A TRANSFORMATION FORMULA RELATING RESOLVENTS OF BEREZIN-TOEPLITZ OPERATORS BY AN INVARIANCE PROPERTY OF BROWNIAN MOTION

BERNHARD G. BODMANN

Abstract. Using a stochastic representation provided by Wiener-regularized path integrals for the semigroups generated by certain Berezin-Toeplitz operators, a transformation formula for their resolvents is derived. The key property used in the transformation of the stochastic representation is that, up to a time change, Brownian motion is invariant under harmonic morphisms. This result for Berezin-Toeplitz operators is obtained in analogy with a well-known technique generating relations among Schrödinger operators that was recently generalized to Riemannian manifolds [Wittich, J. Math. Phys. 41 (2000), 244].

1. Introduction

The idea for the results presented here is taken from a transformation formula relating resolvents of certain Schrödinger operators [DK79, Bla82, CS90]. Recently, Wittich [Wit00] proved this formula and a generalization in the setting of Riemannian manifolds with the help of an invariance property of Brownian motion under harmonic morphisms.

An analogous strategy applied to the probabilistic representation of Berezin-Toeplitz semigroups by Wiener-regularized path integrals [Bod, Sec. 4] gives a relationship between resolvents of different Berezin-Toeplitz operators whenever the base manifolds of two holomorphic line bundles have Kähler structures that are locally conformally equivalent. Such relationships can link operators that model quantum mechanical systems with qualitatively different degrees of freedom, for example those described by various Lie groups [Per86, BMM96].

An intermediate step in the derivation of the main result is to relate resolvents of Schrödinger operators arising as perturbations of the Bochner Laplacian in complex line bundles. The technique of a stochastic time change in path-integral representations of such operators is already known [Sto94]. Here, we combine harmonic morphisms with an appropriate time change to exploit the invariance property of Brownian motion and to establish a relationship between resolvents of Schrödinger operators. The main result is then derived from the intermediate step by a monotone limit of certain Schrödinger operators.

Unfortunately, the rigidity of harmonic morphisms only allows a rather trivial result when the base manifolds have complex dimension higher than one [Fug78]. Therefore, we restrict the discussion to Riemann surfaces.

Date: November 13, 2021.

Key words and phrases. Berezin-Toeplitz operators, Wiener-regularized path integrals, harmonic morphisms; 81S10, 58D30 (MSC 2000).
This paper is organized as follows: In Section 2, we fix the notation and briefly explain the relevant background information. Section 3 contains the statement of the transformation formula as the main result and a few illustrating examples, followed by the proof.

2. Basic Definitions and Concepts

2.1. Hilbert spaces of square-integrable, holomorphic sections.

**Definition 2.1.** Let $\mathcal{M}$ be a Riemann surface, that is, a complex manifold of dimension one, and let $\mathcal{L}$ be a holomorphic line bundle over $\mathcal{M}$, equipped with a Hermitian metric $h = \{h_x\}_{x \in \mathcal{M}}$ on its fibers. To be precise, for each base point $x \in \mathcal{M}$ there is a sesquilinear metric $h_x : \mathcal{L}_x \times \mathcal{L}_x \to \mathbb{C}$ on the associated fiber $\mathcal{L}_x$ and by convention, each $h_x$ is conjugate linear in the first argument. Given a measure $m$ on $\mathcal{M}$, we may define an inner product $$(\psi, \phi) := \int_{\mathcal{M}} h_x(\psi(x), \phi(x)) \, dm(x) \quad (1)$$ for sufficiently regular sections $\psi$ and $\phi$.

The linear space of sections in $\mathcal{L}$ will be denoted as $\Gamma_\mathcal{L}(\mathcal{M})$. The subspace of square-integrable sections on a complex line bundle $\mathcal{L}$ over a base manifold $\mathcal{M}$ is denoted by

$$L^2(hm) := \left\{ \psi \in \Gamma_\mathcal{L}(\mathcal{M}) : \int_{\mathcal{M}} h_x(\psi(x), \psi(x)) \, dm(x) < \infty \right\}. \quad (2)$$

When $\mathcal{L}$ is a holomorphic line bundle, we define the generalized Bergman space $L^2_{hol}(hm)$ as the space of all holomorphic sections in $L^2(hm)$.

**Remarks 2.2.** Equipped with the previously defined inner product, the space $L^2(hm)$ containing all square-integrable sections becomes a Hilbert space in the usual way by identifying sections that do not differ up to sets of measure zero.

If $\mathcal{L}$ is a holomorphic line bundle and $m$, interpreted as a volume form, as well as $h$ are everywhere non-degenerate and smooth, then the generalized Bergman space $L^2_{hol}(hm)$ is a space of functions that may be identified with a Hilbert-subspace of $L^2(hm)$ [Bod, Sec. 2].

**Examples 2.3.** Unless otherwise noted, the generalized Bergman spaces cited as examples may be obtained from unitary irreducible Lie group representations as described by Onofri [Ono75].

1. **Fock-Bargmann space.** Let $\mathcal{M} = \mathbb{C}$ and $\mathcal{L} = \mathcal{M} \times \mathbb{C}$. The volume measure on $\mathcal{M}$ is simply the normalized Lebesgue measure $dm = d^2z/\pi = dz_1dz_2/\pi$, where $z_1, z_2 \in \mathbb{R}$ denote the real and imaginary parts of $z = z_1 + iz_2$. Every vector at $z \in \mathcal{M}$ can be thought of as a pair $(z, u)$, $u \in \mathbb{C}$. The Hermitian metric on the fibers of $\mathcal{L}$ is defined over a base point $z \in \mathbb{C}$ by $h_z((z, u), (z, v)) = e^{-|z|^2} uv$. The space $L^2_{hol}(hm)$ obtained in this setting is infinite-dimensional, since the sections $z \mapsto (z, z^n)$, $n \in \mathbb{N}$ are square-integrable and pairwise orthogonal. This space is related to a representation of the Heisenberg-Weyl group [Bar61].

2. **Barut-Girardello space.** The preceding example with instead the Hermitian metric $h_z((z, u), (z, v)) = e^{-|z|^2} uv$ produces a space that is also infinite-dimensional and relates to $SU(1, 1)$ [BG71].
(3) **Generalized Bergman space over powers of the tautological bundle.** Let \( \mathcal{E}^x = \mathbb{C}^2 \setminus \{0\} \) and \( \mathcal{M} = \mathbb{C}P^1 \) that is obtained by identifying two nonzero vectors \( w = (w^{(1)}, w^{(2)}) \) and \( w' = (w'^{(1)}, w'^{(2)}) \) whenever they are collinear, \( w = cw' \) for some \( c \in \mathbb{C} \setminus \{0\} \). The equivalence class of \( w \) will be written as \([w]\). We choose local coordinates given on the set \( U_1 := \{ |w| : w \in \mathbb{C}^2, w^{(1)} \neq 0 \} \) by \( \phi_1([w]) := w^{(2)}/w^{(1)} \) and analogously on \( U_2 \) by flipping \( w^{(1)} \) and \( w^{(2)} \). Filling in the missing zero in each fiber \( L_w \), that is, the vectors belonging to the equivalence class \([w]\), would yield the so-called tautological line bundle. We consider, more generally, powers of this bundle by picking an integer \( k \in \mathbb{Z} \) and specifying the transition functions between the local trivializations over \( U_1 \) and \( U_2 \) as \( t_{1,2} := (w^{(2)}/w^{(1)})^k \) and \( t_{2,1} \) with \( w^{(1)}, w^{(2)} \) exchanged. When \( k < 0 \), there is no global holomorphic section, which makes a nontrivial \( L^2_{\text{hol}}(hm) \) impossible. However, if \( k \) is zero or a positive integer, the vector space of holomorphic sections is \( k + 1 \)-dimensional \([\text{Wel}80]\). One may turn it into a Hilbert space by using the Hermitian metric from the inner product on \( \mathbb{C}^2 \) and choose the measure \( m \) to be invariant under the action of complex automorphisms on the base manifold. The action of \( SU(2) \) on \( \mathbb{C}^2 \) yields for different choices of \( \mathcal{G} \) the unitarily inequivalent irreducible \( SU(2) \) representations on the corresponding generalized Bergman spaces \([\text{Per}86]\).

(4) **A space of theta functions as generalized Bergman space.** Given a lattice \( \mathcal{G} := \{ t_1 \epsilon_1 + t_2 \epsilon_2 : t \in \mathbb{Z}^2 \} \) with spacings \( \epsilon_{1,2} \in \mathbb{C} \) that are linearly independent over \( \mathbb{R} \), we consider \( \mathcal{M} \) as the quotient \( \mathbb{C}/\mathcal{G} \). The underlying identification is understood as the equivalence relation \( z \sim z' \) between \( z \) and \( z' \) in \( \mathbb{C}^n \) whenever \( z = z' + t_1 \epsilon_1 + t_2 \epsilon_2 \) for some choice of \( t \in \mathbb{Z}^2 \). The resulting compact manifold is called a complex torus. Consider the space of holomorphic functions \( \phi \) on \( \mathbb{C} \) that satisfy \( \phi(z + \epsilon_1) = e^{-i\pi \omega_{12}} e^{\frac{i\pi}{2} \epsilon_{1}\epsilon_{2} z^2} \phi(z) \) and \( \phi(z + \epsilon_2) = e^{-i\pi \omega_{12}} e^{\frac{i\pi}{2} \epsilon_{2}\epsilon_{1} z^2} \phi(z) \), with \( \omega_{1,2} \in \mathbb{R} \) and \( \frac{1}{2} \text{Im} (\epsilon_{1}\epsilon_{2}) = k \in \mathbb{Z} \) which is seen to be a space of dimension \(|k|\) \([\text{Bel}61, \text{Per}86]\). After picking a fundamental domain, one may identify function values at equivalent points along the boundary as coinciding vectors in a holomorphic fiber bundle. The Hermitian metric from Example 1 is compatible with this identification, so with \( m \) the restriction of the Lebesgue measure to a fundamental domain one arrives at a generalized Bergman space that does not result directly from the Lie-group setting considered by Onofri \([\text{Ono}75]\), but may be understood as an induced representation, see \([\text{Mau}68, \text{Ch. VIII}]\) or \([\text{Mac}88]\), because this space would result from a change in the inner product of Example 1 and by demanding that the functions be invariant under the representation of a discrete abelian subgroup of the Heisenberg-Weyl group.

2.2. **Berezin-Toeplitz operators defined via quadratic forms.** In the remaining text, we assume that the Hermitian metric \( h \) and \( m \) interpreted as a volume form are smooth and non-degenerate to ensure that \( L^2_{\text{hol}}(hm) \) is complete. In addition, from now on all manifolds are tacitly assumed to be path-wise connected.
Definition 2.4. Given the Hilbert space $L^2_{hol}(hm)$ and a real-valued function $f : \mathcal{M} \to \mathbb{R}$, we consider the sesquilinear form

$$T_f : \mathcal{Q}(T_f) \times \mathcal{Q}(T_f) \to \mathbb{C}$$

$$(\psi, \phi) \mapsto \int_\mathcal{M} f(x) h_x(\psi(x), \phi(x)) dm(x)$$  \hspace{1cm} (3)

with form domain

$$\mathcal{Q}(T_f) := \left\{ \psi \in L^2_{hol}(hm) : \int_\mathcal{M} |f(x)| h_x(\psi(x), \psi(x)) dm < \infty \right\}. \hspace{1cm} (5)$$

When referring to $T_f$ as a quadratic form, it is really the function $\psi \mapsto T_f(\psi, \psi)$ that is meant.

Definition 2.5. Whenever a real-valued function $f : \mathcal{M} \to \mathbb{R}$ gives rise to a semibounded closed form $T_f \geq c, c \in \mathbb{R}$, it is associated with a unique self-adjoint operator $T_f$ satisfying $(\sqrt{\mathcal{I} - c} \psi, \sqrt{\mathcal{I} - c} \psi) = T_f(\psi, \psi) - c(\psi, \psi)$ for all $\psi \in \mathcal{Q}(T_f)$. In the context of generalized Bergman spaces, we call $T_f$ a self-adjoint Berezin-Toeplitz operator and the function $f$ its symbol.

Remarks 2.6. In analogy with the well-known KLMN theorem, see [Sim71] or [RS75, Thm. X.17], it is sufficient for the closedness and semiboundedness of $T_f$ when the negative part $f^- := \max\{-f, 0\}$ of $f$ can be incorporated as a perturbation of $T_f^+, f^+ := \max\{f, 0\}$, with relative form bound strictly less than one. Even in this case it may be, due to singularities of $f$, that $T_f$ is not densely defined and that in consequence, $T_f$ is self-adjoint only on the closure of $\mathcal{Q}(T_f)$ in $L^2_{hol}(hm)$.

The definition in terms of quadratic forms does not provide any direct information about the domain of $T_f$ or how it operates on a given section. However, at least for bounded symbols $f$ we can give a more concrete description in which $T_f$ acts by its integral kernel.

Definition 2.7. A Schwartz kernel in a complex line bundle $\mathcal{L}$ is a family of linear mappings $\{S(x, y) : \mathcal{L}_y \to \mathcal{L}_x\}_{x, y \in \mathcal{M}}$, that is, $S(x, y)$ is linear in vectors with base point $y$ and has as its values vectors at $x$. If $S(x, y)$ is jointly continuous in $x$ and $y$, then it can be interpreted as continuous section in the bundle $\mathcal{L} \otimes \mathcal{L}^* \to \mathcal{M} \otimes \mathcal{M}$, where $\mathcal{L}^*$ is the dual bundle associating with each $x \in \mathcal{M}$ the space of complex linear forms on $\mathcal{L}_x$.

Remarks 2.8. If $hm$ is smooth and nowhere degenerate, then the identity operator on $L^2_{hol}(hm)$ a sesqui-analytic Schwartz kernel $K(x, y)$ [Bod, Prop. 7]. Moreover, then any bounded operator $B$ on $L^2_{hol}(hm)$ possesses a sesqui-analytic Schwartz kernel $B(x, y)$ that is characterized by the equation $h_x(u, B(x, y)v) = (K(\cdot, x)u, BK(\cdot, y)v), u \in \mathcal{L}_x$ and $v \in \mathcal{L}_y$, and the image of $\psi \in L^2_{hol}(hm)$ is expressed as

$$B(\psi)(x) = \int_\mathcal{M} B(x, y) \psi(y) dm(y). \hspace{1cm} (6)$$

If $f$ is a bounded function, then the Schwartz kernel of the operator $T_f$ is given by $h_x(u, T_f(x, y)v) = (K(\cdot, x)u, fK(\cdot, y)v)$ where the inner product is in $L^2(hm)$.

Since the right-hand side of equation (6) is defined even for $\psi \in L^2(hm)$, any bounded operator extends naturally via its integral kernel to all of $L^2(hm)$. From this perspective, $K(x, y)$ is the integral kernel of the identity operator on $L^2_{hol}(hm)$, which extends to that of an orthogonal projection operator, henceforth called $K$, that maps $L^2(hm)$ onto $L^2_{hol}(hm)$. 
2.3. Holomorphic maps between Riemann surfaces. In this subsection, we prepare the setting of the main result by discussing how a holomorphic map onto a Riemann surface may be used to pull back the structures needed to define a generalized Bergman space over the domain of the map.

Definition 2.9. Given a Riemann surface $M$, any metric $g$ on $M$ that is compatible with the almost complex structure $J$ on $M$ is called a conformal metric. By default, all metrics considered are smooth.

Remark. In a local coordinate system $z : U \to \mathbb{C}$, the compatibility requirement implies that $g$ has the form
\[
g = \frac{\gamma^2(z)}{2} (dz \otimes d\bar{z} + d\bar{z} \otimes dz)
\]
with a conformal scaling function $\gamma : \mathbb{C} \to \{ r > 0 \}$. The associated Dirichlet-Laplacian $\Delta$ is locally expressed as
\[
\Delta = \frac{4}{\gamma^2(z)} \frac{\partial^2}{\partial z \partial \bar{z}},
\]
where the abbreviations $\partial/\partial z := \frac{1}{2}(\partial/\partial z_1 - i \partial/\partial z_2)$ and $\partial/\partial \bar{z} := \frac{1}{2}(\partial/\partial z_1 + i \partial/\partial z_2)$ have been used. By inspection of (8), any linear combination of a holomorphic or antiholomorphic function is harmonic and vice versa.

As an aside, we remark that any smooth metric $g$ on an oriented surface allows a complex analytic atlas for which $g$ is conformal [Jos97, Thm. 3.11.1].

Definition 2.10. Let $M$ and $M'$ be Riemann surfaces with conformal metrics $g$ and $g'$, respectively. A mapping $\Phi : M' \to M$ is called conformal if it is a local diffeomorphism and $g_{\Phi(x')} (\Phi_* X', \Phi_* X') = \lambda^2(x') g'(X', X')$ holds for all $X' \in T_x M', x' \in M'$ with a strictly positive dilatation function $\lambda : M' \to \{ r > 0 \}$.

Lemma 2.11. Given two Riemann surfaces $M$ and $M'$ with conformal metrics $g$ and $g'$ and a holomorphic map $\Phi$ from $M'$ onto $M$, then $\Phi$ is conformal on the set where $\Phi^{-1}$ is non-zero. In addition, $\Phi$ is a harmonic morphism. This means, a local harmonic function $f : U \to \mathbb{C}$ defined on a chart domain $U \subset M$ pulls back to a harmonic function on $\Phi^{-1}(U)$.

Proof. To prove this local property, it is enough to consider the special case when both domains $U$ and $\Phi^{-1}(U)$ are open subsets of the complex number plane. The conformality of $\Phi$ results from that of the metrics and because $\Phi$ satisfies the Cauchy-Riemann differential equations. By the chain rule
\[
\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} f \circ \Phi = \left( \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} f \right) (\Phi(z)) \frac{\partial \Phi}{\partial z} \frac{\partial \bar{\Phi}}{\partial \bar{z}}
\]
and the local form (8) of the Dirichlet Laplacians associated with $g$ and $g'$, $\Phi$ is a harmonic morphism. \qed

Definition 2.12. Let $M$ and $M'$ be Riemann surfaces and suppose $M$ is the base manifold of a holomorphic line bundle $L$. Given a holomorphic map $\Phi$ from $M'$ onto $M$, then the pull-back operation creates a bundle $L'$ with fibers $L_{x'} := \pi^{-1}(\Phi(x')) \subset \bigcup_{x \in M} \mathcal{L}_{\Phi(x')}$. The sections $\psi : M \to L$ transfer to $L'$ by $\Phi^* \psi : x' \mapsto \psi(\Phi(x'))$, and the Hermitian structure $h$ on $L$ pulls back to the fibers of $L'$ by $\Phi^* h_{x'} := h_{\Phi(x')}$. 


Let $\nabla$ denote the unique connection on $\mathcal{L}$ that is compatible with the complex and Hermitian structures [Wel80, Ch. III]. Then its pull-back, satisfying the identity $(\Phi^*\nabla)|_X \Phi^*\psi := \Phi^*(\nabla_{\Phi^*X}\psi)$ for smooth vector fields $X$ and sections $\psi$, is in turn compatible with the complex and Hermitian structures present on $\mathcal{L}'$. In addition, the curvature form of $\Phi^*\nabla$ is the pull-back of the curvature on $\mathcal{L}$.

3. A Transformation Formula for Resolvents of Berezin-Toeplitz Operators

3.1. Main result.

**Definition 3.1.** Let $\mathcal{L}$ be a holomorphic line bundle with a smooth, non-degenerate Hermitian metric $h$ on the fibers and a volume measure $m$ on the base manifold. We denote the resolvent of a self-adjoint Berezin-Toeplitz operator $T_f$ as $G^h_{f-c} := (T_f - c)^{-1}$ for any $c \in \mathbb{C}$ outside of the spectrum of $T_f$. For such $c$ in the resolvent set, $G^h_{f-c}$ is by definition a bounded operator and via its integral kernel it extends according to Remarks 2.8 to all of $L^2(hm)$. In addition, if $T_f$ is not densely defined as mentioned in Remarks 2.6, we define $G^h_{f-c}$ to be zero outside the closure of the domain $Q(T_f)$ in $L^2_{hol}(hm)$ of the sesquilinear form $T_f$ corresponding to $T_f$.

In short, this extension is characterized by $G^h_{f-c} = G^h_{f-c}K_f$ where $K_f = K_f^*K_f$ is the orthogonal projector onto the closed subspace $\overline{Q(T_f)}$.

**Definition 3.2.** Let $\mathcal{M}$ be a Riemannian complete manifold and $-\Delta + C^\infty_c(\mathcal{M})$ the self-adjoint negative Dirichlet-Laplace-Beltrami operator on $\mathcal{M}$. A real-valued function $q : \mathcal{M} \to \mathbb{R}$ belongs to the Kato class $\mathcal{K}$ if the following condition is satisfied:

\[
\lim_{t \to 0} \sup_{x \in \mathcal{M}} \int_0^t (e^{s\Delta}|q|(x)) \, ds = 0 .
\]

Whenever this property holds only locally, which means for all products $\chi \lambda q \in \mathcal{K}$ with characteristic functions $\chi$ of compact sets $\Lambda$ in $\mathcal{M}$, we write $q \in K_{loc}$. If a real-valued function $q = q^+ - q^-$, $q^\pm \geq 0$, satisfies $q^+ \in K_{loc}$ and $q^- \in K$ then it is called Kato decomposable, symbolized as $q \in K_\pm$.

**Remarks 3.3.** If the Ricci curvature of $\mathcal{M}$ is bounded below, then bounds on the heat kernel imply the inclusion $K_{loc} \subset L^1_{loc}(m)$, where $m$ is the Riemannian volume measure [Dav85, Dav88, Dav89]. In this case, a real-valued Kato-decomposable function $f : \mathcal{M} \to \mathbb{R}$ defines a semibounded Berezin-Toeplitz operator $T_f$ on $L^2_{hol}(hm)$ that is obtained from the closure $T_f \upharpoonright K_f(C^\infty_c(\mathcal{M}))$ where $C^\infty_c(\mathcal{M})$ denotes the space of smooth, compactly supported sections [Bod, Thm. 29].

**Theorem 3.4.** Let $\mathcal{M}$ and $\mathcal{M}'$ be two Riemannian complete surfaces equipped with conformal metrics $g$ and $g'$, and suppose $\mathcal{M}$ is the base manifold of a holomorphic line bundle $\mathcal{L}$. Furthermore, let $\Phi : \mathcal{M}' \to \mathcal{M}$ be a holomorphic, surjective mapping with dilatation function $\lambda$, that is, $g' = \lambda^2 \Phi^* g$. The pull-back bundle $\mathcal{L}' := \Phi^* \mathcal{L}$ is thought of as being equipped with the Hermitian metric $h' := \Phi^* h$. Assume that the Ricci curvatures of $\mathcal{M}$ and $\mathcal{M}'$ are bounded below and that the functions $f : \mathcal{M} \to \mathbb{R}$ and $\lambda^2 f \circ \Phi : \mathcal{M}' \to \mathbb{R}$ are Kato decomposable such that the corresponding Berezin-Toeplitz operators can be defined via semibounded quadratic forms on $L^2_{hol}(hm)$ and $L^2_{hol}(h'm')$, where $m'$ is the natural volume with respect to $g'$. 
If $\psi \in L^2_{ho}(hm)$ and $\lambda^2 \psi \circ \Phi \in L^2(h'm')$, then there is a relationship between the resolvents
\[
(G^{hm}_{f-c})_\Phi(x') = (G^{h'm'}_{\lambda^2 f})_\Phi(x')
\]
for such $c \in \mathbb{C}$ that satisfy the operator inequalities $T_f > \Re c$ and $T_{\lambda^2 f} > \Re c$. Hereby, the extension convention according to Remarks 2.8 is implicit, since $\lambda^2 \psi \circ \Phi$ is not holomorphic unless $\lambda$ is constant. If additionally $\psi \circ \Phi \in L^2_{ho}(h'm')$, then this equation reads
\[
(G^{hm}_{f-c})_\Phi(x') = (G^{h'm'}_{\lambda^2 f})_\Phi(x')
\]

**Remarks 3.5.** It is a nontrivial issue to establish that the values for $c$ allowed by the two operator inequalities in the assumptions of the preceding theorem form a nonempty open set in $\mathbb{C}$. The first inequality is easy to satisfy for sufficiently negative values of $\Re c$. However, the only generally sufficient condition known to the author to ensure the second inequality is $T_{\lambda^2} \geq \varepsilon > 0$, with again a sufficiently negative $\Re c$. This can be deduced, for example, if $\mathcal{M}$ is compact, since the singular points of $\Phi$ are isolated [Fug78] and thus $T_{\lambda^2} \geq \inf \{ \int \rho \lambda^2 \, dm : \rho \geq 0, \int \rho \, dm = 1 \} > 0$.

It may happen that $c' = \lambda^2 f \circ \Phi$ is constant, see the following examples. In this case, formula (12) simplifies to
\[
(G^{hm}_{f-c})_\Phi(x') = (G^{h'm'}_{\lambda^2 f})_\Phi(x')
\]
and if $T_{\lambda^2}$ is known, the left-hand side can be computed explicitly. It is interesting to note that in a sense, $f$ and $\lambda^2$ switch roles: The constant in the resolvent becomes a multiplier of the emerging symbol $\lambda^2$ and the former symbol $f$ gets turned into a constant. In this special case, the validity of the second operator inequality in the assumptions can then be established in yet another way: If $0 \neq f \in \mathcal{K}$ then $T_f$ is bounded and one may flip the sign of $f$ to ensure that the resulting constant $c'$ is strictly positive, and as a consequence equation (12) holds for any $c$ from the open half-plane $\{ c : \Re c < T_f \}$.

Once an open set for the allowed values of $c$ in equation (13) is established, one can extend this set by analytic continuation of the resolvents. We follow the nice exposition by Wittich [Wit00] of the facts collected from Kato’s book [Kat76]. Hereby, it is important that for $u \in \mathcal{K}_\pm$, $\{ T_{\xi u} \}$ forms a holomorphic family of type B with $\xi \in \mathbb{C}, \Re \xi > 0$ [Kat76, Sec. VII.2] and that for $\xi$ in a smaller, compact set $\mathcal{C}$ in the right-half plane with nonempty interior, the numerical ranges of all $T_{\xi u}$ are contained in a common sector of complex numbers with real parts bounded below by a constant $\zeta < 0$ [Kat76, Thm. VII.4.2]. In addition, the set $\{ (\xi, c) \in \mathbb{C}^2 : \xi \in \mathcal{C}, \Re c < \zeta \}$ is a set of holomorphy for $(\xi u - c)^{-1}$ [Kat76, Ch. V]. Since the resolvents of holomorphic families of type B have unique analytic continuations, one may then extend the set of values of $c$ to all values for which both resolvents in equation (13) exist [Kat76, Rem. VII.1.6].

At first, the generalization of the theorem to higher dimensions of $\mathcal{M}$ and $\mathcal{M}'$ seems straightforward. Unfortunately, one does not gain more generality by restating it this way, because if $\dim_{\mathcal{C}} \mathcal{M}' = \dim_{\mathcal{C}} \mathcal{M} \geq 2$, then the harmonic morphism $\Phi$ is necessarily a local isometry up to an overall rescaling by a constant [Fug78]. Thus, the result in higher dimensions concerns solely covering maps of Kähler manifolds. The case $\dim_{\mathcal{C}} \mathcal{M}' > \dim_{\mathcal{C}} \mathcal{M}$ is not interesting in the quantization context.
because the pull back of the curvature would be degenerate; in other words, the correspondence principle would lead to a classical system with a degenerate symplectic form.

Finally, we remark that the statement of the theorem does not refer directly to the probabilistic elements used in the proof given hereafter. It would be nice to find an alternative derivation of the claimed relationship with a purely analytic argument.

**Examples 3.6.** The following examples have the virtue that, at least in special cases, one may verify the formulas by other means than the probabilistic derivation.

1. A whole class of examples is given whenever \((\mathcal{M}', \Phi)\) defines a covering of \(\mathcal{M}\). Then there is a unique complex structure on \(\mathcal{M}'\) such that \(\Phi : \mathcal{M}' \to \mathcal{M}\) is holomorphic [Go98, Prop. 5.8.3], and the conformal metric \(g\), the bundle \(\mathcal{L}\), and the Hermitian metric \(h\) may be pulled back to give a complex manifold \(\mathcal{M}'\) equipped with \(g'\) that is the base manifold of the Hermitian holomorphic line bundle \(\mathcal{L}' = \Phi^*\mathcal{L}\). The dilatation is just \(\lambda = 1\) because the map \(\Phi\) is a local isometry. Thus, the formula

\[
(G^{hm}_f c \psi)(\Phi(x')) = (G^{h'm'}_f \Phi c \circ \Phi)(x')
\]

relates the resolvent of \(T_f\) to that of its periodic extension \(T_f \circ \Phi\) on the covering space \(\mathcal{M}'\). For a more concrete example of this situation and the details of a relation between operators on the spaces in Examples 2.3.1 and 2.3.4, see [BK01].

2. Let \(\mathcal{M} = \mathcal{M}' = \mathbb{C}\) be equipped with the standard metric \(g = g' = \frac{1}{2}(dz \otimes d\overline{z} + d\overline{z} \otimes dz)\) and take \(\mathcal{L}\) and \(\mathcal{L}'\) to be the trivial bundles \(\mathbb{C} \times \mathbb{C}\). Suppose \(\mathcal{L}\) is equipped with the Hermitian metric from the Barut-Girardello space described in Examples 2.3.2. The mapping \(\Phi : z \mapsto z^2\) from \(\mathcal{M}'\) to \(\mathcal{M}\) then pulls back the Hermitian metric so that \(\mathcal{L}'\) becomes the bundle underlying the Fock-Bargmann space, Examples 2.3.1. The square of the dilatation is given by \(\lambda^2(z) = 4|z|^2\). Just as mentioned in Remarks 3.5, choosing \(f(z) = c'/4|z|\) with a constant \(c' \in \mathbb{C}\) then leads to \(\lambda^2(z)f(\phi(z)) = c'\). In short, the resolvent relation

\[
(G^{hm}_{c - c' \phi |z|^2} \psi)(\Phi(z)) = (G^{h'm'}_{c' - c|z|^2} T_{|z|^2} \psi \circ \Phi)(z)
\]

is derived, a priori valid for \(c'\) with a sufficiently negative real part and \(c\) with a sufficiently large positive real part. Indeed, one may verify that the spectra of the Berezin-Toeplitz operators that are related here are given by \(\text{spec}(T_{|z|^2}) = \{ \frac{c'}{4n^2 + 4} : n \in \mathbb{N}\} \cup \{ 0 \}\) and \(\text{spec}(T_{c|z|^2}) = c\mathbb{N}\), due to their known eigensections that are just the monomials \(z \mapsto (z, z^n), n \in \mathbb{N}\). Of course, the eigensection decomposition offers another method to verify the resolvent relation.

The mapping \(\Phi : z \mapsto z^2\) is of the Clifford type [Bai90] and represents the direct analog on two-dimensional Euclidean space of the harmonic morphism discussed in the example of [Wit00]. There, the resolvent of the Coulomb system in dimension 3 is related to that of the harmonic oscillator in dimension 4. In a similar vein, here the components \(\Phi_1(z) = z_1^2 - z_2^2\) and \(\Phi_2(z) = 2z_1z_2\) of the mapping \(\Phi\) may be interpreted as quadratic forms on \(\mathbb{R}^2\), and the symmetric matrices corresponding to these quadratic forms...
are the basis of a two-dimensional Clifford algebra over \( \mathbb{R} \). According to the common scheme behind both examples, the anticommutation relations observed by the matrices imply that \( \Phi \) is harmonic and horizontally conformal at all non-singular points. Since \( \Phi \) is also surjective, it is a harmonic morphism [Fug78].

(3) Choose \( \mathcal{M} = \mathcal{M}' = \mathbb{CP}^1 \). We will focus on one chart which maps all but one point of each manifold stereographically to the complex plane. In these local coordinates let the Riemannian conformal metrics be given as \( g = g' = \frac{1}{2} \frac{1}{(1+|z|^2)^2} (dz \otimes d\bar{z} + d\bar{z} \otimes dz) \). Let \( \mathcal{M} \) be the base manifold of the bundle described in Examples 2.3.3, so in a local trivialization the holomorphic, square-integrable sections in \( L^2_{\text{hol}}(hm) \) are given by polynomials of maximal degree \( k \in \mathbb{Z}^+ \), where the Hermitian metric is \( h_z((z,u),(z,v)) = \frac{1}{(1+|z|^2)^2} |\bar{v} - \bar{u}|^2 \). Take the conformal map \( z \mapsto \alpha z + \beta \) with \( \alpha, \beta \in \mathbb{C} \) and \( \alpha \neq 0 \) that fixes the point excluded from the chart domain. Then the square of the dilatation is \( \lambda^2(z) = |\alpha|^2 \frac{(1+|z|^2)^2}{(1+|\alpha z + \beta|^2)^2} \). The pull-back of the Hermitian metric \( h_z'((z,u),(z,v)) = \frac{1}{(1+|\alpha z + \beta|^2)^2} |\bar{v} - \bar{u}|^2 \) together with the Riemannian volume \( m' \) define a new Hilbert space \( L^2_{\text{hol}}(h'm') \). Taking \( f(z) = c' \frac{(1+|z|^2)^2}{|\alpha|^2 (1+|z-\beta|^2/|\alpha|^2)^2} \) yields the resolvent relationship

\[
(G_{hm}' \frac{(1+|z|^2)^2}{|\alpha|^2 (1+|z-\beta|^2/|\alpha|^2)^2} - c \psi)(\Phi(z)) = \left(G_{h'm'} \frac{(1+|z|^2)^2}{c' |\alpha|^2 (1+|z-\beta|^2/|\alpha|^2)^2} \psi \circ \Phi\right)(z),
\]

valid again for \( c' \) with a sufficiently negative real part and \( c \) with a sufficiently large positive real part. In case \( \beta = 0 \) the eigensections are again given by monomials and one may verify that the eigenvalues of the corresponding Berezin-Toeplitz operators are inverses to each other, \( \text{spec}(T_f) = \{|\alpha|^{2n} \mathcal{F}_1(k, n+1, 2+k; 1-\alpha) : n = 0, 1, \ldots k\} \) and \( \text{spec}(T_{\lambda^2}) = \{(|\alpha|^{2n} \mathcal{F}_1(k, n+1, 2+k; 1-\alpha))^{-1} : n = 0, 1, \ldots k\} \). Indeed, this inverse relationship is to be expected also for \( \beta \neq 0 \) by observing that the conformal map may simply be interpreted as a coordinate transformation that turns the sesquilinear form of one operator into the inner product on the other space.

3.2. Assembly of the proof. The major ingredients of the proof of Theorem 3.4 are a representation of resolvents of Berezin-Toeplitz operators in the form of so-called Wiener-regularized path integrals and an invariance property of Brownian motion under harmonic morphisms that implies a simple substitution rule in the path-integral representation. Before the final assembly, we explain the ingredients.

3.2.1. Probabilistic representation of resolvents of Berezin-Toeplitz operators.

Definition 3.7. We adopt the usual terminology: An almost surely continuous process \( B \) with values in the Riemannian manifold \( \mathcal{M} \) is called Brownian motion with diffusion constant \( D > 0 \) if for every smooth function \( \phi \in C^\infty(\mathcal{M}) \), the difference

\[
M_t := \phi \circ B_t - \phi \circ B_0 - \int_0^t D\phi \circ B_s \, ds
\]

is a real-valued continuous local martingale \( M \).
A probability measure governing Brownian motion with a diffusion constant $D > 0$ and almost surely fixed starting point $B_0 = x$ will be denoted as $E^D_x$. The expectation with respect to this probability measure is written as $E^D_x$.

**Remark.** A complete Riemannian manifold $M$ with Ricci curvature bounded below is Brownian complete, that is, for a fixed diffusion constant $D > 0$, a Brownian motion $B$ starting at any $x \in M$ has an infinite explosion time [Eme89, Ch. V].

**Proposition 3.8.** Let $L$ be a Hermitian holomorphic line bundle over a Riemann surface $M$ that is Riemannian complete with Ricci curvature bounded below. Denote by $m$ the natural volume measure on $M$ and by $E^D_x$ a family of Brownian-motion measures having a common diffusion constant $D > 0$. Let the real-valued function $\rho$ on $M$ be specified by the curvature term $\rho(x) = (\nabla_Z \nabla_Z - \nabla_Z \nabla_Z - \nabla_{\nabla Z - \nabla Z}) \psi(x)$ with any smooth, locally non-vanishing section $\psi$ and an arbitrary choice of a holomorphic tangent vector field $Z$ that is normalized at $x$, that is, $g_x(Z, Z) = 1$ in terms of the bilinear extension of $g$ to the complexified tangent space.

If $f, \rho \in C^0$ then the image of a section $\psi \in L^2_{hol}(hm)$ under the semigroup $e^{-tT_f} K_f$ is for $t > 0$ given by the ultra-diffusive limit of a Brownian-motion expectation,

$$ (e^{-tT_f} K_f) \psi(x) = \lim_{D \to \infty} E^D_x \left[ e^{-\int_0^t (D \rho(B_r) + f(B_r)) dr} H_{B_1}^{-1}(\nabla \psi(B_1)) \right]. \tag{18} $$

The inverse $H_{B_1}^{-1}$ of the stochastic horizontal transport appearing in this so-called Wiener-regularized path integral is associated with the compatible connection $\nabla$ and therefore preserves the length of a transported vector. The reverse transport can either be understood by appealing to localized expressions [Sch80], i.e. one restricts to a subspace of the probability space by introducing exit times of local coordinate patches and then reformulates the reverse horizontal transport in a local trivialization, or one interprets (18) as a shorthand for

$$ h_x(u, e^{-tT_f} K_f \psi) = \lim_{D \to \infty} E^D_x \left[ e^{-\int_0^t (D \rho(B_r) + f(B_r)) dr} h_{B_1}(H_{B_1} u, \psi(B_1)) \right] \tag{19} $$

with an arbitrary reference vector $u \in L_x$.

**Proof.** The detailed proof is given in [Bod, Thm. 45]. We summarize the key ingredients.

To begin with, we consider a version of the Feynman-Kac formula for perturbations of the semigroup generated by the Bochner Laplacian [Bis81, Ch. IX] and the limiting argument in [Bod, App. C] that permits a larger class of perturbations. In combination with a Weitzenboeck-type formula one then concludes that the pre-limit expression on the right-hand side of equation (18) is the image of $\psi$ under the semigroup generated by the Schrödinger operator $S_{D,f}^{(0,*)} := -D \Delta^{(0,*)} + f$. This operator is understood as the form sum $-D \Delta^{(0,*)} + f$ that is perturbed by the negative part $f^-$. The non-negative operator $-\Delta^{(0,*)}$, in turn, is the form closure of a differential operator initially defined by $-\Delta^{(0,*)} \psi(x) = -\nabla_Z \nabla_Z \psi(x) - \nabla_{\text{Cov}} \nabla \psi(x)$ on smooth, compactly supported sections $\psi$, whereby Cov denotes the Levi-Civita connection and $Z$ is again an arbitrary locally non-vanishing, holomorphic vector field that is normalized at $x$. 


Now the proof is seen to be equivalent to showing the pointwise identity
\[
(e^{-tT_fK_f}\psi)(x) = \lim_{D \to \infty} e^{-tS_{D,f}^{(0,\bullet)}}\psi(x). \tag{20}
\]
This is implied by continuity properties of both sides and strong convergence of \(e^{-tS_{D,f}^{(0,\bullet)}}\) to \(e^{-tT_fK_f}\) in the limit \(D \to \infty\), a result of monotone convergence of the forms associated with \(\{S_{D,f}^{(0,\bullet)}\}_{D>0}\) and the fact that \(-\Delta^{(0,\bullet)}\) vanishes only on holomorphic sections and is otherwise strictly positive. \(\square\)

**Lemma 3.9.** If for some \(D > 1\), \(\Re c < S_{D,f}^{(0,\bullet)}\), then in analogy with (20) we obtain the integral representation for the resolvent
\[
G_{f-c}^{hm}\psi(x) = \lim_{D \to \infty} \int_0^\infty e^{-tS_{D,f}^{(0,\bullet)}} + tc\psi(x) dt. \tag{21}
\]

**Proof.** The strong convergence
\[
G_{f-c}^{hm}\psi = \lim_{D \to \infty} \int_0^\infty e^{-tS_{D,f}^{(0,\bullet)}} + tc\psi dt \tag{22}
\]
is again a result of the monotone convergence of forms associated with \(S_{D,f}^{(0,\bullet)}\). To obtain the pointwise equality, we apply the point-evaluation functional \(\vartheta_u = h_x(u,\cdot)\) to the integrand of (21), use the self-adjointness and semigroup property to obtain
\[
h_x(u, e^{-tS_{D,f}^{(0,\bullet)}} \psi(x)) = (e^{-tS_{D,f}^{(0,\bullet)}} (-, x)u, e^{-t(1 - \frac{1}{D})S_{D,f}^{(0,\bullet)}} \psi), \tag{23}
\]
and integrate over \(t \in [0, \infty)\), which yields a pointwise expression for the image of \(\psi\) under the resolvent of \(S_{D,f}^{(0,\bullet)}\):
\[
h_x(u, (S_{D,f}^{(0,\bullet)} - c)^{-1}\psi(x)) = \int_0^\infty (e^{-tS_{D,f}^{(0,\bullet)}} (-, x)u, e^{-t(1 - \frac{1}{D})S_{D,f}^{(0,\bullet)}} + tc\psi) dt. \tag{24}
\]
The uniform boundedness of \(e^{-t(1 - \frac{1}{D})S_{D,f}^{(0,\bullet)}}\) in \(D\) and the strong convergence of the function \(e^{-tS_{D,f}^{(0,\bullet)}} (-, x)u\) to \(e^{t\Delta^{(0,\bullet)}} (-, x)u\) [Bod, Lem. 44] imply that in the limit \(D \to \infty\) the integral on the right-hand side of equation (24) converges to
\[
h_x(u, G_{f-c}^{hm}\psi(x)) = h_x(e^{t\Delta^{(0,\bullet)}} (-, x)u, G_{f-c}^{hm}\psi) \tag{25}
\]
which proves the claimed identity. \(\square\)

### 3.2.2. An invariance property of Brownian motion.

**Definition 3.10.** Let \(B\) be a Brownian motion on a Riemann surface \(M\). An additive functional of Brownian motion is a stochastic process \(A\) given in the form
\[
A_{\sigma} := \int_0^\sigma q(B_s) ds \tag{26}
\]
with a non-negative function \(q : M \to \mathbb{R}^+\).

If \(A\) is everywhere finite, increasing without jumps, and if \(\lim_{\sigma \to \infty} A_{\sigma} = \infty\) with probability one, then we define a stochastic time change by the inverse \(\tau\) of \(A\), in other words, \(A_{\tau(t)} = t\) for all \(t \geq 0\).
Lemma 3.11. Let $\mathcal{M}$ and $\mathcal{M}'$ be two Riemann surfaces with conformal metrics $g$ and $g'$. Suppose both manifolds are Riemannian complete with Ricci curvatures bounded below. Given $B'$ a Brownian motion on $\mathcal{M}'$ and $\Phi : \mathcal{M}' \to \mathcal{M}$ a surjective holomorphic mapping having the dilatation function $\lambda : \mathcal{M}' \to \mathbb{R}^+$, then the additive functional $A_\tau := \int_0^\tau \lambda^2(B_s')ds$ satisfies the finiteness and limit conditions in the preceding definition of the stochastic time change $\tau$ and $\{B_t := \Phi(B'_{\tau(t)})\}_{t \geq 0}$ defines a Brownian motion $B$ on $\mathcal{M}$.

Proof. Essential to the proof is that since $\Phi$ is holomorphic and surjective, its singular points are isolated and thus by an argument of Fuglede [Fug78] polar. Therefore, $\lambda(B')$ is an a.s. strictly positive, continuous process and $A$ is increasing. Implicit in the assumptions is that $\mathcal{M}'$ is Brownian complete, so $\{\Phi(B'_{\tau(t)})\}_{t \geq 0}$ is some stochastic process with values in $\mathcal{M}$ that may possibly have finite values for its explosion time. However, since $\Phi$ is a harmonic morphism, it relates the generators of Brownian motion by a conformal scaling operation [CO83] as in the localized version (9). Therefore, the time-changed process is by the characterization (17) seen to be a Brownian motion on $\mathcal{M}$ so that due to the lower bound on the Ricci curvature it has an infinite explosion time.

3.2.3. Implementing the substitution rule in the probabilistic representation.

Proof of Theorem 3.4. Since we can always absorb the constant $z$ into the definition of $f$, we will for convenience of notation assume $z = 0$. Using the probabilistic representation, we have

$$
(G^h_{f \psi})(\Phi(x')) = \lim_{D \to \infty} \int_0^\infty \mathbb{E}^D_{\Phi(x')} \left[ e^{-\int_0^t (D\rho + f)(B_s)ds} H_{\Phi(B_{\tau(s)})}^{-1} \psi \Phi(B_{\tau(s)}) \right] dt .
$$

whenever $f$ leads to a Berezin-Toeplitz operator the real part of which is bounded below by zero. By the invariance property of Brownian motion, we can replace $B_{\tau(s)}$ with $\Phi(B_s)$,

$$
(G^h_{f \psi})(\Phi(x')) = \lim_{D \to \infty} \int_0^\infty \mathbb{E}^D_{\Phi(x')} \left[ e^{-\int_0^t (D\rho \circ \Phi + f \circ \Phi)(B_{\tau(s)})ds} H_{\Phi(B_{\tau(s)})}^{-1} \psi \circ \Phi(B_{\tau(s)}) \right] dt .
$$

Interchanging the integration with the expectation and substituting gives

$$
(G^h_{f \psi})(\Phi(x')) = \lim_{D \to \infty} \mathbb{E}^D_{\Phi(x')} \left[ \int_0^\infty e^{-\int_0^t \lambda^2(B_{\tau(s)})d\sigma} (D\rho \circ \Phi + f \circ \Phi)(B_{\tau(s)})d\sigma \right] \lambda^2(B_{\tau(s)}) H_{\Phi(B_{\tau(s)})}^{-1} \psi \circ \Phi(B_{\tau(s)})d\tau ,
$$

where we have used that, as a consequence of the local properties under the pull-back operation explained in Definition 2.12, the horizontal transport and pull-back operations commute in the following sense: Suppose we lift a (Brownian) path in $\mathcal{M}$ to $\mathcal{M}'$. Then first horizontally transporting a vector $u$ through the fibers along the path in $\mathcal{M}$ and lifting it to $\mathcal{L}'$ gives the same as lifting it first and then horizontally transporting along the lifted path in $\mathcal{M}'$.

After reversing the order of integration again, one may interpret the resulting expression as the integral representation of a new resolvent, in short

$$
(G^h_{f \psi})(\Phi(x')) = (G^h_{f \psi})_{\lambda}(\Phi(x'),)
$$

with $\lambda := \int_0^\infty e^{-\int_0^t \lambda^2(B_{\tau(s)})d\sigma} (D\rho \circ \Phi + f \circ \Phi)(B_{\tau(s)})d\sigma$.
because of the identity $\rho'(x') = \lambda^2(x')\rho(\Phi(x'))$ between the curvature terms that results from the conformal relationship between $g'$ and $g$ and the definition of the pull back of the bundle curvature. However, to ensure the validity of this integral representation one depends on the assumption of the second operator inequality in the statement of Theorem 3.4.

Acknowledgements. Thanks are extended to Michael Aizenman and John Klauder for helpful remarks and to Hélène Rey for sharing her enthusiasm about mathematical physics, an unparalleled source of encouragement and inspiration.

References

[Bai90] P. Baird, Harmonic morphisms and circle actions on 3- and 4-manifolds, Ann. Inst. Fourier (Grenoble) 40 (1990), no. 1, 177–212.
[Bar61] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform, Part I, Comm. Pure Appl. Math. 14 (1961), 187–214.
[Bel61] R. Bellman, A brief introduction to theta functions, Holt, Rinehart and Winston, New York, 1961.
[BG71] A. O. Barut and L. Girardello, New “coherent” states associated with non-compact groups, Commun. Math. Phys. 21 (1971), 41–55.
[Bis81] J.-M. Bismut, Mécanique aléatoire, Lecture Notes in Mathematics, no. 866, Springer, Berlin, 1981.
[BK01] B. G. Bodmann and J. R. Klauder, Path-integral quantization for a toroidal phase space, Coherent States, Quantization and Gravity, Proceedings of the XVII Workshop on Geometric Methods in Physics, Białowieża, Poland, 1998 (M. Schlichenmaier, A. Strasburger, S. T. Ali, and A. Odzijewicz, eds.), Warsaw University Press, 2001, documented as quant-ph/9902003, pp. 3–10.
[Bla82] P. Blanchard, Transformations of Wiener integrals and the desingularization of the Coulomb problem, Proceedings of the Workshop on Stochastic Processes in Quantum Theory and Statistical Physics, Marseille, 1981 (S. Albeverio, Ph. Combe, and M. Sirigue-Collin, eds.), Springer Lecture Notes in Physics, vol. 173, Springer, Berlin, 1982, pp. 19–28.
[BMM96] D. Bar-Moshe and M. S. Marinov, Berezin quantization and unitary representation of Lie groups, Topics in statistical and theoretical physics (R. L. Dobrushin, R. L. Minlos, M. A. Shubin, and A. M. Vershik, eds.), Amer. Math. Soc. Transl., vol. 177, AMS, Providence (R. I.), 1996.
[Bod] B. G. Bodmann, Construction of self-adjoint Berezin-Toeplitz operators on Kähler manifolds and a probabilistic representation of the associated semigroup, preprint, Princeton University, 2002, documented as math-ph/0207026.
[CO83] L. Csink and B. Øksendal, Stochastic harmonic morphisms: Functions mapping the paths of one diffusion into the paths of another, Ann. Inst. Fourier 33 (1983), 219–240.
[CS90] D. P. L. Castrigiano and F. Stéark, Intrinsic clock as a tool for path integration: an example, White noise analysis, mathematics and applications (T. Hida, H.-H. Kuo, J. Plotho, and L. Streit, eds.), World Scientific, Singapore, 1990, pp. 49–65.
[Dav85] E. B. Davies, $L^1$ properties of second order elliptic operators, Bull. London Math. Soc. 17 (1985), 417–436.
[Dav88] E. B. Davies, Gaussian upper bounds for the heat kernel of some second-order operators on Riemannian manifolds, J. Funct. Anal. 80 (1988), 16–32.
[Dav89] E. B. Davies, Heat kernels and spectral theory, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1989.
[DK79] I. H. Durr and H. Kleinert, Solution of the path integral for the H-atom, Phys. Lett. B 84 (1979), 185–188.
[Eme89] M. Emery, Stochastic calculus in manifolds, Universitext, Springer, Berlin, 1989.
[Fug78] B. Fuglede, Harmonic morphisms between Riemannian manifolds, Ann. Inst. Fourier (Grenoble) 28 (1978), 107–144.
[Gold98] S. I. Goldberg, Curvature and homology, Dover Publications, Mineola, NY, 1998, Revised reprint of the 1970 edition.
[Jos97] J. Jost, *Compact Riemann surfaces*, Universitext, Springer, Berlin, 1997.

[Kat76] T. Kato, *Perturbation theory for linear operators*, second ed., Classics in Mathematics, Springer, 1976.

[Mac88] G. W. Mackey, *Induced representations and the applications of harmonic analysis*, Harmonic analysis (Luxembourg, 1987), Springer, Berlin, 1988, pp. 16–51.

[Mau68] K. Maurin, *General eigenfunction expansions and unitary representations of topological groups*, Polska Akademia Nauk Monografie Matematyczne, vol. 48, Polish Scientific, Warsaw, 1968.

[Ono75] E. Onofri, *A note on coherent state representations of Lie groups*, J. Math. Phys. **16** (1975), 1087–1089.

[Per86] A. Perelomov, *Generalized coherent states and their applications*, Texts and Monographs in Physics, Springer, Berlin, 1986.

[RS75] M. Reed and B. Simon, *Methods of modern mathematical physics*, vol. II, Fourier analysis, self-adjointness, Academic Press, New York, 1975.

[Sch80] L. Schwartz, *Semi-martingales sur des variétés, et martingales conformes sur des variétés analytiques complexes*, Lecture Notes in Mathematics, no. 780, Springer, Berlin, 1980.

[Sim71] B. Simon, *Quantum mechanics for Hamiltonians defined as quadratic forms*, Princeton University Press, Princeton, N. J., 1971, Princeton Series in Physics.

[Sto94] S. N. Storchak, *Reparametrization of paths in path integrals on vector bundles*, Theoret. Math. Phys. **101** (1994), no. 3, 1422–1429, russ. orig: Teoret. Mat. Fiz. **101** (1994), no. 3, 374–383.

[Wel80] R. O. Wells, *Differential analysis on complex manifolds*, Graduate Texts in Mathematics, vol. 65, Springer, 1980.

[Wit00] O. Wittich, *A transformation of a Feynman-Kac formula for holomorphic families of type B*, J. Math. Phys. **41** (2000), 244–259.

337 Jadwin Hall, Department of Physics, Princeton University, Princeton, NJ 08544, USA

E-mail address: bgb@princeton.edu