GLOBAL EXISTENCE AND BOUNDEDNESS OF WEAK SOLUTIONS TO A CHEMOTAXIS-STOKES SYSTEM WITH ROTATIONAL FLUX TERM

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Abstract. In this paper, the three-dimensional chemotaxis-stokes system

$$\begin{align*}
 n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (nS(x, n, c) \cdot \nabla c), \quad x \in \Omega, \quad t > 0, \\
 c_t + u \cdot \nabla c &= \Delta c - nf(c), \quad x \in \Omega, \quad t > 0, \\
 u_t + \nabla P &= \Delta u + n\nabla \phi, \quad x \in \Omega, \quad t > 0, \\
 \nabla \cdot u &= 0, \quad x \in \Omega, \quad t > 0,
\end{align*}$$

posed in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary is considered under the no-flux boundary condition for $n$, $c$ and the Dirichlet boundary condition for $u$ under the assumption that the Frobenius norm of the tensor-valued chemotactic sensitivity $S(x, n, c)$ satisfies

$$S(x, n, c) \leq n^{l-2} \tilde{S}(c)$$

with $l > 2$ for some non-decreasing function $\tilde{S} \in C^2((0, \infty))$. In present work, it is shown that the weak solution is global in time and bounded while $m > m^*(l)$, where

$$m^*(l) = \begin{cases} 
 l - \frac{2}{3}, & \text{if } \frac{11}{7} \geq l > 2, \\
 \frac{7}{3}l - \frac{28}{15}, & \text{if } l > \frac{31}{17}.
\end{cases}$$

1. Introduction

This paper deals with the global existence of weak solutions to the chemotaxis-stokes system with rotational flux

$$\begin{align*}
 n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (nS(x, n, c) \cdot \nabla c), \quad x \in \Omega, \quad t > 0, \\
 c_t + u \cdot \nabla c &= \Delta c - nf(c), \quad x \in \Omega, \quad t > 0, \\
 u_t + \nabla P &= \Delta u + n\nabla \phi, \quad x \in \Omega, \quad t > 0, \\
 \nabla \cdot u &= 0, \quad x \in \Omega, \quad t > 0,
\end{align*}$$

(1.1)

in a bounded convex domain $\Omega \subset \mathbb{R}^3$. Here the chemotaxis sensitivity $S(x, n, c)$ is a matrix-valued function in $\mathbb{R}^{3 \times 3}$ satisfying $|S(x, n, c)| \leq n^{l-2} \tilde{S}(c)$ with $l > 2$ and nondecreasing function $\tilde{S}$.

As described in [6], the model was arisen to describe the behavior of swimming aerobic bacteria in situations where besides their chemotactically biased movement toward oxygen as their nutrient, a buoyancy-driven effect of bacterial mass on the fluid motion is not negligible. In the system (1.1), density denoted by $n = n(x, t)$, affects the fluid motion, as represented by its velocity field $u = u(x, t)$ and the associated pressure $P = P(x, t)$, through buoyant forces. Moreover, it is assumed that both cells and oxygen, the latter with concentration $c = c(x, t)$, are transported by the fluid and diffuse randomly, that the cells partially direct their movement toward increasing concentration of oxygen, that the latter is consumed by the cells.

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Keller-Segel model. In 1970s, Keller and Segel in [10] proposed a mathematical system

\[
\begin{aligned}
  n_t &= \nabla \cdot (D(n)\nabla n) - \nabla \cdot (S(n)\nabla c), & x \in \Omega, & t > 0, \\
  c_t &= \Delta c - c + n, & x \in \Omega, & t > 0,
\end{aligned}
\]  

(1.2)

to indicate the collective motion of cells, usually bacteria or amoebae, that are attracted by a chemical substance and are able to emit it. For a general introduction to chemotaxis, see [16]. In this model, \( n \) is the density of a biological organism and \( c \) is the concentration of a chemical substance produced by the biological organism. The mathematical properties of (1.2) have been extensively studied by many authors. The most peculiar features of (1.2) are the existence, blow-up and asymptotic behavior to the non-radial or radial solutions under some suitable initial data \( n_0(x) \). For instance, Ciéslak and Winkler in [4] have established the solution is global bounded in a two-dimensional bounded domain under the assumption that \( D(s) \) is decaying exponentially and \( \frac{S(s)}{D(s)} \leq Ks^\alpha \) is fulfilled with \( \alpha \in (0, 1) \). It has also been proven that there exist global bounded solutions when \( S(n) \leq C(1+n)^{-\alpha} \) for all \( s \geq 1 \) and some \( \alpha > 1 - \frac{2}{N} \) by Horstmann and Winkler in [9], whereas the solutions may blow up in the radial case with \( S(n) > cn^{-\alpha} \) for all \( \alpha < 1 - \frac{2}{N} \) \( (N \geq 2) \). Related models with prevention of overcrowding, see [8], volume effects [1, 11, 27], with logistic source [31] or involving more than one chemo-attractant have also been studied in [32, 13, 30]. Recently, the focus of the investigation to chemotaxis model has extended to the chemotaxis-(Navier-)Stokes system.

Chemotaxis coupled with fluid. Chemotaxis is an oriented immigration of living cells or bacteria under the effects of chemical gradients. Some aerobic bacteria such as Bacillus subtilis often live in thin fluid layer near solid-air-water contact, in which the swimming bacteria move towards higher concentration of oxygen according to mechanism of chemotaxis and meanwhile the movement of fluid is influenced by the gravitation force generated by itself. Thus, models with cell-fluid interaction become more complicated than fluid-free case as in (1.2) since it does not only account for chemotaxis and diffusion, but also includes viscous fluid dynamics. Considering that the motion of the fluids is determined by the incompressible (Navier-)Stokes equations, the corresponding chemotaxis-fluid model was proposed as follows:

\[
\begin{aligned}
  n_t + u \cdot \nabla n &= \nabla \cdot (D(n)\nabla n) - \nabla \cdot (S(x,n,c)\nabla c), & x \in \Omega, & t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c + f(n,c), & x \in \Omega, & t > 0, \\
  u_t + k(u \cdot \nabla)u + \nabla P &= \Delta u + n\nabla \phi, & x \in \Omega, & t > 0, \\
  \nabla \cdot u &= 0, & x \in \Omega, & t > 0,
\end{aligned}
\]

(1.3)

where \( n, c, u, P \) and \( \phi \) are defined as before, \( k \in \mathbb{R} \) denotes the strength of nonlinear fluid convection. Moreover,

\[
f(n,c) = -c + n
\]

(1.4)

when the production of cells or bacteria dominates the chemoattractant. Otherwise, if the signal is consumed, rather than produced, by the cells, the function \( f(n,c) \) is defined by

\[
f(n,c) = -ng(c)
\]

(1.5)

for some known \( g(c) \) denoting the consumption rate of the oxygen by the cells. From a viewpoint of mathematical analysis, this system couples the known obstacles from the theory of fluid equations to the typical difficulties arising in the study of chemotaxis system. Despite this challenge, the well-posedness to the system (1.3) with (1.4) or (1.5) have been addressed under various assumptions on the scalar function \( S, f \) and \( \phi \). For instance, Liu and Wang [14] have proved that the solution to the system (1.3) is global in time and bounded for \( k \neq 0 \) and \( N = 3 \) or \( k = 0 \) and \( N \in \{2,3\} \).
under assumptions that (1.4) and

$$S(x, n, c) = \frac{\xi_0}{(1 + \mu c)^2}$$

are satisfied. When the \(f(n, c)\) is defined by (1.5), Winkler [26] has established the global existence of weak solution in a three-dimensional domain when \(S(x, n, c) = D(n) \equiv 1\) and \(k \neq 0\). Based on the method applied in the paper mentioned just before, Zhang and Li [33] further shows the same result for \(m > \frac{2}{3}\) when considering the porous media diffusion \(D(n) = n^{m-1}\). If the \(k = 0\), the solutions to the system (1.3) coupled with (1.5) and porous media diffusion are bounded in time when \(m > \frac{2}{3}\) and \(S(x, n, c) \equiv 1\) are fulfilled. For more literatures related to this model, we can refer to [5, 7, 15, 19, 20, 21] and the reference therein.

**Chemotaxis with rotational flux.** Experiments find that the oriented migration of bacteria or cells may not be parallel to the gradient of the chemical substance, but may rather involve rotational flux components. This requires the sensitivity function \(S\) in (1.3) to be a matrix possibly containing nontrivial off-diagonal entries (see [29] and [28] for detailed model derivation) such as appearing e.g. in the prototype

$$S = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha > 0, \quad \beta \in \mathbb{R}, \quad (1.6)$$

in the two-dimensional case. This generalization results in considerable mathematical difficulties due to the fact that chemotaxis systems with such rotational fluxes lose some energy structure, which has served as a key to the analysis for scalar-valued \(S\). In [25], Winkler investigate the classical Keller-Segel model with tensor-value sensitivity

$$\begin{cases} n_t = \Delta n - \nabla \cdot (nS(x, n, c) \nabla c), & x \in \Omega, \quad t > 0, \\ c_t = \Delta c - nc, & x \in \Omega, \quad t > 0, \end{cases}$$

and obtained a generalized solution under the assumption that \(S(x, n, c) \leq CS_0(c)\) for some non-decreasing function \(S_0\) in \([0, \infty)\). Thereafter, this kind of chemotaxis sensitivity is also applied to chemotaxis model coupled with fluid. For instance, under the assumption that \(S(x, n, c) \leq C(1 + n)^{-\alpha}\) and \(k = 0\) are satisfied, Wang and Xiang [22, 23] established the existence of global bounded solutions to system (1.3) with (1.4) for arbitrary large initial data in a 2D and 3D bounded domain respectively when \(D(n) \equiv 1\), Peng and Xiang [17] further shows the same results with \(m + 2\alpha > 2\) and \(m > \frac{4}{3}\) when the porous media type diffusion \(D(n) = mn^{m-1}\) is considered in a 3D bounded domain.

If the signal is consumed, the global classical solution and its large time behavior under smallness assumption on \(\|c_0\|_{L^\infty(\Omega)}\) in a bounded domain \(\Omega \in \mathbb{R}^N\) (\(N = \{2, 3\}\)) are also been obtained by Cao and Lankeit in [2, 3] when \(k = 0\) and \(|S(x, n, c)| \leq CS_0(c)\) are fulfilled with some proper \(S_0\). Winker [24] establish the solutions to the system with porous media diffusion are also global bounded and converge to the integral equilibrium when \(m > \frac{7}{4}\). For more related works to the system (1.3) [12, 24] can be referred to.

From the introduction to the chemotaxis system with tensor-valued sensitivity above, we can infer that the existing results are only focused on the case \(S(x, n, c) \leq C\) as \(n \to \infty\). Motivated by the work [18], it is meaningful for us to investigate the system (1.3) with nonlinear diffusion \(D(n) = mn^{m-1}\) and tensor-valued chemotactic sensitivity \(S(n, c) \leq Cn^{l-2}S(c)\) for \(l > 2\). In order to formulate our results, we specify the precise mathematical setting: we shall subsequently consider the system (1.1) under the boundary conditions

$$\nabla n \cdot \nu = \nabla c \cdot \nu = 0, \quad u = 0, \quad x \in \partial \Omega, \quad t > 0 \quad (1.7)$$
and the initial conditions
\[ n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x) \quad u(x, 0) = u_0(x), \quad x \in \Omega \quad (1.8) \]
in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary under the assumption that
\[ \begin{cases} n_0 \in L^\infty(\Omega) \text{ is nonnegative}, \\ c_0 \in W^{1,\infty}(\Omega), \text{ is nonnegative}, \\ u_0 \in D(A^\alpha), \quad \alpha \in \left(\frac{3}{4}, 1\right). \end{cases} \quad (1.9) \]
Moreover, we let
\[ |S(x, n, c)| := \max_{i,j \in \{1, 2\}}\{s_{ij}(x, n, c)\}, \text{ with } s_{ij} \in C^2(\overline{\Omega} \times [0, \infty) \times [0, \infty)), \quad (1.10) \]
and
\[ |S(x, n, c)| \leq Cn^{l-2}\tilde{S}(c) \text{ with } l > 2 \quad (1.11) \]
for all $(x, n, c) \in \overline{\Omega} \times [0, \infty) \times [0, \infty)$, where $\tilde{S}$ is a nondecreasing function on $[0, \infty)$.

As for the time independent gravitational potential $\phi$ and $f$ in (1.1), we require that
\[ \phi \in W^{1,\infty}(\overline{\Omega}), \quad (1.12) \]
\[ f \in C^1([0, \infty)) \text{ is nonnegative.} \quad (1.13) \]

Before stating our main result, let us briefly introduce the definition of weak solution to the system (1.1)

**Definition 1.1.** We say that $(n, c, u, P)$ is a global weak solution of (1.1) associated to initial data $(n_0, c_0, u_0)$ if
\[ \begin{align*}
&n \in L^1_{\text{loc}}([0, \infty) \times \overline{\Omega}), \\
c \in L^\infty(\overline{\Omega} \times [0, \infty)) \cap L^1_{\text{loc}}([0, \infty), W^{1,1}(\Omega)), \\
u \in \left( L^1_{\text{loc}}([0, \infty); W^{1,1}(\Omega)) \right)^3
\end{align*} \]
such that $n \geq 0$ and $c \geq 0$ a.e in $\Omega \times [0, \infty)$,
\[ \begin{align*}
nc \in L^1_{\text{loc}}([0, \infty) \times \Omega), \quad u \otimes u \in \left( L^1_{\text{loc}}([0, \infty) \times \Omega) \right)^{3 \times 3}, \quad \text{and} \\
nS(x, n, c) \nabla c, nu, \text{ and } cu \text{ belong to } \left( L^1_{\text{loc}}([0, \infty) \times \Omega) \right)^3,
\end{align*} \]
that $\nabla \cdot u = 0$ a.e in $\Omega \times (0, \infty)$, and that
\[ \begin{align*}
- \int_0^\infty \int_\Omega n\varphi_t dx dt - \int_\Omega n_0 \varphi(\cdot, 0) dx &= - \int_0^\infty \int_\Omega \nabla n^m \cdot \nabla \varphi dx dt + \int_0^\infty \int_\Omega nS(x, n, c) \nabla c \cdot \nabla \varphi dx dt \\
&\quad + \int_0^\infty \int_\Omega nu \cdot \nabla \varphi dx dt \quad (1.14)
\end{align*} \]
for all $\varphi \in C^\infty_0([0, \infty) \times \overline{\Omega})$,
\[ \begin{align*}
- \int_0^\infty \int_\Omega c\varphi_t dx dt - \int_\Omega c_0 \varphi(\cdot, 0) dx &= - \int_0^\infty \int_\Omega \nabla c \cdot \nabla \varphi dx dt - \int_0^\infty \int_\Omega nf(c) \varphi dx dt \\
&\quad + \int_0^\infty \int_\Omega cu \cdot \nabla \varphi dx dt \quad (1.15)
\end{align*} \]
for all $\varphi \in C^\infty_0([0, \infty) \times \overline{\Omega})$ as well as
\[ \begin{align*}
- \int_0^\infty \int_\Omega u \cdot \zeta dx dt - \int_\Omega u_0 \cdot \zeta(\cdot, 0) dx &= - \int_0^\infty \int_\Omega \nabla u \cdot \nabla \zeta dx dt + \int_0^\infty \int_\Omega n \nabla \phi \cdot \zeta dx dt \quad (1.16)
\end{align*} \]
hold for all $\zeta \in C^\infty_0(\overline{\Omega} \times (0, \infty), \mathbb{R}^3)$. 
Our main result reads as follows.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary. Suppose that initial data and $S$ satisfies (1.9)-(1.11) with some $m > m^*(l)$, where

$$m^*(l) = \begin{cases}   l - \frac{5}{6}, & \text{if } \frac{31}{12} \geq l > 2, \\   \frac{7}{5}l - \frac{28}{15}, & \text{if } l > \frac{31}{12}. \end{cases} \quad (1.17)$$

Then, there exists a global weak solution $(n, c, u)$ to the system (1.1) satisfying Definition 1.1. And this solution is bounded in $\Omega \times (0, \infty)$ in the sense that with some $C > 0$ we have

$$\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1, \infty}(\Omega)} + \|u(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C \quad \text{for all } t > 0. \quad (1.18)$$

Furthermore, $c$ and $u$ are continuous in $\overline{\Omega} \times [0, \infty)$ and

$$n \in C^0_{w-*}([0, \infty); L^\infty(\Omega)); \quad (1.19)$$

that is, $n$ is continuous on $[0, \infty)$ as an $L^\infty(\Omega)$-valued function with respect to the weak-* topology.

**Remark 1.1.** If $l = 2$ in the assumption (1.11), the result above is consistent with [24].

This paper is organized as follows. In Section 2, we approximate the problem by a well-posed system (see (2.4) later). Section 3 is devoted to study the boundedness of regularized problem, we will see that the bounds are independent of the way we regularized the problem. Thus, upon the appropriate uniform estimates, we can let $\varepsilon \to 0$ to obtain limit functions of the system (2.4). This procedure is done in section 4, and also these limit functions are shown to solve (1.1) in weak sense as defined in Definition 1.1.

### 2. Approximation

Our main methods is to apply the classical solution of an appropriately regularized system of (1.1) to approximate the weak solution defined as in Definition 1.1. Following the same approximation procedure as in [29], we can find a family of functions $\{\rho_\varepsilon\}$ for any $\varepsilon \in (0, 1)$ such that

$$\rho_\varepsilon \in C^\infty_0(\Omega) \quad \text{with } 0 \leq \rho_\varepsilon \leq 1 \quad \text{in } \Omega \text{ and } \rho_\varepsilon \not\to 1 \quad \text{in } \Omega \text{ as } \varepsilon \searrow 0, \quad (2.1)$$

and define

$$S_\varepsilon(x, n_\varepsilon, c_\varepsilon) = \rho_\varepsilon(x)S(x, n, c), \quad x \in \overline{\Omega}. \quad (2.2)$$

Then, we have $S_\varepsilon(x, n_\varepsilon, c_\varepsilon) = 0$ on $\partial \Omega$ and

$$|S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \leq n_\varepsilon^{-2}\bar{S}(c_\varepsilon) \quad \text{for all } x \in \Omega, n_\varepsilon > 0, c_\varepsilon > 0. \quad (2.3)$$

Now, we consider the regularized system of (1.1) as follows:

$$\begin{aligned}   n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon &= \Delta (n_\varepsilon + \varepsilon)^m - \nabla \cdot (n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon), \quad (x, t) \in \Omega \times (0, T), \\
   c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon &= \Delta c_\varepsilon - n_\varepsilon f(c_\varepsilon), \quad (x, t) \in \Omega \times (0, T), \\
   u_{\varepsilon t} + \nabla P_\varepsilon &= \Delta u_\varepsilon + n_\varepsilon \nabla \phi, \quad (x, t) \in \Omega \times (0, T), \\
   \nabla \cdot u_\varepsilon &= 0, \quad (x, t) \in \Omega \times (0, T), \\
   \nabla n_\varepsilon \cdot \nu &= \nabla c_\varepsilon \cdot \nu = 0, \quad u_\varepsilon = 0, \quad (x, t) \in \partial \Omega \times (0, T), \quad (2.4) \\
   n_\varepsilon(x, 0) = n_0(x), \quad c_\varepsilon(x, 0) = c_0(x), \quad u_\varepsilon(x, 0) = u_0(x), \quad x \in \Omega, \end{aligned}$$

The first lemma concerns the local solvability of the regularized system (2.4) in classical sense. Without lose of generality, the proof is based on well-established methods involving the Schauder fixed point theorem, the standard regularity theory of parabolic equations and the Stokes system. For more details, we can refer to [25].
Lemma 2.1. For all $\varepsilon \in (0, 1)$, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that initial data $(n_0, c_0, u_0)$ satisfies (1.9), and $S$ fulfills (1.10)-(1.11). Then there exist a maximal existence time $T_{\text{max}, \varepsilon} \in (0, \infty]$ and functions
\[
n_{\varepsilon} \in C^0(\Omega \times [0, T_{\text{max}, \varepsilon}]) \cap C^{2,1}(\Omega \times (0, T_{\text{max}, \varepsilon})),
\]
\[
c_{\varepsilon} \in C^0(\Omega \times [0, T_{\text{max}, \varepsilon}]) \cap C^{2,1}(\Omega \times (0, T_{\text{max}, \varepsilon})),
\]
\[
u_{\varepsilon} \in C^0(\Omega \times [0, T_{\text{max}, \varepsilon}]) \cap C^{2,1}(\Omega \times (0, T_{\text{max}, \varepsilon})),
\]
\[
P_{\varepsilon} \in C^1(\Omega \times [0, T_{\text{max}, \varepsilon}]),
\]
such that $(n_{\varepsilon}, c_{\varepsilon}, \nu_{\varepsilon}, P_{\varepsilon})$ is a classical solution of (2.4) in $\Omega \times (0, T_{\text{max}, \varepsilon})$, and such that $n_{\varepsilon}$ and $c_{\varepsilon}$ are nonnegative. Moreover, if $T_{\text{max}, \varepsilon} < \infty$,
\[
\|n_{\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} + \|c_{\varepsilon}(\cdot, t)\|_{W^{1, \infty}(\Omega)} + \|A^\alpha u_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)} \rightarrow \infty
\]
as $t \rightarrow T_{\text{max}, \varepsilon}$, where $\alpha$ is defined in (1.9).

Therefore, in order to prove the global existence of the regularized problem, it is sufficient for us to show the boundedness for each term in (2.5) under the assumption that $T_{\text{max}, \varepsilon} < \infty$. The following lemma is immediately obtained upon observation.

Lemma 2.2. For any $\varepsilon \in (0, 1)$, let $(n_{\varepsilon}, c_{\varepsilon}, \nu_{\varepsilon}, P_{\varepsilon})$ be a classical solution to (2.4). It follows that
\[
\int_\Omega n_{\varepsilon}(\cdot, t)dx = \int_\Omega n_0dx, \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}),
\]
and
\[
\|c_{\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}).
\]

Proof. The mass conservation (2.6) is a straightforward consequence of an integration of the first equation of (2.4) over $\Omega$. Since $n_{\varepsilon}$ and $c_{\varepsilon}$ are nonnegative, an application of the maximum principle to the second equation in regularized system yields (2.7). \hfill \Box

Lemma 2.3. Let $p \in [1, \infty)$ and $r \in [1, \infty]$ be such that
\[
\begin{cases}
    r < \frac{3p}{2-p}, & \text{if } p \leq 3, \\
    r \leq \infty, & \text{if } p > 3.
\end{cases}
\]
Then for all $K > 0$ there exists $C_1 = C(p, r, K)$ such that for some $\varepsilon \in (0, 1)$ we have
\[
\|n_{\varepsilon}(\cdot, t)\|_{L^p(\Omega)} \leq K \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}),
\]
then
\[
\|Du_{\varepsilon}(\cdot, t)\|_{L^r(\Omega)} \leq C_1 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}).
\]

Proof. The proof of this lemma is based on some fundamental estimates of semigroup, and the details can be seen in [24] which is omitted here. \hfill \Box

Lemma 2.4. Let $p > 1$, (1.11) is satisfied. Then for each $\varepsilon \in (0, 1)$ we have
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega (n_{\varepsilon} + \varepsilon)^pdx + \frac{2m(p-1)}{(m+p-1)^2} \int_\Omega |\nabla (n_{\varepsilon} + \varepsilon)\|^{m+p-1}_{2}dx \\
\leq C_0^2(p-1) \int_\Omega (n_{\varepsilon} + \varepsilon)^{p+2l-m-3} |\nabla c_{\varepsilon}|^2dx \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}),
\]
where
\[
C_0 = \tilde{S}(\|c_0\|_{L^\infty(\Omega)})
\]
with $\tilde{S}$ defined as in (1.11).
Proof. We multiply the first equation in (2.4) with \((n_\varepsilon + \varepsilon)^{p-1}\) \((p > 1)\) and integrate by parts over \(\Omega\). Since \(\nabla u_\varepsilon\) and \(S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\) vanish whenever \(x \in \partial \Omega\), this yields

\[
\frac{1}{p} \frac{d}{dt} \int_\Omega (n_\varepsilon + \varepsilon)^p dx = - \int_\Omega \nabla (n_\varepsilon + \varepsilon)^m \cdot \nabla (n_\varepsilon + \varepsilon)^{p-1} dx \\
+ \int_\Omega (n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon) \cdot \nabla (n_\varepsilon + \varepsilon)^{p-1} dx \\
= -m(p - 1) \int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3} |\nabla (n_\varepsilon + \varepsilon)|^2 dx \\
+ (p - 1) \int_\Omega n_\varepsilon (n_\varepsilon + \varepsilon)^{p-2} S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla (n_\varepsilon + \varepsilon) dx
\]

for all \(t \in (0, T_{max, \varepsilon})\). Define

\[
C_0 = \bar{S}(\|c_\varepsilon\|_{L^\infty(\Omega)}).
\]

Due to the nonnegativity of \(n_\varepsilon\), (2.3) and (2.7), we have \(|S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \leq C_0 n_\varepsilon^{l-2}\). Furthermore, by applying Young’s inequality, we derive that

\[
\frac{1}{p} \frac{d}{dt} \int_\Omega (n_\varepsilon + \varepsilon)^p dx \\
\leq - \frac{4m(p - 1)}{(m + p - 1)^2} \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{m+p-1}|^2 dx + (p - 1)C_0 \int_\Omega (n_\varepsilon + \varepsilon)^{p+l-3} \nabla c_\varepsilon \cdot \nabla (n_\varepsilon + \varepsilon) dx \\
\leq - \frac{2m(p - 1)}{(m + p - 1)^2} \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{m+p-1}|^2 dx + \frac{C_0^2 (p - 1)}{2m} \int_\Omega (n_\varepsilon + \varepsilon)^{p+2l-m-3} |\nabla c_\varepsilon|^2 dx
\]

for all \(t \in (0, T_{max, \varepsilon})\). Thus, the proof of this lemma is completed. \(\square\)

Lemma 2.5. Let \(q > 1\) and \(\varepsilon \in (0, 1)\). Then we have

\[
\frac{1}{2q} \frac{d}{dt} \int_\Omega |\nabla c_\varepsilon|^{2q} dx + \frac{3(q - 1)}{8} \int_\Omega |\nabla c_\varepsilon|^{2(q-2)} |\nabla |\nabla c_\varepsilon| |^2 dx + \frac{1}{4} \int_\Omega |\nabla c_\varepsilon|^{2(q-1)} |D^2 c_\varepsilon|^2 dx \\
\leq (2 - 3 + \sqrt{3})^2 f_1^2 \int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2q-2} dx + (2q + 1) \int_\Omega |\nabla c_\varepsilon|^{2q} |D u_\varepsilon| dx
\]

(2.10)

for all \(t \in (0, T_{max, \varepsilon})\), where

\[
f_1 = \|f\|_{L^\infty(0, \|c_\varepsilon\|_{L^\infty(\Omega)})}.
\]

Proof. The omitted detail computation of this lemma can be seen in [24]. \(\square\)

Lemma 2.6. For any \(\varepsilon \in (0, 1)\). Let \(m \geq 1\) and suppose that \(p > 1\) satisfies

\[
p > \frac{7 - 3m}{3}.
\]

Then, we have

\[
\frac{d}{dt} \int_\Omega |A^{\frac{1}{2}} u_\varepsilon|^2 dx + \int_\Omega |Au_\varepsilon|^2 dx \leq \eta \int_\Omega |\nabla n_\varepsilon|^{\frac{m+p-1}{2}} dx + C_2 \quad \text{for all } t \in (0, T_{max, \varepsilon}),
\]

(2.12)

for each \(\eta > 0\) and some positive constant \(C_2 = (p, m, \eta)\).

Proof. We apply the Holmholz projection \(\mathcal{P}\) to the third equation in (2.4), and then test the resulting equation

\[
\partial_t u_\varepsilon + Au_\varepsilon = \mathcal{P}(n\nabla \phi)
\]
with $Au_\varepsilon$ to obtain
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |Au_\varepsilon|^2 \, dx + \int_\Omega |Au_\varepsilon|^2 \, dx = \int_\Omega Au_\varepsilon \cdot \mathcal{P}(n_\varepsilon \nabla \phi) \, dx \\
\leq \frac{1}{2} \int_\Omega |Au_\varepsilon|^2 \, dx + \frac{1}{2} \int_\Omega |(n_\varepsilon \nabla \phi)|^2 \, dx \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}).
\] (2.13)

Furthermore, the Gagliardo-Nirenberg inequality yields that
\[
\int_\Omega |(n_\varepsilon \phi)|^2 \, dx \leq \|\phi\|_{W^{1, \infty}(\Omega)}^2 \left| (n_\varepsilon + \varepsilon) \right|^{\frac{m+p-1}{2}} \left| L^{n \varepsilon + p - 4} \right|^{\frac{4}{m+p-1}} \left| \nabla n_\varepsilon \right|^{\frac{m+p-1}{2}} \left| L^{n \varepsilon} \right|^4 \\
\leq C_3 \|\phi\|_{W^{1, \infty}(\Omega)}^2 \|\nabla n_\varepsilon\|^{\frac{m+p-1}{2}} L^{n \varepsilon + p - 4} + C_4 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}),
\] (2.14)
is fulfilled for some $C_3 = C(p, \eta) > 0$ and $C_4 = C(p, \eta) > 0$ due to the (2.6).

Additionally, by the assumption (2.11), we can see that
\[
\frac{4}{m + p - 1} < 6,
\]
and
\[
\frac{6}{3m + 3p - 4} < 2
\]
are valid. Thus, the claimed inequality (2.12) results from (2.13) and (2.14) by a second application of Young’s inequality. \[\square\]

We next plan to estimate the right-hand sides in the above inequality appropriately by using suitable interpolation arguments along with the basic priori information provided by Lemma 2.2. Here, we introduce an auxiliary interpolation lemma, which will play an important role in making efficient use of the known $L^\infty$ bound for $c_\varepsilon$.

**Lemma 2.7.** Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, that $q \geq 1$ and that
\[
\lambda \in [2q + 2, 4q + 1].
\] (2.15)
Then there exists $C_5 > 0$ such that for all $\varphi \in C^2(\overline{\Omega})$ fulfilling $\varphi \cdot \frac{\partial c_\varepsilon}{\partial \nu} = 0$ on $\partial \Omega$ we have
\[
\|\nabla \varphi\|_{L^\lambda(\Omega)} \leq C_5 \|\nabla \varphi\|_{L^{\infty}(\Omega) \cap D^2 \varphi} \|\varphi\|_{L^{2 \lambda - 6}(\Omega)} + C_5 \|\varphi\|_{L^{\infty}(\Omega)}.
\] (2.16)

**Proof.** The proof of this lemma can be found in [24] for details. \[\square\]

The term on the right-hand of Lemma 2.4 can be estimated as follows.

**Lemma 2.8.** Let $m \geq l - 1$, $q > 1$. Assume that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ is a solution to (2.4) on $[0, T_{\text{max}, \varepsilon})$. Suppose that $p > \max\{1, m - 2l + 3\}$ satisfies
\[
p < \left(2m - 2l + \frac{8}{3}\right) q + m - 2l + 3.
\] (2.17)
Then for all \( \eta > 0 \) there exists \( C_6 = C(p,q,\eta) > 0 \) with the property that for all \( \varepsilon \in (0,1) \),
\[
\int_{\Omega} (n_{\varepsilon} + \varepsilon)^{p+2l-m-3} |\nabla c_{\varepsilon}|^2 \, dx \leq \eta \int_{\Omega} |\nabla (n_{\varepsilon} + \varepsilon)^{\frac{m+1}{2}}|^2 \, dx + \eta \int_{\Omega} |\nabla c_{\varepsilon}|^{2q-2} |D^2 c_{\varepsilon}|^2 \, dx + C_6 \quad (2.18)
\]
for all \( t \in (0,T_{\text{max},\varepsilon}) \).

Proof. For all \( t \in (0,T_{\text{max},\varepsilon}) \), we apply the Hölder inequality with exponents \( \frac{2p+1}{q} \) and \( q+1 \) to obtain
\[
\int_{\Omega} n_{\varepsilon}^{p+2l-m-3} |\nabla c_{\varepsilon}|^2 \, dx \leq \left( \int_{\Omega} n_{\varepsilon}^{\frac{p+2l-m-3(q+1)}{q}} \, dx \right)^{\frac{q}{q+1}} \left( \int_{\Omega} |\nabla c_{\varepsilon}|^{2q+2} \, dx \right)^{\frac{1}{q+1}} \]
\[
= \frac{2p+1}{p+m-1} \left( \int_{\Omega} L \frac{2(p+2l-m-3)(q+1)}{(p+m-1)q} \, dx \right)^{\frac{1}{2}} \cdot \|\nabla c_{\varepsilon}\|^2_{L^2(\Omega)} .
\]
By applying the Gagliardo-Nirenberg inequality and the mass conservation of \( n_{\varepsilon} \) (2.6), there exist positive constants \( C_7 = C(p,q) \) and \( C_8 = C(p,q) \) satisfying
\[
\left\| \frac{m+p-1}{2} n_{\varepsilon}^{\frac{2(p+2l-m-3)}{p+m-1}} \right\|_{L^2(\Omega)} \leq C_7 \left\| \nabla n_{\varepsilon}^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)} \left\| \frac{2(p+2l-m-3)}{p+m-1} n_{\varepsilon}^{\frac{1}{2} - \frac{1}{3}} \right\|_{L^2(\Omega)} \leq C_8 \left\| \nabla n_{\varepsilon} \right\|_{L^2(\Omega)} + C_8 \quad \text{for all } t \in (0,T_{\text{max},\varepsilon}),
\]
while \( \alpha \in (0,1) \) is determined by
\[
\frac{(p+m-1)q}{2(p+2l-m-3)(q+1)} = \left( \frac{1}{2} - \frac{1}{3} \right) \cdot \alpha + \frac{2}{m+p-1} (1 - \alpha),
\]
and
\[
\frac{2(p+2l-m-3)(q+1)}{(m+p-1)q} < 6
\]
is needed due to the embedding \( W^{1,2}(\Omega) \hookrightarrow L^q(\Omega) \). Because of \( m > l-1 \) and \( p > m-2l+3 \), we can see that
\[
6(m+p-1)q - 2(p + m + 2l - 3)(q+1) = 2(p - m + 2l - 3)(2q - 1) + 12(m - l + 1)q > 0,
\]
which indicates
\[
\frac{2(p+2l-m-3)(q+1)}{(m+p-1)q} - 6 < 0,
\]
then (2.22) is satisfied. According to the identity (2.21), we have
\[
\alpha := \frac{3(m+p-1)}{3m+3p-4} \left( 1 - \frac{q}{(p+2l-m-3)(q+1)} \right) \in (0,1),
\]
Then, by inserting \( \alpha \) into (2.20) we see that
\[
\left\| \frac{m+p-1}{2} n_{\varepsilon}^{\frac{2(p+2l-m-3)}{p+m-1}} \right\|_{L^2(\Omega)} \leq C_9 \left( \int_{\Omega} \left\| \nabla n_{\varepsilon}^{\frac{m+p-1}{2}} \right\|^2 \, dx + 1 \right)^{\frac{3(p-m+2l-3)(q+1)-q}{3m+3p-4}(q+1)}
\]
for some \( C_9 = C(p,q) > 0 \).
Next, we apply the Lemma 2.7 to estimate \( \|\nabla c_\varepsilon\|_{L^{2q+2}(\Omega)}^2 \) as follows:

\[
\|\nabla c_\varepsilon\|_{L^{2q+2}(\Omega)}^2 \leq C_{10} \left( \int_\Omega |\nabla c_\varepsilon|^{q-1} |D^2 c_\varepsilon| \right) \frac{2}{L^2(\Omega)} \|c_\varepsilon\|_{L^\infty(\Omega)}^{q-1} + \|c_\varepsilon\|_{L^\infty(\Omega)}^2 \leq C_{11} \left( \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 \, dx + 1 \right)^{\frac{1}{q+1}} \text{ for all } t \in (0, T_{\text{max}, \varepsilon}). \tag{2.24}
\]

for some positive constants \( C_{10} = C(p, q) \) and \( C_{11} = C(p, q) \) by choosing \( \lambda = 2q + 2 \) in (2.16).

Thus, combining with (2.19), (2.20) and (2.24) and employing the Young inequality, we can find that

\[
\int_\Omega (n_\varepsilon + \varepsilon)^{p+2l-m-3} |\nabla c_\varepsilon|^2 \, dx \leq C_{10} C_{11} \left( \int_\Omega \left| \nabla n_\varepsilon \right|^{m+p-1} + 1 \right)^{\frac{3[p-m+2l-3)(q+1)-q]}{q(3m+3p-4)} \times \left( \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 \, dx + 1 \right)^{\frac{1}{q+1}} \leq \eta \left( \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 \, dx + 1 \right) + C_{12} \left( \int_\Omega \left| \nabla n_\varepsilon \right|^{m+p-1} + 1 \right)^{\frac{3(p-m+2l-3)(q+1)-q}{q(3m+3p-4)}} \tag{2.25}
\]

for all \( t \in (0, T_{\text{max}, \varepsilon}) \) and \( \eta > 0 \), where the positive constant \( C_6 \) is related to \( \eta, p \) and \( q \). Otherwise, by (2.17), we can obtain that

\[
3 \left[ (p+2l-m-3)(q+1)-q \right] - q(3m+3p-4) = 3[p - (2m - 2l + \frac{8}{3})q + (p + 2l - 3)] > 0,
\]

which implies that

\[
\frac{3[p-m+2l-3)(q+1)-q]}{q(3m+3p-4)} < 1.
\]

Thus, we can employ the Young inequality again to the second term on right-hand side in (2.25) to finished the proof of this lemma finally. \( \square \)

We are in position to estimate the terms on the right-hand side in (2.10). The following three lemmas are cited from lemma 3.10, lemma 3.11 and lemma 3.12 in [24] respectively, the proof details of which are omitted here.

**Lemma 2.9.** Let \( m \geq 1 \) and \( q > 1 \). Assume that \( p > 1 \) satisfies

\[
p > \frac{3q - 3m + 4}{3}. \tag{2.26}
\]

Then, for each \( \eta > 0 \) and a positive constant \( C_{13} = C(p, q, \eta) \), the classical solution to the system (2.4) have the property

\[
\int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2q-2} \, dx \leq \eta \int_\Omega \left| \nabla n_\varepsilon \right|^{m+p-1} + 1 \right|^2 \, dx + \eta \int_\Omega |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 \, dx + C_{13} \text{ for all } t \in (0, T_{\text{max}, \varepsilon}). \tag{2.27}
\]

**Lemma 2.10.** Let \( m \geq 1 \) and \( r > \frac{3}{2} \) and suppose that \( q \geq r - 1 \) is such that

\[
(4 - 2r)q \leq r - 1. \tag{2.28}
\]
Then, for all \( \eta > 0 \) and each \( K > 0 \), there exists \( C_{14} = C(q, r, \eta, K) > 0 \) such that if for some \( \varepsilon \in (0, 1) \) we have
\[
\|Du_{\varepsilon}(\cdot, t)\|_{L^r(\Omega)} \leq K \quad \text{for all } t \in (0, T_{\max, \varepsilon}),
\] (2.29)
then
\[
\int_{\Omega} |\nabla c_{\varepsilon}|^{2q}|Du_{\varepsilon}|dx \leq \eta \int_{\Omega} |\nabla c_{\varepsilon}|^{2q-2}|D^2c_{\varepsilon}|^2dx + C_{14} \quad \text{for all } t \in (0, T_{\max, \varepsilon}).
\] (2.30)

**Lemma 2.11.** Let \( m \geq 1 \), and suppose that \( r \in (1, \frac{3}{2}] \) and
\[ q \in \left(1, \frac{2r+3}{3}\right). \] (2.31)
Then for each \( \eta > 0 \) and \( K > 0 \) one can find \( C_{15} = C(q, r, \eta, K) > 0 \) such that if there exist \( \varepsilon \in (0, 1) \) fulfilling
\[
\|u_{\varepsilon}\|_{L^r(\Omega)} \leq K \quad \text{for all } t \in (0, T_{\max, \varepsilon}),
\] (2.32)
then
\[
\int_{\Omega} |\nabla c_{\varepsilon}|^{2q}|u_{\varepsilon}|dx \leq \eta \int_{\Omega} |\nabla c_{\varepsilon}|^{2q-2}|D^2c_{\varepsilon}|^2dx + \eta \int_{\Omega} |Au_{\varepsilon}|^2dx + C_{15} \quad \text{for all } t \in (0, T_{\max, \varepsilon}).
\] (2.33)

3. Combining previous estimates

Now if \( m > l - 1 \), then the conditions on \( p \) in Lemmas 2.8 and 2.9 can be fulfilled simultaneously for any choice of \( q > 1 \) and \( l > 2 \). Thus, resorting to such \( m \) allows for combining the above results to derive an ODI. And we note that all constants appear in this section is independent of \( \varepsilon \).

**Lemma 3.1.** Assume that \( m > l - 1 \). Let \( r \geq 1 \) and \( q > 1 \) satisfy
\[
\begin{cases}
q < \frac{3+2r}{3}, & \text{if } r \leq \frac{3}{2}, \\
(4-2r)q \leq r - 1, & \text{if } r > \frac{3}{2},
\end{cases}
\] (3.1)
and assume that \( p > \max\{l - 1, m - 2l + 3\} \) be such that
\[
\frac{3q - 3 + 4}{3} < p < \left(2m - 2l + \frac{8}{3}\right) q + m - 2l + 3.
\] (3.2)
Then for all \( K > 0 \) one can find a constant \( C_{16} = C(p, q, r, K) > 0 \) such that if for some \( \varepsilon \in (0, 1) \) and \( T_{\max, \varepsilon} > 0 \) we have
\[
\|Du_{\varepsilon}(\cdot, t)\|_{L^r(\Omega)} \leq K \quad \text{for all } t \in (0, T_{\max, \varepsilon}),
\] (3.3)
then
\[
\frac{d}{dt} \left\{ \int_{\Omega} (n_{\varepsilon} + \varepsilon)^pdx + \int_{\Omega} |\nabla c_{\varepsilon}|^{2q}dx + \int_{\Omega} |A_{\varepsilon}^2u_{\varepsilon}|^2dx \right\} + \frac{1}{C_{16}} \cdot \left\{ \int_{\Omega} |\nabla n_{\varepsilon}|^{\frac{m+p-8}{4}}dx + \int_{\Omega} |\nabla c_{\varepsilon}|^{2q-2}|D^2c_{\varepsilon}|^2dx + \int_{\Omega} |Au_{\varepsilon}|^2dx \right\} \leq C_{16}.
\] (3.4)

**Proof.** To obtain the ODI inequality (3.4), we only need to combine Lemma 2.4-2.6 with Lemma 2.8-2.11 by choosing a suitable \( \eta > 0 \). 

Assuming the boundedness property of \( u_{\varepsilon} \), upon a further analysis of (3.4) we can estimate \( n_{\varepsilon} \) in \( L^\infty((0, \infty); L^p(\Omega)) \) for certain \( p \in (1, \infty) \).
Lemma 3.2. Let \( m > l - 1 \), and assume that \( r \geq 1 \) and \( p > \max\{l-1,m-2l+3\} \) are such that there exists \( q > 1 \) for which (3.1) and (3.2) hold. Then for all \( K > 0 \) there exists \( C_{17} = C(p,q,r,K) > 0 \) with the property that if \( \varepsilon \in (0,1) \) is such that 
\[
\|Du_\varepsilon\|_{L^r(\Omega)} \leq K \quad \text{for all } t \in (0,T_{\text{max},\varepsilon}),
\]
then we have 
\[
\int_\Omega n_\varepsilon^p(\cdot,t)dx \leq C_{17} \quad \text{for all } t \in (0,T_{\text{max},\varepsilon}).
\]

Proof. In this lemma, we can derive the consequence from (3.4) which is almost the same to the lemma 3.14 in [24], then we omit the details here. \( \square \)

Now by virtue of the mass conservation of \( n_\varepsilon \) (2.6), a first application of Lemma 2.3 warrants that the assumption (3.5) in the above lemma is fulfilled for some suitably small \( r > 1 \). Adjusting the parameter \( q \) properly, we thereby arrive at the following result which may be viewed as an improvement of the regularity property implied by (2.6) because the \( 5m^* (l) - 6l + \frac{25}{3} > 1 \) is fulfilled for any \( l > 2 \).

Lemma 3.3. Let \( m > m^*(l) \) with \( m^*(l) \) defined as in (1.17). Then for all \( p > \max\{l-1,m-2l+3\} \) satisfying

\[
p < 5m - 6l + \frac{25}{3},
\]

one can find \( C_{18} = C(p) > 0 \) such that whenever \( \varepsilon \in (0,1) \), we have 
\[
\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C_{18} \quad \text{for all } t \in (0,T_{\text{max},\varepsilon}).
\]

Proof. We first observe that \( m^*(l) \geq l - 1 \) and 
\[
\max\{l-1,m-2l+3\} < p < 5m - 6l + \frac{25}{3},
\]

\[
\frac{7}{3} - m < 5m - 6l + \frac{25}{3}
\]
are satisfied with \( m > m^*(l) \) for all \( l > 2 \).

Now since \( p < 5m - 6l + \frac{25}{3} \) by the (3.7), we have 
\[
\frac{3(p + 2l - m - 3)}{6m - 6l + 8} < 2.
\]

Moreover, the function 
\[
g(m) = 18m^2 - 18lm + 9m + 6l - 5
\]
is increasing when \( m > \frac{l}{2} - \frac{1}{4} \) by differentiating the function above. Therefore, the assumption \( m > l - \frac{5}{6} \) ensures that for all \( l > 2 \)
\[
-18(m - l + \frac{5}{6})p \leq 18m^2 - 18lm + 9m + 6l - 5
\]
due to \( g(l - \frac{5}{6}) = 0 \) since \( l - \frac{5}{6} > \frac{l}{2} - \frac{1}{4} \) is fulfilled, which is equivalent to the inequality 
\[
\frac{3(p - m + 2l - 3)}{6m - 6l + 8} < \frac{3p + 3m - 4}{3},
\]
then the above inequality is also satisfied with \( m \) chosen above.

According to (3.12), (3.13) and the fact that 
\[
\frac{3p + 3m - 4}{3} > 1
\]
by $p > \frac{7}{3} - m$, we can fix $q \in (1, 2)$ fulfilling
\[ \frac{3(p + 2l - m - 1)}{6m - 6l + 8} < q < \frac{3p + 3m - 4}{3}, \tag{3.15} \]
where the left inequality asserts that
\[ p < 5m - 6l + \frac{25}{3}, \]
and the right inequality guarantees that
\[ p > \frac{3q - 3m + 4}{3}, \]
altogether meaning that (3.2) is satisfied. Due to $q < 2$, we can finally pick $r \in [1, \frac{3}{2})$ sufficiently close to $\frac{2}{3}$ such that $r > \frac{3q - 3}{2}$, so that
\[ q < \frac{2r + 3}{3}, \]
ensuring that also (3.1) is valid. Then in view of (2.6), Lemma 2.3 assets that
\[ \|Du_{\varepsilon}\|_{L^r(\Omega)} \leq C_{19} \text{ for all } t \in (0, T_{max, \varepsilon}) \]
with some $C_{19} > 0$, whence according to the choices of $r$, $q$, and $p$, we may apply Lemma 3.1 to find a $C_{18} = C(p)$ such that (3.8) is satisfied. This proves the Lemma. \qed

In a second step, on the basis of the knowledge just gained, we may apply the Lemma 2.3 and once more combine the outcome thereof with Lemma 3.1 to obtain bounds for $n_{\varepsilon}$ in any space $L^\infty((0, \infty); L^p(\Omega))$.

**Lemma 3.4.** Let $m > m^*(l)$ with $m^*(l)$ denoted in (1.17). Then, for all $p > 1$, there exists $C_{20} = C(p) > 0$ such that for each $\varepsilon \in (0, 1)$ we have
\[ \int_\Omega n_{\varepsilon}^p(\cdot, t)dx \leq C_{20} \text{ for all } t \in (0, T_{max, \varepsilon}). \tag{3.16} \]

**Proof.** In this lemma, we only need to prove that there definitely exist some $p > p_0$ satisfying (3.16) for any $p_0 > \max\{l - 1, m - 2l + 3\}$ with some positive constant $C$.

For this purpose, given such $p_0$ we first fix $q > 1$ satisfying
\[ q > \frac{3(p_0 + 2l - m - 3)}{6m - 6l + 8} \tag{3.17} \]
and observe that then since $m > l - \frac{5}{6}$ we have
\[ 3q - 3m + 4 - (6m - 6l + 8)q - 3m + 6l - 9 = (-6m + 6l - 5)q + (-6m + 6l - 5) < 0 \]
and hence
\[ \frac{3q - 3m + 4}{3} < \left(2m - 2l + \frac{8}{3}\right)q + m - 2l + 3. \]

As (3.17) ensures that moreover
\[ \left(2m - 2l + \frac{8}{3}\right)q + m - 2l + 3 > \frac{6m - 6l + 8}{3} \cdot \frac{3(p_0 + 2l - m - 3)}{6m - 6l + 8} + m - 2l + 3 = p_0, \]
we can therefore pick some $p > p_0$ fulling
\[ \frac{3q - 3m + 4}{3} < p < p_0 < \left(2m - 2l + \frac{8}{3}\right)q + m - 2l + 3. \tag{3.18} \]

Now in order to verify (3.16) for these choices of $p$ and $q$, we fist use use the fact that
\[ 5m - 6l + \frac{25}{3} > \frac{19}{12} \]
for all $l > 2$ under the assumption that $m > \max \{ l - \frac{5}{6}, \frac{7}{3}l - \frac{38}{15} \}$. Then, we can infer from the Lemma 3.3 that there exists some $C_{21} > 0$ fulfilling

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{21} \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}).$$

Since $\frac{3.45}{3.23} = \frac{57}{17} > 3$, Lemma 2.3 yields $C_{22} > 0$ satisfying

$$\|u_\varepsilon(\cdot, t)\|_{L^3(\Omega)} \leq C_{22}$$

for any $t \in (0, T_{\text{max}, \varepsilon})$, which also implies that the condition $(4 - 2r)q \leq r - 1$ in (3.1) is trivially satisfied, thanks to (3.18) we may invoke Lemma 3.2 to establish (3.16).

By the application to some general semigroup estimates and the standard parabolic regularity arguments, we can derive that the classical solution to the system (2.4) is global in time, at the same time some boundeness results can be obtained.

**Lemma 3.5.** Let $m > m^*(l)$ with the definition of $m^*(l)$ in (1.17). Then the solution to the (2.4) is global in time for all $\varepsilon \in (0, 1)$ and also bounded as follows:

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty_{\text{loc}}(0, \infty; L^\infty(\Omega))} \leq C_{23} \quad (3.19)$$

and

$$\|c_\varepsilon(\cdot, t)\|_{L^\infty_{\text{loc}}(0, \infty; W^{1, \infty}(\Omega))} \leq C_{23} \quad (3.20)$$

as well as

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty_{\text{loc}}(0, \infty; W^{1, \infty}(\Omega))} \leq C_{23}. \quad (3.21)$$

**Proof.** First, the validity of estimate (3.16) for any $p > 3$ allows for an application of Lemma 2.3 to $r = \infty$ to infer that $Du_\varepsilon$ is bounded in $L^\infty(\Omega \times (0, T_{\text{max}, \varepsilon}))$, then we can reach that

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty_{\text{loc}}(0, \infty; W^{1, \infty}(\Omega))} \leq C_{24} \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}). \quad (3.22)$$

Next, we establish the boundedness of $\|n_\varepsilon\|_{L^\infty(\Omega \times (0, T_{\text{max}, \varepsilon}))}$ for all $\varepsilon \in (0, 1)$. According to the well-known estimates for the Neumann heat semigroup in $\Omega$, we can invoke the variation-of-constants formula for $n_\varepsilon$ and $\nabla \cdot u_\varepsilon = 0$ to find that there exists a constant $C_{25} > 0$ fulfilling

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|e^{t\Delta}n_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)\Delta}\nabla(n_\varepsilon S(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon + n_\varepsilon u_\varepsilon)(\cdot, s)\|_{L^\infty(\Omega)} ds \quad (3.23)$$

$$\leq \|n_0\|_{L^\infty(\Omega)} + C_{25} \int_0^t (t - s)^{-\frac{1}{2} - \frac{q}{2}} e^{-\lambda_1(t-s)} \|n_\varepsilon S(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon + n_\varepsilon u_\varepsilon\|_{L^q(\Omega)} ds. \quad (3.24)$$

Thus, for any given $3 < q < r$, we can find that

$$\|n_\varepsilon S(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon\|_{L^q(\Omega)} \leq C_0 \|n_\varepsilon^{l-1}(l-1)\|_{L^q(\Omega)} \nabla c_\varepsilon(\cdot, t)\|_{L^{q^*}(\Omega)} \leq C_{26}. \quad (3.25)$$

and

$$\|n_\varepsilon u_\varepsilon\|_{L^q(\Omega)} \leq \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \|n_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \leq C_{27} \quad (3.26)$$

for all $t \in (0, T_{\text{max}, \varepsilon})$ by using the Hölder inequality, Lemma 3.4 and the boundedness result for $u_\varepsilon$. Then, we can infer from (3.23)-(3.26) that

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{26}, \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}) \quad (3.27)$$

is true because $q > 3$ ensures that $\int_0^\infty (t-s)^{-\frac{1}{2} - \frac{q}{2} + \frac{q}{q^*}} e^{-\lambda_1(t-s)} ds$ is finite.
Moreover, it can be derived from an well-known arguments parabolic regularity theory that
\[ \| c_\varepsilon(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq C_{27} \quad \text{for all } t \in (0, T_{\max, \varepsilon}) \]  
(3.28)
is satisfied due to (3.27).

We now turn to estimate for \( A^\alpha u_\varepsilon \). Applying the fractional power \( A^\alpha \) (\( \alpha \) is given in (1.9)) to the variation-of-constants formula
\[ u_\varepsilon(\cdot, t) = e^{-tA}u_0 + \int_0^t e^{(t-s)A} \mathcal{P}(n_\varepsilon(\cdot, s)\nabla \varphi) ds, \quad \text{for all } t \in (0, T_{\max, \varepsilon}), \]
we can arrive at
\[ \| A^\alpha u_\varepsilon(\cdot, t) \|_{L^2(\Omega)} \leq \| A^\alpha e^{-tA}u_0 \|_{L^2(\Omega)} + \int_0^t \| A^\alpha e^{(t-s)A} \mathcal{P}(n_\varepsilon(\cdot, s)\nabla \varphi) \|_{L^2(\Omega)} ds \]  
(3.29)
For the first term on the right-hand side, we can easily obtain that
\[ \| A^\alpha e^{-tA}u_0 \|_{L^2(\Omega)} = \| e^{-tA}A^\alpha u_0 \|_{L^2(\Omega)} \leq C_{29} \]  
(3.30)
is valid for some positive constant \( C_{18} \) and all \( t \in (0, T_{\max, \varepsilon}) \) due to (1.9).

Since the operator \( \mathcal{P} \) is bounded from \( L^2(\Omega) \) to \( L^2(\Omega) \), we can estimate the last term on the right-hand side as follows:
\[ \int_0^t \| A^\alpha e^{(t-s)A} \mathcal{P}(n_\varepsilon(\cdot, s)\nabla \varphi) \|_{L^2(\Omega)} ds \leq C_{30} \| \varphi \|_{W^{1,\infty}(\Omega)} \| n_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} \int_0^\infty (t-s)^\alpha e^{-\lambda(t-s)} ds \leq C_{31} \]  
(3.31)
with some \( C_{30} > 0 \) and \( C_{31} > 0 \).

Substituting (3.30) and (3.31) into (3.29), we can conclude that there exists a \( C_{21} > 0 \) such that
\[ \| A^\alpha u_\varepsilon(\cdot, t) \|_{L^2(\Omega)} \leq C_{32}, \quad \text{for all } t \in (0, T_{\max, \varepsilon}). \]  
(3.32)
Then, combining (3.27), (3.28) and (3.32), we thus can claim that \( T_{\max, \varepsilon} = \infty \) and that the classical solution \( (n_\varepsilon, c_\varepsilon, u_\varepsilon) \) is global in time. Therefore, we can furthermore find that the solution to the system (2.4) satisfies (3.19)-(3.21) by arguing similarly as above. 

As one further class of a priori estimates, let us finally also note straightforward consequence of Lemma 3.5 for uniform H"older regularity properties of \( c_\varepsilon, \nabla c_\varepsilon \) and \( u_\varepsilon \).

**Lemma 3.6.** Let \( m \) be denoted as the above lemma. Then there exists \( \theta \in (0, 1) \) such that for some \( C > 0 \) we have
\[ \| c_\varepsilon \|_{C^\theta(\gamma} \leq C \quad \text{for all } t \geq 0, \]  
(3.33)
and
\[ \| \nabla c_\varepsilon \|_{C^\theta(\gamma} \leq C(\tau) \quad \text{for all } t \geq \tau, \]  
(3.34)
as well as
\[ \| u_\varepsilon \|_{C^\theta(\gamma} \leq C \quad \text{for all } t \geq 0, \]  
(3.35)
with any \( \tau \in (0, \infty) \).

**Proof.** The proof of this lemma is on the standard parabolic regularity theory and some standard semigroup estimation techniques, which is omitted here. For the details, we can refer to the lemma 3.18 and lemma 3.19 in [24].
4. PROOF OF THE THEOREM 1.1

In this section, we use the classical solution of the regularized system \((2.4)\) to approximate the weak solution we defined in Definition 1.1 above. At first, several necessary boundedness results are established in the following lemmas.

**Lemma 4.1.** Let \(m > m^*(l)\) with \(m^*(l)\) given in (1.17). Then, for each \(\varepsilon \in (0,1)\), the global classical solution \((n_\varepsilon, c_\varepsilon, u_\varepsilon, p_\varepsilon)\) satisfies the following inequalities

\[
\int_0^\infty \int_\Omega n_\varepsilon f(c_\varepsilon)\,dx\,dt \leq \int_\Omega c_0\,dx,
\]

and

\[
\int_0^\infty \int_\Omega |\nabla c_\varepsilon|^2\,dx \leq \frac{1}{2} \int_\Omega c_0^2\,dx,
\]

as well as

\[
\int_0^\infty \int_\Omega |\nabla (n_\varepsilon + \varepsilon)|^2\,dx \leq M_1, \quad \text{for any } \eta > \frac{m}{2}.
\]

**Proof.** The inequality (4.1)-(4.2) can be obtained straightforwardly just by testing the second equation in \((2.4)\) with 1 and \(c_\varepsilon\) over \(\Omega\) respectively, and using the nonnegativity of \(n_\varepsilon, \nabla \cdot u_\varepsilon = 0\).

We now establish the inequality (4.3). According to computation of the Lemma 2.4 and the boundedness of \(n_\varepsilon\) (3.19), we can obtain that

\[
\frac{1}{p} \int_\Omega (n_\varepsilon + \varepsilon)^p\,dx + \frac{2m(p-1)}{(m+p-1)^2} \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{\frac{m+p-1}{2}}|^2\,dx
\leq \frac{C_0^2(p-1)}{2m} (\|n_\varepsilon\|_{L^\infty(\Omega)} + 1)^{p+2l-m-3} \int_\Omega |\nabla c_\varepsilon|^2\,dx
\]

for \(p > 1\). Then, an integration over \((0, t)\) yields that

\[
\frac{1}{p} \int_\Omega (n_\varepsilon + \varepsilon)^p\,dx + \frac{2m(p-1)}{(m+p-1)^2} \int_0^t \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{\frac{m+p-1}{2}}|^2\,dx
\leq \frac{C_0^2(p-1)}{2m} (\|n_\varepsilon\|_{L^\infty(\Omega)} + 1)^{p+2l-m-3} \int_0^t \int_\Omega |\nabla c_\varepsilon|^2\,dx
\]

\[
+ \frac{1}{p} \int_\Omega (n_0 + 1)^p\,dx
\]

\[
\leq M_1
\]

for each \(t \in (0, \infty)\).

For any \(p > 1\), we can infer from (4.2) that the (4.3) is satisfied by setting

\[
\eta = \frac{m+p-1}{2},
\]

which furthermore shows

\[
\eta > \frac{m}{2}.
\]

In particularly, we can obtain the boundedness of \(\|\nabla (n_\varepsilon + \varepsilon)^m\|_{L^2_{loc}(0,\infty;L^2(\Omega))}\) by choosing \(\eta = m\). □

**Lemma 4.2.** Let \(\varepsilon \in (0,1), T \in (0, \infty)\) and \(m\) be given as above lemmas. Suppose that \((n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)\) is a classical solution to the regularized system \((2.4)\) on \([0,T]\). There exists a \(\varepsilon\) independent constant \(M_1(T) > 0\) such that

\[
\left\| \frac{\partial}{\partial t} (n_\varepsilon + \varepsilon)^\gamma \right\|_{L^1(0,T),W^{1,2}_0(\Omega)^*} \leq M_2
\]

(4.4)
for all $\gamma \geq \{1, \frac{m}{2} - l + 2\}$.

**Proof.** On differentiation and integration by parts in (2.4), we see that for each $\varphi \in C_0^\infty(\Omega)$ we have

$$
\frac{1}{\gamma} \int_\Omega \frac{\partial}{\partial t}(n_e + \varepsilon) \varphi \, dx = - \int_\Omega \nabla (n_e + \varepsilon)^m \cdot \nabla ((n_e + \varepsilon)^{\gamma-1}) \, dx - \int_\Omega (u_e \cdot \nabla n_e) \cdot ((n_e + \varepsilon)^{\gamma-1}) \, dx
$$

$$
+ \int_\Omega (n_e S_e(x, n_e, c_e) \nabla c_e) \cdot \nabla ((n_e + \varepsilon)^{\gamma-1}) \, dx
$$

$$
= I_1 + I_2 + I_3, \quad \text{for all } t \in (0, T), \quad (4.5)
$$

In order to estimate the $\{I_i\}_{i=1,2,3}$ above, we apply the Hölder inequality and Cauchy inequality to obtain that

$$
I_1 = - m(\gamma - 1) \int_\Omega (n_e + \varepsilon)^{m+\gamma-1} \nabla n_e \cdot \nabla \varphi \, dx - \int_\Omega (n_e + \varepsilon)^{m+\gamma-2} \nabla n_e \cdot \nabla \varphi \, dx
$$

$$
= - \frac{4m(\gamma - 1)}{(m + r - 1)^2} \int_\Omega \|\nabla (n_e + \varepsilon)\|^{m+\gamma-2} \varphi \, dx - \frac{m}{m + \gamma - 1} \int_\Omega \nabla (n_e + \varepsilon)^{m+\gamma-1} \cdot \nabla \varphi \, dx
$$

$$
\leq \frac{4m(\gamma - 1)}{(m + r - 1)^2} \|\varphi\|_{L^\infty(\Omega)} \int_\Omega \|\nabla (n_e + \varepsilon)\|^{m+\gamma-2} \varphi \, dx + \frac{m}{m + \gamma - 1} \int_\Omega \|\nabla (n_e + \varepsilon)^{m+\gamma-1}\| \nabla \varphi \, dx
$$

$$
\leq \frac{4m(\gamma - 1)}{(m + r - 1)^2} \|\varphi\|_{L^\infty(\Omega)} \int_\Omega \|\nabla (n_e + \varepsilon)\|^{m+\gamma-2} \varphi \, dx + \|\nabla (n_e + \varepsilon)^{m+\gamma-1}\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} \quad (4.6)
$$

and

$$
I_2 = - \frac{1}{\gamma} \int_\Omega (u_e \cdot \nabla (n_e + \varepsilon)^\gamma) \cdot \varphi \, dx = - \frac{1}{\gamma} \int_\Omega (n_e + \varepsilon)^\gamma u_e \cdot \nabla \varphi \, dx
$$

$$
\leq \frac{1}{\gamma} (\|n_e\|_{L^\infty(\Omega)} + 1)^\gamma \|u_e\|_{L^\infty(\Omega)} \|\nabla \varphi\|_{L^\infty(\Omega)} \quad (4.7)
$$

as well as

$$
I_3 = \int_\Omega \nabla (n_e S_e(x, n_e, c_e) \nabla c_e) \cdot \nabla (n_e + \varepsilon)^{\gamma-1} \, dx + \int_\Omega (n_e + \varepsilon)^{\gamma-1}(n_e S_e(x, n_e, c_e) \nabla c_e) \cdot \nabla \varphi \, dx
$$

$$
\leq C_0 \|\nabla c_e\|_{L^1(\Omega)} \int_\Omega \|n_e + \varepsilon\|^{l+\gamma-1} \|\nabla (n_e + \varepsilon)^{\gamma-1}\| \varphi \, dx + C_0 \int_\Omega (n_e + \varepsilon)^{l+\gamma-2} \|\nabla c_e \cdot \varphi\| \, dx
$$

$$
\leq \frac{(\gamma - 1)C_0}{l + \gamma - 1} \|\nabla c_e\|_{L^1(\Omega)} \int_\Omega \|\nabla (n_e + \varepsilon)^{l+\gamma-2}\| \varphi \, dx + C_0 \int_\Omega (n_e + \varepsilon)^{l+\gamma-2} \|\nabla c_e \cdot \varphi\| \, dx
$$

$$
\leq C_0 \|\nabla c_e\|_{L^\infty(\Omega)} \|\nabla (n_e + \varepsilon)^{l+\gamma-2}\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} + C_0 \|n_e\|_{L^\infty(\Omega)} + 1)^{l+\gamma-2} \|\nabla c_e\|_{L^\infty(\Omega)} \|\nabla \varphi\|_{L^\infty(\Omega)}. \quad (4.8)
$$

Combining (4.6)-(4.8) with (4.5), we can see that there exists some constant $C_{22} > 0$ satisfying

$$
\left| \int_\Omega \frac{\partial}{\partial t}(n_e + \varepsilon)^\gamma \, dx \right| \leq C_{33} \left( \|\nabla (n_e + \varepsilon)^{m+\gamma-1}\|_{L^2(\Omega)}^2 + \|\nabla (n_e + \varepsilon)^{m+\gamma-1}\|_{L^2(\Omega)} \right)
$$

$$
+ \|\nabla (n_e + \varepsilon)^{l+\gamma-2}\|_{L^2(\Omega)} + 1 \|\varphi\|_{W^{1,\infty}(\Omega)}.
$$

According to the Lemma 4.1 and the embedding $W^{3,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, we can claim that (4.4) is true when the following relation is fulfilled:

$$
\begin{cases}
  m + \gamma - 1 > \frac{m+\gamma-1}{2} \geq \eta, \\
  l + \gamma - 2 \geq \eta.
\end{cases} \quad (4.9)
$$
that is
\[ \gamma > \max\{1, \frac{m}{2} - l + 2\}. \quad (4.10) \]
The proof of this lemma is finished.

**Lemma 4.3.** Let \( m \) be given as above lemmas, the there exists \( C > 0 \) such that
\[ \| \partial_t n_\varepsilon(\cdot, t) \|_{(W^{2,2}_0(\Omega))^*} \leq C \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \quad (4.11) \]
In particular,
\[ \| n_\varepsilon(\cdot, t) - n_\varepsilon(\cdot, s) \|_{(W^{2,2}_0(\Omega))^*} \leq C|t - s| \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \quad (4.12) \]

**Proof.** The proof of this lemma can be seen in [24] lemma 3.23.

**Lemma 4.4.** Let \( m \) be given as in theorem 1.1. Then there exists a subsequence \((\varepsilon_j)_{j \in \mathbb{N}}\) such that \( \varepsilon_j \searrow 0 \) as \( j \to \infty \) and that
\[ n_{\varepsilon_j} \to n \quad \text{a.e in } \Omega \times (0, \infty), \quad (4.13) \]
\[ n_{\varepsilon_j} \overset{*}{\rightharpoonup} n \quad \text{in } L^\infty(\Omega \times (0, \infty)), \quad (4.14) \]
\[ n_{\varepsilon_j} \rightharpoonup n \quad \text{in } C^0_{\text{loc}}([0, \infty); (W^{2,2}_0(\Omega))^*), \quad (4.15) \]
\[ \nabla (n_{\varepsilon_j} + \varepsilon_j)^m \rightharpoonup \nabla n^m \quad \text{in } L^2_{\text{loc}}((0, \infty); L^2(\Omega)), \quad (4.16) \]
\[ c_{\varepsilon_j} \rightharpoonup c \quad \text{in } C^0_{\text{loc}}(\Omega \times [0, \infty)), \quad \text{a.e in } \Omega \times (0, \infty), \quad (4.17) \]
\[ \nabla c_{\varepsilon_j} \rightharpoonup \nabla c \quad \text{in } C^0_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.18) \]
\[ \nabla c_{\varepsilon_j} \overset{*}{\rightharpoonup} \nabla c \quad \text{in } L^\infty(\Omega \times (0, \infty)), \quad (4.19) \]
\[ u_{\varepsilon_j} \rightharpoonup u \quad \text{in } C^0_{\text{loc}}(\Omega \times [0, \infty)), \quad (4.20) \]
\[ D u_{\varepsilon_j} \overset{*}{\rightharpoonup} D u \quad \text{in } L^\infty(\Omega \times [0, \infty)), \quad (4.21) \]
with some triple \((n, c, u)\) which is a global weak solution of (1.1) in the sense of Definition 1.1.
Moreover, \( n \) satisfies
\[ n \in C^{0}_{\omega-*}([0, \infty); L^\infty(\Omega)) \quad (4.22) \]
and
\[ \int_{\Omega} n(x, t)dx = \int_{\Omega} n_0(x)dx \quad \text{for all } t > 0. \quad (4.23) \]

**Proof.** The proof of this lemma is based on the Aubin-Lions lemma, Arzelà-Ascoli theorem and the boundedness results obtained in the above lemmas. The proof details of this lemma is almost the same to Lemma 4.2 in [24] after some small modification, therefore it is omitted here.

**Proof of Theorem 1.1.** In this part, we shall prove the limits \((n, c, u)\) mention above is a weak solution of problem (1.1). As usual, testing the first equation in (2.4) by \( \varphi \in C^\infty_0(\Omega \times [0, \infty)) \), we can see that
\[
- \int_0^\infty \int_{\Omega} n_{\varepsilon} \varphi_t dxdt = \int_{\Omega} n_0 \varphi(\cdot, 0) dx - \int_0^\infty \nabla (n_{\varepsilon} + \varepsilon)^m \cdot \nabla \varphi dxdt + \int_0^\infty \int_{\Omega} n_{\varepsilon} u_{\varepsilon} \cdot \nabla \varphi dxdt + \int_0^\infty \int_{\Omega} n_{\varepsilon} S_{\varepsilon}(x, n_{\varepsilon}, c_{\varepsilon}) \nabla c_{\varepsilon} \cdot \nabla \varphi dxdt \quad \text{for all } \varepsilon \in (0, 1). \quad (4.24)
\]
Thus, by (4.13)-(4.17)(4.20) and the definition of matrix-valued function \( S_{\varepsilon} \), the (1.14) can be obtained by passing to the limit in each term of the identity above. Along with a similar procedure
applied to the second and the third equation in the system (2.4), we can also deduce (1.15) and (1.16).

For the boundedness results (1.18) to the weak solution \((n, c, u)\), we can deduce from (3.19)-(3.21) that
\[
\|n\|_{L^\infty((0,T);L^\infty(\Omega))} \leq \liminf_{j \to \infty} \|n_{\varepsilon_j}\|_{L^\infty((0,T);L^\infty(\Omega))} < \infty,
\]
and
\[
\|c\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} \leq \liminf_{j \to \infty} \|c_{\varepsilon_j}\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} < \infty,
\]
as well as
\[
\|cu\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} \leq \liminf_{j \to \infty} \|u_{\varepsilon_j}\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} < \infty
\]
for all \(T < \infty\). This completes the proof the Theorem 1.1. \(\square\)

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