From Maxwell Stresses to Photon-like Objects through Frobenius Curvature Geometrization of Local Physical Interaction

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Abstract

This paper aims to review our recent results on exploring the capabilities of nonquantum field theory as a possible tool for describing single photon-like objects, considered as massless time-stable spatially finite physical entities with compatible translational-rotational dynamical structure. It consists of five chapters and includes 22 sections and subsections.

In Chapter 1 (Introduction) we present briefly some remarks on Maxwell vacuum equations and our notions concerning the two basic concepts in physics: physical object and interaction, with an accent on the view that physical interaction necessarily implies energy-momentum exchange, and on the suggestion that the energy-momentum exchanging subsystems of a general field should be described rather by $(F, \ast F)$, than by $(E, B)$.

Chapter 2 (Nonrelativistic considerations) begins with a reasoning on the status, right understanding and appropriate use of the Coulomb force law in electrostatics, leading to the conclusion that the usual way of introducing static electric field as local object directly from the Coulomb force law violates in definite sense the local conservation laws in the frame of Maxwell electrodynamics. Then we concentrate on the role, significance and eigen properties of the Maxwell stress tensor in nonrelativistic terms. This part of the paper culminates in writing down nonlinear field equations for the vacuum electromagnetic fields paying due respect of the Newton view on the sense of dynamical equations as local balance relations of conserved quantities carried by two appropriately defined subsystems of the general electromagnetic field, and presenting some important properties of the nonlinear solutions. Finally, we discuss briefly the contents of this part of the paper.

Chapter 3 (Relativistic considerations) of the paper makes use of the relativistic formalism. We begin with presenting the notion of photon-like object(s) (PhLO). The existing corresponding relativistic description of PhLO in the frame of Extended Electrodynamics is briefly recalled. In terms of integrability and nonintegrability properties of distributions (differential systems) on a manifold a principle of geometric interaction between two nonintegrable distributions is formulated and a corresponding physical interpretation is given. This principle is further substantially used in building mathematical description of single PhLO, considered as composed of two individualized and interacting subsystems, and realizing a special kind of dynamical equilibrium. The mathematical model is built on the assumptions that Frobenius integrability can be made to correspond to physical time-stability, and the nonintegrability of subdistributions of integrable distribution to correspond to local physical interaction between subsystems of a time-stable continuous physical system. Two approaches are considered: direct use of the Frobenius theorem and the corresponding curvature being a measure of nonintegrability, and the recently developed approach known as “non-linear connections”. Both these approaches make use of the corresponding curvatures for generating appropriate quantities describing local physical interaction, i.e. local energy-momentum exchange

A corresponding concept of electromagnetic strain is defined and the basic stress-energy-momentum relations, obtained before, are represented in electromagnetic strain terms.

Chapter 4 (Equations of motion for PhLO.Solutions) deals with various views on dynamical equations for PhLO and gives illustrative examples of appropriate solutions.

Finally, Chapter 5 (Retrospect) gives a retrospective discussion of the contents of the paper.
Contents

1 Introduction
  1.1 Preliminary remarks ................................................. 2
  1.2 Physical Objects and Interactions ................................. 4

2 Nonrelativistic Considerations
  2.1 How to understand the Coulomb force ? ............................. 8
  2.2 A non-physical view on Maxwell stress tensor ...................... 12
  2.3 Notes on the eigen properties of Maxwell stress tensor .......... 14
  2.4 Nonlinear equations for the electromagnetic field ................. 16
    2.4.1 Some Properties of the nonlinear solutions .................. 21
  2.5 Discussion ....................................................... 22

3 Relativistic Considerations
  3.1 The Extended Electrodynamics approach ............................ 25
  3.2 The Notion of Photon-like Object(s) ............................... 28
    3.2.1 Introduction .................................................. 28
    3.2.2 The notion of photon-like object .............................. 29
  3.3 Curvature of Distributions and Physical Interaction ............... 31
    3.3.1 The general idea for geometrization of local physical interaction .
    3.3.2 Frobenius integrability, curvature and local physical interaction 32
  3.4 PhLO Dynamical Structure in Terms of Frobenius Curvature ........ 33
  3.5 PhLO Dynamical Structure in Terms of Non-linear Connections ....... 38
    3.5.1 Projections and algebraic curvatures ............................. 38
    3.5.2 Nonlinear connections ........................................ 39
    3.5.3 Photon-like nonlinear connections ................................ 39
    3.5.4 Electromagnetic PhLO in terms of non-linear connections .......... 43
  3.6 Electromagnetic PhLO in terms of electromagnetic strain .......... 45

4 Equations of motion for PhLO. Solutions
  4.1 The approach based on the notion for PhLO ........................ 48
  4.2 The Lagrangian Approach .......................................... 49
  4.3 Equations of motion in terms of translational-rotational compatability .
  4.4 Photon-like Solutions .............................................. 50

5 Retrospect .............................................................. 53
Chapter 1

Introduction

1.1 Preliminary remarks

As it is well known the vacuum Maxwell equations (zero charge density: \( \rho = 0 \)) do not admit spatially finite time-stable solutions of photon-like type, i.e. solutions, having compatible translational-rotational dynamical structure and propagating as a whole along straight lines in the space with the fundamental velocity "c" - the speed of light in vacuum and without dispersion. This is in corresponding degree due to the fact that in the frame of Maxwell vacuum equations every component \( U(x, y, z, t) \) of the electric \( E \) and magnetic \( B \) fields, satisfying corresponding smoothness conditions, necessarily satisfies the D’Alembert wave equation \( \Box U = 0 \), and according to the Poisson’s theorem for this equation, every spatially finite initial condition \( U(x, y, z, 0) = \varphi(x, y, z); \frac{\partial U}{\partial t}(x, y, z, 0) = \psi(x, y, z) \), where \( \varphi \) and \( \psi \) are finite functions satisfying definite differentiability conditions, blows up radially and goes to infinity with the speed of light \([1,2,3]\). So, in such a case, through every spatial point outside such an initial condition pass fore-front and back-front, and after this the corresponding point forgets about what has happened. Hence, photon-like objects require new kind of description.

On the other hand the more than a century successful usage of Maxwell equations in describing various electromagnetic physical systems and processes undoubtedly suggests that their adequateness to reality is hardly understood and used fully. For example, one of the crucially important relations that follows from Maxwell equations, and that has been successfully used throughout all these years, is the Poynting energy-momentum balance equation in vacuum

\[
\frac{\partial}{\partial t} \left( \frac{E^2 + B^2}{2} \right) = -c \text{div}(E \times B),
\]

where \( c \) is the velocity of light in vacuum. Considered from the above mentioned point of view it turns out that this Poynting equation admits time-stable, spatially finite and propagating along straight lines solutions of the following kind

\[
E = [u(x, y, ct + \varepsilon z), p(x, y, ct + \varepsilon z), 0], \quad B = [\varepsilon p(x, y, ct + \varepsilon z), -\varepsilon u(x, y, ct + \varepsilon z), 0], \quad \varepsilon = \pm 1,
\]

where \( u \) and \( p \) are arbitrary functions, so they can be chosen finite. This observation suggests to look deeper and more carefully into the structures and assumptions used for mathematical interpretation of the experimental electric-magnetic induction discoveries made in the 19th century. In other words, which relations, and on what grounds, should be defined as basic, so that the further deduced equations and relations to give reasonable and physically acceptable results when viewed from the viewpoint for spatial finiteness and time stability. Finding the right way
to choosing adequate mathematical objects and corresponding equations seems specially im-
portant when we try to describe the intrinsic dynamical properties of such spatially finite and
time stable field objects. Therefore, it seems very important to have the right notion about
concepts like physical object, intrinsic property, dynamical property, identifying characteristics,
dynamical structure, admissible changes, field equations, etc.

The idea to extend the vacuum Maxwell equations in such a way that spatially finite time
stable and straight-line propagating solutions to be incorporated is, of course, not new [4].
Moreover, a general principle concerning all theoretical physics was formulated by Born and
Infeld [5], stating: a satisfactory theory should avoid letting physical quantities become infinite.
Other recent efforts in this direction in the frame of electrodynamics may be found in [6,7,8,9].
We are not going to analyze here all these various approaches, what however deserves to be
noted is that the new equations offered therein have not usually direct physical sense of energy-
momentum balance relations as the Newton law in mechanics has. And this is true also for
the very Maxwell equations. The above mentioned example with the Poynting relation clearly
suggests a more serious and physically motivated respect to be paid to this aspect of the problem
when some basic relation in a physical theory is postulated. For example, should we consider
the very electric and magnetic fields \((E, B)\) as force fields in case of no charges present, or
the divergence \(\nabla_i M^{ij}\) of Maxwell’s stress tensor \(M^{ij}\) [10] should be accepted in the pure field
case as force field? Each of these three vector fields generates integral lines, so, which integral
lines should be considered as force-lines, and what is a force-line in case of absence of charged
particles? Clearly, since the general concept of force, considered as local object, is local energy-
momentum exchange, more reliable seems to be the divergence \(\nabla_i M^{ij}\), so, the Newton-like
equations of motion in the vacuum case should read \(\nabla_i M^{ij} = 0\). In other words, the implied
energy-momentum exchange between the electric and magnetic components of the field should
be represented by relations having direct energy-momentum exchange sense, and not by the
local versions of the so called induction laws

\[
\text{rot} E + \frac{\partial B}{\partial \xi} = 0, \quad \text{rot} B - \frac{\partial E}{\partial \xi} = 0,
\]

where \(\xi = ct\). In fact, we can NOT observe and verify directly these relations since we have
not corresponding devices, we verify them indirectly through observing corresponding energy-
momentum changes, i.e. state-behavior changes, of charged particles. In other words, we have a
system having two subsystems: field and charged particles, and, assuming the energy-momentum
local conservation law, we make energy-momentum change conclusions about the field through
the corresponding mechanical energy-momentum changes of the charged particles. Therefore,
when we want to understand and describe intra-field energy-momentum exchanges, i.e. in case
of no charged objects presented, we must have in mind some preliminary pictures about the
structure and possible more or less individualized and time-stable substructures among which
this energy-momentum inter-exchange should take place. Assuming the electric and magnetic
components as such substructures we find that the absence of well defined local interaction energy
between these two components (recall that the energy density of the field is \(\frac{1}{2}(E^2 + B^2)\)) seriously
complicates this problem: the two Maxwell equations above imply some energy exchange between
\(E\) and \(B\), so how it is performed if there is NO local interaction energy, moreover, neither \(E\)
nor \(B\) are capable to carry separately momentum and angular momentum, so, which are the
corresponding energy-momentum exchanging substructures of the field during propagation?

The relativistic development of classical electrodynamics introduced new point of view: ade-
quate mathematical objects that represent such two substructures of the general vacuum field
are not \(E\) and \(B\), but two differential 2-forms \(F_{(E,B)}\) and \(*F_{(-B,E)}\) on Minkowski space-time, so,
from the new point of view, any internal energy-momentum exchange should take place between
F and *F. However, the "new" field equations \( \mathbf{d} \mathbf{F} = 0, \mathbf{d} \ast \mathbf{F} = 0 \), although in terms of \( \mathbf{F} \) and *\( \mathbf{F} \), keep the old viewpoint, and in a definite sense they forbid such internal exchange (see Sec.3.1). This study is directed to find corresponding new equations.

The paper is organized as follows. In Sec.1.2 we present some general considerations concerning the mathematical description of a physical system. Sections 2.1-2.5 are devoted to defining the model in pre-relativistic terms. Sections 3.1-3.6 give relativistic approach, making use of Frobenius curvature, non-linear connections and relativistic strain. Sections 4.1-4.4 present various views on dynamical equations and give solutions of photon-like nature. Finally, Ch. 5 gives a retrospect of this study.

1.2 Physical Objects and Interactions

From a definite point of view every physical system is characterized by two kinds of properties. The first kind of properties we call identifying, they identify the system throughout its existence in time, so, the corresponding physical quantities/relations must show definite conservation/constancy properties (with respect to the identification procedure assumed). Without such experimentally established properties we could not talk about physical objects/systems at all. The second kind of properties (which may be called kinematical) characterize the time-evolution of the system, the corresponding quantities are time-dependent, and the corresponding evolution is consistent with the conservative/constant character of the identifying properties/quantities. In this sense, the equations of motion of a physical system can be considered as relations determining the admissible time-changes of these quantities. For example, the mass \( m \) of a classical particle is an identifying quantity, while the velocity \( \mathbf{v} \) is a kinematical one. This view implies, of course, that the external world acts on the system under consideration also in an admissible way, i.e. an assumption is made that the system survives, the interaction with the outside world does not lead to its destruction.

In theoretical physics we usually make use of quantities which are functions of the identifying and of the kinematical characteristics of the system and call them dynamical quantities. A well known example is the momentum \( \mathbf{p} \) of a particle: \( \mathbf{p} = m \mathbf{v} \). Of crucial importance for the description of admissible changes are the conservative dynamical quantities, i.e. those which may pass from one physical system to another with NO LOSS. For example energy and momentum are such quantities, moreover, they are universal in the sense that every physical object carries non-zero energy-momentum and, vice versa, every quantity of energy-momentum is carried by some physical object. So, if a definite quantity of energy-momentum passes from one object to another, this same quantity of energy-momentum can be expressed in terms of the characteristics of the two objects/systems, and the two expressions to be equalized. This allows to describe interaction between, or among, physical objects. Thus we have a consistent with the requirement for "identification through conservation"-way to write equations of motion, and this is the way used by Newton to write down his famous equations \( \dot{\mathbf{p}} = \mathbf{F} \), where \( \mathbf{F} \) carries information about where the momentum change of the particle has gone, or has come from. This also clarifies the physical sense of the concept of force as a change of momentum, or as a change of energy-momentum in relativistic terms. Paying due respect to Newton we shall call some equations of motion of Newton type if on the two sides of "\( = \)" stay physical quantities of energy-momentum change, or energy-momentum density change in the case of continuous systems. Note that, written down for the vector field \( \mathbf{p} \), i.e. in terms of partial derivatives, the above Newton equation looks like \( \nabla_\mathbf{p} \mathbf{p} = m \mathbf{F} \), where the left hand side means performing two steps: first, determining the "change quantity" \( \nabla \mathbf{p} \), second, projecting \( \nabla \mathbf{p} \) on \( \mathbf{p} \), and the right hand side may be expressed as a function of the characteristics of both: the particle and the external
physical environment.

If there is no energy-momentum (or energy-momentum density) change, then putting the corresponding expression equal to zero, e.g. \( \nabla_p p = 0 \), we obtain the "free particle" or "free field" equations. In such a case we just declare that only those changes are admissible which are consistent with the (local and integral) energy-momentum conservation.

We note that an initial extent of knowledge about the system we are going to describe mathematically is presupposed to be available, so that the assumptions made to be, more or less, well grounded. This knowledge is the base that generates corresponding insight and directs our attention to the appropriate mathematical structures. This is exclusively important when we deal with continuous, or field, physical objects/systems.

To illustrate our idea, let's consider a many-component continuous system, i.e. such that each component is assumed to be time-stable and recognizable during the system's existence. Accordingly, the \textbf{wholeness+structural integrity} of the system should be mathematically represented by some complex \( \Phi \) of interconnected fields: \( \Phi = \{ \Phi_a \}, a = 1, 2, \ldots \), where each \( \Phi_a \) represents some individualized subsystem and may be also many-component one: \( \Phi_a = \{ \Phi_a^1, \Phi_a^2, \ldots \} \). In view of the above considerations, if our system is free, the steps to follow are:

1. Specify and consider the mathematical model-object \( \Phi = \{ \Phi_a^1, \Phi_a^2, \ldots \}, a = 1, 2, \ldots \) which is chosen to represent the \textbf{wholeness+structural integrity} of the physical system considered;
2. Define the change-objects \( D(\Phi_a) \), considered as due to internal interactions;
3. "Project" \( D(\Phi_a) \) on \( \Phi_a/\Phi_b, b \neq a \), by means of some (in most cases bilinear) map \( \Psi \);
4. The projections \( \Psi(D(\Phi_a), \Phi_b) \) and \( \Psi(D(\Phi_b), \Phi_a) \) obtained, we interpret physically as local energy-momentum exchange between the individualized subsystems described by \( \Phi_a \) and \( \Phi_b \): \( \Psi(D(\Phi_a), \Phi_b) \) is the energy-momentum that \( \Phi_a \) transfers to \( \Phi_b \), and \( \Psi(D(\Phi_b), \Phi_a) \) is the energy-momentum that \( \Phi_b \) transfers to \( \Phi_a \).
5. The subsystem described by \( \Phi_a \) may, or may not, keep its energy-momentum unchanged during the internal interaction. Correspondingly, in the first case we'll have \( \Psi(D(\Phi_a), \Phi_a) = 0 \), and in the second case we shall have \( \Psi(D(\Phi_a), \Phi_a) \neq 0 \).
6. The sum \( \sum_a \Psi(D(\Phi_a), \Phi_a) \) should be equal to zero, meaning that our system \( \Phi = \{ \Phi_a \} \) conserves locally its energy-momentum.

The zero value of the projection \( \Psi(D(\Phi_a), \Phi_a) \) is interpreted in the sense that the identifying characteristics of \( \Phi_a \) have not been disturbed, or, the change \( D(\Phi_a) \) is qualified as \textit{admissible}. This consideration shows the importance of knowing how much and in what way(s) a given physical system is \textit{potentially able} to lose, or gain energy-momentum (locally or globally), without losing its identity.

It is always very important to take care of the physical sense of the quantities that we put on the two sides of the relation \( A = B \). Mathematically, from set theory point of view [13], \( A \) and \( B \) denote the same element, which element may be expressed in different terms, e.g. the real number 2 can be expressed as \( 3 - 1 = 6/3 = \frac{d}{dx}(2x + const) \) and also in many other ways. From physical point of view, however, we must be preliminary sure that \( A \) and \( B \) denote the same thing \textit{qualitatively} and \textit{quantitatively}, i.e. \( A \) and \( B \) must denote the same physical quantity. This is specially important when the equation we want to write down constitutes some basic relation. And the point is not the physical dimension of the two sides to be the same: any two quantities by means of an appropriate constant can be made of the same physical dimension, but this is a formal step. The point is that the physical nature of the physical quantity on the two sides must be the same, and this should be well understood and correspondingly guaranteed beforehand.

For example, it is quite clear that on the two sides of the Newton’s law \( \dot{p} = F \) stays the well defined for any physical system quantity "change of momentum" since the \textit{momentum} quantity
is a universal one. For a counterexample, which physical quantity stays on the two sides of the Poisson equation $\Delta U = k\rho$, $k = \text{const}$? On one hand, such a quantity is expressed through $\Delta U$ and, since $\text{grad}U$ is usually interpreted as force, $\Delta U$ appears as a "change of force" characteristic of the field $U$ since it is essentially defined by the second derivatives of $U$. On the other hand, the same quantity is expressed through $k\rho$ and appears as a characteristic of the mass particles, so, do we know such a quantity? The same question can be raised for one of the Maxwell equations: $\text{rot} \mathbf{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{E} = \frac{4\pi}{c^2} \mathbf{j}$.

In the case of classical particles momentum is always represented as the product $mv$ and this is carried to fluid mechanics (continuous mass distribution) as $\mu(x, y, z; t) \mathbf{v}(x, y, z; t)$, where $\mu$ is the invariant mass density. A similar quantity is introduced in electrodynamics as electric current density $\mathbf{j} = \rho(x, y, z; t) \mathbf{v}(x, y, z; t)$, where $\rho$ is the electric charge density. The energy-momentum exchange between the field energy-momentum and the mechanical energy-momentum of the available charged particles is described by the force field $\mathbf{F}$ in our view should be: $\mathbf{F}$.

The entire propagational behaviour of the field could be understood and modeled by means of the force field $\mathbf{F}$ and the propagation of the free EM-field is available, so that energy-momentum internal exchanges should necessarily take place, how these processes and the energy-momentum density $\rho$, $\mathbf{j}$ should exist, a result that has been proved in studying the photoeffect phenomena about 30 years after Maxwell’s death. As we already mentioned, there exists a sufficiently good force field defined by Maxwell in terms of the divergence of his stress tensor $M^{ij}$, which definition works quite well also out of and away from any media built of, or containing, charged mass particles. So, in the frame of the theory at the end of 19th century if we ask the question: if there are NO charged particles and the time-dependent EM-field cannot transfer energy-momentum to them by means of the force field $\mathbf{F}$, and the propagation of the free EM-field is available, so that energy-momentum internal exchanges should necessarily take place, how these processes and the entire propagational behaviour of the field could be understood and modeled?, the right answer in our view should be: turn to $M^{ij}$ and consider carefully the divergence $\nabla_i M^{ij}$ terms as possible force fields generating corresponding force lines along which energy-momentum is internally and locally transported between/among subsystems. As will be seen further in the paper, such a look on the issue would necessarily lead Maxwell and his followers to the prediction that real, free, spatially finite and time-stable formations of electromagnetic field nature having compatible translational-rotational dynamical structure should exist, a result that has been proved in studying the photoeffect phenomena about 30 years after Maxwell’s death.

We consider as a remarkable achievement of Maxwell the determination of the correct expressions for the energy density of the electromagnetic field through the concept of stress $\sigma^{ij}$. His electromagnetic stress tensor $M^{ij}$ still plays an essential role in modern electromagnetic theory as a part of the modern relativistic stress-energy-momentum tensor. However, by some
reasons, Maxwell did not make further use of the computed by him divergence $\nabla_i M^{ij}$ of the stress tensor (and called by him "force field" [10]) for writing down Newton type equations of motion for a free electromagnetic field through equalizing different expressions for the same momentum change. Probably, he had missed an appropriate interpretation of the vector $c \mathbf{E} \times \mathbf{B}$ (introduced by Poynting 5 years after his death and called "electromagnetic energy flux" [14]).

In connection with the above considerations the following more general question arises: does theoretical physics make the right step allowing static force fields to be written on the right hand side of Newton equation of motion in mechanics? Every static field, whatever is its nature and origin, necessarily conserves locally and globally all its physically meaningful characteristics, so its energy is also conserved and is not at disposal to other physical systems. Moreover, static fields do NOT propagate, so they do NOT carry momentum, therefore, there is NO WAY other physical systems to lose or gain momentum at the expense of external static fields. Hence, trying to pay respect to the momentum conservation through writing down Newton type dynamical equations $\dot{\mathbf{p}} = \mathbf{F}$ with static force field $\mathbf{F}$, we, in fact, violate it: due respect requires due usage, so, in our view, no selfrespecting theory should allow dynamical equations with static $\mathbf{F}$!

This concerns not only mechanics, and in fact, not only static fields: propagating composite time-stable physical systems are supposed to consist of individualized time-stable subsystems capable to interact, i.e. to carry and exchange energy-momentum during propagation. Do we respect always this natural principle in physical theories? Consider for example the two vector vacuum Maxwell equations: $\mathbf{B} = -c \text{rot } \mathbf{E}, \mathbf{E} = c \text{rot } \mathbf{B}$. These equations imply that there is mutual physical influence between $\mathbf{E}$ and $\mathbf{B}$, which we understand physically as energy-momentum exchange between the individualized $\mathbf{E}$ and $\mathbf{B}$ components: $\mathbf{E}$ acts upon $\mathbf{B}$ and $\mathbf{B}$ acts upon $\mathbf{E}$. Now, according to the equations, each of these two components propagates and keeps its individualization during propagation, so, it should be able to carry momentum. However, the energy concept of the theory excludes nonzero interaction energy between these two components to exist since the energy density $w$ is given by the sum of the energies carried by $\mathbf{E}$ and $\mathbf{B}$: $w = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$, so, how does energy exchange take place? Further, the momentum concept in the theory is defined and experimentally proved quantitatively to be given by $\frac{1}{c}(\mathbf{E} \times \mathbf{B})$, so neither of the assumed in the theory electric and magnetic components is allowed to carry momentum separately. In view of this, how e.g. in the plane wave solutions, where the relations $\mathbf{E}^2 = \mathbf{B}^2$ and $\mathbf{E}^2 + \mathbf{B}^2 = 2|\mathbf{E} \times \mathbf{B}|$ always hold, the implied by the above equations internal energy-momentum exchange between the two subsystems mathematically identified as $\mathbf{E}$ and $\mathbf{B}$ is performed? May be we have not made the right mathematical identification of the subsystems?

The suggestion that we should come to is that we must very carefully make significant conclusions and assumptions about the dynamical structure of the physical system under consideration, especially in case of continuous and spatially propagating composite systems. In this work we shall try to follow the rule that an adequate choice of mathematical structures in the theory must correspond to a sufficiently well recognized, identified and interacting physical structures. In the spirit of this, our hope is that, in general, an isolated time-stable nonstatic continuous physical system should correspond to a completely integrable distribution $\Delta$ on an appropriately defined manifold, its interacting subsystems $\Sigma_1, \Sigma_2, ...$ should correspond to NONintegrable subdistributions $\Delta_1, \Delta_2, ...$ of $\Delta$, and the very interaction, i.e. local energy-momentum exchange, between any two subsystems $\Sigma_i \leftrightarrow \Sigma_j$ can be described mathematically in terms of the corresponding Frobenius curvatures $\Omega_{ij}$.
Chapter 2

Nonrelativistic Considerations

2.1 How to understand the Coulomb force?

Usually, the Coulomb force field is introduced starting with the Coulomb force law: \( f = \frac{qQ}{r^2} \), where \( q \) and \( Q \) are the charges of two small bodies (usually considered as point-particles) and \( r \) is the euclidean distance between them. Two fields \( E_Q \) and \( E_q \), considered as generated correspondingly by any of the two charges \( Q \) and \( q \), are defined by the relations

\[
E_Q = \lim_{q \to 0} \left( \frac{f}{q} \right) \quad \text{and} \quad E_q = \lim_{Q \to 0} \left( \frac{f}{Q} \right).
\]

Now the so defined quantities \( E_Q \) and \( E_q \) are considered as vector fields, i.e. local objects, defined outside the regions occupied by the source objects of charge magnitudes \( Q \) and \( q \), and are interpreted as force-fields acting on other unit charges, hence, the force acting on the \( q \)-charge is \( qE_Q \) and the force acting on the \( Q \)-charge is \( QE_q \).

The mechanical behaviour of the \( q \)-particle in the reference frame connected with the \( Q \)-particle is defined by the Newton law \( \dot{p}_q = qE_Q \), where \( p_q \) is the mechanical momentum of the \( q \)-particle in this reference frame (clearly, \( p_Q = 0 \)). This "dynamical" equation presupposes that the change of the mechanical momentum of the \( q \)-particle comes from (or goes to) the corresponding change of the momentum carried by the field \( E_Q \) in accordance with the universal momentum conservation law. However, such a justification assumes that the field \( E_Q \) carries non-zero energy and momentum and is capable to exchange them with other physical objects. How much is, for example, the field momentum? This question requires corresponding definition of the field momentum. This directs our attention to the theory based on Maxwell vacuum equations. Maxwell theory, however, gives three objections to this understanding of the physical situation:

- the field \( E_Q(x, y, z) \), considered as local object, i.e. vector field, satisfies the vacuum equations \( \text{rot} \; E_Q = 0, \ \text{div} \; E_Q = 0 \) outside its source, and according to the theory, every such solution field conserves its energy, momentum and angular momentum;
- the static nature of the field requires no time-changes of any field characteristic;
- the field momentum density in the theory is proportional to the Poynting vector, so, neither the electric field \( E \) nor the magnetic field \( B \) are allowed to carry momentum separately.

In general, every vacuum solution of Maxwell equations conserves its energy, momentum and angular momentum, so, \textbf{NO vacuum solution \((E, B)\) should be allowed to participate directly as force generating agent in the expression} \( qE + \frac{q}{c} \nu \times B \).

We see that for vacuum fields the usual setting "charged particle in external field" does not work: the field can not afford any chance to the "test particle" to gain locally momentum from...
In view of this, how to understand the Coulomb force law from theoretical point of view in the frame of Maxwell theory?

In order to answer this question we make the following considerations. First, some clarifications concerning the structure and admissible changes of the physical situation. We have two mass particles carrying electric charges \( q \) and \( Q \). The two masses “generate” two gravitational fields which are further neglected as physical factors. The two charges “generate” two electric fields: \( \omega_q \) - denoted further just by \( \omega \), and \( \Omega_q \) - denoted further just by \( \Omega \). The whole system is static and time stable, so, the two fields and the two particles considered as mechanical objects, exist consistently with each other.

**Remark.** We put the term “generate” in commas intentionally, because in this case the charge-field configuration we consider as the real one, i.e. no charged particle can exist without such a field, and no such a field can exist without charged particle. In the theory the opposite of the usually stated idea that charges generate fields is realized: not the charged particle “generates” field, but the field “generates” charge through the Gauss theorem, so, both the charge and the field aspects of the situation should be paid equal respect.

Since we consider electrostatic situation, no magnetic fields are assumed to be present. The admissible changes, by assumption, do NOT lead to destruction of any of the objects. Paying now due respect to the Gauss theorem we have to assume that each of the two fields is NOT defined inside the small region that its source occupies. Therefore, the two fields \( \omega \) and \( \Omega \) are defined on the topologically non-trivial space \( \Sigma = \mathbb{R}^3 \setminus (W_q \cup W_Q) \), where \( W_q \) and \( W_Q \) are the two small nonintersecting regions, treated further as two balls with boundaries \( S_q^2 \) and \( S_Q^2 \), occupied by the two particles. How to specify the mathematical nature of \( \omega \) and \( \Omega \)?

The topology of \( \Sigma \), which **must be kept unchanged**, and the assumed spherical symmetry of each of the two fields with respect to (the centers of) \( W_q \) and \( W_Q \) suggest to choose the fields \( \Omega \) and \( \omega \) as spherically symmetric representatives of the 2-dimensional cohomology group of \( \Sigma \). We introduce two spherical coordinate systems \((r, \theta, \varphi)\) and \((\bar{r}, \bar{\theta}, \bar{\varphi})\), originating at the centers of \( W_Q \) and \( W_q \) respectively, so, any two spherically symmetric with respect to the centers of \( W_Q \) and \( W_q \) 2-forms will look as follows:

\[
\Omega(r, \theta, \varphi) = h(r) \sin \theta d\theta \wedge d\varphi, \quad \omega(\bar{r}, \bar{\theta}, \bar{\varphi}) = \tilde{h}(\bar{r}) \sin \bar{\theta} d\bar{\theta} \wedge d\bar{\varphi}.
\]

Being representatives of corresponding cohomology classes \( \Omega \) and \( \omega \) must satisfy \( d\Omega = 0 \) and \( d\omega = 0 \), so, \( h(r) = \text{Const} \) and \( \tilde{h}(\bar{r}) = \text{const} \). We denote \( \text{Const} = Q \) and \( \text{const} = q \). Now, the euclidean Hodge star operator \( * \) and the euclidean identification of vectors and covectors give

\[
E_Q = *\Omega(r, \theta, \varphi) = \frac{Q}{r^2} dr, \quad E_q = *\omega(\bar{r}, \bar{\theta}, \bar{\varphi}) = \frac{q}{\bar{r}^2} \bar{d}\bar{r}.
\]

Going further we note that at every point of \( \Sigma \) the real field configuration is built of two physical fields of the *same physical nature*, therefore, the resulted stress should depend on the local mutual interference/interaction between the two stress generating fields \( \Omega \) and \( \omega \). The point is how to model mathematically this local interaction of the two fields? At this moment the Maxwell stress tensor \( M^i_j = Z_i Z^j - \frac{1}{2} Z^2 \delta^j_i = \frac{1}{4} \Omega(Z)_{mn} \Omega(Z)^{mn} \delta^j_i - \Omega(Z)_{im} \Omega(Z)^{jm} \), \( \Omega(Z) = *\Omega(Z) \), which is defined by any vector field \( Z \), could help us as follows.

Mathematically, the tensor \( M \) can be considered as a quadratic map from the vector fields on \( \Sigma \) to \((1, 1)\)-tensors, i.e. to the linear maps in the linear space of vector fields. Each of our two fields generates such \((1, 1)\)-tensor field: \( M(\omega) \) and \( M(\Omega) \). Recall now that every quadratic map \( \Phi \) between two linear spaces generates a bilinear map \( T_\Phi \) according to \( T_\Phi(x, y) = \Phi(x + y) - \Phi(x) - \Phi(y) \), where \((x, y)\) are corresponding variables. So, in our case we can define corresponding bilinear map.
Identifying the vector fields and 1-forms on Σ by means of the euclidean metric $e$: $\mathbf{E}^i = e^{ij} \mathbf{E}_j$, for the two Maxwell stress tensors, expressing here stresses of topological origin, we have

$$M_q \equiv M(\mathbf{E}_q) = \mathbf{E}_q \otimes \mathbf{E}_q - \frac{1}{2} \mathbf{E}_q^2 \, id_{T \Sigma}, \quad M_Q \equiv M(\mathbf{E}_Q) = \mathbf{E}_Q \otimes \mathbf{E}_Q - \frac{1}{2} \mathbf{E}_Q^2 \, id_{T \Sigma}.$$

The corresponding bilinear map $T$ will be

$$T(\mathbf{E}_q, \mathbf{E}_Q) = \mathbf{E}_q \otimes \mathbf{E}_Q + \mathbf{E}_Q \otimes \mathbf{E}_q - \mathbf{E}_q \cdot \mathbf{E}_Q \, id_{T \Sigma}. \quad (2.1)$$

In components we have correspondingly

$$(M_q)^i_j = (E_q)_i (E_q)_j - \frac{1}{2} (E_q)^2 \delta^i_j = \frac{1}{4} \omega_{mn} \omega^{mn} \delta^i_j - \omega_{im} \omega^{jm},$$

$$(M_Q)^i_j = (E_Q)_i (E_Q)_j - \frac{1}{2} (E_Q)^2 \delta^i_j = \frac{1}{4} \Omega_{mn} \Omega^{mn} \delta^i_j - \Omega_{im} \Omega^{jm},$$

$$T^i_j(\mathbf{E}_q, \mathbf{E}_Q) = (E_q)_i (E_Q)_j + (E_Q)_i (E_q)_j - \mathbf{E}_q \cdot \mathbf{E}_Q \delta^i_j.$$ 

The tensor field $(-\frac{1}{4\pi} T)$ may be called mutual stress tensor, or interaction stress tensor. In $T^* \Sigma \otimes T \Sigma$ we have the trace form $tr$, and on $\Sigma$ we have the standard volume form $\omega_0 = dx \wedge dy \wedge dz$. So we can form the object $tr \otimes \omega_0$. By definition, the quantities

$$w = (tr \otimes \omega_0)(-\frac{1}{4\pi} T) = -\frac{1}{4\pi} <tr, T>, \omega_0, \quad \text{and} \quad U = \int_{\Sigma} w \quad (2.2)$$

will be called interaction energy density and interaction energy for $\omega$ and $\Omega$. Clearly, $w$ and $U$ may be positive, zero, or negative. Further we shall follow the rule that an isolated (quasi-static) physical system of this kind tends to configurations with minimal value $U_{\text{min}}$ of the integral interaction energy $U$, hence, an intrinsically induced drifting between two allowed static configurations should satisfy the relation $\delta U < 0$.

In order to compute $U$ we compute first $w$ and obtain

$$w = \frac{1}{4\pi} \mathbf{E} \cdot \mathbf{E} \omega_0 = \frac{1}{8\pi} (\Omega \wedge \ast \omega + \omega \wedge \ast \Omega) = -\frac{1}{8\pi} \left[ d\left(\frac{q}{r} \Omega\right) + d\left(\frac{Q}{r} \omega\right) \right].$$

Making use of the Stokes theorem the integral of $w$ over $\Sigma$ is transformed to 2-dimensional surface integral over the boundary $\partial \Sigma$ of $\Sigma$: $\partial \Sigma = S^2_r, p = \infty \cup S^2_q \cup S^2_Q$. On $S^2_\infty$ the corresponding integral has zero value. So, in the induced on $\partial \Sigma$ orientation, and denoting by $R_q$ and $R_Q$ the radiiuses of $S^2_q$ and $S^2_Q$ respectively, we have

$$U = \frac{q}{2} \frac{1}{4\pi R^2_q} \int_{S^2_q \cup S^2_Q} R^2_q \frac{Q \sin \theta \, d\theta \wedge d\varphi}{r} + \frac{Q}{2} \frac{1}{4\pi R^2_q} \int_{S^2_q \cup S^2_Q} R^2_q \frac{q \sin \theta \, d\theta \wedge d\varphi}{r}.$$ 

On $S^2_q$ we have $\bar{r} = \text{const}$ and $\int_{S^2_q} \Omega = 0$. Similarly, on $S^2_Q$ we have $r = \text{const}$ and $\int_{S^2_Q} \omega = 0$. Notice further that $\frac{1}{r}$ is a harmonic function, so, at every point $p \in \Sigma$ it can be represented by its average value on the corresponding 2-sphere centered at $p$. Now, the first integral reduces to integral over the 2-sphere $S^2_\Sigma$ and it is equal to $\frac{q Q}{2 R}$; similarly, the second integral reduces to integral over the 2-sphere $S^2_q$ and has the same value, $\frac{q Q}{2 R}$, where $R$ is the euclidean distance between the centers of the two spheres. Thus, the computation gives finally $U = \frac{q Q}{R}$.

Now, according to the above mentioned rule that $\delta U < 0$, and that $q$ and $Q$ do not change, for the case $q, Q > 0$ we obtain $\delta U = -\frac{q Q}{2 R} \delta R < 0$, so $\delta R > 0$, i.e. repulsion should be expected;
and for the case \( qQ < 0 \) we obtain \( \delta U = -\frac{qQ}{R^2}\delta R < 0 \), so \( \delta R < 0 \), i.e. attraction should be expected.

The above consideration clearly suggests the idea that the Coulomb force law originates from available interaction between the two fields \( \mathbf{E}_Q \) and \( \mathbf{E}_q \) under quasistatic changes of the integral interaction energy \( U \) leading to minimization of \( U \). In fact, if \( \delta U \) changes then the change \( \delta U = -\frac{qQ}{R^2}\delta R \) must be carried away mechanically by the \((q,m)\)-particle: \( \delta \mathbf{p} = \delta U \), since there is no other physical factor in the system considered. So, the Coulomb force can be understood as an integral characteristic of the system, therefore its field, i.e local, interpretation may be reconsidered. On the other hand, in the corresponding spherical coordinates, \((Q*\omega)\) and \((q*\Omega)\) look very much as \( \delta U \), but this first-sight resemblance should not mislead us. The difference is quite serious: \((Q*\omega)\) and \((q*\Omega)\) are 1-forms, local objects by definition, while \( \frac{qQ}{R^2}\delta R \) is not local object, \( R \) is not the coordinate \( r \) and, contrary to \( dr \), \( \delta R \) is not 1-form on \( \Sigma \): we should not try to obtain local objects just through noting some possible change tendencies of integral characteristics of a system. We may allow ourselves to call \((\omega)\) and \((\Omega)\), or \((*\omega)\) and \((*\Omega)\), Coulomb fields but NOT Coulomb force fields because they can NOT generate any direct local change of momentum, since as we mentioned earlier, these fields are static and they conserve their energy, momentum and angular momentum. The entire local force is given in the theory by the divergence of the Maxwell stress tensor which is a nonlinear object, namely, a bilinear combination of the field components and their derivatives and all its terms are mutually compensated in the static case.

As we saw, the Coulomb force gets an admissible interpretation as an integral characteristic of the system describing some realizable integral tendencies to minimization of the integral interaction energy \( U \) of the two fields at the expense of the kinetic energy of the two particles. Surely, \( \omega \) and \( \Omega \) carry some local physical information but in a quite indirect manner: except spherical symmetry (which, of course, is not specific only for electric fields) any of these two local objects can not clarify the physical nature of the local changes in the space when charged particles are around. In other words, from local point of view, we could not identify \( \omega \) as electric field. Any topologically nontrivial region of the kind "\( \mathbb{R}^3 \) minus a point" generates such fields, so, the electric nature of the field can be proved only by means of additional procedures concerning the integral structure of the system.

The topological interpretation of \( \omega \) and \( \Omega \) also suggests that the description is rather integral than local: although \( \omega \) and \( \Omega \) are local objects, in fact they are just specially chosen representatives of integral characteristics of the physical system considered: they specify the topology of the space where the two fields are defined. For another example, the Newton gravitation force law looks the same except the different interpretation of the corresponding topological numbers as masses. Following the same argument, the Newton gravitation force law is of integral nature and shows similar tendencies except that the masses are always positive numbers, so, the corresponding interaction energy should be always negative, which does not allow repulsion. But this integral difference says too little about the local nature of the two physically different field structures.

**Remark.** As for the relativistic formulation, \( \omega \) and \( \Omega \) can also be correctly defined and used, just the topology of the space where the two fields are defined very slightly changes: \( \Sigma \rightarrow \Sigma \times \mathbb{R} \).

The above consideration makes us think that, from theoretical point of view, the Maxwell stress tensor field is the right object in terms of which local force fields must be defined, namely, through computing its divergence. If the field is static and free then this divergence is zero and represents physically admissible quasistatic local changes, i.e. corresponding equations of motion, and any additional conditions must be consistent with this zero divergence. In our static
case we have \( d\omega = 0 \), \( d \ast \omega = 0 \), so this divergence is zero:

\[
\nabla_i M^i_k dx^k = \left[ (\ast \omega)^i (d \ast \omega)_{ik} + \frac{1}{2} \omega^{ij} (d \omega)_{ijk} \right] dx^k = \text{rot} \mathbf{E} \times \mathbf{E} + \mathbf{E} \text{div} \mathbf{E} = 0,
\]

where \( \frac{1}{2} \omega^{ij} (d \omega)_{ijk} dx^k = \mathbf{E} \text{div} \mathbf{E} \), \( (\ast \omega)^i (d \ast \omega)_{ik} dx^k = (\text{rot} \mathbf{E}) \times \mathbf{E} \), \((x^1, x^2, x^3) = (x, y, z)\), and vectors and covectors are identified through the euclidean metric. Hence, we can not gain energy from the field \( \mathbf{E} \) in local way. Therefore, the Coulomb force has not a local nature.

The situation seriously changes when we are going to consider independent and self-consistent time-dependent and time-stable spatially finite physical objects of electromagnetic nature, namely, we have no such topologically motivated suggestions to choose adequate mathematical objects been able to represent appropriately the corresponding physical stresses. Hence, the mathematical model must be created on the basis of assumptions of quite different nature, for example: requirements for definite and appropriately defined integrability properties representing the object’s time stability; experimentally proved and traditionally assumed straight-line propagation of the energy-density; orthogonality of the electric and magnetic components of the field suggesting absence of local interaction energy between the electric and magnetic components or their new versions; some notion for internal energy redistribution during time-evolution, etc. In our view, in such cases, the eigen and other algebraic properties of the corresponding stress-energy-momentum tensor field should play a basic role.

In view of this in the next two chapters we consider in a more detail from formal point of view the general Maxwell stress tensor as a starting physically meaningful theoretical object.

2.2 A non-physical view on Maxwell stress tensor

The mathematical identities have always attracted the attention of theorists, in particular, those identities which involve the derivatives of the objects of interest (differential identities). A well known such example is the Bianchi identity satisfied by any connection components: this identity is a second order system of (in general, nonlinear) partial differential equations. The gauge interpretation of classical Maxwell electrodynamics, as well as the Yang-Mills theory, substantially make use of this identity. Such identities are of particular importance when on the two sides of \( "=\)" stay correctly (i.e. in a coordinate free way) defined expressions.

The physical reason to consider a couple of vector fields on \( \mathbb{R}^3 \) (or a couple of spatial-directed vector fields on \( \mathbb{R}^3 \times (time - coordinate) \)) as mathematical representation of a free time dependent physical field comes from the observation that, being free, it propagates translationally along straight lines, so the identification properties of the field should be searched inside the 2-dimensional distribution that is orthogonal to these straight lines.

We begin with the well known differential relation satisfied by the square of every vector field \( V \) on the euclidean space \( \mathbb{R}^3 \). Our attention is directed to the square of \( V \) just because of the experimentally suggested assumption that \( V^2 \) measures the energy-density of each of the electric and magnetic components.

Let \( \mathbb{R}^3 \) be related to the standard coordinates \((x^1 = x, y, z), i = 1, 2, 3\); we denote by \( "\times"\) the vector product, and make use of the \( \nabla\)-operator:

\[
\frac{1}{2} \nabla(V^2) = V \times \text{rot} V + (V, \nabla)V = V \times \text{rot} V + \nabla_V V.
\]

Clearly, on the two sides of this relation stay well defined quantities, i.e. quantities defined in a coordinate free way. The first term on the right hand side of this identity accounts for the rotational component of the change of \( V \), and the second term accounts mainly for the
translational component of the change of \( V \). Making use of component notation we write down the last term on the right side as follows (summation over the repeated indices):

\[
(\nabla_V V)^j = V^i \nabla_i V^j = \nabla_i (V^i V^j) - V^j \nabla_i V^i = \nabla_i (V^i V^j) - V^j \text{div} V.
\]

Substituting into the first identity, and making some elementary transformations we obtain

\[
\nabla_i \left( V^i V^j - \frac{1}{2} \delta^{ij} V^2 \right) = \left[ (\text{rot} V) \times V + V \text{div} V \right]^j,
\]

where \( \delta^{ij} = 1 \) for \( i = j \), and \( \delta^{ij} = 0 \) for \( i \neq j \) are the euclidean metric components. If now \( W \) is another vector field it must satisfy the same above identity:

\[
\nabla_i \left( W^i W^j - \frac{1}{2} \delta^{ij} W^2 \right) = \left[ (\text{rot} W) \times W + W \text{div} W \right]^j.
\]

Summing up these two identities we obtain the new identity

\[
\nabla_i M^{ij} \equiv \nabla_i \left( V^i V^j + W^i W^j - \delta^{ij} \frac{V^2 + W^2}{2} \right) = \left[ (\text{rot} V) \times V + V \text{div} V + (\text{rot} W) \times W + W \text{div} W \right]^j. \quad (2.3)
\]

We note the invariance of \( M^{ij} \) with respect to the transformations \( (V, W) \rightarrow (V \cos \alpha - W \sin \alpha, V \sin \alpha + W \cos \alpha) \) where \( \alpha (x, y, z) \) is an arbitrary real function. With respect to the slightly more general transformation \( (V, W) \rightarrow (V a - W b, V b + W a) \) where \( (a, b) \) are real nonzero functions, we obtain \( M(V, W) \rightarrow (a^2 + b^2) M(V, W) \). Hence, the transformations \( (V, W) \rightarrow (V a - W b, V b + W a) \) do not change the eigen directions structure of \( M^{ij} \).

The expression inside the round brackets on the left of (2.3), denoted by \( M^{ij} \), looks formally the same as the introduced by Maxwell tensor from physical considerations concerned with the electromagnetic stress energy properties of continuous media in presence of external electromagnetic field. This allows to call formally any such tensor Maxwell stress tensor generated by the two vector fields \( (V, W) \). The term ”stress” in this general mathematical setting is not topologically motivated as in the Coulomb case, but could be justified in the following way.

Every vector field on \( \mathbb{R}^3 \) generates corresponding flow by means of the trajectories started from some domain \( U_o \subset \mathbb{R}^3 \): at the moment \( t > 0 \) the domain \( U_o \) is diffeomorphically transformed to a new domain \( U_t \subset \mathbb{R}^3 \). Having two vector fields on \( \mathbb{R}^3 \) we obtain two compatible flows, so, the points of any domain \( U_o \subset \mathbb{R}^3 \) are forced to accordingly move to new positions.

We emphasize the following moments: first, the identity (2.3) is purely mathematical; second, on the two sides of (2.3) stay well defined coordinate free quantities; third, there is no \( V \leftrightarrow W \) interaction stress: the full stress is a sum of the \( V \)-stress and \( W \)-stress.

Physically, we say that the corresponding physical medium that occupies the spatial region \( U_o \) and is parametrized by the points of the mathematical subregion \( U_o \subset \mathbb{R}^3 \), is subject to compatible and admissible physical ”stresses” generated by physical interactions mathematically described by the couple of vector fields \( (V, W) \), and these physical stresses are quantitatively described by the corresponding physical interpretation of the tensor \( M^{ij} (V, W) \).

We note that the stress tensor \( M^{ij} \) in (2.3) is subject to the divergence operator, and if we interpret the components of \( M^{ij} \) as physical stresses, then the left hand side of (2.3) acquires in general the physical interpretation of force density. Of course, in the static situation as it is given by relation (2.3), no energy-momentum propagation is possible, so at every point the forces mutually compensate: \( \nabla_i M^{ij} = 0 \). If propagation is allowed then the force field is NOT zero: \( \nabla_i M^{ij} \neq 0 \), and we may identify the right hand side of (2.3) as a real time-change of
appropriately defined momentum density $\mathbf{P}$. So, assuming some expression for this momentum density $\mathbf{P}$ we are ready to write down corresponding field equation of motion of Newton type through equalizing the spatially directed force densities $\nabla_i M^{ij}$ with the momentum density changes along the time coordinate, i.e. equalizing $\nabla_i M^{ij}$ with the $\partial t$-derivative of $\mathbf{P}$, where $c = \text{const}$ is the translational propagation velocity of the momentum density flow of the physical system $(V, W)$. In order to find how to choose $\mathbf{P}$ we have to turn to the intrinsic physical properties of the field, so, it seems natural to turn to the eigen properties of $M^{ij}$, since, clearly, namely $M^{ij}$ is assumed to carry the physical properties of the field.

2.3 Notes on the eigen properties of Maxwell stress tensor

We consider $M^{ij}(\mathbf{E}, \mathbf{B})$ at some point $p \in \mathbb{R}^3$ and assume that in general the vector fields $\mathbf{E}$ and $\mathbf{B}$ are linearly independent, so $\mathbf{E} \times \mathbf{B} \neq 0$. Let the coordinate system be chosen such that the coordinate plane $(x, y)$ to coincide with the plane defined by $\mathbf{E}(p), \mathbf{B}(p)$. In this coordinate system $\mathbf{E} = (E_1, E_2, 0)$ and $\mathbf{B} = (B_1, B_2, 0)$, so, identifying the contravariant and covariant indices through the Euclidean metric $\delta^{ij}$ (so that $M^{ij} = M_{ij} = M_{ji}$), we obtain the following nonzero components of the stress tensor:

\[
M_1^1 = (E^1)^2 + (B^1)^2 - \frac{1}{2}(E^2 + B^2); \quad M_1^2 = M_1^2 = E^1 E_2 + B_1 B^2;
\]

\[
M_2^2 = (E^2)^2 + (B^2)^2 - \frac{1}{2}(E^2 + B^2); \quad M_3^3 = -\frac{1}{2}(E^2 + B^2).
\]

Since $M_1^1 = -M_2^2$, the trace of $M$ is $\text{Tr}(M) = -\frac{1}{2}(E^2 + B^2)$. The eigen value equation acquires the simple form $[(M_1^1)^2 - (\lambda)^2] + (M_2^2)^2(M_3^3 - \lambda) = 0$. The corresponding eigen values are

\[
\lambda_1 = -\frac{1}{2}(E^2 + B^2); \quad \lambda_{2,3} = \pm \sqrt{(M_1^1)^2 + (M_2^2)^2} = \pm \frac{1}{2}\sqrt{(I_1)^2 + (I_2)^2},
\]

where $I_1 = B^2 - E^2$, $I_2 = 2E \cdot B$. The corresponding to $\lambda_1$ eigen vector $Z_1$ must satisfy the equation $\mathbf{E}(\mathbf{E} \cdot Z_1) + \mathbf{B}(\mathbf{B} \cdot Z_1) = 0$, and since $(\mathbf{E}, \mathbf{B})$ are linearly independent, the two coefficients $(\mathbf{E} \cdot Z_1)$ and $(\mathbf{B} \cdot Z_1)$ must be equal to zero, therefore, $Z_1 
= 0$ must be orthogonal to $\mathbf{E}$ and $\mathbf{B}$, i.e. $Z_1$ must be colinear to $\mathbf{E} \times \mathbf{B}$.

The other two eigen vectors $Z_{2,3}$ satisfy correspondingly the equations

\[
\mathbf{E}(\mathbf{E} \cdot Z_{2,3}) + \mathbf{B}(\mathbf{B} \cdot Z_{2,3}) = \left[ \pm \frac{1}{2}\sqrt{(I_1)^2 + (I_2)^2} + \frac{1}{2}(E^2 + B^2) \right] Z_{2,3}.
\]

Taking into account the easily verified relation

\[
\frac{1}{4}[(I_1)^2 + (I_2)^2] = \left( \frac{E^2 + B^2}{2} \right)^2 - |E \times B|^2, \quad \text{so} \quad \frac{E^2 + B^2}{2} - |E \times B| \geq 0,
\]

we conclude that the coefficient before $Z_{2,3}$ on the right is always different from zero, therefore, the eigen vectors $Z_{2,3}(p)$ lie in the plane defined by $(\mathbf{E}(p), \mathbf{B}(p))$, $p \in \mathbb{R}^3$. In particular, the above mentioned transformation properties of the Maxwell stress tensor $M(V, W) \rightarrow (a^2 + b^2) M(V, W)$ show that the corresponding eigen directions do not change under the transformation $(V, W) \rightarrow (V a - W b, V b + W a)$.

The above consideration suggests that the intrinsically determined potential dynamical abilities of propagation of the field are: translational along $(\mathbf{E} \times \mathbf{B})$, and rotational inside the plane defined by $(\mathbf{E}, \mathbf{B})$. 

14
It is natural to ask now under what conditions the very \( E \) and \( B \) may be eigen vectors of \( M(E,B) \)? Assuming \( \lambda_2 = \frac{1}{2}\sqrt{(I_1)^2 + (I_2)^2} \) and \( Z_2 = E \) in the above relation (*) and having in view that \( E \times B \neq 0 \) we obtain that \( E(E^2) + B(E \times B) \) must be proportional to \( E \), so \( E \cdot B = 0 \), i.e. \( I_2 = 0 \). Moreover, substituting now \( I_2 = 0 \) in that same relation we obtain
\[
E^2 = \frac{1}{2}(B^2 - E^2) + \frac{1}{2}(E^2 + B^2) = B^2, \quad \text{i.e.,} \quad I_1 = 0.
\]
The case "-" sign before the square root, i.e. \( \lambda_3 = -\frac{1}{2}\sqrt{(I_1)^2 + (I_2)^2} \), leads to analogical conclusions just the role of \( E \) and \( B \) is exchanged.

**Corollary.** \( E \) and \( B \) may be eigen vectors of \( M(E,B) \) only if \( I_1 = I_2 = 0 \).

The above notices suggest to consider in a more detail the case \( \lambda_2 = -\lambda_3 = 0 \) for the vacuum case. We shall show, making use of the Lorentz transformation in 3-dimensional form that, if these two relations do not hold then under \( E \times B \neq 0 \) the translational velocity of propagation is less then the speed of light in vacuum \( c \). Recall first the transformation laws of the electric and magnetic vectors under Lorentz transformation defined by the 3-velocity vector \( v \) and corresponding parameter \( \beta = \frac{v}{c}, v = |v| \). If \( \gamma \) denotes the factor \( 1/\sqrt{1 - \beta^2} \) then we have
\[
E' = \gamma E + \frac{1 - \gamma}{v^2} v(E \cdot v) + \frac{\gamma}{c} v \times B,
\]
\[
B' = \gamma B + \frac{1 - \gamma}{v^2} v(B \cdot v) - \frac{\gamma}{c} v \times E.
\]
Assume first that \( I_2 = 2E \cdot B = 0 \), i.e. \( E \) and \( B \) are orthogonal, so, in general, in some coordinate system we shall have \( E \times B \neq 0 \).

If \( I_1 > 0 \), i.e. \( |E| < |B| \), we shall show that the conditions \( E' = 0, v \cdot B = 0, \infty > \gamma > 0 \) are compatible. In fact, these assumptions lead to \( \gamma v \cdot E + (1 - \gamma)(E \cdot v) = 0 \), i.e. \( E \cdot v = 0 \). Thus, \( c|E| = v|B||\sin(v,B)| \), and since \( v \cdot B = 0 \) then \( |\sin(v,B)| = 1 \). It follows that the speed \( v = c|\frac{E}{B}| < c \) is allowed.

If \( I_1 < 0 \), i.e. \( |E| > |B| \), then the choice \( B' = 0 \) and \( v \cdot E = 0 \) analogically lead to the conclusion that the speed \( v = c|\frac{B}{E}| < c \) is allowed.

Assume now that \( I_2 = 2E \cdot B \neq 0 \). We are looking for a reference frame \( K' \) such that \( E' \times B' = 0 \), while in the reference frame \( K \) we have \( E \times B \neq 0 \). We choose the relative velocity \( v \) such that \( v \cdot E = v \cdot B = 0 \). Under these conditions the equation \( E' \times B' = 0 \) reduces to
\[
E \times B + \frac{v}{c}(E^2 + B^2) = 0, \quad \text{so,} \quad \frac{v}{c} = |E \times B|/(E^2 + B^2).
\]
Now, from the above mentioned inequality \( E^2 + B^2 - 2|E \times B| \geq 0 \) it follows that \( \frac{v}{c} < 1 \).

Physically, these considerations show that under nonzero \( I_1 \) and \( I_2 \) the translational velocity of propagation of the field, and of the field energy density of course, will be less than \( c \). Hence, the only realistic choice for the vacuum case (where this velocity is assumed by definition to be equal to \( c \)), is \( I_1 = I_2 = 0 \), which is equivalent to \( E^2 + B^2 = 2|E \times B| \). Hence, assuming \( |Tr(M)| \) to be the energy density of the field, the names "electromagnetic energy flux" for the quantity \( cE \times B \), and "momentum" for the quantity \( \frac{1}{c}E \times B \), seem well justified without turning to any field equations.

These considerations show also that if \( I_1 = 0 \), i.e. \( |E|^2 = |B|^2 \) during propagation, then the electric and magnetic components of the field carry always the same energy density, so, a local mutual energy exchange between \( E \) and \( B \) is not forbidden in general, but, if it takes place, it must be *simultaneous* and in *equal quantities*. Hence, under zero invariants \( I_1 = 0 \)
and \( I_2 = 2\mathbf{E} \cdot \mathbf{B} = 0 \), internal energy redistribution among possible subsystems of the field is allowed but such an exchange should occur without available interaction energy because the full energy density is always equal to the sum of the energy carried by the electric and magnetic components of the field. However, the required time stability and propagation with velocity \( c \) of the field suggest/imply also available internal momentum exchange since under these conditions the energy density is always equal to the momentum magnitude \( |\mathbf{E} \times \mathbf{B}| \), and \( \mathbf{E} \) and \( \mathbf{B} \) can not carry momentum separately. Moreover, besides \( (\mathbf{E}, \mathbf{B}) \), another subsystem of the field has to be constructed out of \( (\mathbf{E}, \mathbf{B}) \) such that both these two subsystems to carry always the same quantity of energy-momentum, so the exchange also must be in equal quantities.

After these preliminary considerations we procede to write down dynamical equations for the field through specializing how the internal local momentum exchange is realized keeping always in mind that the free field energy density propagates translationally with the speed of light, so the relations

\[
I_1 = I_2 = 0, \quad \text{i.e.} \quad \mathbf{E}^2 + \mathbf{B}^2 = 2|\mathbf{E} \times \mathbf{B}|
\]

must always hold.

### 2.4 Nonlinear equations for the electromagnetic field

We are going to consider time dependent fields, and begin with noting once again that the assumption that the energy density of the field coincides with \( |\text{Tr}(\mathbf{M})| = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \) presupposes that there is NO interaction energy between the electric and magnetic components of the field: the full stress tensor (and the energy density, in particular) is a sum of the stress tensors determined separately by \( \mathbf{E} \) and \( \mathbf{B} \). Of course, this does NOT mean that there is no energy exchange between the electric and magnetic components, but if such an exchange takes place, it must occur simultaneously and in equal quantities.

Now, following the above stated idea that the field momentum density could be responsible for such an internal energy-momentum exchange, we have to find at least two appropriate subsystems of the field which subsystems are NOT the electric \( \mathbf{E} \) and magnetic \( \mathbf{B} \) ones, but are constructed out of them. Note that the assumption that the field momentum is given by \( \frac{1}{c} \mathbf{E} \times \mathbf{B} \), i.e. it is a bilinear function of the electric and magnetic components, and that the local energy is always equal to \( |\mathbf{E} \times \mathbf{B}| \), suggests that the electromagnetic momentum of the field is of interaction nature. The point now is to get some clarification how such a local momentum exchange (and the corresponding energy exchange) takes place and to find appropriate mathematical representatives of the corresponding partners realizing such special kind of energy exchange, since neither \( \mathbf{E} \) nor \( \mathbf{B} \) are able to carry momentum separately (although each of them may carry energy independently of the other).

In view of the above we shall assume that the field keeps its identity through adopting some special and appropriate dynamical behavior according to its intrinsic capabilities. Hence, the corresponding dynamical/field equations must be consistent with the intrinsic stress-energy-momentum nature of the field. So, our basic assumption is that the Maxwell stress tensor \( \mathbf{M}(\mathbf{E}, \mathbf{B}) \) should play the basic role, and its zero-divergence in the static case should suggest how to determine the allowed dynamics.

Recalling that any member of the family

\[
(\mathcal{E}, \mathcal{B}) = (\mathbf{E} \cos \alpha - \mathbf{B} \sin \alpha; \ \mathbf{E} \sin \alpha + \mathbf{B} \cos \alpha), \quad \alpha = \alpha(x, y, z, t),
\]

generate the same Maxwell stress tensor, the most natural assumption should read like this: the field \( (\mathcal{E}, \mathcal{B}) \) is looking for an energy-momentum exchanging partner inside the
\(\alpha(x, y, z, t)\)-family of \((E, B)\)-couples, and any such couple identifies itself through appropriate (local) interaction, defining in this way corresponding dynamical behavior.

Replacing \((V, W)\) in \((2.3)\) with \((E, B)\) we obtain

\[
\nabla_i M^{ij} = \nabla_i \left( E^i E^j + B^i B^j - \delta^{ij} \frac{E^2 + B^2}{2} \right) = \left[ (\text{rot} E) \times E + \text{Ediv} E + (\text{rot} B) \times B + B\text{div} B \right]^{ij}.
\]

As we mentioned, in the static case, i.e. when the vector fields \((E, B)\) do not depend on the time coordinate \(\xi = ct\), NO propagation of field momentum density \(P\) should take place, so, at every point, where \((E, B) \neq 0\), the stress generated forces must mutually compensate, i.e. the divergence \(\nabla_i M^{ij}\) should be equal to zero: \(\nabla_i M^{ij} = 0\). In this static case Maxwell equations

\[
\text{rot} E + \frac{\partial B}{\partial \xi} = 0, \quad \text{rot} B - \frac{\partial E}{\partial \xi} = 0, \quad \text{div} E = 0, \quad \text{div} B = 0 \quad (*)
\]

give: \(\text{rot} E = \text{rot} B = 0; \text{div} E = \text{div} B = 0\), so, all static solutions to Maxwell equations determine a sufficient, but NOT necessary, condition that brings to zero the right hand side of \((2.6)\) through forcing each of the four vectors there to get zero values.

In the non-static case, i.e. when \(\frac{\partial E}{\partial t} \neq 0; \frac{\partial B}{\partial t} \neq 0\), time change and propagation of field momentum density should take place, so, a full mutual compensation of the generated by the Maxwell stresses at every spatial point local forces may NOT be possible, which means \(\nabla_i M^{ij} \neq 0\) in general. These local forces generate time-dependent momentum propagation at the spatial points. Therefore, if we want to describe this physical process of field energy-momentum density time change and spatial propagation we have to introduce explicitly the dependence \(P(E, B)\).

If we follow the classical (nonrelativistic) way of consideration and denote by \(\mathbf{\tilde{F}}\) the vector field with components \(\mathbf{\tilde{F}}^j = \nabla_i M^{ij}\), we can write down the force flow across some finite 2-surface \(S\) in the usual (and widely spread in almost all textbooks) way as \(\int_S \mathbf{\tilde{F}} \cdot ds\) (from modern point of view we should write \(i_\mathbf{\tilde{F}}(dx \wedge dy \wedge dz)\) instead of \(\mathbf{\tilde{F}} \cdot ds\) under the integral, where \(i_\mathbf{\tilde{F}}\) denotes the inner product between the vector field \(\mathbf{\tilde{F}}\) and the volume form \(dx \wedge dy \wedge dz\), i.e. to make use of the Poincare isomorphism between vector fields and 2-forms on \(\mathbb{R}^3\)). This flow generates changes of the momentum density flow across \(S\) which should be equal to \(\frac{d}{dt} \int_S P(E, B) \cdot ds\). We obtain

\[
\frac{d}{dt} \int_S P(E, B) \cdot ds = \int_S \mathbf{\tilde{F}} \cdot ds.
\]

The explicit expression for \(P(E, B)\), paying due respect to J.Poynting [13], and to J.J.Thomson, H.Poincare, M. Abraham [15], and in view of the huge, a century and a half available experence, has to be introduced by the following

**Assumption:** The entire field momentum density is given by \(P := \frac{1}{c^2} E \times B\).

According to the **Assumption** and the above interpretation of the relation \(\nabla_i M^{ij} \neq 0\), and in view of the arbitrariness of the 2-surface \(S\) we come to the vector differential equation

\[
\frac{\partial}{\partial \xi} (E \times B) = \mathbf{\tilde{F}}, \quad \xi \equiv ct, \quad (**)\]

which according to relation \((2.6)\) is equivalent to

\[
\left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) \times E + \text{Ediv} E + \left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) \times B + B\text{div} B = 0. \quad (2.7)
\]
This last equation (2.7) we write down in the following equivalent way:

\[
\left( \text{rot} \, \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} + \mathbf{B} \text{div} \, \mathbf{B} = - \left[ \left( \text{rot} \, \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) \times \mathbf{B} + \mathbf{E} \text{div} \, \mathbf{E} \right]. \tag{2.8}
\]

The above relation (**) and the corresponding differential relation (2.7)/(2.8) we consider as mathematical adequate in momentum-change terms of the electric-magnetic and magnetic-electric induction phenomena in the charge free case. We recall that these induction phenomena are described in what we call "Faraday-Maxwell theory" by the following well known integral and differential equations

\[
\frac{d}{d\xi} \int_S \mathbf{B} \cdot d\mathbf{s} = - \int_S \text{rot} \, \mathbf{E} \cdot d\mathbf{s} \rightarrow \frac{\partial \mathbf{B}}{\partial \xi} = - \text{rot} \, \mathbf{E}, \quad \text{(the Faraday induction law)},
\]

\[
\frac{d}{d\xi} \int_S \mathbf{E} \cdot d\mathbf{s} = \int_S \text{rot} \, \mathbf{B} \cdot d\mathbf{s} \rightarrow \frac{\partial \mathbf{E}}{\partial \xi} = \text{rot} \, \mathbf{B}, \quad \text{(the Maxwell displacement current law)}.
\]

We stress once again that these last Faraday-Maxwell relations have NO direct energy-momentum change-propagation (i.e. force flow) nature, so they could not be experimentally verified in a direct way. Our feeling is that, in fact, they are stronger than needed. So, on the corresponding solutions of these equations we'll be able to write down formally adequate energy-momentum change expressions, but the correspondence of these expressions with the experiment will crucially depend on the nature of these solutions. As we already mentioned, the nature of the free solutions (with no boundary conditions) to Maxwell vacuum equations with spatially finite and smooth enough initial conditions requires strong time-instability (the Poisson theorem for the D’Alembert wave equation). And time-stability of time-dependent vacuum solutions usually requires spatial infinity (plane waves), which is physically senseless. Making calculations with spatially finite parts of these spatially infinite solutions may be practically acceptable, but from theoretical viewpoint assuming these equations for basic ones seems not acceptable since the relation "time stable physical object - exact free solution" is strongly violated.

Before to go further we write down the right hand side bracket expression of (2.8) in the following two equivalent ways:

\[
\left[ \left( \text{rot} \, \mathbf{B} + \frac{\partial (-\mathbf{E})}{\partial \xi} \right) \times \mathbf{B} + (-\mathbf{E}) \text{div} \, (-\mathbf{E}) \right] ; \left[ \left( \text{rot} \, (-\mathbf{B}) + \frac{\partial \mathbf{E}}{\partial \xi} \right) \times (-\mathbf{B}) + \mathbf{E} \text{div} \, \mathbf{E} \right]. \tag{2.9}
\]

These last two expressions (2.9) can be considered as obtained from the left hand side of (2.8) under the substitutions \((\mathbf{E}, \mathbf{B}) \rightarrow (\mathbf{B}, -\mathbf{E})\) and \((\mathbf{E}, \mathbf{B}) \rightarrow (-\mathbf{B}, \mathbf{E})\) respectively. Hence, the field \((\mathbf{E}, \mathbf{B})\) chooses as a partner-field one of the fields \((-\mathbf{B}, \mathbf{E})\), or \((\mathbf{B}, -\mathbf{E})\).

We may resume this in the following way:

An adequate mathematical representation of a time dependent free electromagnetic field requires a collection of two fields : \([(\mathbf{E}, \mathbf{B}); (-\mathbf{B}, \mathbf{E})]\), or \([(\mathbf{E}, \mathbf{B}); (\mathbf{B}, -\mathbf{E})]\).

We could also say that a real free field consists of two interacting subsystems described by two partner-fields inside the \(\alpha(x, y, z, t)\)-family

\[(\mathcal{E}, \mathcal{B}) = (\mathbf{E} \cos \alpha - \mathbf{B} \sin \alpha; \mathbf{E} \sin \alpha + \mathbf{B} \cos \alpha)\]

giving the same Maxwell stress-energy tensor. Each partner-field has electric and magnetic components, and each partner-field is determined by the other through \((\pm \frac{\alpha}{2})\) - rotation-like transformation. Both partner-fields carry the same energy-momentum and minimize the relation \(I_1^2 + I_2^2 \geq 0\). This view and relation (2.7/2.8) suggest, in turn, that the intrinsic dynamics of free real time-dependent electromagnetic
fields could be considered as establishing and maintaining local energy-momentum exchange partnership between two fields called above partner-fields. and, since $(\mathbf{E}, \mathbf{B})$ and $(-\mathbf{B}, \mathbf{E})$ carry always the same stress-energy-momentum, the allowed inter-exchange is necessarily simultaneous and in equal quantities, so, each partner-field conserves its energy-momentum.

In order to find how much is the locally exchanged energy-momentum we are going to interpret the equation (2.8) in accordance with the view on equations of motion as stated in Sec.2.2. Our object of interest $\Phi$, representing the integrity of a real time dependent electromagnetic field, is the couple $[\mathbf{E}, \mathbf{B}]; (-\mathbf{B}, \mathbf{E})$ (the other case $[\mathbf{E}, \mathbf{B}]; (\mathbf{B}, -\mathbf{E})$ is considered analogically). In view of the above considerations our equations should directly describe admissible energy-momentum exchange between these recognized two subsystems, i.e. from formal point of view, between the two partner-fields. Hence, we have to define the corresponding change-objects $D(\mathbf{E}, \mathbf{B})$ and $D(-\mathbf{B}, \mathbf{E})$ for each partner-field, and their self-"projections" and their mutual "projections".

The change object $D(\mathbf{E}, \mathbf{B})$ for the first partner-field $(\mathbf{E}, \mathbf{B})$ we naturally define as

$$D(\mathbf{E}, \mathbf{B}) := \left( \text{rot}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi}; \text{div}\mathbf{B} \right).$$

The corresponding "projection" of $D(\mathbf{E}, \mathbf{B})$ on $(\mathbf{E}, \mathbf{B})$

$$\mathcal{P} \left[ D(\mathbf{E}, \mathbf{B}); (\mathbf{E}, \mathbf{B}) \right] = \mathcal{P} \left[ \left( \text{rot}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi}; \text{div}\mathbf{B} \right); (\mathbf{E}, \mathbf{B}) \right]$$

is suggested by the left hand side of (8) and we define it by :

$$\mathcal{P} \left[ \left( \text{rot}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi}; \text{div}\mathbf{B} \right); (\mathbf{E}, \mathbf{B}) \right] := \left( \text{rot}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} + \mathbf{B}\text{div}\mathbf{B}.$$

For the second partner-field $(-\mathbf{B}, \mathbf{E})$, following the same procedure we obtain:

$$\mathcal{P} \left[ D(-\mathbf{B}, \mathbf{E}); (-\mathbf{B}, \mathbf{E}) \right] = \mathcal{P} \left[ \left( \text{rot}(-\mathbf{B}) + \frac{\partial \mathbf{E}}{\partial \xi}; \text{div}\mathbf{E} \right); (-\mathbf{B}, \mathbf{E}) \right] =$$

$$= \left( \text{rot}(-\mathbf{B}) + \frac{\partial \mathbf{E}}{\partial \xi} \right) \times (-\mathbf{B}) + \mathbf{E}\text{div}\mathbf{E} = \left( \text{rot}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) \times \mathbf{B} + \mathbf{E}\text{div}\mathbf{E}.$$

Hence, relation (2.7) looks like

$$\mathcal{P} \left[ D(\mathbf{E}, \mathbf{B}); (\mathbf{E}, \mathbf{B}) \right] + \mathcal{P} \left[ D(-\mathbf{B}, \mathbf{E}); (-\mathbf{B}, \mathbf{E}) \right] = 0.$$

The accepted two-component view on a real time dependent electromagnetic field allows in principle admissible energy-momentum exchange with the outside world through any of the two partner-fields. Hence, the above calculations suggest to interpret the two sides of (2.8) as momentum quantities that each partner-field $(\mathbf{E}, \mathbf{B})$, or $(-\mathbf{B}, \mathbf{E})$, is potentially able to give to some other physical object without destroying itself, and these quantities are expressed in terms of $\mathbf{E}, \mathbf{B}$ and their derivatives only. In the case of free field, since no energy-momentum is lost by the field, there are two possibilities: first, there is NO energy-momentum exchange between the two partner-fields, second, each of the partner-fields changes its energy-momentum at the expense of the other through simultaneous and in equal quantities exchanges. Such kind of mutual exchange is in correspondence with the mathematical representatives of the
two subsystems: the partner-fields \((E, B)\) and \((-B, E)\) being members of the above mentioned \(\alpha(x, y, z, t)\)-family, obviously carry the same energy \(\frac{1}{2}(E^2 + B^2)\) and momentum \(\frac{1}{2\epsilon}(E \times B)\). If we denote by \(\Delta_{11}\) and by \(\Delta_{22}\) the allowed energy-momentum changes of the two component-fields, by \(\Delta_{12}\) the energy-momentum that the first partner-field receives from the second partner-field, and by \(\Delta_{21}\) the energy-momentum that the second partner-field receives from the first partner-field, then according to the energy-momentum local conservation law we may write the following equations:

\[
\Delta_{11} = \Delta_{12} + \Delta_{21}; \quad \Delta_{22} = -(\Delta_{21} + \Delta_{12}),
\]

which is in accordance with the equation (2.8): \(\Delta_{11} + \Delta_{22} = 0\).

We determine now how the mutual momentum exchange between the two partner-fields \(P_{(E,B)} \equiv P_{(-B,E)}\), or, \(P_{(E,B)} \equiv P_{(B,-E)}\) is performed, i.e. the explicit expressions for \(\Delta_{12}\) and \(\Delta_{21}\), keeping in mind that \(|P_{(E,B)}| = |P_{(-B,E)}| = |P_{(B,-E)}| = \frac{1}{2}P_{[[E,B];(-B,E)]}||\). The formal expressions are easy to obtain. In fact, in the case \(P_{(E,B)} \rightarrow P_{(-B,E)}\), i.e. the quantity \(\Delta_{21}\), we have to "project" the change object for the second partner-field given by

\[
D(-B, E) := \left( \text{rot}(-B) + \frac{\partial E}{\partial \xi}; \text{div}E \right)
\]

on the first partner-field \((E, B)\). We obtain:

\[
\Delta_{21} = \left( \text{rot}(-B) + \frac{\partial E}{\partial \xi} \right) \times E + B \text{div} E = - \left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) \times E + B \text{div} E. \tag{2.10}
\]

In the reverse case \(P_{(-B,E)} \rightarrow P_{(E,B)}\), i.e. the quantity \(\Delta_{12}\), we have to project the change object for the first partner-field \((E, B)\) given by

\[
D(E, B) := \left( \text{rot}E + \frac{\partial B}{\partial \xi}; \text{div}B \right)
\]

on the second partner-field \((-B, E)\). We obtain

\[
\Delta_{12} = \left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) \times (-B) + E \text{div} B = - \left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) \times B + E \text{div} B. \tag{2.11}
\]

So, the internal local momentum balance is governed by the equations

\[
\left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) \times E + B \text{div} B = - \left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) \times B + E \text{div} B - \left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) \times E + B \text{div} E, \tag{2.12}
\]

\[
\left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) \times B + E \text{div} E = \left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) \times E - B \text{div} E + \left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) \times B - E \text{div} B. \tag{2.13}
\]

These two vector equations (2.12)-(2.13) we consider as natural Newton type field equations. According to them the intrinsic dynamics of a free electromagnetic field is described by two couples of vector fields, \([E, B]; (-B, E)\), or \([E, B]; (B, -E)\), and this intrinsic dynamics could be interpreted as a direct energy-momentum exchange between two appropriately individualized subsystems mathematically described by these two partner-fields.
A further natural specialization of the above two vector equations (2.12)-(2.13) could be made if we recall that this internal energy-momentum exchange realizes a special kind of dynamical equilibrium between the two partner-fields, namely, the two partner-fields necessarily carry always the same energy and momentum: \( M_{ij}(E, B) = M_{ij}(\mathcal{E}, \mathcal{B}) \), so each partner-field must conserve its momentum: \( \Delta_{11} = \Delta_{22} = 0 \). In such a dynamical situation each partner-field loses as much as it gains during any time period, so, equations (2.12)-(2.13) reduce to

\[
\Delta_{11} \equiv \left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) \times B + B \text{div} B = 0, \tag{2.14}
\]

\[
\Delta_{22} \equiv \left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) \times E + E \text{div} E = 0, \tag{2.15}
\]

\[
\Delta_{12} + \Delta_{21} \equiv \left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) \times B - E \text{div} B + \left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) \times E - B \text{div} E = 0. \tag{2.16}
\]

Equation (2.16) fixes, namely, that the exchange of energy-momentum density between the two partner-fields is simultaneous and in equal quantities, i.e. a permanent dynamical equilibrium between the two partner-fields holds: \( P_{(E,B)} \rightleftharpoons P_{(\mathcal{E},\mathcal{B})} \), or, \( P_{(E,B)} \rightleftharpoons P_{(B,E)} \).

Note that, if equations (2.14) and (2.15) may be considered as field-equivalents to the zero force field (eqn. (2.14)) and its dual (eqn. (2.15)), this double-field viewpoint and the corresponding mutual energy-momentum exchange described by equation (2.16) are essentially new moments. Equations (2.14)-(2.16) also suggest that the corresponding fields are able to exchange energy-momentum with other physical systems in three ways. If such an exchange has been done, then the exchanged energy-momentum quantities can be given in terms of the characteristics of the other physical system (or in terms of the characteristics of the both systems) and to be correspondingly equalized to the left hand sides of equations (2.14)-(2.16) in accordance with the local energy-momentum conservation law.

### 2.4.1 Some Properties of the nonlinear solutions

Clearly, all solutions to Maxwell pure field equations (*) are solutions to our nonlinear equations (2.14)-(2.16), we shall call these solutions linear, and will not be interested of them just because the notion for stress-energy-momentum partnership between \((E,B)\) and \((-B,E)\) is missing. Therefore, we shall concentrate on those solutions of (2.14)-(2.16) which satisfy the conditions

\[
\text{rot} E + \frac{\partial B}{\partial \xi} \neq 0, \quad \text{rot} B - \frac{\partial E}{\partial \xi} \neq 0, \quad \text{div} E \neq 0, \quad \text{div} B \neq 0.
\]

These solutions we call further nonlinear (among them there are no constant ones as it is in the class of linear ones). We note some of the properties they have.

1. \( E \cdot B = 0; \)

2. \( \left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) \cdot B = 0; \) \( \left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) \cdot E = 0. \)

From these two relations the classical Poynting energy-momentum balance equation follows.

The above two properties are obvious from equations (2.14) and (2.15).

3. If \((E,B)\) defines a solution then \((E',B') = (aE - bB; bE + aB)\), where \(a,b \in \mathbb{R}\), defines also a solution. This property is immediately verified through substitution.

4. \( E^2 = B^2. \)
To prove this, we first multiply equation (2.14) on the left by $E$ and equation (2.16) by $B$ (scalar products). Then we make use of the above properties 1 and 2, of the vector algebra relation $X.(Y \times Z) = Z.(X \times Y)$, and of the assumed nonlinear values of the divergences of $E$ and $B$.

Properties 1. and 4. say that all nonlinear solutions to (2.14)-(2.16) are null fields, i.e. the two well known relativistic invariants $I_1 = B^2 - E^2$ and $I_2 = 2E.B$ of the field are zero, and this property leads to optimisation of the inequality $I_1^2 + I_2^2 \geq 0$ (recall the eigen properties of Maxwell stress tensor, Sec. 2.3), which in turn guarantees $\alpha(x, y, z, t)$-invariance of $I_1$ and $I_2$.

5. $B \left( \text{rot} \ B - \frac{\partial E}{\partial R} \right) - E \left( \text{rot} \ E + \frac{\partial B}{\partial R} \right) = B . \text{rot} B - E . \text{rot} E = 0$.

To prove this property we first multiply (vector product) (2.14) from the right by $E$, recall property 1., then multiply (scalar product) from the left by $E$, recall again $E.B = 0$, then multiply from the right (scalar product) by $B$ and recall property 4.

Property 5. suggests the following consideration. If $V$ is an arbitrary vector field on $\mathbb{R}^3$ then the quantity $V . \text{rot} V$ is known as local helicity and its integral over the whole (compact) region occupied by $V$ is known as integral helicity, or just as helicity of $V$. Hence, property 5. says that the electric and magnetic components of a nonlinear solution generate the same helicities. If we consider (through the euclidean metric) $E$ as 1-form on $\mathbb{R}^3$ and denote by $d$ the exterior derivative on $\mathbb{R}^3$, then $E \wedge dE = E . \text{rot} E \, dx \wedge dy \wedge dz$, so, the zero helicity says that the 1-form $E$ defines a completely integrable Pfaff system: $E \wedge dE = 0$. The nonzero helicity says that the 1-form $E$ defines non-integrable 1d Pfaff system, so the nonzero helicity defines corresponding curvature. Therefore the equality between the $E$-helicity and the $B$-helicity suggests to consider the corresponding integral helicities $\int_{\mathbb{R}^3} E \wedge dE = \int_{\mathbb{R}^3} B \wedge dB$ (when they take finite nonzero values) as a measure of the spin properties of the solution.

6. Example of nonlinear solution:

$$E = \left[ \phi(x, y, ct + \varepsilon z) \cos(-\kappa \frac{z}{l_o} + \text{const}), \phi(x, y, ct + \varepsilon z) \sin(-\kappa \frac{z}{l_o} + \text{const}), 0 \right];$$

$$B = \left[ \varepsilon \phi(x, y, ct + \varepsilon z) \sin(-\kappa \frac{z}{l_o} + \text{const}), -\varepsilon \phi(x, y, ct + \varepsilon z) \cos(-\kappa \frac{z}{l_o} + \text{const}), 0 \right],$$

where $\phi(x, y, ct + \varepsilon z)$ is an arbitrary positive function, $l_o < \infty$ is an arbitrary positive constant with physical dimension of length, and $\varepsilon$ and $\kappa$ take values $\pm 1$ independently. Modifying the helicity 3-forms to $\kappa \frac{4\varepsilon l_o}{c} E \wedge dE = \kappa \frac{4\varepsilon l_o}{c} B \wedge dB$, then the corresponding 3d integral gives $\kappa T E$, where $\kappa = \pm 1, T = 4l_o/c$ and $E = \int_{\mathbb{R}^3} \phi^2 dx dy dz$ is the integral energy of the solution.

2.5 Discussion

The main idea of this part of the paper is that carrying out the Newton way for writing down dynamical equations for particles in mechanics to writing down dynamical equations for continuous field systems should naturally result to nonlinear partial differential equations even in non-relativistic theories. Moreover, clarifying the sense of the information included in these dynamical equations according to the Newton approach, we come to the conclusion formulated in the Introduction, namely, we have to mathematically describe those changes of the object considered which are qualified as admissible and consistent with the system’s identification and with the local energy-momentum balance relations. In the case of "free" systems these relations represent the local energy-momentum exchange/conservation properties of the system. The energy-momentum characteristics are chosen because of their two important properties: they are physically universal and conservative. This means that every physical object carries
nonzero energy-momentum and, vice versa, every quantity of energy-momentum is carried by some physical object. Hence, if a physical object loses/gains some quantity of energy-momentum then some other physical object necessarily gains/loses the same quantity of energy-momentum. If this viewpoint is assumed, then the problem of finding appropriate dynamical equations for an object reduces mainly to: first, getting knowledge of the potential abilities of the object considered to lose and gain energy-momentum; second, to create adequate mathematical quantities describing locally these abilities.

The electromagnetic field, considered as a continuous physical object of special kind, gives a good example in this direction since, thanks to Maxwell’s fundamental and summarizing works, all the information needed is available. The notices of Poynting [13], and Thomson, Poincare and Abraham [14], showing the importance of the (deduced from Maxwell equations) vector \( \frac{1}{c} \mathbf{E} \times \mathbf{B} \) from local energy-momentum propagation point of view, has completed the resource of adequate and appropriate mathematical objects since it appears as natural complement of Maxwell stress tensor, and allows to write down dynamical field equations having direct local energy-momentum balance sense. However, looking back in time, we see that this viewpoint for writing down field equations has been neglected, theorists have paid more respect and attention to the "linear part" of Maxwell theory, enjoying, for example, the exact but not realistic, and even physically senseless in many respects, plane wave solutions in the pure field case.

Therefore, not so long after the appearance of Maxwell equations the photoeffect experiments showed the nonadequateness of the linear part of Maxwell theory as a mathematical model of electromagnetic fields producing realistic model-solutions of free time-dependent fields. Although the almost a century long time development of standard quantum and relativistic quantum theories that followed, a reasonable model-solutions describing individual photons, considered as basic, spatially finite and time-stable objects, these theories have not presented so far. Nobody doubts nowadays that photons really exist, and this very fact suggests to try first classical field approach in finding equations admitting 3d-finite and time stable solutions with appropriate properties.

The historical perspective suggests to follow the 4-potential approach, but modern knowledge and experience, and even the Maxwell stress tensor achievements, suggest some different views. In fact, we have all reasons to consider the microobjects as real as all other physical objects, so, no point-like charges and infinite field model-solutions should be considered as adequate. Since the 4-potential approach in Maxwell theory does not allow spatially finite and time stable pure field solutions with photon-like structure and behavior its interpretation as a basic concept does not seem to be appreciable. Also, the 4-potential approach excludes many solutions of the charge free Maxwell equations. For example, in relativistic terms the well known field \( F = \frac{\mathbf{a}}{r^2} dr \wedge d\xi \), \( dF = 0 \), has global 4-potential, and its Minkowski-dual \( *F = q \sin \theta \, d\theta \wedge d\varphi \), \( d*F = 0 \), has NO global 4-potential. Now, the 2-parameter family of 2-forms \((\mathbf{3},*,\mathbf{3}) = (aF - b*F; bF + a*F), a,b \in \mathbb{R} \), gives an infinite number of solutions to Maxwell equations \( d\mathbf{3} = 0, d*\mathbf{3} = 0 \) admitting NO global 4-potential. This suggests the view that the 4-potential can be used as a working tool (wherever it causes no controversies) but not as a basic concept.

In conclusion, paying due respect to the Newton view on dynamical equations and to the local energy-momentum conservation law we based our approach on the Maxwell stress tensor and on the Poynting vector as natural quantities carrying the physically meaningful energy-momentum characteristics of the electromagnetic field. The natural description in these terms leads to the assumption that any real time-dependent electromagnetic field consists of two interacting subsystems mathematically represented by the two partner-fields: \([E(\mathbf{B})], [-(\mathbf{B}, \mathbf{E})]/[(\mathbf{E}, \mathbf{B})], (\mathbf{B}, -\mathbf{E})] \), or any couple \([E, \mathbf{B}],[-(\mathbf{B}, \mathbf{E})]/[(\mathbf{E}, \mathbf{B})], (\mathbf{B}, -\mathbf{E})] \) inside the considered \( \alpha(x, y, z, t) \)-family of fields. These two partner-fields carry always the same stress-energy-momentum, and a dy-
namical equilibrium between these two subsystems is realized through a simultaneous mutual energy-momentum exchange in equal quantities. The equations obtained represent formally this dynamical equilibrium, i.e. they show that **partner-fields identify/recognize each other through appropriate local energy-momentum exchange partnership minimizing the quantity** \( \frac{1}{2} \sqrt{I_1^2 + I_2^2} = \frac{1}{2} (E^2 + B^2) - |E \times B| \geq 0 \). Accordingly, all nonlinear solutions have zero invariants \( I_1 = I_2 = 0 \), and can not be constant. Among these zero-invariant nonlinear solutions there are time-stable and spatially finite ones with helical spatial structure, having photon-like properties and behavior. An analog of the Planck relation \( E = h\nu \) holds for these solutions, where the constant \( h \) appears as an integral helicity of such a solution.
Chapter 3

Relativistic Considerations

3.1 The Extended Electrodynamics approach

The generalization of Classical Electrodynamics (CED) known as Extended Electrodynamics (EED) [11], starts with the conviction that CED surely carries inside the potential ability to be extended in such a way, that spatially finite and time-stable solutions of photon-like nature to be incorporated, and it exploits mainly two ideas: the idea for a direct local energy-momentum exchange sense of the new dynamical equations, and the well known dual symmetry (mentioned above) of the vacuum CED-equations and local conservation laws in the frame of relativistic formalism. Let’s consider first the elementary physical approach.

Maxwell vacuum equations

\[
\text{rot} E + \frac{\partial B}{\partial \xi} = 0, \quad \text{rot} B - \frac{\partial E}{\partial \xi} = 0, \quad \text{div} E = 0, \quad \text{div} B = 0 \quad (\ast)
\]

clearly suggest that a free electromagnetic field has two vector components: electric \( E \) and magnetic \( B \). On one hand, from physical viewpoint, these equations imply also interaction, i.e. energy-momentum exchange between the electric and magnetic components of the electromagnetic field, and on the other hand, the energy density expression \( \frac{1}{2}(E^2 + B^2) \) in the theory does not contain interaction energy term: the full energy density is the sum of the electric and magnetic energy densities. Now, since any of these two components can NOT carry momentum separately (the field momentum is given by \( \frac{1}{c}E \times B \)), then \( E \) and \( B \) can exchange only energy and NO momentum. But the field propagates translationally along null straight-line directions with the speed of light "\( c \)", so it necessarily carries momentum being numerically equal (in energy units) to the energy-density, the so called electromagnetic energy flux. Hence, the field energy is of entirely dynamical nature and any internal energy exchange between subsystems necessarily implies corresponding momentum exchange. Therefore, from energy-momentum exchange point of view, we should be interested in finding such subsystems of the field, which are able to carry and exchange simultaneously both energy and momentum.

In order to come to appropriate mathematical representatives in relativistic terms of such subsystems we recall that under null character of the local energy-momentum, i.e. when \( T_{\mu\nu}T^{\mu\nu} = 0 \), the translational propagation requires zero invariants: \( I_1 = B^2 - E^2 = 0 \) and \( I_2 = 2E \cdot B = 0 \), which in terms of the relativistic 2-form formalism is equivalent respectively to \( I_1 = \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} = 0 \) and \( I_2 = \frac{1}{2}F_{\alpha\beta}(\ast F)^{\alpha\beta} = 0 \), where \( \ast \) is defined by the Minkowski (pseudo)metric: \( \alpha \land \ast \beta = -\eta(\alpha, \beta)\sqrt{\det(\eta_{\mu\nu})}dx^1 \land dx^2 \land dx^3 \land dx^4 \). Now, recall the well known identity, being in force for any two 2-forms \( F \) and \( G \) in Minkowski space-time \((M, \eta)\):

\[
\frac{1}{2}F_{\alpha\beta}G^{\alpha\beta}\delta_\mu = F_{\mu\sigma}G^{\nu\sigma} - (\ast G)_{\mu\sigma}(\ast F)^{\nu\sigma}.
\]
Under $G = F$ and $\frac{1}{F} F_{\alpha\beta} F^{\alpha\beta} = 0$, it follows $F_{\mu\sigma} F^{\nu\sigma} = (F_{\mu\sigma}(F)^{\nu\sigma}$. In view of the canonical stress-energy-momentum tensor of the field (we omit the coefficient $\frac{1}{4\pi}$)

$$T^\nu_{\mu} = -\frac{1}{2} \left[ F_{\mu\sigma} F^{\nu\sigma} + (F)_{\mu\sigma}(F)^{\nu\sigma} \right] =$$

$$= \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \delta_{\mu}^{\nu} - F_{\mu\sigma} F^{\nu\sigma} = \frac{1}{4} (F)_{\alpha\beta}(F)^{\alpha\beta} \delta_{\mu}^{\nu} - (F)_{\mu\sigma}(F)^{\nu\sigma},$$

satisfying the Rainich condition $T_{\mu\nu} T^{\nu\sigma} = \frac{1}{4} T_{\alpha\beta} T^{\alpha\beta} \delta_{\mu}^{\nu} = \frac{1}{4} (I_1^2 + I_2^2) \delta_{\mu}^{\nu} (= 0$ in our case), this physically means that $F(E, B)$ and $*(F(-E, B))$ can carry both energy and momentum, moreover, which is very important, they carry always the same stress-energy-momentum. Formally, this is partially hidden in the obvious invariance of $T^\nu_{\mu}$ with respect to $F \rightarrow *F$. Now, since there is no other physical object participating in the energy-momentum exchange, we come to the conclusion that for adequate mathematical representatives of the two subsystems we are looking for, namely $F$ and $*F$ can be chosen. Moreover, since they necessarily carry always the same stress-energy-momentum, they may exchange locally energy-momentum only simultaneously and in equal quantities. Therefore, in view of $\nabla_{\nu} T^\nu_{\mu} = F^{\alpha\beta}(dF)_{\alpha\beta\mu} + (F)^{\alpha\beta}(dF)^{\alpha\beta}_{\mu\beta} = 0$, $\alpha < \beta$, the most natural dynamical equations should read

$$F^{\alpha\beta}(dF)_{\alpha\beta\mu} = 0, \quad (F)^{\alpha\beta}(dF)^{\alpha\beta}_{\mu\beta} = 0, \quad (F)^{\alpha\beta}(dF)_{\alpha\beta\mu} + F^{\alpha\beta}(dF)^{\alpha\beta}_{\mu\beta} = 0, \quad \alpha < \beta,$$

where the first two equations require that $F$ and $*F$ conserve the energy-momentum they carry, and the third equation establishes the local dynamical equilibrium between $F$ and $*F$;

$(F)^{\alpha\beta}(dF)_{\alpha\beta\mu}$ and $F^{\alpha\beta}(dF)_{\alpha\beta\mu}$ denote respectively the allowed from the conservation laws energy-momentum gains and losses of $*F$ and $F$, which gains and losses are forbidden by the old equations $dF = 0$, $d * F = 0$.

Extended Electrodynamics gives the following mathematical picture of this field dynamics. Recall that if $(F, *F)$ is a CED vacuum solution, i.e. $dF = 0$, $d * F = 0$, then the combinations $\mathcal{F} = a F - b * F$, $\mathcal{F}^* = b F + a * F$, where $(a, b)$ are two arbitrary real numbers, also give a CED vacuum solution and, since on Minkowski space the reduced to 2-forms Hodge star $*$ satisfies the relation $* s^2 = -s dA_{3}(M)$, we obtain $\mathcal{F}^* = * \mathcal{F}$. The two corresponding energy tensors are related by $T(\mathcal{F}, \mathcal{F}^*) = (a^2 + b^2) T(F, *F)$. Recall the real representation of complex numbers $z = aI + bJ$ where $I$ is the unit matrix in $\mathbb{R}^2$ and $J$ is the standard complex structure matrix in $\mathbb{R}^2$ with columns $(0,-1); (1,0)$. So, we obtain an action of the linear group of matrices $\alpha = aI + bJ$ on the CED vacuum solutions. This is a commutative group $G$ and its Lie algebra $\mathcal{G}$ just adds the zero $(2 \times 2)$ matrix to $G$, and $(I, J)$ define a natural basis of $\mathcal{G}$. So, having a CED vacuum solution, we have in fact a 2-parameter family of vacuum solutions. Hence, we can define a $\mathcal{G}$-valued 2-form $\Omega$ on $M$ by $\Omega = F \otimes I + * F \otimes J$, and the equation $d\Omega = 0$ is equivalent to $dF = 0$, $d * F = 0$.

Consider the new basis $(I', J')$ of $\mathcal{G}$ given by

$$I' = (aI + bJ), \quad J' = (-bI + aJ).$$

Accordingly, the "new" solution $\Omega'$, i.e. the old solution in the new basis of $\mathcal{G}$, will be

$$\Omega' = F \otimes I' + * F \otimes J' = F \otimes (aI + bJ) + *(F \otimes (-bI + aJ) = (a F - b * F) \otimes I + (b F + a * F) \otimes J.$$
are zero: \( I_1 = I_2 = 0 \), then all the above transformations keep unchanged these zero-values of \( I_1 \) and \( I_2 \). In fact, under such a transformation \((F, *F) \rightarrow (F, F^*)\) the two Lorentz invariants transform to \((I'_1, I'_2)\) in the following way:

\[
I'_1 = (a^2 - b^2) I_1 + 2ab I_2, \quad I'_2 = -2ab I_1 + (a^2 - b^2) I_2,
\]

and the determinant of this transformation is \((a^2 + b^2)^2 \neq 0\). So, a null field stays a null field under these transformations. Moreover, NO non-null field can be transformed to a null field by means of these transformations, and, conversely, NO null field can be transformed to a non-null field in this way. Hence, the Lorentz invariance and the dual invariance of \( I_1 \) and \( I_2 \) hold simultaneously only in the null-field case. This observation distinguishes once again the null-field case.

In order to come to the new equations we recall that every bilinear map \( \varphi : G \times G \rightarrow W \), where \( W \) is some linear space with basis \( \{e_i, i = 1, 2, \ldots\} \), defines corresponding product in the \( G \)-valued differential forms by means of the relation

\[
\varphi(\Omega_1 \otimes e_i, \Omega_2 \otimes e_j) = \Omega_1^i \otimes \Omega_2^j \varphi(e_i, e_j).
\]

Now, let \( \varphi = \land \), where \( \land \) is the symmetrized tensor product in \( G \). We consider the expression

\[
\land(\Omega, *d\Omega) = (F \land *dF) \otimes I \land I + (*F \land *d*F) \otimes J \land J + (F \land *dF \otimes * + F \land *dF) \otimes I \land J.
\]

The vacuum EED equations are \( \land(\Omega, *d\Omega) = 0 \), or equivalently,

\[
F \land *dF = 0, \quad (*F) \land *F = 0, \quad F \land *dF + (*F) \land *dF = 0.
\]

In terms of the codifferential \( \delta = *d* \) these equations look like

\[
\delta * F \land F = 0, \quad \delta F \land *F = 0, \quad \delta F \land F - \delta *F \land F = 0.
\]

In components we obtain correspondingly

\[ F^\alpha\beta(dF)_{\alpha\beta\mu} \equiv (*F)_{\mu\nu}(\delta *F)^\nu = 0, \quad (*F)^\alpha\beta(d*F)_{\alpha\beta\mu} \equiv F^\mu\nu(\delta F)^\nu = 0, \quad \alpha < \beta; \]

\[
(*F)^\alpha\beta(dF)_{\alpha\beta\mu} + F^\alpha\beta(d*F)_{\alpha\beta\mu} \equiv (\delta *F)^\nu F_{\nu\mu} + (\delta F)^\nu(*F)_{\nu\mu} = 0, \quad \alpha < \beta.
\]

It is easy to see that these equations are equivalent to the equations \( \Delta_{11} = \Delta_{22} = 0, \quad \Delta_{12} + \Delta_{21} = 0 \) as given in terms of \((E, B)\) in Sec.2.4. Moreover, all nonlinear solutions to these EED vacuum equations, i.e., those satisfying \( dF \neq 0, d * F \neq 0 \), have zero invariants: \( I_1 = I_2 = 0 \) (for the case EED in presence of media see [11]).

As for the energy-momentum tensor \( T_{\mu\nu} \) of the vacuum solutions, considered as a symmetric 2-form on \( M \), it is defined in terms of \( \Omega \) as follows:

\[
T(X, Y) = \frac{1}{2} * g[i(X)\Omega, *i(Y)\Omega] = -\frac{1}{2} X^\mu Y^\nu \left[F_{\mu\nu} F^\rho^\sigma + (*F)_{\mu\sigma}(F^\nu)_\rho^\sigma\right] = X^\mu Y^\nu T_{\mu\nu},
\]

where \((X, Y)\) are two arbitrary vector fields on \( M \), \( g \) is the metric in \( G \) defined by \( g(\alpha, \beta) = \frac{1}{2} tr(\alpha \cdot \beta^*), \) and \( \beta^* \) is the transposed to \( \beta \). Note that \( g(I, J) = g(aI + bJ, -bI - aJ) = 0 \), which eliminates the corresponding coefficient in \( T(X, Y) \), which reads \( F_{\mu\sigma}(F^\rho^\sigma) + (*F)_{\mu\sigma} F^\rho^\sigma = \frac{1}{2} F_{\alpha\beta}(F^\alpha)^\sigma F^\mu_{\sigma} \delta^\nu_{\mu}, \) so, in a \( g \)-ORTHONORMAL basis of \( G \) this coefficient will appear. Now, since \( \frac{1}{2} F_{\alpha\beta}(*F)^\rho_{\alpha\beta} = 2(E, B) \), choosing \( g \)-ORTHONORMAL basis in \( G \) corresponds to mutual orthogonality.
of \((E, B)\) in this context, and choosing \(g\)-nonorthogonal basis of \(G\) will formally result in some interaction between \(E\) and \(B\). 

Finally, recall the generalization of Lie derivative \(\mathcal{L}_K\) with respect to the \(k\)-vector \(K\), acting in the exterior algebra of differential forms according to the formula \(\mathcal{L}_K = i(K) \omega - (-1)^k d(i(K))\) [12]. Then, in view of the relations \(F_{\mu \nu} F^{\mu \nu} = (\ast F)_{\mu \nu} F^{\mu \nu} = 0\), the above equations acquire the form

\[
\mathcal{L}_{\bar{F}} F = 0, \quad \mathcal{L}_{\ast \bar{F}} (\ast F) = 0, \quad \mathcal{L}_{\bar{F}} (\ast F) + \mathcal{L}_{\ast \bar{F}} F = 0,
\]

where \(\bar{F}\) and \(\ast F\) are the \(\eta\)-corresponding 2-vectors. In terms of \(\Omega\) and \(\bar{\Omega} = \bar{F} \otimes e_1 + \ast F \otimes e_2\) these three equations can be represented as one relation as follows:

\[
\mathcal{L}_{\Omega} \bar{\Omega} = \mathcal{L}_{\bar{F}} F \otimes e_1 \lor e_1 + \mathcal{L}_{\ast \bar{F}} \ast F \otimes e_2 \lor e_2 + (\mathcal{L}_{\bar{F}} \ast F + \mathcal{L}_{\ast \bar{F}} F) \otimes e_1 \lor e_2 = 0.
\]

The above consideration is based on the assumption that the \(G\)-valued 2-form \(\Omega(F, \ast F)\) represents mathematically the wholeness + structural integrity of the electromagnetic field through the equations \(\mathcal{L}_{\Omega} \bar{\Omega} = 0\), and direct physical motivation for such an assumption was not given. In order to motivate looking for photon-like solutions of these equations we now present our notion for photon-like object(s), and further we show how this notion leads to corresponding mathematics by means of which we could separate the desired subclass of solutions.

### 3.2 The Notion of Photon-like Object(s)

#### 3.2.1 Introduction

At the very dawn of the 20th century Planck [16] proposed and a little bit later Einstein [17] appropriately used the well known and widely used through the whole last century simple formula \(E = h \nu\), \(h = \text{const} > 0\). This formula marked the beginning of a new era and became a real symbol of the physical science during the following years. According to the Einstein’s interpretation it gives the full energy \(E\) of really existing light quanta of frequency \(\nu = \text{const}\), and in this way a new understanding of the nature of the electromagnetic field was introduced: the field has structure, which contradicts the description given by Maxwell vacuum equations. After De Broglie’s [18] suggestion for the particle-wave nature of the electron obeying the same energy-frequency relation, one could read Planck’s formula in the following way: there are physical objects in Nature the very existence of which is strongly connected to some periodic (with time period \(T = 1/\nu\) process of intrinsic for the object nature and such that the Lorentz invariant product \(ET\) is equal to \(h\). Such a reading should suggest that these objects do NOT admit point-like approximation since the relativity principle for free point particles requires straight-line uniform motion, hence, no periodicity should be allowed.

Although the great (from pragmatic point of view) achievements of the developed theoretical approach, known as quantum theory, the great challenge to build an adequate description of individual representatives of these objects, especially of light quanta called by Lewis photons [19], is still to be appropriately met since the efforts made in this direction, we have to admit, still have not brought satisfactory results. Recall that Einstein in his late years recognizes [20] that ”the whole fifty years of conscious brooding have not brought me nearer to the answer to the question ”what are light quanta”, and now, half a century later, theoretical physics still needs progress to present a satisfactory answer to the question ”what is a photon”. We consider the corresponding theoretically directed efforts as necessary and even urgent in view of the growing amount of definite experimental skills in manipulation with individual photons, in particular, in connection with the experimental advancement in the ”quantum computer” project. The dominating modern theoretical view on microobjects is based on the notions and
concepts of quantum field theory (QFT) where the structure of the photon (as well as of any other microobject) is accounted for mainly through the so called structural function, and highly expensive and delicate collision experiments are planned and carried out namely in the frame of these concepts and methods (see the 'PHOTON' Conferences Proceedings, some recent review papers: [21-24]). Going not in details we just note a special feature of this QFT approach: if the study of a microobject leads to conclusion that it has structure, i.e. it is not point-like, then the corresponding constituents of this structure are considered as point-like, so the point-likeness stays in the theory just in a lower level.

According to our view on PhLO we follow here an approach based on the assumption that the description of the available (most probably NOT arbitrary) spatial structure of photon-like objects can be made by continuous finite/localized functions of the three space variables. The difficulties met in this approach consist mainly, in our view, in finding adequate enough mathematical objects and solving appropriate PDE. The lack of sufficiently reliable corresponding information made us look into the problem from as general as possible point of view on the basis of those properties of photon-like objects which may be considered as most undoubtedly trustful, and in some sense, identifying. The analysis made suggested that such a property seems to be the available and intrinsically compatible translational-rotational dynamical structure, so we shall focus on this property in order to see what useful for our purpose suggestions could be deduced and what appropriate structures could be constructed. All these suggestions and structures should be the building material for a step-by-step creation of a self-consistent system.

From physical point of view this should mean that the corresponding properties may combine to express a dynamical harmony in the inter-existence of appropriately defined subsystems of a finite and time stable larger physical system.

3.2.2 The notion of photon-like object

We begin with recalling our view that any notion of a physical object must unify two kinds of properties of the object considered: identifying and kinematical. The identifying properties being represented by quantities and relations, stay unchanged throughout the existence, i.e. throughout the time-evolution, of the object, they represent all the intrinsic structure and relations. The kinematical properties describe those changes, called admissible, which do NOT lead to destruction of the object, i.e. to the destruction of any of the identifying properties. Correspondingly, physics introduces two kinds of quantities and relations, identifying and kinematical. From theoretical point of view the more important quantities used turn out to be the dynamical quantities which, as a rule, are functions of the identifying and kinematical ones, and the joint relations they satisfy represent the necessary interrelations between them in order this object to survive under external influence. This view suggests to introduce the following notion of Photon-like object(s) (we shall use the abbreviation ”PhLO” for ”Photon-like object(s)”):

PhLO are real massless time-stable physical objects with an intrinsically compatible translational-rotational dynamical structure.

We give now some explanatory comments, beginning with the term real. First we emphasize that this term means that we consider PhLO as really existing physical objects, not as appropriate and helpful but imaginary (theoretical) entities. Accordingly, PhLO necessarily carry energy-momentum, otherwise, they could hardly be detected by physical means. Second, PhLO can undoubtedly be created and destroyed, so, no point-like and infinite models are reasonable: point-like objects are assumed to have no structure, so they cannot be destroyed since there is no available structure to be destroyed; creation of infinite physical objects (e.g. plane waves)
requires infinite quantity of energy to be transformed from one kind to another during finite time-periods, which seems also unreasonable. Accordingly, PhLO are spatially finite and have to be modeled like such ones, which is the only possibility to be consistent with their "created-destroyed" nature. It seems hardly reasonable to believe that PhLO can not be created and destroyed, and that spatially infinite and indestructible physical objects may exist at all. **Third**, "spatially finite" implies that PhLO may carry only finite values of physical (conservative or non-conservative) quantities. In particular, the most universal physical quantity seems to be the energy-momentum, so the model must allow finite integral values of energy-momentum to be carried by the corresponding solutions. **Fourth**, "spatially finite" means also that PhLO propagate, i.e. they do not "move" like classical particles along trajectories, therefore, partial differential equations should be used to describe their evolution in time.

The term "massless" characterizes physically the way of propagation in terms of appropriate dynamical quantities: the integral 4-momentum $p$ of a PhLO should satisfy the relation $p_{\mu}p^{\mu} = 0$, meaning that its integral energy-momentum vector must be isotropic, i.e. to have zero module with respect to Minkowski (pseudo)metric in $\mathbb{R}^4$. If the object considered has spatial and time-stable structure, so that the translational velocity of every point where the corresponding field functions are different from zero must be equal to $c$, we have in fact null direction in the space-time intrinsically determined by a PhLO. Such a direction is formally defined by a null vector field $\vec{\zeta}$, $\vec{\zeta}^2 = 0$. The integral trajectories of this vector field are isotropic (or null) straight lines as is traditionally assumed in physics, except in presence of gravity. It follows that with every PhLO a null straight line direction is necessarily associated, so, canonical coordinates $(x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct)$ on $\mathbb{R}^4$ may be chosen such that in the corresponding coordinate frame $\vec{\zeta}$ to have only two non-zero components of magnitude 1: $\vec{\zeta}^{\mu} = (0, 0, -\varepsilon, 1)$, where $\varepsilon = \pm 1$ accounts for the two directions along the coordinate $z$ (further such a coordinate system will be called $\vec{\zeta}$-adapted and will be of main usage). It seems important to emphasize that our PhLO propagates as a whole along the $\vec{\zeta}$-direction, so the corresponding energy-momentum tensor field $T_{\mu\nu}(x, y, z, \xi)$ of the model must satisfy the corresponding local isotropy (null) condition, namely, $T_{\mu\nu}T^{\mu\nu} = 0$ (summation over the repeated indices is throughout used).

The term "translational-rotational" means that besides translational component along $\vec{\zeta}$, the propagation necessarily demonstrates some rotational (in the general sense of this concept) component in such a way that both components are compatible and exist simultaneously, and this is an intrinsic property. It seems reasonable to expect that such kind of dynamical behavior should require some distinguished spatial shapes. Moreover, if the Planck relation $E = h\nu$ must be respected throughout the evolution, the rotational component of propagation should have time-periodical nature with time period $T = \nu^{-1} = h/E = \text{const}$, and one of the two possible, left or right, orientations. It seems reasonable also to expect spatial periodicity of PhLO, which somehow to be related to the time periodicity.

The term "dynamical structure" means that the propagation is supposed to be necessarily accompanied by an internal energy-momentum redistribution, which may be considered in the model as energy-momentum exchange between (or among) some appropriately defined subsystems. It could also mean that PhLO live in a dynamical harmony with the outside world, i.e. any outside directed energy-momentum flow should be accompanied by a parallel inside directed energy-momentum flow.

Finally, note that if the time periodicity and the spatial periodicity should be consistent with each other somehow, the simplest integral feature of such compatibility would seem like this: the spatial size $\lambda$ along the translational component of propagation is equal to $cT$: $\lambda = cT$, where $\lambda$ is some finite positive characteristic constant of the corresponding solution. This would mean that every individual PhLO determines its own length/time scale.
It is important to note now the following. We don’t know what mathematical objects are appropriate for describing PhLO, so, our first task is to come to such mathematical objects having in view what we mean under PhLO. The next Section is devoted namely to find mathematical structures that are adequate enough to the above introduced notion for PhLO and carrying rich enough flexibility to meet all requirements for a field theory of spatially finite and time-stable physical objects with dynamical structure. Our hope is that the ideas and concepts connected with the Frobenius integrability theory seem to represent the most adequate part of mathematics for this purpose.

3.3 Curvature of Distributions and Physical Interaction

3.3.1 The general idea for geometrization of local physical interaction

We begin with a short motivation for this choice of mathematics directed to the readers already acquainted with Frobenius integrability theory, and right after this we shall carefully introduce the necessary mathematics.

Any physical system with a dynamical structure is characterized by some internal energy-momentum redistributions, i.e. internal energy-momentum fluxes, during evolution. Any time-stable compatible system of energy-momentum fluxes (as well as fluxes of other interesting for the case physical quantities subject to change during evolution, but we limit ourselves just to energy-momentum fluxes here) can be considered mathematically as generated by a compatible system of vector fields. A physically isolated and interrelated time-stable system of energy-momentum fluxes can be considered to correspond directly or indirectly to a completely integrable distribution \( \Delta \) of vector fields (or differential system [25]) according to the principle: some local objects can generate integral object. Every nonintegrable distribution on a manifold defines its own curvature form (given further in the section). Let \( \Delta_1 \) and \( \Delta_2 \) be two nonintegrable distributions on the same manifold with corresponding curvature forms \( \Omega_1 \) and \( \Omega_2 \), each of them carries couples of vector fields inside their distributions outside \( \Delta_1 \) and \( \Delta_2 \) correspondingly, i.e. \( \Omega_1(Y_1,Y_2) \neq 0 \) is out of \( \Delta_1 \) and \( \Omega_2(Z_1,Z_2) \neq 0 \) is out of \( \Delta_2 \), where \( (Y_1,Y_2) \) live in \( \Delta_1 \) and \( (Z_1,Z_2) \) live in \( \Delta_2 \). Let now \( \Delta_1 \) and \( \Delta_2 \) characterize two locally interacting physical systems, or two locally interacting subsystems of a larger physical system. It seems reasonable to assume as a working tool the following geometrization of the concept of local physical interaction: two nonintegrable distributions \( \Delta_1 \) and \( \Delta_2 \) on a manifold will be said to interact infinitesimally (or locally) if some of the nonzero values of the corresponding two curvature forms \( \Omega_1/\Omega_2 \) live respectively in \( \Delta_2/\Delta_1 \).

The above geometric concept of infinitesimal interaction is motivated by the fact that, in general, an integrable distribution \( \Delta \) may contain various nonintegrable subdistributions \( \Delta_1, \Delta_2, \ldots \) which subdistributions may be associated physically with interacting subsystems of a larger time stable physical system. Any physical interaction between 2 subsystems is necessarily accompanied with available energy-momentum exchange between them, this could be understood mathematically as nonintegrability of each of the two subdistributions of \( \Delta \) and could be naturally measured directly or indirectly by the corresponding curvatures. For example, if \( \Delta \) is an integrable 3-dimensional distribution represented by the vector fields \( (X_1, X_2, X_3) \) then we may have, in general, three non-integrable, i.e. geometrically interacting, 2-dimensional subdistributions \( (X_1, X_2), (X_1, X_3), (X_2, X_3) \). Finally, some interaction with the outside world can be described by curvatures of distributions (and their subdistributions) in which elements of \( \Delta \) and vector fields outside \( \Delta \) are involved (such processes will not be considered in this paper).

The above considerations launch the general idea to consider the concept of Frobenius curvature as a natural and universal mathematical tool for describing local physical interaction between/among the relatively stable subsystems of the physical world. In other words, the Frobe-
nious curvature appears as appropriate mathematical tool describing formally the possible ability two continuous systems to recognize each other as physically interacting partners.

Two formal aspects of the above idea exist. The first applies directly the Frobenius integrability machinery [25], while the second one (been developed recently) is known as nonlinear connections [26]. We consider now briefly the first one.

### 3.3.2 Frobenius integrability, curvature and local physical interaction

A $p$-dimensional distribution $\Delta_p$ on a $n$-dimensional manifold $M^n$ is defined by associating to each point $x \in M^n$ a $p$-dimensional subspace of the tangent space at this point: $\Delta^p_x \subset T_x M^n$, $x \in M^n$, $1 \leq p < n$. Let the system of vector fields $\{X_1, X_2, \ldots, X_p\}$ represent this distribution, so $\{X_1(x), X_2(x), \ldots, X_p(x)\}$, $x \in M^n$, $1 \leq p < n$, satisfy $X_1(x) \wedge X_2(x) \wedge \ldots \wedge X_p(x) \neq 0$, $x \in M^n$, and represent a basis of $\Delta^p_x$. According to the Frobenius integrability theorem (further all manifolds are assumed smooth and finite dimensional and all objects defined on $M^n$ are also assumed smooth) $\Delta_p$ is completely integrable, i.e. through every point $x \in M^n$ passes a $p$-dimensional submanifold $N^p$ such that all elements of $\Delta_p$ are tangent to $N^p$, iff all Lie brackets $[X_i, X_j]$, $i, j = 1, 2, \ldots, p$, are representable linearly through the very $X_i, i = 1, 2, \ldots, p : [X_i, X_j] = C^k_{ij} X_k$, where $C^k_{ij}$ are functions. Clearly, an easy way to find out if a distribution is completely integrable is to check if the exterior products

$$[X_i, X_j] \wedge X_1(x) \wedge X_2(x) \wedge \ldots \wedge X_p(x), \ x \in M^n; \ i, j = 1, 2, \ldots, p$$

are identically zero. If this is not the case (which means that at least one such Lie bracket "sticks out" of the distribution $\Delta_p$) then the corresponding coefficients, which are multilinear combinations of the components of the vector fields and their derivatives, represent the corresponding curvatures. We note finally that if two subdistributions contain at least one common vector field it seems naturally to expect interaction.

In the dual formulation of Frobenius theorem in terms of differential 1-forms (i.e. Pfaff forms), having the distribution $\Delta_p$, we look for $(n-p)$-Pfaff forms $(\alpha^1, \alpha^2, \ldots, \alpha^{n-p})$, i.e. a $(n-p)$-codistribution $\Delta^*_p$, such that $(\alpha^m, X_j) = 0$, and $\alpha^1(x) \wedge \alpha^2(x) \wedge \ldots \wedge \alpha^{n-p}(x) \neq 0$, $m = 1, 2, \ldots, n-p, \ j = 1, 2, \ldots, p, x \in M^n$. Then the integrability of the distribution $\Delta_p$ is equivalent to the requirements

$$d\alpha^m \wedge \alpha^1 \wedge \alpha^2 \wedge \ldots \wedge \alpha^{n-p} = 0, \ m = 1, 2, \ldots, (n-p),$$

where $d$ is the exterior derivative.

Since the idea of curvature associated with, for example, an arbitrary 2-dimensional distribution $(X, Y)$ is to find out if the Lie bracket $[X, Y](p)$ has components along vectors outside the 2-plane defined by $(X_p, Y_p)$, in our case we have to evaluate the quantities $\langle \alpha^m, [X, Y] \rangle$, where all linearly independent 1-forms $\alpha^m$ annihilate $(X, Y) : \langle \alpha^m, X \rangle = \langle \alpha^m, Y \rangle = 0$. In view of the formula

$$d\alpha^m(X, Y) = X(\langle \alpha^m, Y \rangle) - Y(\langle \alpha^m, X \rangle) - \langle \alpha^m, [X, Y] \rangle = -\langle \alpha^m, [X, Y] \rangle$$

we may introduce explicitly the curvature 2-form for the distribution $\Delta(X) = (X_1, \ldots, X_p)$. In fact, if $\Delta(Y) = (Y_1, \ldots, Y_{n-p})$ define a distribution which is complimentary (in the sense of direct sum) to $\Delta(X)$ and $(\alpha^m, X_i) = 0$, $(\alpha^m, Y_n) = \delta^m_n$, i.e. $(Y_1, \ldots, Y_{n-p})$ and $(\alpha^1, \ldots, \alpha^{n-p})$ are dual bases, then the corresponding curvature 2-form $\Omega_{\Delta(X)}$ should be defined by

$$\Omega_{\Delta(X)} = -d\alpha^m \otimes Y_m, \text{ since } \Omega_{\Delta(X)}(X_i, X_j) = -d\alpha^m(X_i, X_j)Y_m = \langle \alpha^m, [X_i, X_j] \rangle Y_m, \ (3.3)$$

32
where it is meant here that $\Omega_{\Delta(X)}$ is restricted to the distribution $(X_1, \ldots, X_p)$. Hence, if we call the distribution $(X_1, \ldots, X_p)$ horizontal and the complimentary distribution $(Y_1, \ldots, Y_{n-p})$ vertical, then the corresponding curvature 2-form acquires the status of \textit{vertical bundle valued 2-form}. We see that the curvature 2-form distinguishes those couples of vector fields inside $\Delta(X)$ the Lie brackets of which define outside $\Delta(X)$ directed flows, and so, do not allowing to find integral manifold of $\Delta(X)$. Clearly, the supposition here for dimensional complementarity of the two distributions $\Delta(X)$ and $\Delta(Y)$ is not essential for the idea of geometrical interaction, i.e. the distribution $\Delta(Y) \neq \Delta(X)$ may be any other distribution on the same manifold with dimension smaller than $(n-p)$, so that $m = 1, 2, \ldots, q < (n-p)$ in general, the important moment is that the two distributions (or subdistributions) can “communicate” \textit{differentially} through their curvature 2-forms.

Hence, from physical point of view, if the quantities $\Omega_{\Delta(X)}(X_i, X_j)$ are meant to be used for building the components of the energy-momentum locally transferred from the system $\Delta(X)$ to the system $\Delta(Y)$, then, naturally, we have to make use of the quantities $\Omega_{\Delta(Y)}(Y_m, Y_n)$ to build the components of the energy-momentum transferred from $\Delta(Y)$ to $\Delta(X)$.

It deserves to note that this formalism allows a dynamical equilibrium between the two systems $\Delta(Y)$ and $\Delta(X)$ to be described: each system to gain from the other as much energy-momentum as it loses, and this to take place at every space-time point. Therefore, if $W_{(X,Y)}$ denotes the energy-momentum transferred locally from $\Delta(X)$ to $\Delta(Y)$, $W_{(Y,X)}$ denotes the energy-momentum transferred locally from $\Delta(Y)$ to $\Delta(X)$, and $\delta W_{(X)}$ and $\delta W_{(Y)}$ denote respectively the local energy-momentum changes of the two systems $\Delta(X)$ and $\Delta(Y)$, then according to the local energy-momentum conservation law we can write

$$\delta W_{(X)} = W_{(Y,X)} + W_{(X,Y)}, \quad \delta W_{(Y)} = -(W_{(X,Y)} + W_{(Y,X)}) = -\delta W_{(X)},$$

i.e. $\Delta(X)$ and $\Delta(Y)$ are \textit{physically compatible}, or \textit{able to interact}. For the case of dynamical equilibrium we have $W_{(X,Y)} = -W_{(Y,X)}$, so in such a case we obtain

$$\delta W_{(X)} = 0, \quad \delta W_{(Y)} = 0, \quad W_{(Y,X)} + W_{(X,Y)} = 0. \quad (3.4)$$

As for how to build explicitly the corresponding representatives of the energy-momentum fluxes, probably, universal procedure can not be offered. The most simple procedure seems to be to ”project" the curvature values $\Omega_{\Delta(X)}(X_i, X_j)$ and $\Omega_{\Delta(Y)}(Y_m, Y_n)$ on the corresponding co-distribution volume forms, i.e. to consider the corresponding inner products $i(\Omega(X_i, X_j))(\alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^{n-p})$. For every special case, however, appropriate quantities constructed out of the members of the introduced distributions and co-distributions must be worked out.

### 3.4 PhLO Dynamical Structure in Terms of Frobenius Curvature

We consider the Minkowski space-time $M = (\mathbb{R}^4, \eta)$ with signature $\text{sign}(\eta) = (-, -, -, +)$ related to the standard global coordinates $(x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct)$, the natural volume form $\omega_\eta = \sqrt{|\eta|} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 = dx \wedge dy \wedge dz \wedge d\xi$, and the Hodge star $*$ defined by $\alpha \wedge *\beta = -\eta(\alpha, \beta) \omega_\eta$.

In view of our concept of PhLO we introduce the null vector field $\tilde{\zeta}$, $\tilde{\zeta}^2 = 0$, which in the $\tilde{\zeta}$-adapted coordinates (throughout used further) is assumed to look as follows:

$$\tilde{\zeta} = -\varepsilon \frac{\partial}{\partial z} + \frac{\partial}{\partial \xi}, \quad \varepsilon = \pm 1. \quad (3.5)$$
Let’s denote the corresponding to $\tilde{\zeta}$ completely integrable 3-dimensional Pfaff system by $\Delta^*(\tilde{\zeta})$. Thus, $\Delta^*(\tilde{\zeta})$ can be generated by any three linearly independent 1-forms $(\alpha_1, \alpha_2, \alpha_3)$ which annihilate $\zeta$, i.e.

$$
\alpha_1(\zeta) = \alpha_2(\zeta) = \alpha_3(\zeta) = 0; \quad \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \neq 0.
$$

Instead of $(\alpha_1, \alpha_2, \alpha_3)$ we introduce the notation $(A, A^*, \zeta)$ and define $\zeta$ to be the $\eta$-corresponding 1-form to $\zeta$:

$$
\zeta = \varepsilon dz + d\xi, \quad \langle \zeta, \tilde{\zeta} \rangle = 0,
$$

where $\langle , \rangle$ is the coupling between forms and vectors.

Now, since $\zeta$ is closed, it defines 1-dimensional completely integrable Pfaff system, so, we have the corresponding completely integrable distribution $(\bar{A}, \bar{A}^*, \zeta) : \langle \zeta, \bar{A} \rangle = \langle \zeta, \bar{A}^* \rangle = 0$. We shall restrict our further study to PhLO of electromagnetic nature according to the following

**Definition:** We shall call a PhLO electromagnetic if the following conditions hold:

1. the vector fields $(\bar{A}, \bar{A}^*)$ have no components along $\zeta$,
2. $(\bar{A}, \bar{A}^*)$ are $\eta$-corresponding to $(A, A^*)$ respectively .
3. $(A, A^*) = 0, \quad \langle A, \bar{A} \rangle = \langle A^*, \bar{A}^* \rangle$.

**Remark.** These relations formalize knowledge from Classical electrodynamics (CED). In fact, our vector fields $(\bar{A}, \bar{A}^*)$ are meant to represent what we call in CED electric and magnetic components of a free time-dependent electromagnetic field, where, as we have mentioned several times, the translational propagation of the field energy-momentum along a fixed null direction with the velocity "c" is possible only if the two invariants $I_1 = B^2 - E^2$ and $I_2 = 2E \cdot B$ are zero, because only in such a case the electromagnetic energy-momentum tensor $T_{\mu \nu}$ satisfies $T_{\mu \nu}T^{\mu \nu} = 0$ and has unique null eigen direction. So it seems naturally to consider this property as intrinsic for the field and to choose it as a starting point. Moreover, in such a case the relation $(I_1)^2 + (I_2)^2 = 0$ is equivalent to $E^2 + B^2 = 2|E \times B|$ and this relation shows that this is the only case when the field momentum can not be made equal to zero by means of frame change.

Together with the fact that the spatial direction of translational energy-momentum propagation is determined by $E \times B$, this motivates to introduce the vector field $\zeta$ in this form and to assume the properties 1-3 in the above definition.

From the above conditions it follows that in the $\tilde{\zeta}$-adapted coordinate system we have

$$
A = u \, dx + p \, dy, \quad A^* = -\varepsilon p \, dx + \varepsilon u \, dy; \quad \bar{A} = -u \frac{\partial}{\partial x} - p \frac{\partial}{\partial y}, \quad \bar{A}^* = \varepsilon p \frac{\partial}{\partial x} - \varepsilon u \frac{\partial}{\partial y},
$$

where $\varepsilon = \pm 1$, and $(u, p)$ are two smooth functions on $M$.

The completely integrable 3-dimensional Pfaff system $(A, A^*, \zeta)$ contains three 2-dimensional subsystems: $(A, A^*)$, $(A, \zeta)$ and $(A^*, \zeta)$. We have the following

**Proposition 1.** The following relations hold:

$$
dA \wedge A \wedge A^* = 0; \quad dA^* \wedge A^* \wedge A = 0;
$$

$$
dA \wedge A \wedge \zeta = \varepsilon \left[ u(p_\xi - \varepsilon p_\zeta) - p(u_\xi - \varepsilon u_\zeta) \right] \omega_0;
$$

$$
dA^* \wedge A^* \wedge \zeta = \varepsilon \left[ u(p_\xi - \varepsilon p_\zeta) - p(u_\xi - \varepsilon u_\zeta) \right] \omega_0.
$$

**Proof.** Immediately verified.

These relations say that the 2-dimensional Pfaff system $(A, A^*)$ is completely integrable for any choice of the two functions $(u, p)$, while the two 2-dimensional Pfaff systems $(A, \zeta)$ and $(A^*, \zeta)$ are NOT completely integrable in general, and the same curvature factor

$$
R = u(p_\xi - \varepsilon p_\zeta) - p(u_\xi - \varepsilon u_\zeta)
$$

34
determines their nonintegrability.

Correspondingly, the 3-dimensional completely integrable distribution (or differential system) \( \Delta(\tilde{\zeta}) \) contains three 2-dimensional subsystems: \((\tilde{A}, \tilde{A}^*)\), \((\tilde{A}, \tilde{\zeta})\) and \((\tilde{A}^*, \tilde{\zeta})\). We have the

**Proposition 2.** The following relations hold (recall that \([X, Y]\) denotes the Lie bracket):

\[
[\tilde{A}, \tilde{A}^*] \wedge \tilde{A} \wedge \tilde{A}^* = 0, \tag{3.7}
\]

\[
[\tilde{A}, \tilde{\zeta}] = (u_\xi - \varepsilon u_z) \frac{\partial}{\partial x} + (p_\xi - \varepsilon p_z) \frac{\partial}{\partial y}, \tag{3.8}
\]

\[
[\tilde{A}^*, \tilde{\zeta}] = -\varepsilon (p_\xi - \varepsilon p_z) \frac{\partial}{\partial x} + \varepsilon (u_\xi - \varepsilon u_z) \frac{\partial}{\partial y}. \tag{3.9}
\]

**Proof.** Immediately verified.

From these last relations (3.7-3.9) it follows that the distribution \((\tilde{A}, \tilde{A}^*)\) is completely integrable, and it can be easily shown that the two distributions \((\tilde{A}, \tilde{\zeta})\) and \((\tilde{A}^*, \tilde{\zeta})\) would be completely integrable only if the same curvature factor

\[
R = u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z) \tag{3.10}
\]

is zero (the elementary proof is omitted).

As it should be, the two projections

\[
\langle A, [\tilde{A}^*, \tilde{\zeta}] \rangle = -\langle A^*, [\tilde{A}, \tilde{\zeta}] \rangle = \varepsilon u(p_\xi - \varepsilon p_z) - \varepsilon p(u_\xi - \varepsilon u_z) = -\varepsilon R
\]

are nonzero and give (up to a sign) the same factor \(R\). The same curvature factor appears, of course, as coefficient in the exterior products \([\tilde{A}^*, \tilde{\zeta}] \wedge \tilde{A}^* \wedge \tilde{\zeta}\) and \([\tilde{A}, \tilde{\zeta}] \wedge \tilde{A} \wedge \tilde{\zeta}\). In fact, we obtain

\[
[\tilde{A}^*, \tilde{\zeta}] \wedge \tilde{A}^* \wedge \tilde{\zeta} = -(\tilde{A}, \tilde{\zeta}) \wedge \tilde{A} \wedge \tilde{\zeta} = -\varepsilon R \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + R \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial \xi}.
\]

On the other hand, for the other two projections we obtain

\[
\langle A, [\tilde{A}, \tilde{\zeta}] \rangle = \langle A^*, [\tilde{A}^*, \tilde{\zeta}] \rangle = \frac{1}{2}[(u^2 + p^2)_\xi - \varepsilon (u^2 + p^2)_z]. \tag{3.11}
\]

Clearly, the last relation (3.11) may be put in terms of the Lie derivative \(L_{\tilde{\zeta}}\) as

\[
\frac{1}{2}L_{\tilde{\zeta}}(u^2 + p^2) = -\frac{1}{2}L_{\tilde{\zeta}}[A, \tilde{A}] = -\langle A, L_{\tilde{\zeta}}\tilde{A} \rangle = -\langle A^*, L_{\tilde{\zeta}}\tilde{A}^* \rangle.
\]

**Remark.** Further in the paper we shall denote \(\sqrt{u^2 + p^2} \equiv \phi\).

**Proposition 3.** There is a function \(\psi(u, p)\) such, that

\[
L_{\tilde{\zeta}}\psi = \frac{u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)}{\phi^2} = \frac{R}{\phi^2}
\]

**Proof.** It is immediately verified that \(\psi = \arctan \frac{z}{u}\) is such one.

We note that the function \(\psi\) has a natural interpretation of phase because of the easily verified now relations \(u = \phi \cos \psi, p = \phi \sin \psi\), and \(\phi\) acquires the status of amplitude, i.e. energy density. Since the transformation \((u, p) \rightarrow (\phi, \psi)\) is non-degenerate this allows to work with the two functions \((\phi, \psi)\) instead of \((u, p)\).
From \textbf{Prop.3} we have
\[ R = \phi^2 L \bar{\zeta} \psi = \phi^2 (\psi \xi - \varepsilon \psi z) \rightarrow L \bar{\zeta} \psi = \frac{R}{T(\partial_x, \partial_x)} = \frac{*(dA \wedge A \wedge A^*)}{T(\partial_x, \partial_x)}, \tag{3.12} \]
where \( T(\partial_x, \partial_x) \) is the coordinate-free definition of the energy density.

This last formula (3.12) shows something very important: at any \( \phi \neq 0 \) the curvature \( R \) will NOT be zero only if \( L \bar{\zeta} \psi \neq 0 \), which admits in principle availability of rotation. In fact, lack of rotation would mean that \( \phi \) and \( \psi \) are running waves along \( \bar{\zeta} \). The relation \( L \bar{\zeta} \psi \neq 0 \) means, however, that rotational properties are possible in general, and some of these properties are carried by the phase \( \psi \). It follows that in such a case the translational component of propagation along \( \bar{\zeta} \) (which is supposed to be available) must be determined essentially, and most probably entirely, by \( \phi \). In particular, we could expect the relation \( L \bar{\zeta} \phi = 0 \) to hold, and if this happens, then the rotational component of propagation will be represented entirely by the phase \( \psi \), and, more specifically, by the curvature factor \( R \neq 0 \), so, the objects we are going to describe may have compatible translational-rotational dynamical structure. Finally, (3.12) may be considered as a definition for the phase function \( \psi \).

We are going now to represent some relations, analogical to the energy-momentum relations in classical electrodynamics, determined by some 2-form \( F \), in terms of the Frobenius curvatures given above.

The two nonintegrable Pfaff systems \((A, \zeta)\) and \((A^*, \zeta)\) carry two volume 2-forms:
\[ G = A \wedge \zeta \quad \text{and} \quad G^* = A^* \wedge \zeta, \]
and the two corresponding distributions define the 2-vectors
\[ \bar{G} = \bar{A} \wedge \bar{\zeta}, \quad \text{and} \quad G^* = \bar{A}^* \wedge \bar{\zeta}. \]
Making use now of the Hodge \(*\)-operator, we can verify the relation: \( G^* = *G \). Now \( G \) and \( G^* \) define the (1,1)-tensor, called stress-energy-momentum tensor \( T_{\mu}^\nu \), according to the rule
\[ T_{\mu}^\nu = -\frac{1}{2} [G_{\mu\sigma} \bar{G}^{\mu\sigma} + (G^*)_{\mu\sigma} (G^*)^{\mu\sigma}], \]
and the divergence of this tensor field can be represented in the form
\[ \nabla_\nu T_{\mu}^\nu = [i(\bar{G})dG]_\mu + [i(G^*)dG^*]_\mu, \]
where \( \bar{G} \) and \( G^* \) coincide with the metric-corresponding contravariant tensor fields, and \( i(\bar{G}) = i(\bar{\zeta}) \circ i(\bar{A}) \), \( i(G^*) = i(\zeta) \circ i(\bar{A}) \), \( i(X) \) is the standard insertion operator in the exterior algebra of differential forms on \( \mathbb{R}^4 \) defined by the vector field \( X \). So, we shall need the quantities
\[ i(\bar{G})dG, \quad i(G^*)dG^*, \quad i(\bar{G}^*)dG, \quad i(\bar{G})dG^*. \]
Having in view the explicit expressions for \( A, A^*, \zeta, \bar{A}, \bar{A}^* \) and \( \bar{\zeta} \) we obtain
\[ i(\bar{G})dG = i(\bar{G}^*)dG^* = \frac{1}{2} L \bar{\zeta} (\phi^2) \cdot \zeta, \tag{3.13} \]
also, we obtain
\[ i(G^*)dG = -i(\bar{G})dG^* =
\[ = [u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)] dz + \varepsilon [u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)] d\xi = \varepsilon R \cdot \zeta. \tag{3.14} \]
If $F$ and $H$ are correspondingly 2 and 3 forms on $M$ we have the relation

$$ *(F \wedge *H) = i(\bar{F})H = F^{\mu
u} H_{\mu\nu\sigma} dx^\sigma, \mu < \nu. $$

Therefore, since $G^* = \ast G$,

$$ i(\bar{G}^*) dG = -*(\delta \ast G \wedge \ast G), \quad -i(\bar{G}) dG^* = -*(\delta G \wedge G), \quad \text{so,} \quad \delta \ast G \wedge \ast G = \delta G \wedge G = \varepsilon \mathbf{R} \ast \zeta. \quad (3.15) $$

In the following formulae we must keep in mind the relations $d\zeta = 0, \langle A, \bar{A}^* \rangle = \langle A^*, \bar{A} \rangle = \langle \zeta, \bar{A}^* \rangle = \langle \zeta, \bar{A} \rangle = 0$, and $(\bar{A})^2 = (\bar{A}^*)^2 = \langle A, \bar{A} \rangle = \langle A^*, \bar{A}^* \rangle = -(u^2 + p^2) = -\Phi^2 = -|A|^2 = -|A^*|^2 = -|\bar{A}|^2 = -|\bar{A}^*|^2$.

In view of these formulae and the required duality in the definition of the curvature form (3.3), the two distributions $(\bar{A}, \zeta)$ and $(\bar{A}^*, \bar{\zeta})$ determine the following two curvature forms $\Omega$ and $\Omega^*$:

$$ \Omega = -\frac{d\bar{A}^*}{|A^*|} \otimes \frac{\bar{A}^*}{|A^*|} = \frac{dA}{|A|} \otimes \frac{A}{|A|}, \quad \Omega^* = -\frac{dA}{|A|} \otimes \frac{A}{|A|} = \frac{dA}{|A|} \otimes \frac{\bar{A}}{|A|}. $$

Denoting $Z_\Omega \equiv \Omega(\bar{A}, \zeta)$, $Z_\Omega^* \equiv \Omega^*(\bar{A}^*, \bar{\zeta})$, $Z_\Omega^{\ast*} \equiv \Omega^{\ast*}(\bar{A}^*, \bar{\zeta})$ and $Z_\Omega^{\ast*} \equiv \Omega^{\ast*}(\bar{A}^*, \bar{\zeta})$ we obtain

$$ Z_\Omega = -\frac{\varepsilon \mathbf{R}}{\phi^2} \bar{A}, \quad Z_\Omega^* = -\frac{\bar{A}^*}{2\phi^2} L_{\zeta} (\phi^2), \quad Z_\Omega^{\ast*} = -\frac{\bar{A}}{2\phi^2} L_{\zeta^*} (\phi^2), \quad Z_\Omega^{\ast*} = -\frac{\varepsilon \mathbf{R}}{\phi^2} \bar{A}. \quad (3.16) $$

The following relations express the connection between the curvatures and the energy-momentum characteristics.

$$ i(Z_\Omega)(A \wedge \zeta) = 0, \quad i(Z_\Omega)(A^* \wedge \zeta) = \varepsilon \mathbf{R} \zeta = -i(G) dG^* = i(G^*) dG, \quad (3.17) $$

$$ i(Z_\Omega^*)(A^* \wedge \zeta) = 0, \quad i(Z_\Omega^*)(A \wedge \zeta) = -\varepsilon \mathbf{R} \zeta = i(G^*) dG^* = -i(G) dG, \quad (3.18) $$

$$ i(Z_\Omega^{\ast*})(A \wedge \zeta) = 0, \quad i(Z_\Omega^{\ast*})(A^* \wedge \zeta) = \frac{1}{2} L_{\zeta} (\phi^2) \zeta = i(G^*) dG = i(G^*) dG^*, \quad (3.19) $$

$$ i(Z_\Omega^{\ast*})(A^* \wedge \zeta) = 0, \quad i(Z_\Omega^{\ast*})(A \wedge \zeta) = \frac{1}{2} L_{\zeta^*} (\phi^2) \zeta = i(G) dG = i(G^*) dG^*. \quad (3.20) $$

It follows from these relations that in case of dynamical equilibrium we shall have

$$ L_{\zeta} (\phi^2) = 0, \quad i(G) dG = 0, \quad i(G^*) dG^* = 0, \quad \varepsilon \mathbf{R} \zeta = 0. \quad (3.37) $$

Resuming, we can say that Frobenius integrability viewpoint suggests to make use of one completely integrable 3-dimensional distribution (resp. Pfaff system) consisting of one isotropic and two space-like vector fields (resp. 1-forms), such that the corresponding 2-dimensional spatial subdistribution $(\bar{A}, \bar{A}^*)$ (resp. Pfaff system $(A, A^*)$) defines a completely integrable system, and the rest two 2-dimensional subdistributions $(\bar{A}, \zeta)$ and $(\bar{A}^*, \bar{\zeta})$ (resp. Pfaff systems $(A, \zeta)$ and $(A^*, \zeta)$) are NON-integrable in general and give the same curvature. This curvature may be used to build quantities, physically interpreted as energy-momentum internal exchanges between the corresponding two subsystems $(\bar{A}, \zeta)$ and $(\bar{A}^*, \bar{\zeta})$ (resp. $(A, \zeta)$ and $(A^*, \zeta)$). Moreover, rotational component of propagation will be available only if the curvature $\mathbf{R}$ is nonzero, i.e. only if an internal energy-momentum exchange takes place. We see that all physically important characteristics and relations, describing the translational and rotational components of propagation, can be expressed in terms of the corresponding Frobenius curvature. We'll see that this holds also for some integral characteristics of PhLO.
3.5 PhLO Dynamical Structure in Terms of Non-linear Connections

3.5.1 Projections and algebraic curvatures

The projections are linear maps \( P \) in a linear space \( W^n \) (under linear space we mean here module over a ring, or vector space over a field) sending all elements of \( W^n \) to some subspace \( P(W^n) \subset W^n \), such that \( P \circ P = P \). Let \((e_1, \ldots, e_p, \ldots, e_n)\) and \((\varepsilon^1, \ldots, \varepsilon^p, \ldots, \varepsilon^n)\) be two dual bases: \( < \varepsilon^i, e_p > = \delta^i_p \), \( i, \nu = 1, \ldots, n \), and let \( N_i^a, i = 1, \ldots, p; a = p+1, \ldots, n \) be the corresponding to \( P \) \([p \times (n-p)]\) matrix of rank \((n-p)\). We define another couple of dual bases:

\[
k_i = (e_i + N_i^a e_a, e_a); \quad \omega^\mu = (e^i, \varepsilon^b - N_b^i \varepsilon^j), \quad j = 1, \ldots, p; \quad a, b = p+1, \ldots, n.
\]

Now the identity map \( id_{W^n} = \omega^\mu \otimes k_\mu \) acquires the form

\[
\begin{align*}
\text{id}_{W^n} & = \omega^\mu \otimes k_\mu = \omega^i \otimes k_i + \omega^b \otimes k_b = \varepsilon^i \otimes (e_i + N_i^a e_a) + (\varepsilon^b - N_b^i \varepsilon^j) \otimes e_b.
\end{align*}
\]

We obtain two projections: \( P_V = (\varepsilon^b - N_b^i \varepsilon^j) \otimes e_b \) and \( P_H = \varepsilon^i \otimes (e_i + N_i^a e_a) \) such that \( \text{Ker} P_V = \text{Im} P_H \) and \( \text{Ker} P_H = \text{Im} P_V \), also, \( P_H = \text{id}_{W^n} - P_V \). Hence, \( W^n = \text{Ker} P_V \oplus \text{Im} P_V = \text{Ker} P_H \oplus \text{Im} P_H \). Usually \( P_V \) is called vertical projection, and \( P_H \) called horizontal projection.

Let now \( \phi \) and \( \psi \) be two arbitrary linear maps in a module \( \mathfrak{M}, \mathfrak{B} : \mathfrak{M} \times \mathfrak{M} \rightarrow \mathfrak{M} \) be a binar map satisfying \( \mathfrak{B}(\mathbf{x} + \mathbf{z}, \mathbf{y}) = \mathfrak{B}(\mathbf{x}, \mathbf{y}) + \mathfrak{B}(\mathbf{z}, \mathbf{y}) \) and \( \mathfrak{B}(\mathbf{x}, \mathbf{y} + \mathbf{z}) = \mathfrak{B}(\mathbf{x}, \mathbf{y}) + \mathfrak{B}(\mathbf{x}, \mathbf{z}) \), and \( (\mathbf{x}, \mathbf{y}, \mathbf{z}) \) be three arbitrary elements of \( \mathfrak{M} \). We consider the expression

\[
\begin{align*}
A(\mathfrak{B}; \phi, \psi)(\mathbf{x}, \mathbf{y}) & = \frac{1}{2} \left[ \mathfrak{B}(\phi(\mathbf{x}), \psi(\mathbf{y})) + \mathfrak{B}(\psi(\mathbf{x}), \phi(\mathbf{y})) + \phi \circ \psi(\mathfrak{B}(\mathbf{x}, \mathbf{y})) + \psi \circ \phi(\mathfrak{B}(\mathbf{x}, \mathbf{y})) \right].
\end{align*}
\]

Assuming \( \phi = \psi \) are projections in \( \mathfrak{M} \) denoted by \( P \), this expression becomes

\[
A(\mathfrak{B}; P)(\mathbf{x}, \mathbf{y}) \equiv P(\mathfrak{B}(\mathbf{x}, \mathbf{y})) + \mathfrak{B}(P(\mathbf{x}), P(\mathbf{y})) - P(\mathfrak{B}(\mathbf{x}, P(\mathbf{y}))) - P(\mathfrak{B}(P(\mathbf{x}), \mathbf{y})).
\]

Denoting the identity map of \( \mathfrak{M} \) by \( \text{id} \) and adding and subtracting \( P[\mathfrak{B}(P(\mathbf{x}), P(\mathbf{y}))] \), after some elementary transformations we obtain

\[
A(\mathfrak{B}; P)(\mathbf{x}, \mathbf{y}) \equiv P \left[ \mathfrak{B} \left[ (\text{id} - P)(\mathbf{x}), (\text{id} - P)(\mathbf{y}) \right] \right] + (\text{id} - P) \left[ \mathfrak{B} \left[ P(\mathbf{x}), P(\mathbf{y}) \right] \right].
\]

Recalling that \( P \) and \( (\text{id} - P) \) project on two subspaces of \( \mathfrak{M} \), the direct sum of which generates \( \mathfrak{M} \), and naming \( P \) as vertical projection denoted by \( V \), then \( (\text{id} - P) \), denoted by \( H \), gets naturally the name horizontal projection. So the above expression gets the final form of

\[
A(\mathfrak{B}; P)(\mathbf{x}, \mathbf{y}) \equiv V \left[ \mathfrak{B} \left[ H(\mathbf{x}), H(\mathbf{y}) \right] \right] + H \left[ \mathfrak{B} \left[ V(\mathbf{x}), V(\mathbf{y}) \right] \right] = \mathcal{R}_P(\mathfrak{B}; \mathbf{x}, \mathbf{y}) + \mathcal{R}_P(\mathfrak{B}; \mathbf{x}, \mathbf{y}).
\]

Hence, the first term on the right, \( \mathcal{R}_P(\mathfrak{B}; \mathbf{x}, \mathbf{y}) \), which may be called \( \mathfrak{B} \)-algebraic curvature of \( P \), measures the vertical component of the \( \mathfrak{B} \)-image of the horizontal projections of \( (\mathbf{x}, \mathbf{y}) \), and then the second term \( \mathcal{R}_P(\mathfrak{B}; \mathbf{x}, \mathbf{y}) \), acquiring the name of \( \mathfrak{B} \)-algebraic cocurvature of \( P \), measures the horizontal component of the \( \mathfrak{B} \)-image of the vertical projections of \( (\mathbf{x}, \mathbf{y}) \).

We carry now this pure algebraic construction to the tangent bundle of a smooth manifold \( M^n \), where the above binar map \( \mathfrak{B} \) will be interpreted as the Lie bracket of vector fields, and the linear maps will be just linear endomorphisms of the tangent/cotangent bundles of
3.5.2 Nonlinear connections

Let now \((x^1, \ldots, x^n)\) be any local coordinate system on our real manifold \(M^n\). We have the corresponding local frames \(\{dx^1, \ldots, dx^n\}\) and \(\{\partial x^1, \ldots, \partial x^n\}\). Let for each \(x \in M\) we are given a projection \(P_x\) of the same constant rank \((n-p)\), i.e. \(p\) does not depend on \(x\), in every tangent space \(T_x(M)\). The space \(\text{Ker}(P_x) \subset T_x(M)\) is usually called \(P\)-horizontal, and the space \(\text{Im}(P_x) \subset T_x(M)\) then is called \(P\)-vertical. Thus, we have two distributions on \(M\) the direct sum of which gives the tangent bundle: \(T(M) = \text{Ker}(P) \oplus \text{Im}(P)\). The above algebraic construction shows that each of these two distributions can be endowed with corresponding \(2\)-form, valued in the other distribution, and depending on the same binary operation in \(TM^n\). As we mentioned, the choice \(\mathfrak{g} = \text{Lie bracket}\) leads to tensor field. Therefore, assuming this choice, we say that \(P\) defines a nonlinear connection on \(M\). Denoting by \(\mathcal{R}\) the so defined curvature \(2\)-form of \(P\) and by \(\tilde{\mathcal{R}}\) the corresponding cocurvature \(2\)-form of \(P\), by \(V_P\) and \(H_P\) the corresponding vertical and horizontal projections, we can write

\[
[P,P](X,Y) = \mathcal{R}(X,Y) + \tilde{\mathcal{R}}(X,Y),
\]

(3.23)

where

\[
\mathcal{R}(X,Y) = V_P([H_P X, H_P Y]), \quad \tilde{\mathcal{R}}(X,Y) = H_P([V_P X, V_P Y]),
\]

\((X,Y)\) are any two vector fields and the Lie bracket is denoted by \([,\]. Recalling the contents of the preceding section, it can be shown that \(\mathcal{R}(X,Y) \neq 0\) measures the nonintegrability of the corresponding horizontal distribution, and \(\tilde{\mathcal{R}}(X,Y) \neq 0\) measures the nonintegrability of the corresponding vertical distribution.

If the vertical distribution is given before-hand and is completely integrable, i.e. \(\tilde{\mathcal{R}} = 0\), then \(\mathcal{R}(X,Y)\) is called curvature of the nonlinear connection \(P\) if there exist at least one couple of horizontal vector fields \((X,Y)\) such that \(\mathcal{R}(X,Y) \neq 0\).

3.5.3 Photon-like nonlinear connections

We assume now that our manifold is \(\mathbb{R}^4\) endowed with standard coordinates \((x^1, x^2, x^3, x^4 = x, y, z, \xi = ct)\), and make some preliminary considerations in order to make the choice of our projection \(P : T\mathbb{R}^4 \rightarrow T\mathbb{R}^4\) consistent with the introduced concept of PhLO. The intrinsically defined straight-line translational component of propagation of the PhLO will be assumed to be parallel to the coordinate plane \((z,\xi)\). Also, \(\frac{\partial}{\partial z}\) and \(\frac{\partial}{\partial y}\) will be vertical coordinate fields, so every vertical vector field \(Y\) can be represented by \(Y = u \frac{\partial}{\partial z} + p \frac{\partial}{\partial y}\), where \((u, p)\) are two functions on \(\mathbb{R}^4\). It is easy to check that any two such linearly independent vertical vector fields \(Y_1\) and \(Y_2\) define an integrable distribution, hence, the corresponding curvature will be zero. It seems very natural to choose \(Y_1\) and \(Y_2\) to coincide correspondingly with the vertical projections \(P(\frac{\partial}{\partial z})\) and \(P(\frac{\partial}{\partial y})\). Moreover, let’s restrict ourselves to PhLO of electromagnetic nature and denote further the vertical projection by \(V\). Then, since this vertical structure is meant to be smoothly straight-line translated along the plane \((z,\xi)\) with the velocity of light,
a natural suggestion comes to mind these two projections \( Y_1 = \nabla (x) \) and \( Y_2 = \nabla (y) \) to be physically interpreted as representatives of the electric and magnetic components. Now we know from classical electrodynamics that the situation described corresponds to zero invariants of the electromagnetic field, therefore, we may assume that \( Y_1 \) and \( Y_2 \) are orthogonal to each other and with the same modules with respect to the euclidean metric in the 2-dimensional space spent by \( \partial_x \) and \( \partial_y \). It follows that the essential components of \( Y_1 \) and \( Y_2 \) should be expressible only with two independent functions \((u, p)\). The conclusion is that our projection should depend only on \((u, p)\). Finally, we note that these assumptions lead to the horizontal nature of \( dz \) and \( d\xi \).

Note that if the translational component of propagation is along the vector field \( \zeta \) then we can define two new distributions: \((Y_1, \zeta)\) and \((Y_2, \zeta)\), which do not seem to be integrable in general even if \( \zeta \) has constant components as it will be in our case. Since these two distributions are nontrivially intersected (they have a common member \( \zeta \)), it is natural to consider them as geometrical images of two interacting physical subsystems of our PhLO. Hence, we must introduce two projections with the same image space but with different kernel spaces, and the components of both projections must depend only on the two functions \((u, p)\).

Let now \((u, p)\) be two smooth functions on \( \mathbb{R}^3 \) and \( \varepsilon = \pm 1 \). We introduce two projections \( V \) and \( \tilde{V} \) in \( T\mathbb{R}^4 \) as follows:

\[
V = dx \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial y} - \varepsilon \, u \, dz \otimes \frac{\partial}{\partial x} - u \, d\xi \otimes \frac{\partial}{\partial x} - \varepsilon \, p \, dz \otimes \frac{\partial}{\partial y} - p \, d\xi \otimes \frac{\partial}{\partial y},
\]

\[
\tilde{V} = dx \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial y} + p \, dz \otimes \frac{\partial}{\partial x} + \varepsilon \, p \, d\xi \otimes \frac{\partial}{\partial x} - u \, dz \otimes \frac{\partial}{\partial y} - \varepsilon \, u \, d\xi \otimes \frac{\partial}{\partial y}.
\]

So, in both cases we consider \((\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\) as vertical vector fields, and \((dz, d\xi)\) as horizontal 1-forms. By corresponding transpositions we can determine projections \( \ast V \) and \( \ast \tilde{V} \) in the cotangent bundle \( T^*\mathbb{R}^4 \).

\[
\ast V = dx \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial y} - \varepsilon \, u \, dx \otimes \frac{\partial}{\partial z} - u \, dx \otimes \frac{\partial}{\partial \xi} - \varepsilon \, p \, dy \otimes \frac{\partial}{\partial z} - p \, dy \otimes \frac{\partial}{\partial \xi},
\]

\[
\ast \tilde{V} = dx \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial y} + p \, dx \otimes \frac{\partial}{\partial z} + \varepsilon \, p \, dx \otimes \frac{\partial}{\partial \xi} - u \, dy \otimes \frac{\partial}{\partial z} - \varepsilon \, u \, dy \otimes \frac{\partial}{\partial \xi}.
\]

The corresponding horizontal projections, denoted by \( (H, \tilde{H}; H^* \tilde{H}^*) \) look as follows:

\[
H = dz \otimes \frac{\partial}{\partial z} + d\xi \otimes \frac{\partial}{\partial \xi} + \varepsilon \, u \, dz \otimes \frac{\partial}{\partial x} + u \, d\xi \otimes \frac{\partial}{\partial x} + \varepsilon \, p \, dz \otimes \frac{\partial}{\partial y} + p \, d\xi \otimes \frac{\partial}{\partial y},
\]

\[
\tilde{H} = dz \otimes \frac{\partial}{\partial z} + d\xi \otimes \frac{\partial}{\partial \xi} - p \, dz \otimes \frac{\partial}{\partial x} - \varepsilon \, p \, d\xi \otimes \frac{\partial}{\partial x} + u \, dz \otimes \frac{\partial}{\partial y} + \varepsilon \, u \, d\xi \otimes \frac{\partial}{\partial y},
\]

\[
H^* = dz \otimes \frac{\partial}{\partial z} + d\xi \otimes \frac{\partial}{\partial \xi} + \varepsilon \, u \, dx \otimes \frac{\partial}{\partial z} + u \, dx \otimes \frac{\partial}{\partial \xi} + \varepsilon \, p \, dy \otimes \frac{\partial}{\partial z} + p \, dy \otimes \frac{\partial}{\partial \xi},
\]

\[
\tilde{H}^* = dz \otimes \frac{\partial}{\partial z} + d\xi \otimes \frac{\partial}{\partial \xi} - p \, dx \otimes \frac{\partial}{\partial z} - \varepsilon \, p \, dy \otimes \frac{\partial}{\partial z} + u \, dy \otimes \frac{\partial}{\partial \xi} + \varepsilon \, u \, dy \otimes \frac{\partial}{\partial \xi}.
\]

The corresponding matrices look like:

\[
V = \begin{bmatrix} 1 & 0 & -\varepsilon \, u & -u \\ 0 & 1 & -\varepsilon \, p & -p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & \varepsilon \, u & u \\ 0 & 0 & \varepsilon \, p & p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]
The projections of the coordinate bases are:

$$V^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\varepsilon u & -\varepsilon p & 0 & 0 \\ -u & -p & 0 & 0 \end{bmatrix}, \quad H^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon u & \varepsilon p & 0 & 0 \\ u & p & 0 & 1 \end{bmatrix},$$

$$\tilde{V} = \begin{bmatrix} 1 & 0 & p & \varepsilon p \\ 0 & 1 & -u & -\varepsilon u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} 0 & 0 & -p & -\varepsilon p \\ 0 & 0 & u & \varepsilon u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{V}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ p & -u & 0 & 0 \\ \varepsilon p & -\varepsilon u & 0 & 0 \end{bmatrix}, \quad \tilde{H}^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p & u & 1 & 0 \\ -\varepsilon p & \varepsilon u & 0 & 1 \end{bmatrix}.$$

The components \( R_{\mu\nu} \) of the two curvature 2-forms \( R \) and \( \tilde{R} \) in coordinate basis are given by \( V_{\rho}^*(H_{\frac{\partial}{\partial x^\rho}, \frac{\partial}{\partial x^\rho}})^{\sigma} \), and the only nonzero components are just

\[ R_{z\xi}^x = R_{34}^1 = -\varepsilon (u_\xi - \varepsilon u_z), \quad R_{z\xi}^y = R_{34}^2 = -\varepsilon (p_\xi - \varepsilon p_z). \]

For the nonzero components of \( \tilde{R} \) we obtain

\[ \tilde{R}_{z\xi}^x = \tilde{R}_{34}^1 = (p_\xi - \varepsilon p_z), \quad \tilde{R}_{z\xi}^y = \tilde{R}_{34}^2 = -(u_\xi - \varepsilon u_z). \]

The corresponding two curvature forms are:

\[ R = -\varepsilon (u_\xi - \varepsilon u_z) dz \wedge d\xi \otimes \frac{\partial}{\partial x} - \varepsilon (p_\xi - \varepsilon p_z) dz \wedge d\xi \otimes \frac{\partial}{\partial y} \]  \( (3.26) \)

\[ \tilde{R} = (p_\xi - \varepsilon p_z) dz \wedge d\xi \otimes \frac{\partial}{\partial x} - (u_\xi - \varepsilon u_z) dz \wedge d\xi \otimes \frac{\partial}{\partial y}. \]  \( (3.27) \)
We obtain (in our coordinate system): $-\frac{1}{2} tr (V \circ H^*) = -\frac{1}{2} tr \left( \tilde{V} \circ \tilde{H}^* \right) = u^2 + p^2$, and

\[
V \left( \left[ H \left( \frac{\partial}{\partial z} \right), H \left( \frac{\partial}{\partial \xi} \right) \right] \right) = \left[ H \left( \frac{\partial}{\partial z} \right), H \left( \frac{\partial}{\partial \xi} \right) \right] = -\varepsilon (u_\xi - \varepsilon u_\xi) \frac{\partial}{\partial x} - \varepsilon (p_\xi - \varepsilon p_\xi) \frac{\partial}{\partial y} \equiv Z_1,
\]

\[
\tilde{V} \left( \left[ \tilde{H} \left( \frac{\partial}{\partial z} \right), \tilde{H} \left( \frac{\partial}{\partial \xi} \right) \right] \right) = \left[ \tilde{H} \left( \frac{\partial}{\partial z} \right), \tilde{H} \left( \frac{\partial}{\partial \xi} \right) \right] = (p_\xi - \varepsilon p_\xi) \frac{\partial}{\partial x} - (u_\xi - \varepsilon u_\xi) \frac{\partial}{\partial y} \equiv Z_2,
\]

where $Z_1$ and $Z_2$ coincide with the values of the two curvature forms $\mathcal{R}$ and $\tilde{\mathcal{R}}$ on the coordinate vector fields $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \xi}$ respectively:

\[
Z_1 = \mathcal{R} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \xi} \right), \quad Z_2 = \tilde{\mathcal{R}} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \xi} \right).
\]

We evaluate now the vertical 2-form $V^* (dx) \wedge V^* (dy)$ on the bivector $Z_1 \wedge Z_2$ and obtain $\varepsilon \mathcal{K}^2$, where

\[
\mathcal{K}^2 = (u_\xi - \varepsilon u_\xi)^2 + (p_\xi - \varepsilon p_\xi)^2.
\]

An important parameter, having dimension of length (the coordinates are assumed to have dimension of length) and denoted by $l_o$, turns out to be the square root of the quantity

\[
\frac{-\frac{1}{2} tr (V \circ H^*)}{\mathcal{K}^2} = \frac{u^2 + p^2}{(u_\xi - \varepsilon u_\xi)^2 + (p_\xi - \varepsilon p_\xi)^2}.
\]

Clearly, if $l_o$ is finite constant it could be interpreted as some parameter of extension of the PhLO described, so it could be used as identification parameter in the dynamical equations and in lagrangians, but only if $(u_\xi - \varepsilon u_\xi) \neq 0$ and $(p_\xi - \varepsilon p_\xi) \neq 0$. This goes along with our concept of PhLO which does not admit spatially infinite extensions. Finally we’d like to note that the right-hand side of the above relation does not depend on which projection $V$ or $\tilde{V}$ is used, i.e. $[\tilde{V}^* (dx) \wedge \tilde{V}^* (dy)] (Z_1 \wedge Z_2) = \varepsilon \mathcal{K}^2$ too, so

\[
l_o^2 = \frac{-\frac{1}{2} tr (V \circ H^*)}{\mathcal{K}^2} = \frac{-\frac{1}{2} tr (V \circ H^*)}{\mathcal{K}^2} = \frac{u^2 + p^2}{(u_\xi - \varepsilon u_\xi)^2 + (p_\xi - \varepsilon p_\xi)^2}, \quad (3.28)
\]

The parameter $l_o$ has the following symmetry. Denote by $V_o = dx \otimes \frac{\partial}{\partial z} + dy \otimes \frac{\partial}{\partial \xi}$, then

$V = V_o + \tilde{V}$ and $\tilde{V} = V_o + \tilde{V}_1$, where, in our coordinates, $V_1$ and $\tilde{V}_1$ can be seen above how they look like. We form now $W = aV_1 - b\tilde{V}_1$ and $\tilde{W} = bV_1 + a\tilde{V}_1$, where $(a, b)$ are two arbitrary real numbers. The components of the corresponding linear maps $P_W = V_o + W$ and $P_{\tilde{W}} = V_o + \tilde{W}$ can be obtained through the substitutions: $u \rightarrow (au + \varepsilon bp)$; $p \rightarrow (\varepsilon bp - ap)$, and, obviously, $P_W$ and $P_{\tilde{W}}$ are projections. Now, $-\frac{1}{2} tr (V \circ H^*)$ transforms to $(a^2 + b^2)(u^2 + p^2)$ and $\mathcal{K}^2$ transforms to $(a^2 + b^2)(u_\xi - \varepsilon u_\xi)^2 + (p_\xi - \varepsilon p_\xi)^2$, so, $l_o (V, \tilde{V}) = l_o (W, \tilde{W})$. This corresponds in some sense to the dual symmetry of classical vacuum electrodynamics. We note also that the squared modules of the two curvature forms $|\mathcal{R}|^2$ and $|\tilde{\mathcal{R}}|^2$ are equal to $(u_\xi - \varepsilon u_\xi)^2 + (p_\xi - \varepsilon p_\xi)^2$ in our coordinates, therefore, the nonzero values of $|\mathcal{R}|^2$ and $|\tilde{\mathcal{R}}|^2$, as well as the finite value of $l_o$ guarantee that the two functions $u$ and $p$ are NOT plane waves. Finally, the phase function may be defined by the relations

\[
L_{\xi \psi} = \frac{<A, Z_2>}{-\frac{1}{2} tr (V \circ H^*)} = -\frac{<A^*, Z_1>}{-\frac{1}{2} tr (V \circ H^*)}.
\]
3.5.4 Electromagnetic PhLO in terms of non-linear connections

Recall that the relativistic formulation of classical electrodynamics in vacuum \((\rho = 0)\) is based on the following assumptions. The configuration space is the Minkowski space-time \(\mathbb{R}^4, \eta\) where \(\eta\) is the pseudometric with \(\text{sign}(\eta) = (-, -, -, +)\) with the corresponding volume 4-form \(\omega_4 = dx \wedge dy \wedge dz \wedge d\xi\) and Hodge star \(*\) defined by \(\alpha \wedge *\beta = -\eta(\alpha, \beta)\omega_4\). The electromagnetic field is described by two closed 2-forms \((\omega, \tilde{\omega})\) and the corresponding two curvature forms give the mathematical realization of this idea:

\[
\begin{align*}
&\star\delta F - 2d\omega = 0, \quad d\star F = 0. \quad \text{The physical characteristics of the field are deduced from the following stress-energy-momentum tensor field}
&\quad T_{\mu\nu}(F_\ast F) = -\frac{1}{2} [F_{\mu\sigma} F^{\nu\sigma} + (\ast F)_{\mu\sigma} (\ast F)^{\nu\sigma}].
\end{align*}
\]

In the non-vacuum case the allowed energy-momentum exchange with other physical systems is given in general by the divergence

\[
\nabla_\nu T^\nu_\mu = \frac{1}{2} [F^{\alpha\beta}(dF)_{\alpha\beta\mu} + (\ast F)^{\alpha\beta}(d\ast F)_{\alpha\beta\mu}] = F^{\mu\nu}(\delta F)^\nu + (\ast F)^{\mu\nu}(\delta \ast F)^\nu,
\]

where \(\delta = \ast d\ast\) is the coderivative. If the field is free: \(dF = 0, d \ast F = 0\), this divergence is obviously equal to zero on the vacuum solutions since its both terms are zero. Therefore, energy-momentum exchange between the two partner-fields \(F\) and \(\ast F\), which should be expressed by the terms \((\ast F)^{\alpha\beta}(dF)_{\alpha\beta\mu}\) and \(F^{\alpha\beta}(d\ast F)_{\alpha\beta\mu}\) is NOT allowed on the solutions of \(dF = 0, d \ast F = 0\). This shows that the widely used 4-potential approach (even if two 4-potentials \(A, A^\ast\) are introduced so that \(dA = F, dA^\ast = \ast F\) locally) to these equations excludes any possibility to individualize two energy-momentum exchanging time-stable subsystems of the field that are mathematically represented by \(F\) and \(\ast F\).

On the contrary, as we have mentioned several times, our concept of PhLO does NOT exclude such two physically interacting subsystems of the field to really exist, and therefore, to be mathematically individualized. The intrinsically connected two projections \(V\) and \(\tilde{V}\) and the corresponding two curvature forms give the mathematical realization of this idea: \(V\) and \(\tilde{V}\) individualize the two subsystems, and the corresponding two curvature 2-forms \(R\) and \(\tilde{R}\) represent the instruments by means of which the available mutual local energy-momentum exchange between these two subsystems could be described. We should not forget that, as we have already emphasized several times, the energy-momentum relations say that we need two 2-forms to begin with.

Recall that our coordinate 1-forms \(dx, dy\) have the following vertical and horizontal projections:

\[
\begin{align*}
&V^\ast(dx) = dx - \varepsilon u \, dz - u \, d\xi, \quad H^\ast(dx) = \varepsilon u \, dz + u \, d\xi, \quad \text{and} \\
&V^\ast(dy) = dy - \varepsilon p \, dz - p \, d\xi, \quad H^\ast(dy) = \varepsilon p \, dz + p \, d\xi.
\end{align*}
\]

We form now the 2-forms \(V^\ast(dx) \wedge H^\ast(dx)\) and \(V^\ast(dy) \wedge H^\ast(dy)\):

\[
\begin{align*}
&V^\ast(dx) \wedge H^\ast(dx) = \varepsilon u \, dx \wedge dz + u \, dx \wedge d\xi, \\
&V^\ast(dy) \wedge H^\ast(dy) = \varepsilon p \, dy \wedge dz + p \, dy \wedge d\xi.
\end{align*}
\]

Summing up these last two relations and denoting the sum by \(F\) we obtain

\[
F = \varepsilon u \, dx \wedge dz + u \, dx \wedge d\xi + \varepsilon p \, dy \wedge dz + p \, dy \wedge d\xi.
\]
we can represent exchange between momentum tensor given above, these last relations show that some real energy-momentum example, from these last relations we see that field \((\mathbf{F}, i)\) and given by
\[
\mathbf{F}_\mu\mathbf{F}^{\mu} = \mathbf{F}_\mu(\mathbf{F}^\mu) = 0.
\]
\(i\) has zero invariants:
\[
\tilde{\mathbf{F}} = \mathbf{F} = A_\star \wedge \zeta, \quad \tilde{\mathbf{F}} = *\mathbf{F} = A_\star \wedge \zeta.
\]
From these last relations we see that \(F \wedge F = 0, F \wedge *F = 0\), i.e. the field \((\mathbf{F}, *\mathbf{F})\) has zero invariants: \(F_{\mu\nu}\mathbf{F}^{\mu\nu} = F_{\mu\nu}(\mathbf{F})^{\mu\nu} = 0\). The following relations are now easy to verify:
\[
V^*(\mathbf{F}) = H^*(\mathbf{F}) = V^*(\mathbf{F}) = H^*(\mathbf{F}) = \tilde{V}^*(\mathbf{F}) = \tilde{V}^*(\mathbf{F}) = \tilde{H}^*(\mathbf{F}) = 0,
\]
i.e. \(F\) and \(*F\) have zero vertical and horizontal projections with respect to \(V\) and \(\tilde{V}\). Since, obviously, \(\zeta\) is horizontal with respect to \(V\) and \(\tilde{V}\), it is interesting to note that \(A\) is vertical with respect to \(\tilde{V}\) and \(A_\star\) is vertical with respect to \(V\): \(\tilde{V}^*(A) = A, V(A_\star) = A_\star\). In fact, for example,
\[
\tilde{V}^*(A) = \tilde{V}^*(u\,dx + p\,dy) = u\tilde{V}^*(dx) + p\tilde{V}^*(dy) = u[dx + p\,dz + \varepsilon p\,d\xi] + p[dy - u\,dz - \varepsilon u\,d\xi] = u\,dx + p\,dy.
\]
We are going to establish now that there is real energy-momentum exchange between the \(F\)-component and the \(*F\)-component of the field. To come to this we compute the quantities \(i(Z_1) F, i(Z_2) *F, \ i(Z_1) *F, \ i(Z_2) F\). We obtain:
\[
i(Z_1) F = i(Z_2) *F = \langle A, Z_1 \rangle \zeta = \langle A_\star, Z_2 \rangle \zeta = \frac{1}{2} [(u^2 + p^2)\xi - \varepsilon (u^2 + p^2)\mu] \zeta =
\]
\[
= \frac{1}{2} F^{\sigma\rho}(dF)_{\sigma\rho\mu}dx^\mu = \frac{1}{2} (\star F)^{\sigma\rho}(dF)_{\sigma\rho\mu}dx^\mu = \frac{1}{2} \nabla_\nu T^\nu_\mu(F, *F),
\]
\[
i(Z_1) *F = -i(Z_2) F = \langle A_\star, Z_1 \rangle \zeta = -\langle A, Z_2 \rangle \zeta = [u(p_\xi - \varepsilon p) - p(u_\xi - \varepsilon u)] \zeta =
\]
\[
= -\frac{1}{2} F^{\sigma\rho}(dF)_{\sigma\rho\mu}dx^\mu = \frac{1}{2}(\star F)^{\sigma\rho}(dF)_{\sigma\rho\mu}dx^\mu.
\]
If our field is free then \(\nabla_\nu T^\nu_\mu(F, \tilde{F}) = 0\). Moreover, in view of the divergence of the stress-energy-momentum tensor given above, these last relations show that some real energy-momentum exchange between \(F\) and \(*F\) takes place: the magnitude of the energy-momentum, transferred from \(F\) to \(*F\) and given by \(i(Z_1) *F = \frac{1}{2}(\star F)^{\sigma\rho}(dF)_{\sigma\rho\mu}dx^\mu\), is equal to that, transferred from \(*F\) to \(F\), which is given by \(-i(Z_2) F = -\frac{1}{2} F^{\sigma\rho}(dF)_{\sigma\rho\mu}dx^\mu\). On the other hand, as it is well known, the \(*\)-invariance of the stress-energy-momentum tensor in case of zero invariants leads to \(F^{\mu\nu} \Phi^{\rho\sigma} = (\star F)^{\mu\alpha}(\star F)^{\rho\sigma}\), so, \(F\) and \(*F\) carry equal and conserved quantities of stress-energy-momentum.

We interpret physically this as follows. The electromagnetic PhLO exist through a special internal dynamical equilibrium between the two subsystems of the field, represented by \(V\) and \(\tilde{V}\), namely, both subsystems carry the same stress-energy-momentum and the mutual energy-momentum exchange between them is always in equal quantities. This individualization does NOT mean that any of the two subsystems can exist separately, independently on the other. Moreover, NO spatial “part” of PhLO should be considered to represent a real physical object.
3.6 Electromagnetic PhLO in terms of electromagnetic strain

The concept of strain is introduced in studying elastic materials subject to external forces of different nature: mechanical, electromagnetic, etc. In nonrelativistic continuum physics the local representatives of the external forces in this context are usually characterized in terms stresses. Since the force means energy-momentum transfer leading to corresponding mutual energy-momentum change of the interacting objects, then according to the energy-momentum conservation law the material must react somehow to the external interference in accordance with its structure and reaction abilities. The classical strain describes mainly the abilities of the material to bear force-action from outside through deformation, i.e. through changing its shape, or, configuration. The term elastic now means that any two allowed configurations can be deformed to each other without appearance of holes and breakings, in particular, if the material considered has deformed from configuration $C_1$ to configuration $C_2$ it is able to return smoothly to its configuration $C_1$.

The general geometrical description [27] starts with the assumption that an elastic material is a continuum $\mathbb{B} \subset \mathbb{R}^3$ which can smoothly deform inside the space $\mathbb{R}^3$, so, it can be endowed with differentiable structure, i.e. having an elastic material is formally equivalent to have a smooth real 3-dimensional submanifold $\mathbb{B} \subset \mathbb{R}^3$. The deformations are considered as smooth maps (mostly embeddings) $\varphi : \mathbb{B} \rightarrow \mathbb{R}^3$. The spaces $\mathbb{B}$ and $\mathbb{R}^3$ are endowed with riemannian metrics $\mathbf{G}$ and $g$ respectively (and corresponding riemannian co-metrics $\mathbf{G}^{-1}$ and $g^{-1}$), and induced isomorphisms $\tilde{\mathbf{G}}$ and $\tilde{g}$ between the corresponding tangent and cotangent spaces. This allows to define linear map inside every tangent space of $\mathbb{B}$ in the following way: a tangent vector $V \in T_x\mathbb{B}$, $x \in \mathbb{B}$, is sent through the differential $d\varphi$ of $\varphi$ to $(d\varphi)_x(V) \in T_{\varphi(x)}\mathbb{R}^3$, then by means of the isomorphism $\tilde{g}$ we determine the corresponding 1-form (i.e. we "lower the index"), this 1-form is sent to the dual space $T^*_x\mathbb{B}$ of $T_x\mathbb{B}$ by means of the dual linear map $(d\varphi)^* : T_{\varphi(x)}^*\mathbb{R}^3 \rightarrow T^*_x\mathbb{B}$, and finally, we determine the corresponding tangent vector by means of the isomorphism $\tilde{\mathbf{G}}^{-1}$ (i.e. we "raise the index" correspondingly). The so obtained linear map

$$C_x := [\tilde{\mathbf{G}}^{-1} \circ (d\varphi)^* \circ \tilde{g} \circ (d\varphi)]_x : T_x\mathbb{B} \rightarrow T_x\mathbb{B}$$

(which is denoted in [27] by $(\mathbf{F}^T\mathbf{F})_x$), extended to the whole $\mathbb{B}$, is called Cauchy-Green deformation tensor field. Now, the combination

$$\mathbf{E}_x := \frac{1}{2}[(\tilde{\mathbf{G}} \circ \mathbf{C} - \mathbf{G})]_x = \frac{1}{2}[(d\varphi)^* \circ \tilde{g} \circ (d\varphi) - \mathbf{G}]_x : T_x\mathbb{B} \times T_x\mathbb{B} \rightarrow \mathbb{R}$$

is called Lagrangian strain tensor field. Note that if we denote by $\varphi^*g$ the induced on $\mathbb{B}$ metric from the metric $g$ (usually euclidean) on $\mathbb{R}^3$ then $\mathbf{E} = \frac{1}{2}(\varphi^*g - \mathbf{G})$.

We could look at the problem also as follows. The mathematical counterparts of the allowed (including reversible) deformations are the diffeomorphisms $\varphi$ of a riemannian manifold $(M, g)$, and every $\varphi(M)$ represents a possible configuration of the material considered. But some diffeomorphisms do not lead to deformation (i.e. to shape changes), so, a criterion must be introduced to separate those diffeomorphisms which should be considered as essential. For such a criterion is chosen the distance change: if the distance between any two fixed points does not change during the action of the external force field, then we say that there is no deformation. Now, every essential diffeomorphism $\varphi$ must transform the metric $g$ to some new metric $\varphi^*g$, such that $g \neq \varphi^*g$. The naturally arising tensor field $\varepsilon = (\varphi^*g - g) \neq 0$ appears as a measure of the physical abilities of the material to withstand external force actions.

Since the external force is assumed to act locally and the material considered gets the corresponding to the external force field final configuration in a smooth way, i.e. passing smoothly
through a family of allowed configurations, we may introduce a localization of the above scheme, such that the isometry doiffeomorphisms to be eliminated. This is done by means of introducing 1-parameter group $\varphi_t, t \in [a, b] \subset \mathbb{R}$ of local diffeomorphisms, so, $\varphi_a(M)$ and $\varphi_b(M)$ denote correspondingly the initial and final configurations. Now $\varphi_t$ generates a family of metrics $\varphi_t^* g$, and a corresponding family of tensors $\varepsilon_t$. According to the local analysis every local 1-parameter group of diffeomorphisms is generated by a vector field on $M$. Let the vector field $X$ generate $\varphi_t$. Then the quantity

$$
\frac{1}{2} L_X g := \frac{1}{2} \lim_{t \to 0} \varphi_t^* g - g,
$$

i.e. one half of the Lie derivative of $g$ along $X$, is called (infinitesimal) strain tensor, or deformation tensor.

**Remark.** Further in the paper we shall work with $L_X g$, i.e. the factor $1/2$ will be omitted.

In our further study we shall call $L_X g$, where $g = \eta$ is the Minkowski (pseudo)metric, just strain tensor. Clearly, the term ”material” is not appropriate for PhLO because no static situations are admissible, our objects of interest are of entirely dynamical nature, so the corresponding relativistic strain tensors must take care of this.

According to the preliminary considerations important vector fields in our approach to describe electromagnetic PhLO are $\zeta, A, \tilde{A}^*$, so, we consider the corresponding three electromagnetic strain tensors: $L_\zeta \eta; L_\tilde{A} \eta; L_{\tilde{A}^*} \eta$.

**Proposition 4.** The following relations hold:

$$
L_\zeta \eta = 0, \quad (L_\tilde{A} \eta)_{\mu\nu} \equiv D_{\mu\nu} = \begin{bmatrix}
2u_x & u_y + p_x & u_z & u_\xi \\
u_y + p_x & 2p_y & p_z & p_\xi \\
u_z & p_z & 0 & 0 \\
u_\xi & p_\xi & 0 & 0
\end{bmatrix},
$$

$$
(L_{\tilde{A}^*} \eta)_{\mu\nu} \equiv D^*_{\mu\nu} = \begin{bmatrix}
-2\varepsilon p_x & -\varepsilon(p_y + u_x) & -\varepsilon p_z & -\varepsilon p_\xi \\
-\varepsilon(p_y + u_x) & 2\varepsilon u_y & \varepsilon u_z & \varepsilon u_\xi \\
-\varepsilon p_z & \varepsilon u_z & 0 & 0 \\
-\varepsilon p_\xi & \varepsilon u_\xi & 0 & 0
\end{bmatrix}.
$$

**Proof.** Immediately verified.

We give now some important from our viewpoint relations.

$$
D(\zeta, \tilde{\zeta}) = D^*(\tilde{\zeta}, \zeta) = 0,
$$

$$
D(\zeta)_\mu dx^\mu \equiv D_{\mu\nu} \zeta^\nu dx^\mu = (u_\xi - \varepsilon u_z)dx + (p_\xi - \varepsilon p_z)dy,
$$

$$
D(\zeta)_\mu \frac{\partial}{\partial x^\mu} \equiv D^\mu_\nu \tilde{\zeta}^\nu \frac{\partial}{\partial x^\mu} = -(u_\xi - \varepsilon u_z)\frac{\partial}{\partial x} - (p_\xi - \varepsilon p_z)\frac{\partial}{\partial y} = -[\tilde{A}, \zeta],
$$

$$
D_{\mu\nu} \tilde{A}^\mu \tilde{\zeta}^\nu = -\frac{1}{2} \left[(u^2 + p^2)\xi - \varepsilon(u^2 + p^2)z\right] = -\frac{1}{2} L_\zeta \phi^2,
$$

$$
D_{\mu\nu} \tilde{A}^\mu \zeta^\nu = -\varepsilon \left[u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)\right] = -\varepsilon R = -\varepsilon \phi^2 L_\zeta \psi.
$$

We also have:

$$
D^*(\tilde{\zeta}) = \varepsilon \left[-(p_\xi - \varepsilon p_z)dx + (u_\xi - \varepsilon u_z)dy\right],
$$

$$
D^*(\tilde{\zeta})_\mu \frac{\partial}{\partial x^\mu} \equiv (D^*)_{\mu\nu} \tilde{\zeta}^\nu \frac{\partial}{\partial x^\mu} = -\varepsilon(p_\xi - \varepsilon p_z)\frac{\partial}{\partial x} + (u_\xi - \varepsilon u_z)\frac{\partial}{\partial y} = [\tilde{A}^*, \tilde{\zeta}],
$$

46
\[
D^*_{\mu \nu} \tilde{A}^\mu \tilde{\zeta}^\nu = -\frac{1}{2} \left[ (u^2 + p^2)\xi - \varepsilon (u^2 + p^2)_{\xi} \right] = -\frac{1}{2} L \xi \phi^2, \\
D^*_{\mu \nu} \tilde{A}^\mu \tilde{\zeta}^\nu = \varepsilon \left[ u(p_{\xi} - \varepsilon p_z) - p(u_{\xi} - \varepsilon u_z) \right] = \varepsilon R = \varepsilon \phi^2 L \xi \psi.
\]

Clearly, \( D(\tilde{\zeta}) \) and \( D^*(\tilde{\zeta}) \) are linearly independent in general:

\[
D(\tilde{\zeta}) \wedge D^*(\tilde{\zeta}) = \varepsilon \left[ (u_{\xi} - \varepsilon u_z)^2 + (p_{\xi} - \varepsilon p_z)^2 \right] dx \wedge dy = \varepsilon \phi^2 (\psi_{\xi} - \varepsilon \psi_z)^2 dx \wedge dy \neq 0.
\]

Recall now that every 2-form \( F \) defines a linear map \( \tilde{F} \) from 1-forms to 3-forms through the exterior product: \( \tilde{F}(\alpha) := \alpha \wedge F \), where \( \alpha \in \Lambda^1(M) \). Moreover, the Hodge \* operator, composed now with \( \tilde{F} \), gets \( \tilde{F}(\alpha) \) back to \( \ast \tilde{F}(\alpha) \in \Lambda^1(M) \). In the previous section we introduced two 2-forms \( G = A \wedge \zeta \) and \( G^* = A^* \wedge \zeta \) and noticed that \( G^* = \ast G \). We readily obtain now

\[
D(\tilde{\zeta}) \wedge G = D^*(\tilde{\zeta}) \wedge G^* = D(\tilde{\zeta}) \wedge A \wedge \zeta = D^*(\tilde{\zeta}) \wedge A^* \wedge \zeta =
\]

\[
= -\varepsilon \left[ u(p_{\xi} - \varepsilon p_z) - p(u_{\xi} - \varepsilon u_z) \right] dx \wedge dy \wedge dz - \left[ u(p_{\xi} - \varepsilon p_z) - p(u_{\xi} - \varepsilon u_z) \right] dx \wedge dy \wedge dz =
\]

\[
= -\phi^2 L \xi \psi (\varepsilon dx \wedge dy \wedge dz + dx \wedge dy \wedge d\xi) = -R \left( \varepsilon dx \wedge dy \wedge dz + dx \wedge dy \wedge d\xi \right),
\]

\[
D(\tilde{\zeta}) \wedge G^* = -D^*(\tilde{\zeta}) \wedge G = D(\tilde{\zeta}) \wedge A^* \wedge \zeta = -D^*(\tilde{\zeta}) \wedge A^* \wedge \zeta =
\]

\[
= \frac{1}{2} \left[ (u^2 + p^2)_{\xi} - \varepsilon (u^2 + p^2)_{\xi} \right] (dx \wedge dy \wedge dz + \varepsilon dx \wedge dy \wedge d\xi).
\]

Thus, recalling relations (3.16)-(3.20), we get

\[
\ast \left[ D(\tilde{\zeta}) \wedge A \wedge \zeta \right] = \ast \left[ D^*(\tilde{\zeta}) \wedge A^* \wedge \zeta \right] = -\varepsilon R \zeta = -i(G^*) dG = i(G) dG^* , \tag{3.36}
\]

\[
\ast \left[ D(\tilde{\zeta}) \wedge A^* \wedge \zeta \right] = \ast \left[ D^*(\tilde{\zeta}) \wedge A \wedge \zeta \right] = \frac{1}{2} L \xi \phi^2 \zeta = i(G) dG = i(G^*) dG^* . \tag{3.37}
\]

The above relations show various dynamical aspects of the energy-momentum redistribution during evolution of our PhLO. In particular, equations (3.36-3.37) clearly show that it is possible the translational and rotational components of the energy-momentum redistribution to be represented in form depending on the \( \zeta \)-directed strains \( D(\tilde{\zeta}) \) and \( D^*(\tilde{\zeta}) \). So, the local translational changes of the energy-momentum carried by the two vector components \( G \) and \( G^* \) of our PhLO are given by the two 1-forms \( \ast \left[ D(\tilde{\zeta}) \wedge A^* \wedge \zeta \right] \) and \( \ast \left[ D^*(\tilde{\zeta}) \wedge A \wedge \zeta \right] \) and the local rotational ones - by the 1-forms \( \ast \left[ D(\tilde{\zeta}) \wedge A \wedge \zeta \right] \) and \( \ast \left[ D^*(\tilde{\zeta}) \wedge A^* \wedge \zeta \right] \). In fact, the form \( \ast \left[ D(\tilde{\zeta}) \wedge A \wedge \zeta \right] \) determines the strain that "leaves" the 2-plane defined by \( (A, \zeta) \) and the form \( \ast \left[ D^*(\tilde{\zeta}) \wedge A^* \wedge \zeta \right] \) determines the strain that "leaves" the 2-plane defined by \( (A^*, \zeta) \). Since the PhLO is free, i.e. no energy-momentum is lost or gained from outside, this means that the two (null-field) components \( G \) and \( G^* \) exchange locally equal energy-momentum quantities: \( \ast \left[ D(\tilde{\zeta}) \wedge A \wedge \zeta \right] = \ast \left[ D^*(\tilde{\zeta}) \wedge A^* \wedge \zeta \right] \). Now, the local energy-momentum conservation law \( \nabla_\nu \left[ G_{\mu \sigma} G^{\nu \sigma} + (G^*)_{\mu \sigma} (G^*)^{\nu \sigma} \right] = 0 \) requires \( L \xi \phi^2 = 0 \), while the corresponding strain-fluxes become zero: \( \ast \left[ D^*(\tilde{\zeta}) \wedge A \wedge \zeta \right] = 0 \), \( \ast \left[ D(\tilde{\zeta}) \wedge A \wedge \zeta \right] = 0 \).

It seems important to note that, only dynamical relation between the local energy-momentum change and strain fluxes exists, so NO analog of the assumed in elasticity theory generalized Hooke law, (i.e. linear relation between the stress tensor and the strain tensor) seems to exist. This clearly goes along with the fully dynamical nature of PhLO, i.e. linear relations exist between the divergence terms of our stress tensor \( \frac{1}{2} \left[ -G_{\mu \sigma} \tilde{G}^{\mu \sigma} - (G^*)_{\mu \sigma} (G^*)^{\nu \sigma} \right] \) and the \( \zeta \)-directed strain fluxes as given by equations (3.36)-(3.37).
Chapter 4

Equations of motion for PhLO.
Solutions

In this chapter we show that appropriate solutions for PhLO can be obtained by solving linear equations.

4.1 The approach based on the notion for PhLO

Every system of equations describing the time-evolution of some physical system should be consistent with the very system in the sense that all identification characteristics of the system described must not change. In the case of electromagnetic PhLO we assume the couple \((F, \tilde{F})\) to represent the field, and in accordance with our notion for PhLO one of the identification characteristics is straight-line translational propagation of the energy-density with constant velocity "c", therefore, with every PhLO we may associate appropriate direction, i.e. a geodesic null vector field \(\zeta, \zeta^2 = 0\) on the Minkowski space-time. On the other hand, the complex of field functions \((F_{\mu\nu}, \tilde{F}_{\mu\nu})\) admits both translational and rotational components of propagation. We choose further \(\zeta = -\varepsilon \frac{\partial}{\partial z} + \frac{\partial}{\partial \xi}\), which means that we have chosen the coordinate system in such a way that the translational propagation is parallel to the plane \((z, \xi)\). For another such parameter we assume that the finite longitudinal extension of any PhLO is fixed and is given by an appropriate positive number \(\lambda\). In accordance with the "compatible translational-rotational dynamical structure" of PhLO we shall assume that no translation is possible without rotation, and no rotation is possible without translation, and in view of the constancy of the translational component of propagation we shall assume that the rotational component of propagation is periodic, i.e. it is characterized by a constant frequency. The natural period \(T\) suggested is obviously \(T = \frac{\lambda}{c}\). An obvious candidate for "rotational operator" is the linear map \(J\) transforming \(F\) to \(\tilde{F}\), which map coincides with the reduced to 2-forms Hodge-* \(\star\). Geometrically, \(\star\) rotates the 2-frame \((A, A^*)\) to \(\tilde{F}\), so if such a rotation is associated with a translational advancement of \(l_o\), then a full rotation should correspond to translational advancement of \(4l_o = \lambda\). The simplest and most natural translational change of the field \((F, \tilde{F})\) along \(\zeta\) should be given by the Lie derivative of the field along \(\zeta\). Hence, the simplest and most natural equations should read

\[\kappa l_o L_{\zeta}(F) = \varepsilon \tilde{F},\]

where \(F\) and \(\tilde{F}\) are given in the preceding section, \(\kappa = \pm 1\) is responsible for left/right orientation of the rotational component of propagation, and \(l_o = \text{const.}\) Vice versa, since \(J \circ J = -id\) and
\( J^{-1} = -J \) the above equation is equivalent to

\[
\kappa l_o L_{\xi}(\tilde{F}) = -\varepsilon \mathcal{F}.
\]

It is easy to show that these equations are equivalent to

\[
\kappa l_o L_{\xi}(V - V_o) = \varepsilon (\tilde{V} - V_o),
\]

where \( V \) is given by (40) and \( V_o = dx \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial y} \) in our coordinates is the identity map in \( Im(V) = Im(\tilde{V}) \). Another equivalent form is given by

\[
\kappa l_o Z_1 = \tilde{A}^*, \quad \text{or} \quad \kappa l_o Z_2 = -\tilde{A},
\]

where \( \tilde{A}^* \) and \( \tilde{A} \) are \( \eta \)-corresponding vector fields to the 1-forms \( A^* \) and \( A \).

### 4.2 The Lagrangian Approach

Appropriate lagrangian for the above equations (\( l_o = \text{const.} \)) is

\[
\mathcal{L} = \frac{1}{4} \left( \varepsilon \kappa l_o \tilde{\zeta}^\sigma \frac{\partial F_{\alpha\beta}}{\partial x^\sigma} - \tilde{F}_{\alpha\beta} \right) \tilde{F}^{\alpha\beta} - \frac{1}{4} \left( \varepsilon \kappa l_o \tilde{\zeta}^\sigma \frac{\partial \tilde{F}_{\alpha\beta}}{\partial x^\sigma} + F_{\alpha\beta} \right) F^{\alpha\beta},
\]

(4.2)

\( F \) and \( \tilde{F} \) are considered as independent, and the relations \( F \wedge F = F \wedge \tilde{F} = \tilde{F} \wedge \tilde{F} = 0 \) lead to

\[
\tilde{\zeta}^\sigma \frac{\partial F_{\alpha\beta}}{\partial x^\sigma} \tilde{F}^{\alpha\beta} = \tilde{\zeta}^\sigma \frac{\partial \tilde{F}_{\alpha\beta}}{\partial x^\sigma} F^{\alpha\beta} = 0.
\]

The corresponding Lagrange equations read

\[
\varepsilon \kappa l_o \tilde{\zeta}^\sigma \frac{\partial \tilde{F}_{\alpha\beta}}{\partial x^\sigma} + F_{\alpha\beta} = 0, \quad \varepsilon \kappa l_o \tilde{\zeta}^\sigma \frac{\partial F_{\alpha\beta}}{\partial x^\sigma} - \tilde{F}_{\alpha\beta} = 0,
\]

(4.4)

so, on the solutions the lagrangian becomes zero: \( \mathcal{L}(\text{solutions}) = 0 \). The stress-energy-momentum tensor, in view of the null character of \( F \) and \( \tilde{F} \), is given by (3.29), where \( *F \) has to be replaced by \( \tilde{F} \). It deserves noting that the above null conditions lead to \( F_{\mu\sigma} F^{\nu\sigma} = \tilde{F}_{\mu\sigma} \tilde{F}^{\nu\sigma} \) and to \( F_{\mu\sigma} \tilde{F}^{\nu\sigma} = 0 \). Hence, the two subsystems represented by \( F \) and \( \tilde{F} \) carry the same stress-energy-momentum, therefore, \( F \equiv \tilde{F} \) energy-momentum exchange is possible only in equal quantities. In our coordinates the above equations reduce to

\[
\kappa l_o (u_\xi - \varepsilon u_z) = -p, \quad \kappa l_o (p_\xi - \varepsilon p_z) = u,
\]

it is seen that the constant \( l_o \) satisfies the above given relation (3.28). From these last equations we readily obtain the relations

\[
(u^2 + p^2)_\xi - \varepsilon (u^2 + p^2)_z = 0, \quad u (p_\xi - \varepsilon p_z) - p (u_\xi - \varepsilon u_z) = \frac{\kappa}{l_o} (u^2 + p^2),
\]

which represent our equations in energy-momentum terms. Now, the substitution \( u = \Phi \cos \psi, \quad p = \Phi \sin \psi \), leads to the relations

\[
L_\xi \Phi = 0, \quad L_\xi \psi = \frac{\kappa}{l_o}.
\]

In terms of nonlinear connections recalling that \( \Phi^2 = -\frac{1}{2} tr(V \circ H^* \circ L_\xi) \) and computing \( \frac{1}{2} tr(V \circ L_\xi \tilde{H}^*) = \varepsilon [u (p_\xi - \varepsilon p_z) - p (u_\xi - \varepsilon u_z)] = \Phi^2 \varepsilon L_\xi \psi \) the last two relations can be equivalently written as

\[
L_\xi [tr(V \circ H^*)] = 0, \quad tr(V \circ L_\xi \tilde{H}^*) = -\varepsilon \frac{\kappa}{l_o} tr(V \circ H^*).\]
4.3 Equations of motion in terms of translational-rotational compatability

In order to look at the translational-rotational compatability as a generating tool for writing equations of motion we recall first the concept of local symmetry of a distribution: a vector field $Y$ is a local (or infinitesimal) symmetry of a $p$-dimensional distribution $\Delta$ defined by the vector fields $(Y_1, \ldots, Y_p)$ if every Lie bracket $[Y_i, Y_j]$ is in $\Delta$: $[Y_i, Y_j] \in \Delta$. Clearly, if $\Delta$ is completely integrable, then every $Y_i$ is a symmetry of $\Delta$, and the flows of these vector fields move the points of each completely integral manifold of $\Delta$ inside this integral manifold, that’s why they are called sometimes internal symmetries. If $Y$ is outside $\Delta$ then it is called shuffling symmetry [28], and in such a case the flow of $Y$ transforms a given completely integral manifold to another one, i.e. the flow of $Y$ ”shuffles” the lists of the corresponding foliation. We are going to show that our vector field $\vec{\zeta} = -\varepsilon \frac{\partial}{\partial z} + \frac{\partial}{\partial x}$ is a shuffling symmetry for the distribution $\Delta_o$ defined by the vector fields $(\vec{A}, \vec{A}^*)$. In fact, $\Delta_o$ coincides with our vertical distribution generated by $(\frac{\partial}{\partial z}, \frac{\partial}{\partial y})$, so it is completely integrable and its integral manifolds coincide with the $(x, y)$-planes. From physical point of view this should be expected in view of the intrinsically required stability of our PhLO under translational propagation along null straight lines: this propagation just transforms the $2$-plane $(x, y)$ passing through the point $(z_1, \xi_1)$ to a parallel to it $2$-plane passing through the point $(z_2, \xi_2)$, and these two points lay on the same trajectory of our field $\vec{\zeta}$.

The corresponding Lie brackets are

$$[\vec{A}, \vec{\zeta}] = (u_\xi - \varepsilon u_z)_x + (p_\xi - \varepsilon p_z)_y,$$

$$[\vec{A}^*, \vec{\zeta}] = -\varepsilon (p_\xi - \varepsilon p_z)_x + \varepsilon (u_\xi - \varepsilon u_z)_y.$$  

We see that $[\vec{A}, \vec{\zeta}]$ and $[\vec{A}^*, \vec{\zeta}]$ are generated by $(\frac{\partial}{\partial z}, \frac{\partial}{\partial y})$, but $\vec{\zeta}$ is outside $\Delta_o$, so our field $\vec{\zeta}$ is a shuffling local symmetry of $\Delta_o$.

We notice now that at each point we have two frames: $(\vec{A}, \vec{A}^*, \partial_z, \partial_\xi)$ and $(\vec{A}, \vec{\zeta}, \partial_z, \partial_\xi)$. Since physically we have internal energy-momentum redistribution during propagation, we could interprete the permanent availability of these two intrinsically connected frames as corresponding mathematical adequate of this physical process. Taking into account that only the first two vectors of these two frames change during propagation we write down the corresponding linear transformation as follows:

$$([\vec{A}, \vec{\zeta}], [\vec{A}^*, \vec{\zeta}]) = (\vec{A}, \vec{A}^*) \frac{\alpha}{\gamma} \frac{\beta}{\delta}.$$  

Solving this system with respect to $(\alpha, \beta, \gamma, \delta)$ we obtain

$$\begin{vmatrix}
\alpha & \beta \\
\gamma & \delta
\end{vmatrix} = \frac{1}{\varepsilon R^2} \begin{vmatrix}
-\frac{1}{2} L_\xi \Phi^2 & \varepsilon R \\
-\varepsilon R & -\frac{1}{2} L_\xi \Phi^2
\end{vmatrix} = \frac{1}{\varepsilon} \begin{vmatrix}
L_\xi \Phi^2 & 1 \\
0 & 1
\end{vmatrix} + \varepsilon L_\xi \psi \begin{vmatrix}
0 & 1 \\
-1 & 0
\end{vmatrix},$$

where $R = u (p_\xi - \varepsilon p_z) - p (u_\xi - \varepsilon u_z)$. If the translational propagation is governed by the conservation law $L_\xi \Phi^2 = 0$, then we obtain that the rotational component of propagation is governed by the matrix $\varepsilon L_\xi \psi J$, where $J$ denotes the canonical complex structure in $\mathbb{R}^2$, and since $\Phi^2 L_\xi \psi = u (p_\xi - \varepsilon p_z) - p (u_\xi - \varepsilon u_z) \neq 0$ we conclude that the rotational component of propagation would be available if and only if $R \neq 0$. We may also say that a compatible translational-rotational dynamical structure is available if the amplitude $\Phi^2 = u^2 + p^2$ is a running wave along $\zeta$ and the phase $\psi = \arg u^2$ is NOT a running wave along $\zeta : L_\xi \psi \neq 0$. Physically this means that the rotational component of propagation is entirely determined by the available internal energy-momentum exchange: $i(F) dF = -i(F) d\bar{F}$.
Let's assume that the phase is given by \( \psi \). This suggests to choose the initial condition of the two spatial directions of translational propagation of the PhLO along the \( T \) can be given by \( \bar{a}, \bar{b} \) means that the two orientations of the rotation, defined by \( \epsilon \kappa \) and \( \epsilon \kappa ET \), following equations for the two functions \( u, p \):

\[
\begin{align*}
u - \epsilon u_z &= -\frac{\kappa}{l_0} p, \quad p - \epsilon p_z &= \frac{\kappa}{l_0} u.
\end{align*}
\]

The quantity \( R = u (p \epsilon - \epsilon p_z) - p (u \epsilon - \epsilon u_z) = \Phi^2 L_z \psi = kl_0^{-1} \phi^2 \) suggests to find an integral characteristic of the PhLO rotational nature. In fact, the two co-distributions \( (A, \zeta) \) and \( (A^*, \zeta) \) define the two (equal in our case) Frobenius 4-forms \( dA \wedge A \wedge \zeta = dA^* \wedge A^* \wedge \zeta \). Each of these two 4-forms is equal to \( \epsilon R \omega_o \). Now, multiplying by \( l_0/c \) any of them we obtain:

\[
\begin{align*}
\frac{l_0}{c} dA \wedge A \wedge \zeta &= \frac{l_0}{c} dA^* \wedge A^* \wedge \zeta = \frac{l_0}{c} \epsilon R \omega_o = \epsilon \kappa \omega_o.
\end{align*}
\]

Integrating over the 4-volume \( \mathbb{R}^3 \times (\lambda = 4l_0) \) (and having in view the spatially finite nature of PhLO) we obtain the finite quantity \( H = \epsilon \kappa ET \), where \( E \) is the integral energy of the PhLO, \( T = \frac{A}{\epsilon} \), which clearly is the analog of the Planck formula \( E = h \nu \), i.e. \( h = ET \). The combination \( \epsilon \kappa \) means that the two orientations of the rotation, defined by \( \kappa = \pm 1 \), may be observed in each of the two spatial directions of translational propagation of the PhLO along the \( z \)-axis: from \(-\infty \) to \(+\infty \), or from \(+\infty \) to \(-\infty \).

Finally, recalling relations (3.15), we can easily see that in case of \( L \psi = kl_0^{-1} \) and \( L \Phi^2 = 0 \) the 3-form \( \delta F \wedge F = \epsilon R \kappa \psi \vee = \frac{\epsilon}{c} \Phi^2 \kappa \) becomes closed: \( d(\delta F \wedge F) = 0 \), which also gives an integral conservation law. In fact, the 3-integral of the reduced on \( \mathbb{R}^3 \) 3-form \( \frac{\delta F}{c} (\delta F \wedge F) \) gives \( \epsilon \kappa ET \), where \( E \) is the integral energy, so, the Planck formula holds.

### 4.4 Photon-like Solutions

We consider the equations obtained in terms of the two functions \( \Phi = \sqrt{u^2 + p^2} \) and \( \psi = \arctan \frac{u}{p} \). The equation for \( \Phi \) in our coordinates is \( \Phi_t - \epsilon \Phi_x = 0 \), therefore, \( \Phi = \Phi(x, y, \xi + \epsilon z) \), where \( \Phi \) is allowed to be spatially finite, as assumed further, or spatially localized function. The equation for \( \psi \) is \( \psi_t - \epsilon \psi_x = \frac{\kappa}{l_0} \). Two families of solutions for \( \psi \), depending on an arbitrary function \( \varphi \) can be given by

\[
\begin{align*}
\psi_1 &= -\frac{\epsilon \kappa}{l_0} z + \varphi(x, y, \xi + \epsilon z), \quad \text{and} \quad \psi_2 = \frac{\kappa}{l_0} \xi + \varphi(x, y, \xi + \epsilon z).
\end{align*}
\]

Since \( \Phi^2 \) is a spatially finite function representing the energy density we see that the translational propagation of our PhLO is represented by a spatially finite running wave along the \( z \)-coordinate. Let’s assume that the phase is given by \( \psi_1 \) and, for simplicity, \( \varphi = 0 \). The form of this solution suggests to choose the initial condition \( u_{t=0}(x, y, \epsilon z), p_{t=0}(x, y, \epsilon z) \) in the following way. Let for \( z = 0 \) the initial condition be located on a disk \( D = D(x, y; a, b; r_o) \) of small radius \( r_o \), the center of the disk to have coordinates \( (a, b) \), and the value of \( \Phi_{t=0}(x, y, 0) = \sqrt{u_{t=0}^2 + p_{t=0}^2} \) to be proportional to some appropriate for the case bump function \( f > 0 \) on \( D \) of the distance \( \sqrt{(x-a)^2 + (y-b)^2} \) between the origin of the coordinate system and the point \( (x, y, 0) \), such that it is centered at the point \( (a, b) \), so, \( f(x, y) = f(\sqrt{(x-a)^2 + (y-b)^2}) \), \( D \) is defined by \( D = \{(x, y)|\sqrt{(x-a)^2 + (y-b)^2} \leq r_o\} \), and \( f(x, y) \) is zero outside \( D \). Let also the dependence of \( \Phi_{t=0} \) on \( z \) be given by \( f \) be the corresponding bump function \( \theta(z; \lambda) > 0 \) of an interval \( (z, z + \lambda) \) of length \( \lambda = 4l_o \) on the \( z \)-axis. If \( \gamma > 0 \) is the proportionality coefficient we obtain

\[
\begin{align*}
u &= \gamma f(x, y; a, b) \theta(ct + \epsilon z; \lambda) \cos(\psi_1), \\
p &= \gamma f(x, y; a, b) \theta(ct + \epsilon z; \lambda) \sin(\psi_1).
\end{align*}
\]
We see that because of the available *sine* and *cosine* factors in the solution, the initial condition for the solution will occupy a 3d-spatial region of shape that is close to a helical cylinder of height $\lambda$, having internal radius of $r_o$ and wrapped up around the $z$-axis. Also, its center will always be $\sqrt{a^2 + b^2}$-distant from the $z$-axis. Hence, the solution will propagate translationally along the coordinate $z$ with the velocity $c$, and, rotationally, inside the corresponding infinitely long helical cylinder because of the $z$-dependence of the available periodical multiples.

On the two figures below are given two theoretical examples with $\kappa = -1$ and $\kappa = 1$ respectively, amplitude function $\Phi$ located inside a one-step helical cylinder $D$ with height of $\lambda$, and phase $\psi = -\varepsilon \kappa \frac{b}{\lambda} = -\varepsilon \kappa \frac{2a}{2\pi}$. The solutions with $\varepsilon = -1$ will propagate left-to-right along the coordinate $z$.

Figure 4.1: Theoretical example with $\kappa = -1$. The translational propagation is directed left-to-right.

Figure 4.2: Theoretical example with $\kappa = 1$. The translational propagation is directed left-to-right.

The curvature $K$ and the torsion $\tau$ of the screwline inside $D$ through the point $(x, y, 0) \in D$ will be

$$K = \frac{\gamma f \theta}{(\gamma f \theta)^2 + b^2}, \quad \tau = \frac{\kappa b}{(\gamma f \theta)^2 + b^2},$$

where $b = \lambda/2\pi = 2l_o/\pi$. The rotational frequency $\nu$ will be $\nu = c/\lambda = c/4l_o$, so we can introduce period $T = 1/\nu$ and elementary action $h = E.T$, where $E$ is the (obviously finite) integral energy of the solution defined as 3d-integral of the energy density $\Phi^2 = (\gamma f \theta)^2$.  

52
Chapter 5

Retrospect

In trying to understand our observational knowledge of the real world we must be able to separate the important structural and behavioral properties of the real objects from those, the changes of which during time-evolution do not lead to annihilation of the objects under consideration. One of the basic lessons that we more or less have been taught is that the physical objects are spatially finite entities, and that for their detection and further study some energy-momentum exchange is necessarily required. So, every physical object necessarily carries energy-momentum and every interaction between two physical objects has such an energy-momentum exchange aspect. The second lesson concerning any interaction is that, beyond its universality, energy-momentum is a conserved quantity, so no loss of it is allowed: it may only pass from one object to another. This means that every annihilation process causes creation process(es), and the full energy-momentum that has been carried by the annihilated objects, is carried away by the created ones. Energy-momentum always needs carriers, as well as every physical object always carries energy-momentum. Hence, the energy-momentum exchange abilities of any physical object provide protection against external influence on one side, and reveal its intrinsic nature, on the other side. Therefore, our knowledge about the entire complex of properties of a physical object relies on getting information about its abilities in this respect and finding corresponding quantities describing quantitatively these abilities.

The spatially finite nature of a physical object implies spatial structure and finite quantity of energy-momentum needed for its creation, so no structureless and infinite objects may exist. The approximations for "point object" and "spatially infinite field", although useful in some respects, seem theoretically inadequate and should not be considered as basic ones. More reliable appears to be the approximation "finite continuous object", which we tried to follow throughout our exposition. This last approximation suggests that the usually nonlinear local energy-momentum conservation relations should be the basic suggesting tool for finding appropriate mathematical models of local nature of such objects. The natural physical sense of the corresponding model equations is not necessarily supposed to be local energy-momentum exchange, but must be consistent with it, and closely connected to it.

Another useful observation is that physical objects are many-aspect entities, they have complicated structure and their very existence is connected with internal energy-momentum exchange among the various structural components. So, the mathematical model objects should be many-component ones, and with appropriate mathematical structure. Of basic help in finding appropriate mathematical objects is having knowledge of the internal symmetry properties of the physical object under consideration. This "step by step" process of getting and accumulating important information about the physical properties of natural objects reflects in the "step by step" process of refining the corresponding mathematical models.
The greatest discovery at the very beginning of the last century was that the notion of electromagnetic field as suggested by Maxwell equations is inadequate: the time dependent electromagnetic field is not an infinite smooth perturbation of the aether, on the contrary, it consists of many individual time-stable objects, called later photons, which are created/destroyed mainly during intra-atomic energy-transition processes. Photons are finite objects, they carry energy-momentum and after they have been radiated outside their atom-creator, they propagate as a whole translationally by the speed of light. Moreover, their propagation is not just translational, it includes rotational component, which is of intrinsic and periodical nature. The corresponding intrinsic action for one period $T$ is $h = ET$, where $E$ is the full energy of the photon, and all photons carry the same elementary intrinsic action $h$. During the entire 20th century physicists have tried to understand the dynamical structure/nature of photons from various points of view, and this process is still going on today. The conviction that a new point of view on the dynamical nature of the field equations is needed is shortly summarized by Ziolkowski [29]: "Finite-energy, diffraction-free beams for the linear free-space wave equation are impossible".

In order to come to a new look at the situation in nonrelativistic terms we made use of the Newton approach: the identifying features of the object considered must be kept unchanged during evolution and the admissible changes most naturally should be expressed by means of specializing the energy-momentum exchange abilities of the object considered, as well as by paying due respect to the available translation-rotation interrelation. We showed that this approach works well in the nonrelativistic $(\mathbf{E}, \mathbf{B})$ formalism and concluded that the relativistic $(\mathbf{F}, \mathbf{\ast F})$ structure is much more adequate to the PhLO dynamical structure than the $(\mathbf{E}, \mathbf{B})$ one.

The basic theoretical idea in the relativistic formalism was to make use of the Frobenius integrability/nonintegrability theorems as an appropriate mathematical machinery: the integrability of a distribution we connected with the time-stability of the basic identification properties of the object considered, and the nonintegrability of the available subdistributions was interpreted physically as internal interaction among the subsystems, where the corresponding curvatures appeared as natural mathematical tools for generating appropriate mathematical images of the local energy-momentum exchange fluxes between any two subsystems.

We introduced a notion of PhLO as a spatially finite physical object with a compatible translational-rotational dynamical structure and propagating translationally with the frame independent velocity of light "$c$". We showed that Frobenius integrability theory possesses all necessary features to meet the physical aspects of this notion. From physical viewpoint, two dynamically interacting subsystems of a PhLO can be individualized, these subsystems carry the same stress-energy-momentum, and they exchange energy-momentum locally always in equal quantities, so they exist in a dynamical equilibrium. The mathematical realization of the two subsystems of a PhLO was made in two ways: through a direct choice of two nonintegrable subdistributions, and by means of a couple of two nonlinear connections $V$ and $\tilde{V}$ with a common image space. Their inter-communication is carried out and guaranteed by the nonzero curvature forms $\Omega$ and $\Omega^*$ in the first case, and by the nonzero curvature forms $\mathcal{R}$ and $\tilde{\mathcal{R}}$ in the second case. The values of these curvature forms define two 1-dimensional space-like subspaces, so, the corresponding two exterior products with the null direction of translational propagation give the mathematical images $F$ and $\tilde{F}$ of the two interacting subsystems. This approach allows to get some information concerning the dynamical nature of the PhLO structure not only algebraically, but also infinitesimally, i.e. through the curvature forms.

While the energy density $\Phi^2$ of a PhLO propagates only translationally along straight isotropic lines, the available interaction of the two subsystems of a PhLO demonstrates itself through a rotational component of the entire propagational behaviour and is available only if the curvature forms are not zero. The mutual energy-momentum exchanges are given by the
inner products of the curvature images with $F$ and $\ast F$. The dynamical equilibrium between $F$ and $\ast F$ is given by $i(\ast F)(dF) = -i(F)(d \ast F)$.

Besides the spatially finite nature of PhLO that is allowed by our model and illustrated with the invariant parameter $l_o$, two basic identifying properties of PhLO were substantially used: straight-line translational propagation with constant speed, and constant character of the rotational component of propagation. The physical characteristics of a PhLO are represented by an analog of the Maxwell-Minkowski stress-energy-momentum tensor. An interesting moment is that $F$ and $\tilde{F}$ have zero horizontal and vertical components with respect to the two nonlinear connections.

It was very interesting to find that some of the basic characteristics of PhLO could be given in terms of the two strain-tensors, i.e. through the Lie derivatives of the Minkowski pseudometric with respect to spatial direction generators of the two nonintegrable subdistributions, so, each of the two nonintegrable subdistributions has its own strain tensor. This gives entirely new viewpoint on PhLO, namely, the PhLO energy-momentum propagates through deformations! The values of each strain tensor on the generators of its subdistribution gives the translational change of the energy density, and its value on the generators of the other subdistribution gives the Frobenius curvature. The corresponding dynamical aspects are given by expressions (3.36)-(3.37).

It seems important to note that the curvature forms are not zero only if the component-functions of the vector fields defining the distributions (or the component-functions of the associated nonlinear connections) are NOT running waves along the translational propagation, e.g. the squares of the curvature forms are equal to $|dF|^2 = |d \ast F|^2 = |\delta F|^2 = |\delta \ast F|^2 = (u_\xi - \varepsilon u_\zeta)^2 + (p_\xi - \varepsilon p_\zeta)^2 \neq 0$. Also, the dually invariant longitudinal size parameter $l_o$ acquires sense only for finite nonzero curvatures.

The equations of motion can be viewed from different viewpoints: as compatibility conditions between the rotational and translational components of propagation, as Lagrange equations for an action principle, as the nonlinear part of the solutions of the vacuum equations of EED, and also as naturally defined transformation of 2-dimensional frames. In all these aspects of the equations of motion the curvature forms play essential role through controlling the inter-communication between $F$ and $\tilde{F}$. Moreover, the Frobenius curvature turns out to be proportional to the energy density, which recalls the main idea of General Relativity from one side, and allows an analog of the famous Planck formula $E = h\nu$ to be introduced, from the other side.

The solutions considered illustrate quite well the positive aspects of our approach. It is interesting to note that the phase terms of these solutions depend substantially only on spatial variables, so, the spatial structure of the solutions considered participates directly in the rotational component of the PhLO dynamical structure.

Our basic conclusion reads: PhLO are complex objects with dynamical structure of special kind, so any mathematical model of PhLO shall need corresponding mathematical structure. According to the results given in this study the basic adequate mathematical structure in case of electromagnetic PhLO is the 3-dimensional distribution/codistribution $(\tilde{A}, \tilde{A}^*, \tilde{\zeta})/(A, A^*, \zeta)$ on Minkowski space-time together with the corresponding interconnections represented by the integrability/nonintegrability properties of its subdistributions.

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