NEAR-FIELD IMAGING OF SOUND-SOFT OBSTACLES IN PERIODIC WAVEGUIDES

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ABSTRACT. In this paper, we introduce a direct method for the inverse scattering problems in a periodic waveguide from near-field scattered data. The direct scattering problem is to simulate the point sources scattered by a sound-soft obstacle embedded in the periodic waveguide, and the aim of the inverse problem is to reconstruct the obstacle from the near-field data measured on line segments outside the obstacle. Firstly, we will approximate the scattered field by some solutions of a series of Dirichlet exterior problems, and then the shape of the obstacle can be deduced directly from the Dirichlet boundary condition. We will also show that the approximation procedure is reasonable as the solutions of the Dirichlet exterior problems are dense in the set of scattered fields. Finally, we will give several examples to show that this method works well for different periodic waveguides.

1. Introduction. The direct and inverse scattering problems in periodic mediums is a popular topic in both optics and mathematics, see [2, 24]. In this paper, we will introduce a direct method to reconstruct a sound soft obstacle embedded in a periodic waveguide. This kind of direct method is first developed in [15]-[18] and [9]. The authors approximate the far-field data by solving a first kind integral equation, and then find out the boundary from the sound soft boundary condition. As the integral equation is ill-posed, a regularization scheme is needed in this algorithm. This method is then extended to many other cases, such as the scattering problems in periodic structures, see [6, 7, 12] or infinite rough surfaces, see [19].

For the scattering problems in a periodic waveguide, both the direct and the inverse scattering problems become more difficult, as the radiation conditions for the infinite domain is not clear except for very special cases (e.g. planar waveguide). To solve the direct scattering problems numerically, mathematicians assume that the so-called Limiting Absorbing Principle (LAP) holds. Based on LAP, a method based on the solution of a second-order characteristic equation, see [14] and [13],
while another method based on a doubling recursive procedure and an extrapolation was introduced in [10] and [11], also see [25] for locally perturbed waveguides. Both of these methods are efficient, and we will employ the second one in our numerical examples.

Mathematicians have developed a number of efficient numerical methods for the inverse scattering problems in planar/periodic waveguides. In [4] and [5], authors introduced the linear sampling methods to reconstruct a sound-soft obstacle or a sound-soft/sound-hard crack embedded in a 2D planar waveguide. While in [1], a factorization method was developed for an inhomogeneous medium in a 3D planar waveguide. In [23], a reciprocity gap method was introduced. In [26], a linear sampling method was applied for the reconstruction of a sound-soft obstacle in a periodic waveguide, and in [3], both the linear sampling method and the factorization method were studied for the problem. For multiple multiscale acoustic/electromagnetic scatterers imbedded in free space or half-space, a number of numerical schemes are introduced to recover the scatterers from a single far-field measurement, see [20, 21, 22].

In this paper, we will reconstruct a sound-soft obstacle $D$ embedded in the periodic waveguide, from the scattered data measured on two line segments $\Gamma_1$ and $\Gamma_2$. If a disk is known to be located in $D$, then we will seek for a function $\phi \in H^{1/2}(\partial C)$, such that the scattered field $u_{\phi}$ with the Dirichlet boundary condition $u_{\phi} = \phi$ provides a good approximation of the measured scattered data. We will prove that we can always find such a function for any given scattered data. This involves a least square method which is severely ill-posed, so a regularization method will be necessary to this numerical scheme. Then from the sound-soft boundary condition, we will find out the boundary of $D$. To obtain a higher resolution, we will use multi-frequency data.

The paper is organized as follows. In Section 2, we will describe the direct scattering problems and its numerical method. In Section 3, the direct method is introduced, and in Section 4, the theoretical results for the direct method is shown. In Section 5, we will discuss in detail about the numerical scheme for our method. In Section 6, some numerical results will be given to show the efficiency of our new method.

2. Direct scattering problems.

2.1. Formulation. The direct scattering problem is to simulate the wave scattered by the sound-soft obstacle $D$ in a periodic waveguide $\Omega$. The upper and lower boundaries $\Sigma_{\pm}$ with same period $L$ are defined by:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{The scattering problem in the periodic waveguide.}
\end{figure}
\[ \Sigma_{\pm} = \{(x, f_\pm(x) : x \in \mathbb{R}\}, \]

where \( f_\pm \) satisfy \( \min_{t \in \mathbb{R}} \{|f_+(t) - f_-(t)|\} > 0 \). The waveguide \( \Omega \) is the infinite periodic domain between \( \Sigma_{\pm} \):

\[ \Omega = \{(x, y) : x \in \mathbb{R}, y \in [f_-(x), f_+(x)]\}. \]

For simplicity, we assume that the boundaries \( \Sigma_{\pm} \) are both sound-hard.

Let \( D \) be an open bounded domain in \( \Omega \) with \( C^2 \)-continuous boundary \( \partial D \). Given the incident field in \( \Omega \), the scattered field \( u^s \) satisfies the following equations

\begin{align*}
(1) & \quad \Delta u^s + k^2 u^s = 0, \quad \text{in } \Omega \setminus \overline{D}, \\
(2) & \quad \partial_n u^s = 0, \quad \text{on } \Sigma_{\pm}, \\
(3) & \quad u^s = f, \quad \text{on } \partial D,
\end{align*}

where \( k \) is the wave number with a positive real part and a non-negative imaginary part, \( f = -u^i \), where \( u^i \) is an incident field that propagating in the homogeneous waveguide.

It is known that for a wavenumber \( k_\epsilon = k + i\epsilon \) with \( k, \epsilon > 0 \), there is a unique solution \( u^s_\epsilon \in H^1(\Omega \setminus \overline{D}) \) to the scattering problem \((1)-(3)\). But for a positive real wavenumber \( k \), as far as we know, only for some special cases (e.g. planar waveguides), there are uniqueness results for scattering problems. To guarantee the well-posedness of the direct problem, we assume that the Limiting Absorbing Principle holds for the problems with positive wavenumbers, it is described as follows.

**Limiting Absorbing Principle (LAP)** For any \( \epsilon > 0 \), \( u^s_\epsilon \in H^1(\Omega \setminus \overline{D}) \) is the scattered field with the wavenumber \( k_\epsilon \), then \( u^s_\epsilon \) converges to some \( u^s_0 \in H^1_{loc}(\Omega \setminus D) \) as \( \epsilon \to 0^+ \). The solution \( u^s \) of \((1)-(3)\) is then defined by \( u^s_0 \).

Suppose \( x_0 \) is a point located in \( \Omega \) outside \( D \), the background Green’s function \( G(x, x_0) \) located at \( x_0 \in \Omega \setminus D \) is used as the incident field. It satisfies the following equations

\begin{align*}
(4) & \quad \Delta G(x, x_0) + k^2 G(x, x_0) = \delta(x - x_0), \quad \text{in } \Omega, \\
(5) & \quad \partial_{\nu x} G(x, x_0) = 0, \quad \text{on } \Sigma_{\pm},
\end{align*}

where \( \delta \) is the Dirac Delta function. Similar to the problem \((1)-(3)\), the unique solution to the equations \((4)-(5)\) is also defined by LAP.

The most important step to solve the problems \((1)-(3)\) and \((4)-(5)\) is to reduce them into the problems in a finite domain. For this kind of problems, several different methods have been developed, based on the numerical simulation of the Dirichlet-to-Neumann (or Scattering-to-Scattering) maps on the boundary of the finite domain, see [13, 14, 10, 11, 25, 26]. In this paper, we will apply the method that based on the doubling recursive procedure and extrapolation to the numerical scheme, for details see [11]. In the next section, we will explain how this method works on our scattering problems.

### 2.2. Numerical solution of direct problems.

For problems defined on unbounded domains, we have to truncate it into a bounded domain. Define two line segments:

\[ \Gamma_j = \{ (h_j, y) : f_-(h_j) \leq y \leq f_+(h_j) \}, \quad j = 1, 2, \]
where \( \Gamma_1 \) lies to the left of \( D \) and \( \Gamma_2 \) to the right of \( D \), and the bounded domain 
\[ \Omega_0 := \Omega \cap [h_1, h_2] \times \mathbb{R} \setminus D. \]

Define the Sommerfeld-to-Sommerfeld (StS) operators \( \Lambda_1, \Lambda_2 \):
\[
\Lambda_1 : (\partial_{x_1} - ik)u|_{\Gamma_1} \rightarrow (\partial_{x_1} + ik)u|_{\Gamma_1},
\]
\[
\Lambda_2 : (\partial_{x_1} + ik)u|_{\Gamma_1} \rightarrow (\partial_{x_1} - ik)u|_{\Gamma_2}.
\]
The StS operators \( \Lambda_1, \Lambda_2 \) can be calculated from the recursive doubling procedure introduced in [10, 11, 25], for any \( k \) that \( \text{Im} k > 0 \). An extrapolation technique (see [11]) can be employed to obtain the operators for real \( k \)'s. Then the scattered field \( u^s \) and Green’s function \( G(x, x_0) \) satisfy the following boundary value problems
\[
\partial_{x_1} u^s = -ik u^s + \Lambda_1 (\partial_{x_1} - ik) u^s \quad \text{on } \Gamma_1,
\]
\[
\partial_{x_1} u^s = ik u^s + \Lambda_2 (\partial_{x_1} + ik) u^s \quad \text{on } \Gamma_2,
\]
and
\[
\partial_{x_1} G(x, x_0) = -ik G(x, x_0) + \Lambda_1 (\partial_{x_1} - ik) G(x, x_0) \quad \text{on } \Gamma_1,
\]
\[
\partial_{x_1} G(x, x_0) = ik G(x, x_0) + \Lambda_2 (\partial_{x_1} + ik) G(x, x_0) \quad \text{on } \Gamma_2,
\]

**Remark 1.** In this paper, we always assume that DtN maps defined on \( \Gamma_1 \) and \( \Gamma_2 \) are well defined and \( I - \Lambda_1 \) and \( I - \Lambda_2 \) are invertible. Denote by \( T_1 \) and \( T_2 \) the DtN maps defined on \( \Gamma_1 \) and \( \Gamma_2 \), then \( T_1 \) and \( T_2 \) satisfy
\[
T_1 = ik(I - \Lambda_1)^{-1}(I + \Lambda_1), \quad T_2 = ik(I - \Lambda_2)^{-1}(I + \Lambda_2).
\]

So \( G(x, y) \) and \( u^s \) satisfy
\[
\partial_{x_1} u^s = T_1 u^s, \quad \partial_{x_1} G(x, y) = T_1 G(x, y), \quad \text{on } \Gamma_1,
\]
\[
\partial_{x_1} u^s = T_2 u^s, \quad \partial_{x_1} G(x, y) = T_2 G(x, y), \quad \text{on } \Gamma_2.
\]

Next, we will discuss the numerical simulation of Green’s function \( G(x, x_0) \). From [4], if \( \Phi(x, x_0) \) is the Green’s function for the Helmholtz equation in \( \mathbb{R}^2 \), then \( G(x, x_0) \) can be split into two parts:
\[
G(x, x_0) = \Phi(x, x_0) + g_{x_0}(x),
\]
where \( g_{x_0} \) is a \( C^\infty \) function in \( \Omega \) and satisfies the following boundary value problem:
\[
\triangle g_{x_0} + k^2 g_{x_0} = 0, \quad \text{in } \Omega_0,
\]
\[
\partial_{\nu} g_{x_0} = -\partial_{\nu} \Phi(x, x_0), \quad \text{on } \Sigma_\pm,
\]
\[
\partial_{\nu} g_{x_0} = T_1 [g_{x_0} + \Phi(x, x_0)] - \partial_{\nu} \Phi(x, x_0) \quad \text{on } \Gamma_1,
\]
\[
\partial_{\nu} g_{x_0} = T_2 [g_{x_0} + \Phi(x, x_0)] - \partial_{\nu} \Phi(x, x_0) \quad \text{on } \Gamma_2.
\]
The the problems (1)-(3) and (4)-(5) are reduced to equivalent problems in finite domains, then standard finite element method could be employed to the numerical solutions to these problems.

3. **Inverse scattering problems: A direct method.** In this section, we will describe the mathematical presentation of the inverse scattering problems. The aim of the inverse problem is to reconstruct the shape of the scatterer \( D \) from the measured scattered data on \( \Gamma_1 \) and \( \Gamma_2 \), which are generated by some incident Green’s functions.

Define the following space
\[
X = \{ u = (u|_{\Gamma_1}, u|_{\Gamma_2}) : u|_{\Gamma_j} \in H^{1/2}(\Gamma_j), \ j = 1, 2 \}.
\]
Then the conjugate space of $X$ is defined by

$$X^* = \{ u = (u|_{\Gamma_1}, u|_{\Gamma_2}) : u|_{\Gamma_j} \in H^{-1/2}(\Gamma_j), j = 1, 2 \}.$$ 

Then the inverse problem could be described as follows.

**Inverse Problems (IP):** given the measured scattered data $u^s := (u^s|_{\Gamma_1}, u^s|_{\Gamma_2})$, find $D_0$ such that the scattered data $u^s_0 := (u^s_0|_{\Gamma_1}, u^s_0|_{\Gamma_2})$ with the same incident field satisfies

$$u^s = u^s_0 \text{ in } X.$$ 

**Assumption 3.1.** Assume that an inner point of $D$ is already known.

The idea of solve (IP) by the Kirsch-Kress method is divided into two steps. If $C$ is a known to be a curve lying in the interior of $D$, the first step is to approximate the scattered field $u$ by a single-layer potential defined on $\partial C$, and the second step is to find the boundary of $D$ from the Dirichlet boundary condition. As the Green’s functions are no longer easy to evaluate in the periodic waveguide, we will avoid the single-layer potential in the first step, and our two-step method will be described in this section.

**Step 1.** Suppose $C$ is a known disk in $D$, for a function $\phi \in H^{1/2}(\partial C)$, define $v_\phi$ as the solution of the scattering problem (17)-(20)

\[
\begin{align*}
\Delta v_\phi + k^2 v_\phi &= 0, & \text{in } \Omega_0 \setminus \overline{C}, \\
v_\phi &= \phi, & \text{on } \partial C, \\
\partial_{\nu} v_\phi &= T_j v, & \text{on } \Gamma_j, j = 1, 2, \\
\partial_{\nu} v_\phi &= 0, & \text{on } \Sigma_{\pm}.
\end{align*}
\]

From the LAP, the well-posedness of the scattering problem (17)-(20) holds, then we can define the operator $S$ by

$$S : H^{1/2}(\partial C) \rightarrow X$$

\[
\phi \mapsto v_\phi = (v_\phi|_{\Gamma_1}, v_\phi|_{\Gamma_2}).
\]

For this step, we will find a suitable function $\phi \in H^{1/2}(\partial C)$, such that $S\phi$ is a good approximation of $u$. In other words, given a small enough $\epsilon > 0$, find a function $\phi \in H^{1/2}(\partial C)$ such that

$$\|S\phi - u\|_X < \epsilon.$$
We will show in Section 4 that \( S \) has a dense range in \( X \), i.e., for any \( \epsilon > 0 \), we can find out such a \( \phi \) satisfies (21).

**Step 2.** After we have found out the function \( \phi \in H^{1/2}(\partial C) \), i.e., \( S\phi \) is a good approximation of \( u \), then the function \( v_\phi \) is supposed to be a global approximation of the scattered field \( u^s \). As the boundary condition (3) shows that \( u^s + u^i = 0 \) on the boundary of \( D \), the boundary is then defined by a curve that minimizes \( |v_\phi + u^i| \). The numerical scheme of this direct method will be shown in Section 5.

4. **Density results of the solutions.** In Section 3, we have described a two-step method for the numerical solution of IP. There might be a question in Step 1 that if the function \( \phi \) exists. In this section, we will answer the question, i.e., we can always find such a function \( \phi \in H^{1/2}(\partial C) \), such that \( S\phi \) is a "good" approximation of \( u \).

The most important theorem in this section is described as follows.

**Theorem 4.1.** The operator \( S \) has a dense range in \( X \).

Before the proof of Theorem 4.1, we need to study the properties of the DtN maps first. From the limiting absorbing principle, we need to focus on the cases that the wavenumbers with positive imaginary parts first.

**Lemma 4.2.** Suppose the wavenumber \( k_\epsilon = k + i\epsilon \) with \( k, \epsilon > 0 \). Then the DtN maps \( T^*_1 \) and \( T^*_2 \) satisfy

\[
T^*_1 w = (T^*_1)^* \overline{w_1}, \quad T^*_2 w = (T^*_2)^* \overline{w_2},
\]

for any \( w_j \in H^{1/2}(\Gamma_1) \), where \( j = 1, 2 \).

**Proof.** Consider \( T^*_1 \). Define the periodic cell by \( \Omega_j := \Omega \cap \left( [h_1 - jL, h_1 - (j - 1)L] \times \mathbb{R} \right) \), and the left boundary \( \Sigma_j := \Omega \cap \left( \{h_1 - jL\} \times \mathbb{R} \right) \), for \( j \in \mathbb{N} \) (note that \( \Sigma_0 = \Gamma_1 \)). The half-guide is then defined by \( \Omega_- = \bigcup_{j=1}^{\infty} \Omega_j \). Given two functions...
w_1, v_1 \in H^{1/2}(\Gamma_1)$, suppose $v, w \in H^1(\Omega_-)$ are two solutions to the problems

$$\begin{align*}
\Delta v + k_\epsilon^2 v &= 0, \quad \Delta w + k_\epsilon^2 w = 0 \quad \text{in } \Omega_-,
\partial_n v &= 0, \quad \partial_n w = 0 \quad \text{on } \Sigma_\pm,
v = v_1, w = v_1 \quad \text{on } \Gamma_1.
\end{align*}$$

Apply the Green’s formula to the function in a finite domain $\Omega_N := \bigcup_{j=1}^N \Omega_j$, we can have the following result

$$\begin{align*}
0 &= \int_{\Omega_N} (\Delta v + k_\epsilon^2 v) w \, dx = \int_{\Omega_N} (\Delta v w - v \Delta w) \, dx \\
&= \int_{\Gamma_1} [\partial_n vw - v \partial_n w] \, ds + \int_{\Sigma_N} [\partial_n vw - v \partial_n w] \, ds.
\end{align*}$$

Then the following equation is satisfied, for any $j \in \mathbb{N}$:

$$\int_{\Gamma_1} [T_j^* vw - v T_j^* w] \, ds = - \int_{\Sigma_N} [\partial_n vw - v \partial_n w] \, ds.$$

From the fact that $T_j$ is a bounded operator from $H^{1/2}(\Sigma_j)$ to $H^{-1/2}(\Sigma_j)$,

$$\left| \int_{\Gamma_1} [T_j^* vw - v T_j^* w] \, ds \right| \leq C \|v\|_{H^{1/2}(\Sigma_j)} \|w\|_{H^{1/2}(\Sigma_j)}.$$

Together with that $v, w \in H^1(\Omega_-)$, $\|v\|_{H^{1/2}(\Sigma_j)}$, $\|w\|_{H^{1/2}(\Sigma_j)}$ tends to 0 as $j$ tends to infinity. Then the left hand side equals to 0, i.e.,

$$<v, (T_j^*)^* w > = <v, T_j^* w >,$$

where $< \cdot, \cdot >$ is a inner product in $L^2(\Gamma_1)$. Then

$$<v, (T_j^*)^* w > = <v, T_j^* w >,$$

i.e., $(T_j^*)^* w = T_j^* w$. For the operator $T_j^*$, the property in (22) could be deduced from the similar procedure. So the lemma is proved.

From LAP, a similar result for positive wavenumbers could be concluded in the following lemma.

**Lemma 4.3.** The DtN maps $T_j$ are corresponds to a positive wavenumber $k$, defined by $T_j = \lim_{\epsilon \to 0^+} T_j^\epsilon$, then

$$\begin{align*}
T_j w &= (T_j)^* \overline{w},
\end{align*}$$

for any $w \in H^{1/2}(\Gamma_j)$, where $j = 1, 2$.

**Proof.** Consider $T_1$ first. For any $v \in H^{-1/2}(\Gamma_1)$, suppose $\epsilon > 0$ and $T_1^\epsilon$ is the DtN map with the wavenumber $k_\epsilon = k + i\epsilon$, with the result in (22),

$$\begin{align*}
<v, (T_1)^* \overline{w} - T_1 \overline{w} > &= <v, (T_1)^* \overline{w} - T_1 \overline{w} > + <v, (T_1 - T_1^\epsilon)^* \overline{w} > - <v, (T_1^\epsilon - T_1^* \overline{w} > \\
&= <(T_1 - T_1^\epsilon^\epsilon) v, \overline{w} > - <v, (T_1^\epsilon - T_1^\epsilon) w >.
\end{align*}$$
From LAP, $\lim_{\epsilon \to 0^+} T^*_1 = T_1$, then
\[
< v, (T_1)^* w - T_1 w > \\
\leq |< (T_1 - T_1^*) v, w >| + |< v, (T_1 - T_1^*) w >| \\
\leq \| T_1 v - T_1^* v \|_{H^{-1/2}(\Gamma_1)} \| w \|_{H^{1/2}(\Gamma_1)} + \| T_1 w - T_1^* w \|_{H^{-1/2}(\Gamma_1)} \| v \|_{H^{1/2}(\Gamma_1)},
\]
the right hand side tends to 0 as $\epsilon \to 0^+$. This implies that
\[
< v, (T_1)^* w - T_1 w > = 0
\]
for any $v \in H^{1/2}(\Gamma_1)$. The lemma is proved. 

With these results, we can prove the main theorem in this section.

**Proof of Theorem 4.1.** To prove that $S$ has a dense range, we can prove that $S^*$ is injective instead. The operator $S^*$ is defined by
\[
S^* : X^* \rightarrow H^{-1/2}(\partial C)
\]
where $w$ is a solution to the problem
\[
\begin{align*}
\Delta w + k^2 w &= 0, & \text{in } \Omega_0 \setminus \overline{C}, \\
\partial_{\nu} w - T_j w &= \psi_j, & \text{on } \Gamma_j, \; j = 1, 2, \\
w &= 0, & \text{on } \partial C, \\
\partial_{\nu} w &= 0, & \text{on } \Sigma_\pm.
\end{align*}
\]
(a) Firstly, we will prove that $S^*$ is defined as above. For any $\phi \in H^{1/2}(\partial C)$, there is a unique solution $u$ for the scattering problem (17)-(20) in the waveguide with the Dirichlet boundary condition $u = \phi$ on $\partial C$. From the definition of $S$,
\[
S\phi = (u|_{\Gamma_1}, u|_{\Gamma_2}).
\]
Then from the Green’s theorem, with the help of the homogeneous Neumann boundary conditions of $w$ and $u$,
\[
0 = \int_{\Omega_0 \setminus \overline{C}} (\Delta w + k^2 w) u \, dx = \int_{\Omega_0 \setminus \overline{C}} (\Delta w u - w \Delta u) \, dx \\
= \left( \int_{\Gamma_1} + \int_{\Gamma_2} - \int_{\partial C} \right) [\partial_{\nu} w u - w \partial_{\nu} u] \, ds,
\]
i.e.,
\[
\int_{\partial C} [\partial_{\nu} w u - w \partial_{\nu} u] \, ds = \left( \int_{\Gamma_1} + \int_{\Gamma_2} \right) [\partial_{\nu} w u - w \partial_{\nu} u] \, ds.
\]
Use the boundary condition $w = 0$ on $\partial C$,
\[
< u, \partial_{\nu} w >_{L^2(\partial C)} = < u, \partial_{\nu} w >_{L^2(\Gamma_1)} - < \partial_{\nu} u, w >_{L^2(\Gamma_1)} + < u, \partial_{\nu} w >_{L^2(\Gamma_2)} - < \partial_{\nu} u, w >_{L^2(\Gamma_2)}
\]
\[
= < u, \partial_{\nu} w - (T_1)^* w >_{L^2(\Gamma_1)} + < u, \partial_{\nu} w - (T_2)^* w >_{L^2(\Gamma_2)}
\]
\[
= < u, \psi_1 >_{L^2(\Gamma_1)} + < u, \psi_2 >_{L^2(\Gamma_2)}.
\]
This means that
\[
< u, \partial_{\nu} w >_{L^2(\partial C)} = < S\phi, \psi >_{(X, X^*)},
\]
i.e., $S^* \psi = \partial_{\nu} w|_{\partial C}$. 

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(b) Then we will prove that $S^*$ is an injection. Suppose for some $\psi \in X^*$, $S^* \psi = 0$. Then $w$ is the solution of the problem (24)-(27) with the boundary condition $\partial_{\nu} w = 0$ on $\partial C$, i.e., on the boundary $\partial C$, the following conditions are satisfied
\[
\partial_{\nu} w = 0, \quad w = 0.
\]
Extend $w$ into $C$ with $w = 0$, then $w$ satisfies the Helmholtz equation in the whole waveguide $\Omega$. This implies that $w = 0$ in $\Omega$, then $\psi = 0$, i.e., $S^*$ is injective. The theorem is proved.

From Theorem 4.1, given data $u^{meas} := (u^s|_{\Gamma_1}, u^s|_{\Gamma_2})$ and a positive number $\varepsilon > 0$, there is a function $\phi_\varepsilon$ such that
\[
(28) \quad \| S \phi_\varepsilon - u^{meas} \|_X < \varepsilon.
\]

5. Numerical implementation of the direct method. From Section 3, the direct method is divided into two steps. The first step is to approximate the scattered field by seeking for the function $\phi$ defined on $\partial C$. The second step is to locate the boundary of the scatterer $D$ from the Dirichlet value.

In the numerical examples in this paper, we use incident Green’s functions with different wavenumbers $k = 1, 2, \ldots, K$ and locations $x_m$ where $m = 1, 2, \ldots, M$, then we denote by $u^{meas}_{k,m} = G_k(x, x_m)$. The measured scattered data on $\Gamma_1$ and $\Gamma_2$ are defined by $u^{meas,1}_{k,m}$ and $u^{meas,2}_{k,m}$. For each $k, m$, we will find out the function $\phi_{k,m} \in H^{1/2}(\partial C)$, such that $u^{meas}_{\phi_{k,m}}$ provides a good approximation of $u^{meas,1}_{k,m}$ and $u^{meas,2}_{k,m}$.

Remark 2. In Section 3, the scattered data are in the space $X$, i.e., $H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma_2)$, and $\phi \in H^{1/2}(\partial C)$. As was proved, $S$ has a dense range in $X = H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma_2)$, from the denseness of $H^{1/2}$-space in $L^2$-space, $X$ is dense in $L^2(\Gamma_1) \times L^2(\Gamma_2)$, thus $S$ has a dense range in $L^2(\Gamma_1) \times L^2(\Gamma_2)$. For simplicity, we will replace $X$ with $L^2(\Gamma_1) \times L^2(\Gamma_2)$ in numerical implementations.

If $\{ \varphi_j \}_{j=0}^{+\infty}$ is a basis for $L^2(\partial C)$. In this paper, we choose $\varphi_j$ as trigonometric functions,
\[
(29) \quad \varphi_j = \begin{cases} 
1 & \text{if } j = 0; \\
\sin(nx) & \text{if } j = 2n; \\
\cos(nx) & \text{if } j = 2n - 1.
\end{cases}
\]

We will seek the functions $\phi_{k,m}$ in form of the following finite dimensional space
\[
(30) \quad S_N := \left\{ \phi = \sum_{j=0}^{N} a^j \varphi_j : a^j \in \mathbb{C} \right\},
\]
i.e., to find $A^N_{k,m} := (a^0_{k,m}, a^1_{k,m}, \ldots, a^N_{k,m})^T$ such that
\[
(31) \quad \phi_{k,m} = \sum_{j=0}^{N} a^j_{k,m} \varphi_j \in S_N.
\]

For the Dirichlet value $\varphi_j$, define $v_j$ as the solution of the scattering problem (17)-(20). As the problem is linear with respect to the Dirichlet values, the scattered
The field corresponds to $\phi_{k,m}$ has the representation

$$v_{k,m} = \sum_{j=0}^{N} \phi_{k,m}^{j} v_{j}. \tag{32}$$

Denote by $\mathbf{x}_{l_{1}}^{1}$, $l_{1} = 1, 2, \ldots, L_{1}$ the measured points located on $\Gamma_{1}$ and $\mathbf{x}_{l_{2}}^{2}$, $l_{2} = 1, 2, \ldots, L_{2}$ located on $\Gamma_{2}$, then the problem is to solve the linear system

$$\Psi A_{k,m}^{N} = U_{k,m}, \tag{33}$$

where

$$\Psi = \begin{pmatrix} v_{0}(\mathbf{x}_{1}^{1}) & \ldots & v_{N}(\mathbf{x}_{1}^{1}) \\ \vdots & \ddots & \vdots \\ v_{0}(\mathbf{x}_{L_{1}}^{1}) & \ldots & v_{N}(\mathbf{x}_{L_{1}}^{1}) \\ v_{0}(\mathbf{x}_{1}^{2}) & \ldots & v_{N}(\mathbf{x}_{1}^{2}) \\ \vdots & \ddots & \vdots \\ v_{0}(\mathbf{x}_{L_{2}}^{2}) & \ldots & v_{N}(\mathbf{x}_{L_{2}}^{2}) \end{pmatrix}, \quad U_{k,m} = \begin{pmatrix} u_{k,m}^{\text{meas},1}(\mathbf{x}_{1}^{1}) \\ \vdots \\ u_{k,m}^{\text{meas},1}(\mathbf{x}_{L_{1}}^{1}) \\ u_{k,m}^{\text{meas},2}(\mathbf{x}_{1}^{2}) \\ \vdots \\ u_{k,m}^{\text{meas},2}(\mathbf{x}_{L_{2}}^{2}) \end{pmatrix}.$$ 

To solve this linear system, we will use a Tikhonov regularization based on SVD decomposition, with a regularization parameter $\gamma > 0$. Then we have found the function $\phi_{k,m}$, and the solution $v_{k,m}$ is then the approximation of the scattered field corresponding to the incident field $u_{k,m}^{i}$. 

With the approximated scattered fields and the Dirichlet boundary conditions,

$$v_{k,m} + u_{k,m}^{i} \approx 0 \quad \text{on } \partial D, \quad k = 1, \ldots, K, \quad m = 1, \ldots, M. \tag{34}$$

The define the function as

$$\mathcal{F}(\mathbf{x}) = \left( \sum_{k=1}^{K} \sum_{m=1}^{M} |v_{k,m}(\mathbf{x}) + u_{k,m}^{i}(\mathbf{x})|^{2} \right)^{-1/2}, \tag{35}$$

then it will blow up when $\mathbf{x}$ tends to the boundary of $D$.

In the next section, we will give several numerical results for our method. 

6. Numerical results for the direct method. We will show the numerical results for four different scatterers embedded in two different waveguides. The first waveguide is shown in Fig 4 with boundaries defined by

$$\Sigma_{+} = \{(x_{1}, 0.1 \sin x + 1)\}, \quad \Sigma_{-} = \{(x_{1}, 0.1 \sin x)\}.$$ 

The second waveguide is shown in Fig 5 with boundaries defined by

$$\Sigma_{+} = \begin{cases} (x_{1}, 1), & 0.5 < x_{1} < 1.5 \\ (x_{2}, 1.2), & -0.5 < x_{1} < 0.5 \end{cases}$$

$$\Sigma_{-} = \{(x_{1}, 0) : x \in (-0.5, 1.5)\}$$

in one period, and then expanded into a periodic waveguide with period 2.

The four scatterers are shown in Fig 6. They are defined by the following parametric equations.

$$a) : (0, 0.5) + 0.2(\cos t, \sin t); \tag{36}$$

$$b) : (0, 0.5) + 0.2\sqrt{\cos^{2}t + 0.125\sin^{2}t}(\cos t, \sin t); \tag{37}$$

$$c) : (0, 0.5) + (0.1(\cos^{3}t + \cos t), 0.1(\sin^{3}t + \sin t)); \tag{38}$$

$$d) : (0, 0.5) + 0.1(2 + 0.3\cos 3t)(\cos t, \sin t). \tag{39}$$
Figure 4. Waveguide 1

Figure 5. Waveguide 2

Figure 6. Four scatterers.
Let \((0, 0.5)\) be the known inner point of \(D\), \(C\) is chosen as a disk with \((0, 0.5)\) its center and a small enough radius \(r = 0.05\).

The parameters in the numerical examples are given:

\[
M = 4, \quad N = 65, \quad L_1 = L_2 = 151, \quad h_1 = -3, \quad h_2 = 3, \quad \gamma = 10^{-5};
\]

\[
k = 10, 15, 20, 25, 30, 35, 40.
\]

We will show two groups of numerical examples in the rest of this paper. For the first group, we fix four source points

\[
(40) \quad x_1 = (-1, 0.5), \quad x_2 = (0, 0.2), \quad x_3 = (1, 0.5), \quad x_4 = (0, 0.8)
\]

and the numerical results are shown in Figure 7-10. For the second group, the four source points are chosen as

\[
(41) \quad x_1 = (-2, 0.2), \quad x_2 = (-2, 0.8), \quad x_3 = (2, 0.2), \quad x_4 = (2, 0.8)
\]

and the numerical results are shown in Figure 11-14.

From the numerical results, our numerical works well for these examples. The second example shows that the non-convex part of the curve is not so clear, this is reasonable for such a problem. If we compare the left pictures and the right ones, the numerical method produces better results for the first waveguide. This may come from the singularity of the second waveguide. As the second waveguide is not
continuous, the numerical scheme for the direct scattering problems may produce larger error. Thus the numerical results for the inverse problems might be not as good as smooth waveguides, such as the first one.
Figure 12. (a)-(b): numerical result for scatter 2 with waveguides.

Figure 13. (a)-(b): numerical result for scatter 3 with waveguides.

Figure 14. (a)-(b): numerical result for scatter 4 with waveguides.
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