Computation of Cusp Bifurcation Point in a Two-Prey One Predator Model using Lagrange Multiplier Method

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Abstract. In this paper we introduce a new method for computing cusp bifurcation point in a system of two prey one predator dynamical system. The method consists of rewriting the system of equations for the equilibria in a constrained optimization problem at where the classical Lagrange Multiplier Method can be applied to derive the fold bifurcation point. By having one extra parameter as free parameter we construct two fold curves which intersect each other non-transversally in a cusp bifurcation point. This result is verified by looking at three section in two-dimensional parameter space by taking one of the parameter constant and study the stability of the equilibria.

1. Introduction

The development of the predator-prey model has a very long history, originated from the work of A.J.Lotka and V.Volterra in 1920’s (see [2] and the references there in). Variation of the model includes, among others: extending the dimension of the system (by considering more species, see for example: [7, 12, 14, 16, 18]) or by using various functional response (see for example [10, 11, 21]). Tripathi et al. [18] studied a dynamical system consisting of two species of prey and one predator. Here, the authors there assume that the prey species can help each other during predation and the predation rate for each of the population of prey is different. Elettreby [7] studied the global stability and persistence in a similar model to [18] without the assumption that different species of prey can help each other in preventing predation. Klebanoff and Hastings [14] investigated the chaotic dynamics of two prey one predator model.

In this paper, we follow the work initiated in [16] in where the classical Lagrange Multiplier Method is applied on finding fold bifurcation in two-prey one predator model. Note that fold bifurcation is a phenomena where two equilibria collide with each other and vanishes as the bifurcation parameter varies. Geometrically one can realize that an equilibrium can be seen as an intersection between the nullclines i.e. the zero level sets of the component of the vector field defining the dynamical system. Since the system depends on parameters, then computation of fold bifurcation is similar to looking for the maximum or minimum value such that the two nullclines intersect each other. Thus the problem of finding fold bifurcation is viewed as a constrained optimization problem, where Lagrange Multiplier Method can be applied.

This method is proposed as an alternative for computing fold bifurcation, which is usually done using numerical continuation software, for example: AUTO [6], XPPAUT [8], or MATCONT [3, 4]. Furthermore, the proposed method computes directly the fold bifurcation point, in contrast with using numerical continuation software where one start with general equilibirum. Another way of computing fold bifurcation can be found in [13, 17, 19, 20]. The latter depends highly on the existence of explicit formula for the equilibrium. In some case, this is not very easy to achieve.
The dynamical system which is considered in [16] (and also here) is three dimensional and it depends on ten parameters. Thus the system of equations for the equilibria is three dimensional. One can reduce it to two dimensional system depending on ten parameters by solving one of the equations for one of the variables. The solution is then substituted into the remaining two equations. As for the parameters, Marwan et al in [16] choose \( b \), which is the group defence parameter, as the bifurcation parameter and fix the other nine parameter. As an extension to this work, we choose \( b \) and \( \delta \), which is the mortality rate of the predator, as free parameters and fix the other parameters.

It is shown in [16] that the system undergoes two fold bifurcations. By continuing the two loci of the fold bifurcations using another parameter, one can construct two fold curves. It is well-known in the literature that those two fold curves might coalesce non-transversally, and then disappear. This phenomena is known as the cusp bifurcation (see for example [15]), which is of co-dimension two. In this paper, our goal is to compute this cusp bifurcation point using the Lagrange Multiplier Method (see [5] for the detail and the proof of the method). This method we are proposing is an alternative for finding the cusp point in a system of two differential equations. However, the computation might still be too hard to do analytically. In that case, we will use a numerical solver for finding those solution.

The paper is organized as follows. In Section 2 we formulate the system of three differential equations which will be analyzed throughout the paper. This is the same system as in [16]. In Section 3 we briefly sketch the method we are using to compute the cusp bifurcation as an extension of the work in [16] which concentrates on finding fold bifurcations. Section 4 is devoted to study the stability of the equilibria in the neighborhood of the cusp bifurcation. This is to further confirm that the eigenvalues behavior near the cusp point match the theory. We end this paper with a few concluding remarks.

2. Model Formulation

Consider a three dimensional system of ordinary differential equations depending on ten parameters, i.e.

\[
\begin{align*}
\dot{x} &= x \left( 1 - \frac{1}{k_1} x - \frac{\mu z}{cx + 1} \right) = x f(x, y, z) \\
\dot{y} &= \alpha x - \beta y - \frac{1}{k_2} y^2 - \frac{\eta y z}{ay^2 + by + 1} = g(x, y, z) \\
\dot{z} &= z \left( -\delta + \frac{x}{cx + 1} + \frac{y}{ay^2 + by + 1} \right) = z h(x, y)
\end{align*}
\]

(2.1)

where \( x, y \) and \( z \) are the population densities of the young prey, the adult prey, and the predator, respectively. Note that the growth rate of the young prey has been scaled to one, while the mortality rates of the adult prey and the predator are \( \beta \) and \( \delta \), respectively. Living in a large group, the adult prey can be able to defense from predator’s attack. The group mechanism defense is done by the adult prey protect their population especially the young prey. As a consequence, the functional response of the young prey is of Holling type II while the the adult prey is of Holling type IV. The conversion rates are \( \mu \) and \( \eta \), respectively. The environment is assumed to react differently to the class of the prey, i.e. the carrying capacity for the young is \( k_1 \) while the adult. is \( k_2 \). We also includes three other parameters, namely \( c, a \) and \( b \) which determines the geometric shape of the functional response functions. All of these parameter take positive value but \( b \) which can be negative, as long as: \( ay^2 + by + 1 \) non-negative for \( y \geq 0 \). System (2.1) is so-called normalized two-prey one predator dynamical system.

In the absence of the adult prey, i.e. \( y = 0 \), system 2.1 equivalents with Bazykin’s system in [1]. If \( x = 0 \), system 2.1 equivalents with a Predator-Prey systems with non-monotonic response function in [9, 21] but in our case, it doesn’t have a biological meaning. For detail derivative of (2.1) see Marwan et al. [16].
The non-trivial equilibria of (2.1) (if exists) satisfies
\[(x - p)^2 - A(y - q)^2 = r, \quad h(x, y) = 0,\] (2.2)
where
\[A = \frac{k_1 \mu}{\eta (c \delta - 1) k_2}, \quad p = \frac{\eta k_1 (c \delta - 1) - k_1 \mu \alpha - \eta \delta}{2 \eta (c \delta - 1)}, \quad q = -\frac{\beta k_2}{2}, \quad r = p^2 - Aq^2 + \frac{k_1 \delta}{c \delta - 1}.
\]
Following [16], we set a value of the parameters: \(\alpha = 0.8, a = 7.29, \beta = 0.75, c = 1, \delta = 0.8, \eta = 0.25, k_1 = 6, k_2 = 4,\) and \(\mu = 0.75.\) Using Lagrange Multiplier Method, two fold bifurcation points are found at \(b = b_1 = -3.68591\) and \(b = b_2 = -2.86798.\)

### 3. Computation of the Cusp Bifurcation

In this paper, we choose \(\delta\) as another free parameter apart from \(b.\) We will follow the procedure in [16] for computing fold bifurcation. Let us look at the second equation in (2.2), i.e. \(h(x, y) = 0.\) It is easy to see that
\[b = G(x, y, \delta) = -ay + \frac{c x + 1}{x(c \delta - 1) + \delta} - \frac{1}{y},\] (3.1)
solves \(h(x, y) = 0.\) Thus, the solution of the constrained optimization problem:

Max/Min: \(G(x, y, \delta)\)

Given: \(F(x, y, \delta) = (x - p)^2 - A(y - q)^2 - r = 0\)

corresponds to a fold bifurcation. Using, Lagrange Multiplier Method, we solve:

\[\nabla G = \lambda \nabla F, \quad F = 0.\] (3.2)

Note that the solution \((x, y, \lambda)\) of (3.2) depends in the free parameter \(\delta.\) Thus, \(y\)-component of the solution (3.2) satisfies:
\[\sum_{i=0}^{10} A_i(\delta) y^i = 0,\] (3.3)
see Appendix for the formula for the coefficient \(A_i.\)

Let \(\{\delta_1, \delta_2, \ldots, \delta_N\} \in [0.7, 1]\) be chosen such that \(0.7 = \delta_1 < \delta_2 < \ldots < \delta_N = 1,\) for large enough natural number \(N.\) For each \(\delta_k, k = 1, 2, \ldots, N\) we solve (3.3) for \(y_j(\delta_k),\) where \(m\) can be 1, or 2. Then we solve \(F(x, y_m(\delta_k), \delta_k) = 0\) for \(m = 1, 2, k = 1, 2, \ldots, N.\) We derive two families of fold bifurcation points (or fold curves) \((x_j, y_j)\). Lastly, we substitute \(x, y_j,\) and \(\delta\) to (3.1), which defines \(b\) as a function of \(\delta.\) These curves in \((\delta, b)\) is plotted in Figure (1.1), the diagram on the left.

In Figure 1.1., the diagram on the right shows the graph of the curve: \((b, y(b))\) for three different values of \(\delta i.e.: \delta_c = 0.734063212067\) (solid curve), \(\delta = \delta_{f_1} = 0.75\) (dashed curve), and \(\delta = \delta_{f_2} = 0.8\) (dotted curve). One can see that the fold points are getting closer to each other as \(\delta\) approaches \(\delta_c\) from above. This is a strong indication of the occurrence of cusp bifurcation.

### 4. Numerical simulation

In this section we will consider three cases, i.e. \(\delta = \delta_{f_1} = 0.8, \delta = \delta_c = 0.734063212067\) and \(\delta = 0.7.\) For each of this cases we will follow the equilibria for the system (2.1) as the parameter \(b\) varies.
Figure 1. The diagram on the left shows two fold curves (solid curves) in \((\delta, b)\) coordinate of system (2.1) which intersect nontransversally at the cusp point at \((\delta_c, b_c) = (0.734063212067, -2.387660586)\).

We have added three lines \(\delta = \delta_c = 0.734063212067\) (solid line), \(\delta = \delta_{f_1} = 0.75\) (dashed line), and \(\delta = \delta_{f_2} = 0.8\) (dotted line). The diagram on right shows three curves \(y(b)\) which solve (3.3), for \(\delta = \delta_c\) (solid curve), \(\delta_{f_1}\) (dashed curve) and \(\delta_{f_2}\) (dotted curve).

4.1. Two fold bifurcations at \(\delta = \delta_{f_2}\)

For \(\delta = \delta_{f_2} = 0.8\), we derive three branches of equilibria as a function of \(b\), namely \(E_1\), \(E_2\) and \(E_3\) in Figure 2.

Figure 2. The branch of continuation equilibria as variation on parameter \(b\) for \(\delta > \delta_c\). Two fold bifurcation points are \(b = b_1\) and \(b = b_2\). When \(b < b_1\), the system (2.1) have one positive equilibrium \(E_1\). Then at \(b = b_1\), there appears a non-hyperbolic equilibria \(E_2 = E_3\). Next when \(b_1 < b < b_2\), this non-hyperbolic equilibria split into \(E_2\) and \(E_3\) while \(E_1\) still exist. At \(b = b_2\), \(E_1\) and \(E_2\) collapse into one non-hyperbolic equilibrium. And next, when \(b > b_2\), the system (2.1) have one positive equilibrium \(E_3\).
| $b$   | Equilibrium | Eigenvalues of the equilibrium                        |
|-------|-------------|------------------------------------------------------|
| $b < b_1$ | $E_1$ | One negative and complex eigenvalues with positive real part |
| $b = b_1$ | $E_1$ | One negative and complex eigenvalues with positive real part |
|        | $E_2 = E_3$ | One zero, one positive, one negative                 |
| $b_1 < b < b_2$ | $E_1$ | One negative and complex eigenvalues with positive real part |
|        | $E_2$ | One negative and complex eigenvalues with positive real part |
|        | $E_3$ | One positive and complex eigenvalues with positive real part |
| $b = b_2$ | $E_1 = E_2$ | One zero, one positive, one negative                 |
|        | $E_3$ | One negative and complex eigenvalues with positive real part |
| $b > b_2$ | $E_3$ | One negative and complex eigenvalues with positive real part |

### 4.2. Cusp bifurcation at $\delta = \delta_c$

For $\delta = \delta_c$, we get the branch of continuation equilibria as variation on parameter $b$.

| $b$   | Equilibrium | Eigenvalues of the equilibrium                        |
|-------|-------------|------------------------------------------------------|
| $b < b_1$ | $E_1$ | One negative and complex eigenvalues with positive real part |
| $b = b_1$ | $E_1 = E_3$ | One zero, one positive, one negative                 |
| $b > b_1$ | $E_3$ | One negative and complex eigenvalues with positive real part |

### 4.3. No fold bifurcation point when $\delta < \delta_c$

For illustration in this interval, we choose $\delta = 0.7$ and we get the branch of continuation equilibria as variation on parameter $b$.

### 5. Concluding remarks

In previous paper [16], Marwan et al. have developed the normalized two-prey one predator model. They found two fold bifurcation points that occur at $b = b_1$ and $b = b_2$ where $b_1 \leq b_2$. We continue their work by adding $\delta$ as a new parameter. Using Lagrange Multiplier Method, we compute a two parameter family of fold bifurcation curves and cusp bifurcation point in parameter space. Two bifurcation curves intersect each other non-transversally at a cusp point $(\delta, b) = (\delta_c, b_1)$ where $b_1 = b_2$. From the bifurcation diagram, we can analyze the number of equilibria and its stability, for details, see section 4.

We still need to check another one and two parameters bifurcation by some continuation software for further stability analysis. This paper focus on the application of Lagrange Multiplier Method for computing fold bifurcation curves and the cusp bifurcation.
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A. Appendix : Coefficients of equation 3.3

\[
\begin{align*}
A_{10}(\delta) &= 53808401250 \delta^3 - 161425203750 \delta^2 + 161425203750 \delta - 53808401250 \\
A_{9}(\delta) &= 484275611250 \delta^3 - 1452826833750 \delta^2 + 1452826833750 \delta - 484275611250 \\
A_{8}(\delta) &= 146478425625 \delta^4 + 807908418000 \delta^3 - 2249266459725 \delta^2 \\
&\quad + 1488893962950 \delta - 194014346850 \\
A_{7}(\delta) &= 878870553750 \delta^4 - 2460970410750 \delta^3 + 8452627497900 \delta^2 \\
&\quad - 13081784678550 \delta + 6211257037650 \\
A_{6}(\delta) &= 2215581629625 \delta^4 - 10644623014950 \delta^3 + 31086867794085 \delta^2 \\
&\quad - 34211693470620 \delta + 11553642062100 \\
A_{5}(\delta) &= 2571269022000 \delta^4 - 13535257269600 \delta^3 + 32715458776080 \delta^2 \\
&\quad - 19417028388480 \delta - 2914297434000 \\
A_{4}(\delta) &= 883829003400 \delta^4 - 4011367429800 \delta^3 + 3643682552856 \delta^2 \\
&\quad + 11312619948000 \delta - 5438335626000 \\
A_{3}(\delta) &= -755036100000 \delta^4 + 3855873980000 \delta^3 - 9487870704000 \delta^2 \\
&\quad + 6182433324000 \delta + 284048100000 \\
A_{2}(\delta) &= -36905670000 \delta^4 + 1705830240000 \delta^3 - 2826383122800 \delta^2 \\
&\quad - 911946600000 \delta + 529043400000 \\
A_{1}(\delta) &= 52920000000 \delta^4 - 268056000000 \delta^3 + 664588000000 \delta^2 - 449452800000 \delta \\
A_{0}(\delta) &= 28224000000 \delta^4 - 131328000000 \delta^3 + 239708160000 \delta^2 
\end{align*}
\]

We show all fold bifurcation curves in Figure -3.-.

![Bifurcation Diagram](image)

Figure -3.- The bifurcation diagram for all fold bifurcation curves. The diagram on the left for \( b \geq 0 \) and the diagram on the right for \( b \leq 0 \). We use some symbols for points of fold bifurcation curves in the plot, i.e.: solid circle, asterisk, box, circle, cross, diagonal cross and diamond. The solid circle curves are the fold bifurcation curves in Figure -1.-.
References

[1] Bazykin A D 1985 Mathematical biophysics of interacting populations *Nauka, Moscow.*

[2] Berryman A A 1992 *The origins and evolution of predator-prey theory* Ecology 73(5) pp 1530-1535.

[3] Dhooge A, Govaerts W and Kuznetsov Y A 2003 MATCONT: a MATLAB package for numerical bifurcation analysis of ODEs *ACM Transactions on Mathematical Software (TOMS)* 29(2) pp.141-164.

[4] Dhooge A, Govaerts W, Kuznetsov Y A, Mestrom W, Riet A M and Sautois B 2006 MATCONT and CL MATCONT: Continuation toolboxes in matlab *Universiteit Gent, Belgium and Utrecht University, The Netherlands.*

[5] Dineen S 2014 *Multivariate Calculus and Geometry* third edition (London: Springer Undergraduate Mathematics Series)

[6] Doedel E J and Oldeman B E 2012 *AUTO-07P: Continuation and Bifurcation Software for Ordinary Differential Equations* (Canada: Concordia University, Montreal)

[7] Elettreby M F 2009 Two-prey one-predator model *Chaos, Solitons & Fractals* 39 5 pp 2018-2027

[8] Ermentrout B 2002 *Simulating, Analyzing, and Animating Dynamical Systems: A Guide to XPPAUT for Researchers and Students* (Philadelphia: SIAM).

[9] Harjanto E and Tuwankotta J M 2016 Bifurcation of periodic solution in a Predator-Prey type of systems with non-monotonic response function and periodic perturbation *International Journal of Non-Linear Mechanics* 85 pp 188-196.

[10] Holling C S 1959 The components of predation as revealed by a study of small mammal predation of the European Pine Sawfly *The Canadian Entomologist* 91 pp 293-320.

[11] Holling, C S 1959 Some characteristics of simple types of predation and parasitism *The Canadian Entomologist* 91 pp 385-398.

[12] Huang Y and Diekmann O 2001 *Predator migration in response to prey density: What are the consequences?* J. of Math. Bio. 11 43 (6) pp 561-581.

[13] Jia Li and Zhien Ma 2009 *Dynamical Modeling and Analysis of Epidemics.* (Singapore : World Scientific Publishing Co. Pte. Ltd.).

[14] Klebanoff A and Hastings A 1994 Chaos in one-predator, two-prey models: cgeneral results from bifurcation theory *Mathematical biosciences* 122(2) pp 221-233.

[15] Kuznetsov Y A 1998 *Elements of Applied Bifurcation Theory* second edition Applied Math. Sciences 112 (New York : Springer).

[16] Marwan, Tuwankotta J M and Haryanto E 2018 Application of Lagrange Multiplier Method for Computing Fold Bifurcation Point In A Two-Prey One Predator Dynamical System *Journal Indonesian Mathematics Society* Vol. 24 No. 02 pp 07-19.

[17] Shigui Ruan, Wanbiao Ma, and Zhixing Hu 2012 Analysis of SIR Epidemic Models with Nonlinear Incidence Rate and Treatment *Mathematical Biosciences* Vol. 238 pp 12-20.

[18] Tripathi J P, Abbas S and Thakur M 2014 Local and global stability analysis of a two prey one predator model with help *Communications in Nonlinear Science and Numerical Simulation* 19(9) pp 3284-3297.

[19] Wang W and Ruan S 2004 Bifurcations in an epidemic model with constant removal rate of the infectives *Journal of Mathematical Analysis and Applications* 291(2) pp 775-793.

[20] Wang W 2006 Backward Bifurcation of an Epidemic Model with Treatment *Mathematical Biosciences* Vol. 201 pp 58.

[21] Zhu H, Campbell S A and Wolkowicz G S 2003 Bifurcation analysis of a predator-prey system with nonmonotonic functional response *SIAM Journal on Applied Mathematics* 63(2) pp 636-682.