Similarity and self-similarity in random walk with fixed, random and shrinking steps

Tushar Mitra\textsuperscript{1}, Tomal Hossain\textsuperscript{1}, Santo Banerjee\textsuperscript{2} and Md. Kamrul Hassan\textsuperscript{1}

\textsuperscript{1} University of Dhaka, Department of Physics, Theoretical Physics Division, Dhaka 1000, Bangladesh
\textsuperscript{2} Department of Mathematical Sciences, Giuseppe Luigi Lagrange, Politecnico di Torino, Corso Duca degli Abruzzi 24, Torino, Italy

In this article, we first give a comprehensive description of random walk (RW) problem focusing on self-similarity, dynamic scaling and its connection to diffusion phenomena. One of the main goals of our work is to check how robust the RW problem is under various different choices of the step size. We show that RW with random step size or uniformly shrinking step size is exactly the same as for RW with fixed step size. Krapivsky and Redner in 2004 showed that RW with geometric shrinking step size, such that the size of the \textit{n}th step is given by \( S_n = \lambda^n \) with a fixed \( \lambda < 1 \) value, exhibits some interesting features which are different from the RW with fixed step size. Motivated by this, we investigate what if \( \lambda \) is not a fixed number rather it depends on the step number \( n \)? To this end, we first generate \( N \) random numbers for RW of \( t = N \) which are then arranged in a descending order so that the size of the \( n \)th step is \( \lambda_n^n \). We have shown, both numerically and analytically, that \( \lambda_n = (1 - n/N) \), the root mean square displacement increases as \( t^{1/4} \) which are different from all the known results on RW problems.

PACS numbers: 61.43.Hv, 64.60.Ht, 68.03.Fg, 82.70.Dd

I. INTRODUCTION

Finding order in the disorder has always been an attractive proposition for physicists. To that end, our, physicists have come up with various elegant ideas like the idea of similarity, self-similarity, scaling, scale-invariance, renormalization group, etc.\textsuperscript{1-3}. These ideas have always been proved to be extremely useful in gaining deep insights into large and complex systems. Many of the physical systems are not static rather they evolve either probabilistically or deterministically with time. Often we have to extrapolate properties of such systems for the infinitely long-time limit by taking data from a few snapshots at short or intermediate time-limit. Sometimes, we also need to know results of experiment or simulation for infinitely large systems. However, in reality we can neither do experiment nor simulation in the computer in such large systems. We can still extrapolate results for infinite systems from the results of a set of data obtained for finite size systems which is known as finite-size scaling\textsuperscript{4-6}. The idea of extrapolation is not new. Even Galileo Galilei did that in his famous thought experiment while thinking of the now known Newton’s first law. Think of a block being pushed along a surface of a table and observe the distance travelled. Now polish the surface of the block and the table and then apply the same force again. The object then will travel further than before. If we continue to polish the surface of the block and that of the table more and more the object will travel further and further. In the end, it can be concluded that if the friction were totally absent the block would continue to move forever and ever preserving the same speed. This is called extrapolation. The idea of similarity, self-similarity, scaling and renormalization group essentially helps us to gain the ability or the power to extrapolate.

In fact, the idea of similarity and self-similarity is the key to understand many natural and man-made phenomena. In a way, the word self-similarity needs no explanation. Perhaps the best way to perceive the meaning of self-similarity is to consider an example. To that end there can be no better example than the vegetable cauliflower that we all know. The cauliflower head contains branches or parts, which when removed and compared with the whole, after blowing it up to a suitable size, are found to be apparently the same. These isolated branches can again be decomposed into smaller parts, which again look very similar to the whole as well as of the branches. Such self-similarity can easily be carried through for about three to four stages by hand. Maybe we can go up to a few more steps if we use a microscope. After that the structures are too small to go for any further dissection. However, from the mathematical point of view, the property of self-similarity may be continued through infinite stages. Self-similarity can also be found in branching patterns of snowflakes and in aggregating colloidal particles which are examples of statistically self-similar fractals\textsuperscript{2}. Our bodies, like our kidneys, lungs, and circulatory systems have a form of self similarity. In fact, it is so widespread in nature that they are extremely easy to find. In fact nature chooses to have self-similar objects through evolution in time. Nature love simplicity. Simple rules when applied over and over again can emerge as an object which may appear mighty complex. Yet, they are effectively simple and efficient for the purpose they are grown because of their self-similar nature.

Random walk problem perhaps is the best known example of self-similarity. It has been found that the dynamics of the random walk problem is governed by diffusion equation suggesting that it is an important class of stochastic process. It has far reaching implications as it provides the connection between the theory of random walk and Brownian motion. Owing to this connection we can simulate the diffusion phenomena in the computer.
To this end, the idea of random walk has been used in physics, computer science, ecology, economics and other seemingly disparate fields. Typically, a random walk is a sequence of successive random steps. On the other hand, the random motion of a heavy particle suspended in a bath of light particles is known as Brownian motion. It can be described by Langevin dynamics, which replace the collisions with the light particles by an average friction force proportional to the velocity and a randomly fluctuating force with zero mean and infinitely short correlation time. Random walk was first described in literature when the influential journal Nature published a discussion between Pearson and Rayleigh in 1905. It is this discussion that attracted physicists like Einstein and Smoluchowski to the subject who later made invaluable contribution to it. Random walk with random steps, specially cases in which the length of the nth step changes systematically with n, has been extensively studied for due reason of course. Firstly, the random walk, where step size shrinks following geometric series, gives rise to a variety of beautiful and unanticipated features which has pedagogical importance. Secondly, there also are a variety of situations where random walks with variable step size are relevant.

In this article, we give a comprehensive account of the theory of self-similarity in the light of Buckingham II theorem. In particular, we demonstrate that dynamic scaling, which has always been regarded as an ansatz and used as a litmus test for self-similarity, is deeply rooted to the Buckingham II-theorem. It is well-known that the random walk problem with fixed step size, which is governed by the famous diffusion equation, exhibits self-similarity. Our primary goal is to verify it by extensive Monte Carlo simulation using the idea of data-collapse. We also show that random walk with random step size follows exactly the same solution which proves how robust the random walk is vis-a-vis diffusion phenomena. We then study the random walk problem for time $t = N$ with shrinking steps such that the size of the nth step is $R_n$ where $R_n$ is defined as follows. We first draw $N$ number of random number within $[0, 1]$ which we then sort in a descending order such that $R_n$ is the nth value. Krapivsky and Redner have extensively studied them for a wide range of fixed $R$ including the case of golden number which gives rise to non-trivial results. We find completely new results for random walk with shrinking steps as we find that the probability distribution function still obeys the dynamic scaling but with different exponent. In particular, we find that the root mean square displacement increases with time as $t^{1/4}$ instead of $t^{1/2}$ for random or fixed step size.

The paper is organized as follows: in Sec. II, we discuss the idea of self-similarity in the light of Buckingham II theorem and cite an example to explain it. In Sec. III, a connection between the dynamic scaling and the Buckingham II theorem is shown. Sec. IV contains a theoretical approach to show that the random walk problem is actually governed by diffusion equation. In Sec. V, we present a solution of the diffusion equation following the prescription of Buckingham II theorem. Numerical results are presented to verify the analytical solution for random walk with various different step size in Sec VI and random walk with shrinking step size in VII. The results are discussed and conclusions drawn in VIII.

II. SELF-SIMILARITY AND BUCKINGHAM II THEOREM

In physics, we often investigate physical problems or phenomena where it is not enough to rely on our naked eyes to judge whether the system possesses self-similarity or not. In fact, it might be not that straightforward either. Besides, scientists in general and physicists in particular always look for instrumental or mathematical tools to test similarity and self-similarity. Often researchers investigate a given phenomena with the aim of finding fundamental rules and laws behind it. Such laws are nothing but relations between a governed quantity, say $a$ and a set of governing parameters, say they are $a_1, a_2, ..., a_n$ upon which $a$ depends. Such relations can always be represented in the form

$$a = f(a_1, a_2, ..., a_n),$$

where the quantities $a_1, a_2, ..., a_n$ are called the governing parameters. It is then often possible to classify the governing parameters $a_1, ..., a_n$ into two groups using the definition of dependent and independent variables. To that end, consider that the parameters $a_{k+1}, ..., a_n$ have independent dimensions in the sense that none of these parameters can be expressed in terms of the power product of the dimensions of the remaining parameters $a_1, a_2, ..., a_k$ which we call dependent variables.

It implies we can define a dimensionless variable for each dependent variable

$$\Pi_i = \frac{a_k}{a_{k+1}^{\frac{1}{\alpha_1}} a_{k+2}^{\frac{1}{\alpha_2}} ... a_n^{\frac{1}{\alpha_n}}},$$

where $i = 1, 2, ..., k$. Similarly, we can also define a dimensionless quantity for the governed parameter $a$ as well

$$\Pi = \frac{a}{a_k^{\frac{1}{\beta_1}} a_{k+1}^{\frac{1}{\beta_2}} ... a_n^{\frac{1}{\beta_n}}}.$$

It essentially can be written as

$$\Pi = F(a_{k+1}, a_{k+2}, ..., a_n, \Pi_1, ..., \Pi_k),$$

where $F$ is defined as another function which depends on $n - k$ number of parameters instead of $n$ number of parameters. Note that $\Pi$ cannot depend on dimensional variables like $a_{k+1}, a_{k+2}, ..., a_n$ since according to the definition of dimensionless quantity its numerical value must remain the same even if we change any or all of the $a_{k+1}, a_{k+2}, ..., a_n$ variables. It means

$$a = a_k^{\frac{1}{\alpha_1}} ... a_n^{\frac{1}{\alpha_n}} \phi(\Pi_1, \Pi_2, ..., \Pi_k).$$
that if we plot $S/c^2$ vs $b/c$ instead of $S$ vs $b$ then all the three curves of (a) collapse superbly on to a single master curve. It implies that for a given numerical value of the ratio $b/c$ the corresponding $S/c^2$ is the same regardless of the size of the sides of the triangle.

This is known as the Buckingham II-theorem \[1\ 2\]. To find out how it helps to understand the idea of similarity and self-similarity we have to first understand the much known geometric similarity and then we can extend it to some physical systems.

We all learn the idea of geometric similarity in high school. Let us re-visit the same using the formalism of the Buckingham II-theorem. Consider that we have a right triangle of sides $a$ (adjacent), $b$ (opposite) and $c$ (hypotenuse). Say, that we want to measure its area $S$. That is, $S$ is the governed parameter and the sides $a$, $b$, $c$ are the governing parameters. We keep the side $a$ fixed and measure the area $S$ as we vary the side $b$. Obviously, the size of the hypotenuse $c$ will change as we vary the side $b$ and hence in analogy with Eq. (1) we can write

$$S = S(b, c).$$

A trivially simple dimensional analysis immediately reveals that one of the two governing parameters can have independent dimension and let us assume that it is the variable $c$ so that $b$ as well as $S$ can be expressed in terms of $c$ alone. We can thus define the following dimensionless variables. One, for the dependent variable $b$

$$\xi = b/c,$$

so that the length $b$ is measured in terms of $c$. The other dimensionless variable is for the governed variable $S$

$$\Pi = S/c^2,$$

so that all the areas are measured in unit of $c^2$. Now, the numerical value of $\Pi$ remains the same regardless of the choice of the unit of measurement of length since it is a dimensionless quantity. However, it still depends on $\xi$ and hence in analogy with Eq. (1) we can re-write Eq. (8) as

$$S = c^2 \phi(b/c).$$

The question is now: What does it has to do with the idea of similarity and self-similarity? Consider that we have two more triangles differing in size of their sides but share the same acute angle $\theta$. Recall that we keep the base or adjacent $a$ fixed and measure area $S$ for each of the three triangles as a function of their sides $b$. If we now plot $S$ as a function of their respective $b$ then we shall have a set of distinct straight lines, see Fig. (1a), one for each different $a$ with slope equal to $a/2$ since we know $S = \frac{a}{2}b$. Let us now express $b$ in unit of $c$ and $S$ in unit of $c^2$. It means that if we now plot $S/c^2$ versus $b/c$ we find that all the distinct plots of $S$ versus $b$ curve collapse into one universal curve as shown in Fig. (1b). That is, the numerical value of $S/c^2$ for a fixed value of $b/c$ or acute angle $\theta$, will be the same no matter how big or how small the triangles are. What is the significance of such data-collapse? The numerical value of $S/c^2$ of all right triangles having the same $b/c$ will coincide. It means all the right triangles which share the same $b/c$ value are similar. Note that $S/c^2$ and $b/c$ are dimensionless quantity. We can extend this idea of geometric similarity to physical phenomena too. We can say two or more systems or phenomena are similar if they differ in the numerical values of their dimensional quantities however the numerical values of the corresponding dimensionless quantities are the same.

III. DYNAMIC SCALING AND BUCKINGHAM II-THEOREM

Many phenomena which physicists often investigate are not static rather evolve probabilistically with time. The resulting systems can often be described by kinetic or rate equation approach. Mathematicians and physicists often look for scaling or self-similar solution to their respective equations which is actually the solution in the long-time limit. In this limit, the solution usually assumes a simple and universal form. In such systems one often is interested to see if certain observable quantity, say $f(x,t)$, exhibits self-similarity or not. To understand what it really means we can apply the Buckingham II-theorem like we have done for right triangle. Assume that one of the two variables, say time $t$ for convenience, is the independent variable so that we can express both $x$ and $f$ in terms of $t$ alone. It means that we have two dimensionless variables $\xi = x/t^z$ and $\phi = f/t^\tau$ where exponents $z$ and $\tau$ are fixed by the dimensional relations $[t^\tau] = [f]$ and $[t^z] = [x]$. Note that the numerical value of $f/t^\tau$ for a given value of $x/t^z$ will be independent of the choice of the unit of measurement of time $t$ but $\phi$ may depend on $x/t^z$ and hence according to Buckingham II-theorem we can write the following

$$f(x,t) \sim t^\tau \phi(x/t^z),$$

where $\phi(\xi)$ is known as the scaling function \[22\ 27\]. Self-similarity means that snapshots at different times
are similar. However, since the same system at different times is similar we regard it as the temporal self-similarity [1].

IV. RANDOM WALK AND DIFFUSION EQUATION

Diffusion is perhaps one of the most ideal examples of natural phenomena that evolves probabilistically with time. Typically diffusion phenomena are associated with the movement of atoms or molecules within a gas, a liquid, or a solid over a more or less long distance. One of the four papers that Einstein wrote in 1905, that raised him to a height that no one has ever reached in the entire history of science, was on the Brownian motion, which embodies the diffusion process. Since then Brownian motion motion vis-a-vis the diffusion process has always been one of the active field of research in almost every branch of science and in physics in particular. A diffusing particle is subjected to a variety of collisions that we can consider random, such that each event that occurs between time steps is statistically independent. Upon multiplying on both sides of Eq. (15) by $x$ and $x^2$ and integrating over the entire range we get

$$x_{\text{rms}}(t) \sim t^{1/2} \quad \text{and} \quad x_{\text{mean}}(t) = 0,$$

respectively where

$$x_{\text{rms}}^2 = \int_{-\infty}^{\infty} x^2 P(x, t) dx,$$  \hspace{1cm} (17)

and

$$x_{\text{mean}}(t) = \int_{-\infty}^{\infty} x P(x, t) dx,$$  \hspace{1cm} (18)

assuming that all the walkers start their walk from $x = 0$.

V. SOLUTION TO DIFFUSION EQUATION

The diffusion equation for the probability density function $P(x, t)$ suggests that it is a function of two variables

$$P = P(x, t),$$  \hspace{1cm} (19)

since time can be re-scaled as $Dt$. In order to solve the diffusion equation let us first invoke the idea of simple dimensional analysis. Within the MLT class, their dimensions are

$$[x] = L \quad [D] = \frac{L^2}{T} \quad [t] = T \quad \text{and} \quad [P] = L^{-1}.$$  \hspace{1cm} (20)

The above dimensional relation implies that the re-scaled time can be chosen to have independent dimension and define the following dimensionless quantities

$$\xi = \frac{x}{\sqrt{Dt}} \quad \text{and} \quad \phi(\xi) = \frac{P(x, t)}{\langle P(x, t) \rangle}.$$  \hspace{1cm} (21)

where the exponent $\theta = -1/2$ is required by the normalization condition $\int_{-\infty}^{\infty} P(x, t) dx = 1$. Following the
Buckingham II-theorem we find that the probability density function \( P(x,t) \) assumes a simple universal scaling form

\[
P(x,t) \sim \frac{1}{\sqrt{4\pi Dt}} \phi(x/\sqrt{Dt}),
\]

where \( \phi(\xi) \) is the dimensionless scaling function. The structure of this scaling form is highly instructive as it greatly simplifies further analysis.

We now substitute Eq. (22) in Eq. (15) and find that the solution of the partial differential equation reduces to the solution of an ordinary differential equation for the function \( \phi(\xi) \) given by

\[
\left[ \frac{d^2}{d\xi^2} + \frac{\xi}{2} \frac{d}{d\xi} + \frac{1}{2} \right] \phi(\xi) = 0.
\]

Solving it subject to the condition \( \int_{-\infty}^{\infty} \phi(\xi) d\xi = 1 \) we find

\[
\phi(\xi) = A \exp[-\xi^2/4],
\]

where \( A \) is the integration constant fixed by the normalization condition. Substituting this into the normalization condition for \( \phi \) immediately gives \( A = 1/\sqrt{4\pi} \) and therefore

\[
P(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{x^2}{4Dt}\right].
\]

We can easily express it as

\[
P(x,t) \sim t^{-1/2} f(x/t^{1/2}),
\]

where the dynamic scaling function

\[
f(z) = \frac{1}{\sqrt{\pi D}} \exp\left[-\frac{z^2}{4D}\right],
\]

and hence Eq. (25) obeys dynamic scaling. Furthermore, note that the solution is symmetric about \( x = 0 \) as it satisfies the condition \( P(x,t) = P(-x,t) \). Using this solution Einstein deduced his famous prediction that the root mean square displacement of Brownian particles is proportional to the square root of time. Besides, the solution is scale-invariant in the sense that it can be brought to itself under the following similarity transformation

\[
P \rightarrow \lambda^{-b/2} P, \quad x \rightarrow \lambda^{b/2} x, \quad t \rightarrow \lambda^2 t,
\]

since it is a generalized homogeneous function.

VI. EXTENSIVE NUMERICAL SIMULATION

The question is: How can we verify the solution, given by Eq. (26), of the diffusion equation \textit{vis-a-vis} the random walk problem? Consider that we ask \( N \) number of walkers to walk starting from the same initial point, which we call origin, along the same line with fair coin in their hands. Each walker is asked to make \( n \) steps. The rules of the random walk are as follows. Before attempting to make a step each walker flips their coin. Respective walker then make a step to the right of unit step size \( \delta = 1 \) if the upper face of the coin appears head and to the left by the same step size if it is tail. Owing to the random nature of the random walk problem the final position of all the walkers will not be the same. To make \( n \) steps each walker has to flip their coin \( n \) times. Say that out of \( n \) outcome, \( n^+ \) of them flipped head and \( n^- \) of them flipped tail and hence the final position \( x_i \) of the \( i \)th walker is obtained by measuring \( x_i = n^+_i - n^-_i \). We then create a data by finding the fraction of the total \( N \) walkers \( P(x,n) \Delta x = m/N \) within the position \( x \) and \( x + \Delta x \) where \( m \) is the number of walkers found within this range. Effectively, \( P(x,n) \Delta x \) represents the probability that the number of walkers is within the position \( x \) and \( x + \Delta x \) at the end of \( n \) steps. If we assume each step is made in one unit time then the number of steps \( n \) is the time \( t \) and if we consider continuum limit then the solution becomes exactly the same as the solution of the diffusion equation.
In Fig. (2a) we first show the plots of \( P(x, t) \) as a function of position \( x \) for a set of different time \( t \). Note that as the walkers walk for a longer time \( t = n \), the probability of finding the walkers at a larger distance increases but this happens at the expense of lowering the peak value \( P_{\text{max}} \) at \( x = 0 \) since the total area under each curve must always be equal to one i.e. \( \int_{-\infty}^{\infty} P(x, t) dx = 1 \). To find out how the peak height \( P_{\text{max}} \) at \( x = 0 \) decreases with time we plot \( \log(P_{\text{max}}) \) versus \( \log(t) \) in Fig. (2b) and find a straight line with slope equal to \( -1/2 \) revealing \( P_{\text{max}} \sim t^{-1/2} \) and hence \( P_{\text{max}}t^{1/2} \) must be a dimensionless quantity. To prove this, we multiply the ordinate of the data of \( P(x, t) \) versus \( x \) of Fig. (2a) by \( t^{1/2} \) and re-plot the result in Fig. (2c). This is equivalent to plotting \( P(x, t)t^{1/2} \) versus \( x \) and find that all the distinct peaks of Fig. (2a) collapse superbly at \( x = 0 \) as shown in Fig. (2c). In this way we have brought all the plots for \( P(x, t) \) on equal footings as a function of \( x \). We now observe that the probability of finding the walker at larger distances increases. To find out how it increases with time we now measure the full width at half maximum \( W \) of \( P(x, t)t^{1/2} \) versus \( x \) plots for different time \( t \) from Fig. (2c). Plots of \( \log(W) \) versus \( \log(t) \) shown in the inset of Fig. (2c) results in a straight line with slope equal to \( 1/2 \) revealing that \( W \sim t^{1/2} \). We can thus conclude that \( W \) is proportional to standard deviation \( \sigma \). In fact, we can re-write the solution in Eq. (25) as

\[
P(x, t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{x^2}{2\sigma^2}\right], \tag{29}
\]

where \( \sigma^2 = 2Dt \) and standard deviation \( \sigma \) is actually the root-mean square displacement. Thus \( W \sim t^{1/2} \) is consistent with our analytical solution given by Eq. (2).

Finally, we plot \( P(x, t)\sqrt{t} \) versus \( x/\sqrt{t} \) and find that all the distinct plots of Fig. (2c) collapse into one universal curve as shown in Fig. (2c). Note that \( W \) bear the same dimension as that of \( x \) and hence \( x/\sqrt{t} \) is a dimensionless quantity.

The distribution function \( P(x, t) \) for different times is distinct. However, if \( P(x, t) \) and \( x \) are measured using \( t^{1/2} \) and \( t^{-1/2} \) as yard-stick and then the plotting of the resulting data is equivalent to plotting \( P(x, t)\sqrt{t} \) versus \( x/\sqrt{t} \). In these self-similar scaling all the data including the data for infinitely long time walk must collapse into one universal curve which is essentially the solution for the scaling function

\[
\phi(\xi) = \frac{1}{\sqrt{4\pi}} e^{-\xi^2/4}, \tag{30}
\]

where \( \xi^2 = x^2/Dt \). Such data collapse means that random walk vis-a-vis the Brownian motions for different times are similar in the same sense two triangles are similar. Using this idea we can extrapolate data for any latter time since the plots from data for all time including the infinitely long time is contained in this universal curve. Random walk or Brownian motion therefore are self-similar in nature since the plots for longer times are similar to the walkers for shorter times.

Now the question is: What if the steps of the walkers were not of the same size but random? To find an answer to such question we have performed extensive numerical simulation for a few different situations like we have done for fixed step size. First, we consider the case where at each step each walker picks a random number \( R \) from the interval \([0, 1]\) and the numerical value of \( R \) is then chosen as the step size at that instant. Each walker then picks another random number to decide the direction of the step in the same way as for its classical counterpart. We have found that the solution for the probability distribution \( P(x, t) \) is exactly the same as it is for fixed step size (see Fig. (3a)).

To prove this we plot \( P(x, t)\sqrt{t} \) versus \( x/\sqrt{t} \) in Fig. (3b) and find that all the distinct plots collapse into one universal curve. It implies that the root mean square displacement grows following Eq. (10). It has far reaching consequences as it suggests that the collision time vis-a-vis the distance travelled by the Brownian particles or random walkers may not be fixed, yet the solution for the distribution function \( P(x, t) \) remains the same. In fact, random walk of fixed step size is almost impossible to find in nature. Finding random walk with random step size the same as that of the random walk with fixed step size shows how robust is the random walk problem.
VII. RANDOM WALK WITH SHRINKING STEP SIZE

We first consider the simplest case for random walk of $N$ shrinking steps. Such problems are interesting because of their connections to the dynamical systems [30–32]. Here, we first draw $N$ number of random number $R_n$ uniformly from $[0,1]$ and arrange them in descending order so that the first number is $R_1$, which is greater than the second number $R_2$ and in general $R_i > R_{i+1}$ etc. To create the random walk we choose the step size for the $n$th step is $R_n$ and then we choose the direction randomly. In Fig. 4(a) we plot distribution function $P(x,t)$ as a function of $x$ for different walking time $t=N$. Like walk with fixed step size we find that $P_{\text{max}} \sim t^{-1/2}$ and the full width at half maxima $W \sim t^{1/2}$. To prove this we plot $P(x,t) t^{1/2}$ versus $x/t^{1/2}$ in Fig. 4(b) and indeed we find all the distinct curves of Fig. 4(b) collapse superbly. Note that we have drawn $N$ random number $R_n$ from the interval $[0,1]$ with uniform probability. Thus in the large $N$ limit, the length of the first step is $R_1 = 1 - 1/N$, the length of the second step is $R_2 = 1 - 2/N$ and in general the length of the $n$th step is $R_n = (1 - n/N)$. Since the direction of steps are taken independently i.e. uncorrelated, the mean-square displacement after the $N$th step, is given by:

$$\langle x^2 \rangle = R_1^2 + R_2^2 + \ldots + R_N^2 = \sum_{n=1}^{N} (1 - n/N)^2.$$ 

(31)

In the limit $N \rightarrow \infty$ we can treat the above sum as an integral which we can easily integrate and find

$$x_{\text{rms}} = \sqrt{\langle x^2 \rangle} = 0.57735 \times N^{1/2} \sim t^{1/2}.$$ 

(32)

We thus see that the dynamics of the random walk with shrinking steps such that the size of the $n$th step is $R_n$ behaves exactly the same way as for fixed and random step size. We have numerically measured the mean square displacement $\langle x^2 \rangle$ using Eq. (31) for different $N$ and plotted $\sqrt{\langle x^2 \rangle} = x_{\text{rms}}$ versus $N^{1/2}$. The resulting graph, as shown in Fig. 4(a), is a straight line with slope exactly at 0.57737 supporting our result given by Eq. (32). On the other, we find that the slope of the plots of full width at half maxima $W$ versus $N^{1/2} = \sqrt{t}$ as shown in Fig. 4(b) equals to 1.3576. It satisfies the known relation $W = \frac{2}{\lambda} \log 2 \sigma$.

In 2003 Krapivsky and Redner studied a few interesting variants of the random walk problem [21]. They considered the case in which the length of the $n$th step

$$S_n = \lambda^n,$$ 

(33)

where $\lambda < 1$ and it is assumed to be a fixed value across the whole journey. They found that the support of the distribution function $P(x,t)$ is a Cantor set for $\lambda < 1/2$. However, for $1/2 < \lambda < 1$ there is countable infinite set of $\lambda$ values for which $P(x,t)$ is singular. Of all the $\lambda$ values, one of the strikingly interesting results have been found if one chooses for $\lambda$ equal to inverse golden number $(\sqrt{5} - 1)/2$. We shall now study the case where the $n$th step size $S_n$ is given by Eq. (33) except now we choose $\lambda$ not a fixed number rather $\lambda = R_n$ where $R_n$s are random.
numbers picked from the interval \([0, 1]\) and arranged in a descending order such that \(R_1 > R_2 > R_3 > \ldots > R_N\). This time we find interesting results which are significantly different from all known cases including the work of Krapivsky and Redner [21].

In Fig. (5a), we show plots of \(P(x, t)\) versus \(x\) for different times. It is clear from the figures that the shape of the probability distribution function curve is different from all the cases where \(\lambda\) assumes a fixed value [21]. It is actually more similar to the classic random walk problem, albeit the exponents are different, than the works of Krapivsky and Redner. Note that in the case of constant \(\lambda\), the value of \(\lambda^n\) decreases with increasing \(n\) if \(\lambda < 1\). In our case, we have first generated \(n\) number of random numbers from the interval \([0, 1]\) and arranged them in a descending order so that the \(n\)th smallest number is \(R_n\). We then choose the size of the \(n\)th step of the walker as \(R_n\) and hence the larger the \(n\) value the smaller the step size. Interestingly, we find that the probability distribution function \(P(x, t)\) looks similar to that of the classical random walk problem. Nevertheless, we then measure the peak height \(P_{\text{max}}\) as a function time \(t\). We then plot \(\log(P_{\text{max}})\) versus \(\log(t)\) in Fig. (5b) and find that it yields a straight line with slope \(-0.25\) instead of \(0.5\) for its classical counterpart. In Fig. (5c) we now plot \(P(x, t)\) versus \(x\) and find that all the peaks of Fig. (5a) collapse at \(x = 0\). We now measure the full width at the half maxima \(W\) of Fig. (5a). Plotting \(\log(W)\) versus \(\log(t)\) in the inset of Fig. (5a) once again gives straight line but with slope \(0.25\). It implies that root-mean square displacement \(x_{\text{rms}}\) increases like \(t^{1/4}\). We now plot \(t^{0.25}P(x, t)\) versus \(xt^{-0.25}\) in Fig. (5c) and find that all the distinct plots of Fig. (5b) collapse into one universal curve. Thus, the random walk with shrinking step size, so that the step size of the \(n\)th step size equals to \(R_n\), exhibits the same self-similar solution as it is for the fixed step size except the exponents are \(0.25\) instead of \(0.5\).

To understand why RW with \(S_n = R_n\) is different from that \(S_n = R_0\) we find the root mean square displacement and see how it behaves with time \(t\). The length of the \(n\)th step is now

\[ R_n = (1 - n/N)^n, \tag{34} \]

which in the limit \(N \to \infty\) can be re-written as

\[ R_n = e^{n \log(1-n/N)} \approx e^{-n^2/N}. \tag{35} \]

We thus see that the step size decreases exponentially with \(n\) where in the case of \(S_n = R_0\) it decreases linearly with \(n\). We also observe that when \(n < \sqrt{N}\) the step size is of the order of one but when \(n > \sqrt{N}\) the step size quickly becomes vanishingly small. Like before we can again calculate the root mean square displacement

\[ \langle x^2 \rangle = R_1^2 + R_2^2 + \ldots + R_N^2, \tag{36} \]

where \(R_1 > R_2 > \ldots > R_N\). In the large \(N\) limit we can write it as

\[ \langle x^2 \rangle = \sum_{n=1}^{N} (1 - n/N)^{2n}, \tag{37} \]

\[ = \sum_{n=1}^{N} (1 - n/N)^{2n/N} \approx \int_1^{N} e^{-2n^2/N} dn, \]

and in the limit \(N \to \infty\) we can write it as

\[ \langle x^2 \rangle = \frac{N}{2} \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2^{3/2}} N^{1/2}. \tag{38} \]

Thus, the width of the distribution \(P(x, N)\) grows as

\[ x_{\text{rms}} = \sqrt{\langle x^2 \rangle} = 0.79162 \times N^{1/4} = 0.79162 \times t^{1/4}, \tag{39} \]

which clearly support our numerical findings. Plotting \(x_{\text{rms}}\) versus \(N^{1/4}\) we in fact find that slope is almost equal to 0.79162 (see Fig. (6a)). On the other hand, plotting of \(W\) versus \(N^{1/4}\) in Fig. (6b) gives the slope equal to 1.86 which once again is consistent with the fact that \(W = (2\sqrt{2}\text{log}2)\sigma\).

**VIII. DISCUSSION**

We have given a comprehensive description of the most studied random walk problem focusing mainly on its self-similar properties and verifying the results numerically. We also highlighted its connection to diffusion processes. It is noteworthy to mention that diffusion is a ubiquitous phenomenon and knowing its connection to random walk helps us to simulate diffusion in the computer. At first, we have discussed the idea of similarity and self-similarity vis-a-vis the dynamic scaling and its deep connection to Buckingham II theorem where the notion of dimensionless quantity plays a significant role. Using the Markov chain identity it has been shown that the dynamics of the random walk problem is governed by the diffusion equation. We then used the idea of Buckingham II theorem.
to obtain solution of the diffusion equation as it provides deep insight into the problem. One of the primary goals of this work is to perform extensive numerical solution to verify the analytical solution. Besides, we show that the random walk problem is robust with respect to step size albeit up to some extent. On the other hand, Krapivsky and Redner studied random walks with geometrically shrinking steps, in which the size of the nth step is considered to be \( \lambda^n \) with \( \lambda < 1 \). In particular they choose \( \lambda = 2^{-m} \) where \( m = 1, 2, 3 \) etc. for which analytical solution is possible and it is non-trivial. Furthermore, they choose \( \lambda = (\sqrt{5} - 1)/2 \) and found highly non-trivial self-similar features.

We have chosen \( \lambda \) equal to a random number within an interval \([0, 1]\) instead of a fixed number. In contrast, we have also studied two variants of the random walk with shrinking step. First, we have chosen the \( n \)th step size equal to \( R_n \) such that \( R_1 > R_2 > R_3 > \ldots > R_N - 1 > R_N \) where \( R_n \) are random numbers drawn from the interval \([0, 1]\). In this case, we have found that the results are exactly the same as for classical fixed step size random walk. Second, we have chosen shrinking step size so that the \( n \)th step size equal to \( R_n^\alpha \) instead of \( R_n \). This is similar to the geometric random walk of Krapivsky and Redner except the fact that they choose a constant value for \( R_n \). Interestingly, results too are very different. We found that the overall features of random walk, such that the \( n \)th step size \( R_n^\alpha \), are the same in the sense that the distribution function \( P(x, t) \) are still Gaussian and they still obey dynamic scaling. However, the peak height \( P(x, t) \) decays like \( t^{-1/4} \) instead of \( t^{-1/2} \) and root mean square displacement increases like \( t^{1/4} \) instead of \( t^{1/2} \). It would be interesting to see what happens if we extend the present work in higher dimension which we intend to do in our future endeavor.

MKH would like to thank Professor Sidney Redner for critical reading of the manuscript and his valuable comments especially in helping to find analytical argument for random walk with algebraically shrinking steps.

[1] G. I. Barenblatt, Scaling, Self-similarity, and Intermediate Asymptotics (Cambridge University Press, 1996).
[2] S. Banerjee, M. K. Hassan, S. Mukherjee and A. Gowrisankar, Fractal Patterns in Nonlinear Dynamics and Applications (CRS press, Taylor & Francis group, New York, 2020).
[3] P. L. Krapivsky, S. Redner and E. Ben-Naim, A Kinetic View of Statistical Physics (Cambridge University Press, New York, 2010).
[4] M. K. Hassan and M. M. Rahman, Phys. Rev. E 92 040101(R) (2015); ibid 94 042109 (2016).
[5] M. K. Hassan, D. Alam, Z. I. Jitu and M. M. Rahman, Phys. Rev. E, 96 050101(R) (2017).
[6] M. M. H. Sabbir and M. K. Hassan, Phys. Rev. E 97 050102(R) (2018).
[7] K. Pearson, Nature 72 294 (1905).
[8] L. Rayleigh, Nature 72 318 (1905).
[9] A. Einstein, Ann. Phys. 19 371 (1906).
[10] M. Smoluchowski, Phys. Zeit. 17 557 (1916).
[11] B. Jessen and A. Wintner, Trans. Am. Math. Soc. 38 48 (1935).
[12] B. Kershner and A. Wintner, Am. J. Math. 57 541 (1935).
[13] A. Wintner, Am. J. Math. 57 827 (1935).
[14] P. Erdös, Am. J. Math. 61 974 (1939); 62 180 (1940).
[15] A. M. Garsia, Trans. Am. Math. Soc. 102 409 (1962); Pac. J. Math. 13 1159 (1963).
[16] E. Barlai and R. Silbey, Chem. Phys. Lett. 310 287 (1999); Phys. Chem. B, 104 342 (2000).
[17] G. H. Weiss and J. E Kiefer, J. Phys. A 16 489 (1983).
[18] M. K. Hassan, M. Z. Hassan and N. I. Pavel, J. Phys. A: Math. Theor. 44 175101 (2011).
[19] M. K. Hassan, L. Islam, S. A. Haque, Physica A 469 23 (2017).
[20] M. K. Hassan, L. Islam, S. A. Haque, Physica A 469 23 (2017).
[21] P. L. Krapivsky and S. Redner, Am. J. Phys. 72 591 (2004).
[22] P. G. J. van Dongen and M. H. Ernst, Phys. Rev. Lett. 54 1396 (1985).
[23] F. Family, T. Vicsek, J. Phys. A: Math and Gen 18 75 (1985).
[24] T. Vicsek, F. Family, Phys. Rev. Lett. 52 1669 (1984).
[25] P. G. J. van Dongen, M. H. Ernst, Phys. Rev. Lett. 54 1396 (1985).
[26] S. M. K. Hassan, M Z. Hassan and N. I Pavel, J. Phys. A: Math. Gen. 44 175101 (2011).
[27] D. Sarker, L. Islam and M. K. Hassan, Chaos, Solitons & Fractals 132 109591 (2020).
[28] L. E. Reichl, A Modern Course in Statistical Physics (Wiley-Interscience Publication, USA, 1998).
[29] G.H. Weiss, Aspects and Applications of the Random Walk, (North-Holland, Amsterdam, 1994).
[30] J. C. Alexander and J. A. Yorke, Ergod. Theory Dyn. Syst. 4 1 (1984)
[31] J. C. Alexander and D. Zagier, J. Lond. Math. Soc. 44 121 (1991).
[32] A. C. de la Torre, A. Maltz, H. O. Ma’rtin, P. Catuogno, and I. García á-Mata, Phys. Rev. E 62 7748 (2000).