Collective motion is a central theme in the emerging field of active matter studies. For physicists, the interest largely lies in the non-trivial cases where the emergence of collective motion can be seen as an instance of spontaneous symmetry-breaking out of equilibrium: without leaders, guiding external fields, or confinement by boundaries, large groups inside which an “individual” can only perceive local neighbors are able to move coherently. After this was realized in the seminal papers of Vicsek et al. and Toner and Tu, much progress has been recorded in the physics community, alongside continuing modeling work in ethology and biology.

Most models consist of self-propelled particles interacting with neighbors defined by “topological” rules, not metric zones, —a situation advocated recently to be relevant for bird flocks, fish schools, and crowds— we use a kinetic approach to obtain well-controlled nonlinear field equations. We show that the density-independent collision rate per particle characteristic of topological interactions suppresses the linear instability of the homogeneous ordered phase and the nonlinear density segregation generically present near threshold in metric models, in agreement with microscopic simulations.

Continuous theory of active matter systems with metric-free interactions

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We derive a hydrodynamic description of metric-free active matter: starting from self-propelled particles aligning with neighbors defined by “topological” rules, not metric zones, —a situation advocated recently to be relevant for bird flocks, fish schools, and crowds— we use a kinetic approach to obtain well-controlled nonlinear field equations. We show that the density-independent collision rate per particle characteristic of topological interactions suppresses the linear instability of the homogeneous ordered phase and the nonlinear density segregation generically present near threshold in metric models, in agreement with microscopic simulations.

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Even though metric interaction zones are certainly of value in cases such as shaken granular media and motility assays where alignment arises mostly from inelastic collisions, it has been argued recently that they are not realistic in the context of higher organisms such as birds, fish, or pedestrians, whose navigation decisions are likely to rely on interactions with neighbors defined using metric-free, “topological” criteria. Statistical analysis of flocks of hundreds to a few thousand individuals revealed that a typical starling interacts mostly with its 7 or 8 closest neighbors, regardless of the flock density. The realistic, data-based, model of pedestrian motion developed by Moussaid et al. relies on the “angular perception landscape” formed by neighbors screening out others.

At a more theoretical level, the study of the Vicsek model with Voronoi neighbors has shown that metric-free interactions are relevant at the collective scale: in particular, the traveling bands mentioned above disappear, leaving only a Toner-Tu-like phase. Below, we show that the introduction of Voronoi neighbors suppresses the density-segregated phase in other variants of the Vicsek model. In spite of the recognized importance of metric-free interactions, no continuous field equations describing the above models are available which would help put the above findings on firmer theoretical ground.

In this Letter, starting from Vicsek-style microscopic models with Voronoi neighbors, we derive nonlinear field equations for active matter with metric-free interactions using a kinetic approach well-controlled near the onset of orientational order. We show that the density-independent collision rate per particle characteristic of these systems suppresses the linear instability of the homogeneous ordered phase and the nonlinear density seg-
regression in agreement with microscopic simulations. We finally discuss the consequences of our findings for the relevance of metric-free interactions.

Let us first stress that with metric-free interactions, say with Voronoi neighbors, the tenet of the Boltzmann equation approach [3], the assumption that the system is dilute enough so that it is dominated by binary collisions, is never justified since, after all, a particle is almost constantly interacting with the same number of neighbors. Here, instead, we introduce an interaction rate per unit time. In the low rate limit, binary interactions dominate, and one can proceed “as usual”.

Our starting point is not a Vicsek-style model, but a microscopic rule in the same universality class: N point particles move at constant speed $v_0$ on a $L \times L$ torus; their heading $\theta$ is submitted to two different dynamical mechanisms, “self-diffusion” and aligning binary “collisions”. In self-diffusion, $\theta$ is changed into $\theta' = \theta + \eta$ with a probability $\lambda$ per unit time, where $\eta$ is a random variable drawn from a symmetric distribution $P_\eta(\eta)$ of variance $\sigma^2$. Aligning “collisions” occur at rate $\alpha$ per unit time with a randomly chosen Voronoi neighbor. Then $\theta$ is changed to $\theta' = \Psi(\theta, \theta_n) + \eta$ where $\theta_n$ is the heading of the chosen neighbor and the noise $\eta$ is also drawn, for simplicity, from $P_\eta(\eta)$. Isotropy is assumed, namely $\Psi(\theta_1 + \phi, \theta_2 + \phi) = \Psi(\theta_1, \theta_2) + \phi[2\pi]$. For the case of ferromagnetic alignment treated in detail below, $\Psi(\theta_1, \theta_2) = \text{arg}(e^{i\theta_1} + e^{i\theta_2})$.

The evolution of the one-particle phase-space distribution $f(r, \theta, t)$ (defined over some suitable coarse-grained scales) is governed by the Boltzmann equation

$$\partial_t f(r, \theta, t) + v_0 \hat{e}(\theta) \cdot \nabla f(r, \theta, t) = I_{\text{diff}}[f] + I_{\text{coll}}[f]$$  \hspace{1cm} (1)

where $\hat{e}(\theta)$ is the unit vector along $\theta$. The self-diffusion integral is

$$I_{\text{diff}}[f] = -\lambda f(\theta) + \lambda \int_{-\pi}^{\pi} d\theta' \int_{-\infty}^{\infty} d\eta P_\eta(\eta) \delta_{2\pi}(\theta' - \theta + \eta) f(\theta')$$  \hspace{1cm} (2)

where $\delta_{2\pi}$ is a generalized Dirac delta imposing that the argument is equal to zero modulo $2\pi$. In the small-$\alpha$ limit, orientations are decorrelated between collisions (“molecular chaos hypothesis”), and one can write:

$$I_{\text{coll}}[f] = -\alpha f(\theta) + \alpha \rho(r, t) \int_{-\pi}^{\pi} d\theta_1 \int_{-\pi}^{\pi} d\theta_2 \int_{-\infty}^{\infty} d\eta \left[ P_\eta(\eta) f(\theta_1) f(\theta_2) \delta_{2\pi}(\Psi(\theta_1, \theta_2) - \theta + \eta) \right]$$  \hspace{1cm} (3)

The main difference with the metric case treated in [12] is the “collision kernel”, which is independent from relative angles and inversely proportional to the local density

$$\rho(r, t) = \int_{-\pi}^{\pi} f(r, \theta, t) \, d\theta.$$  \hspace{1cm} (4)

Note that, in agreement with basic properties of models with metric-free interactions, Eq. (1), together with definitions (2-3), is left unchanged by an arbitrary normalization of $f$ (and thus of $\rho$) and thus does not depend on the global density $\rho_0 = N/L^2$. Furthermore, a rescaling of time and space allows to set $\lambda = v_0 = 1$, a normalization we adopt in the following.

Equations for hydrodynamic fields are obtained by expanding $f(r, \theta, t)$ in Fourier series, yielding the Fourier modes $\hat{f}_k(x, \theta, t) = \int_{-\pi}^{\pi} d\theta f(x, \theta, t) e^{ik\theta}$ where $\hat{f}_0$ and $\hat{f}_{\pm k}$ are complex conjugates, $\hat{f}_0 = \rho$, and the real and imaginary parts of $\hat{f}_1$ are the coordinates of the momentum vector $\mathbf{w} = \rho \mathbf{P}$ with $\mathbf{P}$ the polar order parameter field. Using these Fourier modes, the Boltzmann equation (1) yields an infinite hierarchy:

$$\partial_t \hat{f}_k + (\nabla \hat{f}_{k-1} + \nabla^* \hat{f}_{k+1}) = (\hat{P}_k - 1 - \alpha)\hat{f}_k + \frac{\alpha}{\rho} \hat{P}_k \sum_{q=-\infty}^{\infty} J_{kq} \hat{f}_{q-\hat{k}}$$  \hspace{1cm} (5)

where the complex operators $\nabla = \partial_\theta + i\partial_\eta$ and $\nabla^* = \partial_\theta - i\partial_\eta$ have been used, the binary collision rate $\alpha$ is now expressed in the rescaled units, $\hat{P}_k = \int_{-\infty}^{\infty} d\eta P_\eta(\eta) e^{ik\eta}$ is the Fourier transform of $P_\eta$, and $J_{kq}$ is an integral depending on the alignment rule $\Psi$. Below, we specialize to the case of ferromagnetic alignment, for which

$$J_{kq} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \cos((q-k/2)\theta).$$  \hspace{1cm} (6)

For $k = 0$ the r.h.s. of Eq. (5) vanishes and one recovers the continuity equation:

$$\partial_t \rho + \nabla \cdot \mathbf{w} = 0.$$  \hspace{1cm} (7)

To truncate and close this hierarchy, we assume the following scaling structure, valid near onset of polar order, assuming, in a Ginzburg-Landau-like approach, small and slow variations of fields:

$$\rho - \rho_0 \sim \epsilon, \quad \hat{f}_k \sim e^{\epsilon k}, \quad \nabla \sim \epsilon, \quad \partial_t \sim \epsilon$$  \hspace{1cm} (8)

Note that the scaling of space and time is in line with the propagative structure of our system [23]. The lowest order yielding non-trivial, well-behaved equations is $e^3$: keeping only terms up to this order, equations for $\hat{f}_{k>2}$ identically vanish, while $\hat{f}_2$, being slaved to $\hat{f}_1$, allows to close the $\hat{f}_1$ equation which reads, in terms of the momentum field $\mathbf{w}$:

$$\partial_t \mathbf{w} + \gamma(\mathbf{w} \cdot \nabla)\mathbf{w} = -\frac{1}{\mu} \nabla \rho + \frac{\kappa}{2} \nabla \mathbf{w}^2$$  \hspace{1cm} (9)

$$+ (\mu - \xi \mathbf{w}^2)\mathbf{w} + \nu \nabla^2 \mathbf{w} - \kappa(\nabla \cdot \mathbf{w})\mathbf{w}$$

Apart from some higher order terms we have discarded here, this equation has the same form as the one derived in [12] for metric interactions, but with different transport coefficients:

$$\begin{align*}
\mu &= \left[ \frac{4\alpha}{\rho} + 1 \right] \hat{P}_1 - (1 + \alpha) \\
\gamma &= \nu \frac{4\alpha}{\rho} \left[ \hat{P}_2 - \frac{2}{3\pi} \hat{P}_1 \right] \\
\kappa &= \nu \frac{4\alpha}{\rho} \left[ \hat{P}_2 + \frac{2}{3\pi} \hat{P}_1 \right] \\
\xi &= \nu \left[ \frac{4\alpha}{\rho} \right]^2 \frac{1}{3\pi} \hat{P}_1 \hat{P}_2
\end{align*}$$  \hspace{1cm} (10)
Note first that, contrary to the metric case, the coefficient $\mu$ of the linear term does not depend on the local density $\rho$; coefficients of the non-linear terms depend on density to compensate the density dependence of $w$. Note further that $\nu$, $\kappa$, and $\xi$ are positive since $0 < \tilde{P}_k < 1$, so that in particular the nonlinear cubic term is always stabilizing. For an easier discussion, we consider now the Gaussian distribution $P_\nu(\eta) = \frac{1}{\sqrt{2\pi}} \exp[-\frac{\eta^2}{2}]$ for which $\tilde{P}_k = \exp[-k^2\sigma^2/2]$. Then $\mu$ is negative for large $\sigma$ (where the trivial $w = 0$ solution is stable with respect to linear perturbations), and changes sign for $2\sigma^2$.

$$\sigma^2_c = 2 \ln \left( \frac{1 + 4\alpha/\pi}{1 + \alpha} \right).$$

(11)

For $\sigma < \sigma_c$, the nontrivial homogeneous solution $\rho = \rho_0$, $w = w_1 \equiv e \sqrt{\mu/\xi}$ (where $e$ is an arbitrary unit vector) exists and is stable to homogeneous perturbations.

We now focus on the linear stability of $w_1$ with respect to arbitrary wavevector $q$. Because we want to discuss later differences between the metric and metric-free cases, we keep a formal $\rho$-dependence of the linear transport coefficient. Linearization around $w_1$ yields

$$\partial_t \delta \rho = -\nabla \cdot \delta w \quad (12)$$

$$\partial_t \delta w = -\gamma (w_1 \cdot \nabla) \delta w - \frac{1}{2} \nabla^2 \delta w + \kappa (\nabla w_1 \cdot \delta w)$$

where primes indicate derivation with respect to $\rho$. Using the ansatz $(\delta \rho(x,t), \delta w(x,t)) = \exp(\alpha t + i q \cdot x) (\delta \rho_1, \delta w_1)$ allows to recast Eq. (12) as an eigenvalue problem for $\sigma$.

We have solved numerically this cubic problem for the metric-free case using coefficients [10] and Gaussian noise in the full $(\alpha, \sigma)$ parameter plane. The resulting stability diagram, presented in Fig. 1 shows that, contrary to the metric case, the homogeneous ordered phase is stable near onset. Like in the metric case [27], there exists an instability region to oblique wavevectors of large modulus rather far from the transition line. Given that microscopic simulations show no sign of similar instabilities, we believe that the existence of this region, situated away from the validity domain of our approximations, is an artifact of our truncation ansatz.

At the nonlinear level, we performed numerical simulations [28] of Eq. 9 (again with coefficients [10] and Gaussian noise) starting from initial conditions with large variations of both $\rho$ and $w$. With parameters $\alpha$ and $\sigma^2$ in the ordered stable region of Fig. 1 we always observed relaxation towards the linearly-stable homogeneous solution $w_1$, albeit after typically long transients. Starting in the unstable region, the solution blows up in finite-time, signaling that indeed our equation is ill-behaved when considered too far away from onset.

The stabilization of the near-threshold region by metric-free interactions can be directly traced back to the absence of $\rho$ dependence of $\mu$, in agreement with remarks in [12, 24] where the long-wavelength instability of $w_1$ was linked to $\mu' > 0$. In the long wavelength limit $q = |q| \ll 1$, the eigenvalue problem can be solved analytically with relative ease. The growth rate $s$ is the solution of the cubic equation

$$s^3 + \beta_2 s^2 + \beta_1 s + \beta_0 = 0$$

(13)

where the coefficients, to lowest orders in $q$, are given by

$$\beta_2 = 2\mu + 2i q \sqrt{\frac{\mu}{\xi}} \cos \phi + 2 q^2 \nu$$

$$\beta_1 = iq \sqrt{\frac{\mu}{\xi}} \cos \phi \left[ 2\gamma \mu + \left( \mu' - \mu \xi \right) \right]$$

$$+ q^2 \left[ 2\mu \nu + 1 - \frac{\mu}{\xi} \left( \gamma \cos^2 \phi + \kappa \sin^2 \phi \right) \right]$$

$$\beta_0 = \mu q^2 \left[ \left( \mu \xi + \frac{\mu'}{\xi} \right) \left( \gamma \cos^2 \phi + \kappa \sin^2 \phi \right) + \sin^2 \phi \right]$$

$$+ iq^3 \sqrt{\frac{\mu}{\xi}} \cos \phi \left( \mu' - \mu \xi \right) + \frac{\gamma}{2}$$

(14)

and $\phi$ is the angle between $q$ and $w_1$ (which has been chosen parallel to the abscissa). Near threshold, where our truncation is legitimate and $\mu \sim \epsilon^2$, two eigenvalues are always stable and linear stability is controlled by the dominant solution real part

$$s_+ \approx \frac{q^2 \cos^2 \phi}{8 \mu} \left( \frac{\mu'}{\mu} \right) + \mathcal{O} \left( \frac{q^4 \mu'}{\mu^2} \right)$$

(15)

with $h(0) = 2$ and $h(\phi) = 1$ otherwise [30]. This expression immediately shows that $s_+ < 0$ in the metric-free case where $\mu' = 0$, confirming the stability of the homogeneous ordered solution $w_1$. This stabilizing effect can

\[\text{FIG. 1: (color online) Continuous theory for self-propelled particles aligning ferromagnetically (Eqs. 14 and 16 with coefficients [10]. Gaussian noise). (a) Phase diagram in the (\alpha, \sigma^2) plane. The solid line marks the order-disorder transition. The homogeneous ordered solution w_1 exists below this line and is linearly stable above the coloured linear instability region. The colour scale (in radians) indicates the most unstable wavevector direction \phi. (b) Modulus q of the most unstable wavevector (green line) and the real part of its corresponding eigenvalue s_+ (dashed red line) as a function of \sigma^2 at \alpha = 2.}\]
be ultimately traced back to the (negative) pressure term $-\nabla \rho$ appearing in Eq. $\Box$. Conversely, in the metric case for which $\mu \sim 0^+$ near threshold, $s_+$ is always positive, yielding the generic long wavelength instability leading to density segregation in metric models.

To sum up at this point: Through a kinetic approach, we have derived a continuous, “hydrodynamic” theory for self-propelled particles aligning with topological neighbors (“metric-free interactions”). Calculations were presented in full for the case of ferromagnetic alignment, for which we showed that the generic long-wavelength instability of the homogeneous ordered solution present in metric models near onset is suppressed. At the nonlinear level, we observed numerically that the density segregated solutions of metric models vanish, so that the homogeneous solution seems to be a global attractor.

These results rest on the independence of the linear coefficient $\mu$ on $\rho$, a direct consequence of the fact that the interaction rate per particle in topological models is fixed only by geometrical constraints and does not grow with local density. This property actually holds for any metric-free system: all the linear coefficients $\mu_k$ appearing in the equations for $f_k$ read

$$\mu_k = \hat{P}_k - 1 - \alpha + \alpha \hat{P}_k (J_{kk} + J_{00})$$

and are thus independent of $\rho$.

Our other general conclusions also extend to the other basic classes of simple, Vicsek-style collective motion: we have in particular worked out the case of polar particles with nematic alignment (“self-propelled rods”) for which $\Psi(\theta_1, \theta_2) = \arg(e^{i\theta_1} + \text{sign(cos(\theta_1 - \theta_2))} e^{i\theta_2})$. While full details will be given in $\Box$, we only sketch here the salient points. Let us first recall that with nematic alignment, the metric model studied at the microscopic level in $\Box$, shows global nematic order. In a large region of parameter space bordering onset, order is segregated to a high-density stationary band oriented along it. We have studied numerically the metric-free version of that Vicsek-style model with Voronoi neighbors. Like for its ferromagnetic counterpart, no segregation in bands is observed anymore, and the transition to nematic order is then continuous (Fig. 2). These properties are well-captured, both in the metric and metric-free case, by a controlled hydrodynamic approach of the type presented here $\Box$. The nematic symmetry of the problem requires to consider three hydrodynamic fields $\Box$, corresponding to the modes $k = 0, 1, 2$ in Eq. $\Box$, with $f_2$ coding for the nematic tensor field $\rho Q$. We have performed the analysis of the $5 \times 5$ linear problem expressing the stability of the homogeneous nematically-ordered solution ($w = 0$, $\rho Q = \text{Cst.}$) appearing at onset in both the metric and non-metric cases $\Box$. Whereas the metric case shows a long wavelength, transversal instability of the homogeneous ordered solution near onset, this solution is linearly stable in the metric-free case. Again, this difference can be traced back to the $\rho$-dependence of the linear coefficients $\mu_k$. At the nonlinear level, simulations indicate that the homogeneous ordered solution is a global attractor in the metric-free case.

Our analysis can also be extended to “non-ballistic” active matter such as the driven granular rods model (“active nematics”) studied in $\Box$ which, for metric interactions, also shows near-threshold phase segregation $\Box$. Simulations of the metric-free microscopic version (with Voronoi neighbors) show no such segregation. In a kinetic approach, because active nematic particles move by non-equilibrium diffusive currents rather than by ballistic motion, the Boltzmann equation has to be replaced by a more general master equation. But it is nevertheless possible to derive a continuous theory which, in the metric-free case, yields a homogeneous ordered phase stable near onset essentially for the same reasons as in the cases presented above $\Box$.

In conclusion, simple, Vicsek-style, models of active matter where self-propelled particles interact with neighbors defined via non-metric rules (e.g. Voronoi neighbors) are amenable, like their “metric” counterparts, to the construction of continuous hydrodynamic theories well-controlled near onset. The relatively simple framework of Vicsek-style models offers a two-dimensional parameter plane which can be studied entirely. More complicated microscopic starting points, for instance where positional diffusion would also be considered, inevitably raise the dimensionality of parameter space. We have shown here that the non-metric theories differ essentially from the metric ones in the independence of their linear coefficients $\mu_k$ on the local density, a property directly linked to the fact that the collision rate per particle is

![FIG. 2: (color online) Vicsek-style model with nematic alignment and topological neighbors, where $N = L^2$ particles move at speed $v_0 = \frac{1}{2}$ on a $L \times L$ torus. Headings and positions are updated at discrete timesteps according to $\theta_{i,j}^{t+1} = \arg\left(\sum_{k,j} \text{sign(cos(}\theta_{i,k} - \theta_{j,k})\text{)} e^{i\theta_{i,j}^t} + n_{i,j}^0 \sigma \xi_{i,j}^t\right)$ and $n_{i,j}^{t+1} = n_{i,j}^t + v_0 e(\theta_{i,j}^{t+1})$ where $e(\theta)$ is the unit vector along $\theta$, the sum is over the $n_{i,j}^t$ Voronoi neighbors of particle $j$ (including $j$ itself), and $\xi_{i,j}^t$ is a random unit vector in the complex plane. Global nematic order parameter $S = \langle \frac{1}{L^2} \sum_{i,j} e^{i2\pi \theta_{i,j}^t} \rangle$ (a) and its Binder cumulant $G$ (b) vs $\sigma$ for $L = 32, 64, 128$ (the arrows indicate increasing sizes), revealing a continuous transition.](image-url)
constant in metric-free systems. We have shown further that the homogeneous ordered phase is linearly stable near onset for metric-free systems, in contrast with the long-wavelength instability present in metric cases.

We finally discuss the relevance (say in the renormalization-group sense) of metric-free interactions in deciding active matter universality classes. Our work has shown that the deterministic continuous theories of metric-free active matter systems are formally equivalent to those of their metric counterparts, except for the density-dependence of the linear coefficients. This could be taken as an indication, in the case of ferromagnetically-aligning particles, that the homogeneous ordered phase is linearly stable. We have shown further that the homogeneous ordered phase is linearly stable near onset for metric-free systems, in contrast with the long-wavelength instability present in metric cases. This quantity is always negative for coefficients (10).

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