Power law violation of the area law in quantum spin chains

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The sub-volume scaling of the entanglement entropy with the system’s size, \( n \), has been a subject of vigorous study in the last decade [1]. The area law provably holds for gapped one dimensional systems [2] and it was believed to be violated by at most a factor of \( \log(n) \) in physically reasonable models such as critical systems.

In this paper, we generalize the spin−1 model of Bravyi et al [3] to all integer spin-\( s \) chains, whereby we introduce a class of exactly solvable models that are physical and exhibit signatures of criticality, yet violate the area law by a power law. The proposed Hamiltonian is local and translationally invariant in the bulk. We prove that it is frustration free and has a unique ground state. Moreover, we prove that the energy gap scales as \( n^{-c} \), where using the theory of Brownian excursions, we prove \( c \geq 2 \). This rules out the possibility of these models being described by a conformal field theory. We analytically show that the Schmidt rank grows exponentially with \( n \) and that the half-chain entanglement entropy to the leading order scales as \( \sqrt{n} \) (Eq. 16). Geometrically, the ground state is seen as a uniform superposition of all \( s \)−colored Motzkin walks. Lastly, we introduce an external field which allows us to remove the boundary terms yet retain the desired properties of the model. Our techniques for obtaining the asymptotic form of the entanglement entropy, the gap upper bound and the self-contained expositions of the combinatorial techniques, more akin to lattice paths, may be of independent interest.

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I. MANY-BODY SYSTEMS

The phase space of a classical system made up of \( n \) particles is \( 6n \) dimensional; each particle contributes 3 spatial and 3 momentum coordinates. Given the initial conditions at some time and the laws of interaction, Hamilton’s equations, in principle, can be integrated and deterministically specify the state at any other time. The state of any given particle is a function of its initial conditions (6 parameters) and the history of the fields impressed upon it from the other \( n-1 \) particles [4]. Hence one can meaningfully talk about the state of any one particle at any time irrespective of the rest of the particles.

Quantum mechanically the size of the phase (Hilbert) space is a multiplicative function of the size of each individual Hilbert spaces which grows exponentially with the number of particles

\[
\psi \rangle \in \bigotimes_{i=1}^{n} \mathcal{H}_i
\]

where \( \mathcal{H}_i \) is the Hilbert space of the \( i^{th} \) particle. Moreover, because of the superposition principle, after it interacts with other particles, for some time, a particle can end up in a state that is entangled. That is, its state for all subsequent times after the interaction may depend on the rest of the system, even if the fields impressed on it diminish by, say, taking the particles far apart. Moreover, the coordinates and momenta cannot simultaneously be specified because of the uncertainty principle. Generally speaking, one requires an exponential number of complex parameters to specify the state at any given time and the Hamiltonian evolution would then update these parameters in time, and in principle, the probabilities of outcomes can deterministically be specified.

These pose a serious challenge in understanding and simulating quantum many-body systems (QMBS); i.e., it is hard to simulate QMBS on a classical computer. One measure of how difficult this task can be, is quantification of the entanglement possessed by its states. Considering a generic state in the Hilbert space, the prospects of finding an efficient description of a random pure state are rather bleak. In the context of black hole evaporation, Page conjectured that if a quantum system of Hilbert space dimension \( mn \) is in a random pure state with respect to Haar measure, then the average entropy of a subsystem of dimension \( m \leq n \) is \( S_{m,n} = (\sum_{k=n+1}^{mn} 1/k) - (m-1)/2n \) [5]. The conjecture was subsequently proved [6, 7]. In the context of a quantum lattice model, a random quantum state drawn from the Haar measure, yields an expected entropy that is extensive (i.e., scales with volume) [1]. Moreover, Hayden et al showed that generically pure states in the Hilbert space are close to maximally entangled [8]. Later Cubitt et al showed that the statement also holds in a Schmidt rank sense [9].

In the last three decades we have come to realize that most physical systems can be well approximated by a far smaller number of parameters, which means that the ground state of interesting physical models are non-generic. It seems that the relevant degrees of freedom for QMBS live in a small subset of the Hilbert space. One can imagine that any given problem has inherent constraints such as underlying symmetries, locality of interaction, etc. that restrict the states to reside on special sub-manifolds.
Since the AKLT model we have come to believe that one dimensional systems are typically easy [10]. Later density matrix renormalization group (DMRG) and its natural representation by matrix product states (MPS) gave successful and systematic recipes for truncating the Hilbert space based on ignoring zero and small singular values in specifying the states [11, 12]. In particular, for gapped one dimensional systems, the MPS ansatz, where the bond dimension is a constant, suffices [2].

DMRG and MPS have been tremendously successful in practice for capturing the properties of QMBS in physics and chemistry [13–16]. One wonders about the limitations of DMRG for one dimensional systems. In particular, are there “reasonable systems” that violate the area law (see below) with a pre-factor that scales faster than \(\log(n)\), which is the correction one expects from relativistic conformal field theory in the context of critical systems [17]? In relativistic conformal field theory the energy gap must scales as \(1/n\).

### A. Quantum criticality, area laws and entanglement entropy

In the last two decades, condensed matter physics has revealed a new type of phase transition that is driven, not by thermal motion, but by quantum fluctuations, i.e., zero point motions due to Heisenberg’s uncertainty principle (see [18] for a review). The hallmark of criticality is divergence of correlation length; in quantum critical systems the order parameter field depends on (imaginary) time as well as space [19]. In quantum systems, criticality is often a byproduct of a vanishing energy gap or long-range entanglement [1 pp. 5-6], where by a gapped system one means that the energy to the first excited state \(\Delta E \geq c\) as the system size tends to infinity, where \(c\) is a positive constant [1].

The interactions in QMBS are usually, to a good approximation, local [20]. This has been exploited to give the so called area laws where locality of interaction gives sub-volume scaling of the entanglement entropy (see [1] for a review). The rigorous proof of an general area law does not exist; however, Hastings proved that it holds for one dimensional gapped systems [2]. Later Wolf et al showed that a finite correlation length, measured in terms of mutual information, implies an area law and that an area law always holds at finite temperature [21]. One expects that an exponential decay of correlations would imply an area law, but the data hiding states seemed to challenge this intuition [22, 23]. Nevertheless, Brandão and Horodecki showed that in 1D, exponential decay of correlations implies an area law [24].

Swingle and Senthil [23], based on scaling arguments, argued that “physically reasonable” local Hamiltonians with unique ground states can violate the area law by at most a \(\log(n)\) factor, which for a spin chain implies that \(\log(n)\) is the maximum expected entanglement entropy scaling. Physically reasonable essentially means not fine-tuned; for example, translational invariance or robustness against perturbations qualify as physically reasonable. There are various examples of fine-tuned Hamiltonians that have larger, even linear, scaling of entanglement entropy with the system’s size (see below). Previously, a rigorous example of a spin–1 chain, translationally invariant, local Hamiltonian with a unique ground state that saturates the \(\log(n)\) bound was proposed [3]. Here we show that the generalization of this model to integer spin \(s > 1\) gives entanglement entropies that, to leading order, scale as \(\sqrt{n}\). Moreover, the other properties such as the uniqueness, frustration freeness of the ground state, locality and translational invariance of the Hamiltonian are preserved.

### B. Frustration free interactions

The Hamiltonian that we will define in this work is frustration free (FF). We say the ground

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1 All finite Hamiltonians are gapped as they have a finite number of eigenvalues (ignoring degeneracies of the ground state). Vanishing of the spectral gap is necessary for a quantum phase transition to occur.
state of a Hamiltonian is FF or *unfrustrated* when it is also a common ground state of all of the local terms.

FF condition is a desirable property to prove as it imposes a local constraint on the ground state(s) and often simplifies the mathematical description of the ground state(s). There are many models such as the Heisenberg ferromagnetic chain, AKLT, and the parent Hamiltonians of MPS that are FF \[10,11,26,27\]. In addition, Hastings proved that gapped local Hamiltonians can be approximated by FF Hamiltonians if one allows for the range of interaction to be \(L \sim \log^2 n\) \[28\].

When a model Hamiltonian is FF, one almost always subtracts the ground state energy from the Hamiltonian to make the ground state energy zero. Then, if one mainly cares about the ground state, the eigenvalues of the excited states can be taken to be one, whereby the local terms become projectors. Therefore, a nice amenity of FF systems is that the ground state is stable against perturbative variations of the Hamiltonian of the form \(H(g) = \sum_k g_k H_{k,k+1}, g_k \geq 0\) \[29\].

Moreover, FF condition naturally generalizes the concept of satisfiability (SAT) in classical complexity theory to the quantum setting \[30\]. The classical SAT problem was generalized by Bravyi \[31\] to the so called quantum SAT or qSAT. The statement of the qSAT problem is: Given a collection of \(m\)–local projectors on \(n\) qubits, is there a state \(\psi\rangle\) that is annihilated by all the projectors? This is equivalent to asking whether the system is FF.

Lastly a physical motivation was given by Verstraete et al \[32\] where they showed that the ground states of FF Hamiltonians can be prepared by dissipation.

II. SUMMARY OF THE RESULTS

A. Previous results

Irani \[33\] showed that for \(d = 21\), it is possible to design translationally invariant Hamiltonians with a gap that is polynomially small, such that the entanglement entropies scale linearly with the size of the subsystem. The construction has a degenerate ground state and is not robust against perturbations. Gottesman and Hastings \[34\] also found an example with \(d = 9\) and a polynomially small gap whose entanglement entropy in some regions scales linearly with the system’s size. In their construction the ground state is not FF, nor is the system translationally invariant.

Bravyi et al \[3\] proposed the first example of a FF translationally invariant spin–1 chain with a local Hamiltonian that has a unique ground state and non-trivial entanglement. The ground state can be seen as a uniform superposition of all Motzkin walks. It was found that the Schmidt rank is \(\chi = n + 1\) and that the entanglement entropy

\[
S(p_m) \approx \frac{1}{2} \log n + \gamma - \frac{1}{2} + \frac{1}{2} (\log 2 + \log \pi - \log 3) \quad \text{nats} \\
\approx \frac{1}{2} \log_2 n + 0.6446547 \quad \text{bits,}
\]

where \(\gamma\) is the Euler constant. Moreover, they proved that the Hamiltonian is gapless and that the energy gap is a polynomial in \(1/n\). In particular, they showed that the upper bound on the gap is \(O(n^{-1/2})\) and the lower bound \(O(n^{-c})\) for some \(c \gg 1\). The exact exponent is not known; however, numerically it seemed to scale as \(\sim n^{-\nu}\), where \(\nu \approx 3\).

B. Main results and organization of this paper

Here we generalize the previous results to all integer spin-\(s\) chains whereby we give a class of such
exactly solvable models. The unique ground state, denoted by \( \mathcal{M}_{2n,s} \), is a uniform superposition of all \( s \)-colored Motzkin walks or equivalently all balanced strings of \( s \) different types of left and right parentheses separated by empty spaces (Section [III]). Our main theorem is:

**Theorem 1.** The colored Motzkin state \( \mathcal{M}_{2n,s} \) is the unique ground state of a spin–\( s \) chain with a local, translationally invariant, FF Hamiltonian. The ground state energy is zero. The spectral gap of the Hamiltonian scales as \( n^{-c} \) where \( c \geq 2 \). The ground state \( \mathcal{M}_{2n,s} \) has a Schmidt rank \( \chi \) and a half-chain entanglement entropy \( S \) that, respectively, are

\[
\chi = \frac{s^{n+1} - 1}{s - 1} \\
S = 2 \log (s) \sqrt{\frac{2\sigma}{\pi}} \sqrt{n} + \frac{1}{2} \log n + \gamma - \frac{1}{2} + \frac{1}{2} (\log 2 + \log \pi + \log \sigma) \text{ nats}
\]

where \( \sigma = \frac{\sqrt{2}}{2\sqrt{s+1}} \) and \( \gamma \) is the Euler constant.

Note that \( s = 1 \) recovers the previous formulas for the Schmidt rank and entanglement entropy. The organization of the paper is as follows.

In Section [III] we
1. Provide a self-contained exposition of the underlying combinatorics of the Motzkin state.
2. Evaluate the Schmidt numbers exactly using Andre’s reflection principle.
3. Calculate the exact and asymptotic values of the Schmidt rank and the entanglement entropy.

In Section [IV] we give the local Hamiltonian whose ground state is \( \mathcal{M}_{2n,s} \) and prove
1. \( H \) is FF and \( \mathcal{M}_{2n,s} \) is the unique ground state.
2. \( O(n^{-2}) \) upper bound on the spectral gap applicable for \( s \geq 1 \).
3. \( n^{-c} \) with a constant \( c \gg 1 \) lower bound on the spectral gap of the ground state.

The tools we use for the upper bound are the theories of Brownian excursions and universality of Brownian motion. For the lower bound we use the projection lemma (i.e., perturbation theory) and statistics of Dyck walks and properties of Markov processes. We present a self contained exposition of these tools and prove the lower bound by proving rapid mixing of an underlying Markov chain using the canonical path method.

### III. THE GROUND STATE AND ITS ENTANGLEMENT

In this section, to adequately describe the ground state, we first introduce the necessary combinatorial background. For now, we only need the state and can postpone the description of the local Hamiltonian that has \( \mathcal{M}_{2n,s} \) (defined below) as its unique ground state to Section [IV].

#### A. Combinatorics: Dyck paths, Catalan numbers and Motzkin walks

**Definition 1.** A Dyck walk (or path) of length \( 2n \) in the \((x, y)\) plane is any path from \((0, 0)\) to \((0, 2n)\) with steps \((1, 1)\) and \((1, -1)\), that never pass below the x-axis. [35] p. 173
The number of all such walks is counted by the Catalan number 
\[ C_n = \frac{1}{n+1} \binom{2n}{n}. \]

Catalan numbers are famous numbers in combinatorics. At the time of writing 212 different combinatorial problems whose solutions are counted by the Catalan numbers have been catalogued by Richard Stanley [36].

We mention in passing that \( \lim_{n \to \infty} C_n \approx \frac{4^n}{\sqrt{\pi n^3}} \), which will be used later to prove the optimality of the proposed canonical path.

Closely related paths are the Motzkin walks [35, p. 238, prob. 6.38]:

**Definition 2.** A Motzkin walk (or path) on \( 2n \) steps is any path from \((0, 0)\) to \((0, 2n)\) with steps \((1, 0)\), \((1, 1)\) and \((1, -1)\) that never pass below the \(x\)-axis.

Comment: A Motzkin walk on \(2n\) steps is a walk, made up of three types of steps: diagonal up, diagonal down and flat. The walker starts at \((x, y) = (0, 0)\) and ends at \((x, y) = (2n, 0)\) such that the walker’s position at any intermediate lattice point \((x_k, y_k)\) has \(y_k \geq 0\) for all \(0 \leq k \leq 2n\). The number of all such walks is counted by the Motzkin numbers \(M_{2n} = \sum_{k=0}^{n} \binom{2n}{2k} C_k\). Please see Fig. 1.

Any Motzkin walk can equivalently be thought of as a string \(t \in \{ (, ), 0 \}^{2n}\) such that the string has a balanced set of right and left parenthesis with some number of zeros in between. Examples of such strings are \((\cdots ( ) \cdots ))\) that are in one to one correspondence with walks that have \(n\) up steps followed by \(n\) down steps or \((0)(0)\cdots(0)\) which corresponds to a zig-zag walk where a step up, a flat step, and step down is concatenated \(2n/3\) times or \(00\cdots0\) which is a flat walk where the position of the walker at any step is \(y_k = 0\) for all \(0 \leq k \leq 2n\).

Now every step up in a Dyck or Motzkin walk has a 'matching' step down to balance it to ultimately give \(y_{2n} = 0\). Suppose there are \(s\) colors available, then a Dyck walk of length \(2m\) can be colored \(s^n\) different ways. For example, take \(s = 2\) then a walk can be \(n\) steps upward with alternating colors of blue and red which then will uniquely determine the coloring of the remaining \(n\) down steps. Combinatorially equivalent to this walk is the string \([ ( ( \cdots [ ( ) \cdots ] ) \cdots ) ]\) where each left parenthesis/bracket has a unique right match.

Any Motzkin walk initially has coordinates \((x, y) = (0, 0)\). In the middle of the chain it will have a coordinate \((x, y) = (m, n)\) for some \(0 \leq m \leq n\); here we denote \(m\) to be the “height” in the middle. To calculate the entropy of a half chain we will need to count the number of Motzkin walks that start at zero \((x, y) = (0, 0)\) and reach height \(m\) in the middle. A theorem due to André (1887) counts related (Dyck like) lattice paths [37] p. 8.

**Theorem.** *(Ballot problem)* Let \(a, b\) be integers satisfying \(1 \leq b \leq a\). The number of lattice paths \(N(p)\) joining the origin \(O\) to the point \((a, b)\) and not touching the diagonal \(x = y\) except at \(O\) is given by

\[ N(p) = \frac{a - b}{a + b} \binom{a + b}{b} \]  

(2)

In other words, given a ballot at the end of which candidates \(P, Q\) obtain \(a, b\) votes respectively, the probability that \(P\) leads \(Q\) throughout the counting of votes is \(\frac{a-b}{a+b}\).

First note that \(a = b + 1\) gives the Catalan numbers. This theorem can also be interpreted as counting the number of Dyck walks that reach a given height for some fixed \(x\)-coordinate.
Figure 1: A Motzkin walk of length $2n$ with $s = 1$. There are $M_{n,m}^2$ such walks with coordinates $(x, y)$: $(0,0), (n, m), (2n, 0)$

$s = 2$

\[
\ell^1\rangle: ( \\
\ell^2\rangle: [ \\
0 \rangle: 0 \\
r^1\rangle: ) \\
r^2\rangle: ]
\]

Figure 2: Labeling the states for $s = 2$.

What is the corresponding count of the height of the Motzkin walks of length $2n$ in the middle (i.e., $x = n$)? Suppose on the half chain, the Motzkin walk has $k$ zeros. The remaining $n - k$ steps in this walk are made up of up and down steps. Let the total number of unmatched left parenthesis be $0 \leq m \leq (n - k)$. Clearly there are $\binom{n}{k}$ ways to put the zeros and there are $n - k - m$ matched parenthesis. Hence, there are a total of $\frac{n-k-m}{2}$ matching pairs of parenthesis on the first $n$ qudits and $m$ unmatched ones, which, for the walk to be a Motzkin walk, will be matched on the second half of the chain.

There are $s^{\frac{n-k-m}{2}}$ ways of coloring the matched pairs on the half chain and $s^m$ ways to color the remaining unmatched up steps. We denote the number of these walks by $M_{n,m,s}^2$; i.e., the total number of micro-states on the left half chain is

\[
\sum_{k=0}^{n-m} \binom{n}{k} s^{\frac{n-k-m}{2}} B_{n-k,m}s^m = s^m M_{n,m,s}
\]

\footnote{Not to be confused with the Motzkin numbers $M_{n,s}$.}
where $B_{n-k,m}$ is the solution of the Ballot problem with height $m$ on $n-k$ walks.

Clearly, the Motzkin walk on the second half starts from height $m$ and will eventually reach coordinates $(x, y) = (2n, 0)$. Therefore, for every walk on the left half chain that reaches the height $m$, there are $M_{n,m,s}$ corresponding walks on the right half that bring it down to zero, i.e., $(x, y) = (2n, 0)$. See Fig. 3 for an example of a 2-color Motzkin walk. Any choice of coloring of the $m$ unmatched left parenthesis on the first half of the chain, uniquely determines the coloring of the second half. Therefore the total number of $s$-colored Motzkin walks reaching height $m$ is $s^m M_{n,m,s}$ and the total number of $s$-colored Motzkin walks of length $2n$ is $N_{n,s} \equiv \sum_{m=0}^{n} s^m M_{n,m,s}$.

In Eq. 2 after using $a + b = m - k + 1$, $a - b = m + 1$ and letting $k \to n - m - 2i$ to take care of parity,

$$B_{n-k,m} = \frac{m + 1}{n - k + 1} \left( \frac{n - k + 1}{2} \right) = \binom{2i + m}{i} - \binom{2i + m}{i - 1}. \quad (4)$$

We substitute this into Eq. 3

$$M_{n,m,s} = \sum_{i=0}^{(n-m)/2} \binom{n}{2i + m} \left[ \binom{2i + m}{i} - \binom{2i + m}{i - 1} \right] s^i \quad (5)$$

$$= (m+1) \sum_{i \geq 0} \frac{(n)!}{(i+m+1)! (n-2i-m)!} s^i$$

$$= \frac{m+1}{n+1} \sum_{i \geq 0} \binom{n+1}{i} \binom{n-2i-m}{i} s^i$$

$$\equiv \sum_{i \geq 0} M_{n,m,s,i}.$$
B. The $s$–colored Motzkin state

For an integer spin-$s$ chain we can label the $d = 2s + 1$ spin states (see Fig. 2 for an example) by generalizing $[3]$ to allow for $s$ types of parenthesis labeled by $\{0, \ell^1, \ell^2, \ldots, \ell^s, r^1, r^2, \ldots, r^s\}$ where $\ell$ means a left parenthesis and $r$ a right parenthesis. We distinguish each type of parenthesis by associating a color from the $s$ colors shown as superscripts on $\ell$ and $r$.

**Definition 3.** The $s$–colored Motzkin state $\mathcal{M}_{2n,s}$ is the uniform superposition of all $s$ colorings of Motzkin walks on $2n$ steps:

$$\mathcal{M}_{2n,s} = \frac{1}{\sqrt{M_{2n,s}}} \sum_{\text{all } s \text{–colored Motzkin walks}} m_p \rangle$$

where $m_p$ is an $s$–colored Motzkin walk and $M_{2n,s}$ is the colored Motzkin number.

**Remark 1.** The nonzero heights in the middle or the number of unmatched parenthesis is the source of mutual information between the two halves of the chain and the source of long-range entanglement entropy of the half chain. For every Motzkin walk reaching height $m$, there are $s^m$ eigenvalues each of size $\frac{M_{2n,m}}{N_{n,s}}$.

The Schmidt decomposition of the ground state in the middle of the chain gives

$$\mathcal{M}_{2n,s} = \sum_{m=0}^{\infty} \sqrt{p_{n,m,s}} \sum_{x \in \{\ell^1, \ell^2, \ldots, \ell^s\}^m} \hat{C}_{0,m,x} \rangle_{1, \ldots, n} \otimes \hat{C}_{m,0,\bar{x}} \rangle_{n+1, \ldots, 2n}$$

where $\hat{C}_{p,q,x}$ is a uniform superposition of all strings in $\{0, \ell^1, \ldots, \ell^s, r^1, \ldots, r^s\}^n$ with $p$ excess right, $q$ excess left parentheses and a particular choice of coloring $x$ of the unmatched parentheses. For every $x$ there is a unique $\bar{x}$ matching set on the second half of the chain which is its mirror image. For example if $s = 2$, one could have $x = \ell^1\ell^2\ell^2$ in which case $\bar{x} = r^2\ell^2\ell^1$.

C. Schmidt rank and entanglement entropy

We now turn to the calculation of the entanglement entropy of the half chain in the ground state. The Schmidt numbers are

$$p_{n,m,s} = \frac{M_{n,m,s}^2}{N_{n,s}}, \quad N_{n,s} = \sum_{m=0}^{n} s^m M_{n,m,s}^2$$

and the entanglement entropy is given by

$$S(\{p_{n,m,s}\}) = -\sum_{m=0}^{n} s^m p_{n,m,s} \log_2 p_{n,m,s}.$$
The Schmidt rank is $\frac{s^{n+1}-1}{s-1} \approx \frac{s^n+1}{s-1}$ because of the geometric sum on $s^m$.

We are interested in asymptotic scaling of $S(\{p_{n,m,s}\})$ with the system size. To this end, we shall in what follows, use tools of asymptotic expansions to evaluate $S(\{p_{n,m,s}\})$.

Let’s look more carefully at

$$M_{n,m,s,i} = (m+1) \left( i + m + 1 \right)^{n-i} \left( n - 2i - m \right)^{s^i}. \quad (9)$$

If it has a saddle point in the $(m,i)$-plane, the point must simultaneously satisfy

$$\frac{M_{n,m,s,i+1}}{M_{n,m,s,i}} = 1, \quad \frac{M_{n,m+1,s,i}}{M_{n,m,s,i}} = 1.$$

The condition $\frac{M_{n,m,s,i+1}}{M_{n,m,s,i}} = 1$ gives $s(n - 2i - m)^2 - i(i + m) \approx 0$, yet $\frac{M_{n,m+1,s,i}}{M_{n,m,s,i}} = 1$ has its maximum at $m = 0$. In solving for $i$, there are two roots; we choose the one that is consistent with the $s = 1$ result, where $i_{sp} \approx \frac{n}{3}$,

$$i_{sp} = \sigma n - \frac{m}{2} + \frac{m}{8\sqrt{s}} \left( \frac{m}{n} \right) + \frac{(4s-1)m}{128s\sqrt{s}} \left( \frac{m}{n} \right)^3 + O \left( n \left( \frac{m}{n} \right)^5 \right) \quad (10)$$

$$\approx \sigma n - \frac{m}{2} + \frac{m}{8\sqrt{s}} \left( \frac{m}{n} \right), \quad \sigma \equiv \frac{\sqrt{s}}{2\sqrt{s} + 1}.$$

Before getting an asymptotic expansion for Eq. [9] we consider an example. We will analyze a trinomial coefficient, where $x + y + z = 0$ (noting that $1 - 2\sigma = \sigma/\sqrt{s}$)

$$\binom{n}{\sigma n + x \quad \sigma n + y \quad (1 - 2\sigma)n + z} \approx \frac{2\pi n}{\sqrt{8\pi^3 (\sigma n + x)(\sigma n + y)(\sigma \sqrt{s})}} \frac{n}{n+y/\sigma} (\frac{n}{n+x/\sigma}) \frac{n\sigma+y}{n\sigma+x} \frac{n\sigma+z}{n\sigma} \frac{n\sigma/\sqrt{s}+z}{n} \quad (11)$$

But,

$$\left( \frac{n}{n+x/\sigma} \right)^{\sigma n + x} = \exp \left\{ - (\sigma n + x) \log \left( 1 + \frac{x}{n\sigma} \right) \right\}$$

$$\approx \exp \left\{ - (\sigma n + x) \left( \frac{x}{n\sigma} - \frac{1}{2} \left( \frac{x}{n\sigma} \right)^2 \right) \right\}$$

$$\approx \exp \left\{ -x - \frac{x^2}{2\sigma n} \right\}.$$
\[
\left( \frac{n}{n + z \sqrt{s}/\sigma} \right)^{\sigma n / \sqrt{s} + z} = \exp \left\{ - \left( \frac{\sigma n}{\sqrt{s}} + z \right) \log \left( 1 + \frac{z \sqrt{s}}{n \sigma} \right) \right\}
\]
\[
\approx \exp \left\{ - \left( \frac{\sigma n}{\sqrt{s}} + z \right) \left( \frac{z \sqrt{s}}{n \sigma} - \frac{1}{2} \left( \frac{z \sqrt{s}}{n \sigma} \right)^2 \right) \right\}
\]
\[
\approx \exp \left\{ - z - \frac{z^2 \sqrt{s}}{2 \sigma n} \right\};
\]

clearly, the expression for \( \left( \frac{n}{n + y / \sigma} \right)^{\sigma n + y} \) \( \approx \exp \left\{ - y - \frac{y^2}{2 \sigma n} \right\}. \) In Eq. 11, inside the square root is approximately \( \sqrt{\frac{2 \pi}{8 \pi^3 \sigma^2 \left( \frac{\sigma}{\sqrt{s}} \right)^n}} \approx \frac{s^{1/4}}{2 \pi \sigma n^{3/2}}. \) Since \( x + y + z = 0, \)
\[
\left( \frac{\sigma n + x}{\sigma n + y} \left( 1 - 2 \sigma \right) n + z \right) \approx \frac{s^{1/4}}{2 \pi n \sigma^{3/2}} \left( \frac{\sigma \sqrt{s}}{\sigma} \right)^n \frac{s^2}{\sigma} \exp \left\{ - \frac{x^2 + y^2 + \sqrt{s} z^2}{2 \sigma n} \right\} \quad (12)
\]

Now we use this result to evaluate Eq. 9 by letting \( i + m = \sigma n + x, i = \sigma n + y \) and \( n - 2i - m = (1 - 2 \sigma) n + z. \) Since the standard of deviation of multinomial distributions scales as \( \sqrt{n}, \) to get a better asymptotic form, we let \( i = i_{sp} + \beta \sqrt{n} \) and \( m = \alpha \sqrt{n}. \) Hence we identify,
\[
x = \left( \beta + \frac{\alpha}{2} \right) \sqrt{n} + \frac{\alpha^2}{8 \sqrt{s}}
\]
\[
y = \left( \beta - \frac{\alpha}{2} \right) \sqrt{n} + \frac{\alpha^2}{8 \sqrt{s}}
\]
\[
z = -2 \beta \sqrt{n} - \frac{\alpha^2}{4 \sqrt{s}}
\]

Making these substitutions we get \( - \frac{x^2 + y^2 + \sqrt{s} z^2}{2 \sigma n} = - \frac{\alpha^2}{4 \sigma} - \frac{\sqrt{s} \beta^2}{\sigma^2} - O \left( n^{-1/2} \right) \) and \( s^{1/2} = s^{\sigma n - \alpha \sqrt{n}}. \) Therefore, using Eq. 12, Eq. 9 becomes

\[
M_{n,m,i,s} = \frac{(m + 1)}{n + 1} \left( \frac{i + m + 1}{i} \frac{n + 1}{n - 2i - m} \right) \approx 
\]
\[
M(n, s, \alpha, \beta) = \frac{(\alpha \sqrt{n})}{2 \pi n \sigma^{3/2}} \left( \frac{\sigma \sqrt{s}}{\sigma} \right)^n \frac{s^{\sigma n - \alpha \sqrt{n}}}{\sigma} \exp \left\{ - \frac{\alpha^2}{4 \sigma} - \frac{\sqrt{s} \beta^2}{\sigma^2} \right\}
\]

We need to evaluate \( M_{n,m,i,s} \) from \( i = 0 \) to \( i = n. \) We approximate this by an integral over \( i \) from 0 to \( \infty. \) Since \( i = \sigma n + \beta \sqrt{n}, \) we have \( di = \sqrt{n} \, d\beta. \) Since the maximum is away from the boundaries we can extend the integration limit to \(-\infty.\) Noting that \( \left( \frac{\sqrt{s}}{\sigma} \right)^n \approx \left( \frac{\sqrt{s}}{\sigma} \right)^n \approx \frac{1}{2 \sqrt{\pi n^3 / 2} \, \sigma^{1/2}} \alpha \, s^{-\alpha \sqrt{n}/2} \) \( \exp \left\{ - \frac{\alpha^2}{4 \sigma} \right\} \), the integration over \( \beta \) gives

\[
M(n, s, \alpha) = \int d\beta \, M(n, s, \alpha, \beta) = \frac{1}{2 \sqrt{\pi n^3 / 2} \, \sigma^{1/2}} \left( \frac{\sqrt{s}}{\sigma} \right)^n \alpha \, s^{-\alpha \sqrt{n}/2} \exp \left\{ - \frac{\alpha^2}{4 \sigma} \right\} \quad (13)
\]
Recall that \( m = \alpha \sqrt{n} \), hence \( s^m M_{n,m,s}^2 \) appearing in Eq. [8] has an extreme point when

\[
\frac{d}{d\alpha} \alpha^2 \exp \left[ -\frac{\alpha^2}{2\sigma} \right] = 0 ,
\]

This happens for \( \alpha = \pm \sqrt{2\sigma} \); we clearly need to take the positive root. For \( s = 1 \), \( \alpha = \sqrt{2/3} \) recovers the previous result [3].

Comment: We pause to interpret the nullity of probability at the minimum \( \alpha = m = 0 \). This corresponds to concatenation of two uniform superpositions of all Motzkin walks in \( n \) steps. Since either half is balanced by itself, this term is not a source of mutual information between the two halves and does not contribute to the entanglement entropy.

We now determine the entropy of the probability distribution \( p_{n,m,s} \). After substituting \( \alpha = m/\sqrt{n} \) in Eq. [13] and noting that normalizations cancel,

\[
S(p_{n,m,s}) = -\frac{1}{T} \sum_{m=0}^{n} \frac{m^2}{n} \exp \left( -\frac{1}{2\sigma} \frac{m^2}{n} \right) \log \left[ \frac{1}{T'} \frac{m^2}{n} s^{-m} \exp \left( -\frac{1}{2\sigma} \frac{m^2}{n} \right) \right]
\]

\[
T = \sum_{m=0}^{n} \frac{m^2}{n} \exp \left( -\frac{1}{2\sigma} \frac{m^2}{n} \right) .
\]

We can approximate this with an integral

\[
S(p_{n,m,s}) \approx -\frac{1}{T'} \int_{m=0}^{\infty} \frac{m^2}{n} \exp \left( -\frac{1}{2\sigma} \frac{m^2}{n} \right) \log \left[ \frac{1}{T''} \frac{m^2}{n} s^{-m} \exp \left( -\frac{1}{2\sigma} \frac{m^2}{n} \right) \right]
\]

\[
T' \approx \int_{m=0}^{\infty} \frac{m^2}{n} \exp \left( -\frac{1}{2\sigma} \frac{m^2}{n} \right) .
\]

In these integrals we restore the substitution \( m = \alpha \sqrt{n} \) to obtain

\[
S(p_{n,m,s}) \approx \frac{1}{2} \log n - \frac{1}{T''} \int_{\alpha=0}^{\infty} d\alpha \ \alpha^2 \exp \left( -\frac{\alpha^2}{2\sigma} \right) \log \left[ \frac{1}{T''} \alpha^2 s^{-\alpha \sqrt{n}} \exp \left( -\frac{\alpha^2}{2\sigma} \right) \right]
\]

\[
T'' = \int_{\alpha=0}^{\infty} d\alpha \ \alpha^2 \exp \left( -\frac{\alpha^2}{2\sigma} \right) ;
\]
the factor $\frac{1}{2} \log n$ occurs because $T' = \sqrt{n} T''$. Therefore,

$$S (\{p_{n,m,s}\}) \approx \frac{1}{2} \log n + \frac{\sqrt{n}}{T''} \log s \int_{\alpha=0}^{\infty} d\alpha \, \alpha^3 \exp \left(-\frac{\alpha^2}{2\sigma}\right) - \frac{1}{T''} \int_{\alpha=0}^{\infty} d\alpha \alpha^2 \exp \left(-\frac{\alpha^2}{2\sigma}\right) \log \left[ \frac{1}{T''} \alpha^2 \exp \left(-\frac{\alpha^2}{2\sigma}\right) \right]$$

The two remaining integrals are just numbers and we can calculate them to obtain the final result

$$S (\{p_{n,m,s}\}) \approx 2 \log (s) \sqrt{\frac{2\sigma}{\pi}} \sqrt{n} + \frac{1}{2} \log n + \gamma - \frac{1}{2} + \frac{1}{2} (\log 2 + \log \pi + \log \sigma) \quad \text{nats} \quad (16)$$

$$= 2 \log_2 (s) \sqrt{\frac{2\sigma}{\pi}} \sqrt{n} + \frac{1}{2} \log_2 n + \left(\gamma - \frac{1}{2}\right) \log_2 e + \frac{1}{2} (1 + \log_2 \pi + \log_2 \sigma) \quad \text{bits.}$$

where $\gamma$ is the Euler gamma number. Note that $s=1$ exactly recovers the previous result (Eq. 1).

Figure 5: Red dots: the exact sum given by Eqs. 5, 7 and 8. Black curve: the asymptotic form (Eq. 16).

Figs. 4 and 5, compare the exact sum given by Eqs. 8 using 5 and 7, with the asymptotic result given by Eq. 16. Fig. 6 shows the ratio of the exact sum (Eq. 8) with the asymptotic formula (Eq. 16).
IV. THE LOCAL HAMILTONIAN AND ITS GAP

A. The Hamiltonian

To build a FF local Hamiltonian whose ground state is $M_{2n,s}$, we first give a local description of the colored Motzkin walks. As in [3], we say two strings $u$ and $v$ are equivalent, denoted by $u \sim v$, if $u$ can be obtained from $v$ by a sequence of following local moves:

$$
\begin{align*}
0r^k & \leftrightarrow r^k0 \\
0\ell^k & \leftrightarrow \ell^k0 \\
00 & \leftrightarrow \ell^k r^k \quad \forall \, k = 1, \ldots, s
\end{align*}
$$

these moves can be applied to any consecutive pair of letters.

Under local moves the matched pairs annihilate one by one and ultimately the string would have some number of excess unmatched right and/or left parenthesis as well as potentially some crossed pairings. For example, below are examples of such “unbalanced” strings when $d = 5$

$$
\begin{align*}
& r^1 r^2 0 \cdots 0 \ell^2 \ell^1 \quad \text{unmatched} \\
& \ell^1 r^2 0 \cdots 0 \ell^2 r^1 \quad \text{crossed} \\
& r^2 r^2 0 \cdots 0 \ell^1 r^2 r^1 0 \cdots 0 \ell^1 \quad \text{unmatched and crossed}
\end{align*}
$$

The equivalent classes of strings that we introduced previously [3] are more complicated now, because now strings can get ‘jammed’ in various ways under local moves. However, any string except the Motzkin path will have a minimum nonzero Hamming weight under local moves.

**Lemma 1.** A string $u$ is a Motzkin path iff it is equivalent to the string of all zeros.

**Proof.** Under local moves all matched pairs annihilate and there will be no substrings with $\ell^k r^k$ or $\ell^k 0 \cdots 0 \ell^k$. If this is the case, and $u$ contains at least one left parenthesis of any color, then we can focus on the rightmost one and denote it by $\ell^k$. We can apply local moves such that there are no
zeros to the right of \( \ell^k \). Then \( \ell^k \) will either be the rightmost letter of \( u \) or it will be followed to its right by an \( r^i \) with \( i \neq k \). In either case the Hamming weight of the string is at least 1. Similarly if after the local moves, the string \( u \) contains at least one right parenthesis, then we can pick the leftmost one and use similar reasoning to show that under the local moves the minimum Hamming weight is at least 1. The only strings that are equivalent to the zero-Hamming weight string \( 0^{2n} \) are the colored Motzkin walks.

We take the ground state to be the uniform superposition of all \( s \)-colored Motzkin walks, i.e., strings \( u \in \{0, \ell^1, \ell^2, \ldots, \ell^s, r^1, \ldots, r^s\}^{2n} \) that are equivalent to \( u_0 = 0^{2n} \). For example, on two qudits, the ground state is

\[
\sim \{00\rangle + \sum_{k=1}^{s} \ell_k r_k \rangle \}
\]

We design a local FF Hamiltonian with projectors as interactions that 'implement' these moves. The Hamiltonian is

\[
H = \Pi_{\text{boundary}} + \sum_{j=1}^{2n-1} \Pi_{j,j+1} + \sum_{j=1}^{2n-1} \Pi_{j,j+1}^{\text{cross}}
\]

where

\[
\Pi_{\text{boundary}} \equiv \sum_{k=1}^{s} \left[ r^k \rangle_1 \langle r^k + \ell^k \rangle_2 n \langle \ell^k \right]
\]

\[
\Pi_{j,j+1} \equiv \sum_{k=1}^{s} \left[ R^k_{j,j+1} \langle R^k + L^k_{j,j+1} \langle L^k + \varphi^k_{j,j+1} \langle \varphi^k \right]
\]

\[
\Pi_{j,j+1}^{\text{cross}} = \sum_{k \neq i} \ell^k r^i_{j,j+1} \langle \ell^k r^i \right]
\]

with

\[
R^k = \frac{1}{\sqrt{2}} \left[ 0 r^k \rangle - r^0 \rangle \right]; \quad L^k = \frac{1}{\sqrt{2}} \left[ 0 \ell^k \rangle - \ell^0 \rangle \right]; \quad \varphi^k = \frac{1}{\sqrt{2}} \left[ 00 \rangle - \ell^k r^k \rangle \right].
\]

The projector \( R^k \rangle \langle R^k \) implements \( 0 r^k \leftrightarrow r^k 0 \), \( L^k \rangle \langle L^k \) implements \( 0 \ell^k \leftrightarrow \ell^k 0 \) and \( \varphi^k \rangle \langle \varphi^k \) implements the interaction term \( 00 \leftrightarrow \ell^k r^k \). The last set of projections \( \Pi_{j,j+1}^{\text{cross}} \) penalize wrongly ordered matching of parenthesis of different types. The rank of the local projectors away from the boundaries is \( s (s + 2) \). The contributions to the rank are \( 2 \binom{s}{2} \) from the penalty terms \( \Pi_{j,j+1}^{\text{cross}} \), \( 2s \) from propagation through the vacuum (i.e., zeros) given by the span of \( L^k \) and \( R^k \), and \( s \) from creation and annihilation of particles given by the span of \( \varphi \).

**Lemma 2.** If a state \( \psi \rangle \) is annihilated by every \( \Pi_{j,j+1} \), then it has the same amplitude on any two strings \( u \) and \( v \) that are equivalent under the local moves (Eq. 17).

**Proof.** If \( u \sim v \) then there exits a sequence of local moves that takes \( u \) to \( v \). Suppose a local move in this sequence is applied to the \( j \) and \( j + 1 \) position of the string to take \( u \) to \( u' \), then \( \langle u| - \langle u'| \rangle_{j,j+1} \) is proportional to a bra of one of the projectors in Eq. 19. If \( \psi \rangle \) is annihilated by all
the projectors then $\langle u | \psi \rangle = \langle u' | \psi \rangle$. Since $v$ is obtained from $u$ by a sequence of such local moves then $\langle u | \psi \rangle = \langle v | \psi \rangle$.

The projectors $\Pi_{j,j+1}$ simply move parenthesis through 0’s (or vacuum) or create or annihilate a pair of parenthesis of some color. Under these moves product states, such as $r^1 \otimes 2^n$, can also be ground states. Moreover, states such as $\ell^1 \otimes r^2 \otimes 2^n$ are also annihilated by every $\Pi_{j,j+1}$. In order to select out $\mathcal{M}_{2n,s}$ from all other states, we impose boundary conditions $\Pi_{\text{boundary}}$ that penalize states that are imbalanced by assigning an energy 1 to any state that starts with a step down ($r^1$) or ends with a step up ($\ell^1$) of any color. In addition, we locally impose $\Pi^{\text{cross}}_{j,j+1}$ to prevent crossed pairing states.

For example, when $s = 2$ the Hamiltonian is

$$H = r^1 \langle r^1 + r^2 \rangle_1 \langle r^1 + \ell^1 \rangle_{2n} \langle r^1 + \ell^2 \rangle_{2n} + \sum_{j=1}^{2n-1} \Pi_{j,j+1} + \sum_{j=1}^{2n-1} \Pi^{\text{cross}}_{j,j+1},$$

where $\Pi^{\text{cross}}_{j,j+1} = \{ \ell^1 r^2 \langle \ell^1 r^2 + \ell^2 r^1 \langle \ell^2 r^1 \}_{j,j+1}$ and $\Pi_{j,j+1}$ is the span of

$$R^1 = \frac{1}{\sqrt{2}} \{ 0 r^1 - r^1 0 \} , \quad R^2 = \frac{1}{\sqrt{2}} \{ 0 r^2 - r^2 0 \} , \quad \varphi^1 = \frac{1}{\sqrt{2}} \{ 00 - \ell^1 r^1 \} ,$$

$$L^1 = \frac{1}{\sqrt{2}} \{ 0 \ell^1 - \ell^1 0 \} , \quad L^2 = \frac{1}{\sqrt{2}} \{ 0 \ell^2 - \ell^2 0 \} , \quad \varphi^2 = \frac{1}{\sqrt{2}} \{ 00 - \ell^2 r^2 \} .$$

It follows that the $s$–colored Motzkin state $\mathcal{M}_{2n,s}$, which is the uniform superposition of all $s$–color Motzkin paths, is the unique ground state of the FF local Hamiltonian $H$.

B. Proof of the gap of the balanced subspace: $O \left( n^{-2} \right)$ upper bound

We shall first give a definition of the balanced subspace. We then assume that we can find a state $\phi$ in the balanced subspace, which has a low energy and a small overlap with the ground state. We will show that this implies that the first excited state has a low energy.

**Definition 4.** (balanced subspace) The balanced subspace is the span of the $s$–colored Motzkin state $\mathcal{M}_{2n,s}$ as defined in Definition 3.

**Remark.** In the balanced subspace, the amplitude of any vector on $\Pi_{\text{boundary}}$ and $\Pi^{\text{cross}}_{j,j+1}$ for all $j$ in the Hamiltonian (Eq. 18) vanishes. The Hamiltonian then is simply $H = \sum_{j=1}^{2n-1} \Pi_{j,j+1}$.

For now we take $s = 1$ and denote $\mathcal{M}_{2n,1}$ by $\mathcal{M}_{2n}$ and the Motzkin number $M_{2n,1}$ by $M_{2n}$. Take the state $\phi$ in the balanced subspace defined by

$$\phi = \alpha_0 \ M_{2n} + \alpha_1 \ e_1 + \alpha_2 \ e_2 + \cdots + \alpha_{(M_{2n}-1)} \ e_{(M_{2n}-1)},$$

(22)
where $e_i$ is the $i^{th}$ excited states of $H$ in the balanced subspace. Clearly, $\sum_{i=0}^{M_2n-1} |\alpha_i|^2 = 1$ and
\[
\langle \phi | H | \phi \rangle = \sum_{i=1}^{M_2n-1} |\alpha_i|^2 \langle e_i | H | e_i \rangle
\]
where we recall that $H = \sum_{j=1}^{2n-1} \{R \langle R + L \rangle \langle L + \phi \rangle |_{j,j+1} \}$ in the balanced subspace. Let us take $\phi$ to have a small overlap with the ground state, say $|\alpha_0|^2 \leq 1/2$. Then
\[
\langle \phi | H | \phi \rangle \geq \sum_{i=1}^{M_2n-1} |\alpha_i|^2 \langle e_1 | H | e_1 \rangle \geq \frac{1}{2} \sum_{i=1}^{M_2n-1} \langle e_1 | H | e_1 \rangle.
\] (23)

We choose $\phi$ to be
\[
\phi \equiv \frac{1}{\sqrt{M_2n}} \sum_{m_p} e^{2\pi i \tilde{\theta} \tilde{A}_p} m_p,
\] (24)
where $\tilde{A}_p$ is the area under the Motzkin walk $m_p$, $\tilde{\theta}$ is a constant we will specify later and the sum is over all Motzkin walks. The overlap with the ground state is
\[
\langle \mathcal{M}_{2n} | \phi \rangle = \frac{1}{M_2n} \sum_{m=0}^{M_2n-1} e^{2\pi i \tilde{\theta} \tilde{A}_p}.
\] (25)

As $n \to \infty$, the random walk converges to a Wiener process [38] and a random Motzkin walk converges to a Brownian excursion [39]. We wish to scale the random walk such that it takes place on $[0, 1]$, which is the standard form and gives $O(1)$ mean and variance.

To this end, we first describe the standard Brownian excursion. Let $B(t)$ be a standard Brownian motion on $[0, 1]$ with $B(0) = 0$. A standard (normalized) Brownian excursion, $B_{\text{ex}}(t)$, on the interval $[0, 1]$ is defined by $B(t)$ conditioned on $B(t) \geq 0$ and $B(1) = 0$ for $t \in [0, 1]$ (see [40]). Let $B_{\text{ex}}(t)$ be a Brownian excursion and [40, p. 84]
\[
B_{\text{ex}} \equiv \int_0^1 B_{\text{ex}}(t) \, dt
\]
the Brownian excursion area. The moments of $B_{\text{ex}}$ are given by
\[
\mathbb{E}[B_{\text{ex}}^k] = \frac{4\pi 2^{-k/2} k!}{\Gamma[(3k - 1)/2]} K_k \quad k \geq 0,
\]
where $K_0 = -\frac{1}{2}$ and
\[
K_k = \frac{3k - 4}{4} K_{k-1} + \sum_{j=1}^{k-1} K_j K_{k-j}, \quad k \geq 1.
\]

It follows that $\mathbb{E}[B_{\text{ex}}] = 4\sqrt{\frac{\pi}{2}} K_1 = \frac{1}{2} \sqrt{\frac{\pi}{2}} \approx 0.626657$ and the standard of deviation $\sigma = \sqrt{5/12 - \pi/8} \approx 0.1548144$. In addition $\mathbb{E}[B_{\text{ex}}^k] \sim 3\sqrt{2} k \left( \frac{k}{12n} \right)^{k/2}$ as $k \to \infty$.

Let $f_A(x)$ be the probability density function of $B_{\text{ex}}$. The analytical form of $f_A(x)$ is [40] see
Figure 7: Left: Plot of the probability density of the area of a Brownian excursion \( f_A(x) \) on \([0,1]\). Right: Fourier transform of \( f_A(x) \) as defined by Eq. [27].

Eq. 92]

\[
f_A(x) = \frac{2\sqrt{6}}{x^2} \sum_{j=1}^{\infty} v_j^{2/3} e^{-v_j} U \left( -\frac{5}{6}; \frac{4}{3}; v_j \right) \quad x \in [0, \infty)
\]

with \( v_j = 2|a_j|^3 / 27x^2 \) where \( a_j \) are the zeros of the Airy function, \( \text{Ai}(x) \), and \( U \) is the confluent hypergeometric function [11]. See Fig. 7 for the plot of \( f_A(x) \).

The total area\(^3\) under all strictly positive Motzkin walks of length \( n \) satisfies a recursion relation \( A_{n+1} = 2A_n + 3A_{n-1} \) [12]; hence \( A_n = \frac{1}{4} (3^{n+1} + (-1)^n) \). By convergence of random walks to Brownian motion \( A_n \), to the leading order, is equal to the total area of Motzkin walks that we are interested in (i.e., non-negative). Since asymptotically \( M_n = \frac{3^n}{n^{3/2}} \sqrt{\frac{27}{4\pi}} (1 + \mathcal{O}(1/n)) \), the expected area is

\[
\mathbb{E}(\tilde{A}_p) = \frac{A_{2n}}{M_{2n}} \approx \sqrt{\frac{2\pi}{3}} n^{3/2}.
\]

We can now solve for the scaling constants. We shall find \( c \) such that \( \mathbb{E} \left[ \tilde{A}_p \right] = cn^{3/2} \mathbb{E}[B_{\text{ex}}] \); therefore, \( \sqrt{\frac{2\pi}{3}} = c \frac{1}{2} \sqrt{\frac{\pi}{2}} \) which gives \( c = 4/\sqrt{3} \).

We take \( \tilde{A}_p = \frac{4}{\sqrt{3}} n^{3/2} x \) and \( \tilde{\theta} = \frac{\sqrt{3}}{4} n^{-3/2} \theta \). With the scaling just performed, most of the probability mass is supported on \( x = \mathcal{O}(1) \).

\(^3\) From convergence to a Brownian motion, we expect the height in the middle to be \( c' \sqrt{n} \), where \( c' \) can be calculated from our previous techniques. Indeed, the expected height of the Motzkin walk in the middle (i.e., at site \( n \)) is

\[
\mathbb{E}[m] = \frac{\sum_{m=0}^{\infty} m M_{n,m}}{\sum_{m=0}^{\infty} M_{n,m}}
\]

where as before we denote the height by \( 0 \leq m \leq n \) and \( M_{n,m} \) is the number of walks that start from zero and end at height \( m \) in \( n \) steps. Using similar derivation leading to Eq. [13] we find

\[
\mathbb{E}[m] \approx \int_0^{\infty} \int_0^{\infty} \frac{\alpha^3 \exp(-3\alpha^2/2)}{\alpha^2 \exp(-3\alpha^2/2)} \sqrt{n} = 2 \sqrt{\frac{2}{3\pi \sqrt{n}}}. 
\]
Evaluation of the sum given by Eq. 25 in the limit gives \(^{4}\)

\[
\lim_{n \to \infty} \langle M_{2n}|\phi \rangle \approx F_A (\theta) \equiv \int_0^\infty f_A (x) e^{2\pi i x \theta} dx .
\]  

(27)

In Eq. 27 taking \(\theta \ll \mathcal{O} (1)\), gives \(\lim_{n \to \infty} \langle M_{2n}|\phi \rangle \approx 1\); however, \(\theta \gg \mathcal{O} (1)\) gives a highly oscillatory integrand that nearly vanishes (see the right figure in Fig. 7). To have a small constant overlap with the ground state, we now show, that \(\theta = \mathcal{O} (1)\). Suppose we choose an area interval \([x_1, x_2]\), where \(f_A (x_1) = f_A (x_2) = y\). We have

\[
\int_0^\infty f_A (x) e^{2\pi i x \theta} dx = \int_0^{x_1} f_A (x) e^{2\pi i x \theta} dx + \int_{x_2}^\infty f_A (x) e^{2\pi i x \theta} dx \\
+ \int_{x_1}^{x_2} [f_A (x) - y] e^{2\pi i x \theta} dx + y \int_{x_1}^{x_2} e^{2\pi i x \theta} dx .
\]

Now if we let \(\theta = \frac{1}{x_2 - x_1}\), then \(y \int_{x_1}^{x_2} e^{2\pi i x \theta} dx = 0\), hence

\[
\int_0^\infty f_A (x) e^{2\pi i x \theta} dx = \int_0^{x_1} f_A (x) e^{2\pi i x \theta} dx + \int_{x_2}^\infty f_A (x) e^{2\pi i x \theta} dx + \int_{x_1}^{x_2} [f_A (x) - y] e^{2\pi i x \theta} dx \\
\leq \int_0^{x_1} f_A (x) dx + \int_{x_2}^\infty f_A (x) dx + \int_{x_1}^{x_2} [f_A (x) - y] dx \\
= 1 - y (x_2 - x_1) .
\]

(28)

We wish to maximize the area of the rectangle \(y (x_2 - x_1)\), so we take \(x_2 - x_1 = \sigma\), which makes \(y = \mathcal{O} (1)\) and \(\tilde{\theta} = \frac{\sqrt{3}}{2\sqrt{5/3-e/2}} n^{-3/2}\).

It is easy to see that \(\sum_j \langle \phi| R_{j,j+1} (R|\phi)\rangle\) is nonzero if and only if it relates two walks that only differ by a local move of type \(0r \leftrightarrow r0\) at the \(j, j+1\) position

\[
\sum_j \langle \phi| R_{j,j+1} (R|\phi)\rangle = \frac{1}{2M_{2n}} \sum_j \sum_{m_p, m_q} e^{2\pi i \tilde{\theta} (\tilde{A}_p - \tilde{A}_q)} \langle m_q| R_{j,j+1} (R| m_p)\rangle
\]

(29)

The change in the area is either one or zero. There are three types of nonzero contributions per \(j, j+1\) in Eq. 29

\[
(m_q)_{j,j+1} = (m_p)_{j,j+1} : \quad e^{2\pi i \tilde{\theta} (\tilde{A}_p - \tilde{A}_q)} \langle m_q| R_{j,j+1} (R| m_p)\rangle = 1
\]

\[
(m_q)_{j,j+1} = 0r \quad (m_p)_{j,j+1} = r0 : \quad e^{2\pi i \tilde{\theta} (\tilde{A}_p - \tilde{A}_q)} \langle m_q| R_{j,j+1} (R| m_p)\rangle = -e^{2\pi \tilde{\theta}}
\]

\[
(m_q)_{j,j+1} = r0 \quad (m_p)_{j,j+1} = 0r : \quad e^{2\pi i \tilde{\theta} (\tilde{A}_p - \tilde{A}_q)} \langle m_q| R_{j,j+1} (R| m_p)\rangle = -e^{-2\pi \tilde{\theta}} .
\]

Note that there is no dependence on the actual values of \(\tilde{A}_p\) and \(\tilde{A}_q\) but only on their difference. Putting it together we find that Eq. 29 gives

\[
\sum_j \langle \phi| R_{j,j+1} (R|\phi)\rangle = \frac{1}{M_{2n}} \sum_j a_j \left[ 1 - \cos \left( 2\pi \tilde{\theta} \right) \right] \approx \frac{1}{M_{2n}} \sum_j 2\pi^2 \tilde{\theta}^2 a_j
\]

where \(a_j\) is the number of strings that have \(0r\) or \(r0\) at the position \(j, j+1\). An entirely similar

^{4} F_A (\theta) is the Fourier transform of the probability density function which is called the characteristic function.
calculation gives
\[\sum_j \langle \phi | L_{j,j+1} | L \phi \rangle = \frac{1}{M_{2n}} \sum_j b_j \left[ 1 - \cos \left( 2\pi \hat{\theta} \right) \right] \approx \frac{1}{M_{2n}} \sum_j 2\pi^2 \hat{\theta}^2 b_j ,\]
\[\sum_j \langle \phi | \varphi_{j,j+1} | \varphi \rangle = \frac{1}{M_{2n}} \sum_j c_j \left[ 1 - \cos \left( 2\pi \hat{\theta} \right) \right] \approx \frac{1}{M_{2n}} \sum_j 2\pi^2 \hat{\theta}^2 c_j ,\]

where \( b_j \) and \( c_j \) are the number of strings that have \( 0\ell, \ell 0 \) and \( 00, \ell r \) at positions \( j, j+1 \) respectively.

Summing up the foregoing equations and using \( \hat{\theta} = \frac{\sqrt{3}}{2\sqrt{5/3-\pi/2}} n^{-3/2} \) we obtain
\[
\langle \phi | H | \phi \rangle = \frac{9\pi^2}{10} \frac{n^{-3}}{3\pi M_{2n}} \sum_j \left( a_j + b_j + c_j \right).
\] (30)

We need to show that \( \frac{a_j + b_j + c_j}{M_{2n}} = O(1) \) to get \( O(n^{-2}) \) upper bound. It is clear that \( c_j = M_{2(n-1)} \) and that \( a_j \approx b_j \approx c_j \). Lastly \( 1/3 \leq M_{2(n-1)}/M_{2n} \leq 1 \); therefore
\[
\langle \phi | H | \phi \rangle = O \left( n^{-2} \right).
\]

If we take a general integer \( s \geq 1 \) then using similar reasoning
\[
\langle \phi | H | \phi \rangle \sim \frac{2\pi^2 sn^{-3}}{M_{2n,s}} \sum_j \left( a'_j + b'_j + c'_j \right) = O \left( n^{-2} \right).
\] (31)

C. Proof of the gap of the balanced subspace: poly \((1/n)\) lower bound

In addition to the 'balanced subspace' above, we define the 'unbalanced subspace' and summarize the proof idea below before presenting the formal proof.

Definition 5. (unbalanced subspace) The space orthogonal to the span of \( M_{2n,s} \).

Remark 2. In the unbalanced subspace, the crossings and/or an overall imbalance can occur.

The summary of the proof is as follows:

- Restrict the Hamiltonian to the balanced subspace, where there are a balanced number of correctly ordered right and left parentheses of each color.
- Identify the terms in the Hamiltonian that implement \( 0r^k \leftrightarrow r^k 0 \) and \( 0\ell^k \leftrightarrow \ell^k 0 \) with \( H_{move} \). Identify the interaction terms that implement \( 00 \leftrightarrow \ell^k r^k \) with \( H_{int} \).
- The Hamiltonian in the balanced subspace is expressed as \( H = H_{move} + H_{int} \).
- Since \( \Delta (H_{move}) \) is known, define \( H_\epsilon \equiv H_{move} + \epsilon H_{int} \) for \( 0 < \epsilon \leq 1 \) and show that \( \Delta (H_\epsilon) \leq \Delta (H) \).
• Use the projection lemma to relate $\Delta(H)$ to $\Delta(H_{\text{move}})$ and the gap of the restriction of $H_{\text{int}}$ to the ground subspace of $H_{\text{move}}$, denoted by $H_{\text{eff}}$.

• Lower bound the gap of $H_{\text{eff}}$ by proving a large spectral gap of a corresponding Markov chain. We do so by proving rapidly mixing using the canonical path technique and ideas from fractional matching in combinatorial optimization.

• Lastly, lower bound the ground states in the unbalanced subspace.

As discussed above in the balanced subspace the Hamiltonian is simply

$$H = \sum_{j=1}^{2n-1} \left\{ \sum_{k=1}^{s} \left[ R^k_{j,j+1} (R^k + L^k)_{j,j+1} (L^k + \varphi^k)_{j,j+1} \right] \right\},$$

where any state automatically vanishes on the boundary terms and the parenthesis are correctly ordered (i.e., non-crossing).

Let $D^s_m$ be the set of Dyck paths of length $2m \leq 2n$ with $s$ colorings and $D^s$ be the union of all $D^s_m$. Let $\mathcal{M}_{2n,s}$ be the set of Motzkin paths of length $2n$, recall that the number of these walks is counted by the colored-Motzkin number $\mathcal{M}_{2n,s} = |\mathcal{M}_{2n,s}|$. Define a Dyck space $H^s_D$ whose basis vectors are Dyck paths $s \in D^s$. Given a Motzkin path $u$ with $2m$ parenthesis and any coloring, let Dyck $(u) \in D^s_m$ be the Dyck path obtained from $u$ by removing zeros. We shall use an embedding $V: H^s_D \to H^s_M$ defined by

$$V|s\rangle = \frac{1}{\sqrt{\binom{2n}{2m}}} \sum_{Dyck(u)=s} u, \quad s \in D^s_m \cap D^s \quad u \in \mathcal{M}_{2n,s}$$

where $\binom{2n}{2m}$ is the number of ways a given Dyck walk of length $2m$ can be embedded into Motzkin walks of length $2n$ each having $2(n-m)$ zeros. It is easily checked that $V^\dagger V = I$; i.e., $V$ is an isometry:

$$\langle t|V^\dagger V|s\rangle = \frac{1}{\binom{2n}{2m}} \sum_{Dyck(u)=s} \sum_{Dyck(v)=t} \langle v|u \rangle = \langle t|s \rangle.$$

**Perturbation Theory**

Similar to our previous work [3], we write the Hamiltonian restricted to the balanced subspace as $H \equiv H_{\text{move}} + H_{\text{int}}$, where
\[
H_{\text{move}} \equiv \sum_{j=1}^{2n-1} \sum_{k=1}^s [R^k]_{j,j+1}(R^k + L^k)_{j,j+1}(L^k)
\]
\[
H_{\text{int}} \equiv \sum_{j=1}^{2n-1} \sum_{k=1}^s \varphi^k_{j,j+1}\varphi^k,
\]
this notation makes explicit that the parentheses move through zeros, or 'vacuum', freely as the local moves indicate (Eq. 17). Yet when a left and a right parenthesis of a given color reach one another they can annihilate to produce a 00 state; alternatively a pair of 00 can spontaneously create a balanced set of parenthesis in correspondence to the local moves (Eq. 17).

Definition 6. Let \( \lambda_1 (H) \) denote the ground state energy of \( H \) and let \( \lambda_2 (H) \) denote the second smallest eigenvalue. We denote the gap of the Hamiltonian \( H \) by \( \Delta (H) \equiv \lambda_2 (H) - \lambda_1 (H) \).

Let us consider the interaction term as a perturbation to \( H_{\text{move}} \) and define a modified Hamiltonian \( H_\epsilon = H_{\text{move}} + \epsilon H_{\text{int}} \) for \( 0 < \epsilon \leq 1 \). \( H_\epsilon \) involves the same projectors, which means that \( \mathcal{M}_{2n,s} \) is a unique ground state of \( H_\epsilon \) as well. It is clear that \( H \succeq H_\epsilon \); therefore \( \Delta (H) \geq \Delta (H_\epsilon) \).

\( H_{\text{move}} \) annihilates states that are symmetric under the local moves \( 0\ell^k \leftrightarrow \ell^k 0 \) and \( 0r^k \leftrightarrow r^k 0 \). Therefore, \( H_{\text{move}} \) coincides with the spin-1/2 quantum Heisenberg chain, whose gap was rigorously calculated to be \( \Delta (H_{\text{move}}) = 1 - \cos \left( \frac{\pi}{n} \right) = \Omega (n^{-2}) \) [26]. Moreover, \( H_{\text{move}} \mathcal{M}_{2n,s} = 0 \), so \( \mathcal{M}_{2n,s} \) is a ground state of \( H_{\text{move}} \) that has a degenerate ground space.

We use perturbation theory to compute the \( \Delta (H_\epsilon) \) for which we need (see the Lemma below [43]) the restriction of \( H_{\text{int}} \) onto the subspace of the Motzkin space. The restriction, denoted by\(^5\)

\[
H_{\text{eff}} \equiv V^\dagger H_{\text{int}} V,
\]
is the process by which we first embed a Dyck walk with a particular coloring assignment into a Motzkin space (i.e., adding zeros), then either cut a peak of a given color (i.e., any \( \ell^k r^k \)) or add a peak of a given color where there are two consecutive zeros. Therefore, \( H_{\text{eff}} \) acts on the Dyck space \( H^s_D \).

**Lemma 3.** The unique ground state of \( H_{\text{eff}} \) is \( D^s \) that satisfies \( \mathcal{M}_{2n,s} \rangle = V | D^s \rangle \) and is given by

\[
D^s = \frac{1}{\sqrt{M_{2n,s}}} \sum_{m=0}^n \sqrt{\binom{2n}{2m}} \sum_{s \in D^s_m} s \rangle.
\]

**Proof.** We proved that \( \mathcal{M}_{2n,s} \) is the unique ground state of \( H = H_{\text{move}} + H_{\text{int}} \). Since \( H | \mathcal{M}_{2n,s} \rangle = HV | D^s \rangle = 0 \), \( H_{\text{eff}} | D^s \rangle = 0 \). We now prove that \( | D^s \rangle \) is the unique ground state. For if it were not, then \( H_{\text{eff}} | D' \rangle = 0 \) for a state \( D' \) other than \( D^s \). But by construction \( H_{\text{move}} V | D \rangle = 0 \), therefore \( (H_{\text{move}} + H_{\text{int}}) V | D \rangle = 0 \). Since \( \mathcal{M}_{2n,s} \) is the unique ground state of \( H \), then we reach a contradiction unless \( D' = D^s \).

\(^5\) In the projection lemma one defines the restriction by \( \tilde{H}_{\text{eff}} = \Pi_{\text{move}} H_{\text{int}} \Pi_{\text{move}} \), where \( \Pi_{\text{move}} = V V^\dagger \) is the projection onto the ground subspace of \( H_{\text{move}} \). Note that \( H_{\text{eff}} = V H_{\text{eff}} V^\dagger \) and \( H_{\text{eff}} = V^\dagger \tilde{H}_{\text{eff}} V \). Their action is equivalent with the distinction that the former acts on Motzkin walks and the latter on Dyck walks.
$H_{eff}$ defines a random Markov process. However, before describing the Markov process, we shall use the Projection Lemma \([43]\), that in our notation reads

**Lemma.** (Projection Lemma \([43]\)) $H_{eff}$ acts on the ground subspace of $H_{move}$. If the spectral gap of $H_{move}$ and $H_{eff}$ are both poly$(1/n)$ then the spectral gap of $H_{e}$ is also poly$(1/n)$ for small enough $\epsilon > 0$. Mathematically,

\[
\epsilon \lambda_1 (H_{eff}) - \frac{O (\epsilon^2) \|H_{int}\|^2}{\Delta (H_{move}) - 2\epsilon \|H_{int}\|} \leq \lambda_1 (H_{e}) \leq \epsilon \lambda_1 (H_{eff}).
\]

(37)

where we take $\Delta (H_{move}) = \Omega (n^{-2}) > 2\epsilon \|H_{int}\|$.

Since $H_{move} \psi \rangle = 0$ so long as $\psi \rangle$ is symmetric under $0\ell^k \leftrightarrow \ell^k 0$ and $0r^k \leftrightarrow r^k 0$, the ground subspace of $H_{move}$ is spanned by $V |s \rangle$ for $|s \rangle \in D^s$ and is degenerate. Therefore we can subtract the Span$(\mathcal{M}_{2n,s} |s \rangle |\mathcal{M}_{2n,s} \rangle)$ from the Hilbert space and consider the Projection Lemma on the orthogonal complement, whereby the subspace with the smallest eigenvalue becomes the gap. From the first inequality we have

\[
\Delta (H_{e}) \geq \epsilon \Delta (H_{eff}) - \frac{O (\epsilon^2) \|H_{int}\|^2}{\Delta (H_{move}) - 2\epsilon \|H_{int}\|}.
\]

(38)

If we choose an $\epsilon \ll n^{-3}$ then $\Delta (H_{move})$ can be considered large with respect to $\epsilon \|H_{int}\|$, which gives

\[
\Delta (H) \geq \Delta (H_{e}) \geq \epsilon \Delta (H_{eff}) - O (\epsilon^2 n^4).
\]

Hence it suffices to prove $\Delta (H_{eff}) \geq n^{-O(1)}$.

**Random walk description**

Let $\pi$ be the induced probability distribution on $D^s$ with entries $\pi (s) = \langle s |D^s \rangle^2$. Define the matrix $P$ by

\[
P = I - \frac{1}{s(2n-1)} \text{diag} \left( \frac{1}{\sqrt{\pi}} \right) H_{eff} \text{diag} \left( \sqrt{\pi} \right)
\]

\[
P (s, t) = \delta_{s,t} - \frac{1}{s(2n-1)} \langle s |H_{eff} |t \rangle \sqrt{\frac{\pi (t)}{\pi (s)}},
\]

(39)

where the second equation explicitly shows the entries.

We claim that $P$ describes a random walk on the set of Dyck paths $D^s$ such that given a pair of Dyck paths $s, t \in D^s$, $P (s, t)$ is a transition probability from $s$ to $t$ and $\pi$ is the unique steady state. One has
1. $P$ is stochastic. We use completeness to prove that the sum of any row is one, i.e.,
$$
\sum_t P(s, t) = 1
$$

$$
\sum_t \left\{ \delta_{s,t} - \frac{\langle s|D^s \rangle^{-1}}{s(2n-1)} \langle s|V^\dagger H_{\text{int}} V|t|D^s \rangle \right\} = 1 - \frac{\langle s|D^s \rangle^{-1}}{s(2n-1)} \langle s|V^\dagger H_{\text{int}} \sum_t |t| \langle t|D^s \rangle
$$
$$
\quad = 1 - \frac{\langle s|D^s \rangle^{-1}}{s(2n-1)} \langle s|V^\dagger H_{\text{int}} |D^s \rangle
$$
$$
\quad = 1 - \frac{\langle s|D^s \rangle^{-1}}{s(2n-1)} \langle s|V^\dagger H_{\text{int}} |M_{2n,s} \rangle = 1,
$$

since the Hamiltonian is FF and the colored-Motzkin state is a zero eigenvector of $H_{\text{int}}$.

2. $P$ has a unique steady state $\pi(s)$ because $\sum_s \pi(s) P(s, t) = \sum_s \left\{ \pi(s) \delta_{s,t} - \frac{1}{s(2n-1)} \langle s|H_{\text{eff}}|t\rangle \sqrt{\pi(s)\pi(t)} \right\} = \pi(t)$.

3. $P$ is reversible, that is $\pi(s) P(s, t) = \pi(t) P(t, s)$ for all $s, t$, as can easily be checked (note that $\langle s|H_{\text{eff}}|t\rangle = \langle t|H_{\text{eff}}|s\rangle$).

4. $P(s, t) = 0$ unless $s$ and $t$ are related by adding or removing a single peak of any color (see the proof of Lemma 4).

5. Since they are related by a similarity transformation, $\Delta(H_{\text{eff}}) = s(2n-1)(1 - \lambda_2(P))$.

**Lemma 4.** $P(s, s) \geq 1/2$. Let $s, t \in D^n$ be any Dyck paths such that $t$ can be obtained from $s$ by adding or removing a single $k_r^r$ pair of any color $k$. Then $P(s, t) = \Omega(1/n^3)$. Otherwise $P(s, t) = 0$.

**Proof.** First we prove that if $s$ and $t$ differ in more than two consecutive positions then $P(s, t) = 0$. More explicitly let $s \neq t$ and $u$ be a Motzkin path in $V|t\rangle$ and $\langle v a Motzkin path in $\langle s|V^\dagger$ then

$$
P(s, t) = -\frac{1}{s(2n-1)}\langle v \left| \sum_{j=1}^{2n-1} I \otimes \left( \sum_{k=1}^{s} \varphi^k_{j,j+1} \langle \varphi^k \rangle \otimes I \right) \right| u \rangle \sqrt{\frac{\pi(t)}{\pi(s)}}.
$$

(40)

In the foregoing equation for any summand $\sum_{k=1}^{s} \varphi^k_{j,j+1} \langle \varphi^k \rangle$, if the two strings $u$ and $v$ differ in any position other than $j, j+1$, then $P(s, t) = 0$. Therefore, $P(s, t)$ is only nonzero when $s$ can be obtained from $t$ by single insertion or removal of a peak (i.e., $00 \leftrightarrow k_r^r$) or vice versa. Next, we evaluate $P(s, s)$

$$
P(s, s) = 1 - \frac{1}{s(2n-1)} \langle s|H_{\text{eff}}|s \rangle
$$

$$
\quad = 1 - \frac{1}{s(2n-1)} \left( \frac{2n}{2m} \right)^{-1} \sum_{\text{Dyck}(u)=s} \sum_{\text{Dyck}(v)=s} \langle v|H_{\text{int}}|u \rangle
$$

but $\langle v|H_{\text{int}}|u \rangle = \frac{s}{2}$ for local moves that take $00 \leftrightarrow 00$ and $\langle v|H_{\text{int}}|u \rangle = \frac{1}{2}$ for moves $\ell_r^k \leftrightarrow \ell_r^k$.

In Eq. 40 we have $\langle s|H_{\text{eff}}|s \rangle \leq \frac{(2n-1)s}{2}$, hence $P(s, s) \geq 1/2$.

Next consider $t \neq s$. If $s \in D_m^n$ then $t \in D_m^n$, which is obtained from $s$ by removing or inserting a peak of any color (i.e., $00 \leftrightarrow \ell_r^k$). Using the definitions in Eqs. 32 and 39.
\[ P(s, t) = -\frac{1}{s(2n - 1)}\sqrt{\left(\frac{2n}{2m}\right)^{-1}\left(\frac{2n}{2(m \pm 1)}\right)^{-1}} \sum_{Dyck(u)=s} \sum_{Dyck(v)=t} \langle u | H_{int} | v \rangle \sqrt{\frac{\pi(t)}{\pi(s)}} \]

where \[ \sqrt{\frac{\pi(t)}{\pi(s)}} = \left| \frac{\langle t | D^s \rangle}{|\langle s | D^t \rangle|} \right| = \sqrt{\left(\frac{2n}{2(m \pm 1)}\right)\left(\frac{2n}{2m}\right)^{-1}}, \]

giving

\[ P(s, t) = -\frac{1}{s(2n - 1)}\left(\frac{2n}{2m}\right)^{-1} \sum_{Dyck(u)=s} \sum_{Dyck(v)=t} \langle u | H_{int} | v \rangle, \]

(41)

First suppose \( t \in D^s_{m+1} \). Let us fix some \( j \in [0, 2m] \) such that \( t \) can be obtained from \( s \) by inserting a pair \( \ell^k r^k \) of a given color \( k \), between \( s_j \) and \( s_{j+1} \). For any string \( u \) such that \( \text{Dyck}[u] = s \) in which \( s_j \) and \( s_{j+1} \) are separated by at least two zeros one can find at least one \( v \) with \( \text{Dyck}(v) = t \) such that \( \langle u | H_{int} | v \rangle = -\frac{1}{2} \). The fraction of strings that are obtained from randomly inserting two consecutive zeros into a string of length \( 2n \) are at least \( \frac{1}{4n^2} \), which is also a lower bound for inserting \( 2(n - m) \) zeros into \( s \). This combined with Eq. (41) and the fact that there are \( s \) different peaks gives

\[ P(s, t) \geq -\frac{1}{8n^3} \langle u | H_{int} | v \rangle = \frac{1}{16n^3}. \]

Now suppose \( t \in D^s_{m-1} \). Let us fix some \( j \in [1, 2m - 1] \) such that \( t \) can be obtained from \( s \) by removing the pair \( s_j s_{j+1} = \ell^k r^k \) for some color \( k \). The fraction of strings \( u \) such that \( \text{Dyck}[u] = s \) and that no zeros are inserted between \( s_j \) and \( s_{j+1} \) are at least \( \frac{1}{2n} \). Similar to above we have

\[ P(s, t) \geq -\frac{1}{4n^2} \langle u | H_{int} | v \rangle = \frac{1}{8n^2}. \]

Hence, in order to prove that the Hamiltonian has a poly \((1/n)\) gap, it suffices to prove that \( P \) has a polynomial gap, \( (1 - \lambda_2(P)) \geq n^{-O(1)} \). We do so by proving that the Markov chain is rapidly mixing [44].

**Rapidly mixing Markov chain: Canonical path technique**

One way to prove that the markov chain has a large spectral gap, is to show that it is rapidly mixing, or equivalently, it has a high conductivity [45]. Showing this ensures that starting from any arbitrary Dyck walk, one can move along the edges of the graph and ultimately reach any other Dyck walk quickly (i.e., in polynomial time).

We prove that \( P \) mixes rapidly and hence has a large gap using the canonical path technique, which ensures that there is a connected path via which one can obtain any \( t \in D^s_m \) from any \( s \in D^k_k \) by a sequence of insertion and removal of peaks such that no intermediate edge is overloaded. Perhaps it is helpful to give a traffic analogy that would illustrate the canonical path technique.
A city is rapidly mixing, or equivalently, has high conductivity if it has a low traffic. We say the city has a low traffic, if one can drive between any two arbitrary houses efficiently. One way to ensure this, is to show that between any two arbitrary chosen houses there are a sequence of roads that connect them such that none of the roads is overly used by other drivers (i.e., none of which is congested). Therefore, one never gets “stuck” in traffic in any intermediate road and consequently reaches the destination quickly.

For the canonical path method, we specify a path $\gamma(s, t)$ between any two arbitrary states of the Markov chain. The canonical path theorem shows that for a reversible Markov chain the spectral gap is

$$1 - \lambda_2 \geq \frac{1}{\rho L}$$

where the maximum edge load $\rho$ is

$$\rho = \max_{(a, b) \in E} \frac{1}{\pi(a) P(a, b) \sum_{(a, b) \in \gamma_{a, t}} \pi(a) \pi(t)}.$$  \hspace{1cm} (43)

The probability distribution $\pi(s)$ is the stationary distribution of the Markov chain, and $L = \max_{(a, t)} |\gamma_{a, t}|$ is the length of the longest canonical path. Thus, if no edge is covered by too many canonical paths, the Markov chain will mix rapidly.

The transition matrix $P$ describes a random walk on the graph of Dyck walks, where two walks $s \in D_k^s$ and $t \in D_m^s$ are connected, i.e., have an edge between them, if $t$ can be obtained from $s$ by insertion/removal of a peak of any color $t^k r^k$. After the proof of Lemma 6, we shall prove rapid mixing between any $s \in D_k^s$ and $t \in D_m^s$. However, to better illustrate the method, for now let $s$ and $t$ be two Dyck walks of length $2n$, then the Markov chain $P$ takes $s$ to $t$ via a sequence of steps that essentially does the following:

1. Pick a position between $1$ and $2n - 1$ on the Dyck path $s$ at random.
2. If there is a peak there, remove it to get a path of length $2n - 2$. Here, a peak is a coordinate on the path that is greater than both of its neighbors.
3. Insert a peak at random position between $0$ and $2n - 2$ with a color randomly chosen out of the $s$ possibilities uniformly.

As shown in Lemma 4, $P(a, b) \geq 1/16 n^3 = \Omega(n^{-3})$. We need to use multi-edges in cases where cutting off two peaks, or inserting a peak at two different positions, gives the same Dyck path. The stationary distribution is uniform, so $\pi(a) = \frac{1}{C_n s^n} \approx \frac{\sqrt{\pi} n^{3/2}}{14 s^{n}}$ where, as before, $C_n$ is the $n^{th}$ Catalan number and $s^n$ corresponds to the $s$ possible colorings of any given Dyck path. Finally in our canonical paths construction, each path will be of maximum length $2n$, and no edge will appear in more than $2n (4s)^{n-1}$ paths.

Putting this together, we get that $\rho \leq \frac{2\sqrt{\pi}}{s} n \frac{1}{2^n}$ and the spectral gap is (see Eq. 43)

$$1 - \lambda_2 \geq \frac{s}{2\sqrt{\pi} n^{\frac{3}{2}}}.$$  \hspace{1cm} (44)

In detail, how can such a canonical path be constructed? Given two $s$—colored Dyck paths, we define the canonical path between them in the following way. We arrange all the Dyck paths of
Figure 8: A tree containing all 2–colored Dyck walks of length 4 or less

Figure 9: Canonical path between 2–colored Dyck walks of length 8. By cutting peaks we shrink the walk on the top left and by inserting peaks we grow the final walk (top right).

length \leq 2n into a tree where the root of the tree is the empty Dyck path (of length 0), and where level \( m \) contains all Dyck paths of length \( 2m \). We will require that any node can be taken to its parent by removing a peak from the Dyck path, and that no node has more than 4s children. For example Fig. 8 gives such a tree containing all Dyck paths of length 4 with two colors.

Now suppose we wish to find the canonical path between two Dyck paths \( s \) and \( t \), where these two Dyck paths have length \( 2n \). By considering the path in the tree from the leaf \( s \) to the root, we obtain a sequence of Dyck paths \( s = s_{2n}, s_{2n-2}, s_{2n-4}, \cdots, s_0 = \emptyset \) and similarly for \( t \). For the \( m^{\text{th}} \) Dyck path in our canonical path we use the concatenation of the two Dyck paths \( s_{2n-2m} \) and \( t_{2m} \). For example, see Fig. 9 for an example canonical path determined using the tree in Fig. 8.

It is clear that the length of this canonical path is at most \( 2n \). Suppose we have an edge \((a, b)\) on our random walk between two Dyck paths. This edge could appear as the \( m^{\text{th}} \) step in a canonical path for \( m = 1, 2, \cdots, n \). If it appears at step \( m \), then this transition corresponds to the transition
between two Dyck paths $s_{2n-2m+2}t_{2m-2} \rightarrow s_{2n-2m}t_{2m}$. This edge will appear on any canonical path between a descendant of $s_{2n-2m+2}$ and a descendant of $t_{2m}$ in our tree. Now, the node $t_{2m}$ has $(4s)^{n-m}$ descendant leaves in our tree, and $s_{2n-2m+2}$ has $4^{m-1}$ descendent leaves in our tree, so there are at most $(4s)^{n-1}$ different pairs $s, t$ for which this transitions is the $m^{th}$ step on the canonical path. Thus, the edge $(a, b)$ lies on at most $2n \cdot (4s)^{n-1}$ canonical paths.

Now, the remaining step in our proof is building the tree. For this, all we need to do is show that we can map the Dyck paths of length $2n$ onto the Dyck paths of length $2n-2$ so that every Dyck path of length $2n-2$ has at most $4s$ pre-images. This mapping will define the edges between the nodes on level $n-1$ and level $n$ of our tree.

In order to build this tree, we use a fractional matching theorem which says that if we can build a fractional matching between paths of length $2n$ and paths of length $2n-2$, then we can find a matching as follows. Consider a matrix $m_{ij}$ with the rows labeled by the colored Dyck paths of length $2n$ and the columns labeled by paths of length $2n-2$. Make $m_{ij} \geq 1$ if column $j$ can be obtained from row $i$ by removing a peak of a given color and $0$ otherwise (see Fig. 10 for examples). We let $m_{ij}$ be the number of ways of getting from path $j$ to path $i$ by removing a peak. Now the definition of fractional matching is a matrix $x_{ij}$ such that $0 \leq x \leq 1$, $\sum_i x_{ij} \leq 4s$, and $\sum_j x_{ij} = 1$. We will build such matrix by induction. In fact, we will show that we can build a matrix $x_{ij}$ where all the column sums are equal and all the row sums are 1. This additional hypothesis will let us use induction to construct the fractional matching.

To construct the fractional matching, we will first put the Dyck paths into a specific order. Recall the Catalan numbers are defined by a recursion

$$C_n = \sum_{m=0}^{n-1} C_mC_{n-m-1},$$
where $C_0 = 1$. Translating this into Dyck paths, each path of length $2n$ can be associated with a pair of paths, of length $2m$ and $2(n - m - 1)$, where $0 \leq m \leq n$.

To take two paths, $P_\alpha$ of length $2m$ and $P_\beta$ of length $2(n - m - 1)$ and obtain a path of length $2n$, add a step up before $P_\beta$ and a step down after $P_\beta$, and concatenate them (See Fig. 11). This is a one-to-one correspondence between pairs of Dyck paths whose length sum to $2n - 2$ and Dyck paths of length $2n$. We have already shown one direction of this mapping. This mapping is reversible because the first path $P_\alpha$ ends at the last point the path hits the $x-$axis before its end.

The construction of colored Dyck walks have a natural correspondence too; they can be defined by what we call the \textit{colored Catalan numbers}

\begin{align}
    s^n C_n = s \sum_{m=0}^{n-1} s^m C_m \left( s^{n-m-1} C_{n-m-1} \right),
\end{align}

where the $s$ multiplying the sum is the number of ways the step up and step down before and after $P_\beta$ can be colored.

Now suppose we have a path of length $2n$ corresponding to a pair of Dyck paths $P_\alpha P_\beta$, where $2\alpha + 2\beta = 2n - 2$. Suppose $\beta \neq 0$. Then, when we remove a peak of a given color, we will either end up with a path corresponding to $P_{\alpha-1} P_\beta$ or $P_\alpha P_{\beta-1}$. If $\beta = 0$, we can also remove the last peak to end up with the path $P_\alpha$.

Thus the matrix $m_{ij}$ breaks into block diagonal pattern, with the columns divided into blocks containing paths of the form $P_\alpha P_\beta$ where $\alpha + \beta = n - 2$, and the rows divided into blocks of the form $P_\alpha P_\beta$ where $\alpha + \beta = n - 1$. Except for the identity matrix added to the column $C_{n-1}$ rows, for each column block there are only two non-zero row blocks, and vice versa. In our construction, we never use the fact that there is an identity added, so we will ignore the existence of this in the following (see Fig. 10).

Let us look at these blocks more closely. The block of rows $P_\alpha P_\beta$ has 0’s except in column blocks $P_{\alpha-1} P_\beta$ or $P_\alpha P_{\beta-1}$. The sub-matrix $P_\alpha P_\beta \times P_{\alpha-1} P_\beta$ is simply the matrix $M_\alpha \otimes I_{C_\beta}$, where $M_\alpha$ is the matrix relating paths of length $2\alpha$ and $2\alpha - 2$ and $I_{C_\beta}$ is an identity matrix of size $C_\beta$ (\beta^{\text{th}} Catalan number). We assume by induction that we have a fractional matching on $M_\alpha$ and $M_\beta$. By taking the tensor product of these and an identity matrix, we obtain a fractional matching on these sub-blocks (see Fig. 10).

Now, we can construct the fractional matching by multiplying the fractional matching for a sub-block $P_\alpha P_\beta \times P_{\alpha-1} P_\beta$ and $P_\alpha P_\beta \times P_\alpha P_{\beta-1}$ by appropriate scalars, so that all the rows add to 1 and all the columns have the same sum. The column sum is $C_n s^n / (C_{n-1} s^{n-1}) < 4s$.

How can we prove this super-tree exists? The proof is based on fractional matching theorem.
and number of other results in linear programming and have been spelled out in \[3\]. In particular, we proved the following useful lemma, where we only had one color (spin $s=1$)\[3\].

Lemma 5. (Bravyi et al \[3\]) Let $D_m$ be the set of Dyck paths of length $2m$. For any $m \geq 1$ there exists a map $f : D_m \rightarrow D_{m-1}$ such that (i) the image of any path $s \in D_m$ can be obtained from $s$ by removing a single fr pair, (ii) any path $t \in D_{m-1}$ has at least one pre-image in $D_m$, and (iii) any path $t \in D_{m-1}$ has at most four pre-images in $D_m$.

This lemma allows us to grow arbitrary long Dyck paths starting from the empty string. Clearly at every level $m$, we can color each $s \in D_m$ walk $s^m$ different ways, whereby we have obtain all the $s-$colored Dyck walks of size $2m$ (i.e., the set $D_m^s$). We can similarly obtain $D_{m-1}$ by all $s$-colorings of every $t \in D_{m-1}$; each $t$ can be colored $s^{m-1}$ different ways.

Lemma 6. Let $D_m^s$ be the set of $s-$colored Dyck paths of length $2m$. For any $m \geq 1$ there exists a map $f : D_m^s \rightarrow D_{m-1}^s$ such that (i) the image of any path $s \in D_m^s$ can be obtained from $s$ by removing a single $\ell^k r^k$, (ii) any path $t \in D_{m-1}^s$ has at least $s$ pre-images in $D_m^s$, and (iii) any path $t \in D_{m-1}^s$ has at most $4s$ pre-images in $D_m^s$.

Proof. On the level $m-1$ there are $s^{m-1}$ copies of any Dyck walk of length $2m-2$, each with a unique coloring assignment. Similarly at the level $m$ there are $s^m$ copies of any Dyck walk of length $2m$ each with a unique coloring. For any fixed coloring at the $m-1$st level and a fixed choice of the color for $\ell^k r^k$ the problem reduces to Lemma 5 so (i) is satisfied. Similarly for (ii), there is a pre-image for any choice of $\ell^k r^k$ so there are at least $s$ such pre-images. To prove (iii) we note that for every choice of coloring of the Dyck walks at the level $m-1$ and a fixed choice of color for $\ell^k r^k$, the problem is identical to the previous case and there are at most $4$ pre-images (Lemma 5). Since there are $s$ choices to color the peak that we remove, there are at most $4s$ pre-images in total.

With these preliminaries, we now return to the proof of rapid mixing time of $P$, whereby we need to prove that the maximum edge load $\rho$ between any two arbitrary paths $s \in D_m^s$ and $t \in D_k^r$ is $n^{O(1)}$ which proves $1 - \lambda_2(P) \geq n^{-O(1)}$.

We define the canonical path $\gamma(s,t)$ such that any intermediate state is the concatenation of two walks $pq$ where $p \in D_r$ is an ancestor of $s$ in the super-tree and $q \in D_r$ is an ancestor of $t$. The canonical path starts with $p = s$, $q = \emptyset$ and alternates between shrinking $p$ by taking steps towards the root and growing $q$ by taking steps away from the root similar to what was discussed above. The path terminates as soon as $p = \emptyset$ and $q = t$. If at some intermediate state $p = \emptyset$ then the subsequent shrinking steps are skipped over while if in some intermediate step $q = t$ then the subsequent growing steps are skipped over. At any intermediate step the length of the concatenated walk $[pq]$ obeys

$$\min(|s|,|t|) \leq |pq| \leq \max(|s|,|t|). \quad (46)$$

Since any $\gamma(s,t)$ has a length that is at most $2n$, it is enough to bound $\rho$. Let the edge with maximum load, denoted by $\rho(m,k,\ell',\ell'')$, be between $a = pq$ to $b$, where as before $s \in D_m^s, \quad t \in D_k^r, \quad p \in D_{\ell'}, \quad q \in D_{\ell''}$.

For the sake of concreteness let $b$ be obtained from $a$ by growing $q$ and shrinking $p$ (the other case is analogous). From Lemma 6 the number of possible descendent strings $s$ from which $p$ is obtained
by shrinking is at most \((4s)^{m-\ell'}\). The number of possible ancestors of \(t\) is at most \((4s)^{k-\ell'}\). Since \(\pi(s) = \pi(t)\) for all \(s \in D_m^{k}\) and \(t \in D_k^{k},\)

\[
\rho(m, k, \ell', \ell'') \leq \frac{(4s)^m \pi(s) \ (4s)^k \pi(t)}{(4s)^{\ell' + \ell''} \pi(a) P(a, b)}. \tag{47}
\]

where by definition \(\pi(w) = \langle w \mid D^s \rangle^2\) and using Eq. 36 we obtain \(\pi(w) = \left(\frac{2n}{2w}\right) / M_{2n,s}\). From Lemma 4 we have \(P(a, b) \geq 1/16n^3\). To bound the right hand side of inequality (47), we first prove that

\[
(4s)^w \pi(w) = \frac{\sigma_w}{\sqrt{\pi w^{3/2}}} \tag{48}
\]

where \(\sigma_w \leq 1\) is the fraction of \(s\)-colored Motzkin paths of length \(w\). Indeed \(\sigma_m = s^w C_w \left(\frac{2n}{2w}\right) / M_{2n,s}\), where \(s^w\) is the number of colorings of the Dyck walks of length \(2w\) counted by the Catalan number \(C_w\). Since \(s^w C_w \approx (4s)^w / \sqrt{\pi w^{3/2}}\), Eq. 48 holds. Hence, we have

\[
\rho(m, k, \ell', \ell'') \leq \frac{16}{\sqrt{\pi}} n^3 \left(\frac{\ell' + \ell''}{m}ight)^{3/2} \frac{\sigma_m \sigma_k}{\sigma_{\ell' + \ell''}}.
\]

Since \(\left(\frac{\ell' + \ell''}{m}\right)\) is at most a polynomial in \(n\), it remains to bound \(\frac{\sigma_m \sigma_k}{\sigma_{\ell' + \ell''}}\). We comment that the maximum edge load is always at least one (for a fully connected graph). As mentioned above, in the canonical path, we add a polynomial number of terms so it suffices to prove that the ratio \(\frac{\sigma_m \sigma_k}{\sigma_{\ell' + \ell''}}\) is small; indeed

\[
\frac{\sigma_m \sigma_k}{\sigma_{\ell' + \ell''}} = \frac{1}{M_{2n,s}} \frac{s^m C_m \left(\frac{2n}{2m}\right) s^k C_k \left(\frac{2n}{2k}\right)}{s^{\ell' + \ell''} C_{\ell' + \ell''} \left(\frac{2n}{2 (\ell' + \ell'')}\right)}.
\]

But \(M_{2n,s} = \sum_{w=1}^n s^w C_w \left(\frac{2n}{2w}\right),\) which includes terms with \(w = m\) and \(w = k\) so we have

\[
\frac{\sigma_m \sigma_k}{\sigma_{\ell' + \ell''}} \leq \frac{1}{s^{\ell' + \ell''} C_{\ell' + \ell''} \left(\frac{2n}{2 (\ell' + \ell'')}\right)} \leq 1.
\]

We conclude that \(\rho \leq n^{O(1)}\), which implies that the spectral gap \(1 - \lambda_2(P) \geq n^{-O(1)}\). This completes the poly \((1/n)\) proof of the lower bound for the gap in the balanced subspace.

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**D. Smallest energy of unbalanced and/or crossed states: poly \((1/n)\) lower bound**

Previously we proved that if we restrict the Hamiltonian to the space where there are an excess number of right or left parenthesis then the smallest eigenvalue is indeed lower bounded by a
polynomial in $1/n$. The problem at hand is different for there are different types of parenthesis and in addition there is the possibility of having mismatches where $\Pi_{\text{cross}}$ does not vanish.

To establish the gap to be a polynomial in $1/n$ we need to lower bound the ground state energy of the Hamiltonian in the unbalanced subspace, where for example a sub string configuration such as ( [ ] ) can occur.

It is sufficient to separately prove lower bounds on: 1. the subspace with only mismatches 2. imbalance subspace without any mismatch. The reason for the sufficiency is that including mismatches to an imbalance space or vice versa can only increase the energy.

**Pure mismatch:** In this case the energy penalties come from $\sum \Pi_{\text{cross}}$. Let us assume there is a single mismatch such as $u_0(u_1 [ u_2 ] u_3 ) u_4$, where $u_1, \ldots, u_4$ are strings in the alphabet $\{0, \ell^1, \ldots, \ell^s, r^1, \ldots, r^s\}$ such that if we ignore the mismatching, the string $u_0(u_1 [ u_2 ] u_3 ) u_4$ would indeed be a colored-Motzkin walk. Moreover, we can assume there is only a single mismatch as in the example just given since having more mismatches results in more penalties and can only increase the energy. Recall that the Hamiltonian is

$$H = \sum_{j=1}^{2n-1} \Pi_{j,j+1} + \sum_{j=1}^{2n-1} \Pi_{\text{cross}}$$

where we can ignore the boundary terms as we are restricting ourselves to only mismatched subspaces. $\Pi_{j,j+1}$ is the hopping Hamiltonian that allows the propagation of parenthesis of any type through the vacuum. In $u_0(u_1 [ u_2 ] u_3 ) u_4$ suppose the first bracket (appearing after $u_1$) is at site $i$ and the first right parenthesis appearing after $u_2$ is at site $j$ and the right bracket between $u_3$ and $u_4$ is on site $k$. If we take the hopping amplitude on sites $i$ and $k$ to be zero then the energy can only decrease and the problem reduces to the case where there is a chain of length $k - i$ with a single excess right parenthesis at site $j$. So the problem formally reduces to the previous problem on a chain of length $k - i$. Therefore the previous polynomial lower bound also lower bounds this case.

**Imbalance subspace without a mismatch:** The Hamiltonian now reads

$$H = \Pi_{\text{boundary}} + \sum_{j=1}^{2n-1} \Pi_{j,j+1}$$

where $\sum_{j=1}^{2n-1} \Pi_{\text{cross}}$ vanishes and therefore can be ignored. We need to lower bound the smallest eigenvalue on strings of type

$$u_0 r_{i_1} u_1 r_{i_2} u_2 \cdots r_{i_k} u_k \ell_{j_1} \cdots v_2 \ell_{j_2} v_1 \ell_{j_1} v_0$$

where $u_i$ and $v_i$ are $s$-colored Motzkin walks and $i_p, j_q$ can take on any values in $\{1, 2, \cdots, s\}$. Since the spectrum of $H$ in this subspace only depends on the total number of excess right and left parenthesis we can focus on having only right imbalanced ones, whereby we simplify the analysis and drop the boundary terms $\sum_{i=1}^s \ell_i^j \ell_i^\ell$ as doing so can only decrease the energy. Below we use a similar argument as before. Given any string $u$ in the imbalanced subspace with only excess right parenthesis, let $\bar{u} \in \{0, \ell^1, \ldots, \ell^s, r^1, \ldots, r^s, x, y\}$ be the string obtained from $u$ by i) replace the first unmatched right parenthesis by $x$ and ii) replace all other unmatched parenthesis in $u$ by
We can define a new Hilbert space $\tilde{H}$ whose basis vectors are $\tilde{u}\rangle$. Consider a Hamiltonian

$$\tilde{H} = x\rangle_1 \langle x + \sum_{j=1}^{2n-1} \Pi_{j,j+1} + \Theta^x_{j,j+1} + \Theta^y_{j,j+1}$$

where $\Theta^x$ and $\Theta^y$ are projectors onto the states $\langle 0x - x0 \rangle$ and $\langle 0y - y0 \rangle$ respectively (with proper normalizations). Since $\langle u|H|v \rangle = \langle \tilde{u}|\tilde{H}|\tilde{v} \rangle$ for any $u,v$ the spectrum of $H$ and $\tilde{H}$ coincide in this subspace. We can further drop $\Theta^y$ terms as doing so only decreases the energy. Therefore, it is sufficient to consider the simplified Hamiltonian

$$H^x = x\rangle_1 \langle x + \sum_{j=1}^{2n-1} \Pi_{j,j+1} + \Theta^x_{j,j+1}$$

which act on $\tilde{H}$ and position of $y$ particles are constants of motion of $H^x$. An entirely a similar argument as in [3] shows that we can only analyze the interval between $1$ and the first $y$-particle, whereby the relevant Hilbert space becomes the span of (as before we denote the set of Motzkin paths of length $k$ by $M_k$)

$$u\rangle \otimes x\rangle \otimes v\rangle, \quad \text{where } u \in M_{j-1}, \quad v \in M_{2n-j}.$$  

To use the projection lemma define

$$H^x_\epsilon = \sum_{j=1}^{2n-1} \Pi_{j,j+1} + \epsilon \left\{ x\rangle_1 \langle x + \sum_{j=1}^{2n-1} \Theta^x_{j,j+1} \right\}$$

and an effective Hopping Hamiltonian $H_{eff}$ can be defined whose ground state lower bounds the ground state of $H^x_\epsilon \leq H^x$. $H_{eff}$ is defined by

$$H_{eff} = 1\rangle \langle 1 + \sum_{j=1}^{2n-1} \Gamma_{j,j+1}$$

where

$$\Gamma_{j,j+1} = \alpha_j^2 j\rangle \langle j + \beta_j^2 j + 1\rangle \langle j + 1 - \alpha_j \beta_j \{ j\rangle \langle j + 1 + j + 1\rangle \langle j \}$$

is a rank-1 projector. The coefficients are now different from the previous case and are given by

$$\alpha_j^2 \equiv \langle \psi_j | \Theta^x_{j,j+1} | \psi_j \rangle = \frac{M_{2n-j-1}}{2s M_{2n-j}}$$

$$\beta_j^2 \equiv \langle \psi_{j+1} | \Theta^x_{j,j+1} | \psi_{j+1} \rangle = \frac{M_{j-1}}{2s M_j}$$

and lastly

$$-\alpha_j \beta_j = -\frac{1}{2s} \sqrt{\frac{M_{2n-j-1}}{M_{2n-j}} \frac{M_{j-1}}{M_j}}$$
where $M_k$ is the $k^{th}$ Motzkin number which is the number of Motzkin walks in $k$ steps. Applying the projection lemma we have $\lambda_1 (H^x) \geq \epsilon \lambda_1 (H_{eff})$ and it suffices to show that $\lambda_1 (H_{eff}) \geq n^{-\Omega(1)}$.

The hopping Hamiltonian without the “repulsive potential” $1\langle 1$ is

$$H_{move} = \sum_{j=1}^{2n-1} \Gamma_{j,j+1}.$$  

This is a FF Hamiltonian with the unique ground state

$$g \sim \sum_{j=1}^{2n} s^{n-\frac{1}{2}} \sqrt{M_{j-1} M_{2n-j}} j.$$  \hspace{1cm} (49)

As before we bound the spectral gap of $H_{move}$ and use the Projection Lemma to lower bound. Let $\pi (j) = \langle j|g \rangle^2$. For any $a, b \in [1, 2n]$ we define

$$P (j, k) = \delta_{j,k} - \langle j|H_{move}|k \rangle \sqrt{\frac{\pi (k)}{\pi (j)}}$$

and a simple algebra shows that

$$P (j, j+1) = \frac{M_{2n-j-1}}{2sM_{2n-j}}$$

and $P (j + 1, j) = \frac{M_{j-1}}{2sM_j}$ are the only off diagonal matrix elements of $P$. Using Lemma 7 in \cite{3} that shows $\frac{1}{3} \leq \frac{M_k}{M_{k+1}} \leq 1$ we conclude that

$$\frac{1}{6s} \leq P (j, j + 1) \leq \frac{1}{2s} \forall j.$$

Consequently the diagonal elements of $P$ are non-negative and it can be considered as a transition matrix. Moreover, using Eq. \cite{49} we conclude that

$$n^{-\Omega(1)} \leq \frac{\pi (k)}{\pi (j)} \leq n^{\Omega(1)} \hspace{1cm} \forall \hspace{0.1cm} 1 \leq j, k \leq 2n.$$

We have $\min_j \pi (j) \geq n^{-\Omega(1)}$. This is sufficient to bound the spectral gap of $P$ as shown in \cite{3}. For example using the canonical paths theorem we get $1 - \lambda_2 (P) \geq \frac{1}{\rho^2}$ with a canonical path that simply moves the $x$–particles from $u$ to $v$. Since the denominator in the maximum edge load given by Eq. \cite{43} is lower bounded by $n^{-\Omega(1)}$ we conclude that the gap of $P$ is polynomially lower bounded and that $\lambda_2 (H_{move}) \geq n^{-\Omega(1)}$.

Lastly, one can apply the Projection lemma to $H_{eff}$ by making $1\langle 1$ a perturbation. The effective first order Hamiltonian will now be constant $\langle 1|g \rangle^2 = \pi (1) \geq n^{-\Omega(1)}$ which proves the bound $\lambda_1 (H_{eff}) \geq n^{-\Omega(1)}$.

\textbf{V. THE MODEL WITHOUT BOUNDARY TERMS}

The model above has a unique ground state because the boundary terms select out the Motzkin state among all other possible ground states. Without the boundaries the local moves preserve the height difference of the walk between the two ends of it, where it does not need to start at zero and end at zero any more. This allows more ground states, whose degeneracy grows quadratically with
a system size $2n$ when $s = 1$ and exponentially when $s > 1$.

For the $s = 1$ case, if we impose periodic boundary conditions, then the superposition of all walks with an excess of $k$ left (right) parentheses is a ground state (more below). This gives us $4n + 1$ degeneracy of the ground state for a chain of length $2n$ for a periodic boundary conditions. Also, on the infinite chain $(-\infty, \infty)$, for any $\alpha \in \mathbb{R}$

$$g_0 \equiv \frac{1}{\left(1 + \alpha^2 + \alpha^{-2}\right)^n} \prod_{j=1}^{2n} \left\{ \alpha \ell_j + 0 \right\} \left( \frac{1}{\alpha} r_j \right)$$

is a tensor product ground state. For a closed chain these states form an over-complete basis for the ground state.

When $s > 1$, each one of the walks with $k$ excess left (right) parenthesis can be colored exponentially many ways; however, they will not be product states. Let us consider the entanglement for $s > 1$ case. Consider an infinite chain $(-\infty, \infty)$ and take $s > 1$. There is a ground state of this system that corresponds to the balanced state, where on average for each of the types of parentheses, the state contains as many $\ell$ as $r$. Suppose we restrict our attention to any block of $n$ consecutive spins. This block contains the sites $j, j+1, \ldots, j+n-1$, which is a section of a random walk. Let us assume that it has initial height $m_j$ and final height $m_{j+n-1}$. Further, let us assume that the minimum height of this section is $m_k$ with $j \leq k \leq j + n - 1$. From the theory of random walks, the expected values of $m_j - m_k$ and of $m_{j+n-1} - m_k$ are $\Theta(\sqrt{n})$. The color and number of any unmatched left parentheses in this block of $n$ spins can be deduced from the remainder of the infinite walk. Thus a consecutive block of $n$ spins has an expected entanglement entropy of $\Theta(\sqrt{n})$ with the rest of the chain. A similar argument shows that any block of $n$ spins has an expected half-block entanglement entropy of $\Theta(\sqrt{n})$.

If we take $s = 1$, where the ground state is a product state (Eq. 50), the $\sqrt{n}$ unmatched parentheses just mentioned can be matched anywhere on the remaining left and right part of the chain. Two consecutive blocks of $n$ spins can be unentangled because the number of parentheses that are matched in the next block is uncorrelated with the number of unmatched parentheses in the first block. However, when $s > 1$ the ordering has to match; even though the number of unmatched parentheses in two consecutive blocks is uncorrelated, the order of the types of unmatched parentheses in them agrees. We thus obtain an expected $\Theta(\sqrt{n})$ amount of entanglement.

A. Presence of an external field

Recall that when $s = 1$, under the local moves strings are partitioned into equivalent classes $C_{p,q}$, with $0 \leq p, q \leq 2n$ such that $0 \leq p + q \leq 2n$. We say two strings $u$ and $v$ are equivalent, denoted by $u \sim v$ if $v$ can be obtained from $u$ by a sequence of local moves. Moreover, by Lemma 2, all strings in a given equivalence class have equal amplitudes if we insist on frustration freeness. Let $C_{p,q}$ be the uniform superposition of all the strings in the equivalent class $C_{p,q}$. If we remove the boundary projectors then the zero energy ground states will be all the state $C_{p,q}$. Consequently, the degeneracy of the ground state is $2n(n + 1)$.

Remark 3. For periodic boundary condition and $s = 1$ the ground state degeneracy is only $4n + 1$ because the ground states are $C_{0,0}$, $C_{0,m}$, and $C_{m,0}$ for $1 \leq m \leq 2n$.

The Hamiltonian without the boundary terms is truly translationally invariant. Below we will propose a model where one can trade off the FF property to retain uniqueness of the ground state and preserve other desirable properties, such as the gap and entanglement entropy scalings.

To do so, we put the system in an external field, where the model is described by the new
Hamiltonian

\[ \hat{H} \equiv H + \epsilon(n) \, F \]

\[ F \equiv \sum_{i=1}^{2n} \{r_i \langle r + \ell \rangle_i \ell \} \]

where, \( H \) is as before but without the boundary projectors and \( \epsilon(n) = \frac{a_n}{2n} \) for \( 0 < \epsilon_0 \ll 1 \); below for simplicity we denote \( \epsilon(n) \) by just \( \epsilon \). It is clear that \( F \) treats \( \ell \) and \( r \) symmetrically. Therefore, the change in the energy as a result of applying an external field depends only on \( m \equiv p + q \) denoted by \( \Delta E_m \). When \( s = 1 \), the degeneracy of the energies after applying the external field will be one for \( m = 0 \), and the ground state will be \( \mathcal{M}_{2n,s} \). The degeneracy is two-fold for a single imbalance \( m = 1 \) (i.e., \( C_{1,0}, C_{0,1} \)), three-fold for two imbalances \( m = 2 \) (i.e., \( C_{2,0}, C_{0,2}, C_{1,1} \)), etc. Since the energies are equal for all \( m \) imbalance states, it is enough to calculate the energy for an excited state with \( m \) imbalances resulting only from excess left parentheses. We denote these states by \( g_m \), where \( 0 \leq m \leq 2n \).

The energy corrections obtained from first order degenerate perturbation theory are

\[ \Delta E_m = \epsilon \langle g_m | F | g_m \rangle \]

and

\[ \Delta E_m = \frac{\epsilon}{N_m} \sum_{i,k} \langle g_m^i | F | g_m^k \rangle \]

where, \( N_m \) is the total number of walks that start at coordinates \((0,0)\) and end at \((2n,m)\) and

\[ g_m \equiv \frac{1}{\sqrt{N_m}} \sum_i g_m^i \]

\[ = \frac{1}{\sqrt{N_m}} \sum_i | \text{state } i \text{ with } m \text{ extra left parenth.} \rangle \]

It is clear that \( 0 < \Delta E_m \leq \epsilon_0 \).

Since only the embedded Dyck walks in the Motzkin state couple to the external field and contribute to the energy corrections, we need to count the number of walks that start from zero and reach coordinates \((2n,m)\).

Remark 4. For now, we pretend that the length of the chains is \( n \) and not \( 2n \). At the end we multiply \( n \) by a factor of 2.

The number of walks of length \( n \) with \( s \) coloring that reach the height \( m \) (i.e., coordinate \((x,y) = (m,n)\)) is denoted here by \( \Gamma(n,m) \). As before, \( \Gamma(n,m) \) is counted by a refinement of the Ballot problem

\[ \Gamma(n,m) \equiv s^m \sum_{k=0}^{n-m} \binom{n}{k} s^{\frac{n-k-m}{2}} B_{n-k,m} \equiv s^m M_{n,m,s} \quad (51) \]

where there are \( \binom{n}{k} \) ways of putting \( k \) zeros, \( B_{n-k,m} \) is the solution of the Ballot problem with height \( m \) on \( n - k \) walks (number of “Dyck” walks on \( n - k \) steps that end at height \( m \)), \( s^m \) ways of coloring the unmatched parenthesis and \( s^{\frac{n-k-m}{2}} \) ways of coloring the matched ones.
We can calculate the energy. Since only the embedded Dyck walks couple to the external field and give positive energy contribution, we have

\[
\langle g_m | F | g_m \rangle = \frac{s^m \sum_{i \geq 0} (m + 2i) M_{n,m,s,i}}{\Gamma(n,m)} = \sum_{i \geq 0} \frac{(m + 2i) M_{n,m,s,i}}{\sum_{i \geq 0} M_{n,m,s,i}}
\]

(52)

where \( M_{n,m,s,i} \) is defined in Eq. 5 and \( m + 2i \) is the number of nonzero terms on the walk (i.e., \( \ell \) and \( r \) terms) – there are \( M_{n,m,s,i} \) of the walks and \( s^m \) cancels.

Remark 5. Another way to interpret this is that \( \langle g_m | F | g_m \rangle \) is the expected length of lattice paths with only step up and down reaching height \( m \) embedded in colored Motzkin paths of length \( n \), where the expectation is taken with respect to a uniform measure over all the walks with \( m \) imbalances and \( s \) colors.

It is not hard to see that the saddle point of \( \sum_{i \geq 0} i M_{n,m,i,s} \) is equal to that of the numerator, which is given by Eq. 10. Eq. 52 after replacing the sum over \( i \) with an integral over \( \beta \), as we did in our entanglement entropy calculation above, and extending to \( \pm \infty \) becomes

\[
\langle g_m | F | g_m \rangle = 2 \sigma + \frac{m}{4 \sqrt{s}} \left( \frac{m}{n} \right) + \frac{(4s - 1)m}{64 s \sqrt{s}} \left( \frac{m}{n} \right)^3 + O \left( m \left( \frac{m}{n} \right)^5 \right) + 2 \sqrt{n} \int d\beta \exp \left( -\frac{\sqrt{s} \beta^2}{\sigma^2} \right)
\]

(53)

Restoring the factor of 2 the new energies induced by an external field of what used to be zero energy states become

\[
\frac{\epsilon_0}{2n} \langle g_m | F | g_m \rangle = 2 \sigma \epsilon_0 + \frac{\epsilon_0}{16 \sqrt{s}} \left( \frac{m}{n} \right)^2 + \frac{(4s - 1) \epsilon_0}{1024 s \sqrt{s}} \left( \frac{m}{n} \right)^4 + O \left( \epsilon_0 \left( \frac{m}{n} \right)^6 \right)
\]

(54)

The physical conclusion is that the Hamiltonian without the boundary projectors, in the presence of an external field, \( F \), has the Motzkin state as its unique ground state with the energy \( 2 \sigma \epsilon_0 \). Moreover, what used to be the rest of the degenerate zero energy states, acquire energies above \( 2 \sigma \epsilon_0 \) that, for first elementary excitations, scale as \( 1/n^2 \). We believe that the gap in the balanced space nevertheless scales as \( 1/n^3 \).

The energy corrections just derived do not mean that the states with \( m \) imbalances will make up for all of the low energy excitations. For example, when \( s > 1 \), in the presence of an external field, the energy of states with a single crossed term will be lower in energy than those with large \( m \) imbalances and no crossings.

Lastly for small \( \epsilon_0 \) the ground state will deform away from the Motzkin state to prefer the terms with more zeros in the superposition. But as long as \( \epsilon_0 \) is small, the universality of Brownian motion guarantees the scaling of the entanglement entropy. It is, however, not yet clear to us whether \( \epsilon_0 \) can be tuned to a quantum critical point where the ground state has a sharp transition from highly entangled to nearly a product state. We suspect that the transition is smooth and that the entanglement continuously diminishes as \( \epsilon_0 \) becomes larger. In particular, in the limit \( \epsilon_0 \gg 1 \),
the effective Hamiltonian is

$$H'' \sim F + \epsilon' H,$$

where \( \epsilon' \) is a small parameter. The ground state of the unperturbed Hamiltonian \( F \) is now simply the product state \( 0 \rangle \otimes 2^n \).

VI. OPEN PROBLEMS

Open problems include:

1. Further investigation of the nature of the excited states.

2. Proof of the poly \((1/n)\) gap for Hamiltonians with interaction terms that create maximally entangled states out of the vacuum, i.e., \( \varphi = \frac{1}{\sqrt{2}} \{ (00) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \ell' r' \} \). The present technique for proving lower bounds would fail as \( P(s, t) \) can become negative.

3. Can a similar model for \( d < 5 \) systems be constructed, where the gap behaves similar to here and the entanglement entropy is long-ranged? Previously, a fermionic \( d = 4 \) model was proposed whose entanglement entropy grows linearly with \( n \) [4]. However, we believe (not yet proved) that the gap is exponentially small for that model. Can other models with \( d < 5 \) be built such that the gap closes slowly with \( n \)?

4. Is \( \sqrt{n} \) entanglement entropy as much as one can get in 'physically reasonable' models [25]? 

5. What does the continuum limit of the class of Hamiltonians proposed here look like?

6. It may be possible to improve the upper bound to be \( O(n^{-3}) \) for the model with boundaries.

7. We think the combinatorial techniques introduced here add to the toolbox of methods for proving the gap of local Hamiltonians. It would be interesting to see other applications of them for example in proving the conjecture in the paper by Breuckmann and Terhal [17].

8. The spin-spin correlation function in the ground state can in principle be calculated using the techniques that were used to calculate entanglement entropies. It would be interesting to know how the correlation functions \( \langle \sigma_i \sigma_k \rangle \) and \( \langle \sigma_i \sigma_{i+1} \sigma_k \sigma_{k+1} \rangle \) scale with \( |i - k| \).

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