Classical solutions and steady states of an attraction–repulsion chemotaxis in one dimension

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We establish the existence of global classical solutions and non-trivial steady states of a one-dimensional attraction–repulsion chemotaxis model subject to the Neumann boundary conditions. The results are derived based on the method of energy estimates and the phase plane analysis.

Keywords: chemotaxis; attraction–repulsion; classical solutions; steady states

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1. Introduction

Chemotaxis describes the directed migration of cells along the concentration gradient of the chemical that is produced by cells. It is a leading mechanism to account for the morphogenesis and self-organization of many biological systems. The prototype of the population-based chemotaxis model, known as the Keller–Segel model, was first proposed by Keller and Segel \cite{Keller1970} in the 1970s to describe the aggregation of cellular slime moulds \textit{Dictyostelium discoideum}. The rudimental structure of the Keller–Segel model is a system of parabolic partial differential equations as follows:

\begin{equation}
\begin{aligned}
    u_t &= D_u \Delta u - \nabla (\chi u \nabla v), \\
    v_t &= D_v \Delta v + f(u, v),
\end{aligned}
\end{equation}

where $u(x, t)$ denotes the cell density and $v(x, t)$ is the chemical concentration, $D_u$ and $D_v$ are positive diffusion coefficients and $\chi > 0$ is called the chemotactic coefficient measuring the strength of influence of the chemical on cells. The Keller–Segel model (1) describes the cell chemotactic movement towards a single chemical (i.e. chemoeattrant), and it has been extensively studied over the past four decades from various perspectives \cite{Keller1970, Wang2011, Wang2013}. However, in
many biological processes, the cells may interact with a combination of repulsive and attractive signalling chemicals to produce various interesting biological patterns, such as the formation of nigrostriatal circuits during development [5], the chick primitive streak formation [3] and many others (see, e.g. [4]). In this paper, we shall consider the following attraction–repulsion chemotaxis model:

\[
\begin{align*}
    u_t &= D_u \Delta u - \nabla (\chi_v u \nabla v) + \nabla (\chi_w u \nabla w), \\
    v_t &= D_v \Delta v + \alpha u - \beta v, \\
    w_t &= D_w \Delta w + \gamma u - \delta w,
\end{align*}
\]  

(2)

where \( D_u, D_v, D_w > 0 \) are diffusion coefficients, \( \chi_v > 0 \) and \( \chi_w > 0 \) are chemotactic coefficients, and \( \alpha, \gamma > 0 \) and \( \beta, \delta \geq 0 \). The model (2) was proposed in [9] to describe the aggregation of microglia observed in Alzheimer’s disease and in [11] to describe the quorum effect in the chemotactic process. In their approaches, it is assumed that there exists a secondary chemical, denoted by \( w \), which behaves as a chemorepellent to mediate the chemotactic response to the chemoattractant \( v \) accordingly. To the best of our knowledge, presently, there is no rigorous result on the chemotaxis model with two opposite chemicals (i.e. chemoattractant and chemorepellent).

The purpose of this paper is to establish the global existence of classical solutions and steady states of Equation (2) in one dimension with Neumann boundary conditions. The results for higher dimensions still remain open.

In one dimension, the system (2) reads

\[
\begin{align*}
    u_t &= D_u u_{xx} - (\chi_v u v_x)_x + (\chi_w u w_x)_x, \\
    v_t &= D_v v_{xx} + \alpha u - \beta v, \\
    w_t &= D_w w_{xx} + \gamma u - \delta w.
\end{align*}
\]  

(3)

With the following scalings

\[
(\tilde{t}, \tilde{v}, \tilde{w}, \tilde{u}, \tilde{\gamma}) = \frac{1}{D_u} (t, v, w, u, \gamma),
\]

system (3) can be reduced to the following system:

\[
\begin{align*}
    u_t &= u_{xx} - (uv_x)_x + (uw_x)_x, \\
    v_t &= D_v v_{xx} + u - \beta v, \\
    w_t &= D_w w_{xx} + \gamma u - \delta w,
\end{align*}
\]  

(4)

where the tilde superscripts have been suppressed for readability.

Letting \( \Omega \) be a bounded open interval in \( \mathbb{R} = (-\infty, \infty) \), we prescribe the initial conditions

\[
u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x),
\]  

(5)

and the Neumann boundary conditions

\[
\frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = \frac{\partial w}{\partial v} = 0, \quad x \in \partial \Omega,
\]  

(6)

where \( v \) denotes the unit outward normal vector to the boundary \( \partial \Omega \).
In this paper, we shall prove the existence of global classical solutions to the model (4)–(6) based on Amann’s theory and the method of energy estimates. We also show the existence of non-trivial steady states of Equation (4) subject to the Neumann boundary conditions (6) for the case $\beta D_w = \delta D_v$ by the phase plane analysis.

Notations: Throughout this paper, $\Omega$ denotes a bounded open interval in $\mathbb{R}$ unless otherwise specified and $C$ denotes a generic constant which can change from one line to another. $L^p = L^p(\Omega)(1 \leq p \leq \infty)$ denotes the usual Lebesgue space in a bounded open interval $\Omega \subset \mathbb{R} = (-\infty, \infty)$ with norm $\|f\|_{L^p} = (\int_{\Omega} |f(x)|^p \, dx)^{1/p}$ for $1 \leq p < \infty$ and $\|f\|_{L^\infty} = \text{ess sup}_{x \in \Omega} |f(x)|$. When $p = 2$, we write $\|f\|_{L^2} = \|f\|$ for notational convenience. $H^l$ denotes the $l$th-order Sobolev space $W^{l,2}$ with norm $\|f\|_{H^l} = \|f\|_l = (\sum_{i=0}^l \|\partial_i^l f\|^2)^{1/2}$. For simplicity, $\|f(\cdot, t)\|_{L^p}$ and $\|f(\cdot, t)\|_l$ will be denoted by $\|f(t)\|_{L^p}$ and $\|f(t)\|_l$, respectively. Moreover, we denote $\|(f, g)\|_{L^p} = \|f\|_{L^p} + \|g\|_{L^p}$ for $1 \leq p \leq \infty$ and $\|(f, g)\|_{H^l} = \|f\|_{H^l} + \|g\|_{H^l}$ for $l = 1, 2, 3, \ldots$.

2. Preliminaries

In this section, we present some inequalities that will be used to derive the required estimates. First, we recall the Gagliardo–Nirenberg inequality for functions that do not vanish at the boundary of $\Omega$ (see [10, Theorem 1]).

**Lemma 2.1** Let $\Omega$ be a open bounded domain in $\mathbb{R}^n$ with a smooth boundary. Then, for any $q \geq 1$, there exists a positive constant $C_q$, which depends on $n, q, \Omega$, such that for all $f \in W^{1,2}(\Omega)$,

$$\|f\|_{L^q} \leq C_q(\|\nabla f\|_{L^q} + \|f\|_{L^1}^{1-a} + \|f\|_{L^1}),$$

where $a = (1 - (1/q))/((1/n) + 1/2)$ and $0 \leq a < 1$.

Letting $n = 1, q = 4$ and $\alpha = 1/2$ in Equation (7) and using the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ for any $a, b \in \mathbb{R}$, we obtain the following inequality:

$$\|f\|_{L^4}^3 \leq C(\|f_x\|_1 \|f\|_{L^1} + \|f\|_{L^2}^2).$$

The following Gronwall-type inequality [15] will be used later.

**Proposition 2.2** Let $\eta(\cdot)$ be a non-negative differentiable function on $[0, \infty)$ satisfying the differential inequality $\eta'(t) + l\eta(t) \leq \omega(t)$, where $l$ is a constant and $\omega(t)$ is a non-negative continuous function on $[0, \infty)$. Then,

$$\eta(t) \leq \left(\eta(0) + \int_0^t e^{l(\tau)} \omega(\tau) \, d\tau\right) e^{-lt}.$$

Alternatively, if for $t \geq 0, \phi(t) \geq 0$ and $\psi(t) \geq 0$ are continuous functions such that the inequality $\phi(t) \leq C \exp(rt) + L \int_0^t \psi(s) \phi(s) \, ds$ holds on $t \geq 0$ with $C$ and $L$ positive constants, then

$$\phi(t) \leq C \exp(rt) \exp(L \int_0^t \psi(s) \, ds).$$
3. Global existence of classical solutions

In this section, we shall establish the global existence of classical solutions of the system (4)–(6). The main result is the following.

**Theorem 3.1** Let \((u_0, v_0, w_0) \in H^2(\Omega)\). Then, there exists a unique global solution \((u, v, w)\) to the system (4)–(6) such that \((u, v, w) \in [C^0(\overline{\Omega} \times [0, \infty); \mathbb{R}^3)]^3 \cap [C^{2,1}(\overline{\Omega} \times (0, \infty); \mathbb{R}^3)]^3\). Moreover, \(u, v, w \geq 0\) if \(u_0, v_0, w_0 \geq 0\).

**Remark 3.2** Theorem 3.1 does not exclude the possibility that the solution may blow up at infinity time.

Theorem 3.1 will be proved by the local existence and the *a priori* estimates as given below.

3.1. Local existence

In this section, we shall apply Amann’s [1] theory to establish the local existence of solutions.

**Theorem 3.3** (local existence) Let \(\Omega\) be a bounded open interval in \(\mathbb{R}\). Then,

(i) for any initial data \((u_0, v_0, w_0) \in [H^1(\Omega)]^3\), there exists a maximal existence time constant \(T_0 \in (0, \infty]\) depending on the initial data \((u_0, v_0, w_0)\), such that the problem (4)–(6) has a unique maximal solution \((u, v, w)\) defined on \(\Omega \times [0, T_0)\) satisfying

\[
(u, v, w) \in [C^0(\overline{\Omega} \times [0, T_0); \mathbb{R}^3)]^3 \cap [C^{2,1}(\overline{\Omega} \times (0, T_0); \mathbb{R}^3)]^3.
\]

(ii) if \(\sup_{0 \leq t < T_0 \cap \Omega} \| (u, v, w)(\cdot, t) \|_{L^\infty} < \infty\) for each \(T > 0\), then \(T_0 = \infty\), namely, \((u, v, w)\) is a global classical solution of the system (4)–(6). Moreover, \(u \geq 0, v \geq 0, w \geq 0\) if \(u_0 \geq 0, v_0 \geq 0, w_0 \geq 0\).

**Proof** Define \(\eta = (u, v, w) \in \mathbb{R}^3\). Then, the system (4) with Equations (5) and (6) can be rewritten as

\[
\eta_t - \nabla \cdot (a(\eta) \nabla \eta) = \mathcal{F}(\eta), \quad \text{in } \Omega \times [0, +\infty),
\]

\[
\frac{\partial \eta}{\partial \nu} = 0, \quad \text{on } \partial \Omega \times [0, +\infty),
\]

\[
\eta(\cdot, 0) = (u_0, v_0, w_0), \quad \text{in } \Omega,
\]

where

\[
a(\eta) = \begin{pmatrix} 1 & -u & u \\ 0 & D_v & 0 \\ 0 & 0 & D_w \end{pmatrix}, \quad \mathcal{F}(\eta) = \begin{pmatrix} 0 \\ u - \beta v \\ y u - \delta w \end{pmatrix}.
\]

It is clear that the eigenvalues of \(a(\eta)\) are all positive, and hence, system (4) is normally elliptic. Then, the local existence result of assertion (i) follows from Theorem 14.6 reported in [1] and (ii) is a consequence of Theorem 15.3 reported in [1]. Finally, the positivity of solutions follows from Theorem 15.1 reported in [1].
3.2. \textit{A priori} estimates

In this section, we are devoted to deriving the \textit{a priori} estimates of solutions obtained in Theorem 3.1 to establish the global existence of solutions. First of all, we observed that the first equation of Equation (4) is a conservation equation. If we denote

\[ \int_{\Omega} u_0(x) \, dx =: m, \]  

(12)

then by integrating the first equation of Equation (4) and using the Neumann boundary conditions (6), we have

\[ \| u(t) \|_{L^1} = \int_{\Omega} u(x, t) \, dx = m. \]  

(13)

Lemma 3.4 Let \((v_0, w_0) \in [L^2(\Omega)]^2\) and Equation (6) hold. Let \((u, v, w)\) be a solution of the problem (4)–(6). Then, for any \(T > 0\), there is a constant \(C\) such that the following inequality holds for any \(0 < t < T\):

\[ \|(v, w)(t)\|_2^2 \leq C, \quad \int_0^t \|(v, w)(\tau)\|_2^2 \, d\tau + \int_0^t \|(v_x, w_x)(\tau)\|_2^2 \, d\tau \leq C(1 + t). \]  

(14)

Proof Multiplication of the second equation of Equation (4) by \(v\) and integration of the resulting equation with respect to \(x\) over \(\Omega\) give rise to

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} v^2 \, dx + \beta \int_{\Omega} v^2 \, dx + D_v \int_{\Omega} v_x^2 \, dx = \int_{\Omega} uv \, dx \leq \|v\|_{L^\infty}. \]

Applying the Sobolev embedding \(H^1 \hookrightarrow L^\infty\), one has that

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} v^2 \, dx + \beta \int_{\Omega} v^2 \, dx + D_v \int_{\Omega} v_x^2 \, dx \leq Cm(\|v\| + \|v_x\|) \leq \frac{D_v}{2} \|v_x\|^2 + \frac{\beta}{2} \|v\|^2 + C, \]

where the Young inequality has been used. Then, it follows that

\[ \frac{d}{dt} \|v\|^2 + \beta \|v\|^2 + D_v \|v_x\|^2 \leq C. \]  

(15)

Applying Gronwall’s inequality, (9) into (15) yield that

\[ \|v\|^2 \leq \left( \|v_0\|^2 + C \int_0^t e^{\beta \tau} \, d\tau \right) e^{-\beta t} \]

\[ \leq \left( \|v_0\|^2 - \frac{C}{\beta} \right) e^{-\beta t} + \frac{C}{\beta} \leq C. \]

Furthermore, the integration of Equation (15) with respect to \(t\) over \([0, t]\) gives

\[ \int_0^t \|v(\tau)\|_2^2 \, d\tau + \int_0^t \|v_x(\tau)\|_2^2 \, d\tau \leq C(1 + t). \]  

(16)

Applying the same procedure to \(w\), we finish the proof. \( \blacksquare \)
Lemma 3.5  Let $u_0 \in L^2(\Omega)$, $(v_0, w_0) \in [H^1(\Omega)]^2$ and $(u, v, w)$ be a solution of the problem (4)–(6). Then, for any $T > 0$, there is a positive constant $C$ such that for any $0 < t < T$, it follows that

$$
\|u(t)\|^2 + \int_0^t \|u_x(\tau)\|^2 \, d\tau + \|(v, w)(t)\|^2_1 + \int_0^t \|(v, w)(\tau)\|^2_2 \, d\tau \leq C(1 + e^{C't}).
$$

(17)

Proof  We multiply the first equation of Equation (4) by $u$ and integrate the resulting equation by parts. Then, by Inequality (8), the Hölder inequality and the Young inequality, we derive that

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx + \int_{\Omega} u_x^2 \, dx = -\frac{1}{2} \int_{\Omega} u^2 v_{xx} \, dx + \frac{1}{2} \int_{\Omega} u^2 w_{xx} \, dx
$$

$$
\leq \frac{1}{2} \|u\|^2_2 \left(\|v_{xx}\| + \|w_{xx}\|\right)
$$

$$
\leq C(m \|u_x\|^2 + m^2)\left(\|v_{xx}\| + \|w_{xx}\|\right)
$$

$$
\leq \frac{1}{2} \|u_x\|^2 + C(\|v_{xx}\| + \|w_{xx}\| + \|v_{xx}\|^2 + \|w_{xx}\|^2).
$$

Using the Cauchy–Schwarz inequality to derive $C(\|v_{xx}\| + \|w_{xx}\|) \leq 2C^2 + \|v_{xx}\|^2 + \|w_{xx}\|^2$, we have

$$
\frac{d}{dt} \int_{\Omega} u^2 \, dx + \int_{\Omega} u_x^2 \, dx \leq C(1 + \|v_{xx}\|^2 + \|w_{xx}\|^2).
$$

(18)

Next, we estimate the right-hand side of Equation (18). To this end, we multiply the second equation of Equation (4) by $-v_{xx}$ and integrate the resulting equation to obtain that

$$
\frac{d}{dt} \int_{\Omega} v^2_x \, dx + \int_{\Omega} v_{xx}^2 \, dx \leq C \int_{\Omega} u^2 \, dx.
$$

(19)

A similar procedure applied to the third equation of Equation (4) leads to

$$
\frac{d}{dt} \int_{\Omega} w^2_x \, dx + \int_{\Omega} w_{xx}^2 \, dx \leq C \int_{\Omega} u^2 \, dx.
$$

(20)

Combining Equations (19) and (20), we have

$$
\frac{d}{dt} \int_{\Omega} (v_{xx}^2 + w_{xx}^2) \, dx + \int_{\Omega} (v_x^2 + w_x^2) \, dx + \int_{\Omega} (v_{xx}^2 + w_{xx}^2) \, dx \leq C \int_{\Omega} u^2 \, dx.
$$

(21)

Then, integration of Equation (21) with respect to $t$ yields that

$$
\|(v_x, w_x)\|^2 + \int_0^t \|(v_x, w_x)(\tau)\|^2 \, d\tau + \int_0^t \|(v_{xx}, w_{xx})(\tau)\|^2 \, d\tau \leq C \left(1 + \int_0^t \|u(\tau)\|^2 \, d\tau\right).
$$

(22)

Now, we integrate Equation (18) with respect to $t$ and obtain that

$$
\|u(t)\|^2 + \int_0^t \|u_x(\tau)\|^2 \, d\tau \leq C(1 + t) + C \int_0^t \|(v_{xx}, w_{xx})(\tau)\|^2 \, d\tau.
$$

(23)
Applying Gronwall’s inequalities (10), (22) and (23), one has that
\[ \|u(t)\|^2 \leq C(1 + t)e^{Ct}. \]  
(24)

Therefore, substitution of Equation (24) back to Equation (22) gives
\[ \|(v_x, w_x)\|^2 + \int_0^t \|(v_x, w_x)(\tau)\|^2 \, d\tau + \int_0^t \|(v_{xx}, w_{xx})(\tau)\|^2 \, d\tau \leq C(1 + t)e^{Ct} \leq Ce^{Ct}, \]
(25)

where we have used the fact that \(0 < t \leq e^{Ct}\) for \(C \geq 1\).

Then, combining Equations (24) and (25) with Equation (16), we get Equation (17). \(\blacksquare\)

With Lemma 3.5, we can derive the following estimates.

**Lemma 3.6** If \((v_0, w_0) \in [H^2(\Omega)]^2\). Let \((u, v, w)\) be a solution of the problem (4)–(6). Then, for any \(T > 0\), there is a positive constant \(C\) such that for any \(0 < t < T\), it holds that
\[ \|(v_{xx}, w_{xx})(t)\|^2 + \int_0^t \|(v_{xx}, w_{xx})(\tau)\|^2 \, d\tau + \int_0^t \|(v_{xxx}, w_{xxx})(\tau)\|^2 \, d\tau \leq C(1 + e^{Ct}). \]

**Proof** Differentiating the second equation of Equation (4) with respect to \(x\) twice and then multiplying the result by \(v_{xx}\), we obtain
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} v_{xx}^2 \, dx + \beta \int_{\Omega} v_{xx}^2 \, dx + Dv \int_{\Omega} v_{xxx}^2 \, dx = \int_{\Omega} u_{xx} v_{xx} \, dx = -\int_{\Omega} u_x v_{xx} \, dx. \]

By the Cauchy–Schwarz inequality, we have \(-u_x v_{xxx} \leq (Dv/2)v_{xx}^2 + (2/Dv)u_x^2\), which when applied to the above identity yields
\[ \|v_{xx}\|^2 + \int_0^t \|v_{xx}(\tau)\|^2 \, d\tau + \int_0^t \|v_{xxx}(\tau)\|^2 \, d\tau \leq C \int_0^t \|u_x(\tau)\|^2 \, d\tau. \]

Then, the application of Lemma 3.5 to the above inequality gives the estimate for \(v\). By the same procedure, we can derive similar estimates for \(w\) and complete the proof. \(\blacksquare\)

Combining Lemmas 3.6 and 3.5 and using the Sobolev embedding \(H^1 \hookrightarrow L^\infty\), we derive that
\[ \|v_x\|_{L^\infty} + \|w_x\|_{L^\infty} \leq C(1 + e^{Ct}). \]  
(26)

Then, we can derive the \(H^1\)-estimates for \(u\).

**Lemma 3.7** Let \(u_0 \in H^1(\Omega)\). Assume that \((u, v, w)\) is a solution of the problem (4)–(6). Then, for any \(T > 0\), there is a positive constant \(C\) such that for any \(0 < t < T\), it has
\[ \|u_x\|^2 + \int_0^t \|u_{xx}(\tau)\|^2 \, d\tau \leq C(1 + e^{Ct}). \]

**Proof** Multiplication of the first equation of Equation (4) by \((-u_{xx})\) and integration of the result yield
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} u_x^2 \, dx + \int_{\Omega} u_{xx}^2 \, dx = \int_{\Omega} u_{xx}(u v_x)_x \, dx - \int_{\Omega} u_{xx}(u w_x)_x \, dx. \]  
(27)
Next, we estimate the terms on the right-hand side of Equation (27). To this end, we first differentiate the second equation of Equation (4) thrice and then multiply the resulting equation by $v_{xxx}$. After integrating the result with respect to $x$ and $t$, we have

$$
\|v_{xxx}\|^2 + \int_0^t \|v_{xxx}(\tau)\|^2 d\tau + \int_0^t \|v_{xxxx}(\tau)\|^2 d\tau \leq C_0 \int_0^t \|u_{xx}(\tau)\|^2 d\tau,
$$

where we have used the Cauchy–Schwarz inequality and $C_0$ is a constant. In virtue of the integration by parts, we deduce that

$$\int_{\Omega} u_{xx}(u v_x)_x dx = \frac{1}{2} \int_{\Omega} u^2 v_{xxx} dx + 3 \int_{\Omega} v_x u_x u_{xx} dx$$

$$\leq \|u\|_{L^4}^2 \|v_{xxx}\| + 3 \|v_x\|_{L^\infty} \int_{\Omega} |u_x u_{xx}| dx,$$

where the Hölder inequality has been used. Then, by the Cauchy–Schwarz inequality and using Equations (28), (17), (8) and (26), one has

$$\int_0^t \int_{\Omega} u_{xx}(u v_x)_x dx \leq C \int_0^t (\|u(\tau)\|_{L^4})^2 d\tau + \frac{1}{8} C_0 \int_0^t \|v_{xxx}(\tau)\|^2 d\tau$$

$$+ C \int_0^t (1 + e^{C\tau}) \int_{\Omega} u_x^2 dx d\tau + \frac{1}{8} \int_0^t \int_{\Omega} u_{xx}^2 dx d\tau$$

$$\leq C \int_0^t (1 + \|u_x(\tau)\|)^2 d\tau + C(1 + e^{Ct}) \int_0^t \|u_x(\tau)\|^2 d\tau$$

$$+ \frac{1}{4} \int_0^t \|u_{xx}(\tau)\|^2 d\tau$$

$$\leq C(1 + e^{Ct}) + \frac{1}{4} \int_0^t \|u_{xx}(\tau)\|^2 d\tau. \quad (29)$$

The same argument as above applied to $w$ also gives rise to

$$\int_0^t \int_{\Omega} u_{xx}(u w_x)_x dx \leq C(1 + e^{Ct}) + \frac{1}{4} \int_0^t \|u_{xx}\|^2 d\tau. \quad (30)$$

Then, integrating Equation (27) with respect to $t$ and applying the inequalities (29) and (30), we complete the proof.

### 3.3. Proof of Theorem 3.1

By Lemmas 3.5 and 3.7 as well as Sobolev embedding $H^1 \hookrightarrow L^\infty$, we have

$$\sup_{0 < t < T_0 \cap T} \|(u, v, w)(t)\|_{L^\infty} \leq C(1 + e^{Ct})$$

for any $T > 0$. That is, for any finite time $t$ with $0 < t < T_0 \cap T$, $\|(u, v, w)(t)\|_{L^\infty}$ is bounded. By the statement (ii) of Theorem 3.3, the maximal existence time constant $T_0$ of the classical solution obtained in Theorem 3.3 must be infinite. The non-negativity of the solution follows from (ii) of Theorem 3.3 directly. Then, the proof of Theorem 3.1 is finished.
4. Steady states

In this section, we study the non-trivial steady states of Equation (4) with homogeneous boundary conditions (6). Steady states of Equation (4) satisfy the system

\[
\begin{align*}
uxx - (uv_x)_x + (uw_x)_x &= 0, \\
D_v v_{xx} + u - \beta v &= 0, \\
D_w w_{xx} + \gamma u - \delta w &= 0.
\end{align*}
\] (31)

The non-trivial steady state of Equation (4) is defined as the solution of Equation (31), where none of \( u, v \) and \( w \) is a constant. In this paper, we only consider the simple case \( \beta/D_v = \delta/D_w = \mu \), which indicates that both the chemoattractant and chemorepellent have the same death rate relative to their diffusions, respectively. The result of the non-trivial steady state for the general system (31) still remains open. By defining \( \phi = v - w \), the system (31) can be transformed as

\[
\begin{align*}
uxx - (u\phi)_x &= 0, \\
\phi_{xx} + \lambda u - \mu \phi &= 0,
\end{align*}
\] (32)

where \( \lambda = 1/D_v - \gamma/D_w \). Then, integrating the first equation of Equation (32) and using the homogeneous boundary conditions (6), we have

\[ u = \eta e^\phi, \]

where \( \eta \) is a positive constant.

We substitute the expression for \( u \) into the second equation of Equation (32) and obtain an elliptic equation for the steady states:

\[ \phi_{xx} = \mu \phi - \lambda \eta e^\phi. \] (33)

This equation of steady states has been extensively investigated when \( \phi \geq 0 \) and \( \lambda > 0 \) (see, e.g. [6,12]). However, in our model, both the variable \( \phi \) and the constant \( \lambda \) can be non-positive. We write Equation (33) as a first-order Hamiltonian system

\[
\begin{align*}
\phi_x &= y, \\
y_x &= \mu \phi - \lambda \eta e^\phi.
\end{align*}
\] (34)

Without loss of generality, we assume \( \Omega = (0, L) \) with \( L > 0 \). Then, the Neumann boundary condition (6) becomes

\[ y(0) = y(L) = 0. \] (35)

Let \((\phi^*, y^*)\) be an equilibrium point of Equation (34). Then, the coefficient matrix of the linearized system of Equation (34) about \((\phi^*, y^*)\) is

\[
M = \begin{bmatrix} 0 & 1 \\ \mu - \lambda \eta e^{\phi^*} & 0 \end{bmatrix}.
\]

It is straightforward to see that the equilibria of Equation (34) satisfy \( y = 0 \) and

\[ \mu \phi = \lambda \eta e^\phi. \] (36)
Then, there are two cases to consider:

(1) When \( \lambda \leq 0 \), namely \( D_w \leq \gamma D_v \), Equation (36) always has a unique solution \( \phi^* < 0 \). The equilibrium \( (\phi^*, 0) \) is a saddle point for the linearized system due to \( \det M = -\mu + \lambda \eta e^{\phi^*} < 0 \). It is also a saddle for the full nonlinear system (34) by the Hartman–Grobman theorem. Since the nonlinear system has the Hamiltonian functional \( H(\phi, y) = y^2/2 - (\mu/2)\phi^2 + \lambda \eta e^\phi \), there are no non-trivial steady-state solutions satisfying the boundary condition (35) by a simple phase plane analysis.

(2) When \( \lambda > 0 \), namely \( D_w > \gamma D_v \), Equation (36) can have zero, one or two solutions depending on the parameters. It is straightforward to check that only the case of two solutions yields the non-trivial steady states. Equation (36) has two solutions \( 0 < \phi_1^* < \phi_2^* \) if and only if \( \mu > \lambda \eta e \), where \( \phi_1^* \) satisfies \( \mu > \lambda \eta e^{\phi_1^*} \) and \( \phi_2^* \) satisfies \( \mu < \lambda \eta e^{\phi_2^*} \) (see Figure 1). It is trivial to check that the equilibrium \( (\phi_1^*, 0) \) is a saddle and the equilibrium \( (\phi_2^*, 0) \) is a centre. Since the interval \([0, L]\) is bounded and the system (34) is Hamiltonian, by the standard phase plane analysis, we can readily show that for each \( L \), there is a non-trivial solution of Equations (34) and (35) which is a closed orbit. The non-trivial steady states are nested around the centre \( (\phi_2^*, 0) \).

Therefore, when \( D_w > \gamma D_v \), there is a non-trivial smooth solution to Equation (33), which satisfies the Neumann boundary condition \( \partial \phi / \partial v = 0 \) at \( x = 0, L \). Hence, the steady-state solution \( u = \eta e^\phi \) exists. Substituting it into the second equation of Equation (31) yields with Equation (6)

\[
-v_{xx} + \mu v = \frac{\eta}{D_v} e^{\phi(x)}, \quad 0 < x < L, \\
\frac{\partial v}{\partial v} = 0, \quad x = 0 \text{ or } L, 
\]

which is a linear elliptic equation with the Neumann boundary condition. Since the non-homogeneous term \( e^\phi \) is smooth for \( x \in (0, L) \), the smooth solution of Equation (37) exists (see, e.g. [2,9]). By the same argument, the solution \( w \) of the third equation of Equation (31) with the Neumann boundary condition can be obtained. In summary, we have the following theorem about the steady states of system (4) subject to the boundary condition (6).

**Theorem 4.1** Let \( \Omega = (0, L) \). Assume \( \beta/D_v = \delta/D_w \). If \( D_w \leq \gamma D_v \), the system (4) with the Neumann boundary conditions (6) has no non-trivial steady states. If \( D_w > \gamma D_v \), then a non-trivial steady state of the system (4) subject to Equation (6) exists for each \( L > 0 \).
From the above analysis and results, we see that the existence of non-trivial steady states of Equation (4) depends on the sign of parameter \( \lambda \) which relates the diffusion coefficients \( D_v \) and \( D_w \). Hence, if other parameters are fixed, the relative diffusivity of the chemoattractant to chemorepellent plays a prominent role in determining the nature of the steady states.

5. Summary

In this paper, we established the existence of global classical solutions and steady states to an attraction–repulsion chemotaxis model in one dimension. Our result does not exclude the possibility that the solution may blow up at infinity time. From the analysis of steady state, we found that the existence of non-trivial steady states depends on the ratio of the chemoattractant diffusion to the chemorepellent diffusion. The existence of global solutions of the attraction–repulsion chemotaxis model in multi-dimensional spaces still remains open, although it is more interesting to investigate. The pattern formation of the attraction–repulsion chemotaxis model is also an interesting issue which would be worthwhile to study in the future. In particular, the difference of the solution behaviour between the classical Keller–Segel model (i.e. attraction chemotaxis model) and the attraction–repulsion chemotaxis model needs to be investigated both analytically and numerically.

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