mahmood, T.; Ullah, K.; Khan, Q.; Jan, N. An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. Neural Comput. Appl. 2019, 31, 7041–7053. [CrossRef]
7. Ashraf, S.; Abdullah, S.; Mahmood, T.; Ghani, F.; Mahmood, T. Spherical fuzzy sets and their applications in multi-attribute decision making problems. J. Intell. Fuzzy Syst. 2019, 36, 2829–2844. [CrossRef]
8. Gündogdu, F.K.; Kahraman, C. Spherical fuzzy sets and spherical fuzzy TOPSIS method. J. Intell. Fuzzy Syst. 2018, 36, 337–352. [CrossRef]
9. Smarandache, F. A Unifying Field in Logics: Neutrosophy: Neutrosophic Probability, Set and Logic; American Research Press: Rehoboth, DE, USA, 1999; pp. 1–141.
10. Wang, H.; Smarandache, F.; Zhang, Y.; Sunderraman, R. Single-Valued Neutrosophic Sets; Infinite Study: Coimbatore, India, 2010; pp. 1–4.
11. Molodtsov, D. Soft set theory-first results. Comput. Math. Appl. 1999, 37, 19–31. [CrossRef]
12. Hashmi, M.R.; Riaz, M.; Smarandache, F. m-polar Neutrosophic Topology with Applications to Multi-Criteria Decision-Making in Medical Diagnosis and Clustering Analysis. Int. J. Fuzzy Syst. 2020, 22, 273–292. [CrossRef]
13. Torra, V. Hesitant fuzzy sets. Int. J. Intell. Syst. 2010, 25, 529–539. [CrossRef]
14. Ye, J. Multiple-attribute Decision-Making Method under a Single-Valued Neutrosophic Hesitant Fuzzy Environment. J. Intell. Syst. 2015, 24, 23–36. [CrossRef]
15. Tanuwijaya, B.; Selvachandran, G.; Son, L.H.; Basset, M.A.; Huynh, H.X.; Pham, V.H.; Ismail, M. A Novel Single Valued Neutrosophic Hesitant Fuzzy Time Series Model: Applications in Indonesian and Argentinian Stock Index Forecasting. IEEE Access 2020, 8, 60126–60141. [CrossRef]
16. Liu, C.F.; Luo, Y.S. New aggregation operators of single-valued neutrosophic hesitant fuzzy set and their application in multi-attribute decision making. Pattern Anal. Appl. 2019, 22, 417–427. [CrossRef]
17. Wang, R.; Li, Y. Generalized Single-Valued Neutrosophic Hesitant Fuzzy Prioritized Aggregation Operators and Their Applications to Multiple Criteria Decision-Making. Information 2018, 9, 10. [CrossRef]
18. Giri, B.C.; Molla, M.U.; Biswas, P. TOPSIS Method for Neutrosophic Hesitant Fuzzy Multi-Attribute Decision Making. Informatica 2020, 31, 35–63. [CrossRef]
19. Xu, X. The SIR method: A superiority and inferiority ranking method for multiple criteria decision making. Eur. J. Oper. Res. 2001, 131, 587–602. [CrossRef]
20. Tam, C.M.; Tong, T.K.L.; Wong, Y.W. Selection of concrete pump using the superiority and inferiority ranking method. J. Constr. Eng. Manag. 2004, 130, 827–834. [CrossRef]
21. Tam, C.M.; Tong, T.K. Locating large-scale harbourfront project developments using SIR method with grey aggregation approach. Constr. Innov. 2008, 8, 120–136. [CrossRef]
22. Chai, J.; Liu, J.N.K. A Novel Multicriteria Group Decision Making Approach with Intuitionistic Fuzzy SIR Method. In Proceedings of the World Automation Congress, Kobe, Japan, 19–23 September 2010.
23. Ma, Z.J.; Zhang, N.; Dai, Y. A novel SIR method for multiple attributes group decision making problem under hesitant fuzzy environment. J. Intell. Fuzzy Syst. 2014, 26, 2119–2130. [CrossRef]
24. Peng, X.; Yang, Y. Some results for Pythagorean fuzzy sets. *Int. J. Intell. Syst.* 2015, 30, 1133–1160. [CrossRef]
25. Rouhani, S. A fuzzy superiority and inferiority ranking based approach for IT service management software selection. *Kybernetes* 2017, 46, 728–746. [CrossRef]
26. Chen, T. A novel PROMETHEE-based outranking approach for multiple criteria decision analysis with Pythagorean fuzzy information. *IEEE Access* 2018, 6, 54495–54506. [CrossRef]
27. Tavana, M.; Zareinejad, M.; Arteaga, F.J.S. An intuitionistic fuzzy-grey superiority and inferiority ranking method for third-party reverse logistics provider selection. *Int. J. Syst. Sci. Oper. Logist.* 2018, 5, 175–194. [CrossRef]
28. Zhao, N.; Xu, Z.; Ren, Z. Hesitant fuzzy linguistic prioritized superiority and inferiority ranking method and its application in sustainable energy technology evaluation. *Inf. Sci.* 2019, 478, 239–257. [CrossRef]
29. Geetha, S.; Narayanaamoorthy, S. Superiority and inferiority ranking method with hesitant Pythagorean fuzzy set for solving MCDM problems. *Malaya J. Mat.* 2020, 1, 11–15.
30. Jian, J.; Zhan, N.; Su, J. A novel superiority and inferiority ranking method for engineering investment selection under interval-valued intuitionistic fuzzy environment. *J. Intell. Fuzzy Syst.* 2019, 37, 6645–6653. [CrossRef]
31. Smarandache, F. New types of Neutrosophic Set/Logic/Probability, Neutrosophic Over-/Under-Set, Neutrosophic Refined Set, and Their Extension to Plithogenic Set/Logic/Probability, with Applications; MDPI: Basel, Switzerland, 2019.
32. Seikh, M.R.; Dutta, S. A Nonlinear Programming Model to Solve Matrix Games with Pay-offs of Single-valued Neutrosophic Numbers. *Neutrosophic Sets Syst.* 2021, 47, 366–383.
33. Saha, A.; Paul, A. Generalized Weighted Exponential Similarity Measures of Single Valued Neutrosophic Sets. *Int. J. Neutrosophic Sci.* 2019, 0, 57–66.
34. Alcantud, J.C.R.; Garcia, G.S.; Akram, M. OWA aggregation operators and multi-agent decisions with N-soft sets, *Expert Syst. Appl.* 2022, 203, 1–17.
35. Sitara, M.; Akram, M.; Riaz, M. Decision-making analysis based on q-rung picture fuzzy graph structures. *J. Appl. Math. Comput.* 2021, 67, 541–577. [CrossRef]
36. Riaz, M.; Riaz, M.; Jamil, N.; Zararsiz, Z. Distance and similarity measures for bipolar fuzzy soft sets with application to pharmaceutical logistics and supply chain management. *J. Intell. Fuzzy Syst.* 2022, 42, 3169–3188. [CrossRef]
37. Farid, H.M.A.; Riaz, M. Some generalized q-rung orthopair fuzzy Einstein interactive geometric aggregation operators with improved operational laws. *Int. J. Intell. Syst.* 2021, 36, 7239–7273. [CrossRef]
38. Zararsiz, Z.; Riaz, M. Bipolar fuzzy metric spaces with application. *Comput. Appl. Math.* 2022, 41, 41–49. [CrossRef]
39. Riaz, M.; Tanveer, S.; Pamucar, D.; Qin, D.S. Topological data analysis with spherical fuzzy soft AHP-TOPSIS for environmental mitigation system. *Mathematics* 2022, 10, 1–36. [CrossRef]
40. Riaz, M.; Farid, H.M.A.; Wang, W.; Pamucar, D. Interval-valued linear Diophantine fuzzy Frank aggregation operators with multi-criteria decision-making. *Mathematics* 2022, 10, 1–36. [CrossRef]
41. Biswas, P.; Pramanik, S.; Giri, B.C. GRA Method of Multiple Attribute Decision Making with Single Valued Neutrosophic Hesitant Fuzzy Set Information. In *New Trends in Neutrosophic Theory and Applications*; Pons Editions: Brussels, Belgium, 2016.
42. Brans, J.P.; Vincke, P.; Mareschal, B. How to select and how to rank projects: The PROMETHEE method. *Eur. J. Oper. Res.* 1986, 24, 228–238. [CrossRef]
43. Cooper, M.C.; Ellram, L.M. Characteristics of supply chain management and the implications for purchasing and logistics strategy. *Int. J. Logist. Manag.* 1993, 4, 13–24. [CrossRef]
44. Manning, L.; Baines, R.; Chadd, S. Quality assurance models in the food supply chain. *Br. Food J.* 2006, 108, 91–104. [CrossRef]
45. Fritz, M.; Schiefer, G. Food chain management for sustainable food system development: A European research agenda. *Agribusiness* 2008, 24, 440–452. [CrossRef]
46. Blandon, J.; Henson, S.; Cranfield, J. Small-scale farmer participation in new agri-food supply chains: Case of the supermarket supply chain for fruit and vegetables in Honduras. *J. Int. Dev.* 2016, 28, 777–800. [CrossRef]
47. Smith, G.; Tatum, J.; Belk, K.; Scanga, J.; Grandin, T.; Sofos, J. Traceability from a US perspective. *Meat Sci.* 2005, 71, 174–193. [CrossRef]
48. Dickinson, D.L.; Bailey, D. Meat traceability: Are US consumers willing to pay for it? *J. Agric. Resour. Econ.* 2002, 27, 348–364.
49. Choi, T.M.; Chiu, C.H.; Chan, H.K. Risk management of logistics systems. *Transp. Res. Logist. Transp. Rev.* 2016, 90, 1–6. [CrossRef]
50. Aramyan, L.H.; Lansink, A.G.O.; Vorst, J.G.V.D.; Van Kooten, O. Performance measurement in agri-food supply chains: A case study. *Supply Chain. Manag.* Int. J. 2007, 12, 304–315. [CrossRef]
51. Bevilacqua, M.; Ciarpata, F.; Giacchetta, G. Business process reengineering of a supply chain and a traceability system: A case study. *J. Food Eng.* 2009, 93, 13–22. [CrossRef]
52. Pagell, M.; Wu, Z. Building a more complete theory of sustainable supply chain management using case studies of 10 exemplars. *J. Supply Chain. Manag.* 2009, 45, 37–56. [CrossRef]
56. Tuncel, G.; Alpan, G. Risk assessment and management for supply chain networks: A case study. *Comput. Ind.* **2010**, *61*, 250–259. [CrossRef]

57. Zhu, Q.H.; Geng, Y.; Fujita, T.; Hashimoto, S. Green supply chain management in leading manufacturers: Case studies in Japanese large companies. *Manag. Res. Rev.* **2010**, *33*, 380–392. [CrossRef]

58. Jacxsens, L.; Luning, P.; Vorst, J.V.D.; Devlieghere, F.; Leemans, R.; Uyttendaele, M. Simulation modelling and risk assessment as tools to identify the impact of climate change on microbiological food safety—The case study of fresh produce supply chain. *Food Res. Int.* **2010**, *43*, 1925–1935. [CrossRef]

59. Friel, S.; Barosh, L.J.; Lawrence, M. Towards healthy and sustainable food consumption: An Australian case study. *Public Health Nutr.* **2014**, *17*, 1156–1166. [CrossRef]

60. Savino, M.M.; Manzini, R.; Mazza, A. Environmental and economic assessment of fresh fruit supply chain through value chain analysis: A case study in chestnuts industry. *Prod. Plan. Control.* **2015**, *26*, 1–18. [CrossRef]

61. Banasik, A.; Kanellopoulos, A.; Claassen, G.; Bloemhof-Ruwaard, J.M.; Vorst, J.G.V.D. Closing loops in agricultural supply chains using multi-objective optimization: A case study of an industrial mushroom supply chain. *Int. J. Prod. Econ.* **2017**, *183*, 409–420. [CrossRef]

62. Sgarbossa, F.; Russo, I. A proactive model in sustainable food supply chain: Insight from a case study. *Int. J. Prod. Econ.* **2017**, *183*, 596–606. [CrossRef]

63. Tzounis, A.; Katsoulas, N.; Bartzanas, T.; Kittas, C. Internet of Things in agriculture, recent advances and future challenges. *Biosyst. Eng.* **2017**, *164*, 31–48. [CrossRef]

64. Feng, F.; Jun, Y.B.; Liu, X.; Li, L. An adjustable approach to fuzzy soft set based decision making. *J. Comput. Appl. Math.* **2010**, *234*, 10–20.

65. Zhong, R.; Xu, X.; Wang, L. Food supply chain management: Systems, implementations, and future research. *Ind. Manag. Data Syst.* **2017**, *117*, 2085–2114. [CrossRef]