Asymptotic properties of the maximum likelihood estimator for nonlinear AR processes with markov-switching

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Abstract

In this note, we propose a new approach for the proof of the consistency and normality of the maximum likelihood estimator for nonlinear AR processes with markov-switching under the assumptions of uniform exponential forgetting of the prediction filter and α-mixing property. We show that in the linear and Gaussian case our assumptions are fully satisfied.

Keywords: Nonlinear autoregressive process, Markov switching asymptotic normality, consistency, hidden Markov chain.

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Switching autoregressive processes with Markov regime can be considered as a combination of hidden Markov models (HMM) and threshold regression models. They have been introduced in an econometric context by Goldfeld and Quandt (1973) and they have become quite popular in the literature since Hamilton (1989) employed them in the analysis of the the rate of growth of USA GNP series for two regimes: one of contraction and another of expansion. This family of models describes the evolution of a time series subject to discrete shifts and the transition is controlled by a HMM.

We consider a nonlinear AR process with markov-switching (abbreviated MS-NAR) \( \{Y_n\}_{n \geq 0} \) defined for integers \( n \geq 1 \) by

\[
Y_n = r(Y_{n-1}, \theta_{X_n}) + e_n, \ Y_n \in \mathbb{R}.
\]

(1)

Here the process \( \{e_n\}_{n \geq 1} \) are i.i.d. random variables and the sequence \( \{X_n\}_{n \geq 1} \) is an homogeneous Markov chain with state space \( \{1, \ldots, m\} \).

Let \( \mathcal{F} = \{r(\cdot, \theta) : \theta \in \Theta\} \) a family of real valued functions defined on \( \mathbb{R}^{m+1} \), indexed by a parameter \( \theta = (\theta_1, \ldots, \theta_m) \in \Theta \) and \( \Theta \) is a compact set of \( \mathbb{R}^m \). We denote by \( A \) the probability transition matrix of the Markov chain \( \{X_n\}_{n \geq 1} \), i.e. \( A = [a_{ij}] \), with \( a_{ij} = \mathbb{P}(X_n = j | X_{n-1} = i) \). The parameter space is the set

\[
\Psi = \left\{ \psi = (\theta, A) : \theta \in \Theta, a_{ij} \in [0, 1] \text{ and } \sum_{j=1}^{m} a_{ij} = 1 \right\}.
\]
We assume that the variable $Y_0$, the Markov chain $\{X_n\}_{n \geq 1}$ and the sequence $\{e_n\}_{n \geq 1}$ are mutually independent. The process $\{X_n\}$, called regime, is not observable and inference has to be carried out in terms of the observable process $\{Y_n\}$.

The consistency of the maximum likelihood estimator for the parameter $\psi$ in the MS-NAR model is given in Krishnarmurthy and Ryden (1998) [3], while the consistency and asymptotic normality are proved in a more general context in the work of Douc et al. (2004) [3]. In the section 2 we prove the consistency and asymptotic normality of the maximum likelihood estimator for functional AR processes with markov-switching under the assumptions of exponential uniform forgetting property for prediction filter and an $\alpha$-mixing property.

1 General properties for MS-NAR model

In this section we review the key properties of the MS-NAR model that we need for proving our results.

1.1 Stability and existence of moments

The study of the stability of the model MS-NAR is relatively complex. In this section we recall known results about the stability of this model given by Yao and Attali [15]. Our aim is to resume the sufficient conditions which ensure the existence and the uniqueness of a stationary ergodic solution for the model, as well as the existence of moments of order $s \geq 1$ of the respective stationary distribution.

1.1.1 Stability and existence of moments

The model is called sublinear if conditions E2 and E3 hold. For the sublinear MS-NAR model, Yao and Attali [15] proved the following result.
Proposition 1.1 Consider a sublinear MS-NAR \( \{Y_n\}_{n \geq 0} \). Under assumptions E1-E7, we have that

i) There exists a unique stationary geometric ergodic solution.

ii) If the spectral radius of the matrix \( Q_s = (\rho_j^s a_{ij})_{i,j=1}^m \) is strictly less than 1, with \( s \) given in E5, then \( \mathbb{E}(|Y_n|^s) < \infty \).

Remark 1.1 The Markov chain is stable under the moment condition \( s \geq 1 \), but for the asymptotic properties of the MLE it will be necessary to assume \( s > 2 \).

Now we introduce some notations:

- \( V_{1:n} \) stands for the random vector \( (V_1, \ldots, V_n) \), and by \( v_{1:n} = (v_1, \ldots, v_n) \) we mean a realization of the respective random vector.

- The symbol \( \mathbb{1}_B(x) \) denotes the indicator function of set \( B \), which assigns the value 1 if \( x \in B \) and 0 otherwise.

- \( p(V_{1:n} = v_{1:n}) \) denotes the density distribution of random vector \( V_{1:n} \) evaluated at \( v_{1:n} \).

We consider the following assumption:

**D1** The random variable \( Y_0 \) admits a density function \( p(Y_0 = y_0) \) with respect to Lebesgue measure.

Under conditions D1 and E6, the random vector \( (Y_{0:n}, X_{1:n}) \) admits the probability density \( p(Y_{0:n} = y_{0:n}, X_{1:n} = x_{1:n}) \) equal to

\[
\Phi(y_n - r(y_{n-1}, \theta_{x_n})) \cdots \Phi(y_1 - r(y_0, \theta_{x_n})) a_{x_{n-1}x_n} \cdots a_{x_1x_2} \mu_{x_1} p(Y_0 = y_0),
\]

with respect to the product measure \( \lambda \otimes \mu_c \), where \( \lambda \) and \( \mu_c \) denote Lebesgue and counting measures respectively. For a proof of this result see Fernández et al [9].

1.2 Strong mixing

A strictly stationary stochastic process \( Y = \{Y_n\}_{n \in \mathbb{Z}} \) is called strongly mixing, if

\[
\alpha_n := \sup \{ |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{M}_\infty^n, B \in \mathcal{M}_\infty \} \to 0, \quad \text{as} \ n \to \infty,
\]

where \( \mathcal{M}_a^b \), with \( a, b \in \mathbb{Z} \), is the \( \sigma \)-algebra generated by \( \{Y_k\}_{k=a:b} \), and is absolutely regular mixing if

\[
\beta_n := \mathbb{E} \left( \text{ess sup} \{ \mathbb{P}(B|\mathcal{M}_\infty^n) - \mathbb{P}(B) : B \in \mathcal{M}_\infty \} \right) \to 0, \quad \text{as} \ n \to \infty.
\]

The values \( \alpha_n \) and \( \beta_n \) are called \( \alpha \)-mixing and \( \beta \)-mixing coefficients respectively. For properties and examples of processes under mixing assumptions, see Doukhan [4]. In general, we have the inequality \( 2\alpha_n \leq \beta_n \leq 1 \).
Note that the $\alpha$-mixing coefficients can be rewritten as:

$$\alpha_n := \sup\{ |\text{cov}(\phi, \xi)| : 0 \leq \phi, \xi \leq 1, \phi \in M^0_{-\infty}, \xi \in M^\infty_n \}. \quad (4)$$

In the case of a strictly stationary Markov process $X$, with state space $(E, B)$, kernel probability transition $A$ and invariant probability measure $\mu$, the $\beta$-mixing coefficients take the following form (see Doukhan [4], section 2.4):

$$\beta_n := \mathbb{E} \left( \sup\{ |A^n(X, B) - \mu(B)| : B \in B \} \right). \quad (5)$$

**Lemma 1.1** Under conditions E1-E7 the process MS-NAR is $\alpha$-mixing with $\alpha$-mixing coefficients decreasing geometrically.

**Proof:** For the proof of this lemma see Fermín et al [9].

**Example 1.1** (Linear autoregressive with Markov switching (MS-AR) nonmixing)

In the case where $r(y, (b_i, \rho_i)^t) = \rho_i y + b_i$, the model is a MS-AR and it is defined by:

$$Y_n = \rho X_n Y_{n-1} + b X_n + e_n. \quad (6)$$

For each $1 \leq i \leq m$, we denote $\theta_i = (b_i, \rho_i)^t$ and

$$\theta = \begin{pmatrix} b_1 & b_2 & \cdots & b_m \\ \rho_1 & \rho_2 & \cdots & \rho_m \end{pmatrix}.$$

More specifically consider the process MS-AR with $\theta_i = (0, \rho_i)^t$ for all $i = 1, \ldots, m$ and such that the random variable $e_1$ follows a Bernoulli distribution with parameter $q$ and $Y_0 = 0$. In this case, we have

$$Y_n = \sum_{k=0}^{n-1} \rho X_k \cdots \rho X_1 e_{k+1},$$

and we adopt the convention that $\rho X_k \cdots \rho X_1 = 1$ for $k = 0$. This process is non $\alpha$-mixing. In fact, according to D. Andrews [11] if $0 < \rho_i \leq 1/2$, for $t \in \mathbb{N}$ there exist some sets $A \in M^0_{-\infty}$, $B_t \in M^\infty_n$, with $\mathbb{P}(A) > 0$, $\mathbb{P}(B_t) \leq c$ for some constant $c < 1$ such that $\mathbb{P}(B_t|A) = 1$, therefore

$$\alpha_t(Y) \geq \mathbb{P}(A \cap B_t) - \mathbb{P}(A)\mathbb{P}(B_t) = \mathbb{P}(A)(\mathbb{P}(B_t|A) - \mathbb{P}(B_t)) \geq \mathbb{P}(A)(1 - c).$$

This implies that $\alpha_t(Y)$ does not tend to 0 as $t \to \infty$ and so $Y$ is a non $\alpha$-mixing process.

**Lemma 1.2** Under conditions E1-E7, the MS-NAR process $\{Y_n\}_{n \geq 0}$ satisfies,

i) For all function $\varphi$ such that $\mathbb{E}(\varphi(Y_k)) < \infty$, we have the strong law of large numbers,

$$\frac{1}{n} \sum_{k=1}^{n} \varphi(Y_k) \to \mathbb{E}(\varphi(Y_1)), \ a.s.$$
ii) Suppose that $\mathbb{E}(\varphi(Y_1)) = 0$, $\mathbb{E}|\varphi(Y_1)|^s < \infty$, for some $s > 2$. Then $\Gamma = \mathbb{E}(\varphi(Y_1)^2) + 2 \sum_{k=1}^{\infty} k + \mathbb{E}(\varphi(Y_1)\varphi(Y_k)) < \infty$ and if $\Gamma \neq 0$,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \varphi(Y_k) \rightarrow \mathcal{N}(0, \Gamma),$$

for $n \rightarrow \infty$, in distribution.

**Proof:** i) This result is a direct consequence of the Corollary 3.1 in Rio [10].

ii) Let $U_k = \varphi(Y_k)$, then $\{U_k\}_{k \geq 0}$ is a strictly stationary sequence and is strongly $\alpha$-mixing, with $\mathbb{E}(|U_k|^r) < \infty$, for $r > 2$. For $\alpha^{-1}(u) = \inf\{k \in \mathbb{N} : \alpha_k \leq u\}$, we have to prove

$$\int_{0}^{1} \alpha^{-1}(u)Q^2(u)du < \infty \quad (7)$$

where $Q$ is the associate quantile function of the process $\{U_k\}$. The condition (7) is implied by

$$\sum_{i \geq 0} (i + 1)^{-2} \alpha_i < \infty, \quad (8)$$

and in our case this is valid, since from geometric $\alpha$-mixing property exist $0 < \zeta < 1$ such that $\alpha_i \leq C\zeta^i$, we have

$$\sum_{i \geq 0} (i + 1)^{-2} \alpha_i \leq C \sum_{i \geq 0} (i + 1)^{-2} \zeta^i < \infty.$$

Thus, we can apply Theorem 4.2. in E. Río [10], obtaining that $\sqrt{n}U_n$ converges in distribution to $\mathcal{N}(0, \Gamma)$. $lacksquare$

2 Maximum likelihood estimation

Using $p_\psi$ as a generic symbol for densities and distributions parameterized for $\psi$. We defined the conditional log-likelihood as $l_n(\psi) = \log p_\psi(Y_{1:n}|Y_0)$ and we can expressed as

$$l_n(\psi) = \sum_{k=1}^{n} \log p_\psi(Y_k|Y_{0:k-1}).$$

We denote by $\psi^*$ the true parameter which is consider as fixed. A maximum likelihood estimator (MLE) is defined by

$$\hat{\psi}_n = \arg \max_\psi l_n(\psi).$$

The MLE is consistent if $\hat{\psi}_n \rightarrow \psi^*$ as $n \rightarrow \infty$ a.s.

The techniques standard used to prove consistency follows the steps:

1. To show that there exists a continuous deterministic function $l(\psi)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} l_n(\psi) = l(\psi) \text{ a.s.}$$
2. To show that \( l(\psi) \) a.s. has a unique maximum at \( \psi = \psi^* \).

3. To conclude that \( \hat{\psi}_n = \arg \max_\psi n^{-1} l_n(\psi) \rightarrow \arg \max_\psi l(\psi) = \psi^* \).

For MS-NAR processes a strong law of large numbers of the log-likelihood is obtained in Rynkiewicz \[12\], Krishnamurthy \[7\] using an additive function of the extended Markov chain \((Y_n, X_n, \mathbb{P}_\psi(\mathbf{X}_k|Y_{0:n}))\). In Douc \emph{et. al.} \[3\] the law of large numbers of the log-likelihood follow from uniform exponential forgetting of the initial distribution for prediction filter.

In this work, following the approach of consistency proof of Handel, chapter 7 in \[13\], for HMM, and joined to the \( \alpha \)-mixing property we obtain a new proof of the consistency for the MLE.

The following lemma shows that we can express \( p_\psi(Y_k|Y_{0:k-1}) \) as a functional of the prediction filter \( \mathbb{P}_\psi(\mathbf{X}_k|Y_{0:n}) \).

**Lemma 2.1** Let \( \delta = \inf_{i,j=1:m} a_{ij} \). Define,

\[
D_{k,l}^\psi = \log \int \int p_\psi(Y_k|Y_{0:k-1}, x_k) a_{x_{k-1}, x_k} \mathbb{P}(x_{k-1}|Y_{l:k-1}) \mu_c(dx_k) \mu_c(dx_{k-1}),
\]

for \( 0 < l < k \). Under assumptions E1 and E7 then \( |D_{k,l}^\psi - D_{k,0}^\psi| \leq 2\delta^{-1}(1 - \delta)^{k-1-l} \).

**Proof:** First, we bound from below the quantities \( \exp(D_{k,0}^\psi) \) y \( \exp(D_{k,l}^\psi) \), by the Fubini Theorem we have

\[
\exp(D_{k,0}^\psi) \geq \delta \int p_\psi(Y_k|Y_{0:k-1}, x_k) \mu_c(dx_k)
\]

and the same for \( \exp(D_{k,l}^\psi) \), thus

\[
\min(\exp(D_{k,0}^\psi), \exp(D_{k,l}^\psi)) \geq \delta \int p_\psi(Y_k|Y_{0:k-1}, x_k) \mu_c(dx_k).
\]

Using inequality \( |\log x - \log y| \leq |x - y|/\min(x, y) \), we estimate

\[
|D_{k,l}^\psi - D_{k,0}^\psi| \\
\leq \frac{\int \int p_\psi(Y_k|Y_{0:k-1}, x_k) a_{x_{k-1}, x_k} (\mathbb{P}_\psi(x_{k-1}|Y_{l:k-1}) - \mathbb{P}_\psi(x_{k-1}|Y_{0:k-1})) \mu_c(dx_k) \mu_c(dx_{k-1})}{\delta \int p_\psi(Y_k|Y_{0:k-1}, x_k) \mu_c(dx_k)} \\
\leq \frac{1}{\delta} \|\mathbb{P}_\psi(X_{k-1} \in \cdot|Y_{l:k-1}) - \mathbb{P}_\psi(X_{k-1} \in \cdot|Y_{0:k-1})\|_T
\]

Applying the Proposition 4.3.26 (iii) in Cappe \emph{et. al.} \[2\], pág 109,

\[
\|\mathbb{P}_\psi(X_k \in \cdot|Y_{l:k}) - \mathbb{P}_\psi(X_k \in \cdot|Y_{0:k})\|_T \leq 2(1 - \delta)^{k-l}
\]

We conclude that \( |D_{k,l}^\psi - D_{k,0}^\psi| \leq 2\delta^{-1}(1 - \delta)^{k-1-l} \).

This lemma shows that the quantity \( D_{k,0}^\psi \), which depends on the observations \( Y_{0:k} \), can be approximated by \( D_{k,l}^\psi \), which is a function of only a fixed number of observations \( Y_{l:k} \).

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Proposition 2.1 Under assumptions E1-E7, suppose $\Psi$ is a compact set and the condition $E_{\omega^*}(p_\omega(Y_k|Y_{k-1},i)) < \infty$, for $i = 1, \ldots, m$. Then $l_n(\psi)$ is a continuous function and $l(\psi) = \lim_{n \to \infty} n^{-1}l_n(\psi)$ exist a.s for each $\psi \in \Psi$.

Proof: The proof is done in two steps. First, we have

$$l(\psi) = \lim_{k \to \infty} E_{\omega^*}(D_k^\psi),$$

exists for every $\psi \in \Psi$.

Second, we show

$$\frac{1}{n} \sum_{k=1}^{n} (D_k^\psi - E_{\omega^*}(D_k^\psi)) \to 0 \text{ a.s.}$$

then conclude

$$\frac{1}{n} l_n(\psi) = \frac{1}{n} \sum_{k=1}^{n} (D_k^\psi - E_{\omega^*}(D_k^\psi)) + \frac{1}{n} \sum_{k=1}^{n} E_{\omega^*}(D_k^\psi) \to l(\psi) \text{ a.s.}$$

Step 1. Let $\Delta_k = E_{\omega^*}(D_k^\psi)$ by Lemma [2.1]

$$|\Delta_{m+n} - \Delta_m| = |E_{\omega^*}(D_{m+n,0}^\psi - E_{\omega^*}(D_{m+n,m}^\psi))| \leq 2\delta^{-1}(1 - \delta)^{m-1}$$

hence $\sup_n |\Delta_{m+n} - \Delta_m| \to 0$ as $m \to \infty$, i.e., $\{\Delta_k\}$ is a Cauchy sequence and therefore convergent. By Cesàros theorem $E_{\omega^*}(l_n(\psi)) = n^{-1}(\sum_{k=0}^{n-1} \Delta_k)$ also converges.

Step 2. According to Proposition 1.1 the sequence $\{D_k^\psi\}_{k \geq 1}$ is $\alpha$-mixing with geometric coefficients $\alpha_k$. We demonstrate that $E(|D_k^\psi|) < \infty$, in fact

$$\int \int p_\psi(Y_k|Y_{0:|k-1},x_k) a_{x_{k-1},x_k} p_\psi(x_{k-1}|Y_{|k-1}) \mu_c(dx_k) \mu_c(dx_{k-1}) \leq \int p_\psi(Y_k|Y_{0:|k-1},x_k) \mu_c(dx_k)$$

and

$$\int p_\psi(Y_k|Y_{0:|k-1},x_k) \mu_c(dx_k) = \sum_{i=1}^{m} p_\psi(Y_k|Y_{|k-1},i) \mu_i \leq m \max_{i=1:m} \{p_\psi(Y_k|Y_{|k-1},i)\} \mu_i$$

under assumption $E(p_\psi^*(Y_k|Y_{|k-1},i)) < \infty$, then $E(|D_k^\psi|) < \infty$ and by Lemma [1.2, i] we obtain

$$\frac{1}{n} \sum_{k=1}^{n} (D_k^\psi - E_{\omega^*}(D_k^\psi)) \to 0 \text{ a.s.}$$

We prove the validity of step three under uniform convergence, $\sup_{\psi \in \Psi} |l_n(\psi) - l(\psi)| \to 0$.

Lemma 2.2 Suppose $\Psi$ is a compact set. Let $l_n : \Psi \to \mathbb{R}$ be a sequence of continuous functions that converges uniformly to a function $l : \Psi \to \mathbb{R}$. Then

$$\hat{\psi}_n = \arg \max_{\psi} l_n(\psi) \to \arg \max_{\psi} l(\psi)$$
Proof: As a continuous function on a compact space attains its maximum, we can find a \( \psi_n \in \arg \max_{\psi} l_n(\psi) \) for all \( n \). Which show using an argument that goes to Wald (1949) that
\[
\lim_{n \to \infty} l(\psi_n) = \sup_{\psi \in \Psi} l(\psi).
\] (9)

Suppose that the sequence \( \{\psi_n\} \) does not converge to the set \( \{\tilde{\psi} : l(\tilde{\psi}) = \max_{\psi \in \Psi} l(\psi)\} \). By compactness there exists a subsequence \( \{\psi'_n\} \subset \{\psi_n\} \) which converges to \( \psi' \notin \{\tilde{\psi} : l(\tilde{\psi}) = \max_{\psi \in \Psi} l(\psi)\} \). But \( l(\psi) \) is continuous, so \( l(\psi'_n) \to l(\psi') < \sup_{\psi \in \Psi} l(\psi) \) and according to (9), this is a contradiction.

Theorem 2.1 Suppose \( \Psi \) is a compact set. Assume that

1. \( \psi = \psi^* \) iff \( P_\psi = P_{\psi^*} \).

2. For all \( i, j \in \{1, \ldots, m\} \) and all \( y, y' \in \mathbb{R} \times \mathbb{R} \) the functions \( \psi \to a_{ij} \) and \( \psi \to p_\psi(Y_1 = y | Y_0 = y', X_1 = i) \) are continuous.

3. There is a \( c < \infty \) such that \( |D_k^\psi - D_k^{\psi'}| \leq c\|\psi - \psi'\| \) for all \( k > 1 \)

Then the maximum likelihood estimate \( \hat{\psi}_n \) is consistent.

Proof: By Theorem 7.5 in Handel [13], the Lipschitz condition 3. and compactness implies that the sequence \( l_n \to l \) a.s uniformly. According to Lemma 2.1
\[
\hat{\psi}_n \to \psi^* = \arg \max_{\psi} l(\psi),
\]
and this value is unique under identifiability.

In the Gaussian and linear case we can prove directly identifiability and equicontinuity. This allows us obtain the consistency of the MLE without assuming a condition of Lipschitz for the parameters.

Example 2.1 (MS-AR gaussian linear)

Let the model defined by (6). Let \( \{\varepsilon_n\} \) are gaussian i.i.d. random variables. Our goal in this example is check that the conditions for consistency apply in this case. In fact, if we assume that for the true model \( \Psi^* \) the vector components \( \{(\alpha_i, b_i, \sigma_i)\}_{i=1}^m \) are different; thus, for every \( n \), there exists a point \( Y_{n-1} \in \mathbb{R} \) such that \( \{(\alpha_i Y_{n-1} + b_i, \sigma_i)\}_{i=1}^m \) are different. Therefore, in agreement with Remark 2.10 of Krishnamurthy and Yin [7] the model is identifiable in the following sense: If \( K \) stands for the Kullback-Leibler divergence \( K(\psi, \psi^*) = 0 \) then, \( \psi = \psi^* \), which proves the identifiability. On the another hand, the Lemma 4.1 in [11] follows that \( \frac{1}{n} \log p_\psi(Y_1^n | Y_0 = y_0) \) is an equicontinuous sequence a.s-\( P_{\psi^*} \). We conclude that in this case the MLE is consistent.

There is a standard technique for prove asymptotic normality of maximum likelihood estimates. The idea is that the first derivatives of a smooth function must vanish at its maximum. If we expand in Taylor series the likelihood gradient around \( \psi^* \), we can write
\[
0 = \nabla_\psi l_n(\hat{\psi}_n) = \nabla_\psi l_n(\psi^*) + \nabla^2_{\psi^2} l_n(\hat{\psi}_n)(\hat{\psi}_n - \psi^*)
\]
where \( \tilde{\psi} = t\hat{\psi}_n + (1-t)\psi^* \). Normalizing this expansion with \( \sqrt{n} \) we obtain

\[
\sqrt{n}(\hat{\psi}_n - \psi^*) = -(\nabla^2 \psi_l_n(\tilde{\psi}))^{-1}(\nabla \psi_l_n(\psi^*))\sqrt{n}.
\]

In order to obtain the asymptotic normality of the maximum likelihood estimator we assume that exist an open neighborhood \( B_r(\psi^*) \) of \( \psi^* \) such that the following statements hold.

**H1** The functions \( \psi \to A \) and \( \psi \to p_\psi(Y_1|Y_0,i) \) are twice continuously differentiable on \( B_r(\psi^*) \).

**H2** There exist functions \( f_0, f_1, f_2 \) such that

\[
\sup_{\psi \in B_r(\psi^*)} \|\nabla \psi p_\psi(y_1|y_0,i)\| \leq f_0(y_1,y_0), \quad \sup_{\psi \in B_r(\psi^*)} \|\nabla^2 \psi p_\psi(y_1|y_0,i)\| \leq f_1(y_1,y_0),
\]

and

\[
\sup_{\psi \in B_r(\psi^*)} \|\nabla \psi p_\psi(y_1|y_0,i)\| \leq f_2(y_1,y_0),
\]

with \( \mathbb{E}(f_s(Y_1,Y_0)) < \infty \), \( s = 0,1 \) and \( \mathbb{E}(f_2(Y_1,Y_0)^r) < \infty \), \( r > 2 \).

**Theorem 2.2** Under assumptions of Theorem 2.1 and H1-H2, assume that \( J(\psi^*) = \text{var}(\nabla \psi l(\psi^*)) \) is non-singular and \( \psi^* \in \bar{\Psi} \). Then, as \( n \to \infty \),

i) \( -(\nabla^2 \psi_l n^{-1}l_n(\tilde{\psi})) \to J(\psi^*), \) in probability.

ii) \( \sqrt{n}\nabla \psi n^{-1}l_n(\psi^*) \to N(0,J(\psi^*)), \) in distribution.

Moreover, we conclude that \( \sqrt{n}(\hat{\psi}_n - \psi^*) \to N(0,J(\psi^*)^{-1}), \) in distribution.

**Proof:** Under H1-H2 if we take \( \varphi() = \frac{\partial^2 \log l_n}{\partial \psi^2} \) the Lemma 2 implies that

\[
\sqrt{n}\nabla \psi n^{-1}l_n(\psi^*) \to N(0,J(\psi^*))
\]

and if \( \varphi() = \frac{\partial n^{-1} \log l_n}{\partial \psi} \), then

\[
-(\nabla^2 \psi_l n^{-1}l_n(\psi^*)) \to J(\psi^*), \ a.s.
\]

For a sequence \( \psi_n \to \psi^* \) we can prove

\[
\lim_{n \to \infty} \frac{1}{n} \nabla^2 \psi_l n(\psi_n) - \frac{1}{n} \nabla^2 \psi_l n(\psi^*) = 0,
\]

in probability.

Let us first observe that \( \frac{1}{n} \nabla^2 \psi_l n(\psi_n) = \frac{1}{n} \sum_{k=1}^{n} \nabla^2 \psi_l p_\psi(Y_k|Y_{0,k-1}) \). Another hand,

\[
\frac{\partial^2 \log p_\psi}{\partial \psi_j \partial \psi_i} = \frac{1}{p_\psi} \frac{\partial^2 p_\psi}{\partial \psi_j \partial \psi_i} - \frac{\partial \log p_\psi}{\partial \psi_j} \frac{\partial \log p_\psi}{\partial \psi_i} \frac{1}{p_\psi^2}
\]
hence,
\[
\frac{\partial^2 \log p_{\psi_n}}{\partial \psi_j \partial \psi_i} - \frac{\partial^2 \log p_{\psi^*}}{\partial \psi_j \partial \psi_i} = \left( \frac{1}{p_{\psi_n}} \frac{\partial^2 p_{\psi_n}}{\partial \psi_j \partial \psi_i} - \frac{1}{p_{\psi^*}} \frac{\partial^2 p_{\psi^*}}{\partial \psi_j \partial \psi_i} \right) + \left( \frac{\partial p_{\psi^*}}{\partial \psi_j} \frac{1}{p_{\psi^*}^2} - \frac{\partial p_{\psi_n}}{\partial \psi_j} \frac{1}{p_{\psi_n}^2} \right)
\]
\[
= T_1 + T_2.
\]
For term $T_2$, by definition of $\psi^*$, $\frac{\partial p_{\psi^*}}{\partial \psi_j} = 0$ and by the Ergodic theorem we have
\[
\lim_{n \to \infty} - \frac{1}{n} \sum_{k=1}^{n} \frac{\partial p_{\psi_n}}{\partial \psi_j} = 0.
\]
For the term $T_1$,
\[
\frac{1}{p_{\psi_n}} \frac{\partial^2 p_{\psi_n}}{\partial \psi_j \partial \psi_i} - \frac{1}{p_{\psi^*}} \frac{\partial^2 p_{\psi^*}}{\partial \psi_j \partial \psi_i} = \frac{1}{p_{\psi_n}} \left( \frac{\partial^2 p_{\psi_n}}{\partial \psi_j \partial \psi_i} - \frac{\partial^2 p_{\psi^*}}{\partial \psi_j \partial \psi_i} \right) + \left( \frac{1}{p_{\psi_n}} - \frac{1}{p_{\psi^*}} \right) \frac{\partial^2 p_{\psi^*}}{\partial \psi_j \partial \psi_i}
\]
Under equicontinuity of the sequence $\{p_{\psi_n}\}_{n \geq 1}$ we have, $p_{\psi_n} \to p_{\psi^*}$ and by conditions E1 and E7 $\frac{1}{p_{\psi_n}} \to \frac{1}{p_{\psi^*}}$, a.s. Using H2 we obtain
\[
\mathbb{E} \left( \frac{\partial^2 p_{\psi^*}}{\partial \psi_j \partial \psi_i} \right) < \infty.
\]
Let
\[
w(r, Y_{0:k}) = \sup_{\psi_n \in B_r(\psi^*)} \left| \frac{\partial^2 p_{\psi_n}(Y_k|Y_{0:k})}{\partial \psi_j \partial \psi_i} - \frac{\partial^2 p_{\psi^*}(Y_k|Y_{0:k})}{\partial \psi_j \partial \psi_i} \right|
\]
proceeding as in Lemma 3 of Vandekerkhove [14], by Markov inequality
\[
\mathbb{P} \left( \frac{1}{k} \sum_{k=1}^{n} \frac{\partial^2 p_{\psi_n}(Y_k|Y_{0:k})}{\partial \psi_j \partial \psi_i} - \frac{1}{k} \sum_{k=1}^{n} \frac{\partial^2 p_{\psi^*}(Y_k|Y_{0:k})}{\partial \psi_j \partial \psi_i} \right) > \epsilon
\]
\[
\leq \mathbb{P} \left( \frac{1}{k} \sum_{k=1}^{n} w(r, Y_{0:k}) > \epsilon - \mathbb{E}(w(r, Y_{0:k})) \right) + \mathbb{P} \left( \psi_n \notin B_r(\psi^*) \right)
\]
\[
\leq \frac{\mathbb{E}(w(r, Y_{0:k}))}{\epsilon - \mathbb{E}(w(r, Y_{0:k}))} + \mathbb{P} \left( \psi_n \notin B_r(\psi^*) \right). \tag{10}
\]
The condition H2 implies that $\mathbb{E}(w(r, Y_{0:k})) \leq 2f_1$. Using the Lebesgue continuity theorem, we obtain that $\mathbb{E}(w(r, Y_{0:k})) \to 0$, as $n \to \infty$. The second term goes to 0 as $n$ to infinity by strong convergence of $\{\psi_n\}$ to $\psi^*$. Hence (10) goes to 0.

Finally, as $\sqrt{n} (\hat{\psi}_n - \psi^*) = - \left( \nabla^2_{\psi^2} (\hat{\psi}) \right)^{-1} \left( \nabla_{\hat{\psi}} l_n (\psi^*) \right) \sqrt{n}$, using i) the first factor in the above expression tends to $J(\psi^*)$. The second factor converges weakly to $N(0, J(\psi^*))$ by ii). Slutsky’s theorem implies that $\sqrt{n} (\hat{\psi}_n - \psi^*) \to N(0, J(\psi^*))^{-1}$.

**Example 2.2** (MS-AR gaussian again)
We employ the asymptotic results obtained to verify the validity of a likelihood test for identifying when the parameter \( \rho \), of a MS-AR is the zero vector. In this case the MS-AR process is a hidden Markov model.

Expanding \( \ln(\rho) \) in Taylor series around \( \hat{\rho} \), we have

\[
-2(\ln(\hat{\rho}) - \ln(0)) = \hat{\rho}^2 \left( -\frac{\partial^2 \ln(\tilde{\rho})}{\partial \rho^2} \right)
\]

and by Theorem 2.1 \( \hat{\rho} \sqrt{J(0)} \to \mathcal{N}(0,1) \) and as \( J(\hat{\rho})/J(0) \to 1 \) then \( \hat{\rho}^2 J(0) \to \chi^2_1 \).

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