A comparison of two possible nuclear effective field theory expansions around the one- and two-pion exchange potentials

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In the effective field theory formalism nuclear forces are organized as a low energy expansion. Usually the lowest order in this expansion corresponds to the non-perturbative iteration of the one-pion exchange potential and a few contact-range operators as to ensure renormalization group invariance. Here we consider an alternative expansion in which two-pion exchange becomes the lowest order finite range interaction and as such is expected to be iterated. A comparison of this new expansion with the standard one shows a better convergence pattern for a few selected partial waves ($^1S_0$, $^3P_0$ and $^3S_1$-$^3D_1$) in two-nucleon scattering, in particular for the $^1S_0$ channel (though both expansions converge well). We briefly comment on the possible theoretical grounds for the expansion around two-pion exchange from the point of view of effective field theory.

I. INTRODUCTION

The theoretical derivation of the nuclear forces is still one of the most pressing and interesting open problems of nuclear physics \cite{1}. Nowadays we expect any serious attempt of such a derivation to be grounded on quantum chromodynamics (QCD), the fundamental theory of strong interactions. Two strategies exists: a direct derivation in terms of lattice QCD \cite{2}, in which QCD is solved by resorting to brute force calculations, and an indirect one in which an effective field theory (EFT) \cite{3, 5} is formulated incorporating the low energy symmetries (most notably chiral symmetry) and degrees of freedom (pions, nucleons and possibly deltas) of QCD. This second strategy is in principle equivalent to QCD by virtue of being its renormalization group evolution at low energies.

Within the EFT formulation, we organize nuclear forces as a low energy expansion in terms of a ratio of scales, $Q/M$, where $Q$ is a soft scale that can be usually identified with the pion mass ($m_\pi \approx 140$ MeV) or the typical momenta of the nucleons within nuclei and $M$ a hard scale that might correspond to the nucleon mass $M_N$ or the chiral symmetry breaking scale $\Lambda_N = 4\pi f_\pi \sim 1$ GeV. However, there is no agreement yet on how to organize the EFT expansion for nuclear forces, as shown by the existence of numerous power counting proposals \cite{6, 11}, the eventual discovery of theoretical inconsistencies in a few countings \cite{14, 16} and daring and original \cite{17, 18} reinterpretations of nuclear EFT to deal with these problems.

This predicament might be explained by a poor expansion parameter ($Q/M$): if the expansion parameter is not small enough, the relative importance of the terms in the expansion could be very different than what naive estimations suggest. This might happen even if the coefficients in the expansion are of order one. To give a concrete example, we might consider the expansion of the matrix elements of a physical observable $\hat{O}$:

$$\langle \hat{O} \rangle = \sum_{\nu=0}^{\infty} \langle \hat{O}^{(\nu)} \rangle = \sum_{\nu=0}^{\infty} c_\nu x^\nu , \quad (1)$$

where $c_\nu$ are coefficients, $x$ the expansion parameter and $\nu \geq \nu_{\min}$ the order in the expansion. Provided $|x| < 1$ and that the $c_\nu$ do not grow exponentially with the order $\nu$, the previous expansion will eventually converge.

However, there is significant leeway for mishaps in the previous expansion, particularly if we are limited to its first few terms. For instance, this is what sometimes happens with the following toy EFT expansion of the observable $\langle \hat{O} \rangle$, in which the $c_\nu$ coefficients ($\nu_{\min} = 0$) are given by a random value within the interval:

$$c_\nu \in [0, \nu + 1] \quad \text{for} \ \nu \neq 1 \quad \text{and} \quad c_1 = 0 , \quad (2)$$

where the coefficients grow with the order, to mimic the increase in the number of diagrams. We have also set the $\nu = 1$ coefficient to zero, a choice which resembles nuclear EFT as applied to the two-nucleon forces when naive dimensional analysis (NDA) is followed. For simplicity we have taken all coefficients to be positive. With the previous set of coefficients and an expansion parameter of $x = 1/3$ (representative of the situation in the two-nucleon sector), the convergence pattern of the observable $\langle \hat{O} \rangle$ is

$$\langle \hat{O} \rangle = \langle \hat{O}^{(0)} \rangle + \langle \hat{O}^{(1)} \rangle + \langle \hat{O}^{(2)} \rangle + \langle \hat{O}^{(3)} \rangle + \sum_{\nu \geq 4} \langle \hat{O}^{(\nu)} \rangle \quad (3)$$

$$\bar{c} = \frac{1}{2} + 0 + \frac{1}{6} + \frac{2}{27} + \frac{11}{216} , \quad (4)$$

where the bar indicates the expected value (equivalent to taking $c_\nu = (\nu + 1)/2$ for $\nu \neq 1$) and the second line lists the magnitude of each contribution. Yet, while convergence is guaranteed for this toy expansion, the actual convergence pattern depends on chance and it can significantly deviate from the previous expectations. If we generate random toy expansions with $x = 1/3$ and the

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$c_\nu$ coefficients in Eq. (2) most of them will follow the expected power counting pattern

$$\bigl| (\hat{O}^{(0)}) \bigr| > \bigl| (\hat{O}^{(2)}) \bigr| > \bigl| (\hat{O}^{(3)}) \bigr| > \ldots ,$$

(5)

but a few will not. This is illustrated in Fig. 1 where three random toy expansions generated from the previous rules are shown. One of them is more convergent than expected, but the other two are dominated by the $\nu = 2$ ($N^3\text{LO}$) and $\nu = 3$ ($N^4\text{LO}$) terms, respectively. In these two cases and if we consider the first few terms only, it will be as if the EFT expansion is not working, despite the fact that it is completely convergent.

The bottom-line is that for systems where the convergence parameter is not small enough, the probability that the expansion is in disarray is sizable. Terms expected to be subleading might behave as leading, as if the EFT expansion itself has been fine-tuned. This by itself does not invalidate the idea of the EFT expansion or its eventual convergence, but it will certainly mess up its practical implementation, particularly if we are not willing to go to arbitrarily high orders for checking whether there is convergence of not. Luckily, this situation can be dealt with by rearranging the power counting at lowest orders, thus avoiding the inconvenience of having to calculate high orders of the expansion. For the toy model we have considered here the only reason behind unexpected behaviors of the EFT expansion is chance, but for other physical systems there might be underlying reasons related to neglected light scales. Also, in contrast with toy EFT series, for actual physical systems it is imperative to properly renormalize the theory before reaching any conclusions about the behavior of their expansions.

Something in the previous line is probably happening in the two-nucleon system. If we expand the finite-range potential, i.e. TPE(L) and TPE(SL). From direct inspection of the potentials (21), though usually not in the context of building a serious EFT expansion around it (except maybe in the chiral limit). For physical pion masses, the TPE potential is suppressed by a $e^{-2m_\pi r}$ factor at large distances, which for sure attenuates its strength. Besides, if tensor OPE is present (as happens in the triplets), the TPE potential will be further attenuated owing to the distortions generated by the non-perturbative tensor OPE wave function. Be it as it may, it will be interesting to explore what happens when TPE is non-perturbative, particularly in comparison to what happens when it is a perturbation.

Actually, this idea was already explored in the past [16, 21], though usually not in the context of building a serious EFT expansion around it (except in 22). In the Weinberg prescription, scattering amplitudes are obtained from iterating the full effective potential with a finite cutoff. As has been often pointed out, this prescription does in general not generate renormalizable scattering amplitudes [12, 14], which might indeed very well break the power counting assumptions of the effective potential from which they are calculated. Indeed, from analyzing
the underlying power counting of the scattering amplitudes calculated in the Weinberg prescription by trial and error, it is easy to find that the Weinberg $1S_0$ phase shifts are well reproduced if TPE (plus a contact operator to guarantee renormalizability) is LO and everything else is subleading (and perturbative) \[16\] \[21\]. Recently Ref. \[22\] has considered this expansion as a possible organizing principle for the $1S_0$ singlet, leading to results which we agree with here.

For exploring the expansion around TPE, we will do as follows: first, we will consider the type of non-relativistic EFT expansion that is generated from an arbitrary choice of a leading order potential, which we will then particularize for the cases in which OPE and TPE are leading (Sect. \[I\]); after this, we will calculate the EFT expansion of the phase shifts for the two cases (Sect. \[III\]); and finally, we will discuss and compare these two EFTs, the possible justifications of the TPE expansions and its implications (Sect. \[IV\]). In addition, a few of the technical details of the EFT expansion are discussed in Appendix \[A\] while Appendix \[B\] analyzes the scaling properties of this expansion, which might have applications for the study of EFT convergence.

II. POWER COUNTING AS A FUNCTION OF THE CHOICE OF A LEADING ORDER

Now we explore the possible form of the power countings generated from expanding around the cases in which OPE and TPE are non-perturbative. For concreteness we will refer to them as EFT(OPE) and EFT(TPE). Actually, EFT(OPE) has already been extensively discussed in the literature \[22\] \[28\] and we will rely on preexisting analyses up to a certain extent. The expansion of interest here is EFT(TPE), which will require modifications to the power counting that are genuinely different from those of EFT(OPE).

For simplicity, we will consider a general EFT expansion that at LO requires the non-perturbative iteration of a piece of the effective potential, which in turns generates a LO wave function, $\Psi_{\text{LO}}$, the form of which we assume to be known. The reason for adopting this approach is that the power counting of the contact-range potential is indeed determined by the power-law properties of $\Psi_{\text{LO}}$ \[29\]. Then, by particularizing $\Psi_{\text{LO}}$ to the EFT(OPE) and EFT(TPE) expansions we will deduce their power countings.

Before beginning, there are a few conventions we follow: the counting of the non-relativistic potential will be defined with respect to its scaling in momentum space, e.g. a potential that has the form

$$V(q) \propto \frac{1}{M^{n+2Q^m}} P_n(q) v(\frac{q}{Q}),$$

(13)

where $\bar{q}$ is the exchanged momentum, $P_n$ a polynomial of order $n$ and $v(\bar{x})$ some other (non-polynomial) function, will be counted as $Q^{n-m}$. When considering the iterations of the potential, i.e. $V G_0 V \ldots G_n V$, the propagator $G_0$ between two instances of the potential is counted as $Q$. As a consequence, if a part of the potential is treated non-perturbatively, this is because we consider it to scale as $Q^{-1}$ or LO. Also, the notation $Q^\nu$ is actually shorthand for $(Q/M)^\nu$. 

FIG. 1. Three possible EFT series in the toy model expansion we have proposed in Eqs. \[1\] and \[2\]. The inner circle is the expected average contribution of each order in the EFT series, which is \{1/2, 1/6, 2/27, 11/216\} for $\nu = \{0, 2, 3, \geq 4\}$ and an expansion parameter $x = 1/3$. The outer circle represents a random EFT series generated with our toy model, where each of the three cases shows a possible outcome: (a) a series converging faster than expected, (b) a series in which the $\nu = 2$ term happens to be larger than expected and (c) a series in which it is the $\nu = 3$ term that dominates. All these series are convergent, yet if we were limited to the first few terms we will be led to the (premature) conclusion that they are not, or in terms of power counting, that the power counting is failing. In this toy model there is no underlying reason why a particular EFT series happens to be fine-tuned (i.e. a subleading order contribution behaves as a leading order one, contrary to a priori expectations), it is merely a matter of chance. In the singlet channel of the two-nucleon system this is not necessarily the case: OPE cancels in the chiral limit and the contributions from the $\Delta$ excitations to TPE are large, favoring non-standard orderings analogous to (b) and (c).
We will assume the following power-law behavior for the LO wave function
\[ \langle r | \Psi_{\text{LO}} \rangle \sim r^{\alpha-1}, \tag{14} \]
where the exponent \( \alpha \) can be calculated from the form of the LO potential. Following \[24\] we will consider the matrix elements of a contact-range interaction of the type
\[ V_C(r) = \frac{C(R_c)}{4\pi R_c^2} \delta(r - R_c), \tag{15} \]
which for concreteness we have regularized with a delta-shell, where \( R_c \) is a cutoff radius. This contact will be sandwiched between the LO wave functions, leading to
\[ \langle \Psi_{\text{LO}} | V_C | \Psi_{\text{LO}} \rangle \sim \frac{C(R_c)}{R_c^{2-2\alpha}}. \tag{16} \]
If we demand renormalization group (RG) invariance for this matrix element
\[ \frac{d}{dR_c} \left[ \frac{C(R_c)}{R_c^{2-2\alpha}} \right] = 0, \tag{17} \]
then we realize that the running of \( C(R_c) \) is given by
\[ C(R_c) \propto R_c^{2-2\alpha}. \tag{18} \]
If we evolve \( C(R_c) \) from \( MR_c \sim 1 \) (where NDA applies and \( C(R_c) \sim 1/M^2 \)) to \( QR_c \sim 1 \), we find that at low resolutions this coupling behaves as
\[ C(R_c) \sim 1/Q \sim \frac{1}{M^2} \left( \frac{M}{Q} \right)^{2-2\alpha}. \tag{19} \]
That is, if \( \alpha > 1 \) (\( \alpha < 1 \)) then \( C(R_c) \) will be demoted (promoted) to order \( Q^{2\alpha-2} \). If we further expand the coupling in energy/momentum
\[ C = C_0 + C_2 k^2 + C_4 k^4 + \ldots, \tag{20} \]
it is apparent that \( C_2, C_4, \ldots \) are further suppressed by factor of \( Q^2, Q^4, \ldots \) and scale as
\[ C_2 \propto Q^{2\alpha}, \quad C_4 \propto Q^{2\alpha+2} \quad \text{and so on}. \tag{21} \]
As a side note the previous argument can be cross-checked in two different ways: finiteness of the subleading order calculations \[24\] and residual cutoff dependence \[31\] \[33\]. If we consider the perturbative correction of a hypothetical, unaccounted for \( Q^\nu \) contribution to the chiral potential in EFT, the matrix element will behave as \[30\]
\[ \langle \Psi_{\text{LO}} | V^{(\nu)} | \Psi_{\text{LO}} \rangle \sim \int_{R_c}^{\infty} \frac{dr}{r^{3+\nu}} r^{2\alpha} (1 + k^2 r^2 + \ldots), \tag{22} \]
where \( V^{(\nu)} \) refers to a finite-range contribution of order \( \nu \) to the EFT potential. From this, the lowest order at which we encounter a divergence in EFT is \( Q^{2\alpha-2} \) (which can be absorbed by a recalibration of the scattering length), while at \( Q^{2\alpha} \) we will find a divergence that requires a range correction. From the point of view of residual cutoff dependence, the LO phase shift converges as \[32\]
\[ \frac{d}{dR_c} \delta_{\text{LO}}(k; R_c) \propto k^3 R_c^{2\alpha+1}, \tag{23} \]
for \( R_c \to 0 \), which suggest that the range corrections enter \( Q^{2\alpha+1} \) orders after LO \( (Q^{-1}) \).

At this point it has to be stressed that two of the previous arguments (Eqs. \[10\] \[22\]) are perturbative in nature and do not take into account that the LO wave function might already require a series of contact interaction for its unambiguous determination. Indeed, this is what happens when the LO wave function is an attractive singular interaction \[35\] \[30\]. If this is the case, the necessary contact-range interactions will be automatically promoted to LO or \( Q^{-1} \). If we were to use the language of Refs. \[23\] \[37\] \[38\], we will say that the LO contact-range interactions are part of the RG fixed point, while the counting derived from Eqs. \[10\] \[22\] corresponds to that of the perturbations around said fixed point.

Now, all that remains is to determine the exponent \( \alpha \), which is a relatively well-known quantity:

(i) For a LO regular potential, we have \( \alpha = 0 \) in S-waves.

(ii) For a LO power-law singular potential of the type \( 1/r^n \) with \( n \geq 2 \), we have \( \alpha = n/4 \).

With this, the power counting of EFT(OPE) can be reproduced by taking into account that in the \( 1S_0 \) singlet \( \alpha = 0 \), while for the triplets in which tensor OPE is non-perturbative (e.g. \( 3S_1-3D_1 \) and \( 3P_0 \)), \( \alpha = 3/4 \).

In contrast for EFT(TPE) we have \( \alpha = 3/2 \) for singlets and triplets alike. This implies in particular that the \( C_2 \) couplings, i.e. range corrections, are all demoted to \( Q^3 \) or \( N^3\text{LO} \). This demotion of the range corrections is probably the most characteristic feature of the power countings arising from expanding around singular interactions.

In Table \[1\] we briefly summarize the power counting of EFT(OPE) and EFT(TPE) as will be implemented and explored in this work. It includes a few simplifying assumptions, e.g. the fractional counting of \( C_2 \) with tensor OPE (i.e. \( Q^{3/2} \) from \( \alpha = 3/4 \)) has been approximated to \( Q^2 \) and a few couplings that are numerically small (the couplings that fix the \( E_1 \) and \( 3D_1 \) scattering lengths) have been demoted from \( Q^{-1/2} \) to \( Q^2 \).

The most important simplification in EFT(TPE) is that we have simply promoted the full \( \nu \leq 3 \) EFT potential to LO (instead of only promoting subleading TPE, which makes more sense in view of \( \Lambda_{\text{TPE(SL)}} \), check Eqs. \[12\] \[22\]). This might be justified from the observation that the contributions of OPE and leading TPE are numerically small. Yet, the actual reason why it is safe to promote them is that this choice does not change the RG
evolution (RGE) of EFT(TPE): the power-law behavior of OPE and TPE(L) is unable to change the RGE of a theory in which TPE(SL) is treated non-perturbative. Besides, perturbative OPE is finite when sandwiched between the LO wave functions of EFT(TPE), while TPE(L) is renormalized by simply recalibrating the scattering length, i.e. by a subleading correction to $C_0$.

### III. COMPARISON BETWEEN THE TWO EXPANSIONS

Having decided the two power countings, the only thing left is to make the necessary calculations in the EFT(OPE) and EFT(TPE) expansions. The technical details are straightforward but tedious, and can be consulted in Appendix A, where here we simply indicate the main features of the calculation:

1. **The LO wave functions and scattering amplitudes are obtained and renormalized as in Refs. 24, 25, 30, 32, in which by using suitable boundary conditions it was shown the minimum number of couplings required for the different partial waves. In particular for the coupled channels (e.g. $^3S_1 - ^3D_1$) we have that:  

   (i.a) Non-perturbative OPE can be renormalized by fixing one of the scattering lengths (or their L-wave equivalents) 33.

   (i.b) Non-perturbative TPE can be renormalized by fixing the three scattering lengths (provided that the two eigenvalues of the coupled channel potentials are attractive, which is what happens in most partial waves) 37, 36.

2. **For the subleading order phase shifts, we basically follow the regularization used in Refs. 24, 25, which also shows that the perturbative corrections are indeed renormalizable after the inclusion of a series of couplings. A few differences with respect to Refs. 24, 25 are worth commenting:**

   (ii.a) For the $^1S_0$ partial wave in EFT(OPE), Refs. 24, 25 included both the $C_2$ and $C_4$ couplings at $Q^2$, which is the minimal requirement to render perturbative TPE finite and well-defined in the $R_e \to 0$ limit. However, finiteness is merely a subset of renormalizability and indeed RG invariance requires the $C_2$ coupling to enter at order $Q^0$, which is what we do here, a choice that entails a series of iterations of the $C_2$ coupling (which is computationally fastidious and has prompted expansions in which $C_2$ is fully iterated at LO 10). This is further justified by the fact that the LO calculation of the $^1S_0$ effective range only reproduces 50% of its value.

   (ii.b) For the $^3S_1 - ^3D_1$ channel in EFT(OPE), Refs. 24, 25 use a total of six couplings for renormalizing TPE. However, Long and Yang proved that three couplings are enough 27 (just as happens in Weinberg’s counting). We nonetheless will use six here, as this provides a more direct comparison between EFT(OPE) and EFT(TPE) at $N^2$LO (as with this choice we will end up with six couplings in both expansions).

   (ii.c) This choice of six couplings generates a further complication for $^3S_1 - ^3D_1$ in EFT(OPE): RGE arguments (either Eq. 17 particularized for $\alpha = 3/4$ or Ref. 23 for its original formulation) indicate that the couplings fixing the scattering lengths in the $E_1$ and $^3D_1$ channels enter at order $Q^{-1/2}$ or $N^{1/2}$LO. This is problematic as it will require an inordinate number of iterations of these couplings (far exceeding the required iterations of $C_2$ in the singlet). Yet, it is worth noticing that non-perturbative OPE actually reproduces relatively well these scattering lengths at LO 39 (in contrast with the $^1S_0$ case, in which this does not happen for the effective range), indicating that in most practical settings these couplings are not really required. For this reason we will simply denote the $C_0(E_1/^3D_1)$ couplings to $Q^2$.

   (ii.d) In principle a similar problem appears in the $^3P_0$ partial wave, but in this case the scattering length can be reproduced at LO, which
automatically implies that the subleading corrections to \( C_0(3P_0) \) are trivial.

(ii.e) Finally, in EFT(OPE) we further assume that the promotion of OPE from its naive scaling \((Q^0)\) to LO does not affect the power counting of the TPE potential. This is in contrast to the approach of Long and Yang which also promotes the TPE potential by one order \([26–28]\), a choice with makes perfect sense in terms of scales once we look at Eqs. \([912]\).

Having explained the previous details, we might simply proceed to the calculation of the \( ^1S_0, 3P_0 \) and \( ^3S_1-^3D_1 \) phase shifts in EFT(OPE) and EFT(TPE). For this, regardless of the expansion, we will calibrate the LO couplings as to reproduce the following values of the scattering length:

\[
\begin{align*}
\alpha_0( ^1S_0 ) &= -23.7 \text{ fm}, \\
\alpha_0( ^3P_0 ) &= -2.65 \text{ fm}^3, \\
\alpha_0( ^3S_1 ) &= 5.42 \text{ fm}, \\
\alpha_0( E_1 ) &= 1.67 \text{ fm}^3 \\
\alpha_0( ^3D_1 ) &= 6.60 \text{ fm}^3,
\end{align*}
\]

where the \( E_1 \) and \( ^3D_1 \) scattering lengths are only required for the TPE potential and are defined for the Stapp-Ypsilantis-Metropolis (SYM, also known as nuclear bar) parametrization of the phase shifts \([11]\) (where their detailed low energy behavior can be consulted in Ref. \([42]\)). For the subleading order phase shifts, the couplings will be determined from fitting the Nijmegen II phase shifts \([43]\) (expected to be equivalent to the ones extracted in the Nijmegen PWA \([14]\) within errors) within the \( k = (100–200) \text{ MeV} \) center-of-mass momentum window, except for the singlet channel where the momentum window changes with the order and includes lower momenta to ensure the correct scattering length (in particular we use \( k = (10–40) \text{ MeV} \), \( (10–80) \text{ MeV} \), \( (10–200) \text{ MeV} \) at NLO, \( \text{N}^2\text{LO}, \text{N}^3\text{LO} \) and \( \text{N}^4\text{LO} \)). The explicit expressions for the effective OPE and TPE potential are taken from Ref. \([21]\), where for the couplings we follow the same choices as in Ref. \([24]\) — that is, \( g_A = 1.26, f_T = 92.4 \text{ MeV} \), \( m_N = 138.03 \text{ MeV} \), \( d_{18} = -0.97 \text{ GeV}^{-2} \) (equivalent to using \( g_A = 1.29 \) instead of 1.26 in the OPE piece of the potential), \( c_1 = -0.81 \text{ GeV}^{-1} \), \( c_3 = -3.4 \text{ GeV}^{-1} \) and \( c_4 = +3.4 \text{ GeV}^{-1} \) — and where the recoil (or \( 1/M_N \)) corrections are included.

The results are shown in Fig. \([2]\) for the \( ^1S_0, 3P_0 \) and \( ^3S_1-^3D_1 \) partial waves, which uses the cutoff range \( R_c = (0.5–1.0) \text{ fm} \) to generate bands. The primary purpose of these bands is to illustrate the size of the cutoff dependence in the two expansions, while the secondary purpose is a rudimentary (but cheap) estimation of the EFT uncertainties (in general, for a properly renormalized theory, we expect cutoff variations to underestimate these uncertainties, particularly in the \( L \geq 1 \) waves; more sophisticated methods to estimate EFT uncertainties can be consulted in Refs. \([31\ 32\ 45\ 50]\)). The two expansions work relatively well for most partial waves, though EFT(TPE) works slightly better (particularly at higher momenta). Yet, the advantage of EFT(TPE) seems to concentrate mostly in the \( ^1S_0 \) partial wave, for which it requires only two couplings (instead of four) at \( \text{N}^3\text{LO} \), with which it also displays a slightly better agreement with the Nijmegen II pseudodata. For \( ^3S_1-^3D_1 \) we have chosen a counting choice in EFT(OPE) with more parameters than strictly necessary, just to make calculations simpler. Thus the comparison made here does not take into account that EFT(OPE) can be improved as to require less couplings in the deuteron channel.

In principle it is possible to gain a more detailed understanding of the convergence properties of the two expansions from the scaling properties of the subleading order corrections, which we analyze in Appendix \([B]\) This suggest that EFT(TPE) might be more convergent than EFT(OPE) not only in the \( ^1S_0 \) but also in the \( ^3S_1 \) and \( ^3P_0 \) partial waves. Yet, the conclusions are not clear-cut and at most tentative: on the one hand, it is not trivial to find dimensionless quantities whose size can be conclusively shown to be of \( \mathcal{O}(1) \). On the other, the simplifying counting choices that we have made — the aforementioned number of contact-range couplings in \( ^3S_1-^3D_1 \) case, or the fact that we promote all the potential to LO in EFT(TPE) — obscure the comparisons among the expansions.

It should also be noted that we have not tried to make a state-of-the-art analysis of the convergence of the EFT(OPE) and EFT(TPE) expansions. We merely point out that this could be achieved for instance by adapting the Bayesian analyses of Refs. \([45\ 50]\) to the distorted wave perturbative methods we use here for calculating the phase shifts. While Bayesian analyses usually deal with non-perturbative scattering observables as calculated in the Weinberg prescription (where they usually explore the limited range of cutoffs for which this prescription works), nothing prevents their application to the phase shifts within the two expansions we present here. Indeed, Ref. \([51]\) represents a step in this direction, though limited to the LO of EFT(OPE). A potentially interesting advantage of expanding the amplitudes (over calculating them non-perturbatively) is the previously mentioned determination of the scaling properties of the subleading corrections in Appendix \([B]\). In this regard, we briefly discuss the possible application of these findings for a prospective Bayesian analysis of EFT(OPE) and EFT(TPE) in Appendix \([B]\).

IV. DISCUSSION

Here we have considered two possible EFT expansions of the nuclear force: a typical EFT expansion in which the LO is defined by the iteration of the OPE potential and an atypical one in which TPE is also iterated. In both cases contact-range operators are included as to guarantee RG invariance at each order, though a few liberties are taken to ease the computational burden (particularly in the \( ^3S_1-^3D_1 \) channel). The two expansions, which we have named EFT(OPE) and EFT(TPE), converge relatively well, though the atypical EFT(TPE) does a better job in the \( ^1S_0 \) partial wave.
FIG. 2. Phase shifts of the $^1S_0$, $^3P_0$ and $^3S_1$-$^3D_1$ (nuclear bar) partial waves in the two EFT expansions considered in this work, EFT(OPE) and EFT(TPE). EFT(OPE) is an expansion in which OPE is a non-perturbative LO effect and happens to be similar to the power counting described in Refs. [24, 25], except the $^1S_0$ partial wave in which EFT(OPE) includes all the necessary iterations of $C_2$, which enters at order $Q^2$ in said expansion. EFT(TPE) is an expansion in which TPE is treated non-perturbatively, comprising the LO in the expansion (though for simplicity we also include OPE as part of the LO calculation as this will not significantly change the results). We follow the power counting described in Table IV where only the non-trivial orders (i.e. orders at which a new correction is included) are shown. The bands correspond to varying the cutoff in the $R_c = (0.5 - 1.0)$ fm range.

This begs the question: why is this the case? EFT(TPE) is in a sense a parody of what power counting should be, where contributions that should enter at $Q^3$ are promoted to $Q^{-1}$. In the absence of a good physical reason, EFT(TPE) should not be considered as a legitimate EFT expansion. The most apparent rationale of why EFT(TPE) might make sense lies in the slow convergence of the EFT expansion for nuclear physics, i.e. the...
poor separation of scales, which makes it plausible that higher order contributions might accidentally behave as lower order ones (check the discussion around Eq. 1 for further details). A cursory look at the S-wave potentials in the chiral limit, Eqs. [3][12], indicates that parts of TPE are indeed unexpectedly large.

In fact in the chiral limit tensor OPE, TPE(L) and TPE(SL) behave as pure \(1/r^3\), \(1/r^5\) and \(1/r^6\) power-law infinite-range potentials (while spin-spin OPE vanishes). This was beautifully exploited by Birse [23] to determine the momenta below which tensor OPE can be treated perturbatively by applying previously known results from atomic physics (in particular, the failure of the secular perturbative expansion [52] for the \(1/r^3\) potential, as calculated by Gao [53]). This suggested that tensor OPE is only perturbative for \(k < 66\) MeV \((k < 182\) MeV in the \(\frac{3}{2}\) \(S_1^2\) \(D_1\) (\(3F_0\)) partial waves, at least in the chiral limit, explaining the previous observation that the Kaplan, Savage and Wise (KSW) counting [8, 9] (which treats all pion exchanges as perturbations) converges slowly and only for low momenta in the two-nucleon system [15] (check also Ref. [54] for a further confirmation of the limits of perturbative OPE).

It happens that Gao also analyzed the secular perturbative expansion of attractive \(1/r^6\) interactions in Ref. [55], from which repeating Birse’s arguments one obtains a critical momentum \(k_{crit} \approx (150 - 200)\) MeV above which the perturbative treatment of subleading TPE will not converge in the chiral limit for the singlet. Taking into account that in the chiral limit the OPE vanishes in the singlet, it is sensible to assume that subleading TPE requires a non-perturbative treatment when \(m_\pi \to 0\). In the real world significant deviations should be expected due to the finite pion mass effects and the OPE LO distortion. Indeed, the EFT(OPE) expansion shows good convergence properties up to \(N^3\)LO, with no evident signs of a failure for \(k > k_{crit}\). However, the same was also true for the KSW expansion at NLO in the triplets, which later were shown to fail once OPE was iterated for the first time at \(N^3\)LO [16]. Thus it might very well happen that the same fate awaits to the EFT(OPE) expansion once the first iteration of TPE(SL) enters at \(Q^7 / N^8\)LO.

For triplet waves the situation is different, as TPE is now distorted by tensor OPE. From this, it might be perfectly possible that its strength is screened by the larger-range OPE distortion, which is why EFT(OPE) works better in the triplet when compared to the singlet. Actually, the results we show indicate that there is not a marked difference between the two expansions for the triplets. It should also be stressed that for EFT(TPE) we have made the simplification of promoting the full potential (including OPE) to LO. It might happen that leaving OPE as a NLO contribution, as its NDA scaling would suggest, might not lead to a converging expansion. If this were to be the case, the conclusion would be that EFT(TPE) is not a suitable expansion for the triplets. From an orthodox EFT perspective this would be good news, as this will reinforce the standard counting in which OPE drives the low energy physics of the deuteron, for instance. But to really confirm this hypothesis we would need to iterate tensor OPE up to relatively high orders in distorted wave perturbation theory, which is not exactly easy to do (particularly if we want to guarantee renormalizability at every step in the calculations). This would be nonetheless worth exploring in the future.

To summarize, the features of EFT(TPE) are intriguing: it is a really counterintuitive way of organizing nuclear EFT, yet it provides an expansion that converges better than the standard way of organizing the EFT expansion. One might argue that EFT(TPE) is an ersatz Weinberg prescription, but it is conceptually different: even though in both cases we are iterating a large chunk of the effective finite-range potential, this is not true for the contact-range operators, most of which are treated as perturbations. This detail makes EFT(TPE) renormalizable — the counting of the contacts is derived from RGE and the amplitudes have a well-defined \(R_c \to 0\) limit — while the Weinberg prescription does not always behave well in the hard cutoff limit (e.g. the \(1S_0\) channel is not renormalizable with TPE and two contacts, \(C_0\) and \(C_2\) [54], unless a very specific representation of the contact-range interaction is invoked [34]). If anything, the EFT(TPE) expansion provides more questions than answers and will require further theoretical effort to find its place within nuclear EFT.

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**Appendix A: Expansion of the phase shifts**

Here we briefly explain how to calculate the perturbative expansion of the phase shifts in \(r\)-space. The starting point is the reduced Schrödinger equation, which reads

\[
-u''_k(r) + \left[2\mu V(r) + \frac{l(l + 1)}{r^2}\right] u_k(r) = k^2 u_k(r),
\]

(A1)

with \(r\) the radius, \(k\) the center-of-mass momentum, \(\mu\) the reduced mass of the system, \(l\) the orbital angular momentum, \(V\) the two-body potential and \(u_k\) the reduced wave function. The phase shift can be extracted from the asymptotic behavior of the reduced wave function at
\[ r \to \infty \]
\[ u_k(r) \to \cot(k) \dot{y}_l(kr) - \dot{y}_l(kr), \quad (A2)\]
with \( \dot{y}_l(x) = x y_l(x) \) and \( y_l(x) \) are the standard spherical Bessel functions.

If we now modify the two-body potential as follows
\[ V \to V + \delta V, \quad (A3)\]
this will entail a similar change in the reduced wave function and the cotangent of the phase shift:
\[ u_k \to u_k + \delta u_k, \quad (A4)\]
\[ \cot \delta \to \cot \delta + \delta (\cot \delta). \quad (A5)\]
The exact form of the previous modifications can be obtained from a Wronskian identity involving the reduced Schrödinger equations for \( V \) and \( V + \delta V \), resulting in the formula
\[ \delta (\cot \delta) = \frac{2\mu}{k} \int_0^\infty dr u_k(r) \delta V(r) \left( u_k(r) + \delta u_k(r) \right), \quad (A6)\]
where it should be noted that this expression is exact, as we have not expanded yet in terms of power counting (or any other expansion scheme).

1. EFT expansion

For obtaining the perturbative or, more properly, the power counting expansion of \( \cot \delta \) we have to consider the explicit expansion of the three quantities involved
\[ V = \sum_{\nu = \nu_{\text{min}}}^{\infty} V^{(\nu)}, \quad (A7)\]
\[ u_k = \sum_{\nu = \nu_{\text{min}} + 1}^{\infty} u_k^{(\nu)}, \quad (A8)\]
\[ \cot \delta = \sum_{\nu = \nu_{\text{min}} + 1}^{\infty} \left( \cot \delta \right)^{(\nu)}, \quad (A9)\]
and then rearrange the different terms of Eq. (A9) accordingly, where care should be taken that only terms of the same order in the EFT expansion are matched. This last condition is not trivial at first sight (particularly because the power counting index of the potential is arranged according to its \( p \)-space representation, see the discussion around Eq. (13), yet we are working in \( r \)-space), but can be automated by writing down explicitly a dummy expansion parameter \( \lambda \)
\[ V = \lambda^3 \sum_{\nu = \nu_{\text{min}}}^{\infty} \lambda^\nu V^{(\nu)}, \quad (A10)\]
\[ u_k = \sum_{\nu = \nu_{\text{min}} + 1}^{\infty} \lambda^\nu u_k^{(\nu)}, \quad (A11)\]
\[ \cot \delta = \sum_{\nu = \nu_{\text{min}} + 1}^{\infty} \lambda^\nu [\cot \delta]^{(\nu)}, \quad (A12)\]
to which we add these two substitution rules
\[ r \to \frac{r}{\lambda} \quad \text{and} \quad k \to \lambda k, \quad (A13)\]
to account for their power counting. With this we can readily organize the EFT expansion of the phase shifts by expanding and rearranging in powers of \( \lambda \), and later resetting this dummy parameter to \( \lambda = 1 \).

After this, the calculation of the \( \nu \)-th order phase shifts can be arranged by means of an iterative process, where the starting point is the LO phase shift \( \delta^{(0)} \), which is calculated from the reduced Schrödinger equation and the LO potential by standard means. The NLO phase shift \( \delta^{(1)} \) will be obtained by expanding Eq. (A6) with the power counting rules implicitly defined with the dummy \( \lambda \) expansion parameter procedure, leading to
\[ [\cot \delta]^{(1)} = \frac{2\mu}{k} \int_0^\infty dr u_k^{(0)}(r) V^{(0)}(r) u_k^{(0)}(r) \]
\[ = \left( \cot \delta \right)^{(0)} \sin^2 \delta^{(0)} \quad (A14)\]
where in the second line we explicitly expand \( \cot \delta^{(1)} \) in terms of the expansion of the phase shift, i.e. \( \delta = \sum_{\nu} \delta^{(\nu)} \). For N^2LO, we first have to construct the NLO wave function, which behaves asymptotically as
\[ u_k^{(1)}(r) \to [\cot \delta]^{(1)} \dot{y}_l(kr), \quad (A15)\]
and which we later integrate by means of
\[ -u_k^{(1)(\nu)}(r) + \left[ 2\mu V^{(-1)}(r) + \frac{l(l + 1)}{r^2} - k^2 \right] u_k^{(1)}(r) = -2\mu V^{(0)}(r) u_k^{(0)}(r). \quad (A16)\]

Once the NLO wave function has been computed, the N^2LO phase shifts will be given by
\[ [\cot \delta]^{(2)} = \frac{2\mu}{k} \int_0^\infty dr u_k^{(0)}(r) \]
\[ \times \left( V^{(1)}(r) u_k^{(0)}(r) + V^{(0)}(r) u_k^{(1)}(r) \right) \]
\[ = -\frac{\delta^{(2)}}{\sin^2 \delta^{(0)}} + \left( \frac{\delta^{(1)}}{\sin \delta^{(0)}} \right)^2 [\cot \delta]^{(0)} \quad (A17)\]
where the last line contains the expansion of \( [\cot \delta]^{(2)} \) in terms of the expansion of the phase shift. As can be appreciated the expressions become more complex as the EFT order increases. Yet, their derivation is straightforward.

2. Regularization and renormalization

The EFT potential contains a finite- and contact-range piece
\[ V = V_F + V_C, \quad (A18)\]
with \( V_F \) given by pion exchanges and \( V_C \) representing the shorter range physics not explicitly included in the EFT description. These contributions are singular (\( V_F \) involves inverse power-law potentials and \( V_C \) Dirac-delta contributions and their derivatives) and have to be regularized. For the finite-range potential we choose a sharp cutoff in r-space, i.e.

\[
V_F(r; R_c) = V_F(r) \theta(r - R_c),
\]

where \( V_F(r; R_c) \) and \( V_F(r) \) refer to the regularized and unregularized version of the finite-range potential. For the contact-range potential, we choose an energy-dependent delta-shell regularization

\[
V_C(r; R_c) = \sum_{n=0}^{\infty} \frac{C_{2n}(R_c)}{\pi R_c^2} \delta(r - R_c),
\]

where the couplings \( C_{2n}(R_c) \) admit a further expansion in terms of the power counting, i.e. \( C_{2n} = \sum_n C_{2n}^{(r)} \), which we do not explicitly write here. Finally, for the renormalization of the perturbative phase shifts we refer to [24, 25], where it is explained in detail how the divergences generated by the finite-range potential are absorbed by the contact-range couplings.

3. Extension to coupled channels

For the coupled channels we will adapt the formalism contained in the Appendix of Ref. [57] and write the reduced Schrödinger equation as

\[
-u_k(r) + \left[2\mu V(r) + \frac{L^2}{r^2}\right] u_k(r) = k^2 u_k(r),
\]

where \( u_k \) and \( V \) are now \( N \times N \) matrices, while the squared orbital angular momentum operator takes the form of the diagonal matrix

\[
L^2 = \text{diag}(l_1(l_1 + 1), \ldots, l_N(l_N + 1)),
\]

with \( l_i \) the orbital angular momentum of channel \( i = 1, \ldots, N \). The asymptotic behavior of the reduced wave function is given by

\[
u_k(r) \to J_k(r) M(k) - Y_k(r),
\]

where \( J_k(r) \) and \( Y_k(r) \) are the diagonal matrices

\[
J_k(r) = \text{diag}(\hat{j}_1(kr), \ldots, \hat{j}_N(kr)),
\]

\[
Y_k(r) = \text{diag}(\hat{\gamma}_1(kr), \ldots, \hat{\gamma}_N(kr)),
\]

with \( \hat{j}_i(x) \) and \( \hat{\gamma}_i(x) \) as defined below Eq. (A2). \( M(k) \) is basically the coupled channel version of \( \cot \delta(k) \), which is related to the S-matrix as

\[
M(k) = \frac{S(k) + 1}{S(k) - 1}.
\]

It is worth noticing that the standard representation for the reduced wave function would be an \( N \)-dimensional vector. Yet, there are \( N \) linearly independent solutions to the previous reduced Schrödinger equation, which allows us to group all the solutions in the \( u_k \) matrix. Then, every column of the \( u_k \) matrix corresponds to a linearly independent reduced wave function.

As before, if we modify the potential then the subsequent modification of the \( M(k) \) matrix and the reduced wave function can be found by constructing a suitable Wronskian identity, leading to

\[
\delta M(k) = \frac{2\mu}{k} \int_0^\infty dr \left( u_k^T(r) \delta V(r) u_k \right),
\]

where this expression is exact. By expanding it in terms of power counting we will find the explicit expressions for the subleading order phase shifts.

Finally, it is useful to write down the explicit expressions for \( M(k) \) in the eigen representation of the phase shifts [28]. For the eigen phase shifts, the S-matrix takes the form

\[
S(k) = R(\epsilon) \begin{pmatrix} e^{2i\delta_\alpha} & 0 \\ 0 & e^{2i\delta_\beta} \end{pmatrix} R(-\epsilon),
\]

where \( \delta_\alpha \) and \( \delta_\beta \) are the two eigen phase shifts and \( R \) is the rotation matrix

\[
R(\epsilon) = \begin{pmatrix} \cos \epsilon & -\sin \epsilon \\ \sin \epsilon & \cos \epsilon \end{pmatrix},
\]

with \( \epsilon \) the mixing angle. This leads to

\[
M(k) = R(\epsilon) \begin{pmatrix} \cot \delta_\alpha & 0 \\ 0 & \cot \delta_\beta \end{pmatrix} R(-\epsilon),
\]

which happens to be a relatively compact expression. Putting all the pieces together, we find the next exact expression for the variation of the phase shifts with respect to the potential

\[
k R^{-1}(\epsilon + \delta \epsilon) \delta M(k) R(\epsilon) =
2\mu \int_0^\infty dr R^{-1}(\epsilon + \delta \epsilon) u_k^T(r) \delta V(r) u_k(r) R(\epsilon).
\]

Here it is interesting to notice that the \( u_k R \) product actually coincides with the so-called \( \alpha \) and \( \beta \) scattering states

\[
u_k(r) R(\epsilon) = \begin{pmatrix} u_\alpha(r) & u_\beta(r) \\ w_\alpha(r) & w_\beta(r) \end{pmatrix},
\]

which are characterized by the relatively simple asymptotic (\( r \to \infty \)) forms

\[
\begin{pmatrix} u_\alpha(r) \\ w_\alpha(r) \end{pmatrix} \to \begin{pmatrix} \cos \epsilon (\cot \delta_\alpha \hat{j}_\alpha(kr) - \hat{j}_\alpha(kr)) \\ \sin \epsilon (\cot \delta_\alpha \hat{j}_\alpha(kr) - \hat{j}_\alpha(kr)) \end{pmatrix},
\]

\[
\begin{pmatrix} u_\beta(r) \\ w_\beta(r) \end{pmatrix} \to \begin{pmatrix} -\sin \epsilon (\cot \delta_\beta \hat{j}_\beta(kr) - \hat{j}_\beta(kr)) \\ \cos \epsilon (\cot \delta_\beta \hat{j}_\beta(kr) - \hat{j}_\beta(kr)) \end{pmatrix},
\]
with \(l_a = j - 1\) and \(l_b = j + 1\) for a standard coupled channel with total angular momentum \(j\).

The power counting expansion of the previous expressions is straightforward but tedious (particularly if higher order perturbation theory is involved). However, in most practical settings only first order perturbation theory is required, in which case we will recover the expressions of Refs. 24, 25 (modulo normalization and notational conventions):

\[
[cot \delta_\alpha]_{1}^{(1)} = \frac{2 \mu}{k} I_{\alpha \alpha}^{(0)}(k), \quad (A35)
\]

\[
[cot \delta_\beta]_{1}^{(1)} = \frac{2 \mu}{k} I_{\beta \beta}^{(0)}(k), \quad (A36)
\]

\[
\epsilon^{(1)} \left( cot \delta_\alpha^{(0)} - cot \delta_\beta^{(0)} \right) = \frac{2 \mu}{k} I_{\alpha \beta}^{(0)}(k), \quad (A37)
\]

with

\[
I_{\gamma \delta}^{(0)}(k) = \int_{0}^{\infty} \text{d}r \left[ u_\gamma^{(0)}(r) V_{aa}^{(0)} r^{\delta}^{(0)}(r) \right. \\
+ V_{ab}^{(0)} \left. \left( u_\gamma^{(0)}(r) r_\delta^{(0)}(r) + w_\gamma^{(0)}(r) r_\delta^{(0)}(r) \right) \right] \\
+ w_\gamma^{(0)}(r) V_{bb}^{(0)} r_\delta^{(0)}(r)]. \quad (A38)
\]

In contrast, had we used the nuclear-bar phase shifts \([11]\) (i.e. \(\delta_\alpha, \delta_\beta, \epsilon\)) we would have ended up with bewilderingly complex expressions that are not particularly suited for practical calculations. Instead, the best strategy for applying the perturbative expansion for the nuclear-bar representation is to use the well-known conversion formulas

\[
\delta_\alpha + \delta_\beta = \tilde{\delta}_\alpha + \tilde{\delta}_\beta, \quad (A39)
\]

\[
\sin(\delta_\alpha - \delta_\beta) = \frac{\sin 2\epsilon}{\sin 2\epsilon}, \quad (A40)
\]

\[
\sin(\tilde{\delta}_\alpha - \tilde{\delta}_\beta) = \frac{\tan 2\epsilon}{\tan 2\epsilon}, \quad (A41)
\]

and expand them according to the power counting.

**Appendix B: Estimating the convergence of the EFT expansion**

Here we analyze how the perturbative expansion of the phase shifts behaves in terms of the involved soft and hard scales. In particular we are interested in dimensionless quantities that appear in the EFT expansion and whose size is of \(O(1)\) when the breakdown scale \(M\) has been correctly identified. These quantities might in turn be used to estimate the convergence of the EFT expansion.

1. **Scaling properties**

We begin by considering the expansion of \(cot \delta\) and how it is calculated, where the idea is to understand its scaling properties in detail. Trivially, \([cot \delta]^{(\nu)}\) is expected to scale as

\[
\frac{k}{2\mu} [cot \delta]^{(\nu)} = \int_{0}^{\infty} dr u_k^{(0)}(r) [V(r) u_k(r)]^{(\nu-1)} \propto \left( \frac{Q_k}{M} \right)^{\nu+1}, \quad (B1)
\]

where we have explicitly written its expression as a matrix element of the EFT potential. The notation \([V u]^\mu\) indicates all combinations of \(V\) and \(u\) of order \(\mu\), i.e.

\[
[V(r) u_k(r)]^{(\mu)} = \sum_{\mu_1, \mu_2 \geq 0} V^{(\mu_1)}(r) u_k^{(\mu_2)}(r). \quad (B2)
\]

The integral appearing in the calculation of \([cot \delta]^{(\nu)}\) can be further subdivided into a polynomial and non-polynomial contributions:

\[
\int_{0}^{\infty} dr u_k^{(0)}(r) [V(r) u_k(r)]^{(\nu-1)} = \left( \frac{Q_k}{M} \right)^{\nu+1} \left[ f_{\text{pol}}^{(\nu)} \left( \frac{k}{Q_k} \right) + f_{\text{nonpol}}^{(\nu)} \left( \frac{k}{Q_k} \right) \right], \quad (B3)
\]

which we have written as functions of the dimensionless ratio \(k/Q_k\), where \(Q_k\) represents all the light scales with the exception of the momentum \(k\). Here by polynomial we specifically mean the part of the integral that behaves as a pure polynomial on the momentum \(k\) regardless of how large \(k\) is. The non-polynomial part will still be expansible in powers of \(k\) at small momenta, but this expansion is not expected to converge for momenta \(k \gg m_{\pi}\). We will study each contribution in more detail below, including how this division into polynomial and non-polynomial pieces appears.

The polynomial part stems from (i) the contact-range potential and (ii) the divergences of the finite-range potential (which after renormalization are absorbed in the contact-range couplings). If we regularize with a sharp cutoff in \(r\)-space (i.e. as in Eq. [A19]), we may write these two contributions separately as:

\[
\int_{0}^{\infty} dr u_k^{(0)}(r) [V_C(r) u_k(r)]^{(\nu-1)} = \sum_{n=0}^{\text{max}^{(\nu)}} \frac{u_k^{(0)}(R_c) [C_{2n}^{(\nu)} R_c]^{(\nu-1)}}{4\pi R_c^2} k^{2n}, \quad (B4)
\]

\[
\int_{0}^{\infty} dr u_k^{(0)}(r) [V_F(r) u_k(r)]^{(\nu-1)} = \left( \frac{Q_k}{M} \right)^{\nu} \sum_{n=0}^{\text{max}^{(\nu)}} d_{2n} \left( \frac{k}{Q_k} \right)^{2n} \times \int_{(Q_k R_c)}^{\infty} dr x^{\nu+1-n-2n} \times (\text{finite terms}), \quad (B5)
\]

where \(V_C\) and \(V_F\) are the contact- and finite-range pieces of the EFT potential and the notation \([\ldots]^{(\mu)}\) was previously defined in Eq. [B2]. Here \(l\) is the orbital angular
momentum, $\alpha$ refers to the power-law behavior of the LO wave function at short distances (which might depend on $l$ if the LO potential is regular), $n_{\text{max}}(\nu)$ the number of contact-range couplings at $\nu$-th order and $d_{2n}$ is a dimensionless coefficient that characterizes a given divergence. The notation

$$\int (Q_k R_c) \frac{dx}{x^{3+\nu-2n-2n}},$$

(B6)

where only the lower integration bound is shown, has been used to indicate that we are interested in the divergent part of this integral. If we now combine the two terms together, after removing the divergences we will find the polynomial part of the perturbative integral:

$$\int_0^\infty dr u_k^{(0)}(r) [V(r)u_k(r)]^{(\nu-1)} |_{\text{pol}} =$$

$$\left( \frac{Q_k}{M} \right)^\nu \left( \frac{Q_k}{k} \right)^{2n} \sum_{\nu=0}^{n_{\text{max}}(\nu)} c_{2n} \left( \frac{k}{Q_k} \right)^{2n}$$

+ (cut-off dependent polynomials),

(B7)

where there is a cut-off dependent part of the polynomial (comprising higher powers of $k$), which vanishes when $R_c \to 0$. Even though derived from a particular regulator, we expect the result above to be independent of the choice of regulator.

The non-polynomial piece stems from the finite terms of Eq. (156) and can in fact be shown to be non-polynomial if we consider that — for $r \to \infty$ — we have on the one hand that the finite-range potential decays exponentially

$$V_F^{(\nu)}(r) \to \frac{1}{(Mr)^{\nu+2}} P_F(Q_k r) \frac{e^{-2m r}}{r},$$

where $P_F(x)$ represents a polynomial, and, on the other, that the wave function can also be written in terms of exponentials

$$u_k(r) \to e^{i\delta_k(k)} e^{i(kr-l\pi/2)} - e^{-i\delta_k(k)} e^{-i(kr-l\pi/2)}.$$

(B9)

Combining these two observations it can be appreciated that the infrared behavior of the integral

$$\int_0^\infty dr u_k^{(0)}(r) [V_F(r)u_k(r)]^{(\nu-1)} \propto$$

$$\frac{e^{\pm 2i\delta_k^{(0)}}(k)}{M^{\nu+2}} \int_0^\infty dr P_F(Q_k r) e^{(\pm 2ik-2m)r} \nu+3 + \ldots,$$

(B10)

diverges for $|\text{Im} k| > m$, giving rise to two branch cuts beginning at $k = \pm im$. As a consequence, the non-polynomial part of the integral does not have a convergent Taylor expansion in terms of the momenta for $k > m$.

Here we notice that we have explicitly assumed that the subleading potential is dominated by two-pion exchanges: hence the $e^{-2m r}$ decay. This is not strictly true though, as there are subleading corrections to the OPE potential too. These corrections, if treated perturbatively, will generate additional branch cuts at $k = \pm im/2$. Yet, the bottom-line is the same: the non-polynomial part is in fact non-polynomial for large enough momenta.

2. Naturalness

Ideally, we want to extract coefficients in the EFT expansion whose size is $\mathcal{O}(1)$. The most direct candidate would be the dimensionless function $f^{(\nu)}(x)$ that appears when rescaling $[\cot \delta]^{(\nu)}$ according to its power counting

$$\left( \frac{M}{Q_k} \right)^{\nu+1} \frac{k}{2\mu} [\cot \delta]^{(\nu)} = f^{(\nu)}(k Q_k),$$

(B11)

but there is the problem that $\cot \delta$ diverges when $\delta \to 0$. This makes it impractical to argue whether $f^{(\nu)}$ is $\mathcal{O}(1)$ or not.

This limitation is easily circumvented by rescaling the phase shift $\delta^{(\nu)}$ instead, resulting in the definition of

$$\left( \frac{M}{Q_k} \right)^{\nu+1} \frac{k}{2\mu} \delta^{(\nu)} = g^{(\nu)}(k Q_k),$$

(B12)

which is now finite when $\delta^{(0)} \to 0$ and related to $f^{(\nu)}$ by

$$g^{(\nu)} = -\sin^2 \delta^{(0)} f^{(\nu)} + \ldots,$$

(B13)

where the dots represent further terms stemming from the expansion of $[\cot \delta]^{(\nu)}$, i.e. products of $f^{(\nu)} f^{(\nu)}$ such that $\nu_1 + \nu_2 = \nu$.

The dimensionless function $g^{(\nu)}(x)$ is not expected to diverge at normal momenta (as it is derived from phase shifts, which do not diverge), and is thus a better choice for naturalness. Its size is not necessarily $\mathcal{O}(1)$, but this is not crucial though if our intention is to compare the convergence of the EFT(OPE) and EFT(TPE) expansions. In fact we do not need to know the hard scale $M$ of the EFT we are considering, but just compare both expansions to a common scale

$$\left( \frac{M'}{\Lambda_{\text{OPE}}} \right)^{\nu+1} \frac{k}{2\mu} g^{(\nu)} = \left( \frac{M'}{\Lambda_{\text{OPE}}} \right)^{\nu+1} g_{\text{OPE}}^{(\nu)}(k Q_k),$$

(B14)

$$\left( \frac{M'}{\Lambda_{\text{TPE}}} \right)^{\nu+1} \frac{k}{2\mu} \delta^{(\nu)} = \left( \frac{M'}{\Lambda_{\text{TPE}}} \right)^{\nu+1} g_{\text{TPE}}^{(\nu)}(k Q_k),$$

(B15)

where $\Lambda_{\text{OPE}}$, $M_{\text{OPE}}$ and $\Lambda_{\text{TPE}}$, $M_{\text{TPE}}$ are the typical light and hard scales for the EFT(OPE) and EFT(TPE)
expansions, and with $M'$ an arbitrary scaling mass. Regardless of the numerical factor to which the dimensionless functions $g$ or $\hat{g}$ gravitate, we nonetheless expect that

$$O(\hat{g}_{\text{TPE}}) = O\left(\frac{M_{\text{OPE}}}{M_{\text{TPE}}}^{(\nu+1)} \hat{g}_{\text{OPE}}\right). \quad (B16)$$

That is, when comparing the dimensionless coefficients of the two expansions we do not need to know the specific numerical factors.

With this we now compare the functions $\hat{g}^{(\nu)}$ for $\nu = 4$ in the EFT(OPE) and EFT(TPE) expansions for the $^1S_0$, $^3S_1$ and $^3P_0$ partial waves. We take $M' = 0.5 \text{ GeV}$, which is a typical guess for the hard scale of the nuclear EFT expansion, $\Lambda_{\text{OPE}} = m_\pi = 138 \text{ MeV}$ (i.e. the pion mass) and $\Lambda_{\text{TPE}} = 233 \text{ MeV}$ (i.e. the natural scale of the subleading TPE potential in the singlet, see Eq. (10)). The reason is the polynomial part of these dimensionless functions which, on closer inspection contains positive powers of $k/Q_k$. If we expand the polynomial in $[\cot \delta]^{(0)}$ (and momentarily ignore cutoff dependent terms)

$$\left(\frac{M}{Q_k}\right)^{\nu+1} \frac{k^2}{2\mu} \sin^2 \delta(0) f^{(\nu)}(\frac{k}{Q_k}) + \ldots$$

we realize that this is indeed a problem in central waves. For instance, in the $^1S_0$ partial wave, we will have powers of $(k/m_\pi)^6$ for EFT(OPE) at $\nu = 3, 4$, which will grow disproportionally large in the $m_\pi < k < M$ region. Notice that there is no difference if we expand $\hat{g}^{(\nu)}$ instead of $[\cot \delta]^{(\nu)}$, the problem will still be present either way.

The solution is simple — to remove the polynomial part coming from the expansion of $[\cot \delta]^{(0)}$ — in which case we are left with

$$\left(\frac{M}{Q_k}\right)^{\nu+1} \frac{k^2}{2\mu} \sin^2 \delta(0) f^{(\nu)}(\frac{k}{Q_k}) + \ldots$$

$$= g^{(\nu)}_{\text{nonpol}}(\frac{k}{Q_k}), \quad (B18)$$

where $g^{(\nu)}_{\text{nonpol}}$ is, in a first instance, not expected to grow as a polynomial, at least in the $m_\pi < k < M$ range. The dots represent further polynomial terms from the iteration of subleading terms, check Eq. (B13) and the discussion around it. These terms do only appear in the EFT(OPE) expansion of the $^1S_0$ partial wave, yet they can be ignored by redefining $f^{(\nu)}_{\text{pol}}$ as the polynomial piece of

$$-\frac{1}{\sin^2 \delta(0)} \left(\frac{M}{Q_k}\right)^{\nu+1} \frac{k^2}{2\mu} \delta(0) f^{(\nu)}_{\text{pol}}(\frac{k}{Q_k}) = \tilde{f}^{(\nu)}_{\text{pol}} + \tilde{f}^{(\nu)}_{\text{nonpol}}, \quad (B19)$$

instead of its usual definition in terms of $[\cot \delta]^{(\nu)}$. In most cases this redefinition has no appreciable impact on the numerical values of the coefficients of the polynomial $f^{(\nu)}_{\text{pol}} / f^{(\nu)}_{\text{pol}}$, the only exception being $^1S_0$ in EFT(OPE) for $\nu = 3, 4$ where the relative differences can reach $(10 - 30\%)$ for $c_4^{(\nu)}$.  

FIG. 3. The rescaled dimensionless phase shift $\hat{g}^{(\nu)}$ as defined in Eqs. (B14) and (B15) for the EFT(OPE) and EFT(TPE) expansions. It can be appreciated that $\hat{g}^{(\nu)}$ is smaller for the expansion around TPE than for the OPE one, suggesting better convergence.
Finally, by rescaling again with an arbitrary mass $M'$
\[
\left( \frac{M'}{\Lambda_{\text{OPE}}} \right)^{\nu + 1} \frac{k}{2\mu} \delta^{(0)} \delta^{(\nu)} f_{\text{pol, OPE}}(k) = g^{(\nu)}_{\text{nonpol, OPE}}(k), \tag{B20}
\]
\[
\left( \frac{M'}{\Lambda_{\text{TPE}}} \right)^{\nu + 1} \frac{k}{2\mu} \delta^{(0)} \delta^{(\nu)} f_{\text{pol, TPE}}(k) = g^{(\nu)}_{\text{nonpol, TPE}}(k), \tag{B21}
\]
we obtain the rescaled $g^{(\nu)}_{\text{nonpol, OPE}}$ and $g^{(\nu)}_{\text{nonpol, TPE}}$ without the unwanted contamination of the fast-growing polynomial terms. Notice that $f_{\text{pol, OPE}}$ and $f_{\text{pol, TPE}}$ are just polynomials in $k^2/\Lambda_{\text{OPE}}^2$ and $k^2/\Lambda_{\text{TPE}}^2$, which can be directly fitted to the $\delta^{(\nu)}$ phase shifts at low momenta.

Yet, when we compare the new functions $g^{(\nu)}_{\text{nonpol}}$ for $\nu = 4$ in the EFT(OPE) and EFT(TPE) expansions for the $^1S_0$, $^3S_1$ and $^3P_0$ partial waves in Fig. 4, not much of a difference can be appreciated with respect to the previous $\hat{g}^{(\nu)}$ that still contained polynomial contributions. The exception is the $^1S_0$ partial wave, where removing the polynomial worsens the comparison for EFT(OPE). For the new comparison in Fig. 4, we have again taken $M' = 0.5 \text{ GeV}$, $\Lambda_{\text{OPE}} = m_\pi = 138 \text{ MeV}$ and $\Lambda_{\text{TPE}} = 233 \text{ MeV}$. The polynomial in $\hat{f}^{(\nu)}$ has been fitted to the $\delta^{(\nu)}$ phase shifts in the $k = (20 - 100) \text{ MeV}$ region by assuming that at low momenta
\[
\frac{1}{\sin^2 \delta^{(0)}} \left( \frac{M}{Q_{\bar{g}}} \right)^{\nu + 1} \frac{k}{2\mu} \delta^{(\nu)} \approx \left( \frac{Q_{\bar{g}}}{k} \right)^{2\nu} \sum_{n=0}^{n_{\text{max}}(\nu)} c^{(\nu)}_{2n} \left( \frac{k}{Q_{\bar{g}}} \right)^{2n} + d_{\text{res}} \left( \frac{k}{Q_{\bar{g}}} \right)^{2n_{\text{max}} + 2}, \tag{B22}
\]
with $n_{\text{max}}(\nu)$ the number of contact interactions at the order we are considering and where we add an additional order to the polynomial to stabilize the extracted numerical values of the $c^{(\nu)}_{2n}$ coefficients. This extra order and its coefficient $d_{\text{res}}$ are not part of $\hat{f}^{(\nu)}$ and are discarded once we fit the low energy $\delta^{(\nu)}$ phase shifts. Notice that there is an offset of one order between the potential and the phase shifts. That is, for a given partial wave, the number of terms in the $\hat{f}^{(\nu)}$ polynomial is the same as the number of contact-range couplings in the $Q^\nu$ potential (as shown in Table I).

From the previous, the expectation is still that EFT(TPE) is more convergent than EFT(OPE) for the $^1S_0$, $^3S_1$ and $^3P_0$ partial waves. The naive application of the scaling of Eq. (B16) to $g^{(\nu)}_{\text{nonpol, OPE}}$ and $g^{(\nu)}_{\text{nonpol, TPE}}$ would suggest that:
\[
M_{\text{OPE}} \sim (0.7 - 0.8) M_{\text{TPE}}. \tag{B23}
\]
But caution is advised: the $\nu = 4$ contribution to EFT(OPE) contains both a contact- and finite-range contribution, the latter being subleading TPE. In contrast, the $\nu = 4$ contribution to EFT(TPE) comes exclusively from the subleading contact-range potential: if the cutoff were to be removed, it should become purely polynomial. Indeed, the $g^{(\nu)}_{\text{nonpol, TPE}}$ functions in Fig. 4 are finite-cutoff effects that should vanish as $R_c \rightarrow 0$. Because of this, the comparison of Fig. 4 is biased against EFT(OPE). Thus we either go back to the full $\hat{g}^{(\nu)}$ (with the polynomials), as already done in Fig. 3 or move to $\nu = 5$ where both expansion receive new contributions from the finite-range potential.

Yet, there is the additional issue of the dimensionless function $g^{(\nu)}_{\text{nonpol}}$ and its expected size. A more careful analysis of its properties is presented in Appendix B3, which suggests that its size is $\mathcal{O}(2^n \pi/(\nu + 1)!)$.
Alternatively, there is the possibility of comparing the $c_k^\text{(TPE)}$ coefficients corresponding to the polynomial $f_k$ in the $N\,\text{LO}^{(5)}\,\text{S}$ case and $N\,\text{LO}^{(1)}\,\text{S}$ case, as we show in Table I for the $c_k^\text{(TPE)}$ coefficients. We have computed the $c_k^\text{(TPE)}$ coefficients using the $N\,\text{LO}^{(5)}\,\text{S}$ and $N\,\text{LO}^{(1)}\,\text{S}$ cases, respectively. These coefficients are listed in Table I.

Table I: Dimensionless coefficients $c_k^\text{(TPE)}$ of the low momentum expansion

| $c_k^\text{(TPE)}$ | $c_k^\text{(OPE)}$ |
|-------------------|-------------------|
| $c_0$             | $c_0$             |
| $c_1$             | $c_1$             |
| $c_2$             | $c_2$             |
| $c_3$             | $c_3$             |

The coefficients $c_k^\text{(TPE)}$ are calculated using the $N\,\text{LO}^{(5)}\,\text{S}$ and $N\,\text{LO}^{(1)}\,\text{S}$ cases, respectively. These coefficients are listed in Table I.

In general, with the information contained in Table I, the $c_k^\text{(TPE)}$ coefficients are smaller than the $c_k^\text{(OPE)}$ coefficients. This is particularly true for the $c_0$ coefficients, which are smaller than the $c_0^\text{(OPE)}$ coefficients. The $c_k^\text{(TPE)}$ coefficients are therefore better than the $c_k^\text{(OPE)}$ coefficients, and the $N\,\text{LO}^{(5)}\,\text{S}$ case is a better approximation to the $N\,\text{LO}^{(1)}\,\text{S}$ case.

In conclusion, the $c_k^\text{(TPE)}$ coefficients are smaller than the $c_k^\text{(OPE)}$ coefficients, and the $N\,\text{LO}^{(5)}\,\text{S}$ case is a better approximation to the $N\,\text{LO}^{(1)}\,\text{S}$ case. This is consistent with the expectation that the $c_k^\text{(TPE)}$ coefficients are smaller than the $c_k^\text{(OPE)}$ coefficients, and the $N\,\text{LO}^{(5)}\,\text{S}$ case is a better approximation to the $N\,\text{LO}^{(1)}\,\text{S}$ case. The $c_k^\text{(TPE)}$ coefficients are therefore better than the $c_k^\text{(OPE)}$ coefficients, and the $N\,\text{LO}^{(5)}\,\text{S}$ case is a better approximation to the $N\,\text{LO}^{(1)}\,\text{S}$ case.

For coefficients that carry new information, their size tends to be $O(1)$, which is actually what we would expect if we have correctly identified the soft and hard scales of the problem. The size of the subleading corrections will be considerably smaller than the naive expectation coming from the power counting.
We can appreciate that $|c_2^{(\nu=4)}(\text{OPE})| \sim (2.1 - 2.3)$ and $|c_4^{(\nu=4)}(\text{OPE})| \sim (2.6 - 2.7)$ are somewhat large for $R_c = (0.1 - 0.2)$ fm, though they can be easily rendered of $\mathcal{O}(1)$ by simply setting $M' = (410 - 430)$ MeV (instead of the original $M' = 500$ MeV), which would be in line with our previous result in Eq. (B23). This does indeed represent a somewhat reduced expansion radius with respect to the OPE(TPE) expansion, but it is still very far away from the (200 - 300) MeV momentum range at which the NLO phase shifts fail in Fig. 5. It is also interesting to notice that $|c_2^{(\nu)}| > |c_2^{(3)}| > |c_2^{(2)}|$ for the singlet in EFT(OPE), which is concerning as this indicates corrections to the effective range that are growing (instead of shrinking) with each order.

However, there is the limitation that the inclusion of subleading TPE is in a sense a “unique” event that happens the moment we reach NLO. The truth is that the EFT(OPE) expansion is fairly convergent till NLO, and then suddenly deteriorates at the next order, a pattern which can be appreciated in Fig. 5. One might wonder how this actually impacts estimates of the breakdown scale that are based on an order-by-order comparison: if subleading TPE, i.e. the $Q^3$ piece of the effective finite-range potential, is the reason for the reduced convergence radius, we will only see its effects at NLO, N$^2$LO, N$^3$LO and so on (i.e. each time at which it is iterated again). Meanwhile, the $c_2^{(\nu)}$ coefficients in between the orders at which a new iteration of subleading TPE appears, e.g. $\nu = 5, 6, 7$, will most probably behave as expected. Indeed, it is perfectly possible that for a subset of the coefficients (the ones at $\nu = 4, 8, 12, \ldots$) we will end up with numbers of $\mathcal{O}(1)$. This is what actually happens to the coefficients in Table III where $|c_2^{(\nu=4)}| \sim (5 - 10) |c_2^{(\nu=3)}|$ for EFT(OPE). Thus, considerations of naturalness if applied in a naive manner might not be able to detect the suspected failure of perturbation theory for subleading TPE.

Finally, it is interesting to notice how the $c_2^{(\nu)}$ coefficients of EFT(TPE) have basically converged in the $R_c = (0.1 - 0.2)$ fm cutoff range, indicating a really smooth cutoff dependence. Indeed, from a comparison of Tables II and III it is apparent that the $c_2^{(\nu)}$ already converged at $R_c = 0.5$ fm for the EFT(TPE) singlet.

### 3. The non-polynomial contributions to the EFT expansion and their expected size

Here we analyze the scaling properties of the non-polynomial contributions to the dimensionless function $f^{(\nu)}$, which is basically the rescaled $\nu$-th order contribution to cot $\delta$, see Eqs. (B3) and (B11) for the relevant definitions. We begin by considering the perturbative integral

$$\int_0^\infty dr u_k^{(0)}(r) [V_F(r)u_k(r)]^{(\nu-1)}, \quad (B24)$$

which, after removing the polynomials and rescaling, generates $f_{\text{nonpol}}^{(\nu)}$. All the terms in $[V_Fu_k]^{(\nu-1)}$ are expected to behave similarly and thus we might simply focus on the easiest one to analyze, that is:

$$\int_0^\infty dr u_k^{(0)}(r) V_F^{(\nu-1)}(r)u_k^{(0)}(r). \quad (B25)$$

If we take into account that the finite-range potential can be written as

$$V^{(\nu)}(r) = \frac{2\pi}{\mu} \frac{1}{M^{\nu+1}} \frac{e^{-2mr}}{r^{\nu+3}} (1 + \mathcal{O}(Q_{gr})), \quad (B26)$$

which is equivalent to Eq. (4) after the substitution of $A_{NN}^{(\nu)}$ by $M$ (plus the assumption that its exponential de-
cay at long distances is driven by TPE), we arrive to
\[
\int_0^\infty dr \left[ u_k^{(0)}(r) \right]^2 V_k^{(\nu-1)}(r) \\
\sim 2\pi \frac{k}{\mu} \left( \frac{k}{M} \right)^\nu \int_0^\infty \frac{dx}{x^{\nu+2}} e^{-\frac{2\pi x}{m/k}} \left( 1 + \mathcal{O}(\frac{Q_k}{k}) \right) \left[ u_k^{(0)} \right]^2 ,
\]
where we have changed the integration variable to \( x = kr \) with the intention of extracting the power-law dependence with respect to the center-of-mass momentum \( k \).

If we now ignore the expansion in terms of \( \frac{Q_k}{k} \) of the finite-range potential, and concentrate in its most singular term, the non-polynomial part of the perturbative integral might be extracted after subtracting the first few terms in the \( k^2 \) expansion of the square function
\[
\int_0^\infty \frac{dx}{x^{\nu+2}} e^{-\frac{2\pi x}{m/k}} \left[ u_k^{(0)} - u_0^{(0)} - 2u_0^{(0)} u_2^{(0)} k^2 - \ldots \right] ,
\]
where for simplicity we have not applied the change of integration variable (from \( r \) to \( x = kr \)) to the wave function and its expansion. We notice here that, after making the subtractions, the perturbative integral is convergent even if we do not further regularize the EFT potential.

Yet, what matters is the general behavior of the integral in terms of \( 2m/k \) (and the order \( \nu \)). For analyzing it, we will momentarily make two simplifying assumptions: (i) that we are dealing with a pure S-wave and (ii) that the LO wave function can be approximated by its asymptotic form, i.e.,
\[
u_k^{(0)} \to \cot \delta^{(0)} \sin(x) - \cos(x) ,
\]
and (iii) that the part of the perturbative integral that dominates at momenta \( k > 2m \) is the one corresponding with the \( \cos x \) part of the wave function. This third condition is equivalent to making the substitution
\[
\left[ u_k^{(0)} \right]^2 \to \cos^2 x ,
\]
in the perturbative integrals, where subtractions can be trivially be taken into account by removing terms in the Taylor expansion of \( \cos^2 x \):
\[
\left( u_k^{(0)} - u_0^{(0)} - 2u_0^{(0)} u_2^{(0)} k^2 - \ldots \right) \to \left( \cos^2 x - 1 + x^2 - \ldots \right) .
\]

Under these conditions and defining the variable \( \lambda = 2m/k \), we might calculate the first term in the Taylor expansion of the perturbative integrals in terms of \( \lambda \). If \( \nu \) is even, with \( 2n = \nu + 2 \), we find for \( n = 1, 2, 3 \) (\( \nu = 0, 2, 4 \)):
\[
\int \frac{dx}{x^2} e^{-\lambda x} \left( \cos^2 x - 1 \right) \\
= -\frac{\pi}{2} + \mathcal{O}(\lambda, \lambda \log \lambda) ,
\]
\[
\int \frac{dx}{x^4} e^{-\lambda x} \left( \cos^2 x - 1 + x^2 \right) \\
= +\frac{\pi}{3} + \mathcal{O}(\lambda, \lambda \log \lambda) ,
\]
\[
\int \frac{dx}{x^6} e^{-\lambda x} \left( \cos^2 x - 1 + x^2 - \frac{x^4}{3} \right) \\
= -\frac{\pi}{15} + \mathcal{O}(\lambda, \lambda \log \lambda) ,
\]
where the general rule for arbitrary even powers is
\[
\int \frac{dx}{x^{2n}} e^{-\lambda x} \left( \cos^2 x - 1 - \sum_{k=1}^{n-1} \frac{(-1)^k k^2}{2(2k)!} x^{2k} \right) \\
= \left( -1 \right)^{n-1} \pi \frac{\log \lambda}{2(2n-1)!} + \mathcal{O}(\lambda, \lambda \log \lambda) .
\]
It is interesting to notice that when we plug this result into Eqs. [B11] and [B27], we generate odd powers of \( k/M \), which cannot originate from a contact-range potential.

If \( \nu \) is odd, with \( 2n + 1 = \nu + 2 \), for \( n = 1, 2, 3 \) (\( \nu = 1, 3, 5 \)) we find
\[
\int \frac{dx}{x^3} e^{-\lambda x} \left( \cos^2 x - 1 + x^2 \right) \\
= - \log \lambda + \mathcal{O}(\lambda, \lambda \log \lambda) ,
\]
\[
\int \frac{dx}{x^5} e^{-\lambda x} \left( \cos^2 x - 1 + x^2 - \frac{x^4}{3} \right) \\
= +\frac{1}{3} \log \lambda + \mathcal{O}(\lambda, \lambda \log \lambda) ,
\]
\[
\int \frac{dx}{x^7} e^{-\lambda x} \left( \cos^2 x - 1 + x^2 - \frac{x^4}{3} + \frac{2x^6}{45} \right) \\
= -\frac{2}{45} \log \lambda + \mathcal{O}(\lambda, \lambda \log \lambda) ,
\]
where the general rule for arbitrary \( n \) is
\[
\int \frac{dx}{x^{2n+1}} e^{-\lambda x} \left( \cos^2 x - 1 - \sum_{k=1}^{n} \frac{(-1)^k k^2}{2(2k)!} x^{2k} \right) \\
= \left( -1 \right)^n \pi \frac{\log \lambda}{2(2n)!} + \mathcal{O}(\lambda, \lambda \log \lambda) .
\]
By plugging the results above into Eqs. [B11] and [B27], we generate even powers of \( k/M \), but multiplied by \( \log(k/2m) \), which again indicate that we are not dealing with a contact-range potential.

Even though the previous analysis is rudimentary in nature, it paints a picture in which the non-polynomial part of the perturbative integral scales as \( \mathcal{O}(1/\nu!) \) as the order \( \nu \) of the EFT expansion is increased. If this were to
be the case, and putting the pieces that we have together, for $\nu$ even we find

$$f^{(\nu)}_{\text{nonpol}}(\frac{Q}{Q_k}) \sim 2\pi \frac{M}{\mu} \left( \frac{k}{Q_k} \right)^{\nu+1} \frac{(-1)^{\nu/2} 2^{\nu}}{\nu!} \left( \log \left( \frac{Q_k}{Q} \right) \right)^2 \times \left[ 1 + \mathcal{O} \left( \frac{Q_k}{Q} \log \left( \frac{Q_k}{k} \right) \right) \right] \sim \pi^2 \frac{2^{\nu}}{(\nu+1)!},$$

(B38)

while for $\nu$ odd we arrive at

$$f^{(\nu)}_{\text{nonpol}}(\frac{Q}{Q_k}) \sim 2\pi \frac{M}{\mu} \left( \frac{k}{Q_k} \right)^{\nu+1} \frac{(-1)^{(\nu+1)/2} 2^{\nu}}{\nu!} \log \left( \frac{Q_k}{k} \right) \times \left[ 1 + \mathcal{O} \left( \frac{Q_k}{Q} \log \left( \frac{Q_k}{k} \right) \right) \right] \sim \pi^2 \frac{2^{\nu}}{(\nu+1)!}. \quad (B39)$$

Of course a more complete analysis should evaluate the size of the additional terms in the potential as well as terms stemming from higher order perturbation theory. Yet, provided that the number of these terms do not scale as $\nu!$, the conclusion is that $f^{(\nu)}_{\text{nonpol}} \sim O(2^\nu \pi/\nu!)$. That is, the non-polynomial pieces tend to generate contributions that are convergent within the EFT expansion and thus they do not provide information about the radius of convergence of the EFT.

4. Comparison with the usual Bayesian analysis

Here we briefly discuss how to incorporate the previous analysis about the scaling properties of perturbative amplitudes into Bayesian techniques, as well as the potential differences with the more standard approaches currently used in the literature [43, 54].

Usually, the starting point of most Bayesian analyses is a series in the form [47]

$$y_k(x) = y_{\text{ref}} \sum_{n=0}^{k} c_n(x) Q^n(x),$$

(B40)

where $y_k(x)$ is the observable quantity we are interested in, $y_{\text{ref}}$ a reference value for this quantity (which serves the purpose of removing the dimensions), $c_n(x)$ a dimensionless coefficient, $Q(x)$ the expansion parameter and $k$ the highest order we are considering in the expansion. The problem we might want to solve could be for instance how to make a reliable estimation of the breakdown scale (which is related to the expansion parameter $Q(x)$) given the assumption that the $c_n(x)$ coefficients are natural, though the application of Bayesian techniques is obviously not limited to this example. Here what we will discuss instead is how to improve over the previous ansatz for the EFT expansion.

If we now apply Eq. (B40) to the phase shift:

$$\delta_{\text{N}}^{\text{max} \, \text{LO}}(k) = \delta_{\text{ref}} \sum_{\nu=0}^{\nu_{\text{max}}} \hat{\delta}^{(\nu)}(k) \left( \frac{Q}{M} \right)^{\nu}, \quad (B41)$$

where $\delta_{\text{ref}}$ is a reference value of the phase shift and $\hat{\delta}^{(\nu)}$ a rescaled phase shift in which we have removed the $(Q/M)$ factors. $Q$ here is usually interpreted as a function of the light scales in the system, $k$ and $m_\pi$, e.g. $(k^{n} + m^2_\pi)/(k^{(n-1)} + m^2_{\pi})$ with $n$ an integer number [48].

Writing the EFT expansion in the previous way is completely correct. Yet, we might instead try to exploit the scaling properties we have derived from analyzing the subleading contributions. If we use again the phase shift as the example, in a first approximation we would obtain

$$\delta_{\text{N}}^{\text{max} \, \text{LO}}(k) = \delta^{(0)} + \frac{2\mu}{k} \sum_{\nu=1}^{\nu_{\text{max}}} g^{(\nu)}(k) \left( \frac{k}{Q_k} \right)^{\nu+1},$$

(B42)

which can be further refined by taking into account the existence of polynomial and non-polynomial terms

$$\delta_{\text{N}}^{\text{max} \, \text{LO}}(k) = \delta^{(0)} + \frac{2\mu}{k} \sum_{\nu=1}^{\nu_{\text{max}}} g^{(\nu)}(k) \left( \frac{k}{Q_k} \right)^{\nu+1} \times \left[ \sum_{n=0}^{c_{\text{max}}^{(\nu)}} c^{(\nu)}_{2n} \left( \frac{k}{Q_k} \right)^{2n} \right] \left( \frac{Q}{M} \right)^{\nu+1}, \quad (B43)$$

where the sum of $g^{(\nu)}_{\text{nonpol}}$ contributions is suspected to be always convergent (or at least their size is not of $O(1)$), and hence they might be ignored. With this new expansion, the problem is now reduced to determining the probability of the breakdown scale $M$ given the assumption that the $c_{2n}^{(\nu)}$ coefficients obtained from this expansion are of natural size.

The main advantage of using the expansion derived from the scaling properties of the EFT amplitudes, i.e. Eq. (B43), is that we do not need the ad-hoc interpolators for $(Q/M)^\nu$ that we would need in the naive expansion of Eq. (B41). The rationale for interpolators such as $Q \to (k^n + m^2_\pi)/(k^{(n-1)} + m^2_{\pi})$ is that we are not sure what is the actual role played by the momentum and the pion mass (or other light scales we might want to add) in the EFT expansion. The analysis of Appendix B1 which resulted in Eq. (B43), effectively solves this problem. Yet, there is an important disadvantage though: with Eq. (B43) we now have an integer (and small) number of coefficients, instead of coefficients on a continuous variable (e.g. the momentum). This somewhat limits the applicability of Bayesian statistics, as there is a smaller set of $O(1)$ quantities from which to extract information about the breakdown scale.
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