Unnormalized Measures in Information Theory

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Abstract—Information theory is built on probability measures and by definition a probability measure has total mass 1. Probability measures are used to model uncertainty, and one may ask how important it is that the total mass is one. We claim that the main reason to normalize measures is that probability measures are related to codes via Kraft’s inequality. Using a minimum description length approach to statistics we will demonstrate with that measures that are not normalized require a new interpretation that we will call the Poisson interpretation. With the Poisson interpretation many problems can be simplified. The focus will shift from probabilities to mean values. We give examples of improvements of test procedures, improved inequalities, simplified algorithms, new projection results, and improvements in our description of quantum systems.

I. INTRODUCTION

In 1933 Kolmogorov gave a firm foundation to probability theory by defining a probability measure as a measure for which the total mass is 1 [1]. After his article, it was possible to study probability theory as a purely mathematical topic and that has given rise to a tremendous number of results with both practical applications and applications in other branches of mathematics.

In his 1933 article it is stated as an axiom that the total probability mass is one. One may look for theorems that justify that the total mass should be one. A number of important theorems are formulated in terms of probability measures, but a closer inspection reveals that they are essentially about measures with finite total mass that may be normalized to be one. For instance any compact group has a unique invariant probability measure [2], but the more general theorem is that any locally compact group has a left invariant measure (Haar measure) that is unique except for a multiplicative constant. Since a compact group has finite Haar measure one can normalize the measure, but the theorem does not tell why one would prefer to normalize the measure so that the total mass is one rather than say two or three.

One of the very few theorems that really gives a preference for measures of total mass one, is Kraft’s Inequality.

Theorem 1 (Modified version of [3, Thm. 1]). Let \( \ell : \mathcal{A} \to \mathbb{R} \) be a function. Then the function \( \ell \) satisfies Kraft’s Inequality if and only if for any \( \epsilon > 0 \) there exists an integer \( n \) and a uniquely decodable fixed-to-variable length block code \( \kappa : \mathcal{A}^n \to \mathbb{B}^* \) such that

\[
\ell (a^n) - \frac{1}{n} \sum_{i=1}^{n} \ell (a_i) \leq \epsilon
\]

where \( \bar{\ell}_\kappa (a^n) \) denotes the length \( \ell_\kappa (a^n) \) divided by \( n \). The uniquely decodable block code can be chosen to be prefix free.

If we define \( \mu (a) = e^{-\ell (a)} \) we get a one-to-one correspondence be abstract codelength functions \( \ell \) and sub-probability measures \( \mu \).

Theorem 2. Let \( \mu \) denote a measure on the discrete alphabet \( \mathcal{A} \) and let \( A \subseteq \mathcal{A} \) and assume that \( \mu (A) < \infty \). Then the abstract code length function \( \ell \) that minimize the mean code length

\[
\sum_{a \in A} \mu (a) \ell (a)
\]

is

\[
\ell (a) = - \ln (\mu (a \mid A))
\]

where

\[
\mu (a \mid A) = \frac{\mu (a)}{\sum_{a \in A} \mu (a)}.
\]

We see that probability measures are naturally associated with the problem of optimizing code length. In particular this theorem justify the use of conditional probabilities even in cases where the measure \( \mu \) is not finite.

Since information theory via Kraft’s inequality may justify that the total mass should be one, we may ask if unnormalized measures are of any use in information theory. In this note we will review to what extend it is really needed to assume that our measures are normalized so that the total mass is 1. New results will be developed, but most of the proofs are omitted. Some of the proofs are simple modifications of previously published proof.

II. THE POISSON INTERPRETATION

If \( P \) and \( Q \) are probability measures and if \( P \) is absolutely continuous with respect to \( Q \) information divergence is defined...
\[
D(P\|Q) = \sum_i p_i \ln \left( \frac{p_i}{q_i} \right) 
\]

which means that information divergence is the difference between mean value of the optimal code length \( \ln \left( \frac{1}{q_i} \right) \) corresponding to \( P \) and the optimal code length corresponding to \( Q \) where both means are calculated with respect to \( P \). Next we will extend divergence from probability vectors to arbitrary positive vectors.

We define a Bernoulli random vector as a random vector, which equals one of the base vectors with probability 1. If \( \vec{X} \) is a Bernoulli random vector then \( E\left( \vec{X} \right) \) is a probability vector.

We say that \( \vec{Y} \) is a Bernoulli sum if \( \vec{Y} \) is a sum of independent Bernoulli random vectors. Let \( Po(\lambda) \) denote the Poisson distribution with mean value \( \lambda \). If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) then the multivariate Poisson distribution \( Po(\lambda) \) is defined as the product measure \( \bigotimes_{i=1}^n Po(\lambda_i) \). With this terminology we can generalize a result from [4]:

**Theorem 3.** The maximum entropy distribution of Bernoulli sums \( \vec{Z} \) satisfying \( E(\vec{Z}) = \vec{\lambda} \) is the multivariate Poisson distribution with mean value \( \vec{\lambda} \).

The idea of thinning is that a proportion of the observations is discarted. In an \( \alpha \)-thinning an observation is kept with probability \( \alpha \). We note that the \( \alpha \)-thinning of the binomial distribution \( bin(n, p) \) is the binomial distribution \( bin(n, \alpha \cdot p) \) and the \( \alpha \)-thinning of the Poisson distribution \( Po(\lambda) \) is the Poisson distribution \( Po(\alpha \cdot \lambda) \). We will extend the notion of thinning of random variables to thinning of random vectors. If \( \vec{Y} = \vec{X}_1 + \vec{X}_2 + \cdots + \vec{X}_n \) then the \( \alpha \)-thinning of the distribution of \( \vec{Y} \) is the distribution of \( \vec{B}_1 \cdot \vec{X}_1 + \vec{B}_2 \cdot \vec{X}_2 + \cdots + \vec{B}_n \cdot \vec{X}_n \) where \( \vec{B}_1, \vec{B}_2, \ldots, \vec{B}_n \) are iid Bernoulli random variables with succes probability \( \alpha \). If the \( P \) is the distribution of \( \vec{Y} \) then the distribution of the \( \alpha \)-thinning of \( \vec{Y} \) is denoted \( T_\alpha(P) \). This vector thinning essentially thins each of the coordinates of the vector independently. These definitions allow us to prove vector versions of results from [5], [6].

**Theorem 4** (Law of thin vectors). Let \( \vec{Z} \) be a random vectors with values in \( \mathbb{N}_0^k \) and with \( E(\vec{Z}) = \vec{\lambda} \). If \( \vec{Z} \) has distribution \( P \), then \( T_{1/n}(P^n) \) has mean value \( \vec{\lambda} \). Further \( T_{1/n}(P^n) \) convergesences to the maximum entropy distribution in total variation. If \( \vec{Z} \) is a Bernoulli sums then \( D\left( T_{1/n}(P^n) \big\| Po(\vec{\lambda}) \right) \to 0 \) and \( H\left( T_{1/n}(P^n) \right) \to H\left( Po(\vec{\lambda}) \right) \) for \( n \to \infty \).

**Theorem 5.** If the probability vectors \( P = \vec{\lambda} \) and \( Q = \vec{\mu} \) are distributions of Bernoulli random vectors, then

\[
D(P\|Q) = D\left( T_{1/n}(P^n) \big\| T_{1/n}(Q^n) \right) 
\]

We have

\[
D\left( Po(\vec{\lambda}) \big\| Po(\vec{\mu}) \right) = \sum_{i=1}^k D\left( Po(\lambda_i) \big\| Po(\mu_i) \right) = \sum_{i=1}^k \lambda_i \ln \left( \frac{\lambda_i}{\mu_i} \right) - (\lambda_i - \mu_i) 
\]

For probability vectors the original formula for divergence is recovered, but the interpretation is different. Information divergence is the difference in mean code length between the code corresponding to \( \bigotimes_i Po(\lambda_i) \) and \( \bigotimes_i Po(\mu_i) \) where the mean is calculated with respect to \( \bigotimes_i Po(\lambda_i) \). With this interpretation we can easily extend the definition of \( D(P\|Q) \) to situations where \( P \) and \( Q \) are general measures rather than probability measures. We will write \( D\left( \vec{\lambda} \big\| \vec{\mu} \right) = \sum_{i=1}^k \lambda_i \ln \left( \frac{\lambda_i}{\mu_i} \right) - (\lambda_i - \mu_i) \) and note that in this formula \( \vec{\lambda} \) and \( \vec{\mu} \) may be arbitrary vectors with positive entries, i.e. \( \vec{\lambda} \) and \( \vec{\mu} \) are arbitrary measures.

On the set of measures we have two basic operations. The measures \( P \) and \( Q \) can be added. The interpretation is that two experiments are performed independently. For each category \( i \) we get counts \( X_i \) and \( Y_i \), and the results are combined by adding the counts to \( X_i + Y_i \). If \( \alpha \in [0, 1] \) then a measure \( P \) can be multiplied by \( \alpha \). The interpretation is that if one has obtained a count \( X_i \) in category \( i \) then \( X_i \) is replaced by \( Z_i \sim bin(X_i, \alpha) \) corresponding to removing each observations with probability \( \alpha \). In information theory this corresponds to concatenation and applying a deletion channel.

The advantage of using Poisson distributions is that this class of distributions is closed under repetition and thinning. The interpretation of the measure \( P \) is that it is a vector \( (p_1, p_2, \ldots, p_k) \) which the mean value of a rando vector of counts with distribution \( \bigotimes_i Po(p_i) \).

**III. TESTING GOODNESS-OF-FIT**

Here we will look at the consequences for testing Good-of-Fit in one of the simplest possible setups. We will test if coin is fair, and we perform an experiment where we count the number of heads \( X \) and the number of tails \( Y \) after tossing the coin a number of times. Our nul-hypothesis is that there is symmetry between heads and tails. Here we will compare the analysis for the case when we have observed \( X = \ell \) and \( Y = m \).

**A. Classical analysis**

Classically one will fix the number of tosses so that \( X+Y = n \), and assume that \( X \) has a binomial distribution with success probability \( p \). The classical nul-hypothesis is that \( p = 1/2 \).
The maximum likelihood estimate of \( p \) is \( \ell/n \). The divergence is

\[
D \left( \text{bin} \left( n, \frac{\ell}{n} \right) \| \text{bin} \left( n, \frac{1}{2} \right) \right) = n \cdot \left( \frac{1}{n} \ln \left( \frac{\ell/n}{1/2} \right) + \frac{m}{n} \ln \left( \frac{m/n}{1/2} \right) \right).
\]

We introduce the signed log-likelihood as

\[
G_n(x) = \begin{cases} 
-2 \cdot D \left( \text{bin} \left( n, \frac{x}{n} \right) \| \text{bin} \left( n, \frac{1}{2} \right) \right)^{1/2}, & \text{if } x < n/2; \\
+2 \cdot D \left( \text{bin} \left( n, \frac{x}{n} \right) \| \text{bin} \left( n, \frac{1}{2} \right) \right)^{1/2}, & \text{if } x \geq n/2.
\end{cases}
\]

In [7, Cor 7.2] it is proved that

\[
\Pr (X < k) \leq \Phi \left( G_n(k) \right) \leq \Pr (X \leq k).
\]

A QQ plot with a Gaussian distribution on the first axis and the distribution of \( G(X) \) on the second axis one gets a stair with horizontal steps each intersecting the line \( x = y \) corresponding to a perfect match between the distribution of \( G(X) \) and a standard Gaussian distribution. If we square \( G(X) \) we get 2 times divergence, which is often called the \( G^2 \)-statistic. Due to symmetry between head and tail the intersection property is also satisfied when the distribution of the \( G^2 \)-statistic is compared with a \( \chi^2 \)-distribution [8]. This is illustrated in Figure 1. Instead of using the Gaussian approximation one could calculate tail probabilities exactly (Fisher’s exact test), but as we shall see below we can do better.

To symmetry between head and tail the intersection property is i.e. the same expression as in the classical analysis. Since \( X \) is binomial given that \( X + Y = n \) we have

\[
\Pr (X < k \mid N = n) \leq \Phi \left( G_n(k) \right) \leq \Pr (X \leq k \mid N = n),
\]

Since the distribution of \( (G_N(X))^2 \) is close to a \( \chi^2 \)-distribution under the condition \( N = n \) the same is true for \( (G_N(X))^2 \) when we take the mean value over \( N \). Since each of the steps intersect the straight line near the mid point of the step the effect of taking the mean value with respect to \( N \) is that the steps to a large extend cancel out as illustrated in Figure 2. If one were testing a nul-hypothesis with less symmetry one will essentially get the same result except that the left tail and the right tail of the signed log-likelihood should be handled separately.

For statistical analysis one should not fix the sample size before sampling. A better procedure is to sample for a specific time so that the sample size becomes a random variable. Often in practice this is how sampling takes place and if the sample size is really random it may even be misleading to analyze data as in the classical setup.

**IV. Quantum states**

We have argued that it is often useful to consider vectors of mean values rather than probability vectors. Since the notion of quantum states and measurements are defined in terms of probability vectors [9] it is relevant to define these concepts in terms of mean values rather than probabilities. The limited space available in this paper does not allow us to go into a proper treatment of this problem. Instead we will just give an example of how the shift in focus get us closer to the experimental reality than using the model based on probabilities.

Often the double slit experiment is presented to illustrate how quantum mechanics differ from classical mechanics. A simplified version of the double slit experiment is the Mach-Zehnder interferometer where photons are emitted by a laser. If the photons can take both paths then interference implies that only Detector 1 will detect photons as illustrated in Figure 3.
one of the paths is blocked then no interference takes place and photons will be detected at both detector 1 and detector 2. In most descriptions of the double slit experiment it is explained that the interference takes place even if “the intensity of the laser is so low that it only emits single photons”. Typically it is stated that the probability for detection at Detector 1 is 1 if both paths are possible, but if one of the paths is blocked then detection of the photon at each of the detectors 1 or 2 have probability 1/2. In an attempt to model quantum mechanics by concepts related to probability theory we get a description that appear paradoxical.

A real laser emits light that can be described by a coherent state and in this case the photon detections follows a Poisson process. This has been used to build cryptographic protocols [10]. A more realistic description of the Mach-Zehnder interferometer is as follows. If both paths are open then detector 1 detects photons according to a Poisson process and detector 2 does not detect any photons due to interference.

The idea of single photons is only imposed in hindsight after the detection has taken place. In order to get a probability measure one condition on the event that a specific total number of photons have been observed. It should be noted that the conditioning is part of the measurement or of the processing of the data after the measurement. It is not part of the preparation of the system. With a correct description in terms of intensities of Poisson processes the paradoxical description involving paths of individual photons dissolves.

Fig. 3. Mach-Zehnder Interferometer. Here both paths are open and Detector 2 does not detect any photons due to interference.

V. PROJECTIONS BASED ON F-DIVERGENCES

Conditioning is a special case an information projection[11], and maximum likelihood estimation is a special case of a reversed information projection. Both information divergence and reverse information divergence are examples of f-divergences, and for this reason our next results are formulated for f-divergences.

Let $f : [0; \infty] \to \mathbb{R}$ denote a convex function such that $f(x) \geq 0$ with equality if $x = 1$. Define

$$f(0) = \lim_{x \to 0} f(x),$$

$$f'(\infty) = \lim_{x \to \infty} \frac{f(x)}{x}.$$  

We introduce the convention that

$$f\left(\frac{x}{\theta}\right) \cdot \theta = f'(\infty) \cdot x.$$  

For finite measures $P$ and $Q$ the $f$-divergence is defined as a number in $[0, \infty]$ given by

$$D_f(P, Q) = \sum_i f\left(\frac{p_i}{q_i}\right) \cdot q_i.$$  

For information divergence $f(x) = x \cdot \ln(x) - (x - 1)$ with $f'(\infty) = \infty$. For reversed information divergence $f(x) = -\ln(x) + x - 1$ with $f'(\infty) = 1$.

Let $C$ denote a convex set and let $Q$ denote a distribution. Then define

$$D(C, Q) = \inf_{P \in C} D_f(P, Q).$$  

A sequence $P_n \in C$ is said to be asymptotically optimal if

$$D_f(P_n, Q) \to D_f(C, Q)$$

for $n \to \infty$.

**Theorem 6.** Assume that $f$ is strictly convex and assume that $C \subseteq M_+$ and $Q \subseteq M_+$. If $D_f(C, Q) < \infty$ then there exists a bounded measure $Q^* \in M_+$ such that for an asymptotically optimal sequence $P_n$ we have $P_n(i) \to Q^*(i)$ for $Q$-almost any $i$. In particular, the measure $Q^*$ is unique $Q$-almost surely.

**Theorem 7.** Assume that $C \subseteq M_+$ and $Q \subseteq M_+$ and $D_f(C, Q) < \infty$. If $f$ is strictly convex and $f'(\infty) = \infty$ then the projection $Q^*$ is a unique probability measure.

**Theorem 8.** Let $Q$ denote a probability measure and let $g : \mathbb{N} \to \mathbb{R}$ denote a positive function and assume that mean value $\sum g(i) \cdot q_i$ is $\mu$. Assume that $0 < \bar{\mu} < \mu$. Let $C$ denote the compact set of measures $P$ for which $\sum g(i) \cdot p_i \leq \bar{\mu}$, and assume that $D_f(C, Q) < \infty$. If $f'(\infty) < \infty$ then the projection of $Q$ on $C$ satisfies

$$\sum g(i) \cdot q_i^* = \bar{\mu},$$

$$\sum q_i^* < \sum q_i.$$  

In addition $Q^*$ is absolutely continuous with respect to $Q$.

The last result demonstrates that unnormalized measures naturally appear as reversed information projections on sets of probability measures. This is of particular relevance for new methods for statistical testing where the tests are based on $E$-values rather than the usual $P$-values [12].
VI. SOME IMPROVED PROJECTION INEQUALITIES

**Theorem 9.** Let \( Q \) denote a measure and let \( X \) denote a random variable with \( E_Q(f(X)) = 0 \). If \( E_Q(f(X)^2) = 1 \) and \( E_Q((f(X))^3) > 0 \). If there exist \( \beta < 0 \) such that \( Z(\beta) = \int \exp(\beta \cdot x) \, dQx < \infty \), then there exists \( \epsilon > 0 \) such that for all measures \( P \) with \( E_p(X) \in [-\epsilon, 0] \) we have the inequality

\[
D(P\|Q) \geq \frac{1}{2} (E(X))^2.
\]

The conditions can be applied for the Gaussian distribution and associated normalized Hermite polynomials. It also holds for Binomial distributions and associated Kravshuk polynomials. This can be used to give bounds on rate on convergence in the Central Limit Theorem [13], [14], [15], in the Law of Thin Numbers [16], [17], [18], [5], [6], [19], and in approximations of hyper geometric distributions by binomial distributions [20]. The theorem improves previous results by not assuming that \( P \) is a probability measure. For Gaussian distributions, Poisson distributions, and binomial distributions one may even drop the condition \( E_p(X) \in [-\epsilon, 0] \) as long as \( E_p(X) \leq 0 \) and the orthogonal polynomials have sufficiently small order, but each case requires special techniques and the proofs are computationally involved, but actually it gives some slight simplifications if we drop the condition that the measures should be normalized.

VII. ALTERNATING MINIMIZATION

Several algorithms in information theory involve information projections, and some of them can be simplified by dropping the condition that we should stick to probability measures. One example is alternating minimization.

Let \( Q \) denote a probability measure and let \( C_1, C_2, \ldots, C_k \) denote a sequence of convex sets of probability measures and let \( C \) denote their intersection. In order to find the information projection of \( Q \) on \( C \) one can find the projection \( Q_1 \) of \( Q \) on \( C_1 \). Then one projects \( Q_1 \) on \( C_2 \) leading a projection \( Q_2 \) and so forth taking the sets \( C_1, C_2, \ldots, C_k \) in cyclic order. Then the sequence of projections \( Q_1, Q_2, \ldots \) will converge to the projection of \( Q \) on \( C \). Different versions of this algorithm have many important applications. Therefore it is useful if we can simplify this algorithm and speed up the rate of conversion.

Assume that each of the sets \( C_i \) are given by a mean value constraint of the form

\[
C_i = \left\{ P \mid \sum_j f_i(j) \cdot p_j = \mu_i \land \sum_j p_j = 1 \right\}.
\]

Let

\[
\tilde{C}_i = \left\{ P \mid \sum_j f_i(j) \cdot p_j = \mu_i \right\}.
\]

Let \( \tilde{C}_0 = \left\{ P \mid \sum_j p_j = 1 \right\} \) and note that \( C_i = \tilde{C}_i \cap \tilde{C}_0 \). Instead of projecting on \( C_i, i = 1, 2, \ldots, k \) in cyclic order one can project on \( \tilde{C}_i, i = 0, 1, \ldots, k \) in cyclic order. This simplifies each of the projections.

In order to accelerate the algorithm we may replace the function \( f_0, f_1, f_2, \ldots, f_k \) by functions that are orthogonal with respect to \( Q \). Such orthogonal functions are easily calculated using the Gram-Schmidt procedure.

VIII. DISCUSSION

As we have seen one may replace probability measures by more general measures without losing the interpretation of information divergence as the mean difference in code length. It may sometimes require that a probability vector \( p_i \) in a probability vector \( P = (p_1, p_2, \ldots, p_k) \) are interpreted as a mean value of a count that is Poisson distributed. That means that instead of having a count that may assume the two values 0 and 1 we get random variables that may assume any values in \( \mathbb{N}_0 \). This is definitely more abstract, which may seem complicate the foundation of probability theory, but often it allows us more freedom if we can more freely switch between considering the probability vector \( (p_1, p_2, \ldots, p_k) \) as describing independent events and as describing mutually excluding events. The translation forth and back is to consider a vector of counts \( (Z_1, Z_2, \ldots, Z_k) \). If they are mutually exclusive and Bernoulli then according to the Law of Thin Numbers the thinned sum of independent copies of this vector is approximately Poisson distributed and if \( (Z_1, Z_2, \ldots, Z_k) \) is Poisson distributed then conditioning on \( Z_1 + Z_2 + \cdots + Z_k = 1 \) leads to a multinomial distribution.

| Standard interpretation | Poisson interpretation |
|-------------------------|------------------------|
| Probability measure     | Measure                |
| Multinomial distribution| Poisson distribution   |
| Probability             | Mean value             |
| KL-divergence           | \( f \)-divergence     |
| Code                    | Conditional probability|
| Product measure         | Sum of measures        |

By switching the focus from probabilities to mean values and integrals we also get in accordance with the following formulation of the Dutch Book Theorem.

**Theorem 10** ([21], [22]). Let \( X_1, X_2, \ldots, X_n \) denote random variables (i.e. functions) on a finite sample space \( \Omega \). Then either there exists positive weights \( s_1, s_2, \ldots, s_n \) such that the linear combination

\[
X = s_1 X_1 + s_2 X_2 + \cdots + s_n X_n
\]

satisfies \( X(\omega) < 0 \) for all \( \omega \in \Omega \) or there exists a measure \( \mu \) on \( \Omega \) such that

\[
\int X(\omega) \, d\mu(\omega) \geq 0
\]

for all \( i = 1, 2, \ldots, n \).

A complete theory about how the ideas presented in this paper can be used not only to extend classical probability theory, but may replace classical probability theory as a foundation of our understanding of randomness, uncertainty, and quantum information theory, is work in progress.
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