Adaptive Detection of Structured Signals in Low-Rank Interference

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Abstract—In this paper, we consider the problem of detecting the presence (or absence) of an unknown but structured signal from the space-time outputs of an array under strong, non-white interference. Our motivation is the detection of a communication signal in jamming, where often the training portion is known but the data portion is not. We assume that the measurements are corrupted by additive white Gaussian noise of unknown variance and a few strong interferers, whose number, powers, and array responses are unknown. We also assume the desired signals array response is unknown. To address the detection problem, we propose several GLRT-based detection schemes that employ a probabilistic signal model and use the EM algorithm for likelihood maximization. Numerical experiments are presented to assess the performance of the proposed schemes.

Index Terms—array processing, adaptive detection, generalized likelihood ratio test, expectation maximization.

I. INTRODUCTION

A. Problem statement

Consider the problem of detecting the presence or absence of a signal $s \in \mathbb{C}^L$ from the measured output $Y \in \mathbb{C}^{M \times L}$ of an $M$-element antenna array. We are interested in the case where $s$ is unknown but structured. A motivating example arises with communications signals, where typically a few “training” samples are known and the remainder (i.e., the “data” samples) are unknown except for their alphabet. We will assume that the signal’s array response $h \in \mathbb{C}^M$ is completely unknown but constant over the measurement epoch and signal bandwidth. The complete lack of knowledge about $h$ is appropriate when the array manifold is unknown or uncalibrated (e.g., see the discussion in [1]), or when the signal is observed in a dense multipath environment (e.g., [2]).

Also, we will assume that the measurements are corrupted by white noise of unknown variance and $N \geq 0$ possibly strong interferers. The interference statistics are assumed to be unknown, as is $N$.

The signal-detection problem can be formulated as a binary hypothesis test [3] between hypotheses $\mathcal{H}_1$ (signal present) and $\mathcal{H}_0$ (signal absent), i.e.,

$$\begin{align*}
\mathcal{H}_1 : \mathbf{Y} &= \mathbf{h}s^H + \mathbf{B}\Phi^H + \mathbf{W} \in \mathbb{C}^{M \times L} \\
\mathcal{H}_0 : \mathbf{Y} &= \mathbf{B}\Phi^H + \mathbf{W} \in \mathbb{C}^{M \times L}.
\end{align*}$$

(1a) (1b)

In (1), $\mathbf{W}$ refers to the noise and $\mathbf{B}\Phi^H$ to the interference. We model $\mathbf{W}$ as additive white Gaussian noise (AWGN) with unknown variance $\nu > 0$. If the array responses of the $N$ interferers are constant over the measurement epoch and bandwidth, then the rank of $\mathbf{B}\Phi^H$ will be at most $N$. As will be discussed in the sequel, we will sometimes (but not always) model the temporal interference component $\Phi^H$ as white and Gaussian.

Communications signals often take a form like

$$s^H = [s_{d}^H, s_{d}^H],$$

(2)

where $s_{d} \in \mathbb{C}^{Q}$ is a known training sequence, $s_{d} \in \mathbb{A}^{L-Q}$ is an unknown data sequence, $\mathbb{A} \subset \mathbb{C}$ is a finite alphabet, and $Q \ll L$. Suppose that the measurements are partitioned as $Y = [Y_t, Y_d]$, conformal with (2). For the purpose of signal detection or synchronization, the data measurements $Y_d$ are often ignored (see, e.g., [2]). But these data measurements can be very useful, especially when the training symbols (and thus the training measurements $Y_t$) are few. Our goal is to develop detection schemes that use all measurements $Y$ while handling the incomplete knowledge of $s$ in a principled manner.

We propose to model the signal structure probabilistically. That is, we treat $s$ as a random vector with prior pdf $p(s)$. Although the general methodology we propose supports arbitrary $p(s)$, we sometimes focus (for simplicity) on the case of statistically independent components, i.e.,

$$p(s) = \prod_{l=1}^{L} p_{l}(s_{l}).$$

(3)

For example, with uncoded communication signals partitioned as in (2), we would use (3) with

$$p_{l}(s_{l}) = \begin{cases} 
\delta(s_{l} - s_{l,t}) & l = 1, \ldots, Q \\
\frac{1}{|\mathbb{A}_l|} \sum_{s_{l} \in \mathbb{A}_l} \delta(s_{l} - s) & l = Q + 1, \ldots, L, 
\end{cases}$$

(4)

where $\delta(\cdot)$ denotes the Dirac delta and $s_{l,t}$ the $l$th training symbol. For coded communications signals, the independent prior (3) would still be appropriate if a “turbo equalization” approach was used, where symbol estimation is iterated with soft-input soft-input decoding. A variation of (2) that avoids the need to know $\mathbb{A}$ follows from modeling $\{s_{l}\}_{l=Q+1}^{L}$ as i.i.d. Gaussian.

The proposed probabilistic framework is quite general. For example, in addition to training/data structures of the form in (2), the independent model (3) covers superimposed training [5], bit-level training [6], constant-envelope waveforms [1], and pulsed signals (i.e., $s^H = [s_{d}^H 0^{M-H}]$) with unknown $s_{d}$ [1]. To exploit sinusoidal signal models, or signals with known spectral characteristics (see, e.g., [1]), the independent model (3) would be discarded in favor of a more appropriate $p(s)$.

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For the case where the entire signal \( s \in \mathbb{C}^L \) is known, the detection problem \([1]\) has been studied in detail. For example, in the classical work of Kelly \([7, 8]\), the interference-plus-noise \( B\Phi^H + W \) was modeled as temporally white and Gaussian with unknown (and unstructured) spatial covariance \( \Sigma > 0 \), and the generalized likelihood ratio test (GLRT) \([3]\) was derived. Detector performance can be improved when the interference is known to have low rank. For example, Gerlach and Steiner \([9]\) assumed temporally white Gaussian interference with known noise variance \( \nu \) and unknown interference rank \( N \) and derived the GLRT. More recently, Kang, Monga, and Rangaswamy \([10]\) assumed temporally white Gaussian interference with unknown \( \nu \) and known \( N \) and derived the GLRT. Other structures on \( \Sigma \) were considered by Aubry et al. in \([11]\). In a departure from the above methods, McWhorter \([12]\) proposed to treat the interference components \( B \in \mathbb{C}^{M \times N} \) and \( \Phi \in \mathbb{C}^L \times N \), as well as the noise variance \( \nu \), as deterministic unknowns. He then derived the corresponding GLRT. Note that McWhorter’s approach implicitly assumes knowledge of the interference rank \( N \). Bandiera et al. \([13]\) proposed yet a different approach, based on a Bayesian perspective.

For adaptive detection of unknown but structured signals \( s \), we are aware of relatively little prior work. Forsythe \([1]\) p.110] describes an iterative scheme for signals with deterministic (e.g., finite-alphabet, constant envelope) structure that builds on Kelly’s GLRT. Each iteration involves maximum-likelihood (ML) signal estimation and least-squares beamforming, based on the intuition that correct decisions will lead to better beamformers and thus better interference suppression. Error propagation remains a serious issue, however, as we will demonstrate in the sequel.

C. Contributions

We propose three GLRT-based schemes for adaptive detection of unknown structured signals \( s \) with unknown array responses \( h \), AWGN of unknown variance \( \nu \), and interference \( B\Phi^H \) of possibly low rank. All of our schemes use a probabilistic signal model \( s \sim p(s) \), under which the direct evaluation of the GLRT numerator becomes intractable. To circumvent this intractability, we use expectation maximization (EM) \([14]\). In particular, we derive computationally efficient EM procedures for the independent prior \([5]\), paying special attention to finite-alphabet and Gaussian cases.

Our first approach treats the interference \( B\Phi^H \) as temporally white Gaussian and makes no attempt to leverage low interference rank, as in Kelly’s approach \([7]\). The full-rank interference model would be appropriate if, say, the interferers’ array responses varied significantly over the measurement epoch. We show that our first approach is a variation on Forsythe’s iterative scheme \([1]\) p.110] that uses “soft” symbol estimation and “soft” signal subtraction, making it much more robust to error propagation.

Our second approach is an extension of our first that aims to exploit the possibly low-rank nature of the interference. As in \([9]–[11]\), the interference is modeled as temporally white Gaussian but, different from \([9]–[11]\), both the interference rank \( N \) and the noise variance \( \nu \) are unknown. More significantly, unlike \([9]–[11]\), the signal \( s \) is assumed to be unknown.

Our third approach also aims to exploit low-rank interference, but it does so while modeling the interference as deterministic, as in McWhorter \([12]\). Unlike \([12]\), however, the interference rank \( N \) and the signal \( s \) are assumed to be unknown. Numerical experiments are presented to demonstrate the efficacy of our three approach.
B. Low-rank Gaussian Interference

The low-rank property of the interference $B\Phi^H$ can be exploited to improve detector performance. Some of the first work in this direction was published by Gerlach and Steiner in [9]. They assumed known noise variance $\nu$ and temporally white Gaussian interference, so that $B\Phi^H + W \sim \mathcal{CN}(0, R + \nu I)$ with unknown low-rank $R \succeq 0$. The GLRT was then posed under the constraint that $\Sigma \in S_\nu \triangleq \{ R + \nu I : R \succeq 0 \}$:

$$\max_{\Sigma \in S_\nu} p(Y|H_1; h, \Sigma) \quad \text{s.t.} \quad \max_{\Sigma \in S_\nu} p(Y|H_0; \Sigma) \geq \eta.$$

(10)

They showed that the GLRT (10) reduces to one of the form (8), but with thresholded eigenvalues $\hat{\lambda}_{i,m} = \max\{\lambda_{i,m}, \nu\}$.

More recently, Kang, Monga, and Rangaswamy [10] proposed a variation on Gerlach and Steiner’s approach [9] where the noise variance $\nu$ is unknown but $N = \text{rank}(R)$ is known and $N < M$. In particular, they proposed the GLRT:

$$\max_{\hat{\Sigma} \in S_N} p(Y|H_1; h, \hat{\Sigma}) \quad \text{s.t.} \quad \max_{\Sigma \in S_N} p(Y|H_0; \Sigma) \geq \eta,$$

(11)

where

$$S_N \triangleq \{ R + \nu I : \text{rank}(R) = N, R \succeq 0, \nu > 0 \}.$$

(12)

Using a classical result from [16], it can be shown that the GLRT (11) simplifies to

$$\prod_{m=1}^M \hat{\lambda}_{0,m} \geq \eta,$$

(13)

with $\{\hat{\lambda}_{i,m}\}_{m=1}^M$ a smoothed version of $\{\lambda_{i,m}\}_{m=1}^M$ from (9):

$$\hat{\lambda}_{i,m} = \begin{cases} \lambda_{i,m} & m = 1, \ldots, N, \\ \hat{\nu}_i & m = N + 1, \ldots, M. \end{cases}$$

(14)

$$\hat{\nu}_i = \frac{1}{M - N} \sum_{m=N+1}^M \lambda_{i,m}. $$

(15)

C. Low-rank Deterministic Interference

The approaches discussed above all model the interference $B\Phi^H$ as temporally white Gaussian. McWhorter [12] instead proposed to treat the interference components $B \in \mathbb{C}^{M \times N}$ and $\Phi \in \mathbb{C}^{L \times N}$ as deterministic unknowns, yielding the GLRT:

$$\max_{B, \Phi, \nu \geq 0} p(Y|H_1; h, B, \Phi, \nu) \quad \text{s.t.} \quad \max_{B, \Phi, \nu \geq 0} p(Y|H_0; B, \Phi, \nu) \geq \eta,$$

(16)

where the interference rank $N$ is implicitly known. It was shown in [12] that the GLRT (16) simplifies to

$$\frac{\hat{\nu}_0}{p_1} = \frac{\sum_{m=0}^M \lambda_{0,m}}{\sum_{m=0}^M \hat{\lambda}_{0,m}} \geq \eta', $$

(17)

using the $\{\lambda_{i,m}\}$ defined in (9). Comparing (17) to (13), we see that both GLRTs involve noise variance estimates $\hat{\nu}$ computed by averaging the smallest eigenvalues. However, (17) discards the largest $N$ eigenvalues whereas (13) uses them in the test.

III. GLRTs via White Gaussian Interference

We now consider adaptive detection via the binary hypothesis test (1) with unknown structured $s \in \mathbb{C}^L$. As described earlier, our approach is to model $s$ as a random vector with prior density $p(s)$.

Our first approach treats the interference $B\Phi^H$ in (1) as temporally white and Gaussian, as in [7], [9]–[11]. In this case, the interference-plus-noise matrix

$$N \triangleq B\Phi^H + W$$

(18)
is temporally white Gaussian with spatial covariance matrix $\Sigma = R + \nu I_M$, where both $R \succeq 0$ and $\nu > 0$ are unknown. For now, we will model $R$ using a fixed rank $N \leq M$. The $N = M$ case is reminiscent of Kelly [7], and the $N < M$ case is reminiscent of Kang, Monga, and Rangaswamy [10]. The estimation of $\nu$ will be discussed in Sec. III-G.

For a fixed rank $N$, the hypothesis test (1) reduces to

$$H_1 : Y = hs^H + \mathcal{CN}(0, I_L \otimes \Sigma)$$

(19a)

$$H_0 : Y = \mathcal{CN}(0, I_L \otimes \Sigma),$$

(19b)

where $h$ and $\Sigma \in S_N$ (defined in (12)) are unknown and $s \sim p(s)$. When $N = M$, note that $\Sigma \in S_N$ reduces to $\Sigma > 0$. The corresponding GLRT is

$$\max_{\hat{\Sigma} \in S_N} p(Y|H_1; h, \hat{\Sigma}) \quad \text{s.t.} \quad \max_{\Sigma \in S_N} p(Y|H_0; \Sigma) \geq \eta.$$

(20)

As a consequence of $s \sim p(s)$, the numerator likelihood in (20) differs from that in (11), as detailed in the sequel.

A. GLRT Denominator

For the denominator of (20), equations (19b) and (12) imply

$$p(Y|H_0; \Sigma) = \frac{\exp(- \text{tr}(Y^H \Sigma^{-1} Y))}{\pi^{L/2} |\Sigma|^{L/2}} = \left[ \frac{\exp(- \text{tr}(\frac{1}{2} Y^H \Sigma^{-1} Y))}{\pi^{M/2} |\Sigma|^{M/2}} \right]^L.$$

(21)

(22)

We first find the ML estimate $\hat{\Sigma}_0$ of $\Sigma \in S_N$ under $H_0$. When $N < M$, the results in [16] (see also [10]) imply that

$$\hat{\Sigma}_0 = V_0 \Lambda_0 V_0^H, \quad \hat{\Lambda}_0 = \text{Diag}(\hat{\lambda}_0, 1, \ldots, \hat{\lambda}_0, M),$$

(23)

where $\{\hat{\lambda}_0, m\}_{m=1}^M$ follow the definition in (14) with $i = 0$. That is, $\{\hat{\lambda}_0, m\}_{m=1}^M$ is a smoothed version of the eigenvalues $\{\lambda_0, m\}$ of the sample covariance matrix $\frac{1}{2} YY^H$ in decreasing order, where the smoothing averages the $M - N$ smallest eigenvalues to form the noise variance estimate $\hat{\nu}_0$, as in (15). When $N = M$, the results in [15] (see also [7]) imply that $\lambda_0, m = \lambda_0, m$, $\forall m$. In either case, the columns of $V_0$ are the corresponding eigenvectors of the sample covariance matrix $\frac{1}{2} YY^H$. Plugging (23) into (22), taking the log, and
rearranging gives
\[
\frac{1}{2} \ln p(Y|H_0; \hat{\Sigma}_0) + M \ln \pi
= - \text{tr} \{ \frac{1}{2} Y Y^H \hat{\Sigma}_0^{-1} \} - \ln |\hat{\Sigma}_0|.
\] (24)

\[
= \sum_{m=1}^M \left( -\lambda_{0,m} - \ln \hat{\lambda}_{0,m} \right)
= \sum_{m=1}^N (1 - \ln \lambda_{0,m}) + \sum_{m=N+1}^M \left( -\lambda_{0,m} - \ln \hat{\lambda}_{0,m} \right).
\] (25)

Since \( \sum_{m=N+1}^M \lambda_{0,m} = \hat{\nu}_0 \), we have
\[
\frac{1}{2} \ln p(Y|H_0; \hat{\Sigma}_0) + M \ln \pi
= -N - \sum_{m=1}^N \ln \lambda_{0,m} + (M-N)(1-\ln \hat{\nu}_0) \tag{26}
= -M - \sum_{m=1}^N \ln \lambda_{0,m} - (M-N) \ln \hat{\nu}_0 \tag{27}
= -M - \sum_{m=1}^N \ln \lambda_{0,m}. \tag{29}
\]

When \( N < M \), note that \( \{\lambda_{0,m}\}_{m=1}^M \) can be computed using only the \( N \) principal eigenvalues of \( \frac{1}{2} Y Y^H \), since
\[
\hat{\nu}_0 = \frac{1}{M-N} \left( \text{tr} \{ \frac{1}{L} Y Y^H \} - \sum_{m=1}^N \lambda_{0,m} \right). \tag{30}
\]

**B. GLRT Numerator**

For the numerator of (20), \( s \sim p(s) \) and (19a) imply
\[
p(Y|H_1; h, \Sigma) = \int p(Y|s, H_1; h, \Sigma) p(s) \, ds \tag{31}
= \int \exp(-\text{tr} \{ (Y - h s^H) \Sigma^{-1} (Y - h s^H) \}) \pi_{ML|\Sigma} \rho_1 \, p(s) \, ds.
\] (32)

Exact maximization of \( p(Y|H_1; h, \Sigma) \) over \( h \) and \( \Sigma \in S_N \) appears to be intractable. We thus propose to approximate the minimization by applying EM [14] with hidden data \( s \). This implies that we iterate the following over \( t = 0, 1, 2, \ldots \):
\[
\hat{h}^{(t+1)}, \hat{\Sigma}_1^{(t+1)} \tag{33}
\]

\[
= \arg \max_{h \in C^M, \Sigma \in S_N} \mathbb{E} \{ \ln p(Y|s, H_1; h, \Sigma) \mid Y; \hat{h}^{(t)}, \hat{\Sigma}_1^{(t)} \}. \tag{34}
\]

The EM algorithm is guaranteed to converge to a local maxima or saddle point of the likelihood [51] [17]. Furthermore, at each iteration \( t \), the EM-approximated log-likelihood increases and lower bounds the true log-likelihood [18].

Because \( p(s) \) is invariant to \( h \) and \( \Sigma \), (33) becomes
\[
\arg \max_{h \in C^M, \Sigma \in S_N} \mathbb{E} \{ \ln p(Y|s, H_1; h, \Sigma) \mid Y; \hat{h}^{(t)}, \hat{\Sigma}_1^{(t)} \} \tag{35}
= \arg \max_{h \in C^M, \Sigma \in S_N} \int \left[ \text{tr} \{ (Y - h s^H) \Sigma^{-1} (Y - h s^H) \} + \ln |\Sigma|^L \right] p(s|Y; \hat{h}^{(t)}, \hat{\Sigma}_1^{(t)}) \, ds.
\]

We first perform the minimization in (35) over \( h \). Since
\[
\text{tr} \{ (Y - h s^H) \Sigma^{-1} (Y - h s^H) \}
= \text{tr} \{ Y^H \Sigma^{-1} Y - h^H \Sigma^{-1} Y s - s^H Y^H \Sigma^{-1} h
+ h^H \Sigma^{-1} h \|s\|^2 \} \tag{36}
\]
the gradient of the cost in (35) w.r.t. \( h \) equals
\[
2 \int \left[ \Sigma^{-1} h \|s\|^2 - \Sigma^{-1} Y s \right] p(s|Y; \hat{h}^{(t)}, \hat{\Sigma}_1^{(t)}) \, ds, \tag{37}
\]
and this gradient is set to zero by
\[
\hat{h}^{(t+1)} = \frac{Y \mathbb{E} \{ s Y; \hat{h}^{(t)}, \hat{\Sigma}_1^{(t)} \} \Sigma}{\mathbb{E} \{ \|s\|^2 \}|Y; \hat{h}^{(t)}, \hat{\Sigma}_1^{(t)} \}} = \frac{Y \hat{s}^{(t)}}{\mathbb{E} \{ \|s\|^2 \}|Y; \hat{h}^{(t)}, \hat{\Sigma}_1^{(t)} \} \tag{38}
\]
which uses the notation
\[
\hat{s}^{(t)} \triangleq \mathbb{E} \{ s Y; \hat{h}^{(t)}, \hat{\Sigma}_1^{(t)} \} \tag{39}
\]
\[
E^{(t)} \triangleq \mathbb{E} \{ \|s\|^2 \}|Y; \hat{h}^{(t)}, \hat{\Sigma}_1^{(t)} \} \tag{40}
\]

Setting \( h = \hat{h}^{(t+1)} \) in (35), we obtain the cost that must be minimized over \( \Sigma \in S_N \):
\[
\begin{align*}
\text{tr} \{ Y^H \Sigma^{-1} Y - \hat{h}^{(t+1)H} \Sigma^{-1 -1} Y \hat{s}^{(t)} - \hat{s}^{(t)H} Y^H \Sigma^{-1 -1} \hat{h}^{(t+1)} + \hat{h}^{(t+1)H} \Sigma^{-1} \hat{h}^{(t+1)} \} + \ln |\Sigma|^L \\
= \text{tr} \{ Y^H \Sigma^{-1} Y - \hat{\Sigma}_1^{(t)H} Y \hat{s}^{(t)} - \hat{s}^{(t)H} Y^H \Sigma^{-1} \hat{h}^{(t+1)} \} + \ln |\Sigma|^L \tag{41}
= \text{tr} \{ Y \tilde{P}^{(t)} \Sigma Y^H \Sigma^{-1} \} + \ln |\Sigma|^L \tag{42}
\end{align*}
\]
where
\[
\tilde{P}^{(t)} \triangleq I_L - \frac{\hat{s}^{(t)H}}{E^{(t)}} \tag{43}
\]
\[
= P^{(t)} + P_{\Sigma}^{(t)} - \frac{\hat{s}^{(t)H}}{E^{(t)}} \|s\|^2 \tag{44}
\]
\[
= P^{(t)} + P_{\Sigma}^{(t)} \frac{E^{(t)} - \|s\|^2}{E^{(t)}} \tag{45}
\]
\[
= P^{(t)} + P_{\Sigma}^{(t)} \frac{\text{tr} \{ \text{Cov} \{ s Y; \hat{h}^{(t)}, \hat{\Sigma}_1^{(t)} \} \}}{E^{(t)}} \tag{46}
\]
Note that \( \tilde{P}^{(t)} \) is a regularized version of the projection matrix \( P^{(t)} \) that equals \( P^{(t)} \) when \( s \) is completely known. In general, however, \( \tilde{P}^{(t)} \) is not a projection matrix. Minimizing (42) over \( \Sigma \in S_N \) is equivalent to maximizing
\[
\exp(-\text{tr} \{ Y \tilde{P}^{(t)} \Sigma Y^H \Sigma^{-1} \}) \frac{1}{\pi^{ML|\Sigma|^L}}. \tag{47}
\]
As with (22), when $N < M$, the results in [16] imply
\begin{equation}
\Sigma^{(t+1)} = V^{(t+1)} \Lambda^{(t+1)} V^{(t+1)\text{T}},
\end{equation}
\begin{equation}
\Lambda^{(t+1)} = \text{Diag}(\lambda_{1}^{(t+1)}, \ldots, \lambda_{M}^{(t+1)})
\end{equation}
\begin{equation}
\Omega_{1,m}^{(t+1)} = \left\{ \begin{array}{ll}
\lambda_{1,m}^{(t+1)} & m = 1, \ldots, N \\
\hat{\nu}_{m}^{(t+1)} & m = N + 1, \ldots, M
\end{array} \right.
\end{equation}
\begin{equation}
\hat{\nu}_{m}^{(t+1)} = \frac{1}{M - N} \sum_{m=N+1}^{M} \lambda_{1,m}^{(t+1)},
\end{equation}
where $\{\lambda_{1,m}^{(t+1)}\}_{m=1}^{M}$ are the eigenvalues of the matrix $\frac{1}{L} Y \bar{P}_{\omega_{1}} Y^\text{T}$ in decreasing order, and the columns of $V^{(t+1)}$ are the corresponding eigenvectors. When $N = M$, we have that $\lambda_{1,m} = \lambda_{1,m} \forall m$.

We have thus derived the EM procedure that iteratively lower bounds [18] the numerator of (20) under a generic signal prior $p(s)$.

C. EM Update under an Independent Prior

The EM updates of $s^{(t)}$ and $E^{(t)}$ in (39)–(40) depend on the specifics of the prior $p(s)$. For any independent prior, as in [3], we can MMSE-estimate the symbols one at a time from the measurement equation
\begin{equation}
y_{t} = h^{(t)} \omega_{t} + \mathcal{CN}(0, \bar{\Sigma}_{1}^{(t)}).
\end{equation}
From $y_{t}$, we obtain a sufficient statistic [3] for the estimation of $s_{1}$ by spatially whitening the measurements via
\begin{equation}
\tilde{y}_{1}^{(t)} = (\tilde{\Sigma}_{1}^{(t)})^{-\frac{1}{2}} y_{t} = (\hat{\Sigma}_{1}^{(t)})^{-\frac{1}{2}} \hat{h}^{(t)} s_{1} + \mathcal{CN}(0, I)
\end{equation}
and then matched filtering via
\begin{equation}
\tilde{r}_{1}^{(t)} = \hat{h}^{(t)\text{T}} (\tilde{\Sigma}_{1}^{(t)})^{-\frac{1}{2}} \tilde{y}_{1}^{(t)} = \xi^{(t)} s_{1} + \mathcal{CN}(0, \xi^{(t)}),
\end{equation}
where
\begin{equation}
\xi^{(t)} \triangleq \frac{1}{\hat{h}^{(t)\text{T}} \hat{\Sigma}_{1}^{(t)} \hat{h}^{(t)}}.
\end{equation}
We find it more convenient to work with the normalized and conjugated statistic
\begin{equation}
r_{1}^{(t)} \triangleq \frac{\tilde{r}_{1}^{(t)*}}{\xi^{(t)}} = s_{1} + \mathcal{CN} \left( 0, \frac{1}{\xi^{(t)}} \right),
\end{equation}
which is a Gaussian-noise-corrupted version of the true symbol $s_{1}$, with noise precision $\xi^{(t)}$.

The computation of the MMSE estimate $\hat{s}_{1}$ from $r_{1}^{(t)}$ depends on the prior $p(s_{1})$. For the Gaussian prior $p(s_{1}) = \mathcal{CN}(s_{1}; \mu_{1}, \nu_{1})$, we have the posterior mean and variance [3]
\begin{equation}
\hat{s}_{1} = \mu_{1} + \frac{\nu_{1}}{\nu_{1} + 1/\xi^{(t)}} (r_{1}^{(t)} - \mu_{1})
\end{equation}
\begin{equation}
\hat{\nu}_{1} = \frac{1}{\xi^{(t)} + 1/\nu_{1}},
\end{equation}
which from (40) implies
\begin{equation}
E^{(t)} = \frac{L}{l=1} E \{ |s_{l}|^2 | Y ; \hat{h}, \hat{\Sigma}_{1}^{(t)} \} = \frac{L}{l=1} \left( |\hat{s}_{l}|^2 + \hat{\nu}_{l} \right).
\end{equation}

**Algorithm 1** EM update under white Gaussian interference

**Require:** Data $Y \in \mathbb{C}^{M \times L}$, signal prior $p(s) = \prod_{l=1}^{L} p(s_{l})$.

1. Initialize $\hat{s}$ and $E > 0$ (see Sec. III-B).
2. repeat
3. $\hat{h} \leftarrow \frac{1}{\nu} Y \hat{s}$
4. $\hat{\Sigma} \leftarrow \frac{1}{\nu} Y Y^\text{T} - \frac{1}{\nu} \hat{s} \hat{h}^\text{T}$
5. Estimate interference rank $N$ (see Sec. III-C).
6. if $N = 0$ then
7. $\hat{\nu}_{1} \leftarrow \frac{1}{\nu} \text{tr}(\hat{\Sigma})$
8. $g \leftarrow \frac{1}{\nu} \hat{h}$
9. else if $N = M$ then
10. $g \leftarrow \hat{\Sigma}_{1} \hat{h}$
11. else
12. $\{ \hat{V}_{1}, \hat{\Sigma}_{1} \} \leftarrow \text{principal_eig}(\hat{\Sigma}, N)$
13. $\hat{\nu}_{1} \leftarrow \frac{1}{\nu} \text{tr}(\hat{\Sigma}) - \text{tr}(\hat{\Sigma}_{1})$
14. $g \leftarrow \frac{1}{\nu} \hat{h} + \hat{V}_{1} (\hat{\Sigma}_{1}^{-1} - \frac{1}{\nu} I_{N}) \hat{V}_{1}^\text{T} \hat{h}$
15. end if
16. $\xi \leftarrow \frac{1}{g} g$
17. $\omega \leftarrow \frac{1}{\xi} Y Y^\text{T} g$ where $\omega \sim \mathcal{CN}(s, I/\xi)$
18. $\hat{s}_{l} \leftarrow E \{ s_{l} | r_{l} ; \omega \} \forall l = 1, \ldots, L$
19. $E \leftarrow \sum_{l=1}^{L} E \{ |s_{l}|^2 | r_{l} ; \omega \}$
20. until Terminated

For the discrete prior $p(s_{l}) = \sum_{k=1}^{K_{1}} \omega_{lk} \delta(s_{l} - d_{lk})$, with alphabet $A_{l} = \{ d_{lk} \}_{k=1}^{K_{1}}$ and prior symbol probabilities $\omega_{lk} \geq 0$ (such that $\sum_{k=1}^{K_{1}} \omega_{lk} = 1 \forall l$), it is straightforward to show that the posterior density is
\begin{equation}
p(s_{l} | r_{l}^{(t)}) = \frac{\sum_{k=1}^{K_{1}} \omega_{lk} \mathcal{CN}(d_{lk} ; r_{l}^{(t)} / 1/\xi^{(t)})}{\sum_{k=1}^{K_{1}} \omega_{lk} \mathcal{CN}(d_{lk} ; r_{l}^{(t)} / 1/\xi^{(t)})},
\end{equation}
and thus the posterior mean and second moment are
\begin{equation}
\hat{s}_{l}^{(t)} = \frac{\sum_{k=1}^{K_{1}} \omega_{lk} d_{lk} \mathcal{CN}(d_{lk} ; r_{l}^{(t)} / 1/\xi^{(t)})}{\sum_{k=1}^{K_{1}} \omega_{lk} \mathcal{CN}(d_{lk} ; r_{l}^{(t)} / 1/\xi^{(t)})},
\end{equation}
\begin{equation}
E \{ |s_{l}^{(t)}|^{2} | Y ; \hat{h}, \hat{\Sigma}_{1}^{(t)} \} = \frac{\sum_{k=1}^{K_{1}} \omega_{lk} |d_{lk}|^{2}}{\sum_{k=1}^{K_{1}} \omega_{lk} \mathcal{CN}(d_{lk} ; r_{l}^{(t)} / 1/\xi^{(t)})},
\end{equation}
which from (40) implies
\begin{equation}
E^{(t)} = \sum_{l=1}^{L} \sum_{k=1}^{K_{1}} \omega_{lk} |d_{lk}|^{2}.
\end{equation}
This EM update procedure is summarized in Alg. 1.

D. Fast Implementation of Algorithm 7

The implementation complexity of Alg. 7 is dominated by the eigenvalue decomposition in line 12 which consumes $O(M^{3})$ operations per EM iteration. We now describe how the complexity of this step can be reduced. Recall that
\begin{equation}
\frac{1}{L} Y Y^\text{T} = V_{0} \Lambda_{0} V_{0}^\text{T},
\end{equation}
as described after [23]. Thus \( \hat{\Sigma} \) in line 4 takes the form
\[
\hat{\Sigma} = V_0 A_0 V_0^H - \frac{1}{2} \bar{h} h^H
\]
(66)
using the definition
\[
\bar{h} \triangleq \sqrt{\frac{2}{T}} V_0^H h.
\]
(68)
The key idea is that the eigen-decomposition of \( A_0 - \bar{h} h^H \)
can be computed in a fast manner due to its diagonal-plus-rank-one structure [19].

We now provide some details. First, define \( R \triangleq \text{rank}(\frac{1}{T} Y Y^H) \), where \( R \leq M \). Without loss of generality, suppose that \( V_0 \) has \( R \) columns and that \( A_0 \in \mathbb{R}^{R \times R} \), and assume that these quantities have been computed before the start of the EM iterations. Then \( \bar{h} \) can be computed in \( O(MR) \) operations, the eigen-decomposition \( Q A_0 Q^H = A_0 - \bar{h} h^H \) can be computed in \( O(R^2) \) operations [19], and the eigenvectors \( V_1 = V_0 Q \) of \( \hat{\Sigma} \) can be computed in \( O(MR^2) \) operations. Since only the \( N \) principal eigenvectors are needed for line 12, the latter reduces to \( O(MRN) \) operations.

E. Evaluation of the GLRT

We now describe what remains of the GLRT. Let us denote the final EM-based estimates of \( s, h \), and \( \Sigma \) under \( H_1 \) as \( \hat{s}, \hat{h}, \) and \( \hat{\Sigma} \), respectively. Notice that
\[
\frac{1}{T} \ln p(Y|H_1; \hat{h}, \hat{\Sigma})
= - \text{tr} \left\{ \frac{1}{T} Y \hat{P} \hat{\Sigma}^{-1} \right\} - \ln |\hat{\Sigma}| - M \ln \pi
= - M - \sum_{m=1}^{M} \ln \hat{\lambda}_{1,m} - M \ln \pi,
\]
(69)
following steps similar to (29). Recalling (20), the log-domain GLRT is obtained by subtracting (29) from (70), yielding
\[
\sum_{m=1}^{M} \ln \frac{\hat{\lambda}_{1,m}}{\lambda_{1,m}} \geq \eta.
\]
(71)
When \( N < M \), this test can be simplified by recalling that the smallest \( M - N \) eigenvalues in \( \{\hat{\lambda}_{1,m}\} \) equal \( \hat{\nu}_i \) for \( i = 0, 1 \). In this case, the log-domain GLRT reduces to
\[
\sum_{m=1}^{N} \ln \frac{\hat{\lambda}_{1,m}}{\lambda_{1,m}} + (M - N) \ln \frac{\hat{\nu}_0}{\nu_1} \geq \eta.
\]
(72)

F. Relation to Forsythe’s Iterative Method

We now connect the above method to Forsythe’s iterative scheme in [1] p.110, which assumes full-rank interference (i.e., \( N = M \)) and positive definite sample covariance, i.e., \( \frac{1}{T} Y Y^H > 0 \). To make this connection, we consider the spatially whitened measurements
\[
Y \triangleq (\frac{1}{T} Y Y^H)^{-\frac{T}{2}} Y.
\]
(73)

Writing lines 3, 4, 16 and 17 of Alg. 1 in terms of the whitened quantities \( \hat{h} \triangleq (\frac{1}{T} Y Y^H)^{-\frac{T}{2}} h \) and \( \hat{\Sigma} \triangleq (\frac{1}{T} Y Y^H)^{-\frac{T}{2}} \hat{\Sigma} (\frac{1}{T} Y Y^H)^{-\frac{T}{2}} \) gives
\[
\hat{h} = \frac{1}{\sqrt{T}} Y \hat{s},
\]
(74)
\[
\hat{\Sigma} \triangleq \frac{1}{T} Y Y^H - \bar{h} h^H
\]
(75)
\[
\xi \triangleq \hat{h} \hat{\Sigma}^{-1} \hat{h}.
\]
(76)
\[
r = \xi^{-1} \frac{1}{T} Y Y^H \xi^{-1} \hat{h}.
\]
(77)
From the construction of \( Y \) and the assumption \( \frac{1}{T} Y Y^H > 0 \), we have \( \frac{1}{T} Y Y^H = I_M \). Thus, applying the matrix inversion lemma to (75) gives
\[
\hat{\Sigma}^{-1} = \hat{I}_M - \left( \frac{T}{\xi} + |\hat{h}|^2 \right)^{-1} \bar{h} \bar{h}^H.
\]
(78)
Plugging (78) into (77), we obtain
\[
r = \frac{Y^H \hat{h}}{|\hat{h}|^2} = \frac{Y^H Y \hat{s} E}{|Y \hat{s}|^2},
\]
(79)
which can be expressed in terms of unwhitened quantities as
\[
r = \frac{Y^H (\frac{1}{T} Y Y^H)^{-1} Y \hat{s} E}{\| (\frac{1}{T} Y Y^H)^{-1} Y \hat{s} \|^2} = \frac{Y^H (\frac{1}{T} Y Y^H)^{-1} Y \hat{s} E}{s^H \hat{\Sigma}^{-1} \frac{1}{T} Y Y^H (\frac{1}{T} Y Y^H)^{-1} Y \hat{s}}.
\]
(80)
\[
E \triangleq \frac{P_{Y^H} \hat{s} E}{\|P_{Y^H} \hat{s} E\|^2}.
\]
(81)
Algorithm [1] prescribes the use of the “soft” symbol estimate \( \hat{s} = E\{s|r; \xi\} \) and the soft squared-norm estimate \( E = E\{|s|^2 |r; \xi\} \) in lines [18][19]. If we replaced these soft estimates with “hard” estimates, i.e., the ML estimate \( \hat{s}_{ML} = \arg \min_{s \in A^c} |r - s|^2 \) and its squared-norm \( E_{ML} = \| \hat{s}_{ML} \|^2 \), then Alg. [1] would become
\[
w \leftarrow (Y Y^H)^{-1} Y \hat{s}_{ML} \| \hat{s}_{ML} \|^2,
\]
(82)
\[
r \leftarrow Y^H w
\]
(83)
\[
\hat{s}_{ML} \leftarrow \arg \min_{s \in A^c} |r - s|^2,
\]
(84)
which is precisely Forsythe’s iterative method from [1] p.110. There, \( w \) is interpreted as a least-squares (LS) beamformer. We have thus shown that Alg. [1] under fixed rank \( N = M \) is a soft version of Forsythe’s iterative method. We will show later, the soft nature of Alg. [1] helps to prevent error propagation.

G. Estimating the Interference Rank \( N \)

We now consider estimation of the interference rank \( N = \text{rank}(R) \). For this, we adopt the standard information-theoretic model-order selection approach described in, e.g., [20], which specifies
\[
\hat{N} = \arg \max_{N=0,\ldots,N_{max}} \ln p(Y|H_1; \hat{\Theta}_N) - J(D(N)),
\]
(85)
where \( J(\cdot) \) is a penalty function, \( \hat{\Theta}_N \) is the ML parameter estimate under rank hypothesis \( N \), and \( D(N) \) is the degrees-
of-freedom (DoF) in the parameters $\Theta_N$. Common choices of $J(\cdot)$ include

$$J(D) = \begin{cases} \frac{D}{2} \ln T & \text{Akaikes Information Criterion (AIC)} \\ \frac{D}{2} \ln T + \frac{GD}{2} & \text{Corrected AIC} \\ \frac{D}{2} \ln T + \ln G & \text{Bayesian Information Criterion (BIC)} \\ \frac{D}{2} \ln T + \ln G & \text{Generalized Information Criterion (GIC)} \end{cases}$$

where $T$ is the number of real-valued measurements and $G > 0$ is a tunable gain. The above BIC rule is the same as that which results from Rissanens Minimum Description Length (MDL) criterion $T$ (see [20]).

For Alg. 1 we have $T = 2ML$ and

$$\Theta_N = \{h, \Sigma\}$$

for $h \in \mathbb{C}^M$ and $\Sigma \in \mathcal{S}_N$, \hspace{1cm} (86)

with $\Sigma$ defined in [12]. Here, the DoF in $h$ equals $2M$ and the DoF in $\Sigma$ equals $(2M-N)N+1$ since the DoF in a $M \times M$ rank-$N$ Hermitian matrix $R$ is $(2M-N)N$ and the DoF in the noise variance $\nu$ is 1. In summary, $D(N) = (2M-N)N+2M+1$. For our numerical experiments, we used GIC with $G = 12.5$.

**H. EM Initialization**

The EM algorithm is guaranteed to converge to a local maxima or saddle point of the likelihood [31]. With a multi-modal likelihood, the initialization of $(\hat{s}, E)$ affects the quality of the final EM estimate. Below, we propose an initialization assuming the training/data structure in (3). That is, $Y = [Y_1, Y_0]$ with

$$Y_1 = hs_1^H + N_1, \quad \text{vec}(N_1) \sim \mathcal{CN}(0, I_Q \otimes \Sigma)$$

$$Y_0 = hs_0^H + N_0, \quad \text{vec}(N_0) \sim \mathcal{CN}(0, I_L \otimes \Sigma),$$

and $s = [s_1^H, s_0^H]^H$. Essentially, we would like to estimate the random vector $s_0 \sim \prod_{l=Q+1}^L p_l(s_l)$ from measurements $Y$ under known $s_1$ but unknown $s_0, h, \Sigma, N$.

Recall that the whitened matched-filter (WMF) outputs

$$r_l = y_l^H \Sigma_{-1}^{-1} h$$

are sufficient statistics [3] for estimating $s_0$. Because $\Sigma$ and $h$ are unknown in our case, we propose to estimate them from the training data $Y_1$ and use the results to compute approximate-WMF outputs of the form

$$\hat{r}_l \triangleq y_l^H \Sigma_{-1}^{-1} \hat{h}_l.$$  \hspace{1cm} (90)

With appropriate scaling $\beta \in \mathbb{C}$, we get an unbiased statistic

$$\beta \hat{r}_l \approx s_l + \mathcal{CN}(0, 1/\hat{\xi})$$

for $l \in \{Q+1, \ldots, L\}$

that can be converted to MMSE symbol estimates $\hat{s}_l$ via (57) or (62), which are suitable for EM initialization. Likewise, the initialization of $E$ can be computed from (59) or (64).

As for the choice of $(\hat{\Sigma}, \hat{h})$ in (20), one possibility is the joint ML estimate of $\Sigma \in \mathcal{S}_N$ and $h \in \mathbb{C}^M$ from the training $Y_1$, assuming known interference rank $N$. The arguments in Sec. [H-E] reveal that these joint-ML estimates equal

$$\hat{h}_l \triangleq \frac{Y_1 s_l}{||s_l||^2}$$

$$\hat{\Sigma}_l^{(N)} \triangleq \text{V_1 Diag}(\hat{\lambda}_1^{(N)}, \ldots, \hat{\lambda}_M^{(N)}) \text{V}_1^H,$$

where

$$\hat{\lambda}_t^{(N)} \triangleq \left\{ \begin{array}{ll} \lambda_{t,m} & m = 1, \ldots, N \\ \frac{1}{M-N} \sum_{m'=N+1}^M \lambda_{t,m'} & m = N+1, \ldots, M \end{array} \right.$$  \hspace{1cm} (93)

such that $\{\hat{\lambda}_{t,m}\}_{m=1}^N$ are the eigenvalues of the sample covariance matrix

$$\hat{\Sigma}_l \triangleq \frac{1}{Q} Y_l P_{s_l} Y_l^H$$

in decreasing order and $V_1$ contains the eigenvectors. When the interference rank $N$ is unknown, the methods in Sec. [H-G] can be used to estimate $N$ from $Y_1$. However, the estimation of the unbiasing gain $\beta$ and the precision $\xi$ in (91) remain challenging.

Instead of rank-$N$ covariance estimation, we propose to use a regularized estimate of the form [22]

$$\hat{\Sigma}_l^{(\alpha)} = (1 - \alpha) \hat{\Sigma}_l + \alpha c I_M, \quad \alpha \in (0, 1],$$

with $\hat{\Sigma}_l$ from (95) and $c \triangleq \text{tr}(\hat{\Sigma}_l)/M$. Since the goal of regularization is robust estimation under possibly few training samples $Q$, we propose to choose $\alpha$ to maximize (post-unbiased) precision $\xi$, where the precision is estimated via leave-one-out cross-validation (LOOCV) [23] on the training data. Our LOOCV approach is similar to the “SEO” scheme from [24] but targets minimum-variance unbiased estimation rather than MMSE estimation and, more significantly, handles non-white interference. Details are provided below.

We first define the leave-one-out training quantities $Y_{-l} \triangleq [y_1, \ldots, y_{l-1}, y_{l+1}, \ldots, y_Q]$ and $s_{-l} \triangleq [s_1, \ldots, s_{l-1}, s_{l+1}, \ldots, s_Q]^T$. From these, we construct the ML $h$-estimate and $\alpha$-regularized sample covariance

$$\hat{h}_{-l} \triangleq \frac{Y_{-l} s_{-l}}{||s_{-l}||^2}$$

$$\hat{\Sigma}_{-l}^{(\alpha)} \triangleq \frac{1}{Q-1} Y_{-l} P_{s_{-l}} Y_{-l}^H + \alpha c I_M,$$

which can be used to form the out-of-sample estimate

$$\hat{r}_{l}^{(\alpha)} \triangleq y_l^H (\hat{\Sigma}_{-l}^{(\alpha)})^{-1} \hat{h}_{-l}.$$  \hspace{1cm} (99)

It can be shown that

$$\hat{h}_{-l} = \hat{h}_l - \frac{s_l}{||s_l||^2} \hat{n}_l$$

for $\hat{n}_l \triangleq y_l - \hat{h}_l s_l$.

Also, using the matrix inversion lemma, it can be shown that

$$(\hat{\Sigma}_{-l}^{(\alpha)})^{-1} = (\hat{\Sigma}_l^{(\alpha)})^{-1} + \frac{(\hat{\Sigma}_l^{(\alpha)})^{-1} \hat{n}_l (\hat{\Sigma}_l^{(\alpha)})^{-1} \hat{n}_l^H}{1 - g_l^{(\alpha)} \hat{n}_l^H (\hat{\Sigma}_l^{(\alpha)})^{-1} \hat{n}_l}$$  \hspace{1cm} (102)
for
\[
\hat{\Sigma}_t^{(\alpha)} \triangleq (1 - \alpha) \frac{Q}{Q - 1} \hat{\Sigma}_t + \alpha c I_M
\]
(103)
\[
g_t^{(\alpha)} \triangleq (1 - \alpha) \frac{1}{Q - 1} \left( 1 + \frac{|s_l|^2}{\|s\|^2 - |s_l|^2} \right).
\]
(104)
Merging (99), (100), and (102), we find that
\[
\hat{r}_t^{(\alpha)} = y_t^H (\hat{\Sigma}_t^{(\alpha)})^{-1} \hat{h}_t + \frac{\hat{y}_t^H (\hat{\Sigma}_t^{(\alpha)})^{-1} \hat{n}_t}{1 - g_t^{(\alpha)} \hat{n}_t^H (\hat{\Sigma}_t^{(\alpha)})^{-1} \hat{n}_t}
\times \left( g_t^{(\alpha)} \hat{n}_t^H (\hat{\Sigma}_t^{(\alpha)})^{-1} \hat{h}_t - \frac{s_l^H}{\|s_l\|^2 - |s_l|^2} \right).
\]
(105)
With the eigen-decomposition \(\hat{\Sigma}_t = V_t \Lambda_t V_t^H\), we have
\[
(\hat{\Sigma}_t^{(\alpha)})^{-1} = V_t \left( (1 - \alpha) \frac{Q}{Q - 1} \Lambda_t + \alpha c I_M \right)^{-1} V_t^H
\]
(106)
which can be used to compute \(\hat{r}_t^{(\alpha)} = \left[ \hat{r}_1^{(\alpha)}, \ldots, \hat{r}_Q^{(\alpha)} \right]^T\) efficiently via
\[
y_t^H (\hat{\Sigma}_t^{(\alpha)})^{-1} \hat{h}_t = \left[ Y_t^H V_t (V_t^H \hat{h}_t \odot \gamma^{(\alpha)}) \right]_l
\]
(107)
\[
\hat{n}_t^H (\hat{\Sigma}_t^{(\alpha)})^{-1} \hat{n}_t = \left[ \hat{N}_t V_t (V_t^H \hat{h}_t \odot \gamma^{(\alpha)}) \right]_l
\]
(108)
\[
\frac{1}{Q} \sum_{l=1}^Q \left| \frac{\beta^{(\alpha)} l^{(\alpha)}}{s_l} \right|.
\]
(111)
After scaling by \(\beta^{(\alpha)}\), the error precision \(\tilde{\xi}^{(\alpha)}\) is
\[
\tilde{\xi}^{(\alpha)} = \frac{1}{Q} \sum_{l=1}^Q \left| \beta^{(\alpha)} r_l^{(\alpha)} - s_l \right|.
\]
(122)
The value of \(\alpha\) can be optimized by maximizing \(\tilde{\xi}^{(\alpha)}\) over a grid of possible values.

IV. GLRT via Deterministic Interference

We now propose a different adaptive detector for \(s \sim p(s)\) that treats the interference \(B \Phi^H\) as a deterministic unknown, rather than as temporally white and Gaussian, as in Sec. III. In particular, it treats \(B \in \mathbb{C}^{M \times N}\) and \(\Phi \in \mathbb{C}^{L \times N}\) as deterministic unknowns, as in [12], for some rank hypothesis \(N < \min\{M, L\}\). The rank hypothesis \(N\) will be adapted as described in Sec. IV-E. However, we first describe the approach under a fixed choice of \(N\). In this case, the binary hypothesis test (1) implies the GLRT
\[
\frac{\max_{h, B, \Phi} \nu > 0}{\max_{B, \Phi, \nu > 0}} p(Y | \mathcal{H}_1; h, B, \Phi, \nu) p(Y | \mathcal{H}_0; B, \Phi, \nu) \geq \eta.
\]
(113)

A. GLRT Denominator

Starting with the denominator of (13), we have
\[
p(Y | \mathcal{H}_0; B, \Phi, \nu) = \prod_{m=1}^M \exp \left( - \frac{\|y_m^H - b_m^H \Phi^H\|^2}{\nu} \right).
\]
(114)
where \(y_m^H\) denotes the \(m\)th row of \(Y\) and \(b_m^H\) denotes the \(m\)th row of \(B\). Due to the factorization in (114), the ML estimate of each \(b_m\) can be individually computed as
\[
\hat{b}_{0,m} \triangleq \operatorname{arg \min}_{b_m} \|y_m - \Phi b_m\|^2 = \Phi^+ y_m,
\]
(115)
where \((\cdot)^+\) denotes the pseudo-inverse, i.e., \(\Phi^+ = (\Phi^H \Phi)^{-1} \Phi^H\). Plugging \(\hat{b}_{0,m}\) into (114) gives
\[
p(Y | \mathcal{H}_0; \hat{B}_0, \Phi, \nu) = \prod_{m=1}^M \exp \left( - \frac{\|y_m^H \Phi^+ y_m\|^2}{\nu} \right)
\]
(116)
which can be used to compute \(\hat{r}_t^{(\alpha)} = \left[ \hat{r}_1^{(\alpha)}, \ldots, \hat{r}_Q^{(\alpha)} \right]^T\) efficiently via
\[
y_t^H (\hat{\Sigma}_t^{(\alpha)})^{-1} \hat{h}_t = \left[ Y_t^H V_t (V_t^H \hat{h}_t \odot \gamma^{(\alpha)}) \right]_l
\]
(107)
\[
\hat{n}_t^H (\hat{\Sigma}_t^{(\alpha)})^{-1} \hat{n}_t = \left[ \hat{N}_t V_t (V_t^H \hat{h}_t \odot \gamma^{(\alpha)}) \right]_l
\]
(108)
\[
\left( \left( Y_t^H V_t \odot \hat{N}_t V_t^H \right) \left( 1 \odot \gamma^{(\alpha)} \right) \right)_l.
\]
(109)
Next we maximize over the noise variance \(\nu > 0\). The negative log-likelihood is
\[
- \ln p(Y | \mathcal{H}_0; \hat{B}_0, \Phi, \nu) = \operatorname{tr} \left( Y P_{\Phi}^+ Y^H \right) / \nu + ML \ln \pi + ML \ln \nu,
\]
(118)
and so zeroing its gradient gives the ML estimate
\[
\hat{\nu}_0 = \frac{1}{ML} \operatorname{tr} \left( Y P_{\Phi}^+ Y^H \right).
\]
(119)
Plugging this back into (118) gives
\[
- \ln p(Y | \mathcal{H}_0; \hat{B}_0, \hat{\nu}_0) = ML (1 + \ln \pi) + ML \ln \left( \frac{1}{ML} \operatorname{tr} \left( Y P_{\Phi}^+ Y^H \right) \right).
\]
(120)
Finally, minimizing this negative log-likelihood over \(\Phi\) is equivalent to minimizing \(\operatorname{tr} \left( Y P_{\Phi}^+ Y^H \right) = \operatorname{tr} \left( Y^H Y \right) - \operatorname{tr} \left( Y P_{\Phi} Y^H \right)\), or maximizing \(\operatorname{tr} \left( Y P_{\Phi} Y^H \right) = \operatorname{tr} \left( P_{\Phi} Y^H Y \right) \Phi \). But since the trace of a matrix is the sum of its eigenvalues, the optimal \(\Phi\) are those whose column space is the span of the dominant eigenvectors of \(Y^H Y\). In summary, the minimized negative log-likelihood equals
\[
- \ln p(Y | \mathcal{H}_0; \hat{B}_0, \hat{\nu}_0, \hat{\nu}_0) = ML (1 + \ln \pi) + ML \ln \left( \frac{1}{ML} \sum_{m=1}^M \lambda_{0,m} \right),
\]
(121)
where \(\lambda_{0,m}\) are the eigenvalues of \(Y^H Y\) in decreasing order, as perceived.\(\Box\)
B. GLRT Numerator

For the numerator of (113), equation (1a) implies
\[
p(Y|\mathcal{H}_1; h, B, \Phi, \nu) = \int p(Y|s, \mathcal{H}_1; h, B, \Phi, \nu) p(s) \, ds
\]
\[
= \int \exp\left(-\frac{\|Y - B\Phi^H s - h s^H\|^2}{(\pi\nu)^{ML}}\right) p(s) \, ds
\]  
(122)

Exact maximization of \( p(Y|\mathcal{H}_1; h, B, \Phi, \nu) \) over \( \Theta \equiv \{h, B, \Phi, \nu\} \)

appears to be intractable. As before, we propose to apply EM

\[\hat{\Theta}^{(t+1)} = \arg \max_{\Theta} \mathbb{E} \{ \ln p(Y, s|\mathcal{H}_1; \Theta) | Y; \hat{\Theta}^{(t)} \} \]. (125)

Because \( p(s) \) is invariant to \( \Theta \), (125) can be rewritten as

\[\hat{\Theta}^{(t+1)} = \arg \min_{\Theta} \int \left[ \frac{\|Y - B\Phi^H s - h s^H\|^2}{\nu} + ML \ln(\pi \nu) \right] \times p(s|Y; \hat{\Theta}^{(t)}) \, ds \]
\[= \arg \min_{\Theta} \int \frac{\|Y - B\Phi^H s - h s^H\|^2}{\nu} p(s|Y; \hat{\Theta}^{(t)}) \, ds + ML \ln(\pi \nu). \]  
(126)

Noting that

\[\|Y - B\Phi^H s - h s^H\|^2 \]
\[= \|Y - B\Phi^H s\|^2 + \|h\|^2 \|s\|^2
\]
\[- h^H (Y - B\Phi^H s - s^H (Y^H - \Phi B^H) h), \]  
(128)

we can rewrite (127) as

\[\hat{\Theta}^{(t+1)} = \arg \min_{\Theta} \left\{ \frac{1}{\nu} \mathbb{E} \left[ \|Y - B\Phi^H s\|^2 + \|h\|^2 \right] \right\}
\[= \arg \min_{\Theta} \left\{ \frac{1}{\nu} \mathbb{E} \left[ (s - \hat{s}^{(t)})^H (Y^H - \Phi B^H) h \right] \right\}
\[+ ML \ln(\pi \nu) \]  
(129)

where, similar to before,

\[\hat{s}^{(t)} \triangleq \mathbb{E}\{s|Y; \hat{\Theta}^{(t)}\}, \]
\[E^{(t)} \triangleq \mathbb{E}\{\|s\|^2|Y; \hat{\Theta}^{(t)}\}. \]  
(130)

We are now ready to minimize (129) over \( \Theta = \{h, B, \Phi, \nu\} \). Zeroing the gradient of the cost over \( h \) yields

\[ \hat{h}^{(t+1)} = \frac{(Y - B\Phi^H \hat{s}^{(t)})}{E^{(t)}}. \]  
(132)

Plugging this back into (129), the term relevant to the optimization of \( B \) and \( \Phi \) becomes

\[\|Y - B\Phi^H\|^2 \]  
\[= \|Y - B\Phi^H \hat{s}^{(t)}\|^2 / E^{(t)} \]
\[= \text{tr} \left\{ (Y - B\Phi^H) \hat{P}_{\pi(s)}^\perp (Y - B\Phi^H)^H \right\}, \]  
(133)

with \( \hat{P}_{\pi(s)}^\perp \) from (43). To optimize (133) over \( B \), we expand

\[ \text{tr} \left\{ (Y - B\Phi^H) \hat{P}_{\pi(s)}^\perp (Y - B\Phi^H)^H \right\} = \text{const} - \text{tr} \left\{ B\Phi^H \hat{P}_{\pi(s)}^\perp Y^H \right\}
\[= \text{tr} \left\{ Y P_{\pi(s)}^\perp \Phi B^H \right\} + \text{tr} \left\{ B\Phi^H P_{\pi(s)}^\perp \Phi B^H \right\}, \]  
(134)

evaluate its gradient, which equals

\[-2 Y \hat{P}_{\pi(s)}^\perp \Phi + 2B\Phi^H \hat{P}_{\pi(s)}^\perp \Phi, \]  
(135)

and set it to zero, yielding

\[\hat{B}^{(t+1)} = Y \hat{P}_{\pi(s)}^\perp \Phi (\Phi^H \hat{P}_{\pi(s)}^\perp \Phi)^{-1}. \]  
(136)

Plugging this back into (133) gives

\[\text{tr} \left\{ (Y - B \hat{B}^{(t+1)} \Phi^H) \hat{P}_{\pi(s)}^\perp (Y - \hat{B}^{(t+1)} \Phi^H)^H \right\} = \text{tr} \left\{ Y (I - \hat{P}_{\pi(s)}^\perp \Phi \Phi^H \hat{P}_{\pi(s)}^\perp \Phi \Phi^H - \hat{B}^{(t+1)} \Phi^H \hat{P}_{\pi(s)}^\perp \Phi \Phi^H) Y^H \right\} \]
\[= \text{tr} \left\{ \hat{Y} (I - \Phi \hat{P}_{\pi(s)}^\perp \Phi \Phi^H \hat{P}_{\pi(s)}^\perp \Phi \Phi^H - \hat{B}^{(t+1)} \Phi^H) \hat{Y}^H \right\} \]
\[= \text{tr} \left\{ \hat{Y} P_{\pi(s)}^\perp \hat{Y}^H \right\} \]
(137)

with \( \hat{Y} \triangleq Y (\hat{P}_{\pi(s)}^\perp \Phi \Phi^H \hat{P}_{\pi(s)}^\perp \Phi \Phi^H) \) and \( \hat{P}_{\pi(s)}^\perp \triangleq (\hat{P}_{\pi(s)}^\perp \Phi \Phi^H \hat{P}_{\pi(s)}^\perp \Phi \Phi^H) \). From (43), note

\[\hat{P}_{\pi(s)}^\perp \]  
(138)

The \( \Phi \) that minimize \( \text{tr} \left\{ \hat{Y} P_{\pi(s)}^\perp \hat{Y}^H \right\} \) are those whose column space equals the span of the \( N \) dominant eigenvectors of \( \hat{Y}^H \hat{Y} \), and so

\[\min_{\Phi} \text{tr} \left\{ \hat{Y} P_{\pi(s)}^\perp \hat{Y}^H \right\} = \sum_{m=N+1}^{M} \lambda_m (\hat{Y}^H \hat{Y}) \]  
(142)

where \( \lambda_m (\hat{Y}^H \hat{Y}) \) is the \( m \)th eigenvalue of \( \hat{Y}^H \hat{Y} \) in decreasing order. These eigenvalues are the same as those of

\[\hat{Y} \hat{Y}^H = \hat{Y} P_{\pi(s)}^\perp \hat{Y}^H. \]  
(143)

Thus, the optimization (129) reduces to

\[\hat{\nu}^{(t+1)} = \arg \min_{\nu} \left\{ \frac{1}{\nu} \sum_{m=N+1}^{M} \lambda_{1,m}^{(t+1)} + ML \ln(\pi \nu) \right\}, \]  
(144)

where \( \{\lambda_{1,m}^{(t+1)}\}_{m=1}^{M} \) are the eigenvalues of \( \frac{1}{\nu} \hat{Y} P_{\pi(s)}^\perp \hat{Y}^H \) in decreasing order. Zerowing the derivative of (144) w.r.t. \( \nu \) yields

\[\hat{\nu}^{(t+1)} = \frac{1}{M} \sum_{m=N+1}^{M} \lambda_{1,m}^{(t+1)}. \]  
(145)

Plugging \( \hat{\nu}^{(t+1)} \) back into the cost expression yields the iteration-(\( t+1 \)) EM-maximized log-likelihood under \( \mathcal{H}_1 \):

\[\ln p(Y|\mathcal{H}_1; \hat{\Theta}^{(t+1)}) = ML(1 + \ln \pi) + ML \ln \left( \frac{1}{M} \sum_{m=N+1}^{M} \lambda_{1,m}^{(t+1)} \right). \]  
(146)
C. EM Update under an Independent Prior

The EM updates \([130]-[131]\) depend on the choice of \(p(s)\). For an independent prior, as in \([3]\), we can compute the MMSE estimate of the \(l\)th symbol using the measurement equation

\[
y_t = \hat{h}^{(t)} s_t^* + \hat{B}^{(t)} \varphi_l^{(t)} + CN(0, \hat{\sigma}_l^{(t)} I),
\]

(147)

where \(\hat{\varphi}_l\) denotes the \(l\)th column of \(\Phi^H\). From \(y_t\), we can obtain the following sufficient statistic \([3]\) for the estimation of \(s_t\) through matched filtering, i.e.,

\[
r_l^{(t)} \triangleq \hat{h}^{(t)H} y_t
\]

(148)

\[
= \|\hat{h}^{(t)}\|^2 s_t^* + \hat{h}^{(t)H} \hat{B}^{(t)} \varphi_l^{(t)} + CN(0, \hat{\sigma}_l^{(t)} \|\hat{h}^{(t)}\|^2).
\]

(149)

We find it more convenient to work with the shifted, conjugated, and normalized statistic

\[
r_l^{(t)} \triangleq \frac{1}{\|\hat{h}^{(t)}\|^2} (r_l^{(t)} - \hat{h}^{(t)H} \hat{B}^{(t)} \varphi_l^{(t)})^*
\]

(150)

\[
= s_t + CN\left(0, \frac{\hat{\sigma}_l^{(t)}}{\|\hat{h}^{(t)}\|^2}\right),
\]

(151)

noting that

\[
r_l^{(t)H} = \frac{1}{\|\hat{h}^{(t)}\|^2} \hat{h}^{(t)H} \left(Y - \hat{\Phi}^{(t)} \hat{\Phi}^{(t)H}\right).
\]

(152)

To efficiently compute (152), we first note that

\[
Y - \hat{\Phi}^{(t)} \hat{\Phi}^{(t)H}
\]

\[
= Y - Y \hat{P}_s^{\perp} \hat{\Phi}^{(t)} \hat{\Phi}^{H} \hat{P}_s^{\perp} \hat{\Phi}^{(t)}\hat{\Phi}^{H}
\]

\[
= Y [I - \hat{P}_s^{\perp} \hat{\Phi}^{H} \hat{P}_s^{\perp} \hat{\Phi}^{(t)}\hat{\Phi}^{H}]
\]

(153)

\[
= Y (\hat{P}_s^{\perp})^* [I - (\hat{P}_s^{\perp})^* \hat{\Phi}^{H} \hat{P}_s^{\perp} \hat{\Phi}^{(t)}\hat{\Phi}^{H}] (\hat{P}_s^{\perp})^{-1/2} - \hat{\Phi}^{H} (\hat{P}_s^{\perp})^{1/2}
\]

(154)

\[
= Y [I - \hat{\Phi}^{H} \hat{P}_s^{\perp} \hat{\Phi}^{(t)}\hat{\Phi}^{H}] (\hat{P}_s^{\perp})^{-1/2}
\]

(155)

\[
= Y \hat{P}_s^{\perp} (\hat{P}_s^{\perp})^{-1/2},
\]

(156)

where we omitted the time index for brevity and defined

\[
\bar{\Phi}^{(t)} \triangleq (\hat{P}_s^{\perp})^{1/2} \hat{\Phi}^{(t)}
\]

(157)

\[
\hat{\Phi}^{(t)} \triangleq Y (\hat{P}_s^{\perp})^{-1/2},
\]

(158)

\[
\hat{\Phi}^{H} \triangleq Y (\hat{P}_s^{\perp})^{-1/2},
\]

(159)

noting that (140), (141) imply

\[
(\hat{P}_s^{\perp})^{-1/2} = I_L + \left(1 - \frac{1}{\zeta}\right) P_s^{(t)}.
\]

(160)

Suppose we take the singular value decomposition (SVD)

\[
\Phi^{(t)} = V^{(t)} D_1^{(t)} U^{(t)H},
\]

(161)

where

\[
\text{diag} (D_1^{(t)}) = \left[\sqrt{\lambda_{1,1}^{(t)}}, \ldots, \sqrt{\lambda_{1,M}^{(t)}}\right]^T
\]

(162)

with \(\lambda_{1,m}^{(t)}\) defined after (144). Then, using the fact that the column space of \(\Phi^{(t)}\) spans the \(N\)-dimensional principal eigenspace of \(\Phi^{(t)H} \Phi^{(t)}\) (as discussed after (139)), we have

\[
Y^{(t)} P_s^{\perp} = Y^{(t)} (I - P_s^{(t)})
\]

(163)

\[
= Y (\hat{P}_s^{(t-1)})^{1/2} - Y^{(t)} D_1^{(t)} U^{(t)H},
\]

(164)

where \(V^{(t)} \in \mathbb{C}^{M \times N}\), \(D_1^{(t)} \in \mathbb{R}^{N \times N}\), and \(U^{(t)} \in \mathbb{C}^{L \times N}\) contain the \(N\) principal components of \(V^{(t)}\), \(D_1^{(t)}\), and \(U^{(t)}\). Plugging this into (157), we get

\[
\hat{\Phi}^{(t)H} = \frac{1}{E(t-1)} \left(Y^{\perp} - \hat{\Phi}^{(t)H} \hat{\Phi}^{(t)}\right) s^{(t-1)}
\]

(165)

\[
= \frac{1}{E(t-1)} \left(Y^{\perp} - \hat{\Phi}^{(t)H} \hat{\Phi}^{(t)}\right) s^{(t-1)}
\]

(166)

Applying (165) to (132) yields

\[
\hat{h}^{(t)} = \frac{1}{E(t-1)} (Y^{\perp} - \hat{\Phi}^{(t)H} \hat{\Phi}^{(t)}) s^{(t-1)}
\]

(167)

and applying (166) to (152) yields

\[
r_l^{(t)H} = \frac{1}{\|\hat{h}^{(t)}\|^2} \left[Y^{(t)} - \hat{\Phi}^{(t)H} \hat{\Phi}^{(t)}\right] (\hat{P}_s^{(t-1)})^{-1/2}.
\]

(168)

We can simplify the previous expression by noting that

\[
\begin{align*}
\left[Y^{(t)} - \hat{\Phi}^{(t)H} \hat{\Phi}^{(t)}\right] (\hat{P}_s^{(t-1)})^{-1/2} \\
&= \left[Y^{(t)} - \hat{\Phi}^{(t)H} \hat{\Phi}^{(t)}\right] (\hat{P}_s^{(t-1)})^{-1/2} I_L + \frac{1 - \zeta(t-1)}{\zeta(t-1)} P_s^{(t-1)}
\end{align*}
\]

(170)

\[
= Y^{(t)} - \hat{\Phi}^{(t)H} \hat{\Phi}^{(t)} + \frac{1 - \zeta(t-1)}{\zeta(t-1)} P_s^{(t-1)}
\]

(171)

\[
\times \left[Y^{(t)} - \hat{\Phi}^{(t)H} \hat{\Phi}^{(t)}\right] s^{(t-1)}
\]

(172)

\[
\times \frac{1 - \zeta(t-1)}{\zeta(t-1)} P_s^{(t-1)}
\]

(173)

\[
= Y^{(t)} - \hat{\Phi}^{(t)H} \hat{\Phi}^{(t)} + \frac{1}{\zeta(t-1)} s^{(t-1)}
\]

(174)

\[
= Y^{(t)} - \hat{\Phi}^{(t)H} \hat{\Phi}^{(t)} + \frac{1}{1 + \zeta(t-1)} \hat{h}^{(t)} s^{(t-1)}
\]

(175)

where (170) used (160); (172) used the fact that \(Y^{(t)} - \hat{\Phi}^{(t)H} \hat{\Phi}^{(t)} = \zeta(t-1) \hat{h}^{(t)} s^{(t-1)}\), as implied by (159); (173) used (168), and (174) used (141). Plugging (174) into (169) then yields

\[
\hat{\Phi}^{(t)} = \frac{1}{\|\hat{h}^{(t)}\|^2} \left[Y^{(t)} - \hat{\Phi}^{(t)H} \hat{\Phi}^{(t)}\right] (\hat{P}_s^{(t-1)})^{-1/2}.
\]

Given \(r_l^{(t)}\), the computation of \(\hat{s}_l^{(t)}\) and \(E^{(t)}\) follows the procedure discussed around (52)-(64). This EM update proce-
the DoF in \( B \) we used GIC with the same approach as described in Sec. III-G. But now the \( \xi \) E. Estimating the Interference Rank

\[
\text{Estimate interference rank } N \text{ (see Sec. IV-E).}
\]

\[
\begin{align*}
\{ \hat{\mathbf{V}}, \hat{\mathbf{D}}, \hat{\mathbf{U}} \} &\leftarrow \text{principal_svd}(\mathbf{Y}, N) \\
\hat{\nu}_1 &\leftarrow \frac{1}{N \mathbb{E}} \| \mathbf{Y} \|_F^2 - \text{tr}(\hat{\mathbf{D}}_1^T \hat{\mathbf{D}}_1) \\
\hat{\mathbf{h}} &\leftarrow \frac{1}{\| \mathbf{s} \|_2^2} \frac{\mathbf{g}}{\xi} \mathbf{V} \hat{\mathbf{D}}_1 \hat{\mathbf{U}}^H \\
\hat{\xi} &\leftarrow \frac{1}{\| \hat{\mathbf{h}} \|_2^2} \\
\hat{r} &\leftarrow \frac{1}{\| \mathbf{s} \|_2^2} \left( \mathbf{Y} \hat{\mathbf{h}} - \mathbf{U} \hat{\mathbf{D}}_1 \mathbf{V}^H \hat{\mathbf{h}} \right) + \frac{1}{1 + \nu} \hat{s}, \text{ where } r \sim \mathcal{CN}(s, \frac{\xi}{\xi} \mathbf{I}) \\
\hat{s}_l &\leftarrow \mathbb{E}(s_l | r_l; \hat{\xi}) \forall l = 1, \ldots, N \\
\hat{E} &\leftarrow \sum_{l=1}^N \mathbb{E}(|s_l|^2 | r_l; \hat{\xi}) \\
\text{until Terminated}
\end{align*}
\]

Algorithm 2: EM update under deterministic interference

\[
\text{Require: Data } \mathbf{Y} \in \mathbb{C}^{M \times L}, \text{ signal prior } p(s) = \prod_{i=1}^L p_i(s_i).
\]

1. Initialize \( \hat{s} \) and \( \hat{E} > 0 \) (see Sec. III-H).
2. repeat
3. \( \xi \leftarrow \sqrt{1 - \| \mathbf{s} \|_2^2} / E \)
4. \( \mathbf{g} \leftarrow \mathbf{Y} \hat{s} / \| \hat{s} \|_2^2 \)
5. \( \mathbf{Y} \leftarrow \mathbf{Y} - (\xi - 1) \mathbf{g} \hat{s}^H \)
6. Estimate interference rank \( N \) (see Sec. IV-E).
7. \( \{ \hat{\mathbf{V}}, \hat{\mathbf{D}}, \hat{\mathbf{U}} \} \leftarrow \text{principal_svd}(\mathbf{Y}, N) \)
8. \( \hat{\nu}_1 \leftarrow \frac{1}{N \mathbb{E}} \| \mathbf{Y} \|_F^2 - \text{tr}(\hat{\mathbf{D}}_1^T \hat{\mathbf{D}}_1) \)
9. \( \hat{\mathbf{h}} \leftarrow \frac{1}{\| \mathbf{s} \|_2^2} \frac{\mathbf{g}}{\xi} \mathbf{V} \hat{\mathbf{D}}_1 \hat{\mathbf{U}}^H \)
10. \( \hat{\xi} \leftarrow \frac{1}{\| \hat{\mathbf{h}} \|_2^2} \)
11. \( \hat{r} \leftarrow \frac{1}{\| \mathbf{s} \|_2^2} \left( \mathbf{Y} \hat{\mathbf{h}} - \mathbf{U} \hat{\mathbf{D}}_1 \mathbf{V}^H \hat{\mathbf{h}} \right) + \frac{1}{1 + \nu} \hat{s}, \text{ where } r \sim \mathcal{CN}(s, \frac{\xi}{\xi} \mathbf{I}) \)
12. \( \hat{s}_l \leftarrow \mathbb{E}(s_l | r_l; \hat{\xi}) \forall l = 1, \ldots, N \)
13. \( \hat{E} \leftarrow \sum_{l=1}^N \mathbb{E}(|s_l|^2 | r_l; \hat{\xi}) \)
14. until Terminated

D. Evaluation of the GLRT

Denoting the final EM estimates by \( \hat{s} \) and \( \hat{\Theta} \), the (EM approximate) GLRT statistic, in the log domain, becomes

\[
\ln p(\mathbf{Y} | \mathcal{H}_1; \hat{\Theta}) - \ln p(\mathbf{Y} | \mathcal{H}_0; \hat{\Theta}) = ML \ln \frac{\hat{\nu}_0}{\hat{\nu}_1},
\]

with \( \nu_0 \) computed from (119) and \( \nu_1 \) computed from Alg. 2.

E. Estimating the Interference Rank

To estimate the interference rank \( N = \text{rank}(\mathbf{R}) \), we adopt the same approach as described in Sec. III-G. But now the DoF \( D(N) \) of the parameters \( \Theta_N \) is different. In particular, the DoF in \( \mathbf{h} \) equals \( 2M \); the DoF in \( \mathbf{B} \Phi^H \), an \( M \times L \) rank-\( N \) complex-valued matrix, equals \( 2(M + L - N)N \); and the DoF in the noise variance \( \nu \) equals 1. In summary, \( D(N) = 2(M + L - N)N + 2M + 1 \). For our numerical experiments, we used GIC with \( G = 1.9 \).

V. NUMERICAL EXPERIMENTS

We now present numerical experiments to evaluate the proposed detectors. Unless otherwise noted, we used \( M = 64 \) array elements, \( L = 1024 \) total symbols, \( Q = 32 \) training symbols, and \( N = 5 \) interferers. (Note that \( Q \ll M \).) The signal \( s \) was i.i.d. QPSK with variance 1, the noise \( \mathbf{W} \) was i.i.d. Gaussian with variance \( \nu \), and the interference \( \Phi \) was i.i.d. Gaussian with variance \( \sigma_\phi^2 / N \), giving a total interference power of \( \sigma_\phi^2 \). For the array, we assumed a uniform planar array (UPA) with half-wavelength element spacing operating in the narrowband regime. For the signal’s array response \( \mathbf{h} \), we assumed that the signal arrived from a random (horizontal,vertical) angle pair drawn uniformly on \([0, 2\pi]^2\). For the \( n \)-th interferer’s array response \( \mathbf{b}_n \), we used the arrival angle corresponding to the \( n \)-th largest sidelobe in \( \mathbf{h} \).

The following detectors were tested. First, we considered several existing methods that used only the training data \( \mathbf{Y}_1 \):

1) the Kang/Monga/Rangaswamy (KMR) approach \( \text{kmr-em} \) with interference rank \( N \) estimated as described in Sec. III-G i.e., “kmr-tr.”
2) McWhorter’s approach \( \text{mcw-tr} \) with interference rank \( N \) estimated as described in Sec. IV-E i.e., “mcw-tr.”
3) Kelly’s full-rank approach \( \text{kel-tr} \), i.e., “kel-tr.”

We also tested the proposed EM-based methods, which use the full data \( \mathbf{Y} \). In particular, we tested
1) Alg. 1 with \( N \) estimated as in Sec. III-G i.e., “kmr-em.”
2) Alg. 2 with \( N \) estimated as in Sec. IV-E i.e., “mcw-em”
3) Alg. 4 with full rank \( N = M \), i.e., “kel-em.”

For the EM algorithm, we used a maximum of 50 iterations but terminated early, at iteration \( i > 1 \), if \( \| s^{(i)} - s^{(i-1)} \| / \| s^{(1)} \| < 0.01 \).

We also tested Forsythe’s iterative method \( \text{I} \) p. 110] by running Alg. 1 with full rank \( N = M \) and hard symbol estimates in lines 18-19 as discussed in Sec. III-G. In addition, we tested a low-rank version of Forsythe’s method by running Alg. 1 with hard estimates and \( N \) estimated as in Sec. III-G.

Finally, we tested Alg. 2 with hard estimates and \( N \) estimated as in Sec. IV-E (denoted by “hard-mcw-em”).

For all methods, detection performance was quantified using the rate of correct detection when the detector threshold \( \eta \) is set to achieve a false-alarm rate of \( 10^{-4} \). All simulation results represent the average of 10,000 independent draws of \( \{ \mathbf{h}, s, B, \Phi, \mathbf{W} \} \).

A. Performance versus SINR

Figure 1 shows detection-rate at false-alarm-rate=10^{-4} versus \( \nu = \sigma_\phi^2 / \nu \) for various detectors. There we see that the proposed EM-based, full-data detectors \( \text{kel-em}, \text{kmr-em}, \text{mcw-em} \) significantly outperformed their training-based counterparts \( \text{kel-tr}, \text{kmr-tr}, \text{mcw-tr} \).

Figure 2 shows the performance of Forsythe’s full-rank iterative method, its low-rank counterpart (i.e., Alg. 1 with hard symbol estimates), and Alg. 2 with hard symbol estimates, under the same data used to create Fig. 1. Comparing the
two figures, we see that the “soft” low-rank methods, kmr-em and mcw-em, outperformed their hard counterparts, forsythe-lowrank and hard-mcw-em. We attribute this behavior to error propagation in the hard detector. Also, we see that the low-rank version of Forsythe’s iterative method significantly outperformed the full-rank version, which is expected since the interference is truly of low rank.

B. Performance versus SIR at fixed SNR

Figure 3 shows detection-rate at false-alarm-rate=$10^{-4}$ versus interference power $\frac{\nu}{Q} = \sigma_i^2/Q$ for various detectors, under $\nu = Q$, $M = 64$, $Q = 32$, $L = 1024$, $N = 5$, i.i.d. QPSK symbols, and UPA geometry. The proposed EM-based, low-rank methods kmr-em and mcw-em gave zero errors over 10 000 realizations.

C. Performance versus training length $Q$

Figure 4 shows detection-rate at false-alarm-rate=$10^{-4}$ versus training length $Q$ for various detectors, under $\nu = Q$, $M = 64$, $Q = 32$, $L = 1024$, $N = 5$, i.i.d. QPSK symbols, and UPA geometry. The proposed EM-based, low-rank methods kmr-em and mcw-em outperform the others for $Q \in [16, 512]$.

D. Performance versus interference rank $N$

Figure 5 shows detection-rate at false-alarm-rate=$10^{-4}$ versus the number of interferers, $N$, for various detectors under $\nu = Q$ and $\sigma_i^2 = QN$. Note that the per-interferer power was fixed at $Q$. Note also that the proposed EM-based, low-rank detectors gave no errors over 10 000 trials. For the other schemes, the error-rate increased with $N$, as expected.

Figure 6 shows the average estimated interference rank $\tilde{N}$ versus the true rank $N$ under $H_1$, using the same data used to construct Fig. 5. There we see that all methods were successful, on average, at correctly estimating the interference rank.

VI. Conclusions

In this paper, we considered the problem of detecting the presence/absence of a structured (i.e., partially known) signal from the space-time outputs of an array. This problem arises when detecting communication signals, where often a few
it would be interesting to consider the detection of multiple signals, as in [2].

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