Elementary excitations in fractional quantum Hall effect from classical constraints

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Abstract
Classical constraints on the reduced density matrix of quantum fluids in a single Landau level termed as local exclusion conditions (LECs) (Yang 2019 \textit{Phys. Rev. B} 100 241302), have recently been shown to characterize the ground state of many fractional quantum Hall (FQH) phases. In this work, we extend the LEC construction to build the elementary excitations, namely quasiholes and quasielectrons, of these FQH phases. In particular, we elucidate the quasihole counting, categorize various types of quasielectrons, and construct their microscopic wave functions. Our extensive numerical calculations indicate that the undressed quasielectron excitations of the Laughlin state obtained from LECs are topologically equivalent to those obtained from the composite fermion theory. Intriguingly, the LEC construction unveils interesting connections between different FQH phases and offers a novel perspective on exotic states such as the Gaffnian and the Fibonacci state.

1. Introduction
Low lying elementary excitations in topologically ordered systems are fascinating objects that capture the essential topological features of the corresponding ground states. In fractional quantum Hall (FQH) systems, these excitations can carry fractionalized charges and obey anyonic or non-abelian braiding statistics [1–4]. On one hand, discerning the nature of these excitations offers important insights into the underlying mechanisms of incompressibility of the FQH states. On the other hand, manipulating these excitations in a controlled manner also holds promises for storing and processing quantum information that is topologically protected [5–8]. Thus, a detailed understanding of the nature of the elementary excitations of FQH states is important from both the fundamental as well as the practical standpoint.

The elementary excitations in FQH systems can be categorized into charged and neutral excitations. The former includes the positively charged quasiholes and the negatively charged quasielectrons. The quasihole excitation can be realized by inserting flux quanta into the incompressible quantum fluid. For FQH states with model Hamiltonians [2, 4, 9–11], the quasihole states usually form a zero energy manifold degenerate with the ground state. The counting of the quasihole states on the sphere encodes the topological order of the FQH phase, from which one can derive the edge modes on the disk or ground state degeneracy on the torus. Whether an FQH state is abelian or non-abelian is also determined by the quasihole properties. In particular, a state is non-abelian if just specifying the positions of the quasiholes does not uniquely determine the wave function for the multi-quasihole state.

Following the terminology used in reference [12], we use the term ‘quasielectrons’, instead of ‘quasiparticles’, for elementary charged excitations carrying fractionalized negative charges in an FQH system. The quasielectrons can be created by adding electrons to or removing fluxes from the FQH fluid. A neutral excitation is composed of a quasielectron and a quasihole. The low-lying branch of the neutral
excitations of FQH fluids forms a collective mode called the magnetoroton mode \cite{13, 14}. A given FQH phase can host different types of quasielectron and neutral excitations \cite{12, 15, 16}. For the FQH state to be incompressible, all the quasielectrons, as well as the neutral excitations, have to be gapped in the thermodynamic limit.

While the construction of the quasiholes states is relatively straightforward, the construction of model states to represent quasielectrons and neutral excitations is much more involved. This is because, in contrast to the ground state and quasiholes, model Hamiltonians for quasielectrons and neutral excitations are not known. For many FQH states, the composite fermion (CF) \cite{17} theory or the Jack polynomial \cite{18} and conformal field theory (CFT) \cite{19} approach can be used to construct excitations involving quasielectrons. These different approaches generically lead to different microscopic wave functions for the same topological state of quasielectrons \cite{20}. It is also worth noting that while for each model FQH phase there is only one type of quasiholes, there can be in general many different types of gapped quasielectrons and neutral excitations \cite{12, 15, 16}. For some exotic non-abelian phases, these gapped excitations are not very well understood.

Previously, one of us introduced the idea of local exclusion conditions (LECs) in the FQH effect and demonstrated that the LECs in conjunction with the requirement of translational invariance can determine the topological properties of many FQH states \cite{21}. An LEC is a well-defined constraint on the reduced density matrix of a quantum Hall fluid. Physically it is a constraint on what we can or cannot measure, thus in a sense, it is a ‘classical constraint’ on a quantum system. Each LEC is specified by a triplet of integers $\hat{c} = \{n, n_e, n_h\}$. An imposition of $\hat{c}$ on a quantum Hall fluid dictates that for any circular droplet within the fluid containing $n$ fluxes, a physical measurement can neither detect more than $n_e$ number of electrons, nor $n_h$ number of holes. Topological indices including filling factor, shift \cite{22} and particle clustering \cite{10} can emerge from quantum Hall fluids satisfying the LECs. Furthermore, LECs also determine the microscopic model wave function of the many-body ground state for these topological phases. One should also note that for many FQH states (Read-Rezayi sequence, Gaffnian, Haffnian, etc.) the number of translationally/rotationally invariant, i.e., $L = 0$, states is exactly one in the truncated basis satisfying the LEC. Such states can plausibly lend themselves to a description in terms of an LEC. However, there are also states, like ground states of hollow-core Hamiltonians or the unprojected 2/5 Jain CF state, for which this is not true and LECs cannot capture such states.

In this work, we show that the LECs can also determine the elementary excitations of the FQH states. For FQH phases where the CF construction or the Jack polynomial/CFT approach is applicable, the model wave functions obtained from LECs agree qualitatively, and semi-quantitatively, with the wave functions obtained using these traditional methods. The LEC approach can also be applied to novel FQH phases that do not lend themselves to a description in terms of CFs or a CFT. Furthermore, the LECs can describe excitations for certain phases, where the ground state is captured by a CFT, but there is no known description of the quasielectrons and neutral excitations. We shall present some examples of such phases below.

The construction of the model wave functions for elementary excitations using the LECs offers a new perspective on the nature of these excitations. In particular, we find that a set of LECs that defines the quasielectrons of one FQH phase could also define the ground state and the quasiholes of a different FQH phase. This not only sheds light on the relationship between different FQH phases but also reveals interesting links between phases that were previously believed to be unrelated. More specifically, we show strong evidence that the Gaffnian state \cite{23} at $\nu = 2/5$ is built from a particular type of quasielectrons of the Laughlin state at 1/3. Similarly, the Laughlin state at 1/3 can be viewed as arising from the condensation of the quasiholes of the Gaffnian state. Another example is the Fibonacci state in the Read–Rezayi series \cite{10} which, following the LECs, can now be understood as consisting of a particular type of quasielectrons of the Moore-Read (MR) state \cite{4} (table 1).

We deploy the spherical geometry \cite{9} for all our calculations. We note that the LECs, being physical constraints, can in principle, be applied to any geometry. However, the spherical geometry is the most convenient one since circular droplets at the north or south poles can be easily defined in this geometry. A nice feature of working on the sphere is that it is also the only geometry without a boundary, in which the topological shifts are well-defined even for finite systems. The two good quantum numbers on the sphere are the total orbital angular momentum $L$ and it is $z$-component $L_z$. For a rotationally invariant Hamiltonian, states with a given $L$ form a $(2L + 1)$-degenerate multiplet with $-L \leq L_z \leq L$. The state with $L_z = L$ is defined to be the highest weight (HW) state in this multiplet. We only consider fully spin polarised quantum fluids here. Furthermore, we only look at the Hilbert space of a single Landau level (LL). We assume that the effects of LL mixing can be captured by more complicated dynamics within a single LL (e.g., by using three-body interactions).
The paper is organized as follows: in section 2 we show how the quasiholes of the FQH states can be constructed with the LEC formalism, and discuss the resulting bulk-edge correspondence between the ground state entanglement spectrum (ES) and the quasihole counting. Then, in section 3 we show how different types of quasielectrons and neutral excitations can be constructed with the LEC formalism. In section 4, we show that the construction of quasielectron states using LECs reveals connections between the ground state entanglement spectrum (ES) and the quasihole counting. Then, in section 3 we show how constructed with the LEC formalism, and discuss the resulting bulk-edge correspondence between the ground state entanglement spectrum (ES) and the quasihole counting. Then, in section 3 we show how constructed with the LEC formalism, and discuss the resulting bulk-edge correspondence between the ground state entanglement spectrum (ES) and the quasihole counting.

| Table 1. Definition of various symbols used in the text. |
|------------------------|------------------------|
| n                      | A single LEC given by a triplet \{n, n_t, n_h\} |
| \(n\)                  | Number of flux quanta |
| \(n_e\)                | Number of electrons |
| \(n_h\)                | Number of holes |
| \(\hat{c}\)            | Constraint on Hilbert space, e.g., \(\hat{c} = \hat{N} = \hat{N}_1 + \hat{N}_2\) |
| \(\mathcal{H}_{N_e,N_h}\) | Hilbert space on sphere with \(N_e\) orbitals and \(N_h\) electrons |
| \(\mathcal{N}_{N_e,N_h}^{\ell}\) | Number of highest weight states in \(\mathcal{H}_{N_e,N_h}\) |
| \(\mathcal{H}_{N_e,N_h}^{\ell}\) | Subset of \(\mathcal{H}_{N_e,N_h}\) satisfying the constraint \(\hat{c}\) |
| \(\mathcal{N}_{N_e,N_h}^{\ell}\) | Number of highest weight states in \(\mathcal{H}_{N_e,N_h}^{\ell}\) |
| \(\mathcal{W}_{N_e,N_h}^{\ell}\) | Subspace of \(\mathcal{H}_{N_e,N_h}^{\ell}\) spanned by all highest weight states, with dimension \(\mathcal{N}_{N_e,N_h}^{\ell}\) |
| \(\mathcal{Q}_{N_e,N_h}^{\ell}\) | (Dressed) quasi-electron Hilbert space of type \(\ell\) |

2. Quasihole state construction

Let us first generalize the LEC construction of the FQH ground states [21] to the quasihole states. The number of orbitals \(N_e\), and the number of electrons \(N_o\) of the ground states for incompressible quantum Hall systems satisfy the relation

\[
N_o = \frac{q}{p} (N_e + S_e) - S_o.
\]

(1)

Here the filling factor \(\nu = p/q\) \((p\) and \(q\) are positive integers), and \(S_e < p, S_o < q\) are integer topological shifts for the electrons and fluxes respectively. The total topological shift could be a rational number \([24–26]\), requiring two integers, and here we use \(S_e, S_o\) as the two integer indices explicitly. Extensive numerical evidence shows that the constraint imposed by one or a combination of LECs \((\text{denoted as } \hat{c})\) on a rotationally invariant quantum Hall fluid determines a set of topological indices \([p, q, S_e, S_o]\) satisfying the following commensurability conditions [21]:

\[
\mathcal{N}_{N_o}^{\ell} = \frac{q}{p} (N_e + S_e) - S_o
\]

(2)

\[
\mathcal{N}_{N_o}^{\ell} = 1, \quad \mathcal{N}_{N_o < N_o}^{\ell} = 0
\]

(3)

The commensurability conditions hold for all values of \(N_o\) subject to the condition that \(N_e + S_e = kp, k \geq 2\). Here, \(\mathcal{N}_{N_o}^{\ell}\) is the number of rotationally invariant \((L = 0)\) states of \(N_e\) electrons in \(N_o\) orbitals that satisfy the constraint specified by \(\hat{c}\). Thus given an allowed \(N_o\), the unique highest density state contains \(N_o^\ell\) orbitals (thus the superscript), is rotationally invariant and satisfies the constraint. Diagonalizing \(L^2\) in the truncated Hilbert space (obtained by removing basis states that do not satisfy \(\hat{c}\) at the north pole or south pole from the full set of basis states) results in the model wave function for the ground state of the FQH system indexed by \([p, q, S_e, S_o]\). Two prominent examples arising from a single LEC are [21]: (i) Laughlin state \([2]\) at \(\nu = 1/(2n - 1)\) which arises from \(\hat{c} = \{n, 1, n\}\) and corresponds to \([p, q, S_e, S_o]\) \(= [1, 2n - 1, 0, 2n - 2]\), and (ii) The Read–Rezayi series \([10]\) at \(\nu = (n - 1)/(n + 1)\) which arises from \(\hat{c} = \{n, n - 1, n\}\) and corresponds to \([p, q, S_e, S_o]\) \(= [n - 1, n + 1, 0, 2]\).

From extensive numerical calculations, we find that \(\mathcal{N}_{N_o}^{\ell}\) is also the number of HW states in the corresponding truncated Hilbert space. If \(\hat{c}\) corresponds to a particular commensurability condition \([p, q, S_e, S_o]\), then \(\mathcal{N}_{N_o}^{\ell} = 1\) implies there is only one HW state even if we scan all possible \(L_e\) sectors. Moreover, this unique HW state always occurs in the \(L_e = 0\) sector and is thus rotationally invariant. The
quasihole states can be naturally obtained by fixing \( N_e \) and letting \( N_o > N_o^d \), which corresponds to inserting flux quanta into the highest density ground state. The quasihole states are the HW states (with \( L = L_z \)) that satisfy the same set of LECs as the ground state. Generically, the number of quasihole states at different values of \( L \) are different from each other. We define \( \mathcal{W}_{N_o,N_e}^c \) to be the subspace of \( \mathcal{H}_{N_o,N_e}^c \) spanned by all the HW states from \( N_o = N_o^d \) to \( N_o = N_o^d + 1 \). \( \mathcal{W}_{N_o,N_e}^c \) is one-dimensional (ground state subspace) while for \( N_o > N_o^d \), \( \mathcal{W}_{N_o,N_e}^c \) is the quasihole subspace.

For FQH states with a CFT description or a model Hamiltonian, the quasihole counting obtained from the LECs scheme matches exactly with the CFT or model Hamiltonian approach. For all states constructed from LECs, there is also an interesting bulk-edge correspondence between the counting of the ground state ES and the edge modes derived from bulk quasihole counting. This holds empirically for all the system sizes we have checked. Such bulk-edge correspondence is well-known in the CFT construction of the FQH states: the counting of levels in the ES of the ground state agrees with the edge state counting of the quasiholes constructed from the same CFT model \([27, 28]\). The LEC construction indicates that such a correspondence holds even for states with no known CFT description, suggesting that the bulk-edge correspondence may be an intrinsic property of the algebraic structure of the truncated Hilbert space (figure 1).

As an example, let us look at different topological phases at filling factor \( \nu = 3/7 \). Consider two different constraints \( \hat{c}_1 = \{5, 3, 5\} \) and \( \hat{c}_2 = \{2, 1, 2\} \lor \{6, 3, 6\} \). The symbol \( \lor \) in the latter means the quantum fluid needs to satisfy either \( \{2, 1, 2\} \) or \( \{6, 3, 6\} \). Both \( \hat{c}_1 \) and \( \hat{c}_2 \) correspond to \( [p, q, S_o, S_o] = [3, 7, 0, 4] \). On one hand, the quasihole counting obtained from these two different constraints is identical. Moreover, the counting of levels in the entanglement spectra of the ground state wave functions obtained from these two constraints also agree with each other [see figures 2(a) and (b)]. On the other hand, the ground state wave functions obtained from \( \hat{c}_1 \) and \( \hat{c}_2 \) have vanishingly small overlap with each other. We conjecture that \( \hat{c}_1 \) and \( \hat{c}_2 \) realize different topological phases, which could be distinguished by analyzing their topological entanglement entropy (TEE) \([29, 30]\) in the thermodynamic limit. At the moment, due to technical challenges, we do not have access to the ground state wave functions of these LECs for very large system sizes, which precludes a reliable extrapolation of their TEEs. A third topological phase at \( \nu = 3/7 \) can be realized by \( \hat{c}_3 = \{3, 2, 3\} \land \{5, 3, 5\} \) [This state was first discovered by R Thomale et al (unpublished).]. The symbol \( \land \) implies that the quantum fluid needs to satisfy both \( \{3, 2, 3\} \) and \( \{5, 3, 5\} \). The topological phase generated by \( \hat{c}_3 \) corresponds to \( [p, q, S_o, S_o] = [3, 7, 1, 5] \), and thus has different shifts, and different quasihole counting [leading to a different ES for the ground state (see figure 2(c))] compared to the phases obtained from \( \hat{c}_1 \) and \( \hat{c}_2 \).

In all three cases, there are no apparent CFT descriptions of the FQH states, but the bulk-edge correspondence holds. In particular, the quasihole counting (and thus the edge state counting) is identical for \( \hat{c}_1 \) and \( \hat{c}_2 \), while \( \hat{c}_3 \) has a different counting. Empirically, this suggests that while different LECs
Figure 2. (a) The ES of the ground state corresponding to $[p, q, S_e, S_o] = [3, 7, 0, 4]$ determined by $\hat{c}_1$ (see text for definition). (b) The ES of the ground state corresponding to $[p, q, S_e, S_o] = [3, 7, 0, 4]$ determined by $\hat{c}_2$ (see text for definition). (c) The ES of the ground state corresponding to $[p, q, S_e, S_o] = [3, 7, 1, 5]$ determined by $\hat{c}_3$ (see text for definition). The subsystem is given by $N_A^e$ and $N_A^o$ (partition of root configuration shown in the lower right corner). The numbers in the plots show the ES counting of the respective $L$ sector.

3. Quasielectron and neutral excitation construction

We will now move on to the quasielectron and neutral excitations, and illustrate the construction for these states in the simplest case of the Laughlin state. Our methodology can be easily extended for other sets of LECs. It is instructive to first recall the construction of single quasielectron states from the Jack polynomial formalism. The starting point in the Jack formalism is the root configuration for the ground state, for example for the $\nu = 1/3$ Laughlin state, the root configuration is 100100100100... , where ... denotes repeated patterns of 100. A single quasielectron at the north pole can be added by flipping the leftmost 0 to 1 (adding one electron), and inserting two fluxes (or 0’s). This results in a net addition of charge $(-e)/3$ ($-e$ is the charge of the electron) consistent with the charge of the quasielectron at $\nu = 1/3$. Different ways determine different topological phases, the quasihole counting is only determined by the shifts and the filling factor.
Figure 3. Energy spectra resolved by the model Hamiltonian containing only $V_1$ pseudopotential interaction, within each excitation type (e.g. $H_{qe}^{(5,2)}$, $H_{qe}^{(6,3)}$, etc.). The energy spectra of the full Hilbert space with the same Hamiltonian (allowing coupling between different excitation types, e.g. between $H_{qe}^{(5,2)}$ and $H_{qe}^{(6,3)}$, etc.) are listed immediately on the right for comparison. (a) $N_e = 8, N_o = 22$, so only neutral excitations are present; (b) $N_e = 8, N_o = 21$, all states containing a single quasielectron (may be dressed by neutral excitations); (c) $N_e = 8, N_o = 20$, all states containing two quasielectrons (may be dressed); (d) $N_e = 8, N_o = 19$, all states containing three quasielectrons (may be dressed). For (b)–(d), only the low-lying part of the spectrum is shown. The legend in (b) is shared by all panels.

of inserting two fluxes leads to different types of quasielectrons as listed below:

$$
\begin{align*}
110001001001001001 & \quad (5, 2) \text{ Type } L = N_e/2 \\
110010001001001001 & \quad (6, 3) \text{ Type } L = N_e/2 + 1 \\
110010010001001001 & \quad (7, 4) \text{ Type } L = N_e/2 + 2 \\
\vdots
\end{align*}
$$

We name the quasielectron types as the (5, 2) Type or (6, 3) Type etc. for reasons that would become apparent later. The solid and empty circles below the root configuration indicate the locations of $-e/3$ quasiparticles and $+e/3$ quasiholes respectively. Here, the quasiparticles (or quasiholes) are located in regions where three consecutive orbitals in the root configuration accommodate more (or less) than a single electron. While a quasihole is an elementary excitation with charge $+e/3$, a quasielectron (not a quasiparticle) is the elementary excitation with charge $-e/3$. The quasielectron has a nontrivial internal structure as a bound state of two quasiparticles and one quasihole. Each of the quasielectron states only consists of the squeezed basis [20] from the respective root configurations shown in equation (4). They can be uniquely determined by the HW condition, with the constraint that the state relaxes back to the Laughlin ground state away from the north pole. Such quasielectron states are identical to the CF construction, where different types of quasielectrons correspond to adding a single CF in different CF levels [15].

The basis squeezed from the (5, 2) Type root configuration manifestly satisfies $c_\alpha = \{2, 1, 2\} \cup \{5, 2, 5\}$, which is the set of LECs that define the Gaffnian ground state and its quasiholes [21]. Thus the (5, 2) Type quasielectron state can also be understood as a condensation of the Gaffnian quasiholes (see section 4). The (5, 2) Type quasielectron is a charged excitation of the Laughlin state (defined by $\{2, 1, 2\}$ [21]) since the quasielectron at the north pole breaks the $\{2, 1, 2\}$ constraint. However, the (5, 2) Type quasielectron still satisfies $c_\alpha$. This justifies the use of our terminology for the various types of quasielectrons in equation (4).

In fact, if we impose $c_\alpha$ on the entire Hilbert space of $L_z = N_e/2$ states, together with the HW condition, we obtain a large number of Gaffnian quasihole states. We can now interpret these states as containing a single quasielectron and potentially multiple neutral excitations (a neutral excitation is composed of a quasielectron and a quasihole) on top of the Laughlin state at $\nu = 1/3$. These different states can be
resolved by diagonalizing the HW subspace with the $V_1$ Haldane pseudopotential interaction (referred to as $V_1$ Hamiltonian from here on) [9].

The two-quasielectron (of $(5, 2)$ Type) states can be constructed by imposing $\hat{c}_\alpha$ on the Hilbert space of $N_e = 3N_e - 4$, and obtaining the HW subspace in different $L_2$ sectors. Analyzing the ground state energy by diagonalizing the $V_1$ Hamiltonian within each subspace, we can extract wave functions and the counting of the two-quasielectron states. This procedure generalizes to multi-quasielectron states, where for $n$ quasielectrons, the target Hilbert space has $N_e = 3N_e - 2 - n$. Moreover, in the Hilbert space of $N_e = 3N_e - 2$, where no fluxes or electrons are added, we can use the same $\hat{c}_\alpha$ to extract the neutral excitations. In particular, the magnetoroton mode [13, 14] is formed by the HW state in the sectors with $L_0 = 2, 3, \ldots, N_e$.

More generally, with $\hat{c}_\alpha = \{2, 1, 2\} \cup \{5, 2, 5\}$, the Hilbert space $W_{N_e,N_o}^{\hat{c}_\alpha}$ can be interpreted either as the Gaffnian quasi-hole manifold (when $N_o > 5N_e/2 - 3$), or as the Laughlin $(5, 2)$ Type quasielectron manifold (when $N_o < 3N_e - 2$). Here, the Laughlin $(5, 2)$ Type quasielectron manifold is defined as the one spanned by states that contain either undressed $(5, 2)$ Type quasielectrons, or dressed $(5, 2)$ Type quasielectrons (e.g. dressed by neutral excitations if $N_o = 3N_e - 2$, or dressed by quasielectrons and neutral excitations if $N_o < 3N_e - 2$).

Each species of quasielectrons in equation (4) is similarly defined by its respective LECs. The $(6, 3)$ Type is defined by $\hat{c}_\gamma = \{2, 1, 2\} \cup \{6, 3, 6\}$, while the $(7, 4)$ Type is defined by $\hat{c}_\beta = \{2, 1, 2\} \cup \{7, 4, 7\}$, and so on. We note that $W_{N_e,N_o}^{\hat{c}_\alpha} \subseteq W_{N_e,N_o}^{\hat{c}_\beta} \subseteq W_{N_e,N_o}^{\hat{c}_\gamma}$ since the constraint $\hat{c}_\alpha$ is stricter than $\hat{c}_\beta$, which in turn is stricter than $\hat{c}_\gamma$. All such Hilbert spaces can be explicitly computed numerically for allowed values of $N_e$ and $N_o$. Let us define $Q_{N_e,N_o}^{(i, j)} := W_{N_e,N_o}^{\hat{c}_i}$ as the Hilbert space spanned by all states containing only $(i, j)$ Type quasielectrons. Similarly, $Q_{N_e,N_o}^{(i, j)}$ is spanned by states that contain at least one $(i, j)$ Type quasielectron. Note that $Q_{N_e,N_o}^{(6,3)}$ can contain some $(5, 2)$ Type quasielectrons, and these $(5, 2)$ Type quasielectrons may or may not be dressed with $\{5, 2\}$ Type or $(6, 3)$ Type neutral excitations. Numerically, $Q_{N_e,N_o}^{(6,3)}$ can be easily determined by projecting out $W_{N_e,N_o}^{\hat{c}_6}$ from $W_{N_e,N_o}^{\hat{c}_5}$. We can also define $Q_{N_e,N_o}^{(7,4)}$ analogously by projecting out $W_{N_e,N_o}^{\hat{c}_7}$ from $W_{N_e,N_o}^{\hat{c}_6}$ and so on. Therefore we have the following relations:

$$Q_{N_e,N_o}^{(i, j)} \subseteq W_{N_e,N_o}^{\hat{c}_i} \iff W_{N_e,N_o}^{\hat{c}_j} \perp W_{N_e,N_o}^{\hat{c}_i},$$

where $\perp$ denotes the two spaces are orthogonal. In this way, the entire Hilbert space of the Laughlin phase can be systematically organized. Each quasielectron manifold $Q_{N_e,N_o}^{(i, j)}$ (where $i$ denotes the LEC type) are physically distinct and orthogonal to each other. This is because different types of quasielectrons have different intrinsic angular momentum, and thus, in principle, could be experimentally distinguished.

In figure 3 we show how the entire Hilbert space could be organized into states containing different types of excitations. While such an organization can be carried out for any Hilbert space (determined by $N_e$ and $N_o$), independent of the microscopic Hamiltonian, it is most relevant physically if the Hamiltonian is dominated by $V_1$ (so that we are in the Laughlin phase). It is important to note that the excitation Hilbert spaces are determined algebraically without resorting to any local operators. Here we use Hamiltonians to show such algebraically determined Hilbert spaces are physically relevant for realistic interactions. Taking $Q_{N_e,N_o}^{(5,2)}$ as an example, all states in this Hilbert space contain $(5, 2)$ Type quasielectron(s) and possibly some Laughlin quasiholes. The microscopic Hamiltonian crucially decides whether these quasiholes can be effectively treated as ‘elementary particles’. The $V_1$ Hamiltonian can be diagonalized within $Q_{N_e,N_o}^{(5,2)}$ to resolve states containing different number of quasiholes. In a multi-quasielectron state, interactions between quasiholes, as well as between quasielectrons and quasiholes, can make it difficult to ascertain the precise number of quasiholes in a particular state. This is because the variational energies are no longer just an integer multiple of the single quasihole creation energy. The deviation, however, is small if the quasiholes are far away from each other.

The magnetoroton mode (one $(5, 2)$ Type quasielectron dressed by one quasihole) can be clearly seen in figure 3(a) for a neutral system. For large momenta when the quasielectron is well-separated from the quasihole, the creation energy of a single quasielectron is roughly equal to the variational energy of the state (since a quasihole far away from other excitations costs negligible energy with the $V_1$ Hamiltonian). This variational energy also matches a single undressed quasi electron state in figure 3(b), as the lowest energy state at $L = N_e/2$. All other higher energy states contain one single quasi electron dressed by $(5, 2)$ Type neutral excitations and therefore contain multiple quasielectrons. In figures 3(c) and (d), the lowest energy states contain two and three undressed quasiholes respectively.
A similar analysis can be done by diagonalizing the $V_1$ Hamiltonian within $Q^{(6,3)}_{N_{e},N_{o}}$ or sectors of other types of quasielectrons, and states containing undressed quasielectrons of different types can also be determined. With $V_1$ Hamiltonian, the creation energies of these quasielectrons are higher than that of the $(5,2)$ Type. Additional branches of neutral excitations discovered in reference [21] can also be constructed just like the magnetoroton mode in $Q^{(5,2)}_{N_{e},N_{o}}$. The key message here is that quasielectrons can be treated as weakly interacting particles for short-range Hamiltonians dominated by $V_1$. Also, different types of undressed quasielectrons have different intrinsic angular momenta since the HW states live in different $L^2$ sectors [12]. Thus a rotationally invariant Hamiltonian cannot couple a single quasielectron of one type to another quasielectron of a different type.

In figure 3 we also show the exact energy spectra of different systems (specified by $N_e$ and $N_o$, with $V_1$ Hamiltonian), comparing them with the energy levels (we denote as variational spectra) obtained from diagonalizing the same $V_1$ Hamiltonian in each quasielectron manifold $Q^{(5,2)}_{N_{e},N_{o}}$ separately. In each sector, the low-lying states predominantly coming from $Q^{(5,2)}_{N_{e},N_{o}}$ match very well with the exact spectra, both in terms of the variational energies and wave function overlaps ($>99\%$). The same is true for the highest energy states, which consist of different types of quasielectrons depending on the system size and the orbital angular momentum. The mismatch between the exact spectra and variational spectra is because the Hamiltonian couples different $Q^{(5,2)}_{N_{e},N_{o}}$, at least for finite systems with $V_1$ interaction. Even though a single undressed quasielectron of one type cannot scatter into another type of undressed quasielectron because they have different quantum numbers, it can scatter into other types of dressed quasielectrons. Matrix elements coupling multiple quasielectrons of different types are non-zero if they carry the same quantum number. Thus with realistic interactions, only a dilute gas of quasielectrons can be resolved in terms of the different quasielectron types. If the interaction is dominated by $V_1$, the low lying quasielectrons are of the $(5,2)$ Type (if they are present). On the other hand, in large $L$ sectors, quasielectrons of other types (e.g. the $(6,3)$ Type) are the low lying ones (since the $(5,2)$ Type is absent). These other types of quasielectrons can be experimentally detected as excitations with large angular momentum transfer from the probing particles (e.g. neutrons or photons) to the FQH fluid [31, 32].

An important comment is in order here about the need for interaction input to resolve microscopic wavefunctions of the quasielectron and neutral excitation states. The LEC formalism determines the manifolds or the sub-Hilbert spaces (spanned by HW states in the truncated Hilbert space) of different types of quasielectrons that differ in their universal topological properties. Such sub-Hilbert spaces are universal, with their properties determined without any interaction input. The microscopic wavefunctions and their variational energies, on the other hand, are non-universal and depend on the details of the microscopic Hamiltonian. Understanding the universal topological properties of these excitations with the LEC formalism could allow a systematic analysis for the robustness of the topological properties with realistic interactions. To this end, in section 5, we have compared the states obtained from the LEC formalism with the exact states of the LLL Coulomb interaction.

4. Connections between different FQH phases

We have shown that a set of LECs can define both the ground state and quasi-hole states of one FQH phase, and at the same time define the quasielectron and neutral excitations manifolds of a different FQH phase. This leads to a new perspective on the nature of many topological phases, as well as the connections between them. Understanding the ground state of one FQH phase as the quantum fluid of the excitations of the (5,2) Type of the Laughlin state at $\nu = 1/3$. Thus this family of quasielectrons and neutral excitations can also be understood as the Gaffnian quasiholes when the quasiholes are uniformly spaced far away from the local defect at which $\tilde{n} = \{2,1,2\}$ is violated. Similarly, the Laughlin state can also be understood as a condensation of the Gaffnian quasiholes. This perspective is familiar in the Jack polynomial formalism (with the root configurations), and in the hierarchical approach where the Laughlin state...
be uniquely defined by the Laughlin state, and the abelian Jain operator in the constraint-satisfying Hilbert space. (and its excitations) are the condensate of the 2ν state. The Fock space representations of the Jain state have a high overlap with the Jain 7 state as well as the LLL Coulomb ground state (see table 2). The Fock space representations of the Jain state are closely connected to the Gaffnian and CF states at ν = 3/7, which is very much reminiscent of the hierarchical or the CF construction. We shall report in detail elsewhere on the connections between the Gaffnian and CF states at ν = 3/7.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
N_e & N_{qe} & \langle\Psi_{7}^{LEC}|\Psi_{3/7}^{LEC}\rangle & \langle\Psi_{7}^{LEC}|\Psi_{3/7}^{LEC}\rangle & \langle\Psi_{7}^{LEC}|\Psi_{3/7}^{LEC}\rangle \\
\hline
9 & 17 & 0.9851 & 0.9866 & 0.9994 [41, 42] \\
12 & 24 & 0.9652 & 0.9720 & 0.9988 \\
15 & 31 & 0.9509 & — & — \\
18 & 38 & 0.9430 & — & — \\
\hline
\end{tabular}
\caption{Overlaps of the ground state for N_e electrons in N_{qe} orbitals of the spherical geometry at the Jain 3/7 flux in the n = 0 LL (obtained by exact diagonalization), \Psi_{7}^{LEC}, with the LEC state, \Psi_{3/7}^{LEC}, generated by the constraint \hat{c}_{0} = \{2, 1, 2\} \lor \{6, 3, 6\} and Jain 3/7 state, \Psi_{3/7}^{LEC}.}
\end{table}

To see how such connections go beyond the Laughlin state, we look at the MR state [4] at ν = 1/3 establishes a connection between the FQH phase at ν = 1/3 and ν = 3/7. This is because the (6, 3) Type quasielectrons and the ν = 3/7 ground state (together with its quasiholes) are all defined by \hat{c}_{3} = \{2, 1, 2\} \lor \{6, 3, 6\} (which is equivalent to \{2, 1, 2\} \lor \{5, 2, 5\} \lor \{6, 3, 6\}). We find that the ν = 3/7 ground state determined by the \hat{c}_{3} LEC has a high overlap with the Jain 3/7 state as well as the LLL Coulomb ground state (see table 2).

| 5, 2 | Type quasielectron |
|-------|---------------------|
| 11100110011001100 . . . |

| 5, 3 | Type neutral excitations (magnetoroton mode) |
|-------|---------------------------------------------|
| 111001011001100 . . . |
| 1110010101001100 . . . |
| . . . |

| 5, 3 | Type neutral excitations (neutral Fermion mode) |
|-------|---------------------------------------------|
| 111000110011001100 . . . |
| 11100101010011001100 . . . |

....
For even and odd number of electrons, the neutral excitations form the well-known magnetoroton mode and the neutral fermion mode respectively. These quasielectron and neutral excitations can be completely defined by $\hat{c}_m = \{3, 2, 3\} \lor \{5, 3, 5\}$. It turns out $\hat{c}_m$ also identifies the Fibonacci state at $\nu = 3/5$, which is the next state after the MR state in the Read–Rezayi sequence [10]. Starting with the MR ground state at $\nu = 1/2$ with $N_e$ electrons and $N_{o} = 2N_e - 2$ orbitals, every time we add one electron and one flux (orbital), we are adding two quasielectrons, each with charge $-e/4$. We continue this process by adding two quasielectrons at a time. The undressed $(5, 3)$ Type quasielectrons can be resolved by first obtaining the HW quasielectron Hilbert space $\mathcal{Q}^{(5,3)}_{N_e, N_o}$ with $\hat{c}_m$, then diagonalizing within this Hilbert space.
with the model three-body Hamiltonian for the MR state. After adding \( N_c \) electrons and \( N_v \) orbitals, the only HW state in \( Q^{(5,3)}_{N_c,N_v} \) is the translationally invariant Fibonacci state, or the state containing 2\( N_c \) quasielectrons of the (5, 3) Type [see figure 4(b)].

5. Comparison of the composite fermion and LEC constructions

In this section, we compare the wave functions obtained from the CF and LEC constructions in detail. We show that for quasiholes and undressed quasielectrons, these two theories, which are microscopically different, lead to model wave functions that agree with each other qualitatively and semi-quantitatively. For the sake of completeness, we will first provide a primer on the CF theory.

A vast majority of the LLL FQHE phenomena is captured in terms of emergent topological particles called CFs, which are bound states of electrons and an even number (2\( p \)) of quantized vortices [17]. CF theory postulates that a system of interacting electrons at filling factor \( \nu = \nu^*/(2p\nu^* \pm 1) \) can be mapped onto a system of weakly interacting CFs at a filling factor \( \nu^* \). In particular, integer filling of CF–LLs (termed ALs), i.e \( \nu^* = n \), leads to FQHE of electrons at \( \nu = n/(2pn \pm 1) \). The mapping to integer quantum Hall effect leads to the following Jain wave functions for interacting electrons in the LLL [17]:

\[
\Psi^\alpha_{\nu^*} = \overline{P_{\nu^*}} \Phi^{\alpha}_{\nu^*}. \tag{7}
\]

Here \( \alpha \) labels the different eigenstates, \( \Phi_{\nu^*} \) is the Slater determinant wave function of electrons at \( \nu^* \) (with \( \Phi_{-\nu^*} = [\Phi_{\nu^*}] \)) where overline denotes complex conjugation) and \( P_{\nu^*} \) implements projection to the LLL. Throughout this work we carry out projection to the LLL using the Jain–Kamilla method [43, 44], details of which can be found in the literature [42, 45–47].

The quasihole and quasielectron excitations of the FQH systems are obtained as composite fermion hole (CFH) and composite fermion particle (CFP) respectively. A CFH is a missing CF in an otherwise full AL, while a CFP is a CF in an otherwise empty AL. Wave functions of the CFP and CFH can be constructed along the lines of equation (7) using the analogy to the particle and hole excitations in the corresponding IQH state. The CF theory is in excellent qualitative as well as quantitative agreement with exact diagonalization studies of the LLL Coulomb problem for both the ground states as well as the excitations [40, 42, 43, 48–50].

We shall focus our attention on the simplest FQH state at \( \nu = 1/3 \), which in the CF theory maps to a \( \nu^* = 1 \) state. The ground state wave function of equation (7) for this case reduces to the Laughlin state [2]. Furthermore, the wave function of the CFH at \( \nu = 1/3 \) is identical to that of the Laughlin quasihole [2] which is identical to the quasihole obtained from the LEC construction. Thus, to compare the CF and LEC constructions we shall mainly focus on the CF wave function. The wave function of the CF at \( \nu = 1/3 \) is not identical to that of the Laughlin quasielectron [2]. However, the CFP and the Laughlin quasielectron have high overlaps with each other for small systems [42] and are believed to describe the same excitation. The orbital angular momentum \( L \) of a single CFP at \( \nu = 1/3 \) is \( L_{\text{CFP}} = N/2 \) [42, 51]. States consisting of multiple CFPs at \( \nu = 1/3 \) are constructed using the analogy to multiple particle states at \( \nu^* = 1 \). The orbital angular momenta of states consisting of multiple CFPs can be ascertained by adding the angular momenta of the constituent CFPs.

The wave functions of equation (7) are most readily evaluated in first quantization using the Metropolis Monte Carlo method [52]. In figure 5 we show the LLL Coulomb spectra of multiple quasielectron states at \( \nu = 1/3 \) obtained from the LEC and CF constructions. We first point out that the angular momentum quantum number for a state with a single quasielectron obtained from the LEC and CF constructions agree with each other. Moreover, their LLL Coulomb energies are close to each other, indicating high overlaps of the corresponding wave functions. Furthermore, the overlaps and energies of multiple quasielectron states constructed from the LEC and CF theories are also in fairly good agreement with each other. The CF wave functions are known to be very accurate for the LLL Coulomb interaction and thus, in general, have lower variational energies than the LEC states. The LEC formalism only defines the universal sub-Hilbert spaces of quasielectrons of different topological phases and is thus not specifically optimized for a particular interaction. The LLL Coulomb interaction leads to scattering between different types of quasielectrons, which is ignored in obtaining the variational energies of the LEC states in figure 5. This leads to higher variational energies in certain cases. We expect shorter-range interactions to reduce scattering between different quasielectron types, thereby lowering the variational energies of the LEC states. Physically this implies the topological properties of the quasielectrons are more robust with such short-range interactions, as compared to the LLL Coulomb one.

It is important to note that the Coulomb interaction, unlike the short-range model \( V_1 \) interaction, is long-ranged. Therefore, a reasonable agreement in the LLL Coulomb energies of the CF and LEC states
Figure 5. Comparison of the lowest LL Coulomb spectra obtained from exact diagonalization (red dashes), CF theory (black dots) and LECs (blue crosses) for systems consisting of one (a), two (b), three (c), four (d) and five (e) quasielectrons at $\nu = 1/3$. The calculations were carried out in the spherical geometry with $N_e$ electrons in $N_o$ orbitals. The CF states are shown schematically in the inset.

suggests that the two seemingly disparate constructions are consistent with each other. Finally, we mention that both the LEC and CF constructions give a good representation of the LLL Coulomb spectra as evidenced from their comparison with the spectra obtained from exact diagonalization [see figure 5]. The qualitative and semi-quantitative agreement between the LEC and the CF construction for the Laughlin quasielectrons has important implications. In particular, the Gaffnian and the CF states at $\nu = 2/5$ turn out to be closely related. This is because the former is made of undressed $(5, 2)$ Type quasielectrons, while the latter is made of undressed CF quasielectrons. Our analysis in this section shows that for undressed quasielectrons, the LEC and CF constructions produce physically equivalent states. When quasielectrons are dressed with neutral excitations, however, the LEC and the CF constructions have qualitatively different predictions in terms of the counting of the states. Typically, the manifold of dressed CF quasielectrons is much larger than the manifold of dressed LEC quasielectrons of the same type. A detailed study of these connections will be presented elsewhere.

6. Conclusions

In this work, we extended the recently developed LEC construction [21] to study the elementary excitations, namely quasiholes, quasielectrons, and neutral excitations, of many FQH phases. The quasihole model wave functions can be generated using the same set of LECs that determine the corresponding ground state. This can be achieved by imposing the HW condition on the truncated Hilbert space containing more orbitals than the ground state (with the same number of electrons as in the ground state). Empirically, we find the edge modes of the FQH phase derived from the quasihole counting matches exactly with the counting of the ES obtained from the ground state. Such agreement is known to hold for the FQH states constructed from a CFT. Our studies indicate that such a bulk-edge correspondence holds more generally, even for states that do not lend themselves to a CFT description.

In contrast to the quasiholes, the set of LECs defining the quasielectrons is different from but closely related to, the set that identifies the ground state. This is reasonable since unlike quasiholes, quasielectrons cost finite energy in the thermodynamic limit which arises from violating the commensurability conditions of the FQH ground state. It is well-known that an FQH phase can have different types of quasielectrons. Here we showed that the quasielectron manifold, which is composed of different types of quasielectrons, can be uniquely determined by the LEC construction. The neutral excitations can be studied analogously as bound states of quasielectrons and quasiholes. The LEC scheme thus allows us to systematically build the entire energy spectrum of an FQH phase with different types of elementary excitations. In particular, the many-body wave functions for the quasielectrons and neutral excitations can be explicitly constructed and uniquely determined once the LEC conditions are specified. However, it is not easy to generically write these wave functions down in a simple first quantized form [53].

We also identified several cases where a set of LECs defining the quasielectrons of one FQH phase also defined the ground state and quasiholes of a different FQH phase. In particular, we explicitly showed that the Gaffnian ground state at $\nu = 2/5$ is made of a particular type of Laughlin quasielectrons. For undressed quasielectrons at $\nu = 1/3$, we provided numerical evidence to show that the LEC and the standard CF...
constructions, which are microscopically different, lead to topologically equivalent states. Given that the abelian Jain $\nu = 2/5$ state is made of CF quasielectrons, we conjecture that just from the wave function itself, the Gaffnian ground state and the Jain $\nu = 2/5$ state are physically indistinguishable. All topological indices that can be extracted from the two wave functions could be identical. Furthermore, the Fibonacci ground state at $\nu = 3/5$ is made of a particular kind of MR quasielectrons. Thus, the LEC construction opens an avenue to find novel connections between various FQH states.

The LEC construction shows that, besides the ground state, the excitations of FQH fluids are also determined by the algebraic structure of the truncated Hilbert space. This is because the quasihole and quasielectron manifold (and thus the neutral excitation manifold) can be defined without referring to microscopic Hamiltonians. Indeed, the topological properties of a particular FQH phase characterize both the ground state as well as the excitations. For the ground state, the topological indices include the filling factor, the topological shifts, the TEE, and particle clustering. For quasiholes and quasielectrons, the topological indices include their fractionalized charge, the quasihole counting, and the topological spins. We have shown here that the LEC construction can uniquely determine all these topological indices, and thus the universal properties of many FQH phases. It would be interesting to further explore how robust these universal topological properties are in the presence of realistic interactions. The study of the interplay between the Hilbert space algebra and a realistic interaction could have crucial experimental ramifications, especially for non-abelian FQH phases.

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