Topological Quantum Gravity of the Ricci Flow

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Based on work with Alex Frenkel and Stephen Randall:

A. Frenkel, P. Hořava and S. Randall,
*Topological Quantum Gravity of the Ricci Flow*,
arXiv:2010.15369[hep-th],

A. Frenkel, P. Hořava and S. Randall,
The Geometry of Time in Topological Quantum Gravity of the Ricci Flow,
arXiv:2011.06230[hep-th],

A. Frenkel, P. Hořava and S. Randall,
*Perelman’s Ricci Flow in Topological Quantum Gravity*,
arXiv:2011.11914[hep-th].
Main Idea

To connect three areas of physics and math:

- topological quantum field theory (of the cohomological type: cf. Witten’s topological Yang-Mills in 4 dimensions [since 1988])

- mathematics of Ricci flows on Riemannian manifolds (of the Hamilton-Perelman type [since 1982])

- nonrelativistic gravity (of the Lifshitz type; [PH, since 2008])

Expected to be useful in both directions.
Ricci Flow: History

Hamilton’s Ricci flow:
Eqn for $g_{ij}(t, x^k)$, a Riemannian metric on spatial manifold $\Sigma^D$,

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}.$$
Ricci Flow: History

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Perelman’s Ricci flow:

$$\dot{g}_{ij} = -2R_{ij} - 2\nabla_i \partial_j \phi,$$

$$\dot{\phi} = -\Delta \phi - R \phi.$$
Ricci Flow: History

Hamilton’s Ricci flow:
Eqn for \( g_{ij}(t, x^k) \), a Riemannian metric on spatial manifold \( \Sigma^D \),

\[
\frac{\partial g_{ij}}{\partial t} = -2R_{ij}.
\]

Perelman’s Ricci flow:

\[
\dot{g}_{ij} = -2R_{ij} - 2\nabla_i \partial_j \phi,
\]

\[
\dot{\phi} = -\Delta \phi - R\phi.
\]

DeTurck’s trick: Apply diffeo generated by \( \xi^i, \xi_i = \partial_i \phi \):

\[
\dot{g}_{ij} = -2R_{ij},
\]

\[
\dot{\phi} = -\Delta \phi + (\partial \phi)^2 - R\phi.
\]
Ricci Flow: History

RHS of Perelman’s Ricci flow follows from a variational principle, Hamilton’s doesn’t.

Perelman’s $\mathcal{F}$-functional:

$$\mathcal{F} = \int d^D x \sqrt{g} e^{-\phi} \left( R + g^{ij} \partial_i \phi \partial_j \phi \right),$$

with variations subjected to a fixed-volume condition:

$$\sqrt{g} e^{-\phi} d^D x = dm, \text{ fixed in time.}$$
Ricci Flow: History

Importance for topology:

- Poincaré conjecture
- Thurston’s geometrization conjecture for 3-manifolds
- New proof of uniformization theorem for 2-manifolds
- Generalized Smale conjecture

Interesting for physics: A theory of gravity, with central role played by concepts of entropy, leading to spacetime singularities with controllable topology change (“Ricci flows with surgery”), for general evolving 3-geometries.
Ricci Flow: Simple Examples

Ricci-flat $\Sigma$
Ricci Flow: Simple Examples

Ricci-flat $\Sigma$  

$\Sigma$ of positive curvature
Ricci Flow: Simple Examples

Ricci-flat $\Sigma$ \hspace{1cm} $\Sigma$ of positive curvature \hspace{1cm} hyperbolic $\Sigma$
Ricci Flow: The Neckpinch (in $D > 2$)

topology change
Ricci Flow: The Neckpinch (in $D > 2$)

$\tau_s$  topology change  Ricci flow with surgery

$t_s + \varepsilon$

$t_s - \varepsilon$
Ricci Flow: The Neckpinch (in $D > 2$)

- topology change
- Ricci flow with surgery
- model of singularity
Gravity with anisotropic scaling
(also known as Hořava-Lifshitz gravity)

Field theory with anisotropic scaling \( (x = \{x^i, i = 1, \ldots D\}) \): 

\[
x \rightarrow \lambda x, \quad t \rightarrow \lambda^z t.
\]

\( z \): dynamical critical exponent – characteristic of RG fixed point.

Many interesting examples: \( z = 1, 2, \ldots, n, \ldots \)
fractions: \( 3/2 \) (KPZ surface growth in \( D = 1 \)), \( \ldots, 1/n, \ldots \)
families with \( z \) varying continuously \ldots

Condensed matter, dynamical critical phenomena, quantum critical systems, \ldots

Goal: Extend to gravity, with propagating gravitons, formulated as a quantum field theory of the metric.
Example: Lifshitz scalar [Lifshitz, 1941]

Gaussian fixed point with $z = 2$ anisotropic scaling:

$$S = S_K - S_V = \frac{1}{2} \int dt \, d^Dx \left\{ \dot{\Phi}^2 - (\Delta \Phi)^2 \right\},$$

($\Delta$ is the spatial Laplacian).

Compare with the Euclidean field theory

$$W = -\frac{1}{2} \int d^d x (\partial \phi)^2.$$

Shift in the (lower) critical dimension:

$$[\phi] = \frac{d - 2}{2}, \quad [\Phi] = \frac{D - 2}{2}.$$
Gravity at a Lifshitz point

Spacetime structure: Preferred foliation by leaves of constant time (avoids the “problem of time”).

Fields: Start with the spacetime metric in ADM decomposition: the spatial metric $g_{ij}$, the lapse function $N$, the shift vector $N_i$.

Symmetries: foliation-preserving diffeomorphisms, $\text{Diff}(M, \mathcal{F})$.

Action: $S = S_K - S_V$, with

$$S_K = \frac{1}{\kappa^2} \int dt \, d^Dx \sqrt{g} N (K_{ij} K^{ij} - \lambda K^2)$$

where $K_{ij} = \frac{1}{N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$ the extrinsic curvature,

and $S_V = \frac{1}{\kappa^2_V} \int dt \, d^Dx \sqrt{g} N \mathcal{V}(R_{ijk\ell}, \nabla_i)$. 
Projectable and nonprojectable theory

$N$, $N_i$ are the gauge fields for the $\text{Diff}(M,\mathcal{F})$ symmetries generated by $\delta t = f(t)$, $\delta x^i = \xi^i(t, x)$. Hence:

1. we can restrict $N(t)$ to be a function of time only: projectable theory.
2. or, we allow $N(t, x)$ to be a spacetime field. New terms, containing $\nabla_i N/N$, are then allowed in $S$ by symmetries: nonprojectable theory.

Spectrum: Tensor graviton polarizations, plus an extra scalar graviton. Three options for the scalar: Live with it, gap it, or eliminate it by an extended gauge symmetry.

Dispersion relation: Nonrelativistic, $\omega^2 \sim k^{2z}$, around this Gaussian fixed point.

Allowed range of $\lambda$: $[0, 1/D, 1]$. 
RG flows

Assume $z > 1$ UV fixed point. Relevant deformations trigger RG flow to lower values of $z$. Example: Lifshitz scalar.

$$S = \frac{1}{2} \int dt d^D x \left\{ \dot{\Phi}^2 - (\Delta \Phi)^2 - \mu^2 \partial_i \Phi \partial_i \Phi - m^4 \Phi^2 \right\},$$

**Multicriticality.** New phases: modulated.

Similarly for gravity:

$$S = \frac{1}{\kappa^2} \int dt d^D x \sqrt{g} N \left\{ K_{ij} K^{ij} - \lambda K^2 - \ldots - \mu^{2z-2} R - M^{2z} \right\}.$$

Flows in IR to $z = 1$ scaling. In the IR regime, $S_V$ is dominated by the spatial part of Einstein-Hilbert.

(The $z > 1$ Gaussian gravity fixed points also emerge in IR in condensed matter lattice models, [Cenke Xu & PH].)
Relevant deformations, RG flows, phases

The Lifshitz scalar can be deformed by relevant terms:

\[ S = \frac{1}{2} \int dt \, d^Dx \left\{ \dot{\phi}^2 - (\Delta \phi)^2 - \mu^2 \partial_i \phi \partial_i \phi + m^4 \phi^2 - \phi^4 \right\} \]

The undeformed \( z = 2 \) theory describes a tricritical point, connecting three phases – disordered, ordered, spatially modulated (“striped”) [A. Michelson, 1976]:

![Diagram showing tricritical point and phases](image-url)
Phase structure in the CDT approach

Compare the phase diagram in the causal dynamical triangulations:

[Ambjørn et al, 1002.3298]

Note: $z = 2$ is sufficient to explain three phases. Possibility of a nontrivial $z \approx 2$ fixed point in $3 + 1$ dimensions?
RG flows in gravity: \( z = 1 \) in IR

Theories with \( z > 1 \) represent candidates for the UV description. Under relevant deformations, the theory will flow in the IR. Relevant terms in the potential:

\[
\Delta S_V = \int dt \, d^D x \sqrt{g} N \left\{ \ldots + \mu^2 (R - 2\Lambda) \right\}.
\]

the dispersion relation changes in IR to \( \omega^2 \sim k^2 + \ldots \)

the IR speed of light is given by a combination of the couplings \( \mu^2 \) combines with \( \kappa, \ldots \) to give an effective \( G_N \).

Sign of \( k^2 \) in dispersion relation is opposite for the scalar and the tensor modes! Can we classify the phases of gravity? Can gravity be in a modulated phase?
Preliminaries: Structure of spacetime

Goal: Topological quantum gravity, localization to Ricci flows. Expect $M$ a foliation, by leaves $\Sigma$ of constant $t$. Take

$$M^{D+1} = I \times \Sigma^D, \quad I \subset \mathbb{R}.$$ 

Topological BRST charge $Q$:

$$Qg_{ij} = \psi_{ij}$$

Antighosts and auxiliary:

$$Q\chi_{ij} = B_{ij}.$$ 

Balanced theory – natural to formulate in $\mathcal{N} = 2$ superspace:

$$G_{ij}(t, x^k, \theta, \bar{\theta}) = g_{ij} + \theta \psi_{ij} + \bar{\theta} \chi_{ij} + \theta \bar{\theta} B_{ij}.$$
Primitive Topological Gravity of Ricci Flow

Supercharges and superderivatives:

\[ Q = \partial_\theta, \quad \bar{Q} = \partial_{\bar{\theta}} + \theta \partial_t, \]
\[ \bar{D} = \partial_{\bar{\theta}}, \quad D = \partial_\theta - \bar{\theta} \partial_t. \]

Superalgebra: \( \{Q, \bar{Q}\} = \partial_t, \quad \{D, \bar{D}\} = -\partial_t. \)

Action: \( S = \frac{1}{\kappa^2} (S_K - S_W), \) with

\[
S_K = \int d^D x \ dt \ d^2 \theta \sqrt{G} \left( G^{ik} G^{j\ell} - \lambda G^{ij} G^{k\ell} \right) \bar{D} G_{ij} D G_{k\ell}
\]
\[
S_W = \int d^D x \ dt \ d^2 \theta \sqrt{G} \left( \ldots + \alpha_R R^{(G)} + \alpha_\Lambda \right).
\]
Localization and Hamilton’s Ricci flow

Action in bosonic components:

\[ S_K = - \int d^D x \, dt \sqrt{g} \left( g^{ik} g^{j\ell} - \lambda g^{ij} g^{k\ell} \right) (B_{ij} - \dot{g}_{ij}) B_{k\ell} \]

\[ S_W = \int d^D x \, dt \sqrt{g} B_{ij} \left\{ \ldots + \alpha_R \left( \frac{1}{2} g^{ij} R - R^{ij} \right) + \alpha_\Lambda \frac{1}{2} g^{ij} \right\}. \]

Localization to solutions of \( B_{ij} = 0 \):

\[ \dot{g}_{ij} = (g_{ik} g_{j\ell} - \tilde{\lambda} g_{ij} g_{k\ell}) \frac{\delta W}{\delta g_{k\ell}}. \]

This is Hamilton’s Ricci flow when we set

\[ \alpha_R = 2, \quad \alpha_\Lambda = 0, \quad \tilde{\lambda} = \frac{1}{D - 2}. \]
Gauge Theory I: Spatial Diffeomorphisms

Physicist’s instinct: Symmetries, in particular gauge symmetries.

Gauging spatial diffeomorphisms: The shift vector $n^i$.

Under $\xi^i(t, x^k)$:

$$\delta n^i = \dot{\xi}^i + \xi^k \partial_k n^i - \partial_k \xi^i n^k.$$  

Morally speaking, $n^i$ plays the role of the gauge field for spatial diffeomorphisms in bosonic gravity (relativistic or not).

In the supersymmetric case, $\xi^i$ becomes a superfield,

$$\Xi^i(t, \theta, \bar{\theta}, x^k) = \xi^i + \ldots$$

Type C, A, B: Chiral, antichiral, balanced.
Shift Superfields

In our $\mathcal{N} = 2$ supersymmetric theory, we must introduce several “shift superfields”:

$$N^i = n^i + \ldots,$$

but also $S^i, \bar{S}^i$, to covariantize supertime derivatives,

$$\begin{align*}
\dot{G}_{ij} &\to \nabla_t G_{ij} = \dot{G}_{ij} - N^k \partial_k G_{ij} - \partial_i N^k G_{kj} - \partial_j N^k G_{ik}, \\
D G_{ij} &\to D G_{ij} = D G_{ij} - S^k \partial_k G_{ij} - \partial_i S^k G_{kj} - \partial_j S^k G_{ik}, \\
\bar{D} G_{ij} &\to \bar{D} G_{ij} = \bar{D} G_{ij} - \bar{S}^k \partial_k G_{ij} - \partial_i \bar{S}^k G_{kj} - \partial_j \bar{S}^k G_{ik},
\end{align*}$$

followed by constraints: $$
\begin{align*}
D S^i &= S^k \partial_k S^i, \\
\bar{D} \bar{S}^i &= \bar{S}^k \partial_k \bar{S}^i,
\end{align*}$$

$$N^i = -\bar{D} S^i - DS^i + \bar{S}^k \partial_k S^i + S^k \partial_k \bar{S}^i.$$
Geometric Interpretation I: Flat Connection on Supertime

Turns out that in retrospect, one can interpret these constraints precisely as equivalent to the condition of vanishing curvatures

$$W = 0$$

where the $W$’s are defined as obstructions against the covariant derivatives

$$\nabla_t, \quad D, \quad \bar{D}$$

satisfying the same algebra as the original $\partial_t, D$ and $\bar{D}$:

$$\{D, \bar{D}\} = -\partial_t, \quad \text{zero otherwise.}$$
Geometric Interpretation II: Super Yang-Mills with $G = \text{Diff}(\Sigma)$

Even more surprisingly, the formulation is identical to the supersymmetric Yang-Mills construction, with:

- The role of spacetime played by the supertime $(t, \theta, \bar{\theta})$,
- The role of the internal gauge group played by the infinite-dimensional $\text{Diff}(\Sigma)$, generated by the Lie algebra elements $\xi^i(x^k)$,
- The role of adjoint index $A$ played by the multi-index $(i, x^k)$

Then $N^i(t, \theta, \bar{\theta}, x^k)$ is $A^A_t(t, \theta, \bar{\theta})$, and $S^i = A^A_\theta$, $\bar{S}^i = A^A_{\bar{\theta}}$.

Constraints in superspace:
Exactly the “conventional constraints” of SYM!

Useful for BCJ?
Action

is again given by

\[ S = \frac{1}{\kappa^2} (S_K - S_W), \]

with

\[ S_K = \int d^D x \, dt \, d^2 \theta \sqrt{G} \left( G^{ik} G^{j\ell} - \lambda G^{ij} G^{k\ell} \right) \bar{D} G_{ij} \bar{D} G_{k\ell} \]

\[ S_W = \int d^D x \, dt \, d^2 \theta \sqrt{G} \left( \ldots + \alpha_R R(G) + \alpha_\Lambda \right). \]

Localization:

The LHS of the flow equation replaces \( \dot{g}_{ij} \) with \( \nabla_t g_{ij} \).

Bonus: We can now perform DeTurck’s trick, if needed.
Gauge Theory II: Time Translations

Now we wish to extend the gauge symmetry to full $\text{Diff}(M, \mathcal{F})$, foliation-preserving diffeos.

To gauge time translations in the bosonic theory, one introduces the lapse function $n$:

$$\delta n = f \dot{n} + \dot{f} n.$$ 

The simplest case: $n(t)$, projectable theory.

To supersymmetrize, we promote $f(t)$ to a superfield,

$$F(t, \theta, \bar{\theta}) = f + \theta \varphi + \bar{\theta} \bar{\varphi} + \theta \bar{\theta} \alpha.$$
Lapse Superfields: Projectable

Covariantize the derivatives. First,

\[ \nabla_t G_{ij} \rightarrow D_t G_{ij} \equiv E \nabla_t G_{ij}. \]

More importantly, the superderivatives are covariantized:

\[ \mathcal{D} G_{ij} \rightarrow \mathcal{D}_\theta G_{ij} \equiv \mathcal{E} \mathcal{D} G_{ij} + \Theta \nabla_t G_{ij}, \]
\[ \mathcal{D} G_{ij} \rightarrow \mathcal{D}_{\bar{\theta}} G_{ij} \equiv \mathcal{E} \mathcal{D} G_{ij} + \bar{\Theta} \nabla_t G_{ij}, \]

followed by constraints:

\[ \mathcal{D} \Theta = -\Theta \dot{\Theta}, \quad \mathcal{D} \bar{\Theta} = -\bar{\Theta} \dot{\bar{\Theta}}, \quad \mathcal{E} = \mathcal{E} = 1, \]

and

\[ E = 1 - \bar{D} \Theta - D \bar{\Theta} - \Theta \dot{\Theta} - \bar{\Theta} \dot{\bar{\Theta}}. \]
The Nonprojectable Theory

Importantly, the construction extends naturally to the case where the lapse superfields $E$, $E$, $\bar{E}$, $\Theta$ and $\bar{\Theta}$ are nonprojectable, i.e., functions of not only supertime coordinates $(t, \theta, \bar{\theta})$ but also of $x^i$.

The constraints just become awfully more complicated; for example,

$$E = E\bar{E} - D\Theta + S^k \partial_k \Theta - D\bar{\Theta} + S^k \partial_k \bar{\Theta}$$

$$- \Theta \left( \dot{\Theta} - N^k \partial_k \Theta \right) - \bar{\Theta} \left( \dot{\bar{\Theta}} - N^k \partial_k \Theta \right),$$

... 

Now we are ready to write down the action.
Action

is again given by

\[ S = \frac{1}{\kappa^2} (S_K - S_W), \]

where now

\[ S_K = \int d^D x \, dt \, d^2 \theta \sqrt{G} N \left( G^{ik} G^{j\ell} - \lambda G^{ij} G^{k\ell} \right) \partial_{\theta} G_{ij} \partial_{\theta} G_{k\ell}, \]

\[ S_W = \int d^D x \, dt \, d^2 \theta \sqrt{G} N \left( \ldots + \alpha_R R^{(G)} + \alpha_\Phi G^{ij} \partial_i \Phi \partial_j \Phi + \alpha_\Lambda \right). \]

(Here we have used \( N = 1/E \) and \( \Phi = -\log N \).

Perelman’s \( F \)-functional is our superpotential, for \( \alpha_R = \alpha_\Phi = 2 \) and \( \alpha_\Lambda = 0 \).

Perelman’s “dilaton” is (minus the log of) the lapse function!
Perelman’s Ricci Flow from Topological Quantum Gravity

Localization in our nonprojectable theory:

\[ e^{\phi} \nabla_t g_{ij} = -\alpha_R R_{ij} + \frac{1}{2} \alpha_R \left[ 1 + (2 - D)\tilde{\lambda} \right] g_{ij} R - \alpha_R \nabla_i \partial_j \phi \]
\[ + \alpha_R \left[ 1 + (1 - D)\tilde{\lambda} \right] g_{ij} \Delta \phi + (\alpha_R - \alpha_\Phi) \partial_i \phi \partial_j \phi \]
\[ + \left\{ \frac{1}{2} \alpha_\Phi \left[ 1 + (2 - D)\tilde{\lambda} \right] - \alpha_R \left[ 1 + (1 - D)\tilde{\lambda} \right] \right\} g_{ij} (\partial \phi)^2 \]
\[ + \frac{1}{2} \alpha_\Lambda (1 - \tilde{\lambda} D) g_{ij}. \]

Lots of junk, which does not look like Perelman’s equation.

First, reframe:

\[ e^{\phi} g_{ij} = \tilde{g}_{ij}, \quad \frac{D}{2} \phi = \tilde{\phi} \]
Perelman’s Fixed-Volume Condition

Recall that Perelman holds a volume element fixed,

\[ e^{-\phi} \sqrt{g} \quad \text{measure fixed in time.} \]

In our frame, this simply becomes:

\[ \nabla_t \sqrt{g} = 0! \]

This suggests to take the limit of

\[ \lambda \to \pm \infty, \quad \text{or} \quad \tilde{\lambda} \to \frac{1}{D}. \]

The fixed-volume condition is realized dynamically, not as a gauge-fixing choice!
Perelman’s Equations

Rewrite theory in Perelma’s variables $\tilde{g}_{ij}, \tilde{\phi}$.

Set $\tilde{\alpha}_R = \tilde{\alpha}_\Phi = 2$, $\lambda = \pm \infty$. Then:

$$\tilde{\nabla}_t \tilde{g}_{ij} - \frac{2}{D} \tilde{g}_{ij} \tilde{\nabla}_t \tilde{\phi} = -2 \tilde{R}_{ij} - 2 \nabla_i \partial_j \tilde{\phi} + \frac{2}{D} \tilde{g}_{ij} \tilde{R} + \frac{2}{D} \tilde{g}_{ij} \tilde{\Delta} \tilde{\phi}.$$  

This is just the sum of the two Perelman equations!

$$\left(\tilde{\nabla}_t \tilde{g}_{ij} + 2 \tilde{R}_{ij} + 2 \nabla_i \partial_j \tilde{\phi}\right) - \frac{2}{D} \tilde{g}_{ij} \left(\tilde{\nabla}_t \tilde{\phi} + \tilde{R} + \tilde{\Delta} \tilde{\phi}\right) = 0.$$  

One can match Perelman’s equations exactly, by performing an alternate gauge-fixing which also fixes time diffeomorphisms.
Perelman’s $\mathcal{W}$-Functional

For shrinking Ricci solitons, Perelman introduces an even more useful $\mathcal{W}$-functional:

$$\mathcal{W} = \int d^D x \sqrt{\tilde{g}} e^{-\tilde{\phi}} \left\{ \tau \left( \tilde{R} - \tilde{g}^{ij} \partial_i \tilde{\phi} \partial_j \tilde{\phi} \right) + \tilde{\phi} - D \right\},$$

and fixes the following volume:

$$\frac{1}{(4\pi \tau)^{D/2}} e^{-\tilde{\phi}} \sqrt{\tilde{g}}.$$

We reproduce that by changing our variables to

$$\tilde{g}_{ij} = e^\phi g_{ij}, \quad \tilde{\phi} = \frac{D}{2} [\phi - \log(4\pi \tau)].$$

Similarly for $\mathcal{W}_+^*$-functional for expanding solitons, introduced by Feldman, Ilmanen and Ni.
Generic Flows

Perelman’s frame
Generic Flows

Perelman’s frame

our frame
Outlook

Exciting connection of three previously disconnected areas:
Topological QFT (of the cohomological type), mathematical theory of Ricci flow, nonrelativistic quantum gravity.

Sets the stage for many intriguing questions, both in physics and in math. Partial list:

- observables and correlation functions,
- probes: branes/strings, Perelman’s $L$-volume and $L$-length, . . .
- Hartle-Hawking wavefunction and initial value problem,
- quantum topology change and Ricci flows with surgery,
- short-distance completeness in $D = 3$ at $z = 2$?
- renormalization group properties, perturbative and not,
- dependence on spatial dimension $D$,
- quantum gravity out of equilibrium, theory in real time . . .