An approximate method for evaluating the fracture process zone near mode II dynamic crack tip

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Abstract. Fracture process zone is the key for understanding the nonlinear fracture of quasi-brittle materials. Evaluation of the shape and size of fracture process zone near mode II dynamic crack tip is still a problem unsolved completely at present. The analytical methods for study on dynamic crack like integral transformation method and Wiener-hopf method are not easy to be used to calculate the stress fields near dynamic crack tip. So far the studies on fracture process zone near dynamic crack tip mostly focused on the experimental and numerical simulations. An approximate method for evaluating the shape and the size of fracture process zone is proposed and the feasibility of the new method is demonstrated by comparing with the results calculated based on the well-known stress fields. The shape and size of the fracture process zone near the mode II dynamic crack tip in dependence on crack propagation velocity are determined based on the approximate method. The results show that the areas of fracture process zone calculated based on the new method are nearly the same with the results calculated based on the well-known stress fields under plane stress condition and plane strain condition by the Von Mises criterion. The approximate method can provide a good reference for determination of the fracture process zone near mode II dynamic crack tip since no analytic method has been found for evaluating the fracture process zone near dynamic crack tip to the authors’ knowledge. The fracture process zone near mode II dynamic crack tip is distributed symmetrically with respect to crack plane and increases with the crack propagation velocity. The area of fracture process zone changes more rapidly when the Rayleigh wave velocity is approached. The areas of fracture process zone calculated under plane stress condition are bigger than these calculated under plane strain condition.

1. Introduction
Fracture Process Zone (FPZ) at crack tip can be viewed as the ensemble of decohesive cells according to Broberg’s cell model [1, 2, 3]. For quasi-brittle materials like rock and concrete, the study on FPZ near dynamic crack tip is very essential, because fracture process zone (FPZ) can significantly affects the strength and stability of the structure [4]. Shear failure is more common than tensile failure because of the confining pressure. This suggests that the investigation of FPZ near mode II crack tip is very necessary for quasi-brittle materials.
The cohesive force model proposed by Barenblatt [5], fictitious crack model developed by Hillerborg [6] and two parameter fracture model introduced by Jenq and Shah [7] can be used to describe FPZ. Equivalent crack fracture model was proposed by Wang [8] to describe the crack tip nonlinearity of quasi-brittle materials based on linear elastic fracture mechanics and superposition principle. Hult and McClintock [9] predicted the elastoplastic boundary of a mode III crack tip zone by using Von Mises criterion, which indicated that the plastic zone was circular in shape. Qiang et al. [10] and Zhang [11] gave the unified solutions to the shape and size of mode I and mode II crack tip plastic zone under small scale yielding by using Yu’s unified yield criterion. The solution can be applied to different kind materials, and the effect of material parameters on FPZ at the crack tip is analyzed. Qing et al. [12, 13] analyzed the length of fracture process zone near mode I crack tip based on the fictitious crack model and the concept of cohesive crack.

The above mentioned studies are limited to the discussion of the FPZ near the static crack tip. The FPZ of the dynamic crack is very complicated due to the inertia effect and the rate effect of mechanical properties of materials. Dai et al. [14] analyzed the formation of the fracture process zone (FPZ) of industrially produced magnesia spinel magnesia refractories using digital image correlation method (DIC). The onset of macro-cracking and location of the crack tip is determined by critical displacement determined from the changes of the FPZ width and length. Tarokh et al. [15] performed experiments on specimens of different sizes with a center notch fabricated from granite of large grain in three points bending and analyzed the relationship between the length and width of the process zone and the specimen size. Freund and Lee [16] analyzed the crack propagation of high strain rate using the rate sensitive cohesive strip model near the crack tip, and showed that the strip length decreases with the increase of crack propagation velocity.

From the above mentioned review, we can see that the characteristics of FPZ depend on material properties, crack propagation velocity, specimen size and other factors. Full analytical solution is difficult to obtain. Therefore, here a simplified approximate method is proposed to calculate the shape and size of the fracture process zone near the mode II dynamic crack tip.

2. The approximate solution to the mode II dynamic crack tip

Consider a Griffith crack with length 2a in an infinite elastic solid moving at constant speed V along the direction of x-axis and the body is subjected to a far-field shearing stress $\tau$ along the direction of x-axis (see the Fig.1). There are two coordinate systems, one is the fixed coordinate systems ($x_1, y, t$), the other one is the one ($x, y, t$) moving with the crack tip [17]. The two coordinate systems have the following relationship

$$x = x_1 - Vt, y = y$$  \hspace{1cm} (1)

![Figure 1. Mode II Griffith crack.](image)

For the plane elastic dynamic problem with stress components $\sigma_x, \sigma_y, \tau_{xy}$, the motion equations are shown in Eq. (2) [18]:
\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= \rho \frac{\partial^2 u}{\partial t^2} \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} &= \rho \frac{\partial^2 v}{\partial t^2}
\end{align*}
\]

(2)

where \( \rho \) is the material density.

The stress-strain relations are:

\[
\begin{align*}
\sigma_x &= \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial u}{\partial x} \\
\sigma_y &= \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial v}{\partial y} \\
\tau_{xy} &= \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)
\end{align*}
\]

(3)

where \( u, v \) are the displacements along \( x \) and \( y \) axes, respectively; and \( \mu \) and \( \lambda \) are Lame’s constants.

Then the equations of deformation compatibility can be derived as follows [18):

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\rho}{\lambda + 2\mu} \frac{\partial^2}{\partial t^2} \right)(\sigma_x + \sigma_y) = 0
\]

(4)

And also the following equations will be satisfied by stress function \( U \) according to Radok [18):

\[
\begin{align*}
\sigma_x &= \frac{\partial^2 U}{\partial y^2} - \frac{\rho}{2\mu} \frac{\partial^2}{\partial t^2} U \\
\sigma_y &= \frac{\partial^2 U}{\partial x^2} - \frac{\rho}{2\mu} \frac{\partial^2}{\partial t^2} U
\end{align*}
\]

(5)

Substituting Eq. (5) into Eq. (4), the following equation can be obtained:

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2}{\alpha_1^2 \partial y^2} + \frac{\partial^2}{\alpha_2^2 \partial y^2} \right) U = 0
\]

(6)

where:

\[
\begin{align*}
\alpha_1 &= \sqrt{1 - \frac{V^2}{c_1^2}} \quad \alpha_2 = \sqrt{1 - \frac{V^2}{c_2^2}} \\
c_1 &= \sqrt{\frac{\mu + 2\mu}{\rho}} \quad c_2 = \sqrt{\frac{\mu}{\rho}}
\end{align*}
\]

(7)

(8)

where \( c_1 \) and \( c_2 \) are longitudinal and transverse wave speeds, respectively.

Eq. (6) can be easily transformed as follows:

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y_1^2} \right) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y_2^2} \right) U = 0
\]

(9)

where [19]:

\[
y_1 = \alpha_1 y, \quad y_2 = \alpha_2 y
\]

(10)
As the accurate stress function of Eq. (9) has not been found, this equation cannot be used to determine the stress fields near crack tip. So an approximate method is proposed to solve the Eq. (9).

Let:

$$y = \sqrt{\frac{2}{\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2}}}$$

then Eq. (9) can be transformed into the following form

$$\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 y^2} + \frac{\partial^4}{\partial y^4}\right)U + \left[\frac{4\alpha_1^2 \alpha_2^2}{(\alpha_1^2 + \alpha_2^2)^2} - 1\right]\frac{\partial^4}{\partial y^4}U = 0$$

that is:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2U + \beta\frac{\partial^4}{\partial y^4}U = 0$$

where:

$$\beta = \frac{4\alpha_1^2 \alpha_2^2}{(\alpha_1^2 + \alpha_2^2)^2} - 1$$

For different crack propagation velocities the corresponding values of $\beta$ are shown in Table 1:

| V/c(m/s) | $\alpha_1$ | $\alpha_2$ | $\alpha_1^2$ | $\alpha_2^2$ | $\beta$ |
|----------|-------------|-------------|--------------|--------------|--------|
| 0        | 1.0000      | 1.0000      | 1.0000       | 1.0000       | 0.0000 |
| 0.19     | 0.9945      | 0.9984      | 0.9891       | 0.9968       | -1.53E-05 |
| 0.38     | 0.9780      | 0.9937      | 0.9565       | 0.9875       | -0.00025 |
| 0.57     | 0.9498      | 0.9859      | 0.9021       | 0.972        | -0.00139 |
| 0.76     | 0.9088      | 0.9748      | 0.8260       | 0.9503       | -0.00490 |
| 0.95     | 0.8533      | 0.9604      | 0.7281       | 0.9224       | -0.01386 |

It can be seen from table 1 that the value of $\beta$ increases with crack velocity. The maximum value $\beta$ of is reached when Relay wave velocity is approached. However, generally the observed in experiments maximum crack propagation velocity is 0.34-0.45 times the Relay wave velocity, and the value of $\beta$ is 0.00025-0.00139. So the value of $\beta$ is very small, and the second term of Eq. (12) can be neglected.

The following equation can be used to explore the approximate solution of Eq. (9):

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2U = 0$$

At this time, Eq. (15) is the biharmonic equation in the coordinate system $(x, \bar{y})$, and has the same form as that for the static crack. So we can find the approximate solution for dynamic crack by the stress function method.

For mode-II static crack governing equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2U_0 = 0$$

the stress function has the following form [17]:

$$U_0(x, y) = -y\text{Re}\left(\int Z_0(z_0)dz_0\right)$$

where Re represents the real part of the integral and the function $Z_0(z_0)$ is given by:
\[ Z_0(c_0) = \frac{\tau(z_0 + a)}{\sqrt{(z_0 + a)^2 - a^2}} \]  

Then the stress field near the crack tip can be expressed by the following equations [17]:

\[
\begin{align*}
\sigma_x &= 2\text{Im}Z_0 - y\text{Re}Z_0' \\
\sigma_y &= -y\text{Re}Z_0' \\
\tau_{xy} &= \text{Re}Z_0 - y\text{Im}Z_0'
\end{align*}
\]  

where Im represents the imaginary part.

So the stress fields can be written as follows according to the stress fields near static crack:

\[
\begin{align*}
\sigma_x &= 2\text{Im}Z - y\text{Re}Z \\
\sigma_y &= -y\text{Re}Z \\
\tau_{xy} &= \text{Re}Z - y\text{Im}Z
\end{align*}
\]  

where:

\[ Z(z) = \frac{\tau(z + a)}{\sqrt{(z+a)^2 - a^2}} \]  

\[ z = x + iy \]

3. Determination of FPZ

The Von Mises criterion is also known as the Octahedral Shear-Stress criterion. It is possible to estimate the extent of the failure zone by applying the Von Mises criterion, which is expressed as follows:

\[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_s^2 \]  

where \( \sigma_s \) is the failure stress in uniaxial tension.

By substituting Eq. (20) into Eq. (23) the implicit equation of FPZ can be easily got. As an example, we take the following material parameters: the longitudinal wave velocity \( c_1 = 5370 \text{m/s} \), shear wave velocity \( c_2 = 3180 \text{m/s} \), the Rayleigh wave velocity is \( c = 2900 \text{m/s} \). Contours of the FPZ constructed by using the approximate method for different crack velocities are shown in Fig. 2.

**Figure 2.** Contours of FPZ constructed by using Von Mises criterion at mode II crack tip (a) Under plane stress conditions (b) Under plane strain conditions (c) Comparison of the FPZs constructed by using Von Mises criterion under plane stress conditions and plane strain conditions (the contour of the FPZ under plane stress condition is represented by line, and the contour of the FPZ under plane strain condition is represented by line and symbol.)

From Fig.2 we notice that the FPZ near mode II dynamic crack tip is distributed symmetrically with respect to crack plane. The size of FPZ vertical to the crack face increases with the crack velocity, and
the size along the direction of crack propagation are almost constant. When crack propagation velocity approaches the Rayleigh wave velocity, the size of FPZ vertical to the direction of the crack face increases significantly, so are the areas of the FPZ. When crack velocity is the same, the contours of FPZ under plane stress conditions are larger than that under plane strain conditions.

4. Validation for the approximate method

In this part the area of fracture process zone calculated based on the known stress fields and the results calculated by the approximate method are compared for validation for the approximate method. The known stress fields are show as follows [20]:

\[
\begin{align*}
\sigma_x &= \frac{K_{11}}{\sqrt{2\pi}} \frac{2 \alpha_2}{4 \alpha_1 \alpha_2 - (1 + \alpha_2)^2} \left[ -\left(1 + 2 \alpha_1^2 - \alpha_2^2\right) \frac{\sin \theta_1}{\sqrt{r_1}} + \left(1 + \alpha_2^2\right) \frac{\sin \theta_2}{\sqrt{r_2}} \right] \\
\sigma_y &= \frac{K_{11}}{\sqrt{2\pi}} \frac{2 \alpha_2 (1 + \alpha_2^2)}{4 \alpha_1 \alpha_2 - (1 + \alpha_2)^2} \left[ \frac{\sin \theta_1}{\sqrt{r_1}} - \frac{\sin \theta_2}{\sqrt{r_2}} \right] \\
\tau_{xy} &= \frac{K_{11}}{\sqrt{2\pi}} \frac{1}{4 \alpha_1 \alpha_2 - (1 + \alpha_2)^2} \left[ 4 \alpha_1 \alpha_2 \frac{\cos \theta_1}{\sqrt{r_1}} - (1 + \alpha_2^2) \frac{\cos \theta_2}{\sqrt{r_2}} \right]
\end{align*}
\] (24)

where:

\[
\begin{align*}
\theta_1 &= \arctan(\frac{\bar{y}}{\bar{x}}), \theta_2 &= \arctan(\frac{\bar{y}}{\bar{x}}) \\
r_1 &= \sqrt{x^2 + \alpha_1^2 y^2}, r_2 = \sqrt{x^2 + \alpha_2^2 y^2} \\
\bar{y} &= \frac{y}{a}, \bar{\theta}_1 = \frac{\theta_1}{a}, \bar{\theta}_2 = \frac{\theta_2}{a} \\
\bar{x} &= x_1 - \alpha t = \bar{r}_1 \cos \bar{\theta}_1 = \bar{r}_2 \cos \bar{\theta}_2
\end{align*}
\] (25)

By substituting Eqs. (24) and (25) into Eq. (23) the implicit equation of FPZ can be easily got. The areas of FPZ calculated by the proposed method and by the well-known stress fields are shown in table 2.

**Table 2.** Area of FPZ for mode-II crack by using Von Mises criterion (plane stress) / cm².

| V/c  | 0.19  | 0.29  | 0.38  | 0.48  | 0.57  |
|------|-------|-------|-------|-------|-------|
| Plane stress | By the known stress field | 0.785 | 0.819 | 0.847 | 0.934 | 1.086 |
|         | By the proposed method     | 0.737 | 0.749 | 0.760 | 0.782 | 0.815 |
| Plane strain | By the known stress field | 0.571 | 0.592 | 0.594 | 0.649 | 0.731 |
|         | By the proposed method     | 0.582 | 0.591 | 0.604 | 0.624 | 0.644 |

It can be seen that the areas of FPZ calculated based on known stress field and the areas calculated based on the new method both increase with the crack velocity under plane stress condition and plane strain condition. The velocity dependence of FPZ calculated based on the known stress field is slightly stronger than that of the FPZ calculated based on the new method, the reason is that the known stress field is an asymptotic expression that ignores the higher-order terms [20]. The results of the two methods in table 2 can verify the correctness of the proposed method for evaluating the FPZ near the dynamic crack tip.
5. Conclusions

Although it is very difficult to find an exact solution of the fracture process zone for dynamic crack, this paper proposed an approximation method which can provide a reference for the solution to the FPZ for dynamic crack. The FPZ for mode II dynamic crack are calculated by the approximation method and following conclusions can be drawn:

- The areas of FPZ calculated based on known stress field and calculated based on the new method are very close, which verifies the feasibility of the new method.
- The new method proposed here is easier to calculate the stress field near dynamic crack tip than the previous analytical methods like integral transformation method and Wiener-hopf method.
- The contours of FPZ at the tip of mode II dynamic cracks determined based on the new method are symmetric with respect to the crack plane. The FPZ distributes not only behind the crack but also ahead of the crack tip. As the crack propagation velocity increases, the area of FPZ near the crack tip increases. When the Rayleigh wave velocity is approached, the area of FPZ increases more rapidly.
- When crack velocity is the same, the contours of FPZ under plane stress conditions are larger than that under plane strain conditions.

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