HOMOGENEOUS EINSTEIN METRICS ON EUCLIDEAN SPACES ARE EINSTEIN SOLVMANIFOLDS

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Abstract. We show that homogeneous Einstein metrics on Euclidean spaces are Einstein solvmanifolds, using that they admit periodic, integrally minimal foliations by homogeneous hypersurfaces. For the geometric flow induced by the orbit-Einstein condition, we construct a Lyapunov function based on curvature estimates which come from real GIT.

A Riemannian manifold \((M^n, g)\) is called Einstein, if its Ricci tensor satisfies \(\text{ric}_g = \lambda \cdot g\), for some Einstein constant \(\lambda \in \mathbb{R}\). As is well known, for \(n \leq 3\) this implies constant sectional curvature. Topological obstructions to the existence of compact Einstein 4-manifolds are known, see [Tho69], [Hit74], [LeB95], [LeB96] and [And10]. In dimensions \(n \geq 5\), compact simply-connected Einstein manifolds with positive Einstein constant are also topologically obstructed, see [Ste92], whereas for negative Einstein constant no such obstruction is known.

Many examples of compact Einstein manifolds have been constructed using bundle, symmetry and holonomy assumptions: see [Bes87], [Wan99], [Wan12], [Joy00] and references therein. Among many others, we mention homogeneous Einstein metrics [WZ86], [BWZ04], Einstein metrics on spheres [Böh98], [BGK05], [FH17], Ricci-flat manifolds with holonomy \(G_2\) and \(\text{Spin}(7)\) [Joy96a], [Joy96b], and Kähler-Einstein manifolds, whose classification could be completed recently [Aub76], [Yau78], [CDS15a], [CDS15b], [CDS15c], [Tia15].

Non-compact Einstein manifolds have non-positive Einstein constant and, in contrast to the compact case, no topological obstruction is known in dimensions \(n \geq 4\). As in the compact case, there exist plenty of examples, see [Bry87], [BS89], [AKL89], [Biq13], [Biq16] and the survey articles mentioned above. These include irreducible non-compact symmetric spaces, and more generally non-compact homogeneous Einstein spaces. The latter are flat for zero Einstein constant by [AK75], and for negative Einstein constant, Alekseevskii’s conjecture states that \(M^n\) is diffeomorphic to Euclidean space \(\mathbb{R}^n\); see [Ale75], [Bes87].

The main result of this paper is

**Theorem A.** Homogeneous Einstein metrics on \(\mathbb{R}^n\) are isometric to Einstein solvmanifolds.

Recall that a simply-connected solvmanifold is a solvable Lie group endowed with a left-invariant metric. Theorem [A] was known for Ricci-flat homogeneous spaces [AK75], for homogeneous \(\mathbb{R}\)-bundles over irreducible Hermitian symmetric spaces, see [BB78] and Remark 3.2 and in dimensions \(n \leq 5\) and \(n = 7\) [AL17], leaving open the case of the 6-dimensional universal cover \(\text{SL}_2(\mathbb{R})^2\) of \(\text{SL}(2, \mathbb{R})^2\). In this direction, we would like to mention that our methods also yield new results beyond Theorem [A] such as the non-existence of left-invariant Einstein metrics on \(\text{SL}_k(\mathbb{R})^k\) for all \(k \geq 2\); see Corollary 6.4.

The main difficulty in proving Theorem [A] is that Euclidean spaces admit extremely different presentations as a homogeneous space \(G/H\). Firstly, \(\mathbb{R}^n\) is a solvmanifold, which in dimension \(n = 3\) already provides uncountably many algebraically distinct choices [Bia98]. Diametrically opposed, it is also a homogeneous space for \(G\) semisimple, such as \(\mathbb{R}^{3k} = \text{SL}_k(\mathbb{R})\), \(k \geq 1\), or \(\mathbb{R}^{2m+1}\) being the universal cover of \(\text{SO}(m, 2)/\text{SO}(m)\), \(m \geq 3\); see Remark 3.2 for further examples. Finally and more generally, semidirect products of the above cases occur.
Next, we restate the following important classification result of Lauret:

**Theorem B** ([Lau10]). Einstein solvmanifolds are standard.

The standard condition, described below, is an algebro-geometric condition introduced and extensively studied by Heber in [Heb98]. For standard Einstein solvmanifolds of fixed dimension, he showed finiteness of the eigenvalue type of the modified Ricci curvature, see below, a result intimately related to the finiteness of critical values of a (real) moment map. Even though a complete classification is out of reach at the moment, see [Lau09], Theorem A together with [Lau10] and [Heb98] would provide a very precise understanding of all non-compact homogeneous Einstein manifolds, provided the Alexseevskii conjecture holds true.

A Ricci soliton \((M^n, g)\) is a Riemannian manifold satisfying \(\text{ric}_g + \mathcal{L}_X g = \lambda \cdot g\), \(\lambda \in \mathbb{R}\), where \(\mathcal{L}_X g\) denotes the Lie derivative of \(g\) in the direction of a smooth vector field \(X\) on \(M^n\). Recall now that a homogeneous Ricci soliton on \(\mathbb{R}^n\) gives rise to a homogeneous Einstein space \((\mathbb{R}^{n+1}, \hat{g})\); see [LL14], [HPW15]. Since by work of Jablonski [Jab15b] Einstein solvmanifolds are strongly solvable spaces, from Theorem A we deduce

**Corollary C.** Homogeneous Ricci solitons on \(\mathbb{R}^n\) are isometric to solvsolitons.

Solvsolitons, introduced in [Lau11], are homogeneous Ricci solitons admitting a transitive solvable Lie group \(S\) of isometries. It is shown in [Jab15a] that \(S\) may be chosen to be simply-transitive, such that the Ricci endomorphism satisfies \(\text{ric}_g - \lambda \cdot \text{Id} = D\), where \(D\) is a derivation of the Lie algebra \(T_g S\). It follows that the corresponding Ricci flow solution is driven by the one-parameter group of automorphisms of \(S\) corresponding to \(D\). Automorphisms are precisely those diffeomorphisms which preserve the space of left-invariant metrics on \(S\).

A simply-connected homogeneous space is diffeomorphic to \(\mathbb{R}^n\) if and only if all its homogeneous metrics have non-positive scalar curvature [BB78]. As a consequence homogeneous Ricci flow solutions on \(\mathbb{R}^n\) always exist for all positive times [Lai15]. By [BL18] and Corollary C it then follows that any parabolic blow-down of such a solution subconverges to a limit solvsoliton in pointed Cheeger-Gromov topology. Notice that for the semisimple Lie group \(SL_\ast (2, \mathbb{R})^k\) this means that for generic left-invariant initial metrics the Lie group structure of their full isometry group changes completely when passing to a limit.

Turning to the proof of Theorem A we would like to mention that all the algebraic structure results for non-compact homogeneous Einstein manifolds developed in [LL14], [HP17] and [AL17] yield for instance no information whatsoever for the presentation \(\mathbb{R}^k = SL_\ast (2, \mathbb{R})^k\). Therefore, a purely algebraic proof of Theorem A is elusive at the moment. To overcome this, we prove that homogeneous Euclidean Einstein spaces admit cohomogeneity-one actions by non-unimodular subgroups \(\bar{G}\) of \(G\), with orbit space \(\mathbb{R}\). We then show periodicity of the corresponding foliation by orbits, meaning that after passing to a quotient, which is still homogeneous, the orbit space becomes \(S^1\), and that the integral of the mean curvature of the orbits over the orbit space vanishes, a condition we call integral minimality. This then reduces the proof of Theorem A essentially to the following second main result of this paper.

**Theorem D.** Suppose that \((M^n, g)\) admits an effective, cohomogeneity-one action of a Lie group \(\bar{G}\) with closed, integrally minimal orbits and \(M^n/\bar{G} = S^1\). If in addition \((M^n, g)\) is orbit-Einstein with negative Einstein constant, then all orbits are standard homogeneous spaces.

A cohomogeneity-one manifold \((M^n, g)\) is called orbit-Einstein with negative Einstein constant, if for \(\lambda < 0\) we have \(\text{ric}_g(X, X) = \lambda \cdot g(X, X)\) for all vectors \(X\) tangent to orbits. Generalizing [Heb98], we say that a homogeneous space \((\bar{G}/\hat{H}, \hat{g})\) is standard, if the Riemannian submersion induced by the free isometric action of the maximal connected normal nilpotent subgroup \(\bar{N} \leq \bar{G}\) on \(\bar{G}/\hat{H}\) has integrable horizontal distribution: see Definition 2.41.
Finally, let us mention that Theorem D immediately implies Theorem B when applying it to the Riemannian product of a circle and an Einstein solvmanifold. Other non-existence results on homogeneous and cohomogeneity-one Einstein metrics include [WZ86], [B¨ oh99b], [Nik00], [Ba05], [JP17], [AL17].

We turn now to the proof of Theorem D. Since the \( \bar{G} \)-orbits form a family of equidistant hypersurfaces in \( M^n \), we write \( g = dt^2 + g_t, t \in \mathbb{R} \), for a smooth curve of homogeneous metrics \( g_t \) on \( \bar{M} = \bar{G}/\bar{H} \). By the Gauß equation, the Riccati equation and [B¨ oh99a], the orbit-Einstein equation (1) on \( (M^n, g) \) with Einstein constant \(-1\) can be considered as the following ‘second order Ricci flow’ on the space of homogeneous metrics on \( \bar{M} \):

\[
\frac{D^2}{dt^2} g_t = -(\text{tr } L_t) \cdot g_t' + 2 \text{ric}_{g_t} + 2g_t.
\]

Here \( \text{tr } L_t \) denotes the mean curvature of an orbit and \( \frac{D}{dt} \) the covariant derivative of the symmetric metric on the symmetric space of \( \bar{G} \)-homogeneous metrics on \( \bar{G}/\bar{H} \).

We decompose the Ricci tensor \( \text{ric} \) of a homogeneous space \( \bar{M} = \bar{G}/\bar{H}, g \) as a sum of two tensors, one tangent to the Diff\( \bar{G} \)(\( \bar{M} \))-orbit through \( \bar{g} \), and another one, the modified Ricci curvature \( \text{ric}^\ast \), orthogonal to it. Here, Diff\( \bar{G} \)(\( \bar{M} \)) denotes the set of those diffeomorphisms of \( \bar{M} \) which preserve the space of \( \bar{G} \)-invariant metrics when acting by pull-back; algebraically these are just the automorphisms of \( \bar{G} \) preserving \( \bar{H} \). Denoting \( \text{scal}^\ast \bar{g} = \text{tr}_{\bar{g}} \text{ric}^\ast \bar{g} \), we set

\[
h(\bar{g}) := \frac{1}{2} \cdot (\text{scal}^\ast \bar{g} - \text{scal}_0) \geq 0,
\]

see Lemma 1.3 and Remark 1.4. Algebraically, \( h = 0 \) if and only if the group \( \bar{G} \) is unimodular.

Using the orbit-Einstein equation for \( (M^n, g) \) and the compactness of the orbit space we establish a maximum principle for the real-valued function \( h(t) := h(g_t) \): see Lemma 1.3 and Lemma 2.5. This yields an upper bound for \( 2h \) given by \( \text{tr } \beta^+ = n - 1 ||\beta||^2 \geq 0 \). Here \( \beta \) is the stratum label of the homogeneous space \( \bar{G}/\bar{H} \), a self-adjoint endomorphism, coming from the Morse-type, Ness-Kirvan stratification of the space of Lie brackets on \( T_\bar{G} \) introduced by Lauret [Lau10]: see Appendix B and BLIN. From this a priori estimate for \( h \) we establish in Lemma 2.6 the existence of a Lyapunov function for the orbit-Einstein equation. Using that the orbits are integrally minimal it follows that this Lyapunov function is periodic, hence constant. As a consequence, several inequalities become equalities and Theorem D follows.

Finally, we indicate how Theorem D implies Theorem A. Using the standard condition for the given homogeneous Einstein space \( (\mathbb{R}^n = G/H, g) \), see [LL14], and all the codimension-one orbits, one shows that the simple factors of a Levi factor \( L \leq G \) are pairwise orthogonal: see Proposition 6.1. Together with the condition \( G/H \simeq \mathbb{R}^n \), this implies that the induced metric on \( L/H \) is awesome, that is, it admits an orthogonal Cartan decomposition. This leads to a contradiction by [Nik00], unless the Levi factor is trivial, in which case \( G \) is solvable.

The article is organized as follows. In Section 1 we discuss the cohomogeneity one Einstein equation and derive the evolution equation for \( h(t) \) induced by (1). In Section 2 we prove Theorem D by establishing a maximum principle for \( h \) and by defining a new Lyapunov function for (1). In Section 3 we introduce the assumption \( M \simeq \mathbb{R}^n \) which will be used from here on, and characterise certain minimal presentations of homogeneous Euclidean spaces and prove their periodicity in Section 4. In Section 5 integral minimality of the so-called Levi presentations is shown, and finally, in Section 6 we prove Theorem A and Corollary C. In Appendix A we address elementary algebraic properties of homogeneous spaces, and Appendix B contains the necessary preliminaries on GIT and the \( \beta \)-endomorphism associated to a homogeneous space.

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1. The cohomogeneity-one Einstein equation

Let \( \tilde{G} \) be a connected Lie group acting on a complete, connected Riemannian manifold \((M^n, g)\) by \((g, p) \mapsto g \cdot p\) for \(g \in \tilde{G}\) and \(p \in M^n\). We assume that the action is proper, isometric, effective and with cohomogeneity one. By [Kos65], properness ensures that the orbits are embedded submanifolds, and that the isotropy subgroups are compact.

Let us in addition assume that all orbits are principal. Then by [Pal61], the orbit space \(M^n/\tilde{G}\) is diffeomorphic to \(\mathbb{R}\) or \(S^1\). Choose \(\gamma : \mathbb{R} \to M^n\) a unit speed geodesic intersecting all orbits orthogonally, and set \(\Sigma_t := \tilde{G} \cdot \gamma(t)\). Let \(N\) denote the unit normal vector field of the foliation \(\Sigma_t\) of \(M^n\) with \(\gamma'(t) = N\gamma(t)\) for all \(t \in \mathbb{R}\). Then, all integral curves of \(N\) are unit speed geodesics intersecting all \(\tilde{G}\)-orbits \(\Sigma_t\) orthogonally.

Infinitesimally, the \(\tilde{G}\)-action induces a Lie algebra homomorphism \(\tilde{g} := \text{Lie}(\tilde{G}) \to \mathfrak{X}(M^n)\) assigning to each \(X \in \tilde{g}\) a Killing field on \((M^n, g)\), also denoted by \(X\), and given by

\[
X_p := \left. \frac{d}{dt} \right|_{t=0} \exp(tX) \cdot p, \quad p \in M^n.
\]

Since the flow of Killing fields consists of isometries, it maps geodesics onto geodesics, and consequently \([X, N] = 0\) where \([\cdot, \cdot]\) denotes the Lie bracket of smooth vector fields on \(M^n\).

Since \(\Sigma_0\) is a principal orbit, the isotropy subgroup \(\mathcal{H} := \tilde{G}\gamma(0)\) at \(\gamma(0)\) fixes all points \(\gamma(t)\), hence \(\mathcal{G}_{\gamma(t)} = \mathcal{H}\) for all \(t\). Set \(\mathfrak{h} := \text{Lie}(\mathcal{H})\) and let \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\) be the `canonical' reductive decomposition: \(\mathfrak{m}\) is the orthogonal complement of \(\mathfrak{h}\) in \(\mathfrak{g}\) with respect to the Killing form \(B_{\tilde{G}}\) of \(\tilde{g}\). By [2] we obtain for all \(t\) an identification

\[
T_{\gamma(t)} \Sigma_t \simeq \mathfrak{m},
\]

which will be used constantly in what follows. In particular, the metric \(g\) on \(M^n\) induces a family of \(\text{Ad}(\mathcal{H})\)-invariant scalar products \((g_t)_{t \in \mathbb{R}}\) on \(\mathfrak{m}\).

Let \(L_t \in \text{End}(\mathfrak{m})\) denote the shape operator of \(\Sigma_t\) at \(\gamma(t)\) evaluated on Killing fields:

\[
g_t(L_tX, Y) = g(\nabla_X N, Y)_{\gamma(t)} = -g(\gamma'(t), \nabla_X Y), \quad X, Y \in \mathfrak{m},
\]

since \(g(N, Y) \equiv 0\). Here \(\nabla\) denotes the Levi-Civita connection of \((M^n, g)\).

**Lemma 1.1.** The scalar products \((g_t)_{t \in \mathbb{R}}\) on \(\mathfrak{m}\) evolve by \(g_t'(\cdot, \cdot) = 2g_t(L_t \cdot, \cdot)\).

**Proof.** Since \(g([X, Y], \gamma'(t))_{\gamma(t)} = 0\) for \(X, Y \in \mathfrak{m}\), \(L_t\) is \(g_t\)-symmetric. Thus,

\[
\frac{d}{dt} g(X, Y)_{\gamma(t)} = g(\nabla_{\gamma'(t)} X, Y) + g(X, \nabla_{\gamma'(t)} Y) = -2g_t(L_t X, Y),
\]

where we have used that \(\nabla X, \nabla Y\) are skew-symmetric. \(\square\)

Let \(\text{ric}_t \in \text{Sym}^2(\mathfrak{m})\) denote the Ricci curvature of \((\Sigma_t, g_t)\) at \(\gamma(t)\), and \(\text{Ric}_t \in \text{End}(\mathfrak{m})\) the \(g_t\)-symmetric Ricci operator defined by

\[
\text{ric}_t(X, Y) = g_t(\text{Ric}_t X, Y), \quad X, Y \in \mathfrak{m}.
\]

**Proposition 1.2.** If \(\text{ric}_g(X, X) = -g(X, X)\) for all \(X \in \mathfrak{m}\), then

\[
L_t' = -(\text{tr} L_t) \cdot L_t + \text{Ric}_t + \text{Id}_{\mathfrak{m}},
\]

**Proof.** The Gauss equation for the hypersurface \(\Sigma_t \subset M^n\) gives

\[
\text{ric}\_g(X, Y) = \text{ric}_t(X, Y) + g(R_N X, Y) + g(L_tX, L_tY) - (\text{tr} L_t) \cdot g(L_tX, Y),
\]

for Killing fields \(X, Y \in \mathfrak{m}\), where \(R_N X = R_{\text{Ind}}(X, N)N\). The claim now follows from the Riccati equation \(\nabla_N L + L + R_N = 0\) using that \(L' = \nabla_N L\); see [EW00]. \(\square\)
A Lie group $\mathcal{G}$ is unimodular if $\text{tr} \text{ad}(X) = 0$ for all $X \in \mathfrak{g}$, $\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}$, $Y \mapsto [X, Y]$. If $\mathcal{G}$ is not unimodular, then it contains a normal, unimodular subgroup $\mathcal{G}_u$ with Lie algebra $\mathfrak{g}_u := \{X \in \mathfrak{g} : \text{tr} \text{ad}(X) = 0\}$. It follows that each $\mathcal{G}$-orbit $(\Sigma_t, g_t)$ is foliated by pairwise isometric $\mathcal{G}_u$-orbits of codimension one with a $\mathcal{G}$-invariant mean curvature vector field. At the point $\gamma(t) \in \mathcal{G}_u$, $\gamma(t) \subset \Sigma_t$ we can consider this mean curvature vector as $H_t \in \mathfrak{m}$, using (3).

An algebraic description of $H_t$ is given in \cite{Bes87} Lemma 7.32:

\[(5) \quad g_t(H_t, X) = \text{tr} \text{ad}(X), \quad \forall X \in \mathfrak{m}.\]

**Lemma 1.3.** The function

\[h : \mathbb{R} \to \mathbb{R} ; \quad t \mapsto \frac{1}{2} g_t(H_t, H_t)\]

satisfies the following evolution equations:

\[H'_t = -2 L_t H_t, \quad h' = -g_t(L_t H_t, H_t), \quad h'' = 2 \|L_t H_t\|^2_t - g_t(L_t' H_t, H_t).\]

Moreover, if $\text{ric}_g(X, X) = -g_t(X, X)$ for all $X \in \mathfrak{m}$ then

\[h'' = 2 \|L_t H_t\|^2_t - (\text{tr} L_t) h' - 2 h - \text{ric}_t(H_t, H_t).\]

**Proof.** Differentiating (5) and using Lemma 1.1 we get $0 = g_t(H'_t, X) + 2 g_t(L_t H_t, X)$ for all $X \in \mathfrak{m}$, which gives the first formula. Using that, we may compute

\[h' = \frac{1}{2} g'_t(H_t, H_t) + g_t(H'_t, H_t) = -g_t(L_t H_t, H_t).\]

Differentiating once more and using Lemma 1.1 again we obtain

\[h'' = -g'_t(L_t H_t, H_t) - g_t(L'_t H_t, H_t) - 2 g_t(L'_t H_t, H_t) = 2 \|L_t H_t\|^2_t - g_t(L'_t H_t, H_t).\]

Finally, the last claim follows from Proposition 1.2. \hfill \square

**Remark 1.4.** For any homogeneous space $(\mathcal{G}/\tilde{H}, \tilde{g})$, the mean curvature vector $H_{\tilde{g}}$ and its norm $h(\tilde{g})$ can be defined as in (5) and Lemma 1.3. Recall now that $\text{Ric}_{\tilde{g}} = \text{Ric}^g - S^g(\text{ad}(H_g))$ and that $\text{tr} S^g(\text{ad}(H_g)) = \|H_g\|^2_g = 2 h(\tilde{g})$: see the proof of Lemma 3.5 in \cite{BL18}. Notice moreover, that $\mathcal{G}$ is unimodular if and only if $h(\tilde{g}) = 0$.

2. **Proof of Theorem D**

In \cite{Heb98} Heber obtained deep structure results for *standard* Einstein solvmanifolds. Building on that, Lauret extended these results to arbitrary Einstein solvmanifolds by showing that they are all standard \cite{Lau10}. The main result of this section is Theorem D which generalizes Lauret’s result to orbit-Einstein cohomogeneity-one manifolds: those satisfying the Einstein condition $\text{Ric}_{\tilde{g}}(v, v) = c g(v, v)$ for all directions $v$ tangent to the group orbits.

Let $(\mathcal{G}/\tilde{H}, \tilde{g})$ be an effective homogeneous space with compact isotropy $\tilde{H}$ and canonical reductive decomposition $\tilde{g} = \mathfrak{h} \oplus \mathfrak{m}$. Extend the $\text{Ad}(\tilde{H})$-invariant scalar product $\tilde{g}$ on $T_{e\tilde{H}} \mathcal{G}/\tilde{H} \simeq \tilde{\mathfrak{m}}$ to an $\text{Ad}(\tilde{H})$-invariant scalar product $\tilde{g}$ on $\tilde{g}$ with $\tilde{g}(\mathfrak{m}, \mathfrak{h}) = 0$.

**Definition 2.1.** We call the homogeneous space $(\mathcal{G}/\tilde{H}, \tilde{g})$ *standard*, if the $\tilde{g}$-orthogonal complement of the nilradical $\tilde{\mathfrak{n}}$—the maximal nilpotent ideal in $\tilde{\mathfrak{g}}$—is a Lie subalgebra of $\tilde{\mathfrak{g}}$.

Recall that $B_3(\tilde{n}, \tilde{g}) = 0$, thus $\tilde{n} \subset \mathfrak{m}$ (see \cite{BL18} Lemma 5.1), and note that this definition does not depend on the way the scalar product is extended. Geometrically, standardness means that the submersion given by the isometric action of the nilradical $\tilde{N} \leq \tilde{G}$ (Lie($\tilde{N}$) = $\tilde{n}$) on $(\mathcal{G}/\tilde{H}, \tilde{g})$ has integrable horizontal distribution.

Turning to the proof of Theorem D recall that we denoted by $\tilde{H} = \mathcal{G}_{\gamma(t)}$ the isotropy at the points $\gamma(t)$ of a normal, unit-speed geodesic, see Section 1. Consider the endomorphism $\beta \in \text{End}(\tilde{g})$ associated to the homogeneous space $\mathcal{G}/\tilde{H}$ after fixing the background metric.
\[ \tilde{g} := g_0, \text{ as explained in Appendix B} \]

We may write the family \((g_t)_{t \in \mathbb{R}}\) of scalar products on \(\tilde{m}\) induced by \(g\) as \(g_t = \tilde{g}_t \cdot \tilde{g}\), for some smooth curve \((\tilde{g}_t)_{t \in \mathbb{R}}\) in \(\text{GL}(\tilde{m})\) with \(g_0 = \text{Id}_{\tilde{m}}\), see (18).

By Lemma A.1 and the discussion around (22) we may even assume that

\[ (\tilde{g}_t)_{t \in \mathbb{R}} \subset Q^\beta_{\tilde{g}}(\tilde{m}), \]

where \(Q^\beta_{\tilde{g}}(\tilde{m}) := Q_\beta \cap \text{GL}(\tilde{m})\), see (21), and \(Q_\beta \subset \text{GL}(\tilde{g})\) is the parabolic subgroup associated to \(\beta\), defined in Appendix B.

**Definition 2.2.** Let

\[ l_\beta : \mathbb{R} \to \mathbb{R}; \quad t \mapsto \text{tr} \left( L_t \beta^t q_t^{-1} \right), \]

where \(\beta^t := (\|\beta\|^2/\beta + \text{Id}_{\tilde{g}})_m \in \text{End}(\tilde{m}).\)

By Theorem B.2 the endomorphism \(\|\beta\|^2/\beta + \text{Id}_{\tilde{g}}\) of \(\tilde{g}\) preserves \(\tilde{m}\) and its kernel contains \(\tilde{n}\).

Moreover, the image of the \(\tilde{g}\)-selfadjoint, positive-semidefinite endomorphism \(\beta^t\) is precisely \(\tilde{n}\). Thus \(\text{tr} \beta^t \geq 0\). Notice also that the shape operator \(L_t\) is symmetric with respect to \(g_t\), but \(q_t^{-1} \cdot L_t \cdot q_t\) is symmetric with respect to the background metric \(g\).

**Lemma 2.3.** If \(M^n/\tilde{G} = S^1\) then \(l_\beta\) is periodic.

**Proof.** Assuming that \(\gamma(T) \in \Sigma_0 = \tilde{G}/\tilde{H}\) for \(T > 0\), it is enough to show \(l_\beta(0) = l_\beta(T)\). Let \(f \in \tilde{G}\) with \(f \cdot \gamma(0) = \gamma(T)\). Since \(f\) normalizes \(\tilde{H}\), we have \(q_T \cdot \tilde{g} = (\text{Ad}_m f) \cdot q_0 \cdot \tilde{g}\) by Lemma A.2, which implies \(q_T^{-1} \cdot (\text{Ad}_m f) \in \text{O}(\tilde{m})\) using \(g_0 = \text{Id}_{\tilde{m}}\). On the other hand, after extending \(q_T\) to all of \(\tilde{g}\) (as in (19)), we have that \(q_T \in Q_\beta\) by (6) and (21). This, together with \(\text{Ad} f \in \text{Aut}(\tilde{g}) \subset Q_\beta\), see Theorem B.2, gives \(B := q_T^{-1} \cdot (\text{Ad} f) \in \text{O}(\tilde{g}) \cap Q_\beta = K_\beta\). In particular \(B\) commutes with \(\beta\). Thus, setting \(A := (\text{Ad}_m f)_m\) we obtain by Corollary A.3

\[ l_\beta(T) = \text{tr}(L_T q_T \beta^t q_T^{-1}) = \text{tr}(AL_\beta A^{-1} q_T \beta^t q_T^{-1}) = \text{tr}(L_\beta B^{-1} \beta^t B) = l_\beta(0), \]

concluding the proof. \(\square\)

We turn now to an apriori estimate for \(h\), which holds for arbitrary homogeneous spaces:

**Lemma 2.4.** For all \(t \in \mathbb{R}\) we have \(\text{ric}_t(H_t, H_t) \cdot \text{tr} \beta^t \geq 4h^2(t)\).

**Proof.** By [AL17] we have

\[ \text{ric}_t(H_t, H_t) = -\|S_t(\text{ad}_m H_t)\|_t^2. \]

Here, \(\text{ad}_m(H_t) = \text{pr}_m \circ \text{ad}(H_t)|_m \in \text{End}(\tilde{m})\), \(\text{pr}_m\) denotes the projection with respect to the decomposition \(\tilde{g} = \tilde{h} \oplus \tilde{m}\), and for an endomorphism \(E \in \text{End}(\tilde{m})\) we denote its \(g_t\)-symmetric part by \(S_t(E) = \frac{1}{2}(E + E^T)\) and its \(g_t\)-norm squared by \(\|E\|_t^2 = \langle E, E \rangle_t = \text{tr}(EE^T)\), transposes taken with respect to the metric \(g_t\). By Cauchy-Schwarz inequality we have

\[ \|S_t(\text{ad}_m(H_t))\|_t^2 \cdot \|q_t \beta^t q_t^{-1}\|_t^2 \geq \langle S_t(\text{ad}_m(H_t)), q_t \beta^t q_t^{-1} \rangle^2. \]

Using that \(q_t \beta^t q_t^{-1}\) and \(S_t(\text{ad}_m(H_t))\) are \(g_t\)-symmetric, we may rewrite the latter as

\[ -\text{ric}_t(H_t, H_t) \cdot \text{tr}((\beta^t)^2) \geq (\text{tr}(\text{ad}_m(H_t) q_t \beta^t q_t^{-1}))^2. \]

A short computation using that \(\text{tr} \beta = -1\) shows now \(\text{tr}((\beta^t)^2) = \text{tr} \beta^t\). On the other hand, Theorem B.2 yields \(\text{ad}(H_t) \in \text{sl}_\beta\), and since \(\text{sl}_\beta\) is an ideal in \(q_\beta\), we have \(q_t^{-1} \cdot \text{ad}(H_t) q_t \in \text{sl}_\beta\). Since \(\text{tr}(E_\beta) = 0\) for all \(E_\beta \in \text{sl}_\beta\) this implies by the very definition of \(\beta^t\) that

\[ \text{tr}_m(q_t^{-1} \cdot \text{ad}_m(H_t) q_t \beta^t) = \text{tr}_{\tilde{h}}(q_t^{-1} \cdot \text{ad}(H_t) q_t \beta^t) = \text{tr} \text{ad}_m(H_t) = 2h(t) \]

using (5) and \([\tilde{m}, \tilde{h}] \subset \tilde{m}\). This shows the claim. \(\square\)
Lemma 2.5 (Maximum principle). Let \((M^n, g)\) be orbit-Einstein with Einstein constant \(-1\) and suppose that \(M^n/\bar{G} = S^1\). Then, \(\text{tr}\beta^+ \geq 2 h(t)\) for all \(t \in \mathbb{R}\).

Proof. Since \(\beta^+ \geq 0\) by Theorem 3.2, \(h(t) \equiv 0 \leq \text{tr}\beta^+\) if \(\bar{G}\) is unimodular: see Remark 1.4. If \(\bar{G}\) is not unimodular, then \(h, \text{tr}\beta^+ > 0\). Since \(M^n/\bar{G} = S^1\), \(h\) is periodic, since it is defined geometrically: see also Remark 1.4. Thus it attains a global maximum \(h(t_0) > 0\), \(t_0 \in [0, T]\).

We deduce from Lemma 1.3 and Lemma 2.4 that

\[
0 \geq 2 \|L_{t_0}H_{t_0}\|_{t_0}^2 - 2h(t_0) - \text{ric}_{t_0}(H_{t_0}, H_{t_0}) \geq 2h(t_0) \left(\frac{2h(t_0)}{\text{tr}\beta^+} - 1\right).
\]

Hence, \(\text{tr}\beta^+ \geq 2h(t_0) \geq 2h(t)\) for all \(t \in \mathbb{R}\), as claimed. \(\square\)

Lemma 2.6 (Lyapunov function). Let \((M^n, g)\) be orbit-Einstein with Einstein constant \(-1\) and suppose that \(M^n/\bar{G} = S^1\). Then

\[
w_\beta: \mathbb{R} \to \mathbb{R}; \quad t \mapsto l_\beta(t) \cdot e^{\int_0^t \text{tr} L_s \, ds}
\]
is non-decreasing. If \(w_\beta\) is constant, then \(\beta^+ \in \text{Der}(\bar{g})\) and \([q_t, \beta] = 0\) for all \(t \in \mathbb{R}\).

Proof. Thanks to Proposition 1.2 and Definition 2.2, we have

\[
l'_\beta(t) = \text{tr} \left( L'_t q_t \beta^+ q_t^{-1} \right) + \text{tr} \left( L_t [q_t \beta^+ q_t^{-1} q_t^{-1}] \right)
\]

\[
= \text{tr} \left( (\text{Ric}_t + \text{Id}_m) q_t \beta^+ q_t^{-1} \right) + \text{tr} \left( (q_t^{-1} L_t q_t) [q_t^{-1} q_t', \beta] \right) - (\text{tr} L_t) \cdot l_\beta(t).
\]

Regarding the second term, first notice that for \(X, Y \in \bar{m}\) we have by Lemma 1.1

\[
2g_t(L_t X, Y) = g'_t(X, Y) = (q_t \cdot \bar{g})(X, Y) = -g_t((q_t^{-1} q_t') X, Y) - g_t(X, (q_t q_t') Y).
\]

Thus, \(L_t + q_t q_t'^{-1}\) is \(\text{so}(\bar{m}, q_t)\) or equivalently \(q_t^{-1} L_t q_t \in \text{so}(\bar{m}, \bar{g})\) for all \(t \in \mathbb{R}\), since \(g_t = q_t \cdot \bar{g}\). Since \(q_t^{-1} L_t q_t\) is \(\bar{g}\)-symmetric for each \(t\) and \(q_t^{-1} q_t' \in q_t \beta \in T_c Q_\beta\) by (11) and (21), after extending these linear maps trivially to \(\text{so}(\bar{g})\), we are in position to apply Lemma 2.3 for \(S = q_t^{-1} L_t q_t\) and \(Q = q_t q_t'^{-1}\). This yields

\[
\text{tr} \left( (q_t^{-1} L_t q_t) [q_t^{-1} q_t', \beta] \right) \geq 0,
\]

with equality if and only if \([q_t^{-1} q_t', \beta] = 0\).

Regarding the first term, notice that by Theorem 3.2 we have

\[
\text{tr} (\text{Ric}_t q_t \beta^+ q_t^{-1}) \geq -2 h(t),
\]

with equality if and only if \(q_t (\frac{1}{\|q_t\|^2} \beta + \text{Id}_q) q_t^{-1} \in \text{Der}(\bar{g})\). Thus,

\[
\text{tr} \left( (\text{Ric}_t + \text{Id}_m) q_t \beta^+ q_t^{-1} \right) \geq -2 h(t) + \text{tr} \beta^+ \geq 0
\]

by Lemma 2.5. We conclude that \(l'_\beta(t) + l_\beta(t) \text{tr} L_t \geq 0\), showing the first claim.

Finally, suppose \(w'_\beta \equiv 0\) and set \(C_t := [q_t, \beta]\) for all \(t \in \mathbb{R}\). Then from equality in (7) we deduce \(C'_t = C_t \cdot (q_t^{-1} q_t')\) with \(C_0 = 0\), \(q_0 = \text{Id}_m\). This shows \(C_t \equiv 0\). From the equality condition in (8) we then obtain \(\beta^+ \in \text{Der}(\bar{g})\). \(\square\)

Proof of Theorem 1.2. We have \(\int_0^T \text{tr} L_s \, ds = 0\), since by assumption the \(\bar{G}\)-orbits are integrally minimal. As a consequence, by Lemma 2.3 and Lemma 2.6 the function \(w_\beta(t)\) is monotone and periodic, therefore constant. The rigidity conditions in Lemma 2.6 imply that \(\beta^+ := \frac{1}{\|q_t\|^2} \beta + \text{Id}_q \in \text{Der}(\bar{g})\) and \([q_t, \beta] = 0\) for all \(t \in \mathbb{R}\). Using that \(0 = [q_t, \beta] = [q_t, \beta^+]\) and \(\bar{n} = \text{Im} \beta^+ = \text{Im} \beta^+\), we see that with respect to \(q_t = q_t \cdot \bar{g}\), the \(g_t\)-orthogonal complement of \(\bar{n}\) in \(\bar{g}\) is a fixed subspace \(\bar{n}^+\) of \(\bar{g}\), for all \(t \in \mathbb{R}\). Since \(\beta^+\) is \(g_0\)-symmetric (and in fact also \(g_t\)-symmetric for all \(t \in \mathbb{R}\)), \(\bar{n}^+ = \ker \beta^+\). But the kernel of a derivation is always a Lie subalgebra, therefore all orbits are standard homogeneous spaces as claimed. \(\square\)
Remark 2.7. When writing \( g_l(\cdot, \cdot) = 2 \bar{g}_l(\cdot, \cdot) \) for a \( \bar{G} \)-invariant background metric \( \bar{g} \) on \( \bar{G}/H \), integral minimality is equivalent to the periodicity of the volume \( V(t) = \det L_t \), because \( \text{tr} L_t = V'(t)/V(t) \). Periodicity of \( V \) is automatically satisfied if \( \bar{G} \) is unimodular, but a true assumption if \( \bar{G} \) is not: one may declare any smooth curve \( \gamma : [0, T] \to M^n \) intersecting all orbits transversally to a normal geodesic if \( \gamma'(T) = (dL_{\bar{g}}) \cdot \gamma'(0) \), see [GYZ08].

3. Euclidean homogeneous spaces

The Euclidean space \( \mathbb{R}^n \) can be presented as an effective homogeneous space \( G/H \) in many different ways. Here and in what follows, \( G \) is a non-compact, connected Lie group and \( H \leq G \) a connected, compact subgroup. Recall that \( H \) is compact if and only if \( G \) is closed in \( \text{Iso}(G/H, g) \) for some (and also any) \( G \)-invariant metric \( g \): see [DM88, Thm.1.1]. For instance, \( \mathbb{R}^n \) is itself an abelian Lie group, and more generally a solvmanifold, that is \( \mathbb{R}^n = G/(e) \) for any simply-connected, solvable Lie group \( G \) of dimension \( n \). Any non-compact irreducible symmetric space has a presentation \( \mathbb{R}^n = L/K \) with \( L \) simple and centerless, and \( K \leq L \) maximal compact. Another possibility is that \( G \) is the universal covering group of \( SL(2, \mathbb{R}) \) and \( H = \{e\} \), or a product of any of the above examples.

The main aim of this section is to characterize certain effective presentations \( \mathbb{R}^n = G/H \), where \( \dim G \) has minimal dimension. Let \( g = \mathfrak{l} \times \mathfrak{r} \) be a Levi decomposition of \( \text{Lie}(G) = \mathfrak{g} \), by which we mean that \( \mathfrak{l} \) is a maximal semisimple Lie subalgebra and \( \mathfrak{r} \) the radical of \( \mathfrak{g} \) (maximal solvable ideal). Let \( L, R \leq G \) denote the connected Lie subgroups of \( G \) with \( \text{Lie}(L) = \mathfrak{l} \) and \( \text{Lie}(R) = \mathfrak{r} \), and suppose that \( \text{Lie}(H) = \mathfrak{h} \subset \mathfrak{l} \). Notice that for non-compact homogeneous Einstein spaces, the structure results yield such a Levi decomposition, see Section 2.

Proposition 3.1. Let \( (\mathbb{R}^n, g) \) be a homogeneous Euclidean space with effective presentation \( \mathbb{R}^n = G/H \), \( G \) connected, \( H \) compact and \( \dim G \) minimal. Assume in addition that there exists a Levi decomposition \( g = \mathfrak{l} \times \mathfrak{r} \) with \( \mathfrak{h} \subset \mathfrak{l} \). Then, \( G \simeq L \ltimes R \), \( H \) is semisimple, and \( L/H \) is an \( \mathbb{R}^r \)-bundle over a Hermitian symmetric space of non-compact type.

A Cartan decomposition \( \mathfrak{l} = \mathfrak{t} \oplus \mathfrak{p} \) for the semisimple Lie algebra \( \mathfrak{l} \) is obtained by letting \( \mathfrak{t} \) (resp. \( \mathfrak{p} \)) be the \(+1\) (resp. \(-1\)) eigenspace of a Cartan involution \( \sigma : \mathfrak{l} \to \mathfrak{l} \), an involutive automorphism of \( \mathfrak{l} \) such that \(-B(\cdot, \sigma \cdot) > 0\), where \( B_\mathfrak{l} \) denotes the Killing form of \( \mathfrak{l} \). The pair \( (\mathfrak{t}, \mathfrak{p}) \) satisfies the bracket relations \( [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{t}, [\mathfrak{t}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t} \). If \( \mathfrak{t} \) has no compact simple ideals, we call \((\mathfrak{t}, \mathfrak{t})\) a symmetric pair of non-compact type. The subalgebra \( \mathfrak{t} \) will be called a maximal compactly embedded subalgebra. Notice however, that the corresponding connected Lie subgroup \( K \leq L \) is compact if and only if \( Z(L) \) is finite [Hel01, Ch.6, Thm.1.1].

A special class of symmetric spaces are the Hermitian symmetric spaces, see [Hel01, Ch. VIII, Thm.6.1] or [Bes87, 7.104]. Irreducible ones correspond to irreducible symmetric pairs \((\mathfrak{l}, \mathfrak{t})\) with \( \dim \mathfrak{t} = 1 \). Writing \( \mathfrak{t} = \mathfrak{z}(\mathfrak{t}) \oplus \mathfrak{t}_{ss} \), we obtain a homogeneous space \( L/K_{ss} \), which is an \( \mathbb{S}^1 \)-bundle over the Hermitian symmetric space \( L/K = \mathbb{R}^k \) (assuming \( L \) has finite center). As a consequence, the universal covering space of \( L/K_{ss} \), still being a homogeneous space, is diffeomorphic to \( \mathbb{R}^{k+1} \) and on Lie algebra level still presented by the pair \((\mathfrak{l}, \mathfrak{t}_{ss})\). As a smooth manifold it is an \( \mathbb{R}^r \)-bundle over \( L/K \). Considering products of such \( \mathbb{R}^r \)-bundles we obtain more complicated presentations \( G/H = \mathbb{R}^n \), which on Lie algebra level can be described as follows:

\[
\mathfrak{l} := l_1 \oplus \cdots \oplus l_r, \quad \mathfrak{h} := t_1^{ss} \oplus \cdots \oplus t_r^{ss} \oplus \mathfrak{z}(\mathfrak{h}), \quad \mathfrak{z}(\mathfrak{h}) \subset \mathfrak{z}(\mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_r)
\]

where \((l_i, t_i) = z(l_i) \oplus t_i^{ss})\) is an irreducible Hermitian symmetric pair of non-compact type, \(1 \leq i \leq r\), and \( \mathfrak{z}(\mathfrak{h}) \) is a proper subalgebra of \( \mathfrak{z}(\mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_r) \). Now \( \mathfrak{h} \) is semisimple if and only if \( \mathfrak{z}(\mathfrak{h}) = 0 \), in which case the homogeneous space \( L/H \) corresponding to such pairs \((l, h)\) will be called an \( \mathbb{R}^r \)-bundle over a non-compact Hermitian symmetric space.
Lemma 3.3. Under the assumptions of Proposition 3.1, we have that two $A(d(k)_{bracket}$ at the same time. Consequently, for any of the three $K$ subgroups with Lie algebras $l$, $h$ is the morphism defining the action. Clearly $m_o \circ \Phi \circ o$ is trivial $O(1,2)$-module, since $d \in O(1,2)$ is a trivial $O(d(K_{ss}))$-module, since $K_{ss} = \{e\}$. It follows that $\dim M \leq h_{ss} = 6$. However, for each $g$ there exists a “Milnor basis” $\mathbb{M}$ of $O(2,\mathbb{R})$ diagonalising both the metric and the bracket at the same time. Consequently, for any of the three $m, o \circ \Phi \circ o$ is trivial $O(2,\mathbb{R})$-module, since $d \in O(2,\mathbb{R})$ is a trivial $O(d(K_{ss}))$-module, since $K_{ss} = \{e\}$. It follows that $\dim M \leq h_{ss} = 6$. However, for each $g$ there exists a “Milnor basis” $\mathbb{M}$ of $O(2,\mathbb{R})$ diagonalising both the metric and the bracket at the same time. Consequently, for any of the three $m, o \circ \Phi \circ o$ is trivial $O(2,\mathbb{R})$-module, since $d \in O(2,\mathbb{R})$ is a trivial $O(d(K_{ss}))$-module, since $K_{ss} = \{e\}$. It follows that $\dim M \leq h_{ss} = 6$. However, for each $g$ there exists a “Milnor basis” $\mathbb{M}$ of $O(2,\mathbb{R})$ diagonalising both the metric and the bracket at the same time. Consequently, for any of the three $m, o \circ \Phi \circ o$ is trivial $O(2,\mathbb{R})$-module, since $d \in O(2,\mathbb{R})$ is a trivial $O(d(K_{ss}))$-module, since $K_{ss} = \{e\}$. It follows that $\dim M \leq h_{ss} = 6$.

We now describe the space $M_{L/K_{ss}}$ of $L$-invariant metrics on $L/K_{ss}$. We will show that for any $g \in M_{L/K_{ss}}$ there exists a Cartan decomposition $l = h \oplus \mathfrak{p}$ with $h \circ \Phi \circ o \circ p \circ \Phi \circ o = eK_{ss}$.

Let $m = m(h) \oplus \mathfrak{p}$ be the canonical complement of $l = [\mathfrak{h}, \mathfrak{p}]$ in $l$, where $h = m(h) \oplus \mathfrak{p}$. Except for the first and the last cases above, $p$ is $Ad(K_{ss})$-irreducible and non-trivial, see [BB78] and [WZ86], and consequently $\dim M_{L/K_{ss}} = 2$ and $h \circ \Phi \circ o \circ p \circ \Phi \circ o$. In the first case, $m = sl(2,\mathbb{R})$ is a trivial $Ad(K_{ss})$-module, since $K_{ss} = \{e\}$. It follows that $\dim M_{L/K_{ss}} = 6$. However, for each $g$ there exists a “Milnor basis” $\mathbb{M}$ of $O(2,\mathbb{R})$ diagonalising both the metric and the bracket at the same time. Consequently, for any of the three $m, o \circ \Phi \circ o$ is trivial $O(2,\mathbb{R})$-module, since $d \in O(2,\mathbb{R})$ is a trivial $O(d(K_{ss}))$-module, since $K_{ss} = \{e\}$. It follows that $\dim M_{L/K_{ss}} = 6$. However, for each $g$ there exists a “Milnor basis” $\mathbb{M}$ of $O(2,\mathbb{R})$ diagonalising both the metric and the bracket at the same time. Consequently, for any of the three $m, o \circ \Phi \circ o$ is trivial $O(2,\mathbb{R})$-module, since $d \in O(2,\mathbb{R})$ is a trivial $O(d(K_{ss}))$-module, since $K_{ss} = \{e\}$. It follows that $\dim M_{L/K_{ss}} = 6$.

The proof of Proposition 3.1 will be clear after the following series of lemmas.

Lemma 3.3. Under the assumptions of Proposition 3.1 we have that $H \leq L$ and $G \simeq L \ltimes R$.

Proof. The universal cover $\tilde{G}$ of $G$ acts transitively and almost-effectively on $(\mathbb{R}^n, g)$, with connected isotropy $\tilde{H}$ at 0, since $\mathbb{R}^n$ is simply-connected. Thus, if $L, R \leq G$ denote the connected Lie subgroups with Lie algebras $l, \mathfrak{r}$ respectively, then $h \subset l$ implies $H \leq L$. Moreover, since $\tilde{G}$ is simply-connected, $\tilde{G} \simeq \tilde{L} \ltimes \tilde{R}$ (Var84, 3.18.13). We have $G = \tilde{G}/ker\tau$, where $\tau : \tilde{G} \to \Iso(\mathbb{R}^n, g)$ is the morphism defining the action. Clearly $\ker\tau \leq H$, thus $G \simeq L \ltimes R$, where $R \simeq \tilde{R}$ is simply-connected and $L \simeq L / \ker\tau$.

Lemma 3.4. Under the assumptions of Proposition 3.1 there exists a Cartan decomposition $l = h \oplus \mathfrak{p}$ such that $h \subset l$. Moreover, $h = [\mathfrak{h}, \mathfrak{p}] \oplus m(h)$ is an abelian extension of $h$, that is, $t_{ss} : = [\mathfrak{h}, \mathfrak{p}] \subset h$, and $(l, \mathfrak{p})$ is a symmetric pair of non-compact type.

Proof. Since $L$ is semisimple, $Ad : L \to GL(l)$ has discrete kernel and $\Lie(Ad(L)) = l$. Moreover, the compact subgroup $Ad(H)$ is contained in a maximal compact subgroup $K$ of the centerless semisimple Lie group $Ad(L)$. By [Hel01, Ch.6, Thm.1.1], $t := \Lie(K)$ is the fixed point set of a Cartan involution. The corresponding Cartan decomposition $l = h \oplus \mathfrak{p}$ satisfies $h \subset l$.

We now show that $[h, t] \subset h$. Let $K, K_{ss} := [K, K]$ be the connected Lie subgroups of $G$ with Lie algebras $l, t_{ss}$ respectively. It is well-known that $G/H = \mathbb{R}^n$ if and only if $H$ is a maximal compact subgroup of $G$, see for instance [LL14 Prop.3.1]. The group $K_{ss}$ is semisimple and has a compactly embedded Lie algebra $t_{ss} \subset l$, therefore it is compact. By uniqueness of maximal compact subgroups, there exists $x \in G$ such that $xK_{ss}x^{-1} \subset H$. At Lie algebra level this implies that $Ad(x)(t_{ss}) \subset h \subset t$. But the Lie algebra $l = t_{ss} \oplus n(h)$ has a unique maximal semisimple ideal, therefore we must have $t_{ss} = Ad(x)(t_{ss})$ and $t_{ss} \subset h$ as claimed.

To prove the last claim let $l_c$ (resp. $l_{nc}$) be the sum of all simple ideals of $l$ of compact (resp. non compact) type, and let $L_c, L_{nc} \leq L$ be the corresponding connected Lie subgroups. The ideal $t_c$ is the unique maximal semisimple ideal of compact type in $l$. Arguing as above, since $L_c$ is compact, there exists $x \in G$ such that $xL_cx^{-1} \leq H$, thus $Ad(x)(t_c) \subset h \subset l$. Then $l_c = Ad(x)(l_c)$ by uniqueness, and $l_c \subset h$. It then follows easily that the connected subgroup of $G$ with Lie algebra $t_{nc} \ltimes \tau$ is closed in $G$ by Lemmas 3.3 and 3.4 and acts transitively on $G/H$. This contradicts minimality of $\dim G$, unless $l_c = 0$, and the conclusion follows.

In order to prove the last part of Proposition 3.1 let us introduce the following class of subgroups of $G$, which will be very important in the sequel.
Proof. By the assumption, it is enough to show that the connected Lie subgroup $\bar{L}$ of $L$ with Lie algebra $l := (a_1 \oplus n_1) \oplus l_2 \oplus \cdots \oplus l_r$ is closed in $L$. Let $L_1 \leq L$ denote the connected Lie subgroup with Lie algebra $l_1$. Since the Iwasawa decomposition holds at the group level, we have $L_1 = K_1 A_1 N_1$ diffeomorphic to $K_1 \times A_1 \times N_1$. In particular, $A_1 N_1$ is closed in $L_1$. Recall that for a connected semisimple Lie group, $L_1$ is closed in $L$ for all $i = 1, \ldots, r$, by Lemma A.4. Then the lemma follows from the fact that $L = A_1 N_1 L_2 \cdots L_r$. □

Lemma 3.7. Under the assumptions of Proposition 3.7, if $I = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition with $\mathfrak{h} \subset \mathfrak{k}$, then $\mathfrak{h} = \mathfrak{k}_{ss}$.

Proof. Thanks to Lemma 3.4 we only need to prove that $\mathfrak{z}(\mathfrak{h}) = 0$. Assume on the contrary that $\mathfrak{z}(\mathfrak{h})$ projects nontrivially onto $\mathfrak{z}(\mathfrak{k}_1)$. This means that there exists $Z \in \mathfrak{z}(\mathfrak{h})$ such that

$$Z = Z_1 + Z_2, \quad Z_1 \in \mathfrak{k}(\mathfrak{t}_1) \setminus \{0\}, \quad Z_2 \in \mathfrak{z}(\mathfrak{t}_2 + \cdots + \mathfrak{t}_r)$$

in the notation of (9). Let $\bar{G} \leq G$ be an Iwasawa subgroup associated to $(t_1, k_1)$. We claim that $g = h + \bar{g}$, which would follow from $t_1 = t_1^{ss} + \mathfrak{z}(t_1) \subset h + \bar{g}$. By Lemma 3.4 it suffices to show that $\mathfrak{z}(t_1) \subset h + \bar{g}$, which follows from (10), since $\dim(\mathfrak{z}(t_1)) = 1$.

Now $\bar{G}$ acts transitively on $G/H$ by the above claim, effectively because $G$ acts effectively, and with compact isotropy, since this is equivalent to $\bar{G}$ being closed in the isometry group of $g$, which holds thanks to Lemma A.4 and the fact that $G$ is closed in $\text{Iso}(\mathbb{R}^n, g)$. But $\dim \bar{G} < \dim G$, and this contradicts the minimality of $\dim G$. □

4. Periodicity

In this section we show that after passing to a quotient we may assume that Iwasawa subgroups act on $\mathbb{R}^n = G/H$ with cohomogeneity one and orbit space $S^1$. The most basic example for periodicity is given by the universal cover $\bar{G} \simeq S^3$ of $G = \text{SL}(2, \mathbb{R})$. Since $\text{SL}(2, \mathbb{R}) = \Gamma \backslash \bar{G}$ for some discrete subgroup $\Gamma \leq Z(\bar{G})$, for any left-invariant metric $\bar{g}$ on $\bar{G}$ there is a left-invariant metric $g$ on $\text{SL}(2, \mathbb{R})$ locally isometric $\bar{g}$. Now from any Iwasawa decomposition $\text{SL}(2, \mathbb{R}) = \text{KAN}$ one gets a solvable subgroup $\bar{G} = \text{AN} \leq \text{SL}(2, \mathbb{R})$ acting on $(M^3 = \text{SL}(2, \mathbb{R}), g)$ with cohomogeneity one and orbit space $K \simeq \text{SO}(2) \simeq S^1$.

Proposition 4.1 (Periodicity). Let $(\mathbb{R}^n = G/H, g)$ be an effective presentation, $H$ compact and $\dim G$ minimal, and let $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{r}$ be a Levi decomposition with $\mathfrak{h} \subset \mathfrak{l}$. Then, there exist a discrete, central subgroup $\Gamma \leq G$ for which $(M^n = G_{\Gamma}/H_{\Gamma}, g_{\Gamma}) := (G/H, g)/\Gamma$, $G_{\Gamma} = G/\Gamma$, $H_{\Gamma} = H/(H \cap \Gamma)$, is a homogeneous space locally isometric to $(\mathbb{R}^n, g)$ such that any Iwasawa subgroup $\bar{G} \leq G_{\Gamma}$ acts on $M^n$ with cohomogeneity one, embedded orbits and orbit space $S^1$.

Proof. By Proposition 3.1 we have a global Levi decomposition $G \simeq L \times R$. Let $\Gamma := Z(G) \cap L$, a central subgroup of $G$ and $L$, hence discrete by semisimplicity. It is not hard to see that the group $G_{\Gamma} := G/\Gamma$ acts transitively on the double coset $\Gamma \backslash G/H$, with isotropy $H_{\Gamma} := H/(H \cap \Gamma)$. Since $\Gamma \leq L$ we clearly also have a global Levi decomposition $G_{\Gamma} \simeq L_{\Gamma} \times R_{\Gamma}$, $L_{\Gamma} := L/\Gamma$. Since $G_{\Gamma}$ and $H_{\Gamma}$ are connected with $\text{Lie}(G_{\Gamma}) = \mathfrak{g}$ and $\text{Lie}(H_{\Gamma}) = \mathfrak{h}$, the space of homogeneous metrics on $G/H$ and $M^n = G_{\Gamma}/H_{\Gamma}$ agrees. Moreover, Iwasawa subgroups $\bar{G}_{\Gamma} \leq G_{\Gamma}$ will act with cohomogeneity one on $M^n$, and by Lemma 3.6 they are closed in $G_{\Gamma}$.
It remains to be shown that the orbit space is $S^1$. To that end, if $\overline{G}_\Gamma \leq \Gamma_G$ is an Iwasawa subgroup associated to $(l_1, t_1)$, let $K_1 \leq L_1 \leq L_{\Gamma}$ be the corresponding connected Lie subgroups with Lie algebras $\mathfrak{k}_1 \subset \mathfrak{t}_1$, respectively. By Proposition 4.1 (see also (9)) we have $\mathfrak{t}_1 = \mathfrak{t}_{1, s}^* \oplus (\mathfrak{t}(1))$ with dim $\mathfrak{t}(1) = 1$. Consider now the action of the connected Lie group $Z(K_1) \subset N_{G}(H_{\Gamma})$ on $M^n$ by right-multiplication. Since this action commutes with the action of $\overline{G}_\Gamma$ by left-multiplication, the direct product $\overline{G}_\Gamma \times Z(K_1)$ acts transitively on $M^n$. Thus, $Z(K_1)$ acts transitively on the orbit space $M^n/\overline{G}_\Gamma$. Finally notice, that the group $\Gamma$ is the kernel of the adjoint representation $\text{Ad} : G \to \text{Aut}(G)$ restricted to $L$. Since $\text{Ad} : L \to \text{End}(g)$ is a faithful representation, by [HN11] Prop. 16.1.7 the group $L_{\Gamma}$ is linearly reductive, implying finite center. By the same reason the center of $L_1 \leq L_{\Gamma}$ is also finite, thus $K_1$ is compact by [HN11] Ch.6, Thm.1.1 and therefore $Z(K_1) \simeq S^1$. \hfill $\Box$

5. Integral minimality

Let $(\mathbb{R}^n = G/H, g)$ satisfy the assumptions of Proposition 4.1 and consider its quotient space $(M^n = \overline{G}_\Gamma/\Gamma_G, g_{\overline{G}_\Gamma})$ given by that result. The proof of Proposition 4.1 yields also a global Levi decomposition $G_\Gamma = L \ltimes R$ with $H_{\Gamma} \leq L$. From now on we also assume that at $o := eH$ we have

$$L \cdot o \perp_{\sigma} R \cdot o.$$  

To ease notation, in what follows we will drop the subscripts $\Gamma$ and write simply $G/H$.

**Definition 5.1.** A homogeneous space $(G/H, g)$ as above satisfying (11) and the conclusion of Proposition 4.1 is said to have a *Levi presentation*.

The main result in this section is the following

**Theorem 5.2** (Integral minimality). Let $(G/H, g)$ have a Levi presentation and choose a simple factor $l_1 \subset I$ and a maximal compactly embedded subalgebra $\mathfrak{t}_1 \subset \mathfrak{l}_1$. Then, there exists an Iwasawa subgroup $\overline{G} \leq G$ associated to $(l_1, t_1)$ which acts on $G/H$ with cohomogeneity one, embedded orbits and orbit space $S^1$. Moreover,

$$\int_{S^1} \text{tr} L_t \, dt = 0,$$

where $\text{tr} L_t$ denotes the mean curvature of the $\overline{G}$-orbits. If $l_1 \simeq \mathfrak{so}(2, \mathbb{R})$, then the orbits of any Iwasawa subgroup associated to $(l_1, t_1)$ are in addition minimal hypersurfaces.

In the case of hyperquadrics $(l_1 \simeq \mathfrak{so}(m, 2)$, see Remark 3.2 it can be shown that the $\overline{G}$-orbits are not minimal hypersurfaces for a generic invariant metric $g$.

The idea is to reduce the proof to the case $G$ simple. To that end, fix a simple factor $I_1 \subset I$ and set $I_{\geq 2} := I_2 \oplus \cdots \oplus I_r$, so that $I = I_1 \oplus I_{\geq 2}$ and $\mathfrak{g}_2 := I_{\geq 2} \ltimes \mathfrak{r} \subset \mathfrak{g}$ is an ideal. Denote respectively by $L_{\geq 2}, G_2 \leq G$ the connected Lie subgroups with Lie algebras $l_{\geq 2}, \mathfrak{g}_2$. Since $G \simeq L \ltimes R$, we have that $G_2 \simeq L_{\geq 2} \ltimes R$, with $L_{\geq 2}$ closed in $L$ by Lemma A.1. Thus, $G_2$ is a closed normal subgroup of $G$. Consider the isometric action of $G_2$ on $G/H$ by left-multiplication. The closed subgroup $G_2$ acts properly on $G/H$, and being normal in $G$, all isotropy subgroups for this action are conjugate. Thus, the orbit space $G_2 \backslash G/H$ is a smooth manifold. It admits a transitive action by $G_2 \backslash G =: L_1$ with compact isotropy $H_1 \simeq H/H \cap G_2$: see [BB78] Prop. 5.

Let $\bar{g}$ be the unique $L_1$-invariant metric on $L_1/H_1$ such that we have a Riemannian submersion

$$\pi : (G/H, g) \to (L_1/H_1, \bar{g}).$$

**Lemma 5.3.** The fibres of $\pi$ are pairwise isometric, minimal submanifolds of $(G/H, g)$.
Proof. Notice that for any \( x \in G \) we have

\[
G_2 \cdot (xH) = x \cdot (x^{-1}G_2x) \cdot o = x \cdot (G_2 \cdot o).
\]

Thus any two \( G_2 \)-orbits are isometric via an ambient isometry. By Lemma 3.7 \( \mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \) is a sum of ideals, with \( \mathfrak{h}_2 \subset G_2 \). Then the isotropy subalgebra of the homogeneous space \( G_2 \cdot o \simeq G_2/(H \cap G_2) \) is \( \mathfrak{h} \cap G_2 = \mathfrak{h}_2 \). Denote by \( \mathfrak{i}_1 = \mathfrak{h}_1 \oplus \mathfrak{m}_1, \mathfrak{g}_2 = \mathfrak{h}_2 \oplus \mathfrak{m}_2 \) the corresponding canonical reductive decompositions. Let \( \{ U_i \} \) be Killing fields in \( \mathfrak{m}_2 \subset \mathfrak{g}_2 \) and let \( \{ N_j \} \) span the normal bundle of \( G_2 \cdot o \) close to \( o \), such that \( \{ U_i \} \cup \{ N_j \} \) at \( o \) form an orthonormal basis of \( T_oG/H \). Then, the mean curvature vector of \( G_2 \cdot o \) at \( o \) is given by

\[
\sum_{i,j} g(\nabla U, N_j, U_i)_{o} N_j = - \sum_{i,j} g(N_j, \nabla U, U_i)_{o} N_j.
\]

Since the last expression is tensorial in \( N_j \), by homogeneity we may replace the \( N_j \) by Killing fields \( X_j \in \mathfrak{g} \) with \( X_j(o) = N_j(o) \). By (11) and \( \forall \in \mathfrak{g}_2 \) we have that \( X_j \in \mathfrak{i} \). Moreover by adding elements in the isotropy \( \mathfrak{h} \subset \mathfrak{i} \) we may assume that \( X_j \in (\mathfrak{m}_1 \oplus \mathfrak{m}_2) \cap \mathfrak{i} \). For each \( j \), formula [Bes97, (7.27)] for the Levi-Civita connection of Killing fields then yields

\[
- \sum g(X_j, \nabla U, U_i) = \sum g([X_j, U_i], U_i) + \sum g([X_j, Z_k], Z_k) = \text{tr}(\text{ad} X_j)_{|\mathfrak{g}_2}.
\]

Here, we have extended \( g \) from \( \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \) to all of \( \mathfrak{g} \) by making \( \mathfrak{h} \perp \mathfrak{m} \), and we denoted by \( \{ Z_k \} \) an orthonormal basis of \( \mathfrak{h}_2 \subset \mathfrak{g}_2 \). Then, the second equality is justified by the fact that \( [X_j, \mathfrak{h}_2] \subset \mathfrak{m}_2 \perp \mathfrak{h}_2 \) using that \( \{ \mathfrak{m}_1, \mathfrak{h}_2 \} = 0 \).

Observe now that since \( \mathfrak{g}_2 \) is an ideal, the map \( \mathfrak{i} \ni X \mapsto (\text{ad} X)_{|\mathfrak{g}_2} \) is a Lie algebra representation. Using that \( \mathfrak{i} \) is semisimple we may write \( X_j = [Y_j, Z_j] \), which implies \( \text{tr}(\text{ad} X_j)_{|\mathfrak{g}_2} = \text{tr}((\text{ad} Y_j)_{|\mathfrak{g}_2}, (\text{ad} Z_j)_{|\mathfrak{g}_2}) = 0 \) and the proof is finished. \( \square \)

Let \( \mathfrak{L}_1 \leq \mathfrak{L}_1 \) be the image of an Iwasawa subgroup \( \mathfrak{G} \subset \mathfrak{G} \) under the quotient \( \pi_2 : \mathfrak{G} \to G_2 \setminus \mathfrak{G} \).

Lemma 5.4. The group \( \mathfrak{L}_1 \) acts on \( (L_1/H_1, \bar{y}) \) with cohomogeneity one, embedded orbits and orbit space \( S^1 \).

Proof. We know by Proposition [11] that the action of \( \mathfrak{G} \) on \( G/H \) has those properties. Therefore, to prove the lemma it suffices to show that \( \pi \) gives a bijection between \( G \)-orbits in \( G/H \) and \( \mathfrak{L}_1 \) orbits in \( L_1/H_1 \). To see this, consider the commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\pi_2} & G_2 \setminus \mathfrak{G} = L_1 \\
\downarrow{\pi_1} & & \downarrow{\pi_1} \\
G/H & \xrightarrow{\pi} & G_2 \setminus \mathfrak{G}/H = L_1/H_1
\end{array}
\]

The map \( \pi_2 \) is a Lie group morphism, and \( \pi_1 \) is \( \mathfrak{L}_1 \)-equivariant (\( L_1 = G_2 \setminus \mathfrak{G} \)). From this one easily deduces that \( \pi \) maps \( G \)-orbits onto \( L_1 \)-orbits: \( \pi(\mathfrak{G} \cdot p) = \mathfrak{L}_1 \cdot \pi(p) \). Now assume that for some \( p, q \in G/H \) \( \pi(p) \) and \( \pi(q) \) belong to the same \( \mathfrak{L}_1 \)-orbit, say \( \pi(p) = \bar{l}_1 \cdot \pi(q) \) for some \( \bar{l}_1 = \pi_2(\bar{x}) \in \mathfrak{L}_1 \), \( \bar{x} \in \mathfrak{G} \), and let us show that \( \mathfrak{G} \cdot p = \mathfrak{G} \cdot q \). Write \( p = xH, q = yH, x, y \in \mathfrak{G} \). By the commutative diagram and \( \mathfrak{L}_1 \)-equivariance of \( \pi_1 \) we have

\[
\pi(xH) = \bar{l}_1 \cdot \pi(yH) = \pi_2(\bar{x} \cdot \pi_1(\pi_2(y))) = \pi_1(\pi_2(\bar{x}y)) = \pi(\bar{x}yH).
\]

This implies that \( p \) and \( \bar{x} \cdot q \) belong to the same \( G_2 \)-orbit. Since \( G_2 \leq \mathfrak{G} \) and \( \bar{x} \in \mathfrak{G} \), it follows that \( \mathfrak{G} \cdot p = \mathfrak{G} \cdot q \). \( \square \)
If \( N \) is a unit normal field to the \( \mathcal{G} \)-orbits in \( G/H \), then \( N \) is basic for the Riemannian submersion \( \pi \) and the corresponding vector field \( \bar{N} \) on \( L_1/H_1 \) is a unit normal field to the \( L_1 \)-orbits. Also, a normal geodesic \( \gamma(t) \) to the \( \mathcal{G} \)-orbits, \( \gamma(0) = eH \), is horizontal for \( \pi \), thus \( \bar{\gamma}(t) = \pi \circ \gamma(t) \) is a normal geodesic to the \( L_1 \)-orbits. The next lemma finally reduces the proof of Theorem 5.2 to the case where \( G \) is simple:

**Lemma 5.5.** The mean curvatures of \( G \cdot \gamma(t) \) and \( \bar{L}_1 \cdot \bar{\gamma}(t) \) coincide for all \( t \in \mathbb{R} \).

**Proof.** An orthonormal basis for \( T_{\gamma(t)}(G \cdot \gamma(t)) \) might be chosen by evaluating the vector fields \( \{X_i\} \cup \{U_j\} \) at \( \gamma(t) \), with \( X_i \) basic for \( \pi \) (horizontal and projectable) and \( U_j \in \mathfrak{g}_2 \) Killing fields.

The mean curvature of \( G \cdot \gamma(t) \) at \( \gamma(t) \) is thus given by

\[
\sum_i g(\nabla_{X_i}N, X_i)_{\gamma(t)} + \sum_j g(\nabla_{U_j}N, U_j)_{\gamma(t)}.
\]

The second sum vanishes thanks to Lemma 5.3. Regarding the first one, notice that being basic, the \( X_i \) are \( \pi \)-related to vector fields \( \bar{X}_i \) in \( L_1/H_1 \), which span the tangent space to \( L_1 \cdot \bar{\gamma}(t) \) at \( \bar{\gamma}(t) \). Hence,

\[
\sum_i g(\nabla_{X_i}N, X_i)_{\gamma(t)} = \sum_i \bar{g}(\nabla_{\bar{X}_i}\bar{N}, \bar{X}_i)_{\bar{\gamma}(t)}
\]

by a well-known property of Riemannian submersions, which is by definition the mean curvature of \( L_1 \cdot \bar{\gamma}(t) \) at \( \bar{\gamma}(t) \). \( \square \)

Before finishing the proof of Theorem 5.2, we briefly recall the definition of an Iwasawa decomposition: see \([Kna02]\) or \([Hel01]\). Let \( L \) be the Lie algebra of a simple Lie group \( L \) and consider a Cartan decomposition \( I = \mathfrak{k} \oplus \mathfrak{p} \). Let \( \theta \in \text{Aut}(g) \) be the corresponding Cartan involution with \( \theta|_\mathfrak{k} = \text{Id}_\mathfrak{k} \), \( \theta|_\mathfrak{p} = -\text{Id}_\mathfrak{p} \) and \( B(X, Y) := -B(X, \theta Y) \) a positive definite inner product on \( \mathfrak{l} \), where \( B = B_\mathfrak{l} \) denotes the Killing form of \( I \).

Choose a maximal abelian subalgebra \( \mathfrak{a} \subset \mathfrak{p} \), and consider the corresponding root space decomposition

\[
\mathfrak{l} = \mathfrak{a} \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{I}_\lambda,
\]

\( \lambda \in \mathfrak{a}^* \) and \( \Lambda \subset \mathfrak{a}^* \) being the set of those nonzero \( \lambda \)'s for which \( \mathfrak{I}_\lambda \neq 0 \). Recall that for \( \lambda \in \Lambda \) we have \( \theta|_{\mathfrak{I}_\lambda} = L_{\lambda} \), from which \( \Lambda = -\Lambda \) follows. Choose a notion of positivity in \( \mathfrak{a}^* \), so that the set of positive elements \( \mathfrak{a}^+_\Lambda \) is invariant under addition, multiplication by positive scalars, and \( \mathfrak{a}^* \setminus \{0\} = \mathfrak{a}^+_\Lambda \cup -\mathfrak{a}^+_\Lambda \). Set \( \Lambda^+ := \Lambda \cap \mathfrak{a}^*_+ \), and consider the nilpotent subalgebra

\[
\mathfrak{n} := \bigoplus_{\lambda \in \Lambda^+} \mathfrak{I}_\lambda.
\]

The subalgebra \( \mathfrak{l} := \mathfrak{a} \oplus \mathfrak{n} \) is solvable, and the corresponding connected Lie subgroup \( \mathcal{L} \) acts on the homogeneous space \( \mathcal{L}/H \) with cohomogeneity one and with all orbits principal. Here we are using that \( \mathcal{L}/K \) is an irreducible non-compact Hermitian symmetric space, thus \( H = [K, K] \). For each \( \lambda \in \Lambda^+ \) we decompose \( \mathfrak{I}_\lambda \oplus \mathfrak{I}_{-\lambda} = \mathfrak{t}_\lambda \oplus \mathfrak{p}_\lambda \), \( \mathfrak{t}_\lambda \subset \mathfrak{k} \) and \( \mathfrak{p}_\lambda \subset \mathfrak{p} \), where \( \mathfrak{t}_\lambda = \{X + \theta X : X \in \mathfrak{I}_\lambda\} \) and \( \mathfrak{p}_\lambda = \{X - \theta X : X \in \mathfrak{I}_\lambda\} \). Set

\[
\mathfrak{p}^+ := \bigoplus_{\lambda \in \Lambda^+} \mathfrak{p}_\lambda,
\]

so that \( \mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}^+ \) is a \( B \)-orthogonal decomposition. Finally, notice that

\[
\mathfrak{t} \oplus \mathfrak{n} = \mathfrak{t} \oplus \mathfrak{p}^+.
\]

**Proof of Theorem 5.2.** By Lemmas 5.4 and 5.5 we may assume that \( G/H = \mathcal{L}/H \) with \( \mathcal{L} = L_1 \) simple and with finite center. Let \( L = \text{KAN} \) be an Iwasawa decomposition and \( \bar{L} = \text{AN} \leq \mathcal{L} \) be the corresponding Iwasawa subgroup. Notice also, that the mean curvature of \( \mathcal{L} \)-orbits is a function on the orbit space \( S^1 \).

Firstly, we assume that \( \mathcal{L} \neq \text{SO}(2, m), m \geq 3 \). We will show in this case, that all \( \mathcal{L} \)-orbits are minimal hypersurfaces. To that end, let \( p = xH \in \mathcal{L}/H \). Using that \( x \in L = \mathcal{L}Z(K)H \), we may
assume without loss of generality that \( p = kH \) with \( k \in Z(K) \). Then at \( p \) the isotropy subgroup is also \( H \), and a reductive decomposition is given by \( I = h \oplus m, \ m = \mathfrak{z}(t) \oplus p \cong T_pL/H \).

Moreover, if in addition \( t \not\cong \mathfrak{sl}(2, \mathbb{R}) \) then \( \mathfrak{z}(t) \) and \( p \) are the two inequivalent irreducible summands for the isotropy representation: see Remark 3.2. In particular, the metric \( g \) preserves \( H \) and \( \Sigma - \) is the unit normal to \( \Sigma - \) and \( \Sigma - \) is a multiple of the Killing form. Notice also that for any \( Z \in \mathfrak{z}(t) \) the linear map \( \mathrm{ad}Z \) preserves \( p \), and \( (\mathrm{ad}Z)|_p \) is skew-symmetric with respect to \( g \).

The mean curvature of \( L \cdot p \) at \( p \) is given by \( \mathrm{tr} \ L_p = \sum_{i=1}^{n-1} g(\nabla X_i, X_i)_p \), where \( N \) denotes the unit normal to \( L \cdot p \) and \( \{X_i\}_{i=1}^{n-1} \) is a \( g_p \)-orthonormal basis for \( L \). The subspace \( \mathfrak{z}(t) \oplus p^+ \subset m \) is the \( g \)-orthogonal complement of \( m \) in \( m \). By (14) we have \( (\mathfrak{z}(t) \oplus n) \cdot p = (\mathfrak{z}(t) \oplus p^+) \cdot p \subset T_pM \). Since \( N(p) \in (1 \cdot p)^+ \subset (a \cdot p)^+ = (\mathfrak{z}(t) \oplus p^+) \cdot p \), we may choose Killing fields \( Z \in \mathfrak{z}(t) \), \( X_n \in n \), such that \( (Z + X_n)(p) = N(p) \). Using the formula \([\text{Bes87}{(7.27)}]\) for the Levi-Civita connection on Killing fields, we compute:

\[
(15) \quad -\mathrm{tr} \ L_p = \sum_{i=1}^{n-1} g([Z, X_i], X_i)_p + \sum_{i=1}^{n-1} g([X_n, X_i], X_i)_p.
\]

The first term vanishes. Indeed, write \( X_i = Z_i + P_i \) where \( Z_i \in t, P_i \in p \). Then \( [Z, X_i] = [Z, P_i] \in p \) and, using that \( P_i \perp \mathfrak{z}(t) \), each summand equals \( ([Z, P_i], P_i) \), which vanishes because \( (\mathrm{ad}Z)|_p \) is skew-symmetric. The second term equals \( \mathrm{tr}(\mathrm{ad} X_n) \), and this vanishes because \( \mathrm{ad} X_n \) is a nilpotent endomorphism.

If on the other hand we had \( t \cong \mathfrak{sl}(2, \mathbb{R}) \) then \( b = 0 \). Choose a Killing field \( X^\perp \in \mathfrak{sl}(2, \mathbb{R}) \) so that \( X^\perp(p) = N(p) \) and notice that since \( \mathfrak{sl}(2, \mathbb{R}) \) is unimodular,

\[
-\mathrm{tr} \ L_p = -\sum_{i=1}^{n-2} g(\nabla X_i, X_i^\perp)_p = \sum_{i=1}^{n-2} g([X^\perp, X_i], X_i)_p = \mathrm{tr} \ (\mathrm{ad} X^\perp)^\perp = 0.
\]

Finally, let us deal with the most complicated case \( t \cong \mathfrak{so}(2, m), \ m \geq 3 \). To see that we may assume that \( L \) is centerless, denote by \( Z \leq L \) its center, a finite normal subgroup contained in \( K \) by [HN11]{Prop. 16.1.7}. Since \( Z \) normalizes \( L \), its action on \( L/H \) by left-multiplication sends \( L \)-orbits to \( L \)-orbits. Thus, the mean curvature \( \mathrm{tr} \ L_t \) considered as a function on the orbit space \( K/H \cong S^1 \) is invariant under the action of \( Z \) on said space. If we would know that on \( Z(\mathfrak{K}/H \cong S^1 \times S^1 \) the mean curvature has mean zero, the same would follow for the mean curvature function on \( K/H \). Then we may work with \( Z \backslash L \) instead.

We claim that in order to prove (12) it is enough to find an isometric involution \( f : L/H \to L/H \) fixing \( o \) with \( df_oN = -N \), which sends \( L \)-orbits to \( L \)-orbits. Indeed, in the notation of Section 1 such an isometry must preserve \( \Sigma_0 = L \cdot o, o = eH = \gamma(0) \), thus \( f(\gamma(t)) = \gamma(-t) \), \( f(\Sigma_t) = \Sigma_{-t} \) and \( df_t(\gamma_t)N_t = -N_{-t} \). Since \( f \) is an isometry of \( L/H \), the mean curvatures of \( \Sigma_t \) and \( \Sigma_{-t} \) with respect to \( N_t \) and \( -N_{-t} \) respectively, coincide. Thus \( \mathrm{tr} \ L_s \) is the mean curvature with respect to \( N_s \), \( \mathrm{tr} \ L_s \) is an odd function on \( S^1 \) and (12) holds.

Let \( L \) be the connected, centerless simple Lie group with Lie algebra \( \mathfrak{so}(2, m) \). Another group with two connected components and the same Lie algebra is given by

\[
\mathbf{SO}(2, m) := \{A \in \mathbf{GL}(2 + m, \mathbb{R}) : \langle Av, Aw \rangle_{2+m} = \langle v, w \rangle_{2+m}, \ \forall v, w \in \mathbb{R}^{2+m}, \ \det A = 1 \}
\]

where \( \langle x, x \rangle_{2+m} = x_1^2 + x_2^2 - x_3^2 - \cdots - x_m^2 \). Thus

\[
\mathfrak{so}(2, m) = \left\{ \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \in \mathbf{gl}_{2+m}(\mathbb{R}), \ A \in \mathfrak{so}(2), \ C \in \mathfrak{so}(m), \ B \in \mathbb{R}^{2 \times m} \right\}
\]

Fix the maximal compact subgroup \( \mathbf{K} = S(O(2)O(m)) \leq \mathbf{SO}(2, m) \) containing the center \( Z \) of \( \mathbf{SO}(2, m) \) by [HN11]{Prop. 16.1.7}. Since \( L = \mathbf{SO}(2, m)/Z, K := K/Z \) is a maximal compact
subgroup of $L$ with Cartan decomposition $I = \mathfrak{k} \oplus \mathfrak{p}$. In the above matrix representation, $\mathfrak{k}$ is characterized by $B = 0$ and $\mathfrak{p}$ by $A = C = 0$.

Let us now choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ as follows:

$$\mathfrak{a} := \left\{ \left( \begin{array}{ccc} 0 & B & 0 \\ B^t & 0 & 0 \\ 0 & 0 & 0 \\ \end{array} \right) \in \mathfrak{p} : B = \left( \begin{array}{ccc} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 0 \\ \end{array} \right), \quad a_i \in \mathbb{R} \right\}.$$ 

Let $I = \bigoplus_{\lambda \in \Lambda} I_\lambda$ be the root space decomposition according to $\mathfrak{a}$, and set $N = \bigoplus_{\lambda \in \Lambda^+} I_\lambda$, for some choice of positive roots $\Lambda^+ \subset \Lambda$. Then, the Iwasawa decomposition \cite[Ch. IX, Thm. 1.3]{He01} asserts a global decomposition $L = KAN$, that is $L$ and $K \times A \times N$ are diffeomorphic, where $A, N$ denote the connected subgroups of $L$ with Lie algebras $\mathfrak{a}, \mathfrak{n}$, respectively. We set $\bar{L} := AN \leq L$, a closed, solvable subgroup acting on $L/H$ with cohomogeneity one and with orbits space $SO(2) \simeq S^1$. Notice that $H = [K, K]$ with $h \simeq so(m)$.

The isotropy representation of $L/H$ has a one-dimensional trivial factor $\mathfrak{z}(\mathfrak{k}) = so(2)$, and $\mathfrak{p}$ is a sum of two irreducible and equivalent $\text{Ad}(H)$-modules of real type, since $m \geq 3$. Thus, given $g$, after rotating the first two coordinates in $\mathbb{R}^{2 \times 1}$ we may assume without loss of generality that the two rows of $B$ in the matrix representation are $g$-orthogonal.

Consider now the diagonal element of $x = SO(2, m)$ whose non-zero entries are given by $x = \text{diag}(-1, 1, -1, 1, \text{Id}_{m-2})$ (and notice that $x \notin SO(2, m)_0$). Since conjugating by $x$ preserves $Z$, it induces an automorphism $\bar{f}$ of $L$ of order two. Also, $x$ normalizes $K$, hence $\bar{f}(K) \subset K$ and therefore $\bar{f}(H) = H$. In this way, $\bar{f}$ induces a diffeomorphism $f$ of $L/H$ with $f \circ f = \text{id}_{L/H}$ fixing $o := eH$, via $f(lH) := \bar{f}(l)H$.

The Lie algebra automorphism $\varphi := df|_e : I \rightarrow I$, which one can identify also with conjugation by $x$ in the matrix representation of $so(2, m)$, fixes the subalgebra $\mathfrak{a}$, and therefore preserves the root spaces and in particular $\mathfrak{n}$. Hence $\bar{f}(L) = L$ and $f(L \cdot p) = L \cdot f(p)$ for all $p \in L/H$. Next, notice that for all $l \in L$ we have $L_1 \circ f = f \circ L_{f^{-1}(l)}$, $L_1(lH) = llH$. Since $(df)_o : (m, g) \rightarrow (m, g)$, $m = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{p}$, is an isometry, we see that $f^* g = g$.

If remains to show that $f$ reverses the orientation of the normal geodesic. To that end notice that the $SO(2)$-orbit through $o$ (also a geodesic) is also preserved by $f$, since $\bar{f}(SO(2)) = SO(2)$. But on Lie algebra level the action of $df|_e$ on $so(2)$ is simply $-\text{Id}_{so(2)}$. Thus the orientation of this curve transversal to the $S$-orbits is reversed. Clearly the same must hold for the normal geodesic, and this completes the proof.

\section{Proof of Theorem \ref{main}}

Let $(\mathbb{R}^n, g)$ be a homogeneous Einstein space with Einstein constant $-1$. Given any effective presentation $G'/H' = \mathbb{R}^n$, by \cite[Thm. 4.6]{LL17} and \cite[Thm. 0.2]{LP17} one can find an ‘improved’ effective presentation $G/H$ with $\dim G \leq \dim G'$ satisfying $\mathfrak{h} \subset \mathfrak{k}$, for some Levi decomposition $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{a}$. Indeed, notice that $\dim \mathfrak{a} \leq \dim \mathfrak{z}(\mathfrak{h})$ in the notation of \cite[§3.2]{LP17}, thus the dimension of the modified transitive group constructed in \cite[Prop.3.14]{LP17} cannot be larger. In addition with respect to this presentation the metric satisfies \cite{La} by \cite[Thm. 2.4]{AL17}.

The above argument ensures that we may take $\dim G$ to be minimal among all transitive groups, and still ensure that \cite{La} is satisfied. By doing so, we can apply Proposition \cite[5.1]{La} and after quotienting by a discrete group of isometries we obtain another Einstein homogeneous space with Einstein constant $-1$, which for simplicity we denote by $(G/H, g)$, on which all Iwasawa subgroups $G \leq G$ act with cohomogeneity one, embedded orbits and orbit space $S^1$. In short, $(G/H, g)$ has a Levi presentation (see Definition \cite{La})..

We turn to the main geometric result in this section: based on Theorem \cite{D}, we show that the simple factors are pairwise orthogonal.
Proposition 6.1. Suppose that \((G/H, g)\) has a Levi presentation and \(\text{ric}_g = -g\). Then, any two simple factors \(l_i, l_j \subset l, i \neq j\), satisfy \(l_i \cdot o \perp_3 l_j \cdot o, o = eH\).

Proof. Let \(i = 1 \text{ and } l_{\geq 2}\) be the unique complementary ideal of \(l_1 \text{ in } l\), so that \(l = l_1 \oplus l_{\geq 2}\). By Proposition 3.4, the isotropy subalgebra decomposes as a direct sum of ideals \(h = h_1 \oplus h_{\geq 2}\) with \((l_1, h_1)\) the infinitesimal data of an \(R\)-bundle over an irreducible Hermitian symmetric space \(L_1/K_1\) of non-compact type, \(h_1 = l_1^*\), \(l_{\geq 2} \subset l_{\geq 2}\). By Lemma 3.3 there exists a maximal compactly embedded subalgebra of \(l_1 \subset l\) containing \(h_1\) with \(h_1 = [l_1, l_1]\).

Let \(\tilde{G} \leq G\) be an Iwasawa subgroup corresponding to \((l_1, l_1)\) as given by Theorem 5.2. Since these \(G\)-orbits are integrally minimal, by Theorem 5.2 they are standard homogeneous spaces. In order to exploit this property, let \(l_1 = t_1 \oplus a_1 \oplus n_1\) be the Iwasawa decomposition for which \(s_1 := a_1 \oplus n_1 \subset \tilde{g}\), and let \(n\) denote the nilradical of \(g\). Then \(n := n_1 \times n\) is the nilradical of \(\tilde{g}\).

Set \(\tilde{u} := n^\perp \subset \tilde{g}\), \(u := n^\perp \subset g\), \(a := n^\perp \subset t\), and recall that in order to define these orthogonal complements we extend the metric \(g\) to all of \(\tilde{g}\) as in Definition 2.1. Thus, \[ g = \left( h_1 \oplus j(l_1) \oplus a_1 \oplus n_1 \oplus l_{\geq 2} \right) \otimes (a \oplus n), \]
where \(\otimes\) denotes vector spaces direct sum. We now claim that

\[(16) \quad n_1 \cdot o \perp l_{\geq 2} \cdot o.\]

To see that, notice that the subspaces \(u, \tilde{u}\) satisfy \(\tilde{u} \subset u \cap \tilde{g} = s_1 \oplus l_{\geq 2} \oplus o\), the last equality thanks to (11). Also, since \((G/H, g)\) is Einstein, it is also standard by [LL14]. Thus, the orthogonal complements \(u\) and \(\tilde{u}\) are both Lie subalgebras of \(g\). Since \(\tilde{u} \simeq \tilde{g} / n\) is reductive (that is, the direct sum of a semisimple ideal and the center), \([u, \tilde{u}]\) is semisimple and isomorphic to a Levi factor \(l_{\geq 2}\) of \(g\) (see [Var84] 3.16.3-4). Moreover, \([u, \tilde{u}] \subset [s_1, s_1] \oplus l_{\geq 2}\) using that \([g, t] \subset n\); see [Var84] 3.8.3. Since \([s_1, s_1] \oplus l_{\geq 2}\) is a Lie algebra direct sum of a solvable and a semisimple ideal, it has exactly one semisimple subalgebra isomorphic to \(l_{\geq 2}\). Hence \([u, \tilde{u}] = l_{\geq 2}\) and in particular, \(n_1 \subset u \perp \tilde{u} \subset l_{\geq 2}\), which shows (16).

We consider now two different cases. Assume first that \(h_1 \neq 0\), that is, \(l_1 \neq \mathfrak{sl}(2, \mathbb{R})\). Then, the reductive complement \(p_1\) of the symmetric pair \((l_1, t_1)\) is an \(\text{Ad}(H)\)-isotypical summand in \(L/H\); see Remark 3.2. Thus, \(p_1 \cdot o \perp l_{\geq 2} \cdot o\). On the other hand, \(n_1 \cdot o \perp l_{\geq 2} \cdot o\) by (16). If we would know that \(n_1 \cdot o \subset p_1 \cdot o\), then from the fact that \(p_1 \cdot o\) is of codimension one inside \(l_1 \cdot o\), we could conclude that \(l_1 \cdot o \perp l_{\geq 2} \cdot o\) and the proof would follow in this case. Suppose on the contrary that \(n_1 \cdot o \subset p_1 \cdot o\), which is equivalent to \(n_1 \oplus h_1 \subset p_1 \oplus h_1\). Since \(a_1 \subset p_1\), this yields \(s_1 \oplus h_1 \subset p_1 \oplus h_1\) and by counting dimensions one gets \(s_1 \oplus h_1 = p_1 \oplus h_1\). Using this several times we observe that \(\{p_1, p_1\} \subset [h_1 \oplus s_1, h_1 \oplus s_1] \subset h_1 + [s_1, s_1] + [h_1, s_1] \subset h_1 + s_1 + [h_1, h_1 \oplus p_1] \subset h_1 + p_1\), contradicting the fact that \(\{p_1, p_1\} = t_1\) (see [HN11] Prop. 13.1.10)).

Finally suppose that \(l_1 \simeq \mathfrak{sl}(2, \mathbb{R})\). Again we have \(n_1 \cdot o \perp l_{\geq 2} \cdot o\) by (16). We may apply this argument to different Iwasawa groups \(\tilde{G}\) defined by considering other Iwasawa decompositions for \(\mathfrak{sl}(2, \mathbb{R})\), which can be obtained by conjugating with elements in \(\text{SO}(2)\). By doing so with three different decompositions whose nilradicals are linearly independent, we conclude that \(l_1 \cdot o \perp l_{\geq 2} \cdot o\), and this finishes the proof. \(\square\)

Recall the following result, which is a particular case of the main theorem in [Nik00]:

Theorem 6.2. [Nik00] Let \((L/H, g)\) be a homogeneous space with \(L\) semisimple and \(H\) compact, and consider a Cartan decomposition \(L = t \oplus p\) with \(h \subset t\) and a reductive complement \(T_aL/H \simeq\)
\[ \mathfrak{m} = \mathfrak{q} \oplus \mathfrak{p} \text{ with } \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{k}. \] Assume that \( g \) is awesome, that is, \( \mathfrak{f} \cdot o \perp \mathfrak{p} \cdot o \) for \( o = e\mathcal{H} \). If in addition \( \text{Ric}_g|_{\mathfrak{p} \times \mathfrak{p}} = c \cdot g|_{\mathfrak{p} \times \mathfrak{p}} \) for some \( c \in \mathbb{R} \), then either \( \mathfrak{h} = \mathfrak{f} \), or there exists \( Z \in \mathfrak{f} \) with \( \text{Ric}_g(Z, Z) > 0 \).

Using this, we are now in a position to prove the following result, which implies Theorem A.

**Theorem 6.3.** If \((\mathbb{G} / \mathcal{H}, g)\) has a Levi presentation and \( \text{ric}_g = -g \), then \( \mathbb{G} \) is solvable.

**Proof.** By Proposition 6.1 and Remark 6.2 the induced metric on \( \mathcal{L} / \mathcal{H} \) is awesome. In the semisimple case \( \mathbb{G} = \mathcal{L} \), awesomeness implies the non-existence of Einstein metrics by Theorem 6.2. To deal with the general case we assume that \( \mathfrak{L} \neq \mathfrak{L} \) and use the structure results [AL17, Theorems 2.1 & 2.4] for non-compact homogeneous Einstein spaces. These results yield that at \( o := e\mathcal{H} \), the induced homogeneous metric \( g_{\mathcal{L} / \mathcal{H}} \) on \( \mathcal{L} \cdot o \simeq \mathcal{L} / \mathcal{H} \) has

\[ \text{Ric}_{g_{\mathcal{L} / \mathcal{H}}} = -g_{\mathcal{L} / \mathcal{H}} + C_{\theta}, \quad C_{\theta}(X, Y) = \frac{1}{4} \text{tr} (\theta(X) + \theta(Y)) (\theta(X) + \theta(Y)^t), \]

where \( \theta : \mathfrak{L} \to \text{End}(\mathfrak{L}) \) is the restriction of the adjoint representation: \( \theta(X) = (\text{ad}_X)|_{\mathfrak{L}} \), and as usual \( \mathcal{L} / \mathcal{H} \simeq \mathfrak{m} \) for some reductive decomposition \( \mathfrak{L} = \mathfrak{h} \oplus \mathfrak{m} \). Now let \( \mathfrak{L}_1 \leq \mathfrak{L} \) be the connected Lie subgroup with Lie algebra \( \mathfrak{L}_1 \), one of the simple ideals in \( \mathfrak{L} \), and set \( \mathcal{H}_1 := \mathfrak{L}_1 \cap \mathcal{H} \). It follows by Proposition 6.1 that \((\mathbb{L} / \mathcal{H}, g_{\mathcal{L} / \mathcal{H}})\) is, at least locally, a Riemannian product of the homogeneous spaces given by the orbits of each of the simple factors. Thus, the induced homogeneous metric \( g_{\mathcal{L}_1 / \mathcal{H}_1} \) on \( \mathcal{L}_1 \cdot o \simeq \mathcal{L}_1 / \mathcal{H}_1 \) has Ricci curvature also given by

\[ \text{Ric}_{g_{\mathcal{L}_1 / \mathcal{H}_1}} = -g_{\mathcal{L}_1 / \mathcal{H}_1} + C_{\theta_1}, \quad \theta_1 := \theta|_{\mathfrak{L}_1}, \quad \mathfrak{L}_1 \to \text{End}(\mathfrak{L}). \]

By a result of Lauret (Proposition A.1 in the Appendix to [AL15]), awesomeness implies \( \theta(X)^t = -\theta(X) \) for all \( X \in \mathfrak{L} \) and \( \theta(X)^t = \theta(X) \) for all \( X \in \mathfrak{P} \), where \( \mathfrak{L} = \mathfrak{F} \oplus \mathfrak{P} \) is the Cartan decomposition with \( \mathfrak{F} \cdot o \perp \mathfrak{P} \cdot o \). In particular \( C_{\theta}|_{\mathfrak{L} \times \mathfrak{L}} = C_{\theta}|_{\mathfrak{F} \times \mathfrak{F}} = 0 \) and

\[ C_{\theta}(X, Y) = \text{tr} (\theta(X)\theta(Y)) \]

for all \( X, Y \in \mathfrak{P} \). Since \( C_{\theta} \) is an \( \text{Ad}(\mathcal{L}) \)-invariant bilinear form on \( \mathfrak{L} \), its restriction to each simple factor is a multiple of the Killing form of that factor. Together with (17), this yields

\[ \text{Ric}_{g_{\mathcal{L}_1 / \mathcal{H}_1}}|_{\mathfrak{L}_1 \times \mathfrak{L}_1} = -g_{\mathcal{L}_1}, \quad \text{Ric}_{g_{\mathcal{L}_1 / \mathcal{H}_1}}|_{\mathfrak{P}_1 \times \mathfrak{P}_1} = a \cdot g_{\mathcal{L}_1}, \quad a \in \mathbb{R}. \]

The second identity allows us to apply Theorem 6.2 to the homogeneous space \((\mathcal{L}_1 / \mathcal{H}_1, g_{\mathcal{L}_1 / \mathcal{H}_1})\), and this contradicts the first identity (recall that \( \mathfrak{L}_1 \) is not contained in the isotropy), by the results of Section 3. Therefore, \( \mathfrak{g} = \mathfrak{r} \) as claimed. \( \square \)

**Proof of Theorem A.** By the observations made at the beginning of this section, given a homogeneous space \((\mathbb{R}^n, g)\) with \( \text{ric}_g = -g \), after passing to a quotient we may assume that it has a Levi presentation. Then by Theorem 6.3 \((\mathbb{R}^n, g)\) is a solvmanifold. \( \square \)

Another immediate consequence of Theorem 6.3 is the following

**Corollary 6.4.** The group \( \text{SL}(2, \mathbb{R})^k \) admits no left-invariant Einstein metrics for any \( k \geq 1 \).

**Proof.** Since the group is semisimple and there is no isotropy, it is clear that the presentation is a Levi presentation. Moreover, by [MIL76] it is well-known that this group does not admit flat metrics. Therefore, by non-compactness any left-invariant Einstein metric would satisfy \( \text{ric}_g = -g \) up to scaling, contradicting Theorem 6.3. \( \square \)

Since the Einstein condition is local, the same statement is of course true for any Lie group locally isomorphic to \( \text{SL}(2, \mathbb{R})^k \), such as its universal cover. It is interesting to remark that this result is in fact not a consequence of Theorem A because there exist left-invariant metrics on \( \text{SL}_n(2, \mathbb{R}) \) which are isometric to a solvmanifold, see [CJ17]. Also, let us point out that this

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also implies non-existence of left-invariant Ricci soliton metrics on these Lie groups, since by [Jab15m, Thm. 1.3] they must be Einstein.

Finally, we prove another application of Theorem A stated in the introduction:

**Proof of Corollary A.** Let \((\mathbb{R}^n, g)\) be a homogeneous Ricci soliton with \(\text{ric}_g + \mathcal{L}_X g = \lambda \cdot g\). For \(\lambda \geq 0\), such solitons are by [Nab10, PW09] quotients of the Riemannian product of a compact homogeneous Einstein manifold and a flat Euclidean space. Since our underlying manifold is \(\mathbb{R}^n\), we may therefore assume \(\lambda < 0\); see [Mil76]. By [Jab14] there exists a presentation \((G/H, g)\) which makes \((\mathbb{R}^n, g)\) an algebraic soliton: the Ricci endomorphism at \(eH\) is given by \(\text{Ric}_g = \lambda \cdot \text{Id} + D\), with \(D\) the projection onto \(g/\mathfrak{h}\). Notice that \(D\) is symmetric, and in particular a normal operator. Using Theorem 3.2 from [HPW15] we obtain a one-dimensional homogeneous Einstein extension \((\mathbb{R}^{n+1}, \tilde{G}/\tilde{H})\) with \(\text{ric}_{\tilde{g}} = \lambda \cdot \tilde{g}\) (with pairwise isometric \(G\)-orbits, because \(G\) is normal in \(\tilde{G}\)).

Theorem A and Theorem 1.3 in [Jab15b] imply now that \((\mathbb{R}^{n+1}, \tilde{g})\) is an algebraic soliton: the Ricci endomorphism at \(e\tilde{H}\) is given by \(\text{Ric}_{\tilde{g}} = \lambda \cdot \text{Id} + \tilde{D}\), with \(\tilde{D}\) the projection onto \(\tilde{g}/\tilde{\mathfrak{h}}\). Notice that \(\tilde{D}\) is symmetric, and in particular a normal operator. Using Theorem 3.2 from [HPW15] we obtain a one-dimensional homogeneous Einstein extension \((\mathbb{R}^{n+1}, \tilde{G}/\tilde{H})\) with \(\text{ric}_{\tilde{g}} = \lambda \cdot \tilde{g}\) (with pairwise isometric \(G\)-orbits, because \(G\) is normal in \(\tilde{G}\)).

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Appendix A. The space of invariant metrics

Let \(G/H\) be a homogeneous space. Fix a background \(G\)-invariant Riemannian metric \(\tilde{g}\) and consider the canonical reductive decomposition \(g = \mathfrak{h} \oplus \mathfrak{m}\), where \(\mathfrak{m}\) is the orthogonal complement of \(\mathfrak{h}\) with respect to the Killing form of \(g\). The space \(\mathcal{M}(G/H)\) of \(G\)-invariant Riemannian metrics on \(G/H\) can be identified with the set of \(\text{Ad}(H)\)-invariant inner products on \(\mathfrak{m}\). The latter is an orbit

\[
\text{GL}^H(\mathfrak{m}) \cdot \tilde{g} \subset \text{Sym}^2(\mathfrak{m}),
\]

where \(\tilde{g}\) denotes also the inner product induced on \(\mathfrak{m} \simeq T_{eH}G/H\) by the background metric. Here \(\text{Sym}^2(\mathfrak{m})\) denotes the space of symmetric bilinear forms on \(\mathfrak{m}\), \(\text{GL}(\mathfrak{m})\) acts on it by the usual change of basis action

\[
(\mathfrak{Q} \cdot b)(\cdot, \cdot) := b(Q^{-1} \cdot, Q^{-1} \cdot), \quad \mathfrak{Q} \in \text{GL}(\mathfrak{m}), \quad b \in \text{Sym}^2(\mathfrak{m}).
\]

and \(\text{GL}^H(\mathfrak{m})\) is the centralizer of \(\text{Ad}(H)|_\mathfrak{m}\) in \(\text{GL}(\mathfrak{m})\). Thus, if \(\mathcal{O}(\mathfrak{m}) \subset \text{GL}(\mathfrak{m})\) denotes the subgroup of \(\tilde{g}\)-orthogonal endomorphisms, and \(\mathcal{O}^H(\mathfrak{m}) = \mathcal{O}(\mathfrak{m}) \cap \text{GL}^H(\mathfrak{m})\), then the set of \(\text{Ad}(H)\)-invariant inner products on \(\mathfrak{m}\) is also a homogeneous space

\[
\mathcal{M}^G(G/H) \simeq \text{GL}^H(\mathfrak{m})/\mathcal{O}^H(\mathfrak{m}).
\]

Moreover, if \(Q \subset \text{GL}^H(\mathfrak{m})\) is a closed subgroup big enough so that \(\text{GL}^H(\mathfrak{m}) = Q\mathcal{O}^H(\mathfrak{m})\) then it still acts transitively on \(\text{GL}^H(\mathfrak{m})/\mathcal{O}^H(\mathfrak{m})\). In this way we also get a presentation

\[
\mathcal{M}^G(G/H) \simeq Q/K_Q,
\]

where \(K_Q = Q \cap \mathcal{O}^H(\mathfrak{m})\) is a maximal compact subgroup of \(Q\).

**Lemma A.1.** Let \((g_t)_{t \in I} \subset \mathcal{M}^G(G/H) \simeq Q/K_Q\) be a smooth family of metrics. Then, there exists a smooth lift \((q_t)_{t \in I} \subset Q\) such that \(g_t = q_t \cdot \tilde{g}\) for all \(t \in I\).
The reductive decomposition induces an inclusion
\[ \text{GL}(m) \simeq \left[ \begin{array}{c} \text{Id}_b \\ \text{GL}(m) \end{array} \right] \subset \text{GL}(g), \]
according to which we set
\[ \text{Aut}^H_m(g) := \text{Aut}(g) \cap \text{GL}^H_m, \]
where Aut(g) denotes the Lie group of automorphisms of the Lie algebra g.

**Lemma A.2.** Let g be a G-invariant metric on G/H, set o := eH and let f ∈ G normalizing H. Then, the scalar products \( g_o, g_{f \circ o} \) induced respectively on m by g after the identifications \( m \simeq T_oG/H, m \simeq T_{f \circ o}G/H \) satisfy
\[ g_{f \circ o}(\cdot, \cdot) = g_o((\text{Ad } f^{-1})|_m \cdot, (\text{Ad } f^{-1})|_m \cdot), \]
with \((\text{Ad } f^{-1})|_m \in \text{Aut}_m^H(g)\).

**Proof.** Since the map f is an isometry, for a Killing field \( X \in m \) with flow \( \Phi_s \) the vector field \( \tilde{X} \) given by \( \tilde{X}_{f \circ p} := df_p X_p \) is again a Killing field, since its flow satisfies \( \tilde{\Phi}_s = f \circ \Phi_s \circ f^{-1} \). As a consequence \( \tilde{X} = (\text{Ad } f)|_m X \in m \). Setting \( p := f \circ o \), we deduce
\[ (df)_o X_o = ((\text{Ad } f)X)_p. \]
Using that f normalizes H we see that the isotropy subgroup at p is also H. In particular, we also have \( m \simeq T_pG/H \) under the usual identification \( X \rightarrow X_p \). Hence, for \( X \in m \) we have
\[ g_o(X_o, X_o) = g_p((df)_o X_o, (df)_o X_o) = g_p((\text{Ad } f)X_p, (\text{Ad } f)X_p) = g_p((\text{Ad } f)|_m X_p, (\text{Ad } f)|_m X_p), \]
and the lemma follows. \( \square \)

An analogous of the above lemma holds of course for any G-invariant tensor.

**Corollary A.3.** Let \((g(t))_{t \in \mathbb{R}}\) be a smooth curve of homogeneous metrics on G/H and \( L_t \) be defined by \( g_t(\cdot, \cdot) = 2g_t(L_t \cdot, \cdot) \). Suppose that there exists \( f \in G \) normalizing H and \( T > 0 \) such that \( g_{T+t} = (\text{Ad } f)|_m \cdot g_t \) for all \( t \in \mathbb{R} \). Then
\[ L_T = (\text{Ad } f)|_m \circ L_0 \circ (\text{Ad } f)|_m^{-1}. \]

**Proof.** We write \( g_t(\cdot, \cdot) = \tilde{g}(G_t \cdot, \cdot) \), where \( \tilde{g} \) is a \( \bar{G} \)-invariant background metric on \( \Sigma_0 \). Then by Lemma A.2 we have for all \( t \in \mathbb{R} \) that \( G_{T+t} = ((\text{Ad } f)|_m^{-1})^T G_t (\text{Ad } f)|_m^{-1} \), the transpose taken with respect to \( \tilde{g} \). Using \( G_t' = 2G_tL_t \) implies the claim. \( \square \)

**Lemma A.4.** Semisimple subgroups of a semisimple Lie group are closed.

**Proof.** Let L be a Lie group whose Lie algebra splits as a direct sum of semisimple ideals \( 1 = l_1 \oplus l_2 \). Let \( Z(L) \) be the center of L and consider the universal cover \( \tilde{L} \) and the centerless quotient \( \tilde{L} := L/Z(L) \). Let \( L_i \) (resp. \( \tilde{L}_i, L_i \)) be the connected Lie subgroup of L (resp. \( \tilde{L}, L \)) with Lie algebra \( l_i, i = 1, 2 \).

It is enough to show that \( L_1 \) is a closed subgroup of L. For the universal cover this is clear, as \( \tilde{L} \) is isomorphic to the direct product \( \tilde{L}_1 \times \tilde{L}_2 \). Notice that \( L \simeq \tilde{L}/Z(\tilde{L}) \), thus \( L_1 \simeq \tilde{L}_1/Z(\tilde{L}_1) \) and \( \tilde{L} \) is isomorphic to the direct product of the centerless subgroups \( \tilde{L}_1 \times \tilde{L}_2 \). Assume on the contrary that \( L_1 \) is not closed in L. This means that its closure \( \tilde{L}_1 \) must have strictly larger dimension, and hence its intersection with \( L_2 \) must have positive dimension. Let \( \pi : L \rightarrow \tilde{L} \) denote the projection onto the quotient. Then \( \pi(L_1) \) is connected and with Lie algebra \( l_i \), hence equal to \( L_i \). On the other hand,
\[ \pi(\tilde{L}_1 \cap L_2) \subset \pi(\tilde{L}_1) \cap \pi(L_2) \subset \overline{\pi(L_1) \cap \pi(L_2)} = \tilde{L}_1 \cap \tilde{L}_2 = \{ e \}. \]
From this, \( L_1 \cap L_2 \subset \mathbb{Z}(L) \), and this is a contradiction since \( \mathbb{Z}(L) \) is discrete. \( \square \)

**APPENDIX B. THE ENDOMORPHISM \( \beta \) ASSOCIATED TO A HOMOGENEOUS SPACE**

Let \( G/H \) be a homogeneous space endowed with a fixed, background \( G \)-invariant Riemannian metric \( \bar{g} \), and assume that \( H \) is compact. Let \( g = h \oplus m \) be the canonical reductive decomposition, and extend \( \bar{g} \) from \( m \) to an \( \text{Ad}H \)-invariant scalar product on all of \( g \), also denoted by \( \bar{g} \), and such that \( \bar{g}(h, m) = 0 \). In this section, the notions of symmetric and orthogonal endomorphisms of \( g \) are considered with respect to \( \bar{g} \) only.

Given any symmetric endomorphism \( \beta \in \text{End}(g) \) we now introduce the associated subgroups \( Q_\beta, SL_\beta \) of \( GL(g) \) following [BL17], and we refer the reader to that article and [BL18] for proofs of the results in this section. Consider the adjoint map \( \text{ad}(\beta) : \text{End}(g) \to \text{End}(g), A \mapsto [\beta, A] \), which is symmetric with respect to the scalar product \( \text{tr} AB^t \) on \( \text{End}(g) \). If \( \text{End}(g)_r \) denotes the eigenspace of \( \text{ad}(\beta) \) with eigenvalue \( r \in \mathbb{R} \) then \( \text{End}(g) = \bigoplus_{r \in \mathbb{R}} \text{End}(g)_r \), and we set

\[
g_\beta := \text{End}(g)_0 = \ker(\text{ad}(\beta)), \quad u_\beta := \bigoplus_{r > 0} \text{End}(g)_r, \quad q_\beta := g_\beta \oplus u_\beta.
\]

**Definition B.1.** We denote respectively by

\[
G_\beta := \{ g \in G : g \beta g^{-1} = \beta \}, \quad U_\beta := \exp(u_\beta) \quad \text{and} \quad Q_\beta := G_\beta U_\beta
\]

the centralizer, the unipotent subgroup and the parabolic subgroup associated with \( \beta \).

Of course their Lie algebras are respectively \( g_\beta, u_\beta \) and \( q_\beta \). The group \( U_\beta \) is nilpotent, and \( G_\beta \) is reductive with Cartan decomposition \( G_\beta = K_\beta \exp(p_\beta) \), where \( K_\beta = O(g) \cap G_\beta \) and \( p_\beta = \{ A \in g_\beta : A = A^t \} \). Notice that \( \beta \in p_\beta \), and that \( h_\beta := \{ A \in p_\beta : \text{tr} A \beta = 0 \} \) is a codimension-one Lie subalgebra in \( p_\beta \). We set

\[
H_\beta := K_\beta \exp(h_\beta), \quad SL_\beta := H_\beta U_\beta, \quad \text{Lie}(SL_\beta) := sl_\beta.
\]

At Lie algebra level we have an orthogonal decomposition \( q_\beta = \mathbb{R} \beta + sl_\beta \).

One of the most important properties \( Q_\beta \) has is that

\[
GL(g) = O(g)Q_\beta.
\]

If in addition \( \beta \) preserves the decomposition \( g = h \oplus m \), then by [BL18] (46) and Appendix A the subgroup

\[
(21) \quad Q_\beta^H := Q_\beta \cap GL^H(m) \subset GL(m)
\]

acts transitively on the space of \( \text{Ad}(H) \)-invariant scalar products on \( m \). The latter is of course in one-to-one correspondence with the space \( \mathcal{M}^G(G/H) \) of \( G \)-invariant metrics. Thus,

\[
(22) \quad \mathcal{M}^G(G/H) = Q_\beta^H : \bar{g}.
\]

**Theorem B.2.** [BL17] [BL18] Given the homogeneous space \( G/H \), there exists a \( \bar{g} \)-symmetric endomorphism \( \beta \in \text{End}(g) \) associated to it, normalized so that \( \text{tr} \beta = -1 \), and satisfying:

1. (Positivity) The endomorphism \( \beta^+ := \beta/||\beta||^2 + \text{Id}_g \) is positive semi-definite, and its kernel is the orthogonal complement of the nilradical of \( g \). In particular, \( \beta^+ \) preserves \( m \).
2. (Automorphism group constraint) We have that \( \text{Aut}(g) \leq SL_\beta \).
3. (Ricci curvature estimate) For any \( G \)-invariant metric \( g = q \cdot \bar{g} \) on \( G/H \), \( q \in Q_\beta^H \), its Ricci endomorphism satisfies

\[
\text{tr} \text{Ric}_g q \beta^+ q^{-1} + g(H_g, H_g) \geq 0,
\]

with equality if and only if \( q \beta^+ q^{-1} \in \text{Der}(g) \). Here \( H_g \in m \) denotes the mean curvature vector of \( (G/H, g) \), and by \( \beta^+ \) we mean \( \beta^+ |_m \), see condition 1.
Proof. Let $\beta$ be the stratum label corresponding to $\mathfrak{g}$, see [BL18 §5]. Recall that for a symmetric endomorphism, the kernel is the orthogonal complement of the image. Thus, property 1. follows from Lemma 5.1 in [BL18]. Property 2. is simply Corollary 4.11 in [BL18]. Regarding 3., in the notation of [BL18] we write $\text{Ric}_g = q \text{Ric}_\mu q^{-1}$, where $\mu = q^{-1} \cdot [\cdot, \cdot]$ and $[\cdot, \cdot]$ is the Lie bracket of $\mathfrak{g}$. We may further decompose

$$\text{Ric}_\mu = \text{Ric}_\mu^* - S(\text{ad}_\mu H_\mu),$$

with $\text{Ric}_\mu^* \perp \text{Der}(\mu)$, and notice that $H_\mu = q^{-1}H_g$. Thus,

$$\text{tr} \text{Ric}_g q \beta^+ q^{-1} + g(H_g, H_g) = \text{tr} \text{Ric}_\mu^* \beta^+ - \text{tr}(\text{ad}_\mu H_\mu)\beta^+ + \bar{g}(H_\mu, H_\mu).$$

Since $\text{ad}_\mu H_\mu \in \text{Der}(\mu) \subset \mathfrak{sl}_\beta \perp \beta$, the second term equals $-\text{tr}\text{ad}_\mu H_\mu = -\bar{g}(H_\mu, H_\mu)$ by definition of $H_\mu$. Thus, the right-hand-side equals $\text{tr} \text{Ric}_\mu^* \beta^+$, which is non-negative by Lemma 6.2 in [BL18]. Finally, the equality condition follows from $\text{Der}(\mu) = q^{-1} \text{Der}(g)q$.

Lemma B.3. Let $S \in \text{End}(\mathfrak{g})$ be a $g$-symmetric endomorphism, and $Q \in \mathfrak{q}_\beta$ be such that $S + Q \in \mathfrak{so}(\mathfrak{g}, \bar{g})$. Then, $\text{tr} S[Q, \beta] \geq 0$, with equality if and only if $[Q, \beta] = 0$.

Proof. Since the map $E \mapsto -E^T$ is an involutive automorphism of the Lie algebra $\text{End}(\mathfrak{g})$, the $\text{ad}(\beta)$-eigenspaces introduced above satisfy in addition $\text{End}(\mathfrak{g})^\perp = \text{End}(\mathfrak{g})_{-\lambda}$ for all $\lambda \geq 0$. From this, it is clear that an endomorphism

$$E := E_0 + \sum_{\lambda > 0} (E_\lambda + E_{-\lambda}), \quad E_\lambda \in \text{End}(\mathfrak{g})_\lambda \forall \lambda,$$

is symmetric if and only if $E_{-\lambda} = E^T_\lambda$ for all $\lambda \geq 0$, and it is skew-symmetric if and only if $E_{-\lambda} = -E^T_\lambda$ for all $\lambda \geq 0$.

By definition of $\mathfrak{q}_\beta$, we may write $Q = \sum_{\lambda > 0} Q_\lambda$, with $Q_\lambda \in \text{End}(\mathfrak{g})_\lambda$. Since $S$ is symmetric we may write it as $S = S_0 + \sum_{\lambda > 0} (S_\lambda + S^*_\lambda)$. In view of the above observation, the condition $S + Q \in \mathfrak{so}(\mathfrak{g}, \bar{g})$ implies $Q_\lambda = -2S_\lambda$, for all $\lambda > 0$. Hence,

$$\text{tr} S[Q, \beta] = -\text{tr}[\beta, Q]S = -\sum_{\lambda > 0} \lambda \text{tr} Q_\lambda S = -\sum_{\lambda > 0} \lambda \langle Q_\lambda, S \rangle = 2 \sum_{\lambda > 0} \lambda \|S_\lambda\|^2 \geq 0.$$ 

Equality occurs if and only if $S_\lambda = 0$ for all $\lambda > 0$, that is, $[S, \beta] = 0$. Moreover, since $Q_\lambda = -2S_\lambda$ for $\lambda > 0$, this is also equivalent to $[Q, \beta] = 0$. \qed

References

[AKL75] Dmitri Alekseevskii and Boris N. Kimelfeld, Structure of homogeneous Riemannian spaces with zero Ricci curvature, Funktional. Anal. i Prilov Zen. 9 (1975), no. 2, 5–11.

[AKL89] Michael T. Anderson, Peter B. Kronheimer, and Claude LeBrun, Complete Ricci-flat Kähler manifolds of infinite topological type, Comm. Math. Phys. 125 (1989), no. 4, 637–642.

[AL15] Romina M. Arroyo and Ramiro Lafuente, Homogeneous Ricci solitons in low dimensions, Int. Math. Res. Not. IMRN (2015), no. 13, 4901–4932.

[AL17] Romina M. Arroyo and Ramiro A. Lafuente, The Alekseevskii conjecture in low dimensions, Math. Ann. 367 (2017), no. 1-2, 283–309.

[Ale75] Dmitri Alekseevskii, Homogeneous Riemannian spaces of negative curvature, Mat. Sb. 96 (1975), 93–117.

[And10] Michael T. Anderson, A survey of Einstein metrics on 4-manifolds, Handbook of geometric analysis, No. 3, Adv. Lect. Math. (ALM), vol. 14, Int. Press, Somerville, MA, 2010, pp. 1–39.

[Aub76] Thierry Aubin, équations du type Monge-Ampère sur les variétés kählériennes compactes, C. R. Acad. Sci. Paris Sér. A-B 283 (1976), no. 3, Aii, A119–A121.

[BB78] Lionel Bérard-Bergery, Sur la courbure des métriques riemanniennes invariantes des groupes de Lie et des espaces homogènes, Ann. Sci. École Norm. Sup. (4) 11 (1978), no. 4, 543–576.

[Bes87] Arthur L. Besse, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987.
HOMOGENEOUS EINSTEIN METRICS ON EUCLIDEAN SPACES ARE EINST EIN SOLVMANIFOLDS

[Joy00] Compact manifolds with special holonomy, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[JP17] Michael Jablonski and Peter Petersen, A step towards the Alekseevskii conjecture, Math. Ann. 368 (2017), no. 1-2, 197–212.

[Kna02] Anthony W. Knapp, Lie groups beyond an introduction, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002.

[Kos65] J.-L. Koszul, Lectures on groups of transformations, Notes by R. R. Simha and R. Sridharan. Tata Institute of Fundamental Research Lectures on Mathematics, No. 32, Tata Institute of Fundamental Research, Bombay, 1965.

[Laf15] Ramiro A. Lafuente, Scalar Curvature Behavior of Homogeneous Ricci Flows, J. Geom. Anal. 25 (2015), no. 4, 2313–2322.

[Lau09] Jorge Lauret, Einstein solvmanifolds and nilsolitons, New developments in Lie theory and geometry, Contemp. Math., vol. 491, Amer. Math. Soc., 2009, pp. 1–35.

[Lau10] Einstein solvmanifolds are standard, Ann. of Math. (2) 172 (2010), no. 3, 1859–1877.

[Lau11] Ricci soliton solvmanifolds, J. Reine Angew. Math. 650 (2011), 1–21.

[LeB95] Claude LeBrun, Einstein metrics and Mostow rigidity, Math. Res. Lett. 2 (1995), no. 1, 1–8.

[LeB96] Four-manifolds without Einstein metrics, Math. Res. Lett. 3 (1996), no. 2, 133–147.

[LL14] Ramiro Lafuente and Jorge Lauret, Structure of homogeneous Ricci solitons and the Alekseevskii conjecture, J. Differential Geom. 98 (2014), no. 2, 315–347.

[Mil76] John Milnor, Curvatures of left-invariant metrics on Lie groups, Adv. Math. 21 (1976), 293–329.

[Nab10] Aaron Naber, Noncompact shrinking four solitons with nonnegative curvature, J. Reine Angew. Math. 645 (2010), 125–153.

[Nik00] Yu. G. Nikonorov, On the Ricci curvature of homogeneous metrics on noncompact homogeneous spaces, Sib. Mat. Zh. 41 (2000), no. 2, 421–429, iv.

[Pal61] Richard S. Palais, On the existence of slices for actions of non-compact Lie groups, Ann. of Math. (2) 73 (1961), 295–323.

[PW09] Peter Petersen and William Wylie, On gradient Ricci solitons with symmetry, Proc. Amer. Math. Soc. 137 (2009), 2085–2092.

[Sto92] Stephan Stolz, Simply connected manifolds of positive scalar curvature, Ann. of Math. (2) 136 (1992), no. 3, 511–540.

[Tho69] John A. Thorpe, Some remarks on the Gauss-Bonnet integral, J. Math. Mech. 18 (1969), 779–786.

[Tia15] Gang Tian, K-stability and Kähler-Einstein metrics, Comm. Pure Appl. Math. 68 (2015), no. 7, 1085–1156.

[Var84] V. S. Varadarajan, Lie groups, Lie algebras, and their representations, Graduate Texts in Mathematics, vol. 102, Springer-Verlag, New York, 1984, Reprint of the 1974 edition.

[Wan99] McKenzie Y. Wang, Einstein metrics from symmetry and bundle constructions, Surveys in differential geometry: essays on Einstein manifolds, Surv. Differ. Geom., vol. 6, Int. Press, Boston, MA, 1999, pp. 287–325.

[Wan12] McKenzie Y.-K. Wang, Einstein metrics from symmetry and bundle constructions: a sequel, Differential geometry, Adv. Lect. Math. (ALM), vol. 22, Int. Press, Somerville, MA, 2012, pp. 253–309.

[WZ86] McKenzie Y. Wang and Wolfgang Ziller, Existence and nonexistence of homogeneous Einstein metrics, Invent. Math. 84 (1986), no. 1, 177–194.

[Yau78] Shing Tung Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I, Comm. Pure Appl. Math. 31 (1978), no. 3, 339–411.

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