Bulk charge distributions on integer and fractional quantum Hall plateaus

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We discuss the charge distributions across the bulk of a two-dimensional electron gas system which is on an integer or fractional quantum Hall plateau. Our analysis is based on a relation, derived from the long wavelength limit of the bulk density-density response function, between the induced charge and leading derivatives of a slowly varying Hall potential. We use the Wiener-Hopf method to solve the coupled response and Poisson equations. Unlike the integer case, treated in previous work, the induced charge in the fractional case can alternate in sign.

I. INTRODUCTION

This paper addresses the change in the charge distribution and the varying Hall field in the interior of a two-dimensional electron gas which accompanies the dissipationless current flow of the integer and fractional quantum Hall effects. This quantity is of interest, in part, because the simple picture of a constant Hall electric field generated by charges localized at the surface of the sample, familiar from the three dimensional case, cannot be replicated with two-dimensional electrostatics. In addition, given the quantized Hall conductivity, a non-constant Hall potential implies a transport current in the bulk of the sample, which is sometimes thought to be in conflict with the edge-state picture of the quantized Hall conductance. The induced charge in the bulk can, in principle, be measured, although accurate measurements on the length scales of interest have proved difficult.

Our view, on which the work presented here is explicitly dependent, is close to that carefully articulated by Thouless and coworkers in a recent series of papers. In particular, there is generically a bulk contribution to the transport current under quantum Hall conditions. In typical macroscopic Hall bar samples, bulk current tends to dominate the total current flow. This view is not in contradiction with the edge state picture of the quantized Hall conductance. The paper is organized as follows. In Section II we derive a relationship between the charge density in the bulk of a two-dimensional (2D) sample carrying a quantized Hall current and leading derivatives of the Hall potential, presumed to be slowly varying on microscopic length scales. These equations are a generalization of those first derived for the integer quantum Hall case in Ref. [1], and permit differences between integer and fractional cases to be discussed. The use of the Wiener-Hopf method, first employed for this problem by Thouless, is discussed in Section III. We use it to express the solution to our coupled response and Poisson equations in terms of a function determined by numerical quadrature. The bulk induced charge distribution depends in part on non-universal aspects of the microscopic physics at the edge of the sample where the bulk response relations do not apply. In the fractional case this leads to the appearance of a free parameter in the solution which results from the Wiener-Hopf analysis. In Section IV we discuss classes of physically sensible solutions and contrast behavior in the integer and fractional quantum Hall cases. We conclude in Section V with a brief summary of the paper.

II. BULK RESPONSE TO A SLOWLY-VARYING POTENTIAL

The quantum Hall effect occurs in a two-dimensional electron system when, at zero temperature, there is a discontinuity in the chemical potential at an electron density which depends on magnetic field strength. At the point of the chemical potential jump, the incompressible electronic ground state generally has no gapless excitations. In this section we discuss the response of electrons in such an incompressible fractional Hall state to a slowly varying static electrostatic potential. Bulk response functions are usually described in terms of their Fourier transforms:

\[ n(q) = \Pi(q) \phi(q), \]

where \( \phi(q) \) includes both the external potential and the potential due to induced charges. In a 2D system the static density-density response function is related to \( \Pi(q) \) by

\[ \chi(q) = \frac{\Pi(q)}{1 - V(q)\Pi(q)}, \]

where \( V(q) = 2\pi e^2/q \) is the 2D Coulomb interaction. The precise form we use for \( \Pi(q) \) below is based on a generalized single mode approximation for \( \chi(q) \) in which the density operator is partitioned into contributions associated with transitions between each pair of Landau levels. In this approximation, which is expected to be reasonably accurate whenever the ground state is
incompressible, we obtain to leading order \([11]\) in \((\hbar \omega_c)^{-1}\) at Landau level filling factors \(\nu \leq 1\)

\[
\chi(q) = \frac{-\nu}{\pi \ell^2} \sum_{n=0}^{\infty} \frac{s_n(q)}{\Delta_n(q)}.
\]

Here \(\Delta_n(q)\) is the collective mode energy for excitations to the \(n\)-th Landau level,

\[
s_n(q) = N^{-1}\langle 0_{\nu} \rho_{0-n}^{\nu} \rho_{0}^{n} | 0_{\nu} \rangle,
\]

\(\omega_c = eB/m^*c\) is the cyclotron frequency, \(\ell = (\hbar c/eB)^{1/2}\) is \(\nu = (2\pi \ell^2)n\) is the density in units of the density which can be accomodated by a single Landau level, \(\rho_n^{\nu,n}\). The contribution to the density operator associated with transitions between Landau levels \(n\) and \(n'\), and \(|0_{\nu}\rangle\) is the many electron ground state when interaction terms which mix Landau levels are neglected. It follows from Ref. [12] that for \(n \neq 0\)

\[
s_n(q) = \frac{\exp(-|q|^2\ell^2/2)}{n!} (q^2\ell^2/2)^n,
\]

independent of the correlations in \(|0_{\nu}\rangle\). \(s_0(q)\), on the other hand, does depend in detail on these correlations although, [13] whenever \(|0_{\nu}\rangle\) is an incompressible state, \(s_0(q) \sim q^2\) at long wavelengths; the term proportional to \(q^2\) in the long-wavelength expansion vanishes. This property is responsible for the difference we discuss below between charge distributions across the interior for integer and fractional incompressible states.

Keeping terms in up to \(q^4\) in a long-wavelength expansion and up to \((\hbar \omega_c)^{-1}\) in a strong-field expansion we obtain the following results for the electron number density in units of the density which can be accommodated by a single Landau level, \(\rho_n^{\nu,n}\).

\[
n(x) = \frac{\nu}{2\pi \hbar \omega_c} \frac{\partial^2 \phi(x)}{\partial x^2} - \frac{C \ell^2}{\epsilon^2 \ell c} \frac{\partial^4 \phi(x)}{\partial x^4}.
\]

where \(\epsilon\) is the dielectric constant of the host semiconductor. This result is obtained by noting that to the stated order in the strong field and long wavelength expansions \(\Pi(q) = \chi(q)\) and by making the replacement valid for slowly varying potentials, \(q \to -i\partial/\partial x\). The first term on the right hand side will always dominate for sufficiently slowly varying densities and was derived originally [3] via a different line of argument valid only for the special case of non-interacting electrons and integer filling factors. The present derivation has generic validity. The second term is retained [12] here because of its larger relative importance at stronger magnetic fields. The coefficient \(C\) in this term vanishes in the case of the incompressible states which occur at integer Landau level filling factors and is not accurately known for all fractional incompressible states. In the case of \(\nu = 1/3\) and \(\nu = 2/3\) it follows from Ref. [12] that

\[
C = e^2/\ell 12\pi \Delta(q \to 0) \sim 0.2.
\]

For other incompressible states with smaller gaps, its value is likely to be larger.

Comparing the two terms motivates the introduction of the length scale \(L\) on which the two terms have comparable importance:

\[
\frac{L}{\ell} = \left[ \frac{2\pi C}{\nu} \frac{\hbar \omega_c}{\epsilon^2 / \ell c} \right]^{1/2}
\]

This length scale diverges in the limit of strong fields but is not extremely long in typical experimental situations; for example for \(\nu = 1/3\) and \(B = 20\) Tesla, \(L \sim 3\ell\). The fact that the induced charge density is proportional to derivatives of the electrostatic potential rather than to the potential itself is a direct consequence of the incompressibility of the many-electron ground state. For compressible states, Thomas-Fermi screening theory gives

\[
n(x) = - (dn/d\mu) \phi(x)
\]

where \(dn/d\mu\) is the thermodynamic density of states of the uniform bulk state. This quantity is zero at zero temperature for incompressible states. However, it will \([14]\) be finite at any finite temperature. We can define a length scale \(L'\) at which the Thomas-Fermi screening term in Eq. [3] and the leading derivative term in Eq. [2] become comparable:

\[
\frac{L'}{\ell} \sim \left[ \frac{k_BT}{\Delta} \right]^{1/2} \exp(\Delta/4k_BT)
\]

where \(\Delta\) is the chemical potential jump at the density of the incompressible state. Our neglect of the Thomas-Fermi screening term implicitly assumes \(\nu'\) is longer than any length of interest so that we require that \(k_BT \ll \Delta/4\).

III. WIENER-HOPF METHOD

Eq. [3] can be rewritten using \(\ell\) as the length unit:

\[
\tilde{\rho}(\vec{x}) = \alpha \frac{\partial^2 \tilde{\phi}(\vec{x})}{\partial \vec{x}^2} - \beta \frac{\partial^4 \tilde{\phi}(\vec{x})}{\partial \vec{x}^4},
\]

where \(\tilde{\rho}(\vec{x}) = \ell^2 n(\vec{x}), \alpha = \nu/(2\pi \hbar \omega_c), \beta = C/(\epsilon^2 / \ell c) = \alpha L^2 / \ell^2\), and \(\vec{x} = x/\ell\). We will drop the tildes used to distinguish dimensionless quantities in Eq. [11] in what follows.

We choose a coordinate system where all points with \(x > 0\) are sufficiently deep in the bulk that Eq. [11] applies. The electrostatic potential appearing here is given by the Poisson equation:

\[
\phi(x) = -\lambda \int_0^\infty dx' \rho(x') \log(|x-x'|) + \phi_{ext}(x),
\]
where $\lambda = 2e^2/\epsilon \ell$. Here $\phi_{\text{ext}}(x)$ specifies the potential created by the electronic charge density for $x < 0$ and by charges external to the electron system. We will initially drop this contribution to Eq. 4; in the end it will implicitly be accounted for in the non-universal boundary conditions that must be imposed in order to fix the charge distribution.

We start by Fourier transforming equation Eq. 12:

$$\phi_-(k) + \phi_+(k) = -\lambda v(k)\rho_+(k),$$

where we define the Fourier transforms as follows:

$$v(k) = \int_{-\infty}^{\infty} dx \ e^{ikx} \log(|x|)$$

$$\rho_+(k) = \int_{0}^{\infty} dx \ e^{ikx} \rho(x)$$

$$\phi_+(k) = \int_{0}^{\infty} dx \ e^{ikx} \phi(x)$$

$$\phi_-(k) = \int_{-\infty}^{0} dx \ e^{ikx} \phi(x).$$

Application of the Wiener-Hopf method requires the continuation of the Fourier transform wavevector to the complex plane: $k = \sigma + i\tau$. To ensure convergence of spatial integrals a convergence factor, $\exp(-a|x|)$, is included in the interaction kernel, so that $v(k)$ is regular and bounded for $|\tau| < a$. $v(k)$ then has branch cuts along the portions of the imaginary axis with $|\tau| > a$. (For completeness, explicit expressions for $v(k)$ are given in the Appendix. No difficulties are encountered in setting $a \to 0$ at the end of the calculation.) The + and − subscripts in the above definitions are used to designate functions which are regular and bounded in the upper and lower halves of the complex $k$-plane, respectively. Eq. 13 contains three unknown functions. Using the response relation for $x > 0$ and integrating the half Fourier transform by parts we obtain

$$\rho_+(k) = \beta\phi''(0) - i\beta k\phi''(0) - (\alpha + \beta k^2)\phi'(0) + (ik + i\beta k^3)\phi(0) - (\alpha k^2 + \beta k^4)\phi_+(k)$$

which can be solved for $\phi_+(k)$:

$$\phi_+(k) = \frac{1}{\alpha k^2 + \beta k^4} [\beta\phi''(0) - i\beta k\phi''(0) - (\alpha + \beta k^2)\phi'(0) + (ik + i\beta k^3)\phi(0) - \rho_+(k)].$$

Here we have chosen $\phi(\infty) = 0$ and assumed that all derivatives of $\phi(x)$ vanish at $\infty$. Note that if we had retained the external potential term, the form of this equation would not have changed. Combining this with Eq. 13 yields

$$A(k)\phi_-(k) + B(k) = \rho_+(k)K(k),$$

where

$$\phi_-(k) + \phi_+(k) = -\lambda v(k)\rho_+(k),$$

$$A(k) = \alpha k^2 + \beta k^4,$$

$$B(k) = \beta\phi''(0) - i\beta k\phi''(0) - (\alpha + \beta k^2)\phi'(0) + (ik + i\beta k^3)\phi(0) + (\alpha k^2 + \beta k^4)\phi_+(k)$$

$$K(k) = 1 - \lambda A(k)v(k).$$

The essential difficulty in the application of the Wiener-Hopf method occurs in making the following factorization of $K(k)$:

$$K(k) = K_+(k)/K_-(k),$$

where $K_+(k)$ is regular and bounded in the upper half plane, $K_-(k)$ is regular and bounded in the lower half plane, and both functions share a common strip of regularity along the real line. In some cases this factorization, which is not in general unique, can be established by inspection and $K_\pm(k)$ specified analytically. Here we resort to a general method with wide applicability. The factors will be specified by a function determined by a numerical integration. The method starts by defining

$$q(k) = \ln[K(k)].$$

Since $K(k)$ has no zeroes, at least near the real line, it follows from the Cauchy integral theorem that, for any positive $\xi < a$, $q(k) = q_+(k) - q_-(k)$ where

$$q_\pm(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\chi \frac{q(\chi + i\xi)}{\chi + i\xi - k}.$$}

By construction, $q_+(k)$ is regular and bounded in the upper half plane and $q_-(k)$ in the lower half plane. This device provides the required factorization since $K(k) = K_+(k)/K_-(k)$ where $K_\pm(k) = \exp[q_\pm(k)]$.

Given the required factors we can reexpress Eq. 20 as

$$A(k)\phi_-(k) + B(k) = \rho_+(k)K_+(k)$$

The left and right hand sides of Eq. 24 are regular and bounded in the lower and upper planes, respectively, and they share a strip of regularity of width $2a$ along the real axis. It follows from analytic continuation that both sides must be equal to an as yet unknown polynomial $P(k)$, which is regular in the whole complex plane. To evaluate the induced charge density in the bulk we require $\rho_+(k)$, and hence $K_+(k)$, only along the real axis. For real $k$, choosing $\xi$ to be infinitesimal in Eq. 23 gives

$$q_\pm(\sigma) = \pm \frac{1}{2} q(\sigma) + ig(\sigma)$$

where

$$g(\sigma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log K(\chi)}{\chi - \sigma} d\chi.$$
$L/\ell = 0$, $\nu = 1/3$ ($L/\ell = 1.7$), and $\nu = 1/5$ ($L/\ell = 3$) at $B = 10$ in the three cases. For the integer case we recover $g(\sigma \to \infty) = -\pi/4$, in agreement with the analytic factorization \[4\] while for the fractional case $g(\sigma \to \infty) = -3\pi/4$; this difference can be traced to the fact that at large $\sigma$, $K(\sigma) \sim \sigma$ in the integer case and $\sim \sigma^3$ in the fractional case.

![Figure 1](image-url)  
**FIG. 1.** Numerically calculated $g(\sigma)$ for $\nu = 1$ ($L/\ell = 0$), $1/3$ ($L/\ell = 1.7$), and $1/5$ ($L/\ell = 3$) at $B = 10T$. The two fractional cases give similar results.

The Wiener-Hopf factorization leads to the following equation which is used below to calculate the induced charge density:

$$
\rho_+(\sigma) = P(\sigma)/K_+(\sigma) = P(\sigma)\{1 + \nu(\epsilon^2/\epsilon l)/(\hbar \omega_c)|\sigma[1 + \sigma^2(L/\ell)^2]\}^{-1/2} \times \exp[-i g(\sigma)],
$$

(27)

The polynomial $P(\sigma)$ can be determined by analysis of the limits $\sigma \to 0$ and $\sigma \to \infty$. In the present case the requirement that $\rho_+(\sigma)$ vanishes at large $|\sigma|$ implies that $P(\sigma) = P_0 + P_1 \sigma$ in the fractional case and that $P(\sigma) = P_0$ in the integer case where $L/\ell = 0$. Taking $\sigma \to 0$ in Eq. \[13\] relates $P_1$ and $P_1$ to $\phi(0)$ and its first three derivatives. Note that $P_1$ is pure imaginary as required in order to obtain $\rho_+(x)$ real.

**IV. INDUCED CHARGE DISTRIBUTIONS**

Inverting the Fourier transform in Eq. \[27\] gives for $x > 0$ $\rho(x) = P_0 \rho_0(x) + iP_1 \rho_1(x)$. These two contributions to the total charge are given by

$$
\rho_0(x) = \frac{1}{\pi} \int_0^{\infty} d\sigma \frac{\cos[g(\sigma) + \sigma x]}{\{1 + \nu(\epsilon^2/\epsilon l)/(\hbar \omega_c)|\sigma[1 + \sigma^2(L/\ell)^2]\}^{1/2}}
$$

(28)

Previous results \[14\] for integer plateaus are recovered by setting $P_1$ and $L$ to zero. Numerical results for $\rho_0(x)$ and $\rho_1(x)$ calculated for $\nu = 1/3$ at $B = 10T$ are shown in Fig. 2 for two values of $L/\ell$. The fact that $\rho_0(0) = 0$ follows from the analyticity of $K_+(\sigma)$ in the upper half-plane. The square root behavior at small $x$, evident in Fig. 2, follows from the fact that $g(\sigma)$ approaches a constant at large $\sigma$ and from the $\sigma^{+3/2}$ behavior of the denominator of the integrands in Eq. \[27\]. Because of the extra factor of $\sigma$ in the numerator of its integrand,

$$
\rho_1(x) = -\frac{1}{\pi} \int_0^{\infty} d\sigma \frac{\sigma \sin[g(\sigma) + \sigma x]}{\{1 + \nu(\epsilon^2/\epsilon l)/(\hbar \omega_c)|\sigma[1 + \sigma^2(L/\ell)^2]\}^{1/2}}
$$

(29)

for small $x$.

![Figure 2](image-url)  
**FIG. 2.** The two contributions, $\rho_0$ and $\rho_1$, (see text) to the total electronic density distribution for $\nu = 1/3$ at $L/\ell = 1.7$ and $L/\ell = 5$. This results were calculated for $B = 10T$.

In Fig. 3 we plot $\rho(x)/iP_1$ for a series of values of $P_0/iP_1$. The ratio $P_0/iP_1$ depends on non-universal details of the microscopic physics at the edge of the sample and cannot be determined by the present analysis. Notice that, unlike the integer case, the charge distribution can change in sign as a function of distance from the edge. The induced charge density in integer case, which also diverges as $x^{-1/2}$ for $x \to 0$, is also shown in Fig. 3 (normalized to $P_0$ in this case).
FIG. 3. Total electronic density distribution, $\rho(x)$ (normalized to $iP_1$) at $\nu = 1/3$ and $B = 10T$ ($L/\ell = 1.7$) for $P_0/iP_1 = 0.0, 0.1, 0.2, 0.3,$ and 0.5 (bottom to top). The charge distribution for the integer case (dashed line) is also shown for comparison (normalized to the coefficient $P_0$).

V. SUMMARY

The work reported in this paper is based on a new derivation of the relationship between the induced charge in the bulk of a quantum Hall sample and derivatives of a slowly varying Hall potential. The derivation is based on gradient expansion of the long-wavelength bulk response functions. It generalizes results obtained previously for the case of the integer quantum Hall effect to the case of interacting electrons, and suggests that at fractional filling factors and especially at very strong magnetic fields, a higher fourth derivative term must be included in the induced density expression. The approach taken here is more surely grounded by microscopic physics than related recent work based on Chern-Simons mean field theories, although the physical conclusions appear to be quite similar. We have solved the coupled Poisson and response equations to determine the self-consistent charge density distribution in the interior of a Hall bar on a quantized plateau. Our solution is based on the Wiener-Hopf method in which the required factorization of the response function is specified in terms of a function which is determined by numerical evaluation of an integral. We find that, in the fractional case, the induced charge density can alternate in sign in the interior of the sample.

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APPENDIX

Including a convergence factor in the integral, the Fourier transform of $v(x)$ at general complex wavevector $k$ is given by:

$$v(k) = 2 \int_0^{\infty} dx \log(x) \cos(kx)e^{-ax}$$

$$= \frac{i}{k} \log\left((1 + ik/a)/(1 - ik/a)\right)$$

For $a \to 0$ and $k = \sigma + i\tau$

$$v(k) = -\frac{\pi |\sigma|}{\sigma^2 + \tau^2} + i\frac{\pi \tau}{\sigma^2 + \tau^2} \text{Sign}(\sigma).$$

$\Re[v(k)]$ is continuous everywhere but $\Im[v(k)]$ is discontinuous along the entire imaginary axis.

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The bulk charge distribution on a fractional Hall plateau has also been discussed recently by Jun'ichi Shiraishi and Mahito Kohmoto, Phys. Rev. B 54, 17667 (1996). These authors base their work on an effective Chern-Simons theory for quantum Hall edges which leads to a bulk response relation which is similar, but not identical, to ours. In our view the present approach, which is soundly based on the known microscopic of fractional Hall states, is more reliable. See also J. Shiraishi, Y. Avishai, and M. Kohmoto, cond-mat/9710309.

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We have dropped terms proportional to \( q^4 \) in \( \Pi(q) \) which vanish for \( (\hbar \omega_c)^{-1} \to 0 \).

The plateau in the quantized Hall conductance in a disordered systems generally includes regimes in which localized charged excitations are present at the Fermi energy. The thermodynamic density of states is then non-zero. Present considerations apply only to weakly disordered systems and in particular require that the thermodynamic density of states at the mid-point of a quantum Hall plateau be small.

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