Redshift drift in uniformly accelerated reference frame

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We construct an alternative uniformly accelerated reference frame based on 3+1 formalism in adapted coordinate. It is distinguished with Rindler coordinate that there is time-dependent redshift drift between co-moving observers. The experimentally falsifiable distinguishment might promote our understanding of non-inertial frame in laboratory.
I. INTRODUCTION

Owe to special relativity, the inertial frames are well tested and understood. The principle of relativity indicates that physical equations remain the same in all inertial frame. For non-inertial frame, there is general principle of relativity. It’s formulated as covariance principle in general relativity. In principle, general relativity should describe well any non-inertial reference frames. However, even uniformly accelerated reference frame is not understood well yet [1]. And different uniformly accelerated reference frames were set up from different points of views [2–6].

Propagation of light in non-inertial frames provides possibilities of testing general relativity referred to non-inertial reference frames in laboratory. The Sagnac effect states that in rotating reference frame, counter-propagating rays, which propagate around a closed path, would take different time intervals [7, 8]. It can be described by Born metric known as a relativistic effect [9–11]. Likewise, does a similar effect exist in uniformly accelerated frame?

As we known in view of inertial observers, accelerated detectors would observe a time dependent redshift of light from co-moving source. Could the redshift drift be observed in the uniformly accelerated reference frame as relativistic effect? And how is it described in general relativity?

In order to answer these questions, space-time metric of the uniformly accelerated reference frame should be given explicitly. Rindler coordinate [3], also named Miller coordinate or Lass coordinate [2], is a commonly used uniformly accelerated reference frame. As a rigid coordinate, it indicates that there is no redshift drift. It seems not consistent with the observations from inertial observers. As Huang [12] suggested that the redshift without drift in Rindler coordinate should be attributed to norm of 4-accelerations that are not the same for all co-moving observers. Minser, Thorne and Wheeler (MTW) [13] derived the Miller coordinate with the hypothesis of locality. It indicates that Rindler coordinate is in fact a local frame. The redshift drift might be higher order effect. All these considerations motivate us to construct an alternative uniformly accelerated reference frame that is different from Rindler coordinate beyond local frame [1, 5, 13–16].

In this paper, we study an adapted coordinate that all co-moving observers have the same norms of 4-acceleration. Explicit metric and coordinate transformation are obtained. The redshift drift in the new uniformly accelerated reference frame is calculated. Using
the new proposed uniformly accelerated frame, we investigate possible Unruh effect and show a non-thermal distribution of Minkowski vacuum perceived by uniformly accelerated observers.

This paper is organized as follows. In section II we review the redshift between co-moving objects in non-relativistic approximation and in Rindler coordinate. There is redshift drift in non-relativistic approximation, while Rindler coordinate shows no redshift drift. In section III we construct a uniformly accelerated reference frame and present features of the frame. In section IV we provide explicit metrics of the accelerated frames. The redshift drift and the possible Unruh effect in the accelerated frame are studied. Finally, conclusions and discussions are summered in section V. Throughout, we use convention that $c = 1$.

II. REDSHIFT DRIFT AND UNIFORMLY ACCELERATED REFERENCE FRAME

We suppose that two light sources A, B and a detector are fixed on a carrier. The light source A is located in a distance $L$ on the right of the detector, while the light source B is located on the left of the detector. The schematic diagram is shown in Figure 1(a). As we know that there is no redshift observed by the detector when the carrier undergoes an inertial motion. It would be different when the carrier undergoes a non-inertial motion (see Figure 1(b) and (c)).

In this section, we would firstly review shortly the redshift in non-relativistic approximation and in Rindler coordinate.

A. Non-relativistic redshift for accelerated observers

As time is absolute in non-relativistic kinematics, the frequency of light is universal in different reference frames. It indicates that redshift calculated in laboratory reference frame is equal to that calculated in reference frame of the moving detector. In a non-relativistic approximation, we can calculate redshift in laboratory reference frame to study what the detector observes.

We consider that light source B is assigned on the left of a detector in the distance of $L$. The carrier undergoes a uniformly accelerated motion to the right. It’s shown in Figure II(b).
FIG. 1: Schematic diagram of redshift and its drift in uniformly accelerated reference frame. Light sources A, B and a detector are fixed on a carrier. Panel (a): There is no redshift observed by the detector when the carrier undergoes an inertial motion. Panel (b): When the carrier moves to the right with a constant acceleration \( a_0 \), detector would observe redshift from source B and blueshift from source A. Panel (c): Observed redshift and blueshift would drift with time, if the carrier remains the uniformly accelerated motion.

The source emits a photon at \( t' \), and the detector observes it at \( t \). In the non-relativistic approximation that \( at \ll 1 \) and \( L \ll 1/a \), the processes can be formulated as

\[
(t - t') - \frac{1}{2} a(t^2 - t'^2) = L, \tag{1}
\]

where \( a \) is acceleration of the carrier. There is difference of time intervals between emitted and received photons. Eq. (1) sets the ratio of the time intervals,

\[
\frac{\Delta t}{\Delta t'} = \frac{1 - at}{1 - at'} = \frac{\sqrt{(1 - at)^2 - 2aL}}{1 - at}. \tag{2}
\]

From Eq. (2), we get the redshift \( z_- \),

\[
z_- \equiv \frac{\Delta t}{\Delta t'} - 1 = \frac{aL}{(1 - at)^2} + O((aL)^2). \tag{3}
\]

It shows that the redshift is time dependent and would get higher with time. Likewise, we consider the light source A located on the right of the detector. It would observe blueshift from the source, which is given by

\[
z_+ \approx -\frac{aL}{(1 + at)^2}. \tag{4}
\]
It shows that the blueshift is also time dependent. As time goes by, the blueshift would get lower. The both redshift and blueshift drift can be illustrated in Figure 1 from the process (b) to (c). In non-relativistic kinematics, it should be understood as Doppler effect, since there seems velocity difference between light source and detector, when the ray is emitted and received. However, in the reference frame of the carrier, the velocity difference might not be perceived. The redshift drift observed in the accelerated frame should be understood as relativistic effect. We would show it in section IV. The situation is similar to the understanding of expansion of the universe.

In most case that $at \ll c = 1$, the redshift and blueshift would lead to the most common version, namely, $z_\pm = \pm \frac{aL}{ct}$.

### B. Redshift in Rindler coordinate

In relativity, the uniformly accelerated motion is commonly described by a constant norm of 4-acceleration of a worldline. In the $t$-$x$ diagram, the uniformly accelerated motion is hyperbolic motion, as space-time trajectory of uniformly accelerated motion is a hyperbola, which can be of the form,

$$x^2 - t^2 = \frac{1}{a}.$$  \hspace{1cm} (5)

The hyperbolic motion can be described by equations as follows,

$$\begin{align*}
\frac{du^0}{d\tau} &= au^1, \\
\frac{du^1}{d\tau} &= au^0,
\end{align*}$$  \hspace{1cm} (6)

where $u^0 \equiv dt/d\tau$, $u^1 \equiv dx/d\tau$, and $\tau$ is proper time. Using normalization condition of $u^{\mu}$, one can find that the norm of 4-acceleration $du^{\mu}/d\tau$ is a constant $a$. A solution of Eq.(6) can be obtained,

$$\begin{align*}
u^0 &= \cosh(a\tau), \\
u^1 &= \sinh(a\tau).
\end{align*}$$  \hspace{1cm} (7)

With specific initial condition, the parametrized trajectory of uniformly accelerated motion can be of the form,

$$\begin{align*}
t &= \frac{1}{a} \sinh(a\tau), \\
x &= \frac{1}{a} \cosh(a\tau).
\end{align*}$$  \hspace{1cm} (8)
Another point of view for uniformly accelerated motion in relativity is from electrodynamics [16]. The equations of motion for charged particle are of the form,

\[ \frac{du^\mu}{d\tau} = \frac{q}{m} F^\mu_{\nu} u^\nu, \]  

(9)

where \( F^\mu_{\nu} \) is electromagnetic tensor, \( m \) and \( q \) is static mass and charge of a particle. We consider a uniform electric field in direction of \( x \)-axis. For simplicity, we ignore other spatial coordinates. The potential is given by \( A_\mu = (E_0 x, 0) \), where \( E_0 \) is strength of electric field.

From the potential, the electromagnetic tensor is of the form,

\[ F^\mu_{\nu} = \eta^{\mu\sigma} F_{\sigma\nu} = \begin{pmatrix} E_0 \\ E_0 \end{pmatrix}. \]  

(10)

The Eq. (9) can be rewritten as

\[
\begin{align*}
\frac{du^0}{d\tau} &= \frac{E_0 q}{m} u^1, \\
\frac{du^1}{d\tau} &= \frac{E_0 q}{m} u^0.
\end{align*}
\]  

(11)

For charged particles, the equations of motion are shown to be the same as hyperbolic motion with acceleration \( a = \frac{E_0 q}{m} \).

Rindler coordinate might be the most commonly used uniformly accelerated frame. The metric is given by

\[ ds^2 = -X^2 dT^2 + dX^2 + dY^2 + dZ^2. \]  

(12)

The coordinate transformation between an inertial frame and Rindler coordinate is of the form,

\[
\begin{align*}
t &= X \sinh(aT), \\
x &= X \cosh(aT), \\
y &= Y, \\
z &= Z.
\end{align*}
\]  

(13)

From Eq. (8), the coordinate transformation suggests \( T \sim \tau \). Namely, coordinate time of a uniformly accelerated frame is in a similar status of proper time of co-moving observers. From this point of view, there are other coordinates that may be regarded as uniformly accelerated frame. The general transformation between Rindler coordinate and an inertial
frame is given by

\[
\begin{align*}
    t &= f(X) \sinh(aT), \\
    x &= f(X) \cosh(aT), \\
    y &= Y, \\
    z &= Z.
\end{align*}
\]

where \( f(X) \) could be understood as different rulers of space. If \( f(X) = \frac{1}{a} + X \), it’s the so-called Müller coordinate. And if \( f(X) = \frac{1}{a} e^{aX} \), it’s the so-called Lass coordinate \([2]\). In general, the metrics are of the form,

\[
ds^2 = -a^2 f^2 dT^2 + (f')^2 dX^2 + dY^2 + dZ^2.
\]

However, it should be noted that the norms of 4-accelerations are not the same for different co-moving observers. It depends on the coordinate position \( X \) of the location of observers. For example that \( f = \frac{1}{a} + X \), the norm of the 4-acceleration is given by

\[
\sqrt{g_{\mu\nu} a^\mu a^\nu} = \frac{1}{f(X)} = \frac{a}{1 + aX}.
\]

If we wish to construct a uniformly accelerated frame based on picture of uniform electric field, the norms of 4-acceleration should be a constant for all observers located at different positions. It led Huang and Guo \([4]\) to construct a new kind of uniformly accelerated frame, in the new frame that the norms of accelerations of co-moving observers are the same constant \( a \). Another understanding of Eq. \((16)\) was given by MTW \([13]\), who derived the Müller coordinate with the hypothesis of locality. At the location where \( X \ll a^{-1} \), the norms of 4-accelerations of different co-moving observers are nearly the same constants. It indicates that Rindler coordinate is in fact a local frame.

The observable feature of Rindler coordinate is the redshift between co-moving observers \([17]\), which can be given by

\[
z_{\pm} = \sqrt{\frac{g_{TT}(X)}{g_{TT}(X')}} = \frac{1 + aX}{1 + aX'} - 1 \approx a(X - X'),
\]

where \( X \) and \( X' \) are the fixed positions of detector and source. The redshift is time-independent. It seems that the redshift in Rindler coordinate is distinguished from that calculated in non-relativistic approximation, Eq. \((3)\).

Huang \([12]\) suggested that the difference is originated from Eq. \((16)\) that norms of 4-accelerations are not the same constant for all co-moving observers. For the new kind of
uniformly accelerated reference frame with the same accelerations of co-moving observers [4], the redshift was shown to be time-dependent.

III. UNIFORMLY ACCELERATED REFERENCE FRAME

To construct an alternative uniformly accelerated reference, we base on 3+1 formalism in adapted coordinate, which is also different from the uniformly accelerated reference frame suggested by Huang [4].

For simplicity, the acceleration of the frame is set along direction of $x$. Coordinate transformation between the uniformly accelerated frame and an inertial frame is expected in the form,

$$\begin{align*}
T &= T(t, x), \\
X &= X(t, x), \\
Y &= y, \\
Z &= z,
\end{align*}$$

(18)

where $T, X, Y$ and $Z$ are coordinates of uniformly accelerated frame and $t, x, y$ and $z$ are coordinates of an inertial frame. The transformation indicates that differential form $dX^\mu$ are integrable and $d^2X^\mu = 0$. For the accelerated frame, we expect that a kind of principle of relativity should be satisfied.

**Axiom 1:** The co-moving observers in uniformly accelerated frame undergo uniformly accelerated motion with respect to inertial frame.

**Axiom 2:** The co-moving observers in inertial frame undergo uniformly accelerated motion with respect to uniformly accelerated frame.

The uniformly accelerated motion in the axioms is formulated as Eq. (6). We would use the axiom 1 for constructing the uniformly accelerated reference frame. It means that the uniformly accelerated observers defined in an inertial frame should move attached to the accelerated reference frame. In the following, we would verify the axiom 2 by the fact that geodesics in uniformly accelerated frame can be formulated as uniformly accelerated motion.
A. Construction of uniformly accelerated reference frame

In 3+1 formalism, 4-velocities $u$ of accelerated observers are normal vectors of space-like hypersurface $\Sigma_T$, which is formulated as

$$u_\mu dx^\mu = -N dT,$$  \hspace{1cm} (19)

where $N$ is so-called lapse function and $N > 0$. The uniformly accelerated frame is adapted to $u$ which is along direction of $x$-axis, namely $u^2 = u^3 = 0$. $T$ is coordinate time of the uniformly accelerated frame. The transformation for $dT$ can be written as

$$dT = \frac{u^0}{N} dt - \frac{u^1}{N} dx.$$  \hspace{1cm} (20)

It suggests that $\frac{\partial_0 T}{N} = \frac{u^0}{N}$ and $\frac{\partial_1 T}{N} = \frac{u^1}{N}$. $N$ as integrating factor ensures that differential form $dT$ is integrable. Because of $d^2 = 0$, Eq. (20) leads to

$$\partial_1 \left( \frac{u^0}{N} \right) + \partial_0 \left( \frac{u^1}{N} \right) = 0.$$  \hspace{1cm} (21)

The transformation for $X$ at present is arbitrary, which can be written as

$$dX = \partial_0 X dt + \partial_1 X dx.$$  \hspace{1cm} (22)

For coordinates $Y, Z$, they are given by $dY = dy$ and $dZ = dz$. From Eqs. (20) and (22), we get inverse transformations for $dt$ and $dx$,

$$\begin{cases}
    dt = \frac{N}{u^0 \partial_0 X + u^1 \partial_1 X} \left( \partial_1 X dt + \frac{u^1}{N} dX \right), \\
    dx = \frac{N}{u^0 \partial_0 X + u^1 \partial_1 X} \left( -\partial_0 X dt + \frac{u^0}{N} dX \right). 
\end{cases}$$  \hspace{1cm} (23)

With the transformations, one can obtain metric of uniformly accelerated reference frame,

$$\begin{aligned}
ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\
&= \frac{N^2}{(u^0 \partial_0 X + u^1 \partial_1 X)^2} \left( -(\partial_1 X)^2 + (\partial_0 X)^2 \right) dT^2 + \frac{1}{N^2} dX^2 - \frac{2}{N} (u^0 \partial_0 X + u^1 \partial_1 X) dT dX \\
&\quad + dY^2 + dZ^2.
\end{aligned}$$  \hspace{1cm} (24)

From the axiom 1, an accelerated observer $u$ should be a co-moving observer of the uniformly accelerated frame, namely,

$$u^0 \partial_0 X + u^1 \partial_1 X = \frac{dX}{d\tau} = 0.$$  \hspace{1cm} (25)
And if set \( \gamma = \frac{1}{u^0 \partial_0 X + u^i \partial_i X} \), by making use of Eq. (25), we can rewrite \( \partial_0 X \) and \( \partial_1 X \) as

\[
\begin{cases}
\partial_0 X = -\frac{u^i}{\gamma}, \\
\partial_1 X = \frac{u^0}{\gamma}.
\end{cases}
\]

Using Eq. (23), (24) and (26), we know that the metric is reduced to

\[
ds^2 = -N^2 dT^2 + \gamma^2 dX^2 + dY^2 + dZ^2.
\]

In the adapted coordinate, the condition axiom 1 guarantees that the metric is always diagonal. It’s different from uniformly accelerated frame suggested by Huang [4].

One may wonder how much it’s related to 3+1 formalism. In general, the metric of 3+1 formalism can be written as

\[
ds^2 = -N^2 dT^2 + \gamma_{ij} (\beta^i dT + dX^i)(\beta^j dT + dX^j),
\]

where \( \gamma_{ij} \) is reduced metric and \( \beta^i \) is so-called shift function. Due to axiom 1, the \( \beta^i \) is shown to be vanished. As we know \( \beta^X = Nu^X = N \frac{dx}{d\tau} = 0 \) and \( u^Y = u^Z = 0 \), it leads to \( \beta^i = 0 \). We can rewrite the metric Eq. (28) as

\[
ds^2 = -N^2 dT^2 + \gamma_{ij} dX^i dX^j.
\]

The metric is diagonal. Comparing it with Eq. (27), one can find that the reduced metric \( \gamma_{ij} \) in the accelerated frame is of the form,

\[
\gamma_{ij} = \begin{pmatrix}
\gamma^2 & 0 \\
0 & 1
\end{pmatrix}.
\]

As \( \gamma \) also functions as integrating factor, we prefer \( \gamma \) to \( \gamma_{ij} \) in our derivation.

In the metric Eq. (27), there are two unknown fields, \( g_{00} \) and \( g_{11} \), which depend on choice of the 4-velocity \( u \). We consider reference frame moving with uniform acceleration. Namely, the \( u \) describes co-moving observers that undergo a uniformly accelerated motion. From the second equation of Eqs. (6), the accelerated motion can be formulated as

\[
\frac{du^1}{d\tau} = \frac{1}{N} \partial_T u^1 = au^0.
\]

By making use of Eq. (26), we rewrite Eq. (21) in the coordinate of uniformly accelerated frame as

\[
\partial_1 \left( \frac{u^0}{N} \right) + \partial_0 \left( \frac{u^1}{N} \right) = \frac{1}{\gamma} \partial_X \frac{1}{N} + \frac{a}{N} = 0.
\]
As \( d^2X = 0 \), it leads to an equation as follows,

\[
\partial_1 \left( \frac{u^1}{\gamma} \right) + \partial_0 \left( \frac{u^0}{\gamma} \right) = \frac{1}{N} \partial_T \left( \frac{1}{\gamma} \right) + \frac{1}{u_0 \gamma^2} \partial_X u^1 = 0. \tag{33}
\]

We rearrange the Eqs. \((31), (32)\) and \((33)\) as

\[
\begin{align*}
\partial_X N &= aN \gamma, \\
\partial_T G &= aN, \\
\partial_X G &= \frac{1}{N} \partial_T \gamma,
\end{align*}
\tag{34}
\]

where we set \( \partial G \equiv \frac{1}{u_0} \partial u^1 \). As \( u^\mu u_\mu = -1 \), it leads to \( G = \text{arcsinh}(u^1) \). At the same time, we notice solution of hyperbolic motion, the second equation of Eqs. \((7)\), which suggests that

\[ G = a \tau. \tag{35} \]

Then, the Eqs. \((34)\) can be rewritten in a natural manner,

\[
\begin{align*}
\partial_X N &= aN \gamma, \\
\partial_T \tau &= N, \\
\partial_X \tau &= \frac{1}{aN} \partial_T \gamma.
\end{align*}
\tag{36}
\]

Solution of Eqs. \((36)\) provide explicit metric of uniformly accelerated reference frame. Expression of coordinate transformation depends on \( N, \gamma \) and proper time \( \tau \). From Eqs. \((20), (26)\) and \((35)\), the coordinate transformation, which takes the form of Eq. \((18)\), can be derived from

\[
\begin{align*}
\mathrm{d}T &= \cosh(a\tau) \mathrm{d}t - \frac{\sinh(a\tau)}{N} \mathrm{d}x, \\
\mathrm{d}X &= -\frac{\sinh(a\tau)}{\gamma} \mathrm{d}t + \frac{\cosh(a\tau)}{\gamma} \mathrm{d}x, \\
\mathrm{d}Y &= \mathrm{d}y, \\
\mathrm{d}Z &= \mathrm{d}z.
\end{align*}
\tag{37}
\]

Besides diagonal, there are other features of the metric from Eqs. \((36)\). Firstly, the metric depends on coordinate time \( T \). If one inserts \( \gamma = \gamma(X) \), the constraint equations would be contradictory. It indicates that the Rindler metric can not be included in our uniformly accelerated frame. Secondly, \( N = 1 \) is not permitted. It means that coordinate time \( T \) of uniformly accelerated frame is not proper time of co-moving observers. In Schwarzschild space-time, one might not require coordinate time of a co-moving observer is proper time, because there is gravity. In uniformly accelerated frame, so does it, because there is fiction force.
From Eqs. (36), we may prove that the metric (Eq. (27)) is a solution of vacuum Einstein equations. We start to check it by calculations of the connection,

\[
\begin{align*}
\Gamma^T_{TT} &= \frac{\partial_r N}{N}, \\
\Gamma^T_{TX} &= \frac{\partial_x N}{N} = a\gamma, \\
\Gamma^X_{TT} &= a\frac{\gamma}{N^2}, \\
\Gamma^X_{XX} &= \frac{N}{\gamma^2} aN^2, \\
\Gamma^X_{TX} &= \frac{\partial_x N}{\gamma}, \\
\Gamma^X_{XX} &= \frac{\partial_x \gamma}{\gamma}, \\
\text{others} &= 0.
\end{align*}
\]

(38)

Non-trivial components of Ricci tensor are given by

\[
\begin{align*}
R_{TT} &= \frac{1}{\gamma^2} \left( N\gamma^2 \frac{\partial_x N}{\gamma} - N\gamma^2 \frac{\partial_T (\frac{\partial_T N}{\gamma})}{N} \right), \\
R_{XX} &= \frac{1}{N^2} \left( -\gamma N^2 \frac{\partial_x N}{\gamma} + \gamma N^2 \frac{\partial_T (\frac{\partial_T N}{\gamma})}{N} \right).
\end{align*}
\]

(39)

From Eqs. (36), one has

\[
\begin{align*}
\partial_T (\frac{\partial_T N}{\gamma}) &= a^2 N\gamma, \\
\partial_X (\frac{\partial_X N}{\gamma}) &= a^2 N\gamma.
\end{align*}
\]

(40)

It leads Ricci tensor to be zero,

\[
R_{\mu\nu} = 0.
\]

(41)

Namely, the metric of uniformly accelerated frame is a solution of vacuum Einstein equations. The checking process seems trivial, as Einstein equation always allows coordinate transformation as gauge symmetry.

B. Inertial motion in uniformly accelerated frame

In the subsection A, we have constructed the metric of uniformly accelerated frame with axiom 1. The axiom presents an equivalent description for uniformly accelerated motion in different reference frames. On the other side, inertial motion also requires the equivalent description. It means that the inertial motion should be formulated as uniformly accelerated motion in a uniformly accelerated frame.

We consider inertial motion in the uniformly accelerated frame at \(T-X\) plane,

\[
\begin{pmatrix}
\frac{dT}{d\tau_0} \\
\frac{dX}{d\tau_0}
\end{pmatrix} = \begin{pmatrix}
\frac{\cosh(a\tau)}{N} & -\frac{\sinh(a\tau)}{N} \\
-\frac{\sinh(a\tau)}{\gamma} & \frac{\cosh(a\tau)}{\gamma}
\end{pmatrix} \begin{pmatrix}
\frac{dt}{d\tau_0} \\
\frac{dx}{d\tau_0}
\end{pmatrix},
\]

(42)
where \((\frac{dt}{d\tau_0}, \frac{dx}{d\tau_0})\) is a constant velocity vector, \(\tau_0\) is proper time of co-moving observers in inertial frame, which is distinguished with \(\tau\) proper time of uniformly accelerated observers. From Eqs. (36) and (42), one can obtain

\[
\begin{cases}
  \frac{d}{d\tau} \left( N \frac{dT}{d\tau_0} \right) = -a \gamma \frac{dX}{d\tau_0}, \\
  \frac{d}{d\tau} \left( \gamma \frac{dX}{d\tau_0} \right) = -aN \frac{dT}{d\tau_0}.
\end{cases}
\]  

(43)

If we set \(v^T = N \frac{dT}{d\tau_0}\) and \(v^X = \gamma \frac{dX}{d\tau_0}\), the Eq. (43) reduce to

\[
\begin{cases}
  \frac{dv^T}{d\tau} = -av^X, \\
  \frac{dv^X}{d\tau} = -av^T.
\end{cases}
\]  

(44)

From Eq. (44), the inertial motion in the uniformly accelerated frame can be formulated as hyperbolic motion with a reverse acceleration. Using Eqs. (42), one can verify that \(\frac{dT}{d\tau_0}\) and \(\frac{dX}{d\tau_0}\) satisfy geodesic equations of the accelerated frame. All these indicate that the axiom 2 is verified. In addition, redefinitions of covariant velocity \((v^T, v^X)\) are insightful. In curvilinear coordinate, it’s exactly standard definition of a vector, where \(N, \gamma\) are so-called Lam coefficients. In tetrad formalism, one may find \(v^a = e^a_{\mu} u^\mu\).

C. Features of the uniformly accelerated reference frame

1. Frenet-Serret frame

The Frenet-Serret frame describes evolution of a frame along observer’s worldline. It can be generally written as

\[
\frac{D}{d\tau} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \kappa & \tau & -\tau & b \\ \kappa & \tau & -\tau & b \\ -\tau & b & -b & \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix},
\]  

(45)

where \(\frac{D}{d\tau}\) is covariant derivative, \(e_\mu\) represent vector bases of the frame, \(\kappa, \tau\) and \(b\) are so-called curvature and torsions of observers’ worldline in Lorentz manifold, respectively.

Our uniformly accelerated frame can be expressed in term of the Frenet-Serret frame in tetrad formalism. Firstly, we rewrite coordinate bases \(\partial_\mu\) as tetrads \(e_a\), which is formulated
as $e_a = e_a^\mu \partial_\mu$. In the uniformly accelerated frame, the tetrads can be given by

$$
\begin{align*}
\begin{cases}
e_0 &= \frac{1}{N} \partial_T, \\
e_1 &= \frac{1}{2} \partial_X, \\
e_2 &= \partial_Y, \\
e_3 &= \partial_Z.
\end{cases}
\end{align*}
\tag{46}
$$

One can find that $\kappa = a$, $\tau = b = 0$ for our uniformly accelerated frame, namely,

$$
\frac{D}{d\tau} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & a \\ a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}.
\tag{47}
$$

The curvature of the moving frame is exactly the constant acceleration $a$ in our uniformly accelerated frame, while the Rindler coordinate can not be described in term of Frenet-Serret frame with tetrads.

### 2. Kinematical quantities

The congruence of co-moving observers $u$ indicates a deformation of space-time. The difference between worldlines of co-moving observers can be described in terms of deviation vector $\chi^\mu$, which satisfies

$$
[u, \chi]^\mu = u^\nu \nabla_\nu \chi^\mu - \chi^\nu \nabla_\nu \chi^\mu = 0.
\tag{48}
$$

For spatial distance of $\chi^\mu$, namely, \(\tilde{\chi}^\mu = \gamma^\mu_\nu \chi^\nu\), the evolution of $\tilde{\chi}^\mu$ indicates spatial deformation of the reference frame,

$$
\tilde{D} \tilde{\chi}^\mu = \chi^\mu \left( \frac{1}{2} \theta \gamma^\mu_\nu + \sigma^\mu_\nu + w^\mu_\nu \right),
\tag{49}
$$

where $\tilde{D} = \gamma^\mu \nabla_\mu$ is spatial covariant derivation along a co-moving observer derived from 3+1 foliation. The $\theta, \sigma^\mu_\nu$ and $w^\mu_\nu$ are so-called kinematical quantities and named after expansion scalar, shear tensor and rotation tensor, respectively.

In the accelerated frame, the co-moving observers $u$ are given by

$$
u_\mu dX^\mu = (-N, 0, 0, 0).
\tag{50}$$
The covariant derivative of observers $u$ can be decomposed as acceleration, expansion scalar, shear and rotation tensor,

$$\nabla_\nu u_\mu = -u_\nu a_\mu + \frac{1}{3}\theta\gamma_{\mu\nu} + \sigma_{\mu\nu} + w_{\mu\nu}, \quad (51)$$

where

$$a^\mu = \nabla_u u^\mu, \quad (52)$$

$$\theta = \nabla_\mu u^\mu, \quad (53)$$

$$\sigma_{\mu\nu} = \frac{1}{2}(\nabla_\mu u_\nu + \nabla_\nu u_\mu + u_\mu \nabla_\nu u_\nu + u_\nu \nabla_\mu u_\mu) - \frac{1}{3}\theta\gamma_{\mu\nu}, \quad (54)$$

$$w_{\mu\nu} = \frac{1}{2}\gamma_{\mu}^\sigma\gamma_{\nu}^\rho(\nabla_{\sigma} u_\rho - \nabla_{\rho} u_\sigma). \quad (55)$$

From Eqs. (29) and (52), the accelerations of co-moving observers are given by $a^\mu = \delta^\mu_1$.

The norms of the accelerations are

$$|a| \equiv \sqrt{g_{\mu\nu}a^\mu a^\nu} = a. \quad (56)$$

It shows that accelerations of any co-moving observers in our uniformly accelerated frame are the same constant acceleration $a$, which is distinguished with Rindler coordinate (Eq. (16)).

We present the expansion scaler, shear tensor, and rotation tensor as the following,

$$\begin{align*}
    w_{\mu\nu} &= 0, \\
    \theta &= \frac{\partial \gamma}{\partial N}, \\
    \sigma_{\mu\nu} &= \frac{1}{3}\theta \begin{pmatrix}
        0 \\
        2 \\
        -1 \\
        1
    \end{pmatrix},
\end{align*} \quad (57)$$

The vanished rotation tensor means that co-moving observers of the frame are Eulerian observers. There is a simultaneous hypersurface $\Sigma_T$ that is orthogonal to all the co-moving observers.
From Eqs. (48), (49), (57), (57) and (57), the evolution of the spatial deviation vector is given by

\[
\begin{bmatrix}
0 \\
\theta \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\tilde{X} \\
\tilde{\chi}
\end{bmatrix}.
\tag{58}
\]

It shows that there is spatial deformation between co-moving worldlines in the direction of axis-\(X\), while in spatial direction referred to \(Y, Z\), there is not a deformation. In the uniformly accelerated frame, it can be understood as non-inertial effect. The fiction force can affect deformation of space.

Evolutions of these kinematical quantities are described by the Raychaudhuri equations. In our uniformly accelerated frame, the equations can be deduced from Eqs. (36). It couldn’t provide any constraints for constructing uniformly accelerated frame.

IV. EXPLICIT SOLUTIONS

Now, we try to obtain solutions of Eqs. (36). Using these solutions, we calculate redshift drift between co-moving objects and possible Unruh effect in the accelerated frame. As there is nothing special in the direction of axis-\(Y\) and \(Z\), we consider two-dimensional metrics for simplicity.

A. Hyperbolic metric and redshift drift

The components of metric tensor turn to be hyperbolic triangle function, when the uniformly accelerated observer \(u\) is function of coordinate time \(t\), namely, \(u = u(t)\). Associating it with Eqs. (36), we get the metric as

\[
ds^2 = -\frac{dT^2}{\sinh^2(-a(T + X))} + \frac{dX^2}{\tanh^2(-a(T + X))}.
\tag{59}\]

As \(N > 0\), it leads to \(-a(T + X) > 0\). Transformation from inertial frame to the accelerated frame is of the form,

\[
\begin{cases}
t = \frac{1}{a\sinh(-a(X+T))}, \\
x = X + \frac{1}{a\tanh(-a(X+T))}.
\end{cases}
\tag{60}
\]
From Eqs. (36), we obtain proper time $\tau$ of co-moving observers in the accelerated frame. We can solve it as function of space-time coordinate,

$$
\tau = \frac{1}{a} \text{arcsinh} \left( \frac{1}{\sinh(-a(T + X))} \right) = \frac{1}{a} \text{arcsinh}(at).
$$

(61)

From Eqs. (59) and (61), it leads to $\sinh(a\tau) = at = N > 0$. If $a > 0$, accessible region of space-time is what with $\tau, t \geq 0$. Namely, the reference frame undergoes uniformly accelerated motion from $t = 0$. The coordinate line of the uniformly accelerated frame in $t$-$x$ plane is presented in Figure 2.

The metric Eq. (59) can describe the reference frame of the carrier in Figure 1, so that, we can reconsider redshift drift in uniformly accelerated reference frame. The reference frame moves with constant acceleration $a$ to the right, and the light source B on the left of the detector co-moves with the carrier. The source emits light that is along a null curve. By making use of the metric, Eq. (59), we give the trajectory of the light,

$$
\frac{1}{\sinh(-a(T + X))} \pm \frac{1}{\tanh(-a(T + X))} \frac{dX}{dT} = 0.
$$

(62)

In this case, only forward propagating light reaches the detector. Namely, the "$-"$ of Eq. (62) is required to be chosen. Then, the trajectory is obtained as

$$
\tanh \left( -\frac{a}{2} (T + X) \right) = -a(X - X') + \tanh \left( -\frac{a}{2} (T' + X') \right),
$$

(63)
where the detector and source are fixed at spatial coordinate \(X\) and \(X'\), where \(X > X'\). From Eq. (63), it takes different time intervals when two light signals are emitted and received. The ratio can be given by

\[
\frac{\Delta T}{\Delta T'} = \frac{\cosh^2 \left( \frac{-a}{2} (T + X) \right)}{\cosh^2 \left( \frac{-a}{2} (T' + X') \right)},
\]

where \(\Delta T'\) and \(\Delta T\) are the emitted and received time intervals, respectively. Using Eq. (59), (61), (63) and (64), we can derive the redshift [18] observed by the detector in the uniformly accelerated reference frame,

\[
z_- \equiv \frac{\Delta \tau}{\Delta \tau'} - 1 = \frac{\sqrt{g_{TT}(T,X)}}{\sqrt{g_{TT}(T',X')}} \frac{\Delta T}{\Delta T'} - 1
\]

\[
= \frac{\tanh \left( \frac{-a}{2} (T' + X') \right)}{\tanh \left( \frac{-a}{2} (T + X) \right)} - 1
\]

\[
= a(X - X') \cosh(a\tau).
\]

(65)

It shows that the redshift drifts with proper time of the detector. There is additional factor \(\cosh(a\tau)\) compared to the result in Rindler coordinate, Eq. (17). As time goes by, the redshift would get higher and finally tends to infinity.

Likewise, we can consider the light source A on the right of the detector in Figure 1. The result shows that there is blueshift, namely \(z < 0\), observed by the detector,

\[
z_+ = -\frac{a(X - X')}{\cosh(a\tau)}.
\]

(66)

In this case \(X < X'\), the blueshift would get lower with time until it’s vanished.

These results, Eqs (65) and (66), are consistent with Huang’s [12] and that in non-relativistic case, qualitatively. There is redshift drift in uniformly accelerated reference frame. Further, we can compare all these results in detail, which is presented in Table I. We recover the speed of light \(c\) in formulations and set \(|X' - X| \equiv L\) for consistence.

In Figure 3, the redshift and blueshift as function of proper time are presented. In the case of non-relativistic case, time \(t\) is absolute. We, therefore, didn’t distinguish it with proper time. The results of Mller coordinate and non-relativistic approximation case are contrasty, which are independent of and sensitive to time, respectively. The redshift calculated with the hyperbolic metric is closed to that calculated in Rindler coordinate. In non-relativity case, it turns to be meaningless when \(at \gtrsim c\). In relativistic case, there is not the limitation, as \(a\tau\) is not a 3-velocity.
TABLE I: Redshift $z_-$ and blueshift $z_+$ between co-moving objects of uniformly accelerated reference frame calculated with different approaches.

|                          | $z_-$                                      | $z_+$                                      |
|--------------------------|--------------------------------------------|--------------------------------------------|
| Miller coordinate [17]   | $\frac{a \ell}{c^2}$                       | $-\frac{a \ell}{c^2}$                     |
| Non-relativity           | $\frac{a \ell}{c^2} (1 - \frac{a \tau}{c})^{-2}$ | $-\frac{a \ell}{c^2} (1 + \frac{a \tau}{c})^{-2}$ |
| Huang’s [12]            | $\frac{a \ell}{c^2} \frac{a \tau}{c}$   | $-\frac{a \ell}{c^2} e^{-\frac{a \tau}{c}}$ |
| Hyperbolic metric       | $\frac{a \ell}{c^2} \cosh \left( \frac{a \tau}{c} \right)$ | $-\frac{a \ell}{c^2} \frac{1}{\cosh \left( \frac{a \tau}{c} \right)}$ |

FIG. 3: Redshift (left panel) and blueshift (right panel) between co-moving objects in uniformly accelerated reference frame as function of proper time.

B. Conformally flat metric and Unruh effect

By making use of Eqs. [36] and constraint $N = \gamma$, we get a conformally flat metric,

$$ds^2 = \frac{1}{a^2(T + X)^2}(-dT^2 + dX^2).$$

(67)
As \( N > 0 \), it also leads to \(-a(T + X) > 0\). Transformation between inertial frame and the accelerated frame is given by
\[
\begin{cases}
    t = \frac{1}{2} \left( -\frac{1}{a^{2}(T + X)} + T - X \right), \\
    x = \frac{1}{2} \left( -\frac{1}{a^{2}(T + X)} - T + X \right).
\end{cases}
\]
(68)

The proper time \( \tau \) of co-moving observers is a function of space-time coordinate,
\[
\tau = \frac{1}{a} \ln \left( -\frac{1}{a(T + X)} \right) = \frac{1}{a} \ln(a(t + x)).
\]
(69)

The coordinate line of the uniformly accelerated frame in \( t-x \) plane is presented in Figure 4.

![Figure 4: The coordinate line of the uniformly accelerated frame in t-x diagram. In the case of \( a > 0 \), the accessible region for the accelerated frame is what with \( t + x > 0 \).](image)

Unruh effect state that in uniformly accelerated frame (Rindler coordinate), the co-moving observers would perceive a thermal distribution of Minkowski vacuum \([19, 21]\). Temperature of the distribution is proportional to the constant acceleration \( a \) in Rindler coordinate. In this subsection, we would use the metric Eq. (67) to calculate the possible Unruh effect. For simplicity, we consider massless boson.
The Klein-Golden equation for massless boson is given by
\[ \nabla_\mu \nabla^\mu \phi = 0, \] (70)
where \( \nabla_\mu \) is covariant derivative. With our metric, the equation of motion can be written as
\[ g^{\mu\nu} \nabla_\nu \nabla_\mu \phi = -\frac{1}{a^2(T + X)^2} (\partial^2_T \phi - \partial^2_X \phi) = 0, \] (71)
namely,
\[ \partial^2_T \phi - \partial^2_X \phi = 0. \] (72)
It’s the same as the Klein-Golden equation in flat space-time. The solution of the equation can be given by
\[ \phi = \frac{1}{\sqrt{2\pi}} \int d^2k \{ \delta(k_0^2 - k^2) \tilde{\phi}(k_0, k) e^{-i(k_0 T - k X)} \} . \] (73)
It can be expanded as
\[ \phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk}{2|k|} \{ \tilde{\phi}(k) e^{-i|k|(T - k X)} + e^{-i(-|k|(T - k X))} \} = \phi_+ + \phi_- , \] (74)
where
\[ \phi_+ = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{dk}{\sqrt{2k}} \{ \tilde{\phi}_+ e^{-ik(T + X)} + \tilde{\phi}_- e^{ik(T + X)} \}. \] (75)
We focus on left-moving sectors \( \phi_+ \) of the field. The different sectors \( \phi_- \) and \( \phi_+ \) would not interact with each other [22]. We quantize \( \phi_+ \) in the form of
\[ \hat{\phi}_+ = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{dk}{\sqrt{2k}} \{ \hat{b}_k e^{-ik(T + X)} + \hat{b}_k^\dagger e^{ik(T + X)} \} , \] (76)
where \( \hat{b}_k \) is ladder operator satisfying canonical communication relations,
\[ [\hat{b}_k, \hat{b}_{k'}^\dagger] = \delta(k - k') , \] (77)
\[ \text{others} = 0 , \]
and
\[ \hat{b}_k |0_A\rangle = 0, \] (78)
where $|0_A\rangle$ is vacuum state in uniformly accelerated reference frame. The mode function is read from the field operator Eq. (76),

$$g_k(T, X) = \frac{1}{\sqrt{4\pi k}} e^{-ik(T+X)}. \quad (79)$$

On the other side, we know the field operator of left-moving sector in flat space-time

$$\hat{\phi}_+ = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dp}{\sqrt{2p}} \{\hat{a}_p e^{-ip(t+x)} + \hat{a}_p^\dagger e^{ip(t+x)}\}, \quad (80)$$

where $\hat{a}_p$ is ladder operator. The canonical communication relations are given by

$$[\hat{a}_p, \hat{a}_p^\dagger] = \delta(p - p') \ ,$$
$$\text{others} = 0. \quad (81)$$

And, one has

$$\hat{a}_p |0_M\rangle = 0, \quad (82)$$

where $|0_M\rangle$ is the Minkowski vacuum state. Mode function is of the form,

$$f_p(t, x) = \frac{1}{\sqrt{4\pi p}} e^{-ip(t+x)}. \quad (83)$$

The ladder operators of accelerated and inertial frame are related with so-called Bogolubov transformation, which is given by

$$\begin{cases}
\hat{a}_p = \int dk \{\alpha_{kp} \hat{b}_k + \beta_{kp}^* \hat{b}_k^\dagger\}, \\
\hat{b}_k = \int dp \{\alpha_{kp}^* \hat{a}_p - \beta_{kp} \hat{a}_p^\dagger\},
\end{cases} \quad (84)$$

where $\alpha_{kp}$ and $\beta_{kp}$ are so-called Bogolubov coefficients satisfying relations,

$$\begin{cases}
\int dk \{\alpha_{kp}\alpha_{kp'}^* - \beta_{kp}^* \beta_{kp'}\} = \delta(p - p') \ , \\
\int dp \{\alpha_{kp}^* \alpha_{kp'} - \beta_{kp} \beta_{kp'}^*\} = \delta(k - k') \ , \\
\int dk \{\alpha_{kp} \beta_{kp'}^* - \beta_{kp}^* \alpha_{kp'}\} = 0 \ , \\
\int dp \{\alpha_{kp}^* \beta_{kp'} - \beta_{kp} \alpha_{kp'}^*\} = 0 .
\end{cases} \quad (85)$$

For mode functions, orthogonal relations can be derived from the so-called Klein-Gordon Inner product,

$$(\phi, \chi) \equiv i \int_\Sigma d\Sigma^\mu \{\phi^* \nabla_\mu \chi - \chi \nabla_\mu \phi^*\} . \quad (86)$$
One can find that
\[
\begin{align*}
(f_p, f_p') &= \delta(p - p') , \\
(f_p, f_p^*) &= 0 , \\
(g_k, g_{k'}) &= \delta(k - k') , \\
(g_k, g_{k'}^*) &= 0 .
\end{align*}
\tag{87}
\]

In different coordinates, the field operator \( \hat{\phi}_+ \) remains the same under Bogolubov transformation. Thus, one can derive Bogolubov transformation for mode function,
\[
g_k = \int dp \{ \alpha_{kp} f_p + \beta_{kp} f_p^* \} . \tag{88}
\]

From Eq. (88), the Klein-Gordon inner product can be used to calculate the Bogolubov coefficients,
\[
\begin{align*}
\alpha_{kp} &= (f_p, g_k), \\
\beta_{kp} &= -(f_p^*, g_k).
\end{align*}
\tag{89}
\]

What the accelerated observers perceive in Minkowski vacuum is formulated as expectation value of occupation number operators \( N_k \) of accelerated observers for Minkowski vacuum state,
\[
\langle 0_M | N_k | 0_M \rangle = \langle 0_M | \hat{b}_k^\dagger \hat{b}_k | 0_M \rangle = \int dp \beta_{pk^*} \beta_{pk} . \tag{90}
\]

It shows that the expectation value only refers to the Bogolubov coefficients \( \beta_{pk} \).

As did in Refs. [22, 23], we would use light-cone coordinate to calculate the Bogolubov coefficients from Eqs. (89) for given null hypersurface. The null coordinates are usually related with radiation closely [24, 25]. Light-cone coordinate of inertial frame and the uniformly accelerated frame are given by \((u, v) = (t - x, t + x)\) and \((U, V) = (T - X, T + X)\), respectively. From Eq. (68) and the light-cone coordinates, the transformation between uniformly accelerated frame and inertial frame is obtained,
\[
\begin{align*}
u &= \frac{1}{a^2 V} .
\end{align*}
\tag{91}
\]

In the light-cone coordinate, the metric of the accelerated frame can be rewritten as
\[
ds^2 = -du dv = \frac{1}{a^2 V^2} dU dV . \tag{92}
\]
We choose the null hypersurface as
\[ \Phi(U, V) \equiv U = \text{constant}. \] (93)

One can find that it’s the only non-trivial null hypersurface for calculating the possible Unruh effect, as others would lead to \( \beta_{pk} \equiv 0 \). The normal vectors of the null hypersurface are null vectors, \( \xi_\nu = -\partial_\nu \Phi = -\delta^0_\nu \). We use \( \lambda \) to parametrize integral curve \((U(\lambda), V(\lambda))\) of \( \xi^\mu(\lambda) \). As the null vector \( \xi^\mu \) is also tangent to the null hypersurface, in the two-dimensional case, \( \lambda \) also can be used to parametrize the null hypersurface. We can solve the integral curve \((U, V)\) as
\[
\begin{cases}
U = \text{constant}, \\
V = \frac{1}{2a^2 \lambda}.
\end{cases}
\] (94)

Using Eq. (94), we get the volume element of the null hypersurface [26],
\[
d\Sigma = \epsilon_{\mu\nu} \xi^\mu \xi^\nu d\lambda = \frac{1}{2a^2 V^2} dV,
\] (95)
where \( \epsilon_{\mu\nu} \) is the Levi-Civita tensor and \( \zeta^\mu \) is an auxiliary null vector satisfying that \( \xi^\mu \zeta_\mu = -1 \) and \( \zeta^\mu \zeta_\mu = 0 \). From Eq. (95), the directed surface element is obtained,
\[
d\Sigma^\mu = -\xi^\mu d\Sigma = \delta^\mu_1 dV.
\] (96)

We choose that \( a > 0 \). For the metric with \( N > 0 \), it leads to \( V < 0 \).

From Eqs. (79), (83), (89), (89) and (96), we can calculate the Bogolubov coefficients,
\[
\alpha_{kp} = (f_p, g_k)
\]
\[
= i \int_{-\infty}^{0} dV \left\{ \frac{1}{\sqrt{4\pi p}} e^{ip\nu} \partial_V \frac{1}{\sqrt{4\pi k}} e^{-ikV} - \frac{1}{\sqrt{4\pi k}} \partial_V \frac{1}{\sqrt{4\pi p}} e^{ip\nu} \right\}
\]
\[
= \frac{1}{4\pi \sqrt{pk}} \left( k \int_{0}^{\infty} dV e^{i(kV + \frac{p}{2V})} - \frac{p}{a^2} \int_{0}^{\infty} dV \left( \frac{1}{V} \right) e^{i(kV + \frac{p}{2V})} \right)
\]
\[
= \frac{1}{\pi a} \int_{1}^{\infty} d\eta \frac{2}{\eta^2 - 1} \left\{ e^{2\sqrt{\frac{p}{pk}} \eta} \right\},
\] (97)
and
\[
\beta_{kp} = -(f^*_p, g_k)
\]
\[
= -i \int_{-\infty}^{0} dV \left\{ \frac{1}{\sqrt{4\pi p}} e^{-ip\nu} \partial_V \frac{1}{\sqrt{4\pi k}} e^{-ikV} - \frac{1}{\sqrt{4\pi k}} \partial_V \frac{1}{\sqrt{4\pi p}} e^{-ip\nu} \right\}
\]
\[
= -\frac{1}{4\pi \sqrt{pk}} \left( \frac{p}{a^2} \int_{0}^{\infty} dV \left( \frac{1}{V} \right) e^{i(kV - \frac{p}{2V})} + k \int_{0}^{\infty} dV e^{i(kV - \frac{p}{2V})} \right)
\]
\[
= -\frac{1}{2\pi a} \int_{-\infty}^{\infty} d\eta \frac{2}{\eta^2 + 1} \left\{ e^{2\sqrt{\frac{p}{pk}} \eta} \right\}.
\] (98)
With tricks that $\eta \to e^{i\epsilon}\infty$ and $\epsilon > 0$ [22], the Bogolubov coefficients can be obtained explicitly,

$$\alpha_{kp} = \frac{1}{\pi a} K_1 \left( -2i \sqrt{\frac{p_k}{a^2}} \right),$$

(99)

and

$$\beta_{kp} = -\frac{1}{2\pi a} \left( 1 + iK_1 \left( 2 \sqrt{\frac{p_k}{a^2}} \right) + \frac{\pi}{2} \left( L_1 \left( 2 \sqrt{\frac{p_k}{a^2}} \right) - I_1 \left( 2 \sqrt{\frac{p_k}{a^2}} \right) \right) \right)
+ \frac{1}{2\pi} G_{3,1}^{1,3} \left( \frac{a^4}{k^2 p^2}, 2 \left| \alpha \frac{1}{2}, \frac{3}{2}, 0 \right| \right),$$

(100)

where $K_1$ and $I_1$ are modified Bessel function of the second and first kind, respectively, $L_1$ is modified Struve function, and $G_{3,1}^{1,3}$ is generalized Meijer G function. It shows that the Bogolubov coefficients $\alpha_{kp}$ and $\beta_{kp}$ are completely different from that in Rindler coordinate [19, 22, 23]. It suggests that expectation value might not be the form of that calculated in Rindler coordinate. We finally obtain the expectation value in our uniformly accelerated frame as follows,

$$\langle 0_M | \hat{N}_k | 0_M \rangle = \int dp \beta_{kp}^* \beta_{kp}$$

$$= \int_0^{\infty} dp \left\{ -\frac{1}{2\pi a} \int_{-\infty}^{\infty} d\eta' \int_{-\infty}^{\infty} d\eta \left\{ e^{\frac{2i}{4\pi^2} \eta} \right\} \right\} \left( -\frac{1}{2\pi a} \int_{-\infty}^{\infty} d\eta'' \int_{-\infty}^{\infty} d\eta''' \left\{ e^{\frac{2i}{4\pi^2} \eta''} \right\} \right)^*$$

$$= \frac{\chi}{k}$$

(101)

where $\chi = 2 \sqrt{\frac{k_p}{a^2}}$, and

$$\Lambda \equiv \frac{1}{8\pi^2} \int_0^{\infty} d\chi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\eta' \left\{ \frac{\chi \eta'}{\sqrt{\eta^2 + 1}} \right\} e^{iz(\eta - \eta')},$$

$$< \frac{1}{8\pi^2} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\chi d\eta d\eta' \left\{ \frac{\chi \eta'}{\sqrt{\eta|\eta'|}} e^{i\chi(\eta - \eta')} \right\} = \frac{1}{4\pi^2} \ln \chi \bigg|_0^{\infty}.$$

(102)

The constant $\Lambda$ is divergent. We use the tricks that $\eta \to e^{i\epsilon}\infty$ to obtain the last equal sign.

Firstly, it shows that the distribution of Eq.(101) is independent of acceleration in the accelerated frame. Secondly, the expectation value of the number operator for Minkowski vacuum state is a non-thermal distribution of $k$. The uniformly accelerated observers can not perceive a temperature in Minkowski vacuum.
V. CONCLUSIONS AND DISCUSSIONS

In this paper, we constructed a new uniformly accelerated reference frame based on 3+1 formalism in adapted coordinate for uniformly accelerated observers $u$. The norms of 4-acceleration of co-moving observers are all the same in the uniformly accelerated reference frame. The inertial motion can be formulated as uniformly accelerated motion in the accelerated frame. And the space-time would be deformed by non-inertial effect. We presented explicit metrics and coordinate transformations. In contrast with Rindler coordinate, the redshift drift between co-moving observers in our accelerated frame is existed. It’s consistent with earlier results [12]. The possible Unruh effect was calculated and shows a non-thermal distribution of Minkowski vacuum perceived by the uniformly accelerated observers.

From our approaches, the constraint equations (36) for uniformly accelerated frame are under-determined. It results in that metric of the frame is non-unique. Degrees of freedom for different measurements and synchronisation conventions are allowed. Firstly, the Eqs. (36) are invariant under transformation $T \rightarrow f(T)$ and $X \rightarrow g(X)$. It’s caused by integrating factor $N$ and $\gamma$ that are not unique, which indicates that time and distance can be measured with different clocks and rulers. Secondly, there are different simultaneous hypersurfaces of the explicit solutions in section IV, which means different synchronisation convention. It’s rather interesting to explore what kinds of conventions are physically operational.

After Huang [12] firstly suggested that the redshift drift can be observed in uniformly accelerated reference frame, we also provided a similar prediction with different approaches. We expect that the results can be verified in future experiments.

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