A SIMPLE CONSTRUCTION OF DERIVED REPRESENTATION SCHEMES

YURI BEREST, GEORGE KHACHATRYAN, AND AJAY RAMADOSS

Abstract. We present a simple algebraic construction of the (non-abelian) derived functors $\mathrm{DRep}^*\!\!\!\!\!
\bullet_n(A)$ of the representation scheme $\mathrm{Rep}_n(A)$, parametrizing the $n$-dimensional representations of
an associative algebra $A$. We construct a related derived version of the representation functor introduced recently
by M. Van den Bergh [vdB] and, as an application, compute the derived tangent spaces $\mathcal{T} DRep^*\!\!\!\!\!
\bullet_n(A)$ to $\mathrm{Rep}_n(A)$. We prove that our construction of $DRep^*\!\!\!\!\!
\bullet_n(A)$ agrees with an earlier construction of derived action spaces, due to I. Ciocan-Fontanine and M. Kapranov [CK];
however, our approach, proofs and motivation are quite different. This paper is mainly a research announcement;
detailed proofs and applications will appear elsewhere.

1. Introduction

Let $A$ be an associative unital algebra over a field $k$. The classical representation scheme, parametrizing
the $n$-dimensional representations of $A$, can be defined as the functor on the category of commutative algebras

$$\mathrm{Rep}_n(A) : \mathrm{Comm Alg}_k \to \mathbf{Sets}, \quad B \mapsto \mathrm{Hom}_\mathrm{Alg}(A, M(n, B)), \quad (1)$$

where $M(n, B)$ denotes the ring of $n \times n$ matrices over the commutative algebra $B$. A natural way to
prove representability of this functor goes back to the work of Bergman [B] and Cohn [C]: the idea is to
extend (1) from $\mathrm{Comm Alg}_k$ to the category of all associative $k$-algebras:

$$\overline{\mathrm{Rep}}_n(A) : \mathrm{Alg}_k \to \mathbf{Sets}. \quad (2)$$

The functor $\overline{\mathrm{Rep}}_n(A)$ is defined by the same formula as $\mathrm{Rep}_n(A)$ in (1), but with $B$ being an associative
algebra. It turns out that $\overline{\mathrm{Rep}}_n(A)$ is representable, and quite remarkably, its representing object $\sqrt[n]{A}$ has a simple and explicit algebraic construction

$$\sqrt[n]{A} = (A \ast_k M(n))^M(n), \quad (2)$$

where $A \ast_k M(n)$ is the free product of $A$ and $M(n) := M(n, k)$ as $k$-algebras (i.e., the coproduct in
the category $\mathrm{Alg}_k$) and $(-)^M(n)$ denotes the centralizer of $M(n)$ in $A \ast_k M(n)$. The algebra $\sqrt[n]{A}$ is thus the
universal coefficient ring for the $n$-dimensional associative representations of $A$; it can be thought of as
the coordinate ring of a noncommutative affine scheme in the sense of [C].

Now, the inclusion functor $\mathrm{Comm Alg}_k \hookrightarrow \mathrm{Alg}_k$ has an obvious left adjoint, which is abelianization:

$$A \mapsto A_1 := A/A[A, A]A,$$ so representability of (1) follows immediately from (2): the commutative algebra

$A_n := \mathbb{k}[\mathrm{Rep}_n(A)]$ representing $\overline{\mathrm{Rep}}_n(A)$ is given by the formula

$$A_n = (\sqrt[n]{A})_1. \quad (3)$$

The aim of this paper is to construct the (non-abelian) derived functors of the functor $A \mapsto \overline{\mathrm{Rep}}_n(A)$
in the sense of Quillen [Q]. To do this, we first extend (1) to the category $\mathrm{DGA}_k$ of differential graded (DG) $k$-algebras, defining $\overline{\mathrm{Rep}}_n(R)$ for a fixed $R \in \mathrm{DGA}_k$ by

$$\overline{\mathrm{Rep}}_n(R) : \mathrm{Comm DGA}_k \to \mathbf{Sets}, \quad B \mapsto \mathrm{Hom}_\mathrm{DGA}(R, B \otimes_k M(n)), \quad (4)$$

This construction has been recently used in [LBW] (see also [L]), from which we borrow the notation $\sqrt[n]{A}$.\footnote{This construction has been recently used in [LBW] (see also [L]), from which we borrow the notation $\sqrt[n]{A}$.}
where \( \text{Comm} \text{DG}_{A_k} \) is the category of commutative differential graded algebras. To represent \( \text{Rep}_n(R) \) we then proceed as in the case of usual algebras: first, we prove representability in the category of all DG algebras, and then by abelianizing, we get representability of \( \text{Rep}_n \) in the category of commutative DG algebras. It turns out that again the natural differential graded analogues of algebras \([2]\) and \([3]\), to wit \( \sqrt{R} := (R *_k M(n))^{[M(n)]} \) and \( R_n := (\sqrt{R})_n \), represent the corresponding functors explicitly.

Given an associative algebra \( A \in \text{Alg}_k \), we now take its almost free DG resolution \( R \to A \) and define
\[
A^*_n := H^\bullet(R_n).
\]

We prove that the assignment \( A \to A^*_n \) is independent of the choice of resolution, and in fact, defines a functor \( D\text{Rep}_n^* : \text{Alg}_k \to \text{GrComm} \text{Alg}_k \) with values in the category of graded commutative algebras, so that \( D\text{Rep}_n^0(A) \cong \text{Rep}_n(A) \) (see Section 2.1 below).

The idea of deriving representation schemes is certainly not new: there are several different (and some, in fact, more general) approaches in the literature: see, e.g., [BCHR], [Tv], and especially [CK]. Our construction is motivated by recent developments in noncommutative geometry (see [KR], [G]), [CEG], [vdB], [vdB1], [BC], which are based on Kontsevich’s idea [K] that the family of representation schemes \( \{ \text{Rep}_n(A) \} \) should be viewed as a ‘good approximation’ of the noncommutative ‘Spec(\( A \))’. This idea turns out to be very fruitful in practice, since it allows one to find correct definitions for various structures on a noncommutative algebra \( A \) by requiring that these structures induce standard geometric structures on all representation spaces \( \text{Rep}_n(A) \). Van den Bergh [vdB] has recently proposed a concrete realization of this principle, by introducing a natural functor \( (-)_n : \text{Bimod}(A) \to \text{Mod}(A, n) \), which transforms noncommutative objects on \( A \) (viewed as bimodules over \( A \)) to the corresponding classical objects on \( \text{Rep}_n(A) \). We will construct a functor \( D(-)_n : D(\text{Bimod}) \to D(\text{Mod}) \) from the derived category of bimodules over \( A \) to the derived category of DG modules over \( R_n \), where \( R \to A \) is an almost free resolution of \( A \). This functor is (essentially) independent of the choice of resolution and should be viewed as a derived functor of \( (-)_n \) in the sense of differential homological algebra (see Section 2.2).

When combined with cohomology, it yields a functor \( (-)_n^* : H^\bullet(D(-)_n : D(\text{Bimod}) \to \text{GrMod}(A^*_n) \), which transforms bimodules over \( A \) to graded modules over \( A^*_n \). Since passing from \( \text{Rep}_n(A) \) to the DG scheme \( \text{Rep}_n(R) \) amounts (in a sense) to a desingularization of \( \text{Rep}_n(A) \), one should expect that \( D(-)_n \) will play a role in geometry of arbitrary (in particular, homologically smooth) algebras similar to the role of Van den Bergh’s functor in geometry of formally smooth algebras. As a simple illustration of this idea, we compute the derived tangent spaces \( T^\bullet_e D\text{Rep}_n(A) \) of \( D\text{Rep}_n^*(A) \) at \( \varrho \in \text{Rep}_n(A) \), which turn out to be isomorphic to Hochschild cohomology of the representation \( \varrho : A \to M(n) \) (see Section 2.3):

\[
T^\bullet_e D\text{Rep}_n(A) \cong T^\bullet_e \text{Rep}_n(A) \cong \text{Der}_k(A, M(n)) \quad \text{and} \quad T^\bullet_e D\text{Rep}_n(A) \cong H^{i+1}(A, M(n)), \ \forall i \geq 1.
\]

Our construction of DRep works naturally in greater generality, when instead of a single \( k \)-vector space, we take a complex \( V \) of finite total dimension. In the special case, when \( V \) is concentrated in degree 0, our results agree with the earlier results of Ciocan-Fontanine and Kapranov [CK]: specifically, if \( R \) is a \( \mathbb{Z} \)-graded DG algebra, the affine DG action scheme \( \text{RAct}(R, V) \) introduced in [CK], Section 3.3, is naturally isomorphic to our DG representation scheme \( \text{Rep}_{\varrho}(R) \). We prove this by verifying that \( \text{RAct}(R, V) \) and \( \text{Rep}_{\varrho}(R) \) satisfy the same universal property, although the a priori constructions of these schemes seem very different. The derived tangent spaces of \( \text{RAct}(R, V) \) have been also computed in [CK], and the result (see loc. cit., Proposition 3.5.4) agrees with [3]. Our method of computing \( T^\bullet_e D\text{Rep}_{\varrho}(A) \) using Van den Bergh’s functor is different from [CK] and apparently quite a bit simpler (cf. Section 2.3 below).

The main advantage of our approach is the explicitness of algebraic constructions, which should allow concrete computations. We will give several examples in the end of the paper (see Section 3): these examples are chosen more or less at random, with a sole goal to illustrate the theory.

**Acknowledgements.** The first author is very grateful to Andrei Okounkov who raised interesting questions related to [BC]; attempts to clarify his questions have been a motivation behind this work. We are also very grateful to Martin Kassabov for several insightful suggestions and to Frank Moore, whose computer DG algebra package for Macaulay2 we have been using extensively. We also thank Frank for his assistance with computations presented in Section 3. The first author was partially supported by the NSF grant DMS 09-01570. The second author acknowledges the support by a NSF
Research Fellowship, and third author is currently funded by the Swiss National Science Foundation (Ambizione Beitrag Nr. PZ00P2-127427/1).

2. Main Results

2.1. Let \( V = [\ldots \to V^i \to V^{i+1} \to \ldots] \) be a complex of \( k \)-vector spaces, with \( \sum \dim_k V^i < \infty \). The graded endomorphism ring \( \text{End}_V \) is then naturally a DG algebra. Using \( \text{End}_V \), we define the functor

\[
\sqrt{} : \text{DGA}_k \to \text{DGA}_k, \quad R \mapsto \sqrt{R} := (R \ast_k \text{End}_V)\text{End}_V,
\]

where \( R \ast_k \text{End}_V \) is the coproduct in the category \( \text{DGA}_k \) and

\[
(R \ast_k \text{End}_V)\text{End}_V := \{ w \in R \ast_k \text{End}_V : [w, m] = 0, \forall m \in \text{End}_V \},
\]

with commutators being taken in the graded sense. For any \( R \in \text{DGA}_k \), we denote by \( R_c := R/R[R,R] \) the abelianization of \( R \), and set

\[
R_V := (\sqrt{R})_2.
\]

The following lemma is a generalization of a classic result of Cohn (see [C], Sect. 6, formula (2)).

Lemma 1. For any \( R, S \in \text{DGA}_k \) and \( B \in \text{Comm DGA} \), there are canonical isomorphisms

(a) \( \text{Hom}_{\text{DGA}}(\sqrt{R}, S) \cong \text{Hom}_{\text{DGA}}(R, S \otimes \text{End}_V) \),

(b) \( \text{Hom}_{\text{Comm DGA}}(R_V, B) \cong \text{Hom}_{\text{DGA}}(R, B \otimes \text{End}_V) \).

The proof of Lemma 1(a) given in [C] for ordinary algebras extends (with some more or less trivial modifications) to all DG algebras, and part (b) is immediate from part (a). It follows from (b) that the commutative DG algebra \( R_V \) represents the functor

\[
\text{Rep}_V(R) : \text{Comm DGA}_k \to \text{Sets}, \quad B \mapsto \text{Hom}_{\text{DGA}}(R, B \otimes \text{End}_V),
\]

and thus should be thought of as the coordinate ring \( k[\text{Rep}_V(R)] \) of an affine DG scheme \( \text{Rep}_V(R) \) (cf. [CK], Sect. 2.2).

Now, recall that every algebra \( A \in \text{Alg}_k \) has an almost free resolution in the category \( \text{DGA}_k \), which is given by a quasi-isomorphism \( R \to A \). Here, \( R \) is a DG algebra, whose underlying graded algebra \( |R| = \oplus_{i \in \mathbb{Z}} R^i \) is free, with \( R^i = 0 \) for all \( i > 0 \). Given two almost free resolutions \( R_1 \to A \) and \( R_2 \to B \) and an algebra map \( f : A \to B \), there is a homomorphism \( \tilde{f} : R_1 \to R_2 \) in \( \text{DGA}_k \), such that \( H^0(\tilde{f}) = f \); moreover, \( \tilde{f} \) is unique up to (multiplicative) homotopy (see [Q], Ch. I, or [CK], Sect. 3.6). Using these (and other) standard results of homotopical algebra, one can verify that the functor \( R \mapsto R_V \) defined by (\ref{eq:rep_v}) preserves quasi-isomorphisms. As often happens with non-abelian derived functors, this verification requires some technical preparations and is not immediate. The consequence is the following theorem, which is our first main result.

Theorem 1. Let \( A \) be an associative unital \( k \)-algebra.

(a) For any almost free resolutions \( R_1 \) and \( R_2 \) of \( A \), there is a quasi-isomorphism \( f : (R_1)_V \to (R_2)_V \).

(b) The assignment \( A \mapsto A^*_V := H^*(R_V) \) defines a functor \( \text{Alg}_k \to \text{GrComm Alg}_k \), which is independent of the choice of almost free resolution \( R \to A \).

(c) If \( V \) is concentrated in degree 0, then \( A^*_V \cong A_V \).

Since \( \text{Spec}(A^*_V) = \text{Rep}_V(A) \) when \( V \) is concentrated in degree 0, Theorem 1(c) implies that the graded scheme \( \text{Spec}(A^*_V) \) should be viewed as a derived representation scheme of \( A \). To simplify the notation, we set \( \text{DRep}_V(A) := \text{Spec}(A^*_V) \) and write \( \text{DRep}_V^*(A) \) instead of \( \text{DRep}_V(A) \) when \( V = k^n \).

2.2. From now on, we assume that \( V \) is concentrated in degree 0. For an algebra \( A \in \text{Alg}_k \), we write \( A_V := k[\text{Rep}_V(A)] \) and let \( \pi_V : A \to A_V \otimes \text{End}_V \) denote the universal representation. Restricting scalars via \( \pi_V \), we can regard \( A_V \otimes \text{End}_V \) as a bimodule over \( A \), or equivalently, as a left module over the enveloping algebra \( A^e := A \otimes A^{\text{opp}} \). Since \( A_V \) is commutative, the image of \( A_V \) under the natural
inclusion $A_V \hookrightarrow A_V \otimes \text{End} V$ lies in the center of this bimodule. Hence, we can regard $A_V \otimes \text{End} V$ as $A^e$-$A_V$-bimodule. Now, following Van den Bergh (see [vdB], Sect. 3.3), we define the additive functor

$$(7) \quad (-)_V : \text{Bimod}(A) \to \text{Mod}(A_V), \quad M \mapsto M \otimes_{A^e} (A_V \otimes \text{End} V).$$

As mentioned in the Introduction, this functor plays a key role in noncommutative geometry of smooth algebras, transforming noncommutative objects on $A$ to the classical geometric objects on $\text{Rep}_V(A)$. Our aim is to construct the higher derived functors of (7), which should replace (7) when $A$ is not smooth. We begin by extending the Van den Bergh functor to the world of DGAs.

Fix $R \in \text{DGA}_k$ and let $\pi : R \to \sqrt{R} \otimes \text{End} V$ denote the universal DG algebra homomorphism corresponding to the identity functor on $\text{Bimod}(V)$ (see Lemma 1(a)). The complex $\sqrt{R} \otimes V$ is naturally a left DG module over $\sqrt{R} \otimes \text{End} V$ and right DG module over $\sqrt{R}$, so restricting the left action via $\pi$ we can regard $\sqrt{R} \otimes V$ as DG bimodule over $R$ and $\sqrt{R}$. Similarly, we can make $V^* \otimes \sqrt{R}$ a $\sqrt{R}$-$R$-bimodule. Using these bimodules, we define the functor

$$(8) \quad \sqrt{-} : \text{DG Bimod}(R) \to \text{DG Bimod}(\sqrt{R}), \quad M \mapsto (V^* \otimes \sqrt{R}) \otimes_R M \otimes_R (\sqrt{R} \otimes V).$$

Now, recall that $R_V := (\sqrt{R})_2$ is a commutative DGA. Using the natural projection $\sqrt{R} \to R_V$, we regard $R_V$ as a DG bimodule over $\sqrt{R}$ and define

$$(9) \quad (-)_2 : \text{DG Bimod}(\sqrt{R}) \to \text{DG Mod}(R_V), \quad M \mapsto M_2 := M \otimes_{\sqrt{R}} R_V,$$

which is nothing but the abelianization functor on bimodules. Combining (8) and (9), we define

$$(10) \quad (-)_V : \text{DG Bimod}(R) \to \text{DG Mod}(R_V), \quad M \mapsto M_V := (\sqrt{M})_2.$$

As suggested by its notation, the functor (10) is a DG extension of (7). In fact, if $R = A$ is a DG algebra with a single nonzero component in degree 0, the category $\text{Bimod}(A)$ can be viewed as a full subcategory of $\text{DG Bimod}(R)$ consisting of bimodules concentrated in degree 0. It is easy to check then that the restriction of (10) to this subcategory coincides with (7).

The next lemma is analogous to Lemma 1 for DG algebras; it holds, however, in greater generality: for homomorphism complexes $\text{Hom}^*$ of DG modules. We recall that, if $R$ is a DG algebra and $M, N$ are DG modules over $R$, $\text{Hom}^*_R(M, N)$ is a complex of vector spaces with $n$-th graded component consisting of all $R$-linear maps $f : M \to N$ of degree $n$ and the $n$-th differential given by $d(f) = d_N \circ f - (-1)^n f \circ d_M$.

**Lemma 2.** There are canonical isomorphisms of complexes

(a) $\text{Hom}^*_R(\sqrt{M}, N) \cong \text{Hom}^*_R(M, N \otimes \text{End} V),$

(b) $\text{Hom}^*_R(M_V, N) \cong \text{Hom}^*_R(M, N \otimes \text{End} V).$

**Example 1.** Let $R$ be a DG algebra. Denote by $\Omega^1 R$ the kernel of the multiplication map $R \otimes R \to R$. This is naturally a DG bimodule over $R$, which, as in the case of ordinary algebras, represents the complex of $k$-linear graded derivations $\text{Der}^*_R(R, M)$ (see, e.g., [Q1], Sect. 3.1 and 3.2). Thus, for any $M \in \text{DG Bimod}(R)$, there is a canonical isomorphism of complexes of vector spaces

$$(11) \quad \text{Der}^*_R(R, M) \cong \text{Hom}^*_R(\Omega^1 R, M).$$

Using (11) and Lemma 2 one can establish canonical isomorphisms

$$(12) \quad \sqrt{\Omega^1 R} \cong \Omega^1(\sqrt{R}), \quad (\Omega^1 R)_V \cong \Omega^1(R_V).$$

This should be compared to [vdB], Proposition 3.3.4.

Next, we recall that if $R$ is a DG algebra, every DG module $M$ over $R$ has a semi-free resolution $F \to M$, which is similar to a free (or projective) resolution for ordinary modules over ordinary algebras (see [FHT], Sect. 2). To construct the derived functors of (7) we now follow the standard procedure in differential homological algebra.

Given an algebra $A \in \text{Alg}_k$ and a complex $M$ of bimodules over $A$, we first pick an almost free resolution $f : R \to A$ in $\text{DGA}_k$ and consider $M$ as a DG bimodule over $R$ via $f$. Then, we pick a
semi-free resolution \( F(R, M) \to M \) in the category \( \text{DG Bimod}(R) \) and apply to \( F(R, M) \) the functor \( [10] \).

The result is described by the following theorem, which is the second main result of this paper.

**Theorem 2.** Let \( A \) be an associative \( k \)-algebra, and let \( M \) be a complex of bimodules over \( A \).

(a) The assignment \( M \mapsto F(R, M)_V \) induces a well-defined functor between the derived categories

\[
D(\cdot)_V : \text{D(Bimod)} A \to \text{D(DG Mod} R_V) ,
\]

which is independent of the choice of the resolutions \( R \to A \) and \( F \to M \) up to auto-equivalence of \( \text{D(DG Mod} R_V) \) inducing the identity on cohomology.

(b) Taking cohomology \( M \mapsto \text{H}^* \text{D}(M)_V \) yields a functor

\[
\text{(-)}^*_V : \text{D(Bimod)} A \to \text{GrMod}(A^*_V) ,
\]

which depends only on the algebra \( A \) and the vector space \( V \).

(c) If \( M \in \text{Bimod}(A) \) is viewed as a 0-complex in \( \text{D(Bimod)} A \), then \( M^0_\text{loc. cit.} \cong M_V \).

The proof of part (a) is standard differential homological algebra; the fact that \( D(\cdot)_V \) is independent of resolutions follows from results of Keller (see, e.g., [Ke1], Sect. 8.4). Part (b) is immediate from (a), and (c) is proved by direct computation.

2.3. We will use the above construction to compute the derived tangent spaces \( T^*_\text{DRep}_V(A) \) at \( a \in \text{Rep}_V(A) \). It is instructive to compare our construction with the ones in [vdB], see loc. cit., Section 3.3.

Recall that if \( X \) is an ordinary affine \( k \)-scheme and \( x \in X \) is a \( k \)-point, the tangent space \( T_x X \) to \( X \) at \( x \) is defined by \( T_x X := \text{Der}_k(k[X], k_x) \). Similarly (cf. [CK], (2.5.6)), if \( X \) is a DG scheme, and \( x \in X^0 \) is a \( k \)-point, the tangent DG space to \( X \) at \( x \) is defined by

\[
T_x X := \text{Der}_k(k[X], k_x) .
\]

Now, if \( X^\bullet := \text{Spec} \text{H}^*(k[X]) \) is the underlying derived scheme of \( X \), the derived tangent space \( T^*_X \) to \( X^\bullet \) is given, by definition, by cohomology:

\[
T^*_X := \text{H}^*_{\text{Der}_k(k[X], k_x)} .
\]

Let \( \rho : R_V \to k \) be the DG algebra homomorphism corresponding to a \( k \)-point in \( \text{DRep}_V^*(A) \), and let \( \varrho : R \to \text{End}(V) \) be the representation corresponding to \( \rho \). Then, we have canonical isomorphisms of complexes of vector spaces

\[
\text{Der}_k(R_V, k) \cong \text{Hom}_{R_V}^* (\Omega^1(R_V), k)
\]

\[
\cong \text{Hom}_{R_V}^* ((\Omega^1 R)_V, k) \quad [\text{see [12]}]
\]

\[
\cong \text{Hom}_{R_V}^* (\sqrt[\wedge \omega] {\Omega^1(R)} \otimes (\sqrt[\wedge \omega])^* R_V, k)
\]

\[
\cong \text{Hom}_{R_V}^* (\sqrt[\wedge \omega] {\Omega^1(R)}, \text{Hom}_{R_V}^* (R_V, k))
\]

\[
\cong \text{Hom}_{R_V}^* (\sqrt[\wedge \omega] {\Omega^1(R)}, k)
\]

\[
\cong \text{Hom}_{\text{End}(V)} (\Omega^1 R, \text{End} V) \quad [\text{see Lemma 2(a)}]
\]

\[
\cong \text{Der}^*_k (R, \text{End} V) ,
\]

which imply

\[
T^*_\text{DRep}_V(A) := \text{H}^* [\text{Der}_k(R_V, k)] \cong \text{H}^* [\text{Der}_k (R, \text{End} V)] .
\]

The following proposition is now an immediate consequence of [BP], Lemma 4.2.1 and Lemma 4.3.2.

**Proposition 1** (cf. [CK]).

\[
T^*_\text{DRep}_V(A) \cong \begin{cases} 
\text{Der}_k(A, \text{End} V) & \text{if } i = 0 \\
\text{H}^{i+1}(A, \text{End} V) & \text{if } i \geq 1
\end{cases}
\]
Remark. As mentioned in the Introduction, in case when $V$ is a single vector space and $R \in \mathsf{DGA}_k$ is almost free, one can show that $\text{Rep}_V(R)$ is isomorphic the DG scheme $\text{RAct}(R, V)$ constructed in [CK]. This implies that $T_i^* \text{DRep}_V(A)$ should be isomorphic to $T_i^* \text{RAct}(R, V)$, which is indeed the case, as one can easily see by comparing our Proposition 1 to [CK], Proposition 3.5.4(b).

3. Examples

In this section, we assume that $V = k^n$. Given an explicit almost free resolution $R \to A$, the DG algebra $R_V = (\sqrt[\mathsf{D}]{R})_\natural$ can be described explicitly. Specifically, let $\{r_i\}_{i \in I}$ be a set of generators of $R$, and let $d_R$ be its differential. Consider a free graded algebra $\tilde{R}$ with generators $\{r_{i,j,k}^{s_k}\}_{i \in I, 1 \leq j, k \leq n}$ of degree $|r_{i,j,k}^{s_k}| = |r_i|$. Define linear functions $\{f_{j,k}\}_{1 \leq j, k \leq n}$ on (tensor) products of generators of $R$ by

$$f_{j,k} : R \to \tilde{R}, \quad r_{t_1} r_{t_2} \ldots r_{t_m} \mapsto \sum r_{t_1}^{s_1} r_{t_2}^{s_2} \ldots r_{t_m}^{s_{m-1}},$$

where the sum is taken over all $1 \leq s_1, \ldots, s_{m-1} \leq n$, and extend this by linearity to the whole of $R$. Now, define a differential $d$ on generators of $\tilde{R}$ by

$$d(r_{i,j,k}^{s_k}) := f_{j,k}(d_R(r_i)),$$

and extend it the Leibniz rule to whole of $\tilde{R}$. This makes $\tilde{R}$ a DG algebra. The abelianization of $\tilde{R}$ is then a free (graded) commutative algebra with generators $r_{i,j,k}^{s_k}$ and the differential induced by the differential of $\tilde{R}$. Then, we have the following result.

**Theorem 3.** There is an isomorphism of DG algebras $\tilde{R} \cong \sqrt[\mathsf{D}]{R}$. Consequently, $R_V \cong \tilde{R}_\natural$.

Using Theorem 3, we can construct explicitly a finite presentation of the graded algebra $A^*_V$, whenever we have a finite almost free resolution of $A$. In practice, for many interesting algebras, such resolutions are available. For example, the DG algebras introduced recently in [GL] and [Ke] provide very interesting (in a sense, canonical) finite resolutions for all 3D Calabi-Yau algebras, including $k[x, y, z]$, $U(\mathfrak{sl}_2)$, Sklyanin algebras, and many others. This allows us to describe the corresponding derived representation schemes quite explicitly, which is rather unusual for derived functors in homotopical algebra.

**Example 2.** Let $U(\mathfrak{sl}_2)$ be the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2(k)$. By [GL], Example 1.3.6, it has a finite DG resolution $R \to U(\mathfrak{sl}_2)$, where $R$ is the free graded algebra

$$R = k\langle x, y, z; X, Y, Z; t \rangle,$$

with generators $x, y, z$ having degree 0; $X, Y, Z$ having degree $-1$ and $t$ having degree $-2$. The differential on $R$ is defined by

$$dX = yz - zy + x, \quad dY = zx - xz + y, \quad dZ = xy - yx + z, \quad dt = [x, X] + [y, Y] + [z, Z].$$

Theorem 3 then implies that

$$R_n = k[x_{ij}, y_{ij}, z_{ij}; X_{ij}, Y_{ij}, Z_{ij}; t_{ij} | 1 \leq i, j \leq n],$$
where the generators $x_{ij}, y_{ij}, z_{ij}$ have degree zero, $X_{ij}, Y_{ij}, Z_{ij}$ have degree $-1$, and $|t_{ij}| = -2$. The differential on $R_n$ is given by

$$
dx_{ij} = dy_{ij} = dz_{ij} = 0,$$

$$dX_{ij} = \sum_{k=1}^{n} (y_{ik}z_{kj} - z_{ik}y_{kj}) + x_{ij},$$

$$dY_{ij} = \sum_{k=1}^{n} (z_{ik}x_{kj} - x_{ik}z_{kj}) + y_{ij},$$

$$dZ_{ij} = \sum_{k=1}^{n} (x_{ik}y_{kj} - y_{ik}x_{kj}) + z_{ij},$$

$$dt_{ij} = \sum_{k=1}^{n} x_{ik}X_{kj} - X_{ik}x_{kj} + y_{ik}Y_{kj} - Y_{ik}y_{kj} + z_{ik}Z_{kj} - Z_{ik}z_{kj}.$$

We now give a simple example when one can actually compute the cohomology of $R_n$.

**Example 3.** Let $A = k[x, y]$ be the (commutative) polynomial algebra in two variables. One can easily check that $A$ has resolution $R := k(x, y; t)$, with generators $x, y$ in degree $0$, $t$ in degree $-1$, and $dt = xy - yx$. In this case, using Gröbner basis techniques, we can compute

$$k[x, y]_2^\bullet = k[x, y]_2(r, s) = \frac{k[x, y]_2}{\langle x_{21}r - y_{21}s, x_{12}r - y_{12}s, (x_{11} - x_{22})r - (y_{11} - y_{22})s, sr, rs, r^2, s^2 \rangle},$$

where the two extra generators $r$ and $s$ have degree $-1$.

Similar presentations can be constructed for other polynomial algebras, using a computer algebra software. For example, $k[x, y, z]_2^\bullet$ has 6 generators in nonzero degrees, all of them in degree $-1$; it has nonzero components only in degrees 0, $-1$, $-2$, and $-3$. The algebra $k[x, y, z]_3^\bullet$ has 16 generators in degree $-1$, and others in lower degrees; regarded as a $k[x, y, z]_2$-module, its minimal generating set has 16 elements in degree $-1$; 56 in degree $-2$; 128 in degree $-3$; 233 in degree $-4$, and more in lower degrees.

**References**

[BP] H.-J. Baues, and T. Pirashvili, *Comparison of MacLane, Shukla and Hochschild cohomology*, J. reine angew. Math. 598 (2006), 25–69.

[BCHR] K. Behrend, I. Ciocan-Fontanine, J. Hwang, and M. Rose, *The derived moduli space of stable sheaves*, preprint, arXiv:1004.1884.

[B] G. M. Bergman, *Coproducts and some universal ring constructions*, Trans. Amer. Math. Soc. 200 (1974), 33–88.

[Be] Yu. Berest, *Calogero-Moser spaces over algebraic curves*, Selecta Math. 14 (2009), 373–396.

[BC] Yu. Berest and O. Chalykh, *$A_\infty$-modules and Calogero-Moser spaces*. J. reine angew. Math. 607 (2007), 69–112.

[CK] I. Ciocan-Fontanine and M. Kapranov, *Derived Quot schemes*, Ann. Sci. ENS 34 (2001), 403–440.

[C] P. M. Cohn, *The affine scheme of a general ring*, Lecture Notes in Math. 753, Springer, Berlin, 1979, pp. 197–211.

[CEG] W. Crawley-Boevey, P. Etingof and V. Ginzburg, *Noncommutative geometry and quiver algebras*, Adv. Math. 209 (2007), 274–336.

[FHT] Y. Felix, S. Halperin, and J.-C. Thomas, *Differential graded algebras in topology*, in *Handbook of Algebraic Topology*, Elsevier, 1995, pp. 829–865.

[G] V. Ginzburg, *Lectures on Noncommutative Geometry*, preprint, arXiv:math.AG/0506603.

[G1] V. Ginzburg, *Calabi-Yau algebras*, preprint, arXiv:math.AG/0612139.

[Ke] B. Keller, *Deformed Calabi-Yau completions*, preprint, arXiv:0908.3499.

[Ke1] B. Keller, * Derived categories and tilting*, in *Handbook of Tilting Theory*, Cambridge University Press, 2007, pp. 49–104.

[K] M. Kontsevich, *Non-commutative smooth spaces*, talk at the Arbeitstagung, MPI, Bonn, June 1999.

[KR] M. Kontsevich and A. Rosenberg, *Noncommutative smooth spaces*, The Gelfand Mathematical Seminars 1996-1999, Birkhäuser, Boston, 2000, pp. 85–108.
[LBW] L. Le Bruyn and G. van de Weyer, *Formal structures and representation spaces*, J. Algebra 247(2) (2002), 616–635.
[L] L. Le Bruyn, *Noncommutative Geometry and Cayley-smooth Orders*, Pure Appl. Math. 290, Chapman & Hall/CRC, Boca Raton, 2008.

[Q] D. Quillen, *Homotopical Algebra*, Lecture Notes in Math. 43, Springer-Verlag, Berlin, 1967.
[Q1] D. Quillen, *Algebra cochains and cyclic cohomology*, Inst. Hautes Etudes Sci. Publ. Math. 68 (1989), 139–174.

[T] B. Toën, *Higher and derived stacks: a global overview*, preprint, [arxiv:math/0604504].

[TV] B. Toën and M. Vaquié, *Moduli of objects in DG categories*, Ann. Sci. ENS 40 (2007), 387–444.

[vdB] M. Van den Bergh, *Noncommutative quasi-Hamiltonian spaces*, Contemp. Math. 450 (2008) 273–299.

[vdB1] M. Van den Bergh, *Double Poisson algebras*, Trans. Amer. Math. Soc. 360 (2008), 5711–5769.