Numerical solution of a logistic growth model for a population with Allee effect considering fuzzy initial values and fuzzy parameters

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Abstract. Predicting the future of population number is among the important factors that affect the consideration in preparing a good management for the population. This has been done by various known method, one among them is by developing a mathematical model describing the growth of the population. The model usually takes form in a differential equation or a system of differential equations, depending on the complexity of the underlying properties of the population. The most widely used growth models currently are those having a sigmoid solution of time series, including the Verhulst logistic equation and the Gompertz equation. In this paper we consider the Allee effect of the Verhulst’s logistic population model. The Allee effect is a phenomenon in biology showing a high correlation between population size or density and the mean individual fitness of the population. The method used to derive the solution is the Runge-Kutta numerical scheme, since it is in general regarded as one among the good numerical scheme which is relatively easy to implement. Further exploration is done via the fuzzy theoretical approach to accommodate the impreciseness of the initial values and parameters in the model.

1. Introduction

Analysis a model that describes the behavior of the population dynamics is important to understanding and predicting the growth of a population. In this paper, we discuss the population model in the form of differential equations that include the possibility of growth and extinction. In this case population growth influenced by the Allee effect. The Allee effect was first described by Warder Clyde Allee in 1930. Allee observed that extinctions may occur to populations of low population densities. Low population density makes it difficult for individuals to interact, which increases the likelihood of their extinction [1].

Moreover, many researchers had assumed that the parameters in their models are constant, in the sense that it is crisp, hence the model is crisp deterministic. But in fact, in real life the parameters are not always certain, sometime they are uncertain and imprecise. This is because the dynamics of the population can be affected by environmental changes such as food availability, birth and death rates, and so on. This situation may cause the parameter of the population model to vary from time, and difficult to estimate. In this case an imprecise estimation is likely to occur. This may cause inaccuracies in making mathematical models, even in classical deterministic approaches. This can be due to the nature
of the state variable involved, such as the impreciseness in the model parameters and in the initial conditions. Some authors have used the fuzzy set theory in their papers in exploring the impreciseness, which among them is in the case of population dynamics [2] [3] and epidemiology problems [4-7]. The fuzzy set theory, introduced by Zadeh in 1965, intended to illustrate some parts of the real world in the mathematical concepts of subjective topics in many cases. It becomes an important tool for a better understanding of some real situations, especially in the dynamics of population [2].

The model given in this paper is the Verhulst model with the Allee effect. We use fuzzy set theory to accommodate the impreciseness in parameters and the initial value. Model parameters and the initial value are assumed to be fuzzy numbers and represented by the triangular membership function. Model parameters and the initial values are given hypothetically and are not based on real data, just to illustrate the idea of the model. The fuzziness of the initial population size and model parameters propagates to the output in the numerical method used to obtain the solution to the model. In this case we use the fourth order Runge-Kutta method.

2. Preliminaries

In this section, we briefly describe the main theoretical tools that will be used in the analysis of the proposing mathematical model for a population with Allee effect considering fuzzy initial values and fuzzy parameters.

2.1. Population growth model with Allee effect

Generally, a population has a slow growth in the beginning, but a rapid acceleration begins at a certain time and finally a phase of stationary growth is achieved, such as in the Verhulst’s logistic model. Later, Warder Clyde Allee observed a fact that in a small population such growth may exist but there may be also the extinction of the species. This observation related to the initial density of the population. Allee concluded that there is a critical point in which if the initial density of the population is below this critical point then the population vanishes and if it is above this critical point then the population has a positive growth phase. In other words, a population with the Allee effect may have a threshold population size, below which the population goes extinct deterministically. This threshold is often called the Allee threshold [8]. The graph of the solution pattern of the population growth with the Allee effect is illustrated with figure 1.

![Figure 1. Illustration of the solution pattern of the population growth model with Allee effect, which is divided into four regions.](image-url)
Figure 1 illustrates the solution pattern of the population growth model with Allee effect, which is divided into four regions. Let \( P \) denotes the number of population densities, \( K \) denotes the carrying capacity, and \( A \) denotes the Allee threshold. Among the simplest equations of the population growth model with Allee effect is formalized by generalizing the known Verhulst’s logistic differential equation, given in the form:

\[
\frac{dP}{dt} = rP \left( 1 - \frac{P}{K} \right) \left( \frac{P}{A} - 1 \right)
\]  

(2.1)

where \( t \) represent time and the constant \( r \) defines the growth rate. In this model, the per capita growth rate \( \left( \frac{dP}{dt} \right) \) is negative if the population size is above the carrying capacity (\( K \)) and positive otherwise.

However, in the presence of the Allee effect, it also decreases below a given population size, and even might be negative below the critical population threshold (\( A \)) which eventually might lead to extinction [1].

From the model, there are three equilibrium solution to the population growth model with the Allee effect. First, is the equilibrium level where the population is zero. The second equilibrium corresponding to the maximum sustainable population \( K \) and the third one corresponding to the minimum sustainable population \( A \), as illustrated in figure 1. These three equilibrium solution divide the region into four distinct subregions. In Region I, where \( P > K \), we should expect that the solutions decay toward \( K \), as we have seen in the original logistic model. In Region II, where \( A < P < K \), we should expect that the solutions rise toward \( K \) eventually, in an asymptotic manner, much as they behave with the logistic model. In Region III, where \( 0 < P < A \), we should expect that the solutions decay toward zero. Finally, in Region IV, where \( P < 0 \), we should expect that the solutions decay toward \(-\infty\). In this case, region IV does not have a biological interpretation. From figure 1, the upper one at the height of \( K \) is a stable equilibrium since any solution that starts near that level converges to that equilibrium; the upper one at the height \( A \) is an unstable equilibrium because any solution that starts near it diverges from that level; the equilibrium at \( P = 0 \) is a semi-stable equilibrium because a solution that starts near it is converges and some are diverges from that level. This stability behavior of the equilibrium points can be analyzed easily through the standard stability analysis via the investigation of the eigen values of the jacobian of the respective equilibrium [9].

Figure 2. Illustration of logistic equation and logistic equation with Allee effect.

The per capita growth of the original logistic equation is always positive while the logistic equation with Allee effect is negative if the population is below the Allee threshold (330 in figure 2). The Allee
effect has strong evidence for ecological systems and sometime related to phenomena like conspecific interactions, rarity and animal sociality. In many situations, environmental conditions which influence the growth of population is subject to imprecise factors. To obtain a more appropriate strategy in managing a population, this impreciseness should be incorporated in the model. One way to do this is by modeling the parameters in the model in a fuzzy form. The aim at the present paper is to investigate the effect of fuzzy initial value and fuzzy parameters in the solution of the Verhulst logistic population model. The solution that we will consider is the numerical solution.

2.2. The fourth-order Runge-Kutta method

The classical Runge-Kutta method of order four for solving differential equation \( \frac{dy}{dt} = y' = f(y) \) with \( y(t_0) = y_0 \) is as follows [10]:

\[
y_{n+1} = y_n + \frac{1}{6}h[k_1 + 2k_2 + 2k_3 + k_4]
\]

(2.2)

with

\[
k_1 = f(y_n),
\]

\[
k_2 = f(y_n + k_1 \frac{h}{2}),
\]

\[
k_3 = f(y_n + k_2 \frac{h}{2}),
\]

\[
k_4 = f(y_n + k_3 h).
\]

All \( k \) values are recursively related. It means \( k_1 \) appears in the equation for \( k_2 \), which appears again in equation \( k_3 \) and so on, with \( y_0 \) is the initial value. This method can be easily to apply to equation (2.1) with \( f(P) = rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{A} - 1\right) \) and \( P(t_0) = P_0 \).

2.3. Fuzzy theory

The basic idea of fuzzy set theory is that real-world phenomena cannot be clearly divided into black and white divisions. Fuzzy set theory provides a mathematical framework that represents impreciseness, unclear and inaccuracies of the abundance of information available. The fuzzy set theory is an extension of the classical set theory. In a classical set theory, the set of membership functions has only two possible memberships, i.e. \( B = \{0, 1\} \). In the fuzzy set theory, the limits set have ambiguity and are not clearly defined. The fuzzy set declares elements to be members of multiple sets with different degrees of membership (\( \mu \)). The set of membership functions is expanded from the set \( B = \{0, 1\} \) which contains only two alternatives, at unit interval \( U = [0, 1] \) which has a number of alternatives. The degree of membership of each element in the fuzzy set can be obtained by the membership function approach [11].

2.3.1. Membership function. There are several functional approaches that can be used to define membership value. The most commonly used are the triangular membership function, the trapezoidal membership function and the bell shape membership function. The following shows the various of membership function of the fuzzy numbers [11]:
Figure 3. Graph of membership function of: (a) representation of the triangular curve; (b) representation of the trapezoidal curve; and (c) representation of the bell shape curve.

The equations for the fuzzy membership numbers presented in the graph of figure 3 are given as follows:

- **Representation of the triangular curve**
  \[
  \mu(x) = \begin{cases} 
  0 & ; x \leq a \\
  \frac{x-a}{b-a} & ; a < x \leq b \\
  \frac{c-x}{c-b} & ; b < x \leq c \\
  0 & ; x \geq c
  \end{cases} \tag{2.3}
  \]

- **Representation of the trapezoidal curve**
  \[
  \mu(x) = \begin{cases} 
  0 & ; x \leq a \\
  \frac{x-a}{b-a} & ; a < x \leq b \\
  1 & ; b < x \leq c \\
  \frac{d-x}{d-c} & ; c < x \leq d \\
  0 & ; x \geq d
  \end{cases} \tag{2.4}
  \]

- **Representation of the bell shape curve**
  \[
  \mu(x) = e^{-\left(\frac{x-r}{\gamma}\right)^2} ; \Re1 \leq x \leq \Re2, \\
  0 \; ; \text{otherwise} \tag{2.5}
  \]

3. **Results and discussion**

The population growth model with the Allee effect used in this paper is given in equation (2.1). Let \( P \) be the initial value of an amount greater than the Allee threshold \( (A) \) and \( Z \) is the initial value for a smaller amount than the Allee threshold \( (A) \). We explore the solution to equation (2.1) for various initial values and parameters. The solution to the model is obtained by using the fourth-order Runge-Kutta method. Assumed that \( P(0) \) is around 3040; \( Z(0) \) is around 2960; \( A \) is around 3000; \( r \) is around 0.0407; and \( K \) is around 8000. To model impreciseness, such as “around” some value number, an initial value
and parameters is fuzzified into a fuzzy form with triangular membership function. For the case of triangular fuzzy number, let \( \tilde{P}(0) = (P(0)_1/P(0)_2/P(0)_3) \) is above \( A \) and \( \tilde{Z}(0) = (Z(0)_1/Z(0)_2/Z(0)_3) \) is below \( A \). Note that we follow the notation in \cite{12} to write the fuzzy numbers. To solve equation (2.1) we do numerically by using Maple program. Technically, in triangular fuzzy number, since we assume that the fuzziness of the input propagates to the output, then we assume that initially there are three initial values as the input to the Runge-Kutta scheme, i.e. \( P(0)_1, P(0)_2, \) and \( P(0)_3 \) correspond to the triangular fuzzy number \( \tilde{P}(0) = (P(0)_1/P(0)_2/P(0)_3) \). The results of the triangular fuzzy number are presented in figures 4 to 7. We stack the graph for consecutive times \( t \), so that the height of the triangular is one. The length of the curve (blue line) for a fixed \( t \) in figure 4 to 7 part b and c is the radius or the length of the domain of the fuzzy number at time \( t \), while the red line is the membership function for those fuzzy numbers.

Figure 4(a) shows the two cases simultaneous graph of solution to \( \tilde{P}(0) \) and \( \tilde{Z}(0) \). If it is given a greater initial value than \( A \), i.e. \( P_1(0) = 3020; P_2(0) = 3040; \) and \( P_3(0) = 3060 \), then the graph of the solution increases and asymptotically towards carrying capacity. It means, if the initial population is greater than the Allee threshold, the population number will grow approaching the carrying capacity. It is eventually stable, no increase or decrease in population numbers. On the other hand, if it is given a smaller initial value than \( A \), i.e. \( Z_1(0) = 2940; Z_2(0) = 2960; \) and \( Z_3(0) = 2980 \), then the graph of solution decreases and asymptotically toward the zero point. It means, if the initial population is smaller than the Allee threshold, the population number will decline to the point where there is no population and the population is potentially extinct. Figure 4(b) shows the graph of triangular membership function of the model solution to 100 iterations. The right graph shows the model solution to the triangular membership function for an initial value greater than \( A \), while the left graph shows the solution to the triangular membership function of a smaller initial value than \( A \). From this figure, it appears that the domain in the triangular membership function is getting larger until the 100th iteration. It means, the possible solution to this model is wider until the 100th iterations. Figure 4(c) shows the graph of triangular membership function of the model solution to 3000 iterations. From this figure, it appears that the domain in the triangular membership function is getting larger until certain iteration, but eventually it closes to a crisp membership function. It shows that the initial values greatly affect the solution obtained from the model with the Allee effect. The figure shows a highly significant difference growth pattern between the populations, depending on their initial values.

![Figure 4](image_url)

**Figure 4.** Numerical solution for fuzzy initial value \( \tilde{P}(0) = (P(0)_1/P(0)_2/P(0)_3) \) of population growth model with Allee effect. (a) Simultaneous graph of solution for \( \tilde{P}(0) \) and \( \tilde{Z}(0) \); (b) Graph of triangular membership function for the model solution with \( T = 100 \); and (c) Graph of triangular membership function for the model solution with \( T = 3000 \).
Figure 5. Numerical solution to fuzzy parameter $\tilde{A} = (A_1/A_2/A_3)$ of population growth model with Allee effect. (a) Simultaneous graph of solution to $P(0)$ and $Z(0)$; (b) Graph of triangular membership function for the model solution to $T = 100$; and (c) Graph of triangular membership function for the model solution to $T = 3000$.

Figure 5(a) shows the two cases simultaneous graph of solution for $P(0)$ and $Z(0)$ with triangular fuzzy number for the Allee thresholds ($\tilde{A}$), i.e. $A_1 = 2980$, $A_2 = 3000$ and $A_3 = 3020$. Figure 5(b) shows the graph of triangular membership function of the model solution to 100 iterations. The right graph shows the model solution to the triangular membership function for an initial value greater than $A$, while the left graph shows the solution to the triangular membership function of a smaller initial value than $A$. Figure 5(c) shows a graph of triangular membership function of the model solution to 3000 iterations. From this figure, it appears that the solution to the model with the triangular membership function obtained is has the same pattern as figure 4. It means, if the Allee threshold is represented in a fuzzy number, then the probability of a change in the number of populations still depends on the number of the initial population. These behaviour is qualitatively the same as figure 4 in the case of triangular fuzzy number for initial population numbers.

The similar pattern as figure 4 and 5 are also shown in figure 6 and 7. Figure 6 shows the two cases simultaneous graph of solution to $P(0)$ and $Z(0)$ with triangular fuzzy number for the carrying capacity ($\tilde{K}$), i.e. $K_1 = 7500$, $K_2 = 8000$ and $K_3 = 8500$. While figure 7 shows the two cases simultaneous graph of solution to $P(0)$ and $Z(0)$ with triangular fuzzy number for the growth rate ($\tilde{r}$), i.e. $r_1 = 0.0357$, $r_2 = 0.0407$ and $r_3 = 0.0457$.

Figure 6. Numerical solution to fuzzy parameter $\tilde{K} = (K_1/K_2/K_3)$ of population growth model with Allee effect. (a) Simultaneous graph of solution to $P(0)$ and $Z(0)$; (b) Graph of triangular membership function for the model solution to $T = 100$; and (c) Graph of triangular membership function for the model solution to $T = 3000$. 
Figure 7. Numerical solution of fuzzy parameter $\bar{r} = (r_1/r_2/r_3)$ of population growth model with Allee effect. (a) Simultaneous graph of solution to $P(0)$ and $Z(0)$; (b) Graph of triangular membership function for the model solution to $T = 100$; and (c) Graph of triangular membership function for the model solution to $T = 3000$.

4. Conclusion
Population growth model used in this paper is the Verhulst logistic model with Allee effect. Model parameters and initial population number are determined hypothetically and not based on real data. The initial value of the population number is divided into 2 parts, first is for the initial value greater than the Allee threshold ($P(0) > A$) and second is the initial value smaller than the Allee threshold ($Z(0) < A$). To accommodate the impreciseness in the model, fuzzy set theory approach is used. In this case, the initial population number and model parameters are assumed to be a triangular fuzzy number. $ar{P}(0) = (P(0)_1/P(0)_2/P(0)_3)$ is triangular fuzzy number for the initial population number, $A = (A_1/A_2/A_3)$ is triangular fuzzy number for the Allee threshold, $\bar{K} = (K_1/K_2/K_3)$ is triangular fuzzy number for the carrying capacity, and $\bar{r} = (r_1/r_2/r_3)$ is triangular fuzzy number for the growth rate. Fuzzy number of initial values and model parameters give the same qualitative behaviour of the solution.

Generally, the initial value greatly affects the solution to the population model with Allee effect. Modeling imprecise initial value around the critical value $A$ with a crisp initial value could cause the extinction of population to a certain possibility degree, depending on the supposedly membership function of the input. From the discussion, it can be concluded that the mathematical model with the fuzzy set theory approach is considered appropriate in accommodating the inadequacy of solving the numerical solution to the existing model.

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