Exponential Mixing for Retarded Stochastic Differential Equations

Jianhai Bao,∗ George Yin,† Le Yi Wang,‡ Chenggui Yuan§

May 7, 2014

Abstract

In this paper, we discuss exponential mixing property for Markovian semigroups generated by segment processes associated with several class of retarded Stochastic Differential Equations (SDEs) which cover SDEs with constant/variable/distributed time-lags. In particular, we investigate the exponential mixing property for (a) non-autonomous retarded SDEs by the Arzelà–Ascoli tightness characterization of the space $C$ equipped with the uniform topology (b) neutral SDEs with continuous sample paths by a generalized Razumikhin-type argument and a stability-in-distribution approach and (c) jump-diffusion retarded SDEs by the Kurtz criterion of tightness for the space $D$ endowed with the Skorohod topology.

Keywords: retarded stochastic differential equation, invariant measure, exponential mixing, uniform metric, Skorohod metric

AMS Subject Classification: 60H15, 60J25, 60H30, 39B82

1 Introduction

Ergodic property of stochastic dynamical systems, which are independent of the past history, has attracted lots of attentions. For stochastic dynamical systems driven by continuous noise processes (e.g. Wiener process and fractional Brownian motion), we refer to [11, 19, 22, 25, 28, 31] and references cited therein. And there has been intense interest in studying dynamical systems subject to discontinuous Markov processes due to their importance both in theory and in applications. There is also extensive literature on ergodicity of SDEs driven by Lévy processes (e.g. Poisson process, $\alpha$-stable process, cylindrical $\alpha$-stable process and subordinate Brownian motion), see e.g. [6, 15, 23, 29], to name a few.
Many physical phenomena should be and in fact have already been successfully modeled by stochastic dynamical systems whose evolution in time is governed by random forces as well as intrinsic dependence of the state on a finite part of its past history. Such models may be identified as retarded (functional) SDEs (see e.g. the monograph \[21\] for more details). Relative to SDEs without memory, the long-term behavior of retarded SDEs is not yet complete. There is a few of literature on investigation in existence of stationary solutions, see e.g. Itô and Nisio \[9\] for retarded SDEs with infinite memory by the Prohorov-Skorohod theory of the totally bounded sets of stochastic processes, Bakhtin and Mattingly \[5\] for retarded SDEs with additive noise and infinite memory by a Lyapunov function approach, Liu \[17\] and Reiβ et al. \[26\] for infinite-dimensional retarded Langevin equations and finite-dimensional semi-linear retarded SDEs driven by jump processes with the diffusion term being independent of the past history, respectively, by the variation of constants formula. For existence of an invariant measure, Es-Sarhir et al. \[7\] and Kinnally and Williams \[14\] considered retarded SDEs with super-linear drift term and positivity constraints, respectively; Bo and Yuan \[4\] investigated reflected SDEs with jumps and point delays. With regard to uniqueness of invariant measures, by an asymptotic coupling method, Hairer et al. \[12\] addressed the open problem of uniqueness of invariant measure for non-degenerate retarded SDEs under some appropriate assumptions which need not guarantee existence of an invariant measure, and Scheutzow \[27\] discussed a very simple linear retarded SDE without the drift term.

In this paper, we shall investigate the exponential mixing property for several class of retarded SDEs which cover SDEs with constant/variable/distributed delays. The content of this paper is organized as follows. Section 2 discusses the exponential mixing property for a class of non-autonomous retarded SDEs by the Arzelà–Ascoli tightness characterization of the space \(C\) equipped with the uniform topology. In Section 3 we proceed to neutral SDEs with continuous sample paths by a generalized Razumikhin-type argument and a stability-in-distribution approach. The last section is devoted to the exponential ergodicity for jump-diffusion retarded SDEs by the Kurtz criterion of tightness for the space \(D\) endowed with the Skorohod topology.

2 Exponential Mixing for Retarded SDEs

We start with some notation. For each integer \(n \geq 1\), let \((\mathbb{R}^n; \langle \cdot, \cdot \rangle, |\cdot|)\) be the \(n\)-dimensional Euclidean space and \(\mathbb{R}^n \otimes \mathbb{R}^m\) denote the totality of all \(n \times m\) matrices endowed with the Frobenius norm \(\|\cdot\|\). For a fixed constant \(\tau > 0\), \(C := C([-\tau, 0]; \mathbb{R}^n)\) stands for the family of all continuous mappings \(\zeta : [-\tau, 0] \mapsto \mathbb{R}^n\) equipped with the uniform norm \(\|\zeta\|_\infty := \sup_{-\tau < \theta < 0} |\zeta(\theta)|\). For any continuous function \(f : [-\tau, \infty) \mapsto \mathbb{R}^n\) and \(t \geq 0\), let \(f_t \in C\) be such that \(f_t(\theta) = f(t + \theta)\) for each \(\theta \in [-\tau, 0]\). As usual, \(\{f_t\}_{t \geq 0}\) is called the segment process of \(\{f(t)\}_{t \geq -\tau}\). Let \(W(t)\) be an \(m\)-dimensional Wiener process defined on a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). The notation \(\mathcal{P}(C)\) denotes the collection of all probability measures on \((C, \mathcal{B}(C))\), \(\mathcal{B}_b(C)\) means the set of all bounded measurable functions \(F : C \to \mathbb{R}\) endowed with the uniform norm \(\|F\|_0 := \sup_{\zeta \in C} |F(\zeta)|\), and \(\mu(\cdot)\) stands for a probability measure on \([-\tau, 0]\). For any \(F \in \mathcal{B}_b(C)\) and \(\pi(\cdot) \in \mathcal{P}(C)\), let \(\pi(F) := \int_C F(\phi) \pi(d\phi)\). Throughout this paper, \(c > 0\) is a generic constant whose value may change from line to line but independent of the time parameters.
In this section, we consider a retarded SDE on \((\mathbb{R}^n, \langle \cdot, \cdot \rangle, | \cdot |)\) in the framework

\[
A1 \quad \text{(2.1)} \quad dX(t) = b(t, X_t)dt + \sigma(t, X_t)dW(t), \quad t > 0
\]

with the initial data \(X_0 = \xi \in \mathcal{C}\), where \(b : [0, \infty) \times \mathcal{C} \rightarrow \mathbb{R}^n\) and \(\sigma : [0, \infty) \times \mathcal{C} \rightarrow \mathbb{R}^n \otimes \mathbb{R}^m\) are measurable and locally Lipschitz with respect to the second variable. Throughout this section, we assume that the initial value \(\xi \in \mathcal{C}\) is independent of \(\{W(t)\}_{t \geq 0}\).

For any \(\phi, \psi \in \mathcal{C}\) and \(t, p \geq 0\), we assume that

(H1) There exist \(\alpha_1 > \alpha_2 > 0\) such that

\[
\mathbb{E}\{|\phi(0) - \psi(0)|^p(2\langle \phi(0) - \psi(0), b(t, \phi) - b(t, \psi) \rangle + \|\sigma(t, \phi) - \sigma(t, \psi)\|^2)\} \\
\leq -\alpha_1 \mathbb{E}|\phi(0) - \psi(0)|^{2+p} + \alpha_2 \sup_{-\tau \leq \theta \leq 0} \mathbb{E}\{|\phi(0) - \psi(0)|^p|\phi(\theta) - \psi(\theta)|^2\};
\]

(H2) There exists \(\alpha_3 > 0\) such that

\[
\mathbb{E}\|\sigma(t, \phi) - \sigma(t, \psi)\|^{2+p} \leq \alpha_3 \sup_{-\tau \leq \theta \leq 0} \mathbb{E}(|\phi(\theta) - \psi(\theta)|^{2+p}).
\]

The following remark shows that there are some examples such that (H1) and (H2).

**Remark 2.1.** Let \(b(t, \phi) = b(t, \phi(0), \phi(-\delta(t)))\) and \(\sigma(t, \phi) = \sigma(t, \phi(0), \phi(-\delta(t)))\) with \(\phi \in \mathcal{C}\), where \(\delta : [0, \infty) \rightarrow [0, \tau]\) is a measurable function. For any \(\phi \in \mathcal{C}\) and \(t \geq 0\), if

\[
2\langle \phi(0) - \psi(0), b(t, \phi(0), \phi(-\delta(t))) - b(t, \psi(0), \psi(-\delta(t))) \rangle \\
+ \|\sigma(t, \phi(0), \phi(-\delta(t))) - \sigma(t, \psi(0), \psi(-\delta(t)))\|^2 \\
\leq -\alpha_1|\phi(0) - \psi(0)|^2 + \alpha_2|\phi(-\delta(t)) - \psi(-\delta(t))|^2,
\]

and

\[
\|\sigma(t, \phi) - \sigma(t, \psi)\|^2 \leq \alpha_3(|\phi(0) - \psi(0)|^2 + |\phi(-\delta(t)) - \psi(-\delta(t))|^2),
\]

then (H1) and (H2) hold respectively for some appropriate constants \(\alpha_1, \alpha_2, \alpha_3 > 0\). On the other hand, for arbitrary \(\phi \in \mathcal{C}\) and \(t \geq 0\), if

\[
2\langle \phi(0) - \psi(0), b(t, \phi) - b(t, \psi) \rangle + \|\sigma(t, \phi) - \sigma(t, \psi)\|^2 \\
\leq -\alpha_1|\phi(0) - \psi(0)|^2 + \alpha_2 \int_{-\tau}^0 |\phi(\theta) - \psi(\theta)|^2 \mu(d\theta),
\]

and

\[
\|\sigma(t, \phi) - \sigma(t, \psi)\|^2 \leq \alpha_3 \left( |\phi(0) - \psi(0)|^2 + \int_{-\tau}^0 |\phi(\theta) - \psi(\theta)|^2 \mu(d\theta) \right),
\]

where \(\mu(\cdot)\) is a probability measure on \([-\tau, 0]\), then (H1) and (H2) are also fulfilled for some \(\alpha_1, \alpha_2, \alpha_3 > 0\). From the previous discussions, we deduce that our framework cover SDEs with constant/variable/distributed delays.
Since $b$ and $\sigma$ are locally Lipschitz, (2.1) admits a unique local strong solution $\{X(t, \xi)\}_{t \geq -\tau}$ with the initial value $\xi \in \mathcal{C}$. Moreover, (H1) guarantees that $\mathbb{P}(\rho = \infty) = 1$, where $\rho := \lim_{n \to \infty} \rho_n$ is the life time of $\{X(t, \xi)\}_{t \geq -\tau}$ with $\rho_n := \inf\{t > 0 : |X(t)| \geq n\}$ for any integer $n \geq 0$. Therefore, (2.1) has a unique strong solution $\{X(t, \xi)\}_{t \geq -\tau}$ under (H1).

For citation convenience, several fundamental inequalities are summarized in the following lemmas.

**Lemma 2.1.** ([9] Lemma 8.1) Let $u, v : [0, \infty) \mapsto \mathbb{R}_+$ be continuous functions and $\beta > 0$. If
\[ u(t) \leq u(s) - \beta \int_s^t u(r)dr + \int_s^t v(r)dr, \quad 0 \leq s < t < \infty, \]
then
\[ u(t) \leq u(0) + \int_0^t e^{-\beta(t-r)}v(r)dr. \]

**Lemma 2.2.** ([9] Lemma 8.2) Let $u : [0, \infty) \mapsto \mathbb{R}_+$ be a continuous function and $\delta > 0, \alpha > \beta > 0$. If
\[ u(t) \leq \delta + \beta \int_0^t e^{-\alpha(t-s)}u(s)ds, \quad t \geq 0, \]
then $u(t) \leq (\delta \alpha)/(\alpha - \beta)$.

**Halanay Lemma 2.3.** ([20] Theorem 2.1) For $a, b > 0$, let $u(\cdot)$ be a nonnegative function such that
\[ u'(t) \leq -au(t) + b \sup_{t-\tau \leq s \leq t} u(s), \quad t > 0 \]
and $u(s) = |\psi(s)|$ is continuous for $s \in [-\tau, 0]$. Then, for $a > b > 0$, there exists $\lambda > 0$ such that
\[ u(t) \leq \left( \sup_{-\tau \leq s \leq 0} u(s) \right) e^{-\lambda t}, \quad t \geq 0. \]

With Lemma 2.1 and Lemma 2.2 in hand, we can obtain a uniform bound of segment process $\{X_t(\xi)\}_{t \geq -\tau}$ with the initial data $\xi \in \mathcal{C}$, which plays a crucial role in investigation on existence of an invariant measure of (2.1). For notation brevity, in the sequel we shall write $X(t)$ and $X_t$ instead of $X(t, \xi)$ and $X_t(\xi)$ respectively.

**Lemma 2.4.** Assume that (H1) and (H2) hold. Then there exists a sufficiently small $\kappa > 0$ such that
\[ \sup_{t \geq -\tau} \mathbb{E}\|X_t(\xi)\|^{2+\kappa} < \infty. \]

**Proof.** For any $\kappa > 0$, by the Itô formula, we obtain that
\[
\rho(t) := \mathbb{E}\|X(t)\|^{2+\kappa} \\
\leq \frac{2 + \kappa}{2} \mathbb{E}\int_0^t |X(s)|^\kappa \{2\langle X(s), b(s, X_s) \rangle + \|\sigma(s, X_s)\|^2\}ds \\
+ |\xi(0)|^{2+\kappa} + \frac{\kappa(2 + \kappa)}{2} \mathbb{E}\int_0^t |X(s)|^\kappa \cdot \|\sigma(s, X_s)\|^2ds \\
=: I_1(t) + I_2(t).
\]
By (H1) and (H2), it is readily to see that there exist $\nu_1 > \nu_2 > 0$ such that
\begin{equation}
\mathbb{E}\{|\phi(0)|^\kappa (2\langle \phi(0), b(t, \phi) \rangle + \|\sigma(t, \phi)\|^2)\} \leq -\nu_1 \mathbb{E}|\phi(0)|^{2+\kappa} + \nu_2 \sup_{-\tau \leq \theta \leq 0} \mathbb{E}(|\phi(0)|^\kappa \cdot |\phi(\theta)|^2) + c
\end{equation}
for any $t \geq 0$ and $\phi \in \mathcal{C}$. This, together with the Young inequality:
\begin{equation}
a^\beta b^{1-\beta} \leq \beta a + (1-\beta)b, \quad a, b > 0, \beta \in (0, 1),
\end{equation}
gives that
\[
I_1(t) \leq \frac{2 + \kappa}{2} \int_0^t \left\{ -\nu_1 \rho(s) + \nu_2 \sup_{-\tau \leq \theta \leq 0} \mathbb{E}(|X(s)|^\kappa \cdot |X(s + \theta)|^2) + c \right\} ds
\leq -\frac{(2 + \kappa)\nu_1}{2} \int_0^t \rho(s) ds + \frac{(2 + \kappa)\nu_2}{2} \int_0^t \left\{ \frac{\kappa}{2 + \kappa} \rho(s) + \frac{2}{2 + \kappa} \sup_{-\tau \leq \theta \leq 0} \rho(r) \right\} ds
\leq -\frac{(2 + \kappa)}{2} \left( \nu_1 - \frac{\nu_2\kappa}{2 + \kappa} - \kappa \right) \int_0^t \rho(s) ds + \int_0^t \{c + \nu_2 r(s)\} ds,
\]
in which $r(t) := \sup_{0 \leq s \leq t} \rho(s)$. According to (H2) and Young’s inequality (2.5), it follows that
\[
I_2(t) \leq \|\xi\|^{2+\kappa} + \frac{\kappa(2 + \kappa)}{2} \int_0^t \left\{ \frac{\kappa}{2 + \kappa} \rho(s) + \frac{2}{2 + \kappa} \mathbb{E} \|\sigma(s, X_s)\|^{2+\kappa} \right\} ds
\leq \|\xi\|^{2+\kappa} + \frac{c\kappa(2 + \kappa)}{2} \int_0^t \{1 + \rho(s) + r(s)\} ds.
\]
Hence, we arrive at
\begin{equation}
\rho(t) \leq \|\xi\|^{2+\kappa} - \lambda_1 \int_0^t \rho(s) ds + \int_0^t \{c + \lambda_2 r(s)\} ds,
\end{equation}
where, for a sufficiently small $\kappa \in (0, 1),$
\[
\lambda_1 := \frac{(2 + \kappa)}{2} \left( \nu_1 - \frac{\nu_2\kappa}{2 + \kappa} - (c + 1)\kappa \right) > \lambda_2 := \nu_2 + \frac{c\kappa(2 + \kappa)}{2}
\]
due to $\nu_1 > \nu_2$. Combining (2.6) with Lemma 2.1 gives that
\begin{equation}
\rho(t) \leq \|\xi\|^{2+\kappa} + \int_0^t e^{-\lambda_1(t-s)} \{c + \lambda_2 r(s)\} ds.
\end{equation}
For a nondecreasing function $u : [0, \infty) \mapsto \mathbb{R}^+$ and any $\lambda > 0$, observe that the integral
\[
\int_0^t e^{-\lambda(t-s)} u(s) ds
\]
due to the fact that
\[
\int_0^t e^{-\lambda(t-s)} u(s) ds = \frac{(1 - e^{-\lambda t})u(0)}{\lambda} + \int_0^t \frac{1 - e^{-\lambda(t-s)}}{\lambda} du(s).
\]
By the nondecreasing property of \( r(t) \) with respect to \( t \), we therefore infer from (2.7) that
\[
    r(t) \leq \|\xi\|^{2+\kappa}_{\infty} + \int_{0}^{t} e^{-\lambda_{1}(t-s)} \{ c + \lambda_{2}r(s) \} ds \leq c + \lambda_{2} \int_{0}^{t} e^{-\lambda_{1}(t-s)} r(s) ds.
\]

Thanks to \( \lambda_{1} > \lambda_{2} \), Lemma 2.2 leads to
\[
    \|\xi\|^{2+\kappa}_{\infty} + \int_{0}^{t} e^{-\lambda_{1}(t-s)} r(s) ds.
\]

Next, for any \( t \geq \tau \), applying the Itô formula, together with the Burkhold-Davis-Gundy inequality and the Young inequality (2.5), we deduce from (2.4) that
\[
    \mathbb{E}\|X_{t}\|^{2+\kappa}_{\infty} \leq \rho(t - \tau) + c \int_{t-\tau}^{t} \{ 1 + \rho(s) + r(s) \} ds
    + (2 + \kappa) \mathbb{E}\left( \sup_{-\tau \leq \theta \leq 0} \left| \int_{t-\tau}^{t+\theta} |X(s)|^{\kappa} \langle X(s), \sigma(s, X) dW(s) \rangle \right| \right)
    \leq \frac{1}{2} \mathbb{E}\|X_{t}\|^{2+\kappa}_{\infty} + \rho(t - \tau) + c \int_{t-\tau}^{t} \{ 1 + \rho(s) + r(s) \} ds.
\]

That is,
\[
    \mathbb{E}\|X_{t}\|^{2+\kappa}_{\infty} \leq 2\rho(t - \tau) + c \int_{t-\tau}^{t} \{ 1 + \rho(s) + r(s) \} ds, \quad t \geq \tau.
\]

Moreover, note that
\[
    \mathbb{E}\|X_{t}\|^{2+\kappa}_{\infty} \leq \|\xi\|^{2+\kappa}_{\infty} + \mathbb{E}\left( \sup_{0 \leq t \leq \tau} |X(t)|^{2+\kappa} \right), \quad t \in [0, \tau].
\]

Following a similar argument to derive (2.9), we deduce that
\[
    \mathbb{E}\|X_{t}\|^{2+\kappa}_{\infty} \leq c + c \int_{0}^{t} \{ 1 + \rho(s) + r(s) \} ds, \quad t \in [0, \tau].
\]

Then the desired assertion (2.2) follows by taking (2.8)-(2.10) into account. \( \square \)

**Definition 2.1.** A probability measure \( \pi(\cdot) \in \mathcal{P}(\mathcal{C}) \) is called an invariant measure of (2.1) if, for arbitrary \( F \in \mathcal{B}_{b}(\mathcal{C}) \),
\[
    \pi(P_{t}F) = \pi(F), \quad t \geq 0,
\]
where \( P_{t}F(\xi) := \mathbb{E}F(X_{t}(\xi)) \).

**Remark 2.2.** If \( \pi(\cdot) \in \mathcal{P}(\mathcal{C}) \) is an invariant measure of (2.1) and the initial segment enjoys the same law, by [1, Lemma 1.1.9, p.14], independence of \( \xi \in \mathcal{C} \) and \( \{W(t)\}_{t \geq 0} \) and smooth property of conditional expectation, one has
\[
    \pi(F) = \int_{\mathcal{C}} \mathbb{E}F(X_{t}(\eta)) \pi(d\eta) = \mathbb{E}(\mathbb{E}(F(X_{t}(\xi))))|_{\mathcal{F}_{0}} = \mathbb{E}(F(X_{t}(\xi))).
\]

Then we conclude that \( X_{t}(\xi) \) shares the law \( \pi \in \mathcal{P}(\mathcal{C}) \), i.e., the law of \( X_{t}(\xi) \) is invariant under time translation.
The main result of this section is stated below.

**Theorem 2.5.** Under (H1) and (H2), (2.11) has a unique invariant measure $\pi(\cdot) \in \mathcal{P}(\mathcal{C})$ and is exponentially mixing. More precisely, there exists $\lambda > 0$ such that

$$
|P_tF(\xi) - \pi(F)| \leq ce^{-\lambda t}, \quad t \geq 0, \quad F \in \mathcal{B}_b(\mathcal{C}), \quad \xi \in \mathcal{C}.
$$

**Proof.** The whole proof of this theorem is divided into the following three steps.

**Step 1: Existence of an Invariant Measure.** The proof on existence of an invariant measure is due to the classical Arzelà–Ascoli tightness characterization of the space $\mathcal{C}$. Recall that $X_t(\xi)$ admits by [21, Theorem 1.1, p.51] the Markovian property although the solution process $X(t, \xi)$ is not Markovian. For arbitrary integer $n \geq 1$, set

$$
\mu_n(\cdot) := \frac{1}{n} \int_0^n \mathbb{P}_t(\cdot, d\xi) dt,
$$

where $\mathbb{P}_t(\xi, \cdot)$ is the Markovian transition kernel of $X_t(\xi)$. By the Krylov-Bogoliubov theorem [10, Theorem 3.1.1, p.21], to show existence of an invariant measure, it is sufficient to verify that $\{\mu_n(\cdot)\}_{n \geq 1}$ is relatively compact. Note that the phase space $\mathcal{C}$ for the segment process $X_t(\xi)$ is a complete separable space under the uniform metric $\| \cdot \|_\infty$ (see e.g. [3, p.220]). Taking [3, Theorem 6.2, p.37] into consideration, we need only show that $\{\mu_n(\cdot)\}_{n \geq 1}$ is tight. Moreover, thanks to [3, Theorem 8.2, p.55], it suffices to claim that

$$
\limsup_{n \to \infty} \mu_n(\varphi \in \mathcal{C} : w_{[-\tau, 0]}(\varphi, \delta) \geq \varepsilon) = 0
$$

for any $\varepsilon > 0$, where $w_{[-\tau, 0]}(\varphi, \delta)$, the modulus of continuity of $\varphi \in \mathcal{C}$ (see e.g. [3, p.54]), is defined by

$$
w_{[-\tau, 0]}(\varphi, \delta) := \sup_{|s-t| \leq \delta, s, t \in [-\tau, 0]} |\varphi(s) - \varphi(t)|, \quad \delta > 0.
$$

In the sequel, for simplicity we write $X(t)$ and $X_t$ instead of $X(t, \xi)$ and $X_t(\xi)$ respectively. Since

$$
I(t, \delta) := \sup_{t \leq v \leq u \leq t + \tau, 0 \leq u - v \leq \delta} |X(u) - X(v)|
$$

$$
\leq \sup_{t \leq v \leq u \leq t + \tau, 0 \leq u - v \leq \delta} \int_v^u |b(s, X_s)| ds + \sup_{t \leq v \leq u \leq t + \tau, 0 \leq u - v \leq \delta} \int_v^u \sigma(s, X_s) dW(s)
$$

$$
=: I_1(t, \delta) + I_2(t, \delta), \quad t \geq \tau,
$$

one has

$$
\mathbb{P}(I(t, \delta) \geq \varepsilon) \leq \mathbb{P}(I_1(t, \delta) \geq \varepsilon/2) + \mathbb{P}(I_2(t, \delta) \geq \varepsilon/2).
$$

For any $\tilde{\varepsilon} \in (0, 1)$, by the Chebyshev inequality and Lemma [24], there exists an $R_0 > 0$ sufficiently large such that

$$
\mathbb{P}(\|X_t\|_\infty > R_0) + \mathbb{P}(\|X_{t+\tau}\|_\infty > R_0) \leq R_0^{-2} \sup_{t \geq -\tau} \left( \mathbb{E}\|X_{t+\tau}\|_\infty^2 + \mathbb{E}\|X_t\|_\infty^2 \right) \leq \tilde{\varepsilon}.
$$
Moreover, since $b$ enjoys locally bounded property, there exists a sufficiently small $\delta_0 > 0$ such that

\[ P(I_1(t, \delta) \geq \varepsilon/2) \leq P(I_1(t, \delta) \geq \varepsilon/2) \quad \|X_t\|_\infty \leq R_0, \quad \|X_{t+\tau}\|_\infty \leq R_0) = 0, \quad \delta < \delta_0. \]

Accordingly, we obtain from (2.13) and (2.14) that

\[ P(I_1(t, \delta) \geq \varepsilon/2) \leq P(I_1(t, \delta) \geq \varepsilon/2) \quad \|X_t\|_\infty \leq R_0, \quad \|X_{t+\tau}\|_\infty \leq R_0) + P(\|X_t\|_\infty \geq R_0) \quad \|X_{t+\tau}\|_\infty \geq R_0) \leq \varepsilon. \]

On the other hand, for $\kappa \in (0, 1)$ such that Lemma 2.4 holds and arbitrary $0 \leq s \leq t$, by the Burkhold-Davis-Gundy inequality, (H2) and Lemma 2.4, it follows that

\[ E \int_s^t \sigma(r, X_r)dW(r) \leq c(t - s)^{\kappa/2} \int_s^t \|X_r\|^{2+\kappa}_{\infty} dr \leq c(t - s)^{1+\kappa/2}. \]

This, combining with the Kolmogorov tightness criterion [13 Problem 4.11, p.64], implies that

\[ \limsup_{\delta \to 0, \tau \to \tau} P(I_2(t, \delta) \geq \varepsilon/2) = 0. \]

Consequently, (2.12) follows from (2.15), (2.16), the arbitrariness of $\varepsilon$, and by noticing that

\[ \mu_n(\varphi \in \mathcal{C} : w_{[-\tau, 0]}(\varphi, \delta) \geq \varepsilon) \leq \frac{2\tau}{n} + \frac{1}{n} \int_{\tau}^{n} P(I(t, \delta) \geq \varepsilon) dt \]

for $n > \tau$. Since $\{\mu_n(\cdot)\}_{n \geq 1}$ is relative compact due to (2.12), there exists a subsequence, still denoted by $\{\mu_n(\cdot)\}_{n \geq 1}$ without confusion, such that $\{\mu_n(\cdot)\}_{n \geq 1}$ converges weakly to some $\pi(\cdot) \in \mathcal{P}(\mathcal{C})$, which indeed is an invariant measure of $X_t(\xi)$ by [10 Theorem 3.1.1, p.21] and recalling from [21 Theorem 3.1, p.67] that the stochastically continuous semigroup $P_t$ is a Feller semigroup, i.e., $P_tf \in C_b(\mathcal{C})$ for any $F \in C_b(\mathcal{C})$ and $t \geq 0$.

**Step 2: Uniqueness of Invariant Measures.** By the Itô formula, it is easy to see that

\[ u(t) := E[X(t, \xi) - X(t, \eta)]^2 \]

\[ = |\xi(0) - \eta(0)|^2 + \int_0^t \{2\langle X(s, \xi) - X(s, \eta), b(s, X_s(\xi)) - b(s, X_s(\eta)) \rangle + \|\sigma(s, X_s(\xi)) - \sigma(s, X_s(\eta))\|^2 \} ds. \]

Differentiating with respect to $t$ on both sides of (2.17), one has from (H1) with $p = 0$ that

\[ u'(t) \leq -\alpha_1 u(t) + \alpha_2 \sup_{-\tau \leq s \leq t} |u(s)|. \]

Then, Lemma 2.3 yields that

\[ \mathbb{E}|X(t, \xi) - X(t, \eta)|^2 \leq \|\xi - \eta\|_{\infty}^2 e^{-\lambda t}, \quad t \geq 0 \]
for some $\lambda > 0$. Next, for any $t \geq \tau$, by the Itô formula, (H1) and the Burkhold-Davis-Gundy inequality, we arrive at

$$\mathbb{E}\|X_t(\xi) - X_t(\eta)\|_\infty^2 \leq \mathbb{E}\|X(t - \tau, \xi) - X(t - \tau, \eta)\|^2 + c\mathbb{E}\int_{t-\tau}^{t} |X(s, \xi) - X(s, \eta)|^2 ds$$

$$+ c \int_{t-\tau}^{t} \sup_{s-\tau \leq \tau \leq s} \mathbb{E}|X(r, \xi) - X(r, \eta)|^2 ds + \frac{1}{2} \mathbb{E}\|X_t(\xi) - X_t(\eta)\|_\infty^2.$$  

This, in addition to (2.18), gives that

$$\mathbb{E}\|X_t(\xi) - X_t(\eta)\|_\infty^2 \leq c\|\xi - \eta\|_\infty^2 e^{-\lambda(t-\tau)} + c \sup_{t-2\tau \leq s \leq t} \mathbb{E}|X(s, \xi) - X(s, \eta)|^2$$

$$\leq ce^{-\lambda t}, \quad t \geq \tau.$$  

Observe that (2.19) still holds for $t \in [0, \tau]$. On the basis of (2.19), we claim that $\pi(\cdot) \in \mathcal{P}(\mathcal{C})$ is the unique invariant measure. Indeed, if $\pi'(\cdot) \in \mathcal{P}(\mathcal{C})$ is also an invariant measure, for any bounded Lipschitz function $f : \mathcal{C} \mapsto \mathbb{R}$, by (2.19) and the invariance of $\pi(\cdot), \pi'(\cdot) \in \mathcal{P}(\mathcal{C})$, it follows that

$$|\pi(f) - \pi'(f)| \leq \int_{\mathcal{C} \times \mathcal{C}} |P_t f(\xi) - P_t f(\eta)| \pi(d\xi) \pi'(d\eta) \leq ce^{-\lambda t}, \quad t \geq 0.$$  

As a result, one gets the uniqueness of invariant measures by taking $t \to \infty$ and applying [8 Proposition 2.2, p.3] and [10 Lemma 7.1.5, p.125] for any $f \in C_b(\mathcal{C})$, the set of all bounded, continuous real-valued functions on $\mathcal{C}$.

**Step 3: Exponential Mixing.** By the invariance of $\pi \in \mathcal{P}(\mathcal{C})$, for any $F \in \mathcal{B}_b(\mathcal{C})$, it follows that

$$|P_t F(\xi) - \pi(F)| \leq \int_{\mathcal{C}} |P_t F(\xi) - P_t F(\eta)| \pi(d\eta).$$

Thus, the desired assertion (2.11) follows by taking (2.19) and [10] Lemma 7.1.5, p.125] into consideration.

**Remark 2.3.** Let $\lambda > 0$ be an appropriate constant and $V : \mathbb{R}^n \mapsto \mathbb{R}_+$ a $C^2$-function. By applying the Itô formula to $e^{\lambda t} V(x)$, under some appropriate conditions, Es-Sarhir et al. [7 Proposition 2.1] and Bo and Yuan [4 Proposition] showed the uniform boundedness of segment processes, where $\lambda > 0$ need to be sufficiently large in the argument of [7 Proposition 2.1]. Based on the uniform boundedness of segment processes, they also discussed existence of an invariant measure for the corresponding Markovian transition semigroup. Although the method adopted therein applies to SDEs with constant delay, it seems not to work for the case of variable time-lag since the differentiable property of delay function is not available. While, Theorem 2.5 covers a wide range of retarded SDEs which include non-autonomous SDEs with constant/variable/distributed delays as their special cases.

### 3 Exponential Mixing for Neutral SDEs

In the previous section, we discuss the exponential ergodicity of the Markovian transition semigroups generated by the associated segment processes for a class of non-autonomous
retarded SDEs. In this section, we proceed to discuss the exponential mixing property for another class of stochastic equation depending on past and present values but that involves derivatives with delays as well as the function itself. Such equations historically have been called neutral SDEs, which have many applications in variational problems, chemical engineering system and optimal stochastic control (see e.g. [18, Chapter 6]). By a close inspection of the argument of Theorem 2.5 seems hard to apply to neutral SDEs although it can deal with the non-autonomous cases. In this section, we shall put forward another method, called stability-in-distribution approach, to cope with the exponential mixing for the neutral SDEs.

Consider a neutral SDE on \( \mathbb{R}^n \)

\[
\text{C1} \quad (3.1) \quad d\{X(t) - G(X_t)\} = b(X_t)dt + \sigma(X_t)dW(t)
\]

with the initial value \( X_0 = \xi \in \mathcal{C} \) which is independent of \( \{W(t)\}_{t \geq 0} \), where \( G : \mathcal{C} \mapsto \mathbb{R}^n \) is measurable and continuous such that \( G(0) = 0 \), and \( b : \mathcal{C} \mapsto \mathbb{R}^n, \sigma : \mathcal{C} \mapsto \mathbb{R}^n \otimes \mathbb{R}^m \) are measurable and locally Lipschitz.

For any \( \phi, \psi \in \mathcal{C} \), we assume that

(A1) There exists \( \kappa \in (0, 1) \) such that \( \mathbb{E}|G(\phi) - G(\psi)| \leq \kappa \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\phi(\theta) - \psi(\theta)|^2 \).

(A2) There exist \( \alpha_1 > \alpha_2 > 0 \) such that

\[
\mathbb{E}\{2(\phi(0) - \psi(0) - (G(\phi) - G(\psi)), b(\phi) - b(\psi)) + \|\sigma(\phi) - \sigma_2(\psi)\|^2\} \\
\leq -\alpha_1 \mathbb{E}|\phi(0) - \psi(0)|^2 + \alpha_2 \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\phi(\theta) - \psi(\theta)|^2.
\]

(A3) There exists \( \alpha_3 > 0 \) such that

\[
\mathbb{E}\|\sigma(\phi) - \sigma(\psi)\|^2 \leq \alpha_3 \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\phi(\theta) - \psi(\theta)|^2.
\]

Under (A1)-(A2), (3.1) has a unique strong solution \( \{X(t, \xi)\}_{t \geq 0} \) with the initial data \( \xi \in \mathcal{C} \). Before the statement of our main result, we first provide a generalized Razumikhin-type theorem (see e.g. [18, Theorem 6.1, p.221]) which guarantees that the segment process admits a uniform bound although the equation (3.1) need not admit an equilibrium.

Raz

**Lemma 3.1.** Let (A1) hold and assume further that there exist \( \delta \geq 0, \lambda > 0 \) such that

\[
\text{eq1} \quad (3.2) \quad \mathbb{E}\{2(\phi(0) - G(\phi), b(\phi)) + \|\sigma(\phi)\|^2\} \leq \delta - \lambda \mathbb{E}|\phi(0) - G(\phi)|^2
\]

provided that, for some \( q > (1 - \kappa)^{-2} \),

\[
\text{eq2} \quad (3.3) \quad \mathbb{E}|\phi(\theta)|^2 < q|\phi(0) - G(\phi)|^2, \quad -\tau \leq \theta \leq 0.
\]

Then there exists \( \gamma < \lambda \) sufficiently small such that

\[
\text{C3} \quad (3.4) \quad \mathbb{E}|X(t)|^2 \leq \frac{\delta/\lambda + e^{-\gamma t}(1 + \kappa)^2\|\xi\|_{\infty}^2}{(1 - \kappa e^{\gamma t/2})^2}, \quad t \geq -\tau.
\]
Proof. By the elemental inequality:

\[ (a + b)^2 \leq a^2/(1 - \varepsilon) + b^2/\varepsilon, \quad a, b \in \mathbb{R}, \varepsilon \in (0, 1), \]

for any \( \gamma > 0 \) and \( t \geq 0 \), we deduce from (A1) that

\[
\sup_{0 \leq s \leq t} \left( e^{\gamma \varepsilon} \mathbb{E}|X(s)|^2 \right) \leq \frac{1}{1 - \varepsilon} \sup_{0 \leq s \leq t} \left( e^{\gamma \varepsilon} \mathbb{E}|X(s) - G(X_s)|^2 \right) + \frac{\kappa^2}{\varepsilon} e^{\gamma \tau} \sup_{-\tau \leq s \leq t} \left( e^{\gamma \varepsilon} |X(s)|^2 \right).
\]

Due to \( q > (1 - \kappa)^{-2} \) and \( \kappa \in (0, 1) \), there exists \( \gamma < \lambda \) sufficiently small such that

\[
\kappa e^{\gamma \tau/2} < 1 \quad \text{and} \quad \frac{e^{\gamma \tau}}{(1 - \kappa e^{\gamma \tau/2})^2} < q.
\]

For \( \gamma > 0 \) sufficiently small such that (3.7) holds, if

\[
e^{\gamma \varepsilon} \mathbb{E}|X(t) - G(X_t)|^2 \leq \frac{\delta}{\lambda} e^{\gamma t} + (1 + \kappa)^2 \|\xi\|_\infty^2, \quad t \geq 0,
\]

then (3.6) gives that

\[
\sup_{-\tau \leq s \leq t} \left( e^{\gamma \varepsilon} \mathbb{E}|X(s)|^2 \right) \leq \frac{1}{1 - \varepsilon} \sup_{0 \leq s \leq t} \left( \frac{\delta}{\lambda} e^{\gamma s} + (1 + \kappa)^2 \|\xi\|_\infty^2 \right) + \frac{\kappa^2}{\varepsilon} e^{\gamma \tau} \sup_{-\tau \leq s \leq t} \left( e^{\gamma \varepsilon} |X(s)|^2 \right).
\]

Thus (3.4) follows by taking \( \varepsilon = \kappa e^{\gamma \tau/2} \). In what follows, under (A1) and (3.2) we verify by a contradiction argument that (3.8) is indeed true for sufficiently small \( \gamma > 0 \) to be determined. Note from (A1) that

\[
|\xi(0) - G(\xi)|^2 \leq (1 + \kappa)^2 \|\xi\|_\infty^2.
\]

If (3.8) is not true, then there exist \( \rho > 0 \) and sufficiently small \( h > 0 \) such that

\[
e^{\gamma \varepsilon} \mathbb{E}|X(t) - G(X_t)|^2 - \frac{\delta}{\lambda} e^{\gamma t} \leq e^{\gamma \rho} \mathbb{E}|X(\rho) - G(X_\rho)|^2 - \frac{\delta}{\lambda} e^{\gamma \rho}
\]

\[= (1 + \kappa)^2 \|\xi\|_\infty^2, \quad 0 \leq t \leq \rho,\]

however,

\[
e^{\gamma \varepsilon} \mathbb{E}|X(t) - G(X_t)|^2 - \frac{\delta}{\lambda} e^{\gamma t} > e^{\gamma \varepsilon} \mathbb{E}|X(\rho) - G(X_\rho)|^2 - \frac{\delta}{\lambda} e^{\gamma \rho}, \quad \rho < t \leq \rho + h.
\]

Taking (3.4) and (3.9) into account, we derive that

\[
\mathbb{E}|X(t)|^2 \leq \frac{\delta/\lambda + e^{-\gamma t}(e^{\gamma \rho} \mathbb{E}|X(\rho) - G(X_\rho)|^2 - \frac{\delta}{\lambda} e^{\gamma \rho})}{(1 - \kappa e^{\gamma \tau/2})^2}, \quad -\tau \leq t \leq \rho,
\]

which, in particular, yields that

\[
\mathbb{E}|X(\rho + \theta)|^2 \leq \frac{\delta/\lambda + e^{-\gamma (\rho + \theta)}(e^{\gamma \rho} \mathbb{E}|X(\rho) - G(X_\rho)|^2 - \frac{\delta}{\lambda} e^{\gamma \rho})}{(1 - \kappa e^{\gamma \tau/2})^2}, \quad -\tau \leq \theta \leq 0,
\]

(3.11)
due to \( \frac{\delta}{\lambda}(1 - e^{-\gamma \theta}) \leq 0 \) for \(-\tau \leq \theta \leq 0\). Thus we get from (3.3), (3.7) and (3.11) that

\[
\mathbb{E}\{2(X(t) - G(X_t), b(X_t)) + \|\sigma(X_t)\|^2\} \leq \delta - \gamma \mathbb{E}|X(t) - G(X_t)|^2, \quad \rho \leq t \leq \rho + h
\]

by virtue of the continuity of sample path, where \( h > 0 \) is sufficiently small. Next, applying the Itô formula and using (3.12) yields that

\[
\mathbb{E}(e^{\gamma(\rho + h)}|X(\rho + h) - G(X_{\rho + h})|^2) - \frac{\delta}{\lambda}e^{\gamma \rho} \leq \mathbb{E}(e^{\gamma \rho}|X(\rho) - G(X_{\rho})|^2) - \frac{\delta}{\lambda}e^{\gamma \rho}.
\]

Finally we conclude that (3.8) holds by the contradiction between (3.10) and (3.13). \(\square\)

Our main result in this section is presented as below.

**Theorem 3.2.** Let (A1)-(A3) hold and \( \kappa \in (0, 1/2) \) and \( \alpha_1 > \alpha_2/(1 - 2\kappa)^2 \). Assume further that

\[
|G(\phi) - G(\psi)| \leq \kappa \|\phi - \psi\|_{\infty}, \quad \phi, \psi \in \mathcal{C}.
\]

Then, (3.11) has a unique invariant measure \( \pi(\cdot) \in \mathcal{P}(\mathcal{C}) \) and is exponentially mixing. That is, there exists \( \lambda > 0 \) such that

\[
|P_tF(\xi) - \pi(F)| \leq ce^{-\lambda t}, \quad t \geq 0, \quad F \in \mathcal{B}(\mathcal{C}), \quad \xi \in \mathcal{C}.
\]

**Proof.** By Yuan et al. [30, Theorem 3.2], if, for a bounded subset \( U \subset \mathcal{C} \),

\[
(N1) \quad \sup_{t \geq 0} \sup_{\xi \in U} \mathbb{E}\|X_t(\xi)\|^2_{\infty} < \infty;
\]

\[
(N2) \quad \lim_{t \to \infty} \sup_{\xi, \eta \in U} \mathbb{E}\|X_t(\xi) - X_t(\eta)\|^2_{\infty} = 0,
\]

then \( \mathbb{P}(t, \xi, \cdot) \) converges weakly to \( \pi(\cdot) \in \mathcal{P}(\mathcal{C}) \). For any \( F \in \mathcal{C}_b(\mathcal{C}) \) and \( t, s \geq 0 \), by the Markovian property of \( \{X_t(\xi)\}_{t \geq 0} \), one has

\[
P_{t+s}F(\xi) = P_sP_tF(\xi).
\]

For fixed \( t \geq 0 \), taking \( s \to \infty \) gives that

\[
\pi(F) = \pi(P_tF)
\]

whenever \( \mathbb{P}(t, \xi, \cdot) \) converges weakly to \( \pi(\cdot) \in \mathcal{P}(\mathcal{C}) \). Hence, (3.11) admits an invariant measure provided that (N1) and (N2) hold respectively. In what follows, we claim that (N1) and (N2) hold under the conditions imposed. Following a similar argument to that of [18, Corollary 6.6, p.227] and taking Lemma 3.1 into consideration, we deduce that there exists \( \gamma > 0 \) sufficiently small such that

\[
\sup_{t \geq -\tau} \mathbb{E}|X(t, \xi)|^2 < \infty \quad \text{and} \quad \mathbb{E}|X(t, \xi) - X(t, \eta)|^2 \leq ce^{-\gamma t}, \quad t \geq 0.
\]

By the Burkhold-Davis-Gundy inequality, (A2)-(A3), for any \( t \geq 2\tau \) we obtain that

\[
\mathbb{E}\left(\sup_{t-\tau \leq s \leq t} |\Lambda(s, \xi)|^2\right) \leq 2\mathbb{E}(|\Lambda(t - \tau, \xi)|^2) + c \int_{t-\tau}^{t} \left\{ 1 + \sup_{-\tau \leq r \leq s} \mathbb{E}|X(r, \xi)|^2 \right\} ds
\]
with \( \Lambda(t, \xi) := X(t, \xi) - G(X_t(\xi)) \), and

\[
\text{eq6} \tag{3.17} \quad \mathbb{E}\left( \sup_{t - \tau \leq s \leq t} |\Gamma(s, \xi, \eta)|^2 \right) \leq 2\mathbb{E}(|\Gamma(t - \tau, \xi, \eta)|^2) + c \int_{t - \tau}^t \mathbb{E}|X(r, \xi) - X(r, \eta)|^2 dr,
\]

where \( \Gamma(t, \xi, \eta) := X(t, \xi) - X(t, \eta) - (G(X_t(\xi)) - G(X_t(\eta))) \). Thus, the inequality \eqref{eq:5}, (A1) and \eqref{eq:15} give that

\[
\text{eq7} \tag{3.18} \quad \delta := \sup_{t \geq \tau} \mathbb{E}\left( \sup_{t - \tau \leq s \leq t} |\Lambda(s, \xi)|^2 \right) < \infty
\]

and

\[
\text{eq8} \tag{3.19} \quad \mathbb{E}\left( \sup_{t - \tau \leq s \leq t} |\Gamma(s, \xi, \eta)|^2 \right) \leq ce^{-\gamma t}, \quad t \geq 2\tau.
\]

For any integer \( n \geq 2 \), note from \eqref{eq:5}, \eqref{eq:14}, and \eqref{eq:18} that

\[
\mathbb{E}\|X_{n\tau}(\xi)\|^2_\infty \leq \frac{1}{\kappa} \mathbb{E}\left( \sup_{(n-1)\tau \leq s \leq n\tau} |G(X_s(\xi))|^2 \right) + \frac{\delta}{1 - \kappa} \\
\leq \kappa \mathbb{E}\|X_{n\tau}(\xi)\|^2_\infty + \kappa \mathbb{E}\|X_{(n-1)\tau}(\xi)\|^2_\infty + \frac{\delta}{1 - \kappa}.
\]

By virtue of an induction argument, due to \( \kappa \in (0, 1/2) \), one derive that

\[
\text{eq12} \tag{3.20} \quad \mathbb{E}\|X_{n\tau}(\xi)\|^2_\infty \leq \frac{\kappa}{1 - \kappa} \mathbb{E}\|X_{(n-1)\tau}(\xi)\|^2_\infty + \frac{\delta}{(1 - \kappa)^2} \left\{ 1 + \frac{\kappa}{1 - \kappa} + \cdots + \left( \frac{\kappa}{1 - \kappa} \right)^{n-1} \right\} \\
\leq \|\xi\|^2_\infty + \frac{\delta}{(1 - \kappa)(1 - 2\kappa)}.
\]

Observe that for any \( t \geq 0 \) there exists an \( n \geq 0 \) such that \( t \in [n\tau, (n + 1)\tau) \) and

\[
\mathbb{E}\|X_t(\xi)\|^2_\infty \leq \mathbb{E}\|X_{n+1}(\xi)\|^2_\infty + \mathbb{E}\|X_{n}(\xi)\|^2_\infty.
\]

Then (N1) follows immediately from \eqref{eq:20}. On the other hand, by \eqref{eq:5}, \eqref{eq:14} and \eqref{eq:19}, for any integer \( n \geq 2 \), it follows that

\[
\mathbb{E}\|X_{n\tau}(\xi) - X_{n\tau}(\eta)\|^2_\infty \leq \frac{1}{\kappa} \mathbb{E}\left( \sup_{(n-1)\tau \leq s \leq n\tau} |G(X_s(\xi)) - G(X_s(\eta))|^2 \right) + \frac{ce^{-n\gamma \tau}}{1 - \kappa} \\
\leq \kappa \mathbb{E}\|X_{n\tau}(\xi) - X_{n\tau}(\eta)\|^2_\infty + \kappa \mathbb{E}\|X_{(n-1)\tau}(\xi) - X_{(n-1)\tau}(\eta)\|^2_\infty + \frac{ce^{-n\gamma \tau}}{1 - \kappa},
\]
Also by an induction argument, we obtain that

\[
\mathbb{E}\|X_{n\tau}(\xi) - X_{n\tau}(\eta)\|_\infty^2 \leq \left(\frac{\kappa}{1 - \kappa}\right)^n \|\xi - \eta\|_\infty^2 + \frac{c}{(1 - \kappa^2)} \left\{ \left(\frac{\kappa}{1 - \kappa}\right)^{n-1} e^{-\gamma \tau} + \left(\frac{\kappa}{1 - \kappa}\right)^{n-2} e^{-2\gamma \tau} + \cdots + e^{-n\gamma \tau} \right\}
\]

\[\leq \left(\frac{\kappa}{1 - \kappa}\right)^n \|\xi - \eta\|_\infty^2 + \frac{e^{-n\gamma \tau} (1 - q^n)}{1 - q} \leq \|\xi - \eta\|_\infty^2 e^{-pn\gamma \tau} + \frac{e^{-n\gamma \tau}}{1 - q} \leq ce^{-(p\wedge 1)n\gamma \tau},\]

where

\[p := \frac{1}{\gamma \tau} \log \left(\frac{1 - \kappa}{\kappa}\right) \quad \text{and} \quad q := \kappa e^{\gamma \tau}/(1 - \kappa) < 1\]

for \(\kappa \in (0, 1/2)\) and \(\gamma > 0\) sufficiently small. Next, for any \(t > 0\), notice that there exists \(n \geq 0\) such that \(t \in [n\tau, (n + 1)\tau)\) and by (3.21) that

\[
\mathbb{E}\|X_t(\xi) - X_t(\eta)\|_\infty^2 \leq \mathbb{E}\|X_{n+1}(\xi) - X_{n+1}(\eta)\|_\infty^2 + \mathbb{E}\|X_n(\xi) - X_n(\eta)\|_\infty^2 \leq ce^{-(p\wedge 1)(n+1)\gamma \tau} + ce^{(p\wedge 1)\gamma \tau} e^{-(p\wedge 1)(n+1)\gamma \tau} \leq ce^{-(p\wedge 1)\gamma t}.
\]

Consequently, (N2) holds. Finally the desired assertion follows by repeating the latter proof of Theorem 2.5. \(\square\)

Remark 3.1. There are some examples such that (A1) and (3.14) hold, e.g., \(G(\phi) = \kappa \phi(-\tau), G(\phi) = \kappa \phi(-\tau(t))\) and \(G(\phi) = \kappa \int_{-\tau}^0 \phi(\theta) \mu(d\theta)\) for \(\phi \in \mathcal{C}\) and \(\kappa \in (0, 1/2)\), where \(\mu(\cdot)\) is a probability measure on \([-\tau, 0]\).

Remark 3.2. By a generalized Razumikhin-type theorem, we give a uniform bound of segment processes associated with neutral SDEs, while the method adopted in Lemma 2.4 seems hard to work because of the appearance of neutral term. Moreover, the Arzelà–Ascoli method utilized in Theorem 2.5 does not apply to neutral SDEs either although it can deal with the non-autonomous retarded SDEs. The trick applied in Theorem 3.2 is call a “stability-in-distribution approach” and our main result, Theorem 3.2 includes neutral SDEs with constant/variable/distributed time-lags.

Remark 3.3. Let \((H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)\) be a real separable Hilbert space and \(V\) a Banach space such that \(V \leftrightarrow H\) continuously and densely. Via the Riesz isomorphism,

\[V \leftrightarrow H \equiv H^* \leftrightarrow V^*,\]

where \(H^*\) and \(V^*\) are the dual space of \(H\) and \(V\) respectively. Stability-in-distribution approach can be applied to a non-linear retarded Stochastic Partial Differential Equation (SPDE) on the Gelfand triple \((V, H, V^*)\)

\[dX(t) = \{A(X(t)) + b(X_t)\}dt + \sigma(X_t)dW(t),\]

(3.22)
where \( A : V \mapsto V' \) is a family of nonlinear monotone and coercive operators. However, the Arzelà–Ascoli approach seems hard to apply to (3.22) because of the monotone and coercive property of \( A \). Therefore, the Arzelà–Ascoli method and the stability-in-distribution approach possess their respective advantages.

4 Exponential Mixing for Retarded SDEs with Jumps

In the last two sections, we investigate the ergodic property of retarded SDEs with continuous sample paths under the uniform topology. In this section, we turn to the case of retarded SDEs with discontinuous paths.

We further need to introduce some additional notation and notions. Let \( D := D([-\tau, 0]; \mathbb{R}^n) \) denote the collection of all càdlàg paths \( f : [-\tau, 0) \mapsto \mathbb{R}^n \). Recall that a path \( f : [-\tau, 0) \mapsto \mathbb{R}^n \) is called càdlàg if it is right-continuous having finite left-hand limits. Let \( \Lambda \) denote the class of increasing homeomorphisms, and

\[
\|\lambda\|^\circ := \sup_{-\tau \leq s < t \leq 0} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| < \infty.
\]

Under the uniform metric \( \|z\|_\infty := \sup_{-\tau \leq \theta \leq 0} \left| z(\theta) \right| \) for each \( z \in \mathcal{D} \), the space \( \mathcal{D} \) is complete but not separable. For any \( \xi, \eta \in \mathcal{D} \), define the Skorhod metric \( d_S \) on \( \mathcal{D} \) by

\[
d_S(\xi, \eta) := \inf_{\lambda \in \Lambda} \{\|\lambda\|^\circ \vee \|\xi - \eta \circ \lambda\|_\infty\},
\]

where \( \eta \circ \lambda \) means the composition of mappings \( \eta \) and \( \lambda \). Under the skorhod metric \( d_S \), \( \mathcal{D} \) is not only complete but also separable (see e.g. [3, Theorem 12.2, p.128]). For the space \( \mathcal{D} \), the uniform metric \( \| \cdot \|_\infty \) may lead to certain misinterpretation of the actual situation while the Skorohod metric \( d_S \) in the mathematical modeling of a certain processes gives a more accurate representation of the processes, allows researchers to perform a correct analysis of a real situation, and make a credible forecast of the possible outcomes of similar processes. For more details on the Skorohod Metric, we refer to [3, Chapter 4]. Let \((\mathcal{Y}, \mathcal{B}(\mathcal{Y}), m(\cdot))\) be a measurable space, \( D_p \) a countable subset of \( \mathbb{R}_+ \) and \( p : D_p \mapsto \mathcal{Y} \) an adapted process taking value in \( \mathcal{Y} \). Then, as in Ikeda and Watanabe [8, p.59], the Poisson random measure \( N(\cdot, \cdot) : \mathcal{B}(\mathbb{R}_+ \times \mathcal{Y}) \times \Omega \mapsto \mathbb{N} \cup \{0\} \), defined on the complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), can be represented by

\[
N((0, t] \times \Gamma) = \sum_{s \in D_p, s \leq t} \mathbf{1}_\Gamma(p(s)), \quad \Gamma \in \mathcal{B}(\mathcal{Y}).
\]

In this case, we say that \( p \) is a Poisson point process and \( N \) is a Poisson random measure. Let \( m(\cdot) := \mathbb{E}N((0, 1] \times \cdot) \). Then, the compensated Poisson random measure

\[
\tilde{N}(dt, dz) := N(dt, dz) - dtm(dz)
\]

is a martingale.

A stochastically continuous Markovian semigroup \( P_t \) is called eventually Feller if \( P_t f \in C_b(\mathcal{E}) \) for any \( F \in C_b(\mathcal{E}) \) and \( t \geq t_0 \), where \( t_0 \geq 0 \) is some constant, and immediately Feller for \( t_0 = 0 \).
Consider a non-autonomous retarded SDE with jump

\[ dX(t) = b(t, X_t)dt + \int_\Gamma \sigma(t, X_{t-}, z)\tilde{N}(dt, dz), \quad t \geq 0 \]

with the initial value \( \xi \in \mathcal{D} \) which is independent of \( N(\cdot, \cdot) \), where \( X_t(\theta) := X((t+\theta)^-) := \lim_{s \uparrow t+\theta} X(s) \) for \( \theta \in [-\tau, 0] \), \( b : [0, \infty) \times \mathcal{D} \times \Omega \rightarrow \mathbb{R}^n \) and \( \sigma : [0, \infty) \times \mathcal{D} \times \Omega \rightarrow \mathbb{R}^n \times \Gamma \rightarrow \mathbb{R}^n \) are progressively measurable.

For any \( \phi, \psi \in \mathcal{D} \) and any \( t \geq 0 \), we assume that

(B1) There exist \( \alpha_1 > \alpha_2 > 0 \) such that

\[
\mathbb{E}\left\{ 2\langle \phi(0) - \psi(0), b(t, \phi) - b(t, \psi) \rangle + \int_\Gamma |\sigma(t, \phi, z) - \sigma(t, \psi, z)|^2 m(dz) \right\} \\
\leq -\alpha_1 \mathbb{E}|\phi(0) - \psi(0)|^2 + \alpha_2 \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\phi(\theta) - \psi(\theta)|^2;
\]

(B2) There exists \( \alpha_3 > 0 \) such that

\[
\mathbb{E}|b(t, \phi) - b(t, \psi)|^2 + \mathbb{E} \int_\Gamma |\sigma(t, \phi, z) - \sigma(t, \psi, z)|^2 m(dz) \leq \alpha_3 \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\phi(\theta) - \psi(\theta)|^2.
\]

The main result in this section is stated as follows.

**Theorem 4.1.** Under (B1)-(B2), (4.2) has a unique invariant measure \( \pi(\cdot) \in \mathcal{P}(\mathcal{D}) \) and is exponentially mixing. More precisely, there exists \( \lambda > 0 \) such that

\[
|P_tF(\xi) - \pi(F)| \leq ce^{-\lambda t}, \quad t \geq \tau, \; F \in \mathcal{B}(\mathcal{D}), \; \xi \in \mathcal{D}.
\]

**Proof.** The whole proof is divided into the following three steps.

**Step 1:** Claim a uniform bound of \( X_t \):

\[ \sup_{t \geq -\tau} \mathbb{E}\|X_t\|^2_\infty < \infty. \]

Following a similar argument to derive (2.8), we derive that

\[ \delta := \sup_{t \geq -\tau} \mathbb{E}\|X(t)\|^2 < \infty. \]

By the Itô formula, for any \( t \geq \tau \) and \( \theta \in [-\tau, 0] \), it follows that

\[
|X(t + \theta)|^2 = |X(t - \tau)|^2 + 2 \int_{t-\tau}^{t+\theta} \langle X(s), b(s, X_s) \rangle ds \\
+ \int_{t-\tau}^{t+\theta} \int_\Gamma |\sigma(s, X_{s-}, z)|^2 N(ds, dz) + 2\Pi(t, t + \theta),
\]

in which

\[
\Pi(t, t + \theta) := \int_{t-\tau}^{t+\theta} \int_\Gamma \langle X(s-), \sigma(s, X_{s-}, z) \rangle \tilde{N}(ds, dz).
\]
Next, due to the Burkhold-Davis-Gundy inequality (see e.g. [24, Theorem 48, p.193]), and the Jensen inequality, we derive that
\[
E \left( \sup_{-\tau \leq \theta < 0} |\Pi(t, t + \theta)| \right) \leq c E \sqrt{[\Pi, \Pi]_{[t-\tau,t]}}
\]
\[
\leq c E \sqrt{\int_{t-\tau}^{t} \int_{\Gamma} |(X(s-), \sigma(s, X_{s-}, z))|^2 N(ds, dz)}
\]
\[
\leq c E \int_{t-\tau}^{t} \int_{\Gamma} |\sigma(s, X_{s-}, z)|^2 N(ds, dz)
\]
\[
\leq \frac{1}{4} E \|X_t\|_{\infty}^2 + c E \int_{t-\tau}^{t} |\sigma(s, X_s, z)|^2 m(dz) ds,
\]
where $[\Pi, \Pi]_{[t-\tau,t]}$ stands for the quadratic variation process (square bracket process) of $\Pi(t, t - \tau)$. Taking (4.5) and (4.6) into consideration and using (B1) and (B2), we arrive at
\[
E\|X_t\|^2 \leq 2E\|X(t - \tau)\|^2 + c \int_{t-\tau}^{t} \left( 1 + \sup_{-\tau \leq r \leq s} E\|X(r)\|^2 \right) ds, \quad t \geq \tau.
\]
This, together with (4.4), leads to (4.3).

**Step 2:** Existence of an invariant measure. For $\theta \in [-\tau, 0]$ and $\tilde{\theta} \in [0, \Delta]$, where $\Delta > 0$ is an arbitrary constant such that $\theta + \Delta \in [-\tau, 0]$. Set $E_s \cdot := E(\cdot|\mathcal{F}_s)$, $s \geq 0$. By the Itô isometry, for any $t \geq \tau$, we obtain from (4.2) that
\[
E_{t+\theta}|X_t(\theta + \tilde{\theta}) - X_t(\theta)|^2 = E_{t+\theta}|X(t + \theta + \tilde{\theta}) - X(t + \theta)|^2
\]
\[
\leq c \int_{t+\theta}^{t+\theta+\Delta} E_{t+\theta} \left\{ |b(s, X_s)|^2 + \int_{\Gamma} |\sigma(s, X_{s-}, z)|^2 m(dz) \right\} ds.
\]
By virtue of (B1)-(B2) and (4.3), there is a $\gamma_0(t, \Delta)$ satisfying
\[
E_{t+\theta}|X(t + \theta + \tilde{\theta}) - X(t + \theta)|^2 \leq E_{t+\theta}\gamma_0(t, \Delta).
\]
Taking expectation and $\limsup_{t \to \infty}$ followed by $\lim_{\Delta \to 0}$, we obtain from (B1)-(B2) and (4.3) that
\[
\lim \limsup_{\Delta \to 0} E\gamma_0(t, \Delta) = 0.
\]
In view of (4.3) and (4.7), combining with [16, Theorem 3, p.47], we conclude that $X_t$ is tight under the Skorohod metric $d_S$. For each integer $n \geq 1$, set
\[
\mu_n(\cdot) := \frac{1}{n} \int_{0}^{n} \mathbb{P}_t(\xi, \cdot) dt,
\]
where $\mathbb{P}_t(\xi, \cdot)$ is the Markovian transition kernel of $X_t(\xi)$. Since $X_t$ is tight under the Skorohod metric $d_S$, for any $\varepsilon > 0$ there exists a compact subset $U \in \mathcal{B}(\mathcal{D})$ such that $\mathbb{P}(X_t \in U) \leq 1 - \varepsilon$. Hence we have $\mu_n(U) \leq 1 - \varepsilon$. That is, $\{\mu_n(\cdot)\}_{n \geq 1}$ is tight. Observe
from Reiß et al. [26] that $P_t$ is eventually Feller. As a result, by the Krylov-Bogoliubov theorem [10, Theorem 3.1.1, p.21], we conclude that \( (4.2) \) has a unique invariant measure $\pi(\cdot) \in \mathcal{P}(\mathcal{D})$, where $\mathcal{D}$ is equipped with the Skorohod topology.

**Step 3:** Exponential Mixing. By the Halanay-type inequality \((2.3)\), one has from (B1) that

\[
\mathbb{E}|X(t, \xi) - X(t, \eta)| \leq c e^{-\lambda t}
\]

for some $\lambda > 0$. Carrying out a similar argument to get \((2.19)\), we derive from \((4.8)\) that

\[
\mathbb{E}\|X_t(\xi) - X_t(\eta)\|_{\infty} \leq c e^{-\lambda t}.
\]

By the definition of $d_S$, note that

\[
d_S(\xi, \eta) \leq \|\xi - \eta\|_{\infty}
\]

by choosing $\lambda(t) \equiv t$ in \((4.1)\). For any bounded Lipschitz function $F : \mathcal{D} \to \mathbb{R}$, by the invariance of $\pi(\cdot) \in \mathcal{P}(\mathcal{D})$, it follows from \((4.10)\) that

\[
|P_t F(\xi) - \pi(F)| \leq \int_{\mathcal{D}} |P_t F(\xi) - P_t F(\eta)| \pi(d\eta) \leq c \mathbb{E} d_S(X_t(\xi), X_t(\eta)) \\
\leq c \mathbb{E}\|X_t(\xi) - X_t(\eta)\|_{\infty}.
\]

Consequently, the desired assertion follows from \((4.9)\) and a monotone class argument for any $F \in \mathcal{B}_b(\mathcal{D})$. \(\square\)

**Remark 4.1.** Since the semigroup $P_t$ generated by the segment process associated with \((4.2)\) is not stochastically continuous for $t \in [0, \tau)$ (see e.g. [26, p.1416]), $P_t$ is not immediately Feller. Hence, $t \geq \tau$ imposed in Theorem 4.1 is natural. However, Theorem 2.5 and Theorem 3.2 hold respectively for any $t \geq 0$. This further shows the different features of retarded SDEs with continuous sample paths and the ones driven by jump processes.

**Remark 4.2.** By a remote start method (or dissipative method), Bao et al. [2] discussed ergodic property for several class of functional SDEs, which cannot cover the equations considered in this paper. In particular, the remote start method applied therein only deals with the autonomous functional SDEs, while the approaches adopted in Theorem 2.5 and Theorem 4.1 even work for non-autonomous cases.

**Remark 4.3.** Since the space $\mathcal{D}$ is complete but not separable under the uniform metric, the Arzelà-Ascoli method adopted in Theorem 2.5 is unavailable for functional SDEs with jumps. Moreover, Kurtz’s criterion used in Theorem 4.1 also applies to infinite-dimensional semi-linear retarded SPDEs driven by jump processes. Hence our method is dimension-free while the trick used in Bo and Yuan [1] is dimensional-dependent.

**References**

[1] Applebaum, D., *Lévy processes and stochastic calculus*, 2nd Ed., Cambridge University Press, Cambridge, 2009.
[2] Bao, J., Yin, G., Yuan, C., Ergodicity for Functional Stochastic Differential Equations, Preprint.

[3] Billingsley, P., *Convergence of probability measures*, J. Wiley & Sons, New York, 1968.

[4] Bo, L., Yuan, C., Invariant measures of reflected stochastic delay differential equations with jumps, arXiv:1301.0442.

[5] Bakhtin, Y., Mattingly, J. C., Stationary solutions of stochastic differential equations with memory and stochastic partial differential equations, *Commun. Contemp. Math.*, 7 (2005), 553–582.

[6] Dong, Z., Xu, L., Zhang, X., Exponential ergodicity of stochastic Burgers equations driven by α-stable processes, arXiv:1208.5804v2.

[7] Es-Sarhir, A., Scheutzow, M., van Gaans, O., Invariant measures for stochastic functional differential equations with superlinear drift term, *Differential Integral Equations*, 23 (2010), 189–200.

[8] Ikeda, N., Watanable, S., *Stochastic Differential Equations and Diffusion Processes*, North-Holland, New York, 1989.

[9] Itô, K., Nisio, M., On stationary solutions of a stochastic differential equation, *J. Math. Kyoto Univ.*, 4–1 (1964), 1-75.

[10] Da Prato, G., Zabczyk, J., *Ergodicity for infinite-dimensional systems*, In: London Mathematical Society, Lecture Note Series, vol. 229, Cambridge University Press, Cambridge, 1996.

[11] Hairer, M., Ergodicity of stochastic differential equations driven by fractional Brownian motion, *Ann. Probab.*, 33 (2005), 703–758.

[12] Hairer, M., Mattingly, J. C., Scheutzow, M., Asymptotic coupling and a general form of Harris’ theorem with applications to stochastic delay equations, *Probab. Theory Related Fields*, 149 (2011), 223–259.

[13] Karatzas I., Shreve, S., *Brownian Motion and Stochastic Calculus*, Graduate Texts in Mathematics, vol. 113, Springer Verlag, New York, 1988.

[14] Kinnally, M. S., Williams, R. J., On existence and uniqueness of stationary distributions for stochastic delay differential equations with positivity constraints, *Electron. J. Probab.*, 15 (2010), 409–451.

[15] Kulik, A. M., Exponential ergodicity of the solutions to SDE’s with a jump noise, *Stochastic Process. Appl.*, 119 (2009), 602–632.

[16] Kushner, H. J., *Approximation and Weak Convergence Methods for Random Processes, with Applications to Stochastic Systems Theory*, MIT Press, Cambridge, MA, 1984.

[17] Liu, K., Stationary solutions of retarded Ornstein-Uhlenbeck processes in Hilbert spaces, *Statist. Probab. Lett.*, 78 (2008), 1775–1783.

[18] Mao, X., *Stochastic differential equations and applications*, 2nd Ed., Horwood Publishing Limited, Chichester, 2008.
[19] Mattingly, J. C., Stuart, A. M., Higham, D. J., Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise, *Stochastic Process. Appl.*, 101 (2001), 185–232.

[20] Mohamad, S., Gopalsamy, K., Continuous and discrete Halanay-type inequalities, *Bull. Austral. Math. Soc.*, 61 (2000), 371–385.

[21] Mohammed, S-E. A., *Stochastic Functional Differential Equations*, Pitman, Boston, 1984.

[22] Odasso, C., Exponential mixing for stochastic PDEs: the non-additive case, *Probab. Theory Related Fields*, 140 (2008), 41–82.

[23] Priola, E., Shirikyan, A., Xu, L., Zabczyk, J., Exponential ergodicity and regularity for equations with Lévy noise, *Stoch. Proc. Appl.*, 1 (2012), 106–133.

[24] Protter, P. E., *Stochastic integration and differential equations*, 2nd Ed., Springer-Verlag, Berlin, 2004.

[25] Rey-Bellet, L., Ergodic properties of Markov processes, Open quantum systems, II, 1–39, Lecture Notes in Math., 1881, Springer, Berlin, 2006.

[26] Reiβ, M., Riedle, M., van Gaans, O., Delay differential equations driven by Lévy processes: stationarity and Feller properties, *Stochastic Process. Appl.*, 116 (2006), 1409–1432.

[27] Scheutzow, M., Exponential growth rate for a singular linear stochastic delay differential equation, [arXiv:1201.2599v1](https://arxiv.org/abs/1201.2599).

[28] Veretennikov, A. Yu., On polynomial mixing bounds for stochastic differential equations, *Stochastic Process. Appl.*, 70 (1997), 115–127.

[29] Wang, J., On the exponential ergodicity of Lévy-driven Ornstein–Uhlenbeck processes, *J. Appl. Probab.*, 49 (2012), 990–1004.

[30] Yuan, C., Zou, J., Mao, X., Stability in distribution of stochastic differential delay equations with Markovian switching, *Systems Control Lett.*, 50 (2003), 195–207.

[31] Zhang, X., Exponential ergodicity of non-Lipschitz stochastic differential equations, *Proc. Amer. Math. Soc.*, 137 (2009), 329–337.