Marginal inequalities of the Rogers-Shephard type*

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Abstract

In convex geometry, a classical and very powerful result is the Rogers-Shephard inequality which states that, for any convex body $K \subset \mathbb{R}^n$,

$$\text{vol}_n(K - K) \leq \binom{2n}{n} \text{vol}_n(K),$$

with equality only when $K$ is an $n$-dimensional simplex, where $\text{vol}_n$ denotes the usual volume in $\mathbb{R}^n$. Recently, the Rogers-Shephard inequality has been generalized to larger classes of measures on $\mathbb{R}^n$.

In this paper we address the following question: given a measure $\mu$ on $\mathbb{R}^n$, does there exist a constant $C > 0$ such that, for any $m$-dimensional subspace $H \subset \mathbb{R}^n$ and any convex body $K \subset \mathbb{R}^n$, the following marginal Rogers-Shephard type inequality holds:

$$\mu((K - K) \cap H) \leq C \sup_{y \in \mathbb{R}^n} \mu(K \cap (H + y))?$$

We show that this inequality is affirmative in the class of measures with radially decreasing densities with the constant $C(n, m) = \binom{n+m}{n}$. Additionally, we prove inequality of a similar type in the class of $s$-concave functions.

1 Introduction

By $(\mathbb{R}^n, |\cdot|)$ we denote the $n$-dimensional real Euclidean space with its usual metric structure. A convex body is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. We will say that a convex body $K \subset \mathbb{R}^n$ is symmetric if, for some $x \in \mathbb{R}^n$, $K - x = -(K - x)$. We represent by $B_n$ the $n$-dimensional Euclidean unit ball, and by $\mathbb{S}^{n-1}$ its boundary. The $n$-dimensional volume of a measurable set $M \subset \mathbb{R}^n$, i.e., its $n$-dimensional Lebesgue measure, is denoted by $\text{vol}_n(M)$. Moreover, we denote by $G_{n,m}$ the set of $m$-dimensional linear subspaces of $\mathbb{R}^n$. For a set $A \subset \mathbb{R}^n$, let $\chi_A$ denotes the characteristic function of $A$.

The Minkowski addition of two sets $A, B \subset \mathbb{R}^n$ is defined by their usual vector sum:

$$A + B = \{a + b: a \in A, b \in B\},$$

and we shall write $A - B$ for $A + (-B)$.

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Connecting the Minkowski addition of convex bodies to their volume is the famed Brunn-Minkowski inequality, one form of which may be stated as follows: given convex bodies \( K, L \subset \mathbb{R}^n \), then
\[
\text{vol}_n(K + L) \geq \text{vol}_n(K)^{\frac{1}{n}} + \text{vol}_n(L)^{\frac{1}{n}}
\]
with equality if and only if \( K \) and \( L \) are homothetic (see [14] for an extensive survey on the Brunn Minkowski inequality). In particular, in the case when \( L = -K \), one has \( \text{vol}_n(K - K) \geq 2^n \text{vol}_n(K) \) with equality only when \( K \) is symmetric. A reverse inequality of this was discovered by Rogers and Shephard in the 1950s; the so called Rogers-Shephard inequality (see [24, Theorem 1]) and [28, Section 10.1]). The Rogers-Shephard inequality reads: given any convex body \( K \subset \mathbb{R}^n \),
\[
\text{vol}_n(K - K) \leq \left(\frac{2n}{n}\right)\text{vol}_n(K),
\]
with equality if and only if \( K \) is an \( n \)-dimensional simplex.

Alternatively, one can view the Minkowski sum in the following way
\[
K + L = \{ x \in \mathbb{R}^n : K \cap (x - L) \neq \emptyset \}.
\]
With this interpretation of the Minkowski sum of sets, given any convex bodies \( K \) and \( L \), the Brunn-Minkowski inequality implies that the function
\[
f(x) = \text{vol}_n(K \cap (x - L))^{\frac{1}{n}}
\]
is concave on \( K + L \).

In recent years, both the Brunn-Minkowski inequality and the Rogers-Shephard inequalities have been studied deeply and generalized extended to larger classes of measures on \( \mathbb{R}^n \). For results on the Brunn-Minkowski inequality see [10, 11, 14, 15, 18, 19, 20, 21, 22, 23], and for generalizations of the Rogers-Shephard inequality see [1, 2, 3, 4, 12, 28].

One of the most famous extensions of the Brunn-Minkowski inequality is the Borell-Brascamp-Lieb inequality (see [10, 11, 14]), which concerns so-called \( \alpha \)-concave measures. A non-negative Borel measure \( \mu \) defined on \( \mathbb{R}^n \) is said to be \( \alpha \)-concave, for some \( \alpha \in [-\infty, \infty] \), if, for all Borel measurable sets \( A, B \subset \mathbb{R}^n \) and any \( \lambda \in [0, 1] \),
\[
\mu((1 - \lambda)A + \lambda B) \geq M_\alpha^\lambda(\mu(A), \mu(B)).
\]
Analogously, a non-negative Borel measurable function \( f \) defined on \( \mathbb{R}^n \) is said to be \( \alpha \)-concave, for some \( \alpha \in [-\infty, \infty] \), if
\[
f((1 - \lambda)x + \lambda y) \geq M_\alpha^\lambda(f(x), f(y))
\]
for all \( x, y \in \mathbb{R}^n \) and \( \lambda \in [0, 1] \). Here \( M_\alpha^\lambda \) denotes the \( \alpha \)-mean of two non-negative numbers:
\[
M_\alpha^\lambda(a, b) = \begin{cases} 
(1 - \lambda)a^\alpha + \lambda b^\alpha)^{\frac{1}{\alpha}}, & \text{if } \alpha \neq 0, \pm\infty, \\
a^{1-\lambda}b^\lambda, & \text{if } \alpha = 0, \\
\max\{a, b\}, & \text{if } \alpha = \infty, \\
\min\{a, b\}, & \text{if } \alpha = -\infty;
\end{cases}
\]

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for $ab > 0$; $M^\lambda_\alpha(a,b) = 0$ when $ab = 0$. A 0-concave function is usually called log-concave whereas a $(-\infty)$-concave function is called quasi-concave. Equivalently, a non-negative function $f$ defined on $\mathbb{R}^n$ is quasi-concave if each of its super-level sets

$$C_t(f) = \{x \in \mathbb{R}^n : f(x) \geq t\|f\|_\infty\}$$

are convex sets for all $0 \leq t \leq 1$. Here

$$\|f\|_\infty = \inf \{t \in \mathbb{R} : \text{vol}_n(\{x \in \mathbb{R}^n : f(x) > t\}) = 0\}$$

denotes the essential supremum of $f$. One form of the Borell-Brascamp-Lieb inequality is stated as follows (see [14, Theorem 10.2] or [5, Proposition 1.4.4]).

**Theorem 1.1 (Borell-Brascamp-Lieb inequality).** Let $\lambda \in [0,1]$ and $-\frac{1}{n} \leq \alpha \leq \infty$. Given non-negative measurable functions $f, g,$ and $h$ defined on $\mathbb{R}^n$ satisfying

$$h((1-\lambda)x + \lambda y) \geq M^\lambda_\alpha(f(x), g(y))$$

for all $x, y \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} h(x)dx \geq M^\lambda_{\frac{\alpha}{n+1}} \left(\int_{\mathbb{R}^n} f(x)dx, \int_{\mathbb{R}^n} g(x)dx\right).$$

(1.2)

Recently, inequality (1.2) was extended, after a suitable change accounting for the lack of translation invariance of general measures, to the setting of measures having radially decreasing densities (see [3, Theorem 1.1]). We say a function $\phi : \mathbb{R}^n \to \mathbb{R}_+$ is radially decreasing if, for each $t \in [0,1]$ and any $x \in \mathbb{R}^n$, one has $\phi(tx) \geq \phi(x)$. Note that a quasi-concave function $f$ that assumes it maximum at the origin is radially decreasing.

Following [12], we consider the following functional analogue of the difference body. Given an $\alpha$-concave function $f : \mathbb{R}^n \to \mathbb{R}_+$, for some $\alpha \in [-\infty, \infty]$, we define

$$\Delta_\alpha f(x) = \sup_{x = x_1-x_2} M_\alpha(f(x_1), f(x_2)),$$

where, for $a, b \geq 0$ with $ab > 0$,

$$M_\alpha(a, b) = \begin{cases} (a^\alpha + b^\alpha)^{\frac{1}{\alpha}}, & \text{if } \alpha \neq 0, \pm \infty, \\ ab, & \text{if } \alpha = 0, \\ \max\{a, b\}, & \text{if } \alpha = \infty, \\ \min\{a, b\}, & \text{if } \alpha = -\infty; \end{cases}$$

and $M_\alpha(a, b) = 0$ when $ab = 0$. The function $\Delta_\alpha f$ is called the difference function. This function is even, $\alpha$-concave for any $\alpha$. For more details on such functions, please see [9, 12, 16, 28]. In [12], Colesanti established the following functional version of inequality (1.1), in the case when $\alpha = \frac{1}{s}$ for some $s \in (-\infty, 0]$:

$$\int_{\mathbb{R}^n} \Delta_{\frac{1}{s}} f(x)dx \leq \left(\frac{2n}{n}\right)^{\frac{s}{n}} \int_{\mathbb{R}^n} f(x)dx,$$

(1.3)
where \( f : \mathbb{R}^n \to \mathbb{R}_+ \) is an integrable \((\frac{1}{s})\)-concave function. Moreover, if one takes \( f = \chi_K \), then (1.1) is recovered.

In [20, Theorem 1] Rudelson found an asymptotic inequality of the same type as (1.1) that bounds the volume of the central section of the difference body of a convex body by an \( m \)-dimensional subspace from above by a constant multiple, depending both on the dimension and sub-dimension, of the maximal parallel section of the original body. This result reads as follows.

**Theorem 1.2** (Rudelson). Given a convex body \( K \) and \( H \in G_{n,m} \), one has

\[
\text{vol}_m((K - K) \cap H) \leq [c\psi(n, m)]^m \sup_{x \in \mathbb{R}^n} \text{vol}_m(K \cap (H + x)),
\]

where \( c > 1 \) is some absolute constant and

\[
\psi(n, m) = \min \left\{ \frac{n}{m}, \sqrt{m} \right\}.
\]

Inequality (1.4) was an important tool in estimating the Banach-Mazur distance between non-symmetric convex bodies (see [5, 25] for more details).

Applying inequality (1.4) to the identity (which follows form Fubini’s theorem)

\[
\frac{\int_H \Delta_{-\infty} f(x)dx}{\|f\|_{\infty}} = \int_0^1 \text{vol}_m((C_t(f) - C_t(f)) \cap H)dt,
\]

we extend inequality (1.3) to marginals of integrable quasi-concave functions.

**Corollary 1.1.** Given any integrable, bounded \((\frac{1}{s})\)-concave function \( f : \mathbb{R}^n \to \mathbb{R}_+ \) with \( s \in (-\infty, 0] \),

\[
\frac{\int_H \Delta_{\frac{1}{s}} f(x)dx}{\|f\|_{\infty}} \leq [c\psi(n, m)]^m \int_0^1 \sup_{y \in \mathbb{R}^n} \text{vol}_m(C_t(f) \cap (H + y))dt \tag{1.5}
\]

for some constant \( c > 0 \).

One may wish to strengthen the inequality appearing in (1.5), in the sense of commuting the integral with the supremum. We address this issue in the case of logarithmically concave functions (cf. Theorem 6.1).

Fix any \( p \in \mathbb{N} \). Given a convex body \( K \), we define the \( p \)-difference body of \( K \) to be the \( np \)-dimensional convex body given by

\[
D_p(K) = \left\{ \bar{x} = (x_1, \ldots, x_p) \in (\mathbb{R}^n)^p : K \cap \left( \bigcap_{i=1}^p (x_i + K) \right) \neq \emptyset \right\}.
\]

Note that \( D_1(K) = K - K \) is the usual difference body of \( K \). These bodies were originally introduced by Schneider in [27], where the convexity of the body \( D_p(K) \) was established as well as the following Rogers-Shephard type inequality for \( D_p(K) \): given a convex body \( K \subset \mathbb{R}^n \),

\[
\text{vol}_{np}(D_p(K)) \leq \binom{np + n}{n} \text{vol}_n(K)^p \tag{1.6}
\]
with equality if and only if $K$ is a simplex.

Given a non-negative Borel measure $\mu$ on $\mathbb{R}^n$ with density $\phi$ and $H \in G_{n,m}$, we define the *marginal* of $\mu$ with respect to the subspace $H$ by

$$\mu(A \cap H) = \int_H \phi(x) \chi_A(x) \, dx$$

for all compact subsets $A$ of $\mathbb{R}^n$.

We prove the following theorem, which generalizes inequality (1.6), and by extension Theorem 1.2 when $\psi(n, m) = \frac{n}{m}$, to the setting of measures with radially decreasing densities.

**Theorem 1.3.** Fix $p \in \mathbb{N}$. For each $i = 1, \ldots, p$ let $\mu_i$ be measure on $\mathbb{R}^n$ with density $\phi_i : \mathbb{R}^n \to \mathbb{R}_+$ that is radially decreasing and assumes its maximum at the origin. Let $\nu = \prod_{i=1}^p \mu_i$ be the associated product measure on $(\mathbb{R}^n)^p$ density $\phi$. For each $i = 1, \ldots, p$ let $H_i \in G_{n,m_i}$ for some $m_i \in \{1, \ldots, n\}$, and set $\bar{H} = H_1 \times \cdots \times H_p$. Then, for any convex body $K \subset \mathbb{R}^n$

$$\nu(D_p(K) \cap \bar{H}) \leq \frac{(m+n)}{\text{vol}_n(K)} \int_K \prod_{i=1}^p \mu_i((y - K) \cap H_i) \, dy,$$

(1.7)

where $m = m_1 + \cdots + m_p$.

By choosing $p = 1$ and letting $m = 1, \ldots, n$ be arbitrary, after an application of Stirling’s formula, we obtain the following extension of Theorem 1.2.

**Corollary 1.2.** Let $\mu$ be a measure on $\mathbb{R}^n$ given by $d\mu(x) = \phi(x) \, dx$, where $\phi : \mathbb{R}^n \to \mathbb{R}_+$ is radially decreasing and assumes its maximum at the origin, and let $H \in G_{n,m}$. Then, for any convex body $K \subset \mathbb{R}^n$,

$$\mu((K - K) \cap H) \leq \binom{n+m}{n} \sup_{y \in \mathbb{R}^n} \mu(K \cap (H + y))$$

$$\leq \left[ \frac{cn}{m} \right]^m \sup_{y \in \mathbb{R}^n} \mu(K \cap (H + y))$$

(1.8)

for some absolute constant $c > 1$.

The organization of the paper is as follows. In Section 2, we provide a proof of Theorem 1.3. In Section 3, we prove inequalities of the type (1.5) for $(\frac{1}{s})$-concave functions, with $s \neq \pm \infty$, in the setting of measures with radially decreasing densities. In Section 4, we discuss symmetry estimates of marginals of $(\frac{1}{s})$-concave functions when $s \neq \pm \infty$ when the measure is taken to have an even quasi-concave density. Section 5 is concerned with proving an inequality of type (1.4) in the class of measures with $(\frac{1}{s})$-concave densities, for $s > -n, s \neq 0$. In Section 6, we prove a variation of Corollary 1.1 for logarithmically concave functions.

## 2 Proof of Theorem 1.3

The proof of Theorem 1.3 follows the general idea of the proof of [3, Theorem 1.1].
Proof of Theorem 1.3. Consider the function $f: \mathbb{R}^n \to \mathbb{R}_+$ given by

$$f(\bar{x}) := f(x_1, \ldots, x_p) = \text{vol}_n \left[ K \cap \left( \bigcap_{i=1}^{p} (x_i + K) \right) \right]. \tag{2.1}$$

We notice that $f$ is supported on $D_p(K)$ and vanishes on the boundary of $D_p(K)$. We claim that $f$ is $(\frac{1}{n})$-concave on its support. Let $\bar{x}, \bar{y} \in D_p(K)$ and $\lambda \in [0, 1]$ be arbitrary. We need to show that

$$f((1 - \lambda)\bar{x} + \lambda\bar{y}) \geq M^\lambda_n(f(\bar{x}), f(\bar{y})). \tag{2.2}$$

In view of the Brunn-Minkowski inequality, in order to prove inequality (2.2), it is necessary only to verify that following inclusion holds:

$$K \cap \bigcap_{i=1}^{p} ((1 - \lambda)x_i + \lambda y_i + K) \supset (1 - \lambda) \left[ K \cap \bigcap_{i=1}^{p} (x_i + K) \right] + \lambda \left[ K \cap \bigcap_{i=1}^{p} (y_i + K) \right]. \tag{2.3}$$

Let $\bar{z} \in (1 - \lambda)[K \cap \bigcap_{i=1}^{p} (x_i + K)] + \lambda[K \cap \bigcap_{i=1}^{p} (y_i + K)]$ be arbitrary. Then $\bar{z} = (1 - \lambda)z + \lambda z'$ for some $z \in K \cap \bigcap_{i=1}^{p} (x_i + K)$ and $z' \in K \cap \bigcap_{i=1}^{p} (y_i + K)$. Using the convexity of $K$, we see that $\bar{z} \in K$. For each fixed $i = 1, \ldots, p$ there exist $k_i, k_i' \in K$ such that

$$z = x_i + k_i \quad \text{and} \quad z' = y_i + k_i',$$

and so $\bar{z} = (1 - \lambda)x_i + \lambda y_i + [(1 - \lambda)k_i + \lambda k_i']$ for every $i$. Consequently, using the convexity of $K$ once again, the inclusion (2.3) follows. Hence, $f$ is $(\frac{1}{n})$-concave on its support, as claimed.

The main goal of the proof is to estimate the following integral from above and below:

$$I(f) := \int_{\bar{H}} f(\bar{x})d\nu(\bar{x}).$$

On the one hand, an application of Fubini’s theorem allows us to write

$$I(f) = \int_{\bar{H}} f(\bar{x})d\nu(\bar{x})$$

$$= \prod_{i=1}^{p} \int_{H_i} \left( \int_{K} \chi_{y - K} (x_i)dy \right) \phi_i(x_i)d\nu$$

$$= \int_{K} \prod_{i=1}^{p} \mu_i((y - K) \cap H_i)dy$$

$$= \text{vol}_n(K) \frac{1}{\text{vol}_n(K)} \int_{K} \prod_{i=1}^{p} \mu_i((y - K) \cap H_i)dy. \tag{2.4}$$

On the other hand, we may consider the function $g: \bar{H} \to \mathbb{R}_+$ given by

$$g(\bar{x}) = f(0) \left( 1 - \frac{|x|}{\rho(\frac{\bar{x}}{|\bar{x}|})} \right)^n, \quad \text{for every } \bar{x} \neq 0$$
and \( f(0) = g(0) \); here \( \rho \) denotes the radial function of the section \( D_\rho(K) \cap \bar{H} \). Since \( g^\frac{1}{n} \) is affine on each radius of the section \( D_\rho(K) \cap \bar{H} \) and satisfies \( g^\frac{1}{n}(0) = f^\frac{1}{n}(0) \) and \( g = 0 \) on the boundary of the section, the concavity condition of \( f \) implies that \( g^\frac{1}{n} \leq f^\frac{1}{n} \) on each radius of \( D_\rho(K) \cap \bar{H} \). Hence, integrating in polar coordinates, we may write

\[
I(f) = \int_S \int_0^{\rho(u)} f(ru)\phi(ru)r^{n-1}drdu
\]

\[
\geq f(0) \int_S \int_0^{\rho(u)} (1 - \frac{r}{\rho(u)})^n r^{n-1}\phi(ru)drdu
\]

\[
= \text{vol}_n(K) \int_S \int_0^{\rho(u)} (1 - \frac{r}{\rho(u)})^n r^{n-1}\phi(ru)drdu,
\]

where \( S = S^{np-1} \cap \bar{H} \).

The final step of the proof consists of finding a constant \( \beta > 0 \) such that

\[
\int_S \int_0^{\rho(u)} (1 - \frac{r}{\rho(u)})^n r^{n-1}\phi(ru)drdu \geq \beta \nu(D_\rho(K) \cap \bar{H}).
\]

If we can find such a constant independent of the chosen direction, the proof will be complete.

Fix an arbitrary direction \( u \in S \) and consider the function \( h \) defined on \((0, \rho(u)]\) given by

\[
h(x) = \beta \int_0^y \phi(r)r^{n-1}dr - \int_0^y \left(1 - \frac{r}{x}\right)^n r^{n-1}\phi(r)dr.
\]

For an inequality of the form (2.6) to hold, we require that \( h \leq 0 \) on \((0, \rho(u)]\). Using the fact that \( h \) is absolutely continuous on each sub-interval \([a, b]\) of \((0, \rho(u)]\) (as a finite product of absolutely continuous functions), we express \( h \) as follows

\[
h(x) = h(a) + \int_a^x h'(s)ds.
\]

Using the fact \( \phi \) is bounded, we may assert that \( h(x) \to 0 \) as \( x \to 0^+ \). Thus, in view of representation (2.7), it would suffice to force the inequality \( h(x) \leq h(a) \) to hold for every \( 0 < a < x \) to find \( \beta > 0 \) satisfying (2.6). For such an inequality to occur, it is necessary only to have that \( h'(y) \leq 0 \) for almost every \( y \in (0, \rho(u)] \).

An application of the Lebesgue differentiation theorem allows us to write

\[
h'(x) = \beta\phi(x)x^{m-1} - n \int_0^x \left(1 - \frac{r}{x}\right)^{n-1} \frac{r^m}{x^2}\phi(r)dr
\]

for almost every \( y \in (0, \rho(u)] \). Since \( \phi \) is decreasing on \((0, \rho(u)]\), to guarantee that \( h'(x) \leq 0 \) almost everywhere \( x \in (0, \rho(u)] \) it is enough to require that

\[
\beta \leq n \int_0^x \left(1 - \frac{r}{x}\right)^{n-1} \frac{r^m}{ym+1}dr,
\]

or equivalently, that \( \beta \leq \left(\frac{n+m}{n}\right)^{-1} \). By choosing \( \beta = \left(\frac{n+m}{n}\right)^{-1} \), we may combine (2.4), (2.3), and (2.6) to complete the proof. \( \square \)
We note that the assumption that each \( \phi_i \) is bounded may be removed in general as it is essential to have the condition \( h(y) \to 0 \) as \( y \to 0^+ \) in the proof of the theorem. One may consider a function that is reasonable on each ray in \( \mathbb{R}^{np} \), however, is possible to explode when we restrict to a subspace. The assumption of boundedness of each density prevents this from occurring.

**Remark 2.1.** In place of the \( p \)-difference body, we may instead consider the following: given convex bodies \( K, L_0, \ldots, L_p \subset \mathbb{R}^n \) with \( \operatorname{int}(K \cap (-L_1) \cap \cdots \cap (-L_p)) \) containing the origin, we define the convex (see (2.3) but with \(-K \) replace with \( L_i \) in the intersection) \( D_p \)-body, \( D_p(K, L_i) \), of these bodies to be the \( np \)-dimensional set given by

\[
D_p(K, L_i) := \left\{ \bar{x} := (x_1, \ldots, x_p) : K \cap \left( \bigcap_{i=1}^{p} (x_i - L_i) \right) \neq \emptyset \right\}.
\]

In this setting (1.4) becomes

\[
\nu_{np}(D_p(K, L_i)) \leq \left( \frac{np + n}{n} \right) \frac{\nu_n(K) \nu_n(L_1) \cdots \nu_n(L_p)}{\nu_n(\bigcap_{i=1}^{p} (-L_i))}.
\]

Let \( k \in \{1, \ldots, n\} \). In the same setting at Theorem 1.3, but replacing the function \( f \) in (2.1) with the function \( \tilde{f} \) given by

\[
\tilde{f}(x_1, \ldots, x_p) = \nu_n \left[ K \cap \left( \bigcap_{i=1}^{p} (x_i - L_i) \right) \right],
\]

we may repeat the proof to obtain the estimate

\[
\nu(D_p(K, L_i) \cap \bar{H}) \leq \frac{\left( \frac{n+m}{n} \right) \int_K \prod_{i=1}^{p} \mu_i(L_i \cap (y + H_i)) dy}{\nu_n(K \cap \bigcap_{i=1}^{p} (-L_i))}, \quad (2.8)
\]

where \( m = m_1 + m_2 + \cdots + m_p \) and \( m_i \in \{1, \ldots, n\} \) for all \( i \).

As an immediate consequence of inequality (2.8), we obtain the following:

**Corollary 2.1.** Let \( \mu \) be a measure on \( \mathbb{R}^n \) given by \( d\mu(x) = \phi(x) dx \), where \( \phi : \mathbb{R}^n \to \mathbb{R}_+ \) is radially decreasing and \( \|\phi\|_\infty = \phi(0) \), \( m \in \{1, \ldots, n\} \), and \( H \in G_{n,m} \). Then, for any convex bodies \( K, L \subset \mathbb{R}^n \) with \( 0 \in \operatorname{int}(K \cap (L)) \),

\[
\mu((K + L) \cap H) \leq \frac{\left( \frac{n+m}{n} \right) \int_K \mu(L \cap (H + y)) dy}{\nu_n(K \cap (L))}.
\]

### 3 Marginal inequalities of the Rogers-Shephard type for \((\frac{1}{s})\)-concave functions \( s \neq \pm \infty \).

In this section we prove marginal inequalities of the Rogers-Shephard type in the setting of measures with radially decreasing densities for \((\frac{1}{s})\)-concave functions with \( s \neq \pm \infty \).
3.1 The case of \((\frac{1}{s})\)-concave functions, \(-\infty < s \leq 0\)

In this section, we prove a functional analogue of (1.4) in the class of integrable \((\frac{1}{s})\)-concave functions, with \(s \in (-\infty, 0]\). The result reads as follows.

**Theorem 3.1.** Let \(\mu\) be a measure on \(\mathbb{R}^n\) given by \(d\mu(x) = \phi(x)dx\), where \(\phi: \mathbb{R}^n \to \mathbb{R}_+\) is radially decreasing and assumes its maximum at the origin, and let \(H \in G_{n,m}\). Consider any bounded, integrable \((\frac{1}{s})\)-concave function \(f: \mathbb{R}^n \to \mathbb{R}_+\) with \(s \in (-\infty, 0]\). Then

\[
\int_H \Delta_{\frac{1}{s}} f(x) d\mu(x) \leq \left( \frac{m+n}{n} \right) \int_0^1 \sup_{y \in \mathbb{R}^n} \mu(C_t(f) \cap (H + y)) dt.
\] (3.1)

As an immediate application of Stirling’s approximation formula, Theorem 3.1 immediately implies that

\[
\int_H \Delta_{\frac{1}{s}} f(x) d\mu(x) \leq \left( \frac{c}{m} \right) m \int_0^1 \sup_{y \in \mathbb{R}^n} \mu(C_t(f) \cap (H + y)) dt
\]

holds for some absolute constant \(c > 1\).

**Proof.** Without loss of generality, we may assume that \(\|f\|_{\infty} = 1\). Using the fact that, for all \(a, b > 0\), we have that \(\left( \frac{a^s}{b} + b^s \right)^s \leq \min\{a, b\}\) whenever \(s \in (-\infty, 0)\), we see that \(\Delta_{\frac{1}{s}} f \leq \Delta_{-\infty} f\) for all \(s \in (-\infty, 0]\). Hence, it suffices only to prove the inequality for the quasi-concave case. After an application of Fubini’s theorem, we may write that

\[
\int_H \Delta_{-\infty} f(x) d\mu(x) = \int_H \int_0^{\Delta_{-\infty} f(x)} dt dx \leq \int_0^1 \mu((C_t(f) - C_t(f)) \cap H) dt.
\] (3.2)

Applying inequality (1.7) with \(p = 1\), we see that

\[
\mu((C_t(f) - C_t(f)) \cap H) \leq \left( \frac{m+n}{n} \right) \sup_{y \in \mathbb{R}^n} \mu(C_t(f) \cap (H + y))
\] (3.3)

holds for each \(t\). Combining (3.2) and (3.3) yields (3.1). \(\square\)

3.2 The case of \((\frac{1}{s})\)-concave functions, with \(0 \leq s < \infty\)

In this section, we prove a version of Theorem 3.1 for \((\frac{1}{s})\)-concave functions with \(s \geq 0\). The result reads as follows.

**Theorem 3.2.** Let \(\mu\) be a Borel measure on \(\mathbb{R}^n\) with radially decreasing density \(\phi: \mathbb{R}^n \to \mathbb{R}_+\) assuming its maximum at the origin, and let \(H \in G_{n,m}\). Then, for any integrable, \((\frac{1}{s})\)-concave function \(f: \mathbb{R}^n \to \mathbb{R}_+\) with \(s \geq 0\),

\[
\int_H \Delta_{\frac{1}{s}} f(x) d\mu(x) \leq \left[ \frac{C(n+s)}{m+s} \right]^{m+s} \sup_{z \in \mathbb{R}^n} \left\{ \int_{H+z} f(x) d\mu(x) \right\}.
\]
The proof of Theorem 3.2 follows the ideas introduced by Klartag in [16] used in his proof of the Borell-Brascamp-Lieb inequality for \((\frac{1}{s})\)-concave functions with \(s \geq 0\).

Let \(s\) be a positive integer. Given any bounded, integrable function \(f : \mathbb{R}^n \to \mathbb{R}_+\), consider the set
\[
A_{f,s} := \left\{ (x,y) : x \in \text{supp}(f), |y| \leq f(x)^{\frac{1}{s}} \right\}.
\]
These sets were originally introduced by Artstein, Klartag and Milman in [6] in their study of the Santalo and functional Santalo inequalities. Let \(\mu\) be a Borel measure on \(\mathbb{R}^n\) having density \(\phi\). Consider the measure \(\nu\) defined on \(\mathbb{R}^n \times \mathbb{R}^s\) given by
\[
d\nu_s(x,y) = \phi(x) \, dx \, dy.
\]
Then, for any \(H \in G_{n,m}\), integrating in polar coordinates gives
\[
\nu_s(A_{f,s} \cap (H \times \mathbb{R}^s)) = \int_{\text{supp}(f) \cap H} \int_{[0,f(x)^{\frac{1}{s}}]} dy \, d\mu(x) = \int_{\text{supp}(f) \cap H} \int_{S^{s-1}} f(x)^{\frac{1}{s}} \, r^{s-1} \, dr \, d\mu(x) = \omega_s \int_{\text{supp}(f) \cap H} f(x) \, d\mu(x),
\]
where \(\omega_s = \text{vol}_s(B_s)\). Moreover, we remark that \(A_{f,s}\) is a convex body if and only if \(f\) is an \((\frac{1}{s})\)-concave function.

With these notions in hand, we are ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** The proof follows via cases: we begin with \(s\) being a positive integer, then pass to the case when \(s\) is a rational number in lowest terms, and finally apply an approximation argument to conclude the proof.

Suppose first that \(s > 0\) is an integer. We begin by noticing that
\[
A_{\Delta f,s} = (A_{f,s} + (-A_{f,s})).
\]
In view of equality (3.4), we may apply inequality (1.8) to conclude
\[
\int_{H} \Delta f(x) \, d\mu(x) \leq \frac{1}{\omega_s} \nu_s((A_{f,s} + (-A_{f,s}) \cap (H \times \mathbb{R}^s)) \leq \left[ \frac{C(n+s)}{m+s} \right]^{m+s} \sup_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^s} \nu_s(A_{f,s} \cap ((H \times \mathbb{R}^s) + (x,y)) = \frac{C(n+s)}{m+s} \sup_{z \in \mathbb{R}^n} \nu_s(A_{f,s} \cap ((H + z) \times \mathbb{R}^s)) = \frac{C(n+s)}{m+s} \sup_{z \in \mathbb{R}^n} \int_{H+z} f(z) \, d\mu(z),
\]
completing the proof in the case when \(s\) is an integer.

Now suppose that \(s = \frac{p}{q}\) is a rational number in lowest terms. Consider the function \(\tilde{f} : (\mathbb{R}^n)^q \to \mathbb{R}_+\) given by
\[
\tilde{f}(\bar{x}) := \tilde{f}(x_1, \ldots, x_q) = \prod_{i=1}^{q} f(x_i),
\]
where
the product of \( q \) independent copies of \( f \). As with \( A_{f,s} \), we define the auxiliary set

\[
B_{f,s} := \left\{(x_1, \ldots, x_q, y) \in (\mathbb{R}^n)^q \times \mathbb{R}^p; x_i \in \text{supp}(f), |y| \leq \prod_{i=1}^{q} f(x_i)^{\frac{1}{p}}\right\}.
\]

We notice that \( B_{f,s} \) is a convex body if and only if \( \bar{f} \) is a \( \left( \frac{1}{p} \right) \)-concave function on its support \( \text{supp}(\bar{f}) = \text{supp}(f) \times \cdots \times \text{supp}(f) \). To this end, let \( \bar{x} = (x_1, \ldots, x_q) \) and \( \bar{y} = (y_1, \ldots, y_q) \in B_{f,s} \) and \( \lambda \in [0,1] \) be arbitrary. Using the \( \left( \frac{1}{p} \right) \)-concavity of \( f \) together with Hölder’s inequality, we may write

\[
\bar{f}((1 - \lambda)\bar{x} + \lambda\bar{y})^{\frac{1}{p}} = \prod_{i=1}^{q} f((1 - \lambda)x_i + \lambda y_i)^{\frac{1}{p}}
\]

\[
\geq \prod_{i=1}^{q} \left((1 - \lambda)f(x_i)^{\frac{1}{p}} + \lambda f(y_i)^{\frac{1}{p}}\right)^{\frac{1}{q}}
\]

\[
\geq \prod_{i=1}^{q} (1 - \lambda)^{\frac{1}{q}} f(x_i)^{\frac{1}{p}} + \prod_{i=1}^{q} \lambda^{\frac{1}{q}} f(y_i)^{\frac{1}{p}}
\]

\[
= (1 - \lambda) f(x)^{\frac{1}{p}} + \lambda \bar{f}(\bar{y})^{\frac{1}{p}},
\]

as desired. Consequently, \( B_{f,s} \) is a convex body.

Consider the measure \( \eta \) on \((\mathbb{R}^n)^q \times \mathbb{R}^p \) given by

\[
d\eta(x_1, \ldots, x_q, y) = \left(\prod_{i=1}^{q} \phi(x_i)x_i\right) dy.
\]

Let \( H_q = H \times \cdots \times H \). In the same spirit as equality \((3.4)\), by using Fubini’s theorem and polar coordinates, we may write

\[
\eta(B_{f,s} \cap (H_q \times \mathbb{R}^p)) = \omega_p \left(\int_{H} f(x)d\mu(x)\right)^{q}.
\]

As we did we \( \bar{f} \), we may check that \( \Delta_{\frac{1}{p}} f \) is \( \left( \frac{1}{p} \right) \)-concave on its support; consequently, \( B_{\Delta_{\frac{1}{p}} f,s} \) is a convex body. Noting that

\[
B_{\Delta_{\frac{1}{p}} f,s} = B_{f,s} + (-B_{f,s}),
\]

we may use \((3.5)\) for \( \Delta_{\frac{1}{p}} f \) and inequality \((1.8)\) to write

\[
\left(\int_{H} \Delta_{\frac{1}{p}} f(x)d\mu(x)\right)^{q} \leq \frac{1}{\omega_p} \nu(B_{f,s} + (-B_{f,s}) \cap (H_q \times \mathbb{R}^p))
\]

\[
\leq \left[C \frac{mq+p}{mp} \right]^{mq+p} \frac{1}{\omega_p} \sup_{(\bar{x},\bar{y}) \in \mathbb{R}^{mq+p}} \nu(B_{f,s} \cap (H_q \times \mathbb{R}^p + (\bar{x}, \bar{y})))
\]

\[
\leq \left[C \frac{mq+p}{mp} \right]^{mq+p} \frac{1}{\omega_p} \sup_{\bar{x} \in \mathbb{R}^{mq}} \nu(B_{f,s} \cap ((H_q + \bar{x}) \times \mathbb{R}^p))
\]

\[
\leq \left[C \frac{mq+p}{mp} \right]^{mq+p} \left(\sup_{z \in \mathbb{R}^n} \int_{H+z} f(x)d\mu(x)\right)^{q}.
\]
Taking the \( q \)th root in (3.6), we obtain
\[
\int_H \Delta_1^s f(x)d\mu(x) \leq \left[ \frac{C(nq+p)}{mq+p} \right]^{mq+p} \sup_{y \in \mathbb{R}^n} \int_{H+y} f(x)d\mu(x),
\]
for some absolute constant \( C > 1 \), as desired.

The case for general \( s \geq 0 \) follows from a standard approximation argument. \( \square \)

We notice that Theorem 1.2 implies for any convex body \( K \subset \mathbb{R}^n \) and any \( H \in G_{n,m} \), where \( m \in \{1, \ldots, n\} \)
\[
\text{vol}_m((K-K) \cap H) \leq (cn^{\frac{1}{m}})^m \sup_{y \in \mathbb{R}^n} \text{vol}_m(K \cap (H+y)) \tag{3.7}
\]
for some absolute constant \( c > 1 \). Repeating the proof of Theorem 3.2 in the case of the Lebesgue measure, but applying inequality (3.7) to \( A_{f,s} \) and \( B_{f,s} \) in place of (1.8), we get the following theorem.

**Theorem 3.3.** Let \( f: \mathbb{R}^n \to \mathbb{R}_+ \) be an integrable \( \left( \frac{1}{s} \right) \)-concave function, for some \( 0 \leq s < \infty \) and let \( H \in G_{n,m} \), \( m \in \{1, \ldots, n\} \). Then
\[
\int_H \Delta_1^s f(x)dx \leq (c(n+s)^{\frac{1}{2}})^{m+s} \sup_{y \in \mathbb{R}^n} \int_{H+y} f(x)dx,
\]
where \( c > 1 \) is some absolute constant.

### 4 Symmetry estimates of marginals of \( \left( \frac{1}{s} \right) \)-concave functions in the case of measures with even quasi-concave densities

In this section we prove estimates for measures of symmetry of a similar nature to [13].

In the case when the density \( \phi \) of the measure \( \mu \) is taken to be an even quasi-concave function, one can hope to reverse inequality (3.1) in a reasonable way. We state this result as follows.

**Theorem 4.1.** Let \( \mu \) be a measure on \( \mathbb{R}^n \) given by \( d\mu(x) = \phi(x)dx \), where \( \phi: \mathbb{R}^n \to \mathbb{R}_+ \) is an even, bounded quasi-concave function and let \( H \in G_{n,m} \). Then, for any bounded, integrable, \( \left( \frac{1}{s} \right) \)-concave function \( f: \mathbb{R}^n \to \mathbb{R}_+ \), where \( s \in (-\infty, 0) \),
\[
\|f\|_\infty^{1/m} \leq \left[ \int_H \Delta_1^s f(2z)d\mu(z) \left/ \int_0^1 \sup_{y \in \mathbb{R}^n} \mu(C_t(f) \cap (H+y))dt \right. \right]^{\frac{1}{m}} \leq \left( n + m \right)^{\frac{1}{n}} \|f\|_\infty^{1/m}. \tag{4.1}
\]

We notice that, immediately from Stirling’s approximation theorem, we may write
\[
\|f\|_\infty^{1/m} \leq \left[ \int_H \Delta_1^s f(2z)d\mu(z) \left/ \int_0^1 \sup_{y \in \mathbb{R}^n} \mu(C_t(f) \cap (H+y))dt \right. \right]^{\frac{1}{m}} \leq \frac{cn^{\frac{1}{m}}}{m} \|f\|_\infty^{1/m}
\]
Before proving Theorem 4.1, we remark on an important connection between the Brunn-Minkowski and Rogers-Shephard inequalities: given any convex body \( K \subset \mathbb{R}^n \)

\[
1 \leq \left( \frac{\text{vol}_n \left( \frac{K-K}{2} \right)}{\text{vol}_n (K)} \right)^{\frac{1}{n}} \leq \left[ \left( \frac{2n}{n} \right) \right]^{\frac{1}{n}} < 2,
\]

where the left-hand side is minimized by symmetric convex bodies and the right-hand side is maximized by simplices. Heuristically, this relation means that every convex body containing the origin is contained in a centrally symmetric convex body with more or less the same volume radius; the volume radius of a convex body is the radius of the Euclidean ball centered at zero that has the same volume. Inequality (4.1) extends (4.2) to the class of quasi-concave functions against marginals of measures with symmetric quasi-concave densities. We list immediate corollaries of similar types.

Taking \( f = \chi_K \) for some convex body \( K \subset \mathbb{R}^n \), and by using the fact that \( K - K \subset K - K \), we have the following corollary:

**Corollary 4.1.** Let \( \mu \) be a measure on \( \mathbb{R}^n \) given by \( d\mu(x) = \phi(x)dx \), where \( \phi: \mathbb{R}^n \to \mathbb{R}_+ \) is an even, bounded quasi-concave function. Then, for any convex body \( K \subset \mathbb{R}^n \),

\[
1 \leq \left( \frac{\mu \left( \frac{K-K}{2} \cap H \right)}{\sup_{y \in \mathbb{R}^n} \mu (K \cap (H + y))} \right)^{\frac{1}{m}} \leq \left( \frac{n + m}{n} \right)^{\frac{1}{n}} \leq \frac{cn}{m}.
\]

for some absolute constant \( c > 1 \).

**Remark 4.1.** We note that, in general, the ratio appearing in (4.3) fails to be bounded for general \( 1 \leq m \leq n \) (see [26, Theorem 2]). However, when \( m \) is of dimension proportional to \( n \), then the this ratio is indeed bounded.

The follow corollary was originally proven in Alonso, Cifre, Roysdon, Yepes, and Zvavitch (see [3, Theorem 2.2]).

**Corollary 4.2.** Let \( \mu \) be a measure on \( \mathbb{R}^n \) given by \( d\mu(x) = \phi(x)dx \), where \( \phi: \mathbb{R}^n \to \mathbb{R}_+ \) is an even, bounded quasi-concave function. Then, for any convex body \( K \subset \mathbb{R}^n \),

\[
1 \leq \left( \frac{\mu \left( \frac{K-K}{2} \right)}{\sup_{y \in \mathbb{R}^n} \mu (-y + K)} \right)^{\frac{1}{2}} \leq \left( \frac{2n}{n} \right)^{\frac{1}{n}} < 4
\]

with equality on the left if and only if \( K \cap \text{supp}(\phi) \) is centrally symmetric. If, in addition, \( \phi \) is continuous at the origin, then there is equality on the right if and only if \( \phi \) is a constant multiple of the Lebesgue measure and \( K \) is an \( n \)-dimensional simplex.

It is useful to note that the left-most inequality appearing in (4.4) fails to hold if either the quasi-concavity assumption or the symmetry assumption on the density \( \phi \) of \( \mu \) are dropped.
To see why the symmetry may not be dropped, we consider the following example. For each 
\( n \in \mathbb{N} \), we define a measure \( \mu_n \) on \( \mathbb{R}^n \) whose density \( \phi_n \) are each defined by 
\[
\phi_n(x) = \begin{cases} 
1, & \text{if } x \in W_n, \\
0, & \text{otherwise}, 
\end{cases}
\]
where \( W_n \) is the wedge formed between the positive \( x \)-axis and the line on the terminal side 
of the angle \( \frac{1}{n} \). Let \( K = \left( \frac{1}{2} \right) B_2 \) so that \( K - K = B_2 \). Then \( \{\mu_n\}_{n \in \mathbb{N}} \) is a sequence of 
measures on \( \mathbb{R}^2 \) with quasi-concave densities such that \( \mu_n(K - K) \to 0 \) as \( n \to \infty \); however, 
as the left-most inequality appearing in (4.4) allows us to shift \( K \), we maintain the measure 
of the body \( K \) even as we send \( n \to \infty \). To see why we need quasi-concavity, we only need 
to consider the same densities \( \phi_n \) but with their support extended to \( -W_n \) for each \( n \).

In view of Corollary 1.1, Theorem 4.1 yields the following result.

**Corollary 4.3.** Given any integrable, bounded \( \left( \frac{1}{s} \right) \)-concave function \( f: \mathbb{R}^n \to \mathbb{R}_+ \) for some 
\( s \in (-\infty, 0] \),
\[
\|f\|_\infty^{1/m} \leq \frac{\int_H \Delta_{-\infty} f(x) dx}{\int_0^1 \sup_{y \in \mathbb{R}^n} \text{vol}_m(C_t(f) \cap (H + y)) dt} \leq c\psi(n, m) \|f\|_\infty^{1/m} \leq cn^\frac{s}{2} \|f\|_\infty^{1/m}.
\]
for some constant \( c > 1 \).

We now proceed to the proof of Theorem 4.1. The proof is analogous to Theorem 2.2 in [3].

**Proof of Theorem 4.1.** Without loss of generality, way may assume that \( \|f\|_\infty = \|\phi\|_\infty = 1 \) 
and that \( f \) is quasi-concave. The right-most inequality appearing in (4.4) follows immediately 
from (3.1). Therefore, we need only prove the left-most inequality. Using Fubini’s theorem, 
we may write
\[
\Delta_{-\infty} f(2z) = \int_0^1 \chi_{C_t(\Delta_{-\infty} f)}(2z) dt \\
= \int_0^1 \chi_{C_t(\Delta_{-\infty} f) / 2}(z) dt \\
= \int_0^1 \chi_{C_t(\Delta_{-\infty} f) / 2}(z) dt.
\]

Another application of Fubini’s theorem yields
\[
\int_H \Delta_{-\infty} f(2z) d\mu(z) = \int_0^1 \chi_{C_t(\Delta_{-\infty} f) / 2}(z) dtd\mu(z) \\
= \int_0^1 \mu \left( \frac{C_t(f) - C_t(f)}{2} \cap H \right) dt.
\]
Fix \( t \in [0,1] \) arbitrarily, and let \( K = C_t(f) \). In view of (4.3), it is suffices to prove that

\[
\sup_{y \in \mathbb{R}^n} \mu(K \cap (H + y)) \leq \mu \left( \frac{K - K}{2} \cap H \right).
\] (4.6)

Fix \( y \in \mathbb{R}^n \) arbitrary. Since \( \phi \) is even and quasi-concave with \( \phi(0) = \|\phi\|_\infty \), the super-level sets \( C_s(\phi) \) are non-empty origin symmetric convex sets. Applying Fubini’s theorem, we observe that

\[
\mu(K \cap (H + y)) = \int_H \phi(x) \chi_{-y+K}(x) \, dx = \int_0^1 \text{vol}_m(K_s(y) \cap H) \, dy \, ds \leq 2^{-m} \int_0^1 \text{vol}_m((K_s(y) - K_s(y)) \cap H) \, dy \, ds,
\] (4.7)

where \( K_s(y) = (-y + K) \cap C_s(\phi) \), and in the last inequality we have used the Brunn-Minkowski inequality together with the fact that \( H \) is a subspace of \( \mathbb{R}^n \). Using the convexity and symmetry of \( C_s(\phi) \), one can see that, for each \( s \in [0,1] \), the following set inclusion holds:

\[
K_s(y) - K_s(y) \subset (K - K) \cap 2C_s(t) = 2 \left( \frac{K - K}{2} \right) \cap C_s(\phi),
\]

which, together with (4.7), implies

\[
\mu(K \cap (H + y)) \leq \int_0^1 \text{vol}_m \left( \left( \frac{K - K}{2} \right) \cap H \right) \, ds = \int_0^1 \text{vol}_m \left( \left( \frac{K - K}{2} \right) \cap H \right) \, ds = \mu \left( \frac{K - K}{2} \cap H \right).
\] (4.8)

Combining (4.7) and (4.8) yields (4.6), as desired. \( \square \)

By repeating the proof of Theorem 3.2 we obtain the following measures of symmetry of marginals of \((\frac{1}{s})\)-concave functions with \( s \geq 0 \).

**Corollary 4.4.** Let \( \mu \) be a non-negative measure on \( \mathbb{R}^n \) have an even, quasi-concave density \( \phi \) with \( \|\phi\|_\infty = \phi(0) \) and let \( H \in G_{n,m} \). Then, for any integrable \((\frac{1}{s})\)-concave function \( f: \mathbb{R}^n \to \mathbb{R}_+ \) with \( s \geq 0 \),

\[
1 \leq \left[ \frac{\int_H \Delta_{\frac{1}{s}} f(2x) \, d\mu(x)}{\sup_{y \in \mathbb{R}^n} \left\{ \int_{H+y} f(x) \, d\mu(x) \right\}} \right]^{\frac{1}{m+s}} \leq c \frac{(n+s)}{m+s},
\]

where \( c > 1 \) is some absolute constant.
5 Sectional Rogers-Shephard type inequalities for measures having \((\frac{1}{s})\)-concave densities, \(-n < s, s \neq 0\)

In this section, we study sectional Rogers-Shephard type inequalities in the setting of measures having \((\frac{1}{s})\)-concave densities, \(-n < s, s \neq 0\), where we choose the binomial constant depending on \(s\) in addition to \(n\) and \(m\).

**Theorem 5.1.** Let \(\mu\) be a measure on \(\mathbb{R}^n\) given by \(d\mu(x) = \phi(x)dx\), where \(\phi\) is a \((\frac{1}{s})\)-concave function, for some \(s \neq 0, s > -n\), with \(\|\phi\|_\infty = \phi(0)\) and let \(H \in G_{n,m}\). Then, for any convex bodies \(K, L \subset \mathbb{R}^n\) with \(0 \in K \cap (-L)\),

\[
\mu((K + L) \cap H) \leq \frac{\mu(K) \sup_{y \in \mathbb{R}^n} \mu(L \cap (H - y))}{(n + s)B(n + s, m)\mu(K \cap (-L))},
\]

where \(B(x, y) = \int_0^1 (1 - t)^{x-1}t^{y-1}dt\) denotes the beta function.

**Proof.** The proof follows along the lines of Theorem 1.3, with a few key modifications. Consider the function \(f : \mathbb{R}^n \rightarrow \mathbb{R}_+\) given by

\[
f(x) = \mu(K \cap (x - L)).
\]

Using the convexity of \(K\) and \(L\), we see that

\[
K \cap (x_\lambda + y_\lambda - L) \supset (1 - \lambda)K \cap (x - L) + \lambda K \cap (y - \lambda L)
\]

holds for all \(x, y \in \mathbb{R}^n\) and \(\lambda \in [0, 1]\), where \(x_\lambda + y_\lambda = (1 - \lambda)x + \lambda y\). This inclusion, together with the Borell-Brascamp-Lieb inequality \((1.2)\), imply that \(f\) is a \((\frac{1}{n+s})\)-concave function on its support.

Applying Fubini’s theorem to the integral \(\int_H f(x)d\mu(x)\), we may write

\[
\int_H f(x)d\mu(x) = \int_H \int_{\mathbb{R}^n} \chi_{K \cap (x - L)}(y)\phi(y)dy\phi(x)dx
\]

\[
= \int_H \int_K \chi_{x-L}(y)\phi(y)dy\phi(x)dx
\]

\[
= \int_H \int_K \chi_{y+L}(x)\phi(y)dy\phi(x)dx
\]

\[
\leq \mu(K) \sup_{y \in \mathbb{R}^n} \mu(L \cap (H - y)).
\]

For a lower bound, we consider the function \(g : (K + L) \cap H \rightarrow \mathbb{R}_+\) given by

\[
g(x) = f(0) \left(1 - \frac{|x|}{\rho \left(\frac{x}{|x|}\right)}\right)^{n+s}
\]

if \(x \neq 0\),

and \(g(0) = f(0)\). Using the concavity condition of \(f\), together with a polar coordinates transformation, we see that

\[
\int_H f(x)d\mu(x) \geq \int_{S_H} \int_0^{\rho(u)} f(0) \left[1 - \frac{r}{\rho(u)}\right]^{n+s} r^{m-1}\phi(ru)drdu.
\]
Since $\phi$ is a $(\frac{1}{s})$-concave function on $\mathbb{R}^n$ with $\phi(0) = \|\phi\|_\infty$, it is radially decreasing; hence, by arguing as in the proof of Theorem 1.3, we see that

$$\int_H f(x) d\mu(x) \geq (n + s)B(n + s, m) \mu((K + L) \cap H). \hspace{1cm} (5.3)$$

Combining (5.2) and (5.3), we obtain

$$\mu((K + L) \cap H) \leq \frac{\mu(K) \sup_{y \in \mathbb{R}^n} \mu(L \cap (H - y))}{(n + s)B(n + s, m) \mu(K \cap (-L))},$$

as desired.

Corollary 5.1. Let $\mu$ be a measure on $\mathbb{R}^n$ given by $d\mu(x) = \phi(x) dx$, where $\phi$ is a $(\frac{1}{s})$-concave, $s \neq 0, s > -n$, function, with $\|\phi\|_\infty = \phi(0)$ and let $H \in G_{n,m}$. Then, for any convex body $K \subset \mathbb{R}^n$,

$$\mu((K - K) \cap H) \leq \sup_{y \in \mathbb{R}^n} \mu(K \cap (H + y)).$$

6 Marginal inequalities for the Rogers-Shephard type for logarithmically concave functions

We begin this section by defining the class of admissible functions,

$$\text{LC}_0 = \left\{ f : \mathbb{R}^n \to \mathbb{R}_+ : f \text{ is log-concave, } f(0) = \|f\|_\infty, 0 < \int f < \infty \right\}.$$

The main theorem of this sections reads as follows.

Theorem 6.1. Let $f \in \text{LC}_0$ and let $H \in G_{n,m}$. Then

$$c \|f\|_\infty^{1/m} \leq \left[ \frac{\int_H \Delta_0 f(x) dx}{\sup_{y \in \mathbb{R}^n} \left\{ \|f\|_\infty \int_{H+y} f(x) dx \right\}} \right]^{\frac{1}{m}} \leq C \|f\|_\infty^{1/m} \psi(n, m) \hspace{1cm} (6.1)$$

where

$$f_{H+y} := \sup_{x_0 \in H+y} f(x_0)$$

and $c > 0$ and $C > 1$ are some absolute constants.

As an immediate consequence of inequality (6.1), we have the following corollary.

Corollary 6.1. Let $f \in \text{LC}_0$ and let $H \in G_{n,m}$. Then

$$c \|f\|_\infty^{1/m} \leq \left[ \frac{\int_H \Delta_0 f(x) dx}{\sup_{y \in \mathbb{R}^n} \left\{ \|f\|_\infty \int_{H+y} f(x) dx \right\}} \right]^{\frac{1}{m}} \leq C \|f\|_\infty^{1/m} n^{1/3}$$

for some absolute constants $c > 0$ and $C > 1$. 

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Before proceeding to the proof of Theorem 6.1, we must first introduce some concepts that are critical to the proof.

To functions \( f \in \mathcal{L}C_0 \), for each \( m \in \{1, \ldots, n\} \), one may associate the following \( m \)-dimensional convex body originally due to Ball (see [7] and [8])

\[
K_m(f) = \left\{ x \in \mathbb{R}^n : \left( \frac{1}{\|f\|_{\infty}} \int_0^\infty mr^{m-1}f(rx)dr \right)^{-\frac{1}{m}} \leq 1 \right\}.
\]

The radial function of \( K_m(f) \) is given by

\[
\rho_{K_m(f)}(u) = \left( \frac{1}{\|f\|_{\infty}} \int_0^\infty mr^{m-1}f(ru)dr \right)^{-\frac{1}{m}},
\]

where \( u \in S^{n-1} \). Moreover, for any \( H \in G_{n,m} \), one has

\[
\int_H f(x)dx = \|f\|_{\infty} \text{vol}_m(K_m(f) \cap H).
\]

Indeed, integrating in polar coordinates we see that

\[
\text{vol}_m(K_m(f) \cap H) = \int_H \chi_{K_m(f)}(x)dx
\]

\[
= m \int_{S^{n-1} \cap H} \int_0^{\rho_{K_m(f)}(u)} r^{m-1}dr du
\]

\[
= \int_{S^{n-1} \cap H} \rho_{K_m(f)}(u)^m du
\]

\[
= \int_{S^{n-1} \cap H} \frac{m}{\|f\|_{\infty}} \int_0^\infty f(rz)r^{m-1}dr du
\]

\[
= \frac{1}{\|f\|_{\infty}} \int_H f(z)dz.
\]

Additionally, we will use the following auxiliary lemmas due to Klartag and Milman (see [17, Lemma 2.2] and [17, Lemma 2.7], respectively).

For \( f \in \mathcal{L}C_0 \) and \( 1 \leq m \leq n \), define the set

\[
L_m(f) = \{ x \in \mathbb{R}^n : f(x) \geq \|f\|_{\infty}e^{-m} \}.
\]

**Lemma 6.1.** Given \( f \in \mathcal{L}^n \) and \( 1 \leq m \leq n \),

\[
K_m(f) \subset L_m(f) \subset cK_m(f)
\]

for some absolute constant \( c > 1 \).

**Lemma 6.2.** Let \( f \in \mathcal{L}C_0 \) and let \( 1 \leq m \leq n \). Then

\[
K_m(\Delta_0 f) \subset c[K_m(f) + (-K_m(f))] \subset c'K_m(\Delta_0 f)
\]

for some absolute constants \( c, c' > 1 \).
The power of the above lemmas is that they allow the replacement of the bodies $K_m(f)$ with certain level sets of a logarithmically concave function $f$. We now proceed to the proof of Theorem 6.1.

**Proof of Theorem 6.1.** Without loss of generality, we may assume that $\|f\|_\infty = 1$. For brevity we set $g = \Delta_0 f$ and consider its $m$-dimensional associated body $K_m(g)$. Since $\|\Delta_0 f\|_\infty = \|f\|_2^2$, we may also assume that $\|g\|_\infty = 1$. As mentioned above, we have that

$$\int_H g(x) dx = \text{vol}_m(K_m(g) \cap H).$$

Using Lemma 6.2, we must have that $K_m(g) \subset c(K_m(f) + (-K_m(f)))$ for some constant $c > 0$. Then, by applying (1.4), it follows that

$$\int_H g(x) dx \leq c^m \text{vol}_m((K_m(f) + (-K_m(f))) \cap H) \leq [c\psi(n,m)]^m \sup_{y \in \mathbb{R}^n} \text{vol}_m(K_m(f) \cap (H + y)).$$

(6.2)

Fix an arbitrary $y \in \mathbb{R}^n$. We must compare $\text{vol}_m(K_m(f) \cap (H + y))$ and $\int_{H+y} f(x) dx$. In view of Lemma 6.1, we observe

$$K_m(f) \cap (H + y) \subset L_m(f) \cap (H + y)$$

$$= \{x \in H + y : f(x) \geq e^{-m}\}$$

$$= L_m(f|_{H+y})$$

$$\subset c'K_m(f|_{H+y})$$

(6.3)

for some absolute constant $c' > 0$. Let $\rho := \rho_{K_m(f|_{H+y})}$. Integrating in polar coordinates,

$$\text{vol}_m(K_m(f|_{H+y})) = \int_{\mathbb{R}^n} \chi_{K_m(f|_{H+y})}(x) dx$$

$$= m \int_{\partial B_y \cap (H+y)} \int_0^{\rho(u)} r^{m-1} dr du$$

$$= \int_{\partial B_y \cap (H+y)} \rho(u)^m du$$

$$= \int_{\partial B_y \cap (H+y)} \frac{m}{f_{H+y}} \int_0^\infty f(rz) r^{m-1} dr du$$

$$= \frac{1}{f_{H+y}} \int_{H+y} f(x) dx,$$

(6.4)

where $B_y = B_n + y$ and $f_{H+y} = \sup_{x_0 \in H+y} f(x_0)$. Combining (6.2), (6.3), and (6.4), we obtain

$$\int_H g(x) dx \leq [c\psi(n,m)]^m \sup_{y \in \mathbb{R}^n} \left\{ \frac{1}{f_{H+y}} \int_{H+y} f(x) dx \right\}.$$

Finally, by taking the $m$th root of both sides, we have proven the upper bound of (6.1).
Now we prove the lower bound of inequality (6.1). Fix an arbitrary $y \in \mathbb{R}^n$. In view of Lemma 6.1, we may apply inclusions similar to (6.3) to conclude that

$$K_m(f \mid_{H+y}) \subset c_1 K_m(f) \cap (H + y)$$

for some absolute constant $c_1 > 0$. Using Lemma 6.2 together with the Brunn-Minkowski inequality, we see that

$$\text{vol}_m(K_m(f \mid_{H+y})) \leq c_1^m \text{vol}_m((K_m(f) - y) \cap H) \leq c_1^m 2^{-m} \text{vol}_m((K_m(f) + (-K_m(f)) \cap H) \leq (c')^m 2^{-m} \text{vol}_m(K_m(g) \cap H)$$

(6.5)

Combining (6.4) and (6.5), we see that, for some constant $\tilde{c} > 0$,

$$\frac{1}{f_{H+y}} \int_{H+y} f(x) dx \leq \text{vol}_m(K_m(g) \cap H) = \int_H g(x) dx.$$

Taking the supremum over all $y \in \mathbb{R}^n$ and taking the $m$th root yields the lower bound of inequality (6.1), completing the proof.

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