A $\mathbb{Z}_2$–Topological Index for Quasi–free Fermions

N. J. B. Aza*, A. F. Reyes-Lega†, and L. A. Sequera M.‡

1Departamento de Física, Universidad de Los Andes

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Abstract

We use infinite dimensional self–dual CAR $C^*$–algebras to study a $\mathbb{Z}_2$–index, which classifies free–fermion systems embedded on $\mathbb{Z}^d$ disordered lattices. Combes–Thomas estimates are pivotal to show that the $\mathbb{Z}_2$–index is uniform with respect to the size of the system. We additionally deal with the set of ground states to completely describe the mathematical structure of the underlying system. Furthermore, the weak$^*$–topology of the set of linear functionals is used to analyze paths connecting different sets of ground states.

Keywords: Operator Algebras, Disordered fermion systems, $\mathbb{Z}_2$–index, ground states.

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*Corresponding author: njavierbuitragoa@gmail.com
†anreyes@uniandes.edu.co
‡la.sequera@uniandes.edu.co
1 Introduction

A considerable number of mathematical results concerning gapped Hamiltonians of fermions has been achieved in recent years. Among the most important ones are topological protection under small perturbations and the persistence of the spectral gap for interacting fermions [Has19, DS19]. We study a $\mathbb{Z}_2$–projection index ($\mathbb{Z}_2$–PI), that is the one introduced long ago by Araki–Evans in their work where the two possible thermodynamic phases of the classical two-dimensional Ising model are characterized using operator algebras technologies [AE83]. Here we deal with disordered free–fermion systems on the lattice within the mathematical framework of self–dual CAR $C^*$–algebras. In particular, their structure is useful to study interacting fermion systems, even with superconducting terms [ABPM21]. To be precise, the $\mathbb{Z}_2$–PI is defined in terms of well–defined basis projections related to a self–adjoint operator, which typically is the Hamiltonian of the system acting on a separable Hilbert space $\mathcal{H}$. See Definition 2 below. Thus, the $\mathbb{Z}_2$–PI can be used to discriminate parity sectors in the set of quasi–free ground states of fermionic systems [EK98, GBVF01].

A very important problem in this context is the classification of topological matter in general. The current classification scheme can be traced back to Dyson’s [Dys62] classical work from 1962. Of course that work did not contemplate topological aspects for such systems, but it provided the setting on which more recent work has been based. Indeed, a completion of this early work was made by Altland and Zirnbauer [AZ97], leading to the identification of new symmetry classes. These ideas were generalized by Kitaev [Kit09] and led to a “periodic table” of topological insulators and superconductors. In that work, Kitaev showed how the classification can be achieved in terms of Bott periodicity and $K$–theory. More recently an exhaustive and complete version of the classification was made by Ryu et al. [RSFL10]. They explore arbitrary dimensions making use again of classifying spaces given by the Cartan symmetric spaces along with Bott periodicity in a more strong way. For further details we refer to the reader to see [HL11, LH10, FMP16, KK18]). This allows them to consider disordered systems and shows the explicit relation between gapped Hamiltonians and Anderson localization phenomena, a very important result for this kind of problem.

The first iconic example of a topological fermionic system is the quantum Hall effect. The observed quantization of the conductivity was explained by Thouless et al. [TKNdN82] and led to the recognition of the important role played by the Chern number. The restrictions on the validity of this result where eventually overcome by Bellissard [BvES94] and collaborators, in what was to become one of the main examples of applications of noncommutative geometry to physics. This was a big step to deal with more realistic models that consider disordered media. In this line of ideas there are more recent works, due to Carey et al [CHM‘06, PS16, BCR16], where Bellissard’s techniques are generalized to deal with a wider class of systems.

For interacting systems rigorous proofs of quantization of conductivity were provided in [GMP16, BDF18]. These studies rely on the study of families of gapped Hamiltonians, such that any two elements on these families can be continuously deformed into one another. The latter was demonstrated rigorously by Bachmann, Michalakis, Nachtergaele and Sims [BMNS12] by studying spectral flow of quantum spin systems under a “quasi–adiabatic” evolution. They proved that such related systems verify the same Lieb–Robinson bounds and in its thermodynamic limit the spectral flow has a cocycle structure for the automorphism in the algebra of observables. By using the dual space of the underlying algebras considered they also studied the ground states associated.

From the point of view of physics, fundamental properties of such systems are deduced from the study of the set of ground states in the thermodynamic limit and zero temperature. Relevant examples include electronic conduction problems (e.g. quantum Hall effect), or the study of different phases of matter. Nevertheless, knowledge of ground states for concrete models is a huge challenge in general. This is due to the fact that there is no general procedure to find the full set of ground states for specific systems. As far as we know, there are very few mathematical physics results about the existence of ground states, in contrast to the theoretical point of view, see [AT85, CNN18]. Instead, one generally
verifies the existence of the ground state energy for specific physical systems\(^1\).

In this paper we focus on the study of \(\mathbb{Z}_2\text{--PI}\) for non–interacting fermion systems. We specifically deal with unique ground states associated to families of gapped Hamiltonians. Note that there is an alternative form of the \(\mathbb{Z}_2\text{--PI}\) \((25)\) in terms of orthogonal complex structures [GBVF01, EK98]. There, the index appears naturally in the proof of the Shale–Stinespring Theorem and is related to the parity of the fermionic Fock ground states. In [CGRL18], this approach to the \(\mathbb{Z}_2\text{--PI}\) was used to study ground states for finite Kitaev chains with different boundary conditions. More recently, for infinite translationally invariant fermionic chains, Bourne and Schulz-Baldes classify ground states using orthogonal complex structures [BSB20]. Furthermore, Matsui [Mat20] uses split–property of infinite chains and its connection with the \(\mathbb{Z}_2\text{--PI}\). Theorem 1 is related, to a certain extent, to the mentioned results. Notice, however, that we do not require translational invariance nor are we making any assumptions about the spatial dimension of the system, as required, for example, in [Mat20]. Although there are many similarities with the \(\mathbb{Z}_2\text{--PI}\) index introduced in [BSB20], they rely on different notions of spectral flow (see the remark in the introduction to [BSB20]). It would be interesting to relate the different approaches to the \(\mathbb{Z}_2\text{--PI}\) index (particularly in view of their generalizations to interacting systems). An important point on which we focus in the present paper is in an analysis of the uniformity of the \(\mathbb{Z}_2\text{--PI}\) with respect to the size of the systems. Physically, \(\mathbb{Z}_2\text{--PI}\) will distinguish if a pair of Hamiltonians are in the same phase of matter or not. Moreover, the \(\mathbb{Z}_2\text{--PI}\) is closely related with the one proposed by Kitaev [Kit01], which was introduced to distinguish the parity of states in quantum wires. However, as already mentioned, our results consider any physical dimension, and has the potential to be studied in the interacting case. In fact, in [AMR] we will report on results about the stability of the \(\mathbb{Z}_2\text{--PI}\) for weakly interacting fermions. Observe that the technical tools in that case differ from the current study and other technologies such as Lieb–Robinson bounds and Renormalization Group Methods will be required. For example, we use similar techniques as in [Oga20, BO21] where are studied indexes on one–dimensional interacting quantum spin systems and [DS19] where is proven the stability of the spectral gap for weakly interacting fermions.

The scope of the current paper is to consider bounded Hamiltonians on separable Hilbert spaces. However, most of the statements can be extended for unbounded Hamiltonians as is done, for example, in [BP16a, NSY18b]. Specifically, in [NSY18b], the authors consider quantum lattice systems such that their results hold for well–defined interactions as well as unbounded zero–range interactions (also known as one–site interactions or on–site terms). In [AMR] we will take into account unbounded zero–range interactions such that we can invoke general Theorems for fermions on lattices as the ones presented in [BP16a]. Moreover, we will tackle all (even unbounded cases, not being limited to semibounded) one–particle Hamiltonians, similar to [BP16b]. Our setting will be closely related to the one of [ABPM21, Sect. 5]. There, we efficiently estimated determinant bounds and summabilities of two–point correlation functions associated with covariances, which depend only on one–particle Hamiltonians. These will be pivotal in the study of analytic properties of generating functions, which in turn describe all the statistical properties of weakly interacting fermion systems on the lattice. Thus, our results will consider any \(\beta \in \mathbb{R}_+ \cup \{\infty\}\), crystal lattice \(\mathbb{Z}^d\) with \(d \in \mathbb{N}\), sufficiently weak interactions, and physical systems not necessarily translationally invariant.

To conclude, our main results are Theorems 1 and 2, as well as the set of Lemmata 1–4: In Theorem 1 we prove that for a differentiable path of gapped self–dual Hamiltonians, a spectral flow automorphism can be constructed, which will allow to show that the relative \(\mathbb{Z}_2\text{–PI}\) of any two Hamiltonians along the path must be equal to one. In Theorem 2 we show how, for a given differentiable path of Hamiltonians connecting gapped Hamiltonians with opposite index, the gap must close somewhere along the path. We furthermore analyse the structure of a family of ground states which is

\(^1\)Ground state energy can be understood as the states associated to the lowest energy of a physical system. For example, Giuliani and Jauslin use rigorous renormalization methods to prove the existence of the ground state energy for the bilayer graphene [GJ16].
induced by the gap closing. From the mathematical point of view, the first part of Theorem 1 is reminiscent of the interacting case \[BMNS12\], however, we additionally state the \(\mathbb{Z}_2\)-PI result, discriminating if a pair of Hamiltonians are equivalent or not on infinite self–dual \(\mathcal{C}^*\)-algebras. In particular, we have in mind differentiable families of operators \(\{H_s\}_{s\in[0,1]}\), e. g., given by the differentiable operator \(H_s \equiv (1 - s)H_0 + sH_1\), for \(s \in [0,1]\), with \(H_0, H_1 \in \mathcal{B}(\mathcal{H})\) bounded operators with the same spectral gap and acting on a separable Hilbert space \(\mathcal{H}\). On the other hand, Theorem 2 deals with subsets of the ground states set. For instance, open spectral gap ground states are considered. As a particular case of the general Theorem 2, we prove that in the weak*-topology, paths connecting states in different topological components implies the existence of a Hamiltonian having 0 as an eigenvalue.

The paper is organized as follows:

- Section 2 presents the mathematical framework of \(\mathcal{C}^*\)-algebras. We introduce self–dual \(\mathcal{C}^*\)-algebras, which were introduced long ago by Araki in his elegant study of non–interacting but non–gauge invariant fermion systems. We recall pivotal properties of general \(\mathcal{C}^*\)-algebras.

- In Section 3 we state the main Theorems, as well as some relevant definitions concerning the \(\mathbb{Z}_2\)-PI and comment on the weak*-topology of the set of states. In particular we discuss the conditions for a system to have pure or mixed states.

- Section 4 is devoted to all technical proofs. We prove the existence of a spectral flow automorphism for self–dual Hilbert spaces, for families of differentiable Hamiltonians. Then, the existence of strong limits for the dynamics, the spectral flow automorphism and the weak*-convergence of ground states are proven. Well–known Combes–Thomas estimates are invoked for families of gapped Hamiltonians, which will permit to analyze two–point correlation functions such that we obtain the trace class properties for relevant unitary operators.

- We finally include Appendix A, providing a general framework of graphs with special attention to disordered models. Appendix B regards on basic statements of the Fock representation of \(\mathcal{C}^*\)-algebras, and how their dynamics are related.

Notation 1.

A norm on the generic vector space \(\mathcal{X}\) is denoted by \(\| \cdot \|_{\mathcal{X}}\) and the identity map of \(\mathcal{X}\) by \(1_{\mathcal{X}}\). The space of all bounded linear operators on \(\mathcal{X}\) is denoted by \(\mathcal{B}(\mathcal{X})\). The unit element of any algebra \(\mathcal{X}\) is always denoted by \(1\), provided it exists of course. The scalar product of any Hilbert space \(\mathcal{X}\) is denoted by \(\langle \cdot, \cdot \rangle_{\mathcal{X}}\) and tr\(\mathcal{X}\) represents the usual trace on \(\mathcal{B}(\mathcal{X})\).

2 Mathematical Framework and Physical Setting

We introduce the mathematical framework based on Araki’s self–dual formalism [Ara68, Ara71]. Our setting considers disorder effects, which come as is usual in physics, i.e., from impurities, crystal lattice defects, etc. Thus, disorder can modeled by (a) a random external potential, like in the celebrated Anderson model, (b) a random Laplacian, i.e., a self–adjoint operator defined by a next–nearest neighbor hopping term with random complex–valued amplitudes. In particular, random vector potentials can also be implemented.

2.1 Self–dual CAR Algebra

If not otherwise stated, \(\mathcal{H}\) always stands for a (complex, separable) Hilbert space. If \(\mathcal{H}\) is finite–dimensional, we will assume it is even–dimensional, i.e., \(\dim \mathcal{H} \in 2\mathbb{N}\). Let \(\Gamma: \mathcal{H} \to \mathcal{H}\) be a
conjugation or antiunitary involution on \( \mathcal{H} \), i.e., an antilinear operator such that \( \Gamma^2 = 1_\mathcal{H} \) and\(^\text{2}\)

\[
\langle \Gamma \varphi_1, \Gamma \varphi_2 \rangle_{\mathcal{H}} = \langle \varphi_2, \varphi_1 \rangle_{\mathcal{H}}, \quad \varphi_1, \varphi_2 \in \mathcal{H}.
\]

The space \( \mathcal{H} \) endowed with the involution \( \Gamma \) is named a self–dual Hilbert space, \( (\mathcal{H}, \Gamma) \), and yields self–dual CAR algebra:

**Definition 1 (Self–dual CAR algebra).**

A self–dual CAR algebra \( s\text{CAR}(\mathcal{H}, \Gamma) \equiv (s\text{CAR}(\mathcal{H}, \Gamma), +, \cdot, \ast) \) is a \( C^* \)–algebra generated by a unit \( 1 \) and a family \( \{B(\varphi)\}_{\varphi \in \mathcal{H}} \) of elements satisfying Conditions 1.–3.:

1. The map \( \varphi \mapsto B(\varphi)^* \) is (complex) linear.
2. \( B(\varphi)^* = B(\Gamma(\varphi)) \) for any \( \varphi \in \mathcal{H} \).
3. The family \( \{B(\varphi)\}_{\varphi \in \mathcal{H}} \) satisfies the CAR: For any \( \varphi_1, \varphi_2 \in \mathcal{H} \),

\[
B(\varphi_1)B(\varphi_2)^* + B(\varphi_2)^*B(\varphi_1) = \langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}} 1.
\]

For a historic overview on self–dual CAR algebras and some of their basic properties see [Ara68, Ara71, Ara87, Ara88, EK98]. Note that by the CAR (1), the antilinear map \( \varphi \mapsto B(\varphi) \) is necessarily injective and contractive. Therefore, \( \mathcal{H} \) can be embedded in \( s\text{CAR}(\mathcal{H}, \Gamma) \).

Conditions 1.–3. of Definition 1 only define self–dual CAR algebras up to Bogoliubov \(*\)– automorphisms\(^\text{3}\) (see (5)). In [ABPM21], an explicit construction of \(*\)–isomorphic self–dual CAR algebras from \( \mathcal{H} \) and \( \Gamma \) is presented. This is done via basis projections [Ara68, Definition 3.5], which highlight the relationship between CAR algebras and their self–dual counterparts.

**Definition 2 (Basis projections).**

A basis projection associated with \( (\mathcal{H}, \Gamma) \) is an orthogonal projection \( P \in \mathcal{B}(\mathcal{H}) \) satisfying \( \Gamma P \Gamma = P^\perp = 1_\mathcal{H} - P \). We denote by \( h_P \) the range \( \text{ran}(P) \) of the basis projection \( P \). The set of all basis projections associated with \( (\mathcal{H}, \Gamma) \) will be denoted by \( p(\mathcal{H}, \Gamma) \).

For simplicity, in the rest of this section, we will assume that \( \mathcal{H} \) is finite–dimensional with even size: \( \dim \mathcal{H} \in 2\mathbb{N} \). For any \( P \in p(\mathcal{H}, \Gamma) \) a few remarks are in order:

\( h_P \) must satisfy the conditions

\[
\Gamma(h_P) = h_P^\perp \quad \text{and} \quad \Gamma(h_P^\perp) = h_P.
\]

Then, by [Ara68, Lemma 3.3], an explicit \( P \in p(\mathcal{H}, \Gamma) \) can always be constructed. Moreover, \( \varphi \mapsto (\Gamma \varphi)^* \) is a unitary map from \( h_P \) to the dual space \( h_P^* \). In this case we can identify \( \mathcal{H} \) with

\[
\mathcal{H} \cong h_P \oplus h_P^*.
\]

and

\[
B(\varphi) \equiv B_P(\varphi) \equiv B(P, \varphi) + B(\Gamma P^\perp \varphi)^*.
\]

Therefore, there is a natural isomorphism of \( C^* \)–algebras from \( s\text{CAR}(\mathcal{H}, \Gamma) \) to the CAR algebra \( \text{CAR}(h_P) \) generated by the unit \( 1 \) and \( \{B_P(\varphi)\}_{\varphi \in h_P} \). In other words, a basis projection \( P \) can be used to fix so–called annihilation and creation operators. For each basis projection \( P \) associated with

\(^2\)We will assume that the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C} \) associated to some Hilbert space \( \mathcal{H} \) is a sesquilinear form on \( \mathcal{H} \) such that is antilinear in its first component while is linear in the second one.

\(^3\)An analogous result for CAR algebra is, for instance, given by [BR03b, Theorem 5.2.5].
\((\mathcal{H}, \Gamma)\), by (3), \(h_P\) can be seen as a one–particle Hilbert space.

Self–dual CAR algebras naturally arise in the diagonalization of quadratic fermionic Hamiltonians (Definition 3), via Bogoliubov transformations defined as follows [Ara68, Ara71]: For any unitary operator \(U \in \mathcal{B}(\mathcal{H})\) such that \(UT = \Gamma U\), the family of elements \(B(U\varphi)_{\varphi \in \mathcal{H}}\) satisfies Conditions (a)–(c) of Definition 1 and, together with the unit 1, generates \(\text{sCAR}(\mathcal{H}, \Gamma)\).

Like in [Ara71, Section 2], such a unitary operator \(U \in \mathcal{B}(\mathcal{H})\) commuting with the antiunitary map \(\Gamma\) is named a Bogoliubov transformation, and the unique \(*\)–automorphism \(\chi_U\) such that

\[
\chi_U(B(\varphi)) = B(U\varphi), \quad \varphi \in \mathcal{H},
\]

is called in this case a Bogoliubov \(*\)–automorphism. Note that a Bogoliubov transformation \(U \in \mathcal{B}(\mathcal{H})\) always satisfies

\[
\det(U) = \det(\Gamma U) = \overline{\det(U)} = \pm 1
\]

If \(\det(U) = 1\), we say that \(U\) is in the positive connected set \(\mathfrak{U}_+\). Otherwise \(U\) is said to be in the negative connected set \(\mathfrak{U}_-\). \(\chi_U(B(\varphi))\) is said to be even (respectively odd) if and only if \(U \in \mathfrak{U}_+\) (respectively \(U \in \mathfrak{U}_-\)).

Clearly, if \(P \in \mathfrak{p}(\mathcal{H}, \Gamma)\), see Definition 2, and \(U \in \mathcal{B}(\mathcal{H})\) is a Bogoliubov transformation, then \(P_U \doteq U^*PU\) is another basis projection. Conversely, for any pair \(P_1, P_2 \in \mathfrak{p}(\mathcal{H}, \Gamma)\) there is a (generally not unique) Bogoliubov transformation \(U\) such that \(P_2 = U^*P_1U\). See [Ara68, Lemma 3.6]. In particular, Bogoliubov transformations map one–particle Hilbert spaces onto one another.

Considering the Bogoliubov \(*\)–automorphism (5) with \(U = -1_\mathcal{H}\), an element \(A \in \text{sCAR}(\mathcal{H}, \Gamma)\), satisfying

\[
\chi_{-1_\mathcal{H}}(A) = \begin{cases} 
A & \text{is called even,} \\
-A & \text{is called odd,}
\end{cases}
\]

Note that the subspace \(\text{sCAR}(\mathcal{H}, \Gamma)^+\) of even elements is a sub–\(C^*\)–algebra of \(\text{sCAR}(\mathcal{H}, \Gamma)\).

It is well–known that in quantum mechanics the even elements are the ones suitable for the description of fermion systems. For example, self–adjoint (even) elements of the CAR algebra which are quadratic in the creation and annihilation operators are used, for instance, in the Bogoliubov approximation of the celebrated (reduced) BCS model. In the context of self–dual CAR algebra, those elements are called \textit{bilinear Hamiltonians} and are self–adjoint bilinear elements:

**Definition 3 (Bilinear elements of self–dual CAR algebra).**

Given an orthonormal basis \(\{\psi_i\}_{i \in I}\) of \(\mathcal{H}\), we define the bilinear element associated with \(H \in \mathcal{B}(\mathcal{H})\) to be

\[
\langle B, HB \rangle = \sum_{i,j \in I} \langle \psi_i, H\psi_j \rangle_\mathcal{H} B(\psi_j) B(\psi_i)^*.
\]

Note that \(\langle B, HB \rangle\) \textit{does not depend} on the particular choice of the orthonormal basis, but does depend on the choice of generators \(\{B(\varphi)\}_{\varphi \in \mathcal{H}}\) of the self–dual CAR algebra \(\text{sCAR}(\mathcal{H}, \Gamma)\), and by (1), bilinear elements of \(\text{sCAR}(\mathcal{H}, \Gamma)\) have adjoints equal to

\[
\langle B, HB \rangle^* = \langle B, H^*B \rangle, \quad H \in \mathcal{B}(\mathcal{H}).
\]

\textit{Bilinear Hamiltonians} are then defined as bilinear elements associated with self–adjoint operators \(H = H^* \in \mathcal{B}(\mathcal{H})\). They include all second quantizations of one–particle Hamiltonians, but also models that are \textit{not} gauge invariant. Important models in condensed matter physics, like in the BCS theory of superconductivity, are bilinear Hamiltonians that are \textit{not} gauge invariant.
Without loss of generality (w.l.o.g.), our analysis of bilinear elements can be restricted to operators \( H \in \mathcal{B}(\mathcal{H}) \) satisfying \( H^* = -\Gamma H \Gamma \), which, in particular, have zero trace, i.e., \( \text{tr}_\mathcal{H}(H) = 0 \). We call such operators self–dual operators:

**Definition 4 (Self–dual operators).**
A self–dual operator on \((\mathcal{H}, \Gamma)\) is an operator \( H \in \mathcal{B}(\mathcal{H}) \) satisfying the equality \( H^* = -\Gamma H \Gamma \). If, additionally, \( H \) is self–adjoint, then we say that it is a self–dual Hamiltonian on \((\mathcal{H}, \Gamma)\).

We say that the basis projection \( P \) (Definition 2) (block–) “diagonalizes” the self–dual operator \( H \in \mathcal{B}(\mathcal{H}) \) whenever

\[
H = \frac{1}{2} \left(P HP - P\Gamma H \Gamma P\right), \quad \text{with} \quad H_P \doteq 2PHP \in \mathcal{B}(\mathcal{H}).
\]

In this situation, we also say that the basis projection \( P \) diagonalizes \( \langle B, HB \rangle \), similarly to [Ara68, Definition 5.1].

By the spectral theorem, for any self–dual Hamiltonian \( H \) on \((\mathcal{H}, \Gamma)\), there is always a basis projection \( P \) diagonalizing \( H \). In quantum physics, as discussed in Section 2.1, \( \mathcal{H} \) is in this case the one–particle Hilbert space and \( H_P \) the one–particle Hamiltonian.

### 2.2 Quasi–Free Dynamics

Bilinear Hamiltonians are used to define so–called quasi–free dynamics: For any \( H = H^* \in \mathcal{B}(\mathcal{H}) \), we define the continuous group \( \{ \tau_t \}_{t \in \mathbb{R}} \) of \(^*\)-automorphisms of \( s\text{CAR}(\mathcal{H}, \Gamma) \) by

\[
\tau_t(A) \doteq e^{-it(B, HB)} A e^{it(B, HB)}, \quad A \in s\text{CAR}(\mathcal{H}, \Gamma), \quad t \in \mathbb{R}.
\]

Provided \( H \) is a self–dual Hamiltonian on \((\mathcal{H}, \Gamma)\) (Definition 4), this group is a quasi–free dynamics, that is, a strongly continuous group of Bogoliubov \(^*\)-automorphisms, as defined in Equation (5). Straightforward computations using Definitions 1 and 3, together with the properties of the antiunitary involution \( \Gamma \), lead to show that

\[
\exp \left( -\frac{z}{2} (B, HB) \right) B (\varphi)^* \exp \left( \frac{z}{2} (B, HB) \right) = B (e^{zH} \varphi)^*,
\]

even for any self–dual operator \( H \) on \((\mathcal{H}, \Gamma)\), all \( z \in \mathbb{C} \) and \( \varphi \in \mathcal{H} \).

Moreover, for \( \{ \tau_t \}_{t \in \mathbb{R}} \), we define the linear subspace

\[
\mathcal{D}(\delta) \doteq \{ A \in s\text{CAR}(\mathcal{H}, \Gamma) : t \mapsto \tau_t(A) \text{ is differentiable at } t = 0 \} \subset s\text{CAR}(\mathcal{H}, \Gamma)
\]

and the linear operator (unique, generally unbounded) \( \delta : \mathcal{D} \to s\text{CAR}(\mathcal{H}, \Gamma) \) by

\[
\delta(A) \doteq \frac{d\tau_t(A)}{dt} \big|_{t=0}.
\]

The operator \( \delta \) is called the generator of \( \tau \) and \( \mathcal{D}(\delta) \) is the (dense) domain of definition of \( \delta \). Here we will assume that \( \delta \) is a symmetric unbounded derivation, i.e., the domain \( \mathcal{D}(\delta) \) of \( \delta \) is a dense \(^*\)-subalgebra of \( \mathfrak{A} \) and, for all \( A, B \in \mathcal{D}(\delta) \),

\[
\delta(A)^* = \delta(A^*), \quad \delta(AB) = \delta(A)B + A\delta(B).
\]

Note that the set of all symmetric derivations on \( \mathcal{D}(\delta) \) can be endowed with a real vector space structure. In fact, for any symmetric derivations \( \delta_1 \) and \( \delta_2 \) and all real numbers \( \alpha_1, \alpha_2 \), the expression

\[
(\alpha_1 \delta_1 + \alpha_2 \delta_2)(A) \doteq \alpha_1 \delta_1(A) + \alpha_2 \delta_2(A), \quad A \in \mathcal{D}(\delta),
\]

gives rise to another symmetric derivation \( \alpha_1 \delta_1 + \alpha_2 \delta_2 \) on \( \mathcal{D}(\delta) \).

\(^4\)Recall that for any separable Hilbert space \( \mathcal{H}, A \in \mathcal{B}(\mathcal{H}) \) and any orthonormal basis \( \{ \psi_i \}_{i \in I} \) of \( \mathcal{H} \) the trace of \( A, \text{tr}_\mathcal{H}(A) \doteq \sum_{i \in I} \langle \psi_i, A\psi_i \rangle_\mathcal{H}, \) does not depend of the choice of the orthonormal basis.
2.3 States

A linear functional \( \omega \in \text{sCAR}(\mathcal{H},\Gamma)^* \) is a “state” if it is positive and normalized, i.e., if for all \( A \in \text{sCAR}(\mathcal{H},\Gamma) \), \( \omega(A^*A) \geq 0 \) and \( \omega(1) = 1 \). In the sequel, \( \mathcal{E} \subseteq \text{sCAR}(\mathcal{H},\Gamma)^* \) will denote the set of all states on \( \text{sCAR}(\mathcal{H},\Gamma) \). Note that any \( \omega \in \mathcal{E} \) is Hermitian, i.e., for all \( A \in \text{sCAR}(\mathcal{H},\Gamma) \), \( \omega(A^*) = \omega(A) \). \( \omega \in \mathcal{E} \) is said to be “faithful” if \( A = 0 \) whenever \( A \geq 0 \) and \( \omega(A) = 0 \). Since \( \text{sCAR}(\mathcal{H},\Gamma) \) is a unital \( C^* \)-algebra, \( \mathcal{E} \) is a weak*-compact convex set, such that its extremal points coincide with the pure states [BR03a, Theorem 2.3.15]. The latter, combined with the fact that \( \text{sCAR}(\mathcal{H},\Gamma) \) is separable allows to claim that the set of states \( \mathcal{E} \) is metrizable in the weak*-topology [Rud91, Theorem 3.16]. Notice that the existence of extremal points is a consequence of the Krein–Milman Theorem. More specifically, if \( E(\mathcal{E}) \) denotes the set of extremal points of \( \mathcal{E} \),

\[
\mathcal{E} = \text{cch} \left( E(\mathcal{E}) \right),
\]

where, for \( \mathcal{X} \) a Topological Vector Space and \( A \subseteq \mathcal{X} \), \( \text{cch}(A) \) refers to the closed convex hull of \( A \). A state \( \omega \in \mathcal{E} \) is said to be a pure state if \( \omega \in E(\mathcal{E}) \). By definition, a mixed state is a state that is not pure. Notice that a pure state can be characterized as a state which cannot be written as a convex linear combination of two different states. Notice in particular that if \( \omega \in \mathcal{E} \) is a state of the form

\[
\omega = \sum_{j=1}^{m} \lambda_j \omega_j,
\]

where \( \{\omega_j\}_{j=1}^{m} \subseteq E(\mathcal{E}) \), \( m \in \mathbb{N} \), and \( \lambda_j \in [0,1] \) for \( j \in \{1,\ldots,m\} \), with \( \sum_{j=1}^{m} \lambda_j = 1 \). Then, if \( \omega \) is pure, it necessarily follows that \( \omega = \omega_1 = \cdots = \omega_m \). As usual, for the state \( \omega \in \mathcal{E} \) on \( \text{sCAR}(\mathcal{H},\Gamma) \), \( (\mathcal{H}_\omega,\pi_\omega,\Omega_\omega) \) denotes its associated cyclic representation: \( \mathcal{H}_\omega \) is the Hilbert space associated to \( \omega \), and is given by the closure of (the linear span) of the set \( \{\pi_\omega(A)\Omega_\omega : A \in \text{sCAR}(\mathcal{H},\Gamma)\}^5 \),

\[
\mathcal{H}_\omega = \overline{\pi_\omega \left( \text{sCAR}(\mathcal{H},\Gamma) \right) \Omega_\omega},
\]

i.e., \( \mathcal{H}_\omega \) is a Hilbert space with scalar product \( \langle \cdot,\cdot \rangle_{\mathcal{H}_\omega} \), \( \pi_\omega \) a representation from \( \text{sCAR}(\mathcal{H},\Gamma) \) into \( \mathcal{B}(\mathcal{H}_\omega) \), the set of bounded operators acting on \( \mathcal{H}_\omega \), and \( \Omega_\omega \in \mathcal{H}_\omega \) is a unit cyclic vector with respect to \( \pi_\omega(\text{sCAR}(\mathcal{H},\Gamma)) \). More specifically, for all \( A \in \text{sCAR}(\mathcal{H},\Gamma) \) we write

\[
\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle_{\mathcal{H}_\omega}.
\]

(\( \mathcal{H}_\omega,\pi_\omega,\Omega_\omega \)) is the so-called GNS construction, which is unique up to unitary equivalence.

If the state \( \omega \in \mathcal{E} \) is mixed, see Expression (14), its associated representation \( (\mathcal{H}_\omega,\pi_\omega) \) is reducible, that is, it can be decomposed as a direct sum \( \pi_\omega = \bigoplus_{j \in J} \pi_\omega_j \) on \( \mathcal{H}_\omega = \bigoplus_{j \in J} \mathcal{H}_j \). Here, \( \{\mathcal{H}_j\}_{j \in J} \) is a countable family of orthogonal Hilbert spaces, by meaning that for two different Hilbert spaces \( \mathcal{H}_i \) and \( \mathcal{H}_j \) of \( \{\mathcal{H}_j\}_{j \in J} \), \( \langle \varphi_1,\varphi_2 \rangle_{\mathcal{H}_j} = 0 \) for all \( \varphi_1 \in \mathcal{H}_1 \) and all \( \varphi_2 \in \mathcal{H}_2 \). The set \( \{\pi_\omega_j\}_{j \in J} \) are representations of \( \text{sCAR}(\mathcal{H},\Gamma) \) on proper subspaces of \( \mathcal{H}_\omega \). In particular if \( \omega \) is pure, its representation \( (\mathcal{H}_\omega,\pi_\omega) \) is irreducible and \( \omega \) is an extremal point \( E(\mathcal{E}) \) of the set of states on \( \text{sCAR}(\mathcal{H},\Gamma) \).

States \( \omega \in \mathcal{E} \) are said to be quasi–free when, for all \( N \in \mathbb{N}_0 \) and \( \varphi_0,\ldots,\varphi_{2N} \in \mathcal{H} \),

\[
\omega(B(\varphi_0)\cdots B(\varphi_{2N})) = 0,
\]

while, for all \( N \in \mathbb{N} \) and \( \varphi_1,\ldots,\varphi_{2N} \in \mathcal{H} \),

\[
\omega(B(\varphi_1)\cdots B(\varphi_{2N})) = \text{Pf} \left[ \omega \left( \bigotimes_{k,l} (B(\varphi_k),B(\varphi_l)) \right) \right]_{k,l=1}^{2N},
\]

\(^5\)For the Topological Vector Space \( \mathcal{X} \), \( \overline{\mathcal{X}} \) denotes its closure.
The tracial state 

\[
\omega_{\mathcal{H}}(A) = \frac{\text{tr}_\mathcal{N} \left( A \exp \left( \frac{i}{2} \langle B, HB \rangle \right) \right)}{\text{tr}_\mathcal{N} \left( \exp \left( \frac{i}{2} \langle B, HB \rangle \right) \right)}, \quad A \in \text{sCAR}(\mathcal{H}, \Gamma),
\]

The state \( \omega_{\mathcal{H}}^{(\beta)} \in \mathcal{E} \) is named the \((\tau_t, \beta)\)-Gibbs state, thermal equilibrium state, or KMS–state, associated with the self–dual (one–particle) Hamiltonian \( H \) on \( (\mathcal{H}, \Gamma) \) at fixed \( \beta \in (0, \infty) \). As is usual, we call to the parameter \( \beta \in (0, \infty) \) the inverse (non–negative) temperature of a physical system. Note that, given \( H \in \mathcal{B}(\mathcal{H}) \), we also can define two particular quasi–free states \( \omega_{\mathcal{H}}^{(0)} \) and \( \omega_{\mathcal{H}}^{(\infty)} \), which satisfy (21) for the convergent sequence \( \{ \beta_n \}_{n \in \mathbb{N}} \subset \mathbb{R}_0 \cup \{ \infty \} \) to a \( \beta \subset \mathbb{R}_0 \cup \{ \infty \} \). The former case is closely related with the tracial state in Definition 5, and corresponds to the infinite temperature. Namely, the state at \( \beta = \lim_{n \to \infty} \beta_n = 0 \) is known as trace state or chaotic state. This particular name comes from the fact that physically it corresponds to the state of maximal entropy which occurs at infinite temperature. Its uniqueness is a well–known property. On the other hand, states at \( \beta = \lim_{n \to \infty} \beta_n = \infty \) are also thermal equilibrium states. More generally, these are defined by:

\[
\Omega_{k,l}(A_1, A_2) \doteq \begin{cases} 
A_1A_2 & \text{for } k < l, \\
-A_2A_1 & \text{for } k > l, \\
0 & \text{for } k = l.
\end{cases}
\]

In Equation (17), \( \text{Pf} \) is the usual Pfaffian defined by

\[
\text{Pf} \left[ M_{k,l} \right]_{k,l=1}^{2N} \doteq \frac{1}{2^N N!} \sum_{\pi \in S_{2N}} (-1)^\pi \prod_{j=1}^N M_{\pi(2j-1), \pi(2j)}
\]

for any \( 2N \times 2N \) skew–symmetric matrix \( M \in \text{Mat}(2N, \mathbb{C}) \). Note that (17) is equivalent to the definition given either in [Ara71, Definition 3.1] or in [EK98, Equation (6.6.9)].

Quasi–free states are therefore particular states that are uniquely defined by two–point correlation functions, via (16)–(17). In fact, a quasi–free state \( \omega \) is uniquely defined by its so–called symbol, that is, a positive operator \( S_\omega \in \mathcal{B}(\mathcal{H}) \) such that

\[
0 \leq S_\omega \leq 1_{\mathcal{H}} \quad \text{and} \quad S_\omega + \Gamma S_\omega \Gamma = 1_{\mathcal{H}},
\]

through the conditions

\[
\langle \varphi_1, S_\omega \varphi_2 \rangle_{\mathcal{H}} = \omega(B(\varphi_1)B(\varphi_2)^*), \quad \varphi_1, \varphi_2 \in \mathcal{H}.
\]

Conversely, any self–adjoint operator satisfying (19) uniquely defines a quasi–free state through Equation (20). In physics, \( S_\omega \) is related to the one–particle density matrix of the system. Note that any basis projection associated with \( (\mathcal{H}, \Gamma) \) can be seen as a symbol of a quasi–free state on \( \text{sCAR}(\mathcal{H}, \Gamma) \). Such state is pure and called a Fock state [Ara71, Lemma 4.3]. Araki shows in [Ara71, Lemmata 4.5–4.6] that any quasi–free state can be seen as the restriction of a quasi–free state on \( \text{sCAR}(\mathcal{H} \oplus \mathcal{H}, \Gamma \oplus (-\Gamma)) \), the symbol of which is a basis projection associated with \( (\mathcal{H} \oplus \mathcal{H}, \Gamma \oplus (-\Gamma)) \). This procedure is called purification of the quasi–free state.

Quasi–free states obviously depend on the choice of generators of the self–dual CAR algebra. Another example of a quasi–free state is provided by the tracial state:

**Definition 5 (Tracial state).**

The tracial state \( \text{tr}_\mathcal{N} \in \mathcal{E} \) is the quasi–free state with symbol \( S_{tr} \doteq \frac{1}{2} 1_{\mathcal{H}} \).
Definition 6 (Ground state).
Let \( \omega \in \mathcal{E} \) be a state on sCAR(\( \mathcal{H} \), \( \Gamma \)) and let \( H \in \mathcal{B}(\mathcal{H}) \) be a self–dual Hamiltonian on (\( \mathcal{H} \), \( \Gamma \)). We say that \( \omega \equiv \omega_H^{(\infty)} \) is a ground state if it satisfies

\[
i \omega(A^* \delta(A)) \geq 0,
\]
for all \( A \in \mathcal{D}(\delta) \). Here \( \delta \) is the generator with domain \( \mathcal{D}(\delta) \), of the continuous group \( \{ \tau_t \}_{t \in \mathbb{R}} \) of \( * \)-automorphisms of sCAR(\( \mathcal{H} \), \( \Gamma \)) given by (10).

From now on, we will denote by \( \mathcal{E}^{(\beta)} \subseteq \mathcal{E} \) the set of all KMS states at inverse temperature \( \beta \in \mathbb{R}^+ \cup \{ \infty \} \), \( \omega_H^{(\beta)} \equiv \omega^{(\beta)} \). For \( \beta \in \mathbb{R}^+ \cup \{ \infty \} \), \( \omega^{(\beta)} \in \mathcal{E}^{(\beta)} \) is \( \tau \) invariant or stationary, i.e., \( \omega^{(\beta)} \circ \tau = \omega^{(\beta)} \). See [BR03b, Propositions 5.3.3 and 5.3.19]. In contrast, the tracial case \( \beta = 0 \) not necessarily is. Then, for \( \beta \in \mathbb{R}^+ \cup \{ \infty \} \), \( \omega \equiv \omega^{(\beta)} \), there is a strongly continuous one–parameter unitary group \((e^{itL_\omega})_{t \in \mathbb{R}}\) with generator \( L_\omega = L_\omega^{(\beta)} \) satisfying \( e^{itL_\omega} \Omega_\omega = \Omega_\omega \) such that for any \( t \in \mathbb{R} \)

\[
\pi_\omega(\tau_t(A)) = e^{-itL_\omega} \pi_\omega(A) e^{itL_\omega} \quad \text{and} \quad e^{itL_\omega} \in \pi_\omega(\text{sCAR}(\mathcal{H}, \Gamma))''.
\]

If any \( A \in \mathcal{D}(\delta) \subseteq \text{sCAR}(\mathcal{H}, \Gamma) \),

\[
\pi_\omega(A) \Omega_\omega \in \mathcal{D}(L) \quad \text{and} \quad L(\pi_\omega(A) \Omega_\omega) = \pi_\omega(\delta(A)) \Omega_\omega.
\]

If \( \omega \) is a ground state, then the generator satisfies \( L_\omega \geq 0 \).

For \( \beta \in \mathbb{R}^+ \), the set \( \mathcal{E}^{(\beta)} \subseteq \mathcal{E} \) forms a weak\(^*\)–compact convex set that also is a simplex\(^6\), while the set of ground states or KMS states at inverse temperature \( \infty \), \( \mathcal{E}^{(\infty)} \subseteq \mathcal{E} \), forms a face \( \mathcal{F} \), i.e., a subset of a compact convex set \( \mathcal{K} \) such that if there are finite linear combinations

\[
\omega = \sum_{j=1}^{n} \lambda_j \omega_j \quad \text{with} \quad \sum_{j=1}^{n} \lambda_j = 1
\]
of elements \( \{ \omega_j \}_{j=1}^{n} \in \mathcal{K} \) and \( \omega \in \mathcal{F} \), then \( \{ \omega_j \}_{j=1}^{n} \in \mathcal{F} \).

Let \( A \) be a self–dual operator on (\( \mathcal{H} \), \( \Gamma \)), such that \( E_{\Sigma}(A) = \chi_{\Sigma}(A) \) defines the spectral projection of \( A \) on the Borel set \( \Sigma \subseteq \mathbb{R} \). Here, \( \chi_{\Sigma} : \Sigma \to \{ 0, 1 \} \) is the so–called characteristic function on \( \Sigma \subseteq \mathbb{R} \), with \( \chi_\mathbb{R}^+ = \chi_{\Sigma} \). For \( H \), a self–adjoint Hamiltonian on (\( \mathcal{H} \), \( \Gamma \)), i.e., \( H = -\Gamma H \Gamma \), we denote by \( E_0 \), \( E_- \) and \( E_+ \), the restrictions of the spectral projections of \( H \) on \( \{ 0 \} \), the negative real numbers \( \mathbb{R}^- \) and the positive real numbers \( \mathbb{R}^+ \), respectively. Using functional calculus we note that

\[
H = \int_{\text{spec}(H)} \lambda \text{d}E_\lambda = \int_{\mathbb{R}} \lambda \text{d}E_\lambda,
\]

where \( \text{spec}(H) \) denotes the spectrum of \( H \). Thus, one verifies that

\[
(22) \quad \Gamma E_\lambda \Gamma = E_{-\lambda} \quad \text{for all} \quad \lambda \in \mathbb{R} \quad \text{and} \quad E_0 + E_- + E_+ = 1_{\mathcal{H}}.
\]

In particular, we have \( \Gamma E_0 \Gamma = E_0 \). However, we strongly will assume throughout this paper that \( E_0 = 0 \) so that the ground state is unique. For details see [AT85][Theorems 3 and 4]. By (22), both \( E_+ \) and \( E_- \) are basis projections in \( \text{p}(\mathcal{H}, \Gamma) \): \( \Gamma E_\pm \Gamma = 1_{\mathcal{H}} - E_\pm \), i.e., ground states can be uniquely characterized by their spectral projections \( E_\pm \). In particular, the symbol \( S_\omega \) in (20) corresponds to the

\(^6\) This is true because one can show that the set of KMS \( \mathcal{E}^{(\beta)} \subseteq \mathcal{E} \) forms a base of the cone which is also a lattice [BR03a, Chapter 4].
spectral projection $E_+$ on the positive real numbers, associated to the self–dual Hamiltonian $H$ on $(H, \Gamma)$ in such a way that ground states are uniquely determined by the two–point correlation function defined by:

$$\omega_{E_+}(B(\varphi_1)B(\varphi_2)^*) = \langle \varphi_1, E_+ \varphi_2 \rangle_\mathcal{H}, \quad \varphi_1, \varphi_2 \in \mathcal{H}.$$  

Thus, for a quasi–free system associated to some self–dual Hamiltonian $H$, the set of all ground states $\mathcal{E}^{(\infty)}_H \equiv \mathcal{E}^{(\infty)}$, is studied via (positive) spectral projections of $H$. Additionally, straightforward calculations show the uniqueness of ground states, even under small perturbations. See [BR03b, Chapter 5] and [Has19] for recent results on the stability of free fermion systems. More generally, for a unital $C^*$–algebra and $\beta \in \mathbb{R}^+$, the quasi–free state is unique, and this also holds for gapped systems, as defined below, for $\beta \to \infty$. We now define:

**Definition 7 (Quasi–free ground states).**

The state $\omega \in s\text{CAR}(\mathcal{H}, \Gamma)^*$ satisfying (19), (20) and (23) it will be called quasi–free ground state. The set of all quasi–free ground states it will denoted by $q\mathcal{E}^{(\infty)} \subset \mathcal{E}^{(\infty)}$.

### 2.4 Gapped Systems

We consider the (possibly unbounded) self–adjoint operator $h = h^* \in \mathcal{L}(\mathbb{H})$ (the linear operators on $\mathbb{H}$), for some separable Hilbert space $\mathbb{H}$, whose spectrum is denoted by $\text{spec}(h) \subset \mathbb{R} \cup \{\infty\}$. Physically, we say that the system described by $\mathbb{H}$ has a gap if whenever we measure the spectrum of the associated Hamiltonian there exists a strictly positive distance $\gamma \in \mathbb{R}^+$ between the two lowest eigenvalues $\delta_1, \delta_2 \in \mathbb{R}$ such that $\delta_2 - \delta_1 > \gamma$, with $\delta_1 = \inf \text{spec}(h)$. The parameter $\gamma$, also called spectral gap, is known to be the difference between the lowest energy of the system and the energy of its first excited state. In Definition 8 below, we formally express this. On the other hand, in the context of fermion systems, Definition 9 is suitable for our interests. Then, introducing the notation $d(X,Y)$ to denote the distance between the sets $X,Y \subset \mathbb{R}$:

$$d(X,Y) \doteq \inf \{d(x,y) : x \in X, y \in Y\},$$

with $d(x,y) = |x - y|$ for $x, y \in \mathbb{R}$, we define:

**Definition 8 (Gapped Hamiltonians).**

Let $\mathbb{H}$ be a (one–particle) Hilbert space and consider $h \in \mathcal{L}(\mathbb{H})$ the (one–particle) Hamiltonian, that is, a self–adjoint operator $h = h^*$, whose spectrum is denoted by $\text{spec}(h) \subset \mathbb{R}$. We will say that $h$ is a gapped Hamiltonian if there are $\Sigma$ and $\bar{\Sigma}$, nonempty and disjoint subsets of $\text{spec}(h)$, such that $\Sigma \cup \bar{\Sigma} = \text{spec}(h)$ and such that $\gamma \doteq d(\Sigma, \bar{\Sigma}) > 0$.

**Remark 1.** In the latter definition $\Sigma \in \mathbb{R}$ can be a Borel set containing the isolated eigenvalue $\delta_1$, which carries the information of the lowest energy associated to the physical system to consider. Note that if $\text{spec}(h)$ is a purely point spectrum (the set of all the eigenvalues associated to $h$) we can define the family of elements of $\bar{\Sigma}$ with indices on $\mathbb{N} \setminus \{1\}$ as the map $\mathcal{E} : \mathbb{N} \setminus \{1\} \to \bar{\Sigma}$, such that $\mathcal{E} = \{\delta_n\}_{n \in \mathbb{N} \setminus \{1\}}$, the rest of eigenvalues of $\mathbb{H}$, given $\mathcal{E}$, belong to $\bar{\Sigma}$.

Definition 8 is completely general and is usually used to study spectrum related to physical systems. Nevertheless, our primary interest is the fermionic case and then we need to consider an alternative expression. In order to find such an expression we recall Definition 4 of a self–dual operator $H \in \mathcal{B}(\mathcal{H})^\Gamma$, where one considers a self–dual Hilbert space $(\mathcal{H}, \Gamma)$, with $\mathcal{H}$ a finite–dimensional Hilbert space with orthonormal basis given by $\{\psi_i\}_{i \in I}$. Hence, for any $H \in \mathcal{B}(\mathcal{H})$ satisfying $H^* = -\Gamma HH\Gamma$ we have:

---

7Following [NSY18b], we could consider unbounded one–site potentials. Thus, we would need to define Hamiltonians on well–defined dense sets on Hilbert spaces. However, for the sake of simplicity, we will omit any mention on densely defined self–adjoint operators. Note that in [BP16a], Bru and Pedra consider unbounded one–site potentials on $C^*$–algebras. In [AMR] we will deal with unbounded one–site potentials.
(i) \( \text{tr}_{\mathcal{H}}(H) = 0 \).

(ii) \( \text{spec}(\mathcal{A}1_{\mathcal{H}} - H) = \lambda - \text{spec}(H) \) for \( \lambda \in \mathbb{C} \).

Both (i) and (ii) are fundamental to study the underlying systems we are considering. Based on Definition 8, items (i) and (ii), and above comments we can define the following:

**Definition 9 (Fermionic Gapped Hamiltonians).**

Let \((\mathcal{H}, \Gamma)\) be a self–dual Hilbert space and consider \( H \in \mathcal{B}(\mathcal{H}) \) be a self–dual Hamiltonian with spectrum denoted by \( \text{spec}(H) \subset \mathbb{R} \). We will say that \( H \) is a **gapped Hamiltonian** if there exists \( \gamma \in \mathbb{R}^+ \) satisfying the **gap assumption**

\[
\gamma = \inf \{ \varepsilon > 0 : [-\varepsilon, \varepsilon] \cap \text{spec}(H) \neq \emptyset \}.
\]

Observe that for fermionic systems Definitions 8 and 9 are equivalent. In fact, in Definition 9, \( \Sigma \in \mathbb{R} \) is a finite interval with \( a \doteq \inf \{ \Sigma \} \) and \( b \doteq \sup \{ \Sigma \} \), \( \Sigma \) is nothing but \( -\Sigma \), so that \( -a \doteq \sup \{ \Sigma \} \) and \( -b \doteq \inf \{ \Sigma \} \). Then, the self–dual formalism permits to consider a symmetric decomposition of the spectrum. Therefore \( \Sigma \) can be understood as a Borel set on \( \mathbb{R}^+ \) related to the positive part of the energy while \( \tilde{\Sigma} \equiv -\Sigma \) its symmetric negative part: the gap \( \gamma \) centered at zero separates these. We finally stress following Definition 9 that denoting by \( \Sigma_0 \) and \( -\Sigma_0 \) the remaining two open sets, their closures respectively are \( \Sigma \) and \( -\Sigma \).

Due to the above reasons, from now on we will only consider fermion systems. Thus, let us now consider the family of self–dual Hamiltonians \( \{ H_s \}_{s \in \mathcal{C}} \subset \mathcal{B}(\mathcal{H}) \) on \((\mathcal{H}, \Gamma)\), where \( \mathcal{C} \) is the compact set \([0, 1]\). In particular, \( \{ H_s \}_{s \in \mathcal{C}} \) will define a differentiable family of self–adjoint operators on \( \mathcal{B}(\mathcal{H}) \). More specifically, for any \( s \in \mathcal{C} \) we will consider that the map \( s \mapsto H_s \) is strongly differentiable so that \( \partial_s H_s \in \mathcal{B}(\mathcal{H}) \). For example, we are particularly interested in the family of differentiable operators \( H_s \doteq (1 - s)H_0 + sH_1 \), for any \( s \in \mathcal{C} \). Other models we are taking into account, is the Anderson model, as discussed in Appendix A. See [BPH14] and [ABPR19]. Following Definition 9 we now define:

**Definition 10 (Phase of Matter).**

Let \( \mathcal{C} \equiv [0, 1] \) and \( \{ H_s \}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H}) \) be a continuous family of self–dual Hamiltonians on \((\mathcal{H}, \Gamma)\). We will say that \( H_s \) is a \( s \)--gapped Hamiltonian if the **gap assumption** in Definition 9 is satisfied for any \( s \in \mathcal{C} \). Define now an equivalence relation on the set of gapped self-dual Hamiltonians, as follows. If \( H^{(0)} \), \( H^{(1)} \in \mathcal{B}(\mathcal{H}) \) are two gapped self-dual Hamiltonians, then \( H^{(0)} \sim H^{(1)} \) if and only if the following conditions are satisfied:

(i) There is a continuous family of self-dual Hamiltonians on \((\mathcal{H}, \Gamma)\), where \( H_s \) is \( s \)--gapped for all \( s \in \mathcal{C} \);

(ii) there is a lower bound \( \gamma \in \mathbb{R}^+ \), independent of \( s \), in the sense that \( \inf_{s \in \mathcal{C}} \gamma_s \geq \gamma > 0 \); and

(iii) \( H_0 = H^{(0)} \) and \( H_1 = H^{(1)} \).

We will refer to such equivalence classes as **phases of matter**, and will use the notation \( \Omega \) to denote a given phase (with an appropriate label, if needed).

Observe that a difference between ground states associated to family of Hamiltonians \( \{ H_s \}_{s \in \mathcal{C}} \) satisfying above definition and the general definition of ground states (Definition 6) is necessary. In fact, one can prove that if the family of Hamiltonians is gapped, then its associated ground states \( \{ \omega_s \}_{s \in \mathcal{C}} \) satisfy:

\[
(i)\omega_s(A^\dagger\delta(A)) \geq \gamma_s(\omega_s(A^\dagger A) - |\omega_s(A)|^2), \quad \text{for any } s \in \mathcal{C} \text{ and } A \in \mathcal{B}(\delta),
\]

with \( \gamma_s \in \mathbb{R}^+, s \in \mathcal{C} \), and \( \inf_{s \in \mathcal{C}} \gamma_s \geq \gamma > 0 \). For details see [Mat13]. In the sequel we will say that states satisfying the above inequality are **gapped ground states**.
3 Main Results

We study gapped Hamiltonians satisfying the following Assumption:

Assumption 1.
Take $\mathcal{C} \equiv [0,1]$. (a) $\mathbf{H} \doteq \{H_s\}_{s \in \mathcal{C}} \subset \mathcal{B}(\mathcal{H})$ is a differentiable family of self–dual Hamiltonians such that $\partial H \doteq \{\partial_s H_s\}_{s \in \mathcal{C}} \subset \mathcal{B}(\mathcal{H})$. (b) For the infinite volume we assume that the sequences of self–dual Hamiltonians $H_{s,L} : \mathcal{C} \to \mathcal{B}(\mathcal{H}_\infty)$ and $\partial_s H_{s,L} : \mathcal{C} \to \mathcal{B}(\mathcal{H}_\infty)$ convergent in norm and pointwise, that is, $\lim_{L \to \infty} H_{s,L} = H_{s,\infty}$ and $\lim_{L \to \infty} \partial_s H_{s,L} = \partial_s H_{s,\infty}$ in the norm sense.

Now, for any self–dual Hilbert space $(\mathcal{H}, \Gamma)$, take $P_1 \in \mathfrak{p}(\mathcal{H}, \Gamma)$ and $P_2 \in \mathfrak{p}(\mathcal{H}, \Gamma)$ basis projections, the “$\mathbb{Z}_2$–projection index” ($\mathbb{Z}_2$–PI) $\sigma : \mathfrak{p}(\mathcal{H}, \Gamma) \times \mathfrak{p}(\mathcal{H}, \Gamma) \to \mathbb{Z}_2$ is the map defined by:

$$\sigma(P_1, P_2) \doteq (-1)^{\dim(P_1 \wedge P_2^\perp)}.$$  

Here, $\wedge$ symbolizes the lower bound or intersection of the basis projections $P_1$ and $P_2$ in $\mathfrak{p}(\mathcal{H}, \Gamma)$. Note that the $\mathbb{Z}_2$–PI defines a topological group with two components. In particular, $\sigma(P_1, P_2)$ gives an equivalence criterion for their associated quasi–free states $\omega_{P_1}$ and $\omega_{P_2}$ restricted to the even part $\text{sCAR}(\mathcal{H}, \Gamma)^+$ of the self–dual $C^*$–algebra $\text{sCAR}(\mathcal{H}, \Gamma)$. See Expression (7). More generally, we know by the Shale–Stinespring Theorem that two Fock representations $\pi_{P_1}$ and $\pi_{P_2}$ associated to $P_1, P_2 \in \mathfrak{p}(\mathcal{H}, \Gamma)$ are unitarily equivalent if and only if $P_1 - P_2 \in \mathcal{J}_2$, i.e., a Hilbert–Schmidt class operator [GBVF01]. See Appendix B, in special Equation (68) and [Ara87, Theo. 6.14]. Then, we analyze the class of Hamiltonians described by last assumption and their connection with topological indexes. We formally state one of the main results of the paper:

Theorem 1 ($\mathbb{Z}_2$–projection Index):
Take $\mathcal{C} \equiv [0,1]$ and let $\mathbf{H} \doteq \{H_{s,\infty}\}_{s \in \mathcal{C}} \subset \mathcal{B}(\mathcal{H}_\infty)$ be a differentiable family of gapped self–dual Hamiltonians on $(\mathcal{H}_\infty, \Gamma_\infty)$, with $\partial \mathbf{H} \doteq \{\partial_s H_{s,\infty}\}_{s \in \mathcal{C}} \subset \mathcal{B}(\mathcal{H}_\infty)$, see Definition 10 and Assumption 1 (b). For any $s \in \mathcal{C}$, $E_{+,s,\infty}$ denotes the spectral projection associated to the positive part of $\text{spec}(H_{s,\infty})$ and consider the $\mathbb{Z}_2$–PI given by (25). Then:

(1) For any $s \in \mathcal{C}$, $H_{0,\infty}$ is unitarily equivalent to $H_{s,\infty}$ via the unitary operator $V_s^{(\infty)}$ on $(\mathcal{H}_\infty, \Gamma_\infty)$ satisfying the differential equation (32).

(2) The Bogoliubov $*-$automorphism $\chi_{V_s^{(\infty)}}$ is inner and maintains its parity, even $V_s^{(\infty)} \in \mathfrak{U}_+^\infty$ or odd $V_s^{(\infty)} \in \mathfrak{U}_-^\infty$ over the family $\mathbf{H}^8$.

(3) For $r, s \in \mathcal{C}$, the $\mathbb{Z}_2$–PI $\sigma(H_{r,\infty}, H_{s,\infty}) \equiv \sigma(E_{+,r,\infty}, E_{+,s,\infty})$ satisfies $\sigma(H_{r,\infty}, H_{s,\infty}) = 1$.

In regard to this Theorem some remarks are in order:
Consider the pair, $P \in \mathfrak{p}(\mathcal{H}, \Gamma)$ a basis projection and $U \in \mathfrak{u}_\pm$ a Bogoliubov transformation as defined in Expression (6). Then, $\dim(\ker(PUP)) \in \mathbb{N}_0$ and $U \in \mathfrak{u}_+$ or $U \in \mathfrak{u}_-$ if $\dim(\ker(PUP))$ is respectively even or odd [Ara87, Theo. 6.3]. According to Theorem 1 (2), it follows that the number

$$\dim(\ker(E_{+,s,\infty}V_s^{(\infty)}E_{+,s,\infty})) \in \mathbb{N}_0$$

is uniform for the family $\mathbf{H}$. Physically, this is in close relation with the number of the particles of the systems described by the family of Hamiltonians $\mathbf{H}$. In fact, consider the even and odd parts $\mathfrak{U}_\pm^\infty \subset \mathfrak{U}_\infty$ of the self–dual CAR $C^*$–algebra associated to the self–dual Hilbert space $(\mathcal{H}_\infty, \Gamma_\infty)$, see Expressions (7) and (48), and consider $\pi_{E_{+,s,\infty}}$, the fermionic Fock representation associated to $E_{+,s,\infty}$ such that can be decomposed: $\pi_{E_{+,s,\infty}} = \pi_{E_{+,s,\infty}}^\perp \oplus \pi_{E_{+,s,\infty}}^{\parallel}$, where $\pi_{E_{+,s,\infty}}^{\parallel}$ is the restriction

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8 Here $U \in \mathfrak{u}_\infty^\infty$ is a Bogoliubov transformation as defined in Expression (6) associated to the self–dual Hilbert space $(\mathcal{H}_\infty, \Gamma_\infty)$.
of $\omega_{E_{+},s,\infty}$ to $\mathfrak{A}$
+$. Then, the GNS representation associated to the vacuum vector $\Omega_{E_{+},s,\infty}$ given by (70), namely, $\pi_{E_{+},s,\infty}$ is identified with $\pi_{E_{+},s,\infty}$ or $\pi_{E_{+},s,\infty}$ depending if $|J|$ in Expression (70) is even or odd [EK98]. Then, the physical meaning of Theorem 1–(3) can be understood by saying that the $Z_2$–PI is 1 for any two self–dual Hamiltonians $H_r, H_s \in \mathbb{H}$, with $r, s \in \mathcal{C}$ (same parity).

**Proof.** (1) For any $A \in \mathcal{B}(\mathcal{H}_{\infty})$ and all $s \in \mathcal{C}$ define the spectral flow automorphism $\kappa_{s} : \mathcal{B}(\mathcal{H}_{\infty}) \rightarrow \mathcal{B}(\mathcal{H}_{\infty})$ by

$$\kappa_{s}(A) = \left(V_{s}^{(\infty)}\right)^{*} A V_{s}^{(\infty)},$$

where $V_{s}^{(\infty)}$ is a linear operator satisfying $V_{0}^{(\infty)} = \pm 1_{\mathcal{H}_{\infty}}$, and the differential equation (32). See Lemmata 1–2 and Corollary 5. In particular, since any Hamiltonian $H_{s,\infty}$ in $\mathbb{H}$ can be written as

$$H_{s,\infty} = \int_{\mathbb{R}} \lambda dE_{\lambda,s,\infty},$$

with $\Gamma E_{\lambda,s,\infty} = E_{-\lambda,s,\infty}$ for all $\lambda \in \mathbb{R}$ and $E_{-\lambda,s,\infty} + E_{+\lambda,s,\infty} = 1_{\mathcal{H}_{\infty}}$, by Lemmata 1–2, (1) follows. By comments around Expression 6, a Bogoliubov $*$–automorphism $\chi_{U}$ on a self–dual CAR–algebra is even or odd if and only if $\det(U) = 1$ or $\det(U) = -1$, respectively. Then, part (2) follows from Corollary 3 and Lemmata 3–4.

(3) Concerning the $Z_2$–PI $\sigma(P_1, P_2)$ we first invoke [EK98, Theo. 6.30 and Lemma 7.17]: (a) $\sigma(P_1, P_2) = \sigma(P_2, P_1)$, (b) If $P_1 - P_2 \in \mathcal{J}_2$, a Hilbert–Schmidt class operator, then $\sigma(P_1, P_2)$ is continuous in $P_1$ and $P_2$ with respect to the norm topology in $\mathcal{B}(\mathcal{H}, \Gamma)$ (c) If $U \in \mathcal{B}(\mathcal{H})$ is a unitary operator such that $U_{T} = \Gamma U$ and $1_{\mathcal{H}} - U$ is a trace class operator, then $\sigma(P, UPU^{*}) = det(U)$. Then we proceed to verify these statements for the family of positive spectral projections $\{E_{+,s,\infty}\}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H}_{\infty})$. By (22)–(23) and comments around it, any positive spectral projection in $\{E_{+,s,\infty}\}_{s \in \mathcal{C}}$ is a basis projection and thus $\{E_{-,s,\infty}\}_{s \in \mathcal{C}} \subset \mathcal{P}(\mathcal{H}_{\infty}, \Gamma)$. W.l.o.g. take $E_{+,r,\infty}$ and $E_{+,s,\infty}$ with $r, s \in \mathcal{C}$.

We then verify (a)–(c) as follows:

(a) Note that $E_{+,r,\infty} \wedge E_{+,s,\infty} = \Gamma_{\infty} \left(E_{+,r,\infty} \wedge E_{+,s,\infty}\right) \Gamma_{\infty}$.

(b) For $L \in \mathbb{R}_{0}^{+}$, we need to verify that $E_{+,r,L} - E_{+,s,L} \in \mathcal{J}_2$ as $L \rightarrow \infty$. Here, $(\mathcal{H}_{\Omega}, \Gamma_{L})$ is the Hilbert space given by the canonical orthonormal basis $\{e_{x}\}_{x \in X_{L}}$ defined by (44) below. Since $E_{+,r,L}$ and $E_{+,s,L}$ are self–adjoint operators on $\mathcal{B}(\mathcal{H}_{\Omega})$ by Lemma 1, there are unitary operators $V_{r}^{(L)}, V_{s}^{(L)}$ such that $E_{+,r,L} = U_{L}(s, r)E_{+,r,L}U_{L}(r, s)$, with $U_{L}(s, r) = V_{s}^{(L)}(V_{r}^{(L)})^{*}$ satisfying (55) and $U_{L}(0, 0) = 1_{\mathcal{H}_{\Omega}}$. Now, for $s, r \in \mathcal{C}$, note from (54) that

$$U_{L_{2}}(s, r) - U_{L_{1}}(s, r) = V_{s}^{(L_{2})}(V_{r}^{(L_{2})})^{*} - V_{s}^{(L_{1})}(V_{r}^{(L_{1})})^{*} = \left(V_{s}^{(L_{2})} - V_{s}^{(L_{1})}\right) \left(V_{r}^{(L_{2})} - V_{r}^{(L_{1})}\right) = V_{s}^{(L)} - V_{r}^{(L)} = \left(V_{s}^{(L_{1})} - V_{r}^{(L_{2})}\right)^{*} - \left(V_{r}^{(L)} - V_{s}^{(L_{2})}\right)^{*}.$$

By Corollary 5, the sequence of linear operators $V_{s}^{(L)}$ converges in norm to some linear operator $V_{s}$ as $L \rightarrow \infty$. Thus by above expression for fixed $s, r \in \mathcal{C}$, $U_{L}(s, r)$ converges in norm to some $U(s, r)$ as $L \rightarrow \infty$. It follows that $E_{+,r,\infty}$ and $E_{+,s,\infty}$ are unitarily equivalent. Similar to Theorem 5 and using Expression (23), the quasi–free ground states $\omega_{E_{+,r,\infty}}$ and $\omega_{E_{+,r,\infty}}$ are unitarily equivalent via a Bogoliubov $*$–automorphism $\chi_{U(s,r)}$, see (5), such that

$$\omega_{E_{+,r,\infty}} = \omega_{E_{+,r,\infty}} \circ \chi_{U(s,r)}.$$

In particular, by the Shale–Stinespring Theorem, the Fock representations $\pi_{E_{+,r,\infty}}$ and $\pi_{E_{+,s,\infty}}$ are unitarily equivalent and therefore the difference $E_{+,r,\infty} - E_{+,s,\infty}$ is Hilbert–Schmidt class. See [GBVF01, Sec. 6.3], [Ara71, Theo. 6] and [Ara87, Theo. 6.14].
(c) By Lemma 5, for any \( r, s \in \mathbb{C} \) and \( L \in \mathbb{R}^d \cup \{ \infty \} \), the operator \( U_L(s, r) \) already defined commutes with \( \Gamma_L \), is a trace–class per unit volume operator and \( \det (U_L(s, r)) = 1 \). Then, by the continuity of the \( \mathbb{Z}_2 \)-PI index on the norm topology on \( p(\mathcal{H}_\infty, \Gamma) \) note that [AE83, Theo. 3]–[EK98, Lemma 7.17]:

\[
\sigma (E_{+, r, \infty}, E_{+, s, \infty}) = \sigma \left( \left( V_r^{(\infty)} \right)^* E_{+, 0, 0} V_r^{(\infty)}, \left( V_s^{(\infty)} \right)^* E_{+, 0, 0} V_s^{(\infty)} \right) = \sigma (E_{+, 0, 0}, U_{r} (r, s) E_{+, 0, 0} U_{s} (s, r)) = \det (U_{s} (r, s)) = 1.
\]

Hitherto in this paper we have been interested in physical systems with open gap, which is the case of systems of last Theorem. In fact, Theorem 1–(1) claims that two self–dual Hamiltonians that belong to the same phase of matter, say \( H_{0, \infty}, H_{1, \infty} \in \Omega \), and that are connected by a differentiable path, can in fact be connected through the spectral flow automorphism \( \kappa_s \) on \( \mathcal{B}(\mathcal{H}_\infty) \). As is usual, a path is nothing but a continuous map \( \gamma: \mathbb{C} \to \mathcal{B}(\mathcal{H}_\infty) \) connecting the initial point \( \gamma_0 = H_{0, \infty} \) and the terminal point \( \gamma_1 = H_{1, \infty} \). Equivalently, we say that \( H_{0, \infty} \) and \( H_{1, \infty} \) are the extremal points of the path. Observe that for any \( s, r \in \mathbb{C} \) with \( H_{s, \infty}, H_{r, \infty} \in \mathcal{H} \subset \Omega \) we can write

\[
H_{s, \infty} = \kappa_{s, r} (H_{r, \infty}), \quad \text{with} \quad \kappa_{s, r} = \kappa_s^{-1} \circ \kappa_r.
\]

In this case, the index is the same for any pair of Hamiltonians along the path. Notice that, by definition, each phase of matter \( \Omega \) is arcwise connected. But different phases correspond to disjoint sets, so that we are also interested in the possibility of having two self–dual Hamiltonians, \( H_{0, \infty} \) and \( H_{2, \infty} \), acting on \( \mathcal{H}_\infty \) but belonging to different phases of matter, in the sense of Definition 10. In this case, if \( H_{0, \infty} \in \Omega \) while \( H_{2, \infty} \notin \Omega \), Theorem 2 (see Corollary 1 too) below shows that the path \( \tilde{\kappa} \) connecting both Hamiltonians closes the gap, meaning that there is a Hamiltonian \( \tilde{H} \in \mathcal{B}(\mathcal{H}_\infty) \) on \( \tilde{\kappa} \) such that \( 0 \) is an eigenvalue of \( \tilde{H} \). Concerning the latter, observe that one can study the gap closing in terms of the self–dual CAR, \( C^* \)-algebra \( \mathfrak{A}_\infty \cong s\text{CAR}(\mathcal{H}_\infty, \Gamma) \), in such a way that we associate to \( H_{0, \infty} \) and \( H_{2, \infty} \) the bilinear elements \( \langle B, H_{0, \infty} B \rangle \) and \( \langle B, H_{2, \infty} B \rangle \) on \( \mathfrak{A}_\infty \) (see Expression (48) below).

We state the second main result of the current paper:

**Theorem 2:**

Take \( \mathcal{C} \equiv [0, 1] \) and let \( \mathcal{H} \equiv \{ H_{s, \infty} \}_{s \in \mathcal{C}} \subset \mathcal{B}(\mathcal{H}_\infty) \) be a differentiable path of self–dual Hamiltonians on \( (\mathcal{H}_\infty, \Gamma) \), with \( \partial \mathcal{H} \equiv \partial_0 H_{s, \infty} \) \( s \in \mathcal{C} \subset \mathcal{B}(\mathcal{H}_\infty) \). Suppose that \( H_{0, \infty} \) and \( H_{1, \infty} \) are gapped, and their corresponding basis projections satisfy \( \sigma (E_{+, 0, \infty}, E_{+, 1, \infty}) = -1 \). Then there exists \( \tilde{s} \in \mathbb{C} \) such that \( 0 \in \text{spec}(H_{\tilde{s}, \infty}) \), i.e., such that \( E_{0, \tilde{s}, \infty} \neq 0 \). Furthermore, the path \( \mathcal{H} \) induces, in a natural way, a family of ground states for \( H_{\tilde{s}, \infty} \) which are generically mixed states. More specifically, we have:

1. The gap closing at \( \tilde{s} \) induces a splitting of the one–particle Hilbert space as the direct sum of two \( \Gamma_\infty \)-invariant subspaces, \( \mathcal{H}_\infty = \mathcal{H}_0 \oplus \mathcal{H}_1 \), as well as a factorization \( \mathfrak{A}_\infty \cong \mathfrak{A}_0 \otimes \mathfrak{A}_1 \), where \( \mathfrak{A}_0 \cong s\text{CAR}(\mathcal{H}_0, \Gamma) \) and \( \mathfrak{A}_1 \cong s\text{CAR}(\mathcal{H}_1, \Gamma) \).

2. The set of ground states associated with the self–dual Hamiltonian \( H_{\tilde{s}, \infty} \) have the form

\[
\tilde{\omega}(a_1 a_0) = \omega_{\tilde{P}_+} (a_1) \phi_0 (a_0),
\]

where \( a_1 \in \mathfrak{A}_1 \), \( \omega_{\tilde{P}_+} \in \mathfrak{E}(\infty) \) is a quasi–free ground state on \( \mathfrak{A}_1 \) associated with a basis projection \( \tilde{P}_+ \in \mathfrak{p}(\mathcal{H}_1, \Gamma) \), and \( a_0 \in \mathfrak{A}_0 \) with \( \phi_0 \in \mathfrak{E}(\infty) \) an arbitrary state on \( \mathfrak{A}_0 \). In particular, the right and left limits \( \lim_{s \to \tilde{s}^\pm} E_{+, s, \infty} \) single out states \( \omega^+_0 \) and \( \omega^-_0 \) on \( \mathfrak{A}_0 \), with \( \omega^+_0 \neq \omega^-_0 \). In this sense, the path \( \mathcal{H} \) induces, in a natural way, the following family of ground states for \( H_{\tilde{s}, \infty} \):

\[
\tilde{\omega}_\lambda(a_1 a_0) = \omega_{\tilde{P}_+} (a_1) (\lambda \omega^+_0 (a_0) + (1 - \lambda) \omega^-_0 (a_0)), \quad \lambda \in [0, 1].
\]

\[\text{End}\]
Proof. In view of Theorem 1, and taking into account the hypothesis \( \sigma(E_{+,0,\infty}, E_{+,1,\infty}) = -1 \), it is clear that the gap assumption cannot hold for all \( s \in C \). Therefore, the set \( N = \{ s \in C : E_{0,s,\infty} \neq 0 \} \) is nonempty. For the sake of simplicity, we assume that \( |N| = 1 \), i.e., \( N \) contains one element, say \( s \). Define now the upper/lower limit (in norm)

\[
E_{s \pm} = \lim_{s \rightarrow s \pm} E_{s,\infty}.
\]

\( E_{s +} \) and \( E_{s -} \) are basis projections, but they are not equal and do not correspond to the positive spectral projection \( E_{+,s,\infty} \). Let \( \omega_{\pm} \in qE^{(\infty)} \) denote the quasi–free state with symbol \( E_{s \pm} \) (cf. Equation (20)). By [Ara68, Lemma 3.6], there exists a Bogoliubov transformation \( U_{s +, s -} \in \mathcal{B}(\mathcal{H}_{\infty}) \) such that \( E_{s +} = U_{s +, s -}^{*}E_{s}U_{s +, s -} \). See Expression (5) and comments around it. Since the gap condition is satisfied for all \( s \in C \setminus \{ s \} \), it is clear that \( \det U_{s +, s -} = -1 \) (in particular, \( 1_{\mathcal{H}_{\infty}} - U_{s +, s -} \) is trace–class) and so \( \sigma(E_{s +}, E_{s -}) = -1 \), by [EK98, Lemma 7.17]. This implies, by [EK98, Theorem 6.30] that the restrictions of \( \omega_{+} \) and \( \omega_{-} \) to the even algebra \( \mathfrak{A}_{\infty} \) lead to inequivalent GNS representations and, therefore, we must have \( \omega_{+} \neq \omega_{-} \). We will use \( \omega_{\pm} \) to construct a family \( \{ \tilde{\omega}_{\lambda} \}_{\lambda \in [0,1]} \) of ground states for \( H_{s} \in H \). We proceed in steps, as follows:

1. Define the following \( \Gamma_{\infty} \)-invariant subspaces of \( \mathcal{H}_{\infty} \):

   \[
   \mathcal{K}_{0} \doteq ((E_{s +} \cap (1_{\mathcal{H}_{\infty}} - E_{s})), \mathcal{H}_{\infty}) \oplus ((E_{s -} \cap (1_{\mathcal{H}_{\infty}} - E_{s})), \mathcal{H}_{\infty})
   \]

   and

   \[
   \mathcal{K}_{1} \doteq ((E_{s +} \cap (1_{\mathcal{H}_{\infty}} - E_{s})), \mathcal{H}_{\infty}) \oplus ((E_{s -} \cap (1_{\mathcal{H}_{\infty}} - E_{s})), \mathcal{H}_{\infty}).
   \]

2. Define the self–dual CAR \( C^{*} \)-algebras \( \mathfrak{A}_{0} \doteq sCAR(\mathcal{K}_{0}, \Gamma_{\infty}) \) and \( \mathfrak{A}_{1} \doteq sCAR(\mathcal{K}_{1}, \Gamma_{\infty}) \). Notice that the restrictions of \( \omega_{+} \) and \( \omega_{-} \) to \( \mathfrak{A}_{1} \) do coincide and correspond to the quasi–free state defined by the basis projection \( P_{+} \doteq E_{s +} \wedge E_{s -} \). We denote this state as \( \omega_{P_{+}} \). On the other hand, the restrictions of \( \omega_{\pm} \) to \( \mathfrak{A}_{0} \) are different (this follows from \( \det U_{s +, s -} = -1 \)). We therefore define \( \omega_{P_{0}} \doteq \omega_{\pm}|_{\mathfrak{A}_{0}} \). Notice that \( P_{+} \) is a basis projection only if regarded as a projection on \( \mathcal{K}_{1} \), for if we define \( P_{-} \doteq (1_{\mathcal{H}_{\infty}} - E_{s +}) \wedge (1_{\mathcal{H}_{\infty}} - E_{s -}) = E_{s -} \wedge (1_{\mathcal{H}_{\infty}} - E_{s +}) + E_{s +} \wedge (1_{\mathcal{H}_{\infty}} - E_{s -}) \) and \( P_{0} \doteq E_{s +} \wedge (1_{\mathcal{H}_{\infty}} - E_{s -}) \), we obtain \( P_{+} + P_{-} + P_{0} = 1_{\mathcal{H}_{\infty}} \).

3. By [BR03b, Prop. 5.3.19], if \( \varphi \in \mathcal{E}(\infty) \) is a ground state and \( f \in C^{\infty} \) has Fourier transform \( \hat{f} \) satisfying \( \text{supp} \hat{f} \subset \mathbb{R}_{-} \), we have

   \[
   \varphi(\tau_{f}(A)^{\ast}\tau_{f}(A)) = 0, \quad \text{for} \quad A \in \mathfrak{A}_{\infty},
   \]

   with \( \tau_{f}(B(\varphi)) = B(\hat{f}(H)\varphi), \varphi \in \mathcal{H}_{\infty}, \) where \( H \in \mathcal{B}(\mathcal{H}_{\infty}) \) is a self–dual Hamiltonian on \( (\mathcal{H}_{\infty}, \Gamma_{\infty}) \) associated with \( \varphi \).

4. Take \( P_{0} \) and \( P_{\pm} \) as defined in step 2. We are now in the situation of [EK98, Proposition 6.37] (see also [AT85]). Following these references, one can show that all ground states for \( H_{s} \) are of the form

   \[
   \tilde{\omega}(a_{1}a_{0}) = \omega_{P_{+}}(a_{1})\phi_{0}(a_{0}),
   \]

   where \( \omega_{P_{+}} \) is the quasi–free state defined in step 2 and \( \phi_{0} \) an arbitrary state on \( \mathfrak{A}_{0} \). We will provide some details for completeness, and will show how the path \( H_{s} \doteq \{ H_{s,\infty} \}_{s \in C} \) induces a particular family of (mixed) ground states for \( H_{s} \).

(a) We first suppose that \( \tilde{\varphi} \in \mathcal{E}(\infty) \) is a ground state for \( H_{s} \). It then follows from step 3, Expression (27), that \( \varphi(B(\varphi)^{\ast}B(\varphi)) = 0, \) whenever \( \varphi \in \tilde{P}_{\pm}\mathcal{H}_{\infty} \). Thus, by the Schwarz inequality one notes that for any \( C \in \mathfrak{A}_{\infty} \) and \( \varphi \in \tilde{P}_{\pm}\mathcal{H}_{\infty} \)

\[
|\tilde{\varphi}(C^{\ast}B(\varphi))|^{2} \leq \tilde{\varphi}(C^{\ast}C)\tilde{\varphi}(B(\varphi)^{\ast}B(\varphi)) = 0.
\]
Consequently, for any \( C \in \mathcal{A}_\infty \) and \( \varphi \in \mathcal{P}_- \mathcal{H}_\infty \),

\[
\tilde{\varphi}(C^*B(\varphi)) = \tilde{\varphi}(B(\varphi)^*C) = \tilde{\varphi}(CB(\varphi)) = \tilde{\varphi}(B(\varphi)^*C) = 0.
\]

For \( \varphi \in \mathcal{H}_\infty \) we write \( \varphi = \varphi_+ + \varphi_- \), with \( \varphi_\pm \in \mathcal{P}_\pm \mathcal{H}_\infty \). Notice that \( \Gamma_\infty \varphi_+ : \varphi_- \in \mathcal{P}_- \mathcal{H}_\infty \) and \( B(\varphi) = B(\Gamma_\infty \varphi_+)^* + B(\varphi_-) \in \mathcal{A}_1 \). Hence, in order to verify (28), by linearity it suffices to consider for \( m, n \in \mathbb{N}_0 \), with \( m + n \in \mathbb{N} \),

\[
a_1 = B(\varphi_1)^* \cdots B(\varphi_m)^* B(\tilde{\varphi}_1) \cdots B(\tilde{\varphi}_n) \in \mathcal{A}_1
\]

for any \( \varphi_i, \tilde{\varphi}_j \in \mathcal{P}_- \mathcal{H}_\infty \), with \( i \in \{1, \ldots , m\} \) and \( j \in \{1, \ldots , n\} \). It then follows from (29) above, on using the CAR relations, that \( \tilde{\varphi}(a_1) = \omega_{\tilde{\varphi}_+}(a_1) \) (cf. [BR03b, Section 5.2]). Equality in (28) follows upon defining \( \phi_0 \equiv \tilde{\varphi}|_{\mathfrak{A}_0} \).

(b) From step 3 and the definitions given in step 2, it follows in particular that \( \omega_\pm(a_1 a_0) = \omega_{\tilde{\varphi}_+}(a_1) \omega_0^+(a_0) \). We can explicitly show that \( \omega_\pm \) and \( \omega_- \) are ground states for \( H_\xi \), by considering the GNS representations of the states \( \omega_{\tilde{\varphi}_+} \), on one hand, and of \( \omega_0^+ \), on the other, the crucial points being the facts that \( H_\xi \) is a positive operator when restricted to \( \tilde{\mathcal{P}}_+ \mathcal{H}_\infty \) and that we have an isomorphism \( \mathfrak{A}_\infty \cong \mathfrak{A}_0 \otimes \mathfrak{A}_1 \). The computations are the same as in the proof of Theorem 3 in [AT85] and we omit the details. Finally, since we already know that \( \omega_+ \neq \omega_- \), for \( \lambda \in [0, 1] \), we can consider a convex linear combination \( \lambda \omega_0^+ + (1 - \lambda) \omega_0^- \) and define \( \tilde{\varphi}_\lambda \) as in Equation (26).

\[\text{End}\]

Remark 2. The gap closing discussed in the theorem above entails the breakdown of the bijective correspondence between the ground state of each Hamiltonian \( H_{\xi, \infty} \) along the path \( \mathcal{H} \equiv \{H_{\xi, \infty}\}_{\xi \in \mathcal{C}} \) and its corresponding spectral projection \( E_{+, \xi, \infty} \). Indeed, if the gap closes at \( \xi \), then the fact that \( E_{0, \xi, \infty} \neq 0 \) implies that there are many different ground states associated to the Hamiltonian \( H_{\xi, \infty} \). Let \( \omega \in \mathcal{E}(\infty) \) be the set of ground states associated with the path \( \mathcal{H} \subset \mathfrak{B} (\mathcal{H}_\infty) \). In particular, let \( \omega_0 \in \mathcal{Q}(\mathcal{E}(\infty)) \) and \( \omega_1 \in \mathcal{Q}(\mathcal{E}(\infty)) \) be the quasi–free ground states associated with the gapped Hamiltonians \( H_{0, \infty} \) and \( H_{1, \infty} \), respectively, such that their corresponding basis projections satisfy \( \sigma(H_{0, \infty}, H_{1, \infty}, -1) = -1 \). Assuming that the gap closing occurs only at \( s = \xi \), one might ask whether it is possible to appropriately choose one of the ground states associated to \( H_{\xi, \infty} \) in order to define a continuous path \( \gamma : \mathcal{C} \rightarrow \mathcal{E}(\infty) \) such that, for each \( s \in \mathcal{C} \), \( \gamma(s) \) is a ground state for \( H_{\xi, \infty} \). As shown below, a straightforward consequence of Theorem 2 is that the answer to this question turns out to be negative.

\[\text{Corollary 1.}\]

Take same assumptions of Theorem 2 and Remark 2. Then, the path \( \gamma : \mathcal{C} \rightarrow \mathcal{E}(\infty) \) from Remark 2 is not continuous.

\[\text{Proof.}\] Under the stated assumptions, it is clear that, for \( s \in \mathcal{C} \setminus \{\tilde{s}\} \), \( \gamma(s) = \omega_{E_{+, s, \infty}} \), the quasi–free ground state associated to the (basis) projection \( E_{+, s, \infty} \). We will show that, irrespective of how \( \gamma(\tilde{s}) \) is chosen, \( \gamma \) has a discontinuity at that point. This is done as follows: First, recall that since \( \mathfrak{A}_\infty \) is separable, the set \( \mathcal{E}(\infty) \subset \mathfrak{A}_\infty^* \) is metrizable in the weak*–topology. In fact, if \( \{A_n\}_{n \in \mathbb{N}} \) is a countable set of operators on \( \mathfrak{A}_\infty \), separating points on \( \mathfrak{A}_\infty^* \), so that \( \|A_n\|_{\mathfrak{A}_\infty} \leq 1 \) for all \( n \in \mathbb{N} \), the metric

\[
\mathcal{d}_{\mathcal{E}(\infty)}(\omega_1, \omega_2) \equiv \sum_{n \in \mathbb{N}} \frac{1}{2n} |\omega_1(A_n) - \omega_2(A_n)|, \quad \omega_1, \omega_2 \in \mathcal{E}(\infty),
\]

induces the weak*–topology on the set of states \( \mathcal{E}(\infty) \). For the sake of simplicity, and without confusion, the metrizable space \( (\mathcal{E}(\infty), \mathcal{d}_{\mathcal{E}(\infty)}) \) is denoted by \( \mathcal{E}(\infty) \). In particular, the open ball with center on \( \omega \in \mathcal{E}(\infty) \) and radius \( \varepsilon \in \mathbb{R}_+^* \) is defined by

\[\mathcal{B}(\omega, \varepsilon) \equiv \{\omega' \in \mathcal{E}(\infty) : \mathcal{d}_{\mathcal{E}(\infty)}(\omega, \omega') < \varepsilon\} \subset \mathcal{E}(\infty).\]
Now, in the proof of Theorem 2 it was shown that \( \omega_+ \neq \omega_- \), where \( \omega_+ = \lim_{s \to \xi^+} \gamma(s) \). Hence, since \( \{ A_n \}_{n \in \mathbb{N}} \) separates points on \( \mathfrak{A}_{\infty}^* \), there exists \( m \in \mathbb{N} \) such that \( \omega_+(A_m) \neq \omega_-(A_m) \), with \( A_m \in \{ A_n \}_{n \in \mathbb{N}} \). Thus, define

\[
\omega \doteq \left| \omega_+(A_m) - \omega_-(A_m) \right| \in \mathbb{R}^+,
\]

and observe that by Expression (30) one has

\[
\delta_{g_{\xi}}(\omega_+, \omega_-) = \sum_{n \in \mathbb{N}} \frac{1}{2^m} \left| \omega_+(A_n) - \omega_-(A_n) \right| \geq \frac{1}{2^m} \omega.
\]

Hence, one can take \( \varepsilon = \frac{1}{2m+1} \omega \) so that \( \mathfrak{B}_+ (\omega_-, \varepsilon) \cap \mathfrak{B}_+ (\omega_+, \varepsilon) = \emptyset \). Furthermore, given that \( \omega_- \) is the left limit of \( \gamma(s) \) at \( \tilde{s} \) and \( \omega_+ \) is its right limit, observe that it is enough to consider \( \varepsilon' < \varepsilon \), such that \( \omega_+ \notin \mathfrak{B}_+ (\omega_-, \varepsilon') \) and \( \mathfrak{B}_+ (\omega_-, \varepsilon') \cap \{ \gamma(s) : s > \tilde{s} \} = \emptyset \). Therefore the path \( \gamma : \mathcal{C} \to \mathfrak{E}^{(\infty)} \) is not continuous at \( s = \tilde{s} \). More precisely, at that point, \( \gamma \) has a discontinuity of the first kind. See [Cho, Def. 13.1].

4 Technical Proofs

4.1 Existence of the spectral flow automorphism

**Lemma 1.**

Take \( \mathcal{C} = [0, 1] \) and let \( \mathcal{H} \) be a family of Hamiltonians as in Assumption 1. For any \( s \in \mathcal{C} \), \( E_{+,s} \) will denote the spectral projection associated to the positive part of \( \text{spec}(H_s) \). Then, for the family of spectral projections \( \{ E_{+,s} \}_{s \in \mathcal{C}} \), there exists a family of automorphisms \( \{ \kappa_s \}_{s \in \mathcal{C}} \) on \( \mathcal{B}(\mathcal{H}) \) satisfying

\[
\kappa_s (E_{+,s}) = E_{+,0}.
\]

**Proof.** The arguments of the proof are completely standard and we state these for the sake of completeness, c.f. [Kat13, BMNS12, NSY18b]. Take \( \mathcal{C} = [0, 1] \) and consider \( \mathfrak{H} = \{ H_s \}_{s \in \mathcal{C}} \in \mathfrak{B}(\mathcal{H}) \) be a differentiable family of self–dual Hamiltonians. Fix \( s \in \mathcal{C} \) and let \( E_{+,s} \) be the spectral projection of \( H_s \) on \( \Sigma_s \). Note that if the automorphism \( \kappa_s : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H}) \) satisfying \( \kappa_s (E_{+,s}) = E_{+,0} \) exists, this implies that it is unitarily implemented by a differentiable unitary operator \( V_s \) defined by

\[
\kappa_s (E_{+,s}) = V_s^* E_{+,s} V_s, \quad \text{with} \quad V_0 = \pm 1_{\mathcal{H}},
\]

and satisfying the differential equation

\[
\partial_s V_s = -i \mathfrak{D}_{g,s} V_s,
\]

where for the gap g, as in Definition 10, \( \mathfrak{D}_{g,s} : \mathcal{C} \to \mathfrak{B}(\mathcal{H}) \) is a pointwise self–adjoint bounded operator. Here, \( \partial_s \) denotes the derivative with respect to \( s \in \mathcal{C} \). Now, for any \( H_s \), we write its spectral projection on \( \Sigma_s \) by

\[
E_{+,s} = \frac{1}{2\pi i} \oint_{\Gamma_s} R_\zeta(H_s) d\zeta,
\]

where, for any \( s \in \mathcal{C} \), \( R_\zeta(H_s) \in \mathfrak{B}(\mathcal{H}) \) is the resolvent set of \( H_s \). In (33), for any \( s \in \mathcal{C} \), \( \Gamma_s \) is a chain, that is, \( \Gamma_s \) is a finite collection of closed rectifiable curves \( \gamma_s \) in \( \mathbb{C} \). In particular, \( \Gamma_s \) surrounds \( \Sigma_s \) and is in the complement of \( \tilde{\Sigma}_s \). By using the second resolvent equation, i.e.,

\[
R_\zeta(A) - R_\zeta(B) = R_\zeta(A)(B - A) R_\zeta(B),
\]

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for any operators $A, B \in \mathcal{B}(\mathcal{H})$ and any $\zeta \notin \text{spec}(A) \cap \text{spec}(B)$, one can show that

$$\partial_s E_{+,s} = -\frac{1}{2\pi i} \oint_{\Gamma_s} R_\zeta(H_s) (\partial_s H_s) R_\zeta(H_s) \, d\zeta,$$

and it follows that the derivative $\partial_s E_{+,s}$ is well–defined on $\mathcal{C}$. A combination of (31)–(32) and $\kappa_{s}(E_{+,s}) = E_{+,0}$ yield us to

$$\partial_s E_{+,s} = -i[\mathcal{D}_{g,s}, E_{+,s}].$$

Additionally, since for any $s \in \mathcal{C}$, $E_{+,s}$ is an orthogonal projection then

$$E_{+,s}^\perp (\partial_s E_{+,s}) E_{+,s}^\perp = E_{+,s} (\partial_s E_{+,s}) E_{+,s} = 0,$$

where for any $s \in \mathcal{C}$, $E_{+,s}^\perp$ denotes the orthogonal complement of $E_{+,s}$, i.e., $E_{+,s}^\perp \doteq 1 - E_{+,s}$. From the latter identity we get the following one

$$\partial_s E_{+,s} = E_{+,s} (\partial_s E_{+,s}) E_{+,s}^\perp + E_{+,s}^\perp (\partial_s E_{+,s}) E_{+,s},$$

and together with (35) and the fact that $E_{+,s}$ and $E_{-,s}$ are basis projections, see (22), we arrive at

$$\partial_s E_{+,s} = -\frac{1}{\pi} \text{Re} \left( \oint_{\Gamma_s} (E_{+,s} R_\zeta(H_s) (\partial_s H_s) R_\zeta(H_s) E_{-,s}) \, d\zeta \right).$$

Here, the self–adjoint operator $\text{Re}(A) \in \mathcal{B}(\mathcal{H})$ is the real part of $A \in \mathcal{B}(\mathcal{H})$, given by $\text{Re}(A) \doteq \frac{1}{2} (A + A^*)$. Similarly, $\text{Im}(A) \in \mathcal{B}(\mathcal{H})$, the imaginary part of $A$, is the self–adjoint operator usually defined by $\text{Im}(A) \doteq \frac{1}{2i} (A - A^*)$.

Then, the existence of the automorphism $\kappa_{s}$ is equivalent to finding the operator $\mathcal{D}_{g,s}$ such that (36) and (37) are satisfied. This is precisely that is done in [BMNS12], and in the present context we explicitly write $\partial_s E_{+,s}$ as

$$\partial_s E_{+,s} = \int_{\Sigma_s} \int_{\Sigma_s} \frac{2}{\mu + \lambda} \text{Re} \left( dE_{\mu,+,s} (\partial_s H_s) \, dE_{-\lambda,+,s} \right),$$

where for any $s \in \mathcal{C}$, $E_{\mu,+,s}$ is a resolution of the identity supported on the positive (negative) part of $\text{spec}(H_s)$, i.e.,

$$E_{+,s} \doteq \int_{\pm \Sigma_s} dE_{\lambda,+,s}.$$

The next step is to verify that the self–adjoint bounded operator

$$\mathcal{D}_{g,s} \doteq \int_{\mathbb{R}} e^{iH_s} (\partial_s H_s) e^{-iH_s} \mathfrak{M}_{g}(t) \, dt,$$

satisfies (36) and (37). Here, $\mathfrak{M}_{g}(t) : \mathbb{R} \to \mathbb{R}$ is an odd function on $L^1(\mathbb{R})$ such that its Fourier transform, $\hat{\mathfrak{M}}_{g} : \mathbb{R} \to \mathbb{R}$ is given for $\mu \neq 0$ by

$$\hat{\mathfrak{M}}_{g}(\mu) \equiv -\frac{1}{\sqrt{2\pi \mu}}.$$

For a complete description of the properties of $\mathfrak{M}_{g}$ see [BMNS12, MZ13, NSY18b]. We now note that for any operator $B \in \mathcal{B}(\mathcal{H})$ and any orthogonal projection $P \in \mathcal{B}(\mathcal{H})$ we get

$$-i[B, P] = i \left( PB(1 - P) - (1 - P)BP \right).$$
In particular, note that by taking $B$ as $\mathcal{D}_{g,s}$ and $P$ as the spectral projection of $H_s$ on $\Sigma_s$, i.e., $E_{+,s}$, we have
\begin{align*}
-i[\mathcal{D}_{g,s}, E_{+,s}] &= i \int_{\mathbb{R}} \int_{\Sigma_s} \int_{\Sigma_s} e^{it(\mu - \lambda)} dE_{\mu+,s} (\partial_s H_s) dE_{\lambda-,s} \hat{\mathcal{W}}_q(t) d\lambda d\mu dt \\
&- i \int_{\mathbb{R}} \int_{\Sigma_s} \int_{\Sigma_s} e^{it(\lambda - \mu)} dE_{\lambda-,s} (\partial_s H_s) dE_{\mu+,s} \hat{\mathcal{W}}_q(t) d\mu d\lambda dt \\
&= i\sqrt{2\pi} \int_{\Sigma_s} \int_{\Sigma_s} dE_{\mu+,s} (\partial_s H_s) dE_{\lambda-,s} \hat{\mathcal{W}}_q(\lambda - \mu) d\lambda d\mu \\
&- i\sqrt{2\pi} \int_{\Sigma_s} \int_{\Sigma_s} dE_{\lambda-,s} (\partial_s H_s) dE_{\mu+,s} \hat{\mathcal{W}}_q(\mu - \lambda) d\mu d\lambda \\
&= \int_{\Sigma_s} \int_{\Sigma_s} \frac{2}{\mu + \lambda} \Re (dE_{\mu+,s} (\partial_s H_s) dE_{\lambda-,s})
\end{align*}
where we have used (38) and that $\hat{\mathcal{W}}_q$ is an odd function.

In particular, the unitary operator $V_s$ satisfying the differential equation (32) commutes with the involution $\Gamma$, i.e., $\Gamma V_s = V_s \Gamma$. In fact, for any $s \in \mathcal{C}$, let $C_s \in \mathcal{B}(\mathcal{H})$ be defined by $C_s = [\Gamma, V_s]$, such that $C_s^* = [V_s^*, \Gamma]$. We would like to show that $C_s = 0$. To do this, observe that the self–adjoint bounded operator $\mathcal{D}_{g,s}$ given by Expression (39), commutes with $\Gamma$. Using (32), after some calculations we have
\[ \partial_s C_s = -i \mathcal{D}_{g,s} C_s \quad \text{and} \quad \partial_s C_s^* = -i C_s^* \mathcal{D}_{g,s}. \]
From the left hand side equation one has $\partial_s C_s^* = i C_s^* \mathcal{D}_{g,s}$, which comparing with the right hand side equation, we obtain $C_s = 0$. We have proven:

**Corollary 2 (Bogoliubov Transformation).**

For any $s \in \mathcal{C} = [0, 1]$, the unitary operator $V_s$ satisfying the differential equation (32) commutes with the involution $\Gamma$, i.e., $\Gamma V_s = V_s \Gamma$, then $V_s$ is a Bogoliubov transformation, see (5).

One primary consequence of Lemma 1 and Corollary 2 is the existence of a strongly continuous family of one–parameter (Bogoliubov) group $\Upsilon_s = \{ \Upsilon_s \} \in \mathcal{F}$ of $*$–automorphisms of $s\mathcal{C}(\mathcal{H}, \Gamma)$, implemented by the Bogoliubov automorphisms $V_s$. To be precise, for the one–parameter unitary group $\{ V_s \} \in \mathcal{F}$ implementing the family of automorphism $\{ \kappa_s \} \in \mathcal{F}$ of Lemma 1 over the family of spectral projections $\{ E_{+,s} \} \in \mathcal{F}$ we are able to show that, for any $s$, the (Bogoliubov) $*$–automorphism
\[ \Upsilon_s (B(\varphi)) = \chi_{V_s^*} (B(\varphi)) = B(V_s^* \varphi), \]
exists, for any $\varphi \in \mathcal{H}$. The latter can be easily verified using bilinear elements, which are described in Definition 3. More generally, for any family of self–dual Hamiltonians $\{ H_s \} \in \mathcal{F}$ as in Assumption 1, we have an associated family of bilinear elements $\{ \langle B, H_s \rangle \} \in \mathcal{F}$ given by,
\[ \Upsilon_s (\langle B, H_s \rangle) = \langle B, H_0 \rangle, \quad \text{for any} \quad s \in \mathcal{C}, \]
(see Definition 3).

As stressed in comments around Expression (25), for any pair $P_1, P_2 \in p(\mathcal{H}, \Gamma)$ there exists a Bogoliubov transformation $U$ relating both, i.e., a unitary operator $U \in \mathcal{B}(\mathcal{H})$ so that $P_2 = U^* P_1 U$, with $U \Gamma U = \Gamma U$. Thus, if $\det(U) = 1$ (det($U$) = -1) we say that $U$ is in the positive (negative) connected component. Following [EK98, Theo. 6.30 and Lemma 7.17] for the special class of $U$ satisfying (i) $U \Gamma U = \Gamma U$ and (ii) $1_{\mathcal{H}} = U$ trace class, the topological index $\sigma(P, U^* PU)$ coincides with $\det(U)$. Note that Lemma 1 tells us about the existence of a family of unitary operators $\{ V_s \} \in \mathcal{F}$ which implements the family of automorphisms $\{ \kappa_s \} \in \mathcal{F}$ on $\mathcal{B}(\mathcal{H})$. However, we need to specify with which kind of Hamiltonians we are dealing. A wide class of fermion systems are those satisfying
Lemma 2 and Proposition 2 below. More concretely, our results will permit to consider disordered fermions systems in which the spectral gap does not close. Note that a suitable control of the properties of \( \{ V_s \}_{s \in \mathcal{C}} \) is closely related to the recently results found by Hastings in [Has19]. Then, as already mentioned we invoke [EK98, Theo. 6.30 and Lemma 7.17] in order to distinguish different physical systems (see Definition 10), which are classified by two components even in the interacting setting, see [NSY18a]. In particular, we have:

**Corollary 3.**
Consider a family of self–dual Hamiltonians \( \mathbf{H} \in \mathcal{B}(\mathcal{H}) \) satisfying Assumption 1. For any \( s \in \mathcal{C} \equiv [0, 1] \), take the Bogoliubov transformation \( V_s \) of Corollary 2. Assume that for any \( s \in \mathcal{C} \), \( \mathbf{1}_\mathcal{H} - V_s \) and \( \mathfrak{D}_{g,s} \in \mathcal{B}(\mathcal{H}) \) are trace class, with \( \mathfrak{D}_{g,s} \) given by (32). Then \( \det (V_s) = \det (V_0) \).

**Proof.** Let \( V_0 = \pm 1_\mathcal{H} \), write \( \det (V_s) - \det (V_0) = \int_0^s \partial_r (\det (V_r)) \, dr \), and apply the Jacobi’s formula of determinants for \( V_r : \partial_r (\det (V_r)) = \det (V_r) \text{tr}_\mathcal{H} (V_r^* (\partial_r V_r)) \) such that:

\[
\det (V_s) - \det (V_0) = -i \int_0^s \text{tr}_\mathcal{H} (\mathfrak{D}_{g,s}) \, dr,
\]

where we have used the differential equation (32) and the cyclic property of the trace. Since \( \mathfrak{D}_{g,s} \) is trace class one can write

\[
\det (V_s) - \det (V_0) = -i \int_0^s \int_\mathbb{R} \det (V_r) \mathfrak{M}_g(t) \text{tr}_\mathcal{H} (\partial_s H_s) \, dt \, dr,
\]

Now, by using that for each \( s \in \mathcal{C} \), \( H_s \) is self–dual Hamiltonian, i.e., \( H_s = -\Gamma H_s \Gamma \) we note that \( \partial_s H_s \) is also self–dual Hamiltonian, i.e., \( \partial_s H_s = -\Gamma (\partial_s H_s) \Gamma \). In particular, \( \text{tr}_\mathcal{H} (\partial_s H_s) = 0 \) (see Definition 4), and the assertion follows.

From now on, we will expose some issues about quasi–free ground states for \( g \in \mathbb{R}^+ \), as in Definition 10, and for \( g = 0 \). The former are called *gapped quasi–free ground states* (see Expression (24) and comments around it), and their uniqueness is guaranteed. Since the set \( \mathcal{C} \) of ground states is metrizable in the weak*–topology, we denote by \( \mathfrak{C}_g \equiv (\mathfrak{C}_g, \partial_g) \) and \( \mathfrak{C}_0 \equiv (\mathfrak{C}_0, \partial_0) \) the metric spaces in the weak*–topology related to the quasi–free ground states for \( g \in \mathbb{R}^+ \) and \( g = 0 \) respectively. In particular, one notes that \( \mathfrak{C}_g \) and \( \mathfrak{C}_0 \) are not homeomorphic since, as we will see in Corollary 4, the representations associated to \( \mathfrak{C}_g \) are reducible whereas those associated to \( \mathfrak{C}_0 \) are not. This is clear from the fact that there is no homeomorphism between one connected metric space and another one that is *disconnected*. Then the representations associated to \( \mathfrak{C}_g \) and \( \mathfrak{C}_0 \) are not physically equivalent as the intuition says. Instead, Corollary 4 below claims that any two gapped quasi–free ground states associated to the quasi–free dynamics of two gapped Hamiltonians on the same \( g \)–phase are unitarily equivalent. Thus, their irreducible representations also are.

In order to prove last statement, recall Expressions (19), (20) and (23), where for any \( s \in \mathcal{C} \equiv [0, 1] \) one knows that for the positive (on \( \Sigma_s \)) spectral projection \( E_{+,s} \in \mathcal{B}(\mathcal{H}) \) associated to the self–dual Hamiltonian \( H_s \) on \( (\mathcal{H}, \Gamma) \) there is a unique quasi–free ground state \( \omega_s \in \mathfrak{C}_g \) such that

\[
\omega_s (B(\varphi_1) B(\varphi_2^*)) = \langle \varphi_1, E_{+,s} \varphi_2 \rangle_{\mathcal{H}}, \quad \varphi_1, \varphi_2 \in \mathcal{H}.
\]

Once again, consider \( \mathbf{H} \), a family of self–dual Hamiltonians satisfying Assumption 1. For any \( s \in \mathcal{C} \), \( \omega_s \) is a gapped quasi–free ground state. By using the family of \( \ast \)–automorphisms \( \{ \kappa_s \}_{s \in \mathcal{C}} \) on \( \mathcal{B}(\mathcal{H}) \) of Lemma 1, with \( V_s \) a unitary operator implementing \( \kappa_s \), we note that

\[
\omega_s = \omega_0 \circ Y_s, \quad s \in \mathcal{C}.
\]

---

In particular \( \mathfrak{C}_g \) is a weak*–compact convex set metrizable in the weak*–topology that can be written as \( \mathfrak{C}_g = \mathfrak{C}_{g,-} \cup \mathfrak{C}_{g,+} \), for \( \mathfrak{C}_{g,-} \) and \( \mathfrak{C}_{g,+} \) nonempty and disjoint metrizable set in the weak*–topology. Here, \( \mathfrak{C}_{g,-} \) and \( \mathfrak{C}_{g,+} \) are associated to the negative and positive components of the unitary operators respectively.
Here, $\Upsilon_s$ is the one–parameter (Bogoliubov) $^*$–automorphism of $\text{sCAR}(\mathcal{H}, \Gamma)$ given by Expression (40). Additionally, let $\omega \doteq \{\omega_s\}_{s \in \mathcal{E}} \subset \mathcal{G} \mathcal{E}$ be a family of gapped quasi–free ground states associated to self–dual Hamiltonians $H$ on some self–dual Hilbert space $(\mathcal{H}, \Gamma)$, with the same assumptions of Lemma 1. The meaning of expression (41) in terms of representations is that the associated (irreducible) GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is unique (up to unitary equivalence): for all $A \in \text{sCAR}(\mathcal{H}, \Gamma)$

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle_{\mathcal{H}_\omega},$$

where the latter notation means that for any two states $\omega_{s_1}, \omega_{s_2} \in \omega$ there exists an isomorphism $J_{s_1, s_2}$ from $\mathcal{H}_{\omega_{s_1}}$ to $\mathcal{H}_{\omega_{s_2}}$ satisfying

$$\pi_{\omega_{s_2}}(A) = J_{s_1, s_2}^* \pi_{\omega_{s_1}}(A) J_{s_1, s_2},$$

i.e., $\pi_{\omega_{s_2}}$ and $\pi_{\omega_{s_2}}$ are unitarily equivalent as well as their associated cyclic vectors $\Omega_{\omega_{s_1}}$ and $\Omega_{\omega_{s_2}}$. Additionally, following Definition 6 and comments around it, there is a strongly continuous one–parameter unitary group $(e^{itL_\omega})_{t \in \mathbb{R}}$ with generator $L_\omega = L^* \geq 0$ satisfying $e^{itL_\omega} \Omega_\omega = \pi_\omega(\tau_t(A)) \Omega_\omega$ and any $A \in \text{sCAR}(\mathcal{H}, \Gamma)$

$$e^{itL_\omega} \pi_\omega(A) \Omega_\omega = \pi_\omega(\tau_t(A)) \Omega_\omega.$$

We summarize the latter with the following Corollary:

**COROLLARY 4.**

Consider a family of self–dual Hamiltonians $H \subset \mathcal{B}(\mathcal{H})$ satisfying Assumption 1. Let $\omega \in \mathcal{G} \mathcal{E}$ be a family of gapped quasi–free ground states associated to $H$. The associated (irreducible) GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is unique (up to unitary equivalence). In particular, any state $\omega_s \in \omega, s \in \mathcal{E}$, is related to $\omega_0 \in \omega$ by Expression (41), namely, $\omega_s = \omega_0 \circ \Upsilon_s$.

## 4.2 Dynamics, ground states and spectral flow automorphism in the Thermodynamic limit

For $d \in \mathbb{N}$, let $\mathbb{Z}^d$ be the Cayley graph as defined in Appendix A, see Expression (61), and let the spin set $\mathcal{S}$, such that $\mathcal{L} = \mathbb{Z}^d \times \mathcal{S}$. Since we are dealing with fermions, w.l.o.g., these can be treated as negatively charged particles. The cases of particles positively charged can be treated by exactly the same methods. Then, in order to take the thermodynamic limit we define the Hilbert spaces $\mathcal{H}_{\mathcal{S}} \doteq \ell^2(\mathcal{S}) \oplus \ell^2(\mathcal{S})^*$ and $\mathcal{H}_L \doteq \ell^2(\Lambda_L; \mathcal{H}_{\mathcal{S}})$ for all $L \in \mathbb{R}_0^+ \cup \{\infty\}$, where $\Lambda_L$ for $L \in \mathbb{R}_0^+ \cup \{\infty\}$ is defined by the increasing sequence of cubic boxes

$$\Lambda_L \doteq \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : |x_1|, \ldots, |x_d| \leq L\} \subset \mathcal{P}_1(\mathbb{Z}^d),$$

of side length $O(L)$. Note that such a sequence is a “Van Hove net”, i.e., the volume of the boundaries$^{10}$ $\partial \Lambda_L \subset \Lambda_L \subset \mathcal{P}_1(\mathbb{Z}^d)$ is negligible w.r.t. the volume of $\Lambda_L$ for $L$ large enough: $\lim_{L \to \infty} \frac{|\partial \Lambda_L|}{|\Lambda_L|} = 0$.

We now fix any antiunitary involution $\Gamma_\mathcal{S}$ on $\mathcal{H}_{\mathcal{S}}$. For any $L \in \mathbb{R}_0^+ \cup \{\infty\}$, we define an antiunitary involution $\Gamma_L$ on $\mathcal{H}_L$ by

$$\Gamma_L \varphi(x) \doteq \Gamma_\mathcal{S}(\varphi(x)), \quad x \in \Lambda_L, \varphi \in \mathcal{H}_L.$$

Then, $(\mathcal{H}_L, \Gamma_L)$ is a local self–dual Hilbert space for any $L \in \mathbb{R}_0^+ \cup \{\infty\}$. Note that $\mathcal{H}_{\mathcal{S}}$ and $\mathcal{H}_L$ are finite–dimensional, with even dimension, whenever $L < \infty$: Let

$$\mathcal{X}_L \doteq \Lambda_L \times \mathcal{S} \times \{-, +\}, \quad L \in \mathbb{R}_0^+ \cup \{\infty\}. \quad (44)$$

$^{10}$By fixing $m \geq 1$, the boundary $\partial \Lambda$ of any $\Lambda \subset \mathbb{Z}^d$ is defined by $\partial \Lambda \doteq \{x \in \Lambda : \exists y \in \mathbb{Z}^d \setminus \Lambda \text{ with } d_\epsilon(x, y) \leq m\}$, where for $\epsilon \in (0, 1], d_\epsilon(x, y) : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ is a well–defined pseudometric related to the distance between $x, y$ in the lattice $\mathbb{Z}^d$ [BP13]. W.l.o.g. we will take the $\epsilon$–Euclidean distance $d_\epsilon(x, y) \doteq |x - y|_\epsilon$. 

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The canonical orthonormal basis \( \{ \mathbf{e}_x \}_{x \in \mathcal{X}_L} \) of \( \mathcal{H}_L \), \( L \in \mathbb{R}_0^+ \cup \{ \infty \} \), now is defined by
\[
\mathbf{e}_x(y) = \delta_{x,y} f_{s,v}, \quad x = (x,s,v) \in \mathcal{X}_L, \quad y \in \Lambda_L,
\]
where \( f_{s,t} = \Gamma \delta_{s,t} \in \mathcal{H}_\mathcal{S} \) and \( f_{s,t}(t) = \delta_{s,t} \) for any \( s, t \in \mathcal{S} \).

Within the self–dual formalism, a lattice fermion system in infinite volume is defined by a self–dual Hamiltonian \( H_\infty \in \mathcal{B}(\mathcal{H}_\infty) \) on \( (\mathcal{H}_\infty, \Gamma_\infty) \), that is, \( H_\infty = H_\infty^* = -\Gamma_\infty H_\infty \Gamma_\infty \). See Definition 4 which is here extended to the infinite–dimensional case. For a fixed basis projection \( P_\infty \) diagonalizing \( H_\infty \), the operator \( P_\infty H_\infty P_\infty \) is the so–called one–particle Hamiltonian associated with the system. To obtain the corresponding self–dual Hamiltonians in finite volume we use the orthogonal projector \( P_{\mathcal{H}_L} \in \mathcal{B}(\mathcal{H}_\infty) \) on \( \mathcal{H}_L \) and define
\[
H_L = P_{\mathcal{H}_L} H_\infty P_{\mathcal{H}_L}, \quad L \in \mathbb{R}_0^+.
\]
By construction, if \( H_\infty \) is a self–dual Hamiltonian on \( (\mathcal{H}_\infty, \Gamma_\infty) \), then, for any \( L \in \mathbb{R}_0^+ \), \( H_L \) is a self–dual Hamiltonian on \( (\mathcal{H}_L, \Gamma_L) \). Note that \( P_{\mathcal{H}_L} \) strongly converges to \( 1_{\mathcal{H}_\infty} \) as \( L \to \infty \).

For the self–dual Hilbert space \( (\mathcal{H}_\infty, \Gamma_\infty) \), the self–dual CAR algebra associated is denoted by \( \mathfrak{A}_\infty \equiv \text{sCAR}(\mathcal{H}_\infty, \Gamma_\infty) \), with generator elements \( 1 \) and \( \{ B(\mathbf{e}_x) \}_{x \in \mathcal{X}_\infty} \) satisfying CAR Expressions of Definition 1. The subalgebra of even elements of \( \mathfrak{A}_\infty \) (see (7)) will be denoted by \( \mathfrak{A}_\infty^+ \) in the sequel. For \( \Lambda \in \mathcal{P}_1(\mathbb{Z}^d) \) and the finite–dimensional (one–particle) Hilbert space \( \mathcal{H}_\Lambda \equiv \ell^2(\Lambda; \mathcal{H}_\infty) \) with involution given by (43), we identify the finite dimensional CAR \( C^* \)–algebra
\[
\mathfrak{A}_\Lambda \equiv \text{sCAR}(\mathcal{H}_\Lambda, \Gamma_\Lambda), \quad \Lambda \in \mathcal{P}_1(\mathbb{Z}^d),
\]
with the \( C^* \)–subalgebra generated by the unit \( 1 \) and \( \{ B(\mathbf{e}_x) \}_{x \in \mathcal{X}_\Lambda} \). Then, we define by
\[
\mathfrak{A}_\infty^{(0)} \equiv \bigcup_{\Lambda \in \mathcal{P}_1(\mathbb{Z}^d)} \mathfrak{A}_\Lambda \subset \mathfrak{A}_\infty,
\]
the normed \( * \)–algebra of local elements, which is dense in \( \mathfrak{A}_\infty \).

From Definition 6 one notes that existence of ground states strongly relies on the existence of the dynamics in the thermodynamical limit. The latter means that the sequence \( \{ \Lambda_L \}_{L \in \mathbb{R}_0^+ \cup \{ \infty \}} \), defined by (42), eventually will contain all the finite subsets, \( \mathcal{P}_1(\mathbb{Z}^d) \) of \( \mathbb{Z}^d \) as \( L \to \infty \). In fact, for any \( H_L = H_L^* \in \mathcal{B}(\mathcal{H}_L) \) one can associate a quasi–free dynamics (10) defining a continuous group \( \{ \tau^L_t \}_{t \in \mathbb{R} \cup \mathbb{R}_0^+} \) of finite volume \( * \)–automorphisms of \( \mathfrak{A}_L \equiv \mathfrak{A}_L^\Lambda \) by
\[
\tau^L_t(A) \equiv e^{-it\langle B, H_L B \rangle} A e^{it\langle B, H_L B \rangle}, \quad A \in \mathfrak{A}_\infty, \quad t \in \mathbb{R}.
\]
See (46) and (48). The associated finite volume generator or finite symmetric derivation is given by (13), namely,
\[
\delta^{(L)}(A) = -i\langle B, H_L B \rangle, \quad A \in \mathfrak{A}_\infty^{(0)},
\]
while, the infinite volume generator or symmetric derivation is
\[
\delta(A) = -i\langle B, H_L B \rangle, \quad A \in \mathfrak{A}_\infty^{(0)}.
\]
For \( L \in \mathbb{R}_0^+ \) and \( \Lambda_L \in \mathcal{P}_1(\mathbb{Z}^d) \), denote by \( \Lambda_L^c \equiv \mathbb{Z}^d \setminus \Lambda_L \) the complement of \( \Lambda_L \). Then, \( \mathfrak{A}_{\Lambda_L^c} \equiv \text{sCAR}(\mathcal{H}_{\Lambda_L^c}, \Gamma_{\Lambda_L^c}) \), will be the \( C^* \)–subalgebra generated by the unit \( 1 \) and \( \{ B(\mathbf{e}_x) \}_{x \in \mathcal{X}_{\Lambda_L^c}} \). The bilinear elements associated to the (border) terms on \( \Lambda_L \) and \( \Lambda_L^c \) are (cf. Definition 3):
\[
\langle B, \partial H_L B \rangle = \sum_{x_1, x_2 \in \mathcal{X}_\infty} \langle \mathbf{e}_{x_2}, \partial H_L \mathbf{e}_{x_1} \rangle_{\mathcal{H}_\infty} B(\mathbf{e}_{x_1}) B(\mathbf{e}_{x_2})^*,
\]
with \( \mathcal{H}_L^c \equiv \mathcal{H}_{\Lambda_L^c} \) and
\[
\partial H_L \equiv P_{\mathcal{H}_L} H_\infty P_{\mathcal{H}_L} + P_{\mathcal{H}_L^c} H_\infty P_{\mathcal{H}_L},
\]
where for any \( \Lambda_L \in \mathcal{P}_1(\mathbb{Z}^d) \), \( P_{\mathcal{H}_L} \in \mathcal{B}(\mathcal{H}_\infty) \) is the orthogonal projector on \( \mathcal{H}_L \), see Expression (46).
Theorem 3 (Infinite volume dynamics):
Assume that the sequence \( \{ H_L \} \) of self–dual Hamiltonians \( H_L \in \mathcal{B}(\mathcal{H}_L) \) strongly converges to \( H_\infty \in \mathcal{B}(\mathcal{H}_\infty) \) so that
\[
\sum_{x \in \mathcal{X}_\infty} \left| \langle e_0, H_\infty e_x \rangle \right| \in \mathbb{R}_0^+.
\]
Then, for \( L \in \mathbb{R}_0^+ \), the continuous group \( \{ \tau_t^{(L)} \} \) with generator \( \delta^{(L)} \) converges strongly to a continuous group \( \{ \tau_t \} \) with generator \( \delta \) as \( L \to \infty \). \( \delta \) is a conservative closed symmetric derivation.

**Proof.** The proof of the statements is completely standard. We present it here for the sake of completeness. This is split in a set of parts:

1. We can combine Expressions (10) and (11) such that for any self–dual Hamiltonian \( H_\infty \in \mathcal{B}(\mathcal{H}_\infty) \) we have
\[
\tau_t^{(L)} (B(\varphi)) = B \left( \left( U_t^{(L)} \right)^* \varphi \right) \quad \text{and} \quad \tau_t (B(\varphi)) = B \left( U_t^\varphi \right).
\]
Here, for \( L \in \mathbb{R}_0^+ \), \( \tau_t^{(L)} = e^{i t H_L} \) and \( \tau_t = e^{i t H_\infty} \) so that
\[
\left\{ U_t^{(L)} = e^{i t H_L} \right\} \quad \text{and} \quad \left\{ U_t \equiv U_t^{(\infty)} = e^{i t H_\infty} \right\}
\]
are the one–parameter unitary groups on \( (\mathcal{H}_\infty, \Gamma_\infty) \) associated to the finite and infinite dynamical systems, respectively. Note that for any \( \varphi \in \mathcal{H}_\infty \), \( B(\varphi) \) is bounded (see Definition 1). Then, using that \( \| B(\varphi) \|_{\mathcal{A}_\infty} \leq \| \varphi \|_{\mathcal{H}_\infty} \), for any \( L_1, L_2 \in \mathbb{R}_0^+ \), with \( L_2 \geq L_1 \), we have
\[
\left\| \tau_t^{(L_2)} (B(\varphi)) - \tau_t^{(L_1)} (B(\varphi)) \right\|_{\mathcal{A}_\infty} \leq \left\| \left( U_t^{(L_2)} - U_t^{(L_1)} \right) \varphi \right\|_{\mathcal{B}(\mathcal{H}_\infty)}.
\]
We can write
\[
U_t^{(L_2)} - U_t^{(L_1)} = \int_0^t \partial_s \left( U_{t-s}^{(L_1)} U_s^{(L_2)} \right) ds = i \int_0^t U_{t-s}^{(L_1)} \left( H_{L_2} - H_{L_1} \right) U_s^{(L_2)} ds,
\]
so that
\[
\left\| \tau_t^{(L_2)} (B(\varphi)) - \tau_t^{(L_1)} (B(\varphi)) \right\|_{\mathcal{A}_\infty} \leq |t| \left\| \left( H_{L_2} - H_{L_1} \right) \varphi \right\|_{\mathcal{B}(\mathcal{H}_\infty)}.
\]
Since the sequence \( \{ H_L \} \) strongly converges to \( H_\infty \) as \( L \to \infty \), the last expression shows that it is a Cauchy sequence of self–adjoint operators. Therefore, the continuous group of \( \tau \)–automorphisms \( \left\{ \tau_t \right\} \), \( L \in \mathbb{R}_0^+ \), strongly converges to \( \{ \tau_t \} \) for all \( t \in \mathbb{R} \).

2. In order to show the existence of the generator, take \( \Lambda \in \mathcal{P}_1(\mathbb{Z}^d) \) and \( A \in \mathcal{A}_\Lambda \) in (49). By the estimate
\[
\| \delta(A) \|_{\mathcal{A}_\infty} \leq 2 \| A \|_{\mathcal{A}_1} \sum_{x_1, x_2 \in \mathcal{X}_\infty, x_2 \neq x_1} \left| \langle e_{x_2}, H_\infty e_{x_1} \rangle \right| \in \mathbb{R}_0^+,
\]
and the hypothesis of the Theorem note that the infinite volume generator, given by Equation (50), is absolutely convergent. By (49), it follows that
\[
\delta(A) = \lim_{L \to \infty} \delta^{(L)}(A), \quad A \in \mathcal{A}^{(0)}_{\mathcal{A}_\infty}.
\]
3. Moreover, for any fixed \( \zeta \notin \text{spec}(\delta^{(L)}) \), the resolvent \( R_\zeta \left( \delta^{(L)} \right) \) converges strongly to \( R_\zeta (\delta) = (\zeta \mathbf{1} - \delta)^{-1} \), the resolvent of \( \delta \). In fact, take \( L \in \mathbb{R}^+ \) and let \( \rho(\delta^{(L)}) = \mathbb{C} \setminus \text{spec}(\delta^{(L)}) \) be the resolvent set of \( \delta^{(L)} \). We know that for any \( \zeta \in \rho(\delta^{(L)}) \) the resolvent \( R_\zeta \left( \delta^{(L)} \right) \) satisfies the identity [EBN+06]

\[
R_\zeta \left( \delta^{(L)} \right) (A) = \int_0^\infty e^{-\zeta t} \tau_t^{(L)} (A) dt, \quad A \in \mathfrak{A}(0).
\]

Note that for any generators \( \delta^{(L_1)}, \delta^{(L_2)} : \mathfrak{A}(0) \to \mathfrak{A}_\infty \) and \( \zeta \in \rho(\delta^{(L_1)}) \cap \rho(\delta^{(L_2)}) \), with \( L_1, L_2 \in \mathbb{R}^+_0 \cup \{\infty\} \) and \( L_2 > L_1 \), we have

\[
R_\zeta \left( \delta^{(L_2)} \right) - R_\zeta \left( \delta^{(L_1)} \right) = \int_0^\infty e^{-\zeta t} \left( \tau_t^{(L_2)} (A) - \tau_t^{(L_1)} (A) \right) dt.
\]

By linearity, we can take \( A \equiv B(\varphi) \), with \( \varphi \in \mathcal{H}_\infty \), use first part of the current Theorem and the Lebesgue’s dominated convergence theorem in order to conclude that \( R_\zeta \left( \delta^{(L_2)} \right) - R_\zeta \left( \delta^{(L_1)} \right) \) is a Cauchy sequence. By above Expression it follows that

\[
\lim_{L \to \infty} \left\| \left( R_\zeta \left( \delta^{(\infty)} \right) - R_\zeta \left( \delta^{(L)} \right) \right) (A) \right\|_{\mathfrak{A}_\infty} = 0,
\]

as desired.

4. By taking into account proof of Theorem 4.8 in [BP16a], and the first and second Trotter–Kato approximation theorems [EBN+06, Chap. III, Sect. 4.8 and 4.9] we claim that \( \mathfrak{A}(0) \subset \text{ran} \left( R_\zeta \left( \delta^{(L)} \right) \right) \) is dense in \( \mathfrak{A}_\infty \) and that \( \delta \) is conservative.

We finally remark, that the proof considered here for the operator \( \delta \) on \( \mathfrak{A} \) with dense domain \( \mathcal{D}(\delta) = \mathfrak{A}(0) \) also works if \( \delta \) is unbounded.

In order to study quasi–free ground states at infinite volume we use:

**Proposition 1.**

Let \( \{H_L\}_{L \in \mathbb{R}^+_0} \in \mathcal{B}(\mathcal{H}_\infty) \) be a sequence of self–dual Hamiltonians on \( \mathcal{H}_\infty \in \mathcal{B}(\mathcal{H}_\infty) \) strongly convergent to \( H_\infty \in \mathcal{B}(\mathcal{H}_\infty) \). For any \( L \in \mathbb{R}^+_0 \cup \{\infty\}, E_{+,L} \) will denote the spectral projection on \( \mathbb{R}^+ \) associated to the self–dual Hamiltonian \( H_L \). If zero is not an eigenvalue of \( H_\infty \), then \( E_+ \) will be the strong limit of the sequence \( \{E_{+,L}\}_{L \in \mathbb{R}^+_0} \), i.e., \( \lim_{L \to \infty} E_{+,L} = E_+ \).

**Proof.** The proof is found in [AE83, Lemma 3.3.]

For any \( L \in \mathbb{R}^+_0 \) let us define the set of local quasi–free ground states by \( q\mathfrak{E}(L) \subset q\mathfrak{E}(\infty) \) on \( \mathfrak{A}_L \subset \mathfrak{A}_\infty \). See Definition 7. To be explicit, for any self–dual Hamiltonian \( H_\infty \in \mathcal{B}(\mathcal{H}_\infty) \) on \( \mathcal{H}_\infty, \Gamma_\infty \) and any orthogonal projection \( P_{\mathfrak{H}_L} \in \mathcal{B}(\mathcal{H}_\infty) \) on \( \mathcal{H}_L \) the local Hamiltonian is given by (46), namely,

\[
H_L = P_{\mathfrak{H}_L} H_\infty P_{\mathfrak{H}_L},
\]

which has an associated local Gibbs state defined by

\[
\varrho_{\mathfrak{A}_L} \left( B(\varphi_{1_L}) B(\varphi_{2,L}^*) \right) = \left\langle \varphi_{1_L}, E_{+,L} \varphi_{2,L} \right\rangle_{\mathfrak{H}_L},
\]

for \( \varphi_{j,L} \in \mathfrak{H}_L, j = \{1,2\} \), where \( E_{+,L} \) denotes the sequence of spectral projections of Proposition 1. Then, using Expressions (17)–(18) the local quasi–free ground state \( \omega_{\mathfrak{A}_L} \in q\mathfrak{E}(L) \) is found to be

\[
\omega_{\mathfrak{A}_L} \left( B(\varphi_{1,L}) B(\varphi_{2,L}^*) B(\varphi_{3,L}^*) B(\varphi_{4,L}^*) \right) = \varrho_{\mathfrak{A}_L} \left( B(\varphi_{1,L}) B(\varphi_{2,L}^*) \right) \omega_{\mathfrak{A}_L} \left( B(\varphi_{3,L}^*) B(\varphi_{4,L}^*) \right),
\]

(53)
where \( \varphi_j, A_L \in \mathcal{H}_{\Lambda_L} \equiv \mathcal{H}_L \), \( j = \{3, 4\} \), cf. [AM03, Section 7.5]. By linearity, for any two even elements \( A \in \mathfrak{A}_L^+ \) and \( B \in \mathfrak{A}_L^{+,c} \), see (7), we get from (53):

\[
\omega_{\Lambda_L} (AB) = \varrho_{\Lambda_L} (A) \omega_{\Lambda_L} (B),
\]

see again [AM03, Section 7.5]. In particular, for \( B = \mathbb{1} \in \mathfrak{A}_\infty \) we have

\[
\omega_{\Lambda_L} (A) = \varrho_{\Lambda_L} (A).
\]

We now state:

**Theorem 4 (Quasi–free ground states):**

The local quasi–free ground state \( \omega_{\Lambda_L} \) converges to

\[
\omega (B(\varphi_1)B(\varphi_2)^*) = \langle \varphi_1, E_+ \varphi_2 \rangle_{\mathcal{H}_\infty},
\]

in the weak*–topology, where \( E_+ \in \mathcal{B}(\mathcal{H}_\infty) \) is the spectral projection on \( \mathbb{R}^+ \) associated to the self–dual Hamiltonian \( H_\infty \in \mathcal{B}(\mathcal{H}_\infty) \), and \( \varphi_1, \varphi_2 \in \mathcal{H}_\infty \).

**Proof.** For the sake of clarity, for any \( L \in \mathbb{R}_0^+ \) denote \( \mathcal{H}_{\Lambda_L} \equiv \mathcal{H}_L \), and \( E_{\Lambda_L} \equiv E_L \). See Expressions (42)–(45) and comments around it. Take \( L_1, L_2 \in \mathbb{R}_0^+ \), with \( L_2 \geq L_1 \) such that \( \Lambda_{L_2} \supseteq \Lambda_{L_1} \). Thus, we analyze the following difference:

\[
D_{\omega_{L_2}, \omega_{L_1}} = \omega_{L_2} \left( B(\varphi_1, L_2)B(\varphi_2, L_2)^* \right) - \omega_{L_1} \left( B(\varphi_1, L_1)B(\varphi_2, L_1)^* \right).
\]

Here, in the way that the set of boxes \( \Lambda_L \) was defined (42), for \( j = \{1, 2\} \) we canonically identify \( \varphi_j, A_{L_j} \in \mathcal{H}_{\Lambda_{L_j}} \) with the element \( \varphi_j, A_{L_j} \oplus 0_{A_{L_j} \setminus \Lambda_{L_j}} \in \mathcal{H}_{\Lambda_{L_j}} \). The spectral projections on \( \mathbb{R}^+ \) are related by \( E_+, \Lambda_{L_2} = E_+, \Lambda_{L_1} \oplus E_+, \Lambda_{L_2} \setminus \Lambda_{L_1} \in \mathcal{B}(\mathcal{H}_{\Lambda_{L_2}}) \). Straightforward calculations yield us to note that \( \lim_{L_1 \to \infty} \lim_{L_2 \to \infty} D_{\omega_{L_2}, \omega_{L_1}} \) equals zero.

\[\end{align}\]

We are now in a position to prove the properties of the family of automorphisms \( \kappa_s : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \), for any \( s \in C \), given by Assumption 1 and Lemma 1, which are associated to a differentiable family of self–dual Hamiltonians \( \mathbb{H} \in \mathcal{B}(\mathcal{H}_\infty) \), see Definition 10. Observe that the existence of such \( \kappa_s \) is closely related to the existence of a differentiable unitary operator \( V_s \) satisfying the non–autonomous differential equation, Expression (32):

\[
\partial_s V_s = -i \mathfrak{D}_{\mathfrak{b}, s} V_s, \quad \text{with} \quad V_0 = \pm 1_{\mathcal{H}},
\]

where \( \{ \mathfrak{D}_{\mathfrak{b}, s} \}_{s \in C} \in \mathcal{B}(\mathcal{H}_\infty) \) is a family of self–adjoint operators that is found to be

\[
\mathfrak{D}_{\mathfrak{b}, s} \equiv \int_\mathbb{R} e^{iH_s \tau} (\partial_s H_s) e^{-iH_s \tau} \mathfrak{W}_0 (t) \, dt,
\]

with \( \mathfrak{W}_0 : \mathbb{R} \to \mathbb{R} \) an integrable odd function the properties of which are summarized in [BMNS12, MZ13] and references therein. In the sequel, for any \( s \in C \), \( V_s, H_s \), and \( \partial_s H_s \) have to be understood as the strong limit of the sequences \( \{ V_s (L) \}_{L \in \mathbb{R}_0^+}, \{ H_s (L) \}_{L \in \mathbb{R}_0^+} \) and \( \{ \partial_s H_s (L) \}_{L \in \mathbb{R}_0^+} \) respectively. We formulate:

**Lemma 2.**

Take \( C = [0, 1] \), fix \( s \in C \) and consider the family of operators satisfying Assumption 1. Then, the sequence of automorphisms \( \{ \kappa_s (L) \}_{L \in \mathbb{R}_0^+} : \mathcal{B}(\mathcal{H}_L) \to \mathcal{B}(\mathcal{H}_L) \) of Lemma 1 on the local self–dual Hilbert space \( (\mathcal{H}_L, \Gamma_L) \) strongly converges on \( C \) to \( \kappa_s : \mathcal{B}(\mathcal{H}_\infty) \to \mathcal{B}(\mathcal{H}_\infty) \). More precisely, for any \( \Lambda \in \mathcal{P}(\mathbb{Z}^d), B \in \mathcal{B}(\mathcal{H}_\Lambda) \) and \( L \in \mathbb{R}_0^+ \) such that \( \Lambda \subset \Lambda_L \) we have

\[
\lim_{L \to \infty} \left\| \kappa_s (B) - \kappa_s (L) (B) \right\|_{\mathcal{B}(\mathcal{H}_\infty)} = 0, \quad \text{for any} \quad s \in C.
\]
Proof. Fix $\Lambda \in \mathcal{P}_t(\mathbb{Z}^d)$ and take $L_1, L_2 \in \mathbb{R}_0^+$, with $L_2 \geq L_1$ such that $\Lambda_{L_2} \supseteq \Lambda_{L_1} \supseteq \Lambda$. We proceed in a similar way as in [BP16a, Lemma 4.4]. Note that with a few modifications of the proof we can arrive at a result that works even in the interparticle case [AMR].

For any $L \in \mathbb{R}_0^+$, let $V_{s}^{(L)}$ be the unitary operator satisfying the differential equation (32), with $V_{0}^{(L)} = \pm 1_{\mathcal{W}}$. For $s, r \in \mathcal{C}$, one defines the unitary element

$$U_L(s, r) = V_{s}^{(L)}(V_{r}^{(L)})^*, \tag{54}$$

which satisfies $U_L(s, s) = 1_{\mathcal{W}}$ for all $s \in \mathcal{C}$ while

$$\partial_s U_L(s, r) = -i \mathfrak{D}_{g,s}^{(L)} U_L(s, r) \quad \text{and} \quad \partial_r U_L(s, r) = i U_L(s, r) \mathfrak{D}_{g,r}^{(L)}. \tag{55}$$

Note that for $B \in \mathcal{B}(\mathcal{H}_L)$ one can write

$$\kappa_s^{(L)}(B) - \kappa_r^{(L)}(B) = \int_0^s \partial_r (U_{L_2}(0, r) U_{L_1}(r, s)) B U_{L_1}(s, r) U_{L_2}(r, 0) dr. \tag{56}$$

Straightforward calculations show us that the derivative inside the integral is

$$i U_{L_2}(0, r) \left( \left( \mathfrak{D}_{g,s}^{(L_2)} - \mathfrak{D}_{g,r}^{(L_1)} \right), U_{L_1}(r, s) B U_{L_1}(s, r) \right) U_{L_2}(r, 0), \tag{56}$$

with $s, r \in \mathcal{C}$, and for $L \in \mathbb{R}_0^+$, and $\Lambda \in \mathcal{P}_t(\mathbb{Z}^d)$, $\Lambda \subset \Lambda_{L_2}$.

On the other hand, for any $s, t \in \mathbb{C}$, $L \in \mathbb{R}_0^+$ such that $\Lambda_{L_2} \supseteq \Lambda$, define the $s$–automorphism $\tilde{\tau}_s^{(L)} : \mathcal{B}(\mathcal{H}_L) \to \mathcal{B}(\mathcal{H}_L)$ by

$$\tilde{\tau}_s^{(L)}(B) = e^{it H_{s,L}} B e^{-it H_{s,L}},$$

with $H_{s,L}$ a self–dual Hamiltonian on $(\mathcal{H}_L, \Gamma_{L})$. Then, for $L_1, L_2 \in \mathbb{R}_0^+$ one can write the following

$$\tilde{\tau}_s^{(L_2)}(B) - \tilde{\tau}_s^{(L_1)}(B) = \int_0^t \partial_u \left( \tilde{\tau}_u^{(L_2)} \circ \tilde{\tau}_{s,t-u}^{(L_1)}(B) \right) du$$

$$= i \int_0^t \tilde{\tau}_u^{(L_2)} \left( [H_{s,L_2} - H_{s,L_1}, \tilde{\tau}_{s,t-u}^{(L_1)}(B)] \right) du,$$

where, for a fix $s \in \mathcal{C}$, the difference $H_{s,L_2} - H_{s,L_1} \in \mathcal{B}(\mathcal{H}_{\infty})$ is given by

$$H_{s,L_2} - H_{s,L_1} = P_{\mathfrak{H}_{L_2}} H_{s,\infty} P_{\mathfrak{H}_{L_2}} \setminus \mathfrak{H}_{L_1} + P_{\mathfrak{H}_{L_2}} \setminus \mathfrak{H}_{L_1} H_{s,\infty} P_{\mathfrak{H}_{L_2}} + P_{\mathfrak{H}_{L_2}} \setminus \mathfrak{H}_{L_1} H_{s,\infty} P_{\mathfrak{H}_{L_2}} \setminus \mathfrak{H}_{L_1},$$

where $P_{\mathfrak{H}_{L_2}} \setminus \mathfrak{H}_{L_1} \equiv P_{\mathfrak{H}_{L_2}} - P_{\mathfrak{H}_{L_1}} \in \mathcal{B}(\mathcal{H}_{\infty})$ is the orthogonal projector on $\mathcal{H}_{L_2} \setminus \mathcal{H}_{L_1}$. Here, $H_{s,\infty}$ is the self–dual Hamiltonian on $(\mathcal{H}_{\infty}, \Gamma_{\infty})$ at infinite volume. It follows that,

$$\left\| \tilde{\tau}_s^{(L_2)}(B) - \tilde{\tau}_s^{(L_1)}(B) \right\|_{\mathcal{B}(\mathcal{H}_{\infty})} \leq \int_0^t \left\| [H_{s,L_2} - H_{s,L_1}, \tilde{\tau}_{s,t-u}^{(L_1)}(B)] \right\| du$$

$$\leq 2 |t| \left\| [H_{s,L_2} - H_{s,L_1}] \right\|_{\mathcal{B}(\mathcal{H}_{\infty})}, \tag{57}$$

By Assumption 1, $\{H_{s,L}\}_{L \in \mathbb{R}_0^+}$ is a sequence of operators which converges in norm to $H_{s,\infty}$ as $L \to \infty$, then last expression is a Cauchy sequence of self–adjoint operators. Hence, for all $t \in \mathbb{R}$, $\tilde{\tau}_s^{(L)}$ converges strongly on $\mathcal{B}(\mathcal{H}_L)$ to $\tilde{\tau}_s$, as $L \to \infty$.

What is important to stress is that the difference $\mathfrak{D}_{g,s}^{(L_2)} - \mathfrak{D}_{g,r}^{(L_1)}$ in Expression (56) can be written as follows

$$\mathfrak{D}_{g,s}^{(L_2)} - \mathfrak{D}_{g,r}^{(L_1)} = \int_{\mathbb{R}} \left( \tilde{\tau}_{s,t}^{(L_2)} \left( \partial_r \{H_{s,L_2}\} \right) - \tilde{\tau}_{s,t}^{(L_1)} \left( \partial_r \{H_{s,L_1}\} \right) \right) \mathfrak{M}_g(t) dt + \int_{\mathbb{R}} \left( \tilde{\tau}_{r,t}^{(L_2)} \left( \partial_r \{H_{r,L_2}\} \right) - \tilde{\tau}_{r,t}^{(L_1)} \left( \partial_r \{H_{r,L_1}\} \right) \right) \mathfrak{M}_g(t) dt.$$
From which one has

\[ \left\| \kappa_s^{(L_2)}(B) - \kappa_s^{(L_1)}(B) \right\|_{\mathcal{B}(\mathcal{H}_\infty)} \leq 2 \left\| B \right\|_{\mathcal{B}(\mathcal{H}_\infty)} \left| s \right| \]

\[ \sup_{r \in \mathbb{C}} \left( \int_{\mathbb{R}} \left| \partial_r \left\{ H_{r,L_2} \right\} - \partial_r \left\{ H_{r,L_1} \right\} \right| \mathfrak{M}_0 (t) \left| dt \right| + \int_{\mathbb{R}} \left| \left( \tilde{z}_{r,t}^{(L_2)} - \tilde{z}_{r,t}^{(L_1)} \right) \right| \partial_r \left\{ H_{r,L_1} \right\} \left| \mathfrak{M}_0 (t) \left| dt \right. \right. \right). \]

Hence, for a fixed \( s \in \mathbb{C} \), by Assumption 1 and Inequality (57) one notes that the right hand side of the last inequality vanishes as \( L_2 \to \infty \) and \( L_1 \to \infty \). Thus, \( \kappa_s^{(L)} \) is a pointwise Cauchy sequence as \( L \to \infty \) and hence the family of automorphism \( \kappa_s^{(L)} \) converges strongly on \( \mathcal{B}(\mathcal{H}_L) \) to \( \kappa_s \) as \( L \to \infty \).

As a consequence we have:

**COROLLARY 5.**

Make the same assumptions as in Lemma 2. Then for any \( s \in \mathbb{C} \), the sequence of unitary operators \( V_s^{(L)} \) converges in norm, as \( L \to \infty \), to some \( V_s \).

**Proof.** As is usual, it is enough to show that the sequence \( V_s^{(L)} \) is a Cauchy sequence. Note that for any \( s \in \mathbb{C} \) and \( L_1, L_2 \in \mathbb{R}_0^+ \) with \( L_2 \geq L_1 \), we can write:

\[ \left( V_s^{(L_2)} \right)^* - \left( V_s^{(L_1)} \right)^* = \int_0^s \partial_r \left( U_{L_2} (0, r) U_{L_1} (r, s) \right) dr, \]

where for any \( s, r \in \mathbb{C} \), \( U_L (s, r) \) is the unitary element defined by (54)–(55). Straightforward calculations yield to

\[ \left( V_s^{(L_2)} \right)^* - \left( V_s^{(L_1)} \right)^* = i \int_0^s U_{L_2} (0, r) \left( \mathfrak{D}_{g,r}^{(L_2)} - \mathfrak{D}_{g,r}^{(L_1)} \right) U_{L_1} (r, s) dr. \]

Proceeding as in (58) we arrive at the desired result. We omit the details.

**LEMMA 3 (UNIFORMITY OF THE DETERMINANT).**

Make the same assumptions as in Lemma 2 and suppose that for any \( s \in \mathbb{C} \) and \( L \in \mathbb{R}_0^+ \cup \{ \infty \}, 1_{\mathcal{H}_L} - V_s^{(L)} \) and \( \mathfrak{D}_{g,r}^{(L)} \) are trace class on \( \mathcal{H}_L \). Then, for any \( s \in \mathbb{C} \), the family of determinants \( \sigma^{(L)}(s) \equiv \det \left\{ V_s^{(L)} \right\} \in \{-1, 1\} \) is uniform for \( L \in \mathbb{R}_0^+ \cup \{ \infty \} \). Moreover, the sequence \( \sigma^{(L)} \in \{-1, 1\} \) converges uniformly on \( \mathbb{C} \).

**Proof.** For \( L \in \mathbb{R}_0^+ \cup \{ \infty \} \) and any \( s, r \in \mathbb{C} \) consider \( U_L (s, r) \), the unitary element defined by (54)–(55). By the Jacobi’s formula of determinants for \( U_L (s, r) \) we have for \( L_1, L_2 \in \mathbb{R}_0^+ \) with \( L_2 \geq L_1 \) that

\[ \left| \det \left( V_s^{(L_2)} \right) - \det \left( V_s^{(L_1)} \right) \right| = \left| \int_0^s \partial_r \left( \det \left( U_{L_2} (0, r) \right) \det \left( U_{L_1} (s, r) \right) \right) dr \right| \]

\[ = \left| \int_0^s \det \left( U_{L_2} (0, r) \right) \det \left( U_{L_1} (s, r) \right) \left( \text{tr}_{\mathcal{H}_L} \left( \mathfrak{D}_{g,r}^{(L_2)} - \mathfrak{D}_{g,r}^{(L_1)} \right) \right) dr \right| \]

Now, similar to the proof of Corollary 3, since \( H_{s,L} \) is self-adjoint for \( s \in \mathbb{C} \) and \( L \in \mathbb{R}_0^+ \cup \{ \infty \} \), \( \partial_s H_{s,L} \) also is. Thus, by Expression (39) and the cyclic property of the trace, it follows that \( \text{tr}_{\mathcal{H}_L} \left( \mathfrak{D}_{g,s}^{(L)} \right) = 0 \), for \( L \in \mathbb{R}_0^+ \). Now, for any \( s \in \mathbb{C} \), note that the sequence of functions \( \left\{ \det \left( V_s^{(L)} \right) \right\}_{L \in \mathbb{N}} \) is equicontinuous and pointwise bounded. By the Ascoli–Arzelà Theorem there exists a uniform convergent subsequence \( \left\{ \det \left( V_s^{(L_n)} \right) \right\}_{n \in \mathbb{N}} \) such that the map \( s \mapsto \det \left( V_s^{(L_n)} \right) \) converges uniformly for \( s \in \mathbb{C} \). By Corollary 3, for any \( s \in \mathbb{C} \) and \( n \in \mathbb{N} \), \( \sigma^{(L_n)}_s \equiv \det \left( V_s^{(L_n)} \right) = \det \left( V_0^{(L_n)} \right) = \pm 1 \).
4.3 Decay estimates of correlations and gapped quasi–free ground states

Fix $\epsilon \in (0, 1]$ and let $(\mathcal{H}_\infty, \Gamma_\infty)$ be the self–dual Hilbert space as defined in subsection 4.2. Moreover, consider the family of self–adjoint operators $\{A_s\}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H}_\infty)$. Thus, for any $s \in \mathcal{C}$ we define the constants

$$ S(A_s, \mu) = \sup_{x_1 \in X_\infty, x_2 \in X_\infty} \left( e^{\mu |x_1 - x_2|^\epsilon} - 1 \right) \left| \langle \varphi_{x_1}, A_s \varphi_{x_2} \rangle \right| \in \mathbb{R}_0^+ \cup \{\infty\}, $$

for $\mu \in \mathbb{R}_0^+$ and

$$ \Delta(A_s, z) = \inf \{|z - \lambda| : \lambda \in \text{spec}(A_s)\}, \quad z \in \mathbb{C}, $$

as the distance from the point $z$ to the spectrum of $A_s$, $X_\infty$ is defined by (44). Here, $\mu$ is not necessarily the same for two different operators $A_{s_1}, A_{s_2} \in \{A_s\}_{s \in \mathcal{C}}$, but in the sequel w.l.o.g. we will assume this. Since the function $x \mapsto (e^{x\epsilon} - 1)/x$ is increasing on $\mathbb{R}_+^*$ for any fixed $r \geq 0$, it follows that

$$ S(A_s, \mu_1) \leq \frac{\mu_1}{\mu_2} S(A_s, \mu_2), \quad \mu_2 \geq \mu_1 \geq 0. \tag{59} $$

We have the following Combes–Thomas estimates:

**Proposition 2 (Combes–Thomas).**

Let $\epsilon \in (0, 1], \mathcal{C} \doteq [0, 1]$, the family of self–adjoint operators $\{A_s\}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H}_\infty)$, $\mu \in \mathbb{R}_0^+$ and $z \in \mathbb{C}$. If $\Delta(A_s, z) > S(A_s, \mu)$ then for any $s \in \mathcal{C}$ and $x = (x, s, v), y = (y, t, w) \in X_\infty$

$$ \left| \langle \varphi_x, (z - A_s)^{-1} \varphi_y \rangle \right| \leq \sup_{s \in \mathcal{C}} \left\{ \frac{e^{-\mu |x - y|^\epsilon}}{\Delta(A_s, z) - S(A_s, \mu)} \right\}. \tag{59} $$

For a proof see [AW15, Theorem 10.5]. Some immediate consequences are summarized as follows:

**Corollary 6.**

Let $\epsilon \in (0, 1], \mathcal{C} \doteq [0, 1]$, the family of self–adjoint operators $\{A_s\}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H}_\infty)$, $\mu \in \mathbb{R}_0^+$ and all $x = (x, s, v), y = (y, t, w) \in X_\infty$. Then,

(a) Let $\eta \in \mathbb{R}_+^*$ such that $\sup_{s \in \mathcal{C}} \{S(A_s, \mu)\} \leq \eta/2, u \in \mathbb{R}$ and $s \in \mathcal{C}$,

$$ \left| \langle \varphi_x, ((A_s - u)^2 + \eta^2)^{-1} \varphi_y \rangle \right| \leq D_{6,(\epsilon)} e^{-\mu |x - y|^\epsilon} \sup_{s \in \mathcal{C}} \left\{ \left( \langle \varphi_x, ((A_s - u)^2 + \eta^2)^{-1} \varphi_x \rangle \right)^{1/2} \langle \varphi_y, ((A_s - u)^2 + \eta^2)^{-1} \varphi_y \rangle \right\}^{1/2}. $$

Moreover, for any function $G(z): \mathbb{C} \to \mathbb{C}$ analytic on $|\text{Im}(z)| \leq \eta$ and uniformly bounded by $\|G\|_\infty$ we have

$$ \langle \varphi_x, G(A_s) \varphi_y \rangle \leq D_{6,(\epsilon)} \|G\|_\infty e^{-\mu \min_{s \in \mathcal{C}} \left\{ \inf_{1 \leq \text{Re}(A_s, \mu)} \frac{\text{Re}(A_s, \mu)}{4S(A_s, \mu)} \right\} |x - y|^\epsilon}. \tag{59} $$

(b) (Gapped Case) For $z \in \mathbb{C}$ such that $\inf_{s \in \mathcal{C}} \Delta(A_s, z) \geq g/2 > 0$, with $g$ as in Definition 10:

$$ \left| \langle \varphi_x, (z - A_s)^{-1} \varphi_y \rangle \right| \leq 4g^{-1} \exp \left( -\mu \min_{s \in \mathcal{C}} \left\{ \inf_{1 \leq \text{Re}(A_s, \mu)} \frac{g}{4S(A_s, \mu)} \right\} \right) |x - y|^\epsilon. \tag{59} $$

Moreover, for $\eta \in (0, g/2]$, and any function $G(z): \mathbb{C} \to \mathbb{C}$ analytic on $z \in \mathbb{R}_0^+ + \eta + i\eta [-1, 1]$ and uniformly bounded by $\|G\|_\infty$ we have

$$ \langle \varphi_x, E_+ G(A_s) E_+ \varphi_y \rangle \leq D_{6,(\epsilon)} \|G\|_\infty e^{-\mu \min_{s \in \mathcal{C}} \left\{ \inf_{1 \leq \text{Re}(A_s, \mu)} \frac{g}{4S(A_s, \mu)} \right\} |x - y|^\epsilon}. \tag{59} $$

29
In all inequalities, the numbers \(D_{6,(a)}, D_{6,(b)}, D_{6,(c)} \in \mathbb{R}^+\) are suitable constants.  

**Proof.** (a) is proven as in [AG98, Theorem 3 and Lemma 3]. (b) The first part is a consequence of Proposition 2 together with Inequality (59). On the other hand, we use Cauchy’s integral formula to write, for all real \(E \in \mathbb{R} \setminus \{\eta\}\),

\[
\chi(\eta, \infty) G(E) = \frac{1}{2\pi i} \int_{\eta}^{\infty} \left( \frac{G(u-i\eta) - G(u+i\eta)}{u-E-i\eta} \right) du - \frac{1}{2\pi} \int_{-\infty}^{\eta} G(\eta + iu) du,
\]

which yields

\[
\chi(\eta, \infty) G(E) = \frac{\eta}{\pi} \int_{\eta}^{\infty} \frac{G(u - i\eta)}{(u-E)^2 + \eta^2} du - \frac{2\eta}{\pi} \int_{\eta}^{\infty} \frac{G(u)}{(u-E)^2 + 4\eta^2} du
\]

\[
+ \frac{1}{2\pi} \int_{0}^{\eta} \frac{G(\eta - iu)}{\eta - iu - E + 2\eta} du + \frac{1}{2\pi} \int_{0}^{\eta} \frac{G(\eta + iu)}{\eta + iu - E - 2\eta} du
\]

\[- \frac{1}{2\pi} \int_{-\eta}^{0} \frac{G(\eta + iu)}{\eta - E + iu} du.
\]

By spectral calculus, together the last equality, part (a) of this Corollary, Inequality (60) and the Cauchy–Schwarz inequality, the result follows. For further details see [ABPM21, Lemma 5.12]. 

At this point it is useful to introduce the normalized trace per unit volume as

\[
\text{Tr}(\cdot) = \lim_{L \to \infty} \frac{1}{\text{dim}(\mathcal{H}_L)} \text{tr}_{\mathcal{H}_L}(\cdot).
\]

We are able to state the following:  

**Lemma 4.**  
Take \(\mathcal{C} \equiv [0,1]\) and consider the family of operators satisfying assumptions of Corollary 6 for \(\{\partial_t H_{s}\}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H}_L)\), \(L \in \mathbb{R}_0^+ \cup \{\infty\}\). Consider the pointwise sequence \(V_s^{(L)}(\cdot) : \mathcal{C} \to \mathcal{B}(\mathcal{H}), \quad L \in \mathbb{R}_0^+ \cup \{\infty\}\), of unitary operators satisfying (32). Then, the sequence \(\{1_{\mathcal{H}_L} - V_s^{(L)}(\cdot)\}_{L \in \mathbb{R}_0^+ \cup \{\infty\}}\) is trace-class per unit volume. Thus, for \(L \in \mathbb{R}_0^+ \cup \{\infty\}\), the family of one–parameter (Bogoliubov) group \(\{\mathcal{Y}_{s}\}_{s \in \mathcal{C}}\) of \(\ast\)–automorphisms on \(\mathfrak{A}_L\) (see (47)), given by (40), is inner.  

**Proof.** For \(s \in \mathcal{C}\) and \(L \in \mathbb{R}_0^+\), let \(W_s^{(L)} \in \mathcal{B}(\mathcal{H}_L)\) be the partial isometry arising from the polar decomposition of \(1_{\mathcal{H}_L} - V_s^{(L)}\)

\[
1_{\mathcal{H}_L} - V_s^{(L)} = W_s^{(L)} \left| 1_{\mathcal{H}_L} - V_s^{(L)} \right|.
\]

From this one can calculate the trace of \(\left| 1_{\mathcal{H}_L} - V_s^{(L)} \right|\) as follows

\[
\text{tr}_{\mathcal{H}_L} \left| 1_{\mathcal{H}_L} - V_s^{(L)} \right| = \sum_{x \in \mathcal{X}_L} \langle e_x, (W_s^{(L)})^* \left( 1_{\mathcal{H}_L} - V_s^{(L)} \right) e_x \rangle_{\mathcal{H}_L}.
\]

Note that for the unitary bounded operator \(V^{(L)}(\cdot)\) on \(\mathcal{H}_L\) we can write \(1_{\mathcal{H}_L} - V_s^{(L)} = -i \int_0^s \mathcal{D}_{\vartheta, r}^{(L)} V^{(L)}(r) dr\). Then, by combining the explicit form of \(\mathcal{D}_{\vartheta, r}^{(L)}\) given by (39), Cauchy–Schwarz inequality, Corollary 6, and other simple arguments we arrive at

\[
\frac{1}{|\Lambda_L|} \left| \text{tr}_{\mathcal{H}_L} \left| 1_{\mathcal{H}_L} - V_s^{(L)} \right| \right| \leq D_{\text{Lem. 4}} |s| \left| \mathcal{G} \right| \int_{\mathbb{R}} \left| \mathcal{M}_s(t) \right| dt \leq \sum_{x \in \mathcal{X}_L} e^{-\mu \inf \left\{ 1, \inf_{r \in \mathcal{C}} \left| \mathcal{A}_{H_+(H, \mu)} \right|, \inf_{r \in \mathcal{C}} \left| \mathcal{A}_{H_+(H, \mu)} \right| \right\} |x|^\mu}.
\]

\(^{11}\)For \(U \in \mathcal{B}(\mathcal{H})\), a Bogoliubov transformation, the Bogoliubov \(\ast\)–automorphism \(\chi_U\) on sCAR(\(\mathcal{H}, \Gamma\)) is inner if and only if \(1_{\mathcal{H}_\infty} - U\) is trace class and \(\text{det}(U) = \pm 1\), see [Ara87, Theorem 4.1].
Then, \(1_{\mathcal{H}_L} - V_s^{(L)}\) is trace class per unit volume, and \(V_s^{(L)} \in \mathcal{B}(\mathcal{H}_L)\) is a Bogoliubov transformation such that \(\det \left( V_s^{(L)} \right) = \pm 1\), for \(L \in \mathbb{R}_0^+ \cup \{\infty\}\). See also Lemma 3. It follows from [Ara87, Theorem 4.1] that the \(\ast\)-automorphism \(\Upsilon_s^{(L)}\) on \(\mathcal{A}_L\) is inner. \(\square\)

A combination of Corollary 5 and Lemma 4 yields to:

**Corollary 7.**

Take some assumptions of Lemma 2. Then, the one–parameter (Bogoliubov) group \(\Upsilon_s^{(L)}\) on \(\mathcal{A}_L^{(0)}\) converges uniformly for \(s \in \mathcal{C}\) as \(L \to \infty\) to the one–parameter (Bogoliubov) group \(\Upsilon_s\) on \(\mathcal{A}_\infty\), thus defining a strongly continuous group on \(\mathcal{A}_\infty\). Moreover, \(\left( \Upsilon_s^{(L)} \right)^{-1}\) exists and strongly converges to \(\Upsilon_s^{-1}\).

**Proof.** Note that the sequence of one–parameter (Bogoliubov) group \(\Upsilon_s^{(L)}\) on \(\mathcal{A}_L^{(0)}\) is Cauchy for any \(B \in \mathcal{A}_L^{(0)}\). We omit the details. Existence of \(\left( \Upsilon_s^{(L)} \right)^{-1}\) is a straight conclusion from Corollary 2, its convergence is immediate. We also omit the details. \(\square\)

In regard to the unitary operator \(U_L(s, r)\) defined by (54)–(55) for any \(r, s \in \mathcal{C}\) and \(L \in \mathbb{R}_0^+ \cup \{\infty\}\) we have the following:

**Lemma 5.**

Take same assumptions of Lemma 2 and consider the unitary operator \(U_L(s, r)\) defined by (54)–(55). For fixed \(r, s \in \mathcal{C}\) we have: (a) The sequence \(\{ 1_{\mathcal{H}_L} - U_L(s, r) \}\) \(L \in \mathbb{R}_0^+ \cup \{\infty\}\) is trace–class per unit volume. (b) \(U_L(s, r)\) commutes with the involution \(\Gamma_L\) for any \(L \in \mathbb{R}_0^+ \cup \{\infty\}\). (c) \(\det (U_L(s, r)) = 1\).

**Proof.** (a) Similar to proof of Lemma 4 for \(r, s \in \mathcal{C}\) and \(L \in \mathbb{R}_0^+\), let \(W_L(r, s) \in \mathcal{B}(\mathcal{H}_L)\) be the partial isometry arising from the polar decomposition of \(1_{\mathcal{H}_L} - U_L(r, s)\), in such a way that we write

\[
\text{tr}_{\mathcal{H}_L} [1_{\mathcal{H}_L} - U_L(r, s)] = \sum_{\xi \in \mathcal{H}_L} \langle \epsilon_x, (W_L(r, s))^* (1_{\mathcal{H}_L} - U_L(r, s)) \epsilon_x \rangle_{\mathcal{H}_L} = \sum_{\xi \in \mathcal{H}_L} \langle \epsilon_x, (W_L(r, s))^* \left( V_r^{(L)} - V_s^{(L)} \right) \left( V_r^{(L)} \right)^* \epsilon_x \rangle_{\mathcal{H}_L},
\]

where we have used (54). Note that we can write \(V_r^{(L)} - V_s^{(L)} = -i \int_s^r \Omega_s^{(L)} V_q^{(L)} dq\). Then, by combining the explicit form of \(\Omega_s^{(L)}\) given by (39), Cauchy–Schwarz inequality, Corollary 6, and other simple arguments we arrive at

\[
\frac{1}{|\lambda_{\mathcal{H}_L}|} |\text{tr}_{\mathcal{H}_L} [1_{\mathcal{H}_L} - U_L(s, r)]| \leq D_{\text{Lem. 5}} |r - s| |\mathcal{G}| \int_{\mathbb{R}} |\mathcal{M}_h(t)| dt \sum_{x \in \mathcal{H}_L} e^{-\mu \inf \left\{ \inf_{\epsilon, r} \left\{ \frac{\epsilon}{\epsilon(\mu, H_r, r)} \right\}, \inf_{\epsilon, r} \left\{ \frac{\epsilon}{\epsilon(\mu, H_r, r)} \right\} \right\}} |x|^r.
\]

Then, \(1_{\mathcal{H}_L} - U_L(s, r)\) is trace–class per unit volume. Part (b) is straightforward from Corollary 2 applied for \(r, s \in \mathcal{C}\) and \(L \in \mathbb{R}_0^+ \cup \{\infty\}\). Part (c) follows from parts (a) and (b) and taking into account Corollary 3 and the uniformity of the determinants of Lemma 3: \(\det \left( V_s^{(L)} \right) = \det \left( V_r^{(L)} \right) = \pm 1\). \(\square\)

For any \(L \in \mathbb{R}_0^+, q \mathcal{E}^{(L, \infty)} \subset q \mathcal{E}^{(\infty)}\) denotes the local quasi–free ground states on \(\mathcal{A}_L \subset \mathcal{A}_\infty\). We postulate:

\(^{12}\)Recall that \(\mathcal{A}_\infty\) is the completeness of the normed \(\ast\)-algebra \(\mathcal{A}_L^{(0)}\) given by (48).
Theorem 5 (Gapped quasi–free ground states):
Take \( \mathcal{C} \equiv [0,1] \) and consider the family of self–dual Hamiltonians satisfying Assumption 1 (b). Fix \( L \in \mathbb{R}^+ \), and let \( \{ \omega_s^{(L)} \}_{s \in \mathcal{C}} \subset \mathbf{qH}^{(L,\infty)} \) be the family of gapped quasi–free ground states associated to the family of Hamiltonians \( \{ H_s^{(L)} \}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H}_L) \). Then,

1. \( \omega_s^{(L)} = \omega_0^{(L)} \circ \Upsilon_s^{(L)} \), for all \( s \in \mathcal{C} \), where \( \Upsilon_s^{(L)} \) is the finite–volume Bogoliubov \(^*-\)automorphism on \( \mathfrak{A}_L \) of Corollary 7.

2. Let \( \omega_s \in \mathbf{qH}^{(\infty)} \) be the weak\(^*\)–limit of \( \omega_s^{(L)} \in \mathbf{qH}^{(L,\infty)} \) and \( \Upsilon_s \) the infinite–volume Bogoliubov \(^*-\)automorphism on \( \mathfrak{A}_\infty \) associated to the sequence \( \Upsilon_s^{(L)} \) of Corollary 7. With respect to the weak\(^*\)–topology, the following three statements are equivalent:

   - (a) \( \lim_{L \to \infty} \omega_s^{(L)} = \omega_s \).
   - (b) \( \lim_{L \to \infty} \omega_s^{(L)} \circ \Upsilon_s = \omega_s \circ \Upsilon_s \).
   - (c) \( \lim_{L \to \infty} \omega_s^{(L)} \circ \Upsilon_s = \omega_s \circ \Upsilon_s \).

Proof. (1) follows from Corollary 4 and Lemma 4. (2) Fix \( s \in \mathcal{C} \). Note that the existence of the weak\(^*\)–limit \( \omega_s \) is consequence of Theorem 4 while the existence of the Bogoliubov \(^*-\)automorphism \( \Upsilon_s \) is a consequence of Corollary 7. Now, take any \( A \in \mathfrak{A}_\infty \) and note that (a) \( \Rightarrow \) (b) because

\[
\left| \omega_s^{(L)} \circ \Upsilon_s(A) - \omega_s \circ \Upsilon_s(A) \right| \leq \left| \omega_s^{(L)} - \omega_s \right| \| A \|_{\mathfrak{A}_\infty}.
\]

(b) \( \Rightarrow \) (c) follows by recognizing \( \omega_s^{(L)} \) and \( \omega_s \) as states and writing

\[
\left| \omega_s^{(L)} \circ \Upsilon_s^{(L)}(A) - \omega_s \circ \Upsilon_s(A) \right| \leq \left| \omega_s^{(L)} - \omega_s \right| \| A \|_{\mathfrak{A}_\infty} + \left| \omega_s^{(L)} \right| \left| \Upsilon_s^{(L)}(A) - \Upsilon_s(A) \right|_{\mathfrak{A}_\infty},
\]

and we have that the left hand side of last inequality is zero. Finally, we note that

\[
\left| \omega_s^{(L)}(A) - \omega_s(A) \right| \leq \left| \omega_s^{(L)} \circ \Upsilon_s^{(L)} - \omega_s \circ \Upsilon_s \right| \| A \|_{\mathfrak{A}_\infty} + \left| \omega_s \circ \Upsilon_s \right| \left| (\Upsilon_s^{(L)})^{-1}(A) - \Upsilon_s^{-1}(A) \right|_{\mathfrak{A}_\infty},
\]

and from Corollary 7, the right hand side of last inequality is zero, thus (c) \( \Rightarrow \) (a).

End

A Disordered models on general graphs

Consider the graph \( \mathfrak{G} \equiv \mathfrak{V} \times \mathfrak{E} \), where \( \mathfrak{V} \) is the so–called set of vertices and \( \mathfrak{E} \) is called set of edges. A graph has the following basic properties:

1. For any, \( v, w \in \mathfrak{V} \), and \( (v, w) \in \mathfrak{V} \times \mathfrak{V} \), \( v \) and \( w \) are called the endpoints of \( (v, w) \in \mathfrak{E} \).
2. For \( v, w \in \mathfrak{V} \), the vertices \( \mathfrak{E} \) set does not contain element of the form \( (v, v) \).
3. Unless otherwise indicated, the edges set is not–oriented: \( (v, w) \in \mathfrak{E} \) iff \( (w, v) \in \mathfrak{E} \).
4. For simplicity, the element \( g \in \mathfrak{G} \) is written as \( g \equiv (v, e) \) for some \( v \in \mathfrak{V} \) and \( e \in \mathfrak{E} \).
5. For \( e \in \{0,1\} \) and any \( v, w \in \mathfrak{V} \), one can endow with \( \mathfrak{G} \) of a pseudometric \( \delta_e : \mathfrak{G} \times \mathfrak{G} \to \mathbb{R}_0^+ \cup \{\infty\} \), that is, an equivalence relation satisfying the metric properties on \( \mathfrak{G} \), except that \( \delta_e(v, w) = 0 \) does not implies that \( v = w \). \( \delta_e \) is closely related to the size of the path with the minimum number of edges joining the vertices \( v \) and \( w \).
6. For \( \mathfrak{G} \), \( \mathcal{P}_f(\mathfrak{G}) \subset 2^\mathfrak{G} \) will denote the set of all finite subsets of \( \mathfrak{G} \).
We refer the reader to [LP17] for a complete discussion about graphs.

Take \( d \in \mathbb{N} \). Among the graphs that physicists consider, the \( d \)-dimensional cubic lattice or crystal \( \mathbb{Z}^d \) is taken as a subset of \( \mathbb{R}^d \) in the following way\(^\text{13}\):

\[
\mathbb{Z}^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d: x_j \in \mathbb{Z} \text{ for any } 1 \leq j \leq d\}.
\]

However, one can take more general assumptions by considering Cayley graphs, which are defined via the group \( \mathbb{G} \equiv (\mathbb{G}, \cdot) \) generated by the subset \( v \equiv (v, \cdot) \). Then, we associate to any element of \( \mathbb{G} \) a vertex of the Cayley graph \( \mathcal{G} \) and the set of edges is defined by

\[
\mathcal{E} \equiv \{(v, w) \in \mathbb{G}^2: v^{-1}w \in v\}.
\]

In the \( \mathbb{Z}^d \) case, the group \( \mathbb{G} \equiv (\mathbb{G}, +) \) is the so-called translation group.

From the physical point of view, mobility or confinement of particles embedded in a graph \( \mathcal{G} = \mathbb{G} \times \mathcal{E} \) will rely on the impurities of the material, crystal lattice defects (as in the \( \mathbb{Z}^d \) case), etc., which usually are modeled (in the simplest case) by random (one–site) external potentials on the set of vertices \( \mathbb{G} \) as follows: We take the probability space \( (\Omega, \mathcal{A}_\Omega, a_\Omega) \), where \( \Omega = [-1,1]^\mathbb{G} \). For any \( v \in \mathbb{G} \), \( \Omega_v \) is an arbitrary element of the Borel \( \sigma \)-algebra \( \mathcal{A}_v \) of the Borel set \([-1,1]\) w.r.t. the usual metric topology. Then, \( \mathcal{A}_\Omega \) is the \( \sigma \)-algebra generated by the cylinder sets \( \prod_{v \in \mathbb{G}} \Omega_v \), where \( \Omega_v = [-1,1] \) for all but finitely many \( v \in \mathbb{G} \). Additionally, we assume that the distribution \( a_\Omega \) is an arbitrary ergodic probability measure on the measurable space \( (\Omega, \mathcal{A}_\Omega) \). I.e., it is invariant under the action

\[
\rho \mapsto \chi^{(\Omega)}_v (\rho) = \chi^{(\Omega)}_v (\rho), \quad v \in \mathbb{G},
\]

of the group \( \mathbb{G} \equiv (\mathbb{G}, \cdot) \) on \( \Omega \) and \( a_\Omega (\mathcal{O}) \in \{0,1\} \) whenever \( \mathcal{O} \in \mathcal{A}_\Omega \) satisfies \( \chi^{(\Omega)}_v (\mathcal{O}) = \mathcal{O} \) for all \( v \in \mathbb{G} \). Here, for any \( \rho \in \Omega, v \in \mathbb{G} \) and \( w \in \mathbb{G} \)

\[
\chi^{(\mathcal{E})}_v (\rho) (w) = \rho (v^{-1}w).
\]

As is usual, \( \mathbb{E} [\cdot] \) denotes the expectation value associated with \( a_\Omega \).

For the Cayley graph \( \mathcal{G} = \mathbb{G} \times \mathcal{E}, \mathcal{H} = l^2(\mathcal{G}, \mathbb{C}) \) will denote a separable Hilbert space associated to \( \mathcal{G} \) with scalar product \( \langle \cdot, \cdot \rangle_\mathcal{H} \) and canonical orthonormal basis denoted by \( \{e_v\}_{v \in \mathbb{G}} \), which is defined by \( e_v (w) = \delta_{v^{-1}w,1_\mathbb{G}} \) for all \( v, w \in \mathbb{G} \), with \( \mathcal{H} \) the generator set of \( \mathbb{G} \) and \( 1_\mathbb{G} \) the unit on \( \mathbb{G} \). For any \( \rho \in \Omega \), one introduces the external potential \( V_\rho \in \mathcal{B}(\mathcal{H}) \) as the self–adjoint multiplication operator on \( V_\rho: \mathbb{G} \rightarrow [-1,1] \). On the other hand, one defines for the compact set \( C = [0,1] \), the family of graph Laplacians \( \{\Delta_{\mathcal{G},s}\}_{s \in C} \) defined for any \( s \in C \) by

\[
[\Delta_{\mathcal{G},s}(\psi)](v) = \text{deg}_{\mathcal{G}}(v) \psi(v) - s \sum_{p \in \mathbb{G}: d_{c}(v,w) = 1} \psi(v^{-1}w), \quad v \in \mathbb{G}, \psi \in \mathcal{H}
\]

where for any \( s \in [0,1], d_c: \mathbb{G} \times \mathcal{G} \rightarrow \mathbb{R}_+^* \cup \{\infty\} \) is a pseudometric on \( \mathcal{G} \). In (62), on the right hand side, \( \text{deg}_{\mathcal{G}}(v) \) is the number of nearest neighbors to vertex \( v \), or degree of \( v \). If \( \{\text{deg}_{\mathcal{G}}(v)\}_{v \in \mathbb{G}} \in \mathbb{N} \) is the same for all \( v \in \mathbb{G} \), we say that the graph is regular.

The random tight–binding (Anderson) model is the one–particle Hamiltonian defined by

\[
h^{(\rho)}_{\mathcal{G},s} \equiv \Delta_{\mathcal{G},s} + \lambda V_\rho, \quad \rho \in \Omega, \lambda \in \mathbb{R}_0^+.
\]

See [AW15] for further details. In [ABPR19], we consider a more general setting such that hopping disorder is present, i.e., we associate to particles a hopping probability on the non–oriented edges \( \mathcal{E} \). In this case, one deals with hopping amplitudes and the probability space \( (\Omega, \mathcal{A}_\Omega, a_\Omega) \) is properly implemented.

\(^{13}\)Because of its spatial symmetric properties: translations, rotations.
B Fermionic Fock space and parity of the vacuum vector

Let \((\mathcal{H}, \Gamma)\) be a self–dual Hilbert space as defined in Section 2.1 and take \(P \in p(\mathcal{H}, \Gamma)\), a basis projection, with range \(\text{ran}(P) = h_P\). For \(n \in \mathbb{N}_0\), let \(h_P^n = \mathbb{C}\) and for \(n \in \mathbb{N}\) define

\[
h_P^n = \text{lin}\{\varphi_1 \otimes \cdots \otimes \varphi_n : \varphi_1, \ldots, \varphi_n \in h_P\}.
\]

The set \(\varphi_1, \ldots, \varphi_n \in h_P\) denotes \(n\) state vectors of a single particle. Thus, the element \(\varphi_1 \otimes \cdots \otimes \varphi_n \in h_P^n\) associate the state of the particle 1 in the state \(\varphi_1\), the particle 2 in the state \(\varphi_2\), and so on [AJP06]. Then, the Fock (Hilbert) space is nothing but

\[
\mathcal{F}(h_P) = \bigoplus_{n \geq 0} h_P^n,
\]

where, as always, this infinite direct sum of Hilbert spaces is the subspace of the product space \(\bigoplus_{n=0}^{\infty} h_P^n\), with elements zero for all but for a finite number of these. An element \(\Upsilon \in \mathcal{F}(h_P)\) is the sequence of functions \(\{Y_n\}_{n \geq 0}\) such that \(Y_0 \in \mathbb{C}\) and \(Y_n \in h_P^n\) for \(n \in \mathbb{N}\) [RS81]:

\[
(64) \quad \Upsilon = \{Y_0, Y_1(\varphi_1^*), Y_2(\varphi_1^*, \varphi_2^*), \ldots\},
\]

with \(Y_n = \varphi_1 \otimes \cdots \otimes \varphi_n \in h_P^n\) and

\[
(65) \quad (\varphi_1 \otimes \cdots \otimes \varphi_n)(\varphi_1^*, \ldots, \varphi_n^*) = \varphi_1^*(\varphi_1) \cdots \varphi_n^*(\varphi_n), \quad \varphi_1^*, \ldots, \varphi_n^* \in h_P^*.
\]

Naturally, the inner product on \(\mathcal{F}(h_P)\) is given by

\[
(\Upsilon, \Phi)_{\mathcal{F}(h_P)} = \sum_{n \geq 0} (\varphi_n, \varphi_n)_{h_P^n}.
\]

for \(\Upsilon, \Phi \in \mathcal{F}(h_P)\). We then define the completely antisymmetric \(n\)-linear form \(\varphi_1 \wedge \cdots \wedge \varphi_n \in \Lambda^n h_P\) as

\[
(66) \quad \varphi_1 \wedge \cdots \wedge \varphi_n = \sum_{\pi \in \mathfrak{S}_n} \varepsilon_\pi \varphi_{\pi(1)} \otimes \cdots \otimes \varphi_{\pi(n)},
\]

where \(\mathfrak{S}_n\) denotes the set of all permutations of \(n \in \mathbb{N}\) elements, with \(\varepsilon_\pi\) equals +1 or −1 if the permutation is even or odd respectively. Note that, for any permutation \(\varepsilon_\pi\) of \(n \in \mathbb{N}\) elements we have

\[
\varphi_1 \wedge \cdots \wedge \varphi_n = \varepsilon_\pi \varphi_{\pi(1)} \wedge \cdots \wedge \varphi_{\pi(n)}, \quad \varphi_1, \ldots, \varphi_n \in h_P.
\]

For \(n \in \mathbb{N}_0\) we use that \(\Lambda^0 h_P = \mathbb{C}\) and for \(n \in \mathbb{N}\) we define

\[
\Lambda^n h_P = \text{lin}\{\varphi_1 \wedge \cdots \wedge \varphi_n : \varphi_1, \ldots, \varphi_n \in h_P\}.
\]

Note that by (65), (66) and the Leibniz formula for determinants we are able to write

\[
(\varphi_1 \wedge \cdots \wedge \varphi_n)(\varphi_1^*, \ldots, \varphi_n^*) = \det \left(\left(\varphi_i^*(\varphi_j)\right)_{ij}^n\right), \quad \varphi_1^*, \ldots, \varphi_n^* \in h_P^*.
\]

Then, for the Hilbert space \(h_P\), here we define the fermionic Fock space by

\[
\wedge h_P = \bigoplus_{n \geq 0} \Lambda^n h_P.
\]

Note that the subspace of \(\wedge h_P\) generated by monomials \(\varphi_1, \ldots, \varphi_n\) of even order \(n \in 2\mathbb{N}_0\) forms a commutative subalgebra, the even subalgebra of \(\wedge h_P\), and it is denoted by \(\wedge_e h_P\). Then, for \(H \in \mathcal{F}(h_P)\)
$\mathcal{B}(\mathcal{H})$ be a self–dual element, $H^* = -\Gamma H \Gamma$, and an orthonormal basis $\{\psi_j\}_{j \in J}$ of $\mathfrak{h}_P$, the bilinear element $\langle \mathfrak{h}_P, H \mathfrak{h}_P \rangle$ on the fermionic Fock space $\Lambda \mathfrak{h}_P$ is defined by

$$
\langle \mathfrak{h}_P, H \mathfrak{h}_P \rangle = \sum_{i,j \in J} \langle \psi_i, H \psi_j \rangle \mathcal{G}(\psi_j) \wedge \psi_i \in \Lambda^+ \mathfrak{h}_P,
$$

and hence we use the exponent function in $\Lambda \mathfrak{h}_P$

$$
e^\zeta = 1 + \sum_{k=1}^{2 \dim \mathfrak{h}_P} \frac{\zeta^k}{k!}, \quad \zeta \in \Lambda \mathfrak{h}_P,
$$

in order to define the Gaussian element $e^{(\mathfrak{h}_P, H \mathfrak{h}_P)} \in \Lambda^+ \mathfrak{h}_P$.

The vacuum vector denoted by $\Omega \in \Lambda \mathfrak{h}_P$ is such that $[\Omega]_0 \doteq 1 \in \mathfrak{h}_P^0$ and $[\Omega]_n \doteq 0 \in \mathfrak{h}_P^n$ for $n \geq 1$, thus, physically $\Omega$ is associated to the state $(1,0,0,\ldots)$ without fermions. See (64). The maps $a : \wedge^n \mathfrak{h}_P \to \wedge^{n-1} \mathfrak{h}_P$ and $a^* : \wedge^n \mathfrak{h}_P \to \wedge^{n+1} \mathfrak{h}_P$ are the so–called “annihilation” and “creation” operators, respectively. For $\varphi, \varphi_1, \ldots, \varphi_n \in \mathfrak{h}_P$ they are defined by

$$
a(\varphi)(\varphi_1 \wedge \cdots \wedge \varphi_n) = \sum_{k=1}^{n} (-1)^{k-1} \langle \varphi, \varphi_k \rangle_{\mathfrak{h}_P} \varphi_1 \wedge \cdots \wedge \hat{\varphi}_k \wedge \cdots \wedge \varphi_n,
$$

$$na^*(\varphi)(\varphi_1 \wedge \cdots \wedge \varphi_n) = \varphi \wedge \varphi_1 \wedge \cdots \wedge \varphi_n
$$

where the symbol $\hat{}$ means that the corresponding coordinate $\varphi_k$ was omitted. Hence, $a(\varphi)\Omega = 0$ and $a^*(\varphi)\Omega = \varphi$ for all $\varphi \in \mathfrak{h}_P$. Here, for $\varphi \in \mathfrak{h}_P$, the involution of $a(\varphi) \in \mathcal{B}(\wedge \mathfrak{h}_P)$, namely $a(\varphi)^* \in \mathcal{B}(\wedge \mathfrak{h}_P)$, is canonically identified with $a^*(\varphi)$, i.e., $a^*(\varphi) \equiv a(\varphi)^*$. Then for $n \in \mathbb{N}$, $P \in \mathfrak{p}(\mathcal{H}, \Gamma)$ and $\varphi_1, \ldots, \varphi_n \in \mathfrak{h}_P$ we can define the element of size $n$ in $\Lambda \mathfrak{h}_P$ by

$$\varphi_1 \wedge \cdots \wedge \varphi_n = a^*(\varphi_1) \cdots a^*(\varphi_n) \Omega \in \Lambda \mathfrak{h}_P.
$$

Additionally, we can show that the canonical anticommutation relations hold

$$a(\varphi_1)a^*(\varphi_2) + a^*(\varphi_2)a(\varphi_1) = \langle \varphi_1, \varphi_2 \rangle_{\mathfrak{h}_P} 1_{\Lambda \mathfrak{h}_P}, \quad a(\varphi_1)a(\varphi_2) + a(\varphi_2)a(\varphi_1) = 0.
$$

Hence, the family of operators $\{a(\varphi)\}_{\varphi \in \mathfrak{h}_P}$ and $1_{\Lambda \mathfrak{h}_P}$ generate a CAR $C^*$–algebra. By [BR03b, Theorem 5.2.5] there is an injective homomorphism between the self–dual CAR algebra $s\mathcal{CAR}(\mathcal{H}, \Gamma)$ and the space of bounded operators acting on the fermionic Fock space $\pi_P : s\mathcal{CAR}(\mathcal{H}, \Gamma) \to \mathcal{B}(\wedge \mathfrak{h}_P)$, which is Fock representation of the CAR algebra. In the finite dimension situation, this homomorphism is even a $^*$–isomorphism of $C^*$–algebras. Explicitly, for any $P \in \mathfrak{p}(\mathcal{H}, \Gamma)$ and $\varphi \in \mathcal{H}$ we have

$$\pi_P(B(\varphi)) = a(P \varphi) + a^*(\Gamma P \varphi), \quad \varphi \in \mathcal{H},
$$

c.f., Expression (4). The Fock state is canonically defined by

$$\omega_P(A) = \langle \Omega, \pi_P(A) \Omega \rangle_{\wedge \mathfrak{h}_P}, \quad A \in s\mathcal{CAR}(\mathcal{H}, \Gamma).
$$

Let now, $\{\psi_j\}_{j \in J}$ be an orthogonal basis of $\mathfrak{h}_P$, we define the element of size $|J|$ in $\Lambda \mathfrak{h}_P$ as

$$\Omega_P \doteq \psi_1 \wedge \cdots \wedge \psi_{|J|} = a^*(\psi_1) \cdots a^*(\psi_{|J|}) \Omega,
$$

where $\Omega$ is the vacuum vector in $\Lambda \mathfrak{h}_P$ while $a^*(\cdot)$ is the associated creation operator on $\Lambda \mathfrak{h}_P$, see Expressions (67). The GNS construction associated to $\Omega_P$ in (70) is written by

$$(\mathcal{H}_{\omega_P}, \pi_{\omega_P}, \Omega_{\omega_P}) \equiv (\mathcal{H}_{\omega_P}, \pi_{\omega_P}, \Omega_P),$$
with Fock state given by (c.f. (69))

$$\omega_P (A) = \langle \Omega_P, \pi_{\omega_P} (A) \Omega_P \rangle_{_{\mathcal{H}_P}}, \quad A \in \mathfrak{A},$$

with (see Equation (15) and comments around it) [EK98, Exer. 6.10]. Now, take the even and odd parts \(s\text{CAR}(\mathcal{H}, \Gamma)^\pm \subset \text{CAR}(\mathcal{H}, \Gamma)\) of the self–dual CAR \(C^*\)-algebra associated to the self–dual Hilbert space \((\mathcal{H}, \Gamma)\), see Expressions (7), and let \(\pi_p\) be the fermionic Fock representation associated to \(P\) given by (68). Note that \(\pi_p\) can be decomposed as two disjoint irreducible representations [Ara87]: \(\pi_p = \pi_p^+ \oplus \pi_p^\pm\), where \(\pi_p^\pm\) is the restriction of \(\omega_P\) to \(s\text{CAR}(\mathcal{H}, \Gamma)^\pm\), and coincides with the restriction of \(\pi_p(\mathfrak{A}_+\pm)\) to the closure \(\Lambda_\pm \mathfrak{h}_P\) of \(\pi_{\omega_P}(s\text{CAR}(\mathcal{H}, \Gamma)^\pm)\). In this way, \((\pi_{\omega_P})_\pm\) is identified with \(\pi_p^\pm\) or \(\pi_p^\mp\) depending if \(|J|\) in Expression (70) is even or odd [EK98].

Observe that we can study the parity of the vacuum vector \(\Omega\) by using Clifford algebra tools. In this framework, instead we need to consider orthogonal complex structures \(J \in \mathfrak{B}(\mathcal{H})\), that is, a linear endomorphism on \((\mathcal{H}, \Gamma)\) satisfying \(J^2 = -1\) and \(J^* = -J\), as well as use suitable isomorphisms between self–dual CAR–algebra \(s\text{CAR}(\mathcal{H}, \Gamma)\) and the Clifford algebra \(\mathcal{C}(\text{Re}(\mathcal{H}))\) generated by \(\text{Re}(\mathcal{H})\) [GBVF01]. Here, \(\text{Re}(\mathcal{H}) = \{ \varphi \in \mathcal{H} : \Gamma \varphi = \varphi \}\), and \(J = i(2P - 1)\text{Re}(\mathcal{H})\), for any \(P \in \mathfrak{P}(\mathcal{H}, \Gamma)\). We also endow to \(\mathcal{C}(\text{Re}(\mathcal{H}))\) with the inner product \(\langle \cdot, \cdot \rangle_{\text{Re}(\mathcal{H})}\), which is defined by a symmetric non–degenerated bilinear form \(S: \text{Re}(\mathcal{H}) \times \text{Re}(\mathcal{H}) \rightarrow \mathbb{R}\), so that \(J(\varphi_1, \varphi_2) \equiv \langle \varphi_1, \varphi_2 \rangle_{\text{Re}(\mathcal{H})}\) for any \(\varphi_1, \varphi_2 \in \text{Re}(\mathcal{H})\). In [CGRL18] the parity of \(\Omega\) was studied via orthogonal complex structures, and it is completely equivalent to that presented in this paper. In fact, one defines a \(\mathbb{Z}_2\) topological index via two orthogonal complex structures \(J_1, J_2\) as follows

$$\Sigma(J_1, J_2) \equiv (-1)^{\dim \ker(J_1 + J_2)},$$

which coincides with the one of Expression (25) [EK98].

## C CAR \(C^*\)-Algebra

Let \(\mathfrak{h}\) be a separable Hilbert space with \(\dim \mathfrak{h} \in \mathbb{N}_0\), and consider the direct sum

$$\mathcal{H} \equiv \mathfrak{h} \oplus \mathfrak{h}^*.$$

Compare with Equation (3). The scalar product on \(\mathcal{H}\) is

$$\langle \varphi, \bar{\varphi} \rangle_{\mathcal{H}} \equiv \langle \varphi_1, \bar{\varphi}_1 \rangle_{\mathfrak{h}} + \langle \varphi_2, \bar{\varphi}_2 \rangle_{\mathfrak{h}}^*, \quad \varphi \equiv (\varphi_1, \varphi_2), \varphi^* \equiv (\bar{\varphi}_1, \bar{\varphi}_2^*) \in \mathcal{H}.$$

Here, \(\varphi^*\) denotes the element of the dual \(\mathfrak{h}^*\) of the Hilbert space \(\mathfrak{h}\) which is related to \(\varphi\) via the Riesz representation. We define the canonical antunitary involution \(\Gamma\) of \(\mathcal{H}\) by

$$\Gamma(\varphi_1, \varphi_2^*) \equiv (\varphi_2, \varphi_1^*), \quad \varphi \equiv (\varphi_1, \varphi_2^*) \in \mathcal{H}.$$

Note that \(\varphi^* = \Gamma \varphi\) for any \(\varphi \in \mathfrak{h} \subset \mathcal{H}\). Then, the CAR algebra \(\text{CAR}(\mathfrak{h})\) and the self–dual CAR algebra \(s\text{CAR}(\mathcal{H}, \Gamma)\) are the same \(C^*\)-algebra, by defining

$$B(\varphi) \equiv B_{\mathfrak{h}}(\varphi) \equiv a(\varphi_1) + a(\varphi_2)^*, \quad \varphi = (\varphi_1, \varphi_2^*) \in \mathcal{H},$$

with \(B_{\mathfrak{h}} \in \mathfrak{B}(\mathcal{H})\) being the basis projection of \((\mathcal{H}, \Gamma)\) with range \(\mathfrak{h}\). See (1) and (4).

We now consider the CAR \(C^*\)-algebra generated by the identity \(1\) and the elements \(\{a(\psi)\}_{\psi \in \mathfrak{h}}\) satisfying the canonical anticommutation relations (CAR): For all \(\psi, \varphi \in \mathfrak{h},\)

$$a(\psi)a(\varphi) = -a(\varphi)a(\psi), \quad a(\psi)a(\varphi)^* + a(\varphi)^*a(\psi) = \langle \psi, \varphi \rangle_{\mathfrak{h}}1.$$
As is usual, \(a(\psi)\) and \(a(\psi)^*\) are called, respectively, annihilation and creation operators of a fermion in the state \(\varphi \in \mathfrak{h}\). One usually consider fermionic Hamiltonians of the form

\[
(72) \quad H = d\Phi(h) + d\Upsilon(g) + W, \quad W = W^* \in \text{CAR}(\mathfrak{h}),
\]

where, for any \(h = h^* \in \mathscr{B}(\mathfrak{h})\) and antilinear operator \(g = -g^*\) on \(\mathfrak{h}\),

\[
d\Phi(h) = \sum_{i,j \in J} \langle \psi_i, h \psi_j \rangle_\mathfrak{h} a(\psi_i)^* a(\psi_j),
\]

\[
d\Upsilon(g) = \frac{1}{2} \sum_{i,j \in J} \left( \langle \psi_i, g \psi_j \rangle_\mathfrak{h} a(\psi_i)^* a(\psi_j)^* + \overline{\langle \psi_i, g \psi_j \rangle_\mathfrak{h}} a(\psi_j) a(\psi_i) \right).
\]

Here, \(\{\psi_j\}_{j \in J}\) is any orthonormal basis of \(\mathfrak{h}\). In (72), \(d\Phi(h) = d\Phi(h)^* \in \text{CAR}(\mathfrak{h})\) is the second quantization of the one–particle Hamiltonian \(h \in \mathscr{B}(\mathfrak{h})\). It represents a gauge invariant model of free fermions. On the other hand, \(d\Upsilon(g) = d\Upsilon(g)^* \in \text{CAR}(\mathfrak{h})\) represents the non–gauge invariant quadratic part of \(H\). Finally, \(W\) encodes the interparticle interaction of the fermion system.

Let now \(P \in \mathfrak{p}(\mathscr{H}, \Gamma)\) be a basis projection with \(\text{ran}(P) = \mathfrak{h}_P\). For any \(h = h^* \in \mathscr{B}(\mathfrak{h}_P)\) and antilinear operator \(g = -g^*\) on \(\mathfrak{h}_P\) are defined the maps \(\varkappa\) and \(\varkappa^*\) by:

\[
\varkappa(h) = \frac{1}{2} (P h P - \Gamma P h \Gamma) \in \mathfrak{B}(\mathfrak{h}_P) \quad \text{and} \quad \varkappa^*(g) = \frac{1}{2} (P g P \Gamma - \Gamma P g P) \in \mathfrak{B}(\mathfrak{h}_P).
\]

Observe that

\[
\varkappa(h)^* = \varkappa(h) = -\Gamma \varkappa(h) \Gamma \quad \text{and} \quad \varkappa^*(g)^* = \varkappa^*(g) = -\Gamma \varkappa^*(g) \Gamma,
\]

thus \(\varkappa\) and \(\varkappa^*\) provide self–dual Hamiltonians, see Definition 4. By (71) we can write

\[
d\Phi(h) + d\Upsilon(g) = -\langle B, [\varkappa(h) + \varkappa^*(g)] B \rangle + \frac{1}{2} \text{tr}_{\mathfrak{h}_P}(h) 1 \in \text{CAR}(\mathfrak{h}_P),
\]

that is, a bilinear element of a self–dual Hamiltonian plus a constant term. For further details see [ABPM21, Sects. 1 and 6.]. For \(W = 0\) in (72), the dynamics is provided by the continuous group \(\{\tau_t\}_{t \in \mathbb{R}}\) of \(*\)–automorphisms of \(\text{CAR}(\mathfrak{h}_P)\) defined by

\[
\tau_t(A) = e^{itH} A e^{-itH}, \quad A \in \text{CAR}(\mathfrak{h}_P),
\]

which, by above comments, can be written as

\[
\tau_t(A) = e^{it(d\Phi(h) + d\Upsilon(g))} A e^{-it(d\Phi(h) + d\Upsilon(g))} = e^{-it \langle B, [\varkappa(h) + \varkappa^*(g)] B \rangle} A e^{it \langle B, [\varkappa(h) + \varkappa^*(g)] B \rangle}.
\]

The latter equation provides exactly the quasi–free dynamics given by Equation (10).

Now, instead of considering \(H\), observe that for any basis projection \(P \in \mathfrak{p}(\mathscr{H}, \Gamma)\) equivalently

\[
(73) \quad - \langle B, [F + G] B \rangle + \text{tr}_{\mathfrak{h}_P}(P F P) 1 \in \text{CAR}(\mathfrak{h}_P),
\]

gives us the description of all free–fermion systems, where \(F \in \mathscr{B}(\mathfrak{h}_P)\) and \(G \in \mathscr{L}(\mathfrak{h}_P)\) are self–dual Hamiltonians on \(\mathscr{H}\). As mentioned in (9), \(F_P \doteq 2P F P\) is the so–called one–particle Hamiltonian, then, w.l.o.g. we can remove the term \(\text{tr}_{\mathfrak{h}_P}(P F P) 1\), by writing (73) as

\[
- \langle B, [\bar{F} + G] B \rangle \in \text{CAR}(\mathfrak{h}_P),
\]

for \(\bar{F} \doteq F - \frac{1}{|I|} \text{tr}_{\mathfrak{h}_P}(P F P) \varkappa(1_{\mathfrak{h}_P})\), with \(|I|\) the cardinality of the Hilbert space \(\mathscr{H}\). For \(H \doteq \bar{F} + G\), a self–dual Hamiltonian, define \(h \doteq 2P_h H P_h\) and \(g \doteq 2P_h H P_h\), in order to describe any quadratic
Fermionic Hamiltonian. In fact, given $P \in p(\mathcal{H}, \Gamma)$ with ran$(P) = \mathfrak{h}_P$ and the self–dual Hamiltonian $H \in \mathcal{B}(\mathcal{H})$ the operators

$$h = 2PHP \in \mathcal{B}(\mathfrak{h}_P) \quad \text{and} \quad g = 2PHP\Gamma \in \mathcal{L}(\mathcal{H}),$$

provide all the possible free–fermion models.

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References

[ABPM21] N. J. B. Aza, J.-B. Bru, de Siqueira W. Pedra, and L. C. P. A. M. Müssnich, Large Deviations in Weakly Interacting Fermions: Generating Functions as Gaussian Berezin Integrals and Bounds on Large Pfaffians., Reviews in Mathematical Physics (2021).

[ABPR19] N. J. B. Aza, J–B Bru, de Siqueira W. Pedra, and A Ratsimanetrimanana, Accuracy of Classical Conductivity Theory at Atomic Scales for Free Fermions in Disordered Media, Journal de Mathématiques Pures et Appliquées (2019).

[AE83] H. Araki and D. E. Evans, On a $C^*$–algebra approach to phase transition in the two–dimensional Ising model, Communications in Mathematical Physics 91 (1983), no. 4, 489–503.

[AG98] M. Aizenman and G. M. Graf, Localization bounds for an electron gas, Journal of Physics A: Mathematical and General 31 (1998), no. 32, 6783.

[AJP06] S. Attal, A. Joye, and C.A. Pillet, Open Quantum Systems I: The Hamiltonian Approach, Lecture Notes in Mathematics, Springer, 2006.

[AM03] H. Araki and H. Moriya, Equilibrium Statistical Mechanics of Fermion Lattice Systems, Reviews in Mathematical Physics 15 (2003), no. 02, 93–198.

[AMR] N. J. B. Aza, L. C. P. A. M. Müssnich, and A. F. Reyes-Legá, A $\mathbb{Z}_2$ Topological Index for Interacting Fermion Systems, To appear.

[Ara68] H. Araki, On the diagonalization of a bilinear Hamiltonian by a Bogoliubov transformation, Publications of the Research Institute for Mathematical Sciences, Kyoto University. Ser. A 4 (1968), no. 2, 387–412.

[Ara71] _____, On quasifree states of CAR and Bogoliubov automorphisms, Publications of the Research Institute for Mathematical Sciences 6 (1971), no. 3, 385–442.

[Ara87] _____, Bogoliubov Automorphisms and Fock Representations of Canonical Anticommutation Relations, Contemp. Math 62 (1987), 23–141.

[Ara88] _____, Schwinger terms and cyclic cohomology, Quantum Theories and Geometry, Springer, 1988, p. 1–22.

[AT85] H. Araki and Matsui T., Ground states of the XY–model, Communications in Mathematical Physics 101 (1985), no. 2, 213–245.
[AW15] M. Aizenman and S. Warzel, Random operators: Disorder effects on quantum spectra and dynamics, volume 168 of, Graduate Studies in Mathematics (2015).

[AZ97] A. Altland and M. R. Zirnbauer, Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures, Physical Review B 55 (1997), no. 2, 1142–1161.

[BCR16] C. Bourne, A. L. Carey, and A. Rennie, A non–commutative framework for topological insulators, Reviews in Mathematical Physics 28 (2016), no. 02, 1650004.

[BDF18] S. Bachmann, W. De Roeck, and M. Fraas, The adiabatic theorem and linear response theory for extended quantum systems, Communications in Mathematical Physics (2018), no. 3, 997–1027.

[BMNS12] S. Bachmann, S. Michalakis, B. Nachtergaele, and R. Sims, Automorphic equivalence within gapped phases of quantum lattice systems, Communications in Mathematical Physics 309 (2012), no. 3, 835–871.

[BP13] J–B Bru and de Siqueira W Pedra, Non–cooperative equilibria of Fermi systems with long range interactions, vol. 224, American Mathematical Soc., 2013.

[BP16a] ______, Lieb–Robinson Bounds for Multi–Commutators and Applications to Response Theory, Springer 13 (2016).

[BP16b] ______, Universal Bounds for Large Determinants from Non Commutative Hölder Inequalities in Fermionic Constructive Quantum Field Theory, Preprint: mp arc 16-16 (2016).

[BPH14] J–B Bru, W de Siqueira Pedra, and C. Hertling, Heat production of Noninteracting fermions subjected to electric fields, Communications on Pure and Applied Mathematics (2014).

[BRO3a] O. Bratteli and D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics 1: C*– and W*-Algebras. Symmetry Groups. Decomposition of States, 2 ed., Operator Algebras and Quantum Statistical Mechanics, Springer, 2003.

[BRO3b] ______, Operator Algebras and Quantum Statistical Mechanics: Equilibrium States. Models in Quantum Statistical Mechanics 2: Equilibrium States. Models in Quantum Statistical Mechanics, Springer, 2003.

[BSB20] C. Bourne and H. Schulz-Baldes, On $\mathbb{Z}_2$-indices for ground states of fermionic chains, Reviews in Mathematical Physics (2020), no. 0, 2050028.

[BO21] C. Bourne and Y. Ogata, The classification of symmetry protected topological phases of one-dimensional fermion systems, Forum of Mathematics, Sigma, vol 9, Cambridge University Press, p. e25. (2021).

[BvES94] J. Bellissard, A. van Elst, and H. Schulz–Baldes, The noncommutative geometry of the quantum Hall effect, Journal of Mathematical Physics 35 (1994), no. 10, 5373–5451.

[CGRL18] J. S. Calderón-García and A. F. Reyes-Lega, Majorana fermions and orthogonal complex structures, Modern Physics Letters A (2018), no. 14, 1840001.

[CHM+06] A. L. Carey, K. C. Hannabuss, V. Mathai, et al., Quantum Hall effect and noncommutative geometry, Journal of Geometry and Symmetry in Physics 6 (2006), 16–37.
[Cho] G. Choquet, Academic Press.

[CNN18] M. Cha, P. Naaijkens, and B. Nachtergaele, The complete set of infinite volume ground states for Kitaev’s abelian quantum double models, Communications in Mathematical Physics 357 (2018), no. 1, 125–157.

[DS19] W. De Roeck and M. Salmhofer, Persistence of exponential decay and spectral gaps for interacting fermions, Communications in Mathematical Physics 365 (2019), no. 2, 773–796.

[Dys62] F. J. Dyson, The Threefold Way. Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics, Journal of Mathematical Physics 3 (1962), no. 6, 1199–1215.

[EBN+06] K.J. Engel, S. Brendle, R. Nagel, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, et al., One–Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics, Springer New York, 2006.

[EK98] D.E. Evans and Y. Kawahigashi, Quantum symmetries on operator algebras, Oxford Mathematical Monographs, ISSN 0964-9174, Clarendon Press, 1998.

[FMP16] D. Fiorenza, D. Monaco, and G. Panati, Z(2) Invariants of Topological Insulators as Geometric Obstructions, Communications in Mathematical Physics (2016), no. 3, 1115–1157.

[GBVF01] J. G. Gracia-Bondía, J. C. Várilly, and H. Figueroa, Elements of Noncommutative Geometry, Birkhäuser advanced texts. Basler Lehrbücher, Birkhäuser, 2001.

[GJ16] A. Giuliani and I. Jauslin, The ground state construction of bilayer graphene, Reviews in Mathematical Physics 28 (2016), no. 08, 1650018.

[GMP16] A. Giuliani, V. Mastropietro, and M. Porta, Universality of the Hall Conductivity in Interacting Electron Systems, Communications in Mathematical Physics (2016), 1–55.

[Has19] M. B. Hastings, The stability of free Fermi Hamiltonians, Journal of Mathematical Physics 60 (2019), no. 4, 042201.

[HL11] M. B. Hastings and T. A. Loring, Topological insulators and C*-algebras: Theory and numerical practice, Annals of Physics (2011), no. 7, 1699–1759, July 2011 Special Issue.

[Kat13] T. Kato, Perturbation Theory for Linear Operators, vol. 132, Springer Science & Business Media, 2013.

[Kit01] A. Y. Kitaev, Unpaired Majorana fermions in quantum wires, Physics-Uspekhi (2001), no. 10S, 131.

[Kit09] A. Kitaev, Periodic table for topological insulators and superconductors, American Institute of Physics Conference Series 1134 (2009), 22–30.

[KK18] H. Katsura and T. Koma, The noncommutative index theorem and the periodic table for disordered topological insulators and superconductors, Journal of Mathematical Physics (2018), no. 3, 031903.

[LH10] T. A. Loring and M. B. Hastings, Disordered topological insulators via C*-algebras, EPL (Europhysics Letters) (2010), no. 6, 67004.
[LP17] R. Lyons and Y. Peres, *Probability on Trees and Networks*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 2017.

[Mat13] T. Matsui, *Boundedness of entanglement entropy and split property of quantum spin chains*, Reviews in Mathematical Physics 25 (2013), no. 09, 1350017.

[Mat20] T. Matsui, *Split Property and Fermionic String Order*, arXiv preprint arXiv:2003.13778 (2020).

[MZ13] S. Michalakis and J. P. Zwolak, *Stability of Frustration-Free Hamiltonians*, Communications in Mathematical Physics 322 (2013), no. 2, 277–302.

[NSY13] B. Nachtergaele, R. Sims, and A. Young, *Lieb-Robinson bounds, the spectral flow, and stability of the spectral gap for lattice fermion systems*, Mathematical Problems in Quantum Physics 117 (2018), 93.

[NSY18a] ______, *Quasi–Locality Bounds for Quantum Lattice Systems. Part I. Lieb–Robinson Bounds, Quasi–Local Maps, and Spectral Flow Automorphisms*, arXiv preprint arXiv:1810.02428 (2018).

[Oga20] Y. Ogata, *A Z(2)-Index of Symmetry Protected Topological Phases with Time Reversal Symmetry for Quantum Spin Chains*, Communications in Mathematical Physics 374 (2020), no. 2, 705–734.

[PS16] E. Prodan and H. Schulz-Baldes, *Bulk and Boundary Invariants for Complex Topological Insulators*, Springer, 2016.

[RS81] M. Reed and B. Simon, *I: Functional Analysis*, Methods of Modern Mathematical Physics, Elsevier Science, 1981.

[RSFL10] Shinsei Ryu, Andreas P Schnyder, Akira Furusaki, and Andreas W W Ludwig, *Topological insulators and superconductors: tenfold way and dimensional hierarchy*, New Journal of Physics 12 (2010), no. 6, 065010.

[Rud91] W. Rudin, *Functional Analysis*, International series in pure and applied mathematics, McGraw–Hill, 1991.

[TKNdN82] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and Md. den Nijs, *Quantized Hall conductance in a two–dimensional periodic potential*, Physical Review Letters 49 (1982), no. 6, 405.