The four postulates of quantum mechanics are three

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The tensor product postulate of quantum mechanics states that the Hilbert space of a composite system is the tensor product of the components' Hilbert spaces. All current formalizations of quantum mechanics that do not contain this postulate contain some equivalent postulate or assumption (sometimes hidden). Here we give a natural definition of composite system as a set containing the component systems and show how one can logically derive the tensor product rule from the state postulate and from the measurement postulate. In other words, our paper reduces by one the number of postulates necessary to quantum mechanics.

In this paper we derive the tensor product postulate (which, hence, loses its status of postulate) from two other postulates of quantum mechanics: the state postulate and the measurement postulate. The tensor product postulate does not appear in all axiomatizations of quantum mechanics: it has even been called “postulate 0” in some literature [1]. A widespread belief is that it is a direct consequence of the superposition principle, and it is hence not a necessary axiom. This belief is mistaken: the superposition principle is encoded into the quantum axioms by requiring that the state space is a linear vector space. This is, by itself, insufficient to single out the tensor product, as other linear products of linear spaces exist, such as the direct product, the exterior/wedge product or the direct sum of vector spaces, which is used in classical mechanics to combine state spaces of linear systems. These are all maps from linear spaces to linear spaces but they differ in how the linearity of one is mapped to the linearity of the others [32]. This belief may have arisen from the seminal book of Dirac [2], who introduces tensor products (Chap. 20) by appealing to linearity. However, he adds the seemingly innocuous request that the product among spaces be distributive (rather, bilinear), which is equivalent to postulating tensor products (or linear functions of them). It is not an innocuous request. For example it does not hold where the composite vector space of two linear spaces is described by the direct product, e.g. in classical mechanics, for two strings of a guitar: it is not distributive. [General classical systems, not only linear ones, are also composed through the direct product.] Of course, Dirac is not constructing an axiomatic formulation, so his ‘sleight of hand’ can be forgiven. In contrast, von Neumann (Chap. VI.2, also 4) introduces tensor products by noticing that this is a natural choice in the position representation of wave mechanics (where they were introduced in 1, 6), and then explicitly postulates them in general: “This rule of transformation is correct in any case for the coordinate and momentum operators [...] and it conforms with the [observable axiom and its linearity principles], we therefore postulate them generally.” More mathematical or conceptually-oriented modern formulations (e.g. 8, 11) introduce this postulate explicitly. An interesting alternative is provided in 12, 13: after introducing tensor products, Ballentine verifies a posteriori that they give the correct laws of composition of probabilities. Similarly, Peres uses relativistic locality 14. While these procedures seemingly bypass the need to postulate the tensor product, they do not guarantee that this is the only possible way of introducing composite systems in quantum mechanics. In the framework of quantum logic, tensor products arise from some additional conditions 15 which (in contrast to what is done here) are not connected to the other postulates. In 16, 17 tensor products were obtained by specifying additional physical or mathematical requirements.

Let us first provide a conceptual overview of our approach. We start from the natural definition of a composite system as the set of two (or more) quantum systems. The composite system is therefore made of system $A$ and (joined with) system $B$ and nothing else. The first key insight is that the first two postulates of quantum theory (introduced below) already assume that the preparation of one system is independent from the preparation of another (statistical independence). In fact, we cannot even talk about a system in the first place if we cannot characterize it independently. The second key insight is that, using the law of composition of probabilities of independent events, we can find a map $M$ that takes the state of the component systems and gives the composite state for the statistically independent case. These insights are enough to characterize mathematically the state space of the composite: the linearity given by the Hilbert space, together with the fact that the composite system is fully described by the observables of $A$ and $B$, allows us to extend the construction from the statistically independent composite states to the general case (that includes entangled states). So the work consists of two interrelated efforts: a physical argument that starts from the first two postulates and leads to the necessary existence of the composition map $M$ and its properties together with a formal argument that shows how $M$ leads to the tensor product.

This map $M$ acts on the state spaces of the subsystems.
Each pure state is identified by a ray \( \psi \), a subspace of the system’s Hilbert space comprising all vectors \( \psi \) differing by their (nonzero) modulus and phase: a one-dimensional complex subspace (a complex plane). In the same way, constraining the observable \( X \) to a particular outcome value \( x_0 \) means identifying the subspace comprising all non-normalized eigenvectors \( |x_0\rangle \) of arbitrary phase such that \( X|x_0\rangle = x_0|x_0\rangle \). The map \( M \) establishes a relationship between the states of the subsystems and the composite, so it is a map between subspaces, not vectors. Therefore, \( M \) acts on the projective spaces, where all vectors within the same ray are “collapsed” into a single point (i.e. a quotient space in the equivalence class), removing the unphysical “overspecification” of the phase and of the modulus. The physical requirements on \( M \) are such that we can find a bilinear map \( m \) between vectors that acts consistently with \( M \) in terms of subspaces. This map \( m \) is the tensor product.

More in detail, the physical requirements of statistical independence, together with the fact that one can arbitrarily prepare the states of the subsystems, imply three conditions on the map \( m \): (H1) totality: the map is defined on all states of the subsystems; (H2) bilinearity: the map is bilinear thanks to the fundamental theorem of projective geometry; (H3) span surjectivity: the span of the image of the map coincides with the full composite Hilbert space. We then prove that, if these three conditions H1, H2 and H3 hold, then the map \( m \) is the tensor product, namely the Hilbert space of the composite system is the tensor product of the components’ Hilbert spaces. The tensor product “postulate” hence loses its status of a postulate. An overview of all these logical implications is given in Fig. 1. The rest of the paper contains the sketch of this argument, including all the physical arguments outlined above. The supplementary material [7] contains the mathematical details.

![FIG. 1: Schematic depiction of the logical implications used in this paper. FTPG stands for “Fundamental Theorem of Projective Geometry”.

We start with the axiomatization of quantum mechanics based on the following postulates (e.g. [3,11]): (a) The pure state of a system is described by a ray \( \psi \) corresponding to a set of non-zero vectors \( |\psi\rangle \) in a complex Hilbert space, and the system’s observable properties are described by self-adjoint operators acting on that space; (b) The probability that a measurement of a property \( X \), described by the operator with spectral decomposition \( \sum_{x,i} x |x_i\rangle \langle x_i| \) (\( i \) a degeneracy index), returns a value \( x \) given that the system is in state \( \psi \) is \( p(x|\psi) = \sum_i |\langle x_i|\psi\rangle|^2 \) (Born rule). (c) The state space of a composite system is given by the tensor product of the spaces of the component systems; (d) The time evolution of an isolated system is described by a unitary operator acting on a vector representing the system state, \( |\psi(t)\rangle = U_t|\psi(t = 0)\rangle \), or, equivalently, by the Schrödinger equation. The rest of quantum theory can be derived from these axioms. While some axiomatizations introduce further postulates, we will be using only (a) and (b) to derive (c), so the above are sufficient to our aims.

Note that we limit ourselves to kinematically-independent systems, where all state vectors \( |\psi\rangle \) in the system’s Hilbert space \( \mathcal{H} \) describe a valid state, unconditioned on anything else. We call this condition “preparation independence” and it should be noted that the tensor product applies only in this case. For example, the composite system of two electrons is not the tensor product, rather the anti-symmetrized tensor product, precisely because the second electron cannot be prepared in the same state of the first. We note that restrictions due to superselection rules arise either from practical (not fundamental) limitations on the actions of the experimenter [15,20] or from the use of ill-defined quantum systems. In the example above, the field is the proper quantum system and the electrons are its excitations. [33]

The definition of a composite system as containing only the collection of the subsystems means that any preparation of both subsystems independently must correspond to the preparation of the composite system. Since states are defined by postulate (a) as rays in the respective Hilbert spaces, there must exist a map \( M : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C} \) that takes a pair of states for the subsystems (\( \mathcal{A} \) and \( \mathcal{B} \) represent the projective space, where each point represents all vectors that identify the same state, and the Cartesian product is the set of all possible pairs) and returns a state in the projective space \( \mathcal{C} \) for the composite. To visualize the geometrical meaning of \( M \) directly within the Hilbert spaces, given a ray (a complex plane) in each of \( \mathcal{A} \) and \( \mathcal{B} \), \( M \) returns a ray (a complex plane) in \( \mathcal{C} \). Our final goal will be to find a map \( m : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C} \) that acts on vectors in the Hilbert spaces \( \mathcal{A} \), \( \mathcal{B} \) and \( \mathcal{C} \) consistently with \( M \). Namely, \( m(a,b) = M(a,b) \) where the underline sign indicates the elements in the projective space. Again geometrically, \( m \) takes a vector in each of \( \mathcal{A} \) and \( \mathcal{B} \), and returns a vector in \( \mathcal{C} \) and we want this to be consistent with \( M \) such that vectors picked from the same rays will return vectors in the same ray. We will prove that the map \( m \) is the tensor product. We focus on pure states here: the argument can be extended to mixed states using standard tools [12].

The map \( M \) must be injective: as above, different states of the subsystems must correspond, by definition of composite system, to different states of the composite.
Moreover, preparation independence implies that $M$, and hence $m$, must be total maps (condition H1): each subsystem of the composite system can be independently prepared and gives rise to a state of the composite. H1 is not sufficient to identify the tensor product: by itself it does not even guarantee that the map $m$ is linear.

Postulate (b) contains the connection between quantum mechanics and probability theory. It must then implicitly contain the axiomatization of probability, e.g. see \[12, 13, 21\]. One of the axioms of probability theory (axiom 4 in \[13\]) asserts that the joint probability events $a$ and $b$ given $z$ is $p(a \land b | z) = p(a | z) p(b | z \land a)$. Consider $p(a \land b | \psi \land b)$ which represents the probability of measurement outcomes $a$ on system $A$ and $b$ on system $B$ given that system $A$ was prepared in $\psi$ and system $B$ in $b$. We have $p(a \land b | \psi \land b) = p(a | \psi \land b \land b) p(b | \psi \land b) = p(a | \psi \land b) p(b | \psi \land b)$. The Born rule tells us that $p(a | \psi \land b) = |\langle a | \psi \rangle|^2$ and that $p(b | \psi \land b) = |\langle b | b \rangle|^2 = 1$, where $|a\rangle$, $|b\rangle$ are the normalized eigenstates relative to outcomes $a$ and $b$, and $|\psi\rangle$ is the normalized state vector. We have:

\[ \begin{align*}
p(a \land b | \psi \land b) &= p(a | \psi) \\
p(a \land b | \phi \land \phi) &= p(b | \phi)
\end{align*} \]

In other words, since the probability for a measurement on one system depends only on its pure state, the Born rule requires that the measurement of one system is independent from the preparation of the other. We call this property “statistical independence” \[22\]. It characterizes the map $M$, since $M(a, b)$ corresponds to the composite state where $A$ and $B$ are prepared in the states $|a\rangle$ and $|b\rangle$. Define $M_b(a) = M(a, b)$. From the Born rule we find

\[ \langle M(a, b) | M(\psi, b) \rangle_C^2 = \langle M_b(a) | M_b(\psi) \rangle_C^2 = |\langle a | \psi \rangle|_A^2, \]

where the first and second terms contain the inner product in the composite space $C$. [This is not a new assumption: it follows from the measurement postulate (b) for the composite system.] This means that, when one subsystem is prepared in an eigenstate of what is measured there, the state space of the other is mapped preserving the square of the inner product. This implies orthogonality and the hierarchy of subspaces are preserved through $M_b$, making $M_b$ a colinear transformation by definition. Geometrically, recall that $M_b$ maps rays to rays. The fact that $M_b$ is colinear means that it also maps higher order subspaces to higher order subspaces (lines to lines, planes to planes, and so on) while preserving inclusion (if a line is within a plane, the mapped line will be within the mapped plane). In this case, the fundamental theorem of projective geometry \[22\] applies, which tells us that a unique semi-linear map $m_b$ that acts on the vectors exists in accordance with $M_b$. Moreover, conservation of probability further constrains it to be either linear or antilinear. This tells us that the corresponding $m$ is either linear or antilinear in the first argument. Namely, if equation \[3\] holds, then

\[ \langle a | \psi \rangle = \langle m(a, b) | m(\psi, b) \rangle \]

or $\langle a | \psi \rangle = \langle m(\psi, b) | m(a, b) \rangle$.

In this setting, the antilinear case \[15\] corresponds to a change of convention (much like a change of sign in the symplectic form for classical mechanics) and can be ignored. Given a Hilbert space, in fact, we can imagine replacing all vectors and all the operators with their Hermitian conjugate, mapping vectors into duals $|\psi\rangle^\dag = |\psi\rangle$. These changes would effectively cancel out leaving the physics unchanged: the two equations $A |w\rangle = B |z\rangle$ and $\langle w | A^\dag = \langle z | B^\dag$ are equivalent. (For example, in his first papers Schrödinger used both signs in his equation: effectively writing two equivalent equations with complex-conjugate solutions \[23\]. Also Wigner pointed out this equivalence \[24\], pg.152). We can repeat the same analysis for the second argument of $m$ to conclude that it is a bilinear map, condition (H2).

The last condition, span surjectivity (H3), follows directly from the definition of a composite system. Since it is composed only of the component systems, for any state $c$ of the composite system, we must find at least one pair $|a\rangle$, $|b\rangle$ such that $p(a \land b | c) \neq 0$. Span-surjectivity follows: namely the span of the map applied to all states in the component systems spans the composite system state space. In other words, the composite does not contain states that are totally independent of (i.e. orthogonal to) the states of the components.

We have obtained the conditions H1, H2 and H3 from the state postulate (a), the measurement postulate (b) and the definitions of composite and independent systems. We now prove that these three conditions imply that the (up to now unspecified) composition rule $m$ is the tensor product. More precisely, given a total, span-surjective, bilinear map $m: A \times B \to C$ that maps the Hilbert spaces $A$, $B$ of the components into the Hilbert space $C$ of the composite and that preserves the square of the inner product, we find that $C$ is equivalent to $A \otimes B$ and that $m = \otimes$.

Proof. Step 1: the bases of the component systems are mapped to a basis of the composite system. Because of totality property (H1) and because the square of the inner product is preserved, we can conclude that, given two orthonormal bases $\{|a_i\rangle\} \in A$ and $\{|b_j\rangle\} \in B$, $|\langle m(a_i, b_j) | m(a_k, b_l) \rangle|^2 = \delta_{ij}\delta_{kl}$, namely $\{|m(a_i, b_j)\rangle\}$ is an orthonormal set in $C$. Moreover, the surjectivity property (H3) guarantees that in $C$ no vectors are orthogonal to this set. This implies that it is a basis for $C$.

Step 2: use the universal property. The tensor product is uniquely characterized, up to isomorphism, by a universal property regarding bilinear maps: given two vector spaces $A$ and $B$, the tensor product $A \otimes B$ and the associated bilinear map $T: A \times B \to A \otimes B$ have the property
than any bilinear map \( m : \mathcal{A} \times \mathcal{B} \to \mathcal{C} \) factors through \( T \) uniquely. This means that there exists a unique \( I \), dependent on \( m \), such that \( I \circ T = m \). In other words, the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{A} \times \mathcal{B} & \xrightarrow{T} & \mathcal{A} \otimes \mathcal{B} \\
\downarrow{m} & & \downarrow{I} \\
\mathcal{C} & & 
\end{array}
\]

Since \( m : \mathcal{A} \times \mathcal{B} \to \mathcal{C} \) is a bilinear operator (property H2), thanks to the universal property of the tensor product we can find a unique linear operator \( I : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C} \) such that \( m(a, b) = I(a \otimes b) \). The set \( \{ I(a_i \otimes b_j) \} \) with \( |a_i\rangle \) and \( |b_j\rangle \) orthonormal bases for \( \mathcal{A} \) and \( \mathcal{B} \) forms a basis for \( \mathcal{C} \), since \( I(a_i \otimes b_j) = m(a_i, b_j) \) and we have shown above that the latter is a basis. Thus,

\[
\langle I(a_i \otimes b_j)| I(a_k \otimes b_l)\rangle_{\mathcal{C}} = \langle m(a_i, b_j)| m(a_k, b_l)\rangle_{\mathcal{C}} = \delta_{ik}\delta_{jl} = \langle a_i \otimes b_j| a_k \otimes b_l\rangle_{\otimes},
\]

where we used the orthonormality of the bases and the fact that \( |a_i \otimes b_j\rangle \) is a basis of the tensor product space \( \mathcal{A} \otimes \mathcal{B} \). Since the function \( I \) is a linear function that maps an orthonormal basis of \( \mathcal{A} \otimes \mathcal{B} \) to an orthonormal basis of \( \mathcal{C} \), \( I \) is an isomorphism (a bijection that preserves the mathematical structure) between \( \mathcal{A} \otimes \mathcal{B} \) and \( \mathcal{C} \). As \( \mathcal{C} \cong \mathcal{A} \otimes \mathcal{B} \) are isomorphic as Hilbert spaces, they are mathematically equivalent: \( c \in \mathcal{C} \) and \( I^{-1}(c) \) represent the same physical object. In this sense, we can loosely say that \( I \) is the identity, as it connects spaces that are physically equivalent. So we can directly use the tensor product to represent the composite state space. This means that the map \( m : \mathcal{A} \times \mathcal{B} \to \mathcal{C} \) is equivalent to the map \( \otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes \mathcal{B} \) in the sense that \( m \circ I^{-1} = \otimes \).

A few comments on the proof: it is based on the universal property of the tensor product, which uniquely characterizes it. In step 1 we show that the bilinear map \( m \) maps subsystems’ bases into the composite system basis. We also know that there exists a tensor product map \( T = \otimes \) that can compose the vectors in \( \mathcal{A} \) and \( \mathcal{B} \). In step 2 we use the universal property: since \( m \) is a bilinear map, we are assured that there exists a unique \( I \) such that \( I \circ T = m \). Since we show that \( I \) is an isomorphism, then \( I \) bijectively maps vectors in \( \mathcal{C} \) onto vectors in the tensor product space. Namely \( m = T = \otimes \).

We conclude with some general comments. The tensor product structure of quantum systems is not absolute, but depends on the observables that are accessible\(^{[19][20]}\). This is due to the fact that an agent that has access to a set of observables will define quantum systems differently from an agent that has access to a different set of observables. Where one agent sees a single system, an agent that has access to less refined observables (and is then limited by some superselection rules) can consider the same system as composed of multiple subsystems.

It has been pointed out before that the quantum postulates are redundant: in\(^{[19][20]}\), it was shown that the measurement postulate (b) can be derived from the others (a), (c), (d). Here instead we have shown how the tensor product postulate (c) can be logically derived from the state postulate (a), the measurement postulate (b) and a reasonable definition of independent systems, and we have described the logical relations among them. Of course, we do not claim that this is the only way to obtain the tensor product postulate from the others.

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**SUPPLEMENTARY INFORMATION FOR "THE FOUR POSTULATELS OF QUANTUM MECHANICS ARE THREE".**

**Mathematical formulation.**

Here we give the mathematical details of the proof sketched in the main paper.

The core idea is that the probability space constrains what the state space for the composite system can be. Therefore we must develop a precise map between events in probability and their correspondents in terms of Hilbert space. Conceptually, the event \( X = x_0 \), the observable \( X \) is equal to the value \( x_0 \), will correspond to the subspace spanned by all eigenstates of \( X \) with eigenvalue \( x_0 \). The event \( \psi \), the system was prepared in state \( \psi \), will correspond to the ray (one-dimensional complex subspace) corresponding to the \( \psi \) vector. Therefore, in general, all events in probability will correspond to subspaces (of different dimensionality) of the Hilbert space. Projective spaces are the right tool to keep track of subspaces. The proof, then, consists of establishing the correct definitions in the space of events, mapping those into elements of the projective space and then, from the projective space, constructing a map on the vector space directly.

Let us establish, then, the notation we will be using to
distinguish the projective space from the Hilbert space itself. If $X$ is a Hilbert space, we denote $X$ the projective space. The projective space is mathematically constructed from the Hilbert space by removing the origin and quotienting by the equivalence relationship $v \sim \lambda v$, $v \in X$ and $\lambda \in \mathbb{C}$. A quantum state is a point in projective space. Each point of the projective space is called a ray, because for a real vector space it would correspond to a line going through the origin, with the origin removed. As we are in a complex space, the ray should be thought as a complex plane without the origin, which is the space of the vectors reachable from a fixed one through multiplication by a complex number. It can also be thought as a subspace of dimension one.

Given a vector $v \in X$, we will denote $\mathbf{v}$ the ray in the projective space corresponding to $v$. Note that $\mathbf{v}$ denotes a quantum state, without having picked a modulus or phase. Given two or more vectors $v_1, \ldots, v_n \in X$, the subspace of $X$ they span (i.e. all the vectors reached by linear combinations) is noted by $Sp(v_1, \ldots, v_n)$. Note that this subspace will correspond to a set of rays in the projective space, which we note as $Sp(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Geometrically, this can be thought as the smallest hyperplane that contains all vectors. Given $v, w \in X$, we can write $P(v|w) = \frac{|\langle v|w \rangle|^2}{\langle v|v \rangle\langle w|w \rangle}$ which corresponds to the probability of observing $v$ given $w$ was prepared. Note that $P(v|w) = P(\lambda v|\mu w)$, with non-null $\lambda, \mu \in \mathbb{C}$, and therefore one can write $P(v|w) = P(|v|w)$ as a function of the rays. Geometrically, this corresponds to the angle between the two complex planes identified by the two vectors.

**Postulate (a).** The state of a quantum system is described by a ray $\mathbf{\psi} = \{\alpha|\psi\rangle\}$ non-null $\alpha \in \mathbb{C}, |\psi\rangle \in \mathcal{H}$ in a separable complex Hilbert space $\mathcal{H}$, and the system’s observable properties are described by self-adjoint operators acting on that space.

**Remark.** All proofs, except one, do not depend on the dimensionality of the space. The exception is proposition [13] for which we prove the finite case by induction and then show that it holds in the limit. This would not work in the non-separable case, since the basis would not be countable.

**Postulate (b).** The probability that a measurement of a property $X$, described by the operator with spectral decomposition $X = \sum_{x,i} x_i^2 |x_i\rangle\langle x_i|$, where $i$ is a degeneracy index, returns a value $x$ depends only on $X$ and on the state of the system $\mathbf{\psi}$ and is given by $P(x|\psi) = \sum_i \frac{|\langle x_i|\psi\rangle|^2}{\langle \psi|\psi \rangle}$ (Born rule).

Given two events $a$ and $b$, for example $X > x_1$ and $X < x_2$, their conjunction $a \wedge b$ is the event where both are true, $x_1 < X < x_2$ in the example. In terms of our Hilbert spaces, both $a$ and $b$ correspond to subspaces and $a \wedge b$ is exactly the intersection of the two, which is also a subspace. We should not confuse $a \wedge b$ with $a/b$: the first refers to either preparation or measurement of both systems in the respective state, while the second corresponds to preparing one system in one state and measuring the other system in the other state [21].

**Definition I.1** (Compatible states). Let $A$ and $B$ be two systems. Let $\mathcal{A}$ and $\mathcal{B}$ be their corresponding state spaces. We say two (pure) states $(a, b) \in \mathcal{A} \times \mathcal{B}$ are compatible iff the respective systems can be prepared in such states at the same time. Formally, the proposition $a \wedge b$ is possible, which means it does not correspond to the empty set in the $\sigma$-algebra of the probability space [23].

**Definition I.2** (Preparation independence). Two systems are said independent iff the preparation of one does not affect the preparation of the other. Formally, all (pure) state pairs $(a, b) \in \mathcal{A} \times \mathcal{B}$ are compatible.

**Proposition I.3.** Given two systems, each prepared independently in their own state, the probability of measuring a value for one system depends only on the preparation of that system. That is, $P(a_1|a_2 \wedge b) = P(a_1|a_2)$.

**Proof.** We first note that, by postulate [B] the probability of measuring a value for one system depends only on the preparation of that system, which means that it is independent of the properties of any other system. Therefore $P(a_1|a_2 \wedge b) = P(a_1|a_2) = (a_1|a_2)(a_2|a_2) = P(a_1|a_2)$.

**Definition I.4** (Composite systems). Let $A$ and $B$ be two systems. The composite system $C$ of $A$ and $B$ is formed by the simple collection of those and only those two systems, in the sense that it satisfies the following two requirements.

1. Every preparation of both subsystems is a preparation of the composite. Formally, let $\mathcal{C}$ be the state space for $C$, there exists a map (not yet specified) $M: \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ such that, for any compatible pair of (pure) states $(a, b) \in \mathcal{A} \times \mathcal{B}$, the proposition $a \wedge b$ is equivalent to the (pure) state $M(a, b) \in \mathcal{C}$ where $M$ returns the state of the composite system where the subsystems were prepared in the given states. In other words, $a \wedge b$ and $M(a, b)$ correspond to the same event in probability space [36].

2. For every preparation of the composite, local projective measurements must have at least one outcome with non-zero probability. Formally, for every $c \in \mathcal{C}$, we can find at least $a \in A$ and $b \in B$ such that $P(a \wedge b|c) \neq 0$.

It is important to understand that these requirements are necessary. Requirement 1 ensures that the composite system is well defined at least when the components are prepared independently. Conceptually, this ensures that the composite system contains all the properties of the components. Note that superselection rules or other
restrictions may prevent the independent preparation of all possible pairs (e.g. two fermions cannot be jointly prepared in the same state). The tensor product is recovered only when all pairs are compatible. Requirement 2 ensures that it does not contain properties that are orthogonal to all the components’ properties, i.e. that the composite system contains only the components. Violation of the second requirement would mean that some composite states would not define all the properties of the subsystems. That is, while the systems A and B by themselves would define observables $O_A$ and $O_B$, when grouped together all values would be assigned zero probability; those observables no longer exist. In this case, the nature of the system would have changed so radically we would no longer call it a composite system.

**Proposition I.5** (Span surjectivity, H3). The map $\mathcal{M} : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ is span surjective, meaning that the span of the image coincides with the whole space. That is $\text{Sp}\{c \in \mathcal{C} | c \in \mathcal{M}(\mathcal{A}, \mathcal{B})\} = \mathcal{C}$.

**Proof.** Consider $I = \{c \in \mathcal{C} | c = \mathcal{M}(\mathcal{A}, \mathcal{B})\}$ and its span. This forms a subspace of $\mathcal{C}$. By requirement 2 of [14], for any $c \in \mathcal{C}$ we can always find $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $\mathcal{P}(a \land b) \neq 0$. This means there is no element in $\mathcal{C}$ that is orthogonal to $\text{Sp}(I)$, therefore $\text{Sp}(I)$ must cover the whole $\mathcal{C}$. $\square$

**Proposition I.6** (Totality, H1). The map $\mathcal{M}$ is in general a partial function. However, if $A$ and $B$ are independent, $\mathcal{M}$ is a total function.

**Proof.** As $\mathcal{M}(a, b)$ is defined only if $(a, b) \in \mathcal{A} \times \mathcal{B}$ is a compatible pair of pure states, it is not defined on pairs that are not compatible. If the two systems are independent, however, all pairs are allowed and $\mathcal{M}$ is a total function. $\square$

**Remark.** As noted in [14] if $a$ and $b$ are incompatible, $a \land b = \emptyset$ corresponds to the impossible event (i.e. the empty set in the $\sigma$-algebra). This is not a state, and therefore $\mathcal{M}(a, b)$ is not defined on incompatible pairs. Therefore independent systems will map each pair to a non-zero element of the tensor product, while systems that are not independent will map incompatible states to the zero vector (e.g. the composite state of two electrons will exclude the cases where both electrons are in the same state).

**Proposition I.7** (Statistical independence). Let $\mathcal{A}$ and $\mathcal{B}$ be the state spaces of two quantum systems and $\mathcal{C}$ be the state space of their composite. The map $\mathcal{M} : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ is such that:

\[
P(\mathcal{M}(a, b) | \mathcal{M}(a_2, b_2)) = P(a | a_2) \tag{S1}
\]

\[
P(\mathcal{M}(a, b) | \mathcal{M}(a_1, b_1)) = P(b | b_1) \tag{S2}
\]

for all $a, a_1, a_2 \in \mathcal{A}$ and $b, b_1, b_2 \in \mathcal{B}$.

**Proof.** By [13] we have $P(a_1 | a_2 \land b) = P(a_1 | a_2)$ and similarly $P(b_1 | a \land b_2) = P(b_1 | b_2)$. Using standard probability rules and remembering that $\mathcal{M}(a, b) = a \land b$ by [14], we have $P(\mathcal{M}(a, b) | \mathcal{M}(a_2, b_2)) = P(a | a_2 \land b) = P(a_2 | a \land b_2) = P(b_1 | a \land b_2) = P(b_2 | a \land b_2)$, since trivially $P(b_2 | b_2) = 1$. Similarly $P(\mathcal{M}(a, b) | \mathcal{M}(a_1, b_1)) = P(a | a_1 \land b_1) = P(b_1 | b_1)$.

**Proposition I.8** (Fundamental theorem of projective geometry). Let $X$ and $Y$ be two separable complex Hilbert spaces and $X'$ and $Y'$ their respective projective spaces. Let $\mathcal{M} : X \rightarrow Y$ be a map such that $P(v | w) = \mathcal{M}(P(v | w))$. Then we can find, up to a total phase, a unique map $m : X \rightarrow Y$ such that $m(v) = \mathcal{M}(v)$. Moreover, $m$ is either linear, $\langle v | w \rangle = (m(v) | m(w))$, or anti-linear, $\langle v | w \rangle = (m(w) | m(v))$.

**Remark.** The above proposition is, for the most part, an adaptation of the fundamental theorem of projective geometry [22]. The conservation of the probability imposes the semi-linear map to be either linear or anti-linear (i.e. conjugate-linear). This is not new, it is essentially Wigner’s theorem, but the proof we offer is insightful as it clearly shows the connection between the construction of the map and the choice of gauge.

**Proof.** First we note that, given an orthonormal basis $\{e_i\}_{i \in I}$ over $X$, we can use $\mathcal{M}$ to construct a corresponding basis over $Y' \subseteq Y$ where $Y' = \mathcal{M}(X)$. In fact, for each $e_i$, pick a unit $u_i \in \mathcal{M}(e_i)$. We have $\delta_{i,j} = |\langle e_i | e_j \rangle|^2 = P(e_i | e_j) = P(e_i, e_j) = P(\mathcal{M}(e_i) | \mathcal{M}(e_j)) = P(u_i | u_j) = |\langle u_i | u_j \rangle|^2$. The set $\{u_i\}_{i \in I}$ spans the entire $Y'$ since for all $y \in Y'$ we can find $x \in X$ and at least one $u_i$ such that $|\langle y | u_i \rangle|^2 = P(y | u_i) = P(\mathcal{M}(y) | \mathcal{M}(u_i)) = P(x | e_i) = |\langle x | e_i \rangle|^2 \neq 0$. Note that we have an arbitrary choice for each $u_i$, since we have to pick a vector from the unit circle (i.e. a phase for each basis vector). This corresponds to a choice of gauge.

We also note that the map is colinear, meaning that if $U_X, V_X \subseteq X$ are two subspaces such that $U_X \subset V_X$, then $U_Y, V_Y \subseteq Y$ such that $\mathcal{M}(U_X) = \mathcal{M}(V_X)$ and $\mathcal{M}(U_Y) \subseteq \mathcal{M}(V_Y)$ are subspaces of $Y$ and $U_Y \subset V_Y$. In fact, take an orthonormal basis $\{e_i\}_{i \in I} \subset X$ such that $\{e_i\}_{i \in I} \subset C \subset L$ are bases for $U_X$ and $V_X$ respectively. An element of $X$ belongs to $U_X$ if and only if it is not orthogonal only to elements of the basis of $U_X$ and belongs to $V_X$ only if it not orthogonal only to elements of the basis of $V_X$. As the map $\mathcal{M}$ preserves orthogonality, these relationships are preserved by the map. Therefore $U_Y$ and $V_Y$ are subspaces of $Y$ such that $U_Y \subset V_Y$.

Additionally we note that, for any colinear map, given two subspaces $U_1, U_2 \subset X$ we have $\mathcal{M}(\text{Sp}(U_1, U_2)) = \text{Sp}(\mathcal{M}(U_1), \mathcal{M}(U_2))$. In fact, $\text{Sp}(U_1, U_2)$ is the smallest subspace containing all vectors in $U_1$ and $U_2$. In the same way, $\mathcal{M}(\text{Sp}(U_1, U_2))$ is the smallest subspace containing all vectors in $\mathcal{M}(U_1)$ and $\mathcal{M}(U_2)$.
the subspace inclusion is preserved by \( M \), we must have
\[
M(Sp(U_1, U_2)) = Sp(M(U_1), M(U_2)).
\]
We now use the gauge freedom to redefine the basis such that for all \( i \) we have
\[
M(e_i) = v_i \quad \text{and} \quad M(c_i + e_i) = v_i + k_i.
\]
Let \( v_1 = u_1 \). This is the only arbitrary choice we make, and corresponds to the choice of a global phase. For each \( i > 1 \), consider \( e_1 + c_i \). This will belong to the subspace \( Sp(e_1, e_i) \). This subspace, when mapped through \( M \), will give us the subspace spanned by \( v_1 \) and \( u_i \). That is, \( M(Sp(e_1, e_i)) = Sp(v_1, u_i) \). This means we can find a unique \( k \in \mathbb{C} \) such that
\[
M(e_1 + e_i) = v_1 + k u_i.
\]
We fix \( v_i = ku_i \). Note that
\[
P(e_1 + e_i) = \frac{1}{2} P(e_1 e_i + e_i) = P(v_1 v_i + ku_i = P(u_i u_i + ku_i).
\]
Therefore \( |k| = 1 \) and \( ku_i = v_i \) is a unit vector.

Now we want to show that \( M(e_1 + e_i) = v_1 + \tau(e_i) v_i \)

where either \( \tau(e_i) = c \) or \( \tau(e_i) = c' \). For each \( i \), consider \( w = e_1 + c_i e_i \in Sp(e_1, e_i) \). Since \( M(w) \in Sp(v_1, u_i) \), there must be a \( \tau(e_i) \) such that \( v_1 + \tau(e_i) u_i = M(w) \). Since we must have \( P(e_1 w) = P(v_1 M(w)) \) and \( P(e_1 + e_i w) = P(v_1 + v_i + ku_i M(w)) \), we must have
\[
|c| = |\tau(e_i)| \quad \text{and} \quad \cos(\arg(c)) = \cos(\arg(\tau(e_i))) \quad \text{for any} \ c.
\]
This means that either \( \tau(e_i) = c \) or \( \tau(e_i) = c' \).

Next we want to show that \( \tau(i, e_i) = \tau(j, e_i) \) for all pairs \( (i, j) \). That is, either we have to take the complex conjugate of all components or of none. Consider \( e_1 + e_j \). We have \( e_1 - e_j \in Sp(e_1, e_j) ) \) and, for any \( c \in \mathbb{C}, \ e_1 - e_j \in Sp(e_1 + c e_j, e_1 + c e_j) \). By construction, we have \( M(e_1 - e_j) \in Sp(v_1, u_i) \) and \( M(e_1 - e_j) \in Sp(u_1, v_i) \in Sp(v_1, u_i) \). Therefore
\[
M(e_1 - e_j) = Sp(v_1, u_i) \cap Sp(v_1, u_i, v_i, v_i + \tau(e_i) v_i) \tau(e_i) v_i - \tau(e_i) v_i.
\]
This means that, for all \( c, \tau(e_i) = \tau(j, e_i) \).

Now we show that for all \( c_2, ..., c_n \in \mathbb{C} \) we have
\[
M(c_1 + c_2 e_2 + ... + c_n e_n) = v_1 + \tau(c_1) v_i + ... + \tau(c_n) v_i.
\]
We prove this by induction. If only the first two components are non-zero, we have
\[
M(c_1 + c_2 e_2) = v_1 + \tau(c_1) v_i + ... + \tau(c_2) v_i.
\]
If we assume
\[
M(c_1 + c_2 e_2 + ... + c_n e_{n-1}) = v_1 + \tau(c_1) v_i + ... + \tau(c_{n-1}) v_{n-1},
\]
then
\[
M(Sp(c_1 + c_2 e_2 + ... + c_n e_{n-1})) = Sp(v_1, c_1 v_i v_2 + ... + c_{n-1} v_{n-1} v_i).
\]
This means that there exists \( k \in \mathbb{C} \) such that
\[
M(c_1 + c_2 e_2 + ... + c_n e_{n-1}) = v_1 + \tau(c_1) v_i + ... + \tau(c_{n-1}) v_{n-1} + k v_i.
\]
We also have
\[
M(Sp(c_1 + c_2 e_2 + ... + c_n e_{n-1})) = Sp(v_1 + \tau(c_1) v_i v_2 + ... + c_{n-1} v_{n-1} v_i).
\]
The only way this can work is if
\[
k = \tau(c_p).
\]
The above works also over a countable sum. That is, for all \( c_2, ..., c_n \in \mathbb{C} \) we have
\[
M(c_1 + c_2 e_2 + ... + c_n e_n) = v_1 + \tau(c_1) v_i + ... + \tau(c_n) v_i.
\]
Let \( X \) be a separable space. Let \( a = \sum_{k=1}^\infty c_k e_k \) such that \( c_i = 1 \). Let \( a_i = \sum_{k=1}^i c_k e_k \) be the sum of the first \( i \) components. We have
\[
\lim_{i \to \infty} a_i = a. \quad \text{Let} \quad b_i = \sum_{j=1}^i \tau(c_j) e_k \quad \text{and} \quad \lim_{i \to \infty} b_i = b. \quad \text{We already know that} \quad b_i = M(a_i) \quad \text{for all finite} \ i. \quad \text{We need to show that} \quad b = M(a).
\]
First note that, given \( a, b \in Y \), \( a = b \) if and only if \( \langle a, c \rangle = \langle b, c \rangle \) for all \( c \in Y \). Therefore \( a = b \) if and only if \( P(a, c) = P(b, c) \) for all \( c \in Y \). For all \( c \in X \) we have
\[
\lim_{i \to \infty} P(a_i, c) = P(a, c), \quad \text{for all} \ c \in Y. \quad \text{We also have} \quad \lim_{i \to \infty} P(a, c) = \lim_{i \to \infty} P(M(a_i), c) = P(b, c) \quad \text{for all} \ c \in Y \quad \text{and} \ c \in X. \quad \text{Note that} \quad M \text{ is bijective over } \mathbb{C} \text{. Therefore} \quad P(M(a), c) = P(b, c) \quad \text{for all} \ c \in Y. \quad \text{And} \quad b = M(a).
\]
We also need to show the above works when there is no component on the first element of the basis. That is, for all \( c_2, ..., c_n \in \mathbb{C} \) we have
\[
M(c_2 e_2 + ... + c_n e_n) = \tau(c_2) v_i + ... + \tau(c_n) v_i.
\]
First note that \( M(c_2 e_2 + ... + c_n e_n) \subset M(Sp(c_2 e_2, e_2 + ... + c_n e_n)) = Sp(v_1, v_i + \tau(c_2) v_i + ... + \tau(c_n) v_i). \quad \text{Also note that} \quad M(c_2 e_2 + ... + c_n e_n) \subset M(Sp(e_1, e_1 + e_2 + ... + c_n e_n)) = Sp(v_1, v_i + \tau(c_2) v_i + ... + \tau(c_n) v_i). \quad \text{The only way this can work is if} \quad M(c_2 e_2 + ... + c_n e_n) = \tau(c_2) v_i + ... + \tau(c_n) v_i. \quad \text{With same reasoning as before, we can extend the sum to the countably infinite case.}
\]
We can now define \( m : X \to Y \) such that \( m(e_i) = v_i \) for all \( i \) and \( m(\sum_{i = 1}^n c_i e_i) = \sum_{i = 1}^n \tau(c_i) v_i. \quad \text{This means} \quad m(\sum_{i = 1}^n c_i e_i) = M(\sum_{i = 1}^n c_i e_i). \quad \text{Moreover, if} \quad \tau(c) = c, \quad \text{we have} \quad m(\sum_{i = 1}^n c_i e_i) = \sum_{i = 1}^n c_i v_i = \sum_{i = 1}^n c_i \sum_{j = 1}^n \frac{d_i}{d_j} v_j. \quad \text{On the other hand, if} \quad \tau(c) = c', \quad \text{we have} \quad m(\sum_{i = 1}^n c_i e_i) = \sum_{i = 1}^n c_i v_i = \sum_{i = 1}^n c_i \sum_{j = 1}^n \frac{d_i}{d_j} v_j. \quad \text{This can be extended to the case where the basis is countable.}
\]

Remark. The fact that the proposition identifies either a linear map or an anti-linear (i.e. conjugate-linear) corresponds, in physics terms, to a choice of convention. As analogies: a change in metric signature in relativity would change the mathematical space but not the physics; in classical phase-space, a change in signature of the symplectic form would change the mathematical space, but not the physics it represents. These choices are widely recognized as a matter of personal preference.

In simple terms, for a Hilbert space, the conjugate vector space is equivalent to the dual space, so we could equivalently choose one or the other. For example, Schrödinger, in the papers in which he introduces the Schrödinger equation, writes it with both signs, as the choice of sign of the imaginary part of the wave function is arbitrary: one sign refers to the Hilbert space, the other to the dual space, namely to the complex-conjugate wave function \( \bar{\psi} \). So we can think of the anti-linear map as one that preserves the inner product but maps ket vectors into bra vectors. Looking ahead, the above result does not exclude a composition map similar to the tensor product, but that maps the kets of one or both subsys-
tems into bras in the composite system. This would only make the representation of the composite physical system more complicated, as we need to keep track of the different conventions in the different subspaces. Therefore, without changing the physics, we can always mathematically redefine the second space so that the resulting map is linear. With this in mind, we will assume that the map between the spaces is linear, which will in turn lead to identifying the tensor product as a unique composition map.

Another way to look at this is that Hermitian operators, and therefore all the physics, are invariant under an anti-linear transformation. In contrast, anti-Hermitian operators will change sign. This changes the connection between the generators and the generated transformations (i.e. while $A$ generates $\exp(\frac{i}{\hbar} H)$ on one space, the mapped $A$ will generate $\exp(-\frac{i}{\hbar} H)$ in the mapped space). Note that the choice of whether to put the minus or not is arbitrary as long as one is consistent across all generators. Similarly, we typically define $[A, B] = AB - BA$ but we could have alternatively chosen $[A, B] = BA - AB$. The anti-linear map is simply a change of that convention.

Note that this unnecessary subtlety could in principle be avoided by reformulating quantum mechanics in terms of quantum states given by density matrices $\rho = |\psi\rangle \langle \psi|$ (which contain both kets and bras), as is done, for example in [8, 26]. In this paper we employed the more familiar formulation in which quantum states are rays in Hilbert space (identified either by kets or bras).

**Proposition I.9 (Bilinearity, H2).** Given $M$ in (3) if we can find an $m : A \times B \to C$ such that for all $(a, b) \in A \times B$, we have $m(a, b) = M(a, b)$, then $m$ must be bilinear. That is:

$$
m(k_1 a_1 + k_2 a_2, b) = k_1 m(a_1, b) + k_2 m(a_2, b) \quad (S3)
$$

$$
m(a, k_1 b_1 + k_2 b_2) = k_1 m(a, b_1) + k_2 m(a, b_2) \quad (S4)
$$

for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$ and $k_1, k_2 \in C$.

**Proof.** If we fix $b \in B$, then we have $M_b : A \to C$ where $M_b(a) = M(a, b)$. By (7) and (8) we can find a linear map $m_b : A \to C$ such that $m_b(a) = M_b(a) = M(a, b)$. As this must map subspace to subspace, we must have $m(a, b) = km_b(a)$ for some $k \in C$. Since $m_b$ is linear, we have $m(k_1 a_1 + k_2 a_2, b) = k_1 m(a_1, b) + k_2 m(a_2, b)$ for any $a_1, a_2 \in A$ and $k_1, k_2 \in C$. We can repeat the argument fixing $a \in A$, and find $m(a, k_1 b_1 + k_2 b_2) = k_1 m(a, b_1) + k_2 m(a, b_2)$ for any $b_1, b_2 \in B$ and $k_1, k_2 \in C$.

**Proposition I.10 (Subsystems’ basis gives composite system basis).** Let $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ be bases of $A$ and $B$ respectively, then a set of unit vectors $\{e_{ij}\}_{(i,j) \in I \times J} \subset C$ such that $e_{ij} \in M(a_i, b_j)$ forms a basis for $C$.

**Proof.** Since $M$ is a map on the projective spaces, it maps spans to spans. Since the span of the basis of $A$ and $B$ is the whole space, then the span of the image of the basis is the whole image of $M$. By (3) the image of $M$ coincides with the whole $C$. Therefore, given $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ bases of $A$ and $B$ respectively, any set of unit vectors $\{e_{ij}\}_{(i,j) \in I \times J} \subset C$ such that $e_{ij} \in M(a_i, b_j)$ spans the whole $C$.

Now consider $P(M(a_i, b_j)|M(a_k, b_l))$. If $i = k$ and $j = l$ we have $P(M(a_i, b_j)|M(a_k, b_l)) = P(M(a_i, b_j)|M(a_i, b_l)) = 1$. If $i \neq k$, we have $P(M(a_i, b_j)|M(a_k, b_l)) = P(ab_i \wedge b_l, ab_k \wedge b_l) \leq P(ab_i | ab_k \wedge b_l)$. By (3) we have $P(ab_i | ab_k) = P(ab_i | ab_i) = 0$ since $a_k$ and $a_k$ are different elements of an orthogonal basis. Therefore we have $P(M(a_i, b_j)|M(a_k, b_l)) = \delta_{ik}\delta_{jl}$ which means $\langle e_{ij}|e_{kl}\rangle = \delta_{ik}\delta_{jl}$.

The elements $e_{ij}$ form a set of orthonormal vectors that span the whole space and are therefore a basis.

**Theorem I.11** (Composite system theorem). The state space of a composite system of independent systems is given by the tensor product of the spaces of the component systems.

**Proof.** We are looking for a map $m : A \times B \to C$ such that, for all $(a, b) \in A \times B$ we have $m(a, b) = M(a, b)$. We saw in (9) that if $m$ exists, it must be bilinear.

Now we show that, if $m$ exists, then $C \cong A \otimes B$ (where $\cong$ indicates an isomorphism) and $m : A \times B \to A \otimes B$ is the standard map from the Cartesian product to the tensor product. As $m : A \times B \to C$ is a bilinear operator, by the universal property of the tensor product we can find a linear operator $\bar{m} : A \otimes B \to C$ such that $m(a, b) = \bar{m}(a \otimes b)$. By (10) the set $\{m(a_i, b_j)\}_{(i,j) \in I \times J}$ forms a basis since $m(a_i, b_j) \in M(a_i, b_j)$ for all $(i, j)$, therefore $\bar{m}(\delta_{ij} b_j)\}_{(i,j) \in I \times J}$ also forms a basis since $\bar{m}(a_i \otimes b_j) = m(a_i, b_j)$. By (10) each $m(a_i, b_j)$ will correspond to a unit vector in $C$. We have $\langle \bar{m}(a \otimes b)|m(a_i \otimes b_j)\rangle_C = \langle m(a_i \otimes b_j)|m(a \otimes b)\rangle_C = \delta_{ik}\delta_{jl} = \langle a_i \otimes b_j|a_i \otimes b_j\rangle_C$. The function $\bar{m}$, then, preserves the inner product across all elements of the basis and is therefore an isomorphism for Hilbert spaces. We have $C \cong A \otimes B$ and $m(a, b) = \bar{m}(a \otimes b) \cong a \otimes b$.

Given that the tensor product map exists and it satisfies all the properties $m$ must satisfy, then $m$ exists and it is the tensor product.

In conclusion, as an aside, we note that in quantum field theory one tends to avoid problems connected with tensor products of infinite dimensional spaces by focusing on algebraic commutation structures, e.g. [27, 28]. In particular, the recent MIP*=RE result [29] implies that, in infinite dimensions, the tensor product is strictly less computationally powerful than the commutation structures, emphasizing the difference among these two structures, at least for the infinite-dimensional case.

Moreover, we note that in our paper we mainly focused on systems where no superselection rules or other restrictions to the state space are present: it is possible to
prepare each subsystem of a composite system in a state that is independent of the other systems (preparation independence). This is the only case in which the tensor product can be properly employed: the Hilbert space of composite systems that have restrictions is not the tensor product of the component spaces, but a subspace of it (e.g. the anti-symmetric subspace for fermions). Typically, this is ignored in the literature, since the tensor product formalism is very convenient and is often used also in these cases, and superselection rules are typically unavoidable [18–20]. A typical example comes from quantum field theory. It is customary in basically all quantum optics literature to treat different states of the radiation field (e.g. the output of two lasers) as independent systems composed through the tensor product. Clearly the electromagnetic field is a single system and an agent who is able to access an optical mode that is a linear combination of the two will give a quantum description for it that cannot easily accommodate tensor products. Similarly, an agent can consider two electrons as two systems, joined with the tensor product, whenever they are distinguishable for all practical purposes (e.g. the electrons are in widely separated physical locations). Yet, in principle, electrons are just excitations of a field, and the ‘true’ quantum system is the field, not the single electrons [30, 31]. So, in quantum field theory, the quantum systems that should be joined through tensor products are the different fields and not the particles, which are just excitations (states) of the fields. In the words of Teller ([30], pg. 22), tensor products can be safely used only if there is a “primitive thisness”, which is captured in the definition of system.

To conclude, we give a schematic outline of the logical implications that led us to the result. This is an expanded version of Fig. 1 of the main paper:

1. \( \text{H1} \) states and observables postulate.
2. \( \text{H2} \) Born rule (measurement postulate).
3. \( \text{Def I.2} \) Preparation independence: systems are independent if the preparation of one does not affect the other.
4. \( \text{H3} \) the outcome probabilities depend only on the inner product.
5. \( \text{Def I.4} \) Composite system definition: A composite system is a collection of the subsystems (i.e. all compatible states give a preparation) and only of the subsystems (i.e. all composite preparations give non-trivial measurements on the subsystems).
6. \( \text{H4} \) + Def I.3 \( \Rightarrow \) \( \text{L5} \) (H3): Span surjectivity (all composite \( C \) are superpositions of \( A \) and \( B \)).
7. \( \text{Def I.2} \) + Def I.4 \( \Rightarrow \) \( \text{L6} \) (H1): Totality (all possible state pairs of the subsystems correspond to a state of the composite).
8. \( \text{H[a]} \) + \( \text{H3} \) + Def I.4 \( \Rightarrow \) \( \text{L7} \) Statistical independence (if one subsystem does not change, the probability on the composite system is given by the probability of the subsystem that changes).
9. \( \text{L8} \) Fundamental theorem of projective geometry (preserving square of inner product leads to unique linear map)
10. \( \text{L7} \) + \( \text{L8} \) \( \Rightarrow \) \( \text{L9} \) (H2) composition map on vector spaces is bilinear.
11. \( \text{H[b]} \) + Def I.4 + \( \text{L5} \) (H3) \( \Rightarrow \) \( \text{L10} \) Basis carries over from subsystems to composite
12. \( \text{L6} \) + \( \text{L9} \) + \( \text{L10} \) \( \Rightarrow \) \( \text{L11} \) the composition map is the tensor product.

**Addendum**

The above proof relies only on the independent preparation of subsystems and not their measurements. However, during the review process, an anonymous referee contributed a sketch for a proof that shows very directly that the state and measurement postulates also imply independence of measurements. The insight is that the mixture created by all possible measurement outcomes on \( B \) must behave overall as a pure state on \( A \). Since pure states are extreme points, this can only happen if every measurement outcome on \( B \) leaves \( A \) in a pure state, which makes the probability factorize.

**Proposition I.12.** Let \( \psi \) and \( \phi \) be two preparations for \( A \) and \( B \). Let \( a \) and \( b \) be two measurements on the respective systems. Then \( P(\psi \land b \land \phi) = P(\psi)P(b, \phi) \).

**Proof.** Consider \( P(\psi \land b \land \phi) \). We can imagine first performing the measurement on \( B \) and then conditioning the result on \( A \). We have \( P(\psi \land b \land \phi) = P(\psi \land \phi)P(b) \) where by “\( b \land \phi \)” we mean that the systems were prepared in \( \psi \) and \( \phi \) respectively and \( b \) was measured. Any preparation on \( A \) can be expressed with a mixed state, and any measurement on \( B \) depends only on \( \phi \), so we have \( P(\psi \land b \land \phi)P(b) = P(\psi \land \phi)P(b) \). We also must have \( \sum \psi P(\psi) = P(\psi) \).

Putting it all together, we have \( P(\psi) = \sum \psi P(\psi) \). Which means \( P(\psi) = P(\psi \land \phi)P(b) \). But the only way that a mixture of mixed states can be equal to a pure state is if all mixed states are the same pure state. Therefore \( P(\psi \land b \land \phi) = P(\psi \land \phi)P(b) \). We finally have \( P(\psi \land b \land \phi) = P(\psi \land \phi)P(b) \). \( \square \)
One can also prove that the measurements on the components are independent as well (see Supplementary material), but we only strictly need preparation here. For example, the dice shows a number that is even and less than two, or the electron is prepared in spin up along x and also along z are impossible events. We will end up proving that the map M leads to the tensor product. A partial function is one that is not defined on the full domain. For example, $\sqrt{x}$ is a partial function since it is defined for $x > 0$. A total function is one that is defined on the full domain. For example, $x^2$ is a total function since it is defined for any x.