On absence of steady state in the Bouchaud-Mézard network model

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Abstract
In the limit of infinite number of nodes (agents), the Itô-reduced Bouchaud-Mézard network model of economic exchange has a time-independent mean and a steady-state inverse gamma distribution. We show that for a finite number of nodes the mean is actually distributed as a time-dependent lognormal and inverse gamma is quasi-stationary, with the time-dependent scale parameter.

Keywords: Bouchaud-Mézard, steady state, inverse gamma, quasi-stationary, lognormal

1. Introduction

Bouchaud-Mézard (BM) network model of economic exchange\cite{1} sparked much interest in econophysics community because it organically predicted a power-law-tailed – inverse gamma (IGa) – distribution for the “wealth”. Recently, questions of relaxation in this model came to the forefront\cite{2,3,4,5}. For constant mean wealth, relaxation times of the cumulants and of the entire distribution were studied in\cite{6}. Here we ascertain that for any finite number of nodes the mean should not be considered a constant and IGa a steady-state distribution.

The fully connected BM network model\cite{1} can be reduced to the following Itô process for a node \(i\) of the total of \(M\) network nodes:
\[
dw_i = \sqrt{2\sigma w_i} dB_i + J(w - w_i) dt
\]
where \(J\) is the connection (exchange) strength (assumed equal between all nodes), \(B_i\) is a Wiener process, \(\sigma\) is the strength of multiplicative stochasticity and
\[
\overline{w} = \frac{1}{M} \sum_i w_i
\]
is the mean over the network.

Our main result is that the quasi-stationary distribution function in a network is given by
\[
P(w) = \text{IGa}(w; \alpha, \beta) = \frac{1}{w^{\Gamma(\alpha)}} \left( \frac{\beta}{w} \right)^{\alpha} e^{-\frac{\beta}{w}}
\]
where \(\alpha = 1 + J/\sigma^2\) is the shape parameter (\(\alpha + 1\) is the exponent of the power-law tail), \(\beta(t) = \overline{w}(t)J/\sigma^2\) is the scale parameter and \(\overline{w}(t)\) is the running mean distributed as lognormal (LN).

\[
P(\overline{w}, t) = \frac{1}{2\sqrt{\pi t} \sigma_M \overline{w}} \exp \left( -\frac{1}{2} \left( \frac{\log(\overline{w}) + \sigma_M^2 t}{\sqrt{2t}} \right)^2 \right)
\]
\[
\sigma_M = \frac{\sigma}{\sqrt{M}} \sqrt{\frac{\alpha - 1}{\alpha - 2}}
\]

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where the distribution is understood as over a system of BM networks. The mean and the variance of the mean are then respectively

$$
\langle w \rangle = 1
$$ (6)

$$
\langle w^2 \rangle - \langle w \rangle^2 = \exp \left( 2\sigma_M^2 t \right) - 1
$$ (7)

Unity in (6) is due to the fact that in the corresponding infinite-node case $w$ in (1) and (3) is constant and can be set to unity by simple rescaling. In our calculations and simulations we use all $w_i = 1$ as the initial condition.

This paper is organized as follows. Secs. 2 and 3 below contain respectively the analytical derivation and its numerical verification. We conclude with the discussion of possible extensions of our work.

2. Analytical derivation

Here we first give a simple derivation of the LN distribution of the mean and then provide a detailed analytical derivation of the variance of the mean that independently verifies the LN result.

2.1. Distribution of the mean

For the BM network (1), from Eqs. (1) and (2)

$$
dw = \sqrt{2\sigma M} \sum_{i=1}^{M} w_i dB_i
$$ (8)

that is the exchange term is eliminated. Notice, that this means that (8) is formally the same as for the completely disconnected network (see Appendix A). Surmising that the distribution of the mean for the BM network is LN, we seek to replace the r.h.s. of (8) with

$$
dw = \sqrt{2\sigma M \mu dB}
$$ (9)

Then squaring and equating the r.h.s. of Eqs. (8) and (9), with the off-diagonal terms being zero, we obtain

$$
\sigma_M^2 = \frac{\sigma^2}{M} \sum_{i=1}^{M} w_i^2
$$ (10)

To determine $\sigma_M$ we made a crucial assumption – which will be confirmed numerically in Sec. 3 – that the quasi-stationary distribution in a network is given by (3). Then replacing $w = \left( \sum_{i=1}^{M} w_i \right) / M$ and $\left( \sum_{i=1}^{M} w_i^2 \right) / M$ with the expectation values of $w$ and $w^2$ computed with the distribution (3), we obtain

$$
\sigma_M = \frac{\sigma}{\sqrt{M}} \sqrt{\frac{\alpha - 1}{\alpha - 2}}
$$ (11)

Since the resultant $\sigma_M$ is time-independent, by Itô calculus (11) becomes

$$
d\ln w = \sqrt{2\sigma_M dB} - \sigma_M^2 dt
$$ (12)

which yields the LN distribution (4).
2.2. Variance

Denoting $\kappa_2$ the variance of the mean $\bar{w}$ and setting $\langle \bar{w} \rangle = 1$, we have

$$\kappa_2 = \langle \bar{w}^2 \rangle - \langle \bar{w} \rangle^2 = \langle \bar{w}^2 \rangle - 1 \quad (13)$$

Then, using (8) and that $\langle \bar{w} \cdot w_i dB_i \rangle = 0$, we obtain

$$d\bar{w}^2 = 2\bar{w} dw + (dw)^2 = 2\sqrt{2} \sigma \bar{w} \sum_{i=1}^{M} w_i dB_i + \left(\sqrt{2} \sigma \bar{w} \sum_{i=1}^{M} w_i dB_i\right)^2 \quad (14)$$

and

$$dk_2 = 2\sigma^2 \left\langle \left( \frac{1}{M} \sum_{i=1}^{M} w_i dB_i \right)^2 \right\rangle = 2\sigma^2 \left\langle \left( \frac{1}{M} \sum_{i=1}^{M} w_i dB_i \right)^2 \right\rangle \quad (15)$$

Since,

$$\left( \frac{1}{M} \sum_{i=1}^{M} w_i dB_i \right)^2 = \frac{1}{M^2} \sum_{i=1}^{M} w_i^2 (dB_i)^2 + \frac{1}{M^2} \sum_{i=1}^{M} \sum_{j=1,j\neq i}^{M} w_i w_j dB_i dB_j \quad (16)$$

then averaging, replacing $(dB_i)^2 = dt$, and eliminating the off-diagonal terms due to $dB_i$ and $dB_j$ being independent, we find

$$dk_2 = 2\sigma^2 \left\langle \left( \frac{1}{M} \sum_{i=1}^{M} w_i dB_i \right)^2 \right\rangle \quad (17)$$

Notice now that

$$\left\langle \left( \frac{1}{M} \sum_{i=1}^{M} w_i^2 \right)^2 \right\rangle = \left\langle \frac{1}{M} \sum_{i=1}^{M} (w_i - \bar{w})^2 \right\rangle + \langle \bar{w}^2 \rangle \quad (18)$$

Per our surmise (3), the variance in each path is $\bar{w}^2 / (\alpha - 2)$, so that (18) yields per (13)

$$\left\langle \frac{1}{M} \sum_{i=1}^{M} w_i^2 \right\rangle = \frac{\langle \bar{w}^2 \rangle}{\alpha - 2} + \langle \bar{w}^2 \rangle = \frac{\alpha - 1}{\alpha - 2} (\kappa_2 + 1) \quad (19)$$

which reduces (17), with the use of (5), to

$$dk_2 = 2\sigma^2 M (k_2 + 1) dt \quad (20)$$

Solving the differential equation with the initial condition of $k_2 = 0$ when $t = 0$, we finally obtain

$$k_2 = \exp \left( 2\sigma^2 M t \right) - 1 \quad (21)$$

3. Numerical simulations

We simulate (1) using $J = 0.1$, $\sigma^2 = 0.05$ and $M = 2^{13}$. The time step is $dt = 2^{-6}$, that is we use $2^6$ steps between two time “ticks”, $\Delta t = 1$. Since in the infinite-node BM system the fixed mean value can be always set to unity, we use $w_i = 1$ as the initial value for all nodes in all networks. For averaging over and for distribution over the networks, we use $2^{11}$ networks.

The left column in Fig. 1 shows segments of typical behavior of the running mean: increasing, oscillating around unity and decreasing. The middle column shows Kolmogorov-Smirnov (KS) statistic for fitting with
Figure 1: Each row corresponds to a different network. Left column: $\beta$ – top (green) line and $\mu$ – bottom (blue) line; Middle column – KS statistic for fitting with IGa distribution; Right column: parameters $\alpha$ – top (red) line and $\beta/\mu$ – bottom (green) line.

Figure 2: Variance of the mean in a single network for two different networks – top (green) and bottom (blue) lines; Average over the networks – middle (red) line.
IGa and the right column the parameters of the fitted IGa. The results are clearly in excellent agreement with our surmise (3).

The top (green) and bottom (blue) in Fig. (2) are examples of the variance of the running mean \( \overline{w}(t) \) in a single network. The middle (red) line is the average over all networks.

Fig. (3) is the test of Eq. (21). At a given time \( t \), we evaluate the variance of \( \overline{w}(t) \) over the networks. The smooth (cyan) line is the theoretical result and the jagged (red) line is the numerical simulation.

![Figure 3: Variance of the instantaneous \( \overline{w}(t) \) over the networks: theoretical result (21) – smooth (cyan) line; numerical simulation – jagged (red) line.](image)

Finally, Fig. 4 shows the KS statistic for fitting the distribution of \( \overline{w}(t) \) over the networks with LN as a function of time.

![Figure 4: KS statistic as a function of time for fitting instantaneous \( \overline{w}(t) \) over networks with LN.](image)

4. Conclusion

We explored relaxation in the Bouchaud-Mézard network model and showed that for any finite number of nodes there is no steady state per se. At long times, the distribution of node values is described by a quasi-stationary IGa distribution with the fixed shape parameter, that defines the exponent of the power-law tail, and variable scale parameter proportional to the running mean. The distribution of the running mean over a system of Bouchaud-Mézard networks is LN(\( \overline{w} - Q/2, \sqrt{Q} \)), where \( Q \) is linear in time and inversely proportional to the number of nodes. For the infinite number of nodes, the problem reduces to the standard IGa process, whose relaxation to the IGa distribution with the constant mean (which can always be set to unity) had been previously explored (3).

An obvious extension of this work is to the partially connected networks – random and regular – which are described by the generalized inverse gamma distribution (2). We have numerical results that parallel our findings for the fully connected networks, namely, that we have a quasi-stationary generalized inverse gamma distribution and a time-dependent heavy-tailed distribution of the mean (possibly also lognormal). However, we do not yet have a clear path to analytical derivation. Extension to relaxation processes in other stochastic processes believed to have steady-state distributions is another obvious direction of this research.
Appendix A. Disconnected network

A disconnected network allows for an independent check of the formalism developed in Sec. 2.2. In a disconnected network, \( J = 0 \),

\[
dw_i = \sqrt{2\sigma w_i} dB_i
\]

we have a LN distribution for each node

\[
P(w, t) = \frac{1}{2\sqrt{\pi t} w} \exp \left( -\frac{1}{2} \left( \frac{\log(w) + \sigma^2 t}{\sqrt{2t}} \right)^2 \right)
\]

with average and variance given by (6) and (7), where \( \sigma_M \) is replaced with \( \sigma \).

Since all nodes are independent, the distribution of the network mean \( \langle w \rangle \) is determined by the average of independent and identically distributed (i.i.d.) LN random variables. Therefore, per the central limit theorem (CLT), one expects the variance of the mean to be \( \approx \exp(2\sigma^2 t)/M \) for large \( M \). All the steps through (18) in Sec. 2.2 are the same here while the rest are now as follows:

\[
\left\langle \frac{1}{M} \sum_{i=1}^{M} w_i^2 \right\rangle = (e^{2\sigma^2 t} - 1) + (\kappa_2 + 1) = e^{2\sigma^2 t} + \kappa_2
\]

\[
d\kappa_2 = \frac{2\sigma^2}{M} (\kappa_2 + e^{2\sigma^2 t}) \, dt
\]

which indeed yields

\[
\kappa_2 = \frac{1}{M - 1} \left( e^{2\sigma^2 t} - e^{2\sigma^2 t/M} \right) \approx \frac{e^{2\sigma^2 t}}{M}
\]

for large \( M \).

The sum of LN distributions can be well approximated as LN distribution \([3]\), so it is tempting to think that the average of \( M \) LN distributions may be approximated by a \( \sqrt{M} \)-narrower LN distribution, which tends to the normal (N) distribution while maintaining the character of the LN heavy tail \([9]\). (Incidentally, the sum of i.i.d. IGa random variables is well approximated by a IGa distribution so it is tempting again to approximate the average of \( M \) IGa distributions as a a \( \sqrt{M} \)-narrower IGa distribution, tending to N while maintaining the character of the IGa fat tail \([10]\). Of course, in the BM model the variables in the sum for the network mean are not independent.) A lognormal LN\((w; -Q/2, \sqrt{Q})\), where \( Q = \log(\kappa_2 + 1) \), has the required variance and unity mean. In the large \( M \) limit, its explicit form is as follows:

\[
P(w, t) = \frac{1}{\sqrt{2\pi \log \left( \frac{M + e^{2\sigma^2 t}}{M} \right)}} w \exp \left( -\frac{1}{2} \frac{\log(w) + \frac{1}{2} \log \left( \frac{M + e^{2\sigma^2 t}}{M} \right)^2}{\log \left( \frac{M + e^{2\sigma^2 t}}{M} \right)} \right)
\]

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