Unitary and Renormalizable Theory of Higher Derivative Gravity

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Abstract. In 3+1 space-time dimensions, fourth order derivative gravity is perturbatively renormalizable. Here it is shown that it describes a unitary theory of gravitons (with/without an additional scalar) in a limited coupling parameter space which includes standard cosmology. The running of gravitational constant which includes contribution of graviton is computed. It is shown that generically Newton’s constant vanishes at short distance in this perturbatively renormalizable and unitary theory.

1. Introduction and Theory

Perturbative Quantum Field Theory (QFT) has been very successful in describing three of the four known forces of nature. It has been used to construct the Standard Model of particle physics, which has been tested to a great accuracy in accelerator experiments. However when the same methods of QFT are applied to the Einstein-Hilbert (EH) theory of gravity, well known problem emerge, namely the theory is plagued with UV divergences. At each order of perturbation theory one has to add new counter terms to cancel the existing divergences, as a result the theory loses predictability. This is due to the fact that in 3+1 dimensions the Newton’s constant has the mass dimension (\text{Mass})^{-2}. Classically EH theory is very successful in describing Cosmology [1, 2]. It was shown that when one studies quantum matter fields on curved background [3], four kind of divergences appear: $\sqrt{g}$, $\sqrt{g}R$, $\sqrt{g}R^2$, $\sqrt{g}R_{\mu\nu}R^{\mu\nu}$. This was a first hint that perhaps if when one augments the EH theory with 4-th order metric derivative terms, one might witness UV renormalizability. Indeed it was shown in [4] that the 4-th order metric derivative gravity is UV renormalizable using $4-\epsilon$ dimensional regularization scheme [7]. The most general action that includes all terms up to 4-th order metric derivative terms is given by,

$$A = \int \frac{d^4x\sqrt{-g}}{16\pi G} \left[ 2\Lambda - R + \frac{\omega R^2}{6M^2} - \frac{R_{\mu\nu}R^{\mu\nu} - \frac{4}{3} R^2}{M^2} \right],$$

where $R$ is the Ricci scalar and $R_{\mu\nu}$ is the Ricci tensor. The last term is proportional to the square of Weyl tensor in 3+1 dimensions up to an Euler characteristic. Here $\omega$ is dimensionless and $M$ has dimensions of mass.

Doing perturbations around the flat background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ($\eta_{\mu\nu} = \{1, -1, -1, -1\}$) one obtains the propagator of the theory [5, 6]. Here we will consider the action $A$ for $\Lambda = 0$ in the Landau gauge $\partial^\mu h_{\mu\nu} = 0$, where the Feynman propagator of theory in momentum space is given...
by,
\[ D_{\mu\nu,\alpha\beta} = \frac{i 16\pi G}{(2\pi)^4} \left( \frac{(2P_2 - P_s)_{\mu\nu,\alpha\beta}}{q^2 + i\epsilon} + \frac{(P_s)_{\mu\nu,\alpha\beta}}{q^2 - M^2/\omega + i\epsilon} - \frac{2(P_2)_{\mu\nu,\alpha\beta}}{q^2 - M^2 + i\epsilon} \right), \tag{2} \]
where \( q \) is the momentum of fluctuating field \( h_{\mu\nu} \). Various spin projectors are \( (P_2)_{\mu\nu,\alpha\beta} = \frac{1}{2} [T_{\mu\alpha}T_{\nu\beta} + T_{\mu\beta}T_{\nu\alpha}] - \frac{1}{3} T_{\mu\nu}T_{\alpha\beta}, \) \( (P_s)_{\mu\nu,\alpha\beta} = \frac{1}{3} T_{\mu\nu}T_{\alpha\beta}, \) where \( T_{\mu\nu} = \eta_{\mu\nu} - q_{\mu}q_{\nu}/q^2 \). The first term of Eq. (2) is the massless spin-2 graviton with two degrees of freedom, the second term is the scalar (Riccion) of mass \( M/\sqrt{\omega} \) and the last term is that of massive spin-2 with mass \( M \) (\( M \)-mode). The last term arises due to the presence of \( R_{\mu\nu}R^{\mu\nu} \) term in the action. If we only had \( F(R) \) type of theory, this term will be absent. From the propagator Eq. (2) we note that the residues at the pole for graviton and Riccion are positive, while for the \( M \)-mode it is negative. The \( M \)-mode renders the theory non-unitary.

We now study the issue of unitarity of higher derivative gravity action given in Eq. (1). The full \( S \)-matrix of theory involve gravitons, Riccions and \( M \)-mode as the external legs. Let's consider a subpart of this \( S \)-matrix which involve only Riccion and graviton as external legs, which means that we are considering scattering process which involve only Riccion and graviton but no \( M \)-mode as external leg. In this subpart of \( S \)-matrix we ask whether this under some conditions remains unitary? In any such scattering process, \( M \)-mode can appear as an intermediate state, which is not present in our subpart of \( S \)-matrix. For intermediate energies \( \mu \) less than \( M \) it cannot occur anyway. In renormalized quantum field theory these mass parameter are also energy dependent. Hence if \( M(\mu) \) is always greater than \( \mu \) then it cannot occur at any intermediate energy, consequently the subpart of \( S \)-matrix can also satisfy unitarity. In this scenario the physical amplitudes do get contributions from the \( M \)-mode but not from the imaginary part of \( M \)-mode i.e. Cutkosky cut corresponding \( M \)-mode vanish identically. Thus in the renormalized theory if \( M^2(\mu)/\mu^2 > 1 \) is satisfied, then we can have unitary theory of only gravitons and Riccions [9]. In the following we will see how this can be realized.

### 2. Renormalization Group Analysis

We choose to parameterize our gravity action as in Eq. (1), so that in the path-integral \( G \) effectively plays the same role as \( h \) (in the absence of matter fields) i.e. the loop expansion is same as perturbation theory in small \( G \). To one-loop beta function of couplings have been computed using the Schwinger-Dewitt technique [8]. In [10, 11] this was done using dimensional regularization in \( 4 - \epsilon \) space-time dimensions. In the Landau gauge the beta function are the following [12]:

\[
\frac{d}{dt} \left( \frac{1}{M^2 G} \right) = -\frac{133}{40\pi}, \tag{3}
\]

\[
\frac{d}{dt} \left( \frac{\omega}{M^2 G} \right) = -\frac{\pi}{3\omega} \left( \omega^2 + 3\omega + \frac{1}{2} \right), \tag{4}
\]

\[
\frac{d}{dt} \left( \frac{1}{G} \right) = \frac{5M^2}{3\pi} \left( \omega - \frac{7}{40\omega} \right), \tag{5}
\]

\[
\frac{d}{dt} \left( \frac{2\Lambda}{G} \right) = \frac{M^4}{2\pi} \left( 5 + \frac{1}{\omega^2} \right) - \frac{4M^2\Lambda}{3\pi} \left( 14 + \frac{1}{\omega} \right), \tag{6}
\]

where \( t = \ln(\mu/\mu_0) \) and all \( \text{rhs} \) contains the leading contribution in \( G \) (\( M^2G \) is also taken to be small), with higher powers coming from higher loops being neglected. Using Eqs. (3 and 4) we solve for the running of \( \omega \),

\[
\frac{d\omega}{dt} = \frac{5M^2 G}{3\pi} \left( \omega^2 + \frac{549}{50} \omega + \frac{1}{2} \right) = \frac{5M^2 G}{3\pi} (\omega + \omega_1)(\omega + \omega_2), \tag{7}
\]

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where $\omega_1 = 0.0457$ and $\omega_2 = 10.9343$. On studying Eq. (7) we find that the RG flow of $\omega$ has two fixed point: $-\omega_1$ and $-\omega_2$, with former being repulsive and later attractive under the UV evolution or increasing $t$. We realize that both these fixed points lie in the unphysical domain. The last equality of Eq. (7) tells that $rhs$ is always positive for $\omega > 0$. This means that $\omega$ is a monotonic increasing function of $t$ and vice-versa. Eq. (3) is easily integrated to express the flow of $M^2 G$ in terms of $t$. This is then plugged in Eq. (7) to obtain,

$$t = T \left[ 1 - \left( \frac{\omega + \omega_2}{\omega + \omega_1} \frac{\omega_0 + \omega_1}{\omega_0 + \omega_2} \right)^{\alpha} \right],$$

(8)

where $T = 10\pi/(133 M_0^2 G_0)$ and $\alpha = 399/50(\omega_2 - \omega_1)$, with subscript 0 meaning that the coupling parameters are evaluated at $t = 0$ or $\mu = \mu_0$. As $\omega$ takes value between zero and infinity, this translates using Eq. (8) into a range for $t$. We note that $t$ takes a minimum value for $\omega = 0$ while a maximum value for $\omega = \infty$.

$$t_{\text{min}} / T = 1 - \left( \frac{\omega_0 \omega_2 + \omega_1}{\omega_0 \omega_1 + \omega_2} \right)^{\alpha} \leq \frac{t}{T} < 1 - \left( \frac{\omega_0 + \omega_1}{\omega_0 + \omega_2} \right)^{\alpha} = t_{\text{max}} / T. \quad (9)$$

Using the evolution equation for $\omega$, one can transform any evolution of coupling in $t$ space to $\omega$ space. This allows analytical expressions for the flow of other couplings. Using Eqs. (5 and 7) we get,

$$\frac{d \ln G}{d \omega} = -\frac{\omega - \frac{7}{40 \omega}}{(\omega + \omega_1)(\omega + \omega_2)}, \quad \frac{G}{G_0} = \frac{\omega_0}{\omega} \left( \frac{1 + \omega_1/\omega}{1 + \omega_1/\omega_0} \right)^{A_1} \left( \frac{1 + \omega_2/\omega}{1 + \omega_2/\omega_0} \right)^{A_2}, \quad (10)$$

where the first equation expresses the running of $\ln G$ in $\omega$ space, while in second equation we write its solution. The first equation tells that during the RG evolution $G$ gets extremised at $\omega = \sqrt{7/40}$, taking the second derivative it is shown that this point is a maxima. We choose this point to be our reference point $\mu_0$ or $t = 0$ and integrate to obtain the second equation, where $A_1 = -0.3473$ and $A_2 = -1.0027$. Eq. (10) show that for large $\omega$ or $t$, $G \sim 1/\omega$, thereby vanishing for large $t$ [16], while for small $\omega$, $G \sim \omega^{7/20}$ reaching a peak at $\omega_0 = \sqrt{7/40}$. Similarly, using Eqs. (4 and 5) we obtain the running of $M^2/\omega$, which along with with Eq. (7) is integrated to give $M^2/\omega = (M_0^2/\omega_0)((1 + \omega_1/\omega)/(1 + \omega_1/\omega_0))^{B_1} ((1 + \omega_2/\omega)/(1 + \omega_2/\omega_0))^{B_2}$, where $B_1 = 1.0802$ and $B_2 = 0.2698$. This tells that as $\omega \rightarrow \infty$, the mass of the $M$-mode also goes to infinity, which means that ultimately it is decoupled from the theory. But does it goes to infinity quick enough so that it is never encountered in the theory, is the question we ask next? To answer this we now analyze $M^2/\mu^2$. It is instructive to note that from Eqs. (3, 5 and 7) we obtain the running of $\ln(M^2/\mu^2)$,

$$\frac{d}{d\omega} \ln \left( \frac{M^2}{\mu^2} \right) = \left( \frac{\omega + 399/50 - \frac{7}{40 \omega}}{(\omega + \omega_1)(\omega + \omega_2)} \right). \quad (11)$$

This shows that $M^2/\mu^2$ reaches a minima for $\omega = \omega_*$ given by $(\omega_* + 399/50 - 7/40 \omega_*) = 6\pi/(5 M_*^2 G_*).$ Hence by demanding $M^2/\mu^2_* = (6\pi/5 M_*^2 G_*)/(\omega_* + 399/50 - 7/40 \omega_*) > 1$, we make the $M$-mode not realizable in the physical GR sector of the theory. This inequality is easily achievable by choosing $\mu^2_* G_*$ appropriately. Perturbative loop expansion requires that $M^2 G$ is small. Therefore $M$ is a sub-Planckian mass, yet the running mass as dictated by interactions makes it physically not realizable even in post Planckian regime.

One can do a similar analysis to answer the question whether Riccion is realizable or not? For this we study the RG evolution of $M^2/(\omega \mu^2)$. From Eqs. (4, 5 and 7) we obtain $d \ln (M^2/(\omega \mu^2)) / d\omega = -(3 + 27/40 \omega + 6\pi/5 M^2 G)/(\omega + \omega_1)(\omega + \omega_2)$, showing that the Riccion
mass relative to $\mu$ decreases monotonically. By a suitable choice we can make the Riccions to be physically realizable or not. So we conclude that there exists unitary physical subspace only with the gravitons or along with Riccions.

To make the running of $\Lambda$ to zero we add two spin-$\frac{1}{2}$ Dirac fields (for detail see [9]) with mass $(5/4)^{1/4}M$ and $M/\sqrt{2}\omega$ [13] (these additional fermionic fields can be interpreted as ghosts normalizing the functional integral). This assures that if initially $\Lambda = 0$, then throughout the RG flow $\Lambda$ will remain zero. The effect of adding the two Dirac fields is just to shift $\omega_0$, $\omega_1$, $\omega_2$ and $\omega_*$, but all conclusions remains unaltered.

In arbitrary gauge the beta function $1/G$ and $\omega/M^2G$ remains unaltered, which means that running of $\omega$ and two fixed points $\omega_1$ and $\omega_2$ remains same. However the beta function of $1/G$ gets modified. In a general gauge it is $d\ln G/dt = -\left(5M^2G/3\pi\right)(\omega + a + b/\omega)$, where $a$ and $b$ depend on the gauge-fixing parameters used, while the leading term is gauge invariant. This means that $\omega_0$ shifts. Doing the same analysis as before, we write the running of $G$ in terms of $\omega$, from which we note that for large $\omega$ or $t$, $G \sim 1/\omega$ is a gauge invariant result, while for small $\omega$, the rate at which $G$ vanishes depend on $a$ and $b$.

### 3. Discussion

By doing the one-loop analysis we have found that there is no solution for $\omega$ from Eq. (8) if $t > t_{\text{max}}$. This is because $\omega$ reaches its maximum value infinity. In the other extreme when $\omega = 0$, $t$ reaches its minimum value $t_{\text{min}}$. For $t < t_{\text{min}}$, $\omega$ become negative i.e. $M^2/\omega$ the Riccion mass square becomes negative signaling the instability of the vacuum.

Finally we conclude that the action $A$, Eq. (1) describes a perturbative quantum gravity as self consistent, renormalizable and unitary theory of gravitons and the curvature cannot become singular, in particular it cannot fluctuate wildly at sub-Planckian [15] or post Planckian regimes consistent with known cosmology. Its predictions asymptotically beyond Planck scale needs to be investigated further.

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[16] $G$ is indeed a nontrivial function of $ln\mu/\mu_0 = t$ through $\omega(t)$, consequently the criticisms alluded in “ Anber M M and Donoghue J F, arXiv:1111.2875 [hep-th].” do not pertain to our analysis.