Research Article
Some Properties for Solutions of Riemann-Liouville Fractional Differential Systems with a Delay

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In this paper, we study properties for solutions of Riemann-Liouville (R-L) fractional differential systems with a delay. Some results on integral inequalities are first presented by Hölder inequality. Then we investigate properties on solutions for R-L fractional systems with a delay by using the obtained inequalities and obtain upper bound of solutions. Finally, an illustrative example is considered to support our new results.

1. Introduction
Fractional differential equations have been studied for several centuries. At first the researches were only on the pure theoretical aspect. In the last few years, more and more fractional differential equations have been applied to described some actual researches, such as mechanics, aerodynamics, chemistry, and the electrodynamics of complex mediums [1–8].

Integral inequalities play an important role in researches not only on properties of solutions for various differential and integral equations [9–12], but also on some fractional differential equations. Recently, some results are obtained on properties of solutions for a fractional differential equation with or without delays. For example, Ma [13] obtained upper bounds for solutions of a class of nonlinear fractional differential systems by a result of two dimensional linear integral inequalities. Ye [14] studied the Henry-Gronwall type retarded integral inequalities and then obtained a certain properties of fractional differential equations with delay. This paper studies some properties for solutions of R-L fractional differential systems with a delay. First, we obtain some results on the integral inequalities by Hölder inequality. Then, using the obtained inequalities, properties are investigated on solutions for R-L fractional systems with a delay, and upper bound of solutions is obtained. Moreover, an illustrative example is studied to show that new results presented in this paper work very well.

2. Main Results
This section is devoted to studying properties of solutions for R-L fractional differential systems with a delay and presents the main result of this paper. First, we give some lemmas on integral inequalities.

Lemma 1 (let \( I = [t_0, +\infty) \) and \( \mathbb{R}_+ = [0, +\infty) \)). Suppose \( a(t), b(t), and c(t) \in C(I, \mathbb{R}_+), \phi(t) \in C([t_0 - \tau, t_0], \mathbb{R}_+), \)
\( a_i(t_0) = \phi_i(t_0), i = 1, 2, \) and \( \tau > 0 \) is a constant. If \( u_i(t) \in C([t_0 - \tau, +\infty), \mathbb{R}_+) \) and
\[
\begin{align*}
u_1(t) &\le a_1(t) + \int_{t_0}^{t} \left[ b_1(s) u_1(s - \tau) + c_1(s) u_2(s - \tau) \right] ds, \\
u_2(t) &\le a_2(t) + \int_{t_0}^{t} \left[ b_2(s) u_1(s - \tau) + c_2(s) u_2(s - \tau) \right] ds,
\end{align*}
\]
\(u_1(t) \leq \phi_1(t), \quad t \in [t_0 - \tau, t_0),\)

\(u_2(t) \leq \phi_2(t), \quad t \in [t_0 - \tau, t_0),\)  

\[\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \leq A(t) + F(t), \quad t \in I, \]  

(2)

\(F(t) = \exp \left\{ \int_{t_0}^{t} H(s) ds \right\} - 1, \)

(3)

\[F(t) = \exp \left\{ \int_{t_0}^{t} H(s) ds \right\} - 1, \]

(4)

Then we have

\[\exp \left\{ - \int_{t_0 + \tau}^{t} H(s) ds \right\} W(t) \leq \exp \left\{ - \int_{t_0 + \tau}^{t} H(s) ds \right\} W(t) A(t - \tau).\]

Hence,

\[\int_{t_0 + \tau}^{t} \left\{ \exp \left\{ - \int_{t_0 + \tau}^{\xi} H(s) ds \right\} W(\xi) \right\} d\xi \leq \int_{t_0 + \tau}^{t} \left\{ \exp \left\{ - \int_{t_0 + \tau}^{\xi} H(s) ds \right\} H(\xi) A(\xi - \tau) \right\} d\xi.\]

That is,

\[\exp \left\{ - \int_{t_0 + \tau}^{t} H(s) ds \right\} W(t) - W(t_0 + \tau) \leq \int_{t_0 + \tau}^{t} \left\{ \exp \left\{ - \int_{t_0 + \tau}^{\xi} H(s) ds \right\} H(\xi) A(\xi - \tau) \right\} d\xi.\]

Then

\[W(t) \leq \exp \left\{ \int_{t_0 + \tau}^{t} H(s) ds \right\} W(t_0 + \tau) + \int_{t_0 + \tau}^{t} \left\{ \exp \left\{ \int_{t_0 + \tau}^{\xi} H(s) ds \right\} H(\xi) A(\xi - \tau) \right\} d\xi.\]

When \(t \in [t_0, t_0 + \tau],\) from (1),

\[W(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix},\]

(5) and (6) can be rewritten as a matrix form

\[W'(t) \leq H(t) A(t - \tau) + H(t) W(t),\]

(7)

\[W'(t) = H(t) A(t - \tau) + H(t) W(t),\]
\[
\int_{t_0}^t \left[ b_1(s) \phi_1(s - \tau) + c_1(s) \phi_2(s - \tau) \right] ds,
\]
\[
\int_{t_0}^t \left[ b_2(s) \phi_1(s - \tau) + c_2(s) \phi_2(s - \tau) \right] ds
\]
\[= \int_{t_0}^t H(s) \Phi(s - \tau) ds. \tag{12}
\]

Let

\[
F(t) = \left\{ \begin{array}{ll}
\exp \left( \int_{t_0}^t H(s) ds \right) \int_{t_0}^{t + \tau} H(s) \Phi(s - \tau) ds + \int_{t_0}^t \exp \left( \int_{\xi}^t H(s) ds \right) H(\xi) A(\xi - \tau) d\xi, & t \in [t_0 + \tau, +\infty), \\
\int_{t_0}^t H(s) \Phi(s - \tau) ds, & t \in [t_0, t_0 + \tau],
\end{array} \right.
\]

Thus,

\[
\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \leq A(t) + W(t) \leq A(t) + F(t), \quad t \in I. \tag{14}
\]

Therefore, (2) is satisfied and the proof is completed. □

Lemma 2. Suppose \( a_i(t), b_i(t), \) and \( c_i(t) \in C(I, \mathbb{R}_+), \phi_i(t) \in C([t_0 - \tau, t_0], \mathbb{R}_+), a_i(t_0) = \phi_i(t_0), i = 1, 2, \) and \( \tau > 0 \) and \( 0 < \beta < 1 \) are constants. If \( u_i(t) \in C([t_0 - \tau, +\infty), \mathbb{R}_+) \) and

\[
u_1(t) \leq a_1(t) + \int_{t_0}^t (t - s)^{\beta - 1} \left[ b_1(s) u_1(s - \tau) + c_1(s) u_2(s - \tau) \right] ds,
\]

\[t \in [t_0, +\infty),\]

\[
u_2(t) \leq a_2(t) + \int_{t_0}^t (t - s)^{\beta - 1}
\]

\[
\cdot \left[ b_2(s) u_1(s - \tau) + c_2(s) u_2(s - \tau) \right] ds,
\]

\[t \in [t_0, +\infty),\]

then,

(i) when \( 1/2 < \beta < 1, \) set

\[
\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} e^{w_1(t)} \\ e^{w_2(t)} \end{bmatrix}
\]

where

\[
W(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \leq A(t) + Q(t), \tag{17}
\]

and we have

\[
Q(t) = \left\{ \begin{array}{ll}
\exp \left( \int_{t_0}^t H(s) ds \right) \int_{t_0}^{t + \tau} H(s) \Psi(s - \tau) ds + \int_{t_0}^t \exp \left( \int_{\xi}^t H(s) ds \right) H(\xi) A(\xi - \tau) d\xi, & t \in [t_0 + \tau, +\infty), \\
\int_{t_0}^t H(s) \Psi(s - \tau) ds, & t \in [t_0, t_0 + \tau],
\end{array} \right.
\]

\[
A(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix},
\]

\[
H(t) = \begin{bmatrix} l_1(t) & m_1(t) \\ l_2(t) & m_2(t) \end{bmatrix},
\]

\[
\Psi(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} e^{-2t} \phi_1^2(t) \\ e^{-2t} \phi_2^2(t) \end{bmatrix}, \tag{19}
\]
and \( a_i(t) = 2e^{-2t}a_i^2(t) \), \( l_i(t) = (\Gamma(2\beta - 1)/2^{2\beta-3})e^{-2t}b_i^2(t) \),
\( m_i(t) = (\Gamma(2\beta - 1)/2^{2\beta-3})e^{-2t}c_i^2(t) \), \( i = 1, 2 \);

(ii) when \( 0 < \beta \leq 1/2 \), let \( p = 1 + \beta, q = 1 + 1/\beta \)

\[
\begin{bmatrix}
  u_1(t) \\
  u_2(t)
\end{bmatrix} =
\begin{bmatrix}
  e^{-\frac{t}{q}}w_1^{1/2}(t) \\
  e^{-\frac{t}{q}}w_2^{1/2}(t)
\end{bmatrix}
\quad (20)
\]

where

\[
\overline{Q}(t) = \begin{cases}
\exp\left(\int_{t_0+\tau}^{t} \bar{H}(s) \, ds\right) \int_{t_0}^{t_0+\tau} \bar{H}(s) \bar{\varphi}(s) \, ds + \int_{t_0}^{t} \bar{H}(s) \, ds, & t \in [t_0 + \tau, +\infty) \\
\int_{t_0}^{t} \bar{H}(s) \bar{\varphi}(s) \, ds, & t \in [t_0, t_0 + \tau),
\end{cases}
\]

\[
\bar{\varphi}(t) = \begin{bmatrix}
  \bar{\varphi}_1(t) \\
  \bar{\varphi}_2(t)
\end{bmatrix} = \begin{bmatrix}
  e^{-\frac{t}{q}}\phi_1^{1/2}(t) \\
  e^{-\frac{t}{q}}\phi_2^{1/2}(t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \bar{a}_1(t) \\
  \bar{a}_2(t)
\end{bmatrix} = \begin{bmatrix}
  2^{\frac{1}{2}}e^{-\frac{t}{q}}a_1^2(t) \\
  2^{\frac{1}{2}}e^{-\frac{t}{q}}a_2^2(t)
\end{bmatrix},
\]

\[
\begin{bmatrix}
  \bar{b}_1(t) \\
  \bar{b}_2(t)
\end{bmatrix} = \begin{bmatrix}
  2^{\frac{1}{2}}e^{-\frac{t}{q}}b_1^2(t) \\
  2^{\frac{1}{2}}e^{-\frac{t}{q}}b_2^2(t)
\end{bmatrix},
\]

\[
\begin{bmatrix}
  \bar{c}_1(t) \\
  \bar{c}_2(t)
\end{bmatrix} = \begin{bmatrix}
  2^{\frac{1}{2}}e^{-\frac{t}{q}}c_1^2(t) \\
  2^{\frac{1}{2}}e^{-\frac{t}{q}}c_2^2(t)
\end{bmatrix},
\]

\[
\begin{bmatrix}
  \bar{\gamma}_1(t) \\
  \bar{\gamma}_2(t)
\end{bmatrix} = \begin{bmatrix}
  2^{\frac{1}{2}}e^{-\frac{t}{q}}\gamma_1^2(t) \\
  2^{\frac{1}{2}}e^{-\frac{t}{q}}\gamma_2^2(t)
\end{bmatrix},
\]

\[
\begin{bmatrix}
  \bar{\lambda}_1(t) \\
  \bar{\lambda}_2(t)
\end{bmatrix} = \begin{bmatrix}
  2^{\frac{1}{2}}e^{-\frac{t}{q}}\lambda_1^2(t) \\
  2^{\frac{1}{2}}e^{-\frac{t}{q}}\lambda_2^2(t)
\end{bmatrix},
\]

\[
\begin{bmatrix}
  \bar{w}_1(t) \\
  \bar{w}_2(t)
\end{bmatrix} = \begin{bmatrix}
  2^{\frac{1}{2}}e^{-\frac{t}{q}}w_1^2(t) \\
  2^{\frac{1}{2}}e^{-\frac{t}{q}}w_2^2(t)
\end{bmatrix},
\]

\[
\begin{bmatrix}
  \bar{z}_1(t) \\
  \bar{z}_2(t)
\end{bmatrix} = \begin{bmatrix}
  2^{\frac{1}{2}}e^{-\frac{t}{q}}z_1^2(t) \\
  2^{\frac{1}{2}}e^{-\frac{t}{q}}z_2^2(t)
\end{bmatrix},
\]

Proof (i) Suppose \( \beta > 1/2 \). By Hölder inequality and \((a + b)^2 \leq 2(a^2 + b^2), \) for \( t \in [t_0 + \tau, +\infty) \) and \( i = 1, 2 \), we have

\[
\begin{align*}
  u_i(t) & \leq a_i(t) + \int_{t_0}^{t} (t - s)^{\frac{1}{2}} e^{-\frac{1}{q}} [b_i(s) u_i(s - \tau) \\
  + c_i(s) u_2(s - \tau)] \, ds + \int_{t_0}^{t} (t - s)^{\frac{1}{2}} e^{-\frac{1}{q}} [b_i(s) u_i(s - \tau) \\
  + c_i(s) u_2(s - \tau)] \, ds \leq a_i(t) + \int_{t_0}^{t} (t - s)^{\frac{1}{2}} e^{-\frac{1}{q}} [b_i(s) u_i(s - \tau) \\
  + c_i(s) u_2(s - \tau)] \, ds \leq a_i(t)
\end{align*}
\]

\[
+ 2^{1/2} \left( \int_{t_0}^{t} (t - s)^{\frac{1}{2}} e^{2t} \, ds \right)^{1/2}
\]

\[
+ c_i^2(s) e^{-2t} u_2^2(s - \tau) \right) \, ds \right)^{1/2} = a_i(t)
\]

\[
+ 2^{1/2} \left( \int_{t_0}^{t} (t - s)^{\frac{1}{2}} e^{2t} \, ds \right)^{1/2}
\]

Let \( w_i(t) = [e^{-t}u_i(t)]^2, \ a_i(t) = 2e^{-2t}a_i^2(t), l_i(t) = (\Gamma(2\beta - 1)/2^{2\beta-3})e^{-2t}b_i^2(t), m_i(t) = (\Gamma(2\beta - 1)/2^{2\beta-3})e^{-2t}c_i^2(t), \) \( i = 1, 2 \). Then, for \( t \in [t_0, +\infty) \), we have

\[
\begin{align*}
  u_i(t) & \leq a_i(t) + \int_{t_0}^{t} (t - s)^{\frac{1}{2}} e^{-2t} u_2^2(s - \tau) \, ds \\
  + m_i(t) w_2(s - \tau) \, ds.
\end{align*}
\]
Set $\psi_i(t) = e^{-2t} \phi_i^2(t)$ $(i = 1, 2)$. For $t \in [t_0 - \tau, t_0]$,

$$w_i(t) \leq \psi_i(t) \quad (i = 1, 2). \quad (27)$$

By (26), (27), and Lemma 1, (17) is satisfied.

(ii) Suppose $0 < \beta \leq 1/2$. Set $p = \beta + 1$ and $q = 1 + 1/\beta$. Then $1/p + 1/q = 1$. By Hölder inequality and $(a + b)^q \leq 2^{1-q}(a^q + b^q)$ $(0 < l < 1)$, for $t \in [t_0 + \tau, +\infty)$ and $i = 1, 2$, we have

$$u_i(t) \leq a_i(t) + \int_{t_0}^{t} (t - s)^{\beta - 1} e^{(t-s)} \left[ b_i(s) w_i(s - \tau) - c_i(s) u_2(s - \tau) \right] ds \leq a_i(t) + \left[ \int_{t_0}^{t} (t - s)^{\beta - 1} e^{(t-s)} \left[ b_i(s) u_1(s - \tau) - c_i(s) u_2(s - \tau) \right] ds \right]^{1/q} \leq a_i(t) \quad (28)$$

Thus,

$$\left[ e^{-t} u_i(t) \right]^q \leq 2^{\beta - 1} \left[ e^{-t} a_i(t) \right]^q + \frac{2^{\beta - 1} \beta^\gamma}{s} \int_{t_0}^{t} \left[ b_i(s) e^{-s} u_1(s - \tau) \right]^q ds + c_i(s) e^{-s} u_2(s - \tau) ds, \quad i = 1, 2. \quad (29)$$

Let $\bar{w}_i(t) = \left[ e^{-t} u_i(t) \right]^q$, $\bar{a}_i(t) = \frac{2^{\beta - 1} \beta^\gamma}{s} \int_{t_0}^{t} \left[ b_i(s) e^{-s} u_1(s - \tau) \right]^q ds + c_i(s) e^{-s} u_2(s - \tau) ds$, $i = 1, 2$. Then, we have

$$\bar{w}_i(t) \leq \bar{a}_i(t) \quad (30)$$

Set $\bar{\psi}_i(t) = e^{-t} \phi_i^2(t)$ $(i = 1, 2)$. For $t \in [t_0 - \tau, t_0]$,

$$\bar{w}_i(t) \leq \bar{\psi}_i(t) \quad (i = 1, 2). \quad (31)$$

By (30), (31), and Lemma 1, (21) is satisfied and the proof is completed. \qed

Now, we introduce some definitions on Riemann-Liouville fractional derivative and fractional primitive.

**Definition 3** (see [2]). The fractional derivative of order $0 < \alpha < 1$ of a function $x(t) \in C(\mathbb{R}, \mathbb{R})$ is given by

$$D_\alpha^x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (s-t)^{-\alpha} x(s) ds. \quad (32)$$

**Definition 4** (see [2]). The fractional derivative of order $0 < \alpha < 1$ of a function $x(t) \in C(\mathbb{R}, \mathbb{R})$ is given by

$$I_\alpha^x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (s-t)^{\alpha-1} x(s) ds. \quad (33)$$

Consider the following Riemann-Liouville fractional differential system with a delay:

$$D_\alpha^x(t) = f(t, x(t - \tau), y(t - \tau)), \quad \alpha \in (0, 1), \quad \tau \in (0, +\infty), \quad t \in [t_0, +\infty), \quad (34)$$

$$D_\alpha^y(t) = g(t, x(t - \tau), y(t - \tau)), \quad \alpha \in (0, 1), \quad \tau \in (0, +\infty), \quad t \in [t_0, +\infty), \quad (35)$$

where $0 < \alpha < 1$, $\tau > 0$, $\xi, \eta$ are constants, and $f, g : [t_0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions. Then, a result can be obtained on the solutions of system (34).

**Theorem 5.** Consider system (34). $f(t, x, y)$ and $g(t, x, y) \in C([t_0, +\infty) \times \mathbb{R}^2, \mathbb{R})$ and satisfy the following condition:

$$|f(t, x, y)| \leq b_1(t) |x| + c_1(t) |y|, \quad (36)$$

$$|g(t, x, y)| \leq b_2(t) |x| + c_2(t) |y|, \quad (37)$$

where $b(t)$ and $c(t) \in C([t_0, +\infty), \mathbb{R}_+), \mathbb{R}_+)$. Then, solutions of system (34) satisfy that

(i) when $1/2 < \alpha < 1$,

$$\begin{align*}
|x(t)| & \leq A(t) + R(t), \\
|y(t)| & \leq A(t) + R(t),
\end{align*} \quad (38)$$

where
\[ R(t) = \left\{ \begin{array}{ll}
\exp \left\{ \int_{t_{0}}^{t} H(s) \, ds \right\} \int_{t_{0}}^{t} H(s) \Psi(s-\tau) \, ds + \int_{t_{0}+\tau}^{t} \left[ \exp \left\{ \int_{\xi}^{t} H(s) \, ds \right\} H(\xi) A(\xi-\tau) \right] d\xi, & t \in [t_{0} + \tau, +\infty), \\
\int_{t_{0}}^{t} H(s) \Psi(s-\tau) \, ds, & t \in [t_{0}, t_{0} + \tau].
\end{array} \right. \tag{39} \]

\[ A(t) = \begin{bmatrix} a_{1}(t) \\
\alpha_{2}(t) \end{bmatrix}, \]
\[ H(t) = \begin{bmatrix} l_{1}(t) & m_{1}(t) \\
l_{2}(t) & m_{2}(t) \end{bmatrix}, \]
\[ \Psi(t) = \begin{bmatrix} e^{-2a_{1}^{2}(t)} \\
e^{-2a_{2}^{2}(t)} \end{bmatrix}, \] \tag{40}

\[ a_{1}(t) = \frac{\xi}{\Gamma(\alpha)} t^{\alpha-1}, \quad a_{2}(t) = \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1} \quad \text{and} \quad \alpha_{i}(t) = 2e^{-2a_{i}^{2}(t)}, \quad i = 1, 2; \]

\( \text{(ii) when } 0 < \alpha \leq 1/2, \)

\[ \tilde{R}(t) = \left\{ \begin{array}{ll}
\exp \left\{ \int_{t_{0}}^{t} \tilde{H}(s) \, ds \right\} \int_{t_{0}}^{t} \tilde{H}(s) \tilde{\Psi}(s-\tau) \, ds + \int_{t_{0}+\tau}^{t} \left[ \exp \left\{ \int_{\xi}^{t} \tilde{H}(s) \, ds \right\} \tilde{H}(\xi) \tilde{A}(\xi-\tau) \right] d\xi, & t \in [t_{0} + \tau, +\infty), \\
\int_{t_{0}}^{t} \tilde{H}(s) \tilde{\Psi}(s-\tau) \, ds, & t \in [t_{0}, t_{0} + \tau].
\end{array} \right. \tag{43} \]

\[ \tilde{A}(t) = \begin{bmatrix} \tilde{a}_{1}(t) \\
\tilde{a}_{2}(t) \end{bmatrix}, \]
\[ \tilde{H}(t) = \begin{bmatrix} \tilde{l}_{1}(t) & \tilde{m}_{1}(t) \\
\tilde{l}_{2}(t) & \tilde{m}_{2}(t) \end{bmatrix}, \]
\[ \tilde{\Psi}(t) = \begin{bmatrix} \tilde{\psi}_{1}(t) \\
\tilde{\psi}_{2}(t) \end{bmatrix} = \begin{bmatrix} e^{-\gamma\psi_{1}^{3}(t)} \\
e^{-\gamma\psi_{2}^{3}(t)} \end{bmatrix}, \] \tag{44}

\[ p = 1 + \alpha, \quad q = 1 + 1/\alpha \quad \text{and} \quad \bar{a}_{i}(t) = \frac{2^{9-1}[e^{a_{i}}(t)]^{9}}, \]
\[ \tilde{l}_{1}(t) = \frac{4^{1/\alpha^{1/\alpha}}(\alpha^{2})/p^{\alpha}}{e^{-\gamma\psi_{1}^{3}(t)}}, \quad \tilde{m}_{i}(t) = \frac{4^{1/\alpha^{1/\alpha}}(\alpha^{2})/p^{\alpha}}{e^{-\gamma\psi_{2}^{3}(t)}}, \quad i = 1, 2. \]

**Proof.** Fractional differential system (34) with a delay can be converted to the following integral equation:

\[ x(t) = \frac{\xi}{\Gamma(\alpha)} t^{\alpha-1} \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_{0}}^{t} (t-s)^{\alpha-1} f(s,x(s-\tau),y(s-\tau)) \, ds, \quad t \in [t_{0}, +\infty), \]

\[ y(t) = \frac{\eta}{\Gamma(\alpha)^{\alpha-1}} \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_{0}}^{t} (t-s)^{\alpha-1} g(s,x(s-\tau),y(s-\tau)) \, ds, \quad t \in [t_{0}, +\infty), \]

\[ x(t) = \frac{\xi}{\Gamma(\alpha)^{\alpha-1}}, \]
\[ y(t) = \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1}, \quad t \in [t_0 - \tau, t_0]. \tag{45} \]

By condition (36), we have

\[ |x(t)| \leq \frac{|\xi|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \cdot |b_1(s)| |x(s-\tau)| + c_1(s) |y(s-\tau)| \, ds, \]

\[ |y(t)| \leq \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \cdot |b_2(s)| |x(s-\tau)| + c_2(s) |y(s-\tau)| \, ds, \tag{46} \]

\[ t \in [t_0, +\infty), \]

\[ |x(t)| = \frac{|\xi|}{\Gamma(\alpha)} t^{\alpha-1}, \]

\[ |y(t)| = \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1}, \]

\[ t \in [t_0 - \tau, t_0]. \]

Set \( a_1(t) = (|\xi|/\Gamma(\alpha)) t^{\alpha-1} \) and \( a_2(t) = (|\eta|/\Gamma(\alpha)) t^{\alpha-1} \). Then, according to Lemma 2, the result is obtained and the proof is completed. \( \square \)

### 3. An Illustrative Example

In this section, we give an illustrative example to show effectiveness of results obtained in this paper.

**Example 1.** Consider the following fractional differential equation:

\[ D^{3/4} x(t) = f(t, x(t-1), y(t-1)), \]

\[ D^{3/4} y(t) = g(t, x(t-1), y(t-1)), \quad t \in [1, +\infty), \tag{47} \]

\[ D^{-1/4} x(t) = D^{-1/4} y(t) = \Gamma \left( \frac{3}{4} \right), \quad t \in [0, 1], \]

where \( f(t, x, y) = g(t, x, y) = (e/\sqrt{1/2})(t-1)^{1/4}[x + y], \quad t \in [1, +\infty). \)

It is obvious that \( |f(t, x, y)| = |g(t, x, y)| \leq (e/\sqrt{1/2})(t-1)^{1/4}|x| + |y|, \quad t \in [1, +\infty). \) From (47) and Theorem 5, we obtain \( a_1(t) = a_2(t) = t^{-1/4}, \alpha_1(t) = \alpha_2(t) = 2t^{-1/2}e^{-2t}, \ell(t) = m(t) = 2\sqrt{t-1}, \quad t = 1, 2. \) Thus, \( A(t) = 2t^{-1/2}e^{-2t} \left[ \begin{array}{c} 1 \\ 1 \end{array} \right], H(t) = 2\sqrt{t-1} \left[ \begin{array}{c} 1 \\ 1 \end{array} \right], \Psi(t) = t^{-1/2}e^{-2t} \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]. \)

Set

\[ \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{\ell/2} w_1(t) \\ e^{\ell/2} w_2(t) \end{bmatrix}. \tag{48} \]

Therefore,

\[ \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \leq A(t) + R(t), \tag{49} \]

where for \( t \in [1, 2], \)

\[ R(t) = \int_{t_1}^{t} H(s) \Psi(s-1) \, ds = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2\ell(s-1)} \, ds \]

\[ = \left( 1 - e^{-2(t-1)} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{50} \]

and for \( t \in [2, +\infty), \)

\[ R(t) = \exp \left( \int_{t_2}^{t} H(s) \, ds \right) \left( \int_{t_2}^{t} e^{2\ell(s-1)} \, ds \right) \]

\[ + \int_{t_2}^{t} \exp \left( \int_{\xi}^{t} H(s) \, ds \right) H(\xi) A(\xi - 1) \, d\xi \leq 2 \]

\[ \exp \left( \frac{4}{3} (t-1)^{3/2} \right) \left[ 2 \left( 1 - e^{2} \right) \right. \]

\[ + 16e^{(4/3)(t-1)^{3/2}} \left. \int_{t_2}^{t} e^{-2\ell} \, d\xi \right] \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] \leq 4 \left[ e^{(4/3)(t-1)^{3/2}} \right. \]

\[ + 4e^{(8/3)(t-1)^{3/2}-4} \left. \right] \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]. \tag{51} \]

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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