String/Flux Tube Duality
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Abstract

I describe Field/String duality as applied to the response of gauge fields to separated quark and antiquark sources. This is a talk contributed to the conference Quark Confinement and the Hadron Spectrum VII, Ponta Delgada, Azores, 2-7 September 2006.

1 Introduction

The physics of quark confinement is generally believed to involve two conjectured properties of quantum Yang-Mills gauge theory: (1) that there is a mass gap $m_G$ (the lightest glueball mass) and (2) that the gauge field responds to a fixed $Q$ source separated from a fixed $\bar{Q}$ source by a distance $L$ by forming a gluonic flux tube (or gluon chain) between $Q$ and $\bar{Q}$ with energy $U(L) \approx T_0 L$ for large $L$.

Although both of these facets of quark confinement are firmly established numerically, an analytic understanding of either is so far unattained. Indeed just proving the mass gap is one of the Clay Institute millennium problems. My message here is that a less daunting path to such an analytic understanding may be the reformulation of Yang-Mills as a String Theory (Field/String Duality) [1–3]. This reformulation can proceed without solving the theory or even without establishing a mass gap. It might well provide both a setting, in which the physics of confinement can be understood by using string variables to construct a tractable model of the gluonic flux tube (gluon chain), and a vehicle for a self-consistent determination of a mass gap.

The AdS/CFT correspondence shows that a string reformulation of QFT is quite independent of confinement and also of the existence of a mass gap. Indeed, the best understood case of Field/String duality is the equivalence of $\mathcal{N} = 4$ supersymmetric SU(N) Yang Mills to IIB superstring theory on AdS$_5 \times$S$_5$ [1]. In this case the finiteness of the $\mathcal{N} = 4$ theory implies conformal invariance which in turn implies a vanishing mass gap and zero string tension. On the string side these features are consequences of the curved AdS background. The string interpretation is particularly transparent when $N \to \infty$ because $g_{\text{string}} \sim 1/N$, so it is a limit in which strings do not break or interact.

From this point of view string theory offers something much more tangible to theoretical physics than a nebulous and quasi-religious “theory of everything”. I believe that it provides a practical theoretical framework for resolving some of the most intriguing conundrums of quantum field theory. In this regard, I offer a new definition of string theory by way of an analogy:

String Theory : $\sum$ (Planar Diagrams) :: Bethe-Salpeter Equation : $\sum$ (Ladder Diagrams)

Just as the sum of ladder diagrams gives a zeroth order Bethe-Salpeter equation so does the sum of planar diagrams give a zeroth order (noninteracting) string theory. In both cases the full QFT can be regained by systematic corrections. For string theory the nonplanar corrections are neatly handled via ’t Hooft’s $1/N$ expansion [4], since the planar approximation is exact in the large $N$ limit with $\lambda = N\alpha_s/\pi$ fixed.

In this talk I shall explain how stringy features appear in the conformally invariant $\mathcal{N} = 4$ case. Though this theory lacks a mass gap and quark confinement, it nonetheless produces a stringy gluonic flux tube between separated color sources. Moreover, we can easily reach interesting conclusions about this flux tube’s physical properties, especially in the strong ’t Hooft coupling limit when the string can be treated semiclassically. This limit makes sense for $\mathcal{N} = 4$ because the coupling does not depend on the scale, and is thus a free parameter. In a string theory formulation of pure Yang-Mills, such a semiclassical limit is not possible because there is no tunable coupling. Nonetheless, the conformal case is an important example because it

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shows a limit in which the planar sum can actually be done. Although there should be a strikingly different outcome for the sum of planar diagrams in QCD, the technical difficulties in the two problems are quite comparable. Solving one should teach us a great deal about solving the other.

2 Separated $Q\bar{Q}$ Sources

At $N = \infty$ the response of the $\mathcal{N} = 4$ theory to separated static color sources is very interesting: a flux tube forms with an excitation spectrum that becomes string-like in the limit of strong ’t Hooft coupling $\lambda \to \infty$. A convenient probe that reveals this excitation spectrum is the expectation of a rectangular $L \times T$ Wilson Loop, $\langle W(L,T) \rangle \sim \sum_n w_n \exp(-TE_n(L))$ as $T \to \infty$. The $L$ dependence of the ground state energy tests the presence of confinement, a test the $\mathcal{N} = 4$ theory fails because it is conformal $E_G(L) \sim -c/L$ at large $L$. But the excitation spectrum $E_n(L)$ at fixed $L$ can look stringy and does at strong ’t Hooft coupling.

First we consider the weak coupling limit of this $Q\bar{Q}$ system, $\lambda \ll 1$. In this limit in Coulomb gauge the planar approximation to the Wilson loop, in pure Yang-Mills theory, is given by multiple instantaneous Coulomb exchanges (See Fig. 1). The Wilson loop ensures $Q\bar{Q}$ are in a singlet. We can therefore read off the singlet energy as the coefficient of $-T$ in the exponential behavior at large $T$. In the case of the sum of diagrams with only Coulomb exchange, this shows $E_{\text{singlet}} = -\pi\lambda/2L$, and this is the only eigenstate that couples. The resolvent shows only a single pole

$$R_0(E,L) \equiv \int_0^\infty dT e^{ET}W_0 \sim \frac{\rho_0}{-E - \pi\lambda/2L}$$

Note that in the $\mathcal{N} = 4$ conformal theory the Wilson loop is modified to include coupling to the scalar fields. This (1) doubles the effect of Coulomb exchange, so we have to understand $\lambda \to 2\lambda$ in this formula, and (2) introduces a continuum for $E > 0$ at lowest order, because the scalar propagator is not instantaneous. A very interesting feature of this continuum as well as that due to planar radiative corrections (e. g. the middle diagram of Fig. 1) is that, since planarity forbids gluon exchanges between parts of the diagram separated by propagating transverse gluons, there is a gap between the discrete ground state and the continuum, summarized by the following form for the planar resolvent (now for the $\mathcal{N} = 4$ theory)

$$R_{\text{planar}}(E,L) \sim \frac{\rho_0}{-E - \pi[\lambda + O(\lambda^2)]/L} + \int_0^\infty dE' \frac{\rho_1(E')}{E' - E}$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Wilson loop with only Coulomb exchange (left), a planar radiative correction (middle), and a nonplanar correction (right).}
\end{figure}
In contrast, nonplanar corrections, suppressed by powers of $1/N$, show no such gap:

$$R_{\text{nonplanar}}(E, L) \sim \frac{1}{N^2} \int_{-\pi \lambda/2L}^{\infty} dE' \frac{p_2(E')}{E' - E}$$

Thus at $N = \infty$ and arbitrarily weak coupling, there is a gap in the $Q\bar{Q}$ system [5].

At strong coupling, $\lambda \gg 1$, the AdS/CFT correspondence gives the excitation spectrum of the $Q\bar{Q}$ system in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory as the semi-classical quantization of a IIB superstring connecting the two sources on the boundary of AdS$_5$. The lightcone action for the worldsheet fields representing the coordinates of AdS$_5$ is [6]

$$S_{\text{AdS}_5}^{\text{ws}} = \int d\tau \int^{p^+}_0 d\sigma \frac{\alpha_s^2}{2} \left[ \dot{x}^2 - e^{2\phi} x^2 + e^{-\phi} \dot{\phi}^2 - e^\phi \phi'^2 \right]$$

$$4\gamma^2 = \frac{R^2 T_0}{\sqrt{\alpha_s N_c/\pi}} = \sqrt{\lambda}$$

The static solution (see Fig. 2) has energy $E_0$ which increases as $-c\sqrt{\lambda}/L$ with separation, showing the absence of a confining force [7]. Nonetheless, this stretched string has an infinite number of stringy excitations.

Semi-classical quantization of the small oscillations about this static solution [8] gives string-like modes with discrete levels just above $E_0$:

$$E_{N_n} - E_0 = \sum N_n \omega_n; \quad \omega_n = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2 L} \xi_n; \quad \xi_n \sqrt{\xi_n^2 - 1} \int_0^1 \frac{t^2 dt}{[1 + \xi_n^2 t^2]^{1/4}} = \frac{n\pi}{2}$$

Where $n = 1, 2, \ldots$. For large $n$, $\omega_n \sim (2\pi)^3(n + 1)/\Gamma(1/4)^4 L$ [9], typical of normal modes of a string with effective tension $T_{\text{eff}} \sim 1/L^2$. We see that $\mathcal{N} = 4$ is teetering on the brink of quark confinement: a stringy object is there for $L$ finite, but a mechanism to prevent $T_{\text{eff}}$ from dropping to 0 when $L \to \infty$ is lacking. Finally, we note that near threshold ($E = 0$) discrete levels accumulate [5] $E_{n+1}/E_n \sim e^{-\pi/\sqrt{4\lambda}}$. This is shown by a semiclassical quantization of the motion of the midpoint of string stretched a distance $D >> L$ from the boundary of AdS, where the string ends reside.

To summarize, we show the energy level diagram for the $\mathcal{N} = 4 \bar{Q}Q$ system at $N = \infty$ for weak and strong 't Hooft coupling (see Fig. 3). The transition from weak to strong coupling is rather mundane: a stronger binding with a deepening gap that eventually supports excited discrete levels that peel off the continuum and move into the gap. $N = \infty$ is essential here: otherwise the continuum goes all the way down to $E_0$. At large finite $N$ this continuum would be dominated by very narrow resonances.

Surprisingly a simple Feynman gauge ladder diagram model [10] shows qualitatively similar physics. The authors of this paper show that the weak coupling energy is recovered and at strong coupling the ground energy $\propto -\sqrt{\lambda}/L$ with a different numerical coefficient than the AdS string. At intermediate coupling [9] this model shows no more bound states for $\lambda < 1/4$. For $\lambda > 1/4$ an infinite number of bound states appear with threshold behavior $E_{n+1}/E_n \sim e^{-\pi/\sqrt{4\lambda}}$; with the same strong coupling behavior as $\mathcal{N} = 4$. The big
Figure 3: Energy spectrum of the $\mathcal{N} = 4$ $Q\bar{Q}$ system for strong and weak coupling at $N = \infty$.

qualitative difference from the exact strong coupling behavior is that the ladder model shows only a single mode of small oscillations instead of an infinite number of stringy modes $\omega_n$. This defect is presumably due to the neglect of all the non-ladder planar diagrams. An accurate treatment of the $\mathcal{N} = 4$ theory at intermediate coupling requires either the proper quantization and solution of IIB superstring on $\text{AdS}_5 \times \text{S}_5$, or a way to sum all planar diagrams. For QCD, the absence of a sensible strong coupling limit has hindered the discovery of its string theory dual. Understanding the $\mathcal{N} = 4$ theory by directly summing planar diagrams could teach us how to do the same with real QCD at $N = \infty$. In particular, the lightcone worldsheet formalism was developed as a way to read off the string dual directly from the sum of planar diagrams [3]. This program also gives a strategy for implementing a gluon chain model of the flux tube [11].

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