Multi-instantons in seven dimensions

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Abstract

We consider the self-dual Yang-Mills equations in seven dimensions. Modifying the t’Hooft construction of instantons in $d = 4$, we find $N$-instanton 7d solutions which depend on $8N$ effective parameters and are $E_6$-invariant.

1 Introduction

The pure Yang-Mills (YM) theory defined in the four-dimensional Euclidean space has a rich and interesting structure even at the classical level. The discovery of regular solutions to the YM field equations, which correspond to absolute minimum of the action (Belavin et al.) [1], has led to an intensive study of such a classical theory. One hopes that a deep understanding of the classical theory will be invaluable when one tries to quantize such a theory.

In the past few years, increased attention has been paid to gauge field equations in space-time of dimension greater than four, with a view to obtaining physically interesting theories via dimensional reduction [2]. Such equations appear in the many-dimensional theory of supergravity, in the low-energy effective theory of $d$-branes, and in M-theory [3]. Using solutions of the YM equations in $d > 4$ makes possible to obtain soliton solutions in these theories [4]. It is known also that the YM theory in $d$ dimensions may

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be reduced to the Yang-Mills-Higgs (YMH) theory in \( k < d \) dimensions [5].

Hence, solutions of the YMH equations in \( d = 4 \) may be obtained from solutions of the YM equations in \( d > 4 \) dimensions.

In Ref. [6], the 4d self-dual Yang-Mills equations was generalized to the higher-dimensional linear relations (CDFN equations)

\[
c_{mnps}F^{ps} = \lambda F_{mn},
\]

where the numerical tensor \( c_{mnps} \) is completely antisymmetric and \( \lambda = \text{const} \) is a non-zero eigenvalue. It is obviously that these equations lead to the full YM equation, via the Bianchi identity. Several self-dual solutions of (1) were found in [7].

The paper is organized as follows. Sections 2 and 3 contain well-known facts about the Cayley-Dickson algebras and their derivations. In Section 4 multi-instanton solutions of the \( G_2 \)-invariant CDFN equations are found. In Section 6 the \( E_6 \)-invariance of these solutions are proved.

## 2 Cayley-Dickson algebras

Let \( A \) be an algebra with an involution \( x \rightarrow \bar{x} \) over a field \( F \) of characteristic \( \neq 2 \). Given a nonzero \( \alpha \in F \) we define a multiplication on the vector space \((A, \alpha) = A \oplus A\) by

\[
(x_1, y_1)(x_2, y_2) = (x_1x_2 - \alpha\bar{y}_2y_1, y_2x_1 + y_1\bar{x}_2).
\]

This makes \((A, \alpha)\) an algebra over \( F \). It is clear that \( A \) is isomorphically embedded into \((A, \alpha)\) and \( \dim(A, \alpha) = 2\dim A \). Let \( e = (0, 1) \). Then \( e^2 = -\alpha \) end \((A, \alpha) = A \oplus Ae\). Given any \( z = x + ye \) in \((A, \alpha)\) we suppose \( \bar{z} = \bar{x} - ye \). Then the mapping \( z \rightarrow \bar{z} \) is an involution in \((A, \alpha)\).

Starting with the base field \( F \) the Cayley-Dickson construction leads to the following sequence of alternative algebras:

1) \( F \), the base field.

2) \( \mathbb{C}(\alpha) = (F, \alpha) \), a field if \( x^2 + \alpha \) is the irreducible polynomial over \( F \); otherwise, \( \mathbb{C}(\alpha) \simeq F \oplus F \).

3) \( \mathbb{H}(\alpha, \beta) = (\mathbb{C}(\alpha), \beta) \), a generalized quaternion algebra. This algebra is associative but not commutative.

4) \( \mathbb{O}(\alpha, \beta, \gamma) = (\mathbb{H}(\alpha, \beta), \gamma) \), a Cayley-Dickson algebra. It is easy to prove that this algebra is nonassociative.
The algebras in 1) – 4) are called composition. Any of them has the non-degenerate quadratic form (norm) \( n(x) = x\bar{x} \), such that \( n(xy) = n(x)n(y) \). The norm \( n(x) \) defines the scalar product

\[
(x, y) = \frac{1}{2}(\bar{xy} + \bar{yx})
\]

that is invariant with respect to all automorphisms of the composition algebra. It is known also that over the field \( \mathbb{R} \) of real numbers, the above construction gives 3 split algebras (e.g., if \( \alpha = \beta = \gamma = -1 \)) and 4 division algebras (if \( \alpha = \beta = \gamma = 1 \)): the fields of real \( \mathbb{R} \) and complex \( \mathbb{C} \) numbers, the algebras of quaternions \( \mathbb{H} \) and octonions \( \mathbb{O} \), taken with the Euclidean norm \( n(x) \). Finally note that any composition algebra is alternative, i.e. in any of them the associator

\[
(x, y, z) = (xy)z - x(yz)
\]

is skew-symmetric over \( x, y, z \). Note also that any simple nonassociative alternative algebra is isomorphic to the Cayley-Dickson algebra \( \mathbb{O}(\alpha, \beta, \gamma) \).

As any finite-dimensional algebra, the Cayley-Dickson algebra may be defined by "a multiplication table" in some fixed basis. For that we consider a real linear space \( A \) equipped with a nondenerate symmetric metric \( g \) of signature \((8, 0)\) or \((4, 4)\). Choose the basis \( 1, e_1, \ldots, e_7 \) in \( A \) such that

\[
g = \text{diag}(1, \alpha, \beta, \alpha\beta, \gamma, \alpha\gamma, \beta\gamma, \alpha\beta\gamma),
\]

where \( \alpha, \beta, \gamma = \pm 1 \). Define the multiplication

\[
e_i e_j = -g_{ij} + c_{ij}^k e_k,
\]

where the structural constants \( c_{ijk} = g_{ks}c_{ij}^s \) are completely antisymmetric and different from 0 only if

\[
c_{123} = c_{145} = c_{167} = c_{246} = c_{275} = c_{374} = c_{365} = 1.
\]

The multiplication (5) transform \( A \) into a real linear algebra. It can easily be checked that \( A \) is isomorphic to \( \mathbb{O}(\alpha, \beta, \gamma) \).

### 3 Derivations

Recall that a derivation of an algebra \( A \) is a linear transformation \( D \) of \( A \) satisfying

\[
(xy)D = (xD)y + x(yD)
\]
for all \( x, y \in A \). The derivations of Cayley-Dickson algebra may be described in intrinsic terms. Namely, let \( A \) be a Cayley-Dickson algebra. Then for any \( x, y \in A \) the mapping

\[
D_{x,y} : z \rightarrow 2[z, [x, y]] + 6(z, x, y)
\]  

is a derivation of \( A \). Therefore we have the linear mapping \( \Lambda^2 \rightarrow \text{Der}A \). Since any Cayley-Dickson algebra is simple, it follows that this mapping is surjective. In addition, the following relations

\[
D_{x,y}z = D_{y,x}z + D_{z,x}y,
\]

are true. Note also that the derivations algebra \( \text{Der}A \) is a simple exceptional Lie algebra of type \( g_2 \).

Since the associator (3) of Cayley-Dickson algebra is skew-symmetric over its arguments, it follows that we can define the completely antisymmetric tensor \( c_{ijkl} \) by

\[
(e_i, e_j, e_k) = 2c_{ijk}^l e_l.
\]

It is easy to prove that this tensor satisfies the following identities:

\[
c_{ijs}c_{kl}^s = g_{ik}g_{jl} - g_{il}g_{jk} + c_{ijkl},
\]

\[
c_{ijps}c_{kl}^ps = 4(g_{ik}g_{jl} - g_{il}g_{jk}) + 2c_{ijkl};
\]

and has the nonzero components:

\[
c_{4567} = c_{2367} = c_{2345} = c_{1357} = c_{1364} = c_{1265} = c_{1274} = 1.
\]

Further, it follows from (2) and (9) that the tensor \( c_{ijkl} \) is invariant with respect to all automorphisms of algebra \( A \). Noting that the group \( \text{Aut}A \) is isomorphic to the Lie group of type \( G_2 \), we see that the tensor \( c_{ijkl} \) is \( G_2 \)-invariant. Finally, rewriting the identity (7) in the form

\[
c_i^{jk} D_{jk} = 0,
\]

where the derivation \( \frac{1}{8} D_{e_i,e_j} \) is denoted by the symbol \( D_{ij} \), we get the following relations

\[
c_i^{k} D_{kl} = -2D_{ij}.
\]
Since the algebra $\text{Der}A$ is a Lie algebra of type $g_2$ (or $g'_2$ in noncompact case), it follows that it may be considered as a subalgebra of the Lie algebra $\text{so}(m, n)$ of type $so(7)$ or $so(3, 4)$. Hence there exists the projector $c^+_{ijkl}$ of one onto the subspace $\text{Der}A$. Usually this projector is chosen in the form (See [7]):

$$c^+_{ijkl} = \frac{1}{6} (2g_{ik}g_{lj} - 2g_{il}g_{jk} - c_{ijkl}).$$  \hfill (13)

In addition, it is easily shown that the derivations

$$D_{ij} = \frac{3}{2} c^+_{ij} E_{kl},$$  \hfill (14)

where $E_{kl}$ are generators of the Lie algebra $\text{so}(m, n)$ satisfying the switching relations

$$[E_{ij}, E_{kl}] = g_{k[i}E_{j]l} - g_{l[i}E_{j]k}.$$

Besides, it follows from (11) that

$$c_{ij} c^+_{kl} = -2c^+_{ijkl}.$$  \hfill (15)

Comparing (14) and (15), we again obtain the identity (12).

4 Solutions

Recall that the self-dual equations has been successfully tackled by the twistor techniques, and in the case of finite action solutions by the algebraic ADHM construction [9]. A generalization of the ADHM construction for the equations (1) which break $SO(4n)$ up to $Sp(1) \times Sp(n)/Z_2$ was found in [10]. However in dimensions 7 and 8 there exists an exceptional $G_2$-covariant (respectively $\text{Spin}(7)$-covariant) duality which is connected with the octonionic algebra. Therefore the search of generalized ADHM construction in $d = 7$ and 8 appears very attractive.

Such attempt was done in the recent paper [11]. In one the generalized ADHM construction in $d = 8$ was built with the help of the algebra $L(\mathbb{O})$ of left multiplications of octonionic algebra $\mathbb{O}$. Unfortunately, calculating the field strength in Section 5 and proving its self-duality the authors incorrect use the equality $L(L(xy)z) = L(xyz)$, where $L(xyz) = x(yz)$ and $x, y, z \in \mathbb{O}$. By associativity of the octonionic algebra it would not be done.
Nevertheless, it is easy to get multi-instanton solutions (but not a generalized ADHM construction) of CDFN equations in seven dimensions. We choose the ansatz $A_m$ in the form:

$$A_m = \frac{\lambda^i y_i}{1 + y^i y} D_{mi},$$  \hspace{2cm} (16)

where $y$ is a column vector with the elements $y_1, \ldots, y_N$ of Cayley-Dickson algebra such that

$$y^\dagger = (y^k_1, \ldots, y^k_N) e_k, \quad y^k_i \in \mathbb{R},$$  

$$\lambda^\dagger = (\lambda_1, \ldots, \lambda_N), \quad \lambda_I \in \mathbb{R}^+,$$  

$$y^k_i = (b^k_{IJ} + \delta_{IJ} x^k) \lambda_J, \quad b^k_{IJ} = b^k_{JI}.$$  

Using the identities (8)–(10), we get the field strength

$$F_{mn} = -\frac{\lambda^i((2 + 2y^i y - y^i y^i) D_{mn} + 3c^+_{mn} i s D_{sj} y^j y^i)}{(1 + y^i y)^2}\lambda,$$

where the tensor $c^+_{ijkl}$ is defined by the equality (13). Now it follows from (12) and (15) that the field strength $F_{mn}$ satisfies the CDFN equations (1) as for Euclidean as for pseudoeucidean metric of the form (4).

This construction of multi-instanton solutions of the CDFN equations may be easy to extend in eight dimensions. It is sufficient to take the projector $f^+_{ijkl}$ of the algebra Lie of type $so(8)$ or $so(4, 4)$ onto the subalgebra $so(7)$ or $so(3, 4)$ respectively in place $c^+_{ijkl}$, to define the elements $D'_{ij}$ of the form (14), and to prove an analog of the identity (15) (See [7]). Then choosing the ansatz $A'_m$ in the form:

$$A'_m = \frac{\lambda^i y_i}{1 + y^i y} D'_{mi},$$  \hspace{2cm} (17)

where the indexes $m, i \in \{0, \ldots, 7\}$, we can obtain the following expression for the field strength:

$$F'_{mn} = -\frac{1}{3} \frac{\lambda^i((6 + 6y^i y - 3y^i y^i) D'_{mn} + 8f^+_{mn} i s D'_{sj} y^j y^i)}{(1 + y^i y)^2}\lambda.$$

Obviously, the $N$-instanton solutions (16) and (17) depend on $8N$ and $9N$ effective parameters respectively, and are a generalization of the t’Hooft solution in $d = 4$ (see e.g. [12]).
5 \( E_6 \)-invariance

Let \( A \) be a real Cayley-Dickson algebra with the involution \( x \rightarrow \bar{x} \), and let \( A_3 \) be the algebra of all \( 3 \times 3 \) matrix with elements of \( A \). Consider the set

\[
J = \{(x_{ij}) \in A_3 \mid (\bar{x}_{ij}) = (x_{ji})\}.
\]

The set \( J \) is a commutative nonassociative algebra with the respect to the product

\[
x \circ y = \frac{1}{2}(xy + yx).
\]

The algebra \( J \) satisfies the identity

\[
(x^2y)x = x^2(yx)
\]

and is said to be an exceptional Jordan algebra.

Denote \( 3 \times 3 \) matrix \((x_{ij})\) with the unique nonzero element \( x_{ij} = 1 \) by the symbol \( \varepsilon_{ij} \) and choose in \( J \) the basis:

\[
\begin{align*}
E_1 &= \varepsilon_{11}, \quad X_1(e_i) = e_i\varepsilon_{23} + \bar{e}_i\varepsilon_{32}, \\
E_2 &= \varepsilon_{22}, \quad X_2(e_j) = e_j\varepsilon_{31} + \bar{e}_j\varepsilon_{13}, \\
E_3 &= \varepsilon_{33}, \quad X_3(e_k) = e_k\varepsilon_{12} + \bar{e}_k\varepsilon_{21},
\end{align*}
\]

where \( e_0 = 1, e_1, \ldots, e_7 \) is the standard basis of \( A \). It can easily be checked that

\[
E_\alpha \circ X_\beta(e_i) = \begin{cases} 
0, & \text{if } \alpha = \beta, \\
\frac{1}{2}X_\beta(e_i), & \text{if } \alpha \neq \beta,
\end{cases}
\]

\[
X_\alpha(e_i) \circ X_\beta(e_j) = \begin{cases} 
\delta_{ij}(E - E_\alpha), & \text{if } \alpha = \beta, \\
\frac{1}{2}X_\gamma(e_j\bar{e}_i), & \text{if } \alpha \neq \beta,
\end{cases}
\]

where \( E \) is the identity \( 3 \times 3 \) matrix, and \((\alpha\beta\gamma) = (123), (231), (312)\).

It is well known (see e.g. [8]) that the derivations algebra Der\( J \) is a simple exceptional Lie algebra of the type \( f_4 \). Since there is an isomorphic enclosure of the algebra \( g_2 \) into \( f_4 \), we can consider (16) as a field that takes its values
in Der.$J$. To prove the $F_4$-invariance of these solutions, we find the trace of the matrix

$$X_\beta = \{(X_\alpha(e_i), X_\beta(e_j), X_\alpha(e_k)) - \frac{1}{2}(X_\alpha(e_i), X_\alpha(e_j), X_\alpha(e_k)) \} \circ X_\beta(e_l),$$

(21)

where $i, j, k \neq 0$, and we do not sum on the recurring indexes. Using (9) and (19)–(20), we prove that

$$X_\beta = \frac{1}{2}c_{ijkl}(E - E_\beta),$$

and hence

$$\text{tr}X_\beta = c_{ijkl}.$$  

Since a trace of matrix in $J$ is invariant with respect to all automorphisms of $J$, we prove the $F_4$-invariance of solutions of the corresponding CDFN equations.

Moreover, it can be proved that the tensor $c_{ijkl}$ is $E_6$-invariant. Indeed, the group $E_6$ is a group of linear transformations of the space $J$ that preserve the norm

$$n(X) = x_{11}x_{22}x_{33} + (x_{12}x_{23})x_{31} + x_{13}(x_{32}x_{21}) - x_{11}x_{23}x_{32} - x_{22}x_{31}x_{13} - x_{33}x_{12}x_{21},$$

where $X = (x_{ij}) \in J$. Choose an element $X$ in the form

$$X = X_1 + X_2 - X_3 + E_1 + E_2,$$

where matrices $E_\alpha$ and $X_\beta$ are defined by the relations (18) and (21) respectively. Then it follows easily that the norm

$$n(X) = c_{ijkl}.$$  

Since the group $F_4$ can be isomorphically enclosed into the group $E_6$, we prove the $E_6$-invariance of the found solutions.

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