ON THE HEAT KERNEL IN COVARIANT BACKGROUND GAUGE

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Abstract

The first three coefficients in an expansion of the heat kernel of a nonminimal nonabelian kinetic operator taken in an arbitrary background gauge in arbitrary space-time dimension are calculated.

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1 Introduction

In this letter we present an explicit calculation of the first three coefficients in the quasiclassical expansion of the heat kernel of the gluon kinetic operator taken in the arbitrary covariant background field gauge. The importance of the heat kernel can hardly be underestimated (see, e.g, the recent review [1]). In quantum field theory the heat kernel is an extremely convenient technical tool for computing the Green functions of the particles propagating in the external fields and of the quantum corrections to the classical action of the theory. In this way one can compute the contributions to the charge renormalization and anomalies (see, e.g., [2]). The attractiveness of the heat kernel method is also due to its important role in the differential geometry, where the heat kernel is an exponentiated covariant Laplacian on some manifold. The expansion of this exponential gives the invariants of the considered manifold. In particle physics the Laplacian on the manifold is nothing else but a kinetic operator of the particle propagating on it. Unfortunately, in the general case one encounters more complicated operators - for example, the gluon kinetic operator in a general covariant background gauge reduces to a covariant Laplacian only in the Feynman gauge - this being the reason for its convenience in the calculations in nonabelian external fields. In a general covariant gauge one deals with the so-called nonminimal operators. The computation of the corresponding heat kernel becomes much more involved, and only a very limited information on its expansion coefficients is available. The first two coefficients for the case of gravity coupled to an abelian field was considered in [3], the trace of the fourth coefficient for Yang-Mills theory in four dimensions was computed in [4], and that in the high derivative gravity in four dimensions in [5].

Below we propose a method allowing an explicit calculation of the heat kernel expansion coefficients for the non-minimal operators in an arbitrary dimensional space-time and compute the first three Seeley coefficients for the nonabelian kinetic operator taken in an arbitrary background gauge. We shall see, that the dependence of the Seeley coefficients on the quantum gauge fixing parameter and the space-time dimension is strongly interrelated.

The plan of the paper is the following. In the first section we introduce the necessary notations and describe the calculational method. In the second one we present the results for the heat kernel expansion coefficients and comment on their relevance to the effective action calculation. We conclude by summarizing the results.
2 Calculational method

We begin with introducing the basic notations. Let us consider a second order elliptic operator $\mathcal{W}$ on the $2\omega$-dimensional manifold $\mathcal{M}$. By definition the heat kernel operator corresponding to $\mathcal{W}$ is obtained by its exponentiation:

$$K(s) = \exp(-\mathcal{W}s)$$ (1)

We shall be interested in the expansion of the trace of the matrix elements of the heat kernel taken at one space-time point. For the second order operator this expansion has a form (see, e.g., [2]):

$$Tr < x|K(s)|x> = \sum_{k=0}^{\infty} b(\mathcal{W}, \omega|x)s^{-\omega+k}$$ (2)

where the coefficients $b_{-\omega+k}$ are the so-called Seeley coefficients. These coefficients are the invariants of the manifold $\mathcal{M}$ and the trace of the heat kernel can be considered as a generating function for these invariants. In the often considered standard case the operator $\mathcal{W}$ is a covariant Laplacian on $\mathcal{M}$. Let us mention that in this case the coefficients $b_{-\omega+k}$ depend on $\omega$ only trivially. We shall consider a more complicated case and analyse the heat kernel expansion for the kinetic operator of Yang-Mills particles taken in an arbitrary covariant background gauge

$$\mathcal{W}_{\mu\nu}^{ab} = -D^2(A)^{ab}\delta_{\mu\nu} - 2f^{acb}G_{\mu\nu}^c - \left(\frac{1}{\alpha} - 1\right)D^{ac}D^{cb}_{\mu\nu}$$ (3)

where $f^{abc}$ are the structure constants of a corresponding Lie algebra, $D_{\mu}(A)$ is a covariant derivative containing the external field potential $A_{\mu}$, and $G_{\mu\nu}$ is a corresponding field strength. The coefficients of the heat kernel expansion

$$Tr < x|e^{-\mathcal{W}_{\mu\nu}^{ab}s}|x> = \sum_{k=0}^{\infty} b_{-\omega+k}(G_{\mu\nu}, \omega, \alpha|x)s^{-\omega+k}$$ (4)

are gauge invariant with respect to the transformations of the external field potentials $A_{\mu}^a$. Our main interest is to trace their dependence on the space-time dimension $2\omega$ and the quantum gauge fixing parameter $\alpha$. To calculate the functional trace in (4) we shall use the basis of plane waves (see, e.g., [7,8]):

$$Tr < x|e^{-\mathcal{W}_{\mu\nu}^{ab}s}|x> = Tr \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} e^{-ipx} < x|e^{-\mathcal{W}_{\mu\nu}^{ab}s}|x> e^{ipx}$$

where the trace is performed over the Lorentz and colour indices. The $\exp[ipx]$ should be pushed through the operator to the left then cancelling $\exp[-ipx]$, all differentiation
operators in $\mathcal{W}$ becoming shifted: $\partial_\mu \rightarrow \partial_\mu + ip_\mu$. Thus we have

$$Tr\; e^{-\mathcal{W}s} = Tr\int \frac{d^4p}{(2\pi)^4} e^{-s\mathcal{W}(\partial_\mu \rightarrow \partial_\mu + ip_\mu)}$$

(5)

where the operator in the right hand side of (5) acts on 1. In the considered case $\mathcal{W}(\partial_\mu \rightarrow \partial_\mu + ip_\mu) = \mathcal{W}_0 - i\mathcal{W}_1 - \mathcal{W}_2$, where we have introduced the following notation:

$$\mathcal{W}_0^{\mu\nu} = p^2 (P^{\mu\nu}_\perp + \frac{1}{\alpha} P^{\mu\nu}_\parallel), \quad P^{\mu\nu}_\perp = \delta^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2}, \quad P^{\mu\nu}_\parallel = \frac{p^{\mu}p^{\nu}}{p^2}$$

and the colour indices can be trivially restored, for example $G^{\mu\nu} \rightarrow G_{ab}^{\mu\nu} = f^{acb}G_c^{\mu\nu}$.

To obtain the expansion of $Tr e^{-(-\mathcal{W}_0 + i\mathcal{W}_1 + \mathcal{W}_2)s}$ in $s$ we use an ordinary perturbation theory:

$$Tr e^{(-\mathcal{W}_0 + i\mathcal{W}_1 + \mathcal{W}_2)s} = Tr K^0_0(s) + \int_0^s ds_1 Tr(K_0(s - s_1)(i\mathcal{W}_1 + \mathcal{W}_2)K_0(s_1)) +$$

$$\int_0^s ds_1 \int_0^{s_1} ds_2 Tr(K_0(s - s_1)(i\mathcal{W}_1 + \mathcal{W}_2)K_0(s_1 - s_2)(i\mathcal{W}_1 + \mathcal{W}_2)K_0(s_2)) + \ldots$$

(7)

where $K_{0}^{\mu\nu}(s)$ is a free propagator

$$K_{0}^{\mu\nu}(s) = e^{-\mathcal{W}_0^{\mu\nu}s} = e^{-sp^2}(P^{\mu\nu}_\perp + e^{-s\beta p^2}P^{\mu\nu}_\parallel)$$

(8)

The expressions for the Seeley coefficients are obtained by collecting the terms of a given order in covariant derivatives.

### 3 Seeley coefficients

In this section we shall compute the first three Seeley coefficients for the operator (3) acting on a $2\omega$-dimensional space-time manifold $R^{2\omega}$. The first coefficient $b_{-\omega}$ is read off from the zeroth order term in the covariant derivatives in (7). We have

$$Tr \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} (e^{-sp^2}P^{\mu\nu}_\perp + e^{-s\beta p^2}P^{\mu\nu}_\parallel) =$$

$$\frac{1}{2^{2\omega} \pi^{\omega}} \frac{N_c^2 - 1}{\left[2\omega - [1 - \alpha]\right]}$$

(9)

where $N_c$ is number of colours thus getting for the first Seeley coefficient

$$b_{-\omega} = \frac{1}{2^{2\omega} \pi^{\omega}} (N_c^2 - 1)(2\omega - [1 - \alpha])$$

(10)
Collecting the terms proportional to $D^2_{\mu}$, we obtain for the second Seeley coefficient

$$b_{-\omega+1} = \frac{1}{2\omega\pi^2}(2\omega - [1 - \alpha^{\omega-1}])\left\{ \frac{\Gamma(\omega) - \frac{1}{2}\Gamma(\omega + 1)}{\Gamma(\omega)} \right\} N_{\mu}A^a_{\mu}(x)A^a_{\mu}(x) = 0 \quad (11)$$

due to a well known relation for the $\Gamma$ functions. This had to be expected because this contribution corresponds to a gluon mass and is therefore not gauge invariant with respect to the external field transformation. Eq. (11) proves it in all dimensions and arbitrary covariant gauge.

The calculation of the term of fourth order in the covariant derivatives is straightforward but very tedious. The computation was performed using the REDUCE package. The result reads

$$b_{-\omega+2} = \frac{1}{2\omega\pi^2}(2 - \frac{2\omega}{12} + \frac{1}{12}[1 - \alpha^{\omega-2}])N_{\mu}G^{a}_{\mu\nu}(x)G^{a}_{\mu\nu}(x), \quad (12)$$

which is remarkably independent of $\alpha$ at $\omega = 2$. Taking $\alpha = 1$ (Feynman gauge) we restore the corresponding expression from [2]. The formula (12) is the main result of our paper.

Now when we have obtained the explicit expressions for the first three Seeley coefficients $b_{-\omega+k}, k = 0, 1, 2$ it is instructive to analyze their dependence on the quantum gauge fixing parameter $\alpha$. From (10-13) we see, that in all the examined orders the $\alpha$-dependent contributions to the heat kernel expansion have a form

$$(1 - \alpha^{\omega-k})s^{-\omega+k} = s^{-\omega+k} - \left(\frac{s}{\alpha}\right)^{-\omega-k} \quad (13)$$

Let us now recall that the (unregularized) contribution to the effective action originating from the kinetic operator has a form

$$W[A] = \frac{1}{2}Tr\log W = -\frac{1}{2} \int_0^\infty ds Tr e^{-Ws} \quad (14)$$

Thus the $\alpha$-dependent contribution to the effective action is proportional to

$$\int \frac{ds}{s}(s^{-\omega+k} - \left(\frac{s}{\alpha}\right)^{-\omega+k}) \quad (15)$$

We see that if (15) would be the final expression, the answer would not depend on $\alpha$ because of the invariance of (15) with respect to the proper time rescaling. However this scaling invariance is explicitly broken by the regularization procedure. At $\omega = 2$ the charge renormalization contribution from (12) will contain contribution proportional to $\log \alpha$ in the finite part (in agreement with [4]). This brings in a question of determining a background Landau gauge ($\alpha = 0$) in this situation. It seems that the only way to do it is to include the substraction of the terms proportional to $\log \alpha$ in the definition of the renormalization procedure (the choice of a scheme). After such a substraction one can take a limit of $\alpha \to 0$. Landau gauge is of a considerable interest because it corresponds to calculation of the quantum gauge invariant effective action [9].

5
4 Conclusion

We have proposed an algebraically convenient procedure of calculating the seeley coefficients for the nonminimal kinetic operator for the Yang-Mills particles taken in the arbitrary covariant background gauge for arbitrary space-time dimension $2\omega$ and have computed the first three coefficients in this expansion. We observe that the answers have a universal structure corresponding to the rescaling of the proper time in some terms by the quantum gauge fixing parameter $\alpha$. The consequences for the computation of the corresponding terms in the effective action are briefly discussed. We think that it would be very interesting to extend this result to higher orders in the heat kernel expansion. Because of the purely algebraical nature of the proposed procedure we can expect this extension to be straightforward (though very tedious). One of the interesting questions in looking at the higher order terms is the interrelation between the quantum gauge fixing parameter and the infrared cutoff that is often introduced to define otherwise infrared divergent higher order contributions to the effective action.

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6 References

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