Quantification of spatial spread in track-based applications

Jacob C. H. Cheung*
Met Office, Exeter, UK

ABSTRACT: Traditionally, spaghetti charts and track ellipses are used in track-based meteorological applications to illustrate the spatial uncertainty of a set of paths on a map. However, the spatial spread of the set is often evaluated by the eye, and as a result, the decisions made can be highly subjective. A unit-less metric based on the Fréchet distance is developed and proposed as a consistent way of quantifying spatial uncertainty in track-based applications. The methodology described is generic and can be configured for use in applications where a measure of the spatial similarity among a set of tracks is required, such as tropical storm track forecasts and flight/ship routing.

KEY WORDS Fréchet distance; trajectory; aviation; storm track; ensemble; spaghetti

Received 1 October 2014; Revised 16 October 2015; Accepted 19 November 2015

1. Introduction

Owing to the chaotic nature of weather, and limits to model parameterizations and observation techniques, inaccuracies remain in the forecasts from the state-of-the-art numerical weather prediction (NWP) systems. Ensemble prediction systems (EPS) have been developed over the past 2 decades and are run operationally nowadays in most weather centres, such as the European Centre for Medium-Range Weather Forecasts (ECMWF), the Met Office and the National Centers for Environmental Prediction (NCEP). The core idea of an EPS is to run multiple forecasts in parallel, each with a small perturbation to the initial conditions and schemes (e.g. stochastic physics, multiple parameterizations) in order to account for model uncertainties (e.g. Toth and Kalnay, 1993; Molteni et al., 1996; Bowler et al., 2008), yielding an ensemble of weather forecasts. Other variants of ensemble are also available. Depending on the application, results from multiple NWP models can be combined to form a superensemble (e.g. Krishnamurti et al., 1999). Alternatively, it is possible to construct a time-lagged ensemble from consecutive runs of an NWP system to quantify the uncertainty in forecasts (e.g. Branković et al., 1990).

The statistical characteristics of the ensemble allow the uncertainty of a forecast to be quantified. Common EPS metrics include ensemble mean, ensemble spread, probability of specific event, quantities, spaghetti charts, postage stamp maps and site-specific meteograms (World Meteorological Organization, 2012).

There are meteorological applications in which existing EPS metrics fail to facilitate decision making. For instance, in tropical cyclone track forecasts, spaghetti charts and track ellipses are often used to estimate uncertainty. The uncertainty of the projected track location decreases with the geospatial convergence of the ensemble track forecast. However, the spatial spread of the tracks is often evaluated by the eye and can therefore be highly subjective. In the present study, a metric is proposed to quantify the spatial spread for a set of trajectories, which could potentially be an important decision support tool (DST) for applications such as flight/ship routing and tropical storm track forecasting.

The metric was developed initially for use in flight-routing applications, which will be the focus of the present study. However, it is important to note that the metric is generic and can be applied to other trajectory-related problems. The present study is structured as follows: Section 2 provides a brief background on flight trajectory prediction (TP); Section 3 describes the methodology; the results and conclusions are presented in Sections 4 and 5, respectively.

2. Trajectory prediction for aviation

TP is a core aspect of air traffic management (ATM). Although, unlike cars, aircraft are not confined to a road, pilots follow pre-defined trajectories whenever possible during the en route phase. The determination of these trajectories is based on factors such as flight intent, airspace capacity and restrictions and weather. In order to minimize fuel costs, airlines and air navigation service providers employ TP algorithms of different complexities, ranging from full six degrees of freedom to point mass and macroscopic models (Mondoloni, 2006), to seek the most efficient route based on wind and temperature. For instance, while the great circle is the shortest route, flying along a longer route with a strong tail wind might be faster. Also, the efficiency of jet engines is affected by environmental temperature. As a result, while the flight intent is clear, trajectories are not finalized until 0–3 days before flight execution, which is when short-range NWP forecasts become available. In aviation, the current industry standard is to use short-range deterministic NWP data as input for TP. This results in a single trajectory with no uncertainty information.

Owing to the chaotic nature of the atmosphere, using deterministic NWP data for TP often leads to re-routing, increasing fuel costs to airlines and pressure on air traffic control (ATC). The idea of ensemble forecasting has been introduced in flight planning recently. For example, the IMET project (Cheung et al., 2014, Cheung et al. 2015) explores different ways of using ensemble meteorological (MET) data in TP, which includes an...
ensemble approach of TP, that is, creating a trajectory from each of the ensemble MET forecasts to give an ensemble of trajectories. Other approaches of estimating uncertainty in TP include the use of a time-lagged ensemble (Lee et al., 2009).

Without radical improvements, it is likely that current ATM systems will fail to cope with the ever-growing demand for air traffic capacity within the next 20 years (Eurocontrol, 2013). For the efficient management of airspace capacity, it is inevitable that uncertainties in TP will need to be accounted for in the future. In the context of ensemble TP, there are two aspects of uncertainty, flight durations and geographical locations of ensemble trajectories. The former concerns how much extra fuel is required for a given flight and the latter is related to effective ATM.

3. Methods

To facilitate ATC, aircraft trajectories are verified en route against the planned trajectory. In this section, an explanation is given of why existing verification methods in the industry are not suitable for ensemble TP. Traditionally, there are two approaches: time-based and trajectory-based verification (e.g. Ryan et al., 2004; Mondoloni and Bayraktutur, 2005). Given that an aircraft follows its planned trajectory perfectly, time-based verification measures the difference between the proposed and actual time of arrival at a certain point along the trajectory. With ensemble TP, each member of the ensemble trajectory is directly related to its corresponding member of the MET forecast, which can be very different from each other. It is therefore possible that each trajectory member takes a completely different route to the same destination. As a result, time-based verification is not relevant to the present study.

Trajectory-based verification requires temporal synchronisation between the planned and actual trajectories. At any given time, this approach measures the along-track and side-track errors as a function of time. Consider Figure 1(a) where the dashed and solid lines indicate the planned and actual trajectories, respectively. The heavy arrow heads denote the times at which verification takes place; the side-track is uniform and there is no along-track error. Figure 1(b) considers the same set of trajectories but with an unexpectedly strong tailwind (headwind) at the start (end) of the actual trajectory. It can be seen that the actual time taken for the flight to arrive at its destination is the same as planned (as indicated by the last set of heavy arrowheads). In this case, even though the set of trajectories are identical to that in Figure 1(a), the along-track error varies along the trajectory. As a result, existing trajectory-based verification methods are not good for the quantification of the geometric spread of ensemble trajectories as they are time-dependent.

It is a common practice to measure the maximal or average Euclidean distance as a metric for the similarity between any two curves. For the purpose of differentiating flight trajectories, the Fréchet distance (Fréchet, 1906) is chosen over the Hausdorff (Hausdorff, 1949) distance as the latter does not account for backtracking is not allowed. This reduces the impact of uneven sampling along trajectories (Figure 2) and is superior over any other metrics that rely on distance calculations based on a waypoint-to-waypoint basis. It is also not a requirement that the trajectory pair in concern has the same number of sampling points.

As measuring the similarity between curves has been a challenge in many applications, research in the field has grown,

**Figure 1.** Schematic showing sets of planned (dashed) and actual (solid) trajectories. The heavy arrow heads indicate times at which trajectory-based verification takes place, and the light double-ended arrows indicate along- and side-track errors. (a) The aircraft (solid) travels at the same speed as planned (dashed). (b) The aircraft (solid) encounters strong tail wind at the beginning and then slows down due to headwinds towards the end.

**Figure 2.** Schematic showing identical sets of trajectory pair. The heavy arrow heads indicate the waypoints on each trajectory. The difference in the number of waypoints and spacing between the top and bottom trajectories is seen in each set. (a) The similarity of the two trajectories is measured by the distances between waypoint pairs and is highly dependent on the number of waypoints in the set. (b) The similarity is measured by the Fréchet distance (shown by the bold, double-ended arrow) and is robust against the sampling frequency. Also, it is no longer a requirement that the trajectory pair has the same number of waypoints.
and variants of the Fréchet distance have been developed. For example, the weak Fréchet distance (Alt and Godau, 1995) allows backtracking, while the homotopic Fréchet distance (Chambers et al., 2008) accounts for obstacles (imagine the man–dog scenario but in a woodland). Considering the nature of flight trajectories, the discrete Fréchet distance as defined in Eiter and Mannila (1994), which is a computationally efficient approximation to the Fréchet distance, is used in the present study. Following the methodology described in Mascret et al. (2006), for a given pair of trajectories \( L_1 \) and \( L_2 \) with \( n \) and \( m \) waypoints, respectively, a Fréchet matrix \( M_F \) of size \( n \times m \) is computed with its element defined as follows:

\[
M_F(i,j) = \max \left( \frac{d_F(L_1(i), L_2(j))}{\min(M_F(i-1,j), M_F(i,j-1), M_F(i-1,j-1))} \right)
\]

where \( 1 \leq i \leq n \), \( 1 \leq j \leq m \) and \( d_F(L_1(i), L_2(j)) \) denotes the Euclidean distance between the \( i^{th} \) waypoint on \( L_1 \) and \( j^{th} \) waypoint on \( L_2 \). After iterating through all the possible combinations of \( i \) and \( j \), the discrete Fréchet distance \( d_F \) between \( L_1 \) and \( L_2 \) is given by the last element in \( M_F \), that is, \( M_F(n, m) \). A step-by-step example of the algorithm is given in Mascret et al. (2006).

It should be noted that \( d_F \) is only defined for a pair of curves and is not applicable for a set of trajectories. Next, the man–dog scenario but with multiple dogs is considered. Each dog is connected to the man via a different leash. Dumitrescu and Rote (2004) pointed out that the distance between any dog–dog or man–dog pair throughout the journey is bounded by the sum of the lengths of the two longest leashes the man holds. While the method described in Dumitrescu and Rote (2004) is a good approximation of the Fréchet distance for a set of curves, it describes only the range but not the distribution of the spread. Figure 3(a) shows an extreme scenario where the Fréchet distance for the set of trajectories equals its upper bound (indicated by the sum of the double-ended arrows), that is, the sum of the length of the two longest leashes the man (solid) holds. Then a scenario is considered in which all the ‘dogs’ (dotted) converge on to the trajectory of the ‘man’ (solid) except for the topmost and bottommost ‘dogs’. The upper bound of the Fréchet distance is the same for both set of trajectories despite the difference in their spatial distribution. In TP, the scenario shown in Figure 3(b) provides more confidence compared with that in Figure 3(a) as the majority of ensemble trajectories converge on to the same route, while the topmost and bottommost trajectories become outliers. For convenience, the expression ‘man’ refers to the leading/reference trajectory and ‘dogs’ refers to the rest of the trajectories in the set for the rest of the present study.

As no existing variant of the Fréchet distance is specifically suitable for the task, a new metric \( \sigma_N \) is proposed:

\[
\sigma_N = \frac{\sigma_{D_F}}{L_{gc}} \times 100\%
\]

where \( \sigma_{D_F} \) is the standard deviation of the discrete Fréchet displacement of each member of the set from a reference trajectory, and \( L_{gc} \) is the length of the great circle between the starting and ending positions. \( D_F \) is the discrete Fréchet displacement and is associated with a sign relative to the reference trajectory:

\[
D_F = I \times d_F
\]

where \( d_F \) is the discrete Fréchet distance as in Equation (1). For simplicity, \( I \) is defined to be 1 if the majority of waypoints of the ‘dog’ are north of the corresponding ones in the reference trajectory and to be \(-1\) if otherwise. Extending the definition of the Fréchet distance to a one-dimensional (1D) vector in the north–south direction is found to be sufficient for a transatlantic ensemble TP in general. The approach is generic, meaning \( I \) can be defined in the east–west direction or any pair of opposite directions.

For an ensemble of size \( k \) with \( m \) waypoints in each member and a reference trajectory with \( \eta \) waypoints, the computing time for \( \sigma_N \) is \( O(kn \eta m) \). There is no restriction on \( m \), that is, it can vary among the ‘dogs’. However, to avoid the issues illustrated in Figure 1, it is advisable to sample the waypoints per unit distance travelled rather than per unit time lapsed.

In the rest of the present study, the choice of the reference trajectory is explored to justify the definition of \( D_F \). The ensemble trajectories presented here are the most efficient routes based on wind and temperature forecasts from the Met Office Global and Regional Ensemble Prediction System (MOGREPS). Unless otherwise stated, the ensemble size is 12 in the scenarios considered here. Equations (3) and (4) are independent of how the ensemble trajectories are generated and can be applied to any set of ensemble flight trajectories.

4. Results

4.1. Choice of ‘man’

As mentioned in Section 1, ensemble NWP involves running perturbed forecasts in conjunction with the deterministic forecast (also known as the control member) to form an ensemble. It follows that TP can be performed recursively using each member of the ensemble NWP as the input, generating an ensemble of trajectories. Each member of the ensemble trajectories correspond to its respective member of the ensemble MET forecast. While it might be intuitive to use the control trajectory (one that corresponds to the deterministic MET forecast) as the ‘man’ for the calculation of \( \sigma_N \), it is not always the best choice. It is demonstrated below that an inappropriate choice of ‘man’ can lead to misleading answers.

Figure 4 shows a 58 h TP from London Heathrow Airport or Heathrow (EGLL) to Miami International Airport (KMIA) valid at 1000Z on 26 March 2014. All the panels share the same set of predicted trajectories. The only difference among the plots is
Figure 4. Ensemble TP for flights from London Heathrow Airport or Heathrow (EGLL) to Miami International Airport (KMIA). Trajectories shown are $t + 58\text{h}$ TP valid at 10Z on 26 March 2014 and the same set of trajectories is used for all subplots. For the calculation of $\sigma_N$, a different member from the same set of ensemble trajectories is used as the reference trajectory (dashed). It follows that the rest of the ensemble members are treated as ‘dogs’ (solid). The resultant value of $\sigma_N$ is displayed at the bottom right in each panel.

Figure 5. Ensemble TP for flights from London Heathrow Airport or Heathrow (EGLL) to Miami International Airport (KMIA). Trajectories shown are $t + 58\text{h}$ TP valid at 10Z on 28 February 2014. The ensemble mean trajectory (dashed) is used as the ‘man’, and all members of the ensemble including the control are treated as ‘dogs’ (solid). The $\sigma_N$ is the normalized standard deviation of the Fréchet displacement defined in Equations (3) and (4). The calculation of $\sigma_{N_{\text{Scalar}}}$ is similar but represents the Fréchet distance.

that a different member of the ensemble trajectory is regarded as the ‘man’ for the calculation of $\sigma_N$. At first glance, there is a convergence of trajectories and an obvious outlier. Depending on the choice of ‘man’, $\sigma_N$ varies from 6.824% to 7.490% except when the outlier is chosen as the ‘man’ (Figure 4 (panel Man: 5)), in which a value of 1.544% is reported for the same set of trajectories.

In the case where the outlier is regarded as the ‘man’ (Figure 4 (panel Man: 5)), all the ‘dogs’ travel closely together while staying away from the ‘man’. Although the leashes are all relatively long, they do not differ much in length. This is reflected in the small value of $\sigma_N$, which is a normalized measure of the standard deviation of the leashes. For completeness, in the cases where any other trajectory is regarded as the ‘man’ (Figure 4 (panel Man: 0–4 and 6–11)), there are 10 short leashes of similar length and one long leash connecting the outlying ‘dog’, hence a larger $\sigma_N$.

From daily observations (not shown), there is no guarantee that the spatial distribution of the ensemble trajectories is centred around the control (deterministic). In fact, the control trajectory is sometimes the obvious outlier. For a given set of trajectories, the uncertainty in $\sigma_N$, that results from the choice of ‘man’ can be removed by using a reference trajectory that is invariant, for example, the mean ensemble trajectory or the great circle. For the rest of the present study, the mean ensemble trajectory will be used as the ‘man’.

4.2. The Fréchet ‘displacement’

The Fréchet distance is originally defined as a scalar. In Equation (4), the concept of the Fréchet distance is extended to a 1D vector, that is, the Fréchet displacement. While the Fréchet distance is effective in measuring the similarity between two curves, an explanation on why the current definition is insufficient for a set of curves is given below. Figure 5 shows...
the TP from EGLL to KMIA valid at 1000Z on 28 February 2014. Members of the set of ensemble trajectories, including the control, are shown as solid lines. It is clear that the predicted trajectories diverge soon after departure, forming roughly two convergences of trajectories. Intuitively, the two groups of trajectories are far away from each other and the spatial distribution is uneven. This is reflected by a large \( \sigma_N \), which works out to be 24.317%. If the Fréchet distance is considered (i.e. \( I = 1 \) in Equation (4)), the proposed metric returns a value of 3.640% (for clarity, this is referred to as \( \sigma_{NScalar} \)), which is significantly different from \( \sigma_N \).

Referring to Figure 5, the dashed line shows the mean trajectory of the ensemble and is used as the ‘man’ for the calculation of both \( \sigma_N \) and \( \sigma_{NScalar} \). As the Fréchet distance is a scalar, the calculation of \( \sigma_{NScalar} \) is equivalent to the scenario as shown in Figure 5 but with the bottom convergence of the ‘dogs’ (solid) flipped about the ‘man’ (dashed) such that all the ‘dogs’ are on the same side of the ‘man’ (i.e. all taking a high latitude route compared to that of the ‘man’); using the Fréchet displacement, however, will result in a larger value for \( \sigma_N \), reflecting the broad spatial coverage of the ensemble trajectories.

4.3. Maximum versus average Fréchet distance

The calculations of \( \sigma_D \) in the present study are based on the discrete Fréchet displacement, \( D_F \). Depending on the application, it is possible to use other variants of the Fréchet distance given that the issues described in Section 4.2 are addressed accordingly. The use of a Fréchet distance variant is examined, namely the average discrete Fréchet distance \( D_{FAvg} \) (Mascret et al., 2006) for flight trajectories.

The original definition of the discrete Fréchet distance, \( D_F \), is essentially the full length of the shortest possible leash required for the man–dog pair to complete the journey. It follows that for a given \( D_F \), there are multiple ways for the man–dog pair to complete their journey as there is no requirement for the leash to be fully stretched at all times. Mascret et al. (2006) defined the average discrete Fréchet distance \( D_{FAvg} \) as the average leash length for the journey, assuming the man–dog pair will try to stay as close to each other as possible at all times.

Figure 6(a) shows an example of two completely different trajectories. The maximum \( D_F \) and average \( D_{FAvg} \) Fréchet distance between the two is 1888 and 972 km, respectively. For flight TP, the resultant trajectories are generally smooth; \( D_{FAvg} \) is found to be generally about half of \( D_F \) as in Figure 6(a), and \( \sigma_N \) is also scaled accordingly (not shown). This does not affect the consistency of using \( \sigma_N \) as a DST in TP because the difference between \( D_F \) and \( D_{FAvg} \) is systematic for smooth trajectories. Moreover, airspace users have the freedom to choose their own threshold of \( \sigma_N \), which depends on both their uncertainty requirement and the variant of the Fréchet distance used.

An extreme case where the two trajectories are identical except that there is an unrealistic detour in one of them (dashed) is shown in Figure 6(b). Despite the fact that the trajectories are largely similar, \( D_F \) is found to be 492 km (~12 times larger than that of \( D_{FAvg} \)). Hence, for trajectories that are not always smooth (e.g. historical flight trajectories), \( D_{FAvg} \) should be used for the calculation of \( \sigma_N \).

For transatlantic ensemble flight TP, the definition in Equations (3) and (4) (i.e. the standard deviation of discrete Fréchet displacement from the mean ensemble trajectory normalized by the length of the great circle) is found to be sufficient to quantify the spatial spread of the trajectories consistently. For instance, \( \sigma_N \leq 1 \% \) is found to be an appropriate indicator of low spatial spread for transatlantic ensemble flights’ TP at the Met Office. The chosen threshold value is application-specific. Users are advised to adjust the threshold to meet their requirements. For demonstration purposes, cases with different levels of spatial spread are shown in Figure 7.

The \( \sigma_N \) is not designed specifically for cluster analysis. However, the concept of discrete Fréchet displacement can be extended by applying clustering techniques to the \( D_F \) values of the ensemble in order to obtain detailed information on the clustering of the trajectory ensemble, which is out of the scope of the present study and is planned for future work.

5. Conclusion

The Fréchet distance is a mathematical tool used commonly to quantify the similarity between a pair of curves. While variants of the Fréchet distance are available, the majority are defined only for a pair of curves and are not applicable for larger set of curves. The aim of the present study is therefore to describe a consistent and comprehensive way of quantifying the similarity amongst a set of trajectories. A new unit-less metric \( \sigma_N \) is proposed and defined as the normalized standard deviation of the discrete Fréchet displacements from an invariant reference trajectory. It is also found that extending the original definition of the Fréchet distance
distance from a scalar to a vector is essential for the avoidance of misleading results.

Although the present study focuses on the application of the proposed metric to flight routing, the methodology described is generic and can be configured to use in other applications where a measure of the similarity amongst a set of trajectories is required.

Acknowledgements

The author would like to thank Piers Buchanan, Edmund Henley, Paul Maisey and Helen Wells for commenting on an early draft of the paper.

References

Alt H, Godau M. 1992. Measuring the resemblance of polygonal curves. In Proceedings of the Eighth Annual Symposium on Computational Geometry. ACM: New York, NY: 102–109.

Alt H, Godau M. 1995. Computing the Fréchet distance between two polygonal curves. Int. J. Comput. Geom. Appl. 5 (1–2): 75–91.

Bowler NE, Arribas A, Mylne KR, Robertson KB, Beare SE. 2008. The MOGREPS short-range ensemble prediction system. Q. J. R. Meteorol. Soc. 134: 703–722.

Branković Ĉ, Palmer TN, Molteni F, Tibaldi S, Cubasch U. 1990. Extended-range predictions with ECMWF models: time-lagged ensemble forecasting. Q. J. R. Meteorol. Soc. 116: 867–912.

Chambers EW, Colin de Verdière E, Erickson J, Lazarz S, Lazarus F, Thite S. 2008. Walking your dog in the woods in polynomial time. In Proceedings of the 24th Annual Symposium on Computational Geometry, held in University of Maryland in College Park, Maryland, June 9–11 2008., held in University of Maryland in College Park, Maryland, June 9–11 2008., ACM: New York, NY: 101–109.

Cheung JCH, Brenguier JL, Heijstek J, Marsman A, Wells H. 2014. Sensitivity of flight durations to uncertainties in numerical weather prediction. In 6th SESAR Innovation Days, Madrid, Spain.

Cheung JCH, Brenguier JL, Halla A, Heijstek J, Marsman A. 2015. Recommendations on trajectory selection in flight planning based on weather uncertainty. In 5th SESAR Innovation Days, Bologna, Italy.

Dumitrescu A, Rote G. 2004. On the Fréchet distance of a set of curves. In Proceedings of the 16th Canadian Conference on Computational Geometry (CCCG’04), Montreal, August 9–11 2004: 162–165.

Eiter T, Mannila H. 1994. Computing discrete Fréchet distance. Tech. Report CD-TR 94/64, Information Systems Department, Technical University of Vienna.

Eurocontrol. 2013. Challenges of growth 2013 – Task 4: European air traffic in 2035. Technical Report, Eurocontrol. https://www.eurocontrol.int/sites/default/files/content/documents/official-documents/reports/201307-challenges-of-growth-summerly-report.pdf (accessed 26 March 2014).

Fréchet M. 1906. Sur quelques points du calcul fonctionnel. Rend. Circ. Mat. Palermo (1884–1940) 22(1): 1–72.

Hausdorff F. 1949. Grundzüge der Mengenlehre. 1914. Viet: Leipzig.

Krishnamurti TN, Kishitawal CM, LaRow TE, Bachiochi DR, Zhang Z, Williford CE, et al. 1999. Improved weather and seasonal climate forecasts from multimodel superensemble. Science 285(5433): 1548–1550.

Lee AG, Weygangt SS, Schwartz B, Murphy JR. 2009. Performance of trajectory models with wind uncertainty. In AIAA Modeling and Simulation Technologies Conference, Chicago, IL.

Mascet A, Devoge T, Le Berre I, Hénaff A. 2006. Coastline matching process based on the discrete Fréchet distance. In SDH 2006 Conference. 12th International Symposium on Spatial Data Handling. Springer-Verlag: Vienna, Austria: 383–400.

Molteni F, Buizza R, Palmer TN, Petroliagis T. 1996. The ECMWF ensemble prediction system: methodology and validation. Q. J. R. Meteorol. Soc. 122(529): 73–119.

Mondoloni S. 2006. Aircraft trajectory prediction errors: including a summary of error sources and data (version 0.2). FAA/EUROCONTROL Action Plan 16. Eurocontrol.

Mondoloni S, Bayraktutar I. 2005. Impact of factors, conditions and metrics on trajectory prediction accuracy. In 6th USA/Europe Air Traffic Management R&D Seminar, Baltimore, MD.

Ryan HF, Paglione MM, Green SM. 2004. Review of trajectory accuracy methodology and comparison of error measurement metrics. AIAA Guidance, Navigation, and Control Conference and Exhibit, 16–19 August 2004, Rhode Island. American Institute of Aeronautics and Astronautics: Reston, VA.

Toth Z, Kalnay E. 1993. Ensemble forecasting. Technical Report CD-TR 94/64, Information Systems Department, Technical University of Vienna.

Thite S. 2008. Walking your dog in the woods in polynomial time. In Proceedings of the 24th Annual Symposium on Computational Geometry, held in University of Maryland in College Park, Maryland, June 9–11 2008., ACM: New York, NY: 101–109.

Toth Z, Kalnay E. 1993. Ensemble forecasting. Technical Report CD-TR 94/64, Information Systems Department, Technical University of Vienna.