Implications of the hidden charm pentaquarks $P_c(X)$ at the LHCb

Ruolin Zhu\textsuperscript{1} *, Xuejie Liu\textsuperscript{1}, Hongxia Huang\textsuperscript{1}, Cong-Feng Qiao\textsuperscript{2,3} †

\textsuperscript{1}Department of Physics and Institute of Theoretical Physics, Nanjing Normal University, Nanjing, Jiangsu 210023, China
\textsuperscript{2}School of Physics, University of Chinese Academy of Sciences, YuQuan Road 19A, Beijing 100049, China
\textsuperscript{3}CAS Center for Excellence in Particle Physics, Beijing 100049, China

Motivated by the very recent observations of hidden charm pentaquarks $P_c(4312)^+$, $P_c(4440)^+$ and $P_c(4457)^+$ of the LHCb Collaboration, we study the spectra of the hidden charm pentaquarks using the non-relativistic constituent quark model. The variational method is employed to solve the Schrödinger equation, where the test functions adopted are symmetric for the light quarks. Based on this model independent approach we investigate the doubly-heavy pentaquark, hidden heavy flavor and doubly-heavy tetraquarks. In our study, the $P_c(4312)^+$ may be assigned as the ground state with spin-parity $\frac{1}{2}^-$ or $\frac{3}{2}^-$, while the $P_c(4440)^+$ and $P_c(4457)^+$ may be assigned as the excited states with $\frac{1}{2}^-$ and $\frac{3}{2}^-$. It is notable that our working framework is quite similar to that of Hydrogen molecule, but with different potential structure. Some optimistic channels for the observation of doubly-heavy pentaquarks and doubly-heavy tetraquarks are proposed.

I. INTRODUCTION

The discovery of exotic states greatly enriches the hadron family and our knowledge of the nature of QCD. Up to date, more than thirty exotic states or candidates, denoted as XYZ states, have been observed in experiment. To understand the properties of those exotic states and find more possible states are urgent tasks in hadron physics. However, it seems the journey of exotic baryon study has just begun.

Very recently, the LHCb Collaboration has reported the observations of hidden charm pentaquarks $P_c(4312)^+$, $P_c(4440)^+$ and $P_c(4457)^+$ by the $P_c(X)^+ \to J/\psi + p^+$ in $\Lambda_b \to J/\psi + p^+ K^-$ decays \cite{1, 2}. The data are consistent with the 2015 results which led to two pentaquarks wide $P_c(4380)^+$ and narrow $P_c(4450)^+$ \cite{3}. New data indicated that the previous $P_c(4450)^+$ shape actually belong to two states $P_c(4440)^+$ and $P_c(4457)^+$. The $P_c(4312)^+$ state is a new hidden charm pentaquarks. Their masses and decay widths are \cite{2}:

\[ M_{P_c(4312)^+} = 4311.9 \pm 0.7^{+6.8}_{-0.6} \text{MeV}, \]
\[ \Gamma_{P_c(4312)^+} = 9.8 \pm 2.7^{+3.7}_{-4.5} \text{MeV}; \]
\[ M_{P_c(4440)^+} = 4440.3 \pm 1.3^{+4.1}_{-4.7} \text{MeV}, \]
\[ \Gamma_{P_c(4440)^+} = 20.6 \pm 4.9^{+8.7}_{-10.1} \text{MeV}; \]
\[ M_{P_c(4457)^+} = 4457.3 \pm 0.6^{+4.1}_{-1.7} \text{MeV}, \]
\[ \Gamma_{P_c(4457)^+} = 6.4 \pm 2.0^{+5.7}_{-1.9} \text{MeV}. \]

There are some theoretical interpretations for these new hidden charm pentaquarks \cite{4–17}. In Ref. 4, the three narrow structures $P_c(4312)^+$, $P_c(4440)^+$ and $P_c(4457)^+$ were identified as the molecular $\Sigma_c^{*}\bar{D}$ with (I=1/2,J=1/2), $\Sigma_c^{*}\bar{D}$ with (I=1/2,J=1/2) and $\Sigma_c^{*}\bar{D}$ with (I=1/2,J=3/2), respectively. In Ref. 5, $P_c(4457)^+$ may be identified as the molecular $\Sigma_c^{*}\bar{D}$ with (I=1/2,J=5/2). In Ref. 6–8, both $J^P = \frac{1}{2}^-$ and $J^P = \frac{3}{2}^-$ are compatible when treating the $P_c(4440)^+$ and $P_c(4457)^+$ as the $\Sigma_c^{*}\bar{D}$ bound state.

For a hidden charm pentaquark system with four quarks and one anti-quark, there are different physical pictures in the literature: meson-baryon molecular model where the energy spectrum had been calculated by a chiral quark model \cite{18}, the coupled channel unitary approach \cite{19, 20}, chiral effective Lagrangian approach \cite{21, 22}, QCD sum rules \cite{23}, the color-screen potential model \cite{24}, and the scattering amplitudes approach \cite{25}. diquark-diquark-antiquark model \cite{26, 30}; compact diquark-triquark model \cite{31, 32}. In these models, the five-body interactions are reduced into quasi two-body or three-body interactions when assuming quarks form clusters. In fact, there is no a priori justification for these clusters in a five quark system. All of them have the possibility before a precise test has been done at experiment.

In this paper, we study the spectra of hidden heavy flavor\textsuperscript{1} pentaquarks, doubly heavy flavor pentaquarks, hidden heavy flavor tetraquarks and doubly heavy flavor tetraquarks in non-relativistic constituent quark model. There is no strict solution for these multi-body

\textsuperscript{1} Throughout the paper, for the heavy flavor we focus on the charm and beauty quarks since the top quark usually decays before forming the bound state; for the light quark we focus on the up and down quarks which hold the Isospin symmetry.
Schrödinger equations. Considering the mass of two heavy flavor quarks are much larger than the light quarks, it is naturally to assume that the light quarks orbit around the two heavy flavor quarks. This picture is very similar to the case of Hydrogen molecule where two electrons orbit around the two protons.

The paper is organized as followed. In Sec. II we give the multi-body Schrödinger equations. In the variational method, we use test functions and obtain the optimal values for the parameters. In Sec. III we give the spectra of hidden heavy flavor pentaquarks, doubly heavy flavor pentaquarks, hidden heavy flavor tetraquarks and doubly heavy flavor tetraquarks. In the end, we summarize and conclude.

II. FORMULAE

A. Five quarks system

In non-relativistic constituent quark model, the Hamiltonian operator for the hidden charm pentaquark \((c\bar{c}q'q'q'')\) or the doubly charm pentaquark \((ccqqq)\) can be written as (natural units with \(\hbar = 1, \ c = 1\))

\[
\hat{H} = \sum_{i=1}^{5} \left( m_i - \frac{\nabla_i^2}{2m_i} \right) + \sum_{j>i=1}^{5} \frac{\lambda_i^c \cdot \lambda_j^c}{4r_{ij}} + \sum_{j>i=1}^{5} \lambda_i^c \cdot \lambda_j^c (b_1 r_{ij} - b_0) + V_S(r_{ij}) + V_L(r_{ij}) \tag{1}
\]

where \(m_i\) and \(r_i\) denote the mass and the position vector for each quark; \(\lambda_i\) is the Gell-mann matrix for SU(3) color group while \(r_{ij} = |r_i - r_j|\) is the distance between two quarks; \(\nabla_i^2\) is the Laplace operator while \(\alpha_s\) is Strong coupling constant. In this Hamiltonian, the first term is the mass and kinetic term; the second is the color Coulomb term; the third is the color linear confining term but with the unknown coefficients \(b_i\); the fourth and the fifth are the spin-dependent and orbital excited terms as in Refs. [32][36]. Therein, the spin-dependent term can be written as

\[
V_S(r_{ij}) = -\frac{3}{8} \sum_{j>i} \left[ \frac{C_{ij}}{m_i m_j} \right] \lambda_i \cdot \lambda_j s_i \cdot s_j v_i \delta(r_{ij}) \tag{2}
\]

where \(s_i = \sigma_i / 2\) is the quark spin operator with the Pauli matrix \(\sigma_i\).

The Schrödinger equation for the multi-body quark states is

\[
\hat{H}\Psi(r_1) = E\Psi(r_1) \tag{3}
\]

The exact solution for this multi-body Schrödinger equation is not clear. Considering the spin-dependent term is usually treated as the source of the hyperfine splitting, it plays less influence to calculate the ground state. The orbital term is also related to the excited states. The information of the color linear confinement term is lack and the coefficients \(b_i\) have large degree of freedom. For the ground states, we major consider the first two terms to calculate the binding energy under the variational method. Then considering the spin-dependent and orbital dependent terms, we will get the total spectra.

Considering the mass of the heavy flavor quark is much larger than that of the light quarks, we will use Born-Oppenheimer approximation for simplification. The total energy for the five quark system will separate into the two heavy flavor quark’s masses, the three light quark masses, the kinetic and potential energies of the three light quarks, and the spin dependent and orbital excited terms. This separation is hold at the leading order of the relativistic velocity of the two heavy flavor quarks. At the leading order of the heavy flavor relativistic velocity, the two heavy flavor quarks are rest. One can assume the distance between two heavy flavor quarks is \(R\), which is a free parameter.

Therein the Hamiltonian operator for the kinetic and potential energies of the three light quarks is

\[
\hat{H}_q = -\frac{1}{2m_q} (\nabla_1^2 + \nabla_2^2 + \nabla_3^2)
+ \sum_{i=1}^{3} \frac{\alpha_s}{4} \lambda_i^c \cdot \lambda_i^c \left( \frac{1}{r_{i1}} + \frac{4}{\alpha_s} b_1 r_{i1} + b_0 \right)
+ \sum_{j>i=1}^{3} \frac{\alpha_s}{4} \lambda_i^c \cdot \lambda_j^c \left( \frac{1}{r_{ij}} + \frac{4}{\alpha_s} b_1 r_{ij} + b_0 \right)
+ \frac{\alpha_s}{4} \lambda_i^c \cdot \lambda_j^c \left( \frac{1}{R} + \frac{4}{\alpha_s} b_1 R + b_0 \right) + V_S(r_{ij}) \tag{4}
\]

For the pentaquark \((c\bar{c}qq'q'')\) where three light quarks orbit around the two rest charm quarks, the wave function can be separated into two part

\[
\Psi(c\bar{c}qq'q'') = \chi_\lambda(\lambda_a, \lambda_b) \otimes \chi_s(s_a, s_b) \otimes \Psi_q(q, q', q'') \tag{5}
\]

where the wave function for the three light quarks is always written into four parts: space-coordinate, flavor, color, and spin subspaces,

\[
\Psi_q(q, q', q'') = R(r_1, r_2, r_3) \otimes \chi_f(f_1, f_2, f_3)
\otimes \chi_\lambda(\lambda_1, \lambda_2, \lambda_3) \otimes \chi_s(s_1, s_2, s_3) \tag{6}
\]

where \(R(r_i), \chi_f(f_i), \chi_\lambda(\lambda_i),\) and \(\chi_s(s_i)\) are the radial, flavor, color, and spin wave functions, respectively.
the Isospin symmetry for the light quarks, \( \chi_f(f_1, f_2, f_3) \) is symmetrical for exchanging two light quarks.

For the pentaquark \((c\bar{c}qq'q'')\), the color structure of \(c\bar{c}\) is \(1_c\) or \(8_c\) representation, while the color structure of \(qq'q''\) is also \(1_c\) or \(8_c\) representation since \(3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 10\). Due to the color singlet for the hidden charm pentaquark, the color structure of \(qq'q''\) can not be \(10\) representation.

For the pentaquark \((c\bar{c}q'q'')\), the color structure of \(cc\) is \(3_c\) or \(6_c\) representation, while the color structure of \(qq'q''\) is also \(3_c\) or \(6_c\) representation since \(3 \otimes 3 \otimes 3 = 3 \oplus 6 \oplus 15\). Due to the color singlet for the doubly charm pentaquark, the color structure of \(qq'q''\) can not be \(15\) representation.

The color bases for the pentaquark \((c\bar{c}qq'q'')\) become \(1^{0+}1^{0+}qq'q''\) and \(8^{0+}8^{0+}qq'q''\). It is not unique for the color bases of the pentaquark \((c\bar{c}qq'q'')\). Other bases such as \(3^{0+}3^{0+}qq'q''\) and \(6^{0+}6^{0+}qq'q''\) are also feasible.

The color operators can be written as

\[
\lambda_i \cdot \lambda_j = \frac{1}{2} (\lambda_i^2 - \lambda_j^2 - 2 \lambda_i \lambda_j),
\]

where \(\lambda_i^2\) is the quadratic Casimir operator, which hold the relation

\[
\lambda_i^2 \chi(\lambda \mu) = \frac{4}{3} (\lambda^2 + \mu^2 + \lambda \mu + 3 \lambda + 3 \mu) \chi(\lambda \mu).
\]

For the color singlet \(1\), \(\chi(\lambda \mu) = \chi(00)\), while the color triplet \(3\), \(\chi(\lambda \mu) = \chi(10)\).

The strict solution for the five-body Schrödinger equation is not clear. In the following, we will adopt the variational method with a test function. The radial wave function of the three light quarks in the ground state of pentaquarks can be assumed as

\[
R(r_1, r_2, r_3) = C_s(R_a(r_1)R_b(r_2)R_c(r_3) + R_b(r_1)R_c(r_2)R_a(r_3)),
\]

where the test function is chosen as \(R_i(r_i) = \beta^3 \exp(-\beta r_i)\) and \(\beta\) is the free parameter. The normalization constant \(C_s\) is expressed as

\[
C_s = \sqrt{\frac{1}{3(1 + 2D^2)}},
\]

wherein the overlap integral \(D\) is defined as

\[
D = \int_0^{2\pi} \int_0^1 \int_0^{\infty} R_a(r) R_b(r) r^2 dr d\cos \theta d\varphi = (1 + R \beta + \frac{1}{3} R^2 \beta^2) \exp(-R \beta).
\]

**B. Four quarks system**

Similarly, the total energy for the four quark system will separate into the two heavy flavor quark’s masses, the two light quark masses, the kinetic and potential energies of the two light quarks, and the spin dependent and orbital excited terms. The Hamiltonian operator for the kinetic and potential energies of the two light quarks
is
\[\hat{H}_q' = -\frac{1}{2m_q}(\nabla_1^2 + \nabla_2^2)\]
\[+ \sum_i \frac{\alpha_s}{4} \lambda_i^c \cdot \lambda_5^c (\frac{1}{r_{1a}} + \frac{4}{\alpha_s} (b_1 r_{1a} + b_0))\]
\[+ \sum_i \frac{\alpha_s}{4} \lambda_i^c \cdot \lambda_6^c (\frac{1}{r_{ib}} + \frac{4}{\alpha_s} (b_1 r_{ib} + b_0))\]
\[+ \frac{\alpha_s}{4} \lambda_5^c \cdot \lambda_6^c (\frac{1}{r_{12}} + \frac{4}{\alpha_s} (b_1 r_{12} + b_0)) + V_S(r_{12}) (12)\]

For the tetraquark \((\bar{c}cqq')\) where two light quarks orbit around the two rest charm quarks, the wave function can be separated into two parts
\[\Psi(\bar{c}cqq') = \chi_\Lambda(\lambda_a, \lambda_b) \otimes \chi_s(s_a, s_b) \otimes \Psi_q(q, q'),\]  
(13)
where
\[\Psi_q(q, q') = R(r_1, r_2) \otimes \chi_f(f_1, f_2)\]
\[\otimes \chi_s(s_1, s_2),\]  
(14)
Due to the Isospin symmetry for the light quarks, \(\chi_f(f_1, f_2)\) is also symmetrical.

For the Tetraquark \((\bar{c}cqq')\), the color structure of \(\bar{c}c\) is \(3_c\) or \(6_c\) representation, while the color structure of \(qq'\) is \(3_s\) or \(6_s\) representation. Considering the anti-symmetrical properties for identical fermions, the spin quantum number of \(\bar{c}c\) is 0 for color triplet and 1 for color anti-sexet, while the spin quantum number of \(qq'\) is 0 for color anti-triplet and 1 for color sexet. For the Tetraquark \((\bar{c}cqq')\), the color structure is \(1_c\) or \(8_c\) representation for both \(\bar{c}c\) and \(qq'\).

The strict solution for the four-body Schrödinger equation is also not clear. The radial wave function of the two light quarks in the ground state of pentaquarks can be assumed as
\[R(r_1, r_2) = C'_s(R_u(r_1)R_b(r_2) + R_b(r_1)R_u(r_2)),\]  
(15)
where the normalization constant \(C'_s\) is expressed as
\[C'_s = \sqrt{\frac{1}{2(1+D^2)}},\]  
(16)

III. DISCUSSIONS

Combining the spin parts, the spin-color bases for the pentaquark \([\bar{c}c][qq')(q'')\) can be written as

| TABLE I: Color matrix elements for five quark system \([\bar{c}(1)c(2)][q(3)q'(4)q''(5)].\) Therein the superscript 1 denotes the color structure of two quarks are symmetrical, while the superscript 2 denotes the color structure of two quarks are antisymmetrical. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \(\hat{O}\) | \(\lambda_1 \cdot \lambda_2\) | \(\lambda_1 \cdot \lambda_3\) | \(\lambda_1 \cdot \lambda_4\) | \(\lambda_1 \cdot \lambda_5\) | \(\lambda_2 \cdot \lambda_3\) | \(\lambda_2 \cdot \lambda_4\) | \(\lambda_2 \cdot \lambda_5\) | \(\lambda_3 \cdot \lambda_4\) | \(\lambda_3 \cdot \lambda_5\) | \(\lambda_4 \cdot \lambda_5\) |
| \(<11 | \hat{O} | 11>\) | \(-\frac{10}{3}\) | 0 | 0 | 0 | 0 | 0 | \(-\frac{8}{3}\) | \(-\frac{8}{3}\) | \(-\frac{8}{3}\) |
| \(<8^1 3^1 | \hat{O} | 8^1 3^1>\) | \(\frac{2}{3}\) | \(-\frac{10}{3}\) | \(-\frac{10}{3}\) | \(\frac{2}{3}\) | \(-\frac{5}{3}\) | \(-\frac{5}{3}\) | \(-\frac{8}{3}\) | \(\frac{4}{3}\) | \(-\frac{5}{3}\) | \(-\frac{5}{3}\) |
| \(<8^1 3^2 | \hat{O} | 8^1 3^2>\) | -1 | -1 | -1 | -1 | \(-\frac{2}{3}\) | \(-\frac{2}{3}\) | -1 | \(-\frac{1}{3}\) | \(-\frac{1}{3}\) | \(-\frac{1}{3}\) |
| \(<8^2 3^1 | \hat{O} | 8^2 3^1>\) | 0 | 0 | 0 | 0 | \(\sqrt{\frac{2}{3}}\) | \(-\sqrt{\frac{2}{3}}\) | 0 | 0 | 0 | 0 |
| \(<8^2 3^2 | \hat{O} | 8^2 3^2>\) | 0 | 0 | 0 | 0 | \(-\frac{2}{\sqrt{3}}\) | \(-\frac{2}{\sqrt{3}}\) | \(2\sqrt{2}\) | 0 | 0 | 0 |

| TABLE II: Color matrix elements for five quark system \([\bar{c}(1)c(2)][q(3)q'(4)q''(5)].\) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \(\hat{O}\) | \(\lambda_1^c \cdot \lambda_2^c\) | \(\lambda_1^c \cdot \lambda_3^c\) | \(\lambda_1^c \cdot \lambda_4^c\) | \(\lambda_1^c \cdot \lambda_5^c\) | \(\lambda_2^c \cdot \lambda_3^c\) | \(\lambda_2^c \cdot \lambda_4^c\) | \(\lambda_2^c \cdot \lambda_5^c\) | \(\lambda_3^c \cdot \lambda_4^c\) | \(\lambda_3^c \cdot \lambda_5^c\) | \(\lambda_4^c \cdot \lambda_5^c\) |
| \(<633 | \hat{O} | 633>\) | \(\frac{4}{3}\) | \(-\frac{5}{3}\) | \(-\frac{5}{3}\) | \(-\frac{10}{3}\) | \(-\frac{2}{3}\) | \(-\frac{5}{3}\) | \(-\frac{10}{3}\) | \(-\frac{8}{3}\) | \(\frac{4}{3}\) | \(\frac{2}{3}\) |
| \(<363 | \hat{O} | 363>\) | \(-\frac{5}{3}\) | \(-\frac{5}{3}\) | \(-\frac{5}{3}\) | \(\frac{2}{3}\) | \(-\frac{2}{3}\) | \(-\frac{5}{3}\) | \(\frac{2}{3}\) | \(\frac{4}{3}\) | \(-\frac{10}{3}\) | \(-\frac{10}{3}\) |
| \(<333 | \hat{O} | 333>\) | \(-\frac{5}{3}\) | \(-\frac{5}{3}\) | \(-\frac{5}{3}\) | \(-\frac{2}{3}\) | \(-\frac{2}{3}\) | \(-\frac{5}{3}\) | \(-\frac{2}{3}\) | \(-\frac{4}{3}\) | \(-\frac{8}{3}\) | \(-\frac{3}{3}\) | \(\frac{4}{3}\) |
| \(<363 | \hat{O} | 363>\) | 0 | -1 | 1 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 |
| \(<633 | \hat{O} | 333>\) | 0 | \(-\sqrt{2}\) | \(-\sqrt{2}\) | \(\sqrt{8}\) | \(\sqrt{2}\) | \(\sqrt{2}\) | \(-\sqrt{8}\) | 0 | 0 | 0 | 0 |
| \(<363 | \hat{O} | 333>\) | 0 | \(\sqrt{2}\) | \(-\sqrt{2}\) | 0 | \(\sqrt{2}\) | \(-\sqrt{2}\) | 0 | 0 | \(-\sqrt{8}\) | \(\sqrt{8}\) |
TABLE III: Spin matrix elements for five quark system $[c(1)c(2)][q(3)q'(4)q''(5)]$. Therein the superscript 1 denotes the spin structure of two quarks are symmetrical, while the superscript 2 denotes the spin structure of two quarks are antisymmetrical.

| $J^P$ | $\tilde{B}$ | $\sigma_1 \cdot \sigma_2$ | $\sigma_1 \cdot \sigma_3$ | $\sigma_1 \cdot \sigma_4$ | $\sigma_1 \cdot \sigma_5$ | $\sigma_2 \cdot \sigma_3$ | $\sigma_2 \cdot \sigma_4$ | $\sigma_2 \cdot \sigma_5$ | $\sigma_3 \cdot \sigma_4$ | $\sigma_3 \cdot \sigma_5$ | $\sigma_4 \cdot \sigma_5$ |
|-------|-----------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $\frac{1}{2}^-$ | $\frac{1}{2}^+ | \tilde{B} | \frac{1}{2}^+$ | 1 | -$\frac{3}{2}$ | -$\frac{3}{2}$ | -$\frac{3}{2}$ | -$\frac{3}{2}$ | -$\frac{3}{2}$ | 1 | 1 | 1 |
| | $\frac{1}{2}^- | \tilde{B} | 0 | -3 | 0 | 0 | 0 | 0 | 0 | 1 | -2 | -2 |
| | $\frac{1}{2}^- | \tilde{B} | 0 | -3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $\frac{1}{2}^+ | \tilde{B} | \frac{1}{2}^+$ | 1 | -$\frac{3}{2}$ | -$\frac{3}{2}$ | -$\frac{3}{2}$ | -$\frac{3}{2}$ | -$\frac{3}{2}$ | 1 | -2 | -2 |
| | $\frac{1}{2}^- | \tilde{B} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -3 | 0 | 0 |
| | $\frac{1}{2}^- | \tilde{B} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $\frac{1}{2}^+ | \tilde{B} | \frac{1}{2}^+$ | 0 | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | 0 | 0 | 0 |
| | $\frac{1}{2}^- | \tilde{B} | 0 | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | 0 | 0 | 0 |
| | $\frac{1}{2}^- | \tilde{B} | 0 | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | 0 | 0 | 0 |
| | $\frac{1}{2}^- | \tilde{B} | 0 | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | 0 | 0 | 0 |
| | $\frac{1}{2}^- | \tilde{B} | 0 | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | 0 | 0 | 0 |
| | $\frac{1}{2}^- | \tilde{B} | 0 | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | 0 | 0 | 0 |
| | $\frac{1}{2}^- | \tilde{B} | 0 | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | -$\sqrt{\frac{3}{2}}$ | 0 | 0 | 0 |

The spin-color bases for the pentaquark ($[cc][qq']q''$) can be written as

$$ |0_{cc}, 1_{qq'}, \frac{1}{2}_{q''}, \frac{1}{2}_{q''}' \rangle \otimes |1_{cc}, 3_{qq'}, 3_{q''}, \frac{1}{2}_{q''}' \rangle_c, $$

$$ |1_{cc}, 1_{qq'}, \frac{1}{2}_{q''}, \frac{1}{2}_{q''}' \rangle_c \otimes |1_{cc}, 3_{qq'}, 3_{q''}, \frac{1}{2}_{q''}' \rangle_c, $$

$$ |0_{cc}, 0_{qq'}, \frac{1}{2}_{q''}, \frac{1}{2}_{q''}' \rangle \otimes |8_{cc}, 6_{qq'}, 3_{q''}, \frac{1}{2}_{q''}' \rangle_c, $$

$$ |1_{cc}, 0_{qq'}, \frac{1}{2}_{q''}, \frac{1}{2}_{q''}' \rangle \otimes |8_{cc}, 6_{qq'}, 3_{q''}, \frac{1}{2}_{q''}' \rangle_c, $$

$$ |0_{cc}, 1_{qq'}, \frac{1}{2}_{q''}, \frac{1}{2}_{q''}' \rangle \otimes |8_{cc}, 3_{qq'}, 3_{q''}, \frac{1}{2}_{q''}' \rangle_c, $$

$$ |1_{cc}, 1_{qq'}, \frac{1}{2}_{q''}, \frac{1}{2}_{q''}' \rangle_c \otimes |8_{cc}, 3_{qq'}, 3_{q''}, \frac{1}{2}_{q''}' \rangle_c. $$

The spin-color bases for the tetraquark ($[cc][qq']$) can be
written as

$$\begin{align*}
|0_{cc}, 0_{qq}\rangle_s &\otimes |1_{cc}, 1_{qq}\rangle_c (or |8_{cc}, 8_{qq}\rangle_c), \\
|0_{cc}, 1_{qq}\rangle_s &\otimes |1_{cc}, 1_{qq}\rangle_c (or |8_{cc}, 8_{qq}\rangle_c), \\
|1_{cc}, 0_{qq}\rangle_s &\otimes |1_{cc}, 1_{qq}\rangle_c (or |8_{cc}, 8_{qq}\rangle_c), \\
|1_{cc}, 1_{qq}\rangle_s &\otimes |1_{cc}, 1_{qq}\rangle_c (or |8_{cc}, 8_{qq}\rangle_c).
\end{align*}$$

The spin-color bases for the tetraquark ([\bar{c}c][qq']) can be written as

$$\begin{align*}
|0_{cc}, 0_{qq}\rangle_s &\otimes |6_{cc}, 6_{qq}\rangle_c, \\
|1_{cc}, 1_{qq}\rangle_s &\otimes |3_{cc}, 3_{qq}\rangle_c.
\end{align*}$$

In our approach, two parameters $\beta$ and $R$ are free. The linear confinement potential is also unknown because of the lack of the information of $b_1$ and $b_0$. For simplicity, we do not consider the linear confinement potential contribution and let the two parameters $\beta$ and $R$ vary. The color and spin matrix elements are given in Tabs. I, II, III, and IV.

The constituent quark masses are chosen as [34, 35, 37, 38]: $m_q = 305$ MeV, $m_c = 1670$ MeV, and $m_b = 5008$ MeV for mesons (Set I); $m_c = 362$ MeV, $m_s = 1721$ MeV, and $m_b = 5050$ MeV for baryons (Set II). Thus the constituent quarks masses threshold become 3.950GeV for charm tetraquarks, 10.63GeV for bottom tetraquarks, 4.52GeV for charm pentaquarks, and 11.19GeV for bottom pentaquarks. The coupling is chosen as $C^{qq}/m_q^2 = 193$MeV, $C^{cc}/m_{cc}m_q = 23$MeV, $C^{qq}/m_{qq}m_q = 128$MeV, and $C^{cc}/m_{cc}m_q = 318$MeV, $C^{cd}/m_{cd}m_q = 699$MeV, $C^{qq}/m_{qq}m_q = 23$MeV, $C^{cc}/m_{cc} = 57$MeV, $C^{c8}/m_{c8} = 31$MeV [39].

Through the calculations, we found that the optimal value for $\beta$ is around 50MeV and the optimal value for $R$ is around 0.1fm for doubly heavy flavor tetraquarks and doubly heavy flavor pentaquarks. For hidden heavy flavor tetraquarks and hidden heavy flavor pentaquarks, the optimal value for $\beta$ is around 100MeV but the optimal value for $R$ is 0 which leads to the divergences. From the above model, the $QQqqq'$ and $QQqqq^{''}$ systems become more attractive than the $QQqqq'$ and $QQqqq^{''}$ systems when the value of the heavy flavor distance $R$ become smaller. To avoid the divergences, we set the value of $R$ around 0.5fm for hidden heavy flavor tetraquarks and hidden heavy flavor pentaquarks.

The spectra of the tetraquarks are given in Tab. V where the values of the parameters are chosen as Set II. Here we only focus on the S-wave states and ignore the orbitally excited states. Combing the LHCb data for the hidden charm pentaquarks, the $P_c(4312)^+$ may be assigned into the ground state with spin-parity $\frac{1}{2}^+$ or $\frac{1}{2}^-$; the $P_c(4440)^+$ and $P_c(4457)^+$ may be assigned into the excited states with $\frac{1}{2}^-$. Besides, the wide resonance $P_c(4380)^+$ may be assigned as the excited state with $\frac{1}{2}^-$, or the ground state with $\frac{3}{2}^-$, or from the interference by two states with $\frac{1}{2}^-$. Of course, one should note that the $P_c(4440)^+$ and $P_c(4457)^+$ state may be assigned into the ground state of $\frac{1}{2}^-$ or the excited states of $\frac{3}{2}^-$ if we employ the parameter values of Set I.

The spectra of the hidden charm tetraquarks are (in GeV)

$$m(c\bar{c}qq') = \begin{cases} 
3.823, 3.949, & J^P = 0^+, \\
3.944, 4.045, 4.112, & J^P = 1^+, \\
4.185, & J^P = 2^+. 
\end{cases}$$

Considering the current data for the hidden charm tetraquarks, the $X(3823)$ state [39] may be assigned as the
TABLE V: Pentaquark spectra (in GeV) within Set II which is widely employed in the analysis of the three-quark baryons. All the masses of the charm pentaquarks will decrease by 273MeV, while all the masses of the beauty pentaquarks will decrease by 255MeV, if we use the Set I which is widely employed in the analysis of the mesons. When the lowest energy of the pentaquarks with certain $J^P$ is higher than the constituent quarks mass threshold, we also give its mass but label it with “*”.

| Constituents                  | $J^P = \frac{1}{2}^-$ | $J^P = \frac{3}{2}^-$ | $J^P = \frac{5}{2}^-$ |
|-------------------------------|-----------------------|-----------------------|-----------------------|
| $ccq^q'((\bar{1}_{bc},3,3_q,3_{q'},c))$ | 4.302.4.420.4.709*   | 4.591*, 4.709*        | 4.709*                |
| $ccq^q''((\bar{8}_{bc},6,3_q,3_{q'},c))$ | 4.574*, 4.592*       | 4.356                |                       |
| $ccq^q''((\bar{8}_{bc},3,3_q,3_{q'},c))$ | 4.485.4.487.4.551*   | 4.576*, 4.583*, 4.607* | 4.626*                |
| $bbq^q'((\bar{1}_{bb},3,3_q,3_{q'},c))$ | 10.91, 10.98, 11.27*  | 11.21*, 11.27*        | 11.27*                |
| $bbq^q''((\bar{8}_{bb},6,3_q,3_{q'},c))$ | 11.15, 11.11         | 10.93                |                       |
| $bbq^q''((\bar{8}_{bb},3,3_q,3_{q'},c))$ | 11.07, 11.15         | 11.16, 11.17         | 11.18                 |
| $ccq^q'((\bar{6}_{cc},3,3_q,3_{q'},c))$ | 4.647*               | 4.527                |                       |
| $ccq^q''((\bar{3}_{cc},6_q,3_{q'},c))$ | 4.610*               | 4.584*               |                       |
| $ccq^q''((\bar{3}_{cc},3_q,3_{q'},c))$ | 4.460.4.559*         | 4.423.4.606*         | 4.666                 |
| $bbq^q'((\bar{6}_{bb},3,3_q,3_{q'},c))$ | 11.30*               | 11.18                |                       |
| $bbq^q''((\bar{3}_{bb},6_q,3_{q'},c))$ | 11.25*               | 11.24*               |                       |
| $bbq^q''((\bar{3}_{bb},3_q,3_{q'},c))$ | 11.13, 11.21*        | 11.09, 11.26*        | 11.30                 |

The spectra of the doubly charm tetraquarks become (in GeV)

$$m(ccq^q) = \begin{cases} 3.963, 3.997, & J^P = 0^-; \\ 3.997, & J^P = 1^-; \\ 4.066, & J^P = 2^-; \end{cases}$$  (22)

Similarly, the spectra of the hidden bottom tetraquarks become (in GeV)

$$m(bbq^q) = \begin{cases} 10.63, 10.66, & J^P = 0^+; \\ 10.70, 10.73, 10.79, & J^P = 1^+; \\ 10.78, & J^P = 2^+. \end{cases}$$  (23)

The spectra of the doubly bottom tetraquarks are (in GeV)

$$m(bbq^q) = \begin{cases} 10.67, 10.68, & J^P = 0^+; \\ 10.69, & J^P = 1^+; \\ 10.72, & J^P = 2^+. \end{cases}$$  (24)

We calculated the spectra of the S-wave multi-quark states with two heavy flavors. The orbitally excited states are not considered here. To hunting for these multi-quark states, the golden channels are useful. For the hidden heavy flavor pentaquarks, one may use $\Lambda_b \rightarrow J/\psi + p^+(\Delta^+) + K^-$, $\Lambda_b \rightarrow J/\psi + n + K^0$, $pp(p) \rightarrow \Upsilon + p^+(\Delta^+) + X$ and $pp(p) \rightarrow \Lambda + \Xi_c + X$. For the doubly heavy flavor pentaquarks, one may use $pp(p) \rightarrow \Lambda_c + D(B) + X$. For the doubly heavy flavor tetraquarks, one may use $pp(p) \rightarrow \Lambda + \Xi_c + X$ and $pp(p) \rightarrow \bar{p} + \Sigma_b + X$.

IV. CONCLUSION

We calculated the spectra of the hidden doubly heavy flavor pentaquarks, hidden heavy flavor and doubly heavy flavor tetraquarks using the variational method. The model we adopted is very similar to Hydrogen molecule but with SU(3) color interactions. The results within Set II indicated that the $P_c(4312)^+$ may be assigned as the ground state with spin-parity $\frac{1}{2}^-$ or $\frac{3}{2}^-$, the $P_c(4440)^+$ and $P_c(4557)^+$ as the excited states with $\frac{1}{2}^-$, and $P_c(4380)^+$ as the interference by two states with $\frac{1}{2}^-$. The Hydrogen molecule-like model indicated that the $QQqq'q''$ and $QQqq''q$ systems become more attractive and stable than the $QQqq'q''$ and $QQqq''q$ systems when the value of the heavy flavor distance $R$ become smaller. Hunting for these exotic states shall enlightening us the nature of the QCD color confinement.

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