On Franklin’s relativistic rotational transformation and its modification

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Abstract

Unlike the Lorentz transformation which replaces the Galilean transformation among inertial frames at high relative velocities, there seems to be no such a consensus in the case of coordinate transformation between inertial frames and uniformly rotating ones. There has been some attempts to generalize the Galilean rotational transformation to high rotational velocities. Here we introduce a modified version of one of these transformations proposed by Philip Franklin in 1922. The modified version is shown to resolve some of the drawbacks of the Franklin transformation, specially with respect to the corresponding spacetime metric in the rotating frame. This new transformation introduces non-inertial observers at non-zero radii on a uniformly rotating disk and the corresponding metric in the rotating frame is shown to be consistent with the one obtained through Galilean rotational transformation for points close to the rotation axis. Using the threading formulation of spacetime decomposition, spatial distances and time intervals based on the spacetime metric in a rotating frame are also discussed.

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I. INTRODUCTION

“There is no relativity of rotation”. This relatively famous quote by Feynman [1] may look as the final word on the discussion of rotation in the context of special relativity. Based on the fact that the presence of acceleration in a uniformly rotating frame, by the equivalence principle, takes us into the realm of general relativity may convince one not to bother with the formulation of rotation in the context of special relativity and look for the resolution of each rotation-based problem in general relativity and in the suitably chosen/constructed solutions of Einstein field equations (which are of course not usually available). On the other hand rotation and rotating frames have always been a source of confusion, the famous example of which is the Ehrenfest’s Paradox [2]. Indeed, looking at the literature [3], one finds out how diverse are ideas on the relativistic physics in rotating frames and consequently how distant we are from establishing a general consensus on the main notions in this subject. So in practice one uses either the Galilean rotational transformation (GRT) or consecutive Lorentz transformations between an inertial (laboratory) frame and comoving inertial frames which are momentarily at rest with respect to the non-inertial rotating frame. The latter could be obtained either by employing the so called hypothesis of locality and the same method led to the Fermi coordinates of an accelerated spinning observer [4], or by reducing a general Lorentz transformation obtained for accelerated spinning frames [5] to the case of rotating frames [6]. Another alternative is the introduction of a relativistic rotational transformation (RRT) which is the main subject of the present paper.

It seems that Ehrenfest’s Paradox is a good opening point to start our discussion on rotation and relativistic rotational transformations. To explain the paradox we consider two frames/observers one at rest (the laboratory observer/frame) and the other one rotating counter-clockwise around it with constant angular velocity $\Omega$ (the rotating observer/frame) measured by/in the inertial (non-rotating) observer/frame. At this point we use frames (set of clocks and extended fiduciary triad axes) and observers interchangeably but to be more precise one should differentiate between them, for a rotating frame is a non-inertial frame but not all observers in a rotating frame are non-inertial. In other words we should distinguish between a centrally rotating observer (i.e at the center of the disk) which is an inertial observer and those at non-zero radii which are non-inertial. We will get back to this point in the next section. Using cylindrical coordinates we denote the spacetime
points in the non-rotating frame with coordinates \((t, r, \phi, z)\) and in the one rotating around the \(z'(z')\)-axis with \((t', r', \phi', z')\) where \(\phi'\) is measured from the \(x'(x')\)-axis. These are related through the Galilean rotational transformation (GRT)

\[
t' = t \quad , \quad r' = r \quad , \quad \phi' = \phi - \Omega t \quad , \quad z' = z
\]

or in its differential form

\[
dt' = dt \quad , \quad dr' = dr \quad , \quad d\phi' = d\phi - \Omega dt \quad , \quad dz' = dz
\]

It is noted that in both the rotating and non-rotating frames the radial distances are measured from the rotation axis. Through the above equation we would like to emphasize on the meaning of the Galilean rotational transformation. Interpreted kinematically, as in the cases of linear Galilean and Lorentz transformations, it introduces a prescription of how the spacetime coordinates of an event in the two frames are related to one another. This interpretation leads to the following relation between the angular velocities of a test particle observed in the two frames (Figure 1)

\[
\omega' = \omega - \Omega
\]

which in turn leads to the well known relation \(E' = E - \mathbf{L} \cdot \mathbf{\Omega}\) between the energies of the particle in the two frames. Usually the problem of rotation and rotating frames is discussed in the context of uniformly rotating rigid disks, in other words the rotating frame is a frame attached to a uniformly rotating incompressible disk whose constant angular velocity is measured in the non-rotating/inertial frame. The above coordinate transformation could also be employed to a uniformly rotating disk and its points (at different times) taken as events whose spacetime coordinates are measured both in the laboratory frame and in the rotating frame attached to the disk. Obviously in this case it is expected that for any point on the disk \(\omega' = 0\) and \(\omega = \Omega\) (Figure 2).

II. EHRENFEST’S PARADOX

Ehrenfest’s paradox is a contradiction that an inertial (laboratory) observer faces in applying special relativistic length contraction to a rotating disk. From an inertial observer’s
point of view the rim of a rotating disk undergoes a length contraction due to its transverse motion with velocity \( v = R \Omega \) and so circumference of a rotating disk (\( P' \)) is shorter than the one non-rotating (\( P \)), i.e. \( P' < P \). On the other hand since the radius of the disk is perpendicular to the direction of the rotational motion of the rim, the same observer will not attribute length contraction to it and so \( R' = R \). Therefore the inertial (laboratory) observer, living in a flat spacetime and thereby using the *Euclidean* prescription for the circumference of a circle, finds out the contradicting result \( P = 2\pi R = 2\pi R' = P' \).

Perhaps it should be left for experiment to decide which relation holds between \( P \) and \( P' \) but nevertheless people have tried hard to find either a theoretical resolution to this paradox or otherwise to invalidate it. An apparently favorite resolution in the literature is based on considering the situation from the point of view of a rotating frame and on the idea, introduced by Einstein [8,9], that the spatial geometry in such a frame is *non-Euclidean*. But, as we will show below, that does not seem to be leading to any kind of resolution of...
FIG. 2: A disk and its frame (solid) rotating around the laboratory frame (dashed) with uniform angular velocity $\Omega$. Coordinates of a point $P$ in the rim are given in the two frames with angular velocities $\omega' = 0$ and $\omega = \Omega$.

the paradox but to a somewhat similar paradox from the rotating frame’s point of view. As pointed out earlier, in the case of a rotating disk one should distinguish between the observer at the center of the disk (calling it the centrally rotating observer/frame) whose spatial coordinates, measured in the non-rotating/laboratory frame, are fixed and those at different non-zero radii which are non-inertial due to the centrifugal force attributed to them by the non-rotating observers and call them orbiting observers/frames. Einstein calls them eccentric observers “relative to whom a gravitational field prevails” [8]. In other words these observers, by the equivalence principle, find themselves and anything fixed with respect to the disk in a gravitational field. Later, elaborating on this matter, it will be shown that rotating observers at non-zero radii are of central importance in our discussion of relativistic rotational transformations but for the purpose of Ehrenfest paradox we only deal with the rotating observer/frame at the center of the disk. From a rotating observer’s point of view the above mentioned non-Euclidean character of the disk geometry could be obtained from
considering the metric of flat spacetime in the rotating frame, as it is the spatial geometry (metric), defined through spacetime metric, which accounts for spatial distances including that of the disk circumference. Using the differential Galilean rotational transformation \( (2) \), the flat spacetime metric in the non-rotating frame

\[
ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2 \tag{4}\]

transforms into \( [10, 11] \)

\[
ds^2 = (c^2 - \Omega^2 r^2) dt^2 - 2\Omega r^2 dt d\phi' - dr^2 - r^2 d\phi'^2 - dz^2 \tag{5}\]

in the rotating frame. It is seen that this metric is applicable for radii less than \( c/\Omega \), corresponding to the so called light cylinder, beyond which \( g_{00} \) becomes negative (with the corresponding points having velocities greater than \( c \)) and hence from physical point of view not of interest \( [10, 11] \).

The famous result, based on special relativistic arguments made by Einstein, that a rotating clock at non-zero radius \( r = R \) runs slower than that sitting at the center of the disk (or very close to it) \( [8, 9] \) is clearly encoded in the above metric, from which we have

\[
d\tau = \sqrt{1 - \frac{\Omega^2 R^2}{c^2}} dt \]

where \( dt \) is the world time recorded by the inertial/laboratory clocks as well as the one at the center of the disk. The above spacetime metric plays the same role for a centrally rotating observer that Rindler spacetime metric

\[
ds^2 = \eta_{ab} dx^a dx^b = (1 + a\bar{x}^1)^2 (d\bar{x}^0)^2 - (d\bar{x}^1)^2 - (d\bar{x}^2)^2 - (d\bar{x}^3)^2 \tag{6}\]

with

\[
x^0 = (a^{-1} + \bar{x}^1) \sinh(a\bar{x}^0) \quad ; \quad x^2 = \bar{x}^2
\]

\[
x^1 = (a^{-1} + \bar{x}^1) \cosh(a\bar{x}^0) \quad ; \quad x^3 = \bar{x}^3 \tag{7}\]

plays for a uniformly accelerating observer with 3-acceleration \( a = (a, 0, 0) \). In other words Rindler metric in the limit \( \bar{x}^1 \ll 1 \) (i.e for points infinitesimally close to the world line of the observer) reduces to the Fermi metric \( [12] \) in the absence of rotation (i.e \( \Omega = 0 \)) while \( (5) \) in the limit \( r \ll 1 \) (i.e infinitesimally close to the centrally rotating observer) reduces to the Fermi metric in the absence of linear acceleration (i.e \( a = 0 \)) \( [6] \). It should be noted that the spacetime of a rotating observer \( (5) \), like Rindler spacetime, is the flat spacetime in a coordinate system which is not maximally extended due to existence of light cylinder in the
former and the horizon in the latter. On the other hand, unlike Rindler spacetime, it is a stationary spacetime (reflected in the presence of its cross term $dtd\phi$) and so one needs to employ a spacetime decomposition formalism to define spatial distances and time intervals in it and on their basis to prescribe suitable measurement procedures. In what follows we will employ the $1 + 3$ or *threading* formulation of spacetime decomposition [10] which is based on sending and receiving light signals between nearby observers (refer to appendix for a brief introduction). Although we are not going to discuss the spacetime measurement procedure here, the employment of the $1 + 3$ formulation makes it clear that, in principle, we are using light signals to measure the relevant physical quantities, namely spatial distance and time intervals. Indeed, as pointed out earlier in another well known approach absed on the hypothesis of locality [4] one can find a coordinate transformation between an inertial frame and a rotating frame [13] which has its own measurement limitations [31]. Based on $1 + 3$ formulation, the spatial line element for the metric (5) is given by [10, 15],

$$dl^2 = dr^2 + dz^2 + \frac{r^2 d\phi^2}{1 - \Omega^2 r^2/c^2}.$$  

(8)

Now for a circle of radius $r = r' = R$ in the $z = constant$ plane the circumference is given by

$$P' = \int_0^{2\pi} dl = \frac{2\pi R}{\sqrt{1 - \Omega^2 R^2/c^2}} = \frac{P}{\sqrt{1 - \Omega^2 R^2/c^2}}$$  

(9)

so that $P' > P$ with $P$ the circumference of a non-rotating disk. Therefore from the rotating observer’s point of view $P$ and $P'$ are also not equal but the relation between the two quantities is just the opposite of that found by the inertial (laboratory) observer based on Lorentz contraction. Interpretation of the above results goes as follows: Although the transformed spacetime is the flat spacetime in disguise, its spatial geometry now has non-zero Gaussian curvature leading to the fact that the ratio of the circumference of a circle to its radius is larger than $2\pi$. We are not going to follow this disagreement on the relation between $P$ and $P'$ from the two observers’ points of view nor discuss further the content of Ehresfest’s paradox but there remains a legitimate question that one might ask and that is: Are we allowed to use the Galilean rotational transformation (II) in all the above considerations? specially noting that the metric in the rotating frame can be employed out to a specific radius given by $c/\Omega$ which decreases as we increase the angular velocity. Our experience
with Lorentz transformations intuitively leads to the expectation that Galilean rotational transformation to be an approximation valid for points near the axis of rotation having small linear velocities. Hence for eccentric observers one needs to replace the Galilean rotational transformation with a relativistic (Lorentz-type) rotational transformation to account for linear velocities comparable to \( c \). Obviously if one could devise such a relativistic rotational transformation, it might be expected that either the transformation (based on its kinematical interpretation) or the spatial line element of the transformed flat spacetime metric lead to a contracted/dilated circumference for a rotating disk or any other circle of a given radius. A comparison between the usual Lorentz transformation (LT), and the Galilean rotational transformation (GRT) is useful at this point. In the case of LT the length contraction is built in into the transformation itself and since the flat spacetime line element is form-invariant under the transformation, the length contraction is not expected to be tractable in the form of the corresponding spatial metric. On the other hand in the case of GRT as we noticed, the transformation (11) is devoid of any length contraction or dilation while the transformed spatial metric (8) leads to the length dilation. One such relativistic rotational transformation was discovered by Philip Franklin a Princeton mathematician in 1922 [16] and some 30 years later by Trocheris [17] and Takeno [18] [32]. It should be noted that this is not the only non-classical rotational transformation and there are other proposals such as the one introduced by Hill [19] (also refer to [20] and references therein). In the present article we will discuss Franklin transformation and its properties including its advantages over the classical transformation and also its drawbacks specially with respect to the corresponding spacetime metric and show how a simple modified version of the transformation could lead to the resolution of some of these drawbacks. Obviously the main criterion for the preference of any non-classical rotational transformation over the classical one should be the verification of its experimental consequences. For the sake of completeness we will give a brief derivation of Franklin transformation in the next section.

III. FRANKLIN TRANSFORMATION

Taking two coordinate frames \( S \) and \( S' \), with \( S' \) uniformly rotating about \( S \), Franklin requires the following plausible conditions and properties to be valid on the relation between the two frames [16]:
1-The velocity of a fixed point in \( S' \) with respect to the point in \( S \) with which it momentarily coincides is independent of the time, and is the same for all points at a given distance from the axis of rotation.

2-For the two concentric circles \( r' = r = \text{Constant} \), the equations of transformation are similar to those for a Lorentz boost (say along the x-direction) with \( r\phi \) the arclength replacing the linear distance (say \( x \)). These two properties lead to the following transformation law

\[
 t' = \gamma(r) \left( t - v(r)r\phi/c^2 \right) ; \quad r' = r \\
 r'\phi' = \gamma(r) \left( r\phi - v(r)t \right) ; \quad z' = z
\]

in which \( \gamma = \frac{1}{\sqrt{1-v(r)^2/c^2}} \) is the Lorentz-type factor with velocity \( v(r) \) to be determined through the last property which is;

3-The velocity of a point at the distance \( r' + \Delta r' \) from the axis with respect to a point at the distance \( r' \) from the axis (both in the system \( S' \)) is given by \( \Omega \Delta r' \). In other words two different points at two different radii with two different rotational velocities are taken as the analogues of two inertial frames moving uniformly with respect to one another.

In effect, the last property is a prescription for velocity composition law, out of which the nontrivial form of the rotational velocity is obtained. For two points \( B \) and \( C \) at radii \( r_B = r \) and \( r_C = r + \Delta r \) with velocities \( v(r) \) and \( v(r + \Delta r) \) (with respect to the point \( A \) at the center of the disk) respectively, the composition law reads

\[
 v_{BC} = \frac{v_{AC} - v_{AB}}{1 - \frac{v_{AC}v_{AB}}{c^2}} \Rightarrow \Omega \Delta r = \frac{v(r + \Delta r) - v(r)}{1 - \frac{v(r + \Delta r)v(r)}{c^2}}
\]

In the limit \( \Delta r \to 0 \) this leads to the velocity relation

\[
 v(r) = c \tanh(\Omega r/c)
\]

Substituting (12) in (10), explicit form of the Franklin transformation (FT) is given by

\[
 t' = \cosh(\Omega r/c)t - \frac{r}{c} \sinh(\Omega r/c)\phi ; \quad r' = r \\
 \phi' = \cosh(\Omega r/c)\phi - \frac{c}{r} \sinh(\Omega r/c)t ; \quad z' = z
\]

For points close to the rotation axis i.e when \( \frac{\Omega r}{c} \ll 1 \) this transformation reduces to the classical Galilean transformation by neglecting terms of order \( \frac{\Omega^2 r^2}{c^2} \) and higher. These transformations form a group and the inverse transformation is given by changing \( \Omega \) to \(-\Omega\).
One of the advantages of this transformation over the old Galilean one is in the definition of the velocity given in (12) which approaches \( c \) at \( r \to \infty \) (i.e. the light cylinder is not at a finite distance but is sent to infinity) and reduces to the Newtonian value \( v = \Omega r \) for points near the axis. Formal Comparison with a pure Lorentz transformation as a hyperbolic rotation, reveals that, it is the linear velocity \( v = \Omega r \) in (12) which now plays the role of some kind of rapidity.

Another obvious difference between Franklin transformation and the Lorentz transformation (LT), when FT is rewritten in the following form,

\[
ct' = \frac{1}{\sqrt{1 - \frac{v(r)^2}{c^2}}}(ct - \frac{v(r)}{c}r\phi) \quad r' = r
\]

\[
r\phi' = \frac{1}{\sqrt{1 - \frac{v(r)^2}{c^2}}}(r\phi - \frac{v(r)}{c}ct) \quad z' = z
\]

is the fact that velocity entering the definition of FT unlike LT is not a constant but an \( r \)-dependent quantity. This will lead to undesirable results in the case of FT when we consider the transformed spacetime metric (i.e in the rotating frame) and the corresponding spatial distances and time intervals. It will be shown that neither will reduce to their expected expressions at small rotational velocities (i.e when \( \frac{\Omega c}{e} \ll 1 \)). But before discussing these issues, it seems appropriate to discuss interpretation of FT as compared to those of GRT and LT.

### A. Interpretation of FT

An important issue about the Franklin transformation which seems to be taken for granted in most of the previous studies, is its interpretation as the transformation of the spacetime coordinates of an event between two frames; a non-rotating (inertial) frame and another one rotating uniformly about their common axis. This is the same usual interpretation attributed to the GRT as illustrated in figure 1. But characteristics of FT would prevent one to easily interpret this transformation as a kinematical one. The main characteristic is the radial dependence of velocity entering the transformation. This velocity distribution is attributed to the rigid arms of the rotating frame (or disk points if the frame is attached to a uniformly rotating rigid disk) and so, taking into account the fact that in FT the non-rotating and rotating frames share the same axis, the transformation of the arclengths in FT (which...
is given in terms of this velocity) is only valid for disk points. By the above reasoning, it seems more reasonable to look at FT as a transformation specially tailored for the problem of a rotating disk in which events are nothing but different points of a rotating disk at different times. In other words one should be cautious in interpreting FT as a kinematical transformation relating coordinates of an event in a rotating frame to that of an inertial non-rotating one. Further restriction of the transformation to the $z = \text{constant}$ plane and the fact that for $r = r' = 0$, i.e for an event (based on a kinematical interpretation) at the origin where the cylindrical coordinate system is degenerate and $v(0) = 0$, FT reduces exactly to GRT justifies our claim. If FT is going to be elevated to a kinematical transformation one needs to modify and reinterpret it. This will be done in later sections when we introduce the modified Franklin transformation (MFT).

IV. SPACETIME METRIC AND SPATIAL GEOMETRY IN THE ROTATING FRAME THROUGH FRANKLIN TRANSFORMATION

Using the inverse of the Franklin transformation in its differential form

\[
\begin{align*}
    cdt &= \cosh(\Omega r/c)cdt' + r \sinh(\Omega r/c)d\phi' + A_1 dr \quad ; \quad dr = dr' \\
    rd\phi &= \cosh(\Omega r/c)rd\phi' + \sinh(\Omega r/c)cdt' + A_2 dr \quad ; \quad dz = dz' \\
    A_1 &= \sinh(\Omega r/c)(\phi' + \Omega t') + \cosh(\Omega r/c)(\frac{\Omega r}{c} \phi') \\
    A_2 &= \sinh(\Omega r/c)(\frac{\Omega r}{c} \phi' - ct'/r) + \cosh(\Omega r/c)(\Omega t')
\end{align*}
\]

and substituting in (5) the spacetime metric in the rotating frame is given by

\[
ds^2 = c^2 dt'^2 - (1 - A_1^2 + A_2^2)dr^2 - r^2 d\phi'^2 - dz^2 + 2c (A_1 \cosh(\Omega r/c) - A_2 \sinh(\Omega r/c)) drdt' + 2 (A_1 \sinh(\Omega r/c) - A_2 \cosh(\Omega r/c)) r drd\phi' \tag{16}
\]

Unlike the cross term in (5) which is the typical $dt'd\phi'$ term representing the rotation the cross terms in the above metric include $drdt'$ and $drd\phi'$ terms and that is why the reduction of this metric form to (5) for $\Omega r/c \ll 1$ is not expected. Further also it should be noted that due to the explicit appearance of $\phi'$ and $t'$ in (16) both the temporal and angular isometries present in (5) are now lost.
A. Spatial distances and time intervals

From the above result on the spacetime metric it is obviously not expected that the spatial geometry corresponding to \((16)\) be reducible to the one given by \((8)\) in the limit \(\Omega r/c \ll 1\). Indeed using the 1 + 3 decomposition (equation \((A2)\)) the spatial metric corresponding to \((15)\) is given by

\[
dl^2 = \left\{\begin{array}{l}
1 - A_1^2 + A_2^2 + 4c^2[A_1 \cosh(\Omega r/c) - A_2 \sinh(\Omega r/c)]^2 \, dr^2 + \\
-2[A_1 \sinh(\Omega r/c) - A_2 \cosh(\Omega r/c)] \, r \, dr \, d\phi' + dz^2 + r^2 \, d\phi'^2
\end{array}\right.
\]

(17)

through which the circumference of a disk with radius \(r = R\) in the \(z = \text{constant}\) plane is given by the Euclidean value \(2\pi R\) compared to the non-Euclidean value \((9)\) obtained through the Galilean transformed spatial metric \((8)\). It should be noted that despite the above fact the Gaussian curvature of the spatial metric is not zero indicating the non-Euclidean nature of the spatial metric \((16)\). It should also be noted from \((16)\) that proper time interval in the rotating frame is given by

\[
d\tau' = c dt' = c \cosh(\Omega r/c) dt = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} dt
\]

(18)

which in the limit \(\frac{\Omega R}{c} \ll 1\) is different from the relation between the two time intervals obtained from the Galilean transformed metric for rotating clocks at nonzero radii i.e \(d\tau' = \sqrt{1 - \frac{\Omega^2 r^2}{c^2}} dt\). On the other hand as we discussed earlier one could relate spatial distances and time intervals not through the metric obtained from Franklin transformation but in the coordinate transformations themselves according to their kinematical interpretation. Obviously using the formal analogy between FT and LT one can obtain relation between spatial distances (arclengths) and time intervals in the two coordinate systems as follows

\[
\Delta t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Delta t
\]

(19)

\[
\Delta l' = r \Delta \phi' = \sqrt{1 - \frac{v^2}{c^2}} R \Delta \phi = \sqrt{1 - \frac{v^2}{c^2}} \Delta l
\]

(20)

corresponding to the time dilation and length contraction respectively. With \(v = c \tanh(\frac{\Omega R}{c})\) at radius \(r = R\), the above results are consistent with what one expects from applying special relativistic length contraction (based on LT) to a rotating disk for \(\frac{\Omega R}{c} \ll 1\). It
seems that once again we are faced with the Ehrenfest’s paradox, in the sense that using the spatial geometry (either spatially Euclidean flat spacetime in an inertial frame or spatially non-Euclidean flat spacetime in a rotating frame given by equation \((17)\)) implies that the circumference of a rotating disk is the same as the one non-rotating whereas employing Franklin transformation, the circumference of a rotating disk is found to be shorter than the one non-rotating.

**B. Angular velocity of a test particle/disk point in the two frames related by FT**

Using the differential Franklin transformation \((15)\) to calculate the rotational frequency in the inertial observer’s frame we find

\[
\omega = \frac{d\phi}{dt} = \frac{\cosh(\Omega r/c) d\phi' + \frac{v r}{c} \sinh(\Omega r/c) + \frac{A_2}{c} dr'}{\cosh(\Omega r/c) dt' + \frac{r}{c} \sinh(\Omega r/c) d\phi' + \frac{A_1}{c} dr'}
\]

from which for the frequency in the rotating frame we have

\[
\omega' = \frac{d\phi'}{dt'} = \frac{\omega \cosh(\Omega r/c) - \frac{v}{r} \sinh(\Omega r/c) + \frac{dr'}{c} (\frac{A_1}{c} \omega - \frac{A_2}{r})}{\cosh(\Omega r/c) - \omega \frac{r}{c} \sinh(\Omega r/c)}
\]

In the limit where \((\Omega r/c) \ll 1\), the above expression reduces to the classical relation \((3)\)

\[
\omega' \approx \omega - \Omega.
\]

**V. MODIFIED FRANKLIN TRANSFORMATION: ITS INTERPRETATION AND THE SPACETIME METRIC IN THE ROTATING FRAME**

As it is obvious from its derivation, Franklin transformation was obtained in close analogy with the usual Lorentz transformation for inertial frames moving with constant velocities relative to one another. Our starting point for modification of Franklin transformation is its main formal difference from the Lorentz transformation which is the dependence of relative velocity on the radial coordinate (i.e \(v \equiv v(r)\)) in \((12)\). It is clear from Franklin’s derivation of \((12)\) that this coordinate-dependent velocity is a direct consequence of applying the relativistic composition law to high rotational velocities. Indeed the nonlinear velocity relation \((12)\) could also be obtained by the requirement that for any two infinitesimally close points on a uniformly rotating rigid rod (divided into \(n\) infinitesimal segments) with angular frequency \(\Omega\) the linear velocity is given by \(\Omega \Delta r\) and then using the relativistic composition
law iteratively to find the velocity at a finite distance along the rod in the limit $n \to \infty$.

Although velocity depends on the radius, since the kinematical transformation is supposed to give the relation between coordinates assigned to events by two observers, an inertial non-rotating one (laboratory observer/frame) and a non-inertial rotating observer at a given radius $R$, going back to the transformation law (by formal analogy with LT) the observer velocity at that radius i.e $v = c \tanh(R\Omega/c)$ should enter the transformation law. Indeed it has already been pointed out in some literature, without further clarification, that Franklin transformation leads to inconsistencies if one neglects the fact that it is determined at $r = constant$ as well as $z = constant$. We have mentioned some of these inconsistencies in previous sections and so by the above argument we introduce the following modified version of Franklin transformation (MFT)

$$t' = \cosh(\Omega R/c)t - \frac{R}{c} \sinh(\Omega R/c)\phi \quad ; \quad r' = r$$
$$\phi' = \cosh(\Omega R/c)\phi - \frac{R}{c} \sinh(\Omega R/c)t \quad ; \quad z' = z$$

(24)

This could be obtained by changing the second and third steps in the derivation of the Franklin transformation by assigning observers to the disk points at a given radius $r = r' = R$, for which the velocity with respect to the inertial observers, using the third step, is found to be $v = c \tanh(R\Omega/c)$. This is indeed a simple but physical modification with profound consequences. To see its effect, first of all we find the equivalent metric by finding the inverse differential transformation which is

$$dt = \cosh(\Omega R/c)dt' + \frac{R}{c} \sinh(\Omega R/c)d\phi' \quad ; \quad dr = dr'$$
$$d\phi = \cosh(\Omega R/c)d\phi' + \frac{R}{c} \sinh(\Omega R/c)dt' \quad ; \quad dz = dz'$$

(25)

and substituting them in the inertial frame’s flat spacetime metric (4) we end up with (taking $\beta = \frac{\Omega R}{c}$)

$$ds^2 = c^2 \cosh^2 \beta(1 - \frac{r^2}{R^2} \tanh^2 \beta)dt'^2 - dr'^2 - r^2 \cosh^2 \beta$$
$$(1 - \frac{R^2}{r^2} \tanh^2 \beta)d\phi'^2 + 2cR \sinh \beta \cosh \beta(1 - \frac{r^2}{R^2})dt'd\phi' - dz'^2$$

(26)

Note that now there is a radial coordinate $r$ as well as a constant radius $R$ which specifies a class of observers sitting at that radius. This will allow a kinematical interpretation of the above modified Franklin transformation. In other words no matter what the constant radius
in (24), this transformations gives a prescription of how the temporal \((t & t')\) and angular \((\phi & \phi')\) coordinates of an event in the two frames are related. Indeed, it is now that one could justify the division of the originally introduced transformation of arclengths \((r'\phi' & r\phi\) for an event at radial coordinate \(r = r')\) by the common radial coordinate leading to the transformation of angular coordinates \(\phi\) and \(\phi'\). It should be noted that spatial coordinate measurements by the inertial as well as the eccentric (non-inertial) observers are made from the axis of rotation as a preferred direction and indeed this is the point where we explicitly distinguish between frames and observers at non-zero radii and only employ the latter. These observers carry their own clocks but use the triad axes of the centrally rotating observer to designate spatial coordinates to events. The presence of \(R\) as a constant in the transformed flat spacetime as given by (25) may look strange but obviously it is no stranger than the appearance of \(\Omega\) in (11) or in (13). Both \(\Omega\) and \(R\) are transformation parameters, one \((\Omega)\) from an inertial frame/observer to a centrally rotating frame/observer and the other \((R)\) from the centerally rotating observer to a set of equivalent rotating observers at radius \(R\) (eccentric observers). Further it should not be forgotten that the spacetime in the rotating frame is always flat whether it is obtained through Franklin transformation or its modified version given by (25) or through Galilean transformation given in (5) and it is only the spatial metric which looses its Euclidean character. Obviously the metric (26) is of interest for radial distances

\[
r \leq \frac{\beta}{|\tanh| \left(\frac{c}{\Omega}\right)},
\]

and in the classical Galilean limit where \(\beta \ll 1\) (i.e close to the rotation axis) it reduces to

\[
ds^2 = c^2(1 - \frac{r^2\Omega^2}{c^2})dt'^2 - dr^2 - r^2(1 - \frac{R^2}{r^2}\beta^2)d\phi'^2 + 2R^2\Omega(1 - \frac{r^2}{R^2})dt'd\phi' - dz^2,
\]

which in turn reduces to the spacetime metric (5) under the extra condition that the radial coordinates of the events under consideration are larger than or equal to \(R\). In other words for observers close to the axis the range \(R \leq r < \frac{c}{\Omega}\) replaces the range \(0 \leq r < \frac{c}{\Omega}\). So, unlike the Franklin transformation, not only the transformation itself but also the metric in rotating frame reduces to the Galilean one in the limit \(\beta \ll 1\). It should be noted that for \(r = R\) in (26), i.e at the radial position of the eccentric observer, the metric reduces to that of a spatially Euclidean flat spacetime (5) of an inertial observer, i.e at \(r = R\) the form of the spacetime metric is invariant under MFT. This is a feature of (28) which is somewhat shared
with the Fermi metric of an accelerated, spinning observer in flat or curved background \cite{35}. In other words for the rotating observer sitting at radius \( R \) there will be no length contraction of small arclengths at that radius. One could think of observers at different radii as the centrally rotating one who has moved radially to that radius and measures the same proper length for arclengths without any contraction. Now setting \( R = 0 \) in (24) and its reduction exactly to GRT while (26) reduces to (4) has a consistent interpretation (in contrast to setting \( r = 0 \) in FT which was shown to lead to contradiction with respect to its kinematical interpretation); it corresponds to the centrally rotating observer who is at rest with respect to the non-rotating inertial observer there, and so their observations are naturally related through GRT. So in our setting of the problem of rotation and rotating frames, we have drastically changed the scenario by introducing non-inertial observers sitting at non-zero radii on the disk and introducing the MFT as the kinematical transformation between the coordinates assigned to events by these observers and the inertial ones.

In the next two subsections we find out how the spatial metric and angular velocity of a test particle/disk point are modified in a rotating frame through MFT.

\[ \text{A. Spatial line element and spatial distances} \]

Using the 1 + 3 approach (Appendix A), the metric (26) could be written in the following form

\[ ds^2 = c^2 \cosh^2 \beta (1 - \frac{r^2}{R^2} \tanh^2 \beta) (dt' - A_\alpha dx'^\alpha)^2 - dl^2 \]

(29)

in which the spatial line element is given by

\[ dl^2 = dr^2 + dz^2 + \left( r^2 \cosh^2 \beta (1 - \frac{R^2}{r^2} \tanh^2 \beta) + R^2 \frac{\sinh^2 \beta (1 - \frac{r^2}{R^2})^2}{(1 - \frac{r^2}{R^2} \tanh^2 \beta)} \right) d\phi'^2 \]

(30)

and the gravitomagnetic potential is

\[ A_\alpha \equiv A_\phi' \delta^\phi'_\alpha = (0, 0, -R \frac{\tanh \beta (1 - \frac{r^2}{R^2})}{(1 - \frac{r^2}{R^2} \tanh^2 \beta)}) \]

(31)

Now one could find the circumference of a circle/disk of radius \( r \) in \( z = \text{constant} \) plane using the above line element as

\[ L_{MFT} = \int dl = \int_0^{2\pi} \left( r^2 \cosh^2 \beta (1 - \frac{R^2}{r^2} \tanh^2 \beta) + R^2 \frac{\sinh^2 \beta (1 - \frac{r^2}{R^2})^2}{(1 - \frac{r^2}{R^2} \tanh^2 \beta)} \right)^{1/2} d\phi' \]

(32)
It is an easy task to show that the above spatial line element reduces to the classical spatial element \((8)\) in the limit of \(\beta \ll 1\). Also it is noted that for an observer at the center of the disk (i.e \(R = 0\)) circumference of the disk is equal to \(2\pi r\) and for an observer sitting at non-zero radius \(R\) a circle at that radius i.e \(r = R\), has the Euclidean circumference \(2\pi R\) as expected from the form-invariance of the metric \((26)\) at that radius. On the other hand using the MFT \((25)\), one obtains the following relation between the differential arclengths (at radius \(R\)) as measured by the rotating and inertial observers

\[
Rd\phi = \cosh(\Omega R/c)Rd\phi'
\]

In other words as in the case of FT, again we are faced with the Ehrenfest paradox in the sense that an arclength of a rotating disk, measured by the inertial observer, is the same as that of the non-rotating disk if spacetime metric is employed but different if MFT is used. The relation between length measurements by the inertial and rotating observers, based on MFT and hypothesis of locality \([28]\), are discussed and compared in \([14]\).

B. Angular velocity of a test particle/disk point in the two frames using MFT

In terms of the kinematical interpretation, the angular velocities of a test particle in the two frames related by the MFT is found by employing the inverse differential rotation \((25)\) so that

\[
\omega = \frac{d\phi}{dt} = \frac{\cosh \beta d\phi' + \frac{R}{c} \sinh \beta dt'}{\cosh \beta dt' + \frac{R}{c} \sinh \beta d\phi'}
\]

leading to

\[
\omega' = \omega(1 + \frac{R}{c} \tanh \beta) - \frac{c}{R} \tanh \beta
\]

in which we used the fact that \(\omega' = \frac{d\phi'}{dt'}\). As in the case of FT, it could be easily seen that in the limit of \(\beta \ll 1\) the above relation reduces to the classical relation \((3)\) which was found through the Galilean transformation. In terms of a rotating disk interpretation, from an inertial frame’s point of view, all points on the disk have the angular velocity \(\frac{d\phi}{dt} = \Omega\) and so the above relation changes into

\[
\omega' = \Omega(1 + \frac{R}{c} \tanh \beta) - \frac{c}{R} \tanh \beta
\]
in other words in MFT, unlike the case of Galilean transformation, for rotating observers
the angular velocities of points on a rotating disk are not zero and depend on the radius.
But the Galilean expectation is restored in the limit of \( \beta \ll 1 \) where \( \omega' = 0 \).

VI. NON-ININVARIANCE OF ELECTROMAGNETISM UNDER (MODIFIED)
FRANKLIN TRANSFORMATION

In some of the studies in the literature discussing the Franklin transformation it is claimed
that this transformation restores the full Lorentz (-type) covariance of electrodynamics \[23\].
Here we show in detail that such a claim is not correct. Under Lorentz transformation
Maxwell equations are invariant in the sense that they retain the same 3-dimensional vector
form in the transformed coordinates, consequently the electromagnetic wave equation which
is obtained from these equations is also form-invariant. In what follows we show that neither
the Maxwell equations nor wave equation are form-invariant under Franklin transformation.
To make life easier we show this in the absence of any EM sources and for the modified
Franklin transformation but the same result (non invariance of electromagnetism) holds for
the original Franklin transformation. From modified Franklin transformation \[24\] we have
the following relation between the partial derivatives

\[
\begin{align*}
\frac{\partial}{\partial t'} &= \cosh \beta \frac{\partial}{\partial t} + \frac{1}{R} \sinh \beta \frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial \phi'} &= R \sinh \beta \frac{\partial}{\partial t} + \cosh \beta \frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial r'} &= \frac{\partial}{\partial r} \\
\frac{\partial}{\partial z'} &= \frac{\partial}{\partial z}
\end{align*}
\]

(A. Non-invariance of wave equation under MFT)

Using the above relations the wave equation in the unprimed coordinates (inertial frame)

\[
\frac{\partial^2 \psi}{\partial t^2} - \frac{1}{r} \frac{\partial}{\partial r}(r \frac{\partial \psi}{\partial r}) - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} - \frac{\partial^2 \psi}{\partial z^2} = 0
\]

transforms into

\[
\left( \frac{r^2 \cosh^2 \beta - R^2 \sinh^2 \beta}{r^2} \right) \frac{\partial^2 \psi}{\partial t'^2} + 2 \left( \frac{(R^2 - r^2) \sinh \beta \cosh \beta}{R r^2} \right) \frac{\partial^2 \psi}{\partial t' \partial \phi'} - \frac{1}{r} \frac{\partial}{\partial r}(r \frac{\partial \psi}{\partial r}) \\
+ \left( \frac{r^2 \sinh^2 \beta - R^2 \cosh^2 \beta}{R^2 r^2} \right) \frac{\partial^2 \psi}{\partial \phi'^2} - \frac{\partial^2 \psi}{\partial z'^2} = 0
\]
under MFT, i.e the wave equation is not form-invariant under MFT. The same result could also be obtained by using the metric corresponding to MFT (Eqn. (26)) and the following general form of the wave equation in a curved background with metric $g_{ij}$

$$\Box \psi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_i} (g^{1/2} g^{ik} \partial \psi / \partial q_k) = 0$$  \hspace{1cm} (40)

where $q_i = t', r, \phi', z$.

**B. Non-invariance of Maxwell equations under MFT**

To obtain (source-free) Maxwell equations for a rotating observer from those in the frame of an inertial observer related through MFT we use the field tensor in the spacetime of a rotating observer (MFT metric) given by:

$$F'_{ij} = \begin{pmatrix}
0 & -\frac{A}{R} E'_{r} & -rE'_{\phi'} & -\frac{A}{R} E'_{z} \\
\frac{A}{R} E'_{r} & 0 & -\frac{A}{R} E'_{r} + \frac{A}{R} B'_{z} & -B'_{\phi'} \\
rE'_{\phi'} & \frac{A}{R} E'_{r} - \frac{A}{R} B'_{z} & 0 & \frac{A}{R} E'_{z} + \frac{A}{R} B'_{r} \\
\frac{A}{R} E'_{z} & B'_{\phi'} & -\frac{A}{R} E'_{z} - \frac{A}{R} B'_{r} & 0
\end{pmatrix}$$  \hspace{1cm} (41)

where

$$A = \sqrt{R^2 \cosh^2 \beta - r^2 \sinh^2 \beta} \quad \text{and} \quad \tilde{A} = (-R^2 + r^2) \sinh \beta \cosh \beta$$  \hspace{1cm} (42)

so that the inhomogeneous equations

$$\frac{1}{\sqrt{g}} \partial_t (\sqrt{g} F'_{ij}) = 0$$  \hspace{1cm} (43)

are given by

$$\partial_t [r(\frac{R}{A} E'_{r} - \frac{\tilde{A}}{rA} B'_{z})] + \partial_{\phi'} (E'_{r} \phi') + \partial_z [r(\frac{R}{A} E'_{z} + \frac{\tilde{A}}{A} B'_{r})] = 0$$  \hspace{1cm} (44)

$$\frac{R}{A} \partial_r E'_{r} - \frac{\tilde{A}}{rA} \partial_r B'_{z} - \frac{A}{rR} \partial_{\phi'} B'_{z} + \partial_r B'_{\phi'} = 0$$  \hspace{1cm} (45)

$$\partial_r E'_{\phi'} + \partial_t (\frac{A}{R} B'_{z}) - \partial_z (\frac{A}{R} B'_{r}) = 0$$  \hspace{1cm} (46)

$$\frac{rR}{A} \partial_r E'_{z} + \frac{\tilde{A}}{A} \partial_r B'_{r} - \partial_r (r B'_{\phi'}) + \frac{A}{R} \partial_{\phi'} B'_{r} = 0$$  \hspace{1cm} (47)

respectively for $j = 0, 1, 2, 3$. Also the homogeneous equations

$$\partial_{[i} F'_{jk]} = 0$$  \hspace{1cm} (48)
These equations are different in form from those obtained in the non-rotating inertial frame which are given by the above equations with \( A = R \) and \( \tilde{A} = 0 \). On the other hand in the limit \( \beta \ll 1 \), where MFT reduces to GRT, from equation (42) we have \( A \approx R \) and \( \tilde{A} \approx 0 \), i.e the above homogeneous equations retain their inertial forms. In other words for points close to the rotation axis, where MFT reduces to GRT, the homogeneous Maxwell equations are form-invariant under GRT, a result first shown by Schiff [26].

The same results as above could also be obtained by first writing the Maxwell equations in the non-rotating inertial frame using the field tensor in flat spacetime in cylindrical coordinates as follows

\[
F_{ij} = \begin{pmatrix}
0 & -E_r & -rE_\phi & -E_z \\
E_r & 0 & rB_z & -B_\phi \\
rE_\phi & -rB_z & 0 & rB_r \\
E_z & B_\phi & -rB_r & 0
\end{pmatrix}
\]  

(53)

and then employ the general relation between the field tensors in the two frames

\[
F_{ij} = \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^m}{\partial x^j} F'_{mn}
\]  

(54)

to relate the primed and unprimed electromagnetic fields and finally replace the unprimed quantities (including partial differentials using Eqn. (37)) by the primed ones. So in general neither wave equation nor the Maxwell equations are invariant under MFT.

VII. DISCUSSION AND SUMMARY

We have discussed a proposed relativistic rotational transformations (dubbed as Franklin transformation) relating coordinates of an inertial non-rotating frame to the one rotating
around their common axis with constant angular velocity $\Omega$ (measured by the inertial observers). Advantages and also drawbacks of this transformation specially with respect to the spacetime metric from the rotating observer’s point of view as well as with its kinematical interpretation are pointed out. By introducing non-inertial observers sitting at non-zero radii we have modified FT and showed how the modified transformation gives rise to a more consistent spacetime metric for these observers. The resulted spacetime metric includes two parameters $\Omega$ and $R$, corresponding to the rotational angular velocity and radial position of these observers. Though a flat spacetime, it has a non-Euclidean spatial line element (found through $1+3$ formulation of spacetime decomposition) leading to non-Euclidean value for the circumference of a rotating disk or any other circle of a given radius. In our setting of the problem of relativistic rotational transformation, there are three different kinds of observers: 

I- Inertial non-rotating (laboratory) observers

II- Centrally rotating (spinning) observer

III- Non-inertial rotating observers at non-zero radii (eccentric observers) who are rotating analogues of Rindler observers. In brief following are the important features of the MFT:

1- Unlike FT it leads to a spacetime metric in the rotating frame which reduces to the spacetime metric obtained by GRT in the corresponding limit (i.e close to the rotation axis).

2- Unlike in FT, the spacetime metric obtained via MFT preserves the temporal and angular isometries present in (5).

3- At $R = 0$ it reduces to the exact GRT as expected from its interpretation.

4- It gives a possible answer to the question: what is the spacetime metric for an eccentric observer on a rotating disk?.

5- Related to the above point, at the position of an eccentric observer (i.e at $r = R$), the spacetime metric is found to be form-invariant (i.e it reduces to the Minkowski metric) and spatially Euclidean, a fact hinting towards a possible relation with Fermi metric and Fermi coordinates. Indeed this seems to be an interesting evidence reinforcing our interpretation of the MFT and its corresponding metric. This could be further investigated by studying the relation between the metric (26) and the Fermi metric and coordinates for an accelerated spinning observer [12]. These matters will be discussed elsewhere [14].

It is also shown explicitly that, against the previous claims, neither Maxwell equations nor wave equation are invariant under FT or MFT.

From the experimental and observational points of view it is expected that application of
a relativistic rotational transformation to known physical effects related to the rotating systems and phenomena should lead to predictions different from those obtained through application of GRT or rotational transformations based on the hypothesis of locality. Some of the example include transverse Doppler effect, Sagnac effect \[25\] and rotational properties of pulsars. For a light source circling a receiver on a rotating disk, transverse Doppler effect will be affected naturally by FT and MFT, due to the nonlinear velocity \(12\) introduced in FT and this could be the most feasible test of the validity of MFT. Also it is expected that employing a relativistic rotational transformation will lead to a relativistic Sagnac effect distinct from the one due to propagation of light in a non-vacuum medium where relativistic velocity addition rule applies. Finally, fastest rotating celestial objects (apart from the supermassive black holes) are pulsars and the fastest pulsar, named PSR J1748-2446ad is located some 28,000 light-years from Earth in the constellation Sagittarius and is spinning at 716 Hertz. If its radius is taken to be 16 km it will have a Galilean linear velocity of 75000km/s i.e about 25% that of light speed at the equator. It is expected that at this rotational velocity a relativistic rotational transformation to be at work and observationally effective.

To look for experimental signatures of departure from GRT or rotational transformations based on hypothesis of locality, other physical effects (mainly electromagnetic in nature) which have already been studied in rotating frames \[27, 28\] should be reconsidered and interpreted in terms of MFT. In this regard, some of the rotational phenomena mentioned above are studied comparatively in \[14\].

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Appendix A: 1 + 3 (threading) formulation of spacetime decomposition and spatial distance

To define spatial metric and spatial distances in a given spacetime (metric) one could choose different spacetime decomposition formalisms. In our study we have employed the 1 + 3 (or threading) formulation of spacetime decomposition. Unlike the 3 + 1 (or foliation)
formulation of spacetime decomposition \[12\] in which spacetime is foliated into constant-time hypersurfaces, in the 1 + 3 formulation it is decomposed into threads tracking history of each spatial point. This formulation of spacetime decomposition starts from the following form for the metric of a stationary spacetime \[10\],

\[
ds^2 = d\tau_{syn}^2 - dl^2 = g_{00}(dx^0 - A_{\alpha} dx^{\alpha})^2 - \gamma_{\alpha\beta} dx^{\alpha} dx^{\beta}, \quad \alpha, \beta = 1, 2, 3 \tag{A1}
\]
in which all the metric components are time-independent i.e the coordinate system is adapted to the timelike Killing vector field of the spacetime \((\xi^a \equiv \delta^a_0 = (1, 0, 0, 0))\). Also \(d\tau_{syn} = \sqrt{g_{00}}(dx^0 - A_{\alpha} dx^{\alpha})\) is the synchronized proper time, \(A_{\alpha} = -\frac{g_{0\alpha}}{g_{00}}\) is the so-called gravitomagnetic potential and

\[
dl^2 = \gamma_{\alpha\beta} dx^{\alpha} dx^{\beta} = (-g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}}) dx^{\alpha} dx^{\beta} \tag{A2}
\]
is the spatial line element (also called the radar distance element) of the 3-space in terms of its three-dimensional spatial metric \(\gamma_{\alpha\beta}\). It is the integral of this line element which gives the spatial distance between two events with spatial coordinates \(x^{\alpha}_i\) and \(x^{\alpha}_f\) \[10, 15\],

\[
L = \int^{x^{\alpha}_f}_{x^{\alpha}_i} dl \tag{A3}
\]
For two simultaneous events at nearby points \(x^{\alpha}\) and \(x^{\alpha} + dx^{\alpha}\) the difference between their coordinate (world) time is given by

\[
\Delta x^{0} = A_{\alpha} dx^{\alpha}, \tag{A4}
\]

This allows one to synchronize clocks in an infinitesimal region of space and also along any open curve. But synchronization of clocks along a closed path is generally not possible, since upon returning to the initial point the world time difference is not zero and in the case of stationary spacetimes is given by the line integral

\[
\Delta x^{0} = \oint A_{\alpha} dx^{\alpha} \tag{A5}
\]
taken along the closed path. Using the above equation the world-time difference for two photons started at the same point but travelling in opposite directions (clockwise and counter clockwise) along a circle of radius \(R\) on a disk rotating with angular velocity \(\Omega\) such that \(\frac{\Omega R}{c} \ll 1\) is given by

\[
\Delta t = 4\pi R^2 \frac{\Omega}{c^2} \tag{A6}
\]
This difference which leads to a phase shift \( \delta \phi = \frac{2\pi c \Delta t}{\lambda} \) could also be obtained through classical reasoning by an inertial non-rotating observer and is the theoretical basis of the so-called Sagnac effect \(^{25}\) or in its modern version, ring laser interferometry.

The 3-velocity of a test particle is defined in terms of the synchronized proper time as follows

\[
v^\alpha = \frac{dx^\alpha}{d\tau_{\text{syn}}} = \frac{dx^\alpha}{\sqrt{g_{00}(dx^0 - A_\alpha dx^\alpha)}}, \tag{A7}
\]

where now using (A1) and (A7) the spacetime line element could be written as follows

\[
ds^2 = d\tau_{\text{syn}}^2 (1 - v^2). \tag{A8}
\]

Now the components of the 4-velocity \( u^i = \frac{dx^i}{ds} \) \((i = 0, 1, 2, 3)\), in terms of the components of the 3-velocity are given by

\[
u^0 = \frac{1}{\sqrt{g_{00}\sqrt{1 - v^2}}} + \frac{A_\alpha v^\alpha}{\sqrt{1 - v^2}} ; \quad v^\alpha = \frac{v^\alpha}{\sqrt{1 - v^2}}, \tag{A9}
\]

where in the comoving frame, \( v^\alpha = 0 \), it reduces to \( u^i = (\frac{1}{\sqrt{g_{00}}} 0, 0, 0) \) as expected. It is this same formulation of spacetime decomposition which allows one to use analogy with electromagnetism and define gravitoelectric and gravitomagnetic fields as follows;

\[
E_g = -\frac{\partial h}{2h} B_g = \nabla \times A. \tag{A10}
\]

In terms of the above fields and in the context of the so-called gravitoelectromagnetism, vacuum Einstein field equations could be rewritten in the following quasi-Maxwell form \(^{29,30}\),

\[
\nabla \times E_g = 0 ; \quad \nabla \cdot B_g = 0 \tag{A11}
\]

\[
\nabla \cdot E_g = 1/2h B_g^2 + E_g^2 \tag{A12}
\]

\[
\nabla \times (\sqrt{h} B_g) = 2E_g \times (\sqrt{h} B_g)v \tag{A13}
\]

\[
^{(3)} R^{\mu\nu} = -E_g^{\mu\nu} + \frac{1}{2} h (B_g^\mu B_g^\nu - B_g^2 \gamma^{\mu\nu}) + E_g^\mu E_g^\nu. \tag{A14}
\]

where \(^{(3)} R^{\mu\nu}\) is the 3-dimensional Ricci tensor of the 3-space constructed from the 3-dimensional metric \( \gamma_{\alpha\beta} \) in the same way that the usual 4-dimensional Ricci tensor \( R^{ab}\) is made out of \( g_{ab}\). The first two equations (A11) are direct consequences of our definitions of gravitoelectric and gravitomagnetic fields and the original ten field equations are now given by those constituted in equations (A12-A14).
It should also be noted that in the above equations all the differential operations are defined in the 3-space with metric $\gamma_{\alpha\beta}$ [10, 29], in particular divergence and curl of a vector are defined as follows

$$\text{div} V = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha}(\sqrt{\gamma} V^\alpha), \quad (\text{curl} V)^\alpha = \frac{1}{2\sqrt{\gamma}} \epsilon^{\alpha\beta\gamma}(\frac{\partial V_\gamma}{\partial x^\beta} - \frac{\partial V_\beta}{\partial x^\gamma}),$$  \quad (A15)

in which $\gamma = \text{det} \gamma_{\alpha\beta}$ and one can show that

$$- g = h\gamma.$$  \quad (A16)
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[31] A comparative study on the measurements of physical phenomena related to rotating frames has been carried out in both Fermi and modified Franklin coordinates (the latter introduced later in this paper) [14].

[32] In some literature this transformation is called Takeno-Trocheris transformation, but due to Franklin’s precedence by almost 30 years and also to highlight his largely ignored work, we will call it Franklin transformation.

[33] It should be noted that $\Omega$ is taken as a constant and such that the integrity of the rotating disk is retained.

[34] Note that the condition $\beta \ll 1$ is equivalent to $R \ll \frac{c}{\Omega}$ whereas the same condition employed in (27) leads to $r \leq \frac{c}{\Omega}$.

[35] Recall the feature of the Fermi metric that on the observer’s worldline it reduces to the Minkowski metric [12].