Pathwise Uniqueness for the Stochastic Heat Equation with Hölder Continuous Drift and Noise Coefficients
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Abstract. We study the solutions of the stochastic heat equation with multiplicative space-time white noise. We prove a comparison theorem between the solutions of stochastic heat equations with the same noise coefficient which is Hölder continuous of index $\gamma > 3/4$, and drift coefficients that are Lipschitz continuous. Later we use the comparison theorem to get sufficient conditions for the pathwise uniqueness for solutions of the stochastic heat equation, when both the white noise and the drift coefficients are Hölder continuous.

1 Introduction and main results
We study the solutions of the stochastic heat equation with space-time white noise. This equation has the form

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2} \Delta u(t,x) + \sigma(t,x,u(t,x)) \dot{W} + b(t,x,u(t,x)), \quad t \geq 0, \quad x \in \mathbb{R}.$$ (1.1)

Here $\Delta$ denotes the Laplacian and $\sigma(t,x,u), b(t,x,u) : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions with at most a linear growth in the $u$ variable. We assume that the noise $\dot{W}$ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$. Equations like (1.1) arise as scaling limits of critical branching particle systems. For example, in the case where $\sigma(t,x,u) = \sqrt{u}$ and $b = 0$, such equations describe the evolution in time and space of the density of the classical super-Brownian motion (see e.g. Section 3.4 of [26]). If $b = b(x) \geq 0$ is a continuous deterministic function with compact support and $\sigma(u) = \sqrt{u}$, then the solution to (1.1) arises as scaling limit of critical branching particle systems with immigration and the limit is known as super-Brownian motion with immigration $b$. In other words, the density of the super-Brownian motion with immigration $b$ satisfies (1.1), $P$-a.s. (see e.g. Section 3.4 of [26]).

In this work we consider the pathwise uniqueness for the solution of (1.1) where $\sigma$ and $b$ are Hölder continuous in $u$ and $W$ is a space-time Gaussian white noise. More precisely $W$ is a mean zero Gaussian process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where $\mathcal{F}_t$ satisfies the usual hypothesis and we assume that $W$ has the following properties. We denote by

$$W_t(\phi) = \int_0^t \int_{\mathbb{R}} \phi(s,y)W(ds,dy), \quad t \geq 0,$$

the stochastic integral of a function $\phi$ with respect to $W$. We denote by $\mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ the space of compactly supported infinitely differentiable functions on $\mathbb{R}_+ \times \mathbb{R}$. We assume that $W$ has the following covariance structure

$$E(W_t(\phi)W_t(\psi)) = \int_0^t \int_{\mathbb{R}} \phi(s,y)\psi(s,y)dyds, \quad t \geq 0,$$

for $\phi, \psi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$. The stochastic heat equation with space-time white noise was studied among many others, by Cabaña [2], Dawson [8], [9], Krylov and Rozovskii [15], [17], [16], Funaki [14], [10] and Walsh [30]. Pathwise uniqueness of the solutions for the stochastic heat equation, when the white noise coefficient $\sigma$ and the drift coefficient are Lipschitz continuous was derived in [30]. In [22], pathwise uniqueness for the solutions of the
stochastic heat equation, where the white noise coefficient is Hölder continuous of index $\gamma > 3/4$, and again the drift coefficient $b$ is Lipschitz continuous was established. The $d$-dimensional stochastic heat equation driven by colored Gaussian noise was also extensively studied. Pathwise uniqueness for the solutions of the stochastic heat equation driven by colored Gaussian noise, with Hölder continuous noise coefficients was studied in [23].

The result in [23] was later improved by Rippl and Sturm in [27]. The method of proof in [22], [23] and [27] is a version of the Yamada-Watanabe argument (see [31]) for infinite dimensional stochastic differential equations.

Pathwise uniqueness of the solutions for the stochastic heat equation (1.1) where $\gamma > 3/4$, and again the drift coefficient $b$ is Lipschitz continuous was also extensively studied. Pathwise uniqueness for the solutions of the stochastic heat equation driven by colored Gaussian noise, with Hölder continuous noise coefficients was studied in [23].

In this work we study the uniqueness property in the sense of pathwise uniqueness. The definition of pathwise uniqueness is given below.

Pathwise uniqueness of the solutions for the stochastic heat equation (1.1), where $b$ is a non-Lipschitz measurable function was studied in [12], [13], [1], [11], [3], [4] among others. In these papers $\sigma$ is a Lipschitz continuous function which satisfies a so-called non-degeneracy condition. For example in [11], $\sigma$ satisfies the following non-degeneracy condition

$$\sigma(t,x,u) \geq \varepsilon > 0, \forall (t,x,u) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R},$$

and $b$ is a $P \times \mathcal{B}(\mathbb{R})$-measurable random field which also satisfies some integrability conditions. The idea of proof in [12], [13], [1] and [11] is as follows: first proving uniqueness in law by Girsanov’s theorem and then using a comparison theorem to get pathwise uniqueness. The proofs also use Malliavin calculus to get estimates on the density of the solutions to (1.1) without drift.

Before we describe in more detail the known uniqueness results for the stochastic heat equation with space-time white noise, we introduce additional notation and definitions.

**Notation.** For every $E \subset \mathbb{R}$, we denote by $C(E)$ the space of continuous functions on $E$. In addition, a superscript $k$, (respectively, $\infty$), indicates that functions are in addition $k$ times (respectively, infinitely often), continuously differentiable. A subscript $c$ indicates that they also have compact support.

For $f \in C(\mathbb{R})$ set

$$\|f\|_\lambda = \sup_{x \in \mathbb{R}} |f(x)| e^{-\lambda |x|}, \; \lambda \in \mathbb{R},$$

and define

$$C_{tem} := \{ f \in C(\mathbb{R}), \|f\|_\lambda < \infty \text{ for every } \lambda > 0 \}.$$  

The topology on this space is induced by the norms $\| \cdot \|_\lambda$ for $\lambda > 0$.

For $I \subset \mathbb{R}_+$ let $C(I,E)$ be the space of all continuous functions on $I$ taking values in topological space $E$ endowed with the topology of uniform convergence on compact subsets of $I$. Hence the notation $u \in C(\mathbb{R}_+, C_{tem})$ implies that $u$ is a continuous function on $\mathbb{R}_+ \times \mathbb{R}$ and

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} |u(t,x)| e^{-\lambda |x|} < \infty, \; \forall \lambda > 0, \; T > 0.$$ (1.4)

In many cases it is possible to show that solutions to (1.1) are in $C(\mathbb{R}_+, C_{tem})$. Let us define a stochastically strong solution to (1.1), which is also called a strong solution to (1.1).

**Definition 1.1** (Definition next to Equation (1.5) in [23]) Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space and let $W$ be a white noise process defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Let $\mathcal{F}_t^W \subset \mathcal{F}_t$ be the filtration generated by $W$. A stochastic process $u : \Omega \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ which is jointly measurable and $\mathcal{F}_t^W$-adapted, is said to be a stochastically strong solution to (1.1) with initial condition $u_0$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, if for all $t \geq 0$ and $x \in \mathbb{R}$,

$$u(t,x) = G_t u_0(x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) \sigma(s,y,u(s,y)) W(ds,dy)$$

$$+ \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) b(s,y,u(s,y)) dy ds, \; P - \text{a.s.}$$ (1.5)

Here

$$G_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \; x \in \mathbb{R}, \; t > 0,$$

and $G_tf(x) = \int_{\mathbb{R}} G_t(x-y)f(y)dy$, for all $f$ such that the integral exists.

In this work we study the uniqueness property in the sense of pathwise uniqueness. The definition of pathwise uniqueness is given below.
Definition 1.2 (Definition before Theorem 1.2 in [22]) We say that pathwise uniqueness holds for solutions of (1.1) in \( C(\mathbb{R}_+, C_{tem}) \) if for every deterministic initial condition, \( u_0 \in C_{tem} \), any two solutions to (1.1) with sample paths a.s. in \( C(\mathbb{R}_+, C_{tem}) \) are equal with probability 1.

Convention. Constants whose values are unimportant and may change from line to line are denoted by \( C_i, M_i \), \( i = 1, 2, \ldots \), while constants whose values will be referred to later and appear initially in say, Equation (i.j) are denoted by \( C_{(i,j)} \).

Next we present in more detail some results on pathwise uniqueness for the solutions of (1.1) driven by space-time white noise which are relevant to us. When \( \sigma \) and \( b \) are Lipschitz continuous, the existence and uniqueness of a strong solution to (1.1) in \( C(\mathbb{R}_+, C_{tem}) \) was proved in [29]. The proof uses the standard tools that were developed in [30] for solutions to SPDEs. In [22], Lipschitz assumptions on \( \sigma \) were relaxed and the following conditions were introduced: for every \( T > 0 \), there exists a constant \( C_{(1.6)}(T) > 0 \) such that for all \( (t, x, u) \in [0, T] \times \mathbb{R}^2 \),

\[
|\sigma(t, x, u)| + |b(t, x, u)| \leq C_{(1.6)}(T)(1 + |u|).
\]

Also, for some \( \gamma > 3/4 \) there are \( R_1, R_2 > 0 \) and for all \( T > 0 \) there is an \( R_0(T) \) so that for all \( t \in [0, T] \) and all \( (x, u, u') \in \mathbb{R}^3 \),

\[
|\sigma(t, x, u) - \sigma(t, x, u')| \leq R_0(T)e^{R_1|x|}(1 + |u| + |u'|)^{R_2}|u - u'|^{\gamma},
\]

and there is \( B > 0 \) such that for all \( (t, x, u, u') \in \mathbb{R}_+ \times \mathbb{R}^3 \),

\[
|b(t, x, u) - b(t, x, u')| \leq B|u - u'|.
\]

Mytnik and Perkins in [22] proved that if \( u_0 \in C_{tem} \), and \( \sigma, b : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R} \) satisfy (1.6), (1.7), and (1.8), then there exists a unique strong solution of (1.1) in \( C(\mathbb{R}_+, C_{tem}) \).

Remark 1.3 It was also proved in [22] (see Equation (2.25)) that if \( b \) and \( \sigma \) satisfy (1.6) then any \( C(\mathbb{R}_+, C_{tem}) \) solution \( u \) to (1.1) satisfies

\[
E\left( \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} |u(t, x)|^p e^{-\lambda |x|} \right) < \infty, \quad \forall \lambda, p > 0.
\]

The bound (1.9) will be very useful in our proofs later.

Now we are ready to present our main results. The first result is a comparison theorem for the solutions of (1.1) with H"older continuous \( \sigma \) and a Lipschitz continuous \( b \). The second result of the paper is the pathwise uniqueness for (1.1) under assumptions (1.6) and (1.7) on \( \sigma \) and while relaxing the Lipschitz assumption on \( b \).

In what follows we assume that the drift coefficient \( b(t, x, u, \omega) : \mathbb{R}_+ \times \mathbb{R}^2 \times \Omega \to \mathbb{R} \) in (1.1) is an \( \{ \mathcal{F}_t^W \} \)-predictable function. We further assume that the constants \( C(T) \) and \( B \) in (1.6), and (1.8) do not depend on \( \omega \). The dependence of \( b \) in \( \omega \) is often suppressed in our notation for the sake of readability.

Theorem 1.4 Assume that \( b_i : \mathbb{R}_+ \times \mathbb{R}^2 \times \Omega \to \mathbb{R}, \ (i = 1, 2) \) and \( \sigma : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R} \) satisfy (1.6), (1.7), (1.8), P-a.s. Let \( u^i(t, \cdot) \) be a \( C_{tem} \)-valued solution of (1.1) associated with the coefficients \( \sigma \) and \( b_i \), having initial condition \( u^i(0, \cdot) = u^i_0(\cdot) \in C_{tem}, \ i = 1, 2 \). Suppose further that

\[
\sigma(t, x, u) \quad \text{and} \quad b_i(t, x, u, \omega) \quad (i = 1, 2) \quad \text{are continuous in} \ (x, u), \ P - \text{a.s.},
\]

\[
b_1(t, x, u, \omega) \leq b_2(t, x, u, \omega), \ \forall t \geq 0, \ x \in \mathbb{R}, \ u \in \mathbb{R}, \ P - \text{a.s.},
\]

and

\[
u^2_0(x) \geq u^2_0(x), \ \forall x \in \mathbb{R}.
\]

Then,

\[
P(u^2(t, \cdot) \geq u^1(t, \cdot), \ \text{for every} \ t \geq 0) = 1.
\]

Remark 1.5 Comparison theorems for the stochastic heat equations when the white noise and drift coefficients \( \sigma \) and \( b \) are Lipschitz continuous were proved in [20] and [22] among others. A weaker version of a comparison theorem was proved in Proposition 3.1 in [21] for the case of a non-Lipschitz \( \sigma \). It was proved in [21] that there exists a probability space on which there is a white noise \( W \) such that (1.15) holds. Note that in Theorem 1.4 the probability space and the white noise are specified in advance.

Before we state our main results we introduce some notation and recall Girsanov’s theorem for the white noise process.
One of the applications of Theorem 1.8 is the pathwise uniqueness of solutions to (1.1) in the special case where uniqueness holds for the solutions of (1.1) with sample path a.s. in $\gamma > \sigma$.

A weak solution to this equation under less restrictive assumptions on the white noise and drift coefficients are non-Lipschitz. This is the main result of this paper. The existence of the following theorem gives a sufficient condition for the pathwise uniqueness of a solution to (1.1) when both

Theorem 1.6 (Theorem 10.2.1 in [5]) If $\{Z(s, x) : (s, x) \in [0, T] \times \mathbb{R}\}$ is such that $E(L_T) = 1$ then

$$\tilde{W}(dt, dx) = Z(t, x)dt dx + W(dt, dx), \ t \in [0, T], \ x \in \mathbb{R},$$

is a space-time white noise under the probability measure $Q$, where $Q$ is defined by

$$\frac{dQ}{dP}|_{F_t^W} = L_T.$$  

Here $L_T$ is the Radon-Nikodym derivative of $Q$ with respect to $P$ restricted to $F_T^W$.

Remark 1.7 The assumption that $E(L_T) = 1$ is often replaced by the assumption that $\{L_t\}_{t \in [0, T]}$ is a martingale with respect to the filtration $F_t^W$. Such assumption is satisfied if, for example Novikov’s condition holds:

$$E \left( \exp \left( \frac{1}{2} \int_0^t \int_{\mathbb{R}} Z(s, x)^2 dx ds \right) \right) < \infty, \ \text{for every } t \in [0, T],$$

see for example Proposition 10.17 in [5].

Before we state our main results we will need some additional definitions.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space and let $W$ be a white noise process defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Let $u_0 \in C_{tem}$. Denote by $S_{u_0}^W$ the class of strong $C(\mathbb{R}_+, C_{tem})$-solutions to (1.1).

Now we will give our basic assumptions on the drift coefficient $b$ in (1.1).

Assumption A. We assume that there exists an $\{F_t^W\}$-predictable function $Z(t, x, u, \omega) : \mathbb{R}_+ \times \mathbb{R}^2 \times \Omega \to \mathbb{R}$ such that

$$b(t, x, u, \omega) = Z(t, x, u, \omega)\sigma(t, x, u), \ \forall t \in [0, T], \ x \in \mathbb{R}, \ u \in \mathbb{R}, \ P - a.s.$$  

The following theorem gives a sufficient condition for the pathwise uniqueness of a solution to (1.1) when both the white noise and drift coefficients are non-Lipschitz. This is the main result of this paper. The existence of a weak solution to this equation under less restrictive assumptions on $\sigma$ and $b$ was proved in [22].

Theorem 1.8 Let $\tilde{W}$ be a space-time white noise. Let $u(0, \cdot) \in C_{tem}(\mathbb{R})$. Let $b : \mathbb{R}_+ \times \mathbb{R}^2 \times \Omega \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ be continuous in $(x, u)$ and satisfy (1.6) and Assumption A, $P$-a.s. Let $\sigma$ satisfy (1.7) for some $\gamma > 3/4$. Assume that for every $u \in S_{u_0}^W$, $Z(t, x, u(t, x))$ from (1.18) satisfies (1.14). Then pathwise uniqueness holds for the solutions of (1.1) with sample paths $a.s.$ in $C(\mathbb{R}_+, C_{tem})$.

One of the applications of Theorem 1.8 is the pathwise uniqueness of solutions to (1.1) in the special case where:

$$\sigma(u) = |u|^p \text{ and } b(u) = -|u|^q, \ \text{for some } 3/4 < p < q \leq 1.$$  

(1.19)
Notation. For \( f \in \mathcal{C}(\mathbb{R}) \) set
\[
C_{\text{rap}} := \{ f \in \mathcal{C}(\mathbb{R}), \| f \|_\lambda < \infty \text{ for every } \lambda < 0 \}.
\]
The topology on this space is induced by the norms \( \| \cdot \|_\lambda \) for \( \lambda < 0 \). Denote by \( C_\text{rap}^+ \) (respectively \( C_\text{tem}^+ \)) the set of nonnegative functions in \( C_\text{rap} \) (respectively \( C_\text{tem} \)).

The existence of a stochastically weak \( C(\mathbb{R}_+, C_\text{tem}^+) \) solution to (1.1) was proved in Theorem 1.1 of [29] for a larger class of \( b \) and \( \sigma \) which also includes our example in (1.19). One can easily show that if \( b \) and \( \sigma \) satisfy (1.19) and \( u(0, \cdot) \in C_\text{rap}^+ \), then any \( C(\mathbb{R}_+, C_\text{tem}^+) \) solution to (1.1) is also in \( C(\mathbb{R}_+, C_\text{rap}^+) \) (the proof follows the same lines as the proof of Theorem 2.5 in [29]). Under the assumption in (1.19) we have
\[
Z(s, u(s, x)) := |u(s, x)|^{q-p}, \ x \in \mathbb{R}, \ s \geq 0, \text{ for every } u \in S_{\text{rap}}^W, \tag{1.20}
\]
and we get that (1.14) is satisfied for every \( u \in S_{\text{rap}}^W \). From the discussion above and Theorem 1.8 we get the following corollary.

\textbf{Corollary 1.9} Let \( \hat{W} \) be a space-time white noise. Let \( u(0, \cdot) \in C_\text{rap}^+ \). Assume that \( \sigma \) and \( b \) are as in (1.19). Then pathwise uniqueness holds for the solutions to (1.7) with sample paths a.s. in \( C(\mathbb{R}_+, C_\text{tem}^+) \).

\textbf{Remark 1.10} Recall that one of the necessary conditions for the pathwise uniqueness theorems in [12], [13], [14] and [15], is the non-degeneracy condition (1.2). One of the by products of Corollary 1.9 is that (1.2) is not a necessary condition for pathwise uniqueness.

The rest of this paper is devoted to the proofs of Theorems 1.4 and 1.8. In Section 2 we prove Theorem 1.4.

\section{Proof of Theorem 1.4}

This section is devoted to the proof of Theorem 1.4. The proof of a comparison theorem for SDEs with non-Lipschitz noise and drift coefficients was carried out by Nakao in [24]. The proof of Theorem 1.4 uses ideas from Nakao’s proof. First, let us introduce the following notation.

\textbf{Notation.} We denote by \((i, j)(b, \sigma)\), equation \((i, j)\) with drift function \( b \) and white noise coefficient \( \sigma \).

We will also use a more general notion of strong solution that was introduced in [19]. Let \( S_1 \) and \( S_2 \) be Polish spaces and let \( \Gamma : S_1 \times S_2 \rightarrow \mathbb{R} \) be a Borel measurable function. Let \( Y \) be an \( S_2 \)-valued random variable with distribution \( \nu \). We are interested in the solution \((X, Y)\) to the equation
\[
\Gamma(X, Y) = 0. \tag{2.1}
\]

\textbf{Definition 2.1 (Definition 2.1 in [19])} A solution \((X, Y)\) to (2.1) is called a strong solution if there exists a Borel measurable function \( F : S_2 \rightarrow S_1 \) such that \( X = F(Y), \ P\text{-a.s.} \)

In our case \( Y \) is the white noise process \( W \) and we only consider strong solutions that are adapted to \( \{\mathcal{F}_t\}_{t \geq 0} \), the natural filtration of the white noise.

\textbf{Remark 2.2} The existence and uniqueness results of Theorems 1.1 and 1.2 in [22], still hold if we assume that the drift coefficient \( (b(t, x, u, \omega) : \mathbb{R}_+ \times \mathbb{R}^2 \times \Omega \rightarrow \mathbb{R} \) in (1.1) is an \( \mathcal{F}_t \)-predictable function which satisfies (1.6) and (1.20), \( P\text{-a.s.}, \) where constants \( C(T), B, \) in (1.3) and (1.5) are independent of \( \omega \). The proof of this generalisation follows directly from the proof in Section 8 of [22] hence it is omitted. We can also get (1.9) for the solutions of (1.1) when \( b \) and \( \sigma \) satisfy the assumptions in Remark 1.3 and where \( b \) is an \( \mathcal{F}_t \)-predictable function as above.

In order to prove Theorem 1.4 we need the following additional notation.

\textbf{Notation.} With a slight abuse of notation set
\[
W(t, x) := \begin{cases} \int_0^t \int_{y=0}^x W(dy, ds), & x \geq 0, \\ -\int_0^t \int_{y=x}^0 W(dy, ds), & x < 0. \end{cases} \tag{2.2}
\]

Note that \( t \mapsto W(t, \cdot) \in C_\text{tem}, \ P\text{-a.s.} \) This can be easily verified by checking the conditions of Lemma 6.3(i) in [24].
Proof of Theorem 1.4. Let $u^i$ be two solutions of (1.1)$(\sigma, b_i)$, $i = 1, 2$, with the same white noise on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, with sample paths in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})$ a.s. Here $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration of the white noise process $W$. Assume that $u^i$, $i = 1, 2$, have the deterministic initial conditions

$$u^i(0, \cdot) = u^i_0(\cdot) \in \mathcal{C}_{tem}, \ i = 1, 2,$$

which satisfy (1.2). Note that by Theorem 1.3 in [22], there exists a unique strong solution to (1.1)$(\sigma, b_i)$, for $i = 1, 2$. By Definition 2.1 this means that for each $i = 1, 2$, there exists a unique measurable function $F_i : \mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem}) \rightarrow \mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})$, such that $u_i = F_i(W)$, $P$-a.s., $i = 1, 2$ are $\mathcal{F}_t$-adapted (if $\tilde{u}_i$ is any other strong solution to (1.1)$(\sigma, b_i)$, then $\tilde{u}_i = F_i(W)$, $P$-a.s. as well).

Note that in order to prove Theorem 1.4 it is sufficient to show

$$F_1(W)(t, x) \leq F_2(W)(t, x), \ \forall t \geq 0, \ x \in \mathbb{R}, \ P - \text{a.s.}$$

Let $\Psi \in \mathcal{C}_c^\infty$ be a symmetric function so that $0 \leq \Psi \leq 1$, $\|\Psi\|_\infty \leq 1$, $\Psi_n(x) = 1$ if $|x| \leq n$ and $\Psi_n(x) = 0$ if $|x| \geq n + 2$. Let

$$\sigma^\alpha(t, x, u) = \int \sigma(t, x, u')G_{2-\alpha}(u-u')\Psi_n(u')du'.$$

By the proof of Theorem 1.1 in [22], $\sigma^\alpha(t, x, u)$ is Lipschitz continuous in $u$ and satisfies (1.6) for every $n \in \mathbb{N}$. Let

$$\begin{cases}
u^\alpha_n(t, x) = G_tu_0(x) + \int_0^t \int \sigma(t-s,y)G_2(s,y)W(ds,dy) \\
+ \int_0^t \int \sigma(t-s,y)b_1(s,y,u_n(s,y))dyds, \ x \in \mathbb{R}, \ t \geq 0, \ P - \text{a.s., } i = 1, 2,
\end{cases}$$

$$u^\alpha_n(0, x) = u_0(x), \ x \in \mathbb{R}, \ i = 1, 2.$$

From Theorem 2.2 in [29] we get that for every $\alpha$ and $n$ there exists unique solution $u^\alpha_n$ to (2.4). Let $Z_n = (u^\alpha_n, u^\alpha_n, W)$. Now argue as in the proof of Theorem 1.1 in [22] that the family of laws $P^{Z_n}$ is tight in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})^3$. Let $\{n_k\}_{k \geq 0}$ be a subsequence such that $\{Z_{n_k}\}_{k \geq 0}$ converges weakly to $Z = (u_1, u_2, W)$. By Skorohod’s theorem there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{P})$ on which the sequence of processes $\{\bar{Z}_{n_k}\}_{k \geq 0} = \{(\bar{u}^\alpha_n, \tilde{u}^\alpha_n, \bar{W}^{n_k})\}_{k \geq 0}$ is defined and converges $\bar{P}$-a.s. to $\bar{Z} = (\bar{u}_1, \bar{u}_2, \bar{W}) \overset{d}{=} (u_1, u_2, W)$, hence $\bar{Z}$ is also in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})^3$. Here $\bar{W}$ is a space-time white noise on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{P})$ and the Brownian sheet $\bar{W}$ is defined analogously to $W$ in [22]. Note that just as in the proof of Theorem 1.1 in [22], $\bar{u}_i$, $i = 1, 2$ are weak solutions to (1.1)$(\sigma, b_i)$ with $\bar{W}$.

Since $b_1(t, x, u) \leq b_2(t, x, u)$, $P$-a.s. and $u^\alpha_1(x) \leq u^\alpha_2(x)$, we have from Corollary 2.4 in [29] that $u^\alpha_1(t, x) \leq u^\alpha_2(t, x)$ for all $t > 0$, $x \in \mathbb{R}$, $P$-a.s., for every $k = 1, 2, \ldots$. We get that

$$\bar{u}_1(t, x) \leq \bar{u}_2(t, x), \ \forall t > 0, \ x \in \mathbb{R}, \ \bar{P} - \text{a.s.}$$

(2.5)

Now recall that by Theorem 1.3 in [22], $u_i$ are unique strong solutions to (1.1)$(\sigma, b_i)$ with $\bar{W}$. Hence, from (2.5) we get

$$F_1(\bar{W})(t, x) \leq F_2(\bar{W})(t, x), \ \forall t \geq 0, \ x \in \mathbb{R}, \ \bar{P} - \text{a.s.}$$

Therefore we conclude that

$$F_1(W)(t, x) \leq F_2(W)(t, x), \ \forall t \geq 0, \ x \in \mathbb{R}, \ P - \text{a.s.}$$

and we are done.

3 Proof of Theorems 1.8

3.1 Auxiliary Lemmas

In this section we prove a few auxiliary lemmas that will be used in the proof of Theorem 1.8. Before we start with the proofs, we recall the distributional form of (1.1).
Let $u$ be a solution of \([1.1]\) on $\Omega \times \mathcal{F}_t, \mathbb{P}$ with sample paths in $\mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem})$, $\mathbb{P}$-a.s. and the initial condition $u(0) = u_0 \in \mathcal{C}_{tem}$. By Theorem 2.1 in \([29]\), \([1.5]\) is equivalent to the distributional form of \([1.4]\). That is,

\[
\langle u(t), \phi \rangle = \langle u_0, \phi \rangle + \frac{1}{2} \int_0^t \left( \langle u(s), \frac{1}{2} \Delta \phi \rangle + \langle b(s, \cdot, u(s, \cdot), \phi) \rangle \right) ds \\
+ \int_0^t \int_\mathbb{R} \sigma(s, x, u(s, x)) \phi(x) \mathbb{W}(ds, dx), \quad \forall t \geq 0, \quad \mathbb{P}$-a.s., $\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}).
\]

**Convention:** Let $f : \mathbb{R}_+ \times \mathbb{R}^2 \times \Omega \to \mathbb{R}$ be an $\{\mathcal{F}_t\}_{t \geq 0}$-predictable function. We say that $f$ is continuous in the $x$ variable if

\[
x \mapsto f(t, x, u) \quad \text{is continuous } \forall (t, u) \in \mathbb{R}_+ \times \mathbb{R}, \quad \mathbb{P}$-a.s.,
\]

and in the $u$ variable if

\[
u \mapsto f(t, x, u) \quad \text{is continuous } \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \quad \mathbb{P}$-a.s.
\]

(3.2)

Let $f_n : \mathbb{R}_+ \times \mathbb{R}^2 \times \Omega \to \mathbb{R}, \ n \geq 1$, be a sequence of $\{\mathcal{F}_t\}_{t \geq 0}$-predictable functions. We say that $\{f_n\}_{n \geq 1}$ is a monotone increasing sequence of functions if

\[
f_n(t, x, u) \leq f_{n+1}(t, x, u), \quad \forall (t, x, u) \in \mathbb{R}_+ \times \mathbb{R}^2, \ n \geq 1, \quad \mathbb{P}$-a.s.
\]

(3.3)

The definition of a decreasing sequence of functions is completely analogous to \([3.3]\). We say that $\{f_n\}_{n \geq 1}$ is a monotone sequence of functions if it is either increasing or decreasing sequence.

Before we start with the proof of Theorem 1.8 we will also need the following crucial lemma.

**Lemma 3.1** Let $b_n : \mathbb{R}_+ \times \mathbb{R}^2 \times \Omega \to \mathbb{R}, \ n \geq 1$, be a sequence of monotone $\{\mathcal{F}_t\}_{t \geq 0}$-predictable functions such that for every $n \geq 1$, $b_n$ is continuous in $u$ the variable, and satisfy for some $T > 0$,

\[
\sup_{n \in \mathbb{N}} |b_n(t, x, u, \omega)| \leq C(T)(1 + |u|), \quad \forall (t, x, u) \in [0, T] \times \mathbb{R}^2, \quad \mathbb{P}$-a.s.
\]

(3.4)

Assume that

\[
\lim_{n \to \infty} b_n(t, x, u) = b(t, x, u), \quad \forall (t, x, u) \in [0, T] \times \mathbb{R}^2, \quad \mathbb{P}$-a.s.,
\]

(3.5)

where $b : \mathbb{R}_+ \times \mathbb{R}^2 \times \Omega \to \mathbb{R}$ is an $\{\mathcal{F}_t\}_{t \geq 0}$-predictable function which is continuous in the $u$ variable. Let $\sigma : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function in $u$ variable which satisfies \([1.6]\). Assume that \([3.1]\)(b, $\sigma$) admits jointly measurable $\mathcal{F}_t$-adapted solution $u_n$ such that

\[
E\left( \sup_{n} \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} |u_n(t, x)|^2 e^{-\lambda |x|} \right) < \infty,
\]

(3.6)

and

\[
u_n(t, x) \to u(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \quad \mathbb{P}$-a.s.,
\]

(3.7)

as $n \to \infty$, where $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is also a jointly measurable and $\mathcal{F}_t$-adapted process. Then $u(t, \cdot)$ satisfies \([3.7]\)(b, $\sigma$) for any $t \in [0, T]$.

**Proof:** Note that from \([3.6]\) and \([3.7]\) it immediately follows that

\[
E\left( \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} |u(t, x)|^2 e^{-\lambda |x|} \right) < \infty.
\]

(3.8)

Let us show that $u$ solves \([5.1]\)(b, $\sigma$). Let $\phi \in \mathcal{C}_c^\infty$, and fix $K > 0$ such that

\[
\text{supp}(\phi) \subset [-K, K].
\]

(3.9)

From \([3.6]\)–\([3.8]\) and dominated convergence we get

\[
\lim_{n \to \infty} E\left( |\langle u_n(t), \phi \rangle - \langle u(t), \phi \rangle| \right) = 0, \quad \forall t \in [0, T],
\]
and
\[
\lim_{n \to \infty} E\left( \left( \frac{1}{2} \int_0^t (u_n(s) - u(s)) \frac{1}{2} \Delta \phi ds \right) \right) = 0, \ \forall t \in [0, T].
\]

Let us show the convergence of the drift term. Note that for every \( t \in [0, T] \),
\[
E\left( \left| \int_0^t \langle b_n(s, \cdot, u_n(s, \cdot)) - b(s, \cdot, u(s, \cdot)), \phi \rangle ds \right| \right) \leq E\left( \left| \int_0^t \langle b_n(s, \cdot, u_n(s, \cdot)) - b(s, \cdot, u_n(s, \cdot)), \phi \rangle ds \right| \right)
+ E\left( \left| \int_0^t \langle b(s, \cdot, u_n(s, \cdot)) - b(s, \cdot, u(s, \cdot)), \phi \rangle ds \right| \right) = I_1 + I_2.
\] (3.10)

From (3.4) and since \( \{b_n\}_{n \geq 1} \) converges pointwise to \( b \), \( P \)-a.s., we have
\[
|b(t, x, u, \omega)| \leq C(T)(1 + |u|), \ \forall (t, x, u) \in [0, T] \times \mathbb{R}^2, \ P - \text{a.s.}
\] (3.11)

From (3.4) and (3.11) we have
\[
\lim_{n \to \infty} \sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} \left| b_n(s, x, u_n(s, x)) - b(s, x, u_n(s, x)) \right| |\phi(x)|
\leq C(T) \sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} ((1 + |u_n(s, x)|)|\phi(x)|)
\leq C(T) \left( 1 + \sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} (|u_n(s, x)|e^{-\lambda|x|}) \right) \left( \sup_{x \in \mathbb{R}} e^{\lambda|x|}|\phi(x)| \right).
\] (3.12)

Since \( \{b_n\}_{n \geq 1} \) is a monotone sequence of continuous functions in the \( u \) variable and \( b \) is continuous in the \( u \) variable, we get from (3.3) that for every \( 0 < M < \infty \) we have
\[
\lim_{n \to \infty} \sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} \left| b_n(s, x, u) - b(s, x, u) \right| = 0, \ \forall (s, x) \in [0, T] \times \mathbb{R}, \ P - \text{a.s.}
\] (3.13)

Use (3.3), (3.4), (3.12), (3.13) and the dominated convergence theorem again to get,
\[
\lim_{n \to \infty} I_1 = 0.
\] (3.14)

For \( I_2 \) we can use again (3.4) and (3.11) to get
\[
\sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} \left( |b(s, x, u_n(s, x)) - b(s, x, u(s, x))| |\phi(x)| \right)
\leq C(T) \sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} ((1 + |u_n(s, x)| + |u(s, x)|)|\phi(x)|)
\leq C(T) \left( 1 + \sup_{s \in [0, T]} \sup_{x \in \mathbb{R}} (|u_n(s, x)| + |u(s, x)|) e^{-\lambda|x|}) \right) \left( \sup_{x \in \mathbb{R}} e^{\lambda|x|}|\phi(x)| \right), \ P - \text{a.s.}
\] (3.15)

Use (3.4), (3.8), (3.9), (3.15) and the dominated convergence theorem again to get,
\[
\lim_{n \to \infty} E\left( \int_0^t \int_{\mathbb{R}} \left| b(s, x, u_n(s, x)) - b(s, x, u(s, x)) \right| |\phi(x)| dx ds \right)
= E\left( \int_0^t \int_{\mathbb{R}} \lim_{n \to \infty} \left| b(s, x, u_n(s, x)) - b(s, x, u(s, x)) \right| |\phi(x)| dx ds \right)
= 0,
\] (3.16)

where the last equality follows from (3.7) and the continuity of \( b \) in the \( u \) variable.

From (3.15) and (3.16) we get
\[
\lim_{n \to \infty} I_2 = 0.
\] (3.17)
From (3.10), (3.14) and (3.17) we get
\[
\lim_{n \to \infty} E\left( \left| \int_0^t \langle b_n(s,\cdot,u_n(s,\cdot)) - b(\cdot,\cdot,u(s,\cdot)),\phi \rangle ds \right| \right) = 0, \ \forall t \in [0,T].
\]

Now let us handle the stochastic integral term. Denote by
\[
M^n_t := \int_0^t \int_\mathbb{R} \sigma(s,x,u_n(s,x))\phi(x)W(ds,dx), \ t > 0, \ n \geq 0, \quad (3.18)
\]
and
\[
M_t := \int_0^t \int_\mathbb{R} \sigma(s,x,u(s,x))\phi(x)W(ds,dx), \ t > 0, \ n \geq 0. \quad (3.19)
\]

From (1.6) we have,
\[
[(\sigma(s,x,u_n(s,x)) - \sigma(s,x,u(s,x)))\phi(x)]^2 \leq C(T)(1 + |u_n(t,x)|^2 + |u(t,x)|^2)\phi(x)^2 \quad (3.20)
\]
From (5.9) we immediately get
\[
(1 + |u_n(t,x)|^2 + |u(t,x)|^2)\phi(x)^2 \leq C(T)\left(1 + \sup_{s \in [0,T]} \sup_{y \in \mathbb{R}} ((|u_n(s,y)|^2 + |u(s,y)|^2)e^{-|y|})\right)\left(\sup_{y \in \mathbb{R}} \phi(y)^2e^{|y|}\right)1_{\{x \in [-K,K]\}}, \ \forall t \in [0,T], \ x \in \mathbb{R}. \quad (3.21)
\]

From (5.6), (5.8), (3.20) and (3.21), dominated convergence and the continuity of \(\sigma\) in the \(u\) variable we have
\[
\lim_{n \to \infty} < M^n - M, >_t = \lim_{n \to \infty} E\left(\int_0^T \int_\mathbb{R} (\sigma(s,x,u_n(s,x)) - \sigma(s,x,u(s,x)))^2 \phi(x)^2 dx ds\right) = E\left(\int_0^T \int_\mathbb{R} \lim_{n \to \infty} (\sigma(s,x,u_n(s,x)) - \sigma(s,x,u(s,x)))^2 \phi(x)^2 dx ds\right) = 0, \ \forall t \in [0,T]. \quad (3.22)
\]

From Theorem 3.1(1) in [18] we get that the sequence of square integrable martingales \(\{M^n\}_{n \geq 1}\) converges to \(M\) in \(L^2\), where \(M\) is also a square integrable martingale, that is
\[
\lim_{n \to \infty} E\left[\left(\int_0^t \int_\mathbb{R} \sigma(s,x,u_n(s,x))\phi(x)W(ds,dx) - \int_0^t \int_\mathbb{R} \sigma(s,x,u(s,x))\phi(x)W(ds,dx)\right)^2\right] = 0, \ \forall t \in [0,T].
\]
and we are done. \(\blacksquare\)

The following lemma is a special case of equation (2.4e) in [28].

**Lemma 3.2** There exist constants \(C_{3.2}^1, C_{3.2}^2 > 0\) such that,
\[
|G_t(x) - G_t(y)| \leq C_{3.2}^1 |x - y| + C_{3.2}^2 |t|^{2/1}, \quad \forall t > 0, \ x, y \in \mathbb{R}.
\]

We will also need the following lemma.

**Lemma 3.3** Let \(T > 0\) and \(\lambda > 0\). There exists a constant \(C_{3.3}(\lambda,T) > 0\) such that,
\[
\int_0^{t+t'} \int_\mathbb{R} e^{\lambda|x|} (G_{t+s}(x' - y) - G_{t+s}(x-y))^2 dy ds \leq C_{3.3}(T,\lambda) e^{\lambda|x|} e^{\lambda|x' - x'|(|t' - t|^{1/2} + |x' - x|)},
\]
\[
\forall 0 \leq t, t' \leq T, \ x, x' \in \mathbb{R}, \quad (3.23)
\]
where \(G_t(x-y) = 0\) for \(t \leq 0\).

**Proof:** Lemma 3.3 appears in [29] (see Lemma 6.2(i)) for the case where \(\lambda = 0\). More details on proof of (3.23) when \(\lambda = 0\), are given in the proof of Theorem 6.7 in Chapter 1.6 of [6]. The proof for the case where \(\lambda > 0\) follows the same lines and hence it is omitted. \(\blacksquare\)
3.2 Proof of Theorem 1.8

Step 1. Construction of a Solution  In this step we construct a solution to (3.1). Later we will show that this solution is the unique solution of (1.1).

We assume that σ satisfies (1.7) and both σ and b are continuous in (x, u) and satisfy (1.6). Let Ψₙ be as in the proof of Theorem 1.4. For any m ∈ N, define

\[ b_m(t, x, u, ω) = \int_R b(t, x, u', ω)G_{2-m}(u - u')Ψ_m(u')du', \quad P - \text{a.s.} \]  \hspace{1cm} (3.24)

Let

\[ \tilde{b}_{n,k} := \wedge_{m=n}^k b_m, \quad n \leq k, \]
\[ \tilde{b}_n := \wedge_{m=n}^{\infty} b_m. \]  \hspace{1cm} (3.25)

Fix an arbitrary T > 0. As a direct consequence of (1.6) we have,

\[ |b_m(t, x, u)| \leq C(T)(1 + |u|), \quad \forall (t, x, u) \in [0, T] \times R \times R, \quad m \in N, \quad P - \text{a.s.} \]  \hspace{1cm} (3.26)

Again from (1.6) and Lemma 3.2 we have

\[ |b_m(t, x, u) - b_n(t, x, u')| \leq C(m)|u - u'|, \quad \forall (t, x, u) \in [0, T] \times R \times R, \quad m \in N, \quad P - \text{a.s.} \]  \hspace{1cm} (3.27)

From (3.26) follows that

\[ b_m(t, x, u) \to b(t, x, u), \]  \hspace{1cm} (3.28)

pointwise for any \((t, x, u) \in [0, T] \times R \times R, \quad P \text{-a.s.}\). From (3.27), (3.29) and (3.30) we can easily get \(\tilde{b}_{n,k}\) is Lipschitz in \(u\) uniformly with respect to \((t, x) \in [0, T] \times R\) and

\[ \tilde{b}_{n,k} \downarrow \tilde{b}_n, \quad \text{as} \ k \to \infty, \]
\[ \tilde{b}_n \uparrow b, \quad \text{as} \ n \to \infty, \]  \hspace{1cm} (3.29)

for any \((t, x, u) \in [0, T] \times R^2, \quad P \text{-a.s.}\).

By Theorems 1.2 and 1.3 in [22], there exists a unique strong \(C(R, C_{tem})\)-valued solution to (1.1)\((\tilde{b}_{n,k}, \sigma)\). Denote by \(\tilde{u}_{n,k}\) the solution of (1.1)\((\tilde{b}_{n,k}, \sigma)\). From Theorem 1.4 we get that the sequence \(\{\tilde{u}_{n,k}\}\) decreases with \(k\), hence it has a \(P \text{-a.s.}\) pointwise limit

\[ u_n := \lim_{k \to \infty} \tilde{u}_{n,k}. \]  \hspace{1cm} (3.30)

Note that \(u_n\) is also jointly measurable, \(F_t\)-adapted process since it is an infimum of the jointly measurable, \(F_t\)-adapted processes \(\{\tilde{u}_{n,k}\}_{k \geq 1}\). Denote by \(\tilde{b}(u) := C(T)(1 + |u|)\) and note that \(\tilde{b}\) is Lipschitz uniformly in \(u\) and satisfies trivially (1.6). From Theorems 1.2 and 1.3 in [22] we get that there exists a unique strong \(C(R, C_{tem})\) solution to (1.1)\((\tilde{b}, \sigma)\) and to (1.1)\((-\tilde{b}, \sigma)\). Denote by \(\tilde{u}\) (\(\tilde{u}\), respectively) the solution to (1.1)\((\tilde{b}, \sigma)\) (the solution to (1.1)\((-\tilde{b}, \sigma)\), respectively).

From Theorem 1.4 and (3.26) we have

\[ \underline{u}(t, x) \leq \tilde{u}_{n,k}(t, x) \leq \bar{u}(t, x), \quad \forall x \in R, \quad t \in [0, T], \quad k, n \in N, \quad k \geq n, \quad P \text{-a.s.} \]  \hspace{1cm} (3.31)

From the fact that \(\underline{u}(t, x), \bar{u}(t, x) \in C(R, C_{tem})\) and (1.9) we get

\[ E\left(\sup_{t \in [0, T]} \sup_{x \in R} (|\underline{u}(t, x)|^p + |\bar{u}(t, x)|^p)e^{-\lambda|x|}\right) < \infty, \quad \forall \lambda, \quad p > 0. \]  \hspace{1cm} (3.32)

Furthermore, from (3.31) and (3.32) we get

\[ E\left(\sup_n \sup_{k \geq n} \sup_{t \in [0, T]} \sup_{x \in R} (|\tilde{u}_{n,k}(t, x)|^p + |u_n(t, x)|^p)e^{-\lambda|x|}\right) < \infty, \quad \forall \lambda, \quad p > 0. \]  \hspace{1cm} (3.33)

Note that for every \(n \geq 1\), the sequence \(\{\tilde{b}_{n,k}\}_{k \geq n}\) is uniformly bounded and equicontinuous in the \(u\) variable, therefore from (3.24) we get that \(\tilde{b}_n\) is continuous in the \(u\) variable. From (3.24), (3.26), (3.28), (3.30) and
Lemma 3.1 we get that \( u_n \) solves (3.1) \((\tilde{b}_n, \sigma)\).

We would like to construct our solution to (3.1) \((\sigma, b)\) as the limit of \( u_n \), as \( n \to \infty \). From Theorem 1.4 we get

\[
\tilde{u}_{n,k} \geq \tilde{u}_{m,k}, \quad \forall m \leq n \leq k, \tag{3.34}
\]

and hence \( u_n \) increases as \( n \) increases. Since \( \{u_n\}_{n \geq 1} \) is an increasing sequence of jointly measurable, \( \mathcal{F}_t \)-adapted processes, we get that \( u_n \) converges pointwise:

\[
u(t,x) := \lim_{n \to \infty} u_n(t,x), \quad \forall (t,x) \in [0, T] \times \mathbb{R}, \quad P \text{-a.s.} \tag{3.35}
\]

and that \( u \) is also jointly measurable, \( \mathcal{F}_t \)-adapted process. Therefore by (3.26), (3.29), (3.33), (3.35) and Lemma 3.1 we get that \( u \) solves (3.1) \((b, \sigma)\). From (3.33) and (3.35) we get that

\[
E \left( \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} |\nu(t,x)|^p e^{-\lambda|x|} \right) < \infty, \quad \forall \lambda, \ p > 0. \tag{3.36}
\]

Step 2. Continuity of the Constructed Strong Solution In this step we prove that the strong solution constructed in Step 1 has a modification which is jointly continuous. Note that the continuity of \( u \) together with (3.36) implies that \( u \in \mathcal{C}(\mathbb{R}_+, \mathcal{C}_{loc}) \). The proof uses ideas from the proof of Theorem 2.2 in [29].

Let

\[
X_{n,k}(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(s,y,\tilde{u}_{n,k}(s,y))W(ds,dy), \quad \forall x \in \mathbb{R}, \ t \geq 0, \ n, k \in \mathbb{N}, \ n \leq k. \tag{3.37}
\]

Let \( \lambda > 0 \) be an arbitrary constant. Apply Burkholder’s inequality and (1.0) to (3.37) to get for every \( p > 1 \),

\[
E((X_{n,k}(t',x') - X_{n,k}(t,x))^2) \\
\leq C(p,T)E\left[ \left( \int_0^{t'v'} \int_{\mathbb{R}} e^{2\lambda|y|} |G_{t'-s}(x' - y) - G_{t-s}(x-y)|^2 + e^{-\lambda|y|}|\tilde{u}_{n,k}(s,y)|^2 dyds \right)^p \right],
\]

\forall 0 \leq t, t' \leq T, \ x, x' \in \mathbb{R}, \ n, k \in \mathbb{N}, \ n \leq k. \tag{3.38}

Use (3.33) and Lemma 3.3 on (3.38) to get that there exists a constant \( C_{3.39}(\lambda, p, T) > 0 \) independent of \( n, k \) such that

\[
E((X_{n,k}(t',x') - X_{n,k}(t,x))^2) \leq C_{3.39}(\lambda, p, T)e^{2\lambda p|x'|}e^{2\lambda p|x-x'|}(|t' - t|^{p/2} + |x' - x|^p),
\]

\forall 0 \leq t, t' \leq T, \ x, x' \in \mathbb{R}, \ n, k \in \mathbb{N}, \ n \leq k. \tag{3.39}

Let

\[
Y_{n,k}(t,x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)b_{n,k}(s,y,\tilde{u}_{n,k}(s,y))dyds, \quad \forall x \in \mathbb{R}, \ t \geq 0, \ n, k \in \mathbb{N}, \ n \leq k. \tag{3.40}
\]

From (3.25) and (3.29) we have

\[
|b_{n,k}(t,x,u)| \leq C_{1.16}(T)(1 + |u|), \quad \forall (x,u) \in \mathbb{R} \times \mathbb{R}, \ t \in [0,T], \ n, k \in \mathbb{N}, \ n \leq k, \quad P \text{-a.s.} \tag{3.41}
\]

Use (3.41), (3.33), Jensen’s inequality and Lemma 3.3 to get that there exists a constant \( C_{3.32}(\lambda, p, T) > 0 \) independent of \( n, k \) such that

\[
E((Y_{n,k}(t',x') - Y_{n,k}(t,x))^2) \leq C(p,T)(1 + E(\sup_{k \geq n} \sup_{s \in [0,T]} \sup_{y \in \mathbb{R}} e^{-2\lambda p|y|}|\tilde{u}_{n,k}(s,y)|^{2p}))
\]

\[
\times \left( \int_0^{t'v'} \int_{\mathbb{R}} e^{2\lambda|y|} |G_{t'-s}(x'-y) - G_{t-s}(x-y)|e^{-\lambda|y|}dyds \right)^{2p}
\]

\[
\leq C(\lambda, p, T)\left( \int_0^{t'v'} \int_{\mathbb{R}} e^{4\lambda|y|} |G_{t'-s}(x'-y) - G_{t-s}(x-y)|^2 e^{-\lambda|y|}dyds \right)^p
\]

\[
\leq C_{3.32}(\lambda, p, T)e^{3\lambda p|x'|}e^{3\lambda p|x-x'|}(|t' - t|^{p/2} + |x' - x|^p), \quad \forall 0 \leq t, t' \leq T, \ x, x' \in \mathbb{R}, \ n, k \in \mathbb{N}, \ n \leq k. \tag{3.42}
\]
Recall that \( u_0 \in \mathcal{C}_{tem} \). Then from Jensen’s inequality and Lemma 3.3 we get that there exists \( C(b, \sigma, T) > 0 \) such that,

\[
|G_t u_0(x) - G_t' u_0(x')|^{2p} = \left| \int_{\mathbb{R}} \left( G_t(x - y) - G_t'(x' - y) \right) e^{2\lambda|y|} u_0(y) e^{-2\lambda|y|} dy \right|^{2p} 
\]

(3.43)

\[
\leq \left( \sup_{y \in \mathbb{R}} |u_0(y)| e^{-\lambda|y|} \right)^{2p} \left( \int_{\mathbb{R}} |G_t(x - y) - G_t'(x' - y)| e^{2\lambda|y|} e^{-\lambda|y|} dy \right)^{2p} 
\]

\[
\leq C(\lambda, p) \left( \int_{\mathbb{R}} |G_t(x - y) - G_t'(x' - y)|^2 e^{4\lambda|y|} e^{-\lambda|y|} dy \right)^p 
\]

\[
\leq C(\lambda, p, T) e^{2\lambda p|x|} e^{3\lambda p|x - x'|} (|t' - t|^{p/2} + |x' - x|^p), \quad \forall 0 \leq t, t' \leq T, x, x' \in \mathbb{R}. 
\]

Recall that \( \tilde{u}_{n,k} \) for any \( n \leq k \) is a solution to (1.1) \((\tilde{b}_{n,k}, \sigma)\), therefore,

\[
\tilde{u}_{n,k}(t, x) = G_t u_0(x) + Y_{n,k}(t, x) + X_{n,k}(t, x), \quad \forall x \in \mathbb{R}, \ t \geq 0, \ n, k \in \mathbb{N}, \ n \leq k. 
\]

(3.44)

From (3.39), (3.40), (3.42), (3.43) and dominated convergence (applied twice) we have

\[
E((u(t', x') - u(t, x))^{2p}) = \lim_{n \to \infty} \lim_{k \to \infty} E((\tilde{u}_{n,k}(t', x') - \tilde{u}_{n,k}(t, x))^{2p}) 
\]

\[
\leq (C(3.39) \lambda, p, T) + C(3.40) \lambda, p, T) + C(3.43) \lambda, p, T) \|e^{3p|x|} e^{3p|x - x'|} (|t' - t|^{p/2} + |x' - x|^p)\|, \quad \forall 0 \leq t, t' \leq T, x, x' \in \mathbb{R}. 
\]

(3.45)

From (3.44) and Kolmogorov’s continuity theorem (see e.g. Theorem 4.3 in Chapter 1 of [1]) we get that there exists a continuous modification of \( u \), and together with (3.36) we get that \( u \in C(\mathbb{R}_+, \mathcal{C}_{tem}) \), \( P\)-a.s.

3. Pathwise Uniqueness of the Solution

In this step we prove the pathwise uniqueness for \( C(\mathbb{R}_+, \mathcal{C}_{tem}) \)-solutions of (1.1) \((b, \sigma)\), starting from the same initial condition.

Recall that \( Z \) was defined in (1.18). We assume throughout this step that for every strong \( C(\mathbb{R}_+, \mathcal{C}_{tem}) \) solution \( v \) of (1.1) \((b, \sigma)\) there exists \( K > 0 \) such that

\[
\int_0^T \int_{\mathbb{R}} Z^2(t, x, v(t, x)) dx ds \leq K, \ P\text{-a.s.} 
\]

(3.46)

This assumption is relaxed in Step 4. Note however that that by Remark 1.7, assumption (3.46) ensures that \( L = \{L_t\}_{t \in [0, T]} \) of (1.15) is a martingale so that in particular \( E(L_t) = 1 \) for every \( t \in [0, T] \).

First we show that uniqueness in law for \( C(\mathbb{R}_+, \mathcal{C}_{tem}) \) solutions of (1.1) \((b, \sigma)\) holds. Let \( v \) be a strong \( C(\mathbb{R}_+, \mathcal{C}_{tem}) \) solution to (1.1) \((b, \sigma)\) on \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\). Note that by Theorem 1.6 we have for every \( \phi \in \mathcal{C}_c \)

\[
\int_0^t \int_{\mathbb{R}} b(s, x, v(s, x)) \phi(x) dx ds + \int_0^t \int_{\mathbb{R}} \sigma(s, x, v(s, x)) \phi(x) W(ds, dx) 
\]

\[
= \int_0^t \int_{\mathbb{R}} Z(s, x) \sigma(s, x, v(s, x)) \phi(x) dx ds + \int_0^t \int_{\mathbb{R}} \sigma(s, x, v(s, x)) \phi(x) W(ds, dx) 
\]

\[
= \int_0^t \int_{\mathbb{R}} \sigma(s, x, v(s, x)) \phi(x) \tilde{W}(ds, dx), \quad \forall t \geq 0, \ P\text{-a.s.}, 
\]

(3.47)

where \( \tilde{W} \) is a white noise process under the probability measure \( Q \), defined by (1.16). It follows from (3.47) that \( v \) solves (3.1) \((0, \sigma)\) under the measure \( Q \).

Notation. We denote by \( EP \) the expectation with respect to the probability measure \( P \).
Note that from (1.15) and (3.46) we have

\[ E^P(L_t^2) = E^P \left[ \exp \left( 2 \int_0^t \int_{\mathbb{R}} Z(s, x, v(s, x)) W(ds, dx) - 2 \int_0^t \int_{\mathbb{R}} Z(s, x, v(s, x))^2 dx ds \right) \right] \times \exp \left( \int_0^t \int_{\mathbb{R}} Z(s, x, v(s, x))^2 dx ds \right) \]

\[ \leq C(K) E^P \left[ \exp \left( 2 \int_0^t \int_{\mathbb{R}} Z(s, x, v(s, x)) W(ds, dx) - \frac{1}{2} \int_0^t \int_{\mathbb{R}} [2Z(s, x, v(s, x))]^2 dx ds \right) \right] \]

\[ \leq C(K), \quad (3.48) \]

where in the last two inequalities we have used (3.46), as well as Remark 1.7 in the last inequality. By Theorem 1.6 we can use (1.16) together with the Cauchy-Schwarz inequality and (3.48) to get for every \( \lambda > 0 \),

\[ E^Q(|v(t, x) - v(t', x'|)^{2p}) = E^P(L_T |v(t, x) - v(t', x'|)^{2p}) \]

\[ \leq \left( E^P(|v(t, x) - v(t', x'|)^{4p}) \right)^{1/2} \left( E^P(L_T^2) \right)^{1/2} \]

\[ \leq C(K) \left( E^P(|v(t, x) - v(t', x'|)^{4p}) \right)^{1/2}, \forall 0 \leq t, t', T, x, x' \in \mathbb{R}, |x - x'| \leq 1. \quad (3.49) \]

Now follow the same lines as in Step 2 with \( v \) instead of \( \tilde{u}_{n,k} \) and by using (1.6) and (1.9) instead of (3.41) and (3.50),

From the preceding paragraph we get that for every \( \lambda > 0 \) and \( p > 2 \) there exists \( C(T, \lambda, p) > 0 \) such that

\[ E^P(|v(t, x) - v(t', x'|)^{4p}) \leq C(T, \lambda, p)(|x - x'|^{2p} + |t - t'|^p e^{6\lambda p|x|}, \]

\[ \forall 0 \leq t, t', T, x, x' \in \mathbb{R}, |x - x'| \leq 1. \quad (3.50) \]

From (3.50) we get for every \( \lambda > 0 \) and \( p > 2, \)

\[ \left( E^P(|v(t, x) - v(t', x'|)^{4p}) \right)^{1/2} \leq C(T, \lambda, p)(|x - x'|^{p} + |t - t'|^{p/2} e^{3\lambda p|x|}, \]

\[ \forall 0 \leq t, t' \leq T, x, x' \in \mathbb{R}, |x - x'| \leq 1. \quad (3.51) \]

From (3.49), (3.51) and Lemma 6.3(i) in [29] we get that under the measure \( Q \), \( v \) has a \( \mathcal{C}(\mathbb{R}^+, C_{tem}) \) version which satisfies (1.13) \((0, \sigma)\). Note that from Theorem 1.6 we also get that \( Q \) is absolutely continuous with respect to \( P \) restricted to \( \mathcal{F}_T^W \), and so together with (3.49) we have

\[ \int_0^T \int_{\mathbb{R}} Z^2(t, x, v(t, x)) dx ds \leq K, \quad Q - a.s. \quad (3.52) \]

We can use (3.52), Remark 1.7 and repeat the same lines as in (3.47) to get that

\[ \frac{dP}{dQ}|_{\mathcal{F}_T^W} = \tilde{L}_T, \quad (3.53) \]

where

\[ \tilde{L}_t = \exp \left( - \int_0^t \int_{\mathbb{R}} Z(s, x, v(s, x)) W(ds, dx) + \frac{1}{2} \int_0^t \int_{\mathbb{R}} Z(s, x, v(s, x))^2 dx ds \right). \]

From Theorem 1.3 in [22] we get the uniqueness in law for \( \mathcal{C}(\mathbb{R}^+, C_{tem}) \) solutions of (1.1) \((0, \sigma)\). Use this uniqueness in law together with (3.53) to get that the law of any strong \( \mathcal{C}(\mathbb{R}^+, C_{tem}) \) solution of (1.1) \((b, \sigma)\) is uniquely determined by the law of \( v \) under the measure \( Q \) and by \( \tilde{L}_T \), therefore the uniqueness in law for (1.1) \((b, \sigma)\) follows. From Theorem 3.14 in [19] we get that the pathwise uniqueness for the solutions of (1.1) \((b, \sigma)\) follows from the existence of a strong solution \( u \) together with the uniqueness in law for (1.1) \((b, \sigma)\).
Step 4. Pathwise Uniqueness in the General Case  In this step we assume that \( b \) and \( \sigma \) satisfy the assumptions of Theorem 1.8. Let \( v^i, \ i = 1, 2 \) be two strong \( \mathcal{C}(\mathbb{R}_+, \mathcal{C}_{tem}) \) solutions of (1.1)(\( b, \sigma \)). Let \( K > 0 \) and define
\[
T_K = \inf \left\{ t \geq 0 : \left( \int_0^t \int_{\mathbb{R}} Z^2(s, x, v^1(s, x)) dx ds \right) \lor \left( \int_0^t \int_{\mathbb{R}} Z^2(s, x, v^2(s, x)) dx ds \right) > K \right\}.
\]
Denote by
\[
b_K(t, x, u, \omega) = b(t, x, u, \omega) \mathbb{1}(t \leq T_K),
\]
and
\[
Z_K(s, x, u, \omega) = Z(s, x, u, \omega) \mathbb{1}(t \leq T_K), \ i = 1, 2.
\]
Here \( b_K \) and \( Z_K \) satisfy (1.18) and (3.46). Note that the restrictions on \( v^i, \ i = 1, 2, \) to \( [0, T_K] \times \mathbb{R} \) are the restrictions of the unique solution to (1.1)(\( b_K, \sigma \)). By the hypothesis of Theorem 1.8 we have \( T_K \to \infty, \ P\text{-a.s.} \) when \( K \to \infty, \) and the pathwise uniqueness follows.

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