Quantum group symmetry and particle scattering in (2+1)-dimensional quantum gravity

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Abstract

Starting with the Chern-Simons formulation of (2+1)-dimensional gravity we show that the gravitational interactions deform the Poincaré symmetry of flat space-time to a quantum group symmetry. The relevant quantum group is the quantum double of the universal cover of the (2+1)-dimensional Lorentz group, or Lorentz double for short. We construct the Hilbert space of two gravitating particles and use the universal R-matrix of the Lorentz double to derive a general expression for the scattering cross section of gravitating particles with spin. In appropriate limits our formula reproduces the semi-classical scattering formulae found by ’t Hooft, Deser, Jackiw and de Sousa Gerbert.

1 Introduction

The extensive literature on (2+1)-dimensional gravity (see [1] for a bibliography) includes numerous hints that quantum groups should play a pivotal role in a satisfactory formulation of the quantised theory. Physically, an important hint comes from the role of the braid group in the classical interaction of gravitating point particles in two spatial dimensions, see e.g. [2]. Since quantum groups yield representations of the braid
group, this suggest a natural place for quantum groups in the quantised theory. At the mathematical level, the possibility of formulating (2+1)-dimensional gravity as a Chern-Simons theory gives a strong clue. The quantisation of Chern-Simons theory with compact gauge groups is a long and beautiful chapter of mathematical physics (not yet closed) in which quantum groups are central protagonists.

The goal of this paper is to establish the central role of a certain quantum group in (2+1)-dimensional quantum gravity. We show that the quantum double of the Lorentz group in 2+1 dimensions (or Lorentz double for short) arises automatically in the quantisation of (2+1)-dimensional gravity with vanishing cosmological constant and demonstrate that it provides a powerful new tool for analysing interesting physics. In this paper we focus on the gravitational scattering of particles. We show that in the quantum theory gravitating particles carry representations of the Lorentz double and that, as a result, the multi-particle Hilbert space carries a representation of the braid group. In a rather precise sense, gravitating particles in two spatial dimensions may thus be thought of as gravitational non-abelian anyons. We show that the scattering is determined by the representation of the braid group.

In some ways this paper is a continuation of [3], where the Lorentz double was first introduced and its role in understanding (2+1)-dimensional gravity was pointed out. Two questions were raised but not answered in that paper. The first concerns the role of the Lorentz double in a deformation quantisation of the classical gravitational phase space. That question was addressed in the Euclidean setting in [4] and here we will be able to adapt the results of that paper to the Lorentzian situation without much difficulty. The second, physically deeper, question concerns the reconstruction of space-time physics (such as scattering) from the rather abstract representation theory of the Lorentz double. This is the main topic of the current paper, which is organised as follows.

We begin in the next section by comparing the interactions of charge-flux composites with those of gravitating particles in 2+1 dimensions, pointing out the topological nature of both. The purpose of that section is to motivate our approach to (2+1)-dimensional quantum gravity and to outline, without much technical detail, some of the gravitational phenomena we hope to explain using quantum groups. The technical development begins in Sect. 3. We review the representation theory of the Poincaré group in 2+1 dimensions and introduce the basic notion of ribbon Hopf algebras. We show how they provide the natural language for discussing spin and statistics of particles in 2+1 dimensions. Using and generalising results of [5], [6] and [7] we give a relativistic treatment of the general Aharonov-Bohm scattering of particles which carry representations of ribbon Hopf algebras. The level of exposition in this section is quite technical and detailed since it provides the basis of our analysis of quantised gravitational interactions in 2+1 dimensions. An important aspect of our treatment is that it takes places entirely in momentum space. In sect. 4 we briefly review the Chern-Simons formulation of (2+1)-dimensional gravity with a particular emphasis on the inclusion of particles. Following the approach pioneered by Fock and Rosly [8] we sketch how the the Poisson structure of the phase space of (2+1)-dimensional gravity can be described in terms of a classical $r$-matrix. Our approach to the quantisation of the phase space is motivated
by the combinatorial quantisation programme developed in [9], [10] and [11] for Chern-Simons theories with compact gauge groups. An important step in that quantisation procedure is the identification of a quantum $R$-matrix (solution of the quantum Yang-Baxter equation) which reduces to the classical $r$-matrix of the Fock-Rosly description in the classical limit. In sect. 5 we introduce the Lorentz double and show that its $R$-matrix has the desired limit. We do not fully work out the combinatorial quantisation programme for (2+1)-dimensional gravity, which requires further technical analysis.

Having identified the Lorentz double as a central ingredient in the quantisation we use it instead as a tool for addressing the physical issues raised in sect. 2. In sect. 6 we construct the theory of gravitationally interacting particles in analogy with the theory of relativistic anyons developed in sect. 3. This analogy is both very powerful and subtle: ordinary anyons carry representations of both the Poincaré group and a ribbon Hopf algebra, but our gravitational anyons only carry representations of the Lorentz double.

The Lorentz double thus plays a dual role, determining both the space-time properties and the braiding of gravitating particles in 2+1 dimensions. To illustrate and test the theory we construct the Hilbert space of two massive particles with arbitrary spins and compute the scattering cross sections. In sect. 7 we discuss our results and draw some conclusions.

Throughout the paper we use units in which the speed of light is 1. We mostly set $\hbar = 1$ and in the gravitational calculations we use the fact that Newton’s constant in (2+1)-dimensional gravity has the dimensions of inverse mass to measure masses in units of $(8\pi G)^{-1}$. The Planck length in 2+1 dimensions is $\ell_P = hG$. Whenever the interplay of the constants $\hbar$ and $G$ in our calculations seemed interesting we have explicitly included them.

## 2 Topological interactions in 2+1 dimensions

### 2.1 An algebraic approach to Aharonov-Bohm scattering

The best known manifestation of non-trivial topological interactions in a physical experiment is the Aharonov-Bohm interaction between an electrically charged particle and a magnetic flux tube. More generally, such interactions occur between flux-charge composites $(\Phi, q)$. Here $\Phi \in [0, 2\pi)$ labels the fractional part of the magnetic flux and $q$ is the integer electric charge. The point of the following brief review is to show how the familiar Aharonov-Bohm interaction of such composites can be understood from an algebraic perspective. While some of the algebraic language may seem unnecessarily complicated in this simple example it will help to highlight those aspects which generalise to the gravitational interaction.

The starting point of the algebraic interpretation is the interpretation of the label $(\Phi, q)$ as the representation label of the “double”

$$D = \mathbb{Z} \times U(1).$$ (2.1)

For what we are about to say we should really replace $U(1)$ by its group algebra, but to keep the discussion as simple as possible we think of $D$ simply as an abelian group. We
write its elements as \((n, \omega), n \in \mathbb{Z} \text{ and } \omega \in [0, 2\pi]\), and the group composition is simply

\[(n_1, \omega_1)(n_2, \omega_2) = (n_1 + n_2, \omega_1 + \omega_2), \quad (2.2)\]

where the addition of the angles should be taken modulo \(2\pi\). Irreducible representations (irreps) of \(D\) are simply tensor products of the one dimensional irreps of \(\mathbb{Z}\) and of \(U(1)\). They are therefore labelled by a angle \(\Phi \in [0, 2\pi]\) and an integer \(q\), which we interpret physically as flux and charge. Writing \(W_{(\Phi,q)}\) for the carrier space of the irrep \((\Phi, q)\), the action of \((n, \omega) \in D\) on \(\phi \in W_{(\Phi,q)}\) is

\[r_{(\Phi,q)}(n, \omega)\phi = e^{i(\Phi n + q\omega)}\phi. \quad (2.3)\]

Physically, one of the most basic observed properties of flux-charge composites are the rules for combining flux-charge composites, often called fusion rules. These are determined by the decomposition of the tensor product into irreps, which in turn is ruled by the co-multiplication of \(D\). This is a map

\[\Delta : D \rightarrow D \otimes D, \quad (2.4)\]

allowing elements of \(D\) to act on \(W_{(\Phi_1,q_1)} \otimes W_{(\Phi_2,q_2)}\). In the present situation, the co-multiplication is so simple (often called group-like) that it is not usually mentioned explicitly:

\[\Delta((n, \omega)) = (n, \omega) \otimes (n, \omega). \quad (2.5)\]

It follows that, at the level of representations,

\[(\Phi_1, q_1) \otimes (\Phi_2, q_2) \simeq (\Phi_1 + \Phi_2, q_1 + q_2) \quad (2.6)\]

showing that fluxes and charges simply add when flux-charge composites are fused.

One of the physically most striking and interesting properties of charge-flux composites is that they can carry fractional spin and that they undergo non-trivial Aharonov-Bohm scattering. To see how this is encoded in the algebraic structure of \(D\) we define the element

\[c = \frac{1}{2\pi} \int d\omega \sum_{n \in \mathbb{Z}} e^{-i\omega n}(n, \omega). \quad (2.7)\]

(In sect. 5 we will see how to make sense of the infinite sum.) Acting with \(c\) on \(\phi \in W_{(\Phi,q)}\) we find

\[r_{(\Phi,q)}(c)\phi = e^{i\Phi q}\phi. \quad (2.8)\]

The eigenvalue \(e^{i\Phi q}\) is precisely the “spin factor” associated to a flux-charge composite. It determines the fractional part of the spin \(s\) carried by the flux-charge composite \((\Phi, q)\) according to the rule

\[e^{2\pi is} = e^{i\Phi q}. \quad (2.9)\]

4
The final ingredient for discussing statistical properties algebraically is the algebraic equivalent of the monodromy operation, which transports one flux-charge composite around each other. This is implemented by the element \( Q \in D \otimes D \) given by

\[
Q = \frac{1}{(2\pi)^2} \sum_n \sum_{n'} \int d\omega \int d\omega' \ e^{-i(\omega n + \omega' n')}(n, \omega') \otimes (n', \omega).
\]  (2.10)

The element \( c \) and the monodromy operation operator \( Q \) are linked by the important relation

\[
\Delta c = Q \otimes c.
\]  (2.11)

This relation, called the generalised spin-statistics relation, shows that our interpretation of \( c \) and \( Q \) is consistent: rotating a system of two flux-charge composites by \( 2\pi \) is the same as rotating the composites individually by \( 2\pi \) and transporting the composites around each other.

Now consider two flux-charge composites \((\Phi_1, q_1)\) and \((\Phi_2, q_2)\) with spins \( s_1 \) and \( s_2 \) both obeying the condition (2.9). By (2.6), the total flux and charge of the combined system is \((\Phi, q) = (\Phi_1 + \Phi_2, q_1 + q_2)\). It follows from (2.11) (and also by direct computation) that the action on an element \( \phi_1 \otimes \phi_2 \in W_{(\Phi_1, q_1)} \otimes W_{(\Phi_2, q_2)} \) is

\[
r_{(\Phi_1, q_1)} \otimes r_{(\Phi_2, q_2)}(Q) \phi_1 \otimes \phi_2 = e^{i(\Phi_1 q_2 + \Phi_2 q_1)} \phi_1 \otimes \phi_2.
\]  (2.12)

If we impose the spin quantisation rule (2.9) also on the spin \( s \) of the combined system it follows that

\[
e^{2\pi i(s - s_1 - s_2)} = e^{i(\Phi_1 q_2 + \Phi_2 q_1)}.
\]  (2.13)

However, \( s - s_1 - s_2 \) is just the orbital angular momentum \( l \) of the relative motion of the two flux-charge composites in the centre of mass frame. We therefore conclude that the orbital angular momentum for the relative motion of flux-charge composites \((\Phi_1, q_1)\) and \((\Phi_2, q_2)\) has to satisfy the quantisation condition

\[
e^{2\pi il} = e^{i(\Phi_1 q_2 + \Phi_2 q_1)},
\]  (2.14)

which implies

\[
l = n + \frac{1}{2\pi} (\Phi_1 q_2 + \Phi_2 q_1).
\]  (2.15)

The important consequence of the spin condition (2.9) and the spin-statistics relation (2.11) is thus that not only the individual spins of flux/charge composites but also the orbital angular momentum of their relative motion has a fractional part. It is this fractional part which is responsible for their non-trivial Aharonov-Bohm scattering.

A careful discussion of the phase-shift analysis of the standard Aharonov-Bohm scattering problem, where a beam of electrons of charge \( q_1 = e \) is scattered off a tube carrying magnetic flux \( \Phi_2 = \Phi \) can be found in [12]. The non-integer eigenvalues \( l = n + \frac{e\Phi}{2\pi} \) of the orbital angular momentum are a basic but crucial input to the discussion. The phase
shift in the \( n \)-the partial wave, or equivalently the restriction of the \( S \)-matrix to the \( n \)-the partial wave depends on \( \frac{e^2}{2\pi} \) and on the sign of \( \lfloor l \rfloor \), the largest integer \( \geq l \), \([12]\):

\[
S^{(l)} = e^{2i\delta} = \begin{cases} 
  e^{-\frac{i}{2}\Phi} & \text{if } \lfloor l \rfloor \geq 0 \\
  e^{\frac{i}{2}\Phi} & \text{if } \lfloor l \rfloor < 0.
\end{cases}
\]  

(2.16)

The resulting normalised cross section in the frame of the flux tube is the famous Aharonov-Bohm cross section

\[
\frac{d\sigma}{d\varphi}(\varphi) = \frac{\hbar}{2\pi M_e v} \sin^2 \left( \frac{e^2}{2} \right) \sin^2 \frac{\varphi}{2}
\]

(2.17)

where \( M_e \) and \( v \) are the mass and the speed of the incident electrons and \( \varphi \) is the scattering angle.

### 2.2 Gravitational scattering in 2+1 dimensions

One reason why gravity in 2+1 dimensions is so much simpler than in 3+1 dimensions is that the Ricci tensor has the same number of independent components as the Riemann tensor. Thus the curvature of space-time is entirely determined, via the Einstein equations, by the energy-momentum tensor. The simplest situation which illustrates this fact, and moreover gives the first hint that what we have summarised in the preceding sections is indeed relevant to (2+1)-dimensional gravity, is the motion of a test particle in the gravitational field of a point particle at rest.

In a local coordinate system with coordinate indices \( a, b = 0, 1, 2 \), and in units where the speed of light is 1 the Einstein equations read

\[
R_{ab} - \frac{1}{2}g_{ab}R = 8\pi GT_{ab}.
\]  

(2.18)

Here \( R_{ab} \) is the Ricci tensor, \( R \) is the Ricci scalar and \( T_{ab} \) is the energy-momentum tensor. As mentioned in the introduction, Newton’s constant \( G \) has the dimension of inverse mass in 2+1 dimensions. For a particle of mass \( M \) we can therefore define the dimensionless quantity

\[
\mu = 8\pi GM,
\]  

(2.19)

which measures the mass in units of \( 1/(8\pi G) \). Suppose a point particle of mass \( M \) is at rest at the origin of a local coordinate system \((x_0, x_1, x_2)\). Einstein’s equation for this situation are

\[
R_{ab} - \frac{1}{2}g_{ab}R = \mu \delta(x_1)\delta(x_2).
\]  

(2.20)

They were first solved in \([13]\). The space-time surrounding the particle is flat, but has a conical singularity at the particle’s position. The cone’s deficit angle is determined by the mass. In terms of polar coordinates \((\rho, \varphi) \in \mathbb{R}^+ \times [0, 2\pi - \mu)\) the spatial line element is just

\[
 ds^2 = (d\rho)^2 + \rho^2 (d\varphi)^2.
\]  

(2.21)
For some purposes another coordinate system \((r, \theta)\) used in \([15]\) is more useful. It describes the cone embedded in flat three-dimensional space in terms of its projection parallel to its axis of symmetry. Defining the rescaling parameter

\[
\alpha = 1 - \frac{\mu}{2\pi},
\]

the projected coordinate are \((r, \theta)\), where \(r = \alpha \rho\) and \(\theta = \varphi/\alpha\), which has the usual angular range \(\theta \in [0, 2\pi)\). The line element now has the form

\[
ds^2 = \alpha^{-2}(dr^2) + r^2(d\theta)^2.
\]

(2.23)

Geodesics are most easily visualised by cutting the cone open, unfolding it and drawing straight lines. Upon gluing the cone back together and looking at the geodesics in the projected coordinates system \((r, \theta)\) one finds that some of the geodesics cross. One can interpret the result by saying that parallel geodesics get deflected by an angle \(\pm \mu/2\alpha\), with the sign only depending on whether they have passed the cone’s apex on the left or on the right but not on the geodesic’s distance from the apex. Such distance-independence is typical of topological interactions and reminiscent of the phase shifts in Aharonov-Bohm scattering. The analogy can be made more convincing by considering the corresponding quantum problem. This was first done in \([14]\) and \([15]\). In both papers the mathematical model is the Schrödinger equation on the cone. For stationary scattering states with energy \(E = \hbar^2 k^2/2M\) this becomes

\[
\left( \left( r \frac{\partial}{\partial r} \right)^2 + \left( \frac{1}{\alpha} \frac{\partial}{\partial \theta} \right)^2 + \left( \frac{k r}{\alpha} \right)^2 \right) \psi = 0.
\]

(2.24)

The basic but crucial input in the partial wave analysis of the scattering problem in \([15]\) is the observation that the orbital angular momentum operator \(-i\frac{1}{\alpha} \frac{\partial}{\partial \theta}\) has non-integer eigenvalues

\[
l = \frac{n}{\alpha}.
\]

(2.25)

This is responsible for non-trivial phase shifts and hence a non-trivial \(S\)-matrix:

\[
S^{(l)} = e^{2i\delta_l} = \begin{cases} e^{-i\mu} & \text{if } [l] \geq 0 \\ e^{i\mu/2} & \text{if } [l] < 0. \end{cases}
\]

(2.26)

While this formula is reminiscent of the Aharonov-Bohm phase shifts (2.16), there are important differences, most notably the increase of the phase shift with the magnitude of the angular momentum \(l\) in (2.26). In the rest of this paper we will show that the gravitational phase shifts nevertheless have an interpretation in terms of spin and statistics, much like the Aharonov-Bohm phase shifts. We begin by looking at a generalisation of the Aharonov-Bohm effect from a more abstract, algebraic perspective.
3 Spin sum rules and relativistic scattering of particles obeying braid statistics

3.1 The Hilbert space of a single relativistic anyon

The Aharonov-Bohm story generalises naturally to particles in 2+1 dimensions carrying representations of ribbon Hopf algebras. We are interested in relativistic particles, and since the consequences of relativistic invariance are crucial in this paper, we briefly review them.

Write $\text{Mink}_3$ for three-dimensional Minkowski space and denote its elements by $x = (x_0, x_1, x_2)$ and $y = (y_0, y_1, y_2)$. The inner product is

$$\eta(x, y) = x_0y_0 - x_1y_1 - x_2y_2.$$  \hfill (3.1)

The group of linear transformations on $\text{Mink}_3$ leaving $\eta$ invariant is the group $O(2,1)$ which has four connected components. The Lorentz group (sometimes called the proper Lorentz group) is the subgroup $SO(2,1)$ of $O(2,1)$ transformations of determinant one. It still has two components, one which preserves the direction of time and one which reverses it. The component containing the identity consists of those $SO(2,1)$ transformations which preserve the direction of time. It is called the orthochronous Lorentz group and we denote it by $L_3^\uparrow$. The orthochronous Lorentz group is infinitely connected. Its double cover $SU(1,1)$ and its universal cover $\tilde{L}_3^\uparrow$ are described in detail in appendix A. We denote elements of $\tilde{L}_3^\uparrow$ by $u, v, w, x...$ and the $L_3^\uparrow$-element associated to $u \in \tilde{L}_3^\uparrow$ by $\Lambda(u)$.

The group of affine transformations on $\text{Mink}_3$ leaving $\eta$ invariant is the semi-direct product $\mathbb{R}^3 \rtimes O(2,1)$ of translations and $O(2,1)$ transformations. For us the identity component of that group is particularly relevant. This is the orthochronous Poincaré group

$$P_3^\uparrow = \mathbb{R}^3 \rtimes L_3^\uparrow;$$  \hfill (3.2)

which we shall often simply refer to as the Poincaré group. The multiplication law for two elements $(a_1, L_1), (a_2, L_2) \in P_3^\uparrow$ is

$$(a_1, L_1)(a_2, L_2) = (a_1 + L_1a_2, L_1L_2).$$  \hfill (3.3)

The universal cover of the orthochronous Poincaré group is

$$\tilde{P}_3^\uparrow = \mathbb{R}^3 \rtimes \tilde{L}_3^\uparrow;$$  \hfill (3.4)

For two elements $(a_1, u_1), (a_2, u_2) \in \tilde{P}_3^\uparrow$ the multiplication rule is

$$(a_1, u_1)(a_2, u_2) = (a_1 + \Lambda(u_1)a_2, u_1u_2).$$  \hfill (3.5)

Following the notation used in the literature on (2+1)-dimensional gravity we denote the Lie algebra of $L_3^\uparrow$ by $so(2,1)$. We denote the Lie algebra of $P_3^\uparrow$ by $iso(2,1)$ and introduce generators $P_a, J_a, a = 0, 1, 2$, satisfying the commutation relations

$$[P_a, P_b] = 0, \quad [J_a, J_b] = \epsilon_{abc}J^c, \quad [J_a, P_b] = \epsilon_{abc}P^c,$$  \hfill (3.6)
where $\epsilon_{abc}$ is the totally antisymmetric tensor in three dimensions, normalised so that $\epsilon_{012} = 1$. The elements $P_0$, $P_1$ and $P_2$ generate translations in time and space, $J_0$ generates spatial rotations and $J_1$ and $J_2$ generate boosts. In the context of (2+1)-dimensional gravity it is important that we can interpret the Lie algebra $so(2, 1)$ as momentum space by associating to a momentum $p = (p_0, p_1, p_2)$ the element $p_0 J_0 + p_1 J_1 + p_2 J_2 \in so(2, 1)$. The Lorentz group acts on this momentum space via the adjoint representation, leaving the inner product $p_2 = \eta(p, p)$ invariant. Thus $J_0$ is also the generator of rotations in momentum space, but with our conventions $[J_0, J_1] = -J_2$ so $J_0$ generates a clockwise rotation in the $p_1p_2$ plane. $J_1$ is the generator of boosts along the 2-axis (because $[J_1, J_0] = J_2$) and $J_2$ is the generator of boosts along the negative 1-axis (because $[J_2, J_0] = -J_1$). For more details on our conventions we refer the reader to appendix A.

According to Wigner’s dictum the fundamental properties mass and spin of a particle in (2+1)-dimensional Minkowski space label irreps of the universal cover of the (2+1)-dimensional Poincaré group. The irreps of the (2+1)-dimensional Poincaré group are described in [16] and the representation theory of the universal cover is studied in some detail in [17]. Some are not physically relevant, but those which are, are indeed labelled by a positive and real parameter $M$, interpreted as the particle’s mass, and another real parameter $s$ interpreted as the particle’s spin. We denote these irreps by $\pi_{Ms}$. Geometrically, $M$ labels an orbit of $\tilde{L}_3^\uparrow$ acting on momentum space $(\mathbb{R}^*)^3$. This orbit is obtained by boosting the energy-momentum vector $(M, 0, 0)$ of a particle at rest, thus sweeping out the mass hyperboloid

$$H_M = \{ p \in (\mathbb{R}^*)^3 \mid p = L(M, 0, 0)^t, L \in \tilde{L}_3^\uparrow \}. \tag{3.7}$$

The mass hyperboloid consists of all momenta $p$ satisfying $p^2 = M^2$ and $p_0 > 0$. Geometrically, it can be identified with the coset $\tilde{L}_3^\uparrow/SO(2) = \tilde{L}_3^\uparrow/\mathbb{R}$. Using the parametrisation (A.14) of Lorentz transformations we can therefore write elements of $H_M$ in terms of a boost parameter $\vartheta$ and an angle $\varphi$ as

$$p = (M \cosh \vartheta, M \sinh \vartheta \cos \varphi, M \sinh \vartheta \sin \varphi)^t. \tag{3.8}$$

The spin $s$ labels an irrep of the centraliser group of the reference momentum $(M, 0, 0)$ on the mass hyperboloid. The centraliser group is the universal cover of the two-dimensional rotation group $\tilde{SO}(2) \cong \mathbb{R}$. Hence the spin $s$ takes arbitrary real values in the representation theory of the universal cover of the Poincaré group. In the representation theory of the Poincaré group itself the spin is an integer. The irreps $\pi_{Ms}$ of $\tilde{P}_3^\uparrow$ can be described in two equivalent ways. Both require a Lorentz invariant measure $dm(p)$ on the mass hyperboloid (3.7). Using the parametrisation (3.8) we can write that measure in terms of the boost parameter $\vartheta \in \mathbb{R}$ and the angle $\varphi \in [0, 2\pi)$ as

$$dm(p) = \frac{M}{8\pi^2} \sinh \vartheta d\vartheta d\varphi. \tag{3.9}$$

As we shall see, the normalisation of the measure is motivated by the usual normalisation of scattering states in relativistic quantum theory. The irreps of the universal cover of
the Poincaré group can be described either in terms of equivariant functions on $\tilde{L}^\uparrow_3$ or in terms of functions on $H_M$ on which Poincaré transformation act via a multiplier representation \cite{10}. For us both points of view are important.

First we consider equivariant functions and define

$$V_{Ms} = \{ \phi: \tilde{L}^\uparrow_3 \to \mathbb{C} | \phi(xr(\psi)) = e^{-i\psi s}\phi(x) \ \forall \psi \in \mathbb{R}, x \in \tilde{L}^\uparrow_3, \int_{H_M} |\phi(x)|^2 \, dm(p) < \infty \}, \quad (3.10)$$

where $r(\psi)$ is the anti-clockwise (mathematically positive) spatial rotation by $\psi$ as defined in \cite{A.18}. Note that integrating $|\phi(x)|^2$ with respect to $dm(p)$ makes sense since it only depends on the projection $p = \Lambda(x)(M,0,0)^t$. Elements $(a,u) \in \tilde{P}^\uparrow_3$ act on this space via

$$\pi_{Ms}(a,u)\phi(x) = \exp(i a \cdot p)\phi(u^{-1}x). \quad (3.11)$$

where $p$ is again the momentum on the mass shell $H_M$ associated to $x \in \tilde{L}^\uparrow_3$ via the projection $p = \Lambda(x)(M,0,0)^t$.

Alternatively, we can describe irreps in terms of the Hilbert space $L^2(H_M, dm)$ of square integrable functions on the mass hyperboloid. The action of $\rho_{Ms}(a,u)$ on an element $\Psi \in L^2(H_M, dm)$ is

$$\rho_{Ms}(a,u)\Psi(p) = \exp(i a \cdot p)\exp(is\psi(u,p))\Psi(\Lambda(u^{-1})p), \quad (3.12)$$

where the phase factor $\exp(is\psi(u,p))$ encodes the spin of the representation $\rho_{Ms}$ and is called the multiplier. To write down an explicit formula for the multiplier and to establish an isomorphism between the representations $\pi_{Ms}$ and $\rho_{Ms}$ one requires a measurable map

$$s: H_M \to \tilde{L}^\uparrow_3 \quad (3.13)$$

satisfying $\Lambda(s(p))(M,0,0)^t = p$, see \cite{10} for details. Geometrically, $s$ is a section of the bundle $\tilde{L}^\uparrow_3 \to H_M$. The phase $\psi(u,p)$ entering the multiplier is defined in terms of $s$:

$$r(\psi(u,p)) = (s(p)^{-1}u s(\Lambda(u^{-1})p)). \quad (3.14)$$

Having provided the general framework for space-time characteristics of particles we turn to the generalisation of the internal charges $(\Phi, q)$ encountered in the Aharonov-Bohm effect. This requires some Hopf algebra technology. We refer the reader to \cite{18} for a full definition of a ribbon Hopf algebra. For our purposes it is sufficient to know that a ribbon Hopf algebra $\mathcal{A}$ combines and generalises the algebraic elements which entered the discussion of flux/charge composites. Thus there is a multiplication rule

$$m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \quad (3.15)$$

and a co-multiplication

$$\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \quad (3.16)$$
which dictates the way tensor products of representations are decomposed. Ribbon Hopf algebras are in particular quasi-triangular Hopf algebras, which means that there exists an invertible element $R \in A \otimes A$, called the universal $R$-matrix, which obeys the quantum Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$  (3.17)

Here $R_{12} = R \otimes 1 \in A \otimes A \otimes A$ and $R_{13}$ and $R_{23}$ are defined similarly. Finally, ribbon Hopf algebras have a special central element $c$ which together with $R$ obeys the generalised spin-statistics relation

$$\Delta c = (R_{21} R) c \otimes c.$$  (3.18)

The element $R_{21} \in A \otimes A$ is obtained from $R$ by changing the order of the tensor product. If $R = \sum_i a_i \otimes b_i$ then $R_{21} = \sum_i b_i \otimes a_i$. The product $R_{21} R$ is the monodromy operator. We introduce the abbreviation

$$Q := R_{21} R.$$  (3.19)

Consider now a particle in two spatial dimensions which is charged with respect to a ribbon Hopf algebra $\mathcal{A}$. By this we mean that it carries an irrep $r_A$ of $\mathcal{A}$. The internal state of the particle is an element of the carrier space $W_A$ of $r_A$ and the kinematic state of the particle is an element of an irrep $V_{Ms}$ of $\tilde{P}_3^\dagger$ labelled by its mass $M$ and its spin $s$. We have kept the following discussion general but restrict attention to unitary irreps and assume that tensor products of representations are reducible into irreps.

The central ribbon element $c$ of the internal symmetry $\mathcal{A}$ acts on $W_A$ via a phase:

$$r_A(c) W_A = e^{2\pi i s_A} W_A.$$  (3.20)

The Poincaré group $\tilde{P}_3^\dagger$ has a centre isomorphic to $\mathbb{Z}$, generated by the $2\pi$-rotation, which we denote by $\Omega$. Being central, $\Omega$, too, acts by a phase:

$$\pi_{Ms}(\Omega) V_{Ms} = e^{2\pi i s} V_{Ms}.$$  (3.21)

The physically allowed states of the particle are determined by demanding the spin quantisation condition

$$e^{2\pi i s_A} = e^{2\pi i s}$$  (3.22)

i.e. $s = s_A + n$, where $n \in \mathbb{Z}$. There is no a priori restriction on the value of $s_A$, showing that particles which carry representations of ribbon Hopf algebra can have arbitrary spin. In the following we refer to them as anyons. Thus, the one-anyon Hilbert spaces are tensor products of the form $V_{Ms} \otimes W_A$ for which $\Omega$ and $c$ give the same phase. In symbols:

$$\mathcal{H}_1 = V_{Ms} \otimes W_A \text{ provided } \pi_{Ms}(\Omega) \phi \otimes v = \phi \otimes r_A(c) v \quad \forall \phi \otimes v \in V_{Ms} \otimes W_A.$$  (3.23)
3.2 The multi-anyon Hilbert space

Now consider \( n \) anyons, each carrying an irrep of the internal symmetry and having masses \( M_i \) and spins \( s_i, i = 1, \ldots, n \), all satisfying the condition (3.22). The spin sum rules for anyons were analysed in the more general framework of algebraic quantum field theory in [5], the multi-anyon Hilbert space was constructed for abelian braiding in [6] and that construction was extended to non-abelian braiding in [7] and [19]. The upshot of these studies is that the multi-particle Hilbert space is not isomorphic to the tensor product of single anyon Hilbert spaces. Instead the \( n \)-anyon space can be described as the space of \( L^2 \)-sections of a certain vector bundle over the space of \( n \) distinct velocities. Let \( q_i = p_i / M_i \) so that for all \( i = 1, \ldots, n \) \( q_i \) lies on the unit mass hyperboloid \( q_i \in M_{1}^{n} \), and define

\[
\mathcal{M}_n = \{(q_1, \ldots, q_n) \in M_{1}^{n} | q_i \neq q_j \quad \text{for} \quad i \neq j \}. \tag{3.24}
\]

This is the configuration space of \( n \) distinct velocities of \( n \) non-identical particles. For identical particles one obtains the corresponding configuration space after dividing by the action of the symmetric group:

\[
\mathcal{N}_n = \mathcal{M}_n / S_n. \tag{3.25}
\]

Both spaces are multiply connected. For identical particles the fundamental group \( \pi_1(\mathcal{N}_n) = B_n \) is the braid group on \( n \) strands, and for non-identical particles the fundamental group \( \pi_1(\mathcal{M}_n) = PB_n \) is the pure braid group on \( n \) strands [18]. Hence there are natural flat \( PB_n (B_n) \) bundles over \( \mathcal{M}_n (\mathcal{N}_n) \). The result of [6], [7] and [19] is that \( n \)-anyon Hilbert spaces are the spaces of \( L^2 \)-sections of associated vector bundles. For our purposes it is useful to recast the construction in the language of ribbon Hopf algebras. For simplicity we restrict attention to two distinct particles.

The space of velocities \( q_1 = p_1 / M_1 \) and \( q_2 = p_2 / M_2 \) of two particles can be conveniently parametrised as follows. We use Lorentz transformations to go to the rest frame of one of the particles, say particle 1. In that frame, particle 2 is not generally at rest, but we can use rotations to make sure that the velocity \( q_2 \) is along the positive 1-axis, i.e. \( q_2 \) is of the form

\[
B(\xi)(1, 0, 0)^t = (\cosh \xi, \sinh \xi, 0)^t, \tag{3.26}
\]

where \( B(\xi) \) is the Lorentz boost along the 1-axis with boost parameter \( \xi \in \mathbb{R}^+ \), i.e. \( B(\xi) = \Lambda(b(\xi)) \) in the notation of (A.19). The combined momentum in the rest frame of particle 1 is again time-like, so we can write it in the form

\[
(M_1 + M_2 \cosh \xi, M_2 \sinh \xi, 0)^t = V(\xi)(M(\xi), 0, 0)^t. \tag{3.27}
\]

Here

\[
M(\xi) = \sqrt{M_1^2 + M_2^2 + 2M_1M_2 \cosh \xi} \tag{3.28}
\]

is the invariant mass of the combined system. \( V(\xi) \) is a Lorentz boost along the 1-axis which relates the rest frame of particle 1 to the centre of mass frame. Both depend on
\(\xi, M_1 \text{ and } M_2\) but we have suppressed the dependence on \(M_1\) and \(M_2\) in our notation because both are kept fixed in the following discussion. The dependence on \(\xi\), however, is crucial. Thus we have an alternative parametrisation of the two-particle momentum space. Instead of specifying the two momenta \(p_1\) and \(p_2\), we specify the “relative” boost parameter \(\xi\), and one overall Lorentz transformation \(L \in L_3^+\) which relates \(p_1\) and \(p_2\) to their values in the centre of mass frame. Explicitly we define

\[
q_1(\xi) = V^{-1}(\xi)(1, 0, 0)^t \quad \text{and} \quad q_2(\xi) = V^{-1}(\xi)B(\xi)(1, 0, 0)^t, 
\]

noting that the two velocities are equal if and only if \(\xi = 0\). Thus we have the following bijection between the two parametrisations of the two-particle momentum space:

\[
I : L_3^+ \times \mathbb{R}^+ \to \mathcal{M}_2 \quad (L, \xi) \mapsto (Lq_1(\xi), Lq_2(\xi)).
\]

The space \(\mathcal{M}_2\) has a natural Lorentz-invariant measure which is important for us. It is the product of the usual Lorentz invariant measures on the mass shell of the two momenta \(p_1\) and \(p_2\). If we parametrise

\[
q_1 = (\cosh \vartheta_1, \sinh \vartheta_1 \cos \varphi_1, \sinh \vartheta_1 \sin \varphi_1)^t \quad q_2 = (\cosh \vartheta_2, \sinh \vartheta_2 \cos \varphi_2, \sinh \vartheta_2 \sin \varphi_2)^t
\]

then the measure is

\[
dm(p_1, p_2) = \frac{M_1 M_2}{(8\pi)^2} \sinh \vartheta_1 \sinh \vartheta_2 d\vartheta_1 d\vartheta_2 d\varphi_1 d\varphi_2.
\]

For later use it is important to express this measure also in terms of the parameters \((L, \xi)\) \((3.30)\). Recall that these parameters allow us to think of the space \(\mathcal{M}_2\) as foliated by \(L_3^+\) orbits, with each orbit parametrised by a positive boost parameter \(\xi\). Since \((3.33)\) is invariant under the (left) action of \(L_3^+\) it is not surprising that the measure is proportional to the standard invariant measure of \(L_3^+\). Parametrising

\[
L = e^{-\theta J_0} e^{-\vartheta J_2} e^{-\varphi J_0}
\]

one finds the following remarkably simple formula

\[
dm(p_1, p_2) = \frac{M_1 M_2}{(8\pi)^2} \sinh \xi \sinh \vartheta d\xi d\theta d\varphi.
\]

The measure also has a simple expression in terms of the total three momentum \(P = p_1 + p_2\), which can be expressed in terms of the parameters \((3.34)\) as

\[
P = (M(\xi) \cosh \vartheta, M(\xi) \sinh \vartheta \cos \theta, M(\xi) \sinh \vartheta \sin \theta),
\]

with \(M(\xi)\) given by \((3.28)\). In terms of \(P\) the measure is simply

\[
dm(p_1, p_2) = \frac{1}{(8\pi)^2 M(\xi)} dP_0 dP_1 dP_2 d\varphi.
\]
We now have all the ingredients to describe the Hilbert space of two distinct relativistic anyons. It follows from refs. [3] and [4] that this Hilbert space can be described as the space of $L^2$-sections of a certain vector bundle over $\mathcal{M}_2$ associated to the principal bundle

$$PB_2 \rightarrow \tilde{\mathcal{M}}_2 \downarrow \mathcal{M}_2.$$  

In order to describe this bundle and its section explicitly we need to lift the bijection (3.30) to the universal covers $\tilde{\mathcal{M}}_2$ and $\tilde{L}_3$. To do this, first note that $\tilde{L}_3$ acts on $\mathcal{M}_2$ and that, since $\tilde{L}_3$ is simply connected, this action lifts to a unique action on $\tilde{\mathcal{M}}_2$. We denote elements of $\tilde{\mathcal{M}}_2$ by $\tilde{q}$ and the action of $u \in \tilde{L}_3$ on $\tilde{q}$ simply by $u\tilde{q}$. For given $\xi \in \mathbb{R}^+$ let $\tilde{q}(\xi)$ be a lift of the reference velocities $(q_1(\xi), q_2(\xi)) \in \mathcal{M}_2$ defined in (3.29). Then define the lifted bijection via

$$I: \tilde{L}_3 \times \mathbb{R}^+ \rightarrow \tilde{\mathcal{M}}_2$$

$$(u, \xi) \mapsto u\tilde{q}(\xi).$$  

Consider now the case of two anyons which carry representations $W_A$ and $W_B$ of the ribbon Hopf algebra $A$. Then the fibre of the vector bundle describing the quantum theory is $W_A \otimes W_B$. A natural choice of generator for $PB_2$ is the closed path generated by a $2\pi$ rotation in $\mathcal{M}_2$. We denote this generator by $\Omega$. Then we define a representation of $PB_2$ on $W_A \otimes W_B$ by representing $\Omega^{-1}$ by $r_A \otimes r_B(Q)$, where $Q$ is the monodromy element defined in (3.19). This representation defines a vector bundle with fibre $W_A \otimes W_B$ associated to (3.38). Sections of this vector bundle are conveniently described in terms of equivariant $W_A \otimes W_B$-valued functions on the cover $\tilde{\mathcal{M}}_2$. The equivariance condition is the requirement that two particle wave functions transform via $r_A \otimes r_B(Q)$ under the action of $\Omega^{-1}$ on $\tilde{\mathcal{M}}_2$. The two-anyon Hilbert space is thus

$$\mathcal{H}_2 = \{ \Psi \in L^2(\tilde{\mathcal{M}}_2, W_A \otimes W_B) | \Psi(\Omega^{-1}\tilde{q}) = r_A \otimes r_B(Q)\Psi(\tilde{q}) \},$$  

where the measure on $\tilde{\mathcal{M}}$ is again given by the formula (3.33) except that the range of $\varphi$ is now all of $\mathbb{R}$.

In the case of two particles, the description of the Hilbert space can be simplified because the structure group $PB_2$ of the bundle is abelian. As a result the bundle splits into a direct sum of line bundles. We can perform this splitting explicitly by using the decomposition of $W_A \otimes W_B$ into irreps in terms of fusion coefficients $N_{AC}^{AB}$:

$$W_A \otimes W_B = \bigoplus_C N_{AC}^{AB}W_C.$$  

It follows from the relation (3.18) that this decomposition of $W_A \otimes W_B$ diagonalises the monodromy operator $Q$. For each of the fusion channels $W_C$, $N_{AC}^{AB} \neq 0$, the eigenvalue of $Q$ is given by the combination of spin factors $e^{2\pi i(s_C - s_A - s_B)}$. Thus we arrive at the alternative description of the two-particle Hilbert space

$$\mathcal{H}_2 = \bigoplus_C N_{AC}^{AB}\{ \Psi \in L^2(\tilde{\mathcal{M}}_2, W_C) | \Psi(\Omega^{-1}\tilde{q}) = e^{2\pi i(s_C - s_A - s_B)}\Psi(\tilde{q}) \}.$$  

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We can simplify our description further by studying the action of the universal cover of the Poincaré group on this space. The action of \((a, u) \in \tilde{P}_3^\uparrow\) on \(\Psi \in L^2(M_2, W_C)\) is
\[
\rho(a, u) \Psi(\tilde{q}) = \exp(i a \cdot (p_1 + p_2)) \exp(i(s_1\psi(u, p_1) + s_2\psi(u, p_2))) \Psi(u^{-1}\tilde{q}). \tag{3.44}
\]
In particular it follows that
\[
\rho(\Omega) \Psi(\tilde{q}) = e^{2\pi i(s_1 + s_2)} \Psi(\Omega^{-1}\tilde{q}). \tag{3.45}
\]
However, since both anyons satisfy the spin quantisation rule (3.22) we can use this relation to rewrite the definition (3.43) as
\[
\mathcal{H}_2 = \bigoplus_C N_{AB}^C \{ \Psi \in L^2(\tilde{M}_2, W_C) | \rho(\Omega) \Psi = e^{2\pi i s_C} \Psi \}. \tag{3.46}
\]
To cast this definition into its final, most useful form we re-write it once more using the classical coordinate transformation (3.39). The idea is to use the pull back
\[
\tilde{I}^* : L^2(\tilde{M}_2, W_C) \to L^2(\tilde{L}_3^\uparrow \times \mathbb{R}^+, W_C), \tag{3.47}
\]
which is an isometry of Hilbert spaces. Moreover, defining a representation of \(\tilde{P}_3^\uparrow\) on \(L^2(\tilde{L}_3^\uparrow \times \mathbb{R}^+, W_C)\) via
\[
\pi(a, u) \Phi(x, \xi) = \exp(i a \cdot P) \Phi(u^{-1}x, \xi), \tag{3.48}
\]
where \(P = \Lambda(x)(M(\xi), 0, 0)^t\) is the total momentum vector (3.36), one checks that the isometry \(\tilde{I}^*\) intertwines between the representations \(\pi\) and \(\rho\) of \(\tilde{P}_3^\uparrow\). In particular we can therefore write the definition (3.46) as
\[
\mathcal{H}_2 = \bigoplus_C N_{AB}^C \{ \Phi \in L^2(\tilde{L}_3^\uparrow \times \mathbb{R}^+, W_C) | \pi(\Omega) \Phi = e^{2\pi i s_C} \Phi \}. \tag{3.49}
\]
This description of the two-anyon Hilbert space has a very simple interpretation. It is a direct sum of spaces of the form
\[
V_{s_C} = \{ \Phi \in L^2(\tilde{L}_3^\uparrow \times \mathbb{R}^+, C) \otimes W_C | \Phi(\Omega^{-1}x, \xi) \Phi = e^{2\pi i s_C} \Phi(x, \xi) \}, \tag{3.50}
\]
each of which satisfy the equality
\[
\pi(\Omega)V_{s_C} = r_C(c)V_{s_C}. \tag{3.51}
\]
This is just the condition we imposed on one-anyon Hilbert spaces (3.23).

The spaces \(V_{s_C}\) are reducible as a representation of \(\tilde{P}_3^\uparrow\). Its irreducible components are labelled by the invariant mass \(M\) and spins \(s = s_C + n\), where \(n \in \mathbb{Z}\). Thus we have the decomposition
\[
V_{s_C} = \int_{M_1 + M_2}^{\infty} dM \bigoplus_{s=s_C+Z} V_{Ms} \otimes W_C. \tag{3.52}
\]
into one-particle Hilbert spaces. Repeating this decomposition for every fusion channel 
\( C \) with \( N_{AB}^C \neq 0 \) we finally obtain the decomposition of the two-particle space into irreps 
of \( \tilde{P}_3^\dagger \) and of the ribbon Hopf algebra \( \mathcal{A} \), each obeying the spin rule (3.51):

\[
\mathcal{H}_2 = \bigoplus_C N_{AB}^C \int_{M_1+M_2}^{\infty} dM \bigoplus_{s=s_C+Z} V_{Ms} \otimes W_C.
\] (3.53)

In each of the spaces \( V_{Ms} \) the quantity

\[
l = s - s_1 - s_2
\] (3.54)

is interpreted as the orbital angular momentum of the relative motion of the two particles
in the centre of mass frame. Its fractional part plays a crucial role in the scattering
theory.

### 3.3 Relativistic anyon scattering

It follows from the quantisation condition

\[
s = n + s_c, \quad n \in \mathbb{Z},
\] (3.55)

that the orbital angular momentum of the relative motion in the fusion channel \( C \) has the form

\[
l = n + \Delta_{AB}^C, \quad n \in \mathbb{Z},
\] (3.56)

where

\[
\Delta_{AB}^C = s_C - s_A - s_B.
\] (3.57)

Note that \( \Delta_{AB}^C \) need not lie in \([0,1)\) so that \([l]\) (the largest integer \( \leq l \)) need not be the same as \( n \) in the parametrisation (3.56). Following an argument of E. Verlinde given in a non-relativistic context in [20] we shall now show how to compute the relativistic scattering of anyons in terms of the numbers \( \Delta_{AB}^C \). We postulate that the \( S \)-matrix is diagonal in each of the components \( V_{Ms} \otimes W_C \) of the two-particle Hilbert space \( \mathcal{H}_2 \). For given \( A, B \) and \( C, N_{AB}^C \neq 0 \), its value depends on the sign of \( l \) and is

\[
S(l) = \begin{cases} 
e^{-\frac{i}{2} \Delta_{AB}^C} & \text{if } [l] \geq 0 \\ ne^{\frac{i}{2} \Delta_{AB}^C} & \text{if } [l] < 0. \end{cases}
\] (3.58)

In preparation for the discussion of gravitational scattering we combine this formula
with the relevant phase space factors to write down a relativistic cross section for anyon
scattering. We adopt the conventions and definitions used in standard textbooks on
quantum field theory such as [21].

As always in scattering theory, we will need to consider generalised energy and momentum eigenstates which are not normalisable and therefore not strictly in the Hilbert spaces defined so far. We use standard ket notation for momentum eigenstates \( |p\rangle \) of a
particle of mass $M$. They can be realised as delta-functions on the mass shell (3.7) with measure (3.9):

$$ |p\rangle = \frac{8\pi^2}{M} \frac{1}{\sinh \vartheta} \delta_\vartheta \delta_\varphi, $$

(3.59)

where $(\vartheta, \varphi)$ are the boost parameter and angle parametrising $p$ via (3.8). These states are normalised so that the inner product of two momentum states is

$$ \langle p'|p \rangle = \frac{8\pi^2}{M} \frac{1}{\sinh \vartheta} (\vartheta') \delta_\vartheta (\varphi'). $$

(3.60)

Similarly we write two particle scattering states as a product of delta functions on $M_2$ with measure (3.33):

$$ |p_1, p_2\rangle = \left(\frac{8\pi^2}{M_1 M_2}\right)^2 \frac{1}{\sinh \vartheta_1} \frac{1}{\sinh \vartheta_2} \delta_{\vartheta_1} \delta_{\varphi_1} \delta_{\vartheta_2} \delta_{\varphi_2}. $$

(3.61)

As in the discussion of classical momenta it is convenient to switch to the parametrisation $(L, \xi)$ (3.30) in terms of an overall Lorentz transformation and a relative boost parameter. A calculation analogous to the conversion of the measure into the form (3.35) leads to the simple formula

$$ |p_1, p_2\rangle = \left(\frac{8\pi^2}{M_1 M_2}\right)^2 \frac{1}{\sinh \xi} \delta_L \delta_\xi \delta_{\vartheta_1} \delta_{\varphi_1} \delta_{\vartheta_2} \delta_{\varphi_2}, $$

(3.62)

where we have used the parametrisation of $L$ given in (3.34). Finally note that in terms of the total momentum $P = p_1 + p_2$ (3.36) we have

$$ |p_1, p_2\rangle = (8\pi^2)^2 M(\xi) \delta_P \delta_{P_1} \delta_{P_2} \delta_{\varphi}. $$

(3.63)

In order to characterise scattering states in the Hilbert space (3.53) we need to specify both the momenta of the anyons and the internal state $v \in W_A \otimes W_B$. Generic scattering states of two particles with momenta $p_1$ and $p_2$ and internal state $v$ are thus of the form

$$ |p_1, p_2; v\rangle = |p_1, p_2\rangle \otimes v. $$

(3.64)

To simplify our presentation we consider the case where $v$ lies entirely in one of the fusion channels $W_C$ with $N^C_{AB} \neq 0$. This assumption and the fact that scattering states have a definite invariant mass $M(\xi)$ means that they are elements of

$$ \bigoplus_{s=s_C+Z} V_{Ms} \otimes W_C. $$

(3.65)

Looking at the formula (3.63) and introducing the abbreviation $|s\rangle$ for the function $e^{-is\psi}$ on $\tilde{L}_3$ in the parametrisation (A.13) we have the expansion

$$ |p_1, p_2; v\rangle = 4(2\pi)^3 M(\xi) \delta_P \delta_{P_1} \delta_{P_2} \sum_{s=s_C+Z} e^{is\varphi} |s\rangle \otimes |v\rangle. $$

(3.66)
This expansion is useful because, according to our postulate (3.58), the $S$-matrix only depends on the values of $s_C, s_A$ and $s_B$ and no other details of the scattering state. With the usual decomposition

$$S = 1 + iT$$

we define on-shell reduced matrix elements with respect to an initial state $|\mathbf{p}_1^i, \mathbf{p}_2^i; \nu^i\rangle$ and a final state $|\mathbf{p}_1^f, \mathbf{p}_2^f; \nu^f\rangle$ in the standard fashion

$$\langle \mathbf{p}_1^f, \mathbf{p}_2^f; \nu^f | T | \mathbf{p}_1^i, \mathbf{p}_2^i; \nu^i\rangle = (2\pi)^3 \delta^3(\mathbf{P}^i - \mathbf{P}^f) \langle \mathbf{p}_1^f, \mathbf{p}_2^f; \nu^f | T | \mathbf{p}_1^i, \mathbf{p}_2^i; \nu^i\rangle,$$

where $\mathbf{P}^i = \mathbf{p}_1^i + \mathbf{p}_2^i$ and $\mathbf{P}^f = \mathbf{p}_1^f + \mathbf{p}_2^f$. The differential cross section in the centre of mass frame is then given by the usual expression

$$d\sigma = \frac{1}{4\sqrt{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - M_1^2 M_2^2}} \int dm(p_1^f, p_2^f) (2\pi)^3 \delta^3(\mathbf{P}^i - \mathbf{P}^f) |\langle \mathbf{p}_1^f, \mathbf{p}_2^f; \nu^f | T | \mathbf{p}_1^i, \mathbf{p}_2^i; \nu^i\rangle|^2. \quad (3.69)$$

With our simple expression of the $S$-matrix (3.58) we find

$$\langle \mathbf{p}_1^f, \mathbf{p}_2^f; \nu^f | T | \mathbf{p}_1^i, \mathbf{p}_2^i; \nu^i\rangle = 4M(\xi)it(\varphi^i - \varphi^f)\langle \nu^f, \nu^i\rangle,$$

where

$$it(\varphi) = \sum_n (S(l) - 1)e^{is\varphi}, \quad (3.71)$$

with $l$ and $s$ expressed in terms of $n$ via (3.55) and (3.56). Explicitly this sum is

$$it(\varphi) = \sum_{n \geq -[\Delta C_{AB}^3]} \left( e^{-\frac{1}{2}\Delta C_{AB}^3} - 1 \right) e^{is\varphi} + \sum_{n < -[\Delta C_{AB}^3]} \left( e^{\frac{1}{2}\Delta C_{AB}^3} - 1 \right) e^{is\varphi}. \quad (3.72)$$

This expression is divergent but can be renormalised in standard fashion [21]. The resulting finite part is

$$\tilde{t}(\varphi) = e^{i\Delta C_{AB}^3}e^{-it^l(\Delta C_{AB})\varphi}e^{-\frac{1}{2}\Delta C_{AB}^3} - e^{\frac{1}{2}\Delta C_{AB}^3} \frac{1 - e^{i\varphi}}{1 - e^{i\varphi}}. \quad (3.73)$$

This quantity contains most of the physics of the scattering process. The angle $\varphi = \varphi^i - \varphi^f$ is the scattering angle in the centre of mass frame. It remains to combine the $\varphi$-dependence with appropriate phase space factors to obtain an expression for the differential cross section.

Using the expression (3.31) for the integration measure and the formula

$$\sqrt{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - M_1^2 M_2^2} = M_1 M_2 \sinh \xi, \quad (3.74)$$

we compute the following simple formula for the differential cross section in the centre of mass frame

$$\frac{d\sigma}{d\varphi}(\varphi, \xi) = \frac{\hbar M(\xi)}{2\pi M_1 M_2 \sinh \xi} |t(\varphi)|^2 |\langle \nu^f, \nu^i\rangle|^2, \quad (3.75)$$

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where we have reinstated Planck’s constant.

Note that in the case where the ribbon Hopf algebra $A$ is abelian and all its irreps one dimensional the formula simplifies to

$$\frac{d\sigma}{d\phi}(\phi, \xi) = \frac{hM(\xi)}{2\pi M_1 M_2 \sinh \xi} \frac{\sin^2(\frac{1}{2}\Delta_{AB}^C)}{\sin^2 \frac{\phi}{2}}.$$  

(3.76)

The standard Aharonov-Bohm scattering discussed in sect. 2.1 is a special case of this formula, obtained by setting $s_A = s_B = 0$ and $s_C = e\Phi$. In order to recover the familiar non-relativistic formula for Aharonov-Bohm scattering of electrons off a flux tube take the limit where the particles with mass $M_1$ (the fluxes) become very heavy and their rest frame becomes the centre of mass frame. Then $M_1 \approx M(\xi)$, $M_2 = M_e$ and $\sinh \xi \approx v_2$, the velocity of the electrons.

4 The Chern-Simons formulation of gravity in 2+1 dimensions

4.1 Einstein gravity as Chern-Simons theory

The possibility of writing general relativity in 2+1 dimensions as a Chern-Simons theory was first noticed in [22]. This observation opened up a new approach to gravity and in particular to its quantisation, which was first systematically explored in [23]. Since then, a vast body of literature has been devoted to the subject. Here we give a brief summary of those aspects which are relevant for us.

In (2+1)-dimensional gravity, space-time is a three-dimensional manifold $M$. In the following we shall only consider space times of the form $M = \mathbb{R} \times \Sigma$, where $\Sigma$ is an orientable two-dimensional manifold. A three-manifold of that form is orientable and hence, by a classic theorem of Stiefel, parallelisable. Thus its tangent bundle is topologically trivial.

In Einstein’s original formulation of general relativity, the dynamical variable is a metric $g$ on $M$. For the Chern-Simons formulation it is essential to adopt Cartan’s point of view, where the theory is formulated in terms of the dreibein of one-forms $e_a$, $a = 0, 1, 2$ and the spin connection one-forms $\omega_a$, $a = 0, 1, 2$. The dreibein is related to the metric via

$$\eta^{ab} e_a \otimes e_b = g,$$

(4.1)

where $\eta^{ab} = \text{diag}(1, -1, -1)$. Indices are raised and lowered with $\eta_{ab}$ and the Einstein summation convention is used throughout. The one-forms $\omega_a$ should be thought of as components of the $L^3$ connection

$$\omega = \omega_a J^a,$$

(4.2)

where $J_a$, $a = 0, 1, 2$ are the generators of the Lie algebra $so(2, 1)$ and satisfy the commutation relations (3.6). The curvature two-form $F_\omega = d\omega + \frac{1}{2}[\omega, \omega]$ can be expanded as $F_\omega = F_\omega^a J_a$, with

$$F_\omega^a = d\omega^a + \frac{1}{2}\epsilon^{abc} \omega^b \wedge \omega^c.$$  

(4.3)
The Einstein-Hilbert action in 2+1 dimension can be written as

$$S_{EH}[\omega, e] = \int_M e_a \wedge F^a_\omega.$$  \hfill (4.4)

In Cartan’s formulation of gravity, both the connection \(\omega_a\) and the dreibein \(e_a\) are dynamical variables and varied independently. Variation with respect to the spin connection yields the requirement that torsion vanishes

$$D_\omega e_a = de_a + \frac{1}{2} \epsilon_{abc} \omega^b e^c = 0.$$ \hfill (4.5)

Variation with respect to \(e_a\) yields the vanishing of the curvature tensor

$$F_\omega = 0.$$ \hfill (4.6)

In 2+1 dimensions this is equivalent to the vanishing of the Ricci tensor, and thus to the Einstein equations in the absence of matter.

An important step in the Chern-Simons formulation of gravity is the combination of the dreibein and the spin connection into a Cartan connection \cite{24}. This is a one-form with values in the Lie algebra \(\text{iso}(2,1)\) for which we defined generators in (3.6). The Cartan connection one-form is

$$A = \omega_a J^a + e_a P^a.$$ \hfill (4.7)

with curvature

$$F = (D_\omega e^a) P_a + (F^a_\omega) J_a$$ \hfill (4.8)

combining the curvature and the torsion of the spin connection.

The final technical ingredient we need in order to establish the Chern-Simons formulation is special to three dimensional space-times. This is a non-degenerate, invariant bilinear form on the Lie algebra \(\text{iso}(2,1)\)

$$\langle J_a, P_a \rangle = \eta_{ab}, \quad \langle J_a, J_b \rangle = \langle P_a, P_b \rangle = 0.$$ \hfill (4.9)

Then the Chern-Simons action for the connection \(A\) on \(M = \Sigma \times \mathbb{R}\) is

$$S_{CS}[A] = \frac{1}{2} \int_M \langle A \wedge dA \rangle + \frac{2}{3} \langle A \wedge A \wedge A \rangle.$$ \hfill (4.10)

A short calculation shows that this is equal to the Einstein-Hilbert action \(4.4\). Moreover, the equation of motion found by varying the action with respect to \(A\) is

$$F = 0.$$ \hfill (4.11)

Using the decomposition \(4.8\) we thus reproduce the condition of vanishing torsion and the three-dimensional Einstein equations, as required.
So far we have only studied the Einstein equations in vacuum. Matter in the form of point particles can be included in a mathematically elegant fashion in the Chern-Simons formulation. We refer the reader to [2] for a discussion and to [3] for further geometrical background. Particles are introduced by marking points on the surface \( \Sigma \) and coupling the particle’s phase space to the phase space of the theory. The phase space of a particle with mass \( M \) and spin \( s \) is the set of energy-momentum vectors \( p \) and generalised angular momenta \( j = (j^0, j^1, j^2) \) satisfying the mass-shell condition \( p^2 = M^2 \) and the spin condition \( p \cdot j = Ms \). Mathematically, this condition defines a co-adjoint orbit \( O_{Ms} \) of \( \text{iso}(2,1)^* \). To make this explicit we write \( P^a \) and \( J^a \) for the basis elements of \( \text{iso}(2,1) \) dual to \( P_a \) and \( J_a \), and we write an element \( \xi \) as

\[
\xi = p^a P^*_a + j^a J^*_a. \tag{4.12}
\]

Using the inner product (4.9) we can identify \( \xi \) with the element

\[
\xi = p^a J_a + j^a P_a \tag{4.13}
\]

in \( \text{iso}(2,1) \). With this identification we can think of the orbit \( O_{Ms} \) as lying in \( \text{iso}(2,1) \). A generic element like (4.13) can be obtained by conjugating the representative element

\[
\hat{\xi} = MJ_0 + sP_0 \tag{4.14}
\]

with a suitable \( P^*_3 \) element \((a, L)\). The orbit \( O_{Ms} \) for \( M \neq 0 \) is diffeomorphic to the tangent bundle of a hyperboloid. Co-adjoint orbits have a canonical symplectic structure, often called the Kostant-Kirillov symplectic structure. For the generic case \( M \neq 0 \) the corresponding Poisson brackets of the coordinate functions \( j_a \) and \( p_a \) are

\[
\{j_a, j_b\} = \epsilon_{abc} j^c, \quad \{j_a, p_b\} = \epsilon_{abc} p^c. \tag{4.15}
\]

In order to introduce \( m \) particles with masses and spins \((M_1, s_1), \ldots (M_m, s_m)\) we mark \( m \) points \( z_1, \ldots, z_m \) on \( \Sigma \) and associate to each point \( z_i \) a co-adjoint orbit \( O_{M_i s_i} \). The coupling of the particle degrees of freedom to the gauge field via minimal coupling is described in [3]. The upshot is that we specify the kinematic state of each particle in terms of representative \( \text{iso}(2,1) \) elements \( \xi_{(i)} = M_i J_0 + s_i P_0 \) and Poincaré elements \((a_i, L_i)\). These determine the curvature of according to

\[
F'(z) + \sum_{i=1}^{m} (a_i, L_i)(8\pi GM_i J_0 + s P_0) (a_i, L_i)^{-1} \delta^2(z - z_i) = 0. \tag{4.16}
\]

Expanding the curvature term as in (4.8) we find that the energy-momentum vectors of the particles act as sources for curvature and their generalised angular momenta act as sources of torsion, in agreement with physical expectations. Note that Newton’s constant appears as a coupling constant in such a way that the curvature only sees the rescaled, dimensionless masses \( \mu_i = 8\pi GM_i \). The following discussion is conducted almost entirely in terms of these.
4.2 Gravitational phase space and its Poisson structure

The phase space of a classical field theory is the space of solutions of the equations of motions, modulo gauge invariance. Adopting the Chern-Simons formulation of (2+1)-dimensional gravity we thus find that the phase space of gravity in 2+1 dimensions is the moduli space of flat $P_3^+$-connection on the surface $\Sigma$. Starting with the classic paper of Atiyah and Bott [25], the moduli space of flat $G$-bundles on $\Sigma$ has been studied extensively for semi-simple, compact Lie groups $G$. A pedagogical description of that space and its symplectic structure can be found in [26]. The rigorous extension of the theory to non-compact groups like $P_3^+$ is a very important but (it seems) largely open problem. In the following we give a brief description of the gravitational phase in the language of moduli spaces of flat connections. We have endeavoured to be conceptually clear but do not enter into technical questions related to the non-compactness of $P_3^+$.

In this paper we are only interested in the gravitational interactions of particles, without the added complication of handles. We therefore specialise to $\Sigma = \mathbb{R}^2$ with $m$ marked points $z_1, ... , z_m$ which are not allowed to coincide, i.e. $z_i \neq z_j$ for $i \neq j$. For us, the most useful description of the moduli space is in terms of representations of the fundamental group of $\mathbb{R}^2 - \{z_1, ... , z_m\}$ with a base point $\ast$ which we choose to be at infinity. The fundamental group $\pi_1(\mathbb{R}^2 - \{z_1, ... , z_m\}, \ast)$ is the group freely generated by $m$ invertible generators $l_1, ... , l_m$. A flat $P_3^+$-connection on $\mathbb{R}^2 - \{z_1, ... , z_m\}$ associates to each generator a holonomy element in $P_3^+$. However, the insertion of the charges at marked points as in (4.16) fixes the holonomy around the $i$-th marked point to be in the $P_3^+$ conjugacy class

$$C_i = \{(a, L) \exp(-\mu_i J_0 - s_i P_0) (a, L)^{-1} | (a, L) \in P_3^+\}.$$ (4.17)

Note that for a generic particle with mass $\mu$ and spin $s$, we can write elements of the associated conjugacy class as

$$(a, L) \exp(-\mu J_0 - s P_0) (a, L)^{-1} = \exp(-p^a J_a - j^a P_a)$$ (4.18)

and have the explicit formula

$$p = (\mu \cosh \vartheta, \mu \sinh \vartheta \cos \varphi, \mu \sinh \vartheta \sin \varphi)$$ (4.19)

for the momentum in terms of the parametrisation (A.14) of the Lorentz transformation $L$. The formula for $j$ is

$$j = a \times p + \frac{s}{\mu} p.$$ (4.20)

Fock and Rosly gave a very explicit description of the Poisson structure of the moduli space of flat $G$-bundles on a Riemann surface in [8]. The application of the Fock-Rosly description to Euclidean gravity is discussed in some detail in [4]. It is not difficult to adapt that description to the Lorentzian situation. To write down the Fock-Rosly Poisson structure on the moduli space of flat $P_3^+$-bundles one requires an element $r \in so(2, 1) \otimes so(2, 1)$ which satisfies the classical Yang-Baxter equation and is such that its
symmetric part agrees with the non-degenerate invariant form $\langle \ , \ \rangle$ used in the definition of the Chern-Simons action (4.10). One checks that
\[ r = P_a \otimes J^a \] (4.21)
satisfies the classical Yang-Baxter equation. The symmetrised part
\[ r^s = \frac{1}{2}(P_a \otimes J^a + J_a \otimes P^a) \] (4.22)
is the invariant form on $iso(2,1)$ which we used in defining (4.10). As explained in [4], this implies that $r$ defines a bi-algebra structure on $iso(2,1)$ which is co-boundary and quasi-triangular.

We do not enter further into a discussion of the Poisson structure on the multi-particle phase space here, which requires separate and detailed study. Instead we adopt the general philosophy of the combinatorial quantisation programme of Chern-Simons theory, see [9], [10], [11] and [4]. Roughly speaking, the classical $r$-matrices play the role of structure constants in the Poisson structure of classical phase space. To quantise, one looks for a quantum $R$-matrix which has (4.21) as a classical limit. This $R$-matrix then plays the role of structure constants of the quantum algebra of observables. In the next section we show that a certain deformation of the group algebra of the Poincaré group provides the required $R$-matrix.

5 The Lorentz double

5.1 Definition

The quantum double of a group $H$ is a quasi-triangular Hopf algebra constructed, via Drinfeld’s double construction, out of the Hopf algebra of functions on $H$. As a vector space the quantum double $D(H)$ is the tensor product of the algebra of functions $F(H)$ on $H$ and the group algebra $\mathbb{C}(H)$. For finite groups this definition makes rigorous sense, but when generalising the construction to locally compact Lie groups $H$, one has to make mathematical sense of the group algebra. Here, different choices are made by different authors. The Lorentz double is the quantum double $D(\tilde{L}_3^\uparrow)$ constructed from the universal cover $\tilde{L}_3^\uparrow$ of the orthochronous Lorentz group. In the definition we give below we follow the approach taken in [27] but generalise it slightly in the way explained in [4] for the case $H = SU(2)$.

Before delving into technical details it is perhaps useful to give a general description of the Lorentz double. As we shall explain, it can be thought of as a deformation of the group algebra of the universal cover $\tilde{P}_3^\uparrow$ of the orthochronous Poincaré group. More precisely, there is a homomorphism from $D(\tilde{L}_3^\uparrow)$ to the group algebra of $\tilde{P}_3^\uparrow$. However the co-algebra structure of $D(\tilde{L}_3^\uparrow)$ differs from that of $\tilde{P}_3^\uparrow$. Physically this difference corresponds to the non-commutative addition rule for momenta in (2+1)-dimensional gravity.

The easiest way to make sense of the group algebra of $\tilde{L}_3^\uparrow$ is to consider generalised
functions on $\hat{L}_3^+$ and to define the product via convolution:

$$f_1 \ast f_2(u) = \int_{\hat{L}_3^+} dv f_1(v) f_2(v^{-1}u). \tag{5.1}$$

As explained in [3] an appropriate class of generalised functions is the set $M(\hat{L}_3^+)$ of measures which are absolutely continuous with respect to the Haar measure $dv$ on $\hat{L}_3^+$ or pure point measures. Such measures can always be written in terms of an $L^1$-function and an $l^1$-sequence $\lambda_i, i \in \mathbb{N}$ as

$$dm = (f + \sum_{i \in \mathbb{N}} \lambda_i \delta_{h_i}) dv. \tag{5.2}$$

In practice this means that we can think of the elements of $M(\hat{L}_3^+)$ as $L^1$-functions, or Dirac $\delta$-functions or a linear combination of both. It is advantageous to include Dirac $\delta$-functions because the unit element is represented by $\delta_e$.

The function algebra of $\hat{L}_3^+$ is represented by a different class of functions on $\hat{L}_3^+$. This time the multiplication is defined by pointwise multiplication of the functions. For this to make sense, the functions should at least be continuous. In practice we take bounded, uniformly continuous functions $C_B(\hat{L}_3^+)$ on $\hat{L}_3^+$.

Putting the ingredients together we define the Lorentz double $D(\hat{L}_3^+)$ as the set of generalised functions $F$ on $\hat{L}_3^+ \times \hat{L}_3^+$ which are bounded and uniformly continuous in the first argument and measures of the form (5.2) in the second argument. We define a norm for such functions

$$||F||_1 = \int_{\hat{L}_3^+} dv \sup_{h \in \hat{L}_3^+} |F(h, v)| \tag{5.3}$$

and equipped with that norm the Lorentz double becomes a Banach algebra.

We refer the reader to [27] and [3] for a complete list of how the algebraic operations of a quasi-triangular Hopf-$\ast$-algebra are implemented in $D(H)$. For our purposes we only need the formulae for the multiplication, the co-multiplication, the ribbon element and the universal $R$-element. The multiplication of two elements $F_1$ and $F_2$ of $D(\hat{L}_3^+)$ is

$$(F_1 \bullet F_2)(h, u) = \int_{\hat{L}_3^+} dw F_1(h, w) F_2(w^{-1}hw, w^{-1}u). \tag{5.4}$$

The co-multiplication $\Delta$ is defined via

$$\Delta F(h_1, u_1, h_2, u_2) = F(h_1 h_2, u_1) \delta_{u_1}(u_2). \tag{5.5}$$

Finally the central ribbon element is

$$c(h, u) = \delta_h(u) \tag{5.6}$$

and the universal $R$-element is

$$R(h_1, u_1, h_2, u_2) = \delta_{h_1}(u_2) \delta_e(u_1). \tag{5.7}$$

Together they satisfy the spin-statistics relation (3.18), as required for a ribbon Hopf algebra.
5.2 The Lorentz double as a deformation of the \( \tilde{P}_3^\uparrow \) group algebra

These formulae may seem complicated and far removed from the Poincaré group \( P_3^\uparrow \) which entered the classical formulation of (2+1)-dimensional gravity. In order to exhibit the intimate relation between \( \tilde{P}_3^\uparrow \) and the Lorentz double \( D(\tilde{L}_3^\uparrow) \) we need to express the algebraic structure of the Poincaré group in a somewhat unusual fashion. Recall the multiplication law (3.5) for two elements \((a_1, u_1), (a_2, u_2) \in \tilde{P}_3^\uparrow \). The idea is to consider the group algebra, written in terms of suitable functions on the group, and then to perform a Fourier transform. As a manifold \( \tilde{P}_3^\uparrow \cong \mathbb{R}^3 \times \tilde{L}_3^\uparrow \), and the Haar measure on \( \tilde{P}_3^\uparrow \) is the product of the Lebesgue measure on \( \mathbb{R}^3 \) and the Haar measure on \( \tilde{L}_3^\uparrow \). Concretely, if \((a, u) \in \tilde{P}_3^\uparrow \) then we use the Haar measure \( d^3a \, du \). We realise the group algebra of \( \tilde{P}_3^\uparrow \) as the set \( M(\tilde{P}_3^\uparrow) \) of bounded measures on \( P_3^\uparrow \) either absolutely continuous with respect to \( da \, du \) or pure point measures. Again representing such measures by generalised functions \( \hat{f}_1, \hat{f}_2 \) (possibly including delta-functions) the multiplication rule is the convolution

\[
(\hat{f}_1 \cdot \hat{f}_2)(a, u) = \frac{1}{2\pi^3} \int_{\mathbb{R}^3 \times \tilde{L}_3^\uparrow} d^3b \, dw \, \hat{f}_1(b, w) \hat{f}_2(\Lambda(w^{-1})(a - b), w^{-1}u). \tag{5.8}
\]

Now we perform a Fourier transform on the first argument of \( \hat{f} \), thus obtaining a function \( f \) on \((\mathbb{R}^3)^* \times \tilde{L}_3^\uparrow \):

\[
f(k, w) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3a \, \exp(-ik \cdot a) \hat{f}(a, w). \tag{5.9}
\]

Note that elements \( k \) of the dual space \((\mathbb{R}^3)^* \) have the dimension of inverse length.

The Fourier transform of a bounded measure on \( \mathbb{R}^3 \) is a bounded, uniformly continuous function on \((\mathbb{R}^3)^* \). After applying the Fourier transform (5.9) to functions in \( M(\tilde{P}_3^\uparrow) \) we thus obtain generalised functions \( f \) on \((\mathbb{R}^3)^* \times \tilde{L}_3^\uparrow \) which are bounded, uniformly continuous functions of the first argument and measures of the form (5.2) with respect to the second. We denote this set by \( \mathbb{C}(\tilde{P}_3^\uparrow) \). The product of two generalised functions \( f_1, f_2 \in \mathbb{C}(\tilde{P}_3^\uparrow) \) is obtained by applying the Fourier transform (5.9) to the convolution product. The result is

\[
(f_1 \circ f_2)(k, u) = \int_{\tilde{L}_3^\uparrow} dw \, f_1(k, w) f_2(\Lambda(w^{-1})k, w^{-1}u). \tag{5.10}
\]

The group-like co-multiplication for \( P_3^\uparrow \) leads to the following co-multiplication for \( f \in \mathbb{C}(\tilde{P}_3^\uparrow) \)

\[
(\Delta f)(k_1, u_1, k_2, u_2) = f(k_1 + k_2, u_1) \delta_{u_1}(u_2). \tag{5.11}
\]

It is now not difficult to write down promised relationship between \( D(\tilde{L}_3^\uparrow) \) and \( \tilde{P}_3^\uparrow \). The idea is to interpret the first argument of an element \( F \in D(\tilde{L}_3^\uparrow) \) as a group valued momentum. To implement this we need a family of exponential maps from the Lie algebra \( so(2,1) \) to \( \tilde{L}_3^\uparrow \). Let \( J_a^\kappa = \kappa J_a \) for a real, positive parameter \( \kappa \), so that

\[
[J_a^\kappa, J_b^\kappa] = \kappa \epsilon_{abc} J_c^\kappa. \tag{5.12}
\]
Then define
\[ \exp_\kappa : (\mathbb{R}^3)^* \to \tilde{L}_3^\dagger, \quad \exp_\kappa(k) = \exp(k_a J_3^a), \] (5.13)
where \( \exp \) is the exponential map defined in (A.20). For the exponential map to make sense, its argument needs to be dimensionless. Since \( k \) has the dimension of inverse length we require \( \kappa \) to be of dimension length. The available constants \( \hbar \) and \( G \) can be combined in an essentially unique fashion to give the dimension length. We set
\[ \kappa = 8\pi G \hbar \] (5.14)
and shall see in the next section that
\[ p = \kappa k \] (5.15)
is physically interpreted as momentum measured in units of \( 8\pi G \).

We use the map (5.13) to define
\[ \text{EXP}^*_\kappa : D(\tilde{L}_3^\dagger) \to \mathbb{C}(\tilde{P}_3^\dagger) \] (5.16)
by pull-back on the first argument:
\[ \text{EXP}^*_\kappa(F)(k, u) = F(\exp_\kappa(-k), u). \] (5.17)
This establishes the promised relationship between the Lorentz double and the Poincaré group algebra. As explained in the Euclidean setting in [4] \( \text{EXP}^*_\kappa \) is an algebra homomorphism but not a homomorphism of Hopf algebras. Rather one should think of \( D(\tilde{L}_3^\dagger) \) as a non-co-commutative deformation of \( \mathbb{C}(\tilde{P}_3^\dagger) \). The details of the argument given in [4] can be adapted to the present situation by switching from the Euclidean metric to the Lorentzian metric \( \eta \). The basic point is that the pull-back of the \( R \)-matrix
\[ R_\kappa = (\text{EXP}^*_\kappa \otimes \text{EXP}^*_\kappa)(R) \] (5.18)
is a deformation of the classical \( r \)-matrix (4.21) in the sense that in the limit \( \kappa \to 0 \)
\[ R_\kappa = 1 \otimes 1 + i \kappa r + \mathcal{O}(\kappa^2). \] (5.19)
As explained at the end of the previous section, this result is an important step in the combinatorial quantisation scheme.

### 5.3 The irreducible representations of the Lorentz double

The irreps of \( D(\tilde{L}_3^\dagger) \) are listed in appendix B, to which we refer the reader for our notational conventions. Here only we discuss the elliptic representations in more detail. These are labelled by pairs \((\mu, s)\), where \( \mu \in \mathbb{R}^+ \) labels an elliptic conjugacy class in \( \tilde{L}_3^\dagger \) and \( s \) labels an irrep of the centraliser \( N_E \) of the element \( r(\mu) \) (A.18) in that conjugacy class. The centraliser group is isomorphic to \( \mathbb{R} \), so all irreps are one dimensional and labelled by a (real-valued) spin \( s \in \mathbb{R} \). The elliptic representations \( \varpi_{\mu s} \) are labelled by...
the mass parameter $\mu \in \mathbb{R}^+$ and the spin parameter $s \in \mathbb{R}$. Their carrier space is the again the space $V_{\mu s}$ defined in (3.10), which also carries the representations of $\tilde{P}_3^\dagger$. The action of an element $F \in D(\tilde{L}_3^\dagger)$ on this space is

$$ (\varpi_{\mu s}(F)\phi)(x) = \int_{\tilde{L}_3^\dagger} dw \, F(xr(\mu)x^{-1},w)\phi(w^{-1}x). \quad (5.20) $$

We introduce the abbreviation

$$ g(\mu, x) = xr(\mu)x^{-1} \quad (5.21) $$

and note that in terms of the parametrisation (A.14) for $L = \Lambda(x)$, it follows from $r(\mu) = \exp(-\mu J_0)$ that

$$ g(\mu, x) = \exp(-p_a J^a), \quad (5.22) $$

where $p$ is the classical momentum (4.19). The first argument of $F$ should therefore be thought of as an exponentiated or group valued momentum. It then follows from (5.15) that the vector $k$ is the de Broglie wave vector associated to $p$.

It is now straightforward to calculate the action of the central ribbon element $c$ (5.6) on any element $\phi \in V_{\mu s}$. Since $c$ is central it acts by a phase. A short calculation shows

$$ \varpi_{\mu s}(c)\phi = e^{is\mu} \phi. \quad (5.23) $$

Just as in the case of the Poincaré representation, one can use the space $L^2(H_{\mu}, dm)$ of square-integrable functions on the mass hyperboloid as the carrier space of elliptic irreps of the Lorentz double. The action $g(F)$ on an element $\Psi \in L^2(H_{\mu}, dm)$ is

$$ g_{\mu s}(F)\Psi(p) = \int_{\tilde{L}_3^\dagger} dw \, F(\exp(-p_a J^a), w) \exp(is\psi(w,p))\Psi(\Lambda(w^{-1})p). \quad (5.24) $$

where the phase factor $\exp(is\psi(w,p))$ is defined via (3.14).

Finally, consider tensor products of irreps of the Lorentz double, studied in a more general context in [28]. Here we only need the action of the universal $R$-element (5.7) in the tensor product representation $\varpi_{\mu_{s_1}} \otimes \varpi_{\mu_{s_2}}$ on some state $\Phi \in V_{\mu_{s_1}} \otimes V_{\mu_{s_2}}$. It is

$$ (\varpi_{\mu_{s_1}} \otimes \varpi_{\mu_{s_2}}(R)\Phi)(x_1, x_2) = \Phi(x_1, g^{-1}(\mu_1, x_1)x_2). \quad (5.25) $$

Thus the effect of $R$ is to rotate or boost the group valued momentum of the second particle by the group valued momentum of the first. Similarly one computes the effect of the monodromy element $Q = R_{21} R$:

$$ (\varpi_{\mu_{s_1}} \otimes \varpi_{\mu_{s_2}}(Q)\Phi)(x_1, x_2) = e^{-is_1 \mu_1 + s_2 \mu_2}\Phi(g^{-1}(x_1, g^{-1}x_2), \quad (5.26) $$

where $g$ is the total group valued momentum of the system:

$$ g = g(\mu_1, x_1)g(\mu_2, x_2). \quad (5.27) $$

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6 Quantum scattering of gravitating particles

As outlined in the introduction, the strategy of this section is to base our study of the quantum theory of gravitating particles in 2+1 dimensions on the analogy with the theory of ordinary anyon scattering developed in sect. 3. There the Hilbert spaces of single and several anyons were constructed out of irreducible representations of the Poincaré group (external or space-time symmetry) and of an appropriate ribbon Hopf algebra (internal symmetry). For gravitating particles external and internal symmetry are both captured by the same object, namely the Lorentz double. It is not clear \emph{a priori} that the bundle construction of the multi-particle Hilbert spaces of sect. 3 have a well-defined analogue in the gravitational context. We shall now show that this is the case, and test our result against the scattering formulae of 't Hooft \cite{14}, Deser, Jackiw and Sousa-Gerbert \cite{13}, \cite{29}. The following discussion is close in spirit to the paper \cite{3} on quantum scattering of gravitating particles. We recover many of the results of that paper but our more general framework allows us to include particles with spin.

### 6.1 One- and two-particle Hilbert spaces

As in discussion of ordinary anyons we begin by looking at a single particle of mass \( \mu \) and spin \( s \). We restrict attention to \( \mu \in (0, 2\pi) \) in order to have a simple classical interpretation of the space-time surrounding the particle as described in sect. 2.2. In the representation theory of \( D(L^+_3) \) there is no need to restrict masses to lie in this range, a point to which we return briefly at the end of this paper. Since external and internal symmetry coincide in gravity, there is no need to introduce separate external and internal Hilbert spaces as in sect. 3. Instead we consider the irrep \( V_{\mu s} \) of the Lorentz double and impose the analogue of the spin condition (3.22). Formally this condition takes the same form as for the relativistic anyons in sect. 3, namely the requirement that the two central elements \( \Omega \) and \( c \) agree when acting on any element in \( V_{\mu s} \). Thus not all combinations of mass and spin are allowed for particles in (2+1)-dimensional gravity. One-particle Hilbert spaces are of the form

\[
\mathcal{H}_1 = V_{\mu s} \quad \text{provided} \quad \varpi_{\mu s}(\Omega)\phi = \varpi_{\mu s}(c)\phi \quad \forall \phi \in V_{\mu s}.
\] (6.1)

Using the expression (5.23) for the eigenvalue of \( c \) we find that in gravity the quantisation condition for the spin depends on the mass:

\[
e^{2\pi is} = e^{i\mu s} \quad \text{or} \quad s = \frac{n}{1 - \frac{\mu}{2\pi}}, \quad n \in \mathbb{Z}.
\] (6.2)

This formula provides the first point of contact between our algebraic approach to quantising (2+1)-dimensional gravity and some of the basic phenomena described in our introductory sect. 2. We find that the spin of particle of mass \( \mu \) satisfies the same quantisation condition as the angular momentum (2.25) of a test particle moving in the conical space-time created by that particle.

Next consider two particles, with masses \( \mu_1 \) and \( \mu_2 \) and spin \( s_1 \) and \( s_2 \), both satisfying the spin quantisation condition (6.2). Thus there exist two integers \( n_1 \) and \( n_2 \) such that

\[
s_1 = \frac{n_1}{1 - \frac{\mu_1}{2\pi}}, \quad s_2 = \frac{n_2}{1 - \frac{\mu_2}{2\pi}}.
\] (6.3)
In order to keep our discussion of the interaction of these particles as simple as possible we restrict attention to \( \mu_1, \mu_2 \in (0, \pi) \) to start with. In order to construct the two-particle Hilbert space we begin with the space \( L^2(\tilde{\mathcal{M}}_2, \mathbb{C}) \) of square-integrable complex-valued functions on the cover of the space of two distinct velocities \((3.24)\). We again write \( q_1 = p_1/\mu_1 \) and \( q_2 = p_2/\mu_2 \) for the individual velocities and \( \tilde{q} \) for elements of the covering space \( \tilde{\mathcal{M}}_2 \). The Lorentz double acts on elements \( \Psi \in L^2(\tilde{\mathcal{M}}_2, \mathbb{C}) \) via

\[
\varrho(F)\psi(\tilde{q}) = \int_{L^3_1} dw \, F(g(\tilde{q}), w) \exp(is_1\psi(w, p_1) + is_2\psi(w, p_2))\Psi(w^{-1}\tilde{q}),
\]

where

\[
g(\tilde{q}) = \exp(-p_1^a J_a)\exp(-p_2^a J_a)
\]

is the total group valued momentum associated to the pair of velocities covered by \( \tilde{q} \).

Not all elements of \( L^2(\tilde{\mathcal{M}}_2, \mathbb{C}) \) are physically allowed. As in the case of anyons we demand that under the (inverse) \( 2\pi \) rotation \( \Omega^{-1} \) physical states transform according to the monodromy operator \( Q \). The formula \((5.26)\) defines the action of \( Q \) on the tensor product of two one-particle irreps. However, there is a natural way to let \( Q \) act on \( L^2(\tilde{\mathcal{M}}_2, \mathbb{C}) \) as well. Using the relation \((3.18)\) and the action \((6.4)\) we define

\[
\varrho(Q)\Psi(\tilde{q}) = e^{-i(s_1\mu_1 + is_2\mu_2)}\varrho(c)\Psi(\tilde{q}).
\]

Then the requirement on physical two-particle states becomes

\[
\varrho(Q)\Psi(\tilde{q}) = \Psi(\Omega^{-1}\tilde{q}).
\]

Since both particles satisfy the spin quantisation condition this is equivalent to

\[
\varrho(c)\Psi = \varrho(\Omega)\Psi.
\]

To solve this condition explicitly we change coordinates from \( \tilde{q} \) to a relative boost and an overall Lorentz transformation, as in our discussion in sect. 3.

In order to keep the notation simple, we use the abbreviation \( g_1 \) and \( g_2 \) for the group valued momenta \((3.21)\) of particle 1 and 2, i.e

\[
g_1 = \exp(-p_1^a J_a) \quad g_2 = \exp(-p_2^a J_a).
\]

The total group valued momentum is the product \( g_1 g_2 \). Here we encounter a subtlety of momenta in (2+1)-dimensional gravity. It is possible for the total group valued momentum not to be in an elliptic conjugacy class of \( \tilde{L}_3^1 \). In order to understand this phenomenon we use a Lorentz transformation to move to the rest frame of particle 1. In that frame the group valued momentum of particle 1 is the rotation element \( g_1 = r(\mu_1) \in \tilde{L}_3^1 \). The second particle will in general be moving in this frame, but using rotations we can assume that it is moving along the 1-axis. Then, with the notation \((A.19)\), the group valued momentum of the second particle in the rest frame of the first is

\[
g_2 = b(\xi)r(\mu_2)b^{-1}(\xi) = \exp(-\mu_2 \cosh \xi J_0 - \mu_2 \sinh \xi J_1).
\]
As first discussed by Gott in [30] the total group valued momentum fails to be in an elliptic class when the second particle is moving too fast relative to the first. Quantitatively, the condition for an element \( g \in \tilde{L}_3^+ \) to be in an elliptic conjugacy class is 
\[-1 < \text{tr}(U(g)) < 1.\]
Applying this to the product \( g_1g_2 \) and evaluating in the rest frame of particle 1 one finds
\[
\cosh \xi \sin \frac{\mu_1}{2} \sin \frac{\mu_2}{2} < 1 + \cos \frac{\mu_1}{2} \cos \frac{\mu_2}{2}.
\] (6.11)

We postpone the very interesting discussion of what happens when this condition is violated. Assuming that it is satisfied we can find a Lorentz transformation \( w(\xi) \) such that
\[
r(\mu_1)b(\xi)r(\mu_2)b^{-1}(\xi) = w(\xi)r(\mu(\xi))w^{-1}(\xi).
\] (6.12)

This condition defines the invariant mass \( \mu(\xi) \) of the two particle system. Under the assumption of the condition (6.11) the range of the invariant mass is restricted:
\[
\mu_1 + \mu_2 < \mu(\xi) < 2\pi.
\] (6.13)

As in our discussion of special relativistic particles in sect. 3 the invariant mass and the Lorentz transformation \( w \) both depend on \( \xi, \mu_1 \) and \( \mu_2 \), but we suppress the dependence on \( \mu_1 \) and \( \mu_2 \) in our notation. Note that the function \( \mu(\xi) \) is different from the (rescaled) non-gravitational invariant mass \( 8\pi G M(\xi) \) defined in (3.28) but approaches it in the limit of small relative speed \( \xi \to 0 \). Note also that, unlike \( V \) in (3.27) the element \( w \in \tilde{L}_3^+ \) will not generally be a boost along the 1-axis. In other words, the momenta \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) in the centre of mass frame do not point in opposite directions in the sense of flat space physics. To see this explicitly, consider the condition of being in the centre of mass frame:
\[
g_1g_2 = r(\mu)
\] (6.14)
for some given invariant mass \( \mu \). Suppose that particle 1 is moving along the 1-axis, i.e.
\[
g_1 = \exp(-\mu_1 \cosh \vartheta_1 J_0 - \mu_1 \sinh \vartheta_1 J_1).
\] (6.15)

Then solving the condition (6.14) for \( g_2 \) we find
\[
g_2 = \exp(-\mu_2 \cosh \vartheta_2 J_0 - \mu_2 \sinh \vartheta_2 \cos \varphi J_1 - \mu_2 \sinh \vartheta_2 \sin \varphi J_2)
\] (6.16)
with \( \varphi = \pi + \mu/2 \) and the masses and rapidities related by
\[
\cos \frac{\mu_2}{2} = \cos \frac{\mu_1}{2} \cos \frac{\mu}{2} + \sin \frac{\mu_1}{2} \sin \frac{\mu}{2} \cosh \vartheta_1
\]
\[
\sin \frac{\mu_1}{2} \sinh \vartheta_1 = \sin \frac{\mu_2}{2} \sin \vartheta_2.
\] (6.17)

In other words, in the centre of mass frame the directions of motion are related by a rotation by \( \varphi = \pi - \mu(\xi)/2 \) instead of the familiar \( \pi \). The momenta thus behave as if they belonged to particles moving in opposite directions on a cone of deficit angle \( \mu(\xi) \).
Despite these differences with the situation in sect. 3 we can again change variables from the momenta of the particles to a relative boost parameter $\xi > 0$ and an overall Lorentz transformation. We define a new set of reference velocities

$$q_1(\xi) = \Lambda(w^{-1}(\xi))(1,0,0)^t \quad \text{and} \quad q_2(\xi) = \Lambda(w^{-1}(\xi)b(\xi))(1,0,0)^t$$  \hspace{1cm} (6.18)

and the bijection

$$K : \mathcal{L}^\uparrow_3 \times \mathbb{R}^+ \rightarrow \mathcal{M}_2$$  \hspace{1cm} (6.19)

$$(L, \xi) \mapsto (Lq_1(\xi), Lq_2(\xi)) = (p_1, p_2).$$  \hspace{1cm} (6.20)

With our assumption that the group valued the total momentum is in an elliptic conjugacy class we can define a linear total momentum $\mathbf{P}$ of the two particle system via

$$\tilde{\exp}(-p_a^1 J_a) \tilde{\exp}(-p_a^2 J_a) = \tilde{\exp}(-P_a^a J_a).$$  \hspace{1cm} (6.21)

In terms of the invariant mass $\mu(\xi)$ and the parameterisation $[3.34]$ of $L$ and we find that the expression for $\mathbf{P}$ has the familiar form

$$\mathbf{P} = (\mu(\xi) \cosh \vartheta, \mu(\xi) \sinh \vartheta \cos \theta, \mu(\xi) \sinh \vartheta \sin \theta).$$  \hspace{1cm} (6.22)

For the application in the quantum theory we again need to lift the map (6.19) to the covers $\tilde{\mathcal{L}}^\uparrow_3 \times \mathbb{R}^+$ and $\tilde{\mathcal{M}}_2$. Imitating the steps leading up to (3.39) we pick a lift $\tilde{\mathbf{q}}(\xi)$ of the reference momentum $(q_1(\xi), q_2(\xi)) \in \mathcal{M}_2$. Then, using the action of $\tilde{\mathcal{L}}^\uparrow_3$ on $\tilde{\mathcal{M}}_2$ we define the bijection

$$\tilde{K} : \tilde{\mathcal{L}}^\uparrow_3 \times \mathbb{R}^+ \rightarrow \tilde{\mathcal{M}}_2$$  \hspace{1cm} (6.23)

$$(u, \xi) \mapsto u \tilde{\mathbf{q}}(\xi).$$  \hspace{1cm} (6.24)

The crucial aspect of the map $K$ and its cover is that it preserves the total group valued momentum in the following sense. Suppose $\tilde{\mathbf{q}} = u \tilde{\mathbf{q}}(\xi)$ covers the pair of velocity vectors $(\mathbf{q}_1, \mathbf{q}_2)$ and that $\tilde{K}(u, \xi) = \tilde{\mathbf{q}}$. The one checks that

$$g(\tilde{\mathbf{q}}) = ur(\mu(\xi))u^{-1},$$  \hspace{1cm} (6.25)

where $g(\tilde{\mathbf{q}})$ is the total momentum associated to $\tilde{\mathbf{q}}$ via (3.34). At the level of representations this relation implies that the pull-back of elements of $L^2(\tilde{\mathcal{M}}_2, \mathbb{C})$ to $L^2(\tilde{\mathcal{L}}^\uparrow_3 \times \mathbb{R}^+, \mathbb{C})$ via $\tilde{K}^*$ intertwines between the Lorentz double action

$$\Pi(F) \Phi(x, \xi) = \int_{\mathcal{L}^\uparrow_3} dw F(xr(\mu(\xi))x^{-1}, w) \Phi(w^{-1}x, \xi)$$  \hspace{1cm} (6.26)

on $L^2(\tilde{\mathcal{L}}^\uparrow_3 \times \mathbb{R}^+, \mathbb{C})$ and the action (3.4) on $L^2(\tilde{\mathcal{M}}_2, \mathbb{C})$.

Proceeding further along the lines of the anyon discussion in sect. 3 we use this interwiner to express our condition (6.8) on physical two-particle states in terms of
elements of $L^2(\tilde{L}_3^+ \times \mathbb{R}^+, \mathbb{C})$. Because of the intertwining property the condition takes the same form

$$\Pi(c)\Phi = \Pi(\Omega)\Phi.$$  \hfill (6.27)

The Hilbert space of physical two particle states is thus

$$\mathcal{H}_2 = \{ \Phi \in L^2(\tilde{L}_3^+ \times \mathbb{R}^+, \mathbb{C})| \Pi(c)\Phi = \Pi(\Omega)\Phi \}. \hfill (6.28)$$

The advantage of expressing the condition (6.27) on the space $L^2(\tilde{L}_3^+ \times \mathbb{R}^+, \mathbb{C})$ is that it can now be solved explicitly. First we decompose $L^2(\tilde{L}_3^+ \times \mathbb{R}^+, \mathbb{C})$ into eigenspaces of the right action of rotations $r(\psi)$:

$$V_s = \{ \Phi \in L^2(\tilde{L}_3^+ \times \mathbb{R}^+, \mathbb{C})|\Phi(xr(\psi), \xi)\Phi = e^{-i\psi s}\Phi(x, \xi) \forall \psi \in \mathbb{R} \}. \hfill (6.29)$$

A priori the eigenvalue $s$ can take arbitrary real values, so we have the direct integral decomposition

$$L^2(\tilde{L}_3^+ \times \mathbb{R}^+, \mathbb{C}) = \int_{s \in \mathbb{R}} ds V_s. \hfill (6.30)$$

Then note that for $\Phi \in V_s$

$$\Pi(c)\Phi = e^{i\mu(\xi)s}\Phi$$ \hfill (6.31)$$

and

$$\Pi(\Omega)\Phi = e^{2\pi is}\Phi. \hfill (6.32)$$

When we impose the equality (6.27) we select those component spaces $V_s$ satisfying the spin selection rule

$$e^{2\pi is} = e^{i\mu(\xi)s}. \hfill (6.33)$$

While this rule looks similar to (3.51) there is an important difference. In gravity, the total spin of the two particle system is quantised in units which depend on the centre of mass energy of the two particles. Solving the condition we find that

$$s = \frac{n}{1 - \frac{\mu(\xi)}{2\pi}}, \quad n \in \mathbb{Z}. \hfill (6.34)$$

The two-particle Hilbert space can thus be decomposed into irreps of the Lorenzt double as follows:

$$\mathcal{H}_2 = \int_{\mu_1 + \mu_2}^{2\pi} d\mu \bigoplus_s V_{\mu s}, \hfill (6.35)$$

where, for given $\mu$, we sum over all $s$ of the form (6.34). Each component space $V_{\mu s}$ thus satisfies the one-particle spin quantisation condition. Recall that the range of

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the invariant mass in the direct integral is determined by our imposition of the Gott condition according to (6.13).

Much of the physics of two gravitating particles is contained in the quantistation condition (6.34). The orbital angular momentum $l$ in the centre of mass frame of the two particles is related to $s$ via

$$l = s - s_2 - s_2.$$  \hspace{1cm} (6.36)

It follows that for spinless particles the orbital angular momentum is quantised according to the rule

$$l = \frac{n}{1 - \frac{\mu(\xi)}{2\pi}}, \quad n \in \mathbb{Z}. \hspace{1cm} (6.37)$$

This rule generalises the formula (2.25) and agrees with the quantisation condition found in [3]. The more general formula (6.36) for particles with spin implies

$$l = \frac{n}{1 - \frac{\mu(\xi)}{2\pi}} - s_1 - s_2 \quad n \in \mathbb{Z}, \hspace{1cm} (6.38)$$

which can be written as

$$l = \frac{\tilde{n}}{1 - \frac{\mu(\xi)}{2\pi}} + \frac{1}{1 - \frac{\mu(\xi)}{2\pi}} \left( \frac{\mu(\xi) - \mu_1}{2\pi} s_1 + \frac{\mu(\xi) - \mu_2}{2\pi} s_2 \right), \quad \tilde{n} \in \mathbb{Z}. \hspace{1cm} (6.39)$$

with the simple shift $\tilde{n} = n - n_1 - n_2$ by the integers characterising the individual spins according to (6.3). This formula should be compared with various conjectured formulae for the quantisation of orbital angular momentum of two gravitating particles with spin. In [31] the situation of one spinning particle moving in the background of one heavy particle is considered and in [29] the authors study the Dirac equation on a “spinning cone”. The quantisation rule implicit in the (conjectured) general scattering cross section found in [29] is

$$l = \frac{\tilde{n}}{1 - \frac{\mu(\xi)}{2\pi}} + \frac{1}{1 - \frac{\mu(\xi)}{2\pi}} (E_2 s_1 + E_1 s_2), \quad \tilde{n} \in \mathbb{Z}, \hspace{1cm} (6.40)$$

where $E_1$ and $E_2$ are the energies of particles 1 and 2. We note that this formula and also the one conjectured in [31] contain the term (6.37) with the spin-dependent correction term containing products of the form “spin of one particle” $\times$ “energy associated with the other particle”. Our formula is also of that form, but the details are different. We stress that our result is obtained by a calculation in a completely relativistic framework which treats both particles on equal footing, whereas the general formulae in [29] and [31] are conjectured generalisations of results obtained by studying the motion of a lighter particle in the background geometry due to a heavier particle. Our result agrees with the results of [29] in the limiting case where one particle, say particle 1, is at rest and the second particle is light compared to the first and moving slowly. In that case $E_1 = \mu_1$ and $\mu \approx \mu_1 + E_2$ so that $s_1(\mu - \mu_1) \approx s_1 E_2$. If $E_2 - \mu_2$ small compared to $\mu_2$ then we also have $s_2(\mu - \mu_2) \approx s_2 E_1$. The relative factor of $2\pi$ is due to a different choice of units in [29].

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6.2 Gravitational scattering revisited

Exploiting further our analogy with non-gravitational anyons in sect. 3 it is now not difficult to compute cross sections for gravitational scattering of massive particles. One important aspect of the calculations in sect. 3 which makes their adaptation to gravity straightforward is that they were formulated entirely in momentum space. We shall see that gravitational scattering in 2+1 dimensions can be studied and computed entirely in momentum space. As always when comparing the outcome of a quantum mechanical calculation with a (thought) experiment we need to use quantum states which have a classical interpretation. Just as in sect. 3 these will be (non-normalisable) momentum eigenstates. Thus, while we perform our calculation in the space (6.28) it is essential for the interpretation of our result that we can relate our computation to two particle momentum eigenstates via (6.23). The two particle scattering states have the same form as in (3.61):

\[
|p_1, p_2\rangle = \left(\frac{8\pi^2}{\mu_1 \mu_2}\right)^{2} \frac{1}{\sinh \vartheta_1} \frac{1}{\sinh \vartheta_2} \delta_{\vartheta_1} \delta_{\vartheta_2}. \tag{6.41}
\]

Following through the steps leading to (3.63) we write this state in terms of the total momentum \(P\) (6.22) and the angle \(\phi\) which gives the spatial orientation of the momenta in the centre of mass frame (recall that the relative spatial angle between the momenta in the centre of mass frame is fixed to be \(\pi - \mu(\xi)/2\)):

\[
|p_1, p_2\rangle = (8\pi^2)^2 \mu(\xi) \delta_{P_0} \delta_{P_1} \delta_{P_2} \delta_{\phi}. \tag{6.42}
\]

For gravitational scattering, the scattering states are completely characterised by the momenta. Fixing the relative boost parameter \(\xi\) and hence invariant mass \(\mu(\xi)\) we introduce the abbreviation

\[
\alpha(\xi) = 1 - \frac{\mu(\xi)}{2\pi}. \tag{6.43}
\]

Scattering states have a definite total invariant mass but contain all allowed spin states. They should be thought of as non-normalisable elements of the direct sum

\[
\bigoplus_{s = \frac{\pi(\xi)}{\alpha(\xi)}, n \in \mathbb{Z}} V_{\mu(\xi)s} \tag{6.44}
\]

appearing in the direct integral decomposition (6.33). Scattering states can therefore be Fourier expanded as

\[
|p_1, p_2\rangle = 4(2\pi)^3 M(\xi) \delta_{P_0} \delta_{P_1} \delta_{P_2} \sum_{n \in \mathbb{Z}} e^{i\frac{n}{2}} |s\rangle. \tag{6.45}
\]

The \(S\)-matrix is again constructed according to the general prescription (3.58). It is diagonal in each of the components \(V_{\mu s}\) of the decomposition (6.33) and its eigenvalues in the space \(V_{\mu s}\) are

\[
S^{(l)} = \begin{cases} 
  e^{-\frac{i}{2}(\mu_s - \mu_1 s_1 - \mu_2 s_2)} & \text{if } [l] \geq 0 \\
  e^{\frac{i}{2}(\mu_s - \mu_1 s_1 - \mu_2 s_2)} & \text{if } [l] < 0,
\end{cases} \tag{6.46}
\]

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where again $l = s - s_1 - s_2$. Note that

$$e^{i(\mu s - \mu_1 s_1 - \mu_2 s_2)} = e^{2\pi i(s - s_1 - s_2)} = e^{2\pi il}$$

(6.47)

so that we have the simple formula

$$S^{(l)} = \begin{cases} e^{-il} & \text{if } [l] \geq 0 \\ e^{il} & \text{if } [l] < 0, \end{cases}$$

(6.48)

with all the physics in the quantisation condition for $s$ (and hence $l$).

Following the manipulations in sect. 3 we define the reaction matrix $T$ and its reduced matrix element as in (3.67) and (3.68) and find that it can be written as

$$\langle p_f^1, p_f^2 | T | p_i^1, p_i^2 \rangle = 4\mu(\xi) t(\xi, \varphi^i - \varphi^f),$$

(6.49)

where the function $t(\xi, \varphi)$ encodes the dependence of the scattering on the scattering angle $\varphi = \varphi^i - \varphi^f$. Note that, unlike its non-gravitational counterpart (3.72) it also depends on the rapidity $\xi$. Before we discuss it in detail we show how it enters the expression for the differential cross section. Mimicking the derivation of phase space factors for anyon scattering in the calculation leading up to (3.75) we arrive at the following expression for the differential cross section in the centre of mass frame:

$$\frac{d\sigma}{d\varphi}(\varphi, \xi) = \frac{\tilde{M}(\xi)\hbar}{2\pi M_1 M_2 \sinh \xi} |t(\xi, \varphi)|^2,$$

(6.50)

where we have re-instated $\hbar$ and written down the masses in physical units i.e. $\mu_1 = 8\pi GM_1$, $\mu_2 = 8\pi GM_2$ and $\mu(\xi) = 8\pi G\tilde{M}(\xi)$. Reverting to the masses expressed in units of $(8\pi G)^{-1}$ and recalling that the Planck length is $\ell_p = \hbar G$ we have the expression

$$\frac{d\sigma}{d\varphi}(\varphi, \xi) = 4\frac{\mu(\xi)\ell_p}{\mu_1 \mu_2 \sinh \xi} |t(\xi, \varphi)|^2.$$

(6.51)

Let us first compute the function $t(\xi, \varphi)$ for the case of spinless particles. In that case $s = l$ and the analogue of the formula (3.71) is

$$it(\xi, \varphi) = \sum_l (S^{(l)} - 1) e^{il\varphi} = \sum_{n \in \mathbb{Z}} \left( e^{-i\frac{|n|}{\alpha(\xi)}} - 1 \right) e^{in\varphi/\alpha(\xi)}.$$

(6.52)

This leads to a scattering cross section (6.51) in agreement with the result of Deser and Jackiw [15] in the limit where one particle is very heavy and the other very light (so that the invariant mass is approximately equal to the mass of the heavy particle). The divergence in $t$ can be regularised in the way described in [15] leaving the finite contribution

$$it(\xi, \varphi) = \frac{1}{2} \left( \cot \frac{\varphi - \pi}{2\alpha(\xi)} - \cot \frac{\varphi + \pi}{2\alpha(\xi)} \right).$$

(6.53)
In our formalism it is easy to deal with spinning particles. The amplitude function \( t \) for particles of spin \( s_1 \) and \( s_2 \) is again given by the general expression (3.71). We find:

\[
\begin{align*}
  i t(\xi, \varphi) &= \sum_{n \in \mathbb{Z}} \left( e^{-i\pi n|l|} - 1 \right) e^{i s_0} \\
  &= \sum_{n \in \mathbb{Z}} \left( e^{-i\pi |s_1 - s_2|} - 1 \right) e^{i s_0} \\
  &= \sum_{n \geq -[-\alpha(s_1 + s_2)]} \left( e^{i\pi(s_1 + s_2)} e^{i \frac{n(\varphi - \pi)}{\alpha}} - e^{i \frac{m_2}{\alpha}} \right) + \\
  &\quad \sum_{n < -[-\alpha(s_1 + s_2)]} \left( e^{-i\pi(s_1 + s_2)} e^{i \frac{n(\varphi + \pi)}{\alpha}} - e^{i \frac{m_1}{\alpha}} \right), \quad (6.54)
\end{align*}
\]

where we have suppressed the explicit dependence of \( \mu \) and \( \alpha \) on \( \xi \) for notational simplicity. The infinite sums an again be treated as described in [29]. The finite part is

\[
\begin{align*}
  i\tilde{t}(\xi, \varphi) &= \frac{1}{2} e^{-i[\alpha(s_1 + s_2)]/2} \left[ e^{-i\pi/2}(-\alpha(s_1 + s_2)) \left( \cot \frac{\varphi - \pi}{2\alpha} - i \right) \\
  &\quad - e^{i\pi/2}(-\alpha(s_1 + s_2)) \left( \cot \frac{\varphi + \pi}{2\alpha} - i \right) \right], \quad (6.55)
\end{align*}
\]

where \( \{x\} = x - [x] \) denotes the fractional part of \( x \). It is very interesting to compare this result with the conjectured universal scattering amplitude given in the summary of [29]. There is a trivial difference in kinematical factors which arises because we work in the center of mass frame and our definition of \( t \) is such that it is stripped of all explicit momentum dependence. The formula given in [29] is structurally similar to ours and we can recover it in an appropriate limit if we can relate our \( \alpha(s_1 + s_2) \) to their \( E_1 s_2 + E_2 s_1 \), where \( E_1 \) and \( E_2 \) are again the energies of particles 1 and 2. The key to this relation lies again in the spin quantisation condition of the individual particles (6.3). Since \( (1 - \frac{\mu}{2\pi}) s_1 \) and \( (1 - \frac{\mu}{2\pi}) s_2 \) are both integers we have

\[
\{-\alpha(s_1 + s_2)\} = \left\{ \frac{(\mu - \mu_1)}{2\pi} s_1 + \frac{(\mu - \mu_2)}{2\pi} s_2 \right\}. \quad (6.56)
\]

Following the argument given in the discussion of the quantisation condition (6.40) this expression reduces to \( \{E_2 s_1 + E_1 s_2\} \) in the limit where particle 1 is very heavy and at rest and particle 2 is a slow and light test particle.

## 7 Discussion and outlook

For gravitating particles in 2+1 dimensions, space-time symmetry and internal symmetry are unified in the Lorentz double. Much of this paper has been devoted to justifying this statement and to exploring its consequences. To end, we highlight and discuss some of our key results.

The Lorentz double allows us to give a precise a description of the Hilbert space for the gravitational system consisting of one or several gravitating particles in 2+1
dimensions. It does so in a way which generalises and makes precise various notions which have appeared in the literature on (2+1)-dimensional gravity. For example, our definition of the one-particle Hilbert space (6.1) captures the idea, expressed e.g. in [2], that one should consider irreps of the universal cover of the Poincaré group and pick out the physically allowed spin values by a suitable mass-dependent condition. The ribbon element of the Lorentz double gives a precise algebraic implementation of that idea. For several particles, the role of the braid group in defining the Hilbert space has been anticipated in many publications, e.g. [2], [32], [33], [34], [35], [36] and [37]. Our formulation (6.28) is in the spirit of these papers. However, drawing on the analogy with ordinary anyons and using the representation of the braid group furnished by the Lorentz double we are able to give a completely general and invariant definition of the Hilbert space of gravitating particles with spins.

Finally we have shown that the interaction of gravitating particles in 2+1 dimensions can be computed by viewing them as gravitational anyons. The interaction of ordinary anyons is dictated by the universal $R$-matrix of a suitable ribbon Hopf algebra and here we have seen that, by drawing a careful analogy, the interactions of gravitating particles can be computed from the $R$-matrix of the Lorentz double. The results agree with older semi-classical calculations in certain limits and generalise them in a simple and very natural fashion. In particular we claim that the combination of (6.55) with (6.51) gives a completely general formula for the gravitational scattering cross section of two spinning particles in the centre of mass frame, provided their relative speed satisfies the condition (6.11).

It is clear that the formalism developed here is powerful enough to handle more general situations than we have discussed. It would be interesting, for example, to study the scattering when the relative speed violates the condition (6.11) and to investigate the interpretations of elliptic representations of the Lorentz double when the total mass exceeds 2$\pi$. Also, our computations could be extended to deal with several particles, possibly moving in universes with non-trivial topology.

It remains an interesting challenge to give a detailed derivation of our results in the framework of combinatorial quantisation of Chern-Simons theory. Recent work on the combinatorial quantisation of Chern-Simons theory with gauge group $SL(2,\mathbb{C})$ (corresponding to Lorentzian gravity with a cosmological positive constant) in [38] and [39] shows how these questions can be addressed but also illustrates the level of technical difficulty involved.

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A The universal cover of the Lorentz group in 2+1 dimensions

Our description of the universal cover of $L^\uparrow_3$ follows essentially that given in [7], which in turn follows [40]. As an intermediate step it is best to consider the double cover $SU(1,1)$. This group is a subgroup of $SL(2,\mathbb{C})$, consisting of all $U \in SL(2,\mathbb{C})$ satisfying

$$U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \text{(A.1)}$$

Elements can be written as

$$U = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

with the condition

$$|\alpha|^2 - |\beta|^2 = 1. \quad \text{(A.3)}$$

For later use we also introduce generators for the Lie algebra of $SU(1,1)$:

$$T_0 = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \quad \text{(A.4)}$$

They satisfy

$$T_a^\dagger = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T_a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{(A.5)}$$

and the algebraic relation

$$T_a T_b = -\frac{1}{4} \eta_{ab} + \frac{1}{2} \epsilon_{abc} T^c, \quad \text{(A.6)}$$

where indices are raised with the Minkowski metric $\eta_{ab} = \text{diag}(1,-1,-1)$ and $\epsilon_{012} = 1$. A useful corollary is the following formula for the exponential

$$\exp(-p^a T_a) = \cos \frac{\mu}{2} - \frac{2}{\mu} \sin \frac{\mu}{2} p^a T_a, \quad \text{(A.7)}$$

where $p$ is a time-like vector with $p^2 = \mu^2$. We also deduce the commutation relations

$$[T_a, T_b] = \epsilon_{abc} T^c \quad \text{(A.8)}$$

and the normalisation

$$\text{tr} (T_a T_b) = -\frac{1}{2} \eta_{ab}. \quad \text{(A.9)}$$

The adjoint action of $SU(1,1)$ on its Lie algebra gives an explicit homomorphism $\Lambda : SU(1,1) \to L^\uparrow_3$. More precisely, we define the $3 \times 3$ matrix $\Lambda(U)$ via

$$U T_a U^{-1} = \sum_{b=0}^{2} \Lambda(U)_{ba} T_b \quad \text{(A.10)}$$

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and find, in terms of the parametrisation \([A.2]\),
\[
\Lambda(U) = \begin{pmatrix}
|\alpha|^2 + |\beta|^2 & -2\text{Re}(\alpha \overline{\beta}) & 2\text{Im}(\alpha \overline{\beta}) \\
-2\text{Re}(\alpha \overline{\beta}) & \text{Re}(\alpha^2 + \beta^2) & -\text{Im}(\alpha^2 - \beta^2) \\
-2\text{Im}(\alpha \overline{\beta}) & \text{Im}(\alpha^2 + \beta^2) & \text{Re}(\alpha^2 - \beta^2)
\end{pmatrix}.
\]
(A.11)

We define generators \(J_a\) of the Lie algebra \(L^+_3\), denoted \(so(2,1)\), via
\[
J_a = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \Lambda(\exp(\epsilon T_a)).
\]
(A.12)

These are used in the main text, where \(so(2,1)\) is physically interpreted as momentum space.

There are other ways of parametrising \(SU(1,1)\) matrices which are used in the
main text. One set of parameters is analogous to the Euler angles used for \(SU(2)\)
matrices. Since elements of the form \(\exp(-\varphi T_0)\) correspond to (anti-clockwise) rotation
in the \(p_1p_2\)-plane and elements of the form \(\exp(-\vartheta T_2)\) correspond to finite boosts along
the 1-axis, we can parametrise a general element \(U \in SU(1,1)\) in terms of two angles \(\varphi \in [0, 2\pi)\), \(\psi \in [0, 4\pi)\) and a boost parameter \(\vartheta \in [0, \infty)\) via
\[
U(\varphi, \vartheta, \psi) = e^{-\varphi T_0} e^{-\vartheta T_2} e^{-\psi T_0}.
\]
(A.13)

Clearly there is a corresponding parametrisation of \(L^+_3\) matrices
\[
L(\varphi, \vartheta, \psi) = e^{-\varphi J_0} e^{-\vartheta J_2} e^{-\psi J_0}
\]
in terms of angles \(\varphi \in [0, 2\pi)\), \(\psi \in [0, 2\pi)\) and a boost parameter \(\vartheta \in [0, \infty)\).

Another parametrisation is useful for understanding the universal cover of \(SU(1,1)\). Let \(D = \{\gamma \in \mathbb{C}| |\gamma| < 1\}\) be the open unit disk. Then elements of \(SU(1,1)\) can be
parametrised in terms of \(\gamma \in D\) and \(\omega \in [0, 4\pi)\) via
\[
U(\gamma, \omega) = (1 - \gamma \overline{\gamma})^{-\frac{1}{2i}} \begin{pmatrix}
\gamma e^{\frac{i}{2} \omega} & \gamma e^{-\frac{i}{2} \omega} \\
\overline{\gamma} e^{-\frac{i}{2} \omega} & \overline{\gamma} e^{\frac{i}{2} \omega}
\end{pmatrix}.
\]
(A.15)

The universal cover \(\tilde{L}^+_3\) can be identified with the set
\[
\{(\gamma, \omega)|\gamma \in D, \ \omega \in \mathbb{R}\}.
\]
(A.16)

The group multiplication is \((\gamma_1, \omega_1)(\gamma_2, \omega_2) = (\gamma_3, \omega_3)\) where
\[
\gamma_3 = \frac{\gamma_2 + \gamma_1 e^{-i\omega_2}}{1 + \gamma_1 \overline{\gamma_2} e^{-i\omega_2}}
\]
\[
\omega_3 = \omega_1 + \omega_2 + \frac{1}{i} \ln \left(1 + \frac{1 + \gamma_1 \overline{\gamma_2} e^{-i\omega_2}}{1 + \gamma_1 \overline{\gamma_2} e^{i\omega_2}}\right)
\]
(A.17)

with the logarithm defined in terms of the power series. We use small latin letters
\(u, v, w, x, \ldots\) to denote elements of \(\tilde{L}^+_3\). We also introduce special names for elements of
two subgroups of \(\tilde{L}^+_3\), both isomorphic to \(\mathbb{R}\). We write
\[
r(\mu) = (0, \mu), \quad \mu \in \mathbb{R},
\]
(A.18)
for elements which get mapped to (anti-clockwise, i.e. mathematically positive) spatial rotations \( \exp(-\mu J_0) \) under the map (A.23) and

\[
b(\xi) = (-\tanh \frac{\xi}{2}, 0), \quad \xi \in \mathbb{R},
\]

for elements which get mapped to boosts \( \exp(-\xi J_2) \) along the 1-axis under (A.23).

In the main text we also require the exponential map from the Lie algebra \( so(2, 1) \) to \( \tilde{L}_3^\uparrow \). Since \( \tilde{L}_3^\uparrow \) is not a matrix group, this exponential map cannot be written in terms of a power series of matrices, but has to be defined in differential geometric terms, as for example in [41]. We denote the exponential map by

\[
\exp : so(2, 1) \to \tilde{L}_3^\uparrow \quad (A.20)
\]

and note that

\[
\exp(-\mu J_0) = r(\mu), \quad \text{and} \quad \exp(-\xi J_2) = b(\xi).
\]

As we shall discuss in detail in appendix B, this map is neither injective nor surjective.

One checks that the map

\[
U : \tilde{L}_3^\uparrow \to SU(1, 1), \quad (\gamma, \omega) \mapsto U(\gamma, \omega), \quad (A.22)
\]

where \( U(\gamma, \omega) \) is defined as in (A.13), defines a group homomorphism with kernel \( \{(0, 4\pi n)|n \in \mathbb{Z}\} \simeq \mathbb{Z} \). Finally defining \( \Lambda(\gamma, \omega) := \Lambda(U(\gamma, \omega)) \) we have the group homomorphism

\[
\Lambda : \tilde{L}_3^\uparrow \to \tilde{L}_3^\uparrow, \quad (\gamma, \omega) \mapsto \Lambda(\gamma, \omega) \quad (A.23)
\]

with kernel \( \{(0, 2\pi n)|n \in \mathbb{Z}\} \simeq \mathbb{Z} \). This kernel is in fact the centre of \( \tilde{L}_3^\uparrow \). Its generator plays an important role in our discussion. It is the rotation by \( 2\pi \), for which we introduce a special name

\[
\Omega = r(2\pi). \quad (A.24)
\]

### B Irreducible representations of \( D(\tilde{L}_3^\uparrow) \)

As explained in the main text, irreps of \( D(\tilde{L}_3^\uparrow) \) are labelled by conjugacy classes in \( \tilde{L}_3^\uparrow \) together with centraliser representations. Here we list the conjugacy classes and the centraliser groups of specified elements in the conjugacy class. We give a physical interpretation of the conjugacy classes and point out whether they lie in the image of the exponential map (A.20). The classification of the conjugacy classes in \( SL(2, \mathbb{R}) \) can be found in [27]. Since \( SL(2, \mathbb{R}) \) is conjugate to \( SU(1, 1) \) in \( SL(2, \mathbb{C}) \), one can translate the list given in [27] into a list of \( SU(1, 1) \) conjugacy classes, from which the conjugacy classes in \( \tilde{L}_3^\uparrow \) can be deduced.
1. Elliptic Representations

These are labelled by elliptic conjugacy classes, consisting of Lorentz transformations which are conjugate to a pure rotation (A.18). There is family of elliptic conjugacy classes labelled by the integers:

\[ E_\mu^n = \{ x\Omega^nx^{-1} | x \in \tilde{L}_3^\uparrow \}, \quad 0 < \mu < 2\pi, \ n \in \mathbb{Z}, \]  

where \( \Omega \) is the \( 2\pi \) rotation introduced in (A.24). The elliptic conjugacy classes can all be reached by exponentiating time-like momenta \( p_0 J_0 + p_1 J_1 + p_2 J_2 \) whose invariant mass \( |p| \) is not an integer multiple of \( 2\pi \). More precisely, defining

\[ E_\mu^n = \{ p_0 J_0 + p_1 J_1 + p_2 J_2 \in \text{so}(2,1) | p^2 = (2\pi n + \mu)^2, \ \text{sign}(p_0) = -\text{sign}(n) \}, \]  

where we take \( \text{sign}(0) \) to be +, the exponential map (A.20) maps \( E_\mu^n \) bijectively onto \( E_\mu^n \).

The centraliser of the \( \tilde{L}_3^\uparrow \) action on \( r(\mu) \) is the same for all \( \mu \neq 2\pi n \). It is the subgroup

\[ N_E = \{ r(\omega) \in \tilde{L}_3^\uparrow | \omega \in \mathbb{R} \}. \]  

This is the universal cover of the group of spatial rotations and is isomorphic to \( \mathbb{R} \).

2. Hyperbolic representations

These are labelled by hyperbolic conjugacy classes consisting of elements conjugate to a pure boost of the form (A.19). There is a family of hyperbolic conjugacy classes, again indexed by the integers

\[ H_\xi^n = \{ x\Omega^nb(\xi)x^{-1} | x \in \tilde{L}_3^\uparrow \}, \quad \xi \in \mathbb{R}^+. \]  

The hyperbolic classes \( H_\xi^0, \xi \in \mathbb{R}^+ \), can be obtained by exponentiating \( \text{so}(2,1) \) elements corresponding to space-like momenta \( p_0 J_0 + p_1 J_1 + p_2 J_2, \ p^2 = -\xi^2 \). The hyperbolic classes \( H_\xi^n \) for \( n \neq 0 \) are not in the image of the exponential map (A.20), showing in particular that the exponential map is not surjective.

The centraliser of the \( \tilde{L}_3^\uparrow \) action on \( b(\xi) \) is the same for all \( \xi \in \mathbb{R}^{>0} \). It is the subgroup

\[ N_H = \{ (\tanh \xi, 2\pi n) \in \tilde{L}_3^\uparrow | \xi \in \mathbb{R}, n \in \mathbb{Z} \}. \]  

Physically this is the group of boosts along the 1-axis and of spatial rotations by an integer multiple of \( 2\pi \). It is isomorphic to \( \mathbb{R} \times \mathbb{Z} \).

3. Parabolic representations

Consider the two elements

\[ v_+ = \left( \frac{1 + i}{2}, -\frac{\pi}{2} \right) \]  

and

\[ v_- = \left( \frac{-1 + i}{2}, \frac{\pi}{2} \right). \]  

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Then define the conjugacy classes

\[ P^n_+ = \{ x\Omega^n v_+ x^{-1} | x \in \tilde{L}^\uparrow_3 \}, \]  

(B.8)

and

\[ P^n_- = \{ x\Omega^n v_- x^{-1} | x \in \tilde{L}^\uparrow_3 \}. \]  

(B.9)

The conjugacy class \( P^0_+ \) can be obtained by exponentiating \( so(2, 1) \) elements corresponding to light-like momenta \( p^0 J_0 + p^1 J_1 + p^2 J_2, p^2 = 0, p_0 > 0 \) and the conjugacy class \( P^0_- \) can be obtained by exponentiating light-like momenta \( p^0 J_0 + p^1 J_1 + p^2 J_2, p^2 = 0, p_0 < 0 \). Physically, they thus correspond to the exponentiated forward and backward light cones. The hyperbolic classes \( P^n_\xi \) for \( n \neq 0 \) are not in the image of the exponential map (A.20), showing again that the exponential map is not surjective.

The centraliser of the \( \tilde{L}^\uparrow_3 \) action on both \( v_+ \) and \( v_- \) is the subgroup

\[ N_P = \left\{ \left( \frac{\lambda}{1 + i\lambda}, \arg(1 - i\lambda) + 2\pi n \right) \in \tilde{L}^\uparrow_3 | \lambda \in \mathbb{R}, n \in \mathbb{Z} \right\}. \]  

(B.10)

Physically this group consists of combined boosts and rotations and of rotations by integer multiples of \( 2\pi \). It, too, is isomorphic to \( \mathbb{R} \times \mathbb{Z} \).

4. Vacuum representations

Finally there are classes, again indexed by the integers, consisting of one element each:

\[ O^n = \{ \Omega^n \}. \]  

(B.11)

Each of these one element classes is the image of the mass hyperboloids

\[ H_n = \{ p^0 J_0 + p^1 J_1 + p^2 J_2 \in so(2, 1) | p^2 = 2\pi |n|, \operatorname{sign}(p_0) = -\operatorname{sign}(n) \}, \]  

(B.12)

under the exponential map, showing that the exponential map is not injective.

The centraliser of \( O^n \) is the entire group \( \tilde{L}^\uparrow_3 \).
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