Witnessing multipartite entanglement by detecting asymmetry

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The characterization of quantum coherence in the context of quantum information theory and its interplay with quantum correlations is currently subject of intense study. Coherence in an Hamiltonian eigenbasis yields asymmetry, the ability of a quantum system to break a dynamical symmetry generated by the Hamiltonian. We here propose an experimental strategy to witness multipartite entanglement in many-body systems by evaluating the asymmetry with respect to an additive Hamiltonian. We test our scheme by simulating asymmetry and entanglement detection in a three-qubit GHZ-diagonal state.

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I. INTRODUCTION

Quantum Information Theory provides important insights on the foundations of Quantum Mechanics, as well as its technological applications. The framework of resource theories characterizes the quantum laws as constraints, and the properties of quantum systems as resources for information processing [1]. In this context, the degree of coherent superposition of a state \( \sum_i \alpha_i |i\rangle \langle i| \alpha_i | i \rangle = 1 \), i.e. coherence (we omit the quantum label, from now on) in a reference basis \( | i \rangle \), is a resource. The crucial question is to determine how to obtain a computational advantage powered by coherence [2–19]. The coherence of a finite-dimensional quantum state \( \rho \) has been defined as its distinguishability from the sets of states which are diagonal in a given basis [14–19]. Yet, to date, there is no operational interpretation for such definition of coherence. A concurrent body of work has linked the coherence of \( \rho \) in a basis \( | h \rangle \) to the degree of uncertainty in a measurement of an observable \( H = \sum_i h |i\rangle \langle i| \) on \( \rho \). Such genuinely quantum uncertainty has been proven to have an operational interpretation, corresponding to the sensitivity of the state to a phase shift generated by \( H \) [2–13]. From a physics perspective, coherence here underpins \( U(1) \)-asymmetry. The asymmetry of a quantum system quantifies its ability to be a reference frame under a phase superselection rule, where \( H \) is the observable whose coherent superpositions are prohibited (e.g. electric charge, energy). In other words, asymmetry is the geometric property of a quantum system which makes it able to break a symmetry generated by an Hamiltonian \( H \).

Further studies bridged the gap between these recent theoretical findings and the experimental implementation of quantum information processing, by providing a strategy to measure the asymmetry of an arbitrary quantum state in the laboratory with the current technology [8] (for coherence witnesses, see [20–22]). These results paved the way for investigating the link between coherence and quantum properties of multipartite systems. In particular, the relationship between coherence and quantum correlations has been explored [6, 9, 13, 16, 18, 19, 23, 24]. In this work, we show how detecting asymmetry in states of multipartite qubit systems allows an experimentalist to verify entanglement with limited resources. Entanglement is a crucial property for quantum information processing [25], e.g. providing speed-up in communication and metrology protocols [26, 27]. Yet, it is hard to be quantified in both theoretical and experimental practice [28–30]. On this purpose, we here introduce an experimentally friendly witness of multipartite entanglement in terms of the asymmetry with respect to an additive Hamiltonian.

The structure of the paper is the following. In Sec. II A, we recall that the quantum Fisher information, a measure of sensitivity of a state to phase shifts employed in quantum metrology [27, 31–33], is an asymmetry quantifier. It is possible to identify a lower bound of it in terms of traces of density matrix powers. We calculate how much the experimentally reconstructed bound deviates from the theoretical quantity (Sec. II B). Also, we express the lower bound for one, two and three-qubit states in terms of finite phase shifts generated by spin observables. These quantities can be evaluated by single qubit interferometry [34–38], as well as local projective measurement schemes [8, 39–44], without performing full state reconstruction. In Sec. III A, we show that the asymmetry lower bound witnesses genuinely multipartite entanglement when measured with respect to an additive multipartite Hamiltonian. We complete the study with a demonstrative example (Sec. III B). We simulate the evaluation of asymmetry and entanglement in a GHZ-diagonal state by a seven-qubit quantum information processor. We draw our conclusions in Sec. IV.

II. MEASURING ASYMMETRY

A. Theoretically consistent measure of asymmetry

Quantum Fisher information

In the resource theory of asymmetry [2–5, 7], the consumable resource is any system whose state is not commuting with a fixed, bounded observable \( H \) with spectral decomposition \( H = \sum_i h |i\rangle \langle i| \). A system in the incoherent

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state $\rho_H$ such that $\rho_H = \sum_{i} c_i |\psi_i\rangle \langle \psi_i|$, where $\{\psi_i\}$ are orthogonal, is free, in the sense that it can be arbitrarily added or discarded in a quantum protocol without affecting the available asymmetry. Free states are invariant under phase rotations generated by $H$: $e^{iH_0} \rho_H e^{-iH_0} = \rho_H$, $\forall H_0 \in \mathbb{R}$. The free operations are the CPTP (completely-positive trace-preserving) maps $\mathcal{E}_H$ which cannot increase the amount of asymmetry in a state. They are identified by maps commuting with the unitary evolution generated by the observable under scrutiny, $e^{iH_0} \mathcal{E}_H(e^{iH_0}) = \mathcal{E}_H(e^{iH_0} e^{iH_0})$, $\forall H, \theta$. Their explicit form is studied in Ref. [3]. Several quantifiers of asymmetry have been proposed [3, 7, 8]. Here we adopt the viewpoint of asymmetry as a measure of the state usefulness in a phase estimation scenario. The symmetric logarithmic derivative quantum Fisher information is indeed a measure of asymmetry [7, 33]. Let us recall its definition. Given the spectral decomposition of a probe state $\rho = \sum_{i} \lambda_i |i\rangle \langle i|$, $\sum_i \lambda_i = 1$, and an observable $H$, the quantum Fisher information $F_H(\rho) = 2 \sum_{i,j} \langle i| H |j\rangle \langle j| H |i\rangle = \sum_{i,j} \langle i| H |j\rangle \langle j| H |i\rangle$, quantifies the sensitivity of the probe to a phase shift $U_H(\theta) = e^{i\theta H}$ generated by $H$, under the assumption that the state changes smoothly [31]. The quantum Fisher information is (four times) the convex roof of the variance, $V_H(\rho) := 4 \langle |\psi_H|^2 |\langle \psi_H|^2 \rangle$, meaning that $F_H(\rho) = \inf_{\rho_\theta} \left( \sum_i p_i V_H(\rho_i) \right)$, where the infimum is taken over all the convex decompositions $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$, such that $\{p_i\}$ form a probability distribution [45, 46]. Moreover, a decomposition saturating the equality always exists. This property implies convexity, $F_H(\rho + (1 - p) \rho') \leq p F_H(\rho) + (1 - p) F_H(\rho')$. The quantum Fisher information is equal to the variance for pure states, $F_H(|\psi\rangle \langle \psi|) = V_H(\rho)$. We recall what implies that the quantum Fisher information is a reliable measure of asymmetry. It satisfies the following criteria:

i) It vanishes if and only if the state is incoherent. Since the quantum Fisher information is convex, for any incoherent state one has $F_H(\rho_H) = F_H(\sum_{i} c_i |\psi_i\rangle \langle \psi_i|) \leq \sum_i c_i F_H(|\psi_i\rangle \langle \psi_i|) = 0$. Also, we observe that $F_H(\rho) = 0 \iff [\rho, H] = 0$, and $[\rho, H] = 0 \iff [\rho, U_H(\theta)] = 0, \forall \theta$, which is a condition satisfied if and only if the state is incoherent.

ii) It cannot increase under symmetric operations. Given $H_AB = H_A \otimes I_B + I_A \otimes H_B$, by Theorem II.1 of Ref. [5], any incoherent map $\mathcal{E}_H(\rho_A)$ admits a Stinespring dilation $\mathcal{E}_H(\rho_A) = Tr_B[V^A_B(\rho_A \otimes \tau_B)V^B_H]|\langle i| H |i\rangle \rangle^\otimes$, where $V^A_B$ is a symmetric unitary with respect to $H_A$ and $[\tau_B, H_B] = 0$. In other words, any symmetric map can be represented by the unitary, symmetric evolution of the system of interest and an ancilla in an incoherent state. One then obtains $F_{H,\theta}(\rho_A) = F_{H,\theta}(\rho_A \otimes \tau_B) = F_{H,\theta}(V^A_B(\rho_A \otimes \tau_B)V^B_H) \geq F_{H,\theta}(Tr_B[V^A_B(\rho_A \otimes \tau_B)V^B_H]) = F_{H,\theta}(\mathcal{E}_H(\rho_A))$.

The proof can be extended to any quantum Fisher information $F_H(\rho) = \sum_{i,j} \langle i| H |j\rangle \langle j| H |i\rangle$, where each of the real-valued functions $f$ identifies a quantization of the classical Fisher information which preserves contractivity under noisy operations, being $F_H(\rho) = F_H(\rho), F(x) = (1 + \chi)/2, \chi \in \mathbb{R} [47]$. The quantum Fisher informations are topologically equivalent, being connected by the chain $2f(0) F_H(\rho) \leq F_H(\rho) \leq F_H(\rho), \forall \rho, \theta$ [48]. Also, the property ii) can be generalized to show that any quantum Fisher information is an ensemble monotone, i.e., it does not increase on average under symmetric operations, $\sum_{\mu} p_{\mu} F_H(\mathcal{E}_{\mu}(\rho)), \forall (p_{\mu}, \mathcal{E}_{\mu}) : \sum_{\mu} p_{\mu} = 1, [\mathcal{E}_{\mu}, U_H(\theta)] = 0, \forall \theta$ [33].

### Asymmetry lower bound

Picking the Fisher information as a measure of asymmetry is useful for experimental purposes. Coherence is not a linear property of a system, so it cannot be directly related to a quantum operator [49]. Also, the quantum Fisher information is usually hard to be computed. Yet, it is possible to build up an observable quantity which provides a nontrivial lower bound:

$$O_H(\rho) \equiv \mathcal{F}_H(\rho),$$

$$O_H(\rho) = -2Tr[\rho \rho_H^2] = 4Tr[\rho^2 H^2 - \rho H^2].$$

As observed in Ref. [33], one has $O_H(\rho) = 2 \sum_{i,j} (\lambda_i - \lambda_j)^2 H_{ij}$. Since $\lambda_i + \lambda_j \leq 1, \forall i, j$, by recalling the expression of the quantum Fisher information, the lower bound holds. For pure states, one has $O_H(\rho) = \mathcal{F}_H(\rho) = 4V_H(\rho)$. The lower bound reliably detects asymmetry, as $O_H(\rho) = 0 \iff \mathcal{F}_H(\rho) = 0$. One may wonder if the quantity $O_H(\rho)$ itself is a consistent measure of asymmetry. For pure states, the lower bound equals the quantum Fisher information, so the answer is positive in such a case. Unfortunately, this does not hold for mixed states. We can see that with a simple example. Given a bipartite state $\rho_{AB} = \rho_A \otimes \rho_B$, let us suppose to measure the asymmetry of the marginal state $\rho_A$ as the uncertainty measuring $H_A$. One obtains $O_{H_A}(\rho_{AB}) = O_{H_A}(\rho_A) Tr[\rho_B^2]$, then, discarding the subsystem $\rho_B$ would increase the asymmetry of the state $\rho_A$, which is manifestly undesirable. One may normalize the quantity by employing $O_H(\rho)/Tr[\rho^2]$ as a measure of asymmetry, yet there would still be a problem. Note that the bound is written (modulo a constant) as an Hilbert-Schmidt norm in the zero shift limit, $O_H(\rho) = 2 \lim_{\mu \to 0} ||U_H(\theta_0)|\rho|U_H(\theta_0)^\dagger - \rho||^2 = 2 \lim_{\mu \to 0} ||U_H(\theta_0)|\rho|U_H(\theta_0)^\dagger - \rho||^2$.

### B. Experimental observability of the asymmetry bound

#### Experimental scheme

As shown in Ref. [8], the asymmetry lower bound is a function of mean values of self-adjoint operators. By applying the Taylor expansion about $\theta = \theta_0$, one has $Tr[\rho U_H(\theta)\rho U_H(\theta)] = \sum_{\mu} p_{\mu} \mathcal{E}_{\mu}(\rho))$.

\[ F_H(\rho) = \sum_{i,j} \langle i| H |j\rangle \langle j| H |i\rangle. \]
two CNOT gates on each copy. Then, one evaluates the purity and the overlap by local Bell measurements, a routine measurement for the lower bound in terms of phase shifts $U_{N+1}(\theta) = e^{i\delta_{h,\theta}}$. For $N = 1$, $H_1 = h$, one has

$$O_{N+1}(\rho) = \text{Tr}[\rho^{2}] - \text{Tr}[\rho U_{N}(\pi)\rho U_{N}^{\dagger}(\pi)].$$

For $N = 2$, $H_2 = h_1 + h_2$:

$$O_{N+1}(\rho) = 3\text{Tr}[\rho^{2}] - 4\text{Tr}[\rho U_{N}(\pi/2)\rho U_{N}^{\dagger}(\pi/2)] + \text{Tr}[\rho U_{N}(\pi)\rho U_{N}^{\dagger}(\pi)],$$

$$U_{N+1}(\theta) = U_{h_1}(\theta)U_{h_2}(\theta).$$

For $N = 3$, $H_3 = h_1 + h_2 + h_3$:

$$O_{N+1}(\rho) = 6\text{Tr}[\rho^{2}] - 4\text{Tr}[\rho U_{N+1}(\pi/2)\rho U_{N+1}^{\dagger}(\pi/2)] + \text{Tr}[\rho U_{N}(\pi/2)\rho U_{N+1}^{\dagger}(\pi/2)] + \text{Tr}[\rho U_{N}(\pi)\rho U_{N+1}^{\dagger}(\pi)],$$

$$U_{N+1}(\theta) = U_{h_1}(\theta)U_{h_2}(\theta)U_{h_3}(\theta).$$

We conjecture that it is possible to iterate the procedure and work out equivalent expressions for an arbitrary number of qubits.

### III. DETECTION OF MULTIPARTITE ENTANGLEMENT VIA ASYMMETRY

#### A. Asymmetry witnesses Entanglement

It is often desirable to consider a high dimensional system as a partition of subsystems. Such a partition is usually dictated by the physical constraints of the problem, for example the spatial separation between the parts of the system. It is then interesting to understand the interplay between asymmetry with respect to a global observable and the relevant case of N-qubit systems, it is possible to extract purity and overlap by local Bell measurements, a routine measurement scheme in optical setups [8, 33, 39–44]. Thus, for systems of arbitrary dimension, the lower bound $O_{N}(\rho)$ can be extracted by the statistics of a limited number of detections, bypassing full state reconstruction.

Closed formula for the asymmetry lower bound in multi-qubit systems

We here provide a closed formula for the asymmetry lower bound in one, two and three-qubit states, with respect to additive Hamiltonians $H_N = \sum_{i=1}^{N} h_i, h_i = 1\text{,}2,\ldots, N, h_i = 1/2\sigma_i$, representing spin-1/2 observables, e.g. the Pauli matrices. By recalling that $e^{i\theta/2} = \cos \theta/2 I_2 + i\sin \theta/2 \sigma$, we get an exact expression for the lower bound in terms of phase shifts $U_{N+1}(\theta) = e^{i\delta_{h,\theta}}$. For $N = 1$, $H_1 = h$, one has

$$O_{N+1}(\rho) = \text{Tr}[\rho^{2}] - \text{Tr}[\rho U_{N}(\pi)\rho U_{N}^{\dagger}(\pi)].$$

For $N = 2$, $H_2 = h_1 + h_2$:

$$O_{N+1}(\rho) = 3\text{Tr}[\rho^{2}] - 4\text{Tr}[\rho U_{N}(\pi/2)\rho U_{N}^{\dagger}(\pi/2)] + \text{Tr}[\rho U_{N}(\pi)\rho U_{N}^{\dagger}(\pi)],$$

$$U_{N+1}(\theta) = U_{h_1}(\theta)U_{h_2}(\theta).$$

For $N = 3$, $H_3 = h_1 + h_2 + h_3$:

$$O_{N+1}(\rho) = 6\text{Tr}[\rho^{2}] - 4\text{Tr}[\rho U_{N+1}(\pi/2)\rho U_{N+1}^{\dagger}(\pi/2)] + \text{Tr}[\rho U_{N}(\pi/2)\rho U_{N+1}^{\dagger}(\pi/2)] + \text{Tr}[\rho U_{N}(\pi)\rho U_{N+1}^{\dagger}(\pi)],$$

$$U_{N+1}(\theta) = U_{h_1}(\theta)U_{h_2}(\theta)U_{h_3}(\theta).$$
TABLE I: Theoretical values of the quantum Fisher information, the observable lower bound defined in Eq. 1, and the conditions witnessing entanglement, Eq. 6, for the spin observables $J_{x,y,z}$ in $\rho'_{ABC}$. The coherence lower bound is an entanglement witness almost as efficient as the quantum Fisher information, being blind to entanglement only for $p \in [0.674, 0.751], [0.646, 0.772]$, and to tripartite entanglement for $p \in [0.813, 0.861]$. Note that a more general sufficient condition for genuine tripartite entanglement is $|\rho_{1,2,3}| > \sqrt{p_{1,2,3}^2 + \sqrt{p_{1,3}p_{2,3}^2} + \sqrt{p_{1,2}p_{3,2}^2}}$, which for GHZ-diagonal states is also a necessary condition [57]. Hence, $\rho'_{ABC}$ is three-partite entangled when $p > 2^{2/3} - 1 \approx 0.587$.

| $J_{x,y,z}$ | $F_{J_{x,y,z}}(\rho'_{ABC})$ | $O_{J_{x,y,z}}(\rho'_{ABC})$ | $F_{J_{x,y,z}}(\rho'_{ABC}) > 3.5$ | $O_{J_{x,y,z}}(\rho'_{ABC}) > 3.5$ |
|-----------|----------------|----------------|----------------|----------------|
| $J_{x}$   | $2p^2[p + 2]$ | $(2p^2 + p^4)/4$ | $p > 1$ | $p > 1$ |
| $J_{y}$   | $2p^2[p + 2]$ | $(2p^2 + p^4 + 2p^3)/4$ | $p > 1$ | $p > 1$ |
| $J_{z}$   | $2p^2[p + 2]$ | $(2p^2 + 3p^3 + 3p^3)/4$ | $p > 0.674, 0.813$ | $p > 0.751, 0.861$ |

the quantum properties of the subsystems. In spite of being a basis-dependent feature, coherence is linked to basis-independent features of multipartite systems as quantum correlations [6, 16, 18, 23, 28]. Here we show that, for an $N$-qubit system, the observable asymmetry bound $O_{\rho}(\rho)$ measured on the global system states witnesses entanglement between the partitions. There are several entanglement witnesses written in terms of the quantum Fisher information. They relate entanglement to the system speed of response to phase shifts generated by additive spin-1/2 Hamiltonians $J_N = \sum_{i=1}^N 1/2\sigma_i$ [32, 46, 54–56]. In particular, a constraint which cannot be satisfied by $k$-separable states of $N$ qubits is $F_{J_{x,y,z}}(\rho) \geq nk^2 + (N-nk)^2$, where $n = \lfloor N/2 \rfloor$. Thus, verifying this relation certifies genuine $k$-partite entanglement [25]. Also, if $F(\rho) = 1/3(F_{J_{x,y,z}}(\rho) + F_{J_{y,z}}(\rho) + F_{J_{x,z}}(\rho)) > 2N/3$, then the state is entangled. Therefore, if there exists a spin basis $\{x,y,z\}$ such that the following conditions are satisfied:

$$O_{J_{x,y,z}}(\rho) > nk^2 + (N-nk)^2,$$

$$O(\rho) = 1/3(O_{J_{x,y,z}}(\rho) + O_{J_{y,z}}(\rho) + O_{J_{x,z}}(\rho)) > 2N/3,$$

a state $\rho$ is respectively genuinely $k$-partite entangled and entangled.

B. A case study

Here we apply our scheme to simulate the non-tomographic detection of asymmetry and entanglement in a three-qubit state. We choose as probe state the GHZ-diagonal state $\rho'_{ABC}$. This allows one to investigate the behavior of the asymmetry lower bound and entanglement witness in the presence of noise in the system. The two copies of the GHZ diagonal state $\rho'_{ABC}$ are obtained by initializing a six qubit processor in $|\rho_{i}\rangle = 1/\sqrt{2}(|i\rangle + p|\sigma_i\rangle)$, $i = 1, \ldots, 6$, and applying Hadamard and CNOT gates as described in Fig. 1.

We measure the asymmetry of the input state with respect to the set of spin Hamiltonians $J_3 = \sum_{i=A,B,C} j_{i,3}$, $J_{3,2} = j_{A,2} \otimes J_{B,2}$, $J_{3,1} = j_{A,1} \otimes J_{B,1}$, $J_{3,0} = j_{A,0} \otimes J_{B,0}$, $j_{A,0} = j_{A,0} + j_{A,1} + j_{A,2} = 1/2\sigma_{A(0)}$, by computing the values of the lower bound, and the approximation defined in Eq. 2, for each observable. Of course, we may obtain the asymmetry with respect to any self-adjoint operator in the three-qubit Hilbert space. This is done by implementing the unitary gate $U_{J_{i}}(\theta) = U_{j_{A}}(\theta) \otimes U_{j_{B}}(\theta) \otimes U_{j_{C}}(\theta)$ on a copy of the state and then building up an interferometric configuration (Fig. 1). Performing the polarisation measurements on the ancillary qubits makes possible to determine $O_{J_{i}}(\rho'_{ABC}).$ We select a small but experimentally plausible phase shift, $\theta = \pi/6$ [33]. Obviously, to evaluate the purity, no gate has to be applied. The purity and overlap values extracted by the quantities $Tr[\rho'_{ABC}U_{J_{i}}(\pi/6)\rho'_{ABC}U_{J_{i}}^\dagger(\pi/6)]$ determine $O_{J_{i}}(\rho'_{ABC})$. No further action is necessary to verify the presence of entanglement through the witnesses in Eq. 6, as the values of $O_{J_{i}}(\rho'_{ABC})$ have been obtained in the previous steps. For $N = 3$, we have $k = 1 \Rightarrow O_{J_{1}}(\rho'_{ABC}) \geq 3, k = 2 \Rightarrow O_{J_{2}}(\rho'_{ABC}) \geq 5,$ and $O(\rho'_{ABC}) > 2.$ The results are summarised in Tab. 1 and Figs. 2, 3.

IV. CONCLUSION

In this work, we provided an experimental recipe to witness multipartite entanglement by detecting asymmetry with respect to an additive Hamiltonian. We employed an experimentally friendly lower bound of the quantum Fisher information to quantify asymmetry, a geometric property of quantum systems underpinning coherence in an observable eigenbasis. The scheme is suitable for detection of asymmetry in large scale quantum registers, as it requires a limited number of measurements regardless the dimension of the system. We showed that in multipartite states the asymmetry lower bound with respect to additive observables is a witness of multipartite entanglement. Our results suggest further lines of investigation. To the best of our knowledge, the lower bound $O_{\rho}$ is the first faithful experimental quantifier of asymmetry for finite dimensional systems. Thus, on the experimental side, we call for a demonstration of our study. Moreover, we observe that a quadratic ($O(N^2)$) sensitivity to phase shifts generated by additive Hamiltonian in $N$-party systems, as measured by the quantum Fisher information, has been associated to another elusive quantum effect, i.e. quantum macroscopicity [58–60]. It is clear that high values of coherence are essential to quantum macroscopicity, yet the interplay between the two concepts still needs to be clarified.
FIG. 2: (Colors Online) – Evaluation of asymmetry in the state $\rho^p_{ABC}$ with respect to the observables $J_{3,\pm x}$ (figures (a), (b), and (c)) as a function of the mixing parameter $p$. The blue dotted line is the quantum Fisher information, here showed for reference, the red dashed line is the bound $O_{J_{3,\pm x}}(\rho^p_{ABC})$, the red continuous line is the approximation $O^p_{J_{3,\pm x}}(\rho^p_{ABC})$ obtained by imposing $\theta = \pi/6$, and the yellow band is the error region, whose extreme values are $O^p_{J_{3,\pm x}}(\rho^p_{ABC}) \pm \Delta O^p_{J_{3,\pm x}}(\rho^p_{ABC})$.

FIG. 3: (Colors Online) – Witnessing entanglement by asymmetry via the inequalities in Eq. 6. (a) Witnessing entanglement in the state $\rho^p_{ABC}$ by computing the quantum Fisher information and the lower bound, as a function of the mixing parameter $p$. The blue dotted line depicts $\mathcal{F}_{J_{3,\pm x}}(\rho^p_{ABC}) \leq 3$, the red dashed line is $O_{J_{3,\pm x}}(\rho^p_{ABC}) \leq 3$, while the red continuous line is $O^p_{J_{3,\pm x}}(\rho^p_{ABC}) \leq 3$. Positive values of such quantities signal entanglement. The yellow band is the error region, bounded by the extreme values $O^p_{J_{3,\pm x}}(\rho^p_{ABC}) \pm \Delta O^p_{J_{3,\pm x}}(\rho^p_{ABC}) \leq 3$. The $J_{3,\pm y}$ cases are not reported as trivially useless, see Tab. 1. (b) Witnessing genuine tripartite entanglement (it is the case $N = 3, k = 2$ of Eq. 6). The blue dotted line depicts $\mathcal{F}_{J_{3,\pm x}}(\rho^p_{ABC}) \leq 5$, the red dashed line is $O_{J_{3,\pm x}}(\rho^p_{ABC}) \leq 5$, while the red continuous line is $O^p_{J_{3,\pm x}}(\rho^p_{ABC}) \leq 5$. The error region (yellow) is bounded by the extreme values of $O^p_{J_{3,\pm x}}(\rho^p_{ABC}) \pm \Delta O^p_{J_{3,\pm x}}(\rho^p_{ABC}) \leq 5$. (c) Witnessing entanglement by computing the average values of the quantum Fisher information and the lower bound over a spin basis $\{x, y, z\}$. The blue dotted line is $\mathcal{F}(\rho^p_{ABC}) \leq 2$, the red dashed line is $\bar{O}(\rho^p_{ABC}) \leq 2$, while the red continuous line is $\bar{O}^p(\rho^p_{ABC}) \leq 2$. The yellow error region is bounded by $\bar{O}^p(\rho^p_{ABC}) \pm \sqrt{\sum_{j \neq i} \Delta O^p_{J_{3,j}}(\rho^p_{ABC}) + 2 \sum_{j \neq i} \Delta O^p_{J_{3,j}}(\rho^p_{ABC})} \leq 5$.

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