Comparison between theoretical four-loop predictions and Monte Carlo calculations in the two-dimensional $N$-vector model for $N = 3, 4, 8$

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We have computed the four-loop contribution to the beta-function and to the anomalous dimension of the $N$-vector model. This allows the determination of the second perturbative correction to various long-distance quantities like the correlation lengths and the susceptibilities. We compare these predictions with new Monte Carlo data for $N = 3, 4, 8$. From these data we also extract the values of various universal nonperturbative constants, which we compare with the predictions of the $1/N$ expansion.

The $N$-vector model describes configurations of classical spins taking values on the unit sphere $S^{N-1} \subset \mathbb{R}^N$. We consider here the standard nearest-neighbor action

$$\mathcal{H}(\sigma) = -\beta \sum_{\{xy\}} \sigma_x \cdot \sigma_y .$$

(1)

The perturbative renormalization group predicts that when $\beta \not\to \infty$, the (infinite-volume) long-distance quantities of lattice theory should give the same results as the continuum theory in the $\overline{\text{MS}}$ normalization, provided that one rescales lengths by the factor

$$\Lambda = e^{-2\pi\beta/(N-2)} \left( \frac{2\pi\beta}{N-2} \right)^{\frac{N-2}{2}} 2^{5/2} \pi^{(N-2)/2} .$$

(2)

For the isovector and isotensor two-point functions

$$G_V(x, y) = \langle \sigma_x \cdot \sigma_y \rangle$$

(3)

$$G_T(x, y) = \langle (\sigma_x \cdot \sigma_y)^2 \rangle - \frac{1}{N}$$

(4)

we shall consider the RG predictions for the correlation lengths

$$\xi_\#(\beta) = \tilde{C}_\# \Lambda^{-1} \left[ 1 + \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \right]$$

(5)

and the susceptibilities

$$\chi_V(\beta) = C_{\chi_V} \Lambda^{-2} \left( \frac{2\pi\beta}{N-2} \right)^{-\frac{N-2}{2}} \left[ 1 + \sum_{i=1}^{\infty} \frac{b_i}{\beta^i} \right]$$

$$\chi_T(\beta) = C_{\chi_T} \Lambda^{-2} \left( \frac{2\pi\beta}{N-2} \right)^{-\frac{N-2}{2}} \left[ 1 + \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \right]$$

(6)

Here the constants $\tilde{C}_\#$ are universal but nonperturbative, while the coefficients $a_i$, $b_i$ and $d_i$ can be determined at the $(i+2)$-th order of the perturbative expansion. There is a prediction [1] for the exponential correlation length (= inverse mass gap) in the isovector channel:

$$\tilde{C}_\#(\nu_{\nu}) = \left( \frac{e}{8} \right)^{1/(N-2)} \Gamma \left( 1 + \frac{1}{N-2} \right) .$$

(7)

As the model should not have bound states, in the isotensor channel one expects

$$\tilde{C}_\#(\nu_{\nu}) = \frac{1}{2} C_{\chi_T(\nu_{\nu})} .$$

(8)

The constants related to the second-moment correlation lengths and the susceptibilities are known analytically only in the large-$N$ expansion.

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namely, through order $1/N$ both in the isovector [2] and isotensor [3] sectors. In particular,

$$\frac{\tilde{C}_{(2)\text{corr}}}{C_{(2)\text{corr}}} = 1 - \frac{0.003225}{N} + O(1/N^2),$$

(10)

so that even for $N = 3$ this ratio differs only marginally from 1 (in good agreement with Monte Carlo simulations [4]).

We can also try an improved expansion parameter [5,6] based on the isovector energy $E_V = \langle \sigma_{\#,0} \cdot \sigma_{1,0} \rangle$; we define

$$\beta_{\text{eff}} \equiv \frac{N - 1}{4(1 - E_V)} = \beta + O(1).$$

(11)

This $\beta_{\text{eff}}$ has the property that at $N \to \infty$ with $\beta \equiv \beta/N$ fixed, there are only exponentially small corrections to the two-loop predictions for correlation lengths and susceptibilities.

A precise confirmation from Monte Carlo simulations of (5) with the correct nonperturbative constant (8)/(10) would be good evidence in favor of the conventional asymptotic-freedom picture [7], which has been criticized [8]. At the last Lattice conference [9] we presented data for the case $N = 3$ at infinite-volume correlation lengths $\xi_\infty$ up to $\approx 10^5$ [10], obtained by using finite-size-scaling extrapolation at fixed $\beta$ [11]. The discrepancy of these data from the three-loop predictions [12,6] was already quite small ($\sim 4\%$), and in the “improved expansion parameter” the agreement was even better. For this reason we considered it worthwhile to compute the next-order perturbative correction, in order to see whether the remaining discrepancy — which is nevertheless larger than the estimated statistical error — can be removed. We therefore computed the four-loop contributions to the beta-function and the anomalous dimension of the field for the lattice model (1); from these we derived the $1/\beta^2$ corrections to the correlation lengths and the general spin-$n$ susceptibility [13]. For example, we have

$$a_2 = \frac{0.0688 - 0.0286 N + 0.007 N^2 - 0.0129 N^3}{(N - 2)^2}.$$

(12)

In Figure 1 we plot $\xi^{(2)}_{\text{estimates}}/\xi^{(2)}_{\text{theor}}$ versus $\beta$ for the $O(3)$ model. Error bars are one standard deviation (statistical error only). There are five versions of $\xi^{(2)}_{\text{theor}}$: standard perturbation theory in $1/\beta$ gives points $\circ$ (2-loop), $\times$ (3-loop) and $\otimes$ (4-loop); “improved” perturbation theory in $1/\beta_{\text{eff}}$ gives points $\square$ (2-loop) and $\bigcirc$ (3-loop).

Figure 1. $\xi^{(2)}_{\text{estimates}}/\xi^{(2)}_{\text{theor}}$ versus $\beta$ for the $O(3)$ model. Error bars are one standard deviation (statistical error only). There are five versions of $\xi^{(2)}_{\text{theor}}$: standard perturbation theory in $1/\beta$ gives points $\circ$ (2-loop), $\times$ (3-loop) and $\otimes$ (4-loop); “improved” perturbation theory in $1/\beta_{\text{eff}}$ gives points $\square$ (2-loop) and $\bigcirc$ (3-loop).

(points $\circ$, $\times$ and $\otimes$) or by the “improved” two-loop and three-loop predictions (points $\square$ and $\bigcirc$). The four-loop truncation of (5) is now fully compatible with our last extrapolated point. It falls roughly halfway between the three-loop truncations in $1/\beta$ and $1/\beta_{\text{eff}}$.

A similar result is obtained for $N = 4$, where the central estimate of our last point (which is at the rather small correlation length $\xi_\infty \approx 155$) differs from the theoretical four-loop prediction by only $4-5\%$; and for $N = 8$ [14], where the central estimate of our last point (which is at $\xi_\infty \approx 650$) shows an extremely good agreement (better than 1%).

We have also tried to take into account higher-loop corrections by using information from the $1/N$-expansion. Let

$$a_n^{(1/N)}(N) = N^{n-1} \bar{a}_n,$$

(13)

be the leading contribution to $a_n$ in the limit $N \to \infty$. We computed $\bar{a}_n$ up to $n = 8$ [13]. How good is the approximation in which only such a term is retained? We can compare with the known
coefficients $a_1$ and $a_2$. For $N = 4$, we have

$$\frac{a_1(4)}{a_1^{(1/N)}(4)} = 3.73; \quad \frac{a_2(4)}{a_2^{(1/N)}(4)} = 2.88;$$

while for $N = 8$, we have

$$\frac{a_1(8)}{a_1^{(1/N)}(8)} = 1.91; \quad \frac{a_2(8)}{a_2^{(1/N)}(8)} = 1.58.$$

The convergence seems slow; indeed, only at $N \gtrsim 50$ are the first two coefficients correct to within 10%. Let us now define $k_n$ by

$$a_n(8) = k_n 8^{-n-1} a_n.$$ 

Already at $\beta = 5.80$ ($\xi \approx 33$) the Monte Carlo value \[14,15\] is in good agreement with the theoretical predictions. Indeed we get

$$\frac{\xi_{MC}/\xi_{th}}{\xi_{8-loop}} = \begin{cases} 0.998 & \text{when } k_n = 1 \forall n \\ 1.001 & \text{when } k_n = 2 \forall n \end{cases}$$

with a statistical error of $\pm 0.002$. For larger values of $\beta$ this ratio remains roughly constant, although the error bars grow. We can thus claim a nice control of (5) for $N = 8$.

Having verified (5), we can now extract from the Monte Carlo data a numerical evaluation of the nonperturbative universal constants $C$. For the limiting ratio $\xi^{(2)}/\xi^{(1)}$ we have the numerical results 3.51 for $N = 3$, 3.14 for $N = 4$, and 2.77 for $N = 8$, with error bars less than $\pm 0.01$, which can be compared with the $1/N$-expansion prediction \[3\]

$$\frac{C_{\xi^{(2)}}}{C_{\xi^{(1)}}} = \sqrt{6} \left[ 1 + \frac{1.1999}{N} + \cdots \right]$$

$$\approx \begin{cases} 3.43 & \text{for } N = 3 \\ 3.18 & \text{for } N = 4 \\ 2.82 & \text{for } N = 8 \end{cases}$$

To extrapolate the asymptotic value of the isovector susceptibility, we found it convenient to study the dimensionless ratio

$$\frac{\chi_V}{(\xi_V^{(2)})^2} = \frac{C_{\chi_V}}{C_{\xi_V^{(2)}}} \left( \frac{2 \pi \beta}{N - 2} \right)^{\frac{8}{15}} \left[ 1 + \sum_{i=1}^{\infty} \frac{c_i}{\beta^i} \right]$$

because our knowledge of the lattice beta and gamma functions at four loops allows the determination of the first three coefficients $c_i$ \[6,13\]. We

Figure 2. Estimate of $\tilde{C}_{\chi_V}$ [from (20)/(8)/(10)] versus $\beta$ for the $O(8)$ model. Error bars (one standard deviation, statistical error only) shown for clarity only on one set of points. There are seven versions of $\chi_V/(\xi_V^{(2)})^2$: standard perturbation theory in $1/\beta$ gives points + (leading), × (with $c_1$), ♦ (c_{1,2}) and ◊ (c_{1,2,3}); “improved” perturbation theory in $1/\beta_{eff}$ gives points □ (leading), ◯ (c_1') and □ (c_1').

then use the exact formula (8)/(10) to estimate $\tilde{C}_{\chi_V}$; see Figure 2 for $N = 8$. We find

$$\tilde{C}_{\chi_V} = \begin{cases} 10.8 \pm 0.8 & \text{for } N = 3 \\ 5.9 \pm 0.1 & \text{for } N = 4 \\ 5.6 \pm 0.1 & \text{for } N = 8 \end{cases}$$

which can be compared with the $1/N$-expression \[2\]

$$\tilde{C}_{\chi_V} = 2\pi \left[ 1 + \frac{4 + 3\gamma_C - 3\gamma_E - 7\log 2}{N} \right]$$

$$\approx \begin{cases} 3.67 & \text{for } N = 3 \\ 4.32 & \text{for } N = 4 \\ 5.30 & \text{for } N = 8 \end{cases}$$

where $\gamma_E$ is Euler’s constant and

$$\gamma_C = \log \frac{\Gamma(1/3)\Gamma(7/6)}{\Gamma(2/3)\Gamma(5/6)}.$$  

Clearly the $O(1/N^2)$ corrections are significant! We have made the wild guess that the exact ex-
pression for $\tilde{C}_{\chi V}$ is

\[
\tilde{C}_{\chi V} = 2\pi \left( \frac{e^{4+3\gamma_e}}{128} \right)^{\frac{1}{N-2}} \Gamma^3 \left( 1 + \frac{1}{N - 2} \right),
\]

in analogy with (8). In Figure 3 we compare our Monte Carlo results with the $1/N$-prediction (22) and the wild guess (25). The result for $N = 3$ is close to the guessed formula but outside the statistical errors, while the values for $N = 4, 8$ follow the Ansatz nicely. It would be interesting to test (25) by computing the $O(1/N^2)$ term in (22).

The agreement with the $1/N$-expansion is much poorer for the isotensor susceptibility: our Monte Carlo data yield

\[
\tilde{C}_{\chi T} = \begin{cases} 
1200 \pm 100 & \text{for } N = 3 \\
23 \pm 2 & \text{for } N = 4 \\
4.7 \pm 0.2 & \text{for } N = 8 
\end{cases}
\]

compared to the $1/N$-expansion result [3]

\[
\tilde{C}_{\chi T} = \pi \left[ 1 - \frac{0.0296}{N} + \ldots \right].
\]

Clearly the $O(1/N^2)$ and higher corrections must have a drastic effect for $N \lesssim 20$.

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