Cooperative oligopoly games with boundedly rational firms

Paraskevas V. Lekeas∗ Giorgos Stamatopoulos†

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Abstract

We analyze cooperative Cournot games with boundedly rational firms. Due to cognitive constraints, the members of a coalition cannot accurately predict the coalitional structure of the non-members. Thus, they compute their value following simple heuristics. In particular, they assign various non-equilibrium probability distributions over the outsiders’ set of partitions. We construct the characteristic function of a coalition and analyze the core of the corresponding games. We show that the core is non-empty provided the number of firms in the market is sufficiently large. Moreover, we show that if two distributions are related via first-order dominance, then the core of the game under the dominated distribution is a subset of the core under the dominant distribution.

Keywords: Cooperative game; externalities; Cournot market; core; bounded rationality

JEL Classification: C71, L2

1 Introduction

The issue of cooperation among firms in oligopolistic markets is a wide-spread phenomenon and constantly attracts the interest of economists. By colluding, firms can restrict output and raise prices in the market, thus extracting a higher surplus from the consumers. From a methodological point of view, economists analyze this market phenomenon using either non-cooperative games (when agreements among firms are non-enforceable by an outside entity) or cooperative games (whenever the signing of enforceable agreements is possible).1 Under the last approach, the focus usually lies on the core of an appropriately defined cooperative game. The core consists of all allocations of total market profits that cannot

∗Talk3, P.O. Box 441 Wilmette, IL 60091 USA; email: plekeas@gmail.com
†Department of Economics, University of Crete, 74100 Rethymno, Crete, Greece; email: gstimato@noc.gr

1For a discussion on legal cartels, we refer the reader to Dick (1996), Motta (2007) and Haucap et.al (2010).
be blocked by any coalition of firms. Non-empty core means that cooperation among all firms in the market is a priori feasible.

When a coalition contemplates breaking-off from the set of the other firms, it has to calculate its payoff. In a market environment such a calculation is not a trivial task, as the coalition’s worth depends on how the non-members act. It particular, it depends on the partition (i.e., the coalition structure) that the outsiders will form. This calls for the formation of beliefs about non-members’ behavior.

Different conjectures about the reaction of the outsiders lead to different coalitional worths and thus to different notions of core. The α and β cores (Aumann 1959) are based on min-max behavior on behalf of the non-members; the γ-core (Chander & Tulkens 1997) is based on the assumption that outsiders play individual best replies to the deviant coalition; the δ-core scenario (Hart & Kurz 1983) assumes that outsiders form a single coalition. Various authors applied these core notions to the study of Cournot markets. Rajan (1989) used the concept of γ-core and showed that it is non-empty for a market with 4 firms. A more general result for any number of firms is provided by Chander (2010). Currarini & Marini (2003) built a refinement of the γ-core by assuming that the deviant coalition acts as a Stackelberg leader in the product market. Zhao (1999) showed that the α and β cores of oligopolistic markets are non-empty.

The seminal work of Ray & Vohra (1999) goes one step further, as the worth of a coalition is deduced via arguments that satisfy a consistency criterion: a deviant coalition takes into account the fact that after its deviation, other deviations might follow, with the newly deviant coalitions thinking in a similar forward way. For games where binding agreements are feasible, Huang & Sjostrom (1998, 2003) and Koczy (2007) developed the recursive core. The recursive core is constructed under the assumption that the members of a coalition compute their value by looking recursively on the cores of the sub-games played among the outsiders.

Predicting the optimal coalitional formation in a game with many players is computationally cumbersome. Sandholm et.al (1999) showed that for an n-player game the number of different coalition structures is $O(n^n)$ and $\omega(n^{2n})$. Hence, computing the coalition structure that the outsiders form is a particularly difficult task (at least, for games with a large number of players). As a matter of fact, the problem of finding the coalition structure that maximizes the sum of all players’ payoffs is $NP$-hard (Sandholm et.al 1999). Even finding sub-optimal solutions requires the search of an exponential number of cases.

The last considerations give the motivation of the current paper. We analyze an n-firm cooperative Cournot oligopoly assuming that no group of firms has the cognitive ability to accurately deduce the partition that the firms outside the group will form. As a result of this, the members of a coalition cannot compute their value with precision. Instead, they compute it by following simple procedures or heuristics.

Clearly, the number of different heuristics one can adopt is very large. Computer scientists, for example, model similar situations via search algorithms that give solutions within certain bounds from the optimal coalition structure (Sandholm et.al 1999, Dang & Jennings 2004). On the other hand, the economists’ toolbox of heuristics includes models with players of various degrees of cognitive abilities (Stahl & Wilson 1994, Camerer 2003, Camerer et.al 2004, Haruvy & Stahl 2007), models with probabilistic choice rules (McKelvey & Palfrey 1995, Chen et.al 1997, Anderson et.al 2002), to name only a few.
In our paper, the heuristics are based on the assignment of non-equilibrium probability distributions over the set of coalition structures that the opponents can form. I.e., when contemplating a deviation from the grand coalition, the members of a coalition make the simplifying assumption that the reactions of the outsiders are governed by various plausible -but not necessarily optimal- probability distributions.

Our benchmark case assumes that the probability of a coalition structure is proportional to the profitability that the structure induces for the outsiders. Namely, a deviant coalition assumes that it is more likely that its opponents will manage to partition themselves according to the more efficient structures. This approach is in the spirit of the logit quantal response approach of McKelvey & Palfrey (1995) in non-cooperative games, where the probability of choosing a strategy depends on its relative payoff, with the probability being positive even if the strategy is inferior.\textsuperscript{2} We derive the characteristic function of a coalition under the logit distribution and we examine the core of the corresponding game. We show that if the number of firms in the market is sufficiently large then the core is non-empty. Hence, bounded rationality supports cooperation among all firms in the market.

In the second part of the paper, we extend our analysis by considering other probability distributions. In particular we consider a pair of distributions which are related via first-order stochastic dominance. If a distribution dominates another one, it gives relatively higher weight to partitions consisting of many coalitions. Our analysis has two goals: first, given that a relatively less concentrated partition hurts a deviant coalition,\textsuperscript{3} we present a novel way of modeling pessimism in a cooperative game. Secondly, we use this machinery to examine the core under a large number of distributions (other than the logit).

We fix a pair of distributions satisfying the first-order dominance property. We show that the core of the game under the dominated distribution is contained in the core of the game under the dominant. In particular, this implies that the core under the logit distribution is contained in the core of the game under any distribution that first-order dominates it. Thus we indirectly show that our game has a non-empty core for a large class of probability distributions.\textsuperscript{4}

In particular, the above inclusion holds for the case of $\gamma$-core. Namely, the core under the logit distribution is contained in the core of the game constructed under the assumption that outsiders form singleton coalitions (in our terminology, the $\gamma$ scenario corresponds to the degenerate distribution that assigns probability one to the singletons partition).

In what follows, we present the basic model in section 2. In sections 3 and 4 we present our results. Section 5 concludes.

\section{The model}

We consider a market with the set $N = \{1, 2, \ldots, n\}$ of firms. Firms produce a homogeneous product. Firm $l$ produces quantity $q_l$ using the cost function $C(q_l) = cq_l$, $l \in N$. The market price $p$ is determined via the inverse demand function $p = p(Q)$ where $Q = q_1 + q_2 + \ldots + q_n$ is the market quantity.

\textsuperscript{2}Hence, we assume that firms are able to rank the partitions according to their profitability but use heuristics when predicting the equilibrium partition.

\textsuperscript{3}This is a typical property of Cournot games.

\textsuperscript{4}These distributions might or might not respect the ranking of partitions' profits.
Assumptions

A1 $\exists Q_0 > 0$ such that $p(Q) > 0$ for $Q < Q_0$ and $p(Q) = 0$ for $Q \geq Q_0$

A2 $p'(Q) < 0$, whenever $p(Q) > 0$

A3 $p'(Q) + qip''(Q) < 0$

where $p'(Q)$ and $p''(Q)$ denote the first and second derivatives of the inverse demand function. The above assumptions\(^5\) are standard and guarantee the existence and uniqueness of Cournot equilibrium (see, for example, Vives 2001).

Let $S \subset N$ denote a coalition of firms with $|S| = s$ members and let $N \setminus S$ denote the complementary set of $S$, where $|N \setminus S| = n - s$. The worth or value of $S$ is the sum of its members’ profits. These profits depend on how the members of $N \setminus S$ partition themselves into coalitions. The set $N \setminus S$ can be partitioned into disjoint subsets in $B_{n-s}$ ways, where $B_{n-s}$ is Bell’s $(n-s)^{th}$ number (Bell 1934).

What matters for $S$ is only the number of the opponent coalitions. Consider for example the case $N = \{1, 2, 3, 4, 5\}$ and $S = \{1\}$. The set of outsiders is $N \setminus S = \{2, 3, 4, 5\}$. Consider the partitions $\{\{2, 3\}, \{4, 5\}\}$ and $\{\{2, 3, 4\}, \{5\}\}$ of outsiders. These partitions are equivalent for $S$ (and so are all partitions with two coalitions) in the sense that both induce the same profit for $S$ (in both cases, $S$ would compete in a triopoly market). More generally, all partitions with $j$ coalitions induce the same profit for $S$, irrespective of how the outsiders are grouped among the $j$ coalitions. We will call these partitions $j$-similar.

Denote the number of $j$-similar partitions by $K_{n-s,j}$, where $K_{n-s,j}$ gives the number of ways to partition a set of $n - s$ objects into $j$ groups, or else the Stirling numbers of the second kind. Then

$$K_{n-s,j} = \frac{1}{j!} \sum_{i=0}^{j} (-1)^i \binom{j}{i} (j-i)^{n-s}$$

The basic assumption that underlies this paper is that the members of $S$ use simple probabilistic models in order to predict the coalitional behavior of the non-members. In particular, the probability of a partition is proportional to the profitability that the partition induces for the outsiders. This approach is in line with the spirit of the logit quantal response model (McKelvey & Palfrey 1995) in non-cooperative games, where the probability of choosing a strategy depends on its relative payoff, with the probability being positive even if the strategy is inferior.

Consider a coalition $S$ with $s$ members and an outsiders’ partition with $j$ coalitions. Let $\Pi_j$ denote the sum of the profits that the $j$ coalitions earn under this partition (this sum is constant over all $j$-similar partitions). Define

$$f_{n,s}(j) = \frac{e^{\Pi_j} K_{n-s,j}}{\sum_{m=1}^{n-s} e^{\Pi_m} K_{n-s,m}}$$

---

\(^5\)Another standard assumption is $p'(Q) - C''(q_l) < 0$; however, this is automatically satisfied in our constant-returns model.
Notice that \( f_{n,s}(j) \in (0, 1) \) and \( \sum_{j=1}^{n-s} f_{n,s}(j) = 1 \). Then, \( f_{n,s}(j) \) gives the total probability that \( S \) assigns to all \( j \)-similar structures.\(^{6}\) Note in (2) that the profitability of each partition is adjusted by the corresponding Stirling number. The results of the paper hold even if such an adjustment does not take place.

**Example**

Let us illustrate the above by considering an example with five firms, \( N = \{1, 2, 3, 4, 5\} \). Assume that the inverse demand is \( p = 1 - Q \) and that \( c = 0 \). Consider a coalition \( S \) with \( s \) members. If the \( n - s \) outsiders form \( j \) coalitions then there are \( j + 1 \) active players in the market. By simple calculations, the total profits of the outside coalitions are

\[
\Pi_j = \frac{j}{(j+2)^2}, \quad j = 1, 2, ..., n-s
\]  

(3)

Consider first a singleton coalition, say \( S = \{1\} \). Then \( B_{n-s} = B_4 = 15 \) and

\[ K_{4,1} = K_{4,4} = 1, \quad K_{4,2} = 7, \quad K_{4,3} = 6 \]

Using (2) and (3) the probabilities that \( S \) assigns to outsiders’ partitions are

\[
f_{5,1}(1) = f_{5,1}(4) = \frac{e^{1/9}}{Z_1}, \quad f_{5,1}(2) = \frac{7e^{1/8}}{Z_1}, \quad f_{5,1}(3) = \frac{6e^{3/25}}{Z_1}
\]

where \( Z_1 = 2e^{1/9} + 7e^{1/8} + 6e^{3/25} \). Consider next a coalition with two members, say \( S = \{1, 2\} \). Then \( B_{n-s} = B_3 = 5 \) and

\[ K_{3,1} = 1, \quad K_{3,2} = 3, \quad K_{3,3} = 1 \]

We then have

\[
f_{5,2}(1) = \frac{e^{1/9}}{Z_2}, \quad f_{5,2}(2) = \frac{3e^{1/8}}{Z_2}, \quad f_{5,2}(3) = \frac{e^{3/25}}{Z_2}
\]

where \( Z_2 = e^{1/9} + 3e^{1/8} + e^{3/25} \). Finally, consider a coalition with three members,\(^7\) \( S = \{1, 2, 3\} \). In this case, \( B_{n-s} = B_2 = 2, \)

\[ K_{2,1} = K_{2,2} = 1 \]

Hence we have

\[
f_{5,3}(1) = \frac{e^{1/9}}{e^{1/9} + e^{1/8}}, \quad f_{5,3}(2) = \frac{e^{1/8}}{e^{1/9} + e^{1/8}}
\]

\(^6\)Later on we discuss other, more general, distributions.

\(^7\)When a deviant coalition has four members, there is no ambiguity about the outsider.
2.1 The game \((N, v^n)\)

In this section we compute the characteristic function of a coalition. We use the \(j\)-similarity and focus for each \(j\) on one representative of the \(j\)-similar structures. So, let \(q_i^j\) denote the quantity that coalition \(i\) chooses, \(i = 1, 2, ..., j\), under a partition with \(j\) members; and let \(q_s\) denote the quantity of the deviant coalition \(S\). The objective function that \(S\) faces is given by

\[
\pi_f(S) = n - s \sum_{j=1}^{n-s} f_{n,s}(j)[p(q_s + \sum_{i=1}^{j} q_i^j) - c]q_s
\]  

(4)

The objective function of coalition \(i\) under a \(j\)-similar partition, \(j = 1, 2, ..., n - s\), is

\[
\pi_i^j = [p(q_s + \sum_{r=1, r \neq i}^{j} q_r^j + q_i^j) - c]q_i^j, \ i = 1, 2, ..., j
\]

Hence the maximization problems to solve for are

\[
\max_{q_s} \pi_f(S)
\]

(5)

and for \(j = 1, 2, ..., n - s\),

\[
\max_{q_i^j} \pi_i^j, \ i = 1, 2, ..., j
\]

(6)

It is easy to show that the solution of the above problems is given implicitly by

\[
q_s = \frac{\sum_{j=1}^{n-s} f_{n,s}(j)p(q_s + jq_i^j) - c}{\sum_{j=1}^{n-s} f_{n,s}(j)p'(q_s + jq_i^j)}
\]

(7)

and for \(j = 1, 2, ..., n - s\),

\[
q_i^j = \frac{p(q_s + jq_i^j) - c}{-p'(q_s + jq_i^j)}, \ i = 1, 2, ..., j
\]

(8)

where we used the fact that for each \(j\), the solution regarding the quantities of outsiders is symmetric, i.e., \(q_1^j = q_2^j = ... = q_j^j\). Using (7) and (8) in (4), we obtain the characteristic function of \(S\). We shall denote this function by \(v^n(S)\) or by\(^8 v^n(s)\). Let \(Q_j = q_s + jq_i^j\). We then have

\[
v^n(S) = \frac{[\sum_{j=1}^{n-s} f_{n,s}(j)p(Q_j) - c]^2}{-\sum_{j=1}^{n-s} f_{n,s}(j)p'(Q_j)}
\]

(9)

\(^8\)In what follows we use the two notations interchangeably, according to the context.
Hence our game is the pair \((N, v^n)\) where \(v^n\) is defined by (9). The value of the grand coalition \(v^n(N)\) is the monopoly profit (this is independent of \(n\) but for notational uniformity we keep the index \(n\)). Denote by \(Q_M\) the monopoly output. Then

\[
v^n(N) = \frac{[p(Q_M) - c]^2}{-p'(Q_M)}
\]

An allocation is a vector \((x_1, x_2, ..., x_n)\) such that \(\sum_{i \in N} x_i = v^n(N)\). The core \(C_f\) of \((N, v^n)\) is the set of all allocations that cannot be blocked by any coalition given distribution \(f_{n,s}\). I.e., the core is the set

\[
C_f = \{(x_1, ..., x_n) : \forall S \text{ with } v^n(S) > \sum_{i \in S} x_i\}
\]

In the next sections we examine the core for various demand and probability functions.

## 3 Results

The first result in this section states the following useful property.

**Lemma 1** For every positive integer \(k\), the equality \(v^n(s) = v^{n+k}(s+k)\) holds.

**Proof** We first notice that the total profits of \(j\) outside coalitions in a market with \(n\) firms when a deviant coalition \(S\) has \(s\) members are equal to their total profits in a market with \(n+k\) firms when \(S\) has \(s+k\) members. This is due to the constant returns assumption. Hence

\[
f_{n,s}(j) = \frac{e^{\Pi_j K_{n-s,j}}}{\sum_{m=1}^{n-s} e^{\Pi_m K_{n-s,m}}} = \frac{e^{\Pi_j K_{n+k-(s+k),j}}}{\sum_{m=1}^{n+k-(s+k)} e^{\Pi_m K_{n+k-(s+k),m}}} = f_{n+k,s+k}(j)
\]

Moreover for each \(j\), \(q_s\) and \(q^j\) are also constant under the two scenarios (again due to constant returns to scale): namely, the quantity of \(S\) when it has \(s\) members and the market has \(n\) firms is equal to its quantity when \(S\) has \(s+k\) members and the market has \(n+k\) firms; and the same holds for \(q^j\) and by consequence for \(Q_j\). Combining this fact with (9) proves the result. 

The intuition behind Lemma 1 is clear. If a coalition has \(s+k\) members in a game with \(n+k\) players, it faces \(n+k-(s+k) = n-s\) outsiders. This is equal to the number of
outsiders that a coalition with $s$ members faces in a game with $n$ players. Hence the two coalitions face the same set of potential coalition structures.

An almost immediate implication of Lemma 1 is the monotonicity of $v^n(s)$ in $s$.

**Lemma 2** For every $n$, $v^n(s)$ is strictly increasing in $s$.

**Proof** The proof will be based on induction. For the base case, $n = 2$, we have to prove that $v^2(2) > v^2(1) > v^2(0)$. Recall that $v^2(2)$ is the monopoly profit and $v^2(1)$ is the profit of one of the two duopolists. Under assumptions A1-A3 we have that the former profit is higher than the latter, i.e., $v^2(2) > v^2(1)$. Moreover, $v^2(1) > 0 = v^2(0)$ and so we have the base case.

Assume for the induction hypothesis that in a game with $n$ players and for an arbitrary $s$, we have that $v^n(s) > v^n(s-1)$. We will prove that $v^{n+1}(s) > v^{n+1}(s-1)$. By Lemma 1 and the induction hypothesis we have that $v^{n+1}(s) = v^n(s-1) > v^n(s-2) = v^{n+1}(s-1)$. Note also that $v^{n+1}(s+1) > v^{n+1}(s)$ (by Lemma 1) and that $v^{n+1}(1) > v^{n+1}(0) = 0$. This completes the proof.

Lemmas 1 and 2 hold under any demand function (that satisfies A1-A3). In what follows, we use these Lemmas to derive conditions for core non-emptiness under certain demand functions. In particular we will focus on the family of functions

$$Q = 1 - p^b, \ b > 0$$

which we borrow from Anderson & Engers (1992). Note that if $b > (>) 1$, demand is concave (convex); and if $b = 1$, demand is linear. In order to derive analytically the solution of the game, we need to set $c = 0$. The solution with respect to quantities and the characteristic function is then given by\(^{10}\)

$$q_s = \frac{\sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^\frac{1}{b}}{\sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^\frac{1}{b}(1 + j + 1/b)}$$

$$q_i = b \psi_{n-s}^{\frac{1}{b}} \frac{\sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^\frac{1}{b}(j + 1/b)}{\sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^\frac{1}{b}(1 + j + 1/b)}, \ i = 1, 2, ..., j$$

$$v^n(S) = \sum_{j=1}^{n-s} f_{n,s}(j) \left( \frac{\sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^\frac{1}{b}(j + 1/b)}{(bj + 1) \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^\frac{1}{b}(1 + j + 1/b)} \right)^{\frac{1}{b}} \cdot q_s$$

\(^9\)See for example Amir and Lambson (2000).

\(^{10}\)The details of the derivations appear in Lemma A1 in the Appendix.
where $\psi_j = \frac{1}{bj + 1}$ [in (13), $q_s$ is given by (11)]. The value of the grand coalition is

$$v^n(N) = \frac{b}{(1 + b)^{1+b}}$$

As a benchmark case, we first present a result for the linear demand ($b = 1$). Afterwards, we discuss the non-linear case ($b \neq 1$).

**Proposition 1** Assume the demand function is given by $Q = 1 - p$. The game $(N, v^n)$ has a non-empty core if $n$ is sufficiently large.

**Proof** Since firms are identical, the core is non-empty if and only if for all $s \leq n$,

$$\frac{v^n(n)}{n} \geq \frac{v^n(s)}{s} \quad (14)$$

It is easy to verify that the inequality does not hold for $3 \leq n \leq 11$. So for these values of $n$ the core is empty.

\[11\] The inequality holds for $n = 12$ (Table 2 in the Appendix). We will prove the rest of the proposition using induction on $n$, where $n \geq 12$.

*Base:* Table 2 in the Appendix establishes the base case ($n = 12$).

*Induction hypothesis:* For all $S : |S| = s \leq n$, $\frac{v^n(n)}{n} \geq \frac{v^n(s)}{s}$.

*Induction step:* We will show that for all $S : |S| = s \leq n + 1$,

$$\frac{v^{n+1}(n + 1)}{n + 1} \geq \frac{v^{n+1}(s)}{s}$$

By Lemma 1 we have that $v^{n+1}(s) = v^{n+1}((s - 1) + 1) = v^n(s - 1)$ and also $v^{n+1}(n + 1) = v^n(n)$. So we have to show that

$$\frac{v^n(n)}{n + 1} \geq \frac{v^n(s - 1)}{s} \quad (15)$$

From the Induction hypothesis we have

$$v^n(n) \geq \frac{n}{s - 1}v^n(s - 1)$$

and thus

$$(s - 1)v^n(n) \geq nv^n(s - 1) \quad (16)$$

Using Lemma 2,

\[11\] For $3 \leq n \leq 11$ it holds that $v^n(1) > \frac{v^n(n)}{n}$ (see Table 3 in the Appendix). The relevant calculations were made using the Maple program and they are available by the authors upon request.
Adding (16) and (17) we have

\[ s v^n(n) > (n + 1) v^n(s - 1) \]

which implies that (15) holds. So we have the proof for \( n + 1 \) and thus the proposition is proved.

The monopoly profit is independent of \( n \). On the other hand, \( v^n(s) \) decreases in \( n \). As a result, for sufficiently large \( n \) the difference \( v^n(n)/n - v^n(s)/s \) becomes positive for all \( s \) and the core is non-empty.

**The case \( b \neq 1 \)**

We now discuss the core for the non-linear demand case. To this end, we will utilize the previous results, i.e., Lemmas 1 and 2, and Proposition 1. Recall that Lemmas 1 and 2 hold for any demand function. Furthermore, among the three steps of the induction proof of Proposition 1, i.e., base step, induction hypothesis and induction step, only the base step depends on the demand function used. Hence when extending Proposition 1 to cases where \( b \neq 1 \) we only need to ensure the validity of the base step. Namely, for a certain value of \( b \) we need to find a number \( n(b) \) which provides the base step of the induction argument (see page 9, proof of Proposition 1).

Table 1 presents pairs \((b^*, n(b^*))\) that have the above property: given \( b^* \) (where \( b^* \neq 1 \)), the number \( n(b^*) \) establishes the base step of the induction process for the demand function \( Q = 1 - p^{b^*} \).

| \( b^* \) | \( n(b^*) \) |
|---|---|
| 0.5 | 3 |
| 0.6 | 5 |
| 0.7 | 6 |
| 0.8 | 7 |
| 0.9 | 9 |
| 1.1 | 15 |
| 1.2 | 19 |
| 1.3 | 24 |
| 1.4 | 32 |
| 1.5 | 42 |
| 1.6 | 57 |
| 1.7 | 78 |
| 1.8 | 107 |
| 1.9 | 147 |
| 2.0 | 205 |

Table 1: \( b^* \) and \( n(b^*) \) for base step of induction.

In other words, given a specific value \( b^* \), the game has a non-empty core for \( n \geq n(b^*) \). The complexity of the model does not allow us to derive analytically the relation between
b* and the corresponding critical value n(b*). From our sample we note though that as b* increases, the core is non-empty less often (as b* increases, the number n(b*) increases). Moreover, if b* is sufficiently low, the core is non-empty for all\textsuperscript{12} n ≥ 3.

4 Extensions

In this section we compare the cores of games that differ with respect to the probability schemes assigned to outsiders’ partitions. In particular, we consider distributions that are related via first-order stochastic dominance. We will show that if a certain distribution dominates at first-order another one, then the core under the dominated distribution is a subset of the core under the dominant distribution. An application of this result is that Proposition 1 holds under any distribution that dominates the distribution defined in (2).

4.1 First-order stochastic dominance

Fix a coalition S. Let \( z_{n,s} \) and \( w_{n,s} \) be two probability distributions over the set of the outsiders’ partitions. Assume that \( z_{n,s} \) dominates \( w_{n,s} \) at first-order, i.e., for all \( j^* \),

\[
\sum_{j=1}^{j^*} z_{n,s}(j) \leq \sum_{j=1}^{j^*} w_{n,s}(j)
\]

Denote by \( C_w \) and \( C_z \) the cores under the two distributions. Assume that \( C_z \neq \emptyset \). We have the following result.

**Proposition 2** Assume the inverse demand \( p(Q) \) is concave. If \( z_{n,s} \) stochastically dominates \( w_{n,s} \) at first-order then \( C_w \subseteq C_z \).

**Proof** We need to introduce some notation. By \( \tilde{q}_j \) we denote the solution of maximization problems (6) (where the probability distributions do not yet play any role\textsuperscript{13}). By \( q_s(w) \) and \( q_s^l(w) \) we denote the simultaneous reduced form solution of (5) and (6) when the probability distribution is \( w_{n,s} \); and by \( q_s(z) \) and \( q_s^l(z) \) we denote the simultaneous solution of (5) and (6) when the probability distribution is \( z_{n,s} \). Finally, \( v_n^u(S) \) and \( v_n^l(S) \) denote the characteristic functions of coalition \( S \) under \( w_{n,s} \) and \( z_{n,s} \) respectively.

Define\textsuperscript{14}

\[
\hat{Q}_j^{s} = \sum_{i=1}^{j} \tilde{q}_i, \quad j = 1, 2, ..., n - s
\]

We shall show that \( \hat{Q}_{j+1}^{s} > \hat{Q}_j^{s}, \ j = 1, ..., n - s - 1 \). To this end we will use a result by Amir & Lambson (2000): they show that total market output in a Cournot market is an increasing function of the number of active firms (provided conditions equivalent to A1-A3 hold). Before applying this result to our model, we need to re-interpret our game: first

\textsuperscript{12}The case \( n = 2 \) clearly has no interest.

\textsuperscript{13}At this stage of computations, \( \tilde{q}_j \) is not the reduced form solution but it depends on \( q_s \).

\textsuperscript{14}To be more precise, we should express \( \tilde{q}_i^l \) and \( \hat{Q}_i^l \) as functions of \( q_s \), but for notational simplicity we drop the term \( q_s \).
notice that for each \( j \), \( q_s \) is a constant. So for each \( j \), we will treat \( S \) as a dummy player and we will consider a market with \( j \) only firms (the \( j \) outside coalitions). In other words, for each \( j \) we treat \( q_s \) as a parameter of the market with \( j \) firms. In such an environment the result of Amir & Lambson (2000) is directly applicable: increasing the number of active firms from \( j \) to \( j + 1 \) results in higher market output, i.e., \( \tilde{Q}^{-s}_{j+1} > \tilde{Q}^{-s}_j \).

Given the above, we next plug \( \tilde{Q}^{-s}_j \) into the objective function of coalition \( S \) under the two probability distributions and we obtain

\[
\tilde{\pi}_w(S) = \sum_{j=1}^{n-s} w_{n,s}(j) [p(q_s + \tilde{Q}^{-s}_j) - c] q_s \equiv \sum_{j=1}^{n-s} w_{n,s}(j) \pi_s(q_s, \tilde{Q}^{-s}_j)
\]

and

\[
\tilde{\pi}_z(S) = \sum_{j=1}^{n-s} z_{n,s}(j) [p(q_s + \tilde{Q}^{-s}_j) - c] q_s \equiv \sum_{j=1}^{n-s} z_{n,s}(j) \pi_s(q_s, \tilde{Q}^{-s}_j)
\]

where

\[
\pi_s(q_s, \tilde{Q}^{-s}_j) \equiv [p(q_s + \tilde{Q}^{-s}_j) - c] q_s, \quad j = 1, 2, ..., n - s
\]

Notice that since \( Q^{-s}_{j+1} > Q^{-s}_j \), we have

\[
\pi_s(q_s, Q^{-s}_j) > \pi_s(q_s, Q^{-s}_{j+1}) \quad (18)
\]

where the last inequality is due to the fact that the profit of a firm in a Cournot market is decreasing in the (individual or aggregate) quantities of the other firms. Notice next that

\[
\tilde{\pi}_w(S) - \tilde{\pi}_z(S) = \left( \sum_{j=1}^{n-s} w_{n,s}(1) - z_{n,s}(1) \right) \pi_s(q_s, Q^{-s}_1) + \sum_{j=2}^{n-s} \left( w_{n,s}(j) - z_{n,s}(j) \right) \pi_s(q_s, Q^{-s}_j)
\]

\[
> \left( \sum_{j=1}^{n-s} w_{n,s}(1) - z_{n,s}(1) \right) \pi_s(q_s, Q^{-s}_1) + \sum_{j=2}^{n-s} \left( w_{n,s}(j) - z_{n,s}(j) \right) \pi_s(q_s, Q^{-s}_j)
\]

\[
= \left( \sum_{j=1}^{n-s} w_{n,s}(1) + w_{n,s}(2) - z_{n,s}(1) - z_{n,s}(2) \right) \pi_s(q_s, Q^{-s}_2)
\]

\[
+ \sum_{j=3}^{n-s} \left( w_{n,s}(j) - z_{n,s}(j) \right) \pi_s(q_s, Q^{-s}_j) \quad (19)
\]

where the inequality is due to (18). Continuing the iterations on \( j \), we eventually have that
\[ \tilde{\pi}_w(S) - \tilde{\pi}_z(S) > \left( \sum_{j=1}^{n-s-1} w_{n,s}(j) - \sum_{j=1}^{n-s-1} z_{n,s}(j) \right) \pi_s(q_s, Q_{n-s-1}^-) \]
\[ + \left( w_{n,s}(n - s) - z_{n,s}(n - s) \right) \pi_s(q_s, Q_{n-s}^-) \]
\[ > \left( \sum_{j=1}^{n-s} w_{n,s}(j) - \sum_{j=1}^{n-s} z_{n,s}(j) \right) \pi_s(q_s, Q_{n-s}^-) = 0 \] (20)

We conclude that for all \( q_s \) and the corresponding \( \tilde{Q}_j^- \), we have that
\[ \sum_{j=1}^{n-s} w_{n,s}(j) \pi_s(q_s, \tilde{Q}_j^-) > \sum_{j=1}^{n-s} z_{n,s}(j) \pi_s(q_s, \tilde{Q}_j^-) \] (21)

In the Appendix we show that if the inverse demand function \( p(Q) \) is concave then \( Q_j^- \) \((w) < Q_j^- \) \((z) \) (Lemma A2 in the Appendix). Notice next that
\[ v_{w}^n(S) = \sum_{j=1}^{n-s} w_{n,s}(j) \pi_s(q_s, Q_j^-) \]
\[ \geq \sum_{j=1}^{n-s} w_{n,s}(j) \pi_s(q_s, Q_j^-) \]
\[ \geq \sum_{j=1}^{n-s} w_{n,s}(j) \pi_s(q_s, Q_j^-) \]
\[ > \sum_{j=1}^{n-s} z_{n,s}(j) \pi_s(q_s, Q_j^-) = v_{z}^n(S) \] (22)

where the first inequality is due to the fact that \( q_s \) is the optimal choice of coalition \( S \) against \( Q_j^- \); the second inequality holds because \( Q_j^- \) \( \leq Q_j^- \) and because the Cournot profit of a firm decreases in the quantities of outsiders; and the third inequality is due to (21). Since \( v_w(S) > v_z(S) \) we conclude that \( C_w \subseteq C_z \).

As an application of Proposition 2, we note that Proposition 1 holds not only under \( f_{n,s} \) but also under any distribution that dominates \( f_{n,s} \) at first order.\(^{15}\)

**Corollary 1** Assume the inverse demand is given by \( p = 1 - Q \). Consider any distribution \( z_{n,s} \) that dominates distribution (2) at first-order. Then \( C_z \neq \emptyset \) for large \( n \).

\(^{15}\)Similar statements hold for the non-linear demand case (recall that Proposition 1 focuses on the linear demand).
Proof If the inverse demand is linear and \( n \) is large (in particular, greater than 12), then \( C_f \neq \emptyset \) (Proposition 1). Since \( z_{n,s} \) dominates distribution (2) at first-order then \( C_f \subseteq C_z \).
Hence the latter core is non-empty.

Compared to the cumulative distribution of \( f_{n,s} \), the cumulative distribution of \( z_{n,s} \) assigns higher probabilities to events that include coalition structures with many coalitions. Clearly, these events are unfavorable for \( S \), hence the use of \( z_{n,s} \) indicates some sort of pessimism on behalf of the members of \( S \). Our analysis in this section can be motivated by resorting to the theory of risk measurement, which often measures risk by assigning relatively high probabilities to unfavorable events (see e.g., Acerbi 2002). Our approach provides a novel way of computing the impact of varying the degree of pessimism or risk in a cooperative game.

Finally we note that a particular distribution that dominates \( f_{n,s} \) is the distribution defined by \( z_{n,s}(j) = 0 \), for \( j = 1, 2, \ldots, n - s - 1 \) and \( z_{n,s}(n - s) = 1 \). This distribution corresponds to the \( \gamma \)-core scenario. It is known that the latter core is non-empty for general Cournot oligopolies (Chander 2009). Let \( C_\gamma \) denote the \( \gamma \)-core. We have the following corollary.

**Corollary 2** The inclusion \( C_f \subseteq C_\gamma \) holds.

The \( \gamma \)-core is based on the worst scenario for \( S \): all \( n - s \) firms remain separate entities. Under \( f_{n,s} \), the singleton coalitions structure is just one of the partitions that \( S \) takes into account. Other, more favorable, partitions occur with positive probability. Hence, the value of \( S \) under \( f_{n,s} \) is higher than its value under \( z_{n,s} \).

## 5 Conclusions

This paper analyzed cooperative Cournot games. The analysis is based on the assertion that when a coalition contemplates a deviation from the grand coalition, it assigns various non-equilibrium distributions on the set of partitions that the outsiders can form. This assumption is justified by imposing cognitive constraints on behalf of the firms in the market. Provided that the number of firms in the market is sufficiently high, the corresponding game has a non-empty core for a large class of probability distributions and demand functions.

Let us mention a few extensions of the current work. The analysis of oligopolistic markets with more general cost functions and/or other modes of competition (e.g., product differentiation, price competition) are natural future directions. Further, the application of the current framework to other economic environments is of interest.

Appendix

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16 We thank an anonymous referee for pointing out this connection.
| s | \(v^n(s)\) | \(v^n(s)/s\) |
|---|---|---|
| 1 | 0.02047 | 0.02047 |
| 2 | 0.02273 | 0.01136 |
| 3 | 0.02544 | 0.00848 |
| 4 | 0.02876 | 0.00719 |
| 5 | 0.03289 | 0.00657 |
| 6 | 0.03815 | 0.00635 |
| 7 | 0.04503 | 0.00680 |
| 8 | 0.05443 | 0.00755 |
| 9 | 0.06795 | 0.00775 |
| 10 | 0.08736 | 0.00873 |
| 11 | 0.11111 | 0.01010 |
| 12 | 0.25 | 0.02083 |

Table 2: values \(v^n(s)\) and \(v^n(s)/s\) with \(n = 12\)

| n | \(v^n(\{i\})\) | \(v^n(n)/n\) |
|---|---|---|
| 3 | 0.08736 | 0.08333 |
| 4 | 0.06795 | 0.06250 |
| 5 | 0.05444 | 0.05000 |
| 6 | 0.04503 | 0.04166 |
| 7 | 0.03815 | 0.03571 |
| 8 | 0.03289 | 0.03125 |
| 9 | 0.02876 | 0.02777 |
| 10 | 0.02544 | 0.02500 |
| 11 | 0.02273 | 0.02272 |

Table 3: values \(v^n(\{i\})\) and \(v^n(n)/n\), \(n \in \{3, 4, ..., 11\}\)

**Lemma A1** Assume the demand function is \(Q = 1 - p^b\). Then the characteristic function is given by (13).

**Proof** The profit function of coalition \(i\) under a partition with \(j\) members is

\[
\pi^j_i = (1 - q_s - \sum_{r=1, r \neq i}^j q_r^j - q_i^j)^b q_i^j, \quad i = 1, 2, ..., j
\]

(23)

Note that

\[
\frac{\partial \pi^j_i}{\partial q_i^j} = 0 \iff b(1 - q_s - q_i^j - \sum_{r=1, r \neq i}^j q_r^j) = q_i^j
\]

(24)

By symmetry, all \(j\) outside coalitions produce the same. So let \(q_i^j = q_i^j\), for all \(r\). Therefore by (24), \(b(1 - q_s - jq_i^j) = q_i^j\) and hence

\[
q_i^j = \frac{b(1 - q_s)}{bj + 1}
\]

(25)
The objective function of the deviant coalition is

\[ \pi(S) = \sum_{j=1}^{n-s} f_{n,s}(j)(1 - q_s - \sum_{i=1}^{j} q_i^j)^{1-b} q_s \]  \hspace{1cm} (26)

Note that \( \frac{\partial \pi(S)}{\partial q_s} = 0 \) if

\[ \sum_{j=1}^{n-s} f_{n,s}(j)(1 - q_s - \sum_{i=1}^{j} q_i^j)^{1-b} = \frac{1}{b} \sum_{j=1}^{n-s} f_{n,s}(j)(1 - q_s - \sum_{i=1}^{j} q_i^j)^{1-b-1} q_s \] \hspace{1cm} (27)

Using (25), (27) becomes

\[ \sum_{j=1}^{n-s} f_{n,s}(j)[\frac{1-q_s}{1+bj}]^{1-b} = \frac{1}{b} \sum_{j=1}^{n-s} f_{n,s}(j)[\frac{1-q_s}{1+bj}]^{1-b-1} q_s \]

and hence

\[ (1-q_s)^{1-b} \sum_{j=1}^{n-s} f_{n,s}(j)[\frac{1}{1+bj}]^{1-b} = \frac{1}{b}(1-q_s)^{1-b-1} \sum_{j=1}^{n-s} f_{n,s}(j)[\frac{1}{1+bj}]^{1-b-1} q_s \]

Define \( \psi_j = \frac{1}{bj+1} \). Then rearranging the above relation gives

\[ q_s \left[ \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} \right] + \frac{1}{b} \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}-1} = \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} \] \hspace{1cm} (28)

Notice that

\[ \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} + \frac{1}{b} \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}-1} = \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} (1 + \frac{1}{b\psi_j}) = \]

\[ \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} (1 + (bj+1)/b) = \sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} (1 + j + 1/b) \] \hspace{1cm} (29)

Using (28) and (29) we get

\[ q_s = \frac{\sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}}}{\sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^{\frac{1}{b}} (1 + j + 1/b)} \] \hspace{1cm} (30)
Using (30), (25) becomes

\[ q_i^j = b \psi_j \frac{\sum_{j=1}^{n-s} \psi_j^s (j + 1/b)}{\sum_{j=1}^{n-s} f_{n,s}(j) \psi_j^s (1 + j + 1/b)} \]  \hspace{1cm} (31)

Plugging (30) and (31) in (26) gives us (13).

Lemma A2 Assume the inverse demand \( p(Q) \) is concave. Then \( Q_j^{-s}(w) < Q_j^{-s}(z) \), \( j = 1, 2, ..., n - s \).

Proof Let \( \tilde{q}_s(w) \) and \( \tilde{q}_s(z) \) denote the quantities that solve respectively the problems \( \max_{q_s} \sum_{j=1}^{n-s} w_{n,s}(j) \pi_s(q_s, Q_j^{-s}) \) and \( \max_{q_s} \sum_{j=1}^{n-s} z_{n,s}(j) \pi_s(q_s, Q_j^{-s}) \). For convenience define the functions \( F(q_s) = \sum_{j=1}^{n-s} w_{n,s}(j) \pi_s(q_s, Q_j^{-s}) \) and \( H(q_s) = \sum_{j=1}^{n-s} z_{n,s}(j) \pi_s(q_s, Q_j^{-s}) \). Hence \( \tilde{q}_s(w) \) and \( \tilde{q}_s(z) \) satisfy respectively the first-order conditions

\[ \frac{\partial F(q_s)}{\partial q_s} = 0 \quad \text{and} \quad \frac{\partial H(q_s)}{\partial q_s} = 0 \]

or equivalently

\[ \sum_{j=1}^{n-s} w_{n,s}(j) p'(q_s + Q_j^{-s})q_s + \sum_{j=1}^{n-s} w_{n,s}(j) p(q_s + Q_j^{-s}) - c = 0 \]  \hspace{1cm} (32)

and

\[ \sum_{j=1}^{n-s} z_{n,s}(j) p'(q_s + Q_j^{-s})q_s + \sum_{j=1}^{n-s} z_{n,s}(j) p(q_s + Q_j^{-s}) - c = 0 \]  \hspace{1cm} (33)

The function \( F(q_s) \) is strictly concave in \( q_s \) (by assumptions A1-A3). Hence \( \tilde{q}_s(w) > \tilde{q}_s(z) \) if and only if \( \frac{\partial F(\tilde{q}_s(z))}{\partial q_s} > 0 \). We have

\[ \frac{\partial F(\tilde{q}_s(z))}{\partial q_s} > 0 \iff \sum_{j=1}^{n-s} w_{n,s}(j) p'(\tilde{q}_s(z) + Q_j^{-s})\tilde{q}_s(z) + \sum_{j=1}^{n-s} w_{n,s}(j) p(\tilde{q}_s(z) + Q_j^{-s}) - c > 0 \]  \hspace{1cm} (34)

Solving for \( q_s(z) \) in (33) and plugging in (34) we have that

\[ \frac{\partial F(\tilde{q}_s(z))}{\partial q_s} > 0 \iff \frac{\sum_{j=1}^{n-s} w_{n,s}(j) p'(q_s(z) + Q_j^{-s})}{\sum_{j=1}^{n-s} z_{n,s}(j) p' \tilde{q}_s(z) + Q_j^{-s}) - c} \]
\[ + \sum_{j=1}^{n-s} w_{n,s}(j)p(\tilde{q}_s(z) + Q_j^{-s}) - c > 0 \] (35)

We now claim that
\[ \sum_{j=1}^{n-s} w_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=1}^{n-s} z_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s}) > -1 \] (36)

To show that above we can equivalently show
\[ \sum_{j=1}^{n-s} w_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=1}^{n-s} z_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s}) > 0 \] (37)

We have
\[ \sum_{j=1}^{n-s} w_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=1}^{n-s} z_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s}) = \]
\[ (w_{n,s}(1) - z_{n,s}(1))p'(\tilde{q}_s(z) + Q_1^{-s}) + \sum_{j=2}^{n-s} w_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=1}^{n-s} z_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s}) > \]
\[ (w_{n,s}(1) - z_{n,s}(1))p'(\tilde{q}_s(z) + Q_2^{-s}) + \sum_{j=2}^{n-s} w_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=1}^{n-s} z_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s}) \]

where the inequality holds because \( w_{n,s}(1) - z_{n,s}(1) \geq 0 \) and because the concavity of price implies that \( p'(\tilde{q}_s(z) + Q_1^{-s}) > p'(\tilde{q}_s(z) + Q_2^{-s}) \) (recall that \( Q_1^{-s} < Q_2^{-s} \)). If we continue this process on iterating \( j \), we end up with (37). Since the latter condition holds, we have that

\[ \sum_{j=1}^{n-s} w_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=1}^{n-s} z_{n,s}(j)p'(\tilde{q}_s(z) + Q_j^{-s}) - c > \]
\[ -\sum_{j=1}^{n-s} z_{n,s}(j)p(\tilde{q}_s(z) + Q_j^{-s}) - c + \sum_{j=1}^{n-s} w_{n,s}(j)p(\tilde{q}_s(z) + Q_j^{-s}) - c = \]
\[ \sum_{j=1}^{n-s} w_{n,s}(j)p(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=1}^{n-s} z_{n,s}(j)p(\tilde{q}_s(z) + Q_j^{-s}) \] (38)
But the last expression can be written as

\[ (w_{n,s}(1) - z_{n,s}(1)p(\tilde{q}_s(z) + Q_1^{-s})) \sum_{j=2}^{n-s} w_{n,s}(j)p(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=2}^{n-s} z_{n,s}(j)p(\tilde{q}_s(z) + Q_j^{-s}) > 0 \]

where the last inequality holds because \( Q_1^{-s} < Q_1^{-s} \). Continuing the iterations on \( j \), we end up with

\[ \sum_{j=1}^{n-s} w_{n,s}(j)p(\tilde{q}_s(z) + Q_j^{-s}) - \sum_{j=1}^{n-s} z_{n,s}(j)p(\tilde{q}_s(z) + Q_j^{-s}) > 0 \] (39)

Combining (35), (38) and (39) we conclude that \( \frac{\partial F(\tilde{q}_s(z))}{\partial q_s} > 0 \) and hence \( \tilde{q}_s(w) > \tilde{q}_s(z) \).

But then \( Q_j^{-s}(w) < Q_j^{-s}(z) \), since \( Q_j^{-s}(w) \) and \( Q_j^{-s}(z) \) emerge from \( \tilde{q}_s \) for \( q_s = \tilde{q}_s(w) \) and \( q_s = \tilde{q}_s(z) \) respectively (see footnote 13) and commodities in a Cournot market are substitutes.

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