RECENT CLASSIFICATION
AND CHARACTERIZATION RESULTS
IN GEOMETRIC TOPOLOGY

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ABSTRACT. We announce a complete topological classification
of the function spaces $C_p(X)$ of Borel class not higher than
2, provided that $X$ is a countable space. We also present a
topological classification of the $k$-dimensional universal pseudo-
doboundaries and pseudointeriors in $\mathbb{R}^k$, and we investigate
under what conditions strong negligibility of $\sigma Z$-sets charac-
terizes Hilbert space manifolds.

1. INTRODUCTION

The work presented in this announcement traces its history back
to Fréchet [18] and Banach [6] who proposed to classify metric
linear spaces according to topological type. For complete spaces
this program was carried out by Anderson [1], Kadec [21], and
Toruńczyk [25]: A Fréchet space is characterized topologically by
its linear dimension (i.e., minimal cardinality of sets with a dense
span). The classification of incomplete linear spaces, however, is
still in the beginning stage. In the case that the space is a so-called
absorber (see §2) characterizations have been developed (Mogilski
[24] and Bestvina and Mogilski [8]). We apply these results to
the classification of certain function spaces. If $X$ is a space, then
$C_p(X)$ denotes the space of continuous, real valued functions on
$X$ endowed with the topology of pointwise convergence. This
function space is metrizable only if $X$ is a countable space (barring
spaces without point separating real valued functions). Therefore
we consider countable completely regular spaces $X$ that are for
obvious reasons also nondiscrete. We show that all $F_{\sigma \delta}$-spaces

Received by the editors September 28, 1989 and, in revised form November
30, 1989.

1980 Mathematics Subject Classification (1985 Revision). Primary 57N15,
57N20.
$C_p(X)$ are homeomorphic to $\sigma^\omega$, the countable product of $l^2 = \{(x_i) \in l^2 : x_i = 0 \text{ for almost all } i\}$. According to Dijkstra et al. [14], $C_p(X)$ cannot be an absolute $G_{\delta\sigma}$-set and hence we obtain a complete classification of the spaces $C_p(X)$ of Borel class not higher than 2. Using similar techniques, we detect other sequence and function spaces homeomorphic to $\sigma^\omega$.

It was observed by Geoghegan and Summerhill [20] that the techniques developed for infinite-dimensional manifolds can be applied to $\mathbb{R}^n$ as well. They constructed $k$-dimensional universal pseudoboundaries $B^n_k$ and pseudointeriors $s^n_k$ in $\mathbb{R}^n$ analogously to the pseudoboundary $B$ and the pseudointerior $s$ of the Hilbert cube. Pseudoboundaries are absorbers and pseudointeriors are their complements. The pseudointeriors $s^n_k$ for $n \geq 2k+1$ are particularly interesting because of their resemblance to the separable Hilbert space $l^2$, which is homeomorphic to the pseudointerior $s$ by a celebrated theorem of Anderson [1]. We show that $B^n_k$ is homeomorphic to $B^n_k$ if and only if $s^n_k$ is homeomorphic to $s^n_k$ if and only if $n = m$ or $n, m \geq 2k+1$.

In the final section we consider Hilbert space manifolds and strongly negligible sets. A strongly negligible subset of a space is a set whose complement is topologically equivalent to the whole space via homeomorphisms that can be chosen arbitrarily close to the identity. This concept is closely related to absorbers (see Chapter 1 in [9]). Anderson shows in [2] that in Hilbert space manifolds the strongly negligible sets coincide with the $\sigma Z$-sets. We discuss the result that this property characterizes Hilbert space manifolds if every compactum is a strong $Z$-set but not if every compactum is merely a $Z$-set.

For background information on infinite-dimensional topology see [7] or [23].

2. Absorbers

The central idea in this note is the concept of a (generalized) absorber. Let $X$ be a space and let $\mathcal{M}$ be a collection of closed subsets that is hereditary and additive. A countable union $A$ of elements of $\mathcal{M}$ is called an $\mathcal{M}$-absorber if for every $D \in \mathcal{M}$ and every collection $\mathcal{U}$ of open subsets of $X$ there is a homeomorphism $h$ of $X$ that is $\mathcal{U}$-close to the identity and that has the property $h(D \cap \bigcup \mathcal{U}) \subseteq A$. This definition is due to West [27].

This idea was extended as follows by Bestvina and Mogilski [8].
Let \( \mathcal{H} \) be a class of spaces that is topological (i.e., homeomorphic images of elements of \( \mathcal{H} \) belong to \( \mathcal{H} \)), additive, and hereditary with respect to closed sets. A subset \( X \) of a topological copy \( E \) of \( l^2 \) is a generalized \( \mathcal{H} \)-absorber if:

1. \( X = \bigcup_{i=1}^{\infty} X_i \), where each \( X_i \in \mathcal{H} \) is a Z-set in \( X \), and
2. given an open cover \( \mathcal{U} \) of \( X \) in \( E \), a set \( D \in \mathcal{H} \), a closed subset \( C \) of \( D \), and a map \( f : D \to \bigcup \mathcal{U} \) such that \( f|C \) is a Z-embedding into \( X \), there exists a Z-embedding \( v : D \to X \) that is \( \mathcal{U} \)-close to \( f \) and that satisfies \( v|C = f|C \).

A closed subset \( S \) of a space \( X \) is called a Z-set if continuous maps from \( X \) into \( X \setminus S \) can be found arbitrarily close to the identity. A Z-embedding is an embedding whose range is a Z-set and a \( \sigma \)Z-set is a countable union of Z-sets.

Bestvina and Mogilski [8] proved that any two generalized \( \mathcal{H} \)-absorbers in \( E \) are homeomorphic. We are especially interested in the case where \( \mathcal{H} \) is the class \( \mathcal{F}_{\sigma \delta} \) of all absolute \( F_{\alpha \delta} \)-spaces. Recall that the first Borel class consists of the absolute \( F_{\sigma \delta} \)-spaces ( = \( \sigma \)-compacta) and the absolute \( G_{\delta} \)-spaces ( = completely metrizable spaces). The second Borel class then consists of \( \mathcal{F}_{\sigma \delta} = \{ \text{countable intersections of elements of } \mathcal{F}_{\sigma} \} \) and \( \mathcal{G}_{\delta \sigma} = \{ \text{countable unions of elements of } \mathcal{G}_{\delta} \} \). In our case the space \( \sigma^\omega \) is the standard generalized \( \mathcal{F}_{\sigma \delta} \)-absorber and we have:

**Proposition 2.1.** A space \( X \) is homeomorphic to \( \sigma^\omega \) if and only if

1. \( X \) is an absolute retract;
2. \( X = \bigcup_{i=1}^{\infty} X_i \), where each \( X_i \in \mathcal{F}_{\sigma \delta} \) is a Z-set in \( X \); and
3. \( X \) is homeomorphic to a space \( Y \) such that \( X_f^{\omega} \subseteq Y \subseteq X^\omega \); and
4. \( X \) contains a closed subset homeomorphic to \( X_f^{\omega} \).

In this proposition \( X_f^{\omega} \) stands for the subset of the product \( X^\omega \) consisting of the sequences \( (x_i) \) that have almost all terms equal to some fixed point in \( X \). Proposition 2.1 can be used to prove the following:

**Theorem 2.2.** The following linear spaces are homeomorphic:

1. \( \sigma^\omega = (l^2_f)^\omega \);
2. \( c_0 = \{(x_i) \in \mathbb{R}^\omega : x_i \to 0\} \) endowed with the topology of coordinatewise convergence;
3. \( \overline{l}^p = \bigcap_{q>p} l^q \), \( (0 \leq p < \infty) \), with the topology of coordinatewise convergence;
\(\hat{L}^p = \bigcap_{q<p} L^q\), \((0 < p \leq \infty)\), with the topology of convergence in (Lebesgue) measure.

Details will appear in [15].

3. Function spaces in the topology of pointwise convergence

Using a slight modification of Proposition 2.1 and some elements of the theory of Borel filters, we prove:

**Theorem 3.1.** Let \(X\) be a nondiscrete, countable, completely regular space such that the function space \(C_p(X)\) is an absolute \(F_{\sigma\delta}\)-set. Then \(C_p(X)\) and \(C^*_p(X)\) are homeomorphic to \(\sigma^\omega\).

Here \(C^*_p(X)\) is the subspace of \(C_p(X)\) consisting of all bounded functions. Details will appear in [17]. Since for every countable metric space \(X\) the function space \(C_p(X)\) is in \(\mathcal{T}_{\sigma\delta}\), this theorem generalizes results of van Mill [22], Baars et al. [5], and Dobrowolski et al. [16]. As an application of Theorem 3.1 we can answer in the negative some questions of Arhangel'skii [3, 4] by producing a countable, completely regular space \(X\) which fails to be a \(b_k\)-space, a \(k\)-space, and an \(\aleph_0\)-space, while the function space \(C_p(X)\) is homeomorphic to the \(C_p\) of the convergent sequence.

4. Classification of finite-dimensional pseudoboundaries and pseudointeriors

Let \(n\) and \(k\) be fixed integers such that \(n \geq 1\) and \(0 \leq k < n\). In addition, let \(\mathcal{M}^n_k\) denote the collection of “tame” \(\leq k\)-dimensional compacta in \(\mathbb{R}^n\). In [20] Geoghegan and Summerhill prove that there exists an \(\mathcal{M}^n_k\)-absorber. This set is called the \(k\)-dimensional universal pseudoboundary of \(\mathbb{R}^n\), and we denote it by \(B^n_k\). The \(k\)-dimensional universal pseudointerior \(s^n_k\) is the complement of \(B^n_{n-k-1}\) in \(\mathbb{R}^n\). If \(n \geq 2k + 1\), then \(s^n_k\) can be seen as a \(k\)-dimensional analogue of Hilbert space in the topological category. These spaces are \(k\)-dimensional (locally) \((k - 1\) )-connected complete spaces which are universal for the \(k\)-dimensional spaces and which share the following properties with Hilbert space: (a) homogeneity, (b) every \(\sigma Z\)-set is strongly negligible, and (c) Toruńczyk’s discrete approximation property. We classify these spaces topologically by deriving the following:

**Theorem 4.1.** \(B^n_k\) is homeomorphic to \(B^n_k\) if and only if \(s^n_k\) is homeomorphic to \(s^n_k\) if and only if \(n = m\) or \(n\), \(m \geq 2k + 1\).
The following consequence is noteworthy. Consider the trefoil (or any other knot) in $\mathbb{R}^3$. Since the trefoil and the unknot are elements of $\mathcal{M}_3$, we may assume that they are subsets of $s^3_1$ or $B^3_1$. Since tame embeddings of $S^1$ are equivalent in $\mathbb{R}^4$, they are equivalent in $s^4_1$ and $B^4_1$ (see [19, Theorem 2.5] and [9, Theorem 1.2.13]). The theorem then implies that the trefoil is unknotted in both $s^3_1$ and $B^3_1$.

The method used in [13] to prove the theorem is strongly “infinite-dimensional” in spirit and is based on techniques similar to those that were used in §2, in [8], and in [24]. In order to show that $B^m_k$ is homeomorphic to $B^n_k$ for $m, n \geq 2k + 1$, we use as a fixed model for these spaces a $k$-dimensional absorber $B^{\omega}_k$ in the Hilbert space $l^2$. The existence of $B^{\omega}_k$ is established in [10]. Let $C$ stand for the cone of $l^2$ and let $\pi$ be the projection $\mathbb{R}^n \times C \to \mathbb{R}^n$. We embed $B^{\omega}_k$ as a $k$-absorber in the topological Hilbert space $\mathbb{R}^n \times C$ in such a way that $\pi(B^{\omega}_k) = B^n_k$. Using a version of Bing’s shrinking criterion that was developed by Toruńczyk [26] for incomplete spaces, we prove that $\pi|B^{\omega}_k : B^{\omega}_k \to B^n_k$ is a near homeomorphism if $n \geq 2k + 1$.

The corresponding statement for $s^m_k$ follows easily. The pseudo-interiors $s^{m}_k$ and $s^{n}_k$ contain embedded copies of $B^{m}_k$ and $B^{n}_k$, respectively. By a classic Theorem of Lavrentiev the homeomorphism between $B^{m}_k$ and $B^{n}_k$ can be extended to a homeomorphism between $G_{\delta}$-subsets $X$ and $Y$ of $s^{m}_k$ and $s^{n}_k$, respectively. Since it can be shown that the complements of $X$ and $Y$ are strongly negligible, we have a homeomorphism between $s^{m}_k$ and $s^{n}_k$.

5. CHARACTERIZING HILBERT SPACE TOPOLOGY IN TERMS OF STRONG NEGLIGIBILITY

In the late 1960s R. D. Anderson introduced the concept of a strongly negligible set to infinite-dimensional topology. He shows in [2] that in Hilbert space manifolds the strongly negligible sets are precisely the $\sigma Z$-sets. Let us denote this property by $SN = \sigma Z$. We investigate under what conditions the property characterizes the Hilbert space manifolds among the complete absolute neighbourhood retracts (ANRs). In [11] we proved the following:

**Theorem 5.1.** A complete ANR is a Hilbert space manifold if and only if $SN = \sigma Z$ and moreover if every compact subset is a strong $Z$-set.
A closed subset \( S \) of a space \( X \) is called a strong \( Z \)-set if there exist continuous maps \( f : X \to X \), arbitrarily close to the identity, such that the closure of \( f(X) \) does not meet \( S \) (cf. the definition of \( Z \)-set in §2). This theorem is sharp in the sense that there exists an absolute retract, not homeomorphic to \( l^2 \), with \( SN = \sigma Z \) and the property that compacta are \( Z \)-sets rather than strong \( Z \)-sets. This counterexample will be described in [12].

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