Generalizations of some results about the regularity properties of an additive representation function

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Abstract

Let $A = \{a_1, a_2, \ldots\}$ ($a_1 < a_2 < \ldots$) be an infinite sequence of nonnegative integers, and let $R_{A,2}(n)$ denote the number of solutions of $a_x + a_y = n$ ($a_x, a_y \in A$). P. Erdős, A. Sárközy and V. T. Sós proved that if $\lim_{N \to \infty} B(A, N) \sqrt{N} = +\infty$ then $|\Delta_1(R_{A,2}(n))|$ cannot be bounded, where $B(A, N)$ denotes the number of blocks formed by consecutive integers in $A$ up to $N$ and $\Delta_l$ denotes the $l$-th difference. Their result was extended to $\Delta_l(R_{A,2}(n))$ for any fixed $l \geq 2$. In this paper we give further generalizations of this problem.

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1 Introduction

Let $N$ denote the set of nonnegative integers. Let $k \geq 2$ be a fixed integer and let $A = \{a_1, a_2, \ldots\}$ ($a_1 < a_2 < \ldots$) be an infinite sequence of nonnegative integers. For $n = 0, 1, 2, \ldots$ let $R_{A,k}(n)$ denote the number of solutions of $a_{i_1} + a_{i_2} + \cdots + a_{i_k} = n$, $a_{i_1} \in A, \ldots, a_{i_k} \in A$, and we put

$$A(n) = \sum_{\substack{a \in A \leq n \atop a \leq n}} 1.$$ We denote the cardinality of a set $H$ by $\#H$. Let $B(A, N)$ denote the number of blocks formed by consecutive integers in $A$ up to $N$, i.e.,

$$B(A, N) = \sum_{\substack{n \leq N \atop n \in A, n-1 \notin A}} 1.$$ If $s_0, s_1, \ldots$ is given sequence of real numbers then let $\Delta_l s_n$ denote the $l$-th difference of the sequence $s_0, s_1, s_2, \ldots$ defined by $\Delta_1 s_n = s_{n+1} - s_n$ and $\Delta_l s_n = \Delta_1(\Delta_{l-1} s_n)$.

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In a series of papers [2], [3], [4] P. Erdős, A. Sárközy and V.T. Sós studied the regularity properties of the function $R_{A,2}(n)$. In [4] they proved the following theorem:

**Theorem A** If $\lim_{N \to \infty} \frac{B(A,N)}{\sqrt{N}} = \infty$, then $|\Delta_1(R_{A,2}(n))| = |R_{A,2}(n+1) - R_{A,2}(n)|$ cannot be bounded.

In [4] they also showed that the above result is nearly best possible:

**Theorem B** For all $\varepsilon > 0$, there exists an infinite sequence $A$ such that

(i) $B(A, N) \gg N^{1/2-\varepsilon}$,

(ii) $R_{A,2}(n)$ is bounded so that also $\Delta_1 R_{A,2}(n)$ is bounded.

Recently, [9] A. Sárközy extended the above results in the finite set of residue classes modulo a fixed $m$.

In [6] Theorem A was extended to any $k > 2$:

**Theorem C** If $k \geq 2$ is an integer and $\lim_{N \to \infty} \frac{B(A,N)}{\sqrt{N}} = \infty$, and $l \leq k$, then $|\Delta_l R_{A,k}(n)|$ cannot be bounded.

It was shown [8] that the above result is nearly best possible.

**Theorem D** For all $\varepsilon > 0$, there exists an infinite sequence $A$ such that

(i) $B(A, N) \gg N^{1/k-\varepsilon}$,

(ii) $R_{A,k}(n)$ is bounded so that also $\Delta_l R_{A,k}(n)$ is bounded if $l \leq k$.

In this paper we consider $R_{A,2}(n)$, thus simply write $R_{A,2}(n) = R_A(n)$. A set of positive integers $A$ is called Sidon set if $R_A(n) \leq 2$. Let $\chi_A$ denote the characteristic function of the set $A$, i.e.,

$$\chi_A(n) = \begin{cases} 1, & \text{if } n \in A \\ 0, & \text{if } n \notin A. \end{cases}$$

Let $\lambda_0, \ldots, \lambda_d$ be arbitrary integers with $\sum_{i=0}^d |\lambda_i| > 0$. Let $\mathbf{\lambda} = (\lambda_0, \ldots, \lambda_d)$ and define the function

$$B(A, \mathbf{\lambda}, n) = \left| \left\{ m : m \leq n, \sum_{i=0}^d \lambda_i \chi_A(m-i) \neq 0 \right\} \right|.$$ 

**Theorem 1.** We have

$$\limsup_{n \to \infty} \left| \sum_{i=0}^d \lambda_i R_A(n-i) \right| \geq \limsup_{n \to \infty} \frac{\left| \sum_{i=0}^d \lambda_i \right|}{2(d+1)^2} \left( \frac{B(A, \mathbf{\lambda}, n)}{\sqrt{n}} \right)^2.$$

The next theorem shows that the above result is nearly best possible:

**Theorem 2.** Let $\sum_{i=0}^d \lambda_i > 0$. Then for every positive integer $N$ there exists a set $A$ such that

$$\limsup_{n \to \infty} \left| \sum_{i=0}^d \lambda_i R_A(n-i) \right| \leq \limsup_{n \to \infty} 4 \sum_{i=0}^d |\lambda_i| \left( \frac{B(A, \mathbf{\lambda}, n)}{\sqrt{n}} \right)^2$$

and

$$\limsup_{n \to \infty} \frac{B(A, \mathbf{\lambda}, n)}{\sqrt{n}} \geq N.$$
Theorem 3. Let $\sum_{i=0}^{d} \lambda_i = 0$. Then we have
\[
\limsup_{n \to \infty} \left| \sum_{i=0}^{d} \lambda_i R_A(n-i) \right| \geq \limsup_{n \to \infty} \frac{\sqrt{2}}{e^2} \frac{\sqrt[3]{B(A, \lambda, n)}}{\sqrt{n}}.
\]

It is easy to see that if $\lambda = (\lambda_0, \lambda_1) = (-1, 1)$ then $B(A, \lambda, n) \geq B(A, n)$ thus Theorem 3 implies Theorem A. It is natural to ask that the exponent of $\frac{B(A, \lambda, n)}{\sqrt{n}}$ in the right hand side can be improved.

Problem 1. Is it true that if $\sum_{i=0}^{d} \lambda_i = 0$ then there exists a positive constant $C(\lambda)$ depends only on $\lambda$ such that for every set of nonnegative integers $A$ we have
\[
\limsup_{n \to \infty} \left| \sum_{i=0}^{d} \lambda_i R_A(n-i) \right| \geq \limsup_{n \to \infty} C(\lambda) \cdot \left( \frac{B(A, \lambda, n)}{\sqrt{n}} \right)^{3/2}.
\]

In the next theorem we prove that the exponent cannot grow over $3/2$.

Theorem 4. Let $\sum_{i=0}^{d} \lambda_i = 0$. For every positive integer $N$ there exists a set $A \subset \mathbb{N}$ such that
\[
N \leq \limsup_{n \to \infty} \left( \frac{B(A, \lambda, n)}{\sqrt{n}} \right) < \infty
\]
and
\[
\limsup_{n \to \infty} \left| \sum_{i=0}^{d} \lambda_i R_A(n-i) \right| \leq \limsup_{n \to \infty} 48(d+1)^{2d+7.5} \sum_{i=0}^{d} |\lambda_i| \left( \frac{B(A, \lambda, n)}{\sqrt{n}} \right)^{3/2} \left( \log \frac{B(A, \lambda, n)}{\sqrt{n}} \right)^{1/2}.
\]

2 Proof of Theorem 1

Since $-\lambda = (-\lambda_0, \ldots, -\lambda_d)$ and clearly
\[
\limsup_{n \to \infty} \left| \sum_{i=0}^{d} \lambda_i R_A(n-i) \right| = \limsup_{n \to \infty} \left| \sum_{i=0}^{d} (-\lambda_i) R_A(n-i) \right|,
\]

$B(A, \lambda, n) = B(A, -\lambda, n)$, therefore
\[
\limsup_{n \to \infty} \left| \sum_{i=0}^{d} \lambda_i \left( \frac{B(A, \lambda, n)}{\sqrt{n}} \right)^2 \right| \geq \limsup_{n \to \infty} \left| \sum_{i=0}^{d} (-\lambda_i) \left( \frac{B(A, -\lambda, n)}{\sqrt{n}} \right)^2 \right|,
\]
thus we may assume that $\sum_{i=0}^{d} \lambda_i > 0$. On the other hand we may suppose that
\[
\limsup_{n \to \infty} \left( \frac{B(A, \lambda, n)}{\sqrt{n}} \right)^2 > 0.
\]
It follows from the definition of the limsup that there exists a sequence $n_1, n_2, \ldots$ such that
\[
\lim_{j \to \infty} \frac{B(A, \lambda, n_j)}{\sqrt{n_j}} = \limsup_{n \to \infty} \frac{B(A, \lambda, n)}{\sqrt{n}}.
\]
To prove Theorem 1 we give a lower and an upper estimation to
\[
\sum_{\ell < n \leq 2n_j} \left( \sum_{i=0}^{d} \lambda_i R_A(n - i) \right). \tag{1}
\]

The comparison of the two bounds will give the result. First we give an upper estimation. Clearly we have
\[
\left| \sum_{\ell < n \leq 2n_j} \left( \sum_{i=0}^{d} \lambda_i R_A(n - i) \right) \right| \leq \sum_{\ell < n \leq 2n_j} \left| \sum_{i=0}^{d} \lambda_i R_A(n - i) \right| \leq 2n_j \max_{\ell < n \leq 2n_j} \left| \sum_{i=0}^{d} \lambda_i R_A(n - i) \right|.
\]

In the next step we give a lower estimation to (1). It is clear that
\[
\sum_{\ell < n \leq 2n_j} \left( \sum_{i=0}^{d} \lambda_i R_A(n - i) \right) = \sum_{\ell < n \leq 2n_j} (\lambda_0 + \ldots + \lambda_d) R_A(n)
\]
\[
- \left( (\lambda_1 + \ldots + \lambda_d) R_A(2n_j) + (\lambda_2 + \ldots + \lambda_d) R_A(2n_j - 1) + \lambda_d R_A(2n_j - d + 1) \right)
\]
\[
+ (\lambda_1 + \ldots + \lambda_d) R_A(\lfloor \sqrt{n_j} \rfloor) + (\lambda_2 + \ldots + \lambda_d) R_A(\lfloor \sqrt{n_j} \rfloor - 1) + \ldots + \lambda_d R_A(\lfloor \sqrt{n_j} \rfloor - d + 1).
\]

Obviously,
\[
R_A(m) = \#\{(a, a') : a + a' = m, a, a' \in A\} \leq 2 \cdot \#\{(a, a') : a + a' = m, a \leq a', a, a' \in A\}
\]
\[
\leq 2 \cdot \#\{(a : a \leq m/2, a \in A\} = 2A(m/2).
\]

It follows that
\[
\sum_{\ell < n \leq 2n_j} \left( \sum_{i=0}^{d} \lambda_i R_A(n - i) \right) \geq (\lambda_0 + \ldots + \lambda_d) \sum_{\ell < n \leq 2n_j} R_A(n) - \left( \sum_{i=0}^{d} |\lambda_i| \right) 2A(n_j) 2d
\]
\[
\geq \left( \sum_{i=0}^{d} \lambda_i \right) \#\{(a, a') : a + a' = n, \sqrt{n_j} < a, a' \leq n_j, a, a' \in A\} - \left( \sum_{i=0}^{d} |\lambda_i| \right) 4dA(n_j)
\]
\[
= \left( \sum_{i=0}^{d} \lambda_i \right) (A(n_j) - A(\sqrt{n_j}))^2 - O(A(n_j)).
\]

The inequality \( \sum_{i=0}^{d} \lambda_i \lambda_A(m - i) \neq 0 \) implies that \( [m - d, m] \cap A \neq 0 \). Then we have \( \{m : m \leq n, \sum_{i=0}^{d} \lambda_i \chi_A(m - i) \neq 0\} \subseteq \cup_{a \leq n, a \in A}[a, a + d] \), which implies that
\[
B(A, \lambda n) \leq |\cup_{a \leq n, a \in A}[a, a + d]| \leq A(n)(d + 1).
\]

By the definition of \( n_j \), there exists a constant \( c_1 \) such that
\[
\frac{B(A, \lambda n_j)}{\sqrt{n_j}} > c_1 > 0.
\]
It follows that $A(n_j) > \frac{c_1}{d+1} \sqrt{n_j}$ and clearly $\sqrt{n_j} \geq A(\sqrt{n_j})$. By using these facts we get that

$$\left( \sum_{i=0}^{d} \lambda_i \right) (A(n_j) - A(\sqrt{n_j}))^2 - O(A(n_j)) = (1 + o(1)) \left( \sum_{i=0}^{d} \lambda_i \right) A(n_j)^2 \geq (1 + o(1)) \left( \sum_{i=0}^{d} \lambda_i \right) \frac{B(A, \Delta, n_j)^2}{(d+1)^2}.$$ 

Comparing the lower and the upper estimations we get that

$$2n_i \max_{\sqrt{n_j} < n \leq 2n_j} \left| \sum_{i=0}^{d} \lambda_i R_A(n - i) \right| \geq \sum_{\sqrt{n_j} < n \leq 2n_j} \left( \sum_{i=0}^{d} \lambda_i R_A(n - i) \right) \geq (1 + o(1)) \sum_{i=0}^{d} \lambda_i \frac{B(A, \Delta, n_j)}{(d+1)^2} \geq (1 + o(1)) \sum_{i=0}^{d} \lambda_i \frac{B(A, \Delta, n_j)}{(d+1)^2}.$$ 

this implies that

$$2n_i \max_{\sqrt{n_j} < n \leq 2n_j} \left| \sum_{i=0}^{d} \lambda_i R_A(n - i) \right| \geq (1 + o(1)) \sum_{i=0}^{d} \lambda_i \frac{B(A, \Delta, n_j)}{(d+1)^2}.$$ 

To complete the proof we distinguish two cases. When

$$\limsup_{n \to \infty} \left( \frac{B(A, \Delta, n)}{\sqrt{n}} \right)^2 < \infty$$

then

$$\max_{\sqrt{n_j} < n \leq 2n_j} \left| \sum_{i=0}^{d} \lambda_i R_A(n - i) \right| \geq (1 + o(1)) \sum_{i=0}^{d} \lambda_i \frac{B(A, \Delta, n_j)}{(d+1)^2} \geq (1 + o(1)) \sum_{i=0}^{d} \lambda_i \frac{B(A, \Delta, n_j)}{(d+1)^2} \geq (1 + o(1)) \sum_{i=0}^{d} \lambda_i \frac{B(A, \Delta, n_j)}{(d+1)^2} \geq (1 + o(1)) \sum_{i=0}^{d} \lambda_i \frac{B(A, \Delta, n_j)}{(d+1)^2},$$

which gives the result.

When

$$\limsup_{n \to \infty} \left( \frac{B(A, \Delta, n)}{\sqrt{n}} \right)^2 = \infty$$

then

$$\limsup_{j \to \infty} \left( \frac{B(A, \Delta, n_j)}{\sqrt{n_j}} \right)^2 = \infty,$$

which implies by (2) that $\limsup_{n \to \infty} \left| \sum_{i=0}^{d} \lambda_i R_A(n - i) \right| = \infty$, which gives the result.
3 Proof of Theorem 2

It is well known \[5\] that there exists a Sidon set \(S\) with
\[
\limsup_{n \to \infty} \frac{S(n)}{\sqrt{n}} \geq \frac{1}{\sqrt{2}},
\]
where \(S(n)\) denotes the number of elements of \(S\) up to \(n\). Define the set \(T\) by removing the elements \(s\) and \(s'\) from \(S\) when \(s - s' \leq (N + 1)(d + 1)\). It is clear that \(T(n) \geq S(n) - 2(N + 1)(d + 1)\) and define the set \(A\) by
\[
A = T \cup (T + (d + 1)) \cup (T + 2(d + 1)) \cup \ldots \cup (T + N(d + 1)).
\]
It is easy to see that \(A(n) \geq S(n) - 2(N + 1)(d + 1)\) and define the set \(A\) by
\[
A = T \cup (T + (d + 1)) \cup (T + 2(d + 1)) \cup \ldots \cup (T + N(d + 1)).
\]
It is easy to see that \(A(n) \geq (N + 1)T(n) - N\). We will prove that \(B(A, \lambda, n) \geq A(n) - d\).

By the definitions of the sets \(T\) and \(A\) we get that if \(a < a'\), \(a, a' \in A\) then \(a - a' \geq d + 1\).

If
\[
\sum_{i=0}^{d} \lambda_i \chi_A(m - i) \neq 0
\]
then there is exactly one term, which is nonzero. Fix an index \(w\) such that \(\lambda_w \neq 0\). It follows that \(\sum_{i=0}^{d} \lambda_i \chi_A(a + w - i) \neq 0\) for every \(a \in A\). Hence,
\[
|B(A, \lambda, n)| \geq \# \{a : a + w \leq n, a \in A\} = A(n - w) \geq A(n) - w \geq A(n) - d
\]
\[
\geq (N + 1)T(n) - N - d \geq (N + 1)S(n) - 2(N + 1)^2(d + 1) - N - d.
\]
Thus we have
\[
\frac{B(A, \lambda, n)}{\sqrt{n}} \geq (N + 1) \frac{S(n)}{\sqrt{n}} - \frac{2(N + 1)^2(d + 1) + N + d}{\sqrt{n}}
\]
and
\[
\limsup_{n \to \infty} \left( \frac{B(A, \lambda, n)}{\sqrt{n}} \right)^2 \geq \frac{(N + 1)^2}{2} \geq N.
\]
By the definition of \(A\), we have
\[
R_A(m) = \sum_{i=0}^{N} \sum_{j=0}^{N} \# \{(t, t') : (t + i(d + 1)) + (t + j(d + 1)) = m, t, t' \in T\}
\]
\[
= \sum_{i=0}^{N} \sum_{j=0}^{N} R_T(m - (i + j)(d + 1)) \leq 2(N + 1)^2.
\]
Then we have
\[
\left| \sum_{i=0}^{d} \lambda_i R_A(n - i) \right| \leq \left( \sum_{i=0}^{d} |\lambda_i| \right) \max_n R_A(n) \leq \left( \sum_{i=0}^{d} |\lambda_i| \right) (N + 1)^2 \leq \limsup_{n \to \infty} 4 \cdot \left( \sum_{i=0}^{d} |\lambda_i| \right) \left( \frac{B(A, \lambda, n)}{\sqrt{n}} \right)^2,
\]
which gives the result.
4 Proof of Theorem 3

In the first case we assume that

\[
\limsup_{n \to \infty} \frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A, \lambda, n)}{\sqrt{n}} < \infty.
\]

We prove by contradiction. Assume that contrary to the conclusion of Theorem 3 we have

\[
\limsup_{n \to \infty} \left| \sum_{i=0}^{d} \lambda_i R_A(n - i) \right| < \limsup_{n \to \infty} \frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A, \lambda, n)}{\sqrt{n}}.
\tag{3}
\]

Throughout the remaining part of the proof of Theorem 3 we use the following notations: \(N\) denotes a positive integer. We write \(e^{2\pi i \alpha} = e(\alpha)\) and we put \(r = e^{-1/N}\), \(z = re(\alpha)\) where \(\alpha\) is a real variable (so that a function of form \(p(z)\) is a function of the real variable \(\alpha : p(z) = p(re(\alpha)) = P(\alpha)\)). We write \(f(z) = \sum_{a \in A} z^a\). (By \(r < 1\), this infinite series and all the other infinite series in the remaining part of the proof are absolutely convergent).

We start out from the integral \(I(N) = \int_0^1 |f(z)(\sum_{i=0}^{d} \lambda_i z^i)|^2 \, d\alpha\). We will give lower and upper bound for \(I(N)\). The comparison of these bounds will give a contradiction.

First we will give a lower bound for \(I(N)\). We write

\[
f(z) \left( \sum_{i=0}^{d} \lambda_i z^i \right) = \left( \sum_{n=0}^{\infty} \chi_A(n) z^n \right) \left( \sum_{i=0}^{d} \lambda_i z^i \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \lambda_0 \chi_A(n) + \lambda_1 \chi_A(n - 1) + \ldots + \lambda_d \chi_A(n - d) \right) z^n.
\]

It is clear that if \(\lambda_0 \chi_A(n) + \lambda_1 \chi_A(n - 1) + \ldots + \lambda_d \chi_A(n - d) \neq 0\), then \((\lambda_0 \chi_A(n) + \lambda_1 \chi_A(n - 1) + \ldots + \lambda_d \chi_A(n - d))^2 \geq 1\). Thus, by the Parseval formula, we have

\[
I(N) = \int_0^1 \left| f(z) \left( \sum_{i=0}^{d} \lambda_i z^i \right) \right|^2 \, d\alpha
\]

\[
= \int_0^1 \left| \sum_{n=0}^{\infty} \left( \lambda_0 \chi_A(n) + \lambda_1 \chi_A(n - 1) + \ldots + \lambda_d \chi_A(n - d) \right) z^n \right|^2 \, d\alpha
\]

\[
= \sum_{n=0}^{\infty} \left( \lambda_0 \chi_A(n) + \lambda_1 \chi_A(n - 1) + \ldots + \lambda_d \chi_A(n - d) \right)^2 \geq e^{-2} \sum_{n \leq N}^{\infty} \sum_{\lambda_0 \chi_A(n) + \lambda_1 \chi_A(n - 1) + \ldots + \lambda_d \chi_A(n - d) \neq 0} 1
\]

\[
= e^{-2} B(A, \lambda, N).
\]
Now we will give an upper bound for $I(N)$. Since the sums $\sum_{i=0}^{d} |\lambda_iR_A(n-i)|$ are nonnegative integers it follows from (3) that there exists an $n_0$ and an $\varepsilon > 0$ such that

$$\sum_{i=0}^{d} |\lambda_iR_A(n-i)| \leq \limsup_{n \to \infty} \frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A, \lambda, n)}{\sqrt{n}} (1 - \varepsilon). \quad (4)$$

for every $n > n_0$. On the other hand there exists an infinite sequence of real numbers $n_0 < n_1 < n_2 < \ldots < n_j < \ldots$ such that

$$\limsup_{n \to \infty} \frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A, \lambda, n)}{\sqrt{n}} (1 - \varepsilon) < \frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A, \lambda, n_j)}{\sqrt{n_j}}. \quad (5)$$

We get that

$$\limsup_{n \to \infty} \frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A, \lambda, n)}{\sqrt{n}} (1 - \varepsilon) < \frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A, \lambda, n_j)}{\sqrt{n_j}}. \quad (5)$$

Obviously, $f^2(z) = \sum_{n=0}^{\infty} R_A(n)z^n$. By our indirect assumption, the Cauchy inequality and the Parseval formula we have

$$I(N) = \int_{0}^{1} \left| f(z) \left( \sum_{i=0}^{d} \lambda_i z^i \right) \right|^2 \, d\alpha \leq \left( \sum_{i=0}^{d} |\lambda_i| \right) \int_{0}^{1} \left| f^2(z) \left( \sum_{i=0}^{d} \lambda_i z^i \right) \right| \, d\alpha$$

$$= \left( \sum_{i=0}^{d} |\lambda_i| \right) \int_{0}^{1} \left| \left( \sum_{n=0}^{\infty} R_A(n)z^n \right) \left( \sum_{i=0}^{d} \lambda_i z^i \right) \right| \, d\alpha = \left( \sum_{i=0}^{d} |\lambda_i| \right) \int_{0}^{1} \left| \sum_{n=0}^{\infty} \left( \sum_{i=0}^{d} \lambda_i R_A(n-i) \right) z^n \right| \, d\alpha$$

$$\leq \left( \sum_{i=0}^{d} |\lambda_i| \right) \left( \int_{0}^{1} \left| \sum_{n=0}^{\infty} \left( \sum_{i=0}^{d} \lambda_i R_A(n-i) \right) z^n \right|^2 \, d\alpha \right)^{1/2} = \left( \sum_{i=0}^{d} |\lambda_i| \right) \left( \sum_{n=0}^{\infty} \left( \sum_{i=0}^{d} \lambda_i R_A(n-i) \right)^2 r^{2n} \right)^{1/2} \cdot$$

In view of (4), (5) and the lower bound for $I(n_j)$ we

$$e^{-2}B(A, \lambda, n_j) < I(n_j) < \left( \sum_{i=0}^{d} |\lambda_i| \right) \left( \sum_{n=0}^{\infty} \left( \sum_{i=0}^{d} \lambda_i R_A(n-i) \right)^2 r^{2n} \right)^{1/2} \cdot$$

$$\leq \left( \sum_{i=0}^{d} |\lambda_i| \right) \left( \sum_{n=0}^{n_0} \left( \sum_{i=0}^{d} \lambda_i R_A(n-i) \right)^2 r^{2n} + \sum_{n=n_0+1}^{\infty} \left( \frac{\sqrt{2}}{e^2 \sum_{i=0}^{d} |\lambda_i|} \frac{B(A, \lambda, n_j)}{\sqrt{n_j}} \right)^2 r^{2n} \right)^{1/2}$$

$$< \left( \sum_{i=0}^{d} |\lambda_i| \right) \left( c_2 + \sum_{n=0}^{\infty} \left( \frac{2}{e^4 (\sum_{i=0}^{d} |\lambda_i|)^2} \frac{B^2(A, \lambda, n_j)}{n_j} (1 - \varepsilon) r^{2n} \right)^{1/2} \right),$$

where $c_2$ is a constant. Taking the square of both sides we get that

$$e^{-4}B^2(A, \lambda, n_j) < \left( \sum_{i=0}^{d} |\lambda_i| \right)^2 \left( c_2 + \sum_{n=0}^{\infty} \left( \frac{2}{e^4 (\sum_{i=0}^{d} |\lambda_i|)^2} \frac{B^2(A, \lambda, n_j)}{n_j} (1 - \varepsilon) \sum_{n=0}^{\infty} r^{2n} \right) \right). \quad (6)$$
It is easy to see that

\[ 1 - e^{-x} = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots > x - \frac{x^2}{2!} = x(1 - \frac{x}{2}) > \frac{x}{x+1} \]

for \(0 < x < 1\). Applying this observation, where \(r = e^{-1/n_j}\) we have

\[ \sum_{n=0}^{\infty} r^{2n} = \frac{1}{1-r^2} = \frac{1}{1-e^{-n_j^2}} < \frac{n_j}{2} + 1. \]

In view of (6) we obtain that

\[ e^{-4}B^2(A, \lambda, n_j) < \left( \sum_{i=0}^{d} |\lambda_i| \right)^2 \left( c_2 + \frac{2}{e^{4(\sum_{i=0}^{d} |\lambda_i|^2)}} \frac{B^2(A, \lambda, n_j)}{n_j} (1 - \varepsilon) \left( \frac{n_j}{2} + 1 \right) \right) \]

\[ < c_3 + e^{-4}B^2(A, \lambda, n_j)(1 - \varepsilon), \]

where \(c_3\) is an absolute constant and it follows that

\[ B^2(A, \lambda, n_j) < c_3e^4 + B^2(A, \lambda, n_j)(1 - \varepsilon), \]

or in other words

\[ B^2(A, \lambda, n_j) < \frac{c_3e^4}{\varepsilon}, \]

which is a contradiction if \(n_j\) is large enough because \(\lim_{j \to \infty} B(A, \lambda, n_j) = \infty\). This proves the first case.

Assume that

\[ \limsup_{n \to \infty} \sqrt{\frac{2}{\sum_{i=0}^{d} |\lambda_i|}} \frac{B(A, \lambda, n)}{\sqrt{n}} = \infty. \]

Then there exists a sequence \(n_1 < n_2 < \ldots\) such that

\[ \limsup_{j \to \infty} \frac{B(A, \lambda, n_j)}{\sqrt{n_j}} = \infty. \]

We prove by contradiction. Suppose that

\[ \limsup_{n \to \infty} \left| \sum_{i=0}^{d} \lambda_i R_A(n - i) \right| < \infty. \]

Then there exists a positive constant \(c_4\) such that \(\left| \sum_{i=0}^{d} \lambda_i R_A(n - i) \right| < c_4\) for every \(n\).

It follows that

\[ e^{-2}B(A, \lambda, n_j) < I(n_j) < \left( \sum_{i=0}^{d} |\lambda_i| \right) \left( \sum_{n=0}^{\infty} \left( \sum_{i=0}^{d} \lambda_i R_A(n - i) \right)^2 r^{2n} \right)^{1/2} < \left( c_4 \sum_{n=0}^{\infty} r^{2n} \right)^{1/2} < c_5 \sqrt{n_j}, \]

thus we have

\[ \frac{B(A, \lambda, n_j)}{\sqrt{n_j}} < c_5 e^2, \]

where \(c_5\) is a positive constant, which is absurd.
5 Proof of Theorem 4

We argue as Sárközy in [9]. In the first step we will prove the following lemma:

Lemma 1. There exists a set \( C_M \subset [0, M(d+1)-1] \) for which \( |R_{C_M}(n) - R_{C_M}(n-1)| \leq 12\sqrt{M(d+1)} \log M(d+1) \) for every nonnegative integer \( n \) and \( B(C_M, \Lambda, M(d+1)-1) \geq \frac{M}{2^e} \) if \( M \) is large enough.

Proof of Lemma 1 To prove the lemma we use the probabilistic method due to Erdős and Rényi. There is an excellent summary about this method in books [1] and [5]. Let \( \mathbb{P}(E) \) denote the probability of an event \( E \) in a probability space and let \( \mathbb{E}(X) \) denote the expectation of a random variable \( X \). Let us define a random set \( C \) with \( \mathbb{P}(n \in C) = \frac{1}{2} \) for every \( 0 \leq n \leq M(d+1)-1 \). In the first step we show that

\[
\mathbb{P}\left( \max_n |R_C(n) - R_C(n-1)| > 12\sqrt{M(d+1)} \log M(d+1) \right) < \frac{1}{2}
\]

Define the indicator random variable

\[
\varphi_C(n) = \begin{cases} 1, & \text{if } n \in C \\ 0, & \text{if } n \notin C. \end{cases}
\]

It is clear that

\[ R_C(n) = 2 \sum_{k < n/2} \varphi_C(k) \varphi_C(n-k) + \varphi_C(n/2) \]

sum of independent indicator random variables. Define the random variable \( \zeta_i \) by \( \zeta_i = \varphi_C(i) \varphi_C(n-i) \). Then we have

\[ R_C(n) = 2X_n + Y_n, \]

where \( X_n = \zeta_0 + \ldots + \zeta_{n-1} \) and \( Y_n = \varphi_C(n/2) \).

Case 1. Assume that \( 0 \leq n \leq M(d+1)-1 \). Obviously, \( \mathbb{P}(\zeta_i = 0) = \frac{3}{4} \) and \( \mathbb{P}(\zeta_i = 1) = \frac{1}{4} \) and

\[ \mathbb{E}(X_n) = \frac{\lfloor n/2 \rfloor}{4}. \]

As \( Y_n \leq 1 \), it is easy to see that the following events satisfy the following relations

\[
\begin{align*}
\{ \max_{0 \leq n \leq M(d+1)-1} |R_C(n) - R_C(n-1)| > 12\sqrt{M(d+1)} \log M(d+1) \} \\
\subset \{ \max_{0 \leq n \leq M(d+1)-1} \left( \left| R_C(n) - \frac{n}{4} \right| + \left| R_C(n-1) - \frac{n-1}{4} \right| > 10\sqrt{M(d+1)} \log M(d+1) \right) \} \\
\subset \{ \max_{0 \leq n \leq M(d+1)-1} \left( \left| R_C(n) - \frac{n}{4} \right| > 5\sqrt{M(d+1)} \log M(d+1) \right) \} \\
= \{ \max_{0 \leq n \leq M(d+1)-1} \left| 2X_n + Y_n - \frac{n}{4} \right| > 5\sqrt{M(d+1)} \log M(d+1) \} \\
\subset \{ \max_{0 \leq n \leq M(d+1)-1} \left| 2X_n - \frac{n}{4} \right| > 4\sqrt{M(d+1)} \log M(d+1) \} \\
= \{ \max_{0 \leq n \leq M(d+1)-1} \left| X_n - \frac{n}{8} \right| > 2\sqrt{M(d+1)} \log M(d+1) \} \\
\subset \{ \max_{0 \leq n \leq M(d+1)-1} \left| X_n - \frac{n^2+1}{24} \right| > \sqrt{M(d+1)} \log M(d+1) \}.
\end{align*}
\]
It follows that
\[
P\left( \max_{0 \leq n \leq M(d+1)-1} |R_C(n) - R_C(n-1)| > 12 \sqrt{M(d+1) \log M(d+1)} \right) \\
\leq P\left( \max_{0 \leq n \leq M(d+1)-1} \left| X_n - \frac{n+1}{4} \right| > \sqrt{M(d+1) \log M(d+1)} \right).
\]
\[
\leq \sum_{n=0}^{M(d+1)-1} P\left( \left| X_n - \frac{n+1}{4} \right| > \sqrt{M(d+1) \log M(d+1)} \right).
\]

It follows from the Chernoff type bound \[1\], Corollary A.1.7. that if the random variable \( X \) has Binomial distribution with parameters \( m \) and \( p \) then for \( a > 0 \) we have
\[
P(|X - mp| > a) \leq 2e^{-2a^2/m}. \tag{7}
\]
Applying (7) to \( \left| \frac{n+1}{2} \right| \) and \( p = \frac{1}{4} \) we have
\[
P\left( \left| X_n - \frac{n+1}{4} \right| > \sqrt{M(d+1) \log M(d+1)} \right) < 2 \cdot \exp\left( -\frac{2M(d+1) \log M(d+1)}{\left| \frac{n+1}{2} \right|} \right)
\]
\[
\leq 2e^{-\frac{M(d+1) \log M(d+1)}{M(d+1)}} = 2e^{-4 \log M(d+1)} = \frac{2}{(M(d+1))^4} \leq \frac{1}{4M(d+1)}. \tag{8}
\]

It follows that
\[
P\left( \{\max_{0 \leq n \leq M(d+1)-1} |R_C(n) - R_C(n-1)| > 12 \sqrt{M(d+1) \log M(d+1)}\} \right) \tag{9}
\]
\[
< \frac{M(d+1)}{4M(d+1)} = \frac{1}{4}.
\]

**Case 2.** Assume that \( M(d+1) \leq n \leq 2M(d+1) - 2 \).

Obviously, \( P(\zeta_i = 0) = \frac{3}{4} \) and \( P(\zeta_i = 1) = \frac{1}{4} \) when \( n - M(d+1) < i < \frac{n}{2} \), and if \( 0 \leq i \leq n - M(d+1) \) then \( \zeta_i = 0 \). Clearly we have
\[
E(X_n) = \frac{\left( 2M(d+1) - 1 - n \right)}{4}.
\]
As $Y_n \leq 1$, it is easy to see that the following relations holds among the events

$$\left\{ \max_{M(d+1) \leq n \leq 2M(d+1)-2} |R_C(n) - R_C(n-1)| > 12 \sqrt{M(d+1) \log M(d+1)} \right\}$$

$$\subseteq \left\{ \max_{M(d+1) \leq n \leq 2M(d+1)-2} \left| R_C(n) - \frac{M(d+1) - \frac{n}{2}}{4} + R_C(n-1) - \frac{M(d+1) - \frac{n-1}{2}}{4} \right| > 10 \sqrt{M(d+1) \log M(d+1)} \right\}$$

$$\subseteq \left\{ \max_{M(d+1) \leq n \leq 2M(d+1)-2} \left( \left| R_C(n) - \frac{M(d+1) - \frac{n}{2}}{4} \right| + \left| R_C(n-1) - \frac{M(d+1) - \frac{n-1}{2}}{4} \right| \right) > 10 \sqrt{M(d+1) \log M(d+1)} \right\}$$

$$\subseteq \left\{ \max_{M(d+1)-1 \leq n \leq 2M(d+1)-2} |R_C(n) - \frac{M(d+1) - \frac{n}{2}}{4}| > 5 \sqrt{M(d+1) \log M(d+1)} \right\}$$

$$= \left\{ \max_{M(d+1)-1 \leq n \leq 2M(d+1)-2} \left| 2X_n + Y_n - \frac{2M(d+1) - n}{4} \right| > 5 \sqrt{M(d+1) \log M(d+1)} \right\}$$

$$\subseteq \left\{ \max_{M(d+1)-1 \leq n \leq 2M(d+1)-2} \left| 2X_n - \frac{2M(d+1) - n}{4} \right| > 4 \sqrt{M(d+1) \log M(d+1)} \right\}$$

$$= \left\{ \max_{M(d+1)-1 \leq n \leq 2M(d+1)-2} \left| X_n - \frac{2M(d+1) - n}{8} \right| > 2 \sqrt{M(d+1) \log M(d+1)} \right\}$$

$$\subseteq \left\{ \max_{M(d+1)-1 \leq n \leq 2M(d+1)-2} \left| X_n - \frac{2M(d+1) - 1 - n}{8} \right| > \sqrt{M(d+1) \log M(d+1)} \right\}$$

It follows that

$$\mathbb{P} \left( \max_{M(d+1)-1 \leq n \leq 2M(d+1)-2} |R_C(n) - R_C(n-1)| > 12 \sqrt{M(d+1) \log M(d+1)} \right)$$

$$\leq \mathbb{P} \left( \max_{M(d+1)-1 \leq n \leq 2M(d+1)-1} \left| X_n - \frac{2M(d+1) - 1 - n}{4} \right| > \sqrt{M(d+1) \log M(d+1)} \right)$$

$$\leq \sum_{n=M(d+1)-1}^{2M(d+1)-2} \mathbb{P} \left( \left| X_n - \frac{2M(d+1) - 1 - n}{4} \right| > \sqrt{M(d+1) \log M(d+1)} \right)$$

Applying (7) for $m = \frac{2M(d+1) - 1 - n}{4}$ and $\lambda = \frac{1}{4}$ we have for $M(d+1) \leq n \leq 2M(d+1) - 2$

$$\mathbb{P} \left( \left| X_n - \frac{2M(d+1) - 1 - n}{4} \right| > \sqrt{M(d+1) \log M(d+1)} \right) < 2 \cdot \exp \left( -\frac{2M(d+1) \log M(d+1)}{2 M(d+1) - 1 - n} \right)$$

$$< 2e^{-\frac{M(d+1) \log M(d+1)}{M(d+1)}} = 2e^{-4 \log M(d+1)} = \frac{2}{(M(d+1))^4} < \frac{1}{4M(d+1)}$$
and by (8) we have
\[ P \left( \left| X_{M(d+1)-1} - \frac{n+1}{4} \right| > \sqrt{M(d+1) \log M(d+1)} \right) < \frac{1}{4M(d+1)}. \]

It follows that
\[ P \left( \max_{M(d+1) \leq n \leq 2M(d+1)-2} |R_C(n) - R_C(n-1)| > 12 \sqrt{M(d+1) \log M(d+1)} \right) < \frac{M(d+1)}{4M(d+1)} = \frac{1}{4}. \]  

By (9) and (10) we get that
\[ P \left( \max_{0 \leq n \leq 2M(d+1)-2} |R_C(n) - R_C(n-1)| > 12 \sqrt{M(d+1) \log M(d+1)} \right) < \frac{1}{2}. \]  

In the next step we show that
\[ P \left( B(C, \lambda, M(d+1) - 1) < \frac{M}{2d+2} \right) < \frac{1}{2}. \]

It is clear that the following events $E_1, \ldots, E_M$ are independent:
\[
E_1 = \left\{ \sum_{i=0}^{d} \lambda_i \theta_{C}(d - i) \neq 0 \right\}, \\
E_2 = \left\{ \sum_{i=0}^{d} \lambda_i \theta_{C}(d + 1 + d - i) \neq 0 \right\}, \\
\vdots \\
E_M = \left\{ \sum_{i=0}^{d} \lambda_i \theta_{C}((m-1)(d+1) + d - i) \neq 0 \right\}.
\]

Obviously, $P(E_i) = P(E_j)$, where $1 \leq i, j \leq M$. Let $p = P(E_1)$. It is clear that there exists an index $u$ such that $\lambda_u \neq 0$. Thus we have
\[ p \geq P(\theta_{C}(0) = 0, \theta_{C}(1) = 0, \ldots, \theta_{C}(u-1) = 0, \theta_{C}(u) = 1, \theta_{C}(u+1) = 0, \ldots, \theta_{C}(d) = 0) = \frac{1}{2d+1}. \]

Define the random variable $Z$ as the number of occurrence of the events $E_j$. It is easy to see that $Z$ has Binomial distribution with parameters $M$ and $p$. Apply the Chernoff bound (7) we get that
\[ P \left( |Z - Mp| > \frac{Mp}{2} \right) < 2 \exp^{-2(Mp/2)^2/M} < 2 \exp^{-\frac{M}{2(2d+2)}} < \frac{1}{2}. \]
if \( M \) is large enough. On the other hand, we have
\[
\frac{1}{2} > \mathbb{P} \left( \left| Z - Mp \right| > \frac{Mp}{2} \right) \geq \mathbb{P} \left( Z < \frac{Mp}{2} \right) \geq \mathbb{P} \left( Z < \frac{M}{2d+2} \right).
\]
Hence,
\[
\mathbb{P} \left( B(C, \Delta, 2M(d+1) - 2) < \frac{M}{2d+2} \right) < \frac{1}{2}.
\]
Let \( \mathcal{E} \) and \( \mathcal{F} \) be the events
\[
\mathcal{E} = \left\{ \max_{0 \leq n \leq 2M(d+1)-2} \left| R_C(n) - R_C(n-1) \right| > 12\sqrt{M(d+1) \log M(d+1)} \right\},
\]
\[
\mathcal{F} = \left\{ B(C, \Delta, M(d+1) - 1) < \frac{M}{2d+2} \right\}.
\]
It follows from (11) and (12) that
\[
\mathbb{P} \left( \mathcal{E} \cup \mathcal{F} \right) < 1,
\]
then
\[
\mathbb{P} \left( \mathcal{E} \cap \mathcal{F} \right) > 0,
\]
therefore there exists a suitable set \( C_M \) if \( M \) is large enough, which completes the proof of Lemma 1.

We are ready to prove Theorem 4. It is well known [5] that there exists a Sidon set \( S \) with
\[
\limsup_{n \to \infty} \frac{S(n)}{\sqrt{n}} \geq \frac{1}{\sqrt{2}},
\]
where \( S(n) \) is the number of elements of \( S \) up to \( n \). Let \( s, s' \in S \) and assume that \( s > s' \). Define \( S_M = S \setminus \{ s, s' \in S : s - s' \leq 2M(d+1) \} \) and let \( A = C_M + S_M \), where \( C_M \) is the set from the lemma.
\[
\left| \sum_{i=0}^{d} \lambda_i R_A(n-i) \right| = \left| \sum_{i=0}^{d} \lambda_i \# \{(a, a') : a + a' = n-i, a, a' \in A \} \right|
\]
\[
= \left| \sum_{i=0}^{d} \lambda_i \# \{(s, c, s', c') : s + c + s' + c' = n-i, s, s', c, c' \in C_M \} \right|
\]
\[
= \left| \sum_{i=0}^{d} \sum_{j=0}^{2M(d+1)} \lambda_i \# \{(s, c, s', c') : c + c' = j, s + s' = n-i-j, s, s', c, c' \in C_M \} \right|
\]
\[
= \left| \sum_{i=0}^{d} \sum_{j=0}^{2M(d+1)} \lambda_i R_{C_M}(j) R_{S_M}(n-i-j) \right|
\]
\[
= \left| \sum_{j=0}^{2M(d+1)} \sum_{i=0}^{d} \lambda_i R_{C_M}(j) R_{S_M}(n-i-j) \right| = \left| \sum_{k=0}^{2M(d+1)+d} \sum_{i=0}^{d} \lambda_i R_{C_M}(k-i) R_{S_M}(n-k) \right|
\]
In the next step we give an upper estimation to $|\sum_{i=0}^{d} \lambda_i R_{C_M} (k - i)|$. We have

$$ |\lambda_0 R_{C_M} (k) + \ldots + \lambda_d R_{C_M} (k - d)| = |\lambda_0 (R_{C_M} (k) - R_{C_M} (k - 1)) + (\lambda_0 + \lambda_1) (R_{C_M} (k - 1) - R_{C_M} (k - 2)) + \ldots + (\lambda_0 + \lambda_1 + \ldots + \lambda_{d-1}) (R_{C_M} (k - d + 1) - R_{C_M} (k - d)) + (\lambda_0 + \lambda_1 + \ldots + \lambda_d) R_{C_M} (k - d)|. $$

Since $\sum_{i=0}^{d} \lambda_i = 0$, the last term in the previous sum is zero. Then we have

$$ |\lambda_0 R_{C_M} (k) + \ldots + \lambda_d R_{C_M} (k - d)| \leq d \left( \sum_{i=0}^{d} |\lambda_i| \right) \max_t |R_{C_M} (t) - R_{C_M} (t - 1)| \leq 12d \sum_{i=0}^{d} |\lambda_i| \sqrt{M(d + 1) \log M(d + 1)}. $$

Then we have

$$ \left| \sum_{i=0}^{d} \lambda_i A(n - i) \right| \leq 48d \sum_{i=0}^{d} |\lambda_i| (M(d + 1))^{3/2} (\log M(d + 1))^{1/2}. $$

We give a lower estimation to

$$ \limsup_{n \to \infty} \frac{B(A, \lambda, n)}{\sqrt{n}}. $$

If $0 \leq v \leq M(d + 1) - 1$ and $\sum_{i=0}^{d} \lambda_i \chi_{C_M} (v - i) \neq 0$ then $\sum_{i=0}^{d} \lambda_i \chi_{A} (s + v - i) \neq 0$ for every $s \in S_M$. Then we have

$$ B(A, \lambda, n) \geq (S_M (N) - 1) B(C_M, \lambda, (M + 1) - 1). $$

Thus we have

$$ \limsup_{n \to \infty} \frac{B(A, \lambda, n)}{\sqrt{n}} \geq \frac{M}{2d + 2.5}. $$

It follows that

$$ \limsup_{n \to \infty} \left| \sum_{i=0}^{d} \lambda_i A(n - i) \right| \leq 48d \sum_{i=0}^{d} |\lambda_i| \left( (M(d + 1))^{3} \log M(d + 1) \right)^{1/2} \leq \limsup_{n \to \infty} 48(d + 1)^{3} 2^{3d + 7.5} \sum_{i=0}^{d} |\lambda_i| \left( \frac{B(A, \lambda, n)}{\sqrt{n}} \right)^{3} \log \frac{B(A, \lambda, n)}{\sqrt{n}} \left( \frac{B(A, \lambda, n)}{\sqrt{n}} \right)^{1/2}, $$

if $M$ is large enough. The proof of Theorem 4 is completed.
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