Free Field Realizations of Affine Current Superalgebras, Screening Currents and Primary Fields

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Abstract

In this paper free field realizations of affine current superalgebras are considered. Based on quantizing differential operator realizations of the corresponding basic Lie superalgebras, general and simple expressions for both the bosonic and the fermionic currents are provided. Screening currents of the first kind are also presented. Finally, explicit free field realizations of primary fields with general, possibly non-integer, weights are worked out. A formalism is used where the (generally infinite) multiplet is replaced by a generating function primary operator. The results allow setting up integral representations for correlators of primary fields corresponding to integrable representations. The results are generalizations to superalgebras of a recent work on free field realizations of affine current algebras by Petersen, Yu and the present author.

PACS: 11.25.Hf

Keywords: Conformal field theory; Lie superalgebra; Affine current superalgebra; Free field realizations

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1 Introduction

Since the work by Wakimoto \[1\] on free field realizations of affine $SL(2)$ current algebra much effort has been made in obtaining similar constructions in the general case, a problem in principle solved by Feigin and Frenkel \[2\]. Recently two independent methods have led to general and explicit solutions \[3, 4, 5\]. The method used by Petersen, Yu and the present author \[4, 5\] gives particularly simple and compact free field realizations and is amenable of generalizing to affine current superalgebras. Much less coherent results have been established so far in the case of superalgebras. Free field realizations are only known in certain particular cases (see e.g. \[6, 7, 8, 9\]). In this paper we present general and explicit free field realizations of affine current superalgebras, generalizing the results in \[4, 5\]. The affine current superalgebras we consider have basic Lie superalgebras as classical counterparts.

Free field realizations enables one in principle to build integral representations for correlators in conformal field theory \[10, 11, 12, 13\]. In a recent series of papers Petersen, Yu and the present author have carried out such a study for conformal field theory based on affine $SL(2)$ current algebra \[14, 15\]. It turns out that screening operators of both the first and the second kinds are crucial for being able to treat the general case of degenerate representations \[16\] and in particular admissible representations \[17\]. In that connection it is also necessary to be able to handle fractional powers of free fields. Well defined rules for that have been established also in \[14, 15\]. A particular interest in these techniques is due to their close relationship with 2D quantum gravity and string theory \[18, 19\].

In order to generalize our work on affine $SL(2)$ current algebra to affine current superalgebras (and eventually to be able to treat superstring theory along the lines of \[18, 19\]), one needs not only free field realizations of the affine currents but also of screening currents and primary fields. In this paper we present such realizations of screening currents of the first kind, only. However, that is expected to be sufficient for treating integrable representations. Nevertheless, the free field realizations we provide of primary fields are valid for general representations. For that purpose we use techniques based on “super-triangular” coordinates on representation spaces, which in turn makes our results very compact.

The reduction of the results presented in this paper on Lie superalgebras and affine current superalgebras to standard bosonic Lie algebras and affine current algebras, is essentially by “fermionic truncation”. By that we mean disregarding all terms involving at least one of the odd parameters: odd roots $\hat{\alpha}$, odd “super-triangular” coordinates $\theta^\alpha$ or fermionic ghost fields $(b_{\hat{\alpha}}, c_{\hat{\alpha}})$, see below. Thus, the results presented in this paper are direct generalizations of similar ones for purely bosonic algebras in \[4, 5\].

The paper is organized as follows. Section 2 serves to fix notation. Some basic Lie superalgebra properties are reviewed and systems of free bosonic and fermionic ghost fields and of free bosonic scalar fields are discussed. We define our “super-triangular” coordinates. A key object is the introduction of a matrix representation of the raising part of the Lie superalgebra depending on those coordinates in the adjoint representation, since the main new results in subsequent sections are expressed in terms of that matrix.

In Section 3 we work out explicitly certain Gauss decompositions leading to differential operator realizations of the Lie superalgebras. This is the first main new result in this
paper. We then discuss some polynomials later to become building blocks in construction of screening currents in Section 5.

In Section 4 the differential operator realization of a Lie superalgebra derived in Section 3, is quantized to a (generalized) Wakimoto free field realization of the corresponding affine current superalgebra. The non-trivial part consists in taking care of multiple contractions by adding anomalous terms to the lowering operators. The full free field realization is the second main new result in this paper.

In Section 5 free field realizations of screening currents of the first kind are provided. This is the third main new result in this paper. The set of screening operators depends on the choice of simple roots. In particular, the numbers of bosonic and fermionic screening currents are equal to the numbers of even and odd simple roots, respectively.

In Section 6 we give a thorough discussion of primary fields using the formalism based on “super-triangular” parameters. Simple and general free field realizations of primary fields with arbitrary, possibly non-integer, weights are derived. This is the fourth and final main new result obtained in this paper.

In Section 7 we compare a subset of the results obtained in this paper with results known in the literature, by working out explicit examples.

Section 8 contains concluding remarks, while classical and quantum polynomial identities following from the differential operator realization and the free field realization, respectively, are listed in Appendix A.

2 Notation

Let \( g^0 \) and \( g^1 \) denote the even and odd parts of the basic Lie superalgebra \( g \) of rank \( r \), see \citen{20} and references therein. \( \Delta = \Delta^0 \cup \Delta^1 \) is the set of roots of \( g \) where \( \Delta^0 (\Delta^1) \) is the set of even (odd) roots. A generic positive even (odd) root is written \( \alpha \in \Delta^0_+ (\dot{\alpha} \in \Delta^1_+) \), and for such a root we write \( \alpha > 0 (\dot{\alpha} > 0) \). An arbitrary positive root is written \( \bar{\alpha} \in \Delta^+ \). A choice of a set of simple roots is written \( \{\beta_i\}_{i=1,...,r} \) in which there are \( r_0 \) even ones and \( r_1 = r - r_0 \) odd ones.

Using the triangular decomposition

\[
g = g^- \oplus h \oplus g^+
= (g^+ \oplus g^-) \oplus h \oplus (g^0_+ \oplus g^0_-) \tag{1}
\]

the raising and lowering even operators are denoted \( E_\alpha \in g^0_+ \) and \( F_\alpha \in g^0_- \) respectively with \( \alpha \in \Delta^0_+ \), and \( H_i \in h \) are the Cartan operators. For the odd generators we use lower case letters: \( e_{\dot{\alpha}} \in g^1_+ \) and \( f_{\dot{\alpha}} \in g^1_- \), where \( \dot{\alpha} \in \Delta^1_+ \). We let \( J_a (J_{\dot{a}}) \) denote an even (odd) generator. \( J_A \) denotes an arbitrary Lie superalgebra generator. The (anti-)commutator algebra may be written

\[
[J_A, J_B] = f_{A,B}^C J_C \tag{2}
\]

where \([\cdot, \cdot]\) is an anti-commutator if both arguments are fermionic, and otherwise a commutator. This is equivalent to

\[
[J_A, J_B] = J_A J_B - (-1)^{\deg(J_A)\deg(J_B)} J_B J_A \tag{3}
\]
where the degree of a bosonic generator is \( \text{deg}(J_a) = 0 \), while of a fermionic generator it is \( \text{deg}(j_a) = 1 \). Sometimes we will indicate the degree by \( p(A) = \text{deg}(J_A) \). Some of the structure coefficients satisfy

\[
\begin{align*}
 f_{\tilde{a}_i, -\tilde{a}_j}^k &= \delta_{ij}\delta_j^k \\
f_{i, \pm\tilde{a}}^A &= \pm\tilde{\alpha}(H_i)\delta_i^A \\
f_{\alpha, \beta}^a &= f_{\alpha, \beta}^\alpha{}^\beta, \quad \text{for } \alpha + \beta \in \Delta^0 \\
f_{\tilde{\alpha}, \beta}^a &= f_{\tilde{\alpha}, \beta}^{\tilde{\alpha}{}^\beta}, \quad \text{for } \tilde{\alpha} + \beta \in \Delta^0 \\
f_{\alpha, \beta}^\tilde{\alpha} &= f_{\alpha, \beta}^{\alpha{}^\beta}, \quad \text{for } \alpha + \tilde{\beta} \in \Delta^1
\end{align*}
\]

while the Jacobi identity reads

\[
[J_A, [J_B, J_C]] = [[J_A, J_B], J_C] + (-1)^{\text{deg}(J_A)\text{deg}(J_B)}[J_B, [J_A, J_C]]
\]

The Chevalley generators comprise the sets of raising and lowering operators corresponding to simple roots and of Cartan generators, a total of \( 3r \) generators. The Cartan-Killing form \( \kappa \) satisfies

\[
\begin{align*}
\kappa_{AB} &= (J_A, J_B) = (-1)^{\text{deg}(J_A)\text{deg}(J_B)}\kappa_{BA} \\
\kappa_{ij} &= G_{ij}, \quad \kappa_{\tilde{\alpha}, \tilde{\beta}} = 0 \text{ unless } \tilde{\alpha} = \tilde{\beta} \\
2h^\nu \kappa_{AB} &= \text{str}(\text{ad}_{J_A}\text{ad}_{J_B}) = (-1)^{\text{deg}(J_B)}\left(f_{A,d}^D f_{B,D}^d - f_{A,d}^D f_{B,D}^d\right)
\end{align*}
\]

where the metric \( G_{ij} \) is related to the Cartan matrix \( A_{ij} = \tilde{\alpha}_j(H_i) \) as \( G_{ij} = A_{ij} \kappa_{\tilde{\alpha}_j, -\tilde{\alpha}_i} \), \( h^\nu \) is the dual Coxeter number of the Lie superalgebra. We shall understand “properly” repeated indices as in (3) to be summed over. In the case of properly repeated root indices \( (\alpha, \beta, \gamma, ...) \) the summation is over the positive (even and/or odd) roots. In the explicit examples discussed in Section 7 we follow the normalization convention of Kac [20], saying that if \( A_{ii} \neq 0 \) then \( A_{ii} = 2 \), and in the \( i \)’th row where \( A_{ii} = 0 \) the first non-vanishing element of the form \( A_{i,i+j}, j \geq 1 \) is 1. It is always possible to rescale the Lie superalgebra generators in order to meet these normalization conditions. The bilinear form on the root space is defined by \( \tilde{\alpha}_i \cdot \tilde{\alpha}_j = (H_{\tilde{\alpha}_i}, H_{\tilde{\alpha}_j}) \), where \( H_i = \kappa_{\tilde{\alpha}_i,-\tilde{\alpha}_i} H_{\tilde{\alpha}_i} \). The Weyl vector

\[
\rho = \rho^0 - \rho^1 = \frac{1}{2} \sum_{\alpha > 0} \alpha, \quad \rho^1 = \frac{1}{2} \sum_{\tilde{\alpha} > 0} \tilde{\alpha}
\]

satisfies \( \rho \cdot \tilde{\alpha}_i = \tilde{\alpha}_i^2/2 \), while the labels \( \Lambda_k \) and \( \lambda_k \) of the weight \( \Lambda \) are defined by

\[
H_k|\Lambda\rangle = \Lambda(H_k)|\Lambda\rangle = \Lambda_k|\Lambda\rangle, \quad \Lambda = \lambda_k \Lambda^k \quad \text{and} \quad \Lambda_i = \lambda_k \Lambda^k(H_i)
\]

Here the particular (fundamental) set of linearly independent weights \( \{\Lambda^k\} \) has the property that the associated highest weight representations \( M_{\Lambda^k} \) are finite dimensional.

Elements in \( g_+^0 \) and \( g_+^1 \) (or \( g_- \)) or vectors in representation spaces (see below) are parametrized using “super-triangular coordinates” denoted by \( x^\alpha \) and (Grassmann odd
variables) \( \theta^\alpha \) respectively, one for each positive even or odd root. Thus we introduce the Lie algebra elements

\[
g_+(x, \theta) = x^\alpha E_\alpha + \theta^\beta \epsilon_\beta \in g_+ \quad , \quad g_-(x, \theta) = x^{\alpha} F_\alpha + \theta^{\beta} \eta_\beta \in g_-
\]

and the corresponding group elements \( G_+(x, \theta) \) and \( G_-(x, \theta) \):

\[
G_+(x, \theta) = e^{g_+(x, \theta)} \quad , \quad G_-(x, \theta) = e^{g_-(x, \theta)}
\]

The matrix representation \( C(x, \theta) \) of \( g_+(x, \theta) \) in the (pseudo-)adjoint representation is introduced as

\[
C_B^A(x, \theta) = -x^\beta f_{\beta,A} - \theta^\beta \eta_{\beta,A}
\]

This does not correspond precisely to the adjoint representation where \( (\text{ad} J_c)_A^B = f_{A,C} B \) such that

\[
(\text{ad} g_+(x, \theta))_A^B = -x^\beta f_{\beta,A} - (-1)^{\rho(A)} \theta^\beta \eta_{\beta,A}
\]

However, as it will be demonstrated in subsequent sections, it is a very convenient matrix representation essentially due to

\[
(\text{ad} g_+(x, \theta))^n (J_A) = [(-C(x, \theta))^n]_A^B J_B
\]

The following notation is used for the (block) matrix elements

\[
C = \begin{pmatrix} C_{++}^+ & 0 & 0 \\
0 & C_{++}^+ & 0 \\
C_{++}^- & C_{++}^- & C_{++}^-
\end{pmatrix}
\]

\( C_{++} \) etc are matrices themselves. In \( C_{++} \) both row and column indices are positive (even or odd) roots, in \( C_{++} \) the row index is a negative (even or odd) root and the column index is a Cartan algebra index, etc. One easily sees that (leaving out the arguments \( x \) and \( \theta \) for simplicity)

\[
(C^n)_+ = (C_+^n), \quad (C^n)_0 = C_0^+(C_+^n)^{-1}, \quad (C^n)_- = (C_-)^n, \quad (C^n)_+^+ = \sum_{l=0}^{n-2} (C_-)^l (C_+^n)^{n-l-1} + \sum_{l=0}^{n-2} (C_-)^l (C_-^0 C_0^+(C_+^n)^{n-l-2})
\]

The block elements may be specified further as in

\[
C_{++} \sim \begin{pmatrix} C_{\alpha}^{\beta} & C_{\alpha}^{\beta} \\
C_{\alpha}^{\beta} & C_{\alpha}^{\beta}
\end{pmatrix} \begin{pmatrix} -x^\gamma f_{\gamma,\alpha}^{\beta} & -\theta^\gamma f_{\gamma,\alpha}^{\beta} \\
-\theta^\gamma f_{\gamma,\alpha}^{\beta} & -x^\gamma f_{\gamma,\alpha}^{\beta}
\end{pmatrix}
\]

We shall use repeatedly that \( C_{\alpha}^{\beta}(x) \) vanishes unless \( \alpha < \beta \), and similarly for the remaining 3 block elements of \( C_{++} \). This corresponds to each block \( (C_{\alpha}^{\beta}, C_{\alpha}^{\beta}, C_{\alpha}^{\beta} \) and \( C_{\alpha}^{\beta} \))
being upper triangular with zeros in the diagonals. Likewise, the block elements of $C_{-}$ are lower triangular. It will turn out that we shall be able to provide remarkably simple universal analytic expressions for most of the objects we consider, using the matrix $C(x, \theta)$.

For the associated affine Lie superalgebra, the operator product expansion, OPE, of the associated currents is

$$J_{A}(z)J_{B}(w) = \frac{\kappa_{AB}k}{(z-w)^{2}} + \frac{f_{A,B}^{C}J_{C}(w)}{z-w}$$

(17)

where regular terms have been omitted. $k$ is the central extension or level of the affine current superalgebra. We use the same notation $J, E, F, H, j, e, f$ for the currents as for the algebra generators. Hopefully, it will not lead to misunderstandings. The associated Sugawara construction is

$$T(z) = \frac{1}{2(k+h^{\vee})}\kappa^{AB} : J_{A}(z)J_{B}(z) :$$

(18)

and has central charge

$$c = \frac{k \text{sdim}(g)}{k+h^{\vee}}$$

(19)

In the mode expansion

$$J_{A}(z) = \sum_{n=-\infty}^{\infty} \hat{J}_{A,n}z^{-n-1}$$

(20)

we use the identification

$$\hat{J}_{A,0} \equiv J_{A} \in g$$

(21)

The standard free field construction (see [1, 2, 3, 4, 5] for affine current algebras and [6, 8] for affine current superalgebras) consists in introducing for every positive even root $\alpha > 0$, a pair of free bosonic ghost fields $(\beta_{\alpha}, \gamma_{\alpha})$ of conformal weights $(1,0)$ satisfying the OPE

$$\beta_{\alpha}(z)\gamma_{\beta}(w) = \delta_{\alpha\beta}z-w$$

(22)

The corresponding energy-momentum tensor is

$$T_{\beta\gamma} = : \partial \gamma^{\alpha} \beta_{\alpha} :$$

(23)

with central charge

$$c_{\beta\gamma} = 2|\Delta_{\alpha}^{0}| = \text{dim}(g^{0}) - r$$

(24)

For every positive odd root $\dot{\alpha} > 0$ one introduces a pair of free fermionic ghost fields $(b_{\dot{\alpha}}, c^{\dot{\alpha}})$ of conformal weights $(1,0)$ satisfying the OPE

$$b_{\dot{\alpha}}(z)c^{\dot{\beta}}(w) = \delta_{\dot{\alpha}\dot{\beta}}z-w$$

(25)

The corresponding energy-momentum tensor is

$$T_{bc} = : \partial c^{\dot{\alpha}} b_{\dot{\alpha}} :$$

(26)
with central charge
\[ c_{bc} = -2|\Delta^-| = -\dim(g^1) \] (27)

For every Cartan index \( i = 1, ..., r \) one introduces a free scalar boson \( \varphi_i \) with contraction
\[ \varphi_i(z)\varphi_j(w) = G_{ij} \ln(z - w) \] (28)

The energy-momentum tensor
\[ T_\varphi = \frac{1}{2} : \partial \varphi \cdot \partial \varphi : - \frac{1}{\sqrt{t}} \rho \cdot \partial^2 \varphi \] (29)

has central charge
\[ c_\varphi = r - \frac{h^\vee \sdim(g)}{k + h^\vee} \] (30)

where the super-dimension \( \sdim(g) \) of the Lie superalgebra \( g \) is defined as the difference \( \dim(g^0) - \dim(g^1) \). In obtaining (30) we have used Freudenthal-de Vries (super-)strange formula [21]
\[ \rho^2 = \frac{h^\vee}{12} \sdim(g) \] (31)

The total free field realization of the Sugawara energy-momentum tensor is \( T = T_{\beta\gamma} + T_{bc} + T_\varphi \) and has indeed central charge (19). In subsequent sections, the combination \( k + h^\vee \) will be abbreviated by \( t \):
\[ t = k + h^\vee \] (32)

One of the new results in this paper will be explicit free field realizations of the currents in a general affine Lie superalgebra in Section 4, based on matrix representations similar to \( C \) in (11).

The vertex operator
\[ V_\Lambda(z) = : e^{\sqrt{t} \Lambda \cdot \varphi(z)} : \] (33)

has conformal weight
\[ \Delta(V_\Lambda) = \frac{1}{2t} (\Lambda, \Lambda + 2\rho) \] (34)

It is also affine primary corresponding to highest weight \( \Lambda \). One of the main new results in this paper will be an explicit general construction of the full multiplet of primary fields, parametrized by the \( x^\alpha \) and \( \theta^{\dot{\alpha}} \) coordinates in Section 6.

### 3 Differential Operator Realization

In this section we discuss differential operator realizations of the Lie superalgebra \( g \) on the polynomial ring \( \mathbb{C}[x^\alpha, \theta^{\dot{\alpha}}] \). The techniques for obtaining such realizations in the case of standard bosonic Lie algebras have been known for some time (see e.g. [22]) and completed in [4, 5]. The procedure and results of [4, 5] is generalized to cover Lie superalgebras in the following. In [8] Ito used similar techniques to obtain differential operator realizations of the *Chevalley generators*, see also Section 7.
The lowest weight vector \( \langle \Lambda | \) in the (dual) representation space is introduced as
\[
\langle \Lambda | F_\alpha = \langle \Lambda | f_\dot{\alpha} = 0 \quad , \quad \langle \Lambda | H_i = \Lambda_i \langle \Lambda |
\]
(35)

An arbitrary vector in this representation space is parametrized as
\[
\langle \Lambda, x, \theta | = \langle \Lambda | G_+(x, \theta)
\]
(36)
The differential operator realization \( \{ \tilde{J}_A(x, \theta, \partial, \Lambda) \} \) with \( \partial_\alpha = \partial x^\alpha \) and \( \partial_{\dot{\alpha}} = \partial \theta^{\dot{\alpha}} \) denoting partial derivatives wrt \( x^\alpha \) and \( \theta^{\dot{\alpha}}, \) is then defined by
\[
\langle \Lambda, x, \theta | J_A = \tilde{J}_A(x, \theta, \partial, \Lambda) \langle \Lambda, x, \theta |
\]
(37)
and it follows immediately that the generators \( \tilde{J}_A(x, \theta, \partial, \Lambda) \) satisfy the Lie superalgebra commutation relations. It is convenient (in particular when considering primary fields in Section 6) to have a similar notation for highest weight vectors
\[
\Lambda, x, \theta \rangle = \Lambda \langle \Lambda \rangle \quad , \quad J_A \Lambda, x, \theta \rangle = -J_A \Lambda, x, \theta \rangle | \Lambda, x, \theta \rangle
\]
(38)
The relation between the two sets of realizations of the Lie superalgebra, \( \{ \tilde{J}_A(x, \theta, \partial, \Lambda) \} \) and \( \{ J_A(x, \theta, \partial, \Lambda) \}, \) is
\[
\tilde{J}_A(x, \theta, \partial, \Lambda) = -J_A \Lambda, x, \theta \rangle
\]
(39)
where the super-adjoint operation in \( \mathfrak{g} \) is defined by
\[
E_{\alpha}^\dagger = F_\alpha \quad , \quad F_{\dot{\alpha}}^\dagger = E_{\dot{\alpha}} \quad , \quad H_i^\dagger = H_i
\]
(40)
In general \( [23], \) a super-adjoint (or grade adjoint) operation is linear, degree (or grade) preserving and satisfies
\[
[J_A, J_B]^\dagger = (-1)^{\text{deg}(J_A)\text{deg}(J_B)} [J_B^\dagger, J_A^\dagger]
\]
(41)
In particular, we have \( (g_+(x, \theta))^\dagger = g_-(x, \theta) \) since \( (\theta^\dot{\alpha}_J A^\dagger = \theta^\dot{\alpha}_J A^\dagger. \) Furthermore, one easily derives the following symmetries of the structure coefficients
\[
\begin{align*}
 f_{-\alpha, \beta}^{\pm\gamma} &= -f_{\alpha, -\beta}^{\mp\gamma} \quad , \quad f_{-\alpha, \beta}^{\pm\gamma} &= \mp f_{\alpha, -\beta}^{\mp\gamma} \quad , \quad f_{-\dot{\alpha}, \beta}^{\pm\gamma} &= f_{\dot{\alpha}, -\beta}^{\mp\gamma} \\
 f_{-\dot{\alpha}, -\beta}^{\mp\gamma} &= -f_{\dot{\alpha}, \beta}^{\pm\gamma}
\end{align*}
\]
(42)
Here all roots are meant to be positive.

The Gauss decompositions of \( \langle \Lambda | G_+(x, \theta) e^{t J_a} \) for \( t \) small and of \( \langle \Lambda | G_+(x, \theta) e^{\mu J_a} \) for \( \mu \) a Grassmann odd parameter, may be written
\[
\begin{align*}
\langle \Lambda | G_+(x, \theta) \exp(t E_\alpha) &= \langle \Lambda | \exp \left( g_+(x, \theta) + t V_\alpha^\beta(x, \theta) E_\beta + t V_\alpha^{\dot{\beta}}(x, \theta) e_{\dot{\beta}} + O(t^2) \right) \\
&= \langle \Lambda | \exp \left( t \left( V_\alpha^\beta(x, \theta) \partial_\beta + V_\alpha^{\dot{\beta}}(x, \theta) \partial_{\dot{\beta}} \right) + O(t^2) \right) G_+(x, \theta)
\end{align*}
\]
(7)
\[ \langle \Lambda | G_+(x, \theta) \exp(tH_i) = \langle \Lambda | \exp \left( tH_i + O(t^2) \right) \]
\[ \cdot \exp \left( g_+(x, \theta) + tV_i^\beta(x, \theta)E_\beta + tV_i^\beta(x, \theta)e_\beta + O(t^2) \right) \]
\[ = \langle \Lambda | \exp \left( t \left( V_i^\beta(x, \theta)\partial_\beta + V_i^\beta(x, \theta)\partial_\beta + \Lambda_i \right) \right) + O(t^2) \]
\[ \cdot G_+(x, \theta) \]
\[ \langle \Lambda | G_+(x, \theta) \exp(tF_\alpha) = \langle \Lambda | \exp \left( tQ^-_\alpha(x, \theta)F_\beta + tQ^-_\alpha(x, \theta)f_\beta + O(t^2) \right) \]
\[ \cdot \exp \left( tP^j_\alpha(x, \theta)H_j + O(t^2) \right) \]
\[ \cdot \exp \left( g_+(x, \theta) + tV^-_\alpha(x, \theta)E_\beta + tV^-_\alpha(x, \theta)e_\beta + O(t^2) \right) \]
\[ = \langle \Lambda | \exp \left( t \left( P^j_\alpha(x, \theta)\Lambda_j \right. \right. \right.
\[ + \left. \left. V^-_\alpha(x, \theta)\partial_\beta + V^-_\alpha(x, \theta)\partial_\beta \right) \right) + O(t^2) \rangle G_+(x, \theta) \]
\[ \langle \Lambda | G_+(x, \theta) \exp(\mu e_\alpha) = \langle \Lambda | \exp \left( g_+(x, \theta) + \mu V^\beta_\alpha(x, \theta)E_\beta + \mu V^\beta_\alpha(x, \theta)e_\beta \right) \]
\[ = \langle \Lambda | \exp \left( \mu \left( V^\beta_\alpha(x, \theta)\partial_\beta + V^\beta_\alpha(x, \theta)\partial_\beta \right) \right) G_+(x, \theta) \]
\[ \langle \Lambda | G_+(x, \theta) \exp(\mu f_\alpha) = \langle \Lambda | \exp \left( \mu Q^-_\alpha(x, \theta)F_\beta + \mu Q^-_\alpha(x, \theta)f_\beta \right) \]
\[ \cdot \exp \left( \mu P^j_\alpha(x, \theta)H_j \right) \]
\[ \cdot \exp \left( g_+(x, \theta) + \mu V^-_\alpha(x, \theta)E_\beta + \mu V^-_\alpha(x, \theta)e_\beta \right) \]
\[ = \langle \Lambda | \exp \left( \mu \left( P^j_\alpha(x, \theta)\Lambda_j \right. \right. \right.
\[ + \left. \left. V^-_\alpha(x, \theta)\partial_\beta + V^-_\alpha(x, \theta)\partial_\beta \right) \right) G_+(x, \theta) \] (43)

It follows that the differential operator realization is of the form
\[ \tilde{E}_\alpha(x, \theta, \partial) = V^\beta_\alpha(x, \theta)\partial_\beta \]
\[ \tilde{H}_i(x, \theta, \partial, \Lambda) = V^\beta_i(x, \theta)\partial_\beta + \Lambda_i \]
\[ \tilde{F}_\alpha(x, \theta, \partial, \Lambda) = V^\beta_\alpha(x, \theta)\partial_\beta + P^j_\alpha(x, \theta)\Lambda_j \]
\[ \tilde{e}_\alpha(x, \theta, \partial) = V^\beta_\alpha(x, \theta)\partial_\beta \]
\[ \tilde{f}_\alpha(x, \theta, \partial, \Lambda) = V^\beta_\alpha(x, \theta)\partial_\beta + P^j_\alpha(x, \theta)\Lambda_j \] (44)

Since \( \tilde{E}_\alpha(x, \theta, \partial, \Lambda) = \tilde{E}_\alpha(x, \theta, \partial) \) and \( \tilde{e}_\alpha(x, \theta, \partial, \Lambda) = \tilde{e}_\alpha(x, \theta, \partial) \) are independent of \( \Lambda \) they may be defined through a Gauss decomposition alone.

We shall now work out explicitly the relevant Gauss decompositions in order to determine the polynomials \( V, P \) and \( Q \). For that purpose we use repeatedly the Campbell-Baker-Hausdorff formula (see e.g. \[ ] for a proof)
\[ e^A e^B = \exp \left\{ A + t \sum_{n \geq 0} \frac{B^n}{n!} (-\text{ad}_A)^n B + O(t^2) \right\} \]
\[ e^A e^{\mu b} = \exp \left\{ A + \sum_{n \geq 0} \frac{B_n}{n!} \left( -\text{ad}_A \right)^n(\mu b) \right\} \quad (45) \]

where \( \mu \) is a Grassmann odd parameter and \( b \) is an odd operator, and where \( B_n \) are the Bernoulli numbers with generating function \( B(u) \)

\[ B(u) = \frac{u}{e^u - 1} = \sum_{n \geq 0} \frac{B_n}{n!} u^n \]

\[ (B(u))^{-1} = \frac{e^u - 1}{u} = \sum_{n \geq 0} \frac{1}{(n+1)!} u^n \quad (46) \]

It turns out that only slight modifications are needed of the techniques employed in the recent work [4, 5] by Petersen, Yu and the present author on purely bosonic free field realizations of affine current algebras. Utilizing the results of that work we find

\[ V_{\tilde{\alpha}}^{\tilde{\beta}}(x, \theta) = [B(C(x, \theta))]_{\tilde{\alpha}}^{\tilde{\beta}} \]
\[ V_{i}^{\tilde{\beta}}(x, \theta) = - [C(x, \theta)]_{i}^{\tilde{\beta}} \]
\[ V_{-\tilde{\alpha}}^{\tilde{\beta}}(x, \theta) = [e^{-C(x, \theta)}]_{-\tilde{\alpha}}^{\tilde{\beta}} [B(-C(x, \theta))]_{\gamma}^{\tilde{\beta}} \]
\[ P_{-\tilde{\alpha}}^{\tilde{\beta}}(x, \theta) = [e^{-C(x, \theta)}]_{-\tilde{\alpha}}^{\tilde{\beta}} \]
\[ Q_{-\tilde{\alpha}}^{\tilde{\beta}}(x, \theta) = [e^{-C(x, \theta)}]_{-\tilde{\alpha}}^{\tilde{\beta}} \quad (47) \]

Note that the expressions are valid for all positive roots and that the summation \( \sum_{\tilde{\gamma} > 0} \) in \( V_{-\tilde{\alpha}}^{\tilde{\beta}}(x, \theta) \) is over both even and odd (positive) roots. Due to the fact that for any given Lie superalgebra the matrix \( C(x, \theta) \) is nilpotent, the formal power series in \( (47) \) all truncate and become polynomials. We refer to [4, 5] for further details. The explicit polynomial expressions in \( (47) \) comprise the first main new result in this paper, since they provide us with explicit differential operator realizations \( (44) \) and \( (39) \) of the Lie superalgebra in question.

A possible generalization of the work [3] by de Boer and Fehér on affine current algebras, may be based on the following (super-)trace

\[ V_{\tilde{\alpha}}^{\tilde{\beta}}(x, \theta) \text{str} \left( G_{-1}(x, \theta) \partial_{\tilde{\gamma}} G_{+}(x, \theta) F_{\tilde{\gamma}} \right) = 2h^{\gamma} \kappa_{\tilde{\alpha}, -\tilde{\gamma}} \quad (48) \]

Here we have used the common notation \( F_{\tilde{\gamma}} \) for the lowering generators corresponding to the positive root \( \tilde{\gamma} \). The expression follows immediately from the realizations \( (43) \) and \( (44) \) of \( \tilde{E}_{\alpha}(x, \theta) \) and \( \tilde{e}_{\alpha}(x, \theta) \). In [3] essentially the purely bosonic counterpart of this trace is introduced as a key object in their explicit Wakimoto construction.

### 3.1 Differential Screening Operators

We shall also be interested in screening currents in Section 5. They are built from certain differential operators \( S_{\tilde{\alpha}} \) to be defined presently. In the case of purely bosonic Lie algebras similar operators are well known [4, 24, 25, 26, 3, 4, 5].
Let $t$ and $\mu$ be Grassmann even and odd parameters, respectively. The operators

$$S_\alpha(x, \theta, \vartheta) = S_\alpha^\beta(x, \theta) \partial_\beta$$

are then defined through the Gauss decompositions

$$e^{tS_\alpha(x, \theta, \vartheta)}G_+(x, \theta) = e^{-tE_\alpha}G_+(x, \theta) \quad , \quad e^{\mu S_\alpha(x, \theta, \vartheta)}G_+(x, \theta) = e^{-\mu E_\alpha}G_+(x, \theta)$$

and we find

$$S_\alpha^\beta(x, \theta) = -[B(-C(x, \theta))]^{\beta}_{\alpha}$$

(51)

It follows from associativity of the Lie supergroup, that these polynomials satisfy the commutator relations

$$[\bar{E}_\alpha(x, \theta, \vartheta), S_\beta(x, \theta, \vartheta)] = 0$$

$$[\bar{H}_i(x, \theta, \vartheta), S_\beta(x, \theta, \vartheta)] = \bar{\beta}(H_i)S_\beta(x, \theta, \vartheta)$$

$$[\bar{F}_\alpha(x, \theta, \vartheta), S_\beta(x, \theta, \vartheta)] = \bar{\beta}(H_j)P^j_\alpha(x, \theta)S_\beta(x, \theta, \vartheta) + Q^{-\gamma}_{\alpha}(x, \theta)f_{\beta, -\gamma}^j \Lambda_j$$

$$- f_{\beta, -\gamma}^\sigma Q^{-\gamma}_{\alpha}(x, \theta)S_\sigma(x, \theta, \vartheta) - f_{\beta, -\gamma}^\sigma Q^{-\gamma}_{\alpha}(x, \theta)S_\sigma(x, \theta, \vartheta)$$

$$[\bar{F}_\alpha(x, \theta, \vartheta), S_\beta(x, \theta, \vartheta)] = \bar{\beta}(H_j)P^j_\alpha(x, \theta)S_\beta(x, \theta, \vartheta) - Q^{-\gamma}_{\alpha}(x, \theta)f_{\beta, -\gamma}^j \Lambda_j$$

$$+ f_{\beta, -\gamma}^\sigma Q^{-\gamma}_{\alpha}(x, \theta)S_\sigma(x, \theta, \vartheta) - f_{\beta, -\gamma}^\sigma Q^{-\gamma}_{\alpha}(x, \theta)S_\sigma(x, \theta, \vartheta)$$

$$[\bar{e}_\alpha(x, \theta, \vartheta), S_\beta(x, \theta, \vartheta)] = 0$$

$$\{\bar{e}_\alpha(x, \theta, \vartheta), S_\beta(x, \theta, \vartheta)\} = 0$$

$$[\bar{f}_\alpha(x, \theta, \vartheta), S_\beta(x, \theta, \vartheta)] = \bar{\beta}(H_j)P^j_\alpha(x, \theta)S_\beta(x, \theta, \vartheta) + Q^{-\gamma}_{\alpha}(x, \theta)f_{\beta, -\gamma}^j \Lambda_j$$

$$- f_{\beta, -\gamma}^\sigma Q^{-\gamma}_{\alpha}(x, \theta)S_\sigma(x, \theta, \vartheta) - f_{\beta, -\gamma}^\sigma Q^{-\gamma}_{\alpha}(x, \theta)S_\sigma(x, \theta, \vartheta)$$

$$\{\bar{f}_\alpha(x, \theta, \vartheta), S_\beta(x, \theta, \vartheta)\} = \bar{\beta}(H_j)P^j_\alpha(x, \theta)S_\beta(x, \theta, \vartheta) - Q^{-\gamma}_{\alpha}(x, \theta)f_{\beta, -\gamma}^j \Lambda_j$$

$$+ f_{\beta, -\gamma}^\sigma Q^{-\gamma}_{\alpha}(x, \theta)S_\sigma(x, \theta, \vartheta) - f_{\beta, -\gamma}^\sigma Q^{-\gamma}_{\alpha}(x, \theta)S_\sigma(x, \theta, \vartheta)$$

$$[S_\alpha(x, \theta, \vartheta), S_\beta(x, \theta, \vartheta)] = f_{\alpha, \beta}^\gamma S_\gamma(x, \theta, \vartheta)$$

$$[S_\alpha(x, \theta, \vartheta), S_\beta(x, \theta, \vartheta)] = f_{\alpha, \beta}^\gamma S_\gamma(x, \theta, \vartheta)$$

$$\{S_\alpha(x, \theta, \vartheta), S_\beta(x, \theta, \vartheta)\} = f_{\alpha, \beta}^\gamma S_\gamma(x, \theta, \vartheta)$$

(52)

Several non-trivial classical polynomial identities (as opposed to quantum polynomial identities, see Section 4 and Appendix A) may be derived from the commutator relations using the differential operator realizations of the Lie superalgebra generators (14) and (52). The identities will be used in subsequent sections, and are listed in Appendix A.

### 4 Wakimoto Free Field Realization

In the case of purely bosonic affine current algebras, it is well known how to obtain free field realizations (based on bosonic ghost pairs ($\beta_{\alpha}, \gamma^\alpha$) and bosonic scalars $\varphi_i$) from
the “fermionic truncation” of the differential operator realization \{\tilde{J}_a\}, simply by the substitutions \[3, 27, 24, 28, 25, 23, 22, 1, 4\]
\[
\partial_\alpha \rightarrow \beta_\alpha(z) \quad , \quad x^\alpha \rightarrow \gamma^\alpha(z) \quad , \quad \Lambda_i \rightarrow \sqrt{i}\partial \varphi_i(z)
\] (53)
followed by adding anomalous terms \(\partial \gamma^\beta(z)F_{\alpha \beta}(\gamma(z))\) to the lowering part. The natural generalization of that to the case of affine current superalgebras is to make the substitutions (see \[3\] for the case of \(OSp(1|2)\), and \[3\] for Chevalley generators in general)
\[
\partial_\alpha \rightarrow \beta_\alpha(z) \quad , \quad x^\alpha \rightarrow \gamma^\alpha(z) \quad , \quad \Lambda_i \rightarrow \sqrt{i}\partial \varphi_i(z)
\]
\[
\partial_\beta \rightarrow b_\beta(z) \quad , \quad \theta^\alpha \rightarrow c^\alpha(z)
\] (54)
in the differential operator realization \{\tilde{J}_a\} and subsequently to add anomalous terms
\[
F^\text{anom}_\alpha(\gamma(z), c(z), \partial \gamma(z), \partial c(z)) = \partial \gamma^\beta(z)F_{\alpha \beta}(\gamma(z), c(z)) + \partial c^\beta(z)F_{\alpha \beta}(\gamma(z), c(z))
\]
\[
f^\text{anom}_\alpha(\gamma(z), c(z), \partial \gamma(z), \partial c(z)) = \partial \gamma^\beta(z)f_{\alpha \beta}(\gamma(z), c(z)) + \partial c^\beta(z)f_{\alpha \beta}(\gamma(z), c(z))
\] (55)
to the lowering operators \(F_\alpha(z)\) and \(f_\alpha(z)\), respectively. We are left with the following form of the free field realization of the affine current superalgebra
\[
H_i(z) = :V_i^\beta(\gamma(z), c(z))b_\beta(z) : + :V_i^\beta(\gamma(z), c(z))b_\beta(z) : + \sqrt{i}\partial \varphi_i(z)
\]
\[
E_\alpha(z) = :V_\alpha^\beta(\gamma(z), c(z))b_\beta(z) : + :V_\alpha^\beta(\gamma(z), c(z))b_\beta(z) :
\]
\[
F_\alpha(z) = :V^\beta_\alpha(\gamma(z), c(z))b_\beta(z) : + :V^\beta_\alpha(\gamma(z), c(z))b_\beta(z) : + \sqrt{i}\partial \varphi_j(z)P_{\alpha \beta}(\gamma(z), c(z)) + \partial \gamma^\beta(z)F_{\alpha \beta}(\gamma(z), c(z)) + \partial c^\beta(z)F_{\alpha \beta}(\gamma(z), c(z))
\]
\[
e_\alpha(z) = :V^\beta_\alpha(\gamma(z), c(z))b_\beta(z) : + :V^\beta_\alpha(\gamma(z), c(z))b_\beta(z) :
\]
\[
f_\alpha(z) = :V^\beta_\alpha(\gamma(z), c(z))b_\beta(z) : + :V^\beta_\alpha(\gamma(z), c(z))b_\beta(z) : + \sqrt{i}\partial \varphi_j(z)P_{\alpha \beta}(\gamma(z), c(z)) + \partial \gamma^\beta(z)f_{\alpha \beta}(\gamma(z), c(z)) + \partial c^\beta(z)f_{\alpha \beta}(\gamma(z), c(z))
\] (56)

Now, the obvious task is to work out unique solutions for the anomalous terms. This we will do in the following, and the result is one of the main new results in this paper since it concludes the (generalized) Wakimoto free field realization of affine current superalgebras \([54]\).

A comparison of the free field realization \([53]\) with \([17]\) yields a set of quantum polynomial identities, listed in \([133]\) in Appendix A. Among those identities are
\[
k\kappa_{\alpha,-\beta} = -\partial_\sigma V^\gamma_\alpha \partial_\delta V^\sigma_- \beta + \partial_\delta V^\gamma_\alpha \partial_\sigma V^\delta_- \beta + V^\gamma_\alpha F_{\beta \gamma}
\]
\[
k\kappa_{\alpha,-\beta} = -\partial_\sigma V^\gamma_\alpha \partial_\delta V^\sigma_- \beta + V^\gamma_\alpha f_{\beta \gamma}
\]
\[
k\kappa_{\alpha,-\beta} = -\partial_\delta V^\gamma_\alpha \partial_\sigma V^\sigma_- \beta + V^\gamma_\alpha F_{\beta \gamma}
\]
\[
k\kappa_{\alpha,-\beta} = -\partial_\delta V^\gamma_\alpha \partial_\sigma V^\sigma_- \beta + \partial_\sigma V^\gamma_\alpha \partial_\delta V^\delta_- \beta + V^\gamma_\alpha f_{\beta \gamma}
\] (57)
following from the OPE’s \(E_\alpha F_\beta, E_\alpha f_\beta, e_\alpha F_\beta\) and \(e_\alpha f_\beta\), respectively. Hence, in order to determine the anomalous terms, we must show that \(V_+^+\) is invertible and find its inverse. However, this follows immediately from \([17]\)
\[
V_+^+(\gamma, c) = [B(C(\gamma, c))]_+^+ = B(C_+^+(\gamma, c))
\] (58)
and we may read off (10)

\[
(V_+^+(\gamma, c))^{-1} = (B(C_+^+(\gamma, c)))^{-1} = \sum_{n \geq 0} \frac{1}{(n+1)!}(C_+^+(\gamma, c))^n
\]

Thus we have

\[
F_{\alpha\bar{\beta}}(\gamma, c) = k \left[ (V_+^+(\gamma, c))^{-1} \right]_{\alpha \beta}^\alpha \kappa_{\alpha,-\alpha} + \left[ (V_+^+(\gamma, c))^{-1} \right]_{\alpha \beta}^\mu \partial_\sigma V_\mu^\gamma(\gamma, c) \partial_\gamma V_\alpha^\alpha(\gamma, c) - \partial_\sigma V_\mu^\gamma(\gamma, c) \partial_\gamma V_\alpha^\alpha(\gamma, c) + \left[ (V_+^+(\gamma, c))^{-1} \right]_{\alpha \beta}^\mu \partial_\sigma V_\mu^\gamma(\gamma, c) \partial_\gamma V_{\bar{\alpha}}^\bar{\alpha}(\gamma, c) - \partial_\sigma V_\mu^\gamma(\gamma, c) \partial_\gamma V_{\bar{\alpha}}^\bar{\alpha}(\gamma, c)
\]

\[
f_{\bar{\alpha}\bar{\beta}}(\gamma, c) = k \left[ (V_+^+(\gamma, c))^{-1} \right]_{\bar{\alpha} \bar{\beta}}^\bar{\alpha} \kappa_{\bar{\alpha},-\bar{\alpha}} + \left[ (V_+^+(\gamma, c))^{-1} \right]_{\bar{\alpha} \bar{\beta}}^\mu \partial_\sigma V_\mu^\gamma(\gamma, c) \partial_\gamma V_{\bar{\alpha}}^\bar{\alpha}(\gamma, c) - \partial_\sigma V_\mu^\gamma(\gamma, c) \partial_\gamma V_{\bar{\alpha}}^\bar{\alpha}(\gamma, c)
\]

where we have used that \( \kappa_{\bar{\alpha},-\bar{\beta}} = 0 \) unless \( \bar{\alpha} = \bar{\beta} \). Note that there are no summations over \( \alpha \) and \( \bar{\alpha} \) in the terms proportional to \( k \). In particular for \( \alpha \) or \( \bar{\alpha} \) a simple root, we find (using the common notation \( F_{\bar{\alpha}\bar{\beta}} \) for the anomalous parts when \( \bar{\alpha} \) is a general positive root)

\[
F_{\bar{\alpha}\bar{\beta}}(\gamma, c) = \frac{1}{2} \delta_{\bar{\alpha},\bar{\beta}} ((2k + h^\gamma)\kappa_{\bar{\alpha},-\bar{\alpha}} - A_{ii})
\]

which is seen to be a constant, independent of \( \gamma \) and \( c \). Here we have used that

\[
\partial_\sigma V_\alpha^\gamma(\gamma, c) \partial_\gamma V_{\bar{\alpha}}^\bar{\alpha}(\gamma, c) - \partial_\sigma V_\alpha^\gamma(\gamma, c) \partial_\gamma V_{\bar{\alpha}}^\bar{\alpha}(\gamma, c) = \frac{1}{2} \delta_{ij} (h^\gamma \kappa_{\bar{\alpha},-\bar{\alpha}} - A_{ii})
\]

5 Screening Currents

A screening current has conformal weight 1 and has the property that the singular part of the OPE with an affine current is a total derivative. These properties ensure that integrated screening currents (screening charges) may be inserted into correlators without altering the conformal or affine Ward identities. This in turn makes them very useful in construction of correlators, see e.g. [10, 11, 12, 13, 14]. The best known screening currents [2, 24, 28, 29, 32, 3, 4, 5] are the ones of the first kind in standard bosonic affine current algebra. Screening currents of the second kind however [33, 28, 4, 5], involve non-integer powers of the ghost fields and therefore have been less studied. The techniques for handling such objects have been developed in [14, 15].

To the best of our knowledge, in the case of superalgebras the only known screening currents are the ones for \( osp(1|2) \) and \( sl(2|1) \). In the case of \( osp(1|2) \), the screening current of the first kind is due to Bershadsky and Ooguri [1] while the screening current of the second kind is due to Ennes et al [34]. In the case of \( sl(2|1) \), only screening currents of the first kind are known and are due to Ito [4], and only for one choice of simple roots,
see Section 7. In this section we will provide universal expressions for screening currents \( s_{\alpha_j}(w) \) of the first kind for generic affine current superalgebras, thus presenting one of the main new results in this paper. Namely, we find the screening currents to be of the form

\[
s_{\alpha_j}(w) = \left( S_{\alpha_j}^\sigma(\gamma(w), c(w))\beta_\sigma(w) + S_{\alpha_j}^\sigma(\gamma(w), c(w))b_\sigma(w) \right) : e^{-\tilde{\alpha}_j(H_\varphi(w))/\sqrt{T}} : = \left( S_{\alpha_j}^\sigma(\gamma(w), c(w))\beta_\sigma(w) + S_{\alpha_j}^\sigma(\gamma(w), c(w))b_\sigma(w) \right) : e^{-\tilde{\alpha}_j(H_\varphi(w))/\sqrt{T}} : \]

where

\[
\partial_\varphi(z)\Lambda \cdot \varphi(w) = \frac{A_{ji}}{z - w} , \quad \partial_\varphi(z)\beta(H \cdot \varphi(w)) = \frac{\beta(H_i)}{z - w} \]

and to produce the following total derivatives

\[
\begin{align*}
H_\beta(z)s_{\alpha_j}(w) &= 0 \\
E_\beta(z)s_{\alpha_j}(w) &= 0 \\
F_\beta(z)s_{\alpha_j}(w) &= \partial_\varphi(w) \left( \frac{(-1)^{p(\alpha_j)} + 1}{z - w} K_{\alpha_j, -\alpha_j} t Q_{\beta}^{-\alpha_j}(\gamma(w), c(w)) : e^{-\tilde{\alpha}_j(H_\varphi(w))/\sqrt{T}} : \right) \\
e_\beta(z)s_{\alpha_j}(w) &= 0 \\
f_\beta(z)s_{\alpha_j}(w) &= \partial_\varphi(w) \left( \frac{(-1)^{p(\alpha_j)} + 1}{z - w} K_{\alpha_j, -\alpha_j} t Q_{\beta}^{-\alpha_j}(\gamma(w), c(w)) : e^{-\tilde{\alpha}_j(H_\varphi(w))/\sqrt{T}} : \right) \\
T(z)s_{\alpha_j}(w) &= \partial_\varphi(w) \left( \frac{1}{z - w} s_{\alpha_j}(w) \right)
\end{align*}
\]

Utilizing the classical polynomial identities \([31]\), the proof is straightforward for the raising operators \( E_\beta \) and \( e_\beta \), for the Cartan operator \( H_i \) and for the energy-momentum tensor \( T \). The last identity merely shows that indeed \( s_{\alpha_j}(w) \) is a conformal primary field with weight 1. For the lowering operators, comparisons of the two sides in \([32]\) yield the following consistency conditions

\[
\begin{align*}
-tK_{\tilde{\alpha}_j, -\alpha_j} Q_{\beta}^{-\tilde{\alpha}_j} &= -(-1)^{p(\tilde{\alpha}_j)(1-p(\beta))} \left( S_{\alpha_j}^\sigma F_{\tilde{\beta}\tilde{\sigma}} + A_{ij} S_{\alpha_j}^\sigma \partial_\gamma P_i^{\tilde{\beta}} + (-1)^{p(\beta)} \partial_\gamma V_{-\tilde{\beta}}^{\gamma} \partial_\gamma S_{\alpha_j}^\sigma - (-1)^{p(\beta)} \partial_\gamma V_{-\tilde{\beta}}^{\gamma} \partial_\gamma S_{\alpha_j}^\sigma \right) \\
-tK_{\alpha_j, -\tilde{\alpha}_j} \partial_\gamma Q_{\beta}^{-\tilde{\alpha}_j} &= (-1)^{p(\tilde{\alpha}_j)(1-p(\beta))} \left( -S_{\alpha_j}^\sigma \partial_\gamma F_{\tilde{\beta}\tilde{\sigma}} - (-1)^{p(\tilde{\gamma})} S_{\alpha_j}^\sigma \partial_\gamma F_{\tilde{\beta}\tilde{\sigma}} \right) \\
+tK_{\alpha_j, -\tilde{\alpha}_j} \partial_\gamma Q_{\beta}^{-\tilde{\alpha}_j} &= (-1)^{p(\tilde{\alpha}_j)(1-p(\beta))} \left( -S_{\alpha_j}^\sigma \partial_\gamma F_{\tilde{\beta}\tilde{\sigma}} - (-1)^{p(\tilde{\gamma})} S_{\alpha_j}^\sigma \partial_\gamma F_{\tilde{\beta}\tilde{\sigma}} \right)
\end{align*}
\]

They are easily verified for \( \tilde{\beta} \) a simple root \( \tilde{\beta} = \tilde{\alpha}_i \), using that

\[
\partial_\gamma V_{-\tilde{\alpha}_i}^{\gamma} \partial_\gamma S_{\alpha_j}^\sigma - \partial_\gamma V_{-\tilde{\alpha}_i}^{\gamma} \partial_\gamma S_{\alpha_j}^\sigma = \frac{1}{2} \delta_{ij}(-1)^{p(\tilde{\alpha}_i)} (h^\gamma_{\tilde{\kappa}_i, -\tilde{\alpha}_i} - A_{ii})
\]

In the case of a non-simple root \( \tilde{\beta} \), we have proven the conditions \([66]\) by induction in addition of roots using various classical and quantum polynomial identities. The strategy is fairly straightforward, though very tedious. First one eliminates terms involving \( F_{\tilde{\beta}\tilde{\sigma}} \),
using in particular the recursion relations \( \{130\} \) and \( \{134\} \) expressing e.g. \( f_{\tilde{\alpha},\tilde{\alpha}'} \tilde{\beta} F_{\tilde{\beta}\tilde{\alpha}} \) in terms of polynomials with indices \( \tilde{\alpha} \) and \( \tilde{\alpha}' \). By induction assumption, such polynomials do satisfy \( \{66\} \), so by substitution we may get rid of terms involving derivatives of the anomalous polynomials. Next one eliminates terms involving \( F_{\tilde{\alpha}\tilde{\alpha}} \), and we are left with a set of relations in \( V, P, \) and \( S \). (The \( Q \)'s in \( \{66\} \) may be expressed in terms of \( P \)'s and \( S \)'s \( \{132\} \)). The proof is then concluded by virtue of the classical polynomial identities. Throughout we may act on the various identities with appropriate differential operators in order to derive further identities.

Note that the numbers of fermionic and bosonic screening currents depend on the underlying Lie superalgebra and the choice of a set of simple roots in that.

6 Primary Fields

The final main new result reported in this paper is the explicit construction in this section of primary fields for arbitrary representations, integral or non-integral. We find it particularly convenient to replace the traditional multiplet of primary fields (which generically would be infinite for non-integrable representations) by a generating function for that, namely the primary field \( \phi_\Lambda(w, x, \theta) \) which must satisfy

\[
J_\Lambda(z) \phi_\Lambda(w, x, \theta) = \frac{-J_\Lambda(x, \theta, \partial, \Lambda)}{z - w} \phi_\Lambda(w, x, \theta)
\]

\[
T(z) \phi_\Lambda(w, x, \theta) = \frac{\Delta(\phi_\Lambda)}{(z - w)^2} \phi_\Lambda(w, x, \theta) + \frac{1}{z - w} \partial \phi_\Lambda(w, x, \theta)
\]

(68)

Here \( J_\Lambda(z) \) are the affine currents, whereas \( J_\Lambda(x, \theta, \partial, \Lambda) \) are the differential operator realizations given in \( \{38\}, \{39\}, \{40\}, \{44\} \) and \( \{47\} \). We shall find the result in the form

\[
\phi_\Lambda(w, x, \theta) = \phi'_\Lambda(\gamma(w), c(w), x, \theta)V_\Lambda(w)
\]

\[
V_\Lambda(w) = : e^{\frac{1}{2} \Lambda \varphi(w) } :
\]

(69)

Indeed, such a field is conformally primary and has conformal dimension \( \Delta(\phi_\Lambda) = \frac{1}{2\ell}(\Lambda, \Lambda + 2\rho) \). In order to comply with \( \{58\} \) for \( J_\Lambda = H_i, \phi'_\Lambda \) is seen to be supersymmetric in \( (x, \theta) \) and \( (\gamma(w), c(w)) \). Below we shall demonstrate this by explicit construction. Due to the fact that the anomalous parts of \( F_\tilde{\alpha}(z) \) and \( f_\tilde{\alpha}(z) \) do not give singular contributions when contracting with \( \phi'_\Lambda \), we are left with the following sufficient conditions on \( \phi'_\Lambda(\gamma(w), c(w), x, \theta) \), a pair for each \( \tilde{\alpha} > 0 \)

\[
(-1)^{\rho(\tilde{\alpha})} \left( V^\beta_\tilde{\alpha}(\gamma, c) \partial_{c^\beta} + V^\beta_\tilde{\alpha}(\gamma, c) \partial_{\gamma^\beta} \right) \phi'_\Lambda = \left( V^{-\beta}_{-\tilde{\alpha}}(x, \theta) \partial_{x^\beta} + V^{-\beta}_{-\tilde{\alpha}}(x, \theta) \partial_{\theta^\beta} \right) \phi'_\Lambda
\]

\[
+ \Lambda_j P_{\tilde{\alpha}j}(x, \theta) \phi'_\Lambda
\]

\[
\left( V^\beta_\tilde{\alpha}(x, \theta) \partial_{x^\beta} + V^\beta_\tilde{\alpha}(x, \theta) \partial_{\theta^\beta} \right) \phi'_\Lambda = \left( V^{-\beta}_{-\tilde{\alpha}}(\gamma, c) \partial_{\gamma^\beta} + V^{-\beta}_{-\tilde{\alpha}}(\gamma, c) \partial_{c^\beta} \right) \phi'_\Lambda
\]

\[
+ \Lambda_j P_{\tilde{\alpha}j}(\gamma, c) \phi'_\Lambda
\]

(70)
where the sign factor is due to (39) and (40). Further, one can use the classical polynomial identities (130) to reduce the numbers of necessary conditions. Let us assume that \( \check{\alpha} > 0 \) is non-simple such that there exist \( \check{\beta} > 0 \) and \( \check{\gamma} > 0 \) satisfying \( f_{\check{\beta}, \check{\gamma}}^{\check{\alpha}} \neq 0 \neq f_{-\check{\beta}, -\check{\gamma}}^{-\check{\alpha}} \). We may then multiply the left hand sides by \( f_{\check{\beta}, \check{\gamma}}^{\check{\alpha}} \) and the right hand sides by \( \frac{f_{\check{\beta}, \check{\gamma}}^{\check{\alpha}}}{f_{-\check{\beta}, -\check{\gamma}}^{-\check{\alpha}}} \). By virtue of (130) it then follows that \( f_{\check{\beta}, \check{\gamma}}^{\check{\alpha}} / f_{-\check{\beta}, -\check{\gamma}}^{-\check{\alpha}} = -1 \), in accordance with (42). In conclusion, there are only 2r sufficient conditions a primary field must satisfy

\[
(-1)^{\nu(\check{a})} \left( V_{\check{\alpha}_i}^\beta (\gamma, c) \partial_{\gamma^\beta} + V_{\check{\alpha}_i}^\check{\beta} (\gamma, c) \partial_{\check{\gamma}^\beta} \right) \phi'_\Lambda = \left( V_{-\check{\alpha}_i}^\beta (x, \theta) \partial_x^{\gamma\beta} + V_{-\check{\alpha}_i}^\check{\beta} (x, \theta) \partial_{\theta^{\check{\gamma}\check{\beta}}} \right) \phi'_\Lambda + \Lambda_j P_{-\check{\alpha}_i}^j (x, \theta) \phi'_\Lambda
\]

\[
\left( V_{\check{\alpha}_i}^\beta (x, \theta) \partial_{x^{\gamma\beta}} + V_{\check{\alpha}_i}^\check{\beta} (x, \theta) \partial_{\theta^{\check{\gamma}\check{\beta}}} \right) \phi'_\Lambda = \left( V_{-\check{\alpha}_i}^\beta (\gamma, c) \partial_{\gamma^\beta} + V_{-\check{\alpha}_i}^\check{\beta} (\gamma, c) \partial_{\check{\gamma}^\beta} \right) \phi'_\Lambda + \Lambda_j P_{-\check{\alpha}_i}^j (\gamma, c) \phi'_\Lambda
\]

where

\[
P_{-\check{\alpha}_i}^j (x, \theta) = \delta^j_i x^{\alpha_j}, \quad P_{-\check{\alpha}_i}^j (x, \theta) = \delta^j_i \theta^{\alpha_j}
\]

(71)

It seems very hard to solve this set of partial differential equations directly. However, in [3] an alternative way to obtain the primary fields was developed in the case of purely bosonic affine current algebras. The analogous construction for affine current superalgebras goes as follows.

First we directly construct primary fields for each basis or fundamental representation \( M_{\Lambda^k} \). Such representation spaces are finite dimensional modules and \( \phi'_{\Lambda^k} (\gamma(w), c(w), x, \theta) \) will be polynomial in \( \gamma(w), c(w), x \) and \( \theta \). Then finally, for a general representation with highest weight \( \Lambda = \lambda_k \Lambda^k \) (see (8)) we use (71) to immediately verify that

\[
\phi'_\Lambda (\gamma(w), c(w), x, \theta) = \prod_{k=1}^r [\phi'_{\Lambda^k} (\gamma(w), c(w), x, \theta)]^{\lambda_k}
\]

(73)

We emphasize here that the labels \( \lambda_k \) may be non-integers, as is required for degenerate representations. We proceed to explain how to construct the building blocks

\[
\phi'_{\Lambda^k} (\gamma(w), c(w), x, \theta)
\]

(74)

The strategy goes as follows. First we concentrate on the case \( w = 0 \) where the object reduces to

\[
\phi'_{\Lambda^k} (\gamma_0, c_0, x, \theta)
\]

(75)

when acting on the highest weight state \( |\Lambda^k\rangle \). \( \gamma_0 \) and \( c_0 \) are the zero modes in the mode expansions

\[
\gamma(w) = \sum_n \gamma_n w^{-n}, \quad \beta(w) = \sum_n \beta_n w^{-n-1} \]

\[
c(w) = \sum_n c_n w^{-n}, \quad b(w) = \sum_n b_n w^{-n-1}
\]

(76)
Conformal covariance requires \( \phi'_{A_k}(\gamma(w), c(w), x, \theta) \) to be obtained just by replacing \( \gamma_0 \) by \( \gamma(w) \) and \( c_0 \) by \( c(w) \). The function \( \phi_{A_k}(\gamma(x), c, x, \theta) \) in turn is obtained from

\[
|\Lambda^k, x, \theta \rangle = G_-(x, \theta)|\Lambda \rangle = \phi_{A_k}(\gamma_0, c_0, x, \theta)|\Lambda \rangle
\]

Indeed, it is a consequence of the formalism, that the primary field constructed this way will satisfy the OPE. The construction is now simply achieved by expanding the state \( |\Lambda^k, x, \theta \rangle \) on an appropriate basis which is convenient to obtain using the free field realization.

Let the orthonormal basis elements in the \( k \)'th fundamental highest weight module \( M_{A_k} \) be denoted \( \{|U, \Lambda \rangle \} \) such that the identity operator may be written

\[
I = \sum_U |U, \Lambda \rangle \langle U, \Lambda |
\]

The state \( |\Lambda^k, x, \theta \rangle \) may then be written

\[
|\Lambda^k, x, \theta \rangle = \sum_U |U, \Lambda \rangle \langle U, \Lambda^k | \Lambda^k, x, \theta \rangle
\]

One of the basis vectors will always be taken to be the highest weight vector \( |\Lambda \rangle \) itself.

A particular basis vector will be of the form

\[
|U, \Lambda \rangle \sim F_{\beta_1} \cdots F_{\beta_n} |\Lambda \rangle \quad , \quad n(U) = n^0(U) + n^1(U)
\]

and the expansion of \( |\Lambda^k, x, \theta \rangle \) will be

\[
|\Lambda^k, x, \theta \rangle = \sum_U \frac{1}{\langle \Lambda | E_{\beta_1} \cdots E_{\beta_n} F_{\beta_1} \cdots F_{\beta_n} |\Lambda \rangle} \cdot F_{\beta_1} \cdots F_{\beta_n} \langle \Lambda | E_{\beta_1} \cdots E_{\beta_n} |\Lambda^k, x, \theta \rangle
\]

Here we have used the common notation \( F_{\beta} (E_{\bar{\beta}}) \) for the lowering (raising) generators corresponding to the positive root \( \beta \). The parameters \( n^0(U) \) and \( n^1(U) \) denote the numbers of even and odd generators (respectively) appearing in the expression \( (80) \). For each term in the sum \( (81) \) we treat the two factors differently. First consider the second factor.

We may use the differential operator realizations to write

\[
\langle \Lambda | E_{\beta_1} \cdots E_{\beta_n} |\Lambda \rangle
\]

\[
= (-1)^{n(U)(n(U) - 1)/2 + n(U)} E_{\beta_1} \cdots E_{\beta_n} (x, \theta, \partial, \Lambda) \cdots E_{\beta_n} (x, \theta, \partial, \Lambda) \langle \Lambda |\Lambda \rangle
\]

\[
= (-1)^{n(U)(n(U) + 1)/2} U(x, \theta, \partial/\partial x, \partial/\partial \theta, \Lambda)
\]

where

\[
U(x, \theta, \partial/\partial x, \partial/\partial \theta, \Lambda)
\]

\[
= \tilde{F}_{\beta_1} \cdots \tilde{F}_{\beta_n} (x, \theta, \partial, \Lambda) \cdots \tilde{F}_{\beta_n} (x, \theta, \partial, \Lambda) \cdot 1
\]

\[
= \begin{bmatrix}
V_{\gamma_1} \cdots V_{\gamma_n} (x, \theta) \partial_{\gamma_1} + \Lambda (H_j) P_{\gamma_1}^j (x, \theta) \cdots \\
V_{\gamma_1} \cdots V_{\gamma_n} (x, \theta) \partial_{\gamma_n} + \Lambda (H_j) P_{\gamma_n}^j (x, \theta)
\end{bmatrix}
\]

\[
\cdot \begin{bmatrix}
\Lambda (H_j) P_{\gamma_1}^j \cdots \Lambda (H_j) P_{\gamma_n}^j (x, \theta) \cdots \\
\Lambda (H_j) P_{\gamma_1}^j \cdots \Lambda (H_j) P_{\gamma_n}^j (x, \theta)
\end{bmatrix}
\]

(83)
In the last step in (82) we used that clearly
\[ \langle \Lambda^k | \Lambda^k, x, \theta \rangle \equiv 1 \] (84)
Actually, the function \( U \) is independent of \( \partial/\partial x \) and \( \partial/\partial \theta \) since the differentiations may easily be carried out.

In the first factor in (81)
\[ F_{\beta_1(u)} \cdots F_{\beta_{n(u)}} | \Lambda^k \] (85)
we use the free field realizations. The state \( | \Lambda^k \rangle \) is a vacuum for the \( \beta, \gamma \) and the \( b, c \) systems, so it is annihilated by \( \gamma_n, n \geq 1 \) and \( \beta_n, n \geq 0 \), and by \( c_n, n \geq 1 \) and \( b_n, n \geq 0 \). The \( F_{\beta_j} \)'s are the zero modes of the affine currents (33). It follows that only \( \gamma_0 \)'s, \( \beta_0 \)'s, \( c_0 \)'s and \( b_0 \)'s need be considered. Also the anomalous terms will not contribute, and we obtain
\[
F_{\beta_1(u)} \cdots F_{\beta_{n(u)}} | \Lambda^k \rangle = \left[ V_{\gamma_1} \right] \left( \gamma_0, c_0 \right) | \beta_1, 0 \rangle + \sum_{n=1}^{\infty} V_{\gamma_n} \left( \gamma_0, c_0 \right) | \beta_{n-1}, 0 \rangle \cdots | \gamma_{n+1}, c_0 \rangle \right]
\]
As before, the function \( U \) is independent of \( \beta_0 \) and \( b_0 \). By the remarks above this completes the construction in general:
\[
\phi_{\Lambda^k}^\prime (\gamma, c, x, \theta) = \sum_{U} \frac{(-1)^{n(U)(n(U)+1)/2}}{\langle \Lambda^k | E_{\beta_1(u)} \cdots E_{\beta_{n(u)}} F_{\beta_1(u)} \cdots F_{\beta_{n(u)}} | \Lambda^k \rangle} U(\gamma, c, x, \theta, \Lambda^k) U(\gamma, c, x, \theta, \Lambda^k) = \sum_{U} \frac{(-1)^{n(U)(n(U)-1)/2}}{\langle \Lambda^k | E_{\beta_1(u)} \cdots E_{\beta_{n(u)}} F_{\beta_1(u)} \cdots F_{\beta_{n(u)}} | \Lambda^k \rangle} U(x, \theta, \Lambda^k) U(\gamma, c, x, \theta, \Lambda^k) (87)
\]
Explicit expressions for the \( V \)'s and the \( P \)'s have already been provided (17).

It remains to account in detail for how to obtain a convenient basis for the fundamental representations. This part will depend on the Lie superalgebra in question, and we anticipate that there are no problems in writing down such a basis. In the case of standard affine \( SL(N) \) current algebra, this has been done in [7]. See Section 7 for explicit constructions of primary fields in particular cases of affine current superalgebras.

7 Examples

In this section we shall discuss several examples of the general constructions provided in previous sections, in order to illustrate how powerful these constructions are and to compare with results known in the literature.
7.1 Chevalley Generators

In [8] Ito has considered realizations of Chevalley generators of both classical Lie superalgebras and of affine current superalgebras. Actually, he considers also the trivial extensions to any raising generator (in the \{\tilde{J}_A\} basis, or lowering generators in the \{J_A\} basis). It is not difficult to verify that our similar results reduce to those of Ito. Indeed, we find the following key objects in the differential operator realizations (see (39), (44) and (17))

$$V_{\tilde{\alpha}}(x, \theta) = \delta_{\tilde{\alpha}} + \sum_{n \geq 1} B_n \frac{1}{n!} (-1)^n f_{\beta_1, \tilde{\alpha}} \gamma_1 f_{\beta_2, \gamma_1} \gamma_2 \cdots f_{\beta_n, \gamma_{n-1}} \beta$$

$$V_i^{\tilde{\beta}}(x, \theta) = -\tilde{\beta}(H_i) \left( x^i \delta_{\tilde{\beta}_1} + \theta^i \delta_{\tilde{\beta}} \right)$$

$$V_{-\tilde{\alpha}}(x, \theta) = f_{\tilde{\alpha}, -\alpha} \left( x^\mu \delta_{\mu} + \theta^\mu \delta_{\mu} \right) + \sum_{n \geq 1} B_n \frac{1}{n!} \beta_1(H_i) f_{\beta_2, \beta_1} \gamma_1 f_{\beta_3, \gamma_1} \gamma_2 \cdots f_{\beta_n, \gamma_{n-1}} \beta$$

$$P_{-\tilde{\alpha}}(x, \theta) = \delta_j \left( x^{\alpha_j} \delta_{\alpha_j} + \theta^{\alpha_j} \delta_{\alpha_j} \right)$$

This is in accordance with [8]. The expression for $V_{-\tilde{\alpha}}(x, \theta)$ is a simple reduction of the following non-trivial rewriting of $V_{\tilde{\alpha}}(x, \theta)$ (17), valid for all positive roots $\tilde{\alpha}$

$$V_{-\tilde{\alpha}}(x, \theta) = \sum_{n \geq 1} \sum_{\tilde{\beta}_1 > 0} \sum_{l=0}^{n-m} B_l \frac{1}{l!(n-l)!} f_{\tilde{\alpha}, -\tilde{\beta}_1} \gamma_1 f_{\beta_2, \gamma_1} \gamma_2 \cdots f_{\beta_n, \gamma_{n-1}} \beta$$

where $m(-\alpha, \beta_1, \ldots, \beta_n)$ is defined for a given sequence of roots $(-\alpha, \beta_1, \ldots, \beta_m, \ldots, \beta_n)$ as the minimum integer for which $-\alpha + \beta_1 + \ldots + \beta_m > 0$. Our proof of the rewriting is based on the lemma stating that in any (formal) expansion of the form

$$\sum_{s \geq 0} \left( \sum_{n \geq 1} b_n x^n \right)^s = \sum_{n \geq 0} a_n x^n$$

the following recursion relation is valid

$$a_0 = 1$$
$$a_n = -\sum_{l=0}^{n-1} b_{n-l} a_l , \quad \text{for } n > 0$$

Proofs of (the bosonic equivalence of) (89) and of the lemma may be found in [8]. In the affine current superalgebras, Ito [8] works out the anomalous terms only for Chevalley generators (corresponding to simple roots). His result is in complete agreement with ours given in (61).
7.2 Case of $OSp(1|2)$

The classical Lie superalgebra $osp(1|2)$, isomorphic to $B(0,1)$, consists of 3 bosonic generators $E$, $H$ and $F$, and 2 fermionic generators $e$ and $f$. This corresponds to the fact that there are exactly 2 positive roots, namely one odd simple root $\alpha$ and one even non-simple root $\alpha = 2\hat{\alpha}$. Thus, the Weyl vector becomes $\rho = \frac{1}{2}\hat{\alpha}$. Let us impose the normalization condition $\alpha^2 = 2$, or equivalently $h^\vee = 3/2$ (following from Freudenthal-de Vries (super-)strange formula), and also use the convention of Kac setting $A_{11} = 2$. This leads to some slightly unconventional normalizations of the Cartan-Killing form

$$\kappa_{\hat{\alpha},-\hat{\alpha}} = \kappa_{\alpha,-\alpha} = 4$$

and of the structure coefficients

$$f_{1,\pm}\hat{\alpha}^\pm = \pm 2$$

$$f_{\hat{\alpha},-\hat{\alpha}}^1 = 1$$

$$f_{\pm}\hat{\alpha}^\pm = \pm 2$$

These parameters may be obtained from the more standard ones (see e.g. [33])

$$[J_3, j_\pm] = \pm \frac{1}{2} j_\pm$$

$$[J_3, J_\pm] = \pm J_\pm$$

$$\{j_+, j_-\} = 2J_3$$

$$[J_+, J_-] = 2J_3$$

$$[j_+, j_-] = \pm 2J_\pm$$

by the substitutions

$$J_+ = \frac{1}{2}E$$

$$J_3 = \frac{1}{4}H$$

$$J_- = \frac{1}{2}F$$

$$j_+ = \frac{1}{\sqrt{2}}e$$

$$j_- = \frac{1}{\sqrt{2}}f$$

The general results in previous sections are easily reduced in the case of $OSp(1|2)$. We find the differential operator realizations

$$\left\{ \begin{array}{ll}
\hat{H}(x, \theta, \partial, \Lambda) = -2\theta \partial_\theta - 4x \partial_x + \Lambda_1 \\
\hat{e}(x, \theta, \partial) = \partial_\theta + \theta \partial_x \\
\hat{f}(x, \partial, \theta, \partial) = -2\theta \partial_\theta - 2\theta x \partial_x + \theta \Lambda_1 \\
\hat{E}(x, \theta, \partial) = \partial_x \\
\hat{F}(x, \theta, \partial, \Lambda) = -4\theta x \partial_\theta - 4x^2 \partial_x + 2x \Lambda_1 
\end{array} \right. \quad \left\{ \begin{array}{ll}
H(x, \theta, \partial, \Lambda) = 2\theta \partial_\theta + 4x \partial_x - \Lambda_1 \\
e(x, \theta, \partial, \Lambda) = -2\theta \partial_\theta - 2\theta x \partial_x + \theta \Lambda_1 \\
f(x, \theta, \partial) = -\partial_\theta - \theta \partial_x \\
E(x, \theta, \partial, \Lambda) = 4\theta x \partial_\theta + 4x^2 \partial_x - 2x \Lambda_1 \\
F(x, \theta, \partial) = -\partial_\alpha 
\end{array} \right.$$  

The Wakimoto free field realization is based on one pair of bosonic ghost fields $(\beta, \gamma)$, one pair of fermionic ghost fields $(b, c)$ and one bosonic scalar field $\varphi = \varphi_1$ satisfying $\varphi(z)\varphi(w) = G_{11} \ln(z-w) = 8 \ln(z-w)$, and is found to be

$$H(z) = -2 : c(z)b(z) : -4 : \gamma(z)\beta(z) : +\sqrt{i}\partial\varphi(z)$$

$$e(z) = b(z) + c(z)\beta(z)$$
\[ f(z) = -2\gamma(z)b(z) - 2c(z) : \gamma(z)\beta(z) : + \sqrt{t}c(z)\partial\varphi(z) + 2(2k + 1)\partial c(z) \]
\[ E(z) = \beta(z) \]
\[ F(z) = -4\gamma(z) : c(z)b(z) : -4 : \gamma^2(z)\beta(z) : + 2\sqrt{t}\gamma(z)\partial\varphi(z) - 4(k + 1)\partial c(z)c(z) + 4k\partial\gamma(z) \]

(97)

This is in accordance with the literature [3]. It is easily verified that indeed (17) is satisfied. The Sugawara energy-momentum tensor becomes

\[ T(z) =: \partial\gamma(z)\beta(z) : + : \partial c(z)b(z) : + \frac{1}{2} : \partial\varphi(z) : \partial\varphi(z) : - \frac{1}{2\sqrt{t}}\hat{\alpha} \cdot \partial^2\varphi(z) \]

(98)

The screening current of the first kind is found to be

\[ s_\hat{\alpha}(w) = (c(w)\beta(w) - b(w)) : e^{-\varphi(w)/(4\sqrt{t})} : \]

(99)

and is also known in the literature [3]. It is not difficult to show that the screening current of the second kind is [34]

\[ \tilde{s}_\hat{\alpha}(w) = (c(w)\beta(w) - b(w)) \beta^{-(k+2)}(w) : e^{\sqrt{t}\varphi(w)/2} : \]

(100)

and that it produces the following total derivatives

\[ F(z)\tilde{s}_\hat{\alpha}(w) = \frac{\partial}{\partial w} \left( \frac{4}{z - w} ((k + 1)c(w)\beta(w) - (k + 2)b(w)) \beta^{-(k+3)}(w) : e^{\sqrt{t}\varphi(w)/2} : \right) \]
\[ f(z)\tilde{s}_\hat{\alpha}(w) = \frac{\partial}{\partial w} \left( \frac{2}{z - w}(-\beta^{-(k+2)}(w)) : e^{\sqrt{t}\varphi(w)/2} : \right) \]

(101)

OPE’s with the remaining affine current generators simply vanish. The primary field is found to be

\[ \phi_\Lambda(w, x, \theta) = (1 + 2\theta c(w) + 4x\gamma(w))^\Lambda_1/2 : e^{\Lambda \cdot \varphi(w)/\sqrt{t}} : \]

(102)

This result has also been obtained in [34], though based on different normalization conventions for the Lie superalgebra parameters.

### 7.3 Case of \( OSp(2|2) \simeq SL(2|1) \simeq SL(1|2) \)

The Lie superalgebra \( A(1, 0) \), which is isomorphic to \( osp(2|2) \), \( C(2) \) and \( sl(1|2) \simeq sl(2|1) \), has rank \( r = 2 \) and 3 positive roots, while the dual Coxeter number is \( h^\vee = 1 \) and \( \dim(g_\hat{\alpha}) = \dim(g_1) = 4 \). First we choose the set of simple roots to consist of one even simple root \( \alpha_1 \) and one odd simple root \( \hat{\alpha}_2 \). The remaining and non-simple root \( \hat{\alpha} = \alpha_1 + \hat{\alpha}_2 \) is then odd. The Weyl vector is \( \rho = -\hat{\alpha}_2 \). From the oscillator realization (see [30])

\[ H_1 = a_1^\dagger a_1 - a_2^\dagger a_2, \quad H_2 = a_2^\dagger a_2 + b^\dagger b \]
\[ E_{\alpha_1} = a_1^\dagger a_2, \quad F_{\alpha_1} = a_2^\dagger a_1 \]
\[ e_{\hat{\alpha}_2} = a_2^\dagger b, \quad f_{\hat{\alpha}_2} = b^\dagger a_2 \]
\[ e_{\hat{\alpha}} = a_1^\dagger b, \quad f_{\hat{\alpha}} = b^\dagger a_1 \]

(103)
where \( a_i^{(t)} \) and \( b^{(t)} \) are fermionic and bosonic oscillators satisfying
\[
[b, b^\dagger] = 1 \quad , \quad \{a_i, a_j^\dagger\} = \delta_{ij} \quad , \quad [b^{(t)}, a_i^{(t)}] = 0 \tag{104}
\]
we find the non-vanishing elements of the Cartan-Killing form to be
\[
G_{11} = 2 \quad , \quad G_{12} = G_{21} = -1 \quad , \quad G_{22} = 0
\]
such that the Cartan matrix is given by \( A_{ij} = G_{ij} \). The remaining non-vanishing structure coefficients are found to be
\[
\begin{align*}
\kappa_{\alpha_1, -\alpha_1} &= \kappa_{\tilde{\alpha}_2, -\alpha_2} = \kappa_{\tilde{\alpha}, -\tilde{\alpha}} = 1 \\
\end{align*}
\]
Now, the differential operator realization \( \{J_A\} \) is worked out to be
\[
\begin{align*}
\tilde{H}_1(x, \theta, \partial, \Lambda) &= -2x^{a_1}\partial_{a_1} + \theta^{a_2}\partial_{\tilde{a}} - \theta^{\tilde{a}}\partial_{\tilde{a}} + \Lambda_1 \\
\tilde{H}_2(x, \theta, \partial, \Lambda) &= x^{a_1}\partial_{a_1} + \theta^{a_2}\partial_{\tilde{a}} + \Lambda_2 \\
\tilde{E}_{a_1}(x, \theta, \partial) &= \partial_{a_1} - \frac{1}{2}\theta^{a_2}\partial_{\tilde{a}} \\
\tilde{E}_{\tilde{a}_2}(x, \theta, \partial) &= \partial_{\tilde{a}_2} + \frac{1}{2}x^{a_1}\partial_{\tilde{a}_2} \\
\tilde{\tilde{E}}_{a_2}(x, \theta, \partial) &= \partial_{a_2} + \frac{1}{2}\theta^{a_2}\partial_{\tilde{a}_2} \\
\tilde{\tilde{E}}_{\tilde{a}_1}(x, \theta, \partial) &= \partial_{\tilde{a}_1} \\
\tilde{\tilde{E}}_{\tilde{a}}(x, \theta, \partial) &= \partial_{\tilde{a}} \\
\tilde{J}_{a_1}(x, \theta, \partial, \Lambda) &= -x^{a_1}x^{a_1}\partial_{a_1} + \left(\frac{1}{2}x^{a_1}\theta^{a_2} - \theta^{\tilde{a}}\right)\partial_{\tilde{a}_2} \\
&\quad - \frac{1}{2}x^{a_1}\left(\frac{1}{2}x^{a_1}\theta^{\tilde{a}_2} + \theta^{\tilde{a}}\right)\partial_{a_2} + x^{a_1}\Lambda_1 \\
\tilde{J}_{\tilde{a}_2}(x, \theta, \partial, \Lambda) &= \left(\frac{1}{2}x^{a_1}\theta^{\tilde{a}_2} + \theta^{\tilde{a}}\right)\partial_{a_1} + \frac{1}{2}\theta^{a_2}\theta^{a_3}\partial_{\tilde{a}_2} + \theta^{a_2}\Lambda_2 \\
\tilde{J}_{\tilde{a}_1}(x, \theta, \partial, \Lambda) &= -x^{a_1}\left(\frac{1}{2}x^{a_1}\theta^{\tilde{a}_2} + \theta^{\tilde{a}}\right)\partial_{a_1} - \theta^{\tilde{a}_2}\theta^{\tilde{a}_3}\partial_{\tilde{a}_2} \\
&\quad + \left(\frac{1}{2}x^{a_1}\theta^{\tilde{a}_2} + \theta^{\tilde{a}}\right)\Lambda_1 - \left(\frac{1}{2}x^{a_1}\theta^{\tilde{a}_2} - \theta^{\tilde{a}}\right)\Lambda_2 \tag{107}
\end{align*}
\]
The alternative realization \( \{J_A\} \) is easily derived from this.

The (generalized) Wakimoto free field realization of the associated affine current superalgebra becomes
\[
\begin{align*}
H_1(z) &= -2 : \gamma(z)\beta(z) : + : c(z)b(z) : - : C(z)B(z) : + \sqrt{t}\partial\varphi_1(z) \\
H_2(z) &= : \gamma(z)\beta(z) : + : C(z)B(z) : + \sqrt{t}\partial\varphi_2(z) \\
E_{a_1}(z) &= \beta(z) - \frac{1}{2}c(z)B(z) \\
F_{a_1}(z) &= - : \gamma^2(z)\beta(z) : + : \left(\frac{1}{2}\gamma(z)c(z) - C(z)\right) b(z) : \\
\end{align*}
\]
\[ e_{\tilde{a}_2}(z) = b(z) + \frac{1}{2} \gamma(z) B(z) \]

\[ f_{a_2}(z) = \left( \frac{1}{2} \gamma(z) c(z) + C(z) \right) \beta(z) : + \frac{1}{2} c(z) : C(z) B(z) : \]

\[ + \sqrt{t} c(z) \partial \varphi_2(z) + (k + 1/2) \partial c(z) \]

\[ e_{\tilde{a}}(z) = B(z) \]

\[ f_{\tilde{a}}(z) = - : \gamma(z) \left( \frac{1}{2} \gamma(z) c(z) + C(z) \right) \beta(z) : - : c(z) C(z) b(z) : \]

\[ + \sqrt{t} \left( \frac{1}{2} \gamma(z) c(z) + C(z) \right) \partial \varphi_1(z) - \sqrt{t} \left( \frac{1}{2} \gamma(z) c(z) - C(z) \right) \partial \varphi_2(z) \]

\[ + \frac{1}{2} (k - 1) \partial \gamma(z) c(z) - \frac{1}{2} (k + 1) \partial c(z) \gamma(z) + k \partial C(z) \]

where we have introduced the simplifying notation

\[ \beta(z) = \beta_{a_1}(z), \quad \gamma(z) = \gamma^{a_1}(z) \]

\[ b(z) = b_{\tilde{a}_2}(z), \quad c(z) = c^{a_2}(z) \]

\[ B(z) = b_{\tilde{a}}, \quad C(z) = c^{a}(z) \]

It is straightforward to verify that this is a free field realization of the affine current superalgebra \( A(1, 0)^{(1)} \) with Cartan-Killing form and structure coefficients given by (103) and (108). The Sugawara energy-momentum tensor is

\[ T(z) = : \partial \gamma(z) \beta(z) : + : \partial c(z) b(z) : + \partial C(z) B(z) : \]

\[ + \frac{1}{2} : \partial \varphi(z) \cdot \varphi(z) : + \frac{1}{\sqrt{t}} \partial_{a_2} \cdot \partial^2 \varphi(z) \]

Owing to the equal numbers of bosonic and fermionic generators, the central charge of the Sugawara tensor vanishes.

The screening currents of the first kind are easily found to be

\[ s_{a_1}(z) = - \left( \beta(z) + \frac{1}{2} c(z) B(z) \right) : e^{-\varphi_1(z)/\sqrt{t}} : \]

\[ s_{\tilde{a}_2}(z) = - \left( b(z) - \frac{1}{2} \gamma(z) B(z) \right) : e^{-\varphi_2(z)/\sqrt{t}} : \]

and it may be checked that indeed they have the required properties. These particular expressions do not seem to have appeared in the literature before.

Finally, the primary field of weight \( \Lambda \) is found to be

\[ \phi_{\Lambda}(w, x, \theta) \]

\[ = \left[ 1 + x^a \gamma(w) + \left( \frac{1}{2} x^{a_1} \theta_{a_2} + \theta^a \right) \left( \frac{1}{2} \gamma(w) c(w) + C(w) \right) \right]^\Lambda_1 \]

\[ \cdot \left[ 1 + \theta_{a_2} c(w) + \left( \frac{1}{2} x^{a_1} \theta_{a_2} - \theta^a \right) \left( \frac{1}{2} \gamma(w) c(w) - C(w) \right) - 2 \theta_{a_2} \theta^a c(w) C(w) \right]^\Lambda_2 \]

\[ : e^{\Lambda \varphi(w)} : \]
This expression follows from the observation that the basis vectors in \(M_{A^1}(A(1,0))\) and \(M_{A^2}(A(1,0))\) are proportional to

\[
|\Lambda^1\rangle, \quad F_{\alpha^1}|\Lambda^1\rangle, \quad f_{\dot{\alpha}^1}|\Lambda^1\rangle = f_{\dot{\alpha}^2}F_{\alpha^1}|\Lambda^1\rangle
\]

and

\[
|\Lambda^2\rangle, \quad f_{\dot{\alpha}^2}|\Lambda^2\rangle, \quad f_{\alpha^2}|\Lambda^2\rangle = -F_{\alpha^1}f_{\dot{\alpha}^2}|\Lambda^2\rangle, \quad f_{\dot{\alpha}^2}f_{\alpha^2}|\Lambda^2\rangle = -f_{\dot{\alpha}^2}f_{\alpha^2}|\Lambda^2\rangle
\]

The explicit expression for the primary field with arbitrary weight in (112) is a new result illustrating the general construction (87) in Section 6.

Let us now turn to the alternative choice of a set of simple roots where both simple roots are odd, \(\dot{\alpha}_1\) and \(\dot{\alpha}_2\). The remaining non-simple root \(\alpha = \dot{\alpha}_1 + \dot{\alpha}_2\) is even. This alternative (purely odd) set of simple roots, \(\dot{\alpha}_1^{(2)}\) and \(\dot{\alpha}_2^{(2)}\), is obtained from the (distinguished) one used above, \(\alpha_1^{(1)}\) and \(\dot{\alpha}_2^{(1)}\), by Weyl reflections (see e.g. 37) associated to the odd root \(\dot{\alpha}_2^{(1)}\). In particular, we find \(\dot{\alpha}_1^{(2)} = \alpha_1^{(1)} + \dot{\alpha}_2^{(1)} = \dot{\alpha}_1^{(1)}\) and \(\dot{\alpha}_2^{(2)} = -\dot{\alpha}_2^{(1)}\) such that \(\alpha^{(2)} = \alpha_1^{(1)}\), and the Weyl vector becomes \(\rho^{(2)} = 0\). This implies that

\[
\begin{align*}
H_1^{(2)} &= H_1^{(1)} + H_2^{(1)}, & H_2^{(2)} &= -H_2^{(1)} \\
E_{\dot{\alpha}_1}^{(2)} &= e_{\alpha_1}^{(1)}, & f_{\dot{\alpha}_1}^{(2)} &= f_{\alpha_1}^{(1)} \\
E_{\dot{\alpha}_2}^{(2)} &= f_{\dot{\alpha}_2}^{(1)}, & f_{\dot{\alpha}_2}^{(2)} &= -e_{\alpha_2}^{(1)} \\
E_{\alpha_1}^{(2)} &= F_{\alpha_1}, & F_{\alpha_1}^{(2)} &= F_{\alpha_1}^{(1)}
\end{align*}
\]

Using this correspondence between the Lie superalgebra generators, one may work out the structure coefficients

\[
\begin{align*}
f_{\dot{\alpha}_1,-\dot{\alpha}_1}^{1} &= 1, & f_{\dot{\alpha}_2,-\dot{\alpha}_2}^{2} &= 1 \\
\alpha(H_1) &= f_{1,\alpha} = 1, & \alpha(H_2) &= f_{2,\alpha} = 1 \\
f_{\alpha,-\alpha}^{1} &= 1, & f_{\alpha,-\alpha}^{2} &= 1 \\
f_{\pm\dot{\alpha}_1,\pm\alpha}^{\pm\dot{\alpha}_2} &= 1, & f_{\pm\dot{\alpha}_1,\pm\dot{\alpha}_1}^{\pm\dot{\alpha}_2} &= \pm 1, & f_{\pm\dot{\alpha}_2,\pm\alpha}^{\pm\dot{\alpha}_1} &= 1
\end{align*}
\]

and in particular the Cartan-Killing form

\[
\begin{align*}
G_{11} &= 0, & G_{12} &= G_{21} = 1, & G_{22} &= 0 \\
\kappa_{\alpha_1,-\alpha_1} &= \kappa_{\dot{\alpha}_2,-\dot{\alpha}_2} = \kappa_{\alpha,-\alpha} = 1
\end{align*}
\]

such that \(A_{ij} = G_{ij}\). Based on these parameters the alternative (fermionic) free field realization becomes

\[
\begin{align*}
H_1(z) &= -:c'(z)b'(z): - :\gamma(z)\beta(z): + \sqrt{t}\partial\varphi_1(z) \\
H_2(z) &= - :c(z)b(z): - :\gamma(z)\beta(z): + \sqrt{t}\partial\varphi_2(z) \\
e_{\alpha_1}(z) &= b(z) + \frac{1}{2}c'(z)\beta(z) \\
f_{\alpha_1}(z) &= -:\left(\frac{1}{2}c(z)c'(z) + \gamma(z)\right)b'(z): -\frac{1}{2}c(z) :\gamma(z)\beta(z): \\
&\quad + \sqrt{t}c(z)\partial\varphi_1(z) + (k + 1/2)\partial c(z)
\end{align*}
\]
$e_{\tilde{\alpha}_2}(z) = b'(z) + \frac{1}{2}c(z)\beta(z)$

$f_{\tilde{\alpha}_2}(z) = \left(\frac{1}{2}c(z)c'(z) - \gamma(z)\right)b(z) - \frac{1}{2}c'(z)\gamma(z)\beta(z) :$

$+ \sqrt{t}c'(z)\partial\varphi_2(z) + (k + 1/2)\partial c'(z)$

$E_\alpha(z) = \beta(z)$

$F_\alpha(z) = -c(z)\gamma(z)b(z) - c'(z)\gamma(z)b'(z) - \gamma^2(z)\beta(z) :$

$- \sqrt{t}\left(\frac{1}{2}c(z)c'(z) - \gamma(z)\right)\partial\varphi_1(z) + \sqrt{t}\left(\frac{1}{2}c(z)c'(z) + \gamma(z)\right)\partial\varphi_2(z)$

$- \frac{1}{2}(k + 1)\partial c(z)c'(z) - \frac{1}{2}(k + 1)\partial c'(z)c(z) + k\partial\gamma(z)$

(118)

where we have introduced the shorthand notation

$b(z) = b_{\tilde{\alpha}_1}(z), \quad c(z) = c^{\tilde{\alpha}_1}(z)$

$b'(z) = b_{\tilde{\alpha}_2}(z), \quad c'(z) = c^{\tilde{\alpha}_2}(z)$

$\beta(z) = \beta_\alpha(z), \quad \gamma(z) = \gamma^\alpha(z)$

(119)

Note that this Wakimoto realization is invariant under the interchanging of $i = 1$ and $i = 2$. The associated Sugawara energy-momentum tensor is

$T(z) =: \partial c(z)b(z) : + : \partial c'(z)b'(z) : + : \partial\gamma(z)\beta(z) : + \frac{1}{2} : \partial\varphi(z) \cdot \partial\varphi(z) :$

(120)

The screening currents of the first kind are

$s_{\tilde{\alpha}_1}(z) = -\left(b(z) - \frac{1}{2}c'(z)\beta(z)\right) : e^{-\varphi_1(z)/\sqrt{t}} :$

$s_{\tilde{\alpha}_2}(z) = -\left(b'(z) - \frac{1}{2}c(z)\beta(z)\right) : e^{-\varphi_2(z)/\sqrt{t}} :$

(121)

Similar realizations ((118) and (121)) in the fermionic basis of simple roots have also been obtained by Ito [7]. More recently, in [9] the relation is discussed between the Wakimoto free field realizations (108) and (118) of the affine currents based on the two inequivalent choices of simple roots.

Finally, the primary field of weight $\Lambda$ becomes

$\phi_\Lambda(w, x, \theta) = \left[1 + \theta^{\alpha_1}c(w) + \left(\frac{1}{2}\theta^{\tilde{\alpha}_1}\theta^{\tilde{\alpha}_2} - x^\alpha\right)\left(\frac{1}{2}c(w)c'(w) - \gamma(w)\right)\right]^{\Lambda_1}$

$\cdot \left[1 + \theta^{\alpha_2}c'(w) + \left(\frac{1}{2}\theta^{\tilde{\alpha}_1}\theta^{\tilde{\alpha}_2} + x^\alpha\right)\left(\frac{1}{2}c(w)c'(w) + \gamma(w)\right)\right]^{\Lambda_2}$

$\cdot : e^{\sqrt{t}\Lambda \varphi(w)} :$

(122)

Again, such an explicit result seems not to be found in the literature. In this case the basis vectors are proportional to

$|\Lambda^1\rangle, \ f_{\tilde{\alpha}_1}|\Lambda^1\rangle, \ F_\alpha|\Lambda^1\rangle = -f_{\tilde{\alpha}_2}f_{\tilde{\alpha}_1}|\Lambda^1\rangle$

(123)
and
\[ |\Lambda^2\rangle , f_{\alpha_2}|\Lambda^2\rangle , F_\alpha|\Lambda^2\rangle = -f_{\alpha_1}f_{\alpha_2}|\Lambda^2\rangle \] (124)

Inspired by the oscillator representation (103) and (115) we may choose the fundamental labels differently, namely as
\[ \Lambda^1(H_1) = 1 , \Lambda^1(H_2) = 0 , \Lambda^2(H_1) = 2 , \Lambda^2(H_2) = -1 \] (125)

as opposed to the diagonal choice above: \( \Lambda^k(H_i) = \delta^k_i \). Based on this new set \( \{ \Lambda^k \} \) we find the primary field
\[ \phi_{\Lambda}(w,x,\theta) = \left[ 1 + \theta^{\alpha_1}c(w) + \left( \frac{1}{2}\theta^{\alpha_1}\theta^{\alpha_2} - x^\alpha \right) \left( \frac{3}{2}c(w)c'(w) - \gamma(w) \right) \right]^{\Lambda_1+2\Lambda_2} \]
\[ \cdot \left[ 1 + 2\theta^{\alpha_1}c(w) - \theta^{\alpha_2}c'(w) + \left( \frac{3}{2}\theta^{\alpha_1}\theta^{\alpha_2} - x^\alpha \right) \left( \frac{3}{2}c(w)c'(w) - \gamma(w) \right) \right]^{-\Lambda_2} \]
\[ \cdot : e^{\frac{1}{4}A_{\Lambda,\varphi}(w)} : \] (126)

Again, such an explicit result seems not to be found in the literature. In this case the basis vectors are proportional to
\[ |\Lambda^1\rangle , f_{\alpha_1}|\Lambda^1\rangle , F_\alpha|\Lambda^1\rangle \] (127)

and
\[ |\Lambda^2\rangle , f_{\alpha_1}|\Lambda^2\rangle , f_{\alpha_2}|\Lambda^2\rangle , F_\alpha|\Lambda^2\rangle \] (128)

As a consistency check it is easily verified that for given (basis independent) labels \( \Lambda_1 \) and \( \Lambda_2 \) of \( \Lambda \), the two primary fields (122) and (126) are identical. This illustrates the freedom in choosing a convenient basis \( \{ \Lambda^k \} \) in weight space.

In [38] differential operator realizations and free field realizations are discussed in the case of the Lie supergroup \( OSp(2|2) \). We note that the realizations obtained there are different from ours (even for the Sugawara tensor) and are based on a different set of defining commutator relations for the Lie superalgebra \( osp(2|2) \). We don’t know the translations between them.

8 Conclusions

In this paper we have provided in particular 4 new results. First, we have derived general differential operator realizations of Lie superalgebras. Second, we have quantized these classical realizations and thereby obtained general free field realizations of affine current superalgebras. In this process the non-trivial part was to take care of multiple contractions by adding anomalous terms to the lowering operators. Third, we have worked out general expressions for screening currents of the first kind and presented proofs of their properties. Fourth, we have provided explicit generating function primary fields for arbitrary representations, based on super-triangular coordinates. Finally, we have compared the results with the literature and found them to be in accordance.

The results allow setting up integral representations for correlators of primary fields corresponding to integrable representations. When screening currents of the second kind
have been worked out, we then have sufficient ingredients for setting up integral representations for correlators in the case of degenerate representations. This would be of interest e.g. in the $G/G$ approach to non-critical strings. In the case of $OSp(1|2)$ the screening current of the second kind is known [34]. We intend to come back elsewhere with a discussion of screening currents of the second kind in more general situations [39] and of correlators for degenerate representations of $OSp(1|2)$ [40].

Very recently there has been rapidly increasing interest in realizations of $q$-deformed Lie (super-)algebras. However, most results only pertain to specific examples (see e.g. [41, 38] and references therein). It would be interesting to try to develop a general scheme for obtaining such realizations.

Acknowledgement

The author wants to express his sincere gratitude towards J.L. Petersen and M. Yu for the collaboration on the recent common paper [3] (and also on [14, 15]) of which the present work is a generalization. He also thanks J.L. Petersen and H.-T. Sato for fruitful discussions on $osp(1|2)$ in the early stages of this work, and L. Fehér for pointing out some references.

A  Polynomial Identities

This appendix is devoted to listing several polynomial identities following from the realizations obtained in Section 3 and Section 4, and used in Section 5 and Section 6. The terminology of classical and quantum polynomial identities originates in our viewpoint of affine current superalgebras being quantizations of the corresponding (classical) Lie superalgebras. Thus, identities obtained from comparison of the differential operator algebra with the Lie superalgebra (anti-)commutator relations, are denoted classical polynomial identities. Similarly, the identities obtained from comparing the free field realization with the defining OPE of the affine current superalgebra, are denoted quantum polynomial identities.

In this appendix we leave out the arguments of the polynomials. All root indices represent positive roots, while $a$ ($\dot{a}$) represents any even root or Cartan index (any odd root).

A.1  Classical Polynomial Identities

From the differential operator realization of the Lie superalgebra we derive the following set of classical polynomial identities

$$\pm \dot{a}(H_i) V_{\dot{a}}^\sigma = V_i^\gamma \partial_{\gamma} V_{\dot{a}}^\sigma - V_{\dot{a}}^\gamma \partial_i V_i^\sigma$$

$$-\ddot{a}(H_i) P_{-\dot{a}}^j = V_i^\gamma \partial_{\gamma} P_{-\dot{a}}^j$$

$$f_{\alpha,\beta}^\gamma V_{\gamma}^\sigma = V_{\dot{a}}^\gamma \partial_{\gamma} V_{\dot{a}}^\sigma - V_{\dot{a}}^\gamma \partial_i V_i^\sigma$$

$$f_{\alpha,-\beta}^\alpha V_{\dot{a}}^\sigma = V_{\dot{a}}^\gamma \partial_{\gamma} V_{-\dot{a}}^\sigma - V_{\dot{a}}^\gamma \partial_i V_i^\sigma$$

$$f_{\dot{a}},-\dot{a}^j = V_{\dot{a}}^\dot{\gamma} \partial_{\dot{\gamma}} P_{-\dot{a}}^j$$
Similarly, from the differential screening operator commutation relations (52) we find

\[
f_{\alpha,-\beta}^{-\gamma}P_{-\gamma}^{j} = V_{\alpha}^{\gamma}\partial_{\gamma}P_{-\beta}^{j}, \quad \text{for} \quad \beta - \alpha \in \Delta_{+}^{1}
\]
\[
f_{\alpha,\alpha}^{-\beta}V_{\beta}^{\gamma} = V_{\alpha}^{\gamma}\partial_{\gamma}V_{\beta}^{\gamma} - V_{\alpha}^{\gamma}\partial_{\gamma}V_{\alpha}^{\gamma}
\]
\[
f_{\alpha,-\alpha}^{-\beta}V_{\beta}^{\gamma} = V_{\alpha}^{\gamma}\partial_{\gamma}V_{-\alpha}^{\gamma} - V_{\alpha}^{\gamma}\partial_{\gamma}V_{\alpha}^{\gamma}
\]
\[
f_{\alpha,-\beta}^{-\beta}P_{-\beta}^{j} = V_{\alpha}^{\gamma}\partial_{\gamma}P_{-\beta}^{j}, \quad \text{for} \quad \alpha - \hat{\alpha} \in \Delta_{+}^{1}
\]
\[
f_{-\alpha,-\beta}^{-\gamma}V_{-\beta}^{\gamma} = V_{-\alpha}^{\gamma}\partial_{\gamma}V_{-\beta}^{\gamma} - V_{-\alpha}^{\gamma}\partial_{\gamma}V_{-\beta}^{\gamma}
\]
\[
f_{-\alpha,-\beta}^{-\beta}P_{-\beta}^{j} = V_{-\alpha}^{\gamma}\partial_{\gamma}P_{-\beta}^{j} - V_{-\alpha}^{\gamma}\partial_{\gamma}P_{-\beta}^{j}
\]
\[
f_{\alpha,-\beta}^{-\gamma}P_{-\gamma}^{j} = V_{\alpha}^{\gamma}\partial_{\gamma}P_{-\beta}^{j} + V_{\alpha}^{\gamma}\partial_{\gamma}P_{-\alpha}^{j}
\]
\[
f_{\alpha,-\beta}^{-\gamma}P_{-\gamma}^{j} = V_{\alpha}^{\gamma}\partial_{\gamma}P_{-\beta}^{j} - V_{\alpha}^{\gamma}\partial_{\gamma}P_{-\alpha}^{j}
\]
\[
f_{\alpha,-\beta}^{-\gamma}P_{-\gamma}^{j} = V_{\alpha}^{\gamma}\partial_{\gamma}P_{-\beta}^{j} + V_{\alpha}^{\gamma}\partial_{\gamma}P_{-\alpha}^{j}
\]
\[
(129)
\]

Of course, this set of relations can be written in a more compact way using obvious properties such as \(\alpha - \hat{\alpha} \neq 0\). In Section 5 and Section 6 the following ("compact") recursion relations prove themselves useful:

\[
f_{\alpha,\beta}^{-\gamma}V_{\gamma}^{\beta} = V_{\alpha}^{\gamma}\partial_{\gamma}V_{\beta}^{\gamma} - (-1)^{p(\hat{\alpha})p(\beta)}V_{\beta}^{\gamma}\partial_{\gamma}V_{\alpha}^{\gamma}
\]
\[
f_{\alpha,\beta}^{-\gamma}V_{\gamma}^{\beta} = V_{\alpha}^{\gamma}\partial_{\gamma}V_{\beta}^{\gamma} - (-1)^{p(\hat{\alpha})p(\beta)}V_{\beta}^{\gamma}\partial_{\gamma}V_{\alpha}^{\gamma}
\]
\[
f_{\alpha,\beta}^{-\gamma}V_{\gamma}^{\beta} = V_{\alpha}^{\gamma}\partial_{\gamma}V_{\beta}^{\gamma} - (-1)^{p(\hat{\alpha})p(\beta)}V_{\beta}^{\gamma}\partial_{\gamma}V_{\alpha}^{\gamma}
\]
\[
f_{\alpha,-\beta}^{-\gamma}P_{-\gamma}^{j} = V_{\alpha}^{\gamma}\partial_{\gamma}P_{-\beta}^{j} - (-1)^{p(\hat{\alpha})p(\beta)}V_{\beta}^{\gamma}\partial_{\gamma}P_{-\alpha}^{j}
\]
\[
f_{\alpha,-\beta}^{-\gamma}P_{-\gamma}^{j} = V_{\alpha}^{\gamma}\partial_{\gamma}P_{-\beta}^{j} - (-1)^{p(\hat{\alpha})p(\beta)}V_{\beta}^{\gamma}\partial_{\gamma}P_{-\alpha}^{j}
\]
\[
(130)
\]

Similarly, from the differential screening operator commutation relations (121) we find

\[
V_{\alpha}^{\gamma}\partial_{\gamma}S_{\beta}^{\sigma} - S_{\beta}^{\sigma}\partial_{\gamma}V_{\alpha}^{\gamma} = 0
\]
\[
V_{\alpha}^{\gamma}\partial_{\gamma}S_{\beta}^{\sigma} - S_{\beta}^{\sigma}\partial_{\gamma}V_{\alpha}^{\gamma} = \tilde{\beta}(H_{\tilde{\gamma}})S_{\beta}^{\sigma}
\]
\[
V_{\alpha}^{\gamma}\partial_{\gamma}S_{\beta}^{\sigma} = \tilde{\beta}(H_{\tilde{\gamma}})P_{-\beta}^{j}\partial_{\gamma}S_{\beta}^{\sigma} - f_{\beta,-\gamma}^{\mu}Q_{-\gamma}^{\mu}S_{\beta}^{\sigma} - f_{\beta,-\gamma}^{\mu}Q_{-\gamma}^{\sigma}S_{\beta}^{\mu}
\]
\[
S_{\beta}^{\sigma}\partial_{\gamma}P_{-\beta}^{j} = -f_{\beta,-\gamma}^{\mu}Q_{-\gamma}^{\mu}
\]
\[
(131)
\]

\[
V_{\alpha}^{\gamma}\partial_{\gamma}S_{\beta}^{\sigma} - (-1)^{p(\hat{\alpha})}S_{\beta}^{\gamma}\partial_{\gamma}V_{\alpha}^{\sigma} = \tilde{\beta}(H_{\tilde{\gamma}})P_{-\beta}^{j}\partial_{\gamma}S_{\beta}^{\sigma} - f_{\beta,-\gamma}^{\mu}Q_{-\gamma}^{\sigma}S_{\beta}^{\mu} - f_{\beta,-\gamma}^{\mu}Q_{-\gamma}^{\mu}S_{\beta}^{\sigma}
\]
\[
S_{\beta}^{\sigma}\partial_{\gamma}P_{-\beta}^{j} = (-1)^{p(\hat{\alpha})}f_{\beta,-\gamma}^{\mu}Q_{-\gamma}^{\mu}
\]
\[
V_{\alpha}^{\gamma}\partial_{\gamma}S_{\beta}^{\sigma} - (-1)^{p(\hat{\beta})}S_{\beta}^{\gamma}\partial_{\gamma}V_{\alpha}^{\sigma} = 0
\]
A.2 Quantum Polynomial Identities

\[
S^\alpha_\alpha \partial_\beta S^\beta_\beta - S^\beta_\beta \partial_\gamma S^\gamma_\gamma = f_{\alpha, \beta} \gamma S^\gamma_\gamma \\
S^\alpha_\alpha \partial_\gamma S^\gamma_\gamma - S^\beta_\beta \partial_\gamma S^\gamma_\gamma = f_{\alpha, \beta} \gamma S^\gamma_\gamma \\
S^\alpha_\alpha \partial_\gamma S^\gamma_\gamma + S^\beta_\beta \partial_\gamma S^\gamma_\gamma = f_{\alpha, \beta} \gamma S^\gamma_\gamma
\]

(131)

In the proof in Section 5 the following are useful “compactified” relations

\[
V^\gamma_{-\alpha} \partial_\gamma S^\gamma_\alpha \gamma - (-1)^{p(\alpha)p(\beta)} S^\gamma_\alpha \partial_\gamma V^\gamma_{-\alpha} = A_{ij} P^i_{-\alpha} S^\gamma_\alpha \gamma \\
S^\gamma_\alpha \partial_\gamma F^i_{-\alpha} = -(-1)^{p(\alpha)(1-p(\bar{\alpha}))} \delta^i_{-\bar{\alpha}} Q^\gamma_{-\bar{\alpha}}
\]

(132)

A.2 Quantum Polynomial Identities

From the field realization of the affine current superalgebra (and using the classical polynomial identities \( [29] \)) we derive the following set of quantum polynomial identities

\[
\bar{\beta}(H_i) F_{\bar{\beta}\gamma} = \bar{\gamma}(H_i) F_{\bar{\gamma}\beta} - V^\gamma_{-\alpha} \partial_{\theta} F^\gamma_{-\bar{\alpha}} \\
0 = tG_{ij} P^i_{-\bar{\beta}} + V^\gamma_{-\bar{\alpha}} \partial_{\theta} V^\gamma_{-\alpha} - \partial_{\theta} V^\gamma_{-\alpha} V^\gamma_{-\bar{\alpha}}
\]

(131)

\[
k_{\kappa_{\alpha, \beta}} = -\partial_{\gamma} V^\gamma_{\alpha} \partial_{\gamma} V^\gamma_{\beta} + \partial_{\theta} V^\gamma_{\alpha} \partial_{\gamma} V^\gamma_{\beta} + V^\gamma_{\alpha} F^\gamma_{\beta} \\
k_{\kappa_{\alpha, \beta}} = -\partial_{\alpha} V^\gamma_{\alpha} \partial_{\gamma} V^\gamma_{\beta} + V^\gamma_{\alpha} f_{\beta\gamma} \\
k_{\kappa_{\alpha, \beta}} = -\partial_{\alpha} V^\gamma_{\alpha} \partial_{\gamma} V^\gamma_{\beta} + V^\gamma_{\alpha} f_{\beta\gamma}
\]

(131)

\[
f_{\alpha, \beta} \gamma \gamma F_{\gamma\theta} = (-1)^{p(\bar{\alpha})p(\bar{\beta})} V^\gamma_{\alpha} \partial_\alpha F^\gamma_{\beta} + \partial_{\gamma} V^\gamma_{\alpha} F^\gamma_{\beta} \\
0 = -\partial_{\alpha} V^\gamma_{\alpha} \partial_{\gamma} V^\gamma_{\beta} + \partial_{\theta} V^\gamma_{\alpha} \partial_{\gamma} V^\gamma_{\beta} + tG_{ij} P^i_{-\alpha} P^j_{-\beta} + V^\gamma_{\alpha} F^\gamma_{\beta} + V^\gamma_{\beta} F^\gamma_{\alpha}
\]

(131)

\[
f_{-\alpha, \beta} \gamma \gamma F_{\gamma\theta} = (-1)^{p(\bar{\alpha})p(\bar{\beta})} \partial_{\alpha} V^\gamma_{\alpha} \partial_{\gamma} V^\gamma_{\beta} + tG_{ij} P^i_{-\alpha} P^j_{-\beta} + V^\gamma_{\alpha} \partial_{\gamma} f_{\beta\gamma} + V^\gamma_{\beta} \partial_{\gamma} f_{\alpha\gamma}
\]

(131)
and as

\[ 0 = - \partial_\mu V^{-\alpha}_{-\alpha} \partial_\gamma V^{\mu}_{-\beta} + (1)^{p(\tilde{\alpha})+p(\tilde{\beta})} \partial_\mu V^{-\alpha}_{-\alpha} \partial_\gamma V^{\mu}_{-\beta} + t G_{ij} P^i P^j \]

(135)

It is not difficult to show that (133) follows from (134).

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