Hamiltonian structure and Darboux’ theorem for families of Generalized Lotka-Volterra systems

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Abstract

This work is devoted to the establishment of a Poisson structure for a format of equations known as Generalized Lotka-Volterra systems. These equations, which include the classical Lotka-Volterra systems as a particular case, have been deeply studied in the literature. They have been shown to constitute a whole hierarchy of systems, the characterization of which is made in the context of simple algebra. Our main result is to show that this algebraic structure is completely translatable into the Poisson domain. Important Poisson structures features, such as the symplectic foliation and the Darboux’ canonical representation, rise as result of rather simple matrix manipulations.

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I. INTRODUCTION

Poisson structures\(^1,2\) (sometimes named *generalized Hamiltonian structures* in the literature) are ubiquitous in all fields of Mathematical Physics, from finite-dimensional dynamical systems\(^3-7\) to field theories:\(^8,9\) Fluid dynamics,\(^10,11\) magnetohydrodynamics,\(^11,12\) plasmas,\(^13-15\) continuous media,\(^15\) condensed matter,\(^16\) etc. Reformulating a given problem in terms of a Poisson structure provides fruitful insight into the behaviour of the system, which may take the form of perturbative solutions,\(^17\) nonlinear stability analysis through the energy-Casimir algorithm\(^7,18\) or the energy-momentum method,\(^19\) bifurcation properties and characterization of chaotic dynamics,\(^20\) integrability results,\(^21\) application of reduction of order procedures\(^2,22\) or explicit determination of new solutions.\(^14,23\)

In the present work, we shall restrict ourselves to finite-dimensional Poisson structures. In terms of local coordinates, a Poisson system defined on an \(n\)-dimensional manifold takes the following form:

\[
\dot{x}_i = \sum_{j=1}^{n} J_{ij} \partial_j H , \quad i = 1, \ldots, n
\]  

(1)

The smooth, real-valued function \(H(x)\) in (1) is a constant of motion of the system, which plays the role of Hamiltonian, and the \(J_{ij}(x)\) are also smooth and real-valued, being the entries of a \(n \times n\) skew-symmetric structure matrix \(J\) which verifies the Jacobi equations:

\[
\sum_{l=1}^{n} (J_{li} \partial_l J_{jk} + J_{lj} \partial_l J_{ki} + J_{lk} \partial_l J_{ij}) = 0
\]  

(2)

Here \(\partial_l\) means \(\partial / \partial x_l\) and indices \(i, j, k\) run from 1 to \(n\). The flow (1) can then be expressed as \(\dot{x}_i = [x_i, H]\), in terms of the Poisson bracket defined by

\[
[F, G] = \sum_{i,j=1}^{n} \frac{\partial F}{\partial x_i} J_{ij} \frac{\partial G}{\partial x_j}
\]
where $F$ and $G$ are smooth real-valued functions defined on the Poisson manifold. Consequently, Poisson structures generalize classical Hamiltonian systems, for which $J$ is the well-known symplectic matrix. In particular, the classical restriction to even-dimensional manifolds is not present in Poisson systems. However, Poisson dynamics preserves the Hamiltonian character of the motion. This is proven by Darboux’ theorem, which states that there exist local coordinates in the neighbourhood of every point of the Poisson manifold, such that the equations of motion take essentially the classical Hamiltonian form. The practical construction of Darboux’ coordinates is, however, a complicated task in general, which has been carried out only for a limited sample of systems.

An important question is that of characterizing a given vector field not in form (1), as an actual Poisson system. In the finite-dimensional case the problem amounts to giving a procedure for decomposing (whenever possible) a smooth function $f(x) : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $\Omega$ is open, as $f(x) = J(x) \cdot \nabla H(x)$, where $J$ is a solution of the nonlinear PDE (2) and $H(x)$ is a real-valued function. This is a nontrivial problem to which important efforts have been devoted in past years in a variety of approaches. The question is well understood in the simplest cases —two and three dimensions— and the existence and determination of at least one Poisson structure is ensured if a first integral is known for the system. In higher dimensions the situation is by no means so clear, and comparable results are still lacking.

The main exception to this absence of results in $n$-dimensional systems is, to our knowledge, given by the Lotka-Volterra equations (LV from now on). They were introduced by Lotka and Volterra in chemical and biological contexts, respectively, and Volterra himself was already aware of the (classical) Hamiltonian nature of some LV systems. However, the main and more
systematic advances are due to Kerner,\textsuperscript{31} who developed a biological analog of classical statistical mechanics for predator-prey systems. More recently, the attention has turned to the search of Poisson structures for more general LV models, since Poisson structures generalize classical Hamiltonian systems while retaining the Hamiltonian nature of the dynamics—in this sense, the Poisson structure of many two and three-dimensional LV systems has been established;\textsuperscript{3,25,27,28} and all three-dimensional biHamiltonian LV systems have been classified by Plank.\textsuperscript{5} Also, in the domain of \( n \)-dimensional LV flows, a first tentative classification of Poisson structures has been carried out.\textsuperscript{6}

Both the relevance of LV equations and the importance of finding their Hamiltonian or Poisson representations, transcends the fact that LV models are appropriate in describing many problems in Biology, Chemistry or Physics. Cairó and Feix,\textsuperscript{32} for example, refer to a fairly long list of systems modelled by LV equations—their ubiquity has even prompted Peschel and Mende\textsuperscript{33} to head their book on the issue with the title: \textit{Do we live in a Volterra World?}

Actually, the significance of LV equations goes beyond a strict modelling context, because they have been shown to be canonical representatives of infinite families, or classes, of very general flows,\textsuperscript{33,34} to which, following Brenig, we shall refer as Generalized Lotka-Volterra (GLV) systems. There is a whole formalism associated to the GLV equations.\textsuperscript{33−36} As we shall see later, the most relevant feature of this formalism is that of permitting the analysis and interpretation of certain properties of the vector field in purely algebraic terms.

Our purpose in this article is to demonstrate how these algebraic properties are of fundamental importance in understanding the Poisson structure
of LV and GLV models. We shall investigate Poisson structures for GLV families of systems. They include as particular elements all LV models which have been the object of interest in the literature in relation to Poisson structures (in works of Volterra, Kerner and Plank). We shall show that the algebraic GLV matrix properties can be translated into the Poisson context and acquire a new significance. The reverse is also true, thus defining a close connection between GLV algebraic properties and the Poisson structure of the system.

Implementing the structure of the GLV formalism on its Poisson counterpart has very interesting consequences. First of all, we are able to include in our scheme systems which are more general than the LV ones. Second, we can take into account larger sets of LV flows than those studied by previous approaches. For example, we are neither limited to LV systems of even dimension, nor cases with a unique fixed point. These are two common restrictions often imposed in the literature,\textsuperscript{6,37} which we obviate at once. Third, we are able to capture important phase-space features in terms of simple properties of constant matrices. Finally, our approach leads to an algorithmic reduction to the Darboux’ form for the equations. And last but not least, our construction is always global.

II. OVERVIEW OF THE GLV FORMALISM

We proceed now to briefly summarize the main features of the GLV formalism. We refer to the reader interested in a more detailed exposition to the original references.\textsuperscript{33–36}
Definition 2.1: A GLV system is a set of ordinary differential equations which is defined in the real positive orthant and complies to the form:

\[
\dot{x}_i = x_i(\lambda_i + \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}}), \quad i = 1 \ldots n
\]

where \(n\) and \(m\) are positive integers, \(m \geq n\), and \(A\), \(B\) and \(\lambda\) are \(n \times m\), \(m \times n\) and \(n \times 1\) real matrices, respectively.

The \(m\) nonlinear terms of the right-hand side of (3) are usually known as quasimonomials. Sometimes we shall group all the coefficients of \(A\) and \(\lambda\) in a single, composite matrix, \(M = (\lambda \mid A)\). We shall assume that matrix \(B\) is of maximal rank. This is a standard case to which every GLV system can be reduced. Notice also that the well-known LV equations

\[
\dot{x}_i = x_i(\lambda_i + \sum_{j=1}^{n} A_{ij}x_j), \quad i = 1 \ldots n
\]

are a particular case of (3) where \(m = n\) and \(B\) is the \(n \times n\) identity matrix.

System (3) is form-invariant under quasimonomial transformations (or QMTs from now on):

\[
x_i = \prod_{k=1}^{n} y_k^{C_{ik}}, \quad i = 1, \ldots, n, \quad |C| \neq 0
\]

Under (4), matrices \(B, A, \lambda\) and \(M\) change to \(B' = B \cdot C\), \(A' = C^{-1} \cdot A\), \(\lambda' = C^{-1} \cdot \lambda\) and \(M' = C^{-1} \cdot M\), respectively, but the GLV format is preserved. Obviously, all GLV systems which can be connected through QMTs share the value of the product \(B \cdot M\). These families of systems are, in fact, classes of equivalence, the product \(B \cdot M\) being a class invariant. The QMTs are diffeomorphisms defined in the positive orthant, and are orientation-preserving iff \(|C| > 0\). Consequently, QMTs preserve the topology of the phase portrait modulo an inversion.
Definition 2.2: A GLV class of equivalence for which \( \text{rank}(M) = r \), and whose members are \( n \)-dimensional and have \( m \) quasimonomials, is denoted as an \((r, n, m)\)-class.

The kind of manipulations in which we shall be interested later will transform a GLV system into another one belonging to the same or, eventually, to a different class. However, these manipulations will affect neither \( r \) nor \( m \), but may change \( n \). We shall always operate, however, in the range \( r \leq n \leq m \). Obviously, a QMT does not modify anyone of these three indexes.

If \( m = n \), we can perform a QMT of matrix \( C = B^{-1} \). The result is another flow for which \( B' = I_n \), that is, an LV system. Such a system can be taken as the canonical representative of the GLV class of equivalence.

In the complementary case \( m > n \), there is no LV system inside the class of equivalence. However, the reduction to the LV form is possible if we perform an embedding, just by adding new variables to system (3).

Definition 2.3: We define a \( p \)-embedding as the result of adding to a GLV system \( p \) new variables in the following way:

\[
\dot{x}_i = 0, \quad x_i(0) = \alpha_i > 0, \quad i = n + 1, \ldots, n + p, \quad 1 \leq p \leq m - n
\]

Let \( A, B, \) and \( \lambda \) be the matrices of the original GLV system. The \( p \)-embedded system is also GLV, and its characteristic matrices are:

\[
\tilde{B} = (B \mid B_{m \times p}^*), \quad \tilde{\lambda} = \begin{pmatrix} \lambda \\ O_{p \times 1} \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} A \cdot E \\ O_{p \times m} \end{pmatrix},
\]

where

\[
E = \text{diag}(e_1, \ldots, e_m), \quad e_j = \left( \prod_{k=n+1}^{n+p} \alpha_k^{\tilde{B}_{jk}} \right)^{-1}, \quad j = 1, \ldots, m
\]

In (5), \( O \) denotes a submatrix of null entries, while \( B_{m \times p}^* \) has arbitrary real entries appropriately chosen for \( \tilde{B} \) to be of maximal rank. The subscripts
such as $m \times p$ indicate the size of the corresponding submatrix ($B^*$ in this case); we shall maintain this notation henceforth.

Notice how the previous operation transforms a GLV system from an $(r, n, m)$-class, with $m > n$, into a GLV system belonging to an $(r, n + p, m)$-class, with $1 \leq p \leq m - n$. The embedded system is topologically equivalent to the original one in the manifold $x_i = \alpha_i, i = n + 1, \ldots, n + p$.

For fixed $p$, infinitely many different $(r, n + p, m)$-classes can be reached by means of an embedding, depending on the entries of matrix $E$. Also, in the particular case $p = m - n$ the target system belongs to an $(r, m, m)$-class, in which an LV representative can be reached by a QMT of matrix $C = \tilde{B}^{-1}$ (since rank$(\tilde{B}) = m$). Thus, the recasting of the original GLV flow into LV form is always possible. For this $m$-dimensional LV system, however, rank$(\tilde{M}_{LV}) < m$, and is not maximum. Whenever this happens in a GLV system, it indicates the existence of quasimonomial constants of motion, as the following proposition shows:

**Proposition 2.4:** In a GLV system (3), rank$(M) = r < n$ if and only if there exist $(n - r)$ functionally independent constants of the motion which are time-independent and have quasimonomial form.

**Proof:** We can assume, without loss of generality, that the $r$ first rows of $M$ are the linearly independent ones. Then, there exist real constants $\gamma_{ki}$, with $i = 1, \ldots, r$, and $k = r + 1, \ldots, n$, such that:

$$M_{kl} = \sum_{i=1}^{r} \gamma_{ki} M_{il} , \quad \forall \ l = 1, \ldots, m + 1$$

From (3), we arrive at:

$$\frac{\dot{x}_k}{x_k} = \sum_{i=1}^{r} \gamma_{ki} \frac{\dot{x}_i}{x_i} .$$
After a simple integration this leads to the set of \((n-r)\) constants of motion:

\[
x_k^{-1} \prod_{i=1}^{r} x_i^{k_i} = c_k,
\]

where the \(c_k\) are real constants given by the initial conditions. The functional independence holds immediately from simple evaluation of the Jacobian. The proof in the opposite sense is straightforward after this. Q.E.D.

Therefore, a \(p\)-embedding in the \(m > n\) case introduces \(p\) quasimonomial constants of motion, which are obviously form-invariant under QMTs. This invariance implies that the quasimonomial constants of motion can always be decoupled from a GLV system by means of an appropriate QMT. When this is done, what we are doing is to reverse the \(p\)-embedding procedure, actually. The first step to show this is the following result:

**Proposition 2.5:** Let \(A^*, B^*\) and \(\lambda^*\) be the matrices of a GLV system belonging to an \((r, n+p, m)\)-class, where \(r \leq n\) and \(0 < p \leq m-n\). Then there exists a quasimonomial transformation that leads to an \((n+p)\)-dimensional GLV system of matrices

\[
\lambda = \begin{pmatrix}
\bar{\lambda} \\
O_{p \times 1}
\end{pmatrix}, \quad A = \begin{pmatrix}
\bar{A} \\
O_{p \times m}
\end{pmatrix}
\]

(7)

**Proof:** We shall omit it, since it is based on simple matrix algebra properties.

In (7), we have decoupled the final \(p\) components of the vector field: Let \(x_1, \ldots, x_{n+p}\) be the variables of the system of matrices (7), and let \(x_{n+i}(0) = \alpha_{n+i} > 0, i = 1, \ldots, p\), be the initial conditions of the decoupled variables. Let us also write \(B = (\bar{B} \mid B'_{m \times p})\) for the matrix of exponents of this system. Then, when we restrict the dynamics to an \(n\)-dimensional flow, the result is another GLV system from an \((r, n, m)\)-class, which is characterized by three
matrices $\hat{B}$, $\hat{\lambda}$ and $\hat{A}$, given by:

$$\hat{B} = \bar{B}, \quad \hat{\lambda} = \bar{\lambda}, \quad \hat{A} = \bar{A} \cdot E^{-1},$$

where again

$$E = \text{diag}(e_1, \ldots, e_m), \quad e_j = \left(\prod_{k=n+1}^{n+p} \alpha_{jk}^B\right)^{-1}, \quad j = 1, \ldots, m \quad (8)$$

**Definition 2.6:** The previous operation transforming a GLV system in an $(r, n+p, m)$-class, with $r \leq n$ and $0 < p \leq m - n$, into a GLV system belonging to an $(r, n, m)$-class, is called a $p$-decoupling.

Even for fixed $n$, there are again infinite possible target classes, due to the arbitrariness in the initial conditions represented by $E$. In any case, there is obviously a conservation of the topological properties of the flow in the process. It is also clear that, in the especial case in which we choose $n = r$, we have the maximum reduction possible by means of this method; otherwise the decoupling is partial. In either case, the simplification is possible because we are, in fact, restricting the dynamics to the level surfaces of quasimonomial constants of motion.

To summarize, we have a multilevel structure of $(r, n, m)$-classes of equivalence, with $n$ ranging in the interval $r \leq n \leq m$. We can transform freely every GLV system inside this scheme by means of the QMTs and the two basic —and opposite— operations: $p$-embeddings and $p$-decouplings, which proceed by the introduction of quasimonomial first integrals, or by the restriction of the system dynamics to their level surfaces, respectively.
III. GLV FAMILIES OF POISSON SYSTEMS

We start by characterizing the systems of interest. In what follows, the superscript $^T$ denotes the transpose of a matrix.

**Theorem 3.1:** Let us consider a GLV system of the form (3) such that

$$\lambda = K \cdot L, \quad A = K \cdot B^T \cdot D,$$

with $K$, $L$ and $D$ matrices of real entries, where $K$ is $n \times n$ and skew-symmetric; $L$ is $n \times 1$; and $D$ is $m \times m$, diagonal and of maximal rank. Then the system has a constant of motion of the form:

$$H = \sum_{i=1}^{m} D_{ii} \prod_{k=1}^{n} x_{B_{ik}} + \sum_{j=1}^{n} L_j \ln(x_j)$$

Moreover, the system is Poisson with Hamiltonian $H$.

**Proof:** The GLV flow complies to the format $\dot{x} = J \cdot \nabla H$, where the Hamiltonian is smooth in the positive orthant and given by $H$ in (10), while $J$ is the smooth matrix

$$J = X \cdot K \cdot X, \quad X = \text{diag}(x_1, \ldots, x_n)$$

That $H$ is the Hamiltonian implies that it is a constant of motion. Q.E.D.

Notice that the first part of the Hamiltonian is associated to the $m$ quasi-monomials of the GLV vector field (in fact, it is a linear combination of them), while the logarithmic terms are closely connected to the linear contributions. The observation that a matrix of the form $X \cdot K \cdot X$ is a structure matrix iff $K^T = -K$ is due to Plank. From now on, we shall denote the systems described by Theorem 3.1 as GLV-Poisson (GLVP).

**Proposition 3.2:** The Poisson structure of GLVP systems is form-invariant under a QMT. After a QMT of matrix $C$, the characteristic matrices of the
transformed Poisson structure are:

\[ K' = C^{-1} \cdot K \cdot (C^{-1})^T, \quad L' = C^T \cdot L, \quad D' = D \]

In particular, both the Hamiltonian and the structure matrix are form-invariant under QMTs.

*Proof:* The simplest proof is the algebraic one. After a QMT we have:

\[ \lambda' = C^{-1} \cdot K \cdot L = C^{-1} \cdot K \cdot (C^{-1})^T \cdot C^T \cdot L = K' \cdot L' \]
\[ \mathcal{A}' = C^{-1} \cdot K' \cdot B^T \cdot D = K' \cdot (B')^T \cdot D' \]

and \( D' = D \). Then, from Theorem 3.1 the new system is also GLVP, and its structure matrix and Hamiltonian are, respectively, \( \mathcal{J}' = Y \cdot K' \cdot Y \), and

\[ H' = \sum_{i=1}^{m} D'_{ii} \prod_{k=1}^{n} y_k^{B'_{ik}} + \sum_{j=1}^{n} L'_{j} \ln(y_j) \]

This demonstrates the result. Q.E.D.

**Corollary 3.3:** The Poisson bracket of a GLVP system is form-invariant under QMTs.

There is an important degree of freedom in the Poisson structure: Let \( N \in \text{Ker}\{K\} \). Then, the GLVP system we obtain does not change if we replace \( L \) by \( L + N \), since from (9) we have \( \lambda = K \cdot L = K \cdot (L + N) \). This is, in fact, a source of ambiguity in the Hamiltonian itself, because the flow is unaltered if we add to \( H \) an extra term of the form:

\[ \phi_N = \sum_{j=1}^{n} N_j \ln(x_j), \quad N \in \text{Ker}\{K\} \quad (12) \]

This degree of freedom is precisely the one associated to the well-known Casimir functions:
Proposition 3.4: Let $r = \text{rank}(K)$ in a given GLVP structure. There is a complete set of $n - r$ functionally independent Casimir functions of the form (12).

Proof: Evidently, $\text{rank}(J) = \text{rank}(K) = \text{constant}$ everywhere in the positive orthant. Then there are exactly $n - \text{rank}(K)$ functionally independent Casimirs. If $N \in \text{Ker}\{K\}$ we have $J \cdot \nabla \phi_N = 0$, and all such $\phi_N$ are Casimirs. But $\dim(\text{Ker}\{K\}) = n - r$: Then we can get a maximal set of independent Casimirs by choosing $n - r$ linearly independent vectors of $\text{Ker}\{K\}$, i.e. a basis of $\text{Ker}\{K\}$. The functional independence can be readily verified in this case. Q.E.D.

Moreover, under a QMT of matrix $C$, $\phi_N$ is form-invariant and changes to $\phi_{N'}$, where $N' = C^T \cdot N$ and $N' \in \text{Ker}\{K'\}$. Thus $\phi_{N'}$ is a Casimir of the transformed system. Let $\{N^{(1)}, \ldots, N^{(n-r)}\}$ be a basis of $\text{Ker}\{K\}$, where $r = \text{rank}(K)$ as before. Then $\{\phi_{N^{(1)}}, \ldots, \phi_{N^{(n-r)}}\}$ is a complete set of Casimirs of the initial system. After a QMT of matrix $C$, the Casimirs are transformed into a new family of Casimirs $\{\phi_{N'^{(1)}}, \ldots, \phi_{N'^{(n-r)}}\}$, where $N'^{(i)} = C^T \cdot N^{(i)}$ for all $i$. In fact, the new set $\{N'^{(1)}, \ldots, N'^{(n-r)}\}$ is also a basis of $\text{Ker}\{K'\}$. Therefore, every QMT carries a complete set of independent Casimirs of the form (12) into its counterpart for the target system. In other words, the symplectic foliation of a GLVP system is a class property.

We can equivalently express the Casimir functions (12) in quasimonomial form as:

$$\phi_N = \prod_{j=1}^{n} x_j^{N_j}, \quad N \in \text{Ker}\{K\}$$

However, we have seen in Proposition 2.4 that the quasimonomial first integrals arise in a purely GLV context, independently of the existence of a Poisson structure of the system. We saw that they are associated to a degen-
eracy in the rank of $M$. Clearly, there must be a close relationship between quasimonomial Casimirs and quasimonomial first integrals in general. We shall now demonstrate that both sets do coincide:

**Theorem 3.5:** Every quasimonomial constant of motion of a GLVP system is a Casimir function.

**Proof:** Notice that a quasimonomial constant of motion can be expressed as:

$$\prod_{j=1}^{n} x_j^{N_j} = \text{constant}, \quad N \in \text{Ker}\{M^T\}$$

Therefore, our statement will be automatically demonstrated if we show that $\text{Ker}\{M^T\} = \text{Ker}\{A^T\}$.

First, we demonstrate that $\text{rank}(A) = \text{rank}(M)$ for GLVP systems, where $M = (\lambda \mid A)$. From (9), we can write symbolically $M = K \cdot (L \mid B^T \cdot D)$. Expressed in this form, it is immediate to see that $\text{rank}(A) = \text{rank}(M)$ by simple inspection. Consequently, $\text{Ker}\{M^T\} = \text{Ker}\{A^T\}$ for GLVP systems, and the theorem will be demonstrated if $\text{Ker}\{A^T\} = \text{Ker}\{K\}$.

It is evident that $\text{Ker}\{K\} \subset \text{Ker}\{A^T\}$. To show that $\text{Ker}\{A^T\} \subset \text{Ker}\{K\}$, note that $\text{rank}(D \cdot B) = n$, and therefore $\text{Ker}(D \cdot B) = \{0\}$. The result is then straightforward, and the theorem is demonstrated. Q.E.D.

**Corollary 3.6:** In every GLVP system, $\text{rank}(M) = \text{rank}(A) = \text{rank}(\mathcal{J}) = \text{rank}(K)$.

Theorem 3.5 will be of fundamental importance in what follows. It summarizes very well the interplay between algebraic and Poisson properties, which is present in GLVP systems. The level surfaces of the Casimir functions yield the global structure of the Poisson system—they constitute the symplectic foliation of the phase-space. We are now able to reconstruct this important feature in terms of a very simple and purely algebraic scheme.
which is summarized in the proof of Proposition 2.4. More concisely, we can say that the symplectic foliation of the system is condensed in the rank properties of the GLV matrix $M$. Conversely, purely algebraic properties of the system in the GLV context now assume a completely new role. This is the case of the quasimonomial first integrals, which now become Casimir functions. Moreover, this parallelism is not only valid for a single system, but it is a class property. Obviously, some logical consequences arise from this interplay: For example, $\text{rank}(M)$ may take any value between 0 and $n$ in a general GLV system. However, from Corollary 3.6 this must be an even number for GLVP systems, an evident restriction if the level surfaces of the quasimonomial first integrals are to be a symplectic foliation.

In Section II, we elaborated in some detail on the techniques for the manipulation of the quasimonomial constants of motion in a pure GLV context. These procedures involved inter-class transformations which might, if desired, eliminate all such first integrals. These ideas acquire completely new implications in the light of the last results: Such manipulations give now the key for restricting the dynamics to the symplectic leaves or, in the opposite sense, to embed the system in Poisson structures of higher dimensionality. We shall devote the next section to give a systematic treatment of these issues.

IV. TRANSFORMATIONS ON THE SYMПLECTIC FOLIATION

According to Section II, three basic procedures are those which allow the manipulation of GLV systems: QMTs, embeddings, and decouplings. In Section III, we have seen how QMTs preserve the GLVP structure. That
is, we have demonstrated that the GLVP structure is a class property. In this section, our aim will be to show that the inter-class operations do also preserve in an appropriate way the same GLVP character. This will complete the Poisson description of these systems and reinforce the parallelism between algebraic and Poisson properties.

We shall first establish the result for the embeddings:

**Proposition 4.1:** If a GLVP system which belongs to an \((r, n, m)\)-class, with \(m > n\), is subjected to a \(p\)-embedding, with \(1 \leq p \leq m - n\), then the resulting system is also GLVP.

**Proof:** Relations \(A = K \cdot B^T \cdot D\) and \(\lambda = K \cdot L\) in the original system, imply that \(\tilde{A} = \tilde{K} \cdot \tilde{B}^T \cdot \tilde{D}\) and \(\tilde{\lambda} = \tilde{K} \cdot \tilde{L}\) in the embedded vector field, where:

\[
\begin{align*}
\tilde{K} &= \begin{pmatrix} K & O_{n \times p} \\ O_{p \times n} & O_{p \times p} \end{pmatrix}, \\
\tilde{L} &= \begin{pmatrix} L \\ L^*_{p \times 1} \end{pmatrix}, \\
\tilde{D} &= D \cdot E
\end{align*}
\]

Here, \(L^*\) is composed of arbitrary entries, \(E\) is given by (6), and \(\tilde{A}, \tilde{B},\) and \(\tilde{\lambda}\) are those in (5). With the help of Theorem 3.1, this proves the result. Q.E.D.

**Corollary 4.2:** Under the same hypotheses of Proposition 4.1, all the members of the \((r, n + p, m)\)-class to which the expanded system belongs, are GLVP systems.

**Corollary 4.3:** Given a GLVP flow belonging to an \((r, n, m)\)-class, every \(m\)-dimensional LV representative of the system is also GLVP.

We shall demonstrate now that we have an analogous situation in the case of the decouplings. For this we need a preliminary result:

**Proposition 4.4:** Given a GLVP system of matrices \(M^*\) and \(K^*\), which belongs to an \((r, n + p, m)\)-class, with \(r \leq n\) and \(0 < p \leq m - n\), then there exists at least one QMT such that the transformed flow has matrices of the
form:

\[ M = \begin{pmatrix} \bar{M} & 0_{p \times (m+1)} \\ 0_{p \times (m+1)} & O_{p 	imes (m+1)} \end{pmatrix}, \quad K = \begin{pmatrix} \bar{K} & O_{n \times p} \\ O_{n \times p} & O_{p \times p} \end{pmatrix} \quad (13) \]

Proof: Let \( C \) be the matrix of the QMT. From the transformation rule

\[ K = C^{-1} \cdot K^* \cdot (C^{-1})^T \]

and the skew-symmetry of \( K \), it is clear that there exists an invertible \( C \) which recasts \( K^* \) in the desired way. The form of \( M \) is a consequence of the form of \( K \), since from (9) we have \( \lambda = K \cdot L \) and \( A = K \cdot B^T \cdot D \). Q.E.D.

This leads to the main result for reductions:

Proposition 4.5: If a GLVP system belonging to an \((r, n+p, m)\)-class, with \( r \leq n \) and \( 0 < p \leq m - n \), is subjected to a \( p \)-decoupling, then the resulting system is also GLVP.

Proof: From Proposition 4.4, we can first transform the GLVP system into another one in the same \((r, n+p, m)\)-class, with matrices like those in (13). We shall denote the rest of matrices of the latter as \( A, B, \lambda, L \) and \( D \) (\( A \) and \( \lambda \) being given by equation (7)). We shall also, for convenience, express \( B \) and \( L \) as composed of submatrices in the usual form \( B = ( \bar{B} \mid \bar{B}'_{m \times p} ) \), and \( L = ( \bar{L} \mid \bar{L}'_{p \times 1} )^T \). As we know from Section II, the matrices of the restricted system are \( \bar{B} = \bar{B}, \bar{\lambda} = \bar{\lambda}, \) and \( \bar{A} = \bar{A} \cdot E^{-1} \), where \( E \) is given by (8). It is not difficult to check that the relations \( A = K \cdot B^T \cdot D \) and \( \lambda = K \cdot L \), imply that \( \bar{A} = \bar{K} \cdot \bar{B}^T \cdot \bar{D} \) and \( \bar{\lambda} = \bar{K} \cdot \bar{L} \) in the reduced flow, where:

\[ \bar{K} = \bar{K}, \quad \bar{L} = \bar{L}, \quad \bar{D} = D \cdot E^{-1} \]

This proves the result. Q.E.D.

Therefore, not only the GLV format itself, but also the proper GLVP structure is preserved when a reduction is carried out. This is consistent with the fact that quasimonomial constants of motion are Casimirs of the Poisson structure. We can then state:
Corollary 4.6: If a GLV system of an \((r, n, m)\)-class is GLVP, then all the \((r, k, m)\)-classes, for \(k = r, \ldots, m\), which can be reached by means of successive embeddings and/or decouplings, are composed of GLVP systems.

Consequently, all the transformations that we have defined within and among the classes preserve both the algebraic GLV properties of the equations and the Poisson format. A decoupling amounts to restricting the system totally or partially to the level surfaces of the Casimirs. An embedding adds new Casimirs to the system, thus increasing its dimensionality. The dynamics on the symplectic leaves remains, on the other hand, always intact. Knowing how to increase the dimension by means of embeddings is important, because it connects every class with the LV format —and vice versa in the case of decouplings. In particular, this tells us how an LV system can be simplified.

In the case in which we perform a maximal decoupling, we obviously arrive at an \((r, r, m)\)-class of GLVP systems, with \(r\) even: The members of this class are symplectic systems, since all the Casimirs have been removed. The previous results not only allow, however, the mere transformation of the vector field into symplectic form: They can also be used to reach the full reduction to the canonical forms of Darboux or Hamilton. This is the issue of the next section.

V. REDUCTION TO DARBOUX’ CANONICAL FORM

We detail here three ways for constructing the Darboux’ canonical form. We shall first address the more general approach, and then comment on two more specific possibilities.
A. General method

Proposition 5.1: Given a GLVP system belonging to an \((r, n, m)\)-class, there exists a quasimonomial transformation such that for the transformed system:

\[ K' = S(r, n - r) = \text{diag}(S_1, S_2, \ldots, S_{r/2}, 0, \ldots, 0), \quad (14) \]

where

\[ S_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \forall i = 1, \ldots, r/2 \]

Proof: From Corollary 3.6, \(\text{rank}(K) = r\) for the original system. After a QMT of matrix \(C\), we have from Proposition 3.2 that \(K' = C^{-1} \cdot K \cdot (C^{-1})^T\), which is a congruence transformation over \(K\). But, since \(K\) is skew-symmetric and of rank \(r\), it is congruent to a matrix of the form (14). Then, the QMT exists. Q.E.D.

We can now state the following result:

Theorem 5.2: Every GLVP system belonging to an \((r, n, m)\)-class can be globally reduced to Darboux’ canonical form inside the positive orthant.

Proof: The proof is constructive. We shall assume that the system has been already transformed in such a way that its matrix \(K\) complies to format (14). We can then introduce the following transformation, which is a global orientation-preserving diffeomorphism inside the positive orthant:

\[ y_i = \ln(x_i), \quad i = 1, \ldots, n \quad (15) \]

We now take into account the equation for the transformation of the structure matrix under general diffeomorphisms, \(y_i = y_i(x)\):

\[ (\mathcal{J}')(ij)(y) = \sum_{k,l=1}^{n} \frac{\partial y_i}{\partial x_k} \mathcal{J}_{kl}(x) \frac{\partial y_j}{\partial x_l} \quad (16) \]
The result, upon applying (15), is:

\[(\mathcal{J}')(ij) = (\mathcal{S}(r, n - r))_{ij}, \quad \forall \ i, j = 1, \ldots, n\]

This transforms the original GLVP into a non-GLVP system that conforms, however, to Darboux’ form: There are obviously \(r/2\) pairs of canonically conjugate variables, and \((n - r)\) trivial Casimirs. The transformation of the Hamiltonian is straightforward. This proves the result. Q.E.D.

B. Decoupling method

Notice that in the case \(r < n\), i.e. when the GLVP system is not symplectic, we can make use of the reduction procedure of the previous section, instead of applying Theorem 5.2 from the very beginning. The result would be another GLVP system, now symplectic, which belongs to an \((r, r, m)\)-class. Making use of Theorem 5.2 on the reduced flow, would lead to a purely Hamiltonian system, since

\[\mathcal{J}' = \mathcal{S}(r, 0) ,\]

which is the classical symplectic matrix of dimension \(r\).

C. Linear transformation method

Assume that we first subject the initial system of an \((r, n, m)\)-class and matrix \(K\), to transformation (15). From (16), we find:

\[\mathcal{J}' = K,\]

The resulting system \(\dot{y} = K \cdot \nabla H'(y)\) is not GLVP, though it is Poisson. A linear change of variables \(w = C \cdot y\), where \(C\) is an invertible \(n \times n\) matrix,
leads to another Poisson system of constant structure matrix,

\[ \dot{w} = (C \cdot K \cdot C^T) \cdot \nabla H''(w) \]

Finally, there exists a \( C \) such that \( C \cdot K \cdot C^T = S(r, n - r) \), and Darboux’ form is achieved. This procedure has already been applied in the literature to certain symplectic LV systems, of even-dimension and with a single fixed point.\(^{37}\)

**VI. EXAMPLE: 3D LOTKA-VOLTERRA EQUATIONS**

**A. Poisson structure**

As an illustration of the previous results, we shall look upon the 3D LV Poisson structure first characterized by Nutku.\(^{27}\) The flow is given by the equations:

\[
\begin{align*}
\dot{x}_1 &= x_1(\rho + cx_2 + x_3) \\
\dot{x}_2 &= x_2(\mu + x_1 + ax_3) \\
\dot{x}_3 &= x_3(\nu + bx_1 + x_2)
\end{align*}
\]

As Nutku has pointed out, this is a Poisson system if

\[ abc = -1, \quad \nu = \mu b - \rho ab \]

In this case, the structure matrix and the Hamiltonian are, respectively:

\[
J = \begin{pmatrix}
0 & cx_1 x_2 & b c x_1 x_3 \\
-c x_1 x_2 & 0 & -x_2 x_3 \\
-b c x_1 x_3 & x_2 x_3 & 0
\end{pmatrix},
\]

obeying to form (11), and

\[ H = ab x_1 + x_2 - a x_3 + \nu \ln(x_2) - \mu \ln(x_3) \]
which is of the form (10). The system is thus GLVP with characteristic matrices:
\[ B = I_{3 \times 3}, \quad M = \begin{pmatrix} \rho & 0 & c & 1 \\ \mu & 1 & 0 & a \\ \nu & b & 1 & 0 \end{pmatrix}, \]
\[ K = \begin{pmatrix} 0 & c & bc \\ -c & 0 & -1 \\ -bc & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} ab & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad L = \begin{pmatrix} 0 \\ \nu \end{pmatrix} \]
Notice that \( \text{rank}(M) = \text{rank}(A) = \text{rank}(K) = \text{rank}(J) = 2 \) inside the positive orthant of \( \mathbb{R}^3 \). There is then one independent Casimir function. By noting that, in \( M \), \( \text{row}(3) = (1/c) \times \text{row}(1) + b \times \text{row}(2) \), we immediately find the quasimonomial first integral:
\[ x_1^{ab} x_2^{-b} x_3 = \text{constant}, \]
which is also a Casimir of the Poisson structure from Theorem 3.5. This way of recovering Casimir functions of the system is certainly more economic than solving the PDE \( J \cdot \nabla \phi = 0 \), which is the usual approach.

B. Darboux’ form: General method

Let us now subject the system to a QMT of matrix:
\[ C = \begin{pmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 1 & b & -1 \end{pmatrix} \]
We arrive to a new GLVP of matrices:
\[ B' = \begin{pmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 1 & b & -1 \end{pmatrix}, \quad M' = \begin{pmatrix} \rho/c & 0 & 1 & 1/c \\ \mu & 1 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ K' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D' = \begin{pmatrix} ab & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad L' = \begin{pmatrix} -\mu \\ \rho/c \end{pmatrix} \]
The Casimir function has been decoupled, and now is just $x'_3$. A change of variables $y_i = \ln(x'_i)$, $i = 1, 2, 3$ yields Darboux’s form, with:

$$
\mathcal{J}(y) = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

and

$$
H(y) = abe^{cy_1} + e^{y_2} - ae^{y_1 + by_2 - y_3} - \mu y_1 + (\rho/c)y_2 + \mu y_3
$$

**C. Darboux’ form: Decoupling method**

Although the previous one is the shortest way to achieve the transformation into Darboux’ form, it may be sometimes more convenient to proceed in a two-step alternative: The first step is the transformation of the system into a symplectic flow. This might be more appropriate in systems of higher dimensions, in which an initial reduction of the dimensionality of the problem may produce the most manageable system. We shall briefly display it for the sake of illustration.

We can first make a QMT of matrix:

$$
C_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1/c & b & -1
\end{pmatrix}
$$

If we then decouple the third variable, assuming for simplicity that its initial condition is $x'_3(0) = 1$, the reduced GLVP system is given by:

$$
\hat{B}' = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
1/c & b
\end{pmatrix}, \quad \hat{M}' = \begin{pmatrix}
\rho & 0 & c & 1 \\
\mu & 1 & 0 & a
\end{pmatrix}, \quad \hat{K}' = \begin{pmatrix}
0 & c \\
-c & 0
\end{pmatrix}, \quad \hat{D}' = \begin{pmatrix}
ab & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -a
\end{pmatrix}, \quad \hat{L}' = \begin{pmatrix}
-\mu/c \\
\nu - \mu b
\end{pmatrix}
$$
We now perform a second QMT, this time acting on the reduced \((2, 2, 3)\)-class:

\[
C_2 = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}
\]

In the resulting flow, we have:

\[
\hat{K}'' = S(2, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

After the final change of variables \(y_i = \ln(x_i'')\), \(i = 1, 2\), we arrive at a Hamiltonian system in which:

\[
H(y) = abe^{cy_1} + e^{y_2} - ace^{y_1} + b y_2 - \mu y_1 + (\rho/c)y_2
\]

Notice that this Hamiltonian can be obtained from (18) with \(y_3 = 0\). This is due to the initial condition we have assumed for \(x_3'\) in the 1-decoupling. We obviously retrieve, up to trivial differences in form, the Darboux system (17-18).

**VII. FINAL REMARKS**

We have seen that there is a close parallelism between the Poisson structure of GLVP flows and the algebraic properties of GLV equations. The deep and unexpected interplay between both aspects of the systems results in an economy in their description. It also establishes an operational framework for their manipulation and simplification. This is, to our knowledge, a novel approach to the treatment of finite-dimensional Poisson structures.

We end this work by giving an evaluation in relation to what has been done previously in the literature. Unfortunately, the only way for doing this is by particularizing the comparison to LV models, for which earlier results
are available. We shall only consider previous approaches which are valid for \( n \)-dimensional LV models, for arbitrary \( n \). We can then say that most Hamiltonian and Poisson LV systems treated in the literature have a matrix \( A \) which is of maximal rank, i.e., they have a single fixed point.\(^6\)\(^{30}\)\(^{37}\) Also, they are often restricted to even dimensionality,\(^30\)\(^{37}\) which usually entails that the system is symplectic (these are, of course, two requirements which are implicit in the classical Hamiltonian studies). None of these restrictions is present, as we have seen, in our models. On the other hand, our treatment joins previous works in what concerns certain requirements on matrix \( A \): For a GLVP Lotka-Volterra system, we have that \( A = K \cdot D \), where \( K \) is skew-symmetric. This implies, as it can be readily seen, that \( D_{ii}A_{ij} = -D_{jj}A_{ji} \), which is exactly the same kind of generalized skew-symmetry which can be found in the works of Kerner\(^37\) and some cases from Plank,\(^6\) for example. Therefore, the scope of our treatment does not differ, in this sense, to that of previous ones. Notice also how our Hamiltonian (10) reduces, in the case of LV systems, to a generalization of the classical Volterra’s constant of the motion:

\[
H_V = \sum_{i=1}^{n} \beta_i (x_i - p_i \ln(x_i)) ,
\]

where \( p_i \) are the coordinates of the (unique) fixed point of Volterra’s systems.

Another interesting issue which we would like to comment here concerns the use of an arbitrary Hamiltonian, while retaining the form (11) for the structure matrix. This leads, of course, to the generation of a wide range of Poisson systems. This procedure can be found, for example, in Plank’s work.\(^6\) We may mention that many of our previous results still hold in this Hamiltonian-independent situation. This is the case, for instance, in the reduction of the system to the Darboux’ form. The reason is that the criterion
to decide whether a system complies to Darboux’ format or not, relies on the form of the structure matrix, exclusively. Consequently, the manipulations to which the system is to be subjected concern the recasting of $J$ in the desired form, $H$ being irrelevant for that case —which is the situation in Section V.

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