Doped orbital Chern insulator, Chern Fermi pockets, and chiral topological pair density wave in kagomé superconductors

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The recent discovery of novel charge density wave (CDW) \textsuperscript{[1,2]} and pair density wave (PDW) \textsuperscript{[3]} in vanadium-based superconductors (SCs) \textit{AV}\textsubscript{3}Sb\textsubscript{5} (A = K, Rb, Cs) \textsuperscript{[4,5]} hints at new correlated and topological quantum states of itinerant electrons on the kagomé lattice. Here, we present a physical mechanism behind the remarkable phenomenology of the kagomé SCs and a new route toward topological superconductivity. We demonstrate that the $2a_0 \times 2a_0$ complex CDW state with circulating currents near the van Hove filling is a doped orbital Chern insulator with emergent Chern Fermi surface pockets. The Chern Fermi pockets, of sizes comparable to those detected by quantum oscillations, carry large Berry curvature and orbital magnetic moment, and account for the observed intrinsic anomalous Hall effect. Intriguingly, the Chern Fermi pockets are connected by $\tfrac{1}{2}Q_{\text{ breathing}}$ in the 3Q directions and produce $\tfrac{1}{2}a_0 \times \tfrac{1}{2}a_0$ PDWs as observed in the superconducting and pseudogap phases \textsuperscript{[3]}. The incipient PDW is a never-before-encountered chiral topological superconductor protected by an intertwined uniform superconductivity due to the hexagonal symmetry. A plethora of correlated and topological states emerge, including Pottsnematic, fractional vortices, and macroscopic phase coherent condensate of four- and six-electron bound states. Our findings reveal a new orbital-based mechanism and a prominent kagomé materials platform for intrinsic chiral topological SCs, fractional vortices, and charge-4e and 6e superconductivity.

The field of transition-metal kagomé lattice materials has leapt forward with the recent discovery of superconductivity in a new family of vanadium-based kagomé metals \textit{AV}\textsubscript{3}Sb\textsubscript{5} (A = K, Rb, Cs) \textsuperscript{[4,5]}. In contrast to the insulating kagomé compounds extensively studied for quantum spin liquids and doped Mott insulators \textsuperscript{[6–8]}, \textit{AV}\textsubscript{3}Sb\textsubscript{5} are nonmagnetic correlated metals with itinerant electrons traversing the unique kagomé lattice structure that geometrically frustrates kinetic motion due to quantum interference. They are complementary to the (Fe, Co, Mn)-based itinerant kagomé magnets that exhibit rich correlated topological phenomena \textsuperscript{[9–14]}, but have remained nonsuperconducting at low temperatures.

All \textit{AV}\textsubscript{3}Sb\textsubscript{5} undergo charge density wave (CDW) \textsuperscript{[1,2]} and pair density wave (PDW) \textsuperscript{[3]} transitions below $T_{\text{cdw}} \sim 78 – 103$ K and superconducting (SC) \textsuperscript{[4]} transitions below $T_c \sim 0.9 – 2.5$ K. Both the diagonal and the off-diagonal long-range ordered states turn out to be highly intriguing and unconventional. While charge order has been confirmed to be a $2a_0 \times 2a_0$ 3Q-CDW in the kagomé plane stacked along the $c$-axis \textsuperscript{[1,3,13–18]}, it surprisingly produces giant anomalous Hall effect (AHE) despite the absence of magnetism \textsuperscript{[13,20]}. Scanning tunneling microscopy (STM) imaging of KV\textsubscript{3}Sb\textsubscript{5} observed that the CDW responds to an applied magnetic field differently when the field direction is reversed along the $c$-axis \textsuperscript{[11]}, suggesting time-reversal symmetry (TRS) breaking. Although the robustness of the STM observation is controversial \textsuperscript{[21,22]}, evidence for TRS breaking has been detected in $\mu$SR \textsuperscript{[23,24]} and optical Kerr rotation \textsuperscript{[25]} experiments. At low temperatures, STM experiments in CsV\textsubscript{3}Sb\textsubscript{5} reveal the coexistence of strong-coupling superconductivity with $2a_0 \times 2a_0$ CDW and $4a_0$ unidirectional charge order \textsuperscript{[2,5]}. Remarkably, a novel pair density wave (PDW) with $\tfrac{1}{2}a_0 \times \tfrac{1}{2}a_0$ period was discovered that spatially modulates the SC gap and coherence peaks \textsuperscript{[3]}. Moreover, the phenomenology of the PDW is striking compared to that in high-$T_c$ cuprates: it is detectable in the vortex core and above $H_{c2}$ emerges as a “mother state” above $T_c$, and is responsible for the observed pseudogap behavior. Here, we find that these highly unusual properties are captured by the physics of doped orbital Chern insulator on the kagomé lattice, and the emergent Chern Fermi surface pockets provide a new mechanism for an unprecedented chiral topological PDW and other intriguing quantum states arising from crystalline symmetry breaking superconductors.

Breathing kagomé lattice

In the AV\textsubscript{3}Sb\textsubscript{5} compounds, V atoms form an ideal kagomé lattice coordinated by Sb atoms and the alkali atoms (A) intercalate between the kagomé layers. Very similar band structures are predicted across the series by the density functional theory (DFT) \textsuperscript{[5,16]}. There is an electron-like Fermi surface (FS) around the center of the hexagonal Brillouin zone (BZ) derived from the Sb $p_z$ orbital, while the actions of the low-energy vanadium $d$-orbitals are located around the zone boundary, forming quasi-2D FS sheets close to the van Hove (vH) singularities. The band dispersions measured by angle-resolved photoemission spectroscopy (ARPES) show overall agreement with the DFT predictions \textsuperscript{[4,26–30]}.

To capture the most essential physics of a $d$ band crossing the Fermi level near the vH point, we consider the one-orbital model on the kagomé lattice as depicted in Fig. 1a with the lattice constant $a_0 \equiv 1$. The two basis vectors are $a_1 = (1, 0)$ and $a_2 = (-\tfrac{1}{4}, \tfrac{\sqrt{3}}{4})$ and the reciprocal lattice vectors $b_1 = \tfrac{4\pi}{\sqrt{3}}(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2})$ and $b_2 = \tfrac{4\pi}{\sqrt{3}}(0, -1)$. The third direction follows $a_3 = -a_1 - a_2$ and $b_3 = -b_1 - b_2$. The location of the three sublattices in a unit cell at $r$ is given by $r_1 = r - \frac{1}{2}a_3$, $r_2 = r$, and $r_3 = r + \frac{1}{2}a_1$. The kagomé lattice is composed...
FIG. 1: **Itinerant fermions on kagomé lattice.** a, Lattice structure. The three sublattices are denoted by red (1), blue (2), and green (3) circles, \( \alpha_{1,2,3} \) are lattice vectors and \( \alpha_{1,2} \) defines the unit cell. b, Band dispersion and DOS. Fermi level at \( \mu = 0 \). c, FS at \( \mu = 0 \). d, FS at \( \mu = 0.19t \) for electron doping above \( v_H \) filling. Dashed cyan curves enclose the 2 \( \times \) 2 reduced BZ with the resulting high symmetry points denoted by \( K \sim \frac{1}{2}K \) and \( \Gamma \sim \frac{1}{2}M \). e, FS in d folded by the 2 \( \times \) 2 BZ boundary.

of corner-sharing up and down triangles (Fig. 1a). A generic tight-binding model can be written as,

\[
H = - \sum_{(\alpha \beta \gamma) \mu r} \left[ t_{\alpha \beta \gamma}^{\mu} (r) c_{\alpha \beta \gamma \mu}^\dagger (r) c_{\alpha' \beta' \gamma' \mu} (r) + h.c. \right] - \mu n_{\alpha \beta \gamma},
\]

where \( c_{\alpha \beta \gamma}^\dagger \) creates an electron on sublattice \( \alpha \) in unit cell \( r \), \( t_{\alpha \beta \gamma}^{\mu} (r) \) are the \( r \)-dependence nearest-neighbor (nn) hopping in the up and down triangles between sublattices \( \alpha \) and \( \beta \), \( \mu \) is the chemical potential, and \( n_{\alpha \beta \gamma} = c_{\alpha \beta \gamma}^\dagger c_{\alpha \beta \gamma} \) is the density operator. The sublattice indices run over \( (\alpha, \beta, \gamma) = (1, 2, 3) \), and spin indices are left implicit.

To reveal the unique geometry of the kagomé lattice, which plays a crucial role in the charge ordered state, we rewrite

\[
H = - \sum_{(\alpha \beta \gamma) \mu r} \left[ t_{\alpha \beta \gamma}^{\mu} (r) \chi_{\alpha \beta \gamma}^{\mu} (r) + t_{\alpha \beta \gamma}^{\mu} (r) \chi_{\alpha' \beta' \gamma' \mu}^\dagger (r) + h.c. \right] - \mu n_{\alpha \beta \gamma},
\]

where \( t_{\alpha \beta \gamma}^{\mu} = \frac{1}{2}(t_{\alpha \beta \gamma}^{\mu} \pm t_{\alpha \beta \gamma}^\dagger) \) and the corresponding symmetric (+) and antisymmetric (−) nn bond operators,

\[
\chi_{\alpha \beta \gamma}^{\mu} (r) = c_{\alpha \beta \gamma}^\dagger c_{\beta' \gamma' \mu}^\dagger \pm c_{\alpha' \beta' \gamma' \mu} c_{\alpha \beta \gamma}^\dagger.
\]

For a kagomé lattice with uniform nn hopping, \( t_{\alpha \beta \gamma}^{\mu} (r) = t \) and \( t_{\alpha \beta \gamma}^{\mu} (r) = 0 \). The corresponding band dispersion and density of states (DOS) are shown in Fig. 1b. A uniform \( t_{\alpha \beta \gamma}^{\mu} (r) = \delta \neq 0 \) produces an intra-cell breathing kagomé lattice [31] without breaking lattice translation symmetry. A spatially modulated hopping \( t_{\alpha \beta \gamma}^{\mu} (r) = \delta \cos (Q \cdot r) \) describes a breathing kagomé lattice with broken translation symmetry and a band ordered CDW with wavevector \( Q \).

**Complex CDW and Chern Fermi pockets**

At band filling \( n_{\phi 1} = 5/12 \) or \( \mu = 0 \), the sublattice-resolved Fermi surface (FS) is the hexagon connecting the \( v_H \) singularity at the \( M \) points of the BZ in Fig. 1c. Theoretical studies at this \( \mu \) value have demonstrated a rich set of instabilities toward correlated states due to the unique sublattice quantum interference effects [32–34]. When extended Coulomb interactions \( V \) play a more important role than the local Hubbard \( U \), the leading instability is a \( 2a_0 \times 2a_0 \) bond ordered CDW. The single-orbital model prediction turns out, remarkably, to be consistent with the triple-\( Q \) CDW observed in the multi-orbital \( AV_3Sb_3 \) [1–3], signifying the essential physics captured by the proximity of the Fermi level to the \( d \)-band \( \mu \) singularity. The first-principle DFT calculations [16] show that the \( 2a_0 \times 2a_0 \) CDW is accompanied by the softening of the phonon breathing mode of the kagomé lattice at the zone boundary \( M \) points (Fig. 1c). This corresponds to the triple-\( Q \) breathing modulation of the antisymmetric bond in Eq. (2) and described by the Hamiltonian

\[
H_{\text{cdw}} = \sum_{(\alpha \beta \gamma) \mu r} \rho_{\gamma} \cos (Q_{\gamma} \cdot r) \chi_{\alpha \beta \gamma}^{\mu} (r) + h.c.,
\]

where the CDW wavevector \( Q_\gamma = \frac{\pi}{\sqrt{3}} G_\alpha \), with the Bragg vector \( G_1 \sim 2b_1,1.3 \). The CDW maintains rotation and inversion symmetry, since the bond \( \chi_{\alpha \beta \gamma} \) points along \( a_\gamma \), which is perpendicular to \( Q \). The star-of-David (SD) and inverse-SD (trihexagonal) CDW configurations are realized for \( \rho_{\gamma} > 0 \) and \( \rho_{\gamma} < 0 \), respectively.

Motivated by the conjecture of spontaneous TRS breaking orbital currents [1], several theoretical model studies [33–42] have explored complex bond ordered triple-\( Q \) CDWs supporting circulating current and plaquette flux, a long-sought after quantum state also relevant for the pseudogap phase in the high-\( T_c \) cuprates [36–38] and the quantum anomalous Hall insulators [35]. It has been shown that the electron-electron and electron-phonon interactions can drive such a complex CDW state [41], which can be described by \( H_{\text{cdw}} \) in Eq. (4) with complex \( \rho_\gamma \) in the single-orbital model. A crucial point to realize is that the actual Fermi level is above the \( v_H \) point of the \( d_{x^2-y^2} \) band at \( M \) in all three realistic \( AV_3Sb_3 \) compounds, as shown by DFT calculations [16] and ARPES measurements [26, 50]. It is thus important to study the complex bond ordered phase described by Eq. (4) close to but not at \( v_H \) filling, which, as we show below, results in a novel metallic triple-\( Q \) CDW state and an emergent primary PDW state detected by STM in CsV3Sb5 recently [3].

We thus study the physics above \( v_H \) filling \( n \gtrsim n_{\phi 1} \) by setting the symmetric hopping \( t_{\alpha \beta \gamma}^{\mu} (r) = t \) uniform as the energy and the chemical potential \( \mu \gtrsim 0.19t \). The sublattice resolved FS contour in Fig. 1d describes that of the \( d_{x^2-y^2} \) band in the DFT [14]. The triple-\( Q \) complex CDW described by \( H_{\text{cdw}} \) amounts to a correlation-induced breathing modulation of the antisymmetric hopping \( t_{\alpha \beta \gamma}^{\mu} (r) = \rho_{\gamma} \cos (Q_{\gamma} \cdot r) \) in Eq. (2). Diagonalizing \( H \) with equal amplitudes \( \rho_\gamma \) in the 3\( Q \) directions \( \rho = (0.1 + 0.3i)(1, 1, 1) \), we obtain the \( 2a_0 \times 2a_0 \) complex CDW state shown in Fig. 2a characterized by three different current-carrying bonds. Ac-
cumulating the link phases around the plaquette produces four gauge-independent plaquette fluxes $\phi_{1,...,4}$ in the shaded 2 x 2 unit cell. The flux is staggered since the total flux $\Phi = \phi_1 + 3\phi_2 + 2\phi_3 + 6\phi_4$ is zero by symmetry. This orbital antiferromagnet breaks TRS but maintains inversion symmetry and gives rise to topological Chern bands on the kagomé lattice.

The band dispersion and the DOS are shown in Fig. 2b in the reduced BZ. The Fermi level crosses the red Chern band with Chern number $C = -3$ near the $M_1$ points, corresponding to electron doping of the Chern insulator at vH filling. The CDW metal is therefore a doped orbital Chern insulator with emergent Chern Fermi pockets (CFPs), as shown in Fig. 2e, residing along $\Gamma$-K and around $M_2$ along $\Gamma$-M direction due to the $2 \times 2$ band folding (Fig. 1e). Each CFP has a volume $\sim 0.56\%$ of the original BZ. Remarkably, FS reconstruction and FS pockets of similar sizes have been detected by quantum oscillation experiments [18, 43].

To understand the intriguing properties of the CFPs, we calculate the Berry curvature and orbital magnetic moment (See Methods) of the hosting $C = -3$ Chern band. Fig. 2d shows that the Berry curvature in momentum space $\Omega(k)$ concentrates heavily on the CFPs. Remarkably, small orbits carrying large Berry curvature have indeed been detected in quantum oscillations [18, 43]. As a result, the CFPs contribute significantly to the AHE in addition to the fully occupied Chern bands in Fig. 2b: $\sigma_{xy} = -2e^2/h - 2e^2/9 \int d^2k \Omega(k) \approx -1.1 e^2/9$.

Using the $c$-axis lattice constant $c \approx 8.95\AA$ [7], the magnitude of the intrinsic anomalous Hall conductivity is estimated \( \sigma_{xy}/c \approx 474\Omega^{-1}\text{cm}^{-1} \), which accounts for the observed large intrinsic AHE ($\sim 500\Omega^{-1}\text{cm}^{-1}$) [19, 20]. Moreover, the CFPs naturally lead to carrier density dependent intrinsic AHE observed by controlled gating in CsV$_2$Sb$_3$ [44].

The calculated orbital magnetic moment $m(k)$ is plotted in Fig. 2e for the $C = -3$ Chern band hosting the CFPs in unit of $\mu_B$ using the lattice constant $a_0 \approx 5.4\AA$ and the hopping $t \approx 0.5eV$, corresponding to a 3eV bandwidth of the $d_{x'y'}$-band [16]. The orbital moment, as large as $-5\mu_B$, concentrates on the CFPs and couples to an applied magnetic field along the $c$-axis via the orbital Zeeman effect $-m(k)\mu_B$. We thus expect strong and $k$-dependent field-induced responses in the electronic structure. Moreover, the intrinsic thermodynamic orbital magnetization is calculated (Methods) to be $M \approx 0.078\mu_B$ per V, which is large enough to be detectable by thermodynamic measurements.

**Complex PDW and vortex-antivortex lattice**

The dynamically generated FS pockets are connected by well-defined momenta due to the band-folding. Fig 3a displays the 12 color coded CFPs. Pockets of the same color are related by the reciprocal lattice vectors of the $2a_0 \times 2a_0$ CDW $Q_{x_{\text{CFP}}} = \frac{1}{2}G$. Those of different colors are, in contrast, connected by new wave vectors $Q_{x_{\text{CFP}}} = \frac{3}{2}G$ and $Q_{x_{\text{CFP}}} = 3G$ in all 3Q directions, enabling new correlated quantum states that can coexist with the $2a_0 \times 2a_0$ CDW.

Note that neither $Q_{x_{\text{CFP}}}$ nor $Q_{3x_{\text{CFP}}}$ leads to full nesting because of the ellipticity of the pockets (Fig. 3b). Moreover, since the pockets are all electron-like, there is no instability in the particle-hole channel. Additional triple-Q CDWs at these wave vectors cannot gap out the pockets (see Methods) and have not appeared experimentally, possibly because they are ineffective at lowering energy. A striped 4$d_0$ charge order was indeed observed by STM on the Sb surface in CsV$_2$Sb$_3$ [2, 3] which is allowed as a unidirectional single-Q CDW in the Ginzburg-Landau (GL) analysis [35].

The path toward a stable correlated ground state can arise in the particle-particle channel through a PDW with finite momentum $Q_{pdw}$ pairing of a spin-down electron at momentum $k$ and a spin-up electron at $-k + Q_{pdw}$. In contrast to a CDW, it does not require the connected CFPs to be electron and hole like to produce a large susceptibility. The outer CFPs in Fig. 3a carry the majority of the spectral weight (Fig. 2e) and are more susceptible to a stabilizing PDW with $\frac{1}{4}d_0 \times \frac{1}{4}d_0$ periodicity (Methods). The energetically favorable triple-Q PDW therefore has the wave vector $Q_{pdw} = 3G$, which connects each of the six outer elliptical pockets in Fig. 3a with its two neighbors.

To reveal the properties of the tripe-Q PDW in connection to the experimental observation of the $\frac{1}{8}d_0 \times \frac{1}{8}d_0$ PDW [3], we consider the simplest onsite, spin-singlet pairing,

\[
H_{pdw} = -\sum_{\alpha} 2\Delta_{pdw}^\alpha(r) c_{\alpha}^\dagger(r) c_{\alpha}^\dagger(r) + h.c.,
\]

where $\Delta_{pdw}^\alpha(r)$ is the triple-Q PDW order parameter on the $\alpha$-sublattice in unit cell $r$. It is a coherent superposition of the six basic PDWs $\Delta_{pdw}^\alpha(r) = \Delta^\eta_{pdw}e^{iQ_{pdw}r}$, with $\eta = 1, 2, 3$ and the amplitude and phase factor $\Delta^\eta_{pdw} = \Delta^\eta e^{i\phi^\eta}$. Following the symmetry
where \( \mathbf{r} \) and \( \mathbf{r}_0 \) are located at the \( C_6 \) inversion center of the CDW in Fig. 2a and the relative phase \( \phi = \ell (\eta - 1) \pi / 3 \) with integer \( \ell \). The spatial distribution of the order parameter shows that \( \ell = 0 \) describes a real PDW on a \( \frac{3}{4}a_0 \times \frac{3}{4}a_0 \) triangular lattice in Fig. 3c. Intriguingly, \( \ell = \pm 1 \) generate complex PDW on an emergent \( \frac{3}{4}a_0 \times \frac{3}{4}a_0 \) kagomé lattice as shown in Fig. 3d for \( \ell = 1 \). The zeros of the PDW order parameter in the middle of the triangles and hexagons are the centers of the single vortex and the double antivortex, respectively. Such a vortex-antivortex lattice, and thus the complex triple-Q PDW, is precisely the conjectured roton-PDW, based on the notion that a roton corresponds to a tightly bound vortex-antivortex pair.

Including the \( H_{pdw} \), the total Bogoliubov-de Gennes (BdG) Hamiltonian can be diagonalized, giving rise to the Fourier peaks in the CDW and PDW field distributions in Fig. 3b. The BdG spectrum in Fig. 3f reveals nearly degenerate quasi-particle excitations in PDW. The PDW gap is expected to open first at the crossings of the FS pockets in Fig. 3b. For large enough PDW amplitudes, the FS pockets in the CDW state are fully gapped, as shown in the left panel of Fig. 3f at \( \Delta_p = 0.1t \). Three gapped quasiparticle bands, stemming from the three-colored CFPs, emerge with the minimum gap loci along the \( \Gamma - K - M - \Gamma \) directions, leaving gapless excitations on the residual FS sections. Fig. 3f (right panel) shows the quasi-
particle dispersion of the PDW$_{0,1}$ for an order of magnitude smaller $\Delta_p = 0.01r$. Surprisingly, the gapless PDW does not appear, and upon zooming in, a full excitation gap is still visible in Fig. 3h. Since the gap developed over the residual FS is small, the low energy DOS (Fig. 3g, bottom panel) shows a V-shaped SC gap inside the PDW pseudogap at very low temperatures, with gap sizes $\sim 0.7\text{meV}$ and $10\text{meV}$ for $t = 0.5\text{meV}$, in agreement with STM observations [3]. The fully gapped PDW state hints at an incipient uniform SC discussed below.

**Intertwined and vestigial order**

**Intertwined CDW with orbital current** A primary PDW induces coexisting and intertwined electronic orders. The secondary order can become the primary vestigial order after the melting of the PDW by quantum and/or thermal fluctuations [47, 51]. We find that the hexagonal symmetry $\sum Q^0_i = 0$ and the intrinsic complex CDW lead to a plethora of novel secondary and vestigial states. The secondary CDWs originate from the bilinear products

$$\rho^{\prime}_{Q^0_i}(r) \propto \Delta_{Q^0_i}(r) \Delta_{Q^0_i}^*(r), \quad \rho^{\prime}_{Q^0_i=\pm Q^0_i}(r) \propto \Delta_{Q^0_i}(r) \Delta_{Q^0_i}^*(r),$$

and carry wavevectors that coincide with those of the intrinsic CDW and primary PDW since $2Q^0_i = Q^0_i + G^\alpha_i$, $Q^0_i + Q^0_i = -Q^0_i$, and $Q^0_i - Q^0_j = Q^0_i + G^\alpha_{ij} + G^\beta_{ij}$, where $G$ and $G^\alpha_{ij}$ are the reciprocal vectors of the original and the two 2D CDW lattice. Importantly, the complex PDW$_{1,1}$ generates a complex triple-Q CDW at $Q^0_i$ with nonzero angular momentum and orbital currents commensurate to those produced by the intrinsic complex CDW. It is thus likely the preferred ground state.

**Fluctuating PDW and pseudogap phase** The PDW state has a global $U(1) \times U(1) \times U(1)$ symmetry under the phase change of the complex order parameter $\Delta = \Delta_{Q^0_i} \rightarrow e^{\nu_i Q^0_i} \Delta_{Q^0_i}$, where $\nu_i$ is the is the global SC phase and $\varphi_{d_i} = Q^0_i \cdot d_i$ the phonon modes associated with the displacement field $d_i$ due to the broken point group symmetry. When the proliferation of vortices in $\varphi_{d_i}(r)$ drives $\langle \Delta_{Q^0_i} \rangle = 0$, the induced CDW, transforming as $\rho_{Q^0_i} \rightarrow e^{\nu_i Q^0_i} \rho_{Q^0_i}$, is unaffected by the SC vortices and becomes a primary vestigial order detectable directly by STM at the intrinsic CDW and PDW wave vectors. The observed pseudogap phases [3], both above $T_c$ and in a strong magnetic field above $H_{c2}$, likely originate from the fluctuating PDW.

**Intertwined BCS instability** Intriguingly, the hexagonal symmetry allows a secondary uniform SC order [46].

$$\Delta \propto \Delta_{Q^0_i}(r) \Delta_{Q^0_i}(r) \Delta_{Q^0_i}(r) = \Delta^3 \propto e^{-(\theta_i + \theta_j + \phi_i)^*}.$$  

(7)

Accordingly, an s-wave SC coexists with PDW$_0$ or a $d_{x^2-y^2} \pm id_{xy}$ SC described by $\Delta \propto \Delta^3(1, e^{2\pi i/3}, e^{4\pi i/3})$ on the three-subslices coexists with the complex roton PDW$_{1,1}$. The induced uniform charge-2 condensate ensures a novel intertwined BCS instability in the triple-Q PDW that fully gaps out the CFPs at arbitrarily small $\Delta_p$ as observed above.

**Chiral $d \pm id$ and nematic SC** The uniform $\Delta_p$ form a two-component complex order parameter $\Delta = (\Delta, \Delta, \Delta)^T$. Transforming as the two-dimensional $E_2$ irreducible representation of the point group $D_6$. The SC ground state is determined by the fourth-order terms in the GL free energy density, $F_2 = u(\Delta^2 + \Delta^2)^2 + \lambda(\Delta^2 - \Delta^2)^2$. For $\lambda < 0$, the ground state is given by $\Delta_p = (1, 0)\Delta$ or $(0, 1)\Delta$. They are the chiral $d_{x^2-y^2} \pm id_{xy}$ states that break TR symmetry but preserve rotation symmetry. For $\lambda > 0$, the lowest energy state is selected by $\Delta_p = (\cos \theta_i, \sin \theta_i)\Delta$, which describes a TR invariant nematic SC with broken rotation symmetry. The nematic director is identified as $\hat{n} = (\cos \theta_i, \sin \theta_i)\Delta$ by writing $\Delta = (\cos \theta_i, \sin \theta_i)(\Delta_1, \Delta_2)^T$, where $\Delta_1, 2$ correspond to $d_{x^2-y^2}$ and $d_{xy}$ pairing. It is determined by the $\Delta_k$ crystal symmetry that allows a sixth-order free-energy term for the nematic state (absent in the chiral phase), $F_6 = \omega (\Delta^2 \Delta^2 + h.c. \propto \omega \cos \theta_i)$, minimized by $\theta_i = \pi n/3 (\omega < 0)$ or $\theta_i = (n + 1/2)\pi/3 (\omega > 0)$ with integer $n$. Rotation symmetry breaking has been detected in both normal and SC states in AV$_3$Sb$_5$ [2, 21, 52].

**Potts nematic order** The two components of the nematic SC transform as $\Delta_p = e^{i\theta_i} \Delta$, where $\theta_i$ is the phase mode of the nematic director, and provide two composite orders: the nematic state $\Psi_n = \Delta \Delta e^{i\theta_i}$, and the charge-4 SC $\Psi_4 = \Delta \Delta e^{2i\theta_i}$. The three discrete values of the phases in $\Psi_n$ indicate it is a $Z_3$ order parameter as in the three-state Potts model and describes a Potts nematic [53]. Since it decouples from the topological defects in the SC phase $\theta_i(r)$, the nematic phase can emerge above $T_c$, and manifest as a Pomeranchuk instability of the FPs in the CDW state [54].

**Charge-4e and 6e SC** Just as the PDW order parameter $\Delta_p$ allows, in addition to SC vortice in $\theta_i(r)$, double-dislocations (vortices) in $\varphi_{d_i}(r)$ and $1/2$-vortices bound to single dislocations [48, 49], the single-valued nematic SC $\Delta_p$ allows nematic vortices of double-disclinations in $\theta_i(r)$ and $1/2$-vortices in $\theta_i(r)$ bound to single-disclinations around which the $\Delta_p$ changes sign. As a result, the proliferation of nematic vortices in $\theta_i(r)$ induced by thermal/quanum fluctuations and random lattice strain suppresses the nematic charge-2e condensate $\langle \Delta_p \rangle = 0$, restores the rotation symmetry $\langle \Psi_n \rangle = 0$, and materializes the long-sought-after charge-4e SC $\langle \Psi_4 \rangle \neq 0$. Charge-4e SC as a vestigial order of multi-component nematic SC was also proposed for twisted bilayer graphene recently [55, 56].

The most striking prediction for the PDW state is the emergence of charge-4e and charge-6e superconductivity,

$$\Delta_{4e} \propto \Delta_{Q^0_i}(r) \Delta_{Q^0_i}(r), \quad \Delta_{6e} \propto \Delta_{Q^0_i}(r) \Delta_{Q^0_i}(r) \Delta_{Q^0_i}(r)$$

(8)

at low temperatures, in addition to the charge-2e condensate in Eq. (7) discussed above. These independent secondary electronic orders describe coexisting condensates of Cooper pairs, four-electron, and six-electron bound states. As the proliferation of dislocations melts the PDW, and that of the disclinations melts the nematic charge-2e condensate, the vestigial charge-4e and charge-6e condensates $\langle \Delta_{4e,6e} \rangle \neq 0$ prevail by confining the fractional $1/2$ and $1/2$-vortices [46, 48, 49]. The kagome SC AV$_3$Sb$_5$ thus provides a remarkable platform for these fundamentally new macroscopic phase coherent states beyond the BCS theory of Cooper pairs.

**Chiral topological PDW**

The PDW ground state with fully gapped CFPs carries an integer topological invariant $N \in \mathbb{Z}$ given by the total Chern number of the occupied BdG quasiparticle bands in Fig. 3e. We obtain $N = 2$ for both PDW$_{0,1}$, indicating two charge neutral chiral edge modes (CEMs) at the sample boundary. Under
understood as a chiral topological SC achieved by the complex 3Q PDW (red and blue) that pairwise localize on the opposite edges. Both states exhibit two CEMs of equal chirality inside the supercell averaged wavefunction amplitudes corresponding to the two in-gap CEMs indicated at \( k_z = \pm \pi/80 \) (red circle and blue square), showing localization at the opposite edges illustrated in a. c. Low-energy quasiparticle spectrum of a uniform \( s \)-wave superconductor with \( \Lambda_s = 0.03\tau \) coexisting with the complex CDW. \( L_y = 96 \) with \( 2 \times 2 \) supercell.

inversion symmetric open boundaries in \( y \)-direction (Fig. 4a), the spectrum of the BdG Hamiltonian is plotted in Fig. 4b-c, as a function of the momentum \( k_x \) in the periodic \( x \)-direction. Both states exhibit two CEMs of equal chirality inside the PDW gap (red and blue) that pairwise localize on the opposite edges (Fig. 4d). For comparison, the spectrum of a uniform \( s \)-wave SC in the complex CDW phase in the absence of the PDW reveals two similar CEMs in Fig. 4e, which can be understood as a chiral topological SC achieved by \( s \)-wave pairing in a doped orbital Chern insulator. We thus conclude that the tripe-Q real and complex PDWs are intrinsic chiral topological PDW superconductors. Note that our simplified BdG Hamiltonian is in class \( C \) due to spin rotation symmetry, and the two CEMs combine to produce a chiral fermion edge mode. When \( SU(2) \) symmetry is broken by spin-orbit coupling and/or magnetic fields, the BdG Hamiltonian belongs to class \( D \), supporting two chiral Majorana edge states and vortex Majorana zero modes. Evidence for zero-bias conductance peaks in vortex cores has been detected by STM.

Our findings reveal that doped orbital Chern insulators with CFPs embody a rich set of correlated and topological states. They can originate from itinerant electrons on the kagome lattice lightly doped away from the vH filling, and capture an essential part of the unconventional phenomenology in \( AV_3Sb_5 \) superconductors. The PDW mechanism for intrinsic chiral topological SC, Potts-nematic, fractional vortices, charge-4e and charge-6e superconductivity via doped orbital Chern insulators is potentially relevant for twisted bilayer graphene and other hexagonal quantum materials.

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