The Convergent Properties of a New Parameter for Unconstrained Optimization

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Abstract. Because of its simplicity, low memory requirement, low computational cost, and global convergence properties, the Conjugate Gradient (CG) method is the most popular iterative mathematical technique for optimizing both linear and nonlinear systems. Some classical CG methods, however, have drawbacks such as poor global convergence and numerical performance in terms of iterations and function evaluations. To address these shortcomings, researchers proposed new CG parameter variants with efficient numerical results and good convergence properties. We present a new conjugate gradient formula $\beta^G_k$ based on the memoryless self-scale DFP quasi-Newton (QN) method in this paper. The proposed new formula fulfills the sufficient descent property and the global convergent condition with any proposed line research. When the exact line search is used, the proposed formula is reduced to the classical HS formula. Finally, we conclude that our proposed method is effective.

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Key Words and Phrases: Conjugate gradient, self-scale DFP, strong Wolfe-Powell line search, sufficient descent property

1. Introduction

For solving the unconstrained minimization problem, the quasi-Newton methods are extremely useful and efficient to solve.

$$\min\ z(x),\ x \in \mathbb{R}^n$$

(1)

where $z : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice continuously differentiable [5]. Broyden [2] introduced the QN family of variable metric formulas in 1970, which is the most efficient technique for minimizing a non-linear function $z(x)$. The following quasi-Newton equation has traditionally been used to update the iterate matrix:

$$\beta_{k+1} v_k = y_k$$

(2)

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We have
\[ v_k = \gamma_k d_k = x_{k+1} - x_k, \quad \text{and} \quad y_k = g_{k+1} - g_k \]
where \( \gamma_k > 0 \) is determined by a suitable line search. Iterative methods are commonly used to solve (1), and the iterative formula is as follows:
\[ x_{k+1} = x_k + \gamma_k d_k, \quad k = 0, 1, 2, 3, \ldots, \quad (4) \]
If \( H_k \) is to be regarded as a close approximation to \( B_k^{-1} \), it follows that:
\[ H_{k+1} y_k = v_k \quad (5) \]
The direction \( d_k \) is obtained by solving the equation the updating matrix \( B_k \) is required to satisfy the equation (2) and the usual quasi-Newton equation (5).
\[ d_k = -H_k g_k \quad (6) \]
The nonlinear conjugate gradient (CG) method is one of the most well-known methods for solving the unconstrained optimization problem (1), which is especially useful when the dimension \( n \) of \( z(x) \) is large [7]. This is because the iteration is simple and requires little memory. The search direction is typically defined as:
\[ d_k = \begin{cases} 
-g_k, & k = 0 \\
-g_k + \beta_k d_{k-1}, & k \geq 1 
\end{cases} \quad (7) \]
\( \beta_k \in \mathbb{R} \), characterized the CG-method. If \( f(x) \) is a strictly convex quadratic function with exact line search, the parameter \( \beta_k \) is typically chosen to reduce the linear CG-method in (4) and (7). [12][11][18][19][10][15][9] define the six pioneering forms of \( \beta_k \).
\[ \beta_{kHS} = \frac{g_k^T y_{k-1}}{y_{k-1}^T d_{k-1}}; \quad \beta_{kFR} = \frac{g_k^T g_k}{g_{k-1}^T g_{k-1}}; \quad \beta_{kPRP} = \frac{g_k^T y_{k-1}}{g_{k-1}^T g_{k-1}}; \]
\[ \beta_{kCD} = \frac{g_k^T g_k}{y_{k-1}^T d_{k-1}}; \quad \beta_{kLS} = \frac{g_k^T y_{k-1}}{-g_{k-1}^T d_{k-1}}; \quad \beta_{kDY} = \frac{g_k^T g_k}{y_{k-1}^T d_{k-1}}; \]
Many of the classic parameters \( \beta_k \) mentioned above have been modified by a group of researchers, and another group has derived or imposed new parameters \( \beta_k \); however, not all of them can be included in a research, for example, see [13][21][16][1][22].
To determine the convergence conditions of above methods, it is usually necessary that the step size \( \gamma_k \) verify some properties, one of which is the strong Wolfe-Powel line search (sWP):
\[ f(x_k + \gamma_k d_k) \leq f(x_k) + \rho \gamma_k d_k \quad (8) \]
\[ |g(x_k + \gamma_k d_k)^T d_k| \leq \sigma |g_k^T d_k| \quad (9) \]
where \( 0 < \sigma < 0.5 < \rho < 1 \) are some fixed parameters. The step-size \( \gamma_k \) plays an essential role when investigating the sufficient descent condition.
and global convergence properties

\[ \lim_{k \to \infty} | | g_k | |^2 = 0 \] (11)

2. New Formulas for \( \beta' \)'s and its Algorithm

The DFP update was first proposed by Davidon, and popularized by Fletcher and Powell. The DFP formula can be expressed as follows:

\[
H_k = H_{k-1} + \frac{v_{k-1}v_{k-1}^T}{y_{k-1}^T y_{k-1}} - \frac{H_{k-1}y_{k-1}y_{k-1}^T}{y_{k-1}^T y_{k-1}} H_{k-1} \] (12)

To scale the Hessian matrix \( H_k \), we will use the self-scaling quasi-Newton method. Oren [17] introduced self-scaling variable metric algorithms, which are defined as

\[
\eta_{k-1} = \frac{v_{k-1}^T y_{k-1}}{| | v_{k-1} | |^2} \] (13)

which is a well-known and effective adaptive formula. Our proposed method’s general strategy is to scale all DFP terms, i.e. update the matrix by self-scaling DFP of the form

\[
H_k = \eta_{k-1} \left[ H_{k-1} + \frac{v_{k-1}v_{k-1}^T}{y_{k-1}^T y_{k-1}} - \frac{H_{k-1}y_{k-1}y_{k-1}^T}{y_{k-1}^T y_{k-1}} H_{k-1} \right] \] (14)

The preceding self-scaling DFP method is transformed into the memoryless self-scaling DFP method when \( H_k \) is substituted for I (i.e. \( H_k \equiv I \), where I is the identity matrix).

As a result, the memoryless DFP formed by:

\[
H_k = \frac{v_{k-1}^T y_{k-1}}{| | v_{k-1} | |^2} \left[ I + \frac{v_{k-1}v_{k-1}^T}{y_{k-1}^T y_{k-1}} - \frac{y_{k-1}y_{k-1}^T}{y_{k-1}^T y_{k-1}} \right] \] (15)

To derive the new formula multiply both sides of equation (15) by \( -g_k \), we have (6) and in the CG (7), hence

\[
-g_k + \beta_k d_{k-1} = \frac{v_{k-1}^T y_{k-1}}{| | v_{k-1} | |^2} \left[ -g_k + \frac{v_{k-1}^T g_k}{y_{k-1}^T y_{k-1}} v_{k-1} - \frac{y_{k-1}^T g_k}{y_{k-1}^T y_{k-1}} y_{k-1} \right] \] (16)

When we multiply both sides of (16) by \( y_{k-1} \), we get

\[
-g_k^T y_{k-1} + \beta_k d_{k-1}^T y_{k-1} = \frac{v_{k-1}^T y_{k-1}}{| | v_{k-1} | |^2} \left[ -g_k^T y_{k-1} + \frac{v_{k-1}^T g_k}{y_{k-1}^T y_{k-1}} v_{k-1}^T y_{k-1} - \frac{y_{k-1}^T g_k}{y_{k-1}^T y_{k-1}} y_{k-1}^T y_{k-1} \right] \]
\begin{align}
\beta_k d_{k-1}^T y_{k-1} &= g_k^T y_k - \frac{\nu_{k-1}^T y_{k-1}}{||\nu_{k-1}||^2} v_{k-1}^T g_k \\
\text{(17)}
\end{align}

Use \( v_{k-1} = \gamma_{k-1} d_{k-1} \) in (17), we have

\[ \beta_k = g_k^T y_k - \frac{\gamma_{k-1} d_{k-1}^T y_{k-1}}{||\nu_{k-1}||^2} v_{k-1}^T g_k \]

\[ \beta_{Gh}^k = g_k^T y_k - g_k^T d_{k-1} - \frac{\gamma_{k-1} d_{k-1}^T y_{k-1}}{||\nu_{k-1}||^2} \]

\[ \text{(18)} \]

It should be noted that if we used exact line search, \( \beta_{Gh}^k = \beta_{HS}^k \).

Now, we will go over the main steps of the algorithm that was used to create the new formula of \( \beta_{Gh}^k \).

2.1. Algorithm A

Given \( x_0 \in \mathbb{R}^n \), and \( \varepsilon > 0 \), set \( k = 0 \).

S1: Put \( d_k = -g_k \), if \( ||g_k|| < \varepsilon \), stop, otherwise continue.

S2: Calculate \( \gamma_k \) by using (8) and (9).

S3: Calculate \( x_{k+1} \) by (4), and \( g_{k+1} \), if \( ||g_{k+1}|| < \varepsilon \), then stop; Otherwise continue.

S4: Calculate \( \beta_{Gh}^k \) by (19) and \( d_{k+1} \) by (7).

S5: If the restarting criteria \( |g_k^T g_{k-1}| \geq 0.2 ||g_k||^2 \) is satisfy, proceed to S1, else put \( k = k + 1 \) go to S2.

2.2. Convergence Property

The following basic assumptions on the objective function are required to determine the global convergence property for algorithm (A).

2.3. Assumption B

i. \( f(x) \) is constrained by the level set from below \( \psi = \{ x \in \mathbb{R}^n, f(x) \leq f(x_0) \} \), where \( x_0 \) represents the starting point. Namely, there exists \( \tau > 0 \) which implies \( ||x_k|| \leq \tau \ \forall x \in \psi \ [3] \).

ii. \( f(x) \) it is a smooth in a specific neighborhood \( N \) of \( \psi \), and its gradient is Lipschitz continuous, i.e, there is a constant \( \mathcal{L} \) greater than zero, such that

\[ ||\nabla f(x) - \nabla f(y)|| \leq \mathcal{L} ||x - y||, \ \forall x, y \in N \]  \[ \text{(19)} \]
Using algorithm (A), there is now a positive constant \( \omega \), resulting in \( 0 < ||g_k|| \leq \omega \), \( \forall x \in \psi \) [14] To attain global convergence, algorithms (A) must be globally converged. To begin, we will look into the new proposed method’s descent property.

**Theorem 1.** Impose that Assumptions (B) hold. Suppose the method of the form (4) and (7) with \( \beta_k \) satisfy (18), and the step size \( \gamma_k \) satisfy sWP line search (8) and (9), then there exists a constant \( \vartheta > 0 \), s.t.

\[
g_k^T d_k \leq -\vartheta ||g_k||^2, \quad \vartheta > 0, \quad \forall k \geq 0 \tag{20}
\]

**Proof.** To begin, the proof is trivial for \( k = 0 \), i.e.

\[
d_0 = g_0 \Rightarrow g_0^T d_0 = -||g_0||^2
\]

Multiplying both sides of (7) by \( g_k^T \), we get

\[
g_k^T d_k = -g_k^T g_k + \left[ \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} - \frac{g_k^T d_{k-1}}{||d_{k-1}||^2} \right] g_k^T d_{k-1}
\]

\[
= -g_k^T g_k + \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} g_k^T d_{k-1} - \frac{||g_k|| \cdot ||d_{k-1}||^2}{||d_{k-1}||^2}
\]

We know that \( g_k^T d_{k-1} \leq d_{k-1}^T y_{k-1} \), and \( g_k^T d_{k-1} \leq ||g_k|| \cdot ||d_{k-1}|| \)

\[
g_k^T d_k \leq -||g_k||^2 + \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} d_{k-1}^T y_{k-1} - \frac{(||g_k|| \cdot ||d_{k-1}||)^2}{||d_{k-1}||^2}
\]

\[
= -||g_k||^2 + g_k^T y_{k-1} - ||g_k||^2
\]

\[
g_k^T y_{k-1} = ||g_k||^2 - g_k^T y_{k-1}
\]

Using the restarting criteria, i.e. \( g_k^T g_{k-1} \leq -0.2 ||g_k||^2 \), yield

\[
g_k^T d_k \leq -2||g_k||^2 + ||g_k|| + 0.2||g_k||^2
\]

\[
g_k^T d_k \leq -0.8||g_k||^2
\]

As a result, (20) holds true for all \( k \). After demonstrating that Algorithm (A) satisfies the descent property, we must demonstrate Algorithm (A) global convergence under assumption (B). We require the following lemmas, which are frequently used to demonstrate global convergence and Zoutendijk [23] provides them.

**Lemma 1.** Let the Assumption (B) be correct. Assume any iteration method (4) and (7), and \( \gamma_k \) obtained by the sWP (8) and (9). If

\[
\sum_{k \geq 1} \frac{1}{||d_k||^2} = \infty \tag{21}
\]

Then

\[
\liminf_{k \to \infty} ||g_k|| = 0 \tag{22}
\]
Theorem 2. Consider that Assumption (B) established. Suppose that the algorithm (A), and $\gamma_k$ is obtained through the sWP and $d_k$ is the descent direction. Then $\liminf_{k \to \infty} ||g_k|| = 0$.

Proof. Because the descending property holds, we now have $d_k \neq 0$. Therefore, lemma (2) is sufficient to show that $||d_k||$ is constrained above. From (2) and (8),

$$||d_k|| = \left| \left| -g_k + \left[ \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} - \frac{d_{k-1}^T g_k}{||d_{k-1}||^2} \right] d_{k-1} \right| \right|$$

Since $|d_{k-1}^T y_{k-1}| \geq m||d_{k-1}|| ||y_{k-1}||$, where $m > 0$, so

$$||d_k|| \leq ||g_k|| + \frac{||g_k|| ||y_{k-1}||}{m||d_{k-1}|| ||y_{k-1}||} - ||g_k||$$

$$\leq \left( \frac{1}{m||d_{k-1}||} \right) ||g_k|| \leq \left( \frac{1}{m \cdot \omega} \right) \omega = \mu$$

$$\Rightarrow \sum_{k \geq 1} \frac{1}{||d_k||^2} \geq \frac{1}{\mu^2} \sum_{k \geq 1} 1 = \infty$$

As a result, (24) applies to all k.

3. Numerical Results

The main task in this section is to report the Algorithm (A performance)’s on a set of test functions. All codes are written with double precision in FORTRAN. We chose twenty unconstrained large-scale optimization test problems. We considered two experiments with different numbers of variables (n=100 and 1000) for each test function. The test problems are from the CUTE [6] library, as well as other large-scale optimization test problems from [4]. To assess the reliability of our algorithms, we used the same test functions to compare them to the well-known routines HS, DY, PRP and LS. All of these algorithms use sWP (8) and (9) line searches with $\delta = 0.001$ and $\sigma = 0.5$, respectively. When the following stopping criterion is met $||g_{k+1}|| \leq 10^{-5}$, all of these methods terminate. Dolan and Mor’e created performance profiling software. [8] was also used to analyze the execution Figures 1 and 2.
Table 1: Comparison between $\beta^\text{HS}$, $\beta^\text{PRP}$, $\beta^\text{LS}$, and $\beta^\text{Gh}$ with $n = 100$

| Test functions | $\beta^\text{Gh}_k$ NOI | NOF | $\beta^\text{HS}_k$ NOI | NOF | $\beta^\text{PRP}_k$ NOI | NOF | $\beta^\text{LS}_k$ NOI | NOF |
|----------------|--------------------------|-----|--------------------------|-----|--------------------------|-----|--------------------------|-----|
| Wood           | 27 64                    |     | 33 73                    |     | 29 67                    |     | 30 69                    |     |
| Wolfe          | 54 101                   |     | 55 103                   |     | 57 114                   |     | 55 115                   |     |
| Rosen          | 38 101                   |     | 34 94                    |     | 35 96                    |     | 35 96                    |     |
| NON            | 28 67                    |     | 30 78                    |     | 32 81                    |     | 31 80                    |     |
| Shallow        | 10 25                    |     | 10 25                    |     | 10 25                    |     | 10 25                    |     |
| ENGVAL 1       | 21 44                    |     | 22 46                    |     | 24 46                    |     | 22 46                    |     |
| Diagonal 2     | 54 207                   |     | 60 213                   |     | 58 214                   |     | 60 221                   |     |
| Ex. BD1        | 20 40                    |     | 22 46                    |     | 19 39                    |     | 23 48                    |     |
| Ex. Wood       | 27 61                    |     | 29 65                    |     | 32 67                    |     | 29 65                    |     |
| Powell         | 31 92                    |     | 37 104                   |     | 39 112                   |     | 41 118                   |     |
| Dixmaanc       | 5 15                     |     | 5 15                     |     | 5 15                     |     | 5 15                     |     |
| DENSCHNF       | 21 44                    |     | 24 51                    |     | 28 60                    |     | 26 56                    |     |
| Dixmaanb       | 29 66                    |     | 33 73                    |     | 26 60                    |     | 27 67                    |     |
| Ex. Rosen      | 29 66                    |     | 30 69                    |     | 31 72                    |     | 30 70                    |     |
| Cubic          | 14 39                    |     | 17 46                    |     | 21 51                    |     | 16 44                    |     |
| Ex. Beal U63   | 10 27                    |     | 10 27                    |     | 10 27                    |     | 10 27                    |     |
| Ex. TET        | 39 81                    |     | 45 98                    |     | 51 117                   |     | 53 117                   |     |
| Gen.Tridiagonal-2 | 116 251          |   | 120 255                  |     | 123 261                  |     | 125 268                  |     |
| Diagonal 6     | 3 9                      |     | 3 9                      |     | 3 9                      |     | 3 9                      |     |
| SUM            | 16 77                    |     | 18 82                    |     | 21 110                   |     | 20 103                   |     |
| **Total**      | **592 1477**             |     | **637 1542**             |     | **653 1643**             |     | **651 1600**             |     |

Table 2: The percentage between PRP, LS, HS, and Gh for $n = 100$

| Measurement   | PRP method | LS method | HS method | Gh method |
|---------------|------------|-----------|-----------|-----------|
| NOI           | 100%       | 99.69%    | 97.55%    | 90.66%    |
| NOF           | 100%       | 97.38%    | 93.85%    | 89.9%     |
Table 3: Comparison between $\beta_{HS}$, $\beta_{PRP}$, $\beta_{LS}$ and $\beta_{Gh}$ with $n = 1000$

| Test functions | $\beta_{Gh}$ | $\beta_{HS}$ | $\beta_{PRP}$ | $\beta_{LS}$ |
|----------------|--------------|--------------|--------------|--------------|
|                | NOI | NOF | NOI | NOF | NOI | NOF | NOI | NOF |
| Wood           | 32  | 74  | 36  | 84  | 35  | 81  | 37  | 87  |
| Wolfe          | 58  | 108 | 59  | 119 | 58  | 116 | 59  | 118 |
| Rosen          | 38  | 101 | 34  | 94  | 35  | 96  | 35  | 96  |
| NON            | 30  | 76  | 33  | 84  | 39  | 94  | 32  | 83  |
| Shallow        | 10  | 25  | 10  | 25  | 10  | 25  | 10  | 25  |
| ENGVAL 1       | 20  | 44  | 22  | 46  | 23  | 46  | 22  | 46  |
| Diagonal 2     | 54  | 207 | 62  | 225 | 58  | 214 | 59  | 219 |
| Ex. BD1        | 21  | 44  | 23  | 48  | 24  | 50  | 23  | 48  |
| Ex. Wood       | 27  | 61  | 30  | 67  | 30  | 67  | 29  | 65  |
| Powell         | 39  | 110 | 41  | 109 | 56  | 129 | 52  | 121 |
| Dixmaanc       | 5   | 15  | 5   | 15  | 5   | 15  | 5   | 15  |
| DENSCHNF       | 27  | 62  | 29  | 67  | 31  | 73  | 30  | 69  |
| Dixmaanb       | 35  | 75  | 39  | 89  | 32  | 67  | 34  | 72  |
| Ex. Rosen      | 31  | 69  | 34  | 75  | 37  | 86  | 39  | 90  |
| Cubic          | 16  | 41  | 18  | 50  | 21  | 59  | 20  | 55  |
| Ex. Beal U63   | 12  | 29  | 12  | 29  | 12  | 29  | 12  | 29  |
| Ex. TET        | 48  | 95  | 54  | 109 | 59  | 118 | 58  | 116 |
| Gen.Tridiagonal-2 | 121 | 232 | 125 | 234 | 127 | 239 | 130 | 241 |
| Diagonal 6     | 3   | 9   | 3   | 9   | 3   | 9   | 3   | 9   |
| SUM            | 19  | 83  | 20  | 87  | 23  | 115 | 22  | 107 |
| **Total**      | 647 | 1571| 689 | 1663| 722 | 1759| 711 | 1691 |

Table 2 shows that the proposed $\beta_{Gh}$ formula outperforms the classic $\beta_{PRP}$, $\beta_{LS}$, and $\beta_{HS}$ formulas in terms of percentage performance. We discovered that the proposed Gh algorithm saves (NOI, 9.34%), (NOF, 10.1%), the LS algorithm saves (NOI, 0.31%), (NOF, 2.62%) and the HS algorithm saves (NOI, 2.45%), (NOF, 6.15%), for $n = 100$.

Table 4 shows that the proposed $\beta_{Gh}$ formula outperforms the classic $\beta_{PRP}$, $\beta_{LS}$, and $\beta_{HS}$ formulas in terms of percentage performance. We discovered that the proposed Gh algorithm saves (NOI, 9.0%), (NOF, 7.1%), the LS algorithm saves (NOI, 1.52%), (NOF, 3.86%) and the HS algorithm saves (NOI, 4.57%), (NOF, 5.46%), for $n = 1000$.

Figures (1) (a) and (b) depict the efficiency of the proposed method in terms of NOI for $n = 100$ and 1000, respectively. Figures (2): (c) and (d)) demonstrate the effectiveness of the suggested method in terms of NOF for $n = 100$ and 1000, respectively.

Table 4: The percentage between PRP, LS, HS, and Gh for $n = 1000$

| Measurement | PRP method | LS method | HS method | Gh method |
|-------------|------------|-----------|-----------|-----------|
| NOI         | 100%       | 98.48%    | 95.43%    | 91.0%     |
| NOF         | 100%       | 96.14%    | 94.54%    | 92.9%     |
4. Conclusion

We present a new parameter $\beta^G_{k}$ based on the memoryless self-scale DFP QN method in this article. Any line search will suffice to ensure adequate descent property. We also demonstrated that the Zoutendijk condition holds and that the method is globally convergent by using some step-length technique. The numerical results demonstrated the proposed algorithm’s efficiency when compared to some standard formulas.
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