ABSTRACT. Concise methods are used to compute the propagator for a non-relativistic particle subject to a potential with \( x^2 \) and \( 1/x^2 \) terms.

KEY WORDS: Harmonic oscillator, Lie algebra, Propagator, SL(2,C)

Time-development is a – if not the – central issue in quantum mechanics. The abundance of papers on the propagator for the harmonic oscillator [1] bears witness to this fact, especially in pedagogical journals [2–7]. Here is yet another paper on this subject, hopefully more concise than the average and at least as useful for purposes of pedagogy. The present paper differs from most of those just cited in that it does not use the path integral formulation of quantum mechanics, nor does it use explicit properties of the energy eigenfunctions for the oscillator. The essential ingredient used here is an algebraic identity for the special linear group in two dimensions, \( SL(2, \mathbb{C}) \). While some of the pedagogical papers cited also use algebraic methods, in particular raising and lowering operators for the oscillator [4,6,7], none exploits Lie group methods so succinctly as the analysis to follow.¹

Consider the following identity for elements of the group \( SL(2, \mathbb{C}) \) that is relevant for this problem. The identity can be realized in terms of canonical operators \( x \) and \( p \) for the quantum oscillator. So expressed, the operator identity is [9]

\[
\exp \left( -\frac{it}{\hbar} \left( \frac{1}{2m} \left( p^2 + \frac{\lambda}{x^2} \right) + \frac{1}{2} m\omega^2 x^2 \right) \right)
= \exp \left( -\frac{im\omega}{2\hbar} x^2 \tan(\omega t/2) \right) \exp \left( -\frac{i}{2m\omega \hbar} \left( p^2 + \frac{\lambda}{x^2} \right) \sin(\omega t) \right) \exp \left( -\frac{im\omega}{2\hbar} x^2 \tan(\omega t/2) \right),
\]

where as usual, \( \hbar, m, \lambda, \) & \( \omega \) are constants, and \( t \) is the time. This identity follows from the elementary commutation relations that provide a realization of the Lie algebra \( sl(2, \mathbb{C}) \).

¹That said, a paper using similar algebraic methods with comparable conciseness, albeit to solve a different problem, is [8].

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\[ x^2, p^2 + \frac{\lambda}{x^2} \] = 2i\hbar (xp + px), \quad [x^2, xp + px] = 4i\hbar (x^2), \quad [p^2 + \frac{\lambda}{x^2}, xp + px] = -4i\hbar \left( p^2 + \frac{\lambda}{x^2} \right), \quad (2)

which in turn follow from \([x, p] = i\hbar\). Upon differentiating both left- and right-hand sides with respect to the time, with the obvious initial condition at \(t = 0\), the identity (1) can be confirmed by re-ordering the various operator expressions through the use of the commutators (2) in conjunction with the expansion

\[ e^G O e^{-G} = O + \sum_{n=1}^{\infty} \frac{1}{n!} [G, \ldots, [G, O]] \quad (3) \]

This straightforward but tedious method to establish (1) is left as an exercise for the conscientious reader.

But perhaps the easiest way to check the identity (1) is to use a fundamental faithful representation of the \(sl(2, \mathbb{C})\) algebra (2) as \(2 \times 2\) traceless matrices. In the oscillator context, such a representation is given by

\[ x^2 \rightarrow \begin{pmatrix} 0 & 2\hbar \\ 0 & 0 \end{pmatrix}, \quad p^2 + \frac{\lambda}{x^2} \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 2\hbar \end{pmatrix}, \quad xp + px \rightarrow \begin{pmatrix} -2i\hbar & 0 \\ 0 & 2i\hbar \end{pmatrix}. \quad (4) \]

Hence the above group elements are realized as the \(2 \times 2\) matrices.

\[ \exp(-i\alpha x^2) \rightarrow \begin{pmatrix} c & c \\ -c^* & -c^* \end{pmatrix}, \quad \exp(-i\beta (p^2 + \frac{\lambda}{x^2})) \rightarrow \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \quad (5) \]

as well as

\[ \exp(-i\alpha x^2) \exp \left(-i\beta \left( p^2 + \frac{\lambda}{x^2} \right) \right) \exp(-i\alpha x^2) \rightarrow \begin{pmatrix} 1 - 4\alpha \beta \hbar^2 - 4i\alpha \hbar (1 - 2\alpha \beta \hbar^2) \\ -2i\beta \hbar & 1 - 4\alpha \beta \hbar^2 \end{pmatrix}, \quad (6) \]

where \(\alpha = \frac{m\omega}{2\hbar} \tan(\omega t/2)\) and \(\beta = \frac{1}{2m\omega} \sin(\omega t)\). The last of these group elements is exactly the same as

\[ \exp \left(-\frac{it}{\hbar} \left( \frac{1}{2m} \left( p^2 + \frac{\lambda}{x^2} \right) + \frac{1}{2} m\omega^2 x^2 \right) \right) \rightarrow \exp \left(-\frac{it}{2m\hbar} \begin{pmatrix} 0 & 0 \\ 2\hbar & 0 \end{pmatrix} - \frac{it}{2\hbar} m\omega^2 \begin{pmatrix} 0 & 2\hbar \\ 0 & 0 \end{pmatrix} \right) \]

\[ = \begin{pmatrix} \cos(\omega t) - im\omega \sin(\omega t) \\ \frac{i}{m\omega} \sin(\omega t) \cos(\omega t) \end{pmatrix} \quad (7) \]
thereby establishing the identity (1). Note that there is no dependence on \( \lambda \) when using this realization of the group elements as \( 2 \times 2 \) matrices [10].

With the identity (1) in hand, the propagator for the Hamiltonian

\[
H(\lambda, \omega) = \frac{1}{2m} \left( p^2 + \frac{\lambda}{x^2} \right) + \frac{1}{2} m \omega^2 x^2 \tag{8}
\]

immediately reduces to the propagator for the same Hamiltonian shorn of the \( x^2 \) term,

\[
H_0(\lambda) = \frac{1}{2m} \left( p^2 + \frac{\lambda}{x^2} \right) \tag{9}
\]

albeit with a simple re-parameterized time, \( t \rightarrow \sin(\omega t)/\omega \). That is to say, the identity (1) gives a relation between localized matrix elements \( \langle x_1 | \cdots | x_2 \rangle \) for complete eigenstates of the operator \( x \), where \( \langle x_1 | x = x_1 \langle x_1 | \) and \( x | x_2 = x_2 \langle x_2 | \), normalized such that \( \int |x_1 \rangle \langle x_1 | dx_1 = 1 \). Namely,

\[
\langle x_1 | \exp(-iH(\lambda, \omega)t/\hbar)|x_2 \rangle = \exp \left( -\frac{im\omega^2}{2\hbar} x_1^2 \tan(\omega t/2) \right) \times \langle x_1 | \exp \left( -iH_0(\lambda) \frac{\sin(\omega t)}{\hbar \omega} \right) |x_2 \rangle \exp \left( -\frac{im\omega^2}{2\hbar} x_2^2 \tan(\omega t/2) \right). \tag{10}
\]

It only remains to compute the matrix element involving \( H_0 \).

The simplest case is \( \lambda = 0 \) for which there is no \( 1/x^2 \) potential. In that case, the problem reduces to the familiar free particle propagator, most easily evaluated by inserting complete sets of momentum eigenfunctions, normalized such that \( \int |p_1 \rangle \langle p_1 | dp_1 = 1 \). The transformation to position eigenstates is then given by plane waves \( \langle x_1 | p_1 \rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp(ip_1 x_1/\hbar) \), and hence

\[
\langle x_1 | \exp \left( -\frac{itp_1^2}{2\hbar} \right) |x_2 \rangle = \frac{1}{2\pi\hbar} \int dp_1 \exp \left( \frac{i}{\hbar} p_1 (x_1 - x_2) - \frac{itp_1^2}{2\hbar} \right) = \sqrt{\frac{m}{2\pi i\hbar}} \exp \left( \frac{im\omega}{2\hbar} (x_1 - x_2)^2 \right). \tag{11}
\]

The final exponential involves the action for a classical path connecting \( x_1 \) and \( x_2 \) in time \( t \), as is well-known.

Re-parameterizing to a periodic time variable, \( t \rightarrow \sin(\omega t)/\omega \), and combining this last relation with (10) then gives immediately the expected result for the harmonic oscillator [11].

\[
\langle x_1 | \exp \left( -\frac{it}{\hbar} \left( \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 \right) \right) |x_2 \rangle = \sqrt{\frac{m \omega}{2\pi i\hbar \sin(\omega t)}} \times \exp \left( -\frac{im\omega}{2\hbar} x_1^2 \tan \left( \frac{\omega t}{2} \right) + \frac{im\omega}{2\hbar \sin(\omega t)} (x_1 - x_2)^2 - \frac{im\omega}{2\hbar} x_2^2 \tan \left( \frac{\omega t}{2} \right) \right) \exp \left( \frac{im\omega}{2\hbar} (x_1^2 + x_2^2) \cot(\omega t) - \frac{im\omega}{\hbar \sin(\omega t)} x_1 x_2 \right). \tag{12}
\]
where \( \cot (2\theta) = -\tan (\theta) + 1/\sin (2\theta) \) was used in the last step. Once again, the final exponential involves the action for a classical path connecting \( x_1 \) and \( x_2 \) in time \( t \), as is well-known.

The situation with \( \lambda \neq 0 \) is more challenging, in part because of singular behavior if the potential is too attractive when \( \lambda < 0 \), as will become apparent in the final result for the propagator to be given below. For this and other reasons, it is convenient to define

\[
\lambda = \hbar^2 \left( n^2 - 1/4 \right),
\]

where \( n \) is a positive, dimensionless number. Clearly, for \( \lambda > 0 \), i.e. \( n > 1/2 \), the potential is repulsive so that a classical particle would be confined to the half-line with \( x > 0 \), and would only be allowed to have positive energy. This is also true for the quantized system if \( \lambda > 0 \), and this will be assumed to be the case in the following. However, the final results will turn out to be valid for \( \lambda \geq -\hbar^2/4 \), i.e. \( n \geq 0 \).

The propagator for \( H_0 \) with \( \lambda > 0 \) can be obtained by a method similar to the plane wave expansion used for \( \lambda = 0 \), only now the relevant set of states involves Bessel functions, \( J_n(kx) \) [12]. This is true because

\[
\sqrt{kx} J_n(kx)
\]

are energy eigenfunctions of \( H_0 \) for all real \( k > 0 \). Explicitly [13],

\[
\frac{\hbar^2}{2m} \left( -\frac{d^2}{dx^2} + \frac{n^2 - 1/4}{x^2} \right) \left( \sqrt{kx} J_n(kx) \right) = \frac{\hbar^2 k^2}{2m} \sqrt{kx} J_n(kx).
\]

Orthogonality and completeness of these eigenfunctions are now expressed as a superfluous pair of equations

\[
\int_0^\infty \sqrt{k_1} J_n(k_1 x) \sqrt{k_2} J_n(k_2 x) \, dx = \delta(k_1 - k_2),
\]

\[
\int_0^\infty \sqrt{k_1} J_n(k_1 x) \sqrt{k_2} J_n(k_2 x) \, dk = \delta(x_1 - x_2).
\]

That is to say, for the system with a repulsive \( 1/x^2 \) potential, the functions \( \sqrt{kx} J_n(kx) \) play a role similar to the plane waves for the free particle.

The propagator for \( H_0 \) is therefore given by an integral involving Bessel function bilinears, namely,

\[
\langle x_1| \exp(-iH_0 t/\hbar)|x_2 \rangle = \int_0^\infty \sqrt{k_1} J_n(k_1 x) e^{-\frac{\hbar k_1^2}{2m}} \sqrt{k_2} J_n(k_2 x) \, dk.
\]

It so happens this integral reduces to a closed form in terms of another, modified Bessel function, \( I_n \) (e.g. see [14]).

\[
\int_0^\infty \sqrt{k_1} J_n(k_1 x) e^{-\frac{\hbar k_1^2}{2m}} \sqrt{k_2} J_n(k_2 x) \, dk
\]

\[
= \frac{m \sqrt{x_1 x_2}}{i \hbar t} I_n \left( \frac{m x_1 x_2}{i \hbar t} \right) \exp \left( \frac{i m}{2 \hbar t} (x_1^2 + x_2^2) \right).
\]
Thus the result for $H_0$, at least when the potential is repulsive, is
\[ \langle x_1 \mid \exp(-iH_0t/\hbar) \mid x_2 \rangle = m\sqrt{x_1x_2}I_n\left(\frac{mx_1x_2}{\imath\hbar}\right)\exp\left(\frac{\imath m(x_1^2 + x_2^2)}{2\hbar}\right). \] (18)
Combining this result with (10) then gives the propagator for $H(\lambda, \omega)\mid_{\lambda=\hbar^2(n^2-1/4)}$ upon re-parameterizing the time, $t \to \sin(\omega t)/\omega$, in agreement with long-known results (again see [14], as well as Section 3.3 in [15]).

\[ \langle x_1 \mid \exp(-iHt/\hbar) \mid x_2 \rangle = m\omega\sqrt{x_1x_2}I_n\left(\frac{m\omega x_1x_2}{\imath\hbar\sin(\omega t)}\right)\exp\left(\frac{\imath m\omega}{2\hbar}(x_1^2 + x_2^2)\cot(\omega t)\right), \] (19)
where $\cot(2\theta) = -\tan(\theta) + 1/\sin(2\theta)$ was once again used in the last step.

It is perhaps reassuring that for short times, and both $x_1, x_2 > 0$,
\[ \lim_{t \to 0} \langle x_1 \mid \exp(-iHt/\hbar) \mid x_2 \rangle \sim \sqrt{\frac{m}{2\pi\hbar t}}\exp\left(\frac{\imath m}{2\hbar t}(x_1 - x_2)^2\right) \] (20)
as follows from the asymptotic behavior of $I_n$. Consequently,
\[ \lim_{t \to 0} \langle x_1 \mid \exp(-iHt/\hbar) \mid x_2 \rangle = \delta(x_1 - x_2). \]

Even more so, it is straightforward to check that
\[ \frac{\hbar^2}{2m} \left( -\frac{\partial^2}{\partial x_1^2} + \frac{n^2 - 1/4}{x_1^2} + \frac{m^2\omega^2x_1^2}{\hbar^2} \right) \langle x_1 \mid \exp(-iHt/\hbar) \mid x_2 \rangle = \imath\hbar\frac{\partial}{\partial t} \langle x_1 \mid \exp(-iHt/\hbar) \mid x_2 \rangle \] (21)
upon using (19) for $\langle x_1 \mid \exp(-iHt/\hbar) \mid x_2 \rangle$. These properties guarantee the time evolution of a wave function $\psi(x, t)$ defined on the half-line is correctly given by $\psi(x_1, t) = \int_0^\infty \langle x_1 \mid \exp(-iHt/\hbar) \mid x_2 \rangle \psi(x_2, 0) dx_2$ [16].

**Appendix: Other Identities**

There are many other $SL(2, \mathbb{C})$ “roads” that lead to a closed form expression for the oscillator propagator (i.e. the “Rome” of this problem). Here we list a few more of them involving the operators $x$ and $p$ with $[x, p] = \imath\hbar$. For simplicity,
we omit the $\lambda/x^2$ potential, but the identities to follow are valid even with that potential term upon substituting $p^2 \rightarrow p^2 + \lambda/x^2$.

In addition to the identity in the main text, (1), there are many relations paired by conjugation plus $t \rightarrow -t$. For example,

\[
\exp\left(-\frac{i t}{\hbar} \left(\frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 \right)\right) = \exp\left(-i \alpha x^2\right) \exp\left(-i \gamma (xp + px)\right) \exp\left(-i \beta p^2\right) \tag{A1a}
\]

with $e^{2\gamma h} = \cos(\omega t)$, $\alpha = \frac{m \omega}{2 \hbar} \tan(\omega t)$, $\beta = \frac{1}{2\hbar m \omega} \tan(\omega t)$;

\[
\exp\left(-\frac{i t}{\hbar} \left(\frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 \right)\right) = \exp\left(-i \beta p^2\right) \exp\left(-i \gamma (xp + px)\right) \exp\left(-i \alpha x^2\right) \tag{A1b}
\]

with $e^{-2\gamma h} = \cos(\omega t)$, $\alpha = \frac{m \omega}{2 \hbar} \tan(\omega t)$, $\beta = \frac{1}{2\hbar m \omega} \tan(\omega t)$;

\[
\exp\left(-\frac{i t}{\hbar} \left(\frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 \right)\right) = \exp\left(-i \alpha x^2\right) \exp\left(-i \gamma (xp + px)\right) \exp\left(-i \beta p^2\right) \tag{A2a}
\]

with $e^{2\gamma h} = \cos(\omega t)$, $\alpha = \frac{m \omega}{2 \hbar} \sin(\omega t) \cos(\omega t)$, $\beta = \frac{1}{2\hbar m \omega} \tan(\omega t)$;

\[
\exp\left(-\frac{i t}{\hbar} \left(\frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 \right)\right) = \exp\left(-i \gamma (xp + px)\right) \exp\left(-i \alpha x^2\right) \exp\left(-i \beta p^2\right) \tag{A2b}
\]

with $e^{-2\gamma h} = \cos(\omega t)$, $\alpha = \frac{m \omega}{2 \hbar} \sin(\omega t) \cos(\omega t)$, $\beta = \frac{1}{2\hbar m \omega} \tan(\omega t)$;

\[
\exp\left(-\frac{i t}{\hbar} \left(\frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 \right)\right) = \exp\left(-i \alpha x^2\right) \exp\left(-i \gamma (xp + px)\right) \exp\left(-i \beta p^2\right) \tag{A3a}
\]

with $e^{2\gamma h} = \cos(\omega t)$, $\alpha = \frac{m \omega}{2 \hbar} \tan(\omega t)$, $\beta = \frac{\sin(\omega t) \cos(\omega t)}{2\hbar m \omega}$;

\[
\exp\left(-\frac{i t}{\hbar} \left(\frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 \right)\right) = \exp\left(-i \gamma (xp + px)\right) \exp\left(-i \alpha x^2\right) \exp\left(-i \beta p^2\right) \tag{A3b}
\]

with $e^{-2\gamma h} = \cos(\omega t)$, $\alpha = \frac{m \omega}{2 \hbar} \tan(\omega t)$, $\beta = \frac{\sin(\omega t) \cos(\omega t)}{2\hbar m \omega}$.
Lie Groups and Propagators Exemplified

These identities immediately lead to the propagator (12), or (19) after restoration of the $\lambda/x^2$ term, upon taking into account the effects of $\exp(-i\gamma(xp + px))$ to rescale position eigenstates. Namely,

$$\exp(-i\gamma(xp + px)) |x_2\rangle = \exp(i\gamma) |x_2\rangle \exp(2i\gamma)$$

(A4)

etc.

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[9] The identity (1) is sometimes called “van Kortryk’s $SU(1,1)$ Lie-group identity” but similar results have appeared from time to time in the physics literature since the 1950s. In particular, see M. Kolsrud (1956) Exact Quantum Dynamical Solutions for Oscillator-Like Systems. Phys. Rev. 104 1186-1188. Kolsrud’s work would probably have received more of the credit that it deserved if only the author had used the word “propagator” at some point in his paper.

[10] As another exercise, the reader is invited to find parameter-dependent matrix realizations of the $SL(2, C)$ group elements and use them to establish the identity (1). Hint: The $sl(2, C)$ algebra (2) is also realized by

$$x^2 \rightarrow \begin{pmatrix} 0 & 2\hbar/\lambda \\ 0 & 0 \end{pmatrix}, \quad p^2 + \frac{\lambda}{x^2} \rightarrow \begin{pmatrix} 0 & 0 \\ 2\hbar \lambda & 0 \end{pmatrix}, \quad xp + px \rightarrow \begin{pmatrix} -2\hbar & 0 \\ 0 & 2\hbar \end{pmatrix}$$

for any $\lambda$. One may now use copies of this realization for various different $\lambda$ including the $\lambda = 1$ matrices of (4), to build larger matrix realizations of the algebra and the group elements.

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For any integer $n$ (including negative values) upon replacing $x \rightarrow r$ the reader may recognize this as the radial Schrödinger equation for a free particle on the plane, with angular momentum $\hbar n$. Similarly, setting $\lambda = \hbar^2 (l + 1)$ for integer $l \geq 0$ would produce the free particle radial equation in three dimensions. So transcribed, the results in this paper may have application to a variety of physical problems: See: S.A. Coon, B.R. Holstein (2002) Anomalies in quantum mechanics: The $1/r^2$ potential. *Am. J. Phys.* **70** 513-519.

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Mind the pestiferous phases of the trigonometric functions in (12) and (19) for $t > 2\pi/\omega$. See:

P.A. Horváthy (1979) Extended Feynman Formula for Harmonic Oscillator. *Int. J. Theor. Phys.* **18** 245-250;

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Also see [3], as well as Sections 3.2 and 5.2 in [15].