Quantum Noise in Amplifiers and Hawking/Dumb-Hole Radiation as Amplifier Noise

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The quantum noise in a linear amplifier is shown to be thermal noise. The theory of linear amplifiers is applied first to the simplest, single or double oscillator model of an amplifier, and then to linear model of an amplifier with continuous fields and input and outputs. Finally it is shown that the thermal noise emitted by black holes first demonstrated by Hawking, and of dumb holes (sonic and other analogs to black holes), arises from the same analysis as for linear amplifiers. The amplifier noise of black holes acting as amplifiers on the quantum fields living in the spacetime surrounding the black hole is the radiation discovered by Hawking. For any amplifier, that quantum noise is completely characterized by the attributes of the system regarded as a classical amplifier, and arises out of those classical amplification factors and the commutation relations of quantum mechanics.

I. INTRODUCTION

Linear amplifiers, devices which take in a signal and produce an output signal of a different amplitude, are ubiquitous, but in general seem to be poorly understood. All produce noise, but again the source of that noise tends to be poorly understood, and seems often based on a case by case analysis. While often an amplifier can have excess noise, caused by some infelicity in its construction, all amplifiers must have a minimum level of noise, set by quantum mechanics. This was recognized by Haus and Mullen[1] and other [2] [3] [4] half a century ago, but the lesson bears repeating. In particular, the noise is often roughly characterized by a temperature. We will see that this is exact– the noise output is thermal. Furthermore, black holes turn out to simply be an unusual instance of one of these amplifiers.

By a linear amplifier I mean a device into which one feeds an input signal \( I(t) \) and out of which comes an output signal

\[
\int A(t-t')I(t')dt'
\]

In principle, that amplification

\( A(t-t') \)

could be a function of both \( t \) and \( t' \) rather than just their difference. Such an amplifier is in general a phase sensitive amplifier, for example one which amplifies the cosine component of the input differently from the output, but I will here be interested in phase insensitive amplifiers as above.

As the simplest model of an amplifier let me first assume that the input and the output are both single modes, defined by the input canonical variables \( x, p \) and output \( Y, P \). They are not time dependent signals– they are simply single degrees of freedom, which change in time only due to the dynamical Hamiltonian for that degree of freedom. The only “signal” is the value of that variable. An example of an amplification would be that the output degree of freedom, represented by \( Y \) and \( P \) is \( A \) times the input.

\[
Y = Ax
\]
\[
P = Ap
\]

No matter what the input, the output is \( A \) times larger. However, it is clear that this amplifier is unrealizable. Any amplifier is some physical device which produces a unitary transformation between the input and the output. In particular, if \( Y \) and \( P \) are conjugate variables, we have

\[
[Y, P] = A^2[x, p] = A^2i\hbar
\]

But clearly this is the correct commutation relation only if \( A^2 = 1 \), and the amplification is trivial. In order to have such an amplifier one cannot have only one input channel. One needs at least two. (In this case a “channel” just

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means another degree of freedom.)

\[ Y = Ax + Bq \]  \hspace{1cm} (4)
\[ P = Ap + E\theta \]  \hspace{1cm} (5)

where \( \theta \) is the conjugate momentum to \( q \). Demanding that \([Y, P] = i\hbar\) then leads to

\[ A^2 + BE = 1 \]

We can always do a canonical transformation of the form \( e^{\eta q} \) and \( e^{-\eta \theta} \) to have

\[ E = -B \]  \hspace{1cm} (6)

so that

\[ A^2 - B^2 = 1 \]  \hspace{1cm} (7)

or \( A = \cosh(\mu) \) and \( B = \sinh(\mu) \).

Clearly if one has two input channels, one also needs two output channels as well. Let me designate the second output channel by \( Z, \tilde{P} \). Then in order that the commutation relations of \( Z \) and \( \tilde{P} \) and \( Y \) and \( P \) be maintained, we need

\[ Z = Aq + Bx \]  \hspace{1cm} (8)
\[ \tilde{P} = A\theta - Bp \]  \hspace{1cm} (9)

(One could also have other canonical transformations of \( Z, \tilde{P} \) which would of course leave the commutation relations the same, but this is the simplest case, and using one of the others do not change the results).

This is more easily expressed in terms of creation and annihilation operators Defining

\[ a = \frac{x + ip}{\sqrt{2}}, \]  \hspace{1cm} (10)
\[ b = \frac{q + i\theta}{\sqrt{2}}, \]  \hspace{1cm} (11)
\[ C = \frac{Y + iP}{\sqrt{2}}, \]  \hspace{1cm} (12)
\[ D = \frac{Z + i\tilde{P}}{\sqrt{2}} \]  \hspace{1cm} (13)

we have

\[ C = \cosh(\mu) a + \sinh(\mu) b^\dagger \]  \hspace{1cm} (14)
\[ D = \cosh(\mu) b + \sinh(\mu) a^\dagger \]  \hspace{1cm} (15)

or

\[ a = \cosh(\mu) C - \sinh(\mu) D^\dagger \]  \hspace{1cm} (16)
\[ b = \cosh(\mu) D - \sinh(\mu) C^\dagger \]  \hspace{1cm} (17)

This is the form of a Bogoliubov transformation.

Note that the commutation relations are maintained if we multiply \( a \) or \( b \) by phase

\[ C = \cosh(\mu)e^{i\nu} a + \sinh(\mu)e^{-i\kappa} b^\dagger \]  \hspace{1cm} (18)
\[ D = \cosh(\mu)e^{i\kappa} b + \sinh(\mu)e^{-i\nu} a^\dagger \]  \hspace{1cm} (19)

or multiply each term overall by a phase factor. In the following I will not follow this complication, since all it does is to make the equations messier.

Let us now assume that the input states of the two input modes, represented by \( a, b \) are thermal states, with input density matrices

\[ \rho_a = N_a e^{-\Lambda_a a^\dagger a} \]  \hspace{1cm} (20)
\[ \rho_b = N_b e^{\Lambda_b b^\dagger b} \]  \hspace{1cm} (21)
\[ N_a \text{ and } N_b \text{ are normalisation factors equal to } \]
\[ N_a = 1 - e^{-\Lambda_a} \]  
(22)

and, if \( a \) has frequency \( \omega_a \), then \( \Lambda_a = \frac{\omega_a}{T_a} \) where \( T_a \) is the temperature of the \( a \) input channel.

The density matrix in terms of the output annihilation and creation operators \( C, D \) is then

\[ \rho = N_a N_b e^{-\Lambda_a a^\dagger a - \Lambda_b b^\dagger b} \]
\[ = N_a N_b e^{-\Lambda_a a^\dagger a - \Lambda_b b^\dagger b} \]
\[ = N_a N_b e^{-\Lambda_a (\cosh(\mu) C^\dagger - \sinh(\mu) D)(\cosh(\mu) C - \sinh(\mu) D^\dagger) - \Lambda_b \cosh(\mu) D^\dagger - \sinh(\mu) C^\dagger (\cosh(\mu) D - \sinh(\mu) C)} \]
\[ = N_a N_b e^{-(\Lambda_a) \cosh^2(\mu) - \Lambda_b \sinh^2(\mu)(C^\dagger C) + (\Lambda_a + \Lambda_b) \cosh(\mu) \sinh(\mu) (C D + C^\dagger D^\dagger)} \]
(26)

Taking the trace over the \( D \) using the complete set of \( m \) particle states \( D^\dagger D|m\rangle = m|m\rangle \), \( \sum_m \langle m|\rho|m\rangle \) and expanding the exponential in a power series, we see that each term must have the same number of \( D \) as \( D^\dagger \) operators. Since each \( D \) in the expansion is multiplied by either a \( D^\dagger \) or a \( C \) and \( D^\dagger \) by \( D \) or \( C^\dagger \), one must thus also have the same number of \( C \) as \( C^\dagger \) operators. The reduced density matrix is thus a function only of \( C^\dagger C \) (after an appropriate number of commutations). It is in fact an exponential function in \( C^\dagger C \) (see appendix) of the form \( e^{-\Lambda e^{C^\dagger C}} \), where we can find \( \Lambda_C \) from

\[ \text{Tr} C^\dagger C N_C e^{-\Lambda_C C^\dagger C} = -N_C \partial_{\Lambda C} \text{Tr}(e^{-\Lambda_C C^\dagger C}) \]
\[ = \frac{e^{-\Lambda_C}}{1 - e^{-\Lambda_C}} \]  
(27)

Thus

\[ \frac{e^{-\Lambda_C}}{1 - e^{-\Lambda_C}} = \text{Tr}(N_a N_b \cosh(\mu)^2 a^\dagger a \sinh(\mu)^2 (b^\dagger b + 1) + \cosh(\mu) \sinh(\mu)(ab + a^\dagger b^\dagger) e^{-\Lambda_a a^\dagger a - \Lambda_b b^\dagger b}) \]
(29)

\[ = \frac{\cosh(\mu)^2}{e^{\Lambda_a} - 1} + \sinh(\mu)^2 \left( \frac{1}{e^{\Lambda_b} - 1} + 1 \right) \]  
(30)

since the \( a, b \) modes are uncorrelated. For small \( \Lambda_{a,b} \) we get

\[ \frac{1}{\Lambda_C} = \frac{\cosh(\mu)^2}{\Lambda_a} + \sinh(\mu)^2 \left( \frac{1}{\Lambda_b} + 1 \right) \]
(31)

If the inputs and outputs all have the same frequency \( \omega \) then \( \Lambda = \frac{\omega}{T} \) and this becomes

\[ T_C = \cosh(\mu)^2 T_a + \sinh(\mu)^2 (T_b + \omega) \]
(32)

and if the amplification is large, so that \( \cosh(\mu) \approx \sinh(\mu) \),

\[ \frac{1}{\Lambda_C} = \frac{\cosh(\mu)^2}{\Lambda_a} \left( \frac{1}{\Lambda_b} + 1 \right) \]
(33)

or

\[ T_C = \cosh(\mu)^2 (T_a + T_b) \]
(34)

(recall that \( \cosh(\mu) \) is the amplification factor \( \Lambda \)).

For \( \Lambda_{a,b} \) large (which corresponds to low temperatures) , we have

\[ \frac{1}{e^{\Lambda_C} - 1} = \cosh(\mu)^2 e^{-\Lambda_a} + \sinh(\mu)^2 \approx \sinh(\mu)^2 \]
(35)

or

\[ \frac{\omega_C}{-2\ln(\tanh(\mu))} \]
(36)

This is exact in the limit as the input temperatures go to zero (\( \Lambda_{a,b} \rightarrow \infty \)). I.e, for low temperatures in the inputs (temperatures much less than the input frequencies) , the output temperature is determined by the amplification and the frequency of the output solely. This is a quantum noise. Assuming the output has frequency \( \omega_C \) we have

\[ T_C = \frac{\omega_C}{-2\ln(\tanh(\mu))} \]  
(37)
Note that the output thermal noise due to the amplification of the vacuum fluctuation is given purely by the amplification $\cosh(\mu)$ and the frequency of the output, both of which are determined purely by the classical behaviour of the amplifier. The quantum behaviour of the amplifier is a "classical" effect, in that it depends only on the classical attributes of the amplifier. That the expression for the temperature includes both $\frac{\hbar}{k_B}$ does not alter the fact that it is completely determined by classical measurements.

If we put the input into a coherent state, $a|\alpha\rangle = \alpha|\alpha\rangle$, $b|\alpha\rangle = 0$, then

$$\langle \alpha|C|\alpha\rangle = \cosh(\mu)\alpha$$  \hspace{1cm} (38)

where $\alpha$ can be as large as desired. I.e., by measuring the output for a classical input, one can determine the parameter of the amplifier which determines the noise output of the amplifier.

Alternatively one could have a situation in which one takes $D$ as the output channel to be measured with $a$ still being the input channel. Then the amplification of $a$ in the $D$ output is

$$\langle \alpha|D|\alpha\rangle = \sinh(\mu)\alpha$$  \hspace{1cm} (39)

I.e., for small $\mu$ the "amplification" goes to zero, rather than to 1.

It is also of interest to note that while the two inputs are, by assumption, statistically independent (no correlations between $a$, $b$) the outputs are not. Even in the vacuum input, we have

$$\langle CD \rangle = \langle 0|\cosh(\mu)\sinh(\mu)(a^\dagger b^\dagger)|0\rangle = \cosh(\mu)\sinh(\mu)$$  \hspace{1cm} (40)

which implies a correlation (entanglement) between the $C$ and $D$ outputs.

That same entanglement implies that we could have "noiseless" (i.e., not altering the signal to noise ratio of the input signal) by choosing an input state which was an entangled state—i.e. such that in the output state, the $C$ and $D$ modes were in a product state. One would then have a noiseless (zero temperature) output. This is in general not possible, because the input signal comes from somewhere that one is not able to correlate its quantum fluctuations with the other input to the amplifier. However in certain situations, in which the signal is a classical signal imposed on a quantum input channel, this may allow one to reduce the noise in a detector by choosing an appropriately correlated set of input channels, as for example in and interferometric gravity wave detection in which the gravity wave signal, a very large "classical" source, affects a quantum input channel in the electromagnetic field in the arms of an interferometer.

II. CONTINUUM

While the above "two mode" analysis is important, it is also instructive to examine a model for a continuous phase insensitive amplifier—i.e., one with a continuous input and an continuous amplified output.

Let us define the Lagrangian

$$L = \frac{1}{2} \int \left[ \dot{\phi}^2 - (\partial_x \phi)^2 - (\dot{\psi}^2 - (\partial_x \psi)^2) + 2\dot{q}(\alpha \dot{\phi} + \bar{\epsilon} \psi)\delta(x - \lambda) \right] dx$$  \hspace{1cm} (41)

$$+ \frac{1}{2}(q^2 + \Omega^2 q^2)$$  \hspace{1cm} (42)

with reflection boundary condition at $x = 0$ of $\partial_x \phi(t,0) = \partial_x \phi(t,0) = 0$. $\lambda$ is assumed to be very small, and we will take the limit as $\lambda \to 0$.

I could have taken $x$ to be a continuous variable with the field propagating from infinity to $+\infty$ and the oscillator located at $x = 0$, but in that case all of the antisymmetric modes for the $\phi$ and $\psi$ fields would not have interacted at all with the oscillator. I take the oscillator at $\lambda$ and take the limit as $\lambda$ goes to zero to ensure that I am properly treating the interaction. The $\delta$ at $x = 0$ is not well defined on the half line $x \geq 0$.

The $\psi$ field has negative definite energy, which is the source for the energy amplification which accompanies the amplifier. (Note that while this particular model amplifies energy that is not necessary for an amplifier, as we will see below)

The equations of motion for the field are

$$\partial_t^2 \phi - \partial_x^2 \phi = \epsilon \dot{\partial}_1 q\delta(x - \lambda)$$  \hspace{1cm} (43)

$$\partial_t^2 \psi - \partial_x^2 \psi = -\bar{\epsilon} \dot{\partial}_1 q\delta(x - \lambda)$$  \hspace{1cm} (44)

$$\partial_t^2 q + \Omega^2 q = -\epsilon \dot{\partial}_1 \phi(t,\lambda) - \bar{\epsilon} \dot{\partial}_1 \psi(t,\lambda)$$  \hspace{1cm} (45)
where \( \Xi = \{ \phi(t,x) + \phi_0(t,-x) + \frac{1}{2} \epsilon \left\{ \begin{array}{c} q(t-x-\lambda) + q(t-x+\lambda) ; x > \lambda \\ q(t+x-\lambda) + q(t-x-\lambda) ; x < \lambda \end{array} \right\} \} \) (46)

\( \psi(t,x) = \psi_0(t-x) + \psi_0(t+x) - \frac{1}{2} \epsilon \left\{ \begin{array}{c} q(t-x-\lambda) + q(t-x+\lambda) ; x > \lambda \\ q(t+x-\lambda) + q(t-x-\lambda) ; x < \lambda \end{array} \right\} \) (47)

\( \partial_t^2 q + \Omega^2 q + \frac{\epsilon^2 - \epsilon_0^2}{2} \partial_t (q(t) + q(t-2\lambda)) = -\epsilon \partial_t (\phi_0(t-\lambda) + \phi_0(t+\lambda)) - \tilde{\epsilon} \partial_t (\psi_0(t-\lambda) + \psi_0(t+\lambda)) \) (48)

where I will only be interested in the limit as \( \lambda \to 0 \). Taking the Fourier transform of the resulting equations where \( q(t) = \int q_\omega e^{-i\omega t} \), we have

\[ q_\omega = 2i\omega \frac{(\epsilon \phi_0(\omega) + \tilde{\epsilon} \psi_0(\omega))}{-\omega^2 - i(\epsilon^2 - \epsilon_0^2)\omega + \Omega^2} \] (49)

We take \( \phi_0(t+x) \) and \( \psi_0(t+x) \) as the ingoing modes (the \( x \) and \( q \) of the above simple two mode analysis) and

\[ \phi_{out} = \psi_0(t-x) + \epsilon q(t-x) \] (50)

\[ \psi_{out} = \psi_0(t-x) - \tilde{\epsilon} q(t-x) \] (51)

are the outgoing modes corresponding to \( C, D \) of the simple two mode analysis.

Thus we find

\[ \phi_{out}(\omega) = \psi_0(\omega) + 2i\omega \epsilon \frac{\epsilon \phi_0(\omega) + \tilde{\epsilon} \psi_0(\omega)}{-\omega^2 - i(\epsilon^2 - \epsilon_0^2)\omega + \Omega^2} \] (52)

The conserved norm for the system is

\[ \langle \Xi', \Xi \rangle = i \left[ \int (\phi^{**} \partial_t \phi - \psi^{**} \partial_t \psi) dx + q^{**} (\partial_t q + \epsilon \phi(t,0) + \tilde{\epsilon} \psi(t,0)) \right. \]

\[ \left. - \left( \int \partial_t q^{**} \phi - \partial_t \psi^{**} \psi) dx + (\partial_t q^{**} + \epsilon \phi^{**}(t,0) + \tilde{\epsilon} \psi^{**}(t,0))q \right) \] (53)

where \( \Xi = \{ \phi, \psi, q \} \) designates a complete solution of the equations of motion at any time \( t \). This norm is conserved by the equations of motion and relates the ingoing modes at \( t \to -\infty \) to the outgoing at \( t \to +\infty \).

Note the sign of the \( \psi \) term in the norm. This arises from the fact that the conjugate momentum for the \( \psi \) field is \( -\partial_t \psi \).

The quantization of the fields is such that positive norm fields are associated with annihilation operators while the negative norm fields are associated with creation operators. In the case of the \( \psi \) field, the vacuum state, annihilated by the annihilation operators

\[ a|0\rangle = 0 \] (55)

is also a maximum energy, rather than a minimum energy state. Also, while as usual, the positive norm states for the \( \phi \) fields are the positive frequency states, \( e^{-i\omega t} \) the positive norm states for the \( \psi \) field are negative frequency states \( e^{i\omega t} \). Thus, the outgoing positive norm \( \phi \) states are linear combinations of the ingoing positive norm \( \phi_0 \) states, and ingoing negative norm \( \psi \) states, and the annihilation operators of the outgoing \( \phi \) field are linear combinations of the annihilation of the ingoing \( \phi \) field and creation operators of the ingoing \( \psi \) field. This is precisely the situation examined in the first section.

The fact that the \( \psi \) field has negative energy is clearly an approximation in any real world situation, as the energy will not go to \( -\infty \) in reality. However, in amplifiers, the system is often set up such that some of the modes of the system are just this type of negative energy modes at least for small enough perturbations of the system. In a laser for example, pumping the atoms to their excited state gives a systems whose small fluctuations are exactly the above sort of negative energy harmonic oscillator. At large amplitudes, those modes will saturate and the system will become non-linear (when a significant portion of the atoms have made the transition from the excited state to the ground state). Thus this model is not a good model for all regimes of such an amplifier, but is a good approximation as long as one is concerned only with its small signal behaviour. Note that that small signal regime can be one in which the excitations of the \( \psi \) field are much much larger than the size of the mean quantum or thermal noise in that field.
The annihilation operators for a specific mode $\phi_i(t-x)$, assumed in the distant future to be far from the origin $x = 0$ is

$$a_{\phi_i} = \langle \phi_i, \Phi \rangle = i \int (\phi_i^* \Pi \Phi(t,x) - i \partial_t \phi_i(t-x)^* \Phi(t,x)) \, dx$$

(56)

where $\Phi$ and $\Pi \Phi$ are the quantum field and conjugate momentum operators in the Heisenberg representation. The annihilation and creation operators obey the usual commutation relation $[a_{\phi_i}, a^\dagger_{\phi_i}] = i \langle \phi_i, \phi_i \rangle$, which, if $\phi_i$ is normalized, is the usual commutation relation for annihilation operators. Similarly, the annihilation operator for a $\psi$ mode

$$a_{\psi_j} = \langle \psi_j, \Phi \rangle = i \int (\psi_j^* \Pi \Psi(t,x) + i \partial_t \psi_j(t-x)^* \Psi(t,x)) \, dx$$

(57)

The relation between the outgoing modes of the $\psi$ and $\phi$ fields to the ingoing are then

$$a_{\phi,\omega,\text{out}} = a_{\phi,\omega,\text{in}} \left[ \frac{-\omega^2 + \Omega^2 - i \omega (\epsilon^2 + \tilde{\epsilon}^2)}{-\omega^2 + \Omega^2 - i \omega (\epsilon^2 - \tilde{\epsilon}^2)} \right] + a_{\psi,\omega,\text{in}}^\dagger \left[ \frac{\epsilon \tilde{\epsilon} \omega}{-\omega^2 + \Omega^2 - i \omega (\epsilon^2 - \tilde{\epsilon}^2)} \right]$$

(58)

$$= A_\omega a_{\phi,\omega,\text{in}} + B_\omega a_{\psi,\omega,\text{in}}^\dagger$$

(59)

where the amplification factors obey

$$|A_\omega|^2 - |B_\omega|^2 = 1$$

(60)

as required.

The amplification $|A_\omega|^2$ is maximized when $\omega = \Omega$ and is there equal to $\left( \frac{\epsilon^2 + \tilde{\epsilon}^2}{\epsilon^2 - \tilde{\epsilon}^2} \right)^2$ and thus to get a large amplification we require that $\epsilon^2 - \tilde{\epsilon}^2 \ll \epsilon^2 + \tilde{\epsilon}^2$.

An interesting situation occurs if we take the central oscillator to be a free particle, so that $\Omega = 0$. The amplification is then roughly constant for a frequency range around $\omega = 0$ until $\omega \approx \epsilon^2 - \tilde{\epsilon}^2$ and then falls at 6dB/octave for frequencies higher than that until $\omega \approx \epsilon^2 + \tilde{\epsilon}^2$ at which frequency the amplification goes to 1. This is just the behaviour one has for many amplifiers—e.g. the gain curve of a transistor amplifier.

The quantum noise temperature for the $\Omega = 0$ case (assuming both of the input modes are at zero temperature) and assuming $\epsilon^2 - \tilde{\epsilon}^2 << 2\epsilon^2$ is given by

$$T = -\frac{\omega}{\ln \left( \frac{|B|^2}{|A|^2} \right)} = -\frac{\omega}{\ln \left( \frac{4\epsilon^2 \tilde{\epsilon}^2}{\omega^2 - (\epsilon^2 + \tilde{\epsilon}^2)} \right)}$$

(61)

$$\approx \left\{ \begin{array}{ll}
\frac{4\epsilon^2 \tilde{\epsilon}^2}{\omega} & : \omega << 2\epsilon^2 \\
\frac{\omega}{\ln \left( \frac{\omega}{\omega^2} \right)} & : \omega >> 2\epsilon^2
\end{array} \right.$$
smaller than \( \omega \) and the number of particles in those modes will be much less than unity (ie, one is in the Wien tail of the distribution).

Note that I have assumed that both the output and input channels are the \( \phi \) field. One could also choose the input channel to be the \( \phi \) field and the output the \( \psi \) field in which case \( B_\omega \) would be the amplification factor instead, and the amplification would fall to zero, rather than 1 for very large frequencies.

What is clear is that this simple model for a continuum amplifier captures many of the features of a real amplifier and can be applied to a wide variety of amplifiers\[7\].

III. BLACK HOLES

One of the more fascinating forms of amplifier is that provided by a black hole\[8\]. Hawking\[9\] showed that the amplification factor instead, and

\[
\phi_{\omega,\text{out}} = \frac{e^{\beta\omega/2}}{\sinh(\beta\omega/2)} \phi_{\omega,\text{in}}^+ + \frac{e^{-\beta\omega/2}}{\sinh(\beta\omega/2)} \phi_{\omega,\text{in}}^- \tag{63}
\]

where \( \phi_{\omega,\text{out}} \) is a positive norm mode escaping from the black hole with frequency \( \omega \), and \( \phi_{\omega,\text{in}}^+ \) is an mode ingoing toward the star which will eventually form the black hole made up entirely of positive norm positive frequency modes of the ingoing the field, and \( \phi_{\omega,\text{in}}^- \) are modes made up entirely of the usual negative norm, negative frequency ingoing modes. The relation between the ingoing and outgoing modes is unusual in that the relation between the ingoing energies and outgoing energies is bizarre.

Let us take as a model for black hole formation the collapse of a spherically symmetric null shell of dust. Before the collapse, spacetime is flat, with null coordinates \( U \) and \( V \) and with "radius" (\( \frac{1}{2\pi} \) of the circumference of the spheres of spherical symmetry) given by \( r = (V - U)/2 \). Let me choose the coordinate \( v \) so that \( V = 0 \) corresponds to the shell of dust. Outside the shell the metric is Schwartzschild, with null coordinates \( u,v \) with \( (v - u)/2 = (r - 2M) + 2M \ln(\frac{r - 2M}{2M}) \), and with the null shell given by \( v = 0 \). Along the null shell the requirement that the circumferential radius be continuous across the shell gives us the relation between the \( U \) and \( u \) coordinates as

\[
u = (+U + 4M) - 4M \ln \left( \frac{(U + 4M)}{4M} \right) \tag{64}
\]

Thus, if we have a wave-packet of the form

\[
\phi_{\text{out}} = S(u)e^{i\omega u} \tag{65}
\]

where \( S \) is a relatively slowly varying envelope concentrated around \( u >> 4M \) we will have that the form for the incoming wave function will be

\[
\phi_{in}(V) \approx S \begin{cases} 4M \ln(\frac{4M - V}{4M}) (\frac{4M - V}{4M})^{-i4M\omega} & ; V < -4M \\ 0 & ; V > -4M \end{cases} \tag{66}
\]

This can be written in terms of the ingoing positive norm modes which are pure linear combinations of the ingoing positive norm modes \( e^{-i\Omega V} \) with \( \Omega > 0 \).

\[
\phi_{a,\omega} = \begin{cases} \frac{e^{2\pi M\omega}}{\sinh(4\pi M\omega)} (\frac{-4M - V}{4M})^{-i4M\omega} & ; V < -4M \\ \frac{e^{-2\pi M\omega}}{\sinh(4\pi M\omega)} (\frac{4M + V}{4M})^{i4M\omega} & ; V > -4M \end{cases} \tag{68}
\]

\[
\phi_{b,\omega} = \begin{cases} \frac{e^{-2\pi M\omega}}{\sinh(4\pi M\omega)} (\frac{-4M - V}{4M})^{i4M\omega} & ; V < -4M \\ \frac{e^{2\pi M\omega}}{\sinh(4\pi M\omega)} (\frac{4M + V}{4M})^{-i4M\omega} & ; V > -4M \end{cases} \tag{69}
\]

where in each case \( \omega > 0 \). These two positive norm modes correspond to the \( a, b \) incoming modes discussed in the first section.

Note that in this case the two types of mode \( a, b \) correspond to different types of the same incoming field \( \phi_{in} \). The frequency of the ingoing mode which goes as \( \frac{(4M + V)}{4M} \) is approximately

\[
\Omega \approx i\partial_V \ln \left( \frac{(4M + V)}{4M} \right) \approx \frac{4M\omega}{4M + V} \approx \omega e^{\pi/4M} \tag{70}
\]
Ie, the frequency of the incoming mode which creates an outgoing mode of frequency $\omega$ at retarded time $u$ is exponential in that retarded time. For example for a retarded time 1 second after a solar mass black hole forms, the incoming frequency corresponding to an outgoing frequency of $\omega$ is about $\omega e^{10^5}$ which is $e^{10^5}$ times a frequency corresponding to the mass of the whole universe. Thus the energy of the incoming modes which are amplified with an amplification factor of $\frac{e^{2M\omega}}{\sqrt{2\sinh(4\pi M\omega)}}$ by the black hole amplifier, have their energy decreased by a factor of $e^{-\frac{u}{4M}}$. Amplification does not imply energy amplification. The black hole, as an amplifier, amplifies the amplitudes (norms) by a thermal amplification factor, but de-amplifies the energy by a term with is exponential in the time after the black hole forms. It is this feature of the black hole as an amplifier that makes it unique.

IV. DUMB HOLES

In 1981 I\cite{10} suggested that many of the features of the black hole particle creation could also be captured by what I have since called Dumb holes– analogs in condensed matter system which have horizons and mimic many of the features of black holes, including the output of quantum noise as the analog of Hawking radiation. In the case of Hawking radiation, if one traces back the the emitted radiation into the past, it is squeezed against the horizon exponentially until one gets back to the time when the black hole was originally formed. Only then can the backward propagating modes escape toward infinity, but with absurdly high frequencies and short wavelengths.

Dumb holes (named after the original usage of the term to mean “unable to speak”, and not the more modern usage to mean “stupid”– ie, “Stumm” not “Dumm” in German, or “Muet” not “Stupide” in French), are systems in which the flow of some background material is such that some wave in that material, which has a non-trivial dispersion relation, to experience a horizon, a surface through which the low frequency components of the wave cannot travel from inside to out because the ingoing velocity of the material equals the outgoing velocity of the wave. Such systems experience quantum noise in the same was as black holes do– ie they emit a thermal spectrum of radiation of that quantized wave whose temperature is determined, instead of by the mass as in the black hole case, by the nature of that background flow at the horizon.

In the case of dumb holes, the same thing happens as for black holes initially (finally since we are tracing the modes backward in time?) and the backward-in-time modes are exponentially squeezed against the horizon. But at sufficiently short wavelength the dispersion relation and in particular the group velocity of the modes changes and the mode escapes from the horizon with very short wavelengths. Depending on the nature of the dispersion relation the outgoing mode can either be dragged in from large distances because the group velocity is now much less than the velocity of the fluid far from the horizon in the case where the dispersion relation makes the group velocity small at high frequencies, or can now travel out from inside in the case in which the dispersion relation has a group velocity at high frequencies much larger than the velocity of the fluid inside the horizon.

If we assume the flow to be stationary, the waves see a time independent situation, and, while the wavelength can change drastically, the frequency is conserved in the lab frame (but not in the fluid frame). In Figure 2 we have a graph of such a dispersion relation in a still fluid, in which the group velocity falls at high frequency. If the fluid moves with some velocity which is smaller than the long wavelength speed of the wave in still water, the dispersion relation looks as in figure 3, while if the fluid is moving with a velocity higher than the long wavelength speed of the wave, figure 4 gives the dispersion relation.

As an example, let us assume we have a 1+1 dimensional wave in a fluid whose still fluid dispersion relation is

$$\omega^2 = F^2(k)$$

In the moving fluid the dispersion relation will be

$$\omega = vk \pm F(k)$$

where the two branches represent the left and right moving waves. We will be interested in the minus sign which will represent waves which are trying to move against the flow.

The Lagrangian for such a fluid could be given by

$$\mathcal{L} = \int (\partial_t \phi - v(x) \partial_x \phi)^2 - (F(i\partial_x)\phi)^2$$

where I have assumed that $F$ is an odd function of $k$.

The norm is again

$$\langle \phi, \phi \rangle = i \int \phi^* (\partial_t \phi - v(x) \partial_x \phi) dx + CC$$
FIG. 2: Figure 2. Dispersion relation for a “subluminal” case in still fluid. (This is in fact the relation for surface waves on water, with $\omega = \sqrt{gktanh(kh)}$)

FIG. 3: Figure 3. Dispersion relation for waves as in figure 1 with the flow rate being ‘subluminal” (ie, slower than the velocity of the waves at zero wave-vector). The horizontal line is for frequency 1, and the three possible wave-vectors are $k_0^+$ which is a positive norm wave, both of whose phase and group velocity is to the right. $k_1^-$ has positive phase velocity and negative group velocity, and has positive norm. $k_2^-$ has negative phase and group velocity and negative norm.

If we assume $v$ to be a constant, and $\phi(t,x) = \phi_ke^{-i\omega t - kx}$ then the norm will be

$$<\phi, \phi> \propto 2(\omega - vk)|\phi_k|^2$$

(75)

The important point to note is that the norm is positive or negative, depending on the sign of $\omega - vk$. Since $\omega$ is given by the above dispersion relation, we have

$$<\phi, \phi> = 2(F(k))|\phi_k|^2$$

(76)
FIG. 4: Figure 4. Dispersion relation for waves as in figure 1 with the fluid flowing faster than the velocity of the still water waves at small wave-number. There is only one possible wave-vector $k_4$ for the given frequency, and it has negative phase and group velocities and negative norm.

I.e., the sign of the norm depends not on the value of $\omega$ but the sign of the still water dispersion function for that mode. Modes with positive $\omega$ can have negative norm, and modes with negative $\omega$ can have positive norm.

Let us imagine that we have a flow where the fast flow occurs to the right and the slow flow to the left. There will be a horizon between the two. Now consider modes with the frequency indicated in the diagrams. The modes which have group velocity toward the horizon are the modes with wave-vectors $k_1$ and $k_2$. We see, that just as in the above continuum model of an amplifier, the ingoing modes with a given positive frequency $\omega$ come in two flavours, the ones with positive norm ($k_0$ and $\phi_0$) and ones with negative norms, but the same positive frequencies ($k_3$ and $\psi_0$). The outgoing modes—travelling away from the horizon, or away from the oscillator in the continuum model—also come in the same two flavours, the positive norm modes ($k_1$ and $\phi_{out}$) and the negative norm outgoing modes ($k_2$ and $\psi_{out}$). In the continuum model the coupling between these modes which leads to amplification is the harmonic oscillator. In this dumb hole model, the coupling between the modes is provided by the non-adiabatic, spatially dependent changes in the background flow given by the changing velocity $v(x)$.

In the case of the dumb hole, but not in the oscillator coupling, the effective temperature of the emitted quantum noise from this amplifier has a constant temperature, independent of frequency, at least for low frequencies. This differs significantly from the continuum amplifier mentioned above where at low frequencies the temperature diverges.\[10\][11][12]

V. APPENDIX

To show that the $C$ state is left in a thermal density matrix after tracing out over the $D$ states, it is easiest to do so if we assume that the state of the system is the vacuum state for the $a$ and $b$ inputs. In this case the condition

$$a|0\rangle_{ab} = b|0\rangle_{ab}$$

becomes

$$\cosh(\theta)C + \sinh(\mu)D|1\rangle_{ab} = 0$$  
$$\cosh(\theta)D + \sinh(\mu)C|1\rangle_{ab} = 0$$

which has solution

$$|0\rangle_{ab} = e^{\tanh(\mu)C^\dagger D^\dagger}|0\rangle_{CD}$$  

\[80\]
where $|0\rangle_{CD}$ is the state annihilated by the $C$, $D$ operators. Tracing over $D$ by using the quanta eigenstates $|m\rangle = (D^\dagger)^m|0\rangle_D$ we have

$$T_{rD}|0\rangle_{ab}|0\rangle_{ab} = \sum_m (m_D) \sum_r \frac{\tanh(\mu)r}{r!} |CD\rangle^r |0\rangle_{CD} \sum_s \frac{\tanh(\mu)s}{s!} |m\rangle_D$$

$$= \sum_m \frac{\tanh(\mu)^2m}{\sqrt{m!}} |0\rangle_C |0\rangle_C \sum_m \frac{C_m}{\sqrt{m!}}$$

$$= \sum_m e^{m(2\ln(\tanh(\mu))}|m\rangle_C \langle m|_C$$

$$= e^{2\ln(\tanh(\mu))}C^\dagger C$$

which is a thermal density matrix with thermal factor $\Lambda = -2\ln(\tanh(\mu))$.\[8\]

It is also clear that if one began with the two mode squeezed state

$$|\xi\rangle = e^{-2\ln(\tanh(\mu))a^\dagger b^\dagger} |0\rangle_{ab}$$

That state in terms of the output modes would just be the vacuum state $|0\rangle_{CD}$ which would minimize the output noise.

To show that if the input state is a thermal state in each of the channels,

$$\rho = e^{\Lambda_a a^\dagger a} e^{\Lambda_b b^\dagger b}$$

the output state of the $C$ channel is also a thermal state, I found it easiest to go use path integrals. Using

$$x = \frac{a^\dagger + a}{\sqrt{2}} \quad p_x = \frac{a^\dagger + a}{\sqrt{2}}$$

$$y = \frac{b^\dagger + b}{\sqrt{2}} \quad p_x = \frac{b^\dagger + b}{\sqrt{2}}$$

we can write the density matrix,

$$\rho = Ne^{-\frac{1}{2} \Lambda_a (p_x^2 + x^2)}$$

between the initial and final eigenstates of $x,y$ operators as We can take as our example the density matrix $\rho = e^{\frac{1}{2} (s^2 + x^2)}$ to get

$$|\langle x'\rangle|^{\rho|x|} = \int (x,y|p_xN)(p_xN(1 - \frac{1}{2} \Lambda_{a}(p_{x}^{2} + x^{2}))\cdots(p_{xN-1}(1 - \frac{1}{2} \Lambda_{a}(p_{x}^{2} + x^{2}))\cdots(\frac{1}{2} \Lambda_{a}(p_{x}^{2} + x^{2}))/N|x_{N-1})\cdots|\langle x_{1}\rangle|^{\rho|x|}$$

$$= \int \sum_{j} e^{ip_{x}(x_{j}-x_{j-1})-\Lambda_{a}x_{j}^{2}} \Pi_{k} d\Pi_{k} d\Pi_{k}$$

$$= \int \sum_{j} e^{ip_{x}(x_{j}-x_{j-1})-\Lambda_{a}x_{j}^{2}} \Pi_{k} d\Pi_{k} d\Pi_{k}$$

Completing the squares in the exponent with respect to $p_x$ and doing the $p_x$ integrals we get

$$\langle x'\rangle^{\rho|x|} = \int e^{-\frac{1}{2} \int_{0}^{1} (x^{2}) + \Lambda x^{2})d\tau}\Pi_{x} d\Pi_{x} d\Pi_{x}$$

where $x(1) = x'$ and $x(0) = x$.

Similarly for the two modes, we have

$$\langle x', y'\rangle^{\rho|x, y|} = N \int_{0}^{1} (e^{\frac{1}{2} \int_{0}^{1} \frac{x^{2}}{s_{a}} + \Lambda_{a} x^{2} + \frac{y^{2}}{s_{b}} + \Lambda_{a} y^{2})d\tau} \Pi_{x} \delta x(\tau) \delta y(\tau)$$

where the path integral is taken over all paths $x(\tau), y(\tau)$ such that $x(0) = x, y(0) = y, x(1) = x', y(1) = y'$.\[94\]
As usual we can do a change of variables of the path integral, such that
\[ \dot{x}(\tau) = x(\tau) - X(\tau) \]
\[ \dot{y}(\tau) = y(\tau) - Y(\tau) \]  
(95) (96)

where \( X(\tau), Y(\tau) \) obey
\[ \dot{X} = \Lambda_a^2 X \]
\[ \dot{Y} = \Lambda_b^2 Y \]  
(97) (98)

and where \( X(0) = x, Y(0) = y, X(1) = x', Y(1) = y' \). The boundary condition on the tilde variables is
\[ \tilde{x}(0) = \tilde{x}(1) = \tilde{y}(0) = \tilde{y}(1) = 0 \]  
(99)

The exponent of the path integral then becomes
\[ S = \frac{1}{2} \int_0^1 \frac{\dot{x}(\tau)^2}{\Lambda_a} + \Lambda_a x(\tau)^2 + \frac{\dot{y}(\tau)^2}{\Lambda_b} + \Lambda_b y(\tau)^2 d\tau \]
\[ = \frac{1}{2} \int \frac{\dot{X}(\tau)^2}{\Lambda_a} + \Lambda_a X(\tau)^2 + \frac{\dot{Y}(\tau)^2}{\Lambda_b} + \Lambda_b Y(\tau)^2 + \frac{1}{2} \int \dot{\tilde{x}}(\tau)^2 + \frac{\dot{\tilde{y}}(\tau)^2}{\Lambda_b} + \Lambda_b \tilde{y}(\tau)^2 \]  
(100) (101)

where the cross terms between \( X, \tilde{x} \) and \( Y, \tilde{y} \) vanish by integration by parts and because \( X, Y \) obey the equations of motion, and because \( \tilde{x}, \tilde{y} \) are zero at the endpoints. Doing an integration by parts on the \( X, Y \) terms, and using that they obey the equations of motion we get that the only contribution to the integral is from the endpoints where we get for the integrand
\[ S = -\frac{1}{2} \left( (x'\dot{X}(1) - x\dot{Y}(0))/\Lambda_a + (y'\dot{Y}(1) - y\dot{Y}(0))/\Lambda_b \right) \]
\[ -\frac{1}{2} \int \dot{\tilde{x}}(\tau)^2 + \Lambda_a \tilde{x}(\tau)^2 + \frac{\dot{\tilde{y}}(\tau)^2}{\Lambda_b} + \Lambda_b \tilde{y}(\tau)^2 d\tau \]  
(102) (103)

where the path integral now is over paths where the endpoints of the tilde variables are all 0. The contribution of the second part (the integration over the tilde variables) to the path integral is independent of the values at the end points \( x, x', y, y' \) so it simply multiplies the path integral by a constant which can be absorbed into the normalisation factor \( N \). The solution for \( X, Y \) with the given boundary conditions is
\[ X = x \frac{\sinh(\Lambda_a (1 - \tau))}{\sinh(\Lambda_a)} + x' \frac{\sinh(\Lambda_a \tau)}{\sinh(\Lambda_a)} \]  
(104)

\[ Y = y \frac{\sinh(\Lambda_b (1 - \tau))}{\sinh(\Lambda_b)} + y' \frac{\sinh(\Lambda_b \tau)}{\sinh(\Lambda_b)} \]  
(105)

which gives as the only non-trivial contribution to the integrand
\[ \rho(x' y'; xy) \propto e^{\tilde{S}} \]  
(106)

with
\[ \tilde{S} = -\frac{1}{2} \left( x'\dot{X}(1) - x\dot{Y}(0) \right) / \Lambda_a + y'\dot{Y}(1) - y\dot{Y}(0) / \Lambda_b \]  
(107)

\[ = -\frac{1}{2} \left( (x^2 + x'^2) \coth(\Lambda_a) - 2xx' 1 / \sinh(\Lambda_a) + (y^2 + y'^2) \coth(\Lambda_b) - 2yy' 1 / \sinh(\Lambda_b) \right) \]  
(108)

and
\[ \rho(x' y'; x, y) = \langle x' y' | \rho | x y \rangle = \tilde{N} e^{\tilde{S}} \]  
(109)

Now, we want to take the trace of this density matrix over all \( D \) states. Defining
\[ Z = \frac{C + C^\dagger}{\sqrt{2}} \]  
(110)

\[ W = \frac{D + D^\dagger}{\sqrt{2}} \]  
(111)
we have that
\[ |x, y\rangle = |\cosh(\mu)Z - \sinh(\mu)W, \cosh(\mu)W - \sin(\mu)Z\rangle \] (112)
and the trace of \( \rho \) over \( D \) becomes
\[
\langle Z' \mid Tr_D \rho \mid Z \rangle = \int \rho(\cosh(\mu)Z' - \sinh(\mu)W, \cosh(\mu)W - \sin(\mu)Z'; \cosh(\mu)Z - \sinh(\mu)W, \cosh(\mu)W - \sin(\mu)Z) dW \] (113)

Since the integrand is an Gaussian exponential in the three variables \( Z, Z', W \), after the integration over \( W \), (since the coefficient of \( W^2 \) is independent of \( Z, Z' \)) the result is also Gaussian in \( Z \) and \( Z' \) and is symmetric in \( Z, Z' \). Ie, it is also a thermal state. Explicit calculation, by completing the squares in the exponent of the integrand for \( W \), shows it is of the form
\[
\langle Z' \mid Tr_D \rho \mid Z \rangle \propto e^{-\frac{(\cosh(\Lambda_C)Z^2 + Z'^2 - 2\sinh(\Lambda_C)ZZ')}{\sinh(\Lambda_C)}} \] (114)
which is again a thermal density matrix with thermal factor \( \Lambda_C \). While one could actually evaluate the terms in order to determine what \( \Lambda_C \) is in terms of \( \mu, \Lambda_a, \Lambda_b \), it is far easier to do this by the procedure in the main section and simply evaluate \( Tr(\rho C^\dagger C) \) to determine the thermal factor.

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