Geometry-Based Conditions for a Quadratic Function: Application to Stability of Time-Varying Delay Systems

TAE H. LEE (Member, IEEE)
Division of Electronic Engineering, Jeonbuk National University, Jeonju 54896, South Korea
e-mail: fesuslee@gmail.com

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Abstract In this paper, novel geometry-based negative conditions are presented for a quadratic function with time-varying delays. The system performance could be optimal by adjusting a parameter value. New stability criteria for the time-varying delayed systems are derived.

Index Terms Matrix-refined-function, quadratic function, stability, time-varying delay.

I. INTRODUCTION

From the several decades, time delays have been a hot issue in both theory and industry because it is unavoidable but poisons systems. Therefore, it is essential to fulfill the stability of the systems on time delays (see [1] and references therein). The combination of Lyapunov-Krasovskii functional (LKF) and linear matrix inequality (LMI) a most preferable method for the stability analysis of time delay systems. In this concept, to derive less conservative stability criteria is the first aim, in other word to form such LMI conditions to be feasible under certain value of time delay as larger as possible (see [2] and references therein). To this end, there are numerous approaches have been developed such as delay partitioning approach [3], free-weighting matrix approach [4], relaxing LMI variable conditions [5], [6], constructing new LKFs [7]–[9], techniques for handling reciprocal convexity [10]–[13].

Since the high efficiency in reducing conservatism of the stability criteria, the development of new integral inequalities which estimate tighter bound of those inequalities has been attracted much attention in the past decade such as Wirtinger-based integral inequality [14], free-weighting matrices one [15], auxiliary function-based one [16], free-matrix-based one [17], Bessel-Legendre-based one [18] (hereafter those are referred to as the advanced integral inequalities). As the result, the use of one of the advanced integral inequalities is indispensible for less conservative stability criteria. The advanced integral inequalities contain the delay dependent vector in their estimated bound.

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For simple explanation, let us consider the following time-varying delay systems:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bx(t-h(t)), \\
x(t) &= \phi(t), \quad t \in [-h, 0]
\end{align*}
\]  

(1)

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( \phi(t) \) is a compatible vector valued initial function, and \( A, B \) are known real constant matrices with appropriate dimensions, \( h(t) \) is the time-varying delay satisfying \( 0 \leq h(t) \leq h \) and \( \dot{h}(t) \leq \mu < 1 \) for \( \forall t \geq 0 \).

The use of the advanced integral inequalities for the system (1) basically contain vectors \( \frac{1}{h(t)} \int_{1-h}^{t} v(s)ds \) or \( \frac{1}{(h-h(t))} \int_{t-h}^{t-h(t)} v(s)ds \) where \( v(t) \) is a certain vector consisted with \( x(t) \). In case of the advanced integral inequalities with higher order terms, they have higher order vectors like \( \frac{1}{h(t)} \int_{1-h}^{t} \int_{u_1}^{t} \cdots \int_{u_{m-1}}^{t} v(s)dsdu_{m-1} \cdots du_1 \) or \( \frac{1}{(h-h(t))} \int_{t-h}^{t-h(t)} \int_{u_1}^{t-h(t)} \cdots \int_{u_{m-1}}^{t-h(t)} v(s)dsdu_{m-1} \cdots du_1 \) in their estimated bound. The utilization of the advanced integral inequalities with higher order terms or together with the augmented LKF and delay-product-type LKF would yield the square of time-varying delay dependent terms, which could formulate LMI conditions in the form of the following quadratic function: \( f(h^2(t), h(t)) = h \dot{h}^2(t) + 2h \dot{h}(t) + C \).

In order to guarantee the negative of the quadratic function, many works check the negative of its’ vertices, i.e. \( f(h^2, h) < 0, f(0, h) < 0, f(h^2, 0) < 0, f(0, 0) < 0 \) [9]. Just a few results on developing negative conditions of a quadratic function have been reported till now (see Section 3 for more discussion) [20]–[25].

This situation motivate us to develop a set of new negative conditions of a quadratic function. The proposing negative
conditions are developed the geometrical information of a quadratic function, and also containing an adjustable parameter to tune the condition to be optimal. Furthermore, a new LKF which is augmented-delay-product-type in addition to the matrix-refined-function based LKF is proposed which make the stability criteria more dependent on the square of the time-varying delay. As the results, the combination of novel conditions for a quadratic function and new LKFs would produce a great synergy effect on reducing conservatism. Finally the outstanding of the proposed methods is described by three numerical examples.

**Notations:** $\mathbb{R}^n$ is the $n$-dimensional Euclidean space. $\mathbb{S}^n$ (respectively, $\mathbb{S}^n_+$) denotes a set of symmetric matrices (respectively, positive definite matrices) with $n \times n$ dimensions. $X > 0$ (respectively, $X \succeq 0$) means that the matrix $X$ is a real symmetric positive definite matrix (respectively, positive semi-definite). $I$ denotes the identity matrix with appropriate dimensions. $\text{diag} \{ \cdots \}$ denotes block diagonal matrix. $\ast$ in a matrix represents the elements below the main diagonal of a symmetric matrix. $\text{Sym}[X]$ indicates $X + X^T$. For $X \in \mathbb{R}^{m \times n}$, $X^\perp$ denotes a basis for the null-space of $X$.

**II. PRELIMINARY STABILITY CRITERION**

To derive our theorem, the following lemma is necessary.

**Lemma 1 [15]:** For an integrable function $x(t)$ in $[a, b]$, any matrices $L$, $M$ and $N$ with appropriate dimensions, and a positive definite matrix $R$, the following inequality holds

$$-\int_a^b x^T(s)Rx(s)ds \leq \text{Sym}[v^T(L\chi_1 + M\chi_2 + N\chi_3)] + (b - a)v^T\left( LR^{-1}L + \frac{1}{3}MR^{-1}M + \frac{1}{3}NR^{-1}N \right)v,$$

where $v$ is any vector, $\chi_1 = \int_a^b x(s)ds$, $\chi_2 = -\chi_1 + 2\kappa_1$, $\chi_3 = \chi_1 - 6\kappa_1 + 12\kappa_2$, $\kappa_1 = \frac{1}{b-a}$, $\kappa_2 = \frac{1}{6(b-a)^2}$.

Now, let us begin to define the following notations which will be used for our main results.

$$\xi_1(t) = \begin{bmatrix} \eta_1^T(t) & \bar{\eta}_1^T(t) \end{bmatrix}^T,$$

$$\xi_2(t) = \begin{bmatrix} x^T(t) & \dot{x}^T(t) \end{bmatrix}^T,$$

$$\xi_3(t) = \begin{bmatrix} x^T(t - h(t)) & \dot{x}^T(t - h(t)) \end{bmatrix}^T,$$

$$\eta_1(t) = \begin{bmatrix} x^T(t) & x^T(t - h(t)) \end{bmatrix}^T,$$

$$\eta_2(t) = \begin{bmatrix} x^T(t - h(t)) & \dot{x}^T(t - h(t)) \end{bmatrix}^T,$$

$$\eta_3(t) = \begin{bmatrix} \frac{v_1^T(t)}{h(t)} & \frac{v_2^T(t)}{h(t)} \end{bmatrix}^T,$$

$$\bar{\eta}_3(t) = \begin{bmatrix} \frac{v_1^T(t)}{h(t)} & \frac{v_2^T(t)}{h(t)} \end{bmatrix}^T,$$

$$\bar{\eta}_h(t) = \dot{h} - h(t),$$

$$h_d(t) = 1 - \dot{h}(t),$$

$$v_1(t) = \int_{-h(t)}^t x(s)ds,$$

$$v_2(t) = \int_{-h(t)}^t x(s)ds,$$

$$v_3(t) = \int_{-h(t)}^t \int_{-h(t)}^t x(s)dsdv,$$

$$v_4(t) = \int_{-h(t)}^t \int_{-h(t)}^t x(s)dsdv,$$

$$\zeta(t) = \left[ \eta_1^T(t), \eta_2^T(t), \eta_3^T(t) \right]^T.$$

In addition, we define $k_i \in \mathbb{R}^{10 \times n}$ ($i = 1, 2, \ldots, 10$) are block entry matrices, e.g. $k_2^T = [0, I_n, [0, \ldots, 0]]$.

For the system (1), let us consider a LKF $V(t) = \sum_{i=1}^8 V_i(t)$ defined as follows:

$$V_1(t) = \xi_1^T(t)(h(t)P_1 + h_d(t)P_2)\xi_1(t),$$

$$V_2(t) = \int_{-h(t)}^t \xi_2^T(t)Q_1\xi_2(t)ds + \int_{-h}^{t-h(t)} \xi_2^T(t)Q_2\xi_2(t)ds,$$

$$V_3(t) = \int_{-h}^{t-h(t)} \dot{x}^T(t)R\dot{x}(s)ds,$$

$$V_4(t) = h(t)\int_{-h(t)}^t \dot{x}^T(t)X_6\dot{x}(s)ds + \xi_3^T(t)(h(t)X_1 + \dot{X}_2)\xi_3(t) + \eta_2^T(t)h_b(t)\xi_2(t) + \dot{\xi}_2^T(t)h_b(t)\xi_2(t),$$

where $P_1 = \begin{bmatrix} P_{11} & * \\ * & P_{13} \end{bmatrix}$, $P_2 = \begin{bmatrix} P_{21} & * \\ * & P_{23} \end{bmatrix}$, $Q_1$, $Q_2$, $R$, are positive definite matrices, $X_1$, $X_4$, $X_6$, $Y_1$, $Y_4$, $Y_6$ are symmetric matrices, $X_2$, $X_3$, $X_5$, $Y_2$, $Y_3$, $Y_5$ are any matrices with appropriate dimensions, and

$$X_1 = X_1 + \frac{1}{3}X_4, \quad Y_1 = Y_1 + \frac{1}{3}Y_4,$$

$$X_2 = \text{Sym}\{[X_3, -X_3, 0] + [-X_5, -X_5, 2X_5]\},$$

$$Y_2 = \text{Sym}\{[Y_3, -Y_3, 0] + [-Y_5, -Y_5, 2Y_5]\},$$

satisfying

$$X = \begin{bmatrix} X_1 & X_2 & X_3 \\ * & X_4 & X_5 \\ * & * & X_6 \end{bmatrix} > 0,$$

$$Y = \begin{bmatrix} Y_1 & Y_2 & Y_3 \\ * & Y_4 & Y_5 \\ * & * & Y_6 \end{bmatrix} > 0.$$
Then, the time derivative of LKF $V(t)$ can be estimated as

$$\dot{V}(t) \leq \sum_{i,j} h_i h_j (C_{ij}(t) + C_{ij}(t_0)),$$

where all notations are defined in the bottom of this page.

By Lemma 1, the integral terms in $V(t)$ and $\dot{V}(t)$ can be further estimated as

$$\int_0^t \ddot{V}(s) \, ds \leq \int_0^t \ddot{V}(s) \, ds \leq \int_0^t \ddot{V}(s) \, ds + \int_0^t \ddot{V}(s) \, ds,$$

where all notations are defined in the bottom of this page.
From Eqs. (7-11), \( \dot{V}(t) \) can be estimated as
\[
\dot{V}(t) \leq \xi^T(t) \left( h^2(t) \mathcal{A}(h(t)) + h(t) \mathcal{B}(h(t)) + \mathcal{C}(h(t)) \right) \xi(t),
\]
where \( \mathcal{A}(\dot{h}(t)) = \sum_{i=1}^2 A_i(\dot{h}(t)), \mathcal{B}(\dot{h}(t)) = \sum_{i=1}^3 B_i(\dot{h}(t)), \) and \( \mathcal{C}(\dot{h}(t)) = \sum_{i=1}^5 C_i(\dot{h}(t)) + k_4 h R k_4^T. \) Then by Finsler’s lemma with a matrix \( \Gamma = \begin{bmatrix} A, B, 0_{n \times n}, -I, 0_{n \times 6n} \end{bmatrix} \), \( \dot{V}(t) < 0 \) is equivalent to
\[
f(h^2(t), h(t), \dot{h}(t)) = (\Gamma^\perp)^T \left( h^2(t) \mathcal{A}(h(t)) + h(t) \mathcal{B}(h(t)) + \mathcal{C}(h(t)) \right) \Gamma^\perp < 0 \quad (12)
\]

**Remark 1:** The newly constructed LKF, \( V_1(t) \), and matrix-refined function based LKF, \( V_4(t) \) which is originated from [26] are produce terms depending on the square of time-varying delay in its’ differentiation. This implies the derived stability criteria based on the LKF are more affected from the technique for the square of time-varying delay, and also may lead less conservatism.

III. MAIN RESULTS

At the previous section, we derived a matrix-valued quadratic function depending on the time-varying delay, square of the time-varying delay, and time derivative of the time-varying delay. In order to guarantee the stability of the system (1), we should prove the negativeness of the function, i.e. \( f(h^2(t), h(t), \dot{h}(t)) < 0 \). This section proposes three stability criteria by using existing conditions for a quadratic function in the first subsection. In second subsection, we propose a stability criteria based on a novel condition for a quadratic function which is main contribution of this paper.

A. STABILITY CRITERIA BY EXISTING NEGATIVE CONDITIONS FOR A QUADRATIC FUNCTION

For a quadratic function, we introduce the following three existing conditions.

**Lemma 2:** Let us consider a quadratic function \( g(x) = ax^2 + bx + c \), where \( a, b, c \in \mathbb{R} \), if \( g(x) \) satisfies the one of conditions among the following A, B, C.

A. [20] (i) \( g(\dot{h}_1) < 0 \), (ii) \( g(h_2) < 0 \), (iii) \( h_2^2 + g(\dot{h}_1) < 0 \)

B. [21] (i) \( g(\dot{h}_1) < 0 \), (ii) \( g(h_2) < 0 \), (iii) \( -\beta^2 h_2^2 + g(\dot{h}_1) < 0 \)

C. [23] (i) \( g(\dot{h}_1) < 0 \), (ii) \( g(h_2) < 0 \), (iii) \( -\beta^2 h_2^2 a + g(\dot{h}_1) < 0 \), (iv) \( -(1 - \beta^2)(h_2 - h_1)^2 a + g(h_2) < 0 \) where \( \beta \in [0, 1] \)

Then \( g(x) < 0 \) for \( \forall x \in [h_1, h_2] \).

Based on Lemma 2, we can obtain the following theorem for Eq. (12):

**Theorem 1:** For given positive constants \( h, \mu, \beta \in [0, 1] \), the system (1) is asymptotically stable if there exist matrices \( P_1, P_2, X, Y \in \mathbb{S}^n_+ \), \( Q_1, Q_2 \in \mathbb{S}^n_+ \), \( R \in \mathbb{S}^n_+ \), \( L_i \in \mathbb{R}^{10 \times n} \) (\( i = 1, 2, 3 \)) satisfying the following LMIs A, B, or C: \( \forall h \in [-\mu, \mu] \)

A. \( \Omega(h(t)) < 0 \), (i) \( 1, 2, 3 \), (13)
B. \( \Omega(h(t)) < 0 \), (i) \( 1, 2, 4 \), (14)
C. \( \Omega(h(t)) < 0 \), (i) \( 1, 2, 5, 6 \), (15)

where
\[
\begin{align*}
\Omega_1(\dot{h}(t)) &= \tilde{\Omega}_1(0, 0, \dot{h}(t)), \\
\Omega_2(\dot{h}(t)) &= \tilde{\Omega}_2(h^2, h, \dot{h}(t)), \\
\Omega_3(\dot{h}(t)) &= \tilde{\Omega}_3(0, h, \dot{h}(t)), \\
\Omega_4(\dot{h}(t)) &= \tilde{\Omega}_4(-h^2, 0, \dot{h}(t)), \\
\Omega_5(\dot{h}(t)) &= \tilde{\Omega}_5(-\beta^2 h^2, 0, \dot{h}(t)), \\
\Omega_6(\dot{h}(t)) &= \tilde{\Omega}_6(h^2, h, \dot{h}(t)), \\
\tilde{\Omega}_1(h_a, h_b, \dot{h}(t)) &= \begin{bmatrix} f(h_a, h_b, \dot{h}(t)) (\Gamma^\perp)^T h M & \ast \\ \ast & -h R_2 \end{bmatrix}, \\
\tilde{\Omega}_2(h_a, h_b, \dot{h}(t)) &= \begin{bmatrix} f(h_a, h_b, \dot{h}(t)) (\Gamma^\perp)^T h L & \ast \\ \ast & -h R_1 \end{bmatrix}, \\
\tilde{\Omega}_3(h_a, h_b, \dot{h}(t)) &= \begin{bmatrix} f(h_a, h_b, \dot{h}(t)) (\Gamma^\perp)^T \tilde{h} & \ast \\ \ast & -h \tilde{R} \end{bmatrix}, \\
\tilde{C}(\dot{h}(t)) &= \sum_{i=1}^2 \tilde{C}_i(\dot{h}(t)) + k_4 h R k_4^T, \\
\mathcal{R}_1(\dot{h}(t)) &= \text{diag} \left\{ R_1(\dot{h}(t)), 3 \tilde{R}_1(\dot{h}(t)), 5 \tilde{R}_1(\dot{h}(t)) \right\}, \\
\mathcal{R}_2(\dot{h}(t)) &= \text{diag} \left\{ R_2(\dot{h}(t)), 3 \tilde{R}_2(\dot{h}(t)), 5 \tilde{R}_2(\dot{h}(t)) \right\}, \\
L &= \begin{bmatrix} L_1 & L_2 & \ldots & L_3 \end{bmatrix}, \\
M &= \begin{bmatrix} M_1 & M_2 & \ldots & M_3 \end{bmatrix}.
\end{align*}
\]

**Proof:** By Lemma 2.A, B, and C with \( h_1 = 0 \) and \( h_2 = h \) and Schur complement, we can easily obtain LMIs (13), (14), and (15), respectively, which imply \( f(h^2(t), h(t), \dot{h}(t)) < 0 \), in other word, \( \dot{V}(t) < 0 \). Therefore, the system (1) is stable by Lyapunov stability theory if the one of LMIs (13), (14), or (15) are held. This completes the proof.

B. STABILITY CRITERIA BY NOVEL NEGATIVE CONDITIONS FOR A QUADRATIC FUNCTION

Among the existing conditions, Lemma 2A, B, C, only Lemma 2B is the geometric information based one. So, we firstly explain it, and then propose a new geometry-based negative conditions of a quadratic function.

Recall a quadratic function \( g(x) = ax^2 + bx + c \) with \( a, b, c \in \mathbb{R} \). First of all, the function \( g(x) \) is a convex function when \( a > 0 \), so in order to show \( g(x) < 0 \) in case of \( a > 0 \) for \( \forall x \in [h_1, h_2] \), \( g(h_1) < 0 \) and \( g(h_2) < 0 \) are only conditions. On the other hand, the function \( g(x) \) is a concave function when \( a < 0 \), for a concave function we consider three cases according to the location of the maximum point: Case i) \( \max_x g(x) \) is in \( x \in (h_2, \infty) \); Case ii) \( \max_x g(x) \) is in \( x \in (-\infty, h_1) \); Case iii) \( \max_x g(x) \) is at \( x \in [h_1, h_2] \). Then the maximum points on \( g(x) \) in \( x \in [h_1, h_2] \) for the above three cases are \( (h_2, g(h_2)), (h_1, g(h_1)) \), and \( (-\frac{b}{2a}, g(-\frac{b}{2a})) \), respectively. Showing these maximum points have a negative value in \( y \)-axis is equivalent to \( g(x) < 0 \) for \( \forall x \in [h_1, h_2] \) in case of \( a < 0 \).
(See Fig. 1) Lemma 2.B implies that if the $y$–axis value of the three points $B_1$, $B_2$, $B_1$ are negative, then $g(x) < 0$ for $\forall x \in [h_1, h_2]$. In which, the points $B_1$ and $B_2$ are points on $g(x)$ at $x = h_1$ and $x = h_2$, respectively, and the point $B_1$ is an intersection point of $y$–axis and a line $l_{h_2}(x)$ which is a tangent line to $g(x)$ at $x = h_2$.

Let us consider two points, $B_1$ and $B_{\text{max}}$ where $B_{\text{max}}$ is maximum point on $g(x)$ for $x \in [h_1, h_2]$. For Case i), Lemma 2.B is best to ensure $g(x) < 0$ for $\forall x \in [h_1, h_2]$ because of $B_1$ is a maximum point of $g(x)$ of a point $X$. However, for Case ii) and iii), $B_1|y > B_{\text{max}}|y$, this implies Lemma 2.B is very conservative to ensure $g(x) < 0$ for $\forall x \in [h_1, h_2]$.

Inspired by Lemma 2.B, we suggest novel geometry-based negative conditions of a quadratic function which is less conservative to ensure $g(x) < 0$ for $\forall x \in [h_1, h_2]$.

**Lemma 3:** For a quadratic function $g(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ with a given scalar $\alpha \in [0, 1]$, if $g(x)$ satisfies (i)-(iv) for $\alpha \in (0, 1)$; (i)-(iii) for $\alpha = 1$; and (i), (ii), (iv) for $\alpha = 0$, then $g(x) < 0$ for $\forall x \in [h_1, h_2]$.

(i) $g(h_1) < 0$
(ii) $g(h_2) < 0$
(iii) $(h_1^2 + a(h_2 - h_1))a + \frac{2h_1a + a(h_2 - h_1)}{2}b + c < 0$
(iv) $(a\alpha h_2^2 + h_1h_2(1 - \alpha))a + \frac{h_2(1+\alpha)+h_1(1-\alpha)}{2}b + c < 0$

**Proof:** (See Fig. 2:) Let us define two tangent lines to $g(x)$ at $x = h_1$ and $x = h_2$ as $l_{h_1}(x)$ and $l_{h_2}(x)$, respectively, and a maximum point on $g(x)$ in $x \in [h_1, h_2]$ as $E_{\text{max}}$. And taking a point $E$ on $g(x)$ in $x \in [h_1, h_2]$ can obtain a tangent line $E_l(x)$ at $E$. Then we can obtain the other two points, $E_1$ and $E_2$, which are intersection points of $E_l(x)$ and $l_{h_1}(x)$, and $E_l(x)$ and $l_{h_2}(x)$, respectively. It is clear that $E_{\text{max}}|y \geq \max(|E_1|y, |E_2|y)$ for Cases i) and ii) in $\forall x \in [h_1, h_2]$. In case of i) and ii), conditions $E_{h_1}|y < 0$ and $E_{h_2}|y < 0$ ensure $g(x) < 0$ for $\forall x \in [h_1, h_2]$. For Case iii), conditions $E_1|y < 0$ and $E_2|y < 0$ guarantee $g(x) < 0$ for $\forall x \in [h_1, h_2]$ because $E_{\text{max}}|y \leq \max(|E_1|y, |E_2|y)$ for $\forall x \in [h_1, h_2]$ in Case iii). Therefore, showing $y$–axis value of four points, $E_{h_1}$, $E_{h_2}$, $E_1$, and $E_2$ are negative is equivalent to $g(x) < 0$ for $\forall x \in [h_1, h_2]$. Now, we define $E = (a(h_2 - h_1) + h_1, g(a(h_2 - h_1) + h_1))$ where $\alpha \in [0, 1]$, then by simple calculation, we can have the equation of three tangent lines and the $y$–axis value of the intersection points, respectively, as

$$
\begin{align*}
l_{h_1}(x) &= (2h_1a + b)x - h_1^2a + c, \\
l_{h_2}(x) &= (2h_2a + b)x - h_2^2a + c, \\
l_{E}(x) &= (2(\alpha h_2 - h_1) + h_1)a + bx \\
&- (\alpha h_2 - h_1 + h_1)^2a + c, \\
E_{h_1}|y &= \frac{2h_1 + a(h_2 - h_1)}{2}, \\
E_{h_2}|y &= \frac{2h_1 + a(h_2 - h_1)}{2}, \\
E_{E}|y &= \frac{2h_1(1 + \alpha)(h_2 - h_1)}{2}. 
\end{align*}
$$

For $\alpha \in (0, 1)$, it is clear that $E_{h_1}|y < 0$, $E_{h_2}|y < 0$, $E_{E}|y < 0$, and $E_{E}|y < 0$ are equivalent to the conditions in Lemma 3. When $\alpha = 0$ or 1, then $l_{E}(x) = l_{h_1}(x)$ and $E_1$ is disappeared or $l_{E}(x) = l_{h_2}(x)$ and $E_2$ is disappeared, respectively. Therefore, for $\alpha = 0$ or 1, it is clear that $E_{h_1}|y < 0$, $E_{h_2}|y < 0$, and $E_{E}|y < 0$ or $E_{E}|y < 0$, respectively, are equivalent to the conditions in Lemma 3. This completes the proof.

**Theorem 2:** For given positive constants $h$, $\mu$, the system (1) is asymptotically stable if there exist matrices $P_1$, $P_2$, $X$, $Y \in \mathbb{S}^n_+$, $Q_1$, $Q_2 \in \mathbb{S}^n_+$, $R \in \mathbb{S}^n_+$, $L_i$, $M_i \in \mathbb{R}^{10n \times n}$.
The proof.

For $\alpha \in (0, 1): \Omega_i(\hat{h}(t)) < 0$, \((i = 1, 2, 7, 8, 16)\)

For $\alpha = 0$ or $1: \Omega_i(\hat{h}(t)) < 0$, \((i = 1, 2, 9, 17)\)

where

$$
\Omega_7(\hat{h}(t)) = \begin{bmatrix}
F_1 & (\Gamma^{-1})^T \frac{\alpha h}{2} L & (\Gamma^{-1})^T (2 - \alpha) h M \\
\ast & - \frac{\alpha h}{2} R_1 & 0 \\
\ast & 0 & - (2 - \alpha) h R_2
\end{bmatrix},
$$

$$
\Omega_8(\hat{h}(t)) = \begin{bmatrix}
F_2 & (\Gamma^{-1})^T (1 + \alpha) h L & (\Gamma^{-1})^T (1 - \alpha) h M \\
\ast & - (1 + \alpha) h R_1 & 0 \\
\ast & 0 & - (1 - \alpha) h R_2
\end{bmatrix},
$$

$$
\Omega_9(\hat{h}(t)) = \begin{bmatrix}
F_3 & (\Gamma^{-1})^T \frac{h}{2} L & (\Gamma^{-1})^T \frac{h}{2} M \\
\ast & - \frac{h}{2} R_1 & 0 \\
\ast & 0 & - \frac{h}{2} R_2
\end{bmatrix},
$$

\begin{align*}
F_1 &= \hat{f} \left( 0, \frac{\alpha h}{2}, \hat{h}(t) \right), \\
F_2 &= \hat{f} \left( \alpha h^2, \frac{(1 + \alpha) h}{2}, \hat{h}(t) \right), \\
F_3 &= \hat{f} \left( 0, \frac{h}{2}, \hat{h}(t) \right).
\end{align*}

Proof: Theorem 2 can be easily proven by Lemma 3 with $h_1 = 0$ and $h_2 = h$ and Schur complement. This completes the proof.

Remark 2: The proposed geometry-based negative conditions for a quadratic function, Lemma 3, could be optimal when we find the tuning parameter $\alpha$ to be $\alpha(h_2 - h_2) = E_{\text{max}}(\alpha)$. In other word, the derived theorem by Lemma 3 could be least conservative in the perspective of the quadratic function.

IV. NUMERICAL EXAMPLES

In this section, the three numerical examples listed in Table 1 are considered to show the effectiveness of the proposed scheme. The simulation results by Theorems 1 and 2 for the cases of various $|\mu| < 1$ and the existing results are listed in Tables 2, 3, and 4 for each examples. For the simulation of Theorem 1.C and Theorem 2, we simulated with tuning parameters $\alpha$ and $\beta$ by varying in 0.01 scale.

As seen in Tables 2, 3, and 4, our Theorem 2 can guarantee the stability of the system under larger maximum allowable upper bound (MAUB) of $h$ than existing ones and also Theorem 1.A, B, C for all cases which can be said that Theorem 2 is less conservative than others. The results in the tables can be summarised in the following aspects.

1) Advantages of the constructed LKF approach

The theory in [26] and Theorem 1.A utilized the negative conditions of a quadratic function, Lemma 2.A, and theories in [27] and [22] and Theorem 1.B used the one of Lemma 2.B. Therefore, the comparison with them can conclude the impact of employed LKFs. The results in the tables imply the constructed LKF in this paper derive less conservative stability criteria than others under the same negative conditions of a quadratic function. Especially, the results by Theorem 2.B are better than one in [22] even though it utilized an extension version of Lemma 2.B.

2) Superiority of the proposed geometry-based negative conditions for a quadratic function

The results by Theorem 1.A, B, C and Theorem 2, the results by Theorem 2 are less conservative than others. Only difference between Theorems 1 and 2 is the
negative conditions of a quadratic function. The newly developed negative conditions of a quadratic function, i.e., Lemma 3, are applied to Theorem 2 but existing ones are applied to Theorem 1. Therefore, we can check the effectiveness of the proposed Lemma 3 by comparing with the results by Theorems 1 and 2. As the result, the proposed geometry-based negative conditions of a quadratic function is more effective to reduce the conservatism of the stability criteria than existing ones regardless of employing more variables.

V. CONCLUSION

The paper was concerned with the development of a new geometry-based negative conditions of a quadratic function in which a tuning parameter was introduced. In addition, a new LKF was established which make LMI conditions to be quadratic function of time-varying delay together with MRF based LKF. Three numerical examples were given to display the effect on reducing conservatism of the stability criteria. The proposed methods would improve the performance of controllers once they applied to controller design problems which would remain for our future task.

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TAE H. LEE (Member, IEEE) received the B.S., M.S., and Ph.D. degrees in electrical engineering from Yeungnam University, Kyongsan, South Korea, in 2009, 2011, and 2015, respectively. He was a Postdoctoral Researcher with Yeungnam University, from 2015 to 2017, and an Alfred Deakin Postdoctoral Research Fellow with the Institute for Intelligent Systems Research and Innovation, Deakin University, Australia, in 2017. He began with Jeonbuk National University, Jeonju-si, South Korea, in September 2017, where he is currently an Assistant Professor. His research interests include complex dynamical networks, sampled-data control systems, chaotic/biological systems, and networked-control systems.