Combinatorics of involutive divisions

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Abstract

The classical involutive division theory by Janet decomposes in the same way both the ideal and the escalier. The aim of this paper, following Janet’s approach, is to discuss the combinatorial properties of involutive divisions, when defined on the set of all terms in a fixed degree $D$, postponing the discussion of ideal membership and related test.

We adapt the theory by Gerdt and Blinkov, introducing relative involutive divisions and then, given a complete description of the combinatorial structure of a relative involutive division, we turn our attention to the problem of membership. In order to deal with this problem, we introduce two graphs as tools: one is strictly related to Seiler’s $L$-graph, whereas the second generalizes it, to cover the case of ”non-continuous” (in the sense of Gerdt-Blinkov) relative involutive divisions. Indeed, given an element in the ideal (resp. escalier), walking backwards (resp. forward) in the graph, we can identify all the other generators of the ideal (resp. elements of degree $D$ in the escalier).

1 Introduction

Denote by $\mathcal{P} := \mathbb{k}[x_1,...,x_n]$ the graded ring of polynomials in $n$ variables with coefficients in the field $\mathbb{k}$, $\text{char}(\mathbb{k}) = 0$ and by $\mathcal{T} := \{x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n} | \gamma = (\gamma_1,...,\gamma_n) \in \mathbb{N}^n\}$ the semigroup of terms generated by the set $\{x_1,...,x_n\}$.

Given a monomial/semigroup ideal $J \subset \mathcal{T}$ and its minimal set of generators $\mathcal{G}(J)$ (also called its monomial basis), Janet introduced in \cite{Janet} both the notion of multiplicative variables and the connected decomposition of $J$ into disjoint cones. Then, he gave a procedure (completion) to produce such a decomposition.

In the same paper, in order to describe Riquier’s \cite{Riquier} formulation of the description for the general solutions of a PDE problem, Janet gave a similar decomposition in terms of disjoint cones, generated by multiplicative variables, also for the related normal set/order ideal/escalier $\mathcal{N}(J) := \mathcal{T} \setminus J$.

Later in \cite{Janet2} \cite{Janet3} \cite{Janet4}, he gave a completely different decomposition (and the related algorithm for computing it) which labelled as involutive and which is behind both Gerdt-Blinkov \cite{GerdtBlinkov} \cite{GerdtBlinkov2} \cite{GerdtBlinkov3} procedure for computing Gröbner bases and Seiler’s \cite{Seiler} theory of involutiveness.

The aim of Janet in these three papers was twofold:

1. to reinterpret, in terms of multiplicative variables and cone decomposition, the solution of PDE problems given by Cartan \cite{Cartan} \cite{Cartan2} \cite{Cartan3}, whence the name involutiveness;
2. to re-evaluate within his theory the notion of *generic initial ideal* introduced by Delassus [6, 7, 8] and the correction of his mistake by Robinson [31, 32] and Gunther [17, 18], who point out that the notion requires $J$ to be Borel-fixed (a modern but identical reformulation was proposed by Galligo [10], who merged the considerations of Hironaka [22] and Grauert [15]; see also [16] and [9]); Janet remarked that all Borel-fixed ideals are involutive, but the converse is false.

More precisely, in his survey [25] Janet presents, as *nouvelle formes canoniques*, the results by Delassus, Robinson and Gunther and compares them with the one which can be deduced from an involutive basis; and in [26, p.62], assuming to have a given a homogeneous ideal $I \subset \mathcal{P}$ within a generic frame of coordinates, he reformulates Riquier’s completion proposing essentially a Macaulay-like construction, iteratively computing the vector-spaces $I_d := \{ f \in I : \deg(f) = d \}$ until Cartan test grants that Castelnuovo-Mumford [29, pg.99] regularity $D$ has been reached. This would allow him to consider the monomial ideal $T(I)$ of the leading terms (in the sense of Gröbner basis theory) w.r.t. a deg-lex term-ordering and obtain the related involutive reduction required by Riquier’s procedure.

The results on involutiveness presented in both papers, however, simply restate the results of [24] which reinterprets Cartan’s result in terms of multiplicative variables; more precisely Janet assumes to have a set of forms of degree $D$ which satisfies Cartan test and directly considers both the monomial ideal

$$T := T(I) \subset T_{\geq D} := \{ t \in T, \deg(t) \geq D \}$$

and the partial *escalier*

$$N := T_{\geq D} \setminus T = \{ \tau \in N(I) : \deg(\tau) \geq D \}$$

decomposing both of them in terms of disjoint cones, generated by multiplicative variables. Actually, he simply explicitly demotes the rôle of the ideal in this construction considering the whole set $T_{\geq D}$ and decomposing it in terms of disjoint cones generated by multiplicative variables, the related set of vertices being the set of all monomials $T_D = \{ \tau \in T : \deg(\tau) = D \}$.

The aim of this paper is to discuss involutiveness following the approach proposed by Janet in [24]; in particular we postpone the discussion of ideal membership and related test only after having performed a deep reconsideration of the combinatorial properties of involutive divisions [12, 13, 14], when defined on the set $T_D$.

To do so, we of course apply the theory of involutive divisions, set up by Gerdt–Binklov [12, 13, 14], but we are forced to slightly adapt it, talking about *relative* involutive divisions, and requiring that the union of all the cones produces the ideal $T_{\geq D}$ and that the cones are disjoint; in fact our setting considers the *single* finite set $T_D$ and thus does not require (as, of course, they need) comparing different divisions.

Moreover, the aim of their theory is to produce a setting for describing and building a Riquier-Janet procedure for computing Gröbner-like bases for ideals; thus they cannot

\[ \text{La proposition est vraie en particulier pour le système involutif constitué par tout les monomes d’ordre} \ [D]. \ [24, \text{p.46} ] \]
assume neither that the division is involutive, \textit{i.e.} that the union of all the cones defined on a set \( U \) produces the semigroup ideal generated by \( U \) (this being in their setting the aim of the procedure) nor uniqueness of involutive divisors, \textit{i.e.} that that all cones are disjoint (the failure of this condition triggering the completion procedure). These two conditions are instead essential to grant that (the implicit procedure is completed and that) a unique decomposition is available both for the given ideal (granting unique reduction) and its associated \textit{escalier} (granting standard Hironaka-like description of canonical forms).

We discuss the combinatorial structure of relative involutive divisions; we begin with the combinatorial formula given by Janet \cite{Janet, Janet2, Janet3} and Gunther \cite{Gunther} p.184 evaluating, for each \( i, 1 \leq i \leq n \), the number \( \sigma_i \) of the cones having \( i \) multiplicative variables and which is, essentially, an adaptation of Vandermonde’s convolution \cite[pg.492]{Vandermonde}; next we prove a set of Lemmata, which allows us to sketch an approach for imposing a relative involutive division structure on \( \mathcal{T}_{\geq D} \) and which will be generalized to a procedure to list all the possible relative involutive divisions up to symmetries.

We further characterize the relative involutive divisions which are Pommaret divisions up to a relabelling of the variables.

Thus, given a complete description of the combinatorial structure of a relative involutive division, we turn our attention to the problem of membership. Let us begin with the trivial remarks that if a term \( u \in \mathcal{T}_D \) is contained (or is a generator) of the monomial ideal \( \mathcal{T} \),

\begin{itemize}
  \item the whole cone whose vertex is \( u \) is contained in the ideal and that
  \item for each non-multiplicative variable \( x \), there is necessarily a term \( v \in \mathcal{T}_D, v \neq u \) s.t. \( xu \) belongs to the cone whose vertex is \( v \) and that such vertex (and cone) necessarily belongs to \( \mathcal{T} \);
  \item conversely, if \( v \) belongs to \( \mathcal{N} \), not only its related cone belongs to \( \mathcal{N} \), but the same holds to \( u \) and its related cone.
\end{itemize}

Moreover if \( \mathcal{T} \) is not trivial then both

\begin{itemize}
  \item the single monomial \( m \) which has no non-multiplicative variables (called “peak” throughout the paper), and its cone necessarily belong to \( \mathcal{T} \) while
  \item for at least a value \( i, 1 \leq i \leq n, x_i^D \in \mathcal{N} \).
\end{itemize}

On the basis of these remarks, we can define on \( \mathcal{T}_D \) a rooted directed graph whose root is \( m \) and where an arrow \( u \rightarrow v \) is given when, for a non multiplicative variable \( x_i \) for \( u, x_i u \) belongs to the cone whose vertex is \( v \). Of course such graph is redundant and our aim is to give a (more compact, non necessarily minimal) directed graph which has the following properties:

\begin{itemize}
  \item if a vertex \( h \) is included in \( \mathcal{T} \) and we walk against the flow towards the “peak” \( m \), we reach all the terms in \( \mathcal{T}_D \) which necessarily belong to \( \mathcal{T} \) too; and
  \item if a vertex \( n \) is included in \( \mathcal{N} \) and we follow the flow towards the “mouths” we reach all the terms in \( \mathcal{T}_D \) which necessarily belong to \( \mathcal{N} \) too.
\end{itemize}
We begin our investigation giving conditions (based on the computation of lcm’s),
which, for each \( t \in T, s \in U \), allows to deduce further elements \( X(t, s) \) (actually the
vertex of the cone containing lcm \( (s, t) \)) which are necessary members of \( T \). Then we
will give analogous lcm-based conditions for \( N \).

Next we specialize our investigation to Pommaret divisions for which we prove that
it is sufficient to adapt Ufnarovsky graph [36, 37, 38, 39] to the commutative case in
order to obtain a graph which has exactly the shape and properties described above.

Unfortunately, in general, a graph with the properties described above cannot exist;
in fact we show an example in \( n \) variables and degree \( d = n - 1 \), in which the \( d 
\)
monomials with \( n - 1 \) multiplicative variables are connected together, via their single
non-multiplicative variables, in a loop which walks around the “peak” \( m \); moreover
these \( n \) monomials having either \( n \) or \( n - 1 \) multiplicative variables are s.t. if one of them
belongs to either \( T \) the same happens for all of them. Moreover, if one of the terms with
\( n - 1 \) multiplicative variables lies in \( N \), then all terms with \( n - 1 \) multiplicative variables
lie in \( N \), but they do not impose any condition on that with \( n \) multiplicative variables.

Thus, in general, it is impossible to produce a graph as the Ufnarovsky-like existing
for Pommaret division and which has the required structure, simply by multiplying
each monomial \( t \) by its non-multiplicable variables \( x \) and the graph is obtained recording
the cone in which \( xt \) belongs.

The only way we are seeing for producing a graph with the required properties is to
build the redundant graph which can be obtained by testing the condition on lcm \((s, t)\)
and extract a minimal subgraph, an approach which in general is NP-complete.

2 Some general notation

Throughout this paper, in connection with monomial ideals, we mainly follow the no-
tation of [28].

We denote by \( \mathcal{P} := k[x_1, \ldots, x_n] \) the graded ring of polynomials in \( n \) variables with co-
efficients in the field \( k \).

The semigroup of terms, generated by the set \( \{x_1, \ldots, x_n\} \) is:

\[
\mathcal{T} := \{x^{\gamma} : = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \mid \gamma := (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n\}.
\]

If \( \tau = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \), then \( \deg(\tau) = \sum_{i=1}^n \gamma_i \) is the degree of \( \tau \) and, for each \( h \in \{1, \ldots, n\} \)
\( \deg_h(\tau) := \gamma_h \) is the \( h \)-degree of \( \tau \).

For each \( d \in \mathbb{N} \), \( T_d \) is the \( d \)-degree part of \( \mathcal{T} \), i.e. \( T_d := \{x^{\gamma} \in \mathcal{T} \mid \deg(x^{\gamma}) = d\} \) and it
is well known that \( |T_d| = \binom{n+d-1}{d} \). For each subset \( M \subseteq \mathcal{T} \) we set \( M_d = M \cap T_d \). The
symbol \( \mathcal{T}(d) \) denotes the degree \( \leq d \) part of \( \mathcal{T} \), namely \( \mathcal{T}(d) := \{x^{\gamma} \in \mathcal{T} \mid \deg(x^{\gamma}) \leq d\} \).

For each term \( \tau \in \mathcal{T} \) and \( x_{j} | \tau \), the only \( v \in \mathcal{T} \) such that \( \tau = x_{j} \nu \) is called \( j \)-th predeces-
sor of \( \tau \).

A semigroup ordering \( < \) on \( \mathcal{T} \) is a total ordering such that \( \tau_1 < \tau_2 \Rightarrow \exists \tau_1 < \tau \tau_2 \), \( \forall \tau, \tau_1, \tau_2 \in \mathcal{T} \). For each semigroup ordering \( < \) on \( \mathcal{T} \), we can represent a polynomial \( f \in \mathcal{P} \) as a
linear combination of terms arranged w.r.t. $<$, with coefficients in the base field $\mathbf{k}$:

$$f = \sum_{\tau \in \mathcal{T}} c(f, \tau) \tau = \sum_{i=1}^{s} c(f, \tau_i) \tau_i : c(f, \tau_i) \in \mathbf{k}^*, \tau_i \in \mathcal{T}, \tau_1 > ... > \tau_s,$$

with $T(f) = Lt(f) := \tau_1$ the leading term of $f$, $Le(f) := c(f, \tau_1)$ the leading coefficient of $f$ and $tail(f) := f - c(f, T(f))T(f)$ the tail of $f$.

A term ordering is a semigroup ordering such that 1 is lower than every variable or, equivalently, it is a well ordering.

Given a term $\tau \in \mathcal{T}$, we denote by $\min(\tau)$ the smallest variable $x_i$, $i \in \{1, ..., n\}$, s.t. $x_i | \tau$ and, analogously, we denote by $\max(\tau)$ the biggest variable appearing in $\tau$ with nonzero exponent.

A subset $J \subseteq \mathcal{T}$ is a semigroup ideal if $\tau \in J \Rightarrow \sigma \tau \in J$, $\forall \sigma \in \mathcal{T}$; a subset $N \subseteq \mathcal{T}$ is a normal set (or order ideal) if $\tau \in N \Rightarrow \sigma \in N \forall \sigma | \tau$. We have that $N \subseteq \mathcal{T}$ is an order ideal if and only if $\mathcal{T} \setminus N = J$ is a semigroup ideal.

Given a semigroup ideal $J \subset \mathcal{T}$ we define $N(J) := \mathcal{T} \setminus J$. The minimal set of generators $G(J)$ of $J$, called the monomial basis of $J$, satisfies the conditions below

$$G(J) := \{ \tau \in J \mid \text{each predecessor of } \tau \in N(J) \}$$

and

$$= \{ \tau \in \mathcal{T} \mid \{ N(J) \cup \{ \tau \} \text{ is an order ideal, } \tau \notin N(J) \} \}.$$

For all subsets $G \subset \mathcal{P}$, $T(G) := \{ T(g), g \in G \}$ and $T(G)$ is the semigroup ideal of leading terms defined as $T(G) := \{ \tau T(g), \tau \in \mathcal{T}, g \in G \}$.

Fixed a term order $<$, for any ideal $I \in \mathcal{P}$ the monomial basis of the semigroup ideal $T(I) = T[I]$ is called monomial basis of $I$ and denoted again by $G(I)$, whereas the ideal $In(I) := (T(I))$ is called initial ideal and the order ideal $N(I) := \mathcal{T} \setminus T(I)$ is called Groebner escalier of $I$.

### 3 Involutively divisions

In this section, following [12] [13], we recall the main definitions and properties of involutive divisions. We also define, as an example of involutive division, Janet and Pommaret divisions; the latter will be very important in what follows, besides being important for its link with generic initial ideals (see section 8).

First of all, we recall the definition of involutive division.

**Definition 1** (Gerdt-Blinkov, [12]). An involutive division $L$ or $L$-division on $\mathcal{T}$ is a relation $|_L$ defined, for each finite set $U \subset \mathcal{T}$, on the set $U \times \mathcal{T}$ in such a way that the following holds for each $u, u_1 \in U$ and $t, t_1 \in \mathcal{T}$

(i). $u |_L t \Rightarrow u | t$;

(ii). $u |_L u$ for each $u \in U$;

(iii). $u |_L u t, u |_L u t_1 \iff u |_L u t t_1$;

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(iv). \( u \mid_L t, u_1 \mid_L t \implies \text{either } u \mid_L u_1 \text{ or } u_1 \mid_L u; \)

(v). \( u \mid_L u_1, u_1 \mid_L t \implies u \mid_L t; \)

(vi). if \( V \subseteq U \) and \( u \in V \) then \( u \mid_L t \text{ w.r.t. } U \implies u \mid_L t \text{ w.r.t. } V. \)

If \( u \mid_L t = uw \), \( u \) is called an involutive divisor of \( t \), \( t \) is called an involutive multiple of \( u \) and \( w \) is said to be multiplicative for \( u \). If \( u \nmid_L t = uw \), \( w \) is said to be non-multiplicative for \( u \). □

This definition, for each set \( U \) and each \( u \in U \), partitions the set of variables in two subsets

- \( M_L(U, u) \), containing the variables \( x_i \) multiplicative for \( u \):
  \[
  x_i \in M_L(U, u) \iff u \mid_L ux_i;
  \]

- \( NM_L(U, u) \), containing the variables \( x_i \) non-multiplicative for \( u \):
  \[
  x_i \in NM_L(U, u) \iff u \nmid_L ux_i.
  \]

Finally, for each involutive division \( L \), each finite set \( U \subset T \) and each \( u \in U \), we denote by \( L(u, U) \) the multiplicative set for \( u \), i.e. the set of all the terms \( w \in T \) which are multiplicative for \( u \):

\[
L(u, U) := \{ w \in T : u \mid_L uw \}.
\]

Remark that condition (iii) implies that each \( L(u, U) \) is completely characterized by the partition

\[
\{x_1, \ldots, x_n\} = M_L(U, u) \sqcup NM_L(U, u)
\]

since

\[
L(u, U) = \{ x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n} : a_i \neq 0 \implies x_i \in M_L(U, u) \}.
\]

With this notation it is easy to realize that the definition of involutive division can be formulated as follows:

**Definition 2** (Gerdt—Blinkov). An involutive division \( L \) or \( L \)-division on \( T \) is the assignment, for each finite set \( U \subset T \) and each \( u \in U \) of a submonoid \( L(u, U) \subset T \) such that the following holds for each \( u, u_1 \in U \) and \( t, w \in T \):

(a). \( t \in L(u, U), t_1 \mid t \implies t_1 \in L(u, U) \),

(b). if \( uL(u, U) \cap u_1L(u_1, U) \neq \emptyset \) then \( u \in u_1L(u_1, U) \) or \( u_1 \in uL(u, U) \);

(c). if \( u_1 = uw \) for some \( w \in L(u, U) \), then \( L(u_1, U) \subseteq L(u, U) \);

(d). if \( V \subseteq U \) then \( L(u, U) \subseteq L(u, V) \) for each \( u \in V \).

The next definitions 3 and 4 state some properties that a set \( U \subseteq T \), w.r.t. an involutive division \( L \), may satisfy.

**Definition 3** (12). A finite set \( U \subset T \) is called
• involutely autoreduced w.r.t. the division $L$ or $L$-autoreduced if it does not contain elements $L$-divisible by other elements in $U$.

• involutive w.r.t. the division $L$ or $L$-involutive if $\forall u \in U, \forall w \in T \exists v \in U$ s.t. $v \mid_L u$.

**Definition 4** ([12]). A set $U$ is called locally involutive with respect to the involutive division $L$ if $\forall u \in U, \forall x_i \in \mathcal{N}_L(u, U) \exists v \in U$ s.t. $v \mid_L u x_i$.

With the following definition, we introduce the concept of *continuity* for an involutive division $L$.

**Definition 5** ([12]). The involutive division $L$ is called continuous if for each finite set $U \subset T$, and each finite sequence of terms in $U$

$$w_1, \ldots, w_j, \ldots, w_j,$$

such that, for each $j < J$, there is $x_{ij} \in \mathcal{N}_L(U, w_j) : w_{j+1} \mid_L w_j \cdot x_{ij}$ the inequality $w_j \neq w_i$ holds for each $j \neq i$.

**Proposition 6** ([12]). If an involutive division $L$ is continuous then local involutivity of any set $U$ implies its involutivity.

**Example 7** (Janet division). Let $U \subset T \subset k[x_1, \ldots, x_n]$ be a finite set of terms, $x_1 < \ldots < x_n$ and $\tau := x_1^{a_1} \ldots x_n^{a_n} \in U$. For each $j, 1 \leq j \leq n$, the variable $x_j$ is said to be multiplicative for $\tau$ w.r.t. $U$ if there is no $\tau' := x_1^{a_1} \ldots x_j^{\beta_j} x_{j+1}^{a_{j+1}} \ldots x_n^{a_n} \in U$ for which $\beta_j > a_j$.

For example, if $U = \{x^2, xy, z^3\} \subset k[x, y, z]$, $x < y < z$, and Janet division $J$ is defined on $U$, then $M_J(x^2, U) = \{x\}, \mathcal{N}_J(x^2, U) = \{y, z\}, M_J(xy, U) = \{y\}, \mathcal{N}_J(xy, U) = \{z\}$.

**Example 8** (Pommeret division). Consider the polynomial ring $k[x_1, \ldots, x_n]$ and suppose the variable ordered as $x_1 < \ldots < x_n$.

Given a term $t = x_j^{a_j} \ldots x_n^{a_n}$ with $a_j > 0$ Pommeret division considers as multiplicative the variables $x_i, i \leq j$ and non-multiplicative the variables $x_k, k > j$.

Notice that for a term $s = x_n^{a_n}, a_n > 0$, all the variables are multiplicative.

For example, if $U = \{x^2, xy, z^3\} \subset k[x, y, z]$, $x < y < z$, and Pommeret division $P$ is defined on $U$, then $M_P(x^2, U) = \{x\}, \mathcal{N}_P(x^2, U) = \{y, z\}, M_P(xy, U) = \{x\}, \mathcal{N}_P(xy, U) = \{y, z\}, M_P(z^3, U) = \{x, y, z\}, \mathcal{N}_P(z^3, U) = \emptyset$.

**Proposition 9** ([12]). Janet and Pommeret monomial divisions are involutive and continuous.

# 4 Relative involutive divisions

We define now a variation of Gerdt-Blinkov involutive division (Definition 1), calling it *relative involutive division* since it is defined specifically for a fixed set of terms $U \subset T$.

In the next section, restricting to the case $U = T_D \subset k[x_1, \ldots, x_n], D \in \mathbb{N}$, i.e. to the set
of all terms of degree \( D \) in \( n \) variables, we will see the criteria for constructing relative involutive divisions on it.

Let us start from the definition of relative involutive division.

**Definition 10.** Let \( U \subset T \) be a finite set of terms. We say that a relative involutive division \( L \) is given on \( U \) if, for each \( u \in U \) a partition

\[
\{x_1, \ldots, x_n\} = M_L(u, U) \cup NM_L(u, U),
\]

is given on the set of variables s.t. denoted

\[
L(u, U) := \{x_1^{a_1} \cdots x_n^{a_n} | a_i \neq 0 \Rightarrow x_i \in M_L(u, U)\},
\]

the following two conditions hold:

1. \( T(U) = \bigcup_{u \in U} uL(u, U) \);
2. \( \forall u, v \in U, uL(u, U) \cap vL(v, U) = \emptyset \).

The set \( M_L(u, U) \) is called (relative) multiplicative variable’s set, \( NM_L(u, U) \) is called (relative) non-multiplicative variable’s set, whereas \( L(u, U) \) is the set of (relative) multiplicative terms. Denoting by \( C_L(u, U) := uL(u, U) \) the (relative) cone of \( u \in U \), conditions 1 – 2 above may be also rewritten as:

1’. \( T(U) = \bigcup_{u \in U} C_L(u, U) \);
2’. \( \forall u, v \in U, C_L(u, U) \cap C_L(v, U) = \emptyset \).

We will write \( uLw \) if \( w = uv \) and \( v \in L(u, U) \) (so that \( w \in C_L(u, U) \)) and we will say that \( u \) is a (relative) involutive divisor of \( w \) and that \( w \) is a (relative) involutive multiple of \( u \).

In order to give a clear comparison between Gerdt-Blinkov involutive division and our relative involutive divisions, we prove the following Lemma.

**Lemma 11.** A relative involutive division \( L \) on a finite set \( U \subset T \) satisfies conditions (i) – (iii) of Definition[7]

**Proof.** Condition (i) is true by definition, whereas condition (ii) comes trivially observing that \( \forall u \in U, 1 \in L(u, U) \).

We prove now (iii) Suppose that \( uLuv \) and \( uLuw \); \( v = x_1^{a_1} \cdots x_n^{a_n}, w = x_1^{b_1} \cdots x_n^{b_n} \), where each variable appearing in \( v \) and \( w \) with nonzero exponent is multiplicative for \( u \). Take now \( vw = x_1^{a_1+b_1} \cdots x_n^{a_n+b_n} \); if for some \( 1 \leq i \leq n \) it holds \( a_i + b_i \neq 0 \) then either \( a_i \) or \( b_i \) (or both) are different from zero. This implies \( x_i \in M_L(u, U) \) and so \( uLuvw \).

Viceversa, suppose \( uLuvw \); \( vw = x_1^{a_1} \cdots x_n^{a_n} \), with, for \( 1 \leq i \leq n, a_i \neq 0 \Rightarrow x_i \in M_L(u, U), v = x_1^{b_1} \cdots x_n^{b_n}, w = x_1^{c_1} \cdots x_n^{c_n}, b_i + c_i = a_i \) for \( i = 1, \ldots, n \).

If, for some \( 1 \leq i \leq n, b_i \neq 0 \), then \( a_i \neq 0 \), so \( x_i \in M_L(u, U) \) and the same holds for the exponents of the variables appearing in \( w \), so \( v, w \in L(u, U) \) and 3. is proved. \( \square \)
As regards conditions (iv)-(v), we see that they trivially hold since their hypothesis can never happen, because we have imposed the relative cones to be disjoint. Moreover, condition (vi) does not make sense in our context, due to the relativity of our involutive division. Indeed, in [12, 13], Gerdt and Blinkov define involutive divisions on $\mathcal{T}$ as “rules” to be applied to any $U \subset \mathcal{T}$, whereas relative involutive divisions only involve a specific $U \subset \mathcal{T}$.

**Remark 12.** Given a relative involutive division $L$ on a finite set $U \subset \mathcal{T}$, we can notice that $U$ turns out to be involutively autoreduced and $L$-involutive according to Gerdt-Blinkov definitions [12, 13]: this trivially follows from conditions 1-2 of Definition 10.

**Remark 13.** We remark that the relative involutive divisions we are defining are not continuous in the sense defined by Gerdt-Blinkov in [12] (see Proposition 6). In our set $U$, in fact, local involutivity does not imply involutivity, i.e. it is not true in general that

(a) $\forall u \in U, \forall x_j \in NM(u, U), 1 \leq j \leq n, \exists v \in U$ s.t. $ux_j \in vL(v, U)$ implies that
(b) $T(U) = \bigcup_{u \in U} uL(u, U)$.

Take for example (see also [12]) $D = 1, n = 3$, so $U := \mathcal{T}_1 = \{x, y, z\}$ and suppose that $M(x, U) = \{x, y\}, M(y, U) = \{y, z\}, M(z, U) = \{x, z\}$.

This way, $xz \in zL(z, U), xy \in xL(x, U)$ and $yz \in yL(y, U)$, so (a) is trivially satisfied. We notice that (b) does not hold, since $xyz \notin xL(x, U) \cup yL(y, U) \cup zL(z, U)$, since $z \notin M(x, U), x \notin M(y, U)$ and $y \notin M(z, U)$.

This example also shows that the completion procedure by Janet does not work for a general relative involutive division. Indeed, the set $U$ is complete according to Janet definition, but $T(U) \neq \bigcup_{u \in U} uL(u, U)$.

Moreover, we notice that if $\forall u, v \in U, \exists w \in U$ s.t. lcm$(u, v) \in wL(w, U)$, this in general does not imply that $T(U) = \bigcup_{u \in U} uL(u, U)$.

**Example 14** (Janet relative involutive division). Consider a set $U \subseteq \mathcal{T} \subset \mathbf{k}[x_1, \ldots, x_n]$ and define on it Janet involutive division $J$, as in example 7 with $x_1 < \ldots < x_n$. We know (see [12]) that condition 2. of Definition 10 is always satisfied, whereas condition 1. is satisfied only if $U$ is involutive or, in Janet’s language [23], complete.

**Example 15** (Pommaré involutive division). Consider a set $U \subseteq \mathcal{T} \subset \mathbf{k}[x_1, \ldots, x_n]$ and define on it Pommaré involutive division $P$, as in example 8. Neither condition 1. nor condition 2. are automatically satisfied for an arbitrarily chosen set $U \subset \mathcal{T}$, as shown in the following examples:

1. if $U = \{x_1, x_2\} \subset \mathbf{k}[x_1, x_2, x_3], x_1 < x_2 < x_3$, then $M_P(x_1, U) = \{x_1\}, NM_P(x_1, U) = \{x_2, x_3\}, M_P(x_2, U) = \{x_1, x_2\}, NM_P(x_2, U) = \{x_3\}$, so $x_1, x_3 \in T(U)$ but it does not belong to $x_1L(x_1, U) \cup x_2L(x_2, U)$.

2. $U = \{x_1, x_2^2\} \subset \mathbf{k}[x_1, x_2], x_1 < x_2$, then $M_P(x_1, U) = \{x_1\}, NM_P(x_1, U) = \{x_2\}, M_P(x_1^2, U) = \{x_1\}, NM_P(x_1^2, U) = \{x_2\}$, so $x_1^2 \in x_1L(x_1, U) \cap x_1^2L(x_1, U)$.
5 Constructing a relative involutive division

In this section, we specialize to the case $U = T_D$ and we give precise criteria to assign relative involutive divisions on $U$ in all possible ways.

First of all, we remark that, in order to satisfy both conditions of Definition\[10] $\forall d \in \mathbb{N}, |T_{D+d}| = \binom{D+d+n-1}{n-1}$ must equal the sum of the terms generated by each element of $U := T_D$. Given a term $u \in U$, s.t. $|M_u(u, U)| = k$ the number of terms in degree $d + D$, generated by $u$, multiplying it only by its multiplicative variables is $m_{d,k} := \binom{d+k-1}{k-1}$, so we need preliminary to partition $U$ into $n$ sets $U_i$, $1 \leq k \leq n$, each consisting of $a_k := |U_k|$, elements so that

$$\sum_{k=1}^{n} a_k m_{d,k} = \binom{D + d + n - 1}{n - 1}$$

for each $d \geq 0$ (1)

and then decide which $k$ multiplicative variables associate to each term $u \in U_k$ belonging to each subset $U_k$, in such a way that the related cones do not intersect while covering the whole ideal $T_{D+D}$.

Historical Remark 16. The notation $\sum_{i=0}^{d} x_i^{(k)}$ was used by Hilbert (with the sum running from $i = 0$ to $i = d$) [21, Th.IV,pg.512] as a notation for his characteristic Function; Janet connects it with (1) setting $a_k := \sigma_k := \#(\tau \in T_D : \min(\tau) = k) = \binom{D + n - 1 - k}{n - k}$

while describing his decomposition on terms of Cartan’s results\[2].

It is then sufficient to note the each term $\tau = \gamma_1 \cdots \gamma_n$, $\sum \gamma_i \geq D$ can be uniquely decomposed as

$$\tau = \nu \omega, \nu = \gamma_1 \cdots \gamma_j^{-\bar{\gamma}_j}, \omega = x_j^{\bar{\gamma}_j} \cdots x_n^{\gamma_n} \in U, \bar{\gamma}_j + \sum_{i=j+1}^{n} \gamma_i = D$$

to elementary deduce the relation

$$\binom{D + d + n - 1}{n - 1} = \sum_{k=1}^{n} \binom{D + n - 1 - k}{n - k} \binom{d + k - 1}{k - 1}.$$

but this would be an historical cheating since the formula was well-known in the circle of Hilbert followers (see for instance [19, p.184]).

In the next proposition we give a direct proof of equation (2).

Proposition 17. With the above notation, for $a_k := \binom{D+n-1-k}{n-k}$, $1 \leq k \leq n$, we get a partition of $|T_{D+D}|$. More precisely (2) holds.

\[Janet is actually defining Pommaret division.\]
Proof. We set $N = n - 1$ and reformulate (2) as \( \binom{D + n}{N} = \sum_{k=0}^{N} \binom{D + n - 1 - k}{N - k} \binom{d}{k} \).

Applying twice upper negation formula \( \binom{l}{k} = (-1)^{k-l} \binom{k}{k-l} \) and the (generalized) Vandermonde’s convolution \( \sum_{k} \binom{r}{k} \binom{s}{n-k} = \binom{r + s}{n} \) to the summation on the right term of the above formula we get

\[
\binom{N + d + D}{N} = (-1)^N \binom{-d - D - 1}{N} \\
= (-1)^N \sum_{k=0}^{N} \binom{-D}{N - k} \binom{-d - 1}{k} \\
= \sum_{k=0}^{N} (-1)^{N-k} \binom{-D}{N - k} (-1)^k \binom{-d - 1}{k} \\
= \sum_{k=0}^{N} \binom{D + N - 1 - k}{N - k} \binom{d + k}{k},
\]

proving the assertion. 

Now we have to prove that $a_k = \binom{D + n - 1 - k}{n - k}, 1 \leq k \leq n$, is the unique choice leading to a partition of the form defined above

**Proposition 18.** The decomposition of equation (2) is unique.

**Proof.** For each $l \in \mathbb{N}$, we first define the polynomial $f_l(x) := \binom{x}{l}$, which is trivially monic of degree $l$ in $x$.

It is clear that \( \binom{D + n}{n-1} = f_{n-1}(d + D) \) and that \( \binom{d + k - 1}{k-1} = f_{k-1}(d) \), so we may rewrite (2) as

\[
f_{n-1}(d + D) = \sum_{k=1}^{n} \binom{D + n - 1 - k}{n - k} f_{k-1}(d),
\]

\[
f_{n-1}(d + D) - f_{n-1}(d) = \sum_{k=1}^{n-1} \binom{D + n - 1 - k}{n - k} f_{k-1}(d).
\]

Since $d$ does not appear in the binomial coefficients \( \binom{D + n - 1 - k}{n - k} \), we may see them as integer coefficients, so we may write

\[
f_{n-1}(d + D) - f_{n-1}(d) = \sum_{k=1}^{n-1} a_k f_{k-1}(d),
\]

where $a_k = \binom{D + n - 1 - k}{n - k}, k = 1, ..., n - 1$.

Suppose now there is another decomposition, so that

\[
f_{n-1}(d + D) - f_{n-1}(d) = \sum_{k=1}^{n-1} b_k f_{k-1}(d),
\]

then we have

\[
a_k = b_k,
\]

for $k = 1, ..., n - 1$.
for some $b_k \in \mathbb{N}$, $k = 1, ..., n - 1$, so that
\[
\sum_{k=1}^{n-1} (a_k - b_k)f_{k-1}(d) = 0 \tag{3}
\]
and consider the maximal $k_0$ s.t. $a_{k_0} \neq b_{k_0}$. We have $T(f_{k-1}(d)) = d^{k-1}$, which appears only once in (3), with coefficient $(a_{k_0} - b_{k_0}) \neq 0$, so $\sum_{k=1}^{n-1} (a_k - b_k)f_{k-1}(d)$ cannot be zero, leading to a contradiction. \qed

The propositions above show that if a relative involutive division for $T_D$ exists, multiplicative variables must be assigned to terms in a way making the condition (2) satisfied.

Anyway, we still have not proved that such a decomposition can be achieved, for each $D \in \mathbb{N}$. Actually, the answer is positive, as we can see in the following example:

**Example 19** (Pommaret division – relative case for $T = T_D$). Consider $U = T_D$, $D \in \mathbb{N}$ and define Pommaret division on $U$, with $x_1 < x_2 < ... < x_n$.

Clearly, $\forall t \in T(t, U)$ and $\forall s \in U, s \neq t, t \not\in sL(s, U)$.

Take now $t_1 = x_1^{\beta_1} \cdots x_n^{\alpha_n} \in T_{x_D}$ i.e. a term in $T$ of degree strictly greater than $D$. Let
\[
s_1 := x_1^{\beta_1} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n}, 1 \leq i \leq n, \beta_i \leq \alpha_i, D = \beta_1 + \sum_{j=i+1}^{n} \alpha_j.
\]

Clearly $s_1 \in U$ is the only element of $U$ which Pommaret-divides $t_1$; thus $\forall t_1 \in T_{x_D}, \exists! s_1 \in U$ s.t. $t_1 \in s_1L(s_1, U)$, i.e. for $U = T_D$, Pommaret division is a relative involutive division.

The terms in $U$ with minimal variable $x_k$ $1 \leq k \leq n$ are exactly the terms of degree $D$ in $x_k, ..., x_n$ which contain $x_k$, so their number is the difference between the terms of degree $D$ in $x_k, ..., x_n$ and those in $x_{k+1}, ..., x_n$ of the same degree:
\[
\left(\frac{D + (n-k + 1) - 1}{n-k + 1} - 1\right) \left(\frac{D + (n-k + 1) - 2}{n-k + 1} - 1\right) = \frac{D + (n-k + 1) - 2}{n-k + 1} + \frac{D + (n-k + 1) - 2}{n-k + 1} = \frac{D + (n-k + 1) - 2}{n-k + 1} - \frac{D + (n-k + 1) - 2}{n-k + 1} = \frac{D + (n-k + 1) - 2}{n-k + 1}.
\]

These terms have exactly $k$ multiplicative variables, so decomposition of equation (2) holds for Pommaret division.

In this particular case, i.e. for $U = T_D$, $D \in \mathbb{N}$, we can also notice that Pommaret division and Janet division coincide as relative involutive divisions, with the same variable ordering.

Indeed, let $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in U$ and suppose $\text{min}(t) = x_i$, so $\alpha_i > 0$, for some $1 \leq i \leq n$.

If $x_j > x_i$ then $\frac{x_j}{x_i} \in U$, so by definition of Janet division, $x_j$ cannot be multiplicative for $t$.

On the other hands, if $x_j \leq x_i$, for $x_j$ not being multiplicative, we should find in $U$ a term $x_j^{\beta_1} \cdots x_n^{\beta_n}$ with
• $\beta_l = \alpha_l, l \geq i$, and $\beta_j > \alpha_j = 0$ if $x_j < x_i$,

• $\beta_l = \alpha_l, l > i$, and $\beta_j \geq \alpha_j + 1$ if $x_j = x_i$.

Due to degree reasons, this is clearly impossible for a term in $U$, so each variable $x_j \leq x_i$ must be multiplicative for $t$.

As an example, consider the terms in $n = 3$ variables and degree $D = 2$, supposing $x < y < z$; Pommaret division is defined

| Terms  | Multiplicative Variables |
|--------|--------------------------|
| $x^2$  | $x$                      |
| $xy$   | $x$                      |
| $y^2$  | $x, y$                   |
| $xz$   | $x$                      |
| $yz$   | $x, y$                   |
| $z^2$  | $x, y, z$                |

We have exactly $\binom{3}{2} = 3$ terms with only one multiplicative variable (namely $x^2, xy, xz$),
\(\binom{3}{1} = 2\) terms with two multiplicative variables (namely $y^2, yz$) and \(\binom{3}{0} = 1\) term with three multiplicative variables (namely $z^2$).

If we define Janet division on the same set we get exactly the same partition into multiplicative and non-multiplicative variables as for Pommaret division. Indeed:

• $x^2$: $x$ is multiplicative for $x^2$; since $xy \in U$ $y$ is not multiplicative and the same goes for $z$, being $xz \in U$.

• $xy$: $x$ is multiplicative for $xy$; since $y^2 \in U$ $y$ is not multiplicative and the same goes for $z$, being $xz \in U$.

• $y^2$: $x, y$ are multiplicative for $y^2$; since $xz \in U$ $z$ is not multiplicative.

• $xz$: $x$ is multiplicative for $xz$; since $yz \in U$ $y$ is not multiplicative and the same goes for $z$, being $z^2 \in U$.

• $yz$: $x, y$ are multiplicative for $yz$; since $z^2 \in U$ $z$ is not multiplicative.

• $z^2$: $x, y, z$ are all multiplicative for $z^2$.

We finally point out that, in this example, the multiplicative sets are all contained one in another:
\[
\{x\} \subset \{x, y\} \subseteq \{x, y, z\}.
\]

We will come back to this fact in section 5.1 while focusing on the main properties of Pommaret relative involutive division.

\*$\circ$*

In order to give all the possible decompositions of $T_D$ in cones, we have now to study the criteria for choosing multiplicative variables to assign to each term in the set $U = T_D$, so that conditions 1., 2. of Definition 10 are satisfied.

First of all, we prove a simple property of pure powers.
Lemma 20. Let $U := T_D$, $D \in \mathbb{N}$ and suppose that $\forall U \in U$ a partition of the variables into multiplicative and non-multiplicative ones $\{x_1, ..., x_n\} = M(U, U) \cup N M(U, U)$ is given s.t., with the above notation, $T(U) = T_{\geq D} = \bigcup_{u \in U} u L(U, U)$. Then $\exists 1 \leq i \leq n, x_i \in M(x_i, U)$.

Proof. Consider a term $x_i^h$, $h > D$ By assumption there is a term $u \in U$ s.t. $x_i^h = uv$ $h > D$ with $v \in L(u, U)$. The only solution is $u = x_i^D$, $v = x_i^{h-D}$ which implies $x_i$ multiplicative for $x_i^D$.

The following proposition gives a direct proof of the obvious fact that one and only one term $t \in T_D$ must have $M(t, U) = \{x_1, ..., x_n\}$.

Proposition 21. Let $U := T_D$, $D \in \mathbb{N}$ and suppose that $\forall U \in U$ a partition of the variables into multiplicative and non-multiplicative ones $\{x_1, ..., x_n\} = M(U, U) \cup N M(U, U)$ is given s.t., with the above notation, $T(U) = T_{\geq D} = \bigcup_{u \in U} u L(U, U)$. Then $\exists t \in U$ s.t. $M(t, U) = \{x_1, ..., x_n\}$.

Proof. Consider the term $v := \text{lcm}(u| u \in U) = x_1^D \cdots x_n^D \in T_{\geq D}$.

Since $T(U) = T_{\geq D} = \bigcup_{u \in U} u L(U, U)$ there must be a term $t \in U$ s.t. $v \in t L(t, U)$:

a) if $t = x_i^D$, for some $i \in \{1, ..., n\}$, then $v = t \prod_{j \in \{1, ..., n\}\backslash\{i\}} x_j^D$, so, $\forall j \in \{1, ..., n\} \backslash \{i\}$, $x_j \in M(t, U)$. Since, by the above Lemma 20, $x_i \in M(t, U)$, we can conclude that $M(t, U) = \{x_1, ..., x_n\}$;

b) if $t = x_1^{\beta_1} \cdots x_n^{\beta_n}$, with $\sum_i \beta_i = D$, s.t. $\exists i_1, i_2 \in \{1, ..., n\}$, $i_1 \neq i_2$ s.t. $\beta_{i_1} \neq 0$ and $\beta_{i_2} \neq 0$, then $\beta_i < D, \forall i \in \{1, ..., n\}$.

In this case, $v = tx_1^{D-\beta_1} \cdots x_n^{D-\beta_n}, D - \beta_i > 0$, for each $i \in \{1, ..., n\}$ and this implies $M(t, U) = \{x_1, ..., x_n\}$.

\[\square\]

The only term $t$ such that $M(t, U) = \{x_1, ..., x_n\}$ is called the peak of $U$.

Remark 22. Note that the Janet-Gunther formula \(2\) already told us that there is exactly a single such $t$.

Remark 23. Consider again the setting of Remark 13.

The non-continuity of the described set can be seen also by observing that there are 10 terms in three variables and degree three. Each term of $T_J$ has 2 multiplicative variables, so it generates only 3 different terms in degree 3 (and there is no intersection between $x L(x, U)$, $y L(y, U)$ and $z L(z, U)$) so, in degree 3, we only get 9 terms, instead of 10 ($x y z$ is exactly the missing one), so this is due to the fact that formula \(2\) does not hold, since there is no peak in $U$.

In the remaining part of this section we prove a criterion for defining a partition into multiplicative and non-multiplicative variables s.t. the resulting cones are disjoint.

Lemma 24. Let $U := T_D$, $D \in \mathbb{N}$ and suppose that $\forall U \in U$ a partition of the variables into multiplicative and non-multiplicative ones $\{x_1, ..., x_n\} = M(U, U) \cup N M(U, U)$ is given.

Let $u, v \in U$ and $w = uv_1 = vw_2 = \text{lcm}(u, v)$. Then

\[u L(U, U) \cap v L(v, U) \neq \emptyset \iff w \in u L(U, U) \cap v L(v, U).\]
Proof. We prove only the non-trivial part of the statement. Suppose \( t \in uL(u, U) \cap vL(v, U) \), so \( t = ut_1 = vt_2 \) with \( t_1 \in L(u, U) \) and \( t_2 \in L(v, U) \).

Since \( w \mid t \) we have \( t = wm \) for some \( m \in T \), so

\[
t = ut_1 = wm = uw_1m.
\]

Dividing by \( u \) we get \( t_1 = w_1m \in L(u, U) \) whence \( w_1 \in L(u, U) \) and \( w \in uL(u, U) \). With an analogous argument for \( v \) we can show that \( w \in vL(v, U) \), concluding our proof.

\( \square \)

Proposition 25. Let \( U = \mathcal{T}_D, D \in \mathbb{N} \) and fix on \( U \) a relative involutive division \( L \).

Let \( t \in U \) s.t. \( M_L(t, U) = \{x_1, \ldots, x_n\} \) and \( A := \{x_{j_1}, \ldots, x_{j_l}\} \subset \{x_1, \ldots, x_n\} \) be the set of all and only the variables appearing in \( t \) with nonzero exponent. Then, \( \forall u \in U \setminus \{t\}, A \not\subset M_L(u, U) \).

Proof. Suppose \( A \subset M_L(u, U) \) and let \( w = \text{lcm}(t, u) = \frac{ut}{\text{GCD}(t, u)} \); it may be regarded as

a) \( w = \frac{ut}{\text{GCD}(t, u)} \): the variables appearing in \( \frac{t}{\text{GCD}(t, u)} \) with nonzero exponent, all belong to \( A \subset M_L(u, U) \), so \( w \in C_L(u, U) \);

b) \( w = \frac{ut}{\text{GCD}(t, u)} \): since \( M_L(t, U) = \{x_1, \ldots, x_n\}, w \in C_L(t, U) \).

Thus we can conclude, since \( C_L(u, U) \cap C_L(t, U) \neq \emptyset \) and so condition (ii) of the definition of relative involutive division is contradicted.

\( \square \)

It is now obvious the following

Corollary 26. Let \( U = \mathcal{T}_D, D \in \mathbb{N} \) and fix on \( U \) a relative involutive division \( L \). Then, there exists one and only one \( t \in U \) s.t. \( M_L(t, U) = \{x_1, \ldots, x_n\} \).

Remark 27. Let \( t \in U \) be the only term s.t. \( M(t, U) = \{x_1, \ldots, x_n\} \) and suppose, given \( i \in \{1, \ldots, n\}, t \neq x_i^{D^h} \). In this case, \( \forall h \geq 0 \) we have \( x_i^{D^h} \notin tL(t, U) \). Indeed, since \( \deg(t) = D \) and \( t \neq x_i^{D^h} \), there exists \( j \in \{1, \ldots, n\} \setminus \{i\} \) s.t. \( x_j \mid t \) and this trivially implies that \( \forall t \in tL(t, U), x_j \mid t \) so \( x_j^{D^h} \notin tL(t, U) \).

This implies that, whatever is the term \( t \in U \) s.t. \( M(t, U) = \{x_1, \ldots, x_n\} \), the limitations on the choices of multiplicative variables imposed by \( t \) cannot affect the (necessary) choice imposed by Lemma 20.

Similarly, we can observe that the assignment \( x_i \in M(x_i^{D^h}, U) \), for all \( 1 \leq i \leq n \) does not impose any condition on the future choices, since \( \forall h \geq 0, x_i^{D^h} \) is not multiple of any \( u \in U \setminus \{x_i^{D^h}\} \).

We see now a criterion for setting a partition on the variables into multiplicative and non-multiplicative ones, so that no intersection between sets of the form \( tL(t, U) \) may arise.

Proposition 28. Let \( U := \mathcal{T}_D, D \in \mathbb{N} \) and suppose that \( \forall u \in U \) a partition of the variables into multiplicative and non-multiplicative ones \( \{x_1, \ldots, x_n\} = M(u, U) \cup NM(u, U) \) is given. Let \( u, v \in U, w := \text{GCD}(u, v) \) and suppose

\[
\bullet \frac{u}{w} = x_1^{\alpha_1} \cdots x_i^{\alpha_i}, \quad \alpha_i > 0, \quad i = 1, \ldots, l;
\]
By Proposition 28, assigning \( M(x, U) \supseteq \{x_{k_1}, ..., x_{k_s}\} \) and \( M(v, U) \supseteq \{x_{j_1}, ..., x_{j_l}\} \) if and only if \( uL(u, U) \cap vL(v, U) \neq \emptyset \).

**Proof.** Suppose \( M(u, U) \supseteq \{x_{k_1}, ..., x_{k_s}\} \) and \( M(v, U) \supseteq \{x_{j_1}, ..., x_{j_l}\} \) and let \( w' := \text{lcm}(u, v) = \frac{w}{a} \); then

1. \( w' = v \frac{w}{a} = v x_{j_1}^{\beta_1} \cdots x_{j_l}^{\beta_l} \in vL(v, U) \), being \( x_{j_1}, ..., x_{j_l} \in M(v, U) \) by hypothesis;

2. \( w' = u \frac{w}{a} = u x_{k_1}^{\beta_1} \cdots x_{k_s}^{\beta_s} \in uL(u, U) \), being \( x_{k_1}, ..., x_{k_s} \in M(u, U) \) by hypothesis.

We can then conclude that \( w' \in uL(u, U) \cap vL(v, U) \neq \emptyset \).

Conversely, let \( uL(u, U) \cap vL(v, U) \neq \emptyset \). By Lemma 24 we deduce that \( w' := \text{lcm}(u, v) = \frac{w}{a} \in uL(u, U) \cap vL(v, U) \). Then \( w' = v \frac{w}{a} = v x_{j_1}^{\beta_1} \cdots x_{j_l}^{\beta_l} \in vL(v, U) \), so \( x_{j_1}, ..., x_{j_l} \in M(v, U) \) and, similarly, \( w' = u \frac{w}{a} = u x_{k_1}^{\beta_1} \cdots x_{k_s}^{\beta_s} \in uL(u, U) \), then \( x_{k_1}, ..., x_{k_s} \in M(u, U) \), allowing us to conclude.

Given \( U := T_D, D \in \mathbb{N} \), suppose that \( \forall u \in U \) a partition of the variables into multiplicative and non-multiplicative ones \( \{x_1, ..., x_n\} = M(u, U) \cup N M(u, U) \) is given, so that the binomial formula (2) holds and it never happens a situation as that described in Proposition 28, then a relative involutive division is assigned, so \( T(U) = \bigcup_{u \in U} C_L(u, U) \) and \( \forall v \in U, C_L(u, U) \cap C_L(v, U) = \emptyset \).

Indeed, the condition \( \forall u, v \in U, C_L(u, U) \cap C_L(v, U) = \emptyset \) immediately follows from Proposition 28, whereas the condition \( T(U) = \bigcup_{u \in U} C_L(u, U) \) comes from the binomial formula (2), observing that the relative cones have been proved to be all disjoint.

We have then found a criterion that, combined to condition 24, allows us to define a relative involutive division.

**Example 29.** Take, for example, \( D = 2 \) and \( n = 3 \). We have \( U := T_2 = \{x^2, xy, y^2, xz, yz, z^2\}, |U| = 6 \).

By Lemma 20 we have to impose \( x \in M(x^2, U) \), \( y \in M(y^2, U) \) and \( z \in M(z^2, U) \).

Then, by Corollary 26 we have to take one and only one element of \( U \) to which assign all the variables as multiplicative; in our example we take \( xy \in U \), so \( M(xy, U) = \{x, y, z\} \).

By Proposition 28 assigning \( M(xy, U) = \{x, y, z\} \) imposes some limitations on the ways we can assign the multiplicative variables to the other terms. More precisely, since \( x \in M(xy, U) \) then \( y \notin M(x^2, U) \) and \( y \in M(xy, U) \) implies \( x \notin M(y^2, U) \).

Moreover, \( z \in M(xy, U) \) implies \( y \notin M(xz, U) \), \( x \notin M(yz, U) \) and \( \{x, y\} \notin M(z^2, U) \).

After these first steps, we have the following configuration

| Terms | Multiplicative Variables |
|-------|-------------------------|
| \( x^2 \) | \( x, x, ? \) |
| \( xy \) | \( x, y, z \) |
| \( y^2 \) | \( x, y, ? \) |
| \( xz \) | \( ?, x, ? \) |
| \( yz \) | \( x, ?, ? \) |
| \( z^2 \) | \( ?, /z \) |
where we denote by the symbol \( ? \) the free variables, i.e. those that can be freely assigned as multiplicative for the term in the same row of the table by \( \times \) the variables which cannot be assigned as multiplicative and by \( / \) the variables that cannot be contemporarily assigned as multiplicative for the term in the same row of the table.

Now, taking into account the limitations stated above, we assign two multiplicative variables to \( xz \in U \): \( M(xz) = \{x, z\} \).

Again this choice will impose some limitations on the future choices: \( x \in M(xz, U) \) so \( z \notin M(x^2) \), whereas \( z \in M(xz, U) \) implies \( x \notin M(z^2, U) \).

Then, after this step, we reach the configuration below:

| Terms | Multiplicative Variables |
|-------|-------------------------|
| \( x^2 \) | \( x, x, x \) |
| \( xy \) | \( x, y, z \) |
| \( y^2 \) | \( x, y, ? \) |
| \( xz \) | \( x, x, x \) |
| \( yz \) | \( x, ?, ?, ? \) |
| \( z^2 \) | \( x, ?, ?, z \) |

We set now \( M(z^2, U) = \{y, z\} \); \( y \in M(z^2, U) \) leads to \( z \notin M(y^2, U) \) and \( z \notin M(yz, U) \), so

| Terms | Multiplicative Variables |
|-------|-------------------------|
| \( x^2 \) | \( x, x, x \) |
| \( xy \) | \( x, y, x \) |
| \( y^2 \) | \( x, y, x \) |
| \( xz \) | \( x, x, x \) |
| \( yz \) | \( x, ?, ?, x \) |
| \( z^2 \) | \( x, y, z \) |

We get \( M(y^2, U) = \{y\} \), which does not impose any relation on the future choices.

Looking at the configuration, we finally get \( M(yz, U) = \{y\} \) and \( M(x^2) = \{x\} \), so we conclude with the configuration below:

| Terms | Multiplicative Variables |
|-------|-------------------------|
| \( x^2 \) | \( x \) |
| \( xy \) | \( x, y, z \) |
| \( y^2 \) | \( y \) |
| \( xz \) | \( x, z \) |
| \( yz \) | \( y \) |
| \( z^2 \) | \( y, z \) |

In Appendix A all the relative involutive divisions of \( T_2 \subset k[x, y, z] \) are displayed, up to a permutation of the variables.

\(^3\)They are not affected by the choice previously made.
5.1 Pommaret relative involutive division

This section is devoted to the study of a complete characterization for Pommaret relative involutive division.

Proposition 30. Let $U = T_{D} = \{u_{1}, ..., u_{l}\}$, $D \in \mathbb{N}$, $l = \binom{D+n-1}{n-1}$ and suppose that a relative involutive division $L$ is defined on $U$ in such a way that there is a relabelling of the terms in $U$ s.t.

\[
M_{L}(u_{1}, U) \subseteq M_{L}(u_{2}, U) \subseteq ... \subseteq M_{L}(u_{l}, U) = \{x_{1}, ..., x_{n}\}.
\]

Let us denote for each $1 \leq i \leq n$, $\overline{u}_{i} = x_{i}^{D}$. Then, up to a reordering and a relabelling of the variables the following properties hold:

1. $M(\overline{u}_{1}, U) \subseteq M(\overline{u}_{2}, U) \subseteq ... \subseteq M(\overline{u}_{n}, U) \subseteq \{x_{1}, ..., x_{n}\}$.

2. the (unique) term $u_{l}$ s.t. $M_{L}(u_{l}, U) = \{x_{1}, ..., x_{n}\}$ has the form $u_{l} = x_{i}^{D}$, for some $1 \leq i \leq n$.

3. for $1 \leq i \leq n$, $M_{L}(\overline{u}_{i}, U) = \{x_{j}|1 \leq j \leq i\}$.

4. under such reordering and relabelling of the variables, $L$ coincides with Pommaret division defined on $U$, i.e.

\[
\forall u \in U, M_{L}(u, U) = \{x_{i} | x_{i} \leq \min(u)\}.
\]

Proof.

1. For a particular reordering we w.l.o.g. have

\[
M_{L}(\overline{u}_{j}, U) \subseteq M_{L}(\overline{u}_{j}, U) \subseteq ... \subseteq M_{L}(\overline{u}_{j}, U).
\]

It is then sufficient to choose a such reordering and to relabel the variables setting $x_{i} := x_{i}$ for each $i$ in order to obtain the claim.

2. By Lemma[20] for each $1 \leq i \leq n$, $x_{i} \in M(\overline{u}_{i}, U)$, so $x_{1} \in M_{L}(\overline{u}_{1}, U)$, $\{x_{1}, x_{2}\} \subseteq M_{L}(\overline{u}_{2}, U)$, $\{x_{1}, ..., x_{n}\} \subseteq M_{L}(\overline{u}_{n}, U)$ so $M_{L}(\overline{u}_{n}, U) = \{x_{1}, ..., x_{n}\}$, so $u_{l} = \overline{u}_{n}$ and then we can conclude.

3. In the proof of 2. we have already shown that, for each $1 \leq i \leq n$, $M_{L}(\overline{u}_{i}, U) \supseteq \{x_{j}|1 \leq j \leq i\}$, so we only need to prove that $\forall 1 \leq i \leq n, \forall i < j \leq n, x_{j} \notin M_{L}(\overline{u}_{i}, U)$.

If $x_{j} \in M_{L}(\overline{u}_{i}, U)$, being $x_{i} \in M_{L}(\overline{u}_{i}, U) \subseteq M(\overline{u}_{i}, U)$, we have $\overline{u}_{i}x_{i}^{D} = u_{j}x_{j}^{D} \in C_{L}(\overline{u}, U) \cap C_{L}(\overline{u}, U)$, leading to a contradiction.

4. By 3., the inequalities stated in 4. are strict.

Taking the ordering $x_{1} < ... < x_{n}$ on the variables we notice that, again by 3., the multiplicative variable sets for pure powers are in accordance with Pommaret division. Then, we only have to prove the assertion for non-pure powers.
Corollary 31. Let \( U = T_D = \{u_1, ..., u_l\}, D \in \mathbb{N}, l = \binom{D+r-1}{n-1} \) and suppose that a relative involutive division \( L \) is defined on \( U \) in such a way that there is a relabelling of the terms in \( U \) s.t.

\[ \forall x_i \in L(u, U), \; j < l. \]

Then there is a reordering of the variables \( x_{j_1} < x_{j_2} < ... < x_{j_n} \), under which \( L \) coincides with Pommaret division defined on \( U \), i.e.

\[ \forall u \in U, M_L(u, U) = \{x_i | x_i \leq \min(u)\}. \]

6 The ideal and the graph

In this section, given \( U = T_D, D \in \mathbb{N} \) and supposed that a relative involutive division \( L \) is defined on \( U \), we deal with defining semigroup ideals generated by a subset of terms
in $U$ and the associated order ideals, in order to be consistent with the cone decomposition induced by $L$.

In particular, given a term $t \in U$, if $t$ is a generator for the semigroup ideal we want to construct, all its multiple must belong to the same semigroup ideal, so we have to consider as generators of the semigroup ideal all the elements in $U$ whose involutive cone contain some multiple of $t$. Clearly, an analogous argument apply to order ideals.

Anyway, we first see the case of semigroup ideals.

Let $U = \mathcal{T}_D$, $D \in \mathbb{N}$ and suppose that a relative involutive division $L$ is defined on $U$.

From now on, for each $s, t \in U$, $X(s, t)$ will denote the unique element in $U$ s.t.
\[
\text{lcm}(s, t) \in C_L(X(s, t), U); \text{it exists by condition 1. (or 1'). of Definition}^{[1]} \text{ whereas its uniqueness is granted by condition 2. (or 2'). of the same definition.}
\]

**Definition 32.** With the above notation, a set $M \subseteq U$ is called compliant if $\forall s, t \in U, t \in M \Rightarrow X(s, t) \in M$.

**Proposition 33.** Let $U = \mathcal{T}_D$, $D \in \mathbb{N}$ and suppose that a relative involutive division $L$ is defined on $U$. Let $M \subseteq U$ and $J = (M)$. Then it holds $J = \bigcup_{l \in M} C_L(l, U) \iff M$ is compliant.

**Proof.** “$\Rightarrow$” With the notation of Definition^{[2]} assuming $t \in M$, we need to prove $X(s, t) \in M$. Since $t \in M \Rightarrow \text{lcm}(s, t) \in J$ and the only cone containing $\text{lcm}(s, t)$ is $C_L(X(s, t), U)$, then $C_L(X(s, t), U) \subseteq J$ and hence $X(s, t) \in M$.

“$\Leftarrow$” We first observe that, clearly, $J \supseteq \bigcup_{l \in M} C_L(l, U)$ since the elements of this union are by definition multiples of some generators of $J$; then we have only to prove that $J \subseteq \bigcup_{l \in M} C_L(l, U)$, i.e. that
\[
\forall u \in J, \exists v \in M \text{ s.t. } u \in C_L(v, U).
\]

If $\deg(u) = D$, then $u \in M$, so $u \in C_L(u, U)$. If $\deg(U) = D + 1$, then, since $U \in J$ and so it is multiple of some generator, there exists $v \in M$ s.t. $u = v x_j$ for some variable $x_j$. If $x_j \in M_L(v', U)$, then $u \in C_L(v', U)$ Otherwise, we consider $x_j \mid v$, $x_j \neq x_j$

Clearly $x_j \mid u$, so we can take $v'' = \frac{u}{x_j}$; we have $v' \in M$, $v'' \in U$ and $\text{lcm}(v', v'') = u$, so by hypothesis, $X(v', v'') \in M$, allowing us to conclude that $u \in \bigcup_{l \in M} C_L(l, U)$.

In order to prove the claim we consider a counterexample of minimal degree. So we assume to have

- $u \in J$, $\deg(u) = D + h + 1$, $h \geq 1$,
- the single element $v \in U$ s.t. $u \in C_L(v, U)$ (whose existence and uniqueness are granted by definition of relative involutive division) and which satisfies $v \in N(J)$ thus giving the required contradiction;
- $m \in L(v, U)$, $\deg(m) = h + 1 : u = vm$;

\footnote{It is always possible to find $x_j \mid v'$, $x_j \neq x_j$, because otherwise $v' = X_0$ and so, by Lemma^{[20]} $x_j \in M_L(v', U)$.}
• Since \( u \in J \), it has a predecessor \( w \in J \), \( \deg(w) = D + h \) and
  
  • A variable \( x_j : u = x_jw \);

  • The element \( w' \in M \) s.t. \( w \in CL(w', U) \) and

  • The related cofactor \( m' \in L(w', U) : w = w'm' \)

  so that we have relation

\[
\frac{u}{x_j} = vm = wx_j = w'm'x_j. \tag{4}
\]

We remark that

1. \( v \in N(J), w' \in J \implies v \neq w' \);

2. Whence \( w \notin CL(v, U) \) whence

3. The minimality of the counterexample implies \( x_j \in NM_L(v, U) \) (otherwise \( w \)
    would be a counterexample of lesser degree); moreover

4. \( w \in CL(w', U), u = wx_j \notin CL(w', U) \implies x_j \in NM_L(w', U) \) whence

5. \( x_j \nmid m \in L(v, U) \) so that

6. \( x_j \mid v \)

And we can define

• \( v' = \frac{1}{x_j} \),

• \( t_{v'} = \frac{v'}{\gcd(w', v')} \),

• \( t_m : m = t_{v'}t_m \)

And get, dividing (4) by \( x_j \),

\[
\frac{u}{x_j} = w = v'm = w't_{v'}t_m = w'm'. \tag{5}
\]

Then

7. \( t_m = 1 \) since, otherwise the term \( vt_{v'} \) would contradict minimality; so

8. \( v'm = w'\frac{v'}{\gcd(w', v')} \) whence

9. \( m = \frac{w'}{\gcd(w', v')} \mid w' \).

10. \( \deg_j(\gcd(w', v')) = \min(\deg_j(w'), \deg_j(v')) \);

11. Since \( \deg_j(\gcd(w', v')) = \deg_j(v') \implies x_j \mid m \) which is false, we have

12. \( \deg_j(\gcd(w', v')) = \deg_j(w') \) and

13. \( \gcd(w', v') = \gcd(w', v) \).
We have then
\[ u = v'x, m = vm = \frac{w'v}{\gcd(w', v')} = \lcm(w', v). \]

Now we have \( w' \in M, v \in U, \lcm(w', v) = u \in C_L(v, U) \) which, by hypothesis implies \( v = X(w', v) \in M \) proving that no counterexample exists.

\[ \square \]

**Remark 34.** Let \( U = T_D \) and suppose a relative involutive division \( L \) to be defined on \( U \). Let \( t \) be the only element s.t. \( M_L(t, U) = \{ x_1, ..., x_n \} \). The above proposition shows that if \( M \neq \emptyset \) necessarily \( t \in M \). Indeed, for each \( u \in U, t = X(u, t) \).

**Proposition 35.** Let \( U = T_D, D \in \mathbb{N} \) and suppose that a relative involutive division \( L \) is defined on \( U \). It holds \( \forall s, t \in U, \exists u \in U \) s.t. \( X(s, u) = t \) if and only if \( X(s, t) = t \).

**Proof.** “\( \Rightarrow \)”

It is trivial with \( u = t \).

“\( \Leftarrow \)’”

Let \( w = \lcm(s, u) \), then \( w = tm \) with \( m \in L(t, U) \).

By definition of least common multiple \( w = \frac{tm}{\gcd(s, u)} \). We denote \( h = gcd(s, u) \); so \( h | u \), \( u = hh' \) and then \( w = \frac{sh'}{h} = sh \). From \( w = tm = sh \), we can deduce that \( s = \frac{tm}{h} \).

We compute now \( l := \lcm(t, s) = \lcm(t, \frac{tm}{h}) \); clearly \( t | l \), but since \( s = \frac{tm}{h} \), then \( l | tm \) so \( l = tm' | tm \) and then \( m' | m \) so \( m' \in L(t, U) \) and finally \( l \in C(t, U) \).

With the above Proposition we can make simpler the verification that a set \( M \) is compliant, giving an equivalent definition of compliance:

**Definition 36.** Let \( U = T_D, D \in \mathbb{N} \) and suppose that a relative involutive division \( L \) is defined on \( U \).

A set \( M \subseteq U \) is called compliant if \( \forall s, t \in U \) it holds
\[ X(s, t) = t, s \in M \Rightarrow t \in M. \]

**Example 37.** Consider now \( n = 3, D = 2 \) and define on \( U = T_2 \) the following division

| Terms | Multiplicative Variables |
|-------|-------------------------|
| \( x^2 \) | \( x, y, z \) |
| \( xy \) | \( y, z \) |
| \( y^2 \) | \( y \) |
| \( xz \) | \( z \) |
| \( yz \) | \( y, z \) |
| \( z^2 \) | \( z \) |

If, with the notation of Proposition \[33\] \( xy \in M \), then \( \lcm(x^2, xy) = x^2y \in C_L(x^2, U) \) so \( x^2 \in M \).

This is the only \( \lcm \) which must be computed ad thus the ideal \( J = (x^2, xy) \) is exactly given by the union \( C_L(x^2, U) \cup C_L(xy, U) \).

Indeed, all the multiples of \( x^2 \) belong to \( C_L(x^2, U) \) so \( \frac{\lcm(x^2, J)}{x^2} = \frac{\lcm(x^2, y)}{x^2} \in L(x^2, U) \) and \( X(x^2, t) = x^2 \) for all \( t \in M \); the multiples of \( xy \) involving only \( y, z \) belong to
$C_L(xy, U)$, so \(\text{lcm}(xy, t) = \frac{t}{\text{gcd}(xy, t)} \in L(xy, U)\) and $X(xy, t) = xy$ for each $t \in M \setminus \{x^2\}$; moreover all the multiples of $xy$ involving $x$ are also multiples of $x^2$ so they belong to $C_L(x^2, U)$. ♦

**Example 38.** Consider $D = n = 3$ and define the following relative involutive division:

| Terms | Multiplicative Variables |
|-------|-------------------------|
| $x^4$ | $x, z$ |
| $x^3y$ | $x$ |
| $xy^2$ | $x$ |
| $y^3$ | $x, y$ |
| $x^2z$ | $z$ |
| $xyz$ | $x, y, z$ |
| $y^2z$ | $y$ |
| $x^2z$ | $z$ |
| $y^2z$ | $y$ |
| $z^3$ | $y, z$ |

With the notation above, suppose $xy^2 \in M$, then:

- $\text{lcm}(xy^2, x^3) = xy^3 \in C_L(y^3, U) \Rightarrow y^3 \in M$
- $\text{lcm}(xy^2,xyz) = xy^2z \in C_L(xyz, U) \Rightarrow xyz \in M$
- $\text{lcm}(y^2z, y^3) = y^3z \in C_L(y^2z, U) \Rightarrow y^2z \in M$
- $\text{lcm}(yz^2, y^2z) = y^2z^2 \in C_L(yz^2, U) \Rightarrow yz^2 \in M$
- $\text{lcm}(x^2y, z^3) = x^3z \in C_L(x^2y, U) \Rightarrow z^3 \in M$
- $\text{lcm}(xy^2, z^3) = xz^3 \in C_L(xy^2, U) \Rightarrow xz^2 \in M$
- $\text{lcm}(x^2z, x^2z) = x^2z^2 \in C_L(x^2z, U) \Rightarrow x^2z \in M$
- $\text{lcm}(x^2, x^2z) = x^3z \in C_L(x^2, U) \Rightarrow x^3 \in M$
- $\text{lcm}(x^3, x^2y) = x^3y \in C_L(x^2y, U) \Rightarrow x^2y \in M$

So, $M = U$ and in this case clearly $J = \bigcup_{l \in M} C_L(l, U)$. ♦

We focus now on order ideals.

**Definition 39.** Let $U = T_D$, $D \in \mathbb{N}$. Suppose that a relative involutive division $L$ is defined on $U$. A set $N \subseteq U$ is called revenant if $\forall t, s \in U, X(t, s) = t, t \in N$ then $s \in N$.

**Proposition 40.** Let $U = T_D$, $D \in \mathbb{N}$. Suppose that a relative involutive division $L$ is defined on $U$ and let $N \subseteq U$. It holds

$$H := \left( \bigcup_{l \in N} C_L(l, U) \right) \cup \{v \in T \mid \deg(v) < D\}$$

is an order ideal, if and only if $N$ is revenant.
oriented graph, whose vertices are the elements of ideals in a simple way. To do so, we define the Ufnarovsky-like graph $G$.

The Pommaret case is rather peculiar and we can construct semigroup ideals

Example 41. Consider $n = 3$, $D = 2$ and the division defined in \cite{37}. Suppose that $x_2 \in \mathbb{N}$, then $\operatorname{lcm}(x_2, z^2) = x_2 z^2 \in C_L(x_2, U)$, so $z^2 \in \mathbb{N}$ and, for each $s \in U \setminus \{x_2, z^2\}$, $\operatorname{lcm}(x_2, s) \notin C_L(x_2, U)$, $\operatorname{lcm}(z^2, s) \notin C_L(z^2, U)$, so $N = \{x_2, z^2\}$ and $(\bigcup_{l \in \mathbb{N}} C_L(l, U)) \cup \{v \in \mathcal{T} \mid \deg(v) < 2\}$ is an order ideal.

\section{Pommaret division and the Ufnarovsky-like graph}

In this section, we focus on the particular case of Pommaret division. We begin by remarking that:

- if the Pommaret relative involutive division $L$ is defined on $\mathcal{T}_D$, then $\forall u_1, u_2 \in U$, $M_L(u_1, U) \subseteq M_L(u_2, U)$ or $M_L(u_2, U) \subseteq M_L(u_1, U)$.

- a relative involutive division $L$ defined on $U = \mathcal{T}_D = \{u_1, ..., u_n\}$, $D \in \mathbb{N}$, $l = \binom{D+n-1}{n-1}$ coincide with Pommaret division w.r.t. some variable reordering if and only if there is a relabelling of the terms in $U$ s.t. $M(u_1, U) \subseteq M(u_2, U) \subseteq ... \subseteq M(u_l, U) = \{x_1, ..., x_n\}$ (see Proposition \cite{30}).

The Pommaret case is rather peculiar and we can construct semigroup ideals/ order ideals in a simple way. To do so, we define the Ufnarovsky-like graph $G_U$, i.e. an oriented graph, whose vertices are the elements of $U = \mathcal{T}_D$, $D \in \mathbb{N}$ and s.t., given $s, t \in U$, there is an edge $t \rightarrow s$ if and only if $\exists x_j \in \mathcal{N}M_L(s, U)$ s.t. $sx_j \in C_L(t, U)$.

Example 42. Consider again $D = 2$, $n = 3$, and suppose that Pommaret division with $x < y < z$ is defined on $U$.

The Ufnarovsky-like graph is:

Proof. \(\Rightarrow\) Consider $t \in \mathbb{N}$, $s \in U$, s.t. $w := \operatorname{lcm}(t, s) \in C_L(t, U) \subset H$.

Since $H$ is an order ideal and $s | w \in H$ then $s \in H$ whence $s \in \mathbb{N}$.

\(\Leftarrow\) Let $w \in H$; we prove that every divisor of $w$ belongs to $H$ as well.

If $\deg(w) \leq D$, its divisors have degree strictly smaller than $D$, so they clearly belong to $H$.

If $\deg(w) > D$, we know that $w \in C_L(t, U)$, for some $t \in \mathbb{N}$, then we may write $w = tu$, with $u \in L(t, U)$. Let $l \mid w$; if $l \mid l$, then $l \in C_L(t, U) \subset H$. Otherwise, if $\deg(l) < D$, then $l \in H$ by definition of $H$, so the only case we have still to examine is the case $t \nmid l$ and $\deg(l) \geq D$. By definition of relative involutive division, $\exists m \in U$ s.t. $l \in C_L(m, U)$, so $l = mv$, $v \in M(m, U)$. Take now $w' := \operatorname{lcm}(m, t) = th'$ with $h' \mid u$, since $th' \mid w = tu$.

This implies $\operatorname{lcm}(m, t) = w' \in C_L(t, U)$ and $X(m, t) = t$ so, by hypothesis, $m \in \mathbb{N}$, allowing us to conclude that $l \in H$.

\(\Box\)
The variables labelling the edges, in the picture represent the non-multiplicative variables involved in drawing the Ufnarovsky-like graph; for example

means that $x^2 \cdot y \in C_L(xy, U)$.

By means of the Ufnarovsky-like graph, we can construct both ideals and order ideals, as shown in what follows.

**Definition 43.** Let $U = T_D$, $D \in \mathbb{N}$. Suppose that the Pommaret relative involutive division $L$ is defined on $U$ and let $G_U$ the related Ufnarovsky-like graph.

- A subset $M \subset U$ is Ufnarovsky-compliant if $\forall s, t \in U$ for which there is an arrow $t \to s$ in $G_U$, $s \in M \Rightarrow t \in M$.

- Let $N \subset U$. $N$ is Ufnarovsky-revenant if $\forall t, s \in U$ for which there is an arrow $t \to s$ in $G_U$, $t \in N \Rightarrow s \in N$.

**Proposition 44.** Let $U = T_D$, $D \in \mathbb{N}$. Suppose that the Pommaret relative involutive division $L$ is defined on $U$ and let $G_U$ the related Ufnarovsky-like graph.

a. Let $M \subset U$ and $J = (M) = \bigcup_{l \in M} C_L(l, U) \iff M$ is Ufnarovsky-compliant.

b. Let $N \subset U$. Then $H := (\bigcup_{l \in N} C_L(l, U)) \cup \{v \in T \mid \deg(v) < D\}$ is an order ideal $\iff N$ is Ufnarovsky-revenant.

**Proof.**

a. “$\Rightarrow$”

Consider $s \in M$, $t \in U$ s.t. there is an arrow $t \to s$; then $\exists x_j \in NM_L(s, U)$ s.t. $sx_j \in C_L(t, U)$.

If, by contradiction, $t \notin M$, being $x_js \in J = \bigcup_{l \in M} C_L(l, U)$, $\exists w \in M$ (and so $w \neq t$) s.t. $x_js \in C_L(w, U)$ and this contradicts condition 2’ of the definition of relative involutive division.

“$\Leftarrow$” Let $t \in U$, $s \in M$ and suppose $w = \text{lcm}(s, t) \in C_L(t, U)$, so

$$w = \frac{st}{l} = th, \ l = \text{GCD}(s, t), \ h = \frac{s}{l} \in L(t, U).$$
We have to prove that $t \in M$, by means of Ufnarovsky-compliance, i.e. by showing that there is a path from $t$ to $s$ in $G_U$.

Notice that $\deg(s) = \deg(t)$, so $\deg(h) = \deg(\frac{y}{l}) = \deg(\frac{z}{l'}) = d$. We can write $h = \frac{y}{l} = y_1 \cdots y_d$ and $t = z_1 \cdots z_d$, with $y_1, z_1 \in \{x_1, \ldots, x_n\}$, $1 \leq i \leq d$, $y_1 \leq \ldots \leq y_d$, $z_1 \leq \ldots \leq z_d$.

Since $h \in L(t, U)$, we have $y_d = \min(h) \leq \min(t)$ and since $l \mid t y_d \leq \min(l)$.

Now take $s_1 := \frac{z_1}{y_1} \in U$. It holds $y_1 \leq \min(t) \leq z_1$. If $y_1 = z_1$, then $y_1 \mid h = \frac{y}{l}$ and $y_1 \mid \frac{t}{l'}$ and this is impossible since GCD($\frac{y}{l}, \frac{t}{l'}$) = 1 by definition of $l$. So we can conclude that $y_1 < z_1$. We prove now that $y_1 = \min(s_1)$.

In principle, $\min(s) \leq y_1$; but if $\min(s) < y_1 = \min(h)$ then $\min(s) \mid l$, $l \mid t$ so $\min(s) \mid l$ and $\min(s) \mid h$, being smaller than the minimal variable appearing in $h$ with nonzero exponent.

Then $\min(t) \leq \min(s) < y_1 \leq y_d$ and this contradicts $h \in L(t, U)$, so $\min(s) = y_1$.

Since, moreover, $y_1 < z_1$, then $y_1 \leq \min(s_1)$, whence $y_1 \in M(s_1, U)$. From condition 2’ of Definition4 and observing that $s_1 y_1 = z_1$ we can deduce that $s_2 \in C_{L(s_1, U)} \frac{1}{y}$ and this implies that have an arrow $s_1 \rightarrow s$ in $G_U$ and so $s_1 \in M$.

For $2 \leq j \leq d$, we define

$$s_j := \frac{s_{j-1} z_j}{y_j}$$

with $z_j := \min(\frac{1}{y_1}, \ldots, \frac{1}{y_j}) = \min(z_j, \ldots, z_d)$ and $y_j := \min(\frac{h}{l}, \ldots, \frac{h}{l'}) = \min(y_j, \ldots, y_d)$.

It holds $y_j \leq \max(h) \leq \min(t) \leq z_j$.

If $y_j = y_j$ then $y_j \mid \frac{t}{l'}$, $h$ but this is impossible since GCD($\frac{y_j}{l'}$, $\frac{h}{l'}$) = 1, by definition of $l$, so we have $z_j > y_j$.

Now we prove that $y_j \leq \min(s_j)$. We know that $\prod_{i=1}^{j-1} y_i \mid s_{j-1}$, so $\min(s_{j-1}) \leq y_j$. If, by contradiction, $\min(s_{j-1}) < y_j$, then $\min(s_{j-1}) \mid h$ and $\min(s_{j-1}) \mid l$ and $l \mid t$, whence $\min(t) \leq \min(s_{j-1}) < y_j \leq \max(h)$, which is impossible, so $\min(s_{j-1}) = y_j < z_j$. Then $y_j \leq \min(s_j)$ and so $y_j \in M_L(s_j, U)$; $z_j > \min(s_{j-1})$ so $z_j \in N M_L(s_{j-1}, U)$.

By $s_j y_j = s_{j-1} z_j$, we get that there is an arrow $s_j \rightarrow s_{j-1}$ in $G_U$, and so $s_j \in M$.

Finally, we observe that if $j = d$ we have $s_j = s_d = t$, so we can conclude. Indeed, we have found a path from $t \in U$ and $s \in M$, so we can deduce that also $t \in M$.

b. “$\Rightarrow$”

Consider $t \in N$, $s \in U$ s.t. there is an arrow $t \rightarrow s$ in $G_U$; there is a variable $x_j \in N M_L(s, U)$ s.t. $sx_j = tv, v \in L(t, U)$. In particular $\deg(sx_j) = D + 1 = \deg(tv)$ and $\deg(t) = D, s\frac{v}{x_j} = x_j \neq x_j$. We may write $sx_j = tx_i$ and so $x_j \mid t$, so $\lcm(s, t) = \lcm(\frac{v}{x_j}, t) = tx_i \in C_L(t, U)$. By Proposition4 we can conclude that $s \in N$.

“$\Leftarrow$” The construction is exactly the same as for part a. The only difference is

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3This could be seen also directly, since we have $\min(s) = y_1 < z_1$

4It is clear that $x_j \neq x_j$, since otherwise $s = t$ and $N M_L(s, U) = N M_L(t, U)$.
that now \( t \in N \) and \( s \in U \), so, once constructed a path from \( t \) to \( s \), we can conclude that each \( s_j \) in the path belongs to \( N \) so \( s \in N \).

\[ \square \]

**Example 45.** Referring to the graph of example 42 we see that if we want to construct an ideal \( J = (M) \) and we suppose \( xz \in M \), then, following the graph, \( yz, z^2 \in M \) and actually \( J = (z^2, yz, xz) = C_L(z^2, U) \cup C_L(yz, U) \cup C_L(xz, U) \). On the other hands, if we want to construct an order ideal and we suppose \( xz \in N \) then \( x^2 \in N \) and \( H = C_L(xz, U) \cup C_L(x^2, U) \cup \{ v \in T \mid \deg(v) < 2 \} \) is actually an order ideal.

\[ \diamond \]

It may seem that the Ufnarovsky-like graph may be enough to treat the general solution. Indeed, in some small examples, as the following Example 46, the Ufnarovsky-like graph solves the problem even if the relative involutive division \( L \) defined is not Pommaret division.

**Example 46.** Consider \( D = 2 \) and \( n = 3 \). There are exactly 6 terms in \( T_2 \), i.e. \( T_2 = \{ x^2, xy, y^2, xz, yz, z^2 \} \). We take the relative involutive division given by

| Terms | Multiplicative Variables |
|-------|-------------------------|
| \( x^2 \) | \( x \) |
| \( xy \) | \( x, y, z \) |
| \( y^2 \) | \( y \) |
| \( xz \) | \( x, z \) |
| \( yz \) | \( y \) |
| \( z^2 \) | \( y, z \) |

The corresponding Ufnarovsky-like graph is

Suppose we want to construct a monomial ideal \( J = (M) \) s.t. \( xz \in M \). Then, all multiples of \( xz \) must belong to \( J \). For those arising by multiplying by \( x, z \) there is nothing to do, since these two variables are multiplicative for \( xz \). For those involving \( y \), we may add \( xy \) to \( M \) (indeed there is an arrow \( xy \to xz \)) and we can conclude since \( M_L(xy, U) = \{ x, y, z \} \).

\[ \diamond \]

**Remark 47.** In the paper [27], Pleksen and Robertz employ the theory of Janet bases to compute resolutions of finitely generated modules over polynomial rings over fields and over rings of linear differential operators with coefficients in a differential field.
Given the polynomial ring $R := k[x_1, ..., x_n]$ over a field $k$, they study the free module $R^q$. First, they consider a multiple-closed set $S$ (i.e. a set of terms in $R^q$ which is closed by the multiplication for terms in $R$) and they take a Janet basis $J(S)$. Then, they introduce as a tool the so-called Janet graph of $J(S)$. The vertex set of the graph is given by $J(S)$ itself. For each $v \in J(S)$ and each $x_i$ that is non-multiplicative for $v$, let $w \in J(S)$ the unique involutive divisor of $v x_i$. Then, there is an edge $v \xrightarrow{x_i} w$.

The paper [34] is dedicated to structural properties of involutive divisions. In particular, it extensively deals with Pommaret bases and its syzygy theory. In connection with the construction of a Groebner basis for the syzygy module of an involutive basis $H$, looking for a way to get an involutive basis for the module, the author applies Janet-Schreyer theorem [23, 33], which requires a suitable ordering of the elements in $H$.

For producing it, the author generalizes the graph constructed by Pleksen and Robertz [27] to an arbitrary involutive division $L$, defining the $L$-graph. This graph is defined exactly in the same way as the Janet graph, with vertex set $T(H)$, only the involutive division is different. From the $L$-graph, he defines the $L$-ordering on $H$, setting $h_a < h_b$ if there is a path from $T(h_a)$ to $T(h_b)$ in the graph.

In the paper [20], the authors study stable ideals, showing that they share many properties with the generic initial ideal. Moreover, they relate Pommaret bases to some invariants associated with local cohomology and exhibit the existence of linear quotients in Pommaret bases. In such context, they take the Pommaret basis $H$ of a monomial ideal and they associate Seiler’s $L$-graph with $L$ given by Pommaret division. This graph is again used to produce an ordering on $H$ (reversing that of [34]).

Finally, in [11], in the context of computing resolutions and Betti numbers, the authors again employ Janet-Schreyer theorem and the $J$-graph, i.e. essentially Pleksen-Robertz’ Janet graph, with no significant modifications.

All these graphs (that are specifications of the most general version, i.e. that defined in [34]) are very close to our Ufnarovsky-like graph. In particular, an $L$-graph is our graph with reversed arrows (and without the mark of the involved non-multiplicative variable, that we have put on each edge and that is only present in [27]). Moreover, we construct it on the set of all terms in some degree $D$ (so including the ideal and the escalier in the same graph), whereas they use the generators of a semigroup ideal.

Our theory is restricted to the case of all monomials of the set $T_D \subset R = k[x_1, ..., x_n]$, which have a fixed degree $D$; in order to cover the theory of [27, 34] [20, 1], which consider an involutive basis $H = \{f_1, ..., f_m\} \subset R^q$ and restrict to the leading terms, lying in $T_{\geq D}^{(m)} = \{te_i, \ deg(t) + \deg(f_i) \geq D\}$, we should extend our theory from $T_D$ to the $T_{\geq D}^{(m)}$.

In the next section, we will see whether Ufnarovsky-like can cover the general case or not.
8 Relative involutive divisions, involutiveness and generic initial ideal

Riquier \cite{30} gave not only S-polynomial-like relations which must be satisfied by principal differential equation systems in order to have solutions but also described the initial conditions as series with the shape

\[ \sum_{l \in \mathbb{N}} \sum_{t} t \in C_l(l, U)c_{l,t}t. \]

Delassus \cite{6, 7, 8} criticized Riquier for not having realized that his result was giving, up to a generic change of coordinates, a canonical form of the solutions. The (wrong) intuition by Delassus was the notion of generic initial ideal. He considered the \((\text{degree})\)-lexicographical ordering \(>\) induced by \(x_1 > \ldots > x_n\) and claimed that, given \(l\) independent forms \(G := \{g_1, \ldots, g_l\}\) in \(n\) variables of degree \(p\), denoting \(I := \mathcal{I}(G)\) the ideal generated by \(G\) and \(T(G) = T(I)\) its initial ideal, there is a monomial ideal \(g_{\text{in}}(I)\) which not only satisfies \(g_{\text{in}}(I) = T(g(I))\) for all generic change of coordinate \(g\) but even that it consisted of the first \(l >\)-maximal monomials in \(T_p\).

As it is well known, and as it was independently discovered by Gunther\cite{17, 18, 19} and Robinson \cite{31, 32}, the result is at the same time false and not complete, but it holds for all Borel-fixed monomial ideals.

Janet \cite{24, 25, 26} in order to apply Cartan’s test and condition applied himself a (Zariski open) generic change of coordinate obtaining the Pommaret involution and the related canonical form which he called involutive.

He stated that each non-trivial Borel-fixed monomial ideal is involutive \textit{id est} with our terminology its relative involutive division is Pommaret.

Example 48. \cite[p.31]{25} Consider \(n=3\) and \(D=2\), supposing \(x < y < z\) and define Pommaret relative involutive division, as follows:

| Terms | Multiplicative Variables |
|-------|--------------------------|
| \(x^2\) | \(x\) |
| \(xy\) | \(x\) |
| \(y^2\) | \(x, y\) |
| \(xz\) | \(x\) |
| \(yz\) | \(x, y\) |
| \(z^2\) | \(x, y, z\) |

The associated Ufnarovsky-like graph is

\footnote{This historical remark is deeply depending on \cite[IV.55]{28}}

\footnote{The notation used by these Hilbert’s followers is not obvious, the more so since Janet systematically reversed the notion of the other researchers; we present here Delassus claim using the present standard notations as described in \cite{16}}

\footnote{id est for all \(g \in U,\ U\) a Zariski open subset \(U \in GL(n)\); of course that time was missing the notion of Zariski sets, but the informal intuition was clear to researchers.}

\footnote{If we consider the (trivially) Borel-fixed monomial ideal \(U = T_D\) of course the statement is trivially counterexampled by any non-Pommaret relative involutive division.}
Suppose we want to construct an ideal and that $xy \in M$; looking at the above graph we see that this implies that also both $y^2$ and $yz$ must belong to $M$. Then also $z^2 \in M$ and we can conclude. We have then $J = (xy, y^2, yz, z^2)$. Even it has been constructed using Pommaret division, $J$ is not Borel-fixed\(^\text{11}\), so it cannot be a gin.

9 A graph for the general solution

In section 7, we have shown how to construct semigroup ideals and the corresponding escaliers in a consistent way w.r.t. the decomposition in cones induced by Pommaret relative involutive divisions. We do so by walking in a simple graph, the Ufnarovsky-like graph.

In this section, we will see that the Ufnarovsky-like graph cannot be used to cover the general case and we will see how to generalize it, in order to cover all the possible relative involutive divisions.

Let us see an example

Example 49. Take $D = 3$ and $n = 4$ so that we get the 20 terms in

$T_3 = \{x^3, x^2y, xy^2, y^3, x^2z, z^3, x^2t, xt^2, t^3, xyt, xzt, xyz, yz^2, y^2z, y^2t, z^2t, yzt, zt^2\}$.

Let us then define the relative involutive division $L$, by specifying the multiplicative variables for each term:

\(^{11}\)Indeed $xy \in J$ but $xz \not\in J$. 
| Terms  | Multiplicative variables |
|--------|--------------------------|
| $x^3$  | $x$                      |
| $x^2y$ | $x, y, t$                |
| $xy^2$ | $y, z$                   |
| $y^3$  | $y, z$                   |
| $x^2z$ | $x, y, z$                |
| $xz^2$ | $z$                      |
| $z^3$  | $z$                      |
| $x^2t$ | $x, z, t$                |
| $xt^2$ | $y, t$                   |
| $t^3$  | $t$                      |
| $xyt$  | $y$                      |
| $xzt$  | $z, t$                   |
| $xyz$  | $z$                      |
| $yc^2$ | $z$                      |
| $yt^2$ | $t$                      |
| $y^2z$ | $z$                      |
| $y^2t$ | $y, t$                   |
| $z^2t$ | $z, t$                   |
| $yzt$  | $x, y, z, t$             |
| $zt^2$ | $t$                      |

and construct the corresponding Ufnarovsky-like graph:
This graph is rather complicated, but we can isolate a part, to look more deeply into that one:

As usually, we denote $J = (M)$ a monomial ideal. If we want $x^2y \in M$, then we need $x^2z \in M$ since $(x^2y)z \in C(x^2z, U)$. Now, if $x^2z \in M$, we must have $x^2t \in M$ and, for having $x^2t \in M$ we only need $x^2y \in M$, so, since each term has only the non-multiplicative variable we are using, it may seem that we can build an ideal considering only that three terms. This is actually impossible, since, for example $x^2yzt \notin C(x^2t) \cup C(x^2z) \cup C(x^2t)$ and, in any case we know that (see Remark 34) $yzt \in M$.

**Remark 50.** The paper [2] provides a new approach to the theory of involutive divisions. The given definition of involutive division is rather more general than that by Gerdt and Blinkov [12, 13], but then the author restricts to a smaller class of involutive divisions, that he calls *admissible*, giving effective criteria (somehow similar to our criteria for relative involutive divisions) for a division to be admissible. Notice that an admissible division is not necessarily a relative involutive division, since (as for [12, 13]) the condition $T(U) = \bigcup_{u \in U} L(u, U)$ is not required.

The problem of constructing an involutive basis for an ideal is then tackled and the underlying idea is that for admissible involutive divisions it is enough to walk backwards in Ufnarovsky-like graph (but Apel does not introduce any graph as a tool).

It can be easily shown that the relative involutive division of example 49 is not admissible.

In order to cover the general case, we define a new oriented graph $G$, such that its nodes are the elements of $U$ and, given $t, s \in U$, there is an edge from $t$ to $s$ if the following two conditions are verified:

1. there are no oriented paths from $t$ to $s$;
2. $\text{lcm}(s, t) \in C_L(t, U)$

We call such a graph a *generalized Ufnarovsky-like graph*.

**Definition 51.** Let $U = T_D$, $D \in \mathbb{N}$ and suppose that a relative involutive division $L$ is defined on $U$. Let $G$ be the generalized Ufnarovsky-like graph.

1. A subset $M \subseteq U$ is $G$-compliant if $\forall t, s \in U$ for which in the oriented graph $G$ there is a path from $t$ to $s$, $s \in M \Rightarrow t \in M$.

2. A subset $N \subseteq U$. is $G$-revenant if $\forall t, s \in U$, for which in the oriented graph $G$ there is a path from $t$ to $s$, $t \in N \Rightarrow s \in N$.

The paper is not involving also Groebner escalier’s decomposition.
Proposition 52. Let $U = \mathcal{T}_D$, $D \in \mathbb{N}$ and suppose that a relative involutive division $L$ is defined on $U$. Let $G$ be the generalized Ufnarovsky-like graph.

a. Let $M \subseteq U$, and $J = (M)$. $J = \bigcup_{l \in M} C_L(l, U) \iff M$ is $G$-compliant.

b. Let $N \subseteq U$, then $H := \bigcup_{l \in N} C_L(l, U) \cup \{v \in T \mid \deg(v) < D\}$ is an order ideal, if and only if $N$ is $G$-revenant.

Proof. a. “$\Rightarrow$”

Consider $s, s', t \in U$, $s \in M$ and suppose $\text{lcm}(s, s') \in C_L(t, U)$. By definition of $G$, there must be an edge from $t$ to $s$ unless they are not already connected by a path, so by the hypothesis, we can conclude that $t \in M$, whence $J = \bigcup_{l \in M} C_L(l, U)$ by Proposition 33.

“$\Leftarrow$”

Let $t, s \in U$, $s \in M$ and suppose there is a path from $t$ to $s$, i.e. $t \rightarrow s_r \rightarrow \ldots \rightarrow s_1 \rightarrow s$ is a path in $G$.

Consider $s_1 \rightarrow s$; by definition of $G$, there is such an edge iff there is a term $s'' \in U$ s.t. $\text{lcm}(s'', s) \in C_L(s_1, U)$. This implies (c.f. Proposition 33) that $s_1 \in M$. Walking backwards this way in the path $t \rightarrow s_r \rightarrow \ldots \rightarrow s_1 \rightarrow s$ of $G$, we get in a finite number of steps that $t \in M$.

b. “$\Rightarrow$”

Let $t \in M$, $s \in U$ and suppose that there is a path from $t$ to $s$, i.e. $t \rightarrow s_r \rightarrow \ldots \rightarrow s_1 \rightarrow s$ is a path in $G$. Since there is an edge $t \rightarrow s_r$ in $G$, then there is $s' \in U$ s.t. $\text{lcm}(s', s) \in C_L(t, U)$ and, by Proposition 35 $\text{lcm}(s', t) \in C_L(t, U)$ and so $s_r \in M$ by 40.

Walking this way in the path $t \rightarrow s_r \rightarrow \ldots \rightarrow s_1 \rightarrow s$ of $G$, e get in a finite number of steps that $s \in M$.

“$\Leftarrow$”

Let us take $t \in M$, $s \in U$ and suppose $\text{lcm}(t, s) \in C_L(t, U)$. This implies that in $G$ there is an edge $t \rightarrow s$ (or, at least, a path between them) and both the possibilities imply that $s \in M$.

Remark 53. The rationale of imposing condition 1. to the definition of generalized Ufnarovsky-like graph is to have a minimal graph, with no redundant paths. The only condition 2. would make us construct a directed graph with many useless arrows. On the other hands, the only way we actually see to concretely construct the generalized Ufnarovsky-like graph associated to a given relative involutive division $L$ is to construct the redundant graph using condition 2. and then eliminate the useless arrows. Looking at the above two propositions, we can notice that what we are interested is the path set of the graph. In other words, condition 1. of its definition is not mandatory, but it is only set in order to avoid to consider useless edges. Considering a graph constructed using only condition 2., we may notice that the graph we defined is a minimum equivalent digraph, i.e. the smallest subgraph preserving the path set.

Finding a minimum equivalent digraph for a general directed graph is an NP-complete
problem \[11\]. It would be hopeful to find the minimal graph in our particular cases with less effort, or at least in some particular cases as we have already done for Pommaret division.

Example 54. Let us take \( n = 3, D = 2 \) and define the following division:

| Terms | Multiplicative Variables |
|-------|-------------------------|
| \( x^2 \) | \( x \) |
| \( xy \) | \( x, y, z \) |
| \( y^2 \) | \( y, z \) |
| \( xz \) | \( x, z \) |
| \( yz \) | \( z \) |
| \( z^2 \) | \( z \) |

The corresponding graph is

![Graph](image)

If \( J = (M) \) and \( xz \in M \), then \( xy \in M \) and \( J = (xz, xy) = C_L(xz, U) \cup C_L(xy, U) \).

On the same way, if we want to compute an order ideal and \( xz \in N \) then both \( x^2 \) and \( z^2 \) must be in \( N \) and, with their three relative involutive cones (and, of course, the elements of degree 0, 1), we have an order ideal.

Example 55. Referring to the relative involutive division defined in example \[49\], we can see that, if we restrict the graph only to the part involved in the argument introduced before, we have

![Graph](image)

As usually, we denote \( J = (M) \) a monomial ideal. If we want \( x^2y \in M \), then we need \( x^2z, yzt \in M \). Now, if \( x^2z \in M \), we must have \( x^2t \in M \) and, for having \( x^2t \in M \) we only need \( x^2y \in M \). In this case \( J = (x^2y, x^2z, x^2t, yzt) \) is exactly the union of the cones of its generators.
A Relative involutive divisions

In this section, we summarize all the possible relative involutive divisions for $U := T_D$, $D = 2, n = 3$ up to symmetries.

We begin remarking that the orbits of $U$ under the symmetric group $S_3$ are \{x^2, y^2, z^2\} and \{xy, yz, zx\}.

It we choose $x^2$ as the “source” with \{x, y, z\} as non-multiplicative variables, the symmetry group becomes $\langle (y, z) \rangle$ and we need to fix one and only one term with 2 non-multiplicative variables among \{xy, xz\}; fixing $xy$, whose multiplicative variables are necessarily \{y, z\} we have no more symmetries but we have still to set a second (and last) element with \{y, z\} with multiplicative variables. We can freely choose any among \{y^2, yz, z^2\}. Thus we obtain

| Terms  | Multiplicative Variables |
|--------|--------------------------|
| $x^2$  | x, y, z                  |
| $xy$   | y, z                     |
| $y^2$  | y, z                     |
| $xz$   | z                        |
| $yz$   | y                        |
| $z^2$  | z                        |

Next if we choose the “source” to be $xy$ we have again the symmetry group $\langle (x, y) \rangle$ and the orbits \{xy\}, \{xz, yz\}, \{x^2, y^2\}, \{z^2\}. We need to choose both an element in \{x^2, xz, z^2\}.

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with \{x, z\} as non multiplicative variables and an element in \{y^2, yz, z^2\} with \{y, z\} as non multiplicative variables; note that the two choices are symmetric; so we can use the last freedom for deciding to first choose the element in \{x^2, xz, z^2\} as the one with \{x, z\} as non multiplicative variables and next adapt the choice of the element with \{y, z\} as non multiplicative variables consequently in \{y^2, yz, z^2\}; we can thus either

- choose respectively \(x^2\) and \(y^2\) obtaining

| Terms | Multiplicative Variables |
|-------|-------------------------|
| \(x^2\) | \(x, z\) |
| \(xy\)  | \(x, y, z\) |
| \(y^2\)  | \(y, z\) |
| \(xz\)  | \(z\) |
| \(yz\)  | \(z\) |
| \(z^2\)  | \(z\) |

- or choose respectively \(xz\) and \(yz\) obtaining

| Terms | Multiplicative Variables |
|-------|-------------------------|
| \(x^2\) | \(x\) |
| \(xy\)  | \(x, y, z\) |
| \(y^2\)  | \(y\) |
| \(xz\)  | \(x, z\) |
| \(yz\)  | \(y, z\) |
| \(z^2\)  | \(z\) |

which of course preserves \((x, y)\) as symmetry group; or

- choosing respectively \(x^2\) and \(yz\) obtaining

| Terms | Multiplicative Variables |
|-------|-------------------------|
| \(x^2\) | \(x, z\) |
| \(xy\)  | \(x, y, z\) |
| \(y^2\)  | \(y\) |
| \(xz\)  | \(z\) |
| \(yz\)  | \(y, z\) |
| \(z^2\)  | \(z\) |

- choose respectively \(x^2\) and \(z^2\) having no more freedom and obtaining
B Generalized Ufnarovsky-like graph for example 49.

Example 49 shows that Ufnarovsky-like graph cannot be used for constructing ideals and Groebner escaliers consistent with every relative involutive division.

In Section 9 we have defined the generalized Ufnarovsky-like graph and we proved that it solves the problem in the general case.

We show now the generalized Ufnarovsky-like graph for example 49.
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