Topological rigidity of automorphism actions on nilmanifolds

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1 Introduction

Let $G$ be connected simply connected nilpotent Lie group and $D$ be discrete uniform subgroup of $G$. Then $X = G/D$ is called a nilmanifold. If $X_1 = G_1/D_1$ and $X_2 = G_2/D_2$ are nilmanifolds then a map $f : X_1 \to X_2$ is said to be a homomorphism if it is induced by a continuous homomorphism from $G_1$ to $G_2$ which maps $D_1$ into $D_2$. Isomorphisms and automorphisms are defined similarly. A map $f : X_1 \to X_2$ is said to be affine if there exists an element $a$ in $G_2$ and a continuous homomorphism $A : G_1 \to G_2$ such that $f(gD_1) = aA(g)D_2$ for all $g$ in $G_1$. If $\Gamma$ is a discrete group and $X$ is a nilmanifold then a $\Gamma$-action $\rho$ on $X$ is said to be an automorphism action if each $\rho(\gamma)$ is an automorphism of $X$.

It is known that any nilmanifold $X = G/D$ carries a unique $G$-invariant probability measure $\lambda_X$. It is easy to see that for any nilmanifold $X$, the measure $\lambda_X$ is invariant under any affine action on $X$. An automorphism action $\rho$ of a discrete group $\Gamma$ on a nilmanifold $X$ is said to be ergodic if for every $\Gamma$-invariant function $f$ in $L^2(X, \lambda_X)$ is a constant almost everywhere.

Suppose $X_1, X_2$ are nilmanifolds and $\rho, \sigma$ are continuous actions of a discrete group $\Gamma$ on $X_1$ and $X_2$ respectively. If $f : X_1 \to X_2$ is a continuous
map and $\Gamma_0$ is a subgroup of $\Gamma$ then $f$ is said to be $\Gamma_0$-equivariant if $f \circ \rho(\gamma) = \sigma(\gamma) \circ f$, $\forall \gamma \in \Gamma_0$. A continuous map $f : X_1 \to X_2$ is said to be almost equivariant if there exists a finite-index subgroup $\Gamma_0 \subset \Gamma$ such that $f$ is $\Gamma_0$-equivariant.

In this paper we show that if $\rho, \sigma$ are automorphism actions on nilmanifolds satisfying certain conditions then every $\Gamma$-equivariant continuous map from $(X_1, \rho)$ to $(X_2, \sigma)$ is an affine map. When $\Gamma = \mathbb{Z}$, $(X_2, \sigma)$ is a factor of $(X_1, \rho)$ and $\rho, \sigma$ are generated by affine transformations, this phenomenon has been studied in [AP], [Wa1] and [Wa2]. In this case a necessary and sufficient condition for existence of a non-affine $\Gamma$-equivariant map is given in [Wa2]. Our methods are however different and applicable in more general situations.

This paper is organized as follows. In section 2 we study structure of continuous equivariant maps from $(X_1, \rho)$ to $(X_2, \sigma)$. In Theorem 1 we give a necessary and sufficient condition for existence of a non-affine almost equivariant map. For any Lie group $G$, by $L(G)$ we denote the Lie algebra of $G$. If $\rho$ is an automorphism action of a discrete group $\Gamma$ on a nilmanifold $X = G/D$, then by $\rho_e$ we denote the $\Gamma$-action on $L(G)$ induced by $\rho$. We prove the following.

**Theorem 1 :** Let $X_1 = G_1/D_1, X_2 = G_2/D_2$ be nilmanifolds and $\rho, \sigma$ be automorphism actions of a discrete group $\Gamma$ on $X_1$ and $X_2$ respectively. Then there exists a non-affine almost equivariant continuous map from $(X_1, \rho)$ to $(X_2, \sigma)$ if and only if the following two conditions are satisfied.

a) There exists a non-constant $\Gamma$-invariant continuous function from $(X_1, \rho)$ to $\mathbb{R}$.  

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b) There exists a nonzero vector $v$ in $L(G_2)$ with finite $\sigma_e$-orbit.

As a consequence we obtain that if either $(X_1, \rho)$ is ergodic or $(X_2, \sigma)$ is expansive then every continuous $\Gamma$-equivariant map from $(X_1, \rho)$ to $(X_2, \sigma)$ is an affine map (see corollary 2.1).

In section 3 we consider the case when $X_1$ is a torus. If $\rho$ is an automorphism action of a discrete group $\Gamma$ on a torus $T^m$ then we denote the induced automorphism action of $\Gamma$ on the dual group $\hat{T}^m$ by $\hat{\rho}$. By $F_\rho$ we denote the subgroup of $\hat{T}^m$ which consists of all elements with finite $\hat{\rho}$-orbit and by $\Gamma_\rho$ we denote the subgroup of $\Gamma$ consisting of all elements which acts trivially on $F_\rho$ under the action $\hat{\rho}$. We prove the following.

**Theorem 2 :** Let $\Gamma$ be a discrete group, $T^m$ be the $m$-torus and $X = G/D$ be a nilmanifold. Let $\rho, \sigma$ be automorphism actions of $\Gamma$ on $T^m$ and $X$ respectively. Then there exists a non-affine continuous $\Gamma$-equivariant map from $(T^m, \rho)$ to $(X, \sigma)$ if and only if the following two conditions are satisfied.

a) $(T^m, \rho)$ is not ergodic.

b) There exists a nonzero vector $v$ in $L(G)$ which is fixed by $\Gamma_\rho$ under the action $\sigma_e$.

In section 4 we consider the case when $\Gamma$ is abelian and $(X_2, \sigma)$ is a topological factor of $(X_1, \rho)$ i.e. there exists a continuous $\Gamma$-equivariant map from $(X_1, \rho)$ onto $(X_2, \sigma)$. Generalizing the corresponding results in [Wa1] and [Wa2] we obtain the following.

**Theorem 3 :** Let $X_1, X_2$ be nilmanifolds and $\rho, \sigma$ be automorphism ac-
tions of a discrete abelian group $\Gamma$ on $X_1$ and $X_2$ respectively. Suppose that $(X_2, \sigma)$ is a factor of $(X_1, \rho)$ and either $X_1 = X_2$ or $X_2$ is a torus. Then there is a non-affine continuous $\Gamma$-equivariant map from $(X_1, \rho)$ to $(X_2, \sigma)$ if and only if $(X_2, \sigma)$ is not ergodic.

2 Almost equivariant maps

In this section we give a necessary and sufficient condition for existence of a non-affine almost equivariant map from $(X_1, \rho)$ to $(X_2, \sigma)$. Throughout this section for $i = 1, 2; X_i = G_i/D_i$ will denote a nilmanifold, $\pi_i$ will denote the projection map from $G_i$ to $X_i$, $e_i$ will denote the identity element of $G_i$ and $\bar{e}_i$ will denote the image of $e_i$ in $X_i$ under the map $\pi_i$. It is known that in this case any homomorphism from $D_1$ to $D_2$ can be extended to a continuous homomorphism from $G_1$ to $G_2$ (cf. [Ma]). We will also use the following fact (cf. [AGH], pp. 54).

**Proposition 2.1** ([AGH]): Let $X = G/D$ be a nilmanifold. For $a = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$ let $I_a$ denote the set defined by

$$I_a = \{ x \mid a_i \leq x_i \leq a_i + 1 \ \forall i = 1, \ldots, n \}.$$ 

Then there exists an invertible linear map $T$ from $\mathbb{R}^n$ to $L(G)$ such that for all $a$ in $\mathbb{R}^n$, the set $\exp \circ T(I_a)$ is a fundamental domain for $X = G/D$.

The following proposition was proved in [Wa2]. (see also [AP]).

**Proposition 2.2**: Let $X_1 = G_1/D_1, X_2 = G_2/D_2$ be nilmanifolds and $f : X_1 \to X_2$ be a continuous map. Let $F : G_1 \to G_2$ be a lift of $f$. Then
there exist a \( g_0 \in G_2 \), a continuous homomorphism \( \theta(f) : G_1 \to G_2 \) and a continuous map \( P(f) : G_1 \to G_2 \) such that

\[
  a) \quad P(f)(e_1) = e_2, \quad P(f)(g \cdot \gamma) = P(f)(g) \quad \forall g \in \Gamma.
\]

\[
  b) \quad F(g) = P(f)(g) \cdot g_0 \cdot \theta(f)(g) \quad \forall g \in \Gamma.
\]

Moreover for a given \( f \), the maps \( \theta(f) \) and \( P(f) \) are unique.

Suppose \( X_1, X_2 \) are nilmanifolds and \( \rho, \sigma \) are automorphism actions of a discrete group \( \Gamma \) on \( X_1 \) and \( X_2 \) respectively. Let \( \overline{\rho} \) and \( \overline{\sigma} \) denote the induced automorphism actions of \( \Gamma \) on \( G_1 \) and \( G_2 \) respectively. Then from the uniqueness part of Proposition 2.2 it is follows that for any \( \Gamma \)-equivariant continuous map \( f \) from \((X_1, \rho)\) to \((X_2, \sigma)\), \( P(f) \) is a \( \Gamma \)-equivariant continuous map from \((G_1, \overline{\rho})\) to \((G_2, \overline{\sigma})\). Now we obtain the following.

**Lemma 2.1 :** Let \( X_1 = G_1/D_1, X_2 = G_2/D_2 \) be nilmanifolds and \( \rho, \sigma \) be automorphism actions of a discrete group \( \Gamma \) on \( X_1 \) and \( X_2 \) respectively. Then there exists a nonaffine continuous \( \Gamma \)-equivariant map from \((X_1, \rho)\) to \((X_2, \sigma)\) if and only if there exists a nonzero continuous \( \Gamma \)-equivariant map \( S \) from \((X_1, \rho)\) to \((L(G_2), \sigma_e)\) such that \( S(e_1) = 0 \).

**Proof :** Suppose there exists a nonaffine \( \Gamma \)-equivariant continuous map \( f \) from \((X_1, \rho)\) to \((X_2, \sigma)\). Let \( P = P(f) : G_1 \to G_2 \) be as defined above. Since \( P(e_1) = e_2 \) and \( P(g \cdot \gamma) = P(g) \) for all \( g \in \Gamma \), there exists a unique continuous map \( Q \) from \( X_1 \) to \( G_2 \) such that \( Q(\overline{\pi_1}) = e_2 \) and \( P = Q \circ \pi_1 \). It is easy to see that \( Q \) is a \( \Gamma \)-equivariant map from \((X_1, \rho)\) to \((G_2, \overline{\sigma})\). Note that since \( G_2 \) is a connected simply connected nilpotent Lie group, the map \( \exp : L(G_2) \to G_2 \) is a diffeomorphism. Hence there is a unique map \( S : X_1 \to L(G_2) \) such that \( Q = \exp \circ S \). Now \( S(\overline{e_1}) = 0 \) and since \( \exp \) is a \( \Gamma \)-equivariant map from
(L(G_2), \sigma_e) to (G_2, \sigma) it is easy to see that $S$ is a $\Gamma$-equivariant map from $(X_1, \rho)$ to $(L(G_2), \sigma_e)$. Since $f$ is a nonaffine map, $P(f) = \exp \circ S \circ \pi_1$ is non-constant i.e. $S$ is a nonzero map.

Now suppose there exists a non-zero $\Gamma$-equivariant continuous map $S$ from $(X_1, \rho)$ to $(L(G_2), \sigma_e)$ such that $S(\bar{e}_1) = 0$. Define a map $f : X_1 \to X_2$ by

$$f(x) = \pi_2 \circ \exp \circ S(x) \ \forall x \in X_1.$$ 

It is easy to check that $f$ is a $\Gamma$-equivariant map from $(X_1, \rho)$ to $(X_2, \sigma)$ and $P(f) = \exp \circ S \circ \pi_1$. Since the map $\exp \circ S$ is non-constant, so is $P$. Now from the uniqueness part of Proposition 2.2 it follows that $f$ is a nonaffine map.

**Lemma 2.2 :** Let $X = G/D$ be a nilmanifold and $V$ be a finite dimensional vector space over $\mathbb{R}$. Let $\Gamma$ be a discrete group and $\rho, \sigma$ be automorphism actions of $\Gamma$ on $X$ and $V$ respectively. Then for any $\Gamma$-equivariant map $S : X \to V$ there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ such that $S \circ \rho(\gamma) = S \ \forall \gamma \in \Gamma_0$.

**Proof :** We define $A, A_1, A_2, \ldots \subset L(G)$ by

$$A_i = \{v \mid \exp(iv) \in D\}, \quad A = \cup A_i.$$ 

If $\pi : G \to G/D$ denotes the projection map then we define $B, B_1, B_2, \ldots \subset X$ by

$$B_i = \pi \circ \exp(A_i), \quad B = \cup B_i.$$ 

From Proposition 2.1 it follows that each $B_i$ is a finite subset of $X$ and $B$ is dense in $X$. Also it is easy to see that each $B_i$ is invariant under the action $\rho$. Therefore for any element $x$ in $B$, the $\rho$-orbit of $x$ is finite. Let $W$ denote
the subspace of $V$ which consists of all elements of $V$ whose $\sigma$-orbit is finite. Since $S$ is $\Gamma$-equivariant and $B$ is a dense subset of $X$, it follows that the image of $S$ is contained in $W$. Now choose a basis $\{w_1, w_2, \ldots, w_l\}$ of $W$. Define $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$ and $\Gamma_0$ by

$$\Gamma_i = \{ \gamma \in \Gamma \mid \sigma(\gamma)(w_i) = w_i \}, \quad \Gamma_0 = \cap \Gamma_i.$$ 

Since each $\Gamma_i \subset \Gamma$ is a subgroup of finite index, so is $\Gamma_0$. Since $\Gamma_0$ acts trivially on $W$ and image of $S$ is contained in $W$, we conclude that $S$ is a $\Gamma_0$-invariant map.

**Proof of Theorem 1:** Suppose there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ and a nonaffine continuous map $f$ from $(X_1, \rho)$ to $(X_2, \sigma)$ which is $\Gamma_0$-equivariant. Then by Lemma 2.1 there exists a nonzero continuous $\Gamma_0$-equivariant map $S$ from $(X_1, \rho)$ to $(L(G_2), \sigma_e)$ such that $S(\bar{e}_1) = 0$. Let $W$ denote the subspace of $L(G_2)$ which consists of all elements of $L(G_2)$ whose $\sigma_e$-orbit is finite. Then from Lemma 2.2 it follows that the image of $S$ is contained in $W$. Hence there exists a nonzero vector $v$ in $L(G_2)$ such that the $\sigma_e$-orbit of $v$ is finite. To prove a) we choose a norm $|| \cdot ||$ on $W$ and define a function $p : W \mapsto \mathbb{R}$ by

$$p(w) = \inf \{ ||\sigma_e(\gamma)(w)|| \mid \gamma \in \Gamma \}.$$ 

Since $\sigma_e$-orbit of any element of $W$ is finite, the map $q = p \circ S$ is a nonconstant continuous $\Gamma$-invariant function from $X_1$ to $\mathbb{R}$.

Now suppose both the conditions a) and b) are satisfied. Then $W$ is a nonzero subspace of $L(G_2)$ and there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ such that the $\sigma_e$-action of $\Gamma_0$ on $W$ is trivial. Let $q : X_1 \mapsto \mathbb{R}$ be a nonconstant continuous $\Gamma$-invariant function from $X_1$ to $\mathbb{R}$ and let $h : \mathbb{R} \mapsto W$
be a continuous map such that the map $h \circ q$ is nonzero and $h \circ q(\bar{e}_1) = 0$. Then $S = h \circ q$ is a nonzero continuous $\Gamma_0$-equivariant map from $(X_1, \rho)$ to $(L(G_2), \sigma_e)$ and $S(\bar{e}_1) = 0$. Applying Lemma 2.1 we see that there exists a nonaffine continuous $\Gamma_0$-equivariant map from $(X_1, \rho)$ to $(X_2, \sigma)$.

Let $(X, d)$ be a metric space and $\rho$ be a continuous action of a group $\Gamma$ on $X$. Then $(X, \rho)$ is said to be expansive if there exists $\epsilon > 0$ such that for any two distinct points $x, y$ in $X$,

$$\text{Sup} \{ d(\rho(\gamma)(x), \rho(\gamma)(y)) \mid \gamma \in \Gamma \} \geq \epsilon.$$ 

Any such $\epsilon$ is called an expansive constant of $(X, \rho)$. It is easy to check that the notion of expansiveness is independent of the metric $d$.

Now as a corollary of Theorem 1 we obtain the following.

**Corollary 2.1** : Let $X_1 = G_1/D_1, X_2 = G_2/D_2$ be nilmanifolds and $\rho, \sigma$ be automorphism actions of a discrete group $\Gamma$ on $X_1$ and $X_2$ respectively. Suppose that either $(X_1, \rho)$ is ergodic or $(X_2, \sigma)$ is expansive. Then every continuous $\Gamma$-equivariant map from $(X_1, \rho)$ to $(X_2, \sigma)$ is an affine map.

**Proof** : If $(X_1, \rho)$ is ergodic then there is no non-constant $\Gamma$-invariant continuous function from $(X_1, \rho)$ to $\mathbb{R}$. Applying Theorem 1 we see that there exists no nonaffine continuous $\Gamma$-equivariant map from $(X_1, \rho)$ to $(X_2, \sigma)$.

Suppose that $(X_2, \sigma)$ is expansive. Choose a metric $d$ on $X_2$ and an expansive constant $\epsilon > 0$ with respect to $d$. Define open sets $U \subset X_2$ and $V \subset L(G_2)$ by

$$U = \{ x \mid d(\bar{e}_2, x) < \epsilon \}, \quad V = (\pi_2 \circ \exp)^{-1}(U).$$
We claim that for every nonzero vector $v$ in $L(G_2)$, the $\sigma_e$-orbit of $v$ is infinite. To see this choose any vector $v_0$ in $L(G_2)$ such that the $\sigma_e$-orbit of $v_0$ is finite. Choose $\alpha > 0$ sufficiently small so that the $\sigma_e$-orbit of $\alpha v_0$ is contained in $V$ and does not intersect the set $\exp^{-1}(D_2) - \{0\}$. Then the $\sigma$-orbit of the element $x_0 = \pi_2 \circ \exp(\alpha v_0)$ is contained in $U$. Since $\bar{e}_2$ is fixed by the action $\sigma$, it follows that $x_0 = \bar{e}_2$ i.e. $v_0 = 0$. Now applying Theorem 1 we see that every continuous $\Gamma$-equivariant map from $(X_1, \rho)$ to $(X_2, \sigma)$ is an affine map.

3 Rigidity of toral automorphisms

In this section we will consider automorphism actions of discrete groups on tori. Suppose $\rho$ is an automorphism action of a discrete group $\Gamma$ on $T^m$. Then $\hat{\rho}$ will denote the automorphism action of $\Gamma$ on $T^m$ defined by

$$\hat{\rho}(\gamma)(\chi) = \chi \circ \rho(\gamma) \quad \forall \chi \in \hat{T}^m, \gamma \in \Gamma.$$ 

It is well known that $(T^m, \rho)$ is ergodic if and only if $\hat{\rho}$ has no nontrivial finite orbit. Recall that if $(T^m, \rho)$ is not ergodic then $F_{\rho} \subset \hat{T}^m$ will denote the subgroup consisting of all elements with finite $\hat{\rho}$-orbit and $\Gamma_{\rho} \subset \Gamma$ will denote the subgroup defined by

$$\Gamma_{\rho} = \{ \gamma \mid \chi \circ \rho(\gamma) = \chi \quad \forall \chi \in F_{\rho} \}.$$ 

Since $F_{\rho}$ is a finitely generated group, it follows that $\Gamma_{\rho} \subset \Gamma$ is a subgroup of finite index.

Lemma 3.1: Suppose $\Gamma$, $\rho$ and $\Gamma_{\rho}$ are as above. Then there exists $\chi_0 \in \hat{T}^m$ and $x_0 \in T^m$ such that

$$\Gamma_{\rho} = \{ \gamma \mid \chi_0 \circ \rho(\gamma)(x_0) = \chi_0(x_0) \}$$
**Proof:** For any $\chi$ in $F_\rho$, let $\Gamma_\chi \subset \Gamma$ denote the stabilizer of $\chi$ under the $\Gamma$-action $\hat{\rho}$. We claim that for any $\chi_1, \chi_2$ in $F_\rho$, there exists a $\chi'$ in $F_\rho$ such that $\Gamma_{\chi'} = \Gamma_{\chi_1} \cap \Gamma_{\chi_2}$. To see this, for $i = 1, 2$ define $A_i \subset \hat{T}^m$ by

$$A_i = \{ \chi_i \circ \rho(\gamma) - \chi_i \mid \gamma \in \Gamma \}.$$ 

Since $\chi_1, \chi_2$ are elements of $F_\rho$, both $A_1$ and $A_2$ are finite. Choose $n$ large enough so that $nA_1 \cap A_2 = \{0\}$. Define $\chi' = n\chi_1 - \chi_2$. Clearly $\Gamma_{\chi_1} \cap \Gamma_{\chi_2}$ is contained in $\Gamma_{\chi'}$. On the other hand if $\gamma \in \Gamma_{\chi'}$ then

$$n(\chi_1 \circ \rho(\gamma) - \chi_1) = \chi_2 \circ \rho(\gamma) - \chi_2.$$

Since $nA_1 \cap A_2 = \{0\}$, this implies that $\gamma \in \Gamma_{\chi_1} \cap \Gamma_{\chi_2}$.

Suppose $\chi_1, \ldots, \chi_d$ is a finite set of generators of $F_\rho$. From the above claim it follows that that there exists a $\chi_0$ in $F_\rho$ such that

$$\Gamma_{\chi_0} = \Gamma_{\chi_1} \cap \cdots \cap \Gamma_{\chi_d} = \Gamma_\rho.$$ 

Let $x_0 \in T^m$ be any element such that the cyclic subgroup generated by $x_0$ is dense in $T^m$. Then it is easy to see that

$$\Gamma_\rho = \Gamma_{\chi_0} = \{ \gamma \mid \chi_0 \circ \rho(\gamma)(x_0) = \chi_0(x_0) \}.$$ 

**Lemma 3.2:** Let $\Gamma_1 \subset \Gamma$ be a subgroup of finite index and $f$ be a $\Gamma_1$-invariant continuous map from $(T^m, \rho)$ to a metric space $(Y, d)$. Then $f$ is $\Gamma_\rho$-invariant.

**Proof:** First let us assume that $(Y, d) = \mathbb{C}$ with the usual metric. Let
\[ \hat{f} : \hat{T}^m \to \mathbb{C} \] be the Fourier transform of \( f \). It is easy to check that \( f \) is invariant under a subgroup \( \Gamma_2 \subset \Gamma \) if and only if \( \hat{f} \) is constant on each \( \Gamma_2 \)-orbit under the action \( \hat{\rho} \). Since \( \Gamma_\rho \) acts trivially on \( F_\rho \) under the action \( \hat{\rho} \), to prove \( \Gamma_\rho \)-invariance of \( f \) it is sufficient to show that \( \hat{f} = 0 \) on \( \hat{T}^m - F_\rho \).

Since \( \Gamma_1 \) is a subgroup of finite index, for any \( \phi \) in \( \hat{T}^m - F_\rho \), the \( \Gamma_1 \)-orbit of \( \phi \) is infinite. Since \( f \) is \( \Gamma_1 \)-invariant, \( \hat{f} \) is constant on the \( \Gamma_1 \)-orbit of \( \phi \). Since \( \sum_{\phi} |\hat{f}(\phi)|^2 < \infty \), we conclude that \( \hat{f}(\phi) = 0 \).

Now let \( (Y, d) \) be any arbitrary metric space and \( f \) be a continuous \( \Gamma_1 \)-invariant function from \( T^m \) to \( Y \). If \( C(Y, \mathbb{C}) \) denotes the set of all continuous functions from \( Y \) to \( \mathbb{C} \), then for each \( g \) in \( C(Y, \mathbb{C}) \) the map \( g \circ f \) is \( \Gamma_1 \)-invariant. Since \( C(Y, \mathbb{C}) \) separates points of \( Y \), from the previous argument it follows that \( f \) is \( \Gamma_\rho \)-invariant.

**Proof of Theorem 2:** Suppose there exists a non-zero continuous \( \Gamma \)-equivariant map from \( (T^m, \rho) \) to \( (X, \sigma) \). Then the condition a) follows from Corollary 2.1. Also from Lemma 2.1 it follows that there exists a non-zero continuous \( \Gamma \)-equivariant map \( S \) from \( (T^m, \rho) \) to \( (L(G), \sigma_e) \). Applying Lemma 2.2 and Lemma 3.2 we see that \( S \) is \( \Gamma_\rho \)-invariant. This implies that the \( \sigma_e \)-action of \( \Gamma_\rho \) on the image of \( S \) is trivial. Now the condition b) follows from the fact that \( S \) is non-zero.

Now suppose the conditions a) and b) are satisfied. Fix a finite subset \( A = \{\gamma_1, \ldots, \gamma_d\} \) of \( \Gamma \) which contains exactly one element of each right coset of \( \Gamma_\rho \). Let \( W \) denote the subspace of \( L(G) \) which is fixed by \( \sigma_e(\gamma) \) for all \( \gamma \) in \( \Gamma_\rho \). For any \( \Gamma_\rho \) invariant map \( h : T^m \to W \), we define a map \( h_A : T^m \to L(G) \) by

\[
h_A = \sum_{\gamma \in A} \sigma_e(\gamma^{-1}) \circ h \circ \rho(\gamma).
\]
Let $\gamma_1$ and $\gamma_2 = \gamma_0 \gamma_1$ be two elements of $\Gamma$ belonging to the same right coset of $\Gamma_\rho$. Since $h$ is $\Gamma_\rho$-invariant and $\Gamma_\rho$-action on $W$ is trivial, it is easy to see that

$$\sigma_e(\gamma_2^{-1}) \circ h \circ \rho(\gamma_2) = \sigma_e(\gamma_1^{-1}) \circ h \circ \rho(\gamma_1).$$

Therefore if $B$ is another set containing exactly one element of each coset of $\Gamma_\rho$ then $h_A = h_B$. Now it is easy to verify that for all $\gamma$ in $\Gamma$,

$$h_A \circ \rho(\gamma) = \sigma_e(\gamma) \circ h \gamma_A.$$ 

Hence $h_A$ is a $\Gamma$-equivariant map from $(T^m, \rho)$ to $(L(T^m), \sigma_e)$. We will show that for a suitable choice of $h$, $h_A$ is nonzero and $h_A(e) = 0$.

Let $\chi_0 \in \hat{T}^m$ and $x_0 \in T^m$ be as in Lemma 3.1. Define $c_0, c_1, \ldots, c_d \in S^1$ by

$$c_0 = 1, \quad c_i = \chi_0 \circ \rho(\gamma_i)(x_0) \quad \forall i = 1, \ldots, d.$$ 

Then $1, c_1, \ldots, c_d$ are distinct. We choose a continuous map $g$ from $S^1$ to $W$ such that

$$g(c_d) \neq 0, \quad g(c_i) = 0, \quad i = 0, \ldots, d - 1.$$ 

Since the map $g \circ \chi_0 : T^m \to W$ is $\Gamma_\rho$-invariant, from the previous argument it follows that the map $S = (g \circ \chi_0)_A$ is a $\Gamma$-equivariant map from $(T^m, \rho)$ to $(L(T)^n, D\sigma)$. Also it is easy to see that $S$ is nonzero and $S(e) = 0$. Now Theorem 2 follows from Lemma 2.1.

The following corollary generalizes earlier results of [AP] and [Wa1].

**Corollary 3.1 :** Let $A$ and $B$ be elements of $GL(m, \mathbb{Z})$ and $GL(n, \mathbb{Z})$ respectively. Let $k_A$ be the smallest positive integer $i$ such that $A^i$ has no
eigenvalue which is a proper root of unity. Then the following two are equivalent.

a) There exists a continuous nonaffine map \( f : T^m \to T^n \) satisfying
\[ f \circ A = B \circ f. \]
b) 1 is an eigenvalue of \( B^k \).

**Proof:** Let \( \Gamma \) be the cyclic group, \( \rho \) be the \( \Gamma \)-action on \( T^m \) generated by \( A \) and \( \sigma \) be the \( \Gamma \)-action on \( T^n \) generated by \( B \). Then after suitable identifications we have,

\[ \hat{T}^m = \mathbb{Z}^m, \quad F_\rho = \{ z \in \mathbb{Z}^m \mid A^i(z) = z \text{ for some } i \}. \]

It is easy to see that \( A^k \) leaves \( F_\rho \) invariant. Since no eigenvalue of \( A^k \) is a proper root of unity, it follows that \( A^k \) leaves \( F_\rho \) pointwise fixed. Suppose \( j \) is another positive integer such that \( A^j \) leaves \( F_\rho \) pointwise fixed. Then it easy to check that \( j \) is a multiple of \( k \). Therefore \( \Gamma_\rho = kA\mathbb{Z} \). It is easy to see that the action \( \sigma|_{\Gamma_\rho} \) has a nonzero fixed point in \( L(\mathbb{R}^n) \) if and only if 1 is an eigenvalue of \( B^k \). Now the given assertion follows from Theorem 2.

4 Rigidity of factor maps

In this section we will consider the case when \( \Gamma \) is abelian and \( X_2 \) is a topological factor of \( X_1 \) i.e. there exists a continuous \( \Gamma \)-equivariant map from \( X_1 \) onto \( X_2 \). We will need the following two results.

**Theorem 4.1** (see [Be], Theorem 5.1): Let \( \Gamma \) be a discrete abelian group, \( T^n \) be the \( n \)-torus and \( \rho \) be an ergodic automorphism action of \( \Gamma \) on \( T^n \). Then there exists an element \( \gamma_0 \) of \( \Gamma \) such that \( \rho(\gamma_0) \) is an ergodic automorphism.
**Theorem 4.2 (see [Pa]):** Let \( X = G/D \) be a nilmanifold and \( \theta \) be an automorphism of \( X \) such that \( \theta \) induces an ergodic automorphism on the torus \( G/[G,G] \circ D \). Then \( \theta \) is an ergodic automorphism of \( X \).

If \( X = G/D \) is a nilmanifold then by \( X^0 \) we denote the torus \( G/[G,G] \cdot D \) and by \( \pi^0 \) we denote the projection map from \( G \) onto \( X^0 \). If \( \rho \) is an automorphism action of a discrete group \( \Gamma \) on \( X \) then \( \rho^0 \) will denote the automorphism action of \( \Gamma \) on \( X^0 \) induced by \( \rho \).

We note the following simple consequence of Theorem 4.1 and Theorem 4.2.

**Proposition 4.1:** Let \( \Gamma \) be a discrete abelian group, \( X = G/D \) be a nilmanifold and \( \rho \) be an automorphism action of \( \Gamma \) on \( X \). Then \( (X, \rho) \) is ergodic if and only if \( (X^0, \rho^0) \) is ergodic.

**Proof:** Let \( q : X \to X^0 \) denote the projection map. Then it is easy to check that \( q \) is a measure preserving \( \Gamma \)-equivariant map from \( (X, \rho) \) to \( (X^0, \rho^0) \). Therefore ergodicity of \( (X, \rho) \) implies ergodicity of \( (X^0, \rho^0) \). On the other hand if \( (X^0, \rho^0) \) is ergodic then by Theorem 4.1 there exists a \( \gamma \) in \( \Gamma \) such that \( \rho^0(\gamma) \) is an ergodic automorphism of \( X^0 \). Applying Theorem 4.2 we see that \( \rho(\gamma) \) is an ergodic automorphism of \( X \) i.e. \( (X, \rho) \) is ergodic.

If \( V \) is a finite dimensional vector space over a field \( K \) then by \( V^* \) we denote the dual of \( V \). If \( \Gamma \) is a discrete group and \( \rho : \Gamma \to GL(V) \) is an automorphism action of \( \Gamma \) on \( V \) then by \( \rho^* \) we denote the action of \( \Gamma \) on \( V^* \) defined
by
\[ \rho^*(\gamma)(q)(v) = q(\rho^*(\gamma)^{-1}v) \quad \forall q \in V^*, v \in V. \]

**Proposition 4.2**: Let \( \Gamma \) be an abelian group and \( V \) be a finite dimensional vector space over \( \mathbb{R} \). Let \( \rho : \Gamma \to GL(V) \) be an automorphism action of \( \Gamma \) on \( V \) such that the induced \( \Gamma \)-action on the dual of \( V \) has a nontrivial fixed point. Then \( \rho \) has nontrivial fixed point in \( V \).

**Proof**: By passing to the complexification we see that it is enough to prove the analogous statement when \( V \) is a finite dimensional vector space over \( \mathbb{C} \). In that case after suitable identifications we can assume that \( V = V^* = \mathbb{C}^n \), \( \rho : \Gamma \to GL(n, \mathbb{C}) \) is a homomorphism and \( \rho^* : \Gamma \to GL(n, \mathbb{C}) \) is the homomorphism defined by \( \rho^*(\gamma) = \rho(\gamma^{-1})^T \). Let us consider the special case when with respect to some basis in \( \mathbb{C}^n \) each \( \rho(\gamma) \) is given by an upper triangular matrix with equal diagonal entries. In this case it is easy to verify that \( \rho \) or \( \rho^* \) has a nonzero fixed vector in \( \mathbb{C}^n \) if and only if for any \( \gamma \) in \( \Gamma \) all the diagonal entries of \( \rho(\gamma) \) are equal to 1. To prove the general case we note that since \( \Gamma \) is abelian, there exist subspaces \( V_1, V_2, \ldots, V_k \) of \( \mathbb{C}^n \) and homomorphisms \( \rho_i : \Gamma \to GL(V_i) \); \( i = 1, \ldots, k \) such that \( \mathbb{C}^n = V_1 \oplus \cdots \oplus V_k \), \( \rho = \rho_1 \oplus \cdots \oplus \rho_k \) and each \( \rho_i \) satisfies the above condition (cf. [Ja], pp. 134).

**Proposition 4.3**: Let \( \sigma \) be an automorphism action of a discrete abelian group \( \Gamma \) on a torus \( T^n \). Then \( (T^n, \sigma) \) is ergodic if and only if there is no nonzero element in \( L(T^n) \) with finite \( \sigma_e \)-orbit.

**Proof**: Since \( T^n = \mathbb{R}^n / \mathbb{Z}^n \), \( L(T^n) \) can be identified with \( \mathbb{R}^n \). Also \( \sigma_e \)
can be realised as a homomorphism from $\Gamma$ to $GL(n, \mathbb{Z})$, the dual action $\sigma_e^*$ can be realised as the homomorphism from $\Gamma$ to $GL(n, \mathbb{Z})$ which takes $\gamma$ to $\sigma_e(\gamma^{-1})^T$ and $\hat{\sigma}$ can be identified with $\sigma_e^*|_{\mathbb{Z}^n}$. Suppose $(T^n, \sigma)$ is ergodic. Let $\Gamma_0 \subset \Gamma$ be a subgroup of finite index. Then no nonzero element of $\mathbb{Z}^n$ is fixed by $\Gamma_0$ under the action $\sigma_e^*$. Since $\sigma_e^*(\gamma) \in GL(n, \mathbb{Z})$ for all $\gamma$, this implies that no nonzero element of $\mathbb{R}^n$ is fixed by $\Gamma_0$ under the action $\sigma_e$. Now suppose $(T^n, \sigma)$ is not ergodic. Then there exists a finite index subgroup $\Gamma_0 \subset \Gamma$ and a nonzero point $z$ in $\mathbb{Z}^n$ such that $z$ is fixed by $\Gamma_0$ under the action $\sigma_e^*$. Now from Proposition 4.1 we conclude that there exists a nonzero element in $\mathbb{R}^n$ which is fixed by $\Gamma_0$ under the action $\sigma_e$.

**Proof of Theorem 3:** Suppose $(X_2, \sigma)$ is not ergodic. By our assumption there exists a continuous $\Gamma$-equivariant map $f$ from $(X_1, \rho)$ onto $(X_2, \sigma)$. If $f$ is nonaffine then there is nothing to prove. Therefore we may assume that there exists a $g_0 \in G_2$ and a continuous homomorphism $\theta : G_1 \to G_2$ such that $f(gD_1) = g_0\theta(g)D_2$ for all $g$ in $G_1$. Since $f$ is surjective and $\Gamma$-equivariant, so is $\theta$. Let $\theta_0$ denote the homomorphism from $X_1^0$ to $X_2^0$ induced by $\theta$. Then $\theta_0$ is surjective. Since $(X_2, \sigma)$ is not ergodic, from Proposition 4.1 it follows that $(X_2^0, \sigma^0)$ is not ergodic. Let $\phi$ be an element of $X_2^0$ such that $\hat{\sigma}^0$-orbit of $\phi$ is finite. Since $\theta^0$ is $\Gamma$-equivariant, it follows that $\hat{\phi}^0$-orbit of $\phi \circ \theta^0$ is also finite, which implies that $(X_1^0, \rho_0)$ is not ergodic. Also for any $\gamma$ in $\Gamma_\rho$,

$$\phi \circ \sigma^0(\gamma) \circ \theta^0 = \phi \circ \theta^0 \circ \rho^0(\gamma) = \phi \circ \theta^0.$$ 

Since $\theta^0$ is a surjective map, this implies that $\phi \circ \sigma^0(\gamma) = \phi$ for all $\gamma$ in $\Gamma_\rho$. Let $\pi_2^0$ denote the projection map from $G_2$ onto $X_2^0$ and let $q$ denote the map
\( \phi \circ \pi_2^0 \circ \exp \). Then \( dq : L(G_2) \to \mathbb{R} \) is an element of the dual of \( L(G_2) \) such that \( dq \circ \sigma_e(\gamma) = dq \) for all \( \gamma \) in \( \Gamma_\rho \). Now from Proposition 4.2 it follows that there exists a nonzero point in \( L(G_2) \) which is fixed by \( \Gamma_\rho \) under the action \( \sigma_e \). Applying Theorem 2 we see that there exists a continuous nonaffine \( \Gamma \)-equivariant map \( h \) from \( (X_1^0, \rho^0) \) to \( (X_2, \sigma) \). If \( \pi_1^0 \) denotes the projection map from \( X_1 \) to \( X_1^0 \) then it is easy to see that \( h \circ \pi_1^0 \) is a continuous nonaffine \( \Gamma \)-equivariant map \( h \) from \( (X_1, \rho) \) to \( (X_2, \sigma) \).

Now suppose \( (X_2, \sigma) \) is ergodic. Since by our assumption either \( (X_1, \rho) = (X_2, \sigma) \) or \( X_2 \) is a torus from Proposition 4.3 it follows that either \( (X_1, \rho) \) is ergodic or there is no non-zero element in \( L(G_2) \) whose \( \sigma_e \)-orbit is finite. Applying Theorem 1 we conclude that every continuous \( \Gamma \)-equivariant map from \( (X_1, \rho) \) to \( (X_2, \sigma) \) is an affine map.

The following examples show that Theorem 3 does not hold if any of the assumptions as in the hypothesis is dropped.

**Example 1**: Let \( \Gamma \) be the cyclic group and \( \rho, \sigma \) be the automorphism actions of \( \Gamma \) on \( \mathbb{R}/\mathbb{Z} \) generated by the identity automorphism and the automorphism \( z \to -z \) respectively. Then it is easy to see that in this case \( \Gamma_\rho = \Gamma \) and no non-zero element of \( L(\mathbb{R}) \) is fixed by \( \Gamma_\rho \) under the action \( \sigma_e \). Now applying Theorem 2 we conclude that there is no nonaffine continuous \( \Gamma \)-equivariant map from \( (S^1, \rho) \) to \( (S^1, \sigma) \). Note that in this case \( \Gamma \) is abelian and neither of the two actions is ergodic.

**Example 2**: Fix \( n \geq 3 \) and define a subgroup \( \Gamma \) of \( GL(n, \mathbb{Z}) \) by

\[
\Gamma = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mid A \in GL(n - 1, \mathbb{Z}), b \in \mathbb{Z}^{n-1} \right\}
\]
Let $\rho$ denote the natural action of $\Gamma$ on $\mathbb{R}^n/\mathbb{Z}^n$. Then it is easy to see that for any $x = (x_1, \ldots, x_n)$ in $L(\mathbb{R}^n)$, the $\rho_e$-orbit of $x$ is unbounded. Applying Theorem 1 we see that there is no nonaffine continuous $\Gamma$-equivariant map from $(T^n, \rho)$ to $(T^n, \rho)$. Note that in this case $(T^n, \rho)$ is not ergodic since the vector $x_0 = (0, \ldots, 0, 1)$ is fixed by the dual action $\rho^*$.

**Example 3:** Suppose $X = G/D$ where $G$ and $D$ are defined by

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}, \quad D = \left\{ \begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \bigg| p, q, r \in \mathbb{Z} \right\}$$

Let $A$ be an ergodic automorphism of $G/D$. If $G_0$ denotes the center of $G$ then it is easy to see that $G_0/G_0 \cap D$ is isomorphic to $S^1$. Hence replacing $A$ by $A^2$ if necessary we may assume that $A$ acts trivially on $G_0$. Define a nilmanifold $X_1$ and an automorphism $A_1$ of $X_1$ by $X_1 = X \times S^1$, $A_1 = A \times Id$. Let $\rho_1$ and $\rho$ denote the automorphism actions of $\mathbb{Z}$ on $X_1$ and $X$ generated by $A_1$ and $A$ respectively. Then $(X, \rho)$ is a factor of $(X_1, \rho_1)$. Let $\pi : X_1 \to S^1$ be the the projection map and $h : S^1 \to L(G_0)$ be any nonzero map such that $h(e) = 0$. Then $h \circ \pi$ is a nonzero $\Gamma$-equivariant map from $(X_1, \rho_1)$ to $(L(G), \rho_e)$ such that $h \circ \pi(e) = 0$. Applying Lemma 2.1 we see that there exists a nonaffine continuous $\Gamma$-equivariant map from $(X_1, \rho_1)$ to $(X, \rho)$. 

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