Asymptotically Tight Bounds on the Time Complexity of Broadcast and its Variants in Dynamic Networks

Antoine El-Hayek\textsuperscript{1}, Monika Henzinger\textsuperscript{2}, Stefan Schmid\textsuperscript{3}

\textsuperscript{1}Faculty of Computer Science, University of Vienna
\textsuperscript{2}IST Austria
\textsuperscript{3}TU Berlin, Germany

ITCS 2023
Information Dissemination in Dynamic Rooted Trees

- The network of each round can be a different rooted tree.
- Each node transmits all I.D.s it has received in previous rounds.
Information Dissemination in Dynamic Rooted Trees

- The network of each round can be a different rooted tree.
- Each node transmits all I.D.s it has received in previous rounds.
Information Dissemination in Dynamic Rooted Trees

- The network of each round can be a different rooted tree.
- Each node transmits all I.D.s it has received in previous rounds.
Information Dissemination in Dynamic Rooted Trees

- The network of each round can be a different rooted tree.
- Each node transmits all I.D.s it has received in previous rounds.
Information Dissemination in Dynamic Rooted Trees

- The network of each round can be a different rooted tree.
- Each node transmits all I.D.s it has received in previous rounds.
Information Dissemination in Dynamic Rooted Trees

- The network of each round can be a different rooted tree.
- Each node transmits all I.D.s it has received in previous rounds.
Information Dissemination in Dynamic Rooted Trees

- The network of each round can be a different rooted tree.
- Each node transmits all I.D.s it has received in previous rounds.
- Broadcast is when 1 I.D. reaches everyone
Information Dissemination in Dynamic Rooted Trees

- The network of each round can be a different rooted tree.
- Each node transmits all I.D.s it has received in previous rounds.
- Broadcast is when 1 I.D. reaches everyone
- How many rounds do we need to ensure Broadcast?
Adversarial Model

- An adversary can choose any network among a set $A$ of predefined networks.
- There’s an objective the adversary tries to delay as much as possible.
- We want to determine the number of rounds $T$ the adversary can delay the objective.

Example for $n - 1$ rounds:

![Diagram](image-url)
Previous Work

- [Charron-Bost, Schiper ‘09] + [Charron-Bost, Függer, Nowak ‘15] : $O(n \log n)$.
- [Zeiner, Schwarz, Schmid ‘19] : $O(n \log n)$ (General Case); $O(kn)$ if $k$ internal nodes or $k$ leaves in each round.
- [Függer, Nowak, Winkler ’20] : $O(n \log \log n)$.

Our Work: $\theta(n)$
Main intuitions

Main Observation

Any I.D. received by the root before the start of a round, is received by at least one new process during the round.
Main Observation
Any I.D. received by the root before the start of a round, is received by at least one new process during the round.

Round 1

Round 2

Round 3
Main intuitions

Main Observation

Any I.D. received by the root before the start of a round, is received by at least one new process during the round.

- If an I.D. has been received by \( n \) roots, then everyone has received the I.D.
- We will keep track of the I.D.s the root has received before each round.
Create a new graph:
- one node for each I.D.
- one node for each round.

For each round $t$, add an edge from every I.D. the root has received, and from every round $t' < t$ if the root of $t$ has received the I.D. of the root of $t'$. 

I.D.s

$n=5$

rounds: 1, 2, 3, 4, ..., $3n$

root: 1, 1, 3, 2, ...

1

2

3

4

1

21

32

43

54

1

21

321

432

543

132

4321

132

4321

21

5432

321

54321

...
Create a new graph:
- one node for each I.D.
- one node for each round.

For each round $t$, add an edge from every I.D. the root has received, and from every round $t' < t$ if the root of $t$ has received the I.D. of the root of $t'$. 

\[
\text{I.D.s} \\
1 \\
2 \\
3 \\
4 \\
n=5 \\
\text{rounds} \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad 3n \\
\text{root} \quad 1 \quad 1 \quad 3 \quad 2 \quad \cdots
\]
Create a new graph:

- one node for each I.D.
- one node for each round.

For each round $t$, add an edge from every I.D. the root has received, and from every round $t' < t$ if the root of $t$ has received the I.D. of the root of $t'$. 

 Antoine El-Hayek, Monika Henzinger, Stefan Schmid

Variants of Broadcast
Create a new graph:

- one node for each I.D.
- one node for each round.

For each round \( t \), add an edge from every I.D. the root has received, and from every round \( t' < t \) if the root of \( t \) has received the I.D. of the root of \( t' \).
Create a new graph:
- one node for each I.D.
- one node for each round.

For each round $t$, add an edge from every I.D. the root has received, and from every round $t' < t$ if the root of $t$ has received the I.D. of the root of $t'$. 
Create a new graph:

- one node for each I.D.
- one node for each round.

For each round $t$, add an edge from every I.D. the root has received, and from every round $t' < t$ if the root of $t$ has received the I.D. of the root of $t'$. 

---

| I.D.s | rounds |
|-------|--------|
| 1     | 1      |
| 2     | 1      |
| 3     | 3      |
| 4     | 2      |
| 5     | 3      |

| root | 1 | 1 | 3 | 2 | ... |
|------|---|---|---|---|-----|

---

| 1   | 21 | 32 | 43 | 54 |
|-----|----|----|----|----|
| 21  | 321| 432| 543| 1  |
| 321 | 4321| 5432| 132| 21 |
| 4321| 54321| 132| 4321| 21 |
| 54321| 132| 21 | 54321| ... |
Create a new graph:
- one node for each I.D.
- one node for each round.

For each round \( t \), add an edge from every I.D. the root has received, and from every round \( t' < t \) if the root of \( t \) has received the I.D. of the root of \( t' \).
Create a new graph:

- one node for each I.D.
- one node for each round.

For each round $t$, add an edge from every I.D. the root has received, and from every round $t' < t$ if the root of $t$ has received the I.D. of the root of $t'$. 
Create a new graph:
- one node for each I.D.
- one node for each round.

For each round $t$, add an edge from every I.D. the root has received, and from every round $t' < t$ if the root of $t$ has received the I.D. of the root of $t'$. 
Create a new graph:
- one node for each I.D.
- one node for each round.

For each round \( t \), add an edge from every I.D. the root has received, and from every round \( t' < t \) if the root of \( t \) has received the I.D. of the root of \( t' \).
Create a new graph:
- one node for each I.D.
- one node for each round.

For each round $t$, add an edge from every I.D. the root has received, and from every round $t' < t$ if the root of $t$ has received the I.D. of the root of $t'$.
Observations:

- If a node has degree at least $n$, then the corresponding I.D. has reached everyone.
- Round $t$ has in-degree at least $t$.
- The total number of edges is larger than $\sum_{t=1}^{3n} t = \frac{9n^2}{2}$.
- We have $4n$ nodes total.
The Upper Bound

An upper bound for Broadcast on rooted trees is $O(n)$. 
The Upper Bound
An upper bound for Broadcast on rooted trees is $O(n)$.

The Lower Bound
A lower bound for Broadcast on rooted trees is $\Omega(n)$.

\textsuperscript{a}Zeiner, M., Schwarz, M., and Schmid, U. (2019). On linear-time data dissemination in dynamic rooted trees. Discrete Applied Mathematics, 255, 307-319.
**k-Broadcast**

**k-Broadcast on k-Rooted Networks**

- **A**: the set of networks on \( n \) processes with \( k \) roots.
- **Objective**: \( k \) I.D.s that has each been received by everyone.
- **We prove** \( T = \Theta(n) \).
**k-Broadcast**

### $k$-Broadcast on $k$-Rooted Networks

- **A**: the set of networks on $n$ processes with $k$ roots.
- **Objective**: $k$ I.D.s that has each been received by everyone.
- **We prove**: $T = \Theta(n)$.

#### Round 1
- **2-Broadcast in 3 rounds.**
- Broadcasters: 1 and 2.

#### Round 2

#### Round 3

**Antoine El-Hayek, Monika Henzinger, Stefan Schmid**

Variants of Broadcast
The Upper Bound

An upper bound for $k$-Broadcast on networks with $k$ roots is $O(n)$.
The Upper Bound
An upper bound for $k$-Broadcast on networks with $k$ roots is $O(n)$.

The Lower Bound
A lower bound for $k$-Broadcast on networks with $k$ roots is $\Omega(n)$. 
Cover of size $k$ on $k$-Forests

- $A$: the set of forests on $n$ processes with $k$ rooted trees.
- Objective: $k$ I.D.s such that everyone has received at least one of them.
- We prove $T = \Theta(n)$. 
**k-Cover**

**Cover of size \( k \) on \( k \)-Forests**

- **\( A \):** the set of forests on \( n \) processes with \( k \) rooted trees.
- **Objective:** \( k \) I.D.s such that everyone has received at least one of them.
- **We prove** \( T = \Theta(n) \).

---

2-Cover in 2 rounds.
Coverers: 1 and 2.
Results

The Upper Bound
An upper bound for Cover of size $k$ on $k$-forests is $O(n)$. 

The Lower Bound
A lower bound for Cover of size $k$ on $k$-forests is $\Omega(n-k)$. 

Antoine El-Hayek, Monika Henzinger, Stefan Schmid
The Upper Bound
An upper bound for Cover of size $k$ on $k$-forests is $O(n)$.

The Lower Bound
A lower bound for Cover of size $k$ on $k$-forests is $\Omega(n - k)$. 
Main Takeaway

In the worst case scenario, when enough connectivity is ensured and when there is no limit on the message sizes, data dissemination is linear.

Future Work:

- Find ways to speed up the objectives by constraining the adversary differently.
- Look at a random adversary rather than a “smart” one.
- Look at applications - Leader election or Consensus.
- Look at message size constraints.