Renormalization Ambiguities in Chern-Simons Theory

M. Asorey, F. Falceto, J. L. López and G. Luzón

Departamento de Física Teórica. Facultad de Ciencias
Universidad de Zaragoza. 50009 Zaragoza Spain

Abstract

We introduce a new family of gauge invariant regularizations of Chern-Simons theories which generate one-loop renormalizations of the coupling constant of the form $k \to k + 2sc_v$ where $s$ can take any arbitrary integer value. In the particular case $s = 0$ we get an explicit example of a gauge invariant regularization which does not generate radiative corrections to the bare coupling constant. This ambiguity in the radiative corrections to $k$ is reminiscent of the Coste-Lüscher results for the parity anomaly in (2+1) fermionic effective actions.

PACS numbers, 11.10 Kk, 11.15.-q, 11.25.Hf
Keywords: Chern-Simons Theories, Topological Field Theories.
Although the Chern-Simons theory is exactly solvable in the canonical formalism, there are still many interesting open questions related to the behavior of the theory in arbitrary three-dimensional manifolds which can only be answered in the covariant formalism. In particular, Witten’s conjectures on the connection of expectation values of gauge invariant observables with topological invariants in arbitrary three-dimensional manifolds can only be formulated in a covariant formalism. Some of the conjectures have been proven by different methods such as surgery of manifolds, operator formalism or Wess-Zumino-Witten model approaches with a few additional assumptions. However, from a field theoretical point of view is still a challenge to prove those conjectures by functional integral methods. One of the major problems of this approach is the presence of ultraviolet divergences which require the introduction of a regularization. All gauge invariant regularization methods considered so far yield a finite radiative correction to the coupling constant $k$ of the form $k \to k + c_v$. This fact has been interpreted as an explanation of the appearance of the expression $k + c_v$ in many formulae in representation theory of the corresponding Kac-Moody algebra. However, we will see in this paper that the claims about the universality of this correction do not hold in more general regularization schemes.

Most of the gauge invariant regularizations introduce scalar self-interacting terms of Yang-Mills type for gluons while keep the scalar character of ghost-gauge field interactions. In the regularizations of this type the effective coupling constant $k$ gets a finite radiative correction of the form $k + c_v$ at one loop and nothing else at two loops. This phenomenon indicates that the effective values of $k$ are very strictly constrained and suggests that this behavior might be universal.

From a pure perturbative viewpoint any restriction on the renormalization of a coupling constant has to be understood in terms of symmetry arguments. In this case, however, there is not such a symmetry to explain the behavior of the quantum correction to $k$. The invariance of the bare action under large gauge transformations implies that $k$ has to be an integer, but this fact does not imposes any restriction on the value of the renormalized coupling constant (see Ref. for a general discussion of the problem in Chern-Simons theory).

In this note we address the question of whether the above behavior is universal or it is regularization dependent. In particular, we find out the existence of ambiguities in the renormalization of the Chern-Simons coupling constant which depend on the regularization scheme. We shall introduce a new family of continuum regularizations of the theory by
adding to the Chern-Simons action new higher covariant derivative pseudoscalar terms and some Pauli-Villars regulators with scalar and pseudoscalar couplings. The regularizations are inspired by the Coste-Lüscher method of analyzing the ambiguities which appear in the effective action of fermionic determinants [11]. Regularizations with similar characteristics can be obtained by geometric regularization, but for the sake of simplicity we shall not consider them here (see [10] for details).

The Chern-Simons action

$$S_0(A) = \frac{ik}{4\pi} \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$ (1)

is invariant under diffeomorphism transformations of the base manifold $M$ and global gauge transformations, provided the coupling constant $k$ is an integer $k \in \mathbb{Z}$. In the functional integral quantization approach there are several possible sources of symmetry breaking. First, to define the functional integral it is necessary to introduce a functional volume element $[\delta A]$ which depends on the Riemannian structure of $M$. Another possible source of symmetry breaking is the gauge fixing condition. A simple BRST analysis shows that both symmetries, gauge and diffeomorphism invariance, can be preserved at the quantum level, at the price of generating a framing anomaly [1][3]. From a less formal point of view there is however another source of symmetry breaking. The existence of ultraviolet divergences in the covariant formalism makes necessary the introduction of a regularization, and depending on the type of regularization some of the classical symmetries may be broken.

Since both symmetries cannot be simultaneously preserved by an ultraviolet regularization we will try to preserve only one of them, namely, gauge invariance.

Most of the regularization methods based on higher covariant derivatives introduce Yang-Mills like terms into the regularized action which modify the pseudoscalar character of the original Chern-Simons action. Here we shall consider a regularization by higher covariant derivatives which preserves this peculiarity of the Chern-Simons action.

Let $\ast$ denote the Hodge operator associated to the oriented Riemannian structure of $M$, and $d_A$ and $\Delta_A = d_A^*d_A + d_Ad_A^*$ the covariant exterior differential and the covariant laplacian operator, respectively. The novelty of the present regularization is that the regularized action

$$S_\Lambda(A) = S_0(A) - \frac{ik}{8\pi\Lambda^2} \left( \ast F(A), (1 + \frac{\Delta_A}{\Lambda^2})^n \ast d_A(1 + \frac{\Delta_A}{\Lambda^2})^n \ast F(A) \right)$$ (2)
contains only pseudoscalar couplings like the original Chern-Simons action. The bracket \((\alpha, \beta)\) of the regulating term in (2) denotes the inner product defined by
\[
(\alpha, \beta) = -2 \int_M \text{tr} \alpha \wedge * \beta
\]
in the space of forms \(\alpha, \beta\) taking values in the Lie algebra of the (simple and compact) structure group \(G\). The new pseudoscalar term was already introduced in Ref. [7] for a different purpose. It is well known however that the method of higher covariant derivatives does not get rid of one loop divergences which have to be eliminated by an additional Pauli-Villars regularization.

The final expression of the regularized functional integral in the Landau gauge \(d^* A = 0\)
\[
Z_A = \int [\delta A] [\delta \phi] [\delta \bar{\psi}] [\delta \psi] [\delta c] [\delta \bar{c}] \exp\{-S_{reg}(A, \bar{c}, c, \phi, \bar{\psi}, \psi)\},
\]
is given in terms of the effective action
\[
S_{reg}(A, \bar{c}, c, \phi, \bar{\psi}, \psi) = S_A(A) + (\bar{c}, d^* d_A c) + \frac{1}{2} \sum_{i=1}^{M_b} (\phi_i, [I + \frac{\lambda_i}{A^2} \Delta_A] m_i \phi_i)
\]
\[
+ \sum_{j=1}^{M_f} (\bar{\psi}_j, [I + \frac{\mu_j}{A^2} \Delta_A] n_j \psi_j).
\]
where besides the Faddeev-Popov ghosts \(\bar{c}, c\) there appear bosonic Pauli-Villars ghosts \(\phi_i\) and grassmanian fermionic ghosts \(\psi_j\) and \(\bar{\psi}_j\). The interactions of the fields \(\phi_i, \psi_j\) and \(\bar{\psi}_j\) with the gauge fields differ from gluon selfinteractions, and therefore the cancellation of the ultraviolet divergences does not proceed as in conventional Pauli-Villars method. In particular, the finite radiative corrections may depend on the prescription used to calculate each diagram. If the manifold \(M\) is a three-dimensional torus, a very natural unambiguous prescription is to introduce an auxiliary momentum pre-cutoff \(|p| < \Omega\) for all propagating modes. The only problem with this prescription is that it breaks gauge invariance and it is therefore necessary to impose some subsidiary conditions to ensure that it is restored when the pre-cutoff is removed \((\Omega \to \infty)\) [12].

The only divergent contributions arise in the two-point function \(\Gamma^{ab}_{\mu \nu}(q)\) and can be cancelled by an appropriate choice of the Pauli-Villars regulators. We will restrict ourselves to the case of a three-dimensional symmetric flat torus \(M = T^3\) with volume \(L^3\). The leading contribution in the large \(L\) limit is \(L\) independent and can be estimated by replacing the infinite sums of propagating momentum modes by conventional integrals.
The radiative corrections to the two-point function generated by one loop of gluons (diagrams (1) and (2) of Fig. 1) are given by

\begin{align}
\Gamma^{(1)}_{\mu\nu}(q) &= \frac{c_v}{6\pi^2} (16n^2 + 24n + 9) \Omega \delta^{ab} \delta_{\mu\nu} \\
&\quad + 2\frac{c_v}{3\pi^2} \Lambda I(n) \delta^{ab} \delta_{\mu\nu} \\
&\quad - \frac{c_v}{32\pi^2} \left[ (4n + 3)^2 + \frac{\pi^2}{2} \right] \delta^{ab} \left( |q| \delta_{\mu\nu} + \frac{q_\mu q_\nu}{|q|} \right) \\
&\quad + O(\Omega^{-1}, L^{-1}, \Lambda^{-1}),
\end{align}

and

\begin{align}
\Gamma^{(2)}_{\mu\nu}(q) &= -\frac{c_v}{3\pi^2} (8n^2 + 14n + 5) \Omega \delta^{ab} \delta_{\mu\nu} \\
&\quad - \frac{2c_v}{3\pi^2} \Lambda I(n) \delta^{ab} \delta_{\mu\nu} \\
&\quad + O(\Omega^{-1}, L^{-1}, \Lambda^{-1}),
\end{align}

with

\[ I(n) = \int_0^\infty dp \frac{[5 + 2(5 + 9n)p^2]p^2 (1 + p^2)^{2n-2}}{1 + p^2(1 + p^2)^{2n}}. \]

Part of the finite contribution which depends on $|q|$ comes from the existence of the pre-cutoff $\Omega$ in every propagator $|p| < \Omega$, $|p + q| < \Omega$ of diagram (1) of Fig. 1.

**Figure 1.** One loop Feynman diagrams contributing to the 2-point function involving gluon loops.
The contributions to the same proper function generated by the Faddeev-Popov ghosts are given by

$$\Gamma_{\mu\nu}^{ab}(q) = -\frac{c_v}{6\pi^2} \Omega \delta^{ab} \delta_{\mu\nu}$$

$$+ \frac{c_v}{32\pi^2} \left[ 1 + \frac{\pi^2}{2} \right] \delta^{ab} \left( |q|\delta_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{|q|} \right) +$$

$$+ O(\Omega^{-1}, L^{-1}, \Lambda^{-1}).$$  \hspace{1cm} (8)

In a similar way the contributions of bosonic Pauli-Villars ghosts $\phi_i$ read

$$\Gamma_{\mu\nu}^{ab}(q) = -\frac{c_v}{3\pi^2} \Omega m_i^2 \delta^{ab} \delta_{\mu\nu}$$

$$- \frac{c_v}{4\pi} \Lambda m_i^2 \lambda_i^{-\frac{1}{2}} \delta^{ab} \delta_{\mu\nu}$$

$$- \frac{c_v}{16\pi^2} m_i^2 \delta^{ab} \left( |q|\delta_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{|q|} \right)$$

$$+ O(\Omega^{-1}, L^{-1}, \Lambda^{-1}),$$  \hspace{1cm} (9)

and

$$\Gamma_{\mu\nu}^{ab}(q) = -\frac{c_v}{6\pi^2} m_i (2m_i + 1) \Omega \delta^{ab} \delta_{\mu\nu}$$

$$+ \frac{c_v}{4\pi} \Lambda m_i^2 \lambda_i^{-\frac{1}{2}} \delta^{ab} \delta_{\mu\nu}$$

$$+ O(\Omega^{-1}, L^{-1}, \Lambda^{-1}).$$  \hspace{1cm} (10)

The Pauli-Villars fermionic ghosts $\psi_j$ and $\bar{\psi}_j$ give a similar contribution with an additional factor $(-2)$.

Therefore the cancellation of linear divergences requires that

$$2 \sum_{j=1}^{M_f} n_j - \sum_{i=1}^{M_b} m_i - 1 - (4n + 1) = 0.$$  \hspace{1cm} (11)

The finite terms generated by the auxiliary pre-cutoff regularization are not transverse as required by Slavnov-Taylor identities. Therefore, the preservation of those identities imposes more constraints on the exponents of the Pauli-Villars regulators,

$$4 \sum_{j=1}^{M_f} n_j^2 - 2 \sum_{i=1}^{M_b} m_i^2 + 1 - (4n + 3)^2 = 0.$$  \hspace{1cm} (12)

It is easy to see that once the condition (11) is satisfied, there are not further divergences provided that $2n - n_j + 1 > 0$ and $2n - m_i + 1 > 0$ for every $i = 1, \cdots M_b$ and $j = 1, \cdots M_f$. These conditions guarantee the absence of subdivergences associated to the
two-point functions of the different ghost fields because the ultraviolet behavior of the
gluon propagator compensates the singularities associated to ghost-gluon interactions. On
the other hand there are not divergences associated to three-point functions. In this
case the potentially logarithmic divergent terms cancel by algebraic reasons and the final
contribution remains finite. In the same way, it is easy to verify that once the condition
(12) is satisfied all the other Slavnov-Taylor identities of the theory are also satisfied.
The reason is that only linear divergences generate anomalous contribution to the Green’s
functions. Anomalous contributions arise only in the Slavnov-Taylor identities which state
the transversality of the two gluon function and was discussed above, and in the identity
which relates the two-point and three-point gluonic functions, which also vanishes if the
condition (12) is satisfied [12]. Of course there always exist an infinite number of solutions
of the constraints (11)(12) but in order to keep only local interactions all the exponents
of the Pauli-Villars interacting terms should be integer numbers. There is however a large
number of solutions satisfying all these requirements. The simplest one $m_1 = 2; m_i =
1, i = 2, \cdots , 8n^2 + 4n + 1; n_j = 2, j = 1, \cdots , 2n^2 + 2n + 1$, is parametrized by an arbitrary
positive integer $n > 0$.

It is obvious from the above calculations that the one-loop effective action does not
contains any pseudoscalar term and therefore in this regularization scheme there are not
perturbative corrections to the wave function normalization or the Chern-Simons coupling
constant. Therefore, we have found a gauge invariant regularization which does not yield
any renormalization effect on the coupling constant at one loop level. In particular, this
implies that the symmetry arguments do not enforce universality in the renormalization
of $k$. This behavior is similar to the one obtained in Ref. [13] by means of a regularization
which is not gauge invariant and presumably the non-renormalization of $k$ is also preserved
up to two loops in the present scheme.

We remark that although the above regularization method is correct from a perturba-
tive viewpoint (the correlations functions are finite and satisfy Slavnov-Taylor identities),
it might not have a non-perturbative meaning because the regularized action is imaginary
and therefore the functional measure is not damped enough for large field configurations.
For such a reason, we shall restrict ourselves to a pure perturbative analysis.

Once we have shown that there is not universality in the renormalization of $k$, we
proceed to generalize the regularization scheme to obtain different renormalizations of
$k$. This generalization can be implemented by analogy with the observation made by
Lüscher and Coste about the existence of a similar ambiguity in the effective action of
2 + 1 massless fermions in the presence of a background gauge field [11]. They showed that the coefficient of the parity anomalous Chern-Simons term of the effective action is determined modulo a shift by $2\pi s$, associated to the number of ghosts fields $s$ involved in a Pauli-Villars regularization or a parameter in different lattice regularizations. In our case this ambiguity can be obtained by means of new Pauli-Villars ghost fields interacting with gauge fields by pseudoscalar couplings. The regularized action in this case would be given by

$$S'_{\text{reg}}(A, \bar{c}, c, \phi, \bar{\psi}, \psi, \xi, \chi, \chi) = S_{\text{reg}}(A, \bar{c}, c, \phi, \bar{\psi}, \psi) + \frac{1}{2} \sum_{r=1}^{N_b} (\xi_r, [I + i\frac{1}{\Lambda} * d_A] \xi_r)$$

$$+ \sum_{s=1}^{N_f} (\bar{\chi}_s, [I + i\frac{\eta_s}{\Lambda} * d_A] \chi_s)$$

(13)

where the new ghost fields $\xi$ are bosonic, whereas $\bar{\chi}$ and $\chi$ are Grassmannian.

The contribution of the new bosonic ghost fields $\xi$ to the two-point function

$$\Gamma^{\nu\rho}_{\mu\sigma}(q) = -\frac{c_v l_i}{6\pi^2} \Omega \delta^{\nu\rho} \delta_{\mu\sigma} - \frac{c_v l_i^2}{12\pi^2 \Lambda^2} \Omega \delta^{\nu\rho} (q^2 \delta_{\mu\nu} - q_{\mu} q_{\nu})$$

$$+ \frac{c_v l_i}{4\pi^2 \Lambda} \Omega \delta^{\nu\rho} \epsilon_{\mu\nu\rho\sigma} + \frac{c_v}{16\pi^2} \delta^{\nu\rho} \left( |q| \delta_{\mu\nu} - q_{\mu} q_{\nu} \right)$$

$$- \frac{c_v l_i}{4\pi |l_i|} \delta^{\nu\rho} \epsilon_{\mu\nu\rho\sigma} + O(\Omega^{-1}, L^{-1}, \Lambda^{-1}),$$

(14)

presents two new types of divergent terms.

Further divergent contributions also arise in the three-point and four-point functions,

$$\Gamma^{abc}_{\mu\nu\sigma}(q, r) = -\frac{ic_v l_i}{12\pi^2 \Lambda} \Omega f^{abc} \epsilon_{\mu\nu\sigma} + \frac{ic_v l_i}{12\pi |l_i|} f^{a\sigma d} \epsilon_{\mu\nu\sigma}$$

$$- \frac{ic_v l_i^2}{36\pi^2 \Lambda^2} \Omega f^{abc} ((2q_{\nu} + r_{\nu}) \delta_{\mu\sigma} - (q_{\mu} + 2r_{\mu}) \delta_{\nu\sigma} - (q_{\sigma} - r_{\sigma}) \delta_{\mu\nu})$$

$$+ O(\Omega^{-1}, L^{-1}, \Lambda^{-1}),$$

(15)

$$\Gamma^{abcd}_{\mu\nu\gamma\sigma}(q, r, s) = \frac{c_v l_i^2}{144\pi^2 \Lambda^2} \Omega \left[ \delta^{ab} \delta^{cd} (\delta_{\mu\gamma} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\gamma} - 2\delta_{\mu\nu} \delta_{\sigma\gamma}) + \delta^{ad} \delta^{bc} (\delta_{\mu\gamma} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\gamma} - 2\delta_{\mu\nu} \delta_{\sigma\gamma}) \right]$$

$$+ O(\Omega^{-1}, L^{-1}, \Lambda^{-1}),$$

(16)

because of the very singular ultraviolet behavior of the longitudinal part of propagators of the new Pauli-Villars fields. Similar divergent contributions are generated by the new
fermionic ghost fields $\chi$. In fact, the contribution of those fields only differ by a factor $(-2)$ of the contributions (14) - (16) of the $\xi$ fields. Notice, however, that there are not further divergences in the contribution of higher point functions.

Thus, if besides the normal condition for the cancellation of linear divergences (11) we impose the additional conditions

$$2N_f = N_b \quad 2 \sum_{s=1}^{N_f} \eta_s - \sum_{r=1}^{N_b} l_r = 0 \quad 2 \sum_{s=1}^{N_f} \eta_s^2 - \sum_{r=1}^{N_b} l_r^2 = 0$$

(17)

on the new Pauli-Villars regulators all new types of divergences cancel out. The dependence of the Pauli-Villars conditions (17) on the inverse of the masses, $l_i/\Lambda$ and $\eta_i/\Lambda$, of the regulating fields is due to the the fact that the leading ultraviolet behavior of the new propagators is governed by the longitudinal modes, which behave as $O(1)$ in the large $q$-momentum limit.

On the other hand, there is no change on the condition (12) which preserves Slavnov-Taylor identities because the new regulating fields are Pauli-Villars fields and once the finiteness conditions (17) are satisfied their contribution is gauge invariant. Since there are not further constraints on the values of the parameters $l_i$ and $\eta_i$ any of the infinite real solutions of equations (17) can be used to complete the definition of the regularization.

Collecting all the new finite contributions to the pseudoscalar part of the effective action we get

$$\frac{c_v}{k} \left\{ \sum_{r=1}^{N_b} \frac{l_r}{|l_r|} - 2 \sum_{s=1}^{N_f} \frac{\eta_r}{|\eta_r|} \right\} S_0(A).$$

(18)

This implies that the renormalized coupling constant $k$ is shifted by an even multiple of the dual Coxeter number $c_v$,

$$k + 2sc_v,$$

(19)

where $2s$ is the even integer defined by the difference of sums of $\pm 1$ in (18). A renormalization with a shift by an odd multiple of $c_v$ can be induced by means of a slight (geometric) generalization of the above regularization. In any case, this result shows that in Chern-Simons theory the degree of ambiguity in the renormalization of the coupling constant is at least as large as in the case of the fermionic parity anomaly. On the other hand, we remark the absence of renormalization of the wave function, which is in contrast with the results obtained by other methods involving Yang-Mills terms with higher covariant derivatives.
In spite of the existence of the ambiguity which breaks the universality properties of the renormalization, it is quite remarkable that the integer character of the coupling constant is always preserved by the above quantum corrections. This fact is not a consequence of the Slavnov-Taylor identities, which are related to invariance under infinitesimal gauge transformations. The analysis of this phenomenon deserves further investigation (see [10]).

Acknowledgements: J. L. L. was supported by a MEC fellowship (FPI program) and G.L. by a CONAI (DGA) fellowship. We also acknowledge to CICYT for partial financial support under grant AEN93-219.
References

[1] E. Witten, Commun. Math. Phys. 121 (1989) 351
[2] S. Axelrod, I.M. Singer, Proceedings of the XXth International Conference on Differential Geometrical Methods in Theoretical Physics, S. Catto and A. Rocha eds., World Sci. (1991)
[3] D. Freed, R. Gompf, Commun. Math. Phys. 141 (1991) 79
[4] D. Bar-Natan, E. Witten, Commun. Math. Phys. 141 (1991) 423
[5] W. Chen, G. W. Semenoff, Y.-S. Wu, Mod. Phys. Lett. A5 (1990) 1833
[6] L. Alvarez-Gaumé, J.M.F. Labastida, A.V. Ramallo, Nucl. Phys. B334 (1990) 103
[7] M. Asorey, F. Falceto, Phys. Lett. B 241 (1990) 31
[8] C.P. Martin, Phys. Lett. B 241 (1990) 513
[9] G. Giavarini, C.P. Martin, F. Ruiz Ruiz, Nucl.Phys. B381(1992) 222
[10] M. Asorey, F. Falceto, J. L. López, G. Luzón, DFTUZ-93.10 preprint
[11] A. Coste, M. Lüscher, Nucl.Phys. B323 (1989) 631
[12] M. Asorey, F. Falceto, Int. J. Mod. Phys.7(1992) 235
[13] E.Guadagnini, M. Martellini, M. Mintchev, Phys.Lett. B 227 (1989) 111