Non-perturbative 2d quantum gravity and hamiltonians unbounded from below

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Abstract

We show how the stochastic stabilization provides both the weak coupling genus expansion and a strong coupling expansion of 2d quantum gravity. The double scaling limit is described by a hamiltonian which is unbounded from below, but which has a discrete spectrum.
1 Introduction

One of the problems of euclidean quantum gravity is that the action is not bounded from below. One might think that this is not a serious problem in two dimensions where the Einstein-Hilbert action is topological:

$$\int d^2 \xi \sqrt{g} R = 2\pi \chi$$  \hspace{1cm} (1)

However, in case one attempts a summation over all genera, as is needed in string theory and as might be needed in quantum gravity, it creates ambiguity. This ambiguity is highlighted by analyzing 2d gravity in the matrix model approach. It is conveniently formulated as follows: In the regularized path integral the class of manifolds to be included can be chosen to be the piecewise linear orientable manifolds, which one can get by gluing together equilateral triangles. The curvature assignment can then be done according to Regge calculus which means that the curvature is assigned to the vertices and is proportional to the number of triangles sharing a given vertex. The gluing of \( K \) labeled oriented triangles can be done in a systematic way by associating with each oriented link a hermitian \( N \times N \) matrix \( \phi_{\alpha\beta} \) (see fig.1)

![Figure 1: Labels of a triangle.](image)

choosing an appropriate gaussian measure

$$d\mu_N(\phi) = c_N \exp\left(-\frac{1}{2} \sum_{\alpha,\beta} |\phi_{\alpha\beta}|^2 \right) \prod_{\alpha \leq \beta} d\text{Re} \phi_{\alpha\beta} \prod_{\alpha < \beta} d\text{Im} \phi_{\alpha\beta},$$  \hspace{1cm} (2)

and taking the action to be proportional to the product of matrices around the oriented triangle:

$$A(\phi) = \phi_{\alpha\beta} \phi_{\beta\gamma} \phi_{\gamma\alpha} = \text{Tr} \phi^3.$$  \hspace{1cm} (3)

In this way all closed triangulated surfaces of the kind mentioned will be generated by the following field theoretical model

$$Z(h, N) = \int e^{hA(\phi)/\sqrt{N}} d\mu_N(\phi).$$  \hspace{1cm} (4)

In fact, by expanding \( e^{hA(\phi)} \), the Wick contractions of the term \( (h/\sqrt{N})^K A(\phi)^K / K! \) will generate all oriented 2D simplicial manifolds consisting of \( K \) triangles. Further
the $1/N$-expansion organizes the surfaces according to powers of $N$. For a surface with Euler number $\chi$ the total power of $N$ is $\chi$.

If we write the continuum euclidean action as

$$S(\lambda, G) = \lambda \int d^2 \xi \sqrt{g} - \frac{1}{2\pi G} \int d^2 \xi \sqrt{g} R$$

(5)

a comparison with (1) leads to the identification of $e^{\chi/G}$ and $N^\chi$:

$$\frac{1}{G} = \log N$$

(6)

Of course the field theory (1) is not well defined since the action is not bounded from below. It should be emphasized that this unboundedness has nothing to do with the cubic action. Had we decided to glue together regular squares the action $\text{Tr} \phi^3$ in (3) would have been changed to a quartic action $\text{Tr} \phi^4$, but it would appear with the wrong sign, since each surface should be assigned a positive weight. However, each term in the $1/N$-expansion is perfectly well defined and has a finite radius of convergence. The difference between the restricted summation provided by each term in the $1/N$-expansion and the unrestricted summation over all surfaces is the following: For a fixed topology, i.e. fixed $\chi$, the number of surfaces which can be constructed by gluing together $K$ equilateral triangles grows exponentially, while the total number of such surfaces grows factorially, i.e. much faster:

$$\mathcal{N}_\chi(K) \sim K^{\gamma_\chi - 3} e^{\mu K}$$

$$\mathcal{N}_{\text{tot}}(K) \geq \text{const. } K!$$

(7)

The exponential bound in (7) leads to a critical point which is the same for all $\chi$ and which allows us to take a scaling limit for $\mu \to \mu_c$ for each individual $\chi$. One gets

$$Z_\chi(\mu, G) = e^{\chi/G} \sum_K \mathcal{N}_\chi(K) e^{-\mu K} \sim \frac{c_\chi e^{\chi/G}}{(\mu - \mu_c)^{\gamma_\chi - 2}}$$

(8)

In (8) $\mu$ is related to $h$ in (3) by

$$h = e^{-\mu}$$

(9)

The so-called double-scaling limit is an attempt to go beyond this expansion, and can be viewed as a formal renormalization of the gravitational constant $G$ in front of the Einstein-Hilbert action. The key observation is that

$$\gamma_\chi = -\frac{5}{4} \chi + 2$$

(10)

which allows us to write

$$Z(\mu, G) \sim \sum_\chi c_\chi e^{\chi/G_R}$$

(11)
where the renormalized gravitational constant is defined as
\[
\frac{1}{G_R} = \frac{1}{G} + \frac{5}{4} \log(\mu - \mu_c) \tag{12}
\]
The remarkable fact that the partition function in the scaling limit is only a function of the renormalized gravitational coupling constant was first observed in [1, 2] in the context of Liouville theory, and in the context of the matrix model in [3, 4, 5].

The factorial growth (7) of the total number of piecewise linear surfaces is reflected in a factorial growth of the coefficients \( c_\chi \) in (11). The series is only an asymptotic series and not even Borel summable since all the coefficients are positive. This is consistent with the unboundedness of the action (3). The origin of the unbounded action (3) is therefore different from the usual problem caused by the conformal mode in euclidean gravity, but it is a problem which will be present even more severely in higher dimensional gravity.

Two possible cures of the unboundedness of the action in euclidian quantum gravity were suggested long time ago. One method is based on stochastic quantization and uses the bounded Fokker-Planck potential [6]. The other method involves a rotation of the integration contour in the functional integral into the complex plane such that the functional integral converges [7]. They were both intended to be used to cure the problem of the conformal mode, but have also been applied to regularize the summation over topology in 2d gravity [8, 9, 10, 11, 12, 13]. The outcome is that the two methods do not agree [15]. In fact the contour rotation leads to a complex partition function and complex correlators, a signal that either the method is no good or the theory itself is incomplete. The stochastic method has no obvious flaws and in the following will will analyze this method in detail.

\section{Stochastic regularization}

The stochastic quantization scheme for a \( d \)-dimensional euclidean field theory with an action \( S[\phi] \) shows that the vacuum expectation value of any operator \( Q \) can be interpreted as the ground state expectation value in a \( (d+1) \)-dimensional quantum theory:
\[
\langle Q \rangle = \frac{1}{Z} \int d\phi \ e^{-S[\phi]/\hbar} Q[\phi] = \int d\phi \ \Psi_0^2[\phi] \ Q[\phi]. \tag{13}
\]
Here
\[
\Psi_0[\phi] = \frac{e^{-S[\phi]/2\hbar}}{\sqrt{Z}} \tag{14}
\]
\footnote{Here it should also be mentioned that yet another suggestion of a non-perturbative regularization can be found in the papers by Morris et al. [16]. This regularization has no obvious flaws either, but does unfortunately not agree with the stochastic method [17]. At the moment we do not have any convincing arguments in favor of one of the the two methods.}
is the ground state of the theory, determined by the Fokker-Planck Hamiltonian

\[ H_{FP} = \int d^d x \left[ -\frac{\delta^2}{\delta \phi^2} + \frac{1}{4\hbar^2} \left( \frac{\delta S}{\delta \phi} \right)^2 - \frac{1}{2\hbar} \frac{\delta^2 S}{\delta \phi^2} \right] \]  (15)

This Hamiltonian is a positive semi-definite operator and can be written as

\[ H_{FP} = \int d^d x R^\dagger(x)R(x), \quad R = i \frac{\delta}{\delta \phi} + i \frac{\delta S}{2\hbar \delta \phi} \]  (16)

and it is readily verified that (14) is an eigenvector of \( H_{FP} \) corresponding to energy \( E = 0 \).

For this to make sense it has been assumed that the action \( S[\phi] \) produces a normalizable wavefunctional by (14). This is clearly not the case if \( S[\phi] \) is unbounded from below. Formally the wavefunctional (14) will still satisfy \( H_{FP} \Psi_0 = 0 \), even for a bottomless action, but it does not qualify as a groundstate as it is not normalizable.

The true ground state will correspond to the lowest eigenvalue of \( H_{FP} \) where the eigenstate is normalizable. If we denote these by \( E_0^{(T)} \) and (again) by \( \Psi_0 \), we can write in the case of a bottomless action:

\[ \Psi_0[\phi] = \frac{e^{-S_{eff}[\phi]/2\hbar}}{\sqrt{Z_{eff}}}, \quad S_{eff} \neq S, \quad E_0^{(T)} > 0. \]  (17)

This equation can serve as a definition of a new stabilized action. The expectation values of observables are defined as in (13):

\[ \langle Q \rangle = \frac{1}{Z_{eff}} \int d\phi Q[\phi] e^{-S_{eff}[\phi]/\hbar} = \langle \Psi_0|Q|\Psi_0 \rangle \]  (18)

and it can be shown [6] that they have the same classical limit, the same perturbative expansion in coupling constants and the same \( 1/N \) expansion as in the ill-defined bottomless theory. Especially the last point will be important for us.

Let us now apply the above formalism to the matrix models describing 2d-gravity. The partition function of 2d-gravity is in the matrix model approach given by

\[ Z = \int d\phi e^{-N \text{Tr} V(\phi/\sqrt{N})} \]  (19)

where \( \phi \) denotes a \( N \times N \) hermitean matrix and the potential \( V(x) \) is characterized by starting with a quadratic term and then containing a number of wrong sign higher power terms \( -g_3 x^3 - g_4 x^4 - \cdots \). As explained above the cubic term corresponds to the gluing of triangles, the quartic term to the gluing of squares etc. The critical behaviour is universal as long as the coupling constants \( g_3, g_4, ... \) are positive and is uniquely determined by the quadratic nature of the maximum of \( V(x) \) for \( x > 0 \). We
will not consider the more general situation, where the coupling constants $g_3, g_4, \ldots$ can have different signs. In that case a fine-tuning of the $g_i$’s with different signs will result in different critical behaviour whenever the first extremum of $V(x)$ for $x > 0$ will be of higher order than two. Such critical behaviour corresponds to non-unitary matter coupled to 2d-gravity. The non-unitarity is not unexpected in view of the surface representation, since it corresponds to gluing certain polygons to the surface with negative weight. We will here restrict ourselves to the simplest potential, which also has the nicest interpretation in terms of geometry: the construction of triangulated surfaces:

$$V(\tilde{x}) = \frac{1}{2} \tilde{x}^2 - g_3 \tilde{x}^3$$

It is convenient by a simple translation and rescaling to use it in the form

$$V(x) = gx - \frac{x^3}{3} \quad (20)$$

Exploiting the assumed $U(N)$ invariance of the vacuum we follow the classical treatment in [18] and diagonalize the hamiltonian (15) with respect to the eigenvalues $x_i$, $i = 1, \ldots, N$ of the rescaled matrix $\phi = \phi/\sqrt{N}$

$$H_{\text{FP}}[\phi] = N \sum_{i=1}^{N} H_{\text{fp}}[x_i] \quad (21)$$

$$H_{\text{fp}}[x] = -\frac{1}{N^2} \frac{d^2}{dx^2} + V_{\text{fp}}(x) \quad (22)$$

$$V_{\text{fp}}(x) = \frac{1}{4}(g - x^2)^2 + x \quad (23)$$

For the simple potential (20) we see that $H_{\text{FP}}[\phi]$ just becomes the sum of $N$ non-interacting single particle hamiltonians. The particles behave as fermions since the expectation value of any $U(N)$ invariant observable, calculated according to (18), is

$$\langle Q[\phi] \rangle = \int \prod_{i=1}^{N} dx_i \Delta^2(\{x_i\}) \Psi_0^2(\{x_i\}) Q(\{x_i\})$$

where $\Delta(\{x_i\})$ comes from the integration over the angular part of the $\phi$ variable, and is given by the Vandermonde determinant:

$$\Delta(\{x_i\}) = \prod_{i<j}(x_i - x_j) \quad (25)$$

In this way, as already observed in [18], the function

$$\Phi(\{x_i\}) \equiv \Delta(\{x_i\}) \Psi_0(\{x_i\})$$

(26)
becomes the totally antisymmetric groundstate wave function of a fermionic system of $N$ non-interacting particles and can therefore be written as the Slater determinant of the $N$ lowest single particle eigenfunctions of $H_{fp}$:

$$\Phi(\{x_i\}) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \chi_1(x_1) & \cdots & \chi_N(x_1) \\ \vdots & \ddots & \vdots \\ \chi_1(x_N) & \cdots & \chi_N(x_N) \end{vmatrix} \tag{27}$$

where

$$H_{fp}\chi_j(x) = E_j\chi_j(x) \tag{28}$$

and where the ground state energy of the total system is

$$E_0^{(T)} = N(E_1 + \cdots + E_N) \tag{29}$$

In the context of the $1/N$-expansion the total energy $E_0^{(T)}$ is zero. As explained above the ground state energy of $H_{FP}$ is trivially zero when the original problem is well defined and (14) is normalizable. Even if the original matrix problem looks unbounded (it involves $\phi^3$ terms, wrong sign $\phi^4$ terms etc) it is actually well defined to any finite order in the $1/N$ expansion as long as the coupling constants are less than the critical coupling constants which determine the radius of convergence of the sum of diagrams to leading order in the $1/N$ expansion. This is reflected in the fact that the distribution of eigenvalues of the matrices to the leading order in $N$ is restricted to a finite interval. The wave function, as given by (14), is therefore normalizable in this leading approximation, and in fact to all finite orders in $1/N$. Using the stochastic regularization one is able to go beyond the $1/N$ expansion and in addition one is able to extend the theory beyond the critical value of the coupling constant. When the coupling constant is beyond the radius of convergence the matrix model is not defined even in the planar approximation. But again stochastic regularization will provide us with a definition of the theory in this region of coupling constant space.

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\textsuperscript{2}This is a general property of semi-definite operators of the form (16), as is well known from supersymmetric quantum mechanics. However, we feel that supersymmetry in this context is somewhat of a red herring. First, the supersymmetry alluded to in the context of matrix models has not yet been given a useful physical interpretation, and next the phenomenon is more general. As an example we can mention the case of ordinary 4d Yang-Mills theory where the operator $R(x)$ in temporal gauge is $i\delta/\delta A(x) - iB(x)$, and an exact solution to the functional Schrödinger operator corresponding to $E_0 = 0$ is $\exp(-N_{cs}[A])$, where $N_{cs}[A]$ denotes the Chern-Simons number of the gauge configuration. However, this solution is not normalizable, as is the case for the true groundstate which has $E_0 > 0$.  


3 The large-N limit

Since the system of $N$ non-interacting particles is fermionic and since we have to perform the summation over the first $N$ eigenvalues, let us denote the energy of the $N$’th level by $E_F$ (the Fermi energy). From (22) we see that large $N$ corresponds to the semiclassical limit $\hbar = 1/N \to 0$. We expect therefore the WKB approximation to be good for all but the lowest eigenvalues, unless special circumstances occur. According to the Bohr-Sommerfeld quantization rule the phase space of the classical theory is related to the $n$’th energy level of $H_{fp}$ by:

$$2\pi n\hbar = \int dp \, dx \, \theta[E_n - (p^2 + V_{fp}(x))]$$

(30)

where $\hbar = 1/N$. This gives the following relation:

$$\frac{n}{N} = \frac{1}{\pi} \int_{x_l(E_n)}^{x_r(E_n)} dx \sqrt{E_n - V_{fp}(x)}$$

(31)

where $x_r(E_n)$ and $x_l(E_n)$ are the classical turning points for a particle with energy $E_n$ moving in the potential $V_{fp}(x)$. The Fermi energy is then determined by

$$1 = \frac{1}{\pi} \int_{x_l(E_F)}^{x_r(E_F)} dx \sqrt{E_F - V_{fp}(x)}$$

(32)

while the energy density is

$$\rho(E) \equiv \frac{\partial n}{\partial E} = \frac{N}{2\pi} \int_{x_l(E)}^{x_r(E)} dx \frac{1}{\sqrt{E - V_{fp}(x)}}.$$  

(33)

We can now simplify the expression (24) if we restrict ourselves to observables of the form

$$\frac{1}{N} \text{Tr} f(\phi) = \frac{1}{N} \sum_{i=1}^{N} f(x_i).$$

(34)

By expanding the Slater determinant and using the orthogonality of the single particle wave functions $\chi_n$ we get

$$\langle \frac{1}{N} \text{Tr} f(\phi) \rangle = \int dx \, u(x) f(x)$$

(35)

$$u(x) \equiv \frac{1}{N} \sum_{i=1}^{N} \chi_i^2(x) ; \quad \int_{-\infty}^{\infty} dx \, u(x) = 1.$$  

(36)

\[3\text{We shall discuss these special regions of coupling constant space later.}\]
The density $u(x)$ can be calculated in the WKB approximation, since the eigenfunctions $\chi_n(x)$ in this approximation in the classically allowed region where the energy $E_n > V_{fp}(x)$ are given by

$$\chi_n(x) \approx \frac{C(E_n)}{(E_n - V_{fp}(x))^{1/4}} \cos \left( N \int_x^\infty dy \sqrt{E_n - V_{fp}(y)} \right)$$

The normalization is fixed by

$$\int_{-\infty}^\infty dx \chi^2(x) = 1$$

where the integration is effectively cut off at the classical classical turning points $x_l(E_n)$ and $x_r(E_n)$. Due to Riemann’s lemma the $\cos^2$ is replaced by $1/2$ in the large $N$ limit and

$$\frac{1}{C^2(E_n)} = \frac{1}{2} \int_{x_l(E_n)}^{x_r(E_n)} \frac{dx}{\sqrt{E_n - V_{fp}(x)}} = \frac{\pi}{N} \rho(E_n)$$

We conclude that $u(x)$ in the large $N$ limit is

$$u(x) \equiv \frac{1}{N} \sum_{i=1}^N \chi_i^2(x) = \frac{1}{2\pi} \int_{V_{fp}(x)}^{E_F} dE \frac{1}{\sqrt{E - V_{fp}(x)}} = \frac{1}{\pi} \sqrt{E_F - V_{fp}(x)}$$

It is clear from (35) that $u(x)$ should be given the interpretation of density of eigenvalues in the original matrix model. Formulae (35)-(40) give the corresponding derivation in the context of stochastic regularization and we see that the finite range of eigenvalues in the large $N$ limit is determined by the classical turning points corresponding to the Fermi energy $E_F$.

Although we know from the general theorems [6] that $u(x)$ has to agree with the eigenvalue density in the original 0-dimensional matrix model, it is interesting to see how this comes about. From the classical work of Brezin et al. [18] we know that the semiclassical eigenvalue density of a matrix model described by a potential $V(x)$ is given by

$$u(x) = \frac{1}{\pi} \sqrt{\frac{1}{k - 1} V''(x) - \frac{1}{4}(V'(x))^2 + P_{k-3}(x)}$$

if $V(x)$ is a polynomial of order $k$. In (41) the polynomial $P_{k-3}(x)$ of order $k - 3$ is fixed by requiring the support of $u(x)$ to be connected, In the case where $k = 3$ we get

$$u(x) = \frac{1}{\pi} \sqrt{P_0 - V_{fp}(x)}$$

This argument was first presented in [10].
and the constant $P_0$ is fixed by the normalization condition for $u(x)$. A glance on (32) allows us to identify $P_0$ with the Fermi energy in the potential $V_{fp}(x)$.

![Diagram](image)

Fig.2 The figure to the left (fig.2a) has $g > g_c$. For the other figure (fig.2b) $g < g_c$. The dashed lines represent the Fermi energy in the two cases.

If we return for a moment to the original matrix model given by the potential $V(x) = gx - x^3/3$ the sum over planar diagrams which constitutes the large $N$ limit has a finite radius of convergence as a power series in the coupling constant $(1/g)^{3/4}$, as explained in the introduction. This radius is determined from the eigenvalue distribution $u(x)$. The power series is convergent as long as the function $\sqrt{V_{fp}(z) - E_F}$, $z$ complex, has a single cut on the real axis. This is only possible if the fourth order polynomial $V_{fp}(x)$ has the form shown in fig. 2a and if $E_F$ coincides with the second minimum of the potential, since in this case

$$V_{fp}(z) - E_F = \frac{1}{4}(z - x_+)^2(z - x_r)(z - x_l)$$

(42)

where $x_+$ denotes the position of the local minimum at the far right, while $x_l$ and $x_r$ are the classical turning points in $V_{fp}$ for the energy $E_F$. It is a remarkable fact that $E_F$ coincides with the local minimum at $x_+$. From fig.2a it is clear that there will be a critical value $g_c$ below which (42) cannot be realized and where $x_+$ will in

\[ ^5 \text{It is a power series in } (1/g)^{3/4} \text{ rather than } g \text{ due to the rescaling and translation } \tilde{x}^2 - \tilde{g}\tilde{x}^3 \to gx - x^3/3. \]
fact split in two complex conjugate zeroes. For this value of $g_c$, $x_+$ and $x_-$ merge to a single point $x_c$ where

$$V'_x(x_c, g_c) = 0, \quad V''_{xx}(x_c, g_c) = 0. \quad (43)$$

We find from (43)

$$g_c = \frac{3}{2^{2/3}}, \quad x_c = \sqrt[3]{g_c}. \quad (44)$$

The planar expansion will be convergent for $g > g_c$ and the eigenvalue distribution for a given $g > g_c$ will be located between $x_l$ and $x_r$. This is true to any finite order in the $1/N$ expansion. Note however that while the original 0-dimensional matrix model made no sense below $g_c$ the stochastically regularized model is perfectly well defined even in this range of coupling constants in agreement with general results [3].

4 The double scaling limit

Let us consider the situation where $g > g_c$ (fig. 2a) and show how the double scaling limit as a function of only one parameter,

$$\hbar^2 = \frac{4g_c^{5/2}}{N^2(g - g_c)^{5/2}} \quad (45)$$

appears in a trivial way as the WKB expansion of the one-particle hamiltonian $H_{fp}$.

If we introduce a scaled variable

$$z = \sqrt{\frac{g_c}{g - g_c}} \frac{x - x_c}{x_c} \quad (46)$$

the Fokker-Planck hamiltonian (22) can be written as

$$H_{fp} = V_{fp}(0; g - g_c) + \frac{4}{g_c^{3/2}} \frac{(g - g_c)^{3/2}}{g_c^{3/2}} h_{fp}(z) \quad (47)$$

$$h_{fp}(z) = -\hbar^2 \frac{d^2}{dz^2} + v_{fp}(z; \sqrt{g - g_c}) \quad (48)$$

$$v_{fp}(z; \sqrt{g - g_c}) = -3z + z^3 + \sqrt{\frac{g - g_c}{g_c}} \left[ -\frac{3}{2} z^2 + \frac{1}{4} z^4 \right]$$

$$= v_{ds}(z) + o(\sqrt{g - g_c}) \quad (49)$$

The constant $V_{fp}(0; g - g_c)$ is a second order polynomial in $g - g_c$. The double scaling limit is defined as $N \to \infty$, $g \to g_c$, $\hbar$ fixed, and we see that the physics in this limit is determined by the hamiltonian

$$h_{ds}(z) = -\hbar^2 \frac{d^2}{dz^2} + v_{ds}(z) \quad (50)$$
To be precise we have dropped the trivial analytic behaviour in \( g - g_c \) present in (47) in the term \( V_{fp}(0; g - g_c) \). Likewise one has to extract a power of \( g - g_c \) in order to get a non-trivial result in the limit \( g \to g_c \), as is well known from the conventional analysis of the 0-dimensional matrix models.

The Hamiltonian (50) admits a WKB expansion in \( \hbar^2 \). The fact that 0-dimensional matrix models allow for such an expansion was a non-trivial result in the original formulation (as well as in the continuum approach using Liouville theory), but by the method of stochastic regularization we get it almost for free, as was also the case for the eigenvalue distribution and the critical point in the large \( N \) limit.

Since the expansion in the double scaling limit is identical to the WKB expansion of (50) it is worth while to formulate explicitly the exact WKB equation corresponding to (50). The leading order was already discussed above, but we can write in general:

\[
\chi(z) = A(z)e^{\pm iS(z)/\hbar} \tag{51}
\]

\[
A(z) = A^{(0)}(z) + \hbar^2 A^{(1)}(z) + \cdots \tag{52}
\]

\[
S(z) = S^{(0)}(z) + \hbar^2 S^{(1)}(z) + \cdots \tag{53}
\]

and the Schrödinger equation

\[
h ds(z)\chi_e(z) = e\chi_e(z) \tag{54}
\]

is equivalent to the exact WKB equations:

\[
S' = c/A^2 \tag{55}
\]

\[
-\hbar^2 A''A^3 + c^2 = (e - v_{ds}(z))A^4. \tag{56}
\]

The constant \( c \) in (55) is determined by the normalization of \( \chi_e \). If we introduce \( R = A^2 \) (56) can be written as

\[
-\hbar^2 \left( R''_e(z)R_e(z) - \frac{1}{2}R'_e(z)^2 \right) + 2c^2 = 2(e - v_{ds}(z))R^2_e(z) \tag{57}
\]

or differentiating in order to get rid of \( c \), if wanted:

\[
-\frac{\hbar^2}{2}R''_e(z) + v'_{ds}(z)R_e(z) = 2(e - v_{ds}(z))R'_e(z). \tag{58}
\]

This is the so-called non-perturbative equation first derived in [14] by means of the Dikii-Gelfand equation. We see that it is nothing but the WKB equation for \( A^2_e(z) \) and in our opinion it seems to have no advantage compared to the original Schrödinger equation (54).
At first sight it appears as if we have gained nothing if we take the double scaling limit as in (50). The problem of dealing with the original unbounded action has been replaced by the problem of how to deal with the hamiltonian (50), which is unbounded from below. However, hamiltonians unbounded from below are perfectly respectable objects from a mathematical point of view, as we will review in the next section. They have not played a significant role in quantum mechanics, due to the lack of situations where they appear in a natural way. It is somewhat paradoxical that one has to go all the way to quantum gravity to find such a situation.

5 Hamiltonians unbounded from below

Consider a hamiltonian like (50), where the potential is unbounded from below. To be more precise we will write

\[ h = -\frac{d^2}{dy^2} + v(y) \]  

where we assume that \( v(y) \rightarrow -\infty \) for \( y \rightarrow -\infty \) so fast that

\[ \int_{-\infty}^{0} \frac{dy}{\sqrt{|v(y)|}} < \infty \]  

This condition means that a classical particle, once it is not trapped in local energy minima of the potential \( v(y) \), will move to \( y = -\infty \) in a finite time since a classical particle with energy \( e \) and hamiltonian \( p^2 + v(y) \) will move from \( y_t \) to \( -\infty \) in time

\[ T = \int_{-\infty}^{y_t} \frac{dy}{\sqrt{e - v(y)}} \]  

provided \( e - v(y) > 0 \) for all \( y \leq y_t \). The dynamical problem is not classical complete, in the sense that we need to specify some boundary conditions at \( y = -\infty \) if we want to be able to address dynamical questions ranging over all times. From this point of view the mathematical aspects of the situation are not much different from the considerations of dynamics in a finite box. The situation transfers to the quantum mechanical case as well. Had (59) been defined in a finite box we would have been forced to impose a boundary condition for each wall in the box in order to get a self-adjoint operator. For each wall there would be a one-parameter family of self-adjoint extensions of the symmetric operator defined by (59) and acting on functions with support which does not include the coordinate of the wall itself. The spectral theory of such operators is well known. In the case of two walls and no singularities of \( v(y) \) the spectrum is purely discrete and extends to infinity for any
of the self-adjoint extensions. What is less known to physicists is that this result extends to potentials unbounded from below and satisfying (60). In fact the usual situation of walls with imposed boundary conditions can be treated as a special case of the potentials unbounded from below, and although this might seem somewhat perverse from the point of view of physics, it is not unnatural from a mathematical point of view, where both situations can be classified as Weil’s circle-limit case of the Sturm-Liouville theory of second order differential operators (see for instance [19] for a recent discussion). The important conclusion is that the spectra of the self-adjoint extensions of (59) are purely discrete and extend from $-\infty$ to $+\infty$, provided (60) is satisfied and the potential behaves in a similar way for $y \to +\infty$ or $v(y) \to \infty$ for $y \to \infty$.

It is not the purpose here to describe in any detail the Sturm-Liouville theory for potentials unbounded from below, but the important points are easily explained. Suppose we want to solve the differential equation

$$h\psi = e\psi$$

(62)

where $h$ is given by (59) and (60). Far to the left the potential goes to $-\infty$ and the WKB-approximation becomes an excellent approximation. In this region we can write:

$$\psi(y; e, \alpha) \sim \frac{1}{(e - v(y))^{1/4}} \cos \left\{ \left( \int_y^{y_t(e)} \sqrt{e - v(x)} \, dx + \alpha \right) \right\}$$

(63)

where $\alpha$ is an arbitrary angle and $y_t(e)$ denotes the classical turning point for a particle coming from the left (we assume that $e$ is so small that this point exists). We see that any solution to (62) is square-integrable at $-\infty$ thanks to the condition (60). However, this class of functions is too large to constitute the domain of a self-adjoint version of $h$ since a partial integration leads to a term:

$$\psi^*_1(y; e_1, \alpha_1) \frac{d\psi_2(y; e_2, \alpha_2)}{dy} - \psi^*_2(y; e_2, \alpha_2) \frac{d\psi_1(y; e_1, \alpha_1)}{dy}$$

(64)

which does not go to zero for $y \to -\infty$. Special choices of $\alpha$ ensure the convergence, namely

$$\alpha(e) = -\int_{-\infty}^{y_t(e)} dy \left[ \sqrt{e - v(y)} - \sqrt{e_0 - v(y)} \right] + \int_{y_t(e)}^{y_t(e_0)} \sqrt{e_0 - v(y)}$$

(65)

where $e_0 \geq e$ (and where we again assume that $e_0$ is chosen such that $y_t(e_0)$ exists). We see that the integrals in (65) give a well defined way to write the formal difference in WKB phases corresponding to energies $e$ and $e_0$:

$$\alpha(e) = \int_{-\infty}^{y_t(e_0)} dy \sqrt{e_0 - v(y)} - \int_{-\infty}^{y_t(e)} dy \sqrt{e - v(y)}$$

(66)
With this choice all functions in (63) have the asymptotic behaviour
\[ \psi_{e_0}(y) \sim \frac{1}{(-v(y))^{1/4}} \cos \left\{ \left( \int_y^{y_{e_0}} \sqrt{e_0 - v(x) \cdot dx} + o((e_0 - e)y/\sqrt{-v(y)}) \right) \right\} \]
and for fixed \(e_0\) (64) goes to zero for \(y \to -\infty\). The different self-adjoint extensions will be characterized by different choices of the parameter \(e_0\). Due to the asymptotic behaviour (67) of the wave functions it is possible to characterize the unbounded hamiltonian as a sequence of ordinary bounded hamiltonians [20]. For \(-y\) sufficiently large all wave functions \(\psi(y; e, e_0)\) will vanish at points \(y_n\) given by:
\[ \int_{y_n}^{y_{e_0}} dy \sqrt{e_0 - v(y)} = \pi(n + 1/2) + o \left( \frac{(e_0 - e)y}{\sqrt{-v(y)}} \right) \]
(68)
Consequently one will get the same result if one replaces the unbounded potential by a potential \(v_n(y)\) cut off by an infinitely high wall at \(y_n\), provided \(|e|\) is not too large. In this way the self-adjoint hamiltonian, unbounded from below and characterized by the parameter \(e_0\), is reached by a sequence of ordinary hamiltonians, bounded from below, and corresponding to the potentials \(v_n(y)\).

This discussion, based on the WKB approximation, can be made mathematical rigorous, but it is worth emphasizing that it can be formulated independently of the WKB expansion. Let \(e\) be given and let \(\psi_1(y; e)\) and \(\psi_2(y; e)\) be two independent solutions to (62) with a wronskian (which of course is independent of \(y\))
\[ w = \psi_2'(y)\psi_1(y) - \psi_2(y)\psi_1'(y). \]
(69)
If we define a class of functions by
\[ f(y) = \psi_1(y; e) \left\{ c_1 + \frac{1}{w} \int_{-\infty}^{y} dx \psi_2(x; e)g(x) \right\} + \psi_2(y; e) \left\{ c_2 - \frac{1}{w} \int_{-\infty}^{y} dx \psi_1(x; e)g(x) \right\} \]
(70)
where \(g\) is any square-integrable function, it is readily seen that
\[ hf = ef + g \quad \text{and} \quad f(y) \to c_1\psi_1(y; e) + c_2\psi_2(y; e) \quad \text{for} \quad y \to -\infty. \]
(71)
Since \(f\) is square-integrable, \(h\) maps \(f\) into a square-integrable function, and using the asymptotic behaviour (71) in (74) it is seen that the class of functions defined by (71) constitutes the domain of a self-adjoint version of \(h\) provided the so-called (complex) limit numbers \(c_1\) and \(c_2\) satisfy
\[ c_1 \cos \alpha_0 + c_2 \sin \alpha_0 = 0 \]
(72)
for some real $\alpha_0$. In this way we recover the WKB results above.

If we assume that the potential also satisfies a condition like (60) for $y \to \infty$ we can repeat the discussion above and get a relation like (72), just with other constants $c'_1, c'_2$ and $\alpha'_0$. In order to solve the eigenvalue equation we have to choose $g$ equal to zero in (70) and we get a matching condition at (say) $y = 0$ for $f$ and $f'$. For fixed $\alpha_0$ and $\alpha'_0$ the matching can only be satisfied for certain discrete values of $e$, leading to a quantization of the energy eigenvalues in much the same way as for a particle in a box of finite width. If the potential instead goes to $+\infty$ for $y \to \infty$ we also get a purely discrete eigenvalue spectrum. The solutions to (62) will in this case consist of exponentially growing and exponentially decaying parts for $y \to \infty$. The linear combination of $\psi_1(y; e)$ and $\psi_2(y; e)$ dictated by (72) will in general contain an exponentially growing part. Only for special discrete values of $e$ will the wave function be exponentially decaying and therefore square-integrable for $y \to \infty$. This last situation will be the one of interest to us.

6 Application to $v_{ds}(z) = -3z + z^3$

Let us apply the formalism of unbounded hamiltonians to the hamiltonian (50) and compare the results with similar results obtained by using the full Fokker-Planck potential. We first discuss the quantization of energies. The potential has a local maximum at $z = -1$ and a local minimum at $z = +1$ with a value $v_{ds}(z = +1) = -2$. The same value of the potential is obtained if $z = -2$, which therefore is the classical turning point when the energy coincides with the local minimum, as is the case for the Fermi energy in the WKB limit for the full Fokker-Planck potential $v_{fp}(z)$ given in (49). Let us denote the Fermi energy of the full Fokker-Planck hamiltonian (48) by $e_f$. It is trivially related to the $E_F$ defined previously by

$$E_F = V_{fp}(0; g - g_c) + x_e^4 \frac{(g - g_c)^{3/2}}{g_c^{3/2}} e_f. \tag{73}$$

As mentioned above the spectrum for $h_{ds}(z)$ is purely discrete. If $\hbar$ is small and $e \leq e_f$ we know that the solutions to the Schrödinger equation are well approximated by the WKB solutions (63) to the left of the classical turning point $z_t(e)$. To the right of the turning point it is given by similar expressions which fall off or increase exponentially. The WKB matching condition to the exponentially decreasing solution is

$$\psi(z; e) \sim \frac{1}{(e - v_{ds}(z))^{1/4}} \cos \left\{ \frac{1}{\hbar} \int_z^{z_e(e)} \sqrt{e - v_{ds}(z')} \, dz' - \pi/4 \right\} \tag{74}$$
and the quantization of energies in the WKB expansion comes about by comparing with the phase requirement \((\text{65})\) which was needed in order that we had a self-adjoint hamiltonian.

If we let \(\epsilon'_0\), the parameter characterizing the self-adjoint extension in the WKB language (not an eigenvalue), be of order one, we get for the \(n'\)th eigenvalue below \(\epsilon'_0\), \(e_n\):

\[
-\pi\hbar(n + 3/4) = \int_{-\infty}^{z_1(\epsilon_n)} dz \left( \sqrt{e_n - v_{ds}(z)} - \sqrt{\epsilon'_0 - v_{ds}(z)} \right) - \int_{z_1(\epsilon_n)}^{z_1(\epsilon'_0)} dz \sqrt{\epsilon'_0 - v_{ds}(z)}.
\]

(75)

We can solve this equation in two regions (assuming as usual that \(\hbar\) is small). Assume first that \(\epsilon'_0 - e_n\) is of the order of \(\hbar\), and denote by \(\epsilon_0\) the first eigenvalue below \(\epsilon'_0\). We get:

\[
e_n = \epsilon_0 - 2\hbar \omega n + o \left( (\hbar n)^{3/2} \right)
\]

(76)

where

\[
\omega = \pi / \int_{-\infty}^{z_1(\epsilon_0)} dz \frac{1}{\sqrt{\epsilon_0 - v_{ds}(z)}}
\]

(77)

has the interpretation as the cyclic frequency in the classical motion out to infinity in a finite time (see (61)). The only arbitrariness present at the energy levels close to the Fermi energy is therefore an all-over displacement of the energy levels. If we choose \(\epsilon'_0\) such that \(\epsilon_0 = \epsilon_f\), the Fermi energy, we get

\[
\omega \equiv \omega_f = \sqrt{3},
\]

(78)

which is in agreement with the similar calculation done in the full Fokker-Planck potential:

\[
e_n = \epsilon_f - 2\hbar \omega_f n + o \left( (\hbar n)^{3/2} \right) + o \left( (g - g_c)^{1/4} \right).
\]

(79)

The main source to the difference between (78) and (79) is the difference in escape time to infinity in the potential \(v_{ds}(z)\) compared to the time it takes to reach the left turning point in the Fokker-Planck potential \(v_{fp}(z)\). This difference is of the order \((g - g_c)^{1/4}\) and vanishes in the double scaling limit.

The above results are even more manifest if we calculate the energy density for small \(\hbar\) where we expect (75) to be a good approximation and where the eigenvalues are dense

\[
\frac{dn_{ds}}{d|e|} = \frac{1}{2\pi\hbar} \int_{-\infty}^{z_1(e)} \frac{dz}{\sqrt{e - v_{ds}(z)}}.
\]

(80)

We see that all reference to \(\epsilon'_0\) drops out. Let us compare this result with the corresponding result for the full potential \(v_{fp}(y)\). From (83) we get by simple rescaling:

\[
\frac{dn_{fp}}{d|e|} = \frac{1}{2\pi\hbar} \int_{z_1(e)}^{z_1'(e)} \frac{dz}{\sqrt{e - v_{fp}(z)}}.
\]

(81)
where $z_r(e)$ and $z_l(e)$ denote the right and left turning points. We have $z_r(e) = z_t(e) + o(\sqrt{g - g_c})$ and one can check that

$$\frac{\hbar}{de}\frac{dn_{fp}}{de} = \frac{\hbar}{de}\frac{dn_{ds}}{de} + o((g - g_c)^{1/4}) \quad (82)$$

provided $e$ is not too close to the bottom of $v_{fp}(z)$, i.e. provided $e \gg -(g - g_c)^{-3/2}$. Again we see that the difference between the eigenvalue densities of $h_{ds}$ and $h_{fp}$ vanishes in the double scaling limit as $(g - g_c)^{1/4}$, and that this is valid for a much larger energy range than indicated by (76) and (79).

The other region where we can solve (75) is for $-e_n >> -e_f \approx 2$. In this region we get

$$e_n = -\left(\frac{\hbar n}{c}\right)^{6/5} \left(1 + o((n\hbar)^{-4/5})\right) \quad (83)$$

where the constant $c$ is

$$c = \int_{-\infty}^{-1} \left(\sqrt{-y^3} - \sqrt{-1 - y^3}\right) + \int_{-1}^{0} \sqrt{-y^3} \quad (84)$$

We can write (83) as

$$e_n \approx -\frac{1}{c^{6/5}} \left(\frac{n}{N}\right)^{6/5} \frac{1}{(g - g_c)^{3/2}} \quad (85)$$

and even in the case of the the Fokker-Planck potential will this expression for the energy be valid all the way down to the $N$'th level below the Fermi energy. However, the sum of these large negative eigenvalues does not contribute to the non-trivial scaling since we have, using (85)

$$\Delta E \sim N(g - g_c)^{3/2} \sum_{e_n}^N \sim N^2. \quad (86)$$

Although the energy eigenvalues of the self-adjoint extensions agree with the eigenvalues of the full Fokker Planck potential in the double scaling limit up to an all-over displacement we need reference to the full Fokker Planck potential in two ways if we want to calculate physical observables like $\langle \text{Tr } \phi \rangle$ or more generally (64). We need it to define the top level (i.e. the Fermi energy) and the bottom level in the summation (60), but we also need it to define a cut-off for large negative $z$ when we calculate expectation values of operators like $\langle \text{Tr } \phi \rangle$. The reason is that although the density $u(z)$ defined from (60) is integrable out to $z = -\infty u_{fp}(z)$ and $u_{ds}(z)$ will differ when we go beyond the left turning point of for the full Fokker Planck potential. Beyond the left turning point,

$$z_l(e) \approx -4\sqrt{g_c} \quad \sqrt{g - g_c} \quad (87)$$
$u_{fp}(z)$ falls of exponentially. It even vanish beyond $z_1(e)$ in the large $N$ limit. This is not the case for $u_{ds}(z)$ and integrals like $\int z^n u_{ds}(z)$ diverge for $\Lambda \to -\infty$. If we introduce a cut-off

$$\Lambda \sim -\frac{1}{\sqrt{g - g_c}} \quad (88)$$

we get however the correct scaling behaviour of our observables. The only features we need to borrow from the full Fokker Planck potential in order to get the correct scaling behaviour of observables in the double scaling limit is thus a Fermi energy of order one below the local maximum of $v_{ds}(z)$ and a cut-off $\Lambda$ in the negative $z$ as given by (88). Note that such a cut-off by an infinite wall is by no means unnatural for the self-adjoint extensions since their wave functions converge to the same oscillating function \((67)\), independent of the eigenvalues $e_n$, for $z \to -\infty$. We can now put the wall at one of the zeros of the oscillating function \((67)\) and this will not influence the spectrum or eigenfunctions. If we are only interested in the non-trivial scaling behaviour of the physical observables which is associated with the asymptotic WKB expansion in $\hbar^2$ all of the self-adjoint extensions can be used. If we however are interested in the full non-perturbative contributions which survive in the double scaling limit, but which cannot be expressed as powers of $\hbar^2$, the different self-adjoint extensions will differ, and we have to use either the full Fokker Planck potential or the unique self-adjoint extension which has an energy eigenvalue exactly equal to the Fermi energy $e_f$ of the full Fokker Planck potential. This will be clear in a moment when we calculate the lowest non-perturbative correction in the limit of small $\hbar$.

Let us finally address the non-perturbative corrections which are not to be found as powers of $\hbar$ within a systematic WKB expansion. When $g > g_c$ we have the situation in fig.2a. We have two wells. A large one (unbounded from below in the double scaling limit) and a smaller one above the Fermi energy to the right. In the limit of small $\hbar$ the tunneling through the barrier will be exponentially suppressed as $e^{-\Gamma/\hbar}$, where $\Gamma$ will be calculated below, and to a first approximation we have independent states in the left well and the right well. However, the situation is remarkably similar to that of a symmetric well, even if there at first sight is nothing symmetric about the situation. First we note that the level spacing is the same on both sides of the barrier. From \((77)\) and \((78)\) we get that the level spacing in the left well is given by $2\hbar \omega_f$ where $\omega_f = \sqrt{3}$. The level spacing in the right well is determined by expanding $h_{ds}(z)$ around the local minimum at $z = 1$ and we get with $\tilde{z} = z - 1$:

$$h_{ds}(\tilde{z}) = -\hbar^2 \frac{d^2}{d\tilde{z}^2} + 3\tilde{z}^2 + \tilde{z}^3 \quad (89)$$

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$$h_{ds}(\tilde{z}) = -\hbar^2 \frac{d^2}{d\tilde{z}^2} + 3\tilde{z}^2 + \tilde{z}^3 \quad (89)$$
and the harmonic frequency which determines the lowest lying levels in the semi-classical approximation is just \( \omega = \sqrt{3} \). Not only is the level spacing the same on both sides of the barrier, but in addition the Fermi energy in the well to the left coincides precisely with the lowest energy in the well to the right.

![Fig.3 The energy levels in the two wells. The lowest energy in well to the right coincides with the Fermi energy in the well to the left if this energy is calculated according to the WKB prescription.]

Until now we have determined the Fermi energy by (32), which in scaled variables reads

\[
\hbar N = \frac{1}{\pi} \int_{z_l(e_f)}^{z_r(e_f)} dz \sqrt{e_f - v_{fp}(z)}. \tag{90}
\]

As remarked above (see (42)) this implies that \( e_f = -2 \), i.e. \( e_f \) coincides with the local minimum of the right well. However, to be precise one should in the context of WKB expansion replace \( N \) by \( N + \frac{1}{2} \) and this leads to a shift of the Fermi energy such that it agrees with the lowest energy in the right well. We have now a situation identical to the standard double well. By tunneling the degenerate energy levels split in two and the lowest one will now have to be identified with the Fermi energy. The shift is non-perturbative in \( \hbar \) and is given by the standard WKB formula:

\[
e_f \rightarrow e_f - \frac{\omega_f}{\pi} \cdot e^{-\Gamma/\hbar} \tag{91}
\]

where the tunneling amplitude \( \Gamma \) is given by

\[
\Gamma = \int_{z_r}^{z_l} dz \sqrt{v_{fp}(z) - e_f} \approx \int_{-2}^{1} dz \sqrt{v_{ds}(z) + 2} = \frac{12\sqrt{3}}{5}. \tag{92}
\]
In (92) \( \tilde{z}_l \) is the position of the left turning point for a classical particle with energy \( e_f \) in the right well, while \( z_r \) as usual denotes the right turning point for a classical motion with energy \( e_f \) in the left well. Again we see that it is possible in this semiclassical calculation to replace \( v_{fp} \) by \( v_{ds} \), in which case the integral can be performed since \( v_{ds}(z) + 2 = (1 - z)^2(2 + z) \).

The lower levels \( e_n, \quad n > 0 \) will receive contributions which are exponentially small in \( 1/\hbar \) compared to the one received by \( e_f \) since they are located below the local minimum of the well to the right. We note that the contribution (91) is identical to the “non-perturbative” ambiguity which comes from the Painleve equation of ordinary gravity (but in the stochastic approach it is of course not an ambiguity). This observation was first made in refs. [21, 22] and is in agreement with the fact that the WKB expansion of the stochastically regularized model agrees with the genus expansion of the original matrix model of pure 2d gravity.

7 The strong coupling expansion

The theory outlined above offers the possibility of a convergent strong coupling expansion. It might be useful to recall a similar situation in the case of the ordinary anharmonic oscillator in quantum mechanics. Let the hamiltonian be given by

\[
H = -\frac{d^2}{dx^2} + x^2 + gx^4, \quad g > 0. \tag{93}
\]

The ordinary perturbation expansion around the harmonic oscillator is only an asymptotic expansion. The reason is that \( gx^4 \) is not a small perturbation for any value of \( g > 0 \) and clearly \( g \to -g \) changes drastically the nature of \( H \). For this reason the expansion in powers of \( g \) is not a convergent power expansion, but only an asymptotic expansion. It is, however, possible to analyze the strong coupling region \( g \to \infty \) by a simple scaling argument:

\[
H(x) = \lambda^{-1/2} \tilde{H}(y), \quad y = g^{1/6}x, \quad \lambda = \frac{1}{g^{2/3}} \tag{94}
\]

\[
\tilde{H}(y) = -\frac{d^2}{dy^2} + y^4 + \lambda y^2 \tag{95}
\]

From this we deduce that \( \tilde{H}(y) \) has a strong coupling expansion, which is analytic in \( \lambda = 1/g^{2/3} \). In fact the potential \( V(y) = y^2 \) satisfies the standard requirement for being an analytic perturbation of

\[
H_0(y) = -\frac{d^2}{dy^2} + y^4
\]
namely that \( V \) is \( H_0 \)-bounded, i.e. (1) the domain \( D(V) \supseteq D(H_0) \) and (2) \( ||V\psi||_2 \leq a||H_0\psi||_2 + b||\psi||_2 \) for some \( a \) and \( b \) and all \( \psi \in D(H_0) \). For an eigenvalue \( E_n \) of the original Hamiltonian \( H \) we can therefore write:

\[
E_n(g(\lambda)) = \lambda^{-1/2} \sum_{k=0}^{\infty} c_{nk} \lambda^{k}
\]  

(96)

where the power series has a finite radius of convergence.

The situation is very similar in the case of the Hamiltonian \( h_{fp}(z) \). In sect. 4 we introduced a scaling which was designed to make contact with the asymptotic genus expansion of ordinary 2d quantum gravity, i.e. the limit \( \hbar \to 0 \). But we can perform the same scaling argument as just outlined for the anharmonic oscillator and derive an expansion for large \( \hbar^2 \), i.e. in the strong coupling regime. If we introduce scaled variables

\[
z = \hbar^{2/5} y, \quad \lambda = \frac{1}{\hbar^{4/5}} = \frac{g - g_c}{g_c} \left( \frac{N}{2} \right)^{4/5}
\]

(97)

we can write

\[
h_{fp}(z) = \lambda^{-3/2} \tilde{h}_{fp}(y)
\]

(98)

\[
\tilde{h}_{fp}(y) = -\frac{d^2}{dy^2} + \tilde{v}_{fp}(y, \lambda, N)
\]

(99)

\[
\tilde{v}_{fp}(y) = y^3 - 3\lambda y + \left( \frac{2}{N} \right)^{2/5} \left[ -\frac{3\lambda}{2} y^2 + \frac{1}{4} y^4 \right]
\]

\[
= \tilde{v}_{ds}(y, \lambda) + \left( \frac{2}{N} \right)^{2/5} \left[ -\frac{3\lambda}{2} y^2 + \frac{1}{4} y^4 \right]
\]

(100)

and the double scaling limit is obtained as before for \( N \to \infty, g \to g_c \), but \( \lambda \) fixed.

The non-trivial information is contained in the Hamilton function \( \tilde{h}_{fp}(y) \) which in the double scaling limit goes to

\[
\tilde{h}_{ds}(y) = -\frac{d^2}{dy^2} + \tilde{v}_{ds} = -\frac{d^2}{dy^2} + y^3 - 3\lambda y
\]

(101)

and we have formally the same situation as for the anharmonic oscillator: \( -3\lambda y \) looks like a small perturbation with respect to \( y^3 \). It is thus natural to expect a strong coupling expansion of the energy eigenvalues of \( h_{ds}(z) \) of the form:

\[
e_n(\hbar) = \lambda^{-3/2} \sum_{k=0}^{\infty} c_{nk} \hbar^k = \hbar^{6/5} \sum_{k=0}^{\infty} c_{nk} \hbar^{-4k/5}
\]

(102)

where the series is convergent. The domain of the perturbation \( v(y) = -3\lambda y \) does not include the domain of \( h_0(y) = -d^2/dy^2 + y^3 \) so we have no rigorous proof of this conjecture, but one should keep in mind that the requirements (1) and (2) mentioned above are only sufficient conditions, not necessary conditions, for analyticity.
8 Discussion and conclusion

We have shown that many of the results of the matrix models are easily and transparently derived by means of stochastic quantization. The asymptotic expansion, called the double scaling limit, is nothing but the WKB expansion of the Fokker-Planck hamiltonian. In addition stochastic quantization provides a non-perturbative definition of 2d quantum gravity. To be entirely correct one should first calculate expectation values of observables using the full Fokker-Planck hamiltonian and afterwards take the double scaling limit $N \to \infty$, $g \to g_c$, with $\hbar \sim N^{-1}(g - g_c)^{-5/4}$ fixed. If we reverse the procedure the double scaling limit of the Fokker-Planck hamiltonian results in a hamiltonian $h_{ds}$ which is unbounded from below and which has a one-parameter family of self-adjoint extensions. One member is picked out by the requirement that the Fermi energy should coincide with the corresponding energy of the full Fokker-Planck potential. We mentioned the possibility of a strong coupling expansion, similar to the one of the anharmonic oscillator.

It is interesting to compare the situation described above with the similar situation present for 2d gravity coupled to a scalar field (i.e. $c = 1$). In this case we again have a matrix model description which, in the double scaling limit, results in a Schrödinger eigenvalue equation:

$$\left[-\hbar^2 \frac{d^2}{dy^2} - y^2\right] \psi_n = \varepsilon_n \psi_n. \quad (103)$$

where $\hbar^{-1} = N(g_c - g)$ is fixed. Again the ground state is fermionic and given by the Slater determinant of the $N$ eigenfunctions counted from the top of the upside-down quadratic potential in (103). One could be tempted to apply the methods outlined above to this hamiltonian, which is unbounded from below, but condition (60) is not satisfied for this potential, and the problem is classical complete: It takes an infinite time for a classical particle to escape to infinity. In the terminology of Weil it is a point-limit case of the Sturm-Liouville theory of second order differential equations. The point spectrum is empty and there is a unique self-adjoint extension. If we cut off the potential at $\pm L$ we get a sequence of Hamiltonians which are bounded from below and which convergences to the self-adjoint extension of the lhs of (103) for $L \to \infty$. The (generalized) eigenfunctions are parabolic cylinder functions and they allow for the calculation of both local [26] and non-local [27] observables within the WKB approximation, but although the self-adjoint extension of hamiltonian given by (103) is unique and defined beyond the WKB expansion, it is still not clear whether it allows a strong coupling expansion for small $\hbar$ as conjectured in [24].
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