RIESZ TRANSFORMS ON COMPACT RIEMANNIAN SYMMETRIC SPACES OF RANK ONE

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Abstract. In this paper we prove mixed norm estimates for Riesz transforms related to Laplace–
Beltrami operators on compact Riemannian symmetric spaces of rank one. These operators are
closely related to the Riesz transforms for trigonometric Jacobi polynomials expansions, where the
type parameter in the Jacobi expansions depends on the level in which the $L^2$ projections are taken.
Thus, the key point is to obtain sharp estimates for the kernel of the Jacobi–Riesz transforms
with control on the parameters, together with a weighted vector-valued extension by means of an
adaptation of Rubio de Francia’s extrapolation theorem. The latter results are of independent
interest.

1. Introduction

Let $M$ be a Riemannian symmetric space of rank one$^1$ and compact type. These spaces are
completely classified. They are the sphere, the real, complex and quaternionic projective spaces and
the Cayley plane. With the Riemannian gradient $\nabla_M$ on $M$ and the corresponding divergence $\text{div}_M$,
one can define the Laplace–Beltrami operator $\tilde{\Delta}_M = \text{div}_M \nabla_M$. Suppose that $u$ is a solution to the
fractional problem

$$(-\Delta_M)^{1/2} u = f, \quad \text{in } M,$$

where $(-\Delta_M)^{1/2}$ is the square root of $-\Delta_M = -\tilde{\Delta}_M + \lambda_M$, and $\lambda_M$ is a positive constant depending on
$M$ (see (2.1) below). The operator $(-\Delta_M)^{1/2}$ could be thought as a first order (nonlocal) differential
operator. Thus, when $f$ is in some functional space $X$ we may expect the gradient of $u$ to be in the
same space $X$, that is, our hope is that

$$|\nabla_M u| = |\nabla_M (-\Delta_M)^{-1/2} f| \in X.$$

The operator

$$R_M f := |\nabla_M (-\Delta_M)^{-1/2} f|,$$

is the Riesz transform on $M$. The consideration above would say that the Riesz transform is bounded
from $X$ to $X$. In this paper we choose as $X$ the mixed norm spaces $L^p(L^2(M))$ that can be defined by
using polar coordinates on $M$.

Here is our first main result.

Theorem 1.1. Let $M$ be a compact Riemannian symmetric space of rank one. Then the Riesz
transform $R_M$ is a bounded operator from $L^p(L^2(M))$ into itself, for all $1 < p < \infty$.

The mixed norm spaces $L^p(L^2(M))$ are defined in terms of appropriate polar coordinates $(\theta, x') \in
(0, \pi) \times S_M$, where $S_M$ is a unit sphere whose dimension depends on the particular manifold $M$, see

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de Francia extrapolation theorem.

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$^1$These are the “model” spaces of Riemannian geometry, cf. Isaac Chavel, Riemannian Symmetric Spaces of Rank
One, Lecture Notes in Pure and Applied Mathematics, Vol. 5. Marcel Dekker, Inc., New York, 1972.
It turns out that, if we denote by $\tilde{\Delta}_{SM}$ the Laplace–Beltrami operator on the sphere $S_M$, then we can write
\[ -\Delta_M = J^{\alpha,\beta} - \rho_M(\theta)\tilde{\Delta}_{SM} , \]
where, for given parameters $\alpha, \beta > -1$, that depend on $M$,
\[ J^{\alpha,\beta} = -\frac{d^2}{d\theta^2} - \frac{\alpha - \beta + (\alpha + \beta + 1) \cos \theta}{\sin \theta} \frac{d}{d\theta} + \left( \frac{\alpha + \beta + 1}{2} \right)^2 , \]
is the trigonometric Jacobi polynomials differential operator, and $\rho_M(\theta)$ is some nonnegative function of $\theta$. Observe that the coordinates separate variables in $M$. Therefore, the eigenvalues $\lambda_n(M)$ of $-\Delta_M$ can be written in terms of products of spherical harmonics and trigonometric Jacobi polynomials whose type parameters depend on $j$, where $j$ varies in a range depending on $n$ and the dimension of $M$. Moreover, the Riesz transform on $M$ in the spaces $L^p(L^2(M))$ is related to the Jacobi–Riesz transform
\[ R^{\alpha,\beta} = \delta(J^{\alpha,\beta})^{-1/2}, \quad \text{where } \delta = \frac{d}{d\theta}, \]
and an auxiliary operator expressed via the Jacobi fractional integral
\[ T_M^{\alpha,\beta} = \sqrt{\rho_M(\theta)(J^{\alpha,\beta})^{-1/2}}, \]
see Section 2. In particular, we need vector-valued estimates for the Jacobi–Riesz transform. A key point is to obtain estimates for the kernel of the Jacobi–Riesz transform with control on the constants. To introduce our second main result we let
\[ d\mu_{\alpha,\beta}(\theta) = (\sin \frac{\theta}{2})^{\alpha + 1} (\cos \frac{\theta}{2})^{\beta + 1} d\theta. \]
Then $J^{\alpha,\beta}$ is a symmetric operator on $L^2(d\mu_{\alpha,\beta}) := L^2((0, \pi), d\mu_{\alpha,\beta}(\theta))$ whose spectral resolution is given by the trigonometric Jacobi polynomials $P_n^{(\alpha,\beta)}(\theta)$ as in (2.3) below. With this we can give a precise meaning to $R^{\alpha,\beta}$ as a bounded operator in $L^p(L^2(d\mu_{\alpha,\beta})$, see (2.5). It is known (see [23]) that $R^{\alpha,\beta}$ is a Calderón–Zygmund operator on the space of homogeneous type $((0, \pi), |\cdot|, d\mu_{\alpha,\beta})$ with associated kernel $K^{\alpha,\beta}(\theta, \varphi)$.

**Theorem 1.2** (Sharp estimates for the Jacobi–Riesz kernel). Let $\alpha, \beta > -1/2$, $a \geq 1$, $b = 0$ or $b \geq 1$, and $j \in \mathbb{N}_0$. Define
\[ u_j(\theta) = (\sin \frac{\theta}{2})^{aj}(\cos \frac{\theta}{2})^{bj}, \quad \theta \in (0, \pi), \quad j = 0, 1, \ldots. \]
Then
\[ |u_j(\theta) u_j(\varphi) K^{\alpha+j,\beta + bj}(\theta, \varphi)| \leq \frac{C_1}{\mu_{\alpha,\beta}(B(\theta, |\theta - \varphi|))}, \quad \theta \neq \varphi, \]
and
\[ |\nabla_{\theta,\varphi}(u_j(\theta) u_j(\varphi) K^{\alpha+j,\beta + bj}(\theta, \varphi))| \leq \frac{C_2}{|\theta - \varphi| \mu_{\alpha,\beta}(B(\theta, |\theta - \varphi|))}, \quad \theta \neq \varphi, \]
with $C_1$ and $C_2$ independent of $j$, where $\mu_{\alpha,\beta}(B(\theta, |\theta - \varphi|))$ is the $d\mu_{\alpha,\beta}$-measure of the interval with center $\theta$ and radius $|\theta - \varphi|$.

As a consequence, and by means of an adaptation of Rubio de Francia’s extrapolation theorem given in Theorem 5.1 below, we get the following weighted vector-valued extension for the Jacobi–Riesz transform.

**Theorem 1.3** (Vector-valued extension for $R^{\alpha,\beta}$). Let $\alpha, \beta > -1/2$, $a \geq 1$, $b = 0$ or $b \geq 1$, $1 < p, r < \infty$ and $u_j$ as in (1.3). Let $A_p^{\alpha,\beta}$ be the class of Jacobi weights defined in (5.1) below. Then there is a constant $C$ such that
\[ \left\| \left( \sum_{j,k=0}^{\infty} |u_j R^{\alpha+j,\beta + bj}(u_j^{-1} f_{j,k})^r \right)^{1/r} \right\|_{L^p(w d\mu_{\alpha,\beta})} \leq C \left\| \left( \sum_{j,k=0}^{\infty} |f_{j,k}|^r \right)^{1/r} \right\|_{L^p(w d\mu_{\alpha,\beta})}, \]
for all $f_{j,k} \in L^p(w d\mu_{\alpha,\beta}) := L^p((0, \pi), w(\theta) d\mu_{\alpha,\beta}(\theta))$ and $w \in A_p^{\alpha,\beta}$. 

Similar results are obtained for the auxiliary operator $T_M^\alpha,\beta$, see Theorem 2.1 and Theorem 2.2. With these results we can then prove Theorem 1.1, see Section 2.

The analysis on Riemannian symmetric spaces has been studied by many authors, see for example [2, 3, 5, 10, 14, 16, 17, 18, 19, 20, 24, 29, 30, 32, 34]. It is remarkable the work on Fourier and Geometric Analysis done by S. Helgason [16, 17, 18, 19, 20]. In [29, 30], T. O. Sherman developed the counterpart of Helgason’s Fourier theory for compact type spaces. More recently, S. Thangavelu considered in [34] holomorphic Sobolev spaces. We studied mixed norm estimates for the fractional integral on $M$ in [14]. The Harmonic Analysis for compact Lie groups is treated in E. M. Stein [31].

Certainly the Lebesgue mixed norm spaces $L^p(L^r)$ are of interest in Harmonic Analysis and Partial Differential Equations. A systematic study was carried out by A. Benedek and R. Panzone in [4]. The spaces $L^p_{\text{rad}}(L^2_{\text{ang}}(\mathbb{R}^d))$ were used by J. L. Rubio de Francia to tackle the problem of almost everywhere convergence of Bochner–Riesz means for the Fourier transform in even dimensions, see [26]. On the other hand, the remarkable paper by M. Keel and T. Tao [21] refers to Strichartz estimates for the Schrödinger and wave equations, which is boundedness of solutions in the mixed norm spaces $L^p_tL^q_x$ ($t$ for time and $x$ for space). These estimates turn out to be useful for proving well-posedness of nonlinear equations.

It is known that the Riesz transforms based on the Laplace–Beltrami operator $\nabla_M(-\tilde{\Delta}_M)^{-1/2}$ are bounded in $L^p(M)$, $1 < p < \infty$, even if $M$ is of non-compact type, see [32, Section 6]. Nothing had been done so far in the mixed norm context. On the other hand, in [23] A. Nowak and P. Sjörgen considered the Riesz transform related to the operator (1.1) and they showed that it is a Calderón–Zygmund operator. Nevertheless, the constants in the kernel estimates in [23] are not explicit and it is not clear from their computations that they can be tracked down explicitly. In turn, we get control on the constants in Theorem 1.2. Notice that when $\alpha = \beta = \lambda - 1/2$, $\lambda > -1/2$, the Jacoby expansions reduce to ultraspherical expansions, see (2.11). A notion of conjugacy for ultraspherical expansions was given by B. Muckenhoupt and E. M. Stein in [22]. The Calderón–Zygmund theory and, in particular, the Riesz transforms in such a context, were studied by D. Buraczewski et al. [7, 8] and J. J. Betancor et al. [6].

The paper is organized as follows. In Section 2 we prove Theorem 1.1 by assuming the validity of Theorems 1.3 and 2.2. Section 3 is devoted to the proof of Theorem 1.2, while Section 4 contains the estimates for the kernel of the operator $T_M^\alpha,\beta$. The computations, though simple and systematic, are rather cumbersome, so we try to keep them to a minimum. Theorems 1.3 and 2.2 share the same proof, so they will be proved at the end of the paper, see Section 5. The last section also contains the adaptation to the Jacobi context of the Rubio de Francia’s extrapolation theorem that we need.

2. The Riesz Transform on $M$

In this section we give the precise definitions of the Riesz transform $R_M$ and the mixed norm spaces $L^p(L^2(M))$ on the compact Riemannian symmetric spaces of rank one $M$. By assuming Theorem 1.3 and Theorem 2.2 we will prove Theorem 1.1 case by case.

As we said in the introduction, the manifolds $M$ we are considering here are completely classified. Indeed, H.-C. Wang [35], showed that the Riemannian symmetric spaces of compact type and rank one are compact two-point homogeneous spaces. They are

1. the sphere $S^d \subset \mathbb{R}^{d+1}$, $d \geq 1$;
2. the real projective space $P_d(\mathbb{R})$, $d \geq 2$;
3. the complex projective space $P_l(\mathbb{C})$, $l \geq 2$;
4. the quaternionic projective space $P_l(\mathbb{H})$, $l \geq 2$;
5. the Cayley plane $P_2(\mathbb{C}_\alpha\mathbb{V})$.

In this way, we are going to take

\begin{equation}
\lambda_{S^d} = \lambda_{P_d(\mathbb{R})} = (\frac{d-1}{2})^2, \quad \lambda_{P_l(\mathbb{C})} = \lambda_{P_l(\mathbb{H})} = \lambda_{P_2(\mathbb{C}_\alpha\mathbb{V})} = (\frac{m+d}{2})^2.
\end{equation}

Here $d = 2, 4, 8$, $m = l - 2, 2l - 3, 3$, for $P_l(\mathbb{C})$, $P_l(\mathbb{H})$ and $P_2(\mathbb{C}_\alpha\mathbb{V})$, respectively. Besides, we will assume that $d \geq 2$ in the case of $S^d$. 

2.1. Preliminaries on Jacobi expansions. The standard Jacobi polynomials of degree \( n \geq 0 \) and type \( \alpha, \beta > -1 \) are given by
\[
P_n^{(\alpha, \beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta} \left( \frac{-1}{2n+1} \right)^n \frac{d^n}{dx^n} \left\{ (1-x)^{n+\alpha}(1+x)^{n+\beta} \right\}, \quad x \in (-1,1),
\]
see [33, (4.3.1)]. These form an orthogonal basis of \( L^2((-1,1), (1-x)^{\alpha}(1+x)^{\beta}dx) \). After making the change of variable \( x = \cos \theta \), we obtain the normalized Jacobi trigonometric polynomials
\[
P_n^{(\alpha, \beta)}(\theta) = d_n^{\alpha, \beta} P_n^{(\alpha, \beta)}(\cos \theta),
\]
where the normalizing factor is
\[
d_n^{\alpha, \beta} = 2^{\alpha+\beta+1/2} \frac{\|P_n^{(\alpha, \beta)}\|_{L^2((-1,1), (1-x)^{\alpha}(1+x)^{\beta}dx)}}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}.
\]
The trigonometric polynomials \( P_n^{(\alpha, \beta)} \) are eigenfunctions of the differential operator \( J^{\alpha, \beta} \) in (1.1). Indeed, we have \( J^{\alpha, \beta} P_n^{(\alpha, \beta)} = \lambda_n^{\alpha, \beta} P_n^{(\alpha, \beta)} \), with eigenvalue \( \lambda_n^{\alpha, \beta} = \left( n + \frac{\alpha + \beta + 1}{2} \right)^2 \). Moreover, the system \( \{ P_n^{(\alpha, \beta)} \}_{n \geq 0} \) forms a complete orthonormal basis of \( L^2(d\mu_{\alpha, \beta}) \), with \( d\mu_{\alpha, \beta} \) as in (1.2). For further references about Jacobi polynomials, see [33, Chapter IV].

2.1.1. Jacobi–Riesz transforms. We have the decomposition
\[
J^{\alpha, \beta} = \delta^* \delta + \left( \frac{\alpha + \beta + 1}{2} \right)^2,
\]
where \( \delta = \frac{d}{dx} \) and \( \delta^* \) is its formal adjoint in \( L^2(d\mu_{\alpha, \beta}) \), that is, \( \delta^* = -\frac{d}{dx} - (\alpha + 1/2) \cot \frac{\theta}{2} + (\beta + 1/2) \tan \frac{\theta}{2} \). The Jacobi–Riesz transform is formally defined as \( R^{\alpha, \beta} := \delta(J^{\alpha, \beta})^{-1/2} \). For a function \( f \in L^2(d\mu_{\alpha, \beta}) \) we can write
\[
f = \sum_{n=0}^{\infty} a_n^{\alpha, \beta}(f) P_n^{(\alpha, \beta)} = \sum_{n=0}^{\infty} \langle f, P_n^{(\alpha, \beta)} \rangle_{L^2(d\mu_{\alpha, \beta})} P_n^{(\alpha, \beta)} \text{ in } L^2(d\mu_{\alpha, \beta}).
\]
Then
\[
R^{\alpha, \beta} f(\theta) = \sum_{n=0}^{\infty} \frac{1}{n + \frac{\alpha + \beta + 1}{2}} c_n^{\alpha, \beta}(f) \sin \theta P_n^{(\alpha, \beta)}(\theta),
\]
where
\[
R^{\alpha, \beta} f(\theta) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(n(n + \alpha + \beta + 1))^{1/2}}{n + \frac{\alpha + \beta + 1}{2}} c_n^{\alpha, \beta}(f) \sin \theta P_{n+1}^{(\alpha + 1, \beta + 1)}(\theta).
\]
It can be checked that \( R^{\alpha, \beta} \) is a bounded operator in \( L^2(d\mu_{\alpha, \beta}) \). We also know that \( R^{\alpha, \beta} \) is a Calderón–Zygmund operator associated with the kernel \( K^{\alpha, \beta}(\theta, \varphi) \) given in (3.5) below. See Section 3. In this section we assume that Theorem 1.3 is true.

2.1.2. The auxiliary operator \( T_{M}^{\alpha, \beta} \). Here we define \( T_{M}^{\alpha, \beta} \) as in the introduction by taking
\[
\rho_{\psi_M}(\theta) = \frac{1}{(\sin \theta)^2}, \quad \rho_{P_M(\mathbb{R})}(\theta) = \rho_{P_M(\mathbb{C})}(\theta) = \rho_{P_M(\mathbb{B})}(\theta) = \rho_{P_M(\mathbb{C} \psi_M)}(\theta) = \frac{1}{(\sin \frac{\theta}{2})^2}.
\]
For \( f \in L^2(d\mu_{\alpha, \beta}) \) we have
\[
T_{M}^{\alpha, \beta} f(\theta) = \sqrt{\rho_M(\theta)} \sum_{n=0}^{\infty} \frac{1}{n + \frac{\alpha + \beta + 1}{2}} c_n^{\alpha, \beta}(f) P_{n+1}^{(\alpha + 1, \beta + 1)}(\theta).
\]
We will prove in Section 4 that \( j u_j T_{M}^{\alpha + a, \beta + b} \) can be seen as a Calderón–Zygmund operator with kernel \( j u_j(\theta)u_j(\varphi) T_{M}^{\alpha + a, \beta + b}(\theta, \varphi) \), see (4.1) below. We will also prove there the following estimate.

**Theorem 2.1.** Let \( \alpha, \beta > -1/2, a = 1 \) or \( a \geq 2, b = 0 \) or \( b \geq 1, j \geq 1 \) and \( u_j \) be as in (1.3). Then
\[
|j u_j(\theta)u_j(\varphi) T_{M}^{\alpha + a, \beta + b}(\theta, \varphi)| \leq \frac{C}{\mu_{\alpha, \beta}(B(\theta, |\theta - \varphi|))}, \quad \theta \neq \varphi,
\]
where \( C \) is a constant depending only on \( \alpha, \beta, a, b, j \).
\[ |j \nabla_{\theta,\varphi}(u_j(\theta)u_j(\varphi)T^\alpha_{M\beta}^\alpha \beta + b_j(\theta, \varphi))| \leq \frac{C_2}{|\theta - \varphi| \mu_{\alpha, \beta}(B(\theta, \theta - \varphi))}, \quad \theta \neq \varphi, \]

with \( C_1 \) and \( C_2 \) independent of \( j \).

In this section we assume the validity of the following consequence of Theorem 2.1.

**Theorem 2.2.** Let \( \alpha, \beta > -1/2, a = 1 \) or \( a \geq 2, b = 0 \) or \( b \geq 1 \) and \( 1 < p, r < \infty \). Let \( u_j \) be as in (1.3). When \( M = S^d \), we also assume that \( \beta > 0 \) and \( b \geq 1 \). Then there exists a constant \( C \) such that

\[
\left\| \left( \sum_{j,k=1}^{\infty} |j u_j T^\alpha_{M\beta}^\alpha \beta + b_j(u_j f_{j,k}|)^r \right)^{1/r} \right\|_{L^p(w \, d\mu_{\alpha, \beta})} \leq C \left\| \left( \sum_{j,k=1}^{\infty} |f_{j,k}|^r \right)^{1/r} \right\|_{L^p(w \, d\mu_{\alpha, \beta})},
\]

for all \( f_{j,k} \in L^p(w \, d\mu_{\alpha, \beta}) \) and all \( w \in A^\alpha_{p, \beta} \).

2.2. **The case of the unit sphere** \( S^d \). Let \( S^d = \{ x \in \mathbb{R}^{d+1} : x_1^2 + \cdots + x_{d+1}^2 = 1 \} \) be the unit sphere in \( \mathbb{R}^{d+1}, d \geq 2 \). We set \( -\Delta_{S^d} = -\Delta_0 + \lambda_{S^d} \) as in the introduction, see also (2.1). It is well-known that \( L^2(S^d) = \bigoplus_{n=0}^\infty \mathcal{H}_n(S^d) \), where \( \mathcal{H}_n(S^d) \) is the set of spherical harmonics of degree \( n \) in \( d + 1 \) variables, see [5, Chapter 3, Section C.1]. Each \( \mathcal{H}_n(S^d) \) is an eigenspace of \( \Delta_{S^d} \) with eigenvalue \(-n(n + d - 1)\). Hence, \(-\Delta_{S^d}(\mathcal{H}_n(S^d)) = (n + d - 1)^2(\mathcal{H}_n(S^d)) \).

We introduce the following coordinates on \( S^d \), known as geodesic polar coordinates. Each point on the sphere can be written as

\[ \Phi(\theta, x') = (\cos \theta, x_1 \sin \theta, \ldots, x_d \sin \theta) \in S^d, \]

for \( \theta \in [0, \pi] \) and \( x' \in S^{d-1} \) (see [11, pp. 69–70], also [30, p. 104] after the change \( t = \cos \theta \)). For a function \( f \) on \( S^d \) let us write \( F = f \circ \Phi \). We have, see [5, pp. 56–57],

\[
\int_{S^d} f(x) \, dx = \int_0^\pi \int_{S^{d-1}} F(\theta, x') \, dx' (\sin \theta)^{d-1} \, d\theta,
\]

By using the coordinates in (2.8), we can see that

\[ -\Delta_{S^d} = -\frac{\partial^2}{\partial \theta^2} - (d - 1) \cot \theta \frac{\partial}{\partial \theta} + \lambda_{S^d} \frac{1}{\sin^2 \theta} \Delta_{S^{d-1}} = J^\alpha(\theta) \Delta_{S^{d-1}}, \]

where \( \Delta_{S^{d-1}} \) is the spherical part of the Laplacian on \( \mathbb{R}^d \), acting on functions on \( \mathbb{R}^{d+1} \) by holding the first coordinate fixed and differentiating with respect to the remaining variables, see [10, Chapter II, Section 5]. In (2.9) we have chosen \( \alpha = \beta = \frac{d-2}{2} \geq 0 \) for the operator \( J^\alpha(\theta) \) of (1.1).

By using the spherical harmonics \( \mathcal{H}_n(S^d) \) via spherical harmonics and the coordinates in (2.8), it is an exercise to see that an orthonormal basis associated to (2.9) is given by

\[ \varphi_{n,j,k}(x) = \psi_{n,j}(\theta) Y_{j,k}^d(x), \quad x = \Phi(\theta, x'), \quad 0 < \theta < \pi, \quad x' \in S^{d-1}. \]

See also [30, Section 3, p. 110]. Here, for \( n \geq 0 \) and \( j = 0, 1, \ldots, n \),

\[ \psi_{n,j}(\theta) = a_{n,j}(\sin \theta)^j C^\lambda_{n-j}(\cos \theta), \]

where

\[ C^\lambda_{n-j}(x) = \frac{\Gamma(\lambda + 1/2) \Gamma(k + 2\lambda)}{\Gamma(2\lambda) \Gamma(k + \lambda + 1/2)} P^{(\lambda-1/2, \lambda-1/2)}(x), \quad \lambda > -1/2, \quad x \in (-1, 1), \]

is an ultraspherical polynomial, see [33, (4.7.1)], and \( a_{n,j} \) is the normalizing constant

\[ a_{n,j} = \| (\sin \theta)^j C^\lambda_{n-j} \|_{L^1(S^2)}^2 = \frac{2^{j-1} (2j + 1) \Gamma(2j + d - 1) \Gamma(n + d/2)}{\Gamma(j + d/2) \Gamma(n + j + d - 1)} d^{j-d/2} \left( j + \frac{d}{2} \right) \]

\[
\times \left( \frac{j!}{\left( \frac{j-d}{2} \right)!} \right)^2, \]

where \( d = \frac{d-2}{2} \).
with $d_{n,d}^2$ as in (2.4). The functions $Y_{j,k}^d$, $k = 1, \ldots, d(j)$, where $d(j) = (2j + d - 2)!/(2j)!j!$, form an orthonormal basis of spherical harmonics on $\mathbb{S}^{d-1}$ of degree $j \geq 0$. The orthogonal projections of $f$ onto the spaces $\mathcal{H}_n(\mathbb{S}^d)$ can be written in coordinates as

$$
\text{Proj}_{\mathcal{H}_n(\mathbb{S}^d)}(f) = \sum_{j=0}^{n} \sum_{k=1}^{d(j)} \langle f, \varphi_{n,j,k} \rangle L^2(\mathbb{S}^d) \varphi_{n,j,k}.
$$

(2.12)

We define the mixed norm space $L^p(L^2(\mathbb{S}^d))$, $1 \leq p < \infty$, as the set of functions $f$ on $\mathbb{S}^d$ such that

$$
\|f\|_{L^p(L^2(\mathbb{S}^d))} := \left( \int_0^\pi \left( \int_{\mathbb{S}^{d-1}} |F(\theta, x')|^2 \rho_{\mathbb{S}^d}(\theta) \sin \theta \, d\theta \right)^{p/2} \sin \theta \, d\theta \right)^{1/p} < \infty.
$$

The spaces $L^p(L^2(\mathbb{S}^d))$ under this norm are Banach spaces.

The Riesz transform on the mixed norm space $L^p(L^2(\mathbb{S}^d))$ are defined in the following way. Let

$$
(-\Delta_{\mathbb{S}^d})^{-1/2} f \equiv (-\Delta_{\mathbb{S}^d})^{-1/2} F = \sum_{n=0}^{\infty} \frac{1}{n + \frac{d^2}{2}} \text{Proj}_{\mathcal{H}_n(\mathbb{S}^d)}(f),
$$

(2.13)

(note the abuse of notation in (2.13)) for $f \in L^2(\mathbb{S}^d)$. Next, notice that $-\Delta_{\mathbb{S}^d} = \delta_{\mathbb{S}^d}^* \delta_{\mathbb{S}^d} + \lambda_{\mathbb{S}^d}$, with

$$
\delta_{\mathbb{S}^d} = x' \frac{\partial}{\partial \theta} + \sqrt{\rho_{\mathbb{S}^d}(\theta)} \nabla_{\mathbb{S}^{d-1}},
$$

and

$$
\delta_{\mathbb{S}^d}^* = -\left( \frac{\partial}{\partial \theta} + (d-1) \cot \theta - \frac{d-1}{\sin \theta} \right) x' + \sqrt{\rho_{\mathbb{S}^d}(\theta)} \text{div}_{\mathbb{S}^{d-1}},
$$

the formal adjoint of $\delta_{\mathbb{S}^d}$ in $L^2(L^2(\mathbb{S}^d)) = L^2(\mathbb{S}^d)$. Observe now that, since $\mathbb{S}^d$ is compact, by [5, p. 130, G.IV.2], and (2.10),

$$
\int_{\mathbb{S}^d} |\nabla_{\mathbb{S}^d} f(x)|^2 \, dx = \int_{\mathbb{S}^d} f(x) \tilde{\Delta}_{\mathbb{S}^d} f(x) \, dx
$$

$$
= \int_0^\pi \int_{\mathbb{S}^{d-1}} F(\theta, x') \delta_{\mathbb{S}^d}^* \delta_{\mathbb{S}^d} F(\theta, x') \sin \theta \, d\theta \, dx'
$$

$$
= \int_0^\pi \int_{\mathbb{S}^{d-1}} |\delta_{\mathbb{S}^d} F(\theta, x')|^2 \sin \theta \, d\theta \, dx'
$$

$$
= \int_0^\pi \int_{\mathbb{S}^{d-1}} (|\delta F(\theta, x')|^2 + \rho_{\mathbb{S}^d}(\theta)|\nabla_{\mathbb{S}^{d-1}} F(\theta, x')|^2) \sin \theta \, d\theta \, dx',
$$

(2.14)

with $\delta = \frac{\partial}{\partial \theta}$ as above, where in the last equality we used the orthogonality property $x' \cdot \nabla_{\mathbb{S}^{d-1}} F(\theta, x') = 0$. Hence, in the coordinates, $|\nabla_{\mathbb{S}^d} f(x)|^2 = |\delta F(\theta, x')|^2 + \rho_{\mathbb{S}^d}(\theta)|\nabla_{\mathbb{S}^{d-1}} F(\theta, x')|^2$. Therefore, by (2.13), in the coordinates we have

$$
|R_{\mathbb{S}^d} f|^2 = |\delta(-\Delta_{\mathbb{S}^d})^{-1/2} F|^2 + \rho_{\mathbb{S}^d}(\theta)|\nabla_{\mathbb{S}^{d-1}}(-\Delta_{\mathbb{S}^d})^{-1/2} F|^2.
$$

**Proof of Theorem 1.1 for $M = \mathbb{S}^d$.** By projecting $F$ on the space of spherical harmonics, we can write

$$
F(\theta, x') = \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} F_{j,k}(\theta) Y_{j,k}^d(x'), \quad \text{for } \theta \in (0, \pi), \ x' \in \mathbb{S}^{d-1},
$$

(2.15)

where

$$
F_{j,k}(\theta) = \int_{\mathbb{S}^{d-1}} F(\theta, x') Y_{j,k}(x') \, dx'.
$$

With this,

$$
\|f\|_{L^p(L^2(\mathbb{S}^d))} = \left( \int_0^\pi \left( \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} |F_{j,k}(\theta)|^2 \right)^{p/2} \sin \theta \, d\theta \right)^{1/p}.
$$

(2.16)
By applying (2.15) and the orthogonality of the spherical harmonics $Y_{j,k}^d$, we can write

$$\langle f, \varphi_{n,j,k} \rangle_{L^2(S^d)} = \langle F_{n,j,k}^{d} Y_{j,k}^d(x') \psi_{n,j,k}(x') \varphi_{n,j,k} \rangle_{L^2(S^d \times (0,\pi) \times (\sin \theta)^{d-1} d\theta \, dx')}$$

$$= \int_0^\pi \int_{S^{d-1}} \sum_{l=0}^{\infty} \sum_{s=1}^{d(l)} F_{l,s}(\theta) Y_{l,s}^d(x') \psi_{n,j,k}(\theta) Y_{j,k}^d(x') \sin \theta)^{d-1} d\theta \, dx'$$

$$= \sum_{l=0}^{\infty} d(l) \int_{S^{d-1}} F_{l,s}(\theta) \psi_{n,j,k}(\theta) (\sin \theta)^{d-1} d\theta \int_{S^{d-1}} Y_{l,s}^d(x') \overline{Y_{j,k}^d(x')} \sin \theta)^{d-1} d\theta \, dx'$$

$$= \int_0^\pi F_{j,k}(\theta) \psi_{n,j,k}(\theta) (\sin \theta)^{d-1} d\theta = \langle F_{j,k}^{d} \psi_{n,j,k} \rangle_{L^2((0,\pi), (\sin \theta)^{d-1} d\theta)}.$$

By using (2.13), (2.12), (2.17),

$$(-\Delta_{S^d})^{-1/2} f(x) = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=1}^{d(j)} \frac{(F_{j,k}^{d} \psi_{n,j,k})_{L^2((0,\pi), (\sin \theta)^{d-1} d\theta)}}{n + \frac{d(j)}{2}} \psi_{n,j,k}(\theta) Y_{j,k}^d(x')$$

$$= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=1}^{d(j)} \frac{(F_{j,k}^{d} \psi_{n+j,j})_{L^2((0,\pi), (\sin \theta)^{d-1} d\theta)}}{n + \frac{2j+d-1}{2}} \psi_{n+j,j}(\theta) Y_{j,k}^d(x')$$

$$= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sum_{k=1}^{d(j)} (F_{j,k}^{d} \psi_{n+j,j})_{L^2((0,\pi), (\sin \theta)^{d-1} d\theta)}}{n + \frac{2j+d-1}{2}} \psi_{n+j,j}(\theta)$$

Therefore, by (2.14), the Riesz transform can be written as

$$|R_{S^d} f(x)|^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} \sum_{n=0}^{\infty} \left( \frac{(F_{j,k}^{d} \psi_{n+j,j})_{L^2((0,\pi), (\sin \theta)^{d-1} d\theta)}}{n + \frac{2j+d-1}{2}} \delta_{n+j,j}(\theta) \right)^2$$

$$+ \rho_{S^d}(\theta) \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=1}^{d(j)} \left( \frac{(F_{j,k}^{d} \psi_{n+j,j})_{L^2((0,\pi), (\sin \theta)^{d-1} d\theta)}}{n + \frac{2j+d-1}{2}} \psi_{n+j,j}(\theta) \right)^2$$

Since

$$\langle F_{j,k}^{d} \psi_{n+j,j} \rangle_{L^2((0,\pi), (\sin \theta)^{d-1} d\theta)} = 2j+1 \left( \delta_{n+j,j} \frac{(\sin \theta)^{-j} F_{j,k}^{d} \mathcal{P}^{(a+j,a+j)}_{n}}{n + \frac{2j+d-1}{2}} \right)^2$$

and $-\Delta_{S^d} Y_{j,k}^d = j(j + d - 2) Y_{j,k}^d$, we have

$$\int_{S^{d-1}} |R_{S^d} f(x)|^2 \, dx'$$

$$= \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} \left( \frac{(\sin \theta)^{-j} F_{j,k}^{d} \mathcal{P}^{(a+j,a+j)}_{n}}{n + \frac{2j+d-1}{2}} \delta_{n+j,j}(\theta) \right)^2$$

$$+ \rho_{S^d}(\theta) \left( \frac{(\sin \theta)^{-j} F_{j,k}^{d} \mathcal{P}^{(a+j,a+j)}_{n}}{n + \frac{2j+d-1}{2}} \psi_{n+j,j}(\theta) \right)^2$$

$$\leq 2 \sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} \left( \frac{(\sin \theta)^{-j} F_{j,k}^{d} \mathcal{P}^{(a+j,a+j)}_{n}}{n + \frac{2j+d-1}{2}} \delta_{n+j,j}(\theta) \right)^2$$

$$+ \rho_{S^d}(\theta) \left( \frac{(\sin \theta)^{-j} F_{j,k}^{d} \mathcal{P}^{(a+j,a+j)}_{n}}{n + \frac{2j+d-1}{2}} \psi_{n+j,j}(\theta) \right)^2$$

$$= 2 \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{(\sin \theta)^{-j} F_{j,k}^{d} \mathcal{P}^{(a+j,a+j)}_{n}}{n + \frac{2j+d-1}{2}} \right)^2$$

$$+ \rho_{S^d}(\theta) \left( \frac{(\sin \theta)^{-j} F_{j,k}^{d} \mathcal{P}^{(a+j,a+j)}_{n}}{n + \frac{2j+d-1}{2}} \right)^2$$

$$= 2 \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{(\sin \theta)^{-j} F_{j,k}^{d} \mathcal{P}^{(a+j,a+j)}_{n}}{n + \frac{2j+d-1}{2}} \right)^2$$

$$+ \rho_{S^d}(\theta) \left( \frac{(\sin \theta)^{-j} F_{j,k}^{d} \mathcal{P}^{(a+j,a+j)}_{n}}{n + \frac{2j+d-1}{2}} \right)^2$$
\[ + \rho_{\delta M}(\theta) \left[ \sum_{j=0}^{\infty} \left( \sin \theta \right)^{-j} F_{j,k} \mathcal{P}_n^{(\alpha+j, \alpha+j)} \right]_{\delta M} \mathcal{P}_n^{(\alpha+j, \alpha+j)} (\sin \theta)^j \mathcal{P}_n^{(\alpha+j, \alpha+j)} (\sin \theta)^{j(j+d-2)} \right] \]

\[ \leq C_d \sum_{j=0}^{d} \sum_{k=1}^{d} \left[ (\sin \theta)^{j} \mathcal{R}^{\alpha+j, \alpha+j} ((\sin \phi)^{-j} F_{j,k})(\theta) \right]^2 + \left| j(\sin \theta)^{j} \mathcal{R}^{\alpha+j, \alpha+j} ((\sin \phi)^{-j} F_{j,k})(\theta) \right|^2 \].

By Theorem 1.3 and Theorem 2.2 with \( a = b = 1 \) and \( w = 1 \), from (2.16), we get

\[ \| R_{\delta M} f \|_{L^p(\mathbb{S}^d)} \leq C \left( \int_0^\pi \left[ \sum_{j=0}^{\infty} \sum_{k=1}^{d} \left( |F_{j,k}(\theta)|^2 \right) \right]^{p/2} (\sin \theta)^{d-1} d\theta \right)^{1/p} \]

\[ \leq C \left( \int_0^\pi \left( \sum_{j=0}^{\infty} \sum_{k=1}^{d} |F_{j,k}(\theta)|^2 \right)^{p/2} (\sin \theta)^{d-1} d\theta \right)^{1/p} = C \| f \|_{L^p(\mathbb{S}^d)} \]

\[ \square \]

2.3. The case of the real projective space \( P_d(\mathbb{R}) \). To deal with the real projective space we just have to consider even functions on the sphere (as done at the end of Section 2 of [30] or in [14, Subsection 3.2], see also [5, Chapter III, Section C.II] or [10, p. 36]). More precisely, a function defined on \( P_d(\mathbb{R}) \) can be identified with an even function on \( \mathbb{S}^d \subset \mathbb{R}^{d+1} \). Indeed, the antipodal map \( s \mapsto \pm s \) from \( \mathbb{S}^d \) to \( P_d(\mathbb{R}) \) is a Riemannian covering. The spherical harmonics in \( \mathbb{S}^d \) satisfy \( Y_{n,j}^{d+1}(\pi) = Y_{n,j}^{d+1}(-\pi) = Y_{n,j}^{d+1}(\pi) \). Then the space \( L^2(\mathbb{S}^d) \) of even functions in \( L^2(\mathbb{S}^d) \) can be decomposed as \( L^2(\mathbb{S}^d) = \mathbb{T}_{n=0}^\infty \mathcal{H}_2n(\mathbb{S}^d) \). Hence, a function on \( P_d(\mathbb{R}) \) is written as

\[ f = \sum_{n=0}^\infty \text{Proj}_{\mathcal{H}_2n(\mathbb{S}^d)}(f) = \sum_{n \geq 0, n \text{ even}} \text{Proj}_{\mathcal{H}_n(\mathbb{S}^d)}(f)(x). \]

Thus, Theorem 1.1 for \( P_d(\mathbb{R}) \) is then established as a particular case of the result for the sphere applied to even functions.

2.4. The case of the projective spaces \( P_t(\mathbb{C}) \), \( P_t(\mathbb{H}) \) and \( P_2(\mathbb{C} \mathcal{A} \gamma) \). In this subsection we use the tools developed in [30, Section 4]. Let \( M \) be any of the projective spaces \( P_t(\mathbb{C}), P_t(\mathbb{H}) \) or \( P_2(\mathbb{C} \mathcal{A} \gamma) \) and take \( -\Delta_M \) as in the introduction, with \( \lambda_M \) as in (2.1). We have the orthogonal direct sum decomposition \( L^2(M) = \mathbb{T}_{n=0}^\infty \mathcal{H}_n(M) \). Each space \( \mathcal{H}_n(M) \) is finite-dimensional and corresponds to the eigenspace of \( \Delta_M \) with respect to the eigenvalue \( -n(n+m+d) \). From here it is readily seen that \( \mathcal{H}_n(M) = \{ f \in C^\infty(M) : -\Delta_M f = (n + \frac{m+d}{2})^2 f \} \).

Now we introduce appropriate polar coordinates on \( M \) by following [30]. Let \( \mathbb{B}^{d+1} \) be the unit ball in \( \mathbb{R}^{d+1} \) and \( \omega(r) := c_\omega r^{-1}(1-r)^m \) for \( 0 < r < 1 \), with \( c_\omega = \frac{\Gamma(m+d+1)}{\Gamma(m+1)} \). According to [30, Lemma 4.15] there is a bounded linear map \( E : L^1(M) \to L^1(\mathbb{B}^{d+1}, \omega(|x|) dx) \) such that for every \( f \in L^1(M) \),

\[ \int_M f d\mu_M = \int_{\mathbb{B}^{d+1}} E(f) \omega(|x|) dx, \]

where \( d\mu_M \) is the Riemannian measure on \( M \). Each \( x \in \mathbb{B}^{d+1}, x \neq 0 \), can be written in polar coordinates as \( x = rx' \), where \( 0 < r = |x| < 1 \) and \( x' \in \mathbb{S}^d \). Then, if we write \( r = (\sin \frac{\theta}{2})^2 \) for \( 0 < \theta < \pi \) and \( F(\theta, x') = E(f)(rx') \) we have that integration over \( M \) reduces to

\[ \int_M f d\mu_M = c_\omega \int_0^\frac{\pi}{2} \int_{\mathbb{S}^d} F(\theta, x') d\mu_M = \int_0^\pi (\sin \frac{\theta}{2})^{2d+1} (\cos \frac{\theta}{2})^{2m+1} d\theta. \]
Let $F_n(B^{d+1})$ be the space of functions on $B^{d+1}$ which are polynomials of degree less than or equal to $n$ in the variables $x$ and $r = |x|$. Define the set $H_n(B^{d+1}, \omega)$ as the orthocompliment of the space $F_{n-1}(B^{d+1})$ in $F_n(B^{d+1})$ with respect to the inner product on $L^2(B^{d+1}, \omega(|x|) \, dx)$. It is shown in [30, Corollary 4.26] that $E(H_n(M)) = H_n(B^{d+1}, \omega)$. With polar coordinates and the trigonometric change as above, the eigenspaces $H_n(B^{d+1}, \omega) = H_n((0, \pi) \times S^d, c_\omega d\mu_{d-1,m} \times dx')$. These spaces are eigenspaces of the differential operator

$$-\Delta_M = -\frac{\partial^2}{\partial \theta^2} - \frac{(d-1-m) + (d+m) \cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} + \lambda_M - \frac{1}{\sin^2 \theta} \Delta_{S^d} = J^{\alpha, \beta} - \rho_{M}(\theta) \Delta_{S^d},$$

where $\alpha = d-1 \geq 1$ and $\beta = m \geq 0$, see [30, Section 4, pp. 135–136]. The corresponding eigenvalues are $(n + \frac{m+d}{2})^2$, see [30, Theorem 4.22]. In this way it is possible to write $L^2((0, \pi) \times S^d, c_\omega d\mu_{\alpha, \beta} \times dx') = \bigoplus_{n=0}^{\infty} H_n((0, \pi) \times S^d, c_\omega d\mu_{\alpha, \beta} \times dx') = \bigoplus_{n=0}^{\infty} \bigoplus_{j=0}^{n} H_{n,j}((0, \pi) \times S^d, c_\omega d\mu_{\alpha, \beta} \times dx')$. A basis of the latter spaces is

$$\varphi_{n,j,k}^M(x) = \psi_{n,j}(\theta) Y_{j,k}^d(x'), \quad x = (\sin \frac{\theta}{2})^2 x', \quad 0 < \theta < \pi, \quad x' \in S^d,$$

where, for $P_{n}(\alpha, \beta)$ as in (2.2),

$$\psi_{n,j}(r) = (-1)^{n-j} \varphi_{n,j}^M(\sin \frac{\theta}{2})^{2j} P_{n-j}^{(d-1+2j,m)}(\cos \theta),$$

and the set $\{Y_{j,k}^d\}_{1 \leq k \leq d(j)}$ is an orthonormal basis of the space of spherical harmonics of degree $j$, and $d(j) := (2j + d - 1)(j+d-1)!/j!(d-1)!$. By virtue of (2.3) and (2.4) the normalizing constant is

$$c_{n,j}^M := ||\varphi_{n,j,k}^M||_{L^2((0, \pi) \times S^d, c_\omega d\mu_{\alpha, \beta} \times dx')} = c_\omega^{-1/2} c_{n-j}^{p+2j, \beta}.$$  

Thus, the projections can be written in these coordinates as

$$(2.18) \quad \text{Proj}_{H_n(M)}(f) = \sum_{j=0}^{n} \sum_{k=1}^{d(j)} \langle F, \varphi_{n,j,k}^M \rangle_{L^2((0, \pi) \times S^d, c_\omega d\mu_{\alpha, \beta} \times dx')} \psi_{n,j,k}^M.$$  

The mixed norm spaces $L^p(L^2(M))$, $1 \leq p < \infty$, are defined as the set of functions $f$ on $M$ for which the norm

$$(2.19) \quad \|f\|_{L^p(L^2(M))} = \left( c_\omega \int_0^\pi \left( \int_{S^d} |F(\theta, x')|^2 \, dx' \right)^{p/2} \left( \sin \frac{\theta}{2} \right)^{2d-1} (\cos \frac{\theta}{2})^{2m+1} \, d\theta \right)^{1/p}$$

is finite. The spaces $L^p(L^2(M))$ are Banach spaces.

Now we pass to the definition of the Riesz transform on the mixed norm space $L^p(L^2(M))$. By (2.18), in the coordinates we let

$$(2.20) \quad (-\Delta_M)^{-1/2} f \equiv (-\Delta_M)^{-1/2} F = \sum_{n=0}^{\infty} \frac{1}{n + \frac{m+d}{2}} \text{Proj}_{H_n(M)}(f).$$

We have $-\Delta_M = \delta_M^* \delta_M + \lambda_M$, with

$$\delta_M = x' x' \frac{\partial}{\partial \theta} + \sqrt{\rho_M(\theta)} \nabla_{S^d},$$

and

$$\delta_M^* = -\left( \frac{\partial}{\partial \theta} + \frac{2m+1}{2} \cot \frac{\theta}{2} - \frac{2d-1}{2} \tan \frac{\theta}{2} \right) x' + \sqrt{\rho_M(\theta)} \text{div}_{S^d}.$$  

Analogously to the case of the sphere $S^d$ in Subsection 2.2, we have

$$\int_M |\nabla_M f(x)|^2 \, dx = c_\omega \int_0^\pi \int_{S^d} \left[ (|\delta F(\theta, x')|^2 + \rho_M(\theta)|\nabla_{S^d} F(\theta, x')|^2) d\theta x' \, d\mu_{d-1,m}(\theta) \right] \, dx'.$$

Therefore, in the coordinates, $|\nabla_M f(x)|^2 = |\delta F(\theta, x')|^2 + \rho_M(\theta)|\nabla_{S^d} F(\theta, x')|^2$. In this way, by (2.20),

$$(2.21) \quad |R_M f|^2 = |(-\Delta_M)^{-1/2} F|^2 + \rho_M(\theta)|\nabla_{S^d} (-\Delta_M)^{-1/2} F|^2.$$
Proof of Theorem 1.1 for $M = P_1(\mathbb{C}), P_1(\mathbb{H}), P_2(\text{Cay})$. Parallel to the case of the sphere, we can write
\[
\|f\|_{L^p(L^2(M))} = \left( c_\omega \int_0^\pi \left( \sum_{j,k=0}^{\infty} \sum_{m=1}^\infty |F_{j,k}(\theta)|^2 \right) \; d\mu_{d-1,m}(\theta) \right)^{1/p},
\]
with
\[
F_{j,k}(\theta) = \int_{\mathbb{S}^d} F(\theta, x') Y_{j,k}^{d-1}(x') \; dx', \quad \theta \in (0, \pi).
\]
Proceeding as in (2.17),
\[
\langle F, \phi^M_{n,j,k}\rangle_{L^2((0,\pi) \times \mathbb{R}^d, c_\omega \mu_{d-1,m} \times dx')} = c_\omega \langle F_{j,k}, \psi^M_{n,j,k}\rangle_{L^2(d\mu_{d-1,m})}.
\]
Hence, by (2.18) and (2.20),
\[
(-\Delta_M)^{-1/2} f(x) = \sum_{j=0}^{\infty} \sum_{k=1}^{d} \sum_{n=0}^{\infty} c_\omega \langle F_{j,k}, \psi^M_{n,j,k}\rangle_{L^2(d\mu_{d-1,m})} \frac{\nu^M_{n,j,k}(\theta)}{n + 2j + m + \frac{d}{2}} + \rho_M(\theta) \left( \sum_{j=0}^{\infty} \sum_{k=1}^{d} \sum_{n=0}^{\infty} c_\omega \langle F_{j,k}, \psi^M_{n,j,k}\rangle_{L^2(d\mu_{d-1,m})} \frac{\nu^M_{n,j,k}(\theta)}{n + 2j + m + \frac{d}{2}} \right) Y_{j,k}^{d+1}(x').
\]
From here and (2.21), we can write
\[
|R_M f(x)|^2 = \left( \sum_{j=0}^{\infty} \sum_{k=1}^{d} \sum_{n=0}^{\infty} c_\omega \langle F_{j,k}, \psi^M_{n,j,k}\rangle_{L^2(d\mu_{d-1,m})} \frac{\nu^M_{n,j,k}(\theta)}{n + 2j + m + \frac{d}{2}} \right)^2 + \rho_M(\theta) \left( \sum_{j=0}^{\infty} \sum_{k=1}^{d} \sum_{n=0}^{\infty} c_\omega \langle F_{j,k}, \psi^M_{n,j,k}\rangle_{L^2(d\mu_{d-1,m})} \frac{\nu^M_{n,j,k}(\theta)}{n + 2j + m + \frac{d}{2}} \right)^2 \nabla_{\mathbb{S}^d} Y_{j,k}^{d+1}(x')^2.
\]
Recall that we are taking $\alpha = d - 1, \beta = m$, so we obtain
\[
c_\omega \langle F_{j,k}, \psi^M_{n,j,k}\rangle_{L^2(d\mu_{d-1,m})} = c_\omega \left( \frac{\sin \frac{\theta}{2}}{2} \right)^{-2j} F_{j,k} \mathcal{P}(\alpha + 2j, \beta)_{n} \left( \sin \frac{\theta}{2} \right)^{2j} \mathcal{P}(\alpha + 2j, \beta)_{n} \theta.
\]
Hence, analogously to the case of the sphere,
\[
\int_{\mathbb{S}^d} |R_M f(x)|^2 \; dx' \leq 2 c_\omega \sum_{j=0}^{\infty} \sum_{k=1}^{d} \sum_{n=0}^{\infty} \left( \frac{\sin \frac{\theta}{2}}{2} \right)^{-2j} F_{j,k} \mathcal{P}(\alpha + 2j, \beta)_{n} \left( \sin \frac{\theta}{2} \right)^{2j} \mathcal{P}(\alpha + 2j, \beta)_{n} \theta \right)^2 + \rho_M(\theta) \left( \sum_{j=0}^{\infty} \sum_{k=1}^{d} \sum_{n=0}^{\infty} \left( \frac{\sin \frac{\theta}{2}}{2} \right)^{-2j} F_{j,k} \mathcal{P}(\alpha + 2j, \beta)_{n} \left( \sin \frac{\theta}{2} \right)^{2j} \mathcal{P}(\alpha + 2j, \beta)_{n} \theta \right)^2 \nabla_{\mathbb{S}^d} Y_{j,k}^{d+1}(x')^2 \leq C c_\omega \sum_{j=0}^{\infty} \sum_{k=1}^{d} \left[ \left( \sin \frac{\theta}{2} \right)^{2j} \mathcal{R}^{\alpha + 2j, \beta} \left( \sin \frac{\theta}{2} \right)^{-2j} F_{j,k} \theta \right]^2 + \left( \sin \frac{\theta}{2} \right)^{2j} \mathcal{T}_M^{\alpha + 2j, \beta} \left( \sin \frac{\theta}{2} \right)^{-2j} F_{j,k} \theta \right]^2.
\]
By Theorem 1.3 and Theorem 2.2 with $a = 2, b = 0$ and $w = 1$, by using (2.19),
\[
\|R_M f\|_{L^p(L^2(M))} \leq C \left( \int_0^\pi \left[ \sum_{j=0}^{\infty} \sum_{k=1}^{d} \left( \sin \frac{\theta}{2} \right)^{2j} \mathcal{R}^{\alpha + 2j, \beta} \left( \sin \frac{\theta}{2} \right)^{-2j} F_{j,k} \theta \right]^2 \; d\mu_{\alpha, \beta}(\theta) \right)^{1/p} \]
\[
+ C \left( \int_0^\pi \left[ \sum_{j=0}^{\infty} \sum_{k=1}^{d} \left( \sin \frac{\theta}{2} \right)^{2j} \mathcal{T}_M^{\alpha + 2j, \beta} \left( \sin \frac{\theta}{2} \right)^{-2j} F_{j,k} \theta \right]^2 \; d\mu_{\alpha, \beta}(\theta) \right)^{1/p}.
\]
where the Jacobi–Poisson kernel is given by

\[ \theta \]

for \( \theta \) in \( (0, \pi) \). Also,

\[ \theta \]

We need a more explicit expression of this kernel when \( \theta \) is close to \( \pi \). Let \( z = z(u, v, \theta, \varphi) := u \sin \frac{\theta}{2} \sin \frac{\varphi}{2} + v \cos \frac{\theta}{2} \cos \frac{\varphi}{2}, \quad \theta, \varphi \in (0, \pi), \quad u, v \in [-1, 1] \).

Consider on \([-1, 1]\] the measure

\[ d\Pi_{\alpha}(u) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} (1 - u^2)^{\alpha - 1/2} du, \]

the analogous definition for \( d\Pi_{\beta}(v) \). The expression for the Jacobi–Poisson kernel is (see [23])

\[ P_{t}^{\alpha, \beta}(\theta, \varphi) = \frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha+\beta+1} \Gamma(\alpha + 1) \Gamma(\beta + 1)} \int_{-1}^{1} \int_{-1}^{1} \sinh \frac{t}{2} \left( \cosh \frac{t}{2} - 1 \right)^{\alpha + \beta + 2} \frac{d\Pi_{\alpha}(u)}{d\Pi_{\beta}(v)}. \]

Now, see [23], for a compactly supported smooth function \( f \), we have

\[ R^{\alpha, \beta} f(\theta) = \int_{0}^{\infty} \delta P_{t}^{\alpha, \beta}(\theta) dt = \int_{0}^{\infty} K^{\alpha, \beta}(\theta, \varphi) f(\varphi) d\mu_{\alpha, \beta}(\varphi), \]

for \( \theta \) outside the support of \( f \). The kernel is given by

\[ K^{\alpha, \beta}(\theta, \varphi) = \int_{0}^{\infty} \delta P_{t}^{\alpha, \beta}(\theta, \varphi) dt. \]

Also, \( K^{\alpha, \beta}(\theta, \varphi) \) satisfies standard Calderón–Zygmund estimates on the space of homogeneous type \((\mathbb{R}^n, | \cdot |, d\mu_{\alpha, \beta})\). The following identity is easy to check:

\[ \int_{0}^{\infty} P_{t}(\theta, \varphi) dt = \frac{\Gamma(\alpha + \beta + 1)}{\pi 2^{\alpha+\beta+1} \Gamma(\alpha + 1/2) \Gamma(\beta + 1/2)} \int_{-1}^{1} \int_{-1}^{1} \frac{(1 - u^2)^{\alpha - 1/2}(1 - v^2)^{\beta - 1/2}}{(1 - z)^{\alpha + \beta + 1}} du dv, \]

Then we get

\[ K^{\alpha, \beta}(\theta, \varphi) = \frac{\Gamma(\alpha + \beta + 2)}{\pi 2^{\alpha+\beta+1} \Gamma(\alpha + 1/2) \Gamma(\beta + 1/2)} \int_{-1}^{1} \int_{-1}^{1} \frac{(1 - u^2)^{\alpha - 1/2}(1 - v^2)^{\beta - 1/2}}{(1 - z)^{\alpha + \beta + 2}} \partial_\theta (1 - z) du dv, \]

where

\[ \partial_\theta (1 - z) = \frac{1}{2} \sin \frac{\theta}{2} \sin \frac{\varphi}{2} + \frac{1}{2} \cos \frac{\theta}{2} \cos \frac{\varphi}{2} - \frac{1}{2} \sin \frac{\theta}{2} \cos \frac{\varphi}{2} - \frac{1}{2} \cos \frac{\theta}{2} \sin \frac{\varphi}{2}. \]
In the same way, the operator \( u_j \mathcal{R}^{\alpha+a_j,\beta+b_j}(u_j^{-1}f)(\theta) \), with \( u_j \) as in (1.3), can be defined by using (2.5) and it can be expressed as an integral operator in the Calderón–Zygmund sense, with associated kernel \( u_j(\theta)u_j(\varphi)\mathcal{K}^{\alpha+a_j,\beta+b_j}(\theta, \varphi) \). Let us establish this as a lemma.

**Lemma 3.1.** Let \( \alpha, \beta > -1/2, a, b \geq 1 \) and \( u_j \) be as in (1.3). Take \( f, g \in C^\infty_c(0, \pi) \) having disjoint supports. Then,

\[
\langle u_j \mathcal{R}^{\alpha+a_j,\beta+b_j}(u_j^{-1}f), g \rangle_{d\mu_{\alpha,\beta}} = \int_0^\pi \int_0^\pi u_j(\theta)u_j(\varphi)\mathcal{K}^{\alpha+a_j,\beta+b_j}(\theta, \varphi)f(\varphi)\overline{g(\theta)} \, d\mu_{\alpha,\beta}(\varphi) \, d\mu_{\alpha,\beta}(\theta).
\]

The proof of Lemma 3.1 follows in a standard way just by writing the left hand side as a Fourier series and checking that it coincides, after changing the order of summation and integration (which can be done thanks to the disjoint supports of \( f \) and \( g \)), with the right hand side.

Now that everything is defined, we will prove Theorem 1.2. The main tool in the proof is the following elementary lemma.

**Lemma 3.2 (Lemma 5.3 in [12]).** Let \( c > -1/2, 0 < B < A, \lambda > 0 \) and \( d \geq 0 \). Then

\[
\int_0^1 \frac{(1-s)^{c+d-1/2}}{(A-Bs)^{c+d+\lambda+1/2}} \, ds \leq C(d) \frac{A^{c+1/2}B^d(A-B)\lambda}{A^c},
\]

where

\[
C(d) = \begin{cases} 
\Gamma(d)\Gamma(\lambda)/\Gamma(d+\lambda), & d > 0, \\
C(c), & d = 0.
\end{cases}
\]

For \( \alpha, \beta > -1/2 \) we can readily check that

\[
\mu_{\alpha,\beta}(B(\theta, |\theta - \varphi|)) \simeq |\theta - \varphi|(\theta + \varphi)^{2\alpha+1}(\pi - \theta + \pi - \varphi)^{2\beta+1}, \quad \theta, \varphi \in (0, \pi).
\]

We will often use the following well known fact [1, eq. 6.1.46]

\[
\frac{\Gamma(z+r)}{\Gamma(z+t)} \simeq z^{r-t}, \quad z > 0, \ r, t \in \mathbb{R}.
\]

The next estimate is easy to see. For \( \eta > 0 \) and \( \gamma \geq 1/2 \), we have

\[
(1-r)^{\eta r^{\gamma-1/2}} \leq \left( \frac{\eta}{\eta + \gamma - 1/2} \right)^{\eta}, \quad \text{when} \ 0 < r < 1.
\]

Finally, we collect several trigonometric identities that will be used throughout the proofs:

\[
\begin{align*}
1 - \cos \frac{\theta}{2} \cos \frac{\varphi}{2} &= \left( \sin \frac{\theta-\varphi}{4} \right)^2 + \left( \sin \frac{\theta+\varphi}{4} \right)^2 \simeq \theta^2 + \varphi^2 \\
1 - \sin \frac{\theta}{2} \sin \frac{\varphi}{2} &= \left( \sin \frac{\theta-\varphi}{4} \right)^2 + \left( \cos \frac{\theta+\varphi}{4} \right)^2 \simeq (\pi - \theta)^2 + (\pi - \varphi)^2 \\
1 - \cos \frac{\theta}{2} \sin \frac{\varphi}{2} &= 2 \left( \sin \frac{\theta-\varphi}{4} \right)^2 \simeq (\theta - \varphi)^2.
\end{align*}
\]

**Proof of Theorem 1.2.** We have to prove (1.4) and (1.5) for \( j \geq 1 \).

**Growth estimates: proof of (1.4).** By taking into account (3.5) and (3.6),

\[
u_j(\theta)u_j(\varphi)\mathcal{K}^{\alpha+a_j,\beta+b_j}(\theta, \varphi) = \frac{\Gamma(\alpha+a_j+\beta+b_j+2)\Gamma(\alpha+a_j+1/2)\Gamma(\beta+b_j+1/2)}{\pi\Gamma(\alpha+a_j+\beta+b_j+2)\Gamma(\beta+b_j+1/2)}(J_1 + J_2 - J_3),
\]

where

\[
J_1 := \sin \frac{\theta-\varphi}{2} \int_{-1}^1 \int_{-1}^1 \frac{(1-u)^{\alpha+a_j-1/2}(1-v)^{\beta+b_j-1/2}}{(1-z)^{\alpha+a_j+\beta+b_j+2}} \, du \, dv,
\]

\[
J_2 := \cos \frac{\theta}{2} \sin \frac{\varphi}{2} \int_{-1}^1 \int_{-1}^1 \frac{(1-u)(1-v)^{\alpha+a_j-1/2}(1-v)^{\beta+b_j-1/2}}{(1-z)^{\alpha+a_j+\beta+b_j+2}} \, du \, dv,
\]

\[
J_3 := \sin \frac{\theta}{2} \cos \frac{\varphi}{2} \int_{-1}^1 \int_{-1}^1 \frac{(1-u)^{\alpha+a_j-1/2}(1-v)(1-v)^{\beta+b_j-1/2}}{(1-z)^{\alpha+a_j+\beta+b_j+2}} \, du \, dv.
\]

Let us proceed with the estimates of each term.
Estimate for $J_1$. Observe that

$$J_1 \leq 2^{1+\alpha+j+\beta+bj} \sin \frac{\theta-\varphi}{2} \int_0^1 \int_0^1 \frac{(1-u)^{\alpha+j-1/2}(1-v)^{\beta+bj-1/2}}{(1-z)^{\alpha+j+\beta+bj+2}} \, du \, dv.$$ 

We apply twice Lemma 3.2, first in the integral in $u$, taking $c = \alpha, d = aj, \lambda = \beta + bj + 3/2, A = 1 - v \cos \frac{\theta}{2} \cos \frac{\varphi}{2}$, $B = \sin \frac{\theta}{2} \sin \frac{\varphi}{2}$, and then in the integral in $v$, taking $c = \beta, d = bj, \lambda = 1, A = 1 - \sin \frac{\theta}{2} \sin \frac{\varphi}{2}$ and $B = \cos \frac{\theta}{2} \cos \frac{\varphi}{2}$. We get

$$\left| \frac{\Gamma(\alpha + aj + \beta + bj + 2)u_j(\theta)u_j(\varphi)}{\Gamma(\alpha + aj + 1/2)\Gamma(\beta + bj + 1/2)2^{\alpha+aj+\beta+bj+2}} J_1 \right| \leq \frac{\Gamma(\alpha + aj + \beta + bj + 2)(aj)\Gamma(\beta + bj + 1/2)(aj + \beta + bj + 3/2)\Gamma(aj + 1/2)}{\Gamma(\alpha + aj + 1/2)\Gamma(\beta + bj + 1/2)(aj + \beta + bj + 3/2)\Gamma(aj + 1/2)(\alpha + aj)^{1/2}} \mu_{\alpha, \beta}(B(\theta, |\theta - \varphi|)).$$

This and (3.8) give (1.4) for $J_1$.

Estimate for $J_2$. As in the first step to estimate $J_1$, one passes the integration in $v$ from $(-1, 1)$ to $(0, 1)$ and uses that $1 - v^2 \leq 2(1 - v)$ in the numerator. Observe that $\cos \frac{\theta}{2} \sin \frac{\varphi}{2} \leq \sin \frac{\theta}{2} \sim \frac{\theta}{2} \leq \frac{\varphi}{2}$. Then, by applying Lemma 3.2 to the remaining integral in $v$ with $c = \beta, d = bj, \lambda = \alpha + aj + 3/2, A = 1 - u \sin \frac{\theta}{2} \sin \frac{\varphi}{2}$ and $B = \cos \frac{\theta}{2} \cos \frac{\varphi}{2}$ we have that $J_2$ is bounded by

$$\frac{\Gamma(bj)\Gamma(\alpha + aj + 3/2)2^{\beta+bj-1/2}}{\Gamma(\alpha + aj + bj + 3/2)} \frac{\theta + \varphi}{(\cos \frac{\theta}{2} \cos \frac{\varphi}{2})^{bj}} \times \int_{-1}^1 \frac{(1-u)(1-u^2)^{\alpha+aj-1/2}}{(1-u \sin \frac{\theta}{2} \sin \frac{\varphi}{2})^{\beta+1/2}(1-u \sin \frac{\theta}{2} \sin \frac{\varphi}{2} - \cos \frac{\theta}{2} \cos \frac{\varphi}{2})^{\alpha+aj+3/2}} \, du.$$ 

Now we make the change of variable $1 + u = 2w$, use (3.9) with $\eta = 1/2$ and $\gamma = \alpha + aj$, and apply Lemma 3.2 taking $c = \alpha + 1/2, d = aj, \lambda = 1/2, A = 1 + \sin \frac{\theta}{2} \sin \frac{\varphi}{2} - \cos \frac{\theta}{2} \cos \frac{\varphi}{2}$ and $B = 2 \sin \frac{\theta}{2} \sin \frac{\varphi}{2}$. Therefore, for $j \geq 1$, we get

$$\left| \frac{\Gamma(\alpha + aj + \beta + bj + 2)u_j(\theta)u_j(\varphi)}{\Gamma(\alpha + aj + 1/2)\Gamma(\beta + bj + 1/2)2^{\alpha+aj+\beta+bj+2}} J_2 \right| \leq \frac{\Gamma(\alpha + aj + \beta + bj + 2)(aj)\Gamma(\alpha + aj + 1/2)(aj + \beta + bj + 3/2)\Gamma(aj + 1/2)}{\Gamma(\alpha + aj + 1/2)\Gamma(\beta + bj + 1/2)(\alpha + aj + bj + 3/2)(\alpha + aj)^{1/2}} \times \mu_{\alpha, \beta}(B(\theta, |\theta - \varphi|)).$$

The latter estimate and (3.8) give (1.4).

Estimate for $J_3$ when $b \geq 1$. Note that $\sin \frac{\theta}{2} \cos \frac{\varphi}{2} \leq \cos \frac{\varphi}{2} = \sin \frac{\pi - \varphi}{2} \sim \frac{\pi - \varphi}{2} \leq \frac{\pi - \varphi + \pi - \theta}{2}$. We follow the same procedure as for $J_2$, but exchanging the role of $u$ and $v$ and the corresponding values for the parameters. Let us explain this. First, by applying Lemma 3.2 to the integral in $u$ with $c = \alpha, d = aj, \lambda = \beta + bj + 3/2, A = 1 - v \cos \frac{\theta}{2} \cos \frac{\varphi}{2}$ and $B = \sin \frac{\theta}{2} \sin \frac{\varphi}{2}, J_3$ is bounded by

$$(3.10) \quad \frac{\Gamma(aj)\Gamma(\beta + bj + 3/2)2^{\alpha+aj-1/2}}{\Gamma(\alpha + aj + bj + 3/2)} \frac{\pi - \theta + \pi - \varphi}{(\sin \frac{\theta}{2} \sin \frac{\varphi}{2})^{aj}} \times \int_{-1}^1 \frac{(1-v)(1-v^2)^{\beta+bj-1/2}}{(1-v \cos \frac{\theta}{2} \cos \frac{\varphi}{2} - \sin \frac{\theta}{2} \sin \frac{\varphi}{2})^{\beta+bj+3/2}} \, dv.$$ 

Next, we make the change of variable $1 + v = 2u$, use (3.9) with $\eta = 1/2$ and $\gamma = \beta + bj$, and apply Lemma 3.2 taking $c = \beta + 1/2, d = bj, \lambda = 1/2, A = 1 + \cos \frac{\theta}{2} \cos \frac{\varphi}{2} - \sin \frac{\theta}{2} \sin \frac{\varphi}{2}$ and $B = 2 \cos \frac{\theta}{2} \cos \frac{\varphi}{2}$. We get the estimate, which is similar to that for $J_2$, by exchanging the parameters $\alpha$ and $\beta$.

Estimate for $J_3$ when $b = 0$. Here we proceed with the integral in $u$ as in the previous case, and then observe that the integral appearing in (3.10) is bounded by the sum of two terms $J_{3,1}$ and $J_{3,2}$, given by

$$J_{3,1} := \frac{C_{\beta}}{(1 - \cos \frac{\theta}{2} \cos \frac{\varphi}{2})^{\alpha+1/2}} \int_0^1 \frac{(1-v)^{\beta+1/2}}{(1 - v \cos \frac{\theta}{2} \cos \frac{\varphi}{2} - \sin \frac{\theta}{2} \sin \frac{\varphi}{2})^{\beta+3/2}} \, dv.$$
and
\[ J_{3,2} := \frac{C_\beta}{(1 - \cos \frac{\theta}{2} \cos \frac{\phi}{2})^{\alpha + 1/2}} \int_0^1 \frac{(1 - v)^{\beta - 1/2}}{(1 + v \cos \frac{\theta}{2} \cos \frac{\phi}{2} - \sin \frac{\theta}{2} \sin \frac{\phi}{2})^{\beta + 1/2}} dv. \]

For \( J_{3,1} \), \( (1 - v)^{\beta + 1/2} \leq (1 - v)^{\beta} \), and we apply Lemma 3.2 with \( c = \beta + 1/2, d = 0, \lambda = 1/2, \alpha = 1 - \sin \frac{\theta}{2} \sin \frac{\phi}{2} \) and \( B = \cos \frac{\theta}{2} \cos \frac{\phi}{2} \). Concerning \( J_{3,2} \), using that \( 1 + v \cos \frac{\theta}{2} \cos \frac{\phi}{2} \geq 1 - \cos \frac{\theta}{2} \cos \frac{\phi}{2} \), since \( \beta - 1/2 > -1 \),
\[ J_{3,2} \leq \frac{C_\beta}{(1 - \cos \frac{\theta}{2} \cos \frac{\phi}{2})^{\alpha + 1/2}(1 - \sin \frac{\theta}{2} \sin \frac{\phi}{2})^{\beta + 1}(1 - \sin \frac{\theta}{2} \sin \frac{\phi}{2} - \cos \frac{\theta}{2} \cos \frac{\phi}{2})^{1/2}}. \]

By plugging the estimates obtained for \( J_1, J_2 \) and \( J_3 \), and taking into account (3.7), we get (1.4).

**Smoothness estimates: proof of (1.5).** Derivative in \( \theta \). We have
\[ \frac{\partial}{\partial \theta} (u_j(\theta)u_j(\varphi)) =: K_1 + K_2, \]
where, for \( J_1, J_2 \) and \( J_3 \) as above,
\[ K_1 = \frac{\Gamma(\alpha + a_j + \beta + bj + 2)}{\pi \Gamma(\alpha + a_j + 1/2)\Gamma(\beta + bj + 1/2)2^{\alpha + a_j + \beta + bj + 2}} \left( \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} - bj \sin \frac{\theta}{2} \cos \frac{\phi}{2} \right) (J_1 + J_2 - J_3) \]
\[ =: \frac{\Gamma(\alpha + a_j + \beta + bj + 2)}{\pi \Gamma(\alpha + a_j + 1/2)\Gamma(\beta + bj + 1/2)2^{\alpha + a_j + \beta + bj + 2}} \sum_{i=1}^6 K_{1,i}, \]
with
\[ K_{1,1} := a_j \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} J_1, \quad K_{1,2} := a_j \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} J_2, \quad K_{1,3} := a_j \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} J_3, \]
\[ K_{1,4} := -bj \frac{\sin \frac{\theta}{2}}{\cos \frac{\phi}{2}} J_1, \quad K_{1,5} := -bj \frac{\sin \frac{\theta}{2}}{\cos \frac{\phi}{2}} J_2, \quad K_{1,6} := bj \frac{\sin \frac{\theta}{2}}{\cos \frac{\phi}{2}} J_3, \]
and
\[ K_2 = \frac{\Gamma(\alpha + a_j + \beta + bj + 2)}{\pi \Gamma(\alpha + a_j + 1/2)\Gamma(\beta + bj + 1/2)2^{\alpha + a_j + \beta + bj + 1}} \frac{\partial}{\partial \theta} K^{\alpha + aj, \beta + bj} (\theta, \varphi). \]

**Estimate for \( K_1 \).** We estimate each term \( K_{1,i}, i = 1, \ldots, 6 \) separately. The treatment may be tedious and long, but it is systematic. The ideas are the same as in the growth estimates, so we only sketch the hints to follow the proofs.

Indeed, the estimate for \( K_{1,1} \) follows by using first Lemma 3.2 to the integral against \( v \) with \( c = \beta, d = bj, \lambda = \alpha + a_j + 3/2, A = 1 - u \sin \frac{\theta}{2} \sin \frac{\phi}{2} \) and \( B = \cos \frac{\theta}{2} \cos \frac{\phi}{2} \). From here, we proceed with the change \( 1 + u = 2w, u = (3.9) \) with \( \eta = 1/2 \) and \( \gamma = \alpha + a_j \) and apply again Lemma 3.2 with \( c = \alpha + 1/2, d = a_j - 1, \alpha = 3/2, A = 1 + \sin \frac{\theta}{2} \sin \frac{\phi}{2} - \cos \frac{\theta}{2} \cos \frac{\phi}{2} \) and \( B = 2 \sin \frac{\theta}{2} \sin \frac{\phi}{2} \). We see that the case \( K_{1,4} \) for \( b \geq 1 \) follows analogously as \( K_{1,1} \), but exchanging the roles of \( u \) and \( v \), in the same way as it was done with \( J_3 \). Observe that if \( b = 0 \), then \( K_{1,4} = 0 \).

Similarly, for \( K_{1,2} \) we use the same procedure as in \( K_{1,1} \), but using (3.9) with \( \eta = 1 \) and \( \gamma = \alpha + a_j \), and Lemma 3.2 with \( \lambda = 1 \) in the second application of the lemma. Again, the case of \( K_{1,6} \) for \( b \geq 1 \) is analogous to \( K_{1,2} \), but exchanging the role of the variables \( u \) and \( v \).

For \( K_{1,5} \) and \( b \geq 1 \) we can follow parallel arguments to the ones for \( K_{1,3} \) in the case \( b \geq 1 \) below, by interchanging the roles of \( u \) and \( v \).

As for \( K_{1,3} \), we distinguish the two cases.

**Estimate for \( K_{1,3} \) when \( b \geq 1 \).** For the \( K_{1,3} \) term, we use Lemma 3.2 in the integral against \( u \) with \( c = \alpha, d = a_j, \lambda = \beta + bj + 3/2, A = 1 - \cos \frac{\theta}{2} \cos \frac{\phi}{2} \) and \( B = \sin \frac{\theta}{2} \sin \frac{\phi}{2} \). Then, we make the change of variable \( 1 + v = 2w \) in the integral against \( v \), and use (3.9) with \( \eta = 1 \) and \( \gamma = \beta + bj \). Finally, we apply again Lemma 3.2 taking \( c = \beta, d = bj, \lambda = 1, A = 1 + \cos \frac{\theta}{2} \cos \frac{\phi}{2} - \sin \frac{\theta}{2} \sin \frac{\phi}{2} \) and \( B = 2 \cos \frac{\theta}{2} \cos \frac{\phi}{2} \).
Estimate for $K_{1,3}$ when $b = 0$. First, notice that $(1-u^2)^{\alpha + aj - 1/2} \leq (1-u^2)^{\alpha + aj - 3/2}$ for $u \in (-1, 1)$. Then we use Lemma 3.2 in the integral against $u$ with $c = \alpha, d = aj - 1, \lambda = \beta + 5/2, A = 1 - v \cos \frac{\theta}{2} \cos \frac{\varphi}{2}$ and $B = \sin \frac{\theta}{2} \sin \frac{\varphi}{2}$. After that, the integral arising in $v$ is

$$
\int_{-1}^{1} \frac{(1-v)(1-v^2)^{\beta-1/2}}{(1-v \cos \frac{\theta}{2} \cos \frac{\varphi}{2})(1-v \cos \frac{\theta}{2} \cos \frac{\varphi}{2} - \sin \frac{\theta}{2} \sin \frac{\varphi}{2})^{\beta+5/2}} dv.
$$

(3.11)

Proceeding as in the case of $J_3$ above, this integral is bounded by the sum of two terms $K_{1,3,1}$ and $K_{1,3,2}$, given by

$$
K_{1,3,1} := \frac{C_\beta}{(1-\cos \frac{\theta}{2} \cos \frac{\varphi}{2})^{\alpha+1/2}} \int_{0}^{1} \frac{(1-v)^{\beta+1/2}}{(1-v \cos \frac{\theta}{2} \cos \frac{\varphi}{2} - \sin \frac{\theta}{2} \sin \frac{\varphi}{2})^{\beta+5/2}} dv
$$

and

$$
K_{1,3,2} := \frac{C_\beta}{(1-\cos \frac{\theta}{2} \cos \frac{\varphi}{2})^{\alpha+1/2}} \int_{0}^{1} \frac{(1-v)^{\beta-1/2}}{(1-v \cos \frac{\theta}{2} \cos \frac{\varphi}{2} - \sin \frac{\theta}{2} \sin \frac{\varphi}{2})^{\beta+5/2}} dv.
$$

For $K_{1,3,1}$, we apply Lemma 3.2 with $c = \beta + 1, d = 0, \lambda = 1, A = 1 - \sin \frac{\theta}{2} \sin \frac{\varphi}{2}$ and $B = \cos \frac{\theta}{2} \cos \frac{\varphi}{2}$. Concerning $K_{1,3,2}$, by using that $1 + v \cos \frac{\theta}{2} \cos \frac{\varphi}{2} \geq 1 - \cos \frac{\theta}{2} \cos \frac{\varphi}{2}$, since $\beta - 1/2 > -1$,

$$
K_{1,3,2} \leq \frac{C_\beta}{(1-\cos \frac{\theta}{2} \cos \frac{\varphi}{2})^{\alpha+1/2}(1-\sin \frac{\theta}{2} \sin \frac{\varphi}{2})^{\beta+3/2}(1-\sin \frac{\theta}{2} \sin \frac{\varphi}{2} - \cos \frac{\theta}{2} \cos \frac{\varphi}{2})}
$$

Estimate for $K_2$. Observe that $\partial^2_\theta (1-z) = z/4$. Then,

$$
\left| \frac{\partial}{\partial \theta} \left( \frac{\partial \theta (1-z)}{(1-z)^{\alpha+aj+\beta+bj+2}} \right) \right| = \left| \frac{\partial^2 \theta (1-z)}{(1-z)^{\alpha+aj+\beta+bj+2}} - \frac{(\alpha + aj + \beta + bj + 2)(\partial \theta (1-z))^2}{(1-z)^{\alpha+aj+\beta+bj+3}} \right|.
$$

From here and (3.5), we have that

$$
|K_2| \leq \frac{\Gamma(\alpha + aj + \beta + bj + 2)u_j(\theta)u_j(\varphi)}{\pi \Gamma(\alpha + aj + 1/2)\Gamma(\beta + bj + 1/2)2^{\alpha+aj+\beta+bj+2}} \times \int_{-1}^{1} \int_{-1}^{1} (1-u^2)^{\alpha+aj-1/2}(1-v^2)^{\beta+bj-1/2} \sum_{i=1}^{7} |K_{2,i}| du dv,
$$

where

$$
K_{2,1} := \frac{z}{4(1-z)^{\alpha+aj+\beta+bj+2}}, \quad K_{2,2} := \frac{(\alpha + aj + \beta + bj + 2)(\sin \frac{\theta}{2} \sin \frac{\varphi}{2})^2}{(1-z)^{\alpha+aj+\beta+bj+3}}
$$

$$
K_{2,3} := \frac{(\alpha + aj + \beta + bj + 2)^2}{(1-z)^{\alpha+aj+\beta+bj+3}} (1-u)^2 (\cos \frac{\theta}{2} \sin \frac{\varphi}{2})^2
$$

$$
K_{2,4} := \frac{(\alpha + aj + \beta + bj + 2)^2}{(1-z)^{\alpha+aj+\beta+bj+3}} (1-v)^2 (\sin \frac{\theta}{2} \cos \frac{\varphi}{2})^2
$$

$$
K_{2,5} := \frac{(\alpha + aj + \beta + bj + 2)^2}{(1-z)^{\alpha+aj+\beta+bj+3}} (1-u) \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{\varphi}{2} \sin \frac{\varphi}{2}
$$

$$
K_{2,6} := \frac{(\alpha + aj + \beta + bj + 2)^2}{(1-z)^{\alpha+aj+\beta+bj+3}} (1-v) \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{\varphi}{2} \sin \frac{\varphi}{2}
$$

$$
K_{2,7} := \frac{(\alpha + aj + \beta + bj + 2)^2}{(1-z)^{\alpha+aj+\beta+bj+3}} (1-u)(1-v) \sin \frac{\theta}{2} \cos \frac{\theta}{2} \sin \frac{\varphi}{2} \cos \frac{\varphi}{2}.
$$

We sketch the proof for each integral containing the term $K_{2,i}, i = 1, \ldots, 7$.

For the integral with $K_{2,1}$ we observe that $z < 1$ and the estimate follows in the same way as for $J_1$ above.

Concerning the integral with $K_{2,2}$, the proof is analogous to the one for $J_2$, but we take $\lambda = \beta + bj + 5/2$ in the first application of Lemma 3.2 and $\lambda = 2$ in the second application.

We pass to $K_{2,3}$. We apply Lemma 3.2 in the integral in $v$ with $c = \beta, d = bj, \lambda = \alpha + aj + 5/2, A = 1 - u \sin \frac{\theta}{2} \sin \frac{\varphi}{2}$ and $B = \cos \frac{\theta}{2} \cos \frac{\varphi}{2}$, and we make the change of variable $1 + u = 2w$ in
the integral in $u$. Then, we use (3.9) with $\eta = 1$ and $\gamma = \alpha + aj$ and apply again Lemma 3.2 taking $c = \alpha + 1, d = aj, \lambda = 1, A = 1 + \sin \frac{\theta}{2} \sin \frac{\pi}{2} - \cos \frac{\theta}{2} \cos \frac{\pi}{2}$ and $B = 2 \sin \frac{\theta}{2} \sin \frac{\pi}{2}$.

The treatment of $K_{2,4}$ with $b \geq 1$ is identical to $K_{2,3}$, but exchanging the roles of $u$ and $v$.

Next we estimate $K_{2,4}$ when $b = 0$. We use Lemma 3.2 in the integral against $u$ with $c = \alpha, d = aj, \lambda = \beta + 5/2, A = 1 - \cos \frac{\theta}{2} \cos \frac{\pi}{2}$ and $B = \sin \frac{\theta}{2} \sin \frac{\pi}{2}$. Observe that the integral arising in $v$ is the same as in (3.11) but with $(1 - v)^2$ in place of $(1 - v)$ in the numerator. This integral is then bounded by $K_{1,3,1}$ and $K_{1,3,2}$ above.

The proof of the estimate involving $K_{2,5}$ is also analogous to the one for $J_2$, but taking $\lambda = \alpha + aj + 5/2$ in the first application of Lemma 3.2 and $\lambda = 3/2$ the second time.

Concerning the integral involving $K_{2,6}$, the case $b \geq 1$ works in the same way as $K_{2,5}$, but exchanging the roles of $u$ and $v$. For the case $b = 0$ one uses the procedure given for $K_{2,4}$, but for the corresponding integral $K_{1,3,2}$, one needs the estimate

\[
\int_0^1 (1 - v)^{\beta - 1/2} \left( 1 + \cos \frac{\theta}{2} \cos \frac{\pi}{2} - \sin \frac{\theta}{2} \sin \frac{\pi}{2} \right) \left( 1 - \cos \frac{\theta}{2} \sin \frac{\pi}{2} - \cos \frac{\theta}{2} \cos \frac{\pi}{2} \right) dv \leq \frac{C_\beta}{(1 - \sin \frac{\theta}{2} \sin \frac{\pi}{2})^{\beta + 1}(1 - \sin \frac{\theta}{2} \sin \frac{\pi}{2} - \cos \frac{\theta}{2} \cos \frac{\pi}{2})^{3/2}}.
\]

For the integral term with $K_{2,7}$, we first make the change $1 + u = 2w$ and use (3.9) with $\eta = 1/2$ and $\gamma = \alpha + aj$. Then, apply Lemma 3.2 taking $c = \alpha + 1/2, d = aj, \lambda = \beta + bj + 2, A = 1 + \sin \frac{\theta}{2} \sin \frac{\pi}{2} - \cos \frac{\theta}{2} \cos \frac{\pi}{2}$ and $B = 2 \sin \frac{\theta}{2} \sin \frac{\pi}{2}$. Later on, we distinguish two cases. If $b \geq 1$ we proceed in the same way with the integral in $v$, that is, first make the change $1 + v = 2w$, then use (3.9) and finally apply Lemma 3.2 with $c = \beta + 1/2, d = bj, \lambda = 1, A = 1 + \cos \frac{\theta}{2} \sin \frac{\pi}{2} - \sin \frac{\theta}{2} \cos \frac{\pi}{2}$ and $B = 2 \cos \frac{\theta}{2} \cos \frac{\pi}{2}$. If $b = 0$, the integral arising in $v$ is bounded by the sum of the following two integrals:

\[
K_{2,7,1} := \frac{C_\beta}{(1 - \cos \frac{\theta}{2} \cos \frac{\pi}{2})^{\alpha + 1}} \int_0^1 \frac{(1 - v)^{\beta + 1/2}}{(1 - \cos \frac{\theta}{2} \cos \frac{\pi}{2} - \sin \frac{\theta}{2} \sin \frac{\pi}{2})^{\beta + 2}} dv,
\]

\[
K_{2,7,2} := \frac{C_\beta}{(1 - \cos \frac{\theta}{2} \cos \frac{\pi}{2})^{\alpha + 1}} \int_0^1 \frac{(1 - v)^{\beta - 1/2}}{(1 + \cos \frac{\theta}{2} \cos \frac{\pi}{2} - \sin \frac{\theta}{2} \sin \frac{\pi}{2})^{\beta + 2}} dv.
\]

For the integral $K_{2,7,1}$, observe that $(1 - v)^{\beta + 1/2} \leq (1 - v)^{\beta}$, and we apply Lemma 3.2 with $c = \beta + 1/2, d = 0, \lambda = 1, A = 1 - \sin \frac{\theta}{2} \sin \frac{\pi}{2}$ and $B = \cos \frac{\theta}{2} \cos \frac{\pi}{2}$. For $K_{2,7,2}$ we have

\[
K_{2,7,2} \leq \frac{C_\beta}{(1 - \cos \frac{\theta}{2} \cos \frac{\pi}{2})^{\alpha + 1}(1 - \sin \frac{\theta}{2} \sin \frac{\pi}{2})^{\beta + 1}(1 - \sin \frac{\theta}{2} \sin \frac{\pi}{2} - \cos \frac{\theta}{2} \cos \frac{\pi}{2})}.
\]

By pasting together all the estimates and taking into account (3.7), we get (1.5).

**Smoothness estimates: proof of (1.5).** Derivative in $\varphi$. We have

\[
\frac{\partial}{\partial \varphi} (u_j(\theta)u_j(\varphi)K^{\alpha + aj, \beta + bj}(\theta, \varphi)) =: L_1 + L_2,
\]

where

\[
L_1 = \frac{\Gamma(\alpha + aj + \beta + bj + 2)u_j(\theta)u_j(\varphi)}{\Gamma(\alpha + aj + 1/2)\Gamma(\beta + bj + 1/2)2^{\alpha + aj + \beta + bj} + 2} \left( aj \frac{\cos \frac{\theta}{2}}{\sin \frac{\pi}{2}} - bj \frac{\sin \frac{\theta}{2}}{\cos \frac{\pi}{2}} \right) (J_1 + J_2 - J_3)
\]

with $J_1, J_2$ and $J_3$ as above, and

\[
L_2 = \frac{\Gamma(\alpha + aj + \beta + bj + 2)u_j(\theta)u_j(\varphi)}{\Gamma(\alpha + aj + 1/2)\Gamma(\beta + bj + 1/2)2^{\alpha + aj + \beta + bj} + 2} \frac{\partial}{\partial \varphi} \left( K^{\alpha + aj, \beta + bj}(\theta, \varphi) \right).
\]

In order to treat $L_1$, we use the symmetry of the expressions on $u$ and $v$ and $\theta$ and $\varphi$, and the estimate (1.5) follows with analogous reasoning as for $K_1$ above. Concerning $L_2$, we have to take into account that $|\partial^2 \varphi(1 - z)| = 1/4 \left| (u \cos \frac{\theta}{2} \sin \frac{\pi}{2} + v \sin \frac{\theta}{2} \sin \frac{\pi}{2}) \right| \leq 1$ and, on the other hand, that $\partial \varphi(1 - z) = 1/2 \sin \frac{\theta}{2} - \frac{1}{2} \cos \frac{\theta}{2} \sin \frac{\pi}{2} \leq 1/2$. Then, with the same ideas as in $K_2$, the proof of the estimate (1.5) follows. We leave details to the interested reader. \qed
4. Kernel estimates for the auxiliary operator $T_{M}^{\alpha,\beta}$

Let us define the operator $T_{M}^{\alpha,\beta}$ in $L^2(d\mu_{\alpha,\beta})$ as in Subsection 2.1.2. By using the Poisson semigroup $P_{t}$ in (3.1) we can see that

$$T_{M}^{\alpha,\beta} f(\theta) = \sqrt{\rho_{M}(\theta)}(J^{\alpha,\beta})^{-1/2} f(\theta) = \int_{0}^{\pi} T_{M}^{\alpha,\beta}(\theta, \varphi) f(\varphi) d\mu_{\alpha,\beta}(\varphi), \quad \text{in } L^2(d\mu_{\alpha,\beta}),$$

where

$$T_{M}^{\alpha,\beta}(\theta, \varphi) = \sqrt{\rho_{M}(\theta)} \int_{0}^{\infty} P_{t}^{\alpha,\beta}(\theta, \varphi) dt.$$

**Lemma 4.1.** Let $\alpha, \beta > -1/2$, $a \geq 1$, $b = 0$ or $b \geq 1$, and let $u_j$ be as in (1.3), $j \in \mathbb{N}$. When $M = S^d$ we also assume that $\beta > 0$ and $b \geq 1$. Then, the operator $J_{j} T_{M}^{\alpha+j,\beta+b}(u_{j-1}^{-1})$ is bounded from $L^2(d\mu_{\alpha,\beta})$ into itself.

**Proof.** Before starting with the estimate, we need an identity for the Jacobi polynomials. By using the identities

$$(n + \frac{\alpha}{2} + \frac{\beta}{2})(1-x)P_{n-1}^{\alpha+1,\beta}(x) = (n + \alpha)P_{n-1}^{\alpha,\beta}(x) - nP_{n}^{\alpha,\beta}(x),$$

see [1, 22.7.15] with $n$ replaced by $n - 1$, and

$$P_{n-1}^{\alpha,\beta}(x) = \frac{1}{n + \beta} \left( (n + \alpha + \frac{\beta}{2})(1-x)P_{n}^{\alpha,\beta}(x) - nP_{n}^{\alpha,\beta}(x) \right),$$

which follows from [1, 22.7.18], we have

$$\alpha P_{n}^{\alpha,\beta}(x) = (n + \alpha)P_{n}^{\alpha-1,\beta}(x) + \frac{n + \beta}{2}(1-x)P_{n-1}^{\alpha+1,\beta}(x).$$

With the relations $d_{n-1}^{\alpha-1,\beta} = A_{n}d_{n}^{\alpha,\beta}$ and $d_{n}^{\alpha+1,\beta} = B_{n}d_{n}^{\alpha,\beta}$, where $d_{n}^{\alpha,\beta}$ is as in (2.4) and

$$A_{n} = \frac{2n + \alpha + \beta}{2n + \alpha + \beta + 1} \frac{n + \alpha}{n + \alpha + \beta}, \quad B_{n} = \frac{2n + \alpha + \beta}{2n + \alpha + \beta + 1} \frac{n + \beta}{n + \alpha + \beta},$$

and the substitution $x = \cos \theta$, we conclude from (4.2) that

$$\frac{\alpha}{\sin \frac{\theta}{2}} P_{n}^{\alpha,\beta}(\theta) = \frac{n + \alpha}{A_{n}} \frac{1}{\sin \frac{\theta}{2}} P_{n}^{\alpha-1,\beta}(\theta) + \frac{n + \beta}{B_{n}} \frac{1}{\sin \frac{\theta}{2}} P_{n-1}^{\alpha+1,\beta}(\theta).$$

Let us start with the estimate in the case $M \neq S^d$. The required $L^2$ estimate can be deduced from the inequality

$$\int_{0}^{\pi} |\alpha T_{M}^{\alpha,\beta} f|^2 d\mu_{\alpha,\beta} \leq C \int_{0}^{\pi} |f|^2 d\mu_{\alpha,\beta},$$

with $C$ a constant independent of $\alpha$ and $\beta$. By (4.3), the left-hand side in (4.4) is bounded by the sum of

$$\int_{0}^{\pi/2} \left| \sum_{n=0}^{\infty} \frac{n + \alpha}{(n + \frac{\alpha + \beta + 1}{2})A_{n}} c_{n}^{\alpha,\beta}(f) \mathcal{P}_{n}^{\alpha-1,\beta} \right|^2 d\mu_{\alpha-1,\beta}$$

and

$$\int_{0}^{\pi/2} \left| \sum_{n=0}^{\infty} \frac{n + \beta}{(n + \frac{\alpha + \beta + 1}{2})B_{n}} c_{n}^{\alpha,\beta}(f) \mathcal{P}_{n-1}^{\alpha+1,\beta} \right|^2 d\mu_{\alpha+1,\beta}.$$

Now, by taking into account that the sequences $\frac{n + \alpha}{(n + \frac{\alpha + \beta + 1}{2})A_{n}}$ and $\frac{n + \beta}{(n + \frac{\alpha + \beta + 1}{2})B_{n}}$ are bounded by a constant independent of $\alpha$ and $\beta$ and the orthogonality, for $\alpha > 0$, of the systems $\{\mathcal{P}_{n}^{\alpha-1,\beta}\}_{n \geq 0}$ and $\{\mathcal{P}_{n}^{\alpha+1,\beta}\}_{n \geq 0}$, both summands are controlled by $\sum_{n=0}^{\infty} |c_{n}^{\alpha,\beta}(f)|^2 = \int_{0}^{\pi} |f|^2 d\mu_{\alpha,\beta}$. 


For the case $M = S^d$, we split $[0, \pi]$ into the intervals $[0, \pi/2]$, $[\pi/2, \pi]$. In the first interval we have $\sin \theta \sim \sin \frac{\theta}{2}$ and in the second one $\sin \theta \sim \cos \frac{\theta}{2}$. Then, with the change of variable $\theta = \pi - w$ in the interval $[\pi/2, \pi]$, we can easily check that
\[
\int_0^\pi |ju_j T_{d_\alpha}^{\alpha + aj, \beta + bj} (u_{-1}^{-1} f)|^2 d\mu_{\alpha, \beta} \sim \int_0^{\pi/2} |ju_j T_{d_\alpha}^{\alpha + aj, \beta + bj} (u_{-1}^{-1} f)|^2 d\mu_{\alpha, \beta} + \int_0^{\pi/2} |ju_j T_{d_\alpha}^{\beta + bj, \alpha + aj} (v_{-1}^{-1} g)|^2 d\mu_{\beta, \alpha}
\]
with $g(w) = f(\pi - w)$, $v_j = (\cos \frac{\theta}{2})^{aj} (\sin \frac{\theta}{2})^{bj}$. Now, the operators appearing in the right hand side above contain the factor $1/\sin \frac{\theta}{2}$ instead of $1/\sin \theta$. So, enlarging the intervals of integration, the boundedness of both integrals is reduced to the previous case.

Proceeding as in Lemma 3.1, it is easily seen that the operators $ju_j T_{d_\alpha}^{\alpha + aj, \beta + bj} (u_{-1}^{-1} f)$ can be associated, in the Calderón–Zygmund sense, with the kernel $ju_j (\theta) u_j (\varphi) T_M^{\alpha + aj, \beta + bj} (\theta, \varphi)$.

Now we proceed with the proof of Theorem 2.1.

**Proof of Theorem 2.1.** We have to prove the growth estimate (2.6) and the smoothness estimate (2.7). We only need to consider the case $\sqrt{\mu_M (\theta)} = \frac{1}{\sin \frac{\theta}{2}}$, that is, $M \neq S^d$. If $M = S^d$ we observe that $(\sqrt{\mu_S (\theta)})^{-1} = \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$, which introduces a singularity in $\theta = 0$ and $\theta = \pi$. In fact, when $\theta \in (0, \pi/2)$, $(\sqrt{\mu_S (\theta)})^{-1}$ behaves as $\sin \frac{\theta}{2}$, and the operator $T_{d_\alpha}^{\alpha, \beta}$ is treated exactly in the same way as $T_M^{\alpha, \beta}$, $M \neq S^d$. If $\theta \in (\pi/2, \pi)$, $(\sqrt{\mu_S (\theta)})^{-1}$ behaves as $\cos \frac{\theta}{2}$, and the proofs are easily adapted in order to cancel this singularity.

**Growth estimates: proof of (2.6).** Taking into account the expression for the kernel (4.1) and the identity (3.4), we get
\[
ju_j (\theta) u_j (\varphi) T_M^{\alpha + aj, \beta + bj} (\theta, \varphi) = \frac{ju_j (\theta) u_j (\varphi) \Gamma (\alpha + aj + \beta + bj + 1)}{\pi \Gamma (\alpha + aj + 1/2) \Gamma (\beta + bj + 1/2) 2^{\alpha + aj + \beta + bj + 2} \sin \frac{\theta}{2}} \times \int_{-1}^{1} \int_{-1}^{1} \frac{1 - u^2)^{\alpha + aj - 1/2} (1 - v^2)^{\beta + bj - 1/2}}{(1 - z)^{\alpha + aj + \beta + bj + 1}} du dv.
\]
The ideas to get the estimate (2.6) are exactly the same as for the Jacobi–Riesz transform estimates in the proof of Theorem 1.2 given in Section 3. In this way, we apply Lemma 3.2 with $c = \beta, d = bj, \lambda = \alpha + aj + 1/2, A = 1 - u \sin \frac{\theta}{2} \cos \frac{\theta}{2}$ and $B = \cos \frac{\theta}{2} \cos \frac{\theta}{2}$, then we make the change $1 + u = 2w$ in the integral in $u$, we use (3.9) with $\eta = 1/2$ and $\gamma = \alpha + aj$ and then again Lemma 3.2 with $c = \alpha + 1/2, d = aj - 1, \lambda = 1/2, A = 1 + \sin \frac{\theta}{2} \sin \frac{\theta}{2} - \cos \frac{\theta}{2} \cos \frac{\theta}{2}$ and $B = 2 \sin \frac{\theta}{2} \sin \frac{\theta}{2}$. With this and (3.8), we obtain (2.6).

**Smoothness estimates: proof of (2.7).** We have
\[
\frac{\partial}{\partial \theta} (ju_j (\theta) u_j (\varphi) T_M^{\alpha + aj, \beta + bj} (\theta, \varphi)) =: N_1 + N_2,
\]
where
\[
N_1 = \frac{ju_j (\theta) u_j (\varphi) \Gamma (\alpha + aj + \beta + bj + 1)}{\pi \Gamma (\alpha + aj + 1/2) \Gamma (\beta + bj + 1/2) 2^{\alpha + aj + \beta + bj + 2} \sin \frac{\theta}{2}} \times \int_{-1}^{1} \int_{-1}^{1} \frac{1 - u^2)^{\alpha + aj - 1/2} (1 - v^2)^{\beta + bj - 1/2}}{(1 - z)^{\alpha + aj + \beta + bj + 1}} du dv =: N_{1,1} + N_{1,2}
\]
and
\[
N_2 = \frac{ju_j (\theta) u_j (\varphi) \Gamma (\alpha + aj + \beta + bj + 1)}{\pi \Gamma (\alpha + aj + 1/2) \Gamma (\beta + bj + 1/2) 2^{\alpha + aj + \beta + bj + 2} \sin \frac{\theta}{2}} \times \frac{\partial}{\partial \theta} \int_{-1}^{1} \int_{-1}^{1} \frac{1 - u^2)^{\alpha + aj - 1/2} (1 - v^2)^{\beta + bj - 1/2}}{(1 - z)^{\alpha + aj + \beta + bj + 1}} du dv.
\]
\[ = \frac{j u_j(\theta) u_j(\varphi) \Gamma(\alpha + aj + \beta + bj + 2)}{\pi \Gamma(\alpha + aj + 1/2) \Gamma(\beta + bj + 1/2) 2^{\alpha+aj+\beta+bj+2}\sin^{\frac{\beta}{2}}(J_1 + J_2 + J_3)}, \]

where \( J_1, J_2 \) and \( J_3 \) are as in Section 3.

For \( N_{1,1} \), we suppose that \( aj \geq 2 \) (observe that if \( aj = 1 \), then \( N_{1,1} = 0 \)). Apply Lemma 3.2 with \( c = \beta, d = bj, \lambda = \alpha + aj + 1/2, A = 1 - u \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} \) and \( B = \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2} \), then we make the change \( 1 + u = 2w \) in the integral in \( u \), use (3.9) with \( \eta = 1 \) and \( \gamma = \alpha + aj \) and then again Lemma 3.2 with \( c = \alpha + 1, d = aj - 2, \lambda = 1, A = 1 + \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} - \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2} \) and \( B = 2 \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} \). Now the idea for \( N_{1,2} \) is to apply Lemma 3.2 with \( c = \alpha, d = aj, \lambda = \beta + bj + 1/2, A = 1 - v \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2} \) and \( B = \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} \), then the change \( 1 + v = 2w \) in the integral in \( v \), use (3.9) with \( \eta = 1 \) and \( \gamma = \beta + bj \) (observe that when \( b = 0, N_{1,2} = 0 \)) and then again Lemma 3.2 with \( c = \beta, d = bj - 1, \lambda = 1, A = 1 + \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2} - \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} \) and \( B = 2 \cos \frac{\varphi}{2} \cos \frac{\theta}{2} \). This concludes the estimate for \( N_1 \).

Finally, we sketch the proofs for \( N_2 \), which involves the terms with \( J_1, J_2 \) and \( J_3 \), although they are routine. For that with \( J_1 \), apply Lemma 3.2 with \( c = \beta, d = bj, \lambda = \alpha + aj + 3/2, A = 1 - u \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} \) and \( B = \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2} \), then the change \( 1 + u = 2w \) in the integral in \( u \), use (3.9) with \( \eta = 1/2 \) and \( \gamma = \alpha + aj \) and then again Lemma 3.2 with \( c = \alpha + 1/2, d = aj - 1, \lambda = 3/2, A = 1 + \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} - \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2} \) and \( B = 2 \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} \). For the term with \( J_2 \), apply Lemma 3.2 with \( c = \beta, d = bj, \lambda = \alpha + aj + 3/2, A = 1 - u \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} \) and \( B = \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2} \), then the change \( 1 + u = 2w \) in the integral in \( u \), use (3.9) with \( \eta = 1 \) and \( \gamma = \alpha + aj \) and then again Lemma 3.2 with \( c = \alpha + 1, d = aj - 1, \lambda = 1, A = 1 + \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} - \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2} \) and \( B = 2 \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} \). The term with \( J_3 \) for \( b \geq 1 \) follows as in the previous case of \( J_2 \), but with the role of \( u \) and \( v \) exchanged. If \( b = 0 \), we first make the change \( 1 + u = 2w \) in the integral in \( u \) and use (3.9) with \( \eta = 1/2 \) and \( \gamma = \alpha + aj \). Then we apply Lemma 3.2 in \( w \) with \( c = \alpha, d = aj - 1/2, \lambda = \beta + 2, A = 1 - \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} \) and \( B = 2 \cos \frac{\varphi}{2} \cos \frac{\theta}{2} \). The integral arising in \( v \) is bounded by the sum of

\[ \frac{C_{\beta}}{(1 - \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2})^{\alpha+1/2}} \int_0^1 (1 - \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2})^{\beta+2} dv \]

and

\[ \frac{C_{\beta}}{(1 - \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2})^{\alpha+1/2}} \int_0^1 (1 + \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2})^{\beta-2} dv. \]

For the first one, we use that \( (1 - v)^{\beta+1/2} \leq (1 - v)^2 \) and apply Lemma 3.2 with \( c = \beta + 1/2, d = 0, \lambda = 1, A = 1 - \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} \) and \( B = \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2} \). The second one is bounded by

\[ \frac{C_{\beta}}{(1 - \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2})^{\alpha+1/2}(1 - \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2})^{\beta+1}(1 - \sin^2 \frac{\varphi}{2} \sin^2 \frac{\theta}{2} - \cos^2 \frac{\varphi}{2} \cos^2 \frac{\theta}{2})}. \]

\[ \square \]

5. THE EXTRAPOLATION THEOREM AND VECTOR-VALUED EXTENSIONS

In this section we prove an adaptation of Rubio de Francia’s extrapolation theorem for the Jacobi expansions and we present the proofs of Theorem 1.3 and Theorem 2.2.

We define the Jacobi–Hardy–Littlewood maximal function as

\[ M_{\alpha, \beta} f(\theta) = \sup_{\delta \in \mathbb{R}} \frac{1}{\mu_{\alpha, \beta}(I)} \int_I |f(\varphi)| d\mu_{\alpha, \beta}(\varphi), \]

where \(|I|\) denotes the length of the interval \( I \subset (0, \pi) \). For \( 1 \leq p < \infty \), we denote by \( A^p_{\alpha, \beta} \) the class of \( A_p \) weights on the space of homogeneous type \( ((0, \pi), d\mu_{\alpha, \beta}(\theta), | \cdot |) \), see [9]. Namely, \( A^p_{\alpha, \beta} \) is the class of nonnegative functions \( w \in L^1_{loc}(d\mu_{\alpha, \beta}) \) for which, if \( 1 < p < \infty \), \( w^{-p'/p} \in L^1_{loc}(d\mu_{\alpha, \beta}) \) and

\[
\left( \frac{1}{\mu_{\alpha, \beta}(I)} \int_I w d\mu_{\alpha, \beta} \right) \left( \frac{1}{\mu_{\alpha, \beta}(I)} \int_I w^{-p'/p} d\mu_{\alpha, \beta} \right)^{p'/p} < \infty.
\]
In the case $p = 1$, $w \in A_{\alpha}^{\alpha,\beta}$ whenever $M_{\alpha,\beta} w(\theta) \leq C w(\theta)$, $d\mu_{\alpha,\beta}$-a.e. As a particular case of [9, Theorem 3], we have that $M_{\alpha,\beta}$ is bounded in $L^p(w \, d\mu_{\alpha,\beta})$, $1 < p < \infty$, and it is of weak type $(1,1)$ for $p = 1$.

**Theorem 5.1** (Extrapolation theorem). Assume that for some family of pairs of nonnegative functions $(f, g)$, for some fixed $1 < r < \infty$ and for all $w \in A_{p}^{\alpha,\beta}$ we have

$$\int_0^\pi g(\theta)^r w(\theta) \, d\mu_{\alpha,\beta}(\theta) \leq C \int_0^\pi f(\theta) w(\theta) \, d\mu_{\alpha,\beta}(\theta),$$

with $C$ non depending on the pair $(f, g)$. Then, for all $1 < p < \infty$ and all $w \in A_{p}^{\alpha,\beta}$ we have

$$\int_0^\pi g(\theta)^p w(\theta) \, d\mu_{\alpha,\beta}(\theta) \leq C \int_0^\pi f(\theta)^p w(\theta) \, d\mu_{\alpha,\beta}(\theta).$$

**Proof.** We follow the proof given by J. Duoandikoetxea in [15, Theorem 3.1]. The main ingredients are the factorization theorem (see [15, Lemma 2.1]) and the construction of the Rubio de Francia weights $RF$ and RH (see [15, Lemma 2.2]) in our context. The first ingredient is available here because of the general factorization theorem proved by Rubio de Francia [25, Section 3]. For the second ingredient we use the operator $M_{\alpha,\beta}$ to construct the weights $RF$ and RH as in Lemma 2.1 of [15]. Then the proof follows the same lines as in [15].

**Remark 5.2.** An analogous result to that of Theorem 5.1 but in the context of Laguerre function expansions was obtained in [13, Theorem 3.9].

**Proof of Theorem 1.3 and Theorem 2.2.** Let $S_j$, $j \geq 1$, be either the operator $u_j R^{\alpha+aj,\beta+bj}(u_j^{-1})$, or the operator $ju_j T_M^{\alpha+aj,\beta+bj}(u_j^{-1})$, for $\alpha, \beta > -1/2$. By Theorems 1.2 and 2.1, and the Calderón–Zygmund theory for spaces of homogeneous type (see [9] and [28], also [27]), we see that $S_j$ is bounded in $L^r(w \, d\mu_{\alpha,\beta})$, for $1 < r < \infty$ and all $w \in A_{p}^{\alpha,\beta}$, uniformly in $j \geq 0$, namely, there exists a constant $C$ independent of $j$ such that

$$\int_0^\pi |S_j f(\theta)|^r w(\theta) \, d\mu_{\alpha,\beta}(\theta) \leq C \int_0^\pi |f(\theta)|^r w(\theta) \, d\mu_{\alpha,\beta}(\theta).$$

In particular, for any sequence of functions $f_{j,k}$ in $L^r(w \, d\mu_{\alpha,\beta})$, we have

$$\int_0^\pi \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |S_j f_{j,k}(\theta)|^r w(\theta) \, d\mu_{\alpha,\beta}(\theta) \leq C \int_0^\pi \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |f_{j,k}(\theta)|^r w(\theta) \, d\mu_{\alpha,\beta}(\theta),$$

for all $w \in A_{p}^{\alpha,\beta}$. Now, in the extrapolation theorem above we make the following choices:

$$g = \left( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |S_j f_{j,k}|^r \right)^{1/r}, \quad f = \left( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |f_{j,k}|^r \right)^{1/r}.$$

With this pair $(f, g)$, the inequality (5.2) is just the hypothesis of Theorem 5.1. Therefore, for any $1 < p < \infty$ and all $w \in A_{p}^{\alpha,\beta}$,

$$\int_0^\pi \left( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |S_j f_{j,k}|^r \right)^{p/r} w(\theta) \, d\mu_{\alpha,\beta}(\theta) \leq C \int_0^\pi \left( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |f_{j,k}|^r \right)^{p/r} w(\theta) \, d\mu_{\alpha,\beta}(\theta).$$

**Final remark.** In the limit case $\alpha = \beta = -1/2$, we put $\Pi_{-1/2} = \frac{1}{2}(\delta_{-1} + \delta_1)$ in (3.2). By continuity arguments, the representation for the Jacobi–Poisson kernel in (3.3) remains valid in the limiting cases $\alpha = -1/2$ or $\beta = -1/2$. Therefore, it can be seen that all the results in this paper concerning the operators $u_j R^{\alpha+aj,\beta+bj}(u_j^{-1})$ and $ju_j T_M^{\alpha+aj,\beta+bj}(u_j^{-1})$ are still valid in these particular cases.
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References

[1] M. Abramowitz and I. A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, National Bureau of Standards Applied Mathematics Series 55, Washington, 1964.

[2] J.-P. Anker, \(L^p\) Fourier multipliers on Riemannian symmetric spaces of the noncompact type, Ann. of Math. (2) 132 (1990), 597–628.

[3] J.-P. Anker and L. Ji, Heat kernel and Green function estimates on noncompact symmetric spaces, Geom. Funct. Anal. 9 (1999), 1035–1091.

[4] A. Benedek and R. Panzone, The space \(L^p\), with mixed norm, Duke Math. J. 28 (1961), 301–324.

[5] M. Berger, P. Gauduchon and E. Mazet, Le Spectre d’une Variété Riemannienne, Lecture Notes in Mathematics 194, Springer-Verlag, Berlin-New York, 1971.

[6] J. J. Betancor, J. C. Fariña, L. Rodríguez-Mesa and R. Testoni, Higher order Riesz transforms in the ultraspherical setting as principal value integral operators, Integral Equations Operator Theory 70 (2011), 511–539.

[7] D. Buraczewski, T. Martínez and J. L. Torrea, Calderón-Zygmund operators associated with ultraspherical expansions, Canad. J. Math. 59 (2007), 1223–1244.

[8] D. Buraczewski, T. Martínez, J. L. Torrea and R. Urban, The Riesz transform associated with ultraspherical polynomials, J. Anal. Math. 98 (2006), 113–143.

[9] A. P. Calderón, Inequalities for the maximal function relative to a metric, Studia Math. 57 (1976), 297–306.

[10] I. Chavel, Eigenvalues in Riemannian Geometry, Pure and Applied Mathematics 115, Academic Press Inc., Orlando, FL, 1984.

[11] I. Chavel, Riemannian Geometry: A Modern Introduction, 2nd ed., Cambridge Studies in Advanced Mathematics 98, Cambridge University Press, Cambridge, 2006.

[12] O. Ciaurri and L. Roncal, Vector-valued extensions for fractional integrals of Laguerre expansions, preprint 2012, arXiv:1212.4715.

[13] O. Ciaurri and L. Roncal, The Riesz transform for the harmonic oscillator in spherical coordinates, preprint 2013, arXiv:1304.0702.

[14] O. Ciaurri, L. Roncal and P. R. Stinga, Fractional integrals on compact Riemannian symmetric spaces of rank one, Adv. Math. 235 (2013), 627–647.

[15] J. Duoandikoetxea, Extrapolation of weights revisited: new proofs and sharp bounds, J. Funct. Anal. 260 (2011), 1886-1901.

[16] S. Helgason, A duality for symmetric spaces with applications to group representations, Advances in Math. 5 (1970), 1–154.

[17] S. Helgason, Differential Geometry and Symmetric Spaces, Pure and Applied Mathematics XII, Academic Press, New York-London, 1962.

[18] S. Helgason, Groups and Geometric Analysis, Pure and Applied Mathematics 113, Academic Press, Inc., Orlando, FL, 1984.

[19] S. Helgason, Radon-Fourier transforms on symmetric spaces and related group representations, Bull. Amer. Math. Soc. 71 (1965), 757–763.

[20] S. Helgason, The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds, Acta Math. 113 (1965), 153–180.

[21] M. Keel and T. Tao, Endpoint Strichartz estimates, Amer. J. Math. 120 (1998), 955–980.

[22] B. Muckenhoupt and E. M. Stein, Classical expansions and their relation to conjugate harmonic functions, Trans. Amer. Math. Soc. 118 (1965), 17–92.

[23] A. Nowak and P. Sjögren, Calderón-Zygmund operators related to Jacobi expansions, J. Fourier Anal. Appl. 18 (2012), 717–749.

[24] G. Olafsson and H. Schlichtkrull, A local Paley–Wiener theorem for compact symmetric spaces, Adv. Math. 218 (2008), 202–215.

[25] J. L. Rubio de Francia, Factorization theory and \(A_p\) weights, Amer. J. Math. 106 (1984), 533–547.

[26] J. L. Rubio de Francia, Transference principles for radial multipliers, Duke Math. J. 58 (1989), 1–19.

[27] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, Calderón-Zygmund theory for operator-valued kernels, Adv. in Math. 62 (1986), 7–48.

[28] F. J. Ruiz and J. L. Torrea, Vector-valued Calderón-Zygmund theory and Carleson measures on spaces of homogeneous nature, Studia Math. 88 (1988), 221–243.

[29] T. O. Sherman, Fourier analysis on compact symmetric space, Bull. Amer. Math. Soc. 83 (1977), 378–380.

[30] T. O. Sherman, The Helgason Fourier transform for compact Riemannian symmetric spaces of rank one, Acta Math. 164 (1990), 73–144.

[31] E. M. Stein, Topics in Harmonic Analysis Related to the Littlewood-Paley Theory, Annals of Mathematics Studies 63, Princeton University Press, Princeton, NJ, 1970.

[32] R. S. Strichartz, Analysis of the Laplacian on the complete Riemannian manifold, J. Funct. Anal. 52 (1983), 48–79.

[33] G. Szegő, Orthogonal Polynomials, fourth edition, American Mathematical Society, Colloquium Publications XXIII, American Mathematical Society, Providence, 1975.

[34] S. Thangavelu, Holomorphic Sobolev spaces associated to compact symmetric spaces, J. Funct. Anal. 251 (2007), 438–462.
[35] H.-C. Wang, Two–point homogeneous spaces, Ann. of Math. (2) 55 (1952), 177–191.

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