CHERN CONJECTURE ON MINIMAL HYPERSURFACES

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Abstract. In this paper, we study \(n\)-dimensional complete minimal hypersurfaces in a unit sphere. We prove that an \(n\)-dimensional complete minimal hypersurface with constant scalar curvature in a unit sphere with \(f_3\) constant is isometric to the totally geodesic sphere or the Clifford torus if \(S \leq 1.8252n - 0.712898\), where \(S\) denotes the squared norm of the second fundamental form of this hypersurface.

1. Introduction

As one knows that it is important to investigate compact minimal hypersurfaces in spheres. By computing the Laplacian of the squared norm \(S\) of the second fundamental form of minimal hypersurfaces in spheres, Simons in [14] proved that for an \(n\)-dimensional compact minimal hypersurface in a unit sphere \(S^{n+1}(1)\), if \(S \leq n\), then \(S \equiv 0\) or \(S \equiv n\). In the landmark papers of Chern, do Carmo and Kobayashi [5] and Lawson [10], they proved that the Clifford torus \(S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})\) for \(1 \leq m \leq n - 1\) are the only compact minimal hypersurfaces in \(S^{n+1}(1)\) with \(S \equiv n\). The following Chern conjecture is important and well-known:

Chern conjecture. For \(n\)-dimensional compact minimal hypersurfaces in \(S^{n+1}(1)\) with constant scalar curvature, \(S > n\), then \(S \geq 2n\).

In 1982, Peng and Terng studied the above Chern conjecture, they proved that for \(n\)-dimensional compact minimal hypersurfaces in \(S^{n+1}(1)\) with constant scalar curvature, \(S > n\), then \(S \geq n + \frac{1}{12n}\). Furthermore, for \(n = 3\), they solved Chern conjecture affirmatively. For \(n \geq 4\), Yang and the first author ([16], [17], [18]) made an important breakthrough. They proved if \(S > n\), then \(S \geq \frac{4n}{3}\). (cf. [2], [7], [8], [9], [12], [13], [15]).

In [4], Cheng and Wei have solved Chern conjecture for \(n = 4\) under the additional condition that \(f_3 = \sum \lambda_i^3\) is constant, where \(\lambda_i\)'s are principal curvatures of \(M^4\). On the other hand, for \(n \geq 4\), Yang and the first author [18] proved the following:

Theorem 1.1. Let \(M^n, n \geq 4\), be an \(n\)-dimensional compact minimal hypersurface in \(S^{n+1}(1)\) with constant scalar curvature. If \(f_3\) is constant, \(S = 0\), or \(S = n\) if \(S \leq n + \frac{2}{3}n\).
Our main purpose in this paper is to study Chern conjecture under the condition that $f_3$ is constant. We improve the result of Yang and Cheng [18] under weaker topology.

**Theorem 1.2.** Let $M^n$ ($n \geq 5$) be an $n$-dimensional complete minimal hypersurface in $S^{n+1}(1)$ with constant scalar curvature. If $f_3$ is constant and $S > n$, then

$$S > 1.8252n - 0.712898.$$ 

**Remark 1.1.** In the above theorem, we only assume that $M^n$ is complete.

2. Preliminary

In this paper, we assume that all manifolds are smooth and connected without boundary. Let $M^n$ be an $n$-dimensional hypersurface in $S^{n+1}(1)$. We choose a local orthonormal frame $\{\vec{e}_1, \cdots, \vec{e}_n, \vec{e}_{n+1}\}$ and the dual coframe $\{\omega_1, \cdots, \omega_n, \omega_{n+1}\}$ in such a way that $\{\vec{e}_1, \cdots, \vec{e}_n\}$ is a local orthonormal frame on $M^n$. Hence, we have

$$\omega_{n+1} = 0$$
on $M^n$. Thus, one has

$$\omega_{n+1,i} = \sum_{j=1}^{n} h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$  

The mean curvature $H$ and the second fundamental form $\bar{\alpha}$ of $M^n$ are defined, respectively, by

$$H = \frac{1}{n} \sum_{i=1}^{n} h_{ii}, \quad \bar{\alpha} = h_{ij} \omega_i \otimes \omega_j \vec{e}_{n+1}.$$  

If $H$ is zero in $M^n$, one calls that $M^n$ is a minimal hypersurface. From the structure equations of $M^n$, Guass equations, Codazzi equations and Ricci formulas are given by

$$R_{ijkl} = (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + (h_{ik} h_{jl} - h_{il} h_{jk}),$$

$$h_{ijkl} = h_{ijkl},$$

$$h_{ijkl} - h_{ijkl} = \sum_{m} h_{im} R_{mjkl} + \sum_{m} h_{mj} R_{imkl},$$

where $h_{ijk} = \nabla_k h_{ij}$ and $h_{ijkl} = \nabla_l \nabla_k h_{ij}$, respectively. For minimal hypersurfaces in $S^{n+1}(1)$ from (2.1), we have

$$r = n(n - 1) - S,$$

where $r$ and $S$ denote the scalar curvature and the squared norm of the second fundamental form of $M^n$, respectively. We define functions $f_3$ and $f_4$ by

$$f_3 = \sum_{i,j,k=1}^{n} h_{ij} h_{jk} h_{ki}$$

and

$$f_4 = \sum_{i,j,k,l=1}^{n} h_{ij} h_{jk} h_{kl} h_{li}$$

respectively. Then, we have, for minimal hypersurfaces,

$$(2.1) \quad \frac{1}{3} \Delta f_3 = (n - S) f_3 + 2C,$$
\[
\frac{1}{4} \Delta f_4 = (n - S) f_4 + (2A + B),
\]
where
\[
C = \sum_{i,j,k} \lambda_i h_{ijk}^2, \quad A = \sum_{i,j,k} \lambda_i^2 h_{ijk}^2, \quad B = \sum_{i,j,k} \lambda_i \lambda_j h_{ijk}^2
\]
and \(\lambda_i\)'s are principal curvatures of \(M^n\). If the squared norm \(S\) of the second fundamental form is constant, we have
\[
\sum_i h_{ii} = \sum_i \lambda_i = 0, \quad S = \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2,
\]
\[
\sum_{i,j,k} h_{ij}^2 = S(S - n), \quad \sum_{p} h_{ijpp} = (n - S) h_{ij},
\]
\[
h_{ijij} - h_{jiji} = (\lambda_i - \lambda_j)(1 + \lambda_i \lambda_j).
\]
By a direct computation, we have
\[
\sum_{i,j,k,l} h_{ijkl}^2 + S(S - n)(2n + 3 - S) + 3(2B - A) = 0,
\]

3. A proof of Theorem 1.2

Defining
\[
u_{ijkl} := \frac{1}{4}(h_{ijkl} + h_{jkl} + h_{kli} + h_{lijk}),
\]
we have
\[
\sum_{i,j,k,l} h_{ijkl}^2 = \sum_{i,j,k,l} u_{ijkl}^2 + \frac{3}{2} \left[ S f_4 - f_3^2 - 2S^2 + nS \right],
\]
Putting \(S - n = tS\) and defining
\[
S f = S f_4 - f_3^2 - \frac{S^3}{n},
\]
The following formulas can be found in [17]
\[
A - B \leq \frac{1}{3}(\lambda_1 - \lambda_2)^2 tS^2,
\]
where \(\lambda_1 = \max \lambda_i\) and \(\lambda_2 = \min \lambda_i\). Taking the orthonormal frame \(\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}\) at each point such that
\[
h_{ij} = \lambda_i \delta_{ij},
\]
we have
\[
S f + \frac{S^3}{n} \equiv S f_4 - (f_3)^2 = S \sum_i \lambda_i^4 - \left( \sum_i \lambda_i^2 \right)^2 = \frac{1}{2} \sum (\lambda_i - \lambda_j)^2 \lambda_i^2 \lambda_j^2.
\]
Since \(S\) and \(f_3\) are constant, we have
\[
\sum_i \lambda_i h_{iik} = 0, \quad \sum_i \lambda_i^2 h_{iik} = 0, \text{ for any } k
\]
and 
\[ \sum_i \lambda_i h_{iikl} = - \sum_{i,j} h_{ijkhij}, \quad \sum_i \lambda_i^2 h_{iikl} = -2 \sum_{i,j} \lambda_i h_{ijkhij}, \text{ for any } k, l. \]

Hence, we have 
\[ \sum_{i,j} h_{ijjj} \lambda_i \lambda_j = -C, \quad \sum_{i,j} h_{ijjj} h_{ijij} \lambda_i \lambda_j = -2B, \quad \sum_{i,j} h_{ijjj} \lambda_i \lambda_j^2 = -A. \]

Defining 
\[ a_{ij} = \sum_m h_{im} h_{mj} - y h_{ij} - \frac{S}{n} \delta_{ij}, \quad i, j = 1, 2, \ldots, n \]

with \( Sy = f_3 \) we have
\[
\sum_{i,j=1}^n a_{ij}^2 = f, \quad \sum_{i,j=1}^n a_{ij} h_{ij} = 0, \quad \sum_{i,j=1}^n a_{ij} \delta_{ij} = 0.
\]

For \( \forall \alpha, \beta, \gamma \in \mathbb{R}, \)
\[
\sum_{i,j,k,l} \left\{ u_{ijkl} + \alpha(a_{ij} h_{kl} + h_{ij} a_{kl}) + \beta h_{ij} h_{kl} + \gamma(h_{ij} \delta_{kl} + \delta_{ij} h_{kl}) \right\}^2 \geq 0,
\]

Because of 
\[
\sum_{i,j,k,l} u_{ijkl} h_{ij} \delta_{kl} = -\frac{t}{2} S^2, \quad \sum_{i,j,k,l} u_{ijkl} h_{ij} h_{kl} = -C,
\]
\[
\sum_{i,j,k,l} u_{ijkl} a_{ij} h_{kl} = -B - \frac{1}{2} A + yC + \frac{t}{2(1-t)} S^2,
\]
we obtain, from (3.3) and (3.4),
\[
\sum_{i,j,k,l} u_{ijkl}^2 \geq 2\alpha(2B + A - 2yC - \frac{t}{1-t} S^2) - 2\alpha^2 Sf + \frac{C^2}{S^2} + \frac{t}{2(1-t)} t S^2
\]

by taking \( \beta = \frac{C}{S^2} \) and \( \gamma = \frac{t}{2(1-t)} \). Since \( f_3 \) is constant, we have 
\[ t S f_3 = 2C \]

and
\[
(3.6) \quad Sf + \frac{t}{1-t} S^2 = Sf_4 - f_3^2 - S^2 = A - 2B.
\]

From
\[
(3.7) \quad \sum_{i,j,k,l} h_{ijkl}^2 = S(S-n)(S-2n-3) + 3(A-2B)
\]
and
\[
\sum_{i,j,k,l} h_{ijkl}^2 = \sum_{i,j,k,l} u_{ijkl}^2 + \frac{3}{2} \left[ Sf_4 - f_3^2 - 2S^2 + nS \right],
\]
we have
\[(3.8) \quad S(S - n)(S - 2n) = \sum_{i,j,k,l} u_{ijkl}^2 + \frac{3}{2}S(S - n) - \frac{3}{2}(A - 2B).\]

**Remark 3.1.** If one can prove
\[\sum_{i,j,k,l} u_{ijkl}^2 + \frac{3}{2}S(S - n) - \frac{3}{2}(A - 2B) \geq 0,\]
Chern conjecture will be solved under condition that \(f_3\) is constant.

Since \(S\) is constant, we know that the Ricci curvature is bounded from below according to the Gauss equations. By applying the Generalized Maximum Principle of Omori [11] and Yau [19] to function \(f_4\), we know that there exists a sequence of \(\{p_m\}_{m=1}^{\infty} \subset M^n\) such that
\[
\lim_{m \to \infty} f(p_m) = \sup f_4, \quad \lim_{m \to \infty} |\nabla f_4(p_m)| = 0, \quad \lim_{m \to \infty} \Delta f_4(p_m) \leq 0
\]
Since \(S\) is constant, we have that, for any \(i, j, k, l\), \(\{\lambda_i(p_m)\}\), \(\{h_{ijk}(p_m)\}\) and \(\{h_{ijkl}(p_m)\}\) are bounded sequences. Thus, we can assume, for any \(i, j, k, l\),
\[
\lim_{m \to \infty} \lambda_i(p_m) = \bar{\lambda}_i, \quad \lim_{m \to \infty} h_{ijk}(p_m) = \bar{h}_{ijk}, \quad \lim_{m \to \infty} h_{ijkl}(p_m) = \bar{h}_{ijkl}
\]
All of the following computations are made for \(\bar{\lambda}_i\), \(\bar{h}_{ijk}\) and \(\bar{h}_{ijkl}\). For simple, we omit \(\bar{\cdot}\). From (2.2), we have
\[
tSf_4 \geq 2A + B.
\]
According to
\[
Sf_4 - f_3^2 - S^2 = A - 2B,
\]
we obtain
\[(3.9) \quad tf_3^2 \geq (2 - t)A + (1 + 2t)B - tS^2.
\]
Since
\[
C^2 = (\sum_{i,j,k} \lambda_i h_{ijk}^2)^2 = \frac{1}{9} \left( \sum_{i,j,k} (\lambda_i + \lambda_j + \lambda_k) h_{ijk}^2 \right)^2
\]
\[(3.10) \quad \leq \frac{1}{3}(A + 2B)tS^2,
\]
Taking
\[(3.11) \quad t_z = \frac{17t^2 - 33t + 24}{1 - t} > 0,
\]
we have
\[(3.12) \quad zC^2 \leq \frac{z}{3}(A + 2B)tS^2.
\]
From (3.5) and taking $\alpha = -\frac{3 - 4t}{2}$, we obtain

$$\sum_{i,j,k,l} u_{ijkl}^2 - \frac{3}{2} (A - 2B) + \frac{3}{2} S(S - n)$$

$$\geq 2\alpha (2B + A) - (2\alpha^2 + \frac{3}{2})(A - 2B) + (-2\alpha t + (1 + z) \frac{t^2}{4}) f^2_3 - z \frac{C^2}{S^2}$$

$$- 2\alpha \frac{t}{1-t} S^2 + 2\alpha^2 \frac{t}{1-t} S^2 + \frac{t}{2(1-t)} tS^2 + \frac{3}{2} tS^2$$

$$\geq 2\alpha (2B + A) - (2\alpha^2 + \frac{3}{2})(A - 2B)$$

$$+ (-2\alpha + (1 + z) \frac{t}{4}) \{(2 - t)A + (1 + 2t)B\} - z \frac{t}{3} (A + 2B)$$

$$- 2\alpha \frac{t}{1-t} S^2 + 2\alpha^2 \frac{t}{1-t} S^2 - (-2\alpha + (1 + z) \frac{t}{4}) tS^2 + \frac{t}{2(1-t)} tS^2 + \frac{3}{2} tS^2$$

$$= \frac{4t^2 - 9t + 3}{3(1-t)} (A - B) - 2t \frac{t}{1-t} S^2,$$

that is,

(3.13) $$S(S - n)(S - 2n) \geq \frac{4t^2 - 9t + 3}{3(1-t)} (A - B) - 2t \frac{t}{1-t} S^2.$$

If $t \leq \frac{9 - \sqrt{33}}{8}$, we have $4t^2 - 9t + 3 \geq 0$. From (3.13), we get

$$t \geq \frac{1}{2} - \frac{t}{(1-t)S'}$$

$$S \geq 2n - \frac{2t}{(1-t)} \geq 2n - \frac{\sqrt{33} - 3}{2},$$

then

$$t \geq \frac{n}{2(n+1)} > \frac{5}{12} > \frac{9 - \sqrt{33}}{8}.$$

It is a contradiction. Hence, we have

$$t > \frac{9 - \sqrt{33}}{8}.$$
Thus, we obtain

\begin{equation}
(3.14) \quad t - \frac{45 - \sqrt{465}}{52} \geq -\frac{36t}{45 + \sqrt{465} - 52tS}.
\end{equation}

We conclude

\begin{equation}
(3.15) \quad t \geq \frac{45 - \sqrt{465}}{52} - \frac{9}{26} \left(\frac{3\sqrt{465}}{31} - 1\right) \frac{1}{S}.
\end{equation}

In fact, if not, we have

\[ t < \frac{45 - \sqrt{465}}{52} - \frac{9}{26} \left(\frac{3\sqrt{465}}{31} - 1\right) \frac{1}{S} < \frac{45 - \sqrt{465}}{52}. \]

Hence, we infer

\[ -\frac{36t}{45 + \sqrt{465} - 52tS} > -\frac{9}{26} \left(\frac{3\sqrt{465}}{31} - 1\right) \frac{1}{S}. \]

We conclude

\[ t > \frac{45 - \sqrt{465}}{52} - \frac{9}{26} \left(\frac{3\sqrt{465}}{31} - 1\right) \frac{1}{S}, \]

which is a contradiction. Thus, (3.15) must hold.

Hence, we get

\[ S - n = tS > \frac{45 - \sqrt{465}}{52} - \frac{9}{26} \left(\frac{3\sqrt{465}}{31} - 1\right), \]

that is,

\begin{equation}
(3.16) \quad S > \frac{\sqrt{465} - 7}{8} n - \frac{9}{4} (1 - \frac{\sqrt{465}}{31}) \approx 1.82048n - 0.684881.
\end{equation}

Furthermore, we give a better estimate on \( t \). In order to do it, for any \( i, j \), we have

\[ -\lambda_i\lambda_j \leq \frac{1}{4}(\lambda_i - \lambda_j)^2. \]

Hence, we get, for \( \lambda_i\lambda_j \leq 0 \) and \( \lambda_i\lambda_k \leq 0 \),

\begin{equation}
(3.17) \quad (|\lambda_i\lambda_j| + |\lambda_i\lambda_k|)^3 \leq 4(|\lambda_i\lambda_j|^3 + |\lambda_i\lambda_k|^3) \leq (\lambda_i - \lambda_j)^2 \lambda_i^2 \lambda_j^2 + (\lambda_i - \lambda_k)^2 \lambda_i^2 \lambda_k^2.
\end{equation}

For three different \( i, j, k \), we know that at least one of \( \lambda_i\lambda_j, \lambda_i\lambda_k \) and \( \lambda_j\lambda_k \) is non-negative. Without loss of generality, we assume \( \lambda_j\lambda_k \geq 0 \) and \( \lambda_i\lambda_j \leq 0, \lambda_i\lambda_k \leq 0 \), then we get from (3.17) and (3.2)

\begin{equation}
(3.18) \quad -\lambda_i\lambda_j - \lambda_i\lambda_k - \lambda_j\lambda_k \leq |\lambda_i\lambda_j| + |\lambda_i\lambda_k| \leq (Sf + \frac{S^3}{n})^\frac{1}{3},
\end{equation}

and

\begin{equation}
(3.19) \quad -2\lambda_i\lambda_j \leq 2(|\lambda_i\lambda_j|^3)^\frac{1}{3} \leq 2\left(\frac{1}{4}(\lambda_i - \lambda_j)^2 \lambda_i^2 \lambda_j^2\right)^\frac{1}{3} \leq 2\left(\frac{Sf + \frac{S^3}{n}}{4}\right)^\frac{1}{3}.
\end{equation}
Since
\[3(A - B) = \sum_{i,j,k} (\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - \lambda_i\lambda_j - \lambda_i\lambda_k - \lambda_j\lambda_k)h_{ijk}^2 \]
\[\quad = \left(\sum_{i\neq j\neq k\neq i} (\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - \lambda_i\lambda_j - \lambda_i\lambda_k - \lambda_j\lambda_k)h_{ijk}^2 \right) + 3 \sum_{i\neq j} (\lambda_i^2 + \lambda_j^2 - 2\lambda_i\lambda_j)h_{iij}^2,
\]
we conclude from (3.18) and (3.19),
\[\quad \leq \left(S + 2\left(\frac{Sf + \frac{S^3}{4n}}{12(1-t)} + \frac{1}{4n}\right)^{\frac{1}{3}}\right)tS^2.
\]
According to
\[S - n = ts, \quad (1-t)Sf \leq \frac{1}{3}(A - B), \quad A - B \leq \frac{2}{3}tS^3\]
we infer from (3.22)
\[3(A - B) \leq \left(S + 2\left(\frac{A - B}{12(1-t)} + \frac{1}{4n}\right)^{\frac{1}{3}}\right)tS^2.
\]
Thus, we can assume that \(3(A - B) \leq a_k tS^3\). We have from (3.23) that \(3(A - B) \leq a_{k+1} tS^3\), where
\[a_1 = 2, \quad a_{k+1} = 1 + 2\left(\frac{1}{36(1-t)}a_k + \frac{1}{4n}\right)^{\frac{1}{3}}.
\]
We next assume that \(t < 0.452115\) and \(n \geq 6\), then we get from (3.24) that
\[a_{k+1} \leq 1 + 2\left(\frac{1}{36(1 - 0.452115)}a_k + \frac{1}{24}\right)^{\frac{1}{3}}.
\]
By a direct calculation, we know
\[a_7 \leq 1.878415.
\]
Hence, we obtain
\[3(A - B) < 1.878415tS^2.
\]
From (3.13) and (3.27), we have
\[ (2t - 1)S > \frac{4t^2 - 9t + 3}{3(1-t)} \times \frac{1.878415}{3}S - \frac{2t}{1 - t}.
\]
Then, we get
\[2.83485(t - 0.452115)(t - 1.26876) < \frac{2t}{S}.
\]
Because \(t < 0.452115\), we know
\[\frac{2t}{2.83485(t - 1.26876)} > -0.390586 \frac{1}{S}.
\]
Thus, we infer
\[ t > 0.452115 - 0.390586 \frac{1}{S}, \]
then
\[ (3.29) \quad S > 1.8252n - 0.712898. \]
If \( t \geq 0.452115 \), then we get
\[ (3.30) \quad S \geq 1.8252n. \]
From (3.16) and (3.29), we know that if \( n = 5 \),
\[ 1.82048n - 0.684881 > 1.8252n - 0.712898. \]
We complete the proof of theorem 1.2.

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