Topological String on $AdS_3 \times \mathcal{N}$

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Abstract

We study the topologically twisted string theory on the general back-ground $AdS_3 \times \mathcal{N}$ which is compatible with the world-sheet $N = 2$ superconformal symmetry and is extensively discussed in the recent works [20]. After summarizing the algebraic structure of the world-sheet topological theory, we show that the space-time (boundary) conformal theory should be also topological. We directly construct the space-time topological conformal algebra (twisted $N = 2$ superconformal algebra) from the degrees of freedom in the world-sheet topological theory. Firstly, we work on the world-sheet of the string propagating near boundary, in which we can safely make use of the Wakimoto free field representation. Secondly, we present a more rigid formulation of space-time topological conformal algebra which is still valid far from the boundary along the line of [11]. We also discuss about the relation between this space-time topological theory and the twisted version of the space-time $N = 2$ superconformal field theory given in [20].
1 Introduction

String theory on $AdS \times N$ back-ground has been getting great importance in the studies of $AdS/CFT$-duality \[1, 2, 3, 4\]. Among them, studies of string theory on $AdS_3$ back-ground have old histories \[3, 5\], in which the several consistencies as the first quantized string theory have been intensively discussed from the points of view of string propagating on a non-compact curved back-ground. More recently, a study emphasizing the relationship between the world-sheet picture of string theory and the description of boundary conformal field theory (CFT) on the $AdS_3$-target \[7, 8\] was initiated by Giveon-Kutasov-Seiberg \[9\], and many subsequent works which refine or extend it were carried out \[10\]-\[23\]. In particular the roles played by the short string sectors, which were lacking in \[3\], were investigated in \[11, 12\].

These approaches from the world-sheet picture of perturbative string have opened up a new perspective to the studies of $AdS_3/CFT_2$-duality. Namely, we can consider quite general back-ground $AdS_3 \times N$, which may not correspond to any brane configuration, as a consistent string vacuum. Although such a back-ground is outside of original idea of Maldacena \[1\], which is based on brane theory, we can still expect the existence of good holographic correspondence with some boundary CFT, because the asymptotic isometries of $AdS_3$ considered by Brown-Hennaux \[7\] should correspond to infinite numbers of conserved charges in perturbative string theory. In fact, in the framework given in the recent works \[20\] (and their related works \[24\]), we must generally assign a fractional value to the level of $SL(2; \mathbf{R})$ WZW describing the $AdS_3$-sector, which corresponds to the brane charge in the usual setting of NS1/NS5.

In this paper we study the topological string on $AdS_3 \times N$ with the suitable $NS B$-field, whose world-sheet fermions have integral spins. Although we again have no origin based on brane theory, it is natural to expect that there exists some space-time (boundary) CFT which is dual to the world-sheet theory of topological string, because of the same reason as above. We show this space-time CFT should be also topological, and the main result of this paper is the construction of space-time topological conformal algebra from the world-sheet point of view. There are two non-trivial points in this construction: The first is the difference of BRST charges between the topological string theory and the usual fermionic string theory. The second point is as follows: In the standard RNS formalism of fermionic string theory, the space-time SUSY is realized by the spin fields, and so they are expected to compose a part of the space-time superconformal algebra. However, in the case of topological string, we cannot
consistently consider the spin fields, since there do not exist the concepts such as NS and R sectors. This fact seems to make it difficult to construct the space-time topological conformal algebra in the analogous form as that for the untwisted string theory.

In order to construct the space-time conformal algebra we first consider the near boundary approximation, in which we can use the Wakimoto free field realizations of affine $SL(2; \mathbb{R})$ current algebra \([25]\). We propose the several conditions which should be satisfied by the correct space-time conformal algebra, and they lead us to a unique answer. Nextly, we work on a general string world-sheet propagating far from the boundary according to the formalism developed in \([11]\). We present a candidate for the space-time algebra in this framework. Although it has rather complicated form, we can prove without using the free field approximations that it indeed generates the correct topological conformal algebra, and it reduces to the previous result by the Wakimoto representation, when taking the near boundary limit.

\section{Topological Twisting of Fermionic String on $AdS_3 \times \mathcal{N}$}

As was discussed in \([20]\), the world-sheet superconformal symmetry of the fermionic string theory on $AdS_3 \times \mathcal{N}$ enhances to $N = 2$, if $\mathcal{N}$ has an affine $U(1)$-symmetry and $\mathcal{N}/U(1)$ defines an $N = 2$ superconformal field theory (SCFT) on world-sheet.

To fix the notations, let us first present the field contents in this fermionic string theory. For more detailed arguments, see \([20]\).

- **$AdS_3$-sector ($j^A, \psi^A$):**

  This sector has an affine $SL(2, \mathbb{R})_k$ symmetry $J^A = j^A - \frac{i}{2} \epsilon_{BC}^A \psi^B \psi^C$, where the bosonic parts of currents $j^A$ generate an affine $SL(2, \mathbb{R})_{k+2}$ algebra and $\psi^A$ ($A = 1, 2, 3$) denote the free fermions in the adjoint representation.

  \begin{align}
  \psi^A(z) \psi^B(0) & \sim \frac{\eta^{AB}}{z}, \quad \eta^{AB} \overset{\text{def}}{=} \text{diag}(+,-) \\
  j^A(z) j^B(0) & \sim \frac{(k+2) \eta^{AB}}{2z^2} + \frac{i \epsilon^{AB}_C}{z} j^C(0), \quad (\epsilon^{123} = 1 \text{ in our convention}) \\
  J^A(z) J^B(0) & \sim \frac{k \eta^{AB}}{2z^2} + \frac{i \epsilon^{AB}_C}{z} J^C(0)
  \end{align}

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This sector has a world-sheet $N = 1$ superconformal symmetry given by the next superconformal current;

$$G_{SL(2,R)} = \sqrt{\frac{2}{k}} \left( \psi^A j_A - i \psi^1 \psi^2 \psi^3 \right).$$  \hfill (2.4)

- $U(1)$-sector \ ($K, \chi$):

We assume the existence of an affine $U(1)$ symmetry; $K(z)K(0) \sim \frac{1}{z^2}$, and its fermionic partner; $\chi(z)\chi(0) \sim \frac{1}{z}$. The $N = 1$ superconformal current in this sector is simply given by $G_{U(1)} = \chi K$.

- $\mathcal{N}/U(1)$-sector:

We assume that this sector can be described by an $N = 2 \text{SCFT}$ $\mathcal{N} = 2 \{T_{\mathcal{N}/U(1)}, G_{\mathcal{N}/U(1)}^\pm, J_{\mathcal{N}/U(1)} \}$.

Since we here consider a critical fermionic string, the total central charge (except the ghost sector) should be equal to $c = 15$ ($\hat{c} \equiv \frac{c}{3} = 5$). Especially, the $N = 2 \text{SCFT}_2$ for the $\mathcal{N}/U(1)$-sector must have the central charge $c = 9 - \frac{6}{k} (\hat{c} = 3 - \frac{2}{k})$, which implies

$$J_{\mathcal{N}/U(1)}(z)J_{\mathcal{N}/U(1)}(0) \sim \frac{3 - \frac{2}{k}}{z^2}. \hfill (2.5)$$

The $N = 2$ structure is realized by considering the following formal decomposition of target space $[20]$ (see also the appendix B of $[3]$);

$$AdS_3 \times \mathcal{N} \sim \frac{SL(2,R)}{U(1)} \times \frac{U(1)^2 \times \mathcal{N}}{U(1)}. \hfill (2.6)$$

We already assumed the $\mathcal{N}/U(1)$ sector has $N = 2$ superconformal symmetry. We can describe the $SL(2,R)/U(1)$ sector by the Kazama-Suzuki coset CFT $[28]$, which is essentially realized by the conformal fields $\{j^+, j^-, \psi^+, \psi^-\}$ ($j^\pm \equiv j^1 \pm ij^2$, $\psi^\pm \equiv \psi^1 \pm i\psi^2$), and the $U(1)^2$ sector is described by the standard “Coulomb Gas” representation of $N = 2 \text{SCFT}$, whose members are two bosonic $U(1)$ currents $\{J^3, K\}$ and two fermions $\{\psi^3, \chi\}$. It is convenient for our later discussions to combine these bosonic currents and define the bosonic scalars $\Phi^\pm$ as follows;

$$i\partial \Phi^\pm = \frac{1}{\sqrt{2}} K \pm \frac{1}{\sqrt{k}} J^3, \hfill (2.7)$$
which have the OPE: \( \partial \Phi^\pm(z) \partial \Phi^\pm(0) \sim 0, \) \( \partial \Phi^\pm(z) \partial \Phi^\mp(0) \sim -\frac{1}{z^2}. \)

Let us introduce the total \( U(1)_R \) current defined by
\[
J_R = J_{SL(2,\mathbb{R})/U(1)} + J_{U(1)^2} + J_{N/U(1)} = \left( \frac{1}{2} \psi^+ \psi^- + \frac{2}{k} \psi^3 \right) + \chi \psi^3 + J_{N/U(1)}. \quad (2.8)
\]

According to the value of its charge, the total \( N = 1 \) superconformal current; \( G_{tot}(z) = G_{SL(2,\mathbb{R})} + G_{U(1)} + (G^+_N/U(1) + G^-_N/U(1)) \) is naturally decomposed to the \( N = 2 \) pieces; \( G_{tot}(z) = G^+_N(z) + G^-_N(z), \) where
\[
G^\pm_{N=2} = \frac{1}{\sqrt{k}} \psi^\pm j^\mp, \quad (2.9)
\]
\[
G^\pm_{SL(2,\mathbb{R})/U(1)} = \frac{1}{\sqrt{2}} (\chi \mp \psi^3) i \partial \Phi^\pm, \quad (2.10)
\]
\[
G^\pm_{U(1)^2} = \frac{1}{\sqrt{2}} (\chi \mp \psi^3) i \partial \Phi^\pm, \quad (2.11)
\]
which defines an \( N = 2 \) SCFT \{\( T^{N=2}, G^\pm_{N=2}, J_R \)\}.

It is a famous fact that we can construct a topological conformal field theory (TCFT\(_2\)) from an arbitrary \( N = 2 \) SCFT\(_2\) by the procedure of “topological twisting” [20, 27]. Roughly speaking, topological twisting means to shift the spins of the fermionic coordinates, namely, to replace the physical fermion system with spin 1/2 with the ghost system with spin \((0, 1)\).

Usually topological twisting is defined with respect to the \( U(1)_R \)-current \( J_R \) in \( N = 2 \) superconformal algebra (SCA) \{\( T^{N=2}, G^\pm_{N=2}, J_R \)\} [27];
\[
T^{\text{top}}_{EY} = T^{N=2} + \frac{1}{2} \partial J_R. \quad (2.12)
\]

\( T^{\text{top}}_{EY} \) indeed has central charge 0, since
\[
J_R(z)J_R(0) \sim \frac{5}{z^2} \equiv \frac{\hat{c}}{z^2}, \quad (2.13)
\]
and can be written in a BRST exact form \( T^{\text{top}}_{EY} = \{Q_s, G_{N=2}^-\}, \) where \( Q_s \) \( \overset{\text{def}}{=} \int dz G_{N=2}^+ \) denotes the BRST charge for TCFT. However, this definition of twisting is not convenient for our purpose, because the fermionic coordinates (and also the \( \beta\gamma \)-system in Wakimoto representation which we will introduce afterward) in the theory have fractional spins with respect to the twisted stress tensor \( T^{\text{top}}_{EY} \) (2.12), and so the boundary conditions for these fields will become subtle. Therefore, we shall slightly modify the definition of topological
twisting. We consider the twisting with respect to the “total fermion number current” \( J_f \) defined by

\[
J_f \overset{\text{def}}{=} \frac{1}{2} \psi^+ \psi^- + \chi \psi^3 + \left( J_{N/U(1)} - \sqrt{\frac{2}{k}} K \right).
\]

The twisted stress tensor \( T^{\top} \overset{\text{def}}{=} T^{N=2} + \frac{1}{2} \partial J_f \) also has zero central charge, because \( J_f(z)J_f(0) \sim \frac{5}{z^2} \) holds, and as we will see below, \( T^{\top} \) can be written in a \( Q_s \)-exact form. With respect to \( T^{\top} \), \( \xi_1 \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \psi^+ \), \( \xi_2 \overset{\text{def}}{=} \frac{1}{\sqrt{2}}(\chi - \psi^3) \) (and also \( \gamma \) in Wakimoto rep.) have conformal weight 0 (we call them “ghosts”), and \( \eta_1 \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \psi^- \), \( \eta_2 \overset{\text{def}}{=} \frac{1}{\sqrt{2}}(\chi + \psi^3) \) (and \( \beta \) in Wakimoto rep.) have conformal weight 1 (“anti-ghosts”). It may be worthwhile to notice that the “fermion number current” for the \( N/U(1) \)-sector \( J_{N/U(1)}' \equiv J_{N/U(1)} - \sqrt{\frac{2}{k}} K \) is in fact the same \( U(1) \)-current as the one made use of in order to define the spin fields in the papers [20] and has an integer level; \( J_{N/U(1)}'(z)J_{N/U(1)}'(0) \sim \frac{3}{z^2} \).

The explicit form of our topologically twisted stress tensor is given as follows;

\[
T^{\top} = \frac{1}{k^2} j^A j_A + \frac{1}{2} K^2 - \frac{1}{\sqrt{2k}} \partial K - \eta_1 \partial \xi_1 - \eta_2 \partial \xi_2 + \left( T_{N/U(1)}^{N=2} + \frac{1}{2} \partial J_{N/U(1)} \right).
\]

This topological theory can be naturally decomposed to the three independent sectors mentioned above;

\[
T^{\top} = T_1 + T_2 + T_{N/U(1)}.
\]

Each of them is described by the “topological conformal algebra” (TCA): \( \{ T, G^{\pm}, J \} \), which is defined as the twisted \( N = 2 \) superconformal algebra and our convention for this algebra is presented in appendix A. The generators belonging to the different sectors (anti-)commute with one another. This is the structure analogous to that for the topologically twisted gauged WZW model [29]. We summarize this aspect as follows;

\[\text{In fact, in the special case } \mathcal{N} = S^3 \times T^4, \text{ one will find that } J_f \text{ can be identified with } \frac{1}{2} \psi^+ \psi^- + \frac{1}{2} \chi^+ \chi^- + \chi^3 \psi^3 + \lambda^{11} \lambda^{22} - \lambda^{12} \lambda^{21}, \text{ where } \{ \chi^\pm, \chi^3 \} \text{ and } \lambda^{\alpha \dot{a}} (\alpha, \dot{a} = 1, 2) \text{ denote the fermionic coordinates along } S^3 \text{ and } T^4 \text{ respectively.} \]
1. $SL(2, \mathbb{R})/U(1)$-sector (twisted Kazama-Suzuki for $SL(2, \mathbb{R})_{k+2}/U(1)$ with $\hat{c} = 1 + \frac{2}{k}$):

$$
\begin{cases}
T_1 &= \frac{1}{k} (j^A j_A + J^3 J^3) - \eta_1 \partial \xi_1 + \frac{1}{k} \partial J^3 \\
G_1^+ &= \frac{1}{\sqrt{k}} \xi_1 j^- \\
G_1^- &= \frac{1}{\sqrt{k}} \eta_1 j^+ \\
J_1 &= \xi_1 \eta_1 + \frac{2}{k} J^3
\end{cases}
$$

(2.17)

2. $U(1)^2$-sector (twisted $N = 2$ Coulomb-Gas representation, $\hat{c} = 1$)

$$
\begin{cases}
T_2 &= -\partial \Phi^+ \partial \Phi^- - \eta_2 \partial \xi_2 - \frac{1}{\sqrt{k}} i \partial^2 \Phi^+ \\
G_2^+ &= i \xi_2 \partial \Phi^+ \\
G_2^- &= i \partial \Phi^- - \frac{1}{\sqrt{k}} \partial \eta_2 \\
J_2 &= \xi_2 \eta_2 - \frac{1}{\sqrt{k}} i \partial \Phi^+
\end{cases}
$$

(2.18)

3. $\mathcal{N}/U(1)$-sector (twisted in the usual sense of [27], $\hat{c} = 3 - \frac{2}{k}$):

$$\{T_{\mathcal{N}/U(1)}(\equiv T_{\mathcal{N}/U(1)}^{N=2} + \frac{1}{2} \partial J_{\mathcal{N}/U(1)}), G_{\mathcal{N}/U(1)}^\pm, J_{\mathcal{N}/U(1)}\}.$$  

By our construction it is obvious that $G_1^\pm$ are equal to $G_{SL(2, \mathbb{R})/U(1)}^\pm$ defined previously (2.10). However, one should remark that $G_2^- \neq G_{U(1)^2}^-$, although $G_2^+ = G_{U(1)^2}^+$ still holds. (Namely, $G_{\text{tot}}^+ \equiv G_1^+ + G_2^+ + G_{\mathcal{N}/U(1)}^+$ is equal to $G_{N=2}^+$, but $G_{\text{tot}}^- \neq G_{N=2}^-$. ) Generally, for an arbitrary parameter $\alpha$,

$$
\begin{cases}
T_2 &= -\partial \Phi^+ \partial \Phi^- - \eta_2 \partial \xi_2 - \frac{\alpha}{\sqrt{k}} i \partial^2 \Phi^+ \\
G_2^+ &= i \xi_2 \partial \Phi^+ \\
G_2^- &= i \partial \Phi^- - \frac{\alpha}{\sqrt{k}} \partial \eta_2 \\
J_2 &= \xi_2 \eta_2 - \frac{\alpha}{\sqrt{k}} i \partial \Phi^+
\end{cases}
$$

(2.19)

compose the TCA with $\hat{c} = 1$. The case of $\alpha = 0$ corresponds to the topological twisting of [27] (2.12) and we have $G_2^- = G_{U(1)^2}^-$. The twist by $J_f$ (2.14) corresponds to $\alpha = 1$, in which we have $G_2^- \neq G_{U(1)^2}^-$ as we remarked above.
We must also consider the ghost system associated with the gauge fixing of diffeomorphism. In the fermionic string theory before twisting, we have the spin \((2, -1)\) \(bc\)-ghosts and the spin \((3/2, -1/2)\) \(\beta\gamma\)-ghosts, which yield an \(N = 1\) SCFT with \(c = -15\) that cancels the central charge of “matter sector”. On the contrary, because the matter sector in the topological string theory has already \(c = 0\), the ghost sector should become a \(c = 0\) CFT, too. This leads us to the system of fermionic ghosts with spin \((2, -1)\) and bosonic ghosts also with spin \((2, -1)\). It is no other than the “topological gravity sector” [30], which also has a structure of twisted \(N = 2\) with \(\hat{c} = -3\). Through this paper we shall use the symbols \(\beta, \gamma\) for the Wakimoto free fields, and describe the topological gravity sector by the standard bosonizations [31] to avoid confusion, that is, we set \(b = e^{-i\sigma}, c = e^{i\sigma}, \beta = \partial\xi e^{-\rho}, \gamma = \eta e^{\rho}\), where \((\xi, \eta)\) is a spin \((0,1)\) fermionic fields, and \(\sigma, \rho\) are scalar fields with the back-ground charge \(Q_\sigma = 3i, Q_\rho = -3\), which have the OPEs; \(\sigma(z)\sigma(0) \sim -\log z, \rho(z)\rho(0) \sim -\log z\). It is also convenient to define \(\phi \equiv -\rho + i\sigma\). The TCA in this sector is generated by the following currents:

\[
\begin{align*}
T_{TG} &= -\frac{1}{2} \partial\phi \partial\phi^* + \frac{3}{2} \partial^2 \phi - \eta \partial\xi \\
G^+_{TG} &= \eta e^{-\phi} \\
G^-_{TG} &= \{ -\partial^2 \xi + \partial\xi (\partial\rho - 2i\partial\sigma) \} e^\phi \\
J_{TG} &= -2\partial\rho + i\partial\sigma
\end{align*}
\] (2.20)

The BRST charge\(^{3}\) for our topological theory is defined by

\[
Q_s \defeq \oint (G^+_1 + G^+_2 + G^+_N/U(1) + G^+_T G)(\equiv \oint (G^+_t + G^+_T G)).
\] (2.21)

The stress tensor in each sector \(T_i\) can be written in the BRST exact form;

\[
T_i = \{ Q_s, G^-_i \},
\] (2.22)

and so does the total stress tensor \(T^{\text{top}}\) (2.15).

The physical states are defined by

\[
(Q_s + \tilde{Q}_s)|_{\text{phys}} = 0,
\] (2.23)

\(^2\)In our convention the back-ground charge \(Q\) is defined by \(c = 1 + 3Q^2\).

\(^3\)Strictly speaking, the precise BRST charge should be \(Q_{BRST} \defeq Q_s + Q_{\text{diff}}\), where \(Q_{\text{diff}}\) is defined by

\[
Q_{\text{diff}} = \oint \left\{ c \left( T^{\text{top}} + \frac{1}{2} T_{TG} \right) - \gamma \left( G^+_t + \frac{1}{2} G^+_T G \right) \right\}.
\]

But we can easily find that \(UQ_s U^{-1} = Q_{BRST}\) holds, where we set \(U = e^{\oint c (G^+_t + \frac{1}{2} G^+_T G)}\), and hence we can consistently regard \(Q_s\) as the BRST charge of the theory.
according to the generalities of TCFT. As is well-known, there are natural two choices for the chiralities of \( \bar{Q}_s \) ("topological A-model" or "topological B-model"). However, the detailed discussions for this subject is beyond the scope of this paper. We shall only focus on the left (or right) mover in the following discussions, and consider the physical condition \((2.23)\) simply as

\[
Q_s |\text{phys}\rangle = 0, \quad |\text{phys}\rangle \sim |\text{phys}\rangle + Q_s |\text{any}\rangle.
\]

The physical states correspond to the chiral primary states in the untwisted \( N = 2 \) theory on world-sheet. Moreover, since

\[
J_0^3 + \sqrt{\frac{k}{2}} K_0 \left( \equiv \oint (j^3 + \xi_1 \eta_1 + \sqrt{\frac{k}{2}} K) \right) = \{ Q_s, \sqrt{k} \oint \eta_2 \} \quad \text{(2.25)}
\]

holds, arbitrary physical states must satisfy

\[
\left( J_0^3 + \sqrt{\frac{k}{2}} K_0 \right) |\text{phys}\rangle = 0 \quad \text{(mod BRST)} , \quad \text{(2.26)}
\]

which is no other than the BPS condition in space-time. It implies that the physical states are also in one-to-one correspondence to the 1/2 BPS states, in other words the chiral primary states in the space-time theory derived from the untwisted world-sheet theory.

From this observation we can immediately conclude that the space-time CFT associated to our topological string must be also topological. In fact, since we now work on the \( AdS_3 \) back-ground, it is natural to expect that the \( Q_s \)-invariant Virasoro operators \( \{ \mathcal{L}_n \}_{n \in \mathbb{Z}} \) of Brown-Hennaux exist, and if so, they must satisfy

\[
[J_0^3, \mathcal{L}_n] = n \mathcal{L}_n, \quad [K_0, \mathcal{L}_n] = 0 , \quad \text{[2.27]}
\]

because of the requirement of \( SL(2; \mathbb{R}) \)-symmetry on \( AdS_3 \). However, this means that \( \mathcal{L}_n |\text{phys}\rangle \) cannot satisfy the condition \((2.26)\) for any physical state \( |\text{phys}\rangle \), \textit{unless all of the} \( \{ \mathcal{L}_n \} \) \textit{are BRST-exact}. In this way, we have found that topological string on \( AdS_3 \) back-ground should lead to the space-time conformal symmetry with the Virasoro operators having the BRST-exact forms, namely, a topological conformal field theory.

At this stage a few natural questions may arise; Does this space-time TCFT have a twisted \( N = 2 \) algebra describing its local symmetry in the same way as the familiar models of TCFT in two dimension? Moreover, if it exists, can we regard it as the twisted version of the space-time \( N = 2 \) algebra constructed in [20]?
The answer for the first question is *Yes*. In the next section we will construct the suitable space-time conformal algebra from the world-sheet field contents. At first glance, the answer for the second question seems also *Yes*, because the most natural choice of \( \mathcal{L}_0 \)-operator is as follows (up to normalization, of course);

\[ \mathcal{L}_0 = - \left( J_0^3 + \sqrt{\frac{k}{2}} K_0 \right), \quad (2.28) \]

which is a BRST-exact operator as seen above. This is indeed the \( \mathcal{L}_0 \)-operator in the twisted version of space-time \( N = 2 \) theory in [20], since \( \mathcal{L}_0^{N=2} = -J_0^3 \), and the \( U(1)_R \) charge operator is \( \mathcal{J}_0^{N=2} = \sqrt{2k}K_0 \). However, this is not the whole story. We must also care about the detailed structure of higher Virasoro generators to obtain this answer completely. In the next section we will discuss this point.

### 3 Topological Conformal Symmetry in Space-time

As we already demonstrated, the space-time conformal theory for the topologically twisted string should also become topological. In this section we construct the TCA in space-time. We first work in the near boundary region, where the string coordinates on \( AdS_3 \) become free fields, and next present the more rigid framework that is still valid far from the boundary.

#### 3.1 Space-time Topological Conformal Algebra in the Near Boundary Approximation

First of all, let us remember the (bosonic) \( AdS_3 \) sector is described in quantum level by the following Lagrangian [3];

\[ \mathcal{L} = \partial \varphi \bar{\partial} \varphi - \sqrt{\frac{2}{k}} R^{(2)} \varphi + \beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma} - \beta \bar{\beta} \exp \left( -\sqrt{\frac{2}{k}} \varphi \right), \quad (3.1) \]

where \( R^{(2)} \) means the curvature on world-sheet. As was clarified in [10], in the near boundary region (\( \varphi \sim +\infty \)), since the interaction term \( \sim \beta \bar{\beta} \exp \left( -\sqrt{\frac{2}{k}} \varphi \right) \) is suppressed, we can safely
make use of the Wakimoto free field representations \cite{Wakimoto} for the AdS₃-sector. The (bosonic) $SL(2; \mathbb{R})$-current can be realized as follows;

\[
\begin{align*}
    j^3 &= \beta \gamma + \sqrt{\frac{k}{2}} \partial \varphi \\
    j^- &= \beta \\
    j^+ &= \beta \gamma^2 + \sqrt{2k} \gamma \partial \varphi + (k + 2) \partial \gamma
\end{align*}
\]

where $\beta(z)\gamma(0) \sim \frac{1}{z}$, $\varphi(z)\varphi(0) \sim -\log z$ in our convention. The stress tensor in this sector is given by

\[
T(z) = -\frac{1}{2} (\partial \varphi)^2 - \frac{1}{\sqrt{2k}} \partial^2 \varphi + \beta \partial \gamma.
\]

It is important to notice that the interaction term $\sim \beta \bar{\beta} \exp \left( -\sqrt{\frac{2}{k}} \varphi \right)$ is so-called “screening charge”, which does not break at least perturbatively the conformal invariance as well as the affine $SL(2; \mathbb{R})$ symmetry on the world-sheet. This implies that the results given in the previous section are still valid in a generic region with $\varphi \sim$ finite, in which the free field approximations to the coordinates $\beta, \gamma, \varphi$ are not reliable. In fact, we did not need the explicit realizations of $SL(2; \mathbb{R})$-currents for the discussions in the previous section.

However, since the Wakimoto rep. is very powerful for the purpose in this section, which is to construct the suitable space-time conformal algebra, we shall first work in the near boundary region $\varphi \sim +\infty$. In the next subsection we will consider the general region according to the framework developed in \cite{Gopakumar:2005pm}.

The goal in this subsection is to construct the full space-time conformal algebra from the degrees of freedom in the world-sheet topological theory in the analogous form as that given in \cite{Gopakumar:2005pm}. The first guess for the answer may be the twisted version of the space-time $N = 2$ SCA for the untwisted string \cite{Shiu:2005} as mentioned in the previous section. Namely, one might suppose that

\[
\mathcal{L}_n = \mathcal{L}^{N=2}_n - \frac{1}{2} (n + 1) \mathcal{J}^{N=2}_n,
\]

where $\mathcal{L}^{N=2}_n, \mathcal{J}^{N=2}_n$ are the stress tensor and the $U(1)_R$ current given in \cite{Shiu:2005}. Unfortunately, although this expectation seems to be natural, it is not correct. This is because the R.H.S of (3.4) is not BRST-invariant in the sense of topological string theory (except $n = 0$). Furthermore we have another serious difficulty: In the untwisted theory the space-time supercurrents are made up of the spin fields. But, in the topological string theory, we have only the fermionic
fields with integral spins (ghost system) on world-sheet, and hence only the periodic boundary condition is allowed. In this case we cannot consider the spin fields, which intertwine the NS-sector with the R-sector in the usual fermionic string theory. Therefore we must look for the candidates of space-time conformal currents only within the ordinary vertex operators (which has no pieces of the spin fields).

We here assume the conditions which the space-time conformal algebra \( \{G_n^\pm, L_n, J_n\} \) in our topological model should satisfy;

1. They should generate a topological conformal algebra (twisted \( N = 2 \) superconformal algebra). Especially, \( \{L_n\} \) generates the Virasoro algebra with zero central charge (irrespective of the value of \( \oint \gamma^{-1} d\gamma \)).

2. \( \{L_n\} \) act on the (bosonic) boundary coordinates as suitable local conformal transformations in the same way as those given in [9].

3. \( \{L_n\} \) and \( \{G_n^+\} \) should be \( Q_s \)-invariant.

4. \( G_0^+ = Q_s = \oint (G_{\text{tot}}^+ + G_{TG}^+) \) holds (up to normalization).

5. All of the generators can be written in the form; \( A_n = \oint dza_n(z) \), where \( a_n(z) \) is a spin 1 primary field on the world-sheet, that is, it is a spin 1 current with no back-ground charge on the world-sheet;

\[
T_{\text{top}}(z) a_n(w) \sim 0 \times \frac{1}{(z-w)^3} + \cdots.
\]

The first and second conditions are quite natural. But there is one important remark; We require the first condition is satisfied “strictly”, that is, not only up to \( Q_s \)-exact terms. Otherwise, we cannot determine the space-time conformal algebra at all, because the Virasoro generators themselves are now \( Q_s \)-exact and hence the algebra becomes trivial identities in the sense of up to \( Q_s \)-exact terms.

One might ask about the third condition: Why do not we impose the \( Q_s \)-invariance on \( G^- \) and \( J \), too? However, it is not peculiar in topological conformal field theory. Usually, \( \{Q_s, G^-\} \) is not zero, rather equal to the stress tensor, and the “ghost number current” \( J \) does not also commute with \( Q_s \). As was already demonstrated, the assumption of the \( Q_s \)-invariance of \( L_n \) inevitably leads to the \( Q_s \)-exactness of it. This implies the topological invariance in the space-time theory.
The fourth condition means that the BRST-charge in the space-time topological theory is equal to the one for the world-sheet theory. This assumption might be too strong. But, from the viewpoint of AdS/CFT-duality, all of the physical observables in the world-sheet theory are expected to become also physical in the sense of space-time. This is the simplest assumption to realize this correspondence.

The fifth condition is more subtle. In the usual (untwisted) string theory, this is nothing but a result of BRST-invariance. However, we cannot derive this condition from the $Q_s$-invariance. Nevertheless the reason why we believe this condition is natural is as follows: In a general $CFT_2$ the back-ground charge term $T(z)a(w) \sim \frac{q}{(z-w)^3} \cdots$ is related with the anomaly term on a curved world-sheet; $\bar{\partial}a(z) \sim qR^{(2)}$. So, if we require that all of the generators in the space-time conformal algebra should be conserved charges even on the curved world-sheet, we need impose this condition on it. This condition is quite important as a guiding principle to look for the correct answer. For example, the space-time $U(1)_R$ current in the untwisted model has a non-zero back-ground charge with respect to the stress tensor $T^\text{top}$ (after the replacement $\psi^+ \to \sqrt{2}\xi_1, \psi^- \to \sqrt{2}\eta_1$, etc.), and hence we cannot take this current as the candidate for the “ghost number current” $J_n$ in the space-time TCA.

These five conditions can lead us to the unique answer (up to normalizations). Let us exhibit this result. We here use the abbreviated notation $\oint \equiv \oint \frac{dz}{2\pi i}$;

$$G^+_n = \left[ Q_s, -\sqrt{k} \oint \gamma^n \hat{J} \right]$$

$$= \oint \left\{ \sqrt{k} \gamma^n (G^+_\text{tot} + G^+_G) - n(\gamma^{n-1}\xi_1 + \gamma^n\xi_2)\hat{J} - (k+1)\gamma^n\partial\xi_2 \right\}$$

$$G^-_n = \oint \left\{ n\eta_1 \gamma^{n+1} - (n+1)\eta_2 \gamma^n \right\}$$

$$L_n = \{ Q_s, \sqrt{k}G^-_n \}$$

$$= \oint \left\{ n(j^- + \eta_1\xi_2)\gamma^{n+1} - (n+1)(j^3 - \eta_1\xi_1 + \sqrt{\frac{k}{2}}K)\gamma^n - n(n+1)\gamma^nR \right\}$$

$$J_n = \oint \gamma^n \hat{J} + \{ Q_s, \sqrt{k} \oint \gamma^n(-\gamma\eta_1 + \eta_2) \}$$

$$= \oint \gamma^n (J_f - \partial\rho + \sqrt{2kK} + nR)$$

(3.5)

where we set

$$\hat{J} = J_f - \partial\rho + \sqrt{\frac{k}{2}}K - \sqrt{\frac{k}{2}}\partial\varphi,$$

(3.6)

$$R = \eta_1\xi_1 + \gamma\eta_1\xi_2 - \gamma^{-1}\eta_2\xi_1 - \eta_2\xi_2.$$

(3.7)
The integrands of these operators are actually spin 1 conformal fields with no background charges with respect to $T_{\text{top}}^{(2,13)}$. After some straightforward and lengthy calculations, we can show they generate a TCA with $\hat{c} = 2k \oint \gamma^{-1}d\gamma$. This $\hat{c}$ is equal to that for the space-time theory for the untwisted string ($c_{N=2} = 6k \oint \gamma^{-1}d\gamma$), although our space-time TCA (3.5) has a form which is quite different from the twisted version of space-time $N=2$ SCFT. In fact, $L_0 = -\left( J_0^3 + \frac{k}{2} K_0 \right)$ holds as we mentioned in the previous section, but in general $L_n \neq L_{N=2}^n - \frac{1}{2}(n+1)J_{N=2}^n$.

We here make one comment: The topological gravity sector (especially, the scalar field $\partial \rho$) seems to play an important role in order to obtain the correct conformal algebra in space-time. In fact, it is not so difficult to show that we cannot construct the conformal algebra which satisfies all the five conditions above without topological gravity sector. Why it is so may be interesting problem. We would like to further discuss this point (from a geometrical point of view, maybe) elsewhere.

Before leaving the arguments in the near boundary approximation, let us discuss the “space-time vacuum” (long string vacuum) which realizes $\oint \gamma^{-1}d\gamma = p \neq 0$ in topological theory. By making use of the familiar bosonizations: $\beta = i\partial Y e^{-X-i\gamma}$, $\gamma = e^{X+iY}$, where $X(z)X(0) \sim -\ln z$, $Y(z)Y(0) \sim -\ln z$, we can immediately make up a $Q_s$-invariant (and not $Q_s$-exact) state having such a property within the large Hilbert space of $\beta \gamma$\cite{GKS}. We set

$$V \overset{\text{def}}{=} \xi_1 e^{-X} e^\phi \equiv \xi_1 e^{-X} e^{-\rho}$$
$$V^{-1} \overset{\text{def}}{=} \eta_1 e^X e^{-\phi} \equiv \eta_1 be^X e^\rho.$$  

They are $Q_s$-invariant: $[Q_s, V^{\pm 1}] = 0$ and $V \cdot V^{-1} = V^{-1} \cdot V = 1$. Moreover they satisfy

$$\left[ \oint \gamma^{-1}d\gamma, \ V^{\pm 1} \right] = \pm V^{\pm 1}. \tag{3.9}$$

It is easy to see we can consider the operator products $V \cdot V \cdots$, $V^{-1} \cdot V^{-1} \cdots$ without any regularization of divergence. Therefore we can introduce the following space-time vacuum in our topological model;

$$|\text{vac}_p \rangle \overset{\text{def}}{=} V^p |0 \rangle \quad (p \in \mathbb{Z}), \tag{3.10}$$

\footnote{It may be worthwhile to emphasize that we must here work on the large Hilbert space rather than the reduced space projected by $\oint e^{iY} = 0$. This is because we need the negative powers of $\gamma$ as well as the positive powers in the original construction of the space-time Virasoro algebra by GKS \cite{GKS}.}
which is manifestly $Q_s$-invariant and satisfies
\[
\oint \gamma^{-1} d\gamma |\text{vac}; p\rangle = p |\text{vac}; p\rangle
\] (3.11)
for any winding number $p \in \mathbb{Z}$.

It is interesting to compare this result with that for the untwisted theory. The operator $V_{\text{untwisted}}$ in the untwisted fermionic string theory should be given by
\[
V_{\text{untwisted}} \overset{\text{def}}{=} \psi + ce^{-X} e^{-\rho}.
\] (3.12)
This operator is BRST-invariant (and not exact) in the framework of the untwisted fermionic string, and also satisfies $[\mathcal{L}_n^{GKS}, V] = 0 \text{ (mod BRST)}$ for $\forall n \geq -1$, where $\mathcal{L}_n^{GKS}$ denote the space-time Virasoro generators given in [9]. The space-time vacuum for $p = 1$ is defined as
\[
|\text{vac}\rangle_{\text{untwisted}} \overset{\text{def}}{=} V_{\text{untwisted}} |0\rangle,
\] (3.13)
which is no other than the $(-1)$-picture version of that introduced in [15]. (Remember that in this paper we write the bosonized superghost as “$\rho$” instead of the usual notation “$\phi$”.)

Since one can again define the operator products $V_{\text{untwisted}} \cdot V_{\text{untwisted}} \cdots$ without any regularization, one might imagine that the space-time vacuum for any $p \in \mathbb{Z}$ can be defined similarly as (3.10). But this naive guess does not work. In fact, we can prove that $V_{\text{untwisted}}^p (\forall p \geq 2)$ becomes BRST-exact, and moreover the operator corresponding to $V^{-1}$ (which is given by replacing $\eta$ with $\psi^-$ in the expression (3.8)) does not become BRST-invariant in the untwisted theory. This observation suggests the claim given in [15]: One can only construct the space-time vacuum for $p = 1$ on the single string world-sheet, and in order to realize the cases $p \geq 2$ one should consider the multi-string system described by $\text{Sym}^p$ orbifold theory. This claim is consistent with the naive estimation of physical degrees of freedom, because the winding number $p$ should correspond to a brane charge (NS1 charge), and is related with the upper bound for the $U(1)_R$ charges of the chiral primary fields in the boundary CFT [32].

On the other hand, as seen above, in the topological string we can make up the space-time vacuum for an arbitrary winding $p$ on a single string world-sheet. It is not a contradiction, since the winding in topological string does not correspond to any brane charge, and we can show that we have no upper bound for the $U(1)_R$ charge related to the value of $p$ in the topological model. The topological gravity sector seems to play an essential role in this feature because of the existence of “gravitational descendants”. Of course, in order to complete
this discussion we need to analyse carefully the detailed structure of physical spectrum in the
topological theory. This analysis will be given in [33].

3.2 Space-time Topological Conformal Algebra in Finite $\varphi$

Now let us consider the more general situation when the string world-sheet exists far from
boundary. In this case the free field description does not work, although we still have an affine
$SL(2; \mathbb{R})$ symmetry. To this aim it is useful to introduce auxiliary parameters $x$, $\bar{x}$, which are
naturally identified with the coordinates on boundary, according to the works [1, 11, 11, 22],
and to write down directly the space-time conformal currents as $\mathcal{T}(x)$, $\mathcal{G}^\pm(x)$, $\mathcal{J}(x)$ instead
of dealing with their Fourier modes. In particular, we shall follow the convention of [11], and
refer to the above papers [3, 11, 11, 22] for the more detailed arguments.

First of all, let us introduce

$$ j(x; z) \overset{\text{def}}{=} 2xj^3(z) - x^2j^-(z) - j^+(z), \quad (3.14) $$

which satisfies the following OPE;

$$ j(x; z)j(y; w) \sim (k + 2)\frac{(y - x)^2}{(z - w)^2} + \frac{1}{z - w} \left\{ (y - x)^2 \partial_y - 2(y - x) \right\} j(y; w). \quad (3.15) $$

The counterpart in the right mover $\bar{j}(\bar{x}; \bar{z})$ is similarly defined. It is also convenient to intro-
duce the notation

$$ \tilde{j}(x; z) \overset{\text{def}}{=} j(x; z) + 2x\sqrt{\frac{k}{2}} K(z). \quad (3.16) $$

These currents (3.14), (3.15) have the weight (1,0) for world-sheet and the weight $(-1, 0)$ for
space-time. (The boundary coordinates $x$, $\bar{x}$ should have the space-time dimensions $(-1, 0)$,
$(0, -1)$ respectively). The primary field in space-time is defined by

$$ \Phi_h(x, \bar{x}; z, \bar{z}) \overset{\text{def}}{=} \frac{1}{\pi} \left( \frac{1}{|x - \gamma(z)|^{2h}e^{\frac{\gamma(z)}{\sqrt{2k}}} + e^{-\frac{\gamma(z)}{\sqrt{2k}}}} \right)^{2h}, \quad (3.17) $$

which is an analog of the bulk-boundary Green function. This operator should have the
following OPE with the current $j(x; z)$;

$$ j(x; z)\Phi_h(y, \bar{y}; w, \bar{w}) \sim \frac{1}{z - w} \left\{ (y - x)^2 \partial_y + 2h(y - x) \right\} \Phi_h(y, \bar{y}; w, \bar{w}), \quad (3.18) $$
which means $\Phi_h$ indeed has the space-time conformal weight $(h, h)$. For our purpose the $h = 1$ case is the most important;

$$j(x; z)\Phi_1(y, \bar{y}; w, \bar{w}) \sim \frac{1}{z - w} \partial_y \left\{ (y - x)^2 \Phi_1(y, \bar{y}; w, \bar{w}) \right\}, \quad (3.19)$$

and it is easy to see

$$\lim_{\varphi \to +\infty} \Phi_1(x, \bar{x}; z, \bar{z}) = \delta^2(x - \gamma(z)). \quad (3.20)$$

We must further prepare the fermionic partners of the above currents. We set

$$\xi(x; z) \equiv \xi_1(z) + x \xi_2(z)$$
$$\eta(x; z) \equiv -x \eta_1(z) + \eta_2(z), \quad (3.21)$$

which satisfy the OPE;

$$\xi(x; z)\eta(y; w) \sim -\eta(x; z)\xi(y; w) \sim \frac{x - y}{z - w}, \quad (3.22)$$

$\xi(x; z)$ should have the world-sheet weight $(0, 0)$ and the space-time weight $(-1, 0)$. $\eta(x; z)$ should have $(1, 0)$ for world-sheet and $(0, 0)$ for space-time.

Under these preparations we can propose the following currents as the suitable topological conformal algebra in space-time $\{G^\pm(x), T(x), J(x)\}$. To avoid the complexity of notations we shall use the abbreviations; $\xi \equiv \xi(x; z), \eta \equiv \eta(x; z), j \equiv j(x; z)$ etc.;

$$G^+(x) = [Q_s, -\sqrt{k}J(x)]$$
$$= -\frac{1}{k + 2} \int d^2z \left\{ (\bar{j}\Phi_1)(\sqrt{k}(G^+_{N/U(1)} + G^+_{T_G}) - \partial_x^2 \bar{j}\xi - (k + 1)\partial_x \partial_x \xi) + \partial_x (\bar{j}\Phi_1) \left( J_{N/U(1)} - \partial_x + (k - 1)\sqrt{\frac{2}{k} K + 2\partial_x} \right) \right.$$  
$$\left. - \partial_x (\bar{j}\Phi_1)(\frac{1}{2} \partial_x \bar{j} \xi + \eta \partial_x \xi + \partial_x \xi) \right\} \quad (3.23)$$

$$G^-(x) = -\frac{1}{k + 2} \int d^2z \{ 2(\bar{j}\Phi_1)\partial_x \eta + \partial_x (\bar{j}\Phi_1)\eta \}$$
$$T(x) = \{ Q_s, \sqrt{k}G^-(x) \}$$
$$= -\frac{1}{k + 2} \int d^2z \left\{ (\bar{j}\Phi_1)(\partial_x^2 \bar{j} + 4\partial_x \eta \partial_x \xi) + \partial_x (\bar{j}\Phi_1)(\frac{1}{2} \partial_x \bar{j} \xi + 3\partial_x \eta \xi + 2\eta \partial_x \xi) + \partial_x^2 (\bar{j}\Phi_1)\eta \xi \right\}$$
$$J(x) = -\frac{1}{k + 2} \int d^2z \left\{ (\bar{j}\Phi_1) \left( J_{N/U(1)} - \partial_x + (k - 1)\sqrt{\frac{2}{k} K + 2\partial_x} \right) + \partial_x (\bar{j}\Phi_1)\eta \xi \right\}$$
In these expressions one should regard the products of operators inserted at the same points on world-sheet as the standard normal product defined by

$$A(z)B(z) \overset{\text{def}}{=} \oint dz \frac{A(w)B(z)}{w - z}.$$  \hspace{1cm} (3.24)

Now, why can we claim that they are the correct currents in the space-time conformal algebra?

First of all, we can derive the Ward identities of these currents \{\(T(x), G^\pm(x), J(x)\}\} by means of the methods outlined in [11] without using the free field approximations. We only have to evaluate the correlators having the forms;

$$\langle \partial x A(x) A'(y) O \ldots \rangle,$$

where \(A(x), A'(y)\) are one of \{\(T(x), G^\pm(x), J(x)\}\}, and \(O \ldots\) means the suitable primary operators in space-time CFT. In these calculations we can make use of the formulas (3.15), (3.19), (3.22) and among other things\(^5\)

$$\Phi_1(x, \bar{x}; z, \bar{z}) \Phi_h(y, \bar{y}; z, \bar{z}) = \delta^2(x - y) \Phi_h(y, \bar{y}; z, \bar{z}).$$  \hspace{1cm} (3.25)

The following identities are also useful;

$$\Phi_1 = \frac{1}{\pi} \partial_x \Lambda,$$

$$\bar{j} \Phi_1 = \frac{k + 2}{\pi} \partial_{\bar{z}} \Lambda,$$

$$\partial_x \Phi_1 = \frac{1}{k + 2} \partial_x (j \Phi_1),$$  \hspace{1cm} (3.26)\hspace{1cm} (3.27)\hspace{1cm} (3.28)

where

$$\Lambda(x, \bar{x}; z, \bar{z}) = \frac{1}{x - \gamma(z)} \frac{|x - \gamma(z)|^2 e^{\sqrt{2} \phi(x)}}{|x - \gamma(z)|^2 e^{\sqrt{2} \phi(x)} + 1},$$  \hspace{1cm} (3.29)

as well as the next elementary identity of delta function;

$$x^m \partial_x^n \delta(x) = \begin{cases} (-1)^m \frac{n!}{(n-m)!} \partial_x^{n-m} \delta(x) & (m \leq n) \\ 0 & (m > n). \end{cases}$$  \hspace{1cm} (3.30)

We shall not present here the explicit forms of the Ward identities among the currents \{\(T(x), G^\pm(x), J(x)\}\} (3.23), but the important point is as follows: These Ward identities are nicely interpreted as the OPE’s among them, and we can prove that they truly generate

\(^5\)This formula (3.25) is justified only by semiclassical calculations in [11]. But, we here assume the validity of this identity in quantum level.
a topological conformal algebra with
\[ \hat{c} = 2kI, \]
\[ I = \frac{1}{(k + 2)^2} \int d^2 z j(x; z) \bar{j}(\bar{x}; \bar{z}) \Phi_1(x, \bar{x}; z, \bar{z}). \] (3.31)

Here \( I \) is actually a constant on the boundary \( \partial_x I = \partial_{\bar{x}} I = 0 \), and under the long string configuration near boundary this constant reduces to \( \lim_{\varphi \to +\infty} I = \oint_{\gamma} \gamma^{-1} d\gamma \). (For the more detailed arguments, especially with respect to the contribution from the short string sector, refer to \([10, 11]\).)

Secondly, we can show that the currents (3.23) reduce to the free field realizations (3.5) in the near boundary limit \( \varphi \to +\infty \). Under this limit we have \( \Lambda \sim \frac{1}{x - \gamma} \), and can recover the results in the free field approximations (3.5); \( \lim_{\varphi \to \infty} \oint dxx^n T(x) = L_n, \lim_{\varphi \to \infty} \oint dxx^n G^+(x) = G^+_n, \lim_{\varphi \to \infty} \oint dxx^n G^-(x) = G^-_n, \) and \( \lim_{\varphi \to \infty} \oint dxx^n J(x) = J_n \).

Here we comment on a non-trivial point to take the limit \( \varphi \to +\infty \). The operator products in the expressions (3.23) are defined by (3.24), but those for the near boundary results (3.5) stand for the normal ordering with respect to the mode expansions of free fields. These two do not coincide in general. Especially, (in the calculation of \( \lim_{\varphi \to \infty} \oint dxx^n G^+(x) \)) it is important to remark the next identity under the limit \( \varphi \to +\infty \);

\[ \partial_x j(x; z) \partial_x \Phi_1(x, \bar{x}; z, \bar{z}) =: \partial_x j(x; z) \partial_x \Phi_1(x, \bar{x}; z, \bar{z}) : + \frac{2}{k + 2} \partial^2_x : (j(x; z) \Phi_1(x, \bar{x}; z, \bar{z})) : \] (3.32)

where L.H.S means the normal product defined by (3.24) and the symbol “: :” in R.H.S means the normal ordered product of free fields.

There is an important consistency check for the space-time currents (3.23). We must show that \( \{ G^\pm(x), T(x), J(x) \} \) are actually “holomorphic”, that is, they have no singular OPE’s with their anti-holomorphic counterparts \( \{ \bar{G}^\mp(\bar{y}), \bar{T}(\bar{y}), \bar{J}(\bar{y}) \} \). It is not a trivial statement in contrast to the case of free fields (3.5), since the expressions of (3.23) include the terms such as \( \sim \partial^a_x (\bar{j} \Phi_1) \) which may have singular OPE’s with the anti-holomorphic currents. For this purpose let us first confirm the next identity;

\[ \{ \bar{Q}_s, A(x) \} = 0, \quad (\forall A(x) \in \{ G^\pm(x), T(x), J(x) \}). \] (3.33)

We start with the next simple identity;

\[ [\bar{Q}_s, \partial^a_x (\bar{j} \Phi_1)] = -\frac{k + 2}{\sqrt{k}} \partial_x \partial^a_x (\bar{c} \Phi_1). \] (3.34)
From this, we obtain

\[ \left[ \bar{Q}_s, \int d^2z \frac{\partial}{\partial \bar{z}} (\bar{j} \Phi_1) F(j, \eta, \xi, \ldots) \right] = \int d^2z \partial \bar{z}(\ast) = 0, \quad (3.35) \]

for an arbitrary polynomial \( F \) of only the holomorphic operators \( j(x; z), \eta(x; z), \xi(x; z), \ldots \), which proves the above identity (3.33).

Nextly, it is not so hard to prove

\[ \partial_x \bar{G}^-(\bar{x}) A(y) \sim 0, \quad \partial_x \bar{T}(\bar{x}) A(y) \sim 0, \quad (\forall A(y) \in \{G^\pm(y), T(y), J(y)\}). \quad (3.36) \]

In the proof for them we can make use of the next identity (and its cousins);

\[ \oint_{\bar{w}} d\bar{z} \Phi_1(x, \bar{x}; z, \bar{z}) (\bar{j}(\bar{g}; \bar{w}) \Phi_1(y, \bar{y}; w, \bar{w})) = -2(\bar{x}-\bar{y}) \delta^2(x-y)-(\bar{x}-\bar{y})^2 \delta \partial \bar{x} \delta^2(x-y) = 0. \quad (3.37) \]

Because of the above results (3.33), (3.36) and also the relations \( \bar{G}^+ = -\sqrt{k}[\bar{Q}_s, \bar{T}], \bar{T} = \sqrt{k}\{\bar{Q}_s, \bar{G}^-\} \), we can also conclude that

\[ \partial_x \bar{G}^+(\bar{x}) A(y) \sim 0, \quad \partial_x \bar{T}(\bar{x}) A(y) \sim 0, \quad (\forall A(y) \in \{G^\pm(y), T(y), J(y)\}), \quad (3.38) \]

which are no other than the claims we should confirm.

4 Conclusions

In this paper we explored the topologically twisted fermionic string theory on the general background \( AdS_3 \times \mathcal{N} \) which is compatible with \( N = 2 \) SUSY on world-sheet. We summarized the structure of the world-sheet topological theory, and showed that the dual space-time (boundary) theory should become also a topological conformal field theory. Our main result is the construction of space-time conformal algebra from the world-sheet picture. Firstly we considered the near boundary approximation, which is described by the Wakimoto free field representation, and secondly we took the more rigid formulation which is still valid far from the boundary according to that given in [11]. The space-time algebra we derived became a twisted \( N = 2 \) superconformal algebra with \( \hat{c} = 2k \oint \gamma^{-1} d\gamma \). Although this \( \hat{c} \) is equal to that of the twisted version of the space-time \( N = 2 \) algebra given in [20], these two are not the same. The generators of the latter are not BRST-invariant in the topological string theory,
and so cannot be regarded as the symmetry of the space-time topological theory which is dual to the world-sheet topological theory. Nevertheless it might be interesting to study further the relation between them. For example, one of the interesting questions is whether or not there exists a good modification, such as a similarity transformation, from the former to the latter.

As we emphasized, the topological string has only the world-sheet fermions with integral spins, and hence we cannot have spin fields. These field contents on world-sheet are analogous to that given in [13], which is based on the free field realizations of affine Lie superalgebra and is deeply connected with the recent "hybrid formalism" [34] for the $AdS_3 \times S^3$ superstring (in the special case with $B_{NS} \neq 0$, $B_{RR} = 0$). Especially, it may be worthwhile to point out that the complex scalar $\phi$ in the topological gravity sector (2.20) has the same world-sheet property as that of the "ghost field" in [34]. It might be interesting to discuss in detail the relationship between the topologically twisted theory and the framework of hybrid formalism.

Analysis on physical spectrum in our topological string theory is significant. As several authors pointed out [4, 22, 23] (see also [5, 6]), there are still several open problems and pathologies in (fermionic) string theory on $AdS_3$, in particular with respect to the analyses on physical spectra. Fortunately, in the topologically twisted theory it seems that the spectrum becomes rather simple and we need not face such pathologies. This is because only the $1/2$ BPS states can be physical in the topological theory. However, there may be a few subtleties in this spectrum, for example, about the field identification rules of chiral primary fields. We will study the spectra in detail for some concrete examples in future work [33].

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A Topological Conformal Algebra

We here present our convention of the topological conformal algebra \( \{L_n, G_n^\pm, J_n\} \) (or \( T(x) = \sum_n \frac{L_n}{x^{n+2}}, \ G^+(x) = \sum_n \frac{G_n^+}{x^{n+1}}, \ G^-(x) = \sum_n \frac{G_n^-}{x^{n+1}}, \ J(x) = \sum_n \frac{J_n}{x^{n+1}} \) ) which are defined from the \( N = 2 \) superconformal algebra (in Ramond sector) by twisting with respect to the \( U(1)_R \)-current [27];

\[
L_n \overset{\text{def}}{=} L_{n}^{N=2} - \frac{1}{2}(n + 1)J_n, \quad (A.1)
\]

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n}, \\
[J_m, J_n] &= \hat{c}m\delta_{m+n,0}, \\
\{G^+_m, G^-_n\} &= L_{m+n} + mJ_{m+n} + \frac{\hat{c}}{2}m(m-1)\delta_{m+n,0}, \\
\{G^+_m, G^+_n\} &= 0, \\
[L_m, G^+_n] &= \{\frac{1}{2}(1 \pm 1)m - n\}G^+_m, \\
[L_m, J_n] &= -nJ_{m+n} - \frac{\hat{c}}{2}m(m+1)\delta_{m+n,0}, \\
[J_m, G^+_n] &= \pm G^+_n, \quad (A.2)
\end{align*}
\]

or equivalently,

\[
\begin{align*}
T(x)T(y) &\sim \frac{2}{(x-y)^2}T(y) + \frac{1}{x-y}\partial_x T(y) \\
J(x)J(y) &\sim \frac{\hat{c}}{(x-y)^2} \\
G^+(x)G^-(y) &\sim \frac{\hat{c}}{(x-y)^3} + \frac{1}{(x-y)^2}T(y) + \frac{1}{x-y}J(y) \\
G^+(x)G^+(y) &\sim 0 \\
T(x)G^+(y) &\sim \frac{\frac{3}{4} \pm \frac{1}{2}}{(x-y)^2}G^+(y) + \frac{1}{x-y}\partial_y G^+(y) \\
T(x)J(y) &\sim -\frac{\hat{c}}{(x-y)^3} + \frac{1}{(x-y)^2}J(y) + \frac{1}{x-y}\partial_y J(y) \\
J(x)G^+(y) &\sim \pm \frac{1}{x-y}G^+(y). \quad (A.3)
\end{align*}
\]

This algebra is completely characterized by only one parameter \( \hat{c} (\equiv \frac{c_{N=2}}{3}) \).
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