Landau level broadening, hyperuniformity, and discrete scale invariance

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We study the energy spectrum of a two-dimensional electron in the presence of both a perpendicular magnetic field and a potential. In the limit where the potential is small compared to the Landau level spacing, we show that the broadening of Landau levels is simply expressed in terms of the structure factor of the potential. For potentials that are either periodic or random, we recover known results. Interestingly, for potentials with a dense Fourier spectrum made of Bragg peaks (as found, e.g., in quasicrystals), we find an algebraic broadening with the magnetic field characterized by the hyperuniformity exponent of the potential. Furthermore, if the potential is self-similar such that its structure factor has a discrete scale invariance, the broadening displays log-periodic oscillations together with an algebraic envelope.

In the presence of a magnetic field, the energy spectrum of noninteracting electrons in two dimensions is known to consist of Landau levels. These discrete energy levels are responsible for many remarkable phenomena among which the celebrated integer quantum Hall effect [1, 2]. Each Landau level has a macroscopic degeneracy which is proportional to the strength of the magnetic field. This degeneracy is expected to be lifted by a generic perturbation leading to a broadening of Landau levels that may have important physical consequences. For instance, plateaus observed in the Hall resistance are directly related to the broadening induced by disorder as realized early by Ando et al. [3–5]. Most studies on Landau level broadening focused on disordered systems (see Ref. 6 for a review) but the role played by periodic potentials has also triggered much attention following the original work of Rauh [7, 8]. Based on a free-electron picture, Rauh’s approach also allows to qualitatively understand the Landau level broadening in the small-field limit of the Hofstadter butterfly for periodic lattices [9, 10], although a quantitative analysis requires a semi-classical treatment [11]. Recently, the Hofstadter butterfly of some quasiperiodic systems has been investigated unveiling an unusual broadening of Landau levels [12] different from the one expected for periodic or disordered systems, hence suggesting a nontrivial mechanism for potentials with a dense set of Bragg peaks.

The goal of the present Letter is to provide a general framework to compute the broadening of Landau levels in the presence of an arbitrary potential. Our main result, given in Eq. (11), relates the variance of the lowest Landau level (LLL) to the structure factor of the perturbing potential (extension to higher-energy Landau levels is straightforward). This simple expression reproduces aforementioned results for disordered and periodic cases, but it also allows to investigate more subtle potentials (see Fig. 1 for a summary of the results). In particular, we find that when the Fourier spectrum of the potential is dense and made of Bragg peaks (as in quasicrystals), the variance of the LLL increases algebraically with the magnetic field [see Eq. (21)] with an exponent characterizing the hyperuniformity of the potential. This notion of hyperuniformity is commonly used to describe sets of points with unusually large suppression of density fluctuations at long wavelengths [13]. We also show that if the potential has a discrete scale invariance [14, 15], then the variance displays log-periodic oscillations together with a power-law envelope. To illustrate these results, we consider three examples of quasiperiodic potentials, for which we compute exactly the hyperuniformity exponent and the period of these oscillations, when it exists.

Landau levels perturbed by a potential.— To begin with, let us remind a few well-known results. The Hamiltonian describing a particle of mass $m$ and charge $e$ in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is given by

$$H_0 = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m}. \quad (1)$$

Here, we consider a two-dimensional system with a magnetic field perpendicular to the plane. Such a field can be described by the symmetric gauge where the vector potential reads $\mathbf{A} = B(-y/2, x/2, 0)$. The spectrum of $H_0$ consists in equidistant energy levels known as Landau levels.

![Sketch of the LLL variance $w^2$ as a function of the magnetic field $B$ for different perturbing potentials. Red: disordered $w^2 \sim B^2$ [3]. Green: hyperuniform with discrete scale invariance $w^2 \sim B^{2+\alpha} + log$-periodic oscillations, where $\alpha$ is the hyperuniformity exponent (this work). Blue: periodic $w^2 \sim e^{-\#/B}$ [8].](image-url)
levels
\[ E_n = \hbar \omega_c (n + 1/2), \forall n \in \mathbb{N}, \]  
(2)

where \( \omega_c = |eB|/m \) is the cyclotron frequency. Each Landau level has a degeneracy proportional to the sample area \( A \) and the magnetic field. In the following, we set \( \hbar = e = 1 \).

Our aim is to study the behavior of the Landau levels in the presence of a time-independent potential. Thus, we consider the following general Hamiltonian
\[ H = H_0 + V(x,y), \]  
(3)

and we assume that the magnitude of the potential is small compared to the Landau level spacing \( \omega_c \gg |V| \). In this regime, we can neglect the coupling between different Landau levels and use degenerate perturbation theory to compute the degeneracy splitting of a single level. Without loss of generality, we assume \( V(x,y) \geq 0 \) and \( B > 0 \) in the following.

**Variance of the LLL.** — For simplicity, in the following, we focus on the LLL corresponding to \( n = 0 \) for which the nonperturbed wavefunctions, in the thermodynamic limit, can be chosen as
\[ \varphi_l(z) = \langle x,y|l \rangle = \frac{1}{\sqrt{2\pi l_B^2}} z^l e^{-|z|^2/4}, \]  
(4)

where \( z = (x + iy)/l_B, l = 0, 1, ..., N_\phi - 1 \) is the angular momentum, \( N_\phi = A/(2\pi l_B^2) \gg 1 \) is the degeneracy of the LLL and \( l_B = 1/\sqrt{B} \) is the magnetic length.

To characterize the broadening of the LLL due to the potential, we consider the variance defined by
\[ w^2 = \frac{1}{N_\phi} \sum_{p=0}^{N_\phi-1} \epsilon_p^2 - \left( \frac{1}{N_\phi} \sum_{p=0}^{N_\phi-1} \epsilon_p \right)^2, \]  
(5)

where \( \epsilon_p \)'s are eigenenergies of \( H \) projected onto the LLL. This variance can be recast as
\[ w^2 = \frac{1}{N_\phi} \sum_{l=0}^{N_\phi-1} \sum_{l' = 0}^{N_\phi-1} \langle |l| |l'| \rangle^2 - \left( \frac{1}{N_\phi} \sum_{l=0}^{N_\phi-1} \langle |l| |l| \rangle \right)^2, \]  
(6)

so that one does not need to compute explicitly the \( \epsilon_p \)'s. Setting \( r(x,y) = r(\cos \theta, \sin \theta) \), a matrix element of the perturbation potential in the LLL basis \( \langle |l|, l = 0, ..., N_\phi - 1 \rangle \) reads
\[ \langle |l| |l'| \rangle = \int \frac{dq}{(2\pi)^2} \frac{\tilde{V}(q)}{2\pi \sqrt{|l|! |l'|!}} \]  
\[ \times \int_{0}^{\infty} \frac{dr}{l_c B} \left( \frac{r}{l_c B} \right)^{1+l+l'} e^{- \frac{r^2}{2l_c^2 B}} \int_{0}^{2\pi} d\theta e^{iqa \theta} e^{i\theta(l-l')}, \]  
(7)

where we introduced the Fourier transform of the potential
\[ \tilde{V}(q) = \int dr e^{-iqr} V(r). \]  
(8)

In the large-\( N_\phi \) (thermodynamical) limit one then gets:
\[ \sum_{l=0}^{\infty} \langle |l| |l| \rangle = \frac{\tilde{V}(0)}{2\pi l_B^2}, \]  
(9)
\[ \sum_{l=0}^{\infty} \sum_{l' = 0}^{\infty} \langle |l| |l'| \rangle^2 = \frac{1}{2\pi l_B^2} \int \frac{dq}{(2\pi)^2} |\tilde{V}(q)|^2 e^{-|q| l_B^2/2}. \]  
(10)

Finally, one obtains the following expression for the variance
\[ w^2 = \int \frac{dq}{(2\pi)^2} S(q) e^{-|q| l_B^2/2}, \]  
(11)

where we introduced the structure factor
\[ S(q) = \frac{\tilde{V}(q)^2}{A} (1 - \delta_{q,0}). \]  
(12)

The variance is therefore essentially equal to the integral of the structure factor over a disk of radius \( l_B^2 \) which is the main result of this Letter. Before discussing the most interesting case of a potential with a dense Fourier spectrum, let us first show that this expression allows one to recover known results for simple potentials.

**Periodic potential.** — For a periodic potential of strength \( V_0 \) with a single spatial frequency \( a^{-1} \)
\[ V(x,y) = V_0 \left[ \cos(2\pi x/a) + \cos(2\pi y/a) \right], \]  
(13)

Eq. (11) leads to
\[ w^2 = V_0^2 e^{- \frac{2\pi^2}{\pi^2}}, \]  
(14)

in agreement with the expression found by Rauh [8]. The generalization to Fourier spectra with a finite set of frequencies is straightforward, even if the potential is no longer periodic. In the zero-field limit, the LLL broadening is exponential and controlled by the smallest frequency. The case of a dense set of frequencies is more subtle.

**Random potential.** — Landau level broadening due to an uncorrelated random potential has been widely studied in the literature [6]. For the simple case of a random potential with zero mean and white-noise correlations
\[ \overline{V(r) \cdot V(r')} = 0, \]  
(15)
\[ \overline{V(r)V(r')} = (V_0 a)^2 \delta(r - r'), \]  
(16)

where the overline denotes the average over disorder realizations, Eq. (11) gives
\[ w^2 = V_0^2 \frac{a^2}{2\pi l_B^2} = V_0^2 \frac{B a^2}{2\pi}, \]  
(17)

in agreement with the result of Ando [5]. This result is very different from the one obtained for a potential with a finite number of frequencies discussed above.
Potential with a dense Fourier spectrum.— The most interesting situation comes from potentials with a dense Fourier spectrum made of Bragg peaks (as found, e.g., in quasicrystals). To aim this, let us consider a general potential

\[ V(\mathbf{r}) = V_0 a^2 \sum_{j=1}^{N} \delta(\mathbf{r} - \mathbf{r}_j), \tag{18} \]

built on a set of \( N \) scattering points located at position \( \mathbf{r}_j \) with a typical density \( a^{-2} \). The random potential discussed above belongs to this family.

Before proceeding further, let us stress that the exponential term in Eq. (11) acts as a smooth cutoff that eliminates wave vectors \( |\mathbf{q}| \gtrsim l_B^{-1} \). In order to analyze the behavior of \( w^2 \), we shall instead consider a sharp cutoff regularization by introducing the integrated intensity function

\[ Z(k) = \int_{|\mathbf{q}|<k} d\mathbf{q} S(\mathbf{q}), \tag{19} \]

so that one has

\[ w^2 \sim Z(k \sim 1/l_B). \tag{20} \]

This approximation clearly misses exponentially small terms so that, for the periodic case discussed previously [see Eq. (14)], it leads to \( w^2 = 0 \). Let us remind that Eq. (20) is valid in the perturbative regime where \( mV_0 \ll B \). In the following, we further focus on the case where \( B \ll 1/a^2 \) since, for many potentials, \( Z \) has a simple behavior in the \( k \sim 1/l_B \ll 1/a \) limit.

The integrated intensity function (also known as the spectral measure [16]), is commonly used to analyze sets of points with nontrivial structure factor [17]. In one-dimensional quasicrystals, \( Z \) is conjectured to have a power-law envelope [16–18] for \( k \ll 1/a \) (we shall see that this is also the case for two-dimensional quasicrystals). Assuming \( Z(k) \sim k^{2+\alpha} \), Eq. (20) leads to

\[ w^2 \sim B \frac{2^{2+\alpha}}{4}, \text{ for } mV_0 \ll B \ll 1/a^2, \tag{21} \]

establishing a relation between the broadening of the LLL and the so-called hyperuniformity exponent \( \alpha \) that characterizes the potential. For \( \alpha > 0 \), the potential is hyperuniform [17] whereas \( \alpha < 0 \) refers to hypo-uniformity (or anti-hyperuniformity [18]). The special case \( \alpha = 0 \) corresponds to a potential with a constant \( S \) such as the random potential considered previously [see Eq. (17)].

Interestingly, if \( Z \) further manifests discrete scale invariance, i.e., if there exists \( \lambda > 1 \) such that

\[ Z(k/\lambda) = Z(k)/\lambda^{2+\alpha}, \tag{22} \]

then one has

\[ Z(k) = k^{2+\alpha} F(\ln k/\ln \lambda), \tag{23} \]

where \( F(x+1) = F(x) \) (see examples below and Refs. 14 and 15 for review). As a result, the LLL variance \( w^2 \) displays log-periodic oscillations together with a power-law envelope in the small-\( B \) limit.

Examples of quasiperiodic potentials.— For illustration, let us consider some potentials of the form given in

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**FIG. 2.** Integrated density function \( Z \) of the Rauzy (left), Ammann-Beenker (center), and Penrose (right) tilings (see insets for illustrations). Top: Log-log plot (blue dots) together with the algebraic envelope \( k^{2+\alpha} \) (green line). Bottom: \( Z(k)/k^{2+\alpha} \) as a function of \( \ln k/\ln \lambda \) (\( \ln k \)) showing 1-periodic (nonperiodic) oscillations for tilings with (without) discrete scale invariance.
Eq. (18) where the points correspond to vertices of two-dimensional quasiperiodic tilings. For each tiling considered below, we computed exactly the structure factor $S$ (see Ref. 19 for details), the hyperuniformity exponent $\alpha$ characterizing the power-law behavior of $Z(k) \sim k^{2+\alpha}$, and the discrete scale invariance factor $\lambda$ defined in Eq. (22) when it exists. Numerical results displayed in Fig. 2 have been obtained by integrating more and more Bragg peaks of smaller and smaller intensities. In each case, we checked that the results were converged in the range considered. Units are taken such that $V_0 = \sqrt{A/(Na^2)}$ and $a = 1$, where $a$ is the edge length of the hypercubic lattice that is used to build the tiling in the standard cut-and-project method [20–23].

Let us first consider the twofold-symmetric Rauzy tiling [24]. The hyperuniformity exponent is $\alpha = 4$ but $Z$ has no discrete scale invariance. By contrast, for the eightfold-symmetric Ammann-Beenker tiling [25–27], the hyperuniformity exponent is $\alpha = 2$ and $Z$ has a discrete scale invariance with $\lambda = 1 + \sqrt{2}$. For the fivefold-symmetric Penrose tilings, the hyperuniformity exponent is $\alpha = 6$, and $Z$ has a discrete scale invariance with $\lambda = \tau^2$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

More generally, for a $d$-dimensional quasicrystal built from the cut-and-project method, we conjecture that $Z(k) \sim k^{d+\alpha}$, where $\alpha = d + 2\beta$ ($\beta \in \mathbb{N}$).

**Substitution tilings and discrete scale invariance.**—As recently conjectured by Oguz et al. [18], the behavior of $Z$ in one-dimensional substitution tilings is determined by the eigenvalues of the substitution matrix. More precisely, for non-periodic binary substitutions associated to a $2 \times 2$ substitution matrix with eigenvalues $\lambda_1 > |\lambda_2| > 0$, one has

$$Z(k/\lambda_1) = Z(k)(\lambda_2/\lambda_1)^2,$$

when $k$ tends to zero, so that

$$Z(k) = k^{1+\alpha}F(\ln k/\ln \lambda_1), \quad \alpha = 1 - 2\frac{\ln |\lambda_2|}{\ln \lambda_1},$$

with $F(x+1) = F(x)$. In two dimensions, it is very likely that the existence of substitution rules also implies the existence of discrete scale invariance for $Z$. This is clearly the case for the Ammann-Beenker and Penrose tilings that, contrary to the Rauzy tiling, can be built by substitution. However, we have not found a simple expression for the hyperuniformity exponent [such as the one given Eq. (25)] for two-dimensional binary substitutions.

**Outlook.**—In this Letter, we obtained a simple relation between the Landau level broadening and the integrated intensity function $Z$ of the perturbing potential. For potentials with a dense Fourier spectrum made of Bragg peaks, this relation implies that the variance of the LLL is driven by the hyperuniformity exponent $\alpha$ [see Eq. (21)]. In the absence of a complete classification of the possible behavior of $Z$, a first step to go beyond would consist in analyzing two-dimensional potentials with singular continuous Fourier spectrum for which one expects more complex behavior of $Z$ as observed in one dimension [16, 18]. For instance, one may find noninteger exponents $\alpha$ or even nonalgebraic decay. It would also be important to bridge the gap between the perturbed free-electron results and the one obtained numerically in tight-binding models [12]. A possible route would be to develop the analog of Wilkinson semi-classical treatment [11] for non-periodic potential.

Finally, let us mention that the magnetic-field dependence of the Landau level broadening induced by disorder has already been measured in graphene [28]. Combining such an experimental device with a nontrivial superlattice potential would allow to measure the behaviors discussed in the present work.

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Fourier transform of a cut-and-project quasicrystal

The cut-and-project (CP) method [20–23] consists in selecting points of a $D$-dimensional periodic lattice if their projection onto the $(D - d)$-dimensional perpendicular space $E_{\perp}$ belongs to a so-called acceptance window. The tiling is then obtained by projecting these selected points onto the complementary $d$-dimensional parallel space $E_{\|}$.

Any vector $\mathbf{v}$ in hyperspace can be decomposed uniquely in terms of its projection onto parallel and perpendicular spaces as

$$\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}.$$  (26)

As explained in the early papers introducing the CP method [21–23], the Fourier transform of quasiperiodic tilings can be computed from the higher-dimensional space it stems from. The main idea is that since points of the tiling are selected from a periodic tiling via an acceptance window, computing the Fourier transform of the tiling essentially amounts to compute the Fourier transform of this acceptance window.

For a tiling with $N$ sites (vertices) at position $\mathbf{R}_{j\|}$ and obtained by the CP method, the microscopic density is

$$n(\mathbf{r}_{\|}) = \sum_{j=1}^{N} \delta(\mathbf{r}_{\|} - \mathbf{R}_{j\|}),$$  (27)

and its Fourier transform is

$$\overline{n}(\mathbf{q}_{\perp}) = \sum_{j=1}^{N} e^{-i\mathbf{q}_{\perp} \cdot \mathbf{R}_{j\|}},$$  (28)

where the sum runs over all sites of the $d$-dimensional tiling considered. The convention that we use is that capital letters (such as $\mathbf{R}_{j\|}$) refer to discrete points and small letters (such as $\mathbf{r}_{\|}$) to a continuum of points.

Let $\mathbf{K}$ be a point of the $D$-dimensional hypercubic lattice and $\mathbf{K}$ a vector of its reciprocal lattice such that $\mathbf{K} \cdot \mathbf{R} = 2\pi \times \text{integer}$. These vectors can be decomposed onto the parallel and perpendicular spaces such that their scalar product reads

$$\mathbf{K} \cdot \mathbf{R} = \mathbf{K}_{\|} \cdot \mathbf{R}_{\|} + \mathbf{K}_{\perp} \cdot \mathbf{R}_{\perp} = 2\pi \times \text{integer}.$$  (29)

Equation (28) is non-zero iff $\mathbf{q}_{\perp} = \mathbf{K}_{\perp}$, in which case it becomes

$$\overline{n}(\mathbf{K}_{\perp}) = \sum_{j=1}^{N} e^{i \mathbf{K}_{\perp} \cdot \mathbf{R}_{j\|}}.$$  (30)

For a quasicrystal built along an irrational plane (parallel space), the points in perpendicular space densely and uniformly fill the acceptance window such that

$$\overline{n}(\mathbf{K}_{\perp}) = N \int_{A_{\perp}} \frac{d\mathbf{r}_{\perp}}{A_{\perp}} e^{i \mathbf{K}_{\perp} \cdot \mathbf{r}_{\perp}},$$  (31)
where the integral is over the acceptance window in perpendicular space and \( A_{\perp} \) is its \((D - d)\)-dimensional volume.

Now, for any vector \( \mathbf{q}_d \) in parallel space, the Fourier transform of the density Eq. (28) reads

\[
\tilde{n}(\mathbf{q}_d) = \sum_{\mathbf{K}} \delta_{\mathbf{q}_d, \mathbf{K}_d} N \int_{A_{\perp}} \frac{d \mathbf{r}_r}{A_{\perp}} e^{i \mathbf{K}_d \cdot \mathbf{r}_r},
\]

(32)

where the sum is performed over all vectors \( \mathbf{K} \) of the reciprocal lattice of the hypercubic lattice. As we are considering a quasicrystal, for any \( \mathbf{K}_d \) there is a unique \( \mathbf{K} \) and therefore \( \mathbf{K}_{\perp} \) is well defined. If \( \{a_j^*: j = 1, ..., D\} \) is a basis of vectors in reciprocal space, then \( \mathbf{K} = \sum_j n_j a_j^* \), where \( n_j \) are integers. Its parallel and perpendicular components are also functions of the same integers:

\[
\mathbf{K} = \mathbf{K}_d(n_1, ..., n_D) + \mathbf{K}_{\perp}(n_1, ..., n_D).
\]

(33)

Therefore, the sum over \( \mathbf{K} \) in Eq. (32) is actually a sum over \( D \) integers \( n_1, ..., n_D \), clearly showing that the Fourier transform is pure point of rank \( D > d \).

Let us define the structure factor in the thermodynamical limit as

\[
S(\mathbf{q}_d) = \frac{\tilde{n}(\mathbf{q}_d)^2}{N} (1 - \delta_{\mathbf{q}_d, \mathbf{0}}).
\]

(34)

For two-dimensional potentials \((d = 2)\) of the form given by Eq. (18), this definition differs from the one given in Eq. (12) by a factor \( V_0^2 a^4 N/\mathcal{A}_d \), which disappears upon choosing units such that \( V_0 = \sqrt{\mathcal{A}_d/Na^2} \).

As explained in Ref. 17, for a spectrum made of a dense set of Bragg peaks (discontinuous \( S \)), the integrated intensity function

\[
Z(k) = \int_{|\mathbf{q}_d| < k} S(\mathbf{q}_d) d\mathbf{q}_d,
\]

(35)

provides a reliable characterization of the point distribution. Here, the integral is performed over a disk of radius \( k \). This function is also known as the spectral measure in Ref 16.

For \( d \)-dimensional tilings built by the CP method, this quantity can be recast in the following form:

\[
Z(k) = \frac{(2\pi)^d}{\mathcal{A}} \sum_{|\mathbf{K}_d| < k} S(\mathbf{K}_d).
\]

(36)

The Rauzy tiling

Fourier Transform

The two-dimensional (generalized) Rauzy tiling has been introduced in Ref. 24. This is a codimension-one tiling built from the cubic lattice \( \mathbf{Z}^3 \) (edge length \( a = 1 \)) with a one-dimensional perpendicular space oriented along the direction \( \mathbf{e}_\perp = (\theta^2, \theta, 1) \) where \( \theta \) is the real (Pisot-Vijayaraghavan) root of the cubic equation \( x^3 = x^2 + x + 1 \). Contrary to the Ammann-Beenker and the Penrose tilings discussed in the next sections, the Rauzy tiling cannot be built by substitution rules.

For such a codimension-one quasicrystal, the acceptance window is a segment of length \( A_{\perp} \) defined as the projection of \( \mathbf{h} = (1, 1, 1) \) onto the perpendicular space. This acceptance window also corresponds to the projection of the unit cube onto the perpendicular space. In this case, Eq. (31) gives:

\[
|\tilde{n}(\mathbf{K}_d)| = N \left| \sin \left( \frac{\mathbf{K}_d \cdot \mathbf{h}_\perp}{2} \right) \right|.
\]

(37)

Structure factor

The structure factor is defined in Eq. (34). Our goal is to analyze the behavior of \( S(\mathbf{q}_d) \) in the limit where \( |\mathbf{q}_d| \) tends to zero. By definition, one has \( S(0) = 0 \) but its behavior for small \( |\mathbf{q}_d| \) is nontrivial since \( S(\mathbf{q}_d) \neq 0 \) only when \( \mathbf{q}_d \) coincides with the parallel component \( \mathbf{K}_d \) of a reciprocal-lattice vector \( \mathbf{K} \) of the cubic lattice. Thus, we are interested in computing the behavior of \( S \) when \( |\mathbf{K}_d| \) goes to 0 for \( \mathbf{K}_d \neq 0 \):

\[
S(\mathbf{K}_d) = N \left| \frac{\sin \left( \frac{\mathbf{K}_d \cdot \mathbf{h}_\perp}{2} \right)}{\mathbf{K}_d \cdot \mathbf{h}_\perp} \right|^2,
\]

(38)

\[
\approx N \left| \frac{\mathbf{K}_d \cdot \mathbf{h}_\perp}{\mathbf{K}_d \cdot \mathbf{h}_\perp} \right|^2,
\]

(39)

where we used the fact that \( \mathbf{K} \) is a reciprocal-lattice vector and \( \mathbf{h} \) a direct-lattice vector. The difficulty comes from the fact that, when \( |\mathbf{K}_d| \) goes to 0, \( |\mathbf{K}_d| \) diverges. So, the goal is to find the relation between these two components.

One way to investigate this issue is to follow the approach proposed in Ref. [17] for the Fibonacci chain. In the \( \mathbf{Z}^3 \) canonical basis, any reciprocal-lattice vector \( \mathbf{K} \) has coordinates \( 2\pi(l, m, n) \) where \( l, m, \) and \( n \) are integers. To analyze the behavior of \( \mathbf{K}_d, \mathbf{h}_\perp, \) and \( \mathbf{K}_d \cdot \mathbf{h}_\perp, \) let us consider the matrix

\[
M = \begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
\]

(41)

that satisfies \( M^3 = M^2 + M + 1 \). The eigenvalues of \( M \) are the Tribonacci constant \( \theta \approx 1.8393 \) and two complex
conjugate eigenvalues $e^{\pm i\phi}/\sqrt{\theta}$ with $\phi \simeq 2.1762$. The eigenvector associated to $\theta$ corresponds to the perpendicular direction $e_\perp$. Since $\theta$ is a Pisot-Vijayaraghavan number, the action of $M^p$ onto any vector $v$, such that $v.e_\perp \neq 0$, drives this vector towards the direction $e_\perp$ in the large-$p$ limit.

Hence, to analyze the behavior of $S$ in the limit

$$K^{(p)}_\parallel \cdot h_\parallel = \frac{(\theta - 1)\theta^{-\frac{p+1}{2}} \left\{ \sin(p\phi) \left[ (1 + \theta^2)m - \theta(l + n) \right] + \sqrt{\theta} \left[ (\theta m - l) \sin((1 + p)\phi) + (\theta n - m) \sin((1 - p)\phi) \right] \right\}}{\sin(\phi)(\theta - 1)\theta + 1},$$

(42)

$$K^{(p)}_\perp \cdot h_\perp = \frac{1}{\sin(\phi)((\theta - 1)\theta + 1)} \left\{ -\theta^p(\theta^4 + \theta^2 + 1)\sin(\phi) \left[ (\theta l - 2\sqrt{\theta}m\cos(\phi) + n \right) \right. \right.$$

(43)

Thus, in the large-$p$ limit, one finds that $K^{(p)}_\parallel \cdot h_\parallel$ vanishes as $\theta^{-p/2}$, $K^{(p)}_\perp \cdot h_\perp$ diverges as $\theta^p$, and $S(K^{(p)})$ behaves as $\theta^{-3p}$. As a result, one finds that

$$S(K^{(p)}) \big|_{K^{(p)} \rightarrow 0} \sim |K^{(p)}|^6,$$

(44)

for all $(l, m, n)$. However, we emphasize that, contrary to the Fibonacci chain Ref. [17], this power-law behavior is modulated by a bounded oscillating function as can be seen in Eqs. (42-43).

**Integrated intensity function**

The integrated intensity function is defined in Eq. (36). The sum over all vector $K^{(p)}_\parallel$ with a norm smaller than $k$ can be decomposed into a sum over all triplets $(l, m, n)$ and their iterated under $M^p$. As a result, one has:

$$Z(k) = \frac{4\pi^2}{A} \sum_{(l, m, n)} \sum_{p=1}^{\infty} S(K^{(p)}_\parallel),$$

(45)

where $p_{(l,m,n)}$ is the smallest integer fulfilling the constraint $|K^{(p)}_\parallel| < k$. As already discussed in the previous section, in the large-$p$ limit,

$$S(K^{(p)}_\parallel) \sim |K^{(p)}_\parallel|^6 f(l, m, n, p),$$

(46)

$$|K^{(p)}_\parallel| \sim \theta^{-p/2} g(l, m, n, p),$$

(47)

where, for a given triplet $(l, m, n)$, $f$ and $g$ are bounded oscillating function of $p$ [see Eqs. (42)-(43)]. Thus, $S$ is bounded both above and below

$$Z^-(k) \leq Z(k) \leq Z^+(k),$$

(48)

where $K^{(p)}_\parallel$ tends to zero [see Eq. (41)], let us consider $K^{(p)}_\parallel = M^pK$. More precisely, we are interested in computing $K^{(p)}_\parallel \cdot h_\parallel$ and $K^{(p)}_\perp \cdot h_\perp$. Keeping in mind that $h_\perp = P_\perp(1, 1, 1)$ and $h_\parallel = (I - P_\perp)(1, 1, 1)$ (where $P_\perp$ is the projector onto the perpendicular space), one can easily compute these quantities. After some algebra, one gets:

$$K^{(p)}_\parallel \cdot h_\parallel =$$

$$K^{(p)}_\perp \cdot h_\perp =$$

(43)

Thus, in the large-$p$ limit, one finds that $K^{(p)}_\parallel \cdot h_\parallel$ vanishes as $\theta^{-p/2}$, $K^{(p)}_\perp \cdot h_\perp$ diverges as $\theta^p$, and $S(K^{(p)})$ behaves as $\theta^{-3p}$. As a result, one finds that

$$S(K^{(p)}) \big|_{K^{(p)} \rightarrow 0} \sim |K^{(p)}|^6,$$

(44)

for all $(l, m, n)$. However, we emphasize that, contrary to the Fibonacci chain Ref. [17], this power-law behavior is modulated by a bounded oscillating function as can be seen in Eqs. (42-43).

**Integrated intensity function**

The integrated intensity function is defined in Eq. (36). The sum over all vector $K^{(p)}_\parallel$ with a norm smaller than $k$ can be decomposed into a sum over all triplets $(l, m, n)$ and their iterated under $M^p$. As a result, one has:

$$Z(k) = \frac{4\pi^2}{A} \sum_{(l, m, n)} \sum_{p=1}^{\infty} \theta^{-3p},$$

(49)

$$c^+(l, m, n) = \max_p f(l, m, n, p) g(l, m, n, p),$$

(50)

$$c^-(l, m, n) = \min_p f(l, m, n, p) g(l, m, n, p).$$

(51)

Interestingly, $Z^\pm(k)/\sqrt{\theta} = Z^\pm(k)/\theta^3$, as can be seen from Eqs. (46)-(47) since dividing $k$ by $\sqrt{\theta}$ simply amounts to change $p_{(l,m,n)}$ into $p_{(l,m,n)} + 1$ in Eq. (49). Such a relation reflects a discrete scale invariance [14] (see also next section) for $Z^\pm$ and implies a power-law envelope

$$Z(k) \sim k^6.$$
can be easily computed. The structure factor is

\[ S(K_\parallel) = N \left| \frac{\sin\left(\frac{K_\parallel \cdot h_\parallel}{2}\right)}{K_\parallel \cdot h_\parallel} \right|^2, \]

where \( h_\parallel \) and \( h_\perp \) are the projections of the vector \( h = (1, 1) \) onto \( e_\parallel = \frac{1}{\sqrt{1+\tau}}(-1, \tau) \) and \( e_\perp = \frac{1}{\sqrt{1+\tau}}(\tau, 1) \), where \( \tau = \frac{1 + \sqrt{5}}{2} \) is the golden ratio. The integrated intensity function is then

\[ Z(k) = \frac{2\pi}{A} \sum_{|K_\parallel|<k} S(K_\parallel), \]

where \( A \) is the total length of the chain. As for the Rauzy tiling, let us consider the matrix

\[ M = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \]

that satisfies \( M^2 = M + 1 \). Eigenspaces of \( M \) correspond to the perpendicular and parallel directions with eigenvalues \( \tau \) and \(-1/\tau\), respectively. The small-\( k \) behavior of \( Z \) is obtained by analyzing sequences \( K^{(p)} = M^p K \) (see Ref. 17). One then gets, in the large-\( p \) limit,

\[ S(K_\parallel^{(p)}) \simeq |K_\parallel^{(p)}|^4 f(l, m), \]

\[ |K_\parallel^{(p)}| \simeq \tau^{-p} g(l, m). \]

However, contrary to the Rauzy tiling, \( f \) and \( g \) do not depend on \( p \). Thus, following the same line of reasoning as above, one straightforwardly gets the discrete scaling relation

\[ Z(k/\tau) = Z(k)/\tau^4. \]

The solution of this equation can be written as

\[ Z(k) = k^4 F(\ln k/\ln \tau), \]

where \( F(x+1) = F(x) \) (for a review on discrete scale invariance, see Ref. 14). As a result, \( Z \) has a power-law envelope together with log-periodic oscillations (see Fig. 3 for illustration). This is in stark contrast with the Rauzy tiling where only \( Z^+ \) and \( Z^- \) obey such a discrete scale invariance but not \( Z \) itself. Practically, to compute \( Z \), we first select a set of points \( K^{(p)} \) with \( p = 0, ..., p_{\text{max}} \), and we compute \( S \) for each corresponding \( K^{(p)} \) (avoiding possible redundancy). \( Z \) is then obtained by summing over these Bragg peaks according to Eq. (54). We check the convergence of the results displayed in Fig. 3 by increasing \( K_{\text{max}} \) and \( p_{\text{max}} \).

The octagonal tiling

Fourier transform

The octagonal (Ammann-Beenker) tiling [25–27] is a codimension-two tiling built from the four-dimensional hypercubic lattice \( \mathbb{Z}^4 \) (edge length \( a = 1 \)). Perpendicular and parallel spaces are spanned by the eigenvectors of the matrix

\[ M = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, \]

associated to eigenvalues \( \lambda_+ = 1 + \sqrt{2} \). This matrix satisfies \( M^2 = 2M + 1 \) and its eigenvalues are twofold degenerate. Here, we choose the following orthonormal eigenbasis

\[ e_{\parallel,1} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \]

\[ e_{\parallel,2} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \]

\[ e_{\perp,1} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \]

\[ e_{\perp,2} = \begin{pmatrix} -1/2 \\ -1/2 \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \]

where the perpendicular (parallel) space is associated to \( \lambda_+ \) (\( \lambda_- \)). The acceptance window is an octagon corresponding to the projection of the four-dimensional unit
for all reciprocal-lattice vector $K$ with components $K_{\|j} = K_\| e_{\|j}$ and $K_{\perp j} = K_\perp e_{\perp j}$. These expressions coincide with the one given in Ref. 29.

**Structure factor**

We are interested in computing the behavior of $S$ when $|K_\| |$ goes to 0 for $K_\| \neq 0$. To this aim, we note that for a vector $K = 2\pi(s, t, u, v)$ [where $(s, t, u, v) \in \mathbb{Z}^4$], one has:

\[
\lambda_+ K_{\perp 1} = \frac{K_{\| 1}}{\lambda_+} + 2\pi(s - t + u), \quad (63) \\
\lambda_+ K_{\perp 2} = \frac{K_{\| 2}}{\lambda_+} + 2\pi(-s - t + u), \quad (64) \\
K_{\perp 1} + K_{\| 1} = 2\pi v, \quad (65) \\
K_{\perp 2} + K_{\| 2} = 2\pi u. \quad (66)
\]

A close inspection of Eq. (62) shows that one has to distinguish three different cases.

**Symmetry axes: $S(K_\|) \sim |K_\||^4$**

As can be seen in Eq. (62), the denominator vanishes if one of the components $K_{\perp 1} = 0$ or when $K_{\perp 1} = \pm K_{\perp 2}$. When $K_\perp$ belongs to these four symmetry axes, the Fourier transform can be recast in a simple form. For simplicity, let us focus on the case where $K_{\perp 2} = 0$ (the other cases being treated similarly) for which Eq. (62) becomes

\[
|\tilde{n}(K_\|)| = \frac{N}{\lambda_+^2 K_{\perp 1}^2} \left| 2 \cos \left( \frac{K_{\perp 1}}{2} \right) - 2 \cos \left( \frac{\lambda_+ K_{\perp 1}}{2} \right) + K_{\perp 1} \sin \left( \frac{\lambda_+ K_{\perp 1}}{2} \right) \right|, \quad (67)
\]

Using Eqs. (63)-(65), one then obtains

\[
S(K_\|) \overset{|K_\|=0}{\sim} \frac{N}{4\lambda_+^2} \frac{|K_{\| 1}|^2}{|K_{\perp 1}|^2}. \quad (68)
\]

As previously, to analyze the behavior of the structure factor for small $|K_\|$, we consider $K^{(p)} = MP K$. By construction, in the large-$p$ limit, the parallel components of $K^{(p)}$ tend to zero as $\lambda_+^p$ and its perpendicular components diverge as $\lambda_+^{-p}$. As a result, $S(K^{(p)})$ behaves as $\lambda_+^{-4p}$ so that, in this case,

\[
S(K_\|) \overset{|K_\|=0}{\sim} |K_\||^4. \quad (69)
\]

This result actually holds for all $K_\perp$ belonging to the four symmetry axes discussed above.

**Generic cases: $S(K_\|) \sim |K_\||^8$ or $S(K_\|) \sim |K_\||^{12}$**

When $K_\perp$ does not belong to the four symmetry axes defined as $K_{\perp 1} = 0$, $K_{\perp 2} = 0$, and $K_{\perp 1} = \pm K_{\perp 2}$, one can again use Eqs. (63-66) to express the structure factor as

\[
S(K_\|) \sim \frac{N}{4\lambda_+^2} \frac{|K_{\| 1}|^2}{|K_{\perp 1}|^2} \frac{(K_{\| 1} K_{\perp 2} + K_{\| 2} K_{\perp 1})(K_{\| 1} K_{\perp 1} - K_{\| 2} K_{\perp 2})}{K_{\perp 1} K_{\perp 2} (K_{\perp 1} - K_{\perp 2}) (K_{\perp 1} + K_{\perp 2})}^2, \quad (70)
\]

which leads to

\[
S(K_\|) \overset{|K_\|=0}{\sim} |K_\||^8. \quad (71)
\]

However, as can be seen in Eq. (70), this leading contribution may vanish for some special $K$. In this case, $S$ is given by the subleading contribution which gives

\[
S(K_\|) \overset{|K_\|=0}{\sim} |K_\||^{12}. \quad (72)
\]

**Integrated intensity function**

To compute the integrated intensity function defined in Eq. (36), we decompose the sum over all vector $K_\|$ with a norm smaller than $k$ as a sum over all quadruplets $(s, t, u, v)$ and their iterated vectors $K^{(p)} = MP K_\|$. One
can then write
\[ Z(k) = \frac{4\pi^2}{\mathcal{A}} \sum_{(s,t,u,v)} \sum_{p=1}^{\infty} S(K^{(p)}_\parallel), \quad (73) \]
where \( p_{(s,t,u,v)} \) is the smallest integer fulfilling the constraint \( |K^{(p)}_\parallel| < k \). As discussed above, the behavior of \( S \) in the large-\( p \) (small-\( |K^{(p)}_\parallel| \)) limit, strongly depends on \( K^{(p)}_\parallel \) [see Eqs. (69-72)]. This is in stark contrast with the Rauzy tiling and the Fibonacci chain where there is the same power-law scaling for all \( K^{(p)}_\parallel \) [see Eqs. (46)-(56)].

However, since we are interested in the small-\( k \) (large-\( p \)) limit, one only keeps the dominant terms in Eq. (73) that come from the symmetry axes and gives
\[ S(K^{(p)}_\parallel) \simeq |K^{(p)}_\parallel|^4 f(s, t, u, v), \quad (74) \]
\[ |K^{(p)}_\parallel| \simeq \lambda_+^{-p} g(s, t, u, v). \quad (75) \]

We emphasize that, as for the Fibonacci chain, \( f \) and \( g \) are functions that do not depend on \( p \), so that one straightforwardly gets the following discrete scaling relation
\[ Z(k/\lambda_+) = Z(k)/\lambda_+^4. \quad (76) \]
The solution of this equation can be written as
\[ Z(k) = k^4 F(\ln k / \ln \lambda_+), \quad (77) \]
where \( F(x+1) = F(x) \). As a result, \( Z \) has a power-law envelope together with log-periodic oscillations.

**The Penrose tiling**

The Penrose rhombus tiling [20, 30] can be built by CP from the five-dimensional hypercubic lattice \( \mathbb{Z}^5 \) (edge length \( a = 1 \)) along a well-known procedure (see, e.g., Ref. 31 for details). For our purpose, let us consider the following orthogonal (non normalized) basis
\[ e_{\parallel,1} = \frac{2}{5}(1, c_2, c_4, c_2, c_2), \]
\[ e_{\parallel,2} = \frac{2}{5}(0, s_2, s_4, -s_4, -s_2), \]
\[ e_{\perp,1} = \frac{2}{5}(1, c_4, c_2, c_2, c_2), \]
\[ e_{\perp,2} = \frac{2}{5}(0, s_4, -s_2, s_2, -s_4), \]
\[ e_{\Delta} = \frac{1}{10}(1, 1, 1, 1) \],

that defines the three subspaces \( E_{\parallel}, E_{\perp} \) and \( \Delta \). Here, we introduced the notation \( c_n = \cos(2\pi n/5) \) and \( s_n = \sin(2\pi n/5) \). A point in \( \mathbb{Z}^5 \) is selected whenever it projects onto the perpendicular space \( E_{\perp} + \Delta \) inside a three-dimensional acceptance window which is the projection of the five-dimensional unit hypercube onto this subspace. Remarkably, selected points only fill five planes perpendicular to \( \Delta \). Thus, the selection step only amounts to consider discrete sections of the acceptance window. Among all possible choices, the fivefold symmetric canonical Penrose tilings (known as star and sun [20]) considered here correspond to the following sections: one point which is the symmetry center of the tiling, two regular pentagons of side \( \sqrt{2/5} \cos(3\pi/10) \), two regular pentagons of side \( 2\sqrt{2/5} \cos(3\pi/10) \), where \( \tau = \frac{1+\sqrt{5}}{2} \) is the golden ratio.

**Fourier transform**

The Fourier transform of the tiling’s vertices is obtained as a weighted sum of the Fourier transform of the four regular pentagons. For any vector \( \mathbf{R} \) of the five-dimensional hypercubic lattice, the five-dimensional reciprocal lattices vectors \( \mathbf{K} \) satisfy
\[ \mathbf{K} \cdot \mathbf{R} = \mathbf{K}_\parallel \cdot \mathbf{R}_\parallel + \mathbf{K}_{\perp} \cdot \mathbf{R}_{\perp} + \mathbf{K}_\Delta \cdot \mathbf{R}_\Delta = 2\pi \times \text{integer}. \quad (79) \]

Then, after some algebra, Eq. (31) leads to:
\[ |\tilde{n}(K_\parallel)| = N \left| \frac{8(s_2-s_4)}{5K_{\perp,2}} \right| \left[ \begin{array}{c}
\cos(c_2K_{\perp,1}+s_2K_{\perp,2}-3K_{\Delta}) + \cos(K_{\perp,1}/2 + (s_4 + s_2)K_{\perp,2} + K_{\Delta}) - \cos(K_{\perp,1} - 3K_{\Delta}) - \cos(2c_4K_{\perp,1} - K_{\Delta}) - \\
5K_{\perp,1} - (6s_2 + 2s_4)K_{\perp,2} \\
\cos(c_2K_{\perp,1} - s_2K_{\perp,2} - 3K_{\Delta}) + \cos(K_{\perp,1}/2 - (s_4 + s_2)K_{\perp,2} + K_{\Delta}) - \cos(K_{\perp,1} - 3K_{\Delta}) - \cos(2c_4K_{\perp,1} - K_{\Delta}) + \\
5K_{\perp,1} + (6s_2 + 2s_4)K_{\perp,2} \\
\cos(c_2K_{\perp,1} - s_2K_{\perp,2} - 3K_{\Delta}) + \cos(K_{\perp,1}/2 - (s_4 + s_2)K_{\perp,2} + K_{\Delta}) - \cos(c_4K_{\perp,1} - s_4K_{\perp,2} - 3K_{\Delta}) - \cos((1+c_2)K_{\perp,1} + s_2K_{\perp,2} - K_{\Delta}) \\
5K_{\perp,1} - (6s_4 - 2s_2)K_{\perp,2} \\
\cos(c_2K_{\perp,1} + s_2K_{\perp,2} - 3K_{\Delta}) + \cos(K_{\perp,1}/2 + (s_4 + s_2)K_{\perp,2} + K_{\Delta}) - \cos(c_4K_{\perp,1} + s_4K_{\perp,2} - 3K_{\Delta}) - \cos((1+c_2)K_{\perp,1} - s_2K_{\perp,2} - K_{\Delta}) \\
5K_{\perp,1} + (6s_4 - 2s_2)K_{\perp,2} \end{array} \right] \],

for all reciprocal-lattice vector \( \mathbf{K} \) with components
\[ K_{\parallel,j} = K_{\parallel,j} e_{\parallel,j}, K_{\perp,j} = K_{\perp,j} e_{\perp,j}, \text{ and } K_{\Delta} = K_{\Delta} e_{\Delta}. \]
Structure factor

We are interested in computing the behavior of $S$ when $|K_\parallel| \to 0$ for $K_\parallel \neq 0$. To this aim, we note that for a vector $K = 2\pi(s, t, u, v, w)$ [where $(s, t, u, v, w) \in \mathbb{Z}^5$], one has

$$S(K_\parallel) \sim |K_\parallel|^{-8} \left| \begin{array}{c} K_{\perp,1}^2 + K_{\perp,2}^2 - 3K_{\Delta} = -c_4 K_{\parallel,1} + s_4 K_{\parallel,2} - 5K_{\Delta} + 2\pi v, \\
c_2 K_{\perp,1} - s_2 K_{\perp,2} - 3K_{\Delta} = -c_4 K_{\parallel,1} - s_4 K_{\parallel,2} - 5K_{\Delta} + 2\pi u, \\
K_{\perp,1}/2 + (s_4 + s_2) K_{\perp,2} + K_{\Delta} = -K_{\parallel,1}/2 + (s_4 - s_2) K_{\parallel,2} + 5K_{\Delta} - 2\pi(u + w), \\
K_{\perp,1}/2 - (s_4 + s_2) K_{\perp,2} + K_{\Delta} = -K_{\parallel,1}/2 - (s_4 - s_2) K_{\parallel,2} + 5K_{\Delta} - 2\pi(v + t), \\
K_{\perp,1} - 3K_{\Delta} = -K_{\parallel,1} - 5K_{\Delta} + 2\pi s, \\
2c_4 K_{\parallel,1} - K_{\Delta} = -2c_2 K_{\parallel,1} - 5K_{\Delta} + 2\pi(t + w), \\
c_4 K_{\parallel,1} + s_4 K_{\perp,2} - 3K_{\Delta} = -c_2 K_{\parallel,1} - s_2 K_{\parallel,2} - 5K_{\Delta} + 2\pi t, \\
c_4 K_{\parallel,1} - s_4 K_{\perp,2} - 3K_{\Delta} = -c_2 K_{\parallel,1} + s_2 K_{\parallel,2} - 5K_{\Delta} + 2\pi w, \\
(1 + c_2) K_{\parallel,1} + s_2 K_{\perp,2} - K_{\Delta} = -(1 + c_4) K_{\parallel,1} + s_4 K_{\perp,2} - 5K_{\Delta} + 2\pi(s + v), \\
(1 + c_2) K_{\parallel,1} - s_2 K_{\perp,2} - K_{\Delta} = -(1 + c_4) K_{\parallel,1} - s_4 K_{\perp,2} - 5K_{\Delta} + 2\pi(s + u). \\
\end{array} \right.$$  

(81) (82) (83) (84) (85) (86) (87) (88) (89) (90)

Keeping in mind that $5K_{\Delta} = \pi(s + t + u + v + w)$, one can finally rewrite Eq. (80) as a function of $K_{\perp,1}$ only. In the limit where $|K_\parallel| \to 0$ one then gets generically

$$S(K_\parallel) \sim N(\sqrt{5} - 2)^2 \frac{(K_{\perp,1}^2 + K_{\perp,2}^2)(K_{\parallel,1}^2 - K_{\parallel,2}^2)K_{\perp,2} \Delta K_{\parallel,2} K_{\perp,1}}{K_{\perp,2}(5K_{\perp,1}^2 - 10K_{\perp,1}^2 K_{\perp,2}^2 + K_{\perp,2}^2)}^2.$$  

(91)

Integrated intensity function

As for the octagonal tiling, we decompose the sum over $K_\parallel$ in the integrated intensity function defined in Eq. (36) as a sum over all quintuplets $(s, t, u, v, w)$ and their iterated under $K_\parallel = M^p K_\parallel$. One can then write

$$Z(k) = \frac{4\pi^2}{A} \sum_{(s, t, u, v, w)} \sum_{p = p(s, t, u, v, w)}^\infty S(K_\parallel^p),$$  

(94)

where $p(s, t, u, v, w)$ is the smallest integer fulfilling the constraint $|K_\parallel^p| < k$. In the small-$k$ (large-$p$) limit, one can check that

$$S(K_\parallel^p) \sim |K_\parallel^p|^{-8} f(s, t, u, v, w),$$  

(95)

for any quintuplet $(s, t, u, v, w) \in \mathbb{Z}^5$. However, contrary to the octagonal tiling, the scaling of $K_\parallel^p$ with $p$ depends on the quintuplet. For quintuplets that do not annihilate the denominator in Eq. (80), one gets:

$$|K_\parallel^p| \sim \tau^{-p} g(s, t, u, v, w),$$  

(96)
or, in other words, $|\mathbf{K}_\parallel^{(p)}/\mathbf{K}_\parallel^{(p+1)}| = \tau$. Importantly, when one of the denominators in Eq. (80) vanishes, one gets a weaker relation since one only has $|\mathbf{K}_\parallel^{(p)}/\mathbf{K}_\parallel^{(p+2)}| = \tau^2$. As a direct consequence, one gets the following discrete scaling relation

$$Z(k/\tau^2) = Z(k)/\tau^{16}. \quad (97)$$

The solution of this equation can be written as

$$Z(k) = k^8 F(\ln k/\ln \tau), \quad (98)$$

where $F(x + 1) = F(x)$. As a result, $Z$ has a power-law envelope together with log-periodic oscillations.