LOCAL BRAID MONODROMIES AND LOCAL FUNDAMENTAL GROUPS OF TANGENTED CONIC-LINE ARRANGEMENTS

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Abstract. This paper is the first part of a series of three papers about the fundamental groups of conic-line arrangements consist of two tangented conics and up to two additional tangented lines.

In this part, we compute the local braid monodromies and the local fundamental groups of the singularities which appear in such arrangements.

1. Introduction

There is an extensive research concerning the topology of line arrangements and curves, and the fundamental groups of their complements, see [1], [3], [7], [10], [12], [13], [17], [18], [21], [26]-[31], and others.

The study of fundamental groups of complements of algebraic curves contributes to other areas of algebraic geometry.

In [15], [16] and [33], we presented invariants of branch curves arising from generic projections of surfaces in $\mathbb{CP}^N$ to $\mathbb{CP}^2$ as distinguishing invariants between different connected components of the moduli space of surfaces of general type. Moreover, in [32], we presented fundamental groups of complements of branch curves as new invariants for surfaces.

This paper is the first part of a series of three papers which study the fundamental group of some types of conic-line arrangements, which may appear as a branch curve of a generic projection of a surface of general type in $\mathbb{CP}^N$ to $\mathbb{CP}^2$ (see for example [14], [20], [32], [33] and other works in progress [1], [8]). The other two parts are [2] and [3].

In particular, we study singularities which appear in conic-line arrangements with two tangented conics and up to two additional tangented lines.

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The paper is organized as follows. Section 2 surveys briefly the notion of braid monodromy and the way to compute it, and it includes the presentation of the van Kampen Theorem which computes the induced relations from the braid monodromy of the singular point. In Section 3 we compute the local braid monodromy and the relations induced by the simple tangency point, created by two tangented conics. Section 4 deals with the local braid monodromy and the relations induced by singular points with three components. Section 5 covers some singular points with four components.

2. THE COMPUTATION OF LOCAL BRAID MONODROMY AND ITS INDUCED RELATIONS

In this section, we briefly survey the notion of braid monodromy of a singular point of a curve, and the way how we deduce its corresponding relations in the fundamental group.

We start with the definition of \( l(\gamma) \) and a \( g\)-base, which will be needed later. Let \( K \) be a finite set in a disk \( D \subset \mathbb{R}^2 \).

**Definition 2.1.** Let \( D \) be a disk. Let \( w_i, \ i = 1, \ldots, n, \) be disjoint disks in \( \text{Int}(D) \). Let \( u \in \partial D \). Let \( \gamma \) be a simple path connecting \( u \) with one of the \( w_i \)s, say \( w_{i_0} \), and does not meet any other \( w_j \). We assign to \( \gamma \) a loop \( l(\gamma) \) (actually an element of \( \pi_1(D - K, u) \)) as follows: let \( c \) be a simple loop equal to the (oriented) boundary of a small neighborhood \( V \) of \( w_{i_0} \) chosen such that \( \gamma' = \gamma - V \cap \gamma \) is a simple path (see Figure 1).

Then: \( l(\gamma) = \gamma' \cup c \cup (\gamma')^{-1} \) (we will not distinguish between \( l(\gamma) \) and its representative in \( \pi_1(D - K, u) \)).

![Figure 1. \( l(\gamma) \)](image)

**Definition 2.2.** Let \( D \) be a disk, \( K \subset D, |K| < \infty \). Let \( u \in D - K \). A set of simple paths \( \{\gamma_i\} \) is a **bush** in \( (D, K, u) \), if \( \forall i, j, \gamma_i \cap \gamma_j = \)
$u$; $\forall i$, $\gamma_i \cap K$ contains only one point, and $\gamma_i$ are ordered counterclockwise around $u$. Let $\Gamma_i = l(\gamma_i) \in \pi_1(D - K, u)$ be a loop around $K \cap \gamma_i$ determined by $\gamma_i$. $\{\Gamma_i\}$ is called a g-base of $\pi_1(D - K, u)$.

An example of a g-base is given in Figure 2.

![Figure 2. A g-base](image)

Here we define the standard g-base which will be used throughout the paper:

**Definition 2.3.** Let $D$ be the unit disk, and let $K \subset D$ be a finite set of points on the real axis. Let $u = (-1, 0)$. Let $\{\gamma_i\}$ be the straight segment which connects $u$ to the $i$th point. Let $\Gamma_i = l(\gamma_i) \in \pi_1(D - K, u)$. $\{\Gamma_i\}$ is called the standard g-base of $\pi_1(D - K, u)$ (see Figure 3).

![Figure 3. A standard g-base](image)

2.1. The braid monodromy and the way to compute it. Let $D$ be again a closed disk in $\mathbb{R}^2$, $K \subset D$ a finite set, $u \in \partial D$. In such a case, we can define the braid group $B_n[D, K]$ ($n = |K|$):
Definition 2.4. Let $\mathcal{B}$ be the group of all diffeomorphisms $\beta$ of $D$ such that $\beta(K) = K$, $\beta|_{\partial D} = \text{Id}|_{\partial D}$. Such a diffeomorphism acts naturally on $\pi_1(D-K, u)$ (since such a diffeomorphism acts on paths in general, and on loops in particular). We say that two such diffeomorphisms are equivalent if they define the same automorphism on $\pi_1(D-K, u)$. The quotient of $\mathcal{B}$ by this equivalence relation is called the braid group $B_n[D, K]$.

An alternative equivalent definition for the braid group was given by Artin [6], by means of generators and relations:

Definition 2.5 (Artin). The braid group $B_n$ is the group generated by $\{\sigma_1, ..., \sigma_{n-1}\}$ subject to the relations:

1. $\sigma_i \sigma_j = \sigma_j \sigma_i$ where $|i - j| \geq 2$.
2. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for all $i = 1, \ldots, n-2$.

Another way to look at braids is via motions of $K$.

Definition 2.6. Let $K' = \{a'_1, a'_2, \ldots, a'_n\}$, $K = \{a_1, a_2, \ldots, a_n\}$. A motion of $K'$ to $K$ in $D$ is a set of $n$ continuous functions $m_i : [0,1] \to D$, $i = 1, \ldots, n$, such that:

1. $\forall i$, $m_i(0) = a'_i$, $m_i(1) = a_i$.
2. $\forall i \neq j$, $m_i(t) \neq m_j(t)$ $\forall t \in [0,1]$.

According to the following proposition, we can define a family of diffeomorphisms induced from the motion (with the assumption that $K = K'$).

Proposition 2.7. Given a motion $\mathcal{R}$, there exists a continuous family of diffeomorphisms $D_{\mathcal{R},t} : D \to D$, $t \in [0,1]$, such that:

1. $D_{\mathcal{R},t}|_{\partial D} = \text{Id}|_{\partial D}$.
2. $\forall t, i$, $D_{\mathcal{R},t}(a'_i) = m_i(t)$.

Definition 2.8. When $K = K'$, the braid induced from a motion $\mathcal{R}$, denoted by $b_\mathcal{R}$, is the braid defined by the diffeomorphism $D_{\mathcal{R},1}$.

Before introducing the definition of braid monodromy, we have to make some more constructions. From now on, we will work in $\mathbb{C}^2$. Let $E$ (resp. $D$) be a closed disk on the $x$-axis (resp. $y$-axis), and let $S$ be a part of an algebraic curve in $\mathbb{C}^2$ located in $E \times D$. Let $\pi_1 : E \times D \to E$ and $\pi_2 : E \times D \to D$ be the canonical projections, and let $\pi = \pi_1|_S : S \to E$. Assume $\pi$ is a proper map, and deg $\pi = n$. Let $N = \{x \in E \mid |\pi^{-1}(x)| < n\}$, and assume $N \cap \partial E = \emptyset$. Now choose $M \in \partial E$ and let $K = K(M) = \pi^{-1}(M)$. By the assumption that deg $\pi = n$ ($\Rightarrow |K| = n$), we can write: $K = \{a_1, a_2, \ldots, a_n\}$. 
Under these constructions, from each loop in \( E - N \), we can define a braid in \( B_n[M \times D, K] \) in the following way:

1. Because \( \deg \pi = n \), we can lift any loop in \( E - N \) with a base point \( M \) to a system of \( n \) paths in \( (E - N) \times D \) which start and end at \( \{a_1, a_2, \cdots, a_n\} \).
2. Project this system into \( D \) (by \( \pi_2 \)), to get \( n \) paths in \( D \) which start and end at the image of \( K \) in \( D \) (under \( \pi_2 \)). These paths actually form a motion.
3. Induce a braid from this motion, as defined in Definition 2.8.

In this way, we can match a braid to each loop. Therefore, we get a map \( \varphi: \pi_1(E - N, M) \to B_n[M \times D, K] \), which is also a group homomorphism, called the \textbf{braid monodromy of} \( S \) \textbf{with respect to} \( E \times D, \pi_1, M \).

Hence, for computing the braid monodromy, one should perform the following steps:

1. Define a loop in the \( x \)-axis which circumscribes the \( x \)-projection of the singular point.
2. Compute the points of the curve \( S \) in each fiber over the loop.
3. Determine the behavior of the points in the fibers which correspond to the loop. This gives the braid monodromy.

Some examples of computing braid monodromies of singular points can be found in [22].

2.2. \textbf{The Lefschetz diffeomorphism.} For computing the braid monodromy of a plane curve, we use the Moishezon-Teicher algorithm (see [22]). The idea of this algorithm is to start with some braid which is associated to the local behavior near the singular point, and then to apply on it some actions induced by the singularities to the right side of that singular point. This actions are called \textit{Lefschetz diffeomorphisms} which are induced by the singular points.

The Lefschetz diffeomorphism can be defined in general (see [22]), but here we define it according to our settings. Let \( P \) be a singular point, and assumed that \( P \) is located at the origin. Let \( T(t), 0 \leq t \leq 1 \), be the path which connects the point \( T(0) = -1 \) to the point \( T(1) = 1 \) along the negative part of the unit circle (see Figure 4).

There exists a continuous family of diffeomorphisms \( \psi(t): \{ -1 \} \times D \to T(t) \times D, \forall t \in [0, 1] \), such that:

1. \( \psi(0) = \text{Id}|_{\{ -1 \} \times D} \)
2. \( \forall t \in [0, 1], \psi(t)(\pi_1^{-1}(\{ -1 \}) \cap S) = \pi_1^{-1}(T(t)) \cap S \).
3. \( \forall y \in \partial D, \psi(t)(\{ -1 \}, y) = (T(t), y) \).
Definition 2.9. The Lefschetz diffeomorphism induced by the singular point $P$ is the diffeomorphism

$$\psi_T = \psi_{(1)} : \{-1\} \times D \rightarrow \{1\} \times D$$

One can see that the computation of the Lefschetz diffeomorphism is similar to the computation of the braid monodromy, in the following sense: in order to compute the braid monodromy, we have to follow the movement of the points of the curve in the fibers over the whole loop, whence for computing the Lefschetz diffeomorphism, this following should be done only for a half of the loop.

2.3. The classical van Kampen Theorem. The van Kampen Theorem gives a finite presentation (by generators and relations) of the fundamental group of complements of curves from the braid monodromy.

Let $S$ be an algebraic curve in $\mathbb{C}^2$ ($\deg S = n$). Let $\pi = \pi_1 : \mathbb{C}^2 \rightarrow \mathbb{C}$ be the canonical projection on the first coordinate. Let $\mathbb{C}_x = \pi^{-1}(x)$, and now define: $K_x = \mathbb{C}_x \cap S$ (By assumption $\deg S = n$, we have that $|K_x| \leq n$).

Let $N = \{x \mid |K_x| < n\}$. Choose now $u \in \mathbb{C}$, $u$ real, such that $x \ll u$, $\forall x \in N$, and define: $B_n = B_n[\mathbb{C}_u, \mathbb{C}_u \cap S]$. Let $\varphi_u : \pi_1(\mathbb{C} - N, u) \rightarrow B_n$ be the braid monodromy of $S$ with respect to $\pi, u$. Also choose $u_0 \in \mathbb{C}_u$, $u_0 \notin S$, $u_0$ below the real line far enough such that $B_n$ does not move $u_0$. It is known that the group $\pi_1(\mathbb{C}_u - S, u_0)$ is free. There exists an epimorphism $\pi_1(\mathbb{C}_u - S, u_0) \rightarrow \pi_1(\mathbb{C}^2 - S, u_0)$, so a set of generators for $\pi_1(\mathbb{C}_u - S, u_0)$ determines a set of generators for $\pi_1(\mathbb{C}^2 - S, u_0)$.

In this situation, the classical van Kampen Theorem says:

**Theorem 2.10** (Classical van Kampen’s Theorem). Let $S$ be an algebraic curve, $u, u_0, \varphi_u$ defined as above. Let $\{\delta_i\}$ be a $g$-base of $\pi_1(\mathbb{C} - N, u)$. Let $\{\Gamma_j \mid 1 \leq j \leq n\}$ ($n = \deg S$) be a $g$-base for $\pi_1(\mathbb{C}_u - S, u_0)$. 

![Figure 4. The path $T(t)$](image)
Then, $\pi_1(C^2 - S, u_0)$ is generated by the images of $\Gamma_j$ in $\pi_1(C^2 - S, u_0)$ and we get a complete set of relations from those induced from

$$(\varphi_u(\delta_i))(\Gamma_j) = \Gamma_j, \quad \forall i \neq j$$

Hence, for computing the relations induced from a singular point, one should follow these steps (see also [22]-[23]):

1. Start from a standard $g$-base.
2. Apply on the $g$-base the corresponding braid (given by the braid monodromy).
3. Present any loop as a composition of the generators of the standard $g$-base.
4. The induced relations are the $n$ equations: for all $j$, $\Gamma_j$ is equal to the presentation of the $j$th loop.

One should notice that all the computations here are local, and when we reach the global computations, we need to replace the generators appear in these relations by a word (which is a conjugation of a standard generator by some standard generators).

3. A singular point with two components

In this section, we compute the local braid monodromy of a tangency point of two conics.

The tangency point between two conics can be presented locally by the equation: $(y + x^2)(y - x^2) = 0$ (see Figure 5).

![Figure 5. The singularity of $(y + x^2)(y - x^2) = 0$ at $(0,0)$](image)

The singular point is $(0,0)$, and the points of the curve in the fiber over $x = 1$ are $y = 1$ and $y = -1$. Now, let us take a loop in $x$-axis based at $x = 1$ and circumscribing 0: $\alpha(t) = e^{2\pi it}$, where $0 \leq t \leq 1$. Lifting it to the curve, we get two paths on the curve: $\alpha_1(t) = (e^{2\pi it}, e^{4\pi it})$ and $\alpha_2(t) = (e^{2\pi it}, -e^{4\pi it})$ where $0 \leq t \leq 1$. Projecting...
the paths to the $y$-axis, we get $y_1(t) = e^{4\pi it}$ and $y_2(t) = -e^{4\pi it}$ where $0 \leq t \leq 1$. By substituting $t = 1$, the local braid monodromy of this point is two counterclockwise full-twists. Hence, we have:

**Theorem 3.1.** The local braid monodromy of the tangency point of two conics is two counterclockwise full-twists (similar to the tangency point of a conic and a line). Hence, we have the same induced relations:

$$(x_1x_2)^2 = (x_2x_1)^2$$

where $\{x_1, x_2\}$ are the generators of the standard $g$-base.

Now, we compute the corresponding Lefschetz diffeomorphism:

**Corollary 3.2.** The Lefschetz diffeomorphism of a tangency point between two tangented conics is a counterclockwise full-twist (similar to the usual tangency point).

The reason for this corollary is since the Lefschetz diffeomorphism is obtained by going on the loop mentioned above from $t = \frac{1}{2}$ to $t = 1$. Along this interval, the two points in the fiber make one counterclockwise full-twist, and hence the Lefschetz diffeomorphism of this singular point is a full-twist.

4. **Singular points with three components**

In this section, we will compute the local braid monodromy of four types of singular points consist of three components (where at most two of them are conics which are tangented). Then, we will compute the relations induced from these singular points by the classical van Kampen Theorem (see [34] and Section 2.3 here).

4.1. **First type.** The local equation of the singularity of the first type is $(2x + y)(y + x^2)(y - x^2) = 0$ (see Figure 6).

![Figure 6](image)

**Figure 6.** The singularity of $(2x+y)(y+x^2)(y-x^2) = 0$ at $(0, 0)$

For computing the braid monodromy, we take a loop around $x = 0$ and look what happens to the points of the curve $(2x + y)(y + x^2)(y - x^2) = 0$ in the fibers above this loop.
Let \( x = e^{2\pi i t} \) where \( 0 \leq t \leq 1 \). For \( t = 0 \) we have that \( x = 1 \) and the points of the curve in the fiber over \( x = 1 \) are \( y = 1, -1 \) and \( y = -2 \). For \( t = \frac{1}{2} \), we have \( x = -1 \), and the points of the curve in the fiber over \( x = -1 \) are \( y = -1, 1 \) and \( y = 2 \). The point \( y = 1 \) in the fiber \( x = -1 \) corresponds to the point \( y = 1 \) in the fiber \( x = 1 \), and similarly the point \( y = -1 \) in the fiber \( x = -1 \) corresponds to the point \( y = -1 \) in the fiber \( x = 1 \). The point \( y = 2 \) in the fiber \( x = -1 \) corresponds to the point \( y = -2 \) in the fiber \( x = 1 \). Hence, we get that from \( t = 0 \) to \( t = \frac{1}{2} \), the points \( y = 1 \) and \( y = -1 \) do a counterclockwise full-twist and the point \( y = -2 \) does a counterclockwise half-twist around the points \( y = 1, -1 \). If we continue to \( t = 1 \), the points \( y = 1 \) and \( y = -1 \) do two counterclockwise full-twists and the point \( y = -2 \) does a counterclockwise full-twist around the points \( y = 1, -1 \) together.

Now, we want to compute the induced relations from the braid monodromy of the singular point. According to van Kampen’s Theorem, one should compute the \( g \)-base obtained by applying the action induced by the local braid monodromy of the singular point on the standard \( g \)-base. This is shown in Figure 7: in the first step, we perform two counterclockwise full-twists of points 2 and 3 which represent \( y = -1 \) and \( y = 1 \) respectively. In the second step, we perform a counterclockwise full-twist of point 1 (which represents the point \( y = -2 \)) around the points 2 and 3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{The \( g \)-base obtained from the standard \( g \)-base by the action of the local braid monodromy}
\end{figure}
Now, by van Kampen’s Theorem, we get the following induced relations from the new g-base (where \(\{x_1, x_2, x_3\}\) are the generators of the standard g-base):

\[
\begin{align*}
(1) & \quad x_1 = x_3 x_2 x_1 x_2^{-1} x_3^{-1} \\
(2) & \quad x_2 = x_3 x_2 x_1 x_3 x_2 x_3^{-1} x_1^{-1} x_2^{-1} x_3^{-1} \\
(3) & \quad x_3 = x_3 x_2 x_1 x_3 x_2 x_3 x_2^{-1} x_3^{-1} x_1^{-1} x_2^{-1} x_3^{-1}
\end{align*}
\]

From Relation (1), we get that \(x_1 x_3 x_2 = x_3 x_2 x_1\). Relation (2) becomes \(x_3 x_2 x_1 x_3 x_2 = x_2 x_3 x_2 x_1 x_3\). Now, we show that Relation (3) is not needed. Simplifying Relation (3) yields \(x_2 x_1 x_3 x_2 x_3 = x_3 x_2 x_1 x_3 x_2\). By Relation (1), this relation is equal to Relation (2), and hence Relation (3) is redundant.

To summarize, we proved the following:

**Theorem 4.1.** The local braid monodromy of the singularity presented locally by the equation \((2x + y)(y + x^2)(y - x^2) = 0\) is: two points (correspond to \(y = 1\) and \(y = -1\)) do two counterclockwise full-twists and the third point (corresponds to \(y = -2\)) do a counterclockwise full-twist around them.

The induced relations of this point are:

\[
\begin{align*}
& x_1 x_3 x_2 = x_3 x_2 x_1 \quad ; \quad x_3 x_2 x_1 x_3 x_2 = x_2 x_3 x_2 x_1 x_3
\end{align*}
\]

where \(\{x_1, x_2, x_3\}\) are the generators of the standard g-base.

Since the Lefschetz diffeomorphism of the singular point is obtained by computing the action only on half of the unit circle (from \(t = \frac{1}{2}\) to \(t = 1\)), we have the following corollary:

**Corollary 4.2.** The Lefschetz diffeomorphism of the singular point presented locally by \((2x + y)(y + x^2)(y - x^2) = 0\) is a counterclockwise full-twist of the points correspond to \(y = 1\) and \(y = -1\), and a half-twist of the point corresponds to \(y = 2\) (which becomes \(y = -2\)) around them.

**Remark 4.3.** One can easily see that the singular point presented locally by the equation \(y(2x + y)(y + x^2) = 0\) (see Figure 8) has the same braid monodromy and the same induced relations as the singular point which we have dealt with in this section, since these two singular points are locally the same (from the topological point of view).

4.2. **Second type.** The local equation of the singularity of the second type is \((2x - y)(y + x^2)(y - x^2) = 0\) (see Figure 9).

Applying almost the same computations we have performed for the first type, we get the following result:
Theorem 4.4. The local braid monodromy of the singularity presented locally by the equation $(2x-y)(y+x^2)(y-x^2) = 0$ is: two points (correspond to $y=1$ and $y=-1$) do two counterclockwise full-twists and the third point (corresponds to $y=2$) do a counterclockwise full-twist around them.

The corresponding induced relations are:

$$x_3x_2x_1 = x_2x_1x_3; \quad x_3x_2x_1x_2x_1 = x_1x_3x_2x_1x_2$$

where \(\{x_1, x_2, x_3\}\) are the generators of the standard $g$-base.

The Lefschetz diffeomorphism of the singular point is a counterclockwise full-twist of the points correspond to $y=1$ and $y=-1$, and a half-twist of the point corresponds to $y=-2$ (which becomes $y=2$) around them.

Remark 4.5. As in the previous section, one can easily see that the singular point presented locally by the equation $y(2x-y)(y+x^2) = 0$ (see Figure 10) has the same braid monodromy and the same induced relations as the singular point which we have dealt with in this section, since these two singular points are locally the same.

4.3. Third type. The local equation of the singularity of the third type is $y(y^2+x)(y^2-x) = 0$ (see Figure 11). One should notice that there is a major difference between this type of singularity and the two previous types: In this singularity there are two “hidden” branch points. That is, at any fiber one has three real points and two complex points (i.e. complex level 2), and in each side of the singularity, the
complex points belong to a different conic (by the singularity, two real points become complex and two complex points become real).

![Figure 10](image)

**Figure 10.** The singularity of \( y(2x - y)(y + x^2) = 0 \) at \((0,0)\)

For computing the braid monodromy, we take a loop around \( x = 0 \) and look what happens to the points of the curve \( y(y^2 + x)(y^2 - x) = 0 \) in the fibers over this loop.

Let \( x = e^{2\pi it} \) where \( 0 \leq t \leq 1 \). For \( t = 0 \) we have that \( x = 1 \) and the points of the curve in the fiber over \( x = 1 \) are \( y = -1, 0, 1, i \) and \( y = -i \). For \( t = \frac{1}{2} \), we have \( x = -1 \), and we get again that the points of the curve in the fiber over \( x = -1 \) are \( y = -1, 0, 1, i \) and \( y = -i \).

By a careful checking, one can see that the point \( y = 1 \) in the fiber \( x = -1 \) corresponds to the point \(-i\) in the fiber \( x = 1 \), and similarly all the points except for \( y = 0 \) in the fiber \( x = -1 \) made a 90° rotation counterclockwise around \( y = 0 \in \mathbb{C} \) from their corresponding points in the fiber \( x = 1 \). The point \( y = 0 \) remains fixed. Now, when we continue to \( t = 1 \), the points of the fiber continue to move counterclockwise around \( y = 0 \). When we reach \( t = 1 \) and reach back the fiber \( x = 1 \), the points in the fiber made a 180° rotation counterclockwise around \( y = 0 \in \mathbb{C} \) from their corresponding points in the initial fiber \( x = 1 \).

Hence, we have:

**Theorem 4.6.** The action of local braid monodromy of the point presented locally by the equation: \( y(y^2 + x)(y^2 - x) = 0 \) is a 180° rotation counterclockwise of the four points (correspond to \( y = 1, -1, i, -i \)) around the point corresponds to \( y = 0 \), as shown schematically in Figure 12.
Now, we want to compute the induced relations of this singular point. By van Kampen’s Theorem, one should compute the $g$-base obtained by applying the action induced by the local braid monodromy of the singular point on the standard $g$-base. Since we have two complex points in the fiber before the action of the braid monodromy and after it, we have to start by rotating the two rightmost points by $90^\circ$ counterclockwise, for representing the two complex points (see Step (1) in Figure 12). Then, we move the two complex points to be $i$ and $-i$ (see Step (2) there). Now, we apply the action of the local braid monodromy (Step (3)), and then we return the two new complex points to the right side, and return them to the real axis by rotating them by $90^\circ$ clockwise (Step (4)).

**Remark 4.7.** One should notice here that when we return the points back to the real axis in Step (4), we rotate the points clockwise. The reason is that we want to compute the pure action of the braid on the $g$-base, and the clockwise rotation cancels the counterclockwise rotation which we have performed in Step (1). In the global computations, we indeed rotate always counterclockwise, since we work there in a model (for simplifying the computations), and we can define the model as we want. We will get the same results in the global computations even if we rotate clockwise the complex points into real points.

Now, by van Kampen’s Theorem, we get the following induced relations from the new $g$-base (where $\{x_1, x_2, x_3, x_4, x_5\}$ are the generators of the standard $g$-base):

1. $x_1 = x_4x_3x_4^{-1}$
2. $x_2 = x_4x_3x_2x_3^{-1}x_4^{-1}$
3. $x_3 = x_4x_3x_2x_1x_2^{-1}x_3^{-1}x_4^{-1}$
4. $x_4 = x_5$
5. $x_5 = x_5x_4x_3x_2x_1x_4x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5^{-1}$

Relation (2) is equivalent to $x_4x_3x_2 = x_2x_4x_3$. Substituting $x_1$ by $x_4x_3x_4^{-1}$ in Relation (3) yields $x_3 = x_4x_3x_2x_4x_3^{-1}x_2^{-1}x_3^{-1}x_4^{-1}$, which is
Figure 13. The g-base obtained from the standard g-base by the action of the local braid monodromy.

equivalent to $x_4x_3x_2x_4x_3 = x_3x_4x_3x_2x_4$. By Relation (2), it can also be written as $x_4x_3x_2x_4x_3 = x_3x_2x_4x_3x_4$.

We will show that Relation (5) is redundant. First, we can cancel $x_5$, so we get: $x_4x_3x_2x_1x_4 = x_5x_4x_3x_2x_1$. Since $x_4 = x_5$, we can cancel another $x_5$, and by substituting $x_1$ by $x_4x_3x_4^{-1}$ and some simplifications we get $x_3x_2x_1x_3x_4 = x_4x_3x_2x_4x_3$, which is equal to Relation (3). Hence, Relation (5) is redundant.

Therefore, we have the following corollary:

**Corollary 4.8.** The induced relations for the singular point presented locally by the equation $y(y^2 + x)(y^2 - x) = 0$ are:

1. $x_4x_3x_2 = x_2x_4x_3$
2. $x_3x_2x_4x_3x_4 = x_4x_3x_2x_4x_3$
3. $x_1 = x_4x_3x_4^{-1}$
(4) \( x_4 = x_5 \)

where \( \{x_1, x_2, x_3, x_4, x_5\} \) are the generators of the standard g-base.

Now, we compute the corresponding Lefschetz diffeomorphism:

**Corollary 4.9.** The Lefschetz diffeomorphism of the singular point presented locally by \( y(y^2 + x)(y^2 - x) = 0 \) is a 90° rotation counterclockwise of the four points around the fixed point \( y = 0 \in \mathbb{C} \).

The reason for this corollary is since the Lefschetz diffeomorphism is obtained by computing the action along the loop mentioned above from \( t = \frac{1}{2} \) to \( t = 1 \). Along this interval, the four points (except for \( y = 0 \)) made a 90° rotation counterclockwise.

4.4. **Fourth type.** This type is slightly different from the previous three types. In the previous types, the line intersects the tangency point, but was not tangented to it. In this type, the additional line and the tangency point have a common tangent.

The local equation of the singularity of this type is \( y(y + x^2)(y - x^2) = 0 \) (see Figure 14).

**Figure 14.** The singularity of \( y(y + x^2)(y - x^2) = 0 \) at \((0, 0)\)

We take a loop around \( x = 0 \) and look what happens to the points of the curve \( y(y + x^2)(y - x^2) = 0 \) in the fibers over this loop.

Let \( x = e^{2\pi it} \) where \( 0 \leq t \leq 1 \). For \( t = 0 \), we have that \( x = 1 \) and the points of the curve in the fiber over \( x = 1 \) are \( y = -1, 0 \) and \( y = 1 \). For \( t = \frac{1}{2} \), we have \( x = -1 \), and we get again that the points of the curve in the fiber over \( x = -1 \) are \( y = -1, 0 \) and \( y = 1 \). One can see that the point \( y = 0 \) is fixed while \( t \) is changing from 0 to \( \frac{1}{2} \), and the other two points, \( y = 1 \) and \( y = -1 \), do a counterclockwise full-twist around the point \( y = 0 \). Hence, when we continue to \( t = 1 \), we get that the point \( y = 1 \) and \( y = -1 \) do two counterclockwise full-twists around the fixed point \( y = 0 \).

As before, by van Kampen’s Theorem, one should compute the g-base obtained by applying the action induced by the local braid monodromy of the singular point on the standard g-base. This is shown in Figure 15.
Figure 15. The $g$-base obtained from the standard $g$-base by the action of the local braid monodromy

Now, by van Kampen’s Theorem, we get the following induced relations from the new $g$-base (where $\{x_1, x_2, x_3\}$ are the generators of the standard $g$-base):

1. $x_1 = x_3x_2x_1x_3x_2x_1x_2^{-1}x_3^{-1}x_2^{-1}x_3^{-1}$
2. $x_2 = x_3x_2x_1x_3x_2x_1x_2^{-1}x_3^{-1}x_1^{-1}x_2^{-1}x_3^{-1}$
3. $x_3 = x_3x_2x_1x_3x_2x_1x_3^{-1}x_2^{-1}x_3^{-1}x_1^{-1}x_2^{-1}x_3^{-1}$

From Relation (1), one gets $(x_3x_2x_1)^2 = (x_1x_3x_2)^2$. Relation (2) can be written $x_2(x_3x_2x_1)^2 = (x_3x_2x_1)^2x_2$. By Relation (1), it becomes $x_2(x_1x_3x_2)^2 = (x_1x_3x_2)^2x_2$. By cancelling the rightmost $x_2$, we get: $(x_2x_1x_3)^2 = (x_1x_3x_2)^2$.

Relation (3) is redundant, since by cancelling $x_3$, one gets: $(x_3x_2x_1)^2 = (x_2x_1x_3)^2$, which is already known.

To summarize, we have proved the following result:

**Theorem 4.10.** The local braid monodromy of the singularity presented locally by the equation $y(y + x^2)(y - x^2) = 0$ is: the three points do two counterclockwise generalized full-twists.

The induced relations of this point are:

$$(x_3x_2x_1)^2 = (x_1x_3x_2)^2 = (x_2x_1x_3)^2,$$

where $\{x_1, x_2, x_3\}$ are the generators of the standard $g$-base.

The corresponding Lefschetz diffeomorphism is presented in the following corollary:

**Corollary 4.11.** The Lefschetz diffeomorphism of the singular point presented locally by $y(y + x^2)(y - x^2) = 0$ is a full-twist of all the three points.

The reason for that corollary is since the Lefschetz diffeomorphism is obtained by going along the loop mentioned above from $t = \frac{1}{2}$ to $t = 1$. In this interval, the three points perform a full-twist.
The arguments of this section can be generalized easily to an arbitrary number of tangented components at the same point with a common tangent:

**Corollary 4.12.** Let \( P \) be a singular point which consists of \( n \) smooth branches with a common tangent at \( P \). Let \( x_i \) be the generator which corresponds to the \( i \)th branch. Then, the local braid monodromy is two generalized full-twists of all the segment from 1 to \( n \). Moreover, the relations induced by this singular point are:

\[
(x_n x_{n-1} \cdots x_1)^2 = (x_{n-1} \cdots x_1 x_n)^2 = \cdots = (x_1 x_n \cdots x_2)^2.
\]

Notice that if the tangented components have a higher order of tangency (i.e. the higher derivatives of the components are also equal up to the \( k \)th derivative, for a given \( k \)), then the braid monodromy will consist of an higher power of a full-twist, and a higher exponent will appear in the relations.

5. **Singular points with four components**

In this section, we will compute the local braid monodromy of five cases of singular points consist of four components, where at most two of them are conics. Then, we will compute the relations induced from these singular points by the classical van Kampen Theorem (see [34] and Section 2.3 here).

A singular point consists of four components in a conic-line arrangements with two tangented conics and up to two additional lines can be of the following two types: take the tangency point of the two conics, and add two lines in the following two ways: one way is to add one line which will be tangented to both conics, and the second line will intersect both conics (and the line) at the singular point. The second way will be to add two intersecting lines at the tangency point of the two conics.

5.1. **Two tangented conics with a tangent line and an intersecting line.** As before, there are three possibilities for the intersecting line: it can locally presented as the line \( y = 2x, y = -2x \) or \( x = 0 \). In spite of the fact that one can move from the first type to the second type by rotating the line, so locally the singularities are equivalent, but from the global point of view these singularities induced different relations in the global fundamental group.

5.1.1. **First type.** The local equation of the singularity of the first type is \( y(2x + y)(y + x^2)(y - x^2) = 0 \) (see Figure [16]).
For computing the braid monodromy, we take a loop around $x = 0$ and look what happens to the points of the curve $y(2x + y)(y + x^2)(y - x^2) = 0$ in the fibers above this loop.

Let $x = e^{2\pi i t}$ where $0 \leq t \leq 1$. For $t = 0$ we have that $x = 1$ and the points of the curve in the fiber over $x = 1$ are $y = 1$, $0$, $-1$ and $y = -2$. For $t = \frac{1}{2}$, we have $x = -1$, and the points of the curve in the fiber over $x = -1$ are $y = 1$, $0$, $-1$ and $y = 2$. The point $y = 1$ in the fiber $x = -1$ corresponds to the point $y = 1$ in the fiber $x = 1$, and similarly the point $y = -1$ in the fiber $x = -1$ corresponds to the point $y = -1$ in the fiber $x = 1$. The point $y = 0$ is fixed, and the point $y = 2$ in the fiber $x = -1$ corresponds to the point $y = -2$ in the fiber $x = 1$. Hence, we get that from $t = 0$ to $t = \frac{1}{2}$, the points $y = -1, 0, 1$ do a counterclockwise generalized full-twist and the point $y = -2$ does a counterclockwise half-twist around these three points. If we continue to $t = 1$, the three points $y = -1, 0, 1$ perform two counterclockwise generalized full-twists and the point $y = -2$ does a counterclockwise full-twist around the other three points.

For computing the induced relations, we use the van Kampen Theorem. Here, one should compute the g-base obtained by applying the action induced by the local braid monodromy of the singular point on the standard g-base. This is shown in Figure 17: in the first step, we perform two counterclockwise generalized full-twists of points 2, 3 and 4 which represent $y = -1, 0$ and $y = 1$ respectively. In the second step, we perform a counterclockwise full-twist of point 1 (which represents the point $y = -2$) around the other three points.

By van Kampen’s Theorem, we get the following induced relations (where $\{x_1, x_2, x_3, x_4\}$ are the generators of the standard g-base):

\begin{align*}
(1) & \quad x_1 = x_4 x_3 x_2 x_1 x_2^{-1} x_3^{-1} x_4^{-1} \\
(2) & \quad x_2 = x_4 x_3 x_2 x_1 x_4 x_3 x_2 x_3^{-1} x_4^{-1} x_2^{-1} x_3^{-1} x_4^{-1} \\
(3) & \quad x_3 = x_4 x_3 x_2 x_1 x_4 x_3 x_2 x_4 x_3 x_2^{-1} x_3^{-1} x_4^{-1} x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1} \\
(4) & \quad x_4 = x_4 x_3 x_2 x_1 x_4 x_3 x_2 x_4 x_2 x_3^{-1} x_4^{-1} x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1}
\end{align*}
From Relation (1), we get that $x_1 x_4 x_3 x_2 = x_4 x_3 x_2 x_1$. Relations (2) and (3) become $x_4 x_3 x_2 x_1 x_4 x_3 x_2 = x_2 x_4 x_3 x_2 x_1 x_4 x_3$ and $x_4 x_3 x_2 x_1 x_4 x_3 x_2 x_3 = x_3 x_4 x_3 x_2 x_1 x_4 x_3 x_2$ respectively. By Relation (2), Relation (3) can be written as:

$$x_4 x_3 x_2 x_1 x_4 x_3 x_2 x_3 = x_3 x_2 x_4 x_3 x_2 x_1 x_4 x_3,$$

and hence we get from Relations (2) and (3):

$$x_4 x_3 x_2 x_1 x_4 x_3 x_2 = x_3 x_2 x_4 x_3 x_2 x_1 x_4 = x_2 x_4 x_3 x_2 x_1 x_4 x_3.$$

We will show now that Relation (4) is redundant. From Relation (4), one can get:

$$x_3 x_2 x_1 x_4 x_3 x_2 x_4 = x_4 x_3 x_2 x_1 x_4 x_3 x_2.$$

Using Relation (1), this relation is equal to Relation (3), and hence it is redundant.

Hence, we have:

**Theorem 5.1.** The local braid monodromy of the singularity presented locally by the equation $y(2x + y)(y + x^2)(y - x^2) = 0$ is: three points (correspond to $y = 1, 0$ and $y = -1$) do two counterclockwise generalized full-twists and the fourth point (corresponds to $y = -2$) do a counterclockwise full-twist around them.
The induced relations from this singular point are:

\[ x_1 x_4 x_3 x_2 = x_4 x_3 x_2 x_1 \quad ; \quad (x_4 x_3 x_2)^2 x_1 = x_3 x_2 x_4 x_3 x_2 x_1 x_4 = x_2 x_4 x_3 x_2 x_1 x_4 x_3, \]

where \( \{x_1, x_2, x_3, x_4\} \) are the generators of the standard g-base.

One should notice here that if we delete the generator \( x_1 \) which corresponds to the line that intersects the three tangented components, we get the following set of relations:

\[ (x_4 x_3 x_2)^2 = (x_3 x_2 x_4)^2 = (x_2 x_4 x_3)^2, \]

as expected (see Section 4.4). Similarly, if we delete the generator \( x_3 \) which corresponds to the tangent line, we get the following set of relations:

\[ x_1 x_4 x_2 = x_4 x_2 x_1 \quad ; \quad x_4 x_2 x_4 x_2 x_1 = x_2 x_4 x_2 x_1 x_4, \]

again as expected (see Section 4.1).

Since the Lefschetz diffeomorphism of the singular point is obtained by computing the action only on half of the unit circle (from \( t = \frac{1}{2} \) to \( t = 1 \)), we have the following corollary:

**Corollary 5.2.** The Lefschetz diffeomorphism of the singular point presented locally by \( y (2x + y)(y + x^2)(y - x^2) = 0 \) is a counterclockwise generalized full-twist of the points correspond to \( y = 1, 0, \) and \( y = -1 \), and a half-twist of the point corresponds to \( y = 2 \) (which becomes \( y = -2 \)).

5.1.2. Second type. The local equation of the singularity of the second type is \( y(2x - y)(y + x^2)(y - x^2) = 0 \) (see Figure 18).

![Figure 18](image_url)

**Figure 18.** The singularity of \( y(2x - y)(y + x^2)(y - x^2) = 0 \) at \( (0,0) \)

Applying almost the same computations we have performed for the previous type, we get:

**Theorem 5.3.** The local braid monodromy of the singularity presented locally by the equation \( y(2x - y)(y + x^2)(y - x^2) = 0 \) is: the points \( y = 1, 0, -1 \) do two counterclockwise generalized full-twists, and the point
y = 2 does a counterclockwise full-twist around the points y = 1, 0, −1 together.

The corresponding induced relations are:

\[ x_4x_3x_2x_1 = x_3x_2x_1x_4 \quad ; \quad (x_3x_2x_1)^2 x_4 = x_2x_1x_3x_2x_1x_4x_3 = x_1x_3x_2x_1x_4x_3x_2, \]

where \{x_1, x_2, x_3, x_4\} are the generators of the standard g-base.

The Lefschetz diffeomorphism is presented in the following corollary.

**Corollary 5.4.** The Lefschetz diffeomorphism of the singular point presented locally by \( y(2x - y)(y + x^2)(y - x^2) = 0 \) is a counterclockwise generalized full-twist of the points correspond to \( y = 1, 0 \) and \( y = -1 \), and a half-twist of the point corresponds to \( y = -2 \) (which becomes \( y = 2 \)).

5.1.3. Third type. The local equation of the singularity of the third type is \( xy(y + x^2)(y - x^2) = 0 \) (see Figure 19).

![Figure 19. The singularity of \( xy(y + x^2)(y - x^2) = 0 \) at (0, 0)](image)

For computing the braid monodromy in this case, we will use the following trick: the three components in the middle \( x(y + x^2)(y - x^2) = 0 \) can be thought for a moment as a “thick” line which is perpendicular to the other line. After this observation, we have two intersecting lines, which their local braid monodromy is a counterclockwise full-twist (even though we have here a thick line \( x = 0 \), which is vertical, one can rotate it a bit for getting two “usual” intersecting lines, without changing the braid monodromy).

The local braid monodromy induced by the “thick” line itself (which consists of the curve \( x(y + x^2)(y - x^2) = 0 \)) has already been computed (Section 4.3): we got there that the local braid monodromy is a 180° counterclockwise rotation of four points around one fixed point at the origin.

Hence, the action of the local braid monodromy of the whole singular point can be summarized as follows:

**Theorem 5.5.** The local braid monodromy of the singular point presented locally by the equation \( xy(y + x^2)(y - x^2) = 0 \) is: first perform a 180° counterclockwise rotation of the points correspond to \( y =
$1, -1, i, -i$ around $y = 0$, and then the point corresponds to $y = 2$ performs a full-twist with a block which consists of all the other points (see Figure 20).

Figure 20. The action of the local braid monodromy

Now, we want to compute the induced relations of this singular point. According to van Kampen’s Theorem, one should compute the $g$-base obtained by applying the action induced by the local braid monodromy of the singular point on the standard $g$-base. Since we have two complex points in the fiber before the action of the braid monodromy and after it, we have to start by rotating the two rightmost points by $90^\circ$ counterclockwise, for representing the two complex points (see Step (1) in Figure 21). Then, we move the two complex points to be over the second to the left real point (see Step (2) there). Now, we apply the action of the local braid monodromy (Steps (3) and (4)), and then we return the two new complex points to the right side, and return them to the real axis by rotating them clockwise by $90^\circ$ (Step (5), see Remark 4.7).

Now, by van Kampen’s Theorem, we get the following induced relations from the new $g$-base (where $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ are the generators of the standard $g$-base):

1. $x_1 = x_5 x_4 x_3 x_4^{-1} x_5^{-1}$
2. $x_2 = x_5 x_4 x_3 x_2 x_3^{-1} x_4^{-1} x_5^{-1}$
3. $x_3 = x_5 x_4 x_3 x_2 x_1 x_2^{-1} x_3^{-1} x_4^{-1} x_5^{-1}$
4. $x_4 = x_5 x_4 x_3 x_2 x_1 x_5 x_4 x_5^{-1} x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1} x_5^{-1}$
5. $x_5 = x_6$
6. $x_6 = x_6 x_5 x_4 x_3 x_2 x_1 x_5 x_4 x_5 x_4^{-1} x_5^{-1} x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1} x_5^{-1} x_6^{-1}$

Relation (2) can be written as: $x_5 x_4 x_3 x_2 = x_2 x_5 x_4 x_3$. Using Relation (1), Relation (3) becomes: $x_5 x_4 x_3 x_2 x_5 x_4 = x_5 x_4 x_3 x_2 x_5 x_4 x_3$.

By Relation (1) and some cancellations, Relation (4) becomes:

$$x_4 = x_5 x_4 x_3 x_2 x_5 x_4 x_3 x_4 x_5^{-1} x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1} x_5^{-1} x_6^{-1}.$$  

By Relation (3), we get:

$$x_4 = x_3 x_5 x_4 x_3 x_2 x_5 x_4 x_5^{-1} x_2^{-1} x_3^{-1} x_4^{-1} x_5^{-1} x_3^{-1}.$$
and hence we have: $x_4 x_3 x_5 x_4 x_3 x_2 x_5 = x_3 x_5 x_4 x_3 x_2 x_5 x_4$.

Now, we will show that Relation (6) is redundant. First, this relation can be written as: $x_5 x_4 x_3 x_2 x_1 x_5 x_4 = x_4 x_3 x_2 x_1 x_5 x_4 x_5$. By Relation (1) and some cancellations, we have: $x_5 x_4 x_3 x_2 x_5 x_4 x_3 x_5 = x_4 x_3 x_2 x_5 x_4 x_3 x_5$. By Relation (3), we have: $x_4 x_3 x_4 x_2 x_5 x_4 x_3 = x_4 x_3 x_2 x_5 x_4 x_3 x_5$. By Relation (2), this relation is equal to Relation (4), and hence Relation (6) is redundant.

Therefore, we get the following result for the set of relations for the singular point:

**Corollary 5.6.** The singular point presented locally by $xy(y + x^2)(y - x^2) = 0$ has the following set of induced relations:

1. $x_5 x_4 x_3 x_2 = x_2 x_5 x_4 x_3$
2. $x_3 x_5 x_4 x_3 x_2 x_5 x_4 x_3 = x_4 x_3 x_4 x_3 x_2 x_5 x_4 x_3 = x_4 x_3 x_5 x_4 x_3 x_2 x_5$
(3) \( x_1 = x_5 x_4 x_3 x_4^{-1} x_5^{-1} \)
(4) \( x_5 = x_6 \)

where \( \{x_1, x_2, x_3, x_4, x_5, x_6\} \) are the generators of the standard \( g \)-base.

One should notice here that if we delete the generator \( x_2 \) which corresponds to the vertical line that intersects the three tangented components, we get the following set of relations:

\[
(x_5 x_4 x_3)^2 = (x_4 x_3 x_5)^2 = (x_3 x_5 x_4)^2,
\]

as expected (see Section 4.3). Similarly, if we delete the generator \( x_4 \) which corresponds to the tangent line, we get the relations of the singular point in Section 4.3 as expected.

Since the Lefschetz diffeomorphism of the singular point is obtained by computing the action only on half of the unit circle (from \( t = \frac{1}{2} \) to \( t = 1 \)), we have the following corollary:

**Corollary 5.7.** The Lefschetz diffeomorphism of the singular point presented locally by \( xy(y + x^2)(y - x^2) = 0 \) is a \( 90^\circ \) counterclockwise rotation of the points correspond to \( y = 1, -1, i, -i \) around \( y = 0 \), and then the point corresponds to \( y = 2 \) do a half-twist with a block which consists of all the other points.

5.2. Two tangented conics with two intersecting lines. Although the following two types are almost a rotation of each other, we need them both, since the second type includes also “hidden” branch points inside the singularity, as we had in Section 4.3

5.2.1. First type. The local equation of the singularity of the first type is \( (2x + y)(2x - y)(y + x^2)(y - x^2) = 0 \) (see Figure 22).

![Figure 22. The singularity of \((2x + y)(2x - y)(y + x^2)(y - x^2) = 0\) at \((0, 0)\)](image)

For computing the braid monodromy, we take a loop around \( x = 0 \) and look what happens to the points of the curve \((2x + y)(2x - y)(y + x^2)(y - x^2) = 0\) in the fibers over this loop.

Let \( x = e^{2\pi i t} \) where \( 0 \leq t \leq 1 \). For \( t = 0 \) we have that \( x = 1 \) and the points of the curve in the fiber over \( x = 1 \) are \( y = 1, -1, 2 \) and
\( y = -2 \). For \( t = \frac{1}{2} \), we have \( x = -1 \), and the points of the curve in the fiber over \( x = -1 \) are \( y = 1, -1, -2 \) and \( y = 2 \). The point \( y = 1 \) in the fiber \( x = -1 \) corresponds to the point \( y = 1 \) in the fiber \( x = 1 \), and similarly the point \( y = -1 \) in the fiber \( x = -1 \) corresponds to the point \( y = -1 \) in the fiber \( x = 1 \). The point \( y = 2 \) in the fiber \( x = -1 \) corresponds to the point \( y = -2 \) in the fiber \( x = 1 \), and similarly the point \( y = -2 \) in the fiber \( x = -1 \) corresponds to the point \( y = 2 \) in the fiber \( x = 1 \). Hence, we get that from \( t = 0 \) to \( t = \frac{1}{2} \), the points \( y = 1 \) and \( y = -1 \) do a counterclockwise full-twist and the points \( y = 2 \) and \( y = -2 \) do a counterclockwise half-twist around the points \( y = 1, -1 \). If we continue to \( t = 1 \), the points \( y = 1 \) and \( y = -1 \) do two counterclockwise full-twists and the points \( y = 2 \) and \( y = -2 \) do a counterclockwise full-twist around the points \( y = 1, -1 \) together.

Hence we have the following result:

**Theorem 5.8.** The local braid monodromy of the singularity presented locally by the equation \((2x + y)(2x - y)(y + x^2)(y - x^2) = 0\) is: two points (correspond to \( y = 1 \) and \( y = -1 \)) do two counterclockwise full-twists and the two other points (corresponds to \( y = 2 \) and \( y = -2 \)) do a counterclockwise full-twist around them.

Now, we want to compute the induced relations from the braid monodromy of the singular point. By van Kampen’s Theorem, we compute the \( g \)-base obtained by applying the action induced by the local braid monodromy of the singular point on the standard \( g \)-base. This is shown in Figure [23]. In the first step, we perform two counterclockwise full-twists of point 2 and 3 which represent \( y = -1 \) and \( y = 1 \) respectively. In the second step, we perform a counterclockwise full-twist of points 1 and 4 (which represent \( y = 2 \) and \( y = -2 \)) around the points 2 and 3.

Now, by van Kampen’s Theorem, we get the following induced relations from the new \( g \)-base (where \( \{x_1, x_2, x_3, x_4\} \) are the generators of the standard \( g \)-base):

1. \( x_1 = x_4x_3x_2x_1x_2^{-1}x_3^{-1}x_4^{-1} \)
2. \( x_2 = x_4x_3x_2x_1x_3x_2x_3^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1} \)
3. \( x_3 = x_4x_3x_2x_1x_3x_2x_3^{-1}x_2^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1} \)
4. \( x_4 = x_4x_3x_2x_1x_4^{-1}x_2^{-1}x_3^{-1}x_4^{-1} \)

From Relation (1), we get \( x_4x_3x_2x_1 = x_1x_4x_3x_2 \). From Relation (4), we get \( x_4x_3x_2x_1 = x_3x_2x_1x_4 \). Relation (2) gets the form \( x_4x_3x_2x_1x_3x_2 = x_2x_4x_3x_2x_1x_3 \).

Now, we will show that Relation (3) is redundant. We can write Relation (3) also as: \( x_4x_3x_2x_1x_3x_2x_3 = x_3x_4x_3x_2x_1x_3x_2 \). By Relation
Step (1)  \hspace{2cm} \text{Step (2)}

Figure 23. The g-base obtained from the standard g-base by the action of the local braid monodromy

(4), the left side becomes $x_3x_2x_1x_4x_3x_2x_3$. Now, we can cancel the leftmost $x_3$, to get: $x_2x_1x_4x_3x_2x_3 = x_4x_3x_2x_1x_3x_2$. By Relation (1), this relation is identical to Relation (2), and hence Relation (3) is redundant.

Hence, we get:

**Corollary 5.9.** The singular point presented locally by the equation $(2x+y)(2x-y)(y+x^2)(y-x^2) = 0$ has the following induced relations:

$$x_4x_3x_2x_1 = x_1x_4x_3x_2 = x_3x_2x_1x_4 \quad ; \quad x_4x_3x_2x_1x_3x_2 = x_2x_4x_3x_2x_1x_3$$

where $\{x_1, x_2, x_3, x_4\}$ are the generators of the standard g-base.

If we delete the generator $x_1$ which corresponds to the line $2x+y = 0$, we get the following set of relations:

$$x_4x_3x_2 = x_3x_2x_4 \quad ; \quad x_4x_3x_2x_3x_2 = x_2x_4x_3x_2x_3$$

as expected (see Section 4.2). We get similar results if we delete the generator $x_4$ which corresponds to the line $2x-y = 0$ (see Section 4.1).

Since the Lefschetz diffeomorphism is a half of the action of the braid monodromy, we have that:
Corollary 5.10. The Lefschetz diffeomorphism of the singular point presented locally by \((2x+y)(2x-y)(y+x^2)(y-x^2) = 0\) is a counterclockwise full-twist of the points \(y = 1\) and \(y = -1\), and a counterclockwise half-twist of the points \(y = 2\) and \(y = -2\) around the points \(y = 1, -1\).

5.2.2. Second type. The local equation of the singularity of the second type is \(y(x+2y)(y^2+x)(y^2-x) = 0\) (see Figure 24).

Figure 24. The singularity of \(y(x+2y)(y^2+x)(y^2-x) = 0\) at \((0,0)\)

There is a major difference between this type of singularity and the previous one: In this singularity there are two “hidden” branch points too (as in Section 4.3). That is, at any fiber, one has four real points and two complex points (i.e. complex level 2), and in each side of the singularity, the complex points belong to a different conic (since by the singularity, two real points become complex and two complex points become real).

For computing the braid monodromy in this case, we will use the same trick we have already used in Section 5.1.3: the two lines in the middle, \(x+2y = 0\) and \(y = 0\), can be thought for a moment as a “thick” line which is perpendicular to the tangent direction of the two tangented conics. We have already computed this case (Section 4.3): we got there that the braid monodromy is a 180° counterclockwise rotation of four points around one fixed point at the origin. After observing this, we should add into account that the “thick” line stands for two lines. Hence, the fixed point at the origin is now decomposed into two close points, which are doing a counterclockwise full-twist (like a usual node). To summarize, we have:

Theorem 5.11. The local braid monodromy of the singularity presented locally by the equation \(y(x+2y)(y^2+x)(y^2-x) = 0\) is: The four points correspond to \(y = 1, -1, i\) and \(y = -i\) do a 180° counterclockwise rotation, and the two points in the center do a counterclockwise full-twist (see Figure 25).

Now, we want to compute the induced relations of this singular point. By van Kampen’s Theorem, we compute the g-base obtained by applying the action induced by the local braid monodromy of the singular
point on the standard g-base. Since we have two complex points in the fiber before the action of the braid monodromy and after it, we have to start by rotating the two rightmost points by 90° counterclockwise, for representing the two complex points (see Step (1) in Figure 25). Then, we move the two complex points to be over the two middle real points (see Step (2) there). Now, we apply the action of the local braid monodromy (Steps (3) and (4)), and then we return the two new complex points to the right side, and return them to the real axis by rotating them by 90° clockwise (Step (5), see Remark 4.7).

Now, by van Kampen’s Theorem, we get the following induced relations from the new g-base (where \(\{x_1, x_2, x_3, x_4, x_5, x_6\}\) are the generators of the standard g-base):

1. \(x_1 = x_5 x_4 x_5^{-1}\)
2. \(x_2 = x_5 x_4 x_3 x_2 x_3^{-1} x_4^{-1} x_5^{-1}\)
3. \(x_3 = x_5 x_4 x_3 x_2 x_3 x_2^{-1} x_4^{-1} x_5^{-1}\)
4. \(x_4 = x_5 x_4 x_3 x_2 x_1 x_2^{-1} x_3^{-1} x_4^{-1} x_5^{-1}\)
5. \(x_5 = x_6\)
6. \(x_6 = x_6 x_5 x_4 x_3 x_2 x_1 x_5^{-1} x_2^{-1} x_3^{-1} x_4^{-1} x_5^{-1} x_6^{-1}\)

From Relation (2), we get \(x_5 x_4 x_3 x_2 = x_2 x_5 x_4 x_3\). Using this relation, Relation (3) becomes: \(x_3 = x_2 x_5 x_4 x_3^{-1} x_5^{-1} x_2^{-1}\), and hence \(x_2 x_5 x_4 x_3 = x_3 x_2 x_5 x_4\). Using Relation (2) again, we get \(x_5 x_4 x_3 x_2 = x_3 x_2 x_5 x_4\).

By Relation (1), Relation (4) can be written: \(x_4 x_5 x_4 x_3 x_2 x_5 = x_5 x_4 x_3 x_2 x_5 x_4\)

Now, we will show that Relation (6) is redundant. Since \(x_5 = x_6\), we can simplify Relation (6) to the following form: \(x_5 x_4 x_3 x_2 x_1 = x_4 x_3 x_2 x_1 x_5\). Now, by Relation (1), we get: \(x_5 x_4 x_3 x_2 x_5 x_4^{-1} = x_4 x_3 x_2 x_5 x_4^{-1} x_5\). By some simplifications and Relation (3), we get Relation (4), and hence Relation (6) is redundant.

Therefore, we get the following result:

**Corollary 5.12.** The singular point presented locally by the equation \(y(x + 2y)(y^2 + x)(y^2 - x) = 0\) has the following set of relations:

1. \(x_1 x_4 x_3 x_2 = x_2 x_5 x_4 x_3 = x_3 x_2 x_5 x_4\)
2. \(x_4 x_5 x_4 x_3 x_2 x_5 = x_5 x_4 x_3 x_2 x_5 x_4\)
Figure 26. The g-base obtained from the standard g-base by the action of the local braid monodromy

\begin{align*}
  (3) & \quad x_1 = x_5 x_4 x_5^{-1} \\
  (4) & \quad x_5 = x_6
\end{align*}

where \( \{ x_1, x_2, x_3, x_4, x_5, x_6 \} \) are the generators of the standard g-base.

If we delete the generators \( x_2 \) or \( x_3 \) which correspond to the lines \( y = 0 \) or \( x + 2y = 0 \) respectively, we get the set of relations as in Section 4.3 as expected.

Since the Lefschetz diffeomorphism is a half of the action of the braid monodromy, we have that:

**Corollary 5.13.** The Lefschetz diffeomorphism of the singular point presented locally by \( y(x+2y)(y^2+x)(y^2-x) = 0 \) is a counterclockwise 90° rotation of the four points around the center (which consists of two
points), and then a counterclockwise half-twist of the two points in the center.

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