Every continuum has a compact universal cover

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Abstract

We define the compact universal cover of a compact, metrizable connected space (i.e., a continuum) $X$ to be the inverse limit of all continua that regularly cover $X$. We show that such covers do indeed form an inverse system with bonding maps that are regular covering maps, and the projection $\hat{\phi} : \hat{X} \to X = \hat{X}/\pi_P(X)$ of the inverse limit is a generalized regular covering map, where $\hat{X}$ is a continuum that is “compactly simply connected” in the sense that $\pi_P(\hat{X}) = 1$. We call $\pi_P(X)$ the “profinite fundamental group” of $X$. We prove a Galois Correspondence for closed normal subgroups of $\pi_P(X)$, uniqueness, universal and lifting properties. As an application we prove that every non-compact manifold that regularly covers a compact manifold has a unique “profinite compactification”, i.e. an imbedding as a dense subset in a compactly simply connected continuum. As part of the proof of metrizability of $\hat{X}$ we show that every continuum has at most $n!$ non-equivalent $n$-fold covers by continua.

1 Introduction

Efforts to generalize the powerful tools of covering space theory beyond Poincaré spaces (connected, locally path connected and semilocally simply connected spaces) go back at least to the work of Kawada in 1950 ([15]). Kawada defined a notion of generalized covering map for topological groups to be a homomorphism having conditions, including totally disconnected kernel, that permit covering maps that are not locally 1-1. Note that his definition was too weak because he did not require that the kernel be a complete topological group, and Berestovskii-Plaut showed in [1] that Kawada’s uniqueness statement for generalized universal covers is incorrect. The situation is remedied in [1] by in effect defining generalized covering homomorphisms to be inverse limits of regular covering homomorphisms in the traditional sense, so that the kernel of the resulting map is prodiscrete and hence complete. This led to the theorem
that every compact, connected, metrizable group has a unique generalized universal cover by a compact group (2), which is a product of connected, simply connected, compact simple Lie groups and universal solenoids. This compact universal covering group is in fact the inverse limit of all traditional covering homomorphisms of the group by finite groups, and therefore [2] is a kind of precursor to the present paper.

Efforts that do not involve the extra structure of a topological group go back at least to Fox in the early 1970's ([7]), in which he strengthened the notion of covering map to what he called an overlay. Yet finding a universal cover in greater generality certainly requires weakening the notion of covering map in natural ways that retain the most significant properties of covering maps: lifting properties; a universal property in a categorical sense (and hence uniqueness); the fact that the universal covering map map is the quotient via a group action by a (topological) group that generalizes the topologically discrete traditional fundamental group; and a Galois correspondence between (topologically closed) normal subgroups of that group and generalized regular covering maps of the original space.

Two independent efforts proving the existence of generalized universal covering maps were published in the same year by Fischer-Zastrow ([6]) and Berestovskii-Plaut ([3]). The results of [6] require that the space in question be path and locally path connected, whereas the results of [3] are applicable to many spaces without these properties. Those spaces include the Warsaw Circle and even some totally disconnected spaces, but not, for example, solenoids.

The first (dyadic) solenoid was introduced by Vietoris in 1927 ([23]) and these objects were extensively generalized by McCord in 1965 ([17]). McCord introduced solenoidal spaces as inverse limits of traditional regular coverings of a Poincaré space parameterized by the natural numbers \( \mathbb{N} \), and showed that the resulting projection map is a principle fiber bundle. Solenoidal spaces gained new prominence as a result of the work of Smale, who showed that solenoids appear as expanding attractors in dynamical systems in 1967 ([22]). Solenoidal spaces and their generalizations (e.g. matchbox manifolds) continue to play a significant role in dynamical systems—see for example [10] and [11]. Given their importance, a generalized covering space theory that includes solenoidal spaces is desirable.

In this paper we define the compact universal cover \( \phi : \hat{X} \to X \) of any continuum (compact, connected, metrizable space) \( X \) to be the projection from the inverse limit \( \hat{X} \) of all continua \( Y \) such that there is a traditional regular covering map \( f : Y \to X \). We call such an \( f \) a cover by a continuum. This construction fits into a broader framework of generalized regular covering maps between uniform spaces ([18]), which we show in the present paper is (unlike traditional regular covering maps between topological spaces), a category (Theorem 52). We will give more precise details later, but a generalized regular covering map (Definition 45) between uniform spaces is the quotient map of a prodiscrete and isomorphic action of a group of uniform homeomorphisms, and in particular it is generally not a local homeomorphism. Equivalently, a generalized regular
covering map resolves as an inverse system of discrete covers, which are the uniform analog of traditional regular covering maps.

To simplify our notation we will denote regular covers of a continuum \( X \) by continua by \( f_i : X_i \to X/G_i \). Of course it is possible (e.g. with the circle) that for distinct \( f_i, f_j, X_i \) is homeomorphic to \( X_j \), but by assumption if \( i \neq j \) then \( f_i \) and \( f_j \) are not equivalent.

Basepoints play an important role in many of our proofs, but in the end our results are basepoint independent up to equivalence. Therefore we will avoid the cumbersome “pointed” notation. When basepoints are involved we will always denote them by \( * \), and unless otherwise stated, all maps are assumed to be basepoint preserving. This approach also naturally leads to stating uniqueness of maps in terms of equivalence classes relative to an action rather than involving basepoints. For example, suppose that we have quotients via actions of bijections \( \pi_1 : X_1 \to X = X_1/G_1 \) and \( \pi_2 : X_2 \to X = X_2/G_2 \) and a function \( f : X_1 \to X_2 \) with some property (e.g. uniformly continuous) such that \( \pi_2 \circ f = \pi_1 \). When we say \( f \) is the unique (mod \( G_2 \)) function with that property such that \( \pi_2 \circ f = \pi_1 \), we mean that the set of functions with those properties is precisely the set of functions of the form \( g \circ f \), where \( g \in G_2 \).

**Definition 1** Given two covers by continua \( f_i : X_i \to X \) and \( f_j : X_j \to X \), we say that \( f_i \leq f_j \) if there is a continuous onto map \( f_{ij} : X_j \to X_i \) such that \( f_j = f_i \circ f_{ij} \).

**Theorem 2 (Existence)** For any continuum \( X \),

1. The set of all covers \( f_i : X_i \to X_i/G_i \) is a directed set.

2. For all \( i, j \) there are unique (mod \( G_i \)) regular covering maps \( f_{ij} : X_j \to X_i \) that are the bonding maps for the inverse system, and compatible homomorphisms \( \theta_{ij} : G_j \to G_i \), i.e. for all \( i, j \) and \( g \in G_j \), \( \theta_{ij}(g) \circ f_{ij} = f_{ij} \circ g \).

3. The projection \( \hat{\phi} : \hat{X} = \lim_{\leftarrow} X_i \to X = \hat{X}/\pi_P(X) \) is a generalized regular covering map with profinite deck group \( \pi_P(X) = \lim_{\leftarrow} G_i \), and \( \hat{X} \) is a continuum.

We call \( \pi_P(X) \) the profinite fundamental group of \( X \). When \( \pi_P(X) = 1 \) we say that \( X \) is compactly simply connected. Note that for a compact manifold \( M \), \( \hat{M} \) is by definition a “weak solenoid”–see [12]. For example, when \( X \) is the circle, \( \hat{X} \) is by definition the so-called universal solenoid.

**Theorem 3 (Galois Correspondence)** If \( f : Y \to X = Y/G \) is a generalized regular covering map by a continuum then \( f \) is equivalent to the induced quotient \( \pi : \hat{X}/K \to (\hat{X}/K)/\langle \pi_P(X)/K \rangle \) for some closed normal subgroup \( K \) of \( \pi_P(X) \). Conversely any such induced quotient for a closed normal subgroup of \( \pi_P(X) \) is a generalized regular covering map of \( X \) by a continuum.
Theorem 4 (Universal Property) If \( f : Y \to X = Y/G \) is a generalized regular covering map by a continuum then there is a unique (mod \( G \)) generalized regular covering map \( f_L : \tilde{X} \to Y \) called the lift of \( f \) such that \( \tilde{f} = f \circ f_L \).

Theorem 5 (Uniqueness) \( \tilde{X} \) is compactly simply connected, and if \( f : Y \to X = Y/G \) is a generalized universal cover by a compactly simply connected continuum \( Y \), then \( f \) is equivalent to \( \tilde{f} \).

Corollary 6 If \( f : Y \to X = Y/G \) is a generalized regular covering map by a continuum then there is a unique (mod \( \pi_1(\tilde{X}) \)) homeomorphism \( h : \tilde{X} \to \tilde{Y} \) such that \( \tilde{\phi}_X = f \circ \tilde{\phi}_Y \circ h \), where \( \tilde{\phi}_X \) and \( \tilde{\phi}_Y \) are the compact universal covering maps of \( X \) and \( Y \), respectively.

When \( X \) is a compact metrizable Poincaré space, following (17), the universal cover \( \tilde{X} \) embeds as a dense subgroup of what we are now calling \( \tilde{X} \) (see also Theorem 46). More generally, given any injective homomorphism \( \eta \) of \( \pi_1(X) \) into a profinite group \( P = \prod \limits_{i \in I} P_i \), we obtain a generalized regular covering map \( f : Y \to X = Y/P \) by taking the compact covers corresponding to the kernels of \( \eta_i := \pi^1 \circ \eta \), where \( \pi^1 \) is the projection onto the finite group \( P_i \). However, Theorem 5 gives us something new: this embedding of \( \tilde{X} \) into a compactly simply connected continuum is independent of the particular space \( X \) that \( \tilde{X} \) covers.

Corollary 7 (Profinite Compactification) Suppose that \( Y \) is a metrizable simply connected Poincaré space such that there is a regular covering map \( f : Y \to X \) for some compact space \( X \). Then \( Y \) embeds as a dense subgroup in a unique compactly simply connected continuum \( \tilde{Y} \).

Theorem 8 Let \( f : Z \to X \) be a continuous function such that \( X \) is a continuum and \( Z \) is a simply connected Poincaré space. Then there is a unique (mod \( \pi_1(X) \)) uniformly continuous function \( f_L : Z \to \tilde{X} \) called the lift of \( f \) such that \( f = \tilde{f} \circ f_L \).

The preceding theorem simply follows from the traditional lifting property of regular covers and the universal property of the inverse limit. This result leads to a homomorphism from \( \pi_1(X) \) into the profinite group \( \pi_1(X) \), which for Peano continua has dense image.

We will now sketch out our proofs. The first problem is to show that covers by continua form a directed set. This problem has a classical solution when \( X \) is a Poincaré space, and hence has a traditional universal cover. In this case one has the traditional Galois Correspondence between regular covering maps of a Poincaré space and induced quotients via normal subgroups of the fundamental group—which in the case of compact covers have finite index. Therefore the partial order of regular covering maps corresponds to reverse inclusion of normal subgroups of \( \pi_1(X) \). Since the intersection of any two normal subgroups of finite index is also a normal subgroup of finite index, the corresponding covering spaces form a directed set. Note that the proof of the latter fact depends on
the existence of a traditional universal cover, but the resolution of generalized regular covering maps as inverse limits of discrete covers shows that they are always principle fibrations (but we have no present use for this observation—being a generalized regular covering map is much stronger).

For a general continuum there is no suitable analog of the classical universal cover; instead, we first prove (Theorem 41) a version of Galois Correspondence for the map $\phi_E : X_E \to X$ introduced by Berestovskii-Plaut in [3], where $X$ is a uniform space and $E$ is an entourage in $X$. We will recall some basics about uniform spaces later, but roughly speaking an entourage is a symmetric neighborhood of the diagonal in $X \times X$ that allows one to uniformly measure “closeness” in the space. When $X$ is compact there is a unique uniform structure compatible with the topology and $E$ may be taken to be any symmetric neighborhood of the diagonal in $X \times X$. The map $\phi_E : X_E \to X$, defined in [3] using discrete homotopy theory, is a kind of “universal covering map at the scale of $E$”. It is a quotient map via a discrete group $\pi(E)(X)$ that is a kind of “fundamental group at the scale of $E$”. The set of all $X_E$ with natural bonding maps $\phi_{EF} : X_F \to X_E$ when the entourage $F$ is contained in $E$, is called the fundamental inverse system of $X$. The projection $\phi : \tilde{X} \to X$ of the inverse limit of this system is, for suitably nice spaces (including Peano continua), a generalized regular covering map called the uniform universal cover ([3], [21]). However, this construction fails to produce a surjective projection, hence a generalized regular covering map, for more exotic continua such as solenoids.

In this paper we show that if $f : Y \to X$ is a cover by a continuum, all sufficiently small entourages $E$ in $X$ are “evenly covered” in a natural sense, and for such $E$, $f$ is equivalent to the induced regular covering map $\pi : X_E/K \to (X_E/K)/(\pi_E(X))/K$ for some uniquely determined normal subgroup $K$ of finite index in $\pi_E(X)$ (Theorem 41). However, there are complications: $X_E$ may be neither compact nor connected— in fact when $X$ is a solenoid, $X_E$ may have uncountably many components. The fact that $X_E$ is not compact plunges us firmly into the world of uniform spaces and requires development of a generalized covering space theory in this broader context—which as mentioned above also has the advantage of being a category. In order to tie these results to the traditional covering space theory for metrizable spaces, we show that given a traditional regular covering map between metrizable spaces, the spaces can be “uniformized” so that the covering maps is a discrete cover of the resulting uniform spaces (Theorem 30). In this way, the results of this paper are applicable more generally to metrizable spaces. To see why this is not a contradiction to the fact that the composition of traditional regular covering maps may not be a regular covering map, see Example 53. A key tool in this paper is the General Chain Lifting Lemma (Lemma 34), the discrete analog of the path and homotopy lifting properties from the traditional theory.

As for lack of connectedness, the first problem is that the traditional notion of equivalence of regular covering maps is too weak for covers by spaces that may not be connected. Instead we use a very natural notion of equivalence of uniform group actions (Definition 15), which is stronger in general but is
equivalent to the traditional definition for generalized regular covering maps between connected spaces, see Example 16 and Corollary 50. The maps \( \phi_E : X_E \to X \) are also what we call “proper discrete covers” (Definition 38) which have a weaker connectivity property that is sufficient to apply the General Chain Lifting Lemma. Finally, we show that when \( K \) is a normal subgroup of \( \pi_E(X) \) of finite index, \( X_E/K \) is compact with finitely many components, and one may then restrict to any component to obtain a regular covering map by a continuum (Proposition 36).

There remains one final complication, to find a countable cofinite inverse system. Doing so allows us to both show that \( \hat{\phi} : \hat{X} \to X \) is surjective, hence a generalized regular covering map, and show that \( \hat{X} \) is metrizable (classical results on inverse systems show that \( \hat{X} \) is compact and connected). The existence of this system is a consequence of the theorem below, the proof of which requires some delicate arguments involving discrete homotopies (see the proof of Theorem 43):

**Theorem 9** If \( X \) is a continuum then there are at most \( n! \) (non-equivalent) regular \( n \)-fold covering maps of \( X \) by continua.

In this context we note that Gumerov \([8]\) gave necessary and sufficient conditions for existence of an \( n \)-fold covering map between solenoids.

## 2 Uniform spaces and discrete homotopy theory

We will use the following notation in this paper if \( W \subset X \times X \) is any symmetric subset containing the diagonal. The \( W \)-ball in \( X \) at \( x \) is \( B(x, W) = \{ y \in X : (x, y) \in W \} \). In a metric space we have the traditional metric balls \( B(x, \varepsilon) = B(x, E_{\varepsilon}) \), where \( E_{\varepsilon} := \{ (x, y) : d(x, y) < \varepsilon \} \) is the “metric entourage” for \( \varepsilon \). Recall that a uniform space consists of a topological space \( X \) together with a collection of symmetric neighborhoods of the diagonal in \( X \times X \) called *entourages*. The set of entourages is called the uniform structure or uniformity of \( X \). The characterizing properties are the “triangle inequality”: for every entourage \( F \) there is an entourage \( E \) such that \( E^2 \subset F \); along with the fact that every symmetric set containing an entourage is an entourage. The uniformity is compatible with the topology if whenever \( U \) is open in \( X \) and \( x \in U \) there is some entourage \( E \) such that \( B(x, E) \subset U \).

The set \( E^k \) can be equivalently described using the notion of an \( E \)-chain, which is a finite sequence \( \alpha = \{ x_0, ..., x_n \} \) such that for all \( i \), \((x_i, x_{i+1}) \in E \). So \( E^k \) is the set of all \((x, y)\) such that there is an \( E \)-chain \( \{ x = x_0, ..., x_k = y \} \). The canonical examples of uniform spaces are metric spaces, with *metric entourages* \( E_\varepsilon \) forming a basis for the uniformity; compact topological spaces, which have a unique uniform structure compatible with the topology, in which entourages are simply all symmetric neighborhoods of the diagonal; and topological groups. Given a topological group and a basis for the topology at the identity \( 1 \), there are uniquely determined “left” and “right” uniformities that are invariant with respect to left or right translation. That is, for any \( U \) in the basis at \( 1 \), the
corresponding entourage $E_U$ in the left uniformity consists of all $(g,h)$ such that $g^{-1}h \in U$. So if $k \in G$ then since $(kg)^{-1}kh = g^{-1}h$, $E_U$ is invariant with respect to left multiplication.

We will use simplified notation for Cartesian products of functions, such as using $g(W)$ to denote $(g \times g)(W)$. For example, $W$ is invariant with respect to a bijection $g : X \to X$ by definition if $g(W) = W$. With this notation, $f : X \to Y$ is uniformly continuous if and only if for every entourage $E$ in $Y$, $f^{-1}(E)$ is an entourage in $X$. We say that $f$ is bi-uniformly continuous if in addition $f$ is surjective and $f(E)$ is an entourage in $Y$ for every entourage $E$ in $X$. A 1-1 bi-uniformly continuous function is called a uniform homeomorphism. In the compact case, with the unique compatible uniformity, continuity and uniform continuity are equivalent.

Uniform spaces are the appropriate setting for discrete homotopy theory as developed in [1, 18, 9, and 21]. We recall some basics now. Given an $E$-chain $\alpha = \{x_0,\ldots,x_n\}$ in a uniform space, a basic move consists of adding or removing a single point, except either endpoint, so that the resulting chain is still an $E$-chain. An $E$-homotopy of an $E$-chain is a finite sequence of basic moves. The $E$-homotopy equivalence class of $\alpha$ is denoted by $[\alpha]_E$. We will frequently use without reference the fact that if $f : X \to Y$ is a (possibly not continuous!) function between uniform spaces and $E,F$ are entourages in $X,Y$, respectively, such that $f(E) \subset F$ then given any $E$-homotopic $E$-chains $\alpha$ and $\beta$ in $X$, $f(\alpha)$ and $f(\beta)$ are $F$-homotopic $F$-chains.

If $\alpha$ is an $E$-loop that is $E$-homotopic to the trivial chain (consisting of its start/end point) then $\alpha$ is called $E$-null. Fixing a basepoint $*$, the set of all $[\alpha]_E$ such that $\alpha$ starts at $*$ is denoted by $X_E$, and $\phi_E : X_E \to X$ is the endpoint map. For any entourage $F \subset E$, in [3] we define $F^*$ to be the set of all ordered pairs $([\alpha]_E,[\beta]_E)$ such that $[\alpha]_E \ast [\beta]_E = [[a,b]]_E$ and $(a,b) \in F$, where $a$ and $b$ are the endpoints of $\alpha$ and $\beta$, respectively. Here $\ast$ denotes concatenation of chains when the endpoint of the first chain is the first point of the second chain, and $[\alpha]_E \ast [\beta]_E$ is the reversal of the chain $\alpha$. Note that $\{a,b\}$ must be an $E$-chain since $F \subset E$. There is a useful equivalent definition of $F^*$ (which was the original definition in [3]), namely $([\alpha]_E,[\beta]_E) \in F^*$ if and only if we may write (up to $E$-homotopy) $\alpha$ as an $E$-chain $\{* = x_0,\ldots,x_n,a\}$ and $\beta$ as an $E$-chain $\{* = x_0,\ldots,x_n,b\}$ with $(a,b) \in F$. Equivalently, $(a,b) \in F$ and we may write $\beta$ as $\alpha \ast \{a,b\} = \{* = x_0,\ldots,x_n,a,b\}$.

In what follows we will not use the extra brackets $\{}$ in our notation, e.g. writing $[a,b]_E$ rather than $\{[a,b]\}_E$. The set of all $F^*$ is a basis for a uniformity on $X_E$. The set of all equivalence classes $[\lambda]_E$ of $E$-loops based at $*$ is denoted by $\pi_E(X)$, which is a group with product induced by concatenation of chains. The group $\pi_E(X)$ acts on $X_E$ by uniform homeomorphisms via preconcatenation: $[\lambda]_E([\alpha]_E) = [\lambda \ast \alpha]_E$, and the entourages $F^*$ for $F \subset E$ are invariant with respect to this action. Then $\phi_E : X_E \to X = X_E/\pi_E(X)$ is the quotient map. Moreover, the restriction of $\phi_E$ to any $B([\alpha]_E,F^*)$ is a bijection onto $B(a,F)$, where $a$ is the endpoint of $\alpha$.

When $F \subset E$ is an entourage, since $F^*$ is an entourage in the uniform space
$X_E$, it is natural to consider the space $(X_E)_{F^*}$. Proposition 23 in [3] naturally identifies $X_F$ with $(X_E)_{F^*}$ for any $F \subset E$ via the mapping

$$[x_0, ..., x_n]_F \mapsto ([x_0]_E, [x_0, x_1]_E, ..., [x_0, ..., x_n]_E)_{F^*}. \quad (1)$$

This formula is useful in various ways. For now, note that when $F = E$ this implies that the map $\phi_{E^*} : (X_E)_{E^*} \to X_E$ is a uniform homeomorphism, hence every $E^*$-loop in $X_E$ is $E^*$-null (this explicit statement has not been made previously, but this argument was used in the proof of Theorem 77 in [3]).

Berestovskii-Plaut also showed that if $X$ is path connected and uniformly semilocally simply connected (we will call such $X$ a “uniform Poincaré space”) then for sufficiently small entourages $E$ having connected balls (and such always exist), $X_E$ is the universal cover of $X$ and $\pi_E(X) = \pi_1(X)$. In general the adjective “uniformly” preceding a traditional local topological property means that there are arbitrarily small entourages $E$ such that the $E$-balls have the property. In compact spaces, local properties are automatically uniformly local with respect to the unique compatible uniform structure (see for example Proposition 68 in [3]).

One may take the inverse limit of the spaces $X_E$ over all entourages with bonding maps $\phi_{EF} : X_F \to X_E$ for $F \subset E$ defined by $\phi_{EF}([\alpha]_F) = [\alpha]_E$ to obtain a space $\tilde{X}$ and projection map $\phi : \tilde{X} \to X$ (here we identify $X$ with $X_{X \times X}$). In [3] this system was referred to as the fundamental inverse system. The inverse limit of the groups $\pi_E(X)$ is called the uniform fundamental group $\pi_U(X)$. When $X$ is weakly chained in the sense of [21] (which is a priori stronger than the notion of “coverable” in [3]), the projection $\phi$ is a generalized regular covering map with deck group $\pi_U(X)$, called the uniform universal cover. In [21] we extended the results of [3] for weakly chained $X$, showing that $\tilde{X}$ is uniformly simply connected in the sense that $\pi_U(X)$ is trivial, and is unique up to uniform homeomorphism. However, some continua are not weakly chained, and for these spaces the projection $\phi$ need not be surjective. For example, for a standard solenoid, $\phi$ maps only onto the path component. Throughout this construction, given a basepoint $\ast$ in $X$, we may choose $[\ast]_E$ for the basepoint in $X_E$ and $([\ast]_E)$ for the basepoint in $\tilde{X}$.

**Remark 10** The idea that “$E$-close $F$-chains are $EF$-homotopic”, has been used in one form or another in several papers involving discrete homotopy theory. This concept is particularly important for counting arguments in compact spaces, and in this paper we will use it in the proof of Theorem 4. For completeness and as an illustration for the unfamiliar reader, we will sketch the argument that $E$-close $E$-chains are $E^2$-homotopic. Suppose that $\alpha = \{x = x_0, x_1, ..., x_{n-1}, x_n = y\}$ and $\beta = \{x = x'_0, x'_1, ..., x'_{n-1}, x'_n = y\}$ are $E$-chains from $x$ to $y$ such that for all $i$, $(x_i, x'_i) \in E$. As the reader can readily verify, the following are $E^2$-homotopy basic moves, meaning that at each stage we always have an $E^2$-chain. For example, since $(x_0, x_1) \in E$ and $(x_1, x'_1) \in E$, $(x_0, x'_1) \in E^2$.

$$\alpha \rightarrow \{x_0, x'_1, x_1, x_2, ..., x_n\} \rightarrow \{x_0, x'_1, x_2, ..., x_n\}$$

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The chain component connected if and only if $X$ and in particular uniformly open sets are also closed. It follows that $X$ and for every entourage $E$.

Moreover, $U \in E$ uniformly $x, y$ may be joined by arbitrarily fine chains in $X$ such that $\exists \lambda$.

Given a uniform space $X$ and entourage $E$, a subset $A$ is called $E$-chain connected if every pair of points in $A$ is joined by an $E$-chain in $A$. A is called chain connected (equivalent to “uniformly connected” in the classical literature) if $A$ is $E$-chain connected for every entourage $E$. In this context we often instead use phrases like “$x$ and $y$ are joined by arbitrarily fine chains in $A$”. In the compact case with the unique compatible uniformity, connected and chain connected are equivalent. While we will not formally use it in this paper, we will define “weakly chained” (21) because the definition is very simple and weakly chained continua are a kind of intermediate space between Poincaré spaces and arbitrary continua, which makes them helpful in the exposition of the current paper. A uniform space $X$ is weakly chained if $X$ is chain connected and for every entourage $E$ there exists an entourage $F$ such that if $(x, y) \in F$, $x, y$ may be joined by arbitrarily fine chains $\alpha$ such that $[\alpha]_E = [x, y]_E$. Weakly chained spaces need not be locally path connected, but as mentioned above, solenoids are not weakly chained.

If $x \in X$, the $E$-chain component $U^E_x$ of $x$ in $X$ is the union of all $E$-chain connected sets in $X$ containing $x$, which is clearly itself $E$-chain connected. Moreover, $U^E_x$ is also the intersection of all sets containing $x$ that are uniformly $E$-open in the the sense that if $z \in U^E_x$ then $B(z, E) \subset U^E_x$, see Lemma 9, 21. Combining these two equivalences we have the following lemma that we will use without reference.

**Lemma 11** Suppose that $X$ is a uniform space and $x \in A \subset X$. If $E$ is an entourage in $X$ then $A = U^E_x$ if and only if $A$ is both $E$-chain connected and uniformly $E$-open.

We say that $U$ is uniformly open if $U$ is uniformly $E$-open for some $E$. Note that the complement of any uniformly $E$-open set is also uniformly $E$-open, and in particular uniformly open sets are also closed. It follows that $X$ is chain connected if and only if $X$ is the only non-empty uniformly open subset of $X$. The chain component of $x \in X$ is defined to be the intersection of all $U^E_x$;
equivalently the chain component of \(x\) is the union of all chain connected sets containing \(x\).

**Remark 12** We note for once and for all that when \(X\) is chain connected, the construction of \(\phi_E : X_E \to X\) is independent, up to equivariant homeomorphism, of choice of basepoint in \(X\) ([3], Remark 18). The proof of this fact is similar to the proof from traditional covering space theory.

A summary of known connectivity properties of \(X_E\) is as follows. If \(X\) is coverable in the sense of [3], which includes if \(X\) is a uniform Poincaré space, then equivalently there is a basis consisting of entourages \(E\) such that \(X_E\) is chain connected (this is automatic if, for example, \(E\) has chain connected balls). If \(X\) is weakly chained then for any entourage, the chain components of \(X_E\) are uniformly open. Weakly chained is \textit{a priori} weaker than coverable and it is presently unknown whether these properties are equivalent in the metrizable case. If \(X\) is a solenoid then the chain components are generally not uniformly open because, as with the solenoid itself, \(X_E\) is locally a product of a Cantor set and an interval, and \(\pi_E(X)\) acts by “permuting” some of the path components (see Example 54).

### 3 Quotients and Covering Maps

We recall the basics of quotients of uniform spaces worked out in [13]. Let \(X\) be a uniform space and \(G\) be a group of bijections of \(X\). \(G\) is said to act \textit{isomorphically} on \(X\) if \(X\) has a basis of entourages \(E\) that are invariant with respect to \(G\); that is \(g(E) = E\) for all \(g \in G\). Quotients may be considered in the more general setting of “equiuniform” actions ([13]), but we do not need such generality here. The quotient space \(X/G\) is defined to be the set of all orbits \(Gx := \{g(x) : g \in G\}\); the map \(\pi : X \to X/G\) defined by \(\pi(x) = Gx\) is called the quotient map. From the definition of isomorphic action it is clear that the maps in \(G\) are all uniform homeomorphisms of \(X\). According to Theorem 11, [13], for an isomorphic action the set of all \(\pi(E)\) such that \(E\) is an entourage in \(X\) is a uniform structure on \(X/G\) that is compatible with the quotient topology. This is called the \textit{quotient uniformity}, and will be the default uniformity on \(X/G\). Then the quotient map \(\pi : X \to X/G\) is bi-uniformly continuous.

Metrizability questions are generally simpler for uniform spaces, which are metrizable if and only if they are Hausdorff and have a countable basis. In particular, if the quotient of an isomorphic action on a metrizable uniform space is Hausdorff, it is metrizable. In what follows, we will use these facts without reference.

**Remark 13** If \(f : X \to Y\) is a bi-uniformly continuous mapping and \(G\) acts isomorphically on \(X\) such that for all \(x \in X\), \(f^{-1}(f(x)) = Gx\) then the mapping \(f(x) \leftrightarrow Gx\) is a uniform homeomorphism and we may identify \(f\) as the quotient map \(f : X \to Y = X/G\). We will use this fact frequently to show that a bi-uniformly continuous function \(f\) is in fact a quotient map. Our shorthand for this statement will be “the point preimages of \(f\) are the orbits of \(G\)”. [13]
Remark 14 We note that as was so colorfully stated in [13], the idea that
general topological quotients of uniform spaces have a compatible uniformity is
“horribly false”. But quotients via suitable group actions provide the uniform-
ness of identifications that is needed to make it work. Under the right circum-
stances we also may work in reverse as exemplified in the definition of $F^*$ in
the previous section: start with a uniform structure on the quotient space and lift
to a uniform structure on the original. The following definition was not stated in [13]
but it will be important in the current paper.

Definition 15 Suppose that $G_i$ acts isomorphically on a uniform space $Y_i$ for
$i = 1, 2$. We say the actions are equivalent if there are a uniform homeomor-
phism $h : Y_1 \to Y_2$ and an isomorphism $\theta : G_1 \to G_2$ that are compatible in the
following sense: for all $g \in G_1$, $h \circ g = \theta(g) \circ h$. The pair $(h, \theta)$ is called an
equivalence of the actions. We say that two quotient maps $\pi_i : Y_i \to Y_i/G_i$ are
equivalent if the actions of $G_i$ on $Y_i$ are equivalent.

Note that the equivalence described above is indeed an equivalence relation.
For example, if $(h, \theta)$ is an equivalence of the actions of $G_1$ on $Y_1$ and of $G_2$ on $Y_2$
then $(h^{-1}, \theta^{-1})$ is an equivalence of actions of $G_2$ on $Y_2$ and $G_1$ on $Y_1$. In
fact, for any $k = \theta(g)$ in $G_2$ and $y = h(x)$ in $Y_2$,
$$h^{-1}(k(y)) = \theta^{-1}(k)(h^{-1}(y)) \iff \theta(g)(h(x)) = h(g(x)).$$

Example 16 Consider the compact space $Y = G \times [0, 1]$, where $G$ is the group
with three elements with the discrete topology, with $G$ acting on $Y$ by cycling the
components, with quotient $[0, 1]$. That is, for $z \in \{0, 1, 2\}$, $g_0((z, x)) =
(z + \omega, x)$. Consider the homeomorphism $h$ that fixes the component $\{0\} \times [0, 1]
and exchanges the other two components, i.e. $h((0, x)) = (2\omega, x)$. Then $h$ is a
covering equivalence in the traditional sense, but to be an equivalence in the
current sense would require $\theta : G \to G$ to be the identity map. But for example
$h_0((0, x)) = h_1((0, x)) = (1, x)$, while $\theta((0, z))(h((0, x))) = \theta((0, z))((0, x)) =
(0, 0)$. So compatibility fails. The next lemma shows that this notion of equivalence is in
general stronger than the traditional one, and we will see (Corollary 50) that the
two notions are equivalent for generalized regular covering maps.

Lemma 17 Suppose that $(h, \theta)$ is an equivalence of isomorphic actions of $G_i$
on $Y_i$ for $i = 1, 2$. Then the mapping $\overline{h}(G_1 y) := G_2 h(y)$ is a uniform homeo-


Proof. We first show that $\overline{h}$ is well-defined. If $z \in G_1 y$ then $z = g(y)$ for
some $g \in G_1$. By definition, $h(z) = \theta(g)(h(y))$ and therefore $h(z) \in G_2 y$ and
$G_2 z = G_2 y$. By symmetry the proof that $\overline{h}$ is a bijection will be complete if we
show that $\overline{h} \circ \pi_1 = \pi_2 \circ h$.

Finally, the fact that $\overline{h}$ is a uniform homeomorphism follows from the definition of the
quotient uniformity and the fact that $h$ is a uniform homeomorphism. ■
For any entourage $E$ in a uniform space $X$ on which $G$ acts isomorphically, let

$$U_E(G) := \{g \in G : (x, g(x)) \in E \text{ for all } x \in X\}$$

and $N_E(G)$ denote the subgroup generated by $S_E(G)$ in $G$. According to Theorem 31 in [18], since the action of $G$ is isomorphic, $N_E(G)$ is a normal subgroup of $G$. If for some entourage $F$, $N_F(G) = 1$, then $G$ is said to act \textit{discretely}. We say that $G$ acts \textit{prodiscretely} if for every entourage $E$ there is some entourage $F$ such that $N_F(G) \subset U_E(G)$. The collection of all $U_E(G)$ where $E$ is an entourage in $X$ forms a basis at the identity for the topology of uniform convergence in $G$, making it a topological group (Theorem 17, [18]). We note that when $X$ is metrizable and therefore has a countable base for its uniform structure, $G$ has a countable base of sets $U_E(G)$ and therefore is metrizable. This topology gives rise, as described in the second section, to the left and right uniformities of uniform convergence.

\textbf{Remark 18} There are some subtleties related to invariance properties of the uniform structure and topology on a group $G$ acting on a uniform space—for more information see the exact statement of Theorem 17 and Notation 18 in [18]. These details are unimportant for this paper due to Lemma 19 in [18], which implies that in the case of isomorphic actions, the left and right uniformities of uniform convergence of $G$ are the same.

We have the following basic results (all references in this paragraph are from [18]): If $X$ is Hausdorff, so is $G$, and if $X$ is complete (as a uniform space, defined analogously to completeness in a metric space) and $G$ is a closed subset of the group of all uniform homeomorphisms of $X$ then $G$ is complete (Theorem 20). In the case of a prodiscrete action we have the following: The set of all $N_E(G)$, where $E$ is an entourage of $X$, is a basis at 1 for the topology of $G$ (Corollary 32); the action of $X$ is free (Lemma 33); and if $G$ is complete then $G$ is a prodiscrete topological group (i.e. topologically the inverse limit of discrete groups) (Corollary 34). If $G$ acts discretely then $G$ is discrete, hence complete, each of its orbits is uniformly discrete, and the action is properly discontinuous (Proposition 22). In fact, a discrete action is essentially a “uniformly properly discontinuous” action. Here “$A$ is uniformly discrete” means that for some entourage $E$, every $E$-ball in the uniform space contains at most one point of $A$.

Suppose $G$ acts isomorphically on $X$ and $H$ is a subgroup of $G$. Then clearly $H$ also acts isomorphically on $X$, and the action of $H$ is discrete (resp. prodiscrete) if the action of $G$ is. When $H$ is a normal subgroup of $G$ then $G/H$ acts isomorphically on $X/H$ (with the quotient uniformity) and we have the induced quotient mapping $\phi : X/H \to (X/H)/G/H = X/G$ (Proposition 28, [18]). The latter equality is via the uniform homeomorphism $(G/H)/(Hx) \leftrightarrow Gx$. Although we proved the following statement for prodiscrete actions in Theorem 31, [18], we did not prove it for discrete actions, so we will do so now:
Lemma 19 If $G$ acts discretely and isomorphically on a uniform space $X$ and $H$ is a normal subgroup of $G$ then $G/H$ acts discretely (and isomorphically) on $X/H$.

Proof. Let $\eta : X \to X/H$ be the quotient map and let $E$ be an invariant entourage such that $N_E(G)$ is trivial. We claim that $N_{\eta(E)}(G/H)$ is also trivial, for which it suffices to prove that $S_{\eta(E)}(G/H)$ contains only the trivial element. If $gH \in S_{\eta(E)}(G/H)$ then by definition $\eta(E)$ contains $(Hx, gH(Hx)) = (Hx, Hg(x))$ for some $x \in X$. This means that for some $h_1, h_2 \in H$,
\[(h_1(x), h_2(g(x))) \in E.\]
Since $E$ is invariant, this means $(x, h_1^{-1}h_2(g(x))) \in E$. That is, $h_1^{-1}h_2g \in N_E(G)$ and therefore is trivial. Therefore $g \in H$ and $Hg$ is trivial. \[\blacksquare\]

Proposition 20 If $G$ is a complete group that acts prodiscretely and isomorphically on a metrizable uniform space $X$ then the orbits of $G$ are closed and $X/G$ is metrizable.

Proof. Let $f : X \to X/G$ be the quotient map. Let $\{E_i\}$ be a countable nested invariant basis for $X$ and recall that the set of all $N_{E_i}(G)$ is a basis at the identity for the topology of uniform convergence on $G$, which means that for every $i$ there is some $j$ such that
\[N_{E_i}(G) \subset U_{E_i}(G).\] (2)
This implies the following: if $g \in G$ has the property that $(g(x_0), x_0) \in E_i$ for some $x_0 \in X$ then $(g(x), x) \in E_i$ for all $x \in X$. Now suppose that $y$ is in the closure of some orbit $Gx$. That is, for every entourage $E_j$, there is some $x_j := g_j(x) \in B(y, E_j)$. But now Inclusion (2) implies that $\{g_j\}$ is Cauchy in the topology of uniform convergence. Since $G$ is complete, $g_j \to g \in G$. This means that for any invariant entourage $E$, $(g(x), g_i(x)) \in E$ for all large $i$. In other words, $g(x) = y$, so $y \in Gx$.

To finish the proof, we need only show that $X/G$ is Hausdorff. Let $x \in X$ and suppose that $y$ is not in $Gx$. We first note that there is some invariant entourage $E$ such that $y \notin B(w, E)$ for every $w \in Gx$. If this were not true, then for every $i$ there would be some $w_i \in Gx$ such that $y \in B(w_i, E_i)$, which is equivalent to $w_i \in B(y, E_i)$. But this implies that $y$ is in the closure of $Gx$, so $y \in Gx$, a contradiction.

Suppose $F$ is an invariant entourage such that $F^2 \subset E$. Let $U := \bigcup_{w \in Gy} B(u, F)$ and $W := \bigcup_{w \in Gx} B(w, F)$. We claim that $U \cap W = \emptyset$. If $z \in U \cap W$ then $z \in B(u, F) \cap B(w, F)$ for some $u \in Gy$ and $w \in Gx$. This means that $(u, w) \in F^2 \subset E$. Since $E$ is invariant, if $g(u) = y$, then $(y, g(w)) \in E$. Since $g(w) \in Gx$, this contradicts our choice of $E$.

To finish the proof, we claim that $f(U) \cap f(W) = \emptyset$. Suppose there is some $a \in f(U) \cap f(W)$. So there exist $a' \in U$ and $a'' \in W$ such that $f(a') = f(a'') = a$.
This last equation implies that there is some \( k \in G \) such that \( k(a') = a'' \). By definition of \( U \) and \( W \), we have that \((a', g_1(y)), (a'', g_2(x)) \in F \) for some \( g_1, g_2 \in G \). Since \( F \) is invariant, this means that \((k(g_1(y)), k(a')) = (k(g_1(y)), a'') \in F \). Since \( k(g_1(y)) \) is in the orbit of \( Y \), this means that \( a'' \in U \cap W \), a contradiction. 

**Proposition 21** Let \( h : Z \to X = Z/G \) be a quotient via an isomorphic action by \( G \), and \( g : Z \to Y \) and \( f : Y \to X \) be bi-uniformly continuous functions such that \( h = f \circ g \). Suppose that for some normal subgroup \( H \) of \( G \), the point preimages of \( g \) are the orbits of \( H \). Then \( f \) is a quotient map equivalent to the induced quotient map \( \pi : Z/H \to (Z/H)/(G/H) \).

**Proof.** By assumption, \( g \) is the quotient map \( g : Z \to Y = Z/H \). We know already that \( G/H \) acts isomorphically on \( Y = Z/H \) via the action \( kH(g(z)) = g(k(z)) \). Therefore we need only show that the point preimages of \( f \) are the orbits of \( G/H \). In fact, \( f(g(x)) = f(g(y)) \iff h(x) = h(y) \), which is equivalent to \( k(x) = y \) for some \( k \in G \), which in turn is equivalent to \( kH(g(x)) = g(y) \).

**Definition 22** As usual, if \( G \) is a group of bijections of a set \( Y \) and \( C \subseteq Y \), the stabilizer subgroup of \( C \) is the subgroup \( S_C \) of \( G \) consisting of all \( g \in G \) such that \( g(C) = C \). We will say that \( C \) has Property ST if whenever \( g \in G \), if \( g(C) \cap C \neq \emptyset \) then \( g \in S_C \).

**Lemma 23** Suppose that a group \( G \) acts isomorphically on a uniform space \( Y \).

1. If \( C \) is any chain component of \( Y \) (resp. \( E \)-chain component of \( Y \) for some invariant entourage \( E \)) then \( C \) has Property ST.

2. If \( C \) is uniformly \( E \)-open for some entourage \( E \) and the restriction \( f_C \) of the quotient map \( f : Y \to X = Y/G \) to \( C \) is surjective then:

   (a) \( f_C \) is bi-uniformly continuous.

   (b) If \( C \) has property ST then \( f_C : C \to X = Y/S_C \) is a quotient map.

**Proof.** For the first part, suppose that for some \( x \) and \( g \in G \), \( g(x) \in C \). For any \( z \in C \) there is an arbitrarily fine chain (resp. \( E \)-chain) \( \alpha \) from \( x \) to \( z \). Then \( g(\alpha) \) is an arbitrarily fine chain (resp. \( E \)-chain) in \( Y \) from \( g(x) \) to \( g(z) \), so \( g(z) \in C \), i.e. \( g(C) \subseteq C \). Since \( g \) is a bijection, for every \( y \in C \) there is some \( w \in Y \) such that \( g(w) = y \). But by what we just showed, \( w \) cannot lie in a different component (resp. \( E \)-chain component) of \( Y \). Therefore, \( C \subseteq g(C) \).

For Part 2a, note that by elementary results \( f_C \) is uniformly continuous with respect to the subspace uniformity. Let \( F \subseteq E \) be an invariant entourage, and note that \( C \) is also uniformly \( F \)-open. We will show that \( f(F) \subseteq f(F \cap (C \times C)) \), showing that the latter is an entourage, proving the claim. Let \( (x, y) \in f(F) \). So there exists \((x', y') \in F \) such that \( f(x'), f(y') = (x, y) \). Since \( f_C \) is surjective there is some \( x'' \in C \) such that \( f(x'') = x \). Therefore there is some \( g \in G \) such that \( g(x') = x'' \). Since \( F \) is invariant, \((x'', g(y')) \in F \). Since \( C \) is uniformly...
\( g(y') \in C \). That is, \((x'', g(y')) \in F \cap (C \times C)\) and since \( f((x'', g(y')) = (x, y), (x, y) \in f(F \cap (C \times C))\).

For Part 2b we need only show that the orbits of \( S_C \) are the point preimages of \( f_C \). Clearly if \( z = g(w) \) for some \( z, w \in C \) and \( g \in S_C \subset G \), \( f(z) = f(w) \).

Conversely, suppose the latter equation holds for \( z, w \in C \). Then there is some \( g \in G \) such that \( g(z) = w \). Since \( z, w \in C \) and \( C \) has property ST, \( g \in S_C \).

### 4 Uniformizing Regular Covering Maps

There is an unfortunate variety of terminologies involving actions on topological spaces by discrete groups, see for example [14] for a discussion. We will say that a group \( G \) of homeomorphisms of a metrizable topological space \( Y \) acts properly discontinuously if (1) for every \( y \in Y \) there is some open set \( U \) such that if \( g \in G \) and \( g(U) \cap U \neq \emptyset \) then \( g = 1 \); and (2) the orbit space \( X = Y/G \) with the quotient topology is Hausdorff.

For spaces that are not Poincaré spaces, the notion of “regular covering map” is problematic since this is typically defined in terms of the traditional mapping from the group of deck transformations into the fundamental group, which requires paths. The equivalent notion, that the deck (covering transformation) group is “transitive on the fibers”, simply means that the map is a quotient map. Therefore we will take our definition of a regular covering map between topological spaces to be that the map is a covering map in the usual sense and also a quotient map via a free action by group of homeomorphisms. Recall that a covering map \( f : X \rightarrow Y \) by definition has the property that every \( x \in X \) is contained in an evenly covered open set \( U \), i.e. \( f^{-1}(U) \) is a disjoint collection of open sets, the restriction to any of which is a homeomorphism onto \( U \).

**Definition 24** Suppose \( G \) is a group of bijections on a set \( Y \). The PD-domain \( P \) of the action is the set of all \((x, y) \in Y \times Y \) such that if \( y = g(x) \) for some \( g \in G \) then \( x = y \). If \( R \) is a symmetric, invariant open set containing the diagonal in \( Y \times Y \) such that \( R^2 \subset P \) then \( R \) is called a root domain.

Put another way, the PD-domain consists of all ordered pairs \((y, z)\) such that if \( y \neq z \), \( z \) does not lie in the orbit of \( y \). Clearly the PD-domain is a symmetric, invariant subset of \( Y \times Y \) containing the diagonal. If the action is free then \( x = y \) is strengthened to \( g = 1 \).

If \( R \) is a root domain then of course any symmetric open set containing the diagonal that is contained in \( R \) is again a root domain.

**Example 25** Consider the action of \( \mathbb{Z} \) on \( \mathbb{R} \) generated by \( x \mapsto x + 1 \). As can easily be checked, the PD-domain \( P \) of this action consists of the plane, removing all lines that are shifts of the diagonal line vertically by an integer different from 0. But it is simpler to look at the \( P \)-balls, which determine \( P \). \( B(0, P) \) consists of \( \mathbb{R} \) with all non-zero integers removed. For simplicity, give \( \mathbb{R} \) its standard metric and suppose that \( R = E_\varepsilon \) is a root domain for some \( \varepsilon > 0 \). Since the metric is geodesic, \( B(x, R^2) = B(x, 2\varepsilon) \), and we must have \( B(0, 2\varepsilon) \subset B(0, P) \),
which forces \( \varepsilon \leq \frac{1}{4} \). That is, even though \( P \) is in a sense very large (it is dense in \( \mathbb{R}^2 \)), the requirement that \( R^2 \subset P \) forces \( R \) to be relatively small.

**Definition 26** Suppose \( G \) acts properly discontinuously on a metrizable topological space \( Y \) with quotient map \( f : Y \to X = Y/G \). A symmetric open set \( E \) containing the diagonal in \( X \times X \) is called evenly covered with respect to a root domain \( R \) if \( E \subset f(R) \). If \( E \) is evenly covered with respect to some \( R \), we simply say that \( E \) is evenly covered.

If \( R_1 \subset R_2 \) are root domains and \( E \) is evenly covered with respect to \( R_1 \) and \( R_2 \) then by elementary set theory

\[
E^*_{R_1} = E^*_{R_2} \cap R_1 \subset E^*_{R_2}.
\]  

**Proposition 27** Let \( G \) be a group of homeomorphisms on a metrizable topological space \( Y \) that acts properly discontinuously with quotient map \( f : Y \to X = Y/G \). Let \( P \) be the PD-domain of the action. Then

1. If \( (x, y) \in P \) and \( x \neq y \) then \( f(x) \neq f(y) \).
2. If \( E \) is a symmetric open set containing the diagonal in \( X \) and \( R \) is a root domain in \( Y \), then \( E^*_R := f^{-1}(E) \cap R \) is a symmetric, invariant open set containing the diagonal in \( Y \), called the lift of \( E \) with respect to \( R \). Moreover, if \( E \) is evenly covered with respect to \( R \) then \( f(E^*_R) = E \).

**Proof.** If \( (x, y) \in P \) and \( f(x) = f(y) \) then there is some \( g \in G \) such that \( g(x) = y \), which is impossible by definition of \( P \) unless \( x = y \).

For the second part, note that if \( W \) is any subset of \( X \times X \) then \( f^{-1}(W) \) is invariant with respect to the action of \( G \). In fact, if \( (x, y) \in f^{-1}(W) \) then \( (f(x), f(y)) \in W \). But for any \( g \in G \), \( f(g(x)) = f(x) \) and \( f(g(y)) = f(y) \). Since \( (f(x), f(y)) \in W \), \( (g(x), g(y)) \in f^{-1}(W) \). Invariance of \( E^*_R \) now follows from the fact that \( R \) is invariant. From elementary set theory, \( f(E^*_R) = f(f^{-1}(E) \cap R) \subset E \cap f(R) \subset E \). Finally, suppose that \( (x, y) \in E \subset f(R) \); that is, there is some \( (x', y') \in R \) such that \( f(x') = x \) and \( f(y') = y \). But since \( (x, y) \in E \), \( (x', y') \in f^{-1}(E) \) and hence \( (x', y') \in E^*_R \). That is, \( (x, y) \in f(E^*_R) \).

**Proposition 28** Let \( G \) be a group of homeomorphisms on a metrizable topological space \( Y \) that acts properly discontinuously with quotient map \( f : Y \to X = Y/G \). If \( E \) is evenly covered in \( X \) with respect to some root domain \( R \), then \( E \)-balls are evenly covered by \( E^*_R \)-balls.

**Proof.** Suppose \( y, y' \in f^{-1}(x) \) and there is some \( w \in B(y, E^*_R) \cap B(y', E^*_R) \). Then \( (y, y') \in (E^*_R)^2 = (f^{-1}(E) \cap R)^2 \subset R^2 \subset P \), where \( P \) is the PD-domain. Since \( f(y) = x = f(y') \), by Proposition 27 \( y = y' \). Now let \( h \) be the restriction of \( f \) to some \( B(y, E^*_R) \) with \( y \in f^{-1}(x) \). By Proposition 27 \( f(B(y, E^*_R)) \subset B(x, E) \). Let \( w \in B(x, E) \). By Proposition 27 there exist \( (y', z') \in E^*_R \) with \( f(y') = x \) and \( f(z') = w \). Since \( f(y') = x = f(y) \) there is some \( g \in G \) such that \( g(y') = y \); letting \( z := g(z') \) we have \( f(z) = f(g(z')) = f(z') = w \). Since \( E^*_R \) is
invariant, \( z \in B(y, E^g_h) \), showing that \( h \) is onto \( B(x, E) \). Finally, suppose that \( z_1, z_2 \in B(y, E^g_h) \) and \( f(z_1) = f(z_2) \). Then \( (z_1, z_2) \in (E^g_h)^2 \subset R^2 \subset P \) and by Proposition 27, \( z_1 = z_2 \).

We will use the next proposition frequently without reference.

**Proposition 29** Suppose \( f : Y \to X = Y/G \) is a quotient map between metric spaces. Then \( f \) is a regular covering map if and only if \( G \) acts properly discontinuously and has a root domain.

**Proof.** Suppose that \( f \) is a regular covering. Since \( Y \) is assumed to be metric, hence Hausdorff, we need only verify the first condition in the definition of proper discontinuous action. Let \( y \in Y \). Then there is some evenly covered open set \( V \) containing \( f(y) \). Let \( U \) be an open subset of \( f^{-1}(V) \) containing \( y \) such that \( U \) contains \( z \) and the restriction of \( f \) to \( U \) is 1-1. If \( z \in g(U) \cap U \) then \( z \) and \( g^{-1}(z) \) both lie in \( U \) and by choice of \( U \), \( z = g^{-1}(z) \) and \( g = 1 \).

For every \( x \in X \) there is some maximal \( \varepsilon_x > 0 \) such that \( B(x, \varepsilon_x) \) is evenly covered. By maximality and the triangle inequality, for any \( x_1, x_2 \in X \),

\[
\varepsilon_{x_1} \geq \varepsilon_{x_2} - d(x_1, x_2).
\]

For \( y \in Y \) there is a unique open set \( U_y \) containing \( y \) such that the restriction of \( f \) to \( U_y \) is a homeomorphism onto \( B(f(y), \varepsilon_{f(y)}) \). Define \( R := \{(w, z) : w, z \in U_y \text{ for some } y \in Y\} \), which is clearly a symmetric open set containing the diagonal. Now suppose that \( g \in G \). Then \( f(g(U_y)) = f(U_y) = B(f(y), \varepsilon_{f(y)}) \) and since the restriction of \( f \) to \( U_y \) is a bijection onto \( B(f(y), \varepsilon_{f(y)}) \), so is the restriction of \( f \) to \( g(U_y) \). That is, \( g(U_y) = U_{g(y)} \) and \( R \) is invariant.

Now suppose that \( (w_1, w_2) \in R^2 \). That is, for some \( z, (w_1, z), (w_2, z) \in R \). This in turn means that for some \( y_1, y_2, (w_1, z) \in U_{y_i} \), for \( i = 1, 2 \). Therefore \( f(w_i), f(z) \in B(y_i, \varepsilon_{f(y)}) \). From Formula (4), we have \( \varepsilon_{f(z)} \geq \varepsilon_{f(y)} - d(f(z), f(y_i)) > \varepsilon_{f(y)} - \frac{\varepsilon_{f(y)}}{3} = \frac{2\varepsilon_{f(y)}}{3} \). We also have \( d(f(w_i), f(z)) < \frac{2\varepsilon_{f(y)}}{3} \) and therefore \( f(w_i) \in B(f(z), \varepsilon_{f(z)}) \). Since the latter ball is evenly covered, there is an open set \( V \) containing \( z \) such that the restriction of \( f \) to \( V \) is 1-1. Since \( U_{y_i} \) contains \( z \) and maps into \( B(f(z), \varepsilon_{f(z)}) \) it must be contained in \( V \). Therefore both \( w_1 \) and \( w_2 \) lie in \( V \) and it is impossible for a non-trivial \( g \in G \) to have the property that \( g(w_1) = w_2 \). That is \( R^2 \) is contained in the PD-domain and by definition \( R \) is a root domain.

The converse is an immediate consequence of Proposition 28.

**Theorem 30** (Uniformizing Regular Covering Maps) Let \( f : Y \to X = Y/G \) be a regular covering map between metrizable spaces. Then:

1. If \( Y \) is given a (compatible) uniform structure such that some root domain is an entourage in \( Y \) (such a uniformity always exists) then
   
   (a) \( Y \) has a countable basis \( \mathcal{R} \) consisting of root domains.
   (b) The set of all \( f(R) \) with \( R \in \mathcal{R} \) is a basis for a uniform structure on \( X \) compatible with the quotient topology.
2. If $X$ has a uniform structure compatible with the quotient topology such that $f(R)$ is an entourage for some root domain $R$ in $Y$ (such a uniformity always exists), then:

(a) There is a countable basis $E$ for the uniformity of $X$ consisting of evenly covered entourages with respect to $R$.

(b) For any root domain $R$, the set of all $E^*_R$ for $E \in E$ is an invariant basis for a (compatible) uniform structure on $Y$, called the lifted uniformity, such that $G$ acts discretely.

(c) The lifted uniformity is unique in the sense that if $R'$ is any other root domain such that $f(R')$ is an entourage in $X$ then the lifted uniformities with respect to $R$ and $R'$ are uniformly equivalent.

**Proof.** For existence in the first part we may use the “fine uniformity” on $Y$; since $Y$ is metrizable this simply consists of all symmetric neighborhoods of the diagonal in $Y \times Y$ (it is easy to check that this is a compatible uniformity).

Root domains are open symmetric subsets containing the diagonal, hence are entourages. For Part 1a, just intersect each set in the original countable basis with a single root domain $R$.

Now suppose that $R \in R$. Since every $f(R)$ is open in the quotient topology, to prove the basis statement we need only prove the “triangle inequality”. Suppose $S$ is a root domain with $S^2 \subset R$. Suppose that $(x, y) \in f(S)^2$; so there exist $(x, z), (y, z) \in f(S)$. This in turn means there are $(x', z'), (y'', z'') \in S$ such that $f(x') = x, f(z') = f(z'') = z$ and $f(y'') = y$. The middle equation means that there is some $g \in G$ such that $g(z'') = z'$, and since $S$ is invariant, letting $y' := g(y'')$ we have $(z', y') \in S$. But then $(x', y') \in S^2 \subset R$. Therefore $(x, y) \in f(R)$, showing that $f(S)^2 \subset f(R)$.

For Part 2, existence follows from Part 1. For Part 2a we simply intersect each set in a countable basis for $X$ with $f(R)$ for some root domain $R$.

Now every $E^*_R$ is a symmetric open set containing the diagonal in $Y \times Y$, and therefore we need only show the following to prove that the set of all $E^*_R$ is a basis for a uniform structure: If $F$ is an evenly covered entourage with $F^2 \subset E$ then $(F^*_R)^2 \subset E^*_R$. Let $(a, b) \in (F^*_R)^2$; so there are $(a, c), (b, c) \in F^*_R$. This means that $(f(a), f(b)), (f(b), f(c)) \in f(F^*_R) = F$ (see Proposition 27.2). In other words, $(f(a), f(b)) \in F^2 \subset E$. That is, $(a, b) \in f^{-1}(E)$, and since $(a, b) \in F^*_R = f^{-1}(F) \cap R$, $(a, b) \in R$, and by definition $(a, b) \in E^*_R$.

Now let $U$ be open in $Y$ and $y \in Y$. Since $f(U)$ is open, there is some evenly covered entourage $E$ such that $B(f(y), E) \subset f(U)$. By Proposition 28 the restriction $h$ of $f$ to $B(y, E^*_R)$ is a bijection onto $B(f(y), E)$. Note that $E^*_R = f^{-1}(E) \cap R$ is open and therefore $V := B(y, E^*_R) \cap U$ is open. Now $W := f(U) \cap f(V)$ is open, contains $f(y)$ and is contained in $B(f(y), E)$. But then there is an entourage $F \subset E$ in $X$ such that $B(f(y), F) \subset W$. Since $h$ is a homeomorphism from $B(y, E^*_R)$ onto $B(f(y), E)$, $B(y, F^*_R) = h^{-1}(B(f(y), F))$ is contained in $U$.

We already know that the sets $E^*_R$ are invariant, so to finish the proof of Part 2b we need only show that $G$ acts discretely, which will follow if we prove that for
any evenly covered \( E \), \( N_{E}^{*}(G) = \{1\} \). But this is immediate from Proposition 25. If for some \( y \in Y \) and \( g \in G \), \( (y, g(y)) \in E_{R}^{*} \) then \( g(y) \in B(y, E_{R}^{*}) \). But \( f(y) = f(g(y)) \) and the restriction of \( f \) to \( B(y, E_{R}^{*}) \) is 1-1, so \( y = g(y) \) and \( g = 1 \).

For Part 2c, by taking intersections we may assume that the two root domains satisfy \( R' \subset R \). Again by taking intersections, we can take for our basis elements \( E_{R}^{*} \) and \( E_{R'}^{*} \) of the lifted uniformities of \( R \) and \( R' \), respectively, using entourages \( E \) that are evenly covered by both \( R \) and \( R' \). That is, \( E \subset f(R') \). Suppose that \( (a, b) \in E_{R}^{*} = f^{-1}(E) \cap R \subset f^{-1}(f(R')) \cap R \), so \( (f(a), f(b)) \in f(R') \) and therefore there exist \( (a', b') \in R' \) such that \( f(a') = f(a) \) and \( f(b') = f(b) \). Since \( R' \) is invariant, as before we may find \( b'' \) such that \( f(b'') = f(b) \) and \( (a, b'') \in R' \). Since \( (a, b) \in R \), \( (b, b'') \in RR' \subset R^{2} \subset P \), where \( P \) is the PD-Domain. By Proposition 24 \( b = b' \) and therefore \( (a, b) \in R' \). That is, \( (a, b) \in E_{R'}^{*} \), showing that \( E_{R}^{*} \subset E_{R'}^{*} \). Since \( R' \subset R \), the reverse inclusion is also true, completing the proof of the theorem. ☐

**Remark 31** We summarize how to uniformize a regular covering map \( f : Y \to X = Y/G \). First give \( Y \) a uniform structure such that some root domain is an entourage. Take the quotient uniformity on \( X = Y/G \), and then lift the quotient uniformity back to \( Y \), from which one obtains a (possibly not strictly) finer uniformity than the original, which has an invariant basis and with respect to which \( G \) acts discretely. The uniformities on \( X \) and \( Y \) are uniquely determined by the original uniformity on \( Y \). When \( Y \) is compact then of course there is only one compatible uniformity on each of \( X \) and \( Y \), but Theorem 23 still provides the important information that one can find an invariant basis with respect to which \( G \) acts discretely.

**Notation 32** In general, root domains, which include all symmetric open sets containing the diagonal that are contained in a given root domain, need not be entourages. When \( Y \) is a uniform space and \( G \) acts discretely and isomorphically on \( Y \), a root domain in \( Y \) that is also an entourage will be called a root entourage.

The next lemma, which we will use without reference, brings us full circle concerning uniformizing properly discontinuous actions.

**Lemma 33** Suppose \( G \) acts discretely and isomorphically on a uniform space \( Y \). Then

1. \( G \) acts properly discontinuously on the topological space \( Y \).
2. \( Y \) has a basis of root entourages.
3. The uniform structure on \( Y \) is the lift of the quotient uniformity on \( X = Y/G \) with respect to any root entourage \( R \) in \( Y \).

**Proof.** For the first part we need only observe that the orbits via a discrete action are uniformly discrete, hence closed, so the quotient is Hausdorff. For the second part, let \( E \) be any invariant entourage in \( Y \) such that \( N_{E}(G) = \{1\} \).
and suppose that \((x, g(x)) \in E\) for some \(x \in X\) and \(g \in G\). Then by definition, \(g \in N_E = \{1\}\), so \(E\) is contained in the PD-domain. But then any invariant entourage \(R\) such that \(R^2 \subset E\) is a root domain. The second part follows.

For the third part let \(f : Y \to X = Y/G\) be the quotient map, suppose that \(R\) is a root entourage in \(Y\) and \(F \subset R\) is an invariant entourage in \(Y\). Since \(f(F)^* = f^{-1}(f(F)) \cap R\) is an intersection of entourages, \(f(F)^*\) is an entourage. We will show \(f(F)^* \subset F\) (in fact they are equal), completing the proof. If \((x, y) \in f(F)^*\) then \((x, y) \in R\) and there is some \((x', y') \in F\) such that \(f(x') = f(x)\) and \(f(y') = f(y)\). Now there exists some \(g \in G\) such that \(g(x') = x\), and since \(F\) is invariant, \((x, g(y')) \in F \subset R\). Therefore \((y, g(y')) \in F^2\), which is contained in the PD-domain. Since \(f(y) = f(y') = f(g(y'))\), Proposition \(27\) implies \(y = g(y')\). That is, \((x, y) \in F\).

5 Discrete Covers

Lemma 34 (General Chain Lifting Lemma) Suppose \(f : Y \to X = Y/G\) is a discrete cover, \(R\) is a root entourage in \(Y\) and \(E\) is an evenly covered entourage with respect to \(R\). If \(\alpha\) is an \(E\)-chain in \(X\) starting at \(x_0\) and \(\bar{x}_0 \in f^{-1}(x_0)\) then there is a unique \(E_R^*\)-chain \(\alpha\) starting at \(\bar{x}_0\), called the lift of \(\alpha\), such that \(f(\alpha) = \alpha\). Moreover, \([\alpha]_E = [\beta]_E\) if and only if \([\alpha]_{E_R^*} = [\beta]_{E_R^*}\), and in particular if \([\alpha]_E = [\beta]_E\) then \(\alpha\) and \(\beta\) end at the same point in \(X\).

Proof. The existence and uniqueness of \(\bar{\alpha}\) is immediate from iteration using the fact that the fact that \(E\)-balls are evenly covered by \(E_R^*\)-balls (Proposition \(25\)). Now suppose that \(\alpha = \{x_0, \ldots, x_n\}\) has unique lift \(\bar{\alpha} = \{\bar{x}_0, \ldots, \bar{x}_n\}\) and \(\beta = \{x_0, \ldots, x_i, x, x_{i+1}, \ldots, x_n\}\); that is, \(\beta\) differs from \(\alpha\) by the basic move of adding a point \(x\). Note that \(x, x_{i+1} \in B(x_i, E)\). Therefore there are unique points \(\bar{x}, \bar{z} \in B(\bar{x}_i, E_R^*)\) such that \(f(\bar{z}) = x_{i+1}\) and \(f(\bar{x}) = x\). By uniqueness, \(\bar{z} = \bar{x}_{i+1}\). On the other hand, the restriction of \(f\) to \(B(\bar{x}, E_R^*)\) is a bijection onto \(B(x, E)\), which contains \(x_i\) and \(x_{i+1}\). Therefore there is some \(\bar{w} \in B(\bar{x}, E_R^*)\) such that \(f(\bar{w}) = x_{i+1}\). But then \((x_{i+1}, \bar{w}) \in (E_R^*)^2 = (f^{-1}(E) \cap R)^2 \subset R^2 \subset P\), where \(P\) is the PD-domain. Since \(f(\bar{x}_{i+1}) = x_{i+1} = f(\bar{w})\), Proposition \(27\) implies that \(\bar{x}_{i+1} = \bar{w}\). Since \((\bar{w}, \bar{x}) \in E_R^*, (\bar{x}_{i+1}, \bar{x}) \in E_R^*\). That is, adding \(\bar{x}\) to \(\bar{\alpha}\) is a basic \(E_R^*\)-move. Removing a point is simply the inverse operation, completing the proof that if \([\alpha]_E = [\beta]_E\) then \([\alpha]_{E_R^*} = [\beta]_{E_R^*}\). The converse follows from the fact that \(f(E_R^*) = E\).

Remark 35 The above lemma is actually the third version of a Chain Lifting Lemma (see \(19\) and \(20\)); we will discuss this in more detail just prior to Theorem \(41\).

Proposition 36 If \(f : Y \to X = Y/G\) is a discrete cover between compact metric spaces then \(G\) is finite. Suppose in addition that \(X\) is connected. Then \(Y\) has finitely many components \(C\), each of which is uniformly open, and the restriction \(f_C\) to any of which surjective. In particular \(f_{C} : C \to X = C/S_C\) is a discrete cover, where \(S_C\) is the stabilizer subgroup of \(C\).
Proof. If \( \{y_i\} \) were an infinite collection of distinct elements of \( G \) then taking some \( y \in Y \), without loss of generality we could suppose that \( y_i = g_i(y) \) is a convergent sequence consisting of distinct points, since \( G \) acts freely. But this is impossible because the action of \( G \) is discrete.

For the next statement we first prove that the restriction of \( f \) to \( C \) is surjective. Let \( R \) be a root entourage of \( Y \) and \( E \) be evenly covered with respect to \( R \). Since \( X \) is connected, if \( w = f(y) \) for some \( y \in C \) then for any \( x \in X \) there are arbitrarily fine chains in \( X \) from \( w \) to \( x \). By the General Chain Lifting Lemma these chains (more precisely those that are finer than \( E \)-chains) must lift to arbitrarily fine chains starting at \( y \) ending at some point in \( f^{-1}(x) \). But the latter set is finite, so there must be a point \( z \in f^{-1}(x) \) such that there are arbitrarily fine chains from \( y \) to \( z \). But this means that \( z \in C \), and since \( f(z) = x \) this shows that the restriction of \( f \) to any component is surjective. Therefore each component contains some point in \( f^{-1}(x) \) and since the latter set is finite, there are only finitely many components.

Since there are only finitely many (compact) components, each component \( C_i \) is open. Define \( U \subset Y \times Y \) to be the union of the sets \( C_i \times C_i \); then \( U \) is an entourage in \( Y \) (\( Y \) is compact) and as is easily checked each component is uniformly \( U \)-open.

The final statement is an immediate consequence of Lemma 23. \( \blacksquare \)

**Proposition 37** Let \( f : Y \to X = Y/G \) be a discrete cover and suppose \( E \) is evenly covered with respect to a root entourage \( R \) in \( Y \). Then for any \( y \in Y \), \( E_{R^1} \)-chain \( \alpha \) from \( * \) to \( y \) in \( Y \), and \( g \in G \), \( g(y) \) is the endpoint of the unique lift of the \( E \)-chain \( f(\alpha) \) to \( Y \) at \( g(\alpha) \).

**Proof.** Since \( E_{R^1} \) is invariant, \( g(\alpha) \) is an \( E_{R^1} \)-chain from \( g(\alpha) \) to \( g(y) \). On the other hand, \( f(g(\alpha)) = f(\alpha) \) and therefore \( g(\alpha) \) is the unique lift of \( f(\alpha) \) to \( Y \) at \( g(\alpha) \). \( \blacksquare \)

**Definition 38** If \( f : Y \to X = Y/G \) is a discrete cover, an entourage \( E \) in \( X \) is called properly covered if for some root entourage \( R \) in \( Y \), if \( E \subset f(R) \) (i.e. \( E \) is evenly covered with respect to \( R \)) and \( Y \) is \( E_{R^1} \)-chain connected. In this case we will say \( E \) is properly covered with respect to \( R \). If \( X \) has a properly covered entourage \( E \) then \( f \) is called a proper discrete cover.

Note that if \( Y \) is chain connected then every evenly covered entourage is properly covered.

**Theorem 39** Let \( f_i : Y_i \to X = Y_i/G_i \) be discrete covers of uniform spaces. Suppose \( f_1 \leq f_2 \), i.e. there is some uniformly continuous surjection \( f : Y_2 \to Y_1 \) such that \( f_2 = f_1 \circ f \). Define \( \theta_f : G_2 \to G_1 \) by letting \( \theta_f(g) \) be the unique element of \( G_1 \) such that \( \theta_f(g)(\alpha) = f(g(\alpha)) \). Then:

1. Suppose that \( R_2 \) is a root entourage in \( Y_2 \) and let \( R_1 := f(R_2) \). Then

   (a) \( R_1 \) is a root entourage in \( Y_1 \).
(b) If \( E \) is evenly covered with respect to \( R_2 \) then \( E \) is evenly covered with respect to \( R_1 \), and \( E^*_R = f(E^*_{R_2}) \).

(c) If \( \alpha \) is any \( E^*_R \)-chain in \( Y_2 \) from \( * \) to \( y \) and \( g \in G_2 \) then \( \theta f(g)(f(y)) = f(g(y)) \), which is the endpoint of the unique lift of the \( E \)-chain \( f^*_2(\alpha) \) to \( Y_1 \) at \( f(g(*)) \).

(d) If \( E \) is properly covered with respect to \( R_2 \) then \( E \) is properly covered with respect to \( R_1 \).

2. \( \theta f \) is a surjective homomorphism, and if \( E \) is properly covered with respect to \( R_2 \) we have the compatibility condition \( f \circ g = \theta f(g) \circ f \).

3. \( f : Y_2 \to Y_1 = Y_2/K_f \) is a discrete cover, where \( K_f \) is a normal subgroup of \( G_2 \).

4. \( f_1 \) is equivalent to the induced quotient \( \pi : Y_2/K_f \to (Y_2/K_f)/(G_2/K_f) \).

**Proof.** We start by showing that \( f \) is bi-uniformly continuous. Let \( R \) be a root entourage in \( Y_1 \). Since \( f \) is uniformly continuous there is a root entourage \( R' \subset \) in \( Y_2 \) such that \( f(R') \subset R \). Suppose that \( F \) is an entourage in \( X \) that is evenly covered with respect to both \( R \) and \( R' \). Since by Theorem \( \exists \) the set of all \( F^*_R \) is a basis for the uniform structure on \( Y_2 \), the proof will be complete if we show that

\[
 f(F^*_R) = F^*_R \tag{5}
\]

and hence \( f(F^*_R) \) is an entourage in \( Y_1 \). Since \( f(R') \subset R \) and \( f_1 \circ f = f_2 \),

\[
 f(F^*_R) \subset f(R') \cap f(f_2^{-1}(F)) \subset R \cap f_1^{-1}(F) = F^*_R. \tag{6}
\]

For the reverse inclusion, let \((c, d) \in F^*_R \), i.e. \( d \in B(c, F^*_R) \). Since \( f \) is surjective there is some \( c' \in Y_2 \) such that \( f(c') = c \). From Proposition \( \exists \) we know that \( F \)-balls are evenly covered by both \( F^*_R \) and \( F^*_R \)-balls. If \( h_1 \) denotes the restriction of \( f_1 \) to \( B(c, F^*_R) \), then the restriction of \( h_1^{-1} f_2 \) to \( B(c', F^*_R) \) is a bijection onto \( B(c, F^*_R) \). Therefore there is some \( d' \in B(c', F^*_R) \) such that \( h_1^{-1}(f_2(d')) = d \).

But by Inclusion \( \exists \), \( f(B(c', F^*_R)) \subset B(c, F^*_R) \) and therefore \( f(d') \in B(c, F^*_R) \).

Since \( h_1 \) is a bijection and \( f_1(f(d')) = f_2(d') = f_1(d) \), it must be that \( f(d') = d \).

Returning to the proof of Part 1a, we now know that \( R_1 \) is an entourage and therefore we need only show that \( R_2^2 \) is contained in the PD-domain of \( Y_1 \). Suppose that \((y, g_1(y)) \in R_1^2 \) for some \( y \in Y_1 \) and \( g_1 \in G_1 \). This means that \((y, z), (z, g_1(y)) \in R_1 \) for some \( z \in Y_1 \). Since \( R_1 = f(R_2) \) this in turn means there exist \((y', z'), (z'', w'') \in R_2 \) such that \( f(y') = y, f(z') = f(z'') = z, \) and \( f(w'') = g_1(y) \). Since

\[
 f_2(z') = f_1(f(z')) = f_1(z) = f_1(f(z'')) = f_2(z''),
\]

there is some \( g_2 \in G_2 \) such that \( g_2(z'') = z' \). Since \( R_2 \) is invariant, letting \( w' := g_2(w'') \), we have that \((w', z') \in R_2 \). Therefore \((y', w') \in R_2^2 \), which is contained in the PD-domain \( P \) of \( Y_2 \). Now

\[
 f_2(y') = f_1(f(y')) = f_1(y) = f_1(g_1(y)) = f_1(f(w'')) = f_2(w'') = f_2(w').
\]
Therefore there is some \( h \in G_2 \) such that \( h(y') = w' \). Since \((y', w') \in P\) this means that \( h = 1 \) and hence \( y' = w' \). This in turn means that \( y = f(y') = f(w') = g_1(y) \), so \( g_1 = 1 \), finishing the proof of Part 1a.

For Part 1b, note that if \( E \subset f_2(R_2) \) then \( E \subset f_1(f(R_2)) \subset f_1(R_1) \). Next note that Equation \((\ref{eq:43})\) depended only on \( f(R') \subset R \) (and the entourage in question being evenly covered by both) and therefore it applies to \( R' := R_2 \) and \( R := R_1 = f(R_2) \).

For Part 1c, note that since \( f_2(\alpha) = f_2(g(\alpha)) \) and \( f_1 \circ f = f_2 \), the \( E^*_{R_1} \)-chain \( f(g(\alpha)) \) is the unique lift of \( f_2(\alpha) \) to \( Y_1 \) at \( f(g(\alpha)) \), which ends at \( f(y) \). On the other hand, by Proposition \( 27 \) \( \theta_f(g)(f(y)) \) is also the endpoint of the unique lift of \( f_1(f(\alpha)) = f_2(\alpha) \) to \( Y_1 \) at \( \theta_f(g)(\alpha) = f(g(\alpha)) \). Therefore, \( \theta_f(g(f(y)) = f(g(y)) \).

If \( E \) is properly covered with respect to \( R_2 \) then since \( f \) is surjective, given \( y_1 \in Y_1 \) we may find \( y_2 \in f^{-1}(y_1) \) and an \( E^*_{R_2} \)-chain \( \alpha \) from \( * \) to \( y_2 \). But then \( f(\alpha) \) is an \( f(E_{R_2}^*) = E^*_{R_1} \)-chain from \( * \) to \( y_1 \). That is, \( E \) is properly covered with respect to \( R_1 \).

For the second part, suppose \( g_1, g_2 \in G_2 \). By definition of \( \theta_f \) and Part 1c, using \( \{ \ast \} \) as an \( E^*_{R_2} \)-chain from \( * \) to \( * \),

\[
\theta_f(g_1(g_2)(\ast)) = f(g_1(g_2(\ast))) = \theta_f(g_1)(f(g_2(\ast))) = \theta_f(g_1)(\theta_f(g_2)(\ast)),
\]

showing that \( \theta_f \) is a homomorphism (since the action is free).

For surjectivity of \( \theta_f \), suppose that \( h \in G_1 \). Since \( f \) is surjective there is some \( x \in Y_2 \) such that \( f(x) = h(\ast) \). Since \( f_2(x) = f_1(f(x)) \) and \( f(h(\ast)) = * \) there is a unique \( h_2 \in G_2 \) such that \( h_2(\ast) = x \). By definition, \( \theta_f(h_1)(\ast) = f(h_1(\ast)) = f(x) = h(\ast) \). Since the action is free, \( \theta_f(h_1) = h \). If \( E \) is properly covered with respect to \( R_2 \) then the \( E^*_{R_2} \)-chain \( \alpha \) in the statement of Part 1c always exists, and compatibility follows.

For the last part, since \( f \) is bi-uniformly continuous we need only show that the point preimages of \( f \) are the orbits of \( K_f \) (see Remark \( 13 \) and Proposition \( 21 \)), where \( K_f \) is the kernel of \( \theta_f \). If \( x = g(y) \) then by Part 1c, \( f(x) = f(g(y)) = \theta_f(g)(f(y)) = f(y) \). Conversely, suppose that \( f(x) = f(y) \). Then \( f_2(x) = f_1(f(x)) = f_1(f(y)) = f_2(y) \) and therefore there is some \( g \in G_2 \) such that \( x = g(y) \).

We will now revisit the situation \( \phi_E : X_E \to X \). By Proposition 16.1 in \( \text{(2)} \), the restriction of \( \phi_E \) to any \( E^* \)-ball is 1-1. Now if \( ([\alpha]_E, [\beta]_E) \in (E^*)^2 \), \( [\alpha]_E \) and \( [\beta]_E \) lie in some \( E^* \)-ball and therefore it is impossible for there to be some \( g \neq 1 \) in \( \pi_E(X) \) with \( g([\alpha]_E) = [\beta]_E \). That is, \( R = E^* \) is a root entourage, and in fact it is a canonical choice of root entourage for \( \phi_E \). Now if \( F \subseteq E \), an immediate consequence of the definition of \( F^* \) is that \( F^* = f^{-1}(F) \cap E^* \).

In other words, in the current terminology, \( F^* = F^*_{E'} \). For simplicity and consistency we will continue to use the notation \( F^* \) for the lift of \( F \) with respect to the root entourage \( E^* \). In this case the Chain Lifting Lemma has the following stronger form \((\text{139})\): If \( \alpha = \{ * = x_0, \ldots, x_n \} \) is an \( F \)-chain in \( X \) with \( F \subseteq E \)
then the unique lift of \( \alpha \) to \( X_E \) at * consists of the \( F^* \)-chain

\[
\tilde{\alpha} = \{[x_0]_E, [x_0, x_1]_E, ..., [x_0, ..., x_n]_E = [\alpha]_E\}.
\]

See also Equation (1), from which this stronger version follows. In particular, \( \tilde{\alpha} \) ends at \( [\alpha]_E \). In order to avoid confusion among what will ultimately amount to three different chain lifting lemmas, we will refer to this collection of statements as the Special Chain Lifting Lemma (for \( \phi_E \)). This statement implies that every \( [\alpha]_E \) is joined to * by the \( E^* \)-chain \( \tilde{\alpha} \), i.e. \( X_E \) is \( E^* \)-connected. We summarize these observations in a corollary that we will use below without reference:

**Corollary 40** If \( X \) is a uniform space and \( E \) is an entourage, then for the map \( \phi_E : X_E \to X \), \( E^* \) (as defined in \([3]\)) is a root entourage and \( E \) is properly covered by the entourage \( E^* \), which is equal to \( E^*_E \), as defined in the current paper.

**Theorem 41** Let \( f : Y \to X = Y/G \) be a discrete cover with root entourage \( R \) and properly covered entourage \( E \). Define \( f_E : X_E \to Y \) as follows. For any \([\alpha]_E \in X_E\), let \( f_E([\alpha]_E) \) be the endpoint of the unique lift \( \tilde{\alpha} \) of the \( E \)-chain \( \alpha \) to \( Y \) at * . Then \( f_E \) is a uniformly continuous surjection such that \( \phi_E = f \circ f_E \). In particular, \( f \leq \phi_E \) and \( f \) is equivalent to the induced quotient \( \pi : X_E/K \to (X_E/K)/(\pi_E(X)/K) \) for some normal subgroup \( K \) of \( \pi_E(X) \).

**Proof.** By the Special Chain Lifting Lemma, \( f_E \) is well defined. Since \( E \) is properly covered, for any \( y \in Y \) there is an \( E^*_R \)-chain \( \beta \) from * to \( y \). Then \( f(\beta) \) is an \( E \)-chain from * to \( f(y) \) in \( X \). But then \( \beta \) is the lift of \( f(\beta) \) to \( Y \) at * and by definition \( f([\beta]_E) = y \). That is, \( f_E \) is surjective. By definition, \( \phi_E = f \circ f_E \).

To prove that \( f \) is uniformly continuous we will show that if \( F \subset E \) then \( f(F^*) \subset F^*_R \) (the latter of which is a basis element in \( Y \) by Lemma \([33]\)). Suppose \(([\alpha]_E, [\beta]_E) \in F^* \); so there are \( E \)-chains \( \alpha = \{* = x_0, ..., x_n = a\} \) and \( \beta = \{* = x_0, ..., x_n = a, b\} \) with \((a, b) \in F \). Now \( f_E([\alpha]_E) \) is the endpoint of the unique lift \( \tilde{\alpha} = \{* = \tilde{x}_0, ..., \tilde{a}\} \) of \( \alpha \) to \( Y \) at *, and by uniqueness the lift \( \tilde{\beta} \) of \( \beta \) to \( Y \) at * must be \( \{* = \tilde{x}_0, ..., \tilde{a}, \tilde{b}\} \) for some \( \tilde{b} \in f^{-1}(b) \). So \( f_E([\alpha]_E, [\beta]_E) = (\tilde{a}, \tilde{b}) \in f^{-1}(F) \). Now \( \tilde{\alpha} \) and \( \tilde{\beta} \) are \( E^*_R \)-chains and therefore \((\tilde{a}, \tilde{b}) \in E^*_R \subset R \). Therefore \( f([\alpha]_E, [\beta]_E) = (\tilde{a}, \tilde{b}) \in f^{-1}(F) \cap R = F^*_R \). This shows that \( f \leq \phi_E \), and the last statement is an immediate consequence of Theorem \([39]\).

**Remark 42** As mentioned previously, \( E^* \) is the canonical choice for a root entourage in \( X_E \). From Theorem \([37]\) and Theorem \([39]\) we now see that if \( f : Y \to X = Y/G \) is a proper discrete cover then for any properly covered entourage \( E \) there is a canonical choice for a root entourage in \( Y \), namely \( R := f(E^*) \), and \( E \) is properly covered with respect to \( R \).

**Theorem 43** Suppose \( X \) is a continuum and \( E \) is an entourage. Then there exists a finitely generated subgroup \( H \) of \( \pi_E(X) \) that is the stabilizer subgroup of a uniformly open (hence closed) set \( J \subset X_E \) containing the basepoint such that the following is true. If \( f : Y \to X/G \) is a discrete cover of \( X \) by a
therefore \( c \) contained in the subgroup \( F \) of \( \pi \), for some smallest \( i \).

\[ \text{Note also that for any } j, \text{ concatenation of } L \text{ is the stabilizer subgroup of } \pi(\pi(X)). \text{ We then see that } L \text{ consists of all } \{ \lambda \} \}

This modified loop as a product of loops, each of which has at most 2 points, each equal to some \( c \). Then compactness and the “\( E^m \)-close \( E^{m+n} \)-homotopic” trick (Remark 10) show that there are only finitely many such loops, i.e. \( L \) is contained in some finitely generated group \( H \). We then see that \( H \) is the stabilizer subgroup of \( \phi(\pi(\pi(X))) \), and define \( J \) to be \( H = \bigcup_{k \in H} k(I) \), which has \( H \) as its stabilizer subgroup. It then follows that

\[ \text{the restriction } \rho \text{ of } \phi \text{ to } J \text{ is a discrete cover of } X \text{ via the action of the finitely generated group } H. \text{ Then Theorem } \ref{thm:covering}\text{ finishes the proof.} \]

Here are the details. We first claim there is a finite set \( \Gamma = \{ \gamma_i \} \} \}

such that every element of \( L \) is a product of elements of \( \Gamma \). In other words, \( L \) is contained in the subgroup \( H \) of \( \pi(X) \) generated by \( \Gamma \).

We subclaim that any \( F \)-loop \( \lambda = \{ * = x_0, ..., x_n = * \} \) is \( F^3 \)-homotopic to an \( F^3 \)-loop

\[ \lambda' = \{ * = c_{i_0}, c_{i_1}, ..., c_{i_{n-1}}, c_{i_n} = * \}. \]

The iterative argument is similar to one in the proof of Theorem 37 in [3], but for completeness and due to changes in notation, we will prove it here. For every \( 0 < j < n \), \( x_j \in B(c_j, F) \) for some \( c_j \). Each of the following moves is a basic \( F^2 \)-move:

\[ \lambda \rightarrow \{ x_0, x_1, x_1, x_2, ..., x_n \} \rightarrow \{ x_0, x_1, c_{i_1}, x_1, x_2, ..., x_n \} \rightarrow \{ x_0, c_{i_1}, x_1, x_2, ..., x_n \}. \]

Note also that for any \( j \), we have \( (c_j, x_j), (x_j, x_{j+1}), (x_{j+1}, c_{i_{j+1}}) \} \}

and \( \{ c_{i_j}, x_{j+1} \} \} \}

\( F \) and \( \{ c_{i_j}, c_{i_{j+1}} \} \} \}

\( F^3 \). Now each of the following is a basic \( F^3 \)-move:

\[ \{ x_0, c_{i_1}, x_2, x_3, ..., x_n \} \rightarrow \{ x_0, c_{i_1}, c_{i_2}, x_2, x_3, ..., x_n \} \rightarrow \{ x_0, c_{i_1}, c_{i_2}, c_{i_3}, x_3, ..., x_n \}. \]

Proceeding iteratively finishes the proof of the subclaim.

Now suppose that \( n > M \), i.e. there is a repeated point in \( \lambda' \). That is, for some smallest \( i_j \) and \( i_k > i_j \), \( c_{i_j} = c_{i_k} \). Then \( \lambda' \) is \( F^3 \)-homotopic to the concatenation of

\[ \lambda'' = \{ * = c_{i_0}, ..., c_{i_j}, c_{i_{j+1}}, ..., c_{i_k} = c_{i_j}, c_{i_{j-1}}, ..., c_{i_0} = * \} \]
\[
\lambda'' = \{\ast = c_{i_0}, \ldots, c_{i_k} = c_{i_{k+1}}, \ldots, c_{i_n} = \ast\}.
\]

Here the discrete homotopies are all “retractions” of the form \([\tau \ast \tau]_{F^3} = [t]_{F^3}\), where \(t\) is the endpoint of \(\tau\). That is, \([\lambda']_{F^3} = [\lambda'']_{F^3}[\lambda''']_{F^3}\), where \(\lambda'''\) has at most \(2M + 1\) points and \(\lambda''\) has strictly fewer points than \(\lambda'''\). Repeating this process shows that \([\lambda]_{F^3}\) is a product of \(F^3\)-equivalence classes of \(F^3\)-loops having at most \(2M + 1\) points. The proof of our claim will be finished if we show that the set \(\Gamma\) of \(E\)-homotopy equivalence classes of such loops is finite. Suppose \(\{\lambda_m\}_{m=1}^\infty\) is a sequence of \(F^3\)-loops each having at most \(2M + 1\) points such that no two are \(F^3\)-homotopic. By duplicating points if necessary (which doesn’t change the \(F^3\)-homotopy class) we can suppose that \(\lambda_m = \{\ast = z_0^m, \ldots, z_{2M}^m = \ast\}\). Taking a subsequence if necessary we may assume that for all \(j, z_j^m \rightarrow z_j^m\) for some \(z_j\). Then \(\lambda = \{\ast = z_0, \ldots, z_{2K} = \ast\}\) is an \(F^3\)-loop and hence is an \(F^3\)-loop. In fact, since \((z_i, z_{i+1})\) is in the closure of \(F^3\), we may find \((z_i', z_{i+1}')\) in \(F^3\) such that \(z_i' \in B(z_i, F)\) and \(z_{i+1}' \in B(z_{i+1}, F)\). But then \((z_i, z_{i+1})\) in \(F^5\).

Now for all \(j\) and all large enough \(m\), \((z_j^m, z_j^m) \in F^E\). Referring again to Remark \(\text{[10]}\) we see that that \(\lambda_m\) is \(F^0\)-homotopic to \(\lambda\) and therefore for all large \(m, n\), \(\lambda_m\) is \(F^0\)-homotopic, hence \(E\)-homotopic, to \(\lambda_n\). This is a contradiction, proving the claim that \(\Gamma\) is finite.

Now define \(I := \phi_{E^F}(X_F)\). We claim that \(I\) is the \(F^*\)-chain component of \(\ast\) in \(X_E\). Suppose that \([\alpha]_E \in I\), which means we can assume that \(\alpha\) is an \(F\)-chain, and suppose that for some \([\beta]_E\), \(([\alpha]_E, [\beta]_E) \in F^*\). This means \([\bar{\alpha} * \bar{\beta}]_E = [a, b]_E\) with \((a, b) \in F\). Note that \([\beta]_E = [\alpha * \{a, b\}]_E\) and \(\alpha * \{a, b\}\) is an \(F\)-chain, showing that \([\beta]_E \in I\). That is, \(I\) is uniformly \(F^*\)-open and therefore contains the \(F^*\)-chain component of \(\ast\) in \(X_E\). On the other hand, if \([\alpha]_E \in I\), then by the Special Chain Lifting Lemma, the lift of \(\alpha\) to \(X_E\) at the basepoint is an \(F^*\)-chain that ends at \([\alpha]_E\). That is, \(I\) is \(F^*\)-chain connected and is therefore the \(F^*\)-chain component of \(\ast\) in \(X_E\).

We next claim that \(L\) is the stabilizer subgroup of \(I\). In fact, if \([\lambda]_E \in L\), this means that \(\lambda\) is an \(F\)-chain and \([\lambda]_E([\ast]) = [\lambda]_E \in I\). The claim now follows from Lemma \(\text{[23]}\).

Now let \(J := H(I) = \{g(x) : g \in H \text{ and } x \in I\}\). Put another way, \(J\) is the union of the translates \(g(I)\) with \(g \in H\). Note that since \(F^*\) is invariant with respect to \(\pi_E(X)\), and \(I\) is uniformly \(F^*\)-open, \(J\) is uniformly \(F^*\)-open. Since \(X\) is (chain) connected, \(\phi_F : X_F \rightarrow X\) is onto, and since \(X = \phi_F(X_F) = \phi_E(\phi_F(X_F)) = \phi_E(I) \subset \phi_E(J)\), the restriction \(\rho\) of \(\phi_E\) to \(J\) is onto.

We next claim that \(H\) is the stabilizer subgroup \(S_J\) of \(J\) and \(H\) has Property ST, so we may apply Lemma \(\text{[23]}\) to conclude that \(\rho : J \rightarrow X = J/H\) is a discrete cover. If \(x = g(y) \in J\) with \(y \in I\) and \(g \in H\), and \(h \in H\), then \(h(x) = hg(y) \in J\), so \(h(J) \subset J\). Now let \(w \in J\). Since \(h\) is a bijection there is some \(z \in X_E\) such that \(h(z) = w\), and hence \(h^{-1}(w) = z\). But by what we just showed, \(z \in J\), showing that \(h = h(J)\), i.e. \(H \subset S_J\).

We next note that \(R := E^* \cap (J \times J)\) is a root domain for \(\rho\) (since as we have previously observed, \(E^*\) is a root domain for \(\phi_E\)). We claim that \(F\) is evenly covered with respect to \(R\). In fact, suppose that \((a, b) \in F\). Since \(\phi_E(F^*) = F\)
there exist \((\alpha_E, \beta_E) \in F^*\) such that \( (a, b) = (\phi_E(\alpha_E, \beta_E)) \). Since \( \rho \) is surjective there is some \( z \in J \) such that \( \rho(z) = a \) and therefore there is some \( g \in \pi_E(X) \) such that \( g(\alpha_E) = z \). Since \( F^* \) is invariant, \((z, g(\beta_E)) \in F^* \). Since \( J \) is uniformly \( F^* \)-open, \( g(\beta_E) \in J \). Therefore \((z, g(\beta_E)) \in F^* \cap (J \times J) \subset R \), showing that \( F \subset \rho(R) \).

Now suppose \( k \in \pi_E(X) \) and \( k(a) = b \) for some \( a, b \in J \) (which includes the case when \( k \in S_J \)). Suppose \( a = h_1(a') \) and \( b = h_2(b') \) with \( h_1, h_2 \in H \) and \( a', b' \in I \). That is, \( k(h_1(a')) = h_2b' \Rightarrow h_2^{-1} \circ k \circ h_1(a') = b' \); so \( h_2^{-1}k h_1 = \lambda \in L \subset H \). That is, \( k = h_2\lambda h_1^{-1} \in H \), proving both that \( H = S_J \) and that \( J \) has Property ST.

Note that since \( F \subset E \), \( F \) is also evenly covered with respect to \( f \). Let \( f_J : J \to Y \) be the restriction of the discrete covering map \( f_E : X_E \to Y \) from Theorem 41. That is, \( f_E([\alpha_E]) = \alpha \) is the endpoint of the lift of \( \alpha \) to \( Y \) at \( * \). We claim that \( f_J \) is surjective. In fact, there is some \( F[G] \)-chain \( \beta \) from \(* \) to any \( y \in Y \). Then \( \alpha := f(\beta) \) is an \( F \)-chain, and \( \alpha \) has a unique lift to an \( F[G] \)-chain \( \tilde{\alpha} \) to \( E \) at \(* \), which ends at \([\alpha_E] \). But since \( I \) is uniformly \( F[G] \)-open, \( \tilde{\alpha} \) must stay in \( I \) and therefore \([\alpha_E] \in I \subset J \). By definition of \( f_E \), \( f_E([\alpha_E]) = y \) and so \( f_J \) is surjective. Since \( f_J \) is uniformly continuous and satisfies \( f \circ f_J = \rho \), we have that \( f \leq \rho \). The proof is now finished by Theorem 39.

**Proof of Theorem 9** Suppose that a continuum \( X \) has \( m > n! \) \( n \)-fold covers by continua. Let \( E \) be an entourage in \( X \) that is evenly covered by all of them. Then the finitely generated group \( H \) from Theorem 13 has \( m > n! \) normal subgroups of index \( n \), contradicting a classical algebraic theorem proved by Hall ([9], Section 2).

**Remark 44** There is a natural question: when is a covering map of the form \( \phi_E : X_E \to X \), i.e. when can we take \( K \) to be the trivial group? This question is mostly answered for compact smooth manifolds [24]: For any compact smooth manifold \( M \) of dimension at least 3, the “entourage covers”, i.e. \( \phi_E : M_E \to M \), where \( E \) is an entourage \( E \) has a natural property that ensures \( M_E \) is connected, are precisely those covering maps corresponding to subgroups \( G \) of \( \pi_1(M) \) that are the normal closures of finite sets (Theorem 8, [24]). For the one and only compact manifold of dimension 1, the only entourage covers are the trivial cover and the universal cover (Example 68, [20]). For compact surfaces, the characterization of entourage covers is an open question.

### 6 Generalized Regular Covering Maps

**Definition 45** If \( G \) is complete and acts isomorphically and prodiscretely (resp. discretely) on a uniform space \( X \) then the quotient map \( \phi : X \to X/G \) is called a generalized regular covering map (resp. a discrete regular covering map or simply a discrete cover) with deck group \( G \).

Note that if \( G \) acts discretely then \( G \), as a (uniformly) discrete group, is automatically complete. For those reading [18] we note that what we more appropriately are calling “generalized regular covering maps” in the present
paper were simply called “covers of uniform spaces” in [18]. We also note that Lemma 39 in that paper inadvertently leaves out the word “closed” prior to “subgroup”.

We will need the following theorem summarizing some results about inverse systems of quotients, much of it derived from [18]. We note that in the special case when \( G \) is the fundamental group of a Poincaré space \( Y, X = \tilde{Y} \), and the indexing set is \( \mathbb{N} \), McCord ([17]) proved Theorem 46.2 (Theorem 5.8) and the first part of Theorem 46.4 (Theorem 5.12). Those proofs in fact do not need the stronger assumptions of [17] and carry over directly to the current setting.

**Theorem 46** Suppose \( G \) acts isomorphically on a uniform space \( X \). Suppose that \( K = \{K_\alpha\} \) is a directed set of normal subgroups of \( G \) ordered by reverse inclusion. Let \( X_\alpha := X/K_\alpha \) and define \( \phi_{\alpha\beta} : X_\beta \to X_\alpha \) by \( \phi_{\alpha\beta}(K_\beta x) = K_\alpha x \). Define \( G_\alpha := G/K_\alpha \) and \( \theta_{\alpha\beta} : G_\beta \to G_\alpha \) by \( \theta_{\alpha\beta}(gK_\beta) = gK_\alpha \). Finally, let \( \phi : X \to \lim_{\leftarrow} X_\alpha := \overline{X} \) and \( \theta : G \to \lim_{\leftarrow} G_\alpha := \overline{G} \) be the unique maps determined by the quotient mappings \( \phi_\alpha : X \to X_\alpha = X/G_\alpha \) and \( \theta_\alpha : G \to G_\alpha \). Then

1. The systems \( \{X_\alpha, \phi_{\alpha\beta}\} \) and \( \{G_\alpha, \theta_{\alpha\beta}\} \) comprise an isomorphic inverse system of quotients. (As defined in [18], this means that the natural compatibility condition \( \phi_{\alpha\beta} \circ g = \theta_{\alpha\beta}(g) \circ \phi_{\alpha\beta} \) is satisfied for all \( \alpha \leq \beta \) and \( g \in G_\beta \).) Moreover, the projections \( \phi^\alpha : \overline{X} \to X_\alpha \) and \( \theta^\alpha : \overline{G} \to G_\alpha \) are surjective for all \( \alpha \).

2. The map \( \phi \) is uniformly continuous with dense image in \( \overline{X} \), with the inverse limit uniformity.

3. The map \( \theta \) is a homomorphism with dense image in \( \overline{G} \), with the inverse limit topology, which is the same as the topology of uniform convergence.

4. If \( G \) acts freely then the following are equivalent:
   
   (a) \( \cap_\alpha K_\alpha = \{1\} \)
   
   (b) The maps \( \phi \) and \( \theta \) are injective.
   
   (c) The map \( \phi \) or the map \( \theta \) is injective.

5. If every collection of orbits \( \{K_\alpha x_\alpha\} \) such that \( K_\alpha x_\alpha \subset K_\beta x_\beta \) whenever \( \beta \leq \alpha \) has non-empty intersection then \( \phi \) and \( \theta \) are surjective. This is in particular always true if the orbits are compact.

6. If \( \phi \) is surjective and for every entourage \( F \) in \( X \) there is some \( K_\alpha \) such that \( K_\alpha \subset U_F(G) \), and \( G \) acts freely, then \( \phi \) is equivalent to the quotient map \( \pi : X \to X/K \), where \( K = \cap_\alpha K_\alpha \). If in addition \( G \) acts discretely (resp. each \( K_\alpha \) is complete and \( G \) acts prodiscretely) then \( \phi \) is a discrete (resp. generalized) regular covering map.

**Proof.** That the systems comprise an isomorphic inverse system of quotients follows from Proposition 47 in [18]. Since for every \( \beta \leq \alpha \), \( \phi_{\alpha\beta} \circ \phi^\beta = \phi_\alpha \) and
\[ \theta_{\alpha \beta} \circ \theta^\beta = \theta_{\alpha} \], the mappings \( \phi \) and \( \theta \) are guaranteed by the universal property of the inverse limit and defined by \( \phi(x) := (K_\alpha x) \) and \( \theta(g) := (gK_\alpha) \). We have for all \( \alpha \), \( \phi^\alpha \circ \phi = \phi_\alpha \) and \( \theta^\alpha \circ \theta = \theta_\alpha \) and since \( \phi_\alpha \) and \( \theta_\alpha \) are surjective, so are \( \phi^\alpha \) and \( \theta^\alpha \). That \( \phi \) is uniformly continuous was proved in Proposition 47.3 in [18].

Part 2 is a well-known (in modern times) consequence of the surjectivity of the maps \( \phi^\alpha \); and the proof used by McCord (Theorem 5.8, [17]) is now standard.

Likewise, that \( \theta \) is a homomorphism with dense image in the inverse limit topology is standard. The proof that the two topologies on \( G \) are the same may be found in the proof of Theorem 44, [18] (see the proof of this statement for the group \( K_\beta \) in the penultimate paragraph).

For the fourth part, McCord’s proof works here as well (this has only to do with inverse limits of sets and abstract groups acting on them by bijections), see also Proposition 45 in [18]. The fifth statement follows from Proposition 45.3 in [18] (plus the well-known fact due to Cantor that the intersection of a nested collection of non-empty compact sets is non-empty).

The sixth part significantly improves Proposition 47.4 in [18]; we adapt the proof here. We start by showing that \( \phi \) is bi-uniformly continuous. Let \( E \) be an entourage in \( X \); we will show that an entourage of the form \( (\phi^\alpha)^{-1}(\phi_\alpha(F)) \) is contained in \( \phi(E) \). Let \( F \) be an entourage such that \( F^3 \subseteq E \) and let \( \alpha \) be such that \( K_\alpha \subseteq U_F(G) \). The elements of \( (\phi^\alpha)^{-1}(\phi_\alpha(F)) \) are of the form \( ((K_\beta x, K_\beta y)) \) with \( (K_\alpha x, K_\alpha y) \in \phi_\alpha(F) \). This means that for some \( h, k \in K_\alpha \), \( (h(x), k(y)) \in F \). Since \( h, k \in U_F(G) \) we also have \( (h(x), x), (k(y), y) \in F \). Therefore \( (x, y) \in F^3 \subseteq E \). But then \((K_\beta x, K_\beta y)) = \phi((x, y)) \in \phi(E) \), completing the proof that \( \phi \) is bi-uniformly continuous.

Next note that by definition, \( \phi(x) = \phi(y) \) if and only if for all \( \alpha \), \( \phi_\alpha(x) = \phi_\alpha(y) \). This in turn is equivalent to the fact that for some \( g_\alpha \in K_\alpha \), \( g_\alpha(x) = y \). Since \( G \) acts freely this implies that \( g_\alpha = g_\beta \) for all \( \alpha, \beta \) and \( g_\alpha \in K_\alpha \) with \( g_\alpha(x) = y \). Conversely, if \( g \in K \) with \( g(x) = y \) then for every \( \alpha, g \in K_\alpha \) and \( g(x) = y \). That is, the orbits of \( K \) are precisely the point pre-images of \( \phi \) and since \( \phi \) is bi-uniformly continuous, the first statement in the sixth part is proved (see Remark [13]). For the very last statement, the only question concerns the completeness of \( K \), which is automatic for discrete actions.

**Definition 47** Let \( X, G, K \) be as in the statement of Theorem [40] and suppose that \( G \in K \) and all groups in \( K \) are complete. Then the resulting inverse system is called the \( K \)-resolution of the quotient \( \pi : X \to X/G \) and \( \overline{X} \) is called the \( K \)-completion of \( X \). We have the following special cases:

1. Let \( K \) be the collection of all closed normal subgroups of \( G \) of finite index. We refer to \( \overline{X} \) as the profinite completion of \( X \) (with respect to \( G \)) and the \( K \)-resolution as the profinite resolution of \( \pi \).

2. When \( \pi : X \to X/G \) is a generalized regular covering map and \( K \) is the collection of \( N_K \) for entourages \( E \) in some basis of \( X \) (which are open hence closed and therefore complete) then \( \phi \) is uniform homeomorphism.

We may identify the quotient map \( \pi : X \to X/G \) with the quotient map
\[ \pi : \overline{X} \to \overline{G} \] and we simply call the \( K \)-resolution the resolution of \( \pi \), and each of the induced quotients is a discrete cover (see Theorem 48 in [13]).

3. When \( K \) is the collection of all closed normal subgroups of \( G \) then we refer to the \( K \)-resolution as the full resolution (although we do not need this concept for this paper).

Example 48 Let \( Z \) act on \( \mathbb{R} \) in the usual way with quotient the circle \( S^1 \). Then the profinite resolution of the quotient \( \pi : \mathbb{R} \to \mathbb{R}/Z = S^1 \) is the quotient \( \pi : \Sigma \to S^1 \) via the action of the profinite completion of \( Z \) on the so-called universal solenoid \( \Sigma \), which is the inverse limit of all compact regular covers of \( S^1 \). This quotient is precisely the "\( K \)-universal cover" of \( S^1 \) considered as a compact topological group in [2] and is also the compact universal cover in the sense of the present paper. See also Example 54.

Proposition 49 Suppose that \( f : Y \to X = Y/G \) is a generalized regular covering map and \( Z \) is a chain connected uniform space. If there are (possibly not basepoint-preserving) uniformly continuous functions \( g, g' : Z \to Y \) such that \( h := f \circ g = f \circ g' \) (mod \( G \)). Moreover, \( g' = g \) if and only if for any choice \( * \) of basepoint in \( Z \) such that \( h \) is basepoint preserving, \( g(*) = g'(*) \).

Proof. We begin with the assumption that we have chosen a basepoint \( * \) in \( Z \) such that \( g \) and \( g' \) are both basepoint preserving, and will show that \( g = g' \).

Suppose first that \( f \) is a discrete covering map and let \( E \) be an evenly covered entourage in \( X \) with respect to a root entourage \( R \) in \( Y \). Let \( F \) be an entourage in \( Z \) such that \( g(F), g'(F) \subset E_R \). For any \( z \in Z \), let \( \alpha \) be be an \( F \)-chain from \( * \) to \( z \). Since \( f(E_R) = E, h(\alpha) \) is an \( E \)-chain in \( X \), which therefore has a unique lift \( \tilde{\alpha} \) to \( Y \) at \( * \). Since \( g(\alpha) \) and \( g'(\alpha) \) are \( E_R \)-chains and hence lifts of \( h(\alpha) \), they, and their endpoints \( g(z) \) and \( g'(z) \), must be equal.

Now suppose that \( f \) is a generalized regular covering map; so the resolution of \( f \) is an inverse system spaces \( \{Y_i, f_{ij}\} \) such that the induced quotients \( f_i : Y_i = Y/K_i \to X = Y_i/G_i \) are discrete covering maps. Let the basepoint \( * \) in \( Y_i \) be \( \pi_i(*_i) \), and note that the projections \( \pi_i : Y \to Y_i \) are surjective and uniformly continuous. We have basepoint-preserving surjective compositions \( g_i := \pi_i \circ g : Z \to Y_i \) and \( g'_i := \pi_i \circ g' : Z \to Y_i \), with \( f_i \circ g_i = f = f_i \circ g'_i \). From what we proved above, \( g_i = g'_i \) for all \( i \). Denoting elements of \( Y \) by \( (y_i) \) with \( y_i \in Y_i \), we have that for all \( i \) and \( z \in Z \), \( g(z) = (g_i(z)) = (g'_i(z)) = g'(z) \).

Next, suppose that \( g \) is basepoint preserving but \( g'(*) = *' \in f^{-1}(*) \). There is a unique \( k \in G \) such that \( k(*) = *' \). Now \( k \circ g'(*) = *' \) and by what we proved above, \( k \circ g' = g \); that is, \( g = g' \) (mod \( G \)). The last part of the proposition is now immediate.

Corollary 50 Let \( f_i : Y_i \to X = Y/G_i \) be generalized regular covering maps. Then the actions of \( G_1 \) and \( G_2 \) are equivalent if and only if there is a uniform homeomorphism \( h : Y_1 \to Y_2 \) such that \( f_2 \circ h = f_1 \) (i.e. \( f_1 \) and \( f_2 \) are equivalent in the classical sense for regular covering maps).
Remark 51 Theorem 52 below answers an open question from 2007 ([3], p. 1751).

Theorem 52 The composition of discrete covers (resp. generalized regular covering maps) between chain connected metrizable uniform spaces is a discrete cover (resp. generalized regular covering map). More precisely, suppose that \( g : Z \to Y = Z/H \) and \( f : Y \to X = Y/G \) are discrete covers (resp. generalized regular covering maps) and let \( h := f \circ g \). Then there exists a complete group \( K \) of uniform homeomorphisms of \( Z \) that contains \( H \) as a normal subgroup such that \( h : Z \to X = Z/K \) is a discrete cover (resp. generalized regular covering map).

Proof. Choose basepoints so that both maps are basepoint-preserving. To begin with we assume that \( f : Y \to X = Y/G \) is any quotient map via a free isomorphic action and \( g : Z \to Y = Z/H \) is a discrete cover. Let \( E \) be an invariant entourage in \( Y \) that is evenly covered with respect to some root entourage \( R \) in \( Z \). We note for once and for all that by Lemma 33 and Theorem 34 the set of all \( E^x \) is a basis for the uniform structure on \( Y \) and the set of all \( E^x_R \) is an invariant basis for the uniform structure on \( Z \). Suppose that \( x \in h^{-1}(*) \). Define a function \( k_x : Z \to Z \) as follows. Since \( x \in h^{-1}(*) \), \( g(x) \in f^{-1}(*) \) and therefore there is a unique \( j_x \in G \) such that \( j_x(*) = g(x) \). For any \( z \in Z \), let \( \beta \) be an \( E^x_R \)-chain from \(* \) to \( z \). Define \( k_x(z) \) to be the endpoint of the unique lift \( \tilde{\beta} \) of \( j_x(g(\beta)) \) to \( Z \) at \( x \). We will show that the set \( K \) of all such \( k_x \) is a group of well-defined uniform homeomorphisms acting freely and isomorphically on \( Z \), which contains \( H \) as a normal subgroup, and such that \( h \) is the quotient map \( h : Z \to X = Z/K \). This will involve a series of claims, beginning with the claim that \( k_x \) is well-defined.

Suppose that \( \beta' \) is another \( E^x_R \)-chain from \(* \) to \( z \). Then \( g(\beta) \) and \( g(\beta') \) end at the same point, and therefore \( k_x(g(\beta)) \) and \( k_x(g(\beta')) \) end at the same point \( y \). Let \( z', z'' \) be the endpoints of the lifts \( \tilde{\beta} \) and \( \tilde{\beta}' \) of \( g(\beta) \) and \( g(\beta') \), respectively, to \( X \) at \( x \). Since \( g(z') = y = g(z'') \), there is some \( m \in H \) such that \( m(z') = z'' \). By Proposition 37, if \( \tau \) is any \( E^x_R \)-chain from \(* \) to \( x \), then \( m(*) \) is the endpoint of the lift \( \kappa \) of \( g(\tilde{\beta} \ast \tau) \) at \( z'' \). By uniqueness, the lift of \( g(\tilde{\beta}) \) at \( z'' \) must be \( \beta' \), which ends at \( x \). By uniqueness again, we have that \( \kappa = \beta' \ast \tau \), which ends at \( * \). That is, \( m(*) = * \), which implies \( m = 1 \) and \( z' = z'' \). It is immediate from the definition of \( k_x \) that \( h = h \circ k_x \), and we also have the following compatibility condition for any \( x \in h^{-1}(*) \) and \( z \in Z \):

\[
g(k_x(z)) = j_x(g(z)). \tag{8}
\]

In fact, both sides of this equation are readily seen to be the endpoint of \( j_x(g(\beta)) \).

Letting \( z' \) be the endpoint of the lift of \( g(\tau) \) to \( Z \) at \(* \), we see by uniqueness of lifts that \( k_{z'} \) is an inverse function to \( k_x \), proving that \( k_x \) is a bijection. We next claim that for any \( x_1 \in h^{-1}(*) \), \( k_{x_1} \circ k_x = k_{k_x(x_1)} \), showing that \( K \) is a group. With \( z \) and \( \beta \) as above, \( \tau \ast \beta \) is an \( E^x_R \)-chain from \(* \) to \( k_x(z) \). Therefore...
\( k_{x_1}(k_x(z)) \) is the endpoint of the lift of
\[
  j_{x_1}(g(\tau \ast \beta)) = j_{x_1}(g(\tau)) \ast j_{x_1}(g(\beta))
\]
to \( Z \) at \( x_1 \). This in turn is the endpoint of the unique lift of \( j_{x_1}(j_x(g(\beta))) = j_{x_1}(g(\beta)) \) \((G \text{ acts freely})\) to \( Z \) at \( k_{x_1}(x) \). On the other hand, \( h(k_{x_1}(x)) = h(x) = \ast \), and therefore \( k_{k_{x_1}(x)}(x) \) is defined. And by definition, \( k_{k_{x_1}(x)}(x) \) is also the endpoint of the lift of \( j_{k_{x_1}(x)}(g(\beta)) \) to \( Z \) at \( k_{x_1}(x) \).

Note that by Proposition 37, \( H \) is characterized as those \( m \in K \) such \( g(m(\ast)) = \ast \). By Proposition 28 the restriction \( g_z \) of \( g \) to any \( B(z, E^*_R) \) is a bijection onto \( B(g(z), E) \). We claim that the restriction \( k^2_x \) of \( k_x \) to any \( B(z, E^*_R) \) satisfies the equation
\[
  k^2_x = g_{k_x(z)}^{-1} \circ j_x \circ g_z.
\]
Since \( E \) is invariant with respect to \( j_x \) this will mean that the restriction of \( k_x \) to any \( E^*_R \)-ball is a bijection onto an \( E^*_R \)-ball, showing that \( E^*_R \) is invariant with respect to \( k_x \). In fact, if \( w \in B(z, E^*_R) \) then since \( k_x \) is well-defined we may use any \( E^*_R \)-chain \( \beta \) from \( \ast \) to \( z \), concatenated with \((z, w)\), as our \( E^*_R \)-chain from \( \ast \) to \( w \). Now the fact that \( k_x(w) = g_{k_x(z)}^{-1} \circ j_x \circ g_z(w) \) is immediate from the definition of \( k_x \).

Now let \( k_x \in K \) and \( m \in H \), and consider \( k^{-1}_x(m(k_x(\ast))) = k^{-1}_x(m(x)) \). Again let \( \tau \) be any \( E^*_R \)-chain from \( \ast \) to \( x \) and let \( \eta \) be an \( E^*_R \)-chain from \( x \) to \( m(x) \). Let \( x' := k^{-1}_x(\ast) \); so for some \( g_{x'} \in G \), \( g_{x'}(\ast) = x' \). By definition, \( k^{-1}_x(m(x)) \) is the endpoint of the lift of \( g_{x'}(g(\tau \ast \eta)) \) at \( \ast \). But note that since \( m \in H \), \( g(\eta) \) is a loop, and therefore so is \( g_{x'}(g(\eta)) \). Since the lift of \( g_{x'}(g(\tau)) \) to \( Z \) at \( x' \) ends at \( \ast \), the lift of \( g_{x'}(g(\eta)) \) to \( Z \) at \( \ast \) ends at \( k^{-1}_x(m(x)) = k^{-1}_x(m(k_x(\ast))) \).

But since \( g_{x'}(g(\eta)) \) is a loop, this shows that \( k^{-1}_x(m(k_x(\ast))) = \ast \) and therefore \( k^{-1}_x m k \in H \), completing the proof that \( H \) is normal in \( K \).

Next note that since each of \( f \) and \( g \) is bi-uniformly continuous, so is \( h \).

We next show that the point preimages of \( h \) are the orbits of \( K \), showing that \( h : Z \to X = Z/K \) is a quotient map. Since \( h \circ g_x = h \) for every \( x \in h^{-1}(\ast) \), the orbits of \( K \) are contained in point pre-images. Conversely, suppose that \( h(z) = h(w) \) and let \( \beta \) be an \( E^*_R \)-chain from \( \ast \) to \( z \). Since \( h(z) = h(w) \) there is some \( j \in G \) such that \( j(g(z)) = g(w) \); let \( x \) be the endpoint of the unique lift of \( j(g(\beta)) \) to \( Z \) at \( w \). By definition of \( g_x \), \( g_x(z) \) is precisely the endpoint of the unique lift of \( j(g(\beta)) \) to \( Z \) at \( x \). But that lift is precisely \( w \), i.e. \( g_x(z) = w \), completing the proof that \( h \) is a quotient map.

Now define \( \theta : K \to G \) by \( \theta(k_x) = j_x \). The compatibility Equation 38 now becomes
\[
  \theta(k_x)(g(z)) = g(k_x(z))
\]
for every \( k_x \in K \) and \( z \in Z \). To see that \( \theta \) is a surjective homomorphism with kernel \( H \), let \( k_1, k_2 \in K \) and \( z \in Z \). Applying compatibility a couple of times we have:
\[
  \theta(k_{x_1}, k_{x_2})(g(z)) = g(k_{x_1}(k_{x_2}(z))) = \theta(k_{x_1})(g(k_{x_2}(z))) = \theta(k_1)(\theta(k_2)(g(z)))
\]
If \( j \in G \), let \( x \in g^{-1}(j(\ast)) \subset h^{-1}(\ast) \). Then \( g(k_x(\ast)) = g(x) = j(\ast) \) and therefore by definition \( \theta(k_x) = j \). Similarly, if \( k_x \in \ker \theta \) then \( g(k_x(\ast)) = \ast \) and therefore \( k_x \in H \).

At this point the only fact that we have used about \( G \) is that it acts freely and isomorphically on \( Y \). For the next steps we will assume that \( H \) acts discretely and impose additional conditions on \( E \) for two cases: \( G \) acts discretely and \( G \) acts prodiscretely. Suppose that \( G \) acts discretely. Then we may take the entourage \( E \) from above to have the property that \( N_E(G) = 1 \). Suppose that \( (z, k(z)) \in E^*_H \) for some \( z \in Z \) and \( k \in K \). Then \( E \) contains \( (g(z), g(k(z))) = (g(z), \theta(k)(g(z))) \) and therefore \( \theta(k) = 1 \). This means that \( k \in H \). But now since \( (z, k(z)) \in E^*_H \), \( k = 1 \). That is, \( N_{E_R^G}(K) = 1 \).

Now suppose that \( G \) acts prodiscretely (and still \( H \) acts discretely) choose an entourage \( F \subset E \) such that \( N_F(G) \subset U_E(G) \). We will prove that

\[
N_{F^*_R}(K) \subset U_{E^*_R}(K),
\]

proving that in this case \( K \) acts prodiscretely. Suppose that \( k_x \in N_{F^*_R}(K) \). That is, \( k_x = s_1 \cdots s_n \), where \( s_i \in S_{F^*_R}(K) \), which in turn means \( (s_i(z_i), z_i) \in F^*_R \) for some \( z_i \in Z \). Therefore \( (\theta(s_i)(g(z_i)), g(z_i)) = (g(s_i(z_i)), g(z_i)) \in F \), which implies that each \( \theta(s_i) \in S_F(G) \) and therefore \( \theta(k_x) \in N_E(G) \subset U_E(G) \). Put another way,

\[
\theta(N_{F^*_R}(K)) \subset U_E(G).
\]

Now suppose that \( z \in Z \). We have \( (g(z), j_x(g(z))) = (g(z), g(k_x(z))) \in E \).

Since the restriction of \( g \) to \( B(z, E^*_R) \) is a bijection onto \( B(g(z), E) \), there is some \( w \in B(z, E^*_R) \) such that \( g(w) = j_x(g(z)) \). From Equation (9), we have:

\[
k_x(z) = g_{k_x(w)}^{-1}(j_x(g(z))) = g_{k_x(w)}^{-1}g(w) = w
\]

That is, \( (z, k_x(z)) \in E^*_R \). Since \( z \) was arbitrary, \( k_x \in U_{E^*_R}(K) \).

We will now show that \( K \) is complete. Since the spaces, hence the groups, are all metrizable, we may use Cauchy sequences to verify completeness. Suppose that \( \{k_{x_i}\} \) is a Cauchy sequence in \( K \); then for every \( E \), \( k_{x_i}^{-1}k_{x_j} \in U_{E^*_R}(K) \) for all large \( i, j \). Applying this to \( \ast \), we have that \( (k_{x_i}^{-1}k_{x_j}(\ast), \ast) \in E^*_R \) for all large \( i, j \). Since \( E^*_R \) is invariant, this means that \( (x_i, x_j) \in E^*_R \) for all large \( i, j \). That is, \( \{x_i\} \) is itself Cauchy. Therefore \( \{g(x_i)\} \) is a Cauchy sequence in the (complete) orbit \( G \ast \) of \( \ast \). That is, \( g(x_i) \to y \in Y \). Since \( g \) is bi-uniformly continuous, this in turn implies that for every open set \( U \) containing the orbit \( H z \) of \( z \in g^{-1}(y) \), there is some \( x_i \in U \). Since \( H z \) is uniformly discrete, we may take arbitrarily small open sets containing only single points of \( H z \). Since \( \{x_i\} \) is Cauchy, one open set about some particular \( w \in H z \) must contain all but finitely many points of \( \{x_i\} \). In other words, \( x_i \to w \). We claim that \( k_{x_i} \to k_w \), which will complete the proof that \( K \) is complete. In fact, for any sufficiently small entourage \( F \), \( (x_i, w) \in F^*_R \). But this means that \( (k_{x_i}, k_w) \in N_{F^*_R}(K) \), which is contained in \( U_{E^*_R}(K) \) by Inclusion (10).

For the final case, in which both \( G \) and \( H \) are complete and act prodiscretely, choose a countable nested basis \( E_i \) for \( Y \) and take the resulting resolution of \( g \).
(see Definition 47), which consists of compatible inverse systems \( \{ Z_i, f_{ij} \}_{i, j \in \mathbb{N}} \) of uniform spaces and \( \{ G_i, \theta_{ij} \}_{i, j \in \mathbb{N}} \) of groups \( G_i \) of uniform homeomorphisms acting discretely on \( Z_i \). Let \( g_i : Z_i \rightarrow Y = Z_i / G_i \) be the quotient map, which is a discrete covering map, and \( h_i := f \circ g_i \), which by the special case we already proved, is equivalent to the quotient map \( h_i : Z_i \rightarrow X = Z_i / K_i \) via some complete group \( K_i \) that acts prodiscretely on \( Z_i \) and contains \( H_i \) as a normal subgroup. That is, we have an inverse system of quotients via isomorphic actions in the sense of [18], and by Proposition 41 of [18], the resulting inverse limit of \( K = \varprojlim K_i \) on \( Z = Z_i \) is a generalized regular covering map. Since \( H_i \) is a normal subgroup of \( K_i \), \( H \) is a normal subgroup of \( K \).

Example 53 As is well-known, there are traditional regular covering maps between path and locally path connected spaces whose composition is not even a covering map; there is a nice illustration of this in [3]. But according to Theorem 37 we can “uniformize” each of these covering maps as discrete covering maps! This does not contradict Theorem 32 because as can be readily discerned from the picture in [4], any attempt to uniformize these two maps creates non-equivalent uniform structures on the middle space—i.e., the spaces involved can never be simultaneously uniformized in a way that the result is a composition of discrete covers.

7 Properties of the Compact Universal Cover

Proof of Theorem 2 Starting with a basepoint \( * \) in \( X_0 := X \) we may choose basepoints \( * \) in each \( X_i \) so that \( f_i \) is basepoint-preserving. When \( f_i \leq f_j \), by definition this means there is a (uniformly) continuous surjection \( f_{ij} : X_j \rightarrow X_i \) such that \( f_j = f_i \circ f_{ij} \). A priori \( f_{ij} \) may not be basepoint-preserving, but \( f_{ij}(*) = *' \) with \( *' \in f_i^{-1}(*) \). Therefore there is some \( g \in G_i \) such that \( g(*) = *' \), and replacing \( f_{ij} \) by \( g \circ f_{ij} \) we may assume that \( f_{ij} \) is basepoint-preserving. By Theorem 39 \( f_{ij} \) is a discrete covering map, and by Proposition 31 is the unique basepoint-preserving such discrete covering map. Suppose that \( f_i \leq f_j \) and \( f_j \leq f_i \). Then \( \iota_j := f_i \circ f_{ij} : X_j \rightarrow X_j \) is basepoint-preserving such that \( f_j \circ \iota_j = f_j \). The identity map on \( X_j \) has the same properties and therefore by Proposition 32 \( \iota_j \) is the identity. The same is true for \( i \), and therefore \( f_{ij} \) and \( f_{ji} \) are inverses. In particular, \( f_i \) and \( f_j \) are equivalent, so \( \leq \) is a partial order.

To see that the set is directed, let \( E \) be evenly covered by both \( f_i \) and \( f_j \). According to Theorem 41 for \( m = i, j, f_m \) is equivalent to the induced quotient map \( \pi : X_E / K_m \rightarrow (X_E / K_m) / (\pi_E(X) / K_m) \) for normal subgroups \( K_i, K_j \) of \( \pi_E(X) \) of finite index. Let \( K := K_i \cap K_j \) and consider the induced quotient \( f : X_E / K \rightarrow (X_E / K) / (\pi_E(X) / K) \) which by Lemma 19 is a discrete covering map. We claim that \( Y := X_E / K \) is compact. In fact, let \( \{ y_i \} \) be a Cauchy sequence in \( Y \); so \( f(y_i) \rightarrow x \) for some \( x \in X \). Taking a tail of the sequence if necessary we can assume that \( \{ f(y_i) \} \) lies in an open set \( U \) containing \( x \) that is evenly covered by sets \( U_i \) in \( Y \). Fixing one \( U_k \) and taking preimages
in $U_k$ we have a convergent subsequence $y'_i \to x'$ for some $x' \in f^{-1}(x)$, with $f(y'_i) = f(y_i)$. For each $i$ there is some $k_i \in \pi_E(X)/K$ such that $k_i(y'_i) = y_i$. Since $\pi_E(X)/K$ is finite, by taking a subsequence if necessary we can assume that $y_i = k(y'_i)$ for all $i$ and some fixed $k \in G$. But $k$ is a homeomorphism and since $\{y'_i\}$ is convergent, so is $\{y_i\}$.

By Proposition 36 in [15] the restriction $f_C$ of $f$ to the component $C$ of the basepoint in $X_E/K$ is a cover of $X$ by a continuum. Since $K \subset K_m$ for $m = i, j$, we have induced quotients $g_m : Y = X_E/K \to X_m = (X_E/K)/(K_m/K)$. Then the restriction $g'_m$ of $g_m$ to $C$ is a uniformly continuous surjection such that $f_C = f_m \circ g'_m$. That is, $f_m \leq f_C$ for $m = i, j$.

By uniqueness, if $f_i \leq f_j \leq f_k$ then $f_{ijk} = f_{ij} \circ f_{jk}$ and therefore we have an inverse system with bonding maps $f_{ij}$. From Theorem 9 the restrictions of the induced quotients $\pi_E(X)/K \to X$ from the prior paragraph form a countable cofinal subsystem. It is a classical result (and easy to prove by iteration) that in this situation the fact that the bonding maps are surjective implies that the projection maps from the inverse limit are also surjective. The inverse limit of compact (Hausdorff), connected spaces with surjective bonding maps is compact and connected. This is mentioned for example in Section 2 of [5] (in that paper “continua” are not assumed to be metrizable but only Hausdorff). Moreover, since the spaces in the (countable!) inverse system are metrizable, so is the inverse limit $\hat{X}$. That is, $\hat{X}$ is a continuum.

For the homomorphisms $\theta_{ij} : G_j \to G_i$ we take the homomorphism from Theorem 39. By definition, $\theta_{ik} = \theta_{ij} \circ \theta_{jk}$ and compatibility is simply Theorem 39 lc. From compatibility and the fact that each $f_{ij}$ is a discrete cover it follows from Theorem 44 in [18] that the projection $\hat{\phi} : \hat{X} \to X$ is a generalized universal cover with deck group $\pi_p(X) = \varprojlim G_i$. Proposition 36 implies that every $G_i$ is finite, so by definition $\pi_p(X)$ is profinite. ■

**Proof of Theorem 3** If $K$ is a closed normal subgroup of $\pi_p(X)$ of finite index then since $\pi_p(X)$ is complete, $K$ is complete. By Proposition 20 $\hat{X}/K$ is Hausdorff, hence a continuum that covers $X$ via the induced quotient.

For the converse, suppose first that $f$ is a discrete cover, so $Y = X_j$ for some $j$ and $f = f_j : X_j \to X = X_j/G_j$. Define $K_{f_j} := \ker \theta^j$, where $\theta^j : \pi_p(X) = \varprojlim G_i \to G_j$ is the projection. We claim that the projection $\hat{\phi}^j : \hat{X} \to X_j$ is equivalent to the quotient $\hat{X} \to \hat{X}/K_{f_j}$. Since $\hat{\phi}^j$ is surjective (see the proof of Theorem 2) hence bi-uniformly continuous, we need only show that $\hat{\phi}^j(x) = \hat{\phi}^j(y)$ if and only if for some $g \in K_{f_j}$, $g(x) = y$. If $\hat{\phi}^j(x) = \hat{\phi}^j(y)$ then $\hat{\phi}(x) = \hat{\phi}(y)$ and therefore there is some $g \in \pi_p(X)$ such that $g(x) = y$. By compatibility,

$$\theta^j(g)(\hat{\phi}^j(x)) = \hat{\phi}^j(g(x)) = \hat{\phi}^j(y) = \hat{\phi}^j(x)$$

and since the action is free, $g \in \ker \theta^j = K_{f_j}$. Conversely, suppose there is some $g \in K_{f_j}$ such that $g(x) = y$. Then

$$\hat{\phi}^j(y) = \hat{\phi}^j(g(x)) = \theta^j(g)(\hat{\phi}^j(x)) = \hat{\phi}^j(x).$$
We may now invoke Proposition 21 to see that $f_j$ is equivalent to the induced quotient $\pi : \hat{X}/K_{f_j} \to X = (\hat{X}/K_{f_j})/(\pi_p(X)/K_{f_j})$, and by Proposition 36 $G_j = \pi_p(X)/K_{f_j}$ is finite, and therefore $K_{f_j}$ has finite index.

For $f$ an arbitrary generalized regular covering map we will utilize the resolution of $f$ (see Definition 47). Since the covers in the resolution are discrete, they constitute a directed subsystem of the inverse system of all coverings of $X$ by compacta indexed by some set $J$, \{${X}_j, f_{ij}$\}$_{j \in J}$ where by the previous part, for each $j$ we have the projection $f^j : \hat{X} \to \hat{X}/K_{f_j} = X_j$ and $f_j : X_j \to X$ is equivalent to the quotient

$$
\pi_j : X_j = \hat{X}/K_{f_j} \to X = (\hat{X}/K_{f_j})/(\pi_p(X)/K_{f_j}) = \hat{X}/\pi_p(X).
$$

Note that each $K_j$, being a subgroup of $\pi_p(X)$, acts freely and has compact orbits. Therefore the proof is complete by Theorem 46. ■

**Proof of Theorem 4.** For existence we may simply take the quotient map $f_L : \hat{X} \to \hat{X}/K_f$ from the Galois Correspondence. Uniqueness (mod $G$) follows from Proposition 49. ■

**Proof of Theorem 5.** If $\hat{X}$ were not compactly simply connected then there would be some non-trivial discrete cover $f : Y \to \hat{X}$ by a continuum. By Theorem 52 $h := \phi \circ f$ is a generalized universal cover of $X$. Now Theorem 4 implies that there is a unique (mod $\pi_p(X)$) generalized covering map $h_L : \hat{X} \to Y$ such that $\hat{\phi} = h \circ h_L$. This implies that $g := f \circ h_L : \hat{X} \to \hat{X}$ satisfies

$$
\hat{\phi} \circ g = \hat{\phi} \circ f \circ h_L = h \circ h_L = \hat{\phi}.
$$

But this makes $g$ a lift of $\hat{\phi}$ itself. Since the identity map is also a lift of $\hat{\phi}$, by uniqueness of Theorem 4, $g$ is equal (mod $\pi_p(X)$) to the identity. Since $h_L$ is onto, this means that $f$ is 1-1, a contradiction.

Now suppose that $f : Y \to X$ is a generalized regular covering map with $Y$ a compactly simply connected continuum. By the Universal Property there is a generalized regular covering map $f_L : \hat{X} \to Y$ such that $f \circ f_L = \hat{\phi}$. But if $f_L$ were not a uniform homeomorphism then $Y$ would have a nontrivial discrete cover by a continuum, meaning that $\pi_p(Y) \neq 1$, a contradiction. ■

**Proof of Corollary 5.** By Theorem 52 $g := f \circ \hat{\phi}_Y : \hat{Y} \to \hat{X}$ is a generalized regular covering map and therefore by the Universal Property there is a lift $g_L : \hat{X} \to \hat{Y}$ such that $g \circ g_L = \hat{\phi}_X$. On the one hand, the Universal Property gives us a unique (mod $G$) lift $f_L : \hat{X} \to Y$ such that $f \circ f_L = \hat{\phi}_X$. Applying it again, there is a generalized regular covering map $(f_L)_L : \hat{Y} \to \hat{X}$ such that $f_L \circ (f_L)_L = \hat{\phi}_Y$. Now $(f_L)_L \circ g_L : \hat{X} \to \hat{X}$ is a generalized regular covering map such that

$$
\hat{\phi}_X \circ ((f_L)_L \circ g_L) = f \circ f_L \circ (f_L)_L \circ g_L = f \circ \hat{\phi}_Y \circ g_L = g \circ g_L = \hat{\phi}_X.
$$

By uniqueness again this means that $(f_L)_L \circ g_L$ is (mod $\pi_p(X)$) equal to the identity. Since $(f_L)_L$ is onto, $g_L$ must be 1-1 and hence must be a uniform homeomorphism. ■

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Example 54 As a final example we will consider what happens with the 2-adic solenoid \( X = \Sigma_2 \), which is the inverse limit of the \( 2^n \) covers of the circle. We already saw in [2] that \( \hat{X} \) (viewed as a compact group) is the "universal solenoid", which is the inverse limit of all covers of the circle by itself, which is precisely how \( \hat{X} \) is defined in the present paper. In [3] we saw that \( \hat{X} = \mathbb{R} \) using the map induced by the generalized universal covering of the circle by \( X \). But it is perhaps useful to examine the two inverse systems themselves and their interaction, beginning with the fundamental inverse system. As is well known, \( X \) is locally a product between a compact real interval and the Cantor set. In particular, we may take for a basis of the unique uniform structure on \( X \) all entourages \( E \) such that every \( B(x, E) \) is homeomorphic to \((-\varepsilon, \varepsilon) \times V\), where \( V \) is an open set in \( C \).

Now the basepoint \( * \) lies on some path component, which is the image of a continuous 1-1 map \( p : \mathbb{R} \to X \) with \( p(0) = * \). As \( t \) increases, the local product structure requires that \( p(t) \) leave the local product neighborhood, but compactness forces it to return as another path component of the local product around \( * \). For some \( t_1 \), \( p(t_1) \) has the same second coordinate as \(*\). Now suppose \( (*, p(t_1)) \in E \). Then \( E \) doesn't "see the gap" between \( * \) and \( p(t_1) \) and there is an \( E \)-loop \( \lambda \) at \( * \) that starts as a subdivision of the segment \( p_1 \) of \( p \) restricted to \([0, t_1] \) and then jumps from \( p(t_1) \) to \( * \). If \( E \) is small enough in the "real direction" then \( \lambda \) cannot be \( E \)-null and so represents a non-trivial element of \( \pi_E(X) \). That is, \( \lambda \) "unrolls" to its lift \( \bar{\lambda} \) to \( X_E \) at the basepoint, ending at \( [\lambda]_E \in \phi_E^{-1}(*) \setminus \{ * \} \) in \( X_E \). Now consider the lift of the segment \( p_1 \) to \( X_E \) at the basepoint, which contains \( \bar{\lambda} \) except for its endpoint \( [\lambda]_E \). Due to the gap that is crossed by \( \lambda \), \( [\lambda]_E \) lies in some path component that does not contain the basepoint in \( X_E \). The segment \( p_1 \) "unrolls" in terms of the uniform structure, but not of course topologically because it is simply connected and it must remain inside the path component of \( * \) in \( X_E \). That is, the non-trivial deck group element \( [\lambda]_E \) takes the path component of the basepoint in \( X_E \) to a different path component.

Returning to the path component of \( * \) in \( X \), the lift of \( p \) at \( * \) can never return to any \( E^* \)-ball because \( \phi_E \) is a bijection on \( E^* \)-balls. This implies that \( X_E \) is not compact (although it is locally compact). \( X_E \) is also not connected, and in fact for sufficiently small \( F \subset E \), \( \phi_E \) is not surjective. That is, as soon as an entourage \( F \) "sees a gap" that \( E \) didn't see" the loop \( \lambda \) can no longer be refined in its \( E \)-homotopy class to an \( F \)-loop. Therefore \( \theta_{EF} : \pi_F(X) \to \pi_E(X) \) is not surjective, hence \( \phi_E \) is not surjective either. This means that some path components of \( X_E \) are not in the image of \( \phi_E : \hat{X} \to X_E \) and in the inverse limit, only the path component of \( * \) "survives". That is, the lift of \( p \) to \( \hat{X} \) at \( * \) is equal to \( \bar{\lambda} \) and \( \phi : \hat{X} \to X \) is a bijection onto the path component. Note that \( \phi \) in this case is uniformly continuous but not bi-uniformly continuous—it "rolls up" \( \mathbb{R} \) into the dense path component of \( * \).

Now any discrete cover of \( X \) by a continuum \( X_i \) is covered by some \( X_E \), which maps into \( X_i \) as a dense subset—and therefore \( X_i \) is a compactification of \( X_E \). The cover \( f_i : X_i \to X \) is induced by a quotient of \( X_E \) by a normal subgroup of finite index and of course by definition \( f_i \) is surjective. In other
words, the inverse system leading to $\hat{X}$ can be viewed as a uniquely determined compactification of the fundamental inverse system, which via compactness "corrects" the problem that the bonding maps of the fundamental inverse system are not surjective in this particular example for any choice of a cofinal sequence of entourages. Some of this analysis applies more generally to matchbox manifolds.

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