Periodicity related to a sieve method of producing primes

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Abstract

In this paper we consider a slightly different sieve method from Eratosthenes’ to get primes. We find the periodicity and mirror symmetry of the pattern.

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1 Notations
Let \( p_1, p_2, \ldots, p_n, \ldots \) be the set of all primes, where \( p_n \) denotes the \( n \)-th prime number. \( \pi(x) \) is the number of primes less than or equal to \( x \). \( \mathbb{N} \) is the set of positive numbers. Let

\[
M_{p_n} = \mathbb{N} - \{2k, 3k, 5k, \ldots, p_nk \mid k \in \mathbb{N}\},
\]

and \( D_{p_n} \) be the set of difference of two consecutive numbers in \( M_{p_n} \), that is,

\[
D_{p_n} = \{d_k \mid d_k = x_{k+1} - x_k, x_k \in M_{p_n}\},
\]

where \( x_i \) is the \( i \)-th number in \( M_{p_n} \).

For example, we list the first few numbers of \( M_3 \) and \( D_3 \).

| \( M_3 \) | 1 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 25 | 29 | 31 | 35 | 37 | \ldots |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( D_3 \) | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | \ldots |

\( \mathcal{P}_3 := \{4, 2\} \) is called the pattern of \( D_3 \) since it occurs periodically. We will prove this fact in the next section.

The number of the elements in the pattern \( \mathcal{P}_3 \) for \( D_3 \) is called the period of \( D_3 \). We write it as \( T_3 = 2 \).

The length of the pattern \( \mathcal{P}_3 \) is defined as the sum of the elements in the pattern. We write it as \( L(\mathcal{P}_3) \).

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2 Periodicity of the pattern

Proposition 2.1. \( M_3 = \{u_n\}_{n=1}^\infty \) has the following property:

\[
\begin{align*}
    u_{2k} &= u_{2k-1} + 4, \\
    u_{2k+1} &= u_{2k} + 2,
\end{align*}
\]

where \( k = 1, 2, \ldots \). In other words, \( \{4, 2\} \) is the pattern of \( D_3 \), so the period is 2.

Proof. \( u_n \) satisfies the equations:

\[
\begin{align*}
    x &\equiv 1(\text{mod } 2), & x &\equiv 1(\text{mod } 2), \\
    x &\equiv 1(\text{mod } 3), & x &\equiv 2(\text{mod } 3).
\end{align*}
\]

The left infers that \( x \equiv 1(\text{mod } 6) \). The right is:

\[
\begin{align*}
    x &= 2k + 1, \\
    x &= 3\ell + 2.
\end{align*}
\]

If \( u_n = 2k + 1 = 3\ell + 2 \), then \( 2k = 3\ell + 1 \), which infers that \( \ell \) must be odd. Let \( \ell = 2h + 1 \), then \( 2k = 3(2h + 1) + 1 = 6h + 4 \). Thus \( 2k + 1 = 6h + 5 = 3\ell + 2 \). Hence, \( u_n \equiv 5(\text{mod } 6) \).  

Theorem 2.2. The period of \( D_p \) is

\[
T_p = (2 - 1)(3 - 1)(5 - 1) \cdots (p - 1).
\]

Proof. We prove this by induction. The case for \( D_3 \) has been proved. For simplicity, we try to explain the procedure by proving the case for \( D_5 \). First, we observe that

\[ M_{p_{n+1}} = M_{p_n} - \{p_{n+1}h \mid h \in M_{p_n}\}. \]

We show this procedure (getting \( M_5 \) from \( M_3 \)) in the following table (see Figure 1).

We need to delete the multiples of 5 in \( M_3 \), i.e.,

\[
\begin{align*}
    5 \cdot 1, & \quad 5 \cdot 5, \\
    5 \cdot 7, & \quad 5 \cdot 11, \\
    5 \cdot 13, & \quad 5 \cdot 17, \\
    5 \cdot 19, & \quad \ldots.
\end{align*}
\]

By Proposition 2.1 \( D_3 \) has pattern \( \{4, 2\} \), hence the positions of the elements needed to delete occur periodically in \( M_3 \). The period is \( 5 \times 6 = 30 \), where 6 is the length of the previous pattern \( \{4, 2\} \).
From another view, we make 5 copies of the pattern \( t_{4, 2} \). Say, 
\( 4, 2, 4, 2, 4, 2, 4, 2, 4, 2 \).

After deleting 5 and 25, we get the string:
\[ (4 + 2), 4, 2, 4, 2, 4, (2 + 4), 2. \]

It is the pattern of \( D_5 \), \( \{6, 4, 2, 4, 2, 4, 6, 2\} \). Hence the period is \( T_5 = 8 = 2 \cdot 5 - 2 = (5 - 1) \cdot T_3 \).

We can also show that the pattern of \( D_7 \) is
\[
10 = (6, 4), 2, 4, 2, 4, 6, 2;
6, 4, 2, 4, 6 = (2, 4), 6, 2;
6, 4, 2, 6 = (4, 2), 4, 6, 8 = (2, 6);
4, 2, 4, 2, 4, 8 = (6, 2);
6, 4, 6 = (2, 4), 2, 4, 6, 2;
6, 6 = (4, 2), 4, 2, 4, 6, 2;
6, 4, 2, 4, 2, 10 = (4, 6), 2.
\]

The period is \( T_7 = 48 = (7 - 1) \cdot (5 - 1) \cdot (3 - 1) \cdot (2 - 1) \).

In fact, \( M_7 \) is obtained by deleting the multiples of 7 in \( M_5 \). We can rearrange the numbers in Figure 1 as in Figure 2.

We need to delete the multiples of 7 in \( M_5 \) (See Figure 3). They are
\[
7 \cdot 1, \ 7 \cdot 7, \ 7 \cdot 11, \ 7 \cdot 13, \ 7 \cdot 17, \ 7 \cdot 19, \ 7 \cdot 23, \ 7 \cdot 29, \ 7 \cdot 31, \ 7 \cdot 37, \ 7 \cdot 41, \ 7 \cdot 43, \ 7 \cdot 47, \ 7 \cdot 49, \ 7 \cdot 53, \ 7 \cdot 59, \ 7 \cdot 61, \ \ldots
\]
Since we have proved that $D_5$ has period pattern $\{6, 4, 2, 4, 2, 4, 6, 2\}$, we can assert that the elements to be deleted above are periodic. Every first item in the period has gap $7 \times 30$, where 30 is exactly the length of previous pattern $\{6, 4, 2, 4, 2, 4, 6, 2\}$. Therefore, the pattern is obtained in the following way. First copy the previous pattern 7 times, then combine some pairs of consecutive numbers to get a new pattern.

$$6, 4, 2, 4, 2, 4, 6, 2$$
$$6, 4, 2, 4, 2, 4, 6, 2$$
$$6, 4, 2, 4, 2, 4, 6, 2$$
$$6, 4, 2, 4, 2, 4, 6, 2$$
$$6, 4, 2, 4, 2, 4, 6, 2$$
$$6, 4, 2, 4, 2, 4, 6, 2$$
$$6, 4, 2, 4, 2, 4, 6, 2$$

Thus, we get the pattern of $D_7$, see (1).

If we construct $M_{11}$, then we will consider the 11 blocks:

$$M_7^{(0)}, M_7^{(1)}, M_7^{(2)}, \ldots, M_7^{(10)},$$

where we use $M_7^{(0)}$ to denote the first block in $M_7$ corresponding the first period of $D_7$. 

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Figure 2: rearranging $M_5$
Generally, when constructing the new pattern for $D_{p_{n+1}}$ from $D_{p_n}$, we first take $p_{n+1}$ copies of previous pattern for $D_{p_n}$. Then the period obeys the recursive formula:

$$T_{p_{n+1}} = p_{n+1} \cdot T_{p_n} - T_p = (p_{n+1} - 1) \cdot T_p.$$  

According to the induction hypothesis,

$$T_{p_n} = \prod_{i=1}^{n}(p_i - 1),$$

we have

$$T_{p_{n+1}} = \prod_{i=1}^{n+1}(p_i - 1).$$

**Corollory 2.3.** $L(P_{p_n}) = \prod_{i=1}^{n}p_i$.

**Proof.** Using $L(P_{p_n}) = p_n L(P_{p_n-1})$.  

**Lemma 2.4.** If $p_n \geq 11$ (i.e., $n \geq 5$), $p_n^2$ is contained in the first period of $M_{p_{n-1}}$.  

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**Figure 3:** Obtain $M_7$ from $M_5$ by deleting the $7h, h \in M_5$.  

| $M$ | $M_5^{(3)}$ | $M_5^{(1)}$ | $M_5^{(2)}$ |
|-----|------------|------------|------------|
| 1   | 79 121 157 193 | 329 367 403 | 463 499 529 |
| 7   | 47 53 59 67 71 73  47 53 59 67 71 73  47 53 59 67 71 73 47 53 59 67 71 73 | 343 347 351 381 419 | 433 437 441 471 511 |
| $M_7^{(3)}$ | 49 53 57 61 65 69 73 49 53 57 61 65 69 73 | 343 347 351 381 419 | 433 437 441 471 511 |
Proof. First,
\[ p_n^2 < p_n + L(P_{p_{n-1}}) \]
\[ \iff p_n(p_n - 1) < p_{n-1} \cdot p_{n-2} \cdots 5 \cdot 3 \cdot 2. \]

It is easy to check that \( p_k = 11 \) satisfies the inequality:
\[ p_k(p_k - 1) < p_{k-1} \cdot p_{k-2} \cdots 5 \cdot 3 \cdot 2. \]

By Bertrand’s postulate \( p_{k+1} < 2p_k \), then it is easy to complete the proof by induction.

Hence, if \( p_n > 11 \), then the numbers between \( p_n \) and \( p_n^2 \) in the pattern \( P_{p_{n-1}} \) are all primes.

**Corollary 2.5.** 3, 5, 7 is the only arithmetic sequence composed by primes with common difference 2.

**Theorem 2.6.** The gap sequence except the last element in the pattern is mirror symmetric.

Proof. For example, when constructing \( D_{11} \) from \( D_7 \), we delete the numbers \( \{11k \mid k \in M_7\} \). We list the divisors \( k \) of the first period. They are

\[
\begin{array}{cccccccccccccccccc}
1 & 11 & 13 & 17 & 19 & 23 & 29 & 31 & 37 & 41 \\
43 & 47 & 53 & 59 & 61 & 67 & 71 & 73 & 79 & 83 \\
89 & 97 & 101 & 103 & *107 & 109 & 113 & 121 & 127 & 131 \\
137 & 139 & 143 & 149 & 151 & 157 & 163 & 167 & 169 & 173 \\
179 & 181 & 187 & 191 & 193 & 197 & 199 & 209
\end{array}
\]

These 48 numbers are mirror complemental. The sum of every two symmetric numbers about * are equal to 210. More precisely, every block in \( M_7 \) are mirror complemental about the center of the block. It infers that the mirror symmetry of the gap sequence except the last element in the pattern \( P_{11} \). Therefore, when constructing \( D_{p_{n+2}} \) from \( D_{p_{n+1}} \), we conclude that the numbers which to be deleted \( \{p_{n+2}k \mid k \in M_{p_{n+1}}\} \) are mirror complemental because the numbers \( k \in M_{p_{n+1}} \) corresponding the first period are mirror complemental by the inductive hypothesis. And by inductive hypothesis, the first \( p_{n+2} \) blocks in \( M_{p_{n+1}} \) are mirror complemental. Therefore, the gap sequence in the pattern \( P_{p_{n+2}} \) except the last number is mirror symmetric.

By the argument above, we easily have

**Corollary 2.7.** The last number in every pattern \( P_{p_n} \) is 2.

**Theorem 2.8.** (i) The multiplicities of skips 2 and 4 are always same and odd. (ii) The central gap is always 4, and the multiplicities of all gaps except 2 and 4 are even.
Proof. (i) In fact, these multiplicities for $k$-prime basis, $k > 1$, are both precisely equal

$$(3 - 2) \cdot (5 - 2) \cdots (p_k - 2),$$

which is always odd as well.

The skip 2 multiplicity is easily obtained by noting that some integer $x = (x_1, x_2, \ldots, x_k)$ is the first relative prime (r-prime) before skip 2, iff $x$ has residues of the form $(1, x_2, x_3, \ldots, x_k)$ where no $x_i$ is 0 or $-2$. Since number of such $k$-tuples with 2 forbidden values on the upper $k - 1$ residues is

$$(3 - 2) \cdot (5 - 2) \cdots (p_k - 2),$$

that is the multiplicity of skip 2.

Similarly, for skip 4, the integer $x$ is the first r-prime before skip 4 iff it has residues of the form $x_1, x_2, x_3, \ldots, x_k$ where no $x_i$ is 0 or $-4$, yielding the above multiplicity with the two forbidden values on $x_2, x_3, \ldots, x_k$.

Note that the above simple method doesn’t work for skips larger than 4 since these can be obtained both via replication (by increasing $k$) and via merging of adjacent skips which follows replication. In contrast, for $k > 1$, skips 2 and 4 can be obtained only via replication (with growing $k$), but not via skip merging during filtering, since skip subsequences 2, 2 cannot occur for $k > 1$ (the replicated residue pattern for $k > 1$ is 2, 4, 2, 4, …).

(ii) Regarding the formula above (2) for multiplicities of gaps 2 and 4, the second statement of (ii) (the multiplicities of all gaps except 2 and 4 are even) is true. This follows easily from the mirror symmetry (which is easiest seen as symmetry gaps with respect to change of signs of residues) and the fact that the central gap (which could straddle the symmetry center and mirror into itself) is always 4. (End gap is always 2, and these are the only 2 exceptions.) Regarding the latter clause, for a k-prime basis $p_1 = 2, p_2 = 3, \ldots, p_k$, the central integer (symmetry axis) is at $C = 3 \cdot 5 \cdots p_k$ and its residue pattern is $C = (1, 0, 0, \ldots, 0)$. The integers around $C$ are then in residue form as follows:

- $C - 2$: $-1$ $-2$ $-2$ $-2$ $\ldots$ $-2$ r-prime
- $C - 1$: $0$ $-1$ $-1$ $-1$ $\ldots$ $-1$ r-composite
- $C$: $1$ $0$ $0$ $0$ $\ldots$ $0$ r-composite
- $C + 1$: $0$ $1$ $1$ $1$ $\ldots$ $1$ r-composite
- $C + 2$: $1$ $2$ $2$ $2$ $\ldots$ $2$ r-prime

which always has 2 r-primes at $C \pm 2$, and 3 r-composites at $C, C \pm 1$, yielding thus gap 4 at the symmetry axis. The mirror symmetry around $C$ then implies the second statement.

Remark 2.9. The largest skip comes in pairs only up to first 8 basis primes (i.e. up to $P_{19}$), but already for the 9 prime basis ($P_{23}$) the largest skip is 40 which occurs 12 times. Beyond the 9 prime basis, the longest pattern occurs in pairs only for 3 more cases: 10, 11 and 13 basis primes i.e. $P_{29}$, $P_{31}$ and $P_{41}$. All other cases have larger multiplicities of the longest skip.

Note also that for the basis with first $k$ primes $p_1 = 2, p_2 = 3, \ldots, p_k$, the trivial low bound for the max skip is $2p_{k-1}$, e.g. for the set $P_{13}(k = 6)$, the low bound is $2 \cdot 11 = 22$ which happens to be the actual longest skip for $P_{13}$.
The general residue pattern for solution meeting this low bound with \( k \) prime basis is \( (0, 0, 0, \ldots, 0, \mp 1, \pm 1) \), where first \( k - 2 \) residues are 0, and last two are +1 and -1 or -1 and +1 (which corresponds to the two solutions). The two longest patterns occur only when this low bound \( 2p_{k-1} \) is met, otherwise there are more than 2 maxima (again there is even number of solutions due to mirror symmetry).

For the 9 prime basis \( (P_{23}) \), the above low bound is \( 2 \cdot 19 = 38 \), while the max skip is 40, hence the low bound defect is \( 40 - 38 = 2 \). The max skip is 100 for \( P_{47} \), while the low bound is \( 2 \cdot 43 = 86 \), yielding defect 14. The max skip is 200 for \( P_{79} \) while the low bound is \( 2 \cdot 73 = 146 \), yielding defect 54. The max skip is 300 for \( P_{107} \) (28 primes basis), while low bound is \( 2 \cdot 103 = 206 \), defect 94. The max skip is 400 (i.e. the first \( \geq 400 \)) for \( P_{139} \) (34 primes basis), while low bound is \( 2 \cdot 137 = 274 \), defect 140. The max skip is 500 (the first \( \geq 500 \)) for \( P_{167} \) (39 primes basis), defect 184, etc.

**Conjecture 2.10.** 6 is the number that occurs the most times in any pattern \( p_n \) when \( n \geq 11 \).

**Remark 2.11.** Let \( t_{p_n} \) denote the multiplicity of 2 in the pattern for \( D_{p_{n+1}} \), then we get a recurrence inequality,

\[
t_{p_{n+1}} \geq t_{p_n} \cdot p_{n+1} - T_{p_n}.
\]

But it is a rough estimate. In fact, by equation (2), we have

\[
t_{p_{n+1}} > T_{p_n}.
\]

### 3 Application to get consecutive primes

Based on the above discussion, to obtain the pattern for \( D_{p_n} \), we first take \( p_n \) copies of the previous pattern for \( D_{p_{n-1}} \). Then, delete the corresponding elements \( p_nh \) in \( M_{p_{n-1}} \), where \( h \in M_{p_{n-1}} \). These elements to be deleted are distributed in the following way. The number of them is \( T_{p_{n-1}} \).

Here \( P_{p_{n-1}} = \{d_{p_{n-1}}^1, d_{p_{n-1}}^2, \ldots, d_{p_{n-1}}^{T_{p_{n-1}}} \} \) is the pattern for \( D_{p_{n-1}} \). The red points are just the elements to be deleted. The blue point \( q_n \) denotes the biggest prime which is less than \( p_n^2 \). The last green point is also deleted. But it is regarded as in the next period.

**Proposition 3.1.** The numbers in the set

\[
\{q \in M_{p_{n-1}} \mid p_n < q < p_n p_{n+1}, \quad q \neq p_n^2\}
\]

are consecutive primes.
If there exist at least one gap 2 in the pattern \( P_{p_n-1} \) located in the subset corresponding to the interval \([p_{n+1}, q_n]\). Then, this gap 2 is kept in the next pattern \( P_{p_n} \), and also in the subset corresponding to the interval \([p_{n+1}, q_n]\).

We look at the columns of Figure 3. Let us consider pattern for \( D_{p_{N_0}} \), and let

\[
m = p_3 p_4 \cdots p_{N_0} + 2,
\]

where \( p_i \) is in the set \{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\} \( \subset [7, 7^2] \). Since \( p_i > 7 \), it is easy to see that 210 and 210 - \( p_i \) are coprime for every \( p_i \). Hence, there are infinitely many primes in the set \( \{a_{ik} | a_{ik} = p_i + (k-1)210\} \) for every \( i \).

**Proposition 3.2.** Any pattern contains a gap 2.

**Proof.** Let us consider pattern for \( D_{p_{N_0}} \), and let

\[
m = p_3 p_4 \cdots p_{N_0} + 2,
\]

where \( p_3 = 5, p_4 = 7, \ldots \) Since \( p_3, p_4, \ldots, p_{N_0} \nmid m \), then if \( 3 \nmid m \), \( m \) will be kept in \( M_{p_{N_0}} \) during the procedures like getting \( M_7 \) from \( M_5, \ldots \) getting \( M_{p_{N_0}} \) from \( M_{p_{N_0-1}} \).

Since \( p_3, p_4, \ldots, p_{N_0} \nmid m \), the only possible divisor less than \( p_{N_0} \) is 3. If \( m = 3k \), then \( 3 \nmid (m+2) \). Note that \( p_3, p_4, \ldots, p_{N_0} \nmid (m+2) \), thus \( m + 2 \) is kept in \( M_{p_{N_0}} \). Then we consider \( m + 4 = p_3 p_4 \cdots p_{N_0} + 6 \). Because \( 3, p_3, p_4, \ldots, p_{N_0} \nmid (m+4) \), \( m + 4 \) is also kept in \( M_{p_{N_0}} \). Thus, 2 is contained in the pattern.

If \( 3 \nmid m \) and \( m \neq 3k + 1 \), then \( 3 \nmid (m+2) \). Thus \( m \) and \( m + 2 \) are both kept in \( M_{p_{N_0}} \).

If \( 3 \nmid m \) and \( m = 3k + 1 \), then \( 3 \nmid (m + 2) \). In this case, we consider \( m + 4 \) and \( m + 6 \) which are not divisible by 3. They are also be kept in \( M_{p_{N_0}} \). \qed

**Theorem 3.3.** Define a characteristic function for each interval \( I_k = [(k-1)c, kc), k = 1, 2, \ldots \) as follows:

\[
\chi_{I_k} = \begin{cases} 
1, & \text{if } I_k \text{ contains one or more primes,} \\
0, & \text{otherwise.}
\end{cases}
\]

Here \( c \) is a positive integer. If \( m \) is large enough, and if \( c \geq 3 \), then

\[
\frac{1}{c} \pi(mc) \leq \sum_{k=1}^{m} \chi_{I_k} < \pi(mc),
\]

that is,

\[
\sum_{k=1}^{m} \chi_{I_k} \approx \frac{m}{\ln m}.
\]
Proof. Let $J_{ki}, i = 1, 2, \ldots, c$ be the subintervals in $I_k$, each $J_{ki}$ has the form $[d, d + 1)$. Then,

$$
\chi_{J_{ki}} = \max_{1 \leq i \leq c} \chi_{J_{ki}}.
$$

Thus,

$$
\sum_{k=1}^{m} \chi_{J_{ki}} = \max_{k=1}^{m} \chi_{J_{ki}} \geq \sum_{k=1}^{m} \frac{1}{c} \sum_{i=1}^{c} \chi_{J_{ki}} = \frac{1}{c} \pi(m c).
$$

On the other hand, it is obvious that

$$
\sum_{k=1}^{m} \chi_{I_k} < \pi(m c) \quad \text{for } c \geq 3.
$$

Yitang Zhang [9] proved the following result for consecutive primes based on the recent work of Goldston, Pintz and Yıldırım [2, 3] on the small gaps between consecutive primes.

$$
\liminf_{n \to \infty} (p_{n+1} - p_n) < H, \quad (3)
$$

where $H = 7 \times 10^5$. Then it was reduced to 4680 by Polymath project [4, 7]. In late 2013, James Maynard and Terry Tao found a much simpler proof of Zhang’s result giving $H = 600$. A further progress based on this work has reduced $H$ to 252 by Polymath project [4]. As of April 14, 2014, one year after Zhang’s announcement, according to the Polymath project wiki, the bound has been reduced to 246.

Hence, for $c \geq 246$, we have

**Proposition 3.4.** There are infinitely many pairs $m_2 > m_1 > 0$, $m_2$ and $m_1$ are large enough, such that

$$
\frac{1}{c} (\pi(m_2 c) - \pi(m_1 c)) \leq \sum_{k=m_1}^{m_2} \chi_{I_k} < \pi(m_2 c) - \pi(m_1 c).
$$

**Conjecture 3.5.** For any $m_2 > m_1 > 0$, $m_2$ and $m_1$ are large enough, and any $c > 2$, we have

$$
\frac{1}{c} (\pi(m_2 c) - \pi(m_1 c)) \leq \sum_{k=m_1}^{m_2} \chi_{I_k} < \pi(m_2 c) - \pi(m_1 c).
$$

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