ON THE COMPRESSIBLE NAVIER-STOKES-KORTEWEG EQUATIONS

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Abstract. In this paper, we consider compressible Navier-Stokes-Korteweg (N-S-K) equations with more general pressure laws, that is the pressure $P$ is non-monotone. We prove the stability of weak solutions in the periodic domain $\Omega = \mathbb{T}^N$, when $N = 2, 3$. Utilizing an interesting Sobolev inequality to tackle the complicated Korteweg term, we obtain the global existence of weak solutions in one dimensional case. Moreover, when the initial data is compactly supported in the whole space $\mathbb{R}$, we prove the compressible N-S-K equations will blow-up in finite time.

1. Introduction. For its importance and application in scientific research, the Navier-Stokes-Korteweg (N-S-K) equations have been studied widely in various fields. The present paper addresses the following N-S-K equations in the periodic domain $\Omega = \mathbb{T}^N$:

$$\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P + \rho u|u| = \text{div}(\mu D(u)) + \nabla(\lambda \text{div} u) + \text{div} K,
\end{cases}$$

where $\rho = \rho(t, x)$, $u = u(t, x)$, $P = P(\rho)$ respectively denote the density, velocity and pressure. $\rho$ and $u$ are periodic in $\Omega$. $D(u) = \frac{1}{2}[\nabla u + \nabla^t u]$ is the strain tensor, $\rho u|u|$ represents the drag term. The Korteweg stress tensor $K$ is given by

$$\text{div} K = \nabla (\rho \kappa(\rho) \Delta \rho + \frac{1}{2}(\kappa(\rho) + \rho \kappa'(\rho))|\nabla \rho|^2) - \text{div}(\kappa(\rho) \nabla \rho \otimes \nabla \rho),$$

where $\kappa$ is coefficient of capillarity. The capillarity coefficient is a regular function and used to describe the phase transition. In the current research, the form of

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pressure $P$ is usually assumed as $P = \rho^\gamma$. Here in the present paper, pressure $P$ is neither the polytropic gas, nor a monotone increasing, but obeys a more general barotropic constitutive law (see [6]):

$$P \in C^1[0, \infty), \quad P(0) = 0, \quad \frac{1}{a} \rho^{\gamma-1} - b \leq P'(\rho) \leq a \rho^{\gamma-1} + b, \quad \text{for all } \rho \geq 0, \quad (2)$$

for three constants $a > 0$, $b \geq 0$, $\gamma > 1$. The pressure is assumed to behave typically as a gamma type law, but close to zero density. From the physical viewpoint, a barotropic fluid with a pressure is not always monotonically increasing in the density by the phase transition. From the mathematical viewpoint, a priori estimate on $\nabla \rho$ allows a more general pressure laws due to the Korteweg term in the energy estimate. So it is interesting and meaningful to consider the N-S-K system in the general pressure law.

We define an auxiliary function $\Pi$:

$$\Pi(\rho) = \rho \int_1^\rho \frac{P(z)}{z^2} \, dz.$$ 

As fluid is assumed to be Newtonian, the two Lamé viscosity coefficients satisfy Stokes’ law:

$$\mu > 0, \quad 2\mu + N\lambda \geq 0.$$ 

It was Van der Waals and Korteweg that first considered compressible fluids model endowed with internal capillarity. The model is later developed by Dunn and Serrin [9], which could describe the variation of density at the interfaces between two phases, generally a liquid-vapor mixture. The N-S-K model includes many classical models, such as the compressible Euler equations if $\mu = \lambda = \kappa = 0$; if $\mu > 0$, $2\mu + N\lambda \geq 0$ and $\kappa = 0$, it reduces to the famous compressible Navier-Stokes equations. As to the compressible Navier-Stokes equations, there are many literatures and results, such as [4, 12, 17, 20, 21, 22, 28] and references cited therein. Recently, Guo and Wang [12] got the optimal time decay rates of the compressible Navier-Stokes equations as viscosity coefficients are constant.

The compressible N-S-K system has recently been extensively studied in fluid mechanics due to its physical importance, complexity, rich phenomena and mathematical challenges. When viscosity coefficients $\mu$, $\lambda$ and capillarity coefficient $\kappa$ are constant, readers can refer to [5, 14, 15, 24, 27] and references cited therein. It is inevitable that the N-S-K system becomes complicated when the coefficients depend on density. For one thing, the coefficients may vanish at vacuum, and the degeneracy of coefficients gives rise to difficulty in analysis in contrast to system with constant coefficients. Moreover, the strongly nonlinear third-order differential operator and dispersive structure of the momentum equation cause many new challenges. Bresch et al [1]-[3] introduced a new entropy estimate, which provided more information about density. More precisely in [3], they dealt with the special Korteweg model: $\mu = \rho$, $\lambda = 0$, $\kappa = 1$ and obtained the stability of weak solutions in $T^N$, $N = 2, 3$. Other types of N-S-K system, reader can see [13, 16, 18].

It is well-known that the existence of weak solutions to compressible fluid with coefficients depend on density is an interesting question. The existence is generally obtained by a constructive method: constructing approximate solutions (e.g., by Galerkin method or finite differences), and then passing to limit by the analysis of stability. Readers can refer to [11, 17, 20] for more details. Many literatures are devoted to studying the stability of weak solutions to various compressible
fluid model, since stability of weak solutions is a basic property which allows us to develop a rigorous existence theory. However, the construction of approximation solution is a rough and arduous issue. Especially, when the initial density contain the vacuum, it is hard to get the lower bound of $\rho$. Therefore, the construction is not a trivial process. Due to the difficulty in the construction of approximation solutions, the mathematical results on the global well-posedness are usually limited to one dimensional space and three-dimensional spherical symmetric case (such as [8, 20]). The global existence of weak solutions in multi-dimensional spaces is still open for the mathematical community.

Comparing with Navier-Stokes equations, we encounter extra difficulties in the process of investigating the N-S-K system with coefficients depend on density. Precisely speaking, the capillarity term in the N-S-K system brings indefinite sign for the integrated dissipation in the entropy estimates. Therefore, the known results for the Navier-Stokes equations can not be applied to the N-S-K equations directly. Luckily, by virtue of a new and interesting Sobolev inequality from [11] in one dimensional space, we overcome the obstacle and obtain the existence of weak solutions. In the present paper, the main task is to study the stability of weak solutions in multi dimensions and establish the existence of weak solution in one dimension. Moreover, if the initial data satisfies some conditions, we prove the N-S-K system will blow-up in finite time.

Precisely, our paper is divided into three parts. In the first part, we discuss the stability of the following N-S-K system in high dimensional case ($N = 2, 3$):

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div}(\rho u) &= 0, \\
\frac{\partial}{\partial t}(\rho u) + \text{div}(\rho u \otimes u) + \nabla P + \rho u |u| &= \text{div}(\mu D(u)) + \rho \nabla \Delta \rho,
\end{align*}
\]

with initial conditions

\[
\rho|_{t=0} = \rho_0 \geq 0, \quad \rho u|_{t=0} = m_0.
\]

Our work has some prominent distinctions with Bresch et al [3]. Firstly, they chose the special test function $\rho \varphi$ and only obtain strong convergence of $\rho u$ with a monotone pressure. In order to deal with nonlinear term convergence, they introduced a cut-off function and used complicated analysis. While in our model, we can choose test function $\varphi$ and get strong convergence of $\sqrt{\rho} u$, moreover simplify the proof of nonlinear terms convergence with the general pressure.

In the second part, we prove the global existence of weak solutions to the N-S-K system in one dimensional space. The form of N-S-K system in one dimensional case is:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + P(\rho))_x + \rho u |u| &= (\mu(\rho) u_x)_x + K[\rho],
\end{align*}
\]

where $K[\rho] := \rho \kappa(\rho) \rho_{xx} + \frac{1}{2} (\rho \kappa'(\rho) - \kappa(\rho)) \rho_x^2$, with initial conditions

\[
\rho|_{t=0} = \rho_0 \geq 0, \quad \rho u|_{t=0} = m_0.
\]

The condition on pressure is the same as (2). Here the viscosity coefficient and capillarity coefficient satisfy

\[
\mu(\rho) = \rho^\alpha, \quad \kappa(\rho) = \rho^\beta,
\]

with $2\alpha - 4 < \beta < 2\alpha - 1$. The relationship between $\alpha$ and $\beta$ will be seen in Lemma 5.1. Under the so-called “strong coercivity conditions”, “tame capillarity conditions” and “non-cavitation condition”, Germain and Lefloch [11] got entropy estimate. Then they used the Galerkin method as Gamba et al.[10] to obtain the
approximate solutions with initial data \( \rho_0 > 0 \) in \( \mathbb{T} \). Differently to what done in [11], we improve and simplify their results. Firstly, we remove many restrictions on coefficients, only remaining the key “strong coercivity conditions” which is used to get entropy estimate. Secondly, we construct approximate solutions with initial density may contain vacuum.

The local existence of compressible fluid can be established by classical fixed point theorem. Under the initial density \( \rho_0 > 0 \), Kotschote [19] proved the local existence of strong solutions to the N-S-K system with coefficients depend on density. Therefore, it is nature and interesting to ask this question that whether the strong solutions will exist globally. The third part is partially answering this question. We are investigating the following N-S-K system:

\[
\begin{aligned}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + \rho \gamma)_x &= (\rho \theta u)_x + K[\rho]_x, \\
\end{aligned}
\]

where \( K[\rho] := \rho^{\beta+1} \rho_{xx} + \frac{1}{2} (\beta - 1) \rho^\beta \rho_x^2 \), with initial conditions

\[
(\rho, u)(x, 0) = (\rho_0(x), u_0(x)).
\]

A great part of the literatures devoted to blow-up results concern the compressible Navier-Stokes equations, such as [29, 30]. As to the N-S-K equations, as far as we know, there are few blow-up results: [25, 26, 31]. Here we consider that the viscosity coefficient and capillarity coefficient depend on the density. Motivated by the work of [29] and [30], we modify the corresponding function, and obtain the blow-up result.

In summary, the paper is organized as follows. In Section 2, we define precisely the notion of weak solutions, regular solution and state our main results. In Section 3, we recall the energy inequality and state the entropy estimate. Section 4 is devoted to the proof of stability. Section 5 gives some insight into the scheme of construction of approximate solutions to system (5), besides, the proof of compactness will be given in Appendices. In Section 6, we will show the blow-up of regular solutions when initial data has compact support.

2. Notations and main result.

2.1. Weak solutions. Before stating the main results, we need to specify the definition of weak solutions which we will address. It is necessary to require that the weak solutions should satisfy the natural energy estimates. From the viewpoint of physics, the conservation laws on mass and momentum should be satisfied at least in the sense of distributions. Based on those considerations, the definition of reasonable global-in-time weak solutions goes as follows.

**Definition 2.1.** We say that \((\rho, u)\) is a weak solution of (3) on \( \Omega \times [0, T] \), with initial conditions (4), if

\[
\begin{align*}
\rho &\in L^\infty(0, T; L^\gamma_0(\Omega) \cap H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \\
\nabla \sqrt{\rho} &\in L^\infty(0, T; (L^2(\Omega))^N), \quad \sqrt{\rho} D(u) \in L^2(0, T; (L^2(\Omega))^{N \times N}), \\
\sqrt{\rho} u &\in L^\infty(0, T; (L^2(\Omega))^N), \quad \rho^\gamma u \in L^3(0, T; (L^3(\Omega))^N),
\end{align*}
\]

with \( \rho \geq 0 \), and \((\rho, \sqrt{\rho} u)\) satisfying

\[
\partial_t \rho + \text{div}(\sqrt{\rho} \sqrt{\rho} u) = 0 \quad \text{in } \mathcal{D}'((0, T) \times \Omega),
\]
and if the following equality holds for all smooth test function $\varphi(t,x)$ with compact support such that $\varphi(T,\cdot) = 0$, and

$$\int_\Omega m_0 \cdot \varphi(\cdot,0)dx + \int_0^T \int_\Omega (\sqrt{\rho}u) \cdot \varphi_t + \sqrt{\rho} \otimes \sqrt{\rho} u \cdot \nabla \varphi + P(\rho) \text{div} \varphi dxdt$$

$$= \int_0^T \int_\Omega (\sqrt{\rho}u) \cdot \sqrt{\rho}|\varphi + \Delta \rho \nabla \rho \cdot \rho + \rho \Delta \rho \text{div} \varphi + \sqrt{\rho} \sqrt{\rho} D(u) \cdot \nabla \varphi dxdt,$$

where the product “A:B” means summation over both indices of the matrices A and B.

Now, we give one of the main results of this paper in the following.

**Theorem 2.2.** Assume that $\gamma > 1$ and $N = 2,3$. Let $(\rho_n, u_n)$ be a sequence of weak solutions of (3), which satisfy the entropy inequalities (11) and (13) (in the §3), with initial data

$$\rho_n|_{t=0} = \rho_0^n(x) \geq 0, \rho_n u_n|_{t=0} = m_0^n(x) = \rho_0^n(x) u_0^n(x),$$

where $\rho_0^n, u_0^n$ are such that

$$\rho_0^n \to \rho_0 \text{ in } L^1(\Omega), \rho_0^n u_0^n \to \rho_0 u_0 \text{ in } L^1(\Omega),$$

and satisfy (with $C$ being constant independent of $n$)

$$\int_\Omega [\rho_0^n \frac{|m_0^n|^2}{2} + \Pi(\rho_0^n) + |\nabla \rho_0^n|^2]dx \leq C, \int_\Omega |\nabla \sqrt{\rho_0^n}|^2 dx \leq C.$$

Then, up to a subsequence, $(\rho_n, \sqrt{\rho_n} u_n)$ converges strongly to a weak solution of (3) satisfying the entropy inequalities (11) and (13).

In particularly in one dimensional case, we can also define the the weak solutions in a similar way as in high dimensional case. Furthermore, we can get global existence of weak solution as the following theorem.

**Theorem 2.3.** Let $\gamma > 1$, $\alpha \in (\frac{3}{4},1)$ and $2\alpha - 4 < \beta < -2$. Assume that the initial data $(\rho_0, u_0)$ in system (5) satisfies

$$\rho_0 \in L^\gamma(\Omega), \frac{\rho_0}{\rho_0^2} \in L^1(\Omega), \frac{|m_0|^2}{\rho_0} \in L^1(\Omega), \frac{(\rho_0^n)_{x}}{\sqrt{\rho_0}} \in L^2(\Omega).$$

Then, there is a global weak solution $(\rho, u)$ of (5).

Before showing the blow-up theorem, we give the definition of regular solution of (7) and (8) as follows.

**Definition 2.4.** A solution of (7) and (8) is called a regular solution in $[0,T] \times \mathbb{R}$ if

(i) $\rho(t,x) \in C^4([0,T] \times \mathbb{R}), \rho > 0$, and $u(t,x) \in C^2([0,T] \times \mathbb{R});$

(ii) $\rho^{\frac{\beta}{1-\theta}}(t,x) \in C^4([0,T] \times \mathbb{R}).$

Now the blow-up theorem can be stated as follows.

**Theorem 2.5.** Let $1 < \theta \leq \gamma$, $\beta \geq -1$ or $-\gamma < \beta < -1$ and $(\rho, u)$ be a regular solution of (7) and (8) on $0 \leq t \leq T$. If the support of initial data $(\rho_0(x), u_0(x))$ is compactly supported and $(\rho_0(x), u_0(x)) \neq 0$, then $T$ must be finite.
Remark 1. In [11] Theorem 1.1, Germain and Lefloch had proved that if viscosity coefficient and capillarity coefficient satisfy “no-cavitation” condition and $\rho_0(x) > 0$, then the N-S-K system admits a global weak solution $\rho(t, x) > 0$ for all $x \in \mathbb{R}$. So in the Definition 2.2, we think that the demand of $\rho > 0$ is reasonable.

3. The energy inequalities and entropy estimates. In this section, we will give the energy equality and state the entropy estimate, which is important in the proof of Theorem 2.1. When deriving a priori estimates, it is customary to assume that all quantities appearing in the equations are as smooth as necessary.

We obtain the usual energy inequalities in a classical way by multiplying the momentum equation with $u$, using the mass equation and integrating by parts,

$$\frac{d}{dt} \int_{\Omega} \left[ \frac{\rho u^2}{2} + \Pi(\rho) + \frac{1}{2} |\nabla \rho|^2 \right] dx + \int_{\Omega} \rho |D(u)|^2 dx + \int_{\Omega} \rho |u|^3 dx \leq 0. \quad (11)$$

Here and in the following, we adopt Ducomet [7] strategy to handle pressure term. From the assumption on pressure (2), we can easily get

$$P(\rho) \geq \frac{1}{a\gamma} \rho\gamma - b\rho,$$

then

$$\Pi(\rho) \geq \frac{1}{a(\gamma - 1)} (\rho\gamma - \rho) - b\rho\ln \rho.$$

Plugging the inequality into (11), we get

$$\int_{\Omega} \left[ \frac{\rho u^2}{2} + \frac{1}{a(\gamma - 1)} \rho\gamma + \frac{1}{2} |\nabla \rho|^2 \right] dx + \int_{0}^{T} \int_{\Omega} \rho |D(u)|^2 dx + \int_{0}^{T} \int_{\Omega} \rho |u|^3 dx \leq C + \frac{1}{a(\gamma - 1)} \int_{\Omega} \rho dx + b \int_{\Omega} \rho \ln \rho dx.\quad (12)$$

As $\gamma > 1$, the integral term of right hand side can be absorbed by the “pressure” term $\rho\gamma$ in the left hand, that is

$$\int_{\Omega} \left[ \frac{\rho u^2}{2} + C_0 \rho\gamma + \frac{1}{2} |\nabla \rho|^2 \right] dx + \int_{0}^{T} \int_{\Omega} \rho |D(u)|^2 dx + \int_{0}^{T} \int_{\Omega} \rho |u|^3 dx \leq C, \quad (12)$$

for a suitable positive $C_0$.

It is well-known that these natural estimates are not enough to prove stability, so we need to investigate further estimates. Fortunately, using the B-D entropy, we get the following lemma which offers the key estimates. For reader’s convenience, we will roughly recall the most important steps from original proof and focus on the new features of the system. More details can be found in [3] and [7].

Lemma 3.1. Under the assumptions of Theorem 2.1, then the following equality holds for smooth solutions $(\rho, u)$ of (3) that

$$\frac{d}{dt} \int_{\Omega} \left[ \frac{1}{2} \rho |u + \nabla \ln \rho|^2 + \Pi(\rho) + \frac{1}{2} |\nabla \rho|^2 \right] dx + \int_{\Omega} \nabla \ln \rho \cdot \nabla P(\rho) dx$$

$$+ \int_{\Omega} \left[ \frac{1}{2} \rho |\nabla u - \nabla^2 u|^2 dx + \int_{\Omega} (|u| \nabla \rho + |\Delta \rho|^2 + \rho |u|^3) dx \right] = 0. \quad (13)$$
Proof. First, we multiply the continuity equation by $|\nabla \ln \rho|^2$ and obtain
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |\nabla \ln \rho|^2 \, dx = - \int_{\Omega} \rho \nabla u : \nabla \rho \otimes \nabla \ln \rho \, dx \\
+ \int_{\Omega} \rho \Delta \ln \rho \text{div} u \, dx + \int_{\Omega} \rho |\nabla \ln \rho|^2 \text{div} u \, dx = 0.
\]
Noting that
\[
\frac{d}{dt} \int_{\Omega} \rho u \cdot \nabla \ln \rho \, dx = \int_{\Omega} \nabla \ln \rho \cdot \partial_t (\rho u) \, dx + \int_{\Omega} \frac{1}{\rho} (\text{div}(\rho u))^2 \, dx.
\]
By simple calculating and adding the two equation, we get
\[
\frac{d}{dt} \int_{\Omega} \left[ \frac{1}{2} \rho |\nabla \ln \rho|^2 + \rho u \cdot \nabla \ln \rho \right] \, dx + \int_{\Omega} \frac{1}{2} \rho |\nabla u - \nabla^t u|^2 \, dx - \int_{\Omega} \rho|D(u)|^2 \, dx \\
+ \int_{\Omega} \nabla \ln \rho \cdot \nabla P(\rho) \, dx + \int_{\Omega} u |\nabla \rho| \, dx + \int_{\Omega} |\Delta \rho|^2 \, dx = 0.
\]
Adding this equality to the energy equality (11), we have (13).

In order to control all terms in (13), we first find that, as in the energy equality (12),
\[
\int_{\Omega} \left[ \frac{1}{2} \rho |u + \nabla \ln \rho|^2 + \Pi(\rho) + \frac{1}{2} |\nabla \rho|^2 \right] \, dx \geq \int_{\Omega} \left[ \frac{1}{2} \rho |u + \nabla \ln \rho|^2 + \alpha \rho^\gamma + \frac{1}{2} |\nabla \rho|^2 \right] \, dx.
\]
Using (2), one has
\[
\int_{\Omega} \nabla P \cdot \nabla \ln \rho \, dx = \int_{\Omega} \frac{P'(\rho)}{\rho} |\nabla \rho|^2 \, dx \\
\geq \frac{1}{a} \int_{\Omega} \rho^{\gamma-2} |\nabla \rho|^2 \, dx - b \int_{\Omega} |\nabla \ln \rho|^2 \, dx.
\]
The term coming from the drag friction gives
\[
|\int_{\Omega} |u| u \nabla \rho| \, dx| = | - \int_{\Omega} \rho(|u| \text{div} u + u \nabla u \frac{u}{|u|}) \, dx| \\
\leq C \left( \int_{\Omega} \rho |u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \rho |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.
\]
Integration in $t$ and plugging all of these bounds into (13), we obtain
\[
\int_{\Omega} \left[ \frac{1}{2} \rho |u + \nabla \ln \rho|^2 + \alpha \rho^\gamma + \frac{1}{2} |\nabla \rho|^2 \right] \, dx + \int_0^T \int_{\Omega} |\Delta \rho|^2 \, dx \\
+ \int_0^T \int_{\Omega} \rho |u|^3 \, dx + \int_0^T \int_{\Omega} \frac{\rho^{\gamma-2}}{a} |\nabla \rho|^2 \, dx \\
\leq b \int_{\Omega} \rho |\nabla \ln \rho|^2 \, dx + C \left( \int_{\Omega} \rho |u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \rho |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.
\]
Using the Gronwall’s inequality and the previous energy estimate, we get
\[
\sup_{(0,T)} \int_{\Omega} |\nabla \sqrt{\rho}|^2 \, dx + \int_0^T \int_{\Omega} |\Delta \rho|^2 \, dx + \int_0^T \int_{\Omega} (\rho |u|^3 + \frac{\rho^{\gamma-2}}{a} |\nabla \rho|^2) \, dx \leq C.
\]
To show the compactness of sequences $(\rho_n, u_n)$ in appropriate function space, we gain enlightenment from the corresponding work of Mellet and Vasseur [23] and sketch the proof.
4. The proof of Theorem 2.1. In this section, we assume \((\rho_n, u_n)\) are the sequences of weak solutions of system (3). Using the above prior estimates and initial conditions (9)-(10), we can obtain the following facts:

\[
\begin{align*}
\|\sqrt{\rho_n} u_n\|_{L^\infty(0,T;L^2(\Omega))} & \leq C, \\
\|\Delta \rho_n\|_{L^2(0,T;L^2(\Omega))} & \leq C, \\
\|\rho_n\|_{L^\infty(0,T;L^2(\Omega))} & \leq C, \\
\|\nabla \sqrt{\rho_n}\|_{L^\infty(0,T;L^2(\Omega))} & \leq C, \\
\|\sqrt{\rho_n} \nabla u_n\|_{L^2(0,T;L^2(\Omega))} & \leq C, \\
\|\nabla \rho_n\|_{L^\infty(0,T;L^2(\Omega))} & \leq C, \\
\|\sqrt{\rho_n} u_n\|_{L^3(0,T;L^3(\Omega))} & \leq C,
\end{align*}
\]

where the constant \(C > 0\) here and in the following is independent of \(n\).

The proof of Theorem 2.1 will be given in the following in a sequence of seven lemmas. In the first two steps, we show the convergence of density and pressure. In the third step, we prove the strong convergence of momentum. In the Step 4, we give the key proof of strong convergence of \(\sqrt{\rho_n} u_n\). The final two steps are the convergence of diffusion term, drag term and capillarity term.

**Step 1. Convergence of \(\sqrt{\rho_n}\)**

**Lemma 4.1.** For every \(\rho_n\) satisfying the mass equation of system (3), we have

\(\sqrt{\rho_n}\) is bounded in \(L^\infty(0,T;H^1(\Omega))\),
\n\(\partial_t \sqrt{\rho_n}\) is bounded in \(L^2(0,T;H^{-1}(\Omega))\).

Then, up to a subsequence, \(\sqrt{\rho_n}\) converges a.e. and

\(\sqrt{\rho_n} \to \sqrt{\rho}\) in \(L^2(0,T;L^2(\Omega))\).

Moreover, we have

\(\rho_n \to \rho\) in \(L^2(0,T;H^1(\Omega))\), \([0.5mm] \Delta \rho_n \to \Delta \rho\) in \(L^2(0,T;L^2(\Omega))\).

**Proof.** The strong convergence of \(\sqrt{\rho_n}\) is done the same as [23].

Then, from the estimates (15) and (19), we can get

\(\rho_n \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)).\)

Next, using the continuity equation and estimate (14) and (15), we notice that

\(\partial_t \rho_n = -\text{div}(\rho_n u_n) = -\text{div}(\sqrt{\rho_n} \sqrt{\rho_n} u_n)\)

which is bounded in \(L^2(0,T;H^{-1}(\Omega))\). Thanks to Aubin’s lemma, the compactness of \(\rho_n\) can be obtained. \(\square\)

**Step 2. Convergence of the pressure**

**Lemma 4.2.** The pressure \(P(\rho_n)\) converges to \(P(\rho)\) strongly in \(L^1((0,T) \times \Omega)\).

**Proof.** Under the assumption (2), we have

\(|P(\rho_n)| \leq C(\rho_n^3 + \rho_n).\)

Then using the method in [23], we can get the result. \(\square\)
Step 3. Convergence of momentum $\rho_n u_n$

Lemma 4.3. Let $m_n = \rho_n u_n$ be a sequence satisfying the momentum equation of system (3). Then we have

$$\rho_n u_n \to m \text{ in } L^2(0,T,L^p(\Omega)), \ 1 \leq p < \frac{3}{2}.$$ 

Proof. From Lemma 4.1 and estimates (14) – (18), we conclude that

$$\rho_n u_n = \sqrt{\rho_n} \cdot \sqrt{\rho_n} u_n \in L^\infty(0,T;L^q(\Omega)), \ q \in [1,\frac{3}{2}],$$

and

$$\nabla(\rho_n u_n) = 2\nabla\sqrt{\rho_n} \cdot \sqrt{\rho_n} u_n + \sqrt{\rho_n} \cdot \sqrt{\rho_n} \nabla u_n \in L^2(0,T;L^1(\Omega)),$$

so we get $\rho_n u_n$ is bounded in $L^2(0,T;W^{1,1}(\Omega))$.

Next, we will show $\partial_t(\rho_n u_n)$ is uniformly bounded in $L^2(0,T;W^{-2,\frac{1}{4}}(\Omega))$. We rewrite the momentum equation

$$\partial_t(\rho_n u_n) = -\text{div}(\rho_n u_n \otimes u_n) - \nabla P(\rho_n) + \text{div}(\rho_n D(u_n))$$

$$- \rho_n |u_n|^2 + \rho_n \nabla \Delta \rho_n.$$

The above estimates provide the following:

$$\text{div}(\rho_n u_n \otimes u_n) = \text{div}(\sqrt{\rho_n} u_n \otimes \sqrt{\rho_n} u_n) \in L^\infty(0,T;W^{-1,1}(\Omega)), $$

$$|\nabla P(\rho_n)| \leq C(|\nabla \rho_n^\gamma| + |\nabla \rho_n|) \in L^\infty(0,T;W^{-1,1}(\Omega)),$$

$$\rho_n |u_n|^2 = \sqrt{\rho_n} |u_n|^2 \sqrt{\rho_n} u_n \in L^\infty(0,T;W^{-1,1}(\Omega)).$$

As to diffusion term, we deal with term $\rho_n \nabla u_n$ instead of $\rho_n D(u_n)$:

$$\rho_n \nabla u_n = \nabla(\rho_n u_n) - u_n \nabla \rho_n$$

$$= \nabla(\sqrt{\rho_n} \cdot \sqrt{\rho_n} u_n) - 2\sqrt{\rho_n} u_n \cdot \nabla \sqrt{\rho_n}.$$

The first term is bounded in $L^\infty(0,T;W^{-1,\frac{1}{4}}(\Omega))$, the second term is bounded in $L^\infty(0,T;L^1(\Omega))$, thus we get $\text{div}(\rho_n \nabla u_n) \in L^\infty(0,T;W^{-2,\frac{1}{4}}(\Omega))$. The estimate

$$\int_0^T \int_\Omega \varphi \rho_n \nabla \Delta \rho_n dx dt$$

$$= - \int_0^T \int_\Omega \Delta \rho_n (\nabla \rho_n \cdot \varphi + \rho_n \text{div} \varphi) dx dt$$

$$\leq ||\Delta \rho_n||_{L^2(0,T;L^2(\Omega))} (||\nabla \rho_n||_{L^\infty(0,T;L^2(\Omega))} ||\varphi||_{L^2(0,T;W^{2,4}(\Omega))}$$

$$+ ||\rho_n||_{L^\infty(0,T;L^2(\Omega))} ||\varphi||_{L^2(0,T;W^{2,4}(\Omega))})$$

$$\leq C||\varphi||_{L^2(0,T;W^{2,4}(\Omega))},$$

for all $\varphi \in L^2(0,T;W^{2,4}(\Omega))$ proves that $\rho_n \nabla \Delta \rho_n$ is uniformly bounded in $L^2(0,T;W^{-2,\frac{1}{4}}(\Omega))$.

Using the Aubin’s lemma, we get the compactness of $\rho_n u_n$. \qed

By virtue of drag term, we can easily obtain more regularity about $\sqrt{\rho_n} u_n$, that is $\rho_n \frac{1}{2} u_n \in L^3(0,T;L^3(\Omega))$, which replace the logarithmic estimate for the velocity:

$$\rho_n \frac{1}{2} u_n \ln(1 + |u_n|^2) \in L^\infty(0,T;L^1(\Omega))$$

in [23]. Following [23], we have a similar lemma.
Step 4. Convergence of $\sqrt{\rho_n}u_n$

**Lemma 4.4.** The sequence $\sqrt{\rho_n}u_n$ converges strongly in $L^2(0,T,L^2(\Omega))$ to $m/\sqrt{\rho}$. In particular, we have $m = 0$ a.e on the set $\{\rho = 0\}$, and there exists a function $u(x,t)$ such that $m = \rho u$ and

$$\rho_n u_n \to \rho u \text{ strongly in } L^2(0,T;L^p(\Omega))(1 \leq p < 3/2),$$

$$\sqrt{\rho_n}u_n \to \sqrt{\rho}u \text{ strongly in } L^2(0,T;L^2(\Omega)).$$

**Proof.** The detail can be seen in [23], with a little change. \hfill \square

The last two points to complete the proof of Theorem 2.1 are the convergence of the diffusion term, drag term and capillarity term.

**Step 5. Convergence of diffusion term**

**Lemma 4.5.** We have

$$\rho_n \nabla u_n \to \rho \nabla u \text{ in } D',$$

$$\rho_n \nabla^t u_n \to \rho \nabla^t u \text{ in } D'.$$

**Proof.** More details refer to [23]. \hfill \square

**Step 6. Convergence of capillarity term and drag term**

**Lemma 4.6.** We have

$$\rho_n \nabla \Delta \rho_n \to \rho \nabla \Delta \rho \text{ in } D'.$$

**Proof.** Let $\varphi$ be a test function, then

$$\int_0^T \int_\Omega \varphi \rho_n \nabla \Delta \rho_n \, dx dt = - \int_0^T \int_\Omega \varphi \nabla \rho_n \Delta \rho_n \, dx dt - \int_0^T \int_\Omega \rho_n \Delta \rho_n \text{div} \varphi \, dx dt.$$

Due to Lemma 4.1, it holds that

$$\rho_n \to \rho \text{ in } L^2(0,T;H^1(\Omega)), \Delta \rho_n \to \Delta \rho \text{ in } L^2(0,T;L^2(\Omega)).$$

So we finally obtain

$$\int_0^T \int_\Omega \varphi \rho_n \nabla \Delta \rho_n \, dx dt \to \int_0^T \int_\Omega \varphi \rho \nabla \Delta \rho \, dx dt$$

$$= - \int_0^T \int_\Omega \rho \Delta \rho \text{div} \varphi \, dx dt - \int_0^T \int_\Omega \varphi \nabla \rho \Delta \rho \, dx dt.$$

\hfill \square

**Lemma 4.7.** We have

$$\rho_n |u_n| u_n \to \rho |u| u \text{ in } D'.$$

**Proof.** Noticing that

$$\rho_n |u_n| u_n = \sqrt{\rho_n} |u_n| \sqrt{\rho_n} u_n.$$

Using Lemma 4.4, we can easily get the result. \hfill \square

Combining all the lemmas, we have finished the proof of Theorem 2.1.
5. Existence of weak solution in one dimension case. In this section, we will present a possible approach to the solvability of system (5). Combining the methods used in [2, 20] and a similar stability analysis in high dimensions, we prove the global existence of weak solution. Therefore, we do not give details here. Instead, we only briefly describe the proof steps and key estimates.

**Step 1. Construction of smooth approximate solution**

In the following, we consider the approximate compressible N-S-K equations:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + P(\rho))_x + \rho_k u |u| &= (\rho u_{\epsilon x})_x + K[\rho]_x,
\end{align*}
\]

where \(K[\rho] := \rho_k \kappa(\rho) \rho_{xx} + \frac{1}{2} (\rho_k \kappa'(\rho) - \kappa(\rho)) \rho_{xx}^2\). The initial \(\rho_0\) and \(m_0\) \(\in C^\infty(\Omega)\). Without the loss of generality, we can assume that \(\int_\Omega \rho_0 = 1\). By the standard arguments in the classical theory of parabolic and hyperbolic equations, we can obtain that there exists some \(T_* > 0\) such that the approximate system has a unique smooth local solution \((\rho_\epsilon, u_\epsilon)\) on \([0, T_*]\). More details refer to Appendix 1.

**Step 2. Extension the local solution to global**

In order to prove that solution \((\rho_\epsilon, u_\epsilon)\) constructed above exists on the whole time \([0, T]\), we need to do two essential things. One is to show the prior estimates which plays an important role in stability analysis; the other is to derive the lower bound and upper bound of density which is the most important and difficult in the construction of approximation solution.

The first mission is done as requiring energy inequality and entropy estimate. We find that the complicate capillarity term brings some troubles in the proceedings of construction of approximation solution.

The classical energy inequality can be obtained as usual:

\[
\int_\Omega \left[ \frac{\rho_\epsilon u_{\epsilon x}^2}{2} + \Pi(\rho_\epsilon) + \frac{1}{2} \kappa(\rho_\epsilon) \rho_{xx}^2 \right] dx + \int_0^T \int_\Omega \mu(\rho_\epsilon) |u_{xx}|^2 dx + \int_0^T \int_\Omega \rho_\epsilon |u_{\epsilon x}|^3 dx \leq C,
\]

using the condition (2) as before, we get

\[
\int_\Omega \left[ \frac{\rho_\epsilon u_{\epsilon x}^2}{2} + \rho_\epsilon^2 + \frac{1}{2} \rho_\epsilon \rho_{xx}^2 \right] dx + \int_0^T \int_\Omega \rho_\epsilon |u_{xx}|^2 dx + \int_0^T \int_\Omega \rho_\epsilon |u_{\epsilon x}|^3 dx \leq C.
\]

It’s obviously to obtain the B-D entropy in the following as we make a few changes in the derive of Lemma 3.1:

\[
\frac{d}{dt} \int_\Omega \left[ \frac{1}{2} \rho_\epsilon |u_{\epsilon x}| + \partial_x \varphi(\rho_\epsilon) \right]^2 + \Pi(\rho_\epsilon) + \frac{1}{2} \kappa(\rho_\epsilon) \rho_{xx}^2 \right) dx + \int_\Omega \partial_x P(\rho_\epsilon) \cdot \partial_x \varphi(\rho_\epsilon) dx \\
+ \int_\Omega [K(\rho_\epsilon)]_x \partial_\rho \varphi(\rho_\epsilon) dx + \int_\Omega \rho_\epsilon u_{\epsilon x} |u_{\epsilon x} | \partial_\rho \varphi(\rho_\epsilon) + \int_\Omega \rho_\epsilon |u_{\epsilon x}|^3 dx = 0,
\]

where \(\varphi'(\rho_\epsilon) = \frac{\rho'_\epsilon}{\rho_\epsilon}\).

We take some technical skills from [7] and [11] to control all the terms.

\[
\int_\Omega \partial_x P \cdot \partial_x \varphi dx = \int_\Omega P'(\rho_\epsilon) \varphi(\rho_\epsilon) |\partial_x \rho_\epsilon|^2 dx \\
\geq \frac{1}{a} \int_\Omega \varphi |\rho_{\epsilon x}|^{-1} |\partial_x \rho_\epsilon|^2 dx - b \int_\Omega \varphi |\partial_x \rho_\epsilon|^2 dx.
\]
From [7] and [23], we can easily get $\mu'(\rho_e) \geq m > 0$. The last term can be controlled as
\[
b \int_{\Omega} \varphi |\partial_x \rho_e|^2 dx = b \int_{\Omega} \rho_e |\partial_x \varphi|^2 (\mu')^{-1} dx \leq \frac{b}{m} \int_{\Omega} \rho_e |\partial_x \varphi|^2 dx.
\]
The term coming from the drag friction gives
\[
\int_{\Omega} \rho_e u_e |u_e| |\partial_x \varphi| = \int_{\Omega} (\mu(\rho_e))_x u_e |u_e| = \int_{\Omega} (\rho^\alpha_e)_x u_e |u_e| dx,
\]
where
\[
\left| \int_{\Omega} (\rho^\alpha_e)_x u_e |u_e| dx \right| = \left| \int_{\Omega} \rho^\alpha_e (u_e |u_e|)_x dx \right| \lesssim \int_{\Omega} \rho^2_e |u_e| |\rho^2_e |\partial_x u_e| dx
\]
\[
\lesssim \int_{\Omega} \rho^2_e |u_e|^2 dx + \int_{\Omega} \rho^2_e |\partial_x u_e|^2 dx
\]
\[
\leq \int_{\Omega} \rho^2_e |u_e|^2 dx + \int_{\Omega} \rho^2_e |\partial_x u_e|^2 dx
\]
\[
\leq \int_{\Omega} \rho^2_e |\partial_x u_e|^2 dx.
\]
The notation $x \lesssim y$ means $x \leq Cy$ for some constant $C > 0$.
Calculating the capillarity term, we find
\[
\int_{\Omega} [K(\rho_e)]_x \partial_x \varphi dx = \int_{\Omega} \frac{\mu(\rho_e)}{\rho_e} \kappa(\rho_e) |\rho_{xx}|^2 - \frac{1}{3} \kappa'' + \frac{2}{3} \kappa' \int_{\Omega} \frac{\mu(\rho_e)}{\rho_e} \rho_x^2 |\partial_x \rho_e|^2 dx.
\]
In [11], the following Sobolev inequality
\[
\int_{T} f^{a} (f_{x})^{2} dx \geq \left( \frac{a - 1}{3} \right)^2 \int_{T} f^{a - 2} (f_{x})^{4} dx
\]
holds for any $a > 1$ and $f \in H^2(\mathbb{T})$.
Utilizing the Sobolev inequality, we can obtain the following lemma.

**Lemma 5.1.** The inequality
\[
\int_{\Omega} [K(\rho_e)]_x \partial_x \varphi dx \geq C_0 \int_{\Omega} (\rho^2_{xx} + \rho^2_x \mu(\rho_e) \rho_x) |\partial_x \rho_e|^2 dx
\]
holds for some constant $C_0 > 0$ if and only if $2\alpha - 4 < \beta < 2\alpha - 1$.

**Proof.** The proof can be found in [11] Theorem 2.2. \qed

**Remark 2.** As $\alpha \in \left[ \frac{3}{4}, 1 \right)$, the condition $2\alpha - 4 < \beta < -2$ in Theorem 2.2 is reasonable.
Using Gronwall’s inequality, we get

\[
\sup_{(0,T)} \int_\Omega \left( \rho^{2\alpha-3}_t \rho^{2}_x + \rho^{2}_t \rho^{2}_x \right) dx + \int_0^T \int_\Omega \rho^{2\gamma-3} dx dt \\
+ \int_0^T \int_\Omega \rho^{2\alpha+\beta-1}_x dx dt + \int_0^T \int_\Omega \rho^{4}_x \rho^{2\alpha+\beta-3} dx dt \leq C.
\]  

We achieve the second aim by introducing the Lagrangian coordinates \((y,t)\), which brings us many conveniences. We introduce

\[ y = \int_0^x \rho_\epsilon(z,t) dz, \quad t = t. \]

In order to obtain the lower bound and upper bound for density \(\rho_\epsilon\), we take the technical as done in [20]. Due to the assumption about \(\rho_0\), we define that

\[ \Omega_L \triangleq (0, L_\epsilon), \text{ with } L_\epsilon = \int_\Omega \rho_0(x) dx = 1. \]

In Lagrangian coordinates, the B-D entropy equality (23) offers that

\[
\sup_{(0,T)} \int_{\Omega_L} \rho^{\beta+1}_t \rho^{2}_y dy = \sup_{(0,T)} \int_{\Omega_L} \rho^{2}_t \rho^{2}_x dx \leq C.
\]

We set \(\beta + 1 = 2\theta - 2\) \((0 < \theta < \frac{1}{2})\), the above inequality is

\[
\sup_{(0,T)} \int_{\Omega_L} (\rho^{\theta}_t)^2 dy \leq C.
\]

The continuity equation in Lagrangian coordinates \(\rho_t + \rho^{2}_x u_y = 0\) yields that

\[
\int_{\Omega_L} \rho^{-1}_t dy = \int_{\Omega} dx = |\Omega|.
\]

Let \(v_\epsilon = \frac{1}{\rho_\epsilon}\), from the above analysis, one has

\[
v_\epsilon(y,t) \leq \int_{\Omega_L} v_\epsilon(y,t) dy + \int_{\Omega_L} (v_\epsilon)^2 |\rho_\epsilon y| dy \\
\leq C + \max_{y \in \Omega_L} (v_\epsilon)^{\theta + \frac{1}{2}} \|\rho^{\theta}_t\|_{L^2} \left( \int_{\Omega_L} v_\epsilon(y,t) dy \right)^{\frac{1}{2}} \\
\leq C + \frac{1}{2} \max_{y \in \Omega_L} v_\epsilon,
\]

which shows that \(\rho_\epsilon > C > 0\). The upper bound can be obtained the same as in [20].

Then using the same standard argument in [20], we can construct the sequences of approximate solutions \((\rho_{\epsilon n}, u_{\epsilon n})\), and

\[
0 < C_1 \leq \rho_{\epsilon n} \leq C.
\]  

Remark 3. In [20], Li et al added the term \(\epsilon \rho^\theta\) \((0 < \theta < \frac{1}{2})\), such that \(\mu_\epsilon(\rho) = \rho^\alpha + \epsilon \rho^\theta\), which offered the low bound of density. Here, the Korteweg term plays the same role. However, if we add the same term \(\epsilon \rho^\theta\), the domains of \(\alpha\) becomes
\( \alpha \in [\frac{2}{3}, \frac{11}{12}) \). The obstacle comes from the drag term. The corresponding B-D entropy estimate is

\[
\frac{d}{dt} \int_\Omega \left[ \frac{1}{2} \mu_\epsilon \rho_\epsilon |u_\epsilon + \partial_x \varphi(\rho_\epsilon)|^2 + \rho_\epsilon^\gamma + \frac{1}{2} \rho_\epsilon^2 \rho_\epsilon^{\gamma \epsilon} \right] dx + \int_\Omega \partial_x P(\rho_\epsilon) \cdot \partial_x \varphi(\rho_\epsilon) dx \\
+ \int_\Omega \rho_\epsilon |u_\epsilon|^3 dx + \int_\Omega \left( \rho_\epsilon^{\gamma \epsilon} + \rho_\epsilon^{1/2} \rho_\epsilon^{1/2} \right) \mu(\rho_\epsilon) \kappa(\rho_\epsilon) dx + \int_\Omega \rho_\epsilon u_\epsilon |u_\epsilon| \partial_x \varphi(\rho_\epsilon) \leq 0,
\]

where \( \mu(\rho_\epsilon) = \rho_\epsilon^\alpha + \epsilon \rho_\epsilon^\beta \).

While in the drag term \( \int_\Omega \rho_\epsilon u_\epsilon |u_\epsilon| \partial_x \varphi dx = \int_\Omega (\rho_\epsilon^\alpha + \epsilon \rho_\epsilon^\beta) x u_\epsilon |u_\epsilon| dx \), and

\[
| \int_\Omega (\epsilon \rho_\epsilon^\gamma) x u_\epsilon |u_\epsilon| dx | = \theta \epsilon | \int_\Omega \rho_\epsilon^{\theta - \frac{2}{3}} \rho_\epsilon^{\frac{2}{3}} u_\epsilon |u_\epsilon| dx |
\]

\[
\leq \epsilon \left( \int_\Omega \rho_\epsilon^{3(\theta - 5 - 3\beta - 3) - 11} dx \right)^\frac{1}{2} \left( \int_\Omega \rho_\epsilon^{3(\theta - 5 - 3\beta - 3) - 11} |u_\epsilon|^3 dx \right)^\frac{3}{2}
\]

\[
= \epsilon \left( \int_\Omega \rho_\epsilon^{3(\theta - 5 - 3\beta - 3) - 11} dx \right)^\frac{1}{2} \left( \int_\Omega \rho_\epsilon^{3(\theta - 5 - 3\beta - 3) - 11} dx \right)^\frac{3}{2} + C
\]

\[
\leq \epsilon \left( \int_\Omega \rho_\epsilon^{3(\theta - 5 - 3\beta - 3) - 11} dx \right)^\frac{1}{2} \left( \int_\Omega \rho_\epsilon^{3(\theta - 5 - 3\beta - 3) - 11} dx \right)^\frac{3}{2} + C
\]

\[
\leq \epsilon \left( \int_\Omega \rho_\epsilon^{3(\theta - 5 - 3\beta - 3) - 11} dx \right)^\frac{1}{2} \left( \int_\Omega \rho_\epsilon^{3(\theta - 5 - 3\beta - 3) - 11} dx \right)^\frac{3}{2} + C.
\]

The second term can be absorbed by the Korteweg term. In order to control the first term \( \int_\Omega \rho_\epsilon^{3(\theta - 5 - 3\beta - 11) - 11} dx \), we need \( 0 < 9\theta - 3\beta - 11 < \gamma \), such that it can be estimated by the “pressure” term. We set \( \frac{\beta}{4} + \frac{11}{12} < \theta \) and recall the condition \( \theta < \frac{1}{2} \), which implies that \( \alpha < \frac{11}{12} \).

**Step 3. Compactness and convergence**

Utilizing the previous analysis, the uniform upper bound and the lower bound for density \( \rho_\epsilon \), one can easily get

\( \rho_\epsilon \) is bounded in \( L^2(0, T; H^2(\Omega)) \).

By a similar discussion in Lemma 4.1, we can have

\( \rho_\epsilon \to \rho \) in \( L^2(0, T; H^1(\Omega)) \).

The other terms’ convergence is analogous with the previous stability analysis in high dimensions, the big differences is the capillarity term which is complicated. But remembering that \( \kappa(\rho) = \rho^\beta \) and strong convergence of \( \rho_\epsilon \), it is easy to get the compactness. The more details can be found in Appendix.

Combining the approximate solution and stability analysis, we can easily obtain Theorem 2.2.

6. **The proof of Theorem 2.3.** In this section, we will prove the blow-up of regular solutions to the N-S-K system when initial data have compact supports. The proof is based on the modified nonlinear function introduced in [29] and a similar method in [30].
Proof. We first prove that the support of any regular solution of (7) and (8) with compact initial data will not change in time i.e. \( \Omega(t) = \Omega(0) \), where \( \Omega(t) = \text{supp}(\rho, u) \). To do this, we let \( w = \rho^{\frac{\theta}{2}} \) and rewrite (7) as

\[
\begin{align*}
  w_t + uw_x + \frac{\theta - 1}{2} wu_x &= 0, \\
  u_t + uu_x + 2\gamma w^{2(\gamma-\theta)} wu_x &= \frac{2\theta}{\theta - 1} w^2 u_x + w^2 u_{xx} + K,
\end{align*}
\]

(25)

where

\[
K = w^{2(\beta+1)} \left[ \frac{4(3-\theta)(2-\theta+\beta)}{(\theta - 1)^3} w^{3} \right] + \frac{2(9-3\theta+2\beta)}{\theta - 1} (\theta - 1)^3 w^2 w_{xx} + \frac{2}{\theta - 1} w^{-1} w_{xxx}.
\]

Let

\[ M = \sup(|w| + |u| + |w^{-1}| + |w_x| + |u_{xx}| + |w_{xxx}| + |u_x| + |u_{xx}|) \]

Then

\[ |w_t| + |u_t| \leq CM(|w| + |u|), \]

which implies by Gronwall’s inequality

\[ |w| + |u| \leq (|w_0| + |u_0|)e^{CMt}. \]

This immediately implies that \( \Omega(t) \subset \Omega(0) \). On the other hand, it is easy to see \( \Omega(t) \supset \Omega(0) \). Therefore, \( \Omega(t) = \Omega(0) \).

Now we introduce the following functional as in [29]:

\[
H(t) = \int_\mathbb{R} x^2 \rho dx - 2(1 + t) \int_\mathbb{R} xu \rho dx + (1 + t)^2 \int_\mathbb{R} (\rho u^2 \\
+ \frac{2}{\gamma - 1} \rho^\gamma + \rho^\beta \rho_x^2) dx.
\]

(26)

By simple calculating and the Cauchy-Schwarz inequality, we deduce

\[
H'(t) = \frac{2(3-\gamma)}{\gamma - 1} (1 + t) \int_\mathbb{R} \rho^\gamma dx + (1 + t) \int_\mathbb{R} \rho^\beta u_x dx \\
- 2(1 + t)^2 \int_\mathbb{R} \rho^\beta u_x^2 dx - (1 + t)(1 + \beta) \int_\mathbb{R} \rho^\beta \rho_x^2 dx \\
\leq \frac{2(3-\gamma)}{\gamma - 1} (1 + t) \int_\mathbb{R} \rho^\gamma dx + \int_\mathbb{R} \rho^\beta dx - (1 + t)^2 \int_\mathbb{R} \rho^\beta u_x^2 dx \\
- (1 + t)(1 + \beta) \int_\mathbb{R} \rho^\beta \rho_x^2 dx.
\]

When \( \beta \geq -1 \), we obtain

\[
H'(t) \leq \frac{2(3-\gamma)}{\gamma - 1} (1 + t) \int_\mathbb{R} \rho^\gamma dx + \int_\mathbb{R} \rho^\beta dx,
\]

while when \( \beta \leq -1 \), we have

\[
H'(t) \leq \frac{2(3-\gamma)}{\gamma - 1} (1 + t) \int_\mathbb{R} \rho^\gamma dx + \int_\mathbb{R} \rho^\beta dx - (1 + t)(1 + \beta) \int_\mathbb{R} \rho^\beta \rho_x^2 dx.
\]

(27)

We tackle the second case, while the first case can be handled the same.
Case 1. When $\theta = \gamma$, we get

\[
H'(t) \leq \frac{3 - \gamma}{1 + t} H(t) + \frac{\gamma - 1}{2(1 + t)^2} H(t) + \frac{-(1 + \beta)}{1 + t} H(t) = \frac{2 - \gamma - \beta}{1 + t} H(t) + \frac{\gamma - 1}{2(1 + t)^2} H(t),
\]

which implies

\[
H(t) \leq H(0)(1 + t)^{2 - \gamma - \beta} e^{-\frac{\gamma - 1}{2(1 + t)^2}}.
\]

From (26), we obtain

\[
\int_{\Omega(t)} \rho' dx \leq \frac{\gamma - 1}{2} H(0)(1 + t)^{-\beta - \gamma} e^{-\frac{\gamma - 1}{2(1 + t)^2}}.
\]

Then using the conservation of mass, we have

\[
\int_{\Omega(t)} \rho_0 dx = \int_{\Omega(t)} \rho dx \\
\leq (\int_{\Omega(t)} \rho^\gamma dx)^{\frac{1}{\gamma}} (\int_{\Omega(t)} t^{\frac{\gamma - 1}{\gamma}} dx)^{\frac{\gamma}{\gamma - 1}} \\
\leq (\Omega(t))^{\frac{\gamma - 1}{\gamma}} \left( H(0) \right)^{\frac{1}{\gamma}} (1 + t)^{\frac{-\beta - \gamma}{\gamma - 1} e^{-\frac{\gamma - 1}{2(1 + t)^2}}},
\]

which implies that $T$ is finite.

Case 2. When $1 < \theta < \gamma$, we get the following inequality by Young’s inequality

\[
H'(t) \leq \frac{2 - \gamma - \beta}{1 + t} \int_{\mathbb{R}} \rho' dx + \frac{\theta}{\gamma} \int_{\mathbb{R}} \rho^\gamma dx + \frac{\gamma - \theta}{\gamma} \Omega(t) \\
\leq \frac{2 - \gamma - \beta}{1 + t} H(t) + \frac{\theta(\gamma - 1)}{2\gamma(1 + t)^2} H(t) + \frac{\gamma - \theta}{\gamma} \Omega(t).
\]

Solving the inequality, we have

\[
H(t) \leq (1 + t)^{2 - \gamma - \beta} e^{-\frac{\theta(\gamma - 1)}{2\gamma(1 + t)^2}} \left( H(0) + \frac{\gamma - \theta}{\gamma} \Omega(0) \int_0^t (1 + s)^{\gamma + \beta - 2} e^{\frac{\theta(\gamma - 1)}{2\gamma(1 + s)^2}} ds \right) \\
\leq (1 + t)^{2 - \gamma - \beta} e^{-\frac{\theta(\gamma - 1)}{2\gamma(1 + t)^2}} \left( H(0) + \frac{\gamma - \theta}{\gamma} \Omega(0) e^{\frac{\theta(\gamma - 1)}{2\gamma}} \int_0^t (1 + s)^{\gamma + \beta - 2} ds \right).
\]

When $\gamma \neq 1 - \beta$, we have

\[
H(t) \leq (1 + t)^{2 - \gamma - \beta} e^{-\frac{\theta(\gamma - 1)}{2\gamma(1 + t)^2}} \left( H(0) - \frac{\gamma - \theta}{\gamma (\gamma + \beta - 1)} \Omega(0) e^{\frac{\theta(\gamma - 1)}{2\gamma}} \right) \\
+ \frac{\gamma - \theta}{\gamma (\gamma + \beta - 1)} \Omega(0) e^{\frac{\theta(\gamma - 1)}{2\gamma}} (1 + t)e^{-\frac{\theta(\gamma - 1)}{2\gamma(1 + t)^2}}.
\]

When $\gamma = 1 - \beta$, we have

\[
H(t) \leq H(0) (1 + t)^{2 - \gamma - \beta} e^{-\frac{\theta(\gamma - 1)}{2\gamma(1 + t)^2}} \\
+ \frac{\gamma - \theta}{\gamma} \Omega(0) e^{\frac{\theta(\gamma - 1)}{2\gamma}} (1 + t)^{2 - \gamma - \beta} e^{-\frac{\theta(\gamma - 1)}{2\gamma(1 + t)^2}} \ln(1 + t).
\]

By similar estimates, we can deduce that $T$ is finite.
Appendix 1. In this section, we prove the local existence of solutions to approximate viscous N-S-K equations in the following. For simplicity, we omit the notation $\epsilon$

\[
\begin{align*}
\left\{ \begin{array}{l}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2 + P(\rho))_x + \rho |u|u = (\rho^\alpha u_x)_x + K[\rho]_x.
\end{array} \right.
\]

As pointed out in [11], we can follow [10] and use the Faedo-Galerkin approximation to complete the construction. Let $T > 0$ and $\{\epsilon_k\}$ be an orthonormal basis of $C^\infty(\Omega)$. Introduce the finite dimensional space $X_n = \text{span}\{e_1, \ldots, e_n\}$, $n \in \mathbb{N}$. The initial data $(\rho_0, u_0) \in C^\infty(\Omega)$ and $\rho_0 \geq \delta > 0$. In order to linearize the N-S-K equations, we set the given velocity $v \in C^0([0, T]; X_n)$. As a consequence, $v$ can be bounded in $C^0([0, T]; C^k(\Omega))$ for any $k \in \mathbb{N}$ and there exists a constant $C$ such that

\[
||v||_{C^0([0, T]; C^k(\Omega))} \leq C||v||_{C^0([0, T]; L^2)}.
\]

The approximate system is defined as follows

\[
\rho_t + (\rho v)_x = 0, \quad \rho|t=0 = \rho_0.
\]

It is well known that the linear transport equation has a unique strong solution as the following proposition.

**Proposition 1.** Given the velocity $v$ with the property

\[
v \in C^0([0, T]; H^m) \cap L^2([0, T]; H^{m+1}) \quad (m \geq 2),
\]

and $0 < \delta \leq \rho_0 \in H^m$. Then, there exists a unique solution $\rho$ such that $\rho \in C^0([0, T]; H^m)$, $\partial_t \rho \in C^0([0, T]; H^{m-1})$.

**Proof.** The proof can be seen in [4].

From the proposition, we can easily get that there exists a classical solution $\rho \in C^0([0, T]; C^3)$ to (28) and

\[
\inf_{x \in \Omega} \rho_0(x) \exp(-\int_0^t |\partial_x v| ds) \leq \rho(x, t) \leq \sup_{x \in \Omega} \rho_0(x) \exp(\int_0^t |\partial_x v| ds).
\]

Since we assume that $\rho_0 \geq \delta > 0$, the density $\rho(x, t)$ has strictly lower bound and upper bound, i.e., there exist constant $m$ and $M$ such that

\[
0 < m \leq \rho(x, t) \leq M, \quad (x, t) \in \Omega \times [0, T].
\]

We introduce the operator $S : C^0([0, T]; X_n) \to C^0([0, T]; C^3(\Omega))$ by $S(v) = \rho$. Since the equation for $\rho$ is linear, $S$ is Lipschitz continuous in the following sense:

\[
||S(v_1) - S(v_2)||_{C^0([0, T]; C^3(\Omega))} \leq C||v_1 - v_2||_{C^0([0, T]; L^2)}.
\]

Next, we wish to solve the momentum equation on the space $X_n$. For given $\rho = S(v)$, we are seeking for a function $u_n \in C^0([0, T]; X_n)$ satisfy

\[
-\int_{\Omega} \rho_0 u_0 \cdot \phi(\cdot, 0) dx = \int_0^T \int_{\Omega} (\rho u_n \cdot \phi_t + (\rho v u_n + P(\rho)) \phi_x - \rho |v| \phi - K[\rho] \cdot \phi_x - \rho^\alpha u_x \phi_x) dx dt
\]

for all $\phi \in C^1([0, T]; X_n)$ such that $\phi(\cdot, T) = 0$. To solve (29), we introduce the following family operators, given a function $\rho \in L^1(\Omega)$ with $\rho \geq \rho_0 > 0$:

\[
M[\rho] : X_n \to X_n^*, \quad \langle M[\rho] u, w \rangle = \int_{\Omega} \rho u \cdot w dx, \quad u, w \in X_n.
\]
There operators are symmetric and positive definite with the smallest eigenvalue
\[
\inf_{||w||_{L^2}} \langle M[\rho]w, w \rangle = \inf_{||w||_{L^2}} \int_{\Omega} \rho |w|^2 dx \geq \inf_{x \in \Omega} \rho \geq \rho.
\]
Hence, since \( X_n \) is finite dimensional, the operators are invertible with
\[
||M^{-1}[\rho]||_{L(X_n^*, X_n)} \leq \rho^{-1},
\]
where \( L(X_n^*, X_n) \) is the set of bounded linear mapping from \( X_n^* \) to \( X_n \). Moreover, \( M^{-1} \) is Lipschitz continuous in the sense
\[
||M^{-1}[\rho_1] - M^{-1}[\rho_2]||_{L(X_n^*, X_n)} \leq C||\rho_1 - \rho_2||_{L^1(\Omega)}
\]
for all \( \rho_1, \rho_2 \in L^1(\Omega) \) such that \( \rho_1, \rho_2 \geq \rho \).

Now, we rewrite the integral equation (29) as an ordinary differential equation on the finite dimensional space \( X_n \):
\[
\frac{d}{dt} (M[\rho(t)]u_n(t)) = N[v, u_n(t)], \quad t > 0, \quad M[\rho_0]u_n(0) = M[\rho_0]u_0,
\]
where \( \rho = S(v) \) and
\[
\langle N[v, u_n], \phi \rangle = \int_{\Omega} \left( - (\rho v u_n + P(\rho))_x - \rho |v|^2 - K[\rho]_x - (\rho^2 u_n)_x \right) \phi dx
\]
for \( \phi \in X_n \). Recall that \( \rho = S(v) \in C^0([0, T]; C^4(\Omega)) \) is bounded from below, so the above integral is well defined. The operator \( N[v, \cdot] \), defined for every \( t \in [0, T] \) as operator from \( X_n^* \) to \( X_n \) is continuous in time. Standard theory for system of ODEs provides the existence of a unique classical solutions to (30), i.e., for given \( v \), there exists a unique solution \( u_n \in C^1([0, T]; X_n) \) to (29).

Integrating (30) over \((0, t)\) yields the following nonlinear equations:
\[
\left(F(u_n) (t)\right)(t) = M^{-1}[(S(u_n)(t)) (M[\rho_0]u_0 + \int_0^t N[u_n, u_n(s)]ds)] \quad \text{in} \quad X_n.
\]
Recalling the estimates for \( S \) and \( M^{-1} \), this equation can be solved by the Banach fixed point theorem on a short time \([0, T_s]\) in the space \( C^0([0, T_s]; X_n) \). In fact, we have even \( u_n \in C^1([0, T_s]; X_n) \). Thus, there exists a unique local-in-time solutions \((\rho_n, u_n)\).

**Appendix 2.** In this appendix, we briefly give proof of the compactness in Theorem 2.2.

**Proposition 2.** There are sequence \( \epsilon_j \) and function \( \rho(x, t) \), such that as \( \epsilon_j \to 0 \) (i.e. \( j \to \infty \)),
\[
\rho_{\epsilon_j} \to \rho \quad \text{in} \quad L^2(0, T; H^1(\Omega)) \cap C((0, T) \times \Omega),
\]
\[
(\rho_{\epsilon_j})_{xx} \to \rho_{xx} \quad \text{in} \quad L^2(0, T; L^2(\Omega)).
\]

**Proof.** It follows from (24) that \( \rho_{\epsilon_j} \) is bounded in \( L^\infty((0, T) \times \Omega) \), and therefore
\[
\rho_{\epsilon_j} u_{\epsilon_j} = \rho_{\epsilon_j}^2 \cdot \frac{1}{2} \cdot \rho_{\epsilon_j}^2 u_{\epsilon_j} \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)).
\]
This together with mass equation in (5), shows that
\[
\partial_t \rho_{\epsilon_j} \quad \text{is bounded in} \quad L^\infty(0, T; W^{-1,2}(\Omega)).
\]
Utilizing the lower bound of density (24) and the entropy estimate (23), we deduce that

\[(\rho_{\epsilon})_{x} \text{ is bounded in } L^{\infty}(0, T; L^{2}(\Omega)),\]
\[(\rho_{\epsilon})_{xx} \text{ is bounded in } L^{2}(0, T; L^{2}(\Omega)).\]

In one dimension space, by the Sobolev embedding \(H^{1}(\Omega) \hookrightarrow L^{\infty}(\Omega)\), we get

\[\rho_{\epsilon} \text{ is bounded in } L^{\infty}(0, T; H^{1}(\Omega)) \cap L^{2}(0, T; H^{2}(\Omega)).\]

Then, the Lions-Aubin’s lemma yields the result. \(\square\)

**Proposition 3.** Let \(\gamma > 1\), the pressure \(P(\rho_{\epsilon})\) converges to \(P(\rho)\) strongly in \(L^{1}((0, T) \times \Omega)\).

**Proof.** The Proposition follows similarly as the previous Lemma 4.2. \(\square\)

For simplicity, we denote \((\rho_{\epsilon}, u_{\epsilon})\) by \((\rho, u)\).

**Proposition 4.** (1) Up to a subsequence, the momentum \(m_{\epsilon} = \rho_{\epsilon}u_{\epsilon}\) converges strongly to some \(m\) in \(L^{2}(0, T; L^{2}(\Omega))\).

(2) The sequence \(\sqrt{\rho_{\epsilon}}u_{\epsilon}\) converges strongly in \(L^{2}(0, T; L^{2}(\Omega))\) to \(m/\sqrt{\rho}\). In particular, we have \(m = 0\) a.e on the set \(\{\rho = 0\}\), and there exists a function \(u(x, t)\) such that \(m = \rho u\).

**Proof.** This proposition can be proved in a similar way as [23]. For completeness, we sketch the proof here.

First, we recall that \(\rho_{\epsilon}u_{\epsilon}\) is bounded in \(L^{\infty}(0, T; L^{2}(\Omega))\). Furthermore, we find that

\[
\partial_{t}(\rho_{\epsilon}u_{\epsilon}) = \rho_{\epsilon}\partial_{x}u_{\epsilon} + u_{\epsilon}\partial_{x}\rho_{\epsilon} = \frac{1}{2}\partial_{x}(\rho_{\epsilon}u_{\epsilon}) + \frac{1}{\alpha}\rho_{\epsilon}^{\alpha-1}\cdot \rho_{\epsilon}^{\frac{1}{2}}\partial_{x}(\rho_{\epsilon}^{\frac{1}{2}}).
\]

Utilizing (22) – (24), it is easy to see

\[\rho_{\epsilon}u_{\epsilon}\text{ is bounded in } L^{2}(0, T; W^{1,1}(\Omega)).\]

Next, we rewrite the momentum equation in (5) as follows:

\[\partial_{t}(\rho_{\epsilon}u_{\epsilon}) = -(\rho_{\epsilon}u_{\epsilon}^{2} + \mathcal{P}(\rho_{\epsilon}))_{x} - \rho_{\epsilon}u_{\epsilon}|u_{\epsilon}| - (\rho_{\epsilon}^{\alpha}u_{\epsilon}u_{\epsilon, x})_{x} - K[\rho_{\epsilon}]_{x}.
\]

The convection term, pressure term and drag term can be handled as before, the difference is the viscosity term and capillarity term. We will show that the term \(\partial_{t}(\rho_{\epsilon}^{\alpha}\partial_{x}u_{\epsilon})\) is uniformly bounded in \(L^{\infty}(0, T; W^{-2,2}(\Omega))\). Note that

\[
\partial_{t}(\rho_{\epsilon}^{\alpha}\partial_{x}u_{\epsilon}) = (\rho_{\epsilon}^{\alpha}u_{\epsilon})_{x} - (\rho_{\epsilon}^{\alpha})_{x}u_{\epsilon} = (\rho_{\epsilon}^{\alpha-\frac{1}{2}}\cdot \rho_{\epsilon}^{\frac{1}{2}}u_{\epsilon})_{x} - \frac{\alpha}{\alpha - \frac{1}{2}}(\rho_{\epsilon}^{\alpha-\frac{1}{2}})_{x}\rho_{\epsilon}^{\frac{1}{2}}u_{\epsilon}.
\]

As the lower bound of \(\rho_{\epsilon}, \rho_{\epsilon}^{\frac{1}{2}}u_{\epsilon} \in L^{\infty}(0, T; L^{2}(\Omega))\) and \((\rho_{\epsilon}^{\alpha-\frac{1}{2}})_{x} \in L^{\infty}(0, T; L^{2}(\Omega))\), we get \(\rho_{\epsilon}^{\alpha}\partial_{x}u_{\epsilon} \in L^{\infty}(0, T; W^{-1,2}(\Omega))\). As to the capillarity term, remembering that \(\kappa(\rho) = \rho^{\beta}\), we calculate \(K[\rho_{\epsilon}] = \rho^{\beta+1}_{\epsilon} \rho_{\epsilon, x} + \frac{\alpha}{2}(\beta-1)\rho_{\epsilon}^{\beta} \rho_{\epsilon, x}^{2}\). From Proposition 7.1, we can deduce that

\[\rho_{\epsilon, xx} \in L^{2}(0, T; L^{2}(\Omega)), \quad \rho_{\epsilon, x} \in L^{\infty}(0, T; L^{2}(\Omega)) \cap L^{2}(0, T; H^{1}(\Omega)).\]
Recalling the Sobolev embedding: \( H^1(\Omega) \hookrightarrow L^\infty(\Omega) \), so we have the following estimate
\[
\int_0^T \int_{\Omega} \varphi K[\rho_n]_x dx dt = -\int_0^T \int_{\Omega} \left[ \rho_n^{\beta+1} \rho_n xx + \rho_n^\beta \rho_n^2 xx \right] \varphi x dx dt \\
\leq C \| \varphi \|_{L^2(0,T;W^{2,2}(\Omega))} \left( \| \rho_n xx \|_{L^2(0,T;L^2(\Omega))} + \| \rho_n \|_{L^\infty(0,T;L^2(\Omega))} \cdot \| \rho_n xx \|_{L^2(0,T;L^\infty(\Omega))} \right) \\
\leq C \| \varphi \|_{L^2(0,T;W^{2,2}(\Omega))}
\]
for \( \varphi \in L^2(0,T;W^{2,2}(\Omega)) \). Therefore, we get \( K[\rho_n]_x \) is uniformly bounded in \( L^2(0,T;W^{-2,2}(\Omega)) \). Combining all the analysis, we conclude \( \partial_t (\rho_n u_n) \) is bounded in \( L^2(0,T;W^{-2,2}(\Omega)) \).

Using aging the Aubin’s lemma, we can acquire the result. The rest proof is akin to Lemma 4.6 in [23], we omit it.

**Proposition 5.** As \( n \to \infty \), we have
\[
\rho_n^\alpha u_n x \to \rho^\alpha u_x \text{ in } \mathcal{D}'((0,T) \times \Omega).
\]

**Proof.** Using the mass equation in (5), we can derive
\[
\rho_n u_n x = -\partial_t \rho_n - \rho_n xx u_n.
\]

Multiplying this equation by \( \alpha \rho_n^{\alpha - 1} \phi \) with \( \phi \) being a test function and integrating, we arrive at
\[
\int_0^T \int_{\Omega} \rho_n^\alpha \partial_x u_n \phi dx dt = -\int_0^T \int_{\Omega} (\rho_n^\alpha) \phi dx dt - \int_0^T \int_{\Omega} (\rho_n^\alpha)_x u_n \phi dx dt \\
= \int_0^T \int_{\Omega} (\rho_n(x,0))^{\alpha} \phi(x,0) dx + \int_0^T \int_{\Omega} \rho_n^{\alpha} \phi dx dt \\
- \int_0^T \int_{\Omega} \rho_n^{\alpha - \frac{1}{2}} (\rho_n^{\alpha - \frac{1}{2}}) x \rho_n^\frac{\alpha}{2} u_n \phi dx dt.
\]

By virtue of the previous propositions and the fact
\[
(\rho_n^{\alpha - \frac{1}{2}}) \to (\rho^{\alpha - \frac{1}{2}}) \text{ weakly in } L^2((0,T) \times \Omega),
\]
that is a consequence of the above entropy estimate (23), we obtain
\[
\int_0^T \int_{\Omega} \phi \rho_n^{\alpha} \partial_x u_n dx dt \to \int_0^T \int_{\Omega} \phi \rho^{\alpha} \partial_x u dx dt.
\]

Now we focus on drag term and the most difficult term-capillarity term.

**Proposition 6.** As \( n \to \infty \), we have
\[
\rho_n |u_n|u_n \to \rho |u|u \text{ in } \mathcal{D}'((0,T) \times \Omega).
\]

**Proof.** Using the previous propositions, we can easily get the result.

**Proposition 7.** As \( n \to \infty \), we have
\[
K[\rho_n]_x \to K[\rho]_x \text{ in } \mathcal{D}'((0,T) \times \Omega).
\]
Proof. As known,

\[ K[\rho_n] = \rho_n^{\beta+1}(\rho_n)_{xx} + \frac{1}{2}(\beta - 1)\rho_n^2(\rho_n)_x. \]

Using the Proposition 7.1, we recall that

\[ \rho_n \to \rho \text{ in } L^2(0, T; H^{-1}(\Omega)) \cap C((0, T) \times \Omega), \]
\[ (\rho_n)_x \to \rho_x \text{ in } L^2(0, T; L^2(\Omega)), \]
\[ (\rho_n)_{xx} \rightharpoonup \rho_{xx} \text{ in } L^2(0, T; L^2(\Omega)). \]

So we can easily get the result.

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