ALMOST EVERYWHERE CONVERGENCE OF FOURIER SERIES ON COMPACT CONNECTED LIE GROUPS

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Abstract. We consider the open problem: Does every square-integrable function $f$ on a compact, connected Lie group $G$ have an almost everywhere convergent Fourier series? We prove a general theorem from which it follows that if the integral modulus of continuity of $f$ is $O(t^\alpha)$ for some $\alpha > 0$ then the Fourier series of $f$ converges almost everywhere on $G$. In particular, the Fourier series of any $\alpha$-Hölder continuous function on $G$ converges almost everywhere. On the other hand, we show that to each countable subset $E$ of $G = SU(2)$ and each $0 < \alpha < 1$ there corresponds an $\alpha$-Hölder continuous function on $SU(2)$ whose Fourier series diverges on $E$.

1. Introduction

The Peter-Weyl theorem suggests the study of the formal Fourier series $\sum d_\lambda(\chi_\lambda \ast f)$ of a square-integrable function $f$ on a compact, connected Lie group $G$. Here the sum is over the equivalence classes of continuous irreducible unitary representations of $G$, $d_\lambda$ is the degree of the representation, and $\chi_\lambda$ is its character. The vast literature of Fourier analysis on $G$ is primarily concerned with mean convergence or divergence of the Fourier series of $f \in L^p(G)$ (e.g. [12, 15, 19, 27, 28]), uniform or absolute convergence of the partial sums if $f$ is smooth (e.g. [7, 15, 17, 21, 22, 24, 25, 29, 30]), almost everywhere convergence or divergence of the partial sums if $f$ is a central function in $L^p(G)$ (e.g. [9, 13, 20]), and uniform, mean, or almost everywhere summability of the partial sums if $f$ belongs to various subspaces of $L^1(G)$ (e.g. [5, 35, 15, 32]). The aim of this work is to advance the study of almost everywhere convergence or divergence of Fourier partial sums of nonsmooth, possibly noncentral functions in $L^2(G)$.

Relying on Jackson’s theorem for compact, connected Lie groups [4] and a general version of the Rademacher-Menshov theorem [18], we show that if $f$ in $L^2(G)$ has an integral modulus of continuity $\Omega(f, \cdot)$ satisfying

$$\int_0^1 \frac{\Omega^2(f, t)}{t} dt < \infty,$$

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then the sequences of polyhedral Fourier partial sums \( \{S_N f(x)\}_{N=1}^{\infty} \) and spherical Fourier partial sums \( \{\widetilde{S}_N f(x)\}_{N=1}^{\infty} \) converge to \( f(x) \) almost everywhere on \( G \). In particular, if \( f \) is an \( \alpha \)-Hölder continuous function on \( G \) for some \( \alpha > 0 \), or more generally if \( \Omega(f, t) = O(t^\alpha) \) for some \( \alpha > 0 \), then these Fourier partial sums of \( f \) converge to \( f(x) \) almost everywhere on \( G \).

On the other hand, consider the two-dimensional special unitary group \( G = SU(2) \). We show that to each \( \alpha \) in \((0,1)\) and each countable subset \( E \) of \( SU(2) \) there corresponds an \( \alpha \)-Hölder continuous function on \( SU(2) \) whose Fourier partial sums diverge at each \( x \) in \( E \). Since it is possible to arrange that such a set \( E \) is dense in \( SU(2) \), the Fourier partial sums of the corresponding function are divergent at infinitely many points in every nonempty open subset of \( SU(2) \), despite the fact that the Fourier partial sums of such a function converge almost everywhere on \( SU(2) \). It is worth noting in the case \( \alpha = 1 \), i.e. when \( f \) is Lipschitz continuous on \( SU(2) \), the Fourier partial sums of \( f \) converge uniformly to \( f \) on \( SU(2) \) [21][22].

It is an open problem whether

\[
\lim_{N \to \infty} S_N f(x) = f(x)
\]

holds almost everywhere for every \( f \) in \( L^2(G) \). A general theorem of Stanton and Tomas [28] for compact, connected, semisimple Lie groups \( G \) shows that if \( p < 2 \) then there correspond an \( f \) in \( L^p(G) \) and a subset \( E \) of \( G \) of full measure such that \((1.1)\) fails for all \( x \) in \( E \). However, Carleson’s celebrated proof of Lusin’s conjecture [3] guarantees \((1.1)\) holds almost everywhere when \( f \) is square-integrable on the circle group \( \mathbb{T} \), and this was extended to \( \mathbb{T}^n \) for \( n \geq 2 \) [11][26][31]. Furthermore, it follows from a result of Pollard [23] on Jacobi series that if \( f \) is a central function in \( L^p(SU(2)) \) for some \( p > 4/3 \), then \((1.1)\) holds almost everywhere. Finally, the Peter-Weyl theorem implies that to every compact, connected Lie group \( G \) and each \( f \) in \( L^2(G) \) there corresponds an increasing sequence \( \{N_j\} \) of positive integers such that

\[
\lim_{j \to \infty} S_{N_j} f(x) = f(x)
\]

for almost every \( x \) in \( G \). This lends some hope for a positive answer to the almost everywhere convergence problem \((1.1)\), as do the results in this paper.

2. Notation and Preliminaries

We use primarily the notation of [2]. Let \( G \) be a compact, connected Lie group with Haar measure \( \mu \), let \( T \) be a fixed maximal torus in \( G \), and let \( \mathfrak{g} \) and \( \mathfrak{t} \) be the Lie algebras of \( G \) and \( T \), respectively. Choose an inner product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \) which is invariant under the adjoint action of \( G \) on \( \mathfrak{g} \). (When \( G \) is semisimple we may choose \( \langle \cdot, \cdot \rangle \) to be \(-B(\cdot, \cdot)\) where \( B \) is the Killing form.) This provides inner products on \( \mathfrak{t} \) and \( \mathfrak{t}^* \) which are invariant under the Weyl group \( W = N(T)/T \). Let \( I = \{H \in \mathfrak{t} : \exp(H) = 1\} \) be the integral lattice, let \( I^* = \{\lambda \in \mathfrak{t}^* : \lambda(H) \in \mathbb{Z} \text{ for all } H \in I\} \) be the lattice
of integral forms, and let \( P \subset I^* \) be the set of real roots of \( G \) with basis \( \{ \alpha_1, \ldots, \alpha_l \} \), \( P_+ \) the set of positive roots, and \( C \subset t^* \) the corresponding Weyl chamber. The dual object \( \hat{G} \) of \( G \) is in one-to-one correspondence with \( \mathcal{C} \cap I^* \). (When \( G \) is semisimple and simply connected, there exist integral forms \( \rho_1, \ldots, \rho_l \in t^* \) satisfying \( 2\langle \rho_j, \alpha_k \rangle / \langle \alpha_k, \alpha_k \rangle = \delta_{jk} \) for \( j, k \in \{1, \ldots, l\} \) and \( \mathcal{C} \cap I^* = \{ n_1 \rho_1 + \cdots + n_l \rho_l : \text{each } n_j \in \mathbb{Z} \text{ and } n_j \geq 0 \}. \)

Fix \( \omega \in C \cap I^* \) and for \( N = 1, 2, 3, \ldots \) set \( \Lambda_N = \{ \lambda \in \overline{C} \cap I^* : \lambda \leq N\omega \} \). The associated polyhedral Dirichlet kernel on \( G \) is \( D_N = \sum_{\lambda \in \Lambda_N} d_{\lambda \chi} \), where \( N = 1, 2, 3, \ldots \), and the convolution products \( S_N f = D_N * f \) define the sequence of associated polyhedral partial sums for the Fourier series of \( f \in L^2(G) \).

Using the Weyl integration formula on \( G \) \cite{2}, the \( N \)th Fourier polyhedral partial sum of \( f \) at \( x \in G \) is given by

\[
S_N f(x) = (D_N * f)(x) = \int_G D_N(y)f(y^{-1}x)d\mu(y)
\]

\[
= \frac{1}{|W|} \int_T \det(E_{G/T} - \Ad_{G/T}(\theta^{-1})) \int_G D_N(y\theta^{-1}y^{-1}x)d\mu(y)d\theta
\]

\[
= \frac{1}{|W|} \int_T \eta(\theta)D_N(\theta)[Q_x f](\theta)d\theta
\]

where \( \eta(\theta) = \det(E_{G/T} - \Ad_{G/T}(\theta^{-1})) \) and \( [Q_x f](\theta) = \int_G f(y\theta^{-1}y^{-1}x)d\mu(y) \).

Let \( \pi(\lambda) \) denote the continuous, irreducible, unitary representation of \( G \) corresponding to \( \lambda \in \overline{C} \cap I^* \), let \( H_{\pi(\lambda)} \) be the finite dimensional subspace of \( L^2(G) \) generated by the coordinate functions of \( \pi(\lambda) \) \cite{16} p.24, and let \( P_\lambda f = d_\lambda(\chi \lambda * f) \) be the orthogonal projection of \( L^2(G) \) onto \( H_{\pi(\lambda)} \). Then

\[
S_N f(x) = \sum_{\lambda \in \Lambda_N} (P_\lambda f)(x) = \sum_{j=0}^N (\Gamma_j f)(x)
\]

with \( \Gamma_n f = \sum P_\lambda f \) where the sum is over all \( \lambda \in \overline{C} \cap I^* \) such that \( (n-1)\omega < \lambda \leq n\omega \). Let \( \delta_h f(x) = f(x) - f(h^{-1}x) \) be the difference operator on \( L^2(G) \) and let the integral modulus of continuity of \( f \in L^2(G) \) be given by

\[
\Omega(f, t) = \sup\{ \| \delta_h f \|_{L^2(G)} : h = \exp(X), X \in \mathfrak{g}, \text{ and } 0 < \|X\| \leq t \};
\]

here \( \| \cdot \| \) denotes the norm on \( \mathfrak{g} \) generated by the inner product \( \langle \cdot, \cdot \rangle \).

An alternate theory for almost everywhere convergence of Fourier series of \( f \in L^2(G) \) can be based on spherical partial sums (cf. \cite{3} and \cite{14}). Let

\[
\rho = \frac{1}{2} \sum_{\alpha \in P_+} \alpha
\]
and for $N = 1, 2, 3, \ldots$ define $\tilde{\Lambda}_N = \{ \lambda \in C \cap I^* : \| \lambda - \rho \| \leq N \}$, the spherical Dirichlet kernel $\tilde{D}_N = \sum_{\lambda \in \tilde{\Lambda}_N} d_{\lambda} \chi_{\lambda}$, and the spherical partial sums $
abla S_N f = \tilde{D}_N * f = \sum_{\lambda \in \tilde{\Lambda}_N} (P_\lambda f) = \sum_{j=0}^N (\tilde{\Gamma}_j f)$ with $\tilde{\Gamma}_n f = \sum P_\lambda f$ where the sum is over all $\lambda \in C \cap I^*$ such that $n - 1 < \| \lambda - \rho \| \leq n$. We see no essential difference between the behavior of polyhedral and spherical partial sums in the results of this work.

3. Almost everywhere convergence results

**Theorem 3.1.** Let $G$ be a compact, connected Lie group and $f \in L^2(G)$. If

$$\int_0^1 \frac{\Omega^2(f,t)}{t} dt < \infty,$$

then $S_N f(x) \to f(x)$ and $\nabla S_N f(x) \to f(x)$ for almost every $x \in G$.

**Proof.** We give the proof for polyhedral Fourier partial sums of $f$; the case of spherical Fourier partial sums is completely analogous. Note that by Jackson’s theorem for compact, connected Lie groups [4], there is a constant $B > 0$ such that

$$\int_0^1 \frac{\Omega^2(f,t)}{t} dt = \sum_{k=0}^{\infty} \int_{2^{-k}}^{2^{-(k+1)}} \frac{\Omega^2(f,t)}{t} dt$$

$$\geq \log(2) \sum_{k=1}^{\infty} \Omega^2(f, 2^{-k})$$

$$\geq B \log(2) \sum_{k=1}^{\infty} (E_{2^k} f)$$

where

$$E_M(f) = \inf \{ \| f - P \|_{L^2(G)} : P \in \bigoplus_{\lambda \in \Lambda_M} H_{\pi(\lambda)} \}$$

is the best approximation of $f$ in $\bigoplus_{\lambda \in \Lambda_M} H_{\pi(\lambda)}$. Since

$$E_{2^k}^2(f) = \| f - S_{2^k} f \|_{L^2(G)}^2$$

$$= \sum_{j=2^{k+1}}^{\infty} \| \tilde{\Gamma}_j f \|_{L^2(G)}^2,$$

it follows that

$$\sum_{k=1}^{\infty} E_{2^k}^2(f) = \sum_{j=2}^{\infty} \| \log_2(j) \|_{L^2(G)}^2$$

$$\geq \frac{\log_2(e)}{2} \sum_{j=2}^{\infty} \log(j) \| \tilde{\Gamma}_j f \|_{L^2(G)}^2.$$
and therefore
\[
\Omega^2(f,t) \geq B \sum_{j=2}^{\infty} \log(j) \|G_j f\|^2_{L^2(G)}.
\]
Consequently, a general version of the Rademacher-Menshov theorem [18] implies
\[
\lim_{N \to \infty} S_N f(x) = \lim_{N \to \infty} \sum_{j=0}^{N} (G_j f)(x) = f(x)
\]
for almost every \(x \in G\).

It should be noted that in [10], Dai established almost everywhere convergence results for Fourier-Laplace series of functions on spheres which are analogous to the preceding theorem.

**Corollary 3.2.** Let \(f \in L^2(G)\). If \(\Omega(f,t) = O(t^\alpha)\) for some \(\alpha > 0\), then \(S_N f(x) \to f(x)\) and \(\tilde{S}_N f(x) \to f(x)\) for almost every \(x \in G\).

**Corollary 3.3.** If \(f \in Lip_\alpha(G)\) for some \(\alpha > 0\), then \(S_N f(x) \to f(x)\) and \(\tilde{S}_N f(x) \to f(x)\) for almost every \(x \in G\).

4. Divergence of Fourier partial sums on a countable subset

To begin this section, we review notation and results of [2, p.84f] and [16, p.125f]. Equip the two-dimensional special unitary group \(SU(2)\) with the left and right translation invariant metric \(d\) given by
\[
d(x,y) = \sqrt{\frac{1}{2} \text{tr}((x-y)(x-y)^*)}.
\]
Let \(0 < \alpha < 1\) and let \(f\) be a real function on \(SU(2)\). If there exists a number \(M > 0\) such that
\[
|f(x) - f(y)| \leq Md^\alpha(x,y)
\]
for all \(x, y \in SU(2)\), then \(f\) is an \(\alpha\)-Hölder continuous function on \(SU(2)\) and we write \(f \in Lip_\alpha(SU(2))\). A real function \(f\) on \(SU(2)\) is central if, for \(\mu\)-almost every \(x \in SU(2)\), \(f(yxy^{-1}) = f(x)\) for all \(y \in SU(2)\). In particular, since every \(x \in SU(2)\) is diagonalizable via a similarity transformation:
\[
yxy^{-1} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \omega(\theta),
\]
where \(y \in SU(2)\) and \(e^{i\theta}\) are the eigenvalues of \(x\), it follows that if \(f\) is central then for \(\mu\)-almost every \(x \in SU(2)\),
\[
f(x) = f(\omega(\theta))
\]
with \(\theta \in [0, \pi]\). With this notation, the Weyl integral formula is explicitly
\begin{equation}
\int_{SU(2)} g(x) d\mu(x) = \frac{2}{\pi} \int_{0}^{\pi} g(\omega(\theta)) \sin^2(\theta) d\theta
\end{equation}
when $g$ is a central function in $L^2(SU(2))$.

Let $\{\pi_n\}_{n=0}^\infty$ denote the family of all (inequivalent) continuous, irreducible, unitary representations of $SU(2)$; here $\pi_n$ has dimension $n + 1$ and its character $\chi_n = \text{trace}(\pi_n)$ is the continuous central function on $SU(2)$ given by

$$\chi_n(x) = \chi_n(\omega(\theta)) = \frac{\sin((n + 1)\theta)}{\sin(\theta)}$$

where $e^{\pm i\theta}$ are the eigenvalues of $x$. It follows that the Dirichlet kernel $\{D_N\}_{N=0}^\infty$ on $SU(2)$ is the sequence of continuous central functions given by

$$(4.2) \quad D_N(x) = D_N(\omega(\theta)) = \sum_{n=0}^{N} (n + 1)\chi_n(\omega(\theta)) = \frac{-1}{2\sin(\theta)} D_{n+1}(\theta)$$

where

$$D_n(t) = 1 + 2\sum_{j=1}^{n} \cos(jt) = \frac{\sin((2n+1)t/2)}{\sin(t/2)}$$

is the Dirichlet kernel on $[-\pi, \pi]$.

**Theorem 4.1.** Let $\alpha \in (0, 1)$ and let $\{x_i\}_{i=1}^\infty$ be any countable subset of $SU(2)$. Then there exists a function $f \in Lip_\alpha(SU(2))$ such that

$$\sup_{N \geq 1} |S_N f(x_i)| = \infty$$

for all $i = 1, 2, 3, \ldots$.

**Proof.** Observe that for $\alpha \in (0, 1)$, $Lip_\alpha(SU(2))$ is a Banach space with norm

$$\|f\|_{Lip_\alpha(SU(2))} = \sup_{x \in SU(2)} |f(x)| + \sup_{x, y \in SU(2), x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|_\alpha}.$$

Fix $x \in SU(2)$ and $n \in \mathbb{N}$, and set $\Phi_n^x(f) = S_n f(x)$. Each $\Phi_n^x$ is a bounded linear functional on $Lip_\alpha(SU(2))$ of norm

$$\|\Phi_n^x\| = \sup \{|S_n f(x) : f \in Lip_\alpha(SU(2)), \|f\|_{Lip_\alpha(SU(2))} \leq 1\} \leq \|D_n\|_{L^1(SU(2))}.$$

Specializing to the case when $x = e$, the identity matrix in $SU(2)$, and $f \in L^2(SU(2))$ is central, we have

$$\Phi_n^e(f) = (f \ast D_n)(e) = \int_{SU(2)} f(y) D_n(y) d\mu(y)$$

$$= \frac{1}{\pi} \int_{0}^{\pi} f(\omega(\theta)) D_n(\omega(\theta)) \sin^2(\theta) d\theta$$

by (4.1). It then follows from (4.2) that

$$\Phi_n^e(f) = \frac{1}{\pi} \int_{0}^{\pi} f(\omega(\theta)) \cos^2\left(\frac{\theta}{2}\right) D_{n+1}(\theta) d\theta$$

$$- \frac{(2n + 3)}{\pi} \int_{0}^{\pi} f(\omega(\theta)) \cos\left(\left(n + \frac{3}{2}\right)\theta\right) \cos\left(\frac{\theta}{2}\right) d\theta.$$
For \( n \geq 2 \), the absolute maxima and minima of the function \( h_n(\theta) = \cos\left(\left(n + \frac{3}{2}\right) \theta\right) \) on \([0, \pi]\) occur at the endpoints of the intervals

\[
I_k = \left[ \frac{2k\pi}{2n+3}, \frac{2(k+1)\pi}{2n+3} \right]
\]

where \( k \in \{0, 1, 2, \ldots, n\} \). Let \( g_n \) be the sawtooth function on \([0, \pi]\) determined by \( g_n\left(\frac{2k\pi}{2n+3}\right) = (-1)^k \) for \( 0 \leq k \leq n+1 \), \( g_n(\pi) = 0 \), and \( g_n \) is piecewise linear between these points. Define a central function \( f_n \) on \( SU(2) \) by \( f_n(\omega(\theta)) = g_n(\theta) \) for \( \theta \in [0, \pi] \). It is easy to see that each \( f_n \) belongs to \( \text{Lip}_\alpha(SU(2)) \); in fact,

\[
\frac{|f_n(x) - f_n(y)|}{d^\alpha(x, y)} \leq \left( \frac{\pi}{2} \right)^\alpha \left( \frac{2\pi}{2n+3} \right)^{1-\alpha} \leq \pi
\]

for all distinct matrices \( x \) and \( y \) in \( SU(2) \).

Since \( g_n(\theta) \cos\left(\left(n + \frac{3}{2}\right) \theta\right) \geq g_n(\theta) \geq 0 \) on each interval \( I_k \) and on \([\frac{2(n+1)\pi}{2n+3}, \pi]\), and since the function \( \theta \mapsto \cos(\theta/2) \) is positive and decreasing on \([0, \pi]\),

\[
\int_{I_k} f_n(\omega(\theta)) \cos\left(\left(n + \frac{3}{2}\right) \theta\right) \cos(\theta/2) d\theta \geq \int_{I_k} g_n^2(\theta) \cos(\theta/2) d\theta
\]

\[
\geq \cos\left( \frac{(k+1)\pi}{2n+3} \right) \int_{I_k} g_n^2(\theta) d\theta
\]

\[
= \frac{2\pi}{3(2n+3)} \cos\left( \frac{(k+1)\pi}{2n+3} \right)
\]

for all \( k \in \{0, 1, 2, \ldots, n\} \) and

\[
\int_{\frac{2(n+1)\pi}{2n+3}}^{\pi} f_n(\omega(\theta)) \cos\left(\left(n + \frac{3}{2}\right) \theta\right) \cos(\theta/2) d\theta \geq 0.
\]

Adding these inequalities we obtain

\[
\int_0^{\pi} f_n(\omega(\theta)) \cos\left(\left(n + \frac{3}{2}\right) \theta\right) \cos\left( \frac{\theta}{2} \right) d\theta \geq \frac{2}{3} \left( \frac{\pi}{2n+3} \right)^n \sum_{k=1}^{n+1} \cos\left( \frac{k\pi}{2n+3} \right)
\]

\[
= \frac{2}{3} \left( \frac{\pi}{2n+3} \right) \left\{ D_{n+1}\left( \frac{\pi}{2n+3} \right) - 1 \right\},
\]

and hence

\[
\left| \frac{(2n+3)}{\pi} \int_0^{\pi} f_n(\omega(\theta)) \cos\left(\left(n + \frac{3}{2}\right) \theta\right) \cos\left( \frac{\theta}{2} \right) d\theta \right| \geq \frac{2}{3} \left\{ D_{n+1}\left( \frac{\pi}{2n+3} \right) - 1 \right\}.
\]
Because the function $\theta \mapsto f_n(\omega(\theta)) \cos^2(\theta/2)$ is uniformly bounded by 1 on $[0, \pi]$,
\[
\left| \frac{1}{\pi} \int_0^\pi f_n(\omega(\theta)) \cos^2(\theta/2) D_{n+1}(\theta) \, d\theta \right| \leq \frac{1}{\pi} \int_0^\pi |D_{n+1}(\theta)| \, d\theta \leq \frac{4}{\pi^2} \log(n + 1) + o(1)
\]
as $n \to \infty$. Consequently
\[
\frac{\|\Phi_n^\varepsilon(f_n)\|}{\|f_n\|_{\text{Lip}_\alpha(SU(2))}} \geq \frac{\frac{2}{3} \left\{ D_{n+1} \left( \frac{\pi}{2n+3} \right) - 1 \right\} - \frac{4}{\pi} \log(n + 1) + o(1)}{1 + \frac{\pi}{2}}.
\]
But $D_{n+1} \left( \frac{\pi}{2n+3} \right) = \left( \sin \left( \frac{\pi}{2(2n+3)} \right) \right)^{-1} \geq \frac{2(2n+3)}{\pi}$ and hence
\[
\|\Phi_n^\varepsilon\| = \sup \left\{ \frac{|\Phi_n^\varepsilon(f)|}{\|f\|_{\text{Lip}_\alpha(SU(2))}} : f \in \text{Lip}_\alpha(SU(2)), \ f \neq 0 \right\}
\]
is asymptotically bounded below by
\[
\frac{\frac{2}{3} (2n + 3) - \frac{4}{\pi} \log(n + 1)}{1 + \frac{\pi}{2}}
\]
as $n \to \infty$. Thus the sequence of bounded linear functionals
\[
\Phi_n^\varepsilon(f) = S_n f(e)
\]
is not uniformly bounded on the Banach space $\text{Lip}_\alpha(SU(2))$ as $n \to \infty$. By the uniform boundedness principle
\[
\sup_{n \geq 1} |S_n f(e)| = \infty
\]
for all $f$ belonging to some dense $G_\delta$ set in $\text{Lip}_\alpha(SU(2))$.

If $z \in SU(2)$, define the left translation operator $L_z$ on $\text{Lip}_\alpha(SU(2))$ by $L_z f(y) = f(zy)$ for all $y \in SU(2)$. For each element of the countable subset $\{x_i\}_{i=1}^\infty$ of $SU(2)$ and each $n \geq 1$, observe that
\[
\frac{|\Phi_n^\varepsilon(L_{x_i}^{-1} f_n)|}{\|L_{x_i}^{-1} f_n\|_{\text{Lip}_\alpha(SU(2))}} = \frac{|\Phi_n^\varepsilon(f_n)|}{\|f_n\|_{\text{Lip}_\alpha(SU(2))}},
\]
so there corresponds a dense $G_\delta$ subset $E_{x_i}$ of $\text{Lip}_\alpha(SU(2))$ such that
\[
\sup_{n \geq 1} |S_n f(x_i)| = \infty
\]
for all $f \in E_{x_i}$. By the Baire category theorem $E = \bigcap_{i=1}^\infty E_{x_i}$ is dense in $\text{Lip}_\alpha(SU(2))$. In particular, $E$ is nonempty and any $f \in E$ gives the desired conclusion. $\square$

A general theorem of Belen’kii [11] guarantees that if $F : [-1,1] \to \mathbb{R}$ satisfies a Dini-Lipschitz condition and the Fourier-Jacobi partial sums
\[
\{s_N^{(\alpha,\beta)}(F; \pm 1)\}_{N=0}^\infty
\]
at \( \pm 1 \) converge for some \( \alpha > -1 \) and \( \beta > -1 \), then the corresponding Fourier-Jacobi series of \( F \) converges uniformly to \( F \) on \([-1, 1]\). This leads directly to the following result on \( SU(2) \).

**Theorem 4.2.** Let \( f \) be a central function in \( \text{Lip}_\alpha(SU(2)) \) for some \( \alpha \in (0, 1) \). Then:

(a) \( S_N f(x) \to f(x) \) uniformly outside any open set containing \( \{e, -e\} \);

(b) \( S_N f(x) \to f(x) \) uniformly on \( SU(2) \) if \( \{S_N f(\pm e)\}_{N=0}^\infty \) converge.

This suggests the question: Can the word “central” be deleted from the hypothesis of Theorem 4.2 and still obtain conclusions (a) and (b)? Theorem 4.1 shows that there is no possibility of such an analogue of Theorem 4.2 for general functions in \( \text{Lip}_\alpha(SU(2)) \) for some \( \alpha \in (0, 1) \). The points of divergence for the Fourier partial sums of such a noncentral function need no longer be at the “poles” \( \pm e \) of the “sphere” \( SU(2) \). According to Theorem 4.1, points of divergence for \( \text{Lip}_\alpha(SU(2)) \) functions can be dense in \( SU(2) \).

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