LIFESPAN OF SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITHOUT GAUGE INVARIANCE

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ABSTRACT. We study the lifespan of solutions for the nonlinear Schrödinger equation
\[(NLS)\quad i\partial_t u + \Delta u = \lambda |u|^p, \quad (t, x) \in [0, T) \times \mathbb{R}^n,\]
with the initial condition, where $1 < p \leq 1 + 2/n$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Our main aim in this paper is to prove an upper bound of the lifespan in the subcritical case $1 < p < 1 + 2/n$.

1. Introduction

In this paper, we study the initial value problem for the nonlinear Schrödinger equation (NLS) with a non-gauge invariant power nonlinearity:
\[(1.1)\quad i\partial_t u + \Delta u = \lambda |u|^p, \quad (t, x) \in [0, T) \times \mathbb{R}^n,\]
with the initial condition
\[(1.2)\quad u(0, x) = \varepsilon f(x), \quad x \in \mathbb{R}^n,\]
where $T > 0$, $1 < p \leq 1 + 2/n$, $u$ is a complex-valued unknown function of $(t, x)$, $\lambda \in \mathbb{C} \setminus \{0\}$, $f$ is a given complex-valued function, $\varepsilon > 0$ is a small parameter.

It is well known that local well-posedness holds for (1.1)-(1.2) in several Sobolev spaces $H^s$ ($s \geq 0$) (see e.g. [1], [11] and the references therein). However, there had been no results about global existence of solutions for (1.1)-(1.2) in the case of $1 < p \leq 1 + 2/n$. It is also well known that when $p \geq p_s$, where $p_s$ is the Strauss exponent (see [8]), “small data global existence result” holds (see also [1]). Recently, in paper [4], blow-up solutions for (1.1)-(1.2) were constructed in the case of $1 < p \leq 1 + 2/n$ under a suitable initial data. To construct blow-up solutions, they have to choose the shape of the initial data, though the size of the data may be small. But since they used a contradiction argument to construct a blow-up solution, the mechanism of the blow-up solution (e.g. estimate of the lifespan, blow-up speed etc.) has not been known. Motivated by their result, we decided to consider the lifespan of the local solution for (1.1)-(1.2). Especially, our main aim of the present paper is to give an upper bound of the lifespan in the subcritical case $1 < p < 1 + 2/n$. We note that the optimality of the lifespan is still open. And, in the critical case $p = 1 + 2/n$, an upper bound of the lifespan is also still not known. We also remark that it is open what happens in the case of $1 + 2/n < p \leq p_s$. (For more recent information of blow-up results of NLS, see e.g. [6], [7], [10] and the references therein.)
2. KNOWN RESULTS AND MAIN RESULT

First, we recall the local existence result for the integral equation in $L^2$-framework:

\[
    u(t) = \varepsilon U(t) f - i\lambda \int_0^t U(t - s)|u|^p\, ds,
\]

which is associated with (1.1)-(1.2), where $U(t) = \exp(it\Delta)$ is the free evolution group to the Schrödinger equation.

**Proposition 2.1** (Tsutsumi [11]). Let $1 < p < 1 + 4/n, \lambda \in \mathbb{C}, \varepsilon \geq 0$ and $f \in L^2$. Then there exist a positive time $T = T(\|f\|_{L^2}, \varepsilon) > 0$ and a unique solution $u \in C([0, T); L^2) \cap L^r_t(0, T; L^\rho_x)$ of (2.1), where $r, \rho$ are defined by $\rho = p + 1$ and $2/r = n/2 - n/\rho$.

The above solution $u$ is called "$L^2$-solution". Our next concern is the estimate of the lifespan. Let $T_\varepsilon$ be the maximal existence time (lifespan) of the solution, that is,

\[
    T_\varepsilon = \sup \{ T \in (0, \infty) ; \text{there exists a unique solution } u \text{ to (2.1)} \}
\]

such that $u \in C([0, T); L^2) \cap L^r_t(0, T; L^\rho_x)$, where $r, \rho$ are as in Proposition 2.1. The lower bound of the lifespan follows from the proposition immediately.

**Corollary 2.2.** Under the same assumptions as in Proposition 2.1, the estimate is valid

\[
    T_\varepsilon \geq C\varepsilon^{1/\omega},
\]

where $\omega = n/4 - 1/(p - 1)$ and $C = C(n, p, \|f\|_{L^2})$ is a positive constant.

The next interest is an upper bound of the lifespan. In [4], it was proved that $T_\varepsilon$ must be finite for suitable initial data. To recall their result, we introduce some notations: $\lambda_1 = \text{Re}\lambda, \lambda_2 = \text{Im}\lambda, f_1 = \text{Re}f$ and $f_2 = \text{Im}f$.

We impose the additional assumptions on the data:

\[
    f_1 \in L^1, \quad \lambda_2 \int_{\mathbb{R}^n} f_1(x)\, dx > 0 \quad \text{or} \quad f_2 \in L^1, \quad \lambda_1 \int_{\mathbb{R}^n} f_2(x)\, dx < 0.
\]

Then the following is valid:

**Proposition 2.3** (Ikeda and Wakasugi [4]). Let $1 < p \leq 1 + 2/n, \lambda \in \mathbb{C} \setminus \{0\}, \varepsilon > 0$ and $f \in L^2$. If $f$ satisfies (2.2), then $T_\varepsilon < \infty$. Moreover, the $L^2$-norm of the local solution blows up in finite time;

\[
    \lim_{t \to T_\varepsilon^-} \|u(t)\|_{L^2} = \infty.
\]

It is well known that the similar result holds for the corresponding nonlinear heat equation and the damped wave equation. Proposition 2.3 can be said to be the NLS version. We will give a proof of the proposition different from [4] in Appendix.

**Remark 2.1.** In [4], in order to prove $T_\varepsilon < \infty$, a contradiction argument based on papers [12], [13] was used. Therefore, an upper bound of the lifespan was not obtained.
Next, we state our main result in this paper, which gives an upper bound of the lifespan. We put the more additional assumption on the data:

\[
\begin{align*}
\text{“} f_1 & \in L^1, \lambda_2 f_1 (x) \geq |x|^{-k}, |x| > 1” \quad \text{or “} f_2 \in L^1, -\lambda_1 f_2 (x) \geq |x|^{-k}, |x| > 1”
\end{align*}
\]

where \( n < k < 2/(p - 1) \), which exists, if \( 1 < p < 1 + 2/n \). We also note that the function of the right hand side of (2.4) belongs to \( L^1 \cap L^2 \). Then the following is valid;

**Theorem 2.4.** Let \( 1 < p < 1 + 2/n, \lambda \in \mathbb{C} \setminus \{0\} \) and \( f \in L^2 \). If \( f \) satisfies (2.4), then there exist \( \varepsilon_0 > 0 \) and positive constant \( C = C(k, p, \lambda) \) such that

\[
T_{\varepsilon} \leq C\varepsilon^{1/\kappa}
\]

for any \( \varepsilon \in (0, \varepsilon_0) \) where \( \kappa \equiv k/2 - 1/(p - 1) \).

**Remark 2.2.** We note that there is a gap between the lower bound (see Corollary 2.2) and the upper bound in \( L^2 \)-framework, that is \( \kappa > \omega \).

Next, we consider the possibility to fill the gap in other frameworks. Especially, we consider the local existence for (2.1) in \( H^1 \cap L^{1+1/p} \)-framework.

Let \( \gamma > \frac{2}{n} \). The following result is valid:

**Proposition 2.5.** Let \( 1 < p < 1 + 4/(n - 2), \lambda \in \mathbb{C}, \varepsilon \geq 0 \) and \( f \in H^1 \cap L^{1+1/p} \). Then there exist a positive time \( T = T(\varepsilon, \|f\|_{H^1 \cap L^{1+1/p}}) \) and a unique solution \( u \in C([0, T); H^1 \cap L^{1+1/p}) \) for (2.1), where \( \rho = p + 1, r \) is given by \( 2/r = n(1/2 - 1/p) \).

This proposition can be proved in the almost same manner as in the proof of Theorem 6.3.2 in [1] (see also [8]). From the proposition, a lower bound of the lifespan also follows immediately. Let \( \bar{T}_{\varepsilon} \) be the maximal existence time of the local solution obtained in Proposition 2.5.

**Corollary 2.6.** Under the same assumptions as in Proposition 2.6, the inequality is valid:

\[
\bar{T}_{\varepsilon} \geq C\varepsilon^{1/\sigma},
\]

where \( \sigma = \frac{1}{\gamma} + \frac{n}{2(p+1)} - \frac{1}{p-1} \) and \( C = C(n, p, \|f\|_{H^1 \cap L^{1+1/p}}) \) is some positive constant.

**Remark 2.3.** The same conclusion as in Theorem 2.4 holds even if \( (T_{\varepsilon}, L^2) \) is replaced by \( (\bar{T}_{\varepsilon}, H^1 \cap L^{1+1/p}) \). In Corollary 2.6 since

\[
\sigma \to \frac{n}{2} \left( 1 - \frac{1}{p+1} \right) - \frac{1}{p-1}
\]

as \( r \to \frac{2}{n} \), we can see that there is also gap between the upper bound and the lower bound of the lifespan in \( H^1 \cap L^{1+1/p} \)-framework, though the lower estimate is improved, i.e. \( \kappa > \sigma > \omega \) as \( r \to \frac{2}{n} \).

**Remark 2.4.** In the critical case \( p = 1 + 2/n \), we do not know an upper bound of lifespan for (L.1)-(L.3).
At the end of this section, we mention the strategy of the proof of Theorem 2.4. We will use a test-function method based on papers [5], [9]. In [5], [9], upper bounds of lifespan for some parabolic equations were obtained. However, their argument does not be applicable to the present NLS directly. Since solutions for NLS are complex-valued, the constant $\lambda$ in front of the nonlinearity is a complex number and especially, the appropriate function spaces for NLS differs from that of those parabolic equations. To overcome these difficulties, we will consider the real part or imaginary part for the equation and reconsider the problem under the suitable function spaces $L^2$ (or $H^1 \cap L^{1+1/p}$) to NLS, so that we can use the local existence theorem.

3. Integral inequalities

In this section, we prepare some integral inequalities. Before doing so, we introduce the non-negative smooth function $\phi$ as follows, which was constructed in the papers [2], [3]:

$$
\phi(x) = \phi(|x|), \quad \phi(0) = 1, \quad 0 < \phi(x) \leq 1 \text{ for } |x| > 0,
$$

where $\phi(|x|)$ is decreasing of $|x|$ and $\phi(|x|) \to 0$ as $|x| \to \infty$ sufficiently fast. Moreover, there exists $\mu > 0$ such that

$$
|\Delta \phi| \leq \mu \phi, \quad x \in \mathbb{R}^n,
$$

and $\|\phi\|_{L^1} = 1$. This can be done by letting $\phi(r) = e^{-r^\nu}$ for $r \gg 1$ with $\nu \in (0, 1]$ and extending $\phi$ to $[0, \infty)$ by a smooth approximation. Let $\theta$ be sufficiently large and

$$
\eta(t) = \eta_{S,T}(t) = \begin{cases} 
0, & \text{if } t > T, \\
(1 - (t - S) / (T - S))^{\theta}, & \text{if } S \leq t \leq T, \\
1, & \text{if } t < S,
\end{cases}
$$

where $0 \leq S < T$. Furthermore, set $\eta_R(t) = \eta(t/R^2)$, $\phi_R(x) = \phi(x/R)$ and $\psi_R(t, x) = \eta_R(t) \phi_R(x)$ for $R > 0$.

First, we reduce the integral equation (2.1) into the weak form.

**Lemma 3.1.** Let $u$ be an $L^2$-solution of (1.1)-(1.2) on $[0, T)$. Then $u$ satisfies

$$
\int_{[0,T) \times \mathbb{R}^n} u(-i \partial_t (\psi_R) + \Delta (\psi_R)) dx dt
= i\varepsilon \int_{\mathbb{R}^n} f(x) \psi_R(0, x) dx + \lambda \int_{[0,T) \times \mathbb{R}^n} |u|^p \psi_R dx dt.
$$

This lemma can be proved in the same manner as the proof of Proposition 3.1 in [4].

Next, we will lead a integral inequality. Hereafter we only consider the case of $\lambda_1 > 0$ for simplicity. The other cases can be treated in the almost same way (see Remark 3.1).

We introduce some functions:

$$
I_R(S,T) = \int_{(S R^2, T R^2) \times \mathbb{R}^n} |u|^p \psi_R dx dt,
$$

$$
J_R = \varepsilon \int_{\mathbb{R}^n} -f_2(x) \phi(x/R) dx.
$$
and
\[
A(S,T) = \left( \int_{[S,T] \times \mathbb{R}^n} |\partial_t \eta(t)|^q \eta(t)^{-1/(p-1)} \phi(x) \, dx \, dt \right)^{1/q},
\]
\[
B(S,T) = \left( \int_{[0,T] \times \mathbb{R}^n} \eta_{S,T}(t) \phi(x) \, dx \, dt \right)^{1/q},
\]
where \( q = p/(p-1) \). By the direct computation, we have
\[
A(S,T) = \theta \{ \theta - 1/(p-1) \}^{-1/q} (T-S)^{-1/p}, \quad B(S,T) = \left( S + \frac{T-S}{\theta + 1} \right)^{1/q}.
\]

We have the following:

**Lemma 3.2.** Let \( u \) be an \( L^2 \)-solution of (1.1)-(1.2) on \([0,T_*]\). Then the inequality holds
\[
\lambda_1 I_R (0, T) + J_R \leq R^s \left\{ I_R (S,T)^{1/p} A(S,T) + \mu I_R (0,T)^{1/p} B(S,T) \right\}
\]
for any \( 0 \leq S < T \) and \( R > 0 \) with \( TR^2 < T_* \), where \( s = -2 + (2 + n)/q \).

**Proof.** Since \( u \) is \( L^2 \)-solution on \([0,T_*]\) and \( TR^2 < T_* \), by Lemma 3.2, we have
\[
\lambda \int_{[0,T R^2] \times \mathbb{R}^n} |u|^p \psi_R \, dx \, dt + i \varepsilon \int_{\mathbb{R}^n} f(x) \psi_R (0, x) \, dx \geq \int_{[0,T R^2] \times \mathbb{R}^n} u (-i \partial_t (\psi_R) + \Delta (\psi_R)) \, dx \, dt.
\]
Note that \( \lambda_1 > 0 \), by taking real part as the above identity, we obtain
\[
\lambda_1 I_R (0, T) + J_R = \int_{[0,T R^2] \times \mathbb{R}^n} \text{Re} \, u (-i \partial_t (\psi_R) + \Delta (\psi_R)) \, dx \, dt 
\]
\[
\leq \int_{[0,T R^2] \times \mathbb{R}^n} |u| \{ |\partial_t (\psi_R)| + |\Delta (\psi_R)| \} \, dx \, dt 
\]
\[
= K^1_R + K^2_R.
\]
We note that \( (\partial_t \eta) (t) = 0 \) except on \((S,T)\). By using the identity
\[
\partial_t \psi_R (t, x) = R^{-2} \phi_R (x) (\partial_t \eta) (t/R^2)
\]
and the Hölder inequality, we can get
\[
K^1_R = R^{-2} \int_{[SR^2,T R^2] \times \mathbb{R}^n} |u| \eta_{R}^{1/p} \left| (\partial_t \eta) (t/R^2) \right| \eta_{R}^{-1/(p-1)} \phi_R \, dx \, dt
\]
\[
\leq R^{-2} I_R (S,T)^{1/p} \left( \int_{[SR^2,T R^2] \times \mathbb{R}^n} \left| (\partial_t \eta) (t/R^2) \right|^q \eta_{R}^{-1/(p-1)} \phi_R \, dx \, dt \right)^{1/q}
\]
\[
= I_R (S,T)^{1/p} A(S,T) \; R^s,
\]
where we have used the changing variables with \( t/R^2 = t' \) and \( x/R = x' \) to obtain the last identity. Next, by the identity \( \Delta (\phi (x/R)) = R^{-2} (\Delta \phi) (x/R) \), the Hölder
inequality and the estimate (3.1), we have
\[
K_R^2 = R^{-2} \int_{[0,TR^2) \times \mathbb{R}^n} |u| \eta(t/R^2) |(\Delta \phi)(x/R)| \, dx \, dt
\leq \mu R^{-2} \int_{[0,TR^2) \times \mathbb{R}^n} |u| \psi_R \, dx \, dt
\leq \mu R^{-2} I_R(0,T)^{1/p} \left( \int_{[0,TR^2) \times \mathbb{R}^n} \psi_R \, dx \, dt \right)^{1/q}
\]
(3.8)
\[
= \mu I_R(0,T)^{1/p} B(S,T) R^s,
\]
where we have used the changing variables again. By combining the estimates (3.6)-(3.8), we have the conclusion. \( \square \)

**Remark 3.1.** We remark the other cases different from \( \lambda_1 > 0 \). For example, when \( \lambda_2 > 0 \), by taking the imaginary part as (3.2), an estimate similar to (4.2) can be obtained.

Next, we give the upper bound of \( J_R \). Let \( \sigma > 0 \) and \( 0 < \omega < 1 \). We introduce the function
\[
\Psi(\sigma,\omega) \equiv \max_{x \geq 0} (\sigma x^\omega - x) = (1 - \omega) x^{\omega-1} \sigma^{1-\omega}.
\]
We also denote \( I_R(T) = I_R(0,T) \), \( A(T) = A(0,T) \), \( B(T) = B(0,T) \) and
\[
D(T) = A(T) + \mu B(T),
\]
for simplicity. The following estimates are valid:

**Lemma 3.3.** Let \( u \) be an \( L^2 \)-solution of (1.1)-(1.2) on \( [0,T_*) \). Then the estimate
\[
J_R \leq \lambda_1 \Psi(D(T) R^s/\lambda_1,1/p)
\]
holds for any \( T > 0, R > 0 \) with \( TR^2 < T_* \), where \( s = -2 + (2+n)/q \). Moreover, if \( T_* = \infty \), that is \( u \) is a global solution, then the inequality is valid:
\[
\limsup_{R \to \infty} R^{-sq} J_R \leq (\mu/\lambda_1)^{1/(p-1)}.
\]

The proof of this lemma was based on that of Theorem 3.3 in [5] and Theorem 2.2 in [9].

**Proof.** Since \( u \) is an \( L^2 \)-solution on \( [0,T_*) \) and using (3.4) with \( S = 0 \), we obtain
\[
J_R \leq R^s D(T) I_R(T)^{1/p} - \lambda_1 I_R(T) \leq \lambda_1 \Psi(D(T) R^s/\lambda_1,1/p),
\]
which is exactly (3.10).

Next, we will prove (3.11) under the assumption \( T_* = \infty \). By (3.9) and (3.10), we have
\[
J_R \leq \lambda_1 \Psi(D(T) R^s/\lambda_1,1/p)
= \lambda_1 (1 - 1/p) (1/p)^{1/p} \{ D(T) R^s/\lambda_1 \}^{1-1/p}
= C_1 R^{sq} D(T)^q,
\]
(3.12)
for any \( T > 0, R > 0 \), where \( C_1 = \lambda_1^{-1/(p-1)} (p-1) (1/p)^{q} \). This inequality implies
\[
\limsup_{R \to \infty} R^{-sq} J_R \leq C_1 \left\{ \inf_{T > 0} D(T) \right\}^{q}.
\]
(3.13)
Next, we will estimate $D(T)$. Set
\[ a_p = \frac{\theta}{\{\theta - 1/ (p - 1)\}^{1/q}}, \quad b_p = \frac{\mu}{(\theta + 1)^{1/q}}. \]
Remembering the identities (3.3), we can rewrite $D(T)$ as
\[ D(T) = a_p T^{-1/p} + b_p T^{1/q}. \]
Since
\[ \min_{T > 0} D(T) = \frac{\mu}{\{\theta - 1/ (p - 1)\}^{1/q}} \left( \frac{1}{p} \right)^{1/q} \theta^{1/q} \equiv C_1, \]
we have
\[ \lim_{T > 0} \min_{T > 0} D(T) = \frac{\mu}{p} (p - 1)^{-1/q}. \]
Finally, by combining (3.13)-(3.16), we obtain (3.11), which completes the proof of the lemma.

4. Upper bound of lifespan

In this section, we give a proof of Theorem 2.4 which implies an upper bound of the lifespan for the local $L^2$-solution. We also consider the case of $\lambda_1 > 0$ only. The other cases can be treated in the almost same manner. When $\lambda_1 > 0$, we may assume that $f_2$ satisfies
\[ f_1 \in L^1, \quad \lambda_2 f_1 (x) \geq |x|^{-k}, \quad |x| > \rho \quad \text{or} \quad f_2 \in L^1, \quad -\lambda_1 f_2 (x) \geq |x|^{-k}, \quad |x| > \rho \]
where $n < k < 2/ (p - 1)$.

Proof. First, we note that by Corollary 2.2, there exists $\varepsilon_0 > 0$ such that $T_\varepsilon > 1$ for any $\varepsilon \in (0, \varepsilon_0)$. Moreover, since $1 < p < 1+2/n$ and $f$ satisfies (2.4), by Proposition 2.3, we also find $T_\varepsilon < \infty$.

Next, we consider the lower bound of $J_R$. By changing variables and (4.1), we have
\[ J_R = \varepsilon R^n \int_{\mathbb{R}^n} -f_2 (Rx) \phi (x) \, dx \]
\[ \geq \varepsilon R^n \int_{|x| \geq 1/R} -f_2 (Rx) \phi (x) \, dx \]
\[ \geq \varepsilon R^{n-k} \int_{|x| \geq 1/R} |x|^{-k} \phi (x) \, dx \]
\[ \geq \varepsilon R^{n-k} \int_{|x| \geq 1/R} |x|^{-k} \phi (x) \, dx = C_k \varepsilon R^{n-k}. \]
for any $R > R_0 > 0$, where $R_0$ is a constant independent of $R$, $\varepsilon$ and defined later.

Next, let $\tau \in (1, T_\varepsilon)$ and $R > R_0$. By using (3.12) with $T = \tau R^{-2}$, we have
\[ \varepsilon \leq C_k^{-1} C_1 \left( R^n D (\tau R^{-2}) \right)^{q} R^{-n+k} \equiv C_2 H (\tau, R), \]
where \( C_2 = C_k^{-1}C_1 \). By (3.14), we can rewrite \( H \) as
\[
(4.3) \quad H(\tau, R) = R^{-n+k} \left( D(\tau R^{-2}) R^\alpha \right) \tau^\alpha = \left\{ a_p \tau^{-1/p} R^{\alpha_1} + b_p \tau^{1/q} R^{-\alpha_2} \right\} \tau^\alpha ,
\]
where \( \alpha_1 = k/q, \alpha_2 = 2 - k/q \).

Now we derive some properties on \( H(\tau, R) \). We assume that we can find a function \( G(\tau) \) satisfying the following two properties: The first one is that for any \( \tau \in (1, T_2) \) and any \( R > R_0 \), \( H(\tau, R) \geq G(\tau) \) and the other one is that for any \( \tau \in (1, T_2) \), there exists \( R_\tau > R_0 \) such that \( H(\tau, R_\tau) = G(\tau) \). Then (4.2) holds for any \( \tau \in (1, T_2) \), \( R > R_0 \) if and only if
\[
(4.4) \quad \varepsilon \leq C_2 G(\tau),
\]
for any \( \tau \in (1, T_2) \). Actually, we can find such function \( G(\tau) \) as follows. Set \( y = R^{\alpha_1 + \alpha_2} = R^2, \beta_1 = \alpha_2/(\alpha_1 + \alpha_2) = \alpha_2/2 \) and \( h(\tau, y) = a_p y^{1-\beta_1} + b_p y^{-\beta_1} \tau \).

Then we can rewrite
\[
H(\tau, R) = \tau^{-1/(p-1)} h(\tau, y)^q.\]

Denote \( \sigma = \sigma(y) = a_p b_p^{-1} (1 - \beta_1) \beta_1^{-1}, g(\tau) = \left\{ a_p y^{1-\beta_1} \sigma \beta_1^{-1} + b_p y^{-\beta_1} \sigma \beta_1 \right\} \tau^{1-\beta_1} \), and
\[
G(\tau) = \tau^{-1/(p-1)} g(\tau)^q.
\]

It is easy to check \( 0 < \beta_1 < 1 \). Then \( \zeta = g(\tau) \) is a convex. Furthermore, \( \zeta = h(\tau, y) \) is a tangent line of \( \zeta = g(\tau) \) at the point of \( (\sigma, g(\sigma)) \). Therefore, we can see that \( h(\tau, y) \geq g(\tau) \), for all \( \tau > 0 \). Hence \( H(\tau, R) \geq G(\tau) \), for any \( \tau, R > 0 \). Here, we choose \( R_0 \) as \( 0 < R_0 < \left\{ \sigma^{-1}(1) \right\}^{1/2} \). Then for any \( \tau \in (1, T_2) \), if we set \( R_\tau = \left\{ \sigma^{-1}(1) \right\}^{1/2}, \) that is,
\[
R_\tau = \left\{ a_p^{-1} b_p \beta_1 (1 - \beta_1)^{-1} \right\} \tau^{1/2} (> R_0),
\]
we have \( H(\tau, R_\tau) = G(\tau) \). On the other hand, by the direct computation, we have
\[
(4.5) \quad G(\tau) = C_3 \tau^\kappa,
\]
where \( \kappa = k/2 - 1/(p-1) \) and \( C_3 = C_3(\theta, p) > 0 \) is constant dependent only on \( \theta, p \). By combining (4.4) and (4.5), we have \( \varepsilon \leq C_4 \tau^\kappa \), with \( C_4 = C_2 C_3 > 0 \). From the assumption \( k < 2/(p-1) \), we obtain \( \kappa < 0 \). Therefore, by (4.5), we can get
\[
\tau \leq C \varepsilon^{1/\kappa}
\]
for any \( \tau \in (1, T_\varepsilon) \), with some \( C > 0 \). Finally, we can get \( T_\varepsilon \leq C \varepsilon^{1/\kappa} \), which completes the proof of the theorem.

5. Appendix

In this Appendix, we give a proof of Proposition 2.3 which was already proved in [14], though the following argument is different. We consider the case of \( \lambda_1 > 0 \) only. In this case, we may assume that
\[
f_2 \in L^1, \quad \int_{\mathbb{R}^n} f_2(x) \, dx < 0.
\]
Proof. We use a contradiction argument. We assume that $T_\varepsilon = \infty$. Then we note that there exists a unique global $L^2$-solution $u$ for (1.1)-(1.2). By the assumption on $f_2$, we can see that $J_R$ is positive for sufficiently large $R > 0$. In fact, due to $f_2 \in L^1$, by Lebesgue’s convergence theorem, we have

$$
\lim_{R \to \infty} J_R = \varepsilon \int_{\mathbb{R}^n} -f_2(x) \, dx > 0.
$$

Thus by (3.4), we obtain

$$
I_R(0,T) \leq C I_R(0,T)^{1/p} R^s,
$$

for any sufficiently large $R$, with some positive constant $C$ independent of $R$, which implies that

$$
(5.1) \quad I_R(0,T) \leq CR^{qs} \leq C,
$$

for any large $R$, due to $qs \leq 0$, (i.e. $1 < p \leq 1 + 2/n$). Therefore, by monotone convergence theorem and letting $R \to \infty$ in (5.1), we have

$$
\int_{[0,\infty) \times \mathbb{R}^n} |u|^p \, dx \, dt < \infty.
$$

In the case of $s < 0$, that is $1 < p < 1 + 2/n$, letting $R$ tend to infinity in (5.1), we obtain

$$
\int_{[0,\infty) \times \mathbb{R}^n} |u|^p \, dx \, dt = 0.
$$

Hence $u = 0$ for a.e $(t,x) \in [0,\infty) \times \mathbb{R}^n$. Finally, letting $R \to \infty$ in (3.4), we get

$$
\int_{\mathbb{R}^n} -f_2(x) \, dx \leq 0,
$$

which contradicts to the assumption on $f$.

Next, we consider the critical case $s = 0$, i.e. $p = 1 + 2/n$. Remembering (5.3), if we choose small $S$ and large $\Theta$ with $T - S$ bounded, we obtain

$$
(5.2) \quad B(S,T) \leq \left( \int_{\mathbb{R}^n} -f_2(x) \, dx \right) / \left\{ 2\mu \left( \int_{[0,\infty) \times \mathbb{R}^n} |u|^p \, dx \, dt \right)^{1/p} \right\}.
$$

By the uniform boundedness (5.1) of $I_R$, keeping $T - S$ bounded, we have

$$
(5.3) \quad \lim_{R \to \infty} I_R(S,T)^{1/p} A(S,T) = 0.
$$

Finally, letting $R \to \infty$ in (3.4), we obtain

$$
\lambda_1 \int_{[0,\infty) \times \mathbb{R}^n} |u|^p \, dx \, dt + \frac{1}{2} \int_{\mathbb{R}^n} -f_2(x) \, dx \leq 0,
$$

which also contradicts to the assumption on $\lambda_1, f_2$. This completes the proof of the proposition. \qed

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