AN ISOPERIMETRIC FUNCTION FOR STALLINGS’ GROUP

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Abstract. We prove that \( n^{7/3} \) is an isoperimetric function for a group of Stallings that is finitely presented but not of type \( \mathcal{F}_3 \).

1. Introduction

In the early 1960s Stallings [9] constructed a group \( S \) enjoying the finiteness property \( \mathcal{F}_2 \) but not \( \mathcal{F}_3 \). (A group is of type \( \mathcal{F}_1 \) when it can be finitely generated, \( \mathcal{F}_2 \) when it can be finitely presented, and more generally \( \mathcal{F}_n \) when it admits an Eilenberg-Maclane space with finite \( n \)-skeleton.) Bieri [2] recognised \( S \) to be

\[
\text{Ker}( F(\alpha, \beta) \times F(\gamma, \delta) \times F(\epsilon, \zeta) \to \mathbb{Z} )
\]

where the map is that from the product of three rank–2 free groups to \( \mathbb{Z} = \langle t \rangle \) which sends all six generators to \( t \), and he showed that using \( (\mathcal{F}_2)^3 \) in place of \( (\mathcal{F}_2)^3 \) gives a family of groups (the Bieri–Stallings groups) of type \( \mathcal{F}_{n-1} \) but not \( \mathcal{F}_n \) [2].

Isoperimetric functions (defined below) for \( S \) have been investigated by a number of authors. Gersten proved that for \( n \geq 3 \), the groups in this family admit quintic isoperimetric functions [6]; this was sharpened to cubic by Baumslag, Bridson, Miller & Short in the case of \( S \) [11 §6]. Bridson [5] argued that whenever \( G_1 \) and \( G_2 \) are finitely presentable groups admitting quadratic isoperimetric functions and epimorphisms \( \phi_i : G_i \to \mathbb{Z} \), if one doubles \( G_1 \times G_2 \) along the kernel of the map \( \phi : G_1 \times G_2 \to \mathbb{Z} \), defined by \( \phi(g_1, g_2) = \phi_1(g_1) + \phi_2(g_2) \), then the resulting group also admits a quadratic isoperimetric function. The Bieri–Stallings groups are examples of such doubles. But Groves found an error in his proof [4, 7], and it seems that Bridson’s approach, in fact, gives cubic isoperimetric functions, generalising the result in [1]. In this article we prove:

Theorem 1. Stallings’ group \( S \) has \( n^{7/3} \) as an isoperimetric function.
If the Dehn function of $S$ is not quadratic (i.e. not $\simeq n^2$, in the sense defined below) then it would be the first example of a subgroup of a CAT(0) group, namely $(F_2)^3$, with Dehn function bounded above by a polynomial, but not $\simeq n^\alpha$ for any $\alpha \in \mathbb{Z}$ — we thank N. Brady for pointing this out. Also, it would be an example of such a Dehn function ‘occurring naturally’ rather than in a group especially constructed for the purpose such as in [3, 8]. If, on the other hand, the Dehn function of $S$ is quadratic then it would show the class of groups with quadratic Dehn functions to be wild enough to contain groups that are not of type $F_3$, fulfilling Bridson’s aim in [5].

Our theorem makes no reference to a specific finite presentation since, as is well–known, if such an isoperimetric function holds for one finite presentation of a group then it holds for all. We will work with the presentation

$$\langle a, b, c, d, s \mid [a, c], [a, d], [b, c], [b, d], s^a = s^b = s^c = s^d \rangle$$

for $S$ of [1, 6], with $s^a = s^b = s^c = s^d$ shorthand for the six defining relations $s^a s^{-b}, s^a s^{-c}, s^a s^{-d}, s^b s^{-c}, s^b s^{-d}, s^c s^{-d}$. One can view $S$ as an HNN-extension of the product of free groups $F(a, b) \times F(c, d)$ with stable letter $s$ commuting with all elements represented by words on $a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}$ of zero exponent-sum. Gersten [6] shows this is related to the expression for $S$ as a kernel (1) via $a = \epsilon \alpha^{-1}, b = \epsilon \beta^{-1}, c = \epsilon \gamma^{-1}, d = \epsilon \delta^{-1}, s = \epsilon \zeta^{-1}$.

Essentially, our strategy for establishing an $n^{7/3}$ isoperimetric function is to interplay two approaches to reducing words $w$ representing 1 to $\varepsilon$. Both involve identifying a suitable subword $s^{\pm 1} \tau s^{\pm 1}$ of $w$, where $\tau = \tau(a, b, c, d)$, then converting $\tau$ to a word $\hat{\tau}$ in which the letters alternate between positive and negative exponent, and then cancelling off the $s^{\pm 1}$ with the $s^{\pm 1}$. Repeating until all $s^{\pm 1}$ have been eliminated gives a word on $a, b, c, d$ that represents 1 in $F(a, b) \times F(c, d)$.

In the first of these two approaches (Algorithm I) all $a^{\pm 1}, b^{\pm 1}$ are shuffled to the start of $\tau$, leaving all the $c^{\pm 1}, d^{\pm 1}$ at the end, and then letters $a^{\pm 1}, c^{\pm 1}$ are inserted to achieve the word $\hat{\tau}$ in the required alternating form. The cost (see below) of converting $\tau$ to $\hat{\tau}$ in this way is potentially great: it can be as much as $\sim \ell^{2}(\tau)$; however control on the length of $\hat{\tau}$ is good: it is always no more than $3\ell(\tau)$.

The second approach (Algorithm III) is to work through $\tau$ from left to right inserting letters $a^{\pm 1}, c^{\pm 1}$ as necessary to achieve alternating form. The cost of this algorithm and the length of its output are heavily dependent on the internal structure of $\tau$, but if $\tau$ possesses certain properties then good bounds can be found.
In both cases the cost of cancelling off the $s^±_1 s^{±}_1$ is $\sim \ell(\hat{\tau})$. Used alone, either approach would lead to a cubic isoperimetric inequality.

**Basic definitions.** $[x, y] := x^{-1}y^{-1}xy$, $x^y := y^{-1}xy$, $x^{-y} := y^{-1}x^{-1}y$. Write $u = u(a_1, \ldots, a_k)$ when $u$ is a word on the letters $a_i^{±1}, \ldots, a_k^{±1}$. The length of $u$ as a word (with no free reductions performed) is $\ell(u)$. The total number of occurrences of letters $a_1^{±1}, \ldots, a_l^{±1}$ in $u$ is $\ell_{a_1, \ldots, a_l}(u)$. Unless otherwise indicated, we consider two words to be the equal when they are identical letter-by-letter.

Given words $w, w'$ representing the same element of a group with finite presentation $\langle A | R \rangle$, one can convert $w$ to $w'$ via a sequence of words $W = (w_i)_{i=0}^m$ in which $w_0 = w$, $w_m = w'$ and for each $i$, $w_{i+1}$ is obtained from $w_i$ by free reduction ($w_i = \alpha a a^{-1} \beta \mapsto \alpha \beta = w_{i+1}$ where $a \in A^{±1}$), by free expansion (the inverse of a free reduction), or by applying a relator ($w_i = \alpha u \beta \mapsto \alpha v \beta = w_{i+1}$ where a cyclic conjugate of $uv^{-1}$ is in $R^{±1}$). The cost of $W$ is the number of $i$ such that $w_i \mapsto w_{i+1}$ is an application-of-a-relator move. If $w$ represents the identity (i.e. is null-homotopic) then $\text{Area}(w)$ is defined to be the minimal cost amongst all $W$ converting $w$ to the empty word $\varepsilon$, and the **Dehn function** $\text{Area} : \mathbb{N} \to \mathbb{N}$ of $\langle A | R \rangle$ is

$$\text{Area}(n) := \max \{ \text{Area}(w) \mid w = 1 \text{ in } \Gamma \text{ and } \ell(w) \leq n \}.$$ 

An isoperimetric function for $\langle A | R \rangle$ is any $f : \mathbb{N} \to \mathbb{N}$ such that there exists $K > 0$ for which $\text{Area}(n) \leq K f(n)$ for all $n$. (The constant $K$ is not used by all authors, but is convenient for us here.)

For $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ when $\exists C > 0, \forall n \in \mathbb{N}, f(n) \leq C g(Cn + C) + Cn + C$, and we say $f \simeq g$ when $f \preceq g$ and $g \preceq f$.

**Article organisation.** We give a number of definitions, lemmas and algorithms in Section 2. In Section 3 we use these to prove Theorem 1.

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## 2. Preliminaries.

Our proof of Theorem 1 will involve three classes of words.

**Definition 2.** (Alternating words.) A word $u = u(a, b, c, d)$ is alternating if it is a concatenation of words $xy^{-1}$ in which $x, y \in \{a, b, c, d\}$.
The reader familiar with van Kampen diagrams and corridors (also known as bands) may find it helpful to note that alternating words are those which, after removing all $aa^{-1}, bb^{-1}, cc^{-1}$ and $dd^{-1}$ subwords, can be read along the sides of $s$-corridors in van Kampen diagrams over $S$.

**Definition 3.** *(Balanced words.)* A word $u = u(a, b, c, d, s)$ is balanced if it has exponent sum zero and in $S$ it represents an element of the subgroup $\langle a, b, c, d \rangle$.

The following algorithm converts a word $u = u(a, b, c, d)$ of exponent-sum zero into an alternating word of a preferred form that represents the same element of $S$.

**Algorithm I.**

Input a word of exponent-sum zero $u = u(a, b, c, d)$.

1. Shuffle the letters $a^{\pm 1}, b^{\pm 1}$ to the start of $u$ and freely reduce to give a word $\mu \lambda$ where $\mu = \mu(a, b)$ and $\lambda = \lambda(c, d)$.
2. Intersperse $c^{\pm 1}$ through $\mu$ to give a word $\overline{\mu} = \overline{\mu}(a, b, c)$, and $a^{\pm 1}$ through $\lambda$ to give $\overline{\lambda} = \overline{\lambda}(a, c, d)$, such that:
   (a) $\overline{\mu}$ and $\overline{\lambda}$ are alternating,
   (b) for all $1 \leq i < \ell(\overline{\mu})/2$, exactly one of the $(2i - 1)$-st and $(2i)$-th letters in $\overline{\mu}$ is $c$ or $c^{-1}$, and
   (c) for all $1 \leq j < \ell(\overline{\lambda})/2$, exactly one of the $(2j - 1)$-st and $(2j)$-th letters in $\overline{\mu}$ is $a$ or $a^{-1}$.
3. Let $\kappa$ be the exponent-sum of $\mu$. Insert $(ac^{-1})^\kappa$ between $\overline{\mu}$ and $\overline{\lambda}$.

Output $\overline{\mu}(ac^{-1})^\kappa \overline{\lambda}$.

**Definition 4.** *(Preferred alternating words.)* A word $v = v(a, b, c, d)$ is in preferred alternating form if it there is some $u$ such that the output of Algorithm I on input $u$ is $v$.

**Lemma 5.** The output $\overline{\mu}(ac^{-1})^\kappa \overline{\lambda}$ of Algorithm I has length at most $3\ell(u)$ and $u$ can be converted to $\overline{\mu}(ac^{-1})^\kappa \overline{\lambda}$ at a cost of at most $10\ell(u)^2$.

Proof. The exponent sum of $\mu$ is $\kappa$ and so that of $\lambda$ is $-\kappa$. So $|2\kappa| \leq \ell(\mu \lambda) \leq \ell(u)$ and $\ell(\overline{\mu}(ac^{-1})^\kappa \overline{\lambda}) \leq \ell(\overline{\mu} \overline{\lambda}) + |2\kappa| \leq 3\ell(u)$.

The (crude) upper bound of $10\ell(u)^2$ on the cost of converting $u$ to $\overline{\mu}(ac^{-1})^\kappa \overline{\lambda}$ holds because both $u$ and $\overline{\mu}(ac^{-1})^\kappa \overline{\lambda}$ can be converted to $\mu \lambda$ by shuffling letters and freely reducing at costs of at most $\ell(u)^2$ and $\ell(\overline{\mu}(ac^{-1})^\kappa \overline{\lambda})^2 \leq (3\ell(u))^2$, respectively. □

The following lemma reveals balanced words to be those representing elements of the subgroup of $F(a, b) \times F(c, d)$ commuting with $s$ in the HNN-presentation of $S$. We denote the centraliser of $s$ in $S$ by $C_S(s)$. 


Lemma 6. A word $u = u(a, b, c, d, s)$ represents an element $g$ of 
\[ \langle a, b, c, d \rangle \cap C_S(s) \]
in $S$ if and only if $u$ is balanced.

Proof. Note that all the relations of presentation \[ \[\text{2}\] \] have exponent-sum zero, so this quantity is preserved whenever a relation is applied to a word. Thus, if two words on the letters $a, b, c, d, s$ represent the same element in $S$ then they have the same exponent sum.

A word $u$ represents an element of $\langle a, b, c, d \rangle \cap C_S(s)$ if and only if there exists an alternating word $v = v(a, b, c, d)$ with $u$ and $v$ representing the same element of $S$. Since any word on the letters $a, b, c, d$ with exponent-sum zero can be converted into an alternating word by an application of Algorithm I, this is if and only if there exists a word $v = v(a, b, c, d)$ with $u = v$ in $S$ and with $v$ having exponent-sum zero. And by the above remark this is if and only if $u$ represents an element of $\langle a, b, c, d \rangle$ and itself has exponent-sum zero. \[ \square \]

Lemma 7. Suppose word $v_0 = v_0(a, b, c, d, s)$ is expressed as $v_0 = \alpha v_1 \beta$ in which $v_1$ is a balanced subword. Then $v_0$ is balanced if and only if $\alpha \beta$ is balanced.

Proof. Induct on $\ell_s(v_0)$, with the base case $\ell_s(v_0) = 0$ immediate and the induction step an application of Britton’s Lemma. Alternatively, this result is an observation on the layout of $s$-corridors in a van Kampen diagram demonstrating that $v_0$ equates to some alternating word in $S$. \[ \square \]

The next lemma concerns the existence of balanced subwords within prescribed length-bounds in balanced words.

Lemma 8. If $\mu = \mu(a, b, c, d, s)$ is a balanced word with $\ell(\mu) \geq 4$, then for all $k \in [4, \ell(\mu)]$ there is a balanced subword $u$ of $\mu$ with $k/2 \leq \ell(u) \leq k$.

Proof. We induct on $\ell(\mu)$. First we identify certain balanced subwords $\alpha$ and $\beta$ in $\mu$.

Case: $\mu$ starts with a letter $x = s^{\pm 1}$. By Britton’s Lemma, $\mu = x\alpha y \beta$ for $y = x^{-1}$ and for some balanced subword $\alpha$.

Case: $\mu$ starts with a letter $x \neq s^{\pm 1}$. Set a counter to 0, then read through $\mu$ from left to right altering the counter as follows. If the letter being read is not $s^{\pm 1}$ then add the exponent of that letter to the counter. If it is $s^{\pm 1}$ then by Britton’s Lemma that $s^{\pm 1}$ is the first letter of a subword $s^{\pm 1} \gamma s^{\mp 1}$ such that $\gamma$ is balanced; hold the counter constant throughout $s^{\pm 1} \gamma s^{\mp 1}$ and then continue as before. As $\mu$ is balanced, the
Algorithm II.

which plays an important role in our analysis of Algorithm III. 

Let \( \ell P = k \) of this algorithm, the group element represented will not be preserved.

Let \( x, y \) be the difference in length of the input and output words, a quantity \( \ell \) with \( \mu = x\alpha y\beta \) in which \( \alpha \) is balanced.

In both cases, as \( \alpha \) is balanced, so is \( x\alpha y \), and hence so is \( \beta \).

Now, in the base case of the induction we have \( \ell(\mu) = 4 \) and so \( k = 4 \), and we can take \( u = \mu \). Indeed, whenever \( \ell(\mu) = k \) we can take \( u = \mu \), so let us assume henceforth that \( \ell(\mu) > k \).

For the induction step, first suppose \( \beta \neq \varepsilon \). If \( \max(\ell(x\alpha y), \ell(\beta)) \geq k \) then as \( \ell(x\alpha y), \ell(\beta) < \ell(\mu) \) we can apply the induction hypothesis to obtain \( u \). If \( \max(\ell(x\alpha y), \ell(\beta)) < k \) then both \( x\alpha y \) and \( \beta \) have length less than \( k \) and, as \( k < \ell(\mu) = \ell(x\alpha y) + \ell(\beta) \), either \( \ell(x\alpha y) \) or \( \ell(\beta) \) is at least \( k/2 \) and so serves as \( u \).

Finally suppose \( \beta = \varepsilon \). If \( \ell(\alpha) \geq k \) then, as \( \ell(\alpha) < \ell(\mu) \), the induction hypothesis gives us \( u \). If \( \ell(\alpha) < k \) then \( \alpha \) serves as \( u \) because \( \ell(\alpha) = \ell(\mu) - 2 > k - 2 \geq k/2 \) since \( k \geq 4 \). \( \square \)

The remainder of this section works towards Algorithm IV which will convert a balanced word \( u = u(a, b, c, d, s) \) into a preferred alternating word \( v \) representing the same element of \( S \).

The next algorithm concerns converting a word \( \tau_0 = \tau_0(a, b, c, d) \) into alternating form by working through it from left to right inserting letters \( a^{\pm 1} \) as needed. In contrast to Algorithm III, which is an elaboration of this algorithm, the group element represented will not be preserved. The purpose of this algorithm is to define a number \( P(\tau_0) \) which will be the difference in length of the input and output words, a quantity which plays an important role in our analysis of Algorithm III.

**Algorithm II.**

Input a word \( \tau = \tau(a, b, c, d) \). Define \( \beta_i \) to be the length-\( (\ell(\tau) - i) \) suffix of \( \tau \). Define \( \tau_0 := \tau \) and \( \alpha_0 := \varepsilon \).

The algorithm will produce a sequence of words \( (\tau_i)_{i=0}^{\ell(\tau)} \) of the form \( \tau_i = \alpha_i\beta_i \), where \( \alpha_i \) is an alternating word or an alternating word concatenated with an \( a, b, c, d \). For \( 0 \leq i < \ell(\tau) \), obtain \( \alpha_{i+1} \) from \( \alpha_i \) as follows. We have \( \beta_i = x\beta_{i+1} \) for some letter \( x \).

Case: \( \ell(\alpha_i) \) is even.

- If \( x \in \{a, b, c, d\} \), then \( \alpha_{i+1} := \alpha_ix \).
- If \( x \in \{a^{-1}, b^{-1}, c^{-1}, d^{-1}\} \), then \( \alpha_{i+1} := \alpha_iax \).

Case: \( \ell(\alpha_i) \) is odd.

- If \( x \in \{a, b, c, d\} \), then \( \alpha_{i+1} := \alpha_i a^{-1}x \).
- If \( x \in \{a^{-1}, b^{-1}, c^{-1}, d^{-1}\} \), then \( \alpha_{i+1} = \alpha_ix \).

Output \( \tau_\ell(\tau) \).
Definition 9. For words $\tau = \tau(a, b, c, d)$, define $P(\tau)$ to be the number of letters $a^{\pm 1}$ inserted by Algorithm II on input $\tau$.

Lemma 10. Let $\Pi$ be a collection of $p$ disjoint alternating subwords of a word $\tau = \tau(a, b, c, d)$, and let $\bar{\tau}$ be the word formed from $\tau$ by removing all the subwords specified by $\Pi$. Then $P(\tau) \leq P(\bar{\tau}) + 2p$.

Proof. For a letter $l \in \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}\}$ write $\chi(l) \in \{\pm 1\}$ for the exponent of $l$. For a word $w = w(a, b, c, d)$ write $w[i]$ for the $i^{th}$ letter of $w$. For $i \in \{2, \ldots, \ell(w)\}$ define

$$d_i(w) = \begin{cases} 1 & \text{if } \chi(w[i]) = \chi(w[i-1]), \\ 0 & \text{if } \chi(w[i]) \neq \chi(w[i-1]), \end{cases}$$

and define

$$d_1(w) = \begin{cases} 1 & \text{if } \chi(w[1]) = -1, \\ 0 & \text{if } \chi(w[1]) = 1. \end{cases}$$

Note that, during the running of Algorithm II on a word $\tau$, an $a^{\pm 1}$ is inserted during the transition from $\tau_{i-1}$ to $\tau_i$ precisely when $d_i(w) = 1$.

Thus $P(\tau) = \sum_{i=1}^{\ell(\tau)} d_i(\tau)$.

By induction, it suffices to prove the lemma in the case $p = 1$. Suppose $\tau = uvw$ and $\bar{\tau} = uw$ for some words $u, v, w$ with $v$ alternating. Note that:

$$d_i(\tau) = d_i(u) = d_i(\bar{\tau}) \quad i = 1, \ldots, \ell(u);$$
$$d_{\ell(u)+i}(\tau) = d_i(v) = 0 \quad i = 2, \ldots, \ell(v);$$
$$d_{\ell(uv)+i}(\tau) = d_i(w) = d_{\ell(u)+i}(\bar{\tau}) \quad i = 2, \ldots, \ell(w).$$

Thus

$$P(\tau) = \sum_{i=1}^{\ell(\tau)} d_i(\tau)$$

$$= \left[ \sum_{i=1}^{\ell(u)} d_i(\bar{\tau}) \right] + d_{\ell(u)+1}(\tau) + d_{\ell(uv)+1}(\tau) + \left[ \sum_{i=2}^{\ell(w)} d_{\ell(u)+i}(\bar{\tau}) \right]$$

$$\leq \left[ \sum_{i=1}^{\ell(\tau)} d_i(\bar{\tau}) \right] + d_{\ell(u)+1}(\tau) + d_{\ell(uv)+1}(\tau)$$

$$\leq P(\bar{\tau}) + 2.$$

□
**Definition 11.** For words \( \sigma = \sigma(a, b, c, d, s) \), define \( Q(\sigma) \) to be the number of times letters \( b^{\pm 1} \) alternate with letters \( d^{\pm 1} \) in \( \sigma \). More precisely, if the word obtained from \( \sigma \) by deleting all letters \( a^{\pm 1}, c^{\pm 1}, s^{\pm 1} \) is \( \mu_1\nu_1\mu_2\nu_2\ldots\mu_q\nu_q \), in which \( \nu_1, \mu_2, \nu_2, \ldots, \nu_{q-1}, \mu_q \neq \varepsilon \) and \( \mu_i = \mu_i(b) \) and \( \nu_i = \nu_i(d) \) for all \( i \), then \( Q(\sigma) = q \).

**Definition 12.** For words \( \sigma = \sigma(a, b, c, d, s) \), define \( R(\sigma) \) to be the maximum over all suffixes \( \beta \) of \( \sigma \) of the absolute value of the exponent sum of \( \beta \).

The following algorithm works through a word \( \tau_0 = \tau_0(a, b, c, d) \) from left to right inserting letters \( a^{\pm 1}, c^{\pm 1} \) without changing the element of \( F(a, b) \times F(c, d) \) it represents. If the exponent-sum of \( \tau \) is zero then the output will be alternating.

**Algorithm III.**

Input a word \( \tau_0 = \tau_0(a, b, c, d) \). Define \( \beta_i \) to be the length-\((\ell(\tau) - i)\) suffix of \( \tau_0 \). Define \( a_0 := \varepsilon \) and \( \Delta_0 := \varepsilon \).

The algorithm will produce a sequence of words \((\tau_i)_{i=0}^{\ell(\tau)}\) of the form \( \tau_i = \alpha_i\Delta_i\beta_i \), where \( \alpha_i \) is an alternating word or an alternating word concatenated with an \( a, b, c, d \), and \( \Delta_i \) is \( a^r \) or \( c^r \) for some \( r \in \mathbb{Z} \).

For \( 0 \leq i < \ell(\tau) \), obtain \( \alpha_{i+1}, \Delta_{i+1} \) from \( \alpha_i, \Delta_i \) as follows. We have \( \beta_i = x\beta_{i+1} \) for some letter \( x \).

**Case (1)** \( \Delta_i = a^r \) for some \( r \in \mathbb{Z} \setminus \{0\} \) and \( \ell(\alpha_i) \) is even.

1.1 If \( x \in \{a, c, d\} \) then \( \alpha_{i+1} := \alpha_i x \) and \( \Delta_{i+1} := \Delta_i \).
1.2 If \( x = b \) then \( \alpha_{i+1} = \alpha_i(ac^{-1})^r x \) and \( \Delta_{i+1} := c^r \).
1.3 If \( x \in \{a^{-1}, c^{-1}, d^{-1}\} \) then \( \alpha_{i+1} := \alpha_i a x \) and \( \Delta_{i+1} := a^{-r} \).
1.4 If \( x = b^{-1} \) then \( \alpha_{i+1} = \alpha_i(ac^{-1})^r cx \) and \( \Delta_{i+1} := c^{-r} \).

**Case (2)** \( \Delta_i = a^r \) for some \( r \in \mathbb{Z} \setminus \{0\} \) and \( \ell(\alpha) \) is odd.

2.1 If \( x \in \{a, c, d\} \) then \( \alpha_{i+1} := \alpha_i a^{-1} x \) and \( \Delta_{i+1} := a^{r+1} \).
2.2 If \( x = b \) then \( \alpha_{i+1} := \alpha_i(ac^{-1})^r c^{-1} x \) and \( \Delta_{i+1} := c^{r+1} \).
2.3 If \( x \in \{a^{-1}, c^{-1}, d^{-1}\} \) then \( \alpha_{i+1} = \alpha_i x \) and \( \Delta_{i+1} := \Delta_i \).
2.4 If \( x = b^{-1} \) then \( \alpha_{i+1} := \alpha_i(c^{-1}a)^r x \) and \( \Delta_{i+1} := c^r \).

When \( \Delta_i = c^r \) for some \( r \in \mathbb{Z} \), obtain \( \alpha_{i+1} \) and \( \Delta_{i+1} \) similarly, but with \( a, b \) interchanging roles with \( c, d \). Call the cases involved \((3.1–3.4)\) and \((4.1–4.4)\).

Output \( \tau_{\ell(\tau)} \).

**Lemma 13.** Suppose \( \tau = \tau(a, b, c, d) \) is a word of exponent sum zero. Then Algorithm III converts \( \tau \) to an alternating word \( \hat{\tau} \) with \( \ell(\hat{\tau}) \geq \ell(\tau) \), with \( Q(\hat{\tau}) = Q(\tau) \), and with
\[
(3) \quad \ell(\hat{\tau}) - \ell(\tau) \leq P(\tau) + 4(R(\tau) + 1)Q(\tau).
\]
Moreover, the cost of transforming $\tau$ to $\hat{\tau}$ is at most

$$(4) \quad (R(\tau) + 2)\ell(\tau) + 2(R(\tau) + 2)^2Q(\tau).$$

Proof. As the exponent sum of each $\tau_{i+1}$ is the same as that of $\tau_i$, it remains at zero throughout the run of the algorithm and $\Delta_{\ell(\tau)}$ must be $\varepsilon$. It follows that $\hat{\tau} = \tau_{\ell(\tau)}$ is alternating.

If one removes all letters $a^{+1}$ and $c^{+1}$ from $\hat{\tau}$ and $\tau$, they become identical words, and so $Q(\hat{\tau}) = Q(\tau)$.

The transformation $\tau_i$ to $\tau_{i+1}$ can be achieved at a cost of at most $|r| + 1$ in Cases $\ast.1, \ast.3$ and at most $(|r| + 1)^2$ in Cases $\ast.2, \ast.4$, where $r$ is the exponent in $\Delta_i$. In every instance,

$$(5) \quad |r| \leq R(\tau) + 1.$$

Suppose removing all letters $a^{+1}$ and $c^{+1}$ from $\tau$ gives $\mu_1 \nu_1 \nu_2 \ldots \mu_q \nu_q$, in which $\nu_1, \mu_2, \nu_2, \ldots, \nu_{q-1}, \mu_q \neq \varepsilon$ and $\mu_i = \mu_i(b)$ and $\nu_i = \nu_i(d)$ for all $i$. By definition, $Q(\tau) = q$. The process described above will carry a power of $c$ through the word from the left until it hits $\nu_1$, when it will be converted to a power of $a$, which will then be carried until it hits $\mu_2$ when it reverts to a power of $c$, and so on. So Cases $\ast.2$ and $\ast.4$ are invoked either $2Q(\tau) - 1$ or $2Q(\tau) - 2$ times depending on whether or not $\nu_q = \varepsilon$. Cases $\ast.1$ and $\ast.3$ are invoked the remaining $\ell(\tau) - 2Q(\tau) + 1$ or $\ell(\tau) - 2Q(\tau) + 2$ times. Combining these estimates we see that the total cost of converting $\tau$ to $\hat{\tau}$ is at most

$$(6) \quad (R(\tau) + 2)(\ell(\tau) - 2Q(\tau) + 2) + (R(\tau) + 2)^2(2Q(\tau) - 1),$$

which, discarding some negative terms and noting that $Q(\tau) \geq 1$, gives (4).

The length estimate (3) comes from counting the letters deposited into $\tau$ in the above process en route to reaching $\hat{\tau}$. They occur in two forms. (Note: we do not consider the powers of $a$ and $c$ carried through the word as deposited.) Firstly, there are the single letters $a^{+1}$ or $c^{+1}$ inserted in Cases 1.3, 1.4, 2.1, 2.2, 3.3, 3.4, 4.1, and 4.2. These total $P(\tau)$. And, secondly, there are the $(ac^{-1})^r$ of Cases 1.2 and 1.4, the $(c^{-1}a)^r$ of 2.2 and 2.4, the $(ca^{-1})^r$ of 3.2 and 3.4, and the $(a^{-1}c)^r$ of 4.2 and 4.4. These cases occur less than $2Q(\tau)$ times and by (5) each inserts a word of length at most $2|r| \leq 2(R(\tau) + 1)$. □

Our next algorithm transforms a balanced word on $a, b, c, d, s$ to a word in preferred alternating form that represents the same element of $S$. 

Algorithm IV.
Input a balanced word \( u = u(a, b, c, d, s) \). Define \( u_0 := u \) and \( L := \ell_s(u)/2 \). Then for \( 0 \leq i < L \) recursively obtain \( u_{i+1} \) from \( u_i \) by the following two steps.

(A) Locate a subword \( s^\pm \tau_is^\pm \) in \( u_i \) such that \( \tau_i = \tau_i(a, b, c, d) \) and has zero exponent sum (which, by Britton’s Lemma, we know exists). Use Algorithm III to transform \( \tau_i \) into alternating form \( \hat{\tau}_i \).

(B) Shuffle the \( s^\pm \) through \( \hat{\tau}_i \) and cancel it with the \( s^\pm \) to give \( u_i+1 \).

This produces \( u_L \), which contains no letters \( s^\pm \). Next –

(C) Reverse every instance of Step A to get a word \( \bar{u} \), which is \( u \) with all letters \( s^\pm \) deleted.

(D) Run Algorithm I on \( \bar{u} \) to give a word \( v \) in preferred alternating form.

Output \( v \).

Lemma 14. Suppose \( u = u(a, b, c, d, s) \) is a balanced word with \( 2 \leq \ell(u) \leq m \). Suppose \( \Pi \) is some collection of at most \( p \) disjoint subwords in \( u \), each in preferred alternating form, such that deleting these subwords leaves a word of length at most \( k \). Then Algorithm IV transforms \( u \) into \( v \) at a cost of no more than

\[
80k^3 + 75k^2p + 16m^2.
\]

Proof. By Lemma 13, performing Step A on \( \tau_i \) costs at most

\[
(7) \quad (R(\tau_i) + 2)\ell(\tau_i) + 2(R(\tau_i) + 2)^2Q(\tau_i).
\]

Now, for all \( i \),

\[
(8) \quad Q(\tau_i) \leq Q(u_i) = Q(u) \leq k + p,
\]

as letters \( b^\pm \) alternate with letters \( d^\pm \) at most once in each preferred–alternating–form subword of \( u \), and transformations as per Lemma 13 do not alter \( Q \).

Note that if \( \sigma' \) is obtained from a word \( \sigma \) by deleting a collection of disjoint alternating subwords, then \( R(\sigma) \leq R(\sigma') + 1 \). We recursively define a collection \( \Pi_i \) of disjoint alternating subwords of \( u_i \) in the letters \( a^\pm, b^\pm, c^\pm, d^\pm \) by \( \Pi_0 := \Pi \) and for \( 0 \leq i < L \),

\[
\Pi_{i+1} := (\Pi_i \setminus \Pi_i') \cup \{\hat{\tau}_i\}.
\]

Let \( \Pi_i' \) be the subset of \( \Pi_i \) consisting of those words which have letters in common with \( \tau_i \). Note that each word in \( \Pi_i' \) is a subword of \( \tau_i \) since it contains no occurrence of a letter \( s^\pm \). Removing the subwords \( \Pi_i' \) from \( \tau_i \) produces a word \( \tau_i' \) whose letters all originate in \( u \) but not in
any of its subwords $\Pi$. If $i \neq j$ then $\tau'_i$ and $\tau'_j$ originate from different letters in $u$ so

$$\sum_{i=0}^{L-1} \ell(\tau'_i) \leq k.$$  

Furthermore, since $L \leq k/2$ and $R(\tau'_i) \leq \ell(\tau'_i)$ one has

$$\sum_{i=0}^{L-1} R(\tau_i) \leq \sum_{i=0}^{L-1} (R(\tau'_i) + 1) \leq \frac{k}{2} + \sum_{i=0}^{L-1} \ell(\tau'_i) \leq \frac{3k}{2}.$$  

Similarly

$$P(\tau_i) \leq \ell(\tau'_i) + 2|\Pi'_i| \leq \ell(\tau'_i) + 2|\Pi_i| \leq \ell(\tau'_i) + 2p + k$$

for all $i$, by Lemma 10 applied to removing the subwords $\Pi'_i$ from $\tau_i$ and noting that $|\Pi_{j+1}| \leq |\Pi_j| + 1$ for all $j$ and that $i \leq L \leq k/2$. It follows from 11 and 9 that

$$\sum_{i=0}^{L-1} P(\tau_i) \leq k + pk + \frac{k^2}{2} \leq 2k^2 + pk.$$  

Now, for all $j$,

$$\ell(u_{j+1}) - \ell(u_j) = \ell(\hat{\tau}_j) - \ell(\tau_j) \leq P(\tau_j) + 4(R(\tau_j) + 1)Q(\tau_j) \leq P(\tau_j) + 4(R(\tau_j) + 1)(k + p),$$

where the first and second inequalities are applications of 3 and 8, respectively. So, for all $i \leq L$,

$$\ell(u_i) \leq m + \sum_{j=0}^{i-1} (P(\tau_j) + 4(R(\tau_j) + 1)(k + p)) \leq m + 2k^2 + pk + (6k + 4i)(k + p) \leq m + 2k^2 + pk + 8k(k + p) \leq m + 10k^2 + 9kp$$

where the first inequality uses $\ell(u_0) = m$ and 13, the second uses 10 and 12, and the third uses the inequality $i \leq L \leq k/2$. We thus find

$$\sum_{i=0}^{L-1} \ell(u_i) \leq \frac{1}{2}km + 5k^3 + \frac{9}{2}k^2p.$$
The total cost of all instances of Step A, including those implemented within Step C, is at most

\[
2 \sum_{i=0}^{L-1} [(R(\tau_i) + 2)\ell(\tau_i) + 2(R(\tau_i) + 2)^2Q(\tau_i)]
\]

\[
\leq 2 \max_i (\ell(\tau_i)) \sum_{i=0}^{L-1} (R(\tau_i) + 2) + 4 \max_i Q(\tau_i) \left[ \sum_{i=0}^{L-1} (R(\tau_i) + 2) \right]^2
\]

\[
\leq 2(m + 10k^2 + 9kp) \left( \frac{3}{2}k + k \right) + 4(k + p) \left( \frac{3}{2}k + k \right)^2
\]

(16) \quad \leq 75k^3 + 70k^2p + 5mk

where the initial estimate comes from (7), and the second inequality uses \( L \leq k/2, \ell(\tau_i) \leq \ell(u_i), (8), (10), \) and (14).

The total cost of all instances of Step B is \( \sum_{i=0}^{L-1} \ell(\hat{\tau}_i) \) which, by (15), is at most \( \frac{1}{2}km + 5k^3 + \frac{9}{2}k^2p \), as \( \ell(\hat{\tau}_i) \leq \ell(u_i) \). As \( \ell(\bar{\tau}) \leq m \), Lemma 5 tells us that the cost of Step D is at most \( 10m^2 \). Summing these three cost estimates gives the total cost as at most

\[
\left( \frac{1}{2}km + 5k^3 + \frac{9}{2}k^2p \right) + \left( 75k^3 + 70k^2p + 5mk \right) + 10m^2
\]

\[
\leq 80k^3 + \frac{149}{2}k^2p + \frac{11}{2}km + 10m^2
\]

(17) \quad \leq 80k^3 + 75k^2p + 16m^2.

where for the final inequality we have used that \( k \leq m \).

3. PROOF OF THEOREM 1

Our final algorithm concerns converts null-homotopic words in \( S \) to \( \varepsilon \). Its cost analysis will establish Theorem 1 (The finitely many \( w \) of length less than 8 are irrelevant for the asymptotics of the Dehn function of \( S \).) It constructs a sequence of null-homotopic words \( (w_i)_{i=0}^t \) and a subset \( T_i \) of \( \{1, 2, \ldots, \ell(w_i)\} \) specifying a collection of the letters of \( w_i \) by their locations.
Algorithm V.
Input a word $w$ of length at least 8 representing 1 in $S$. Let $n := \ell(w)$.
Input a parameter $k \in [4, n]$.
Define $w_0 := w$ and let $T_0$ be the empty set. For successive $i$ such that $T_i \neq \{1, 2, \ldots, \ell(w_i)\}$, obtain $w_{i+1}$ and $T_{i+1}$ from $w_i$ and $T_i$ by performing the following steps.

(i) Let $\bar{w}_i$ be the word obtained from $w_i$ by deleting the letters in the positions $T_i$. If $\ell(\bar{w}_i) \leq k$ then define $u := w_i$. If $\ell(\bar{w}_i) > k$ then let $\bar{u}$ be a balanced subword of $\bar{w}_i$ with $k/2 \leq \ell(\bar{u}) \leq k$ and take $u$ to be the longest subword of $w_i$ which reduces to $\bar{u}$ when all the letters specified by $T_i$ are removed. In either case, $u$ is a balanced subword of $w_i$ of which between $k/2$ and $k$ letters are not in positions in $T_i$.

(ii) Use Algorithm IV to convert $u$ to a word $v$ in preferred alternating form. Obtain $w_{i+1}$ from $w_i$ by replacing $u$ with $v$. Let $T_{i+1}$ be the locations in $w_{i+1}$ of the letters of $v$ and of the letters that originate in $w_i$ (but not in $u$) and have locations in $T_i$.

Define $l := i + 1$ where $i$ is the value in the final run of Steps (i) and (ii).

(iii) Reduce $w_l$ to $\varepsilon$ by shuffling the letters $a^{\pm 1}, b^{\pm 1}$ to the start of the word and then freely reducing.

Notes. Steps (i) and (ii) are repeated at most $((n - k)/(k/2)) + 1 = (2n/k) - 1$ times and so $l \leq 2n/k$.

The reason $w_i = w_{i+1}$ for all $0 \leq i < l$ is that the words $u$ and $v$ in Step (ii) represent the same element of $S$. One effect of each instance of Step (ii) is to remove any letters $s^{\pm 1}$ in $u$. In particular none of the letters of $w_i$ specified by $T_i$ are $s^{\pm 1}$ and so there are no $s^{\pm 1}$ in $w_i$.

For all $i$, the letters of $w_i$ specified by $T_i$ comprises $\leq i$ subwords, each in preferred alternating form. (The number of these subwords rises by at most one with each run of Steps (i) and (ii).)

The existence of the $u$ of Step (i) follows from Lemmas 7 and 8 as follows. As $w_i$ is null-homotopic and hence balanced, Lemma 7 applies and tells us that $\bar{w}_i$ is also balanced, since $T_i$ specifies a number of alternating subwords in $w_i$. As $\ell(\bar{w}_i) > k \geq 4$, Lemma 8 applies and tells us that $\bar{w}_i$ contains a balanced subword $\bar{u}$ with $k/2 \leq \ell(\bar{u}) \leq k$.

An appeal to Lemma 7 tells us that $u$ is balanced. In the final case, where $u$ is $w_{l-1}$, we see that $u$ is balanced because it is null-homotopic.

The viability of Step (iii) follows from the facts that $w_l$ is null-homotopic in $S$ and contains no letters $s^{\pm 1}$, and so is null-homotopic.
in
\[ F(a, b) \times F(c, d) = \langle a, b, c, d \mid [a, c], [a, d], [b, c], [b, d] \rangle. \]

Cost analysis. For all \( i \), the letters of \( w_i \) specified by \( T_i \) comprise a number of subwords in preferred alternating form. The number of such subwords specified by \( T_i \) is at most \( i \leq l \leq 2n/k \), since the number increases by at most one for each transition \( T_j \) to \( T_{j+1} \). Each specified subword has length at most three times the length of a corresponding subword of \( w \), by Lemma 5. Thus \( \ell(w_i) \leq 3\ell(w) \) and so, for each \( i \), the subword \( u \) of step (i) has \( \ell(u) \leq 3\ell(w) \). Applying Lemma 14 with \( p = 2n/k \) and \( m = 3n \) tells us that the cost of each instance of Step (ii) is at most
\[ 80k^3 + 150nk + 144n^2. \]

Multiplying by \( l \leq 2n/k \) and adding \((3n)^2\), which is an upper bound on the cost of Step (iii) as \( \ell(w_i) \leq 3n \), gives the estimate
\[ \text{Area}(w) \leq 160nk^2 + 309n^2 + 288n^3. \]

So \( k = n^{2/3} \), which is compatible with the condition \( k \geq 4 \) since we assumed that \( n \geq 8 \), gives our \( n^{7/3} \) isoperimetric function. \( \square \)

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