STABILITY OF SOLUTIONS OF CERTAIN EXTENDED RICCI FLOW SYSTEMS

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Abstract. We consider four extended Ricci flow systems—that is, Ricci flow coupled with other geometric flows—and prove dynamical stability for certain classes of stationary solutions of these flows. The systems include Ricci flow coupled with harmonic map flow (studied abstractly and in the context of Ricci flow on warped products), Ricci flow coupled with both harmonic map flow and Yang-Mills flow, and Ricci flow coupled with heat flow for the torsion of a metric-compatible connection. The methods used to prove stability follow a program outlined by Guenther, Isenberg, and Knopf, which uses maximal regularity theory for quasilinear parabolic systems and a result of Simonett.

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1. Introduction

A fundamental problem in the study of differential equations is to determine the asymptotic behavior of solutions. This problem is central to the application of differential equations to Riemannian geometry. For example, Eells and Sampson demonstrated the existence of harmonic maps by proving that solutions of the harmonic map flow converge [6]. Hamilton placed strong restrictions on the topology
of three-manifolds admitting positive Ricci curvature by proving that solutions of Ricci flow converge [8] (see also [2,9]).

One way to phrase this problem is in terms of stability of stationary solutions to the system of equations in question: do solutions with initial data near a fixed point converge to that fixed point? For Ricci flow, whose fixed points include Einstein and Ricci-flat metrics, there are many stability results.

To mention a few of these results in the compact case, Ye proved that Einstein metrics with certain curvature pinching properties are stable [27]; Guenther, Isenberg and Knopf proved that certain flat and Ricci-flat metrics are stable [7], and some of these results were improved by Sezum [19]; using the results of Sezum, Dai, Wang, and Wei proved that Kähler-Einstein metrics with non-positive scalar curvature are stable [5]; Knopf and Young proved that hyperbolic space forms are stable [11]; Wu proved that complex hyperbolic space is stable [26]. We should note that these authors use various techniques and obtain stability relative to various topologies on the space of metrics.

The purpose of this paper is to describe stability of solutions of certain extended Ricci flow systems, which often arise from natural geometric contexts involving Riemannian manifolds with additional structure. First, we consider Ricci flow coupled with the harmonic map flow; see Section 2. If \( \phi : (\mathcal{M}, g) \rightarrow (\mathcal{N}, \gamma) \) is a map of Riemannian manifolds, the harmonic-Ricci flow is the coupled system

\[
\begin{align*}
\partial_t g &= -2 \mathrm{Rc} + 2c \nabla \phi \otimes \nabla \phi \\
\partial_t \phi &= \tau_{g,\gamma} \phi
\end{align*}
\]

where \( \tau_{g,\gamma} \phi \) is the harmonic map Laplacian (or tension field) of \( \phi \), \( d\phi \otimes d\phi \) is \( \phi^* \gamma \), and \( c \) is a (possibly time-dependent) coupling constant. This was introduced in the case \( \mathcal{N} = \mathbb{R} \) in [13], and the general case was addressed in [17]. This flow also arises naturally in certain contexts; see Section 2.1 for examples.

For this flow we demonstrate the stability of fixed points \((g, \phi)\), where \( g \) is an Einstein metric of negative curvature, and \( \phi \) is constant.

**Theorem 1.2.** Let \( \phi : (\mathcal{M}^n, g) \rightarrow (\mathcal{N}, \gamma) \) be a map of Riemannian manifolds. Suppose that \( \mathcal{M} \) is compact and orientable, \( g \) is Einstein with negative sectional curvature (assumed to be constant when \( n = 2 \)), and \( \phi \) is constant. Then for any \( \rho \in (0,1) \), there exists \( \theta \in (\rho,1) \) such that the following holds.

There exists a \((1 + \theta)\) little-Hölder neighborhood \( U \) of \((g, \phi)\) such that for all initial data \((\tilde{g}(0), \tilde{\phi}(0)) \in U\), the unique solution \((\tilde{g}(t), \tilde{\phi}(t))\) of curvature-normalized harmonic-Ricci flow (2.2) exists for all \( t \geq 0 \) and converges exponentially fast in the \((2 + \rho)\)-Hölder norm to a limit \((g_\infty, \phi_\infty)\). In this limit, \( \phi_\infty \) is constant, \( g_\infty = g \) when \( n \geq 3 \), and \( g_\infty \) has constant negative sectional curvature when \( n = 2 \).

We next consider Ricci flow on warped products; see Section 3. Warped products \( \mathbb{R} \times \mathcal{F}^m \), where \( \mathcal{F}^m \) is a positively-curved Einstein manifold, were used by Simon in a construction of metrics with pinching singularities [20]. More recently, Lott and Sezum studied Ricci flow on compact warped products \( \mathcal{M}^3 = B^2 \times S^1 \) and proved several stability-type results [16]. Tran has also considered Ricci flow on warped products \( B^n \times \mathcal{F}^m \) where \( \mathcal{F} \) is Ricci-flat [24]. Here, we consider fibers \((\mathcal{F}^m, \gamma)\) such that \( \gamma \) is \( \mu \)-Einstein. The resulting flow equations on \((B \times \mathcal{F}, g + e^{2\phi} \gamma)\) are

\[
\begin{align*}
\partial_t g &= -2 \mathrm{Rc} + 2m d\phi \otimes d\phi \\
\partial_t \phi &= \Delta \phi - \mu e^{-2\phi}
\end{align*}
\]
This flow is a modified version of [1.1] with a one-dimensional target. Not surprisingly, the behavior here depends strongly on the sign of $\mu$. When $\mu \leq 0$, we obtain a result similar to Theorem 1.2 above.

**Theorem 1.4.** Let $g = g + e^{\phi \gamma} \rho$ be a warped product metric on $\mathcal{M} = \mathcal{B} \times \mathcal{F}$, where $\mathcal{B}$ is orientable and compact, and $(\mathcal{F}, \gamma)$ is $\mu$-Einstein with $\mu \leq 0$. Suppose that $g$ is Einstein with negative sectional curvature (assumed to be constant when $n = 2$). Then for any $\rho \in (0, 1)$, there exists $\theta \in (\rho, 1)$ such that the following holds.

There exists a $(1 + \theta)$ little-Hölder neighborhood $\mathcal{U}$ of $g$ such that for all initial data $\mathcal{g}(0) \in \mathcal{U}$, the unique solution $\mathcal{g}(\tau)$ of curvature-normalized warped product Ricci flow (3.19) exists for all $t \geq 0$ and converges exponentially fast in the $(2 + \rho)$-Hölder norm to a limit metric $\mathcal{g}_{\infty} = g_{\infty} + e^{2\phi_{\infty}} \gamma$. In this limit, $\phi_{\infty} = \phi$, $g_{\infty} = g$ when $n \geq 3$, and $g_{\infty}$ has constant negative sectional curvature when $n = 2$.

Next, we consider the locally $\mathbb{R}^N$-invariant Ricci flow; see Section 4. This flow was introduced by Lott as a means for proving that if $(\mathcal{M}, g(t))$ is a compact, Type III Ricci flow solution with diameter $O(\sqrt{t})$, then the pull-back solution $(\mathcal{M}, \mathcal{g}(t))$ on the universal cover converges to a homogeneous Ricci soliton [15]. For this, Lott considered a class of “twisted” principal $\mathbb{R}^N$-bundles. Certain metrics on such bundles $\mathbb{R}^N \to \mathcal{M} \times \mathcal{N} \to B^n$ can be represented locally as $g = (g, A, G)$, where $g$ is a metric on the base, $A$ is an $\mathbb{R}^N$-valued 1-form corresponding to a connection on $\mathcal{M}$, and $G$ is an inner product on the fibers. Ricci flow on these locally $\mathbb{R}^N$-invariant metrics decomposes into a Ricci flow-type equation for $g$, a Yang Mills flow-type equation for $A$, and a heat-type equation for $G$:

$$
\partial_t g_{\alpha\beta} = - 2 R_{\alpha\beta} + \frac{1}{2} G^{ik} G_{ij} \nabla_\alpha G_{\beta k} + g^{\gamma\delta} G_{ij} (dA)^i_{\alpha\gamma} (dA)^j_{\beta\delta}, \\
(1.5) \quad \partial_t A^{i}_\alpha = - (\delta dA)^i_{\alpha} + G^{ij} G_{jk} (dA)^k_{\beta\alpha},
$$

$$
\partial_t G_{ij} = \Delta G_{ij} - G^{kl} \nabla_\alpha G_{ik} \nabla^\alpha G_{lj} - \frac{1}{2} g^{\alpha\delta} G_{ik} G_{jk} (dA)^k_{\alpha\beta} (dA)^l_{\gamma\delta}.
$$

An important ingredient in the proof of Lott’s theorem is a set of stability results for this system, proved by Knopf in the cases $N = 1$ or $N = 1, 2$. We extend some of those results to arbitrary dimensions, and to a more general class of fixed points.

**Theorem 1.6.** Let $g = (g, A, G)$ be a locally $\mathbb{R}^N$-invariant metric of the form (4.1) on a product $\mathbb{R}^N \times B^n$, where $B$ is compact and orientable. Suppose that $A$ vanishes and $G$ is constant, and that $g$ is Einstein with negative sectional curvature (assumed to be constant when $n = 2$). Then for any $\rho \in (0, 1)$, there exists $\theta \in (\rho, 1)$ such that the following holds.

There exists a $(1 + \theta)$ little-Hölder neighborhood $\mathcal{U}$ of $g$ such that for all initial data $\mathcal{g}(0) \in \mathcal{U}$, the unique solution $\mathcal{g}(\tau)$ of curvature-normalized locally $\mathbb{R}^N$-invariant Ricci flow (4.2) exists for all $t \geq 0$ and converges exponentially fast in the $(2 + \rho)$-Hölder norm to a limit metric $\mathcal{g}_{\infty} = (g_{\infty}, A_{\infty}, G_{\infty})$. In this limit, $A_{\infty}$ vanishes, $G_{\infty}$ is constant, $g_{\infty} = g$ when $n \geq 3$, and $g_{\infty}$ has constant negative sectional curvature when $n = 2$.

Finally, we consider the connection Ricci flow; see Section 5. This flow was introduced by Streets as a geometric interpretation of renormalization group flow on $(\mathcal{M} \times g)\ B$-field included, which takes the form of Ricci flow coupled with heat flow for a closed three-form [22] (see also [18, 23]). Here, $n \geq 3$. One can interpret the $B$-field strength as the torsion $\tau$ of a metric compatible connection,
and Ricci flow in this setting becomes Ricci flow for \( g \) coupled with heat flow for the torsion:

\[
\partial_t g = -2 \text{Rc} + \frac{1}{2} \mathcal{H} \\
\partial_t \tau = \Delta \tau
\]

where \( \mathcal{H}_{ij} = g^{pq} g^{rs} \tau_{ipr} \tau_{jqs} \). There are also certain other assumptions on \( \tau \) that we will make precise in Section 5.1.

We show that the flow is stable when the metric \( g \) is Einstein with negative sectional curvature and the connection is the Levi-Civita connection of \( g \), that is, the torsion vanishes.

**Theorem 1.8.** Let \((\mathcal{M}^n, g)\) be a compact, orientable Riemannian manifold with \( n \geq 3 \). Consider a metric-compatible connection \( \nabla \) on \( \mathcal{M} \) with torsion \( \tau \). Suppose that \( g \) is Einstein with negative sectional curvature and that \( \tau = 0 \). Then for any \( \rho \in (0, 1) \), there exists \( \theta \in (\rho, 1) \) such that the following holds.

There exists a \((1 + \theta)\)-little-Hölder neighborhood \( \mathcal{U} \) of \((g, \tau)\) such that for all initial data \((\tilde{g}(0), \tilde{\tau}(0)) \in \mathcal{U}\), the unique solution \((\tilde{g}(t), \tilde{\tau}(t))\) of curvature-normalized connection Ricci flow (5.1) exists for all \( t \geq 0 \) and converges exponentially fast in the \((2 + \rho)\)-Hölder norm to \((g, \tau) = (g, 0)\).

The proofs of these theorems follow the same general outline.

1. Modify the flow so that the fixed points are more easily studied. This involves pulling back the flow by diffeomorphisms, and the fixed points include Einstein metrics together with other objects relevant to the flow in question (e.g., maps or connections).
2. Compute the linearization of the modified flow, and prove linear stability at the fixed points. This actually involves a second modification (by a DeTurck trick) to make the flow strictly parabolic.
3. Prove dynamical stability by setting up the appropriate Hölder spaces and applying a theorem of Simonett. (See Appendix A for the statement of the theorem.)

The technique described here was introduced by Guenther, Isenberg, and Knopf [7], and has subsequently been used to prove several other results. For example, as mentioned before, Knopf proves stability for certain solutions of locally invariant \( \mathbb{R}^N \)-invariant Ricci flow [10]. Young considers Ricci flow coupled with Yang-Mills flow and proves stability of certain solutions [28]. Wu, as cited above, uses these methods to prove that complex hyperbolic space is stable under Ricci flow [26]. He also adapts the method for use in the non-compact setting.

Step (3) in this technique relies on the maximal regularity theory of Da Prato and Grisvard [4], which exploits the smoothing properties of quasilinear parabolic operators. The actual stability then follows from a (quite general) theorem of Simonett, which is based on this maximal regularity theory [21]. The theorem also has the feature of giving stability even in the presence of center manifolds. The first analysis of center manifolds in problems relating to Ricci flow appeared in [7].

**Remark 1.9.** All of these theorems are true when the Einstein manifold is replaced by a two-dimensional sphere with constant positive sectional curvature. Unfortunately, the techniques used here do not generalize to higher dimensions for positively curved manifolds. See [10] Remark 2].
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2. Harmonic-Ricci flow

2.1. Setup and examples. Let us provide background for the coupled flow \([1.1]\). Let \((\mathcal{M}, g)\) be a closed Riemannian manifold, with \((\mathcal{N}, \gamma)\) a closed target manifold. Let \(\phi : \mathcal{M} \to \mathcal{N}\) be a smooth map. The Levi-Civita covariant derivative \(\nabla^{TM}\) of the metric \(g\) on \(\mathcal{M}\) induces a covariant derivative \(\nabla^{TM^*}\) on the cotangent bundle, which satisfies

\[
\nabla_X^{TM^*} \omega(Y) = X(\omega(Y)) - \omega(\nabla_X^{TM} Y).
\]

By requiring a product rule and compatibility with the metric, we also have covariant derivatives on all tensor bundles

\[
T^p_q(\mathcal{M}) = (T^*\mathcal{M})^\otimes p \otimes (T\mathcal{M})^\otimes q.
\]

The Levi-Civita covariant derivative \(\nabla^{TN}\) of the metric \(\gamma\) on \(\mathcal{N}\) induces a covariant derivative \(\nabla^{\phi^*TN}\) on the pull-back bundle \(\phi^*TN \to \mathcal{M}\), given by

\[
\nabla^{\phi^*TN}_X Y = \nabla^{TN}_{\phi^*_X Y},
\]

for \(X \in C^\infty(T\mathcal{M})\) and \(Y \in C^\infty(T\mathcal{N})\). As before, we get a covariant derivative on all tensor bundles over \(\mathcal{M}\) of the form

\[
T^p_q(\mathcal{M}) \otimes T^r_s(\phi^*\mathcal{N}) = (T^*\mathcal{M})^\otimes p \otimes (T\mathcal{M})^\otimes q \otimes (\phi^*T^*\mathcal{N})^\otimes r \otimes (\phi^*TN)^\otimes s.
\]

We refer to them simply as \(\nabla\). Related quantities are decorated with the metric name, if necessary, e.g., \(g^n\). In local coordinates \((x^i)\) on \(\mathcal{M}\) and \((y^\lambda)\) on \(\mathcal{N}\),

\[
\nabla \phi = \phi_\ast = \partial_i \phi^\lambda \, dx^i \otimes \partial_\lambda|_\phi \in C^\infty(T^*\mathcal{M} \otimes \phi^*TN).
\]

Similarly, we have

\[
\nabla^2 \phi = (\partial_i \partial_j \phi^\lambda - g^{ij} \Gamma^k_{ij} \partial_k \phi^\lambda + (\nabla \phi)^\lambda_{\mu \nu} \partial_i \phi^\mu \partial_j \phi^\nu ) \, dx^i \otimes dx^j \otimes \partial_\lambda|_\phi
\]

\[
\in C^\infty(T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes \phi^*TN).
\]

The harmonic map Laplacian (or tension field) of \(\phi\) with respect to \(g\) and \(h\) is

\[
\tau_{g,\gamma} \phi = \text{tr}_g \nabla^2 \phi
\]

\[
= g^{ij} \left( \partial_i \partial_j \phi^\lambda - g^{ik} \Gamma^k_{ij} \partial_k \phi^\lambda + (\nabla \phi)^\lambda_{\mu \nu} \partial_i \phi^\mu \partial_j \phi^\nu \right) \partial_\lambda|_\phi
\]

\[
\in C^\infty(\phi^*TN).
\]

Additionally,

\[
\nabla \phi \otimes \nabla \phi = \phi_{\lambda \mu} \partial_i \phi^\lambda \partial_j \phi^\mu \, dx^i \otimes dx^j = \phi^* h
\]

is a symmetric \((2,0)\)-tensor on \(\mathcal{M}\), and we define

\[ S = \text{Rc} - c \nabla \phi \otimes \nabla \phi \]

where \(c = c(t) \geq 0\) is a coupling function.

Now we recall the flow \([1.1]\): if \(\phi : (\mathcal{M}, g) \to (\mathcal{N}, \gamma)\) is a map of Riemannian manifolds, the harmonic-Ricci flow is the coupled system

\[
\frac{\partial_t g}{\partial_t \phi} = -2S = -2 \text{Rc} + 2c \nabla \phi \otimes \nabla \phi
\]

\[
\frac{\partial_t \phi}{\partial_t \phi} = \tau_{g,\gamma} \phi
\]
We will call this hrf for short, although it is also sometimes called the \((RH)\_c\) flow. We will assume that \(c(t)\) is non-increasing. As mentioned above, this flow was introduced in [17] and is a generalization of one studied in [13].

Now we consider some examples of the flow. In studying expanding Ricci solitons on homogeneous spaces, Lott considered as a model a special type of vector bundle. Let \(\mathcal{M}\) be an \(\mathbb{R}^n\)-vector bundle with flat connection, flat metric \(G\) on the fibers, and Riemannian base \((B^n, g)\). Assume that the connection preserves fiberwise volume forms. Lott showed that the soliton equation becomes a pair of equations. One is a soliton-like equation for \(g\). The other is an equation for \(G\), which says that, interpreted as a map \(G : B \rightarrow (\text{SL}(N, \mathbb{R})/\text{SO}(N), \gamma)\), \(G\) is harmonic. Here, \(\gamma\) is the natural metric induced by \(G\); see (4.3).

In fact, more is true. Ricci flow on such bundles is the coupled flow
\[
\partial_t g = -2 \text{Rc} + \frac{1}{4} \nabla G \otimes \nabla G
\]
\[
\partial_t G = \tau_{g,\gamma} G
\]
which is the harmonic-Ricci flow on \((B, g)\) with map \(G\) and \(c = 1/8\); see [25]. This is in fact a special case of the coupled system considered in Section 4.

All known 3D and 4D homogeneous spaces admitting expanding Ricci solitons have this bundle structure, so the corresponding Ricci flow solutions are harmonic-Ricci flow solutions.

Ricci flow on a warped product \((\mathcal{M}^n \times S^1, g + e^{2u}d\theta)\), after modification by diffeomorphisms, takes a special form:
\[
\partial_t g = -2 \text{Rc} + 2 du \otimes du
\]
\[
\partial_t u = \Delta u
\]
This is harmonic-Ricci flow with target \(\mathbb{R}\) and \(c = 1\), and was studied by Lott and Sešum [16]. It is also a special case of the system considered in Section 3.

2.2. Stability. In this section we follow the outline given in the introduction to prove Theorem 1.2.

We transform the system into one whose fixed points include pairs \((g, \phi)\) with \(g\) Einstein and \(\phi\) constant. Suppose that \((\overline{g}(\overline{t}), \overline{\phi}(\overline{t}))\) is a solution of (1.1) (with time parameter \(\overline{t}\)). Let \(s(\overline{t})\) be a function with positive anti-derivative \(\sigma\), and consider
\[
(\overline{g}(\overline{t}), \overline{\phi}(\overline{t})) \mapsto (g(t), \phi(t)),
\]
where
\[
g = \sigma^{-1} \overline{g}, \quad \phi = \overline{\phi}, \quad t = \int_{\overline{t}_0}^{\overline{t}} \frac{dr}{\sigma(r)}
\]
for \(\overline{t}_0 \in I\), the interval of existence of the solution. A straightforward calculation shows that this transformation results in the modified flow
\[
\partial_t g = -2 \text{Rc} + 2e \nabla \phi \otimes \nabla \phi - sg
\]
\[
\partial_t \phi = \tau_{g,\gamma} \phi
\]
We want to describe the fixed points of this system, with the proper choice of \(s\).

Lemma 2.3. Suppose that \((\mathcal{M}^n, g_0)\) is compact and Einstein, with \(\text{Rc}(g_0) = \lambda g_0\). Let \(\phi_0 : (\mathcal{M}, g) \rightarrow (N, \gamma)\) be a constant map. Setting \(s = -2\lambda\) makes \((g_0, \phi_0)\) a stationary solution of (2.2)
With these choices, we call (2.2) the curvature-normalized harmonic-Ricci flow system.

Consider a fixed point \((g_0, \phi_0)\) of the flow (2.2) on \(M^n\). From Lemma (2.3), assume that \(g_0\) is Einstein with \(\text{Rc}(g_0) = \lambda g_0\) and \(\phi_0\) is constant. To analyze the stability near this fixed point, we must compute the linearization of the flow. Let \((\tilde{g}(\epsilon), \tilde{\phi}(\epsilon))\) be a variation of \((g_0, \phi_0)\) such that

\[
\begin{align*}
\tilde{g}(0) &= g_0, & \partial \epsilon \big|_{\epsilon=0} \tilde{g}(\epsilon) &= h, \\
\tilde{\phi}(0) &= \phi_0, & \partial \epsilon \big|_{\epsilon=0} \tilde{\phi}(\epsilon) &= \psi,
\end{align*}
\]

for some symmetric \((2, 0)\)-tensor \(h\) and variational vector field \(\psi \in C^\infty(\phi^*TN)\). More explicitly, \(\tilde{\phi}(x, \epsilon) = \exp_{\phi(x)}(\epsilon \psi(x))\). Let \(\Delta\) denote the Lichnerowicz Laplacian acting on symmetric \((2, 0)\)-tensor fields. Its components are

\[
\Delta h_{ij} = \Delta^\ell h_{ij} + \nabla_i (\delta h)_{j} + \nabla_i (\delta h)_{j} + \nabla_i \nabla_j \text{tr} g_0 h + 2 \lambda h_{ij}.
\]

Lemma 2.5. The linearization of (2.2) at a fixed point \((g_0, \phi_0)\) with \(\text{Rc}(g_0) = \lambda g_0\) and \(\phi_0\) constant acts on \((h, \psi)\) by

\[
\begin{align*}
\partial_t h_{ij} &= \Delta^\ell h_{ij} + \nabla_i (\delta h)_{j} + \nabla_i (\delta h)_{j} + \nabla_i \nabla_j \text{tr} g_0 h + 2 \lambda h_{ij}, \\
\partial_t \psi^\alpha &= \Delta \psi^\alpha,
\end{align*}
\]

where \(\Delta\) is the Lichnerowicz Laplacian and \(\Delta\) is the Laplacian acting on functions.

Proof. With a variation as in (2.4), we must compute

\[
\begin{align*}
\partial \epsilon \big|_{\epsilon=0} (\partial \tilde{g}(\epsilon)), & \quad \partial \epsilon \big|_{\epsilon=0} (\partial \tilde{\phi}(\epsilon)).
\end{align*}
\]

Such computations involve standard variational formulas for geometric objects like \(g^{-1}, \Gamma, \text{Rc}, \text{and } R\). See [3] Section 3.1, for example. The first equation is considered in [10] Lemma 3. For the second equation, using the coordinate expression for the tension field from (2.1), it is easy to see that

\[
\partial \epsilon \big|_{\epsilon=0} (\partial h \tilde{\phi}(\epsilon))^\alpha = \partial \epsilon \big|_{\epsilon=0} (\delta g, h \tilde{\phi}(\epsilon))^\alpha
\]

\[
= g^{ij} (\partial_i \partial_j \psi^\alpha - g_i \Gamma^k_{ij} \partial_k \psi^\alpha)
\]

\[
= \Delta \psi^\alpha,
\]

as desired. \(\Box\)

We next use the DeTurck trick to make the linearized system (2.2) strictly parabolic. That is, we pull back by diffeomorphisms generated by carefully chosen vector fields, which has the effect of subtracting a Lie derivative term from both equations in (2.2). To this end, define a vector field depending on \(g(t)\) by

\[
W^k = g^{ij} (\Gamma^k_{ij} - g_{ij} \Gamma^k_{ij}), \quad k = 1, \ldots, n.
\]

Let \(F_t\) be diffeomorphisms generated by \(W(t)\), with initial condition \(F_0 = \text{id}\). The one-parameter family \((F^*_t g(t), F^*_t \phi(t))\) is the solution of the curvature-normalized harmonic-Ricci DeTurck flow. A stationary solution \((g_0, \phi_0)\) of (2.2) with \(\text{Rc} = \lambda g_0\) and \(\phi_0\) constant is then also a stationary solution of the curvature-normalized harmonic-Ricci DeTurck flow.
Lemma 2.8. The linearization of the curvature-normalized harmonic-Ricci DeTurck flow at a fixed point \((g_0, \phi_0)\) with \(Rc = \lambda g_0\) is the autonomous, self-adjoint, strictly parabolic system

\[
\begin{align*}
\partial_t h &= L_0 h = \Delta h + 2\lambda h \\
\partial_t \phi &= L_1 \phi = \Delta \phi,
\end{align*}
\]

where \(L_1 = \Delta\) satisfies \((\Delta \psi)^a = \Delta (\psi^a)\).

Proof. The curvature-normalized Ricci-DeTurck flow is obtained by what amounts to subtracting a Lie derivative from the right side of (4.2):

\[
\begin{align*}
\partial_t g &= -2 \text{Rc} + 2c \nabla \phi \otimes \nabla \phi - L_W g, \\
\partial_t \phi &= \tau_{g,h} \phi - L_W \phi,
\end{align*}
\]

with initial data \((g(0), \phi(0))\), so we must compute the linearization of this Lie derivative, as in Lemma 4.8.

Take a variation \((\tilde{g}(\epsilon), \tilde{\phi}(\epsilon))\) as before. It is well-known that

\[
\partial_{\epsilon=0} (L_W \tilde{g})_{ij} = \nabla_i (\delta h)_j + \nabla_j (\delta h)_i + \nabla_i \nabla_j \text{tr}_{g_0} h.
\]

Subtracting this from (2.6a) gives (2.9a). For the second equation,

\[
L_W \phi = \nabla \phi(W) = \partial_i \phi^a W_i \partial_a | \phi,
\]

and it is easy to see that \(W(\tilde{g}(0)) = 0\), so

\[
\partial_{\epsilon=0} \partial_i \phi^a \tilde{W}_i = 0,
\]

giving (2.9b). \(\square\)

Now assume that \((g_0, \phi_0)\) is a fixed point of the curvature-normalized harmonic-Ricci-DeTurck flow with \(\phi_0\) constant and \(g_0\) a \(\lambda\)-Einstein metric with negative sectional curvature.

Recall that a linear operator \(L\) is weakly (strictly) stable if its spectrum is confined to the half plane \(\text{Re} z \leq 0\) (and is uniformly bounded away from the imaginary axis).

We thank Peter Petersen for the idea behind the following estimate.

Lemma 2.10. Suppose that \(g\) is \(\lambda\)-Einstein, and that there exists \(K < 0\) such that \(\sec \leq K\). Then if \(Lh := \Delta h + 2\lambda h\), we have \((Lh, h) \leq K(n-2)\|h\|^2 < 0\).

Proof. First, write a symmetric 2-tensor \(h\) as \(h = g(\tilde{h}, \cdot)\) and let \(\{e_i\}\) be an orthonormal basis of eigenvectors for \(\tilde{h}\). That is, \(\tilde{h}(e_i) = \lambda_i e_i\). Now, we write part of \(\langle Lh, h \rangle\) in components with respect this basis. Using that \(g\) is \(\lambda\)-Einstein, we have

\[
\sum_{i,j,k,\ell} R_{ijk\ell} h^{it} h^{jk} - \sum_{i,j,k} R_{ij} h^{ij} h^{ij} + 2\lambda \sum_{i,j} h_{ij} h^{ij} = \sum_{i,j} R_{ij} \lambda_i \lambda_j + \lambda \sum_{i,j} h_{ij} h^{ij} = \sum_{i,j} \sec(e_i, e_j) \lambda_i \lambda_j + \sum_{i,j} \sec(e_i, e_j) \lambda_i^2 = \frac{1}{2} \sum_{i,j} \sec(e_i, e_j) 2\lambda_i \lambda_j + \frac{1}{2} \sum_{i,j} \sec(e_i, e_j) \lambda_i^2 + \frac{1}{2} \sum_{i,j} \sec(e_i, e_j) \lambda_j^2 = \frac{1}{2} \sum_{i,j} \sec(e_i, e_j) (\lambda_i + \lambda_j)^2.
\]
Now, integrating by parts and using Koiso’s Bochner formula together with the above equation,

$$(Lh, h) = -\|\nabla h\|^2 + \int R_{ijkl} h^i h^j h^k h^l + \frac{1}{2} \sum_{i,j} \sec(e_i, e_j)(\lambda_i + \lambda_j)^2$$

$$= -\frac{1}{2} T^2 - \|\delta h\|^2 + K(n - 2)\|h\|^2 + K\|\text{tr } h\|^2$$

$$\leq K(n - 2)\|h\|^2,$$

as desired. □

**Lemma 2.11.** Let $$(g_0, \phi_0)$$ be such that $$\phi_0$$ is constant and $$g_0$$ is Einstein with negative sectional curvature. Then the linear system (2.9a)-(2.9b) has the following linear stability properties:

If $$n = 2$$, then the operator $$L_0$$ is weakly stable. On an orientable surface $$\mathcal{B}$$ of genus $$\gamma \geq 2$$, the null eigenspace is the $$(6\gamma - 6)$$-dimensional space of holomorphic quadratic differentials.

If $$n \geq 3$$, the operator $$L_0$$ is strictly stable.

The operator $$L_1$$ is weakly stable. Its null eigenspace is the space of constant variations $$\psi \in C^\infty(\phi^* T\mathcal{N})$$, whose dimension is equal to $$\dim \mathcal{N}$$.

**Proof.** The statements about $$L_0$$ in dimension 2 follows from [10, Lemma 5], and the statement in dimension $$n \geq 3$$ follows from Lemma 2.10. That the operator $$L_1 = \Delta$$ is weakly stable follows from integrating by parts. □

We now turn to the proof of the the main theorem. See the appendix for the statement of Simonett’s theorem, [10, Section 2] for a more detailed description of its application as used here, and [21] for the original statement.

If $$\mathcal{V} \to \mathcal{M}$$ is a vector bundle, let $$\mathfrak{h}^{r+\rho}(\mathcal{V})$$ denote the completion of the vector space $$C^\infty(\mathcal{V})$$ with respect to the $$r + \rho$$ little-Hölder norm. For fixed $$0 < \sigma < \rho < 1$$, consider the following densely and continuously embedded spaces:

$$E_0 := \mathfrak{h}^{0+\sigma}(S^2\mathcal{M}) \times \mathfrak{h}^{0+\sigma}(\phi^* T\mathcal{N})$$

$$\cup$$

$$X_0 := \mathfrak{h}^{0+\rho}(S^2\mathcal{M}) \times \mathfrak{h}^{0+\rho}(\phi^* T\mathcal{N})$$

$$\cup$$

$$E_1 := \mathfrak{h}^{2+\sigma}(S^2\mathcal{M}) \times \mathfrak{h}^{2+\sigma}(\phi^* T\mathcal{N})$$

$$\cup$$

$$X_1 := \mathfrak{h}^{2+\rho}(S^2\mathcal{M}) \times \mathfrak{h}^{2+\rho}(\phi^* T\mathcal{N})$$

For fixed $$1/2 \leq \beta < \alpha < 1$$, define the continuous interpolation spaces

$$X_{\beta} := (X_0, X_1)_{\beta}, \quad X_{\alpha} := (X_0, X_1)_{\alpha}.$$

For fixed $$0 < \epsilon \ll 1$$, let $$\mathcal{G}_{\beta}$$ be the open $$\epsilon$$-ball around $$(g_0, \phi_0)$$ in $$X_\beta$$, and define $$\mathcal{G}_{\alpha} := \mathcal{G}_{\beta} \cap X_{\alpha}$$.

**Proof of Theorem 1.2.** We follow the proof of Theorem 1 in [10], which consists of four steps. First, one must show that the complexification of the operator in (2.9a)-(2.9b) is sectorial. This holds exactly as in [10]. With this established, once checks that conditions (1)-(7) of Simonett’s theorem hold, and this follows exactly
as in [10, Lemmas 1 and 2]. The second step is then to apply Simonett’s theorem (Theorem A.1). Third, in the cases of weak linear stability, one proves the uniqueness of a smooth center manifold consisting of fixed points of the flow \( \varphi \). Our case is in some sense a special case of that considered in [10]. In particular, the \( n = 2 \) case for the metric \( g \) is handled. Note that fixed points of this flow still coincide with those of the curvature-normalized Ricci-DeTurck flow. Finally, one proves convergence of the metric. Again, the arguments of [10] carry through without modification. \( \square \)

3. Ricci flow on warped products

3.1. Setup. Consider a warped product metric \( g = g + e^{2\phi} \gamma \) on a manifold \( M = B \times F \), where \( \phi \in C^\infty(B) \). Let \( \mathcal{H}, \mathcal{V} \subset T M \) denote the horizontal and vertical distributions, respectively. The Ricci curvature of \( g \) is

\[
\begin{align*}
\mathcal{R}_g \big|_{\mathcal{H} \times \mathcal{H}} &= \mathcal{R}_g \big|_{\mathcal{H} \times \mathcal{H}} = \mathcal{R}_g - m \text{Hess}(\phi) - m \, d\phi \otimes d\phi \\
\mathcal{R}_g \big|_{\mathcal{H} \times \mathcal{V}} &= 0 \\
\mathcal{R}_g \big|_{\mathcal{V} \times \mathcal{V}} &= \mathcal{R}_g - e^{2\phi}(\Delta \phi + m |d\phi|^2) \gamma.
\end{align*}
\]

(3.1)

See [1], for example. We wish to describe the Ricci flow equation on the warped product \( (M, g) \) in terms of the evolution of \( g \) and the warping function \( \phi \). A similar approach was taken in [16, 24].

**Proposition 3.2.** Let \( (M = B \times F, g(t)) \) be a solution to Ricci flow, with \( M \) closed and \( g(0) = g(0) + e^{2\phi(0)} \gamma(0) \) a warped product. If \( (F, h) \) is \( \mu \)-Einstein, then \( \gamma(0) = \gamma \) is constant under the flow, \( g(t) \) is a warped product, and the evolutions of \( g \) and \( \phi \) are given by

\[
\begin{align*}
\partial_t g &= -2 \mathcal{R}_g + 2m \text{Hess}(\phi) + 2m \, d\phi \otimes d\phi \\
\partial_t \phi &= \Delta \phi + m |d\phi|^2 - \mu e^{-2\phi}
\end{align*}
\]

(3.3)

**Proof.** Define \( \bar{g}(t) = g(t) + e^{2\phi(t)} h \), where \( g(t) \) and \( \phi(t) \) are solutions of (3.3) with \( g(0) = g_0 \) and \( \phi(0) = \phi_0 \). Using (3.1), we see that the evolution of \( \bar{g}(t) \) is

\[
\begin{align*}
\partial_t \bar{g}(t) &= \partial_t g(t) + \partial_t (e^{2\phi(t)} h) \\
&= -2 \mathcal{R}_g + 2m \text{Hess}(\phi) + 2m \, d\phi \otimes d\phi + 2e^{2\phi(t)} \partial_t \phi(t) \gamma \\
&= -2 \mathcal{R}_g \big|_{\mathcal{H} \times \mathcal{H}} + 2e^{2\phi(t)} (\Delta \phi + m |d\phi|^2 - \mu e^{-2\phi}) \gamma \\
&= -2 \mathcal{R}_g \big|_{\mathcal{H} \times \mathcal{H}} - \mu h + 2e^{2\phi(t)} (\Delta \phi + m |d\phi|^2) \gamma \\
&= -2 \mathcal{R}_g \big|_{\mathcal{H} \times \mathcal{H}} - 2 \mathcal{R}_g \big|_{\mathcal{V} \times \mathcal{V}} \\
&= -2 \mathcal{R}_g,
\end{align*}
\]

since \(-2\mu \gamma = -2 \gamma \mathcal{R}_g\). This means \( \bar{g}(t) \) solves Ricci flow with \( \bar{g}(0) = g_0 + e^{2\phi_0} \gamma \). By uniqueness of solutions of Ricci flow, for any solution \( g(t) \) of Ricci flow with \( g(0) = g_0 + e^{2\phi_0} h \), we must have \( \bar{g}(t) = g(t) \). This means the warped product structure is preserved and the components of \( g(t) = g(t) + e^{2\phi(t)} \gamma \) must satisfy (3.3). \( \square \)
We can simplify the system (3.3). Consider the (horizontal) vector field $X = -m\nabla \phi$. Then we compute the following Lie derivatives,

\[ \mathcal{L}_X g = -2m \text{Hess}(\phi), \]
\[ \mathcal{L}_X \phi = -m|d\phi|^2. \]

Puling back by diffeomorphisms generated by $X$ amounts to adding these Lie derivatives to the equations in (3.3). From this we obtain the system

\[ \partial_t g = -2R^g + 2m d\phi \otimes d\phi \]
\[ \partial_t \phi = \Delta \phi - \mu e^{-2\phi} \]

which we recognize as (1.3).

**Remark 3.4.** The system (1.3) does not involve $(\mathcal{F}, \gamma)$ at all, except for the Einstein constant $\mu$. Considering that system abstractly (that is, outside the context of Ricci flow), we can therefore allow $\mathcal{F}$ to be non-compact.

**3.2. Estimates.** We wish to understand the asymptotic behavior of $\phi(t)$ and $d\phi(t)$ when $(g(t), \phi(t))$ is a solution of (1.3). For this, we need the evolution of various geometric quantities under the flow (1.3). The following evolution equations can be proved in a manner similar to those for (1.1) with 1-dimensional target, as found in [13].

**Lemma 3.5.** Let $(g(t), \phi(t))$ be a solution of (1.3). We have the following evolution equations.

\[ \partial_t \partial_i \phi = \Delta \partial_i \phi - R^g_i \partial_i \phi + 2\mu e^{-2\phi} \partial_i \phi \]
\[ \partial_t |d\phi|^2 = \Delta |d\phi|^2 - 2|\nabla^2 \phi|^2 - 2m|d\phi|^4 + 4\mu e^{-2\phi}|d\phi|^2 \]

We will use the estimates from the lemma with the following version of the Maximum Principle.

**Theorem 3.6.** Suppose $g(t)$ is a family of metrics on a closed manifold $\mathcal{M}^n$, $X(t)$ is a time-dependent vector field on $\mathcal{M}$, and $F : \mathbb{R} \times [0, T) \to \mathbb{R}$ is a Lipschitz continuous function. Consider the semi-linear heat equation

\[ \partial_t u = \Delta_{g(t)} u + \langle X(t), \nabla u \rangle + F(u, t), \]

and the corresponding ordinary differential equation

\[ \frac{d}{dt} U = F(U, t), \]

for functions $u : M \times [0, T) \to \mathbb{R}$, $U : [0, T') \to \mathbb{R}$.

Let $u(x, t)$ be a $C^2$ solution of (3.7), and let $U_1$ and $U_2$ solve (3.8) with $U_1(0) = \min_{\mathcal{M}} u(x, 0)$ and $U_2(0) = \max_{\mathcal{M}} u(x, 0)$, respectively. In particular,

\[ U_1(0) \leq u(x, 0) \leq U_2(0) \]

for all $x \in \mathcal{M}$. Then as long as these functions exist,

\[ U_1(t) \leq u(x, t) \leq U_2(t), \]

for all $x \in \mathcal{M}$. 

Lemma 3.9. Suppose that \((g(t), \phi(t))\) is a solution of (1.3) on \(B \times F\) with \(B\) compact and \((F, h)\) \(\mu\)-Einstein with \(\mu = -1/2\). Then there are constants \(d_1, d_2, C\) such that
\[
e^{2d_1} + t \leq e^{2\phi} \leq e^{2d_2} + t
\]
(3.10)

\[
0 \leq |d\phi(x, t)|^2 \leq \frac{C}{(t + 1)^2}
\]
(3.11)

Proof. The ODE associated with the evolution of \(\phi\) is
\[
d\frac{d\phi}{dt} = F(U) = -\mu e^{-2\phi}.\]

For initial data \(U(0) = d\), this has solution \(U(t) = \frac{1}{2} \log(e^{2d} - 2\mu t)\). Let \(U_1\) and \(U_2\) solve the ODE with initial data \(d_1 = \min_B \phi(x, 0)\) and \(d_2 = \max_B \phi(x, 0)\), respectively. Then
\[
\frac{1}{2} \log(e^{2d_1} - 2\mu t) = U_1(t) \leq \phi(x, t) \leq U_2(t) = \frac{1}{2} \log(e^{2d_2} - 2\mu t)
\]
(3.12)

for as long as the functions exist, and for all \(x \in B\). Translating this into a statement about the metric \(g\), we see that
\[
e^{2d_1} - 2\mu t \leq e^{2\phi} \leq e^{2d_2} - 2\mu t.
\]
(3.13)

We may rescale the metric \(h\) so that the Einstein constant is \(\mu \in \{-\frac{1}{2}, 0, \frac{1}{2}\}\), and this gives us three cases:

\[
e^{2d_1} + t \leq e^{2\phi} \leq e^{2d_2} + t
\]
(3.14)

\[
e^{2d_1} \leq e^{2\phi} \leq e^{2d_2},
\]
(3.15)

\[
e^{2d_1} - t \leq e^{2\phi} \leq e^{2d_2} - t.
\]

Note that the first case is expanding and the third reaches zero in finite time.

Now let us find bounds on \(|d\phi|^2\). We can do this in the case that \(\mu < 0\) (say \(\mu = -\frac{1}{2}\)). First, from the behavior of \(\phi\) above, we have
\[
-2e^{-2\phi} \leq \frac{-2}{t + b},
\]
where \(b = e^{2d_2}\). Then \(|d\phi|^2\) evolves according to
\[
\partial_t |d\phi|^2 = \Delta |d\phi|^2 - 2|\nabla^2 \phi|^2 - 2m|d\phi|^4 - 2e^{-2\phi}|d\phi|^2 \leq \Delta |d\phi|^2 - \frac{2}{t + b}|d\phi|^2.
\]

That is, \(v = |d\phi|^2\) is a subsolution of \(\partial_t u = \Delta u + F(u, t)\), where
\[
F(u, t) = -\frac{2}{t + b} u.
\]

The corresponding ODE has solution
\[
U(t) = \frac{b^2 U_0}{(t + b)^2}.
\]

Taking \(U_0 = \max_B |d\phi(x, 0)|^2\), the Maximum Principle gives
\[
0 \leq |d\phi(x, t)|^2 \leq \frac{b^2 U_0}{(t + b)^2} \leq \frac{C}{(t + 1)^2}
\]
(3.16)

for all time and all \(x \in B\), for some \(C > 0\). Note that \(|d\phi|^2\) is therefore integrable in time. \(\square\)
3.3. **Stability.** In this section we follow the outline given in the introduction to prove Theorem 1.4.

The warping function $\phi$ grows in a controlled way: when $\mu = -\frac{1}{2}$, the lower and upper bounds of $\phi$ on $B$ both go to infinity by (3.10), but we also have $|d\phi|^2 \to 0$ as $t \to \infty$ by (3.11). It is natural, therefore, to hope that $\phi$ converges to a constant function, but the growth condition implies that this constant should be $\infty$. This means that some kind of normalization is needed for $\phi$ (in addition to $g$).

Therefore, let $(\bar{g}(\bar{t}), \bar{\phi}(\bar{t}))$ solve (1.3) (with time parameter $\bar{t}$) and let $s(\bar{t})$ be a function with positive anti-derivative $\sigma$. Consider the transformation

$$(\bar{g}(\bar{t}), \bar{\phi}(\bar{t})) \mapsto (g(t), \phi(t)),$$

where

$$g = \sigma^{-1} \bar{g}, \quad \phi = \bar{\phi} - \frac{1}{2} \log \left(e^{2\bar{\phi}_{avg}(0)} + t\right) + \bar{\phi}_{avg}(0), \quad t = \int_{t_0}^{T} dr \sigma(r),$$

and

$$\bar{\phi}_{avg}(t) = \frac{\int_{B} \bar{\phi}(t) \, d\mu(t)}{\int_{B} d\mu(t)}.$$

Note that $\phi$ is simply a translate of $\bar{\phi}$. As such, $d\phi = d\bar{\phi}$, and so all spatial derivative behavior of $\phi$ is the same as that of $\bar{\phi}$. Additionally, from (3.10) it is easy to see that

$$\lim_{t \to \infty} \phi(x, t) = \bar{\phi}_{avg}(0)$$

for all $x \in B$.

A computation shows that this transformation results in the system

$$\partial_t g = -2\text{Rc} + 2m d\phi \otimes d\phi - sg,$$

$$\partial_t \phi = \Delta \phi + \frac{\sigma e^{-2(\phi - \bar{\phi}_{avg}(0))} - 1}{e^{2\bar{\phi}_{avg}(0)} + t}.$$

(3.17)

Note that if $s$ is constant, then $\sigma$ is linear in $t$, and we can arrange for $\sigma/(e^{2\bar{\phi}_{avg}(0)} + t)$ to be constant. Namely, we set $\sigma(t) = (e^{2\bar{\phi}_{avg}(0)} + t)s$. This makes (3.17) an autonomous system.

**Lemma 3.18.** Suppose that $(B^n \times F^m, g_0 + e^{2\phi_0} \gamma)$ is a warped product with $(B, g)$ compact and $\lambda$-Einstein and $(F, \gamma) = -\frac{1}{2} \lambda$-Einstein. Let $\phi_0 : \mathcal{M} \to \mathbb{R}$ be a constant map. Setting $s = -2\lambda$ makes $(g_0, \phi_0)$ a stationary solution of (3.17).

With these choices, we call (3.17) the curvature-normalized warped product Ricci flow system:

$$\partial_t g = -2\text{Rc} + 2m d\phi \otimes d\phi + 2\lambda g,$$

$$\partial_t \phi = \Delta \phi - \frac{\lambda}{2} (e^{-2(\phi - \bar{\phi}_{avg}(0))} - 1).$$

(3.19)

Consider a fixed point $(g_0, \phi_0)$ of the flow (3.19) on $B^n$. From Lemma (3.18), we can assume that $g_0$ is Einstein with $\text{Rc}(g_0) = \lambda g_0$, $\phi_0$ is constant, and $s = -2\lambda$.

To analyze the stability near a fixed point, we must compute the linearization of the flow. Let $(\tilde{g}(\epsilon), \tilde{\phi}(\epsilon))$ be a variation of $(g_0, \phi_0)$ such that

$$\tilde{g}(0) = g_0, \quad \partial_{\epsilon=0} \tilde{g}(\epsilon) = h,$$

$$\tilde{\phi}(0) = \phi_0, \quad \partial_{\epsilon=0} \tilde{\phi}(\epsilon) = \psi$$

(3.20)
for $\phi \in C^\infty(B)$.

**Lemma 3.21.** The linearization of (3.19) at a fixed point $(g_0, \phi_0)$ with $Rc = \lambda g_0$ and $\phi_0$ constant acts on $(h, \psi)$ by

\[
\begin{align*}
\partial_t h_{ij} &= \Delta_t h_{ij} + \nabla_i (\delta h)_j + \nabla_j (\delta h)_i + \nabla_i \nabla_j \text{tr}_{g_0} h + 2\lambda h_{ij}, \\
\partial_t \psi &= \Delta \psi + 2\lambda \psi,
\end{align*}
\]  

(3.22a) \hspace{1cm} (3.22b)

where $\Delta_t$ is the Lichnorowicz Laplacian and $\Delta$ is the Laplacian acting on functions.

**Proof.** The first equation is the same as in Lemma 2.5, so we only consider the second equation. Let

\[
\begin{align*}
\left. f_{g, \phi}(t) = f(g(t), \phi(t), t) = \frac{\int_B \phi(t) dV_{g(t)}}{\int_B dV_{g(t)}} \right|_{t=0}
\end{align*}
\]

So, given a one-parameter family of metrics and functions $(g(t), \phi(t))$, $f$ computes the average value of $\phi$ with respect to $g$, both taken at the parameter $t$. Therefore we have

\[
\partial_t \phi = \Delta \phi - \frac{\lambda}{2} \left( \exp \left( -2(\phi - f_{g, \phi}(0)) \right) - 1 \right).
\]

In the linearization of the right side, we need to compute the $\epsilon$-derivative of $f$:

\[
\begin{align*}
\left. \partial_\epsilon \right|_{\epsilon=0} f_{g, \phi}(0) &= \left. \frac{\int_B \phi(t) dV_{g(t)}}{\int_B dV_{g(t)}} \right|_{t=0} \\
\left. \frac{\int_B \phi(t) dV_{g(t)}}{\int_B dV_{g(t)}} \right|_{t=0} &= 0
\end{align*}
\]

Now, $\Delta$ is linear, and the desired equation follows from the chain rule. \qed

As before, we use the DeTurck trick to make the linear (3.19) system strictly parabolic. Define a vector field depending on $g(t)$ by

\[
W^k = g^{ij} \left( \Gamma^k_{ij} - g_0 \Gamma^k_{ij} \right), \quad k = 1, \ldots, n.
\]

Let $F_t$ be diffeomorphisms generated by $W(t)$, with initial condition $F_0 = \text{id}$. The one-parameter family $(F^*_t g(t), F^*_t \phi(t))$ is the solution of the curvature-normalized warped product Ricci-DeTurck flow. A stationary solution $(g_0, \phi_0)$ of (3.19) with $Rc = \lambda g_0$ and $\phi_0$ constant is then also a stationary solution of the curvature-normalized warped product Ricci-DeTurck flow.

**Lemma 3.24.** The linearization of the curvature-normalized warped product Ricci-DeTurck flow at a fixed point $(g_0, \phi_0)$ with $Rc = \lambda g_0$ is the autonomous, self-adjoint, strictly parabolic system

\[
\begin{align*}
\partial_t h &= L_0 h = (\Delta_t + 2\lambda \text{id}) h, \\
\partial_t \psi &= L_1 \psi = (\Delta + 2\lambda \text{id}) \psi
\end{align*}
\]

(3.25a) \hspace{1cm} (3.25b)

**Proof.** This is essentially the same as the proof of Lemma 2.8. \qed

Now assume that $(g_0, \phi_0)$ is a fixed point of the curvature-normalized warped product Ricci-DeTurck flow with $\phi_0$ constant and $g_0$ an Einstein metric with negative sectional curvature.

**Lemma 3.26.** Let $(g_0, \phi_0)$ be such that $\phi_0$ is constant and $g_0$ has constant sectional curvature. Then the linear system (3.25a) + (3.25b) has the following stability properties:
If \( n = 2 \), then the operator \( L_0 \) is weakly stable. On an orientable surface \( B \) of genus \( \gamma \geq 2 \), the null eigenspace is the \((6\gamma - 6)\)-dimensional space of holomorphic quadratic differentials.

If \( n \geq 3 \), then the operator \( L_0 \) is strictly stable.

The operator \( L_1 \) is strictly stable.

Proof. The statements about \( L_0 \) follow from those about \( L_0 \) from Lemma 2.11.

The operator \( L_1 = \Delta + 2\lambda \text{id} \), involving the Laplacian acting on \( C^\infty(B) \), is strictly stable:

\[
(L_1 \psi, \psi) = \int_B (\Delta \psi + 2\lambda \psi) \psi \, d\mu_g = -\|\nabla \psi\|^2 + 2\lambda \|\psi\|^2 \leq 0,
\]

since \( \lambda < 0 \). We have equality exactly when \( \psi \) is the zero function. \( \square \)

We now turn to the proof of the main theorem. Again, see the appendix for the statement of Simonett’s theorem. Recall that if \( V \to M \) is a vector bundle, then \( h^{r+\rho}(V) \) denotes the completion of the vector space \( C^\infty(V) \) with respect to the \( r + \rho \) little-Hölder norm, and for brevity, let \( h^{r+\rho}(B) \) denote the corresponding completion of \( C^\infty(B) \).

For fixed \( 0 < \sigma < \rho < 1 \), consider the following densely and continuously embedded spaces:

\[
E_0 := h^{0+\sigma}(S^2B) \times h^{0+\sigma}(B) \\
\cup \\
X_0 := h^{0+\rho}(S^2B) \times h^{0+\rho}(B) \\
\cup \\
E_1 := h^{2+\sigma}(S^2B) \times h^{2+\sigma}(B) \\
\cup \\
X_1 := h^{2+\rho}(S^2B) \times h^{2+\rho}(B)
\]

For fixed \( 1/2 \leq \beta < \alpha < 1 \), define the continuous interpolation spaces

\[
X_\beta := (X_0, X_1)_\beta, \quad X_\alpha := (X_0, X_1)_\alpha.
\]

For fixed \( 0 < \epsilon \ll 1 \), let \( G_\beta \) be the open \( \epsilon \)-ball around \((g_0, \phi_0)\) in \( X_\beta \), and define \( G_\alpha := G_\beta \cap X_\alpha \).

Proof of Theorem 1.4. The \( \mu = 0 \) case follows from directly Theorem 1.2, and modulo the details of the Hölder space setup, the proof for \( \mu < 0 \) is the same as the proof of Theorem 1.2 so we omit the details. \( \square \)

Remark 3.27. One may compare Theorem 1.4 with [16, Theorem 1.1]. Those authors prove convergence of warped product Ricci flow solutions when the base has dimension two, although different techniques are involved.

Remark 3.28. Following Perelman (for the Ricci flow) and List and Muller (for harmonic-Ricci flow), the flow (1.3) is the gradient flow of a certain energy functional. Namely, given a metric \( g \) on \( B \) and functions \( \phi, f : B \to \mathbb{R} \), the energy functional is

\[
\mathcal{F}(g, \phi, f) = \int_M (R - m|d\phi|^2 + m\mu e^{-2\phi} + |df|^2)e^{-f} \, dV.
\]
4. Locally $\mathbb{R}^N$-invariant Ricci flow

4.1. Setup. The manifolds that we will consider in this section have a special bundle structure. Let $B$ be a connected, oriented manifold, and let $E \xrightarrow{\pi} B$ be a flat $\mathbb{R}^N$-vector bundle. We consider $M$ to be a principal $\mathbb{R}^N$-bundle over $B$, twisted by $E$. That is, there exists a smooth map

$$E \times B \rightarrow M$$

that, over each point $b \in B$, gives a free and transitive action that is consistent with the flat connection on $E$. This means that if $U \subset B$ is such that $E_U \rightarrow U$ is trivializable, then $\pi^{-1}(U)$ has a free $\mathbb{R}^N$ action. Let $M$ have a connection $\mathcal{A}$ such that $\mathcal{A}|_{\pi^{-1}(U)}$ is an $\mathbb{R}^N$-valued connection. If we assume that $M$ also has a flat connection itself, then $\mathcal{A}$ is globally an $\mathbb{R}^N$-valued 1-form.

We will use this bundle structure to describe local coordinates for $M$. Let $U \subseteq B$ be an open set such that $E_U \rightarrow U$ is trivializable and has a local section $\sigma: U \rightarrow \pi^{-1}(U)$. Additionally, let $\rho: \mathbb{R}^n \rightarrow U$ be a parametrization of $U$, with coordinates $x^\alpha$, and let $e_i$ be a basis for $\mathbb{R}^N$. Then we obtain coordinates $(x^\alpha, x^i)$ on $\pi^{-1}(U)$ via

$$(x^\alpha, x^i) \mapsto e_i \cdot \sigma(\rho(x^\alpha))$$

where $\cdot$ denotes the free $\mathbb{R}^N$ action described above.

Let $g$ be a Riemannian metric on $M$ such that the $\mathbb{R}^N$-action is a local isometry. With respect to the coordinates above, one may write

$$g = \sum_{\alpha, \beta=1}^n g_{\alpha\beta} dx^\alpha dx^\beta + \sum_{i,j=1}^N G_{ij}(dx^i + \sum_{\alpha=1}^n A^{i\alpha} dx^\alpha)(dx^j + \sum_{\beta=1}^n A^{j\beta} dx^\beta)$$

(4.1)

$$= g_{\alpha\beta} dx^\alpha dx^\beta + G_{ij}(dx^i + A^{i\alpha} dx^\alpha)(dx^j + A^{j\beta} dx^\beta).$$

We will write this informally as $g = (g, A, G)$, where $g(b) = g_{\alpha\beta}(b) dx^\alpha dx^\beta$ is locally a Riemannian metric on $U \subseteq B$, $A(b) = A^{i\alpha}(b) dx^\alpha$ is locally the pullback by $\sigma$ of a connection on $\pi^{-1}(U) \rightarrow U$, and $G(b) = G_{ij}(b) dx^i dx^j$ is an inner product on the fiber $\mathcal{M}_b$.

4.2. Stability. In this section we follow the outline given in the introduction to prove Theorem 1.6.

In [15], Lott considered metrics of the form (4.1) that evolve under Ricci flow, which are called locally $\mathbb{R}^N$-invariant solutions. He showed that the Ricci flow equation for $(M, g)$ becomes three equations: one for each of $g$, $A$, and $G$ (see [15, Equation (4.10)]). To study the asymptotic stability of this system, Knopf transformed it into an equivalent one that has legitimate fixed points (see [10, 15]).
Equation (1.3)]. Let \( s(t) \) be a function. Then the transformed system is

\[
\begin{align*}
(4.2a) \quad & \partial_t g_{\alpha\beta} = -2R_{\alpha\beta} + \frac{1}{2} G^{ik} G_{ij} \nabla \alpha G_{ij} \nabla \beta G_{kl} + g^{ij} G_{ij} (dA)^i_{\alpha\gamma} (dA)^j_{\beta\delta} + s g_{\alpha\beta}, \\
(4.2b) \quad & \partial_t A^i_\alpha = - (\delta dA)^i_\alpha + g^{\beta\gamma} G^{ij} \nabla \gamma G_{jk} (dA)^k_\alpha - \frac{1}{2} s A^i_\alpha, \\
(4.2c) \quad & \partial_t G_{ij} = \Delta G_{ij} - g^{\alpha\beta} G^{kl} \nabla \alpha G_{ik} \nabla \beta G_{lj} - \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} G_{ik} G_{j\ell} (dA)^k_{\alpha\beta} (dA)^\ell_{\gamma\delta}.
\end{align*}
\]

We call this system a \textit{curvature-normalized locally \( \mathbb{R}^n \)-invariant Ricci flow}. Also, we recall that \( \nabla = \frac{\partial}{\partial x} \), \( (dA)^i_\alpha = \nabla_\alpha A^i_\beta - \nabla_\beta A^i_\alpha \), \( (\delta dA)^i_\alpha = -g^{\beta\gamma} \nabla \gamma (dA)^i_\beta \), and \( \Delta G_{ij} = g^{\alpha\beta} \nabla_\alpha \nabla_\beta G_{ij} = g^{\alpha\beta} \left( \frac{\partial^2}{\partial x^\alpha \partial x^\beta} G_{ij} - \Gamma^i_{\alpha\beta} \frac{\partial}{\partial x^\gamma} G_{ij} \right) \), where \( \Gamma \) represents the Christoffel symbols of \( g \).

The case where the bundle connection is flat (i.e., \( A \) vanishes) was studied in [14], in the context of structures that arise from certain expanding Ricci solitons on low-dimensional manifolds. There and in the more general setting, certain Ricci flow solutions give rise to a (twisted) harmonic map \( G : B \to \text{SL}(N, \mathbb{R})/\text{SO}(N) \) (the target being the space of symmetric positive-definite bilinear forms of fixed determinant, with its usual metric) together with a “soliton-like” equation relating the metrics \( g \) and \( G \). These are the \textit{harmonic-Einstein equations}.

We will need a byproduct of this fact. Write \( \mathcal{S}_N = \text{SL}(N, \mathbb{R})/\text{SO}(N) \). The tangent space \( T_G \mathcal{S}_N \) at \( G \in \mathcal{S}_N \) consists of symmetric bilinear forms with no trace. There is a Riemannian metric on \( T_G \mathcal{S}_N \) defined by

\[
\bar{g}_G(X, Y) = \text{tr}(G^{-1} X G^{-1} Y) = G^{ij} X_{jk} G_{k\ell} Y_{\ell i}.
\]

The tension field of \( G : B \to \mathcal{S}_N \), with respect to the metrics \( g \) and \( \bar{g} \), has components

\[
(4.4) \quad (\tau_{g, \bar{g}} G)_{ij} = \Delta G_{ij} + g^{\alpha\beta} \sum_{p < q < r < s} \left( \bar{g}_N \Gamma \circ G \right)_{pq, rs} \nabla \alpha G_{pq} \nabla \beta G_{rs}.
\]

**Proposition 4.5.** The evolution equation for \( G \) from (4.2) is a modified harmonic map flow for \( G : B \to \mathcal{S}_N \). More precisely,

\[
\partial_t G_{ij} = (\tau_{g(t), \bar{g}} G)_{ij} - \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} G_{ik} G_{j\ell} (dA)^k_{\alpha\beta} (dA)^\ell_{\gamma\delta}.
\]

See [25] for a proof and further discussion.

Now, we want to describe the fixed points of (4.2), with the proper choice of \( s \).

**Lemma 4.6.** Suppose that \( (\mathcal{M} = \mathbb{R}^n \times B, g) \) is a twisted principal \( \mathbb{R}^n \)-bundle with locally \( \mathbb{R}^n \)-invariant metric \( g = (g, A, G) \). Suppose that \( B \) is compact and Einstein, with \( \text{Rc}(g_0) = \lambda g_0, A = 0, \) and \( G \) constant. Setting \( s = -2\lambda \) makes \( (g_0, 0, G_0) \) a stationary solution of (4.2).

With these choices, we call (4.2) the \textit{curvature-normalized locally \( \mathbb{R}^n \)-invariant Ricci flow} system.

Consider a fixed point of the flow (4.2) on a Riemannian product \( (\mathbb{R}^n \times B, g) \). From Lemma 4.6 we can assume that \( g \) is Einstein with \( \text{Rc} = \lambda g \), \( A \) is identically zero, and \( G \) is constant. Also, \( c = 0 \) and \( s = -2\lambda \).

To analyze the stability near a fixed point, we must compute the linearization of the flow. Write \( g_0 = (g_0, 0, G_0) \) for such a fixed point. Let \( \tilde{g}(\epsilon), \tilde{A}(\epsilon), \tilde{G}(\epsilon) \)
be a variation of $g$ such that
\begin{equation}
\tilde{g}(0) = g_0, \quad \partial_{\epsilon}|_{\epsilon=0} \tilde{g} = h = (h, B, F).
\end{equation}
More explicitly,
\begin{align*}
\tilde{g}(0) &= g_0, \quad \partial_{\epsilon}|_{\epsilon=0} \tilde{g} = h \in S^2 B, \\
\tilde{A}(0) &= A_0 = 0, \quad \partial_{\epsilon}|_{\epsilon=0} \tilde{A} = B \in C^\infty(T^* B \otimes \mathbb{R}^N), \\
\tilde{G}(0) &= G_0, \quad \partial_{\epsilon}|_{\epsilon=0} \tilde{G} = F \in C^\infty(G^* T S_N)
\end{align*}
where $\tilde{G}(\epsilon, b) = \exp G(b)\epsilon F(b)$ as in (2.4).

**Lemma 4.8.** The linearization of (4.2) at a fixed point $g_0 = (g_0, 0, G_0)$ with $R_c = \lambda g_0$ and $G_0$ constant acts on $h = (h, B, F)$ by
\begin{align}
\partial_t h_{\alpha\beta} &= \Delta_t h_{\alpha\beta} + \nabla_\alpha (\delta h)_\beta + \nabla_\beta (\delta h)_\alpha + \nabla_\alpha \nabla_\beta \text{tr}_{g_0} h + 2\lambda h_{\alpha\beta}, \\
\partial_t B^i_\alpha &= - (\delta dB)^i_\alpha + \lambda B^i_\alpha, \\
\partial_t F_{ij} &= \Delta F_{ij},
\end{align}
where $\Delta_t$ is the Lichnorowicz Laplacian and $\Delta$ is the Laplacian acting on functions.

**Proof.** This is the same as the proof of Lemma 2.5, except for the second equation. With a variation of $g$ as in (4.7), we have
\begin{equation}
\partial_{\epsilon}|_{\epsilon=0} \left( \frac{1}{2} \delta A^i_\alpha \right) = \lambda B^i_\alpha,
\end{equation}
which is desired.

As before, we use the DeTurck trick to make the linear (4.2) system strictly parabolic. Define a vector field depending on $g(t)$ by
\begin{align}
W^\gamma &= g^{\alpha\beta}(\Gamma^\gamma_{\alpha\beta} - g^{\alpha\gamma} \Gamma^\gamma_{\alpha\beta}), \\
W^k &= (\delta A)_k, \quad k = 1, \ldots, N.
\end{align}
Let $\psi_t$ be diffeomorphisms generated by $W(t)$, with initial condition $\psi_0 = \text{id}$. The one-parameter family of metrics $\psi_t^* g(t)$ is the solution of the curvature-normalized $\mathbb{R}^N$-invariant Ricci–DeTurck flow. A stationary solution $g_0 = (g_0, 0, G_0)$ of (4.2) with $R_c = \lambda g_0$ and $G_0$ constant is then also a stationary solution of the curvature-normalized Ricci–DeTurck flow.

**Lemma 4.11.** The linearization of (4.2) at a fixed point $g_0 = (g_0, 0, G_0)$ with $R_c = \lambda g_0$ is the autonomous, self-adjoint, strictly parabolic system
\begin{align}
\partial_t h &= L_2 h = \Delta_t h + 2\lambda h, \\
\partial_t B &= L_1 B = \Delta_1 h + \lambda B, \\
\partial_t F &= L_0 F = \Delta F,
\end{align}
Here, $-\Delta_1 = d\delta + \delta d$ denotes the Hodge–de Rham Laplacian acting on 1-forms and $L_0 = \Delta$ satisfies $(\Delta F)_{ij} = \Delta(F_{ij})$. 
Proof. This is the same as the proof of Lemma 2.8 except for the second equation. We have
\[(\mathcal{L}_W g)_{\alpha i} = (d\delta A)^i_{\alpha},\]
and so
\[\partial_\epsilon |_{\epsilon = 0} (\mathcal{L}_W \tilde{g})_{\alpha i} = (d\delta B)^i_{\alpha}.\]
Subtracting this from (4.9b) gives (4.12b). □

Now assume that \(g_0 = (g_0, 0, G_0)\) is a fixed point of the curvature-normalized \(R^N\)-invariant Ricci–DeTurck flow with \(G_0\) constant and \(g_0\) an Einstein metric with negative sectional curvature.

Lemma 4.13. Let \(g_0 = (g_0, 0, G_0)\) be a metric of the form (4.1) such that \(G_0\) is constant and \(g_0\) has negative sectional curvature. Then the linear system (4.12a)-(4.12c) has the following stability properties:

If \(n = 2\), then the operator \(L_0\) is weakly stable. On an orientable surface \(B\) of genus \(\gamma \geq 2\), the null eigenspace is the \((6\gamma - 6)\)-dimensional space of holomorphic quadratic differentials.

If \(n \geq 3\), then the operator \(L_0\) is strictly stable.

The operator \(L_1\) is strictly stable.

The operator \(L_2\) is weakly stable. Its null eigenspace is the space of constant variations \(F \in C^\infty(G^*T S_N)\), whose dimension is equal to \(\dim S_N\).

Proof. This follows from Lemma 2.11. □

We now turn to the proof of the main theorem. Again, see the appendix for the statement of Simonett’s theorem. Recall that if \(V \to M\) is a vector bundle, then \(h^{r+\rho}(V)\) denotes the completion of the vector space \(C^\infty(V)\) with respect to the \(r + \rho\) little-Hölder norm. For fixed \(0 < \sigma < \rho < 1\), consider the following densely and continuously embedded spaces:

\[E_0 = h^{0+\sigma}(S^2 B) \times h^{0+\sigma}(T^* B \otimes \mathbb{R}^N) \times h^{0+\sigma}(G^* T S_N)\]

\[X_0 := h^{0+\rho}(S^2 B) \times h^{0+\rho}(T^* B \otimes \mathbb{R}^N) \times h^{0+\rho}(G^* T S_N)\]

\[E_1 := h^{2+\sigma}(S^2 B) \times h^{2+\sigma}(T^* B \otimes \mathbb{R}^N) \times h^{2+\sigma}(G^* T S_N)\]

\[X_1 := h^{2+\rho}(S^2 B) \times h^{2+\rho}(T^* B \otimes \mathbb{R}^N) \times h^{2+\rho}(G^* T S_N)\]

For fixed \(1/2 \leq \beta < \alpha < 1\), define the continuous interpolation spaces

\[X_\beta := (X_0, X_1)_\beta, \quad X_\alpha := (X_0, X_1)_\alpha.\]

For fixed \(0 < \epsilon \ll 1\), let \(G_\beta\) be the open \(\epsilon\)-ball around \(g_0\) in \(X_\beta\), and define \(G_\alpha := G_\beta \cap X_\alpha\).

Proof of Theorem 1.6. Modulo the details of the Hölder space setup, this is the same as the proof of Theorem 1.2 so we omit the details. □
5. Connection Ricci flow

5.1. Setup. Let \((\mathcal{M}^n, g)\) be a Riemannian manifold with \(n \geq 3\). Suppose that \(\tau\) is a \((2, 1)\)-tensor on \(\mathcal{M}\), and consider the \((3, 0)\)-tensor \(H\) with components \(H_{ijk} = g_{k\ell} \tau^\ell_{ij}\). We can think of \(\tau\) as the torsion of a connection \(\nabla\) that is compatible with \(g\), and we say that \(\tau\) is \textit{geometric} if \(H \in \Omega^3(\mathcal{M})\) and \(dH = 0\). Define a \((2, 0)\)-tensor \(\mathcal{H}\) (as above) by

\[
\mathcal{H}_{ij} = g^{pq} g^{rs} H_{ipr} H_{jq} = g_{k\ell} g_{rs} \tau^\ell_{ip} \tau^s_{jq}.
\]

The Ricci curvature of \(\nabla\) is a \((2, 0)\)-tensor on \(\mathcal{M}\), but it is not symmetric. Therefore, consider the symmetric and anti-symmetric parts, denoted by \(Rc^\otimes\) and \(Rc^\wedge\), respectively:

\[
Rc^\otimes = gRc + \frac{1}{2} \mathcal{H}, \quad Rc^\wedge = -\frac{1}{2} \mathcal{d}^* H
\]

where \(gRc\) is the Ricci curvature of the Levi-Civita connection of \(g\) and \(\mathcal{d}^* H\) has components \((\mathcal{d}^* H)_{ij} = -g^{lm} g_{ij} H_{mij}\).

Now, one can consider the evolution of a connection \(g(t) \nabla + \tau(t)\) on \(\mathcal{M}\) in terms of the metric \(g\) and geometric torsion \(\tau\) of \(\nabla\),

\[
\partial_t g = -2 Rc^\otimes, \quad \partial_t H = 2 \mathcal{d} Rc^\wedge
\]

From the expressions for the symmetric and anti-symmetric parts of \(Rc\), these equations become

\[
\partial_t g = -2 Rc + \frac{1}{2} \mathcal{H}, \quad \partial_t H = \Delta_g H
\]

which is exactly the flow (1.7) (although we will use \(H\) in place of \(\tau\), for clarity). It is easy to check that the property that \(\tau\) is geometric is preserved under the flow, and that the flow enjoys short-time existence and uniqueness of solutions.

We will consider this flow where \(g\) is a metric and \(H\) is \textit{any} closed three-form, not necessarily dual to the torsion of a connection. Such a coupling arises in physics, for example as the renormalization group flow with \(B\)-field.

Our goal is to show that this flow is stable when \(g\) is Einstein with negative sectional curvature and \(H = 0\). In the context of connection Ricci flow, this is stability at the Levi-Civita connection of \(g\), that is, where \(\tau = 0\).

5.2. Stability. In this section we follow the outline given in the introduction to prove Theorem 1.8.

We transform the system into one whose fixed points include pairs \((g, H)\) with \(g\) Einstein and \(H\) vanishing. Suppose that \((g(t), H(t))\) is a solution of (1.7) (with time parameter \(\ell\)). Let \(s(\ell)\) be a function with positive anti-derivative \(\mathcal{A}\), and consider

\[
(g(t), H(t)) \rightarrow (g(t), H(t)),
\]

where

\[
g = \sigma^{-1} g, \quad \phi = \sigma, \quad t = \int_{t_0}^T dr \frac{\sigma(r)}{\sigma(r)}
\]
for $\mathcal{I} \in I$, the interval of existence of the solution. A straightforward calculation shows that this transformation results in the modified flow

\begin{align}
\partial_t g &= -2 \text{Rc} + \frac{1}{2} \mathcal{H} - sg \\
\partial_t \mathcal{H} &= \Delta_d \mathcal{H} - s \mathcal{H}
\end{align}

(5.1)

Now, we want to describe the fixed points of (5.1), with the proper choice of $s$.

**Lemma 5.2.** Suppose that $(\mathcal{M}, g_0)$ is compact and Einstein, with $\text{Rc}(g_0) = \lambda g_0$, and that $H_0 = 0$. Setting $s = -2\lambda$ makes $(g_0, H_0)$ a stationary solution of (5.1)

With these choices, we call (5.1) the curvature-normalized connection Ricci flow system.

Consider a fixed point of the flow (5.1) on $\mathcal{M}$. From Lemma (5.2) we can assume that $g_0$ is Einstein with $\text{Rc} = \lambda g_0$, $H_0 = 0$, and $s = -2\lambda$.

To analyze the stability near $(g_0, H_0)$, we must compute the linearization of the flow. Let $(\tilde{g}(\epsilon), \tilde{H}(\epsilon))$ be a variation of $(g_0, H_0)$ such that

\begin{align}
\tilde{g}(0) &= g_0, \quad \partial_\epsilon |_{\epsilon=0} \tilde{g} = h \in S^2 \mathcal{M}, \\
\tilde{H}(0) &= H_0 = 0, \quad \partial_\epsilon |_{\epsilon=0} \tilde{H} = \eta \in \Omega^3_{\text{closed}}(\mathcal{M}).
\end{align}

(5.3)

**Lemma 5.4.** The linearization of (5.1) at a fixed point $(g_0, H_0)$ with $\text{Rc} = \lambda g_0$ and $H_0 = 0$ acts on $(h, \eta)$ by

\begin{align}
\partial_t h_{\alpha\beta} &= \Delta_d h_{\alpha\beta} + \nabla_\alpha (\delta h)_{\beta} + \nabla_\beta (\delta h)_{\alpha} + \nabla_\alpha \nabla_\beta H_0 + 2\lambda h_{\alpha\beta}, \\
\partial_t \eta &= \Delta_d \eta,
\end{align}

(5.5)

where $\Delta_d$ is the Lichnorowicz Laplacian and $\Delta_d$ is the Laplace-Beltrami operator.

**Proof.** This is the same as the proof of Lemma 2.5 except for the second equation, which is easy to check. $\Box$

As before, we use the DeTurck trick to make the linear system strictly parabolic. Define a vector field depending on $g(t)$ by

\begin{equation}
W^\gamma = g^{\alpha\beta} (\Gamma^\gamma_{\alpha\beta} - g_0 \Gamma^\gamma_{\alpha\beta}), \quad \gamma = 1, \ldots, n
\end{equation}

(5.6)

Let $F_t$ be diffeomorphisms generated by $W(t)$, with initial condition $F_0 = \text{id}$. The one-parameter family $(F^*_t g(t), F^*_t H(t))$ is the solution of the curvature-normalized connection Ricci–DeTurck flow. A stationary solution $(g_0, H_0)$ of (5.1) with $\text{Rc} = \lambda g_0$ and $H_0 = 0$ is then also a stationary solution of the curvature-normalized connection Ricci–DeTurck flow.

**Lemma 5.7.** The linearization of (5.1) at a fixed point $(g_0, H_0)$ with $\text{Rc} = \lambda g_0$ and $H_0 = 0$ is the autonomous, self-adjoint, strictly parabolic system

\begin{align}
\partial_t h &= \mathbf{L}_2 h = \Delta_d h + 2\lambda h \\
\partial_t \eta &= \mathbf{L}_1 \eta = \Delta_d \eta + 2\lambda \eta
\end{align}

(5.8)

**Proof.** This is the same as the proof of Lemma 2.8 $\Box$

Now assume that $(g_0, H_0)$ is a fixed point of the curvature-normalized connection Ricci–DeTurck flow with $g_0$ an Einstein metric with negative sectional curvature and $H_0 = 0$. 
Lemma 5.9. Let \((g_0, H_0)\) be such that \(g_0\) has negative sectional curvature and \(H_0 = 0\). Then the linear system \((5.8a)-(5.8b)\) has the following stability properties.

The operator \(L_0\) is strictly stable.

The operator \(L_1\) is strictly stable.

**Proof.** The statement for \(L_0\) follows as in Lemma 2.11. The statement about \(L_1\) follows from integration by parts. \(\square\)

We now turn to the proof of the main theorem. Again, see the appendix for the statement of Simonett’s theorem. Recall that if \(V \rightarrow M\) is a vector bundle, then \(h^{r+\rho}(V)\) denotes the completion of the vector space \(C^\infty(V)\) with respect to the \(r+\rho\) little-Hölder norm. For fixed \(0 < \sigma < \rho < 1\), consider the following densely and continuously embedded spaces:

\[
\begin{align*}
E_0 &:= h^{0+\sigma}(S^2 M) \times h^{0+\sigma}(\Omega^3_{\text{closed}}(M)) \\
\cup \\
X_0 &:= h^{0+\rho}(S^2 M) \times h^{0+\rho}(\Omega^3_{\text{closed}}(M)) \\
\cup \\
E_1 &:= h^{2+\sigma}(S^2 M) \times h^{2+\sigma}(\Omega^3_{\text{closed}}(M)) \\
\cup \\
X_1 &:= h^{2+\rho}(S^2 M) \times h^{2+\rho}(\Omega^3_{\text{closed}}(M))
\end{align*}
\]

For fixed \(1/2 \leq \beta < \alpha < 1\), define the continuous interpolation spaces

\[
X_\beta := (X_0, X_1)_\beta, \quad X_\alpha := (X_0, X_1)_\alpha.
\]

For fixed \(0 < \epsilon \ll 1\), let \(G_\beta\) be the open \(\epsilon\)-ball around \((g_0, \tau_0)\) in \(X_\beta\), and define \(G_\alpha := G_\beta \cap X_\alpha\).

**Proof of Theorem 1.8.** Modulo the details of the Hölder space setup, this is the same as the proof of Theorem 1.2, so we omit the details. Note, however, that it only uses a special case of Simonett’s theorem, since there are no center manifolds. \(\square\)

**Appendix A. Stability Theorem**

We use the following version of Simonett’s Stability Theorem. Please see [7, 10] for versions of the theorem, and [21] for the most general statement.

**Theorem A.1** (Simonett). Assume the following conditions hold:

1. \(X_1 \hookrightarrow X_0\) and \(E_1 \hookrightarrow E_0\) are continuous dense inclusions of Banach spaces. For fixed \(0 < \beta < \alpha < 1\), \(X_\alpha\) and \(X_\beta\) are continuous interpolation spaces corresponding to the inclusion \(X_1 \hookrightarrow X_0\).

2. There is an autonomous quasilinear parabolic equation

\[
\partial_\tau \hat{g}(\tau) = Q(\hat{g}(\tau)), \quad (\tau \geq 0),
\]

with the property that there exists a positive integer \(k\) such that for all \(\hat{g}\) in some open set \(G_\beta \subseteq X_\beta\), the domain \(D(L_{\hat{g}})\) of the linearization \(L_{\hat{g}}\) of \(Q\) at \(\hat{g}\) contains \(X_1\) and the map \(\hat{g} \mapsto L_{\hat{g}}|_{X_1}\) belongs to \(C^k(G_\beta, L(X_1, X_0))\).

3. For each \(\hat{g} \in G_\beta\), there exists an extension \(\tilde{L}_{\hat{g}}\) of \(L_{\hat{g}}\) to a domain \(\tilde{D}(\hat{g})\) that contains \(E_1\) (hence is dense in \(E_0\)).
Let $L_{\mathbf{g}}^C$ denote the linearization of the operator $L_{\mathbf{g}}$ of (A.2) at a stationary solution $\mathbf{g}$ of (A.2). Suppose there exists $\lambda_0 > 0$ such that the spectrum $\sigma$ of $L_{\mathbf{g}}^C$ admits the decomposition $\sigma = \sigma_0 \cup \{0\}$, where 0 is an eigenvalue of finite multiplicity and $\sigma_0 \subseteq \{z : \Re z \leq -\lambda_0\}$. If Assumptions 2.7 hold, then:

1. For each $\alpha \in [0, 1]$, there is a direct-sum decomposition $X_\alpha = X_\alpha^\circ \oplus X_\alpha^C$, where $X_\alpha^C$ is the finite-dimensional algebraic eigenspace corresponding to the null eigenvalue of $L_{\mathbf{g}}^C$.

2. For each $r \in \mathbb{N}$, there exists $d_r > 0$ such that for all $d \in (0, d_r]$, there exists a bounded $C^r$ map $\gamma^r_{\mathbf{g}} : B(\mathbf{X}_1, \mathbf{g}, d) \to \mathbf{X}_1^r$ such that $\gamma^r_{\mathbf{g}}(\mathbf{g}) = 0$ and $D\gamma^r_{\mathbf{g}}(\mathbf{g}) = 0$. The image of $\gamma^r_{\mathbf{g}}$ lies in the closed ball $B(\mathbf{X}_1, \mathbf{g}, d)$. Its graph is a local $C^r$ center manifold $\Gamma_{\text{loc}}^r = \{(h, \gamma^r_{\mathbf{g}}(h)) : h \in B(\mathbf{X}_1, \mathbf{g}, d)\} \subset \mathbf{X}_1$ satisfying $T_{\mathbf{g}}\Gamma_{\text{loc}}^r \simeq \mathbf{X}_1^r$. Moreover, $\Gamma_{\text{loc}}^r$ is invariant for solutions of (A.2) as long as they remain in $B(\mathbf{X}_1, \mathbf{g}, d) \times B(\mathbf{X}_1, 0, d)$.

3. Fix $\lambda \in (0, \lambda_0)$. Then for each $\alpha \in (0, 1)$, there exist $C > 0$ and $\delta \in (0, 1)$ such that for each initial datum $\mathbf{g}(0) \in B(\mathbf{X}_\alpha, \mathbf{g}, d)$ and all times $\tau \geq 0$ such that $\tilde{\mathbf{g}}(\tau) \in B(\mathbf{X}_\alpha, \mathbf{g}, d)$, the center manifold $\Gamma_{\text{loc}}^r$ is exponentially attractive in the stronger space $\mathbf{X}_1$ in the sense that

$$
\|\pi^\circ \tilde{\mathbf{g}}(\tau) - \gamma^r_{\mathbf{g}}(\pi^\circ \tilde{\mathbf{g}}(\tau))\|_{\mathbf{X}_1} \leq C_{\alpha, r} e^{-\lambda_0 \tau} \|\pi^\circ \mathbf{g}(0) - \gamma^r_{\mathbf{g}}(\pi^\circ \mathbf{g}(0))\|_{\mathbf{X}_\alpha}.
$$

Here, $\tilde{\mathbf{g}}(\tau)$ is the unique solution of (A.2), while $\pi^\circ$ and $\pi^C$ denote the projections onto $X^\circ_\alpha \cong (X^0_1, X^0_0)_\alpha^C$ and $X^C_\alpha$, respectively.

**References**

[1] Arthur L. Besse, *Einstein manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition.

[2] Christoph Böhm and Burkhard Wilking, *Manifolds with positive curvature operators are space forms*, Ann. of Math. (2) 167 (2008), no. 3, 1079–1097.

[3] Bennett Chow and Dan Knopf, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs, vol. 110, American Mathematical Society, Providence, RI, 2004.

[4] Giuseppe Da Prato and Pierre Grisvard, *Equations d'évolution abstraites non linéaires de type parabolique*, Ann. Mat. Pura Appl. (4) 120 (1979), 329–396.

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1. If $X$ is a Banach space with subspace $Y$ and $L : D(L) \subseteq X \to X$ is linear, then $L^Y$, the part of $L$ in $Y$, is defined by the action $L^Y : x \mapsto Lx$ on the domain $D(L^Y) = \{x \in D(L) : Lx \in Y\}$.

2. Note that $L_{\mathbf{g}}$ is the operator that appears in Assumption 2.
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[5] Xianzhe Dai, Xiaodong Wang, and Guofang Wei, On the variational stability of Kähler-Einstein metrics, Comm. Anal. Geom. 15 (2007), no. 4, 669–693.
[6] James Eells Jr. and J. H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964), 109–160.
[7] Christine Guenther, James Isenberg, and Dan Knopf, Stability of the Ricci flow at Ricci-flat metrics, Comm. Anal. Geom. 10 (2002), no. 4, 741–777.
[8] Richard S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geom. 17 (1982), no. 2, 255–306.
[9] __________, Four-manifolds with positive curvature operator, J. Differential Geom. 24 (1986), no. 2, 153–179.
[10] Dan Knopf, Convergence and stability of locally $\mathbb{R}^N$-invariant solutions of Ricci flow, J. Geom. Anal. 19 (2009), no. 4, 817–846.
[11] Dan Knopf and Andrea Young, Asymptotic stability of the cross curvature flow at a hyperbolic metric, Proc. Amer. Math. Soc. 137 (2009), no. 2, 699–709.
[12] Norihito Koiso, Nondeformability of Einstein metrics, Osaka J. Math. 15 (1978), no. 2, 419–433.
[13] Bernhard List, Evolution of an extended Ricci flow system, Comm. Anal. Geom. 16 (2008), no. 5, 1007–1048.
[14] John Lott, On the long-time behavior of type-III Ricci flow solutions, Math. Ann. 339 (2007), no. 3, 627–666.
[15] __________, Dimensional reduction and the long-time behavior of Ricci flow, Comment. Math. Helv. 85 (2010), no. 3, 485–534.
[16] John Lott and Natasa Sesum, Ricci flow on three-dimensional manifolds with symmetry, Comm. Math. Helv., to appear.
[17] Reto Müller, Ricci flow coupled with harmonic map flow, Ann. Sc. Ec. Norm. Sup. (4) 45 (2012), no. 1, 101–142.
[18] T. Oliynyk, V. Suneeta, and E. Woolgar, A gradient flow for worldsheet nonlinear sigma models, Nuclear Phys. B 739 (2006), no. 3, 441–458. MR2214659 (2006m:81185)
[19] Natasa Sesum, Linear and dynamical stability of Ricci-flat metrics, Duke Math. J. 133 (2006), no. 1, 1–26.
[20] Miles Simon, A class of Riemannian manifolds that pinch when evolved by Ricci flow, Manuscripta Math. 101 (2000), no. 1, 89–114.
[21] Gieri Simonett, Center manifolds for quasilinear reaction-diffusion systems, Differential Integral Equations 8 (1995), no. 4, 753–796.
[22] Jeffrey Streets, Regularity and expanding entropy for connection Ricci flow, J. Geom. Phys. 58 (2008), no. 7, 900–912.
[23] Andrew Strominger, Superstrings with torsion, Nuclear Phys. B 274 (1986), no. 2, 253–284. MR851702 (87m:81177)
[24] Hung Tran, Harnack estimates for Ricci flow on a warped product (2012), available at arXiv:1211.6448
[25] Michael Bradford Williams, Results on coupled Ricci and harmonic map flows (2010), available at arXiv:1012.0291
[26] Haotian Wu, Stability of complex hyperbolic space under curvature-normalized Ricci flow, Geom. Dedicata, to appear.
[27] Rugang Ye, Ricci flow, Einstein metrics and space forms, Trans. Amer. Math. Soc. 338 (1993), no. 2, 871–896.
[28] Andrea Young, Stability of Ricci Yang-Mills flow at Einstein Yang-Mills metrics, Comm. Anal. Geom. 18 (2010), no. 1, 77–100.

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