On reconstruction of the Hilbert space from conceptual variables.

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In [1] the general problem of deriving the Hilbert space formulation in quantum theory is discussed from the point of view of conceptual variables. A conceptual variable is any variable defined by a person or by a group of persons. These variables may be inaccessible, i.e., impossible to assign numerical value to by experiments or by measurements, or accessible. One basic assumption might be that a group \( G \) of actions on a space of at least one maximally accessible variable is defined, and then accessible functions of these variables are introduced. By using group representation theory on the groups defined on a class of accessible variables, basic Hilbert space formalism is restored under the assumption that the observer or the set of observers has two relevant maximally accessible variables in his (their) mind(s). Again, the notion of relevance is precisely defined here. Operators are connected to each variable, and in the discrete case the possible values of the variables are given by the eigenvalues of the operators. The entire approach may be connected to a general epistemic interpretation of quantum theory. In this paper the main results from [1] are made more precise and more general. It turns out that the conditions of the main theorem there can be weakened in two essential ways: 1) No measurements need to be assumed, so the result is also applicable to general decision situations; 2) States can have arbitrary phase factors. Some consequences of this approach towards quantum theory are also briefly discussed here.

I. INTRODUCTION

The foundation of and the interpretation of quantum mechanics has been discussed since the theory was initially developed more than 100 years ago. The discussion has been particularly intense in the last decade.

As is well known, the theory was originally two theories, the matrix theory of Heisenberg and the wave theory of Schrödinger. These were shown to be mathematically equivalent, and von Neumann gave a well known and highly respected foundation in his famous book. This mathematical foundation has been reproduced in dozens, perhaps hundreds, of textbooks.

According to von Neumann, it is all based upon essentially two axioms, the Hilbert space axiom and the axiom describing Born’s law. The basic notions were unit vectors describing states and selfadjoint operators describing physical variables. This abstract foundation has been the basis for thousands of papers. The theory has in all respects been shown to be empirically valid.

In 2018 I published a book on Springer [1] indicating a somewhat different foundation. An error was discovered in the book, and in October this year I published a revised version. In this book, the Hilbert space axiom and the Born formula were shown to be theorems that could be derived from much simpler notions and assumptions.

The book was explicitly about what was called epistemic processes, processes to achieve knowledge through experiments or through measurements, and the Hilbert space theorem was there limited to that setting. But it was indicated that quantum theory also had wider applications; in particular, quantum decision theory was described.

In the first version of the present paper, published earlier in November, I showed that the Hilbert space theorem could be generalized. So far, I have received few reactions on the book and on the paper, but the physicist Bjørn Solheim recently said to me that, as he saw it, the Hilbert space theorem could not be seen as a mathematical theorem.

The purpose of this revised version is twofold: 1) To give a better description of the notions and assumptions behind the theorem, 2) To show that it really is a mathematical theorem.

In Section II of this paper the background is sketched, and in Section III the basic definitions are given. For the plan of the rest of the paper, see Section III.

II. BACKGROUND AND ASSUMPTIONS.

The basic notion of [1] and the present paper is that the essence of quantum theory in some given context can be described in terms of conceptual variables in the mind of an observer or in the joint minds of a group of communicating observers. In the epistemic process situation these are called epistemic conceptual variables, or simply e-variables. I will start with a simple example.

Let us assume that we are given some object A, and ask ‘What is the weight of object A?’ Then \( \mu = \text{‘weight of A’} \) is an e-variable. We can use a scale to obtain a very accurate estimate of \( \mu \). Or we can use several independent measurements, and use the mean of those as a more accurate estimate. In the latter case it is common to introduce a statistical model where \( \mu \) is a parameter of that model. But in my view the e-variable concept is a more fundamental notion. The variable \( \mu \) exists before any statistical model is introduced. Most people will agree that \( \mu \) exists in some sense. In this example the e-variable has some ontic basis, but my claim is that this need not always be the case in all epistemic processes. Even in this case the existence of \( \mu \) as a real number may be discussed. For instance, the question ‘Is \( \mu \) rational or irrational?’ is rather meaningless.

This example is carried over to a general measurement situation. The crucial assumption is that e-variables have a dou-
ble existence: One existence which in some sense is connected to the real world, and one existence in the mind of a relevant observer or a group of observers.

This assumption can in some way be said to be inspired by statistical theory: Basic statistical theory relies on statistical models of data, given some parameters. These models are formulated in attempts to describe the real situation at hand as well as possible. If necessary, they can be tested against data. The models are always formulated in the mind of some researcher or in the joint minds of a group of researchers. In particular, the concept of parameter in statistics, a concept that was introduced in an essential way by Ronald Fisher, has a basis which is related to this mind/these minds. Unfortunately, the word ‘parameter’ also has other meanings in the physical literature; therefore it is avoided here, and it is also avoided in physical contexts in [1].

A fundamental problem in quantum theory is the question of whether or not it is possible to derive the Hilbert space formulation from other, more intuitive axioms and notions. In [1] this problem was approached by using basic group theory together with the notion of accessible and inaccessible conceptual variables. At the outset, a conceptual variable in some given setting is any variable defined by a person or a group of persons. In a typical quantum context the setting may be one where an observer seeks some knowledge about a physical system, either through experiments or through some measurement. In [1] and [2] this is called an epistemic process. The variable may then simply be a physical variable as defined by the observer. Essentially different situations are quantum models of cognition and decision (see [3]), where the variables may have a psychological origin.

One limitation of the derivation in [1] is that, basically, an epistemic process is assumed there, so that general applications to cognitions and decisions are not covered. Here we will try to improve this, in particular, we will make no assumption that there exists some measurement of the basic accessible conceptual variable. This has important consequences: The whole theory may be generalized to situations determined by cognition and decision; see [3] and [18].

Another limitation was that the proof of the basic Theorem 4.3 in [1] was based on the following simple version of quantum theory: Each (maximally accessible; see below) state is given by a single unit vector in a Hilbert space. In reality the state is determined by a group of unit vectors with arbitrary phase factors. This will be tried taken into account here.

III. ON ACCESSIBILITY AND BEYOND

A conceptual variable is called accessible if it is possible for an agent to assign a numerical value to it by a suitable empirical process; it is called inaccessible if this is not possible. In a physical setting an inaccessible variable might be a vector (position, momentum) or the total spin vector of a particle. A psychological case may be a decision situation where the decision variable is so comprehensive that the agent is unable to make a decision. As in [1], a given inaccessible variable is called $\phi$, and is thought to vary in some space $\Omega_\phi$. A group of transformations $K$ may be supposed to be defined on $\Omega_\phi$.

Accessible functions $\theta = \theta(\phi)$ were studied in [1] and [2], and $\theta$ was assumed to vary on a space $\Omega_\theta$. In a physical situation, $\theta$ can, for instance be the spin component in some given direction; in a decision situation, it can be a simpler decision variable that enables the agent to make a partial decision. If $\phi = (\theta_1, \theta_2)$, where $\theta_1$ and $\theta_2$ are accessible, but $\phi$ is inaccessible, we say that $\theta_1$ and $\theta_2$ are complementary.

Two further notions are of interest:

Definition 1. The variable $\theta$ is maximally accessible if the following holds: If $\theta = f(\xi)$ for a function $f$ which is not one-to-one, but has a surjective inverse, then $\xi$ is inaccessible.

Definition 2. The variable $\theta$ is called permissible if the following holds: $\theta(\phi_1) = \theta(\phi_2)$ implies $\theta(k\phi_1) = \theta(k\phi_2)$ for all $k \in K$.

If $\theta$ is permissible, one can define a group $G$ of actions $g$ on $\Omega_\theta$ such that

$$ (g\theta)(\phi) = \theta(k\phi); \ k \in K. \quad (1) $$

The mapping from $K$ to $G$ defined by (1) is a group homomorphism. The left and right invariant measures under $K$ on $\Omega_\theta$ and correspondingly under $G$ on $\Omega_\theta$ may be defined under weak conditions.

Note that Definition 1 and Definition 2 are precise mathematical definitions, and can be taken as a basis for a precise mathematical theory. The only assumed axiom is that connected to the introduction of the accessibility-notion.

The point of departure in [1] is group representation theory and the theory of coherent states. Let $G$ be a group on $\Omega_\theta$, not necessarily induced by $K$ as above. Assume that $G$ is transitive, has a trivial isotropy group, and that $\theta$ is a maximally accessible variable. Introduce an irreducible unitary representation $U(g)$ on a Hilbert space $\mathcal{H}$ and coherent states $|\theta\rangle = U(g)|\theta_0\rangle$ whenever $\theta = g\theta_0$, $|\theta_0\rangle$ is some fixed vector in $\mathcal{H}$ and $\theta_0 \in \Omega_\theta$ is fixed. From this, a resolution of the identity

$$ I = \int |\theta\rangle\langle\theta| \, d\rho(\theta) \quad (2) $$

is proved in [1]. Here $\rho$ is a left-invariant measure on $\Omega_\theta$, which under weak assumptions [1] will exist. Using this, an operator for the variable $\theta$ is defined as

$$ A = \int \theta|\theta\rangle\langle\theta| \, d\rho(\theta). \quad (3) $$

If several accessible variables are to be discussed, it is useful to write the resolution of the identity in equivalent form

$$ I = \int |\theta(\phi)\rangle\langle\theta(\phi)| \, d\mu(\phi), \quad (4) $$
where $\mu$ is a left-invariant measure on $\Omega_\theta$ if such a measure exists. The operator for $\theta$ can then be written

$$A = \int \theta(\phi)|\theta(\phi)\rangle\langle\theta(\phi)|d\mu(\phi),$$

(5)

If $G$ is not transitive at the outset, a model reduction is first made to an orbit of $G$ on $\Omega_\theta$. This was motivated in [1] by a similar procedure from statistics.

In order that the above development shall lead to a useful theory, several assumptions must be made: 1) $G$ is transitive on $\Omega_\theta$, and $\theta$ is maximally accessible; 2) The representation involved is irreducible; 3) There can be established a one-to-one correspondence between the coherent states $|\theta\rangle = U(g)|\theta_0\rangle$, the group elements $g$ and the values $\theta = g\theta_0$ of the conceptual variable. In Section V below, the assumption 2) will be dropped, but instead it will be assumed that two related maximally accessible variables are present.

Some of the main results of [1] are the following: If $A$ has a discrete spectrum, the possible values of $\theta$ are the eigenvalues of $A$, and $G$ is a subgroup of the group of permutations on these values. Furthermore, $\theta$ is maximally accessible if and only if each eigenspace of $A$ is one-dimensional.

Different (complementary) $\theta$ give different resolutions of the identity. If each such $\theta$ is discrete and maximally accessible, each eigenvector can be interpreted as connected to the question ‘What is the value of $\theta$?’ together with a sharp answer to it. I will also consider operators for non-maximally accessible $\theta$; then the eigenspaces of $A$ have a similar interpretation. The generality of such interpretations for all state vectors was discussed and also taken up as an open question in [4].

The purpose of this paper is to discuss some of this theory in more detail. The remainder of this paper is organized as follows: In Section IV the assumptions behind the theory in [1] are briefly reviewed. Section V gives the main theorem of [1] in a weaker and more precise form, and this theorem is proved in Section VI. In Section VII the properties of the constructed operators are presented, and section VIII treats the spin case. Section IX discusses the epistemic interpretation, and Section X provides a brief final discussion.

The result of Section V implies a completely new approach to quantum theory. This result can be seen as fundamental for the ideas promoted in [1]. The discussion here could ideally replace parts of Chapter 4 in that book, but this Chapter is fairly satisfactory as long as we limit ourselves to epistemic processes.

**IV. TECHNICAL ASSUMPTIONS AND BASIC THEORY**

$\Omega_\phi$ and $\Omega_\theta$ are assumed to be equipped with topologies, and all functions are assumed to be Borel measurable. The groups are assumed to be locally compact. As is common, it is assumed that the group operations $(g_1, g_2) \mapsto g_1g_2$, $(g_1, g_2) \mapsto g_2g_1$ and $g \mapsto g^{-1}$ are continuous. Furthermore, it is assumed that the action $(g, \theta) \mapsto g\theta$ is continuous. An additional assumption, which ensures the existence of invariant measures on $\Omega_\theta$, is the following: Every inverse image of compact sets under the mapping $(g, \theta) \mapsto (g\theta, \theta)$ should be compact. This ensures that a left-invariant measure $\rho$ on $\Omega_\theta$ exists.

Basic group representation theory was reviewed in [1]. It was assumed that the groups $K$ and $G$ have representations that give square-integrable coherent state systems (see [5]). It was assumed further that $G$ is transitive on $\Omega_\theta$ and that the isotropy group is trivial. Then, we can fix $\theta_0 \in \Omega_\theta$, and represent every $\theta \in \Omega_\theta$ by $\theta = g\theta_0$, and from this there is a one-to-one correspondence between $\theta$ and $g$. Let $U$ be a relevant irreducible unitary representation of $G$ on a Hilbert space $\mathcal{H}$. Fix a vector in $\mathcal{H}$ and call it $|\theta_0\rangle$. The coherent states are then given by $U(g)|\theta_0\rangle$, and I define $|\theta\rangle = U(g)|\theta_0\rangle$ when $\theta = g\theta_0$. From this, the resolution of the identity $\mathbb{1}$ is proved in [1], and it is natural to define the operator $A$ corresponding to the variable $\theta$ by $A$.

In general the operator $A$ of $\mathbb{1}$ is unbounded, and is defined on the space of $|u\rangle \in \mathcal{H}$ for which $\langle u|A|u\rangle$ converges.

By the spectral theorem, $A$ has a spectral decomposition. As is proved in [1], if $A$ has a discrete spectrum, the possible values of $\theta$ are the eigenvalues of the operator.

To arrive at quantum theory from this basis, we need a rule to calculate probabilities, the Born rule. A recent review of various derivations of the Born rule is given in [6]. In [1] the rule is derived from three assumptions: 1) a focused likelihood principle, that follows from the ordinary likelihood principle of statistical inference; 2) an assumption of rationality as formulated by the Dutch book principle. 3) The variables involved are maximally accessible. In 2) it is not necessarily assumed that the actor(s) involved is (are) perfectly rational; it is enough that they have ideals which can be modelled in terms of an abstract perfectly rational being.

**V. THE MAIN THEOREM**

Theorem 4.3 in [1] was fundamental for that book. Here we want to prove a more precise and in several respects weaker version of that result:

**Theorem 1.** Consider a situation where there are two maximally accessible conceptual variables $\theta$ and $\xi$ in the mind of an actor or in the minds of a communicating group of actors. Make the following assumptions:

(i) On one of these variables, $\theta$, there can be defined group actions from a transitive group $G$ with a trivial isotropy group and with a left-invariant measure $\rho$ on the space $\Omega_\theta$.

(ii) There exists a unitary irreducible representation $U(\cdot)$ of the group $G$ defined on $\theta$ such that the coherent states $U(g)|\theta_0\rangle$ are in one-to-one correspondence with the values of $\theta$.

(iii) The two maximally accessible variables $\theta$ and $\xi$ can both be seen as functions of an underlying inaccessible variable $\phi \in \Omega_\phi$. There is a transformation $k$ acting on $\Omega_\phi$ such that $\xi(\phi) = \theta(k\phi)$.

Then there exists a Hilbert space $\mathcal{H}$ connected to the situation, and to every accessible conceptual variable there can be associated a unique symmetric operator on $\mathcal{H}$.
It is important that the theorem is no longer limited to an epistemic process; the variables \( \theta \) and \( \eta \) can also be general decision variables, variables connected to sets of prospects as defined in [18].

The condition (iii) seems to be crucial. Two variables \( \theta \) and \( \eta \) satisfying this condition are said to be related. If no such underlying variable \( \phi \) can be found, the two variables are said to be essentially different.

The Hilbert space \( \mathcal{H} \) is of course the Hilbert space associated with the representation (ii).

The condition (ii) is rather technical, but essential for the proof of Theorem 1. The following result may be helpful.

**Proposition 1.** Let \( \rho \) be the left-invariant measure on \( \Omega_\theta \), and assume that

\[
\int |f(g\theta) - f(\theta)|d\rho(\theta) \neq 0
\]

for every \( f \in L^2(\Omega_\theta, \rho) \) and every \( g \in G \) different from the unit element.

Then condition (ii) holds for the case where \( \mathcal{H} \) is a subspace of \( L^2(\Omega, \rho) \) and \( U \) is any irreducible subrepresentation of the regular representation given by \( U(g)f(\theta) = f(g^{-1}\theta) \).

**Proof.** We need to prove that \( U(g_1)f(\theta) \neq U(g_2)f(\theta) \) whenever \( g_1 \neq g_2 \). In the \( L^2 \)-space language and in terms of the regular representation this means that

\[
\int |f(g_2^{-1}\theta) - f(g_1^{-1}\theta)|d\rho(\theta) \neq 0
\]

But this follows from (6), since by left invariance \( d\rho(\theta) = d\rho(g_1^{-1}\theta) \).

A consequence of this is that one in the discrete case may drop the assumption (ii) when \( U(\cdot) \) is irreducible.

**VI. PROOF OF THE MAIN THEOREM.**

**A. Basic construction**

In this subsection I will take as a point of departure Chapter 2 in Perelomov [5], which discusses coherent states for arbitrary Lie groups. Let \( G \) be a transitive group acting on the space \( \Omega_\theta \) associated with some conceptual variable \( \theta \) and \( T(g) \) its unitary representation acting on the Hilbert space \( \mathcal{H} \).

I will assume that \( G \) has a trivial isotropy group, so that the elements \( g \) of \( G \) are in one-to-one correspondence with the values of \( \theta \).

As in [5] (and in [1]) I will choose a fixed vector \( |\theta_0\rangle \) in \( \mathcal{H} \), and consider the set \( \{ |\theta_i\rangle \} \), where \( |\theta_i\rangle = T(g_i)|\theta_0\rangle \) with \( g_i \) corresponding to the value \( \theta_i \). It is not difficult to see that two vectors \( |\theta_1\rangle \) and \( |\theta_2\rangle \) correspond to the same state, i.e., differ by a phase factor \( \exp(i\alpha)|\theta_2\rangle, \exp(i\alpha)|\theta_0\rangle = 1 \), only if \( T(g_2^{-1}g_1)|\theta_0\rangle = \exp(i\alpha)|\theta_0\rangle \), where \( g_1 \) corresponds to \( \theta_1 \) and \( g_2 \) corresponds to \( \theta_2 \). Suppose \( E = \{ e \} \) is a subgroup of the group \( G \), such that its elements have the property

\[
T(e)|\theta_0\rangle = \exp[i\alpha(e)]|\theta_0\rangle.
\]

When the subgroup \( E \) is maximal, it will be called the isotropy subgroup for the state \( |\theta_0\rangle \). More precisely, it is the isotropy subgroup of the group \( T(G) \) corresponding to this state.

The construction shows that the vectors \( |\theta\rangle \) corresponding to a value \( \theta \) and thus to an element \( g \in G \), for all the group elements \( g \) belonging to a left coset class of \( G \) with respect to the subgroup \( E \), differ only in a phase factor and so determine the same state. Choosing a representative \( g(x) \) in any equivalence class \( x \), one gets a set of states \( \{|\theta_{g(x)}\rangle\} \), where \( x \in X = G/E \). Again, using the correspondence between \( g \) and \( \theta \), I will write these states as \( \{|\theta(x)\rangle\} \), or in a more concise form \( \{|x\rangle\}, |x\rangle \in \mathcal{H} \).

**Definition 3.** The system of states \( \{ |\theta\rangle = T(g)|\theta_0\rangle \} \), where \( g \) corresponds to \( \theta \) as above, is called the coherent-state system \( \{ T(\cdot)|\theta_0\rangle \} \). Let \( E \) be the isotropy subgroup for the state \( |\theta_0\rangle \). Then the coherent state \( |\theta(g)\rangle \) is determined by a point \( x = x(g) \) in the coset space \( G/E \) corresponding to \( g \) and to \( |\theta(g)\rangle \) is defined by \( |\theta(g)\rangle = \exp(i\alpha)|x\rangle \), \( |\theta_0\rangle = |0\rangle \).

**Remark.** The states corresponding to the vector \( |x\rangle \) may also be considered as a one-dimensional subspace in \( \mathcal{H} \), or as a projector \( P_x = |x\rangle\langle x| \), \( \dim P = 1 \), in \( \mathcal{H} \). Thus the system of coherent states, as defined above, determines a set of one-dimensional subspaces in \( \mathcal{H} \), parametrized by points of the homogeneous space \( X = G/E \).

The general properties of the coherent state system were studied more closely in [5], and I refer to that discussion here. The important consequence for me is the following: Assume that the representation \( T(\cdot) \) above is irreducible, and that there is a measure on \( G \) which is invariant under left and right shifts. Assume also that convergence conditions are satisfied. Then there exists a measure \( \mu \) in \( X \) such that we have a resolution of unity

\[
\int d\mu(x)|x\rangle\langle x| = I.
\]

(See equation (2.3.5) in [5].) The argument behind (7) is essentially the same that I used in [1] to prove the resolution of the identity (where I only assumed left invariance of the measure on \( G \)): Define an operator \( B = \int dx|x\rangle\langle x| \) and show by using Schur’s lemma that \( B \) must be a constant times the identity operator \( I \).

Recall now two crucial assumptions behind the construction above:

1) \( \{ T(g) \} \) can be chosen as an irreducible representation of the group \( G \) on some Hilbert space \( \mathcal{H} \).

2) There is a one-to-one correspondence between the elements \( g \in G \), the variable values \( \theta \) and the coherent states \( T(g)|\theta_0\rangle \).

If \( G \) is abelian, it only possible to satisfy 1) if \( \mathcal{H} \) is one-dimensional, giving a trivial theory.
But, nevertheless, assuming in general 1) and 2), one can now, again under convergence conditions, define a unique operator in $\mathcal{H}$ corresponding to the conceptual variable $\theta$:

$$A^\theta = \int \theta(x)|x\rangle\langle x|d\mu(x).$$ (10)

Properties of such operators are as studied in [1].

### B. Two maximal conceptual variables

For many concrete examples, it is not possible to find a non-Abelian group satisfying 1) and 2) above, and then the theory of the previous subsection is just trivial. For this case, study two conceptual variables $\theta$ and $\xi$.

Assume that the variables $\theta$ and $\xi$ are maximally accessible, and suppose that there exists a transformation $k$ such that $\xi(\phi) = \theta(k\phi)$. Let $G$ be a transitive group acting on $\theta$, and let $\mathcal{H}$ be the transitive group acting on $\xi$, defined by $h\xi(\phi) = g^1 \theta(k\phi)$ when $\xi(\phi) = \theta(k\phi)$, where $g\in G$, an independent copy of $G$. This gives a isomorphism between $G$ and $\mathcal{H}$.

Let $N$ be the group acting on $\psi = (\theta, \xi)$ generated by $G$ and $H$ and a single element $j$ defined by $j\psi = (\xi, \theta)$ and $j\theta = \xi$. For $g\in G$, define $gj\psi(\phi) = (g\theta(k\phi), \theta(\phi))$ when $\xi = \theta(k\phi)$, and for $h\in H$ define $h\psi = (\xi(\phi), h\xi(k^{-1}\phi))$ when $\theta(\phi) = \xi(k^{-1}\phi)$. Since $G$ and $H$ are transitive on the components, and since through $j$ one can choose for a group element of $N$ to act first arbitrarily on the first component and then arbitrarily on the second component, $N$ is transitive on $\psi$. Also, $N$ is non-Abelian: $gj \neq jg$.

I want to fix some Hilbert space $\mathcal{H}$, and define a representation $U(\cdot)$ of $G$ on this Hilbert space with the property that if we fix some vector $|v_0\rangle \in \mathcal{H}$, then the vectors $U(g)|v_0\rangle$ are in one-to-one correspondence with the group elements $g \in G$ and hence with the values $g\theta_0(\theta)$ for some fixed $\theta_0$.

For each element $g\in G$ there is an element $h = jgj \in H$ and vice versa. Note that $j \cdot j = e$, the unit element. Let $U(j) = J$ be some unitary operator on $\mathcal{H}$ such that $J \cdot J = I$. Then for the representation $U(\cdot)$ of $G$, there is a representation $V(\cdot)$ of $H$ given by $V(jgj) = JU(g)J$. These are acting on the same Hilbert space $\mathcal{H}$ with vectors $|v\rangle$, and they are equivalent in some concrete sense.

Note that $J$ must satisfy $JU(jgj) = U(g)J$. By Schur’s Lemma this demands $J$ to be an isomorphism or the zero operator if the representation $U(\cdot)$ is irreducible. In the reducible case a non-trivial operator $J$ exists, however.

In this case there exists at least one proper invariant subrepresentation $U_0$ acting on some vector space $\mathcal{H}_0$, a subspace of $\mathcal{H}$, and another proper invariant subrepresentation $U_0'$ acting on a vector space $\mathcal{H}_0'$. Fix $|v_0\rangle \in \mathcal{H}_0$ and $|v_0'\rangle \in \mathcal{H}_0'$, and then define $J|v_0\rangle = |v_0'\rangle$, $J|v_0'\rangle = |v_0\rangle$ and $J|v\rangle = |v\rangle$ for any $|v\rangle \in \mathcal{H}$ which is orthogonal to $|v_0\rangle$ and $|v_0'\rangle$.

Now we can define a representation $W(\cdot)$ of the full group $N$ acting on $\psi = (\theta, \xi)$ in the natural way: $W(g) = U(g)$ for $g \in G, W(h) = V(h)$ for $h \in H, W(j) = J$, and then on products from this.

If $U$ is irreducible, then also $V$ is an irreducible representation of $H$, and we can define operators $A^\theta$ corresponding to $\theta$ and $A^\xi$ corresponding to $\xi$ as in (10). If not, we need to show that the representation $W$ of $N$ constructed above is irreducible on $\mathcal{H}$.

**Lemma 1.** $W(\cdot)$ as defined above is irreducible.

**Proof.** Assume that $W(\cdot)$ is reducible, which implies that both $U(\cdot)$ and $V(\cdot)$ are reducible, i.e., can be defined on a lower-dimensional space $\mathcal{H}_0$, and that $E = W(j)$ also can be defined on this lower-dimensional space. Let $R(\cdot)$ be the representation of $G$ restricted to vectors $|v\rangle \in \mathcal{H}_0$ orthogonal to $\mathcal{H}_0$. Fix some vector $|v_0\rangle$ in this orthogonal space; then consider the vectors in this space given by $R(g)|v_0\rangle$. Note that the vectors orthogonal to $\mathcal{H}_0$ together with the vectors in $\mathcal{H}_0$ span $\mathcal{H}$, and the vectors $U(g)|v_0\rangle$ in $\mathcal{H}$ are in one-to-one correspondence with $\theta$. Then the vectors $R(g)|v_0\rangle$ are in one-to-one correspondence with a subvariable $\theta^1$. And define the representation $S(\cdot)$ of $H$ by $S(jg|j\theta\rangle = R(g)$ and vectors $S(h)|v_0\rangle$, where $|v_0\rangle$ is a fixed vector of $\mathcal{H}$, orthogonal to $\mathcal{H}_0$. These are in one-to-one correspondence with a subparameter $\xi^1$ of $\xi$.

Given a value $\theta$, there is a unique element $g\theta \in G$ such that $\theta = g\theta_0(\theta)$. (It is assumed that the isotropy group of $G$ is trivial.)

From this look at the fixed vector $S(jg|j\theta\rangle)|v_0\rangle$. By what has been said above, this corresponds to a unique value $\xi^1$, which is determined by $g\theta$, and hence by $\theta$. But this means that a specification of $\theta$ determines the vector $(\theta, \xi^1)$, contrary to the assumption that $\theta$ is maximally accessible. Thus $W(\cdot)$ cannot be reducible.

This shows that there is a group $N$ acting on $(\theta, \xi)$ and an irreducible representative $W$ of this group acting on some Hilbert space $\mathcal{H}$. Hence the conclusion (2) holds if $G$ above is replaced by $N$ and the irreducible representation is $W(\cdot)$. That is, the crucial assumption 1) is now satisfied. It is left to prove that the Hilbert space $\mathcal{H}$ can be chosen so that condition 2) holds in this case.

Now $N$ is generated by $G, H$ and a group $L$ with two elements $l, \text{ the identity element and } j$. Define a binary variable $\lambda$ such that $\lambda = 0$ if $l$ is the identity, $\lambda = 1$ if $l = j$. First I will establish a one-to-one correspondence between the values of $(\theta, \xi, \lambda)$ and the elements $n$ of the group $N$. Now these elements are partly constructed from group elements $g$ acting on $\theta$, and since $G$ is assumed to be transitive with a trivial isotropy group, there is a one-to-one correspondence between $g$ and $\theta$. The group $H$ constructed in the proof is also transitive on $\Omega_\xi$ with a trivial isotropy group, so there is a one-to-one correspondence between the values $\xi$ and the group elements $h\in H$. Finally, there is a one-to-one correspondence between $\lambda$ and $l$. But $n \in N$ as acting on $(\theta, \xi, \lambda)$ is given as $n = (g, h, j)$, so the required one-to-one correspondence is established.

It is only left to prove that under suitable assumptions the vectors $W(n)|v_0\rangle$ are in one-to-one correspondence with the group elements $n$, and hence with $\xi = (\theta, \xi, \lambda)$. Once this
is proved, by the construction of subsection A one can define coherent states \(|\psi\rangle = W(n)|\psi_0\rangle \in \mathcal{H}\) and a left invariant measure \(\nu(\psi)\) such that

\[
\int |\psi\rangle \langle \psi| \nu(d\psi) = I.
\] (11)

As above, with a change of notation, \(\psi\) are elements of the homogeneous space \(\Psi = N/M\), where \(M\) is the isotropy subgroup of the group \(W(N)\) corresponding to the initial state \(|\psi_0\rangle\). In other words it is the maximal sub-group \(M = \{m\}\) such that

\[
W(M)|\psi_0\rangle = \exp[\alpha(m)]|\psi_0\rangle.
\] (12)

Now \(N\) is a group acting on \((\theta, \xi, \lambda)\) constructed from the groups \(G\) acting on \(\theta\) and \(H\) acting on \(\xi\) and an element \(j\) such that \(j(\theta, \xi) = (j\theta, j\xi)\).

We want to characterize the elements \(z\) of \(N/M\). As in \(8\), the elements of \(M\) are such that \(12\) holds.

**Lemma 2.** We can write \(z = (x,y,l)\), where \(x\) is an element of the homogeneous space \(X = G/E\), \(y\) is an element of the homogeneous space \(Y = H/F\), and \(l\) takes the values 0 and 1. Here \(E\) is the isotropy subgroup of the group \(U(G)\), and \(F\) is the isotropy subgroup of the group \(V(H)\), both corresponding to the initial state \(|\psi_0\rangle \in \mathcal{H}\).

**Proof.** For \(e \in E\) we have

\[
W(e)|\psi_0\rangle = \exp[i\alpha(e)]|\psi_0\rangle,
\] (13)

while for \(f \in F\) we have

\[
W(f)|\psi_0\rangle = \exp[i\alpha(f)]|\psi_0\rangle.
\] (14)

Normalizing \(\alpha(m)\) for each \(m\) such that \(\alpha(j) = 0\), \(12\) follows from \(13\) and \(14\).

Now the cosets \(nM\) are found by first specializing \(n\) to \(g\) and \(h\), respectively. \(j\) have \(j(\theta, \xi) = (j\theta, j\xi)\), and the cosets \(gM\) and \(hM\) are given in terms of \(x\) and \(y\). It follows that \(z = (x,y,l)\) if we in addition allow \(jM\) to interchange \(x\) and \(y\). \(\square\)

**C. The operators**

By the symmetry of the situation, and since \(G\) and \(H\) act independently on \(\theta\) and \(\xi\) in the construction of the group \(N\), we conclude that the measure \(\nu\) in \(11\) can be written as \(\nu(d\psi) = \rho(dx)|\psi\rangle \langle \psi| \omega(l)\) for some marginal measure \(\rho\), where \(\omega\) is any measure on the two variables 0 and 1. Then we can define

\[
P(x) = \sum_{l} \int_{Y} |\psi\rangle \langle \psi| \rho(dy)|\omega(l)|
\] (15)

and

\[
A^\theta = \int_{X} \theta(x) P(x) \rho(dx).
\] (16)

Similarly:

\[
Q(y) = \sum_{l} \int_{X} |\psi\rangle \langle \psi| \rho(dx) \omega(l),
\] (17)

and

\[
A^\xi = \int_{Y} \xi(y) Q(y) \rho(dy).
\] (18)

It follows from \(11\) that \(A^\theta = I\) when \(\theta\) is identically equal to 1.

This define the operators for every pair of maximaly accesible conceptual variables \(\theta\) and \(\xi\). The operators for variables that are not maximal, are found by using the spectral theorem, taking as a point of departure the operator for a corresponding maximal variable (cp. equations (4.28) and (4.30) in \[1\]).

**VII. PROPERTIES OF THE OPERATORS**

To continue the theory, we prove the important result (cp. Theorem 4.2 in \[1\]):

**Theorem 2.** For any transformation \(t\) of \(\Omega_\phi\) and any unitary representation \(V\) on which \(V(t)\) is defined, the operator \(V(t)^*A^\theta V(t)\) is the operator corresponding to \(\theta^\prime(\phi) = \theta(t\phi)\).

**Proof.** By \(16\) we have

\[
V(t)^*A^\theta V(t) = \int_{X} \theta(x) P(x) \rho(dx),
\] (19)

where

\[
P_t(x) = \sum_{l} \int_{Y} |\psi(t^{-1} \xi)| \langle \psi(t^{-1} \xi)| \rho(dy) \omega(l).
\] (20)

Here \(|\psi(t^{-1} \xi)|\) are the ket vector is constructed such that they are in one-to-one correspondence with \(t^{-1} \phi = t^{-1}(\theta, \xi)\) and hence with \(t^{-1}(g,h)\). Note that \(t^{-1} \phi\) and \(\theta = \theta(\phi)\), \(\xi = \xi(\phi)\) defines \(t^{-1}\xi\). Write the group elements \(t^{-1}(g,h)\) as \((g', h')\), constituting groups \(G'\) and \(H'\) acting on \(\theta\) and \(\xi\), respectively. Let \(E'\) be the subgroup of \(G'\) constructed as in \(8\), and let \(F'\) be the corresponding subgroup of \(H'\). Write \(X' = G'/E'\) and \(Y' = H'/F'\). The elements of these cosets may be defined as \(x' = t^{-1} x\) and \(y' = t^{-1} y\), respectively. By the left invariance of the measure \(\rho\), \(19\) and \(20\) can be written as

\[
V(t)^*A^\theta V(t) = \int_{X} \theta(x) P_t(x) \rho(d t^{-1} x),
\] (21)
where

\[ P_l(x) = \sum_f \int \psi(t^{-1} \xi) \langle \psi(t^{-1} \xi) | \rho(dt^{-1}y) \omega(l) \rangle. \]  

(22)

The last integral may be written

\[ P_l(x) = \sum_f \int \psi(\theta(t^{-1} \phi), \xi(\phi')) \langle \psi(\theta(t^{-1} \phi), \xi(\phi')) | \rho(dy') \omega(l) \rangle, \]  

(23)

and we can write \( P_l(x) = P(t^{-1} x) \). This is inserted into (21), and using left-invariance of the measure again, this gives that the operator \( V(t^{-1}) A^\theta V(t) \) is associated with the conceptual variable \( \theta(x) \), which also may be written as \( \theta(t \phi) \).

By using this result in the same way as Theorem 4.2 is used in [1], a rich theory follows. I will limit me here to the case where \( \theta \) is a discrete conceptual variable. Then one can show:

1. The eigenvalues of \( A^\theta \) coincide with the values of \( \theta \).
2. The variable \( \theta \) is maximally accessible if and only if the eigenvalues of \( A^\theta \) are non-degenerate.
3. For the maximal case the following holds in a given context: a) For a fixed \( \theta \) each question ‘What is the value of \( \theta \)?’ together with a sharp answer ‘\( \theta = \theta' \)’ can be associated with a normalized eigenvector of the corresponding \( A^\theta \). b) If there is in the context a set \( \{ \theta^a; a \in \mathcal{A} \} \) of maximally accessible conceptual variables (these must by the results of Section IV related to each other) one can consider all ket vectors that are normalized eigenvectors of some operator \( A^{\theta^a} \). Then each of these may be associated with a unique question-and-answer as above.

VIII. THE SPIN CASE

Let \( \phi \) be the total spin vector of a particle, and let \( \theta^a = \| \phi \| \cos(\phi, a) \) be the component of \( \phi \) in the direction \( a \). If we have a coordinate system, the components \( \theta^x, \theta^y \) and \( \theta^z \) are of special interest. As noted in [1], these components are not permissible if \( K \) is the rotation group for some fixed \( | \phi | \).

Consider a Stern-Gerlach experiment with a beam of particles in the \( y \) direction. Write \( \phi = (\phi^x, \theta^y) \) where \( \phi^x \) is the spin component in the \( xz \)-plane. Let \( K_0 \) be the group of rotations of \( \phi^x \) in this plane for fixed \( | \phi^x | \).

**Proposition 2.** Any component \( \theta^a \) in the \( xz \)-plane is a permissible function of \( \phi^x \) under \( K_0 \).

**Proof.** It is sufficient to consider \( \theta^x \). Let \( k_0 \in K_0 \) and \( \theta^x(\phi^x_1) = \theta^x(\phi^x_1) \). The group element \( k_0 \) rotates the \( x \)-axis to a new direction \( x_0 \) and \( \phi^x_1 \) to \( \phi^x_1 \) and \( k_0 \phi_0^x \) to \( k_0 \phi_0^x \). By the geometry, the angle \( (x, \phi^x_1) \) must be equal to angle \( (x_0, k_0 \phi_0^x) \), and angle \( (x, \phi^x_1) \) must be equal to angle \( (x_0, k_0 \phi_0^x) \). Because the cosines of the unrotated angles are equal by assumption, the cosines of the rotated angles must also be equal.

Unit vectors in a plane as quantum state vectors have been discussed extensively in [8]. These vectors can be seen as elements of a real 2-dimensional Hilbert space, and they are uniquely determined by the angle they form with the \( x \)-axis. Using this as a point of departure, density matrices and quantum operators (matrices) were defined in [8]. The authors noted that all real 2-dimensional matrices were generated by the unit matrix and the two real Pauli spin matrices \( \sigma_1 \) and \( \sigma_2 \).

Spin coherent states and their connection to the group \( SU(2) \) are treated for instance in [5] and [9]. I will follow parts of [5] without going into details. It is crucial that any irreducible representation \( D \) is given by a nonnegative integer or half-integer \( r \): \( D(k) = D'(k), dimD' = 2r + 1 \). In the representation space \( \mathcal{H} = \mathcal{H}' \), the canonical basis \( |r; m \rangle \) exists, where \( m \) runs from \(-r \) to \( r \) in unit steps. The infinitesimal operators \( A^x = A^x \pm A^y, A^0 = A^z \) of the group representation \( V' \) satisfy the commutation relations

\[ [A^0, A^\pm] = \pm A^\pm, \quad [A^-, A^+] = -2A^0. \]

(24)

The operators \( A^x, A^y \) and \( A^z \) are related to infinitesimal rotations around the \( x \)-axis, \( y \)-axis and \( z \)-axis, respectively, and may be shown to be identical to the operators associated with \( \theta^x, \theta^y \) and \( \theta^z \), as considered in the previous sections here. We take \( A = (A^x, A^y, A^z) \). The representation space vectors \( |r; m \rangle \) are eigenvectors for the operators \( A^0 \) and \( A^2 = (A^x)^2 + (A^y)^2 + (A^z)^2 \):

\[ A^0 |r; m \rangle = m |r; m \rangle, \quad A^2 |r; m \rangle = (r + 1) |r; m \rangle. \]

(25)

The operator \( \exp[i\omega(n \cdot A)] \), \( ||n|| = 1 \), describes the rotation by the angle \( \omega \) around the axis directed along \( n \). In [5] this was used to describe the coherent states \( D(k) |\phi_0 \rangle \) in various ways. The ket vector \( |\phi_0 \rangle \) may be taken as \( |r; m \rangle \), for a fixed \( m \); the simplest choice is \( m = -r \).

The spin case is discussed more thoroughly in [1]. Among other things, the discretization of spin components is motivated from a general principle of model reduction.

IX. THE EPISTEMIC INTERPRETATION

Consider a physical system, and an observer or a set communicating observers on this system. The physical variables which can be measured in this setting are examples of accessible conceptual variables, and are called e-variables in [1] and [2].

The approach here should be compared to parts of various recent derivations of quantum theory from a set of postulates (see, for instance, [12] and [13]). As is stated in [14], there is a problem connecting these derivations to the many different interpretations of quantum theory. By contrast, the derivation presented here is tied to a particular interpretation: A general epistemic interpretation. This is also elaborated on in [1].

A maximally accessible variable \( \theta^a \) admits values \( \theta^a \) that are single eigenvalues of the operator \( A^a \), uniquely determined from \( \theta^a \). Let \( |a; j \rangle \) be the eigenvector associated with this eigenvalue. Then \( |a; j \rangle \) can be connected to the question.
What is the value of \( \theta^a \)? together with the sharp answer \( \theta^a = a_i^b \). A general ket vector \( |\psi\rangle \in \mathcal{H} \) is always an eigenvector of some operator associated with a conceptual variable. It is natural to conjecture that this operator always can be selected in such a way that the accessible variable is maximally accessible. Then \( |\psi\rangle \) is in a natural way associated with a question-and-answer pair. It is of interest that Höhn and Wever [11] recently derived quantum theory for sets of qubits from such question-and-answer pairs; compare also the present derivation. Note that the interpretation implied by such derivations is relevant for both the preparation phase and the measurement phase of a physical system.

One may also consider linear combinations of ket vectors from this point of view. Let the maximally accessible variable \( \theta^b \) admits values \( a_i^b \) that are single eigenvalues of the operator \( A^b \), uniquely determined from \( \theta^b \). Let \( |b;i\rangle \) be the eigenvector associated with this eigenvalue. Then one can write

\[
|b;i\rangle = \left( \sum_j (a;j) \langle a;j\rangle |b;i\rangle \right) = \sum_j (a;j) |b;i\rangle |a;j\rangle. \tag{26}
\]

This state can be interpreted in terms of a question ‘What is the value of \( \theta^b \)?’ with the sharp answer \( \theta^b = a_i^b \). But if one tries to ask questions about \( \theta^a \) for a system where the observer or the set of observers is in this state, the answer is simply ‘I (we) do not know’.

From a general point of view it may be considered of some value to have an epistemic interpretation which is not necessarily tied to a strict Bayesian view (see for instance [15] on this). Under an epistemic interpretation, one may also discuss various “quantum paradoxes” like Schrödinger’s cat, Wigner’s friend and the two-slit experiment.

Example 1. Schrödinger’s cat. The discussion of this example concerns the state of the cat just before the sealed box is opened. Is it half dead and half alive?

To an observer outside the box the answer is simply: “I do not know”. Any accessible e-variable connected to this observer does not contain any information about the status of life of the cat. But on the other hand – an imagined observer inside the box, wearing a gas mask, will of course know the answer. The interpretation of quantum mechanics is epistemic, not ontological, and it is connected to the observer. Both observers agree on the death status of the cat once the box is opened.

Example 2. Wigner’s friend. Was the state of the system only determined when Wigner learned the result of the experiment, or was it determined at some previous point?

My answer to this is that at each point in time a quantum state is connected to Wigner’s friend as an observer and another to Wigner, depending on the knowledge that they have at that time. The superposition given by formal quantum mechanics corresponds to a ‘do not know’ epistemic state. The states of the two observers agree once Wigner learns the result of the experiment.

Example 3. The two-slit experiment. This is an experiment where all real and imagined observers communicate at each point of time, so there is always an objective state.

Look first at the situation when we do not know which slit the particle goes through. This is a ‘do not know’ situation. Any statement to the effect that the particles somehow pass through both slits is meaningless. The interference pattern can be explained by the fact that the particles are (nearly) in an eigenstate in the component of momentum in the direction perpendicular to the slits in the plane of the slits. If an observer finds out which slit the particles goes through, the state changes into an eigenstate for position in that direction. In either case the state is an epistemic state for each of the communicating observers, which might indicate that it in some sense can be seen as an ontological state. But this must be seen as a state of the screen and/or the device to observe the particle, not as an ontological state of the particle itself.

X. DISCUSSION

Some of the results above were limited to accessible conceptual variables taking a finite number of values. But the notion of conceptual variables is much more general, and can thus be considered in more general settings.

This notion of conceptual variables also has links to other interpretations of quantum theory. Take for instance the classical Bohm interpretation, constructed from particle trajectories plus a pilot wave. These constructions may be seen as conceptual variables, but at least the full trajectory must be inaccessible. Or consider the many worlds interpretation, where the variables associated with the different worlds must be considered as conceptual variables, but only one world is accessible at each point in time.

The arguments developed here are based on focusing (choice of \( a \)) and symmetry. These concepts are not confined to the microscopic world. This is consistent with the fact that quantum theory has recently been applied to cognitive models and to certain social and economic models (see [3] and [16] and references there). In fact it has been claimed that the quantum structure is ubiquitous, see [17].

Further philosophical consequences of these results are discussed in Chapter 6 of [1]. In general it is referred to [1] for further discussions.

It is quite fundamental that the proof of Theorem 1 as given here is correct. This theorem not only serves as a basis for the derivation of quantum theory as discussed in [1], it has also now been used in a for discussion of the Bell theorem, the Bell experiment and a general assertion of our limitation of human beings when making decisions [19]. I am currently working with using the same theorem as a point of departure for discussing when a quantum state also has an ontologic interpretation and the more specular question of whether all quantum states can be said to have ontological interpretations [20]. Note that in [1] and in the present paper, the initial interpretation of states is epistemological.

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