A Cautionary Note on Cosmological Magnetic Fields

Luís F. A. Teodoro, Declan A. Diver and Martin A. Hendry

Department of Physics and Astronomy
Kelvin Building
University of Glasgow
Glasgow, G12 8QQ, Scotland, UK

(Dated: March 21, 2022)

This note is concerned with potentially misleading concepts in the treatment of cosmological magnetic fields by magnetohydrodynamical (MHD) modelling. It is not a criticism of MHD itself but rather a cautionary comment on the validity of its use in cosmology. Now that cosmological data are greatly improved compared with a few decades ago, and even better data are imminent, it makes sense to revisit original modelling assumptions and examine critically their shortcomings in respect of modern science. Specifically this article argues that ideal MHD is a poor approximation around recombination, since it inherently restricts evolutionary timescales, and is often misapplied in the existing literature.

PACS numbers: 98.80.-k, 98.80.Jk, 91.25.Cw, 94.20.wc, 94.30.cs, 95.30.Qd

I. INTRODUCTION

The role of the magnetic field in the evolution of the Universe has been a challenging problem for many decades, retaining a frustrating degree of speculation despite remarkably inventive and ingenious mathematical modelling. Definitive progress to date has been hampered by the ambivalence of observational data: as yet, contradictory hypotheses continue to remain possibilities for as long as the definitive observation remains elusive.

However, the quantity and precision of appropriate cosmological data are soon to be revolutionised with the imminent construction of dedicated instruments such as SKA, and the era of precision diagnosis of cosmological magnetic fields is not far away. It is surely appropriate then to re-examine the theoretical basis upon which the modelling of cosmological magnetic fields is based, in preparation for these new data.

A favourite mathematical framework for modelling the evolution of magnetism near the decoupling era (and sometimes for the early universe) is magnetohydrodynamics (MHD), which blends the continuum properties of the matter in the Universe with a restricted set of electromagnetic concepts, yielding a hybrid magneto-fluid context in which to study the influence of matter on magnetic fields, and vice-versa. Such MHD models are often imbued with perfect electrical conductivity (that is, zero electrical resistivity), and are almost invariably single-fluid, since this combination allows significant simplification of mathematical modelling, without apparently sacrificing too much of the underlying physics. It has to be mentioned that many of the cosmological MHD models used in this context have additional properties not encountered in standard plasma modelling; these primarily arise from the incorporation of Universal expansion, and sometimes from accommodating radiation pressure as an extra force term.

In this short article we examine critically the physical underpinning of the magnetofluid Universe as a suitable model for cosmological magnetogenesis and evolution, and find such models to be flawed in basic concept. We conclude that MHD, and its variants, are quite unsuited to modelling the critical phenomena; instead we believe that consistency demands that we use a relativistic kinetic theory instead [see [1] for a gas kinetics (but not plasma) treatment in a Friedman-Lemaître-Robertson-Walker (FLRW) space-time]. The next sections explain our reasoning, by highlighting the inadequacies of the magnetofluid approach, ranging over issues such as the calculation of conductivity, the meaning of the current density, the interpretation of mass density and the allied velocity field, and finally comments on the juxtaposition of relativity and electromagnetism with an MHD plasma description.

II. THE RELEVANCE OF MHD TO COSMOLOGY

Magnetohydrodynamics is a fluid model, in which aspects of electromagnetism are incorporated into standard hydrodynamics to give an electrically conducting continuum that can generate a self-field, and respond to an applied one. Single fluid MHD cannot support charge separation, and perfectly conducting (or ideal) MHD has zero electric field in the rest frame of the plasma.

MHD is the favoured mathematical model for the description of the cosmic plasma behaviour but caution is required in the interpretation of the results. MHD is an excellent plasma model in many disparate contexts, but care has to be taken to ensure that conclusions are not drawn that cannot be sustained by the restricted physics implicit in MHD, particularly where the context requires elements of relativity.
In cosmological MHD there may be at best a conflict in terminology, and at worst, flawed physics. In order to be clear about the problems, we elaborate below on general issues in which the physical interpretation is perhaps at odds with the mathematical framework.

One fundamental aspect in cosmological descriptions involving plasma is as follows: in the literature it is assumed that all the universe is ionised, meaning the plasma density and baryon density are identical. This may be reasonable before recombination (assuming that dark matter can be ignored) but not after! Whilst this might appear at first a trivial statement, the implications can be significant if care is not taken to identify how the mass densities of constituent fluids contribute to the relevant physical forces.

Below we present a list of issues in which the use of MHD may lead to inconsistencies in the interpretation of the results. These are not in any particular order, and we don’t claim that it is a exhaustive list. However, each topic highlights a specific conceptual shortcoming associated with the use of MHD in cosmological descriptions currently in the literature.

A. The Perfectly Conducting Assumption

One of the great strengths of single-fluid MHD is the simplicity in the field evolution, particularly given the ideal case in which the plasma has zero resistivity. Given that the single-fluid model immediately rules out charge separation, the further imposition of a perfectly conducting fluid allows a very simple Ohm’s law to be placed at the heart of the physics:

\[ \mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \tag{1} \]

where \( \mathbf{E} \), \( \mathbf{u} \) and \( \mathbf{B} \) are respectively the plasma electric field, bulk fluid velocity and magnetic flux density (loosely, magnetic field). One consequence is immediate: the electric field can only be produced as the result of a frame change, and is therefore entirely prescribed by the velocity and magnetic fields, having no independent evolution. (There are no induced fields, for example.) This is because any applied electric field invokes a plasma dynamical response in terms of the particle current density \( \mathbf{J} \) consistent with the relation

\[ \mathbf{J} = \sigma \mathbf{E} \tag{2} \]

where \( \sigma \) is the conductivity tensor; not a scalar, since the magnetic field changes the transport properties of mobile species for cross-field motion compared to the isotropic case. Clearly then unlimited particle currents can be sustained in the perfect conductivity case by vanishingly small electric fields; since such currents are not seen, then the Universal electric field must be zero except for that arising from a frame translation of the magnetic field. Whilst this is perfectly acceptable within the classical concept of MHD plasmas, it pays to consider carefully the implications of this in a cosmological context.

In the early Universe (at least pre-decoupling), the plasma (charged baryons and electrons) co-exists with a radiation field that interacts with it via the Thompson scattering of photons by the free electrons (to identify but one process). Viewed classically (that is, non quantum-mechanically) in terms of the propagation of the electric field disturbances, the plasma is a dielectric, since interaction between the two continua produces a different response than if the radiation simply propagated in a classical vacuum: the plasma has a refractive index different from unity, the classical vacuum value. A non-vacuum refractive index implies a finite conductivity. This can be seen from cold plasmas, for example, in which the dielectric tensor \( K \) and conductivity tensor \( \sigma \) are intimately related

\[ K = I + \frac{i}{\epsilon_0 \omega} \sigma \tag{3} \]

where \( I \) is the unit tensor, \( \omega \) is the frequency of the electromagnetic radiation, and \( \epsilon_0 \) is the vacuum permittivity. Note that the kinetic plasma treatment is similar. Hence if the plasma refractive index is not unity (the classical vacuum value), then the conductivity of the plasma must be finite, and anisotropic: particle transport parallel to the magnetic field is different from that perpendicular to it.

Of course, in the very early universe where current carriers are effectively massless then the conductivity could be very high \([4, 5]\); however, the applicability of MHD in such energetic conditions is very much open to question. The transition from perfect to finite conductivity is subtle, but critical: finite conductivity brings evolutionary and topological constraints that are simply missing from perfect conductivity; moreover a resistive plasma has additional interdependence of the fields compared with an ideal one.

Note that in the recombination era, the conductivity must evolve as the charged particle density changes, making any assumption of perpetual perfect conductivity even less credible (see the discussion later in this section).

Hence a more appropriate general form of Ohm’s Law is the simplest resistive one,

\[ \mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J} \tag{4} \]

where \( \eta = \sigma^{-1} \) is the plasma resistivity, also a tensor in a magnetised plasma.

It is a cosmological convention to calculate electrical conductivity in terms of the effect of photon scattering on electron transport, using a simple scalar expression for the conductivity based on a single interaction time \( \tau \)[6]:

\[ \sigma = \frac{n_e e^2 \tau}{m_e} \tag{5} \]

in which \( \tau = 1/(n_e \sigma_T) \), with \( n_e \) is the photon number density and \( \sigma_T \) is the Thomson scattering cross-section, and \( n_e \) is the free-electron number density. This yields
a post-recombination conductivity that is approximately $10^{-3}$ times that for the pre-recombination case [7]. Taking the plasma perspective, the ratio of electrical conductivities before and after recombination is based on the Spitzer electrical conductivity for a fully ionized plasma at a temperature of 10 eV [8], and the scattering of 0.1 eV electrons by neutral hydrogen [9, 10]. This ratio is

$$\frac{\sigma_{\text{pre}}}{\sigma_{\text{post}}} \approx 0.016/x_e$$

where $x_e$ is the ionization fraction. This yields a similar drop in conductivity as in the Thomson scattering case if $x_e \approx 10^{-5}$ [11]. Since the Spitzer conductivity is similar in value to that calculated by considering only the Thomson scattering of photons by electrons then this plasma calculation shows that the pure plasma treatment is equally valid, though the nature of the BGK approximation in calculating the resistivity Eq. (5) might not be wholly valid in either context, since it is unlikely that a single relaxation time can account for all scattering processes. Either way, the resistivity is finite on either side of recombination, and significantly lower afterwards, underlining the dangers in a perfectly conducting description.

B. The Total Current

In the present article we use the $1 + 3$ covariant formalism introduced by [12, 13], in which a general class of homogeneous space-times are considered. The usual $\nabla$ and $t$ operators are generalised appropriately.

One of the crucial approximations in MHD is that the current density is given exclusively in terms of the magnetic field curvature:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Note that $\mathbf{J}$ is not related directly to the bulk flow velocity $\mathbf{u}$, since the latter is a mass-weighted averaged quantity that is independent of charge. Hence in MHD, the current density is a dependent variable, and can be eliminated everywhere in favour of the magnetic field.

However, full electromagnetism demands that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

where $c$ is the speed of light, and $t$ is the time. The additional term on the right is the displacement current, omitted from MHD because the rapid time evolution of electromagnetic effects is not incorporated in such a fluid description: there are other comparable time-dependent terms in the accompanying plasma equations that have also been omitted and so, for mathematical consistency, the displacement current must be dropped.

The true significance of adopting Eq. (7) instead of Eq. (8) lies in the timescale for the evolution of the magnetic field. With Eq. (7) we have

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} - \frac{2}{3} \Theta \mathbf{B}$$

where we have used Eq. (4) for the most general case. $\Theta$ denotes $3a/a$, where $a$ is the expansion factor. Eliminating the current density using Eq. (7) yields

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B} - \frac{2}{3} \Theta \mathbf{B}$$

assuming the simplest case of constant resistivity. This parabolic equation shows that the magnetic field evolves on the resistive diffusion time-scale (notwithstanding the feedback term from the dynamo contribution, the first term on the right-hand side); cosmological expansion will allow an additional cosmological time-scale.

However, if $\mathbf{J}$ is eliminated using Eq. (8) instead, the governing equation for the magnetic field is higher-order, and hyperbolic:

$$-\frac{\eta}{\mu_0 c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} + \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B} - \frac{2}{3} \Theta \mathbf{B}$$

This introduces new time-scales for field evolution, including wave propagation, damping and growth, to be balanced against the dynamo term and any cosmological input.

Granted, magnetic and electric fields that so arise may be rapidly varying: this means that the overall mathematical framework must be able to accommodate fast time-scale evolution, a point to which we will return in a different context later in this article.

Note that in some very early work (e.g. [14]) the magnetic structure was modelled without any intrinsic time evolution, and only varied on cosmological time scales; other more recent work omits the spatial structure of the magnetic field (e.g. [7]). Neither of these approaches is correct: the true picture must have the correct balance between spatial and temporal derivatives in order to reflect the correct physics.

C. Density considerations

One serious concern with unified fluid descriptions of the universe around recombination is the concept of the mass density. Care must be taken to distinguish between the density of charged and non-charged particles; the latter are also divided between baryonic matter and dark matter. Only the density of plasma (that is, the electrically conducting and magnetised fluid) can appear in those plasma equations that are concerned with exclusively plasma effects. For example, neutral matter density cannot contribute to current densities. There are several examples in the literature [15, 16, 17] where only baryonic matter density enters in the model equations with no distinction between neutrals and current-carrying species. Where baryonic matter is predominantly charged, so that the baryonic and plasma matter densities are approximately the same, this is fine,
but where neutral matter (e.g., hydrogen atoms and molecules) co-exists with plasma, this clearly cannot be correct, and care must be taken to distinguish between species. This latter case leads for example to an Alfvén speed, \( c_A = B/\sqrt{\mu_0 \rho} \), that incorrectly uses the total mass density, instead of just the plasma mass density; the magnetic field cannot be influenced directly by the motion of neutral matter and vice-versa. This becomes particularly important when the Alfvén speed is used to estimate magnetic field strengths in the post recombination era (for example, \([16]\)).

Momentum exchange between the fluids is obviously physically correct, and is a primary method of causing indirect interaction between the magnetic field and neutral particles, but is predicated on making the distinction between the different fluid types in the model \([18, 19]\).

In particular, the pressures of neutral matter and plasma evolve under different physical conditions. Hence, incorporating the magnetic field into the Jeans length requires a careful definition of the concept of magnetic and kinetic pressure. The latter is influenced by both neutral matter and plasma; the former is only directly supplied by the plasma. To get a true holistic picture, the neutral gas and plasma components must be identified as separate species from the outset, with the latter obeying additional force terms, but with each potentially interacting through gravitation, or explicit coupling terms.

D. The Global Velocity Field

This problem is related to the density issue above. Since the plasma and the neutral components respond to different dynamical equations (notwithstanding coupling) then clearly the velocity fields are different. In principle the neutrals need not have the same velocity as the plasma since they do not react directly to magnetic forces. Moreover, the bulk motion of neutrals makes no direct contribution to the evolution of magnetic field. Of course, should the neutral gas and the plasma be coupled in some way, then the evolution of one will affect the other, but this is somewhat different from assuming that the plasma and neutral components are somehow locked together.

In the light of comments about the densities and velocities it is appropriate to mention how plasma and non-plasma can be coupled to give an overall collective response. Plasmas and neutrals can be coupled through momentum transfer \([18, 19]\), in which each fluid exerts a drag on the other by virtue of relative motion. Ionisation and recombination can also be considered as coupling mechanisms, in that species are converted from neutral to plasma and vice-versa. However single fluid MHD is not a good model for such processes, since charge separation is a basic pre-requisite but is impossible in single-fluid MHD plasma. Also, dark matter, neutrals and plasma move under the common self-gravitating potential presented by their respective mass densities, leading to dynamical equations for each species that are separate but coupled, emphasising that care is required in identifying what is meant by velocity at a given point in space since we have to distinguish between different species.

E. Relativity, MHD and Photons

In many cosmological contexts relativity is a key element of the physical model, including the plasma. However, there is a problem here if the plasma is described by a standard fluid MHD model. Given that MHD is exclusively concerned with low frequency, long wavelength phenomena that are not electromagnetic in nature, it is far from obvious how an MHD plasma description can be incorporated into a Lorentz invariant model of the whole ensemble: MHD cannot be Lorentz invariant, since the displacement current has been omitted, and so the current density in MHD is not a Lorentz invariant.

Furthermore, photons and free electrons are formally irreconcilable with an MHD prescription since MHD is not fully electromagnetic and is single fluid: electron and ions are combined in an averaged description. Two-fluid MHD is a possible approach but requires, amongst other things, a more sophisticated Ohm’s law than appears in the literature. (Recall that single fluid MHD cannot sustain charge separation.) Hence, photon scattering using Thomson cross-sections cannot be rigorously quantified in a single MHD context since the electron number density cannot formally be deduced independently, and the physics of the scattering processes involving photons and free electrons is not consistent with the exclusively long time-scale processes that are valid in MHD. Any formal attempt to combine both approaches in a unified treatment of the damping of MHD waves (for example in \([17]\)) is compromised by this mismatch.

On a more general point, full electromagnetic boundary conditions are not appropriate for MHD since the lack of displacement current means the MHD is pre-Maxwell; this should also be taken into account in any scattering description.

III. CONCLUSIONS

In this article we have illustrated some shortcomings of the magnetofluid modelling of the universe when used to determine magnetic field contributions to cosmic development. MHD necessarily can only describe slow timescale effects, simply because that is its mathematical and physical basis. It is no surprise that MHD descriptions yield slow evolutionary behaviour; the model is incapable of delivering anything else.

Assessing the influence of the magnetic field in post-decoupling cosmic structure based on such models is therefore at best approximate, and at worst very misleading. It is now timely to revisit the fundamental basis
of magnetogenesis and evolution in order to extract the best possible interpretation of the observations.

This article is designed to provoke discussion: we cannot fully exploit the true information content of cosmological data if the community persists in attempting to model physical phenomena in an inappropriate mathematical framework. Some of these contentious assumptions are highlighted here. We do not offer detailed alternatives in this article, since the construction of new modelling environments is a challenge for us all.

LFAT acknowledges the financial support of the Leverhulme Trust. DAD and MAH are grateful to PPARC for research funding.

[1] J. Bernstein. *Kinetic theory in the expanding universe.* Cambridge and New York, Cambridge University Press, 1988, 157 p., 1988.
[2] T. J. M. Boyd and J. J. Sanderson. *Plasma dynamics.* Plasma dynamics, by T.J.M. Boyd and J.J. Sanderson. London, Nelson, 1969. Series : Applications of mathematics series ISBN: 177616113, 1969.
[3] T.H. Stix. *Waves in Plasmas.* American Institute of Physics, New York, U.S.A., rev. edition, 1992.
[4] J. Ahonen and K. Enqvist. Electrical conductivity in the early universe. *Physics Letters B,* 382:40–44, 1996.
[5] Gordon Baym and Henning Heiselberg. Electrical conductivity in the early universe. *Phys. Rev. D,* 56(8):5254–5259, 1997.
[6] D Grasso and HR Rubinstein. Magnetic fields in the early universe. *Physics Reports-Review Section of Physics Letters,* 348(3):165 – 266, 2001.
[7] C. Caprini and P. G. Ferreira. Constraints on the electrical charge asymmetry of the universe. *JCAP,* 0502:006, 2005.
[8] D.A. Diver. *A Plasma Formulary for Physics, Technology and Astrophysics.* Wiley-VCH, Berlin, Germany, 1st edition, 2001.
[9] Rainer Hippler, Sigismund Pfau, Martin Schmidt, and Karl H. Schoenbach. *Low Temperature Plasma Physics.* Lecture Notes in Computational Science and Engineering. Wiley-VCH, Berlin, 1st edition edition, 2001.
[10] S. J. Buckman and A. V. Phelps. Vibrational excitation of D2 by low energy electrons. *Journal of Chemical Physics,* 82:4999–5011, 1985.
[11] T. Padmanabhan. *Structure formation in the Universe.* Cambridge University Press, Cambridge, U.K., 1st edition, 1993.
[12] G. F. R. Ellis. Relativistic Cosmology. In E. Schatzman, editor, *Cargese Lectures in Physics,* 1973.
[13] G. F. R. Ellis and M. Bruni. Covariant and gauge-invariant approach to cosmological density fluctuations. *Phys. Rev. D,* 40:1804–1818, 1989.
[14] I. Wasserman. On the origins of galaxies, galactic angular momenta, and galactic magnetic fields. *Astrophys. J.,* 224:337–343, 1978.
[15] K. Subramanian and J. D. Barrow. Microwave Background Signals from Tangled Magnetic Fields. *Physical Review Letters,* 81:3575–3578, 1998.
[16] K. Subramanian and J. D. Barrow. Magnetohydrodynamics in the early universe and the damping of nonlinear Alfvén waves. *Phys. Rev. D,* 58(8):083502, 1998.
[17] K. Jedamzik, V. Katalinić, and A. V. Olinto. Damping of cosmic magnetic fields. *Phys. Rev. D,* 57:3264–3284, 1998.
[18] M. Marklund, P. K. S. Dunsby, G. Betschart, M. Servin, and C. G. Tsagas. Charged multifluids in general relativity. *Classical and Quantum Gravity,* 20:1823–1834, 2003.
[19] D. A. Diver, H. E. Potts, and L. F. A. Teodoro. Gas-plasma compressional wave coupling by momentum transfer. *New Journal of Physics,* 8:265, 2006.