STATISTICAL STABILITY OF MOSTLY EXPANDING DIFFEOMORPHISMS

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Abstract. We study how physical measures vary with the underlying dynamics in the open class of $C^r$, $r > 1$, strong partially hyperbolic diffeomorphisms for which the central Lyapunov exponents of every Gibbs $u$-state is positive. If transitive, such a diffeomorphism has a unique physical measure that persists and varies continuously with the dynamics.

A main ingredient in the proof is a new Pliss-like Lemma which, under the right circumstances, yields frequency of hyperbolic times close to one. Another novelty is the introduction of a new characterization of Gibbs $cu$-states. Both of these may be of independent interest.

The non-transitive case is also treated: here the number of physical measures varies upper semi-continuously with the diffeomorphism, and physical measures vary continuously whenever possible.

1. Introduction

The present work deals with the question of continuity of physical measures in the setting of partially hyperbolic diffeomorphisms whose central direction is mostly expanding. In [6] we define mostly expanding center for three bundle partial hyperbolicity as being the property that all the Gibbs $u$-states of the diffeomorphism have positive central Lyapunov exponents. This is a stronger notion than the original one from [1] but carries the advantage of being robust. More precisely, in [6] we proved:

Theorem 1.1. Let $f : M \to M$ be a $C^r$, $r > 1$, partially hyperbolic diffeomorphism on a compact manifold. Suppose that every Gibbs $u$-state of $f$ has positive central Lyapunov exponent. Then there exists a $C^r$ neighborhood $U$ of $f$ such that every $g \in U$ has a finite number of physical measures whose basins together cover a full Lebesgue measure set in $M$. 

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Throughout this work, by *Gibbs u-states* we mean invariant probabilities absolutely continuous with respect to Lebesgue measure along the partition in strong unstable manifolds and, as usual, by *physical measure* we mean a Borel probability $\mu$ for which the basin

$$B(\mu) = \{ x \in M : \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} \to \mu \}$$

(1.1)

has positive Lebesgue measure.

We point out that for mostly expanding diffeomorphisms, the basin of a physical measure is an open set, modulo a Lebesgue null set. This means in particular that one cannot have, as one can in the analogous case of mostly contracting center [13], the phenomenon of intermingled basins of attraction. It also means that transitivity is sufficient to guarantee uniqueness of the physical measure.

Our main Theorem shows that such diffeomorphisms are statistically stable.

**Theorem A.** Let $f : M \to M$ be a $C^r$, $r > 1$, transitive partially hyperbolic diffeomorphism of type $TM = E^u \oplus E^c \oplus E^s$ such that every Gibbs u-state has positive central Lyapunov exponents. Then there is a $C^r$ neighborhood $\mathcal{U}$ of $f$ such that every $g \in \mathcal{U}$ has a unique physical measure $\mu_g$. Moreover $\mu_g$ varies continuously with $g$ in the weak* topology.

In the terminology of [10, 9], Theorem A gives condition for the stable ergodicity for dissipative mostly expanding diffeomorphisms. The reader can see [11, 17, 16] and the references therein for an extensive discussion.

We also consider the possibility of mostly expanding diffeomorphisms with more than one physical measures. In this case, we obtain results analogous to those in the mostly contracting case [5].

**Theorem B.** Let $f : M \to M$ be a $C^r$, $r > 1$, partially hyperbolic diffeomorphism of type $TM = E^u \oplus E^c \oplus E^s$ (not necessarily transitive) such that every Gibbs u-state has positive central Lyapunov exponents. Then the number of physical measures depends upper semicontinuously on $g$ and physical measures vary continuously in the weak* topology on any subset $\mathcal{C} \subset \mathcal{U}$ on which the number of physical measures is constant.

Theorem A is in fact a corollary of Theorem B, but it is by far the case of greatest interest and therefore deserves to be in the spotlight.

Some comments on terminology is pertinent. By statistical stability we usually mean a situation where all physical measures persist and vary continuously with small perturbations on the dynamics. That means that at the situation in Theorem A is statistically stable, whereas the situation in Theorem B most likely is not. (It would indeed be surprising if mostly expanding diffeomorphisms could be robustly non-transitive.) On the
other hand, it is possible to weaken the notion of statistical stability. Thus we say that a
diffeomorphism $f : M \to M$ is ($C^r$) weakly statistically stable if, given any neighborhood
$U$ of the closed convex hull of the physical measures of $f$, there exists a $C^r$ neighborhood
$U$ of $f$ such that, given any $g \in U$, every physical measure of $g$ belongs to $U$.

**Theorem C.** Let $f : M \to M$ be a $C^r$, $r > 1$, partially hyperbolic diffeomorphism of
type $TM = E^u \oplus E^c \oplus E^s$ such that every Gibbs $u$-state has positive central Lyapunov
exponents. Then $f$ is weakly statistically stable.

Statistical stability in the purely non-uniformly expanding context has been dealt with
earlier, notably in [3] and [2]. In these works, the authors obtain statistical stability by
assuming certain uniformity of the tail behavior of return maps. In the setting of partially
hyperbolic diffeomorphisms, a similar result was proved in [20]. Briefly speaking, the
author considers a sequence $f_n$ converging to $f$ in the $C^r$ topology, for some $r > 1$ and
a sequence $\mu_n$ of physical measures of each $f_n$, respectively. The sequence $\mu_n$ is assumed
to converge to a measure $\mu$, and the author proves that $\mu$ is the sum of measures $\nu + \eta$, with $\nu$ non-zero, such that $\nu$ is a combination of physical measures of $f$. To this end, it is
shown that the $\mu_n$ can be decomposed into $\nu_n + \eta_n$ with $|\nu_n|$ bounded away from zero, and
such that $\nu_n$ has a disintegration along center-unstable manifolds with uniform bounds on
the densities of its conditional measures. Therefore $\nu_n$ accumulates on a measure $\nu$ with
the same properties. By tacitly assuming uniformity of tail behaviour of return maps, the
author concludes that $\mu = \nu$.

The main novelty in our approach is that we are able to get rid of any assumptions
about tail behaviour. The price to pay is that we need our stronger version of mostly
contracting center, i.e. that every Gibbs $u$-state has positive central Lyapunov exponents.
Our strategy is essentially the same as in [20], but with the important improvement that
$|\nu_n|$ can be taken to be not only bounded away from zero, but arbitrarily close to one.
The magic occurs because the following new version of the classical Pliss Lemma:

**Lemma A (Pliss-Like Lemma).** Let $L < \gamma < \Gamma$ and suppose that $a_1, \ldots, a_N$ are numbers
such that $a_i \geq L$ for every $1 \leq i \leq N$. Let $\kappa > 0$ be a number such that
\[
\# \{i \in \{1, \ldots, N\} : a_i < \Gamma\} \leq \kappa N \tag{1.2}
\]
and write $\theta = 1 - \kappa \frac{\Gamma - L}{\Gamma - \gamma}$. Then there exist $1 < n_1 < n_2 < \ldots < n_m \leq N$, with $m \geq \theta N$, such that
\[
\sum_{j=n+1}^{n_i} a_j \geq \gamma (n_i - n) \tag{1.3}
\]
for every $1 \leq i \leq m$ and every $0 \leq n < n_i$.\]
In a recent paper [21], Yang proved that having positive central Lyapunov exponents with respect to every Gibbs $u$-state is a $C^1$ open property. It is therefore natural to ask whether the physical measures vary continuously with the dynamics in the $C^1$ topology. We do not know.

In another recent work [14] the authors prove existence and finiteness of physical measures for partially hyperbolic diffeomorphisms $f$ with dominated splitting $TM = E^u \oplus E^{cu} \oplus E^{cs}$ (with uniform expansion in $E^u$) satisfying a mixture of mostly contracting and mostly expanding behaviour. That is, every Gibbs $u$-state has positive Lyapunov exponents in the $E^{cu}$ bundle and negative Lyapunov exponents in the $E^{cs}$ bundle. We expect that all results in the present work extend to such a setting.

Here is an outline of our arguments.

(i) Compactness of the set of Gibbs $u$-states provides us with uniform bounds, in a robust fashion, on the Lyapunov exponents of these.

(ii) We use Pliss-like Lemma A which allows us to prove that an iterate of a mostly expanding diffeomorphism has hyperbolic times with frequency arbitrarily close to one. This is a considerable improvement on the positive but possibly small frequency of hyperbolic times used in most arguments with a similar flavor.

(iii) The abundance of hyperbolic times given by our Pliss-like Lemma is used to prove that, in our setting, limits of Gibbs $cu$-states are Gibbs $cu$-states. This convergence is tricky to prove rigorously. We overcome this difficulty by introducing a useful characterization of Gibbs $cu$-states which does not directly involve disintegration of the measure.

(iv) Ergodic Gibbs $cu$-states are physical measures.

(v) Finally, distinct ergodic Gibbs $cu$-states cannot get too close to each others; therefore they must either stay apart or collapse into one ergodic Gibbs $cu$-states. This gives upper semi-continuity.

2. Some background

2.1. Dominated splitting and partial hyperbolicity. Let $M$ be a closed Riemannian manifold. We denote by $\| \cdot \|$ the norm obtained from the Riemannian structure and by $m$ the normalized volume measure on $M$ induced by the Riemannian structure. We often refer to $m$ as “the Lebesgue measure on $M$”. Moreover, if $D$ is a submanifold of $M$ we denote by $\text{vol}_D$ the volume measure on $D$ induced by the Riemannian structure and by $m_D$ its normalization, i.e. $m_D = \text{vol}_D / |\text{vol}_D|$. A diffeomorphism $f : M \to M$ has a dominated splitting $F < G$ if there is a $Df$-invariant decomposition $TM = F \oplus G$ into complementary subbundles of $TM$ of constant
dimensions, and $N \geq 1$ such that
\[ \|Df^N|F_x\| \cdot \|Df^{-N}|G_{f^N(x)}\| < 1 \] (2.1)
for every $x \in M$. Any such splitting is necessarily continuous.

A diffeomorphism $f : M \to M$ is \textit{partially hyperbolic} if there exists a continuous $Df$-invariant splitting
\[ TM = E^s \oplus E^c \oplus E^u, \]
such that $E^s < (E^c \oplus E^u)$ and $(E^s \oplus E^c) < E^u$ are both dominated splittings and, moreover, there exists $N \geq 1$ such that $\|Df^N|E^s\| < 1$ and $\|Df^{-N}|E^u\| < 1$.

We denote by $\mathcal{PH}^r$, $r \geq 1$, the set of $C^r$ partially hyperbolic diffeomorphisms. The set $\mathcal{PH}^r$ is open in the $C^r$ topology.

2.2. Gibbs $u$-states, Gibbs $cu$-states, and physical measures. An $f$-invariant probability measure $\mu$ is a Gibbs $u$-state if the conditional measures of $\mu$ with respect to the partition into local strong-unstable manifolds are absolutely continuous with respect to Lebesgue measure along the corresponding local strong-unstable manifold.

We denote by $\mathcal{G}^u(f)$ the subset of $u$-measures for $f$. For future reference, we list some relevant properties of $u$-measures.

\textbf{Proposition 2.1.} [Pesin, Sinai; [15]] If $f$ is a $C^r$ partially hyperbolic diffeomorphism, with $r > 0$, then there exists a Gibbs $u$-state. More precisely, if $D$ is a $u$-dimensional disk inside a strong unstable leaf, then every accumulation point of the sequence of probability measures
\[ \mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f^k_mD \]
is a $u$-measure with densities with respect to the volume measure along the strong unstable leaves is uniformly bounded away from zero and infinity.

Clearly, convex combinations of Gibbs $u$-states are Gibbs $u$-states. Recall that if $\nu$ is any $f$-invariant measure, the limit
\[ \mu_x = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} \]
exists and is ergodic $\nu$-almost everywhere, and
\[ \int \varphi \, d\mu = \int \left( \int \varphi \, d\mu_x \right) \, d\mu \]
for every continuous function $\varphi : M \to \mathbb{R}$.5
Proposition 2.2 ([8, Remark 11.5]). Let \( f : M \to M \) be a \( C^r \) partially hyperbolic diffeomorphism, with \( r > 1 \). If \( \mu \) is a Gibbs \( u \)-state, then \( \mu_x \) is a Gibbs \( u \)-state for \( \mu \)-almost every \( x \). In other words, every Gibbs \( u \)-state \( \mu \) is a convex combination of ergodic Gibbs \( u \)-states.

Proposition 2.3 ([8, Section 11.2.3]). Let \( f : M \to M \) be a \( C^r \) partially hyperbolic diffeomorphism, with \( r > 1 \). Then the set \( \mathcal{G}^u(f) \) is a closed convex subset of the set of \( f \)-invariant measures. Moreover, given any sufficiently small \( C^2 \) neighborhood \( U \) of \( f \), the set
\[
\mathcal{G}^u(U) = \{(g, \mu) : g \in U \text{ and } \mu \in \mathcal{G}^u(g)\}
\] is closed in \( U \times \mathcal{M}(M) \).

Given a mostly expanding diffeomorphism \( f : M \to M \), denote by
\[
\lambda^c(f, \cdot) : M \to \mathbb{R}
\]
\[
x \mapsto \liminf_{n \to \infty} \frac{1}{n} \log \| (Df^n|_{E^c_x})^{-1} \|
\] the minimum central Lyapunov exponents and the integrated minimum central Lyapunov exponents, respectively. We say that \( f \) has positive central Lyapunov exponents with respect to the invariant measure \( \mu \) if \( \lambda^c(f, x) > 0 \) \( \mu \)-almost everywhere.

Proposition 2.4 ([6, Proposition 3.4]). The function
\[
\hat{\lambda}^c : \mathcal{G}^u(ME) \to \mathbb{R}
\]
is lower semicontinuous.

Following [1], we say that an invariant measure \( \mu \) is a Gibbs \( cu \)-state (or \( cu \)-measure) if the conditional measures of \( \mu \) along the corresponding local center-unstable manifolds are almost everywhere absolutely continuous with respect to Lebesgue measure on these manifolds. We denote by \( \mathcal{G}^{cu}(f) \) the subset of \( cu \)-measures for \( f \). Every Gibbs \( cu \)-state is in fact a Gibbs \( u \)-state with positive central Lyapunov exponents, although the converse is not true (see [6]).

Proposition 2.5 ([6, Lemma 4.4]). Let \( f : M \to M \) be a \( C^r \) (\( r > 1 \)) mostly expanding diffeomorphism. Then the set of physical measures coincides with the set of ergodic Gibbs \( cu \)-states.
A central result in this paper is the following Gibbs cu-states version of Proposition 2.3.

**Theorem 2.6.** Let $f$ be a $C^2$ diffeomorphism with mostly expanding central direction, and let $U$ be a $C^2$ neighborhood of $f$, small enough so that every $g \in U$ is mostly expanding. Then the set

$$G^{cu}(U) = \{(g, \mu) : g \in U \text{ and } \mu \in G^{cu}(g)\}$$

is closed in $U \times \mathcal{M}(M)$.

Sections 3 and 4 are entirely dedicated to the proof of Theorem 2.6. Notice that it is not a direct analogue of Proposition 2.3 because there is the extra hypothesis that $f$ is mostly expanding.

### 3. Uniform estimates of non-uniform hyperbolicity

The apparently paradoxical title of this section reflects much of the spirit of non-uniform hyperbolicity in the presence of dominated splittings and partial hyperbolicity. Unlike 'genuine' non-uniformly hyperbolic systems, in which the angle between stable and unstable bundles may be arbitrarily small, these often allow some form of robustness. An important manifestation of such robustness properties is that the measure of sets on which certain degrees of hyperbolicity hold may be uniformly bounded away from zero or even uniformly close to one.

#### 3.1. A Pliss-like Lemma.

The notion of hyperbolic times was introduced by Alves in [4] and has been intimately linked with the so called Pliss’ Lemma [18]. This is because Pliss’ Lemma guarantees that an orbit on which a diffeomorphism is, say, asymptotically expanding in some direction, will have hyperbolic times on a set of iterates that correspond to a positive frequency. Many of the difficulties related to hyperbolic times are that the frequency of hyperbolic times provided by the Pliss’ Lemma is only positive, but not necessarily close to one. This is in fact the main difficulty in the current work, and we overcome it by replacing the Pliss’ Lemma by a different one, which in our situation can be used to show that the frequency of hyperbolic times is indeed close to one, upon possibly replacing the diffeomorphism by one of its iterates.

**Proof of Lemma A.** Just as in Mañé’s proof of Pliss’ Lemma, we define a function $S : \{0, \ldots, N\} \to \mathbb{R}$ by taking $S(0) = 0$ and $S(n) = \sum_{j=1}^{n} a_j - n\gamma$ for $1 \leq n \leq N$. Defining $1 < n_1 < \cdots < n_m \leq N$ as the maximal sequence such that $S(n_i) \geq S(n)$ holds for every $0 \leq n < n_i$ and $i = 1, \ldots, m$, one may easily check that the $n_i$ satisfy (1.3). It remains is to show that $m \geq \theta N$. 

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We set \( F = \{ i \in \{1, \ldots, N\} : a_i < \Gamma \} \) and write \( \{1, \ldots, N\} \setminus \{ n_1, \ldots, n_l \} \) as the finite union \( \bigcup_{\alpha \in \Lambda} I_{\alpha} \) of pairwise disjoint intervals in \( \mathbb{N} \). Note that
\[
\sum_{i \in I_{\alpha}} a_i < |I_{\alpha}| \gamma
\]
for every \( \alpha \in \Lambda \), for else the maximality of the sequence \( n_i \) would be violated. We can bound \( a_i \) from below by either \( L \) or \( \Gamma \), depending on whether or not \( i \) belongs to \( F \). Therefore
\[
\sum_{i \in I_{\alpha}} a_i = \sum_{i \in I_{\alpha} \cap F} a_i + \sum_{i \in I_{\alpha} \cap F^c} a_i \geq |I_{\alpha} \cap F| L + |I_{\alpha} \cap F^c| \Gamma. \tag{3.2}
\]
Combining (3.1) and (3.2) we obtain
\[
|I_{\alpha} \cap F| L + |I_{\alpha} \cap F^c| \Gamma < |I_{\alpha}| \gamma. \tag{3.3}
\]
Using the identity \( |I_{\alpha}| = |I_{\alpha} \cap F| + |I_{\alpha} \cap F^c| \), rearranging terms, and summing over \( \alpha \), (3.3) becomes
\[
(\Gamma - L) \sum_{\alpha \in \Lambda} |I_{\alpha} \cap F| > (\Gamma - \gamma) \sum_{\alpha \in \Lambda} |I_{\alpha}|. \tag{3.4}
\]
Recall that \( \{ I_{\alpha} : \alpha \in \Lambda \} \) is the family of intervals in the complement of the sequence \( n_j \) in \( \{1, \ldots, N\} \). In particular,
\[
\sum_{\alpha \in \Lambda} |I_{\alpha}| = N - m. \tag{3.5}
\]
Moreover,
\[
\sum_{\alpha \in \Lambda} |I_{\alpha} \cap F| \leq |F| \leq \kappa N. \tag{3.6}
\]
Combining (3.4) with (3.5) and (3.6) gives
\[
(\Gamma - L)\kappa N > (\Gamma - \gamma)(N - m). \tag{3.7}
\]
Rearranging terms in (3.7) shows that \( m > N\theta \) \( \square \).

3.2. Abundance of hyperbolic times. We recall (see [1]) that \( n \) is a \( \sigma \)-hyperbolic time for \( x \in M \) if
\[
\prod_{j=n-k+1}^{n} \| Df_j^{-1}\| E_{f_{j}(x)}^\mathbb{C} \leq \sigma^k \quad \text{for all } 1 \leq k \leq n. \tag{3.8}
\]
We fix some positive \( \sigma \) with
\[
0 < \log \sigma^{-1} < \inf_{\mu \in \mathcal{G}^\infty(f)} \hat{\lambda}^c(f, \mu) \tag{3.9}
\]
and write
\[
\tau_x^\ell(f) = \{ n \in \mathbb{N} : n \text{ is a } \sigma^\ell \text{ hyperbolic time for } x \text{ under } f^\ell \}. \tag{3.10}
\]
Lemma 3.1. Given a mostly expanding diffeomorphism $f : M \to M$, a Gibbs $u$-state $\mu$ of $f$ and an arbitrary $\epsilon > 0$, there exists a neighborhood $U$ of $(f, \mu)$ in $G_u(M, E)$ and some natural number $\ell_0$ such that for every $(g, \nu) \in U$, and every $\ell \geq \ell_0$, there is a set $A \subset M$ with $\nu(A) > 1 - \epsilon$ such that

$$\liminf_{N \to \infty} \frac{|\tau^g_\ell(x) \cap \{1, \ldots, N\}|}{N} \geq 1 - \epsilon$$

(3.11)

for every $x \in A$.

Before proving Lemma 3.1 we need an auxiliary result. There is a well known characterization of weak* convergence of probability measures on a compact metric space, saying that a sequence of measures $\mu_n$ converges to $\mu$ if and only if $\liminf_{n \to \infty} \mu_n(U) \geq \mu(U)$ whenever $U$ is an open set. In other words, the function

$$\mathcal{M}(M) \ni \mu \mapsto \mu(U) \in \mathbb{R}$$

(3.12)

is lower semi-continuous whenever $U \subset M$ is open. Lemma 3.2 can be seen as a slight variation of that.

Lemma 3.2. For any $\varphi \in C^0(M, \mathbb{R})$ denote by $U_\varphi$ the (open) set on which $\varphi$ is positive. Then the map

$$C^0(M, \mathbb{R}) \times \mathcal{M}(M) \ni (\varphi, \mu) \mapsto \mu(U_\varphi) \in \mathbb{R}$$

(3.13)

is lower semi-continuous in the product topology on $C^0(M, \mathbb{R}) \times \mathcal{M}(M)$.

The proof is straightforward but included for the sake of completeness.

Proof. Fix some pair $(\varphi, \mu) \in C^0(M, \mathbb{R}) \times \mathcal{M}(M)$ and an arbitrary $\epsilon > 0$. We need to show that there is some open neighborhood $U$ of $(\varphi, \mu)$ in $C^0(M, \mathbb{R}) \times \mathcal{M}(M)$ such that $\nu(U_\varphi) > \mu(U_\varphi) - \epsilon$ for every $(\phi, \nu) \in U$.

By regularity of $\mu$ there is some compact set $C \subset U_\varphi$ such that $\mu(C) > \mu(U_\varphi) - \epsilon$. Since $\varphi$ is positive on $C$, it follows by compactness that we can find some number $\beta$ that satisfies $0 < \beta < \inf_{x \in C} \varphi(x)$. Observe that $U_{\varphi - \beta} \supset C$.

Let $U$ be the open ball of radius $\beta$ around $\varphi$ in $C^0(M, \mathbb{R})$. Thus if $\phi \in U$ and $x \in U_{\varphi - \beta}$ we have $\phi(x) > \varphi(x) - \beta > 0$. Hence

$$U_\phi \supset U_{\varphi - \beta} \supset C$$

(3.14)

for every $\phi \in U$.

Let $\rho : M \to [0, 1]$ be a continuous function satisfying

$$\rho|C = 1$$

(3.15)

$$\rho|U_{\varphi - \beta} = 0.$$ 

(3.16)
In particular \( \mu(U_\phi) \geq \mu(U_{\phi-\beta}) \geq \int \rho \, d\mu \geq \mu(C) > \mu(U_\varphi) - \epsilon \) for every \( \phi \in \mathcal{U} \). Now let \( \mathcal{U} \) be the open neighborhood of \( \mu \) in \( \mathcal{M}(M) \) defined by

\[
\mathcal{U} = \{ \nu \in \mathcal{M}(M) : \int \rho \, d\nu > \mu(U_\varphi) - \epsilon \}.
\]

Then, if \( (\phi, \nu) \in \mathcal{U} \times \mathcal{U} \) we have

\[
\nu(U_\phi) \geq \nu(U_{\phi-\beta}) \geq \int \rho \, d\nu > \mu(C) > \nu(U_\varphi) - \epsilon
\]

so the proof follows by taking \( \mathcal{U} = \mathcal{U} \times \mathcal{U} \). \( \square \)

**Proof of Lemma 3.1.** We write \( \gamma = \log \sigma^{-1} \) and fix some \( \Gamma \) with

\[
\gamma < \Gamma < \inf_{\mu \in G_u} \lambda^c(f, \mu).
\]

We also fix some \( L < \inf_{x \in M} \log \|Df^{-1}|E_{cu}|^{-1} \). Consider the family

\[
U^t_g = \{ x \in M : \frac{1}{t} \log \| (Dg^t|E_{ct}^{-1})^{-1} \|^{-1} > \Gamma \}
\]

of opens sets in \( M \). Because \( \frac{1}{t} \log \| (Df^t|E_{ct}^{-1})^{-1} \|^{-1} \) converges \( \mu \)-almost everywhere to some limit larger than \( \Gamma \) we must have

\[
\lim_{t \to \infty} \mu(U^t_g) = 1.
\]

Take \( \kappa = \epsilon \frac{\Gamma - \gamma}{1 - L} \) and choose \( t \) so that

\[
\mu(U^t_g) > 1 - \epsilon \kappa.
\]

It follows from Lemma 3.2 that the set

\[
\mathcal{U} = \{(g, \nu) : \nu(U^t_g) > 1 - \epsilon \kappa \text{ and } \inf_{x \in M} \log \| Dg^{-1}|E_{cu}^{-1} \|^{-1} > L \}
\]

is open in \( \mathcal{G}_u(ME) \).

Pick any pair \( (g, \nu) \in \mathcal{U} \). We shall prove that \( (g, \nu) \) satisfies the conclusion of Lemma 3.1. Consider the function

\[
F(x) = \lim_{n \to \infty} \frac{1}{n} \#\{ 0 \leq k \leq n - 1 : g^{\ell k}(x) \in U^t_g \}
\]

of the frequency of visits to the set \( U^t_g \). By Birkhoff’s Ergodic Theorem it is well defined \( \nu \)-almost everywhere and satisfies

\[
\int F \, d\nu = \nu(U^t_g) > 1 - \epsilon \kappa.
\]
Let $A = \{ x \in M : F(x) > 1 - \kappa \}$. Chebyshev’s inequality gives

$$\nu(M \setminus A) = \nu(\{ x : 1 - F(x) \geq \kappa \}) \leq \frac{1}{\kappa} \int 1 - F \ d\nu$$  \hspace{1cm} (3.26)\

$$< \frac{\epsilon \kappa}{\kappa} = \epsilon. \hspace{1cm} (3.27)$$

In other words, $\nu(A) > 1 - \epsilon$ and the proof will be complete once we have proved that

$$\liminf_{N \to \infty} \frac{|\tau^{\ell}(g) \cap \{1, \ldots, N\}|}{N} \geq 1 - \epsilon$$  \hspace{1cm} (3.28)

for every $x \in A$. To this end, suppose that $N_0$ is such that

$$\frac{1}{N} \#\{0 \leq k \leq N - 1 : g^{\ell k}(x) \in U^\ell_g\} > 1 - \kappa$$  \hspace{1cm} (3.29)

for every $N \geq N_0$. Let $a_i = \frac{1}{\ell} \log \| (Dg^{\ell}|E_{g^{\ell}(i-1)(x)})^{-1} \|^{-1}$. Then (3.29) implies that

$$\#\{i \in \{1, \ldots, N\} : a_i < \Gamma\} \leq \#\{i \in \{1, \ldots, N\} : a_i \leq \Gamma\} < \kappa N. \hspace{1cm} (3.30)$$

We can therefore conclude, from Lemma A, that there exist $1 < n_1 < \ldots < n_m \leq N$ such that

$$\sum_{j=n+1}^{n_i} \frac{1}{\ell} \log \| (Dg^{\ell}|E_{g^{\ell}(i-1)(x)})^{-1} \|^{-1}$$  \hspace{1cm} (3.31)

$$= \frac{1}{\ell} \log \prod_{j=n+1}^{n_i} \| Dg^{-\ell}|E_{g^{\ell}(i)(x)} \|^{-1} \geq \gamma(n_i - n). \hspace{1cm} (3.32)$$

for every $1 \leq i \leq m$ and every $0 \leq n < n_i$. Writing $k = n_i - n$ and remembering that $\gamma = \log \sigma^{-1}$, (3.32) may be more conveniently expressed by

$$\prod_{j=n_i-k+1}^{n_i} \| Dg^{-\ell}|E_{g^{\ell}(i)(x)} \| \leq \sigma^{\ell k} \hspace{1cm} (3.33)$$

for every $1 \leq i \leq m$ and every $1 \leq k \leq n_i$. That is, each $n_i$ is a $\sigma^\ell$ hyperbolic time for $x$ under $g^\ell$. \hfill \Box

### 3.3. Pesin blocks of uniform measure.

We now change our focus a bit. Instead of considering hyperbolic times of a given point $x$, we consider the set

$$\Lambda^n_\ell(f) = \{ x : \prod_{j=0}^{k-1} \| Df^{-\ell}|E_{f^{-\ell j}(x)} \| \leq \sigma^{\ell k} \forall 1 \leq k \leq n \}$$

of points which are hyperbolic time iterates of some other point. We are particularly interested in the set $\Lambda_\ell(f) = \bigcap_{n \geq 1} \Lambda^n_\ell(f)$, which we call a Pesin block of $f$. 

Remark. The Pesin blocks $\Lambda_\ell(f)$ are different from the Pesin blocks $\text{Bl}(\ell, f^{-1})$ considered by Avila and Bochi in [7]. For example, for points in $\Lambda_\ell(f)$, the Lyapunov exponent in the $E^{cu}$ bundle is bounded below by a fixed number $\log \sigma^{-1}$, whereas for points in $\text{Bl}(\ell, f^{-1})$, they are bounded below by $1/\ell$. Our notion is therefore more restrictive, and suitable to a situation where Lyapunov exponents are almost everywhere bounded away from zero with respect to a relevant set of measures (which is not the case in [7]). A main ingredient in our work is that $\Lambda_\ell(f)$ has large $\mu$-measure for large $\ell$ and $\mu \in G^u(f)$ in a way which is uniform in a neighborhood of $f$ (see Lemma 3.4). It is for this reason that we have proved the Pliss-like Lemma (Lemma A). Avila and Bochi obtain similar results for the set $\text{Bl}(\ell, f^{-1})$ using a very elegant application of the Maximal Ergodic Theorem. The current work could perhaps be made a few pages shorter by working with $\text{Bl}(\ell, f^{-1})$ rather than $\Lambda_\ell(f)$ and making use of their results. However, we think that our estimates on the size of $\Lambda_\ell(f)$ is of independent interest, as well as being more intuitive for those who are used to arguments involving Pliss’ Lemma.

**Lemma 3.3.** Given $f : M \to M$ mostly expanding, $\mu \in G^u(f)$ and $\epsilon > 0$, there exist a neighborhood $U$ of $(f, \mu)$ in $G^u(ME)$ and an integer $\ell_0$ such that $\nu(\Lambda_\ell(g)) > 1 - \epsilon$ for every $(g, \nu) \in U$ and every $\ell \geq \ell_0$.

**Proof.** Fix $(f, \mu) \in G^u(ME)$ and $\epsilon > 0$ arbitrarily. Choose some $\epsilon' > 0$ small enough that $(1 - \epsilon')^2 > 1 - \epsilon$. Lemma 3.1 guarantees the existence of an open neighborhood $U$ of $(f, \mu)$ in $G^u(ME)$ and a positive integer $\ell$ such that, given any $(g, \nu) \in U$, there is some set $A \subset M$, with $\nu(A) > 1 - \epsilon'$, such that

$$\liminf_{N \to \infty} \frac{\left| \tau_\ell^x(g) \cap \{1, \ldots, N\} \right|}{N} > 1 - \epsilon'$$

(3.34)

for every $x \in A$. We will prove that if $(g, \nu)$ belongs to $U$, then $\nu(\Lambda_\ell^n(g)) > 1 - \epsilon$. Let

$$A_n = \{x \in M : \inf_{k \geq n} \frac{\left| \tau_\ell^x(g) \cap \{1, \ldots, k\} \right|}{k} > 1 - \epsilon'\}.$$  

(3.35)

Note that $A_n$ is an increasing sequence of measurable sets and, by our choice of $U$ and $\ell$, we have that $\nu(\bigcup_{n \in \mathbb{N}} A_n) > 1 - \epsilon'$. Therefore, we can (and do) fix some $N$ such that $\nu(A_N) > 1 - \epsilon'$. Likewise, let

$$B_n = \{x \in M : \frac{\left| \tau_\ell^x(g) \cap \{1, \ldots, n\} \right|}{n} > 1 - \epsilon'\}.$$  

(3.36)

The sequence $B_N$ does not have to be increasing, but we have $B_n \supset A_n$ for every $n \in \mathbb{N}$ so that, in particular, $\nu(B_N) > 1 - \epsilon'$.

Observe that

$$\sum_{n=1}^{N} g^n \circ \chi_{\Lambda_\ell^n(g)}(x) = \left| \tau_\ell^x \cap \{1, \ldots, N\} \right|$$

(3.37)
for every \( x \in M \). Consequently

\[
\nu(\Lambda^\ell_n(g)) = \int \chi_{\Lambda^\ell_n(g)} \, d\nu \quad (3.38)
\]

\[
= \int \frac{1}{N} \sum_{n=1}^{N} g^{nm} \circ \chi_{\Lambda^\ell_n(g)} \, d\nu \quad (3.39)
\]

\[
\geq \int_{B_N} \frac{1}{N} \sum_{n=1}^{N} g^{n\ell} \circ \chi_{\Lambda^\ell_n(g)} \, d\nu \quad (3.40)
\]

\[
\geq \int_{B_N} 1 - \epsilon' \, d\nu > (1 - \epsilon')^2 > 1 - \epsilon. \quad (3.41)
\]

Recall that \( \Lambda^\ell(g) = \bigcap_n \Lambda^\ell_n(g) \), and that the \( \Lambda^\ell_n(g) \) form a nested decreasing sequence in \( n \). The proof follows readily. \( \square \)

**Lemma 3.4.** Let \( f : M \to M \) be a \( C^r \) \((r > 1)\) mostly expanding diffeomorphism. Given any \( \epsilon > 0 \), there exists \( \ell_0 \) and a \( C^r \) neighbourhood \( U \) of \( f \) such that \( \nu(\Lambda^\ell(g)) > 1 - \epsilon \) for every \( \ell \geq \ell_0 \), \( g \in U \) and \( \nu \in G^u(g) \).

**Proof.** Fix \( \epsilon > 0 \). Since \( G^u(f) \) is compact, it follows from Lemma 3.3 that there are open sets \( U_1 \times U_1, \ldots, U_m \times U_m \subset G^u(ME) \) and integers \( \ell_1, \ldots, \ell_m \) such that

- \( G^u(f) \subset U_1 \times U_1 \cup \ldots \cup U_m \times U_m \), and
- \( \nu(\Lambda^\ell(g)) > 1 - \epsilon \) whenever \( (g, \nu) \in U_i \times U_i \) for some \( i = 1, \ldots, m \).

Let \( U = U_1 \cap \ldots \cap U_m \) and \( U = U_1 \cup \ldots \cup U_m \). Then

\[
G^u(f) \subset U \times U \subset \bigcup_{i=1}^{m} U_i \times U_i. \quad (3.42)
\]

It follows from Proposition 2.3 that, upon possibly reducing \( U \), we may (and do) suppose that \( G^u(g) \subset U \times U \) for every \( g \in U \). Let \( \ell = \prod_i \ell_i \). Given any \( g \in U \) and any \( \nu \in G^u(g) \) there exists some \( i = 1, \ldots, m \) such that \( (g, \nu) \in U_i \times U_i \). Since \( \Lambda^\ell(g) \supset \Lambda^\ell_i(g) \) we have \( \nu(\Lambda^\ell(g)) > 1 - \epsilon \). \( \square \)

### 3.4. Unstable manifolds and uniform densities.

We state the relevant properties of unstable manifolds.

**Theorem 3.5.** Let \( f : M \to M \) be a \( C^r \) mostly expanding diffeomorphism, with \( r > 1 \). Then there is a \( C^r \) neighborhood \( U \) of \( f \) such that the following holds:

(i) Given any \( \ell \in \mathbb{N} \) there exists \( r_\ell > 0 \) such that for every \( g \in U \) and \( x \in \Lambda^\ell(g) \), there is a \( C^r \) embedded disk \( W^{cu}_{r_\ell}(x) \) of dimension \( \dim E^{cu} \) and of radius \( r_\ell \), centered at \( x \), such that \( T_y W^{cu}_{r_\ell} = E^{cu}_y \) for every \( y \in W^{cu}_{r_\ell} \).
(ii) If $y \in W^{cu}_{\ell}(x)$ then $d(f^{-n}(x), f^{-n}(y)) \leq \sigma^{n/2}$ for every $n \geq 0$.

(iii) The disk $W^{cu}_{\ell}(x)$ depends continuously on $x$ in the $C^1$ topology.

(iv) For every $\ell \in \mathbb{N}$ there exists $\delta = \delta_\ell$ such that if $g \in \mathcal{U}$, $x \in M$, and $y, z \in \Lambda_\ell(g) \cap B_\delta(x)$, then either $W^{cu}_{\ell}(y) \cap W^{cu}_{\ell}(z) = \emptyset$ or $W^{cu}_{\ell}(y) \cap B_{2\delta}(x) = W^{cu}_{\ell}(z) \cap B_{2\delta}(x)$.

The existence of an unstable manifold of uniform size on sets with uniform hyperbolic estimates is rather folkloric in smooth ergodic theory. It can be proved using the machinery of [12] (see [19, Proposition 6.9]), proved directly (as in [7, Theorem 4.7]), or by iterating Pesin’s unstable manifolds (as in [9]). Items (ii) and (iii) are proved in a similar setting in [1]. Item (iv) follows from uniqueness of Pesin’s local unstable manifolds and from the bounded geometry that results from the fact that the $W^{cu}_{\ell}(x)$ are tangent to $E^{cu}$.

3.5. **Lamination bundles.** The notion of Gibbs $cu$-states involves disintegration of a measure along Pesin’s unstable manifolds. This is often done somewhat carelessly. For our present purpose, it is necessary to be more precise about exactly on what sets and the disintegration is taking place. Broadly speaking, two ways to disintegrate a measure along unstable manifolds appear in the literature. One of them, used in [1] and the works influenced by it, uses a so-called foliated box. In such a box, unstable manifolds are graphs of functions from one Euclidean ball to another Euclidean ball. This approach is often practical under the presence of dominated splittings. The other approach, often used in the more general setting of non-uniform hyperbolicity, one considers the union of unstable manifolds of points in the intersection of a Pesin block with a small ball, i.e. a set of the form

$$P = \bigcup_{y \in B_\delta(x) \cap \Lambda_\ell(f)} W^{cu}_{\ell}(y)$$

(The notion of Pesin block and unstable can vary depending on the context, but we use $\Lambda_\ell(f)$ which is appropriate in our setting.) This set may be partitioned into pieces of unstable manifolds. It is not partitioned, however, by the collection $\mathcal{P} = \{W^{cu}_{\ell}(y) : y \in B_\delta(x) \cap \Lambda_\ell(f)\}$, for this family is not pairwise disjoint. (It is surprising that this is ignored in most treatments of the subject.) On the other hand, the set $P' = P \cap B_\delta(x)$ is partitioned by $\mathcal{P}' = \{P \cap B_\delta(x) : P \in \mathcal{P}\}$ if $\delta$ is sufficiently small. Unfortunately, elements of $\mathcal{P}'$ may be arbitrarily small, which is inconvenient for us. We therefore prefer to work with the set $P \cap B_{2\delta}(x)$. Let us be more explicit and set up some notation.

Let

$$Q(\ell, \delta; x) = \{W^{cu}_{\ell}(y) \cap B_{2\delta}(x) : y \in \Lambda_\ell(f) \cap B_\delta(x)\}$$

and

$$Q(\ell, \delta; x) = \bigcup_{D \in Q(\ell, \delta; x)} D.$$

The existence of an unstable manifold of uniform size on sets with uniform hyperbolic estimates is rather folkloric in smooth ergodic theory. It can be proved using the machinery of [12] (see [19, Proposition 6.9]), proved directly (as in [7, Theorem 4.7]), or by iterating Pesin’s unstable manifolds (as in [9]). Items (ii) and (iii) are proved in a similar setting in [1]. Item (iv) follows from uniqueness of Pesin’s local unstable manifolds and from the bounded geometry that results from the fact that the $W^{cu}_{\ell}(x)$ are tangent to $E^{cu}$.
For sufficiently small \( \delta \) (depending on \( \ell \)), but uniform in a neighbourhood of \( f \), if \( y, z \in \Lambda_\ell(g) \cap B_\delta(x) \), then either \( W^{cu}_{\ell}(y) \cap W^{cu}_{\ell}(z) = \emptyset \) or \( W^{cu}_{\ell}(y) \cap B_{2\delta}(x) = W^{cu}_{\ell}(z) \cap B_{2\delta}(x) \). In particular, \( Q(\ell, \delta; x) \) is a partition of \( Q(\ell, \delta; x) \) for sufficiently small \( \delta \) and every \( x \in M \). In this case we say that the set \( Q(\ell, \delta; x) \) is a **lamination bundle**.

Suppose that \( Q = Q(\ell, \delta; x) \) is a lamination bundle. Then \( Q = Q(\ell, \delta; x) \) is clearly a measurable partition in the sense of Rokhlin. We may therefore decompose the restriction of any measure \( \mu \in \mathcal{M}(M) \) to \( Q \) with respect to the partition \( Q \), i.e. to find a measurable family of probability measures \( \{\mu_D : D \in Q\} \) and a measure \( \hat{\mu} \) on \( Q \) with \( |\hat{\mu}| = \mu(Q) \) such that

\[
\int_Q \varphi \, d\mu = \int_Q \left( \int \varphi \, d\nu_D \right) \, d\hat{\nu}(D) \tag{3.46}
\]

for every continuous \( \varphi : M \to \mathbb{R} \).

We give a precise definition of Gibbs \( cu \)-states in the language of lamination bundles.

**Definition 3.6.** Let \( f : M \to M \) be a partially hyperbolic diffeomorphism and let \( \mu \) be an \( f \)-invariant Borel probability. We say that \( \mu \) is a Gibbs \( cu \)-state (or \( cu \)-measure) if, given any lamination bundle \( Q \) with associated partition \( Q \), we have \( \mu_D \ll \nu_D \) \( \hat{\nu} \)-almost everywhere, where \( \mu_D \in \mathcal{M}(M) \), \( D \in Q \) and \( \hat{\mu} \in \mathcal{M}(Q) \) are such that (3.46) holds for every continuous \( \varphi : M \to \mathbb{R} \).

If \( \mu \) is a \( cu \)-state for \( f \), it is known *a fortiori* that in every lamination bundle above, and \( \hat{\mu} \)-almost every \( D \in Q \), the density

\[
\rho_D(x) = \prod_{n=0}^{\infty} \frac{\det(Df^{-1}|E^{cu}_{f^{-n}(x)})}{\det(Df^{-1}|E^{cu}_{f^{-n}(y)})}. \tag{3.47}
\]

The limit (3.47) is bounded above and away from zero by constants that depend only on \( \ell \) in a neighborhood of \( f \). In particular, given any \( \ell \) and sufficiently small \( \delta > 0 \), there are a neighborhood \( U \) of \( f \) and \( L > 0 \), such that for every \( g \in U \), every \( \mu \in \mathcal{G}^{cu}(g) \) and every \( x \in M \), we have

\[
\int \varphi \, d\nu \leq L \int_{Q(\ell, \delta; x)} \left( \int \varphi \, dm_D \right) \, d\hat{\nu} \leq \sup_{D \in Q(\ell, \delta; x)} L|\nu| \int \varphi \, dm_D. \tag{3.48}
\]

**4. Proof of Theorem 2.6**

4.1. **A characterization of Gibbs \( cu \)-states.** As we outlined in the introduction, proving statistical stability of mostly expanding diffeomorphisms involves proving that a limit of Gibbs \( cu \)-states is a Gibbs \( cu \)-state. Although this may be intuitively clear to many readers in light of Lemma 3.4, a direct proof using lamination bundles (or foliated boxes) would be clumsy and difficult to make rigorous. We intend to give a cleaner proof by introducing a "disintegration-free" characterization of Gibbs \( cu \)-states.
Theorem 4.1. Let \( f : M \to M \) be a partially hyperbolic diffeomorphism with splitting \( TM = E^s \oplus E^c \), and let \( \mu \) be a Gibbs \( u \)-state such that \( \mu \) has positive Lyapunov exponents in \( E^c \) for \( \mu \)-almost every \( x \). Let \( \mathcal{W}_\ell(f) = \{ W^{cu}_\ell(x) : x \in \Lambda_\ell(f) \} \). Then the following are equivalent:

(i) \( \mu \) is a Gibbs \( cu \)-state.

(ii) Given any \( \epsilon > 0 \) and sufficiently large \( \ell \), there exists \( K > 0 \) such that

\[
\int \varphi \, d\mu < \epsilon + K \sup_{W \in \mathcal{W}_\ell(f)} \int \varphi \, dm_W
\]

for every continuous function \( \varphi : M \to [0, 1] \).

This characterization is rather subtle and logically intricate. It is the result of experimentation with many similar notions. It can be stated in symbolic form like this:

\[
\mu \in G^{cu}(f) \iff \forall \epsilon > 0 \ \exists \ell_0 > 0 \ \forall \ell \geq \ell_0 \ \exists K > 0 \\
\forall \varphi \in C^0(M, [0, 1]) \ \exists W \in \mathcal{W}_\ell(f) : \\
\int \varphi \, d\mu < \epsilon + K \int \varphi \, dm_W.
\]

It is an expression of quantifier rank equal to six and must be dealt with very carefully. We believe that it reflects an inherent intricacy of the notion of Gibbs \( cu \)-states (or SRB measures more generally) which is not always appreciated. It also explains why a carefully written proof of convergence of Gibbs \( cu \)-states is harder than one may think.

Proof that (i) implies (ii) in Theorem 4.1. Suppose that \( \mu \in G^{cu}(f) \) and fix some \( \epsilon > 0 \). Choose \( m_0 \) as in Lemma 3.3 and let \( m \geq m_0 \). Choose \( \delta > 0 \) small enough so that, given any \( x \in M \), \( Q(\ell, \delta; x) \) is a lamination bundle with associated partition \( Q(\ell, \delta; x) \).

Choose \( x_1, \ldots, x_m \) such that \( \{ B_\delta(x_i) : i = 1, \ldots, m \} \) is a cover of \( M \). Let \( \mathcal{Q}_i = Q(\ell, \delta; x_i) \) and \( Q_i = Q(\ell, \delta; x_i) \) for \( 1 \leq i \leq m \). Let \( L \) be such that for every \( i \in \{1, \ldots, m\} \) and \( \hat{\mu}_i \)-almost every \( D \in Q(\ell, \delta; x_i) \), the density of \( \mu_D \) with respect to \( m_D \) is bounded above by \( L \). Then

\[
\int \varphi \, d\mu_D \leq L \int \varphi \, dm_D
\]

for \( \hat{\mu}_i \)-almost every \( D \in \mathcal{Q}_i \). Let \( S \) be an upper bound for \( \{ |\text{vol}_D|^{-1} : D \in \mathcal{Q}_i, 1 \leq i \leq m \} \) and let \( T \) be an upper bound for \( \{ |\text{vol}_W| : W \in \mathcal{W}_\ell(f) \} \).
We have
\[
\int_{Q_i} \varphi \, d\mu = \int_{Q_i} \left( \int_D \varphi \, d\mu_D \right) \, d\hat{\mu}_i(D) \quad (4.3)
\]
\[
\leq \hat{\mu}_i(Q_i) L \sup_{D \in Q_i} \int \varphi \, dm_D \quad (4.4)
\]
\[
= \mu(Q_i) L \sup_{D \in Q_i} \left( \int \varphi \, d\text{vol}_D / \text{vol}_D \right) \quad (4.5)
\]
\[
\leq LS \sup_{D \in Q_i} \int \varphi \, d\text{vol}_D. \quad (4.6)
\]

Notice that \( \Lambda_{\ell}(f) \subset Q_1 \cup \ldots \cup Q_m \) and that \( \mu(\Lambda_{\ell}(f)) > 1 - \epsilon \). Hence
\[
\int \varphi \, d\mu < \epsilon + \sum_{i=1}^m \int_{Q_i} \varphi \, d\mu \quad (4.7)
\]
\[
\leq \epsilon + mLS \sup_{D \in Q_1 \cup \ldots \cup Q_m} \int \varphi \, d\text{vol}_D \quad (4.8)
\]
\[
\leq \epsilon + mLST \sup_{W \in W_{\ell}(f)} \int \varphi \, dm_W \quad (4.9)
\]

The proof follows by taking \( K = mLST \). □

The converse statement in Theorem 4.1 is harder to prove. We need an auxiliary result.

**Lemma 4.2.** Let \( \ell \in \mathbb{N}, x \in M, \) and \( \delta > 0 \) be small enough so that \( Q = Q(\ell, \delta; x) \) is a lamination bundle with associated partition \( Q = Q(\ell, \delta; x) \). Let \( \nu \) be a Borel measure on \( Q \) and let
\[
\nu = \int \nu_D \, d\hat{\nu}(D) \quad (4.10)
\]
be its disintegration with respect to \( Q \) and write \( \eta = \int \nu m_D \, d\hat{\nu}(D) \). Let \( \phi : M \to [0, 1] \) be a continuous function. Then, given any \( b > \int \phi \, d\eta \), there exists a continuous function \( \xi : M \to [0, 1] \) with \( 1 - \xi \) supported in \( B_{2\delta}(x) \) such that
\[
\nu(\{ x : \xi(x) < 1 \}) < \sqrt{b} \quad (4.11)
\]
and
\[
\int_D \phi \xi \, d\nu_D < \sqrt{b} \quad (4.12)
\]
for every \( D \in Q \).

**Proof.** Let \( K \) be the set of those \( D \in Q \) for which \( \int \phi \, dm_D \geq \sqrt{b} \) and let \( K = \bigcup_{D \in K} D \). By Chebyshev’s inequality we have
\[
\nu(K) = \hat{\nu}(K) \leq \frac{1}{\sqrt{b}} \int_K \left( \int_D \phi \, dm_D \right) \, d\hat{\nu}(D) \leq \frac{1}{\sqrt{b}} \int \phi \, d\eta < \sqrt{b}. \quad (4.13)
\]
In particular, the set \( L = Q \setminus K \) has \( \nu \)-measure larger than \( |\nu| - \sqrt{b} \). For \( 0 < \tau < 2\delta \), write \( L_\tau = L \cap \overline{B}_\tau(x) \) and \( K_\tau = K \cap \overline{B}_\tau(x) \). Fix \( \tau \) large enough that \( \nu(L_\tau) > |\nu| - \sqrt{b} \).

It follows from (iii) in Theorem 3.5 that the map \( x \mapsto \int \psi \, dm_D(x) \) is continuous on \( Q \), where \( D(x) \) is the element of \( Q \) that contains \( x \). Thus \( K_\tau \) is closed. Let \( U \) be an open neighborhood of \( K_\tau \) with \( U \subset B_{2\delta}(x) \) such that \( \nu(U \setminus Q) > |\nu| - \sqrt{b} \). Let \( \xi : M \to [0, 1] \) be a continuous function such that \( \xi = 0 \) on \( K_\tau \) and \( \xi = 1 \) on the compliment of \( U \). Then \( \nu(\{ x : \xi(x) < 1 \}) \leq \nu(U) < \sqrt{b} \).

We claim that \( \int \phi \, dm_D < \sqrt{b} \) for every \( D \in Q \). Indeed, if \( D \notin K \), then \( \int \phi \, dm_D < \sqrt{b} \) by the very definition of \( K \). If on the other hand \( D \in B \), then \( \xi \) vanishes on \( W \), so \( \int \phi \xi \, dm_D = 0 \).

\[ \square \]

**Proof that (ii) implies (i)** in Theorem 4.1. Suppose that \( \mu \) is not a Gibbs \( cu \)-state. Then there is some measurable set \( X \subset M \) with \( \kappa = \mu(X) > 0 \) for which the following happens: For every \( Q(\ell, \delta; x) \) with \( \delta \) sufficiently small, if \( \hat{\mu} \) is the factor measure of \( \mu|Q(\ell, \delta; x) \) we have \( m_D(X) = 0 \) for \( \hat{\mu} \)-almost every \( D \in Q(\ell, \delta; x) \). Let \( \epsilon = \kappa/2 \). We must prove that, given any \( \ell_0 \in \mathbb{N} \), there exists \( \ell \geq \ell_0 \) with the property that for every \( K > 0 \) it is possible to find a continuous function \( \phi : M \to [0, 1] \) such that

\[ \int \phi \, d\mu \geq \epsilon + K \sup_{W \in \mathcal{W}_i(f)} \int \phi \, dm_W. \]  \hfill (4.14)

To this end, fix \( \ell_0 > 0 \) arbitrarily. Thereafter choose \( \ell \geq \ell_0 \) large enough so that \( \mu(\Lambda_\ell(f)) > 1 - \epsilon \). Let \( \sigma < \sigma' < 1 \) and define \( \Lambda_\ell'(f) \) just like \( \Lambda_\ell(f) \) (see Section 3.4) but with \( \sigma \) replaced by \( \sigma' \). Similarly, define \( Q'(\ell, \delta; x) \) and \( \mathcal{Q}'(\ell, \delta; x) \) like \( Q(\ell, \delta; x) \) and \( \mathcal{Q}(\ell, \delta; x) \) but with \( \Lambda_\ell(f) \) replaced by \( \Lambda_\ell'(f) \). It follows from item (ii) of Theorem 3.5 that if \( \ell_1 \) is a sufficiently large multiple of \( \ell \), and if \( W \in \mathcal{W}_i(f) \) then \( W \subset \Lambda_\ell'(f) \). Choose \( \delta > 0 \) such that for every \( x \in M \), \( Q'(\ell_1, \delta; x) \) is a lamination bundle with associated partition \( \mathcal{Q}'(\ell_1, \delta; x) \). Choose \( x_1, \ldots, x_m \) such that \( M = \bigcup_{i=1}^m B_\delta(x_i) \). For ease of notation, write \( Q'_i = Q'(\ell_1, \delta; x_i) \) and \( \mathcal{Q}'_i = \mathcal{Q}'(\ell_1, \delta; x_i) \). Moreover, for \( 1 \leq i \leq m \) let \( \hat{\mu}_i \) be the factor measure of \( \mu|Q'_i \) with respect to the partition \( \mathcal{Q}'_i \) and denote by \( \eta_i \) the measure \( \int_{Q'_i} m_D \, d\hat{\mu}_i(D) \). Let \( \eta = \sum_i \eta_i \). Let \( S \) be an upper bound for \( \{|\text{vol}_D|^{-1} : D \in \mathcal{Q}'_i, 1 \leq i \leq m\} \) and let \( T \) be an upper bound for \( \{|\text{vol}_W| : W \in \mathcal{W}_i(f)\} \).

Fix \( K > 0 \) arbitrarily and choose \( 0 < \alpha < \epsilon/2 \) such that \( K\alpha < \epsilon - 2\alpha \). Note that \( \mu|X \) and \( \eta \) are mutually singular measures. Therefore we can find a continuous function \( \phi : M \to [0, 1] \) such that

\[ \int \phi \, d\mu > \kappa - \alpha \]  \hfill (4.15)
and
\[ \int \phi \, d\eta < \left( \frac{\alpha}{mST} \right)^2. \] (4.16)

Taking \( b = \frac{\alpha^2}{(mST)^2} \) in Lemma 4.2, we can find a continuous functions \( \xi_i : M \to [0, 1] \) such that \( \mu(\{x : \xi_i(x) < 1\}) < \frac{\alpha}{mST} \) and
\[ \int \phi \xi_i \, dm_D < \frac{\alpha}{mST} \] (4.17)
for every \( D \in Q'_i \). Let \( \varphi = \phi \cdot \xi_1 \cdots \xi_m \). Then
\[ \int \varphi \, d\mu > \kappa - \alpha - m \frac{\alpha}{mST} \geq \kappa - 2\alpha, \]
and
\[ \int \varphi \, dm_D < \frac{\alpha}{mST} \] (4.19)
for every \( 1 \leq i \leq m \) and every \( D \in Q'_i \).

Let \( W \in W_\ell(f) \). Recall that \( W \subset \Lambda_{\ell_1}(f) \) (by our choice of \( \ell_1 \)). Therefore, if \( W \cap B_\delta(x_i) \neq \emptyset \) then \( W \cap B_\delta(x_i) \subset D_i \) for some \( D_i \subset Q'_i \). We have
\[ \int \varphi \, dm_W = \frac{1}{|\text{vol}_W|} \int \varphi \, d\text{vol}_W \]
\[ \leq S \sum_i \int \varphi \, d\text{vol}_{D_i} \]
\[ \leq ST \sum_i \int \varphi \, dm_{D_i} \leq \alpha. \]
Hence
\[ K \int \varphi \, dm_W + \epsilon \leq K\alpha + \epsilon < \kappa - 2\alpha < \int \varphi \, d\mu, \]
which finishes the proof. \( \square \)

4.2. Concluding the proof of Theorem 2.6. Everything done so far in section 4 (briefly speaking, our Pliss-like Lemma and our characterization of Gibbs \( cu \)-states) have been for the purpose of proving Theorem 2.6. The proof is based on the observation that the quantities \( \ell \) and \( K \) in Theorem 4.1 are uniform in a neighborhood of \( f \).

Let \( f \) be a \( C^r \) mostly expanding diffeomorphism \( (r > 1) \). Consider a sequence \((f_n, \mu_n) \in \mathcal{G}^{cu}(ME)\) such that \( f_n \) converges in \( C^r \) to some mostly expanding diffeomorphism \( f \) and \( \mu_n \) converges weakly* to some measure \( \mu \). To prove Theorem 2.6 we must establish that \( \mu \) is a \( cu \)-measure. We do that by showing that the inequality (4.1) passes to the limit. The following straightforward Lemma is useful.
Lemma 4.3. Let \( f : M \to M \) be a \( C^r \) (\( r > 1 \)) mostly expanding diffeomorphism, \( \ell \in \mathbb{N} \), and suppose that \( f_n \) is a sequence of \( C^r \) mostly expanding diffeomorphisms converging to \( f \). Then any accumulation point of \( \Lambda_\ell(f_n) \) belongs to \( \Lambda_\ell(f) \). That is,

\[
\bigcap_n \bigcup_{k \geq n} \Lambda_\ell(f_n) \subset \Lambda_\ell(f). \tag{4.24}
\]

Proof. The proposition amounts to saying that the set

\[
\mathcal{C} = \{(f, x) : f \in ME, x \in \Lambda_\ell(f)\} \tag{4.25}
\]

is a closed subset of \( ME \times M \) in the product topology, where \( ME \) is the set of mostly expanding \( C^r \) diffeomorphisms (endowed with the \( C^r \) topology). But \( \mathcal{C} = \bigcap_{n \geq 0} \mathcal{C}_n \) where \( \mathcal{C}_n = \{(f, x) : x \in \Lambda_\ell(f_n)\} \), and \( \mathcal{C}_n \) are clearly closed sets in \( ME \times M \).

\[ \square \]

Fix \( \epsilon > 0 \) arbitrarily and let \( \ell_0 \) be as in Lemma 3.4. Fix \( \ell \geq \ell_0 \) arbitrarily. Since \( f_n \) converges to \( f \), we have \( \mu_n(\Lambda_\ell(f_n)) > 1 - \epsilon \) for sufficiently large \( n \), say \( n \geq n_0 \). According to Theorem 4.1, it suffices to prove that there exists \( K > 0 \) such that, given any continuous function \( \varphi : M \to [0, 1] \), we have

\[
\int \varphi \, d\mu_n < \epsilon + K \sup_{W \in W_\ell(f)} \int \varphi \, d\text{vol}_W. \tag{4.26}
\]

Consider the sets

\[
Q^n(\ell, \delta; x) = \{W^\text{cu}_\ell(y) \cap B_{2\delta}(x) : y \in \Lambda_\ell(f_n) \cap B_\delta(x)\} \tag{4.27}
\]

and

\[
Q^n(\ell, \delta; x) = \bigcup_{D \in Q^n(\ell, \delta; x)} D. \tag{4.28}
\]

Choose \( \delta > 0 \) sufficiently small so that \( Q^n(\ell, \delta; x) \) are lamination bundles for every \( x \in M \). Choose points \( x_1, \ldots, x_m \) so that \( M \subset B_\delta(x_1) \cup \ldots \cup B_\delta(x_m) \). Write \( Q^n_i = Q^n(\ell, \delta; x_i) \) and \( Q^n = Q^n(\ell, \delta; x_i) \) for \( 1 \leq i \leq m \) and let

\[
\mu_n|Q^n_i = \int_{Q^n_i} \left( \int \varphi \, d\mu_D \right) \, d\tilde{\mu}_n^i(D) \tag{4.29}
\]

be the Rokhlin disintegration of the restriction of \( \mu_n \). Let \( L > 1 \) be such that, for every \( n \geq n_0 \), every \( i \in \{1, \ldots, m\} \) and \( \tilde{\mu}_n^i \)-almost every \( D \in Q^n_i \), the density of \( \mu_D \) with respect to \( m_D \) is bounded above by \( L \). Let \( K = mL \). Now consider any continuous function \( \varphi : M \to [0, 1] \). For every \( n > n_0 \) and every \( i \in \{1, \ldots, m\} \) we have

\[
\int_{Q^n_i} \varphi \, d\mu_n = \int_{Q^n_i} \left( \int \varphi \, d\mu_D \right) \, d\tilde{\mu}_n^i(D) \leq L \mu(Q^n_i) \sup_{D \in Q^n_i} \int \varphi \, d\mu_D \tag{4.30}
\]

Consider the sets

\[
Q^n(\ell, \delta; x) = \{W^\text{cu}_\ell(y) \cap B_{2\delta}(x) : y \in \Lambda_\ell(f_n) \cap B_\delta(x)\} \tag{4.27}
\]

and

\[
Q^n(\ell, \delta; x) = \bigcup_{D \in Q^n(\ell, \delta; x)} D. \tag{4.28}
\]

Choose \( \delta > 0 \) sufficiently small so that \( Q^n(\ell, \delta; x) \) are lamination bundles for every \( x \in M \). Choose points \( x_1, \ldots, x_m \) so that \( M \subset B_\delta(x_1) \cup \ldots \cup B_\delta(x_m) \). Write \( Q^n_i = Q^n(\ell, \delta; x_i) \) and \( Q^n = Q^n(\ell, \delta; x_i) \) for \( 1 \leq i \leq m \) and let

\[
\mu_n|Q^n_i = \int_{Q^n_i} \left( \int \varphi \, d\mu_D \right) \, d\tilde{\mu}_n^i(D) \tag{4.29}
\]

be the Rokhlin disintegration of the restriction of \( \mu_n \). Let \( L > 1 \) be such that, for every \( n \geq n_0 \), every \( i \in \{1, \ldots, m\} \) and \( \tilde{\mu}_n^i \)-almost every \( D \in Q^n_i \), the density of \( \mu_D \) with respect to \( m_D \) is bounded above by \( L \). Let \( K = mL \). Now consider any continuous function \( \varphi : M \to [0, 1] \). For every \( n > n_0 \) and every \( i \in \{1, \ldots, m\} \) we have

\[
\int_{Q^n_i} \varphi \, d\mu_n = \int_{Q^n_i} \left( \int \varphi \, d\mu_D \right) \, d\tilde{\mu}_n^i(D) \leq L \mu(Q^n_i) \sup_{D \in Q^n_i} \int \varphi \, d\mu_D \tag{4.30}
\]
Notice that $\mu_n(Q^n_i) > \mu_n(A_\ell(f_n) \cap B_\delta(x_i))$ for every $1 \leq i \leq m$ and every $n \geq n_0$. Since $M = B_\delta(x_1) \cup \ldots \cup B_\delta(x_m)$, this gives us the estimate $\mu_n(Q^n_1 \cup \ldots \cup Q^n_m) > 1 - \epsilon$. It follows that
\[
\int \varphi \, d\mu_n < \epsilon + \sum_{i=1}^{m} L \mu(Q^n_i) \sup_{D \in Q^n_i} \int_D \varphi \, dm_D \leq Lm \max_{1 \leq i \leq m} \sup_{D \in Q^n_i} \int \varphi \, dm_D.
\] (4.31)
For every $n > n_0$, choose a disk $D_n \in Q^n_1 \cup \ldots \cup Q^n_m$ such that
\[
\int \varphi \, dm_{D_n} = \max_{1 \leq i \leq m} \sup_{D \in Q^n_i} \int \varphi \, dm_D.
\] (4.32)
Each disk $D_n$ is contained in some $W_n \in W_\ell(f_n)$. By Lemma 4.3, $W_n$ accumulates on some disk $W \in W_\ell(f)$. Therefore
\[
\int \varphi \, d\mu \leq \epsilon + mL \int \varphi \, dm_W.
\] (4.33)
In particular,
\[
\int \varphi \, d\mu < \epsilon + K \cdot \sup_{W \in W_\ell(f)} \int \varphi \, dm_W,
\] (4.34)
where $K = mL + 1$ is independent of $\varphi$.

5. PROOF OF MAIN THEOREM S

We are now in a position to prove Theorem s A, B, and C.

5.1. **Proof of Theorem C.** Let $f$ be mostly expanding and denote its physical measures by $\nu_1, \ldots, \nu_k$. Let $f_n$ be a sequence of diffeomorphisms converging to $f$ in the $C^r$ topology for some $r > 1$. For large enough $n$, each $f_n$ is mostly expanding so we may as well suppose that each $f_n$ is mostly expanding. Let $\mu_n$ be a sequence of physical measures for $f_n$ respectively. Upon possibly taking a subsequence, we may suppose that the sequence $\mu_n$ converges to some measure $\mu$. Theorem 2.6 tells us that $\mu$ is a $cu$-measure. We know that $cu$-measures are convex combinations of ergodic $cu$-measures and that ergodic $cu$-measures are physical measures. It follows that $\mu$ is a convex combination of physical measures of $f$. This completes the proof of Theorem C.

5.2. **Proof of Theorem B.** The proof of upper semi-continuity of the number of physical measures is by contradiction. Suppose, therefore, that upper semi-continuity on the number of physical measures does not hold. That means that there exists some mostly expanding diffeomorphism $f$, and a sequence $f_n \to f$ of mostly expanding diffeomorphisms converging to $f$ in the $C^r$ topology, all of which have a number of physical measure larger than that of $f$. In other words, if the physical measures of $f$ are $\mu_j$, $j \in J$ for some finite set $J$, there are measures $\nu^n_i$, $i \in I$ for some finite set $I$ with $|I| = |J| + 1$ such that

(i) $\nu^n_i$ is a physical measure for $f_n$ for every $n$,
(ii) \( \nu^n_i \neq \nu^n_i' \) for every \( n \) and every \( i, i' \in I \), with \( i \neq i' \).

Upon taking possibly taking an appropriate subsequence, we may also assume that

(iii) for each \( i \in I \), there exist non-negative numbers \( \alpha_{i,j}, j \in J \) with \( \sum_{j \in J} \alpha_{i,j} = 1 \), such that \( \nu^n_i \to \sum_{j \in J} \alpha_{i,j} \mu_j \).

To get a contradiction, we shall prove that each column in the matrix \( (a_{i,j}) \) can have at most one positive element. Since the number of rows is larger than the number of columns, this implies that \( (a_{i,j}) \) must have a row of zeros, contradicting \( \sum_{j \in J} a_{i,j} = 1 \).

To see why each column of \( a_{i,j} \) can have at most one positive element, let \( P = \{(i, j) \in I \times J : \alpha_{i,j} > 0\} \) and \( \alpha = \min\{\alpha_{i,j} : (i, j) \in P\} \). Recall that each point in the Pesin block \( \Lambda_{\ell}(f_n) \) has an unstable manifold of a fixed size \( r_{\ell} \). Moreover, from Lemma 3.3 we know that it is possible to choose \( \ell \) such that \( \nu^n_i(\Lambda_{\ell}(f_n)) > 1 - \alpha/2 \) for every large \( n \).

The angle between \( E^s \) and \( E^{cu} \) is bounded away from zero in a robust manner. Hence there is some \( \rho > 0 \) such that for every large \( n \), and any point \( x \in \Lambda_{\ell}(f_n) \), the set \( \Gamma(f_n, \ell, x) = \bigcup_{y \in W^{cu}_{r_{\ell}}(x)} W^s(f_n, y) \) contains the ball \( B_{\rho}(x) \).

We cover the supports of \( \mu_j \) by balls \( B_{\rho/2}(x_{j,k}), k \in K \), where \( K \) is some finite set. For sufficiently large \( n \) we have

(i) \( \nu^n_i(B_{\rho/2}(x_{j,k})) > 0 \) for every \( (i, j, k) \in P \times K \), and
(ii) \( \nu^n_i(\Lambda_{\ell}(f_n) \cap B_{\rho/2}(x_{j,k})) > 0 \) for every \( (i, j) \in P \) and some \( k \in K \).

Thus given any \( i \in I \) choose \( j \in J \) such that \( \alpha_{i,j} > 0 \) and \( k \in K \) such that (ii) holds.

Since \( \nu^n_i \) is an ergodic Gibbs \( cu \)-state, there is some \( x \in B_{\rho/2}(x_{j,k}) \) such that \( B(\mu^n_i) \) has full leaf volume in \( W^{cu}_{r_{\ell}}(x) \). Therefore, by absolute continuity of the stable foliation, \( B(\nu^n_i) \) has full volume in \( \Gamma(f_n, \ell, x) \). By our choice of \( \rho \), we have \( \Gamma(f_n, \ell, x) \supset B_\rho(x) \supset B_{\rho/2}(x_{j,k}) \) so, in particular, \( B(\nu^n_i) \) has full volume in \( B_{\rho/2}(x_{i,k}) \).

Now take any \( i' \in I \) different from \( i \). We claim that \( (i', j) \notin P \). Indeed, if it were not so, then by (i) we would have \( \nu^n_{i'}(B_{\rho/2}(x_{j,k})) > 0 \) for sufficiently large \( n \). Therefore, there would be some \( \ell' \) and some \( x' \in B_{\rho/2}(x_{j,k}) \) such that \( B(\nu^n_{i'}) \) has full leaf volume in \( W^{cu}_{r_{\ell'}} \). Again, by absolute continuity of the stable foliation, that would imply that \( B(\nu^n_{i'}) \) has positive volume in \( B_{\rho/2}(x_{j,k}) \). But that is absurd, since \( B(\nu^n_{i'}) \) has full volume in \( B_{\rho/2}(x_{j,k}) \) and basins of distinct physical measures are disjoint. Thus we have proved that each column in the matrix \( (a_{i,j}) \) has at most one non-zero entry and the proof of upper semi-continuity of the number of physical measures is complete.

It remains to prove statistical stability in its most general setting. To this end, suppose that \( f_n \) is a sequence of mostly expanding diffeomorphisms converging to a mostly expanding diffeomorphism \( f \) and that each \( f_n \) and \( f \) all have the same number of physical measures. We use the notation above, so that the physical measures of \( f \) are \( \mu_j, j \in J \)
and those of $f_n$ are $\nu^n_i, i \in I$. The difference now is that $|I| = |J|$. By taking subsequences we may assume that $\nu^n_i \to \sum\limits_{j \in J} \alpha_{i,j} \mu_j$ for some non-negative numbers $\alpha_{i,j}$ with $\sum\limits_{j \in J} \alpha_{i,j} = 1$. It was proved above that in this case, each column of the $|I| \times |J|$ matrix $A = (a_{i,j})_{(i,j) \in I \times J}$ has at most one positive element. Now, $A$ is a square matrix and the sum of the entries in each row is 1. In particular each row has at least one positive entry. Therefore $A$ must be a permutation matrix, i.e. one for which each column and each row has exactly one entry equal to 1 and all other entries are zero. Define the map $\tau : I \to J$ so that $\tau(i)$ is the unique element of $A$ such that $a_{i,j} = 1$. Then $\nu^n_i$ converges to $\mu_{\tau(i)}$ for every $i \in I$. That completes the proof of statistical stability.

5.3. **Proof of Theorem A.** Let $f : M \to M$ be as in Theorem A. Then Theorem 1.1 says that $f$ has a finite number of physical measures, whose basin of attraction cover Lebesgue almost every point in $M$. Now, since $f$ is mostly expanding, the basin of each physical measure is open, up to a zero Lebesgue measure set (see e.g. [6, Lemma 4.5]). Thus it follows from the assumption of transitivity that $f$ has exactly one physical measure. Now, according to Theorem B, the number of physical measures varies upper semi-continuously on $f$. Hence there is a $C^r$ neighborhood $U$ of $f$ such that every $g \in U$ has a unique physical measure $\mu_g$ whose basin has full Lebesgue measure in $M$. Moreover, Theorem B implies that the map $\text{Diff}^r(M) \ni g \mapsto \mu_g \in \mathcal{M}(M)$ is indeed continuous.

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