Lax Connections in $T\bar{T}$-deformed Integrable Field Theories

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Abstract

In this work, we try to construct the Lax connections of $TT$-deformed integrable field theories in two different ways. With reasonable assumptions, we make ansatz and find the Lax pairs in the $T\bar{T}$-deformed affine Toda theories and the principal chiral model by solving the Lax equations directly. This way is straightforward but maybe hard to apply for general models. We then make use of the dynamical coordinate transformation to read the Lax connection in the deformed theory from the undeformed one. We find that once the inverse of the transformation is available, the Lax connection can be read easily. We show the construction explicitly for a few classes of scalar models, and find consistency with the ones in the first way.

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1 Introduction

The $T\bar{T}$-deformation of two-dimensional field theories \cite{1,2} has recently attracted much attention. It is a kind of solvable irrelevant deformation, and induces a flow in the space of field theories that satisfies the differential equation

$$\partial_t \mathcal{L}^{(t)} = \text{det} \left( T^{(t)}_{\mu\nu} \right),$$

(1)

where $T^{(t)}_{\mu\nu}$ is the stress-energy tensor and $T\bar{T} = -\pi^2 \text{det} T^{(t)}_{\mu\nu}$. A remarkable property of this flow is that it preserves integrability if the undeformed theory is integrable. In the original paper \cite{1}, the preservation of integrability under a $T\bar{T}$-deformation or its generations has been supported by showing that the infinite conserved charges of the undeformed theory are still conserved under the flow. Another piece of evidence for integrability is from the fact that the S-matrix in the deformed theory is only modified by adding a CDD-like factor \cite{1,3}. A word of caution is that the solvability of the
deformed theories does not rely on integrability crucially and it can be understood from various aspects [5, 6, 7, 8, 9, 10]. Nevertheless, integrability may provide additional convenient handles on the theory.

As it is well-known, integrability can be described in other frameworks, such as the Lax pair formulation and the Bäcklund transformation formulation. In particular, the existence of the Lax connection is usually taken as the hallmark of classical integrability, and it also paves the way to quantization [4]. However the Lax connection is notoriously difficult to find. Most of the time it needs the art of guess and trial and error. In this work, we will derive the Lax connections of several $T\bar{T}$-deformed integrable theories with two different methods. The first method is kind of straightforward. We start from some reasonable ansatz, and find the connection by imposing equations of motion and solving the Lax equation. This method is suggestive but could be limited to specific models. The second method is more systematic, and it relies on the fact that the $T\bar{T}$-deformation could be realized as a dynamical coordinate transformation [10]. It is reminiscent of the method in deriving the Lax connections of $\gamma$-deformed superstring theory [12]. This similarity is also expected, considering the fact that the holographic $T\bar{T}$-deformation [13], or known as the single trace $T\bar{T}$-deformation, as well as the $\gamma$ deformations, can be related to a TsT deformation [14, 15]. The difference is that the single trace $T\bar{T}$-deformation is a field redefinition while the $T\bar{T}$-deformation of field theories is a change of coordinates.

The paper is organized as follows: In Sect. 2 and Sect. 3, we derive the Lax connection of (affine) Toda field theories and principle chiral model directly with reasonable ansatz; in Sect. 4 we first review the dynamical coordinate transformation approach of $T\bar{T}$-deformation, and then reproduce the results obtained in section 2 and 3, and in the end we derive the Lax connection for a $T\bar{T}$-deformed non-relativistic non-linear Schrödinger theory; in Sect. 5 we end up with conclusions.

2 $T\bar{T}$-deformed (affine) Toda field theories

In this section we consider the (affine) Toda field theories and their $T\bar{T}$-deformations. For the (affine) Toda field theories, the integrability can be studied from the point of view of the Lax connections. As examples of $N$ scalar theories, the $T\bar{T}$-deformed Lagrangians of these models have been derived in [16]. Here we are going to derive the deformed Lax connections with some reasonable ansatz.
2.1 Undeformed theories

The Lagrangian of a rank-$r$ affine Toda field theory is given by

\[ L^{(0)} \equiv \partial \vec{\phi} \cdot \bar{\partial} \vec{\phi} + V \]  

(2)

with

\[ V = -\frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta \vec{\alpha}_i \cdot \vec{\phi}} \]  

(3)

where \( \vec{\phi} \) is a vector field of \( r \) components, the set of integer number \( \{n_i\} \) characterizes the theory, \( \{\vec{\alpha}_i, i = 1, \ldots, r\} \) are positive simple roots of the underlying Lie algebra and \( \vec{\alpha}_0 = -\sum_i n_i \vec{\alpha}_i \). If in the summation the term \( i = 0 \) is omitted then the theory reduces to the conformal Tada field theory. The generators of the Cartan subalgebra \( \vec{H} = \{H_a, a = 1, 2, \ldots, r\} \), and the simple roots \( \{E_{\vec{\alpha}_i}, E_{-\vec{\alpha}_i}, i = 0, 1, \ldots, r\} \) satisfy the standard commutation relations

\[
[H_a, H_b] = 0, \quad \left[ \vec{H}, E_{\pm \vec{\alpha}_i} \right] = \pm \vec{\alpha}_i E_{\pm \vec{\alpha}_i}, \quad \left[ E_{\vec{\alpha}_i}, E_{-\vec{\alpha}_j} \right] = \delta_{ij} \frac{2 \vec{\alpha}_j \cdot \vec{H}}{|\vec{\alpha}_j|^2},
\]

(4)

\[
\left[ E_{\vec{\alpha}_i}, E_{\vec{\alpha}_j} \right] = \begin{cases} N_{\vec{\alpha}_i + \vec{\alpha}_j} E_{\vec{\alpha}_i + \vec{\alpha}_j}, & \text{if } \vec{\alpha}_i + \vec{\alpha}_j \text{ is a root}, \\ 0, & \text{if } \vec{\alpha}_i + \vec{\alpha}_j \text{ is not a root}. \end{cases}
\]

The equations of motion are simply given by

\[ 2\partial \bar{\partial} \vec{\phi} - \frac{\delta V}{\delta \vec{\phi}} = 0, \]  

(5)

and the Lax connections are

\[
L = -\frac{\beta}{2} \partial \vec{\phi} \cdot \vec{H} - \lambda \sum_{i=0}^{r} m_i e^{\beta \vec{\alpha}_i \cdot \vec{\phi}}/2 E_{\vec{\alpha}_i},
\]

\[
\bar{L} = \frac{\bar{\partial} \vec{\phi}}{2} \cdot \vec{H} + \frac{1}{\lambda} \sum_{i=0}^{r} m_i e^{\beta \vec{\alpha}_i \cdot \vec{\phi}}/2 E_{-\vec{\alpha}_i},
\]

(6)

where \( \lambda \in \mathbb{C} \) is the spectral parameter and \( m_i^2 = \frac{1}{4} |\vec{\alpha}_i|^2 m^2 n_i \). For a classical integrable system, the equations of motion are equivalent to the Lax equation

\[ \partial L - \bar{\partial} \bar{L} - [L, \bar{L}] = 0. \]  

(7)

1For a review on Toda field theory, see [18].
For later convenience we introduce two new combinations

\[ E_+ = \sum_{i=0}^{r} m_i e^{\beta \tilde{\alpha}_i \tilde{\phi}/2} E_{\tilde{\alpha}_i}, \quad E_- = \sum_{i=0}^{r} m_i e^{\beta \tilde{\alpha}_i \tilde{\phi}/2} E_{-\tilde{\alpha}_i} \]  

(8)

satisfying

\[ [E_+, E_-] = -\frac{\beta}{2} \bar{\nabla} V \cdot \bar{H}, \quad \left[ \bar{H}, E_\pm \right] = \pm \frac{2}{\beta} \bar{\nabla} E_\pm \]  

(9)

where \( \bar{\nabla} f \) denotes \( \frac{\delta f}{\delta \tilde{\phi}} \). Then the Lax connections (6) can be rewritten as

\[ L = -\frac{\beta}{2} \partial \tilde{\phi} \cdot \bar{H} - \lambda E_+, \]  

\[ \bar{L} = \frac{\beta}{2} \bar{\partial} \tilde{\phi} \cdot \bar{H} + \frac{1}{\lambda} E_. \]  

(10)

2.2 \( \bar{T}\bar{T}\)-deformed theories

The \( \bar{T}\bar{T}\)-deformed Lagrangian of a \( N \)-scalar theory is [16, 17]

\[ \mathcal{L}^{(t)} = \frac{V}{1 - tV} + \frac{1}{2t(1 - tV)}(\Omega_T - 1) \]  

(11)

where

\[ \Omega_T = \sqrt{1 + Y + Z}, \quad Y = 4t(1 - tV) \left( \partial \tilde{\phi} \cdot \bar{\partial} \tilde{\phi} \right), \]  

\[ Z = -4t^2(1 - tV)^2 \left[ \left( \partial \tilde{\phi} \cdot \partial \tilde{\phi} \right) \left( \partial \tilde{\phi} \cdot \bar{\partial} \tilde{\phi} \right) - \left( \partial \tilde{\phi} \cdot \bar{\partial} \tilde{\phi} \right)^2 \right]. \]  

(12)

The equations of motion are given by

\[ \bar{A}_e \equiv \partial_\mu \frac{\delta \mathcal{L}^{(t)}}{\delta \partial_\mu \tilde{\phi}} - \frac{\delta \mathcal{L}^{(t)}}{\delta \tilde{\phi}} = 0. \]  

(13)

Substituting (11) and (12) into (13), one can get

\[ \bar{A}_e = \partial_\mu \left\{ \frac{1}{\Omega_T} \left[ \partial \tilde{\phi} - 2t(1 - tV) \left( \partial \tilde{\phi} \left( \partial \tilde{\phi} \cdot \bar{\partial} \tilde{\phi} \right) - \bar{\partial} \tilde{\phi} \left( \partial \tilde{\phi} \cdot \bar{\partial} \tilde{\phi} \right) \right) \right]\right\} \]  

\[ + \bar{\partial} \left\{ \frac{1}{\Omega_T} \left[ \bar{\partial} \tilde{\phi} - 2t(1 - tV) \left( \bar{\partial} \tilde{\phi} \left( \partial \tilde{\phi} \cdot \bar{\partial} \tilde{\phi} \right) - \partial \tilde{\phi} \left( \partial \tilde{\phi} \cdot \bar{\partial} \tilde{\phi} \right) \right) \right]\right\} \]  

\[ - \frac{\bar{\nabla} V}{4\Omega_T(1 - tV)^2} \left[ (\Omega_T + 1)^2 - Z \right]. \]  

(14)
Given these equations of motions we propose a simple ansatz for the Lax connection:

\[
L = -\frac{\beta}{2} \vec{a}_1 \cdot \vec{H} - \lambda b_1 E_+ + \frac{1}{\lambda} c_1 E_-,
\]

\[
\bar{L} = \frac{\beta}{2} \vec{a}_2 \cdot \vec{H} - \lambda b_2 E_+ + \frac{1}{\lambda} c_2 E_-,
\]

where \(\vec{a}_1, b_1, c_1, \vec{a}_2, b_2, c_2\) are the functions of \(\vec{\phi}\) and their derivatives, and will be determined by imposing the Lax equation and the equations of motion. Notice that in our ansatz \((15)\) the Lax connection depends uniformly on the simple roots \(E_{\vec{\alpha}}\). Plugging \((15)\) into \((7)\) directly gives rise to a set of linear differential equations \(\tilde{A}_H, A_+, A_-\), corresponding to the components \(\vec{H}, E_+\) and \(E_-\), respectively. In principle \(\tilde{A}_H, A'_+, A'_-\), should vanish separately. However because the terms like \(\vec{\nabla} E_{\pm}\) are not uniformly dependent on the simple roots \(E_{\vec{\alpha}}\), we would require that the coefficients before the terms like \(\vec{\nabla} E_{\pm}\) vanish separately. Consequently we obtain five sets of linear equations

\[
\begin{align*}
\tilde{A}_H &\equiv \partial \vec{a}_2 + \bar{\partial} \vec{a}_1 - \vec{\nabla} V (b_1 c_2 - b_2 c_1) = 0 \\
A_+ &\equiv -\partial b_2 + \bar{\partial} b_1 = 0 \\
A_- &\equiv \partial c_2 - \bar{\partial} c_1 = 0 \\
\tilde{A}_{p+} &\equiv -\partial \vec{\phi} b_2 + \bar{\partial} \vec{\phi} b_1 - (a_1 b_2 + a_2 b_1) = 0 \\
\tilde{A}_{p-} &\equiv \partial \vec{\phi} c_2 - \bar{\partial} \vec{\phi} c_1 - (a_1 c_2 + a_2 c_1) = 0
\end{align*}
\]

To solve these equations we make another assumption that they can be written as linear combinations of the equations of motion, i.e.

\[
\begin{align*}
\tilde{A}_H = f_H \vec{A}_e, & \quad A_+ = f_+ \cdot \vec{A}_e, \quad A_- = f_- \cdot \vec{A}_e, \quad \tilde{A}_{p+} = f_{p+} \vec{A}_e, \quad \tilde{A}_{p-} = f_{p-} \vec{A}_e.
\end{align*}
\]

To ensure the equivalence between the Lax equation \((16)\) and the equations of motion \((13)\), there should be no common zero of \(f_H, f_+, f_-, f_{p+}, f_{p-}\). Indeed we are making quite strong assumptions here, but we will show that the consistent solution does exist.

For the undeformed theory, by \((10)\), one can find that

\[
\begin{align*}
f_H = 1, & \quad f_{p+} = 0, \quad f_{p-} = 0,
\end{align*}
\]

and there is no \(f_+, f_-\) terms. We assume that \((18)\) is still true for the deformed theory and observe that if we take

\[
\tilde{f}_+ = -t \bar{\partial} \vec{\phi},
\]

\[
\tilde{f}_- = t \bar{\partial} \vec{\phi},
\]

\[
\tilde{f}_{p+} = -t \bar{\partial} \vec{\phi},
\]

\[
\tilde{f}_{p-} = t \bar{\partial} \vec{\phi},
\]
then

\[ \vec{f} \cdot \vec{A}_e = -\partial \left[ \frac{t}{\Omega_T} \left( \vec{\partial} \vec{\partial} \right) \right] + \vec{\partial} \left[ \frac{(\Omega_T + 1)^2 - Z}{4\Omega_T(1 - tV)} \right] \]  

(20)
suggesting that we can identify

\[ b_1 = \frac{(\Omega_T + 1)^2 - Z}{4\Omega_T(1 - tV)}, \quad b_2 = \frac{t}{\Omega_T} \left( \vec{\partial} \vec{\partial} \right) \]  

(21)
up to some constants which can be fixed to be zero after considering other equations. Similarly, by taking \( \vec{f} = -t\vec{\partial} \vec{\phi} \), we can read off \( c_1 \) and \( c_2 \)

\[ c_1 = \frac{t}{\Omega_T} \left( \vec{\partial} \vec{\partial} \right), \quad c_2 = \frac{(\Omega_T + 1)^2 - Z}{4\Omega_T(1 - tV)}. \]  

(22)
Finally from \( \vec{f}_H \vec{A}_e \), we fix all the remaining functions in our ansatz

\[ \vec{a}_1 = \frac{1}{\Omega_T} \left[ \partial \vec{\phi} - 2t(1 - tV) \left( \vec{\partial} \vec{\partial} \left( \vec{\partial} \vec{\phi} \right) - \vec{\partial} \vec{\phi} \left( \vec{\partial} \vec{\phi} \right) \right) \right], \]

\[ \vec{a}_2 = \frac{1}{\Omega_T} \left[ \partial \vec{\phi} - 2t(1 - tV) \left( \vec{\partial} \vec{\partial} \left( \vec{\partial} \vec{\phi} \right) - \vec{\partial} \vec{\phi} \left( \vec{\partial} \vec{\phi} \right) \right) \right]. \]  

(23)
Plugging (21) (22) and (23) into (16), one can check that (16) is indeed equivalent to the equations of motion (13).

To summarize, the Lax connections of the \( T \bar{T} \)-deformed (affine) Toda field theories are of the forms (15) with the functions being given by (21) (22) and (23). We want to stress that after we assume (18) and (19) the solutions can be read off directly without solving any other equations.

**2.3 Examples**

To compare with the existing results in the literature, let us consider some specific examples. The first one is the Liouville field theory which corresponds to Toda field theory of \( sl_2 \) Lie algebra with the parameters being

\[ \beta = \frac{1}{2}, \quad m_0 = 0, \quad m_1 = -\frac{\sqrt{\mu}}{2}. \]  

(24)
The undeformed Lagrangian is

\[ \mathcal{L}^{(0)} = \partial \phi \bar{\partial} \phi - \mu e^\phi. \]  

(25)
The $T\bar{T}$-deformed Liouville field theory was studied in [19] where infinite conserved currents were constructed from some ansatz without using the Lax connection. From the discussion in the last subsection, we can present the deformed Lax connections explicitly

\[
L = -\frac{1}{4\Omega_T}H + \sqrt{\mu\lambda} Be^{\phi/2} E_{\alpha_1} - \frac{\sqrt{\mu}}{\lambda} (\partial\phi)^2 Ce^{\phi/2} E_{-\alpha_1},
\]

\[
\bar{L} = \frac{1}{4\Omega_T}H - \sqrt{\mu/\lambda} Be^{\phi/2} E_{-\alpha_1} + \sqrt{\mu\lambda}(\bar{\partial}\phi)^2 Ce^{\phi/2} E_{\alpha_1},
\]

where

\[
B = \frac{8\Omega_T}{8\Omega_T(1-tV)}, \quad C = \frac{t}{2\Omega_T}
\]

\[
\Omega_T = \sqrt{1 + 4t(1-tV)} (\partial\phi\bar{\partial}\phi).
\]

Let the generators of $sl_2$ Lie algebra be

\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_{\alpha_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

then if we take the undeforming limit, $t \to 0$, the Lax connections become

\[
L = \begin{pmatrix} -\frac{1}{4}\partial\phi & \frac{\sqrt{\mu}}{2}Ce^{\phi/2} \\ 0 & \frac{1}{4}\partial\phi \end{pmatrix}, \quad \bar{L} = \begin{pmatrix} \frac{1}{4}\bar{\partial}\phi & 0 \\ \frac{\sqrt{\mu}}{2}Ce^{\phi/2} & -\frac{1}{4}\bar{\partial}\phi \end{pmatrix},
\]

which are the Lax connections of the Liouville field theory.

Our next example is the sine-Gordon model which corresponds to the affine Toda field of affine $sl_2$ algebra with parameters

\[
\beta = \frac{i}{2}, \quad m_0 = m_1 = -\frac{i}{2}, \quad n_0 = n_1 = 1, \quad \alpha_0 = -2, \quad \alpha_1 = 2.
\]

The undeformed Lagrangian is given by

\[
\mathcal{L}^{(0)} = \partial\phi\bar{\partial}\phi - 2\cos\phi
\]

By setting

\[
E_{\alpha_0} = E_{-\alpha_1}, \quad E_{-\alpha_0} = E_{\alpha_1},
\]

we find that the deformed Lax connections are

\[
L = -\frac{i}{4\Omega_T}H + \left( i\lambda Be^{i\phi/2} + \frac{1}{i\lambda} (\partial\phi)^2 Ce^{-i\phi/2} \right) E_{\alpha_1} + \left( i\lambda Be^{-i\phi/2} + \frac{1}{i\lambda} (\partial\phi)^2 Ce^{i\phi/2} \right) E_{-\alpha_1},
\]

\[
\bar{L} = \frac{i}{4\Omega_T}H + \left( \frac{1}{i\lambda} Be^{-i\phi/2} + i\lambda(\bar{\partial}\phi)^2 Ce^{i\phi/2} \right) E_{\alpha_1} + \left( \frac{1}{i\lambda} Be^{i\phi/2} + i\lambda(\bar{\partial}\phi)^2 Ce^{-i\phi/2} \right) E_{-\alpha_1}
\]
which are the same as the ones found in [17]. In the undeforming limit, \( t \to 0 \), the Lax connections reduce to the Lax connections of sine-Gordon model

\[
L = \left( \begin{array}{cc}
-\frac{i}{4} \partial \phi & \frac{\lambda}{2} e^{i\phi/2} \\
\frac{\lambda}{2} e^{-i\phi/2} & \frac{1}{4} \partial \phi
\end{array} \right), \quad \bar{L} = \left( \begin{array}{cc}
\frac{i}{4} \partial \phi & \frac{1}{2\lambda} e^{-i\phi/2} \\
\frac{1}{2\lambda} e^{i\phi/2} & -\frac{i}{4} \partial \phi
\end{array} \right).
\]

(34)

3 Principal chiral model

In this section we consider the principal chiral model (PCM) which is an integrable sigma model. The \( TT \)-deformed Lagrangian of PCM has been obtained in [16, 17]. We will use a similar strategy used in the last section to derive the deformed Lax connection.

3.1 Undeformed theory

A principal chiral model (PCM) is a field theory whose field takes values in some Lie group manifold. Its action is

\[
S_0 = \int dx^2 g^{\mu\nu} \text{Tr} \left( g^{-1} \partial_\mu g g^{-1} \partial_\nu g \right), \quad g \in G.
\]

(35)

Usually the Lie group is chosen to be semisimple but we will leave it to be arbitrary since our interest is on integrability. The model has symmetry group \( G_L \times G_R \). The equation of motion of PCM is just

\[
\partial_\mu \left( g^{-1} \partial_\mu g \right) = 0,
\]

(36)

which is equivalent to the current conservation equation

\[
\partial_\mu j^\mu = 0, \quad j^\mu = g^{-1} \partial_\mu g.
\]

(37)

Here \( j^\mu \) is the conserved current corresponding to the \( G_R \) symmetry. In addition to (37), the conserved current also satisfies the flatness condition:

\[
\partial_0 j_1 - \partial_1 j_0 = -[j_0, j_1].
\]

(38)

The equations (37) and (38) are equivalent to the Lax equation with the Lax connections

\[
L_0 = -\frac{1}{\lambda^2 + 1} (\lambda j_1 + j_0),
\]

\[
L_1 = -\frac{1}{\lambda^2 + 1} (-\lambda j_0 + j_1),
\]

(39)

where \( \lambda \in \mathbb{C} \) is the spectral parameter.

\footnote{In the section, we consider the theory in the flat space and take the Euclidean signature, that is, \( g^{\mu\nu} = \text{diag}(1, 1) \). The coordinate is \((x^0, x^1)\) and the Levi-Civita symbol is \( \epsilon^{0,1} = -\epsilon^{1,0} = 1 \).}
3.2 $T\bar{T}$-deformed theory

The $T\bar{T}$-deformed Lagrangian of PCM is given by \[ [16] \]

$$L_{PCM}^{(t)} = \frac{1}{2t} (-1 + \Omega_P), \quad \text{(40)}$$

where

$$\Omega_P = \sqrt{1 + 4t \text{Tr} (g^{-1} \partial_\mu gg^{-1} \partial_\mu g) + 8t^2 \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \text{Tr} (g^{-1} \partial_\mu gg^{-1} \partial_\mu g) \text{Tr} (g^{-1} \partial_\gamma gg^{-1} \partial_\gamma g)} \quad \text{(41)}$$

The equation of motion $A_{ePCM} = 0$, can also be cast into a form of conservation law:

$$A_{ePCM} \equiv \partial_\mu \frac{\delta L_{PCM}^{(t)}}{\delta \phi} - \frac{\delta L_{PCM}^{(t)}}{\delta \phi} = 2(\partial_\mu J^\mu) g^{-1} \quad \text{(42)}$$

Here the conserved current $J^\mu$ is defined as

$$J^\mu = \frac{1}{\Omega_P} (j^\mu + 4t \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \text{Tr} (j_\nu j_\rho)) \quad \text{(43)}$$

which satisfies the following useful identities

$$[J_0, J_1] = [j_0, j_1],$$

$$[J_0, j_0] = \frac{1}{\Omega_P} 4t \text{Tr} (j_0 j_1) [j_0, j_1],$$

$$[J_0, j_1] = \frac{1}{\Omega_P} (1 + 4t \text{Tr} (j_1 j_1)) [j_0, j_1],$$

$$[J_1, j_0] = -\frac{1}{\Omega_P} (1 + 4t \text{Tr} (j_0 j_0)) [j_0, j_1],$$

$$[J_1, j_1] = -\frac{1}{\Omega_P} 4t \text{Tr} (j_0 j_1) [j_0, j_1]. \quad \text{(44)}$$

Notice that the current $j_\mu$ still satisfies the flatness condition \([38]\) so the reasonable ansatz for the Lax connections could be that they are the linear combination of the new conserved current \([43]\) and $j_\mu$:

$$L_0 = a_0 J_1 + b_0 j_0 + c_0 j_1,$$

$$L_1 = a_1 J_0 + b_1 j_0 + c_1 j_1, \quad \text{(45)}$$

where $a_0, a_1, b_0, b_1, c_0, c_1$ are some constants to be determined. Again we assume that the Lax equation is linearly dependent on the equation of motion:
\[ A_{LPCM} \equiv \partial_0 L_1 - \partial_1 L_0 - [L_0, L_1], \quad A_{LPCM} = A_{ePCM} \cdot f_{PCM} \quad (46) \]

For the undeformed theory, using (39) and the definition of \( A_{LPCM} \) and \( A_{ePCM} \), we can get

\[ f_{PCM} = \frac{1}{2} \frac{\lambda}{\lambda^2 + 1} g \quad (47) \]

Assuming that (47) is still true in the deformed case, we end up with

\[ A_{LPCM} = \frac{\lambda}{\lambda^2 + 1} \partial_\mu J^\mu. \quad (48) \]

Plugging (45) into (46) and matching it with (48) we can read off

\[ a_0 = -a_1 = -\frac{\lambda}{\lambda^2 + 1}, \quad b_0 = c_1 = -\frac{1}{\lambda^2 + 1}, \quad b_1 = c_0 = 0, \quad (49) \]

where we have used the identity, \( \partial_0 j_1 - \partial_1 j_0 = -[j_0, j_1] \).

In summary, the Lax connection of the \( TT^\star \)-deformed PCM is given by

\[ L_0 = -\frac{1}{\lambda^2 + 1}(\lambda J_1 + j_0), \]
\[ L_1 = -\frac{1}{\lambda^2 + 1}(-\lambda J_0 + j_1), \quad (50) \]

where \( J_\mu \) has been defined by (43). This result is expected considering the identities (44). Given the Lax connection, we can define the monodromy matrix as the holonomy along a constant time slice

\[ M(x^0; \lambda) = \mathcal{P} \exp \left( \int_{-\infty}^{\infty} dx^1 L_1(x^0, x^1, \lambda) \right). \quad (51) \]

The set of (non-local) infinite conserved charges can be generated by expanding the monodromy matrix with respect to the spectral parameter as

\[ M(\lambda) = \exp \left( \sum_{n=1}^{\infty} \frac{Q_n}{z^n} \right) \quad (52) \]
\[ = 1 + \frac{1}{\lambda} \int_{-\infty}^{+\infty} dx^1 J_0 - \frac{1}{\lambda^2} \left( \int_{-\infty}^{+\infty} dx^1 j_1 - \int_{-\infty}^{+\infty} dx^1 \int_{-\infty}^{+\infty} dy^1 J_0(x) J_0(y) \right) + \mathcal{O}(\frac{1}{\lambda^3}). \]

For the undeformed PCM these non-local charges span the classical Yangian algebra \[20]. Under the \( TT^\star \)-deformation, the algebra gets deformed in a very complicated way.
4 Lax connections from dynamical coordinate transformation

The solvability of $T\bar{T}$-deformation can be understood in various ways. From the point of view of integrability\textsuperscript{3}, the most transparent approach is to realize the $T\bar{T}$-deformation as dynamical coordinate transformation. It was shown in [10, 11, 3], the $T\bar{T}$ deformation can be interpreted as a space-time deformation. In Euclidean signature the deformed and undeformed space-time are related via the following (state dependent or dynamical) coordinate transformation

\[
\begin{align*}
  dx^\mu &= \left( \delta^\mu_\nu + t \bar{T}^\mu_\nu(y) \right) dy^\nu, \quad y = (y^1, y^2), \\
  dy^\mu &= \left( \delta^\mu_\nu + t (T^{(\tau)})^\mu_\nu(x) \right) dx^\nu, \quad x = (x^1, x^2),
\end{align*}
\]

with $\bar{T}^\mu_\nu = -\epsilon^\rho_\mu \epsilon^\sigma_\nu T^\rho_\sigma$ and $(T^{(\tau)})^\mu_\nu = -\epsilon^\mu_\rho \epsilon^\nu_\sigma (T^{(\tau)})^\rho_\sigma$, where $T = T^{(0)}$ and $T^{(\tau)}$ are the undeformed and deformed stress-energy tensor in the coordinates $y$ and $x$, respectively. Using this map we can obtain the solutions of the deformed equation of motions as

\[
\phi^{(\tau)}(x) = \phi^{(0)}(y(x)).
\]

Apart from the solutions of equation of motions, the deformed conserved currents can also be obtained from the undeformed ones by using the above coordinate transformations\textsuperscript{11}. First let us switch to complex coordinates defined by

\[
\begin{align*}
  z &= x^1 + ix^2, \quad \bar{z} = x^1 - ix^2, \\
  w &= y^1 + iy^2, \quad \bar{w} = y^1 - iy^2.
\end{align*}
\]

Starting from the 1-forms in the $w$ coordinates

\[
\begin{align*}
  \mathcal{J}_k &= T_{k+1}(w)dw + \Theta_{k-1}(w)d\bar{w}, \\
  \bar{\mathcal{J}}_k &= \bar{T}_{k+1}(w)d\bar{w} + \bar{\Theta}_{k-1}(w)dw,
\end{align*}
\]

where $T_{k+1}$, $\Theta_{k-1}$ and their complex conjugates are the higher conserved currents of underformed theory. Under the change of coordinates, we have

\[
\begin{pmatrix}
  dw \\
  d\bar{w}
\end{pmatrix} = \mathcal{J}^T \begin{pmatrix}
  dz \\
  d\bar{z}
\end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix}
  \partial w & \partial \bar{w} \\
  \partial \bar{w} & \partial w
\end{pmatrix}. \tag{59}
\]

\textsuperscript{3}Here specifically by integrability we mean there exist infinite conserved charges.
where \( \partial \) and \( \bar{\partial} \) denote the derivative with respect to \( z \) and \( \bar{z} \), respectively. Now the Jacobian is of the form

\[
J = \frac{1}{\Delta(w)} \begin{pmatrix}
1 + 2t\Theta_0(w) & -2tT_2(w) \\
-2t\bar{T}_2(w) & 1 + 2\bar{\Theta}_0(w)
\end{pmatrix}
\]  

(60)

with

\[
\Delta(w) = (1 + 2t\Theta_0(w))(1 + 2\bar{\Theta}_0(w)) - 4t^2T_2(w)\bar{T}_2(w).
\]  

(61)

Substituting (59) and (60) into (58) one can read off the components of the currents in the \( z \) coordinates:

\[
T_{k+1}(z, t) = \frac{T_{k+1}(w(z)) + 2t(T_{k+1}(w(z))\Theta_0(w(z)) - \Theta_{k-1}(w(z))T_2(w(z)))}{\Delta(w(z))},
\]  

(62)

\[
\Theta_{k-1}(z, t) = \frac{\Theta_{k-1}(w(z)) + 2t(\Theta_{k-1}(w(z))\bar{\Theta}_0(w(z)) - T_{k+1}(w(z))\bar{T}_2(w(z)))}{\Delta(w(z))}.
\]  

(63)

In a similar way, we can read the Lax connection of the deformed model. If the Lax connection of the undeformed model is

\[
L(w, \bar{w}) = \mathcal{L} dw + \bar{\mathcal{L}} d\bar{w}
\]  

(64)

one can expect the deformed Lax pair should be given by

\[
L = \mathcal{L}(z, \bar{z}) dz + \bar{\mathcal{L}}(z, \bar{z}) d\bar{z} = \mathcal{L}(w, \bar{w}) \left( \frac{\partial w}{\partial z} dz + \frac{\partial w}{\partial \bar{z}} d\bar{z} \right) + \bar{\mathcal{L}}(w, \bar{w}) \left( \frac{\partial \bar{w}}{\partial z} dz + \frac{\partial \bar{w}}{\partial \bar{z}} d\bar{z} \right),
\]  

(65)

which leads to the transformation law on the Lax connections:

\[
\mathcal{L}(z, t) = \frac{\mathcal{L}_w(w(z)) + 2t(\mathcal{L}_w(w(z))\Theta_0(w(z)) - \mathcal{L}_{\bar{w}}(w(z))T_2(w(z)))}{\Delta(w(z))},
\]  

(66)

\[
\bar{\mathcal{L}}(z, t) = \frac{\bar{\mathcal{L}}_{\bar{w}}(w(z)) + 2t(\bar{\mathcal{L}}_{\bar{w}}(w(z))\bar{\Theta}_0(w(z)) - \bar{\mathcal{L}}_{\bar{w}}(w(z))\bar{T}_2(w(z)))}{\Delta(w(z))}.
\]  

(67)

In the following, we will check the above relations in a free scalar theory and the sine-Gordon model whose deformed Lax pairs are explicitly given in the literature [17]. Moreover, we will try to reproduce the Lax connections of affine Toda field theory and PCM which we found in previous sections.
4.1 Free scalar

Consider the free scalar with Lagrangian
\[ L(w) = \partial_w \phi \partial_{\bar{w}} \phi. \] (68)

The model is integrable with trivial Lax pair
\[ L_w = \partial_w \phi, \quad L_{\bar{w}} = -\partial_{\bar{w}} \phi \] (69)
such that the Lax equation
\[ \partial_{\bar{w}} L_w - \partial_w L_{\bar{w}} = 2 \partial_w \partial_{\bar{w}} \phi = 0 \] (70)
coinciding with the equation of motion. The stress-energy tensor is simply
\[ T_2(w) = -\frac{1}{2} (\partial_w \phi)^2, \quad \Theta_0(w) = 0, \quad \Delta = 1 - 4t^2 T_2(w) \bar{T}_2(w), \] (71)
which leads to the following transformation
\[ \partial_w \phi = \partial \phi - \frac{1}{4\tau} \left( \frac{-1 + \Omega_T}{\partial \phi} \right)^2 \bar{\partial} \phi, \quad \partial_{\bar{w}} \phi = \bar{\partial} \phi - \frac{1}{4\tau} \left( \frac{-1 + \Omega_T}{\partial \phi} \right)^2 \partial \phi, \] (72)
with \( \Omega_T = \sqrt{1 + 4t \partial \phi \bar{\partial} \phi} \). Therefore the deformed Lax connection is given by
\[ L(z, \tau) = \frac{\partial_w \phi(w(z)) + 2t \partial_{\bar{w}} \phi(w(z)) T_2(w(z))}{1 - 4t^2 T_2(w(z)) \bar{T}_2(w(z))} = \frac{\partial \phi}{\Omega_T}, \] (73)
\[ \bar{L}(z, \tau) = -\frac{\partial_w \phi(w(z)) - 2t \partial_{\bar{w}} \phi(w(z)) \bar{T}_2(w(z))}{1 - 4t^2 T_2(w(z)) \bar{T}_2(w(z))} = -\frac{\bar{\partial} \phi}{\Omega_T}. \] (74)
Indeed the Lax equation matches the equation of motion of the \( T\bar{T}\)-deformed free scalar:
\[ \partial \left( \frac{\bar{\partial} \phi}{\Omega_T} \right) + \bar{\partial} \left( \frac{\partial \phi}{\Omega_T} \right) = 0. \] (75)

4.2 Sine-Gordon model

Next, we turn to the \( T\bar{T}\)-deformed sine-Gordon model, whose Lax pair has been given in \textsuperscript{17}. As a first step, we need to find the Jacobian \textsuperscript{60} which is determined by the stress-energy tensor in the \( w \) space-time. The Lagrangian of the sine-Gordon model is given by adding the potential\textsuperscript{4}
\[ V = 4 \sin^2 \left( \frac{\phi}{2} \right) \] (76)
\footnote{Here we have chosen the convention used in \textsuperscript{17} in order to make the comparison.}
to the free scalar Lagrangian \[68\]. From the standard procedure one can find the expression of the stress-energy tensor
\[
T_2(w) = -\frac{1}{2}(\partial_w \phi)^2, \quad \tilde{T}_2(w) = -\frac{1}{2}(\partial_{\bar{w}} \phi)^2, \quad \Theta_0(w) = -2 \sin^2(\frac{\phi}{2})
\] (77)
which leads to the following transformations
\[
\partial_w \phi = \frac{-1 + \Omega_T}{2t \bar{\partial} \phi}, \quad \partial_{\bar{w}} \phi = \frac{-1 + \Omega_T}{2t \partial \phi}, \quad \Omega_T = \sqrt{1 + 4t(1 - tV)\partial \phi \bar{\partial} \phi}.
\] (78)
Recall that the undeformed Lax connection is
\[
L_w = -\frac{i}{4} \partial_w \phi H + \frac{\lambda}{2} e^{i\frac{\phi}{2}} E_+ + \frac{\lambda}{2} e^{-i\frac{\phi}{2}} E_-,
\]
\[
\bar{L}_{\bar{w}} = \frac{i}{4} \partial_{\bar{w}} \phi H + \frac{1}{2\lambda} e^{-i\frac{\phi}{2}} E_+ + \frac{1}{2\lambda} e^{i\frac{\phi}{2}} E_-.
\] (80)
The deformed Lax connection can be expanded with respect these three generators as
\[
\mathcal{L}(z, t) = \mathcal{L}^0 H + \mathcal{L}^+ E_+ + \mathcal{L}^- E_-, \quad \bar{\mathcal{L}}(\bar{z}, \bar{\tau}) = \bar{\mathcal{L}}^0 H + \bar{\mathcal{L}}^+ E_+ + \bar{\mathcal{L}}^- E_-. \quad (81)
\]
From the transformation \[66\], we have the deformed Lax connections
\[
\mathcal{L}^0 = -\frac{i\partial \phi}{4\Omega_T}, \quad \bar{\mathcal{L}}^0 = \frac{i\bar{\partial} \phi}{4\Omega_T},
\]
\[
\mathcal{L}^+ = \frac{e^{-i\frac{\phi}{2}} (\partial \phi)^2 t}{\lambda} \frac{(\Omega_T + 1)^2}{2\Omega_T (1 - tV)} + \lambda e^{i\frac{\phi}{2}} \frac{(\bar{\partial} \phi) t}{2\Omega_T} + \frac{\lambda}{2} e^{-i\frac{\phi}{2}} (\Omega_T + 1)^2, \quad \bar{\mathcal{L}}^+ = \lambda e^{i\frac{\phi}{2}} \frac{(\bar{\partial} \phi)^2 t}{2\Omega_T} + \frac{\lambda}{2} e^{-i\frac{\phi}{2}} (\Omega_T + 1)^2,
\]
\[
\mathcal{L}^- = \frac{e^{i\frac{\phi}{2}} (\partial \phi)^2 t}{\lambda} \frac{(\Omega_T + 1)^2}{2\Omega_T (1 - tV)} + \lambda e^{-i\frac{\phi}{2}} \frac{(\bar{\partial} \phi) t}{2\Omega_T} + \frac{\lambda}{2} e^{i\frac{\phi}{2}} (\Omega_T + 1)^2, \quad \bar{\mathcal{L}}^- = \lambda e^{-i\frac{\phi}{2}} \frac{(\bar{\partial} \phi)^2 t}{2\Omega_T} + \frac{\lambda}{2} e^{i\frac{\phi}{2}} (\Omega_T + 1)^2,
\]
which coincide with the ones found in \[17\].

4.3 Liouville field theory

The Lagrangian of the classical Liouville field theory is
\[
\mathcal{L}(w) = \partial_w \phi \partial_{\bar{w}} \phi - \mu e^\phi, \quad V = -\mu e^\phi
\] (83)
with the Lax connection
\[
\mathcal{L}_w = -\partial_w \phi H + 2\lambda \sqrt{\mu e^\phi} E_+, \quad \mathcal{L}_{\bar{w}} = \partial_{\bar{w}} \phi H - \frac{1}{2\lambda} \sqrt{\mu e^\phi} E_-.
\] (84)
The field transformation is also given by (78). Decomposing the Lax connection as (81) again we find

\[ \mathcal{L}^0 = -\frac{\partial \phi}{\Omega_T}, \quad \bar{\mathcal{L}}^0 = \frac{\bar{\partial} \phi}{\Omega_T}, \]

\[ \mathcal{L}^+ = \frac{\lambda \sqrt{\mu} e^\phi (1 + \Omega_T)^2}{2\Omega_T(1 - tV)}, \quad \bar{\mathcal{L}}^+ = \frac{2t\lambda \sqrt{\mu} (\bar{\partial} \phi)^2}{\Omega_T}, \]

\[ \mathcal{L}^- = \frac{-t\sqrt{\mu} e^\phi (\partial \phi)^2}{2\lambda \Omega_T}, \quad \bar{\mathcal{L}}^- = \frac{-\sqrt{\mu} e^\phi (1 + \Omega_T)^2}{8\lambda \Omega_T(1 - tV)}. \]

They differ from the ones in (25) up to numerical factors, due to different conventions.

With these deformed Lax connection one can derive infinite conserved charges. On the other hand, the (anti)-holomorphic currents are simply given by taking powers of the modified traceless stress-energy tensor

\[ T_{2n} = ((\partial_w \phi)^2 - 2\partial_w^2 \phi)^n, \quad \bar{T}_{2n} = ((\bar{\partial}_w \phi)^2 - 2\bar{\partial}_w^2 \phi)^n. \]

From (78) and (62) one can read the deformed currents

\[ T_{2n}(z) = -\frac{\Omega_T + (2t(1 - tV)\partial \phi \bar{\partial} \phi + 1)}{2\Omega_T(1 - \tau V)} T_{2n}(w(z)), \]

\[ \Theta_{2n}(z) = \frac{t(\bar{\partial} \phi)^2}{\Omega_T} T_{2n}(w(z)). \]

The explicit expressions of these currents have been derived in [19] using a different method.

### 4.4 \( N \) bosonic scalars with arbitrary potential

To construct the deformed Lax connections for the (affine) Tode field theories, let us first consider the \( N \) free scalars with arbitrary potential [16] [17]

\[ \mathcal{L}_N = \sum_i \partial_w \phi_i \partial_w \phi_i + V(\phi_i). \]

From the relationships

\[ \frac{\partial x^1}{\partial y^1} = 1 + tT_{2}^2(\mathbf{y}), \quad \frac{\partial x^2}{\partial y^2} = 1 + tT_{1}(\mathbf{y}), \quad \frac{\partial x^1}{\partial y^1} = \frac{\partial x^2}{\partial y^2} = -tT_{1}(\mathbf{y}), \]

we can compute the inverse of the Jacobian

\[ \mathcal{J}_N^{-1} = \begin{pmatrix} \partial_w z^1 & \partial_w z^2 \\ \partial_z z^1 & \partial_z z^2 \end{pmatrix} = \begin{pmatrix} 1 - tv & -t \sum_i (\partial_w \phi_i)^2 \\ -t \sum_i (\partial_w \phi_i)^2 & 1 - tv \end{pmatrix}. \]
The main technical difficulty of this method is to solve \( \partial_w \phi_i \) and \( \bar{\partial}_{\bar{w}} \phi_i \) from

\[
\begin{pmatrix}
\partial_w \phi_i \\
\bar{\partial}_{\bar{w}} \phi_i
\end{pmatrix} = J^{-1}_N \begin{pmatrix}
\partial \phi_i \\
\bar{\partial} \phi_i
\end{pmatrix}
\]

in terms of \( \partial \phi_i \) and \( \bar{\partial} \phi_i \). For this particular example we find the following solution

\[
\partial_w \phi_i = \frac{1}{2t} \bar{\partial} \phi_i (-1 + \Omega_T) + \bar{t} \frac{\partial B}{\partial \bar{\partial} \phi_i}, \quad \bar{\partial}_{\bar{w}} \phi_i = \frac{1}{2t} \partial \phi_i (-1 + \Omega_T) + t \frac{\partial B}{\partial \partial \phi_i},
\]

with

\[
\bar{t} = t(1 - tV), \quad \Omega_T = \sqrt{1 + 4\bar{t}(L^{(0)} - \bar{t}B)}, \quad (93)
\]

\[
L^{(0)} = \sum_{i=1}^N \partial \phi_i \bar{\partial} \phi_i, \quad B = \sum_{i=1}^N (\partial \phi_i)^2 \sum_{j=1}^N (\bar{\partial} \phi_j)^2 - \left( \sum_{i=1}^N \partial \phi_i \bar{\partial} \phi_i \right)^2, \quad (94)
\]

\[
K = \sum_{i}^N (\partial \phi_i)^2, \quad \bar{K} = \sum_{i}^N (\bar{\partial} \phi_i)^2. \quad (95)
\]

On the other hand the stress-energy tensor is given by

\[
K_w = \sum_{i}^N (\partial_w \phi_i)^2, \quad \bar{K}_{\bar{w}} = \sum_{i}^N (\bar{\partial}_{\bar{w}} \phi_i)^2, \quad (96)
\]

\[
T_2 = -\frac{1}{2} K_w, \quad \tilde{T}_2 = -\frac{1}{2} \bar{K}_{\bar{w}}, \quad \Theta_0 = -\frac{1}{2} V. \quad (97)
\]

Therefore the deformed Lax connection is directly given by (66)

\[
\mathcal{L} = \frac{(1 - \tau V) L_w + \tau K_w L_{\bar{w}}}{(1 - \tau V)^2 - \tau^2 K_w \bar{K}_{\bar{w}}}, \quad \bar{\mathcal{L}} = \frac{(1 - \tau V) \bar{L}_{\bar{w}} + \tau \bar{K}_{\bar{w}} \bar{L}_w}{(1 - \tau V)^2 - \tau^2 K_w \bar{K}_{\bar{w}}}. \quad (98)
\]

Using the identities (91) we can find the relation between \( K_w, \bar{K}_{\bar{w}} \) and \( K \) and \( \bar{K} \)

\[
K_w = (1 - \tau V)^2 K + \tau^2 K_w^2 K - 2\tau K_w (1 - \tau V) L^{(0)}, \quad (99)
\]

\[
\bar{K}_{\bar{w}} = (1 - \tau V)^2 \bar{K} + \tau^2 \bar{K}_{\bar{w}}^2 K - 2\tau \bar{K}_{\bar{w}} (1 - \tau V) L^{(0)}. \quad (100)
\]

These are quadratic equations, whose solutions are\(^5\)

\[
K_w = \frac{2\bar{t}L^{(0)} + 1 - \Omega_T}{2\tau^2 K}, \quad \bar{K}_{\bar{w}} = \frac{2\bar{t}L^{(0)} + 1 - \Omega_T}{2\tau^2 K}. \quad (101)
\]

\(^5\)There are two branches of solutions, here we only keep the one which is consistent with our results in previous section. The other branch gives equivalent result up to a gauge transformation.
where we used the identity
\[ B = K \tilde{K} - \mathcal{L}^{(0)} \mathcal{L}^{(0)}. \] (102)

Substituting (101) into (98) gives
\[ L = -\frac{\Omega_T}{\Omega_T} + (2\tilde{t} L^{(0)} + 1) \mathcal{L}_w - \frac{tK}{\Omega_T} \mathcal{L}_w, \quad \bar{L} = -\frac{\Omega_T}{\Omega_T} + (2\tilde{t} \bar{L}^{(0)} + 1) \mathcal{L}_{\bar{w}} - \frac{t\bar{K}}{\Omega_T} \mathcal{L}_{\bar{w}}. \] (103) (104)

For the affine Toda theories, whose Lax connections are known, it is straightforward to read the deformed Lax connection from (104). They turn out to be in match with the ones we derived previously in section 2.2. Furthermore, we can use the relations to derive the deformed Lax connection of PCM if we make the following identification
\[ j_{\mu} = j_{\mu}^i T_i, \quad j_{\mu}^i \equiv \partial_{\mu} \phi_i \] (105)
where \( T_i \) are the generators of the Lie algebra with the Killing metric \( \text{Tr}(T_i T_j) = \delta_{ij} \).

### 4.5 Nonlinear Schrödinger model

As our last example, let us consider the \( TT^-\)-deformed nonlinear Schrödinger model which is a non-relativistic complex field theory. The \( TT^-\)-deformed Lagrangian was recently derived in [21, 22, 23]. Here we derive the deformed Lax connection from the dynamical coordinate transformation.

For the undeformed model, the Lagrangian is
\[ \mathcal{L}_{NS}(y_1, y_2) = \frac{i}{2} (\bar{q} \partial y_1 q - q \partial y_1 \bar{q}) - \frac{\partial y_2 q \partial y_2 \bar{q}}{2m} - g|q\bar{q}|^2, \] (106)
which has the following equations of motion
\[ -i\partial_{y_1} q = \frac{1}{2m} \partial^2 y_2 q - 2gq^2 \bar{q}, \quad i\partial_{y_1} \bar{q} = \frac{1}{2m} \partial^2 y_2 \bar{q} - 2gq^2, \] (107)
and the stress-energy tensor
\[ T_{y_2y_2} = -\frac{1}{m} \partial y_2 q \partial y_2 \bar{q} - \mathcal{L}_{NS}(y_1, y_2), \quad T_{y_2y_1} = -\frac{1}{2m} (\partial y_2 q \partial y_1 q + \partial y_2 q \partial y_1 \bar{q}), \quad T_{y_1y_2} = \frac{i}{2} (\bar{q} \partial y_2 q - q \partial y_2 \bar{q}), \quad T_{y_1y_1} = \frac{i}{2} (\bar{q} \partial y_1 q - q \partial y_1 \bar{q}) - \mathcal{L}_{NS}(y_1, y_2). \] (108) (109)
The corresponding Lax connection is

\begin{align}
U_{y_2} &= -i\lambda\sigma_3 + i\sqrt{2gmQ}, \\
V_{y_1} &= -\frac{i\lambda^2}{m}\sigma_3 + i\sqrt{\frac{2g}{m}\lambda Q} + \sqrt{\frac{g}{2m}}\partial_{y_2}Q\sigma_3 + igQ^2\sigma_3, 
\end{align}

where

\begin{align}
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}.
\end{align}

Solving (89) one can find the following rules of transformation [23]:

\begin{align}
\partial_{y_1}q &= \frac{2m(B - S)\partial_{x_1}\bar{q} + 2\bar{t}A\bar{C}'}{2tA^2}, \quad \partial_{y_2}q = \frac{2m(B - S)}{2tA}, \\
\partial_{y_1}\bar{q} &= \frac{2m(B - S)\partial_{x_1}q - 2\bar{t}A\bar{C}'}{2tA^2}, \quad \partial_{y_2}\bar{q} = \frac{2m(B - S)}{2tA},
\end{align}

where we have defined

\begin{align}
\bar{t} &= t(1 + tv), \quad C = \partial_{x_2}\bar{q}\partial_{x_1}q - \partial_{x_1}\bar{q}\partial_{x_2}q, \quad B = 1 + \frac{it}{2}(\bar{q}\partial_{x_1}q - q\partial_{x_1}\bar{q}), \\
A &= \partial_{x_2}q + \frac{it}{2}qC, \quad \bar{A} = \partial_{x_2}\bar{q} + \frac{it}{2}\bar{q}C, \quad S = \sqrt{B^2 - \frac{2t}{m}A\bar{A}}.
\end{align}

Substituting (107) and (113) into (66), and after some manipulations we end up with final results of the deformed Lax connection

\begin{align}
\begin{pmatrix} V_{x_1} \\ U_{x_2} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} V_{y_1} \\ U_{y_2} \end{pmatrix}
\end{align}

where

\begin{align}
J_{11} &= \frac{tB(B + S)}{2St}, \quad J_{12} = \frac{-t(A\partial_{x_1}\bar{q} + \bar{A}\partial_{x_1}q)}{2mS}, \\
J_{21} &= \frac{it^2(B + S)(\bar{q}\partial_{x_2}q - q\partial_{x_2}\bar{q})}{4St}, \quad J_{22} = \frac{2tA\bar{A}}{2mS(B - S)} - \frac{t(A\partial_{x_2}\bar{q} + \bar{A}\partial_{x_2}q)}{2mS}.
\end{align}

5 Conclusion

In this work we constructed the Lax connections of several $T\bar{T}$-deformed integrable models in two different ways, and found consistent picture. The first way is based on proper ansatz, which assumes that the Lax equation is linearly dependent on the equation of
motion. In the discussion, we also assumed that some proportional functions or parameters are invariant under the deformation. We obtained the Lax connections for the affine Toda theories and the principal chiral model. The method is suggestive, but its potential is not clear to us.

The other way relies on the dynamical coordinate transformation between the $T\bar{T}$-deformed theory and its ancestor. The method is systematic but maybe difficult to implement in some models due to the complexity of the dynamical coordinate transformation. We showed the power of the coordinate transformation in several models, including the free scalar theory, sine-Gordon model, Liouville field theory, $N$-scalar theory and non-linear Schrödinger model.

We want to stress that the dynamical coordinate transformation is not a diffeomorphism. Because the coordinate transformation depends on the dynamical fields, the inverse of the transformation could not be obtained in a closed form. Actually we tried to derive the Lax connection of $T\bar{T}$-deformed KdV equation. In this case the coordinate transformation depends on the higher order derivatives so the closed form of the inverse of the transformation is unlikely to exist. It is interesting to investigate the effectiveness of the coordinate transformation in other models, for example, the fermionic ones [24].

Besides the two methods discussed in this work, it would be interesting to study the Lax connection from other aspects on $T\bar{T}$-deformation, say from the light–cone gauge approach in [8].

Given the explicit form of the Lax connection, there are various of applications. The first one is to construct infinite conserved charges as we show for the PCM. The expression (52) indicates the conserved charges get deformed in a very complicated way and it would be very interesting to study how the algebra is deformed. The other application is to construct the solitonic surfaces following [10]. Most importantly we hope our construction of Lax connection can shed light on the quantization of the $T\bar{T}$-deformation.

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References

[1] F. A. Smirnov and A. B. Zamolodchikov, “On space of integrable quantum field theories,” Nucl. Phys. B 915, 363-383 (2017) doi:10.1016/j.nuclphysb.2016.12.014 [arXiv:1608.05499 [hep-th]].

[2] A. Cavaglià, S. Negro, I. M. Szécsényi and R. Tateo, “$T\bar{T}$-deformed 2D Quantum Field Theories,” JHEP 10, 112 (2016) doi:10.1007/JHEP10(2016)112 [arXiv:1608.05534 [hep-th]].

[3] S. Dubovsky, V. Gorbenko and M. Mirbabayi, “Asymptotic fragility, near AdS$_2$ holography and $T\bar{T}$,” JHEP 09, 136 (2017) doi:10.1007/JHEP09(2017)136 [arXiv:1706.06604 [hep-th]].

[4] E. K. Sklyanin, “Quantum version of the method of inverse scattering problem,” J. Sov. Math. 19, 1546-1596 (1982) doi:10.1007/BF01091462

[5] A. B. Zamolodchikov, “Expectation value of composite field $T$ anti-$T$ in two-dimensional quantum field theory,” [arXiv:hep-th/0401146 [hep-th]].

[6] J. Cardy, “The $T\bar{T}$ deformation of quantum field theory as random geometry,” JHEP 10, 186 (2018) doi:10.1007/JHEP10(2018)186 [arXiv:1801.06895 [hep-th]].

[7] S. Dubovsky, V. Gorbenko and G. Hernández-Chifflet, “$T\bar{T}$ partition function from topological gravity,” JHEP 09, 158 (2018) doi:10.1007/JHEP09(2018)158 [arXiv:1805.07386 [hep-th]].

[8] S. Frolov, “$T\bar{T}$ Deformation and the Light-Cone Gauge,” Proc. Steklov Inst. Math. 309, 107-126 (2020) doi:10.1134/S0081543820030098 [arXiv:1905.07946 [hep-th]].

[9] N. Callebaut, J. Kruthoff and H. Verlinde, “$T\bar{T}$ deformed CFT as a non-critical string,” JHEP 04, 084 (2020) doi:10.1007/JHEP04(2020)084 [arXiv:1910.13578 [hep-th]].

[10] R. Conti, S. Negro and R. Tateo, “The $T\bar{T}$ perturbation and its geometric interpretation,” JHEP 02, 085 (2019) doi:10.1007/JHEP02(2019)085 [arXiv:1809.09593 [hep-th]].
[11] R. Conti, S. Negro and R. Tateo, “Conserved currents and $T\bar{T}$ irrelevant deformations of 2D integrable field theories,” JHEP 11, 120 (2019) doi:10.1007/JHEP11(2019)120 [arXiv:1904.09141 [hep-th]].

[12] S. Frolov, “Lax pair for strings in Lunin-Maldacena background,” JHEP 05, 069 (2005) doi:10.1088/1126-6708/2005/05/069 [arXiv:hep-th/0503201 [hep-th]].

[13] A. Giveon, N. Itzhaki and D. Kutasov, “$T\bar{T}$ and LST,” JHEP 07, 122 (2017) doi:10.1007/JHEP07(2017)122 [arXiv:1701.05576 [hep-th]].

[14] M. Baggio and A. Sfondrini, “Strings on NS-NS Backgrounds as Integrable Deformations,” Phys. Rev. D 98, no.2, 021902 (2018) doi:10.1103/PhysRevD.98.021902 [arXiv:1804.01998 [hep-th]].

[15] A. Sfondrini and S. J. van Tongeren, “$T\bar{T}$ deformations as $T$ $s$ $T$ transformations,” Phys. Rev. D 101, no.6, 066022 (2020) doi:10.1103/PhysRevD.101.066022 [arXiv:1908.09299 [hep-th]].

[16] G. Bonelli, N. Doroud and M. Zhu, “$T\bar{T}$-deformations in closed form,” JHEP 06, 149 (2018) doi:10.1007/JHEP06(2018)149 [arXiv:1804.10967 [hep-th]].

[17] R. Conti, L. Iannella, S. Negro and R. Tateo, “Generalised Born-Infeld models, Lax operators and the $T\bar{T}$ perturbation,” JHEP 11, 007 (2018) doi:10.1007/JHEP11(2018)007 [arXiv:1806.11515 [hep-th]].

[18] E. Corrigan, “Recent developments in affine Toda quantum field theory,” [arXiv:hep-th/9412213 [hep-th]].

[19] M. Leoni, “$T\bar{T}$ deformation of classical Liouville field theory,” JHEP 07, no.07, 230 (2020) doi:10.1007/JHEP07(2020)230 [arXiv:2005.08906 [hep-th]].

[20] N. J. MacKay, “On the classical origins of Yangian symmetry in integrable field theory,” Phys. Lett. B 281, 90-97 (1992) [erratum: Phys. Lett. B 308, 444-444 (1993)] doi:10.1016/0370-2693(92)90280-H.

[21] B. Chen, J. Hou and J. Tian, “Note on Non-relativistic $T\bar{T}$-deformation,” [arXiv:2012.14091 [hep-th]].

[22] D. Hansen, Y. Jiang and J. Xu, “Geometrizing non-relativistic bilinear deformations,” [arXiv:2012.12290 [hep-th]].
[23] P. Ceschin, R. Conti and R. Tateo, “TT-deformed Nonlinear Schrödinger,” \texttt{arXiv:2012.12760 [hep-th]}. 

[24] E. A. Coleman, J. Aguilera-Damia, D. Z. Freedman and R. M. Soni, “$T\bar{T}$-deformed actions and (1,1) supersymmetry,” JHEP 10, 080 (2019) doi:10.1007/JHEP10(2019)080 \texttt{arXiv:1906.05439 [hep-th]}. 
