The 2D Disk Structure with Advecitive Transonic Inflow–Outflow Solutions around Black Holes

Rajiv Kumar\(^1\)\(^2\) and Wei-Min Gu\(^1\)\(^2\)

\(^1\)Department of Astronomy, Xiamen University, Xiamen, Fujian 361005, People’s Republic of China; kumar@xmu.edu.cn
\(^2\)Jiujiang Research Institute of Xiamen University, Jiujiang 332000, People’s Republic of China; guwm@xmu.edu.cn

Received 2018 March 11; revised 2018 May 4; accepted 2018 May 6; published 2018 June 19

Abstract

We solved analytically viscous two-dimensional (2D) fluid equations for accretion and outflows in spherical polar coordinates \((r, \theta, \phi)\) and obtained explicitly flow variables in \(r\)- and \(\theta\)-directions around black holes (BHs). We investigated global transonic advection-dominated accretion flow (ADAF) solutions in an \(r\)-direction on an equatorial plane using Paczyński–Wiita potential. We used radial flow variables of ADAFs with symmetric conditions on the equatorial plane as initial values for integration in the \(\theta\)-direction. In the study of 2D disk structure, we used two azimuthal components of viscous stress tensors—namely, \(\tau_{r\theta}\) and \(\tau_{\theta\phi}\). Interestingly, we found that the whole advective disk does not participate in outflow generation, and the outflows form close to the BHs. Normally, outflow strength increased with increasing viscosity parameter \((\alpha_1)\), mass-loss parameter \((s)\), and decreasing gas pressure ratio \((\beta)\). The outflow region increased with increasing \(s, \alpha_1\) for \(\tau_{r\theta}\) and decreasing \(\alpha_2\) for \(\tau_{\theta\phi}\). The \(\tau_{\theta\phi}\) is effective in angular momentum transportation at high latitude and outflows collimation along an axis of symmetry, since it changes polar velocity \((v_\theta)\) of the flow. The outflow emission is also affected by the ADAF size and decreases with it. Transonic surfaces formed for both inflows \((v_r < 0, \text{ very close to BH})\) and outflows \((v_r > 0)\). We also explored no outflows, outflows, and failed outflows regions, which mainly depend on the viscosity parameters.

Key words: accretion, accretion disks – black hole physics – hydrodynamics

1. Introduction

An accretion disk is associated with many astrophysical objects (e.g., compact objects; black holes, neutron stars, and white dwarfs) and young stellar objects. Accreting gas onto these objects can generate radiation and bipolar outflows/jets due to an extraction of its gravitational energy. These objects are often associated with non-relativistic to relativistic bipolar jets. In particular, the relativistic jets have been observed around accreting black hole candidates (BHs)—for instance, active galactic nuclei (AGNs) and black hole X-ray binaries (BHXBs). The jets from the AGN M87 emerge from an extremely small central region of a source within \(100r_S\) (Joner et al. 1999), but recent observation shows an even smaller region, less than \(10r_S\) (Doeleman et al. 2012), where \(r_S = 2GM_*/c^2\) is a Schwarzschild radius. AGNs and BHXBs are believed to harbor supermassive BHs \(\sim 10^8-10^9M_\odot\) and stellar mass BHs \(\sim 10M_\odot\) at the center, respectively. \(M_\odot\) denotes the mass of the Sun. Moreover, the BHXBs also show typically two types of spectral states in their observations: one high soft state, which is radiatively efficient and dominated by thermal radiation with blackbody spectrum in soft X-ray regime, and a second low hard state, which is radiatively inefficient and dominated by non-thermal radiation with some power law spectrum in the hard X-ray regime (Remillard & McClintock 2006). These two states are also connected with many intermediate states and, interestingly, bipolar jets and quasi-periodic oscillations are associated with the hard state in the BHXBs (Gallo et al. 2003; Fender et al. 2004). However, such changes in the spectral states are yet to be observed for the AGNs. Since the timescales of AGNs and BHXBs can be scaled by the mass of the BHs, but inner boundary conditions are same, the basic physics of both kind of objects can be similar (McHardy et al. 2006). In this context, there are several theoretical and numerical studies on accretion processes with Keplerian/sub-Keplerian flows, and that can play an important role in the generation of soft spectrum (Novikov & Thorne 1973; Shakura & Sunyaev 1973; Abramowicz et al. 1988) and hard spectral states (Sunyaev & Titarchuk 1980; Chakrabarti & Titarchuk 1995; Narayan & Yi 1995; Narayan & Yi 1995a; Molteni et al. 1996a, 1996b; Igumenshchev & Abramowicz 1999, 2000; Ohsuga et al. 2005; Okuda et al. 2007; Das et al. 2014; Yang et al. 2014; Bu et al. 2016a, 2016b; Lee et al. 2016; Jiang et al. 2017; Kumar & Chattopadhyay 2017). The mechanism of jet generation and its evolution is still unclear and a topic of active research in the fields of theory and observations.

The analytical study of the 2D disk with outflows began with a relaxation of vertical hydrostatic equilibrium in the disk by Narayan & Yi (1995a). They used self-similar ADAF solutions (Narayan & Yi 1994) in the radial direction with symmetry conditions on the equatorial plane and solved the flow variables along the polar direction. However, they could not get actual outflow solutions because they assumed the mass accretion rate is independent of radial distance and thus polar velocity, \(v_\theta = 0\). They have found in their solutions that the Bernoulli parameter is positive; thus the outflows may form close to a rotation axis. Subsequently, Xu & Chen (1997) have included the \(v_\theta \) non-zero in their study and found accretion and ejection solutions. After that, theoretical studies of the 2D disk with self-similar solutions along the radial directions have been carried out by many authors with the outflows (Xue & Wang 2005; Gu et al. 2009; Jiao & Wu 2011; Gu 2012, 2015) and without outflows (Habibi et al. 2017). Similar studies have also been done in the magneto-hydrodynamics regime (Mosallanezhad et al. 2016; Samadi & Abbassi 2016; Zeraatgari et al. 2018). In addition, an analytical study for
the outflows with self-similar solutions in one-dimensional flow has done by Blandford & Begelman (1999), assuming the mass accretion rate varies with some power of the radial distance. These outflow solutions are known as adiabatic inflow–outflow solutions (ADIOS). Further, they have presented their work with a family of two-dimensional self-similar solutions for the outflows (Blandford & Begelman 2004). Simultaneously, many numerical simulations for the investigations of accretion-ejection have been also done with optically thin hot flows by Stone et al. (1999), Igumenshchev & Abramowicz (1999, 2000), Yuan et al. (2012a, 2012b), Bu et al. (2016a, 2016b), a review by Yuan & Narayan (2014), and with optically thick, hyper accreting flows by Ogusa et al. (2005), Okuda et al. (2007), Yang et al. (2014), Jiang et al. (2014, 2017), Jiao et al. (2015). There are a few more models for explanation of jet generation by the extraction of rotational energy of Kerr BHs (Blandford & Znajek 1977), by anchoring of matter with magnetic lines (Blandford & Payne 1982) and by shock generated extra-thermal gradient force in the post-shock region (Molteni et al. 1996a, 1996b; Kumar & Chattopadhyay 2013; Chattopadhyay & Kumar 2016; Lee et al. 2016; Kumar & Chattopadhyay 2017).

The self-similar solution has gained popularity for the analytical studies of the 2D disk with/without outflows in both the hydrodynamics (HD) and magneto-hydrodynamics (MHD) regimes because it simplifies fluid equations in the radial direction, which makes ODEs with only polar derivatives or independent of radius and radial derivatives. Gu (2012, 2015) has mentioned that the outflows form naturally from the advective accretion disk for both optically thin and thick gas media. Moreover, the hot flows with bremsstrahlung, synchrotron emissivity, and Comptonization of soft photons can give rise to the hard spectrum of the BHCs (Chakrabarti & Titarchuk 1995; Narayan & Yi 1995b; Yuan & Narayan 2014).

The bipolar jets have been seen during the hard state with radiatively inefficient flows around the BHCs (Remillard & McClintock 2006). Therefore, we assumed the inner part of the disk is hot advective and also assumed radiatively inefficient accretion flows for the study of jet generations around the BHs. In addition, we assumed a two zone configuration of the accretion disk (Esin et al. 1997; Das & Sharma 2013): one inner part, with hot sub-Keplerian advective radiative inefficient accretion flows (RIAFs), and one outer part, which is geometrical thin and cool Keplerian optically thick (Shakura–Sunyaev disk). For the time being, we did not consider radiative emissivities in the flow since this work mainly focuses on the study of jet generation in the 2D disk, and thus we leave a full consideration of radiative advective flows for future work.

The present paper is based on the study of the structure of an accretion disk in the 2D flow with outflows and extension of previous studies (Narayan & Yi 1995a; Xu & Chen 1997; Blandford & Begelman 1999; Xue & Wang 2005; Xie & Yuan 2008; Jiao & Wu 2011). In this paper, there are the following components that vary from the previous 2D analytical studies, since we want to study inflow–outflow structure close to the BH: First, we used pseudo-Newtonian potential (Paczynski & Wiita 1980), which incorporates general relativistic effects very close to the BH. Second, we considered two azimuthal components of viscous stress tensor (Stone et al. 1999; Yuan et al. 2012b) from out of nine components (Xue & Wang 2005), since it is mostly believed that the azimuthal component of magnetic stress is more important in the angular momentum transfer by the study of magneto-rotational instability (MRI) simulations (Balbus & Hawley 1998). The present study is axisymmetric, with the 2D HD rotating flow, so we used anomalous shear stress, which can approximate the magnetic stress, following Stone et al. (1999) and Yuan et al. (2012b), assuming two-azimuthal components of the viscosity are non-zero. Their effects on the disk structures have also been discussed in the simulation by Yang et al. (2014). Third, we have investigated global transonic ADAF solutions on the equatorial plane, with their flow variables used as the boundary conditions for the integration of differential equations in $\theta$-direction. When doing this, the fluid ODEs are still dependent on the radius of the disk and other radial derivatives, unlike when using the self-similar assumptions (Xue & Wang 2005; Jiao & Wu 2011). In present study, our main interest is to investigate the inflow–outflow structure close to the BHs with changing various flow parameters—namely, disk viscosity parameters ($\alpha_1$ and $\alpha_2$), grand specific energy ($E$), gas pressure ratio ($\beta$), and mass-loss parameter ($s$) in the fluid flows. This paper is structured as follows: Section 2 focuses on model fluid equations and assumptions, Section 3 provides a solution procedure, Section 4 provides numerical results, and the final section presents conclusions of our study.

2. Model Fluid Equations and Assumptions

We considered viscous hydrodynamic fluid equations for advective accretion-outflow solutions with steady-state and axisymmetric in the spherical polar coordinates ($r$, $\theta$, $\phi$). We assumed pseudo-Newtonian geometry $\Phi = -GM_{bh}(r - rs)$ (Paczynski & Wiita 1980) around the Schwarzschild BHs. For time being, we are ignoring the magnetic field in the accretion and outflows. We represented the viscous fluid equations and the flow variables in a geometrical unit system and chose $2G = M_{bh} = c = 1$, where $M_{bh}$, $G$, and $c$ are mass of the BH, universal gravitational constant, and speed of the light, respectively. Therefore, units of a length, length velocity (or sound speed), energy, specific angular momentum, mass, density, pressure, and time are $GM_{bh}/c^2$, $c$, $M_{bh}c^2$, $GM_{bh}/c$, $M_{bh}$, $c^3/(8G^2M_{bh}^2)$, $c^3/(8G^3M_{bh}^3)$, and $GM_{bh}/c^3$, respectively. We also assumed that the two components of viscous stress tensor are effective in the $r - \theta$ plane, which are $\tau_{r\theta}$ and $\tau_{\theta\phi}$, as in Stone et al. (1999). Thus, the conserved form of the fluid equations in the 2D become as follows: the continuity equation,

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho \sin \theta \dot{v}_\theta) = 0,$$

(1)

the components of Navier–Stokes equation,

$$v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta \partial v_r}{r \partial \theta} - \frac{v_r^2 + v_\theta^2}{r} + \frac{1}{\rho} \frac{\partial P}{\partial r} - F_r = 0,$$

(2)

$$v_\theta \frac{\partial v_\theta}{\partial r} + \frac{v_\theta \partial v_\theta}{r \partial \theta} - \frac{v_r v_\theta}{r} - \frac{v_\theta^2 \cot \theta}{r} + \frac{1}{r \rho} \frac{\partial P}{\partial \theta} = 0,$$

(3)

$$v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta \partial v_\phi}{r \partial \theta} + \frac{v_\phi v_r}{r} (v_r + v_\theta \cot \theta) = \frac{1}{r \rho} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{\partial \tau_{\theta\phi}}{\partial \theta} + 2 \tau_{\theta\phi} \cot \theta \right].$$

(4)
and the energy equation,
\[
\rho \left[ \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \rho \frac{F}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\rho}{r} \frac{\partial p}{\partial \theta} \right] = f Q^+, \tag{5}
\]
where \( Q^+ = \tau_{\phi \phi} / \eta_1 + \tau_{b0} / \eta_2 \) is viscous heating rate and \( f \) is advection factor (Narayan & Yi 1995a). For simplicity we assumed \( f \) is fixed. Since the values of \( f \) should not be arbitrary, for brevity we used only \( f = 1 \) for highly advective flow, in spite of radiation-dominated or gas-dominated flow. However, \( f \) should be determined with relevant radiation mechanisms. \( P = \rho g + \rho_{\text{rad}} \) is total pressure, \( p_g = \rho \Theta / \beta \) is gas pressure, and \( \rho_{\text{rad}} \) is radiation pressure, which could be due to blackbody emissivity for optically thick medium (Abramowicz et al. 1988) or bremsstrahlung and synchrotron emissivity for optically thin medium (Narayan & Yi 1995b). \( \Theta = k_B T / (\mu m_e c^2) \) is dimensionless temperature of the fluid, \( \beta = \mu m_p / m_e \), where \( k_B, \mu, m_p, \) and \( m_e \) are the Boltzmann constant, mean molecular weight of the gas, mass of the proton, and mass of the electron, respectively. We assumed \( \mu = 0.5 \) for fully ionized flow. \( F_t = -d \Phi / dr \) is the central attractive force around the BH. The specific internal energy (Kato et al. 2008; Jiao & Wu 2011) is
\[
\epsilon = \frac{P_g}{\rho (\gamma - 1)} + 3 \frac{\rho_{\text{rad}}}{\rho (\gamma_{\text{eff}} - 1)} \tag{6} \quad \text{(or EoS)},
\]
where \( \gamma \) is known as the adiabatic index and defined as the ratio between heat capacities. \( \gamma_{\text{eff}} = 1 / [N \beta + 3(1 - \beta)] + 1 \) is the polytropic index, and \( \beta = \rho e / \rho \) is the gas pressure ratio. The two-azimuthal components of the viscous stress tensors are written as
\[
\tau_{\phi \phi} = \eta_1 \left( \frac{\partial v_\theta}{\partial r} - \frac{v_\rho}{r} \right) \quad \text{and} \quad \tau_{b0} = \eta_2 \left( \frac{\partial v_r}{\partial \theta} - \nu_\phi \cot \theta \right), \tag{7}
\]
where \( \eta_1 = \alpha_1 P / \Omega_K \) and \( \eta_2 = \alpha_2 P / \Omega_K \) are coefficients of viscosity and \( \Omega_K = 1 / (\sqrt{2}r(r - 1)) \) is Keplerian angular velocity on the equatorial plane. Note \( \alpha_1 \) and \( \alpha_2 \) are the Shakura–Sunyaev viscosity parameters. The flow variables in the \( r - \theta \) plane are defined (Xue & Wang 2005) as
\[
\begin{align*}
\text{Mass density:} & \quad \rho(r, \theta) = \rho(\theta) \rho_2(\theta = \pi/2, r), \\
\text{Radial velocity:} & \quad v_r(r, \theta) = v_1(\theta) v_2(\theta = \pi/2, r), \\
\text{Polar velocity:} & \quad v_\rho(r, \theta) = v_\rho(\theta) v_2(\theta = \pi/2, r), \\
\text{Azimuthal velocity:} & \quad v_\phi(r, \theta) = v_\phi(\theta) v_2(\theta = \pi/2, r), \\
\text{Fluid temperature:} & \quad \Theta(r, \theta) = \Theta(\theta) \Theta_2(\theta = \pi/2, r),
\end{align*}
\tag{8}
\]
where the flow variables with “\( \theta \)” in brackets represent variation along the \( \theta \)-direction for a given \( r \); they are called polar flow variables, and corresponding derivatives will be polar flow derivatives. The flow variables with “\( r \)” in brackets are represented by variation along the radial direction, and they are called radial flow variables; corresponding derivatives will be radial flow derivatives. Here, \( v_2 = 1 / \sqrt{2}r \), as appears in previous studies (Xue & Wang 2005; Jiao & Wu 2011) and corresponding radial derivatives. Using the above definitions in Equations (1)–(5), we get ordinary differential equations (ODEs) of the 2D flows:
\[
\rho v_r \left[ \frac{2}{r} + \frac{1}{\rho_2} \frac{d \rho_2}{d \theta} + \frac{1}{v_2} \frac{dv_2}{d \theta} \right] + \rho_0 \left[ \frac{1}{\rho_1} \frac{d \rho_1}{d \theta} + \frac{1}{v_1} \frac{dv_1}{d \theta} + \cot \theta \right] = 0 \tag{9}
\]
\[
v_{r1} v_r \left[ \frac{dv_2}{dr} + v_2 \frac{dv_{11}}{d \theta} - \frac{v_1^2 + v_2^2}{r^2} \right] + \Theta \frac{1}{\beta} \frac{d \rho_2}{d \theta} + \frac{\Theta_1 \rho_2}{\beta^2} + F_r = 0, \tag{10}
\]
\[
v_r v_{r1} \frac{dv_1}{dr} + v_1 v_{r2} \frac{dv_{11}}{d \theta} + v_r v_{r0} - \frac{v_1^2 + v_2^2}{r^2} \cot \theta \tag{11}
\]
\[
\frac{v_r v_{r1} v_r}{d \theta} + \frac{v_2 v_{r2}}{d \theta} + \frac{v_{r1} v_r}{d \theta} + \frac{v_{r2} v_r}{d \theta} + \frac{v_2 v_{r0}}{d \theta} + \frac{v_1 + v_{r0}}{\rho_1 \cot \theta} \tag{12}
\]
\[
v_1 \Theta \left[ \frac{N_{\text{eff}} \Theta_2}{\Theta_1} d \Theta_2 / d \theta - \frac{1}{\rho_2} \frac{d \rho_2}{d \theta} \right] + \frac{v_0 \Theta}{\Theta_1} \frac{N_{\text{eff}} \Theta_1}{\Theta_2} d \Theta_1 / d \theta - \frac{1}{\rho_1} \frac{d \rho_1}{d \theta} = \beta \bar{f} Q^+, \tag{13}
\]
where \( \bar{f} = (d v_{11} / d \theta - v_{11} \cot \theta) \) and \( N_{\text{eff}} = 1 / (\gamma_{\text{eff}} - 1) \) is an effective polytropic index. We have solved Equations (9)–(13) in an explicit way, following a similar methodology as used in previous papers (Xue & Wang 2005; Jiao & Wu 2011). Since we are avoiding self-similar solution definitions along the radial direction, we first need to find out the radial flow variables with corresponding derivatives of the ADAF on the equatorial plane (detailed equations are presented in Appendix A); we then obtain polar flow variables using symmetric boundary conditions on the equatorial plane and finally integrate the above equations along the polar direction. Before doing so, we assume the symmetric properties with boundary conditions described in the next subsection.

2.1. Boundary Conditions for Inflow–Outflow

In order to solve ODEs (9)–(13) in the \( \theta \)-direction, we used symmetric boundary conditions at \( \theta = \pi/2 \) from the rotation axis, which are obtained from the reflection symmetry, following the previous studies (Xue & Wang 2005; Jiao & Wu 2011):
\[
v_{r1}(\pi/2) = 0 = \frac{d \rho_1(\pi/2)}{d \theta} = \frac{d \Theta_1(\pi/2)}{d \theta} = \frac{d v_1(\pi/2)}{d \theta} = \frac{d v_{11}(\pi/2)}{d \theta}; \quad \rho_1(\pi/2) = 1. \tag{14}
\]
\( v_0 \) is an evaporation velocity for the generation of outflows, so we assumed before the outflow at \( \theta = \pi/2 \) that it is zero but
becomes non-zero immediately, when matter goes upward from the equatorial plane. Therefore we treated $d v_\theta / (\pi / 2) / d \theta$ as non-zero on the equatorial plane, as represented below in Equation (18). Moreover, the outflows are started from the equatorial plane, so here we assumed total flow density at $\pi / 2$ is equal to the inflow density; thus $\rho = \rho_2$, which implies $\rho_1(\pi / 2) = 1$ over all the radius. Here the $\rho_2 = \rho_k$ is changing with the radius and also depends on $\theta_e$ and mass accretion rate ($\dot{m}$), as represented below in Equation (15). But the disk structure is independent of $\theta_e$ and $\dot{m}$. By using the above definitions, we obtained explicitly the fluid equations in the pure radial direction at $\theta = \pi / 2$ (Appendix A). Since gas can evaporate from the accretion disk to infinity (Narayan & Yi 1995a; Esin et al. 1997; Gu 2015), we assumed mass loss in the continuity equation (25), which is defined as (Blandford & Begelman 1999)

$$ M_{\text{in}} = -4\pi r^2 \rho_k v_{\text{re}} \cos \theta_e = M_{\text{K}} \left( \frac{r}{r_b} \right)^\gamma, \quad (15) $$

where $s$ is an exponent and called the mass-loss parameter, $r_b$ is the radial distance from the BHs when the disk started outflows from the equatorial plane and other quantities, and $\rho_k$, $v_{\text{re}}$ and $\theta_e$ are as denoted in Appendix A. According to the simulation paper by Ohsuga et al. (2005), $s$ is not a constant in the disk, but its average value has been estimated around 1. Since there are limitations in the analytical approach, we assumed “s” as a parameter and a constant for a particular solution. $M_{\text{K}}$ is the mass accretion rate at radius $r_b$. Here, $M_{\text{K}} = \dot{m} M_{\text{Edd}}$ and $\dot{m}$ are the dimensionless mass accretion rate. $M_{\text{Edd}} = 1.4 \times 10^{17} (M_{\text{BH}} / M_{\odot}) (2GM/c^2)$ is the Eddington mass accretion rate in the geometrical unit. Here, $s = 0$ corresponds to constant accretion rate, meaning no mass loss from the disk. Since we want to study the outflows, “s” should be greater than zero.

Now Equation (15) after differentiation can be written as

$$ \frac{2}{r} + \frac{1}{\rho_k} \frac{d \rho_k}{dr} + \frac{1}{v_{\text{re}}} \frac{dv_{\text{re}}}{dr} = s \frac{1}{\rho_k} \quad (16) $$

Since we assumed that the radial components of flow variables and their derivatives are the same for all values of the polar angle at or above the equatorial plane for a particular radius, then Equation (9) with the help of Equation (16) becomes

$$ v_{\theta, s} + \frac{1}{\rho_k} \frac{d \rho_k}{d \theta} + \frac{1}{v_{\text{re}}} \frac{dv_{\text{re}}}{d \theta} + \cot \theta = 0. \quad (17) $$

On solving the fluid Equations (10)–(13), (17) with (14) at $\theta = \pi / 2$, we still need one more boundary condition in order to get flow variables. So we assumed $d^2 v_\theta ((\pi / 2) / d \theta^2 = 0$ from the following as Equation (14). Thus, the polar flow variables on $\theta = \pi / 2 = 90$ are estimated from Equations (10)–(13), (17) with Equation (14), and after some simplifications, we get

$$ a_e v_\theta^2(90) + b_e v_\theta(90) - F_e = 0, \quad \Theta_1(90) = \frac{x_0}{x_2} v_\theta(90), \quad \Theta_2(90) = \frac{x_3}{x_4} v_\theta(90) $$

$$ v_{\theta, s}(90) = \frac{\sqrt{x_3} v_\theta(90)}{x_4} \quad \frac{dv_{\text{re}}}{d \theta} = -\frac{x_3}{x_2} v_{\text{re}}^2, \quad (18) $$

where $a_e = v_\theta^2, b_e = -v_\theta^2 x_3 / (x_4), x_0 = x_2, x_3 = x_4, x_0 = x_2, x_1 = (\Theta_1^2 + (\rho_2 / \rho_1) \rho_1^2) / (\Theta_2 x_3), x_4 = x_3 \Theta_2 v_{\text{re}} / (\Omega_{\text{K}} f_\delta r),$ and $x_2 = x_3 (N_{\text{eff}} \Theta_1 - (\Theta_1 / \rho_2) \rho_1^2), x_4 = x_3 \Theta_2 (v_{\text{re}} - v_\theta / r)^2 / \Omega_{\text{K}}.$ Here, $v_{\text{re}} = v_{\text{re}}, v_{\text{re}} = v_{\text{re}}, \Theta_1 = \Theta_e, \rho_2 = \rho_k,$ and corresponding radial derivatives $v_{\theta} = dv_{\theta} / dr, \lambda_1 = d \lambda_1 / d r v_{\text{re}} = (d \lambda_1 / dr - v_{\text{re}}) / r, \Theta_1 = d \Theta_1 / d r, \rho_1 = d \rho_1 / d r.$ Calculated from transonic ADAF solutions (Narayan et al. 1997; Lu et al. 1999) on the equatorial plane from Equations (34)–(36). Here, subscript “$e$” denotes values of the flow variables on the equatorial plane. In the next subsection we will discuss a solution procedure to find critical points (CPs) and ADAF solutions.

### 3. Solution Procedure

Since the BH accretion is necessarily transonic because of the nature of gravity around central objects, we first define and find out the CP of the accretion flow in following subsections.

#### 3.1. CP Conditions

The CP is a point of discontinuity of the differential equation and mathematically is defined as the $dv_{\text{re}} / dr \rightarrow 0 / 0$ form. So, the CP conditions are obtained from Equation (35):

$$ N = 0 \implies \left( \frac{v_{\text{re}}}{r_c} \right)^2 + F_{\text{re}} + 2(a_e c_k)^2 = 0 \quad (19) $$

and

$$ D = 0 \implies \left( v_{\theta}^2 - a_e^2 \right) k = 0. \quad (20) $$

Here, subscript “$e$” denotes the flow quantities at the CP and the radial velocity gradient at CPs obtained by the l’Hospital rule. We found CPs by satisfying Equations (19) and (20) together, with the help of the integration of Equations (34)–(36), for a given set of parameters ($E, \lambda_0, \gamma, \alpha_1, and \beta$), with detailed explanations in Appendix B. We integrated the differential Equations (34)–(36) from horizon to outward with the help of Equation (32). For this, we used a very nice technique for the calculation of asymptotic flow variables very close to the horizon, which is described in the next subsection.

#### 3.2. Method to Find Asymptotic Flow Variables

To explore for the CP location, we used the same methodology as described in many papers (Becker et al. 2008; Kumar & Chattopadhyay 2013, 2014; Kumar et al. 2014; Chattopadhyay & Kumar 2016). Using the Frobenius expansion to calculate the asymptotic value of $\lambda_e$ for the differential Equation (36), the expression is

$$ \lambda_e = \lambda_0 + \zeta (r - r_s)^{\lambda}, \quad r \to r_s, \quad (21) $$

where $\zeta$ and $\Lambda$ are constants, to be determined by Equation (36) using Equation (21):

$$ \lim_{r \to r_s} \frac{d \lambda_e}{dr} = \lim_{r \to r_s} \frac{2\lambda_e - \gamma_{\text{eff}} v_{\text{re}} \Omega_{\text{K}} \zeta (r - r_s)^{\lambda}}{\alpha_1 a_e^{\lambda}}. \quad (22) $$

Here, we assumed $v_{\text{re}} = \delta v_{\text{eff}}$ for limit $r \to r_s$ and $(d \lambda_e / dr)_{r \to r_s} = 0$, where $v_{\text{eff}} = 1 / (r - r_s)^{1 / 2}$ is free-fall velocity and $\delta < 1$. The value of $\delta$ will be obtained in iterations by
satisfying the conditions (19), (20). Using expressions of $v_{\text{rs}}$ and $\Omega_K$ in Equation (22), the above equation can be written as

$$\lim_{r \to r_s} \frac{\gamma_{\text{eff}} \delta (r-r_s) \lambda}{\alpha_3 a_{\text{se}} \sqrt{2} (r-r_s)^{3/2}} = \frac{2 \lambda_0}{r_s}$$

(23)

When eliminating all $(r-r_s)$ terms from the above equation, we require $\Lambda = 3/2$. Thus we get $\zeta = 2 \sqrt{2} \alpha_1 \lambda_0 a_{\text{se}} / (\gamma_{\text{eff}} \delta)$. For a choice of $\delta$ value, we obtained a value of $\zeta$, and then we calculated flow variables very close to the horizon, say $r = r_{\text{in}} = 1.001$. Now we can obtain values of $\lambda_s, v_{\text{rs}}$ and $\Theta_k$ at $r_{\text{in}}$ with the help of Equations (21) and (32), and then we can integrate outward fluid Equations (34)–(36) from $r_{\text{in}}$. The detailed method for finding CPs and disk structure is described in Appendix B.

4. Numerical Results

We analytically solved the 2D fluid equations by assuming explicitly radial fluid equations on the equatorial plane and first integrated along the radial direction, say at $r$ and then immediately at the same $r$, we solved along the polar direction from the equatorial plane (details in Appendix B). Since there are many analytical studies on the 2D disk structure using ADAF self-similar assumptions on the equatorial plane (Narayan & Yi 1995a; Xu & Chen 1997; Blandford & Begelman 1984; Xue & Wang 2005; Jiao & Wu 2011), we investigated the ADAF solutions on the equatorial plane for the calculations of the polar flow variables. Thus we first represent the ADAF solutions in the coming subsection and later, in the next subsection, with complete inflow–outflow solutions. In the present work, we used both extreme values of $\gamma$ or $\gamma_{\text{eff}}$: $\gamma = 5/3$, where $\gamma_{\text{eff}}$ depends on $\beta$ (which may change the disk flow variables and structure with changing $\beta$), and $\gamma = 4/3$, where $\gamma_{\text{eff}} = \gamma$ for any value of $\beta$. Here the mass inflow density and pressure of the gas have been calculated with the mass accretion rate $\dot{m} = 0.1$ and $M_{\text{bh}} = 10 M_\odot$ for all the solutions of this paper.

4.1. ADAFs Solutions

We used flow parameters to find the transonic accretion solutions on the equatorial plane: $E$, $\lambda_0$, $\gamma$, $\beta$, and $\alpha_1$. In Figure 1, we represented typical ADAF solutions as previously shown by Narayan et al. (1997) and Lu et al. (1999), which are plotted with different values of $\beta$ in the first column, values of viscosity parameter ($\alpha_1$) in the second column, and values of the grand specific energy ($E$) of the flow in the third column, which changes $r_s$. Here $r_s$ is the outer boundary of the ADAF or assumed transition radius from the Keplerian to the sub-Keplerian flows of the two zone configuration of the disk. The distribution of specific angular momentum ($\lambda_s$), bulk velocity ($v_{\text{rs}}$) with sound speed ($a_{\text{se}}$), and the Bernoulli parameter $B_e$ are plotted in panels 1(a), (d), and (g), panels 1(b), (e), and (h), and panels 1(c), (f), and (i), respectively. In panel 1(a), the value of $\lambda_s$ is lowest for $\beta = 1$ and increases with decreasing $\beta$, when keeping other parameters fixed. Since $\lambda_s$ is lower for $\beta = 1$, $v_{\text{rs}}$ and $a_{\text{se}}$ are higher in panel 1(b), $B_e$ is lower when $\beta$ is higher in panel 1(c), since the $\lambda_s$ is low, which is not compensated for by high values of $v_{\text{rs}}$ and $a_{\text{se}}$. In the second column of Figure 1, we changed $\alpha_1$ and kept the other parameters the same. In panel 1(d), values of $\lambda_s$ are higher for $\alpha_1 = 0.01$. Since angular momentum transported less for lower viscosity when $r_s$ or $E$ is the same, $v_{\text{rs}}$ and $a_{\text{se}}$ are lower in panel 1(e) and $B_e$ is also low in panel 1(f), which is not compensated by higher $\lambda_s$. In the third column of Figure 1, we plotted curves with different $E$, changing $r_s$ and keeping other parameters fixed. Here, the distributions of $\lambda_s$ are almost the same close to the BH and higher when approaching $r_s$ for lower values of $E$ in panel 1(g). The expected variation of $v_{\text{rs}}$ and $a_{\text{se}}$ is low for corresponding high values of $\lambda_s$ around $r_s$ in panel 1(h). $B_e$ is lower for lower $E$ values in panel 1(i), so this may indicate that for shorter $r_s$, the possibility of the outflows may be weak. Here, $r_s$ for different $E = -0.02$, -0.01, 0.001, and -0.00001 versus 28, 62, 705, and 74,240, respectively. We get the same power law scaling with the radius for $a_{\text{se}}$ and $\lambda_s$ as in Narayan et al. (1997) and almost independent of $r_s$, but the scaling for $v_{\text{rs}}$ is changing significantly with $r_s$ (or $E$), as in Figure 1(h)—for example, $v_{\text{rs}} \propto r^{-1}$ for $r_s \approx 700$, $v_{\text{rs}} \propto r^{-0.7}$ for $r_s \approx 70,000$, and $v_{\text{rs}} \propto r^{-0.5}$ for $r_s \approx 2 \times 10^6$ (as seen in Narayan et al. 1997). Interestingly, these scaling rules are also the same for the total of $v_{\text{rs}}, a_{\text{se}}$, and $\lambda_s$ ($=r v_{\text{rs}}$) on the equatorial plane, when calculating the 2D structures. Moreover, all the sub-Keplerian solutions have $\Theta > 1$ in the vicinity of the BHs, but around $r_s$ have $\Theta \ll 1$, and mass density will be higher at $r_s$ since $v_{\text{rs}} \sim 0$, so before the transition radius, we believe that the flow was Keplerian. The sub-Keplerian flows are showing positive $B_e$ in the intermediate values of $r_s$, which thus may give rise to outflows (Narayan & Yi 1995a). Therefore we used these sub-Keplerian hot flows for the generation of the outflows and investigated the 2D disk structures as presented in the next subsection.

4.2. Inflow–Outflow Solutions

Here outflow solutions above the equatorial plane in $\theta$-directions are calculated only up to a sonic surface when the outflow Mach number ($M = |v|/a_s$) becomes equal to one, where $v = v_r + v_0$ is the total velocity of fluid and $a_s^2 = \gamma_{\text{eff}} P/\rho$ is the sound speed. Since the sonic surface raises a kind of discontinuity in the analytical integration of differential equations, integrations are invalid after the sonic surface without using any proper methodology to solve the discontinuity. Therefore, the fate of these outflows after the sonic surface is unknown in this study, but we can predict that these disks may have strong outflows on the basis of the transonic nature of the outflows and that $M$ will reach very large quantities after crossing the sonic surface. Here we used parameters $s$ and $\alpha_2$, which are non-zero when ODEs are integrated along the polar directions. All the 2D disk figures with velocity vectors and density contours are plotted up to the radius size ($r_{\text{fs}}$), where outflows are starting to generate from the disk. Here we redefine a few flow variables in terms of their physical units (e.g., the flow density $\rho = f_p$, the gas pressure $p_g = f_{p_g}$, and the flow temperature $T = \Theta_f T_f$). Also, $f_p = 7.75 \times 10^{10}(\text{m}/\text{g}) \text{ cm}^{-3}$ and $f_{p_g} = 6.98 \times 10^{37}(\text{m}/\text{g}) \text{ cm}^{-2}$, where $m$ and $m$ are the accretion rates in units of the Eddington accretion rate and mass of the BH in unit of the solar mass, respectively, and $f_T = 5.93 \times 10^9 \text{ K}$.

Figure 2 represents the 2D disk structures with velocity vector fields and density contours. The first row of Figure 2 is plotted with different $\beta = 1$ (panel 2(a)), 0.5 (panel 2(b)), and 0.1 (panel 2(c)), which are corresponding radial input solutions of the first column of Figure 1. Disk thickness is high for panel 2(a) in the same row. The disk thickness decreases
with decreasing $\beta$ toward panel 2(c), which is radiation-dominated for $\beta = 0.1$. Since the flow is accelerated more with decreasing $\beta$, the gas formed the disk surface at lower latitude and the radiation-dominated flows may have strong outflows. Moreover, the inflow matter becomes supersonic close to the horizon after the “+” symbol blue-colored line (where $v_i < 0$ and $M = 1$). Here, the disk surface ($v_i = 0$ and $M < 1$) and outflow sonic surface ($v_i > 0$ and $M = 1$) are represented with solid black lines and dotted green lines, respectively. The disk surface (solid line) separated the inflow and outflow regions in the disk. The velocity vectors are mostly directed toward the BH in the inflow region, and in the outflow region, they are going out. Here we find that the gas or radiation pressure dominated flows have outflows, which is consistent with simulations for both gas and radiation pressure supported flows. The case with advection-dominated and $\beta < 1$

Figure 1. Variations of the radial flow variables with radial distance, $\log(r)$. Panels show the variation of $\log(\lambda_d)$ (a), (d), (g), $\log(v_{re}$ and $a_{se}$) (b), (e), (h), and $B_e$ (c), (f), (i). Panels (a)–(c) are plotted for parameters $E = -0.001$, $\alpha_1 = 0.1$, $\gamma = 5/3$ with different $\beta = 1$ (dotted red), 0.5 (dashed black), and 0.1 (long-dashed blue). Panels (d)–(f) are plotted for energy parameters $E = -0.001$ with different $\alpha_1 = 0.01$ (dotted, red), 0.1 (dashed, black), and 0.2 (long-dashed, blue). Panels (g)–(i) are plotted for viscosity parameter $\alpha_1 = 0.1$ with different $E = -0.02$ (dotted, red), $-0.01$ (dashed, black), and $-0.00001$ (long-dashed, blue). The second and third columns are plotted for the same $\gamma = \gamma_{eff} = 4/3$ and $\beta = 1$. The solid curve (cyan color) represents the Keplerian angular momentum distribution in panels (a), (d), (g).

resembles high accretion rate flows with high luminous BH sources. The case with advection-dominated and $\beta \approx 1$ resembles low accretion rate with low luminous BH sources. A second row of Figure 2 is plotted with different viscosity parameters, $\alpha_1 = \alpha_2 = 0.01$ (panel 2(d), 0.1 (panel 2(e)), and 0.2 (panel 2(f)), for the input parameters corresponding to solutions of the second column of Figure 1. In panel 2(d), which is plotted with $\alpha_1 = \alpha_2 = 0.01$, we get two kinds of the sonic surfaces above the disk surface (solid black): one for the outflow, where $v_i > 0$ (dotted green), and the other for the fail outflows (“+” blue), when the solutions fail to make sonic transition in the outflow and matter velocity again becomes $v_i < 0$. Corresponding to these two surface regions, the detailed variations of flow variables along the $\theta$-direction are presented below in Figure 3. In the second row, we increased the value of viscosity parameters, and panel 2(e) is plotted with
$\alpha_1 = \alpha_2 = 0.1$; then we get smooth inflow and outflow surfaces. The outflow strength and region are also increased. If we further increased the viscosity, $\alpha_1 = \alpha_2 = 0.2$ (panel 2 (f)), then we have almost no outflows with 2D disk inflow structure. Typical behavior of the flow variables of the second row panels is presented in Figure 4 with some fixed radius. In
In these cases, the outflows strongly depend on the viscosity, and with changing viscosity we can get no outflows, weak, and strong outflows. They may explain the various states of the BHCs, since the viscous time and cooling timescales are changed with changing viscosity or mass accretion rate parameter (Das & Sharma 2013). Both parameters also changed the distribution of angular momentum in the flow (Kumar & Chattopadhyay 2014), so the flowing matter becomes Keplerian or sub-Keplerian. The last row represents a 2D structure corresponding to solutions of the last column of Figure 1. Here we found that the outflow region and strength depend on the transition radius, and both are increased with increasing \( r_t \). The \( r_t \) depends on the \( E \). For small \( r_t \), the outflow region is small and strength is also weak due to low local energy of the flow, when compared with panels 2(g)-(i) in the row, which have transition radius \( r_t = 28 ~ (E = -0.02) \), \( 62 ~ (E = -0.01) \), and \( 74240 ~ (E = -0.00001) \), respectively. If we compare this with panel 2(e), which has \( r_t = 705 ~ (E = -0.001) \), and panel 2(i), then they give almost the same disk structures because both have small flow energy difference. So, for the small advective disk \( (r_t < 100 ~ r_S) \), the outflow region and strength are much affected by changing the ADAF disk size, and accordingly the Keplerian disk size is also changed, as explained by Esin et al. (1997). However, in Esin et al. (1997), the ADAF size is decreased by increasing the mass accretion rate, which also changed the distribution of local energy of the flow, just as \( E \) did in the present paper.

In Figure 2, the inflow matter formed the sonic surface very close to the BH \( (r < 4 ~ r_S) \), and hence matter enters into the BH supersonically. Moreover, the inflow region away from the BH \( (r > 10r_S) \) also becomes supersonic before the outflows start to generate from the disk, which is marked by a black open square symbol (\( \Box \)) in the second and third columns of Figure 2, except for panel (f). But the inflow matter is always subsonic very close to the equatorial plane before inner CP or sonic surface \( (r < 4r_S) \). These supersonic regions have locally higher density, lower \( T \), high \( v_r \), and high rising \( v_\theta \), so total local velocity \( v = v_r + v_\theta \) is high; therefore arrow length is large in
these regions. We will present more detail of variations of the flow variables in Figure 5. Interestingly, the inflow matter is supersonic before making outflows. As the matter is moving inward, the flow variables are changed very quickly and the flow becomes subsonic. This supersonic to subsonic transition of the inflow matter along the radial direction may hint at the possibility of the occurrence of shock transition in the flow, although this transition is not as sharp as accretion shocks studied in the literature (Fukue 1987; Chakrabarti 1989; Becker et al. 2008; Kumar et al. 2013, 2014; Chattopadhyay & Kumar 2016; Lee et al. 2016; Kumar & Chattopadhyay 2017). The outflows occurred close to the BH because the thermal pressure and rotation velocity are increasing very quickly and the local energy (as the profile of $B_{\theta}$ in Figures 1(c), (f) and (i)) becomes sufficient to generate bipolar outflows. We see the disk surface with a changing slope at every radius, which is unlike the previous studies of 2D disk structure with similar assumptions (Jiao & Wu 2011) presenting the inflow disk surface with a constant slope. We also investigated no outflows, outflows, and failed outflow regions of the 2D flow, which depend on the disk parameters.

The above descriptions of Figure 2 are based on observations of the velocity fields and the outflow size. For more detailed study of these figures, we plotted typical flow variables along the $\theta$-direction with some fixed radii. These solutions are drawn for fixed radius $r = 6$, with different viscosity parameters, $\alpha_1 = \alpha_2 = 0.01$ (solid, red), 0.1 (dotted, blue), and 0.2 (dashed, black), and other parameters correspond to the second row of Figure 2.

![Figure 4](image_url)

**Figure 4.** Variations of flow variables with polar angle, $\theta$. Panels show variations of $v_{\theta}$ (a), $v_\theta$ (b), $v_r$ (c), $p$ (d), $P_g$ (e), and $T$ (f). These solutions are drawn for fixed radius $r = 6$, with different viscosity parameters, $\alpha_1 = \alpha_2 = 0.01$ (solid, red), 0.1 (dotted, blue), and 0.2 (dashed, black), and other parameters correspond to the second row of Figure 2.
multi-valued solution, meaning \( v_r \) has the same value at various \( \theta \); outflow solution (dotted blue, \( r = 6 \)), which is also multi-valued; and radially outward outflow solution (dashed black, \( r = 7 \)), meaning \( v_\theta \to 0 \) at high latitude. Because these outflows are mainly driven by combinations of centrifugal force and gradient of pressure (gas or radiation) force, and because the behavior of temperature and angular velocity vary with the radial distance, outflows and disk structure changed with the radius. If we see panel 3(a), initially the \( v_\theta < 0 \) for inflow and at some “\( \theta \)” becomes zero, which gives the disk surface, and \( v_\theta > 0 \) gives outflow region. In panel 3(b), the polar velocities (\( v_\theta < 0 \)) are increasing with decreasing “\( \theta \),” but the dashed black curve can again turn back and approach zero because \( v_\theta \) is starting to decreasing at high latitude (panel 3(c)). The \( |v_\theta| \) is higher for lower radius solutions, and this behavior may be due to corresponding higher rotation velocity, as in panel 3(c). The gas density \( \rho \) (panel 3(d)) and pressure \( \bar{p}_g \) (panel 3(e)) are monotonically decreasing toward the axis due to the expansion of gas above the equatorial plane. The behavior of temperature is also not monotonic; it is decreasing and increasing toward axis in panel 3(f) due to multi-valued nature of \( v_r \). And the nature of \( v_\theta \) mostly depends on the \( v_\phi \) and therefore on the viscosity. Although the dependence and multi-valued nature of velocities are very complicated, they mostly depend on the viscosity, which we will see in the next figure. The solution corresponding to \( r = 4 \) (solid red) is failed outflow (or failing to make transonic outflow solution, where \( v_r \) becomes again less than zero) due to very fast decreasing \( v_\phi \) (panel 3(c)), although \( T \) is increasing but did not produce sufficient pressure gradient force to maintain \( v_\phi \) as positive and resulting matter falls back toward the BH. The solution corresponding to \( r = 4 \) also feels more gravity than other solutions with higher values of \( r \). If we compare this kind of solution (solid red) with no outflows analytical MHD solutions (Zeraatgari et al. 2018), in both cases \( v_\theta \) decreases very quickly at high latitude. Thus, the variation of \( v_\theta \) plays a key role in generating the outflows. The black dashed curve behaves in similar ways as the bipolar accretion-outflow solutions presented by Xu & Chen (1997).

Figure 5. Variations of flow variables with polar angle, \( \theta \). Panels show variations of \( v_r \) (a), \( v_\theta \) (b), \( v_\phi \) (c), \( \bar{\rho} \) (d), \( \bar{p}_g \) (e), and \( T \) (f). These solutions are drawn with different radii, \( r = 3.1 \) (solid, red), 8.5 (dotted, blue), and 10.5 (dashed, black), and the disk parameters are the same as panel (c) of Figure 2.
Figure 4 is drawn for the fixed radius $r = 6$ with different viscosity parameters and used the same parameters corresponding to the second row of Figure 2. In panel 4(a), $|v|$ is high with high viscosity in the inflow region, since $\lambda_5$ is low (Figure 1(d)); therefore $v_0$ is also low in panel 4(c). The $v_0 > 0$ is again high in the outflow region corresponding to the same value of $\theta$ due to high acceleration for high viscosity, if we compare curves with solid red ($\alpha_1 = \alpha_2 = 0.01$) and dotted blue ($\alpha_1 = \alpha_2 = 0.1$). Since $|v|$ is also high for high viscosity (panel 4(b)), the total outflow velocity ($v$) is high in both curves. The $v_0$ (panel 4(c)) and $T$ (panel 4(f)) are monotonically increasing and decreasing, respectively, for higher viscosity as compared with a multi-valued curve for low viscosity (solid red), since higher viscosity makes flow hotter as $T$ is higher (panel 4(f)) and also transports more angular momentum somehow, which accounts for the smooth variation of $v_0$ (panel 4(c)). The solution corresponding to $\alpha_1 = \alpha_2 = 0.2$ (dashed black) is not producing outflow due to low $v_0$, as all three solutions have the same gravity pull at $r = 6$. The $\bar{p}$ (panel 4(d)) and $\bar{p}_s$ (panel 4(e)) are decreasing smoothly with $^\circ\theta$. Here $\bar{p}$ is higher for low viscous solution and therefore $\bar{p}_s$ is also higher, since the inflow $|v|$ is low. Interestingly, the gas density and pressure do not become zero at high latitude, specifically in high viscosity solutions, because integrations are terminated at the outflow sonic surfaces.

Three curves of Figure 5 are plotted from three different regions of Figure 2(c), which are on inner (no outflow), middle (outflow), and outer (supersonic inflow $r > 10 \bar{r}_s$) regions. The solution for the no outflow (solid red, $r = 3.1$) region is close to the BH and experiences more gravity, and the combined fluid centrifugal force and pressure gradient force are not sufficient to defend gravity. So, the matter is not able to cross the disk surface and it is always the case that $v_0 < 0$ (panel 5(a)). The next middle with outflow region solution (dotted blue, $r = 3.5$) has appropriate forces, which make $v_0$ positive and give outflows. From the outer part of the disk, the solutions from this region show very different behavior. We look at the dashed black curve (for $r = 10.5$) matter starts expanding subsonically upward with increasing $v_0$ and $v_0$ along the $\theta$—direction. At the same time, $v_1$ (panel 5(a)) and $T$ (panel 5(f)) are decreasing or increasing simultaneously, and the resulting flow becomes supersonic and subsonic in the inflow region. In the same region $\bar{p}$ is also increasing or decreasing (panel 5(d)). In this region, velocities, densities, and gas pressure gradients are high from the solutions of the other two regions and also cooler. These kinds of inflow supersonic regions above the equatorial plane are seen in the second and third columns of Figure 2, except panel (f), which is surrounded by the black square symbol line. These kinds of regions are not found with the low (panel 2(d)) or high (panel 2(f)) viscosity and low $r_i$ (panel 2(g)) solutions. Here the disk viscosity parameters roughly categorized as $\sim 0.01$ are low and $\geq 0.1$ is high.

The outflow solutions have a good qualitative agreement with radial self-similar adopted 2D HD (Xue & Wang 2005; Jiao & Wu 2011) and MHD (Mosallanezhad et al. 2016; Samadi & Abbassi 2016) flows. Since the disk vertical thickness of our solutions depends on the radius and flow parameters, there is a possibility that the disk thickness can be matched with previous studies for some suitable flow parameters. If we compare results for 2D disk structure with no outflow (Narayan & Yi 1995a; Zenaatgar et al. 2018), then our disk vertical thickness is low since the integration is terminated at the sonic surface. For the no outflow disk structure (Figure 2(f)), the thickness increased with the increasing viscosity. Moreover, the nature of the flow variable profiles along the $\theta$-direction is mostly consistent with the simulation by Yang et al. (2014).

4.2.1. Effect of $\tau_{b0}$ and $s$

In this subsection, we turn to the effects on the 2D disk structure with variation of $\alpha_2$ for $\tau_{b0}$ and $s$. Here we consider three cases: the first from Figure 2(a), which is plotted with $\gamma = \gamma_{eff} = 5/3$; the second from Figure 2(d), with low viscosity and weak outflows; and the third from Figure 2(f), with high viscosity with no outflows. All the cases are presented in Figure 6 with variation of $\alpha_2$ and keeping other parameters fixed for each case. The first case of Figure 2(a) is represented with two viscosity parameters, $\alpha_2 = 0.15$ (panel 6(a)) and 0.05 (panel 6(b)). Panel 6(a) has a disk structure with no outflows because of an increased transfer of angular momentum due to high $\alpha_2$ and as a result, $v_0$ decreases significantly at high latitude and the variation of flow variables becomes similar to the dashed black curve of Figure 4. So, all the matter will fall supersonically onto the BH after crossing the sonic surface. Panel 6(b) is plotted with $\alpha_2 = 0.05$, which gives outflow solutions, and the outflow region also increased from Figure 2(a). The second case from Figure 2(d) is represented with $\alpha_2 = 0.02$ (panel (c)) and $\alpha_2 = 0.005$ (panel (d)) of Figure 6. When compared with Figure 2(d), the outflow region and strength are increased in panel 6(b) but decreased in panel 6(d). In panel 6(c), $v_0$ is very small from the value of $v_1$, so velocity vectors are almost parallel to the equator and the matter seems to go back at high latitude. The last case of Figure 2(f) is again drawn with two different viscosities: $\alpha_2 = 0.1$ (panel 6(e)) and 0.05 (panel 6(f)). In both panels 6(e) and (f), outflow region is increased with decreasing $\alpha_2$ from Figure 2(f). From this study, we can say that disk structure depends on $\tau_{b0}$ and also depends on the value of viscosity parameters. Since for low viscosity ($\alpha_1$), say $\sim 0.01$, the outflows are increasing with increasing $\alpha_2$ and for high viscosity ($\alpha_1$), say $\geq 0.1$, the outflows are increasing with decreasing $\alpha_2$ when keeping $\alpha_1$ fixed.

In Figure 7, we represent the variation of three velocities and flow temperatures with $\theta$ at a fixed $r = 10$ for different $\alpha_2$ values, which are taken from Figures 2(f), 6(e), and (f). Note $\alpha_2 = 0.2$ (solid red) does not show the outflow. Since variation and values of $v_0$ are less (panel 7(c)) and $T$ is also decreasing above the equatorial plane (panel 7(d)), the combined effect of the outflow driving forces is not enough to make $v_0$ positive. When we decreased $\alpha_2 = 0.1$ (dotted blue) and 0.05 (dashed black), the $v_0$ value became high and rose faster at high latitude (panel 7(c)). So $v_1$ becomes positive and gives outflow. Similar behavior of $v_0$ has also been found in the simulation with the inclusion of $\tau_{b0}$ (Yang et al. 2014), which decreases $v_0$ at high latitude. Moreover, $v_1$ (panel 7(a)) and $v_0$ (panel 7(b)) are increasing faster with decreasing $\alpha_2$, so the outflow strength is also increased.

Now we turn and change the value of $s$, keeping other parameters fixed, to study its effects on the disk structure. Here we take the case of Figure 6(e) and change $s$ as presented in Figure 8. In both panels of Figure 8, the outflow region and strength are increased with increasing $s$ values, since the outflows are significantly affected by the viscosity parameters and mass-loss parameter. Therefore, we want to see the
variation of local energy of the inflow–outflow, and the definition of the local energy of the flow is

\[ B(r, \theta) = B = \frac{v_r^2}{2} + \frac{v_\theta^2}{2} + h + \Phi. \]  

(24)

This is a modified Bernoulli energy parameter for the 2D flow, which is similar to the local energy defined in Appendix A as the Bernoulli parameter \( B_e \) on the equatorial plane, when \( s \) is zero. Here, \( h \) is specific enthalpy.

In Figure 9, we are presented with variations of the outer boundary of the outflows (\( r_0 \)) with \( \alpha_2 \) (panel 9(a)), \( s \) (panel 9(b)), and the Bernoulli parameter \( B \) with \( \theta \) in panels 9(c) and (d). (Other details are written in the caption.) The two curves, the solid line (red) and the dotted line (blue), in panel 9(a) are represented with different flow constants of motion \( E = -0.02 \) and \( -0.001 \), respectively, so they have different \( r_0 \) for \( E = -0.02 \) and \( 705 \) for \( E = -0.001 \). The curve with lower \( r_0 \) has a small outflow region, which is more clear toward lower values of \( \alpha_2 \) in panel 9(a). Here, \( r_0 \) being higher means more matter going out from the disk or higher mass outflow rate. Again, in the same panel 9(a), another curve with a dashed (black) line is plotted with the same \( E \) as the curve dotted (blue), but both have different \( \alpha_1 = 0.1 \) and 0.2. Other flow parameters are the same for both curves. We found that the outflows are high with higher \( \alpha_1 \) for the same \( \alpha_2 \). Since higher \( \alpha_1 \) means higher temperature and increased kinetic energy, local specific energy of the flow is increased as seen in Figure 1(f). Here the outflow region is increased with decreasing \( \alpha_2 \), but disk thickness is decreased, as seen in Figure 6. For high \( \alpha_2 > 0.15 \), we did not find the outflows, but we have the inflow 2D disk structure as seen in Figure 2(f). In panel 9(b), the outflow region is increased with increasing \( s \) and \( \alpha_1 \). The two curves, solid and dashed lines, maximize around \( s \approx 1.8 \) and decrease with further increasing \( s \). For \( s \gtrsim 1.8 \), the gas pressure variation along the radial direction becomes almost flat on the equatorial plane. In panels 9(c), curves with variations of the \( B \) are plotted for the same solutions of Figure 7. The solid red curve \( (\alpha_2 = 0.2) \) is decreased with decreasing \( \theta \) and gives no outflow solution, as in 2(f). In the other two curves \( (\alpha_2 = 0.1 \text{ and } 0.05) \), \( B \) is increased with decreasing \( \theta \) at high latitude and outflows are provided. In panel 9(d), \( B \) is increased with decreasing \( \theta \) and increasing \( s = 1.0 \) (solid red), 1.5 (dotted blue), 2.0 (dashed black). Thus, the outflow region and strength are increased with increasing \( s \), as seen in Figure 8.
5. Summary and Discussion

We have explicitly obtained radial fluid Equations (34)–(36) on the equatorial plane and used them with symmetric conditions (14) for solving ODEs (10)–(13) and (17) along the θ-direction. First, we obtained radial flow variables with their derivatives by integrating the radial fluid equations. Second, we integrated ODEs along the polar direction by using obtained polar flow variables at \( \theta = \pi/2 \). These two integrations are run one by one at each step of \( r \); after repeatedly doing so, we got complete 2D disk structure of the flow. We found two distinct regions in the 2D flow for the viscosity \( \alpha_1 > 0.01 \); one is the inflow region when \( v_r < 0 \) around the equatorial plane and the second is the outflow region when \( v_r > 0 \) above the inflow region. Both regions are separated by the disk surface with \( v_r = 0 \). For \( r \leq 4 \), we found only the inflow region, and flow is supersonic. For low viscosity \( \alpha_1 \lesssim 0.01 \), we also found failed outflow regions in the outflows part above the disk surface. The failed outflows mean the flow radial velocity again becomes negative \( (v_r < 0) \) at high latitude. Here the outflow regions are plotted up to the sonic surface (when \( M = 1 \), since integration is problematic due to the discontinuity at \( M = 1 \) when solving the equations along the polar direction. The inflow disks also show supersonic regions just above the equatorial plane, and it appears away from the BHs \( r > 10 \), which depends on the flow parameters (the second and third columns of Figure 2, except panel f, and Figures 6(b), (e), and (f)). This region is surrounded by the sonic surface in the disk but does not have proper CPs or discontinuity in the flow, which means integration passes smoothly at this surface. These kinds of supersonic regions in the inflow portion are not formed when the disk has no outflows with high \( \alpha_2 > 0.15 \) (Figures 2(f), 6(a)) or when the disk has outflows with low \( \alpha_1 \lesssim 0.01 \) (Figures 2(d), 6(c), and (d)) or \( s > 1.7 \) (Figure 8).

Figure 7. These solutions are plotted with a changing viscosity parameter, \( \alpha_2 = 0.2 \) (solid red), 0.1 (dotted blue), and 0.05 (dashed black) at the same value of \( r = 10 \), and other parameters are the same as in panel (f) of Figure 2.
Our results show the inflow and outflow regions for a certain range of the viscosity parameters, and they also depend on other disk parameters, which is consistent with some analytical (Xu & Chen 1997; Xue & Wang 2005; Jiao & Wu 2011; Gu 2012, 2015) and numerical simulation studies (Ohsuga et al. 2005; Okuda et al. 2007; Yuan et al. 2012a, 2012b; Yang et al. 2014; Jiao et al. 2015; Yuan et al. 2015). However, our solutions and disk structure are quite different from the previous analytical studies on 2D disk flow with self-similar conditions (e.g., the size and shape of the disks, behavior of the solutions and supersonic regions). Still, some basic properties and solutions are similar qualitatively. This is clear due to differences in boundary conditions on the equatorial plane; we also used two viscous stress components in our model fluid equations of motion. In self-similarity, all the radial flow variables (velocities and sound speed) are the same, so the Mach number is one and constant at every radius, but in our case, this happened only at the CP of the transonic ADAF solutions (Figure 1). Moreover, values of these radial variables have not changed with the viscosity parameter and $\beta$, as in Jiao & Wu (2011). Due to these reasons, the outflow structure is different from other analytical 2D flow studies. Most of the common flow parameters of our studies show outflows—for example, range of the $s$ from 0.2 to 2, range of $\alpha_1$ from 0.01 to 0.2, and $\alpha_2 \lesssim 0.15$—although the lower range of the $s$ depends on the $\alpha_1$ but the upper limit $s=2$ is used here, since the gas pressure profile becomes flat along radial direction on the equatorial plane for $s \sim 1.8$. For high $\alpha_2 > 0.15$, we did not get outflow solutions but have 2D advective disk structure (Figures 2(f) and 6(a)). Moreover, for high $\alpha_1 > 0.1$ and low $\alpha_2 < 0.01$ with high $s > 1$, we may get a larger outflow region or almost a whole ADAF disk, which can be predicted from the study of Figures 9(a) and (b). The outflow region is also larger for high $E$ of the flows (the last row of Figures 2 and 9(a) and (b)). The disk structure is not much affected by the variations of $\gamma$ or $\beta$, but the outflow strength is high for radiation-dominated flows ($\beta < 1$). Although the main features of the outflows are roughly consistent with the simulation results but the vertical thickness, the inflow and outflow regions of most of the solutions are small from the simulations by Yuan et al. (2012a, 2012b), with Yang et al. (2014) using two non-zero azimuthal components of the anomalous shear stress tensor. These differences may arise due to an analytical approach, assuming constant mass-loss parameter ($s$) throughout the disk, the integration problem after the sonic surfaces, and assumed boundary conditions at the equatorial plane. Moreover, the radial power law index of $\alpha_1$ and $v_\phi$ around the equatorial plane are roughly close to the simulation (Yuan et al. 2012b), although the scaling for $v_\phi$ and $\rho \propto r^{s-2}/v_\phi$ mostly depend on $r_1$ (or $E$), as explained in Figure 1, and $s$, respectively.

The outflows are more favorable for $\alpha_2 \geq \alpha_1$ when low $\alpha_1 \sim 0.01$ and $\alpha_2 < \alpha_1$ when high $\alpha_1 \gtrsim 0.1$, as cases have presented here. All disk parameters may have outflow solutions but need to find suitable outflow structure parameters, like $\alpha_2$ and $s$. Similarly, we also found that $\alpha_2 > 0.15$, which has the 2D disk structure but no outflows for any value of $s$. Since $s$ and $\alpha$ are fixed here for a particular solution, we have varied possible values of $s$ and $\alpha$ for the 2D disk structure. When doing this, we found the range of outflows region for $s$ and $\alpha_2$ (Figure 9). We have 2D disk structure, and no outflows for $s = 0$ equates to no mass loss from the disk or matter is bound with the disk, even when $v_\phi$ is non-zero with $v_\phi < 0$. The transonic surface formed for the outflows ($v_\phi > 0$, above the inflow disk surface), at this surface outflow density $\gtrsim 20\%$ from the local equatorial plane density. So the outflow solutions can cross the transonic surface and give supersonic outflows, but the present analytical study is limited to the outflow sonic surface.

The present study is done with one kind of input accretion solutions (ADAF-thin) on the equatorial plane for calculation of the 2D disk structure. We found that only the inner region of the advective disk participates in the outflow generation, and size is around a few tens of the Schwarzschild radius, which is consistent with the observed size of the outflow region around M87 (Junor et al. 1999; Doeleman et al. 2012). Our studies also support the two zone configuration theory of the fluid flows (Esin et al. 1997; Das & Sharma 2013), since outflow region and strength are changed with changing viscosity parameters, $r_1$ or $E$ and $s$. The $\tau_{\phi\phi}$ component of viscosity affects the outflows and decreases angular momentum above the equatorial plane (Figure 7), which has also been seen in the simulation.
Yang et al. 2014, and variations of the polar flow variables are also roughly consistent with this simulation. These kinds of analytical studies are worth pursuing for detailed studies of flow solutions and the disk structure with various flow parameters that characterized the flow. This study also gives the idea about no outflows (for very low or high viscosity), outflows (for moderate viscosity), and failed outflows (for low viscosity). Incidentally, the shape of our disk structure is similar to the variations of $B_\theta$ along the radial direction (last row of Figure 1), which supports the idea of positive local energy for generation of the outflows (Narayan & Yi 1995a; Blandford & Begelman 2004). Moreover, we also found the supersonic region in the inflow to be distant from the BH before the outflows occur in some cases. At the boundary of this region, as matter moves inward with sharply rising temperatures and decreasing bulk velocity along the radial direction with resulting flow, a transition occurs from supersonic to subsonic. These kinds of transitions may present the possibility of shocks in the inflows, as studied by many authors (Chakrabarti 1989; Becker et al. 2008; Lee et al. 2016; Kumar & Chattopadhyay 2017).

Studies of ODEs and techniques for solving ODEs are a worthwhile pursuit for future research. One can use a variety of advective solutions on the equatorial plane, such as an ADAF-thick disk (Lu et al. 1999), slim disk (Abramowicz et al. 1988), and shock solution (Kumar & Chattopadhyay 2014, 2017), with relevant cooling mechanisms in the study of two-dimension flows, and these can also be compared for further analysis.

We thank Tuan Yi for helpful discussion during the creation of this paper. This work was supported by the National Basic
Research Program of China (973 Program) under grant 2014CB845800, and the National Natural Science Foundation of China under grants 11573023 and 11333004. We thank the anonymous referee for their helpful comments and suggestions.

Appendix A
Fluid Equations on the Equatorial Plane

Here we made two more assumptions in order to solve fluid equations on the equatorial plane: the first where all $\partial / \partial \theta = 0$ and the second where $v_r = v_{\theta r}$, $v_\theta = 0$, $v_\phi = v_{\phi c}$, $\Theta = \Theta_c$, $\rho = \rho_c$, then, Equations (1)–(5) are written as, the continuity equation,

$$\frac{1}{r^2} \frac{d}{dr} (r^2 \rho_c v_{\theta c}) = 0$$

(25)

Navier–Stokes equations are $r$-component:

$$v_{\theta c} \frac{dv_{\theta c}}{dr} - \frac{v_{\theta c}^2}{r} + \frac{1}{\rho_c} \frac{d\rho_c}{dr} - F_r = 0$$

(26)

$\phi$-component:

$$v_{\phi c} \frac{dv_{\phi c}}{dr} + v_{\theta c} \frac{v_{\theta c}}{r} v_{\phi c} = \frac{1}{\rho_c} \frac{d}{dr} (r^3 \tau_{\phi c})$$

(27)

Energy generation equation:

$$\rho_c \varepsilon \left[ v_{\theta c} \frac{dv_{\theta c}}{dr} - \frac{v_{\theta c}^2}{r} + \frac{1}{\rho_c} \frac{d\rho_c}{dr} \right] = f Q^+_{\theta c},$$

(28)

where $Q^+_{\theta c} = \tau_{\phi c}^2 / \alpha_k$ is a viscous heating rate, and the subscript “$\varepsilon$” represents the quantities for accretion flow on the equatorial plane. Here, $\tau_{\phi c} = n_k r (d K / dr)$ is a $r$-component of the viscous stress tensor, and $n_k = \alpha_1 \rho_c / \Omega_k = \alpha_1 \rho_c a_{\varepsilon c}^2 / (\gamma_{\text{eff}} \Omega k)$ is a coefficient of viscosity. The definition of adiabatic sound speed from the first law of thermodynamics with Equation (6) is obtained as $a_{\varepsilon c} = \sqrt{\gamma_{\text{eff}} / \rho_c}$. Integrating Equations (25) and (27), respectively, we assumed wedge accretion flow with $\theta_c$ angle around the equatorial plane,

$$M_{\text{in}} = -4 \pi r^2 \rho_c v_{\theta c} \cos \theta_c,$$

(29)

and assumed $\lambda = \lambda_0$ as matter approaches to $r \to r_s$ and $\tau_{\phi c \text{horizon}} = 0$ with the help of Equation (29),

$$\tau_{\phi 0} = - \rho_c v_{\theta c} (\lambda_c - \lambda_0) / r,$$

(30)

where, $\lambda_c = r v_{\theta c}$ and $\lambda_0$ are specific angular momentum of the flow and specific angular momentum at the horizon, respectively. Here we assumed a constant $\theta_c$ with a value of $\pi / 3$ from the rotation axis, which is close to the disk thickness of almost all the results of this paper. Integrating Equation (26) with the help of Equations (27) and (28), we get the energy constant

$$E = \frac{v_{\theta c}^2}{2} + h_c - \frac{\lambda_c^2}{2r^2} + \frac{\lambda_c \lambda_0}{r^2} - \int \Lambda_c dr + \Phi.$$  

(31)

This is known as the generalized specific energy of the flow (Kumar & Chattopadhyay 2014) and is a constant of motion for dissipative advective flows, even in the presence of cooling. Here, $\Lambda_c = (1 - f)(\lambda_c - \lambda_0)(d Q / dr)$ and $h_c = e_c + \frac{P_c}{\rho_c}$ is specific enthalpy of the flow. If we use $f = 1$, then above equation becomes

$$E = \frac{v_{\theta c}^2}{2} + h_c - \frac{\lambda_c^2}{2r^2} + \frac{\lambda_c \lambda_0}{r^2} + \Phi.$$  

(32)

This is known as the grand specific energy of the flow (Gu & Lu 2004; Becker et al. 2008; Kumar & Chattopadhyay 2013; Kumar et al. 2014) and is a constant of motion for the viscous flow. If we again take inviscid flow, then $\lambda_c = \lambda_0$, and thus the above equation becomes

$$B_c = \frac{v_{\theta c}^2}{2} + h_c + \frac{\lambda_c^2}{2r^2} + \Phi.$$  

(33)

This is the local specific energy of the flow and known as the Bernoulli parameter. We can use these energy constants for the calculation of flow variables at CP with two CP conditions (Kumar et al. 2013; Kumar & Chattopadhyay 2017) or at the horizon with few assumptions in order to find flow variables close to the horizon (Becker et al. 2008; Kumar & Chattopadhyay 2013; Kumar et al. 2014; Kumar & Chattopadhyay 2014; Chattopadhyay & Kumar 2016). Now, simplifying Equations (26), (28), and (30) with the help of Equations (25), (6) and using the expression of $\tau_{\phi 0} c$, we get

$$\frac{d}{dr} = - \frac{\beta_{\text{eff}}}{N_{\text{eff}}} \left[ \frac{a_{\varepsilon c}^2}{v_{\theta c}} \left( \frac{d v_{\theta c}}{dr} + \frac{2}{r} \right) + f \Lambda_c^+ \right],$$

(34)

where $\Lambda_c^+ = Q_{\theta c}^+ / (\rho_c v_{\theta c})$,

$$\frac{d v_{\theta c}}{dr} = \frac{v_{\theta c}}{r} + F_c + \frac{2 a_{\varepsilon c}^2}{v_{\theta c}} + \frac{f_{\text{eff}} \Lambda_c^+}{N_{\text{eff}}} = N_c^+ / D,$$

(35)

$$\frac{d \lambda_c}{dr} = \frac{2 \lambda_c}{r^2} + r^2 \frac{d Q}{dr} \text{ and } \frac{d Q}{dr} = - \frac{\gamma_{\text{eff}} v_{\theta c}^2 \Omega_k (\lambda_c - \lambda_0)}{\alpha_1 a_{\varepsilon c}^2 r^2}.$$  

(36)

To find complete accretion solutions, we have to solve all three differential Equations (34)–(36) using specified flow parameters —namely, $E$, $\lambda_0$, $\alpha_1$, $\beta$, and $\gamma$. Since BH accretion is transonic in nature, we have to find the location of the sonic point, but for dissipative flow sonic location, this is not known a priori. The equations and detailed methodology for sonic point calculation have been explained in Section 3 and Appendix B, respectively.

Appendix B
Steps for Solving ODEs

In order to get the complete inflow–outflow structure from our model equations, first we have to find the CP for the transonic ADAF solution. Here we used the Runge–Kutta 4th order method to solve the differential equations along the $r$- and $\theta$-direction. The whole solution procedure is divided into the following steps:

1. CP location: For given flow parameters $E$ (or $E_c$), $\lambda_0$, $\alpha_1$, $\gamma$, and $\beta$, we obtained CP from the iteration method by changing the $\delta$ in the following two parts.

   a. Obtaining $\lambda_{\ast e}, v_{\theta e}, \Theta_e$ at $r_{\ast n}$: When we combined Equations (21) and (32) with the value of $\Lambda$ and expression of $\zeta$ at $r_{\ast n} = 1.001$, we got a polynomial in $a_{\varepsilon c}$ or $\Theta_e$. Now, supplying the parameters $E$, $\lambda_0$, $\alpha_1$, $\gamma$, and $\beta$, then we solved the polynomial for $\Theta_e$ at $r_{\ast n}$ for the first choice $\delta = 1$. Once $\Theta_e$
was obtained at \(r_{\text{in}}\), other quantities \(v_{\text{re}}\) and \(\lambda_e\) are easily obtained with the help of \(v_{\text{ff}}\) and Equation (21).

**Part II. Finding \(r_c\):** We now can integrate differential Equations (34)–(36) outward from \(r_{\text{in}}\) by using \(\Theta_c\), \(v_{\text{re}}\), and \(\lambda_e\) simultaneously, checking the sonic point (Equations (19) and (20)). If sonic conditions are not satisfied, then we reduce the value of \(\delta < 1\) and repeat part I. This solution procedure is repeated until sonic conditions are satisfied. To ensure this, we obtained CP location \((r_c)\) for the given flow parameters.

**Step 2. ADAF solution:** Once \(r_c\) is obtained, we integrated Equations (34)–(36) outward along the radial direction for a given \(\lambda_0\) with other disk parameters. Then we investigated outer boundaries of the ADAF solution (Narayan et al. 1997; Lu et al. 1999) again by iteration method, changing \(\lambda_0\) by repeating the entirety of step 1. Once \(\lambda_0\) was obtained for the ADAF solution and corresponding \(r_c\), then we went for the calculation of the 2D disk structure.

**Step 3. 2D solution:** Here we supplied two more additional parameters \(\alpha_2\) and \(s\) for the calculation of the polar flow variables of the disk structure with the outflows. We again divided the procedure into two parts.

**Part I. Obtaining \(r_o\):** When we obtained \(r_o\) for the ADAF solution, we integrated the radial fluid Equations (34)–(36) from \(r_c\) along the \(r\)-direction, outward with some step size \((dr)\), and then at the same \(r\), we integrated the polar fluid Equations (10)–(13), (17) from \(\theta = \pi/2\) along the \(\theta\)-direction toward the rotation axis with some step size \((d\theta)\). At each step size \((dr)\) of \(r\) we obtained the polar variables at \(\theta = \pi/2\) from Equation (18) with the help of radial flow variables and derivatives of the ADAF. Now we again integrated radial equations at \(r + dr\) and then integrated polar equations from \(\pi/2\) to 0. These two integrations run one by one until the radius \(r = r_o\), where the outflows or 2D disk solutions exist, and if these do not exist, then we stop the integrations. Now that we know the location of \(r_o\), we can make a matrix for the density contour and velocity vectors plot, which is described in the next part.

**Part II. \(N_r \times N_\theta\) Matrix:** We choose \(N_r = N_\theta = 256\) for plotting the complete disk structure using IDL (Interactive Data Language). The radial distance from \(r_S\) to \(r_h\) and angular distance \(\theta = \pi/2\) to 0 are divided into 256 parts and obtain radial and polar step size for integration—for example, radial integration step size \(dr = (r_S – r_h)/(N_r – 1)\) and polar integration step size \(d\theta = \pi/2/(N_\theta – 1)\). These two integrations are run one by one as described in part I, first along \(r\) with one step size \(dr\) and then second along \(\theta\) up to \(\theta = 0\) at the same \(r\). Next we increase \(r\) by size \(dr\), and then we repeat the same integrations and repeatedly do so up to \(r_o\), when we obtain all matrix elements and therefore the complete 2D disk structure.