ON BURES DISTANCE OVER STANDARD FORM
\textit{vN}-ALGEBRAS

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Abstract. In case of standard form \textit{vN}-algebras, the Bures distance is the natural distance between the fibres of implementing vectors at normal positive linear forms. Thereby, it is well-known that to each two norm al positive linear forms implementing vectors exist such that the Bures distance is attained by the metric distance of the implementing vectors in question. We discuss to which extent this can remain true if a vector in one of the fibres is considered as fixed. For each nonfinite algebra, classes of counterexamples are given and situations are analyzed where the latter type of result must fail. In the course of the paper, an account of those facts and notions is given, which can be taken as a useful minimum of basic \textit{C}*-algebraic tools needed in order to efficiently develop the fundamentals of Bures geometry over standard form \textit{vN}-algebras.

1. Basic settings and results

1.1. Definitions and conventions. Throughout the paper, a distance function \(d_B\) on the positive cone \(M^*_+\) of the bounded linear forms \(M^*\) over a unital \textit{C}*-algebra \(M\) will be considered. For normal states on a \textit{W}*-algebra \(d_B\) agrees with the Bures distance function \cite{12}. Therefore, henceforth also \(d_B\) will be referred to as Bures distance function. We start by defining \(d_B(M|\nu,\varrho)\) between \(\nu,\varrho\in M^*_+\).

**Definition 1.1.**

\[d_B(M|\nu,\varrho) = \inf\{\pi,\phi\in S_{\pi,M}(\nu),\psi\in S_{\pi,M}(\varrho) \|\psi - \phi\|\}.

Instead of \(d_B(M|\nu,\varrho)\) the notation \(d_B(\nu,\varrho)\) will often be used. For unital \(*\)-representation \(\{\pi,\mathcal{K}\}\) of \(M\) on a Hilbert space \(\mathcal{K}\) \(\langle \cdot,\cdot \rangle\) and for \(\mu \in M^*_+\) we let

\[S_{\pi,M}(\mu) = \{\chi \in \mathcal{K} : \mu(\cdot) = \langle \pi(\cdot)\chi,\chi \rangle\}.

(1.1)

In case of \(S_{\pi,M}(\mu) \neq \emptyset\) this set will be referred to as \(\pi\)-fibre of \(\mu\). The above infimum extends over all \(\pi\) relative to which both \(\pi\)-fibres exist and, within each such representation, \(\varphi\) and \(\psi\) may be varied through all of \(S_{\pi,M}(\nu)\) and \(S_{\pi,M}(\varrho)\), respectively. The scalar product \(\mathbb{C} \times \mathbb{C} \ni (\chi,\eta) \mapsto \langle \chi,\eta \rangle \in \mathbb{C}\) on the representation Hilbert space by convention is supposed to be linear with respect to the first argument \(\chi\), and antilinear in the second argument \(\eta\), and maps into the complex field \(\mathbb{C}\). Let \(\mathbb{C} \ni z \mapsto \bar{z}\) be the complex conjugation, and be \(\Re z\) and \(|z|\) the real part and absolute value of \(z\), respectively. The norm of \(\chi \in \mathcal{K}\) is given by \(\|\chi\| = \sqrt{\langle \chi,\chi \rangle}\). For the relating operator and \textit{C}*-algebra theory, the reader is referred to the standard monographs, e.g. \cite{10,28,21}.

For both the \textit{C}*-norm of an element \(x \in M\) as well as for the operator norm of a concrete bounded linear operator \(x \in B(\mathcal{K})\) the same notation \(\|x\|\) will be used.

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and the involution (*-operation) respectively the taking of the hermitian conjugate of an element \( x \) is indicated by the transition \( x \mapsto x^* \). The notions of hermiticity and positivity for elements are defined as usual in \( \mathbb{C} \)-algebra theory, and \( M_h \) and \( M_p \) are the hermitian and positive elements of \( M \), respectively. The null and the unit element/operator in \( M \) and \( B(K) \) will be denoted by \( 0 \) and \( 1 \). For notational purposes mainly, in short recall some fundamentals relating (bounded) linear forms which subsequently might be of concern in context of Definition 1.1. Recall that the topological dual space \( M^* \) of \( M \) is the set of all those linear functionals (linear forms) which are continuous with respect to the operator norm topology. Equipped with the dual norm \( \| \cdot \|_1 \), which is given by \( \| f \|_1 = \sup \{ |f(x)| : x \in M, \| x \| \leq 1 \} \) and which is referred to as the functional norm, \( M^* \) is a Banach space. For each given \( f \in M^* \), the hermitian conjugate functional \( f^* \in M^* \) is defined by \( f^*(x) = \overline{f(x^*)} \), for each \( x \in M \). Remind that \( f \in M^* \) is hermitian if \( f = f^* \) holds, and \( f \) is termed positive if \( f(x) \geq 0 \) holds, for each \( x \in M \). Relating positivity, the basic fact is that a bounded linear form over \( M \) is positive if, and only if, \( \| f \|_1 = f(1) \).

For each \( f \in M^*_p \), there exists a cyclic *-representation \( \pi_f \) of \( M \) on some Hilbert space \( K_f \), with cyclic vector \( \Omega \in K_f \), and obeying \( f(x) = \langle \pi_f(x) \Omega, \Omega \rangle \), for all \( x \in M \) (Gelfand-Neumark-Segal theorem). Considering that construction in the special case with \( f = \nu + \varrho \) will provide a unital *-representation \( \pi = \pi_f \) where the \( \pi \)-fibres of \( \nu \) and \( \varrho \) both exist (we omit the details, all of which are standard). Thus Definition 1.1 makes sense, in any case of \( \nu, \varrho \in M^*_p \).

In conjunction with the Bures distance \( d_B \) there appears the functor \( P \) of the (\( \ast \)-algebraic) transition probability \( [34] \). For given \( \mathbb{C}^* \)-algebra \( M \) and positive linear forms \( \nu, \varrho \in M^*_p \) the definition reads as follows:

**Definition 1.2.** \( P_M(\nu, \varrho) = \sup_{\pi \in \mathcal{S}_{\pi, M(\nu)}, \varphi \in \mathcal{S}_{\pi, M(\varrho)}} |\langle \psi, \varphi \rangle|^2 \).

The range of variables in the supremum is the same as in Definition 1.1. With the help of \( P_M \) the following formula for \( d_B \) is obtained:

\[
d_B(M|\nu, \varrho)^2 = \left\{ \| \nu \|_1 - \sqrt{P_M(\nu, \varrho)} \right\} + \left\{ \| \varrho \|_1 - \sqrt{P_M(\nu, \varrho)} \right\}.
\]

Thus, properties of \( d_B \) can be obtained from properties of \( P \), and vice versa.

**Remark 1.3.** In [32], in generalizing from the commutative case which had been studied extensively in [24], expressions \( d \) and \( \rho \) instead of \( d_B \) and \( \sqrt{P} \) were considered for normal states on a \( W^* \)-algebra, but with the infima extending over all faithful representations \( \{ \pi, K \} \) where both fibres exist, accordingly.

### 1.2. Some basic results on Bures distance.

Let \( \nu, \varrho \in M^*_p \), and be \( \{ \pi, K \} \) a unital *-representation of \( M \) such that the \( \pi \)-fibres of \( \nu \) and \( \varrho \) both exist. Suppose \( \varphi \in \mathcal{S}_{\pi, M(\nu)} \) and \( \psi \in \mathcal{S}_{\pi, M(\varrho)} \). Let

\[
\pi(M)' = \{ z \in B(K) : z\pi(x) = \pi(x)z, \forall x \in M \}
\]

be the commutant vN-algebra of \( \pi(M) \), and be \( \mathcal{U}(\pi(M)') \) the group of unitary operators of \( \pi(M)' \). Define a linear form \( h_{\psi, \varphi}^{\pi, \nu} \) as follows:

\[
\forall z \in \pi(M)' : h_{\psi, \varphi}^{\pi, \nu}(z) = \langle z\psi, \varphi \rangle.
\]

For \( \chi \in K \) let orthoprojections \( p_\pi(\chi), p_p(\chi) \) be defined as the orthoprojections projecting from \( K \) onto the the closed linear subspaces \( \pi(M)'' \chi \) and \( \pi(M)\chi \), respectively. It is standard that \( p_\pi(\chi) \in \pi(M)'' \) and \( p_p(\chi) \in \pi(M)' \) hold, with the
double commutant \( vN \)-algebra

\[
\pi(M)'' = (\pi(M)')'
\]
of \( \pi(M) \). By the Kaplansky–von Neumann theorem \( \pi(M) \) is strongly dense within \( \pi(M)'' \), and therefore one always has

\[
(1.4) \quad \overline{\pi(M)\chi} = \overline{\pi(M)''\chi},
\]

which is useful to know. Relating \([1,1]\), for each \( \chi \in \mathcal{S}_{\pi,M}(\mu) \) the following is true:

\[
(1.5) \quad \mathcal{S}_{\pi,M}(\mu) = \left\{ v\chi : \; v^*v = p'_s(\chi), \; v \in \pi(M)' \right\}.
\]

Recall some useful facts about (1.3). Assume

\[
(1.6a) \quad h_{\psi,\varphi}^\pi = |h_{\psi,\varphi}^\pi| \langle (\cdot)v_{\psi,\varphi}^*\psi, \varphi \rangle
\]

the polar decomposition of the normal linear form \( h_{\psi,\varphi}^\pi \). According to the polar decomposition theorem for normal linear forms in \( vN \)-algebras, a partial isometry \( v = v_{\psi,\varphi}^\pi \) in and a normal positive linear form \( g = |h_{\psi,\varphi}^\pi| \) over \( \pi(M)' \) and obeying

\[
(1.6b) \quad h_{\psi,\varphi}^\pi g = g (\langle (\cdot)v \rangle)
\]

exist, and both are unique subject to \( v^*v = s(g) \), with the support orthoprojection \( s(g) \) of \( g \). Thus especially

\[
(1.6ba) \quad |h_{\psi,\varphi}^\pi| = h_{\psi,\varphi}^\pi \langle (\cdot)v_{\psi,\varphi}^* \psi, \varphi \rangle = \langle (\cdot)v_{\psi,\varphi}^* \psi, \varphi \rangle,
\]

from which in view of the above the following can be obtained:

\[
(1.6bb) \quad v_{\psi,\varphi}^\pi v_{\psi,\varphi}^* = s\langle h_{\psi,\varphi}^\pi \rangle \leq p'_s (\varphi).
\]

Analogously, by polar decomposition of \( h_{\psi,\varphi}^\pi = h_{\varphi,\psi}^\pi \), one has \( v_{\phi,\psi}^\pi = v_{\psi,\varphi}^\pi \) and

\[
(1.6c) \quad |h_{\psi,\varphi}^\pi| = h_{\varphi,\psi}^\pi \langle (\cdot)v_{\psi,\varphi}^* \varphi, \psi \rangle = \langle (\cdot)v_{\psi,\varphi}^* \varphi, \psi \rangle,
\]

from which analogously

\[
(1.6cb) \quad v_{\psi,\varphi}^\pi v_{\phi,\psi}^\pi = s\langle h_{\phi,\psi}^\pi \rangle \leq p'_s (\psi)
\]

is obtained. In particular, with the Murray-von Neumann equivalence ‘\( \sim \)’ one has

\[
(1.6d) \quad s\langle h_{\psi,\varphi}^\pi \rangle \sim s\langle h_{\phi,\psi}^\pi \rangle.
\]

We are going to comment now on some of the fundamentals of Bures geometry.

1.2.1. Basic algebraic facts on Bures distance. From \([1,1]\) \textbf{COROLLARY 1}, \textbf{COROLLARY 2}, eqs. (5), (6) and \textbf{THEOREM 3} the following facts are known:

\textbf{Lemma 1.4.}

\[
\sqrt{P_M(\nu, \varphi)} = \| h_{\psi,\varphi}^\pi \|_1 = \sup_{u \in \mathcal{U}(\pi(M)')} \| h_{\psi,\varphi}^\pi (u) \| = \inf_{x \in M_+ \text{ invertible}} \sqrt{\nu(x)\varphi(x^{-1})}.
\]

Each unitary orbit \( \mathcal{U}(\pi(M)')\chi \) is a special subset of the \( \pi \)-fibre \([1,1]\). Thus, the first two items of the following result at once can be seen from Lemma 1.4.

\textbf{Theorem 1.5.} Suppose \{\( \pi, \mathcal{K} \)\} is a unital *-representation such that the \( \pi \)-fibres of \( \nu, \varphi \in M_+^\pi \) exist. For given \( \varphi \in \mathcal{S}_{\pi,M}(\nu) \) and \( \psi \in \mathcal{S}_{\pi,M}(\varphi) \) the following hold:

1. \( d_B(\pi(\nu),\varphi) = \inf_{u \in \mathcal{U}(\pi(M)')} \| u\varphi - \varphi \| ; \)
2. \( d_B(\pi(\nu),\varphi) = \inf_{u \in \mathcal{S}_{\pi,M}(\varphi)} \| \psi' - \varphi \| ; \)
3. \( d_B(\pi(\nu),\varphi) = \| \psi - \varphi \| \leftrightarrow h_{\psi,\varphi}^\pi \geq 0. \)

Relating \([1,1]\), note that \( h_{\psi,\varphi}^\pi \geq 0 \) is equivalent to \( \langle \psi, \varphi \rangle = h_{\psi,\varphi}^\pi (1) = \| h_{\psi,\varphi}^\pi \|_1. \) The latter is equivalent to \( d_B(\pi(\nu),\varphi) = \| \psi - \varphi \| \), by \([1,1]\) and Lemma 1.4.
Remark 1.6. 1. By Theorem 13, $d_B$ can be calculated in each $\pi$ where both fibres exist. Thus, on a $W^*$-algebra $M$ and for normal states, the universal $W^*$-representation can be used. Hence, $d_B$ and $\sqrt{F}$ agree with $d$ and $\rho$ of 12 there, see Remark 13.

2. For $\mu \in M_+^*$ and $\pi$ with $S_{\pi,M}(\mu) \neq \emptyset$, there is a unique normal positive $\mu_\pi$ on $\pi(M)^\prime\prime$, with $\mu = \mu_\pi \circ \pi$. By (1.4) and (1.5) then $S_{\mathrm{id},\pi,M}(\mu_\pi) = S_{\pi,M}(\mu)$, with the trivial representation id of $\pi(M)^\prime\prime$ on $\mathcal{K}$.

3. For a $vN$-algebra $M$ and $\pi = \mathrm{id}$, the notation of the subscript/superscript ‘$\pi$’ will be omitted, and then $S_M(\nu)$, $h_{\psi,\varphi}$, $v_{\psi,\varphi}$ and $\rho(\varphi)$, $\rho'(\varphi)$ instead of $S_{\pi,M}(\nu)$, $h_{\psi,\varphi}$, $v_{\psi,\varphi}$ and $p_\pi(\varphi)$, $p_\pi'(\varphi)$, with $\pi = \mathrm{id}$, respectively, will be in use and $S_M(\nu) \neq \emptyset$ will be referred to as fibre of $\nu$, simply.

There are examples where Lemma 1.4 can be made more explicit. Let $\tau$ be a (lower semicontinuous) trace on $M_+$ (see e.g. in [10, 6.1.]). Then one has *-ideals $\mathcal{L}^2(M, \tau) = \{ x \in M : \tau(x^*x) < \infty \}$ and $\mathcal{L}^1(M, \tau) = \{ x \in M : x = y^*z, \ y, z \in \mathcal{L}^2(M, \tau) \}$. It is known that $\mathcal{L}^1(M, \tau)$ is the complex linear span of the hereditary positive cone $\mathcal{L}^1(M, \tau)_+ = \{ x \in M_+ : \tau(x) < \infty \}$, and thus $\tau$ uniquely extends to an invariant positive linear form $\tilde{\tau}$ onto $\mathcal{L}^1(M, \tau)$. That is, $\tilde{\tau}(xy) = \tilde{\tau}(yx)$, for either $x, y \in \mathcal{L}^2(M, \tau)$, or $x \in M$ and $y \in \mathcal{L}^1(M, \tau)$. By uniqueness, $\tau$ can be identified with $\tilde{\tau}$, and then is referred to as trace $\tau$ on $M$. In this sense, for $z \in \mathcal{L}^2(M, \tau)$ and $y \in \mathcal{L}^1(M, \tau)$, we have a unique positive linear form $\tau^z$ and bounded linear form $\tau_y$ on $M$, with $\tau^z(x) = \tau(z^*xz)$ and $\tau_y(x) = \tau(yx)$, respectively, for each $x \in M$.

Let $I(M, \tau) = \{ x \in M : \tau(x^*x) = 0 \}$. This is a *-ideal in $\mathcal{L}^2(M, \tau)$. Consider the completion $\mathcal{L}^2(M, \tau)$ of the space of equivalence classes $\eta_z$ modulo $I(M, \tau)$ under the inner product $\langle y_z, \eta_y \rangle = \tau(y^*x)$, $x, y \in \mathcal{L}^2(M, \tau)$. By standard arguments the latter is well-defined and extends to a scalar product on $\mathcal{L}^2(M, \tau)$, which then is a Hilbert space. Let $\pi : M \in X \mapsto \pi(x)$ and $\pi'(x) : M \in X \mapsto \pi'(x)$ be the *-representations of $M$ over $\mathcal{L}^2(M, \tau)$ which can be uniquely given through $\pi(x)\eta_y = \eta_{xy}$ and $\pi'(x)\eta_y = \eta_{yx}$, respectively, for each $y \in \mathcal{L}^2(M, \tau)$ and all $x \in M$. Then, for each $z \in \mathcal{L}^2(M, \tau)$ and $x \in M$ one has $\tau^z(x) = \langle \pi(x)\eta_z, \eta_z \rangle$. Note that $\pi' \in \pi(M)^\prime$. In carefully analyzing $\pi$ and $\pi'$ in case of a $vN$-algebra $M$, and in applying Lemma 1.4 with the mentioned *-representation $\pi$ and using that a normal trace is lower semicontinuous, one can prove the following important special cases.

Example 1.7. Let $M$ be a $vN$-algebra, with normal trace $\tau$. For $x, y \in \mathcal{L}^2(M, \tau)$ and $a = |x|^2 = xx^*$, $c = |y|^2 = yy^*$, one has $\tau^x = \tau_a$ and $\tau^y = \tau_b$, and the following formulae hold:

\begin{equation}
\sqrt{PM(\tau^x, \tau^y)} = \tau(|x^*y|) = \tau(|\sqrt{a\sqrt{c}}|) = \sqrt{PM(\tau_a, \tau_b)}.
\end{equation}

Let $s(\tau)$ be the support of $\tau$, that is, $z = s(\tau) \perp$ is the central orthoprojection obeying $I(M, \tau) = Mz$, see [28, 1.10.5.]. The following is useful in this context:

\begin{equation}
\tau(|\sqrt{a\sqrt{c}}|) = \tau(\sqrt{a\sqrt{c}}) \iff \sqrt{a\sqrt{c}}s(\tau) \geq 0.
\end{equation}

In the special case of $M = \mathcal{B}(\mathcal{H})$ the formula (1.4a) is known since [34]. With some more effort one even succeeds in proving (1.4a) without assuming normality. Relating (1.4b), this is a special case of the fact saying that $\tau(x) = \tau(|x|)$ is equivalent with $x s(\tau) \geq 0$, and which can be followed by reasoning about polar decomposition of $x$ and invariance of $\tau$ (we omit the details).
1.2.2. Metric aspects of the Bures distance. From each of the first two items of Theorem 1.5 together with Definition 1.1 and (1.3) it gets obvious that $d_B$ is symmetric in its arguments, obeys the triangle inequality, and is vanishing between $\nu$ and $\varrho$ if, and only if, $\nu = \varrho$. Thus, $d_B$ yields another distance function on $M_1^*$. By Theorem 1.5, $d_B$ is the natural distance function, in each space of $\pi$-fibres. Also, $d_B$ is topologically equivalent over bounded subsets to the functional distance $d_1$, which is defined as $d_1(\nu, \varrho) = \|\nu - \varrho\|_1$, for $\nu, \varrho \in M_1^*$. For quantitative estimates, see Proposition 1, formula (1.2)]. We do not prove these fact but instead remark that, in case of normal states/positive linear forms on $W^*$-algebras, this equivalence results from the estimates given in [12, 8, 9], essentially. The extension from there to unital C$^*$-algebras and their positive linear forms can be easily achieved by means of the first two items of the following auxiliary result.

Lemma 1.8. Let $\nu, \varrho \in M_1^*$, and be $\{\pi, K\}$ such that the $\pi$-fibres of $\nu$ and $\varrho$ both exist. Then, the following hold:

1. $\|\nu - \varrho\|_1 = \|\nu_\pi - \varrho_\pi\|_1$;
2. $d_B(M\nu, \varrho) = d_B(\pi(M)''\nu_\pi, \varrho_\pi)$.

Also, for $\nu, \varrho \in M_1^*$ one has

3. $P_M(\nu, \varrho) = 0 \iff \nu \perp \varrho$,

and for each $\mu, \sigma \in M_1^*$ obeying $0 \leq \sigma \leq \varrho$, $0 \leq \mu \leq \nu$,

4. $P_M(\mu, \sigma) + P_M(\delta \nu, \delta \varrho) \leq P_M(\nu, \varrho)$

is fulfilled, with $\delta \nu = \nu - \mu$, $\delta \varrho = \varrho - \sigma$.

Proof. In order to see (1) recall first that, since $\pi$ is a *-homomorphism, according to basic C$^*$-theory, the unit ball $M_1$ of $M$ is related to the unit ball $\pi(M)_1$ of $\pi(M)$ through $\pi(M)_1 = \pi(M_1)$. From this by the Kaplansky density theorem

\[
\left(\pi(M)''\right)_1 = \pi(M)_1^{\text{st}} = \pi(M)_1^{\text{st}} \text{ (strong closure)}
\]

is seen. Owing to normality of $\nu_\pi - \varrho_\pi$ we then may conclude as follows:

\[
\|\nu - \varrho\|_1 = \sup_{x \in M, \|x\| \leq 1} |\nu_\pi(\pi(x)) - \varrho_\pi(\pi(x))| = \sup_{y \in \pi(M), \|y\| \leq 1} |\nu_\pi(y) - \varrho_\pi(y)| = \|\nu_\pi - \varrho_\pi\|_1.
\]

To see (2), apply Theorem 1.5 to $\nu_\pi$ and $\varrho_\pi$ in respect of the identity representation of the vN-algebra $\pi(M)''$. Under these premises Theorem 1.5 yields

\[
d_B(\pi(M)''\nu_\pi, \varrho_\pi) = \sup_{\psi \in \pi(M)'} \|\psi - \varphi\|, \quad \text{for each } \psi \in \pi(M)''\nu_\pi(\varphi) \text{ and } \varphi \in \pi(M)''\varrho_\pi(\varphi).
\]

In view of Remark 1.6 this is $d_B(M\nu, \varrho)$ as asserted by Theorem 1.5 when applied with respect to $M$, $\pi$, $\nu$ and $\varrho$.

Relating (3), note that for each invertible $x \in M_1$ obviously $\nu(x)g(x^{-1}) = \{\nu - \mu\}(x)(\varrho - \sigma)(x^{-1}) + \nu(x)\sigma(x^{-1}) + \mu(x)\varrho(x^{-1})$ holds. From this by Lemma 1.4 and positivity of all terms the assertion follows.

To see (4), note that $P_M(\nu, \varrho) = 0$ is equivalent to $h^{\nu, \varrho}_\psi = 0$, by Lemma 1.4, which is the same as $\langle \psi, y\varphi \rangle = 0$, for all $x, y \in \pi(M)'$. Hence, $P_M(\nu, \varrho) = 0$ is equivalent to $p_\pi(\psi) \perp p_\pi(\varphi)$. Therefore, since $p_\pi(\psi) = s(\nu_\pi)$ and $p_\pi(\varphi) = s(\nu_\pi)$ holds, with the support orthopositions $s(\nu_\pi)$ and $s(\varrho_\pi)$ of the normal positive linear forms $\nu_\pi$ and $\varrho_\pi$ (see Remark 1.3), $P_M(\nu, \varrho) = 0$ is equivalent to orthogonality of $\nu_\pi$ with $\varrho_\pi$. The latter is the same as $\|\nu_\pi - \varrho_\pi\|_1 = \|\nu_\pi\|_1 + \|\varrho_\pi\|_1$. In view of (1) equivalence of $P_M(\nu, \varrho) = 0$ to $\|\nu - \varrho\|_1 = \|\nu\|_1 + \|\varrho\|_1$ follows. \qed
Relating Theorem 1.9 (3), a remarkable fact firstly acknowledged in [9] says that $d_{B}(M|\nu,\varrho)=\|\nu-\varrho\|$ can happen, for each $\nu,\varrho\in M_{+}^{*}$, see also [3, Appendix 7].

**Theorem 1.9.** Suppose $\{\pi,K\}$ is such that the $\pi$-fibres of $\nu,\varrho\in M_{+}^{*}$ both exist. There are $\varphi_{0}\in S_{\pi,M}(\nu)$ and $\psi_{0}\in S_{\pi,M}(\varrho)$ with $d_{B}(M|\nu,\varrho)=\|\psi_{0}-\varphi_{0}\|$.

**Proof.** For $\varphi\in S_{M}(\nu)$, $\psi\in S_{M}(\varrho)$ consider $h_{\psi,\varphi}^{\pi}$ as defined in (1.3). Then $\|h_{\psi,\varphi}^{\pi}\|_{1}=\|\|h_{\psi,\varphi}^{\pi}\|\|_{1}=\|\nu,\varphi\|$.

It suffices to show that partial isometries $v,w\in\pi(M)^{\prime}$ exist which obey $v^{\ast}v\geq p_{\pi}(\varphi)$, $w^{\ast}w\geq p_{\pi}(\psi)$ and $h_{\psi,\varphi}^{\pi}(v^{\ast}w)=h_{\psi,\varphi}^{\pi}(v^{\ast}w)$. In fact, in view of (1.3), for $\psi_{0}=w\psi$ and $\varphi_{0}=v\varphi$ one then has $\psi_{0}\in S_{\pi,M}(\varrho)$ and $\varphi_{0}\in S_{\pi,M}(\nu)$. By the above and in twice applying Lemma 1.4 we may conclude as follows:

$$h_{\varphi_{0},\varphi_{0}}^{\pi}(1)=h_{\varphi_{0},\varphi_{0}}^{\pi}(v^{\ast}w)=h_{\varphi_{0},\varphi_{0}}^{\pi}(v^{\ast}w)=\|h_{\psi,\varphi}^{\pi}\|_{1}=\sqrt{P_{M}(\nu,\varrho)}=\|h_{\psi,\varphi}^{\pi}\|_{1}.$$ 

Hence $h_{\varphi_{0},\varphi_{0}}^{\pi}(1)=\|h_{\psi,\varphi}^{\pi}\|_{1}$, from which $h_{\psi,\varphi}^{\pi}\geq 0$ follows. In view of Theorem 1.9, the constructed $\varphi_{0}$ and $\psi_{0}$ can be taken to meet our demands.

Recall that for a $vN$-algebra $N$ on a Hilbert space $\mathcal{H}$ there is a largest central orthoprojection $z\in N\cap N^{\prime}$ such that the $vN$-algebra $(N^{\prime})z$ over $z\mathcal{H}$ is finite. Thus, in case of $z\neq 1$, $(N^{\prime})z^{\perp}$ is properly infinite over $z^{\perp}\mathcal{H}$. Moreover, on $c\text{K}$ one has

$$(N^{\prime})c=((N^{\prime})c)^{\prime},$$

for each central orthoprojection $c$ (the outer commutant refers to $c\text{K}$). Applying this to $N=\pi(M)^{\prime}$ with $c=z$ or $c=z^{\perp}$, and having in mind Remark 1.4 (2), we see that we can content ourselves with constructing the partial isometries in question only for $vN$-algebras with either finite or properly infinite commutants.

In line with this, let $M$ be the $vN$-algebra in question, with $M^{\prime}$ either finite or properly infinite, and let $\nu$ and $\varrho$ be implemented by vectors $\varphi$ and $\psi$. We then will make use of the notational conventions of Remark 1.4 (3), and are going to construct $v$ and $w$ under these premises now.

Suppose $M^{\prime}$ to be finite. By (1.6d) we have $s(\|h_{\psi,\varphi}^{\pi}\|)=s(\|h_{\psi,\varphi}^{\pi}\|)$. In a finite $vN$-algebra, by another standard fact, see [23, 2.4.2.], this condition implies the following to hold, in any case:

$$(1.5)\quad s(\|h_{\psi,\varphi}^{\pi}\|)^{\perp}=s(\|h_{\psi,\varphi}^{\pi}\|)^{\perp}.$$ 

Hence, there is $m\in M^{\prime}$ obeying $mm^{\ast}=s(\|h_{\psi,\varphi}^{\pi}\|)^{\perp}$ and $mm^{\ast}=s(\|h_{\psi,\varphi}^{\pi}\|)^{\perp}$. Define $w=v_{\psi,\varphi}^{\ast}+m^{\ast}$. Then $w$ is unitary, $w\in\mathcal{U}(M^{\prime})$, with $ws(\|h_{\psi,\varphi}^{\pi}\|)=v_{\psi,\varphi}^{\ast}$. Since by polar decomposition $h_{\psi,\varphi}(w)=h_{\psi,\varphi}(v_{\psi,\varphi}^{\ast})$ is fulfilled, $v=1$ can be chosen.

Suppose a properly infinite $M^{\prime}$. Then $p\in M^{\prime}$ with $p\sim p^{\perp}\sim 1$ exists. By (1.6b), $p^{\prime}(\varphi)-s(\|h_{\psi,\varphi}^{\pi}\|)\in M^{\prime}$ is an orthoprojection. Hence $\{p^{\prime}(\varphi)-s(\|h_{\psi,\varphi}^{\pi}\|)\}^{\perp}$ is fulfilled, $v\in 1$ can be chosen.

Suppose a properly infinite $M^{\prime}$. Then $p\in M^{\prime}$ with $p\sim p^{\perp}\sim 1$ exists. By (1.6b), $p^{\prime}(\varphi)-s(\|h_{\psi,\varphi}^{\pi}\|)\in M^{\prime}$ is an orthoprojection. Hence $\{p^{\prime}(\varphi)-s(\|h_{\psi,\varphi}^{\pi}\|)\}^{\perp}$ is fulfilled, $v\in 1$ can be chosen.
Remark 1.10. From Lemma 1.8 the convexity/subadditivity properties of $P$ are obvious: $P$ is subadditive. On the other hand, by the scaling behavior of $P$ on the cone $M^*_\pi$ together with Theorem 1.7 and formula (1.2), it is easily inferred that $d^2_P$ (resp. $\sqrt{P}$) is jointly convex (resp. jointly concave), on each convex subset of $M^*_\pi$, see also in [3], and [4, Remark 2(3)] for some additional superadditivity property of $\sqrt{P}$.

1.2.3. Minimal pairs of positive linear forms. Let $\varphi_0$ and $\psi_0$ in the $\pi$-fibres of $\nu, \varrho \in M^*_\pi$ be chosen as to satisfy the hypothesis of Theorem 1.9. Define $\varrho^\perp \in M^*_\pi$ as follows:

\begin{equation}
\eta \in S_{\pi, M}(\varrho^\perp), \text{ with } \eta = s(h^\pi_{\psi_0, \varrho_0})^\perp \psi_0.
\end{equation}

Owing to $s(h^\pi_{\psi_0, \varrho_0})^\perp \in \pi(M)'$ and $\psi_0 \in S_{\pi, M}(\varrho)$, $0 \leq \varrho^\perp \leq \varrho$. Also $h^\pi_{\eta, \varrho_0} = 0$, since $h^\pi_{\eta, \varrho_0}(z) = h^\pi_{\psi_0, \varrho_0}(zs(h^\pi_{\psi_0, \varrho_0})^\perp) = 0$, for each $z \in \pi(M)'$. By Lemma 1.4 $P_M(\nu, \varrho^\perp) = 0$, and thus $\varrho^\perp$ and $\nu$ are mutually orthogonal, by Lemma 1.8(3).

On the other hand, let $\sigma \in M^*_\pi$ with $\sigma \leq \varrho$ be orthogonal to $\nu$. Then, by standard facts there exists $z \in \pi(M)'$ with $\|z\| \leq 1$ and $z\varrho_0 \in S_{\pi, M}(\sigma)$. Also, by Lemma 1.8(3), $P_M(\nu, \sigma) = 0$. Hence, for $\eta' = z\varrho_0$, $h^\pi_{\eta', \varrho_0} = 0$ by Lemma 1.4. Thus in particular $h^\pi_{\psi_0, \varrho_0}(z^*z) = 0$. By positivity of $h^\pi_{\psi_0, \varrho_0}$ and $\|z\| \leq 1$ one concludes $z^*z \leq 1 - s(h^\pi_{\psi_0, \varrho_0})$. In view of (1.6) then $\sigma \leq \varrho^\perp$ follows. Thus we arrive at the following result, which goes back to [3], see [3, §2] and [4] for applications.

Theorem 1.11. For each given pair $\{\nu, \varrho\}$ of positive linear forms on a unital $C^*$-algebra, among all positive linear forms $\sigma$ with $\sigma \perp \nu$ and $\sigma \leq \varrho$ there exists a largest element, $\sigma = \varrho^\perp$.

For given pair $\{\nu, \varrho\}$ there has to exist also a largest positive linear form subordinate to $\nu$ and orthogonal to $\varrho$, which we call $\nu^\perp$. Thus, for each pair $\{\nu, \varrho\}$ of positive linear forms both $\nu^\perp$ and $\varrho^\perp$ have an invariant meaning. The derivation of the previous result then also shows that, provided the $\pi$-fibres of $\nu$ and $\varrho$ both exist, with respect to the pair $\{\nu_\pi, \varrho_\pi\}$ over $\pi(M)'$ the following must be valid:

\begin{equation}
(\varrho_\pi)^\perp = (\varrho^\perp)_\pi, (\nu_\pi)^\perp = (\nu^\perp)_\pi.
\end{equation}

On the other hand, since $\rho - \rho^\perp \geq 0$ holds, we may also consider the pair $\{\nu, \varrho - \rho^\perp\}$, and we may ask for $(\rho - \rho^\perp)^\perp$ and $\nu^\perp$, where $\perp'$ now refers to the $\perp'$-operation with respect to the pair $\{\nu, \varrho - \rho^\perp\}$.

Lemma 1.12. For each pair $\{\nu, \varrho\}$ the following facts hold:

1. $(\varrho - \rho^\perp)^\perp = 0$, $\nu^\perp = \nu^\perp$;
2. $d_B(\nu, \varrho)^2 = d_B(\nu - \nu^\perp, \varrho - \rho^\perp)^2 + d_B(\nu^\perp, \varrho^\perp)^2$;
3. $P_M(\nu, \varrho) = P_M(\nu - \varrho^\perp, \varrho - \rho^\perp)$.

Especially, the pair $\{\nu - \nu^\perp, \varrho - \rho^\perp\}$ is the least element of all pairs $\{\theta, \theta'\}$ of positive linear forms with $\theta \leq \nu$ and $\theta' \leq \varrho$, and which obey $P_M(\theta, \theta') = P_M(\nu, \varrho)$.

Proof. By (1.7), Lemma 1.8 and Theorem 1.11 we may content ourselves with proving the assertions for a $\nu N$-algebra $M$ and normal positive linear forms $\nu$ and $\varrho$ with nontrivial fibres. For normal forms orthogonality simply means orthogonality of the respective supports. Hence, $0 \leq \sigma \leq \varrho - \rho^\perp$ and $\sigma \perp \nu$ imply $\sigma + \rho^\perp \leq \varrho$ and $\sigma + \rho^\perp \perp \nu$. By maximality of $\rho^\perp$ then $\sigma = 0$, and thus the first of the relations of (1.7) is seen. Note that $\nu^\perp \leq \nu$, with $\nu^\perp \perp \rho - \rho^\perp$. From the former owing to
\( g^\perp \perp \nu \) then \( g^\perp \perp \nu^\perp \) is seen, and thus the two orthogonality relations may be summarized into \( \nu^\perp \perp (g - g^\perp) + g^\perp = g \). Hence, in view of Theorem 1.11 then \( \nu^\perp \leq \nu^\perp \) follows. On the other hand, by definition, \( \nu^\perp \) is subordinate to \( \nu \) and must also be orthogonal to \( \nu - g^\perp \leq \nu \). Hence, \( \nu^\perp \geq \nu^\perp \) by Theorem 1.11 when applied to \( \nu, \nu - g^\perp \). In summarizing the second relation of \( \nu \) follows.

Let \( \psi_0 \in S_M(\nu) \) and \( \varphi_0 \in S_M(\nu) \) be chosen as in Theorem 1.9. By Theorem 1.5(3) and 1.6 one has \( s^\perp \psi_0 \in S_M(\nu^\perp) \), with \( s = s(h_{\psi_0, \varphi_0}) \in M' \). Note that \( h_{\psi_0, \varphi_0} = h_{\psi_0, \varphi_0}(s)|s = h_{\psi_0, \varphi_0}. \) By Theorem 1.3(3) \( d_B(\nu, \varphi - g^\perp) = ||\varphi_0 - s\psi_0|| \) follows. By orthogonality of \( g^\perp \) with \( \nu \) we especially must have \( s^\perp \psi_0 \perp \varphi_0 \). Hence also \( s^\perp \psi_0 \perp (\varphi_0 - s\psi_0) \), and thus we obtain \( d_B(\nu, \varphi)^2 = ||\varphi_0 - s\psi_0 - s^\perp \psi_0||^2 = d_B(\nu, \varphi - g^\perp)^2 + ||s^\perp \psi_0||^2 = d_B(\nu, \varphi - g^\perp)^2 + ||g^\perp||_1 \).

We may apply this to \( \{g - g^\perp, \nu\} \) accordingly, with the result \( d_B(g - g^\perp, \nu)^2 = d_B(g - g^\perp, \nu - \nu^\perp)^2 + ||\nu^\perp||_1 \). By (1) and by symmetry of \( d_B \), substitution of the latter into the former relation yields \( d_B(\nu, \varphi)^2 = d_B(\nu - \nu^\perp, g - g^\perp)^2 + ||\nu^\perp||_1 + ||g^\perp||_1 = d_B(\nu^\perp, g^\perp)^2 \), from which (3) follows. In view of (1), (3) is seen. Finally, by Lemma 1.3(3), \( P_M(\theta, \theta') \leq P_M(\nu, \theta') \leq P_M(\nu, \varphi) \) as well as \( P_M(\theta, \theta') \leq P_M(\nu, \theta') \) and \( P_M(\nu, \varphi - \theta') \leq P_M(\nu, \varphi) \) must be fulfilled. Therefore, \( P_M(\theta, \theta') = P_M(\nu, \varphi) \) implies \( P_M(\nu, \varphi - \theta') = 0 \). By Lemma 1.3(3) then \( \nu \perp (g - \theta') \) follows. Since \( 0 \leq (g - \theta') \leq \varphi \) holds, in view of Theorem 1.11 \( g^\perp \geq (g - \theta') \) follows. Hence, \( \theta' \geq g - \varphi \). Proceeding analogously for \( \theta \) instead of \( \theta' \) yields \( \theta \geq \nu - \varphi^\perp \). Thus also the final assertion is true.

Definition 1.13. Refer to \( \{\omega, \sigma\} \) as minimal pair if \( \omega^\perp = \sigma^\perp = 0 \) holds, and refer to \( \{\nu - \nu^\perp, \varphi - \varphi^\perp\} \) as \( \{\nu, \varphi\} \)-associated minimal pair.

Remark 1.14. It is obvious from the last part of Lemma 1.12 that the \( \{\nu, \varphi\} \)-associated minimal pair is the unique minimal pair \( \{\theta, \theta'\} \) obeying \( \theta \leq \nu, \theta' \leq \varphi \) and \( P_M(\theta, \theta') = P_M(\nu, \varphi) \), see \[3, Lemma 2.1\].

There is yet another remarkable uniqueness result in context of these structures:

Theorem 1.15. \[3, Theorem 3.1\] For given pair \( \{\nu, \varphi\} \) of positive linear forms on a unital \( C^* \)-algebra \( M \), there is a unique \( g \in M^* \) obeying

1. \( g(1) = \sqrt{P_M(\nu, \varphi)} \),
2. \( |g(y^*x)| \leq \{\nu - \nu^\perp\}(y^*y)\{\varphi - \varphi^\perp\}(x^*x), \ x, y \in M \).

2. Forthcoming Results

2.1. Setting the problems. The problems of this paper naturally came along when analyzing in context of the hypothesis of Theorem 1.3 an example recently noticed in \[\text{[26]}\] and some other optimization problem, see \[\text{[3, Theorem 6(2)]\].

Problem 2.1. 1. Despite the given proof of Theorem 1.9 we are missing a true criterion along which one can decide whether or not, for given individual vector \( \varphi_0 \) in the \( \pi \)-fibre of \( \nu \), the condition in the hypothesis of Theorem 1.3 could be satisfied at all, by some vector \( \psi_0 \) in the \( \pi \)-fibre of \( \varphi \).

2. Let \( \mathcal{S}_{\pi, M}(\nu|\varphi) \) be the subset of all \( \varphi_0 \) in the \( \pi \)-fibre of \( \nu \) for which the question in \[\text{[1]}\] can be affirmatively answered. Then, provided \( \varphi_0 \in \mathcal{S}_{\pi, M}(\nu|\varphi) \) is fulfilled, is there a formula giving a particular \( \psi_0 \) to that \( \varphi_0 \)?
3. Especially, by Theorem 1.9 the question is left open whether \( S_{\pi,M}(\nu|\varrho) = S_{\pi,M}(\nu) \) might happen, on a given unital C*-algebra.

4. And finally, provided for some \( M \) the answer in (3) can be in the negative, what about the structure of those pairs \( \{\nu,\varrho\} \) with \( S_{\pi,M}(\nu|\varrho) \neq S_{\pi,M}(\nu) \)?

Thereby, in line with Remark 1.8(3) and Lemma 1.8(3), it is reasonable to content ourselves with considering these questions for \( vN \)-algebras with normal positive linear forms which can be implemented by vectors. In making reference to some ideas about hereditarity (we omit all these details), we can even be assured that without essential loss of generality we might confine ourselves to considering \( d_B \) over normal positive linear forms over a standard form \( vN \)-algebra \( M \). Therefore, the problems in questions will be dealt with and answered examplarily at least for such cases. Then also the notational conventions of Remark 1.6(3) tacitly will be in use, and \( S_{\pi,M}(\nu|\varrho) \) instead of \( S_{\pi,M}(\nu) \) will be written.

**Remark 2.2.** On a finite standard form \( vN \)-algebra \( M \), problems (1), (2) and (3) trivialize, for then always

\[
S_M(\nu) = S_M(\nu|\varrho) = \{u\varphi_0 : u^*u = 1, u \in M'\}
\]

holds, with arbitrary \( \varphi_0 \in S_M(\nu) \). In fact, for finite \( M, M' \) is finite, too. Since in a finite \( vN \)-algebra each partial isometry admits unitary extensions, for each orthoprojection \( q \in M' \) one has \( \{v \in M' : v^*v = q\} = U(M')q \). Especially then also \( U(M') = \{v \in M' : v^*v = 1\} \) follows. Hence, in view of (1.3) each fibre is the \( U(M') \)-orbit of each of its vectors. From this and Theorem 1.9 then (2.1) follows.

**2.2. The general criterion.** The first two problems of Problem 2.1 can be answered under the rather general settings of a unital C*-algebra \( A \), a pair \( \{\nu,\varrho\} \) of positive linear forms, and \( \varrho^\perp \) as determined by Theorem 1.11. Under the same premises as in Theorem 1.3 we have the following result.

**Theorem 2.3.** A vector \( \varphi_0 \in S_{\pi,M}(\nu) \) belongs to \( S_{\pi,M}(\nu|\varrho) \) if, and only if, there are \( \eta \in S_{\pi,M}(\varrho^\perp) \) and \( \psi \in S_{\pi,M}(\varrho) \) satisfying

\[
(2.2a) \quad p'_{\pi}(\eta) \leq s(|h_{\psi,\varphi_0}^\pi|) \perp.
\]

If this condition is fulfilled, then the special vector

\[
(2.2b) \quad \psi_0 = v_{\psi,\varphi_0}^\pi \psi + \eta
\]

belongs to \( S_{\pi,M}(\varrho) \), that is, \( d_B(\nu,\varrho) = \|\psi_0 - \varphi_0\| \) is satisfied. Thereby, for given \( \varphi_0 \) and \( \eta \), (2.2a) remains true for any \( \psi \in S_{\pi,M}(\varrho) \), and \( \psi_0 \) as given by formula (2.2b) is the same, for each such \( \psi \).

For the proof, it is useful to take notice of the following auxiliary results first.

**Lemma 2.4.** Let \( \nu,\varrho \in M_{\pi,\nu} \) and \( \varphi \in S_{\pi,M}(\nu), \psi \in S_{\pi,M}(\varrho) \). The following hold:

1. \( p'_{\pi}(v_{\psi,\varphi}^{\pi,\psi}) = s\left(|h_{\psi,\varphi}^{\pi,\psi}|\right) \);
2. \( \forall \psi \in S_{\pi,M}(\varrho) : |h_{\psi,\varphi}^{\pi,\psi}| = |h_{\psi,\varphi}^{\pi,\psi}|, v_{\psi,\varphi}^{\pi,\psi} = wv_{\psi,\varphi}^{\pi,\psi}, \) with \( w \in \pi(M)' \) obeying
   \( w^*w = p'_{\pi}(\psi), \) \( \tilde{\psi} = w\psi \);
3. \( v_{\psi,\varphi}^{\pi,\psi}, s\left(|h_{\psi,\varphi}^{\pi,\psi}|\right) \psi \in S_{\pi,M}(\varrho - \varrho^\perp), s\left(|h_{\psi,\varphi}^{\pi,\psi}|\right) \psi \in S_{\pi,M}(\varrho^\perp) \).

**Proof.** Note that \( s\left(|h_{\psi,\varphi}^{\pi,\psi}|\right) \leq p'_{\pi}(v_{\psi,\varphi}^{\pi,\psi}, \) by (1.6ba), and \( p'_{\pi}(v_{\psi,\varphi}^{\pi,\psi}) \leq s\left(|h_{\psi,\varphi}^{\pi,\psi}|\right) \) by (1.6b1). Hence (1) follows. Relating (3), remark that by (1.3) \( w \in \pi(M)' \) obeying
   \( w^*w = p'_{\pi}(\psi) \) and \( \tilde{\psi} = w\psi \) exists and is unique. In view of this and (1.6c1),
\[ u = w \psi_{v,\phi}^\pi \] is a partial isometry of \( \pi(M)' \), with \( u^*u = \psi_{v,\phi}^\pi \psi_{v,\phi}^\pi = s(\|h_{v,\phi}^\pi\|) \). From this and \( h_{v,\phi}^\pi = h_{v,\phi}^\pi(\cdot)w = |h_{v,\phi}^\pi(\cdot)u| \) by uniqueness of the polar decomposition the validity of (2) follows.

Let \( \psi_0, \phi_0 \) be as in Theorem 1.3. By Theorem 2.3, \( h_{\psi_0,\phi_0}^\pi \geq 0 \), and according to (1.3), \( s(h_{\psi_0,\phi_0}^\pi)\psi_0 \in \pi(M) (\pi - \phi^\perp) \). By (1) and one has \( w, u \in \pi(M)' \), with \( w^*w = p_\pi(\psi_0), \psi = w\psi_0, u^*u = p_\pi(\phi_0), \phi = u\phi_0 \) and obeying \( \psi_{v,\phi}^\pi = w\psi_{v,\phi_0}^\pi, v_{\phi,\phi}^\pi = wv_{\phi,\phi_0}^\pi \). The partial isometry \( v_{\alpha,\beta} \) of the polar decomposition of \( h_{\alpha,\beta} \) has to obey \( v_{\alpha,\beta}^*v_{\alpha,\beta} = v_{\beta,\alpha} \). Hence, \( v_{\psi,\phi}^\pi = v_{\psi,\phi_0}^\pi = wv_{\psi,\phi_0}^\pi = uv_{\psi,\phi_0}^\pi w^* \) follows. From this together with the other properties of \( w, u \) one gets

\[ v_{\psi,\phi}^\pi = uv_{\psi,\phi_0}^\pi \psi_0. \]

By \( h_{\psi_0,\phi_0}^\pi \geq 0 \) one has \( v_{\psi,\phi}^\pi = s(h_{\psi_0,\phi_0}^\pi) \). Thus, from \( u^*u = p_\pi(\phi_0) \geq s(h_{\psi_0,\phi_0}^\pi) \) in view of \( s(h_{\psi_0,\phi_0}^\pi)\psi_0 \in \pi(M) (\pi - \phi^\perp) \) and (2.3) also \( v_{\psi,\phi}^\pi \psi \in \pi(M) (\pi - \phi^\perp) \) follows. In view of (1.6c) and (1.5) then \( s(h_{\psi_0,\phi_0}^\pi) \psi \in \pi(M) (\pi - \phi^\perp) \) is obtained. From this (3) follows.

**Proof.** (of Theorem 2.3) Suppose \( \eta \in \pi(M)(\pi^\perp) \) satisfies (2.2a) for some \( \psi \in \pi(M)(\pi) \), and given \( \phi_0 \in \pi(M)(\pi) \). By Lemma 2.3, the same premises then hold with respect to any other vector \( \psi \) in the \( \pi \)-fibre of \( \gamma \). Also, by Lemma 2.4, one has \( p_\pi(\psi_{v,\phi_0}^\pi) = s(h_{\psi_0,\phi_0}^\pi) \). From this and (2.2a) in view of Lemma 2.4, then \( \psi_0 \in \pi(M)(\pi) \) follows. By Theorem 1.1, \( \pi^\perp \perp \psi \). This is the same as \( p_\pi(\eta) \perp p_\pi(\phi_0) \), and thus in view of (1.6b) and by construction of \( \psi_0 \), in accordance with (2.2b) \( h_{\psi_0,\phi_0}^\pi = h_{\psi_0,\phi_0}^\pi(\cdot)h_{\psi_0,\phi_0}^\pi = |h_{\psi_0,\phi_0}^\pi| \geq 0 \). By Theorem 1.3, we see \( d_B(\nu, \phi) = \|\psi_0 - \phi_0\| \), for \( \psi_0 \) of (2.2b). Note that our situation with \( \psi_0, \phi_0 \) and \( \psi_0 \) can be easily adapted to a context where relation (2.3) can be used. In fact, choosing \( \psi = \phi_0 \) there, in the notation of the previous proof one has \( u = p_\pi(\phi_0) \), and in view of \( p_\pi(\psi_{v,\phi_0}^\pi) = s(h_{\psi_0,\phi_0}^\pi) \) (see above and (1.6b)) the relation (2.3) yields \( \psi_{v,\phi_0}^\pi = \psi_{v,\phi_0}^\pi \psi_0 \), for each other \( \psi \in \pi(M)(\pi) \). Hence, \( \psi_0 \) of (2.2b) does not depend on the special \( \psi \), provided \( \phi_0, \eta \) are held fixed.

To see that (2.2a) is necessary, let \( \psi_0, \phi_0 \) obey \( d_B(\nu, \phi) = \|\psi_0 - \phi_0\| \). By Theorem 1.1, \( h_{\psi_0,\phi_0}^\pi \geq 0 \), and thus \( s(h_{\psi_0,\phi_0}^\pi)\psi_0 \in \pi(M)(\pi^\perp) \), by Lemma 2.4. Obviously, for \( \psi = \psi_0 \) the condition (2.2a) can be satisfied with \( \eta = s(h_{\psi_0,\phi_0}^\pi)\psi_0 \). In view of the above this completes the proof.

### 2.3. Specific applications

In the following, \( M \) will be a von Neumann algebra acting in standard form on a Hilbert space \( \mathcal{H} \), with cyclic and separating vector \( \Omega \in \mathcal{H} \). The previous results then make sense with respect to the duality representation, and can be considered for each pair \( \{\nu, \phi\}, \nu, \phi \in M_{*+} \) (normal positive linear forms).

#### 2.3.1. Modular cone and implementing vectors

Recall a few basic facts from modular theory. Let \( S_\Omega, F_\Omega, \Delta_\Omega \) and \( J_\Omega \) be the modular operations of the pair \( \{M, \Omega\} \) with their usual meanings, and which are associated to the respective actions of \( M \) and \( M' \), with their respective dense standard domains of definition \( D(S_\Omega), D(F_\Omega) \) and \( D(\Delta_\Omega) \) within \( \mathcal{H} \). Remind that \( S_\Omega \) and \( F_\Omega \) are the closures of the (usually unbounded but closable) antilinear operators which arise from the action of the hermitian conjugation \( M \Omega \ni x \Omega \mapsto x^* \Omega \) and \( M' \Omega \ni y \Omega \mapsto y^* \Omega \) on \( M \) and \( M' \), respectively. The polar decomposition of \( S_\Omega \) reads as \( S_\Omega = J_\Omega \Delta_\Omega^{1/2} \), where \( \Delta_\Omega = S_\Omega^2 S_\Omega \) is the modular operator, which is linear, and the modular conjugation \( J_\Omega \), which is antiunitary and selfadjoint, and therefore obeys \( J_\Omega^2 = 1 \).
An important feature is that by $M \ni x \mapsto J_\Omega x J_\Omega \in M'$ a $^*$-antiisomorphism between $M$ and $M'$ is given, $J_\Omega M J_\Omega = M'$ (Tomita's theorem). Especially, for each $\varphi \in \mathcal{H}$ one therefore has $p(\varphi) H = J_\Omega M J_\Omega \varphi = J_\Omega M \{J_\Omega \varphi\}$. Thus, by idempotency of $J_\Omega$ and Tomita’s theorem, the following intertwining relation is seen:

$$p(\varphi) J_\Omega = J_\Omega p'(J_\Omega \varphi).$$

Through the following setting a self-dual cone in $\mathcal{H}$ can be associated with $\{M, \Omega\}$:

$$(2.5a) \quad \mathcal{P}_\Omega^\pm = \{x J_\Omega x J_\Omega \Omega : x \in M \} = \{y J_\Omega y J_\Omega \Omega : y \in M' \}.$$  

Recall that self-duality means that the following holds:

$$p_\Omega^\pm = \{\psi \in \mathcal{H} : \langle \psi, \xi \rangle \geq 0, \forall \xi \in p_\Omega^\pm \}.$$  

The elements of this natural positive cone are (pointwise) invariant under the action of the modular conjugation $J_\Omega$, and it is known from modular theory that $p_\Omega^\pm$ (the modular cone) is generating for the real linear space $\mathcal{H}_{sa}^\Omega$ of all vectors fixed under the action of the modular conjugation:

$$(2.5b) \quad \mathcal{H}_{sa}^\Omega = \{\psi \in \mathcal{H} : J_\Omega \psi = \psi\} = p_\Omega^\pm - p_\Omega^\pm.$$  

Remark that each vector $\psi \in \mathcal{H}_{sa}^\Omega$ can be uniquely decomposed as $\psi = \xi_+ - \xi_-$, with $\xi_+, \xi_- \in p_\Omega^\pm$ and $\xi_+ \perp \xi_-$. Most importantly, to each $\nu \in M_{++}$ there is exactly one vector $\xi_\nu \in p_\Omega^\pm \cap S_M(\nu)$, and it is known since the work of [10, 14, 18] that the mapping

$$M_{++} \ni \nu \mapsto \xi_\nu \in p_\Omega^\pm$$

is onto and obeys the estimate

$$(2.6b) \quad \|\xi_\nu - \xi_\varphi\| \leq \|\nu - \varphi\|_1 \leq \|\xi_\nu - \xi_\varphi\| \cdot \|\xi_\nu + \xi_\varphi\|,$$

for any two $\nu, \varphi \in M_{++}$, which shows that both cones $M_{++}$ and $p_\Omega^\pm$ are mutually homeomorphic when considered with the topologies which are induced by the respective uniform topologies of $M_*$ and $\mathcal{H}$, respectively.

Finally, remark that in modular theory, by basic facts known from [10], to each two cyclic and separating vectors $\Omega, \Omega'$ of $M$ there is a unique $U(\Omega', \Omega) \in \mathcal{U}(M')$ such that

$$(2.7a) \quad p_{\Omega'} = U(\Omega', \Omega)p_{\Omega}.$$  

Thereby, $U(\Omega', \Omega) = 1$ is fulfilled if, and only if, $\Omega' \in p_{\Omega}^\pm$ holds. and the unitary $U(\Omega', \Omega)$ obeys the following chain rule:

$$(2.7b) \quad U(w\Omega', \Omega) = wU(\Omega', \Omega), \quad \forall w \in \mathcal{U}(M').$$

**Example 2.5.** Let $\mathcal{B}(\mathcal{H})$ act by multiplication from the left on Hilbert-Schmidt operators $\mathcal{K}$ over the separable Hilbert space $\mathcal{H}$ (call the resulting $vN$-algebra $M$). Let $\tau$ be the unique normal trace induced on $M$ by the standard trace ‘tr’. Then, since $\mathcal{K}$ is already complete under the Hilbert-Schmidt norm $\| \cdot \|_2$, which corresponds to the scalar product $\langle x, y \rangle = \text{tr} y^* x$, for $x, y \in \mathcal{K}$, in terms of Example 1.7, $\mathcal{K}$ can be identified with $L^2(M, \tau) = L^2(M, \tau)$, and the usual trace class operators uniquely correspond to $L^1(M, \tau) = L^1(M, \tau)$, see [29]. $\Omega \in \mathcal{K}$ is cyclic and separating for $M$ if, and only if, $s(\|\Omega\|) = s(\|\Omega^*\|) = 1$. In particular, each positive Hilbert-Schmidt operator with full support can be taken for $\Omega$. Also, each two
Lemma 2.6. \( \forall \) the following properties hold:

Let combining together the last two relations.
Therefore in view of the above assumption of Lemma 2.4 can be fulfilled, for each modular vector \( \varphi \), that is, whenever \( \varphi_0 \in P^\perp_{\tilde{\Omega}} \) holds, for some cyclic and separating vector \( \Omega \in \mathcal{H} \) of \( M \). In fact, if \( \{M, \Omega\} \) is a standard form algebra over \( \mathcal{H} \), then the following is true:

Lemma 2.6. \( \forall \nu, \varrho \in M_{++} : p'(\xi_{\varrho}^+) \leq s(|h_{\xi_{\nu}, \xi_\varrho}|)^+ \).

Proof. Let \( \varrho^+ \) be defined as in Theorem 1.11. Then, by \( \nu \perp \varrho^+ \) one especially has \( P(\xi_{\varrho}^+) \leq p(\xi_{\nu}^+) \). Hence, in view of (2.5b) the intertwining relations (2.4) can be applied and show that \( J_{\Omega}p(\xi_{\nu}^+)J_{\Omega} = p'(\xi_{\varrho}^+) \) and \( J_{\Omega}p(\xi_{\nu})J_{\Omega} = p'(\xi_{\varrho}) \) are fulfilled, and therefore in view of the above \( p'(\xi_{\nu}^+) \leq p'(\xi_{\varrho}^+) \) to hold. On the other hand, we have \( s(|h_{\xi_{\nu}, \xi_\varrho}|) \leq p'(\xi_{\varrho}), \) by (1.6bb). The assertion now will follow upon combining together the last two relations.

In line with formula (2.2b), for given \( \{M, \Omega\} \) consider the vector-valued map

(9.9a) \( M_{++} \times M_{++} \ni \{\nu, \varrho\} \mapsto \psi_{\Omega}^\nu(\varrho) \in S_M(\varrho) \subset \mathcal{H} \)

which in accordance with Theorem 2.3 can be defined by

(9.9b) \( \psi_{\Omega}^\nu(\varrho) = \psi_{\xi_{\nu}, \xi_\varrho}^\nu + \psi_{\xi_{\nu}, \xi_\varrho}^\nu, \forall \psi \in S_M(\varrho). \)

In accordance with Theorem 2.3 the hypothesis of Theorem 1.9 in the modular context then may be supplemented by the following additional information:

**Theorem 2.7 (\([\mathcal{D}]\)).** \( \forall \nu, \varrho \in M_{++} : d_B(\nu, \varrho) = \|\psi_{\Omega}^\nu(\varrho) - \xi_{\nu}\| \).

Much about structure and meaning of the map (2.9a) is known. In context of the following result we only mention the probably most important properties.

**Lemma 2.8.** For each \( \nu, \varrho \in M_{++} \) and cyclic and separating vector \( \Omega \in \mathcal{H} \) the following properties hold:

1. \( \psi_{\Omega}^\nu(\varrho) \) is the unique \( \tilde{\psi} \in S_M(\varrho) \) with \( d_B(\nu, \varrho) = \|\tilde{\psi} - \xi_{\nu}\| \iff \varrho^+ = 0 \);
2. \( \psi_{\Omega}^\nu(\varrho) = \psi_{\Omega}^\nu(\varrho - \varrho^+) + \xi_{\varrho}^+ \);  
3. \( \psi_{\Omega}^\nu(\varrho) \in \mathcal{H}_{\Omega}^{\psi_{\Omega}} \iff \psi_{\Omega}^\nu(\varrho - \varrho^+) \in \mathcal{P}_{\Omega}^{\psi_{\Omega}} \iff \psi_{\Omega}^\nu(\varrho - \varrho^+) \in \mathcal{H}_{\Omega}^{\psi_{\Omega}} \).

**Proof.** In the following, we let \( \psi = \psi_{\Omega}^\nu(\varrho), \ h = h_{\psi, \xi_\varrho} \). To prove (1), note first that, for \( \tilde{\psi} \) in the fibre of \( \varrho \), with \( d_B(\nu, \varrho) = \|\tilde{\psi} - \xi_{\nu}\| \), by Theorem 2.7 (\([\mathcal{D}]\)), also \( \tilde{h} = h_{\tilde{\psi}, \xi_\varrho} \geq 0 \). Also, for \( w \in M' \) with \( w^*w = p'(\psi) \) and \( \tilde{\psi} = w\psi \) in accordance with Lemma 2.4 (\([\mathcal{D}]\)) we see that \( h = \tilde{h} \) and \( s(h) = ws(h) \) have to be fulfilled. Now, suppose \( \varrho^0 \neq 0 \). Then \( s(\tilde{h})^+ \tilde{\psi} \in S_M(\varrho^+) \) cannot vanish. Hence, for each \( t \in [0, 2\pi] \), also \( \tilde{\psi}_t = s(\tilde{h})^t \tilde{\psi} + \exp(-it)s(\tilde{h})^t \tilde{\psi} \in S_M(\varrho^+) \), with \( h_{\psi_t, \xi_\varrho} = h_{\tilde{\psi}, \xi_\varrho} \geq 0 \). Hence, \( d_B(\nu, \varrho) = \|\tilde{\psi}_t - \xi_{\nu}\| \) for all \( t \in [0, 2\pi] \), by Theorem 1.3 (\([\mathcal{D}]\)). For \( t \neq 0 \) obviously \( \tilde{\psi}_t \neq \tilde{\psi} \). By Theorem 2.7 this especially applies to \( \tilde{\psi} = \psi, \) and thus for \( \varrho^+ = 0 \) nonuniqueness follows. On the other hand, in case of \( \varrho^0 = 0 \), in view of the
of the natural positive cone then implies orthogonality of the associated

\[ p'(\psi) = s(h). \]

The above condition on \( w \) in this case yields \( p'(\psi) = s(h) \) = \( w \), and therefore \( \psi = \psi \), that is, uniqueness holds.

To see (3), note first that \( \psi = s(h) \psi + \xi_\nu \) holds, by (2.91). Let \( \psi' = \psi_{\nu}^\sigma (\varrho - \varrho^\perp) \) and \( \nu' = \tilde{h}(s(h) \psi + \xi_\nu) \). Obviously we then have \( \nu' = h \geq 0 \). By Lemma 2.4.3 and Lemma 1.12 (1), \( s(h) \psi \in S_M(\varrho - \varrho^\perp) \) and \( (\varrho - \varrho^\perp)^\perp = 0 \) have to be fulfilled. Owing to \( h \geq 0 \), \( d_B(\nu, \varrho - \varrho^\perp) = \| s(h) \psi - \xi_\nu \| \), by Theorem 1.5 (3). Owing to \( (\varrho - \varrho^\perp)^\perp = 0 \), (3) can be applied to the pair \( \{ \nu, \varrho - \varrho^\perp \} \), and then yields \( \psi' = s(h) \psi \). Substituting the latter into \( \psi = s(h) \psi + \xi_\nu \), then gives (3).

Note that since \( P^2_\Omega \) is a (convex) cone, the first two of the \(' \Rightarrow \)'-implications and the third \(' \Rightarrow \)'-implication within (3), as well as the implication \( \psi_{\nu}^\sigma (\varrho - \varrho^\perp) \in \mathcal{H}^\perp \Rightarrow \psi_{\nu}^\sigma (\varrho) \in \mathcal{H}^\perp \), follow from (2.54) and (2.4). Owing to this and \( P^2_\Omega \subset \mathcal{H}^\perp \), to see the remaining other implications, it suffices to prove the implication \( \psi_{\nu}^\sigma (\varrho) \in \mathcal{H}^\perp \Rightarrow \psi_{\nu}^\sigma (\varrho - \varrho^\perp) \in \mathcal{P}^2_\Omega \). In line with this, assume \( \psi \in \mathcal{H}^\perp \). Formula (3) in accordance with (2.50) then tells us that \( \psi' \in \mathcal{H}^\perp \) holds. Also, in view of Lemma 2.4.3 and \( (\varrho - \varrho^\perp)^\perp = 0 \), one has \( \psi'(\psi') = s(h) \leq p'(\xi_\nu) \). Now, remind the fact mentioned on in context of (2.51) and saying that in this case \( \psi' = \eta_\nu - \eta_\nu \) has to be fulfilled with vectors \( \eta_\nu, \eta_\nu \in P^2_\Omega \), with \( \eta_\nu \perp \eta_\nu \). Let \( \nu_\pm \in M^\pm_+ \) be the positive linear forms implemented by \( \eta_\nu \). By uniqueness of the map of (2.6a) one has \( \xi_\nu = \eta_\nu \), and by orthogonality \( \eta_\nu \perp \eta_\nu \) and since \( \| \nu_\pm \|_1 = \| \nu_\pm \|_1 \) is fulfilled, from estimate (2.6b) the relation \( \| \nu_+ - \nu_- \|_1 = \| \nu_+ \|_1 + \| \nu_- \|_1 \) can be inferred. Hence, \( \nu_+ \perp \nu_- \), that is \( p(\eta_\nu) \perp p(\eta_\nu) \) holds. With the help of (2.50) and (2.4) then also \( \psi'(\eta_\nu) \perp \psi'(\eta_\nu) \). Hence, \( \varrho - \varrho^\perp = \nu_+ + \nu_- \). The latter implies \( p(\eta_\nu) \perp p(\psi) \). Since by assumption \( J_\Omega \psi' = \psi' \), holds, in reasoning with (2.4) once more again from this in view of the above \( \psi'(\nu_\pm) \leq p'(\psi) = s(h) \leq p'(\xi_\nu) \) can be seen. But then especially \( \psi'(\eta_-) \psi' = \eta_- \), from which \( h_{\psi, \xi_\nu}(\psi'(\eta_-)) = \eta_- \xi_\nu \) has to be followed. Now, by Theorem 2.7 and Theorem 1.5 (3), when applied to the pair \( \{ \nu, \varrho - \varrho^\perp \} \), positivity of \( h_{\psi, \xi_\nu} \) can be seen. In line with the previously derived then especially \( \langle \eta_- \xi_\nu \rangle \geq 0 \) has to hold, for \( \eta_- \xi_\nu \in P^2_\Omega \). Since the scalar product between vectors of the natural positive cone has to be always a nonnegative real, \( \langle \eta_- \xi_\nu \rangle = 0 \) is inferred to hold. As argued above, orthogonality among vectors of the natural positive cone then implies orthogonality of the associated \( p^- \)- and \( p^\perp \)-projections, respectively. Thus especially \( \psi'(\eta_-) \perp \psi'(\xi_\nu) \). This in view of \( \psi'(\eta_-) \leq p'(\xi_\nu) \) implies \( \psi'(\eta_-) = 0 \). Hence \( \eta_- = 0 \), and thus \( \psi' \in P^2_\Omega \) holds, and which is (3).

Without proof mention some additional facts about (2.91), see 2.3 3 4 5 6.

Lemma 2.9. Let \( \Omega \) be a cyclic and separating vector of \( M \). Let \( \nu, \varrho \in M^\perp_+ \), and let \( f \in M_\ast \) be defined by \( f = \langle \varphi | M \psi_{\nu}^\sigma (\varrho), \xi_\nu \rangle \), with \( \psi_{\nu}^\sigma (\varrho) \) of (2.91). The following properties hold:

1. \( \psi_{\nu}^\sigma (\varrho) \) is continuous at each \( \{ \nu, \varrho \} \) with \( \varrho^\perp = 0 \);
2. \( \psi_{\nu}^\sigma (\varrho) \in \mathcal{P}^2_\Omega \iff f = f^\ast \iff f \geq 0 \).

The normal linear form \( f \) defined in the premises of the previous Lemma is a quite interesting structure. First of all, by definition of \( f \) and according to Theorem 2.7, Theorem 1.3 (3) and Lemma 1.4 one has

\[
(2.10) \quad f(1) = \langle \psi_{\nu}^\sigma (\varrho), \xi_\nu \rangle = h_{\psi_{\nu}^\sigma (\varrho), \xi_\nu}(1) = \sqrt{P_M(\nu, \varrho)}. 
\]
Secondly, since orthogonality of $g^\perp$ to $\nu$ also implies $p'(\xi_{g^\perp}) \perp p'(\xi_\nu)$, from (2.9b) especially also follows that $f$ can be rewritten as following:

\[ f = \langle (\cdot)_M \psi'_{\Omega}(g), \xi_\nu \rangle = \langle (\cdot)_M v_{\xi_\nu, \xi_{g^\perp}, \xi_\nu}, \xi_\nu \rangle = \langle (\cdot)_M v^*_{\xi_\nu, \xi_{g^\perp}, \xi_\nu}, s(h^*_{\xi_{g^\perp}, \xi_\nu}) \xi_\nu \rangle, \]

where also (1.6b) and the relation $h^*_{\xi_{g^\perp}, \xi_\nu}$ have been taken into account. Note that according to Lemma 2.4 (3), $v^*_{\xi_{g^\perp}, \xi_\nu} \in S_M(g - g^\perp)$ and $s(h^*_{\xi_{g^\perp}, \xi_\nu}) \xi_\nu \in S_M(\nu - \nu^\perp)$ hold. But then, (2.10a) and (2.10b) prove that $f$ obviously obeys the conditions (1) and (2) in Theorem 1.13. Hence, by Theorem 1.13, $g = f$ is identified.

On the other hand, by an analogous line of reasoning, $f' = \langle (\cdot)_M \xi_\nu, \psi'_{\Omega}(\nu) \rangle$ is obeying conditions Theorem 1.13 (1)–(2). Hence, by Theorem 1.13 again, $g = f'$. We may summarize this as follows.

**Lemma 2.10.** For $\nu, g \in M_{++}$, the unique $g \in M^*$ given by Theorem 1.13 is

\[ g = f = \langle (\cdot)_M \psi'_{\Omega}(g), \xi_\nu \rangle = \langle (\cdot)_M \xi_\nu, \psi'_{\Omega}(\nu) \rangle. \]

In particular, from this follows that $f$ for a given pair $\{\nu, g\}$ is uniquely determined by the associated minimal pair, see Definition 1.13/Remark 1.14.

2.3.3. Bures distance and commutation of states. Those pairs $\{\nu, g\}$ of normal positive linear forms and fulfilling $\psi'_{\Omega}(g) \in \mathcal{P}_\Omega^2$ deserve special interest.

**Definition 2.11.** Suppose $\nu, g \in M_{++}$, and be $\Omega \in \mathcal{H}$ a cyclic and separating vector. $g$ is said to commute with $\nu$ if $\psi'_{\Omega}(g) = \xi_{g^\perp}$ is fulfilled.

We are going to show that commutation is a symmetric relation.

**Lemma 2.12.** For each $\nu, g \in M_{++}$ the following are equivalent:

1. $g$ commutes with $\nu$;
2. $\nu$ commutes with $g$.

**Proof.** In view of the symmetry of the assertion, we may content ourselves with proving e.g. that (1) implies (2). Suppose (1). In line with this and Definition 2.11, $\psi'_{\Omega}(g) = \xi_{g^\perp}$. By Theorem 2.7 and Theorem 1.13 (1), we then have $h = h_{\xi_\nu, \xi_{g^\perp}} \geq 0$. From this with the help of Tomita’s theorem and owing to (2.5b), for each $x \in M$, and corresponding $y \in M'$ with $y = J_\Omega x J_\Omega$, we infer that $f(x^*x) = \langle x^*x J_\Omega \psi'_{\Omega}(g), J_\Omega \xi_\nu \rangle = \langle \xi_\nu, J_\Omega x^*x J_\Omega \psi'_{\Omega}(g) \rangle = \langle \xi_\nu, J_\Omega (y^*y) \xi_{g^\perp} \rangle = h(y^*y) \geq 0$. According to Lemma 2.10 we then also have $\langle (\cdot)_M \xi_{g^\perp}, \psi'_{\Omega}(\nu) \rangle = f \geq 0$. This is the same as $\langle (\cdot)_M \psi'_{\Omega}(\nu), \xi_{g^\perp} \rangle \geq 0$. The conclusion of Lemma 2.3 (4) when applied to $\{g, \nu\}$ then yields $\psi'_{\Omega}(\nu) = \xi_\nu$, that is, also $\nu$ commutes with $g$. \(\square\)

By Theorem 2.7, in case of commutation the estimate $d_B(\nu, g) \leq \|\xi_\nu - \xi_{g^\perp}\|$ turns into an equality. The conjecture is that the latter behavior is equivalent to commutation between $\nu$ and $g$, generally (though this is plausible it is not an obvious matter). For those pairs $\{\nu, g\}$ on a general $M$ however, which will be of interest throughout the rest of the paper, the conjecture can be justified easily. This will be done now (see also § 6, Theorem 6.15 for the case of minimal pairs).

**Lemma 2.13.** Assume $\nu, g \in M_{++}$ with either $\nu^\perp = 0$ or $g^\perp = 0$. The following are equivalent:

1. $g$ commutes with $\nu$;
2. $d_B(\nu, g) = \|\xi_\nu - \xi_{g^\perp}\|$. 
Proof. The implications (1) \implies (2) follows at once from Definition 2.11 and Theorem 2.7. Let \( \nu \) be fulfilled, and be \( g^\perp = 0 \). Then, by Theorem 2.7 and Lemma 2.8 (1), \( \psi_\nu^g(\varrho) = \xi_\varrho \), and (1) is seen. In case of \( \nu^\perp = 0 \) the same reasoning by Theorem 2.7 and Lemma 2.8 (1) for the pair \( \{ g, \nu \} \) will yield \( \psi_\nu^g(\nu) = \xi_\nu \), which in view of Lemma 2.12 amounts to (1) again.

On the other hand, in important special cases of \( M \), like matrix algebras and, more generally, \( vN \)-algebras of type I, the conjecture can be shown easily, for each pair of normal positive linear forms. For the type-I\( \infty \)-factor a proof will be given.

Example 2.14. Suppose \( M \simeq B(\mathcal{H}) \) and \( \tau \) as in Example 2.5, and suppose \( \Omega > 0 \). Let \( \nu = \tau_a, \varrho = \tau_c \), with positive trace class operators \( a \) and \( c \). Suppose \( \varrho \) commutes with \( \nu \). By (1.1), (2.8b), (1.4a) and Theorem 2.7, Definition 2.11 then amounts to

\[
\tau(\sqrt{a} \sqrt{c}) = \sqrt{P_M(\nu, \varrho)} = \tau(\sqrt{a} \sqrt{c}^\perp).
\]

Since \( \tau \) is faithful, by (1.4f) the previous is equivalent to commutation \( ac = ca \).

On the other hand, suppose \( \nu = \tau_a, \varrho = \tau_c \), with positive trace class operators \( a \) and \( c \) which are commuting, in the sense of operator theory. Then, since \( M \) corresponds to right actions of bounded linear operators on Hilbert-Schmidt operators, in view of (2.8b) we have

\[
(\ast) \quad h(\cdot) = h_{\xi_\varrho, \xi_\nu}(\cdot) = \tau(\sqrt{a} \sqrt{c}^\perp) \geq 0.
\]

It follows that \( s(h) \) corresponds to right multiplication with \( s(\sqrt{a} \sqrt{c}) = s(a)s(c) \) (remind that for commuting \( a, c \) also \( s(a) \) and \( s(c) \) must commute). In accordance with (1.6) we then have that \( \eta = s(h)^\perp \xi_\varrho = \sqrt{c}(s(a)s(c))^\perp \geq 0 \) must be implementing for \( g^\perp \). In view of (2.8a) we then even have \( \xi_\varrho^\perp = \sqrt{c}(s(a)s(c))^\perp \). But then, the formula (2.9b) in view of (\ast) reads as

\[\psi_\nu^g(\varrho) = s(h)\xi_\varrho + \xi_\varrho^\perp = \sqrt{c}s(a)s(c) + \sqrt{c}(s(a)s(c))^\perp = \sqrt{c} = \xi_\varrho,\]

that is, \( \varrho \) commutes with \( \nu \) in the sense of Definition 2.11. Hence, \( ac = ca \) is equivalent to commutation of \( g \) with \( \nu \). Since according to (2.8b), (1.4b) and as remarked above \( ac = ca \) is also equivalent to \( d_B(\nu, \varrho) = \|\xi_\nu - \xi_\varrho\| \), and since by Lemma 2.13 commutation is symmetric in \( \nu \) and \( \varrho \), we may summarize as follows:

For \( M \simeq B(\mathcal{H}) \) and normal positive linear forms \( \nu = \tau_a \) and \( \varrho = \tau_c \), commutation of \( \nu \) with \( g \) (resp. \( \varrho \) with \( \nu \)) in the sense of Definition 2.11 is equivalent to commutation of the operator densities \( a \) and \( c \), and both are also equivalent to the occurrence of the relation \( d_B(\nu, \varrho) = \|\sqrt{a} - \sqrt{c}\|_2 = \|\xi_\nu - \xi_\varrho\| \).

Finally, note that owing to eqs. (2.7) in the case at hand we must have equivalence of commutation to \( d_B(\nu, \varrho) = \|\xi_\nu - \xi_\varrho\| \), for each cyclic and separating \( \Omega \) (and not just for only those with \( \Omega > 0 \)). This especially proves that the assertion of Lemma 2.13 remains true, for each pair \( \{ \nu, \varrho \} \) of normal positive linear forms on a factor \( M \) of type \( I_\infty \) (finite factors were dealt with in (3)).

Without proof mention yet some generalities on ‘commutation’, see [2, 3, 4, 5].

Remark 2.15. 1. Following [4, p. 325], in a C*-algebraic context ‘commutation’ might be defined by requiring \( g \geq 0 \), for \( g \) defined in Theorem 1.15. In line with Lemma 2.3 (4) and Lemma 2.10, in the standard form case this then amounts to Definition 2.11.

2. Note that the notion of commutation also does not depend on the special \( \Omega \) used (cf. (2.7a) for the reasons of this covariance, see [10]). It extends
to arbitrary normal positive linear forms the notion of ‘commutation’ in the faithful case, and which refers to commutation of the respective modular *-automorphism groups, see 

3. According to Lemma 2.12 and Lemma 2.8 (3) one infers that commutation must be equivalent to commutation between the positive linear forms of the associated minimal pair. Note that from Lemma 2.8 (3) it follows that both $\nu = (\nu - \nu^*) + \nu^*$ and $\nu = (\nu - \nu^*) + \nu^*$ have to be orthogonal decompositions, in this situation.

4. By standard facts originating from [13] it is not hard to see that, for given $\nu, \varrho \in M_+$, $j(\nu \varrho) = j^2(\nu \varrho) = \frac{1}{2} ||J_\Omega \psi(\nu) - \psi J_\Omega(\nu)||^2$ proves to be independent from the special choice of the cyclic and separating vector $\Omega$. But then, in view of (2.51), Lemma 2.8 (3) and Lemma 2.12, $j(\nu \varrho) = 0$ (resp. $j(\nu \varrho) = 0$) if, and only if, $\nu$ commutes with $\varrho$. Thus, $j(\nu \varrho)$ has some invariant meaning which in view of the definition of the above notion of commutation might serve as a quantitative measure estimating how far from ‘commuting’ with $\varrho$ the form $\nu$ is.

2.3.4. A characterization of $S_M(\nu | \varrho)$. We are going to characterize $S_M(\nu | \varrho)$ for a standard form $\nu \varrho$N-algebra in terms of either extension properties of partial isometries of $M$, or by comparison properties of certain characteristic orthoprojections in respect of the Murray-von Neumann comparability relation ‘$\succ$’, respectively.

**Theorem 2.16.** Let $\nu, \varrho \in M_+$, and be $\psi = \psi \varrho(\varrho) = h = h_{\psi\varrho(\varrho)}$. Then, $h \geq 0$, and for $u \in M'$ with $u^* u = p(\xi \varrho)$ the following conditions are mutually equivalent:

1. $u \xi \varrho \in S_M(\nu | \varrho)$;
2. $\exists w \in M'$, $w^* w = p'(\psi) : u s(h) = w s(h)$;
3. $u s(h) u^* = p'(\psi) - s(h)$. 

**Proof.** By Theorem 2.7 and Theorem 2.11 one has $h \geq 0$. Thus $v_{\psi \xi \varrho} = s(h)$. By the last part of Theorem 2.3 and (2.22) we then see $\psi = s(h) \psi + \xi \varrho$. Hence $s(h) \psi = \xi \varrho$. Also, since $|h_{\xi \varrho} | = |h_{\psi \xi \varrho} | = |h| = h$ holds, Lemma 2.4 (3) may be applied with respect to $\nu^* \varrho$ and then yields $s(h) \xi \varrho \in S_M(\nu^* \varrho)$. Note that in accordance with Theorem 2.11 especially also $\nu^* \varrho \perp \varrho$ must be fulfilled. This condition is equivalent to $\langle x s(h)^\perp \psi, w s(h)^\perp \xi \varrho \rangle = 0$, for all $x, y \in M'$. Now, let $u, w \in M'$ be chosen in accordance with (3). By the previous $s(h)^\perp \psi \perp s(h)^\perp \xi \varrho$ and $w s(h)^\perp \psi \perp s(h)^\perp \xi \varrho$ have to be fulfilled. Also, owing to $u^* w \geq s(h)$ and $u^* u \geq s(h)$ and by (2) the following orthogonality relations can be followed:

$$\psi = s(h) \psi - s(h)^\perp \xi \varrho \perp s(h)^\perp \xi \varrho \perp s(h)^\perp \xi \varrho$$

Hence, both $\{ s(h) \psi - s(h)^\perp \xi \varrho \}$, $s(h)^\perp \psi$, $s(h)^\perp \xi \varrho$ and $\{ w s(h) \psi - s(h)^\perp \xi \varrho \}$, $s(h)^\perp \psi$, $s(h)^\perp \xi \varrho$ are orthogonal systems of vectors, and therefore we may conclude as follows:

$$||w \psi - u \xi \varrho||^2 = ||w s(h) \{ \psi - s(h) \} + s(h)^\perp \psi - s(h)^\perp \xi \varrho \|^2$$

$$= ||w s(h) \{ \psi - s(h) \} ||^2 + ||s(h)^\perp \psi ||^2 + ||s(h)^\perp \xi \varrho ||^2$$

$$\leq ||s(h) \{ \psi - s(h) \} ||^2 + ||s(h)^\perp \psi ||^2 + ||s(h)^\perp \xi \varrho ||^2$$

$$= ||s(h) \{ \psi - s(h) \} + s(h)^\perp \psi - s(h)^\perp \xi \varrho ||^2$$

Hence, in view of Theorem 2.15 $||w \psi - u \xi \varrho|| \leq d_B(\nu, \varrho)$ must be fulfilled. Since $w \psi \in S_M(\varrho)$ and $u \xi \varrho \in S_M(\nu)$ hold, from this by Theorem 1.7 equality is inferred, $d_B(\nu, \varrho) = ||w \psi - u \xi \varrho||$. That is, (3) has to be true.
Suppose (1) is fulfilled. By definition of $S_M(\nu|\varrho)$ and in view of (1.1) there has to exist $w \in M'$ with $w^*w = p'(\psi)$ such that $d_B(\nu, \varrho) = \|w\psi - u_{\xi_\varrho}\|$ holds. In view of Theorem 2.3 and Lemma 2.4, then $h_{w\psi,u_{\xi_\varrho}} \geq 0$. Hence, $h(u^*(\cdot)w) \geq 0$, on $M'$. From $h \geq 0$ with $s(h) = p'(\xi_\varrho)$ we then infer that $h_{w\psi,u_{\xi_\varrho}} = h(u^*(\cdot)w) = h(u^*(\cdot)w^*w) \geq 0$. Let $q \in M'$ be the orthoprojection $q = us(h)u^*$. Then $qu = us(h)$, and thus the previous formula reads as $h_{w\psi,u_{\xi_\varrho}} = h(u^*(\cdot)(wu^*)q)u = g((\cdot)wu^*)q) \geq 0$, with $g = h(u^*(\cdot)u) \geq 0$. Note that $s(q) = q$, and also $quwu^* = us(h)p'(\psi)s(h)u^* = us(h)u^* = q$. Thus $wu^*q$ is a partial isometry in $M'$ with initial projection $s(q)$ and obeying $g((\cdot)wu^*)q) \geq 0$. By uniqueness of the polar decomposition then $wu^* = s(q) = s(\varrho)$ follows. Thus $s(\varrho) = q$ exists. By definition of $s(\varrho) = wu^*qu = qu = us(h)h$ follows, which is the condition in (1). Thus, in view of the above (1) is equivalent to (2).

Note that (3) is equivalent to $v^*v = p'(\psi) - s(h)$ and $vv^* \leq us(h)u^*$. For some $v \in M'$. This is the same as requiring $v^*v = p'(\psi) - s(h)$ with $v^*us(h) = 0$. In defining $w = us(h) + v$ we then have $w^*w = s(h)u^*us(h) + p'(\psi) - s(h) = s(h) + p'(\psi) - s(h) = p'(\psi)$, with $us(h) = us(h)$. On the other hand, for each $w$ with $w^*w = p'(\psi)$ and $us(h) = us(h)$ one has that $w(p'(\psi) - s(h)\psi^* \leq us(h)u^*$. Hence, $v = w\{p'(\psi) - s(h)\}$ is a partial isometry, with $v^*v = p'(\psi) - s(h)$ and $vv^* \leq us(h)u^*$. Thus (3) is equivalent to (2).}

\begin{remark}
Suppose $M$ is a unital C*-algebra, and $\pi$ is a unital *-representation such that the $\pi$-fibres of given two positive linear forms $\nu, \varrho$ exist. Then, due to the general character of Theorem 2.3 and Lemma 2.4, and since owing to Theorem 1.9 in the $\pi$- fibre of $\nu$ a vector $\varphi_0$ and corresponding $\eta$ and associated $\psi_0$ obeying eqs. (2.2) exist, Theorem 2.16 can be seen as a special case of a (less specific) result obtained for unital C*-algebras and characterizing $S_{\pi,M}(\nu|\varrho)$ in terms of either extension properties of partial isometries of $\pi(M)'$, or by comparison properties of certain orthoprojections within the $\nu$-algebra $\pi(M)'$, respectively. It is obvious from the previous proof how this more academic variant should read then (with $\varphi_0, \eta$ and $\psi_0$ instead of $\xi_\varrho, \xi_\varrho^\perp$ and $\psi$).
\end{remark}

2.3.5. Modular vectors and the structure of $S_M(\nu|\varrho)$. Start with some generalities about $S_M(\nu)$ and $S_M(\nu|\varrho)$, and which have to be fulfilled, for each $\nu \in M_{++}$ and each pair $\{\nu, \varrho\} \subset M_{++}$, respectively.

Note that according to eqs. (2.3) and with respect to a (particular) given standard form action $\{M, \Omega\}$ the following has to be fulfilled:

$$U(M')\xi_\nu = S_M(\nu) \cap \left\{ \cup_{\Omega'} P^\perp_{\Omega'} \right\}.$$  

Thereby, $\Omega'$ may extend over the cyclic and separating vectors of $M$ within $\cal H$. Due to (2.11) we will refer to $U(M')\xi_\nu$ as set of modular vectors of $S_M(\nu)$. First of all, what will be done is to relate $S_M(\nu)$ and $S_M(\nu|\varrho)$ topologically to the subset of modular vectors of the fibre of $\nu$.

It is plain to see that Theorem 2.16 (2) can be satisfied by each partial isometry $u \in M'$ which can be obtained from an isometry $v \in M'$ as $w = vp'(\xi_\nu)$. In fact, owing to $v^*v = 1$ and $s(h) \leq p'(\psi) \wedge p'(\xi_\nu)$, $w = vp'(\psi)$ can be taken there. In view of Theorem 2.16, and since unitaries are special isometries, we thus have the following net of inclusions to hold, in each case of $\nu, \varrho \in M_{++}$:

$$U(M')\xi_\nu \subset \{ w\xi_\nu : w^*w = 1, w \in M' \} \subset S_M(\nu|\varrho) \subset S_M(\nu).$$

In case of finite $M'$ (which occurs iff $M$ is finite) equality has to occur throughout, since then the whole fibre is made of modular vectors, see Remark 2.2. On the other
hand, in case of properly infinite $M'$ by standard facts it is known that the strong
(operator) closure of the unitary group $\mathcal{U}(M')$ is the set of isometries of $M'$. But
then, since a general nonfinite $M'$ can be (centrally) decomposed into two parts
which are finite and properly infinite, respectively, the above facts relating the finite
and properly infinite case of $M'$ and (2.12) fit together into the following result ($^{st}$
indicates the closure with respect to the strong operator topology on $M$):

\begin{equation}
(2.13a) \quad \mathcal{U}(M')\xi_v \subset \{ w\xi_v : w^*w = 1, w \in M' \} = \overline{\mathcal{U}(M')^{st}} \xi_v \subset S_M(\nu | g).
\end{equation}

On the other hand, from (2.12) and by closedness of the fibre we also infer

\begin{equation}
(2.13b) \quad \overline{\mathcal{U}(M')^{st}} \xi_v \subset \overline{\mathcal{U}(M')\xi_v} \subset S_M(\nu).
\end{equation}

Finally, for completeness, note that also the fibre itself is related to the modular
vectors. Let, for a subset of vectors $\Gamma \subset H$ and real $r > 0$, $\Gamma[r]$ be defined by
$\Gamma[r] = \{ \varphi \in \Gamma : \| \varphi \| = r \}$. Then, the formula for $S_M(\nu)$ reads as follows (‘conv’
stands for the operation of taking the convex hull):

\begin{equation}
(2.13c) \quad S_M(\nu) = \text{conv} \overline{\mathcal{U}(M')\xi_v} \left[ \sqrt{\| \nu \|_1} \right].
\end{equation}

In fact, according to [17], especially $(M')_1 = \text{conv} \overline{\mathcal{U}(M')}$ (uniform closure) must
be fulfilled. Hence, in view of (1.3), $S_M(\nu) \subset \text{conv} \overline{\mathcal{U}(M')\xi_v} \subset \text{conv} \overline{\mathcal{U}(M')\xi_v}$
be followed. On the other hand, suppose $\varphi \in H$, with $\| \varphi \| = \| \nu \|_1 = \nu(1)$ and
$\varphi = \lim_{n \to \infty} \varphi_n \in \text{conv} \mathcal{U}(M')\xi_v$. Let us define $\nu_n \in M_{s+}$
by the condition that $\varphi_n \in S_M(\nu_n)$. Then, by the previous especially $\varphi_n = a_n\xi_v$, with
$a_n \in (M')_1$, and thus $\nu_n \leq \nu$, for each subscript. But then $\| \nu - \nu_n \|_1 = \nu(1) - \nu_n(1) = \| \varphi \|^2 - \| \varphi_n \|^2$. In view of the above
$\nu = \lim_{n \to \infty} \nu_n$, follows, in the uniform sense. Hence, conv$\overline{\mathcal{U}(M')\xi_v} \left[ \sqrt{\| \nu \|_1} \right] \subset S_M(\nu)$ holds. Taking together the
previous inclusions yields (2.13c).

Note that if $\nu$ is faithful, that is, $1 = s(\nu) = p(\xi_v)$ is fulfilled, then owing to (2.4)
one also has $p'(\xi_v) = 1$, and then in view of (1.3), (2.12) and eqs. (2.13) we infer

\begin{equation}
(2.14) \quad S_M(\nu | g) = \overline{\mathcal{U}(M')^{st}} \xi_v = \overline{\mathcal{U}(M')\xi_v} = \text{conv} \overline{\mathcal{U}(M')\xi_v} \left[ \sqrt{\| \nu \|_1} \right] = S_M(\nu),
\end{equation}

for each $g \in M_{s+}$. Thus, from point of view of Problem 2.14(1) those pairs with
faithful $\nu$ will not be of any interest.

In contrast to the previous, the following observation likely yields the most im-
portant class of pairs $\{ \nu, g \}$ where an easy to handle formula for $S_M(\nu | g)$ can be
derived, and which is nontrivial insofar as (2.14) then need not be true.

To explain the formula, suppose in the notations of Theorem 2.16, with $\psi = \psi'_g(\varphi)$
and $h = h_{\psi'_g(\varphi), \xi_v}$, for given $\{ \nu, g \}$ the conditions

\begin{equation}
(2.15a) \quad p'(\psi) = 1, \quad s(h) = p'(\xi_v)
\end{equation}

to be fulfilled. Then, by Theorem 2.16[3]-[4], for given $u \in M'$ with $w^*w = 1$
and $u\xi_v \in S_M(\nu | g)$ we find $w \in M'$ with $w^*w = 1$ such that $u = wp'(\xi_v) =
us(h) = ws(h) = wp'(\xi_v)$. Hence, $u\xi_v = wp'(\xi_v)\xi_v = u\xi_v$. Thus, in this case
$S_M(\nu | g) \subset \{ w\xi_v : w^*w = 1, w \in M' \}$, and then in view of (2.13c) the formula

\begin{equation}
(2.15b) \quad S_M(\nu | g) = \{ w\xi_v : w^*w = 1, w \in M' \} = \overline{\mathcal{U}(M')^{st}} \xi_v
\end{equation}
can be followed. In view of (2.13c) the closure of the latter set then reads as

\begin{equation}
(2.13b) \quad S_M(\nu | g) = \overline{\mathcal{U}(M')\xi_v}.
\end{equation}
Example 2.18. The easiest way to cope with the premises of (c) is to assume that ρ is faithful, and that ν is commuting with ρ.

In fact, for faithful ρ, by Theorem 1.11 ν⊥ = 0 with respect to ρ, for each ν ∈ M+. Remind that h ≥ 0, by Theorem 2.16. Hence, \( \{ p'(ξ_ν) - s(h)\} ξ_ν = s(h)\xi_ν = 0 \), by Lemma 2.4[3]. Thus p'(ξ_ν) = s(h). By assumption and Lemma 2.12, ψ = ξ_ν. Hence, in view of (2.4) and (2.5b), faithfulness of ρ, which means that ψ is separating, implies the modular vector ψ to be also cyclic, p'(ψ) = 1. Thus, the conditions of (c) are satisfied and, as mentioned above, both formulae of eqs. (2.13) then must be fulfilled.

Remark 2.19. It is easy to see that (2.15a) is a special case of the formula

\[
S_M(ν|ρ) = \{ wξ_ρ : w^*w = p'(ψ), w ∈ M' \}
\]

which holds provided ν⊥ = 0 is fulfilled. But note that in general it is difficult to say something definite about p'(ψ) without having some additional informations on the pair \{ν, ρ\} and which go beyond ν⊥ = 0. Even for a faithful ρ, which implies ν⊥ = 0 e.g., in general only p'(ψ) ∼ 1 can be followed. Note that from (2.15a) especially follows that under the premises of Example 2.18 the modular vectors in the fibre of ν are dense within SM(ν|ρ).

2.3.6. When does SM(ν|ρ) provide the whole fibre? We continue with conditions under which Problem 2.3[3] can be affirmatively answered, for a pair ν, ρ ∈ M+, and which will lead us far beyond the finite case or the situation known from (2.14).

Lemma 2.20. Let ρ⊥ = 0, or be s(ν) finite. Then SM(ν|ρ) = SM(ν).

Proof. Suppose ρ⊥ = 0 first. By Lemma 2.4[3] then \{ p'(ψ) - s(h) \} ψ = s(h)⊥ψ = 0. Hence p'(ψ) = s(h), and thus the condition of Theorem 2.16[3] can be fulfilled in a trivial way, for each u ∈ M', u*u = p'(ξ_ν). In view of (1.3) the fact then follows.

If the support orthoprojection s(ν) = p(ξ_ν) is finite, then an application of (2.4) yields that also p'(ξ_ν) is finite (and vice versa), and so has to be also each other p'(χ), for χ ∈ SM(ν). Let χ = uξ_ν, u*u = p'(ξ_ν), with u ∈ M'. Since also z = p'(ξ_ν) ∈ M' is finite, by 2.2.4[1] u ∈ zM'*z extends to a unitary u1 ∈ zM'z. The partial isometry w = (u1 + z^⊥)p'(ψ) then obeys us(h) = ws(h) and w^*w = p'(ψ), and thus Theorem 2.16 can be applied.

Remark 2.21. Under the same premises on π as in Remark 2.19, there must be a C*-variant of that part of Lemma 2.20 which refers to the assumption ρ⊥ = 0, accordingly. This together with (1.7) will show that for each such π by the condition ρ⊥ = 0 then SM_π,M(ν|ρ) = SM_π,M(ν) will be implied.

Another observation provides conditions under which existence of cyclic elements in a fibre allows us to decide on whether or not SM(ν|ρ) = SM(ν) would hold.

Lemma 2.22. Suppose ν, ρ ∈ M+. The following assertions are valid.

1. Suppose ρ⊥ ≠ 0 and ν⊥ = 0. If a cyclic vector χ in the fibre of ν exists, then each such vector obeys χ ∉ SM(ν|ρ). Hence then SM(ν|ρ) ≠ SM(ν).

2. In the factor case of M, if SM(ν|ρ) ≠ SM(ν) holds, then ρ⊥ ≠ 0 and there exists a cyclic vector χ in the fibre of ν.

Proof. According to Lemma 2.4[3] and by positivity of h the assumptions on ρ⊥ and ν⊥ equivalently read as \{ p'(ψ) - s(h) \} ψ = s(h)⊥ψ ≠ 0 and \{ p'(ξ_ν) - s(h) \} ξ_ν =
$s(h)^{-1}\xi_\nu = 0$. Thus $p'(\psi) > s(h)$ and $p'((\xi_\nu) = s(h)$. Suppose a cyclic $\chi$ in the fibre of $\nu$ to exist. Then, in view of (3.4) there is $u \in M'$ with $u^* u = p'((\xi_\nu) and $\chi = u\xi_\nu$. Thus especially $u u^* = p'((\chi) = 1$ and $p'(\psi) > s(h)$. It is obvious that Theorem 2.16 cannot be satisfied by $u$. Thus $\chi = u\xi_\nu \notin S_M(\nu|\varrho)$, which is (3).

To see (2), suppose now $M$ to be a factor, and $S_M(\nu|\varrho) \neq S_M(\nu)$. By Lemma 2.20 we then have $\varrho^\perp \neq 0$ and infinite $s(\nu)$. Remember that, $M$ being a standard form $\nu$:N-algebra, $M$ admits faithful normal positive linear forms. Hence $M$ is a countably decomposable factor. Thus there can exist only one equivalence class of infinite orthoprojections in $M$, and therefore $p(\xi_\nu) = s(\nu) \sim 1$ must hold. From the latter in view of (2.4) and (2.5b) the relation $p'(\xi_\nu) \sim 1$ follows. Thus, a cyclic vector has to exist in $S_M(\nu)$, and $\varrho^\perp \neq 0$ by the above.

Since in the factor case and with faithful $\varrho$ the condition $\nu^\perp = 0$ is fulfilled for any $\nu \in M_{++}$, in this case Lemma 2.23 and (3) then provide sufficient and necessary conditions for $S_M(\nu|\varrho) \neq S_M(\nu)$ to occur, or equivalently, for Problem 2.1(3) to have an affirmative answer. Thus, this case also the opposite direction of the implication in Lemma 2.20 holds true.

**Theorem 2.23.** Let $M$ be a factor which acts in standard form. For $\nu, \varrho \in M_{++}$ with faithful $\varrho$, $S_M(\nu|\varrho) = S_M(\nu)$ holds if, and only if, $\varrho^\perp = 0$ or $s(\nu)$ is finite.

**Proof.** As mentioned, under the given premises $\nu^\perp = 0$. Thus Lemma 2.22 equivalently says that $S_M(\nu|\varrho) = S_M(\nu)$ holds iff $\varrho^\perp = 0$ or $p'(\xi_\nu) \neq 1$. The latter is equivalent to $s(\nu) = p(\xi_\nu) \neq 1$, by (2.4) and (2.5b). Since $M$ is a countably decomposable factor, this is the same as asserting $s(\nu)$ to be finite. \qed

**Remark 2.24.** For finite dimensional or commutative $M$ a test on $\varrho^\perp \neq 0$ can be achieved easily. In fact, in all these cases $\varrho^\perp \neq 0$ proves equivalent to the condition $s(\varrho) \wedge s(\nu^\perp \neq 0$. Thus, in these cases for the question on $\varrho^\perp \neq 0$ only the mutual position of the respective support orthoprojections within the projection lattice of the algebra matters, refer to [13] for that.

Unfortunately, in the infinite dimensional, noncommutative cases of $M$, the condition $s(\varrho) \wedge s(\nu^\perp \neq 0$ in general does not suffice any longer to imply $\varrho^\perp \neq 0$. For convenience of the reader we include a counterexample. Thereby, in view of (1.7) for the effect to appear it cannot be of relevance whether $M$ acts in standard form or not, and thus for simplicity e.g. the algebra $M = B(\mathcal{H})$ of all bounded linear operators over some separable, infinite-dimensional Hilbert space will be considered.

**Example 2.25.** Let $\{\varphi_k : k \in \mathbb{N}\}$ be a complete orthonormal system of vectors within $\mathcal{H}$, and $\psi \in \mathcal{H}$ be a unit vector with $\langle \psi, \varphi_k \rangle \neq 0$, for all $k \in \mathbb{N}$. Let pure normal states $\nu_k$ be defined as $\varphi_k \in S_M(\nu_k)$, and for $0 < \beta < 1$ define $\varrho \in M_{++}$ by $\varrho = \sum_{k \in \mathbb{N}} \beta^k |\psi, \varphi_k \rangle \langle \psi, \varphi_k |$. Let $\nu \in M_{++}$ be any normal positive linear form with support orthoprojection $s(\nu) = p_\psi^\perp$, where $p_\psi$ is the orthoprojection onto $C \psi$.

Thus, $\varrho$ is faithful and $\nu$ has support with codimension one. We now consider $\varrho^\perp$ in the sense of Theorem 1.11 for $\{\nu, \varrho\}$. By orthogonality with $\nu$ then $\varrho^\perp = \gamma \cdot \nu_\psi$, where $\nu_\psi$ is the vector state of $\psi$, and $\gamma \geq 0$, real. On the other hand, $\varrho^\perp \leq \varrho$ has to be fulfilled. Hence, for each $k \in \mathbb{N}$ especially,

$$\gamma \cdot |\langle \psi, \varphi_k \rangle|^2 = \varrho^\perp (p_{\varphi_k}) \leq \varrho(p_{\varphi_k}) = \beta^k |\langle \psi, \varphi_k \rangle|^2 .$$

From this $\gamma \leq \beta^k$ follows, for all $k \in \mathbb{N}$. Thus $\gamma = 0$, and $\varrho^\perp = 0$ is seen. But note that $s(\varrho) \wedge s(\nu^\perp = p_\psi$ in this case.
2.3.7. Fibres and centralizers. In line with Problem 2.3.1, now we are going to look for conditions under which on a standard form vN-algebra $M$ and with fixing some individual vector $\varphi_0$ in the fibre of $\nu$ the hypothesis of Theorem 1.9 could be violated. To that aim, we start looking for additional conditions on the support of $\nu$ under which $g^\perp$ can be constructed more explicitly, in each case of $g$. In making reference to the notion of the centralizer vN-algebra $M^\circ$, which reads as

$$M^\circ = \{ x \in M : g(xy) = g(yx), \forall y \in M \},$$

such conditions can be easily formulated.

Lemma 2.26. Suppose $\nu, g \in M_{++}$, with $s(\nu) \in M^\circ$. Then $g^\perp = g(s(\nu)^\perp(\cdot))$.

Proof. Owing to $s(\nu) \in M^\circ$ both $g(s(\nu)^\perp(\cdot))$ and $g(s(\nu)(\cdot))$ have to be normal positive linear forms which sum up to $g$. Especially, $\omega = g(s(\nu)^\perp(\cdot)) = g(s(\nu)^\perp(s(\nu)^\perp))$ then is subordinate to $g$ and is orthogonal to $\nu$ by construction. On the other hand, $s(g^\perp) \leq s(\nu)^\perp$ also implies $g^\perp = g(s(\nu)^\perp(s(\nu)^\perp))$. Hence, $\omega - g^\perp = \{ g - g^\perp \} s(\nu)^\perp(s(\nu)^\perp) \geq 0.$ In view of Theorem 1.11 then $\omega = g^\perp$ follows. \qed

Now, constructions where Lemma 2.26 can be applied, can be easily achieved.

Lemma 2.27. Suppose $\nu, g \in M_{++}$, with $s(\nu) \in M^\circ$, $s(\nu) < s(g)$ and $s(\nu) \sim 1$. Then $S_M(\nu|g) \neq S_M(\nu)$.

Proof. In view of Lemma 2.26 and $s(\nu) < s(g)$, $g^\perp = g(s(\nu)^\perp(\cdot)) \neq 0$ follows. On the other hand, $s(\nu) \leq s(g)$ implies $\nu^\perp = 0$. Moreover, in view of (2.4) and (2.5b) from $p(\xi_v) = s(\nu) \sim 1$ the relation $p(\xi_v) \sim 1$ is obtained. Thus a cyclic vector has to exist in $S_M(\nu)$. Application of Lemma 2.26 then yields the result. \qed

Clearly, in order that examples in accordance with these premises could exist at all, $M$ has to be supposed to be infinite. Moreover, for factors and faithful $g$, the assertion of Lemma 2.26 even can be strengthened.

Lemma 2.28. On an infinite factor $M$, for $\nu, g \in M_{++}$, with faithful $g$, and with $s(\nu) \in M^\circ$, $S_M(\nu|g) \neq S_M(\nu)$ happens if, and only if, $s(\nu)$ is infinite and $s(\nu) \neq 1$.

Proof. As in the previous proof, $s(\nu) \neq 1$ and $s(\nu) \in M^\circ$ for faithful $g$ imply $g^\perp \neq 0$. Note that the latter condition implies $s(\nu) \neq 1$, by triviality. Hence, for faithful $g$ and $s(\nu) \in M^\circ$, $s(\nu) \neq 1$ is equivalent to $g^\perp \neq 0$. But then the assertion is a consequence of Theorem 2.26. \qed

In order to see along the lines of Lemma 2.26 that $S_M(\nu|g) \neq S_M(\nu)$ can happen, first recall some further facts from modular and operator theory.

Remark 2.29. 1. In case of a standard form vN-algebra $\{ M, \Omega \}$, $\varrho$ defined by $\Omega \in S_M(\varrho)$ is faithful. A fundamental result of modular theory then says that $M^\circ$ is the same as the fixed-point algebra of the modular $^*$-automorphism group of $\varrho$:

$$M^\circ = \{ x \in M : \Sigma_t^\varrho(x) = x, \forall t \in \mathbb{R} \},$$

where the modular $^*$-automorphism group $\{ \Sigma_t^\varrho \}$ of $\varrho$ for all $t \in \mathbb{R}$ acts on elements $x \in M$ as $\Sigma_t^\varrho(x) = \Delta_t^\varrho \varrho \Delta_t^{-1}$. As consequence of this one has

$$\left( M^\circ \right)_h^\Omega = M_h^\Omega \cap M_h^\Omega,$$

$$\left( M^\circ \right)_h^\Omega \subset \mathcal{H}_{sa}^\Omega,$$
with $\mathcal{H}_{sa}^\Omega$ of (2.5), and where $(\cdot)_h$ refers to the Hermitian portion of the algebra in question. For these facts we refer the reader to [23, 24]. Remark that (2.17a) remains true with positive portions instead of Hermitian parts, accordingly.

2. Relating the structure of $M^\theta$, if $M$ is properly infinite, by the duality theorem of Connes and Takesaki [14, 33], see also [22], there are a semifinite but infinite-dimensional $vN$-algebra $N$ and a $\sigma$-weakly continuous group $(\alpha_t)_{t \in \mathbb{R}}$ of $^*$-automorphisms of $N$ such that $M$ is isomorphic to a so-called $W^*$-crossed product, $N \bigotimes_{\alpha_0} \mathbb{R} \simeq M$. Thereby, one knows that $N$ can be isomorphically identified with the fixed-point algebra of $\{\Sigma_\ell^R\}$. Hence, in view of (1) then $M^\theta \simeq N$, and thus $M^\theta$ especially obeys $M^\theta \neq \mathbb{C} 1$.

3. Note in context of (2.17a) that if $\Omega$ is separating for a $vN$-algebra $N$, then according to (1) the condition $\varphi \in N_+\Omega$ is equivalent to the existence of a densely defined, positive, selfadjoint linear operator $A$ which is affiliated with $N$, $A \eta N$, and which obeys $\varphi = A\Omega$. On the other hand one knows that the latter is also equivalent to $\langle (\cdot) \rangle_N (\varphi, \Omega) \geq 0$.

4. Especially, if for faithful $\varphi$ a cyclic and separating $\Omega \in \mathcal{S}_M(\varphi)$ is chosen, then owing to $\langle (\cdot) \rangle_M \psi^\varphi_T(\nu), \Omega \rangle \geq 0$, the previous facts ensure existence of a unique positive selfadjoint linear operator $T \eta M$ with $\psi^\varphi_T(\nu) = T \Omega$. In view of the well-known Radon-Nikodym theorem [27] and following [23, 24] then is referred to as generalized Sakai’s Radon-Nikodym operator of $\nu$ with respect to $\varphi$. In this case $j(\cdot \vert \varphi) = \psi^\varphi_T$ of Remark 2.15 then may be written as $j(\mu \vert \varphi) = \frac{1}{2} \| J_\Omega T^*J_\Omega \Omega - T \Omega \|^2$. Thus, in the case at hand $j(\cdot \vert \varphi)$ amounts to be the same as the skewinformation $I(\varphi, T)$ of the generalized Radon-Nikodym operator $T$ with respect to $\varphi$ in the sense of [15], see [11, 35, 36, 37] for background informations, and which together with the other facts from Remark 2.15 might facilitate the understanding of the notion of commutation also in those cases which go beyond Example 2.14 see also [3, Definition 3, Lemma 2, Theorem 2].

Close this part by two auxiliary results, which are versions of more general assertions, and which ensure applicability of Lemma 2.27 in our cases of interest. In the following, for each $a \in M$ and $\mu \in M_+$, we let a normal positive linear form $\eta^a$ be defined by $\eta^a(x) = \mu(a^* x a)$, for all $x \in M$.

**Lemma 2.30.** Let $M$ be a standard form $vN$-algebra, and be $\varphi \in M_+$. Suppose $\mu = \varphi^a$, for some $a \in M_+$. Then $\psi^\varphi_T(\nu) = x_\xi_\mu$ is fulfilled. Moreover, in case that $\varphi$ is faithful, $\nu$ is commuting with $\varphi$, and only if, $x \in (M^\theta)_+$ holds.

**Proof.** Let $\Omega$ be a cyclic and separating vector for $M$. Note that by assumption on $\nu$, $x_\xi_\mu \in \mathcal{S}_M(\varphi)$. Also, by positivity of $x$, $h \eta x_\xi_\mu = h \eta x_\xi_\mu \geq 0$. From this then $s(h) = p'(\sqrt{x_\xi_\mu}) \leq p'(x_\xi_\mu)$ follows. On the other hand, in any case, $p'(x_\xi_\mu) \leq p'(\sqrt{x_\xi_\mu})$ has to be fulfilled. Thus we conclude $s(h) = p'(x_\xi_\mu)$. In view of Theorem 1.4 and Lemma 2.4 then $d_B(\nu, \varphi) = \| x_\xi_\mu - x_\xi_\mu \| \geq 0 \nu^\perp$ follow. In the case at hand Lemma 2.30 can be applied and yields $\psi^\varphi_T(\nu) = x_\xi_\mu$.

Suppose $\varphi$ is faithful. Then we can choose a cyclic and separating vector $\Omega \in \mathcal{S}_M(\varphi)$, and will work in the corresponding standard form action of $M$. Suppose that $\nu$ commutes with $\varphi$. By Definition 2.11 since $\xi_\mu = \Omega$ holds, and since commutation means $\psi^\varphi_T(\nu) = \xi_\mu$, we have $\langle (\cdot) \rangle_M (\xi_\mu, \Omega) = h_{\xi_\mu - \eta_\xi_\mu} \geq 0$.

Hence, $\xi_\mu \in M_+ \Omega \cap M_+ \Omega$ by
Remark 2.23. As has been explained in context of (2.17a), this is the same as \( \xi_\nu \in (M^\varrho)_+ \Omega \). Also, since \( M^\varrho = M \) implies \( M' \subset (M^\varrho)' \), \( \Omega \) must be separating for \( M^\varrho \). Hence, by Remark 2.29(b) and with \( N = M^\varrho \), \( \xi_\nu = A\Omega \) follows, with densely defined, positive, selfadjoint linear operator \( A \) which is affiliated with \( M^\varrho \subset M \). In view of the above then \( \psi^\varrho_\Omega(\nu) = x\Omega = A\Omega \). Since \( \Omega \) is separating, from this \( x = A \) follows, that is, \( x \in (M^\varrho)_+ \).

On the other hand, for \( x \in M^\varrho \), \( x \geq 0 \), in view of the above formula \( \psi^\varrho_\Omega(\nu) = x\Omega \) and (2.17b), \( \psi^\varrho_\Omega(\nu) \in \mathcal{H}_\Omega^\varrho \) follows. Applying Lemma 2.32 accordingly yields \( \psi^\varrho_\Omega(\nu) \in \mathcal{P}_\Omega^\varrho \). Hence, \( \nu \) commutes with \( \varrho \).

Remark 2.31. According to Remark 2.29(b), if \( \varrho \) is faithful, and \( \Omega \in \mathcal{S}_M(\varrho) \) is a cyclic and separating vector, then for each \( \nu \in M_{++} \) one has \( \psi^\varrho_\Omega(\nu) = x\Omega \), with \( x \) densely defined, affiliated with \( M \), selfadjoint positive linear operator \( x \). Thus, bi- linearly the same conclusions as in the previous proof we see that \( \nu \) will be commuting with \( \varrho \) if \( x = x^* \geq 0 \) is affiliated with \( M^\varrho \).

Lemma 2.32. Let \( M \) be a standard form \( \nu \)-\( N \) algebra, and be \( \varrho \in M_{++} \) faithful. For each \( \nu \in M_{++} \) commuting with \( \varrho \) one has \( s(\nu) \in M^\varrho \).

Proof. Let us consider the standard form action of \( M \) with respect to cyclic and separating \( \Omega \in \mathcal{S}_M(\varrho) \). By assumption such a vector has to exist. As mentioned in the previous proof, \( \psi^\varrho_\Omega(\nu) = \xi_\nu \) implies (in fact is equivalent to) \( \xi_\nu = A\Omega \), with densely defined, positive, selfadjoint linear operator \( A \) which is affiliated with \( M^\varrho \). Thus \( s(\nu) \leq \varrho \), where \( \varrho \) is the orthoprojection onto the closure of the range of \( A \). By spectral calculus, if \( \{ E(\lambda) \} \subset M^\varrho \) is the (left-continuous) spectral resolution of \( A \), then \( \varrho = \text{l.u.b.}\{ E(\lambda) - E(0^+) : \lambda > 0 \} \in M^\varrho \). Also, owing to \( AE(\lambda) \in (M^\varrho)^+ \), for \( \nu_\lambda \) with \( E(\lambda)\xi_\nu = AE(\lambda)\Omega \in \mathcal{S}_M(\nu_\lambda) \), we have \( \nu_\lambda \leq \nu \) and \( s(\nu_\lambda) = E(\lambda) - E(0^+) \). Hence \( \varrho \leq s(\nu) \). Thus \( \varrho = s(\nu) \) follows. In view of the above then \( s(\nu) \in M^\varrho \). □

2.3.8. Conclusions in the standard form case. Start with a mechanism for creating pairs of positive linear forms which are relevant in context of Problem 2.1 (4), and Conclusions in the standard form case.

Lemma 2.33. Let \( M \) be an infinite standard form \( \nu \)-\( N \) algebra, and be \( \varrho \in M_{++} \) faithful. For each \( \sigma \in M_{++} \), with \( s(\sigma) \sim 1 \) there is \( a \in M \) with \( \mathcal{S}_M(\sigma^a | \varrho) = \mathcal{S}_M(\sigma^a) \).

In particular, there is an orthoprojection \( z \in M^\varrho \) such that \( M \simeq zMz \) holds, and \( \mathcal{S}_M(\varrho | \mu) \neq \mathcal{S}_M(\mu) \) is fulfilled, for each \( \mu \in M_{++} \) with support \( s(\mu) = z \).

Proof. Let \( M = Mc + Mc^\perp \), with orthoprojection \( c \in M \cap M' \), be the canonical decomposition of \( M \) into a finite component \( Mc \) (which might be vanishing) and properly infinite component \( Mc^\perp \). By central decomposition techniques, the properly infinite case can be reduced to an appropriately defined field \( \{ M_\lambda \} \) of properly infinite ‘subfactors’ \( M_\lambda \) acting over direct integral Hilbert subspaces \( \mathcal{H}_\lambda \subset c^\perp \mathcal{H} \), in the sense that each \( x \in Mc^\perp \) and \( \omega \in (Mc^\perp)_+ \) can be written as appropriately defined central integrals \( x = \int \oplus \omega_\lambda d\mu(\lambda) \) and \( \omega = \int \oplus \omega_\lambda d\mu(\lambda) \), with some \( \omega_\lambda \in M_\lambda \) and \( \omega_\lambda \in (M_\lambda)_+ \), such that \( \omega(x) \) can be written as \( \omega(x) = \int \omega_\lambda(x_\lambda) d\mu(\lambda) \) over terms \( \omega_\lambda(x_\lambda) \), with \( \omega_\lambda = \omega_\lambda M_\lambda \), and with respect to some central measure \( \mu(\lambda) \). Thereby, the index \( \lambda \) refers to the central decomposition which is isometrically defined relative to some direct integral decomposition \( c^\perp \mathcal{H} = \int \oplus \mathcal{H}_\lambda d\mu(\lambda) \) of the Hilbert space \( c^\perp \mathcal{H} \), see e.g. [31, Chapter I., 5., 10. Corollary] for the details. Since
all decompositions refer to the center and \(q\) is faithful over \(M\), for \(q_\lambda = q|_{M_\lambda}\) we will have \((M^\alpha c^\perp)_\lambda = M^\alpha_\lambda \subset M^\perp_\lambda\), with faithful \(q_\lambda\) over properly infinite subfactor \(M_\lambda\).

By Remark 2.29, \(M^\alpha_\lambda\) cannot be trivial. Therefore, there is an orthoprojection \(0 < c_\lambda < 1_\lambda\), with \(c_\lambda \in (M^\alpha c^\perp)_\lambda\). As orthoprojection of the infinite factor \(M_\lambda\), at least one of \(c_\lambda\) and \(c^\perp_\lambda\) must be infinite. We make a choice, and call this infinite projection \(z_\lambda\). Since \(M\) is countably decomposable by assumption, each of \(M_\lambda\) is a countably decomposable infinite factor. Hence, we then must have \(z_\lambda < 1_\lambda\) and \(z_\lambda \sim 1_\lambda\). Define an orthoprojection \(z\) in \(M\) as follows:

\[
z = c + \int \oplus z_\lambda d\mu(\lambda).
\]

One then has \(z \in M^\alpha\), \(z < 1\), and \(z \sim 1\) with respect to \(M\). But then, by assumption on \(\sigma\), there is \(a \in M\) with \(a^*a = s(\sigma)\) and \(aa^* = z\). It is easily seen that for each such \(a\) then \(s(\sigma^a) = z\) must be fulfilled. Hence, \(s(\sigma^a) \in M^\alpha\) with \(s(\sigma^a) < 1 = s(q)\) but \(s(\sigma^a) \sim 1\), for each such \(a\). By Lemma 2.27 with \(\nu = \sigma^a\) then \(S_M(\sigma^a|q) \neq S_M(\sigma^a)\) follows, for each \(a \in M\) with \(a^*a = s(\sigma)\) and \(aa^* = z\).

Now, let \(v \in M\) be a partial isometry with \(v^*v = z\) and \(vv^* = 1\). Then, the map \(\pi : M \ni x \mapsto v^*xv \in zMz\) is a *-isomorphism between \(M\) and the hereditary vN-subalgebra \(zMz\). Thus, for \(\mu\) running through all \(\mu \in M^\alpha_+\), with \(s(\mu) = z\), \(\sigma = \mu^v\) is running through all faithful \(\sigma \in M^\alpha_+\). In view of \(s(\sigma) = 1\), by the above conclusion for each such \(\sigma\) and with \(a = v^*\) the result is \(S_M(\sigma^v|q) \neq S_M(\sigma^v)\). But note that \(\sigma^v = \sigma(v^*(\cdot)v^*v) = \mu^v(v^*(\cdot)v^*v) = \mu(v^*v(\cdot)v^*v) = \mu(z(\cdot)z) = \mu\). That is, \(S_M(\mu|q) \neq S_M(\mu)\) must be fulfilled, for each \(\mu \in M^\alpha_+\), with \(s(\mu) = z\).

Remark 2.34. Note that under the premises of Lemma 2.33 there have to exist both, \(\nu \in M^\alpha_+\) which commute/do not commute with \(q\), but which obey \(S_M(\nu|q) \neq S_M(\nu)\). In fact, according to Lemma 2.30 and Lemma 2.33, \(\nu = \rho^z\) provides a special positive linear form commuting with \(q\) and obeying \(S_M(\nu|q) \neq S_M(\nu)\). But according to Lemma 2.33 the latter remains true also for any other \(\nu = \rho^z\), with invertible \((\text{within } zMz)\) \(x \in (zMz)_+\). If each of these functionals were commuting with \(q\) then, by Lemma 2.30, \(x \in M^\alpha\) for all those \(x\). By taking the closure we had \((zMz)_+ \subset M^\alpha\), and therefore \(zMz \subset M^\alpha\). But then \(q|_{zMz}\) would be a faithful tracial normal positive linear form over \(zMz\). This is a contradiction, since owing to \(zMz \simeq M\), \(zMz\) cannot be finite.

Example 2.35. If \(M\) is a factor of type \(I_\infty\), things around Lemma 2.33 read quite explicit. In fact, in terms of Example 2.14 we then have \(q = \tau_e\), with uniquely determined, positive definite \(\tau\)-trace class operator \(c\). By Example 2.14, in this case \(M^\alpha = \{x \in M : xc = cx\}\). Then, orthoprojections \(z \in M\) obeying \(zc = cz\), \(\tau(z) = \infty\) and \(z \neq 1\) must exist and, by Lemma 2.28 and in line with the last part of the proof of Lemma 2.33, this is the set of all orthoprojections \(z\) with respect to which the hypothesis of Lemma 2.33 holds, and therefore each \(\nu \in M^\alpha_+\) with support in this set provides an example with \(S_M(\nu|q) \neq S_M(\nu)\).

In view of Problems 2.10/4, we may summarize all that also as follows.

Theorem 2.36. Let \(\{M, \Omega\}\) be a standard form vN-algebra, \(\varrho \in M^\alpha_+\) faithful.

1. \(S_M(\nu|q) = S_M(\nu)\) holds for all \(\nu \in M^\alpha_+\) if, and only if, \(M\) is finite.

2. For infinite \(M\), there exists \(\nu \in M^\alpha_+\) obeying

\[
d_B(\nu, q) = \|\xi_\nu - \xi_q\|
\]
such that \( S_M(\nu | \varrho) \neq S_M(\nu) \).

3. For \( \nu \in M_+ \) satisfying (2.18), \( S_M(\nu | \varrho) = \{ u \xi_\nu : u^* u = 1, u \in M' \} \) holds.

Proof. By Lemma 2.13, condition (2.18) means commutation of \( \nu \) with \( \varrho \), in the sense of Definition 2.11. Thus (1) follows along with Example 2.18 and formula (2.15a). Relating (1), by Remark 2.2 we yet know that \( S_M(\nu | \varrho) = S_M(\nu) \) is always fulfilled on a finite \( M \). On the other hand, for infinite \( M \), according to Lemma 2.33 and Remark 2.34 there exists \( \nu \) commuting with \( \varrho \) and obeying \( S_M(\nu | \varrho) \neq S_M(\nu) \), which in view of Lemma 2.13 yields (3). Since a \( \nu \)-algebra is either finite or infinite, this then also completes the proof of (1). □

In the factor case, for mutually commuting normal positive linear forms one of which is faithful, things considerably simplify (see Example 2.35), and then Theorem 2.36 (2) can be supplemented by the following detail.

**Lemma 2.37.** Let \( M \) be a factor acting in standard form. Suppose \( \nu \in M_+ \) obeys condition (2.18) with faithful \( \varrho \in M_+ \). Then, \( S_M(\nu | \varrho) = S_M(\nu) \) holds if, and only if, \( \nu \) is faithful or the support orthoprojection \( s(\nu) \) is finite.

Proof. By Lemma 2.13 and Lemma 2.32, condition (2.18) implies \( s(\nu) \in M^\varrho \). Lemma 2.22 then may be applied, and yields \( S_M(\nu | \varrho) = S_M(\nu) \) iff either \( s(\nu) \) is finite or \( s(\nu) = 1 \), cf. also Remark 2.24. □

**Example 2.38.** For a type-III-factor \( M \), things around commutation are less explicit than in the type-I\( \infty \)-case. A characterization like (2.18) which reads in terms of a geometrical condition in context of the Bures distance then possibly is the best one can do. By Lemma 2.13, Theorem 2.36 (2) and Lemma 2.37 tell us that for faithful \( \varrho \) there is \( \nu \in M_+ \) obeying (2.18) and \( S_M(\nu | \varrho) \neq S_M(\nu) \), and the latter remains true for any \( \nu \) with (2.18), unless either \( \nu \) is faithful or is vanishing.

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