Using the method of a priori energy estimates, energy dissipation is proved for the class of hereditary fractional wave equations, obtained through the system of equations consisting of equation of motion, strain and fractional order constitutive models, that include the distributed-order constitutive law in which the integration is performed from zero to one generalizing all linear constitutive models of fractional and integer orders, as well as for the thermodynamically consistent fractional Burgers models, where the orders of fractional differentiation are up to the second order. In the case of non-local fractional wave equations, obtained using non-local constitutive models of Hooke- and Eringen-type in addition to the equation of motion and strain, a priori energy estimates yield the energy conservation, with the reinterpreted notion of the potential energy.

This article is part of the theme issue ‘Advanced materials modelling via fractional calculus: challenges and perspectives’.

1. Introduction

Fractional wave equations, describing the disturbance propagation in a viscoelastic or non-local material, are
obtained through the system of equations consisting of the equation of motion corresponding to a one-dimensional deformable body

$$\partial_t \sigma(x,t) = \rho \partial_t u(x,t), \quad (1.1)$$

where $u$ and $\sigma$ are displacement and stress, assumed to be functions of space $x \in \mathbb{R}$ and time $t > 0$, with $\rho$ being constant material density strain for small local deformations

$$\varepsilon(x,t) = \partial_x u(x,t), \quad (1.2)$$

and the constitutive equation connecting stress and strain, which can model either hereditary or non-local material properties.

Using a priori energy estimates, the aim is to investigate energy-conserving properties of hereditary and non-local wave equations, obtained using known fractional constitutive equations of linear viscoelasticity as hereditary type models, as well as previously developed fractional non-local models. Hereditary materials are modelled by the fractional-order constitutive equations of a viscoelastic body including distributed-order model containing fractional differentiation orders up to the first order, as well as fractional Burgers models also containing the differentiation orders up to the second order. Energy dissipation is expected for hereditary wave equations, since the thermodynamical requirements on the model parameters impose dissipativity of such constitutive models. On the other hand, non-local materials, modelled by the non-local Hooke law and fractional Eringen stress gradient model, are not expected to dissipate energy.

Hereditary effects in a viscoelastic body are modelled either by the distributed-order constitutive equation

$$\int_0^1 \phi_0(\alpha) 0D^\alpha_t \sigma(x,t) \, d\alpha = \int_0^1 \phi_0(\alpha) 0D^\alpha_t \varepsilon(x,t) \, d\alpha, \quad (1.3)$$

where $\phi_0$ are constitutive functions or distributions and where fractional differentiation orders do not exceed the first order, or by the thermodynamically consistent fractional Burgers models, where fractional differentiation orders are up to the second order. The fractional Burgers models are represented by unified models belonging to two classes: the first class is represented by the unified constitutive equation

$$(1 + a_1 0D^\alpha_t + a_2 0D^\alpha_t + a_3 0D^\gamma_t) \sigma(x,t) = (b_1 0D^\alpha_t + b_2 0D^\beta_t) \varepsilon(x,t), \quad (1.4)$$

while the second one is represented by

$$(1 + a_1 0D^\alpha_t + a_2 0D^\beta_t + a_3 0D^\beta_t) \sigma(x,t) = (b_1 0D^\beta_t + b_2 0D^\gamma_t) \varepsilon(x,t), \quad (1.5)$$

where $a_1, a_2, a_3, b_1, b_2 > 0$, $\alpha, \beta, \mu \in [0,1]$, with $\alpha \leq \beta$, $\gamma \in [0,2]$, and $\eta \in \{\alpha, \beta\}$. The operator of Riemann–Liouville fractional derivative $0D^\xi_t$ of order $\xi \in [n, n+1]$, $n \in \mathbb{N}_0$, used in constitutive models (1.3), (1.4) and (1.5), is defined by

$$0D^\xi_t y(t) = \frac{d^{n+1}}{dt^{n+1}} \left( \int_0^t \frac{t^{-(\xi-n)}}{\Gamma(1-(\xi-n))} y(t) \right), \quad t > 0,$$

see [1], where $\ast_t$ denotes the convolution in time: $f(t) \ast_t g(t) = \int_0^t f(t') g(t-t') \, dt', \, t > 0$.

Non-locality effects in a material are described either by the non-local Hooke law

$$\sigma(x,t) = E \frac{|x|^{-\alpha}}{2\Gamma(1-\alpha)} \ast_x \varepsilon(x,t), \quad (1.6)$$

or by the fractional Eringen constitutive equation

$$\sigma(x,t) - \ell^\alpha D^\alpha_x \sigma(x,t) = E \varepsilon(x,t), \quad (1.7)$$

where $E$ is Young modulus, $\ell$ is non-locality parameter and $D^\alpha_x$ is defined as

$$D^\alpha_x y(x) = \frac{|x|^{1-\alpha}}{2\Gamma(2-\alpha)} \ast_x \frac{d^2}{dx^2} y(x), \quad \text{for } \alpha \in (1,2) \quad (1.8)$$
and
\[ D_x^\alpha y(x) = \frac{|x|^{2-\alpha} \text{sgn} x}{2\Gamma(3-\alpha)} \star_x \frac{d^3}{dx^3} y(x), \quad \text{for } \alpha \in (2,3), \] (1.9)

with \( \star_x \) denoting the convolution in space: \( f(x) \star_x g(x) = \int_{-\infty}^{\infty} f(x')g(x-x') \, dx', \quad x \in \mathbb{R}. \)

The Cauchy problem on the real line \( x \in \mathbb{R} \) and \( t > 0 \) is considered, so the system of governing equations (1.1), (1.2), and one of the constitutive equations (1.3), or (1.4), or (1.5), or (1.6), or (1.7) is subject to initial and boundary conditions
\[ u(x,0) = u_0(x), \quad \frac{\partial}{\partial t} u(x,0) = v_0(x), \] (1.10)
\[ \sigma(x,0) = 0, \quad \varepsilon(x,0) = 0, \quad \frac{\partial}{\partial t} \sigma(x,0) = 0, \quad \frac{\partial}{\partial t} \varepsilon(x,0) = 0 \] (1.11)
\[ \text{and} \quad \lim_{x \to \pm\infty} u(x,t) = 0, \quad \lim_{x \to \pm\infty} \sigma(x,t) = 0, \] (1.12)

where \( u_0 \) is the initial displacement and \( v_0 \) is the initial velocity. The initial conditions (1.11) are needed for the hereditary constitutive equations: distributed-order constitutive equation (1.3) needs (1.11)\(_{1,2}\) and fractional Burgers models (1.4), (1.5) require all initial conditions (1.11), while non-local constitutive models (1.6) and (1.7) do not need any of the initial conditions (1.11).

The distributed-order constitutive model (1.3) generalizes integer and fractional order constitutive models of linear viscoelasticity having differentiation orders up to the first order, since it reduces to the linear fractional model
\[ \sum_{i=1}^{n} a_i D_i^\alpha \sigma(x,t) = \sum_{j=1}^{m} b_j D_j^\beta \varepsilon(x,t), \] (1.13)

with model parameters \( a_i, b_j > 0 \) and \( \alpha_i, \beta_j \in [0,1], \quad i = 1, \ldots, n, \quad j = 1, \ldots, m \), if the constitutive distributions \( \phi_\sigma \) and \( \phi_\varepsilon \) in (1.3) are chosen as
\[ \phi_\sigma(\alpha) = \sum_{i=1}^{n} a_i \delta(\alpha - \alpha_i), \quad \phi_\varepsilon(\alpha) = \sum_{j=1}^{m} b_j \delta(\alpha - \beta_j), \]

where \( \delta \) denotes the Dirac delta distribution. Moreover, the power-type distributed-order model
\[ \int_{0}^{1} a^\alpha D^\alpha \sigma(x,t) \, d\alpha = E \int_{0}^{1} b^\alpha D^\alpha \varepsilon(x,t) \, d\alpha \] (1.14)
is obtained from (1.3) as the genuine distributed-order model, if constitutive functions \( \phi_\sigma \) and \( \phi_\varepsilon \) in (1.3) are chosen as
\[ \phi_\sigma(\alpha) = a^\alpha, \quad \phi_\varepsilon(\alpha) = E b^\alpha, \]

with model parameters \( E, a, b > 0 \) ensuring dimensional homogeneity.

Thermodynamical consistency of linear fractional constitutive equation (1.13) is examined in [2], where it is shown that there are four cases of (1.13) when the restrictions on model parameters guarantee its thermodynamical consistency, while power-type distributed-order model (1.14) is considered in [3] and revisited in [2], where the conditions \( E > 0 \) and \( 0 \leq a \leq b \), guaranteeing the model’s thermodynamical consistency, are obtained. Four cases of thermodynamically acceptable models corresponding to (1.13) are given in appendix A.

Fractional wave equations, corresponding to the system of governing equations (1.1), (1.2) and distributed-order constitutive model (1.3), are considered for the Cauchy problem in [4], generalizing the results of [5,6], where, respectively, the fractional Zener model and its generalization
\[ (1 + a_0 D_0^\alpha) \sigma(x,t) = E(1 + b_0 D_0^\beta) \varepsilon(x,t), \quad 0 \leq a \leq b, \quad \alpha \in [0,1] \] (1.15)
\[ \sum_{i=1}^{n} a_i D_i^\alpha \sigma(x,t) = \sum_{i=1}^{n} b_i D_i^\beta \varepsilon(x,t), \quad 0 \leq \alpha_1 \leq \cdots \leq \alpha_n < 1, \quad \frac{a_1}{b_1} \geq \cdots \geq \frac{a_n}{b_n} \geq 0 \]
are considered as special cases of (1.13). Considering the wave propagation speed, it is found in
[4] that the finite wave speed, as well as the infinite, is the property of both solid-like and fluid-like
materials. Solid-like and fluid-like materials differ in the creep test, representing the deformation
response of a material to a sudden but later constant stress, where the deformation for the first
type of material is bounded for large time in contrast to the second type of material that has
unbounded deformation for large time.

Eight thermodynamically consistent fractional Burgers models, formulated in [7], all
describing fluid-like material behaviour are divided into two classes. The first class, represented
by (1.4), contains five models, such that the highest fractional differentiation order of strain is
\(\mu + \eta \in [1,2]\), with \(\eta \in [\alpha, \beta]\), while the highest fractional differentiation order of stress is either
\(\gamma \in [0,1]\) in the case of Model I, with \(0 \leq \alpha \leq \beta \leq \gamma \leq \mu \leq 1\) and \(\eta \in [\alpha, \beta, \gamma]\), or \(\gamma \in [1,2]\) in the
case of Models II–V, with \(0 \leq \alpha \leq \beta \leq \mu \leq 1\) and \((\eta, \gamma) \in ((\alpha, 2\alpha), (\alpha, \alpha + \beta), (\beta, \alpha + \beta), (\beta, 2\beta))\). Note
that the fractional differentiation order of stress is less than the differentiation order of strain
regardless on the interval \([0,1]\) or \([1,2]\). The second class, represented by (1.5), contains three
models, such that \(0 \leq \alpha \leq \beta \leq 1\) and \(\beta + \eta \in [1,2]\), with \(\eta = \alpha\), in the case of Model VI; \(\eta = \beta\)
in the case of Model VII; and \(\alpha = \eta = \beta\), \(\bar{a}_1 = a_1 + a_2\), and \(\bar{a}_2 = a_3\) in the case of Model VIII.
Note that considering the interval \([0,1]\), the highest fractional differentiation orders of stress
and strain are equal, which also holds true for the orders from interval \([1,2]\). The explicit
forms of Models I–VIII, along with corresponding thermodynamical restrictions, can be found in
appendix B.

The fractional Burgers wave equation, represented by the governing equations (1.1), (1.2), and
either (1.4) or (1.5), is solved for the Cauchy problem in [8]. The wave propagation speed is
found to be infinite for models belonging to the first class, given by (1.4), contrary to the case
of models of the second class (1.5), that yield finite wave propagation speed. Moreover, numerical
examples indicated that at the wavefront there might exist a jump from finite to a zero value of
placement, obtained as the fundamental solution of the fractional Burgers equation.

The non-local Hooke Law (1.6) is introduced in [9] through the non-local strain measure and
used with the classical Hooke Law as a constitutive equation for modelling wave propagation
in non-local media, while in [10] the constitutive equation including both memory and non-
local effects is constructed using fractional Zener model (1.15) and non-local Hooke Law (1.6),
further to be used in describing wave propagation in non-local viscoelastic material. The tools of
microlocal analysis are employed in [11] to investigate properties of this memory and non-local
type fractional wave equation.

Generalizing the integer-order Eringen stress gradient non-local constitutive law, the fractional
Eringen model (1.7) is postulated in [12], where the optimal values of the non-locality parameter
and order of fractional differentiation are obtained with respect to the Born–Kármán model of
lattice dynamics. Furthermore, wave propagation, as well as propagation of singularities, in non-
local material described by the fractional Eringen model (1.7) is analysed in [13].

The energy estimates for proving existence and uniqueness of the solution to the three-
dimensional wave equation corresponding to material of fractional Zener type using the Galerkin
method are considered in [14,15], while the three-dimensional wave equation as a singular
kernel integrodifferential equation, with kernel being the relaxation modulus unbounded at the
initial time, is analysed in [16]. The positivity of Green’s functions corresponding to a three-
dimensional integrodifferential wave equation, which has a completely monotonic relaxation
modulus as a kernel, is established in [17], while the exponential energy decay of the nonlinear
viscoelastic wave equation under the potential well is analysed in [18] assuming Dirichlet
boundary conditions.

In the case of the one-dimensional wave equation, written as the integrodifferential equation
including the relaxation modulus assumed as a wedge continuous function, the solution existence
and uniqueness analysis is performed in [19], while [20] aimed to underline the similarities
between a rigid heat conductor having heat flux relaxation function singular at the origin
and a viscoelastic material having relaxation modulus unbounded at the origin. In [21,22],
one-dimensional wave propagation characteristics, such as wave propagation speed and wave
attenuation, are investigated without and with the Newtonian viscosity component present in the completely monotonic relaxation modulus. The extensive overview of wave propagation problems in viscoelastic materials can be found in [23–25].

In [26], the transient effects, i.e. short-lived seismic wave propagation through viscoelastic subsurface media, are considered and asymptotic expansions of the solutions via Buchen–Mainardi algorithm method introduced in [27] are obtained. The same method is used in [28] in the case of waves in fractional Maxwell and Kelvin–Voigt viscoelastic materials. Dispersion, attenuation, wave fronts and asymptotic behaviour of solution to viscoelastic wave equation near the wavefront are studied in [29–31].

The survey of acoustic wave equations aiming to describe the frequency-dependent attenuation and scattering of acoustic disturbance propagation through complex media displaying viscous dissipation is presented in [32], while the frequency responses of viscoelastic materials are reviewed in [33]. The approach of solving time- and/or space-fractional partial differential equations numerically using spectral methods is adopted in [34,35].

The existence and uniqueness of solutions to three-dimensional wave equation with the Eringen model as a constitutive equation is studied in [36] and it is found that the problem is in general ill-posed in the case of smooth kernels and well posed in the case of singular, non-smooth kernels. Considering the longitudinal and shear waves propagation in non-local medium, the influence of geometric nonlinearity is investigated in [37]. Combining viscoelastic and non-locality characteristics of the medium, wave propagation and wave decay are studied in [38] under the source positioned at the end of a semi-infinite medium.

2. Hereditary fractional wave equations expressed through relaxation modulus and creep compliance

Relaxation modulus and creep compliance, representing material properties in stress relaxation and creep tests, are used in order to formulate the fractional wave equation corresponding to the system of governing equations (1.1), (1.2) and (1.3), or (1.4) or (1.5).

Relaxation modulus \( \sigma_{sr} \) (creep compliance \( \varepsilon_{cr} \)) is the stress (strain) history function obtained as a response to the strain (stress) assumed as the Heaviside step function \( H \). According to the material behaviour in stress relaxation and creep tests at the initial time-instant, one differentiates the materials having either finite or infinite glass modulus \( \sigma_{sr}^{(g)} = \sigma_{sr}(0) \), implying the finite or zero value of the glass compliance \( \varepsilon_{cr}^{(g)} = \varepsilon_{cr}(0) \). The wave propagation speed, obtained as

\[
\sqrt{\frac{\sigma_{sr}^{(g)}}{\varepsilon_{cr}^{(g)}}} = \frac{1}{\sqrt{\frac{\varepsilon_{cr}^{(g)}}{\sigma_{sr}^{(g)}}}},
\]

in [4] for the distributed-order constitutive model (1.3) and in [8] for the fractional Burgers models (1.4) and (1.5), is the implication of these material properties. On the other hand, according to the material behaviour in stress relaxation and creep tests for large time, one differentiates fluid-like materials, having equilibrium compliance \( \varepsilon_{cr}^{(e)} = \lim_{t \to \infty} \varepsilon_{cr}(t) \) infinite and therefore the equilibrium modulus \( \sigma_{sr}^{(e)} = \lim_{t \to \infty} \sigma_{sr}(t) \) zero, from solid-like materials, having both equilibrium compliance and finite equilibrium modulus. The overview of asymptotic properties for viscoelastic materials described by constitutive models (1.3), (1.4) and (1.5) is presented in table 1.

In order to express the constitutive equations (1.3), (1.4) and (1.5) either in terms of the relaxation modulus or in terms of the creep compliance, the Laplace transform with respect to time

\[
\tilde{f}(s) = \mathcal{L}[f(t)](s) = \int_0^\infty f(t) e^{-st} \, dt, \quad \text{Re} \, s > 0,
\]

is applied to (1.3), (1.4) and (1.5), so that

\[
\Phi_\sigma(s)\tilde{\sigma}(x, s) = \Phi_\varepsilon(s)\tilde{\varepsilon}(x, s), \quad \text{Re} \, s > 0,
\]

(2.1)
Table 1. Summary of model properties.

| model     | material type | wave speed | \( \sigma_{sr}^{[\theta]} \) | \( \xi_{\alpha}^{[\theta]} \) | \( \sigma_{st}^{[\theta]} \) | \( \xi_{\alpha}^{[\theta]} \) |
|-----------|---------------|------------|-----------------|-----------------|-----------------|-----------------|
| power-type| solid-like    | finite     | \( \frac{b_0}{a_0} \) | \( a_0 \) | \( b_1 \) | \( a_1 \) |
| Case I    |               |            | \( \frac{b_0}{a_0} \) | \( a_0 \) | \( b_1 \) | \( a_1 \) |
| Case II   |               | infinite   | \( \infty \) | \( 0 \) | \( b_0 \) | \( a_0 \) |
| Case III  | fluid-like    | finite     | \( \frac{b_1}{a_1} \) | \( a_1 \) | 0 | \( \infty \) |
| Case IV   |               | infinite   | \( \infty \) | 0 | 0 | \( \infty \) |
| Models I–V|               |            | \( \infty \) | 0 | 0 | \( \infty \) |
| Models VI–VIII|        | finite  | \( \frac{b_1}{a_1} \) | \( a_1 \) | 0 | \( \infty \) |

is obtained assuming zero initial conditions (1.11), with

\[ \Phi_\sigma(s) = \int_0^1 \Phi_\sigma(\alpha) s^\alpha \, d\alpha \quad \text{and} \quad \Phi_\xi(s) = \int_0^1 \Phi_\xi(\alpha) s^\alpha \, d\alpha, \]

(2.2)
in the case distributed-order constitutive model (1.3), reducing to

\[ \Phi_\sigma(s) = \sum_{i=1}^n a_i s^{\alpha_i}, \quad \Phi_\xi(s) = \sum_{j=1}^m b_j s^{\beta_j} \]

(2.3)
and

\[ \Phi_\sigma(s) = a s - \frac{1}{\ln(as)}, \quad \Phi_\xi(s) = E \frac{b s - 1}{\ln(bs)}, \]

for linear fractional constitutive equation (1.13) and power-type distributed-order model (1.14), respectively, as well as with

\[ \Phi_\sigma(s) = 1 + a_1 s^{\alpha_1} + a_2 s^{\beta_2} + a_3 s^{r_3}, \quad \Phi_\xi(s) = b_1 s^{\mu_1} + b_2 s^{\mu_2}, \]

(2.4)
and

\[ \Phi_\sigma(s) = 1 + a_1 s^{\alpha_1} + a_2 s^{\beta_2} + a_3 s^{r_3}, \quad \Phi_\xi(s) = b_1 s^{\mu_1} + b_2 s^{\mu_2}, \]

(2.5)
in the case of fractional Burgers model of the first, respectively second, class given by (1.4) and (1.5).

The Laplace transform of relaxation modulus and creep compliance

\[ \tilde{\sigma}_{sr}(s) = \frac{1}{s} \Phi_\sigma(s) \quad \text{and} \quad \tilde{\xi}_{\alpha}(s) = \frac{1}{s} \Phi_\xi(s) \]

(2.6)
are, respectively, obtained by using the Laplace transform of constitutive equation (2.1) for \( \tilde{\xi}(x,s) = \mathcal{L}[H(t)](s) = 1/s \) and \( \tilde{\sigma}(x,s) = \mathcal{L}[H(t)](s) = 1/s \), so that (2.6) used in (2.1) yielded the Laplace transform of constitutive equation (2.1) expressed either in terms of relaxation modulus, or in terms of creep compliance as

\[ \frac{1}{s} \tilde{\sigma}(x,s) = \tilde{\sigma}_{sr}(s) \tilde{\xi}(x,s) \quad \text{or} \quad \frac{1}{s} \tilde{\xi}(x,s) = \tilde{\xi}_{\alpha}(s) \tilde{\sigma}(x,s), \]

(2.7)
providing six equivalent forms of the hereditary constitutive equation: three expressed in terms of relaxation modulus

\[ \int_0^t \sigma(x,t') \, dt' = \sigma_{sr}(t) \ast_t \xi(x,t), \]

(2.8)

\[ \sigma(x,t) = \sigma_{sr}(t) \ast_t \xi(x,t) + \tilde{\sigma}_{sr}(t) \ast_t \xi(x,t) \]

(2.9)
and

\[ \sigma(x,t) = \sigma_{sr}(t) \ast_t \tilde{\xi}(x,t), \]

(2.10)
obtained by the Laplace transform inversion in (2.7)\textsubscript{1} and three expressed in terms of creep compliance

\begin{equation}
\int_0^t \varepsilon(x,t') \, dt' = \varepsilon_{cr}(t) \ast_{t} \sigma(x,t),
\end{equation}

(2.11)

\begin{equation}
\varepsilon(x,t) = \varepsilon_{cr}^{(q)}(x,t) + \dot{\varepsilon}_{cr}(t) \ast_{t} \sigma(x,t)
\end{equation}

(2.12)

and

\begin{equation}
\varepsilon(x,t) = \varepsilon_{cr}(t) \ast_{t} \partial_t \sigma(x,t),
\end{equation}

(2.13)

obtained by the Laplace transform inversion in (2.7)\textsubscript{2}, with \dot{f}(t) = (d/dt)f(t) and by using \((d/dt)(f(t) \ast_{t} g(t)) = f(0)g(t) + \dot{f}(t) \ast_{t} g(t)\), along with the initial conditions on stress and strain (1.11).

Therefore, the equivalent forms of hereditary fractional wave equation expressed in terms of relaxation modulus

\begin{equation}
\rho v_{0}(x) + \sigma_{cr}(t) \ast_{t} \partial_{xx} u(x,t),
\end{equation}

\begin{equation}
\rho v_{0}(x) + \sigma_{cr}(t) \ast_{t} \partial_{xx} u(x,t)
\end{equation}

and

\begin{equation}
\rho v_{0}(x) + \sigma_{cr}(t) \ast_{t} \partial_{xx} u(x,t),
\end{equation}

(2.14)

(2.15)

are, respectively, obtained by differentiation of (2.8), (2.9) and (2.10) with respect to the spatial coordinate and by the subsequent use of equation of motion (1.1) and strain (1.2) in such obtained expressions, including the initial condition (1.10)\textsubscript{2}, while the equivalent forms of hereditary fractional wave equation expressed in terms of creep compliance

\begin{equation}
\rho \dot{\varepsilon}_{cr}(t) \ast_{t} \partial_{tt} u(x,t) = \int_0^t \partial_{xx} u(x,t') \, dt',
\end{equation}

\begin{equation}
\rho \dot{\varepsilon}_{cr}(t) \ast_{t} \partial_{tt} u(x,t) = \partial_{xx} u(x,t)
\end{equation}

and

\begin{equation}
\rho \dot{\varepsilon}_{cr}(t) \ast_{t} \partial_{tt} u(x,t) = \partial_{xx} u(x,t),
\end{equation}

(2.16)

are, respectively, obtained by differentiation of (2.11), (2.12) and (2.13) with respect to the spatial coordinate and by the subsequent use of equation of motion (1.1) and strain (1.2) in such obtained expressions.

3. Relaxation modulus and creep compliance

Starting from the distributed-order viscoelastic model (1.3) having differentiation order below the first order, the conditions for the relaxation modulus to be completely monotonic and simultaneously the creep compliance to be Bernstein function are derived by the means of the Laplace transform method. It is shown that these conditions for relaxation modulus and creep compliance in cases of linear fractional models (1.13) and power-type distributed-order models (1.14) are equivalent to the thermodynamical requirements implying four thermodynamically acceptable cases of linear fractional models (1.13), listed in appendix A, and the power-type model (1.14), with \(E > 0\) and \(0 \leq a \leq b\). These properties of creep compliance and relaxation modulus are proved to be of crucial importance in establishing dissipativity of the hereditary fractional wave equation. Recall that the completely monotonic function is a positive, monotonically decreasing convex function, or more precisely function \(f\) satisfying \((-1)^n f^{(n)}(t) \geq 0\), \(n \in \mathbb{N}_0\), while the Bernstein function is a positive, monotonically increasing, concave function, or more precisely a non-negative function having its first derivative completely monotonic.

The responses in creep and stress relaxation tests of thermodynamically consistent fractional Burgers models (1.4) and (1.5) are examined in [39], where it is found that the requirements for the relaxation modulus to be completely monotonic and creep compliance to be the Bernstein function are more restrictive than the thermodynamical requirements. Conditions guaranteeing the thermodynamical consistency of fractional Burgers models and narrower conditions
guaranteeing monotonicity properties of the relaxation modulus and creep compliance are given in appendix B.

The relaxation modulus, corresponding to the distributed-order viscoelastic model (1.3), takes the form

\[
\sigma_{sr}(t) = \sigma_{sr}^{(e)} + \frac{1}{\pi} \int_{0}^{\infty} \frac{K(\rho)}{|\Phi(\rho e^{i\alpha})|^2} \frac{e^{-\rho t}}{\rho} \, d\rho, \quad \text{with}
\]

\[
\sigma_{sr}^{(e)} = \lim_{t \to \infty} \sigma_{sr}(t) = \lim_{s \to 0} (s\sigma_{sr}(s)) = \lim_{s \to 0} \Phi_{e}(s)
\]

and

\[
K(\rho) = \text{Re} \, \Phi(\rho e^{i\alpha}) \ln \Phi_{e}(\rho e^{i\alpha}) - \text{Im} \, \Phi(\rho e^{i\alpha}) \text{Re} \, \Phi_{e}(\rho e^{i\alpha})
\]

where functions \( \Phi \) and \( \Phi_{e} \) are defined by (2.2), while the creep compliance may be represented either by

\[
\varepsilon_{cr}(t) = \varepsilon_{cr}^{(e)} - \frac{1}{\pi} \int_{0}^{\infty} \frac{K(\rho)}{|\Phi_{e}(\rho e^{i\alpha})|^2} \frac{e^{-\rho t}}{\rho} \, d\rho, \quad \text{with}
\]

\[
\varepsilon_{cr}^{(e)} = \lim_{t \to \infty} \varepsilon_{cr}(t) = \lim_{s \to 0} (s\varepsilon_{cr}(s)) = \lim_{s \to 0} \Phi_{e}(s),
\]

for solid-like materials, or by

\[
\varepsilon_{cr}(t) = \frac{1}{\pi} \int_{0}^{\infty} \frac{K(\rho)}{|\Phi_{e}(\rho e^{i\alpha})|^2} \frac{1 - e^{-\rho t}}{\rho} \, d\rho,
\]

for fluid-like materials, where function \( K \) is given by (3.3). The calculation of relaxation modulus (3.1) and creep compliances (3.4) and (3.6) is performed in appendix C.

The equilibrium modulus \( \sigma_{eq}^{(e)} \) has either zero or finite non-zero value, as seen from table 1, hence the relaxation modulus (3.1) has the same form regardless of the material type, while the equilibrium compliance \( \varepsilon_{eq}^{(e)} \) has either finite value for solid-like materials (power-type distributed-order constitutive equation (1.14) and Cases I (A 1) and II (A 2)), or infinite value for fluid-like materials (Cases III (A 3) and IV (A 4)), as summarized in table 1, implying the need for expressing the creep compliance either in form (3.4), or in the form (3.6).

The function \( K \), calculated by (3.3), for linear fractional models (1.13) and power-type distributed-order model (1.14) takes the respective forms

\[
K(\rho) = -\sum_{i=1}^{n} \sum_{j=1}^{m} a_{i} b_{j} \rho^{a_{i}+b_{j}} \sin \left( \frac{(a_{i} - b_{j})\pi}{2} \right)
\]

and

\[
K(\rho) = E\pi \left( \frac{a\rho + 1}{|\ln(a\rho)| + i\pi^2} \right) \left( \frac{b\rho + 1}{|\ln(b\rho)| + i\pi^2} \right) \ln \frac{b}{a},
\]

obtained by substitution \( s = \rho e^{i\alpha} \) in (2.3). By requiring non-negativity of function \( K \), the conditions on model parameters guaranteeing that the relaxation modulus (3.1) is completely monotonic, while creep compliances (3.4) and (3.6) are Bernstein functions are derived, since the non-negativity of \( K \) implies

\[
\sigma_{sr}(t) \geq 0 \quad \text{and} \quad (-1)^{k} \frac{d^{k}}{dt^{k}} \sigma_{sr}(t) = \frac{1}{\pi} \int_{0}^{\infty} \frac{K(\rho)}{|\Phi_{e}(\rho e^{i\alpha})|^2} \rho^{k-1} e^{-\rho t} \, d\rho \geq 0, \quad k \in \mathbb{N}, \ t > 0,
\]

for relaxation modulus (3.1) and

\[
\varepsilon_{cr}(t) \geq 0 \quad \text{and} \quad (-1)^{k} \frac{d^{k}}{dt^{k}} \varepsilon_{cr}(t) = \frac{1}{\pi} \int_{0}^{\infty} \frac{K(\rho)}{|\Phi_{e}(\rho e^{i\alpha})|^2} \rho^{k} e^{-\rho t} \, d\rho \geq 0, \quad k \in \mathbb{N}_{0}, \ t > 0,
\]

with \( \varepsilon_{cr}(t) = (d/dt) \varepsilon_{cr}(t) \), for the creep compliances (3.4) and (3.6). Note that \( \varepsilon_{cr}(t) \geq 0 \) in the case of (3.4) implies that the creep compliance \( \varepsilon_{cr}(t) \) monotonically increases from \( \varepsilon_{cr}^{(e)} = \lim_{s \to \infty} \Phi_{e}(s) / \Phi_{e}(s) \) to \( \varepsilon_{cr}^{(e)} = \lim_{s \to 0} \Phi_{e}(s) / \Phi_{e}(s) \) for \( t > 0 \), thus being a non-negative function since \( \Phi_{e} \) and \( \Phi_{e} \) are non-negative functions.
By requiring non-negativity of function $K$, given by (3.7), one re-obtains all four cases of linear fractional model (1.13), listed in appendix A along with the explicit forms of corresponding function $K$, since by (3.7) function $K$ is up to the multiplication by the positive function exactly the loss modulus, see [2, eqn. (2.9)], whose non-negativity requirement for all (positive) frequencies yielded four thermodynamically consistent classes of linear fractional models (1.13). In the case of function $K$ given by (3.8), the thermodynamical requirements $E > 0$ and $0 \leq a \leq b$ guarantee non-negativity of function $K$.

The relaxation modulus (3.1) and creep compliances (3.4) and (3.6) are obtained in appendix C under the following assumptions.

(A 1) Functions $\Phi_\sigma$ and $\Phi_\varepsilon$ given by (2.2), except for $s = 0$, have no other branching points and also $\Phi_\sigma(s) \neq 0$ and $\Phi_\varepsilon(s) \neq 0$ for $s \in \mathbb{C}$, implying the non-existence of poles of functions $\Phi_\sigma(s)/\Phi_\sigma(s)$ and $\Phi_\varepsilon(s)/\Phi_\sigma(s)$ in the complex plane.

(A 2) In order to obtain the relaxation modulus (3.1), functions $\Phi_\sigma$ and $\Phi_\varepsilon$ (2.2) should satisfy

$$\frac{1}{R} \left| \frac{\Phi_\sigma(Re^{i(\pi/2)})}{\Phi_\sigma(Re^{i\pi/2})} \right| \to 0 \quad \text{and therefore} \quad \frac{\Phi_\sigma(Re^{i\varphi})}{\Phi_\sigma(Re^{i0})} e^{Rt \cos \varphi} \to 0, \quad \text{as } R \to \infty,$$

for $\varphi \in (-\pi, -\pi/2) \cup (\pi/2, \pi)$.

(A 3) In order to obtain the creep compliance (3.4), functions $\Phi_\sigma$ and $\Phi_\varepsilon$ (2.2) should satisfy

$$\frac{1}{R} \left| \frac{\Phi_\sigma(Re^{i(\pi/2)})}{\Phi_\varepsilon(Re^{i(\pi/2)})} \right| \to 0 \quad \text{and therefore} \quad \frac{\Phi_\sigma(Re^{i\varphi})}{\Phi_\varepsilon(Re^{i0})} e^{Rt \cos \varphi} \to 0, \quad \text{as } R \to \infty,$$

for $\varphi \in (-\pi, -\pi/2) \cup (\pi/2, \pi)$.

(A 4) In order to obtain the creep compliance (3.6), functions $\Phi_\sigma$ and $\Phi_\varepsilon$ (2.2) should satisfy

$$\left| \frac{\Phi_\sigma(Re^{i\varphi})}{\Phi_\varepsilon(Re^{i\varphi})} \right| \to 0 \quad \text{and therefore} \quad R \left| \frac{\Phi_\sigma(Re^{i\varphi})}{\Phi_\varepsilon(Re^{i\varphi})} \right| e^{Rt \cos \varphi} \to 0, \quad \text{or } p_0 = 0, \quad \text{as } R \to \infty,$$

for $\varphi \in (-\pi, -\pi/2) \cup (\pi/2, \pi)$, as well as

$$r \left| \frac{\Phi_\sigma(Re^{i\varphi})}{\Phi_\varepsilon(Re^{i\varphi})} \right| \to 0, \quad \text{as } r \to 0,$$

for $\varphi \in (-\pi, \pi)$.

Assumption (A 1) is satisfied for linear fractional models (1.13) as well as for the power-type model (1.14), due to the fractional differentiation orders belonging to the interval between zero and one. For thermodynamically acceptable cases of linear fractional models (1.13), listed in appendix A, and for the power-type model (1.14) assumption (A 2) is satisfied, since either $|\Phi_\varepsilon(Re^{i0})|/|\Phi_\sigma(Re^{i0})| \sim C$ or $|\Phi_\varepsilon(Re^{i0})|/|\Phi_\sigma(Re^{i0})| \sim C/R^\delta$, as $R \to \infty$, with $C$ being constant and $\delta \in (0, 1)$, see table 2. As already anticipated, constitutive equations corresponding to the solid-like materials (power-type distributed-order constitutive equation (1.14) and Cases I (A 1) and II (A 2)) satisfy assumption (A 3), while constitutive equations corresponding to the fluid-like materials (Cases III (A 3) and IV (A 4)) satisfy assumption (A 4), (table 2).

4. Energy dissipation for hereditary materials

A priori energy estimates stating that the kinetic energy at arbitrary time-instant is less than the initial kinetic energy are derived in order to show the dissipativity of the hereditary fractional wave equations. The material properties at initial time-instant, differing the materials with finite and infinite wave propagation speed, prove to have a decisive role in choosing the form of fractional wave equation and the form of energy estimates as well. In proving dissipativity properties of the hereditary fractional wave equations, the key point is that the relaxation
modulus is a completely monotonic function. Similarly, the energy estimate involving the creep compliance is based on the fact that the creep compliance is the Bernstein function.

(a) Materials having finite glass modulus

The energy estimate for the fractional wave equation expressed in terms of relaxation modulus (2.14) corresponds to materials that have finite glass modulus and thus finite wave speed as well, i.e. materials described by the power-type distributed-order model (1.14), Case I (A 1), and Case III (A 3) of the linear constitutive model (1.13), as well as materials described by the fractional Burgers models VI–VIII (B 6), (B 7), (B 8).

Namely, by multiplying the fractional wave equation (2.14) by \( \partial_t u \) and by subsequent integration with respect to the spatial coordinate along the whole domain \( \mathbb{R} \) and with respect to time over interval \([0, t]\), where \( t > 0 \) is the arbitrary time-instant, one has

\[
\frac{1}{2} \rho \left\| \partial_t u(\cdot, t) \right\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \alpha_s \left\| \partial_x u(\cdot, t) \right\|_{L^2(\mathbb{R})}^2 = \frac{1}{2} \rho \left\| v_0(\cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \int_{\mathbb{R}} (\partial_s \partial_x u(x, t') + \alpha \partial_x u(x, t')) \partial_t u(x, t') \, dx \, dt',
\]

(4.1)

where the change of kinetic energy (per unit square) of a viscoelastic (infinite) body is obtained as

\[
\rho \int_0^t \int_{\mathbb{R}} \partial_x u(x, t') \partial_t u(x, t') \, dx \, dt' = \frac{1}{2} \rho \int_0^t \int_{\mathbb{R}} \partial_t \left( \partial_x u(x, t') \right)^2 \, dx \, dt' = \frac{1}{2} \rho \int_0^t \left\| \partial_t u(\cdot, t') \right\|_{L^2(\mathbb{R})}^2 \, dt',
\]

(4.2)

using the initial condition \((1.10)_2\), while the potential energy (per unit square) of a viscoelastic (infinite) body follows from

\[
\int_0^t \int_{\mathbb{R}} \partial_x u(x, t') \partial_t u(x, t') \, dx \, dt' = \int_0^t \left( \left\| \partial_x u(x, t') \partial_t u(x, t') \right\|_{L^2(\mathbb{R})} \right) \int_{-\infty}^{x-\infty} \partial_x u(x, t') dx \, dt' = \frac{1}{2} \rho \int_0^t \left| \partial_t u(\cdot, t') \right|^2_{L^2(\mathbb{R})} \, dt' = \frac{1}{2} \rho \left\| \partial_t u(\cdot, t) \right\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \left\| \varepsilon(\cdot, 0) \right\|_{L^2(\mathbb{R})}^2 - \frac{1}{2} \left\| \partial_x u(\cdot, t) \right\|_{L^2(\mathbb{R})}^2 (4.3)
\]

using the initial condition \((1.11)_2\) and integration by parts along with the boundary conditions \((1.12)_2\) combined with the constitutive equation (2.9) and strain (1.2) yielding \( \lim_{x \to \pm \infty} \partial_t u(x, t) = 0 \).
The last term on the right-hand side of (4.1) is transformed as
\[
\int_0^t \int_\mathbb{R} (\sigma_{sr}(t') \ast \sigma_{xu}(x, t')) \partial_{xu} u(x, t') \, dx \, dt' \\
= \int_0^t \int_\mathbb{R} \partial_x (\sigma_{sr}(t') \ast \partial_x u(x, t')) \partial_{xu} u(x, t') \, dx \, dt' \\
= \int_0^t \left[ \left( (\sigma_{sr}(t') \ast \partial_x u(x, t')) \partial_{xu} u(x, t') \right)_{t'=0}^\infty - \int_\mathbb{R} (\sigma_{sr}(t') \ast \partial_x u(x, t')) \partial_{xu} u(x, t') \, dx \right] \, dt' \\
= \int_0^t \int_\mathbb{R} \left( (\sigma_{sr}(t') \ast \partial_x u(x, t')) \partial_{xu} u(x, t') \right) \, dx \, dt' \\
= \int_\mathbb{R} \left( (\sigma_{sr}(t') \ast \partial_x u(x, t')) \partial_{xu} u(x, t) \right) \, dx - \int_0^t \int_\mathbb{R} \partial_x ((\sigma_{sr}(t') \ast \partial_x u(x, t')) \partial_{xu} u(x, t') \, dx \, dt'
\]

after the partial integration with respect to spatial coordinate and time, using previously derived boundary condition \(\lim_{x \to \pm \infty} \partial_x u(x, t) = 0\), so that (4.1) reads
\[
\frac{1}{2} \rho \| \partial_t u(\cdot, t) \|^2_{L^2(\mathbb{R})} + \frac{1}{2} \sigma_{sr}(t) \| \partial_x u(\cdot, t) \|^2_{L^2(\mathbb{R})} + \int_0^t \int_\mathbb{R} \partial_x ((\sigma_{sr}(t') \ast \partial_x u(x, t')) \partial_{xu} u(x, t') \, dx \, dt' \\
= \frac{1}{2} \rho \| \partial_t u_0(\cdot) \|^2_{L^2(\mathbb{R})} + \int_\mathbb{R} ((\sigma_{sr}(t') \ast \partial_x u(x, t')) \partial_{xu} u(x, t) \, dx.
\]

(4.4)

Using lemma 1.7.2 in [40], see also [41, eq. (9)], stating that
\[
\int_0^t \int_\mathbb{R} \partial_x (k(t') \ast \partial_x u(x, t')) \, dx \, dt' \geq \frac{1}{2} k(t) \| \partial_x u(\cdot, t) \|^2_{L^2(\mathbb{R})} + \int_0^t k(t') \| u(\cdot, t') \|^2_{L^2(\mathbb{R})} \, dt',
\]

provided that \(k\) is a positive decreasing function for \(t > 0\), the third term on the left-hand side of (4.4) is estimated by
\[
\int_0^t \int_\mathbb{R} \partial_x ((\sigma_{sr}(t') \ast \partial_x u(x, t')) \partial_{xu} u(x, t') \, dx \, dt' \\
\geq \frac{1}{2} (\sigma_{sr}(t) \ast \| \partial_x u(\cdot, t) \|^2_{L^2(\mathbb{R})} + \frac{1}{2} \int_0^t (\sigma_{sr}(t')) \| \partial_x u(\cdot, t') \|^2_{L^2(\mathbb{R})} \, dt',
\]

since \(-\sigma_{sr}\) is completely monotonic and thus positive decreasing function for \(t > 0\), while the second term on the right-hand side of (4.4) is estimated by
\[
\int_\mathbb{R} ((\sigma_{sr}(t') \ast \partial_x u(x, t)) \partial_{xu} u(x, t) \, dx = \int_0^t (\sigma_{sr}(t - t')) \int_\mathbb{R} \partial_x u(x, t') \partial_{xu} u(x, t) \, dx \, dt' \\
\leq \int_0^t (\sigma_{sr}(t - t')) \int_\mathbb{R} \left( \frac{\partial_x u(x, t')}{2} + \frac{\partial_x u(x, t)}{2} \right) \, dx \, dt' \\
\leq \frac{1}{2} (\sigma_{sr}(t) \ast \| \partial_x u(\cdot, t) \|^2_{L^2(\mathbb{R})} + \frac{1}{2} \| \sigma_{sr}(t') - \sigma_{sr}(t) \| \partial_x u(\cdot, t') \|^2_{L^2(\mathbb{R})} \, dt' \\
\]

transforming (4.4) into
\[
\frac{1}{2} \rho \| \partial_t u(\cdot, t) \|^2_{L^2(\mathbb{R})} + \frac{1}{2} \sigma_{sr}(t) \| \partial_x u(\cdot, t) \|^2_{L^2(\mathbb{R})} + \frac{1}{2} \int_0^t (\sigma_{sr}(t') \ast \partial_x u(\cdot, t')) \| \partial_x u(\cdot, t') \|^2_{L^2(\mathbb{R})} \, dt' \leq \frac{1}{2} \rho \| \partial_0 u(\cdot) \|^2_{L^2(\mathbb{R})}.
\]

(4.6)

The energy estimate (4.6) clearly indicates the dissipativity of fractional wave equation (2.14), since the kinetic energy at any time-instant \(t > 0\) is less than the kinetic energy at initial time-instant \(t = 0\), due to the positive terms on the left-hand side of energy estimate (4.6).
(b) Materials having infinite glass modulus

The energy estimate for the fractional wave equation expressed in terms of relaxation modulus (2.15) correspond to materials that have an infinite glass modulus and thus infinite wave speed as well, i.e. materials described by Case II (A 2) and Case IV (A 4) of the linear constitutive model (1.13), as well as materials described by the fractional Burgers models I–V (B 1), (B 2), (B 3), (B 4), (B 5).

Namely, by multiplying the fractional wave equation (2.15) by $\partial_t u$ and by subsequent integration with respect to the spatial coordinate along the whole domain $\mathbb{R}$ and with respect to time over interval $[0, t]$, one has

$$
\frac{1}{2} \rho \| \partial_t u(\cdot, t) \|^2_{L^2(\mathbb{R})} = \frac{1}{2} \rho \| v_0(\cdot) \|^2_{L^2(\mathbb{R})} + \int_0^t \int_{\mathbb{R}} (\sigma_{sr}(t') \ast \partial_{\nu \chi} u(x, t')) \partial_t u(x, t') \, dx \, dt',
$$

(4.7)

where the change of kinetic energy is obtained according to (4.2). The second term on the right-hand side of (4.7) transforms into

$$
\int_0^t \int_{\mathbb{R}} (\sigma_{sr}(t') \ast \partial_{\nu \chi} u(x, t')) \partial_t u(x, t') \, dx \, dt' = \int_0^t \int_{\mathbb{R}} \partial_\nu (\sigma_{sr}(t') \ast \partial_{\nu \chi} u(x, t')) \partial_t u(x, t') \, dx \, dt' = \int_0^t \int_{\mathbb{R}} \left( (\sigma_{sr}(t') \ast \partial_{\nu \chi} u(x, t')) \partial_t u(x, t') \right|_{x \rightarrow -\infty}^{x \rightarrow +\infty} \, dx \, dt' = - \int_0^t \int_{\mathbb{R}} (\sigma_{sr}(t') \ast \partial_{\nu \chi} u(x, t')) \partial_{\nu \chi} u(x, t') \, dx \, dt',
$$

after the partial integration with respect to spatial coordinate, using the boundary condition (1.12) rendering $\lim_{x \rightarrow \pm \infty} \sigma_{sr}(t) \ast \partial_{\nu \chi} u(x, t) = 0$, obtained by combining the constitutive equation (2.10) and strain (1.2), so that (4.7) reads

$$
\frac{1}{2} \rho \| \partial_t u(\cdot, t) \|^2_{L^2(\mathbb{R})} + \int_0^t \int_{\mathbb{R}} (\sigma_{sr}(t') \ast \partial_{\nu \chi} u(x, t')) \partial_{\nu \chi} u(x, t') \, dx \, dt' = \frac{1}{2} \rho \| v_0(\cdot) \|^2_{L^2(\mathbb{R})},
$$

(4.8)

The energy estimate (4.8) clearly indicates the dissipativity of fractional wave equation (2.15), since the kinetic energy at any time-instant $t > 0$ is less than the kinetic energy at initial time-instant $t = 0$, due to the positivity of the second term on the right-hand side of (4.8), thanks to the relaxation modulus $\sigma_{sr}$ being completely monotonic and consequently of the positive type kernels satisfying

$$
\int_0^t \int_0^{t'} \sigma_{sr}(t' - t'') \partial_{\nu \chi} u(x, t'') \partial_{\nu \chi} u(x, t') \, dt'' \, dt' \geq 0,
$$

as also used in [15].

(c) Energy estimates using fractional wave equation (2.16)

The energy estimate for the fractional wave equation expressed in terms of creep compliance (2.16) corresponds to all materials described by the power-type distributed-order model (1.14), as well as to all materials described by the fractional Burgers models I–V (B 1), (B 2), (B 3), (B 4), (B 5).

Multiplying the fractional wave equation (2.16) by $\partial_t u$ and by subsequent integration with respect to the spatial coordinate along the whole domain $\mathbb{R}$ and with respect to time over interval $[0, t]$, one has

$$
\frac{1}{2} \rho \varepsilon_t (\partial_t u(\cdot, t)) \| \partial_t u(\cdot, t) \|^2_{L^2(\mathbb{R})} + \frac{1}{2} \| \partial_\nu u(\cdot, t) \|^2_{L^2(\mathbb{R})} + \rho \int_0^t \int_{\mathbb{R}} (\varepsilon_{cr}(t') \ast \partial_{\nu \chi} u(x, t')) \partial_t u(x, t') \, dx \, dt' = \frac{1}{2} \rho \varepsilon_t (\partial_t u(\cdot, t)) \| v_0(\cdot) \|^2_{L^2(\mathbb{R})},
$$

(4.9)
where the changes of kinetic and potential energy are obtained according to (4.2) and (4.3), respectively. The last term on the left-hand side of (4.9) is estimated as

\[
\int_0^t \int_\mathbb{R} (\dot{e}_{cr}(t') \ast \partial_t u(x,t')) \partial_t u(x,t') \, dx \, dt' = \int_0^t \int_\mathbb{R} (\partial_t (\dot{e}_{cr}(t') \ast \partial_t u(x,t'))) - v_0(x) \dot{e}_{cr}(t') \partial_t u(x,t') \, dx \, dt'
\]

using \( f(t) \ast g(t) = (d/dt)(f(t) \ast g(t)) - f(t)g(0) \), transforming (4.9) into

\[
\frac{1}{2} \rho \dot{e}_{cr}(t) \| \partial_t u(\cdot, t) \|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \| \partial_t u(\cdot, t) \|_{L^2(\mathbb{R})}^2 + \rho \int_0^t \int_\mathbb{R} \partial_t (\dot{e}_{cr}(t') \ast \partial_t u(x,t')) \partial_t u(x,t') \, dx \, dt'
\]

The last term on the left-hand side of (4.10) is estimated as

\[
\int_0^t \int_\mathbb{R} \partial_t (\dot{e}_{cr}(t') \ast \partial_t u(x,t')) \partial_t u(x,t') \, dx \, dt'
\]

according to (4.5), since \( \dot{e}_{cr} \) is completely monotonic, while the second term on the right-hand side of (4.10) is estimated by

\[
\int_0^t \int_\mathbb{R} v_0(x) \dot{e}_{cr}(t') \partial_t u(x,t') \, dx \, dt' = \int_0^t \dot{e}_{cr}(t') \int_\mathbb{R} v_0(x) \partial_t u(x,t') \, dx \, dt'
\]

\[
\leq \int_0^t \dot{e}_{cr}(t') \int_\mathbb{R} \left( \frac{(v_0(x))^2}{2} + \frac{\partial_t u(x,t')}{2} \right) \, dx \, dt'
\]

\[
\leq \frac{1}{2} (\dot{e}_{cr}(t) - \dot{e}_{cr}^{(2)}) \| v_0(\cdot) \|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \int_0^t \dot{e}_{cr}(t') \| \partial_t u(\cdot, t') \|_{L^2(\mathbb{R})}^2 \, dt',
\]

transforming (4.10) into

\[
0 \leq \frac{1}{2} \rho \dot{e}_{cr}(t) \| \partial_t u(\cdot, t) \|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \rho \dot{e}_{cr}(t) \ast \| \partial_t u(\cdot, t) \|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \| \partial_t u(\cdot, t) \|_{L^2(\mathbb{R})}^2 \leq \frac{1}{2} \rho \dot{e}_{cr}(t) \| v_0(\cdot) \|_{L^2(\mathbb{R})}^2
\]

or equivalently to

\[
0 \leq \frac{1}{2} \rho \frac{1}{\dot{e}_{cr}(t)} \partial_t (\dot{e}_{cr}(t) \ast \| \partial_t u(\cdot, t) \|_{L^2(\mathbb{R})}^2) + \frac{1}{2} \dot{e}_{cr}(t) \| \partial_t u(\cdot, t) \|_{L^2(\mathbb{R})}^2 \leq \frac{1}{2} \rho \| v_0(\cdot) \|_{L^2(\mathbb{R})}^2
\]

using \( f(t) \ast g(t) = (d/dt)(f(t) \ast g(t)) - f(t)g(0) \).

The energy estimate (4.11) is not appropriate for showing dissipativity of the fractional wave equation (2.16), since one cannot identify the kinetic energy on the left-hand-side of (4.11), although it figures on the right-hand side of (4.11).

## 5. Energy conservation for non-local materials

A priori energy estimates yield the conservation law for both of the examined non-local fractional wave equations, stating that the sum of kinetic energy and non-local potential energy does not change in time. Non-local potential energy is proportional to the square of fractional strain, obtained by convoluting the classical strain with the constitutive model dependent non-locality kernel, i.e. non-local potential energy in a particular point depends on the square of strain in all other points weighted by the non-locality kernel.
Eliminating stress and strain from the equation of motion (1.1), the non-local Hooke Law (1.6) and strain (1.2), the non-local Hooke-type wave equation is obtained in the form

$$\rho \partial_t u(x, t) = \frac{E}{\ell^{1-\alpha}} \frac{|x|^{-\alpha}}{2\Gamma(1-\alpha)} \partial_\alpha \partial_x u(x, t), \quad \alpha \in (0, 1),$$

(5.1)

transforming into

$$\rho \partial_t \hat{u}(\xi, t) = -E \frac{\sin \frac{\alpha \pi}{\ell^{1-\alpha}}}{\alpha^{1+\alpha}} |\xi|^{1+\alpha} \hat{u}(\xi, t),$$

(5.2)

after application of the Fourier transform with respect to the spatial coordinate

$$\hat{f}(\xi) = \mathcal{F}[f(x)](\xi) = \int \mathcal{F} f(x) e^{-i\xi x} \, dx, \quad \xi \in \mathbb{R},$$

where $\mathcal{F}[|x|^{-\alpha}/2\Gamma(1-\alpha)](\xi) = \sin(\alpha \pi/2)/|\xi|^{1-\alpha}$ is used along with other well-known properties of the Fourier transform.

Multiplying the non-local Hooke-type wave equation in Fourier domain (5.2) with $\partial_t \hat{u}$ and by subsequent integration over the whole domain $\mathbb{R}$, one obtains

$$\partial_t \left( \frac{1}{2} \rho \|\partial_t \hat{u}(\xi, t)\|^2_{L^2(\mathbb{R})} + \frac{1}{2} E \frac{\sin \frac{\alpha \pi}{\ell^{1-\alpha}}}{\alpha^{1+\alpha}} \|\xi\|^{(1+\alpha)/2} \hat{u}(\xi, t)\|^2_{L^2(\mathbb{R})} \right) = 0,$$

(5.3)

yielding the conservation law

$$\partial_t \left( \frac{1}{2} \rho \|\partial_t u(\xi, t)\|^2_{L^2(\mathbb{R})} + \frac{1}{2} E \frac{\sin \frac{\alpha \pi}{\ell^{1-\alpha}}}{\alpha^{1+\alpha}} \|(-\Delta)^{(1+\alpha)/4} u(\xi, t)\|^2_{L^2(\mathbb{R})} \right) = 0,$$

(5.4)

i.e.,

$$\frac{1}{2} \rho \|\partial_t u(\xi, t)\|^2_{L^2(\mathbb{R})} + \frac{1}{2} E \frac{\sin \frac{\alpha \pi}{\ell^{1-\alpha}}}{\alpha^{1+\alpha}} \|(-\Delta)^{(1+\alpha)/4} u(\xi, t)\|^2_{L^2(\mathbb{R})} = \text{const},$$

by the Parseval identity $\|f\|^2_{L^2(\mathbb{R})} = \|\mathcal{F} f\|^2_{L^2(\mathbb{R})}$, as well as by the Fourier transform of fractional Laplacian (in one dimension) $\mathcal{F}[(-\Delta)^{s} f(x)](\xi) = |\xi|^{2s} \hat{f}(\xi)$, with $s \in (0, 1)$, since $(1 + \alpha)/2 \in (1/2, 1)$. The fractional strain, being proportional to $(-\Delta)^{(1+\alpha)/4} u$ in (5.4), has a lower differentiation order than the classical strain $\partial_t u$, since $(1 + \alpha)/4 \in (1/4, 1/2)$.

However, the conservation law (5.4) may also take another form

$$\partial_t \left( \frac{1}{2} \rho \|\partial_t u(\xi, t)\|^2_{L^2(\mathbb{R})} + \frac{1}{2} E \frac{\sin \frac{\alpha \pi}{\ell^{1-\alpha}}}{\alpha^{1+\alpha}} \left\| \frac{\text{sgn} x}{|x|^{(1+\alpha)/2}} \partial_x u(x, t) \right\|^2_{L^2(\mathbb{R})} \right) = 0,$$

(5.5)

i.e.

$$\frac{1}{2} \rho \|\partial_t u(\xi, t)\|^2_{L^2(\mathbb{R})} + \frac{1}{2} E \frac{\sin \frac{\alpha \pi}{\ell^{1-\alpha}}}{\alpha^{1+\alpha}} \left\| \frac{\text{sgn} x}{|x|^{(1+\alpha)/2}} \partial_x u(x, t) \right\|^2_{L^2(\mathbb{R})} = \text{const},$$

where

$$c_\alpha = \frac{\sin \frac{\alpha \pi}{\ell^{1-\alpha}}}{\alpha^{1+\alpha}}$$

is a positive constant, if the term $|\xi|^{(1+\alpha)/2} \hat{u}(\xi, t)$ in (5.3) is rewritten as

$$|\xi|^{(1+\alpha)/2} \hat{u}(\xi, t) = -i \frac{\text{sgn} \xi}{|\xi|^{(1+\alpha)/2}} (\xi \hat{u}(\xi, t)) = \mathcal{F} \left[ \frac{\text{sgn} x}{|x|^{(1+\alpha)/2}} \right](\xi) \frac{2\Gamma \left( 1 - \frac{1+\alpha}{2} \right)}{\cos \left( \frac{(1+\alpha)\pi}{4} \right)} \mathcal{F}[\partial_x u(x, t)](\xi),$$

where the Fourier transform $\mathcal{F}[\text{sgn} x/|x|^\beta](\xi) = -2i \Gamma(1 - \beta) \cos(\beta \pi/2) (\text{sgn} \xi/|\xi|^{1-\beta})$, with $\beta \in (0, 1)$, is used.

The energy estimates (5.4) and (5.5) clearly indicate the energy conservation property of the non-local Hooke-type wave equation (5.1), if the potential energy is reinterpreted to be proportional to the square of fractional strain, expressed either in terms of fractional Laplacian, or in terms of classical strain convoluted by the non-locality kernel of power type.
(b) Materials described by the fractional Eringen model

Fractional Eringen wave equation

$$\rho \partial_t u(x, t) = E H_\alpha(x) \ast_x \partial_x u(x, t), \quad \alpha \in (1, 3),$$

with

$$H_\alpha(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos(\xi x)}{1 + (\xi |\xi|)^\alpha \left| \cos \frac{\alpha \pi}{2} \right|} \, d\xi = \mathcal{F}^{-1} \left[ \frac{1}{1 + (\xi |\xi|)^\alpha \left| \cos \frac{\alpha \pi}{2} \right|} \right](x), \quad (5.6)$$

is found as the inverse Fourier transform of

$$\rho \partial_t \hat{u}(\xi, t) = -\frac{\xi^2}{1 + (\xi |\xi|)^\alpha \left| \cos \frac{\alpha \pi}{2} \right|} \hat{u}(\xi, t), \quad (5.7)$$

obtained by eliminating $\hat{\sigma}$ and $\hat{\epsilon}$ from the system of equations in the Fourier domain

$$i\xi \hat{\sigma}(\xi, t) = \rho \partial_t \hat{u}(\xi, t), \quad i\xi \hat{\epsilon}(\xi, t) = i\xi \hat{u}(\xi, t)$$

and

$$\left( 1 + (\xi |\xi|)^\alpha \left| \cos \frac{\alpha \pi}{2} \right| \right) \hat{\sigma}(\xi, t) = E \hat{\epsilon}(\xi, t),$$

respectively, consisting of the Fourier transforms of equation of motion (1.1), strain (1.2) and fractional Eringen model (1.7), where the Fourier transform of both (1.8) and (1.9), yielding $\mathcal{F}[D_0^\alpha f(x)](\xi) = -|\xi|^\alpha \left| \cos (\alpha \pi / 2) \right| \hat{f}(\xi)$, is used.

Multiplying the fractional Eringen wave equation in Fourier domain (5.7) with $\partial_t \hat{u}$ and by subsequent integration over the whole domain $\mathbb{R}$, one obtains

$$\partial_t \left( \frac{1}{2} \rho \left\| \partial_t \hat{u}(\xi, t) \right\|_{L^2(\mathbb{R})}^2 \right) + \frac{1}{2} E \left\| \hat{h}_\alpha(\xi) \ast \hat{u}(\xi, t) \right\|_{L^2(\mathbb{R})}^2 = 0, \quad (5.8)$$

$$\hat{h}_\alpha(\xi) = -i \frac{\text{sgn} \xi}{\sqrt{1 + (\xi |\xi|)^\alpha \left| \cos \frac{\alpha \pi}{2} \right|}},$$

so that the conservation law

$$\partial_t \left( \frac{1}{2} \rho \left\| \partial_t u(x, t) \right\|_{L^2(\mathbb{R})}^2 \right) + \frac{1}{2} E \left\| h_\alpha(x) \ast_x \partial_x u(x, t) \right\|_{L^2(\mathbb{R})}^2 = 0,$$

i.e.

$$\frac{1}{2} \rho \left\| \partial_t u(x, t) \right\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} E \left\| h_\alpha(x) \ast_x \partial_x u(x, t) \right\|_{L^2(\mathbb{R})}^2 = \text{const.} \quad (5.9)$$

follows from (5.8) by the Parseval identity and inverse Fourier transform of $\hat{h}_\alpha$, given by

$$h_\alpha(x) = \frac{1}{\pi} \int_0^\infty \frac{\sin(\xi x)}{\sqrt{1 + (\xi |\xi|)^\alpha \left| \cos \frac{\alpha \pi}{2} \right|}} \, d\xi.$$

The energy estimate (5.9) clearly indicates the energy conservation property of the fractional Eringen wave equation (5.6), if the potential energy is again reinterpreted to be proportional to the square of fractional strain, expressed in terms of classical strain convoluted by the non-locality kernel $h_\alpha$.

6. Conclusion

Energy dissipation and conservation properties of fractional wave equations, respectively, corresponding to hereditary and non-local materials, are considered by employing the method of $a$ priori energy estimates. More precisely, in the case of hereditary fractional wave equations it is obtained that the kinetic energy at an arbitrary time-instant is less than the initial kinetic energy, while in the case of non-local fractional wave equations it is obtained that the sum of kinetic energy and non-local potential energy does not change in time, with the non-local potential...
energy being proportional to the square of fractional strain, obtained by convoluting the classical strain with the constitutive model dependent non-locality kernel.

Hereditary fractional models of viscoelastic material having differentiation orders below the first order are represented by the distributed-order viscoelastic model (1.3), more precisely by the linear fractional model (1.13) and power-type distributed-order model (1.14), while thermodynamically consistent fractional Burgers models (1.4) and (1.5) represent constitutive models having differentiation orders up to the second order. In order to formulate the hereditary wave equation, in addition to the equation of motion (1.1) and strain (1.2), the hereditary constitutive model expressed in terms of material response in stress relaxation and creep test is used, leading to six equivalent forms of the hereditary wave equation, three of them expressed in terms of relaxation modulus and the other three expressed in terms of creep compliance. It is found that the hereditary wave equation expressed in terms of relaxation modulus, either as (2.14) for materials having finite glass modulus and thus finite wave speed as well, or as (2.15) for materials having infinite glass modulus and thus infinite wave speed as well, leads to the physically meaningful energy estimates either (4.6) or (4.8) corresponding to energy dissipation. Therefore, the form of energy estimate depends on the material properties at the initial time-instant defining the wave propagation speed, rather than the material properties for large time differing the solid- and fluid-like materials. The energy estimate (4.11), implied by the hereditary wave equation expressed in terms of creep compliance (2.16), did not prove to have physical meaning.

The monotonicity property of the relaxation modulus, being a completely monotonic function, and creep compliance, being a Bernstein function, are the key points in proving dissipativity properties of the hereditary fractional wave equations. It is shown that the requirement for relaxation modulus to be completely monotonic, i.e. creep compliance to be Bernstein function, is equivalent with the thermodynamical conditions for linear fractional model (1.13) and power-type distributed-order model (1.14), while in the case of the fractional Burgers models these monotonicity requirements are more restrictive than the thermodynamical requirements, as found in [39].

Non-local Hooke and Eringen fractional wave equations, given by (5.1) and (5.6), are respectively obtained by coupling non-local constitutive models of Hooke- and Eringen-type, (1.6) and (1.7), with the equation of motion (1.1) and strain (1.2). A priori energy estimates (5.4) and (5.5) for non-local Hooke and energy estimate (5.9) for the fractional Eringen wave equation imply the energy conservation, with the reinterpreted notion of the potential energy, being in a particular point dependent on the square of strain in all other points weighted by the model-dependent non-locality kernel. In particular, in the energy estimates (5.4) the non-local potential energy is proportional to the fractional strain, represented by the action of fractional Laplacian on the displacement.

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Appendix A. Thermodynamically consistent linear fractional models

Linear fractional model (1.13), containing fractional differentiation orders below the first order, reduces to the following four thermodynamically consistent model classes, which are listed below, along with corresponding thermodynamical constraints and explicit forms of function $K$, given by (3.3).
Case I: Models having the same number and orders of fractional derivatives of stress and strain

\[
\sum_{i=1}^{n} a_i \partial_t^{\alpha_i} \sigma(t) = \sum_{i=1}^{n} b_i \partial_t^{\beta_i} \epsilon(t),
\]

\[
0 \leq \alpha_1 < \cdots < \alpha_n < 1 \quad \text{and} \quad \frac{a_1}{b_1} \geq \frac{a_2}{b_2} \geq \cdots \geq \frac{a_n}{b_n} \geq 0
\]

and

\[
K(\rho) = - \sum_{i=1}^{n} \sum_{j=i+1}^{n} (a_i b_j - a_j b_i) \rho^{\alpha_i + \alpha_j} \sin \frac{(\alpha_i - \alpha_j)\pi}{2};
\]

Case II: Models having some extra derivatives of strain in addition to the same number and orders of fractional derivatives of stress and strain

\[
\sum_{i=1}^{n} a_i \partial_t^{\alpha_i} \sigma(t) = \sum_{i=1}^{n} b_i \partial_t^{\alpha_i} \epsilon(t) + \sum_{i=n+1}^{m} b_i \partial_t^{\beta_i} \epsilon(t),
\]

\[
0 \leq \alpha_1 < \cdots < \alpha_n < \beta_{n+1} < \cdots < \beta_m < 1 \quad \text{and} \quad \frac{a_1}{b_1} \geq \frac{a_2}{b_2} \geq \cdots \geq \frac{a_n}{b_n} \geq 0
\]

and

\[
K(\rho) = - \sum_{i=1}^{n} \sum_{j=i+1}^{n} (a_i b_j - a_j b_i) \rho^{\alpha_i + \alpha_j} \sin \frac{(\alpha_i - \alpha_j)\pi}{2} - \sum_{i=1}^{n} \sum_{j=n+1}^{m} a_i b_j \rho^{\alpha_i + \beta_j} \sin \frac{(\alpha_i - \beta_j)\pi}{2};
\]

Case III: Models having some extra derivatives of stress in addition to the same number and orders of fractional derivatives of stress and strain

\[
\sum_{i=1}^{n-m} a_i \partial_t^{\alpha_i} \sigma(t) + \sum_{i=n-m+1}^{m} a_i \partial_t^{\alpha_i} \sigma(t) = \sum_{j=1}^{m} b_j \partial_t^{\alpha_{n+m-j}} \epsilon(t),
\]

\[
0 \leq \alpha_1 < \cdots < \alpha_{n-m+1} < \cdots < \alpha_n < 1 \quad \text{and} \quad \frac{a_{n-m+1}}{b_1} \geq \frac{a_{n-m+2}}{b_2} \geq \cdots \geq \frac{a_n}{b_m} \geq 0,
\]

\[
K(\rho) = - \sum_{i=1}^{n-m} \sum_{j=1}^{m} a_i b_j \rho^{\alpha_i + \alpha_{n-m-j}} \sin \frac{(\alpha_i - \alpha_{n-m-j})\pi}{2}
\]

and

\[
- \frac{1}{2} \sum_{j=1}^{m} \sum_{i=n-m+j+1}^{n} (a_i b_j - a_{n-m+j} b_{i-(n-m)}) \rho^{\alpha_i + \alpha_{n-m-j}} \sin \frac{(\alpha_i - \alpha_{n-m-j})\pi}{2};
\]

Case IV: Models having all orders of fractional derivatives of stress and strain different

\[
\sum_{i=1}^{n} a_i \partial_t^{\alpha_i} \sigma(t) = \sum_{i=1}^{m} b_i \partial_t^{\beta_i} \epsilon(t),
\]

\[
0 \leq \alpha_1 < \cdots < \alpha_n < \beta_1 < \cdots < \beta_m < 1, \quad \text{with} \quad \alpha_i \neq \beta_j, \quad \text{for} \ i \neq j
\]

and

\[
K(\rho) = - \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \rho^{\alpha_i + \beta_j} \sin \frac{(\alpha_i - \beta_j)\pi}{2}.
\]

Appendix B. Fractional Burgers models

Thermodynamically consistent fractional Burgers models are listed below, along with corresponding thermodynamical constraints, as well as with the constraints on monotonicity of relaxation modulus and creep compliance, narrowing down the thermodynamical requirements and guaranteeing that relaxation modulus is completely monotonic, while creep compliance is Bernstein function.
Model I:

\[(1 + a_1 D_i^\alpha + a_2 D_i^\beta + a_3 D_i^{2\alpha}) \sigma(t) = (b_1 D_i^\mu + b_2 D_i^{\mu+\alpha}) \varepsilon(t),\]

\[0 \leq \alpha \leq \beta \leq \gamma \leq \mu \leq 1, \quad 1 \leq \mu + \eta \leq 1 + \alpha, \quad \frac{b_2}{b_1} \leq a_1 \left| \frac{\sin \left(\frac{(\mu-\eta)\pi}{2}\right)}{\cos \left(\frac{(\mu+\eta)\pi}{2}\right)} \right|, \]

and

\[\frac{b_2}{b_1} \leq a_1 \left| \frac{\sin \left(\frac{(\mu-\eta)\pi}{2}\right)}{\cos \left(\frac{(\mu+\eta)\pi}{2}\right)} \right|, \]

with \( (\eta, \iota) \in \{(\alpha, 1), (\beta, 2), (\gamma, 3)\}; \)

Model II:

\[(1 + a_1 D_i^\alpha + a_2 D_i^\beta + a_3 D_i^{2\alpha}) \sigma(t) = (b_1 D_i^\mu + b_2 D_i^{\mu+\alpha}) \varepsilon(t),\]

\[\frac{1}{2} \leq \alpha \leq \beta \leq \mu \leq 1, \quad \frac{a_3}{a_1} \left| \frac{\sin \left(\frac{(\mu-\beta)\pi}{2}\right)}{\sin \left(\frac{\mu\pi}{2}\right)} \right| \leq \frac{b_2}{b_1} \leq a_1 \left| \frac{\cos \left(\frac{(\mu-\alpha)\pi}{2}\right)}{\cos \left(\frac{(\mu+\alpha)\pi}{2}\right)} \right|, \]

and

\[\frac{a_3}{a_1} \left| \frac{\sin \left(\frac{(\mu-\beta)\pi}{2}\right)}{\sin \left(\frac{\mu\pi}{2}\right)} \right| \leq \frac{b_2}{b_1} \leq a_1 \left| \frac{\cos \left(\frac{(\mu-\alpha)\pi}{2}\right)}{\cos \left(\frac{(\mu+\alpha)\pi}{2}\right)} \right|; \]

Model III:

\[(1 + a_1 D_i^\alpha + a_2 D_i^\beta + a_3 D_i^{2\alpha}) \sigma(t) = (b_1 D_i^\mu + b_2 D_i^{\mu+\alpha}) \varepsilon(t),\]

\[0 \leq \alpha \leq \beta \leq \mu \leq 1, \quad \alpha + \beta \geq 1, \quad \frac{a_3}{a_2} \left| \frac{\sin \left(\frac{(\mu-\beta-\alpha)\pi}{2}\right)}{\sin \left(\frac{(\mu+\beta+\alpha)\pi}{2}\right)} \right| \leq \frac{b_2}{b_1} \leq a_1 \left| \frac{\cos \left(\frac{(\mu-\alpha)\pi}{2}\right)}{\cos \left(\frac{(\mu+\alpha)\pi}{2}\right)} \right|, \]

and

\[\frac{a_3}{a_2} \left| \frac{\sin \left(\frac{(\mu-\beta-\alpha)\pi}{2}\right)}{\sin \left(\frac{(\mu+\beta+\alpha)\pi}{2}\right)} \right| \leq \frac{b_2}{b_1} \leq a_1 \left| \frac{\cos \left(\frac{(\mu-\alpha)\pi}{2}\right)}{\cos \left(\frac{(\mu+\alpha)\pi}{2}\right)} \right|; \]

Model IV:

\[(1 + a_1 D_i^\alpha + a_2 D_i^\beta + a_3 D_i^{2\alpha}) \sigma(t) = (b_1 D_i^\mu + b_2 D_i^{\mu+\beta}) \varepsilon(t),\]

\[0 \leq \alpha \leq \beta \leq \mu \leq 1, \quad 1 - \alpha \leq \beta \leq 1 - (\mu - \alpha), \quad \frac{a_3}{a_1} \left| \frac{\sin \left(\frac{(\mu-\beta-\alpha)\pi}{2}\right)}{\sin \left(\frac{(\mu-\alpha+\beta)\pi}{2}\right)} \right| \leq \frac{b_2}{b_1} \leq a_2 \left| \frac{\cos \left(\frac{(\mu-\beta)\pi}{2}\right)}{\cos \left(\frac{(\mu+\beta)\pi}{2}\right)} \right|, \]

and

\[\frac{a_3}{a_1} \left| \frac{\sin \left(\frac{(\mu-\beta-\alpha)\pi}{2}\right)}{\sin \left(\frac{(\mu-\alpha+\beta)\pi}{2}\right)} \right| \leq \frac{b_2}{b_1} \leq a_2 \left| \frac{\cos \left(\frac{(\mu-\beta)\pi}{2}\right)}{\cos \left(\frac{(\mu+\beta)\pi}{2}\right)} \right|; \]

Model V:

\[(1 + a_1 D_i^\alpha + a_2 D_i^\beta + a_3 D_i^{2\beta}) \sigma(t) = (b_1 D_i^\mu + b_2 D_i^{\mu+\beta}) \varepsilon(t),\]

\[0 \leq \alpha \leq \beta \leq \mu \leq 1, \quad \frac{1}{2} \leq \beta \leq 1 - (\mu - \alpha), \quad \frac{a_3}{a_2} \left| \frac{\sin \left(\frac{(\mu-\beta)\pi}{2}\right)}{\sin \left(\frac{\mu\pi}{2}\right)} \right| \leq \frac{b_2}{b_1} \leq a_2 \left| \frac{\cos \left(\frac{(\mu-\beta)\pi}{2}\right)}{\cos \left(\frac{(\mu+\beta)\pi}{2}\right)} \right|, \]

and

\[\frac{a_3}{a_2} \left| \frac{\sin \left(\frac{(\mu-\beta)\pi}{2}\right)}{\sin \left(\frac{\mu\pi}{2}\right)} \right| \leq \frac{b_2}{b_1} \leq a_2 \left| \frac{\cos \left(\frac{(\mu-\beta)\pi}{2}\right)}{\cos \left(\frac{(\mu+\beta)\pi}{2}\right)} \right|; \]
Model VI:

\[(1 + a_1D_t^\alpha + a_2D_t^\beta + a_3D_t^{\alpha+\beta}) \sigma(t) = (b_1D_t^\alpha + b_2D_t^{2\beta}) \varepsilon(t), \quad (B6)\]

\[0 \leq \alpha \leq \beta \leq 1, \quad \alpha + \beta \geq 1, \quad \frac{a_3}{a_2} \leq \frac{b_2}{b_1} \leq a_1 \left| \frac{\cos \left( \frac{\beta - \alpha}{2} \pi \right)}{\cos \left( \frac{\beta + \alpha}{2} \pi \right)} \right|, \]

and

\[\frac{a_3}{a_2} \leq \frac{b_2}{b_1} \leq a_1 \left| \frac{\sin \left( \frac{\beta - \alpha}{2} \pi \right)}{\sin \left( \frac{\beta + \alpha}{2} \pi \right)} \right| \leq a_1 \left| \frac{\cos \left( \frac{\beta - \alpha}{2} \pi \right)}{\cos \left( \frac{\beta + \alpha}{2} \pi \right)} \right|; \]

Model VII:

\[(1 + a_1D_t^\alpha + a_2D_t^\beta + a_3D_t^{2\beta}) \sigma(t) = (b_1D_t^\alpha + b_2D_t^{2\beta}) \varepsilon(t), \quad (B7)\]

\[0 \leq \alpha \leq \beta \leq 1, \quad \frac{1}{2} \leq \beta \leq \frac{1 + \alpha}{2}, \quad \frac{a_3}{a_2} \leq \frac{b_2}{b_1} \leq \frac{1}{\left| \cos(\beta \pi) \right|}, \]

and

\[\frac{a_3}{a_2} \leq \frac{b_2}{b_1} \leq \frac{a_2}{2 \cos^2(\beta \pi)} \left( 1 - \sqrt{1 - \frac{4a_3 \cos^2(\beta \pi)}{a_2^2}} \right) \leq \frac{b_2}{b_1} \leq \frac{a_2}{\left| \cos(\beta \pi) \right|}; \]

Model VIII:

\[(1 + \bar{a}_1D_t^\alpha + \bar{a}_2D_t^{2\alpha}) \sigma(t) = (b_1D_t^\alpha + b_2D_t^{2\alpha}) \varepsilon(t), \quad (B8)\]

\[\frac{1}{2} \leq \alpha \leq 1, \quad \frac{\bar{a}_2}{\bar{a}_1} \leq \frac{b_2}{b_1} \leq \frac{1}{\left| \cos(\alpha \pi) \right|} \]

and

\[\frac{\bar{a}_2}{\bar{a}_1} \leq \frac{\bar{a}_1}{2 \cos^2(\alpha \pi)} \left( 1 - \sqrt{1 - \frac{4\bar{a}_2 \cos^2(\alpha \pi)}{\bar{a}_1^2}} \right) \leq \frac{b_2}{b_1} \leq \frac{\bar{a}_1}{\left| \cos(\alpha \pi) \right|}; \]

Appendix C. Calculation of relaxation modulus and creep compliances

The relaxation modulus (3.1) and creep compliance (3.4) are calculated by the Laplace transform inversion formula

\[f(t) = \mathcal{L}^{-1} \left[ \tilde{f}(s) \right](t) = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{\Gamma_0} \tilde{f}(s) e^{st} \, ds, \quad (C1)\]

with \(\Gamma_0\) being the Bromwich path, respectively, applied to the relaxation modulus in complex domain

\[\tilde{\sigma}_{\alpha\varepsilon}(s) = \frac{1}{s} \Phi_\sigma(s), \quad \text{Re} s > p_0 \geq 0, \quad (C2)\]

and creep compliance in complex domain

\[\tilde{\varepsilon}_{\alpha\varepsilon}(s) = \frac{1}{s} \Phi_\varepsilon(s), \quad \text{Re} s > p_0 \geq 0, \quad (C3)\]

see also (2.6), while the creep compliance (3.6) is obtained as

\[\varepsilon_{\alpha\varepsilon}(t) = \int_0^t f_{\alpha\varepsilon}(t') \, dt', \quad \text{with} \]

\[f_{\alpha\varepsilon}(t) = \mathcal{L}^{-1}[\tilde{f}_{\alpha\varepsilon}(s)](t) = \frac{1}{\pi} \int_0^{\infty} \frac{K(\rho)}{\Phi_\varepsilon(\rho e^{i\pi})} e^{-\rho t} \, d\rho \]

and

\[\tilde{f}_{\alpha\varepsilon}(s) = \frac{\Phi_\sigma(s)}{\Phi_\varepsilon(s)}, \quad (C4)\]

from the creep compliance in complex domain (C3) by the use of the Laplace transform inversion formula (C1).
Assuming (A 1), the Cauchy integral theorem

\[ \lim_{R \to \infty} \int_{\Gamma_{3}} e^{st} \, ds = 0, \quad \text{(C 5)} \]

where \( \Gamma \) is a closed curve containing the Bromwich path \( \Gamma_{0} \), chosen as in figure 1, is used in order to calculate the inverse Laplace transform by (C 1).

The integrals along contours \( \Gamma_{3} \) (parametrized by \( s = \rho e^{i\pi}, \rho \in (r, R) \)) and \( \Gamma_{5} \) (parametrized by \( s = \rho e^{-i\pi}, \rho \in (r, R) \)) in the case of relaxation modulus (3.1) read

\[ \lim_{r \to 0} \int_{\Gamma_{3}} \tilde{\sigma}_{sr}(s) e^{st} \, ds = \int_{0}^{\infty} \frac{1}{\rho e^{i\pi}} \frac{\Phi_{e}(\rho e^{i\pi})}{\Phi_{\sigma}(\rho e^{i\pi})} e^{-\rho t} e^{i\pi} \, d\rho = -\int_{0}^{\infty} \frac{\Phi_{e}(\rho e^{i\pi})}{\Phi_{\sigma}(\rho e^{i\pi})} e^{-\rho t} \, d\rho, \]

and

\[ \lim_{r \to 0} \int_{\Gamma_{5}} \tilde{\sigma}_{sr}(s) e^{st} \, ds = \int_{0}^{\infty} \frac{1}{\rho e^{-i\pi}} \frac{\Phi_{e}(\rho e^{-i\pi})}{\Phi_{\sigma}(\rho e^{-i\pi})} e^{-\rho t} e^{-i\pi} \, d\rho = \int_{0}^{\infty} \frac{\Phi_{e}(\rho e^{-i\pi})}{\Phi_{\sigma}(\rho e^{-i\pi})} e^{-\rho t} \, d\rho, \]

while for the creep compliance in the form (3.4), the integrals are

\[ \lim_{r \to 0} \int_{\Gamma_{3}} \tilde{\epsilon}_{cr}(s) e^{st} \, ds = \int_{0}^{\infty} \frac{1}{\rho e^{i\pi}} \frac{\Phi_{e}(\rho e^{i\pi})}{\Phi_{\sigma}(\rho e^{i\pi})} e^{-\rho t} e^{i\pi} \, d\rho = -\int_{0}^{\infty} \frac{\Phi_{e}(\rho e^{i\pi})}{\Phi_{\sigma}(\rho e^{i\pi})} e^{-\rho t} \, d\rho, \]

and

\[ \lim_{r \to 0} \int_{\Gamma_{5}} \tilde{\epsilon}_{cr}(s) e^{st} \, ds = \int_{0}^{\infty} \frac{1}{\rho e^{-i\pi}} \frac{\Phi_{e}(\rho e^{-i\pi})}{\Phi_{\sigma}(\rho e^{-i\pi})} e^{-\rho t} e^{-i\pi} \, d\rho = \int_{0}^{\infty} \frac{\Phi_{e}(\rho e^{-i\pi})}{\Phi_{\sigma}(\rho e^{-i\pi})} e^{-\rho t} \, d\rho, \]

and for the creep compliance in the form (3.6) their form is

\[ \lim_{r \to 0} \int_{\Gamma_{3}} \tilde{f}_{cr}(s) e^{st} \, ds = \int_{0}^{\infty} \frac{\Phi_{\sigma}(\rho e^{i\pi})}{\Phi_{e}(\rho e^{i\pi})} e^{-\rho t} e^{i\pi} \, d\rho = \int_{0}^{\infty} \frac{\Phi_{\sigma}(\rho e^{i\pi})}{\Phi_{e}(\rho e^{i\pi})} e^{-\rho t} \, d\rho, \]

and

\[ \lim_{r \to 0} \int_{\Gamma_{5}} \tilde{f}_{cr}(s) e^{st} \, ds = \int_{0}^{\infty} \frac{\Phi_{\sigma}(\rho e^{-i\pi})}{\Phi_{e}(\rho e^{-i\pi})} e^{-\rho t} e^{-i\pi} \, d\rho = -\int_{0}^{\infty} \frac{\Phi_{\sigma}(\rho e^{-i\pi})}{\Phi_{e}(\rho e^{-i\pi})} e^{-\rho t} \, d\rho, \]
where, according to (2.2), one has $\Phi_{\sigma}(\tilde{s}) = \Phi_{\sigma}(s)$ and $\Phi_{\epsilon}(\tilde{s}) = \Phi_{\epsilon}(s)$ (bar denotes the complex conjugation), so that

\begin{align*}
\lim_{R \to \infty} \int_{\Gamma_3 \cup \Gamma_5} \tilde{\sigma}_{st}(s) e^{st} \, ds &= \int_0^{\infty} \left( \frac{\Phi_{\epsilon}(\rho e^{i\tau})}{\Phi_{\sigma}(\rho e^{i\tau})} - \frac{\Phi_{\sigma}(\rho e^{i\tau})}{\Phi_{\epsilon}(\rho e^{i\tau})} \right) \frac{e^{-\rho t}}{\rho} \, d\rho \\
&= -2i \int_0^{\infty} \frac{K(\rho)}{|\Phi_{\sigma}(\rho e^{i\tau})|^2} \frac{e^{-\rho t}}{\rho} \, d\rho, \quad (C6)
\end{align*}

\begin{align*}
\lim_{R \to \infty} \int_{\Gamma_3 \cup \Gamma_5} \tilde{\varepsilon}_{cr}(s) e^{st} \, ds &= \int_0^{\infty} \left( \frac{\Phi_{\sigma}(\rho e^{i\tau})}{\Phi_{\epsilon}(\rho e^{i\tau})} - \frac{\Phi_{\epsilon}(\rho e^{i\tau})}{\Phi_{\sigma}(\rho e^{i\tau})} \right) \frac{e^{-\rho t}}{\rho} \, d\rho \\
&= 2i \int_0^{\infty} \frac{K(\rho)}{|\Phi_{\epsilon}(\rho e^{i\tau})|^2} \frac{e^{-\rho t}}{\rho} \, d\rho, \quad (C7)
\end{align*}

\begin{align*}
\lim_{R \to \infty} \int_{\Gamma_3 \cup \Gamma_5} \tilde{\varphi}_{cr}(s) e^{st} \, ds &= \int_0^{\infty} \left( \frac{\Phi_{\sigma}(\rho e^{i\tau})}{\Phi_{\epsilon}(\rho e^{i\tau})} - \frac{\Phi_{\epsilon}(\rho e^{i\tau})}{\Phi_{\sigma}(\rho e^{i\tau})} \right) e^{-\rho t} \, d\rho \\
&= -2i \int_0^{\infty} \frac{K(\rho)}{|\Phi_{\epsilon}(\rho e^{i\tau})|^2} e^{-\rho t} \, d\rho, \quad (C8)
\end{align*}

respectively, with

\[ K(\rho) = \frac{1}{2i} (\Phi_{\sigma}(\rho e^{i\tau}) \Phi_{\epsilon}(\rho e^{i\tau}) - \Phi_{\sigma}(\rho e^{i\tau}) \Phi_{\epsilon}(\rho e^{i\tau})) \]

giving (3.3). According to the Cauchy integral theorem (C5), the integrals (C6) and (C7) yield the second terms in relaxation modulus (3.1) and creep compliance (3.4), while the integral (C8) yields function $f_{cr}$ in (C4), since the integrals along $\Gamma_1, \Gamma_2, \Gamma_6, \Gamma_7$ tend to zero as $R \to \infty$ and $r \to 0$, and the integral along $\Gamma_3$ is non-zero in cases of (3.1) and (3.4), and zero in the case of (C4).

The contour $\Gamma_1$ is parametrized by $s = p + iR, p \in (0, p_0)$, with $R \to \infty$, so that the integrals

\begin{align*}
\int_{\Gamma_1} \tilde{\sigma}_{st}(s) e^{st} \, ds &= \int_{p_0}^{0} \frac{1}{p + iR} \Phi_{\epsilon}(p + iR) e^{(p + iR)t} \, dp, \\
\int_{\Gamma_1} \tilde{\varepsilon}_{cr}(s) e^{st} \, ds &= \int_{p_0}^{0} \frac{1}{p + iR} \Phi_{\sigma}(p + iR) e^{(p + iR)t} \, dp, \\
\int_{\Gamma_1} \tilde{\varphi}_{cr}(s) e^{st} \, ds &= \int_{p_0}^{0} \frac{\Phi_{\sigma}(p + iR)}{\Phi_{\epsilon}(p + iR)} e^{(p + iR)t} \, dp
\end{align*}

are estimated as

\begin{align*}
\left| \int_{\Gamma_1} \tilde{\sigma}_{st}(s) e^{st} \, ds \right| &\leq \int_{p_0}^{0} \frac{1}{|p + iR|} \left| \Phi_{\epsilon}(p + iR) \right| e^{pt} \, dp, \\
\left| \int_{\Gamma_1} \tilde{\varepsilon}_{cr}(s) e^{st} \, ds \right| &\leq \int_{p_0}^{0} \frac{1}{|p + iR|} \left| \Phi_{\sigma}(p + iR) \right| e^{pt} \, dp, \\
\left| \int_{\Gamma_1} \tilde{\varphi}_{cr}(s) e^{st} \, ds \right| &\leq \int_{p_0}^{0} \left| \frac{\Phi_{\sigma}(p + iR)}{\Phi_{\epsilon}(p + iR)} \right| e^{pt} \, dp.
\end{align*}

Assuming $s = \rho e^{i\varphi}$, since $R \to \infty$, one obtains $\rho = \sqrt{p^2 + R^2} \sim R$ and $\varphi = \arctan(R/p) \sim \pi/2$, the previous expressions become

\[ \lim_{R \to \infty} \int_{\Gamma_1} \tilde{\sigma}_{st}(s) e^{st} \, ds = \lim_{R \to \infty} \int_{0}^{p_0} \frac{1}{R} \left| \Phi_{\epsilon}(R e^{i\varphi}) \right| e^{pt} \, dp = 0, \]
due to (3.2), (3.5) and assumption (A 4). Analogously, it can be proved that the
integration along $\Gamma_7$ tends to zero as well.

The integrals along contour $\Gamma_2$, parametrized by $s = \Re \psi$, $\varphi \in (\pi/2, \pi)$, with $R \to \infty$, are

$$\left| \int_{\Gamma_2} \tilde{s}(s) e^{st} ds \right| \leq \lim_{R \to \infty} \left| \int_{0}^{\pi/2} \frac{1}{R} \Phi_\sigma(\Re \psi) \frac{\Phi_\sigma(\Re \psi)}{\Phi_\sigma(\Re \psi)} e^{Re^{i\psi}} d\varphi \right| = 0,$$

and

$$\left| \int_{\Gamma_2} \tilde{c}(s) e^{st} ds \right| \leq \lim_{R \to \infty} \left| \int_{0}^{\pi/2} \frac{1}{R} \Phi_\sigma(\Re \psi) \frac{\Phi_\sigma(\Re \psi)}{\Phi_\sigma(\Re \psi)} e^{Re^{i\psi}} iRe^{i\psi} d\varphi \right| = 0,$$

respectively, yielding the estimates

$$\left| \int_{\Gamma_2} \tilde{s}(s) e^{st} ds \right| \leq \lim_{R \to \infty} \left| \int_{0}^{\pi} \frac{1}{R} \Phi_\sigma(\Re \psi) \frac{\Phi_\sigma(\Re \psi)}{\Phi_\sigma(\Re \psi)} e^{Re^{i\psi}} d\varphi \right| = 0,$$

and

$$\left| \int_{\Gamma_2} \tilde{c}(s) e^{st} ds \right| \leq \lim_{R \to \infty} \left| \int_{0}^{\pi} \frac{1}{R} \Phi_\sigma(\Re \psi) \frac{\Phi_\sigma(\Re \psi)}{\Phi_\sigma(\Re \psi)} e^{Re^{i\psi}} iRe^{i\psi} d\varphi \right| = 0.$$

due to assumptions (A 2), (A 3) and (A 4), respectively. By the similar arguments, the integral
along $\Gamma_6$ tends to zero as well.

Parametrization of the contour $\Gamma_4$ is $s = \Re e^{i\psi}$, $\varphi \in (-\pi, \pi)$, with $r \to 0$, so that

$$\left| \int_{\Gamma_4} \tilde{s}(s) e^{st} ds \right| \leq \lim_{r \to 0} \left| \int_{-\pi}^{\pi} \frac{1}{\Re e^{i\psi}} \Phi_\sigma(\Re e^{i\psi}) \frac{\Phi_\sigma(\Re e^{i\psi})}{\Phi_\sigma(\Re e^{i\psi})} e^{Re^{i\psi}} d\varphi \right| = 0,$$

and

$$\left| \int_{\Gamma_4} \tilde{c}(s) e^{st} ds \right| \leq \lim_{r \to 0} \left| \int_{-\pi}^{\pi} \frac{1}{\Re e^{i\psi}} \Phi_\sigma(\Re e^{i\psi}) \frac{\Phi_\sigma(\Re e^{i\psi})}{\Phi_\sigma(\Re e^{i\psi})} e^{Re^{i\psi}} iRe^{i\psi} d\varphi \right| = 0,$$

respectively, yield

$$\lim_{r \to 0} \left| \int_{\Gamma_4} \tilde{s}(s) e^{st} ds \right| = \lim_{r \to 0} \left| \int_{-\pi}^{\pi} \frac{1}{\Re e^{i\psi}} \Phi_\sigma(\Re e^{i\psi}) \frac{\Phi_\sigma(\Re e^{i\psi})}{\Phi_\sigma(\Re e^{i\psi})} d\varphi \right| = -2\pi i \sigma_\sigma(e),$$

$$\lim_{r \to 0} \left| \int_{\Gamma_4} \tilde{c}(s) e^{st} ds \right| = \lim_{r \to 0} \left| \int_{-\pi}^{\pi} \frac{1}{\Re e^{i\psi}} \Phi_\sigma(\Re e^{i\psi}) \frac{\Phi_\sigma(\Re e^{i\psi})}{\Phi_\sigma(\Re e^{i\psi})} d\varphi \right| = -2\pi \sigma_\sigma(e),$$

and

$$\lim_{r \to 0} \left| \int_{\Gamma_4} \tilde{c}(s) e^{st} ds \right| \leq \lim_{r \to 0} \left| \int_{-\pi}^{\pi} \frac{1}{\Re e^{i\psi}} \Phi_\sigma(\Re e^{i\psi}) \frac{\Phi_\sigma(\Re e^{i\psi})}{\Phi_\sigma(\Re e^{i\psi})} d\varphi \right| = 0,$$

due to (3.2), (3.5) and assumption (A 4).

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