Two (Known) Results About Graphs with No Short Odd Cycles

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Abstract

Consider a graph with $n$ vertices where the shortest odd cycle is of length $> 2k + 1$. We revisit two known results about such graphs:

(A) Such a graph is almost bipartite, in the sense that it can be made bipartite by removing from it $O((n/k) \log(n/k))$ vertices. While this result is known [GKL97] – our new proof seems to yield slightly better constants, and is (arguably) conceptually simpler.

To this end, we state (and prove) a version of CKR partitions [CKR04, FRT04] that has a small vertex separator, and it might be of independent interest. While this must be known in the literature, we were unable to find a reference to it, and it is included for the sake of completeness.

(B) While such graphs can be quite dense (e.g., consider a the bipartite clique, which has no odd cycles), they have a large independent set. Specifically, we prove that such graphs have independent sets of size $\geq (1 - o(1)) n^{k/(k+1)}$. Again, this result is known and is implied by the work of Shearer [She95], but our proof is simpler and (seems to) yield a better constant.

1. Graphs with no short odd cycles are almost bipartite

We start by proving a variant of a result of Fakcharoenphol et al. [FRT04] about CKR partitions [CKR04]. Fakcharoenphol et al. proved a bound on the probability of an edge to be cut by the random partitions of CKR (and thus the number of such edges being cut), while we are interested (somewhat imprecisely) in the number of boundary vertices. In particular, we show that a graph with $n$ vertices can be broken into disconnected clusters of diameter at most $\Delta$, by removing only $O((n/\Delta) \log(n/\Delta))$ vertices. The proof is slightly simpler, and the constants are slightly better than Fakcharoenphol et al. [FRT04], and is included for the sake of completeness – while we were unable to find a reference for it in the literature, we assume this result is known. See Theorem 1.3 for the precise statement.

We then use this to prove that graphs that have only long odd cycles can be converted into bipartite graphs by removing “few” vertices. See Theorem 1.4 below for the precise statement.

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1.1. Partitions with small separating set

Let \( G = (V, E) \) be an unweighted and undirected graph with \( n \) vertices. A partition of \( G \) is a set \( P = \{C_1, \ldots, C_m\} \) of disjoint subsets of \( V \) such that \( \bigcup_{C_i \in P} C_i = V \). The sets \( C_i \) are clusters. For \( x \in V \) and a partition \( P \), let \( P(x) \) denote the unique cluster of \( P \) containing \( x \).

In the following, for a vertex \( v \in V \), and an integer \( R \), let \( b(v, R) = \{ x \in V \mid d(v, x) \leq R \} \) be the ball of radius \( r \) centered at \( v \), where \( d(v, x) \) is the shortest path distance in \( G \) between \( v \) and \( x \) (i.e., it is the minimum number of edges on a path between \( v \) and \( x \) in \( G \)).

**Constructing the partition.** Let \( \Delta = 4\delta \) be a prescribed parameter, for some integer \( \delta \). Choose, uniformly at random, a permutation \( \pi \) of \( V \) and a value
\[
R \in \left[ \Delta/4 : \Delta/2 \right] = \{ \Delta/4, \Delta/4 + 1, \ldots, \Delta/2 \}.
\]
The partition is now defined as follows: A vertex \( x \in V \) is assigned to the cluster \( C_y \) of \( y \), where \( y \) is the first vertex in the permutation within distance \( \leq R \) from \( x \). Formally,
\[
C_y = \{ x \in V \mid x \in b(y, R) \text{ and } \pi(y) \leq \pi(z) \text{ for all } z \in V \text{ with } x \in b(z, R) \},
\]

Let \( P = \{C_y\}_{y \in G} \) denote the resulting partition. In words, once we fix the radius of the clusters \( R \), we start scooping out balls of radius \( R \) centered at the vertices of the random permutation \( \pi \). At the \( i \)th stage, we scoop out only the remaining mass at the ball centered at \( x_i \) of radius \( R \), where \( x_i \) is the \( i \)th vertex in the random permutation.

1.1.1. Properties

An edge \( uv \in E \) is a cross edge of a partition \( P \) if \( u \) and \( v \) belong to different clusters of \( P \). A vertex \( v \) of a partition \( P \) computed above is a guard if \( u \in C_y \in P \), and \( d(y,u) = R \). Let \( \text{guards}(P) \) be the set of all the vertices of \( V \) that are guards.

**Lemma 1.1.** Let \( P \) be a partition computed by the above scheme. We have the following:

(A) For any \( C \in P \), we have \( \text{diam}(C) \leq \Delta \), where \( \text{diam}(C) \) is the diameter of \( C \).

(B) Let \( uv \) be a cross edge of \( P \). Then \( u \) is a guard or \( v \) is a guard.

**Proof:**

(A) A cluster is a subset of a ball of radius \( R \), and as such its diameter is at most \( 2R \leq \Delta \).

(B) Assume \( u \in C_x \) and \( v \in C_y \), and \( x \) was before \( y \) in the permutation (the other case is handled symmetrically). If \( u \) is not a guard, then \( d(x, u) < R \). But then \( d(x, v) \leq d(x, u) + 1 \leq R \), and \( v \) would be in \( C_x \). A contradiction. \( \blacksquare \)

**Lemma 1.2.** Let \( G = (V, E) \) be an undirected and unweighted graph over \( n \) vertices, and let \( \Delta = 4\delta \) be a prescribed parameter (for some integer \( \delta \)), and let \( P \) be the random partition of \( G \) generated by the above scheme. Then, for any vertex \( x \in V \), we have that
\[
P[x \in \text{guards}(P)] \leq \frac{4}{\Delta} \ln \left| \frac{\left| b(x, \Delta/2) \right|}{\left| b(x, \Delta/4 - 1) \right|} \right|.
\]

**Proof:** Let \( U = b(x, \Delta/2) \), \( M = |U| \), and \( m = |b(x, \Delta/4 - 1)| \). Arrange the vertices of \( U \) in increasing distance from \( x \), and let \( w_1, \ldots, w_M \) denote the resulting order. Let \( \mathcal{E}_k \) for the event that \( w_k \) is the first vertex in \( \pi \) such that \( x \in C_{w_k} \) and \( d(w_k, x) = R \). Observe that if \( x \in \text{guards}(P) \), then one of the events \( \mathcal{E}_1, \ldots, \mathcal{E}_M \) must occur.

Let \( b(x, \Delta/4 - 1) = \{w_1, \ldots, w_m\} \). Note that if \( w_k \in b(x, \Delta/4 - 1) \), then \( \mathbb{P}[\mathcal{E}_k] = 0 \) since \( R > \Delta/4 - 1 \), which implies \( d(w_k, x) < R \). As such, for \( i = 1, \ldots, m \), we have \( \mathbb{P}[\mathcal{E}_1] = \cdots = \mathbb{P}[\mathcal{E}_m] = 0 \).
Observe that \( \mathbb{P}[\mathcal{E}_k] = \mathbb{P}[\mathcal{E}_k \cap (R = d(w_k, x))] = \mathbb{P}[R = d(w_k, x)] \cdot \mathbb{P}[\mathcal{E}_k | R = d(w_k, x)] \). Since \( R \) is uniformly distributed in the interval \( \left[ \Delta/4 : \Delta/2 \right] \), we have that \( \mathbb{P}[R = d(w_k, x)] \leq 1/(\Delta/4 + 1) \leq 4/\Delta \).

To bound \( \beta_k = \mathbb{P}[\mathcal{E}_k | R = d(w_k, x)] \), we observe that \( w_1, \ldots, w_{k-1} \) are closer (or of the same distance) to \( x \) than \( w_k \). Thus, if any of them appear before \( w_k \) in \( \pi \), then \( \mathcal{E}_k \) does not happen. Thus, \( \beta_k \) is bounded by the probability that \( w_k \) is the first to appear in \( \pi \) out of \( w_1, \ldots, w_k \). This probability is \( 1/k \), and thus \( \beta_k \leq 1/k \). We have that

\[
\mathbb{P}[\mathcal{E}_k] = \mathbb{P}[R = d(w_k, x)] \mathbb{P}[\mathcal{E}_k | R = d(w_k, x)] \leq \frac{4}{\Delta} \cdot \frac{1}{k}.
\]

We conclude that \( \mathbb{P}[x \in \text{guards}(P)] = \sum_{k=1}^{M} \mathbb{P}[\mathcal{E}_k] = \sum_{k=m+1}^{M} \mathbb{P}[\mathcal{E}_k] \leq \frac{4}{\Delta} \sum_{k=m+1}^{M} \frac{1}{k} \leq \frac{4}{\Delta} \ln \frac{M}{m}. \)

We thus get the following result, which is variant of the result of Fakcharoenphol et al. \[FRT04\].

**Theorem 1.3.** Let \( G = (V, E) \) be an unweighted and undirected graph with \( n \) vertices. Given a parameter \( \Delta = 4\delta \), for some positive integer \( \delta \), one can randomly partition the graph into clusters \( C_1, \ldots, C_t \), such that:

(A) Every cluster has a center vertex \( c_i \), such that for all \( x \in C_i \), we have \( d(c_i, x) \leq \Delta/2 \) (and thus, \( \text{diam}(C_i) \leq \Delta \)).

(B) There is a set of vertices \( X \), of expected size \( \leq \frac{4n}{\Delta} \ln \frac{4n}{\Delta} \), such that there is no edge between a vertex of \( C_i \setminus X \) and \( C_j \setminus X \), for all \( i \neq j \).

**Proof:** The construction is described above. We take \( X = \text{guards}(P) \), which by Lemma 1.1, is the desired separating set. In a connected graph, for any vertex \( x \in V \), we have \( |b(x, \Delta/2)| \leq n \) and \( |b(x, \Delta/4 - 1)| \geq \Delta/4 \). As such, by linearity of expectation and Lemma 1.2, we have

\[
\mathbb{E}[|\text{guards}(P)|] = \sum_{x \in V} \mathbb{P}[x \in \text{guards}(P)] \leq n \cdot \frac{4}{\Delta} \ln \frac{|b(x, \Delta/2)|}{|b(x, \Delta/4 - 1)|} \leq \frac{4n}{\Delta} \ln \frac{4n}{\Delta}. \]

### 1.2. Application: Graphs with no short odd cycles are almost bipartite

Odd cycles have an interesting hereditary property – if there is an odd cycle \( C \) that is not simple (or it has a chord), then there must be a shorter simple odd cycle in the graph:

(I) If \( C \) repeats a vertex twice, then it can be decomposed into two shorter cycles. One of these cycles must be odd.

(II) If \( C \) repeats an edge twice, then it repeats a vertex twice, and (I) applies.

(III) If there is an edge between two non-adjacent vertices of \( C \), then it can be split in a similar fashion.

Applying this argument repeatedly results in a simple odd cycle that is a sub-cycle of \( C \). Note, that this hereditary property does not hold for even length cycles.

In particular, using the above partition theorem, we get the following result of Győri et al. \[GKL97\] – our new proof seems to yield slightly better constants, and is conceptually simpler if Theorem 1.3 is a given.
Theorem 1.4. Let $G = (V, E)$ be an undirected graph over $n$ vertices, such that all the odd cycles in $G$ are of length $> 2k + 1$, for some integer $k$. Then, there is a set $X$ of at most $\frac{n}{\lceil k/2 \rceil} \ln \frac{n}{\lceil k/2 \rceil}$ vertices, such that removing $X$ from $G$ results in a bipartite graph.

Proof: Assume $G$ is connected, as otherwise we apply the argument below to each connected component.

Let $\delta = \lceil k/2 \rceil$, $r = 2\delta$, and let $\Delta = 4\delta$. Compute the partition $P$ of Theorem 1.3. This breaks the graph into $t$ clusters $C_1, \ldots, C_t$, where all the vertices of $C_i$ are in distance at most $r$ from its center $c_i$.

Observe that there are no two vertices $x, y \in C_i$, such that $t = d(c_i, x) = d(c_i, y)$ and $xy$ is an edge in the graph. If there is such an edge, then consider the cycle $\sigma = p(c_i, x) + xy + p(y, c_i)$, where $p(u, v)$ denotes the shortest path in $G$ between $u$ and $v$. The cycle $\sigma$ is an odd cycle in $G$ of length $d(c_i, x) + 1 + d(y, c_i) = 2t + 1 \leq 2r + 1 = 4\lceil k/2 \rceil + 1 \leq 2k + 1$, which is a contradiction.

An odd cycle can not be fully contained inside a cluster $C_i$, since it must contain an edge with both endpoints having the same distance to $c_i$, which is by the above impossible. Thus, all odd cycles in $G$ must use a cross edge of $P$. In particular, removing the set of vertices $X = \text{guards}(P)$ from $G$, implies by Lemma 1.1, that the remaining graph $G' = G \setminus X$ has no cross edges, and thus no odd cycles. Namely, $G'$ is bipartite. Theorem 1.3 implies that $E[|X|] \leq \frac{4n}{\Delta} \ln \frac{4n}{\Delta} = \frac{4n}{4\lceil k/2 \rceil} \ln \frac{4n}{4\lceil k/2 \rceil}$. \hfill \blacksquare

Corollary 1.5. Let $G = (V, E)$ be an undirected graph with $n$ vertices, such that all the odd cycles in $G$ are of length $> \varepsilon n$. Then one can remove $O(\varepsilon^{-1} \log \varepsilon^{-1})$ vertices from $G$ and make the graph bipartite. More precisely, if $\varepsilon n \geq 40$, then one can remove at most $(5/\varepsilon) \ln(5/\varepsilon)$ vertices and make the graph bipartite.

Proof: Let $k = \lceil (\varepsilon n - 1)/2 \rceil$. By Theorem 1.4, there is a set $X$ of size

$$\frac{n}{\lceil k/2 \rceil} \ln \frac{n}{\lceil k/2 \rceil} \leq \frac{n}{\varepsilon n/4 - 2} \ln \frac{n}{\varepsilon n/4 - 2} \leq \frac{5}{\varepsilon} \ln \frac{5}{\varepsilon},$$

since $\varepsilon n \geq 40$, such that removing $X$ from $G$ results in a bipartite graph. \hfill \blacksquare

2. Graphs with no short odd cycles have large independent sets

Interestingly, graphs with no short odd cycles have large independent sets, and these sets can be computed by a simple algorithm. The starting point is the greedy algorithm for independent set in a graph, which works by finding a vertex of low degree, removing itself and its neighbors, and repeating this till the graph is exhausted. The idea is to do better by considering not only the direct neighbors of a vertex, but rather inspecting the neighborhood of the vertex till a certain depth. The layers of the neighborhood of a vertex must grow quickly for each one of them not to be a good independent set to harvest, implying that sooner or later a good layer would be encountered.

The following result is known and is implied by the work of Shearer [She95], but our version is simpler and (seems to) yield a better constant.

Lemma 2.1. Let $k > 1$ be an integer parameter, and let $G = (V, E)$ be an undirected graph with $n$ vertices and $m$ edges, with all odd cycles in $G$ being of length $> 2k + 1$, for some $k \geq 1$. Then, $G$ has an independent set of size $(1 - o(1))n^{k/(k+1)}$, which can be computed in $O(n + m)$ time.

Proof: The algorithm repeatedly deletes vertices from the graph $G$ and add some of them to the computed independent set. For any vertex $v$ in the current graph $G$, let $L_i(v)$ be the set of vertices of $G$ at distance exactly $i$ from $v$ (as such, $L_{-1}(v) = \emptyset$ and $L_0(v) = \{v\}$). Let $d_i(v) = |L_i(v)|$ – in particular,
$d_0(v) = 1$, and $d_1(v) = d(v)$ is the degree of $v$. Let $X$ be a set of independent vertices that is initially empty.

Set $K = \lceil n^{1/(k+1)} \rceil$. The algorithm repeatedly picks a vertex $v$, and computes a BFS tree from $v$ level by level. For $j > 0$, after computing the $j$th level of the BFS tree (i.e., $L_j(v)$), the algorithm checks if $d_j(v) \leq Kd_{j-1}(v)$. If so, the algorithm adds $L_{j-1}(v)$ to the independent set $X$, and removes the vertices of $L_1(v) \cup L_{j-1}(v) \cup \cdots \cup L_0(v)$ from $G$. The algorithm does this till the graph is exhausted.

Observe, that if the algorithm harvested the set $L_{j-1}(v)$ to the independent set, then
\[
d_j(v) \leq Kd_{j-1}(v) \quad \text{and} \quad d_l(v) > Kd_{\ell-1}(v) \quad \forall \ell = 0, \ldots, j - 1.
\]

This implies that $d_{j-1}(v) > K^{j-1}$. As such, if $L_{k+1}(v)$ is being added by the algorithm to the output set (i.e., $j = k + 2$), this would imply that $d_{k+1}(v) > K^{k+1} \geq n$, which is impossible.

As such, the sets being added to the output set, are of the form $L_i(v)$, for $i \leq k$. Such a set is independent, as otherwise, there would be an edge $xy \in E$ between two vertices of $L_i(v)$, but that would imply an odd cycle in $G$ of length $\leq 2k + 1$. This readily implies that the computed set is indeed an independent set in the original graph.

As for the size of the independent set computed, observe that when $L_{j-1}(v)$ is being added to the output, the number of vertices being deleted is of size
\[
d_j(v) + d_{j-1}(v) + d_{j-2}(v) + \cdots + d_0(v) \leq \left( K + 1 + \frac{1}{K} + \frac{1}{K^2} + \cdots \right) d_{j-1}(v) \leq (K + 2)d_{j-1}(v)
\]

Namely, each vertex in the output independent set, pays for at most $K + 2$ vertices in the graph, and the computed independent set is of size $\geq n/(K + 2) = cn^{k/(k+1)}$, where
\[
c = \frac{n^{1/(k+1)}}{K + 2} \geq \frac{n^{1/(k+1)}}{n^{1/(k+1)} + 3} = 1 - \frac{3}{n^{1/(k+1)} + 3} = 1 - o(1).
\]

As for the running time of the algorithm, note that the BFS computation, implemented carefully, takes time proportional to the number of edges and vertices being removed. Thus the overall running time of the algorithm is linear.

Bock et al. [BFMR14] claim an $O(n^{2.5})$ time algorithm to find an independent set of size $\geq n^{k/(k+1)}/3$ for graphs with all odd cycles in $G$ being of length $> 2k + 1$. In comparison, the above improves both the running time and the approximation quality.

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