Control theorem and functional equation of Selmer groups over $p$-adic Lie extensions

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Abstract
Let $K_\infty$ be a $p$-adic Lie extension of a number field $K$ which fits into the setting of non-commutative Iwasawa theory formulated by Coates–Fukaya–Kato–Sujatha–Venjakob. For the first main result, we will prove the control theorem of Selmer group associated to a motive, which generalizes previous results by the second author and Greenberg. As an application of this control theorem, we prove the functional equation of the dual Selmer groups, which generalizes previous results by Greenberg, Perrin-Riou and Zábrádi. Especially, we generalize the result of Zábrádi for elliptic curves to general motives. Note that our proof is different from the proof of Zábrádi even in the case of elliptic curves. We also discuss the functional equation for the analytic $p$-adic $L$-functions and check the compatibility with the functional equation of the dual Selmer groups.

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Introduction

Let us fix a prime number $p$ throughout the paper. Let $K$ be a number field and let $K^c$ be a finite extension of $\mathbb{Q}_p$ whose ring of integers is denoted by $\mathcal{O}$. For any $p$-adic Lie group $G$, we define $\Lambda(G) := \varprojlim_U \mathbb{Z}_p[G/U]$ to be the completed group ring where $U$ varies over open normal subgroups of $G$. We denote by $\Lambda_{\mathcal{O}}(G)$ the base extension $\Lambda(G) \otimes_{\mathbb{Z}_p} \mathcal{O}$.

Let $A \cong (K/\mathcal{O})^{ad}$ be a discrete Galois representation of the absolute Galois group $G_K$ of $K$ and a $p$-adic Lie extension $K_\infty/K$. In the first part of the paper, we prove the control theorem of the Selmer group of $A$ over $K_\infty$. Control theorem was originally studied by Mazur [25] for the behavior of the Selmer groups of ordinary elliptic curves when $K_\infty/K$ is the cyclotomic $\mathbb{Z}_p$-extension. The work was generalized in two directions: from elliptic curves to general Galois representations [26] and from the cyclotomic $\mathbb{Z}_p$-extension to general non-commutative $p$-adic Lie-extensions [15]. Here, we study a bi-product of these two different generalizations of [15,26]. To study the most general setting, we will prepare the notation and fix the setting below. Then, we will formulate and prove a control theorem with precise description of the kernel and the cokernel of the restriction map.

First, we introduce error terms $E_{0,U}^A$ and $E_{1,U}^A$ which will appear in the control theorem and in the theorem of functional equation. Let $\mathcal{U}$ be the set of open subgroup of $G = \mathrm{Gal}(K_\infty/K)$. For each $U \in \mathcal{U}$, we denote by $K_U$ the fixed field $(K_\infty)^U$. We will denote by $u$ a prime in $K_U$ and by $v$ a prime in $K$. We denote by $D_{U,u}$ (resp. $I_{U,u}$) the decomposition group (resp. inertia group) of $G_{K_U}$ at $u$.

Let $\Gamma_{cyc}$ be the Galois group of the cyclotomic $\mathbb{Z}_p$-extension $\mathrm{Gal}(K_{cyc}/K)$ of $K$. For any continuous character $\rho : \Gamma_{cyc} \rightarrow \mathbb{Z}_p^\times$, we denote by $A_\rho$ the twist $A \otimes \rho$ of the Galois representation $A$ by $\rho$. Let $P_U$ be the set of primes $u$ of $K_U$ such that the image of $I_{U,u} \subset G_{K_U}$ via $G_{K_U} \twoheadrightarrow U = \mathrm{Gal}(K_\infty/K_U)$ is infinite. Under the assumption that $V = \varprojlim_n A_{p^n} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ satisfies the condition (Pan) which will be stated in Definition 1.3, we define error terms $E_{0,U}^{A_\rho} = \bigoplus_{u \in P_U} E_{U,u}^{A_\rho}$ and $E_{1,U}^{A_\rho} = \bigoplus_{u \in P_U, u \nmid p} E_{U,u}^{A_\rho}$ by putting:
\[ E_{U,u}^{A,\rho} = \begin{cases} \text{Ker} \left[ H^1(I_{U,u}, A_{\rho})^{D_{U,u}} \rightarrow H^1(I_{\infty,w}, A_{\rho})^{D_{\infty,w}} \right] & \text{if } u \nmid p, \\ \text{Ker} \left[ H^1(I_{U,u}, A_{\rho}/F^+_V A_{\rho})^{D_{U,u}} \rightarrow H^1(I_{\infty,w}, A_{\rho}/F^+_V A_{\rho})^{D_{\infty,w}} \right] & \text{if } u \mid p. \end{cases} \] (1)

where \( F^+_V A_{\rho} \) is the filtration induced by the filtration \( F^+_V V_p = F^+_V V \otimes \rho \) given by the condition (Pan) through the surjection \( V_{\rho} \rightarrow A_{\rho} \). When the twisting representation \( \rho \) is trivial, we denote \( E_{U,u}^{A,\rho}, E_{0, U}^{A,\rho} \) and \( E_{1, U}^{A,\rho} \) by \( E_{U,u}^{A}, E_{0, U}^{A} \) and \( E_{1, U}^{A} \) respectively. We further define:

\[ E_0^{A,\rho} := \lim_U E_{0, U}^{A,\rho} = \lim_U \bigoplus_{u \in P_U} E_{U,u}^{A,\rho}, \] (2)

\[ E_1^{A,\rho} := \lim_U E_{1, U}^{A,\rho} = \lim_U \bigoplus_{u \in P_U, u \nmid p} E_{U,u}^{A,\rho}. \] (3)

Under the condition (Pan) on \( V = \lim_{\rightarrow U} A_{\rho} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \), we will define the Selmer group \( \text{Sel}_{A_{\rho}}^{Gr}(K_U) \) for \( A \) over \( K_U \) whose precise definition is recalled in Definition 1.11. We denote the inductive limit \( \lim_{\rightarrow U} \text{Sel}_{A_{\rho}}^{Gr}(K_U) \) by \( \text{Sel}_{A_{\rho}}^{Gr}(K_{\infty}) \). We are interested in the following functorial restriction map:

\[ \text{res}_{A_{\rho},U}^{A,\rho} : \text{Sel}_{A_{\rho}}^{Gr}(K_U) \rightarrow \text{Sel}_{A_{\rho}}^{Gr}(K_{\infty})^{G_K} \] (4)

and we have the following control theorem for these restriction maps:

**Theorem 0.1** (Control Theorem) Let \( K_{\infty} \) be a p-adic Lie Galois extension of \( K \) which fits into the setting (G), (K) of Definition 1.1 (see Sect. 1). Let \( V \cong K^{\text{gend}} \) be a p-adic representation of \( G_K \) satisfying the condition (Pan) of Definition 1.3 (see Sect. 1). We fix a \( G_K \)-stable \( \mathcal{O} \)-lattice \( T \subset V \) and we put \( A := V/T \).

Then, the following statements hold for any continuous character \( \rho : \Gamma_{\text{cyc}} \rightarrow \mathbb{Z}_p^\times \) and for the restriction map \( \text{res}_{A_{\rho},U}^{A,\rho} \):

1. (a) Assume the condition (A) of Definition 1.6 (see Sect. 1) for \( A = V/T \). Then the group \( \text{Ker}(\text{res}_{A_{\rho},U}^{A,\rho}) \) is a finite group whose order is bounded independently of \( U \in \mathcal{U} \).

   (b) Assume the condition (Red) of Definition 1.1. Then the group \( \text{Ker}(\text{res}_{A_{\rho},U}^{A,\rho}) \) is a finite group when \( U \) varies in \( \mathcal{U} \). Further, the inverse limit \( \lim_{\leftarrow U} \text{Ker}(\text{res}_{A_{\rho},U}^{A,\rho}) \) is a finitely generated \( \mathbb{Z}_p \)-module.

2. (a) Assume the condition (A) of Definition 1.6 and the condition (V_q) of Definition 1.3 for each \( q (\neq p) \) dividing \( P_U \). Assume one of the following situations:

   (i) The condition (A_p) of Definition 1.6 holds.

   (ii) The condition (Sol_p) of Definition 1.1 and the conditions (Ord) and (V_p) of Definition 1.3 hold.

Then the group \( \text{Coker}(\text{res}_{A_{\rho},U}^{A,\rho}) \) is a finite group when \( U \) moves in \( \mathcal{U} \). Further, we have a natural map \( E_{0, U}^{A,\rho} \rightarrow \text{Coker}(\text{res}_{A_{\rho},U}^{A,\rho}) \) whose kernel and cokernel are finite groups and their orders are bounded independently of \( U \in \mathcal{U} \).
(b) Assume the condition (Red) of Definition 1.1 and the condition (V_q) of Definition 1.3 for each q (\neq p) dividing P_\U. Assume one of the following situations

(i) The condition (A_p) of Definition 1.6 holds.

(ii) The condition (Sol_p) of Definition 1.1 and the conditions (Ord) and (V_p) of Definition 1.3 hold.

Then the group Coker(res_{\rho,\U}^{A}) is a finite group when U moves in \U. Further, we have a natural map \( E_0^{A,\rho} = \lim U E_0^{A,\rho} \to \lim U Coker(res_{\rho,\U}^{A}) \) whose kernel and cokernel are finitely generated \( \Z_p \)-modules.

The proof of Theorem 0.1 will be given in Sect. 2.

Remark 0.2 In a non-commutative \( p \)-adic Lie extension \( K_\infty/K \), we need to calculate and estimate the growth of the kernel and cokernel of the natural restriction map in the Control Theorem carefully in the proof of the algebraic functional equation because, even if these kernel and cokernel are finite at every intermediate field, they might not be universally bounded. Also, to calculate the contribution of the local terms in the Control Theorem, we need to keep in mind that the primes over \( p \) as well as the primes above \( \ell \neq p \) can be infinitely ramified.

In the latter part of the paper, we prove the algebraic functional equation (the functional equation of the Selmer group of \( A \) over \( K_\infty/K \)) by applying Theorem 1. Let us set \( H := \text{Gal}(K_\infty/K_{\text{cyc}}) \) and let \( \mathfrak{M}_H(G) \) be the category of finitely generated \( \Lambda_\mathcal{O}(G) \)-modules \( M \) such that \( M/M(p) \) are finitely generated over \( \Lambda_\mathcal{O}(H) \) where \( M(p) \) is a union of all \( p^n \)-torsion subgroups \( M[p^n] \) of \( M \). We denote by \( K_0(\mathfrak{M}_H(G)) \) the Grothendieck group of \( \mathfrak{M}_H(G) \). For an object \( M \) in \( \mathfrak{M}_H(G) \), the class of \( M \) in \( K_0(\mathfrak{M}_H(G)) \) is denoted by \( [M] \). Given a \( \Lambda_\mathcal{O}(G) \)-modules \( M, M^i \) is a \( \Lambda_\mathcal{O}(G) \)-module where \( M^i = M \) as an \( \mathcal{O} \)-module but \( G \) acts via the involution \( m.g = g^{-1}m \) for all \( g \in G \).

**Theorem 0.3 (Algebraic Functional Equation)** Let \( K_\infty/K, V, A \) be as in the setting of Control Theorem (Theorem 0.1). Let \( \text{Sel}^{\text{Gr}}_A(K_\infty) \) denote the dual Greenberg Selmer group of \( A \) over \( K_\infty \). Assume that \( \text{Sel}^{\text{Gr}}_A(K_\infty) \) and \( \text{Sel}^{\text{Gr}}_A(K_\infty)^{\vee} \) are in \( \mathfrak{M}_H(G) \) where we set \( A^* = \text{Hom}(A, \mathcal{K}/\mathcal{O}) \otimes_\mathcal{O} \mathcal{K}/\mathcal{O} \). Assume that all of the following conditions are satisfied simultaneously.

1. For every continuous character \( \rho : \Gamma_{\text{cyc}} \to \Z_p^* \), the groups \( \text{Ker}(\text{res}_{\rho,\U}^{A^*}) \) and \( \text{Coker}(\text{res}_{\rho,\U}^{A^*}) \) are finite groups for each \( U \in \U \).
2. Either one of the following two conditions are satisfied.
   
   (a) The order of \( \text{Ker}(\text{res}_{\U}^{A^*}) \) is bounded independently of \( U \in \U \). Further, we have a natural map \( E_0^{A^*} \to \text{Coker}(\text{res}_{\U}^{A^*}) \) whose kernel and cokernel are finite groups and their orders are bounded independently of \( U \in \U \).

   (b) The inverse limit \( \lim U \text{Ker}(\text{res}_{\U}^{A^*}) \) is a finitely generated \( \Z_p \)-module. Further, we have a natural map \( E_0^{A^*} \to \lim U \text{Coker}(\text{res}_{\U}^{A^*}) \) whose kernel and cokernel are finitely generated \( \Z_p \)-modules. Moreover, hypothesis (Van) of Definition 1.1 holds.
(3) Either the hypothesis (a) or the hypothesis (b) of Proposition 4.7 holds.

(4) Either the condition \((A_p)\) of Definition 1.6 or the condition \((\text{Van}_p)\) of Definition 1.1 holds.

Then we have the following equality in \(K_0(\mathcal{M}_H(G))\):

\[
\left[ \text{Sel}^{\mathfrak{Gr}}_{A}(K_{\infty}) \right]^{\vee} + \left[ E^{A*(1)}_1 \right] = \left[ \left( \text{Sel}^{\mathfrak{Gr}}_{A*(1)}(K_{\infty}) \right)^{\vee} \right]^{1}
\]

where \(E^A_1 := \lim_{\rightarrow U} E^A_1, U\) is an exceptional term defined by the Eq. (3).

A quite interesting phenomenon in the non-commutative case observed in the above theorem is that the functional equation is not really symmetric as in the commutative case and it has some error terms, which we call the exceptional divisor. The proof of Theorem 0.3 will be given in Sect. 5.

We denote by \(\text{Sel}^{\mathfrak{BK}}_{A, \rho}(K_U)\) the Bloch–Kato Selmer group for \(A\) over \(K_U\) whose definition is similar to \(\text{Sel}^{\mathfrak{Gr}}_{A, \rho}(K_U)\) replacing the local condition \(H^1_{\mathfrak{Gr}}(K_{U, a}, A)\) of Definition 1.11 by the local condition \(H^1_f(K_{U, a}, A)\) of Bloch–Kato given in [4, Section 3]. We denote the inductive limit \(\lim_{\rightarrow U} \text{Sel}^{\mathfrak{BK}}_{A, \rho}(K_U)\) by \(\text{Sel}^{\mathfrak{BK}}_{A, \rho}(K_{\infty})\). It is known that \(\text{Sel}^{\mathfrak{BK}}_{A, \rho}(K_U)\) (resp. \(\text{Sel}^{\mathfrak{Gr}}_{A, \rho}(K_{\infty})\)) is a subgroup of \(\text{Sel}^{\mathfrak{Gr}}_{A, \rho}(K_U)\) (resp. \(\text{Sel}^{\mathfrak{Gr}}_{A, \rho}(K_{\infty})\)).

**Remark 0.4** (Algebraic Functional equation for Bloch–Kato Selmer group) Let us keep the hypotheses (and the notation) of Theorem 0.3 above. Then \(\text{Sel}^{\mathfrak{BK}}_{A, \rho}(K_{\infty})\) and \(\text{Sel}^{\mathfrak{Gr}}_{A*(1)}(K_{\infty})\) are in \(\mathcal{M}_H(G)\) and we have the following equality in \(K_0(\mathcal{M}_H(G))\):

\[
\left[ \text{Sel}^{\mathfrak{BK}}_{A}(K_{\infty}) \right]^{\vee} + \left[ E^{A*(1)}_1 \right] = \left[ \left( \text{Sel}^{\mathfrak{BK}}_{A*(1)}(K_{\infty}) \right)^{\vee} \right]^{1}
\]

This duality for Bloch–Kato Selmer groups over \(K_{\infty}\) follows from the proof of Theorem 0.3 (see Sect. 5).

We give an example where all the conditions of Theorems 0.1 and 0.3 are satisfied simultaneously. Also see Example 1.10.

**Example 0.5** We consider the false-Tate curve extension \(K_{\infty} = \mathbb{Q}(\mu_{p^{\infty}}, a^{1/p^{\infty}})\) appearing in Example 1.7.

Let \(f\) be a normalized eigen cusiform of weight \(k \geq 2\) whose conductor is prime to \(p\) and assume that \(a_p(f)\) is a \(p\)-adic unit as in Example 1.7. We take a lattice \(T \subset V_f(j)\) for an integer \(j\) satisfying \(1 \leq j \leq k - 1\) and put \(A = T \otimes \mathbb{Q}_p / \mathbb{Z}_p\). Then, from the discussion of Example 1.5(3), Example 1.7(3) and Example 4.10, we have the following:

(1) For any continuous character \(\rho : \Gamma_{\text{cyc}} \to \mathbb{Z}_p^{\times}\), \(\text{Ker}(\text{res}^{A*(1)}_{\rho, U})\) is a finite group whose order is bounded independently of \(U \in \mathcal{U}\) and \(\text{Coker}(\text{res}^{A*(1)}_{\rho, U})\) is a finite group for any \(U\).

Further, the kernel and cokernel of the natural map \(E_0^{A*(1)} \longrightarrow \lim_{\leftarrow U} \text{Coker}(\text{res}^{A*(1)}_{U})\) are finite groups of bounded order, independent of \(U\).
Moreover, if $\text{Sel}^\text{Gr}_A(K_\infty)\vee$ and $\text{Sel}^\text{Gr}_{A^*(1)}(K_\infty)\vee$ are in $\mathcal{M}_H(G)$, then

$$\left[\text{Sel}^\text{Gr}_A(K_\infty)\vee\right] + \left[E^A_{1^*}(1)\right] = \left[\left(\text{Sel}^\text{Gr}_{A^*(1)}(K_\infty)\vee\right)\xi\right] \text{ in } K_0(\mathcal{M}_H(G))$$

where $E^A_{1^*}$ is defined in (2).

Whenever $\text{Sel}^\text{Gr}_A(\mathbb{Q}(\mu_{p^\infty}))\vee$ is a finitely generated $\mathbb{Z}_p$-module, then $\text{Sel}^\text{Gr}_A(K_\infty)\vee$ and $\text{Sel}^\text{Gr}_{A^*(1)}(K_\infty)\vee$ are in $\mathcal{M}_H(G)$.

**Remark 0.6** (Historical comments) Our result is a generalization of [14,29] and that of [3,34,35]. The last three papers establish functional equation of Selmer groups of $p$-ordinary elliptic curves over certain $p$-adic Lie extensions $K_\infty/K$ including false-Tate extensions and $GL_2$-extensions. We allow more general Galois representations and more general $p$-adic Lie extensions compared to the above mentioned articles. There is also a related work of Hsieh [17] on algebraic functional equation over CM fields.

**Remark 0.7** (Technical comments) The method of proof in our work is different from preceding results [3,34,35] even in the case of Selmer groups for elliptic curves which were already studied by them. In their works, functional equations are proved as follows: for each intermediate extension $k$ of $K_\infty/K$ which is finite over $K$, we have the algebraic functional equation over the cyclotomic $\mathbb{Z}_p$-extension $k_{\text{cyc}}$, obtained by Greenberg and Perrin-Riou. Then they take the limit when $k$ varies and obtain the desired functional equation over $K_\infty$.

In our method, in some sense, the argument becomes simpler. For each finite intermediate extension $k$ of $K_\infty/K$, we have a natural duality pairing of Selmer group thanks to the global duality theorems of Galois cohomology and this natural pairing is perfect assuming that these Selmer groups are of finite order [12]. By a technique called a twisting lemma developed in our previous article with Zábrádi [20], we are reduced to the situation where Selmer groups are of finite order at every finite intermediate extension $k$ of $K_\infty/K$. Under this situation of finiteness, we can just take the limit of this pairing with respect to $k$ with help of Control theorem of Selmer group (Theorem 0.1) and we obtain the desired result for Theorem 0.3.

In the philosophy of Iwasawa theory, the characteristic class of the Selmer group is expected to be an algebraic counterpart of the analytic $p$-adic $L$-function whose existence is hypothetical in general. Iwasawa Main Conjecture predicts an equality in $K_0(\mathcal{M}_H(G))$ between the characteristic class of the Selmer group and the class defined by the analytic $p$-adic $L$-function.

Now, let us consider a specific situation, where we study the Selmer group and $p$-adic $L$-function associated to a normalized, $p$-ordinary eigen cuspform $f$ of even weight $k \geq 2$ and level $\Gamma_0(N)$ such that $N$ is square-free and the conductor $N_f$ of $f$ is not divisible by $p$ as discussed in Example 3 of Sect. 3. We take $V = V_{f,p}(k/2)$, the $p$-adic Galois representation for $f$ at the critical value $k/2$ and take a lattice $T$ of $V = V_{f,p}(k/2)$ and we set $A = T \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Now consider a ‘false-Tate curve’ extension $K_\infty = \mathbb{Q}(\mu_\infty, a^{1/p^\infty})$ for some $p$-power free natural number $a$ (see Example 1.7). Let $\mathcal{L}_p(V_{f,p}(k/2))$ be the analytic $p$-adic $L$-function, whose existence is conjectural. At
every Artin representation $\eta$ of $\text{Gal}(K_\infty/\mathbb{Q})$, it makes sense to evaluate $L_p(V_{f,p}(\frac{k}{2}))$ at $\eta$ but the evaluated value $\eta(L_p(V_{f,p}(\frac{k}{2})))$ is only well-defined modulo multiplication by a $p$-adic unit. Similarly, for any $M$ in $\mathcal{M}_H(G)$, $\eta([M])$ makes sense up to a $p$-adic unit (see Sect. 6.)

In “Appendix A”, we discuss a general conjectural framework concerning the conjectural existence and the interpolation property for expected $p$-adic $L$-functions [see (93)] associated to general $p$-ordinary motives. As for the interpolation property satisfied by $L'_p(V_{f,p}(\frac{k}{2}))$, we refer the reader to the conjectural interpolation in (75).

**Theorem 0.8** (Compatibility between the algebraic side and the analytic side) Let us assume the setting above of an ordinary normalized eigen elliptic cuspform $f$ of even weight $k \geq 2$ and square-free level $\Gamma_0(N)$ with a false-Tate extension $K_\infty/\mathbb{Q}$. Assume that a conjectural $p$-adic $L$-function $L_p(V_{f,p}(\frac{k}{2})) \in K_1(\Lambda(\lambda(G),G))$ with the interpolation property (75) exists. Then for any Artin representation $\eta$ of $\text{Gal}(K_\infty/\mathbb{Q})$ we have the equality

$$\frac{\eta\left(\left|\text{Sel}_{A}(K_\infty)^{\vee}\right|\right)}{\eta\left(\left|\text{Sel}_{A}(K_\infty)^{\vee}\right|\right)} = \frac{\eta\left(L_p(V_{f,p}(\frac{k}{2}))\right)}{\eta\left(L_p(V_{f,p}(\frac{k}{2}))\right)}$$

modulo multiplication by a $p$-adic unit.

Theorem 0.8 is restated at Theorem 6.4 and we prove it there. A key point of the proof is to calculate the evaluations of the exceptional divisor at every Artin character $\eta$ of $\text{Gal}(K_\infty/\mathbb{Q})$ in this specific situation. Recall that Iwasawa Main Conjecture mentioned above (see Conjecture A.4 for details) predicts equalities $[\text{Sel}_{A}(K_\infty)^{\vee}] = \delta(L_p(V_{f,p}(\frac{k}{2})))$ and $[\text{Sel}_{A^{(1)}}(K_\infty)^{\vee}] = \delta(L_p(V_{f,p}(\frac{k}{2})))$ in $K_0(\mathcal{M}_H(G))$ where $\delta = \delta_G$ is the standard map $K_1(\Lambda(\lambda(G),G)) \rightarrow K_0(\mathcal{M}_H(G), \Lambda(\lambda(G),G)) = K_0(\mathcal{M}_H(G))$ (see (92)). Thus the equality in (5) holds true if Iwasawa Main Conjecture holds true. A remarkable point of Theorem 0.8 is that we prove (5) without assuming Iwasawa Main Conjecture, thus supporting the validity of Iwasawa Main Conjecture.

**Outline of the paper**

In Sect. 1, we fix the setting of our work. We list up some conditions on $p$-adic Lie extensions $K_\infty/K$ and on its Galois group $\text{Gal}(K_\infty/K)$. We also list up some conditions on Galois representations $V$ with geometric origin with which we define the Selmer group. For these conditions, we give some examples to explain when they are satisfied and when they are not.

In Sect. 2, we prove Control Theorem (Theorem 0.1). We remark that Control theorem in non-commutative case is more complicated than its commutative counterpart. Sometimes it happens that kernels and cokernels are finite but unbounded.

As remarked after Theorem 0.3, the exceptional divisor plays an important role in the statement of the algebraic functional equation. In Sect. 3, we calculate some examples of the exceptional divisor. Using general results obtained in previous section, we determine the exceptional divisor for some non-commutative extensions where $V$
is associated to an elliptic curve or a modular form. We also prove that the exceptional divisor is trivial when the extension $K_\infty/K$ is commutative.

In Sect. 4, we prove the vanishing of some higher extension groups associated to Selmer groups in the category $\mathcal{M}_H(G)$. Main result of the section is Proposition 4.7 and we use some criterion of Ardakov and Wadsley for the triviality of the class of a module of finite cardinality in certain Grothendieck groups.

In Sect. 5, we finally give the proof of the algebraic functional equation (Theorem 0.3) using the vanishing result of the previous section.

In Sect. 6, we first prove the analytic functional equation for the conjectural analytic $p$-adic $L$-function of an ordinary normalized eigen elliptic cuspform $f$ of even weight $k \geq 2$ at the central critical value, over the false-Tate curve extension. Then we prove that the algebraic functional equation is compatible with the analytic functional equation, as predicted by the Iwasawa Main Conjecture. The most important technical point is to calculate of the evaluation of the error terms on the algebraic functional equation, as predicted by the Iwasawa Main Conjecture. The most important technical point is to calculate of the evaluation of the error terms on the algebraic functional equation at every Artin representation of $\text{Gal}(K_\infty/\mathbb{Q})$ and to check that they match with the ‘error terms ’ of the analytic functional equation, given by the Euler factors at certain bad primes.

In “Appendix A”, we first recall some basic conjectures on the complex $L$-function associated to a motive $M$. Then we formulate the conjecture on the existence of analytic $p$-adic $L$-function $L_p(V)$ in a non-commutative situation when $V$ is associated to a $p$-ordinary motive.

**Notation**

Given a left $\Lambda_\mathcal{O}(G)$-modules $M$, $M'$ is a right $\Lambda_\mathcal{O}(G)$-module where $M' = M$ as an $\mathcal{O}$-module but $G$ acts via the involution $m.g = g^{-1}m$ for all $g \in G$. We extend this action of $G$, $\mathcal{O}$-linearly to make $M'$ a right $\Lambda_\mathcal{O}(G)$-module. For a discrete module $A$ which is isomorphic to a direct sum of a finite copies of $\mathcal{O}/\mathcal{O}$, we set $A^* = \text{Hom}(A, \mathcal{O}/\mathcal{O}) \otimes \mathcal{O}/\mathcal{O}$. For a finite dimensional $\mathcal{K}$-vector space $V$, we denote its $\mathcal{K}$-linear dual by $V^*$. For a discrete $\mathcal{O}$-module $M$, we denote the Pontryagin dual $\text{Hom}_{\text{cont}}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ by $M^\vee$.

Unless otherwise specified, all modules over $\Lambda_\mathcal{O}(G)$ are considered as left modules. However, given a left $\Lambda_\mathcal{O}(G)$-module $M$, we view $\text{Ext}^i_{\Lambda_\mathcal{O}(G)}(M, \Lambda_\mathcal{O}(G))$ as a right $\Lambda_\mathcal{O}(G)$-module and $\text{Ext}^i_{\Lambda_\mathcal{O}(G)}(M', \Lambda_\mathcal{O}(G))$ again as a left $\Lambda_\mathcal{O}(G)$-module.

Note that, there are two different categories for $\mathcal{M}_H(G)$; that of left $\Lambda_\mathcal{O}(G)$-modules and that of right $\Lambda_\mathcal{O}(G)$-modules. By abuse of notation, we denote both to them by the same notation $\mathcal{M}_H(G)$. The abelian group $K_0(\mathcal{M}_H(G))$ for the category $\mathcal{M}_H(G)$ for left $\Lambda_\mathcal{O}(G)$-modules and the category $\mathcal{M}_H(G)$ for right $\Lambda_\mathcal{O}(G)$-modules are canonically identified. This last fact is verified by the isomorphism $K_0(\mathcal{M}_H(G)) \cong K_1(\Lambda_\mathcal{O}(G)_{S_\mathcal{O}})/\text{Image}(K_1(\Lambda_\mathcal{O}(G)))$ (see (92) for details) and this is implicitly used in Theorem 0.3 and Proposition 5.7.

1 Setting

We fix the setting of our global Galois extension. We denote by $\chi_{\text{cyc}} : G_\mathbb{Q} \longrightarrow \mathbb{Z}_p^\times$, the $p$-adic cyclotomic character.
Definition 1.1 We consider the following conditions for a compact $p$-adic Lie group $G$ and a Galois extension $K_\infty/K$ of $K$:

(K) Only finitely many primes of $K$ are ramified in the extension $K_\infty$ of $K$ and $\text{Gal}(K_\infty/K)$ is isomorphic to $G$. The group $G$ does not have any elements of order $p$.

(G) The cyclotomic $\mathbb{Z}_p$-extension $K_{\text{cyc}}$ of $K$ is contained in $K_\infty$ so that $H := \text{Gal}(K_\infty/K_{\text{cyc}})$ is a closed subgroup of $G$ and $G/H$ is isomorphic to $\Gamma_{\text{cyc}} = \text{Gal}(K_{\text{cyc}}/K)$.

(RED) The Lie algebra $\text{Lie}(G)$ attached to the $p$-adic Lie group $G$ by Lazard is reductive.

(VAN) Let $G$ and $H$ be as in the condition (G) above. Then $[\mathbb{Z}_p] = 0$ in $K_0(\mathcal{M}_H(G))$.

Under the condition (K), let $\mathcal{U}$ be the set of all open subgroups of $G$ and we denote by $K_U$ the fixed subfield of $K_\infty$ by $U \in \mathcal{U}$. For every finite prime $v$ of $K$, we denote the decomposition group of $G$ at $v$ by $G_v$, We will denote by $u$ a prime in $K_U$ lying above $v$.

(VAN$_p$) Let $G$ and $H$ be as in the condition (G) above. For any prime $v$ in $K$ dividing $p$, set $H_v := H \cap G_v$. Then $[\mathbb{Z}_p] = 0$ in $K_0(\mathcal{M}_{H_v}(G_v))$.

Remark 1.2 An example of infinite extension $K_\infty/K$ whose Galois group is isomorphic to a compact $p$-adic Lie group is often obtained as a trivializing extension of $p$-adic representation of $G_K$ which has geometric origin. For such an extension $K_\infty/K$, the statement of the condition (K) saying that only finitely many prime ramifies, is known by proper-smooth base change theorem of etale cohomology. The statement of the condition (K) saying that $\text{Gal}(K_\infty/K)$ is a $p$-adic Lie group, is known by the fact that a closed subgroup of $GL_d(\mathbb{Z}_p)$ is a compact $p$-adic Lie group. In many arithmetic situations (see Examples 1.7, 1.8 and 1.9) the condition that $\text{Gal}(K_\infty/K)$ has no element of order $p$ is also satisfied.

There are some examples of $K_\infty/K$ which do not have geometric origin. For example, a false-Tate extension $K(\mu_{p^\infty}, a^{1/p^\infty})$ seems to be not obtained as the trivializing extension of a $p$-adic Galois representation of geometric origin. But, it is also easy to see that false-Tate curve extensions satisfy the condition (K).

We then introduce some conditions for the setting of our $p$-adic Galois representation of the absolute Galois group $G_K$ of $K$.

Definition 1.3 We consider the following conditions for a $p$-adic representation $V \cong K^{\mathbb{Q}_d}$ on which we have a continuous and $\mathbb{K}$-linear action of the absolute Galois group $G_K$:

(Geo) The representation $V$ is geometric in the sense that is is obtained by $p$-adic realization of certain pure motive of homogeneous weight over $K$.

(Pan) For any prime $v$ of $K$ over $p$, $V$ restricted to $G_{K_v}$ is a Hodge-Tate representation which has a $G_{K_v}$-stable $\mathbb{K}$-subspace $F_v^+ V \subset V$ such that the following isomorphisms hold:

$$F_v^+ V \otimes_{\mathbb{K}} \mathbb{C}_p \cong \bigoplus_{n \geq 1} \mathbb{C}_p(n)^{\oplus r_n},$$

$$(V/F_v^+ V) \otimes_{\mathbb{K}} \mathbb{C}_p \cong \bigoplus_{n \leq 0} \mathbb{C}_p(n)^{\oplus r_n},$$
where \( r_n \) are non-negative integers almost all of which are zero.

(Ord) For every prime \( v \) of \( K \) dividing \( p \), the action \( \rho|_{G_{K_v}} \) is quasi-ordinary in the sense of [27, Definition 3.1]. That is, we have a \( G_{K_v} \)-stable decreasing filtration \( \text{Fil}_k V \subset V \) such that the action of \( G_{K_v} \) on \( (\text{Fil}_k V/\text{Fil}_{k+1} V) \otimes \chi_{\text{cyc}}^{-1} \) is trivial on an open subgroup of the inertia subgroup \( I_v \).

(V\(_p\)) Assume the condition (Ord). For every prime \( v \) of \( K \) dividing \( p \), the generalized eigenvalues of the Frobenius element in \( G_{K_v}/I_v \) acting on \( ((\text{Fil}_k V/\text{Fil}_{k+1} V) \otimes \chi_{\text{cyc}}^{-1})_v \) are not roots of unity.

(V\(_q\)) For every prime \( v \) of \( K \) dividing a prime number \( q \neq p \), and for any open subgroup \( W_v \subset G_{K_v} \), we have \( V^{W_v} = V^*(1)^{W_v} = 0 \).

**Remark 1.4** The condition (Ord) for \( V \) implies the condition (Pan) for \( V \) by setting \( F^+_v := \text{Fil}_1^k \).

Under the assumption that the extension extension \( K_\infty/K \) satisfies the condition (K) and the \( p \)-adic representation \( V \) satisfies the condition (Ord), we consider the following condition:

(Sol\(_p\)) (a) For every prime \( v \) of \( K \) dividing \( p \) and for every open normal subgroup \( U \) of \( G, U_v := U \cap G_v \) is a solvable \( p \)-adic Lie group. That is, \( G_v \) is expressed as a successive extension of \( p \)-adic Lie group of dimension one.

(b) Moreover, for each open subgroup \( U \) of \( G \), for each prime \( u \) of \( K_U = (K_\infty)^U \) dividing \( p \) and for each \( i > 0 \) and integer \( j \), we have

\[
\text{Hom}(I^{(i)}_{U,u}/I^{(i+1)}_{U,u}, \text{Fil}_v^j V/\text{Fil}_v^{j+1} V)^{D_{U,u}} = 0 \tag{6}
\]

where \( D_{U,u} \) and \( I^{(i)}_{U,u} \) are respectively the decomposition subgroup and the inertia subgroup of \( G_{K_v} \) and \( D_{U,u} \) acts on \( I^{(i)}_{U,u}/I^{(i+1)}_{U,u} \) by conjugation.

In the example below, let us discuss the validity of the conditions for some Galois representations \( V \) listed in Definition 1.3.

**Example 1.5** (1) Let \( B \) be an abelian variety of dimension \( d \) over \( K \) and we define a \( p \)-adic Galois representation \( V \cong \mathbb{Q}_p^{2d} \) to be \( V = T_p B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) where \( T_p B \) is the \( p \)-Tate module of \( B \). By definition, \( V \) satisfies the condition (Geo). The condition (Pan) is satisfied if \( B \) has semistable reduction and the abelian variety part of the special fiber \( B_v \) of the Néron model is ordinary for any prime \( v \) over \( p \). The condition (V\(_p\)) is satisfied if and only if \( B \) satisfies the condition (Pan) and \( B \) has good reduction at every prime \( v \) of \( K \) dividing \( p \). The condition (V\(_q\)) for \( q \neq p \) holds always true for this example.

(2) Let \( f \) be a normalized eigen elliptic cuspform of weight \( k \geq 2 \) and level \( \Gamma_1(N) \) for a natural number \( N \) divisible by \( p \). Let \( K \) be a finite extension \( \mathbb{Q}_p((a_n(f))) \) of \( \mathbb{Q}_p \) obtained by adjoining Fourier coefficients of \( f \). By Deligne and Shimura, we have a continuous \( p \)-adic Galois representation \( V_f \cong \mathbb{K}^{\oplus 2} \) of \( G_{\mathbb{Q}} \) which is unramified outside \( N_\infty \) on which the trace of the action of \( \text{Frob}_q \) for \( q \nmid N \) is equal to \( a_q(f) \). We take \( V \) to be an appropriate Tate twist \( V_f(j) \) of \( V_f \). By the work of Scholl, \( V \) satisfies the condition (Geo). The condition (Pan) is always
satisfied for \( j \leq 0 \) and for \( j \geq k \). For \( 1 \leq j \leq k - 1 \), the condition (Pan) is satisfied if \( a_p(f) \in \mathcal{K} \) is a \( p \)-adic unit (see \([33, \text{Thm} \ 2.1.4]\)). The condition \((V_p)\) is satisfied if \( V \) satisfies the condition (Pan) and the local automorphic representation \( \pi_{f,p} \) associated to \( f \) at \( p \) is an unramified principal series representation. The condition \((V_q)\), for \( q \neq p \), holds always true for \( 1 \leq j \leq k - 1 \).

(3) Let \( B \) be an elliptic curve over \( \mathbb{Q} \) with good ordinary reduction at a prime \( p \). For an odd integer \( d \geq 1 \), let \( V_d \) be the \( d \)-th symmetric power of \( V_pB \). The critical Tate twists \( V_d(j) \) are listed in \([26, \text{Lemma} \ 3.3]\), for every \( d \in \mathbb{N} \). In particular, when \( d \) is odd, then \( V_d\left(-\frac{d+1}{2}\right) \) is the only critical twist. Accordingly, we set \( V \) to be the critical Tate twist \( V_d\left(-\frac{d+1}{2}\right) \), where \( d \) is odd. As explained in the proof of \([26, \text{Prop} \ 3.4]\), the representation \( V \) satisfies (Ord) and hence the (Pan) condition. Let \( q \) be an integer prime to \( p \). The elliptic curve \( B \) has either potentially good reduction or potentially multiplicative reduction at \( q \). When \( B \) has potentially good reduction at \( q \), the eigenvalues of the action of Frobenius element at a prime \( v|q \) on \( V \) has complex absolute value \( \neq 1 \). Hence we have \( V^{G_{\mathbb{Q}q}} = 0 \) for \( V = V_d(j) \). When \( B \) has potentially multiplicative reduction at \( q \), there is a prime \( v|q \) such that \( V^I_v \) is of dimension one. The eigenvalues of the action of Frobenius element at \( v \) on \( V^I_v \) has complex absolute value \( \neq 1 \). Hence we also have \( V^{G_{\mathbb{Q}q}} = 0 \) for \( V = V_d(j) \). Thus the condition \((V_q)\) holds for \( V = V_d(j) \).

We introduce some conditions concerning both the setting of our \( p \)-adic Galois representation \( V \) and the setting of our \( p \)-adic Lie extension \( K_{\infty} \). When \( V \) satisfies (Pan), we define \( F_v^+A \) and \( A/F_v^+A \) to be the image of \( F_v^+V \) and \( V/F_v^+V \), respectively in \( A = V/T \).

**Definition 1.6** Let \( K_{\infty} \) a fixed \( p \)-adic Lie extension of \( K \). We consider the following conditions for a \( p \)-adic representation \( V \cong \mathcal{K}^{\otimes d} \) on which we have a continuous and \( K \)-linear action of the absolute Galois group \( G_K \):

(A) The groups \( A^{G_{K_{\infty}}} \) and \( (A^*)^{G_{K_{\infty}}} \) are finite.

(A\_p) Assume the condition (Pan) of Definition 1.1. For every prime \( v \) of \( K \) dividing \( p \) and for every prime \( w \) of \( K_{\infty} \) over \( v \), the groups \( A^{G_{K_{\infty}, w}} \), \((A^*)^{G_{K_{\infty}, w}} \), \((A/F_v^+A)^{G_{K_{\infty}, w}} \), \((A^*/F_v^+A^*)^{G_{K_{\infty}, w}} \) are finite. For every prime \( v \) of \( K \) dividing \( p \), we have \( H^1_f(K_v, A) = H^1_g(K_v, A) \) where \( H^1_f \) and \( H^1_g \) is the local condition defined in \([4, \S 3]\).

In the example below, let us discuss the validity of the conditions for some Galois representations \( V \) listed in Definition 1.3 restricting ourselves to a case where \( G = \text{Gal}(K_{\infty}/K) \) is a semidirect product \( \mathbb{Z}_p \rtimes \mathbb{Z}_p \).

**Example 1.7** Let us consider the case \( K = \mathbb{Q} \) and \( K_{\infty} = \bigcup_{n \geq 1} K(\mu_{p^n}, \sqrt[p^n]{a}) \) where \( a \) is a \( p \)-power free integer which is prime to \( p \). Any prime \( q \) of \( \mathbb{Q} \) dividing \( a \) is finitely decomposed in \( K_{\infty}/\mathbb{Q} \) and the decomposition group \( G_q \) of \( G \) is open in \( G \). (see \([24, \text{Lemma} \ 3.9]\).) The extension \( \text{Gal}(K_{\infty}/K) \) satisfies (K), (G) of Definition 1.1 and (Sol\_p\_a) of Definition 1.3, but does not satisfy (Red) of Definition 1.1.

(1) Let \( B \) be an elliptic curve over \( \mathbb{Q} \) which is good, ordinary at \( p \). We set \( T = T_pB \), \( V = T \otimes \mathbb{Q}_p \), \( A = T \otimes \mathbb{Q}_p/\mathbb{Z}_p \). Then \( G_{\mathbb{Q}_p} \) acts on \( A/F_v^+A \) by a certain
unramified character \( \alpha \) of infinite order. The group \((A/F_v^+ A)^{G_{K_{\infty,w}}}\) is finite for every prime \( w \) of \( K_{\infty} \) dividing \( p \) since the residue field of \( K_{\infty,w} \) is finite. Also, \( G_{\mathbb{Q}_p} \) acts on \( F_v^+ A \) by \( \chi_{\text{cyc}} \alpha^{-1} \). Since \( K_{\infty,w} \) contains \( \mathbb{Q}(\mu_p) \) for every prime \( w \) of \( K_{\infty} \), \((F_v^+ A)^{G_{K_{\infty,w}}}\) is finite by the same reason as \((A/F_v^+ A)^{G_{K_{\infty,w}}}\). Hence we have \( \# A^{G_{K_{\infty,w}}} < \infty \). We thus verified the first statement of the condition \((A_p)\) of Definition 1.6. We can also verify that the second statement of the condition \((A_p)\) holds true by using results on the comparisons of \( H_1^f(\mathbb{Q}_p, A) \) and \( H_1^f(\mathbb{Q}_p, A)^+ \) in [4, §3]. Since we have \( A^{G_{K_{\infty}}} \subset A^{G_{K_{\infty,w}}} \), we conclude \( \# A^{G_{K_{\infty}}} < \infty \), which implies the condition \((A)\) of Definition 1.6.

(2) Let \( f \) be a normalized eigen elliptic cuspform of weight \( k \geq 2 \) and level \( \Gamma_1(N) \) for a natural number \( N \) divisible by \( p \) such that \( a_p(f) \) is a \( p \)-adic unit. The Galois representation \( V_f \) associated to \( f \) satisfies the condition (Pan). If we take a lattice \( T \subset V_f \) and put \( A = T \otimes \mathbb{Q}_p/\mathbb{Z}_p \), the action of \( \hat{G}_{K_v} \) on \( F_v^+ A \) and \( A/F_v^+ A \) is explicitly known by [33, Thm 2.1.4]. By a similar argument as in (1), we can verify the the condition \((A_p)\) as well as the condition \((A)\) of Definition 1.6 if we assume that the conductor of \( f \) is not divisible by \( p \).

(3) Let us again consider the example \( V_d \) with \( d \) odd, positive as in the setting of Example 1.5(3) and \( V = V_d(f) \) be a Tate twist. We take a Galois stable lattice \( T \subset V \) and put \( A = T \otimes \mathbb{Q}_p/\mathbb{Z}_p \), the action of \( \hat{G}_{K_v} \) on \( F_v^+ A \) and \( A/F_v^+ A \) is explicitly known. Since the residue field of \( K_{\infty,w} \) is finite, by a similar argument as in this Example 1.7(1), we verify that \( A^{G_{K_{\infty,w}}} \) and \( (A/F_v^+ A)^{G_{K_{\infty,w}}} \) are both finite for \( w | p \) and \( H_1^f(\mathbb{Q}_p, A) = H_1^f(\mathbb{Q}_p, A) \). Thus \((A_p)\) and the condition \((A)\) of Definition 1.6 both hold.

Now, for either one of (1), (2) or (3) above, it is important to twist by a continuous character \( \rho : \Gamma_{\text{cyc}} \rightarrow \mathbb{Z}_p^\times \) as given in Introduction. We note that we have \((A/F_v^+ A)^{G_{K_{\infty,w}}} \cong (A/F_v^+ A)^{G_{K_{\infty,w}}} \otimes \rho \) and \( A^{G_{K_{\infty}}} \cong (A^{G_{K_{\infty}}}) \otimes \rho \) thanks to the condition \((G)\) of Definition 1.1. Hence, for any continuous character \( \rho : \Gamma_{\text{cyc}} \rightarrow \mathbb{Z}_p^\times \), the conditions \((A_p)\) and \((A)\) of Definition 1.6 also hold true for \( A_p \) in all of (1), (2) and (3) above.

**Example 1.8** Let \( K \) be an arbitrary number field and \( K_{\infty} = \bigcup_{n \geq 1} K(E_{p^n}) \) where \( E \) is a non-CM elliptic curve over \( K \). The group \( G = \text{Gal}(K_{\infty}/K) \) is known to be an open subgroup of \( GL_2(\mathbb{Z}_p) \) and the extension \( K_{\infty}/K \) always satisfies \((G)\) of Definition 1.1 and also satisfies \((K)\) of Definition 1.1 for \( p \geq 5 \). We have different situations according to the reduction of \( E \) at \( p \).

(S) Suppose that \( E \) has good supersingular reduction at a prime \( v \) over \( p \). Then decomposition group \( G_v \) is open subgroup of \( G \). Hence the extension \( K_{\infty}/K \) satisfies \((\text{Red})\). Let us take a Galois representation \( A \) of \( \hat{G}_K \) as in either Example 1.7(1) or Example 1.7(2). In this case, it is known that the residue field at the primes of \( K \) over \( p \) does not extend in \( K_{\infty}/K \). Hence we verify the conditions \((A_p)\) and \((A)\) of Definition 1.6 by the argument as in the previous example. Moreover, for any continuous character \( \rho : \Gamma_{\text{cyc}} \rightarrow \mathbb{Z}_p^\times \), the conditions \((A_p)\) and \((A)\) also hold true for \( A_p \) in both cases of Example 1.7 above.

(O) Suppose that \( E \) has ordinary reduction at a prime \( v \) over \( p \). The decomposition group \( G_v \) of \( G \) is contained in a Borel subgroup of \( G \). Hence the extension
$K_\infty/K$ satisfies $(\text{Sol}_p)(a)$ of Definition 1.3. Let us take a Galois representation $A = B_p \cong G_K$ for another elliptic curve $B$ over $\mathbb{Q}$ as in Example 1.7 (1). In this case, the condition $(A_p)$ of Definition 1.6 does not hold. The condition $(A)$ of Definition 1.6 is satisfied if $B$ is non-CM and not isogenous to $E$.

Let $v$ be a prime over $p$ of $K$. In this case, the action of $\hat{G}_{\mathbb{Q}_p}$ on graded pieces on $V = T_pB \otimes \mathbb{Q}_p$ are given by characters $\eta, \eta'$ such that $\eta'$ is unramified and $\eta \eta'$ is equal to the $p$-adic cyclotomic character. For every prime $w$ over $p$ of $K_\infty$, we have successive extensions $K_w = K(0) \subset K(1) \subset K(2) \subset K_\infty, w$, where $K(i)$ is the field corresponding to a successive filtration $D(0) \supset D(1) \supset D(2) \supset D(3)$ of $G_v$ which exists according to $(\text{Sol}_p)(a)$. We can take $K(i)$ so that $K(i)/K(0)$ is an unramified isogenous $\mathbb{Z}_p$-extension, $K(2)/K(1)$ is the local cyclotomic $\mathbb{Z}_p$-extension. In this case, $I(0)/I(1)$ is finite and the conjugate action of the decomposition group $G_{K_v}$ on $I(1)/I(2)$ is trivial since $K(2)$ is defined on $K_v$. The conjugate action of the decomposition group $G_{K_v}$ on $I(2)/I(3)$ is given by $\eta(\eta')^{-1}$.

Now, if we take $V = T_pB \otimes \mathbb{Q}_p$ with $B$ a non CM elliptic curve which is isogenous to $E$, the action of $G_{K_v}$ on $V_{\text{Fil}}^0 V/\text{Fil}^1 V$ and $\text{Fil}^1 V/\text{Fil}^2 V$ are given by characters $\eta$ and $\eta'$ respectively. Now, since the value of the unramified character $\eta'$ is not a root of unity, we have

$$\text{Hom}(I_{U,v}^{(i)}/I_{U,v}^{(i+1)}, \text{Fil}^1_v V/\text{Fil}^1_v V)^{D_{U,v}} = 0$$

for every $i = 0, 1, 2$ and $j = 0, 1$, which verifies $(\text{Sol}_p)(b)$ of Definition 1.3.

**Example 1.9** Let $F$ be a totally real field of degree $d$ over $\mathbb{Q}$ and $K$ be a CM field which is a quadratic extension of $F$. Let us consider a Galois extension $K_\infty$ of $K$ with $G = \text{Gal}(K_\infty/K) \cong \mathbb{Z}_p^{d+1}$. The group $G$ is abelian and satisfies $(K), (G), (\text{Red})$ in Definition 1.1. Let $E$ be an elliptic curve defined over $\mathbb{Q}$ with good ordinary reduction at all primes above $p$ in $K$. Then the condition $(V_p)$ of Definition 1.3 is satisfied.

If the condition $(A_p)$ (respectively $(A)$) of Definition 1.6 holds true for $A = B_p \cong G_K$, then for any continuous character $\rho : \Gamma_{\text{cyc}} \rightarrow \mathbb{Z}_p^\times$, the condition $(A_p)$ (resp. $(A)$) also holds true for $A_p$. Moreover $(V_p)$ of Definition 1.3 holds for $A_p$ as well.

Though the condition $(A_p)$ for $A = B_p \cong G_K$ of Definition 1.6 was always satisfied for the cyclotomic $\mathbb{Z}_p$ extension of $K$, the condition $(A_p)$ for $A = B_p \cong G_K$ may not be satisfied for $\mathbb{Z}_p^{d+1}$-extension $K_\infty$ of $K$. However the condition $(\text{Sol}_p)$ in Definition 1.3 is satisfied in our situation. To check $(\text{Sol}_p)$, we note that the action of the decomposition subgroup of $K_v$ on successive quotients of inertia subgroups by conjugation is trivial for a prime $v$ in $K$ above $p$ in this case. Hence we have

$$\text{Hom}(I_{U,v}^{(i)}/I_{U,v}^{(i+1)}, V/\text{Fil}^1_v V)^{D_{U,v}} = \text{Hom}(I_{U,v}^{(i)}/I_{U,v}^{(i+1)}, \text{Fil}^1_v V)^{D_{U,v}} = 0$$

that for $V = T_pB \otimes \mathbb{Q}_p$ and for every $i$ and for every $U$ by a similar argument as in Example 1.8.

We end examples on this section by explicitly citing another example where Theorem 1 and Theorem 2 hold.
Example 1.10 We consider the false-Tate curve extension $K_\infty = \mathbb{Q}(\mu_{p^\infty}, a^{1/p^\infty})$ appearing in Example 1.7. Let us again consider the example of the $d$-th symmetric power $V_d$ with $d$ odd positive as in the setting of Example 1.5(3) and let $V$ be the unique critical twist $V_d(j)$ where $j = \frac{-d+1}{2}$. Set and $A = V/T$ as in the setting of Examples 1.5(3) and 1.7(3). Then, by discussions in Examples 1.5(3), 1.7(3) and 4.10, we deduce

1. For any continuous character $\rho : \Gamma_{cyc} \to \mathbb{Z}_p^\times$, $\text{Ker}(\text{res}_{\rho, U}^{A^*(1)})$ is a finite group whose order is bounded independently of $U \in \mathcal{U}$. $\text{Coker}(\text{res}_{\rho, U}^{A^*(1)})$ is a finite group for any $U$.

Further, the kernel and cokernel of the natural map $E_0^{A^*(1)} \longrightarrow \lim_{\leftarrow U} \text{Coker}(\text{res}_{U}^{A^*(1)})$ are finite groups of bounded order, independent of $U$.

2. Moreover, if $\text{Sel}^\text{Gr}_A(K_\infty)^\vee$ is in $\mathcal{M}_H(G)$, then

$$\left[\text{Sel}^\text{Gr}_A(K_\infty)^\vee\right] + \left[E_1^{A^*(1)}\right] = \left[\left(\text{Sel}^\text{Gr}_A(K_\infty)^\vee\right)^{1}\right] \text{ in } K_0(\mathcal{M}_H(G))$$

where $E_1^A$ is defined in (2).

3. Whenever $\text{Sel}^\text{Gr}_A(\mathbb{Q}(\mu_{p^\infty}))^\vee$ is a finitely generated $\mathbb{Z}_p$-module, then $\text{Sel}^\text{Gr}_A(K_\infty)^\vee$ is in $\mathcal{M}_H(G)$.

Now, we define a Selmer group. Let us fix a $G_K$-stable $\mathcal{O}$-lattice $T$ of $V$ and denote $V/T$ by $A$.

Definition 1.11 Let $K_\infty$ be a $p$-adic Lie Galois extension of $K$ which fits into the setting (G) and (K) of Definition 1.1. Let $V \cong K^{\otimes d}$ be a $p$-adic representation of $G_K$ satisfying the conditions (Geo) and (Pan) of Definition 1.3. We fix a $G_K$-stable $\mathcal{O}$-lattice $T \subset V$ and we put $A := V/T$. Let $\Sigma$ be a finite set of primes of $K$ which contain all primes of $K$ dividing $p^\infty$, all primes of $K$ where the representation $V$ is ramified and all primes of $K$ where the infinite extension $K_\infty/K$ is ramified. For every $U \in \mathcal{U}$, we denote by $\Sigma_U$ (respectively $\Sigma^{(\infty)}_U$) the set of primes of $K_U$ which are above primes of $K$ in $\Sigma$ (respectively the set of infinite primes of $K_U$).

1. For prime $u \in \Sigma_U \setminus \Sigma^{(\infty)}_U$, we define the local condition $H^1_{\text{Gr}}(K_{U,u}, A)$ as follows.

If $u \in \Sigma_U \setminus \Sigma^{(\infty)}_U$ does not divide $p$, we define $H^1_{\text{Gr}}(K_{U,u}, A)$ to be:

$$H^1_{\text{Gr}}(K_{U,u}, A) = \text{Ker}\left[H^1(K_{U,u}, A) \longrightarrow H^1(K_{U,u}^{ur}, A)^{\text{Gal}(K_{U,u}^{ur}/K_{U,u})}\right].$$

If $u \in \Sigma^{(p)}_U$, we define $H^1_{\text{Gr}}(K_{U,u}, A)$ to be:

$$H^1_{\text{Gr}}(K_{U,u}, A) = \text{Ker}\left[H^1(K_{U,u}, A) \longrightarrow H^1(K_{U,u}^{ur}, A/F_v^+)^{\text{Gal}(K_{U,u}^{ur}/K_{U,u})}\right].$$
(2) We define the Greenberg’s Selmer group \( \text{Sel}_{A}^{Gr}(K_U) \) as follows.

\[
\text{Sel}_{A}^{Gr}(K_U) = \ker \left[ H^1(K_{\Sigma}/K_U, A) \longrightarrow \prod_{u \in \Sigma_U \setminus \Sigma_U^{(\infty)}} \frac{H^1(K_{U,u}, A)}{H^1_{Gr}(K_{U,u}, A)} \right].
\]

**2 Proof of the control theorem**

In this section, we prove the control theorem of Selmer group in a non-commutative \( p \)-adic Lie extension (Theorem 0.1).

**Proof of Theorem 0.1** To ease the burden of notation, throughout the proof of this theorem, we will write \( \text{res}_{\rho,U} := \text{res}_{\rho,U}^A \). Let \( v \) be a prime of \( K \) in \( \Sigma \) and \( u \) be a prime of \( K_U \) in \( \Sigma_U \) such that \( u \mid v \). Also \( w \) will denote a prime in \( K_{\infty} \) such that \( w \mid u \mid v \). We denote a decomposition subgroup of \( G \) at \( v \) by \( G_v \subset G \). For each \( v \in \Sigma \) and for each \( U \in \mathcal{U} \), we denote \( U \cap G_v \) by \( U_{v} \). [General strategy for both the assertion (1) and the assertion (2)]

We have the following commutative diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Sel}_{A}^{Gr}(K_U) \\
& & \downarrow \text{res}_{\rho,U} \\
0 & \longrightarrow & H^1(K_{\Sigma}/K_U, A_{\rho}) \\
& & \downarrow \text{log}_{\rho,\infty} \\
& & \prod_{u \in \Sigma_U} \frac{H^1(K_{U,u}, A_{\rho})}{H^1_{Gr}(K_{U,u}, A_{\rho})} \\
& & \downarrow \text{res}_{\rho,U}' \\
& & \prod_{w \in \Sigma_{\infty}} \frac{H^1(K_{\infty,w}, A_{\rho})}{H^1_{Gr}(K_{\infty,w}, A_{\rho})}^{G_K}.
\end{array}
\]

(7)

By the snake lemma applied to the above diagram, the group \( \ker(\text{res}_{\rho,U}) \) is a subgroup of \( \ker(\text{res}_{\rho,U}') \). For each \( U \in \mathcal{U} \), if the group \( \ker(\text{res}_{\rho,U}') \) is a finite group, \( \ker(\text{res}_{\rho,U}) \) is a finite group whose order is bounded by \#\( \ker(\text{res}_{\rho,U}') \). Thus, when the inverse limit \( \lim_{U} \ker(\text{res}_{\rho,U}') \) is a finitely generated \( \mathbb{Z}_p \)-module, so is the group \( \lim_{U} \ker(\text{res}_{\rho,U}) \). When the order of the group \( \ker(\text{res}_{\rho,U}') \) is bounded independently of \( U \in \mathcal{U} \), so is the order of \( \ker(\text{res}_{\rho,U}) \).

Also, by the snake lemma applied to the above diagram, the group \( \text{Coker}(\text{res}_{\rho,U}) \) is an extension of \( \text{Coker}(\text{res}_{\rho,U}') \) by a subquotient of \( \text{Ker}(\text{res}_{\rho,U}) \). Under the assumption that \( \text{Sel}_{A}^{Gr}(K_{\infty}) \) is a finitely generated torsion module over \( \mathcal{O}[[\text{Gal}(K_{\infty}/K)]] \), the map \( \lim_{U} (\text{log}_{\rho,U}) \) is known to be surjective. We omit the proof of this surjectivity, which follows from the global duality theorem of Galois cohomology and some standard arguments and we refer to Lemma 4.11 and Corollary 4.12 of [28] for the argument of this technique for Greenberg type Selmer group. Hence, in order to have the desired conclusion on \( \text{Coker}(\text{res}_{\rho,U}) \), it suffices to study \( \text{Coker}(\text{res}_{\rho,U}') \) and \( \text{Ker}(\text{res}_{\rho,U}') \).
Thus, from now on, we concentrate on the study of $\text{Ker}(\text{res}'_{\rho, U})$, $\text{Coker}(\text{res}'_{\rho, U})$ and $\text{Ker}(\text{res}''_{\rho, U})$. It is sufficient to show the following statements to complete the proof of the theorem.

(α) For any $U \in \mathcal{U}$, $\text{Ker}(\text{res}'_{\rho, U})$ is a finite group whose order is uniformly bounded when $U$ varies in $\mathcal{U}$ in the case (A) in Definition 1.6 and $\text{Ker}(\text{res}'_{\rho, U})$ is a finite group when $U$ moves in $\mathcal{U}$ in the case (Red) in Definition 1.1. Moreover, in the case (Red), $\varprojlim_U \text{Ker}(\text{res}'_{\rho, U})$ is a finitely generated $\mathbb{Z}_p$-module.

(β) For any $U \in \mathcal{U}$, $\text{Coker}(\text{res}'_{\rho, U})$ is a finite group whose order is uniformly bounded when $U$ varies in $\mathcal{U}$ in the case (A) and $\text{Coker}(\text{res}'_{\rho, U})$ is a finite group when $U$ moves in $\mathcal{U}$ in the case (Red). Moreover, in the case (Red), $\varprojlim_U \text{Coker}(\text{res}'_{\rho, U})$ is a finitely generated $\mathbb{Z}_p$-module.

(γ) For any $U \in \mathcal{U}$, $\text{Ker}(\text{res}''_{\rho, U})$ is a finite group which is isomorphic to $(E_{0, U}^A)'$ modulo uniformly bounded errors when $U$ varies in $\mathcal{U}$.

[Argument for the points (α) and (β)]

Firstly, by the Inflation-Restriction sequence, we have

$$\text{Ker}(\text{res}'_{\rho, U}) \cong H^1(K_\infty/K_U, (A \otimes \rho)^{GK_\infty}), \quad (8)$$

$$\text{Coker}(\text{res}'_{\rho, U}) \cong H^2(K_\infty/K_U, (A \otimes \rho)^{GK_\infty}). \quad (9)$$

Suppose that we are in the case (A). Let us consider the order $d_i(U) := \#H^i(U, \mathbb{Z}/p\mathbb{Z})$. Since $(A \otimes \rho)^{GK_\infty}$ is a finite $p$-group and $U$ is a pro-$p$ group, $U$-module $(A \otimes \rho)^{GK_\infty}$ is obtained by taking successive extensions of $\mathbb{Z}/p\mathbb{Z}$ for $\text{ord}_p\#(A \otimes \rho)^{GK_\infty}$ times. Hence we have $\#H^i(U, (A \otimes \rho)^{GK_\infty}) = d_i(U) \cdot \text{ord}_p\#(A \otimes \rho)^{GK_\infty}$ for $U \in \mathcal{U}$ and $i \in \{1, 2\}$.

Recall the following lemma from [15, Lemma 2.1]. The proof can be found in [15] and it essentially follows from the results of [11].

**Lemma 2.1** Let $G$ be a $p$-adic Lie group of dimension $d$. Then $H^1(Z, \mathbb{Z}/p\mathbb{Z})$ and $H^2(Z, \mathbb{Z}/p\mathbb{Z})$ are finite and their orders are bounded when $Z$ varies in closed subgroups of $G$.

By Lemma 2.1, the number $d_i(U)$ is finite and bounded when $U \in \mathcal{U}$ varies. Thus, by (8) and (9), we showed that $\text{Ker}(\text{res}'_{\rho, U})$ and $\text{Coker}(\text{res}'_{\rho, U})$ are finite groups whose orders are bounded independently of $U \in \mathcal{U}$ when we are in the case (A).

Suppose that we are in the case (Red). Recall that we have the following exact sequence

$$0 \longrightarrow ((A \otimes \rho)^{GK_\infty})_{\text{div}} \longrightarrow (A \otimes \rho)^{GK_\infty} \longrightarrow (A \otimes \rho)^{GK_\infty}/((A \otimes \rho)^{GK_\infty})_{\text{div}} \longrightarrow 0.$$

Taking a part of the long exact sequence of Galois cohomology associated to this short exact sequence, we have the following exact sequence

$$H^i(K_\infty/K_U, ((A \otimes \rho)^{GK_\infty})_{\text{div}}) \longrightarrow H^i(K_\infty/K_U, (A \otimes \rho)^{GK_\infty})$$
for $i = 1, 2$. In order to show the desired statements for $\text{Ker}(\text{res}^1_{\rho, U})$, $\text{Coker}(\text{res}^1_{\rho, U})$, 
$\lim\limits_U \text{Ker}(\text{res}^1_{\rho, U})$ and $\lim\limits_U \text{Coker}(\text{res}^1_{\rho, U})$ in the case (Red), it suffices to show similar statements for groups in the left end and the right end of (10).

Since the group $(A \otimes \rho)^{G_{K_\infty}} / ((A \otimes \rho)^{G_{K_\infty}})_{\text{div}}$ is a finite $p$-group, we can show that the group $H^i(K_\infty / K_U, (A \otimes \rho)^{G_{K_\infty}} / ((A \otimes \rho)^{G_{K_\infty}})_{\text{div}})$ in the right end is finite and bounded by the same argument using Lemma 2.1 as in the case (A) above. Let us handle the group $H^i(K_\infty / K_U, (A \otimes \rho)^{G_{K_\infty}})_{\text{div}}$ of the left end in (10). By taking a part of the long exact sequence of Galois cohomology associated to the short exact sequence

$$0 \to ((A \otimes \rho)^{G_{K_\infty}})_{\text{div}}[p] \to ((A \otimes \rho)^{G_{K_\infty}})_{\text{div}} \to ((A \otimes \rho)^{G_{K_\infty}})_{\text{div}} \to 0,$$

we have a surjection $H^i(K_\infty / K_U, (A \otimes \rho)^{G_{K_\infty}})_{\text{div}}[p] \to H^i(K_\infty / K_U, (A \otimes \rho)^{G_{K_\infty}})_{\text{div}}[p]$ for $i = 1, 2$. Since the argument using Lemma 2.1 as in the case (A) also works for $H^i(K_\infty / K_U, (A \otimes \rho)^{G_{K_\infty}})_{\text{div}}[p]$, we prove that the group $H^i(K_\infty / K_U, (A \otimes \rho)^{G_{K_\infty}})_{\text{div}}[p]$ is finite and bounded when $U$ varies in $U$.

The $\mathbb{Z}_p$-rank the cofree part of $H^i(K_\infty / K_U, (A \otimes \rho)^{G_{K_\infty}})_{\text{div}}$ is equal to the $\mathbb{Q}_p$-rank of the Lie algebra cohomology $H^i(g, V^{G_{K_\infty}})$ where $g$ is the Lie algebra of the $p$-adic Lie group $G$. Since $g$ is reductive by the condition (Red) and $V^{G_{K_\infty}}$ is semisimple as a representation of $g$, we have $H^i(g, V^{G_{K_\infty}}) = 0$ by [16, Thm 10]. This shows that $H^i(K_\infty / K_U, (A \otimes \rho)^{G_{K_\infty}})_{\text{div}}$ is finite for $i = 1, 2$ and that $\lim\limits_U H^i(K_\infty / K_U, (A \otimes \rho)^{G_{K_\infty}})_{\text{div}}$ is a finitely generated $\mathbb{Z}_p$-module for $i = 1, 2$.

[Argument for the point $\gamma'$ : contribution outside $p$]

Let us calculate $\text{Ker}(\text{res}^1_{\rho, U})$ when $U$ varies. First, we take a prime $u \in \Sigma_U$ which is not over $p$ and we calculate the local contribution to $\text{Ker}(\text{res}^1_{\rho, U})$ at $u$. Since we have an isomorphism

$$\frac{H^1(I_{U, u}, A_\rho)}{H^1_{\text{Gr}}(K_{U, u}, A_\rho)} \cong H^1(I_{U, u}, A_\rho)^{D_{U, u}},$$

it suffices to calculate

$$E^1_{U, u} := \text{Ker} \left[ H^1(I_{U, u}, A_\rho)^{D_{U, u}} \to H^1(I_{\infty, u}, A_\rho)^{D_{\infty, u}} \right]$$

where $w$ is a prime of $K_\infty$ which is over $u$.

We note that, if $u$ is not in the set $P_U$ defined in Introduction, $E^1_{U, u}$ is finite and bounded when $U$ varies since we have $I_{U, u} = I_{\infty, w}$ for sufficiently small $U$. From now on, we assume that $u \not\in P_U$. 

In order to bound the group $E_{U,u}^{A_\rho}$, we consider the following commutative diagram:

\[
\begin{array}{ccc}
H^1(D_{U,u}, A_\rho) & \longrightarrow & H^1(D_{\infty,u}, A_\rho) \\
\downarrow & & \downarrow \\
H^1(I_{U,u}, A_\rho)^{D_{U,u}} & \longrightarrow & H^1(I_{\infty,u}, A_\rho)^{D_{\infty,u}}.
\end{array}
\]  

(13)

We calculate the kernels and the cokernels of the maps in the diagram (13) by Hochschild-Serre spectral sequence. First, the group $E_{U,u}^{A_\rho}$ is the kernel of the lower horizontal map of the diagram by definition. On the other hand, the kernel of the upper horizontal map of the diagram is isomorphic to $H^1(U_u, (A_\rho)^{D_{\infty,u}})$ where $U_u = \text{Gal}(K_{\infty,w}/K_{U,u})$. The kernel of the left vertical map of the diagram (13) is $H^1(D_{U,u}/I_{U,u}, (A_\rho)^{I_{U,u}})$. The kernel of the right vertical map of the diagram (13) is $H^1(D_{\infty,u}/I_{\infty,u}, (A_\rho)^{I_{\infty,u}})$, which is trivial since the profinite group $D_{\infty,u}/I_{\infty,u}$ has no pro-$p$-part but the coefficient $(A_\rho)^{I_{\infty,u}}$ is $p$-primary.

Thus we have the following exact sequence

\[
0 \longrightarrow H^1(D_{U,u}/I_{U,u}, (A_\rho)^{I_{U,u}}) \longrightarrow H^1(U_u, (A_\rho)^{D_{\infty,u}}) \longrightarrow E_{U,u}^{A_\rho} \longrightarrow 0.
\]  

(14)

Now, we have the following lemma.

**Lemma 2.2** Assume that the condition $(V_q)$ of Definition 1.3 holds for a prime $q$ not dividing $p$. Then the group $E_{U,u}^{A_\rho}$ is finite for every open subgroup $U$ of $G$ and for every prime $u$ of $K_U$ which does not divide $p$.

**Proof** Note that since the kernel of Artin representation $\rho$ is open in $G_K$, the condition $(V_q)$ of Definition 1.3 holds for $A$ if and only if the condition $(V_q)$ holds for $A_\rho$. By the condition $(V_q)$ and by the local Tata duality, $H^0(D_{U,u}, A_\rho)$ and $H^2(D_{U,u}, A_\rho)$ are finite groups. Then, by Euler-Poincaré characteristic formula, the group $H^1(D_{U,u}, A_\rho)$ must be finite. Since we have $U_u = D_{U,u}/D_{\infty,u}$ by definition, we have an Inflation-Restriction exact sequence:

\[
0 \longrightarrow H^1(U_u, (A_\rho)^{D_{\infty,u}}) \longrightarrow H^1(D_{U,u}, A_\rho) \longrightarrow H^1(D_{\infty,u}, A_\rho)^{D_{U,u}}
\]  

(15)

Thus $H^1(U_u, (A_\rho)^{D_{\infty,u}})$ is finite. By (14), this implies that $E_{U,u}^{A_\rho}$ is finite. This completes the proof. $\square$

[Argument for the point ($\gamma$) : contribution over $p$]

Next, we take a prime $u \in P_U$ which is over $p$ and we will show that the local contribution to $\text{Ker}(\text{res}_p^\gamma)$ from all primes $u$ of $K_U$ dividing $p$ is finite assuming either the condition $(A_p)$ of Definition 1.6 or the condition $(\text{Sol}_p)$ of Definition 1.3.

We choose a prime $u \in \Sigma_U$ which is over $p$. Since we have an isomorphism

\[
\frac{H^1(K_{U,u}, A_\rho)}{H^1_\text{Gr}(K_{U,u}, A_\rho)} \cong H^1(I_{U,u}/F^+_v A_\rho)^{D_{U,u}},
\]  

(16)
it suffices to calculate the kernel of
\[ H^1(I_{U,u}, A_{\rho}/F_v^+A_{\rho})^{D_{U,u}} \longrightarrow H^1(I_{\infty,w}, A_{\rho}/F_v^+A_{\rho})^{D_{\infty,w}} \]  
(17)

where \( w \) is a prime of \( K_{\infty} \) which is over \( u \mid v \).

**Case** \((A_{\rho})\): Let us consider the following commutative diagram:

\[
\begin{array}{ccc}
H^1(D_{U,u}, A_{\rho}/F_v^+A_{\rho}) & \longrightarrow & H^1(D_{\infty,w}, A_{\rho}/F_v^+A_{\rho}) \\
\downarrow & & \downarrow \\
H^1(I_{U,u}, A_{\rho}/F_v^+A_{\rho})^{D_{U,u}} & \longrightarrow & H^1(I_{\infty,w}, A_{\rho}/F_v^+A_{\rho})^{D_{\infty,w}}.
\end{array}
\]

The vertical homomorphisms of the diagram are surjective by Inflation-Restriction sequence and by the fact that \( D_{U,u}/I_{U,u} \) and \( D_{\infty,w}/I_{\infty,w} \) are pro-cyclic groups. By the condition \((A_{\rho})\) of Definition 1.6, the kernel of the left vertical map is finite and bounded independently of \( U \in \mathcal{U} \). Hence, the kernel of (17) is finite and bounded independently of \( U \in \mathcal{U} \) if and only if the kernel of the map (18) below is finite and bounded independently of \( U \in \mathcal{U} \):

\[ H^1(D_{U,u}, A_{\rho}/F_v^+A_{\rho}) \longrightarrow H^1(D_{\infty,w}, A_{\rho}/F_v^+A_{\rho}). \]  
(18)

By definition, \( D_{U,u}/D_{\infty,w} \) is isomorphic to the decomposition group \( U_u \) of \( U \subset G \) and \( D_{\infty,w} \) is identified with \( G_{K_{\infty,w}} \). The kernel of the map (18) is hence isomorphic to \( H^1(U_u, (A_{\rho}/F_v^+A_{\rho})^{G_{K_{\infty,w}}}) \) by Inflation-Restriction sequence. Since the group \((A_{\rho}/F_v^+A_{\rho})^{G_{K_{\infty,w}}} \) is finite by the condition \((A_{\rho})\), we prove that \( H^1(U_u, (A_{\rho}/F_v^+A_{\rho})^{G_{K_{\infty,w}}}) \) is finite and bounded independently of \( U \in \mathcal{U} \) similarly as the earlier argument using Lemma 2.1.

**Case** \((\text{Sol}_{\rho})\): In this case, let us consider a decreasing filtration on \( U_u = \text{Gal}(K_{\infty,w}/K_{U,u}) \) starting from \( U_u^{(0)} = U_u \):

\[ U_u^{(0)} \supset U_u^{(1)} \supset \cdots \supset U_u^{(r-1)} \supset U_u^{(r)} = 0 \]

such that \( U_u^{(i)}/U_u^{(i+1)} \) is isomorphic to \( \mathbb{Z}_p \) up to an extension by finite groups for \( i = 0, \ldots, r - 1 \), which is required by the condition \((\text{Sol}_{\rho})\) of Definition 1.3. By our hypothesis that \( G \) has no element of order \( p \), order of these finite groups are prime to \( p \), which do not contribute to the cohomology with \( p \)-primary coefficients. Via the surjection \( D_{U,u}^{(0)} := D_{U,u} \twoheadrightarrow U_v \), the pull-back of the above filtration induces the following filtration:

\[ D_{U,u}^{(0)} \supset D_{U,u}^{(1)} \supset \cdots \supset D_{U,u}^{(r-1)} \supset D_{U,u}^{(r)} = D_{\infty,w} \]  
(19)

such that we have \( D_{U,u}^{(i)}/D_{U,u}^{(i+1)} \cong U_u^{(i)}/U_u^{(i+1)} \) for \( i = 0, \ldots, r - 1 \). Via the injection \( I_{U,u} \hookrightarrow D_{U,u} \), the pull-back of this filtration induces the following filtration:

\[ I_{U,u}^{(0)} \supset I_{U,u}^{(1)} \supset \cdots \supset I_{U,u}^{(r-1)} \supset I_{U,u}^{(r)} = I_{\infty,w} \]  
(20)
such that we have \( I_{U,u}^{(i+1)} / I_{U,u}^{(i)} = D_{U,u}^{(i+1)} / D_{U,u}^{(i)} \) for \( i = 1, \ldots, r - 1 \). As is explained below, only the initial graded piece \( I_{U,u}^{(0)} / I_{U,u}^{(1)} \) depends on the situation, either a finite group of order prime to \( p \) or isomorphic to \( \mathbb{Z}_p \) up to an extension by finite groups. For the first case where \( I_{U,u} \) is of finite index in \( D_{U,u} \), we have \( I_{U,u}^{(0)} / I_{U,u}^{(1)} \) is isomorphic to \( \mathbb{Z}_p \) up to an extension by finite groups. For the second case where \( I_{U,u} \) is of infinite index in \( D_{U,u} \), \( I_{U,u}^{(0)} / I_{U,u}^{(1)} \) is finite group of order prime to \( p \). Note that the kernel of \( (17) \) is finite and bounded when \( U \in \mathcal{U} \) varies if the kernel of

\[
H^1(I_{U,u}^{(i)}, A_p/F_v A_\rho)^{D_{U,u}} \rightarrow H^1(I_{U,u}^{(i+1)}, A_p/F_v A_\rho)^{D_{U,u}} \tag{21}
\]

is finite and bounded when \( U \in \mathcal{U} \) varies for each \( i \). By Inflation-Restriction sequence, the kernel of \( (21) \) is isomorphic to

\[
H^1(I_{U,u}^{(i)}, I_{U,u}^{(i+1)}, (A_p/F_v A_\rho)^{I_{U,u}^{(i+1)}})^{D_{U,u}}. \tag{22}
\]

It is clear that the condition (Ord) (resp. \( (V_p) \)) in Definition 1.3 holds true for \( A \) if and only if it holds true for \( A_\rho \). By the conditions (Ord) and \( (V_p) \), \( A_p/F_v A_\rho \) is a successive extension of the representation of type \( (\mathcal{K}/\mathcal{O})(\alpha \chi_{\text{cyc}}^m \psi) \) where \( \alpha \) is an unramified character of \( G_{K,v} \), \( m \) an integer, \( \psi \) a finite order character of \( I_{U,u} \).

Let us discuss the case \( i = 0 \), where we only need to discuss the first case where \( I_{U,u} \) is of finite index in \( D_{U,u} \) since \( I(0)/I(1) \) is a finite group of order prime to \( n \) for the second case. Since \( I_{U,u}^{(0)}/I_{U,u}^{(1)} \) is the Galois group of the local cyclotomic \( \mathbb{Z}_p \)-extension of \( K_{U,u} \) in this case, we have \( (A_p/F_v A_\rho)^{I_{U,u}^{(1)}} = A_p/F_v A_\rho \) for every \( i \geq 0 \). Let us study the group \( H^1(I_{U,u}^{(i)}/I_{U,u}^{(i+1)}, (A_p/F_v A_\rho)^{I_{U,u}^{(i+1)}})^{D_{U,u}} \) for \( i = 0 \). By looking at graded pieces, we will study the group

\[
H^1(I_{U,u}^{(0)}/I_{U,u}^{(1)}, (\mathcal{K}/\mathcal{O})(\alpha \chi_{\text{cyc}}^m \psi))^{D_{U,u}}. \tag{23}
\]

If \( m \neq 0 \), the pro-cyclic group \( I_{U,u}^{(0)}/I_{U,u}^{(1)} \) acts non-trivially on \( (\mathcal{K}/\mathcal{O})(\alpha \chi_{\text{cyc}}^m \psi) \) and hence \( H^1(I_{U,u}^{(0)}/I_{U,u}^{(1)}, (\mathcal{K}/\mathcal{O})(\alpha \chi_{\text{cyc}}^m \psi))^{D_{U,u}} = 0 \). If \( m = 0 \), \( I_{U,u}^{(0)}/I_{U,u}^{(1)} \) acts trivially on \( (\mathcal{K}/\mathcal{O})(\alpha \psi) \) and hence we have

\[
H^1(I_{U,u}^{(0)} / I_{U,u}^{(1)}, (\mathcal{K}/\mathcal{O})(\alpha \psi))^{D_{U,u}} = \text{Hom}(I_{U,u}^{(0)} / I_{U,u}^{(1)}, (\mathcal{K}/\mathcal{O})(\alpha \psi))^{D_{U,u}}.
\]

The action of \( D_{U,u} \) on \( I_{U,u}^{(0)}/I_{U,u}^{(1)} \) is a conjugate action. Since the action of \( D_{U,u} \) on \( I_{U,u}^{(0)} / I_{U,u}^{(1)} \) is trivial, the \( D_{U,u} \)-invariant part of this group is finite and bounded when \( U \in \mathcal{U} \) varies by the assumption \( (V_p) \) in Definition 1.3. We thus prove that, for \( i = 0 \), the kernel of the map in \( (21) \) is finite and bounded when \( U \in \mathcal{U} \) varies.

Next, we discuss the case \( i > 0 \). The condition \( (6) \) implies that, for every \( i \) and \( j \), \( \text{Hom}(I_{U,u}^{(i)}/I_{U,u}^{(i+1)}, \text{Fil}_v A / \text{Fil}_{v+1} A)^{D_{U,u}} \) is finite and bounded when \( U \in \mathcal{U} \) varies. Hence, for every \( i \), \( \text{Hom}(I_{U,u}^{(i)}/I_{U,u}^{(i+1)}, A / F_v A)^{D_{U,u}} \) is finite and bounded when \( U \in \mathcal{U} \).
varies. Since the kernel of the Artin representation $\rho$ is open in $G_K$, this implies that the group
\[
\text{Hom}(I_{U,u}^{(i)} / I_{U,u}^{(i+1)}, A_{\rho} / F_v^+ A_{\rho})^{D_{U,u}}.
\]
is finite and bounded when $U \in \mathcal{U}$ varies. This completes the proof of the control theorem.

**Remark 2.3** Only the part of the condition $(A_p)$ of Definition 1.6, stating the groups $(A/F_v^+ A)^{G_{K_{\infty,u}}}$ are finite, is used in the proof of control theorem.

In (2) and (3), we defined terms $E_0^A$ and $E_1^A$ which contributes to the functional equation of Theorem 0.3. Later, we will calculate these terms in some specific cases. At the moment, we will show that these two terms represent the same class in $K_0(\mathfrak{M}_H(G))$ under suitable hypotheses.

**Proposition 2.4** We keep the hypotheses and setting of Theorem 0.1 (Control Theorem). Assume further that either $(A_p)$ of Definition 1.6 or $(\text{Van}_p)$ of Definition 1.1 is satisfied. Then we have $[E_0^A] = [E_1^A]$ in $K_0(\mathfrak{M}_H(G))$.

**Proof** Recall that $P_U$ is the set of primes $u$ of $K_U$ such that the image of $I_{U,u} \subset G_{K_U}$ via $G_{K_U} \twoheadrightarrow U = \text{Gal}(K_{\infty}/K_U)$ is infinite by the equation (1). Recall also that we defined the terms $E_0^A$ and $E_1^A$ as follows:
\[
E_0^A := \lim_{U} E_{0,u}^A = \lim_{U \in P_U} \bigoplus_{u \in P_U} E_{0,u}^A \quad \text{and} \quad E_1^A := \lim_{U} E_{1,u}^A = \lim_{U \in P_U \setminus \{P_G\}} \bigoplus_{u \in P_U \setminus \{P_G\}} E_{1,u}^A.
\]
Let us fix a prime $v$ in $K$ such that $v \mid p$. Let $U$ be an open subgroup of $G$ and $u$ a prime in $K_U$ dividing $v$. By the definition of $P_U$, we have $u \in P_U$, $v \in P_G$.

First, let us assume that $(A_p)$ is satisfied. In this case, the proof of control theorem shows that $E_0^A$ is finite and uniformly bounded independent of $U$ and $u$. Using the induced module, we derive $\cdots \Longrightarrow E_0^{A,s} \cong \Lambda_{\infty}(G) \otimes_{\Lambda_{\infty}(G_v)} M_v^A$, where $M_v^A$ is a finite module of $p$-power cardinality. If $v$ is finitely decomposed in the extension $K_{\infty}$ i.e. if $[G : G_v] < \infty$, then we have $\cdots \Longrightarrow E_0^{A,s} \cong \Lambda_{\infty}(G) \otimes_{\Lambda_{\infty}(G_v)} M_v^A$ is finite and uniformly bounded. Hence $[\lim_{U \in P_U} E_{0,u}^A] = [\Lambda_{\infty}(G) \otimes_{\Lambda_{\infty}(G_v)} M_v^A] = 0$ in $K_0(\mathfrak{M}_H(G))$ by Lemma 4.2(a). On the other hand, if $v$ is infinitely decomposed in $K_{\infty}$ i.e. if $G_v$ is of infinite index in $G$, then, we have $[M_v^A] = 0$ in $K_0(\mathfrak{M}_H(H_v))$ by Lemma 4.2(a) where $H_v = H \cap G_v$.

As $\Lambda(G) \otimes_{\Lambda(G_v)} -$ is a well-defined map from $\mathfrak{M}_H(G_v) \longrightarrow \mathfrak{M}_H(G)$, hence we deduce that $[\Lambda_{\infty}(G) \otimes_{\Lambda_{\infty}(G_v)} M_v^A] = [\lim_{U} E_{0,u}^A] = 0$ in $K_0(\mathfrak{M}_H(G))$. Thus from the definition of $E_0^A$, $E_1^A$, we deduce that $[E_0^A] = [E_1^A]$ in $K_0(\mathfrak{M}_H(G))$.

Next, let us assume that $(A_p)$ is not satisfied but the assumption $(\text{Van}_p)$ is satisfied. Using the same notation as above, we have $\cdots \Longrightarrow E_0^{A,s} \cong \Lambda_{\infty}(G) \otimes_{\Lambda_{\infty}(G_v)} M_v^A$, where $M_v^A$ is now a finitely generated $\mathbb{Z}_p$-module. However by the assumption $(\text{Van}_p)$ and Lemma 4.2(a), we have $[M_v^A] = 0$ in $K_0(\mathfrak{M}_H(H_v))$. Thus, by an argument similar to the above case, we deduce that $[\lim_{U} E_{0,u}^A] = 0$ when (i) $v$ is finitely decomposed in $K_{\infty}$ and also when (ii) $v$ is infinitely decomposed in $K_{\infty}$ thanks to the hypothesis $(\text{Van}_p)$. Hence once again from the definition of $E_0^A$, $E_1^A$, we deduce that $[E_0^A] = [E_1^A]$ in $K_0(\mathfrak{M}_H(G))$. This completes the proof of the proposition. \[\Box\]
3 Examples of the error term of the algebraic functional equation

In this section, we calculate the exceptional divisor \( \lim_{U} E_{0,U}^{A} = \lim_{U} \oplus_{v \in P_{U}} E_{U,u}^{A} \) in the setting of Examples 1.7, 1.8 and 1.9.

**Example 3.1** First, we consider the case of Example 1.7 i.e. \( K_{\infty} = \bigcup_{n \geq 1} \mathbb{Q}(\mu_{p^{n}}, \sqrt[p^{n}]{a}) \), \( K = \mathbb{Q} \) and \( A = B_{p^{\infty}} \) where \( B \) is an elliptic curve defined over \( \mathbb{Q} \) with good ordinary reduction at \( p \). Recall that all primes of \( \mathbb{Q} \) are finitely decomposed in the false-Tate extension \( K_{\infty} \) as stated in Example 1.7. Note that in this case the condition \( (A_{p}) \) is satisfied. Hence we only need to calculate error terms corresponding primes not dividing \( p \), as by Proposition 2.4 we have, \( [E_{0}^{A}] = [E_{1}^{A}] \) in \( K_{0}(\mathcal{M}_{H}(G)) \) where \( E_{1}^{A} := \lim_{U} E_{1,U}^{A} = \lim_{U} \oplus_{u \in P_{U}, u \nmid p} E_{U,u}^{A} \). Following notation of [24], we define

\[
P_{0} = \{ \text{primes } q \text{ in } \mathbb{Q} : q \mid a \text{ but } q \nmid p \},
\]

\[
P_{1} = \{ q \in P_{0} : B \text{ has split multiplicative reduction above } q \text{ over } \mathbb{Q}(\mu_{p^{\infty}}) \},
\]

\[
P_{2} = \{ q \in P_{0} : B \text{ has good reduction at } q \text{ and } B_{p^{\infty}}(K_{v}) \neq 0 \}.
\]

In addition, for each \( U \) and \( i = 0, 1, 2 \), we define \( P_{i,U} \) to be the set of primes of \( K_{U} \) dividing \( P_{i} \). Note that we have \( P_{0} = P_{G} \setminus \{ p \} \) and \( P_{0,U} = P_{U} \setminus \{ \text{primes } u \text{ in } K_{U} : u \mid p \} \) by definition and by [24, Lemma 3.9]. Recall that we have

\[
\left[ \lim_{U} E_{1,U}^{A} \right] = \left[ \lim_{U} \oplus_{u \in P_{U}, u \nmid p} E_{U,u}^{A} \right] = \left[ \oplus_{v \in P_{0}} \text{Ind}_{G_{v}}^{G} \lim_{U} \oplus_{u \in P_{U}, u \mid v} E_{U,u}^{A} \right]. \tag{24}
\]

Hence, it suffices to calculate \( \lim_{U} \oplus_{u \in P_{U}, u \mid v} E_{U,u}^{A} \). We also define

\[
P_{00} = \{ q \in P_{0} : A^{D_{\infty,w}} \neq 0 \text{ for any prime } w \text{ of } K_{\infty} \text{ over } q \}.
\]

By the exact sequence (14), we have

\[
E_{U,u}^{A} = 0 \text{ for each prime } u \in P_{0,U} \setminus P_{00,U}. \tag{25}
\]

[23, Proposition 5.1(iii)] proves that \( P_{00,U} = P_{1,U} \cup P_{2,U} \).

For a prime \( u \in P_{2,U} \), the inertia group \( I_{U,u} \) acts trivially on \( A \) and the eigenvalues of Frobenius element in \( D_{U,u}/I_{U,u} \) are non-trivial. Hence the first term \( H^{1}(D_{U,u}/I_{U,u}, A^{I_{U,u}}) \) of the exact sequence (14) is zero and we have \( E_{U,u}^{A} \cong H^{1}(U_{u}, A^{D_{\infty,w}}) \). Since we have \( A^{D_{\infty,w}} = B_{p^{\infty}}(K_{\infty,w}) = B_{p^{\infty}} \) (see [23, Proposition 5.1(i)])], we obtain \( E_{U,u}^{A} \cong H^{1}(U_{u}, A) \). By a similar argument, the restriction map of Galois cohomology induces an isomorphism \( H^{1}(U_{u}, A) \cong H^{1}(I_{U,u}, A)^{D_{U,u}} \). Since \( I_{U,u} \) acts trivially on \( A \), we have \( H^{1}(I_{U,u}, A)^{D_{U,u}} \cong \text{Hom}(I_{U,u}, A)^{D_{U,u}} \). Thus we obtained

\[
E_{U,u}^{A} \cong \text{Hom}(I_{U,u}, A)^{D_{U,u}}.
\]
Note that, the conjugate action of $D_{U,u}$ on $I_{U,u}$ is given by the $p$-adic cyclotomic character. Hence we have

$$\bigoplus_{q \in P_2} \lim_{\leftarrow U} \bigoplus_{u \in P_{2,U}} E^{A}_{U,u} \cong \bigoplus_{q \in P_2} \lim_{\leftarrow U} \bigoplus_{u \in P_{2,U}} B_{p^\infty}(-1)^{D_{U,u}} \cong \bigoplus_{q \in P_2} \text{Ind}_{G_{q}}^G T_p B(-1).$$

(26)

For a prime $q \in P_{1,U}$, we have $A^{D_{\infty,w}} = B_{p^\infty}(K_{\infty,w}) = B_{p^\infty}$ by a similar argument as the previous case. Since $D_{\infty,w}$ has no non-trivial pro-$p$ quotient, the third term of (15) is zero. Hence (14) gives the following exact sequence:

$$0 \longrightarrow H^1(D_{U,u}/I_{U,u}, H^0(I_{U,u}, A)) \longrightarrow H^1(D_{U,u}, A) \longrightarrow E^{A}_{U,u} \longrightarrow 0. \quad (27)$$

Since our representation $A = B_{p^\infty}$ satisfies the condition $(V_q)$ of Definition 1.3 for any prime $q \neq p$, $H^0(D_{U,u}, A)$ and $H^2(D_{U,u}, A)$ are finite. Thus we have

$$H^1(D_{U,u}, A) \cong H^1(D_{U,u}, T)^{\vee} \cong H^0(D_{U,u}, A)^{\vee}.$$  

(28)

Putting these together, we can calculate the inverse limit of the middle term of (27),

$$\bigoplus_{q \in P_1} \lim_{\leftarrow U} \bigoplus_{u \in P_{1,U}} H^0(D_{U,u}, A)^{\vee} \cong \bigoplus_{q \in P_1} \lim_{\leftarrow U} \bigoplus_{u \in P_{1,U}} B_{p^\infty}(K_{U,u})^{\vee} \cong \bigoplus_{q \in P_1} \text{Ind}_{G_{q}}^G T_p B(-1).$$

(29)

On the other hand, we calculate the projective limit of the left most term of (27) with respect to the corestriction maps when $U$ is changing:

$$\bigoplus_{q \in P_1} \lim_{\leftarrow U} \bigoplus_{u \in P_{1,U}} H^1(D_{U,u}/I_{U,u}, A^{I_{U,u}}).$$

By assumption, $B$ has split multiplicative reduction above $q$ over $\mathbb{Q}(\mu_{p^\infty})$. Hence, for any sufficiently small $U$, $B$ has split multiplicative reduction above $q$ over $K_U$. Since we calculate the projective limit of some modules related to $U$, we may and we do assume without loss of generality, that $B$ has split multiplicative reduction above $q$ over $K_U$ for any $U$ below. Since $q \in P_1$, the action of $G_q$ on $A$ has a non-split filtration

$$0 \longrightarrow F_q^{+}A \longrightarrow A \longrightarrow A/F_q^{+}A \longrightarrow 0$$

where $F_q^{+}A$ is cofree $\mathbb{Z}_p$-module of corank one on which $G_q$ acts by the $p$-adic cyclotomic character $\chi_{\text{cyc}}$. Taking the $I_{U,u}$-invariant of the above short exact sequence, we obtain

$$0 \longrightarrow F_q^{+}A \longrightarrow A^{I_{U,u}} \longrightarrow F_{U,u} \longrightarrow 0,$$

(30)
where $F_{U,u}$ is finite. Since $F_q^+A = \mathbb{Q}_p/\mathbb{Z}_p(1)$ as $G_q$-module, we have $H^1(D_{U,u}/I_{U,u}, F_q^+A) = 0$. We also have $H^2(D_{U,u}/I_{U,u}, F_q^+A) = 0$ since $D_{U,u}/I_{U,u}$ is infinite procyclic. From equation (30), we obtain

$$H^1(D_{U,u}/I_{U,u}, A_{U,u}^+ \cong H^1(D_{U,u}/I_{U,u}, F_{U,u}) \cong \text{Hom}(D_{U,u}/I_{U,u}, F_{U,u}).$$

The last isomorphism is true as $D_{U,u}/I_{U,u}$ acts trivially on $A/F_q^+A = \tilde{B}_{p\infty}$. Recall that $A$ is isomorphic to $\left(\mathbb{Q}_q^×/q\mathbb{Z}\right)[p^{\infty}]$ and $B_p^n$ fits into the exact sequence

$$0 \rightarrow \mu_{p^n} \rightarrow B_p^n \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow 0.$$ 

When $K_{U,u}$ is equal to $k_n = \mathbb{Q}_q(\mu_{p^n}, a^{1/p^n})$ we denote $F_{U,u}$ by $F_n$, which is a subgroup of the quotient $\mathbb{Z}/p^n\mathbb{Z}$. There exists a constant $\delta$ such that the order of $F_n$ is $p^{n-\delta}$ when $n$ is sufficiently large. The degree of the extension $k_{n+1}/k_n$ is $p^2$ and the norm map for the extension $k_{n+1}/k_n$ maps $F_{n+1}$ to a subgroup of order $p^2\#F_{n+1}$ in $F_n$. From this observation, we see that see that

$$\bigoplus_{q \in P_1} \lim_{U \in P_1, U \ni q} H^1(D_{U,u}/I_{U,u}, A_{U,u}^+) \cong \bigoplus_{q \in P_1} \lim_{U \in P_1, U \ni q} F_{U,u} \cong 0. \quad (31)$$

Using (31) in (27), the contribution of the split multiplicative primes to the error term is given by

$$\bigoplus_{q \in P_1} \lim_{U \in P_1, U \ni q} E_{U,u}^A \cong \bigoplus_{q \in P_1} \text{Ind}_G^{G_t} T_p B(-1). \quad (32)$$

By (25), (26) and (29), we have the following description of error terms in $K_0(\mathcal{M}_H(G))$:

$$[E_0^A] = [E_1^A] \cong \left[ \bigoplus_{q \in P_1 \cup P_2} \text{Ind}_G^{G_t} T_p B(-1) \right].$$

In this case of Example 1.7(1), our error term is exactly the same error term as that of [34, Theorem 6.2, Equation (6.29)]. Note in this case, $B$ has good ordinary reduction at $p$. Then, using Coates–Greenberg’s theory of deeply ramified extension [6, Proposition 4.3 and Proposition 4.8], it follows that $\text{Sel}^{G_t}_A(K_\infty) = S_{p\infty}^{G_t}(B/K_\infty)$, where $S_{p\infty}(B/K_\infty)$ is the classical Selmer group of $B$ over $K_\infty$. Thus our result is consistent with that of [34].

**Example 3.2** Next, let us consider the case of Example 1.8 (O) i.e. $K_\infty = \bigcup_{n \geq 1} \mathbb{Q}(E_{p^n})$ where $E$ is a non-CM elliptic curve over $\mathbb{Q}$ and $V = T_p B \otimes \mathbb{Q}_p$ is the Galois representation associated to an elliptic curve $B$. We assume that $B$ is isogenous to $E$. As in the previous example, we define

$$\mu_{p^n} \rightarrow B_p^n \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow 0.$$
\[ P_0 = \{ \text{primes } q : q \neq p \text{ and the } q\text{-adic valuation of } j(B) \text{ is negative} \}, \]
\[ P_1 = \{ q \in P_0 : B \text{ has split multiplicative reduction above } q \text{ over } \mathbb{Q}(\mu_p) \}. \]

The primes in \( P_0 \) are precisely those prime where \( B \) has potentially multiplicative reduction. Thus \( P_0 \) is nothing but primes not divisible by \( p \) at which the inertia subgroup of \( \text{Gal}(K_{\infty}/\mathbb{Q}) \) is infinite. Note that there is no analogue of \( P_2 \) of the false-Tate case of Example 1 in this case.

We note that primes in \( P_1 \) are not finitely decomposed in this example. However, the arguments and conclusions of the false-Tate case above for a prime of \( q \) in \( P_1 \) remain valid even if \( q \) is infinitely decomposed. Thus, following the same argument as in the false-Tate case above, we deduce the following description of error terms in \( K_0(\mathcal{M}_H(G)) \):

\[ [E_1^A] \cong \bigoplus_{q \in P_1} \Ind_{G_q}^G T_p B(-1). \]

Note by Example 4.8(5), the condition \((\text{Van}_p)\) of Definition 1.1 is satisfied in this case. Hence by Proposition 2.4, the error term is given by

\[ [E_0^A] = [E_1^A] \cong \bigoplus_{q \in P_1} \Ind_{G_q}^G T_p B(-1). \]

In this case of Example 1.8(O), our error term above is the same error term as that of [35, Theorem 5.2, Equation (5.23)]. Since \( B \) has good, ordinary reduction at \( p \), again using [6, Proposition 4.3 and Proposition 4.8], it follows that \( \text{Sel}^{\text{Gr}}_A(K_{\infty}) \) is the same as the classical Selmer group \( S_p^{\infty}(B/K_{\infty}) \). Hence, our result is consistent with that of [35].

**Example 3.3** Let \( f \) be a normalized eigen elliptic cuspform of even weight \( k \geq 2 \) and level \( \Gamma_0(N) \) such that \( N \) is square-free and the conductor \( N_f \) of \( f \) is not divisible by \( p \). Let us assume that \( a_p(f) \) is a \( p \)-adic unit. We define \( V \) to be the Tate-twist \( V_f(\frac{k}{2}) \) and we take a lattice \( T \subset V \). We set \( A = T \otimes \mathbb{Q}_p/\mathbb{Z}_p \). Let \( K_{\infty}/\mathbb{Q} \) be the false-Tate curve extension as in Example 1 above. In this case, we define

\[ P_0 = \{ \text{prime in } \mathbb{Q} : q \neq p, \ q \mid a \}, \]
\[ P_{00} = \{ q \in P_0 : A^{D_{\infty,w}} \neq 0 \text{ for any prime } w \text{ of } K_{\infty} \text{ over } q \}. \]

By the same reason as (25), we have

\[ E_{U,u}^A = 0 \text{ for each prime } u \in P_{0,U} \setminus P_{00,U}. \quad (33) \]

In order to give the analogues of \( P_1 \) and \( P_2 \) of Example 1, we prepare some notation. Let us denote by \( \pi_{f,q} \) the local automorphic representation at the prime \( q \) associated to \( f \). Since we assume that the Nebentypus character of \( f \) is trivial and the conductor \( N_f \) of \( f \) is square-free, when prime \( q \) divides \( N_f \), \( \pi_{f,q} \) is a special representation and
we have \( \pi_{f,q} \cong \pi(\delta) | \frac{k}{2} - 1, \delta | \frac{k}{2} \) where \( \delta \) is a unramified quadratic character of \( \mathbb{Q}_q^\times \) and \( | \cdot | \) is the valuation on \( \mathbb{Q}_q \). Now, we define

\[
P_1 = \{ q \in \mathbb{P}_0 : q | N_f \text{ and the quadratic character } \delta \text{ associated to } \pi_{f,q} \text{ is trivial on } \mathbb{Q}_q(\mu_p)^\times \}, \]
\[
P_2 = \{ q \in \mathbb{P}_0 : q \nmid N_f \text{ and } A^{D_{\infty,v}} \neq 0 \text{ for any prime } w \text{ of } K_\infty \text{ over } q \}.
\]

By definition, for any sufficiently small \( U \), \( \delta \) is trivial regarded as a Galois character of \( D_{U,u} \). Since we calculate the projective limit of some modules related to \( U \), we may and we do assume without loss of generality, that \( \delta \) is trivial regarded as a Galois character of \( D_{U,u} \) for any \( U \) below. For \( q \in P_1 \), the action of any open subgroup \( D_{U,u} \) of \( G_q \) on \( A \) has a non-split filtration

\[
0 \longrightarrow F_q^+ A \longrightarrow A \longrightarrow A/F_q^+ A \longrightarrow 0
\]

where \( F_q^+ A \) (resp. \( A/F_q^+ A \)) is a cofree \( \mathbb{Z}_p \)-module of corank one on which \( D_{U,u} \) acts by the \( p \)-adic cyclotomic character \( \chi_{\mathbb{Q}_c} \) (resp. \( \chi_{\mathbb{Q}_c} \)). Hence, we have \( P_{00,U} = P_{1,U} \cup P_{2,U} \). By a similar argument as in Example 1, we prove the same type of results as (26) and (29). Hence, the error term is given by

\[
[E_0^A] = [E_1^A] = \left( \bigoplus_{q \in P_1 \cup P_2} \text{Ind}_{G_q}^G T(-1) \right) \text{ in } K_0(\mathfrak{M}_H(G)).
\]

**Example 3.4** Let us discuss commutative examples. Let us keep the same setting as Example 1.9 where \( G \) is isomorphic to \( \mathbb{Z}^{d+1}_p \) and \( A \) is isomorphic to \( E_\infty \) with \( E \) an elliptic curve with good ordinary reduction at all primes of \( K \) above \( p \). Now we have two cases regarding the decomposition subgroup \( G_v \) of \( G \) at a prime \( v \) in \( K \) dividing \( p \). Note as \( K_{\text{cyc}} \subset K_\infty \), we necessarily have dimension of \( G_v \), as a \( p \)-adic Lie group, is at least 1.

In the first case, if \( K_v \) is equal to \( \mathbb{Q}_p \), then for a prime \( w \) in \( K_\infty \) over \( w \), \( E_\infty(\mathbb{Q}_p) = E_\infty(\mathbb{Q}_p) \subset E_\infty(G_v) \) is finite by Imai’s theorem [18] (Note that Imai’s theorem can be applied since we assume that \( E \) has good reduction at \( v \)). Thus in this case condition \( (A_p) \) is satisfied.

On the other hand, if \( K_v \) is a non-trivial extension of \( \mathbb{Q}_p \), then the dimension of \( H_v := H \cap G_v \) is at least 1. Thus \( \mathbb{Z}_p[[G_v]] \cong \mathbb{Z}_p[[T_1, T_2, ..., T_r]] \) for some \( r \) with \( r \geq 2 \). Thus \( \mathbb{Z}_p \) is a pseudo-null \( \mathbb{Z}_p[[G_v]] \)-module and consequently, \( (V_{an}) \) hypothesis is satisfied.

Hence, by Proposition 2.4, we have \( [E_0^A] = [E_1^A] \) in \( K_0(\mathfrak{M}_H(G)) \) no matter how \( K_v \) is equal to \( \mathbb{Q}_p \) or not. Also, since \( \mathbb{Z}^{d+1}_p \) extension of a number field is unramified outside \( p \), we have \( E_{U,u}^A = 0 \) for sufficiently small \( U \) any \( u \in K_U \) with \( u \nmid p \). It follows that \( E_1^A = 0 \). Thus the functional equation takes the expected simple form

\[
[Sel_{\mathcal{A}}^G(K_\infty)\check{\cdot}] = [(Sel_{\mathcal{A}}^G(K_\infty)\check{\cdot})] \text{ in } K_0(\mathfrak{M}_H(G)).
\]
4 Higher extension groups of Selmer groups

In this section, we study the higher extension groups $\text{Ext}^i_{A(G)}(\text{Sel}^{\Gamma}(K_\infty)^\vee, \Lambda(G))$, which appear in the proof of our functional equation given in the next section. Recall $G$ has a closed normal subgroup $H$ such that $G/H \cong \Gamma \cong \mathbb{Z}_p$. We prepare a few lemmas and a proposition in the beginning which is used later.

Lemma 4.1 For any $X \in \text{SL}_2(\mathbb{Z}_p)$, its centralizer

$$C_{\text{SL}_2(\mathbb{Z}_p)}(X) = \{Y \in \text{SL}_2(\mathbb{Z}_p) | YY^{-1} = X\}$$

is infinite. Consequently, the dimension of $C_{\text{SL}_2(\mathbb{Z}_p)}(X)$ as $p$-adic Lie group is $\geq 1$.

Proof Take any matrix $X$ in $\text{SL}_2(\mathbb{Z}_p)$. For a diagonal matrix $X = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_p)$, we have $\{ (\lambda \ 0) \ | \lambda \in \mathbb{Z}_p^\times \} \subset C_{\text{SL}_2(\mathbb{Z}_p)}(X)$, which shows that $C_{\text{SL}_2(\mathbb{Z}_p)}(X)$ is infinite in this case. Next, we consider the case where $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_p)$ is not a diagonal matrix, that is, at least one of $b$ or $c$ is nonzero. Then for each $n$ consider

$$X_n := I_{2 \times 2} + p^n X = \begin{pmatrix} 1 + p^n a & p^n b \\ p^n c & 1 + p^n d \end{pmatrix}.$$  

Then $\det(X_n) \in 1 + p\mathbb{Z}_p$ and set $Y_n = \frac{1}{\det(X_n)} X_n^2 \in \text{SL}_2(\mathbb{Z}_p)$. For all $n$, we have $Y_n X = XY_n$ and we have the presentation

$$Y_n = \begin{pmatrix} (1 + p^n a)^2 + p^{2n} bc & p^n b(2 + p^n a + p^n d) \\ p^n c(2 + p^n a + p^n d) & (1 + p^n d)^2 + p^{2n} bc \end{pmatrix}. $$

Since the $p$-adic valuation of at least one of the non-diagonal entries of $Y_n$ goes to infinity as $n$ goes to infinity, the set $\{Y_n\}_{n \in \mathbb{N}}$ is an infinite subset of $\text{SL}_2(\mathbb{Z}_p)$. This completes the proof. \qed

Let $\sigma$ be a uniformizer of $\mathcal{O}$. For any $p$-adic Lie group $G$, we define $\Omega(G) := \lim_{\leftarrow} \mathcal{O}/(\sigma)[G/U]$ to be the completed group ring where $U$ varies over open normal subgroups of $G$. We denote by $K_0(\Omega(G))$ the Grothendieck group of the ring $\Omega(G)$.

Proposition 4.2 (a) Let $G$ be a compact $p$-adic Lie group $G$ without any element of order $p$ such that $G$ has a quotient $\Gamma$ isomorphic to $\mathbb{Z}_p$. Let $M$ be a $\Lambda_\mathcal{O}(G)$-module which is of finite cardinality. Then, $[M] = 0$ in $K_0(\mathfrak{M}_H(G))$.

(b) Let $H$ be a $p$-adic Lie group without any element of order $p$. Assume that either one of the following two conditions hold:

(i) The centralizer of every element in $H$ is infinite.

(ii) The group $H$ is pro-$p$. 

Then we have \([\mathcal{O}/(\varpi)] = 0\) in \(K_0(\Omega(H))\).

**Proof of Part (a)** This statement is proved in [3, Lemma 2.2]. However, we will give a detailed proof. Since any \(\Lambda_{\mathcal{O}}(G)\)-module of finite cardinality is isomorphic to a successive extension of the \(\Lambda_{\mathcal{O}}(G)\)-module \(\mathcal{O}/(\varpi)\), we reduce the proof to the case \(M = \mathcal{O}/(\varpi)\). As \(G\) has no element of order \(p\), \(\Omega(G)\) has finite global dimension [2, Proposition(e) §3.3]. It follows that every finitely generated \(\Omega(G)\)-module can be represented in the Grothendieck group \(K_0(\Omega(G))\). In particular, \([\mathcal{O}/(\varpi)][\Gamma]\) ∈ \(K_0(\Omega(G))\). Thus \([\mathcal{O}/(\varpi)] = 0\) in \(K_0(\Omega(G))\). Further, [2, §5.5] shows that \(K_0(\Omega(G)) \cong K_0(D)\), where \(D\) denote the category of all finitely generated \(p\)-torsion \((G)\)-modules. The result follows by considering the natural homomorphism (see [2, §1.4]) from \(K_0(D) \rightarrow K_0(\mathcal{M}_H(G))\). □

**Proof of part (b)** First, we discuss the case with the condition (i). Using works of Serre [31], it is shown in [1, §1.2 & §1.3, Page 32] that \([\mathcal{O}/(\varpi)] = 0\) in \(K_0(\Omega(H))\) if the dimension (as a \(p\)-adic Lie group) of the centralizer of every element of \(H\) is at least one. Thus we deduce \([\mathcal{O}/(\varpi)] = 0\) in \(K_0(\Omega(H))\) in the first case.

Next, we discuss the case with the condition (ii). If \(H\) is also pro-\(p\), then Grothendieck group \(K_0(\Omega(H))\) is isomorphic to \(\mathbb{Z}\) and the class of any finitely generated torsion \(\Omega(H)\)-module in \(K_0(\Omega(H))\) is zero. In particular, we obtain \([\mathcal{O}/(\varpi)] = 0\). □

By using Lemma 4.1 and Proposition 4.2, we immediately obtain the following corollary.

**Corollary 4.3** Let \(H = \text{SL}_2(\mathbb{Z}_p)\). Then we have \([\mathcal{O}/(\varpi)] = 0\) in \(K_0(\Omega(H))\).

**Remark 4.4** In [2, Example 9.6], they considered a pro-Dihedral group \(H = \mathbb{Z}_p \times \mathbb{Z}/2\mathbb{Z}\) and showed that \([\mathcal{O}/(\varpi)] \neq 0\) in \(K_0(\Omega(H))\).

**Lemma 4.5** [19, Lemma 2.4] Let \(G\) be a \(p\)-adic Lie group and \(U\) be an open subgroup of \(G\). Then the restriction map induces a functorial isomorphism of \(\Lambda_{\mathcal{O}}(U)\)-modules,

\[
\text{Ext}^r_{\Lambda_{\mathcal{O}}(U)}(M, \Lambda_{\mathcal{O}}(G)) \cong \text{Ext}^r_{\Lambda_{\mathcal{O}}(U)}(M, \Lambda_{\mathcal{O}}(U))
\]

for every \(\Lambda_{\mathcal{O}}(G)\)-module \(M\).

Let \(P_{H \cap U}\) be the set of primes in \(K_{U,K_{\text{cyc}}} = K_{\infty}^{H \cap U}\) lying above \(P_U\) and let \(D_{H \cap U, \tilde{u}}\) (resp. \(I_{H \cap U, \tilde{u}}\)) denote the decomposition (resp. inertia subgroup) of \(H \cap U = \text{Gal}(K_{\infty}/K_UK_{\text{cyc}})\) at \(\tilde{u}\) where \(\tilde{u}\) is a prime in \(K_UK_{\text{cyc}}\) over \(u\). Set

\[
E_{H \cap U, \tilde{u}}^{A^*(1)} = \begin{cases} 
\ker \left[ H^1(I_{H \cap U, \tilde{u}}, A^*(1))^{D_{H \cap U, \tilde{u}}} \rightarrow H^1(I_{\infty, w}, A^*(1))^{D_{\infty, w}} \right] & \text{if } u \nmid p, \\
\ker \left[ H^1(I_{H \cap U, \tilde{u}}, A^*(1))^{F_p A^*(1)} \rightarrow H^1(I_{\infty, w}, A^*(1))^{F_p A^*(1)} \right] & \text{if } u \mid p.
\end{cases}
\]

(34)

Recall that, for a finitely generated \(\mathcal{O}[[\Gamma_U]]\) module \(M\), we denote the maximal \(\mathcal{O}[[\Gamma_U]]\) pseudonull (i.e. finite) submodule of \(M\) by \(M_{\text{null}}\). Then we have the following lemma.
Lemma 4.6 Let $U$ vary over the open normal subgroups $G$, then

$$\lim_{\tilde{u}} \left( \bigoplus_{\tilde{u} \in P_{H \cap U}} \left( E^{A^* (1)}_{H \cap U, \tilde{u}} \right)^\vee \right)_{\text{null}} = 0 \text{ in } K_0(\mathcal{M}_H(G)),$$

if either the condition $(A_p)$ or the condition $(\text{Van}_p)$ is satisfied.

Proof First, we discuss $E^{A^* (1)}_{H \cap U, \tilde{u}}$ for $\tilde{u} \nmid p$. We study the following diagram.

$$
\begin{array}{ccc}
H^1(D_{H \cap U, \tilde{u}}, A^*(1)) & \longrightarrow & H^1(D_{\infty, w}, A^*(1)) \\
\downarrow & & \downarrow \\
H^1(I_{H \cap U, \tilde{u}}, A^*(1))^{D_{H \cap U, \tilde{u}}} & \longrightarrow & H^1(I_{\infty, w}, A^*(1))^{D_{\infty, w}}.
\end{array}
$$

By the same reason as that we gave on the diagram (13), the left vertical homomorphism of the diagram are surjective and the right vertical homomorphism of the diagram is isomorphic. Since the group $E^{A^* (1)}_{H \cap U, \tilde{u}}$ is the kernel of the lower horizontal homomorphism of the diagram, we have a surjection map

$$\text{Ker}[H^1(D_{H \cap U, \tilde{u}}, A^*(1)) \longrightarrow H^1(D_{\infty, w}, A^*(1))] \rightarrow E^{A^* (1)}_{H \cap U, \tilde{u}},$$

(36)

applying the snake lemma to the diagram (35). By the Inflation-Restriction sequence, we have

$$H^1(D_{H \cap U, \tilde{u}}/D_{\infty, w}, (A^*(1))^{G_{K_{\infty, w}}}) \cong \text{Ker}[H^1(D_{H \cap U, \tilde{u}}, A^*(1)) \longrightarrow H^1(D_{\infty, w}, A^*(1))].$$

(37)

By the fact that $v$ is not dividing $p$ and by the hypothesis that $G$ has no element of order $p$, for any prime $\tilde{u}$ of $K_{U} K_{\text{cyc}}$ above $v$, the group $D_{H \cap U, \tilde{u}}/D_{\infty, w}$ is either a finite group of order prime to $p$ or isomorphic to a product of $\mathbb{Z}_p$ and a finite group of order prime to $p$. In the former case, the group (37) is trivial and there is nothing to discuss. In the latter case, the largest finite quotient of the group (37) is a quotient of a finite group $(A^*(1))^{G_{K_{\infty, w}}}/((A^*(1))^{G_{K_{\infty, w}}})_{\text{div}}$. By (37), $(E^{A^* (1)}_{H \cap U, \tilde{u}})_{\text{null}}$ is a finite group for any prime $v$ of $K$ not dividing $p$ and for any prime $\tilde{u}$ of $K_{U} K_{\text{cyc}}$ above $v$. Let $v$ be a prime of $K$ not dividing $p$. Then, $(E^{A^* (1)}_{H \cap U, \tilde{u}})_{\text{null}}$ and $(E^{A^* (1)}_{H \cap U, \tilde{u}})^\vee_{\text{null}}$ are isomorphic to each other for two different primes $\tilde{u}$ and $\tilde{u}'$ of $K_{U} K_{\text{cyc}}$ above $v$. We conclude that

$$(E^{A^* (1)}_{H \cap U, \tilde{u}})^\vee_{\text{null}} \text{ is finite and uniformly bounded independently of } U \text{ and } \tilde{u}. \tag{38}$$

Thus, we get the following isomorphism by using induced module and by (38)

$$\lim_{\tilde{u}} \left( \bigoplus_{\tilde{u} \in P_{H \cap U \setminus \{p\}}} \left( E^{A^* (1)}_{H \cap U, \tilde{u}} \right)^\vee \right)_{\text{null}} \cong \bigoplus_{v} \Lambda_{\mathcal{O}(G)} \otimes \Lambda_{\mathcal{O}(G_v)} M_v$$

where $v$ runs through primes of $K$ such that the inertia subgroup of $G_v$ is infinite and $M_v$ a $\Lambda_{\mathcal{O}(G_v)}$-module of finite cardinality.
Recall the cyclotomic \( \mathbb{Z}_p \) extension \( K_{\text{cyc}} \) is finitely decomposed at every prime of \( K \); hence \( \Lambda(G) \otimes_{\Lambda(G_v)} - \) is a well-defined map from \( \mathcal{M}_{H_v}(G_v) \rightarrow \mathcal{M}_H(G) \) for \( v \in P_G, v \nmid p \). Then applying Proposition 4.2(a), we obtain

\[
\left[ \lim_{U} \left( \bigoplus_{\tilde{u} \in P_H \cap U, \tilde{u} \mid p} (E^{A^*_v}_H(U, \tilde{u})^{\ast}) \right) \right] = [\Lambda_{\mathcal{O}}(G) \otimes_{\Lambda_{\mathcal{O}}(G_v)} M_v] = 0 \text{ in } K_0(\mathcal{M}_H(G)). \tag{39}
\]

Next, we discuss \( E^{A^*_v}_H(U, \tilde{u}) \) for \( \tilde{u} \mid p \). Recall that we have

\[
E^{A^*_v}_H(U, \tilde{u}) \cong H^1(I_H \cap U, \tilde{u}, A^*_w(1) / (A^*_v(F_v^+ A^*_w(1)))^{I_{\infty, v}} D_{H \cap U, \tilde{u}} \tag{40}
\]

by definition. Let us consider the following diagram.

\[
\begin{array}{ccc}
H^1(D_{H \cap U, \tilde{u}}, A^*_w(1) / (F_v^+ A^*_w(1))) & \longrightarrow & H^1(D_{\infty, w}, A^*_w(1) / (F_v^+ A^*_w(1))) \\
\downarrow & & \downarrow \\
H^1(I_{H \cap U, \tilde{u}}, A^*_w(1) / (F_v^+ A^*_w(1))^{D_{H \cap U, \tilde{u}}} & \longrightarrow & H^1(I_{\infty, w}, A^*_w(1) / (F_v^+ A^*_w(1)))^{D_{\infty, w}}.
\end{array} \tag{41}
\]

Since the groups \( D_{\infty, w} / I_{\infty, w} \) and \( D_{H \cap U, \tilde{u}} / I_{H \cap U, \tilde{u}} \) are procyclic group and thus have cohomological dimension one, the left vertical homomorphism and the right vertical homomorphism of the diagram are surjective. Since the group \( E^{A^*_v}_H(U, \tilde{u}) \) is the kernel of the lower horizontal homomorphism of the diagram, we have a map

\[
\text{Ker}[H^1(D_{H \cap U, \tilde{u}}, A^*_w(1) / (F_v^+ A^*_w(1))) \rightarrow H^1(D_{\infty, w}, A^*_w(1) / (F_v^+ A^*_w(1)))] \rightarrow E^{A^*_v}_H(U, \tilde{u}), \tag{42}
\]

whose cokernel is a quotient of the kernel of the right vertical homomorphism of (41) by applying the snake lemma to the diagram (41). We note that the kernel of the right vertical homomorphism of (41) is a finite group for any prime \( w \) of \( K_{\infty} \) dividing \( p \) if the condition \((A_{\tilde{u}})\) is satisfied. By the Inflation-Restriction sequence, we have

\[
H^1(D_{H \cap U, \tilde{u}} / D_{\infty, w}, (A^*_w(1) / (F_v^+ A^*_w(1)))^{G_{K_{\infty, w}}) \equiv \text{Ker}[H^1(D_{H \cap U, \tilde{u}}, A^*_w(1) / (F_v^+ A^*_w(1))) \rightarrow H^1(D_{\infty, w}, A^*_w(1) / (F_v^+ A^*_w(1))]. \tag{43}
\]

We note that this group is a finite group for any prime \( v \) of \( K \) dividing \( p \) and for any prime \( \tilde{u} \) (resp. \( u \)) of \( K_{U \cup \text{cyc}} \) (resp. \( K_{\infty} \)) above \( v \) if the condition \((A_{\tilde{u}})\) is satisfied. Let \( v \) be a prime of \( K \) dividing \( p \). Then, this group for \( \tilde{u} \) and \( w \) dividing \( v \) and group for \( \tilde{w} \) and \( w' \) dividing \( v \) are isomorphic to each other. This group is finite and uniformly bounded independently of \( U, \tilde{u} \) and \( w \) if the condition \((A_{\tilde{u}})\) is satisfied. By applying these facts to (42), we conclude that

\[
(E^{A^*_v}_H(U, \tilde{u}))_{\text{null}} \text{ is finite and uniformly bounded independently of } U, \tilde{u} \text{ and } w \tag{44}
\]

if the condition \((A_{\tilde{u}})\) is satisfied.
As in the previous case, using induced modules, we can deduce from (44) that

\[
\lim_{\lambda \to \Lambda} \left( \bigoplus_{\Lambda \in \Lambda \cap \Lambda_p} (E^{A^*(1)}_{H \cap U})^{\vee}_{\text{null}} \right) \cong \bigoplus_{v} \Lambda_{\Lambda} (G) \otimes \Lambda_{\Lambda} (G_v) \cdot M_v
\]

where \( v \) runs through primes of \( K \) dividing \( p \) and \( M_v \) is a \( \Lambda_{\Lambda} (G_v) \)-module of finite cardinality for each \( v \). Once again, since \( \Lambda (G) \otimes \Lambda (G_v) \) is a well-defined map from \( M_{H_v}(G_v) \to M_H(G) \) for \( v | p \), it follows Proposition 4.2(a) that

\[
\left[ \lim_{\lambda \to \Lambda} \left( \bigoplus_{\Lambda \in \Lambda \cap \Lambda_p} (E^{A^*(1)}_{H \cap U})^{\vee}_{\text{null}} \right) \right] = 0 \text{ in } K_0(M_H(G)). \tag{45}
\]

When the condition \((A_p)\) is not satisfied, \((E^{A^*(1)}_{H \cap U})^{\vee}_{\text{null}}\) might not be a finite group. However, \( \left( \bigoplus_{\Lambda \in \Lambda \cap \Lambda_p} (E^{A^*(1)}_{H \cap U})^{\vee}_{\text{null}} \right) \) is finite for each \( U, \Lambda \) and \( \text{lim}_{\lambda \to \Lambda} \) finite through primes of \( K \) dividing \( p \) and \( M_v \) is a finitely generated \( \mathbb{Z}_p \)-module. Hence by Proposition 4.2 and the condition \((\text{Van}_p)\), the same conclusion as in (45) continues to hold. This completes the proof of the lemma. \( \square \)

**Proposition 4.7** Let us assume that \( \text{Sel}^{Gr}_{A^*(1)}(K_{\infty})^{\vee} \in M_H(G) \) and either the condition (a) or the condition (b) below holds.

(a) The following two conditions are satisfied simultaneously.

(i) The condition (A) in Definition 1.6 is satisfied.

(ii) For each \( n \geq 1 \), the cohomology group \( H_n(H \cap U, \text{Sel}^{Gr}_{A^*(1)}(K_{\infty})^{\vee}) \) is finite for any \( U \in \mathcal{U} \) and the order of this cohomology group is bounded independently of \( U \in \mathcal{U} \).

(b) The following two conditions are satisfied simultaneously.

(i) The assumption (Van) in Definition 1.1 holds.

(ii) For each \( n \geq 1 \), the cohomology group \( H_n(H \cap U, \text{Sel}^{Gr}_{A^*(1)}(K_{\infty})^{\vee}) \) is finite for any \( U \in \mathcal{U} \) and \( \lim_{U} H_n(H \cap U, \text{Sel}^{Gr}_{A^*(1)}(K_{\infty})^{\vee}) \) is a finitely generated \( \mathbb{Z}_p \)-module.

Then we have \([\text{Ext}^n_{\Lambda_{\Lambda} (G)}(\text{Sel}^{Gr}_{A^*(1)}(K_{\infty})^{\vee}, \Lambda_{\Lambda} (G))] = 0 \text{ in } K_0(M_H(G)) \) for any \( n \geq 3 \).

We continue to assume that \( \text{Sel}^{Gr}_{A^*(1)}(K_{\infty})^{\vee} \in M_H(G) \) and that either the condition (a) or the condition (b) above holds. In addition, we also assume either the condition \((A_p)\) of Definition 1.6 or the condition \((\text{Van}_p)\) of Definition 1.1 holds.

Then we have \([\text{Ext}^2_{\Lambda_{\Lambda} (G)}(\text{Sel}^{Gr}_{A^*(1)}(K_{\infty})^{\vee}, \Lambda_{\Lambda} (G))] = 0 \text{ in } K_0(M_H(G)) \).

**Proof** We will first prove that \([\text{Ext}^n_{\Lambda_{\Lambda} (G)}(\text{Sel}^{Gr}_{A^*(1)}(K_{\infty})^{\vee}, \Lambda_{\Lambda} (G))] = 0 \text{ in } K_0(M_H(G)) \) for any \( n \geq 3 \).

Let \( d \) be the dimension of \( G \) as a \( p \)-adic Lie group. Then the global dimension of \( \Lambda_{\Lambda} (G) \) is equal to \( d + 1 \).
Set $H_U = H \cap U$ and $\tilde{\Gamma}_U = G/H_U = \text{Gal}(K_{H_U}^\infty/K)$. There is a Grothendieck spectral sequence

$$E_2^{s,t} = \text{Ext}^s_{\mathcal{O}[[\tilde{\Gamma}_U]]}(H_t(H_U, \text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee), \mathcal{O}[[\tilde{\Gamma}_U]]) \Rightarrow E_\infty^{s,t} = \text{Ext}^{s+t}_{\mathcal{O}(G)}(\text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee, \mathcal{O}[[\tilde{\Gamma}_U]])$$

Also, for any $n$, we have

$$\text{Ext}^n_{\mathcal{O}(G)}(\text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee, \mathcal{O}[[G]]) \cong \lim_{\mathcal{U}} \text{Ext}^n_{\mathcal{O}(G)}(\text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee, \mathcal{O}[[\tilde{\Gamma}_U]])$$

(46)

Since $K_{H_U}^\infty = K_U K_{\text{cyc}}^\infty = K_U K_{\text{cyc}}$, we have $[K_{H_U}^\infty : K_{\text{cyc}}] < \infty$ for any $U$. Thus we have $[\tilde{\Gamma}_U : \Gamma_U] < \infty$ where $\Gamma_U := \text{Gal}(K_U K_{\text{cyc}}/K_U)$. By Lemma 4.5, we have,

$$\text{Ext}^n_{\mathcal{O}[[\tilde{\Gamma}_U]]}(M, \mathcal{O}[[\tilde{\Gamma}_U]]) \cong \text{Ext}^n_{\mathcal{O}[[\Gamma_U]]}(M, \mathcal{O}[[\Gamma_U]])$$

for any $\mathcal{O}[[\tilde{\Gamma}_U]]$-module $M$. We have $E_2^{s,t} = 0$ for $s \geq 3$ and for any $t$ since $\mathcal{O}[[\Gamma_U]]$ is a regular local ring of Krull dimension two. We have that $H_t(H_U, \text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee)$ is a torsion $\mathcal{O}[[\Gamma]]$-module for any $t$ thanks to [5, Lemma 3.1] and our assumption that $\text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee$ is in $\mathfrak{M}_H(G)$. Thus we obtain, $E_2^{0,t} = 0$ for any $t$. Putting these together, we have $E_2^{s,t} = 0$ unless $s = 1, 2$.

Consequently, we obtain $E_2^{1,t} = E_\infty^{1,t}$ and $E_2^{2,t} = E_\infty^{2,t}$ for any $t$. This implies the following exact sequence for any $n$:

$$0 \longrightarrow E_2^{1,n-1} \longrightarrow \text{Ext}^n_{\mathcal{O}(G)}(\text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee, \mathcal{O}[[\tilde{\Gamma}_U]]) \longrightarrow E_2^{2,n-2} \longrightarrow 0.$$  (47)

Now we assume the condition (a) of this proposition. Then $\text{Ext}^n_{\mathcal{O}(G)}(\text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee, \mathcal{O}[[\tilde{\Gamma}_U]])$ is finite and uniformly bounded independent of $U$ for any $n \geq 3$ by our assumption (a)(ii) in this Proposition 4.7. Now applying Lemma 4.2(a), from equation (46), we obtain $[\text{Ext}^n_{\mathcal{O}(G)}(\text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee, \mathcal{O}(G))] = 0$ in $K_0(\mathfrak{M}_H(G))$ for any $n \geq 3$.

On the other hand, if we have the assumption (b), then using the hypothesis (b)(ii), we get $\text{Ext}^n_{\mathcal{O}(G)}(\text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee, \mathcal{O}(G))$ is a finitely generated $\mathbb{Z}_p$-module for $n \geq 2$. Thus we are done by assumption (Van) in hypothesis (b)(i).

Thus it only remains to show $[\text{Ext}_2^2(\text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee, \mathcal{O}(G))] = 0$ in $K_0(\mathfrak{M}_H(G))$ to complete the proof of the Proposition 4.7.

By our assumption, $H_1(H_U, \text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee)$ is finite whether we are in case (a) or (b); and hence $E_2^{1,1} = 0$. On the other hand,

$$E_2^{2,0} = \text{Ext}^2_{\mathcal{O}[[\Gamma_U]]}(\text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee_{H_U}, \mathcal{O}[[\Gamma_U]]) \cong (\text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee_{H_U})_{\text{null}}.$$  (48)

the maximal $\mathcal{O}[[\Gamma_U]]$ pseudo-null (i.e. finite) submodule of $\text{Sel}_{A^*((1))}^\text{Gr}(K_\infty)^\vee_{H_U}$. 
By (46), (47) and (48), we obtain

\[ \text{Ext}^2_{\Lambda_\infty(G)}(\text{Sel}_{A^*(1)}^\text{Gr}(K_\infty)^\vee, \Lambda_\infty(G)) = [\lim_{U} (\text{Sel}_{A^*(1)}^\text{Gr}(K_\infty)_{H_U})_{\text{null}}]. \]

We fix an open subgroup \( U \subset G \) as above for the moment. Then, we have a natural restriction map:

\[ \text{Sel}_{A^*(1)}^\text{Gr}(K_U K_{\text{cyc}}) \longrightarrow \text{Sel}_{A^*(1)}^\text{Gr}(K_\infty)^{H \cap U}. \]  

(49)

Further, by applying the Pontryagin dual functor to (49), we get the map \( \phi_{U, \text{cyc}} \)

\[ \text{Sel}_{A^*(1)}^\text{Gr}(K_\infty)^{H_U} \phi_{U, \text{cyc}} \longrightarrow \text{Sel}_{A^*(1)}^\text{Gr}(K_U K_{\text{cyc}})^{\vee}. \]  

(50)

Next, using (50), it suffices to show

\[ [\lim_{U} \text{Ker}(\phi_{U, \text{cyc}})]_{\text{null}} = 0 \]  

and

\[ [\lim_{U} \text{Sel}_{A^*(1)}^\text{Gr}(K_U K_{\text{cyc}})]_{\text{null}} = 0 \]  

in \( K_0(\mathbb{M}_H(G)). \)  

(51)

Taking direct limit of the natural restriction map in the diagram (7) over all open normal subgroups \( W \) of \( G \) with \( H \cap U \subset W \subset U \) and then using a snake lemma on that diagram, we obtain an exact sequence

\[ H^2(H \cap U, A^{G_{K_\infty}})^{\vee} \longrightarrow \text{Ker}(\phi_{U, \text{cyc}}) \longrightarrow \lim_{W} \text{res}_{W}^{''} \]  

(52)

where \( \text{res}_{W}^{''} := \text{res}_{W}^{'''} \) is the local restriction map defined in diagram (7) with trivial \( \rho \).

We now assume the hypothesis (a) in this proposition. In addition, we also assume either \( (A_p) \) or \( (\operatorname{Van}_p) \) holds.

First we consider \( \text{Sel}_{A^*(1)}^\text{Gr}(K_U K_{\text{cyc}})_{\text{null}} \). By a result of [22],

\[ \text{Sel}_{A^*(1)}^\text{Gr}(K_U K_{\text{cyc}})_{\text{null}} \cong \lim_n \text{Ker}(\text{Sel}_{A^*(1)}^\text{Gr}((K_U K_{\text{cyc}})_{\Gamma_U^n}) \longrightarrow \text{Sel}_{A^*(1)}^\text{Gr}(K_U K_{\text{cyc}})_{\Gamma_U^n}). \]

Then by the proof of cyclotomic control theorem (see for example [27]), the cardinality of the finite group \( \text{Sel}_{A^*(1)}^\text{Gr}(K_U K_{\text{cyc}})_{\text{null}} \) is uniformly bounded independently of \( U \) by the cardinality of the module \( (A^*(1))^{G_{K_\infty}} \), which is finite by the hypothesis (A) in condition (a)(i) of this proposition. Also, thanks to the hypothesis (A), the cardinality of \( H^2(H \cap U, A^{G_{K_\infty}}) \) is finite and uniformly bounded independently of \( U \) following the proof of Theorem 1.

Using this in equation (51), it remains to establish \( \lim_{U} (\lim_{W} \text{res}_{W}^{''} \) is uniformly bounded independently of \( U \) to complete the proof of the proposition in this case. Recall the notation \( \left( \bigoplus_{\tilde{u} \in P_{H \cap U}} E_{H \cap U, \tilde{u}}^{A^*(1)} \right)_{\text{null}} \) from Lemma 4.6.
Now we further assume condition \((A_p)\) holds. Notice that \(\lim_U \left( \lim_W \text{res}''^W \right)\) differs from \(\lim_U \left( \bigoplus_{\tilde{a} \in P_{H \cap U}} E_{H \cap U, \tilde{a}}^{A^*_+(1)} \right)\) by a finite module whose order is uniformly bounded independent of \(U\). Then applying Lemma 4.6, it follows that \(\left[ \lim_U \left( \bigoplus_{\tilde{a} \in P_{H \cap U}} E_{H \cap U, \tilde{a}}^{A^*_+(1)} \right) \right] = 0\) in \(K_0(\mathcal{M}_H(G))\).

The proof for the case with the assumption \((b)\) works similarly with suitable modifications as follows:

In this case, we will get \(\lim_U \text{Sel}^{G_\infty}_{\Lambda_\infty}(K_U K_{\text{cyc}})^\vee\) and \(\lim_U H^2(H \cap U, A^{G_K})^\vee\) are finitely generated \(\mathbb{Z}_p\)-modules. Thus their class will vanish in \(K_0\) group by the hypothesis \((\text{Van})\) in condition \((b)(i)\). Finally, thanks to the additional hypothesis \((\text{Van}_p)\), in this case by applying Lemma 4.6, we again deduce \(\left[ \lim_U \left( \lim_W E_{W}^{A^*_+(1)} \right) \right] = 0\) in \(K_0(\mathcal{M}_H(G))\).

This shows that \([\text{Ext}^2_{\Lambda_\infty(G)}(\text{Sel}^G_{\Lambda_\infty}(K_\infty)^\vee, \Lambda_\infty(G))] = 0\) and completes the proof of the proposition. \(\square\)

**Example 4.8** We discuss the hypotheses \((\text{Van})\) for \(K_0(\mathcal{M}_H(G))\) and \((\text{Van}_p)\) for \(K_0(\mathcal{M}_{H_v}(G_v))\) of Definition 1.1. Recall that \(G = \text{Gal}(K_\infty/K), H = \text{Gal}(K_\infty/K_{\text{cyc}})\) and for a prime \(p, G_v\) is the decomposition subgroup of \(G\) at \(p\) and \(H_v := H \cap G_v\).

1. For the false-Tate curve extension in Example 1.7, \([\mathbb{Z}_p] \neq 0\) in \(K_0(\mathcal{M}_H(G))\) (\([34]\)).

2. Let \(G = H \times \Gamma\) be a \(p\)-adic Lie group with \(H\) as in Proposition 4.2(b). Then we have \([\mathbb{Z}_p] = 0\) in \(K_0(\mathcal{M}_H(G))\). We give a brief explanation of this. By our choice of \(\hat{H}\) from Proposition 4.2(b), we have \([\mathbb{F}_p] = 0\) in \(K_0(\Omega(H))\). Note that there is an isomorphism between \(K_0(\Lambda(H))\) and \(K_0(\Omega(H))\) [5, Lemma 4.1] via the natural map sending \([M]\) to \([M/pM]\). Thus we deduce that \([\mathbb{Z}_p] = 0\) in \(K_0(\Lambda(H))\). As \(H\) has no element of order \(p\), \(\Lambda(H)\) has finite global dimension. Thus any finitely generated \(\Lambda(H)\)-module defines a class in \(K_0(\Lambda(H))\). As \(G = H \times \Gamma\), any finitely generated \(\Lambda(H)\)-module belongs to the category \(\mathcal{M}_H(G)\). Hence there is a natural map from \(K_0(\Lambda(H))\) to \(K_0(\mathcal{M}_H(G))\) sending the class of a finitely generated \(\Lambda(H)\)-module \(M\) to \([M]\). Using the composite map from \(K_0(\Omega(H))\) to \(K_0(\mathcal{M}_H(G))\) sending \([\mathbb{F}_p]\) to \([\mathbb{Z}_p]\), the result follows.

3. Let \(G = \text{GL}_2(\mathbb{Z}_p)\). Set \(H = \text{SL}_2(\mathbb{Z}_p) := \{g \in \text{GL}_2(\mathbb{Z}_p) \mid \det(g)^{p-1} = 1\}\). Then we have \([\mathbb{Z}_p] = 0\) in \(K_0(\mathcal{M}_H(G))\). We follow the proof of [35, Proposition 4.2] and provide some details. Note that we have the homomorphism \(\text{GL}_2(\mathbb{Z}_p) \to \mathbb{Z}_p\) where \(\theta(g) = \det(g)^{p-1}\). Then ker(\(\theta\)) = \(\text{SL}_2(\mathbb{Z}_p)\). As \(p\) is odd, any \(\lambda \in 1 + p\mathbb{Z}_p\) has a unique \(2(p - 1)\)-th root \(\frac{1}{2(p - 1)}\sqrt[p-1]{\lambda}\) exists in \(1 + p\mathbb{Z}_p\). Now we have a section of \(\theta\) given by \(1 + p\mathbb{Z}_p \to G\) where \(\psi(\lambda) = \begin{pmatrix} p - 1 & 0 \\ 0 & p - 1 \end{pmatrix}\). Using this we also see that \(\theta\) is surjective and \(G/H = \)
GL₂(ℤ_p)/SL₂(ℤ_p) ≅ 1 + pℤ_p. Note that by definition of ψ, Γ := ψ(1 + pℤ_p)

is a normal subgroup of GL₂(ℤ_p) with Γ ∩ SL₂(ℤ_p) = I₂. Thus we have shown that GL₂(ℤ_p) = \{g ∈ GL₂(ℤ_p) | det(g)^{p−1} = 1\} × Γ ≅ SL₂(ℤ_p) × 1 + pℤ_p.

It is a general fact that for any p-adic Lie group H without an element of order p, the dimension of the centralizer (as a p-adic Lie group) of an element x ∈ H is the same as the ℚ_p-vector space dimension of \{g ∈ Lie(H) | Ad(x)(g) = g\} [1, Proof of Lemma 8.6], (cf. loc. cit. for the definition of Lie(H) and Ad(−)).

We use this fact for SL₂(ℤ_p) and keep in mind that SL₂(ℤ_p) is a finite index subgroup of SL₂(ℤ_p). By Lemma 4.1 and Corollary 4.3, we deduce [ℤ_p] = 0 in K₀(Ω(SL₂(ℤ_p))). Finally using Example 4.8(2), we deduce that [ℤ_p] = 0 in K₀(M₁(SL₂(ℤ_p))(GL₂(ℤ_p))).

(4) Let us consider the case in Example 1.8, Case (O) with E a non-CM elliptic curve isogenous to B with good ordinary reduction at all primes above p in K; for the extension K∞ = K(E_p∞) of a number field K with G = Gal(K∞/K) and H := Gal(K(E_p∞)/Kcyc), the assumption (Van) is satisfied in K₀(M₁(H(G))).

In this case, by Serre’s open image theorem G is of finite index in GL₂(ℤ_p) and also H has a finite index subgroup which is of finite index in SL₂(ℤ_p). Thus for any module M ∈ M₁(SL₂(ℤ_p))(GL₂(ℤ_p)), the map K₀(M₁(SL₂(ℤ_p))(GL₂(ℤ_p))) → K₀(M₁(H(G))) sending [M] → [M] is well-defined. Taking M = ℤ_p, the assumption (Van) is satisfied in K₀(M₁(H(G))) from the discussion in Example 4.8(3).

(5) Let us again consider the case in Example 1.8, Case (O) with E a non-CM elliptic curve isogenous to B with good ordinary reduction at all primes above p in K. We have the extension K∞ = K(E_p∞) of a number field K with G = Gal(K∞/K) and H := Gal(K(E_p∞)/Kcyc). For any prime v in K above p let G_v be the decomposition subgroup of G at v and set H_v := H ∩ G_v. Then [ℤ_p] = 0 in K₀(M₁(H_v(G_v))) for every prime v in K above p. This can be explained by a suitable modification of the Example 4.8(3) as follows. As E has good ordinary reduction at v by Serre’s work (see for example [7, Page 196]) G_v is a 3-dimensional p-adic Lie group which is a finite index subgroup of the Borel subgroup B of all upper triangular matrices in GL₂(ℤ_p). Thus restricting the map θ in Example 4.8(3) to B and noting that determinant of a matrix in B is given by the cyclotomic character, we see θ|B is still surjective on 1 + pℤ_p. Moreover, the section ψ : 1 + pℤ_p → B works as B contains all diagonal matrices. As G_v and H_v are respectively finite index subgroups of B and \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} ∈ GL₂(ℤ_p) \mid det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{p−1} = 1\}, we deduce [ℤ_p] = 0 in K₀(M₁(H_v(G_v))) following the discussions in Example 4.8(3) and Example 4.8(4).

In this case, we further deduce [Λ(G) ⊗_Λ(G_v) ℤ_p] = 0 in K₀(M₁(H(G))) for every prime v in K above p as Λ(G) ⊗_Λ(G_v) − is a well-defined map from M₁(H_v(G_v)) → M₁(H(G)). Thus the statement follows from the above discussion.
Remark 4.9 Proposition 4.7 is a generalization of [3, Proposition 3.6]. We explain our natural motivation for the assumption in this proposition which is not mentioned in the existing literature. See also Remark 4.11.

Let $A = E_{p' \infty}$, where $E$ is an elliptic curve over $\mathbb{Q}$ with good ordinary reduction at $p$. The condition that $H_i(H \cap U, \text{Sel}_{A^+}^\text{Gr}(K_\infty)^\vee)$ is finite for any $U \in \mathcal{U}$ reflects the fact that conjectural $p$-adic $L$-function has no poles.

More precisely, let $\mathcal{L}_p(V_pE)$ be the conjectural $p$-adic $L$-function of $E$ over $K_\infty$ (see “Appendix A” and Sect. 6). Then one would like to ensure the evaluation of the $p$-adic $L$-function at any Artin representation $\eta$ of $\text{Gal}(K_\infty/\mathbb{Q})$ has the property $\eta(\mathcal{L}_p(V_pE)) \neq \infty$.

Assume that $\text{Sel}_{A^+}^\text{Gr}(K_\infty)^\vee \in \mathcal{M}_H(G)$ and let $\xi_{\text{Sel}_{E_{p' \infty}}^\text{Gr}(K_\infty)^\vee} \in K_1(\Lambda(\mathcal{O}(G)_{S^\vee})$ be the pre-image of $[\text{Sel}_{A^+}^\text{Gr}(K_\infty)^\vee] \in K_0(\mathcal{M}_H(G))$ under the natural connecting map $K_1(\Lambda(\mathcal{O}(G)_{S^\vee}) \rightarrow K_0(\mathcal{M}_H(G)) \rightarrow 0.$ (see Sect. 6). Then $H_i(H \cap U, \text{Sel}_{A^+}^\text{Gr}(K_\infty)^\vee)$ is finite for any $U \in \mathcal{U}$ ensures $\eta(\xi_{\text{Sel}_{E_{p' \infty}}^\text{Gr}(K_\infty)^\vee}) \neq \infty$ for any Artin representation $\rho$ as above [5, Theorem 3.8]. By Iwasawa Main conjecture, (see Conjecture A.4) $\delta(\mathcal{L}_p(V_pE)) = [\text{Sel}_{A^+}^\text{Gr}(K_\infty)^\vee]$ in $K_0(\mathcal{M}_H(G))$. Thus our hypothesis will conjecturally ensure that $\eta(\mathcal{L}_p(V_pE)) \neq \infty$, as claimed.

Example 4.10 We continue here with the set up and notation of Example 1.7 related to the false-Tate curve extension. Let $A$ be as given in Case (1), (2) or (3) of Example 1.5. If $\text{Sel}_{A}^\text{Gr}(\mathbb{Q}(\mu_{p' \infty}))^\vee$ is a (finitely generated) torsion $\Lambda(1)^\vee$-module then $\text{Sel}_{A}^\text{Gr}(K_\infty)^\vee$ is a (finitely generated) torsion $\Lambda(G)$-module (cf. [24, second proof of Theorem 2.8]). The proof of [8, Lemma 2.5] easily extends in all of our cases and we deduce $H_i(H \cap U, \text{Sel}_{A}^\text{Gr}(K_\infty)^\vee) = 0$. Also as $H$ has $p$-cohomological dimension $= 1$, $H_i(H \cap U, -)$ also vanish for $i \geq 2$.

Thus whenever $\text{Sel}_{A}^\text{Gr}(K_\infty)^\vee \in \mathcal{M}_H(G)$, then all the conditions of Proposition 4.7(a) are satisfied and we deduce $[\text{Ext}^i_{\Lambda(G)}(\text{Sel}_{A^+}^\text{Gr}(K_\infty)^\vee, \Lambda(G))] = 0$ in $K_0(\mathcal{M}_H(G))$ for every $i \geq 2$.

Moreover, if we assume $\text{Sel}_{A}^\text{Gr}(\mathbb{Q}(\mu_{p' \infty}))^\vee$ is a finitely generated $\mathbb{Z}_p$-module, then $\text{Sel}_{A}^\text{Gr}(K_\infty)^\vee$ is finitely generated over $\Lambda(H)$ and in particular $\in \mathcal{M}_H(G)$ (cf. [24, Theorem 3.1(i)]).

Remark 4.11 In the false-Tate situation of Example 4.10, if we assume $\text{Sel}_{A}^\text{Gr}(K_\infty)^\vee$ has no non-zero $\Lambda(G)$ pseudo-null submodule, we have $[\text{Ext}^i_{\Lambda(G)}(\text{Sel}_{A^+}^\text{Gr}(K_\infty)^\vee, \Lambda(G))] = 0$ in $K_0(\mathcal{M}_H(G))$ even without invoking Proposition 4.7.

To prove this, we set $E^i(-) := \text{Ext}^i_{\Lambda(G)}(-, \Lambda(G))$.

1. We have $E^i(-) = 0$ for every $i > 3$ since the $p$-cohomological dimension of $G$ is 2.

Also, the following results are known by [32]:

2. We have $E^i(E^i(-)) = 0$ when $j < i$ ([32, Proposition 3.5(iii)(a))]

3. For a $\Lambda(G)$-module $N$ with no non-zero pseudo-null submodule, we have $E^i(E^i(N)) = 0$ for every $i \geq 2$ ([32, Proposition 3.5(i)(c)]).
Combining these three facts above, for a $\Lambda(G)$-module $N$ with no non-zero pseudo-null submodule, we have $E^i(E^3(N)) = 0$ for every $i \geq 0$ and $E^i(E^2(N)) = 0$ for every $i \neq 3$. Hence we deduce $E^3(N)$ and $E^2(N)$ are both finite [30, Proposition 2.6(5)]. Consequently, we have $[E^2(N)] = 0$ and $[E^3(N)] = 0$ in $K_0(\mathcal{M}_H(G))$ by Lemma 4.2(a).

Thus, whenever the maximal pseudo-null submodule of the $\Lambda(G)$-module $\text{Sel}_{A}^{G}(K)\vee$ is trivial, we can apply the above argument to $N = \text{Sel}_{A}^{G}(K)\vee$ to obtain

$$[\text{Ext}_{\Lambda(G)}^i(\text{Sel}_{A}^{G}(K)\vee, \Lambda(G))] = 0 \text{ in } K_0(\mathcal{M}_H(G)) \text{ for every } i \geq 2.$$ 

We remark that [24, Theorem 2.6(ii), Theorem 3.1(ii)] discusses a variety of assumptions under which maximal pseudo-null submodule of the $\Lambda(G)$-module $\text{Sel}_{A}^{G}(K)\vee$ is trivial.

However, it seems that this argument based on the non-existence of pseudo-null submodules will work only in such special $p$-adic Lie extensions like false-Tate curve extension and it can not be generalized to higher Ext groups of the Selmer group associated to a general $p$-adic Lie group $G$ with a large $p$-cohomological dimension. In a general situation, we need Proposition 4.7.

**Example 4.12** Next we discuss the vanishing of higher ext groups for the Selmer groups considered in Example 1.8. We carry the same setting from Example 1.8. Let us fix $p \geq 5$ and assume we are in the Case (O) and take $E = B$, a non-CM elliptic curve over $\mathbb{Q}$ with good ordinary reduction at $p$. Here $A = E_{p\infty}$. When $\text{Sel}_{A}^{G}(K)\vee$ is torsion over $\Lambda(G)$, it follows that $H_i(H \cap U, \text{Sel}_{A}^{G}(K)\vee) = 0$ for every $i \geq 1$ [8, Lemma 2.5 and Remark 2.6]. Then all the conditions of Proposition 4.7(b) are satisfied and we deduce $[\text{Ext}_{\Lambda(G)}^i(\text{Sel}_{A}^{G}(K)\vee, \Lambda(G))] = 0 \text{ in } K_0(\mathcal{M}_H(G)) \text{ for every } i \geq 2$.

In this setting, it is conjectured in [5, Conjecture 5.1] that $\text{Sel}_{A}^{G}(K)\vee \in \mathcal{M}_H(G)$ and in particular it is a finitely generated torsion $\Lambda(G)$-module.

Whenever $\text{Sel}_{A}^{G}(K)\vee$ is a finitely generated $\mathbb{Z}_p$-module, then it follows that $\text{Sel}_{A}^{G}(K)\vee$ is finitely generated over $\Lambda(H)$ and in particular $\in \mathcal{M}_H(G)$ [5, Proposition 5.6].

Note that, even in Case (S), the proof of [5, Proposition 5.6] extends and we can deduce $\text{Sel}_{A}^{G}(K)\vee \in \mathcal{M}_H(G)$ if $\text{Sel}_{A}^{G}(K)\vee$ is finitely generated over $\mathbb{Z}_p$.

**Example 4.13** Let us keep the same setting as Example 1.9 where $G$ is isomorphic to $\mathbb{Z}_p^{d+1}$ and $A$ is isomorphic to $E_{p\infty}$ for $E$ an elliptic curve with good ordinary reduction at all primes of $K$ above $p$.

For any $M \in \mathcal{M}_H(G)$, $\text{Ext}_{\Lambda(G)}^i(M, \Lambda(G))$ is a pseudonull $\Lambda(G)$-module for every $i \geq 2$ (see [32, §3]). Then, since $\Lambda(G)$ is commutative and $\text{Ext}_{\Lambda(G)}^i(M, \Lambda(G))$ is a pseudonull $\Lambda(G)$-module, we have $[\text{Ext}_{\Lambda(G)}^i(M, \Lambda(G))] = 0 \text{ in } K_0(\mathcal{M}_H(G))$ for every $i \geq 2$. Recall that we have $A \cong A^\vee(1)$ and hence $\text{Sel}_{A}^{G}(K)\vee \cong \text{Sel}_{A}^{G}(K)\vee$. Thus, if we know $\text{Sel}_{A}^{G}(K)\vee \in \mathcal{M}_H(G)$, then we deduce $[\text{Ext}_{\Lambda(G)}^i(\text{Sel}_{A}^{G}(K)\vee, \Lambda(G))] = 0 \text{ in } K_0(\mathcal{M}_H(G)) \text{ for every } i \geq 2$. 

By the argument of the proof of Lemma 4.6, we can show that the kernel of the map

$$(\text{Sel}_A^{\text{Gr}}(K_{\infty})^\vee)_H \rightarrow \text{Sel}_A^{\text{Gr}}(K_{\text{cyc}})^\vee$$

is a finite generated $\mathbb{Z}_p$-module. Then, by Nakayama’s lemma, we have $\text{Sel}_A^{\text{Gr}}(K_{\infty})^\vee \in \mathcal{M}_H(G)$ if $\text{Sel}_A^{\text{Gr}}(K_{\text{cyc}})^\vee$ is a finitely generated $\mathbb{Z}_p$-module.

### 5 Proof of the algebraic functional equation (Theorem 0.3)

In this section, we will prove the functional equation of the Selmer group. We will use the control theorem and the vanishing of the higher extension groups of the Selmer group as discussed respectively in Sects. 2 and 4. We first discuss a few results which will go into the proof of Theorem 0.3.

Recall that $\mathcal{M}_H(G)$ is the category of finitely generated $\Lambda_\mathcal{O}(G)$-module such that $\mathcal{M}_{p^\infty}$ is finitely generated over $\Lambda_\mathcal{O}(H)$. Since $\text{Sel}_{A_{\rho}}^{\text{BK}}(K_U)$ is a subgroup of $\text{Sel}_A^{\text{Gr}}(K_U)$, we define a module $C_{A_{\rho}}$ as follows:

$$C_{A_{\rho}} = \text{Coker} \left[ \text{Sel}_{A_{\rho}}^{\text{BK}}(K_U) \rightarrow \text{Sel}_A^{\text{Gr}}(K_U) \right].$$  \hspace{1cm} (53)

**Lemma 5.1** Assume either the condition $(A_{\rho})$ of Definition 1.6 or the condition $(\text{Van}_p)$ of Definition 1.1. Then, we have $\left[ (\lim_{U} C_{A_{\rho}}^U)^\vee \right] = 0$ and $\left[ \lim_{U} C_{A_{\rho}}^U \right] = 0$ in $K_0(\mathcal{M}_H(G))$.

**Proof** The comparison between the Selmer group of Bloch–Kato and the Selmer group of Greenberg is known by [12, Theorem 3] and [26, Proposition 4.2]. Recall the sequence

$$0 \rightarrow \text{Sel}_{A_{\rho}}^{\text{BK}}(K_U) \rightarrow \text{Sel}_A^{\text{Gr}}(K_U) \rightarrow C_{A_{\rho}}^U \rightarrow 0.$$

By the proof of [26, Proposition 4.2], $C_{A_{\rho}}^U$ is a finite group which is subquotient of

$$\bigoplus_{u|p} \left( ((A_{\rho}(1))^{D_u})^\vee \oplus (A_{\rho}/F_u^{+}A_{\rho})^{D_u} \oplus (F_u^{+}A_{\rho}(1))^{D_u} \right)$$

$$\oplus \bigoplus_{u \nmid p} \left( ((A_{\rho}(1))^{I_u}/D_u) \oplus (((A_{\rho}(1))^{I_u})^{\text{div}}) \right)$$  \hspace{1cm} (54)

where $u$ runs through the set of primes of $K_U$ over $p$ on the first line and $u$ runs through the set of primes of $K_U$ away from $p$ on the second line. (54) is reformulated as follows
\[
\bigoplus_{v \mid p} \text{Ind}_{G \rightarrow U}^{G \times U} \left\{ \left( (A_{\rho}^p(1)D_{u(v)})^{\nu} \right) \oplus \left( (A_{\rho}/F_{w(v)}^+ A_{\rho})D_{u(v)} \right) \oplus \left( (F_{w(v)}^+ A_{\rho}^s(1)D_{u(v)}) \right) \right\} \\
\bigoplus_{v \mid p} \text{Ind}_{G \rightarrow U}^{G \times U} \left\{ \left( (A_{\rho}^p(1)I_{\text{u}(v)})D_{w(v)} \right) / \left( (A_{\rho}^s(1)I_{\text{u}(v)})D_{w(v)_{\text{div}}} \right) \right\} \quad (55)
\]

where \( v \) runs through the set of primes of \( K \) over \( p \) on the first line and \( v \) runs through the set of primes of \( K \) away from \( p \) on the second line. For each \( v \), we choose a prime \( u(v) \) of \( K_u \) which is over \( v \) and we denote by \( G_v, U \) the image of the decomposition group \( G_v \subset G \) into \( G/U \). The term (55) is independent of the choices of \( u(v) \)’s. Hence, \( (C_{A_{\rho}})_{\text{U}} := (\text{lim}_{U} C_{A_{\rho}}^{I})^{\nu} \) is a subquotient of the module the Pontryagin dual of the injective limit of (55) with respect to open normal subgroups \( U \) of \( G \). Note that the injective limit of \( (A_{\rho}^p(1))D_{u(v)}^{\nu} \) is the Pontryagin dual of the inverse limit of \( (A_{\rho}^p(1))D_{u(v)} \), which is trivial since the order of the group \( (A_{\rho}^p(1))D_{u(v)} \) is finite and uniformly bounded by the assumption \( (A_{\rho}) \).

Hence the module the Pontryagin dual of the injective limit of (55) with respect to open normal subgroups \( U \) of \( G \) is as follows

\[
\bigoplus_{v \mid p} \text{Ind}_{G \rightarrow U}^{G \times U} \left\{ \left( (A_{\rho}/F_{w(v)}^+ A_{\rho})D_{u(v)} \right) \oplus \left( (F_{w(v)}^+ A_{\rho}^s(1)D_{u(v)}) \right) \right\} \\
\bigoplus_{v \mid p} \text{Ind}_{G \rightarrow U}^{G \times U} \left\{ \left( (A_{\rho}^p(1)I_{\text{u}(v)})D_{w(v)} \right) / \left( (A_{\rho}^s(1)I_{\text{u}(v)})D_{w(v)_{\text{div}}} \right) \right\} \quad (56)
\]

where \( v \) runs through the set of primes of \( K \) over \( p \) on the first factor and \( v \) runs through the set of primes of \( K \) away from \( p \) on the second factor. Inside the sums, \( w = w(v) \) is a prime of \( K_{\infty} \) which is over \( v \) for each \( v \). Note the the groups does not depend on the choices of \( w = w(v) \).

By the condition \( (A_{\rho}) \), the group \( (A_{\rho}/F_{w(v)}^+ A_{\rho})D_{u(v)} \oplus (F_{w(v)}^+ A_{\rho}^s(1))D_{u(v)} \) is finite for each place \( v \mid p \) of \( K \). By applying Lemma 4.2(a) to this \( G_v \)-module of finite cardinality, we obtain

\[
\left[ \left( (A_{\rho}/F_{w(v)}^+ A_{\rho})D_{u(v)} \oplus (F_{w(v)}^+ A_{\rho}^s(1))D_{u(v)} \right) \right]^{\nu} = 0
\]

in \( K_0(\text{M}_{H_v}(G_v)) \). Hence we have

\[
\left[ \left( \bigoplus_{v \mid p} (G_v \rightarrow U) \left\{ \left( (A_{\rho}/F_{w(v)}^+ A_{\rho})D_{u(v)} \right) \oplus \left( (F_{w(v)}^+ A_{\rho}^s(1))D_{u(v)} \right) \right\} \right]^{\nu} = 0
\]

in \( K_0(\text{M}_H(G)) \). It is not difficult to see that the group \( (A_{\rho}^s(1)I_{w(v)})D_{w(v)} / ((A_{\rho}^s(1)I_{w(v)})D_{w(v)}_{\text{div}} \) is finite for each place \( v \nmid p \) of \( K \) and the group is trivial if the action of \( I_v \) on \( A_{\rho} \) is trivial. Hence, by using Lemma 4.2(a) to these \( G_v \)-modules of finite cardinality, we obtain

\[
\left[ \left( \bigoplus_{v \mid p} (G_v \rightarrow U) \left\{ \left( (A_{\rho}^s(1)I_{w(v)})D_{w(v)} / ((A_{\rho}^s(1)I_{w(v)})D_{w(v)_{\text{div}}} \right) \right\} \right]^{\nu} = 0
\]

in \( K_0(\text{M}_H(G)) \). This complete the proof for \( \left( \text{lim}_{U} C_{A_{\rho}}^{I} \right)^{\nu} = 0 \).
As for \( \lim C_{U}^{A_{\rho}} \), it is isomorphic to \( \left( \lim_{U} \left( C_{U}^{A_{\rho}} \right)^{\vee} \right)^{\vee} \). Note that the module \( (C_{U}^{A_{\rho}})^{\vee} \) is a subquotient of

\[
\bigoplus_{v \in \mathbb{P}} \operatorname{Ind}_{G_{v}/U}^{G} \left( (A_{\rho}^{+}(1))^{D_{w(v)}} \right) \oplus \left( (A_{\rho}/F_{u(v)}A_{\rho})^{D_{w(v)}} \right)^{\vee} \oplus \left( (F_{u(v)}^{-1}A_{\rho}^{+}(1))^{D_{w(v)}} \right)^{\vee}
\]

\[ \oplus \bigoplus_{v \in \mathbb{P}} \operatorname{Ind}_{G_{v}/U}^{G} \left( ((A_{\rho}^{+}(1))^{{I}_{w(v)}}D_{u(v)})/(((A_{\rho}^{+}(1))^{{I}_{w(v)}}D_{u(v)})_{\text{div}} \right)^{\vee} \right\}. \tag{57}

Under the assumption \( (A_{\rho}) \), a similar discussion as the discussion given before (56) implies that \( \lim_{U} C_{U}^{A_{\rho}} \) is a subquotient of

\[
\left( \bigoplus_{v \in \mathbb{P}} \operatorname{Ind}_{G}^{G_{v}} \left( (A_{\rho}^{+}(1))^{D_{w(v)}} \right) \right)^{\vee}, \tag{58}
\]

where \( v \) runs through the set of primes of \( K \) over \( p \) and \( w = w(v) \) is a prime of \( K_{\infty} \) which is over \( v \) for each \( v \). Since \( (A_{\rho}^{+}(1))^{D_{w(v)}} \) is finite for each \( v \) by the assumption \( (A_{\rho}) \), the same discussion as the case of \( \left[ \lim_{U} C_{U}^{A_{\rho}} \right]^{\vee} \) implies that

\[
\left[ \left( \bigoplus_{v \in \mathbb{P}} \operatorname{Ind}_{G}^{G_{v}} \left( (A_{\rho}^{+}(1))^{D_{w(v)}} \right) \right)^{\vee} \right] = 0
\]

in \( K_{0}(M_{H}(G)) \). This completes the proof for the second statement \( \left[ \lim_{U} C_{U}^{A_{\rho}} \right] = 0 \) in \( K_{0}(M_{H}(G)) \) under the condition \( (A_{\rho}) \).

Finally we remark that, when \( (A_{\rho}) \) does not hold, the groups \( (A_{\rho}^{+}(1))^{D_{w(v)}} \), \( (A_{\rho}/F_{u(v)}A_{\rho})^{D_{w(v)}} \), \( (F_{u(v)}^{-1}A_{\rho}^{+}(1))^{D_{w(v)}} \) for primes of \( K \) over \( p \) and the groups \( ((A_{\rho}^{+}(1))^{{I}_{w(v)}}D_{u(v)})/(((A_{\rho}^{+}(1))^{{I}_{w(v)}}D_{u(v)})_{\text{div}} \) for primes of \( K \) away from \( p \) are a successive extension of finite groups and a copies of \( \mathbb{Q}/\mathbb{Z}_{p} \). If we assume \( (\text{Van}_{p}) \), then

\[
\left[ \left( \bigoplus_{v \in \mathbb{P}} \operatorname{Ind}_{G}^{G_{v}} \left( \mathbb{Q}/\mathbb{Z}_{p} \right) \right)^{\vee} \right] = 0 \quad \text{in} \quad K_{0}(M_{H}(G)).
\]

Hence, even in the case where \( (A_{\rho}) \) does not hold, the same argument applies as far as the assumption \( (\text{Van}_{p}) \) holds. This completes the proof. \( \square \)

Next, we recall the following theorem from [20]:

**Theorem 5.2** (Main Theorem of [20]) Let \( G \) be a compact \( p \)-adic Lie group and let \( H \) be its closed subgroup such that \( G/H \) is isomorphic to \( \Gamma \). Let \( M \) be a \( \Lambda_{\mathcal{O}}(G) \)-module which is finitely generated over \( \Lambda_{\mathcal{O}}(H) \).

Then there exists a continuous character \( \rho : G \rightarrow \mathbb{Z}_{p}^{\times} \) such that \( M(\rho)_{U} \) is finite for every open normal subgroup \( U \) of \( G \).

Next, we recall the definition of \( G \)-Euler characteristic:

**Definition 5.3** Let \( G \) be a compact \( p \)-adic Lie group without any element of order \( p \). For a finitely generated \( \Lambda_{\mathcal{O}}(G) \)-module \( M \), we say that the \( G \)-Euler characteristic of \( M \) exists if the homology groups \( H_{i}(G, M) \) are finite for every \( i \geq 0 \). When the \( G \)-Euler characteristic of \( M \) exists, then set \( \chi(G, M) := \prod_{i} (\# H_{i}(G, M))^{(-1)^{i}} \).
Extending Theorem 5.2 of [20], the following theorem was established in [21]:

**Theorem 5.4** Let $G$ be a compact $p$-adic Lie group without any element of order $p$ and let $H$ be its closed subgroup such that $G/H$ is isomorphic to $\Gamma$. Let $M$ be a $\Lambda_\mathcal{O}(G)$-module which is in $\mathfrak{M}_H(G)$ i.e. $M/M(p)$ is finitely generated over $\Lambda_\mathcal{O}(G)$. Then there exists a continuous character $\rho : \Gamma \longrightarrow \mathbb{Z}_p^\times$, such that $\chi(U, M(\rho))$ exists for every open normal subgroup $U$ of $G$.

By applying Theorem 5.4, we deduce the following corollary.

**Corollary 5.5** We assume that $\text{Sel}^{\text{Gr}}_{A^*(1)}(K_\infty)^\vee$ is in $\mathfrak{M}_H(G)$. Then there exists a continuous character $\rho : \Gamma \longrightarrow \mathbb{Z}_p^\times$, such that both $(\text{Sel}^{\text{Gr}}_{A^*(1)}(K_\infty)^\vee(\rho))_U$ and $H_1(U, \text{Sel}^{\text{Gr}}_{A^*(1)}(K_\infty)^\vee(\rho))$ are finite.

Next we prove the following lemma:

**Lemma 5.6** Let $M$ be a finitely generated $\Lambda_\mathcal{O}(G)$-module and assume that $H_1(U, M)$ is finite for an open normal subgroup $U$ of $G$. Then we have a $\mathcal{O}[G/U]$-module isomorphism $\text{Ext}^1_{\Lambda_\mathcal{O}(G)}(M, \mathcal{O}[G/U]) \cong \text{Ext}^1_{\mathcal{O}[G/U]}(M_U, \mathcal{O}[G/U])$.

**Proof** To simplify the notation through this proof, we write $\Lambda := \Lambda_\mathcal{O}(G)$. By the definition of Ext functor, $M$ fits into an exact sequence $0 \longrightarrow R \overset{\theta}{\longrightarrow} P \longrightarrow M \longrightarrow 0$ where $P$ is a projective $\Lambda_\mathcal{O}(G)$-module and $R$ is a $\Lambda_\mathcal{O}(G)$-module. This gives rise to a second exact sequence $0 \longrightarrow \frac{R_U}{\text{Image}(H_1(U, M))} \overset{\theta_U}{\longrightarrow} P_U \longrightarrow M_U \longrightarrow 0$.

By our assumption, we have $\text{Hom}_{\mathcal{O}[G/U]}(\text{Image}(H_1(U, M)), \mathcal{O}[G/U]) = 0$. Then we can compute the extension groups appearing in the statement of this lemma as follows:

$$
\text{Ext}^1_{\Lambda}(M, \mathcal{O}[G/U]) \cong \frac{\text{Hom}_{\Lambda}(R, \mathcal{O}[G/U])}{\theta([\text{Hom}_{\mathcal{O}[G/U]}(P, \mathcal{O}[G/U])])} \cong \frac{\text{Hom}_{\mathcal{O}[G/U]}(R_U, \mathcal{O}[G/U])}{\theta_U([\text{Hom}_{\mathcal{O}[G/U]}(P_U, \mathcal{O}[G/U])])},
$$

$$
\text{Ext}^1_{\mathcal{O}[G/U]}(M_U, \mathcal{O}[G/U]) \cong \frac{\text{Hom}_{\mathcal{O}[G/U]}(\frac{R_U}{\text{Image}(H_1(U, M))}, \mathcal{O}[G/U])}{\theta_U([\text{Hom}_{\mathcal{O}[G/U]}(P_U, \mathcal{O}[G/U])])} \cong \frac{\text{Hom}_{\mathcal{O}[G/U]}(R_U, \mathcal{O}[G/U])}{\theta([\text{Hom}_{\mathcal{O}[G/U]}(P_U, \mathcal{O}[G/U])])}.
$$

This completes the proof of the lemma.

Finally, we recall following well known result in homological algebra (see for example [34, Proposition 6.1] for the proof):

**Proposition 5.7** Let $M$ be a module in $\mathfrak{M}_H(G)$. Then $\text{Ext}^i_{\Lambda_\mathcal{O}(G)}(M, \Lambda_\mathcal{O}(G)) \in \mathfrak{M}_H(G)$ for every $i \geq 1$ and we have the following equality in $K_0(\mathfrak{M}_H(G))$.

$$
[M] = \sum_{1 \leq i \leq \dim G+1} (-1)^{i+1} [\text{Ext}^i_{\Lambda_\mathcal{O}(G)}(M, \Lambda_\mathcal{O}(G))].
$$

In particular, if we have $[\text{Ext}^i_{\Lambda_\mathcal{O}(G)}(M, \Lambda_\mathcal{O}(G))] = 0$ in $K_0(\mathfrak{M}_H(G))$ for every $i \geq 2$, then we have

$$
[M] = [\text{Ext}^1_{\Lambda_\mathcal{O}(G)}(M, \Lambda_\mathcal{O}(G))] \text{ in } K_0(\mathfrak{M}_H(G)).
$$
We are now ready to prove Theorem 2.

**Proof of Theorem 0.3 (Functional Equation)** We claim the following equality in $K_0(\mathcal{M}_H(G))$

\[
[(\text{Sel}_{A}^{\text{Gr}}(K_{\infty})^\vee)^i] + [(E_1^{A+}(1))^i] = [\text{Ext}_{\Lambda_{\mathcal{O}}(G)}^{i} (\text{Sel}_{A^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee, \Lambda_{\mathcal{O}}(G))].
\]  

(59)

First, we will complete the proof of Theorem 0.3 assuming (59). By applying Proposition 5.7 to $M = \text{Sel}_{A^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee$ and by using (59), we deduce the following equality in $K_0(\mathcal{M}_H(G))$:

\[
[(\text{Sel}_{A}^{\text{Gr}}(K_{\infty})^\vee)^i] + [(E_1^{A+}(1))^i] = [\text{Sel}_{A^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee] + \sum_{i \geq 2} (-1)^i [\text{Ext}_{\Lambda_{\mathcal{O}}(G)}^{i} (\text{Sel}_{A^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee, \Lambda_{\mathcal{O}}(G))].
\]  

(60)

By our assumption (3) in Theorem 0.3, either the condition 4.7(a) or the condition 4.7(b) is satisfied. Then by Proposition 4.7, $[\text{Ext}_{\Lambda_{\mathcal{O}}(G)}^{1} (\text{Sel}_{A^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee, \Lambda_{\mathcal{O}}(G))] = 0$ for $i \geq 3$. Moreover by assumption (4) of Theorem 2, either ($A_{\rho}$) or (Van$_{p}$) is satisfied. Then from Proposition 4.7, we can deduce $[\text{Ext}_{\Lambda_{\mathcal{O}}(G)}^{2} (\text{Sel}_{A^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee, \Lambda_{\mathcal{O}}(G))] = 0$ as well. From the above discussion, we deduce from (60),

\[
[(\text{Sel}_{A}^{\text{Gr}}(K_{\infty})^\vee)^i] + [(E_1^{A+}(1))^i] = [\text{Sel}_{A^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee].
\]  

(61)

Twisting this by $\iota$, Theorem 0.3 is an immediate consequence of (61).

In the rest of the proof, we are left to establish the claim in (59). By our assumption in Theorem 0.3, $\text{Sel}_{A^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee$ is in $\mathcal{M}_H(G)$. Thus by Corollary 5.5, there exists a continuous character $\rho : G \rightarrow \Gamma \rightarrow \mathbb{Z}^\infty_p$ such that $\left( \text{Sel}_{A^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee(\rho) \right)_U$ and $H_1 \left( U, \text{Sel}_{A^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee(\rho) \right)$ are both finite for any open normal subgroup $U$ of $G$. Since the character $\rho$ is trivial on $G_{K_{\infty}}$, we deduce that $\text{Sel}_{A^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee(\rho) \cong \text{Sel}_{A_{\rho}^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee$. Thus, $\left( \text{Sel}_{A_{\rho}^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee \right)_U$ and $H_1 \left( U, \text{Sel}_{A_{\rho}^{+}(1)}^{\text{Gr}}(K_{\infty})^\vee \right)$ are both finite. By our hypothesis (1) in Theorem 0.3, the kernel and the cokernel of the natural restriction map $\text{Sel}_{A_{\rho}^{+}(1)}^{\text{Gr}}((K_U)) \rightarrow \text{Sel}_{A_{\rho}^{+}(1)}^{\text{Gr}}(K_{\infty})$ are both finite for any $U$ in $\mathcal{U}$. Hence we have

\[
\text{Sel}_{A_{\rho}^{+}(1)}^{Gr}(K_U) \text{ is finite for any open subgroup } U \subset G.
\]  

(62)

Since the Bloch–Kato Selmer group $\text{Sel}_{A_{\rho}^{+}(1)}^{BK}(K_U)$ is a subgroup of $\text{Sel}_{A_{\rho}^{+}(1)}^{Gr}(K_U)$ by definition, (62) implies that

\[
\text{Sel}_{A_{\rho}^{+}(1)}^{BK}(K_U) \text{ is finite for any open subgroup } U \subset G.
\]  

(63)
Then, by the generalized Cassels-Tate pairing established for Bloch–Kato Selmer groups by Flach (cf. [12, Theorem 1, Theorem 3], [29, Theorem 3.1.1]), we have an isomorphism

$$\text{Sel}_{A_p}^{BK}(K_U) \cong \text{Sel}_{A_p^*(1)}^{BK}(K_U),$$

(64)

for any $U$. Using the isomorphism in (64), together with the natural restriction map and the natural inclusion of a Bloch–Kato Selmer group into a Greenberg Selmer group, we get an $\mathcal{O}[G/U]$-linear map $\text{Sel}_{A_p^*(1)}^{Gr}(K_U) \rightarrow \text{Sel}_{A_p^*(1)}^{Gr}(K_\infty)^U$ by the composition as follows:

$$\text{Sel}_{A_p}^{Gr}(K_U) \rightarrow \text{Sel}_{A_p}^{BK}(K_U) \sim \text{Sel}_{A_p^*(1)}^{BK}(K_U) \rightarrow \text{Sel}_{A_p^*(1)}^{Gr}(K_U) \rightarrow \text{Sel}_{A_p^*(1)}^{Gr}(K_\infty)^U.$$

(65)

Taking the inverse limit with respect to the norm map over $U \in \mathcal{U}$, we get a $\Lambda_{\mathcal{O}}(G)$-linear map

$$\text{Sel}_{A_p^*(1)}^{Gr}(K_\infty)^U \xrightarrow{\phi^1_\rho} \lim_{\mathcal{U} \in \mathcal{U}} \text{Sel}_{A_p^*(1)}^{Gr}(K_\infty)^U.$$

(66)

where $\phi^1_\rho$ is by the composition as follows:

$$\text{Sel}_{A_p}^{Gr}(K_\infty)^U \rightarrow \text{Sel}_{A_p}^{BK}(K_\infty)^U \sim \lim_{U \in \mathcal{U}} \text{Sel}_{A_p^*(1)}^{BK}(K_U) \rightarrow \lim_{U \in \mathcal{U}} \text{Sel}_{A_p^*(1)}^{Gr}(K_U) \rightarrow \lim_{U \in \mathcal{U}} \text{Sel}_{A_p^*(1)}^{Gr}(K_\infty)^U.$$

As $\text{Sel}_{A_p^*(1)}^{Gr}(K_\infty)^U$ is finite for any $U$, we have

$$\lim_{U \in \mathcal{U}} \left( \text{Sel}_{A_p^*(1)}^{Gr}(K_\infty)^U \right)^\iota = \lim_{U \in \mathcal{U}} \text{Tor}_\mathcal{O} \left( \text{Sel}_{A_p^*(1)}^{Gr}(K_\infty)^U \right)^\iota$$

$$\sim \lim_{U \in \mathcal{U}} \text{Ext}^1_{\mathcal{O}[G/U]} \left( \text{Sel}_{A_p^*(1)}^{Gr}(K_\infty)^U, \mathcal{O}[G/U] \right) \text{ (see [Pe, Line 05, Page 732])}$$

$$\sim \lim_{U \in \mathcal{U}} \text{Ext}^1_{\Lambda_{\mathcal{O}}(G)} \left( \text{Sel}_{A_p^*(1)}^{Gr}(K_\infty)^U, \Lambda_{\mathcal{O}}(G) \right) \text{ (by Lemma 4.5)}$$

$$\sim \lim_{U \in \mathcal{U}} \text{Ext}^1_{\Lambda_{\mathcal{O}}(G)} \left( \text{Sel}_{A_p^*(1)}^{Gr}(K_\infty)^U, \Lambda_{\mathcal{O}}(G) \right) \text{ (by Lemma 5.6)}$$

$$\sim \text{Ext}^1_{\Lambda_{\mathcal{O}}(G)} \left( \text{Sel}_{A_p^*(1)}^{Gr}(K_\infty)^U, \Lambda_{\mathcal{O}}(G) \right).$$
Thus, we obtain a $\Lambda_O$-linear isomorphism

$$\lim_{U \in \mathcal{U}} (\text{Sel}_{A^*_p(1)}^{Gr}(K_\infty)^U)^\iota \overset{\phi^2_\rho}{\cong} \text{Ext}^1_{\Lambda_O(G)}(\text{Sel}_{A^*_p(1)}^{Gr}(K_\infty)^\vee, \Lambda_O(G)).$$

(67)

By composing $\phi^2_\rho$ in (67) with $\phi^1_\rho$ of (66) twisted by $\iota$, we have a $\Lambda_O(G)$-linear map

$$(\text{Sel}_{A^*_p(1)}(K_\infty)^\iota)^\phi \overset{\phi_\rho}{\cong} \text{Ext}^1_{\Lambda_O(G)}(\text{Sel}_{A^*_p(1)}^{Gr}(K_\infty)^\vee, \Lambda_O(G)).$$

(68)

Now using the first vanishing assertion of Lemma 5.1, we have the following equality:

$$[\text{Sel}_{A^*_p(1)}^{Gr}(K_\infty)^\vee] = [\text{Sel}_{A^*_p(1)}^{BK}(K_\infty)^\vee] \text{ in } K_0(\mathcal{M}_H(G)).$$

(69)

Also, we have the following equality by taking the limit of (64) for $U \in \mathcal{U}$:

$$[\text{Sel}_{A^*_p(1)}^{BK}(K_\infty)^\vee] = [\lim_{U \in \mathcal{U}} \text{Sel}_{A^*_p(1)}^{BK}(K_U)] \text{ in } K_0(\mathcal{M}_H(G)).$$

(70)

Moreover, by the second vanishing assertion of Lemma 5.1, we deduce

$$[\lim_{U \in \mathcal{U}} \text{Sel}_{A^*_p(1)}^{BK}(K_U)] = [\lim_{U \in \mathcal{U}} \text{Sel}_{A^*_p(1)}^{Gr}(K_U)] \text{ in } K_0(\mathcal{M}_H(G)).$$

(71)

Using these observations in (68), we deduce

$$[\text{Ker}(\phi_\rho)] = \left[\lim_{U \in \mathcal{U}} (\text{Ker}(\text{res}_{A^*_p(1)}^{A^*_p(1)}))^\iota\right],$$

$$[\text{Coker}(\phi_\rho)] = \left[\lim_{U \in \mathcal{U}} (\text{Coker}(\text{res}_{A^*_p(1)}^{A^*_p(1)}))^\iota\right] \text{ in } K_0(\mathcal{M}_H(G)).$$

As $\rho$ is trivial on $G_{K_\infty}$, we obtain by applying $- \otimes O \rho^{-1}$ to (68)

$$\left(\text{Sel}_{A^*_p(1)}^{Gr}(K_\infty)^\vee\right)^\iota \overset{\phi}{\cong} \text{Ext}^1_{\Lambda_O(G)}(\text{Sel}_{A^*_p(1)}^{Gr}(K_\infty)^\vee, \Lambda_O(G)).$$

(72)

Thus we have

$$[\text{Ker}(\phi)] = \left[\lim_{U \in \mathcal{U}} (\text{Ker}(\text{res}_{U}^{A^*_p(1)}))^\iota\right] \text{ and } [\text{Coker}(\phi)]$$

$$= \left[\lim_{U \in \mathcal{U}} (\text{Coker}(\text{res}_{U}^{A^*_p(1)}))^\iota\right] \text{ in } K_0(\mathcal{M}_H(G)).$$

(73)
(1) First, we assume the condition 2(a) in Theorem 0.3 holds. Then, it follows that 
\[ \lim_{U \in \mathcal{U}} \ker(\text{res}^{A^*(1)}_U) \] is finite in cardinality. Hence by Lemma 4.2(a), we deduce from (73) that
\[ [\ker(\phi)] = 0 \text{ in } K_0(\mathcal{M}_H(G)). \] (74)

Moreover, from (73), the condition 2(a) in Theorem 0.3 implies that \([\ker(\phi)]\) and \([\left(E_0^{A^*(1)}(1)\right)]\) in \(K_0(\mathcal{M}_H(G))\) differ by the class of a module which is finite in cardinality. Thus again using Lemma 4.2(a), we deduce from (73) that \([\ker(\phi)] = \left(E_0^{A^*(1)}(1)\right)\) in \(K_0(\mathcal{M}_H(G))\). Further, the assumption (4) in Theorem 0.3 says that either (A_p) or (Van_p) holds. By Proposition 2.4, we have \([\left(E_0^{A^*(1)}(1)\right)] = \left(E_1^{A^*(1)}(1)\right)\) in \(K_0(\mathcal{M}_H(G))\). The claim in (59) now follows from (74) and (72).

(2) Next, we assume the condition 2(b) in Theorem 0.3 holds. The condition 2(b) says that \(\lim_{U \in \mathcal{U}} \ker(\text{res}^{A^*(1)}_U)\) is a finite generated \(\mathbb{Z}_p\)-module. Then due to our assumption (Van) of Definition 1.1, we deduce from (73), \([\ker(\phi)] = 0\). Also, in this case from (73), \([\ker(\phi)]\) and \([\left(E_0^{A^*(1)}(1)\right)]\) in \(K_0(\mathcal{M}_H(G))\) differ by the class of a finitely generated \(\mathbb{Z}_p\)-module. Thanks to the assumption (Van) of Definition 1.1, which is part of hypothesis 2(b) of Theorem 0.3, we get from (73) that \([\ker(\phi)] = \left(E_0^{A^*(1)}(1)\right)\) in \(K_0(\mathcal{M}_H(G))\).

Finally, we again obtain \([\left(E_0^{A^*(1)}(1)\right)] = \left(E_1^{A^*(1)}(1)\right)\) in \(K_0(\mathcal{M}_H(G))\) by Proposition 2.4. Thus the claim in (59) holds also in this case. This completes the proof of Theorem 0.3.

\[ \square \]

**Proof of Remark 0.4:** Note we have \([\text{Sel}_A^{Gr}(K_\infty)\mathcal{V}] = [\text{Sel}_A^{BK}(K_\infty)\mathcal{V}]\) in \(K_0(\mathcal{M}_H(G))\) due to the first vanishing result, for trivial \(\rho\) in Lemma 5.1 given by \([\lim_{U \in \mathcal{U}} \text{Coker}(\text{res}^{A^*(1)}_U)\mathcal{V}] = 0\) in \(K_0(\mathcal{M}_H(G))\). In addition, by the proof of Lemma 5.1, we also have \([\lim_{U \in \mathcal{U}} \text{Coker}(\text{res}^{A^*(1)}_U)\mathcal{V}] = 0\) in \(K_0(\mathcal{M}_H(G))\) due to our assumption that either (A_p) or (Van_p) holds. As a result of this, the equality \([\text{Sel}_A^{Gr}(K_\infty)\mathcal{V}] = [\text{Sel}_A^{BK}(K_\infty)\mathcal{V}]\) in \(K_0(\mathcal{M}_H(G))\) holds true. Applying the involution \(i\) and using the functional equation of the Greenberg Selmer group, we finally deduce that
\[ [\text{Sel}_A^{BK}(K_\infty)\mathcal{V}] + [E_1^{A^*(1)}] = [(\text{Sel}_A^{BK}(K_\infty)\mathcal{V})]^i \text{ in } K_0(\mathcal{M}_H(G)). \]

\[ \square \]

### 6 Compatibility of the algebraic and the conjectural analytic functional equation

In this section, we explain how the functional equation of the conjectural \(p\)-adic \(L\)-function looks like and then we check the compatibility of this conjectural functional
equation with the algebraic functional equation. We will consider a special case of $V = V_{f,p}(\frac{k}{2})$ for a normalized eigen elliptic cuspidal form $f$ of even weight $k \geq 2$ and level $\Gamma_0(N)$ whose $p$-th Fourier coefficient $a_p(f)$ is a $p$-adic unit such that $N$ is square-free and the conductor $N_f$ of $f$ is not divisible by $p$ as discussed in Example 3 of Sect. 3. Here $V_{f,p}$ is the $p$-adic Galois representation for $f$. We take a lattice $T \subset V$ and we set $A = T \otimes \mathbb{Q}_p / \mathbb{Z}_p$. We set $K_{\infty}$ to be a false-Tate curve extension in the setting of Example 1.7. Recall that $K = \mathbb{Q}$, $K_{\infty} = \mathbb{Q}(\mu_{p\infty}, a^{1/p\infty})$, $G = \text{Gal}(K_{\infty}/\mathbb{Q}) \cong \mathbb{Z}_p^\times \times \mathbb{Z}_p$, $H = \text{Gal}(K_{\infty}/\mathbb{Q}_{cyc}) \cong \mathbb{Z}_p^\times$ and $\Gamma = G/H = \text{Gal}(\mathbb{Q}_{cyc}/\mathbb{Q})$. Recall from Sect. 3,

$$P_0 := \{ q \text{ prime in } \mathbb{Q} : q | a \text{ but } q \nmid p \}.$$ 

Also, we define $P_1$ and $P_2$ to be the subsets of $P_0$ given as in Example 3 of Sect. 3.

Let $\eta$ be an Artin representation whose representation space over $\mathbb{C}$ is denoted by $W_\eta$. As a special case of the general definition of complex $L$-function given in (90) in “Appendix”, we have $L(f, \eta, s) := \prod_q P_q(f, \eta, q^{-s})^{-1}$ where we choose any prime $\ell \neq q$ and define $P_q(f, \eta, T) = \det \left( 1 - \text{Frob}_q^{-1} T \mid (V_{f,\ell} \otimes \mathbb{Q}_\ell W_{\eta,\ell})^I_q \right)$ where $V_{f,\ell}$ is the $\ell$-adic Galois representation for $f$ and $W_{\eta,\ell}$ is an Artin representation which is isomorphic to $W_\eta$ but defined over an $\ell$-adic field.

Let us fix complex periods $\Omega_+(f), \Omega_-(f) \in \mathbb{C}^\times$ such that $\frac{L(f, \chi, \frac{k}{2})}{\Omega_{\text{sgn}(\chi)}(f)} \in \mathbb{Q}_f[\chi]$ for any Dirichlet character $\chi$ where $\text{sgn}(\chi)$ is the signature of $\chi(-1)$. Here $\mathbb{Q}_f$ is the Hecke field of $f$ which is a finite extension of $\mathbb{Q}$ obtained by adjoining Fourier coefficients of $f$ to $\mathbb{Q}$. A priori, the periods $\Omega_+(f), \Omega_-(f)$ are defined only up to multiplication of elements of $\mathbb{Q}_f$. However, we can normalize them so that $\Omega_+(f), \Omega_-(f)$ are defined up to multiplication of units of the ring of integers of $\mathbb{Q}_f$ localized at a prime over $p$.

As a special case of Conjecture A.3 when $V = V_{f,p}(\frac{k}{2})$ and $K_{\infty}$ as above, the conjectural $p$-adic $L$-function $L_p(V_{f,p}(\frac{k}{2})) = L_p(V_{f,p}(\frac{k}{2})); \{ \Omega_+(f) \}) \in K_1(\Lambda_{\mathcal{O}}(G)_{S_v})^1$ depending on the fixed periods exists and it is characterized by

$$\eta^*(L_p(V_{f,p}(\frac{k}{2}))) = \epsilon_p(\eta) \alpha_p^{-C_p(\eta)} P_p(\eta, \alpha_p^{-1}) \times \frac{L_p(f, \eta, \frac{k}{2})}{\Omega_+(f)^{d_+}(\eta) \Omega_-(f)^{d_-}(\eta)} \left(75\right)$$

where $\eta^*$ denotes the contragradient representation of $\eta$. Here $\mathcal{O}$ is the ring of integers of some finite extension of $\mathbb{Q}_p^2$, $p^{C_p(\eta)}$ is the $p$-part of the conductor $\eta$ at $p$, $\epsilon_p(\eta)$ is the local $\epsilon$ factor of $\eta$ at $p$ and $P$ is the set of primes of $\mathbb{Q}$ which are infinitely ramified

---

1 The $p$-adic $L$-function $L_p(V_{f,p}(\frac{k}{2}))$ essentially depends on the choice of periods $\Omega_+(f), \Omega_-(f)$. However, below, we remove the dependence from the notation to ease the notation.

2 Note that, according to the work of Fukaya-Kato [13, Theorem 2.2.26] on Tamagawa number conjecture, it is suggested in [5, Page 203] that the coefficient ring of Iwasawa algebra for the conjectural $p$-adic $L$-function be enlarged to a finite extension $\mathcal{O}$ even when $\mathbb{Q}_f = \mathbb{Q}$. 

in $K_{\infty}/\mathbb{Q}$ and $L_P(f, \eta, \frac{k}{2}) := \prod_{q \notin P} P_q(f, \eta, q^{-\frac{k}{2}})^{-1}$. One has that $P = P_0 \cup \{p\}$ (see [24, Lemma 3.9]).

If the weight $k$ is larger than two, the complex $L(f, \eta, s)$ has critical points other than $s = \frac{k}{2}$ and the interpolation property of the $p$-adic $L$-function $L_p(V_{f,p}(\frac{k}{2}))$ covers more specializations than (75) corresponding to other critical points. However, since we assume that $f$ is ordinary, the interpolation property (75) is enough to characterize $L_p(V_{f,p}(\frac{k}{2}))$.

On the other hand, recall that the twisted complex $L$-function of $f$ has the following conjectural functional equation,

$$\Lambda(f, \eta, s) = \omega(f, \eta) \Lambda(f, \eta^*, k - s),$$  

(76)

where $\Lambda(f, \eta, s)$ is the completed $L$-function which is obtained by multiplying certain $\Gamma$-factor to $L(f, \eta, s)$. The conjectural functional equation of complex $L$-function in (76) is a special case of Conjecture A.1 when $V = V_{f,p}(\frac{k}{2}) \otimes W_{\eta,p}$.

Then putting $s = \frac{k}{2}$ in (76) and using the definition of $\Lambda(f, \eta, \frac{k}{2})$, we deduce the following equation:

$$L(f, \eta, \frac{k}{2}) = \omega(f, \eta)L(f, \eta^*, \frac{k}{2}).$$  

(77)

Recall that $P$ is the set of primes of $\mathbb{Q}$ which are infinitely ramified in $K_{\infty}/\mathbb{Q}$. By (75), we have

$$\eta^*(L_p(V_{f,p}(\frac{k}{2}))) = \epsilon_p(\eta)\alpha_p^{-C_p(\eta)} \times \frac{P_p(\eta^*, \alpha_p^{-1})}{P_p(\eta, \beta_p^{-1})} \times \frac{L(f, \eta, \frac{k}{2})}{\Omega_+(f)^{d_+(\eta)} \Omega_-(f)^{d_-(\eta)}} \prod_{q \notin P} P_q(f, \eta, q^{-\frac{k}{2}}),$$

$$\eta(L_p(V_{f,p}(\frac{k}{2}))) = \epsilon_p(\eta^*)\alpha_p^{-C_p(\eta^*)} \times \frac{P_p(\eta, \alpha_p^{-1})}{P_p(\eta^*, \beta_p^{-1})} \times \frac{L(f, \eta^*, \frac{k}{2})}{\Omega_+(f)^{d_+(\eta^*)} \Omega_-(f)^{d_-(\eta^*)}} \prod_{q \notin P} P_q(f, \eta^*, q^{-\frac{k}{2}}).$$

Since we have $d_+(\eta) = d_+(\eta^*)$ and $C_p(\eta) = C_p(\eta^*)$, the following identity is obtained by using (77) and the last two expressions above:

$$\frac{\eta^*(L_p(V_{f,p}(\frac{k}{2})))}{\prod_{q \notin P_0} P_q(f, \eta, q^{-\frac{k}{2}})} = \frac{\eta(L_p(V_{f,p}(\frac{k}{2})))}{\prod_{q \notin P_0} P_q(f, \eta^*, q^{-\frac{k}{2}})} \omega(f, \eta) \frac{\epsilon_p(\eta)}{\epsilon_p(\eta^*)} \frac{P_p(f, \eta, p^{-\frac{k}{2}})}{P_p(f, \eta^*, p^{-\frac{k}{2}})} \frac{P_p(\eta, \alpha_p^{-1})}{P_p(\eta^*, \alpha_p^{-1})} \frac{P_p(\eta, \beta_p^{-1})}{P_p(\eta^*, \beta_p^{-1})}.  

(78)$$
In (78), we have used $P_0 = \mathcal{P}\setminus \{p\}$. Further, as we have $\eta(\mathcal{L}_p(V_{f,p}(\frac{k}{2}))^\prime) = \eta^*(\mathcal{L}_p(V_{f,p}(\frac{k}{2})))$, we get the following identity from (78):

$$\frac{\eta(\mathcal{L}_p(V_{f,p}(\frac{k}{2}))^\prime)}{\prod_{\mathcal{P}\in \mathcal{P}_0} P_{\mathcal{P}}(f, \eta, q^{-\frac{k}{2}})} = \frac{\eta(\mathcal{L}_p(V_{f,p}(\frac{k}{2}))))}{\prod_{\mathcal{P}\in \mathcal{P}_0} P_{\mathcal{P}}(f, \eta^*, q^{-\frac{k}{2}})} \cdot \omega(f, \eta) \frac{\epsilon_p(\eta)}{\epsilon_p(\eta^*)} \frac{P_p(f, \eta, p^{-\frac{k}{2}})}{P_p(f, \eta^*, p^{-\frac{k}{2}})} \frac{P_p(\eta^*, \alpha_p^{-1})}{P_p(\eta, \alpha_p^{-1})} \frac{P_p(\eta^*, \beta_p^{-1})}{P_p(\eta, \beta_p^{-1})}. \tag{79}$$

Note that $\frac{\epsilon_p(\eta)}{\epsilon_p(\eta^*)}$ is a $p$-adic unit. Now we discuss that the factors at $p$ in (79). First of all, note that if $\eta$ is the trivial Artin representation of $G$, then the factors at prime $p$ in (79) are the same for $\eta$ and $\eta^*$.

First, we check that the contributions of factors at $p$ is trivial in this particular case of a false-Tate extension. Recall that, by the $\ell$-independence of an Euler factor, we have

$$P_p(f, \eta, p^{-\frac{k}{2}}) = \det(1 - \text{Frob}_p^{-1} p^{-\frac{k}{2}} | (V_{f,\ell}(\frac{k}{2}) \otimes W_{\eta,\ell})^I),$$

$$P_p(\eta, \alpha_p^{-1}) = \det(1 - \text{Frob}_p^{-1} \alpha_p^{-1} | (W_{\eta,\ell})^I),$$

$$P_p(\eta, \beta_p^{-1}) = \det(1 - \text{Frob}_p^{-1} \beta_p^{-1} | (W_{\eta,\ell})^I),$$

for any inertia invariant $\ell \neq p$ where $W_{\eta,\ell}$ is defined in the paragraph preceding (75). Note that the inertia invariant subspace $W_{\eta,\ell}$ is trivial for any non-trivial Artin representation $\eta$ of $G$. The representation $V_{f,\ell}(\frac{k}{2})$ is unramified at $p$ since the conductor $N_f$ of $f$ is not divisible by $p$. Hence the inertia invariant space $(V_{f,\ell}(\frac{k}{2}) \otimes W_{\eta,\ell})^I$ is trivial. Thus we have $P_p(f, \eta, p^{-\frac{k}{2}}) = P_p(\eta, \alpha_p^{-1}) = P_p(\eta, \beta_p^{-1}) = 1$ for any non-trivial Artin representation $\eta$ of $G$. By combining the discussions above with (79), we have proved the following result.

**Proposition 6.1** Assume Conjectures A.2 and A.3 for $V = V_{f,p}(\frac{k}{2})$ and $G = \text{Gal}(\mathbb{K}_\infty/\mathbb{Q})$ where $\mathbb{K}_\infty$ is a false-Tate curve extension as considered in Example 1.7 and Sect. 3.

Then the $p$-adic L-function $\mathcal{L}_p(V_{f,p}(\frac{k}{2})) \in K_1(\Lambda(\mathcal{O}(G))_{S^*})$ satisfies the following interpolation property:

$$\eta(\mathcal{L}_p(V_{f,p}(\frac{k}{2}))^\prime) = \omega(f, \eta) \frac{\prod_{q \in \mathcal{P}_0} P_q(f, \eta, q^{-\frac{k}{2}})}{\prod_{q \in \mathcal{P}\setminus \{p\}} P_q(f, \eta^*, q^{-\frac{k}{2}})} \tag{80}$$

for every Artin representation $\eta$ of $G$. Here $S$ and $S^*$ are given by (91) and $\mathcal{O}$ is the ring of integers of a certain finite extension of $\mathbb{Q}_p$.

**Proposition 6.2** Let us consider the setting of Sect. 3 where $\mathbb{K}_\infty$ is a false-Tate curve extension. We take a lattice $T$ of $V = V_{f,p}(\frac{k}{2})$ and we set $A = T \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Let $\xi_{E_1} \in K_1(\Lambda(\mathcal{O}(G))_{S^*})$ be the characteristic element of the preimage of the class $[E_1] \in K_0(\mathcal{M}_H(G))$ of the error term.
\[ E_1 = E_1^A \cong \bigoplus_{q \in P_1 \cup P_2} \text{Ind}_{G_{q}}^{G} T(-1) \]

of the functional equation of Selmer groups and \( O \) is the ring of integers of a certain finite extension of \( \mathbb{Q}_p \).

Then, for every Artin representation \( \eta \) of \( G \), we have the equality:

\[
\eta(\xi_{E_1}) = \prod_{q \in P_0} \frac{P_q(f, \eta, q^{-\frac{k}{2}})}{P_q(f, \eta^*, q^{-\frac{k}{2}})}
\]

modulo multiplication by \( p \)-adic units.

Before proving Proposition 6.2, we give some preparations.

Let \( \theta_n \) denote the Artin representation of \( G = \text{Gal}(K_{\infty}/\mathbb{Q}) \) is obtained by inducing a character \( \phi_n \) of exact order \( p^n \) of \( \text{Gal}(\mathbb{Q}(\mu_{p^n}, a^{1/p^n})/\mathbb{Q}(\mu_{p^n})) \) to \( \text{Gal}(\mathbb{Q}(\mu_{p^n}, a^{1/p^n})/\mathbb{Q}) \). Then \( \theta_n \) is irreducible, and every irreducible Artin representation \( \eta \) of \( G \) is either of the form \( \eta = \psi \) or of the form \( \eta = \theta_n \psi \) for an integer \( n \geq 1 \), where \( \psi \) is a 1-dimensional character of \( \text{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) \) (see [10, Page 212]).

For \( q \in P_2 \), let \( C \) be a free \( O \)-module of finite rank equipped with a continuous unramified action of \( G_{\mathbb{Q}_q} \). We recall that \( H_q := H \cap G_q \) is completely unramified at \( q \) and \( \Gamma_q := G_q/H_q \) is unramified at \( q \). We identify \( H_q \) with inertia subgroup and we note that \( G_q/H_q \) is topologically generated by \( \text{Frob}_q \). Hence, \( G_q \) acts naturally on \( C \) through the quotient \( G_q/H_q \), which allows us to regard \( C \) as a \( \Lambda_O(G_q) \)-module. With the above explanation, we have a characteristic element \( \xi_q^q \) of \( P \) in \( K_1(\Lambda_O(G_q), S^*) \). This especially gives us \( \xi_q^q_{T(-1)} \in K_1(\Lambda_O(G_q), S^*) \) for every \( q \in P_2 \) since \( T(-1) \) is unramified for every \( q \in P_2 \). Here \( O \) can be taken to be the ring of integers of a \( p \)-adic field which contains the eigenvalues of the action of \( \text{Frob}_q \) on \( T \). When \( q \in P_1 \), the action of \( G_{\mathbb{Q}_q} \) on \( T(-1) \) factors through the quotient \( G_{\mathbb{Q}_q} \to G_q \). This allows us to regard \( T(-1) \) as a \( \Lambda(O_q) \)-module and we have a characteristic element \( \xi_q^q \) of \( T(-1) \) in \( K_1(\Lambda(O_q), S^*) \) also for \( q \in P_1 \).

Let \( q \in P_2 \). Let us denote by \( \Lambda(O_q)^{\varepsilon} \) a free \( \Lambda_O(G_q) \)-module of rank one on which the decomposition group \( G_{\mathbb{Q}_q} \) acts on \( \Lambda(O_q) \) by the tautological action through the surjection \( G_{\mathbb{Q}_q} \to G_q \). Let \( h_q \) be a topological generator of \( H_q \). We have the exact sequence of \( \Lambda_O(G_q)[G_{\mathbb{Q}_q}] \)-modules:

\[
0 \to (h_q - 1)(\Lambda_O(G_q)^{\varepsilon}/(\text{Frob}_q - 1)\Lambda_O(G_q)^{\varepsilon}) \otimes_{\Lambda_O(G_q)} C \\
\to (\Lambda_O(G_q)^{\varepsilon}/(\text{Frob}_q - 1)\Lambda_O(G_q)^{\varepsilon}) \otimes_{\Lambda_O(G_q)} C \to C \to 0.
\]

Let us denote the first term and the second term of the above exact sequence by \( M \) and \( N \) respectively. Then \( M, N \) are in \( K_0(\mathfrak{M}_H(G)) \) and \( \xi_q^q \in K_1(\Lambda_O(G_q), S^*) \) is equal to \( \xi_M^q \in K_1(\Lambda_O(G_q), S^*) \) modulo multiplication by an element of \( K_1(\Lambda_O(G_q)) \). Let us calculate the evaluations of \( \xi_M^q \) and \( \xi_N^q \) at characters \( \psi \) and \( \theta_n \psi \).

We need the following lemma.
Lemma 6.3 Let $C$ be a free $O$-module of rank one equipped with a continuous unramified action of $G_{\mathbb{Q}_p}$. Assume that the eigenvalue of $\text{Frob}_q$ acting on $C$ is $x$.

(1) Let $M = (h_q - 1) (\Lambda(G_q)\mathcal{O}/(\text{Frob}_q - 1)\Lambda(G_q)) \otimes_{\Lambda(G_q)} C$. Then we have $\psi(\xi_M^q) = \psi(\text{Frob}_q) - qx$ modulo a multiplication by unit in $O^\times$.

(2) Let $N = (\Lambda(G_q)\mathcal{O}/(\text{Frob}_q - 1)\Lambda(G_q)) \otimes_{\Lambda(G_q)} C$. Then we have $\psi(\xi_N^q) = \psi(\text{Frob}_q) - x$ modulo a multiplication by a unit in $O^\times$.

Proof of Lemma 6.3 Since the second statement is easier, we start from the proof of the assertion (2). In this case $N$ is a cyclic $\Lambda(G_q)$-module whose annihilator ideal is a principal ideal generated by $\text{Frob}_q - x \in \Lambda_{\mathcal{O}}(G_q)$. So $\xi_N^q \in K_1(\Lambda_{\mathcal{O}}(G_q)\mathcal{S}_q^\times)$ has a representative in $\text{Frob}_q - x \in (\Lambda_{\mathcal{O}}(G_q)\mathcal{S}_q^\times)^\times$. This implies that $\psi(\xi_N^q) = \psi(\text{Frob}_q - x) = \psi(\text{Frob}_q) - x$, modulo a multiplication by a unit in $O^\times$, which proves the assertion (2).

For the assertion (1), be simple calculation, we can check that the annihilator ideal of $M$ is a principal ideal generated by $\text{Frob}_q - \frac{(h_q)^q - 1}{h_q - 1} x \in \Lambda_{\mathcal{O}}(G_q)$. Since $\psi(h_q) = 1$, we have $\psi(\frac{(h_q)^q - 1}{h_q - 1}) = q$ which implies the equality of the assertion (1) modulo a multiplication by a unit in $O^\times$. 

Proof of Proposition 6.2 In order to show the desired equality, we start by calculating the analytic local terms $\prod_{q \in P_0} P_q(f, \eta, q^{-\frac{k}{2}}) / P_q(f, \eta^*, q^{-\frac{k}{2}})$.

We can check that, when $\eta$ is of the form $\eta = \theta_n \psi$ for an integer $n \geq 1$, $(W_\eta)^q$ is trivial for any $q \in P_0$. Since $p \neq 2$, we have $\prod_{q \in P_0} P_q(f, (\theta_n \psi)^*, q^{-\frac{k}{2}}) = 1$. When $\eta$ is of the form $\eta = \psi$, we have

$$\frac{P_q(f, \psi, q^{-\frac{k}{2}})}{P_q(f, \psi^*, q^{-\frac{k}{2}})} = \begin{cases} 1 - \delta_q(\text{Frob}_q) \psi(\text{Frob}_q) q^{-1} & q \in P_1, \\ 1 - \alpha_q(\text{Frob}_q) \psi(\text{Frob}_q) q^{-1} & q \in P_2, \\ 1 - \beta_q(\text{Frob}_q) \psi(\text{Frob}_q) q^{-1} & q \in P_0 \setminus (P_1 \cup P_2) \end{cases}$$

where $\alpha_q$ and $\beta_q$ are eigenvalues of the action of $\text{Frob}_q$ on $V_{f,p}(\chi)$ and $\delta_q$ is a unramified quadratic character of $\mathbb{Q}_q^\times$ which is attached to the special representation $\pi_{f,q}$ at $q$ (see Example 3 of Sect. 3).

Now we pass to the algebraic side. We have $E_1 \cong \bigoplus_{q \in P_1 \cup P_2} \text{Ind}_{G_{\mathcal{O}}(G_q)}^{G_0} T(-1)$. Recall that, $[E_1]$ denotes the image of $E_1$ in $K_0(M_{H_0}(G_0))$ and the characteristic element $\xi_{E_1}$ denotes any preimage of $[E_1]$ in $K_1(\Lambda_{\mathcal{O}}(G_q)\mathcal{S}_q^\times)$ via the surjection from $K_1(\Lambda_{\mathcal{O}}(G_q)\mathcal{S}_q^\times) \rightarrow K_0(M_{H}(G))$ (cf. (92)). Here $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$ containing the eigenvalues of the action of $\text{Frob}_q$ on $V_{f,p}(\chi)$, for all $q \in P_2$. Thus $\xi_{E_1}$ is defined up to an element of $K_1(\Lambda_{\mathcal{O}}(G))$. We have
\[ K_1(\Lambda_{O}(G)_{S}) \cong \frac{\Lambda_{O}(G)_{S}^{\times}}{[\Lambda_{O}(G)_{S}^{\times}, \Lambda_{O}(G)_{S}^{\times}]} \] by [5, Theorem 4.4] and there is a surjective map

\[ \frac{\Lambda_{O}(G)_{S}^{\times}}{[\Lambda_{O}(G)_{S}^{\times}, \Lambda_{O}(G)_{S}^{\times}]} \rightarrow K_1(\Lambda_{O}(G)_{S}) \rightarrow 1. \quad (83) \]

Let us recall that the following diagram, where vertical maps are induced by natural inclusion, commute:

\[
\begin{array}{ccc}
K_1(\Lambda_{O}(G_{q})_{S}) & \longrightarrow & K_0(\mathfrak{M}_{H_{q}}(G_{q})) \\
\downarrow & & \downarrow \\
K_1(\Lambda_{O}(G)_{S}) & \longrightarrow & K_0(\mathfrak{M}_{H}(G)).
\end{array}
\]

Hence, the evaluation at \( \eta \) of the characteristic element \( \xi_{E_{q}} \in K_1(\Lambda_{O}(G)_{S}) \) is the product of the evaluations at \( \eta \) of the characteristic elements \( \xi_{T_{p}}^{q}(-1) \in K_1(\Lambda_{O}(G_{q})_{S}) \) at primes in \( P_1 \) and \( P_2 \). In order to complete the proof of the proposition, it suffices to show the equalities

\[
\begin{aligned}
\left\{
\begin{array}{ll}
\theta_{\eta} \psi(\xi_{T_{p}}^{q}(-1)) = 1 & q \in P_1 \cup P_2, \\
\psi(\xi_{T_{p}}^{q}(-1)) = \frac{1-\delta_{q}(\text{Frob}_{q})\psi(\text{Frob}_{q})q^{-1}}{1-\delta_{q}(\text{Frob}_{q})\psi(\text{Frob}_{q})q^{-1}} & q \in P_1, \\
\psi(\xi_{T_{p}}^{q}(-1)) = \frac{(1-\alpha_{q}\psi(\text{Frob}_{q})q^{-1})(1-\beta_{q}\psi(\text{Frob}_{q})q^{-1})}{1-\alpha_{q}\psi(\text{Frob}_{q})q^{-1}(1-\beta_{q}\psi(\text{Frob}_{q})q^{-1})} & q \in P_2,
\end{array}
\right.
\end{aligned}
\]  

(84)

modulo multiplications by a \( p \)-adic unit where \( \delta_{q} \) is as defined in (82). In the rest of the proof, we prove the equation (84).

Let \( q \in P_2 \). Note that we have \( \alpha_{q} \beta_{q} = q^{-1} \). Since \( \text{Frob}_{q} \) acts on \( \mathbb{Z}_{p}(1) \) by \( q^{-1} \), the action of \( \text{Frob}_{q} \) on \( V_{f,p}(\xi) \) has eigenvalues \( \beta_{q}^{-1} = \frac{\alpha_{q}}{q}, \alpha_{q}^{-1} = \beta_{q}q \). For a prime \( q \in P_2, \mathcal{O} \) contains \( \alpha_{q} \) and \( \beta_{q} \), by definition. Then the unramified \( G_{O_{q}} \)-module \( T \) has a decomposition \( T \otimes_{\mathbb{Z}_{p}} \mathcal{O} \cong C_{\alpha_{q}} \oplus C_{\beta_{q}} \) where \( C_{\alpha_{q}} \) and \( C_{\beta_{q}} \) are rank one \( \mathcal{O} \)-modules on which \( \text{Frob}_{q} \) acts by multiplication of \( \beta_{q}^{-1} \) and \( \alpha_{q}^{-1} \) respectively. By definition, the characteristic element \( \xi_{T_{p}E_{q}}^{q} \) is equal to \( \xi_{C_{\alpha_{q}}}^{q} \xi_{C_{\beta_{q}}}^{q} \). When \( \eta \) is of the form \( \eta = \psi \), we obtain:

\[
\eta(\xi_{T_{p}}^{q}(-1)) = \psi(\xi_{C_{\alpha_{q}}}^{q})\psi(\xi_{C_{\beta_{q}}}^{q})
\]

\[
= \frac{\psi(\text{Frob}_{q}) - \beta_{q}^{-1}\psi(\text{Frob}_{q}) - \alpha_{q}^{-1}}{(\psi(\text{Frob}_{q}) - q\beta_{q}^{-1}\psi(\text{Frob}_{q}) - q\alpha_{q}^{-1})}
\]

\[
= \frac{(1-\alpha_{q}\psi(\text{Frob}_{q})q^{-1})(1-\beta_{q}\psi(\text{Frob}_{q})q^{-1})}{(1-\alpha_{q}\psi(\text{Frob}_{q})q^{-1})(1-\beta_{q}\psi(\text{Frob}_{q})q^{-1})} \times q^{-1}
\]
by applying Lemma 6.3. Let us consider the case where \( \eta \) is of the form \( \eta = \theta_n \psi \). In this case, we also have

\[
\eta(\xi^q_{T(-1)}) = \theta_n \psi(\xi^q_{\theta}) \theta_n \psi(\xi^q_{\phi}) = 1.
\]

Hence it suffices to prove that \( \theta_n \psi(\xi^q_{\theta}) = 1 \) and \( \theta_n \psi(\xi^q_{\phi}) = 1 \). By the definition of \( \theta_n \) explained after the statement of Proposition 6.2, \( \theta_n \) is an induced representation of a character \( \phi_n \) of \( \text{Gal}(\mathbb{Q}_q(\mu_{p^n}, m^{1/p^n})/\mathbb{Q}_q(\mu_{p^n})) \) to \( \text{Gal}(\mathbb{Q}_q(\mu_{p^n}, m^{1/p^n})/\mathbb{Q}_q) \) which is realized on the space

\[
W = V_{\phi_n} \otimes_{\mathcal{K}} V_{\psi} \otimes_{\mathcal{K}} \mathcal{K}[\text{Gal}(\mathbb{Q}_q(\mu_{p^n})/\mathbb{Q}_q)]^\mathbb{Z}
\]

where \( V_{\phi_n} \) is a representation space of \( \phi_n \) which is a one-dimensional vector space over the coefficient field \( \mathcal{K} \) on which \( \text{Gal}(\mathbb{Q}_q(\mu_{p^n}, m^{1/p^n})/\mathbb{Q}_q(\mu_{p^n})) \) acts and \( V_{\psi} \) is a representation space of \( \phi_n \) which is a one-dimensional vector space over \( \mathcal{K} \) on which \( \text{Gal}(\mathbb{Q}_q(\mu_{p^n})/\mathbb{Q}_q) \) acts. Let us calculate \( \theta_n \psi(\xi^q_{\theta}) \). In this situation, an element \( g \in G_q, \theta_n \psi(g) \) is represented by a matrix of degree \( [\mathbb{Q}_q(\mu_{p^n}) : \mathbb{Q}_q] \) which is regarded as an element of \( \text{End}_{\mathcal{K}}(W \otimes_{\mathcal{O}} C_{\alpha}) \). Note also that the evaluation of elements in \( K_1(\mathcal{O}(G_q)_{\mathbb{Q}_q}) \) at Artin representation \( \eta \) of \( G \) is defined [5, §3] through the Morita equivalence \( K_1(M_n(-)) = K_1(-) \) which is known to be obtained by taking the determinant of the matrix. As in the proof of Lemma 6.3, we calculate the ratio of the determinant of the matrix in \( \text{End}_{\mathcal{K}}(W \otimes_{\mathcal{O}} C_{\alpha}) \) induced by an annihilator \( \text{Frob}_q - (h_q^q - 1) \in \Lambda_{\mathcal{O}}(G_q) \) of the cyclic module

\[
(h_q - 1) \left( \Lambda_{\mathcal{O}}(G_q)^{\mathbb{Z}} / (\text{Frob}_q - 1) \Lambda_{\mathcal{O}}(G_q)^{\mathbb{Z}} \right)
= \left( \Lambda_{\mathcal{O}}(G_q)^{\mathbb{Z}} / (\text{Frob}_q - (h_q^q - 1)) \Lambda_{\mathcal{O}}(G_q)^{\mathbb{Z}} \right)
\]

and the determinant of the matrix in \( \text{End}_{\mathcal{K}}(W \otimes_{\mathcal{O}} C_{\alpha}) \) induced by an annihilator \( \text{Frob}_q - 1 \in \Lambda_{\mathcal{O}}(G_q) \) of the cyclic module \( \left( \Lambda_{\mathcal{O}}(G_q)^{\mathbb{Z}} / (\text{Frob}_q - 1) \Lambda_{\mathcal{O}}(G_q)^{\mathbb{Z}} \right) \). The matrix representing \( \theta_n \psi(\text{Frob}_q) \) is a permuting matrix the basis labeled by elements of \( \text{Gal}(\mathbb{Q}_q(\mu_{p^n})/\mathbb{Q}_q) \) twisted by the eigenvalues of these elements acting on \( P_{\alpha} \). The matrix representing \( \theta_n(\frac{(h_q)^{p-1}}{h_q - 1}) \) is a diagonal matrix whose diagonal entries are given by \( \phi_n(\frac{(h_q)^{p-1}}{h_q - 1}) \) when \( g \) runs through elements of \( \text{Gal}(\mathbb{Q}_q(\mu_{p^n})/\mathbb{Q}_q) \). Note that the product of \( \phi_n(\frac{(h_q)^{p-1}}{h_q - 1}) \) for \( g \in \text{Gal}(\mathbb{Q}_q(\mu_{p^n})/\mathbb{Q}_q) \) equals to 1. Now the determinants of two matrices which appear in the calculation of \( \theta_n \psi(\xi^q_{\theta}) \) are both equal to \( (\psi(\text{Frob}_q) \alpha_q)^{\mathbb{Q}_q(\mu_{p^n})/\mathbb{Q}_q} - 1 \), which implies that the ratio is equal to 1, hence we have \( \theta_n \psi(\xi^q_{\theta}) = 1 \). Since we prove \( \theta_n \psi(\xi^q_{\phi}) = 1 \) exactly in the same way, we finally obtain:\n
\[
\eta(\xi^q_{T(-1)}) = \theta_n \psi(\xi^q_{\theta}) \theta_n \psi(\xi^q_{\phi}) = 1.
\]
Let $q \in P_1$. Then the action of $G_{\mathbb{Q}_q}$ on $T(-1)$ is not unramified, but we can define $\xi_{T(-1)}^q \in K_1(\Lambda_{\mathcal{O}}(G_q) \Sigma_q^*)$ as explained earlier, with $\mathcal{O}$ as before. Since we have an exact sequence

$$0 \rightarrow (\Lambda_{\mathcal{O}}(G_q)^2) / (\text{Frob}_q - 1) \Lambda_{\mathcal{O}}(G_q)^2) \otimes_{\Lambda_{\mathcal{O}}(G_q)} \mathbb{Z}_p \otimes \delta_q \rightarrow (\Lambda_{\mathcal{O}}(G_q)^2) / (\text{Frob}_q - 1) \Lambda_{\mathcal{O}}(G_q)^2) \otimes_{\Lambda_{\mathcal{O}}(G_q)} T(-1) \rightarrow (\Lambda_{\mathcal{O}}(G_q)^2) / (\text{Frob}_q - 1) \Lambda_{\mathcal{O}}(G_q)^2) \otimes_{\Lambda_{\mathcal{O}}(G_q)} \mathbb{Z}_p(-1) \otimes \delta_q \rightarrow 0$$

in the category $\mathcal{M}_H(G)$, we calculate $\eta(\xi_{T(\mathcal{C}_{\mathcal{G}_q})}^q)$ to be $\eta(\xi_{\mathbb{Z}p \otimes \delta_q}^q) \eta(\xi_{\mathbb{Z}p(-1) \otimes \delta_q}^q)$. The calculation of $\eta(\xi_{\mathbb{Z}p \otimes \delta_q}^q)$ and $\eta(\xi_{\mathbb{Z}p(-1) \otimes \delta_q}^q)$ for $q \in P_1$ goes exactly in the same way as that of $\eta(\xi_{\mathcal{G}_q}^q)$ and $\eta(\xi_{\mathcal{G}_q}^q)$ for $q \in P_2$ and this completes the proof of Proposition 6.2.

For any $M$ in $\mathcal{M}_H(G)$ and for any Artin representation $\eta$ of $G$, we introduce a notation $\eta([M]) := \eta(\xi_{M})$, which is well defined up to a $p$-adic unit.

**Theorem 6.4** Let us assume the setting above of an ordinary normalized eigen elliptic cuspform $f$ of even weight $k \geq 2$ and square-free level $\Gamma_0(N)$ with a false-Tate extension $K_{\infty}/\mathbb{Q}$. We take a lattice $T$ of $V = V_{f,p}(\frac{k}{2})$ and we set $A = T \otimes \mathbb{Q}_p / \mathbb{Z}_p$. Assume that a conjectural $p$-adic L-function $L_p(V_{f,p}(\frac{k}{2})) \in K_1(\Lambda_{\mathcal{O}}(G) \Sigma^*)$ with the interpolation property (75) exists. Recall from Sect. 3 that $E_1^A \cong \bigoplus_{q \in P_1 \cup P_2} \text{Ind}_G^A \xi_{\mathcal{G}_q}^q T(-1)$ is the error term of the algebraic functional equation. Then, for any Artin representation $\eta$ of $\text{Gal}(K_{\infty}/\mathbb{Q})$,

$$\eta(\xi_{E_1^A}) = \prod_{q \in P_0} \frac{P_q(f, \eta, q^{-\frac{k}{2}})}{P_q(f, \eta^*, q^{-\frac{k}{2}})}$$

modulo multiplication by a $p$-adic unit. Hence, for any Artin representation $\eta$ of $\text{Gal}(K_{\infty}/\mathbb{Q})$, we have the equality:

$$\frac{\eta([\text{Sel}^\text{BK}_A(K_{\infty})^\vee])}{\eta([\text{Sel}^\text{BK}_A(K_{\infty})^\vee]^t)} = \frac{\eta(L_p(V_{f,p}(\frac{k}{2})))}{\eta(L_p(V_{f,p}(\frac{k}{2})))}$$

modulo multiplication by a $p$-adic unit.

**Proof** The first assertion of the theorem has been established in the proof of Proposition 6.2. So we prove only the second part. Recall our second main theorem (Theorem 2) applied to this setting of an ordinary normalized eigen elliptic cuspform over the false-Tate curve extension implies the following

$$[\text{Sel}^\text{Gr}_A(K_{\infty})^\vee] + [E_1^A] = [\left(\text{Sel}^\text{Gr}_A(K_{\infty})^\vee\right)^t] \text{ in } K_0(\mathcal{M}_H(G)).$$

(87)
Clearly, (87) can be reformulated as follows:

\[ \xi_{\text{SelGr}}(K_\infty) \vee \xi_{E_1} = \xi(\text{SelGr}(K_\infty))^{\vee} \]  

(88)

as elements of \( K_1(\Lambda_{G}(G)_{S^*}) \) up to an element of \( K_1(\Lambda_{G}(G)) \). On the other hand, from Proposition 6.1, we have

\[ \eta(\mathcal{L}_p(\mathcal{V}_f, p(\frac{k}{2}))) = \eta(\mathcal{L}_p(\mathcal{V}_f, p(\frac{k}{2}))) \prod_{q \in \mathcal{P}_0} \frac{P_p(f, \eta, q^{-\frac{k}{2}})}{P_p(f, \eta^*, p^{-\frac{k}{2}})} \]

modulo multiplication by a \( p \)-adic unit. Now by Proposition 6.2, we know \( \eta(\xi_{E_1}) = \prod_{q \in \mathcal{P}_0} \frac{P_q(f, \eta, q^{-\frac{k}{2}})}{P_q(f, \eta^*, q^{-\frac{k}{2}})} \), up to a \( p \)-adic unit. Thus we have

\[ \eta(\mathcal{L}_p(\mathcal{V}_f, p(\frac{k}{2}))) = \eta(\mathcal{L}_p(\mathcal{V}_f, p(\frac{k}{2}))) \eta(\xi_{E_1}) \]  

(89)

up to a \( p \)-adic unit. Evaluating (88) at \( \eta \) and then comparing it with (89), we obtain (86) up to a unit in a ring of integers of a finite extension of \( \mathbb{Q}_p \). This completes the proof of the theorem.

Remark 6.5  The results of this section are a generalization of the corresponding results of Zábrádi. More precisely, the compatibility between the algebraic and the conjectural analytic functional equation over the false-Tate curve extension for an elliptic curve \( E \) over \( \mathbb{Q} \) with good, ordinary reduction at \( p \) was established in [34, Prop. 7.3]. With the same assumption on \( E \), the compatibility over \( \mathbb{Q}(E_{p^\infty}) \) is discussed in [35, Proposition 7.2].

Remark 6.6  Fukaya-Kato has a formulation of the conjectural functional equation of analytic \( p \)-adic zeta function [13, Theorem 4.4.7]. Though they do not study the functional equation on the algebraic side, their functional equation of \( p \)-adic zeta function combined with their main conjecture in [13, Theorem 4.2.22], which generalizes the main conjecture of [5] (see Conjecture A.4), implies a functional equation in the algebraic side.

We remark that the algebraic object in the main conjecture of Fukaya-Kato is Selmer complex \( SC(T, T^0) \) rather than the Selmer group. However, as remarked in [13, 4.5.3, Page 82], we have the equality \([SC(T, T^0)] = [\text{SelGr}_{E_{p^\infty}}(K_{\infty})^{\vee}] \) in \( K_0(H_{G}(G)) \) in the setting and hypotheses of Theorem 6.4. Thus the conjectural functional equation of Fukaya-Kato coincides with the conjectural functional equation in Proposition 6.1 or with [34, Equation 7.12] and via the Iwasawa main conjecture in A.4, shows the compatibility of our algebraic functional equation (as well as of [34]) with that of [13, Theorem 4.2.7].

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Appendix A: Conjectural existence of analytic $p$-adic $L$-functions

In this section, we state a conjectural existence of analytic $p$-adic $L$-function for a given Galois representation $V$. For technical reasons, we restrict ourselves to the situation $K = \mathbb{Q}$ in the analytic side. Also, in order to talk about the algebraic part of special values of Hasse–Weil $L$-functions, we fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ simultaneously.

Let $V$ be a $p$-adic Galois representation of $G_{\mathbb{Q}}$ which satisfies the condition (Geom) stated in Introduction. Then the $L$-function $L(s, V)$ given by the following Euler product is convergent on $\Re(s) > \frac{w}{2} + 1$:

$$L(s, V) = \prod_{q \neq p} \det(1 - \text{Frob}_q^{-1}q^{-s}; V^{I_q})^{-1} \times \det(1 - \varphi p^{-s}; D_{\text{pst}}(V)^{ip})^{-1}$$  \hfill (90)

where $w$ is the weight of motive whose $p$-adic étale realization gives $V$, $D_{\text{pst}}(V)$ is a potentially stable filtered module of Fontaine on which the operator $\varphi$ is acting. It is known that each Euler factor is a polynomial whose coefficients are in $\mathbb{Q}$ and the Euler product is absolutely convergent on $\Re(s) > 1 + \frac{w}{2}$ due to Deligne.

We recall the following well-known conjecture which is a folklore.

**Conjecture A.1** Let $V \cong \mathcal{K}^d$ be a $p$-adic Galois representation of $G_{\mathbb{Q}}$ which satisfies the condition (Geom) stated in Introduction. Then the following statements hold.

1. The $L$-function $L(s, V)$ is meromorphically continued to the whole $\mathbb{C}$-plane with at most finitely many poles. Further, if $V$ does not contain any direct summand which is isomorphic to Tate twists of the trivial representation $\mathbb{Q}_p$, the $L$-function $L(s, V)$ is holomorphic on the whole $\mathbb{C}$-plane.

2. We have the following functional equation:

$$L(s, V) = a(V) \frac{w+1}{2}^{-s} \epsilon(V)L(1-s, V^*(1))$$

where $a(V)$ (resp. $\epsilon(V)$) is the Artin conductor (resp. epsilon factor) for $V$ respectively, which we do not explain here.

**Conjecture A.2** (Deligne) Let $V \cong \mathcal{K}^{2d}$ be a $p$-adic Galois representation of $G_{\mathbb{Q}}$ of even dimension which satisfies the condition (Geom) stated in Introduction. For simplicity, we assume that $V$ does not contain any Artin representations as its subquotient. Further we assume the following conditions

(i) The motive which corresponds to $V$ is critical in the sense of Deligne [9].

(ii) Conjecture A.1 holds true for $V$.

Then there exist two constants $\Omega_+(V), \Omega_-(V)$ such that, for any Artin representation $W$ of $G_{\mathbb{Q}}$ with values in $\mathcal{K}$ such that the motive which corresponds to $V \otimes W$ is critical in the sense of Deligne and Conjecture A.1 holds true for $V \otimes W$, we have

$$\frac{L(0, V \otimes_{\mathcal{K}} W)}{\Omega_+(V)^{dd_+(W)} \Omega_-(V)^{dd_-(W)}} \in \overline{\mathbb{Q}}$$
where $d_+(W)$ (resp. $d_-(W)$) is the dimension of the eigenspace of complex conjugate with eigenvalue $+1$ (resp. $-1$).

Based on the conjectural existence of $L$-values and periods, the conjectural existence of analytic $p$-adic $L$-function was formulated by [5, Conj. 5.7] (for elliptic curves) and [13, Thm 4.2.26] (for general motives).

**Conjecture A.3** Let $V \cong \mathbb{K}^{2d}$ be a $p$-adic Galois representation of $G_{\mathbb{Q}}$ of even dimension which satisfies the condition (Geom) stated in Introduction. For simplicity, we assume that $V$ does not contain any Artin representations as its subquotient. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ and define

$$S = \left\{ f \in \Lambda_\mathcal{O}(G) \left| \frac{\Lambda_\mathcal{O}(G)}{\Lambda_\mathcal{O}(G)f} \text{ is a finitely generated } \Lambda_\mathcal{O}(H)-\text{module} \right. \right\}. \quad (91)$$

Set $S^* = \bigcup_{n \geq 0} p^n S$. Then $S$ and $S^*$ are left and right Ore set in $\Lambda_\mathcal{O}(G)$ [5, §2, §3]. In particular the localization $\Lambda_\mathcal{O}(G)_{S^*}$ is well-defined and we have an exact sequence of $K$-groups

$$K_1(\Lambda_\mathcal{O}(G)) \to K_1(\Lambda_\mathcal{O}(G)_{S^*}) \xrightarrow{\delta_G} K_0(\Lambda_\mathcal{O}(G), \Lambda_\mathcal{O}(G)_{S^*}) = K_0(\mathcal{M}_H(G)) \to 0. \quad (92)$$

Given $M \in \mathcal{M}_H(G)$, we denote by $[M]$ its class in $K_0(\mathcal{M}_H(G))$ and the preimage of $[M]$ in $K_1(\Lambda_\mathcal{O}(G)_{S^*})$ by $\xi_M$. Note that, under the assumption that $G$ has no element of order $p$, the surjectivity of $\delta_G$ was proved in [5, Proposition 3.4].

Also given an element $\xi \in K_1(\Lambda_\mathcal{O}(G)_{S^*})$ and an Artin representation $\eta$ of $G$, [5, §3] associates a canonical evaluation map which gives rise to an element $\eta(\xi) \in \overline{\mathbb{Q}}_p \cup \{\infty\}$.

Further we assume the following conditions

(i) The motive which corresponds to $V$ is critical in the sense of Deligne.

(ii) Conjecture A.1 and Conjecture A.2 hold true for $V$.

(iii) The representation $V$ is crystalline at $p$ in the sense of Fontaine.

(iv) The representation $V$ is of Panchishkin type at $p$ in the sense that $V$ has a $G_{\mathbb{Q}_p}$-stable filtration $F^+_p V \subset V$ of dimension $d$ such that the Hodge-Tate weights of $F^+_p V$ (resp. $V/F^+_p V$) are all negative (resp. non-negative).

For any non-trivial Artin representation $\eta$ of $G$ and for any prime number $q$, we define the Euler factor at $q$ as follows:

$$P_q(\eta, X) = \begin{cases} \det(1 - \text{Frob}_q^{-1} X | (W_\eta)^I_q) & q \neq p, \\ \det(1 - \varphi X | D_{\text{crys}}(W_\eta)) & q = p. \end{cases}$$

Then, we have an analytic function $\mathcal{L}_p(V) \in K_1(\Lambda_\mathcal{O}(G)_{S^*})$, where $\mathcal{O}$ is the ring of integers of a certain finite extension of $\overline{\mathbb{Q}}_p$, such that
\[
\eta^* (L_p(V)) = \epsilon_p (W^*_\eta)^{-d (\alpha^{(1)} \ldots \alpha^{(d)}) - C_p(\eta)} \times \prod_{i=1}^{d} \frac{P_p(\eta^*, (\alpha^{(i)})^{-1})}{P_p(\eta, (\beta^{(i)})^{-1})} \times \frac{L_p(0, V \otimes_K W^*_\eta)}{\Omega_+(V)^{dd_p(W^*_\eta)} \Omega_-(V)^{dd_p(W^*_\eta)}}
\]

for any non-trivial Artin representation \(\eta\) of \(G\), where \(\epsilon_p (W^*_\eta)\) is the local epsilon factor at \(p\) for \(W^*_\eta\), \(C_p(\eta)\) is the \(p\)-order of the conductor of \(\eta\), \(\alpha^{(1)}, \ldots, \alpha^{(d)}\) (resp. \(\beta^{(1)}, \ldots, \beta^{(d)}\)) is the eigenvalues of \(\varphi\)-operator acting on \(D_{\text{cris}}(F_p^+ V)\) (resp. \(D_{\text{cris}}(V/F_p^+ V)\)). Here \(\eta^*\) denotes the contragradient representation of \(\eta\). Also \(P\) denotes the set of primes \(q\) of \(\mathbb{Q}\) such that the image of \(I_{\mathbb{Q}_q}\) in \(G\) is infinite and \(L_p(s, V \otimes_K W^*_\eta)\) means the \(L\)-function which is obtained by removing the Euler factors at every prime \(q \in P\) from the \(L\)-function \(L(s, V \otimes_K W^*_\eta)\).

We can also state the main conjecture in our setting assuming Conjectures A.1, A.2 and A.3.

**Conjecture A.4** (Iwasawa Main Conjecture) Assume Conjectures A.1, A.2 and A.3. Then via the natural map from \(K_1(\Lambda_{\mathcal{O}}(G)_{S^*}) \xrightarrow{\delta} K_0(\mathfrak{M}_H(G))\), the image of \(L_p(V)\) in \(K_0(\mathfrak{M}_H(G))\), \(\delta(L_p(V))\) has the property,

\[
\delta(L_p(V)) = [\text{Sel}_{A}^{BK}(K_{\infty}) \lor] \in K_0(\mathfrak{M}_H(G)).
\]

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