Tokamak MHD equilibria with reversed magnetic shear and sheared flow

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Abstract

Analytic solutions of the magnetohydrodynamic equilibrium equations for a cylindrically symmetric magnetically confined plasma with reversed magnetic shear, \( s < 0 \), and sheared flow are constructed by prescribing the safety factor-, poloidal velocity- and axial velocity- profiles consistently with experimental ones. On the basis of the solutions obtained in most of the cases considered it turns out that an increase of \( |s| \) and of the velocity components result in larger absolute values for the radial electric field, \( E_r \), its shear, \( |dE_r/dr| \equiv |E'_r| \), and the \( E \times B \) velocity shear, \( \omega_{E \times B} = |d/dr(E \times B/B^2)| \), which may play a role in the formation of Internal Transport Barriers (ITBs) in tokamaks. In particular for a constant axial magnetic field, \( \omega_{E \times B} \) at the point where \( E'_r = 0 \) is proportional to \( 1 - s \). Also, \( |E'_r| \) and \( \omega_{E \times B} \) increase as the velocity shear takes larger values. The results clearly indicate that \( s < 0 \) and sheared flow act synergetically in the formation of ITBs with the impact of the flow, in particular the poloidal one, being stronger than that of \( s < 0 \).
1. Introduction

Understanding Internal Transport Barriers (ITBs) in plasmas is very important for the advanced tokamak scenarios [2], [3]. The ITBs usually are associated with reversed magnetic shear profiles [4], [5] and their main characteristics are steep pressure profiles in the barrier region [6] and radial electric fields associated with sheared flows [7], [8]. The mechanism responsible for the formation of ITBs is far from completely understood. It is believed that the flow, the radial electric field, its shear and the \( \omega_{E \times B} \) velocity shear,

\[
\omega_{E \times B} = \left| \frac{d}{dr} \frac{E \times B}{B^2} \right|
\]

play a role in the barrier formation by mode decorrelation thus resulting in a reduction of the outward particle and energy transport [3], [9], [10].

The experimental evidence up to date has not made clear whether the reversed magnetic shear, \( s < 0 \), or the sheared flow (toroidal or poloidal) are more important for the ITBs formation. In some experiments the safety factor profile is considered as the crucial quantity (e.g. [11]) while according to others the necessity of reversed magnetic shear is questionable (e.g. [9]). On the other hand, the flow—either toroidal [12] or poloidal [13], [14]—may be important in the formation of ITBs. Also, it has been argued that the toroidal velocity may be more important than the poloidal one (see for example Ref. [13]). It should be noted, however, that only few direct measurements of the poloidal velocity have been performed; this velocity is usually calculated by means of neoclassical theory [12].

The aim of the present work is to contribute to the answer of the above mentioned open questions by studying magnetohydrodynamic (MHD) cylindrical equilibria with reversed magnetic shear and sheared flow. The study can be viewed as an extension of a previous one on tokamak equilibria with incompressible sheared flows and monotonically increasing q-profiles in connection with certain characteristics of the L-H transition [15]. The work is conducted through the following steps: The profiles of certain free quantities, including the safety factor and the velocity components are first prescribed and then exact equilibrium solutions are constructed self consistently. This is the subject of Sec. 2. In Sec. 3 on the basis of the solutions obtained the equilibrium properties are examined and the impact of \( s < 0 \) and the flow on \( E_r, E'_r \) and \( \omega_{E \times B} \) is evaluated. The conclusions are summarized in Sec. 4.

2. Cylindrical equilibria with reversed magnetic shear

The equilibrium of a cylindrical plasma with flow satisfies (in convenient units) the relation

\[
\frac{d}{dr} \left( P + \frac{B_\theta^2 + B_z^2}{2} \right) + (1 - M_\theta^2) \frac{B_\theta^2}{r} = 0
\]

stemming from the radial component of the force-balance equation \( \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla P \) with the aid of Ampère’s law. Here, \( P \) is the plasma pressure; \( B_\theta \) and \( B_z \) are the poloidal and axial components of the magnetic field, respectively; \( M_\theta^2 = (v_\theta^2/\rho)/B_\theta^2 \) is the square of the Mach number defined as the ratio of the poloidal velocity to the poloidal-magnetic-field Alfvén velocity. Because of the symmetry any equilibrium quantity depends only on the radial distance \( r \) and the axial velocity \( v_z \) as well as the velocity shear do not appear
in (2); also, the flow is incompressible. In addition to $v_z$ four out of the five quantities in (2) can be prescribed.

On account of typical experimental ITB profiles we prescribed the quantities $q$, $B_z$, $v_\theta$, $v_z$ and $\varrho$ as follows:

strongly reversed shear profile (SRS) (Fig. 1)

$$ q(\rho) = q_c \left( 1 - \frac{3\Delta q}{q_c} \frac{r_0^2}{r_{\text{min}}^2} \rho^2 + \frac{2\Delta q}{q_c} \frac{r_0^3}{r_{\text{min}}^3} \rho^3 \right) $$ (3)

or alternatively weakly reversed shear profile (WRS)

$$ q(\rho) = q_c \left( 1 - \frac{2\Delta q}{q_c} \frac{r_0}{r_{\text{min}}} \rho + \frac{\Delta q}{q_c} \frac{r_0^2}{r_{\text{min}}^2} \rho^2 \right) $$ (4)

where $\rho = r/R_0$ with $r_0$ defining the plasma surface, $q_c = q(r = 0)$, $r_{\text{min}}$ is the position of minimum $q$, and $\Delta q = q_c - q_{\text{min}}$. The SRS-profile (3) does exhibit a maximum at the plasma center $r = 0$ in addition to the minimum one at $r = r_{\text{min}}$ and has stronger magnetic shear in the central region just inside the $q_{\text{min}}$ position than that of the WRS one. It should be clarified, however, that the WRS profile (4), which does not have an extremum on the magnetic axis $r = 0$, has been chosen in order to simplify the calculations though the physical situation may not be well represented in the immediate vicinity of the magnetic axis;

$$ B_z = B_{z0} \left[ 1 + \delta (1 - \rho^2) \right]^{1/2} $$ (5)

where $B_{z0}$ is the vacuum magnetic field and the parameter $\delta$ is related to the magnetic properties of the plasma, i.e. for $\delta < 0$ the plasma is diamagnetic;

Gaussian-like poloidal velocity profile

$$ v_\theta = 4v_{\theta 0} \rho (1 - \rho) \exp \left[ -\frac{(\rho - \rho_{\text{min}})^2}{h} \right] $$ (6)

where the parameter $h$ determines its broadness and $v_{\theta 0}$ is the maximum of $v_\theta$; either peaked axial velocity profile

$$ v_z = v_{z0} (1 - \rho^3)^3 $$ (7)

or Gaussian-like $v_z$ profile similar to that of (6); and the density profile

$$ \varrho = \varrho_0 (1 - \rho^3)^3. $$ (8)

The following quantities can then be calculated: the poloidal magnetic field $B_\theta = \epsilon \rho B_z/q$ where $\epsilon = r_0/R_0$ is the inverse aspect ratio with $2\pi R_0$ associated with the length of the plasma column; the magnetic shear $s = (r/q)(dq/dr)$; the current density via Ampere’s law; the electric field via Ohm’s law; its shear $E_r'$ and $\omega_E \times B$ by (11). Also, integration of (2) so that $P(r = r_0) = 0$ yields the pressure. The calculations have been performed analytically by developing a programme for symbolic computations [16] in connection with Ref. [17]. This also allowed us to examine conveniently purely poloidal flows, purely axial flows, $z$-pinch configurations or $\theta$-pinch configurations as particular cases. The analytic expressions which can be derived readily by the programme are generally lengthy and will not be given explicitly here. Some concise and instructive expressions will only be presented in the next section along with typical profiles for the calculated quantities supporting the results obtained.
3. Results

We have set the following values for some of the parameters: \( B_{z0} = 1 \) Tesla, \( q_0 = 8.35 \times 10^{-8} \text{kgr/m}^3 \) corresponding to \( n_0 = 5 \times 10^{19} \) particles/m\(^3\), \( \rho_{\text{min}} = 0.5, \epsilon = r_0/R_0 \approx 1/3, \delta = -0.0975 \), \( q_{\text{min}} = 2 \), \( \max v_\theta = 1 \times 10^4 \) m/sec and \( \max v_z = 1 \times 10^5 \) m/sec; Consequently, it is guaranteed that \( M_\theta^2 \approx M_z^2 \), where \( M_z^2 = (v_z^2)/B_z^2 \), a scaling typical in tokamaks because \( B_z \approx 10 B_\theta \) and \( v_z \approx 10 v_\theta \). It is noted here that since in tokamaks \( M_\theta < 0.1 \) the flow term in (2) is perturbative around the “static” equilibrium \( M_\theta = 0 \). Also, the choice \( q_{\text{min}} = 2 \) was made because according to experimental evidence for \( q_{\text{min}} < 2 \) strong MHD activity destroys confinement possibly due to a double tearing mode \([18]\). A similar result was found numerically for one-dimensional cylindrical equilibria with hollow currents in Ref. [19]. The impact of the magnetic shear and flow on the equilibrium, in particular on the quantities \( E_r, E_\theta \) and \( \omega_{E\times B} \), was examined by varying the parameters \( q_c, \Delta q, h, v_{z0}, \) and \( v_{\theta0} \) [Eqs. (3), (4), (6) and (7)].

For reversed magnetic shear profiles we came to the following conclusions:

1. Pressure

Substitution of \( B_\theta \) and its derivative in terms of \( q \) and \( s \) in (2) yields

\[
P' = -B_zB'_z \left[ 1 + \left( \frac{r_0}{R_0} \right)^2 \right] + r_0 \rho \left[ M_\theta^2 + (s - 2) \right] \left( \frac{B_z}{R_0q} \right)^2 \, .
\]

(9)

For \( s < 0 \), increase of \( |s| \) makes the pressure profile steeper (see also Fig. 2). Equation (9) also implies that the pressure profile becomes steeper when the plasma is more diamagnetic, i.e. when \( B'_z \) related to the parameter \( \delta \) in (5) takes larger values.

2. Current density

- The axial current density profile becomes hollow and, irrespective of the reversal of the magnetic shear, there is a critical distance \( \rho_{\text{cr}} \) outside the \( q_{\text{min}} \) position at which \( J_z \) becomes negative (Fig. 3). In particular, for \( B_z = B_{z0} = \) const. one obtains

\[
J_z = \frac{1}{r} \frac{d}{dr} (r B_\theta) = \frac{B_{z0}}{R_0q} (2 - s)
\]

(10)

Consequently, for \( s > 2 \), \( J_z \) reverses. The radial distances at which \( J_z = 0 \) for the SRS [Eq. (3)] and the WRS [Eq. (4)] \( q \)-profiles, respectively, are

\[
\rho_{\text{cr}}^{\text{SRS}} = \rho_{\text{min}} \left( \frac{q_c}{\Delta q} \right)^{1/3}
\]

and

\[
\rho_{\text{cr}}^{\text{WRS}} = \rho_{\text{min}} \frac{q_c}{\Delta q}.
\]

Therefore, the position of \( \rho_{\text{cr}} \) is shifted towards the center as \( s \) takes lower negative values. It is noted here that equilibrium toroidal current density reversal for monotonically increasing \( q \)-profiles was reported in Ref. [20] (Fig. 3 therein).
• Very large values of $\Delta q$ on the order of $10^2$ result in the formation of $j_z$ profiles with "holes" in the central region, $j_z \approx 0$ inside the $\rho_{min}$ position as demonstrated in Fig. 4, a result consistent with experimental evidence ([21], [22]).

• The total axial current $I_z = 2\pi r_0 B_\theta(r_0)$ for SRS profiles is smaller than that for WRS profiles.

3. $E_r$ and $E'_r$

• Typical $E_r$ profiles exhibit an extremum in the region around $q_{min}$ and vanish at $\rho = 0$ and $\rho = 1$ in agreement with experimental ones [7], [23]. Profiles with more than one extrema are also possible in the case of peaked $v_z$ profiles, localized $v_\theta$ ones and $v_z v_\theta > 0$ as demonstrated in Fig. 5. Experimental profiles of this kind were reported in Ref. [23] (Fig. (9) therein).

• The main contribution to $E_r$ comes from the velocity, to which is proportional, and particularly from the poloidal one (Fig. 6).

• $E_r$ is sensitive to the relative orientation of $v_z$, $v_\theta$ and $B_z$; in particular, for $v_z v_\theta < 0$ $|E_r|$ is larger than that for $v_z v_\theta > 0$. (Fig. 7). Similar results hold for $E'_r$ (Fig. 8) and $\omega_{E \times B}$.

• For extended velocity profiles with $v_z \neq 0$, an increase of $|s|$ results in an increase of $|E_r|$ (Fig. 9), $|E'_r|$ and $\omega_{E \times B}$. If $v_z = 0$, however, $|s|$ has no impact on $|E_r|$ and $|E'_r|$, as can be seen by inspection of $E = v \times B$, and very weak impact on $\omega_{E \times B}$. This result indicates that the presence of $v_z$ "activates" the impact of $s$ on $E_r$, $E'_r$ and $\omega_{E \times B}$.

• An increase of the velocity shear nearly does not affect or even decreases the maximum $|E_r|$ (Fig. 10) but increases $|E'_r|$ (Fig. 11).

4. $\omega_{E \times B}$

• A typical profile of $\omega_{E \times B}$ has two large local maxima at the positions where the edges of the barrier are expected to be located in addition to other two smaller local ones (Fig. 12). In most of the cases considered the maximum in the $s < 0$ region is slightly larger than that in the $s > 0$ region. (see Fig. 12). In particular, for $B_z = \text{const.}$ at the point where $E'_r = 0$ one obtains:

$$\omega_{E \times B} = \left| \frac{(1 - s)(e^{q v_z q} - v_\theta)}{R_0 q \left[ 1 + (e^{q v_z q})^2 \right]} \right|$$

Eq. (11) implies the following:

1. $\omega_{E \times B}$ depends on the relative sign of $v_z$, $v_\theta$ and $B_z$, a result which we confirmed by $\omega_{E \times B}$ profiles obtained via the symbolic computation programme.

2. The factor $(1 - s)$ indicates that $\omega_{E \times B}$ for nearly shearless stellarator equilibria may be lower than that for tokamak equilibria with $s < 0$. 

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3. Despite the scaling $v_z \approx 10v_\theta$, for tokamak pertinent parametric values the contributions of $v_z$ in connection with the term $\epsilon \rho v_z/q$- and $v_\theta$ to $\omega_{E\times B}$ are of the same order of magnitude, a result indicating the importance of the poloidal velocity.

- For extended velocities (large values of the parameter $h$ or/and peaked $v_z$- profile) a percentage increase of $|s|$ in the barrier region results:
  1. in approximately the same percentage increase of $\omega_{E\times B}$ if the velocity is purely axial (Fig. 13).
  2. nearly does not affect the value of $\omega_{E\times B}$ if $v_\theta \neq 0$.

- An increase of the flow shear (variation of the parameter $h$ from 0.1 to 0.001) causes a mean percentage increase of $\omega_{E\times B}$ as large as 0.7 of that of the flow shear. (Fig. 14).

- The impact of a variation of the $v_\theta$-shear on $\omega_{E\times B}$ is stronger than of the same variation of the $v_z$-shear.

- The maximum increase of $\omega_{E\times B}$ is caused by $v_\theta$ in the case of non-vanishing peaked $v_z$ profiles.

- Inspection of $v_{E\times B} = E \times B/B^2$ and $|\mathbf{B}|$ implies that $\omega_{E\times B}$ for a $z$-pinch is equal to that for an equilibrium with purely axial flow. The same equality is valid for a $\theta$-pinch in comparison with an equilibrium with purely poloidal flow. In addition, it holds that

$$\omega'_{E\times B-z\text{-pinch}} \approx 10\omega'_{E\times B-\theta\text{-pinch}}. \quad (12)$$

4. Conclusions

The self consistent study of cylindrical equilibria with reversed magnetic shear and sheared flow presented in the previous sections led to the following conclusions:

1. For reversed magnetic shear profiles ($s < 0$):
   - The larger values of $|s|$ the steeper the pressure profile.
   - The axial current density profile become hollow.
   - Strong reversed shear profiles formed by appropriately large values of $\Delta q$ are associated with "hole" axial current density profiles.

   These results are consistent with experimental ones.

2. Irrespective of the sign of $s$ the axial current density can reverse in the outer plasma region, the reversal point being shifted towards the plasma core as $s$ takes lower negative values.

3. An increase of either $|s|$ or the velocity results generally in an increase of $|E_r|$, $|E'_r|$ and $\omega_{E\times B}$.

4. An increase of the velocity shear results in an increase of $|E'_r|$ and $\omega_{E\times B}$. 
5. For a given value of $|s|$, $\omega_{E \times B}$ takes slightly larger values in the $s < 0$ region than in the $s > 0$ region.

6. $E_r$, $E'_r$ and $\omega_{E \times B}$ are sensitive to the relative orientation of $v_\theta$, $v_z$ and $B_z$. In particular, they take larger values for $v_z v_\theta < 0$ rather than for $v_z v_\theta > 0$.

7. The presence of $v_z$ activates $s < 0$, in the sense that for $v_z = 0$, $E_r$ and $E'_r$ are $s$-independent. Also, for $v_z = 0$, $s < 0$ has very weak impact on $\omega_{E \times B}$.

8. The impact of the poloidal flow and its shear on $E_r$, $E'_r$ and $\omega_{E \times B}$ is stronger than that of the axial flow and the magnetic shear.

Presuming that $E_r$, $E'_r$ and $\omega_{E \times B}$ are of relevance to the ITBs formation, the above results clearly indicate that the reversed magnetic shear and the sheared flow have synergistic effects on this formation with the flow, in particular the poloidal one, and its shear playing an important role.
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Figure captions

Fig. 1: SRS and WRS safety factor profiles associated with Eqs. (3) and (4), respectively. It is noted that the finite slope of the WRS curve at $\rho = 0$ may not represent well the physical situation in the immediate vicinity of the magnetic axis.

Fig. 2: WRS pressure profiles for $\Delta q = 4$ and $\Delta q = 14$.

Fig. 3: Toroidal current density profiles for $\Delta q = 4$. It is noted that the finite slope of the WRS curve at $\rho = 0$ may not represent well the physical situation in the immediate vicinity of the magnetic axis.

Fig. 4: Toroidal current density profile for WRS, $q_c = 102$ and $\Delta q = 100$ that demonstrates the current "hole" in the core region.

Fig. 5: Electric field profile for WRS with $v_z$ peaked and $v_\theta$ localized having three local extrema.

Fig. 6: Two $E_r$-profiles the one with $v_z = 0$ and the other with $v_\theta = 0$ for SRS and Gaussian-like velocity profiles. $E_r$ is normalized with respect to its value at $\rho = 0.5$ for $v_z = 0$.

Fig. 7: Two $E_r$-profiles for $v_z$ peaked and SRS, the one with $v_\theta \cdot v_z > 0$ and the other with $v_\theta \cdot v_z < 0$. The profiles are normalized with respect to the first case at $\rho = 0.5$.

Fig. 8: Two profiles of $E'_r$ with $v_z$ peaked and SRS, the one with $v_\theta \cdot v_z > 0$ and the other with $v_\theta \cdot v_z < 0$. The profiles are normalized with respect to the first case at $\rho = 0.3$.

Fig. 9: Profiles of $E_r$ with peaked axial and extended poloidal velocities for WRS and two different values of $\Delta q$. The profiles are normalized with respect to the case with $\Delta q = 4$ at $\rho = 0.5$.

Fig. 10: Profiles of $E_r$ with $v_z = 0$ for SRS and either extended ($h = 0.1$) or localized ($h = 0.001$) poloidal velocity. The profiles are normalized with respect to the first case at $\rho = 0.5$.

Fig. 11: Two profiles of $E'_r$ for SRS with $v_z = 0$ the one for extended ($h = 0.1$) and the other for localized ($h = 0.001$) poloidal velocities. The profiles are normalized with respect to the second case at $\rho = 0.55$.

Fig. 12: Typical $\omega_{E\times B}$-profile for WRS, peaked axial and localized poloidal velocities.

Fig. 13: Profiles of $\omega_{E\times B}$ for WRS, peaked axial velocity, and either $\Delta q = 4$ or $\Delta q = 14$. The profiles are normalized with respect to the first case at $\rho = 0.45$.

Fig. 14: $\omega_{E\times B}$-profile for SRS, Gaussian-like axial and poloidal velocity components both either extended ($h = 0.1$) or localized ($h = 0.001$). The profiles are normalized with respect to the first case at $\rho = 0.3$. 
Figure 1:

Figure 2:

Figure 3:
Figure 4:

\[ \frac{J_z(\rho)}{J_z(0)} \]

Figure 5:

\[ \frac{E_r(\rho)}{E_r(0.5)} \]

Figure 6:

\[ |E_{r-norm}(\rho)| \]

\[ v_\theta = 0 - \]
\[ v_z = 0 \ldots \]
\[ |E_{r-norm}(\rho)| \]

\[ v_\theta \cdot v_z > 0 \quad - \quad v_\theta \cdot v_z < 0 \quad \cdots \]

Figure 7:

\[ E'_{r-norm}(\rho) \]

\[ v_\theta \cdot v_z > 0 \quad - \quad v_\theta \cdot v_z < 0 \quad \cdots \]

Figure 8:

\[ |E_{r-norm}(\rho)| \]

\[ \Delta q = 4 \quad - \quad \Delta q = 14 \quad \cdots \]

Figure 9:
Figure 10:

\[ |E_{r-norm}(\rho)| \]

\[ h = 0.1 - h = 0.001 \cdots \]

Figure 11:

\[ E'_{r-norm}(\rho) \]

\[ h = 0.1 - h = 0.001 \cdots \]

Figure 12:

\[ \frac{\omega_{E \times B}(\rho)}{\omega_{E \times B}(0.45)} \]
Figure 13:

\[ \omega_{E \times B - \text{norm}}(\rho) \]

\[ \Delta q = 4 - \Delta q = 14 \cdots \]

Figure 14:

\[ \omega_{E \times B - \text{norm}}(\rho) \]

\[ h = 0.1 - h = 0.001 \cdots \]