Sensitivity Sampling Over Dynamic Geometric Data Streams 
with Applications to $k$-Clustering

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Abstract

Sensitivity based sampling is crucial for constructing nearly-optimal coreset for $k$-means / median clustering. In this paper, we provide a novel data structure that enables sensitivity sampling over a dynamic data stream, where points from a high dimensional discrete Euclidean space can be either inserted or deleted. Based on this data structure, we provide a one-pass coreset construction for $k$-means clustering using space $\tilde{O}(k \text{poly}(d))$ over $d$-dimensional geometric dynamic data streams. While previous best known result is only for $k$-median [BFL+17], which cannot be directly generalized to $k$-means to obtain algorithms with space nearly linear in $k$. To the best of our knowledge, our algorithm is the first dynamic geometric data stream algorithm for $k$-means using space polynomial in dimension and nearly optimal in $k$.

We further show that our data structure for maintaining coreset can be extended as a unified approach for a more general classes of $k$-clustering, including $k$-median, $M$-estimator clustering, and clusterings with a more general set of cost functions over distances. For all these tasks, the space/time of our algorithm is similar to $k$-means with only poly$(d)$ factor difference.

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1 Introduction

Clustering is one of the central problems in modern research of computation, including algorithmic design, machine learning and many more. Among all the clustering methods, $k$-means is one of the most important approaches for clustering geometric datasets. The first $k$-means algorithm dates back to the 1950s. Over the last half of the century, varies works have studied this problem (see [Jai10] for a complete survey). It also inspired tremendous other variants of clustering methods, e.g. affinity propagation, mean-shift, spectral clustering, mini batch $k$-means and many more (e.g., [Scu10, CM02, SM00, MS01, NJW02, VL07]). This problem has also been studied in different settings and computational models, e.g. distributed computing, parallel computing, map-reduce, streaming setting and even quantum computing. $k$-means clustering has been successfully used in various research topics, e.g., database, data-mining, computer vision, geostatistics, agriculture, and astrophysics (e.g., [SM00, GKL15, LIY15, MS01]).

In 2004, Indyk [Ind04] introduced the model for dynamic geometric data streams, in which a set of geometric points from a $d$-dimensional discrete space $[\Delta]^d$ are updated dynamically, for some large $\Delta$, i.e., the data stream is of the form insert($p$) and delete($p$) for $p \in [\Delta]^d$. For a fixed dimension $d$, Frahling and Sohler [FS05] develop the first efficient streaming $(1 + \epsilon)$-approximation algorithms for $k$-means, $k$-median, and as well as other geometric problems over dynamic data streams. In their paper, they propose an algorithm to maintain a coreset of size $k\epsilon^{-O(d)}$ for obtaining a $(1 \pm \epsilon)$ approximation of $k$-clustering, including $k$-means and $k$-median. A coreset for clustering is a small weighted point set that summarizes the original dataset. Solving the clustering problem over the coreset provides an approximate solution for the original problem. It is one of the fundamental tool for solving $k$-means clustering problem.

A very recent work by Braverman, Frahling, Lang, Sohler & Yang [BFL+17] provides a data structure for maintaining a coreset for $k$-median in dynamic stream using space polynomial in $d$. However their technique cannot be extended to $k$-means, since the technique relies heavily on the fact that $k$-median cost function is a form of sum of absolutes, i.e., $\sum_{p \in P} \text{dist}(p, C)$, where $\text{dist}(\cdot, \cdot)$ is the Euclidean metric and $C$ is a set of points. Later we show a hard example (in Section C) that the error based on their design is unbounded for $k$-means, which is a form of sum of squares, i.e., $\sum_p \text{dist}^2(p, C)$. In this paper, we first show that by combining the coreset framework of [Che09] together with the grid structure constructed in [BFL+17] gives a data structure for maintaining a $k$-means coreset over dynamic stream that use space polynomial in $d$. Such a data structure uses space $\tilde{O}(k^2 \text{poly}(d))$. Although the space is not optimal in $k$, but it is already the first (to the best of our knowledge) of its kind obtaining space polynomial in $d$.

To obtain an algorithm with space nearly optimal in $k$ (i.e., linear in $k$), new idea has to be introduced. In 2011, [FL11] introduced a revolutionary coreset framework for constructing coreset in batch-setting and insertion-only stream. Their coreset framework is by sampling data points based on the “sensitivity” of each point. It is defined as the maximum percentage change of the cost function over all possible clustering solutions after removing the point from the dataset. Very recently, [BFL16] improves upon this framework and gives a better insertion-only streaming algorithm. One of our major contributions is to show that the sensitivity-based sampling scheme is achievable even in dynamic update streams. Hence we obtain another algorithm that is not only space nearly optimal in $k$ but also polynomial in dimension $d$.

We further show that our data structure for maintaining coreset can be extended as a unified approach for a more general classes of $k$-clustering, including $k$-median, $M$-estimator clustering, and clusterings with a more general set of cost functions over a distance. For all these tasks, the space/time of our algorithm is similar to $k$-means with only a $\text{poly}(d)$ factor difference. Here an $M$-estimator represents a specific function over the distance, e.g., the Huber norm [Hub64] is specified...
by a parameter $\tau > 0$, and its measure function $H$ is given by $H(x) = x^2/2\tau$, if $|x| \leq \tau$ and $H(x) = |x| - \tau/2$ otherwise. Recently, $M$-estimators attracted interests in a variety of computing fields (e.g., [NYWR09, CW15b, CW15a, SWZ17b]).

1.1 Related Works

It is well known that finding the optimal solution for $k$-means is NP-hard even for $k = 2$ or $d = 2$ [DFK+04, ADHP09, MNV09, Vat09]. The most success algorithm used in practice is Lloyd’s algorithm, which is also known as “the” $k$-means method [Llo82]. Since $k$-means can’t be solved in polynomial time, there has several works trying to understand the “local search method” for $k$-means. Kanungo et al. [KMN+02] proved that a very simple local search heuristic is able to obtain a polynomial-time algorithm with approximation ratio $9 + \epsilon$ for any fixed $\epsilon > 0$ for $k$-means in Euclidean space. Very recently, two groups improved the ratio to $1 + \epsilon$ independently, Friggstad, Rezapour and Salavatipour [FRS16] showed that, for any error parameter $\epsilon > 0$, the local search algorithm that considers swaps of up to $d^{O(d)} \cdot (1/\epsilon)^{O(d/\epsilon)}$ centers at a time will produce a solution using exactly $k$ centers whose cost is at most $(1 + \epsilon)$-factor greater than the optimum solution. Cohen-Addad, Klein and Mathieu [CAKM16] proved that the number of swapped centers is $\text{poly}(1/\epsilon)$.

There is a line of works targeting insertion-only streaming $k$-means or $k$-median. For example, [BS80, GMMO00, COP03, BDMO03, AHPV04, HP04, HPK05, Che09, FL11, FS12, AMR+12, BFL16] and many others have developed and improved streaming algorithms for computing a solution for $k$-means and $k$-median approximately. Recent years, there have been lots of interest in dynamic streaming algorithms for other settings, e.g. [BYJK+02, FKM+05, Bas08, KL11, AGM12a, AGM12b, GKP12, GKK12, AGM12b, BKS12, CMS13, CMS13, AGM13, McG14, BGS15, BHNT15, BS16, ACD+16, ADK+16, BWZ16, KLM+17, SWZ17a, SWZ17b]. In addition, there are many works related to $k$-median or $k$-means in different settings, e.g., [IP11, BIP+16, BCMN14, CCGG98].

1.2 Problem Setting

In this section, we formally define the problem of interests. We consider about the streaming setting, where the space of an algorithm is limited, i.e., we cannot store all the input points. In the dynamic data stream setting, we allow points to be deleted. Formally, let $Q$ denote a set of points on high dimensional grids $\Delta^d$, initialized as an empty set. At time $t$, we observe a tuple $(p_t, o_t)$ where $p_t \in \Delta^d$ and $o_t \in \{\text{insertion}, \text{deletion}\}$ meaning we insert a point to or delete a point from $Q$. Note that some of the points we observed in the stream might not belong to $Q$ at the end of the stream since deletion is allowed. After one-pass of the data-stream, we want to output a small multi-set of points (i.e., of size $o(|Q|)$) $S$ (associated with some weight-function $w$ to each point) which summarizes the the important properties of the ground-truth points set $Q$. Also we require our algorithm to have a fixed memory budget at any update time. This rules out the naïve approach of storing $Q$ in memory explicitly. Formally speaking, we require $S$ to be an $\epsilon$ $k$-means coreset for $Q$, which satisfies,

$$
\forall Z \subset \Delta^d, |Z| = k : (1 - \epsilon) \text{cost}(Q, Z) \leq \text{cost}_w(S, Z) \leq (1 + \epsilon) \text{cost}(Q, Z),
$$

\(^1\)We restrict the setting to be on the integer grid for the sake of representation simplicity. Our algorithm and analysis can be trivially extended to non-integer grid setting.
where
\[
\text{cost}(Q, Z) = \sum_{p \in Q} \min_{z \in Z} \text{dist}^2(p, z) \quad \text{and} \quad \text{cost}_w(S, Z) = \sum_{p \in Q} w_p \min_{z \in Z} \text{dist}^2(p, z)
\]

For other different clustering objectives, e.g., M-estimator clustering, our problem setting is roughly the same except the \(\text{dist}^2(\cdot)\) function is changed to the corresponding function.

### 1.3 Our Results

**k-Means** For a discrete geometric point set \(P \subset [\Delta]^d\), the \(k\)-means problem is to find a set \(C^* \subset [\Delta]^d\) with \(|C^*| = k\) such that the function \(\text{cost}(C) := \sum_{p \in P} \text{dist}^2(p, C)\) is minimized, where \(\text{dist}(p, C) = \min_{c \in C} \text{dist}(p, c)\) describing the minimal distance from \(p\) to a point in \(C\). We develop the first \((1 + \epsilon)\)-approximation algorithm for the \(k\)-means clustering problem in dynamic data streams that uses space polynomial in the dimension \(d\). To the best of our knowledge, all previous algorithms for \(k\)-means in the dynamic streaming setting required space exponentially in the dimension. Formally, our main theorem states,

**Theorem 1.1** (k-means). Fix \(\epsilon, \delta \in (0, 1/2)\), \(k, \Delta \in \mathbb{N}_+\), let \(L = \log \Delta\). There is a data structure supporting insertions and deletions of a point set \(Q \subset [\Delta]^d\), maintaining a weighted set \(S\) with positive weights for each point, such that with probability at least \(1 - \delta\), at any time of the stream, \(S\) is an \(\epsilon\)-coreset for \(k\)-means of size \(\epsilon^{-2} k^2 \cdot \log(1/\delta) \cdot \text{poly}(d, L)\). The data structure uses \(\tilde{O}(\epsilon^{-2} k^2 \cdot \log(1/\delta) \cdot \text{poly}(d, L))\) bits in the worst case\(^2\). For each update of the input, the algorithm needs \(\text{poly}(d, 1/\epsilon, L, \log k)\) time to process. After one pass, it outputs the coreset in time \(\text{poly}(d, k, L, 1/\epsilon, \log 1/\delta)\).

Note that, for the \(k\)-means problem, \([CEM+15]\) shows that one can always do the random projection to reduce \(d\) to \(O(k/\epsilon^2)\). However since random projection does not preserve the grid-like structure, it remains a caveat to use random projection method in dynamic geometric stream. Nevertheless, the most interesting setting would be that when \(d \leq O(k/\epsilon^2)\) and \(d \gg \log k\). The theorem is restated in Theorem E.5 in Section E and the proof is presented therein.

We further present another one-pass algorithm with different trade-offs in \(k\) and \(\epsilon\). This algorithm has space nearly linear in \(k\) while still polynomial in \(d\). The dependence on \(k\) is essentially nearly optimal (for fixed \(\epsilon\) and \(d\)! The guarantee of our result is presented in the following theorem.

**Theorem 1.2.** Fix \(\epsilon, \delta \in (0, 1/2)\), \(k, \Delta \in \mathbb{N}_+\), let \(L = \log \Delta\). There is a data structure supporting insertions and deletions of a point set \(Q \subset [\Delta]^d\), maintaining a weighted set \(S\) with positive weights for each point, such that with probability at least \(1 - \delta\), at any time of the stream, \(S\) is an \(\epsilon\)-coreset for \(k\)-means of size \(k \cdot \log(1/\delta) \cdot \text{poly}(d, L, 1/\epsilon)\),\(^3\) The data structure uses \(\tilde{O}(k \cdot \log^2(1/\delta) \cdot \text{poly}(d, L, 1/\epsilon))\) bits in the worst case. For each update of the input, the algorithm needs \(\text{poly}(d, 1/\epsilon, L, \log k)\) time to process. After one pass, it outputs the coreset in time \(\text{poly}(d, k, L, 1/\epsilon, \log 1/\delta)\).

**M-estimator Clustering and More General Cost Functions** Further we show that our algorithm and analysis on \(k\)-means can be extended to general functions (including M-estimator) over distances, i.e., a function that satisfies approximately sub-additivity: there is a fixed constant \(C > 0\),

\[
\forall x, y \geq 0, M(x + y) \leq C \cdot (M(x) + M(y)).
\]

\(^2\)For any function \(f\), we define \(\tilde{O}(f)\) to be \(f \cdot \log^{O(1)}(f)\).

\(^3\)The exact dependence of \(\epsilon\) is determined by the coreset framework in [FL11] and [BFL16], we show that \(\epsilon^{-3}\) dependence is sufficient. See the Section H for details.
Notice that the above equation automatically holds when $M(\cdot)$ is a non-decreasing function satisfying

$$\forall x > 0, M(x) > 0 \text{ and } \forall c > 0, M(cx) \leq f(c)M(x)$$

where $f(c) > 0$ is a bounded function. In this case, we are aiming to solve

$$\min_{Z \subset [\Delta]^d, |Z| \leq k} \sum_{z \in Z} \min_{z \in Z} M(\text{dist}(q, z))$$

for a given point set $Q$.

Although M-estimator clustering problem has been studied by [FS12, BFL16], their algorithm can only be applied in insertion only streaming model. We show the first dynamic streaming algorithms for this problem. Our data structure maintains a coreset with size similar to $k$-means, with only slightly different dependence on the dimension $d$. More precisely, if we extend the algorithm and the analysis of Theorem 1.1 to the M-estimator clustering setting, then we can get an algorithm with success probability at least $1 - \delta$ which can output an $\epsilon$-coreset for M-estimator clustering of size $\epsilon^{-2}k^2 \cdot \log(1/\delta) \cdot \text{poly}(d, L)$. The data structure uses $O(\epsilon^{-2}k^2 \cdot \log^2(1/\delta) \cdot \text{poly}(d, L))$ bits in the worst case, where the poly($d$) factor depends on the M-estimator function. For each update of the input, the algorithm needs $\text{poly}(d, 1/\epsilon, L, \log k)$ time to process and outputs the coreset in time $\text{poly}(d, k, L, 1/\epsilon, \log 1/\delta)$. We can also generalize the algorithm and the analysis of Theorem 1.2 to the M-estimator clustering setting. In this case, we can get a coreset of size $k \cdot \log(1/\delta) \cdot \text{poly}(d, L, 1/\epsilon)$, and the data structure uses $O(k \cdot \log^2(1/\delta) \cdot \text{poly}(d, L, \epsilon))$ bits in the worst case.

In Section F, we show the details of how to generalize the algorithm and analysis of Theorem 1.1 to M-estimator clustering setting.

**Maintaining Approximate Solutions for Max-Cut and Average Distance** We also show that our data-structure can be used to maintain an approximate solution for the Max-Cut and Average-Distance problems, where the first problem asks to find a cut of the streaming point set $Q$, such that the sum of distances of all possible pairs across the cut are maximized; the later problem asks to estimate average distance over all pairs. Our data structure supports maintaining a $1/2$-approximation to Max-Cut over all times of the stream, and estimating the cost of the cut up to $(1 \pm \epsilon)$ factor. For Average-Distance, our data structure supports maintaining a $(1 \pm \epsilon)$-approximation over all times of the stream. Furthermore, our data structure maintains approximate solutions for the generalized version of these problems, i.e., the M-estimator over distances. The data structure for these problems uses space polynomial in $1/\epsilon, d, \log \Delta, \log 1/\delta$, where $\delta$ is the failure probability. The formal results are presented in Section G.

### 1.4 Our Techniques

#### 1.4.1 $O(k^2)$-Space Algorithm with Chen’ Framework

We first show a data structure design whose underlying framework is based on [Che09]. Denote $\mathcal{X}_k = \{C \subset [\Delta]^d : |C| \leq k\}$ as the set of all $k$-sets. For a finite point set $P \subset [\Delta]^d$, an $\epsilon$-coreset for $k$-means of $P$ is a weighted point set $S \subset [\Delta]^d$ such that

$$\forall C \in \mathcal{X}_k : |\text{cost}(P, C) - \text{cost}(S, C)| \leq \epsilon \text{cost}(P, C)$$

where $\text{cost}(S, C) = \sum_{s \in S} w_s \text{dist}(s, C)^2$, where $w_s$ is the weight of point $s$. Our coreset algorithm can be viewed as a combination of several techniques in both clustering algorithms and dynamic
streaming algorithms. From a high level, we apply an oblivious grid structure over the point sets as used in [FS05] and [BFL+17] to form an implicit partition of the point sets. This partition satisfies the crucial property as required by the coreset framework of Chen [Che09]. We then build a dynamic streaming algorithm to simulate the random sampling of Chen’s framework. These combinations are highly non-trivial. We highlight several difficulties and how we resolve it in this paper.

- Firstly, a naive application of the grid structure in [FS05] gives a coreset with size exponentially depending on \( d \). To resolve this problem, we randomly shift the grid structure and show that the exponential dependence on \( d \) becomes polynomial.

- Secondly, in Chen’s framework, points are stored in memory, one can do straightforward sampling from the dataset. However, for us, the memory of the algorithm is limited and points can even be deleted. We do not know any of the sampling parameters before the stream coming. But these parameters are crucial to implement Chen’s framework. We resolve this problem by applying a clever data structure, which is similar to the K-set data structure as in [Gan05]. This data structure has a small memory budget, and can output the set of points under insertions and deletions if the size of final point set is smaller than the memory budget. Without knowing any parameters of the true dataset, we guess polylogarithmically many possibilities of all the parameters in Chen’s framework. We show that if the guessed parameters are close to the true parameters, the data sample is guaranteed to be smaller than the memory budget. This set of points form a coreset of the dataset and will be output by the sampling data structure.

Next we elaborate the details of each framework that we have been applying. We start with the introduction of Chen’s coreset framework.

**Chen’s Coreset Framework** The bottom level of our algorithm is the framework of Chen [Che09]. As shown in Figure 1, the core idea is to find a partition of the dataset \( P = P_1 \cup P_2 \cup \ldots \cup P_t \), for some \( t \leq \alpha k \) and \( \sum_{i=1}^{t} |P_i| \text{diam}(P_i)^2 \leq \beta \text{OPT} \), where \( \text{OPT} \) is the optimal cost of \( k \)-means on \( P \). We denote such a partition as an \((\alpha, \beta)\)-partition. If an \((\alpha, \beta)\)-partition is known, then one can sample \( \tilde{O}((\beta^2 k / \epsilon^2)) \) points uniformly at random from each part \( P_i \) such that the error of estimating the cost of the contribution of \( P_i \) is \( \epsilon |P_i| \text{diam}(P_i)^2 / \beta \). More formally, denote \( \mathcal{X}_k = \{ C \subset [\Delta]^d : |C| \leq k \} \) as the set of all \( k \) centers and \( S_i \) as the set of samples for \( P_i \). Assigning each point in \( S_i \) with a weight \( |P_i|/|S_i| \), then by a Hoeffding bound, we expect, with high probability,

\[
\forall C \in \mathcal{X}_k : |\text{cost}(P_i, C) - \text{cost}(S_i, C)| \leq \epsilon |P_i| \text{diam}(P_i)^2 / \beta.
\]

Combining the samples of each part, we obtain that \( S = \bigcup_{i=1}^{t} S_i \) is a coreset for \( k \)-means of \( P \). Notice that size of the coreset is \( \tilde{O}(\alpha k^2 \beta^2) \).

**Obtaining An \((\alpha, \beta)\)-partition in a Dynamic Stream** In an offline setting, i.e., the case that all points are stored in memory, obtaining an \((\alpha, \beta)\)-partition is quite straightforward, i.e., by using Indyk’s \((\alpha, \beta)\)-bi-criterion algorithm [Ind00a]. But it becomes challenging once the algorithm has limited memory and points can be deleted. We overcome this difficulty by applying a similar grid structure as used in [FS05] and [BFL+17]. As shown in Figure 1(b), we build \( \log \Delta \) many grids over the dataset. Each higher layer refines its parent layer by splitting each cell into \( 2^d \) many sub-cells. We stress that the grid structure is oblivious to the point set. Because of this crucial property, we are able to insert and delete points from the point set, and the grids stay intacted. Similar to [FS05], one can show that for each level \( i \), the number of cells containing more than \( O(\text{OPT} \cdot 4^i / \Delta^2 / k) \)
Figure 1: (a) The coreset framework of Chen [Che09]. The point set is partitioned into a set of sets, which are called a \((\alpha, \beta)\)-partition. Points are then sampled from each partition. See texts for details. (b) The grid structure over the point set. From top to bottom, we have four levels of grids. Each higher level partitions a cell in the parent level into \(2^d\) many sub-cells, where \(d\) is the dimension of \(d\).

The size of this partition is bounded by \(2^{O(d)} \cdot k\), independent of the number of points in \(P\). We shall call these cells as heavy cells, which also form a tree. Notice that the grid structure has only \(\log \Delta\) levels, the number of heavy cells is \(O(\log \Delta \cdot 2^{O(d)} \cdot k)\). Since each heavy cell has diameter \(\sqrt{d} \cdot \Delta / 2^i\) and the non-heavy children of a heavy cell in level \(i\) contains at most \(O(2^d \cdot OPT \cdot 4^i / \Delta^2 / k)\) points, thus in each level, the number of points in non-heavy children \(\cdot (\text{diameter of the heavy cell})^2\) is bounded by

\[
O(2^d \cdot OPT \cdot 4^i / \Delta^2) \cdot (\sqrt{d} \cdot \Delta / 2^i)^2 = O(2^d \cdot OPT / k).
\]

Applying the definition of the \((\alpha, \beta)\)-partition, we then show that these non-heavy children of heavy cells form a \((\alpha, \beta)\)-partition, here \(\alpha = O(2^d \cdot \log \Delta)\) and \(\beta = O(2^d \cdot \log \Delta)\). The majority of their work is showing how a dynamic streaming algorithm can maintain such a partition. We stress that their algorithm does not apply Chen’s framework, and thus their finally \(\epsilon\)-coreset is of size \(\tilde{O}(k / \epsilon^{O(d)})\) and the algorithm uses \(\tilde{O}(k^2 / \epsilon^{O(d)})\) space.

**Removing Exponential Dependence on \(d\)** In the last paragraph, we show that an \((\alpha, \beta)\)-partition can be implicitly obtained by simply building a grid structure over the dataset. However, the size of this partition is of exponential in \(d\). To get rid of this exponential dependence, we apply a random shift to the grid structure, as shown in Figure 2. We now explain why a random shift brings down the dependence on \(d\). As shown in the last paragraph, a heavy cell is defined by the number of points in it. Let us fix an optimal solution of \(k\)-means, i.e., a set of \(k\)-centers \(z_1, z_2, \ldots, z_k\). A random shift guarantees that, with high probability, any of the \(z_i\)s is “far” away from a boundary of a cell. Conditioning on this event, for the cells that do not contain an optimal center, every point in it contributes to the cost at least the distance of the optimal center to the boundary of a cell, which is “far”. Therefore, we are able to bound the number of cells containing no center but containing too many points. With the \(k\) cells containing a optimal center, the total number of
randomly shift

only small number of cells containing many points

Figure 2: Random shift of a grid brings down the number of heavy cells. In the left panel, we have a worst case alignment of points and grids that many cells contain lots of points. In the right panel, after the random shift, only two cells are containing many points.

“heavy cell”s is bounded by $O(\alpha \cdot k)$, where $\alpha = O(d \cdot \log \Delta)$. With additional tricks, we show an implicit $(\alpha, \beta)$-partition can be constructed from the heavy cells, where $\beta = (\text{poly}(d \log \Delta))$.

Sample Maintenance and Rejection Data Structure  With the random-shifted grid structure, we aim to sample points from the $(\alpha, \beta)$-partition, as it does in Chen’s construction. However, without knowing the parameters of the point sets, i.e., the optimal cost $\text{OPT}$, we have to guess its value. We guess logarithmically many possibilities of $\text{OPT}$, i.e., $o_i = 2^i$ for $i = 1, 2, \ldots, \log(d\Delta^d)$. For each guess, we run an sampler and attempt to get a coreset. Notice that, all these guesses and sampler based on Chen’s construction are oblivious to the input point set. Later we show that if the guess $o_i \leq \text{OPT}$, then the above mentioned samples (with the sampling probability and heavy cells defined by $o_i$) form an $\epsilon$-coreset to the point set. However if $o_i / \text{OPT}$ is too small, the coreset is too large. We construct a new dynamic set-point sampler data structure that has a fixed memory budget and rejects large data samples but preserve data samples with a small number of points.

With this data structure, we can pick the smallest $o_i$, whose resulting samples are preserved by the data structure. As a result, these samples are the correct coreset.

Now we elaborates the details of the data structure. This data structure supports insertions and deletions over pairs of the form $(C, p)$, where $C$ represents a set and $p \in [\Delta]^d$ is point. The data structure supports querying all the points in sets with small number of points, at any time of the stream. Suppose there are at most $\alpha$ non-empty sets and at most $\theta$ sets containing at least $\beta$ points and the rest of the sets containing $\gamma$ points. Then with a memory budget of $[\Theta(\theta \beta + \gamma)]$, the algorithm is able to maintain the number of points in each cell, and the $\gamma$ points in cells with less than $\beta$ points. The high level idea is to use two level of pair-wise hash functions. In the first level, we hash the ID of the set (i.e., the name or the coordinate if the set is a cell) to a universe $[\Theta(\alpha)]$ and the point to a universe $[\Theta(\beta)]$, then we hash the pair of hash values of the set ID and the point to a universe $[\Theta(\theta \beta + \gamma)]$. It can be shown that if a point $c$ from a set $C$ with less than $\beta$ points, then the third hash value is unique to all other pairs, with at least constant probability. Use this unique hash value, plus a sanity checker based on parity we can recover each bit of the point and the ID of the set. By repeating $\log 1/\delta$ times, we can recover all the $\gamma$ points from sets with less than $\beta$ points.
1.4.2 \(\tilde{O}(k)\)-Space Algorithm (Nearly Optimal) with Sensitivity-based Sampling

Next we show our techniques for improving the space complexity from \(\tilde{O}(k^2)\) to \(\tilde{O}(k)\). The high level idea of our data structure is to simulate a sensitivity-based sampling using an oblivious linear sketch. Our sampling scheme takes the advantages of most of the data structures constructed in the last section. To begin, we first briefly review the coreset framework of [FL11] and [BFL16].

A Brief Review of the Coreset Framework in [FL11] and [BFL16] In [FL11] and [BFL16], they have proposed a framework called “sensitivity” sampling. Since techniques are similar for distance functions other than \(\ell_2^2\), we take \(k\)-means for an example. Let \(Q\) be the set of points, let \(\mathcal{X}^k \subset [\Delta]^{d \times k}\) be the set of all possible \(k\)-centers. Then the sensitivity of a point \(q \in Q\) is defined as

\[
s(q) = \max_{Z \in \mathcal{X}^k} \frac{\text{dist}^2(q, Z)}{\sum_{p \in Q} \text{dist}^2(p, Z)}.
\]

Namely, \(s(q)\) denotes the maximum possible change in the cost function of any \(k\)-set \(Z\) after removing the point \(q\) in \(Q\). It can be easily shown that \(\sum_q s(q) \approx k\). Suppose one designs a uniform upper bound \(s'(\cdot)\) such that \(\forall q \in Q : s'(q) \geq s(q)\), and denote a probability distribution \(\mathcal{D}\) over \(Q\) as each point gets a probability \(s'(q)/\left(\sum_{p \in Q} s'(p)\right)\). The framework in [FL11] and [BFL16] shows that, and let \(R\) be a set of i.i.d. samples sampling from \(\mathcal{D}\) with \(|R| \geq \sum_{q \in Q} s'(q)/\epsilon^2\) and assign each point \(q\) in \(Q\) a weight \(|R|^{-1}\), then \(R\) is with high probability a \((1 \pm \epsilon) k\)-means coreset for \(Q\). The proof is by a establishing connection between the VC-dimension theory and the coreset theory. If \(\sum_{q \in Q} s'(q)\) is not too different from \(\sum_{q \in Q} s(q)\), then a \(\tilde{O}(k)\)-sized coreset is constructed. Vaguely speaking, the reason that sensitivity sampling removes a \(k\) factor from our \(k^2\)-construction is by constructing a new functional space (from the the \(\geq k\) samples), whose VC-dimension is \(O(d)\) instead of \(O(kd)\).

Sensitivity Sampling Over Dynamic Data Stream Analogously to the framework described in the last section, we describe an algorithm that simulates the sensitivity sampling over the dynamic stream. As we have shown, to correctly assign the sampling probability, all we need is a upper bound on the sensitivity. For each point \(p \in [\Delta]^d\), if \(p\) is in the set \(Q\) at the end of the stream, then it must have a corresponding sensitivity. Denote its true sensitivity as \(s(p)\). We want to design a “good” upper bound \(s'(p)\) of the sensitivity of each point \(p\) such that

1. the sum of \(s'(p)\) is not too large;
2. we are able to i.i.d. sample \(m\) points from the final undeleted points such that each point is chosen with probability proportional to \(s'(p)\) in each sample, where the magnitude of \(m\) is influenced by the sum of \(s'(p)\);
3. at the end of the stream, we are able to approximate \(s'(p)\) for each given point \(p\).

The first property ensures that \(m\) will not be too large which means that the size of the coreset is small. The second property ensures that we can finally obtain the points in the coreset by implementing such sampling procedure. The third property shows that we can obtain the weights of the points in the coreset.

In the previous section, we gave the concepts of “heavy cell” and the \((\alpha, \beta)\) partition based on the random shifted grid. Precisely, if a cell in the \(i^{th}\) level which contains at least \(T_i\) number of points, then we say the cell is “heavy” where \(T_i\) is a threshold parameter which is set to be \(\Theta\left(\frac{\nu^2}{k} \cdot \frac{\OPT}{(\Delta/2)^2}\right)\).
in our work. As discussed in the previous section, all the non-“heavy” cells whose parent cell is “heavy” formed a partition. We call such non-“heavy” cell as a partition cell. Since the cells formed a partition, it is easy to see, for each undeleted point, it must belong to a unique partition cell. Furthermore, if the unique partition cell which contains point \( p \) is in the \( i \)th level of the grid, then we say \( p \) is a partition point in the \( i \)th level. Then we show that all the points which are partition points in the \( i \)th level have a universal sensitivity upper bound which is \( \Theta(d^3/T_i) \). Furthermore, if we set the sensitivity upper bound of \( p \) as \( s'(p) = \Theta(d^3/T_i) \), it is easy to argue that \( \sum_{p \in Q} s'(p) \) cannot be too large. The reason is that

\[
\sum_{p \in Q} s'(p) = \sum_{\text{level } i} (\# \text{ of partition points in the } i\text{th level}) \cdot \Theta(d^3/T_i)
\leq \sum_{\text{level } i} (\# \text{ of heavy cells in the } i\text{th level}) \cdot \Theta(d^3)
= (\# \text{ of heavy cells}) \cdot \Theta(d^3)
\leq \alpha \cdot k \cdot \Theta(d^3),
\]

where the first inequality follows by that each level \( i \) heavy cell contains at least \( T_i \) number of points. The second inequality follows by the last section, and \( \alpha \) is as the same as mentioned in the last section.

Since \( \sum_{p \in Q} s'(p) \) is not large, we know that the size of the coreset will be small. Also notice that when we know a point \( p \) is a partition point in the \( i \)th level, then we already know \( s'(p) \). By last section, we know that we can have a streaming algorithm which can find out all the heavy cells which means that we can know the shape of the whole partition at the end of the stream. Thus, for any given point \( p \), we can determine its partition cell, and thus know the \( s'(p) \) which means that for each point in the coreset, we could calculate its weight. Now, the problem remaining is to get \( m \) i.i.d. sample points, and for each sample, each point \( p \) is chosen with probability proportional to \( s'(p) \).

A challenge to implement the sampling procedure is that the sampling scheme here is different from the sampling procedure described in the previous section. Note that in the previous section, the sampling scheme is that we independently determine each point should be chosen or not, but here we need to get \( m \) independent samples where each sample is a point in \( Q \), and the probability that \( p \) is chosen is proportional to \( s'(p) \). We cannot directly use the sampling scheme described in the previous section. However we can handle this issue by two-stage sampling. A good property of our \( s'(p) \) is that all the partition points in the \( i \)th level have the same \( s'(p) \). To sample a point \( p \) with probability proportional to \( s'(p) \) is equivalent to firstly sample a level \( i \), where each level \( j \) is sampled with probability proportional to (\# of partition points in the \( j \)th level) \cdot s' \( \text{where } s'_j = s'(p) \) for point \( p \) which is a partition point in the \( j \)th level; then we uniformly sample a partition point in the \( i \)th level. To implement the first sampling stage, we just need to know the number of partition points in each level. Fortunately, we can achieve this by using the streaming algorithm described by the last section. Now let us focus on the second stage. In this stage, we want to implement the uniform sampling oracle over all the partition points in the \( i \)th level. We cannot apply the traditional \( \ell_0 \) sampler here since we do not have any information of the partition points in each level at the beginning of the stream. To achieve our goal, we should be more carefully. We firstly subsample all the points in the \( i \)th level, we just use the streaming data structure described in the last section to maintain all the survived points. We show that by using the data structure we are able to recover all the survived partition points. Then for all the survived partition points in the \( i \)th level, we then uniformly choose a survivor as the output. A potential problem is that if in the \( i \)th level, the number of partition points is small, then it is highly possible that none of the partition
Algorithm 1 A Meta-Algorithm for Point Sampling From a Dynamic Data Stream

```plaintext
procedure PointSampler(P)  \triangleright reads points set P in the data-stream
  Let \( \mathcal{O} = \{1, 2, 4, 8, \ldots, 2^{\lceil d \log(d \Delta) \rceil} \}, \mathcal{L} = \{-1, 0, 1, \ldots, \lceil \log \Delta \rceil \} \).
  Choose randomly shifted \(|\mathcal{L}|\) layers grids
  Create \(|\mathcal{O}|\) independent KSet instances (with limited memory budget) for each layer \( l \in \mathcal{L} \)
  For each \((o, l)\), create a set of hash functions \( \mathcal{H}_{t, o} \), each \( h \in \mathcal{H}_{t, o} \) is a function maps \( [\Delta]^d \rightarrow \{0, 1\} \)
  for each update of a point \( p \in [\Delta]^d \) in the data-stream do
    Create \(|\mathcal{H}_{t, o}|\) copies of \( p \) as the form \((p, i)\) for each \( h_i \in \mathcal{H}_{t, o} \)
    for \((o, l) \in \mathcal{O} \times \mathcal{L} \) and \( h_i \in \mathcal{H}_{o,l} \) do
      Update \((o, l)\)-th sketch: if \( h_i(p) = 1 \) then \( \text{KSet}_{o,l}.update(p, i) \)
    end for
  end for
  Choose the smallest \( o^\star \in \mathcal{O} \) such that all the \( \{\text{KSet}_{o^\star, l}\}_{l \in \mathcal{L}} \) succeed.
  Output the sampled point sets and grid cells given by \( \{\text{KSet}_{o^\star, l}\}_{l \in \mathcal{L}} \)
end procedure
```

point can be survived. But this will not introduce a problem since if the number of partition points in the \( i \)th level is small, the probability that level \( i \) is chosen in the first stage is small. More precisely, suppose we need \( m \) i.i.d. samples in total, then with high probability the number of times that the first sampling stage samples level \( i \) is about \( m \cdot \left( \text{# of partition points in the } i \text{th level} \right) \cdot s_i^1 / \left( \sum_j \left( \text{# of partition points in the } j \text{th level} \right) \cdot s_j^1 \right) \). Suppose for each level \( i \), we prepared \( 1.1m \) many uniform sampling oracles, we just need to guarantee that the number of success oracle is at least \( m \cdot \left( \text{# of partition points in the } i \text{th level} \right) \cdot s_i^1 / \left( \sum_j \left( \text{# of partition points in the } j \text{th level} \right) \cdot s_j^1 \right) \), then it is guaranteed to have sufficient many samples. In our work, we show that, for the uniform sampling oracle in level \( i \), we just need the drop probability to be \( \text{poly}(d, \epsilon^{-1}, \log(\Delta), \log(1/\delta)) \cdot \frac{1}{kT_i} \), it is enough to achieve our goal. Thus, we have enough success uniform samplers for each level.

1.4.3 Max-Cut and Average-Distance

We use our coreset data structure to obtain solutions for max-cut and average distance over a dynamic dataset. The basic idea is to use the 1-means coreset as a proxy for estimating pairwise distance. We reduce the Max-Cut instance and Average-Distance instance to a instance of estimating the distance of a point to a subset of the original point set. Hence 1-means coreset is sufficient for this case.

1.5 Meta Algorithms and a Roadmap

All our algorithms share a similar meta-structure to sample points from a dynamic data stream. The meta-algorithm is presented in Algorithm 1. This meta algorithm is an oblivious linear sketch over the input data set (i.e., the algorithm does not need to know the actual data set). If we view the data set as a binary vector in \( \{0, 1\}^{\Delta^d} \), then our meta-algorithm converts the vector linearly into \( \Theta(d \log \Delta) \) binary vectors in \( \{0, 1\}^{\Delta^d} \) and \( \Theta(d \log \Delta) \) vectors in \( \mathbb{Z}^{\Delta^d} \), i.e., it gives \( \Theta(d \log \Delta) \) level of points samples and counts of the number of points in grid cells of each level. These output vectors are very sparse, and hence can be stored in limited memory budget.

In the meta algorithm, we first build random shifted grids \( G_{-1}, G_0, G_1, \ldots, G_L \) where \( L = \log \Delta \) and each grid \( G_i \) refines its parent by splitting each cell into \( 2^d \) cells evenly. Then for each pair of \((o, l)\) where \( o \in \{1, 2, 4, \ldots, 2^{\lceil d \log(d \Delta^{d+1}) \rceil}\}, l \in \{-1, 0, \ldots, L\} \), we choose a set of random hash
functions $\mathcal{H}_{l,o,e,\delta}$, where each $h_i \in \mathcal{H}_{l,o,e,\delta}$ is a function $h_i : [\Delta]^d \rightarrow \{0, 1\}$ such that $\forall p \in [\Delta]^d$, $\Pr[h_i(p) = 1]$ is defined according to the specific tasks (i.e., for the $\tilde{O}(k^2)$ and $\tilde{O}(k)$ algorithms, we chose different sampling probability), and we initialize a point maintainer KSet (see Algorithm 2). Each KSet data structure has a limited memory budget. The KSet data structure can succeed only if the number of points and grid cells sampled (by the hash functions $h_i$) is under the memory budget (otherwise, the information stored by KSet is treated as garbage). When we scan the stream, for each insert/delete operation, suppose the point is $p$, we first check whether $h_i(p) = 1$. If it is 0, we just ignore this operation. Otherwise, for each level $l$ find the cell $C \in G_l$ which contains $p$, and for each $o$, we use the information of $C, p$ and the operation type (ins/del) to update the point maintainer which corresponds to $(o, l)$. We then build our coreset from the succeeded KSet instances with the smallest parameter $o$.

We provide some basic notation in Section A. Section B states some definitions and useful tools from previous works. In Section D, explain how to construct a coreset using randomized grid structure and hash functions over the universe $[\Delta]^d$. In Section E, we show how to maintain the coreset in a dynamic stream. Section F presents the general result for M-estimator and also improvement compared to previous $k$-median result. Section G shows how to extend our $k$-means to some other geometric problems. We provide improved one-pass algorithm for $k$-means based on sensitivity sampling in Section H.

1.6 Concluding Remark

In this paper we present two algorithm for obtaining $\epsilon$-coreset for the $k$-means problem in high-dimensional dynamic geometric data streams. Our first algorithm is a one-pass algorithm and takes space $\tilde{O}(k^2 \epsilon^{-2} \text{poly}(d, \log \Delta))$ based on Chen’s framework [Che09]. Our second algorithm is a one-pass algorithm and takes space $\tilde{O}(k \text{poly}(d, \log \Delta, 1/\epsilon))$ based on sensitivity sampling. Both of the algorithm take space polynomial in $d$. To the best of our knowledge, these are the first results for obtaining $k$-means coreset using space polynomial in $d$. In particular, our second algorithm is nearly optimal with regard to parameter $k$. Furthermore, the coresets output by our algorithms consist only positive weighted points. One can run her favorite offline algorithms to obtain the desired approximated solutions. Both our algorithm can be extended to a much general set of cost functions, e.g., the $M$-estimator.

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A Notation

For an \( n \in \mathbb{N}_+ \), let \([n]\) denote the set \( \{1, 2, \cdots, n\} \). Given integers \( m \leq n \), we denote \([m, n] = \{m, m+1, m+2, \ldots, n\}\) as the integer interval.

For any function \( f \), we define \( \tilde{O}(f) \) to be \( f \cdot \log^{O(1)}(f) \). In addition to \( O(\cdot) \) notation, for two functions \( f, g \), we use the shorthand \( f \lesssim g \) (resp. \( f \gtrsim g \)) to indicate that \( f \leq Cg \) (resp. \( g \geq Cf \)) for an absolute constant \( C \). We use \( f \tilde{=} g \) to mean \( cf \leq g \leq Cf \) for constants \( c, C \).

B Preliminaries

B.1 Definitions of Dynamic Streaming Model for \( k \)-Clustering Problems

In this section, we give the definition of the input form and the computational model.

Definition B.1 (Dynamic streaming model for \( k \)-clustering). Let \( P \subset [\Delta]^d \) initially be an empty set.

In the dynamic streaming model, there is a stream of update operations such that the \( q \)th operation has the form \((\text{op}_q, p_q)\) where \( \text{op}_q \in \{\text{ins}, \text{del}\}, p_q \in [\Delta]^d \) which means that the set \( P \) should add a point \( p_q \) or should remove the point \( p_q \). An algorithm is allowed 1-pass/2-pass over the stream. At the end of the stream, the algorithm stores some information regarding \( P \). The space complexity of an algorithm in this model is defined as the total number of bits required to describe the information the algorithm stores during the stream.

B.2 Definitions of \( k \)-means Clustering

For representation simplicity, we restrict the dataset to be from the discrete space \([\Delta]^d\) for some large integer \( \Delta \) and \( d \). The extension of our results to \( \mathbb{R}^d \) is straightforward. We will discuss this issue more in the concluding remarks. We use \( \text{dist}(\cdot, \cdot) \) as the Euclidean distance in space \([\Delta]^d\), i.e., for any \( p, q \in [\Delta]^d \),

\[
\text{dist}(p, q) = \|p - q\|_2,
\]

where \( \|\cdot\| \) is the \( \ell_2 \) norm. We further extend the distance definition to set and point, and set to set. Formally, let sets \( P, Q \subset [\Delta]^d \) and point \( p \in [\Delta]^d \), then

\[
\text{dist}(p, Q) = \text{dist}(Q, p) = \min_{q \in Q} \text{dist}(p, q) \quad \text{and} \quad \text{dist}(P, Q) = \min_{p \in P, q \in Q} \text{dist}(p, q).
\]

We also denote \( \text{diam}(Q) \) diameter of \( Q \), i.e., the largest distance of any pair of points in \( Q \).

Definition B.2 (\( k \)-means clustering). Given an input point set \( Q \subset [\Delta]^d \), the \( k \)-means clustering problem is to find a set of \( k \) points \( Z \subset [\Delta]^d \), such that the following objective is minimized.

\[
\text{cost}(Q, Z) = \sum_{q \in Q} \text{dist}^2(q, Z).
\]

Each point of \( Z \) is also called a center. Note that a set \( Z \) defines a partition of the point set \( Q \) by assigning each point \( q \in Q \) to the closest center in \( Z \) (ties are broken arbitrarily). We use \( \text{OPT} \) to denote the minimum cost of the \( k \)-means problem.

Definition B.3 (Coreset for \( k \)-means). Given an input point set \( Q \subset [\Delta]^d \), an \( \epsilon \)-coreset \( S \) of \( Q \) is a multiset, usually of smaller size, and summarizes the important structures of \( Q \). The solution of the optimization problem on \( S \) is an approximate solution on \( Q \). Formally, let \( S = \)}
\{(s_1, w_1), (s_2, w_2), \ldots\} be an $\epsilon$-coreset for $Q$, where each $s_i \in [\Delta]^d$ and $w_i \in \mathbb{R}$ is the weight of $s_i$. Then $S$ satisfies
\[ \forall Z \subset [\Delta]^d, |Z| = k : |\text{cost}(S, Z) - \text{cost}(Q, Z)| \leq \epsilon \text{cost}(Q, Z), \]
where
\[ \text{cost}(S, Z) := \sum_{s_i \in S} w_i \text{dist}^2(s_i, Z). \]

**Definition B.4** (Coreset for $k$-means in dynamic stream). Given a point set $P \subset [\Delta]^d$ described by a dynamic stream, an error parameter $\epsilon \in (0, 0.5)$, and an failure probability parameter $\delta \in (0, 1)$, the goal is to design an algorithm in the dynamic streaming model (Definition B.1) which can with probability at least $1 - \delta$ output an $\epsilon$-coreset for $k$-means (Definition B.3) with minimal space.

**B.3 Definitions of $M$-estimator Clustering**

Our coreset framework can also be extended to arbitrary clustering of using $M$-estimators.

**Definition B.5** ($M$-estimator Clustering). We define function $M : \mathbb{R} \to \mathbb{R}$ to be a $M$-estimator. We define
\[ \text{cost}_M(Q, Z) = \sum_{q \in Q} \min_{z \in Z} M(\text{dist}(q, z)). \]
The goal of a $M$-estimator clustering to solve the following optimization problem.
\[ \min_{Z \subset [\Delta]^d, |Z| \leq k} \text{cost}_M(Q, Z). \]

**Definition B.6** (Coreset for $M$-estimator). Given an input point set $Q \subset [\Delta]^d$. An $\epsilon$-coreset $S$ of $Q$ is a multiset, usually of smaller size, and summarizes the important structures of $Q$. The solution of the optimization problem on $S$ is an approximate solution on $Q$. Formally, let $S = \{(s_1, w_1), (s_2, w_2), \ldots\}$ be an $\epsilon$-coreset for $Q$, where each $s_i \in [\Delta]^d$ and $w_i \in \mathbb{R}$ is the weight of $s_i$. Then $S$ satisfies that $\forall Z \subset [\Delta]^d$ with $|Z| = k$,
\[ |\text{cost}_M(S, Z) - \text{cost}_M(Q, Z)| \leq \epsilon \text{cost}_M(Q, Z), \]
holds, where $\text{cost}_M(S, Z) := \sum_{s_i \in S} w_i \min_{z \in Z} M(\text{dist}(s_i, z))$.

**Definition B.7** (Coreset for $M$-estimator clustering in dynamic stream). Given a point set $P \subset [\Delta]^d$ described by a dynamic stream, an error parameter $\epsilon \in (0, 0.5)$, and an failure probability parameter $\delta \in (0, 1)$, the goal is to design an algorithm in the dynamic streaming model (Definition B.1) which can with probability at least $1 - \delta$ output an $\epsilon$-coreset for $k$-center $M$-estimator clustering (Definition B.6) with minimal space.

**B.4 Basic Probability Tools**

**Lemma B.8** (Bernstein inequality [HPM04]). Let $X_1, X_2, \ldots, X_n$ be independent zero-mean random variables. Suppose that $|X_i| \leq b$ almost surely, for all $i \in [n]$. Let $\sigma^2$ denote $\sum_{j=1}^{n} \mathbb{E}[X_j^2]$. Then for all positive $t$,
\[ \Pr \left[ \sum_{i=1}^{n} X_i > t \right] \leq \exp \left( -\frac{t^2}{2\sigma^2 + 2bt/3} \right). \]
B.5 Tools from Previous Work

**Theorem B.9** ([LNNT16]). Given parameters $k \geq 1$, $\epsilon \in (0, 1/2)$, $\delta \in (0, 1/2)$. There is a randomized (one-pass) algorithm that uses $O((k+1/\epsilon^2) \log n / \delta \cdot \log m)$ bits of space, requires $O(\log n)$ time per update, needs $O(k+1/\epsilon^2)$ poly($\log n$) decoding time. For all $i \in [n]$, let $f_i \in [-\text{poly}(n), \text{poly}(n)]$ denote the frequency of $i$ at the end of the data stream. Without loss of generality, we assume that $f_1 \geq f_2 \geq \cdots \geq f_n$. The algorithm is able to output a set $H$ with size $O(k+1/\epsilon^2)$ such that, with probability $1 - \delta$,

- Property (I) : for all $(i, \hat{f}_i) \in H$, $f_i^2 \geq \sum_{j=1}^n f_j^2 / k - \epsilon^2 \sum_{j=k+1}^n f_j^2$;
- Property (II) : for all $i \in [n]$, if $f_i^2 \geq \sum_{j=1}^n f_j^2 / k + \epsilon^2 \sum_{j=k+1}^n f_j^2$, then $(i, \hat{f}_i) \in H$;
- Property (III) : for all $(i, \hat{f}_i) \in H$, $|\hat{f}_i - f_i| \leq \epsilon (\sum_{j=k+1}^n f_j^2)^{1/2}$.

C Why Do Previous Techniques Fail?

**[FIS05]** [FIS05] is one of the early works using sampling procedure to solve dynamic streaming geometric problem. They show that it is possible to use point samples over a dynamic point set as a subroutine to solve several geometric problem, e.g. Euclidean Minimum Spanning Tree. However, they only show how to implement the uniform sampling by using counting distinct elements and subsampling procedure as subroutines. In this work, we require assigning points different “importance”. The bottom level sampling scheme of ours is similar to theirs, but ours requires a much complicated framework to implement the importance sampling over the point sets.

**[Ind04]** This paper uses a critical observation to estimate the cost of $k$-median, that is: let $Z$ be a set of centers, and $P$ be the point sets then $\text{cost}(Z, P) = \int_0^\infty |P - B(Z, r)|dr$, where $B(Z, r)$ is union of neighborhoods (of radius $r$) of every point in $Z$. Then this integration is approximated by a summation with logarithmic levels, i.e., $\int_0^\infty |P - B(Z, r)|dr = \sum_{i=1}^\infty |P - B(Z, r^{i+1})|(r^{i+1} - r^i)$, where $r^i = O(\epsilon(1 + \epsilon)^i)$. The critical part is to estimate $|P - B(Z, r^{i-1})|$. This paper constructed a counting data structure based on grids with side length $O(r^i)$. Then every input point is snapped to a grid point. To obtain sufficiently accurate counts for each $|P - B(Z, r^{i-1})|$, the data structure needs to query $|Z|/\epsilon^d$ many grid points per $Z$. Such a data structure is implemented using pair-wise independent hash functions, and uses memory $O(|Z|/\epsilon^{O(d)})$.

Notice that this paper gives only an approximation of the cost. It is not a coreset. Therefore, to obtain the $k$-median solution, an exhaustive search is used. Furthermore, their technique fails to extend to $k$-means, which lacks the integration formula of the cost function.

**Why Does [FS05] require exponential dependence in $d$, $(1/\epsilon)^d$?** The grid structure of our paper is inspired by [FS05]. In [FS05], they first build a deterministic quadtree-like structure: Consider a big cube containing the entire data set. This cube is treated as the root cell of the tree. Going down a level, they partition the parent cell into $2^d$ subcells. Then each leaf of the tree contains at most one single point of the dataset. Notice that the cell side length decrease geometrically as the level increase. They mark a cell as “heavy” if the cell containing enough points that moving all points in the cell to the center of the cell incurs too much error in the cost of an optimal $k$-means/median solution. Since the side length (or diameter) of cells decreases as level increases, the number of points required to have this effect becomes larger. Eventually, all cells are non-heavy after some level. As such we also have a tree of heavy cells. For each heavy cell, the coreset is constructed by assigning each point in the non-heavy children to its center. It turns out if we want an epsilon-approximated coreset, the threshold of the non-heavy cell is exponential in $d$,
i.e., each non-heavy cell in level $i$ cannot contain more than $\tilde{O}(e^{O(d)} \cdot \text{OPT} / 2^i)$ points, where $\text{OPT}$ is the optimal cost. This small threshold gives $\tilde{O}(1/e^{O(d)})$ many heavy cells.

**Why do the previous insertion-only streaming coreset construction algorithms fail for dynamic stream model?** Many of the previous insertion only streaming coreset construction algorithms (e.g., [FS12]) heavily depend on a merge-reduce technique, i.e. read some points in the stream, construct a coreset, then read another part, construct a new coreset, and merge the two coresets. They repeat this procedure until the stream ends. This technique works well in insertion only streaming model. But once some points get deleted, the framework fails immediately. Though [BFL16] shows a new framework other than merge-reduce, their algorithm relies on a non-deletion only streaming model. But once some points get deleted, the framework fails immediately. Though [BFL16] shows a new framework other than merge-reduce, their algorithm relies on a non-deletion only streaming model. Though [BFL16] shows a new framework other than merge-reduce, their algorithm relies on a non-deletion only streaming model.

**Why Do [BFL+17] fail to extend to $k$-means?** Though some $k$-median coreset construction techniques can be easily extended to $k$-means coreset construction (see e.g. [BFL16, FS12]), their construction can only be implemented in insertion only streaming model.

[BFL+17] showed a $k$-median coreset construction in dynamic streaming model. But their construction cannot be extended to $k$-means coreset construction directly. Their $k$-median algorithm heavily relies on writing the cost of each point as a telescope sum. For example, we consider the 1-median problem. Let $z$ be a candidate center point and $p \in P$ be a point, then \( \text{dist}(p, z) = \text{dist}(p, z) - \text{dist}(c_p^{L-1}, z) + \text{dist}(c_p^{L-1}, z) - \text{dist}(c_p^{L-2}, z) + \cdots - \text{dist}(c_p^0, z) \), where each $c_p^i$ is the center of the cell in the $i$-th level containing $p$. Therefore, the total 1-median cost $\sum_{p \in P} \text{dist}(p, z)$ of point set $P$ on $z$ can be split into $L$ pieces, i.e., $\sum_{p \in P} (\text{dist}(c_p^i, z) - \text{dist}(c_p^{i-1}, z))$ for each $i \in [L]$. [BFL+17] estimates the cost of each $L$ pieces by sampling points, i.e., let $S^i$ be the samples in the $i$-th level, then the estimator to $\sum_{p \in P} (\text{dist}(c_p^i, z) - \text{dist}(c_p^{i-1}, z))$ is $\sum_{p \in S^i} (\text{dist}(c_p^i, z) - \text{dist}(c_p^{i-1}, z)) / \zeta_p$, where $\zeta_p$ is the probability that the point $p$ is sampled in the samples $S^i$. A crucial observation is that we have $| \text{dist}(c, z) - \text{dist}(c^{i-1}, z) | \leq \Delta/2^i$ – the cell size of level $i$ which is independent from the location of $z$. Since this nice upper bound on $| \text{dist}(c, z) - \text{dist}(c^{i-1}, z) |$, [BFL+17] then can apply Bernstein’s inequality to show the high concentration of the estimator $\sum_{p \in S^i} (\text{dist}(c_p^i, z) - \text{dist}(c_p^{i-1}, z)) / \zeta_p$ with only $\tilde{O}(1/e^2)$ samples per level. But this framework does not work for 1-means even though one can still write the telescope sum structure as $\sum_p (\text{dist}^2(c_p^i, z) - \text{dist}^2(c_p^{i-1}, z))$, and we can still setup the estimator as $\sum_{p \in S^i} (\text{dist}^2(c_p^i, z) - \text{dist}^2(c_p^{i-1}, z)) / \zeta_p$. But $| \text{dist}^2(c, z) - \text{dist}^2(c^{i-1}, z) |$ is not upper bounded by the cell size. Instead, the magnitude is depending on the location $z$. For example, it can be as large as $| \text{dist}^2(c_p^i, z) - \text{dist}^2(c_p^{i-1}, z) | \geq \Delta$. If we apply the Bernstein’s inequality here, then since the upper bound of $| \text{dist}^2(c, z) - \text{dist}^2(c^{i-1}, z) |$ is larger than $\Delta$, if we still sample the points per level, one may need $\tilde{\Omega}(2^i)$ samples, which can be as large as $\Delta^d$. Thus no space saving is possible.

### D Coreset Construction for $k$-means Based on Chen’s Framework

To formally introduce our coreset construction, we here explain the high level outline. We impose a randomly shifted grid structure over the universe $\Delta^d$. This is done independently of dataset. The grid structure can be stored in memory with only negligible amount of space. We then prove there are good properties of the randomly shifted grid, for a fixed point set. Next we show how to extract a coreset using the help of the grid and hash functions over the universe. After that we formally prove our construction.
Figure 3: Telescope sum [BFL+17] fails for k-means. In the k-median problem, for a fixed set of center point \( Z \), the total cost can be written as a telescope sum \( \sum_{p \in P} (\text{dist}(c_p, Z) - \text{dist}(c_{p-1}, Z)) \). For each piece, \(|\text{dist}(c_p^i, Z) - \text{dist}(c_{p-1}^i, Z)|\) is always upper bounded by \( \text{dist}(c_{p-1}^i, p) \) which is independent from the choice of \( Z \). However, in the k-means problem, the telescope sum of the total cost is \( \sum_{p \in P} (\text{dist}(c_p, Z)^2 - \text{dist}(c_{p-1}^i, Z)^2) \). For each piece, the upper bound of \(|\text{dist}(c_p, Z)^2 - \text{dist}(c_{p-1}^i, Z)^2|\) may depend on the location of \( Z \), and it can be at least \( \Delta \) in the worst case.

D.1 Definitions and Properties

Without loss of generality, we assume the dataset is from \([\Delta]^d\) and \( \Delta = 2^L \) for some positive integer \( L \). Otherwise, we can enlarge \( \Delta \) to the closest power of 2. The space \([\Delta]^d\) is partitioned by the grid structure recursively as follows. The first level of the grid contains a single cell, which is taken as the entire space. For each higher level, we refine our partition by splitting each cell into \( 2^d \) equal square sub-cells. In the finest level, i.e., the \( L \)-th level, each cell contains a single point. We further randomly shift the boundary of the grids to achieve certain properties, which we will show. Formally, our grid structure is defined as follows.

**Definition D.1 (Grids and cells).** Let \( g_0 = \Delta \). Let \( v \) be a vector chosen uniformly at random from \([0, g_0 - 1]^d\). Partition the space \( \{1, 2, \cdots, \Delta\}^d \) into a regular Cartesian grid \( G_0 \) with side-length \( g_0 \) and translated so that a vertex of this grid falls on \( v \). Each cell \( C \subset [\Delta]^d \) of this grid can be expressed as

\[
[v_1 + n_1 g_0, v_1 + (n_1 + 1)g_0] \times \cdots \times [v_d + n_d g_0, v_d + (n_d + 1)g_0] \subset [\Delta]^d
\]

for some \((n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d\). (Note that each cell is cartesian product of batch of continuous interval)

For \( i \geq 1 \), we define the regular grid \( G_i \) as the grid with side-length \( g_i = g_0/2^i \) aligned such that each cell of \( G_{i-1} \) contains \( 2^d \) cells of \( G_i \). The finest grid is \( G_L \) where \( L = \lceil \log_2 \Delta \rceil \); the cells of this grid therefore have side-length at most 1 and thus contain at most a single input point.

Each grid forms a partition of the point-set \( Q \). There is a \( d \)-ary tree such that each vertex at depth \( i \) corresponds to a cell in \( G_i \) and this vertex has \( 2^d \) children which are the cells of \( G_{i+1} \) that it contains. For convenience, we define \( G_{-1} \) as the entire dataset and it contains a single cell.

**Center Cells** Next, we show that the randomly shifted grid structure has very good properties. First, we fix an optimal \( k \)-set \( Z^* = \{z_1^*, z_2^*, \cdots, z_k^*\} \subset [\Delta]^d \) as the optimal \( k \)-means solution for the input dataset \( Q \), i.e.,

\[
\text{cost}(Q, Z^*) = \min_{Z \subset [\Delta]^d \mid |Z| = k} \sum_{q \in Q} \text{dist}^2(q, Z) = \text{OPT}.
\]
Then we call a cell $C$ in level $G_i$ a center cell if it is close to some centers in $Z^*$. Formally
\[ \text{dist}(C, Z^*) = \min_{q \in C; z^* \in Z^*} \text{dist}^2(q, z^*) \leq \frac{g_i}{2d}, \]
where $g_i = 2^{-i} \Delta$ is the side length of the cell and $d$ is the dimension of the space. We show that there are only a small number of center cells.

**Lemma D.2.** Let $\xi$ denote the event that the number of center cells of all grids is upper bounded by $3kL/\rho$. Then event $\xi$ holds with probability at least $1 - \rho$.

**Proof.** Fix an $i$ and consider a grid $G_i$. For each optimal center $z_j^*$, denote $X_{j,\alpha}$ the indicator random variable for the event that the distance to the boundary in dimension $\alpha$ of grid $G_i$ is at most $g_i/(2d)$. Since in each dimension, if the center is close to a boundary, it contributes a factor at most 2 to the total number of center cells. It follows that the number of cells that have distance at most $g_i/(2d)$ to $z_j^*$ is at most
\[ N = 2 \sum_{\alpha=1}^{d} X_{j,\alpha}. \]
We denote $Y_{j,\alpha}$ to be $2X_{j,\alpha}$, then
\[ \mathbb{E}[N] = \mathbb{E} \left[ \prod_{\alpha=1}^{d} Y_{j,\alpha} \right] = \prod_{\alpha=1}^{d} \mathbb{E}[Y_{j,\alpha}]. \]
Using $\Pr[X_{j,\alpha} = 1] \leq (2g_i/(2d))/g_i = 1/d$, we obtain
\[ \mathbb{E}[Y_{j,\alpha}] \leq \mathbb{E}[1 + X_{j,\alpha}] = 1 + \mathbb{E}[X_{j,\alpha}] \leq 1 + 1/d. \]
Thus $\mathbb{E}[N] = \prod_{\alpha=1}^{d} \mathbb{E}[Y_{j,\alpha}] \leq (1 + 1/d)^d \leq e$. Thus the expected number of center cells in a single grid is at most $(1 + 1/d)^d k \leq ek \leq 3k$. By linearity of expectation, the expected number of center cells in all grids is at most $ekL$. By Markov’s inequality, the probability that we have more than $3kL/\rho$ center cells in all grids is at most $\rho$. \hfill \Box

**Heavy Cells** The property of small number of center cells allows us to bound the number of cells that containing too many points. Next we introduce the notion of heavy cells, in which there are a large number of input points. Ideally, we would use OPT, the optimal cost of the $k$-means solution to define our heavy cells. However, OPT is not known to us. We use an arbitrary guess $o$ to OPT to define the heavy cells.

**Definition D.3** (Heavy Cell). Let $o > 0$ be fixed number. For each $i \in \{0, 1, \cdots, L - 1\}$, a cell in $G_i$ is called $o$-heavy if it contains at least $T_i(o) = \frac{d^2}{g_i} \frac{o^0}{3k}$ points. The cell in $G_{-1}$ is always heavy, and no cell in $G_L$ is called heavy. Here we call $T_i(\cdot)$ the threshold function in level $i$. $o$ is omitted if it is clear from the context.

Since the side length $g_i$ is decreasing as $i$ increases, the thresholding function $T_i(o)$ is increasing as $i$ increases. In particular, $T_{i+1}(o)/T_i(o) = 4$ for any $o > 0$ and $i \in \{0, 1, \cdots, L - 1\}$. Therefore, we have the following lemma.

**Lemma D.4.** For $0 \leq i \leq L$, if a cell $C$ in $G_i$ is heavy, then its parent cell $C'$ in $G_{i-1}$ must be heavy. If a cell $C'$ in $G_{i-1}$ is not heavy, then any of its children cells in $G_i$ cannot be heavy.
The next lemma bounds the number of heavy cells.

**Lemma D.5.** If the number of center cells is at most $3kL/\rho$, then the number of $o$-heavy cells is at most $16kL\OPT/(\rho o)$.

**Proof.** For a non-center heavy cell $C$, the contribution of the input points in $C$ to the optimal $k$-means cost is at least $\frac{g^2}{4\rho} T_i(o) \geq op/(12k)$, since each of them is of distance at least $g_i/(2d)$ to an optimal center in $Z^\ast$. Thus there are at most $12k\OPT/(o \rho)$ many non-center heavy cells in a grid. In total there are at most $12k\OPT/(o \rho) + 1$ many non-center heavy cells, where the +1 term comes from $G_{-1}$. Since there are at most $3kL/\rho$ many center cells, the total number of heavy cells is at most

$$1 + (3kL/\rho)(1 + 4 \OPT/o) \leq 16kL\OPT/(\rho o).$$

The heavy cells allows us to construct the coreset. The number of heavy cells essentially gives us a bound of how many samples we need obtain from the original dataset.

### D.2 Recursive Partition Approach

Recall that in our coreset construction, we first partition the input datasets into $L + 1$ disjoint subsets. And then we sample our points separately from these partitions. To prove Theorem D.14, we first refine the partition conceptually as follows.

**Definition D.6 (o-Partition).** Suppose parameter $o > 0$. Let $Q \subset [\Delta]^d$ be the input set. For each level $0 \leq i \leq L$ we define a set of sets $\mathcal{P}_i$ as follows. Initialize $\mathcal{P}_i$ as an empty set. For each heavy cell $C$ in $G_{i-1}$, we group the non-heavy children cells of $C$ in the following manner. First note
\( T_i(o) = 4T_{i-1}(o) \), so by non-heaviness each child cell contains at most \( 4T_{i-1}(o) \) points of \( Q \). If all the non-heavy children together contain less than \( T_i(o) \) points of \( Q \), we take all the points of \( Q \) containing in them as a single set, and add it to \( P_i \). Otherwise, we make groups such that each group of these non-heavy children cells contains at least \( T_i(o) \) points and at most \( 3T_i(o) \) points of \( Q \). We put the points of \( Q \) in each of these groups as a set and add it to \( P_i \). Our full partition of \( Q \) is defined as \( P = P_0 \cup P_1 \cup \ldots \cup P_L \).

For each \( i \in \{0,1,\ldots,L\} \), we write set \( P_i \) in the following way,

\[
P_i = \{P_{i,1}, P_{i,2}, \ldots, P_{i,|P_i|}\}.
\]

For each \( j \in [|P_i|] \), if set \( P_{i,j} \) contains at least \( T_i(o) \) points, then say set \( P_{i,j} \) is a heavy set. We use \(|P_{i,j}|\) to denote the number of points of \( Q \) that contained in partition \( P_{i,j} \).

**Remark D.7.** We remark that the algorithm and coreset construction does not need to know and find such a partition. The partition we given here is only for analysis purposes. Also note that for each \( i \in \{0,1,\ldots,L\} \), the sets in \( P_i \) give a partition for \( Q_i \), i.e., the set of points falling in non-heavy children of a heavy cell.

**Claim D.8 (Upper bound \(|P_{i,j}|\)).** For each \( i \in \{0,1,\ldots,L\} \), we write set \( P_i \) in the following way,

\[
P_i = \{P_{i,1}, P_{i,2}, \ldots, P_{i,|P_i|}\}.
\]

Then for each \( j \in [|P_i|] \), we have \(|P_{i,j}| \leq 3T_i(o)\).

**Proof.** It directly follows by Definition D.6.

**Fact D.9.** For a partition \( P \) of points set \( Q \), we have

\[
|P| = \sum_{i=0}^{L} |P_i|, \quad \text{and} \quad \sum_{i=0}^{L} \sum_{j=1}^{|P_i|} |P_{i,j}| = |Q|
\]

**Definition D.10.** We say a partition \( P(d,g) \) is a \( \alpha, \beta \) partition if

\[
|P| \leq \alpha k \quad \text{and} \quad \sum_{i=0}^{L} \sum_{j=1}^{|P_i|} |P_{i,j}|d_{g_{i-1}}^2 \leq \beta \text{OPT}
\]

**Lemma D.11.** Given parameters \( \alpha, \alpha, \beta \) with \( 0 < \alpha \leq \text{OPT} \), \( \alpha = 36L \text{OPT} / (\rho o) \), \( \beta = 120d^3 L \), and conditioned on the number of center cells is at most \( 3kL/\rho \). Then the \( o \)-partition \( P \) satisfies

\[
|P| \leq \alpha k \quad \text{and} \quad \sum_{i=0}^{L} \sum_{j=1}^{|P_i|} |P_{i,j}|d_{g_{i-1}}^2 \leq \beta \text{OPT}.
\]

**Proof.** Without loss of generality, we assume \( L = \log \Delta \) is sufficiently large, i.e., \( L \geq 9 \). We first bound the cardinality of \( P \). For the partition sets, we split them into three groups. The first group are these sets containing a center cell. Since there are at most \( 3kL/\rho \) center cells, thus the number of such sets is bounded by \( 3kL/\rho \).

The second group are these non-heavy sets, i.e., these sets containing less than \( T_i \) points. The number of these sets are bounded by the heavy cells since each heavy cell can produce at most one of such a set. Lemma D.5 shows that the number of heavy cells is at most \( 16kL \text{OPT} / (\rho o) \).
The last group are the remaining sets. Each of these sets contain at least \( T_{i-1} \) many points by construction. For any point in such a set, its distance to any center is at least \( g_i/(2d) \). Thus the contribution to \( \text{OPT} \) is at least \( T_i g_i^2/(4d^2) \). Thus the number of such sets per level is bounded by \( 12k \text{ OPT} / (\rho o) \). In total, we can upper bound \( |P| \) in the following sense,

\[
|P| \leq 16kL \text{ OPT} / (\rho o) + 12kL \text{ OPT} / (\rho o) \leq 30kL \text{ OPT} / (\rho o).
\]  

(2)

Next, we show

\[
\sum_{i=0}^{L} \sum_{j=1}^{|P_i|} |P_{i,j}| (g_{i-1} \sqrt{d})^2 \leq \sum_{i=0}^{L} \sum_{j=1}^{|P_i|} 3T_i(o) (g_{i-1} \sqrt{d})^2 \\
\leq \sum_{i=0}^{L} \sum_{j=1}^{|P_i|} 4d^3 \rho o / k \\
\leq |P| \cdot 4d^3 \rho o / k \\
\leq 120d^3 L \text{ OPT}.
\]

where the first step follows by \( |P_{i,j}| \leq 3T_i(o) \) (Claim D.8), the second step follows by \( T_i(o) \leq \frac{4d^3 \rho o}{3k} \), the last step follows by Eq. (2).

D.3 Bounding the Close Parts

Given a set \( Z \) of \( k \) points, we partition each \( P_i \) for \( 0 \leq i \leq L \) into \( r = O(\log(\Delta \sqrt{d})) \) parts based on their distance to \( Z \). Formally, let

\[
P_i^0(Z) = \{ P \in P_i : \text{dist}(P, Z) \leq 2\sqrt{d} g_{i-1} \}.
\]

(3)

and for \( 0 < j \leq r \)

\[
P_i^j(Z) = \{ P \in P_i : 2^j \sqrt{d} g_{i-1} < \text{dist}(P, Z) \leq 2^{j+1} \sqrt{d} g_{i-1} \}.
\]

(4)

Since the distance is upper bounded by \( \Delta \sqrt{d} \), thus \( r = \lceil \log(\Delta \sqrt{d}) \rceil \) is sufficient to partition \( P_i \). It is also easy to see that for any \( i \in [0, L] \) and \( j \geq 1 \) we have

\[
\max_{p \in P, P \subset P_i^j(Z)} \text{dist}(p, Z) \leq (2^j + 1) \sqrt{d} g_{i-1}.
\]

Based on this definition, we first bound the error in \( P_i^0(Z) \) of the coreset in the next lemma.

Lemma D.12 (Cost of close parts). Suppose we are given an input point set \( Q \), its \((\alpha, \beta)\)-partition \( P_0 \cup P_1 \cup \ldots \cup P_L \) and an arbitrary fixed set of \( k \) points \( Z \subset \Delta^d \). For each \( 0 \leq i \leq L \), let \( P_i^0(Z) \) be defined in (3) as the partition sets that is in distance \( \sqrt{d} g_{i-1} \) to \( Z \). Let \( P_i^0 \) denote all the points in \( P_i^0(Z) \). Let multiset \( S_i^0 \) be the independent sample of points in \( P_i^0 \) with sampling probability \( \zeta \) and weight \( 1/\zeta \). If

\[
\zeta \geq \min \left[ \frac{256}{\epsilon^2} \cdot \frac{\beta^2}{kL \cdot T_i(o)} \cdot \ln \left( \frac{2}{\delta} \right), 1 \right]
\]

for some \( \epsilon, \delta \in (0, 1/2) \) and \( o \in (0, \text{OPT}) \), then, with probability at least \( 1 - \delta \),

\[
|\text{cost}(P_i^0, Z) - \text{cost}(S_i^0, Z)| \leq \frac{\epsilon}{2\beta} \cdot \max(|P_i^0|, kL \cdot T_i(o)) \cdot d g_{i-1}^2.
\]

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Proof. First we recall that

$$\text{cost}(P_i^0, Z) = \sum_{p \in P_i^0} d(p, Z)^2.$$  

Let $I_p$ be the indicator that a point $p \in P_i^0$ is sampled. Then

$$\text{cost}(S_i^0, Z) = \sum_{p \in P_j} \frac{I_p}{\zeta} d(p, Z)^2.$$  

The difference can be written as,

$$|\text{cost}(P_i^0, Z) - \text{cost}(S_i^0, Z)| \leq \left| \sum_{p \in P_i^0} (1 - \frac{I_p}{\zeta}) d(p, Z)^2 \right|.$$  

We have that

$$E \left[ \sum_{p \in P_i^0} (1 - \frac{I_p}{\zeta}) d(p, Z)^2 \right] = 0$$  

and for each $p \in P_i^0$,

$$E \left[ (1 - \frac{I_p}{\zeta})^2 d(p, Z)^4 \right] = \frac{(1 - \zeta) d(p, Z)^4}{\zeta} \leq \frac{16d^2 g_i^{2/3}}{\zeta}.$$  

Note that, $|(1 - \frac{I_p}{\zeta}) d(p, Z)^2| \leq d(p, Z)^2 / \zeta$. In order to apply Bernstein’s inequality, Lemma B.8, we compute $\sigma^2$ and $b$ as follows,

$$\sigma^2 = \sum_{p \in P_i^0} E \left[ (1 - \frac{I_p}{\zeta})^2 d(p, Z)^4 \right] \leq |P_i^0| \frac{16d^2 g_i^{2/3}}{\zeta} \quad \text{and} \quad b = \max_{p \in P_i^0} \frac{d(p, Z)^2}{\zeta} \leq \frac{4d^2 g_i^{2/3}}{\zeta}.$$  

If $|P_i^0| \geq kL T_i(o)$, we choose $t = \frac{\epsilon}{2\beta} \cdot |P_i^0| g_i^{2/3}$ and calculate

$$\frac{t^2}{4\sigma^2} = \frac{|P_i^0| \epsilon^2 \cdot \zeta}{256\beta^2} \quad \text{and} \quad \frac{t^2}{4bt/3} = \frac{3|P_i^0| \epsilon \cdot \zeta}{32\beta}.$$  

By Bernstein’s inequality, we have that

$$\Pr \left[ \left| \sum_{p \in P_i^0} (1 - \frac{I_p}{\zeta}) d(p, Z)^2 \right| > t \right] \leq 2 \exp \left( -\frac{t^2}{2\sigma^2 + 2bt/3} \right) \leq \exp \left( -\min \left[ \frac{t^2}{4\sigma^2}, \frac{t^2}{4bt/3} \right] \right) \leq \delta.$$
If \(|P_i^0| \leq kLT_i(o)|, we choose \( t = \frac{\epsilon}{2\beta} \cdot kLT_i(o)d_{g_{i-1}}^2 \) and calculate
\[
\frac{t^2}{4\sigma^2} = \frac{kLT_i(o)}{|P_i^0|} \cdot \frac{kLT_i(o)\epsilon^2}{256\beta^2} \quad \text{and} \quad \frac{t^2}{4bt/3} = \frac{3kLT_i(o) \cdot \zeta}{32\beta}
\]
Thus
\[
\Pr \left[ \sum_{p \in P_i^0} \left(1 - \frac{I_p}{\zeta}\right)d(p,Z)^2 \right] > t \leq \delta.
\]
We thus conclude the proof. \( \square \)

### D.4 Bounding the Far Parts

The goal of this section is to prove Lemma D.13. We show that the error of the coreset \( S \) over each \( P_i^j \) for \( i \in [0,L], j \in [1,r] \) is still bound.

**Lemma D.13** (Cost of far parts). Suppose we are given an input point set \( Q \), its \((\alpha,\beta)\)-partition \( P_0 \cup P_1 \cup \ldots \cup P_L \) and an arbitrary fixed set of \( k \) points \( Z \subset [\Delta]^d \). For each \( 0 \leq i \leq L \) and \( j \in [r] \) where \( r = \lceil \log(\Delta \sqrt{d}) \rceil \), let \( P_i^j(Z) \) be defined in (4) as the partition sets that is in distance \((2^{j-1}\sqrt{d_{g_{i-1}}},2^{j}\sqrt{d_{g_{i-1}}})\) to \( Z \). Let \( P_i^j \) denote all the points in \( P_i^j(Z) \). Let multiset \( S_i^j \) be the independent sample of points in \( P_i^j \) with sampling probability \( \zeta \) and weight \( 1/\zeta \). If
\[
\zeta \geq \min \left[ \frac{256}{\epsilon^2} \cdot \frac{\beta^2}{T_i(o)} \cdot \ln \left( \frac{2}{\delta} \right), 1 \right]
\]
for some \( \epsilon, \delta \in (0,1/2) \) and \( o \in (0,\text{OPT}) \), then, with probability at least \( 1 - \delta \),
\[
| \text{cost}(P_i^j, Z) - \text{cost}(S_i^j, Z) | \leq \epsilon \cdot \text{cost}(P_i^j, Z) + \frac{2\epsilon}{\beta^2} \text{cost}(Q, Z).
\]

**Proof.** The proof is similar to that of Lemma D.12 except that we need to consider the partition sets containing too few number of points. We additionally partition \( P_i^j(Z) = P_{i,1}^j(Z) \cup P_{i,2}^j(Z) \) and denote \( P_{i,1}^j \) and \( P_{i,2}^j \) as the points in \( P_{i,1}^j(Z) \) and \( P_{i,2}^j(Z) \) such that
\[
\forall P \in P_{i,1}^j(Z), |P| \geq \frac{T_i(o)}{\beta^2} \quad \text{and} \quad \forall P' \in P_{i,2}^j(Z), |P'| < \frac{T_i(o)}{\beta^2}.
\]
Similarly we denote \( S_{i,1}^j \) and \( S_{i,2}^j \) as the sampled points in each set. For each \( r \in \{0,1\} \), we recall that
\[
\text{cost}(P_{i,r}^j, Z) = \sum_{p \in P_{i,r}^j} d(p,Z)^2.
\]
Let \( I_p \) be the indicator that a point \( p \in P_{i,r}^j \) is sampled. Then
\[
\text{cost}(S_{i,r}^j, Z) = \sum_{p \in P_{i,r}^j} I_p \frac{d(p,Z)^2}{\zeta}.
\]

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The difference can be written as,
\[ |\text{cost}(P_{i,r}^j, Z) - \text{cost}(S_{i,b}^j, Z)| \leq \sum_{p \in P_{i,r}^j} (1 - \frac{I_p}{\zeta})d(p, Z)^2. \]

We have that
\[ E\left[ \sum_{p \in P_{i,r}^j} (1 - \frac{I_p}{\zeta})d(p, Z)^2 \right] = 0 \]

and for each \( p \in P_{i,r}^j \),
\[ E\left[ (1 - \frac{I_p}{\zeta})^2d(p, Z)^4 \right] = \frac{(1 - \zeta)d(p, Z)^4}{\zeta} \leq \frac{2^{4j}d^2g_i^4}{\zeta}. \]

Note that, \( |(1 - \frac{I_p}{\zeta})d(p, Z)^2| \leq d(p, Z)^2 / \zeta \).

In order to apply Bernstein’s inequality, Lemma B.8, we compute \( \sigma^2 \) and \( b \) as follows,
\[ \sigma^2 = \sum_{p \in P_{i,r}^j} E\left[ (1 - \frac{I_p}{\zeta})^2d(p, Z)^4 \right] \leq |P_{i,r}^j| \frac{2^{4j}d^2g_i^4}{\zeta} \]

and
\[ b = \max_{p \in P_{i,r}^j} \frac{d(p, Z)^2}{\zeta} \leq \frac{(2^{j+1} + 1)^2dg_i^2}{\zeta}. \]

**Case 1.** \( r = 1 \).

For \( r = 1 \), we consider two cases:

**Case 1a.** \( |P_{i,1}^j| \geq |T_i(o)| \), then we choose \( t = \frac{\varepsilon}{8}|P_{i,1}^j|2^{2j}dg_i^2 \) and compute,
\[ \frac{t^2}{4\sigma^2} = \frac{|P_{i,1}^j| \varepsilon^2 \cdot \zeta}{256\beta^2} \quad \text{and} \quad \frac{t^2}{4bt/3} \geq \frac{|P_{i,1}^j| \varepsilon \cdot \zeta}{64\beta}. \]

By Bernstein’s inequality, we have
\[ \Pr \left[ \sum_{p \in P_{i,1}^j} \left( 1 - \frac{I_p}{\zeta} \right)d(p, Z)^2 > t \right] \leq 2 \exp \left( - \min \left[ \frac{t^2}{4\sigma^2}, \frac{t^2}{4bt/3} \right] \right) \leq \delta. \]

Since \( \text{dist}(P, Z) \geq 2^{j-1}dg_{i-1} \) for any \( P \in P_{i,1}^j \), with probability at least \( 1 - \delta \), we have
\[ |\text{cost}(P_{i,1}^j, Z) - \text{cost}(S_{i,1}^j, Z)| \leq \epsilon \text{cost}(P_{i,1}^j, Z). \]

**Case 1b.** \( \frac{|P_{i,1}^j| |T_i(o)|}{\beta^2} \leq |P_{i,1}^j| < |P_{i,1}^j| |T_i(o)| \), then we choose
\[ t = \frac{\epsilon}{8\beta} \cdot \sqrt{|P_{i,1}^j| \cdot |P_{i,1}^j| \cdot T_i(o) \cdot 2^{2j} \cdot dg_i^2} \leq \frac{\epsilon}{8}|P_{i,1}^j|2^{2j}dg_i^2. \]
We compute,
\[
\frac{t^2}{4\sigma^2} = \frac{|P_{i,1}^j| T_i(o) \epsilon^2 \cdot \zeta}{256\beta^2} \quad \text{and} \quad \frac{t^2}{4bt/3} = \sqrt{|P_{i,1}^j||P_{i,1}^j| T_i(o) \cdot \epsilon \cdot \zeta}
\]
By Bernstein’s inequality, we have that
\[
\Pr \left[ \left| \sum_{p \in P_{i,1}^j} \left( 1 - \frac{I_p}{\zeta} \right) d(p, Z)^2 \right| > t \right] \\
\leq 2 \exp \left( - \min \left[ \frac{t^2}{4\sigma^2}, \frac{t^2}{4bt/3} \right] \right) \\
\leq \delta.
\]
Thus with probability at least 1 – δ, we have
\[
\left| \text{cost}(P_{i,1}^j, Z) - \text{cost}(S_{i,1}^j, Z) \right| \leq \epsilon \text{cost}(P_{i,1}^j, Z).
\]

**Case 2. r = 2.**
Lastly, we consider \( b = 2 \). Notice that each of these set contains at most \( T_i(o)/\beta^2 \) points. We now choose \( t = \frac{\epsilon}{8\beta^2} |P_{i,2}^j| T_i(o) 2^{2j} d g_{i-1}^2 \), and compute
\[
2bt/3 = \frac{2}{3} \frac{\epsilon}{8\beta^2} |P_{i,2}^j| T_i(o) \cdot 2^{4j} d^2 g_{i-1}^2 \cdot \frac{1}{\zeta} \geq \frac{1}{12} \epsilon \sigma^2
\]
which implies
\[
\exp \left( - \frac{t^2}{2\sigma^2 + 2bt/3} \right) \\
\leq \exp \left( - \frac{t^2}{16bt + 2bt/3} \right) \\
\leq \exp(-\frac{\epsilon t}{18b}) \\
\leq \exp(-\frac{\epsilon^2 \cdot \zeta \cdot T_i(o)}{144}) \\
\leq \frac{\delta}{2}.
\]
where the second step follows by \( 16 + 2\epsilon/3 \leq 18 \). By taking a union bound, with probability 1 – δ,
\[
\left| \sum_{p \in P_{i,2}^j} \left( 1 - \frac{I_p}{\zeta} \right) d(p, Z)^2 \right| \leq \frac{\epsilon}{8\beta^2} |P_{i,2}^j| T_i(o) 2^{2j} d g_{i-1}^2.
\]
By the construction, each part \( P \in P_{i,2}^j \), it corresponding to a unique heavy cell \( C_P \) in \( G_{i-1} \). Since \( \text{dist}(P, Z) \geq 2^j \sqrt{d g_{i-1}} \) for \( j \geq 1 \), thus \( \text{dist}(C_P \cap Q, Z) \geq 2^{j-1} \sqrt{d g_{i-1}} \). Therefore
\[
\frac{\epsilon}{8\beta^2} |P_{i,2}^j| T_i(o) \cdot 2^{2j} d g_{i-1}^2 \leq \frac{\epsilon}{8\beta^2} \sum_{P \in P_{i,2}^j} 4|C_P \cap Q| \cdot 4 \cdot 2^{2(j-1)} d g_{i-1}^2 \leq \frac{2\epsilon}{\beta^2} \text{cost}(Q, Z).
\]
This concludes the proof. \( \square \)
Figure 5: (a) We show an example of partitioning the set of points from top to bottom. There are four levels in total. The top level is the first level, and the bottom level is the fourth level. The number of blocks is decreasing, when the levels is increasing. Each blue box is a heavy cell. In the first level, there is a single blue block. In the second level, we partition it into four cells. Only two of them are heavy (blue) cells. In the third level, we partition the each of heavy cells (in the second level) into four cells. One has one heavy (blue) cell, and the other has two (blue) heavy cells. (b) We show an example of sampling points from those non-heavy (green) cells. In the second level, there are two green cells, we sample 16 (red) points from each of them. In the third level, there are five green cells, we sample 8 (red) points from each of them. In the fourth (bottom) level, there are 12 green cells, we sample 4 (red) points from each of them.

D.5 Our Coreset Construction

To explain our coreset construction, we first defined a probability of sampling in each level $G_i$, for $i \in [0, L]$, as follows,

$$
\pi_i(o, \epsilon, \delta) := \min \left[ \frac{9 \times 256 \epsilon^2}{\epsilon^2} \cdot \frac{(144d^3L)^2}{T_i(o)} \cdot \ln \left( \frac{2(L + 1)\Delta^{d^k}}{\delta} \right), 1 \right],
$$

where $\epsilon, \delta \in (0, 1/2)$ and $o > 0$ are fixed parameters. We now define $L + 1$ sets of weighted samples as follows. Denote the input point set $Q \subset [\Delta]^d$. For $0 \leq i \leq L$, let $C_i \subset G_i$ be the set of non-heavy cells, whose parent cell in $G_{i-1}$ is $o$-heavy. Let $Q_i$ be the set of points falling in a cell of $C_i$. Notice that by Lemma D.4, $Q_0, Q_1, \ldots, Q_L$ are disjoint sets and form a partition of $Q$.

Coreset Construction Let $S_i$ be a multiset, obtained by sampling each point $q$ in $Q_i$ independently with probability $\pi_i(o, \epsilon, \delta)$. Each point in $S_i$ has weight $1/(\pi_i(o, \epsilon, \delta))$. Then our weight set is obtained as

$$
S(o, \epsilon, \delta) = S_0 \cup S_1 \cup \ldots \cup S_L.
$$
We omit \((o, \epsilon, \delta)\) if it is clear from the context.

Notice that our coreset set construction is determined by the parameter \(o, \epsilon\) and \(\delta\). We show in the next theorem the construction above gives an \(\epsilon\)-coreset for \(k\)-means on \(Q\).

**Theorem D.14.** Suppose \(0 < o \leq \text{OPT}\), and we given a point set \(Q \in [\Delta]^d\). Let \(S(o, \epsilon, \delta)\) be multiset defined in (6). Then conditioning on event \(\xi\) holds, where we only have \(ekL/\rho\) center cells, with probability at least \(1 - \delta\), \(S(o, \epsilon, \delta)\) is an \(\epsilon\)-coreset of \(k\)-means on \(Q\) and

\[
E[|S|] = O(e^{-2}k^2d^7L^4\log(1/\delta) \cdot \text{OPT} \cdot \rho^{-1}).
\]

**Proof.** Let \(\epsilon' = \epsilon/3\). Take the sampling probability

\[
\zeta = \pi(o, \epsilon, \delta) = \min\left[\frac{256}{\epsilon^2} \cdot \frac{\beta^2}{T_i(o)} \cdot \ln\left(\frac{2(L + \log d)^2\Delta dk}{\delta}\right), 1\right].
\]

Let \(S = S(o, \epsilon, \delta)\) be the obtained multiset.

For any fixed set \(Z \in [\Delta]^d\) of \(k\) points, let \(P_i^j(Z)\) be the subsets of partitions defined in (3) and (4) for \(i \in [0, L]\) and \(j \in [r]\) where \(r = \lceil \log(\Delta \sqrt{d}) \rceil\). Let \(P_i^j \subset Q\) be the corresponding point sets falling in a part of \(P_i^j\). Let \(S_i^j\) be the corresponding sampled set of \(P_i^j\). Notice that

\[
S = \bigcup_{i \in [0, L], j \in [r]} S_i^j.
\]

By Lemma D.12, Lemma D.13 and with a union bound over all possible \(i \in [0, L]\) and \(j \in [r]\), with probability at least \(1 - \frac{\delta}{\Delta d}\),

\[
|\text{cost}(S_i^j, Z) - \text{cost}(P_i^j, Z)| \leq \frac{\epsilon'}{2\beta} \cdot |P_i^j| \cdot T_i(o) \cdot d g_{i-1}^2
\]

and

\[
|\text{cost}(P_i^j, Z) - \text{cost}(S_i^j, Z)| \leq \epsilon' \cdot \text{cost}(P_i^j, Z) + \frac{2\epsilon'}{\beta^2} \text{cost}(Q, Z).
\]

Conditioning on the above inequalities, we have

\[
|\text{cost}(S, Z) - \text{cost}(Q, Z)| \leq \sum_{i \in [0, L]} \frac{\epsilon'}{2\beta} \cdot |P_i^0| \cdot T_i(o) \cdot d g_{i-1}^2
\]

\[+ \sum_{i \in [0, L], j \in [r]} \left( \epsilon' \cdot \text{cost}(P_i^j, Z) + \frac{2\epsilon'}{\beta^2} \text{cost}(Q, Z) \right).
\]

By Lemma D.11,

\[
|P_i^0| \leq \frac{12eLk \text{OPT}}{\rho o}
\]

thus

\[
\sum_{i \in [0, L]} \frac{\epsilon'}{2\beta} \cdot |P_i^0| \cdot T_i(o) \cdot d g_{i-1}^2 \leq \frac{\epsilon'}{2\beta} \cdot 144d^3L \text{OPT} = \epsilon' \text{OPT}.
\]

Since \(\beta = 144d^3L\), therefore

\[
(L + 1)(L + \log d/2)/\beta^2 \leq \frac{1}{2}.
\]

Hence have

\[
\sum_{i \in [0, L], j \in [r]} \frac{2\epsilon'}{\beta^2} \text{cost}(Q, Z) \leq \epsilon' \text{cost}(Q, Z).
\]
Overall, we have
\[ |\text{cost}(S, Z) - \text{cost}(Q, Z)| \leq 3\epsilon' \text{cost}(Q, Z). \]
Notice that \( 3\epsilon' = \epsilon \). Since there are at most \( \Delta^{dk} \) possible \( Z \), by a union bound, we have for all \( Z \subset [\Delta]^d \) with \( |Z| \leq k \),
\[ |\text{cost}(S, Z) - \text{cost}(Q, Z)| \leq \epsilon \text{cost}(Q, Z). \]

Next we bound the number of points sampled. By Lemma D.11, there are at most
\[ |\mathcal{P}| \leq \frac{12ekL \text{OPT}}{\rho} \]
parts in the \( o \)-partition. Each parts contains at most \( 3T_i(o) \) points. Therefore, in expectation,
\[
\mathbb{E}(|S|) = \sum_{i \in [0,L]} |\mathcal{P}_i| \cdot 3T_i(o) \cdot \pi_i(o, \epsilon, \delta)
\leq \frac{12ekL}{\rho} \cdot \frac{\text{OPT}}{o} \cdot 9 \cdot 256 \cdot (3 \times 144d^3L)^2 \ln \left( \frac{2(L + 1) \Delta^{dk}}{\delta} \right)
= O \left( \frac{k^2d^7L^4}{\epsilon^2} \log \left( \frac{1}{\delta} \right) \cdot \frac{\text{OPT}}{o} \right).
\]

\[ \square \]

**Remark D.15.** The coreset can be constructed independently and oblivious to the point set, i.e., using hash functions \( h_{o,i} : [\Delta]^d \rightarrow \{0,1\} \), such that the probability of \( h_{o,i}(p) = 1 \) is exactly \( \pi_i(o, \epsilon, \delta) \). Later, we pick the set of points with hash value 1, when it is available. Then we construct the coreset as in 6.

Observing that in our proof, the only requirement to bound the cost of partition. Thus we have the following proposition.

**Proposition D.16.** Suppose \( 0 < o \leq \text{OPT} \), and we given a point set \( Q \subset [\Delta]^d \). Let \( S(o, \epsilon, \delta) \) be multiset defined in (6). Then conditioning on event that there exists an \( o \)-partition (see Definition D.6) satisfying
\[ |\mathcal{P}| \leq \alpha k \quad \text{and} \quad \sum_{i \in [0,L]} \sum_{P \in \mathcal{P}_i} |P| \cdot T_i(o) \cdot d g_{i-1}^2 \leq d^3L \text{OPT} \]
then with probability at least \( 1 - \delta \), \( S(o, \epsilon, \delta) \) is an \( \epsilon \)-coreset of \( k \)-means on \( Q \) and
\[ \mathbb{E}(|S|) = O(\epsilon^{-2}k^2d^7L^3 \alpha \log 1/\delta). \]

## E Coreset Over the Dynamic Data Stream

### E.1 Streaming Coreset Construction

As a first step, we modify our coreset construction slightly such that an streaming algorithm can maintain the coreset construction. The only difficulty is that we cannot obtain the number of points in a cell exactly. We overcome this difficulty by modifying our heavy cell definitions as follows.

**Definition E.1** \((o, \epsilon)-\text{Heavy Cell Scheme})\). Fixing a number \( \epsilon \in [0,1] \), a set of input points \( Q \subset [\Delta]^d \) and a grid structure \( \mathcal{G} = \{G_{-1}, G_0, \ldots, G_L\} \) over \([\Delta]^d \) defined in Definition D.1. We call a procedure an \((o, \epsilon)\)-heavy cell scheme if it satisfies,
1. \( \forall C \in G_{-1}, C \) is o-heavy; \( \forall C \in G_{L}, C \) is not heavy;

2. for \( i \in [0, L] \), if a cell \( C \in G_{i} \) with \( |C \cap Q| \geq T_{i}(o) \), then \( C \) is identified as o-heavy;

3. for \( i \in [-1, L] \), if a cell \( C \) with \( |C \cap Q| < (1 - \epsilon)T_{i}(o) \), then \( C \) is identified as not heavy;

4. for \( i \in [-1, L - 1] \) a cell \( C \in G_{i} \) is identified not heavy, then all its children cells \( C' \in G_{i+1} \) is identified not heavy;

As one can easily verify that the heavy cells in Definition D.1 is given by a \((o,0)\)-heavy cell scheme. In particular, we have the following theorem.

**Theorem E.2.** Suppose \( 0 < o \leq \text{OPT}, 0 \leq \epsilon \leq 1 \), and we are given a point set \( Q \in [\Delta]^{d} \). Let \( S(o, \epsilon, \delta) \) be multiset defined in (6) using heavy cells defined by any \((o,\epsilon)\)-heavy cell scheme. Then conditioning on event \( \xi \) holds, where we only have \( ekL/\rho \) center cells, with probability at least \( 1 - \delta \), \( S(o, \epsilon, \delta) \) is an \( \epsilon \)-coreset of \( k \)-means on \( Q \) and

\[
E[|S|] = O(\epsilon^{-2}k^{2}d^{2}L^{4}\log(L/\delta) \cdot \text{OPT} \cdot o^{-1}).
\]

### E.2 The Dynamic Point-Cell Storing Data Structure

In this section, we introduce a data structure that maintains the coreset in a dynamic data stream. To begin, we first introduce the K-Set data structure, which we use as a sub-routine in our algorithm.

**Lemma E.3** (K-Set). Given parameter \( M \geq 1, N \geq 1, k \geq 1, \delta \in (0,1/2) \). There is a data structure that requires \( O(k(\log M + \log N) \log(k/\delta)) \) bits and is able to process a data stream \( S \) (which contains both insertion and deletion). The data structure processes each stream operation \( \pm i \) \((i \in [N]) \) in \( O(\log(k/\delta)) \) time. For each time \( t \), let \( M_{t} \) denote the sum of the frequency of each the element. Let \( M = \max_{t} M_{t} \). At each point of time \( t \), it supports an operation \( \text{QUERY} \), if the number of distinct elements is \( \leq k \), then with probability at least \( 1 - \delta \) it returns the set of items and the frequencies of each items. It returns \( \emptyset \) otherwise.

Our coreset critically relies on the data structure defined in Algorithm 2. It defines a data structure that supports inserting and deleting cell-point pairs. For example, each input is of the form \((C, p, op)\), where \( C \) is a cell in the grid, identified as its ID (including the information of its level and coordinates), \( p \in C \) and \( op \) is + or − representing insertion or deletion. It has the following guarantees.

**Lemma E.4.** There is a data structure (procedure \( \text{SAMPLESTREAM}^{+} \) in Algorithm 2) that supports insertion and deletions on cell-point pairs from \([2L\Delta^{d}] \times [\Delta]^{d} \) and a query operation. At any time point, the query operation returns either of the follow two results,

1. \( \forall C \subseteq G \), \( f \) and \( S = \{S_{C} : \forall C \in G \} \), where \( G \) is all the cells undeleted, \( f \) is a function encoding the number of points in a cell in \( G \), and \( \forall C \in G, S_{C} \) is the set of points in \( C \) if \( |C| \leq \beta \);

2. \( \emptyset \).

If conditioning on the following three events,

(1) there are at most \( \alpha \) non-empty cells;

(2) there are at most \( \theta \) cells that each of them contains more than \( \beta \) points;

(3) the number of points in all the other cells that each containing most \( \beta \) points is at most \( \gamma \), then the algorithm outputs \( \emptyset \) with probability at most \( \delta \).

The algorithm uses

\[
O((\theta\beta + \gamma)\log(\Delta^{d}) \cdot \log((\theta\beta + \gamma)/\delta)),
\]

bits of space in the worst case.
Proof. We first show that conditioning on the three events, our algorithm outputs the set of points with high probability. We then show that if the three events do not happen, the output of the algorithm always matches the requirement of the statement.

Let $\mathcal{C}$ denote the set of all cells, let $P$ denote the set of all points. We also identify each cell $C \in \mathcal{C}$ by its coordinate. We define hash function $h : \mathcal{C} \rightarrow [a_1 \cdot \alpha]$, $g : [P] \rightarrow [a_2 \cdot \beta]$ and $f : [a_1 \cdot \alpha] \times [a_2 \cdot \beta] \rightarrow [a_3(\theta \beta + \gamma)]$, where constant $a_1, a_2, a_3 \geq 1$ (will be decided later) and $h, g, f$ are pair-wise independence function. For each $w \in [a_3(\theta \beta + \gamma)]$, $j \in [2 \log(\Delta^d)]$ and $z \in \{0, 1\}$, we define counter

$$\text{counter}(w, j, z) \in \{0, 1\}.$$ 

We initialize the all the entries of the counter to be 0. Note the number of counters is $2(a_3(\theta \beta + \gamma) \log(\Delta^d))$. For each update $(C, p, \text{operation})$, we update the update the counter in the following way,

$$\text{counter}(f(h(C), g(p)), j, (C, p)_j) \leftarrow \text{counter}(f(h(C), g(p)), j, (C, p)_j) \pm 1 \bmod 2, \forall j \in [2 \log(\Delta^d)],$$

where $(C, p)_j$ is the $j$-th bit of the pair $(C, p)$, “+” is corresponding to insertion, and “−” is corresponding to deletion. We also initialize an K-Set structure to store at most $\alpha$ cells, and update the cell $C$ to it.

For each $w \in [a_3(\theta \beta + \gamma)]$ we define counter

$$\text{size}(w) \in \{0, 1, \ldots, \Delta^d\}.$$ 

We initialize the all the entries of the counter to be 0. Note the number of counters is $a_3(\theta \beta + \gamma)$. For each update $(C, p, \text{operation})$, we update the counter in the following way,

$$\text{size}(f(h(c), g(p))) \leftarrow \text{size}(f(h(c), g(p))) \pm 1,$$

where “+” is corresponding to insertion, and “−” is corresponding to deletion.

For a fixed cell $C$, for any cell $C' \neq C$, we have

$$\Pr_{h \sim H}[h(C) = h(C')] = \Theta(1/(a_1 \alpha))$$

We define random boolean variable $X_{C'}$ such that $X_{C'} = 1$ if $h(C') = h(C)$ and $b$ is non-empty; $X_{C'} = 0$ otherwise. We define $X = \sum_{C' \in \mathcal{C} \setminus \{C\}} X_{C'}$, and compute $\mathbb{E}[X] = \Theta(1/a_1)$. By Markov’s inequality,

$$\Pr[X \geq 1] \leq \mathbb{E}[X]/1 \leq \Theta(1/a_1).$$

By choosing $a_1$ to be sufficiently large constant, we obtain with probability at least 0.99, there is no non-empty cell $C' \in \mathcal{C} \setminus \{C\}$, satisfying that $h(C') = h(C)$.

For a fixed $x \in [a_1 \alpha]$, conditioned on only one non-empty cell $c$ is hashed into this bin and $c$ has at most $\beta$ points. We consider a fixed $y \in [a_2 \beta]$, suppose there is a point $p$ such that $g(p) = y$. For any other point $p'$ (belong to the same cell $C$), since $g$ is a pair-wise independence hash function, then we have

$$\Pr_{g \sim C}[g(p) = g(p')] = \Theta(1/(a_2 \beta)).$$
We define random variable $Y_p$ such that $Y_p = 1$ if $g(p) = g(p')$; $Y_p = 0$ otherwise. We define $Y = \sum_{p' \in C \cap P \setminus \{p\}} Y_{p'}$, and compute $\mathbb{E}[Y] \leq \Theta(1/a_2)$. By Markov’s inequality,

$$\Pr[Y \geq 1] \leq \mathbb{E}[Y]/1 \leq \Theta(1/a_3).$$

By choosing $a_3$ to be sufficiently large constant, we obtain with probability at least 0.99, there is no $p' \in P \cap C \setminus \{p\}$, satisfying that $g(p) = g(p')$.

Moreover, there are at most $(\theta \beta + \gamma)$ distinct pairs $(c,p)$. Repeating the above argument, we observe that for give pair $(h(C), g(p))$, with probability at least 0.99, no other pairs $(h(C'), g(p'))$ has the same hash value $f(h(C), g(p))$.

We say an entry $(C, p)$ is good, if the cell $C$ is non-empty and contains at most $\beta$ points. We can show for any good entry $(C, p)$, one can recover point $p$ with probability at least 0.9. Since the total number of good pairs is at most $\gamma$, thus repeating the above procedure $\Theta(\log(\gamma/\delta))$ times, we have for any good entry $(C, p)$, we can recover point $p$ with probability at least $1 - \delta/\gamma$. By taking a union bound over at most $\gamma$ pairs, we have with probability at least $1 - \delta/2$, we can recover all the good entries $(C, p)$. By taking a union with the event that the K-Set data-structure is working correctly (with probability at least $1 - \delta/2$), we obtain the probability $1 - \delta$.

It remains to show that if any of the three events not happening our algorithm never outputs an incorrect result. If the number of cells is greater than $\alpha$, then either the K-set structure fails, or we know the number of cells are greater than $\alpha$, thus the algorithm always outputs $\emptyset$. If the remaining events does not happen or there are two good pairs collide in their hash values, the algorithm can always detect it by the counters. Thus the points recovered are always from the stream. A proper pruning detects any possible error.

Overall, for a single repeat, the number of bits used by hash function is $O(\log(\Delta^d))$, the number of bits used by counter counter$_i()$ is $O(\alpha_2(\theta \beta + \gamma))\log(\Delta^d)$, the number of bits used by counter size$_i()$ is $O(\alpha_3(\theta \beta + \gamma))\log(\Delta^d)$. The number of bits for K-Set is $O(\alpha \log(\Delta^d)\log(\alpha/\delta))$. Putting it all together, the number of bits used over all repeats is

$$O((\theta \beta + \gamma)\log(\Delta^d) \cdot \log((\theta \beta + \gamma)/\delta)).$$

\[\Box\]

### E.3 Main Algorithm

**Theorem E.5** ($k$-means). Fix $\epsilon, \delta \in (0, 1/2)$, $k, \Delta \in \mathbb{N}_+$, let $L = \log \Delta$. There is a data structure supporting insertions and deletions of a point set $P \subset [\Delta]^d$, maintaining a weighted set $S$ with positive weights for each point, such that with probability at least $1 - \delta$, $S$ is an $\epsilon$-coreset for $k$-means of size

$$O(\epsilon^{-2}k^2d^7L^4\log(1/\delta)).$$

The data structure uses

$$\tilde{O}(\epsilon^{-2}k^2d^8L^5 \cdot (dkL + \log(1/\delta)) \cdot \log(1/\delta))$$

bits in the worst case. For each update of the input, the algorithm needs $\text{poly}(d, 1/\epsilon, L, \log k)$ time to process and outputs the coreset in time $\text{poly}(d, k, L, 1/\epsilon, 1/\delta, \log k)$ after one pass of the stream.
Proof. Let the point set at the end of the insertion and deletion operations be $Q \subset [\Delta]^d$. Without loss of generality, we assume the optimal cost for $k$-means is at most $\text{OPT}$ and $\text{OPT} > 0$. Indeed we if $\text{OPT} = 0$, then the set $Q$ contains at most $k$ points. Thus we can always use a simple $K$-set structure to dynamically maintain $k$ points. The correctness of such an algorithm is guaranteed by Lemma E.3. Further we observe that $\text{OPT} \geq 1$, since each pair of points is of distance at least 1. To show our data structure, we first pick a constant number $\rho > 0$ and use it as the failure probability. Later we show we can boost it to arbitrarily small $\delta > 0$. Furthermore, we prove our theorem by showing a $a\delta$ failure probability, where $a$ is some absolute constant. We can easily boost the probability by only losing a constant factor in space.

The Data Structure Let $\rho > 0$ be a constant, i.e., take $\rho = 0.01$. Our data structure first initializes a randomized grid structure $G_{-1}, G_0, G_1, \ldots, G_L$ over $[\Delta]^d$, as in Definition D.1. Then we guess logarithmic many possible values for $\text{OPT}$, i.e., let $O = \{1, 2, 4, 8, \ldots, \text{poly log}((\Delta^d+2)\cdot d)\}$. For each guess of $o$, we initialize a SAMPLESTREAM$^\top$ data structure $SS_o$ with parameters $\alpha, \beta, \gamma, \theta, \delta$, where $\alpha, \beta, \gamma, \theta$ to be determined.

For each $o$ and level $i \in [-1, L]$, we initialize a new hash function $h_{o,i} : [\Delta]^d \to \{0, 1\}$ such that $\forall p \in [\Delta]^d : \Pr[h_{o,i} = 1] = \pi_i(o, \epsilon, \delta)$ for some $\epsilon, \delta \in (0, 0.5)$, where $\pi_i(\cdot, \cdot, \cdot)$ is defined in (5). For representation simplicity, we first assume that $h_{o,i}$s are fully independent hash functions. Later we show that we can de-randomize them by using limited independence hash functions.

For each $o \in O$ and $i \in [-1, L]$, and each point $p$ in operation, let $C_i(p)$ be the cell containing $p$ in $G_i$. We also identify $C_i(p)$ with its ID. For all $i \in [-1, L]$, we first compute its hash value $h_{o,i}(p)$, if $h_{o,i}(o) = 1$, we update $(C_i(p), p)$ to the data structure $SS_o$ with the corresponding operations on
Algorithm 2 Better Data Structure with Small Number of Bits

1: procedure SampleStream⁺
2: procedure Init(Δ, d, α, β, γ, θ, δ)
3:  \(a_1 \leftarrow \Theta(1), a_2 \leftarrow \Theta(1), a_3 \leftarrow \Theta(1), R_{\text{repeats}} \leftarrow \Theta((\theta\beta + \gamma)/\delta))\)
4:  Initialize K-Set data-structure : KS.Init(\(\alpha, \Delta^d, \delta/\alpha\))
5:  \(\mathcal{C} \leftarrow \{\text{the set of all cells (i.e., the } L + 2 \text{ levels of grids)}\).\)
6: for \(i = 1 \rightarrow R_{\text{repeats}}\) do
7:  Choose \(h_i : \mathcal{C} \rightarrow [a_1a]\) to be pair-wise independent hash function
8:  Choose \(g_i : [P] \rightarrow [a_2\beta]\) to be pair-wise independent hash function
9:  Choose \(f_i : [a_1a] \times [a_2\beta] \rightarrow [a_3(\theta\beta + \gamma)]\) to be pair-wise independent hash function
10: Initialize counter : counter, \((w, j, z) \leftarrow 0, \forall w \in [a_3(\theta\beta + \gamma)], j \in [2\log(\Delta^d)], z \in \{0, 1\}\)
11: Initialize counter : size, \((w) \leftarrow 0, \forall w \in [a_3(\theta\beta + \gamma)]\)
12: end for
13: end procedure
14: procedure Update(C, p, op) \(\triangleright C, p\) the cell and the point, op \(\in \{-, +\}\)
15:  KS.Update(C, op)
16:  for \(i = 1 \rightarrow R_{\text{repeats}}\) do
17:  for \(j = 1 \rightarrow \lceil \log \Delta^d \rceil\) do
18:     \(x \leftarrow h_i(C)\) \(\triangleright \text{Hash cell into bins}\)
19:     \(y \leftarrow g_i(p)\) \(\triangleright \text{Hash point into bins}\)
20:     \(w \leftarrow f_i(x, y)\)
21:     \(z \leftarrow j\)-th bit of the bit representation of \((C, p)\)
22:     counter, \((w, j, z) \leftarrow \text{counter, } (w, j, z) \mod 2\) \(\triangleright \text{Update counter}\)
23:     size, \((w) \leftarrow \text{size, } (w) \mod 2\) \(\triangleright \text{Update counter}\)
24:  end for
25: end for
26: end procedure
27: procedure Query()
28:  for \(i = 1 \rightarrow R_{\text{repeats}}\) do
29:     \(S_i \leftarrow \emptyset\)
30:     if KS.Check() \(\neq \text{Fail}\) or KS.size \(\leq \alpha\) then
31:      \(\bar{C}, f \leftarrow \text{KS.Query()}\)
32:      for \(c \in \bar{C}\) do
33:       \(S_{i, c} \leftarrow \emptyset\), \(x \leftarrow h_i(C)\)
34:       for \(y \in [b\beta]\) do
35:        \(w \leftarrow f_i(x, y)\)
36:        if size, \((w) = 1\) and \(\forall j \in [2\log(\Delta^d)]\), \(\text{counter}(w, j, 0) + \text{counter}(w, j, 1) = 1\) then
37:         \((C, p) \leftarrow \text{BitToNumber(counter}(w, *, *))\),
38:         \(S_{i, c} \leftarrow S_{i, c} \cup \{(C, p)\}\)
39:      end if
40:     end for
41:     else
42:      return \(\emptyset\)
43:     end if
44:  end for
45:  \(S \leftarrow S \cup S_i\)
46: end for
47: \(\bar{S} \leftarrow \text{Merge}(S_1, S_2, \cdots, S_{\text{repeats}})\)
48: if \(\exists C \in \bar{C}, f(C) \leq \beta : |S_C| \neq f(C)\) then
49:  return \(\emptyset\)
50: else
51:  return \(\bar{C}, f, S\)
52: end if
53: end procedure
54: end procedure
together with the operations in Definition D.14 give a description of retrieving the coreset. The complete data structure is presented in Algorithm 3.

**Retrieve The Coreset** Next we describe how to retrieve the coreset from the data maintained in the above data structure. First we find the smallest \( o^* \in O \) such that \( SS_{o^*} \) does not return \( \emptyset \). From \( SS_{o^*} \), we obtain a set of cells \( \tilde{C} \) together with a function \( \tilde{f} \) that encodes the number of points in each cell in \( \tilde{C} \). We also obtain a set \( SC \) for each \( C \in \tilde{C} \cap G_i \) that is either \( \emptyset \) or \( SC = C \cap Q_{o,i} \), where \( Q_{o,i} = \{ q \in Q : h_{o,i}(q) = 1 \} \). For each \( C \in \tilde{C} \), we estimate the number of points in \( C \), \( |C \cap Q| \), by \( \tilde{f}(C)(1+\epsilon/2)/\pi_i(o, \epsilon, \delta) \). With this, we identify all the heavy cells in \( \tilde{C} \) by following Definition D.14. After this, we initialize our multiset \( S = \emptyset \). For each non-heavy cell \( C \) in \( \tilde{C} \), we assign a weight \( 1/\pi_i(o, \epsilon, \delta) \) to each of point in \( SC \). After this, add all the points in \( SC \) to \( S \). This completes the description of retrieving the coreset. The complete data structure is presented in Algorithm 3.

**Correctness with Constant Probability** We first show that by chosen a constant \( \rho \), i.e., for a \( \rho > 0 \), if the event \( \xi \) happens, i.e., there are at most \( ekL/\rho \) center cells in the grid, then with probability at least \( 1 - \delta \), our data structure maintains an \( \epsilon \)-coreset \( S \) for \( k \)-means. Notice that \( \Pr[\xi] \geq 1 - \rho \). To show the correctness of the data structure, we show that there exists an \( 1 \leq o \leq OPT \), the output \( S \) is an \( \epsilon \)-coreset for \( k \)-means on \( Q \).

**Heavy Cell Identification** First, for an \( 1 \leq o \leq OPT \), we show that with probability at least \( 1 - \delta \), for all \( i \in [0, -L] \) and for all \( C \in G_i \), the estimator of points in a cell,

\[
\tilde{c}_{o,i} = |Q_{o,i} \cap C|/\pi_i(o, \epsilon, \delta) \cdot (1 + \epsilon/2),
\]

together with the operations in Definition D.14 give an \( (o, \epsilon) \)-heavy cell scheme (Definition E.1) by identifying heavy cells from level \(-1 \) to \( L \) and removing those heavy cells with parent cells identified as non-heavy.

Here

\[
\pi_i(o, \epsilon, \delta) = O \left( \frac{1}{\epsilon^2} \cdot \frac{(d^3L)^2}{T_i(o)} \cdot \left( dkL + \log \frac{1}{\delta} \right) \right)
\]

is defined in (5). Indeed it is suffice to show that for all heavy cells \( C \in G_i \), \( |C \cap Q_{o,i}|/\pi_i(o, \epsilon, \delta) \) is an \( (1 \pm \epsilon/2) \)-approximation to \( |Q_{o,i} \cap C| \). Observing that

\[
\mathbb{E} \left[ |C \cap Q_{o,i}| \right] \geq O \left( \frac{1}{\epsilon^2} \cdot (d^3L)^2 \cdot \left( dkL + \log \frac{1}{\delta} \right) \right),
\]

by the Chernoff bound and an union bound over all heavy cell cells, we obtain the desired precision of estimation.

**Correct Parameters of SampleStream** In this section, we show that there exists an \( 1 \leq o \leq OPT \) such that with probability at least \( 1 - \delta \), an \textsc{SampleStream} instance \( SS_o \) returns the set of points in non-heavy cells together with all the estimates of the number of points in heavy cells. To show this, it is suffice to show that an \( OPT/2 \leq o \leq OPT \) satisfies the above statement with probability at least \( 1 - \delta \).

Firstly, we show that with probability at least \( 1 - \delta/4 \) there are at most \( \alpha \) (to be determined) non-empty cells in all grids containing a sampled point (i.e., \( h_{o,i}(p) = 1 \)).
Indeed, for each point in a non center cell, it contributes to the \(OPT\) with at least \(\frac{g_i^2}{4d^2}\) (see Definition (1)). Therefore, there are at most

\[
\frac{OPT}{g_i^2/(4d^2)} = \frac{4d^2}{g_i^2} OPT = 4T_i(o) \cdot \frac{ek}{\rho} \cdot \frac{OPT}{o}
\]

points in non-center cells per level of grid, where \(T_i(o) = \frac{d^2}{g_i^2} \cdot \frac{eo}{ek}\) is the thresholding function. Let \(Q'_o,i\) be the set of points in non-center cells of level \(i\) with \(h_{o,i}\) value 1, i.e., the set of sampled points from non-center cells. Therefore,

\[
E[|Q'_o,i|] = 4T_i(o) \cdot \frac{ek}{\rho} \cdot \frac{OPT}{o} \cdot \pi_i(o, \epsilon, \delta) = O \left( \frac{kL}{\epsilon^2} \cdot (d^3L)^2 \cdot (dkL + \log \frac{1}{\delta}) \right),
\]

where we use the fact that \(OPT/o \leq 2\). By the Chernoff bound and a union bound over all grids, there are at most

\[
O \left( \frac{kL}{\epsilon^2} \cdot (d^3L)^2 \cdot (dkL + \log \frac{1}{\delta}) \right)
\]

points from non-center cells being sampled. Hence there are at most this number of non-center cells containing a sampled point. Together with the number of center cells, there are at most

\[
\alpha = O \left( \frac{kL}{\epsilon^2} \cdot (d^3L)^2 \cdot (dkL + \log \frac{1}{\delta}) \right)
\]

(7)

cells containing a sample point. Next we pick

\[
\beta = \Theta \left( \frac{1}{\epsilon^2} \cdot (d^3L)^2 \cdot (dkL + \log \frac{1}{\delta}) \right)
\]

(8)

By the Chernoff bound and a union bound over all cells, with probability at least \(1 - \delta/4\), we have that each non-heavy cell containing at most \(\beta/2\) sample points. Next, since that are at most \(O(kL/\rho)\) heavy cells, we set

\[
\theta = \Theta \left( \frac{kL}{\rho} \right).
\]

(9)

Then there are at most \(\theta\) cells containing more than \(\beta\) sample points. Lastly, we pick \(\gamma = \alpha\) as an upper bound on the number of points in cells with at most \(\beta\) sample points. As such, conditioning on the above events, by Lemma E.4, with probability at least \(1 - \delta\), the data structure SS\(_o\) outputs the desired set of points and the cells with counts. And by Theorem E.2, the multiset is an \(\epsilon\)-coreset for \(k\)-means on \(Q\).

**Boost Probability from** \(1 - \rho - O(\delta)\) **to** \(1 - O(\delta)\)  The probability boosting procedure is by independently repeating the above procedure in parallel \(u = O(\log \frac{1}{\delta})\) times and take the output as follows. For each \(r \in [u]\), suppose in the \(r\)-th repeat, the output guess (if exists) of OPT is \(o_r\), the SAMPLESTREAM structure is SS\(_o\), the multiset is \(S^r\) and the number of heavy cells returned by SS\(_o\) is \(\alpha'_r\). Based on the heavy cell scheme, we also estimate the number of parts as \(\alpha''_r\), i.e., estimates the number of points in non-heavy children of a heavy cell, and divide \(T_i(o)\) to obtain \(\alpha''_r\). Consider all the repeats with

\[
\alpha''_r \leq a_1 kL
\]

for some large constant \(a_1\). Let the set of repeats satisfying this requirement be \(R\).
Let \( r^* = \arg \min_{r \in R} o^r \). Then the final output is taken as \( S_{r^*} \).

To show that the overall failure probability of such an operation is at most \( O(\delta) \), the first claim is that, if \( \exists o^r \leq \text{OPT} \) and \( a_{o^r}^r \leq a_1 kL \), then in the repeat \( r \), with probability at least \( 1 - \delta \), the number of parts in an \( o^r \)-partition (see Definition D.6) is at most \( O(kL) \). This simply follows from the Chernoff bound, since with probability at least \( 1 - \delta \), the number of points in a set with more than \( T_i(o) \) points can be accurately estimated. Thus it is straightforward to verify that this partition satisfies the requirement of Proposition D.16. Thus, by Proposition D.16, with probability at least \( 1 - O(\delta) \) we obtain the desired coreset.

Next, we show that there exists an \( r \in [u] \), and \( o^r \leq \text{OPT} \). Since each repeat is independent, then with probability at least \( 1 - O(\delta) \), there exists an \( r \) with at most \( ekL/\rho \) centers cells the grid of the \( r \)-th repeat. Conditioning on this event, the \( r \)-th repeat outputs the multiset with \( o^r \leq \text{OPT} \) with probability at least \( 1 - O(\delta) \). This concludes the probability boosting.

### Space Bound and Random Bits

Lastly we show the space algorithm. Each `SAMPLESTREAM` data structure uses

\[
O \left( \alpha \log \left( \frac{\alpha}{\delta} \right) \cdot dL \right) = O \left( \frac{kL}{e^2} \cdot (d^3L)^2 \cdot dL \cdot \left( dkL + \log \frac{1}{\delta} \right) \cdot \left( \log(kLd/e) + \log \log \frac{1}{\delta} \right) \right)
\]

bits of space. For all \( O(d \log \Delta) \) repeats, the overall space complexity is

\[
\tilde{O} \left( \frac{kd^2L^3}{e^2} \cdot (d^3L)^2 \cdot \left( dkL + \log \frac{1}{\delta} \right) \cdot \log \frac{1}{\delta} \right) = \tilde{O} \left( \frac{kd^8L^5}{e^2} \cdot \left( dkL + \log \frac{1}{\delta} \right) \log \frac{1}{\delta} \right)
\]

Lastly for the independent hash function we use, we can de-randomize them by the method in [FS05] by using \( z \)-wise independent hash functions for sufficiently large \( z \), or use a pseudo-random generator as it in [Ind00b] and then apply a auxiliary algorithm argument as it in [BFL+17]. In either case, one can show that the space for the random bits is not dominating. This completes the proof.

### F General Clustering Problem

#### F.1 M-Estimator Clustering

Our framework can be extended easily to other clustering problems, e.g., we consider the following \( M \)-Estimator clustering problem, fixing \( Q \subset [\Delta]^d \),

\[
\min_{Z \subset [\Delta]^d | |Z| \leq k} \sum_{p \in Q} M(\text{dist}(p, Z))
\]

Here \( M(\cdot) \) is non-decreasing function satisfies

\[
\forall x > 0, M(x) > 0 \quad \text{and} \quad \forall c > 0, M(cx) \leq f(c)M(x)
\]

where \( f(c) > 0 \) is a bounded function.
Algorithm 3 Data Structure

1: procedure SampleKMeans
2:   procedure Init($k, \Delta, d, \epsilon, \delta, \delta$)
3:     $O \leftarrow \{1, 2, \cdots, \text{poly}(d, \Delta^d)\}$
4:     $\rho \leftarrow 0.01,$
5:     Construct $G_{-1}, G_0, \ldots, G_L$ \> grids defined in Section D.1 using parameter $\rho$
6:     for $o \in O$ do
7:       for $i = 0 \rightarrow L - 1$ do
8:         Construct hash function $h_{o,i} : [\Delta]^d \rightarrow \{0, 1\}$ s.t. $\Pr[h_{o,i}(p) = 1] = \pi_i(o, \epsilon, \delta)$
9:       \> $\alpha, \theta, \gamma$ and $\beta$ are determined in (7), (8) and (9)
10:      $\text{RecordPoints}_o \leftarrow \text{SampleStream.Init}(\Delta, d, \alpha, \beta, \gamma, \theta, \delta)$
11:     end for
12:   end for
13: end procedure
14: procedure Update($p, op$) \> $op \in \{-, +\}$
15:   for $o \in O$ do
16:     for $i = 0 \rightarrow L - 1$ do
17:       if $h_{o,i}(p) = 1$ then
18:         $c \leftarrow \text{Cell}(p)$ in $G_i$
19:         $\text{RecordPoints}_o, \text{Update}(c, p, op)$
20:       end if
21:     end for
22:   end for
23: end procedure
24: procedure Query()
25:   for $o \in O$ do
26:     $\tilde{C}^o, \tilde{f}^o, S^o \leftarrow \text{RecordPoints}_o, \text{Query}()$
27:   end for
28:   $o^* \leftarrow \arg\min_{o \in O} \{S^{o,i} \neq \emptyset, \forall i \in \{0, 1, \cdots, L - 1\}\}$ \> $S^o = \{S^{o,0}, S^{o,1}, \cdots, S^{o,L-1}\}$
29:   $S \leftarrow \text{PointsToCoreset}(\tilde{C}^{o^*}, \tilde{f}^{o^*}, S^{o^*})$
30:   return $S$
31: end procedure
32: end procedure

Theorem F.1 (M-Estimator Clustering). Fix $\epsilon, \delta \in (0, 1/2)$, $k, \Delta \in \mathbb{N}_+$, let $L = \log \Delta$. There is a data structure supporting insertions and deletions of a point set $P \subset [\Delta]^d$, maintaining a weighted set $S$ with positive weights for each point, such that with probability at least $1 - \delta$, $S$ is an $\epsilon$-coreset for M-estimator clustering problem defined in (10) of size $O(\epsilon^{-2}k^2 d^n L^4 \log(1/\delta))$, where $\eta$ is an absolute constant depending only on $M$. The data structure uses

$$O(\tilde{\Omega}(\epsilon^{-2}k^2 d^n L^4 \cdot (dkL + \log(1/\delta)) \cdot \log(1/\delta)))$$

bits in the worst case. For each update of the input, the algorithm needs $\text{poly}(d, 1/\epsilon, L, \log k)$ time to process and outputs the coreset in time $\text{poly}(d, k, L, 1/\epsilon, 1/\delta, \log k)$ after one pass of the stream.
Proof. We can simply repeat the proof for $k$-means. In the proof, we need to modify heavy cell
definition, i.e., the thresholding function $T_i(o)$,

$$T_i(o) = \frac{1}{M(g_i/(2d))} \cdot \rho \cdot o \cdot \frac{1}{ek}.$$

In the proof of Theorem D.14, we need to modify $\beta$ to be,

$$\beta = \Theta \left( \frac{L \cdot f(\sqrt{d})}{f(1/(2d))} \right).$$

The rest of the proof follows by replacing $\text{dist}^2(\cdot)$ with $M(\text{dist}(\cdot))$. We can verify that only the
dependence on $d$ changes. \hfill \Box

F.2 Improvements Over $k$-median

In this section, we show that our newly developed techniques can improve over the $k$-median con-
struction in [BFL+17]. In particular, we have the following guarantee.

**Theorem F.2** ($k$-median). Fix $\epsilon, \delta \in (0, 1/2)$, $k, \Delta \in \mathbb{N}_+$, let $L = \log \Delta$. There is a data structure
supporting insertions and deletions of a point set $P \subset [\Delta]^d$, maintaining a weighted set $S$ with
positive weights for each point, such that with probability at least $1 - \delta$, $S$ is an $\epsilon$-coreset for $k$-
median of size

$$\epsilon^{-2}k \cdot \text{poly}(d, L) \cdot \log(1/\delta).$$

The data structure uses

$$\tilde{O}(\epsilon^{-2}k \cdot \text{poly}(d, L) \cdot \log(1/\delta))$$

bits in the worst case. For each update of the input, the algorithm needs $\text{poly}(d, 1/\epsilon, L, \log k)$ time
to process and outputs the coreset in time $\text{poly}(d, k, L, 1/\epsilon, 1/\delta, \log k)$ after one pass of the stream.

**Proof.** Using the same construction in [BFL+17]. The improvements can be stated as follows.
Instead of take a union over the each level for union bounding the center cells, we bound the overall
center cells. This saves an $L$ factor in the space. However doing so can only be achieved by using
our newly constructed SampleStream procedure, which additionally saves $L$ factor.

Furthermore, we boost the failure probability from $\rho$ to arbitrary $\delta > 0$ using the procedure in
proof Theorem E.5. \hfill \Box

**Remark F.3.** Using our newly developed technique, we save space complexity for $O(1/\delta)$ factor,
comparing with that in [BFL+17]. Furthermore, our newly developed stream sampling algorithm
also saves space up to $\text{poly}(dL)$ factors.

G Applications

In this section, we show our dynamic data structure can be used to approximately maintain a
solution for many problems in the dynamic streaming setting.
G.1 A Dynamic Streaming Approximation to Max-CUT

In this section, we show that our coreset construction can be used to obtain a 1/2-approximation of Max-CUT of the form as

$$\max_{C_1 \cup C_2} \sum_{p \in C_1} \sum_{q \in C_2} M(\text{dist}(p, q))$$

(11)

where $M(\cdot)$ is an $M$-estimator and $C_1$ and $C_2$ partition the streaming point set $P \subset [\Delta]^d$. Formally, we obtain the following result.

**Theorem G.1 (Max-CUT).** Fix $\epsilon, \delta \in (0, 1/2)$, $\Delta \in \mathbb{N}_+$, let $L = \log \Delta$. There is a data structure supporting insertions and deletions of a point set $P \subset [\Delta]^d$, and outputs a 1/2-solution with cost estimation up to a $(1 \pm \epsilon)$ factor with probability at least $1 - \delta$. The data structure uses $O\left(\frac{1}{\epsilon^2} \cdot \text{poly}(d, L) \cdot \log \frac{1}{\delta}\right)$ bits in the worst case. For each update of the input, the algorithm needs $\text{poly}(d, 1/\epsilon, L)$ time to process and outputs the coreset in time $\text{poly}(d, L, 1/\epsilon, 1/\delta)$ after one pass of the stream.

**Proof.** To solve Max-CUT, we simply use a random solution, i.e. use a hash function to maintain two random cuts $C_1$ and $C_2$. Let $\text{OPT}_{\text{CUT}}$ be the optimal value for Max-CUT. It is a standard result that,

$$\mathbb{E} \left[ \sum_{p \in C_1} \sum_{q \in C_2} M(\text{dist}(p, q)) \right] \geq \frac{\text{OPT}_{\text{CUT}}}{2}.$$  

To approximate the cost, we use the coreset to obtain two $(\epsilon/2)$-coresets for 1-$M$-clustering on both $C_1$ and $C_2$. Let the coresets be $S_1$ and $S_2$. Then we show

$$\sum_{p' \in S_1} \sum_{q' \in S_2} M(\text{dist}(p', q')) = (1 \pm \epsilon) \sum_{p \in C_1} \sum_{q \in C_2} M(\text{dist}(p, q)).$$

Indeed,

$$\sum_{p' \in S_1} \sum_{q' \in S_2} M(\text{dist}(p', q')) = (1 \pm \epsilon/2) \sum_{p' \in S_1} \sum_{q \in C_2} M(\text{dist}(p', q))$$

$$= (1 \pm \epsilon/2)^2 \sum_{p \in C_1} \sum_{q \in C_2} M(\text{dist}(p, q)).$$

as desired. The probability boosting for the solution is standard. \qed

G.2 A Dynamic Streaming Approximation to Average Distance

For similar proof of the Max-CUT, we obtain a dynamic estimation for average distance,

$$\frac{1}{|Q| - 1} \sum_{p, q \in Q} M(\text{dist}(p, q))$$

where $M(\cdot)$ is an $M$-estimator.

**Theorem G.2 (Average Distance).** Fix $\epsilon, \delta \in (0, 1/2)$, $\Delta \in \mathbb{N}_+$, let $L = \log \Delta$. There is a data structure supporting insertions and deletions of a point set $P \subset [\Delta]^d$, and outputs a $(1 \pm \epsilon)$ approximation to the average distance of $P$, with probability at least $1 - \delta$. The data structure uses $O\left(\frac{1}{\epsilon^2} \cdot \text{poly}(d, L) \cdot \log \frac{1}{\delta}\right)$ bits in the worst case. For each update of the input, the algorithm needs $\text{poly}(d, 1/\epsilon, L)$ time to process and outputs the coreset in time $\text{poly}(d, L, 1/\epsilon, 1/\delta)$ after one pass of the stream.
Proof. The proof is similar to that of Theorem G.1 by constructing a corset for 1-M-Clustering on $Q$. 

\section*{H $\tilde{O}(k)$ Space Algorithm Based on Sensitivity Sampling}

\subsection*{H.1 Definitions and Preliminaries}

\textbf{Definition H.1 ($\gamma$-important partition).} Given a set of points $Q \subset [\Delta]^d$, and parameter $\gamma \in (0, 1/10)$. Let $P_{i,j}$ be the partition sets defined for each level $i \in [-1, L]$. Let $P$ denote the partition, 

$$P = \bigcup_{i=0}^{L} P_i = \bigcup_{i=0}^{L} \bigcup_{j=1}^{P_i} \{P_{i,j}\}.$$

We define sets $R, \overline{R} \subset \{0, 1, \ldots, L\}$ as follows

$$R = \left\{ i \mid \sum_{j=1}^{P_i} |P_{i,j}| \leq \gamma \cdot T_i(o) \right\}, \quad \text{and} \quad \overline{R} = \left\{ i \mid \sum_{j=1}^{P_i} |P_{i,j}| > \gamma \cdot T_i(o) \right\}.$$

We define important partition and non-important partition as follows

$$P^I = \bigcup_{i \notin R} P_i \quad \text{and} \quad P^N = \bigcup_{i \in R} P_i.$$

Let $Q^I$ denote the set of $\gamma$-important points of $Q$ that contained in $\gamma$-important partition $P^I$. Let $Q^N$ denote the set of $\gamma$-non-important points of $Q$ that contained in $\gamma$-non-important partition $P^N$.

If in the context, the value of $\gamma$ is clear, we sometimes will omit $\gamma$, and call $Q^I$ as important points, and call $P^I$ as important partition.

\textbf{Theorem H.2 (Off-line Algorithm, [FL11]).} Given a set of points $Q \subset [\Delta]^d$, for each point $p \in Q$, let $s(p)$ denote the sensitivity of point $p$ which is defined as follows

$$s(p) = \max_{Z \in [\Delta]^d, |Z| = k} \frac{\text{dist}^2(p, Z)}{\sum_{q \in Q} \text{dist}^2(q, Z)}.$$

Let $A$ denote a sampling procedure defined in the following way: it repeats $m$ independent samples; for each sample, it chooses point $p$ with probability $s'(p)/t'$ where $s'(p) \in [s(p), 1]$, $t' = \sum_{p \in Q} s'(p)$; and for each sample, if some point $p$ got chosen, we associated weight $w(p) = t'/(ms'(p))$ to that point. Let $(S, w(\cdot))$ denote the set of points and weights outputted by $A$. If $m \geq \Omega((t'\epsilon^{-2}(\log |Q| \log t' + \log(1/\delta))))$, then with probability at least $1 - \delta$, $(S, w(\cdot))$ is an $(1 \pm \epsilon)$-coreset of size $m$ for $Q$.

\subsection*{H.2 Reducing Original Problem to Important Points}

In this section, we show that if we want to output a coreset for the original point set, we only need to find the coreset for those important points. It means that we can drop all the points which are in the “non-important” level.

\textbf{Lemma H.3.} Given a set of points $Q \subset [\Delta]^d$ associated with partition $P$ defined as follows,

$$P = P_0 \cup P_1 \cup \cdots \cup P_L.$$

Let $P^N \subset Q$ denote the set of non-important points associated with partition $P^N$ defined as follows,

$$P^N = \bigcup_{i \in \overline{R}} P_i,$$
where $R = \{ i \mid \sum_{j=1}^{[P_i]} |P_{i,j}| \leq \gamma T_i(o) \}$ and $\gamma \leq \epsilon/(200Ld^3\rho)$. Let $Q^I$ denote the set of points that contained by non-important partitions $P^I$. Then for all $Z \subset [\Delta]^d$ with $|Z|$, we have

(I) $\text{cost}(Q, Z) \geq \text{cost}(Q^I, Z)$

(II) $\text{cost}(Q, Z) \leq (1 + \epsilon) \text{cost}(Q^I, Z)$

Proof. Proof of (I). It is trivially true, because $Q^N$ is a subset of $Q$.

Proof of (II). We consider a $i$ which satisfies

$$\sum_{j=1}^{[P_i]} |P_{i,j}| \leq \gamma \cdot T_i(o),$$

and fix a $j \in [P_i]$. Let $c(P_{i,j})$ denote the cell in $(i - 1)$-th level which contains all the points in partition $P_{i,j}$. For each point $p \in P_{i,j}$, let $N_{i,j}$ denote the set of important points ($\subset Q^I$) that contained by cell $c(P_{i,j})$, there exists a point $q \in N_{i,j}$ satisfies that

$$\text{dist}_2^2(p, Z) \leq 2 \text{dist}_2^2(p, q) + 2 \text{dist}_2^2(q, Z) \leq 2dg_{i-1}^2 + 2 \text{dist}_2^2(q, Z) \leq 2dg_{i-1}^2 + \frac{1}{|N_{i,j}|} \sum_{q \in N_{i,j}} \text{dist}_2^2(q, Z) \quad (12)$$

where the first step follows by triangle inequality, the second step follows by definition of the grids, the last step follows by an averaging argument.

According to definition of $N_{i,j}$,

$$|N_{i,j}| \geq T_{i-1}(o) - \gamma T_i(o) \geq \frac{1}{2} T_{i-1}(o).$$
We can lower bound \( \text{cost}(Q, Z) \) in the following sense,

\[
\text{cost}(Q, Z) = \text{cost}(Q^I, Z) + \text{cost}(Q^N, Z)
\]

\[
= \text{cost}(Q^I, Z) + \sum_{i \in R} \sum_{j=1}^{\left| \mathcal{P}_{i,j} \right|} \sum_{p \in \mathcal{P}_{i,j}} \text{dist}^2(p, z)
\]

\[
\leq \text{cost}(Q^I, Z) + 2 \sum_{i \in R} \sum_{j=1}^{\left| \mathcal{P}_{i,j} \right|} \left| \mathcal{P}_{i,j} \right| \cdot \left( d_{g_{i-1}}^2 + \frac{1}{\left| N_{i,j} \right|} \sum_{q \in N_{i,j}} \text{dist}^2(q, Z) \right)
\]

\[
\leq \text{cost}(Q^I, Z) + 2 \sum_{i \in R} \sum_{j=1}^{\left| \mathcal{P}_{i,j} \right|} \left| \mathcal{P}_{i,j} \right| \cdot \left( d_{g_{i-1}}^2 + \frac{2}{T_{i-1}(o)} \sum_{q \in Q^I} \text{dist}^2(q, Z) \right)
\]

\[
\leq \text{cost}(Q^I, Z) + 2 \sum_{i \in R} \sum_{j=1}^{\left| \mathcal{P}_{i,j} \right|} \left| \mathcal{P}_{i,j} \right| \cdot \left( d_{g_{i-1}}^2 + \frac{2}{T_{i-1}(o)} \sum_{q \in Q^I} \text{dist}^2(q, Z) \right)
\]

\[
\leq \text{cost}(Q^I, Z) + 4L^2 T_i(o) \cdot \left( d_{g_{i-1}}^2 + \frac{2}{T_{i-1}(o)} \sum_{q \in Q^I} \text{dist}^2(q, Z) \right)
\]

\[
= \text{cost}(Q^I, Z) + 4L^2 T_i(o) \cdot \left( d_{g_{i-1}}^2 + \frac{2}{T_{i-1}(o)} \text{cost}(Q^I, Z) \right)
\]

\[
\leq \text{cost}(Q^I, Z) + 4L^2 \gamma (d^3 \rho_0 / (3k) + 8 \text{cost}(Q^I, Z))
\]

\[
\leq \text{cost}(Q^I, Z) + 4L^2 \gamma (d^3 \rho_0 / (3k) + 8 \text{cost}(Q^I, Z)).
\]

where the second step follows by definition of cost, the third step follows by Eq. (12), the fourth step follows by \( |N_{i,j}| \geq T_{i-1}(o)/2 \), the fifth step follows by \( N_{i,j} \subset Q^I \), the sixth step follows by \( \sum_{i=1}^{\left| \mathcal{P}_{i,j} \right|} \leq \gamma T_i(o) \), the seventh step follows by \( |R| \leq L + 1 \leq 2L \), the ninth step follows by \( T_i(o) = 4T_{i-1}(o) \) and \( T_i(o) = d^3 \rho_0 / (3g_i^3 k) \), and the last step follows by \( o \leq \text{OPT} \leq \text{cost}(Q, Z) \).

It implies that

\[
\frac{\text{cost}(Q, Z)}{\text{cost}(Q \setminus Q^N, Z)} \leq \frac{1 + 32L \gamma}{1 - 4L \gamma d^3 \rho_0 / (3k)} \leq \frac{1 + \epsilon / 4}{1 - \epsilon / 4} \leq 1 + \epsilon,
\]

where the second step follows by \( \gamma \leq \epsilon / (200Ld^3 \rho) \), and the last step follows by \( \gamma < 1/2 \).

Therefore, we complete the proof.

**Lemma H.4.** Given a point set \( Q \) associated with partition \( \mathcal{P} \) and parameter \( o \in (0, \text{OPT}] \). For each \( p \in Q \), there must exist a unique partition \( \mathcal{P}_{i,j} \) that contains \( p \). Let \( Q^I \subset Q \) denote the set of

**H.3 Sampling Scores of Important Points**

In this section, we focus on the sensitivity of important points. We first give a good upper bound of sensitivity of important points. And then we show the sum of the upper bound of the sensitivities is small.

**Lemma H.4.** Given a point set \( Q \) associated with partition \( \mathcal{P} \) and parameter \( o \in (0, \text{OPT}] \). For each \( p \in Q \), there must exist a unique partition \( \mathcal{P}_{i,j} \) that contains \( p \). Let \( Q^I \subset Q \) denote the set of
\(\gamma\)-important points, where \(\gamma \leq \epsilon/(200Ld^3 \rho)\). Then we have, for all \(Z \subset |\Delta|\) with \(|Z| = k\), for all \(p \in Q^I\),

\[
\frac{\text{dist}^2(p, Z)}{\sum_{q \in Q^I} \text{dist}^2(q, Z)} \leq 60 \frac{d^3}{T_i(o)}.
\]

**Proof.** Fix a \(p \in Q^I\). Let \(\mathcal{P}_{i,j}\) denote the partition that contains this point. Let \(c(\mathcal{P}_{i,j})\) denote the cell in \(i-1\)-th level which contains all the points in partition \(\mathcal{P}_{i,j}\). Let \(N_{i,j}\) denote the set of non-important points belong to cell \(c(\mathcal{P}_{i,j})\). It is easy to observe that

\[
\exists p' \in N_{i,j}, \text{dist}^2(p', Z) \leq \frac{1}{|N_{i,j}|} \sum_{q \in N_{i,j}} \text{dist}^2(q, Z)
\]  

(13)

We have

\[
\frac{\text{dist}^2(p, Z)}{\sum_{q \in Q^I} \text{dist}^2(q, Z)} \leq 2 \frac{\text{dist}^2(p, p')}{\sum_{q \in Q^I} \text{dist}^2(q, Z)} + 2 \frac{\text{dist}^2(p', Z)}{\sum_{q \in Q^I} \text{dist}^2(q, Z)}
\]

\[
\leq 2 \frac{\sum_{q \in N_{i,j}} \text{dist}^2(q, Z)}{|N_{i,j}|} + \frac{1}{|N_{i,j}|} \sum_{q \in Q^I} \text{dist}^2(q, Z)
\]

\[
\leq 2 \frac{1}{|N_{i,j}|} + 2 \frac{\sum_{q \in Q^I} \text{dist}^2(q, Z)}{\text{OPT}}
\]

\[
\leq 2 \frac{1}{|N_{i,j}|} + \frac{4 \sum_{q \in Q^I} \text{dist}^2(q, Z)}{\text{OPT}}
\]

\[
\leq 2 \frac{1}{|N_{i,j}|} + \frac{16 \sum_{q \in Q^I} \text{dist}^2(q, Z)}{\text{OPT}}
\]

\[
\leq 2 \frac{1}{|N_{i,j}|} + \frac{48 \sum_{q \in Q^I} \text{dist}^2(q, Z)}{T_i(o)k \text{OPT}}
\]

\[
\leq 10 \frac{1}{T_i(o)} + \frac{48 \sum_{q \in Q^I} \text{dist}^2(q, Z)}{T_i(o)k \text{OPT}}
\]

\[
\leq 10 \frac{1}{T_i(o)} + 48 \frac{d^3 \rho o}{T_i(o)k \text{OPT}}
\]

\[
\leq 10 \frac{1}{T_i(o)} + 48 \frac{d^3 \rho o}{T_i(o)k \text{OPT}}
\]

\[
\leq 60 \frac{d^3}{T_i(o)}
\]

where the first step follows by triangle inequality, the second step follows by Eq. (13) and \(p' \in c(\mathcal{P}_{i,j})\), where the fifth step follows by \(\sum_{q \in Q^I} \text{dist}^2(q, Z) \geq (1 - \epsilon) \text{OPT} \geq \text{OPT}/2\) (By Lemma H.3), the sixth step follows by \(g_{i-1}^2 \leq 4g_i^2\), the seventh step follows by \(T_i(o) = \frac{d^3 \rho o}{T_i(o)k \text{OPT}}\), the eighth step follows by \(T_i(o) = 4T_{i-1}(o) \leq 5|N_{i,j}|\) (according partition and \(\gamma\)-important partition, see Definition D.6 and Definition H.1), the ninth step follows by \(\rho \in (0, 1), 1/k \leq 1, o \leq \text{OPT}\).

Thus, we complete the proof. \(\square\)

**Lemma H.5.** Given a point set \(Q\) associated with partition \(\mathcal{P}\) of parameter \(o \in [0, \text{OPT}]\). Let \(Q^I \subset Q\) denote the set of \(\gamma\)-important points, where \(\gamma \leq \epsilon/(200Ld^3 \rho)\). For each important level
\( i \in \overline{R} \), let \( \sum_{j=1}^{\left|P_i\right|} |P_{i,j}| = a_i \cdot T_i(o) \). Then we have, for all \( Z \subset \Delta^d \) with \( |Z| = k \), for all \( p \in Q^t \),

\[
\sum_{p \in Q^t} \frac{\text{dist}^2(p, Z)}{\sum_{q \in Q^t} \text{dist}^2(q, Z)} \leq 60d^3 \cdot \sum_{i \in \overline{R}} a_i \cdot 3.
\]

**Proof.**

\[
\sum_{p \in Q^t} \frac{\text{dist}^2(p, Z)}{\sum_{q \in Q^t} \text{dist}^2(q, Z)} \leq \sum_{P_{i,j} \in P_{i,j} \cap Q^t} 60 \frac{d^3}{T_i(o)}
\leq \sum_{P_{i,j}} |P_{i,j}| \cdot 60 \frac{d^3}{T_i(o)}
\leq 60d^3 \cdot \sum_{i \in \overline{R}} a_i \cdot 3.
\]

where the first inequality follows by Lemma H.4.

**H.4 Algorithm**

The goal of this Section is to prove Theorem H.6,

**Theorem H.6** (k-means, linear in \( k \)). Fix \( \epsilon, \delta \in (0, 1/2), k, \Delta \in \mathbb{N}_+, \) and let \( L = \log \Delta \). There is a data structure supporting insertions and deletions of a point set \( P \subset \Delta^d \), maintaining a weighted set \( S \) with positive weights for each point, such that with probability at least \( 1 - \delta \), \( S \) is an \( \epsilon \)-coreset for k-means of size

\[
\epsilon^{-3}k \text{poly}(d, L, \log(1/\delta)).
\]

The data structure uses

\[
\epsilon^{-3}k \text{poly}(d, L, \log(1/\delta))
\]

bits in the worst case. For each update of the input, the algorithm needs \( \text{poly}(d, 1/\epsilon, L, \log k) \) time to process and outputs the coreset in time \( \text{poly}(d, k, L, 1/\epsilon, 1/\delta, \log k) \) after one pass of the stream.

**Remark H.7.** For each level \( i \), there are three kinds of points, (1) points in partition, (2) points in heavy cell (these points belong to the partition of some level \( j \in \{i + 1, \ldots, L\} \)), (3) points in non-partition and non-heavy cell (these points belong to the partition of some level \( j \in \{0, 1, \ldots, i-1\} \)).

We first introduce some auxiliary concentration results.

**Theorem H.8** ([BR94]). Let \( k \) be an even integer, and let \( X \) be the sum of \( n \) \( k \)-wise independent random variables taking values in \([0, 1]\). Let \( \mu = E[X] \) and \( a > 0 \). Then we have

\[
\Pr \left[ |X - \mu| > a \right] \leq 8 \cdot \left( \frac{k\mu + k^2}{a^2} \right)^{k/2}.
\]
Algorithm 4 Sensitivity-Based Sampling

1: procedure ORACLE()
2:    procedure INIT(i,o,G_i) \[ G_i \text{ is the } i\text{-th grid} \]
3:        m ← c \max(d^3L\kappa^{-3}(Ld + \log(1/\delta)), d^3L\kappa^{-2}(dL \log(dLk) + \log(1/\delta))) \] for some constant c
4:        RECORDPOINTS ← SAMPLESTREAM\^\pi . INIT(\Delta, d, c_1kL + c_3mL, c_2m/k, c_3mL, c_1kL, \delta/(d^{\Delta d}c^4) >
5:        c_1, c_2, c_3, c_4 \text{ are four constants}
6:        for j = 1 \rightarrow 100m do
7:            \[ \text{Construct independent } (\log(1/\delta))-\text{wise independent hash } h_j : |\Delta|^d \rightarrow \{0, 1\} \text{ s.t.} \]
8:            \[ \Pr_{h_j \sim H}[h_j(p) = 1] = 1/T_i(o)/k \]
9:            \[ \text{Return } \text{FAIL if RECORDPOINTS returns FAIL} \]
10:        end for
11:    end procedure
12: end procedure
13: procedure Query(j')
14:    \( \tilde{C}, \tilde{f}, S_1, S_2, \ldots, S_{100m} \leftarrow \text{RECORDPOINTS.QUERY()} \)
15:    \( S_j \text{ contains the samples of the form } (p, j) \)
16:    Return FAIL if RECORDPOINTS returns FAIL
17:    Identify heavy cells using the samples from \( S_1, S_2, \ldots, S_{100m} \)
18:    \( \tilde{S}_j \leftarrow \emptyset \)
19:    for \( C \in \tilde{C} \) do
20:       if \( C \) is a partition cell in this level then
21:          \( \text{Partition cell: a cell is not heavy, but its parent is heavy; The criterion is based on } \tilde{f}(C) \)
22:          \( \tilde{S}_j \leftarrow \tilde{S}_j \cup S_{j,C} \)
23:       end if
24:    end for
25:    Let \( j \) denote the \( j' \)-th non-empty \( \tilde{S}_j \). If no such \( j' \) exists, return FAIL
26:    return A uniform sample from \( \hat{S}_j \)
27: end procedure
28: procedure EstNumPts( )
29:    Denote the same \( \hat{S}_j \) as in QUERY
30:    Return 0 if any step in QUERY returns FAIL
31:    return \( \sum |\cup_j \hat{S}_j| \cdot T_i(o) \cdot k/(100m) \)
32: end procedure
33: end procedure
34: end procedure

Lemma H.9. Let \( \pi \in [0,1] \) be a fixed value. Suppose we have \( m \) independent \( (4 \log(1/\delta))\)-wise hash functions \( h_1, h_2, \ldots, h_m : |\Delta|^d \rightarrow \{0, 1\} \) with that \( \forall i \in [m], p \in [\Delta]^d : \Pr[h_i(p) = 1] = \pi \). Let set \( S \subset [\Delta]^d \) and denote \( h_i(S) = \{ s \mid h_i(s) \neq 0, s \in S \} \). For each \( i \in [m], s \in S \), we define \( X_{i,s} \) to be \( h_i(s) \). Let \( X = \sum_{i=1}^{m} \sum_{s \in S} X_{i,s} \). Then with probability at least \( 1 - \delta \),

\[ |X - m\pi|S| \leq \epsilon m\pi|S| \]

provided \( m \geq \frac{c \log \delta(1/\delta)}{\pi|S|^2} \) for some sufficiently large constant \( c \geq 4 \).
Algorithm 5
1: procedure SampleKMeansLinear()
2:     procedure Init(k, Δ, d, ϵ, δ)
3:         O ← {1, 2, 4, 8, 16 · · · , poly(d, ∆d)}
4:         ρ ← 0.01,
5:         Construct G−1,G0, · · · ,GL  
6:         Initialize ORACLEo,l ← ORACLE.Init(l, o, Gi)
7:     end procedure
8:     procedure Update(p, op)
9:         for o ∈ O do
10:             for l = 0 → L do
11:                 ORACLEo,l.Update(p, op)
12:             end for
13:         end for
14:     end procedure
15: end procedure

Algorithm 6
1: procedure SampleKMeansLinear()
2:     procedure Query()
3:         for o ∈ O do
4:             for i = 0 → L do
5:                 Aoi,i, flago,i ← ORACLEo,i.EstNumPts()
6:                 So,i ← ∅
7:             end for
8:             γ ← ϵ/(200Ld3ρ)
9:             R ← {i | Ai,i > γ · Ti(o)}  
10: m ← c max{d3Lke−3(Ld + log(1/δ)), d3Lke−2(dL log(dLk) + log(1/δ))} for some constant c
11: for j = 1 → m do
12:     Sample a level i ∈ R with probability Aoi,i/ Ai,i′
13:     q, flag ← ORACLEo,i.Query(j)
14:     if flag ≠ Fail then
15:         wq ← (∑i′∈R Ai,i′/Ti′(o))/(m/Ti(o))  
16:         So,i ← So,i ∪ (q, wq)
17:     else
18:         So,i ← ∅
19:         break
20:     end if
21: end for
22: o∗ ← arg mino∈O{So,i ≠ ∅, flago,i ≠ Fail, ∀i ∈ {0, 1, · · · , L − 1}}
23: return So∗  
24: end procedure
25: end procedure

Proof. It is obvious that

\[ X = \sum_{i=1}^{m} \sum_{s \in S} X_{i,s} = \sum_{i=1}^{m} \sum_{s \in S} h_i(s) = \sum_{i=1}^{m} |h_i(S)|. \]

Then we have each \( X_{i,s} \) is a \((\log(1/δ))\)-wise independent random variable. Since \( m \geq \frac{c \log(1/δ)}{\pi|S|^2} \), we
have that
\[ em\pi|S| \geq c\log(1/\delta) \quad \text{and} \quad c^2m\pi|S| \geq ck. \]

Let \( k = 4\log(1/\delta) \). Thus by Lemma H.8, with probability at least
\[ 1 - 8((k\mu + k^2)/(em\pi|S|)^{k/2})^2 \geq 1 - (c/2)^{k/2} \geq 1 - \delta, \]
we have
\[ |X - m\pi|S|| \leq em\pi|S| \]
provided \( c \geq 4 \).

**Lemma H.10.** Let \( k \geq 1 \) be some fixed value. Suppose we have \( 100m \) independent pairwise hash functions \( h_1, h_2, \ldots, h_{100m} : [\Delta]^d \to \{0, 1\} \) with that \( \forall i \in [100m], p \in [\Delta]^d : |S| \cdot \Pr[h_i(p) = 1] = 1/k. \)

Let set \( S \subset [\Delta]^d \) and denote \( h_i(S) = \{s : h_i(s) \neq 0, s \in S\} \). Let \( X = \sum_{i=1}^{m} 1_{|h_i(S)| > 0} \). Then with probability at least \( 1 - \delta \),
\[ X > m/k, \]
provided \( m/k \geq c\log(1/\delta) \) for some sufficiently large constant \( c \).

**Proof.** Denote, for \( j = 1, 2, \ldots, 10m/k, \)
\[ Y_j = \sum_{i=\lfloor 10(j-1)k+1 \rfloor}^{\lfloor 10jk \rfloor} |h_i(S)|. \]

Thus \( E[Y_j] = 10 \) and \( \text{Var}[Y_j] \leq 10. \) Thus by Chebyshev’s inequality, we have
\[ \Pr[|Y_j - 10| \geq 5] \leq 10/25 = 0.4. \]
Thus \( E[X] \geq 4m/k \). Denote
\[ X_j = \sum_{i=\lfloor 10(j-1)k+1 \rfloor}^{\lfloor 10jk \rfloor} 1_{|h_i(S)| > 0}. \]

Then each \( X_j \) is an independent random variable. The lemma follows from applying Chernoff bound over \( \sum X_j \).

**Claim H.11.** For each \( i \in \mathcal{P}_i \), let \( \sum_{j=1}^{P_i} |P_{i,j}| = a_i \cdot T_i(o). \) Conditioned on \( \text{StreamSampling}^+ \) doesn’t fail, then with probability at least \( 1 - \delta \), for all \( o, i \), the number of non-empty \( \text{ORACLE}_{o,i} \cdot \hat{S}_1, \text{ORACLE}_{o,i} \cdot \hat{S}_1, \ldots, \text{ORACLE}_{o,i} \cdot \hat{S}_{100m} \) is at least \( ma_i/k \).

**Proof.** Since \( i \in \mathcal{P}_i \), and \( \gamma = \epsilon/(200Ld^3\rho) \), we know that \( a_i \geq \epsilon/(200Ld^3\rho) \) by Definition H.1. Notice that the expectation of \( |\text{ORACLE}_{o,i} \cdot \hat{S}_1| = a_i/k \). Since \( ma_i/k \geq c' \log(\Delta^d/\delta) \) for a sufficient large constant \( c' \), then by Lemma H.10 and taking union bound over all \( o, i \), we have with probability at least \( 1 - \delta \), the number of non-empty \( \text{ORACLE}_{o,i} \cdot \hat{S}_1, \text{ORACLE}_{o,i} \cdot \hat{S}_1, \ldots, \text{ORACLE}_{o,i} \cdot \hat{S}_{100m} \) is at least \( ma_i/k \).

**Claim H.12.** For each \( i \in \mathcal{P}_i \), let \( \sum_{j=1}^{P_{i,j}} |P_{i,j}| = a_i \cdot T_i(o). \) For the smallest valid \( o \in (0, \text{OPT}] \), i.e., \( o^* \), with probability at least \( 1 - \delta \), \( \sum_{i' \in \mathcal{P}} a_{i'} > 100k \).
Proof. Let us fix an \( o \in O, i \in \{0, \cdots, L-1\} \).

\[
\mathbb{E}[\text{memory size for ORACLE}_{o,i}] \\
\leq \frac{100m}{kT_i(o)} \cdot (\#\text{partition points} + \#\text{non-partition and non-heavy points}) \\
+ \frac{c_2m}{k} \cdot \#\text{heavy cells in level } i \\
= \frac{100m}{kT_i(o)} \left( a_i \cdot T_i(o) + \sum_{i' < i} a_i T_{i'}(o) \right) + \frac{c_2m}{k} \cdot \#\text{heavy cells in level } i \\
\leq \frac{100m}{k} \sum_{i=0}^{L} a_i + \frac{c_2m}{k} \cdot \#\text{heavy cells in level } i \\
\leq \frac{m}{k} \cdot (100 + c_2) \left( \sum_{i=0}^{L} a_i \right).
\]

If \( o \) is the smallest and \( \sum_{i=0}^{L} a_i \leq 100k \), then let \( o' = o/2 \). Since the probability that each point is sampled blow up by twice, the expected memory only blows up by twice. Since \( m \) is large enough, with probability at least \( 1 - \delta \), the actual memory used by \( o' \) is at most twice of the expectation of used memory size. It means \( o' \) is a smaller valid guess, thus it leads to a contradiction to \( o \) is the smallest valid guess. \( \square \)

Claim H.13. Given parameters \( \gamma \in (0, 1/10) \) and the smallest valid guess \( o \in (0, \text{OPT}) \). Let \( \overline{R} = \{ i \in \{0, 1, \cdots, L\} \mid \sum_{j=1}^{\mid P_i \mid} |P_{i,j}| > \gamma \cdot T_i(o) \} \). For each \( i \in \overline{R} \), let \( \sum_{j=1}^{\mid P_i \mid} |P_{i,j}| = a_i \cdot T_i(o) \). Let \( m \) denote the number of samples sampled from all the levels (Algorithm 6). Then with probability at least \( 1 - \delta \), the number of samples sampled from level \( i \) is at most \( ma_i/(50k) \).

Proof. Recall that \( m \) is number of samples in Algorithm 6. We can compute the expectation of number samples choosing level \( i \),

\[
\mathbb{E}[\#\text{samples choosing level } i] = m \cdot \frac{\sum_{p \in P_i} d^3 / T_i(o)}{\sum_{\tilde{p} \in \overline{R}} \sum_{p \in P_{i'}} d^3 / T_{i'}(o)} = m \cdot \frac{a_i}{\sum_{i' \in \overline{R}} a_{i'}}.
\]

Since \( m \) is sufficiently large, by using Chernoff bound and taking union bound over all \( o, i \), we have

\[
\Pr \left[ \exists o, i, \#\text{samples choosing level } i \geq m \cdot \frac{a_i}{\sum_{i' \in \overline{R}} a_{i'}} \cdot \frac{1}{2} \right] \leq \delta,
\]

Further using Claim H.12, we have

\[
m \cdot \frac{a_i}{\sum_{i' \in \overline{R}} a_{i'}} \cdot \frac{1}{2} \leq \frac{ma_i}{50k}.
\]

Thus, with probability \( 1 - \delta \), for all \( o, i \), the number of samples sampled from level \( i \) is at most \( ma_i/(50k) \). \( \square \)
Lemma H.14. At the end of one-pass of stream, for each fixed \( o \leq \text{OPT} \), for each \( i \in \{0,1,\cdots,L\} \), we know \( \mathcal{P}_i \) is important or not. If \( \mathcal{P}_i \) is \( \gamma \)-important, then we can output \( A_{o,i} \) such that

\[
(1 - \epsilon) \sum_{j=1}^{|\mathcal{P}_i|} |\mathcal{P}_{i,j}| \leq A_{o,i} \leq (1 + \epsilon) \sum_{j=1}^{|\mathcal{P}_i|} |\mathcal{P}_{i,j}|
\]

with probability \( 1 - \delta \).

Proof. Note that in the algorithm each point is copied \( 100m \) times. And in the \( i^{th} \) level, each point is sampled with probability \( 1/(T_i(o)k) \). Since each heavy cell has at least \( T_i(o) \) number of points, then according to Lemma H.9, the number of points in the heavy cell which are sampled is at least \( (1 - \epsilon)100m/k \). Thus, at the end of the stream, we are able to identify the heavy cells which means that for each given point, we are able to determine whether it is a partition point, a heavy cell point or a non-partition non-heavy cell point in the \( i^{th} \) level.

Now consider a level \( i \) which is important, which means that the number of partition points is at least \( \gamma T_i(o) \). Recall that \( \gamma = \epsilon/(200Ld^2\rho) \). Then since \( m \) is sufficiently large, we can apply Lemma H.9 again such that the number of partition points in the level \( i \) which are sampled is in the range \( (1 \pm \epsilon)100m/(kT_i(o)) \cdot \sum_{j=1}^{|\mathcal{P}_i|} |\mathcal{P}_{i,j}| \). Thus, we have \( (1 - \epsilon) \sum_{j=1}^{|\mathcal{P}_i|} |\mathcal{P}_{i,j}| \leq A_{o,i} \leq (1 + \epsilon) \sum_{j=1}^{|\mathcal{P}_i|} |\mathcal{P}_{i,j}| \).

Claim H.15. Given parameters \( \gamma \in (0,1/10) \) and a valid guess \( o \in (0,\text{OPT}] \). Let \( \mathcal{R} = \{ i \in \{0,1,\cdots,L\} \mid \sum_{j=1}^{|\mathcal{P}_i|} |\mathcal{P}_{i,j}| > \gamma \cdot T_i(o) \} \). For each \( i \in \mathcal{R} \), let \( \sum_{j=1}^{|\mathcal{P}_i|} |\mathcal{P}_{i,j}| = a_i \cdot T_i(o) \). Then with probability at least \( 1 - \delta \), \( \sum_{i \in \mathcal{R}} a_i \leq O(kL) \).

Proof. We prove by contradiction. Suppose \( \sum_{i \in \mathcal{R}} a_i \geq \omega(kL) \). According to Lemma H.9, the number of partition points stored in our data structure will be at least \( \omega(kL) \) which contradicts to the total space used by our data structure.

Now in the following, we give the whole proof of our main theorem.

proof of Theorem H.6. It is easy to see the total space used is small: we actually maintained \( |O| \times L \) number of oracles, and by Lemma E.4, each oracle uses space at most \( k\epsilon^{-3}\text{poly}(L,d,\log k, \log d, \log(1/\delta)) \) bits.

Now, let us look at the correctness. Firstly, we can argue that the algorithm will output a valid \( o^* \) with \( o^* < \text{OPT} \). Let \( o \in [\text{OPT}/2,\text{OPT}] \), we look at level \( i \), the total number of non-heavy cell points is at most \( 3kL/\rho \cdot T_i(o) + \text{OPT} / (g_t^2/(2d)^2) \) where the first term is all the non-heavy cell points which are in the center cells, and the second term is all the non-heavy cell points which are not in the center cells. Since \( \text{OPT} / (g_t^2/(2d)^2) \leq 24kT_i(o) \) then we have that the total number of non-heavy cell points in the level \( i \) is at most \( 27kLT_i(o) \). Thus, due to Lemma H.9, the total number of sampled non-heavy cell points is at most \( O(mL) \). Furthermore, due to Lemma D.5, the total number of heavy cells is at most \( O(kL) \). Thus, all the oracles will not FAIL. Thus, the algorithm will choose an \( o^* \) which is at most \( \text{OPT} \).

Since \( o^* \) is a valid guess, then according to Claim H.15, Lemma H.5 and Theorem H.2, \( m \) samples is enough to get a good coreset. In the following of the proof, our goal is to prove that our \( m \) samples is actually good.

According to Lemma H.14, we can identify all the important levels. And according to Lemma H.3, we only need to show that these \( m \) samples provide a coreset of the important points.
According to Lemma H.14, we can estimate the number of partition points in each important level up to an approximation factor within $1 \pm \epsilon$. Thus, we can sample each important level $i$ with probability proportional to

$$\frac{\sum_{p \in \mathcal{P}_i} d^3 / T_i(o)}{\sum'_{i' \in \mathcal{R}} \sum_{p' \in \mathcal{P}_{i'}} d^3 / T_{i'}(o)}$$

up to a factor $1 \pm \epsilon$. Now the only thing we remaining to prove is that we can implement uniform sampling over all the partition points for each important level. Since $o^*$ is the smallest valid guess of $o$, Claim H.13 shows that the number of uniform samples needed from level $i$ is upper bounded by $ma_i/(50k)$. Due to Claim H.11, the number of uniform samples the corresponding oracle can provided is at least $ma_i/k$. Thus, we can get enough uniform samples.

Thus, we are able to sample $m$ i.i.d. samples such that each point $p$ is chosen with probability proportional to

$$(1 \pm \epsilon) \frac{d^3 / T_i(o)}{\sum'_{i' \in \mathcal{R}} \sum_{p' \in \mathcal{P}_{i'}} d^3 / T_{i'}(o)}$$

where $p$ is the partition point in the level $i$. By applying Lemma H.4, Theorem H.2 and Lemma H.3, we complete the proof of the correctness. \qed