Ghost-free Massive Gravity with a General Reference Metric

S. F. Hassan
Department of Physics & The Oskar Klein Centre, Stockholm University, AlbaNova University Centre, SE-106 91 Stockholm, Sweden
E-mail: fawad@fysik.su.se

Rachel A. Rosen
Physics Department and Institute for Strings, Cosmology, and Astroparticle Physics, Columbia University, New York, NY 10027, USA
E-mail: rar2172@columbia.edu

Angnis Schmidt-May
Department of Physics & The Oskar Klein Centre, Stockholm University, AlbaNova University Centre, SE-106 91 Stockholm, Sweden
E-mail: angnis.schmidt-may@fysik.su.se

ABSTRACT: Theories of massive gravity inevitably include an auxiliary reference metric. Generically, they also contain an inconsistency known as the Boulware-Deser ghost. Recently, a family of non-linear massive gravity actions, formulated with a flat reference metric, were proposed and shown to be ghost free at the complete non-linear level. In this paper we consider these non-linear massive gravity actions but now formulated with a general reference metric. We extend the proof of the absence of the Boulware-Deser ghost to this case. The analysis is carried out in the ADM formalism at the complete non-linear level. We show that in these models there always exists a Hamiltonian constraint which, with an associated secondary constraint, eliminates the ghost. This result considerably extends the range of known consistent non-linear massive gravity theories. In addition, these theories can also be used to describe a massive spin-2 field in an arbitrary, fixed gravitational background. We also discuss the positivity of the Hamiltonian.

KEYWORDS: massive gravity.
1. Introduction and summary

Generically, theories of massive gravity are plagued by the Boulware-Deser (BD) ghost instability at the non-linear level \[1, 2\]. Recently, significant progress has been made towards constructing massive gravity theories that avoid this instability. In addition to the metric \(g_{\mu\nu}\), theories of massive gravity inevitably include another rank-2 symmetric tensor \(f_{\mu\nu}\), henceforth called the reference metric. This is due to the fact that the interaction terms that can be formed from the metric alone, \(\text{tr } g = 4\) and \(\text{det } g\), cannot be used to construct a mass term. Most of the recent work has focused on the case of a flat reference metric,
essentially, $f_{\mu \nu} = \eta_{\mu \nu}$. In particular, a two-parameter family of actions was proposed in \cite{3, 4} for this case by demanding the absence of the BD ghost in what’s known as the decoupling limit. One of these actions was demonstrated to be ghost-free more generally at fourth order in perturbation theory in \cite{4}. The full two-parameter family of actions were then shown to be free of the BD ghost instability at the complete non-linear level in \cite{3} based on the reformulation given in \cite{4}. For complementary work see \cite{5, 6, 7}.

In this paper we consider non-linear massive gravity actions constructed with a general $f_{\mu \nu}$ and extend the proof of the absence of the BD ghost given in \cite{3} to this case. This generalization is motivated by several considerations. First, there is no reason to insist that a theory of massive gravity always refer to a flat reference metric. For example, one may also consider dS or AdS metrics. Second, forcing $f_{\mu \nu}$ to be flat constrains the classical solutions of the metric $g_{\mu \nu}$. For example, non-Minkowski homogeneous and isotropic spacetimes were argued to be excluded in \cite{10}. A possible resolution of this problem is to allow for more general $f_{\mu \nu}$ \cite{6}. A third motivation is that, from a theoretical standpoint, it is more satisfying to promote $f_{\mu \nu}$ to a dynamical field with its own kinetic term than to have a “frozen-in” reference metric. The resulting theory would resemble the bi-metric construction of \cite{11, 12}. For a dynamical $f_{\mu \nu}$ to be consistent, it is important to first verify that the mass term which was ghost free for flat $f_{\mu \nu}$ remains so for a general $f_{\mu \nu}$.

It should be emphasized that although the discussion in this paper is formulated in the context of massive gravity, the analysis applies equally well to generic massive spin-2 fields. For example, $g_{\mu \nu}$ could also represent a neutral massive spin-2 meson in a fixed gravitational background $f_{\mu \nu}$.

At the linear level, massive gravity theories with a few simple non-flat $f_{\mu \nu}$’s have already been considered. The linear Fierz-Pauli theory of massive gravity \cite{13, 14} in a flat background has been generalized to linear massive gravities in de Sitter and anti de Sitter spacetimes \cite{15, 16, 17, 18}, and to FRW backgrounds \cite{19, 20, 21}. In these constructions $f_{\mu \nu}$ plays the role of the background FRW metric. Our work explicitly shows that it is possible to construct non-linear extensions of these theories that are free from the BD ghost instability even for a general $f_{\mu \nu}$. Namely, the non-linear ghost-free massive gravity actions proposed in \cite{4} remain ghost-free when constructed with respect to a general $f_{\mu \nu}$.

In this work we consider the massive actions of \cite{4} as reformulated and extended to general $f_{\mu \nu}$ in \cite{6}. In this reformulation, the two-parameter family is regarded as an extension of a simpler, “minimal” massive action. Each free parameter is associated with a higher level of non-linear complexity. The simplicity of the minimal model is instrumental in constructing the proof of the absence of the BD instability. Moreover, for this model the constraint equations can be solved explicitly, making it possible to study issues such as the positivity of the Hamiltonian. Once the proof of the absence of the BD ghost is constructed for the minimal model, we find that the exact same construction holds for the more complicated two-parameter family of actions.

Our analysis is based on the ADM formulation of gravity \cite{22}. In the ADM language, the BD ghost is a consequence of the absence of the Hamiltonian constraint. We show that in the models considered here, such a constraint exists. With an associated secondary constraint (see \cite{23}), this is enough to eliminate the BD ghost mode.
The paper is organized as follows: Section 2 starts with a review of non-linear massive gravity with a general reference metric $f_{\mu\nu}$. We discuss the Boulware-Deser ghost problem in non-linear massive gravity and present precise criteria for avoiding it. We then review the specific two-parameter family of actions considered in this paper. In section 3 we show the absence of the BD ghost in the minimal massive action by obtaining the Hamiltonian constraint and arguing for the existence of an associated secondary constraint. We then discuss the positivity of the Hamiltonian. In section 4, the proof of the absence of the BD ghost is extended to the complete two-parameter family of massive actions and the Hamiltonian constraint is determined. The results are briefly discussed in section 5. In the appendix we review the ghost issue in linear and non-linear massive gravities, including the original analysis of Boulware and Deser [1].

2. Review of massive gravity and the Boulware-Deser ghost problem

In this section we discuss the ghost problem in massive gravity, reviewing the Boulware-Deser argument and the caveat by which it can be avoided. We also review the two-parameter family of potentially ghost-free massive gravity actions formulated with respect to a general $f_{\mu\nu}$.

2.1 General structure of non-linear massive gravity

A generic covariant massive gravity action for the metric $g_{\mu\nu}$ is obtained by adding a non-derivative potential term $V(g^{-1}f)$ to the Einstein-Hilbert action [1],

$$S_m = M_p^2 \int d^4x \sqrt{-g} \left[ R(g) - m^2 V(g^{-1}f) \right].$$  \hspace{1cm} (2.1)

$f_{\mu\nu}$ is a non-dynamical rank 2 tensor that is needed to construct generally covariant, non-derivative functions of the metric. The coupling of the metric $g_{\mu\nu}$ to matter is taken to be the same as in GR in order to preserve the weak equivalence principle. Below, we will have more to say about the role of $f_{\mu\nu}$.

Such a generic non-linear massive gravity action (2.1) typically contains a ghost, i.e., a physical mode with negative kinetic energy which in the quantum theory results in negative probability states. The origin of the ghost is easy to understand (see the appendix for a detailed review of the ghost problem). In general relativity, a scalar component of the metric is potentially a ghost, but is eliminated by the equations of motion. The addition of a potential energy term to the GR action generally results in this component becoming an independent dynamical degree of freedom. Then the theory has six propagating modes: the five polarizations of the massive spin-2 graviton and the ghost.

At the linear level, the ghost problem is avoided by the mass term proposed by Fierz and Pauli [13, 14]. A necessary (but not sufficient) requirement for the action (2.1) to be ghost-free is that, when expanded to quadratic order in metric fluctuations $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$, the potential $V$ should reproduce the Fierz-Pauli (FP) mass term in the background $\bar{g}_{\mu\nu}$, provided one takes $\bar{g}_{\mu\nu} = f_{\mu\nu}$,

$$S_{FP} = -\frac{1}{4} M_p^2 m^2 \int d^4x \sqrt{-\bar{g}} \left[ h_{\nu\mu} h_{\mu}^{\nu} - (h_{\mu}^{\mu})^2 \right].$$  \hspace{1cm} (2.2)
where, $h_{\mu}^{\nu} = \tilde{g}_{\mu\rho}h_{\rho\nu}$. At the linear level, this particular choice of relative coefficients between the terms decouples the ghost by making it infinitely massive. However, it was shown by Boulware and Deser [1, 2] that the ghost sixth mode generally reappears at the non-linear level, leading to speculation that a non-linear theory of massive gravity may not exist. We will discuss this further below.

For the purpose of identifying the massive excitations, $f_{\mu\nu}$ must be equated to a background metric, $f_{\mu\nu} = \tilde{g}_{\mu\nu}$. Then the background field equations reduce to the GR equations with a shifted cosmological constant and $f_{\mu\nu}$ is a solution for a given source, say, $\tilde{T}_{\mu\nu}$. For this reason $f_{\mu\nu}$ is often referred to as a “background metric”. But, given $f$, the non-linear theory will also have classical solutions in which $g$ differs appreciably from $f$ in some regions of spacetime, see for example, [24, 25, 26, 27]. Any such solution can be regarded as a background $\tilde{g}$ with fluctuations $h'_{\mu\nu}$ around it, although the action for these fluctuations may no longer have the Fierz-Pauli form (2.2). Thus at the non-linear level, one could consider fluctuations around background metrics other than $f_{\mu\nu}$. For this reason we refer to $f_{\mu\nu}$ as the “reference metric”, rather than a background metric. The physical metric of spacetime is still $g_{\mu\nu}$.

Of the ten components of $f_{\mu\nu}$, four are gauge degrees of freedom, removable by gauge fixing general coordinate transformations. This is made explicit in the parameterization,

$$f_{\mu\nu} = \frac{\partial \phi^a}{\partial x^\mu} \tilde{f}_{ab} \frac{\partial \phi^b}{\partial x^\nu}. \tag{2.3}$$

The $\phi^a$ are interpreted as Stückelberg fields or as Goldstone modes associated with the breaking of general covariance [28]. The remaining six components contained in $\tilde{f}$ are non-dynamical. Possible choices for $f_{\mu\nu}$ are:

**Flat reference metric:** Most of the recent work on massive gravity has focused on $\tilde{f}_{\mu\nu} = \eta_{\mu\nu}$. In the unitary gauge this gives $f_{\mu\nu} = \eta_{\mu\nu}$. For this choice, (2.2) is the original ghost free Fierz-Pauli mass term [13, 14] for metric fluctuations around flat spacetime. Later, the generic instability of the non-linear theory (2.1) was shown by Boulware and Deser [1, 2] for this case, although their analysis also applies to general $f$. The actions recently proposed in [3, 4] also belong to this class, where the fields $\phi^a$ (2.3) played an important role in the construction. The absence of the Boulware-Deser instability at the complete non-linear level was proved for these actions in [7].

**FRW reference metric:** The quadratic action (2.2) is also known to be free of the Boulware-Deser ghost instability when $f_{\mu\nu}$ is a de Sitter or anti de Sitter metric [15, 16, 17, 18], or more generally, an FRW [19, 20, 21] metric. However, consistent non-linear extensions of such quadratic actions had so far remained undetermined.

**General non-dynamical reference metric:** In this paper we demonstrate the consistency of the non-linear massive actions proposed in [4], when extended to general $f_{\mu\nu}$ [6]. Such an extension is not only natural, but is also necessary to obtain a larger and potentially more viable class of solutions.

---

1 It turns out that, in some regions of parameter space, these theories may suffer from instabilities quite distinct from the Boulware-Deser problem, even at the linear level. However, these do not necessarily reflect an inconsistency of the theory [19, 24, 2].
Dynamical reference metric: It is appealing to complete the theory by including dynamics for $f_{\mu\nu}$ [11, 12]. That this can be done consistently in the context of bi-metric theories of gravity will be demonstrated in an accompanying work [29].

2.2 The ADM formulation of general relativity

The physical content of gravity and its propagating modes are easily identified in the ADM formulation [22] which is based on a $3+1$ decomposition of the metric,

$$ N \equiv (-g^{00})^{-1/2}, \quad N_i \equiv g_{0i}, \quad \gamma_{ij} \equiv g_{ij}. \quad (2.4) $$

The $N$ and $N_i$ are the lapse and shift functions respectively. In this parameterization,

$$ g^{\mu\nu} = N^{-2} \begin{pmatrix} -1 & N^j \\ N^i & N^2 \gamma_{ij} - N^i N^j \end{pmatrix}, \quad (2.5) $$

where, $N^j = \gamma^{jk} N_k$ and $\gamma^{ij} \gamma_{jk} = \delta^i_k$. Denoting the momentum canonically conjugate to $\gamma_{ij}$ by $\pi^{ij}$, the Einstein-Hilbert action in terms of these variables becomes (we ignore all boundary terms in what follows),

$$ S = M_p^2 \int d^4x \left[ \pi^{ij} \partial_t \gamma_{ij} + N R^0 + N_i R^i \right]. \quad (2.6) $$

The $R^\mu$ are functions of $\gamma_{ij}$ and $\pi^{ij}$ but are independent of the $N_\mu = (N, N_i)$,

$$ R^0 = \sqrt{\det \gamma} \left[ R(\gamma) + \frac{1}{\det \gamma} \left( \frac{1}{2} \pi^2 - \pi^{ij} \pi_{ij} \right) \right], \quad R^i = 2 \sqrt{\det \gamma} \nabla_j \left( \frac{\pi^{ij}}{\sqrt{\det \gamma}} \right). \quad (2.7) $$

The six components of $\gamma_{ij}$ are potentially propagating modes in the sense that their equations of motion obtained from (2.6), as well as those for their conjugate momenta $\pi^{ij}$, involve time derivatives (so that the Euler-Lagrange equations for $\gamma_{ij}$ are second order in time). Since a single propagating mode involves a field component and its canonically conjugate momentum, the six potentially propagating modes are described by the 12 functions $(\gamma_{ij}, \pi^{ij})$. However, in the theory defined by (2.6) not all of these are independent. To see this, note that the $N_\mu$ appear linearly as Lagrange multipliers, hence their equations of motion are four constraints (the “Hamiltonian” and “momentum” constraints) on the remaining fields,

$$ R_0(\gamma, \pi) = 0, \quad R_i(\gamma, \pi) = 0. \quad (2.8) $$

These constraints can be used along with the four general coordinate transformations to eliminate eight of the 12 functions, in favor of two remaining pairs. These pairs are the two propagating modes of GR, describing the two polarization states of the massless graviton at the non-linear level. In particular, the scalar ghost is not part of the physical spectrum [22].

Of the 12 equations of motion for $(\gamma_{ij}, \pi^{ij})$, four reduce to Bianchi identities while another four determine the $N_\mu$. The remaining four equations describe the propagating modes.
2.3 The Boulware-Deser ghost

Boulware and Deser used the ADM formalism to study the physical content of non-linear massive gravity and argued that, generically, massive gravity has six propagating modes. The sixth mode is the ghost that was avoided in the linear FP action but reappears at the non-linear level [1, 2]. They also found that the non-linear theory had a pathological non-positive Hamiltonian. Let us summarize their analysis here.

In ADM parameterization, the massive gravity action (2.1) becomes,

\[
S_m = M_p^2 \int d^4x \left[ \pi^{ij} \partial_t \gamma_{ij} + NR^0 + N_i R^i - m^2 V'(\gamma, N, N_i, \bar{f}_{\mu\nu}) \right],
\]

(2.9)

where \( V' = \sqrt{\det \gamma} NV \) and coordinate transformations have been used to set \( f_{\mu\nu} = \bar{f}_{\mu\nu} \) (2.3). Since \( V' \) is a non-linear function of the \( N_\mu \), the lapse and shift are no longer Lagrange multipliers and their equations of motion,

\[
R^\mu(\gamma, \pi) = m^2 V^\mu(\gamma, N, N_i, \bar{f}), \quad \text{with} \quad V^\mu = \frac{\partial V'}{\partial N_\mu},
\]

(2.10)

no longer constrain \( \gamma_{ij} \) and \( \pi^{ij} \). Instead, these equations can be solved for the \( N_\mu \) in terms of \( (\gamma_{ij}, \pi^{ij}) \). After eliminating the \( N_\mu \) in this way, one is left with twelve equations for the twelve dynamical variables \( (\gamma_{ij}, \pi^{ij}) \), hence the theory contains six propagating modes. In particular the sixth, ghost mode that was avoided in the linear FP theory has re-emerged as a propagating mode. Boulware and Deser [1] argued that, since in massive gravity \( V' \) is always non-linear in the \( N_\mu \), the sixth mode cannot be avoided.

As an explicit example of a non-linear mass term, [1] considered the FP mass (2.2) with \( f_{\mu\nu} = \eta_{\mu\nu} \), where now \( h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \) is no longer treated as a small fluctuation. This analysis is reviewed in the appendix. They found that in the linear approximation the Hamiltonian constraint eliminates the ghost, hence the linear FP theory is indeed consistent. However, at the non-linear level, for the mass term considered, there is no such constraint. Thus, to reiterate, the Boulware-Deser ghost instability is due to the loss of the Hamiltonian constraint at the non-linear level. Finally, in [1, 2] it was concluded that,

- The massive theory has six rather than five degrees of freedom, and hence contains a ghost.
- The Hamiltonian of the massive theory is not positive definite.
- In the limit \( m \to 0 \) the Hamiltonian diverges, hence this limit does not exist.

There is, however, a caveat in the arguments of Boulware and Deser. As pointed out in [3], avoiding the Boulware-Deser instability does not, in fact, strictly require linearity of the theory in the \( N_\mu \). Rather, it is enough that one combination of the four \( N_\mu \) equations of motion (2.10) becomes a constraint on the \( (\gamma_{ij}, \pi^{ij}) \). Based on this observation, we formulate the criteria for the absence of the ghost in the following subsection.
2.4 Criteria for the absence of the Boulware-Deser ghost

The caveat in the Boulware-Deser argument can be stated as follows. While the potential 
\[ V(g^{-1}f) \equiv V(N, N_i, \gamma, f) \] is a non-linear function of the \( N_\mu \), suppose there exist potentials \( V \) for which the \( N_\mu \) equations of motion depend only on three combinations of the \( N_\mu \),

\[ n_r = n_r(N, N^i, \gamma), \quad r = 1, 2, 3. \] (2.11)

That is, the \( N_\mu \) equations (2.10) take the generic form,

\[ R^\mu(\gamma, \pi) = m^2 \tilde{V}^\mu(\gamma, n_r). \] (2.12)

Then, in principle, three of these equations can be used to determine the \( n_r \) in terms of \((\gamma, \pi)\). Substituting the result into the remaining equation gives a constraint on the \((\gamma, \pi)\) that may have the right form to eliminate the ghost field. Finally, one also needs a second constraint to eliminate the variable canonically conjugate to the ghost field.

The linear Fierz-Pauli theory is linear in the lapse but not in the shift. Thus the \( N_i \) equation of motion provides a modified Hamiltonian constraint while the \( N_i \) equations are not constraints, but determine the \( N_i \) in terms of \((\gamma, \pi)\). It is natural to expect that this feature extends to a ghost-free non-linear theory, especially since the ghost is a scalar. Thus the functions \( n_i(N, N_j, \gamma) \) can be regarded as the counterparts of \( N_i \) in the massive theory, consistent with the 3-dimensional general covariance maintained in the ADM formulation. Assuming that the functional relationship is invertible (as it should be), one can determine the \( N_i \) as functions of the \( n_i \): \( N_i(N, n_j, \gamma) \). Then the massive action (2.9) can be expressed in terms of the combinations \( n_i \),

\[ S[N, N_i] = \tilde{S}[N, n_j(N, N_i, \gamma)]. \] (2.13)

Now consider the \( N_\mu \) equations of motion,

\[ \frac{\delta S}{\delta N_i^j} \equiv \frac{\delta \tilde{S}}{\delta n_j^i} \bigg|_{N} \frac{\delta n_j^i}{\delta N_i^j} = 0, \quad \frac{\delta S}{\delta N} \equiv \frac{\delta \tilde{S}}{\delta N} \bigg|_{n} + \frac{\delta \tilde{S}}{\delta n_j^i} \bigg|_{N} \frac{\delta n_j^i}{\delta N} = 0, \] (2.14)

where the subscript on the vertical bar indicates the variable held fixed in the process of variation. This leads to the equivalent equations,

\[ \frac{\delta \tilde{S}}{\delta n_j^i} \bigg|_{N} = 0, \quad \frac{\delta \tilde{S}}{\delta n} \bigg|_{n} = 0, \] (2.15)

which are linear combinations of the \( N_\mu \) equations. Based on these equations one can now formulate a nested set of criteria for the existence of a Hamiltonian constraint.

1. As described above, for a constraint to exist, the \( n_i \) equations of motion should depend on \( N_\mu \) only through the three combinations \( n_i \). Then they can be used to determine the \( n_i \) in terms of \((\gamma_{ij}, \pi^{ij})\).

2. The \( N \) equation also must involve only the \( n_i \) and be independent of \( N \) so that, given the \( n_i^j \) solution, it becomes a constraint on \((\gamma_{ij}, \pi^{ij})\). For this to be the case, the massive action in the form \( \tilde{S} \), i.e., when regarded as a functional of \( N \) and \( n_i \), must be linear in \( N \).
3. The action \( \tilde{S} \) also contains the term \( N_i R^i \) where \( N_i = N_i(N, n_j, \gamma) \). Linearity of \( \tilde{S} \) in \( N \) then implies that the expression for \( N_i \) in terms of \( n_j \) must be linear in \( N \).

Note that the minimal coupling of the metric to matter is linear in the lapse and shift. If this were not the case, the constraints of even massless GR would be violated. Thus criteria specified here are not modified by the presence of the minimal matter coupling.

In this paper we show that, for the massive gravity theories described in the next subsection, once requirements 2 and 3 are satisfied, then 1 follows automatically. This guarantees the existence of a Hamiltonian constraint associated with the \( N \) equation of motion. We also argue for the existence of a non-trivial secondary constraint (subsequently proven in [23]) as, simply, the non-linear extension of the known secondary constraint in the linear FP theory (see, e.g., [30]). These two constraints eliminate the canonical pair corresponding to the Boulware-Deser ghost, reducing the number of propagating modes from six to five.

### 2.5 Non-linear massive gravity actions with general \( f_{\mu \nu} \)

In principle, the above criteria might be used to construct a theory of massive gravity that does not suffer from the Boulware-Deser instability. In practice, this was not so straightforward.\(^2\) Potentially ghost free actions were first constructed for \( \bar{f}_{\mu \nu} = \eta_{\mu \nu} \) following a very different perturbative argument. It was observed in [28] that the \( \phi^a \) in (2.3) are the Goldstone bosons associated with the breaking of general covariance by the mass term. Then for \( \bar{f}_{\mu \nu} = \eta_{\mu \nu} \) and \( g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \), an analogy with the Goldstone-vector boson equivalence theorem in gauge theory implies that, in the high energy limit, the dynamics of massive gravity is mirrored in the dynamics of the Goldstone sector, particularly, in the “longitudinal” mode of the \( \phi^a \) fluctuations. Thus the ghost of massive gravity appears as a ghost in this longitudinal mode. Being a scalar field in flat spacetime, this is a much easier setup to investigate. One may attempt to constrain \( V \) using this correspondence [32].

The breakthrough came with the work of de Rham and Gabadadze [3] and de Rham, Gabadadze and Tolley [4] who used this approach to construct a two-parameter family of massive actions for \( f_{\mu \nu} = \eta_{\mu \nu} \). The two free parameters are denoted by \( \alpha_3 \) and \( \alpha_4 \). These actions were shown to be ghost free in this high energy limit, the “decoupling limit”. To go beyond the decoupling limit, [4] carried out an ADM analysis of the \( \alpha_3 = \alpha_4 = 0 \) model to quartic order in the metric perturbations and demonstrated the absence of the Boulware-Deser ghost to that order.

The presentation of the actions given in [4] is convenient for a perturbative analysis. However, the expressions that multiply the parameters \( \alpha_n \) contain mixtures of terms with different levels of non-linear complexity. To demonstrate the absence of the Boulware-Deser ghost at the full non-linear level it is helpful to use the reformulation of [6] in which different levels of non-linearity are disentangled. Using this reformulation, the absence of the Boulware-Deser ghost at the non-linear level was proven for \( \bar{f}_{\mu \nu} = \eta_{\mu \nu} \) in [5].

\(^2\)In hindsight, these criteria are powerful enough that they can determine the complete form of the non-linear action as will be discussed elsewhere [31].
In this paper we extend the ghost analysis to any general, non-dynamical $f_{\mu\nu}$ using the presentation of massive gravity actions given in [6]. Let us briefly review this formulation. The basic building block of non-linear massive gravity is the square-root matrix $\sqrt{g-1}f$ [4, 6], where $\sqrt{g-1}f\sqrt{g-1}f = g^{\mu\lambda}f_{\lambda\nu}$. The terms appearing in the massive action are identified as elementary symmetric polynomials of the eigenvalues of this matrix. They sum up to a “deformed determinant” (for related ideas see [25, 27]). The antisymmetry property of this structure allows one to generalize the reference metric from flat to any $f_{\mu\nu}$ and still remain within the same 2-parameter family of actions.

The simplest non-linear massive action with zero cosmological constant is given by [6],

$$S_{\text{min}} = M_p^2 \int d^4x \sqrt{-g} \left[ R - 2m^2 \left( \text{tr} \sqrt{g-1}f - 3 \right) \right]. \quad (2.16)$$

We will refer to this as the minimal massive action. The most general non-linear massive action can be written as

$$S = M_p^2 \int d^4x \sqrt{-g} \left[ R + 2m^2 \sum_{n=0}^{3} \beta_n e_n(\sqrt{g-1}f) \right]. \quad (2.17)$$

The $e_k(\mathcal{X})$ are elementary symmetric polynomials of the eigenvalues of $\mathcal{X}$. For a generic $4 \times 4$ matrix they are given by,

$$e_0(\mathcal{X}) = 1,$$
$$e_1(\mathcal{X}) = [\mathcal{X}],$$
$$e_2(\mathcal{X}) = \frac{1}{2}([\mathcal{X}]^2 - [\mathcal{X}^2]),$$
$$e_3(\mathcal{X}) = \frac{1}{6}([\mathcal{X}]^3 - 3[\mathcal{X}][\mathcal{X}^2] + 2[\mathcal{X}^3]),$$
$$e_4(\mathcal{X}) = \frac{1}{24}([\mathcal{X}]^4 - 6[\mathcal{X}]^2[\mathcal{X}^2] + 3[\mathcal{X}^2]^2 + 8[\mathcal{X}][\mathcal{X}^3] - 6[\mathcal{X}^4]),$$
$$e_k(\mathcal{X}) = 0 \quad \text{for} \quad k > 4,$$

where the square brackets denote the trace. Of the four $\beta_n$, two combinations are related to the mass and the cosmological constant, while the remaining two combinations are free parameters. Setting the cosmological constant to zero and the parameter $m$ as the mass, the four $\beta_n$ are parameterized in terms of the $\alpha_3$ and $\alpha_4$ of [4] as (for $n = 0, \ldots, 4$),

$$\beta_n = (-1)^n \left( \frac{1}{2}(4 - n)(3 - n) - (4 - n)\alpha_3 + \alpha_4 \right). \quad (2.19)$$

The minimal action corresponds to $\beta_2 = \beta_3 = 0$ supplemented by $\beta_0 = 3$, $\beta_1 = -1$ to get a zero cosmological constant contribution from the potential term. We will start with this minimal theory in the analysis that follows.

### 3. Absence of the BD ghost in the minimal massive action

In this section we show that the minimal non-linear massive gravity action (2.16) satisfies the criteria outlined in section 2.4 and thus there exists a Hamiltonian constraint on the dynamical variables. In addition, we argue for the existence of an associated secondary constraint. Thus this action does not suffer from the Boulware-Deser ghost instability. We solve the constraints explicitly and discuss the positivity of the Hamiltonian.
3.1 Enforcing the criteria for the existence of the Hamiltonian constraint

In the ADM formulation the Lagrangian for the minimal massive action (2.16) becomes,

\[ L_{\text{min}} = \pi^{ij} \partial_t \gamma_{ij} + NR^0 + N^i R_i - 2m^2 \sqrt{\det \gamma} N \left( \text{tr} \sqrt{g^{-1} f} - 3 \right), \]  

(3.1)

where, using the parameterization (2.5) for \( g^{\mu \nu} \), one gets,

\[ N^2 g^{-1} f = \begin{pmatrix} -f_{00} + N^i f_{i0} & -f_{0j} + N^i f_{ij} \\ N^2 \gamma^{il} f_{l0} - N^i (-f_{00} + N^l f_{l0}) & N^2 \gamma^{il} f_{lj} - N^i (-f_{0j} + N^l f_{lj}) \end{pmatrix}. \]  

(3.2)

Since the action contains the square root of this matrix, it is highly non-linear in the \( N_\mu \).

Hence it could potentially propagate a ghost sixth mode according to the Boulware-Deser argument. But, as discussed in section 2.4, this can be avoided if the four \( N_\mu \) equations of motion happen to depend only on three combinations of the lapse and shift, say \( n^i(N_\mu) \), leaving a single constraint to eliminate the sixth, ghost mode.

We show now that this is indeed the case for the minimal action (3.1). First, we identify the appropriate functions \( n^i \). This is achieved by imposing the criteria for the absence of ghost discussed in section 2.4. In fact, we will only have to impose criteria 2 and 3. Then 1 follows automatically.

Criterion 3 requires that the expression for \( N_i \) in terms of the \( n_i \) be linear in \( N \),

\[ N^i = c^i + N d^i. \]  

(3.3)

The \( c^i \) and \( d^i \) are functions of \( n^i \) and \( \gamma_{ij} \) but are independent of \( N \). They will be determined in what follows by demanding that the action, when written in terms of \( n_i \) and \( N \), must be linear in \( N \). Using (3.3), (3.2) takes the form

\[ N^2 g^{-1} f = E_0 + N E_1 + N^2 E_2 \]  

(3.4)

where the matrices \( E_0 \), \( E_1 \) and \( E_2 \) are independent of \( N \). To write them compactly, define,

\[ a_\mu = -f_{0\mu} + c^i f_{i\mu}. \]  

(3.5)

Then one gets,

\[ E_0 = \begin{pmatrix} a_0 & a_j \\ -a_0 c^i & -c^i a_j \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 \\ (\gamma^{il} - d^i d^l) f_{l0} & (\gamma^{il} - d^i d^l) f_{lj} \end{pmatrix}, \quad E_2 = \begin{pmatrix} N^2 \gamma^{il} f_{l0} & N^2 \gamma^{il} f_{lj} \\ -d^l f_{l0} - (d^i f_{l0} c^i + a_0 d^i a_j) & -(d^l f_{lj} + d^i a_j) \end{pmatrix}. \]  

(3.6)  

(3.7)

Criterion 2 of section 2.4 requires that the mass term, when written in terms of \( N \) and \( n_i \), must be linear in \( N \). For the minimal massive action, this is satisfied if the matrix \( \sqrt{g^{-1} f} \) has the following form\(^3\),

\[ N \sqrt{g^{-1} f} = A + N B, \]  

(3.8)

\(^3\)Note that this is more restrictive than requiring the linearity of the \( \text{tr}(N \sqrt{g^{-1} f}) \) in \( N \), but leads to simple systematics that satisfy requirement 1 automatically.
where the matrices $A$ and $B$ are independent of $N$. Demanding that this expression (3.8) be consistent with $N^2 g^{-1} f$ as given by (3.4), determines $A$ and $B$ as well as $c^i$ and $d^i$. Explicitly, comparing (3.8) and (3.4) gives,

$$
A^2 = \mathbb{E}_0, \quad B^2 = \mathbb{E}_2, \quad \text{and} \quad AB + BA = \mathbb{E}_1.
$$

(3.9)

Let us consider the first two equalities in (3.9). Using (3.6) it is easy to verify that these imply,

$$
A = \frac{1}{\sqrt{x}} \begin{pmatrix} a_0 & a_j \\ -a_0 c^i & -c^i a_j \end{pmatrix}, \quad B = \sqrt{x} \begin{pmatrix} 0 & 0 \\ D_i^k (3f^{-1})^{kl} f_{0l} & D^i_j \end{pmatrix}.
$$

(3.10)

Here $3f_{ij} \equiv f_{ij}$ and we have introduced,

$$
x \equiv a_0 - c^i a_l, \quad \sqrt{x} D_{ij} \equiv \sqrt{(\gamma^{il} - d^i d^l)} f_{ij}.
$$

(3.11)

The expression for $A = \sqrt{x} \mathbb{E}_0$ follows since $\mathbb{E}_0$ is a projection operator, $\mathbb{E}_0^2 = x \mathbb{E}_0$, hence $A = \mathbb{E}_0 / \sqrt{x}$. In the expression for $B$, the square-root matrix $D$ is defined with an extra factor of $\sqrt{x}$ for later convenience.

Before proceeding further, we note a very important property of the matrix $D$. According to (3.11) it has the form $D = \sqrt{S \, 3f}$ where both $S$ and $3f$ are symmetric matrices. By rewriting $D$ as $\sqrt{1 + (S \, 3f - 1)}$ and then expanding in powers of $(S \, 3f - 1)$, it becomes obvious that $(3f D)^T = 3f D$. In terms of components, this means

$$
f_{ik} D^k_j = f_{jk} D^k_i.
$$

(3.12)

This identity will be used often in the following analysis.

Now let us consider the third equality in (3.9). Using (3.10), one can compute $AB + BA$ and compare the result with $\mathbb{E}_1$ in (3.7). Using (3.5) and the property of $D$ given in (3.12), one obtains the following relation between $c^i$ and $d^i$,

$$
d^i = D^i_k \left[ c^k - (3f^{-1})^{kl} f_{0l} \right].
$$

(3.13)

Guided by the case of flat $f_{\mu\nu}$ considered in [5], we introduce the variables $n^i$ so that$^4$

$$
n^i = c^k - (3f^{-1})^{kl} f_{0l}.
$$

(3.14)

Then, (3.13) reduces to,

$$
d^i = D^i_k n^k.
$$

(3.15)

Substituting for $d^i$ in (3.11) gives a matrix equation for $D$,

$$
\sqrt{x} D = \sqrt{(\gamma^{-1} - D n n^T D^T)} 3f,
$$

(3.16)

$^4$This choice simplifies some of the equations, but is not unique. For instance, we could have also chosen $n^i = c^i$. For a different choice, see section 3.5 below.
which will be solved below in terms of \( n^i \). Thus \( \rho^i, d^i \) and \( D^i_j \) can be determined entirely in terms of the \( n^i \) and \( \gamma_{ij} \). This proves that indeed there exist modified shift variables \( n^i \) in terms of which \( N \sqrt{g^{-1} f} \) is linear in \( N \) and given by (3.8).

In the proof that follows, we need only the condition (3.16), and not its explicit solution. However, it is important that this solution exists and can be used to show that the relation (3.23) is invertible. Thus we take a moment to derive the solution. Squaring both sides of (3.16) and moving the \( D \)-dependent terms to one side gives, after using (3.12),

\[
D^i_k Q^j_k D^j_k = \gamma^{ij} f_{jk}, \quad \text{with,} \quad Q^j_k = x \delta^j_k + n^i n^m f_{mj}.
\] (3.17)

On multiplying both sides by \( Q \), this becomes \((D Q)^2 = (\gamma^{-1} f)Q\). Taking the square root and rearranging gives,

\[
D = (\sqrt{\gamma^{-1} f} Q) Q^{-1}.
\] (3.18)

The inverse matrix \( Q^{-1} \) is easily obtained by noting that \((nn^T f)^2 = (n^T f n)(nn^T f)\). Then one finds

\[
Q^{-1} = \frac{1}{x} (1 - M^{-2} nn^T f),
\] (3.19)

where \( M^2 = -f_{00} + f_{0k}(\gamma^{-1})^{kl} f_{l0} \) is the lapse function of \( f_{\mu\nu} \). Equations (3.18) and (3.19) give the explicit solution for \( D \) in terms of \( n^i \).

Before moving on, let us point out that our final expressions naturally involve the ADM parameters of \( f_{\mu\nu} \). In analogy with the ADM parameterization of \( g_{\mu\nu} \), we define,

\[
M \equiv (-f^{00})^{-1/2}, \quad M_i \equiv f_{0i}, \quad ^3 f_{ij} \equiv f_{ij}.
\] (3.20)

We also define \( M^i = (\gamma^{-1})^{ij} M_j \). Then, the variables \( a_\mu \) defined in (3.7) become,

\[
a_0 = M^2 + n^i M_i, \quad a_i = ^3 f_{i0} n^i,
\] (3.21)

and in terms of the lapse \( M \), the \( x \) of (3.11) is simply,

\[
x = M^2 - n^k f_{kl} n^l.
\] (3.22)

To recapitulate, we have identified three variables \( n^i \) such that the Lagrangian (3.1) written in terms of these variables is linear in \( N \) and hence satisfies criteria 2 and 3 for the existence of a Hamiltonian constraint, as outlined in section 2.4. The functions \( n^i \) are related to the lapse and shift variables of \( g_{\mu\nu} \) and \( f_{\mu\nu} \) through\(^5\),

\[
N^i = n^i + M^i + N D^i_k n^k,
\] (3.23)

where the matrix \( D \) is given by (3.18) and (3.19) above. Note that \( n^i \) parameterize the difference between the shift functions of \( g_{\mu\nu} \) and \( f_{\mu\nu} \). In the following section we will show that, with this choice of \( n^i \), criteria 1 of section 2.4 is automatically satisfied.

\(^5\)For \( f_{\mu\nu} = \eta_{\mu\nu} \), in a perturbative relation between \( N_i \) and \( n_i \) was used to demonstrate the absence of the BD ghost to quartic order in the fluctuations. To compare this perturbative relation with our result we expand (3.23) to quartic order around flat spacetime using (3.18). After lowering the indices and rescaling \( n_i \) by 2, one gets,

\[
N_i = n_i + \frac{1}{2} \delta N n_i - \frac{1}{4} \delta N \gamma^{ij} n_j + \frac{1}{4} (-\gamma^{ij} + \frac{3}{4} \gamma^{ij} n^k n_k) n_j.
\]

In (3.18) the third term comes with a coefficient 1/8 and the fourth term is absent. Thus it appears that at low orders this relation is not unique.
3.2 The Hamiltonian constraint in the minimal massive theory

Now we consider the $N$ and $n^i$ equations of motion (2.15) and show that they do not depend on $N$. Hence they give rise to a Hamiltonian constraint. Incorporating (3.8), (3.10), (3.15) and (3.23) into the minimal massive theory (3.1), leads to an action in terms of the $n^i$ which is linear in $N$, meeting requirements 2 and 3 of section 2.4,

$$L_{\text{min}} = \pi^{ij} \partial_t \gamma_{ij} + NR^0 + R_i \left[ n^i + M^i + N D^i_k n^k \right]$$

$$-2m^2 \sqrt{\text{det} \gamma} \left[ \sqrt{x} + N \sqrt{x} \text{tr} D - 3N \right]. \quad (3.24)$$

Thus the $N$ equation of motion (2.15) is independent of $N$ by construction. We now show that the $n^i$ equations are also independent of $N$ as required by criterion 1. To get the $n^i$ equations, one needs,

$$\frac{\partial}{\partial n^k} \sqrt{x} \frac{\partial}{\partial n^k} = - \frac{1}{\sqrt{x}} n^j f_{ji} \frac{\partial n^i}{\partial n^k}, \quad (3.25)$$

$$\frac{\partial}{\partial n^k} (\sqrt{x} \text{tr} D) = - \frac{1}{\sqrt{x}} n^j f_{ji} \frac{\partial (D^i_l n^l)}{\partial n^k}. \quad (3.26)$$

The first line easily follows from (3.22). In the second line, we have first used (3.16) to re-express the left hand side in terms of the square root matrix and then $\delta \text{tr} \sqrt{E} = \frac{1}{2} \text{tr}(\sqrt{E^{-1}} \delta E)$ to evaluate the derivative. On using (3.13), the right hand side follows. Then, varying $L_{\text{min}}$ in (3.24) with respect to $n^k$ gives,

$$\left( R_i + 2m^2 \sqrt{\text{det} \gamma} n^j f_{ji} \frac{\partial}{\partial n^k} \right) \left[ \frac{\partial}{\partial n^k} (n^i + N D^i_j n^j) \right] = 0. \quad (3.27)$$

Note that the expression in the square brackets is the Jacobian matrix $\frac{\partial n^i}{\partial n^j}$ of (3.23). From (2.14) it is then obvious that the expression in the parenthesis is indeed $\frac{\partial L_{\text{min}}}{\partial N^i}$. The Jacobian matrix is generically invertible as can be checked, for example, perturbatively by using the expression in footnote (3). Hence one gets the $n^i$ (or $N^i$) equations of motion,

$$\sqrt{x} R_i + 2m^2 \sqrt{\text{det} \gamma} f_{ij} n^j = 0, \quad (3.28)$$

which are independent of $N$ as advertised. Using (3.22), these determine $n^i$ in terms of $\gamma_{ij}$ and the conjugate momenta $\pi^{ij}$ (contained in $R_i$),

$$n^i = \frac{-M}{\sqrt{4m^4 \text{det} \gamma + R_k (f^{-1})^k_l R^l}} (f^{-1})^{ij} R_j. \quad (3.29)$$

As a consistency check, note that this implies,

$$x = \frac{4m^4 \text{det} \gamma M}{4m^2 \text{det} \gamma + R R^T f^{-1} R} > 0, \quad (3.30)$$

so that $\sqrt{x}$ is real on the constraint surface.

The $N$ equation of motion is,

$$R^0 + R_i D^i_j n^j - 2m^2 \sqrt{\text{det} \gamma} \left[ \sqrt{x} D^k_k - 3 \right] = 0. \quad (3.31)$$
The $n^i$ that appear explicitly and through $\gamma (3.22)$ and $\pi^{ij}$ can be eliminated using the solution (3.29). Thus the $N$ equation becomes a constraint on the 12 components of $\gamma_{ij}$ and $\pi^{ij}$ and reduces the number of canonical variables to 11. This is the Hamiltonian constraint of the minimal massive action.

To see that this constraint has the correct form to eliminate the ghost, we adapt a parameterization of [22] for $\gamma_{ij}$ in terms of the six functions $\gamma^{TT}_{ij}(2)$, $\gamma^T_{ij}(2)$, $\gamma^L(1)1$ and $\gamma^T(1)$,

$$\gamma_{ij} = \gamma^{TT}_{ij} + \partial_i \gamma_{j} + \partial_j \gamma_{i} + \gamma^T_{ij}.$$  \hspace{1cm} (3.32)

Here, $\gamma^{TT}_{ij} = \frac{1}{2}(\delta_{ij} - \nabla^2 \partial_i \partial_j)\gamma^T$ and $\gamma_i = \gamma^T_i + \partial_i \gamma^L$. As the notation implies, $\gamma^{TT}_{ij}$ is traceless, transverse and $\gamma^T_i$ is transverse; hence the above counting. $\gamma^T$ is the trace of the transverse part of $\gamma_{ij}$. The flat space limit indicates that $\gamma^{TT}_{ij}$, $\gamma^T_{j}$ and $\gamma^L$ carry the massive spin-2 graviton while $\gamma^T$ describes the ghost. From the analysis of [22] one can see that $\gamma^T$ appears in $R^0$ in the right way to be eliminated by (3.31), in analogy with GR. One more constraint is needed to remove the canonically conjugate variable.

### 3.3 The secondary constraint and the absence of the BD ghost

Now we argue that the Hamiltonian constraint gives rise to a secondary constraint (for an explicit proof see [23]) so that the 12 dimensional phase space of $\gamma_{ij}$ and $\pi^{ij}$ has only 10 degrees of freedom, corresponding to the five polarizations of the massive graviton.

Before proceeding note that after integrating out the shifts $N^i$, the Lagrangian (3.24) remains linear in the lapse $N$,

$$\mathcal{L}_{\text{min}} = \pi^{ij} \partial_t \gamma_{ij} - H_0(\gamma_{ij}, \pi^{ij}, f) + N C(\gamma_{ij}, \pi^{ij}, f).$$ \hspace{1cm} (3.33)

From this one can read off a Hamiltonian, ignoring the usual ADM contribution that can be expressed as a boundary term, $H = \int d^3 x (H_0 - NC)$.

A secondary constraint is obtained by demanding that the Hamiltonian constraint (3.31), now summarized as $C = 0$, is independent of time on the constraint surface. In the Hamiltonian formulation this condition is expressed in terms of a Poisson bracket,

$$\frac{d}{dt} C = \{C, H\} \approx 0$$ \hspace{1cm} (3.34)

with $H$ as given above. Now, if $\{C(x), C(y)\} \neq 0$, this condition becomes an equation for $N$ and does not impose any constraint on the $\gamma_{ij}$ and $\pi^{ij}$. If this were true, as argued to be the case in [33], then a second constraint would not exist. However, in [23] it has been shown through explicit calculation that $\{C(x), C(y)\} \approx 0$. Then the condition $dC/dt = 0$ is independent of $N$ and thus becomes a second constraint on $\gamma_{ij}$ and $\pi^{ij}$ (with $H_0 = \int d^3 x H_0$),

$$C_2 \equiv \{C, H_0\} \approx 0,$$ \hspace{1cm} (3.35)

provided the expression for $C_2$ does not vanish identically. That this is the case can be easily shown perturbatively. By construction, the Lagrangian (3.33) reproduces the Fierz-Pauli Lagrangian at lowest order in the fields and for $f_{\mu\nu} = \eta_{\mu\nu}$. Hence,

$$H_0 \simeq H_0^{FP} + \mathcal{O}(\gamma^3, \pi^3, \delta f), \quad C \simeq C^{FP} + \mathcal{O}(\gamma^2, \pi^2, \delta f).$$ \hspace{1cm} (3.36)
Therefore, one should find that,

\[ C_{(2)} \simeq C_{(2)}^{FP} + O(\gamma^2, \pi^2, \delta f), \quad (3.37) \]

where it is known that \( C_{(2)}^{FP} \) is neither identically zero nor equal to \( C_{(2)}^{FP} \) (see, for example, [30]). Moreover, as can be seen from the Fierz-Pauli structure, enforcing \( dC_{(2)}/dt = 0 \) will result in an equation for \( N \) and no further constraints are generated. For details see [23]. To summarize, this proves the existence of a secondary constraint that is needed to completely eliminate the propagating Boulware-Deser ghost mode.

3.4 The positivity of the Hamiltonian and open issues

In general relativity, the Hamiltonian corresponding to the ADM Lagrangian [20] is expressible as a boundary term [22]. The boundary expression for the Hamiltonian can also be derived in a more general setup [34, 35] by considering the Gibbons-Hawking boundary terms that have been suppressed in [26]. This boundary expression for the Hamiltonian is independent of the mass term and remains unchanged in the massive theory. However, the mass term gives an extra bulk contribution \( H_m \) to the Hamiltonian due to the reduction in the number of constraints.

In the massive gravity actions that they analyzed, Boulware and Deser found that the contribution of the mass term to the Hamiltonian \( H_m \) was generically not positive and moreover, it diverged in the limit \( m \to 0 \) (as reviewed in the appendix). The pathological behavior of the Hamiltonian was given as a reason for disregarding massive gravity [1, 2].

Here we consider the contribution of the mass term to the Hamiltonian in the minimal massive action with a general reference metric. This can be easily computed from \( L_{min} \) in (3.24) upon imposing the Hamiltonian constraint (3.31). Using (3.29) and (3.30) this contribution becomes,

\[ H_{min,m} = M \left( 4m^4 \det \gamma + R_k (3f^{-1})^{kl} R_l - R_i M^i \right), \quad (3.38) \]

where, \( M \) and \( M^i \) are the lapse and shift functions of \( f_{\mu\nu} \). Note that for \( f_{\mu\nu} = \eta_{\mu\nu} \) \((M = 1, M^i = 0)\), which was the case considered by Boulware and Deser, this is clearly positive and well behaved in the limit \( m \to 0 \), avoiding the pathologies observed in [1]. The same applies to any reference metric \( f_{\mu\nu} \) for which \( M^i = 0 \) and \( M > 0 \).

In general, when \( M^i \neq 0 \), the last term in \( H_{min,m} \) appears problematic, but at least in some cases this is a gauge artifact. For example, even for a flat \( f_{\mu\nu} = \partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab} \), if we do not choose the physical gauge \( \phi^a = x^a \), the Hamiltonian is not manifestly positive. But clearly, in this case, the problem is a gauge artifact. It seems that in all situations where one can choose a gauge with \( M^i = 0 \), the Hamiltonian remains positive.

However, while the minimal massive action with a general reference metric is free of Boulware-Deser instability, it seems that it may not always have a positive Hamiltonian. In fact, given the known instabilities of massive gravity in, say, de Sitter and anti de Sitter backgrounds, we do not expect a massive gravity Hamiltonian to be positive for all possible \( f_{\mu\nu} \) in all regions of parameter space. However, precisely because of the \( R_i M^i \) structure, the theory is potentially linear in the lapse and shift functions of \( f_{\mu\nu} \), provided the constraints...
do not introduce a non-linear dependence through the elimination of the ghost mode. This opens up the possibility of consistently promoting $f_{\mu\nu}$ to a dynamical variable. This will be discussed in detail in [29].

3.5 An alternative set of variables

It is not apparent that the equations of section 3.1 depend in a simple way on the lapse $M$ of $f_{\mu\nu}$. However, the solution (3.29) shows that the $n^i$ are linear in $M$, so that $x$ is proportional to $M^2$ and the matrix $D$ goes as $1/M$. This motivates working with an alternative set of variables,

$$n^i = M \hat{n}^i, \quad D^i_j = \hat{D}^i_j/M.$$  

which makes the simple dependence of the theory on $M$ manifest even beyond the minimal model. In term of these, (3.23) takes the form,

$$N^i = M \hat{n}^i + M^i + N \hat{D}^i_k \hat{n}^k,$$  

and

$$x = M^2 \hat{x}, \quad \hat{x} = 1 - \hat{n}^T f \hat{n}.$$  

The defining equation for $\hat{D}$, and hence the matrix $\hat{D}$ itself, is independent of $M$,

$$\sqrt{x} \hat{D} = \sqrt{(\gamma^{-1} - \hat{D} \hat{n} \hat{n}^T \hat{D}^T)} f.$$  

Then it follows that, expressed in terms of hatted variables, the matrix $B$ in (3.10) is independent of $M$. The matrix $A$ depends on $M$ in a more complicated way through $a_0 = M^2 + M \hat{n}^l M_l$, $a_i = M \hat{n}^l M_l$ and $c^i = M \hat{n}^i + M^i$. However, the most general massive action (2.17) contains $A$ only in the combinations $\text{tr}(A)$, $\text{tr}(A B)$ and $\text{tr}(A B^2)$ (see the next section). It is easy to verify that all these are linear in $M$. Thus, on using the hatted variables, the action (2.17) becomes linear in both $N$ (see below) and $M$.

4. Absence of the BD ghost in the complete 2-parameter massive action

We now demonstrate the absence of the Boulware-Deser ghost in the full two-parameter generalization of the minimal massive theory with a general reference metric $f_{\mu\nu}$. We show that the Hamiltonian constraint is maintained even in this case. The analysis is more involved but it turns out that the variables identified in the minimal massive model can be used in these more general cases without any modifications. The argument for the existence of the secondary constraint parallels the discussion for the minimal model. The details are given in [23].

4.1 The 2-parameter action in terms of the new variables

We now show that, using the same $n^i$ of the previous section (3.23), the general massive action (2.17) turns out to be linear in $N$ thereby satisfying criteria 2 and 3 of section 2.4. In the ADM parameterization, the general Lagrangian is given by

$$\mathcal{L} = \pi^{ij} \partial_t \gamma_{ij} + NR^0 + N^i R_i + 2m^2 \sqrt{\gamma} N \left( \sum_{n=0}^{3} \beta_n e_n \left( \sqrt{g^{-1} f} \right) \right).$$  

(4.1)
To check if this Lagrangian is linear in the lapse \( N \), let us express the potential in terms of the matrices \( A \) and \( B \) of (3.8). Since \( e_n \sim (\sqrt{g}^{-1}f)^n \), it might seem that negative powers of \( N \) should appear in the Lagrangian. However, due to the property \( \text{tr}(A^k) = (\text{tr} A)^k \), these terms cancel amongst themselves. The potential terms (2.18) then give,

\[
Ne_0(\sqrt{g}^{-1}f) = N, \\
Ne_1(\sqrt{g}^{-1}f) = \text{tr} A + N \text{tr} B, \\
Ne_2(\sqrt{g}^{-1}f) = \text{tr} \text{tr} A B - \text{tr} A B + \frac{1}{2} N \left[ (\text{tr} B)^2 - \text{tr} B^2 \right], \\
Ne_3(\sqrt{g}^{-1}f) = \text{tr} B^2 - \text{tr} A B + \frac{1}{2} \text{tr} A \left[ (\text{tr} B)^2 - \text{tr} B^2 \right] + \frac{1}{6} N \left[ (\text{tr} B)^3 - 3 \text{tr} B^2 + 2 \text{tr} B^3 \right].
\]  

These terms are at most linear in the lapse \( N \) and thus satisfy criterion 2. The \( e_0 \) term just contributes to a cosmological constant while the \( e_1 \) term was already considered in the previous section for the minimal action. So the terms that remain to be investigated are \( e_2 \) and \( e_3 \).

### 4.2 The Hamiltonian constraint in the 2-parameter theory

We now compute the \( n^i \) equations of motion and show that they do not depend on the lapse \( N \). The \( R_i N^i \) term in the action contributes a term \( R_i J^i_k \) to the equations of motion, where,

\[
J^i_k = \frac{\partial N_i}{\partial n^k} = \frac{\partial}{\partial n^k} \left( n^i + N D^i_j n^j \right),
\]

is the Jacobian matrix of (3.22). Since the Jacobian contains the lapse \( N \), the only way for the \( n^i \) equations of motion to be independent of \( N \) is if the contribution of the potential terms is also proportional to the Jacobian. This must happen separately for the \( e_2 \) and \( e_3 \) terms. We show now that this is the case.

In the following, we employ matrix notation where \( n \) is a column vector with elements \( n^i \) and \( n^T \) is its transpose. To vary with respect to \( n^k \), consider the \( A \) terms first. Using (3.10), these terms are,

\[
\text{tr} A = \sqrt{x}, \\
\text{tr} A B = -n^T 3f D n, \\
\text{tr} A B^2 = -\sqrt{x} n^T 3f D^2 n = -\frac{1}{\sqrt{x}} n^T 3f (\gamma^{-1} - D n n^T D^T) 3f n.
\]

The last expression is written in two equivalent ways using (3.16). Varying these gives

\[
\frac{\partial}{\partial n^k} \text{tr} A = -\frac{1}{\sqrt{x}} n^T 3f \frac{\partial n}{\partial n^k}, \\
\frac{\partial}{\partial n^k} \text{tr} A B = -n^T 3f D \frac{\partial n}{\partial n^k} - n^T 3f \frac{\partial (D n)}{\partial n^k}, \\
\frac{\partial}{\partial n^k} \text{tr} A B^2 = \frac{1}{\sqrt{x}} (n^T 3f D^2 n) n^T 3f \frac{\partial n}{\partial n^k} - 2\sqrt{x} n^T 3f D \frac{\partial (D n)}{\partial n^k} - 2\sqrt{x} n^T 3f D^2 \frac{\partial n}{\partial n^k} \]

\[+ \frac{2}{\sqrt{x}} (n^T 3f D n) n^T 3f \frac{\partial (D n)}{\partial n^k}.
\]
From the last equality we derive the relation,

\[
\left[\sqrt{x}D + \frac{1}{\sqrt{x}}Dnn^T3f \right] \frac{\partial (Dn)}{\partial n^k} = \left[\sqrt{x}D^2 + \frac{1}{\sqrt{x}}(n^T3fD^2n) \right] \frac{\partial n}{\partial n^k}.
\] (4.6)

This relation is very useful. We need all derivatives in \(\partial(Ne_3)/\partial n^k\) to appear in the combination \(\text{[1.3]}\). However on direct substitution we will find some \(\partial(Dn)/\partial n^k\) terms without the factor \(N\). Equation (4.6) allows us to re-express these in terms of \(\partial(n)/\partial n^k\).

Now consider the \(\mathcal{B}\) terms. Using (3.11) these are,

\[
\text{tr}\mathcal{B} = \sqrt{x}\text{tr}D, \quad \text{tr}\mathcal{B}^2 = \sqrt{x}^2\text{tr}D^2, \quad \text{tr}\mathcal{B}^3 = \sqrt{x}^3\text{tr}D^3.
\] (4.7)

The variations can be written as,

\[
\frac{\partial}{\partial n^k}\text{tr}\mathcal{B} = -\frac{1}{\sqrt{x}}n^T3f \frac{\partial(Dn)}{\partial n^k}, \quad \frac{\partial}{\partial n^k}\text{tr}\mathcal{B}^2 = -2n^T3fD \frac{\partial(Dn)}{\partial n^k}, \quad \frac{\partial}{\partial n^k}\text{tr}\mathcal{B}^3 = -3\sqrt{x}n^T3fD^2 \frac{\partial(Dn)}{\partial n^k}.
\] (4.8)

Combining all these results gives

\[
\frac{\partial}{\partial n^k}Ne_1(\sqrt{x}^{-1}f) = -\frac{1}{\sqrt{x}}n^T3f \frac{\partial}{\partial n^k}(n + NDn),
\]

\[
\frac{\partial}{\partial n^k}Ne_2(\sqrt{x}^{-1}f) = n^T3f[D - \frac{1}{2} \text{tr}D] \frac{\partial}{\partial n^k}(n + NDn),
\] (4.9)

\[
\frac{\partial}{\partial n^k}Ne_3(\sqrt{x}^{-1}f) = -\sqrt{x}n^T3f[D^2 - D\text{tr}D + \frac{1}{2}[(\text{tr}D)^2 - \text{tr}(D^2)]] \frac{\partial}{\partial n^k}(n + NDn),
\]

where (4.6) was used to simplify the last expression. Note that the right hand sides are proportional to \(J_{ik} = \partial N^i/\partial n^k\) \(\text{[1.3]}\) which was a requirement for the \(n^i\) equations of motion to be independent of \(N\). So, finally, varying the general action \(\text{[4.1]}\) with respect to \(n^i\) gives the \(N\)-independent equations of motion,

\[
R_i - 2m^2\sqrt{\gamma}\frac{n^i3f_{ij}}{\sqrt{x}}\left[\beta_1 \delta_i^j + \beta_2 \sqrt{x}(\delta_i^jD_m^m - D_i^j) \right. \\
+ \beta_3 \sqrt{x}^2 \left\{ \frac{1}{2} \delta_i^j(D_m^m D_n^m - D_n^m D_m^m) + D_i^m D_m^m - D_i^m D_m^m \right\} \right] = 0. \] (4.10)

In principle, these equations can be solved to determine the \(n^i\) in terms of \(R_i\) at least perturbatively, although unlike the minimal massive model, an explicit non-linear solution may be difficult to obtain.

The \(N\) equation of motion is,

\[
R^0 + R_i D_j^i n^j + 2m^2\sqrt{\gamma}\left[\beta_0 + \beta_1 \sqrt{x} D_i^j + \frac{1}{2} \beta_2 \sqrt{x}^2 (D_i^j D_j^j - D_i^j D_j^j) \right. \\
+ \frac{1}{4} \beta_3 \sqrt{x}^3 (D_i^j D_j^k D_k^i - 3D_i^j D_j^k D_k^i + 2D_j^k D_k^i D_i^j - D_i^j D_j^k D_k^i) \left. \right] = 0. \] (4.11)

Eliminating the \(n^i\) in favour of \(R^i\) converts this into the Hamiltonian constraint on \(\gamma_{ij}\) and \(\pi^{ij}\). This and its associated secondary constraint are enough to eliminate the ghost. The
existence of a non-trivial secondary constraint follows from an argument similar to one for the minimal model in the previous section. For the explicit computations see [23].

This demonstrates the existence of a two-parameter family of non-linear theories of massive gravity with general $f_{\mu\nu}$ that do not suffer from the Boulware-Deser ghost instability.

5. Discussions

In this work we have shown that the recently proposed non-linear massive gravity theories do not suffer from the Boulware-Deser ghost instability for an arbitrary non-dynamical reference metric. This is a generalization of the work in [3] which showed the absence of the BD ghost for a flat reference metric. To reiterate, the appearance of the BD ghost is due to the absence of a Hamiltonian constraint. We have shown that the massive actions (2.17) contain such a constraint and an associated secondary constraint and hence are free from this instability.

The theory discussed here need not necessarily be interpreted as a theory of massive gravity, which may or may not be consistent with observations. It also has an alternative interpretation as a ghost free theory of a massive spin-2 field $g_{\mu\nu}$ (say, a meson) in a fixed gravitational background given by $f_{\mu\nu}$.

Much of the recent analysis of the ghost issue in massive gravity has relied on the St"uckelberg formulation and a flat $f_{\mu\nu}$. In the decoupling limit, this formulation provides a powerful tool for studying the ghost content of the theory because of the Goldstone-vector boson equivalence theorem. The recently proposed massive gravity actions, which we have shown to be ghost-free even for a general $f_{\mu\nu}$, were constructed by demanding the absence of the BD ghost in the St"uckelberg formulation and in the decoupling limit alone [3, 4].

However, away from the decoupling limit, the equivalence theorem is no longer valid and, even for a flat $f_{\mu\nu}$, the ghost analysis in the St"uckelberg formulation becomes significantly more involved. A full analysis of the constraints and gauge conditions of the theory must be performed to obtain the physical spectrum in order to identify the ghost. Some recent work which does not take these issues into account has suggested that, in the St"uckelberg formulation (or relatedly, using a helicity decomposition), the BD ghost inevitably reappears away from the decoupling limit, at higher orders in perturbation theory [36, 38]. These results are in contradiction with the conclusions of [3] and its generalization in the present work. However, a thorough analysis of the ghost issue in the perturbative St"uckelberg framework was performed in [8, 9]. This showed that when the constraints are taken into account, the BD ghost is indeed absent and the apparent discrepancy between the perturbative St"uckelberg approach and the non-linear analysis of [3] disappears.

By generalizing the ghost analysis to general $f_{\mu\nu}$, the results of this paper open up the possibility of finding new and interesting classical solutions to massive gravity theories. Moreover, it is known at the linear level that massive gravity in FRW-type backgrounds may contain instabilities that are distinct from the Boulware-Deser ghost. As argued in [19, 20, 21], these problems might be avoided in the full non-linear theory in a dynamical
process. The actions studied here provide a non-linear setup in which this issue might be investigated. Finally, the results of this paper provide a first step towards promoting $f_{\mu\nu}$ to a dynamical variable and thus creating a consistent bimetric theory of gravity (see, [23]).

An interesting possibility is the realization of ghost free massive gravity within string theory setups. While this is not obvious at the level of fundamental string, there is evidence that linear massive gravity in AdS background arises within the AdS/CFT framework [39, 40]. It is interesting to check if AdS/CFT could also reproduce the correct non-linear generalizations described in this paper.

Acknowledgements: We would like to thank F. Berkhahn, S. Deser, J. Enander, S. Hofmann, J. Kluson, B. Sundborg, and M. von Strauss for discussions and comments on the draft. We are especially grateful to F. Berkhahn for comments and suggestions on the presentation of section 2.4. The majority of this work was completed while R.A.R. was supported by the Swedish Research Council (VR) through the Oskar Klein Centre. R.A.R. is currently supported by NASA under contract NNX10AH14G.

A. Appendix: Further review of the ghost problem

A.1 Absence of ghost in general relativity

In field theory, a ghost refers to a physical mode with negative kinetic energy. In the quantum theory this results in states with negative probability. When the action is not in diagonal form in the fields and particularly in the presence of constraints and gauge symmetries, the physical content of the theory may not be directly discernible. To identify the ghost in such cases, one has to first determine the physical degrees of freedom with canonical kinetic terms. Alternatively, the ghost appears in the 2-point function as a mass pole with negative residue\(^6\). This comes handy when propagators are known.

That metric fluctuations in a modified theory of gravity could easily contain a ghost can be inferred by investigating linearized general relativity. Decomposing the metric fluctuations $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ in terms of its traceless transverse ($h^\perp_{\mu\nu}$), transverse vector ($a^\perp_{\mu}$), longitudinal ($\phi$) and scalar ($s$) parts,

$$h_{\mu\nu} = h^\perp_{\mu\nu} + \partial_\mu a^\perp_\nu + \partial_\nu a^\perp_\mu + \frac{1}{4} \eta_{\mu\nu} s$$

(A.1)

one can easily check that the quadratic Einstein-Hilbert action depends only on the five components of $h^\perp$ and the sixth scalar mode $s$,

$$S_{EH} = \frac{1}{4} M_p^2 \int d^4x \left[ h^\perp_{\mu\nu} \Box h^\perp_{\mu\nu} - \frac{3}{8} s \Box s \right].$$

(A.2)

The other modes drop out due to invariance under $\delta h_{\mu\nu} = -\partial_\rho \xi^\rho$. Obviously $s$ has a negative kinetic term and is potentially a ghost, but in general relativity it does not

\(^6\)For fields $\phi_a$ with propagators $G_{ab}$, interacting with external sources $J^a$, the transition amplitude $\langle 0, \text{out} | 0, \text{in} \rangle_J \sim e^{i \int J^a G_{ab} J^b}$ will be less than 1 if $J^a$ excite $|0, \text{in}\rangle$ into particle states of positive probability. But this could exceed unity in the presence negative probability states, implying a positive exponent, $i(2\pi) \text{Res}(J G J, k^0 = |k^0|) > 0$. [1].
survive as a propagating mode. In more detail, $s$ contributes to the graviton propagator $i\Delta^{(0)}$ as a ghost. Indeed it contributes the second term in the saturated $k$-space propagator,

$$T^{\mu\nu} \Delta^{(0)}_{\mu\nu\rho\sigma} T^{*\rho\sigma} = \frac{1}{k^2} \left[ T^{\perp\mu\nu} T^{*\perp}_{\mu\nu} - \frac{1}{6} T T^* \right]$$

(A.3)

which obviously has a negative residue at the $k^2 = 0$ pole. However, for $k^2 = 0$, and only for this value, the ghost is cancelled by a similar contribution from the first term, coming from the helicity zero component of the spin-2 field $h^\perp$. This is discussed below. The cancellation is peculiar to the massless theory and does not take place otherwise.

**A.2 Avoiding the ghost in Fierz-Pauli massive gravity**

Metric fluctuations around flat spacetime are made massive by adding the Fierz-Pauli mass term to the linearized EH action [13, 14, 41],

$$S_{FP} = S_{EH}[h] - \frac{1}{4} M^2 \int d^4x \left[ h^\mu h^\nu - a (h^\mu)^2 \right] .$$

(A.4)

where, $h^\mu = \eta^{\mu\rho} h_{\rho\nu}$. In this form, the mass term is not gauge invariant and depends on the $a_\mu$ and $\phi$ of (A.1). It modifies the massless propagator (A.3) by shifting the mass poles to $k^2 = -m^2$ and $k^2 = -m^2_s = \frac{1 - 4a}{1 - \alpha} m^2$. Then, on using $k_{\mu}T^{\mu\nu} = 0$ (see for example, [12]),

$$T^{\mu\nu} \Delta^{(m)}_{\mu\nu\rho\sigma} T^{*\rho\sigma} = \left[ \frac{1}{k^2 + m^2} T^{\perp\mu\nu} T^{*\perp}_{\mu\nu} - \frac{1}{6} \frac{1}{k^2 + m^2_s} T T^* \right] .$$

(A.5)

Then, as shown below, there is no way of cancelling the wrong sign contribution from the scalar part against the spin-2 part and all 6 modes (including the ghost) contribute to the propagation. The only way out is to tune $a = 1$ so that $m^2_s = \infty$. This decouples the ghost keeping only the 5 healthy polarizations of the massive graviton.

It was pointed out by Boulware and Deser [1] that this method of avoiding the ghost cannot be easily implemented beyond the linear theory. Then, generically the theory will contain 6 propagating modes indicating the reappearance of the 6th mode that was removed at the linear level by setting $a = 1$. We discuss this below.

**A.3 Unitarity analysis of the saturated propagator**

The treatment here follows [13]. In our conventions, a negative residue of the saturated propagator $T^{\mu\nu} \Delta_{\mu\nu\rho\sigma} T^{*\rho\sigma}$ implies the presence of a ghost. Given a 4-vector $k^\mu$ we can construct a set of 4 linearly independent vectors,

$$k^\mu = (k^0, \vec{k}), \quad \vec{k}^\mu = (-k^0, \vec{k}), \quad \epsilon^\mu_r = (0, \epsilon_r^0) \quad \text{for} \quad (r = 1, 2)$$

(A.6)

such that $\epsilon^r \cdot \epsilon_s = \delta_{rs}$ and $\vec{k} \cdot \epsilon_r = 0$. Note that $(k\vec{k}) \equiv k_{\mu}\vec{k}^\mu = (k^0)^2 + |\vec{k}|^2 > 0$ and $k^2 = \vec{k}^2$.

In this basis, a generic symmetric tensor can be expanded as

$$T_{\mu\nu}(k) = ak^{\mu}k^{\nu} + b\vec{k}^{\mu}\vec{k}^{\nu} + \frac{1}{2} \epsilon^{rs}(\epsilon_r^\mu \epsilon_s^\nu + \epsilon_r^\nu \epsilon_s^\mu) + \frac{1}{2} d(k^{\mu}k^{\nu} + k^{\nu}k^{\mu}) + \frac{1}{2} c^r (k^{\mu} \epsilon_r^\nu + k^{\nu} \epsilon_r^\mu)$$

$$+ \frac{1}{2} f^r (\vec{k}^{\mu} \epsilon_r^\nu + \vec{k}^{\nu} \epsilon_r^\mu)$$

(A.7)
The conservation equation \( k^\mu T_{\mu \nu} = 0 \) implies \( b = ak^4/(k\bar{k})^2 \), \( d = -2ak^2/(k\bar{k}) \) as well as \( f^r = -k^2/(k\bar{k})e^r \). The propagator (A.3) in Einstein-Hilbert theory takes the standard from on using \( T_{\mu \nu} = 1/3 (\eta_{\mu \nu} - k^\mu k^\nu) T \). Then in the above parameterization of \( T_{\mu \nu} \) the residue at the zero mass pole is positive, hence the theory is ghost free,

\[
\left[ T^{\mu \nu} T^*_{\mu \nu} - \frac{1}{3} T T^* \right]_{k^2 = 0} = \frac{1}{2} |c_{11} - c_{22}|^2 + 2 |c_{12}|^2 > 0.
\]

(A.8)

In the massive theory, the propagator has the generic momentum space form (A.5). The \( T T^* \) term is due to spin-2 exchange and gives a positive residue at the mass pole,

\[
\left[ T^{\mu \nu} T^*_{\mu \nu} - \frac{1}{3} T T^* \right]_{k^2 = -m^2} = \frac{4}{3} |a k^2 (1 - \frac{k^2\bar{k}^2}{(k\bar{k})^2}) + c/2|^2 + \frac{1}{2} |c_{11} - c_{22}|^2 + 2 |c_{12}|^2 \\
+ \frac{1}{2} (|e|^2 + |e^2|^2) k^2 (\frac{\bar{k}^2}{kk}) - 1) \right]_{k^2 = -m^2} > 0 \quad (A.9)
\]

The first line is manifestly positive. In the second line, \( k^2 (\frac{\bar{k}^2}{kk}) - 1)_{k^2 = -m^2} = m^2 (\frac{m^2}{m^2 + 2|k|^2} + 1) > 0 \), hence the overall positivity.

The \( T T^* \) term in (A.5) is due to the exchange of the scalar mode \( s \) of mass \( m_s \) which is a ghost for any finite mass since,

\[
-\frac{1}{6} T T^* \bigg|_{k^2 = -m^2} = -\frac{1}{6} a k^2 (1 - \frac{k^2\bar{k}^2}{(k\bar{k})^2}) - c \bigg|_{k^2 = -m^2} < 0 \quad (A.10)
\]

Only for \( k^2 = 0 \) this cancels against a contribution in the \( T^\perp T^\perp \) term resulting in the healthy GR expression above. The only other possibility to get rid of this ghost is to take \( m_s \to \infty \), for fixed \( m \). This decouples the scalar ghost and retains only the healthy spin-2 contribution (A.9). This is the Fierz Pauli massive gravity for \( a = 1 \).

### A.4 The Boulware-Deser analysis

As a specific example, [1] considers a FP-type mass (2.2) where \( h_{\mu \nu} = g_{\mu \nu} - \eta_{\mu \nu} \) is not treated as a small perturbation. Then, in ADM variables,

\[
h^\mu_{\nu'} h^\nu_{\mu} - (h_{\mu})^2 = (h_{ij})^2 - h^2 + 2h(1 - N^2 + N_i N_j \gamma^{ij}) - 2N_i N_j \quad (A.11)
\]

where \( h_{ij} = \gamma_{ij} - \delta_{ij} \) and \( h = h_{ii} \). This is obviously non-linear in both \( N \) and \( N^i \). The equations of motion for these can be solved to give,

\[
N = -\frac{R^0}{m^2 h}, \quad N_i = \frac{1}{m^2} \left[ (h \gamma^{-1} - \Gamma^{-1})^{-1} \right]_{ij} R^j \quad (A.12)
\]

There is no constraint on the remaining variables and the theory contains 6 propagating modes, including the ghost. Substituting back in the action one can compute the Hamiltonian density \( \mathcal{H} = \pi_{ij} \partial_t \gamma_{ij} - \mathcal{L} \) (ignoring the boundary contribution that is the same as in GR),

\[
\mathcal{H} = \frac{m^2}{4} \left[ (h_{ij})^2 - h^2 \right] + \frac{1}{2} h m^2 + \frac{1}{2} \frac{(R^0)^2}{m^2 h} - \frac{1}{2} \frac{1}{m^2} R^i \left[ (h \gamma^{-1} - \Gamma^{-1})^{-1} \right]_{ij} R^j \quad (A.13)
\]
noted that the corresponding Hamiltonian is not always positive and that it diverges in the limit $m \to 0$. Hence the conclusion that gravity cannot have a finite range.

It is instructive to see how the ghost disappears in the linear FP limit. In this case, expanding around a flat background, $\delta N = N - 1$, $\delta N_i = N_i$ and $h_{ij} = \gamma_{ij} - \delta_{ij}$ are small perturbations and to quadratic order the FP mass term is linear in $\delta N$,

\[ h^\mu_{\nu} h^\nu_{\mu} - (h^\mu_{\mu})^2 = (h_{ij})^2 - (h)^2 - 4h \delta N - 2\delta N_i \delta N_i \quad (A.14) \]

The $N_i$ equations $R^i(h, \delta \pi) = m^2 \delta N^i$ determine the $\delta N_i$. But the $\delta N$ equation $P^0(h, \delta \pi) = m^2 h_{ij}$ is independent of lapse and shift and becomes the modified Hamiltonian constraint. The requirement that this constraint is maintained under time evolution, results in a non-trivial secondary constraint on the $(h_{ij}, \delta \pi^j)$. These 2 constraints reduce the number of independent $(h, \delta \pi)$ components from 12 to 10, implying the existence of only 5 propagating modes and no ghost mode, consistent with the manifestly covariant analysis of the propagator.

References

[1] D. G. Boulware and S. Deser, Phys. Rev. D 6, 3368 (1972).
[2] D. G. Boulware, S. Deser, Phys. Lett. B40 (1972) 227-229.
[3] C. de Rham and G. Gabadadze, Phys. Rev. D 82, 044020 (2010) [arXiv:1007.0443 [hep-th]].
[4] C. de Rham, G. Gabadadze and A. J. Tolley, arXiv:1011.1232 [hep-th].
[5] S. F. Hassan and R. A. Rosen, arXiv:1106.3344 [hep-th].
[6] S. F. Hassan and R. A. Rosen, arXiv:1103.6055 [hep-th].
[7] C. de Rham, G. Gabadadze and A. J. Tolley, arXiv:1107.0710 [hep-th].
[8] C. de Rham, G. Gabadadze and A. Tolley, arXiv:1107.3820 [hep-th].
[9] C. de Rham, G. Gabadadze, A. J. Tolley, arXiv:1108.4521 [hep-th].
[10] G. D’Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava and A. J. Tolley, arXiv:1108.5231 [hep-th].
[11] C. J. Isham, A. Salam and J. A. Strathdee, Phys. Rev. D 3, 867 (1971).
[12] A. Salam and J. A. Strathdee, Phys. Rev. D 16, 2668 (1977).
[13] M. Fierz, Helv. Phys. Acta 12 (1939) 3.
[14] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A 173 (1939) 211.
[15] A. Higuchi, Nucl. Phys. B 282 (1987) 397.
[16] S. Deser and A. Waldron, Phys. Lett. B 508 (2001) 347 [arXiv:hep-th/0103255].
[17] M. Porrati, Phys. Lett. B 498, 92 (2001) [arXiv:hep-th/0011152].
[18] I. I. Kogan, S. Mouslopoulos and A. Papazoglou, Phys. Lett. B 503, 173 (2001) [arXiv:hep-th/0011138].
[19] L. Grisa and L. Sorbo, Phys. Lett. B 686 (2010) 273 [arXiv:0905.3391 [hep-th]].
[20] F. Berkhahn, D. D. Dietrich and S. Hofmann, JCAP 1011 (2010) 018 [arXiv:1008.0644 [hep-th]].

[21] F. Berkhahn, D. D. Dietrich and S. Hofmann, arXiv:1104.2534 [hep-th].

[22] R. L. Arnowitt, S. Deser and C. W. Misner, arXiv:gr-qc/0405109.

[23] S. F. Hassan and R. A. Rosen, arXiv:1111.2070 [hep-th].

[24] E. Babichev, C. Deffayet and R. Ziour, Phys. Rev. D 82 (2010) 104008 [arXiv:1007.4506 [gr-qc]].

[25] T. M. Nieuwenhuizen, arXiv:1103.5912 [gr-qc].

[26] K. Koyama, G. Niz and G. Tasinato, arXiv:1103.4708 [hep-th].

[27] K. Koyama, G. Niz and G. Tasinato, arXiv:1104.2143 [hep-th].

[28] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, Annals Phys. 305 (2003) 96 [arXiv:hep-th/0210184].

[29] S. F. Hassan and R. A. Rosen, arXiv:1109.3515 [hep-th].

[30] K. Hinterbichler, arXiv:1105.3735 [hep-th].

[31] S. F. Hassan and A. Schmidt-May, In preparation.

[32] P. Creminelli, A. Nicolis, M. Papucci and E. Trincherini, JHEP 0509, 003 (2005) [arXiv:hep-th/0505147].

[33] J. Kluson, arXiv:1109.3052 [hep-th].

[34] T. Regge and C. Teitelboim, Annals Phys. 88 (1974) 286.

[35] J. D. Brown and J. W. York, Phys. Rev. D 47 (1993) 1407 [arXiv:gr-qc/9209012].

[36] L. Alberte, A. H. Chamseddine and V. Mukhanov, JHEP 1104, 004 (2011) [arXiv:1011.0183 [hep-th]].

[37] A. H. Chamseddine and V. Mukhanov, arXiv:1106.5868 [hep-th].

[38] S. Folkerts, A. Pritzel and N. Wintergerst, arXiv:1107.3157 [hep-th].

[39] E. Kiritsis, JHEP 0611 (2006) 049 [hep-th/0608088].

[40] O. Aharony, A. B. Clark and A. Karch, Phys. Rev. D 74 (2006) 086006 [hep-th/0608089].

[41] H. van Dam and M. J. G. Veltman, Nucl. Phys. B 22 (1970) 397.

[42] S. F. Hassan, S. Hofmann and M. von Strauss, JCAP 1101 (2011) 020 [arXiv:1007.1263 [hep-th]].

[43] F. C. P. Nunes and G. O. Pires, Phys. Lett. B 301 (1993) 339.