On the existence of $D$–solutions of the steady–state Navier–Stokes equations in plane exterior domains

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Mathematical Subject classification (2010): 35Q30, 76D03, 76D05, 76D07.

Keywords: steady state Navier–Stokes equations, two–dimensional exterior Lipschitz domains, boundary–value problem, existence and asymptotic behavior of $D$–solutions.

Abstract

We prove that the steady–state Navier–Stokes problem in a plane Lipschitz domain $\Omega$ exterior to a bounded and simply connected set has a $D$–solution provided the boundary datum $a \in L^2(\partial\Omega)$ satisfies
\[
\frac{1}{2\pi} \left| \int_{\partial\Omega} a \cdot n \right| < 1.
\]
If $\Omega$ is of class $C^{1,1}$, we can assume $a \in W^{-1/4,4}(\partial\Omega)$. Moreover, we show that for every $D$–solution $(u, p)$ of the Navier–Stokes equations it holds
\[
\nabla p = o(r^{-1}), \quad \nabla_k p = O(r^{-3/2}), \quad \nabla_k u = O(r^{-3/4}),
\]
for all $k \in \mathbb{N} \setminus \{1\}$ and for all positive $\epsilon$, and if the flux of $u$ through a circumference surrounding $\partial \Omega$ is zero, then there is a constant vector $u_0$ such that
\[
u = u_0 + o(1).
\]
1 Introduction

Let

\[ \Omega = \mathbb{R}^2 \setminus \overline{\Omega'}, \]

with \( \Omega' \) bounded and simply connected Lipschitz domain\(^1\). As is well–known, the steady–state Navier–Stokes problem in \( \Omega \) is to find a solution \((u, p)\) of the system \((2)\)

\[ \begin{align*}
\Delta u - u \cdot \nabla u &= \nabla p & \text{in } \Omega, \\
\text{div } u &= 0 & \text{in } \Omega, \\
u = a & \text{ on } \partial \Omega,
\end{align*} \]

where \( u, p \) and \( a \) are respectively the velocity, the pressure and the boundary datum. In \([26]\) we removed the classical zero flux condition

\[ \int_{\partial \Omega} a \cdot n = 0 \]

for the existence of a solution of system \((2)\)^3. Indeed, by following the well–known approach of invading domains of J. Leray \([21]\), we proved existence of a solution \((u_\ell, p_\ell) \in D^{1,2}(\Omega) \times L^2_{\text{loc}}(\Omega)\) of problem \((2)\), we shall call Leray solution, provided \( a \in W^{1/2,2}(\partial \Omega) \) and

\[ \frac{\kappa}{2\pi} \left| \int_{\partial \Omega} a \cdot n \right| < 1, \]

with

\[ \kappa = \sup_{\|\varphi\|_{D^{1,2}(\Omega)}} \left| \int_{\mathbb{R}^2} (\log r) \text{div} (\varphi \cdot \nabla \varphi) \right| < +\infty. \]

By well–known results of D. Gilbarg & H.F. Weinberger \([16]\) and G.P. Galdi \([12]\) \( u_\ell \) is known to be bounded in a neighborhood of infinity \( \mathbb{R} C_{R_0} \) and there is a (unknown) constant vector \( u_0 \) such that

\[ \begin{align*}
p_\ell(x) &= o(1), \\
u_\ell(x) &= u_0 + o(1)
\end{align*} \]

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\(^1\)See Remark 4.7.

\(^2\)As is always possible, we assume throughout the kinematical viscosity coefficient equal to 1.

\(^3\)See \([11]\), Ch. IX, and \([12]\).
and
\[ \nabla u_\ell = O(r^{-3/4} \log r). \]
Moreover, in [11] it is proved that
\[ \nabla_k p\ell(x) = o(1), \]
\[ \nabla_k u\ell(x) = o(1), \]
for all \( k \in \mathbb{N} \).

In a recent paper [28] we improve (7) by showing that
\[ \nabla_k p\ell(x) = O(r^{-1/2}), \]
\[ \nabla_k u\ell(x) = O(r^{-1/2}), \]
for all \( k \in \mathbb{N} \) and for every positive \( \epsilon \). Let us note that (5)_1, (7) and (8) hold for every solution \((u, p)\) of (2) such that [17]:
\[ \int_{C_{R_0}} |\nabla u|^2 < +\infty, \]
for some \( C_{R_0} \supseteq \Omega' \), we shall call \( D \)-solution. Moreover, (5)_1 is replaced by the weaker one [17]:
\[ |u(x)|^2 = o(\log r) \]
If \( u \) vanishes on \( \partial \Omega \), C.J. Amick proved that \( u \) is bounded so that by the results of [12], [17]
\[ u = u_0 + o(1), \]
with \( u_0 \) constant vector. If \( u_0 \neq 0 \) L.I. Sazonov [35] showed that \( u \) is physically reasonable in the sense of R. Finn and D.R. Smith [9], [37] so that it behaves at infinity (almost) as the solution of the Oseen problem. To the best of our knowledge this is the state of the art of the problem of the existence and asymptotic behavior at infinity of a \( D \)-solution.

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4 We set \( \nabla_k \varphi = \nabla \cdots \nabla_k \times \text{n-times} \varphi, \nabla_0 \varphi = \nabla \varphi, \nabla_0 \varphi = \varphi \).  
5 For a \( D \)-solution (6) is replaced by \( \nabla u = O(r^{-3/4} \log^{3/8} r) \) [17].  
6 The existence of a \( D \)-solution \((u_\ell, p_\ell)\) can be also find by a technique of H. Fujita [10] (see also [11]). Due to the lack of a uniqueness theorem we cannot compare the two solutions. However, if \( u_\ell \) has zero outflow through \( \partial C_{R_0} \), then \( u_\ell \) is bounded (see Theorem 2).  
7 See Remark 3.3.  
8 For \( u_0 \neq 0 \) by different approaches and under suitable smallness assumption on the data R. Finn & D.R. Smith [9] (see also [30]) and G.P. Galdi [11] proved existence of a \( D \)-solution of (2) which takes the value \( u_0 \) at infinity.
In this paper we continue the study started in [26] on system (2) with a threefold main purpose:

• to prove that \( \kappa \leq 1 \) and to get the results of [26] by weakening the hypotheses on the boundary datum; to be precise we shall only assume \( \Omega \) Lipschitz and \( a \in L^q(\partial \Omega) \), \( q \geq 2 \), and prove existence of a \( D \)-solution of equations (2)_{1,2} which takes the boundary value \( a \) in the sense of the nontangential convergence for \( q > 2 \); if \( \Omega \) is of class \( C^{1,1} \), we can assume \( a \in W^{-1/4,4}(\partial \Omega) \).

• to observe that Amick’s result (9) on the boundedness of a \( D \)-solution holds under the sole hypothesis that the flux of \( u \) through \( \partial C_{R_0} \) is zero;

• starting from the results of [28] to show that for every \( D \)-solution \( (u, p) \)

\[
\nabla p(x) = o(r^{-1})
\]

and\(^9\)

\[
\nabla_k p(x) = O(r^{-3/2}), \\
\nabla_k u(x) = O(r^{-3/4}),
\]

for all \( k \in \mathbb{N} \setminus \{1\} \). Moreover, by means of the classical Hamel solutions we observe that (10) is sharp.

\[\text{Notation – A domain (open connected set) } \Omega \text{ of } \mathbb{R}^2 \text{ is said to be of class } C^{k,\alpha} \text{ if for every } \xi \in \partial \Omega, \text{ there exists a neighborhood of } \xi \text{ in } \partial \Omega \text{ which can be expressed as a graph of a function of class } C^{k,\alpha}; \text{ for } k = 0 \text{ and } \alpha = 1 \Omega \text{ is said to be Lipschitz. We shall use a standard vector notation, as in [11]: } \{o, (e_1, e_2)\} \text{ is a cartesian reference frame of } \mathbb{R}^2 \text{ with origin } o \text{ and } \{e_1, e_2\} \text{ orthonormal basis of } \mathbb{R}^2; \{o, (e_\alpha, e_\beta)\} \text{ is the polar coordinate system with origin at } o; \ x = (x_1, x_2) = (r, \theta) \text{ denotes the generic point of } \mathbb{R}^2; \text{ with } r = |x|, x = x - o = re_\theta; \text{ if } u \text{ is a vector field in } \mathbb{R}^2, \text{ by } (u_1, u_2) \text{ and } (u_\alpha, u_\beta) \text{ we denote the cartesian and polar components of } u \text{ respectively, and we set } (\nabla u)_i = \partial u_i / \partial x_i, \partial_x u = e_\theta \cdot \nabla u, \partial_\theta u = e_\rho \cdot \nabla u, \nabla^\perp = (-\partial_2, \partial_1). \text{ } C_R \text{ is the disk of radius } R \text{ centered at } o; \text{ also, we set } T_R = C_{2R} \setminus \{o\}, \Omega_R = \Omega \cap C_R; \text{ if } \Omega_1 \text{ and } \Omega_2 \text{ are two domains, } \Omega_1 \subset \Omega_2; \text{ if } \Omega \text{ is the exterior domain } (1) \text{ we denote by } R_0 \text{ a positive constant such that } \Omega \setminus C_{R_0}; \text{ the symbol } c \text{ will be reserved to denote a positive constant whose numerical value is unessential to our purposes. We use a standard notation to denote (scalar, vector or second-order tensor) function spaces, as in [11] and, in particular } D^{k,\alpha}(\Omega) \text{ denotes the Banach space of all fields } \varphi \in L^1_{loc}(\Omega) \text{ such that } \|\nabla \varphi\|_{L^1(\Omega)} < +\infty; \text{ } D^{1,2}_L(\Omega) = \{\varphi \in L^1_{loc}(\Omega) : \sqrt{\nabla \varphi} \|_{L^2(\Omega)} < +\infty\}, \text{ where } \delta = \delta(x) \text{ is a function equal to the distance of } x \text{ from } \partial \Omega \text{ in a neighborhood of } \partial \Omega \text{ and to } 1 \text{ in } C_{R_0}; \text{ } H^q \text{ (} q > 0 \text{) stands for the Hardy space in } \mathbb{R}^2 \text{ [39]. The symbol } V_x, \text{ where } V(\subset L^1_{loc}(\Omega)) \text{ stands for the subset of } V \text{ of all vector fields } u \text{ such that}
\]

\[^9\text{As will appear clear from the proof the quantity } r^\circ \text{ can be replaced by a suitable power of } \log r \text{ depending on } k.\]

4
\[ \int_{\Omega} u \cdot \nabla \varphi = 0, \text{ for all } \varphi \in C_0^\infty(\Omega). \]

Let \( \varphi \) be a function in \( \Omega \). Let \( \{ \gamma(\xi) \}_{\xi \in \partial \Omega} \) be a family of circular finite (not empty) triangles with vertex on \( \partial \Omega \) such that \( \gamma(\xi) \setminus \{ \xi \} \subset \Omega \):

\( \varphi(x) \) is said to converge nontangentially at the boundary if

\[ \varphi(\xi) = \lim_{x \to \xi \atop (x \in \gamma(\xi))} \varphi(x) \nrightarrow \varphi(\xi) \]

for almost all \( \xi \in \partial \Omega \).

The (Landau) symbols \( f(x) = o(g(x)) \) and \( f(x) = O(g(x)) \) \((g > 0)\) mean respectively that \( \lim_{r \to +\infty} (f/g) = 0 \) and \( f/g \) is bounded in a neighborhood of infinity. If \( \varphi \in L^1(\Omega) \) \([\text{or } \varphi \in \partial \Omega]\) we use the symbol

\[ \int_{\Omega} \varphi \left[ \int_{\partial \Omega} \varphi \right] \]

to denote the integral of \( \varphi \) over \( \Omega \) \([\text{on } \partial \Omega]\).

## 2 Some Lemmas

Throughout the paper we shall consider the domain \( \Omega \) defined by (1) and, as is always possible, we assume that \( C_1 \subset \Omega' \).

Let us start by recalling some well–known results concerning the Stokes problem

\[ \begin{align*}
\Delta u &= \nabla p \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega, \\
u &= a \quad \text{on } \partial \Omega,
\end{align*} \]

we shall use in the sequel.

It is well–known that if \( a \in L^2(\partial \Omega) \), then (11) has an analytical \( D \)–solution in \( \Omega \) \([8], [33], [34]\) expressed by

\[ u = v + \sigma, \]

with

\[ \sigma(x) = -\frac{e_r}{2\pi r} \int_{\partial \Omega} a \cdot n \]

and \( n \) outward (with respect to \( \Omega \)) unit normal to \( \partial \Omega \), such that \( u \) tends nontangentially to \( a \) and

\[ \int_{\partial \Omega} v \cdot n = 0. \]

As is well–known, since \( \Omega \) is Lipschitz such a family of triangles certainly exists.
It is unique in the class of the so-called very weak solutions \cite{27}, \cite{34}. Moreover, there is a constant vector $u_0$\footnote{$u_0$ is determined by $a$ through well-known compatibility conditions (see, e.g., \cite{11}, \cite{27}, \cite{33}, \cite{34}).} such that \cite{24}, \cite{33}, \cite{34}

\begin{equation}
\nabla_k (u - u_0) = O(r^{-1-k}),
\nabla_k p = O(r^{-2-k})
\end{equation}

and

\begin{equation}
\int_{\Omega} \delta(|\nabla u|^2 + |p|^2) \leq c \int_{\partial\Omega} |a|^2.
\end{equation}

Since $D^{1,2}_\delta(\Omega_R) \hookrightarrow W^{1/2,2}(\Omega_R) \hookrightarrow L^4(\Omega_R)$ \cite{18}, by (14), we have in particular that $u - u_0 \in L^4(\Omega)$. Moreover, it holds (see, e.g., \cite{24}, \cite{33}, \cite{34}, \cite{36})

(i) if $a \in L^q(\partial\Omega)$, $q \in [2, +\infty]$, then $u \in W^{1/q,q}_{\text{loc}}(\Omega)$.

There are two positive scalars $\mu_0(<1)$ and $\varepsilon$ depending only on $\partial\Omega$ such that

(ii) if $a \in C^{0,\mu}(\partial\Omega)$, $\mu \in (0, \mu_0)$, then $u \in C^{0,\mu}_{\text{loc}}(\Omega)$; if $\Omega$ is of class $C^1$, we can take $\mu_0 = 1$;

(iii) if $a \in W^{1,1/q,q}(\partial\Omega)$, $q \in [2, 2 + \varepsilon)$, then $(u, p) \in W^{1/q}_{\text{loc}}(\Omega) \times L^q_{\text{loc}}(\Omega)$; if $\Omega$ is of class $C^1$, we can take $q \in (1 + \infty)$;

(iv) if $a \in W^{1,q}(\partial\Omega)$, $q \in (2 - \varepsilon, 2 + \varepsilon)$, then $(u, p) \in W^{1,q}_{\text{loc}}(\Omega) \times W^{1/q}_{\text{loc}}(\Omega)$; if $\Omega$ is of class $C^1$, we can take $q \in (1 + \infty)$. Moreover, if $a \in W^{1,2}(\partial\Omega)$, then

\begin{equation*}
\int_{\Omega} [|\nabla u|^2 + |p|^2 + \delta(|\nabla u|^2 + |p|^2)] \leq c \int_{\partial\Omega} (|a|^2 + |\nabla a|^2).
\end{equation*}

The above results allow us to prove

**Lemma 1.** Let $\Omega$ be a Lipschitz exterior domain of $\mathbb{R}^2$. If $a \in L^2(\partial\Omega)$, then there is a field $h \in C_\infty(\Omega) \cap D^{1,2}_{\delta}(\Omega)$ which tends nontangentially to $a$ on $\partial\Omega$, vanishes outside a disk and satisfies

\begin{equation}
\|h\|_{L^4(\Omega)} \leq c \|h\|_{D^{1,2}_\delta(\Omega)} \leq c \|a\|_{L^2(\partial\Omega)}.
\end{equation}

Moreover, if $a$ is more regular, then also $h$ is more regular according to (i)–(iv).
Proof – Let \( g \) be a \( C^\infty \) cut-off function in \( \mathbb{R}^2 \), equal to 1 in \( \mathcal{C}_R \) and to zero outside \( \mathcal{C}_2 \mathcal{R} \) with \( \mathcal{R} > \mathcal{R}_0 \). Since by (13) \[
abla \int_{T_{\mathcal{R}}} (g v) = 0,
\] the problem \[
abla \omega + \nabla (g v) = 0 \text{ in } T_{\mathcal{R}}
\] admits a solution \( \omega \in C^\infty_0 (T_{\mathcal{R}}) \) (see also [11] Ch.III). It is clear that the field

\[
\begin{cases}
v, & \text{in } \Omega_{\mathcal{R}}, \\
\omega + g v, & \text{in } T_{\mathcal{R}}, \\
0, & \text{in } \mathbb{R}^2 \setminus \Omega_{2 \mathcal{R}},
\end{cases}
\]

satisfies all the properties stated in the Lemma. □

If \( \Omega \) is of class \( C^{1.1} \) and \( a \in W^{-1/q,q}(\partial \Omega) \) \((q > 1)^{12}\), then (11) admits the solution (12) where \( v \) is a simple layer potential (plus a constant vector \( u_0 \)) with a density in \( W^{-1-1/q,q}(\partial \Omega) \) \([6,34]\). The boundary datum is taken in the sense of the unique continuous extension map from \( W^{-1-1/q,q}(\partial \Omega) \) into \( W^{1/q,q}(\partial \Omega) \) of the trace operator of the classical simple layer potential from \( W^{1/q,q}(\partial \Omega) \) to \( W^{1+1/q,q}(\partial \Omega) \) \([6]\). Moreover, \( u \in L^q_{\text{loc}}(\Omega) \) satisfies (14). Therefore, by proceeding as we did in the proof of Lemma 1 and taking also into account the regularity properties of the classical layer potentials \([22,34]\), we have

Lemma 2. Let \( \Omega \) be a exterior domain of \( \mathbb{R}^2 \) of class \( C^{1.1} \). If \( a \in W^{-1/q,q}(\partial \Omega) \), \( q \geq 4 \), then there is a divergence free extension \( h \in C^\infty(\Omega \cap L^q(\Omega)) \) of \( a \) in \( \Omega \), expressed by (17), which satisfies

\[
\|h\|_{L^q(\Omega)} \leq C\|a\|_{W^{-1/q,q}(\partial \Omega)}.
\]

Moreover,

- if \( a \in C^{1,\mu}(\partial \Omega), \mu < 1 \), then \( h \in C^{1,\mu}(\overline{\Omega}) \);
- if \( \Omega \) is of class \( C^k \), \( k \geq 2 \) and \( a \in W^{k-1/q,q}(\partial \Omega), \mu < 1 \), then \( h \in W^{k+2,q}(\Omega) \).

The following elementary but basic Lemma was first proved in [17]. We give a simple proof of a slight generalization.

\[^{12}W^{-1/q,q}(\partial \Omega) \text{ is the dual space of } W^{1-1/q,q'}(\partial \Omega).\]
Lemma 3. Let \( w, z \in W^{1,2}_\sigma(C_R) \). Then

\[
\left| \int_{C_R} w \cdot \nabla z \cdot \frac{e_r}{r} \right| \leq \left\{ \int_{C_R} |\nabla w|^2 \int_{C_R} |\nabla z|^2 \right\}^{1/2}.
\]  

**Proof** – Set \( \bar{\varphi}(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(r, \theta) \).

Since
\[
\int_0^{2\pi} \bar{w}_1 \partial_\theta z_2 = \int_0^{2\pi} \bar{w}_2 \partial_\theta z_1 = 0,
\]
a simple computation yields

\[
\int_{C_R} w \cdot \nabla z \cdot \frac{e_r}{r} = \int_0^R \frac{1}{\rho} \int_0^{2\pi} \left[(w_1 - \bar{w}_1) \partial_\theta z_2 - (w_2 - \bar{w}_2) \partial_\theta z_1 \right].
\]

From Schwarz’s, Wirtinger’s and Cauchy’s inequalities we have

\[
\left| \int_0^{2\pi} (w_1 - \bar{w}_1) \partial_\theta z_2 \right| \leq \left\{ \int_0^{2\pi} |w_1 - \bar{w}_1|^2 \int_0^{2\pi} |\partial_\theta z_2|^2 \right\}^{1/2},
\]

\[
\left| \int_0^{2\pi} (w_2 - \bar{w}_2) \partial_\theta z_1 \right| \leq \left\{ \int_0^{2\pi} |\partial_\theta w_1|^2 \int_0^{2\pi} |\partial_\theta z_2|^2 \right\}^{1/2},
\]

Therefore, (19) follows from (20), taking into account that \(|\partial_\theta w| \leq r|\nabla w|\). \( \square \)

Note that if \( w, z \in D^{1,2}_\sigma(\mathbb{R}^2) \), letting \( R \to +\infty \) in (19) yields

\[
\left| \int_{\mathbb{R}^2} w \cdot \nabla z \cdot \frac{e_r}{r} \right| \leq \left\{ \int_{\mathbb{R}^2} |\nabla w|^2 \int_{\mathbb{R}^2} |\nabla z|^2 \right\}^{1/2}.
\]

If \( w, z \in W^{1,2}_{\sigma, \Omega} \), then the zero extensions of \( w \) and \( z \) belongs to \( D^{1,2}_\sigma(\mathbb{R}^2) \). Therefore, from (21) it follows

\[
\left| \int_{\Omega} w \cdot \nabla z \cdot \frac{e_r}{r} \right| \leq \left\{ \int_{\Omega} |\nabla w|^2 \int_{\Omega} |\nabla z|^2 \right\}^{1/2}.
\]

Lemma 3 allows to quickly prove
Theorem 1. It holds

(23) \[ \sup_{\| \varphi \|_{H^{2,2}(\mathbb{R}^2)} = 1} \left| \int_{\mathbb{R}^2} (\log |x|) \text{div} (\varphi \cdot \nabla \varphi) \right| \leq 1. \]

Proof – It is sufficient to prove (23) in \( C_{\infty_{0}}(\mathbb{R}^2) \). In this case

\[ \left| \int_{\mathbb{R}^2} (\log |x|) \text{div} (\varphi \cdot \nabla \varphi) \right| = \left| \int_{\mathbb{R}^2} \varphi \cdot \nabla \varphi \cdot \frac{e_r}{r} \right| \]

and (23) follows from Lemma 3. \( \square \)

Lemma 4. Let \((u, p)\) be a a solution to \((2)_{1,2}\). Then for all \(k \in \mathbb{N}_0\) and for all \(C_1(x) \subseteq \Omega\)

(24) \[ |\nabla^k p(x)| \leq c \left\{ \int_{0}^{2\pi} |\nabla^k p|(|x|, \theta) + \sum_{j=1}^{k+1} \|\nabla_j u\|_{L^2(C_1(x))}^2 \right\}. \]

Proof – We follow [12] (Lemma 3.10). Setting \(u_1 = \nabla u\) and \(p_1 = \nabla p\), the pair \((u_1, p_1)\) is a solution of the equations

(25) \[ \Delta u_1 - u_1 \cdot \nabla u - u \cdot \nabla u_1 = \nabla p_1, \]

\[ \text{div} u_1 = 0. \]

Let \(x = (r, \theta)\) and let \((r', \theta')\) be a polar coordinate system centered at \(x\). Multiplying (25) scalarly by \(x'/r'^2\) and integrating over \(C_1(x)\), we have

\[ p_1(r, \theta) = p_1(r', \theta') + \int_{C_1(x)} |u_1 \cdot \nabla u + u \cdot \nabla u_1| \frac{x'}{r'^2} \]

Hence, making use of Lemma 3, it follows

(26) \[ |p_1(r, \theta)| \leq \frac{1}{2\pi} \left\{ \int_{0}^{2\pi} |p_1(r', \theta')| + \|\nabla u\|_{L^2(C_1(x))} \|\nabla u_1\|_{L^2(C_1(x))} \right\}. \]

Multiplying (26) by \(r'\) and integrating over \(r' \in [0, 1]\) and \(\theta \in [0, 2\pi]\) show that

(27) \[ \|p_1\|_{L^1(C_1(x))} \leq c \left\{ \int_{0}^{2\pi} |p_1(r, \theta)| + \|\nabla u\|_{L^2(C_1(x))} \|\nabla u_1\|_{L^2(C_1(x))} \right\}. \]
Moreover, multiplying (26) by $r'$ and integrating over $r' \in [0, 1]$ and $\theta' \in [0, 2\pi]$ yield
\[
|p_1(x)| \leq c \left\{ \|p_1\|_{L^1(C_1(x))} + \|\nabla u\|_{L^2(C_1(x))} \|\nabla u_1\|_{L^2(C_1(x))} \right\}.
\]
Therefore, putting together (27)–(28) and using Cauchy’s inequality we find
\[
|p_1(x)| \leq c \left\{ \int_0^{2\pi} |p_1(r, \theta)| + \|\nabla u\|_{L^2(C_1(x))} \|\nabla u_1\|_{L^2(C_1(x))} \right\}
\leq c \left\{ \int_0^{2\pi} |p_1(r, \theta)| + \|\nabla u\|_{L^2(C_1(x))} + \|\nabla u_1\|_{L^2(C_1(x))} \right\},
\]
and (24) is proved for $k = 1$. The proof for $k = 0$ follows the same steps. Iterating such a procedure as many times as we need, we then prove (24). □

**Lemma 5.** [11] Let
\[
v(x) = \int_{\mathbb{R}^2} \frac{1}{|x - y|^\lambda |y|^{\mu}} da_y,
\]
with $\lambda < 2$, $\mu < 2$. If $\lambda + \mu > 2$, then
\[
v(x) = c x^{2-\lambda-\mu},
\]
for a suitable constant $c = c(\lambda, \mu)$.

**Lemma 6.** [39] If $f \in \mathcal{H}^1$, then the problem
\[
\Delta p = f \quad \text{in } \mathbb{R}^2, \quad \lim_{x \to \infty} p(x) = 0
\]
admits the unique solution
\[
p(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y) \log |x - y| da_y \in D^{2,1}(\mathbb{R}^2) \cap D^{1,2}(\mathbb{R}^2).
\]

**Lemma 7.** [5] If $u \in D^{1,2}_p(\mathbb{R}^2)$, then $\nabla u \cdot \nabla u^T \in \mathcal{H}^1$.

### 3 Asymptotic behavior of $D$–solutions

Let us recall that by $D$–solution we mean an analytical pair $(u, p)$ which satisfies equations (2) subscripts $1, 2$ and
\[
\int_{\mathcal{B}_{C_{n_0}}} |\nabla u|^2 < +\infty,
\]
for some \( C_{R_0} \supset \Omega' \).

We deal now with the asymptotic properties of a \( D \)-solution. To this end we need the following classical results of D. Gilbarg and H.F. Weinberger [17].

**Lemma 8.** If \((u, p)\) is a \( D \)-solution, then

\[
\lim_{x \to +\infty} p(x) = 0
\]

and

\[
u = o(\sqrt{\log r}).
\]

Moreover

\[
\nabla u = O(r^{-3/4} \log^{9/8} r).
\]

Also, it holds [28]

**Lemma 9.** If \((u, p)\) is a \( D \)-solution, then

\[
\nabla_k p(x) = O(r^{\epsilon - 1/2}),
\]

\[
\nabla_k u(x) = O(r^{\epsilon - 1/2}),
\]

for all \( k \in \mathbb{N} \).

The following theorem extends to more general boundary data a classical result of C.J. Amick [2], D. Gilbarg and H.F. Weinberger [17] and G.P. Galdi [12].

**Theorem 2.** Let \((u, p)\) be a \( D \)-solution. If

\[
\int_{\partial C_{R_0}} u_{R_0} = 0,
\]

then there is a constant vector \( u_0 \) such that

\[
u = u_0 + o(1).
\]

**Proof** – From (2) and (34) it follows that there is a regular function \( \psi \) such that \( u = \nabla \psi \). Therefore, we can repeat the argument of Section 2.1 of [2] to see that there is a curve connecting a point of \( \partial C_{R_0} \) to infinity along which the Bernoulli function \( \Phi = p + \frac{1}{2} |u|^2 \) is monotone decreasing ((b) of Theorem...
11) and this is sufficient to assert that $u$ is bounded (Theorem 12). Hence by Theorem 4 of [17] there is a constant vector $u_0$ such that

$$\lim_{r \to +\infty} \int_{0}^{2\pi} |u(r, \theta) - u_0|^2 = 0.$$  

Since $\nabla u \in L^q(C_{R_0})$ for all $q \geq 2$ (see, e.g., [11] Lemma X.3.2), (36) implies (35) by virtue of Lemma 3.10 of [12]. □

The following theorem concerns the asymptotic behavior of the derivatives of a $D$–solution.

**Theorem 3.** If $(u, p)$ is a $D$–solution, then

(37) \[ p(x) \in D^{1,2}(\overline{C_{R_0}}), \]

(38) \[ \nabla p(x) = o(r^{-1}) \]

and

(39) \[ \nabla_{k+1} p(x) = O(r^{\epsilon-3/2}), \]

\[ \nabla_k u = O(r^{\epsilon-3/4}), \]

for every positive $\epsilon$ and for every $k \in \mathbb{N}$. Moreover, if $u$ satisfies (34), then $p \in D^{1,2}(\overline{C_{R_0}})$.

**Proof of (37), (38).**

Let

$$\gamma(x) = \frac{e_r}{r} \int_{\partial C_{R_0}} u_{R_0}$$

and set

$$u = v + \gamma.$$

Let $g$ be a regular cut–off function in $\mathbb{R}^2$, vanishing in $C_R$ and equal to 1 outside $C_{2R}$, with $R \gg R_0$. Since

$$\int_{\partial C_R} v \cdot n = 0,$$

the problem

$$\text{div} \ h + \text{div} (gv) = 0 \quad \text{in} \ T_R$$

12
has a solution \( h \in C_0^\infty(T_{\bar{R}}) \) [11]. From (2) it follows that the function \( Q = g^2p \) is a solution of the equation

\[
\Delta Q + \sum_{i=1}^{3} \text{div } h_i + \varphi = 0 \quad \text{in } \mathbb{R}^2,
\]

where

\[
\varphi \in C_0^\infty(T_{\bar{R}})
\]

and

\[
\begin{align*}
    h_1 &= (gv + h) \cdot \nabla (gv + h), \\
    h_2 &= 2gv \cdot \nabla (g\gamma), \\
    h_3 &= g\gamma \cdot \nabla (g\gamma).
\end{align*}
\]

By virtue of (30) equation (40) has a unique solution \( Q \) which by Lemma 5 is expressed by

\[
2\pi Q(x) = -\sum_{i=1}^{3} \int_{\mathbb{R}^2} (\log |x - y|) \text{div } h_i(y) da_y - \int_{\mathbb{R}^2} \varphi(y) \log |x - y| da_y
\]

\[
= \sum_{i=1}^{4} Q_i.
\]

By Lemma 7 \( \text{div } h_1 \in \mathcal{H}^1 \) so that Lemma 6 implies that \( Q_1 \in D^{2,1}(\mathbb{R}^2) \cap D^{1,2}(\mathbb{R}^2) \). Hence it follows in particular that

\[
\lim_{x \to +\infty} Q_1(x) \in \mathbb{R}.
\]

Now, letting \( R \to +\infty \) in the relation

\[
\int_{C_R} (\log |x - y|) \text{div } [(gv) \cdot \nabla (g\gamma)] da_y = \int_{\partial C_R} (\log |x - \zeta|) (v \cdot \nabla \gamma \cdot e_R)(\zeta) ds_\zeta + \int_{C_R} |g\cdot \nabla (g\gamma)(y) \cdot (x - y)| da_y,
\]

\[
\int_{C_R} (\log |x - y|) \text{div } [(g\gamma) \cdot \nabla (g\gamma)] da_y = \int_{\partial C_R} (\log |x - \zeta|) (\gamma \cdot \nabla \gamma \cdot e_R)(\zeta) ds_\zeta + \int_{C_R} |g\gamma \cdot \nabla (g\gamma)(y) \cdot (x - y)| da_y.
\]
and taking into account the behavior at infinity of \( \mathbf{v} \) and \( \gamma \), we have

\[
Q_2(x) = -2 \int_{\mathbb{R}^2} \frac{[g \mathbf{v} \cdot \nabla (g \gamma)](y) \cdot (x - y)}{|x - y|^2} \, da_y,
\]

\[
Q_3(x) = - \int_{\mathbb{R}^2} \frac{[g \gamma \cdot \nabla (g \gamma)](y) \cdot (x - y)}{|x - y|^2} \, da_y.
\]

Now, for large \( |x| \)

\[
\int_{\mathbb{R}^2} \frac{[g \mathbf{v} \cdot \nabla (g \gamma)](y) \cdot (x - y)}{|x - y|^2} \, da_y = \int_{\mathbb{R}^2 \setminus C_1(x)} \frac{[g \mathbf{v} \cdot \nabla (g \gamma)](y) \cdot (x - y)}{|x - y|^2} \, da_y
\]

\[
+ \int_{\partial C_1(x)} \frac{(\mathbf{v} \cdot \nabla \gamma)(y) \cdot (x - y)}{|x - y|^2} \, da_y
\]

\[
= \int_{\mathbb{R}^2 \setminus C_1(x)} \frac{[g \mathbf{v} \cdot \nabla (g \gamma)](y) \cdot (x - y)}{|x - y|^2} \, da_y
\]

\[
- \int_{\partial C_1(x)} (\log |x - y|)(\mathbf{v} \cdot \nabla \gamma)(\zeta) \cdot \mathbf{n}(\zeta) \, da_\zeta
\]

\[
+ \int_{C_1(x)} (\log |x - y|)(\nabla \mathbf{v} \cdot \nabla \gamma^T)(y) \, da_y.
\]

Hence

\[
\nabla Q_2(x) = -2 \nabla \int_{\mathbb{R}^2 \setminus C_1(x)} \frac{[g \mathbf{v} \cdot \nabla (g \gamma)](y) \cdot (x - y)}{|x - y|^2} \, da_y
\]

\[
+ 2 \int_{\partial C_1(x)} \frac{(x - y)(\mathbf{v} \cdot \nabla \gamma)(\zeta) \cdot \mathbf{n}(\zeta)}{|x - y|^2} \, da_\zeta
\]

\[
- 2 \int_{C_1(x)} \frac{(x - y)(\nabla \mathbf{v} \cdot \nabla \gamma^T)(y)}{|x - y|^2} \, da_y.
\]

Likewise, since

\[
\nabla Q_3(x) = -2 \nabla \int_{\mathbb{R}^2 \setminus C_1(x)} \frac{[g \gamma \cdot \nabla (g \gamma)](y) \cdot (x - y)}{|x - y|^2} \, da_y
\]

\[
+ 2 \int_{\partial C_1(x)} \frac{(x - y)(\gamma \cdot \nabla \gamma)(\zeta) \cdot \mathbf{n}(\zeta)}{|x - y|^2} \, da_\zeta
\]

\[
- 2 \int_{C_1(x)} \frac{(x - y)(\nabla \gamma \cdot \nabla \gamma^T)(y)}{|x - y|^2} \, da_y.
\]
taking into account the asymptotic properties of \( v, \nabla v \) and \( \gamma \), (43), (44) and (45) imply

\[
Q_2(x), \ Q_3(x) = O(r^{\epsilon - 1})
\]

and

\[
\nabla Q_2(x), \ \nabla Q_3(x) = O(r^{\epsilon - 2}),
\]

for all positive \( \epsilon \). By virtue of (30)\(^{13}\)

\[
\int_{\mathbb{R}^2} \varphi = 0
\]

so that

\[
\nabla_k Q_4(x) = O(r^{-1-k}),
\]

for all \( k \in \mathbb{N} \), and (37) is proved. By the basic calculus and (7)_1

\[
\int_0^{2\pi} |\nabla Q_1|(R, \theta) = \left| \int_R^{+\infty} \partial_x \nabla Q_1 \right| \leq \frac{C}{R} \int_{\mathbb{S}^1} |\nabla_2 Q_1|.
\]

Hence

\[
\lim_{R \to +\infty} \left\{ R \int_0^{2\pi} |\nabla Q_1|(R, \theta) \right\} = 0.
\]

Then, (38) follows from Lemma 4, taking into account (47), (48), (49) and that \( p(x) = Q(x) \) for large \( |x| \).

**Proof of (39).**

Since

\[
\Delta p = -\nabla u \cdot \nabla u^T \quad \text{in} \quad \mathbb{C} \mathbb{C}_{R_0},
\]

writing the Stokes formula in \( S_R \cap \mathbb{C} \mathbb{C}_{R_0} \), taking the gradient, letting \( R \to +\infty \)

\(^{13}\)Otherwise \( p(x) \) behaves at infinity as \( \log r \).
and taking into account (30), (33), we have

\[
2\pi \nabla p(x) = - \int_{\partial C_{R_0}} \frac{(x - \zeta) \partial_s p(\zeta)}{|x - \zeta|^2} \, ds_\zeta + \nabla \int_{\partial C_{R_0}} \frac{p(\zeta)(x - \zeta) \cdot e_{R_0}}{|x - \zeta|^2} \, ds_\zeta
\]

\[
- \int_{\Delta C_{R_0}} \frac{(x - y)(\nabla u \cdot \nabla u^T)(y)}{|x - y|^2} \, da_y
\]

(50)

\[
= - \int_{\Delta C_{R_0}} \frac{(x - y)(\nabla u \cdot \nabla u^T)(y)}{|x - y|^2} \, da_y + \psi(x)
\]

\[
= - \int_{\Delta C_{R_0} \setminus C_1(x)} \frac{(x - y)(\nabla u \cdot \nabla u^T)(y)}{|x - y|^2} \, da_y + \psi(x)
\]

\[
+ \int_{\partial C_1(x)} (\log |x - \zeta|)(\nabla u \cdot \nabla u^T)(\zeta)n(\zeta) \, ds_\zeta
\]

\[
- \int_{C_1(x)} (\log |x - y|)(\nabla u \cdot \nabla u^T)(y) \, da_y + \psi(x),
\]

with

\[
\nabla_k \psi(x) = O(r^{-1-k}).
\]

Hence, taking the gradient, it follows

(51)

\[
2\pi \nabla^2 p(x) = \sum_{i=1}^3 J_i + O(r^{-2}),
\]

where

\[
J_1 = - \nabla \int_{\Delta C_{R_0} \setminus C_1(x)} \frac{(\nabla u \cdot \nabla u^T)(y)(x - y)}{|x - y|^2} \, da_y,
\]

\[
J_2 = - \int_{C_1(x)} (x - y) \otimes \nabla(\nabla u \cdot \nabla u^T)(y) \, da_y,
\]

\[
J_3 = \nabla \int_{\partial C_1(x)} (\log |x - \zeta|)(\nabla u \cdot \nabla u^T)(\zeta)(x - \zeta) n(\zeta) \, ds_\zeta.
\]

Setting \(|x| = R (R > R_0)\) and after portioning \(\mathbb{C}C_{R_0}\) into \(\mathbb{C}C_{R_0} \cap C_{R/2}, \mathbb{C}C_{R/2}\) and taking into account (32), we get

\[
|J_1(x)| \leq \frac{c}{R^2} \int_{\mathbb{C}C_{R_0} \cap C_R} |\nabla u|^2 + c \int_{\mathbb{C}C_{R}} r^{-7/2} \leq c|x|^{-3/2}
\]
for some positive constant $c$ independent of $R$. Also, by (32)–(33) it is readily seen that $J_2(x), J_3(x) = O(r^{-3/2}).$ Hence (39) follows for $k = 2$. The proof of (39)$_1$ for general $k$ is obtained by iterating the above argument.

The proof of (39)$_2$ follows the above steps, taking into account that by (31), (32) and (38) $\Delta u = u \cdot \nabla u + \nabla p = O(r^{-3/4} \log^{13/8} r)$. Hence (39) follows for $k = 2$. The proof of (39)$_1$ for general $k$ is obtained by iterating the above argument.

Remark 3.1 - Note that under assumption (34) in (41) $Q_2 = Q_3 = 0$ so that $p \in D^{2,1}(\mathbb{C} R_0)$ [28].

Remark 3.2 - The first basic result in [17] assures that

\[
\nabla_2 u \in L^2(\mathbb{C} R_0). 
\]

Hence, taking into account (31) and (2)$_1$, it follows that $\nabla p/\sqrt{\log r} \in L^2(\mathbb{C} R_0)$. This is sufficient to say that (40) has the unique solution (41). Therefore, (30) follows from (42), (46) and (48). In this way we gave an alternative proof of (30) based only on (52). Note that from (52) and (38) it follows that $u \cdot \nabla u \in L^2(\mathbb{C} R_0)$.

Remark 3.3 - The asymptotic results in Theorem 3 are new in the case where $u$ is unbounded or tends to zero at large distance. Indeed, if $u$ tends to $e_1$ (say) at infinity, L.I. Sazonov [35] showed that $(u, p)$ is physically meaningful in the sense of R. Finn an D.R. Smith [9], [37]. Therefore the solution enjoys the following summability properties (see, e.g., [11] Ch. X)

\[
\begin{align*}
\partial u & \in L^q(\Omega), \ u \in L^{q-1}(\Omega), \ p \in L^{q-1}(\mathbb{C} R_0), \ \forall \ q > 3, \\
\partial_2 u & \in L^s(\mathbb{C} R_0), \ \forall \ s > 3/2,
\partial_1 u, \nabla u, \nabla p \in L^t(\mathbb{C} R_0), \ \forall \ t > 1.
\end{align*}
\]

We can say just a little bit more about the second derivatives of $p$. Assuming for simplicity $u = 0$ on $\partial\Omega$, the solution $p$ of the equation

\[
\Delta p + \text{div}(u \cdot \nabla u) = 0
\]

can be written

\[
p(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} (\log |x - y|) \text{div}(u \cdot \nabla u)(y) \text{d}a_y + \varpi(x) = Q(x) + \varpi(x),
\]

$^{14}$By Theorem 2 this could happen only if $\int_{\partial C R_0} u_r \neq 0$. 

17
where \( \varpi(x) \) is a simple layer harmonic potential with a density having zero integral mean over \( \partial \Omega \). By (53) and Theorem II.2 of [5] \( \text{div} (u \cdot \nabla u) \in H^t \), for all \( t > 2/3 \). Therefore, by well–known results about singular integrals (see, e.g., [39] p. 136) we have that

\[
\nabla_2 p(x) = \nabla_2 Q(x) + O(r^{-3}),
\]

with \( \nabla_2 Q(x) \in H^t \), for all \( t > 2/3 \).

\[ \diamond \]

Remark 3.4 - It is worth noting that (38) is sharp in the sense that, in general, it cannot be replaced by

\[
\nabla p = O(r^{-1-\epsilon}),
\]

for some positive \( \epsilon \). Indeed, the pairs

\[
u_r = \frac{\gamma}{r}, \quad u_\theta = \alpha \left( \frac{1}{r} - r^{\gamma+1} \right),
\]

\[
p = -\frac{\gamma^2 + \alpha^2}{2r^2} - \frac{2\alpha^2 r^\gamma}{r^\gamma + 2} + \frac{\alpha^2}{2r^\gamma + 2} r^{2(\gamma+1)},
\]

with \( \gamma \) and \( \alpha \) arbitrary constants, \( \gamma + 1 \neq 0 \), define the Hamel solutions (1916) of the Navier–Stokes equations (see [20] p. xi). For \( \gamma + 1 = -\epsilon/4 \) \( \leq 0 \) (55) is a \( D \)–solution which does not satisfy (54).

For \( \gamma < -1 \) (55) gives a family of \( D \)–solutions of the Navier–Stokes problem in \( \mathbb{C} \) with boundary datum

\[
u_r = \gamma, \quad u_\theta = 0.
\]

Therefore, at least for \( \gamma + 1 < 0 \), problem (2) with the condition at infinity

\[
\lim_{r \to +\infty} u(x) = \mathbf{0}
\]

does not admit a uniqueness theorem in the class of \( D \)–solutions. Let us recall that if \( \varphi \in D^{1,q}(\mathbb{C} \cap R_0), \ q \in [1,2), \) then there is a constant \( \varphi_0 \) such that (see [11] Lemma II.5.2)

\[
\int_0^{2\pi} |\varphi(r, \theta) - \varphi_0|^q \leq c(q) \int_{\mathbb{C} \cap R_0} |\nabla \varphi|^q.
\]

Therefore (55) shows that also (37) is sharp in the sense that it cannot be replaced by \( p \in D^{1,q}(\mathbb{C} \cap R_0) \) for some \( q < 2 \). Moreover, in contrast with (53)
a $D$–solution vanishing at infinity and with nonzero outflow cannot belong to any $D^{1,q}(\mathbb{C}C_{R_0})$ for $q < 2$.

Note that

$$\gamma + 1 < 0 \Rightarrow |\gamma| = \frac{1}{2\pi} \left| \int_{\partial \Omega} u \cdot n \right| > 1$$

so that for $\Omega' = C_1$ the solution of Theorem 6 is not a Hamel solution. \hfill \diamond

**Remark 3.5** - From Theorem 3 it follows that

(58) \[ \partial_r \int_0^{2\pi} u_r^2(r, \theta) = O(r^{-1} \log r) \]

and if $u$ is bounded, then\(^{15}\)

(59) \[ \partial_r \int_0^{2\pi} u_r^2(r, \theta) = o(r^{-1}). \]

Indeed, in the polar coordinate system $(r, \theta)$ \((2)_{1,2}\) read

\[
\frac{1}{r} \partial_p p + u_r \partial_r u_r + \frac{u_\theta}{r} \partial_\theta u_r - \frac{u_r^2}{r} = 0
\]

(60) \[ \frac{1}{r} \partial_p p + u_r \partial_r u_\theta + \frac{u_\theta}{r} \partial_\theta u_\theta + \frac{u_r u_\theta}{r} = 0
\]

\[ \frac{u_r}{r} + \partial_r u_r + \frac{1}{r} \partial_\theta u_\theta = 0. \]

Integrating (60) over $\theta \in (0, 2\pi)$ and taking into account (60)\(_3\), we get

(61) \[ \partial_r \int_0^{2\pi} (p + u_\theta^2)(r, \theta) = \frac{1}{r} \int_0^{2\pi} (u_\theta^2 - u_r^2)(r, \theta) \]

Hence (58) follows by (31) and (38).

Multiply (60) by $r$ and integrate over $C_R \setminus C_{R_0}$. Then, we have

(62) \[ \frac{1}{R} \int_{C_R \setminus C_{R_0}} \left( \frac{p + u_\theta^2}{r} \right) = \int_0^{2\pi} (p + u_\theta^2)(R, \theta) - \frac{R_0}{R} \int_0^{2\pi} (p + u_\theta^2)(R_0, \theta). \]

If $u$ is bounded, then $u(r, \theta)$ tends uniformly (in $\theta$) to a constant vector as $r \to +\infty$. Then (62) implies that

\[ \lim_{r \to +\infty} u_r^2(r, \theta) = \lim_{r \to +\infty} u_\theta^2(r, \theta) \]

and (59) follows from (61), taking into account (38). \hfill \diamond

\(^{15}\)By the example in Remark 3.4 relation (58) is sharp if $u = o(1)$. 19
4 Existence theorems

We are now in a position to prove our general existence theorems of a $D$–solution for problem (2).

**Theorem 4.** Let $\Omega$ be an exterior Lipschitz domain of $\mathbb{R}^2$ and let

\[ a \in L^2(\partial\Omega). \]

If

\[ \frac{1}{2\pi} \left| \int_{\partial\Omega} a \cdot n \right| < 1, \]

then system (2) has a Leray solution $(u, p) \in D_1^{1,2}(\Omega) \times L^2_{2,\text{loc}}(\Omega)$ such that (30) holds uniformly and

\[ u = u_0 + o(1), \]

with $u_0$ constant vector; it satisfies (38), (39) and if $a$ and/or $\partial\Omega$ are more regular, then so does $(u, p)$ according to the regularity results (i) – (iv) for the solutions of the Stokes problem; in particular, if $a \in L^q(\partial\Omega)$ ($q > 2$), then $u \rightharpoonup a$. Moreover, there are positive constants $\epsilon$ and $\mu_0 < 1$ depending on $\Omega$

such that

(i) if $a \in C^{0,\mu}(\partial\Omega)$, then $u \in C^{0,\mu}_{\text{loc}}(\Omega)$ for $\mu \in [0,\mu_0]$;

(ii) if $a \in W^{1-1/q,q}(\partial\Omega)$, $q \in (\max\{4/3,2-\epsilon\}, 2+\epsilon)$, then $(u, p) \in W^{1,q}_{\text{loc}}(\Omega) \times L^q_{\text{loc}}(\Omega)$; if $\Omega$ is of class $C^1$ we can take $\mu_0 = 1$ and $q \in [4/3, +\infty)$;

(iii) $a \in W^{1,2}(\partial\Omega)$, then $(u, p) \in D_2^{1,2}(\Omega) \times D_1^{1,2}(\Omega)$.

**Proof.** We look for a solution of (2) in the form $u = w + h$, with $w \in D_1^{1,2}(\Omega)$ and $h$ defined by (17). As is well–known [4], [33], [34], under assumption (64) the system

\[ \Delta w - (h + w) \cdot \nabla (h + w) + \Delta \zeta = \nabla Q \quad \text{in } \Omega_k, \]

\[ \text{div } w = 0 \quad \text{in } \Omega_k, \]

\[ w = 0 \quad \text{on } \partial\Omega_k \]

(for all $k > k_0 > R_0$) has a solution $w_k \in W^{1,2}_{\sigma,\partial}(\Omega_k)$ we extend to all $\mathbb{R}^2$ by setting $w_k = 0$ in $\mathbb{R}^2 \setminus \Omega_k$. Of course, $w_k$ satisfies the equation

\[ \int_{\Omega} \nabla w_k \cdot \nabla \varphi = \int_{\Omega} (h + w_k) \cdot \nabla \varphi \cdot (h + w_k) - \int_{\Omega} \nabla \zeta \cdot \nabla \varphi, \]
for all $\varphi \in W^{1,2}_{\sigma,0}(\Omega_k)$.

Let us show that if (64) holds, then there is a positive number $c_0$ independent of $k$ such that

$$\int_{\Omega} |\nabla w_k|^2 \leq c_0. \quad (68)$$

To prove (68) we use a well–known reasoning of J. Leray (see also [4] and [11] section VIII.7). If (68) is not true, then we can find a sequence of solutions $\{w'_k\}_{k \in \mathbb{N}}$ such that

$$\lim_{k \to +\infty} J_k^2 = \lim_{k \to +\infty} \int_{\Omega} |\nabla w'_k|^2 = +\infty.$$ 

In virtue of (67) the field

$$w_k = \frac{w'_k}{J_k}$$

satisfies

$$\frac{1}{J_k} \int_{\Omega} \nabla \varphi \cdot \nabla w_k = \int_{\Omega} w_k \cdot \nabla \varphi \cdot w_k + \frac{1}{J_k} \int_{\Omega} h \cdot \nabla \varphi \cdot w_k$$

$$+ \frac{1}{J_k} \int_{\Omega} w_k \cdot \nabla \varphi \cdot h + \frac{1}{J_k} \int_{\Omega} (h \cdot \nabla \varphi \cdot h - \nabla w_k \cdot \nabla \zeta), \quad (69)$$

for all $\varphi \in W^{1,2}_{\sigma,0}(\Omega_k)$. Since $\|\nabla w_k\|_{L^2(\Omega)} = 1$, by the compactness theorem of F. Rellich from $\{w_k\}_{k \in \mathbb{N}}$ we can extract a subsequence, we denote by the same symbol, which converges strongly in $L^q_{\text{loc}}(\Omega)$, for all $q \in (1, +\infty)$, and weakly in $D^{1,2}_0(\Omega)$ to a field $w \in D^{1,2}_{\sigma,0}(\Omega)$, with $\|\nabla w\|_{L^2(\Omega)} \leq 1$. Letting $k \to +\infty$ in (69), we see that the field $w$ is a weak solution of the Euler equations

$$w \cdot \nabla w + \nabla Q = 0 \quad \text{in } \Omega,$$

$$\text{div } w = 0 \quad \text{in } \Omega,$$

$$w = 0 \quad \text{on } \partial \Omega, \quad (70)$$

for some pressure field $Q \in W^{1,q}_{\sigma,0}(\Omega)$, $q \in [1, 2)$, constant on $\partial \Omega$ [19]. Now, choosing $\varphi = w'_k$ in (69) we get

$$1 = \int_{\Omega} w_k \cdot \nabla w_k \cdot \sigma + \int_{\Omega} w_k \cdot \nabla w_k \cdot (h - \sigma)$$

$$+ \frac{1}{J_k} \int_{\Omega} (h \cdot \nabla w_k \cdot h - \nabla w_k \cdot \nabla \zeta). \quad (71)$$
By (22)

\[ \left| \int_{\Omega} w_k \cdot \nabla w_k \cdot \sigma \right| \leq \frac{1}{2\pi} \left| \int_{\partial \Omega} a \cdot n \right| \int_{\Omega} |\nabla w_k|^2 \leq \frac{1}{2\pi} \left| \int_{\partial \Omega} a \cdot n \right|. \]

Therefore (71) yields

\[ 1 - \frac{1}{2\pi} \left| \int_{\partial \Omega} a \cdot n \right| \leq \int_{\Omega} w_k \cdot \nabla w_k \cdot (h - \sigma) \]

\[ + \frac{1}{J_k} \int_{\Omega} (h \cdot \nabla w_k \cdot h - \nabla w_k \cdot \nabla \zeta). \]

Hence, taking into account that by (16)

\[ \left| \int_{\Omega} h \cdot \nabla w_k \cdot h \right| \leq \left\{ \int_{\Omega} |h|^4 \int_{\Omega} |\nabla w_k|^2 \right\}^{1/2} \leq c, \]

\[ \left| \int_{\Omega} \nabla w_k \cdot \nabla \zeta \right| = \left| \int_{T_n} \nabla w_k \cdot \nabla \zeta \right| \leq \left\{ \int_{\Omega} |\nabla w_k|^2 \int_{T_n} |\nabla \zeta|^2 \right\}^{1/2} \leq c, \]

and letting \( k \to +\infty \) in (72), it follows

\[ 1 - \frac{1}{2\pi} \left| \int_{\partial \Omega} a \cdot n \right| \leq \int_{\Omega} w \cdot \nabla w \cdot \zeta. \]

Taking into account that \( Q \) is constant on \( \partial \Omega \) (say \( Q_0 \)) and \( \zeta \) is divergence free in \( \mathbb{R}^2 \) we have

\[ \int_{\Omega} w \cdot \nabla w \cdot \zeta = - \int_{\partial \Omega} \zeta \cdot \nabla Q = -Q_0 \int_{\partial \Omega} \zeta \cdot n = 0. \]

Since, under assumption (64), (73) and (74) are incompatible, we conclude that (68) is true. Therefore, by the compactness theorem of F. Rellich from \( \{w_k\}_{k \in \mathbb{N}} \) we can extract a subsequence which converges strongly in \( L^q_{\text{loc}}(\Omega) \) and weakly in \( D^{1,2}(\Omega) \) to a field \( w \in D^{1,1}_{\text{reg}}(\Omega) \) that a well–known argument shows to be a solution of equations (66) (see, e.g., [38] Ch. 5).

\[ (65) \] is proved in [12], [16], while (30), (38), (39) are consequence of the fact that \( u \) is a \( D \)-solution. As far as the boundary datum is concerned, let us note that \( u = h + w \) attains \( a \) in the following sense

\[ h \rightharpoonup a, \quad \text{tr}_{\partial \Omega} w = 0, \]
where \( \text{tr}_{\partial \Omega} \) stands for the trace operator in the Sobolev space \( D^{1,2}_0(\Omega) \). If \( a \in L^q(\partial \Omega) \) (\( q > 2 \)) then \( h \in L^{2q}(\Omega) \) so that by well–known estimates about solution of the Stokes problem \( w \in W^{1,s}_{\text{loc}}(\Omega) \), for some \( s > 2 \). Hence by Sobolev’s lemma it follows that \( w \) is continuous in \( \overline{\Omega} \) and \( u \xrightarrow{nt} a \). Of course, if \( a \in C(\partial \Omega) \), then \( u \in C^\infty(\Omega) \cap C(\overline{\Omega}) \). Moreover, (1)–(4) are consequence of (1)–(4).

It is not difficult to see that the above argument can be repeated for boundary data \( a \in W^{-1/q,q}(\partial \Omega) \), \( q \geq 4 \), provided we make use of the divergence free extension of \( a \) defined in Lemma 2 and assume that

\[
|\langle a, n \rangle| < 2\pi.
\]

Indeed, the following theorem holds.

**Theorem 5.** Let \( \Omega \) be an exterior domain of \( \mathbb{R}^2 \) of class \( C^{1,1} \). If \( a \in W^{-1/q,q}(\partial \Omega) \), \( q \geq 4 \), satisfies (75), then (2) has a D–solution

\[
u \in L^q_{\text{loc}}(\overline{\Omega}) \cap L^\infty(C_{R_0}).
\]

Moreover,

- if \( a \in C^{1,\mu}(\partial \Omega) \), \( \mu \in (0,1) \), then

\[
(u,p) \in C^{1,\mu}_{\text{loc}}(\overline{\Omega}) \times C^{0,\mu}_{\text{loc}}(\overline{\Omega}),
\]

- if \( \Omega \) is of class \( C^k \) (\( k \geq 2 \)) and \( a \in W^{k-1/q,q}(\partial \Omega) \), then

\[
(u,p) \in W^{k,q}_{\text{loc}}(\overline{\Omega}) \times W^{k-1,q}_{\text{loc}}(\overline{\Omega}).
\]

Let \( \Omega \) be polar symmetric, i.e,

\[
(x_1, x_2) \in \Omega \Rightarrow (-x_1, -x_2) \in \Omega.
\]

If \( a \) is polar symmetric, i.e,

\[
a(\zeta) = -a(-\zeta),
\]

for all \( \zeta \in \partial \Omega \), then the field \( h \) can be constructed polar symmetric and we can find a polar symmetric solution of (66). As a consequence, the solution \( (u,p) \) in Theorem 4 satisfies the symmetry properties

\[
u(x) = -u(-x),
\]

\[
p(x) = p(-x).
\]
for all \( x \in \Omega \). Since by (77),
\[
\int_0^{2\pi} u(R, \theta) = 0,
\]
for all \( R > R_0 \), by Poincaré’s inequality we get
\[
\int_{T_R} |u|^2 \leq cR^2 \int_{T_R} |\nabla u|^2,
\]
with \( c \) independent of \( R \). Therefore, by the trace theorem and (78)
\[
\int_0^{2\pi} |u|^2(R, \theta) \leq c \left\{ \frac{1}{R^2} \int_{T_R} |u|^2 + \int_{T_R} |\nabla u|^2 \right\} \leq c \int_{\mathcal{C}_R} |\nabla u|^2,
\]
with \( c \) independent of \( R \). Hence it follows
\[
\lim_{R \to +\infty} \int_0^{2\pi} |u|^2(R, \theta) = 0.
\]
By virtue of the results of [12], [17], (80) is sufficient to conclude that
\[
\lim_{r \to +\infty} u(r, \theta) = 0,
\]
uniformly in \( \theta \). Therefore we can state

**Theorem 6.** Let \( \Omega \) be a polar symmetric exterior Lipschitz domain of \( \mathbb{R}^2 \). If \( a \in L^2(\partial \Omega) \) is polar symmetric and satisfies (64), then (2), (81) has a Leray solution which satisfies (1)–(11) and (30), (65), (38), (39). If \( \Omega \) is of class \( C^{1,1} \), then we can assume \( a \in W^{-1/4,4}(\partial \Omega) \).

It is evident that (79) holds for every polar symmetric \( D \)-solution. Hence it follows

**Theorem 7.** A polar symmetric \( D \)-solution tends to zero at infinity.

The Hamel solutions (55) are polar symmetric and for \( \gamma < -1 \) have finite Dirichlet integrals. Since we can choose \( \gamma \) close to \( -1 \) as we want, we see that Theorem 7 is sharp in the sense that (at least for \( \gamma < -1 \)) a polar symmetric solution cannot tend to zero at infinity as \( r^{-\varepsilon} \) for some positive \( \varepsilon \). Note that by virtue of (57) these considerations do not apply to the \( D \)-solution of Theorem 6.

**Remark 4.6** - Existence of a solution of (2) with less regular boundary data (say in \( L^1(\partial \Omega) \) and \( W^{-1/q,q}(\partial \Omega) \)) have been studied by several authors for
bounded and regular domains with connected boundaries (see [1] [14], [15], [34] and the references therein). As far as Lipschitz domains are concerned, to the best of our knowledge problem (2) (with $L^q(\partial \Omega)$ data) has been considered only for bounded domains in [31], [33], [34] under a restriction on the flux, in [7] for small data and in [32] for domains symmetric with respect to the $x_1$ axis, $a_1$ pair function of $x_2$ and $a_2$ odd function of $x_2$. In [30] the classical Finn-Smith theorem [9] has been proved for Lipschitz domains and boundary data in $L^\infty(\partial \Omega)$.

As we said in the introduction, there is another technique, based on a Galerkin’s type scheme and due to H. Fujita [10] to prove existence of a $D$–solution of (2), we shall call Fujita solution. It reduces the problem to find the uniform estimate

$$\int_{\Omega} |\nabla w|^2 \leq c,$$

for every solution $w \in D^{1,2}_0(\Omega)$ with compact support in $\Omega$ of the system\(^{17}\)

$$\begin{align*}
\Delta w - (h + w) \cdot \nabla (h + w) + \Delta \zeta &= \nabla Q \quad \text{in } \Omega, \\
\text{div } w &= 0 \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

(82)

where now $h$ has the form (17) with $\zeta = \nabla^\perp (g_{\delta_0}, \eta)$, $v = \nabla^\perp \eta$ and $g_{\delta_0}$ Leray–Hopf cut–off function of the regularized distance $g(x)$ equal to 1 for $g(x) \leq c_1 \delta_0$ and vanishing for $g(x) \geq c_2 \delta_0$. A straightforward calculation yields the relation

$$\int_{\Omega} |\nabla w|^2 \leq \int_{\Omega} w \cdot \nabla w \cdot e_r + c(\delta_0) \left\{ \int_{\partial \Omega} |a|^2 + \left[ \int_{\partial \Omega} |a|^2 \right]^2 \right\}.$$ 

By a classical procedure we have (see, e.g., [11], [40])

$$\left| \int_{\Omega} w \cdot \nabla w \cdot \zeta \right| \leq \alpha(\delta_0) \int_{\Omega} |\nabla w|^2,$$

with

$$\lim_{\delta_0 \to 0} \alpha(\delta_0) = 0.$$ 

Moreover, by (22)

$$\left| \int_{\Omega} w \cdot \nabla w \cdot \frac{e_r}{r} \right| \leq \int_{\Omega} |\nabla w|^2.$$ 

\(^{17}\)Clear expositions of this approach can be find in [11], [20], [40].
so that if (64) holds, then there is a constant $c$ independent of $w$ such that

$$
\int_{\Omega} |\nabla w|^2 \leq c \left\{ \int_{\partial \Omega} |a|^2 + \left[ \int_{\partial \Omega} |a|^2 \right]^2 \right\}.
$$

Therefore, taking also into account (15), we have

**Theorem 8.** Let $\Omega$ be an exterior Lipschitz domain of $\mathbb{R}^2$. If $a$ satisfies (63) and (64) then (2) has a Fujita solution $(u, p)$ such that

$$
\int_{\Omega} \delta |\nabla u|^2 \leq c \left\{ \int_{\partial \Omega} |a|^2 + \left[ \int_{\partial \Omega} |a|^2 \right]^2 \right\}.
$$

**Theorem 9.** Let $\Omega$ be an exterior domain of $\mathbb{R}^2$ of class $C^{1,1}$. If $a \in W^{-1/4,4}(\partial \Omega)$, satisfies (75), then (2) has a $D$–solution $u \in L^4_{\text{loc}}(\Omega)$.

It is quite evident that the Fujita solutions enjoys all the regularity properties as those of the Leray solution. The only substantial difference is that the latter is always bounded while by Theorem 2 we know that the former is bounded for zero outflow.

**Remark 4.7 -** It is not difficult to see that Theorems 2 – 9 can be stated for the system

$$
\begin{align*}
\Delta u - u \cdot \nabla u - \nabla p &= f & \text{in } \Omega, \\
\text{div } u &= 0 & \text{in } \Omega, \\
u &= a & \text{on } \partial\Omega
\end{align*}
$$

in the more general exterior domain

$$
\Omega = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \Omega_i, \quad \Omega_i = \bigcup_{i=1}^m \Omega_i, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j,
$$

with $\partial \Omega_i$ Lipschitz and connected, provided

$$
f \in H^1(\Omega) \tag{87}
$$

---

18 Lemma 1 continues to hold for the domain (86); in such a case

$$
\sigma = -\frac{1}{2\pi} \sum_{i=1}^m \frac{(x-x_i)}{|x-x_i|^2} \int_{\partial \Omega_i} a \cdot n,
$$

where $x_i$ is a fixed point of $\Omega_i$. 

26
vanishes outside a bounded set\textsuperscript{19}, is polar symmetric in Theorems \textit{6, 7} and
\[
\frac{1}{2\pi} \sum_{i=1}^{m} \left| \int_{\partial \Omega_i} a \cdot n \right| < 1
\]
Under assumption (87) \((u, p)\) satisfies (85) almost everywhere in \(\Omega\) and \(u\) is continuous in \(\Omega\) [3]. Moreover, if \(\text{div } f \in H^1(\Omega)\), then \(p\) is continuous in \(\Omega\). Moreover, (84) becomes
\[
\|u\|_{D_1^2(\Omega)} \leq c \left\{ \|a\|_{L^2(\partial \Omega)} + \|f\|_{H^1} + \left[ \|a\|_{L^2(\partial \Omega)} + \|f\|_{H^1} \right]^2 \right\}.
\]
5 A uniqueness theorem

Uniqueness of a \(D\)–solution converging to a nonzero vector at infinity\textsuperscript{20} is a complicated question and only in few cases we know how to determine small uniqueness classes (see [9] and [11] Ch. X). We aim at observing now as uniqueness could be linked with the boundary data at least in particular situations: \textit{the potential flows}.

Let us consider the harmonic simple layer potential with density \(\psi\textsuperscript{21}\)
\[
v(x) = \frac{1}{2\pi} \int_{\partial \Omega} \psi(\zeta) \log |x - \zeta| d\zeta
\]
and the Navier–Stokes problem
\[
\Delta u - u \cdot \nabla u = \nabla p \quad \text{in } \Omega,
\]
\[
\text{div } u = 0 \quad \text{in } \Omega,
\]
\[
\begin{align*}
\Delta u &= 0, \quad u = a \quad \text{on } \partial \Omega, \\
\end{align*}
\]
with the boundary datum
\[
a(\xi) = \nabla v(\xi) + e_1.
\]
\textsuperscript{19}This is not necessary for the existence of a \(D\)–solution to (85). It is worthy to note that for the validity of Theorem 2 it is sufficient that (34) holds for a circumference surrounding \(\Omega\).
\textsuperscript{20}Recall that uniqueness does not hold when the \(D\)–solution is zero at infinity at least for large Reynolds numbers (see Remark 3.4).
\textsuperscript{21}It can be proved that every harmonic function \(u\) in a Lipschitz exterior domain \(\Omega\) of \(\mathbb{R}^2\) such that \(u = o(r)\) and \(\text{tr}_{\partial \Omega} u \in W^{1,2}(\partial \Omega)\), is expressed by (88) for some \(\psi \in L^2(\partial \Omega)\).
Note that by
\[ \int_{\partial \Omega} a \cdot n = \int_{\partial \Omega} \partial_n v = \int_{\partial \Omega} \psi. \]

The pair
\[ (\nabla v + e_1, -\frac{1}{2}|\nabla v|^2 - \partial_1 v) \]
is a \( D \)-solution to (89), (90). By what we said above, we could have other \textit{a priori} different \( D \)-solutions of (89)–(90), as the Finn–Smith, Galdi, Leray and Fujita solutions. Let shows that if
\[ \int_{\partial \Omega} |\psi| < 2\pi, \]
then all these solutions coincide. Indeed, the following theorem holds.

**Theorem 10.** Let \( \Omega \) be an exterior Lipschitz domain of \( \mathbb{R}^2 \). If \( \psi \in L^2(\partial \Omega) \) satisfies (92), then (91) is unique in the class of all \( D \)-solutions.

**Proof.** Let \((u + w, p + Q)\) be another \( D \)-solution to (2), (90). Then \((w, Q)\) satisfies the equation
\[ \Delta w - (u + w) \cdot \nabla w - w \cdot \nabla u = \nabla Q \quad \text{in } \Omega, \]
\[ \text{div } w = 0 \quad \text{in } \Omega, \]
\[ w = 0 \quad \text{on } \partial \Omega, \]
\[ w = o(1). \]

Let \( g(r) \) be a regular function, equal to 1 in \( C_R \), vanishing outside \( C_{2R} \) and such that \( |\nabla g| \leq cR^{-1} \). Then by a standard computation we get
\[ \int_{\Omega} g|\nabla w|^2 = \int_{\Omega} \left[ \frac{1}{2}|w|^2(u + w) + (u \cdot w + Q)w \right] \cdot \nabla g \]
\[ + \int_{\Omega} gw \cdot \nabla w \cdot u. \]
By Hölder’s inequality and (53)

\[
\int_{\Omega} |w|^2 (u - e_1) \cdot \nabla g \leq \frac{c}{R} \left\{ \int_{T_R} |u - e_1|^4 \right\}^{1/4} \left\{ \int_{T_R} |w|^4 \right\}^{1/4} \left\{ \int_{T_R} \right\}^{1/2} = o(1),
\]

\[
\int_{\Omega} |w|^2 \partial_1 g \leq \frac{c}{R} \left\{ \int_{T_R} |w|^4 \right\}^{1/2} \left\{ \int_{T_R} \right\}^{1/2} = o(1),
\]

\[
\int_{\Omega} |w|^2 w \cdot \nabla g \leq \frac{c}{R} \left\{ \int_{T_R} |w|^6 \right\}^{1/3} \left\{ \int_{T_R} \right\}^{1/2} = o(1),
\]

\[
\int_{\Omega} Q w \cdot \nabla g \leq \frac{c}{R} \left\{ \int_{T_R} |w|^4 \int_{T_R} |Q|^4 \right\}^{1/4} \left\{ \int_{T_R} \right\}^{1/2} = o(1).
\]

Likewise,

\[
\int_{\Omega} (u \cdot w) w \cdot \nabla g = o(1).
\]

Moreover,

\[
\int_{\Omega} g w \cdot \nabla w = \int_{T_R} w_1 w \cdot \nabla g \leq \frac{c}{R} \left\{ \int_{T_R} |w|^4 \right\}^{1/2} \left\{ \int_{T_R} \right\}^{1/2} = o(1)
\]

By (22)

\[
\int_{\Omega} w \cdot \nabla w \cdot \nabla v = \int_{\Omega} v \nabla w \cdot \nabla w^T = \frac{1}{2\pi} \int_{\partial \Omega} \psi(\zeta) \int_{\Omega} \nabla w \cdot \nabla w^T \log |x - \zeta| \, da_x
\]

\[
= \frac{1}{2\pi} \int_{\partial \Omega} \psi(\zeta) \int_{\Omega} \nabla w \cdot \nabla w \leq \frac{\|\psi\|_{L^1(\partial \Omega)}}{2\pi} \int_{\Omega} |\nabla w|^2.
\]

for all \(w \in D_{\sigma,2}^{1,2}(\Omega)\). Therefore, letting \(R \to +\infty\) in (94), we have

\[
(2\pi - \|\psi\|_{L^1(\partial \Omega)}) \int_{\Omega} |\nabla w|^2 \leq 0.
\]

Hence uniqueness follows at once. \(\square\)

**Remark 5.8** - Note that if

\[
v(x) = \frac{\mu \log r}{2\pi},
\]

29
then

\[
\int_{\partial \Omega} \mathbf{a} \cdot \mathbf{n} = \mu
\]

and (92) takes the weaker form.

\[|\mu| < 2\pi.\]

\[\diamond\]

**Remark 5.9** - When \(\partial \Omega\) is connected and \(\mathbf{a} = 0\), \(u_0 = e_1\), a solution of the equations

\[
\Delta \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} = \nabla p \quad \text{in } \Omega, \\
\text{div } \mathbf{u} = 0 \quad \text{in } \Omega, \\
\mathbf{u} = \mathbf{0} \quad \text{on } \partial \Omega, \\
\lim_{r \to +\infty} \mathbf{u}(x) = e_1
\]

(95)

represents the translational motion (with velocity \(-e_1\)) of an object in a Navier–Stokes fluid assumed to be at rest at infinity. As we remarked in this paper, problem (95) is completely open. By the Leray argument we know that the sequence of solutions of the systems

\[
\Delta \mathbf{u}_k - \mathbf{u}_k \cdot \nabla \mathbf{u}_k = \nabla p_k \quad \text{in } \Omega_k, \\
\text{div } \mathbf{u}_k = 0 \quad \text{in } \Omega_k, \\
\mathbf{u}_k = \mathbf{0} \quad \text{on } \partial \Omega_k, \\
\mathbf{u}_k = e_1 \quad \text{on } \partial C_k
\]

(96)

converges to a \(D\)-solution to (95)1,2,3 and there is a constant vector \(\mathbf{u}_0\) such that [16]

\[\lim_{r \to +\infty} \mathbf{u}(r, \theta) = \mathbf{u}_0,\]

uniformly on \(\theta\). However, we do not know \(\mathbf{u}_0\) so that in principle it could be zero and the Leray construction could even yield the trivial solution, as it happens for the Stokes paradox (see Section 6). C.J. Amick excluded this possibility for domains of class \(C^3\), symmetric with respect to the \(x_1\)-axis [2] (see also [12]). This result has been recently extended to symmetric Lipschitz domains in [29].
6 Some remarks on the Stokes paradox

More in general, introducing the Reynolds number \( \lambda = vl/\nu \), with \( v, l \) reference velocity and reference length, and \( \nu \) kinematical viscosity of the fluid, the steady–state Navier–Stokes problem in an exterior Lipschitz domain \( \Omega \) of \( \mathbb{R}^2 \) writes

\[
\begin{align*}
\Delta u - \lambda u \cdot \nabla u - \nabla p &= f & \text{in } \Omega, \\
\text{div } u &= 0 & \text{in } \Omega, \\
u = a &\quad \text{on } \partial \Omega, \\
\lim_{r \to +\infty} u(x) &= u_0.
\end{align*}
\]

(97)

Of course, for \( \lambda = 0 \) (97) reduces to the Stokes problem

\[
\begin{align*}
\Delta u - \nabla p &= f & \text{in } \Omega, \\
\text{div } u &= 0 & \text{in } \Omega, \\
u = a &\quad \text{on } \partial \Omega, \\
\lim_{r \to +\infty} u(x) &= u_0.
\end{align*}
\]

(98)

In this section we aim at comparing the known results for systems (97), (98). It is well–known a \( D \)–solution of (98)1,2,3 exists and converges to a constant vector, but contrary to what happens in the nonlinear case, we know that (98) has a solution if and only if \( a \in L^2(\partial \Omega) \), \( f \in H^1 \) and \( u_0 \) satisfy the compatibility condition [13], [34]

\[
\int_{\partial \Omega} (a - u_0) \cdot T(h_i, p_i) \cdot n + \int_{\Omega} f \cdot h_i = 0, \quad i = 1, 2,
\]

(99)

with \((h_i, p_i)\) solution of\(^{22}\)

\[
\begin{align*}
\Delta h_i &= \nabla p_i & \text{in } \Omega, \\
\text{div } h_i &= 0 & \text{in } \Omega, \\
h_i &= e_i & \text{on } \partial \Omega, \\
h_i &= o(r).
\end{align*}
\]

(100)

For instance, if \( \partial \Omega \) is an ellipse, then \([T(h_i, p_i) \cdot n](\xi) = (\xi \cdot n(\xi))e_i \) [23] and for \( f = 0 \) (99) writes

\[
\int_{\partial \Omega} (a - u_0)_i(\xi \cdot n(\xi)) = 0, \quad i = 1, 2.
\]

\(^{22}\)The solutions of (100) span a linear space of dimension two and every \( h \) behaves at infinity as \( \log r \).
In particular, if \( u \) is a \( D \)-solution of (98)\(_{1,2,3}\), then for large \( R \)

\[
\frac{1}{2\pi} \int_0^{2\pi} u_i(R, \theta) \, d\theta
\]

is constant and gives the vector to which \( u \) tends at infinity (Picone’s mean theorem at infinity).

Since

\[
\int_{\partial \Omega} T(h_i, p_i) \cdot \mathbf{n} \neq 0,
\]

from (99) it follows that if \( a = f = 0 \), then the only solution to (98) is the trivial one so that necessarily \( u_0 \) must be zero (Stokes’ paradox). The results of R. Finn & D.R. Smith \([9]\) (see also \([11, 30]\)) and C.J. Amick \([2]\) (see Remark 5.9) allow us to state

**Theorem 11.** Let \( \Omega \) be an exterior Lipschitz domain of \( \mathbb{R}^2 \). If \( \lambda \) is sufficiently small or \( \Omega \) is symmetric with respect to an axis, then the Stokes paradox holds if and only if \( \lambda = 0 \).

Of course, a \( D \)-solution of (97) must satisfy (99) whenever the integrals

\[
\int_{\Omega} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{h}
\]

make sense. In particular, taking into account that if \( \Omega \) is polar symmetric, then \( h(x) = h(-x) \) for all \( x \in \Omega \), we see that the solution of Theorem 6 satisfy (99).

**References**

[1] C. Amrouche and M.A. Rodríguez-Bellido: Stationary Stokes, Oseen and Navier-Stokes equations with singular data, *Arch. Rational Mech. Anal.* online DOI: 10.1007/s00205-010-0340-8 (2010).

[2] C.J. Amick: On Leray’s problem of steady Navier–Stokes flow past a body, *Acta Math.* 161 (1988), 71–130.

[3] A. Bensoussan and J. Frehse: *Regularity results for nonlinear elliptic systems and applications*, Springer (2002).

[4] W. Borchers and K. Pileckas: Note on the flux problem for stationary Navier–Stokes equations in domains with multiply connected boundary, *Acta App. Math.* 37 (1994), 21–30.
[5] R.R. Coifman, J.L. Lions, Y. Meier and S. Semmes: Compensated compactness and Hardy spaces, *J. Math. Pures App.* IX Sér. 72, 247–286 (1993).

[6] R. Dautray and J.L. Lions: *Mathematical Analysis and Numerical Methods for Science and Technology; Vol. 4: Integral Equations and Numerical Methods*, Springer (2000).

[7] M. Dindoš and M. Mitrea: The stationary Navier-Stokes system in non-smooth manifolds: the Poisson problem in Lipschitz and $C^1$ Domains, *Arch. Rational Mech. Anal.* 174 (2004), 1–47.

[8] E. B. Fabes, C. E. Kenig and G. Verchota C.: Boundary value problems for the Stokes system on Lipschitz domains, *Duke Math. J.* 57 (1988), 769–793.

[9] R. Finn and D.R. Smith: On the stationary solutions of the Navier–Stokes equations in two dimensions, *Arch. Rational Mech. Anal.* 25 1967 26–39.

[10] H. Fujita: On the existence and regularity of the steady-state solutions of the Navier-Stokes theorem, *J. Fac. Sci. Univ. Tokyo* 9 (1961), 59-102.

[11] G.P. Galdi: *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, vol. I, II revised edition, Springer Tracts in Natural Philosophy (ed. C. Truesdell) 38, 39, Springer–Verlag (1998).

[12] G.P. Galdi: Stationary Navier-Stokes problem in a two-dimensional exterior domain, in *Stationary partial differential equations* Vol. I, 71–155, Handb. Differ. Equ., North-Holland Elsevier Science, (2004), 71-156.

[13] G.P. Galdi and C. G. Simader: Existence, uniqueness and $L^q$ estimates for the Stokes problem in an exterior domain, *Arch. Rational Mech. Anal.* 112 (1990), 291–318.

[14] G.P. Galdi, C. G. Simader and H. Sohr: A class of solutions to stationary Stokes and Navier-Stokes equations with boundary data in $W^{-1/{q}', q}$, *Math. Ann.* 331, 41–74 (2005).

[15] Y. Giga: Analyticity of the semigroup generated by the Stokes operator in $L_r$ spaces, *Math. Z.* 178 (1981), 297–329.

[16] D. Gilbarg and H. Weinberger: Asymptotic properties of Leray’s solutions of the stationary two-dimensional Navier–Stokes equations, *Russian Math. Surveys* 29 (1974), 109–123.

[17] D. Gilbarg and H. Weinberger: Asymptotic properties of steady plane solutions of the Navier–Stokes equations with bounded Dirichlet integral, *Ann. Scuola Norm. Sup. Pisa* (4) 5 (1978), 381–404.

[18] D. Jerison and C.E. Kenig: The inhomogeneous Dirichle problem in Lipschitz domains, *J. Funct. Anal.* 130 (1995), 161–219.

33
L.V. Kapitanskii and K. Pileckas, On spaces of solenoidal vector fields and boundary value problems for the Navier-Stokes equations in domains with noncompact boundaries, Trudy Mat. Inst. Steklov, 159 (1983), 5–36. English Transl.: Proc. Math. Inst. Steklov, 159 (1984), 3–34.

O.A. Ladyzhenskaia: The Mathematical theory of viscous incompressible fluid, Gordon and Breach (1969).

J. Leray: Étude de diverses équations intégrales non linéaire et de quelques problèmes que pose l’hydrodynamique, J. Math. Pures Appl. 12 (1933), 1–82.

Miranda, C.: Partial differential equations of elliptic type, Springer-Verlag, 1970.

Maremonti, P., Russo, R. & Starita, G.: On the Stokes equations: the boundary value problem. Quad. Mat. 4 (1999), 69–140.

M. Mitrea and M. Taylor: Navier–Stokes equations on Lipschitz domains in Riemannian manifolds, Math. Ann. 321 (2001), 955–987.

K. Pileckas: On spaces of solenoidal vectors, Trudy Mat. Inst. Steklov 159 (1984), 137–149; English Transl.: Proc. Steklov Math. Inst. 159 (1984), 141–154.

A. Russo: A note on the two-dimensional steady-state Navier–Stokes problem, J. Math. Fluid Mech. 2007 on line, 11 (2009) 407–414.

A. Russo: The steady-state Navier–Stokes equations in dimension two, PhD thesis, Department of Mathematics, Second University of Naples (2009).

A. Russo: On the asymptotic behavior of $D$–solutions of the plane steady–state Navier–Stokes equations, to appear on Pacific J. Math. 246 (2010), 253–256.

A. Russo: On symmetric Leray solutions of the stationary Navier–Stokes equations, Ricerche Mat., to appear.

A. Russo: On a classical theorem of plane viscous hydrodynamics, preprint Dipartimento di Matematica, Seconda Università di Napoli (3) 2010, http://www.dimat.unina2.it

A. Russo and G. Starita: On the existence of steady-state solutions to the Navier–Stokes system for large fluxes, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 7 (2008), 171–180.

A. Russo and G. Starita: A mixed problem for the steady Navier-Stokes equations, Math. Comput. Model. 49 (2009), 681–688.

R. Russo: On the existence of solutions to the stationary Navier–Stokes equations, Ricerche Mat. 52 (2003), 285–348.

R. Russo: On Stokes’ problem, in Advances in Mathematical Fluid Mechanics, R. Rannacher and A. Sequeira Eds, DOI 10.1007/978-3-642-04068-9, 473–511 Springer-Verlag Berlin Heidelberg (2010).

34
[35] L.I. Sazonov: On the asymptotic behavior of the solution of the two-dimensional stationary problem of the flow past a body far from it, (Russian) Mat. Zametki 65 (1999), 246-253; English translation in Math. Notes 65 (1999), 202-207. 3, 17

[36] Z. Shen: A note on the Dirichlet problem for the Stokes system in Lipschitz domains, Proc. Amer. Math. Soc. 123 (1995), 801–811. 6

[37] D.R. Smith: Estimates at infinity for stationary solutions of Navier-Stokes equations, Arch. Rational Mech. Anal. (1965) 20 341-372. 3, 17

[38] H. Sohr: Navier Stokes equations. An Elementary Functional Analytical Approach, Birkhäuser Verlag (2001). 22

[39] E. Stein: Harmonic analysis: real–variables methods, orthogonality and oscillatory integrals, Princeton University Press (1993). 4, 10, 18

[40] R. Temam: Navier–Stokes equations, North–Holland (1977). 25