Discrete symmetries of isomonodromic deformations of order two Fuchsian differential equations

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Abstract
In the present work we investigate the group structure of the Schlesinger transformations for isomonodromic deformations of order two Fuchsian differential equations. We perform these transformations as isomorphisms between the moduli spaces of the logarithmic \( sl(2) \)-connections with fixed eigenvalues of the residues at singular points. We give a geometrical interpretation of the Schlesinger transformations and perform our calculations using the techniques of the modifications of bundles with connections. In order to illustrate the result we present classical examples of symmetries of the hypergeometric equation, the Heun equation and the sixth Painlevé equation.

1 Introduction
In this paper we discuss discrete transformations of the moduli spaces of logarithmic \( sl(2) \)-connections with singularities at distinct points \( \{x_1, \ldots, x_n\} \) on the Riemann sphere \( \mathbb{P}^1 \) with fixed eigenvalues (\( sl(2) \)-orbits) of the residues. We are interested in the group structure of isomorphisms between such moduli spaces. Every such connection performs a differential equation with regular singularities on \( \mathbb{P}^1 \) and the eigenvalues correspond to the local parameters of solutions the equation. Our discrete transformations act on the parameters of the equation and on its solutions preserving their local monodromies. In other words one can take a fuchsian differential equation of order two with singularities at \( \{x_1, \ldots, x_n\} \) and put it into isomonodromic analytical family in the following way. If \( Y(z) \) is the fundamental solution of this equation then one can take

\[
\partial_2 Y(z) \cdot Y(z)^{-1} = \sum_i \frac{B_i(x_1, \ldots, x_n)}{z - x_i},
\]

simultaneously with the condition for \( \{B_i\} \)

\[
dB_i(x_1, \ldots, x_n) = \sum_j [B_j, B_i] d \log(x_i - x_j)
\]
called the Schlesinger equation; the isomonodromic system is usually called the Schlesinger
system. We consider an initial data space of such isomonodromic deformation of $n$ points
on $\mathbb{P}^1$ and the discrete symmetries of this system define a structure of discrete system
with discrete time variables. The initial data space of Schlesinger system is isomorphic
to the coarse moduli space $M_n$ of collections $(\mathcal{L}, \nabla, \phi; \lambda_1, ..., \lambda_n)$, with a rank 2 bundle
$\mathcal{L}$ on $\mathbb{P}^1$, a connection $\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1_{\mathbb{P}^1}(x_1 + ... + x_n)$ and the horizontal isomorphism
$\phi : \det \mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1}$; the eigenvalues of the residues of the connection $\nabla$ at $x_i$, $i = 1, ..., n$ are
$(\lambda_i, -\lambda_i)$.

Our main result is the calculation of the appropriate group of discrete transformations.
For these purposes we develop a geometric techniques of the modifications (11) of vector
bundles with connections in the second and the third sections. In the original work [20] L.
Schlesinger consider these transformations of the systems of isomonodromic deformations;
the algebraic aspects of Schlesinger transformations are developed in the paper
of M. Jimba and T. Miwa [15] (see also [16]) but without paying attention to the group
structure. Besides, in classical works [20] and [15] they discuss the monodromy representa-
tion in $GL(N)$. In this work we give the beautiful geometric interpretation of Schlesinger
transformations following the work of D. Arinkin and S. Lysenko [1] and consider the
$SL(2)$-case which one can easily generalise to the classical case of $GL(2)$-representations;
we investigate the structure of the appropriate group which is more delicate than in $GL(2)$-
case. Our basic instrument for calculations is the technique of modifications of bundles,
or F-sheaves. The original idea of the modifications appeared in the works [10] of Erich
Hecke as correspondances between the spaces of modular forms called the Hecke corre-
spondances. Further this construction was applied for the description of moduli spaces
of vector bundles over a curve of arbitrary genus in [22]. In the work [5] of V. Drinfeld
the onstructions of F-sheaves for an arbitrary global field is presentes; it is for proving the
global Langlands hypothesis for $GL(2)$ and they are called Frobenius-Hecke sheaves or
"shtukas". Recently modification are widely used in mathematical physics (see [8], [14],
[16], [17], [23]).

The group of discrete transformations of $M_n$ is turn out to be isomorphic to the affine
Weyl group of the root system of type $C_n$:

$$W(\hat{C}_n) \simeq T \times ((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n),$$

where $n$ is the number of singular points $x_i$; the group acts in a way of a braid group. Let
us explain the structure of this group in terms of our problem. $S_n$ is generated by per-
mutations of the singular points which present the isomorphisms between the appriate
initial data spaces

$$M_n(\mathcal{L}, \nabla; \phi : \det \mathcal{L} \simeq \mathcal{O}; \lambda_1, ..., \lambda_n) \simeq M_n(\mathcal{L}, \nabla; \phi : \det \mathcal{L} \simeq \mathcal{O}; \lambda_1, ..., \lambda_j, ..., \lambda_i, ..., \lambda_n).$$

The translation part of the group $T$ acts by discrete shifts of eigenvalues of residues of the
appropriate connection; moreover there are short and long shifts. These shifts perform the
following isomorphisms

$$M_n(\mathcal{L}, \nabla; \phi : \det \mathcal{L} \simeq \mathcal{O}; \lambda_1, ..., \lambda_n) \simeq M_n(\mathcal{L}, \nabla; \phi : \det \mathcal{L} \simeq \mathcal{O}; \lambda_1, ..., \lambda_i + \frac{1}{2}, ..., \lambda - \frac{1}{2}, ..., \lambda_n)$$

and

$$M_n(\mathcal{L}, \nabla; \phi : \det \mathcal{L} \simeq \mathcal{O}; \lambda_1, ..., \lambda_n) \simeq M_n(\mathcal{L}, \nabla; \phi : \det \mathcal{L} \simeq \mathcal{O}; \lambda_1, ..., \lambda_k + 1, ..., \lambda_n)$$

respectively. Further we can change the signs of the eigenvalues of the residues at these
points; otherwise we perform a composition of noncorrelated 'local' Weyl transpositions
at the $sl(2)$-orbits of the residues of the connection at $x_i$:

$$\sigma^i : \begin{pmatrix} \lambda_i & 0 \\ 0 & -\lambda_i \end{pmatrix} \rightarrow \begin{pmatrix} -\lambda_i & 0 \\ 0 & \lambda_i \end{pmatrix};$$

such transformations generate the normal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ of the finite group $W(C_n)$ and for $(\epsilon_1, ..., \epsilon_n) \in (\mathbb{Z}/2\mathbb{Z})^n$ the appropriate isomorphism is

$$\mathcal{M}_n(\mathcal{L}, \nabla; \phi : \det \mathcal{L} \simeq \mathcal{O}; \lambda_1, ..., \lambda_n) \simeq \mathcal{M}_n(\mathcal{L}, \nabla; \phi : \det \mathcal{L} \simeq \mathcal{O}; \epsilon_1 \lambda_1, ..., \epsilon_n \lambda_n).$$

We can combine such transpositions with the permutations of the points and also get the element of order four. Besides, if we combine a pair of modifications with the such local transpositions we get the finite reflection; it describes the structure of the semidirect product of our group. From the other hand the translation part is isomorphic to the integral lattice, generated by the

$$C_n = \langle \pm 2\epsilon_i, \pm \epsilon_i \pm \epsilon_j \rangle$$

in the standard basis of $\mathbb{R}^n$ and our group of transformations is isomorphic to the automorphism group of this lattice.

In order to clarify the above result we analyse some examples in the 4th, 5th and the 6th sections; they are three classical examples of hypergeometric equation ([2], [14]), Heun’s equation ([11], [2], see also [6], [7]) and the sixth Painlevé equation ([19], [1], [18]). This part of the paper also has the meaning of bibliographical review of classical works on Fuchsian differential equations with three and four singularities. Cases of more than four points are much more complicated for analytical calculations and the appropriate groups of symmetries were almost unstudied in classical literature.

In the case of three regular singularities a Fuchsian differential equation of order two on $\mathbb{P}^1$ is equivalent to the hypergeometric equation. Its $W(\hat{C}_3)$ discrete symmetries were studied by K. Gauß and E. Kummer; our presentation of this subject corresponds to [2].

Next step after the hypergeometric equation is the Fuchsian differential equation of order two with four singularities called the Heun equation ([2]); our presentation follows the original work [11] of Karl Heun. The appropriate calculations in the classical sense of K. Gauß and E. Kummer were thoroughly made by K. Heun; these results are performed in [11] in the form of 192 Heun’s relations analogous to the 24 Kummer relations of hypergeometric functions and we present these results in the fifth section.

One can write the Schlesinger equation for the coefficients of Heun’s equation and in the special case of the Schlesinger system with four regular singularities with $sl(2)$-monodromies get sixth Painlevé equation $P_{VI}$. Discrete symmetries of $P_{VI}$ were studied in [19], [18], [12], [1] and the group is $W(\hat{D}_4)$ which is isomorphic to the extension of the Weyl group $W(D_4)$ with the group $\mathfrak{S}_3$ of automorphisms of Dynkin graph $D_4$.

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2 Modifications of rank N bundles with connections

Let $\mathcal{L}$ be a rank N bundle on $\mathbb{P}^1$ with a connection $\nabla$ and suppose $x \in \mathbb{P}^1$. Denote $V := \mathcal{L}_x$ and let $U \subset V$ be a $k$-dimensional subspace. Let us not differ $\mathcal{L}$ and the sheaf of its sections and consider the following modifications of $\mathcal{L}$

$$(x, U)^{\text{low}}(\mathcal{L}) := \{ s \in \mathcal{L} \mid s(x) \in U \},$$

$$(x, U)^{\text{up}}(\mathcal{L}) := (x, U)^{\text{low}}(\mathcal{L}) \otimes \mathcal{O}(x),$$

which are called the lower and the upper modification respectively. Let us denote the lower modification by $\tilde{\mathcal{L}} := (x, U)^{\text{low}}(\mathcal{L})$ and consider the natural map $\tilde{\mathcal{L}}_x \rightarrow \mathcal{L}_x$; its image is $U$. Put $\tilde{U} := \ker(\tilde{\mathcal{L}}_x \rightarrow \mathcal{L}_x)$ then

$$(x, \tilde{U})^{\text{up}} \tilde{\mathcal{L}} = \mathcal{L}.$$ 

However, the lower and the upper modifications imply the following exact sequences

$$0 \rightarrow (x, U)^{\text{low}}(\mathcal{L}) \rightarrow \mathcal{L} \rightarrow \delta_x \otimes \mathcal{L}_x / U \rightarrow 0,$$

$$0 \rightarrow \mathcal{L} \rightarrow (x, U)^{\text{up}} \mathcal{L} \rightarrow \delta_x \otimes U \otimes \mathcal{O}(x) |_x \rightarrow 0$$

respectively, where $\delta_x$ is a sky-scraper sheaf with the support at $x$.

Roughly speaking, if we have local decomposition $V = U \oplus \tilde{U}$ of $\mathcal{L} \simeq V \otimes \mathcal{O}$ then

$$(x, U)^{\text{low}}(\mathcal{L}) = U \otimes \mathcal{O} \bigoplus \tilde{U} \otimes \mathcal{O}(-x),$$

$$(x, U)^{\text{up}}(\mathcal{L}) = U \otimes \mathcal{O}(x) \bigoplus \tilde{U} \otimes \mathcal{O}.$$ 

In other words we change our bundle rescalling the basis of sections in the neighbourhood of a point $x$; if the local basis is

$$\{s_1(z), \ldots, s_N(z)\}$$

with $U \otimes \mathcal{O} \simeq \{s_1(z), \ldots, s_k(z)\}$ and $\tilde{U} \otimes \mathcal{O} \simeq \{s_{k+1}(z), \ldots, s_N(z)\}$ then the basis of the lower modification of the bundle is generated by the sections

$$\{s_1(z), \ldots, s_k(z), (z - x) s_{k+1}(z), \ldots, (z - x) s_N(z)\},$$

and of the upper one by

$$\{(z - x)^{-1} s_1(z), \ldots, (z - x)^{-1} s_k(z), s_{k+1}(z), \ldots, s_N(z)\}.$$ 

Consequently, in the punctured neighbourhood we may rewrite the action of modifications with the following glueing matrices.

$$(x, U)^{\text{low}} = \begin{pmatrix} 1_k & 0 \\ 0 & (z - x) \cdot 1_{N-k} \end{pmatrix}, \quad (x, U)^{\text{up}} = \begin{pmatrix} (z - x)^{-1} 1_k & 0 \\ 0 & 1_{N-k} \end{pmatrix},$$

where $1_m$ is the identity $(m \times m)$-matrix. Matrix presentation of the modifications is supposed to be quite obvious and further we widely use it.

Now we discuss the action of modifications on a connection with logarithmic singularities on $\mathbb{P}^1$ and we need the following
Definition. We say $\mathcal{M}$ is a module with a support $S$ on the algebraic curve $X$ if we have a finite set $S = \{x_1, ..., x_n\} \subset X$ and a positive integer $n_i$ for each point $x_i$ from $S$. Sometimes we identify a module with the appropriate effective divisor $\sum n_i \cdot x_i$; for our purposes we consider the module $\mathcal{M} = \sum x_i$.

Let us take a look how the modifications change the connection. Suppose we start from some connection $\nabla$ on $L$ and

$$\nabla : \mathcal{L} \rightarrow \mathcal{L} \otimes \Omega^1(\mathcal{M});$$

that means that $\nabla$ has simple poles at the support $S$ of the module $\mathcal{M}$. Let $x \in S$ be a singular point of $\nabla$ and $U \subseteq V$ is a $\text{Res}_x \nabla$-invariant subspace, i.e. $(\text{Res}_x \nabla)(U) \subseteq U$ and let us modify the bundle in this subspace. At first note that the lower and the upper modifications at any point $x \in \mathbb{P}^1$ change the determinant

$$\text{det}(x, U)^\text{low} \mathcal{L} = \text{det} \mathcal{L} \otimes \mathcal{O}(-x \cdot \text{dim } V / U), \quad \text{det}(x, U)^\text{up} \mathcal{L} = \text{det} \mathcal{L} \otimes \mathcal{O}(x \cdot \text{dim } U).$$

For example consider the lower modification $\tilde{\mathcal{L}}$ with the connection

$$\nabla' : \tilde{\mathcal{L}} \rightarrow \tilde{\mathcal{L}} \otimes \Omega(\mathcal{M}), \quad \text{pr}, \tilde{\mathcal{L}} \otimes \Omega(\mathcal{M})$$
on $\tilde{\mathcal{L}}$ and on the determinant bundle we get the connection

$$\text{tr} \nabla' = \text{tr} \nabla + \frac{dz}{z-x} \cdot \text{dim } U.$$ 

It is principal that we modify the pairs $(\mathcal{L}, \nabla)$ in $(\text{Res}_x \nabla)$-invariant subspaces of $V \subseteq \mathcal{L}_x$; otherwise we raise the order of a pole of the connection. In fact, using the matrix presentation let us write the action of the modification of the bundle in a noninvariant subspace at $x = 0$:

$$\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} d + \begin{pmatrix} \lambda & \epsilon \\ z & \frac{z}{x} \end{pmatrix} \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = d + \begin{pmatrix} \lambda & \epsilon \\ z & \frac{z}{x^2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\lambda + 1 \end{pmatrix},$$

where $z$ is a local parameter.

Further we consider a rank 2 bundle $\mathcal{L}$ with $\text{sl}(2)$-connection $\nabla$ hence we suppose that $\text{dim } U=1$. Moreover, we shall perform pairs of the lower and the upper modifications at points $x_i$ and $x_j$ respectively to get the bundle $\mathcal{L}''$ with the same determinant

$$\text{det}\mathcal{L}'' = \text{det} \mathcal{L} \otimes \mathcal{O}(x_j - x_i) \simeq \text{det}\mathcal{L}.$$ 

for this purpose we have to fix a set of compatible isomorphisms $\mathcal{O} \simeq \mathcal{O}(x_i - x_j)$ such that

$$\mathcal{O} \simeq \mathcal{O}(x_i - x_j) \otimes \mathcal{O}(x_j - x_k) \simeq \mathcal{O}(x_i - x_k).$$

Nevertheless if we start from a $\text{sl}(2)$-connection $\nabla$ then after such procedure we get the connection

$$\nabla'' = \nabla + P_{U_i} \frac{dz}{z-x_i} - P_{U_j} \frac{dz}{z-x_j},$$

where
where \( P_s \) are the projections on the appropriate \( \text{Res} \nabla \)-invariant subspaces; it is the \( gl(2) \)-connection. In order to return to the \( sl(2) \)-connection again we have to add the suitable 1-form

\[
\tilde{\nabla}'' = \nabla'' + \frac{1}{2} \left( \frac{dz}{z - x_j} - \frac{dz}{z - x_i} \right).
\]

This construction performs the nontrivial transformation of \( sl(2) \)-Schlesinger system of fuchsian type; henceforward it will be one of ours basic instruments. Precise statements and explanations we perform in next section.

3 \( sl(2) \)-connections with singularities on \( \mathbb{P}^1 \)

Let us describe our initial data following [1]. Fix a collection \( \{\lambda_\alpha\}_{\alpha=1}^n \) of numbers and the module \( \mathfrak{M} \) with the support \( S \) at distinct points \( \{x_1, \ldots, x_n\} \) on \( \mathbb{P}^1 \). There is a three-dimensional group of linear transformations acting on \( \mathbb{P}^1 \) so let us suppose that number of the points \( n \geq 3 \). Suppose \( \mathcal{L} \) be a rank 2 bundle on \( \mathbb{P}^1 \) with fixed horizontal isomorphism \( \phi : \bigwedge^2 \mathcal{L} \cong \mathcal{O} \) and with a connection \( \nabla \) with singularities at \( \mathfrak{M} = \sum x_i \); eigenvalues of \( \text{Res}_{x_i} \nabla \) are \((\lambda, -\lambda)\). Besides, put the following eigenvalue-condition

\[
\sum \epsilon_i \lambda_i \notin \mathbb{Z}, \quad (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in (\mathbb{Z}/2\mathbb{Z})^4,
\]

which guarantees the irreducibility of the pair “bundle \( \mathcal{L} \) with the connection \( \nabla'' \)” and implies the stability of this pair. Hence, we fix \( sl(2) \)-orbits of residues of the connection. Denote the eigenspaces of \( \text{Res}_{x_i} \nabla \)

\[
\ell^+ := \ker(\text{Res}_{x_i} \nabla \mp \lambda_i).
\]

For two points \( x_i, x_j \in S \) consider the modified \( SL(2) \)-bundle

\[
\mathcal{L}'' = (x_j, \ell^+_j)^{up} \circ (x_i, \ell^-_i)^{low} \mathcal{L}
\]

with modified logarithmic connection \( \nabla'' \) defined above. Define an operation \( (\downarrow \uparrow)_{ij} \) on pairs of \( SL(2) \)-bundles with connections

\[
(\downarrow \uparrow)_{ij} : (\mathcal{L}, \nabla) \mapsto (\mathcal{L}'', \nabla'' + \omega_{ij}), \quad \omega_{ij} = \frac{1}{2} \left( \frac{dz}{z - x_j} - \frac{dz}{z - x_i} \right).
\]

More precisely, we get a nontrivial transformations of the coarse moduli space \( \mathcal{M}_n \) of rank 2 bundles with fixed horizontal isomorphism and logarithmic connection with fixed eigenvalues of residues on \( \mathbb{P}^1 \). Let us calculate the correspondence between the eigenvalues under the above isomorphism between such moduli spaces with different eigenvalues of the residues; precise statement is the following.

**Proposal.** Modified pair \((\mathcal{L}'', \nabla'')\) is an element of the coarse moduli space \( \mathcal{M}_n \). The eigenvalues of \( \text{Res}_{x_i} \nabla'' \) are

\[
\{\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + \frac{1}{2}, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_j - \frac{1}{2}, \ldots, \lambda_n\}
\]

for the case of a pair of modifications at distinct points \( x_i, x_j \in S \); if a pair of modifications is at one point \( x_k \in S \) then the eigenvalues are

\[
\{\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + 1, \ldots, \lambda_n\}.
\]
Proof. The first part of the statement have been proved. Let us compare the eigenvalues of the residues of $\nabla$ and modified connection $\nabla''$.

$$\begin{bmatrix} \lambda_i & \lambda_j \\ -\lambda_i & -\lambda_j \end{bmatrix} \xrightarrow{(x_i, x_j)^{-} \circ (x_i, x_j)^{+}} \begin{bmatrix} \lambda_i + 1 & \lambda_j - 1 \\ -\lambda_i & -\lambda_j \end{bmatrix} + \omega_{ij} \rightarrow \begin{bmatrix} \lambda_i + 1 - \frac{1}{2} & \lambda_j - 1 + \frac{1}{2} \\ -\lambda_i - \frac{1}{2} & -\lambda_j + \frac{1}{2} \end{bmatrix}.$$  

therefore we get the shifts of eigenvalues

$$\lambda_i \rightarrow \lambda_i + \frac{1}{2}, \quad \lambda_j \rightarrow \lambda_j - \frac{1}{2}.$$  

In the case of modifications at one point $x_k$

$$(x_i, x_i)^{low} \circ (x_i, x_i)^{up} : \begin{bmatrix} \lambda_k \\ -\lambda_k \end{bmatrix} \rightarrow \begin{bmatrix} \lambda_k + 1 \\ -\lambda_k - 1 \end{bmatrix}$$  

we have zero 1-form $\omega_{kk}$ hence the shift is long:

$$\lambda_k \rightarrow \lambda_k + 1 \quad \blacksquare$$  

Such modifications of pairs $(L, \nabla)$ are the infinite-order (affine) elements of the appropriate group of transformations of $M_n$; and we see that the translation part contains both short and long shifts. Besides this affine symmetry we have evident finite symmetries that are the permutations of singular points $\{x_i\} \in S$ and local Weyl transpositions. It is significant that our group of transformations is generated by the elements of order two i.e. reflections. There is a nice classification of such groups found by H. S. M. Coxeter ([4]) and it alleviates the description of our group. Let us arrange the notations. Denote $(i \ j)$ the permutation of distinct points $x_i, x_j$ of support $S$ of module $\mathcal{M}$; moreover we have the local transpositions

$$\sigma^i : \left( \begin{array}{cc} \lambda_i & 0 \\ 0 & -\lambda_i \end{array} \right) \rightarrow \left( \begin{array}{cc} -\lambda_i & 0 \\ 0 & \lambda_i \end{array} \right)$$  

from the Weyl group $W(SL(2))$ at each point $x_i \in S$. We encode our group of transformations of moduli space $M_n$ with the Coxeter graph of type $\tilde{C}_n$ using the Coxeter classification. Finite part of the group generates by the reflections; we encode the generator with the vertice of the graph and the edges of the graph correspond to the relations between the generators in the following way.

1. Permutational part $S_n$ generates by the $(i \ j)$; the composition of two neighboured transpositions $(i \ j)(j \ k) = (i \ j \ k)$ is the element of order three and we denote this relation with the graph $(i \ j) \leftrightarrow (j \ k)$. If a composition of two transpositions has the order two then we don’t put an edge between them.

2. $(\mathbb{Z}/2\mathbb{Z})^n$ generates by the local Weyl elements $\sigma^i$ and the composition $\sigma^i \circ (i \ j)$ is the element of order four; we encode this relation with the Coxeter graph $\sigma^i \iff (i \ j)$.

3. Translational part $T$ generates by the pairs of modifications at distinct points or at one point and there are short and long affine shifts respectively.

Henceforward we can give the following

**Statement.** We can present our group with the following affine $\tilde{C}_n$ Coxeter graph.

$$(\updownarrow)_1 \circ \sigma^1 \implies (\updownarrow)_2 \circ (1 2) \implies (\updownarrow)_{23} \circ (2 3) \implies \ldots \implies (\updownarrow)_{n-1, n} \circ (n-1 \ n) \iff (\updownarrow)_n \circ \sigma^n$$  

In further sections we analyse this result considering special cases of the module $\mathcal{M} = x_1 + x_2 + x_3$ and $\mathcal{M} = x_1 + x_2 + x_3 + x_4$. These cases are correspond to the hypergeometric differential equation and Heun’s equation; they were studied in classical works.
of K. Gauß, E. Kummer and K. Heun and correspond to the case of $gl(2)$-connections. Let us note that the above calculations can be easily generalised to the $gl(2)$-case. The isomorphisms between the moduli spaces of the pairs $(\mathcal{L}, \nabla)$ with fixed eigenvalues of the residues will be presented by the same transformations but without the 1-form $\omega_{ij}$ and all the shifts of eigenvalues of $Res_{x_i} \nabla$ are long. If the eigenvalues of the connection at a singular point are $(\mu_i, \nu_i)$ and $\widetilde{\mathcal{M}}_n$ denotes the appropriate moduli space of $gl(2)$-connections then the isomorphisms are

$$\widetilde{\mathcal{M}}_n(\mathcal{L}, \nabla; (\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)) \cong \widetilde{\mathcal{M}}_n(\mathcal{L}, \nabla; (\mu_1, \nu_1), \ldots, (\mu_i \pm 1, \nu_i), \ldots, (\mu_n, \nu_n)),$$

$$\widetilde{\mathcal{M}}_n(\mathcal{L}, \nabla; (\mu_1, \nu_1), \ldots, (\mu_i, \nu_i), \ldots, (\mu_n, \nu_n)) \cong \widetilde{\mathcal{M}}_n(\mathcal{L}, \nabla; (\mu_1, \nu_1), \ldots, (\nu_i, \mu_i), \ldots, (\mu_n, \nu_n)),$$

$$\widetilde{\mathcal{M}}_n(\mathcal{L}, \nabla; (\mu_1, \nu_1), \ldots, (\mu_i, \nu_i), \ldots, (\mu_n, \nu_n)) \cong \widetilde{\mathcal{M}}_n(\mathcal{L}, \nabla; (\mu_1, \nu_1), \ldots, (\mu_j, \nu_j), \ldots, (\mu_n, \nu_n)) \cong$$

$$\cong \widetilde{\mathcal{M}}_n(\mathcal{L}, \nabla; (\mu_1, \nu_1), \ldots, (\mu_j, \nu_j), \ldots, (\mu_i, \nu_i), \ldots, (\mu_n, \nu_n))$$

Also note that we perform the last permutational isomorphism of the singular points by the group of linear transformations on the Riemann sphere; this group is three-dimensional and it is convenient to use this representation for permuting three and four points on the Riemann sphere.

4 Classical example: $W(\hat{C}_3)$-symmetries of the hypergeometric equation

At first let us remind the interplay between the meromorphic connections on $\mathbb{P}^1$ and general differential equations with singularities. Let us illustrate our geometric construction in terms of systems of differential equations in the sense of the fundamental work [3] of A. Bolibruch. Consider a vector bundle $\mathcal{E}$ and the covariant derivative $\nabla_v$ for some vector field $v$ on $\mathbb{P}^1$ with zeroes at $\mathfrak{M}$; then on $U = \mathbb{P}^1 - S$ we can trivialize our bundle and hence

$$\nabla_v|_U \simeq \partial_v.$$

For any singular point $x_i \in S$ we take its neighbourhood $V_i$ and we also have a trivialization of our bundle and use duality between vector fields and 1-forms. Finally we have $\nabla_v = \partial_v - \omega$, where $\omega$ is the appropriate 1-form, defined by $v$. If the singularities of $\omega$ are only simple poles then the trivializing maps at $V_i$ will be the residues and one can express

$$\nabla = \frac{d}{dz} - \sum_i \frac{res_{x_i} \omega}{z - x_i}.$$

A section $h(z)$ of the $GL(2)$-bundle $\mathcal{E}$ is called horizontal with respect to $\nabla_v$ if $\nabla_v(h) \equiv 0$; precisely, $\partial_v s = \omega h$. In terms of covariant derivative along the vector field $\frac{\partial}{\partial z}$ we have system of differential equations

$$dY = \omega Y, \text{ where } \omega = B(z)dz.$$

Changing of the basis of sections

$$Y' = gY, \text{ for } g \in GL(2)$$

we also change the 1-form-valued $(2 \times 2)$-matrix $\omega$:

$$\omega' = dg \cdot g^{-1} + g \cdot \omega \cdot g^{-1};$$
such transformations are called gauge transformations. We suppose that all the singularities of $B(z)$ are simple poles and we have the action of the monodromy: for the loop $\gamma$ around some singular point the analytical continuation along $\gamma$ gives $Y \to Y \cdot g_\gamma$, where $g_\gamma$ is the monodromy matrix. Suppose that $z = 0$ is the singular point of $B(z)$ and $\gamma$ is the appropriate element of fundamental group then the behavior of the solution $Y(z) = (y_1(z), y_2(z))$ of our system in the neighbourhood of the singularity at $z = 0$ is described by the following

**Fact.** The fundamental solution $Y(z)$ admits the following presentation

$$Y(z) = U(z) \cdot z^A \cdot z^E,$$

where $A := \text{diag}(\sigma_0, \tau_0)$ where $\sigma_0$ and $\tau_0$ are the exponents of the components $y_1(z)$ and $y_2(z)$ of the solution in the neighbourhood of $z = 0$:

$$\text{exponent of } y(z) := \sup \{ k \in \mathbb{Z} | \forall s < k \frac{y(z)}{|z|^s} \to 0, z \to 0 \};$$

the matrix $U(z)$ is holomorphic and invertible in the neighbourhood of $z = 0$. Consider the gauge transformation of special type:

$$g_0 := U(z) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} U(z)^{-1}$$

then its action

$$g_0 \cdot Y(z) = U(z) z^{A'} z^E,$$

where $A' = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \tau_0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

shifts the exponents of the solution. For arbitrary simple pole $x_i$ of $B(z)$ —

$$g_i := U(z-x_i) \begin{pmatrix} z-x_i & 0 \\ 0 & 1 \end{pmatrix} U(z-x_i)^{-1} \quad \text{or} \quad U(z-x_i) \begin{pmatrix} (z-x_i)^{-1} & 0 \\ 0 & 1 \end{pmatrix} U(z-x_i)^{-1}. $$

In this sense we may understand the modifications of pairs $(\mathcal{L}, \nabla)$ as singular gauge transformations and we have to check that they do not change our system but reparametrise it.

Suppose we have general linear differential equation on the Riemann sphere $\mathbb{P}^1$ of the order two which have three simple poles at $x_1, x_2, x_3 \in \mathbb{P}^1$. Then the set of all its solutions may be encoded in the

Riemann scheme of this equation

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \tau_1 & \tau_2 & \tau_3 \end{pmatrix}.$$

$\{\sigma_i, \tau_i\}$ are the exponents of the solutions at $\{x_i\}$ respectively; otherwise, if $y_1, y_2$ are the independent solutions at $z = a$ then

$$y_1 \sim (x-a)^{\sigma_a}, \quad y_2 \sim (x-a)^{\tau_a}.$$

They satisfy the Fuchs relation

$$\sum_{i=1}^{3} (\sigma_i + \tau_i) = 1;$$
it is analogous to the determinant isomorphism \( \phi \) from the third section and it means that we handle with the bundle with trivial determinant on \( \mathbb{P}^1 \).

It is well known that the differential equation is uniquely determined by its Riemann scheme:

\[
\frac{d^2 y}{dz^2} + \left( \sum_{i=1}^{3} \frac{1 - \sigma_i - \tau_i}{z - x_n} \right) \frac{dy}{dz} + \left( \sum_{i=1}^{3} \sigma_i \tau_i (x_n - x_{n+1})(x_n - x_{n+2}) \right) \frac{y}{(z - x_1)(z - x_2)(z - x_3)} = 0,
\]

where \( x_4 = x_1 \) and \( x_5 = x_2 \).

The fact is that every such equation one may reduce to the hypergeometric equation

\[
z(1-z)\frac{d^2 y}{dz^2} + [c - (a + b + 1)z] \frac{dy}{dz} - ab \cdot y = 0.
\]

\( a, b, c \in \mathbb{C} \setminus \{ -1, -2, \ldots \} \) are the parameters of the equation; the condition on them has the same meaning as the eigenvalues-condition from the third section. One can suppose that the singularities \( \{x_1, x_2, x_3\} = \{0, 1, \infty\} \); on the language of the Riemann schemes it means we can reduce the Riemann scheme to the special case

\[
\begin{pmatrix}
0 & 1 & \infty \\
\sigma_0 & \sigma_1 & \sigma_\infty \\
\tau_0 & \tau_1 & \tau_\infty
\end{pmatrix} = z^{\sigma_0}(z - 1)^{\sigma_1} \begin{pmatrix}
0 & 1 & \infty \\
0 & 0 & \sigma_\infty + \sigma_0 + \sigma_1 \\
\tau_0 - \sigma_0 & \tau_1 - \sigma_1 & \tau_\infty + \sigma_0 + \sigma_1
\end{pmatrix};
\]

in terms of previous chapters we perform

\[
\begin{Bmatrix}
\lambda_0 & \lambda_1 & \lambda_\infty \\
-\lambda_0 & -\lambda_1 & -\lambda_\infty
\end{Bmatrix} \longrightarrow \begin{Bmatrix}
0 & 0 & \lambda_0 + \lambda_1 + \lambda_\infty \\
-2\lambda_0 & -2\lambda_1 & -\lambda_0 - \lambda_1 - \lambda_\infty
\end{Bmatrix},
\]

Further we compare our calculations of \( \hat{C}_3 \)-group from the previous chapter with classical calculations of K.-F. Gauß and E. Kummer.

A solution with the exponent 0 at \( z = 0 \) of the equation is the hypergeometric function

\[
F(\alpha, \beta, \gamma | z) = \sum_n \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!},
\]

where \( (a)_n := \frac{\Gamma(a + n)}{\Gamma(a)} \) and \( \Gamma(x) \) is Euler’s gamma-function.

Further we investigate the symmetries of the equation and present them in the form of relations between hypergeometric functions with different parameters; then we analyse these relations in the sense of our result of the previous sections.

Immediately one can notice that this solution admits the obvious symmetry \( \beta \leftrightarrow \alpha \). Of course the parameters \( \alpha, \beta, \gamma \) determine the characteristic exponents at \( 0, 1, \infty \) by the rule

\[
\begin{pmatrix}
0 & 1 & \infty \\
0 & 0 & \alpha \\
1 - \gamma & \gamma - \alpha - \beta & \beta
\end{pmatrix}.
\]

That means that we have two independent solutions of our equation with the exponents \( \{0, 1 - \gamma\} \) at 0

\[
y_0^{hol} = F(\alpha, \beta, \gamma | z), \quad y_0^{1-\gamma} = z^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma | z);
\]

the first is holomorphic (exponent 0) and the other has a monodromy (exponent 1-\( \gamma \)).

Permuting 0, 1, and \( \infty \) by Möbius transformations:

\[
(0 \ 1) : z \to 1 - z, \quad (0 \ \infty) : z \to \frac{1}{z}, \quad \text{and} \quad (1 \ \infty) : z \to \frac{z}{z-1};
\]
we get solutions at the other points
\[ y_{1}^{\text{hol}} = F(\alpha, \beta, \alpha + \beta + 1 - \gamma | 1 - z), \quad f_{1}^{\gamma - \alpha - \beta} = (1 - z)^{\gamma - \alpha - \beta}F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1 | 1 - z), \]
\[ y_{\infty}^{\alpha} = z^{-\alpha}F(\alpha, \alpha + 1 - \gamma, \alpha + 1 - \beta | \frac{1}{z}), \quad f_{\infty}^{\beta} = z^{-\beta}F(\beta, \beta + 1 - \gamma, \beta + 1 - \alpha | \frac{1}{z}). \]
From this one can deduce all 24 Kummer’s solutions ([2]).
Let us consider the action of modifications on these solutions without adding 1-form \( \omega_{ij} \) but we do not change symbol of modification.

\[
(\uparrow \downarrow)_{1:} \begin{pmatrix} 0 & 1 & \infty \\
1 - \gamma & \gamma - \alpha - \beta & \beta \\
0 & 0 & \alpha 
\end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & \infty \\
1 - \gamma & \gamma - \alpha - \beta & \beta \\
0 & 0 & \alpha + 1 
\end{pmatrix}
\]
i.e. \( \alpha \rightarrow \alpha + 1 \).

This shift of \( \alpha \) does not change our hypergeometric equation. In other words the appropriate Riemann’s schemes are equivalent:

\[
\begin{pmatrix} 0 & 1 & \infty \\
1 - \gamma & \gamma - \alpha - \beta & \beta \\
0 & 0 & \alpha 
\end{pmatrix} \approx \begin{pmatrix} 0 & 1 & \infty \\
1 - \gamma & \gamma - \alpha - \beta & \beta \\
0 & 0 & \alpha + 1 
\end{pmatrix}.
\]

Actually
\[
\alpha z^{\alpha - 1}F(\alpha + 1, \beta, \gamma) = \frac{\partial}{\partial z}z^{\alpha}F(\alpha, \beta, \gamma), \quad \text{hence, } \quad F(\alpha + 1) = F(\alpha) + z \frac{\partial}{\partial z}F(\alpha);
\]
and for \( \alpha' := \alpha + 1 - \gamma \) we have
\[
\alpha' z^{\alpha' - 1}F(\alpha' + 1) = \alpha' z^{\alpha - \gamma}F(\alpha') + z^{\alpha'} \frac{\partial}{\partial \alpha}F(\alpha').
\]
So for another solution \( F_{2}(\alpha') := z^{1-\gamma}F(\alpha + 1 - \gamma) \)
\[
F_{2}(\alpha' + 1) = \frac{\alpha}{\alpha'} F_{2}(\alpha') + \frac{z}{\alpha} \frac{\partial}{\partial \alpha}F_{2}(\alpha')
\]
and finally
\[
\begin{cases} F(\alpha + 1, \beta, \gamma | z) \\
z^{1-\gamma}F(\alpha + 1 - \gamma + 1, \beta + 1 - \gamma, 2 - \gamma, | z) \end{cases} = \begin{cases} F(\alpha, \beta, \gamma | z) + \frac{z}{\alpha} \frac{\partial}{\partial z}F(\alpha, \beta, \gamma | z) \\
\frac{\alpha}{\alpha + 1 - \gamma} F_{2} + \frac{z}{\alpha + 1 - \gamma} \frac{\partial}{\partial z}F_{2} \end{cases} ;
\]
it is called the Gauß relation.
So one can express \( y_{1:}(\alpha - 1) \) in terms of \( y_{1:}(\alpha) \) and \( \frac{\partial}{\partial z}y_{1:}(\alpha) \) and make suitable reparametrization of the hypergeometric equation. As usual it is the affine symmetry of the equation.
Therefore the modifications together with the 24 Kummer’s symmetries, associated with the Möbius transformations and with obvious symmetry
\[
\sigma^{\infty} : \quad \alpha \mapsto \beta
\]
produce all discrete symmetries of the differential equation on \( \mathbb{P}^{1} \) of the order two which have three simple poles at \( x_{1}, x_{2}, x_{3} \in \mathbb{P}^{1} \). These discrete symmetries were just studied in classical works of K. Gauß and E. Kummer. One can check their appropriate relations with the help of the suitable \( \tilde{C}_{3} \) Dynkin diagram:
\[
(\uparrow \downarrow)^{0}_{0} \circ \sigma^{0} \implies (01) \iff (1 \infty) \iff (\uparrow \downarrow)^{+}_{\infty} \circ \sigma^{\infty}
\]
Let us explain this diagram. We use a pair of modifications at the same point and do not add 1-form $\omega_{ij}$. The illustration of the symbols in terms of the Riemann scheme is the following.

$$(\uparrow\downarrow)^+_0 : \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1 - \gamma & \gamma - \alpha - \beta & \beta \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & \infty \\ 0 + 1 & 0 & \alpha \\ 1 - \gamma - 1 & \gamma - \alpha - \beta & \beta \end{pmatrix} =$$

$$= z \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha + 1 \\ 1 - \gamma - 2 & \gamma - \alpha - \beta & \beta + 1 \end{pmatrix}, \text{ i.e. } \begin{cases} \alpha \rightarrow \alpha + 1 \\ \beta \rightarrow \beta + 1 \\ \gamma \rightarrow \gamma + 2 \end{cases};$$

$$(\uparrow\downarrow)^-_\infty : \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1 - \gamma & \gamma - \alpha - \beta & \beta \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha + 1 \\ 1 - \gamma & \gamma - \alpha - \beta & \beta - 1 \end{pmatrix}, \text{ i.e. } \begin{cases} \alpha \rightarrow \alpha + 1 \\ \beta \rightarrow \beta - 1 \end{cases}.$$  

One can easily check these Coxeter relations in the same way using the Kummer and Gauß relations between corresponding solutions.

5 Another example: $W(\hat{C}_4)$-symmetries of the Heun equation

Next step after hypergeometric equation is the Heun equation. Precisely, it can be shown that any Fuchsian equation of the second order with four singularities can be reduced to Heun's equation:

$$\frac{d^2 y}{dz^2} + \left[ \frac{\gamma + \delta}{z - 1} + \frac{\epsilon}{z - a} \right] \frac{dy}{dz} + \frac{\alpha \beta z - q}{z(z - 1)(z - a)} y = 0;$$

the Fuchs relation is

$$\alpha + \beta - \gamma - \delta + 1 - \epsilon = 0.$$  

We assume that the singularities are $\{0, 1, a, \infty\}$ and the Riemann scheme is

$$\begin{pmatrix} 0 & 1 & a & \infty \\ 0 & 0 & 0 & \alpha \\ 1 - \gamma & 1 - \delta & 1 - \epsilon & \beta \end{pmatrix} | q,$$

where $q$ is the auxiliary parameter; the action of linear transformations on the scheme is analogous to the previous case of hypergeometric equation:

$$\begin{pmatrix} 0 & 1 & a & \infty \\ \sigma_0 & \sigma_1 & \sigma_\infty & q \end{pmatrix} = z^{\sigma_0}(z - 1)^{\sigma_1}(z - a)^{\sigma_\infty} \times$$

$$\times \begin{pmatrix} 0 & 1 & a & \infty \\ 0 & 0 & 0 & \sigma_\infty + \sigma_0 + \sigma_1 + \sigma_a \\ \tau_0 - \sigma_0 & \tau_1 - \sigma_1 & \tau_a - \sigma_a & \tau_\infty + \sigma_0 + \sigma_1 + \sigma_a \end{pmatrix} \begin{pmatrix} q \end{pmatrix}.$$

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It should be noted that in the case of four singularities there is no one-to-one correspondence between the Riemann schemes and Fuchsian equations that is why we need the auxiliary parameter $q$ and every transformation of the equation acts it; the transformation of the auxiliary parameter must ascertained by explicit calculation. The appropriate computation will be needed in the next section and we shall make it with the help of geometric interpretation of the Fuchsian differential equation with four singularities.

Then as usual we consider the following linear transformations of projective line $z \rightarrow 1 - z$, $z \rightarrow \frac{1}{z}$, $z \rightarrow \frac{a}{z}$, $z \rightarrow z - a$, $z \rightarrow \frac{1 - a}{z - a}$ and their compositions; for each point there are six such expressions, for example, six possible values of $a$ are $1 - a$, $\frac{1}{a}$, $\frac{1}{1 - a}$, $\frac{a}{a - 1}$, $\frac{a - 1}{a}$.

There are 24 such transpositions at all and they just have the meaning of permutations of four singular points.

Following K. Heun we denote $y_{0}^{\text{hol}} = F(a | \alpha, \beta, \gamma, \delta | z)$ the holomorphic solution at $z = 0$; it is also admits the symmetry $\sigma^\infty : \alpha \leftrightarrow \beta$. The nonholomorphic is $y_{0}^{1 - \gamma} = z^{1 - \gamma} F(a | \alpha + 1 - \gamma, \beta + 1 - \gamma, \gamma, \delta | z)$; so let us consider the action of linear transformations on it. We can get local solutions at other singularities

\[
y_{1}^{\text{hol}} = F(1 - a | \alpha, \beta, \gamma, \delta | 1 - z), \quad y_{1}^{1 - \delta} = (z - 1)^{1 - \delta} (z - a)^{\alpha} F(a | \alpha + 1 - \delta, \gamma + 1 - \beta, \delta, 2 - \gamma | \frac{a(z - 1)}{z - a});
\]

\[
y_{a}^{\text{hol}} = F\left(\frac{1 - a}{a} | \alpha, \beta, \alpha + \beta - \gamma - \delta + 1, \delta, 2 - \gamma | \frac{a - x}{a}\right),
\]

\[
y_{a}^{1 - \epsilon} = (x - 1)^{\gamma + \delta - \alpha - \beta} F\left(\frac{1}{1 - a} | \gamma + \delta - \beta, \gamma + \delta - \alpha, \delta, \gamma + \delta - \alpha - \beta + 1 | \frac{x - 1}{x - a}\right);
\]

\[
y_{a}^{\alpha} = z^{\alpha} F\left(\frac{1}{a} | \alpha, \alpha + 1 - \gamma, \beta + 1 - \alpha, \delta | \frac{1}{z}\right), \quad y_{a}^{\beta} = z^{\beta} F\left(\frac{1}{a} | \beta, \beta + 1 - \gamma, \beta + 1 - \alpha, \delta | \frac{1}{z}\right)
\]

and all 192 expressions for these solutions. In such way we calculate the action of the finite part of our group on the solutions and together with the obvious symmetry $\sigma^\infty : \alpha \leftrightarrow \beta$

we describe the finite part of our group of order $|W(C_{4})| = |(Z/2Z)^{4} \rtimes S_{4}| = 192 \times 2$.

To check the action of the affine part we need a relation on shifted and non-shifted
functions \( F(a \mid z) \) analogous to the Gauß relation for the hypergeometric function; it is the following.

\[
[(\epsilon - 1) - \frac{\alpha \beta}{\gamma} q] F(a, q' \mid \alpha, \beta, \gamma, \delta + 1 \mid z) = \]

\[
= (\epsilon - 1) F(a, q \mid \alpha, \beta, \gamma, \delta \mid z) + (z - a) \frac{\partial}{\partial z} F(a, q \mid \alpha, \beta, \gamma, \delta \mid z),
\]

where the modified auxiliary parameter is

\[
q' = q + a \frac{\gamma + \delta}{\alpha \beta} - \frac{\gamma}{\alpha \beta};
\]

we give the explanation of the modification auxiliary parameter in the next section using beautiful geometric interpretation and hamiltonian presentation of the isomonodromic deformation of the Heun equation. In hamiltonian presentation auxiliary parameter plays the role of the inverse to the moment variable.

The above transformation relates the solution with shifted parameters

\[
\gamma \to \gamma + 1, \quad \delta \to \delta + 1, \quad \epsilon \to \epsilon - 1
\]

to the linear combination of non-shifted solution and its derivative. So one can reparametrise the equation and assure that the affine action of the \( W(\hat{C}_4) \) does not change the equation.

The simple computation is the following; we have to look after the second solution:

\[
\frac{\partial}{\partial z} z^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, \gamma, \delta) =
\]

\[
= (1 - \gamma) z^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, \gamma, \delta) + z^{1-\gamma} F'(\alpha + 1 - \gamma, \beta + 1 - \gamma, \gamma, \delta)
\]

so we substitute it to the above relation and finally get a reparametrisation for a pair of solutions at \( z = 0 \)

\[
\left\{
\begin{aligned}
F(q' \mid \alpha, \beta, \gamma + 1, \delta + 1 \mid z) \\
F_2(q' \mid \alpha + 1 - \gamma, \beta + 1 - \gamma, \gamma + 1, \delta + 1 \mid z)
\end{aligned}
\right\}
\]

\[
= \left\{
\begin{aligned}
\frac{\epsilon - 1}{\epsilon - 1 - \frac{\alpha \beta}{\gamma} q} F(q \mid \alpha, \beta, \gamma, \delta \mid z) + \frac{z - a}{\epsilon - 1 - \frac{\alpha \beta}{\gamma} q} \frac{\partial}{\partial z} F(q \mid \alpha, \beta, \gamma, \delta \mid z) \\
\frac{\epsilon - 1}{\epsilon - 1 - \frac{\alpha \beta}{\gamma} q} F_2(q \mid \alpha, \beta, \gamma, \delta \mid z) + \frac{z - a}{\epsilon - 1 - \frac{\alpha \beta}{\gamma} q} \frac{\partial}{\partial z} F_2
\end{aligned}
\right\},
\]

where \( F_2 = z^{1-\gamma} F(q \mid \alpha + 1 - \gamma, \beta + 1 - \gamma, \gamma, \delta \mid z) \). If we combine it with the action of permutational part we get the reparametrisations for local solutions at other points.

Just like in the case of hypergeometric equation one can combine 192 permutational and translational Heun’s relations with the transposition \( \sigma^\infty : \alpha \leftrightarrow \beta \) and get \( W(\hat{C}_4) \) group of symmetries.

### 6 The isomonodromic deformation of the Heun equation: sixth Painlevé equation

And finally let us consider the equation of isomonodromic deformation of the fuchsian differential equation of order two with four singularities on the projective line.
There is another way of considering our problem; we say that an algebraic differential equation of order two satisfies the Painlevé property if it is free from movable branch points. Sixth Painlevé equation $P_{VI}$ is the general differential equation of order two on $\mathbb{P}^1$ with at most four regular singularities and without movable branch points; as usual we may assume that singular points are 0, 1, $t$, $\infty$. $P_{VI}$ is the following.

\[
\frac{d^2 x}{dt^2} = \frac{1}{2} \left( \frac{1}{x + 1/x - 1} \right) \left( \frac{dx}{dt} \right)^2 - \left( \frac{1}{t + 1/t - 1} \right) \frac{dx}{dt} + \frac{x(x-1)(x-t)}{t^2(t-1)^2} \left( \frac{\alpha - \beta t}{x^2} + \frac{t-1}{(x-1)^2} + \frac{1}{2} - \delta \right) \frac{t(t-1)}{(x-t)^2}. \]

Time variable is the double ratio of the singular points: $t = [0, 1, t, \infty]$. The parameters $\alpha, \beta, \gamma, \delta$ represent the eigenvalues $\lambda_0, \lambda_1, \lambda_t, \lambda_\infty$ of the residues of logarithmic connection in the following way.

\[
\alpha = \frac{1}{2} \lambda_\infty^2, \quad \beta = \frac{1}{2} \lambda_0^2, \quad \gamma = \frac{1}{2} \lambda_1^2, \quad \delta = \frac{1}{2} \lambda_t^2.
\]

For our purposes the Hamiltonian form of $P_{VI}$ is more suitable; it is the following.

\[
\begin{align*}
\frac{dx}{dt} &= \frac{\partial H}{\partial p}, \\
\frac{dp}{dt} &= -\frac{\partial H}{\partial x},
\end{align*}
\]

with the Hamiltonian

\[
H = \frac{1}{t(t-1)}[x(x-1)(x-t)p^2 - \{\lambda_0(x-1)(x-t) + \\
+ \lambda_1(x-t) + (\lambda_t - 1)x(x-1)\}p + \lambda(x-t)],
\]

where $\lambda = \frac{1}{4}[(\lambda_0 + \lambda_1 + \lambda_t - 1)^2 - \lambda_0^2]$. The group of symmetries is isomorphic to $W(\hat{D}_4)$. Its structure is similar to the one in general case and in the case of hypergeometric equation.

Moduli space $\mathcal{M}_4$ admits a group $W(\hat{D}_4)$ of transformations, that is generated only by short shifts or only by pairs of modifications at distinct points. The consideration of this subgroup of $W(\hat{C}_4)$ is quite natural in the following geometrical interpretation ([1]). The singularities of our connection is at the support of the module $\mathcal{M} := 0 + 1 + t + \infty$ on $\mathbb{P}^1$. The initial data space of the isomonodromic system $P_{VI}$ is isomorphic to the noncompact complex surface

\[
\{(x, p) \mid (x, p)^{up} : \mathcal{O} \oplus \mathcal{T}(-\mathcal{M}) \to \mathcal{O} \oplus \mathcal{O}(-\infty)\};
\]

locally it is isomorphic to $\mathcal{T}^*(\mathbb{P}^1 - S)$; the structure of this surface was thoroughly studied in [6, 7] (see also [2]) and [1]. Then moment variable $p$ has the meaning of the direction of the upper modification of bundle $(\mathcal{O} \oplus \mathcal{T}(-\mathcal{M}))$ at point $x$; in the above notation the hamiltonian $H$ corresponds to $(x, p)^{up} : \mathcal{O} \oplus \mathcal{T}(-\mathcal{M}) \to \mathcal{O} \oplus \mathcal{O}(-t)$ that is not important due to our symmetries. Precisely, the action of the pairs of modifications at distinct points is obvious; it is the following

\[
\mathcal{O} \oplus \mathcal{O}(-\infty) \to \mathcal{O} \oplus \mathcal{O}(-x_i), \quad x_i = 0, 1, t.
\]

In matrix presentation

\[
(x, p)^{up} = \begin{pmatrix}
\omega & \eta \\
\omega & \eta \\
\end{pmatrix},
\]

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where \( \omega \in \text{Hom}(\mathcal{T}(-\mathcal{M}), \mathcal{O}) \simeq \Omega(\mathcal{M}) \), nd \( \eta \in \Omega(\mathcal{M} - \infty) \).

The determinant \( \det(x, p)^{up} \) has a simple zero at \( x \), so
\[
\eta(x) = 0, \quad \omega(x) = p \, dx.
\]

This construction is used in the work [21] of E. Sklyanin for separating the variables in \( \text{sl}(2) \)-Schlesinger system with \( n \) singularities.

For example, the shifts of the parameters for a pair of modifications at points 0 and 1 are
\[
p \rightarrow p + \Delta p; \quad \Delta p = \frac{1}{x - 1} - \frac{1}{x}, \quad \text{nd} \quad x \quad \text{does not changed};
\]

these transformation preserves the hamiltonian:
\[
H'' = \frac{1}{t(t - 1)}[x(x - 1)(x - t)(p + \Delta p)^2 - \{(\lambda_0 + \frac{1}{2})(x - 1)(x - t) +
\]
\[
+ (\lambda_1 - \frac{1}{2})x(x - t) + (\lambda_1 - 1)x(x - 1)(p + \Delta p) + k(x - t)] = H + 2p\frac{1}{2}(x - t) +
\]
\[
+ \frac{1}{4} \frac{x - t}{x(x - 1)} - \frac{1}{2} \left[ \frac{\lambda_0 x - t}{x} + \lambda_1 \frac{x - t}{x - 1} + (\lambda_1 - 1) \right] - \frac{1}{2}(x - t) - \frac{1}{2} \frac{x - t}{x(x - 1)} = H.
\]

These \( \hat{D}_4 \)-symmetries we can extend with the group \( \hat{S}_3 \) of automorphisms of the graph \( D_4 \) and get the affine group \( W(\hat{F}_4) \) of symmetries of the \( PVI \) equation. The group \( W(\hat{F}_4) \) contains the group \( W(\hat{C}_4) \) that we get in the general case. However the algebraic stack \( \mathcal{M}_4 \) admits unusually large group of symmetries because of the exisitance of exeptional graph automorphisms.

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