EXTENSIONS OF PRINCIPAL $\mathbb{G}_a$-BUNDLES OVER $\mathbb{A}_2^2$

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Abstract. The aim of this article is to make a first step towards the classification of complex normal affine $\mathbb{G}_a$-threefolds. We consider the special case where the algebraic quotient is $\mathbb{A}_2^2$ and the quotient morphism restricts to a principal $\mathbb{G}_a$-bundle over the punctured plane $\mathbb{A}_2^2 := \mathbb{A}_2^2 \setminus \{0\}$.

1. Introduction

Given a complex normal affine variety $X$ with an algebraic $\mathbb{G}_a$-action, the algebraic quotient $X/\mathbb{G}_a$ is by a theorem of Zariski [Nag59, p. 45] known to exist if $\dim X \leq 3$. In case the quotient exists it is of dimension $\dim X - 1$ unless the $\mathbb{G}_a$-action is the trivial one. Let $\pi: X \to X/\mathbb{G}_a$ be the quotient morphism, and denote by $V \subset X/\mathbb{G}_a$ the maximal open subset such that $\pi|^{-1(V)}: \pi^{-1}(V) \to V$ is a principal $\mathbb{G}_a$-bundle; this set is always nonempty.

If $X$ is a surface the quotient morphism is surjective and $V$ is affine. In particular $\pi^{-1}(V) \subset X$ is equivariantly isomorphic to $V \times \mathbb{G}_a$, where $\mathbb{G}_a$ acts by translation on the second factor. It is shown in [Fie94] that complex normal affine $\mathbb{G}_a$-surfaces are classified by the quotient $X/\mathbb{G}_a$ and neighbourhoods of the fibers of the points in $X/\mathbb{G}_a \setminus V$.

If $X$ is a threefold, the situation is more involved. In this article we investigate the $\mathbb{G}_a$-threefolds which have algebraic quotient $X/\mathbb{G}_a \cong \mathbb{A}_2^2$ and for which the quotient morphism becomes a principal $\mathbb{G}_a$-bundle when restricted to $\pi^{-1}(A_2^2 \setminus \{0\})$. An affine extension of a principal $\mathbb{G}_a$-bundle over $\mathbb{A}_2^2$ is of exactly this kind.

Definition 1.1. An affine extension of a principal $\mathbb{G}_a$-bundle $\pi: P \to \mathbb{A}_2^2$ is a normal affine $\mathbb{G}_a$-variety $\hat{P}$ together with a morphism $\hat{\pi}: \hat{P} \to \mathbb{A}_2^2$ and a $\mathbb{G}_a$-equivariant dominant open embedding $\iota: P \hookrightarrow \hat{P}$ with $\iota(P) = \hat{\pi}^{-1}(\mathbb{A}_2^2 \setminus \{0\})$, which makes the following diagram commute.

\[
\begin{array}{ccc}
P & \xrightarrow{\iota} & \hat{P} \\
\pi \downarrow & & \downarrow \hat{\pi} \\
\mathbb{A}_2^2 & \rightarrow & \mathbb{A}_2^2
\end{array}
\]

We will use the notation $E = \hat{\pi}^{-1}(0)$, $A = \mathcal{O}(P)$ and $B = \iota^*(\mathcal{O}(\hat{P})) \subset A$ for the exceptional fiber, the regular functions on $P$ and the subalgebra of regular functions on $P$ that extend to $\hat{P}$, respectively.

Remark 1.2. If $\hat{P}$ is an affine extension of $P$, then $\hat{\pi}: \hat{P} \to \mathbb{A}_2^2$ is the algebraic quotient morphism; this follows from the fact that $\mathcal{O}(A_2^2) = \mathcal{O}(\mathbb{A}_2^2)$.

Theorem 1. For an affine extension $\hat{P}$ of the trivial bundle $P = \mathbb{A}_2^2 \times \mathbb{G}_a$ the morphism $\iota$ extends to a morphism $j: \mathbb{A}_2^2 \times \mathbb{G}_a \to \hat{P}$, which is either an open embedding or contracts $0 \times \mathbb{G}_a$ to a singular point of $\hat{P}$. In the first case either $j$ is an isomorphism or $E = \hat{\pi}^{-1}(0)$.

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We present an example of the first kind in Example 2. We note that the condition \( j: \mathbb{A}^2 \times \mathbb{G}_a \to \hat{P} \) contracts \( 0 \times \mathbb{G}_a \) to a singular point \( x_0 \in \hat{P} \), the unique \( \mathbb{G}_a \)-fixed point in the exceptional fiber \( E = \hat{P} \). Furthermore all the irreducible components \( E_i \to E \) of the exceptional fiber contain \( x_0 \), and for each \( i \), \( E_i \setminus \{ x_0 \} \) is \( \mathbb{G}_a \)-bundle over a rational curve. Finally, we also have \( E = \hat{P} \mathbb{G}_a \).

Theorems 1 and 2 both about extensions of the trivial bundle, are proven in section 2, where we also give examples of such extensions. In section 3 we consider nontrivial principal \( \mathbb{G}_a \)-bundles over \( \mathbb{A}^2 \), and prove Proposition 3.2 which says that such bundles are affine; indeed they can be realized as pullbacks of the most "basic" nontrivial bundle

\[
\text{SL}_2 \to \mathbb{A}^2_\times, \quad \left( \begin{array}{cc} x & u \\ y & v \end{array} \right) \mapsto \left( \begin{array}{c} x \\ y \end{array} \right).
\]

Recall that \( \mathbb{G}_a \) embeds in \( \text{SL}_2 \) as the upper-triangular unipotent matrices; the action of \( \mathbb{G}_a \) on \( \text{SL}_2 \) is then given by right multiplication. In section 4 we study the graded algebra \( \text{gr}_D(O(\hat{P})) \), as a tool for the classification of affine extensions. We also observe that the exceptional fiber \( E := \hat{n}^{-1}(0) \) is purely two dimensional for any affine extension \( \hat{P} \) of \( P \) (unless \( \hat{P} = P \) is the "trivial" extension of \( P \)). In section 5 we construct two families of extensions of \( P = \text{SL}_2 \), in particular we find a sequence of extensions \( \{ \hat{P}_n \mid n \in \mathbb{N} \} \) with the following property.

**Theorem 3.** Any affine extension \( \hat{P} \) of \( P = \text{SL}_2 \) is an affine \( \mathbb{G}_a \)-equivariant modification of some \( \hat{P}_n \).

In the two constructed families of affine extensions of \( \text{SL}_2 \), the exceptional fiber consists of \( \mathbb{G}_a \)-fixed points only. In the end of section 5 we also give examples of affine extensions of \( \text{SL}_2 \) with a free action of \( \mathbb{G}_a \) on the exceptional fiber, as well as with a 1-dimensional fixed point set (the case of isolated fixed points for a \( \mathbb{G}_a \)-action is not possible).

In the short final section we show that extensions of \( \text{SL}_2 \) induce extensions of \( P \) for any nontrivial bundle over \( \mathbb{A}^2 \).

## 2. Extensions of the Trivial \( \mathbb{G}_a \)-Bundle

Let \( P := \mathbb{A}^2 \times \mathbb{G}_a \) be the trivial \( \mathbb{G}_a \)-bundle, let \( A := O(P) = \mathbb{C}[x, y, s] \) be its algebra of regular functions, and let \( \hat{P} \) be an affine extension. Using the canonical isomorphism \( O(\mathbb{A}^2 \times \mathbb{G}_a) \cong O(\mathbb{A}^2 \times \mathbb{G}_a) = \mathbb{C}[x, y, s] \), we can define a morphism \( j: \mathbb{A}^2 \times \mathbb{G}_a \to \hat{P} \) by the condition \( j^* = \tau^* : O(\hat{P}) \to \mathbb{C}[x, y, s] \). Note that \( j: \mathbb{A}^2 \times \mathbb{G}_a \to \hat{P} \) is an extension of \( \tau: \mathbb{A}^2 \times \mathbb{G}_a \to \hat{P} \), and that \( B := \tau^*(O(\hat{P})) \subset A \) is a subalgebra.

The facts that the \( \mathbb{G}_a \)-action on \( \mathbb{A}^2 \times \mathbb{G}_a \) extends to \( \hat{P} \) and that the morphism \( \hat{P} \to \mathbb{A}^2 \) is locally trivial over \( \mathbb{A}^2 \) can be stated in algebraic terms as follows. The algebra \( B \) is invariant with respect to the locally nilpotent derivation \( \partial_s := \frac{\partial}{\partial s}: A \to A \) which corresponds to the \( \mathbb{G}_a \)-action on \( \mathbb{A}^2 \), furthermore \( \mathbb{C}[x, y] \subset B \), and \( B_x = A_x, B_y = A_y \) holds for the localizations with respect to the coordinate functions \( x \) and \( y \).
Proof of Theorem. The morphism \( j : \mathbb{A}^2 \times \mathbb{G}_a \to \hat{P} \) is equivariant since \( x \) is, and it follows that the restriction \( j|_{0 \times \mathbb{G}_a} \) is either injective or constant with image \( x_0 \) for some \( x_0 \in \hat{P} \). In the first case we may even conclude that all its tangent maps are injective: since \( j|_{0 \times \mathbb{G}_a} \) is not constant, that restriction has at least somewhere an injective tangent map, and then the equivariance gives that its tangent map everywhere is injective.

Finally we use the fact that \( \mathbb{A}^2 \cong \mathbb{A}^2 \times \{ x \} \leftrightarrow \mathbb{A}^2 \times \mathbb{G}_a \twoheadrightarrow \hat{P} \twoheadrightarrow \mathbb{A}^2 \) is the identity in order to conclude that the tangent maps of \( j \) along \( 0 \times \mathbb{G}_a \) are injective as well. It follows (using an embedding into some \( \mathbb{A}^n \) and the implicit function theorem), that \( j(\mathbb{A}^2 \times \mathbb{G}_a) \) is a three-dimensional smooth complex submanifold, and hence \( j(\mathbb{A}^2 \times \mathbb{G}_a) \subset \hat{P}_{\text{reg}} \).

Applying Zariski’s Main theorem we obtain, the morphism \( j \) being birational with finite fibers and \( \hat{P}_{\text{reg}} \) being normal, that \( j : \mathbb{A}^2 \times \mathbb{G}_a \to \hat{P}_{\text{reg}} \) is an open embedding and thus \( j : \mathbb{A}^2 \times \mathbb{G}_a \to \hat{P} \) as well.

Furthermore, \( E_0 = \hat{P} \setminus j(\mathbb{A}^2 \times \mathbb{G}_a) \) is purely two dimensional, being the complement of an affine open set. In the second case, the point \( x_0 \) is a singularity: if not, we could take a non-vanishing three form \( \omega \) on some neighbourhood \( V \) of \( x_0 \). Its pullback \( j^*(\omega) \) would be a three form on the smooth threefold \( j^{-1}(V) \), with zero set \( j^{-1}(V) \cap (0 \times \mathbb{G}_a) \), but this is impossible for dimension reasons.

We now describe a way of constructing smooth extensions of the trivial \( \mathbb{G}_a \)-bundle which are not \( \mathbb{A}^2 \times \mathbb{G}_a \). Let \( M_1 = M_2 = \mathbb{A}^2 \), let \( U_1 = U_2 = \mathbb{A}^1 \times \mathbb{A}^1 \) and let \( M \) be the prevariety which is obtained by glueing \( M_1 \) and \( M_2 \) along \( U_1 \) and \( U_2 \) via the morphism \( U_1 \to U_2, (x, y) \mapsto (x, xy) \). Denote by \( \mathbb{A}^1 \) the affine line with two origins, i.e. the prevariety obtained by glueing \( X_1 = X_2 = \mathbb{A}^1 \) along \( V_1 = V_2 = \mathbb{A}^1 \) via the identity morphism \( V_1 \to V_2 \). Then there is a natural morphism \( \varphi : M \to \mathbb{A}^1 \) given by \( M_i \to X_i, (x, y) \mapsto x \) for \( i = 1, 2 \). Now take any nontrivial principal \( \mathbb{G}_a \)-bundle \( Q \to \mathbb{A}^1 \) and let \( \hat{P} := \varphi^*(Q) \); then \( \hat{P} \) is an affine \( \mathbb{G}_a \)-variety since \( Q \) is, and the natural morphism \( \hat{P} \to Q \) is affine. The principal \( \mathbb{G}_a \)-bundle \( \hat{P} \to M \) has trivializations over the affine subsets \( \varphi^{-1}(X_i) = M_i \), for \( i = 1, 2 \), and the trivial principal \( \mathbb{G}_a \)-bundle is embedded into \( \hat{P} \) via the canonical embedding \( \mathbb{A}^2 \times \mathbb{G}_a \to M_2 \times \mathbb{G}_a = \mathbb{A}^2 \times \mathbb{G}_a \). The quotient morphism is \( \hat{P} \to M \to \mathbb{A}^2 \), where the second arrow is the identity on \( M_2 \subset M \) and \( (x, y) \mapsto (x, xy) \) on \( M_1 \subset M \).

Example 2.1. We determine \( \hat{P} \) explicitly in a special case. Let \( Q_i = X_i \times \mathbb{G}_a \), for \( i = 1, 2 \), and let \( Q \) be the principal \( \mathbb{G}_a \)-bundle which is obtained by glueing \( Q_1 \) and \( Q_2 \) along \( V_1 \times \mathbb{G}_a \) and \( V_2 \times \mathbb{G}_a \) via the morphism \( V_1 \times \mathbb{G}_a \to V_2 \times \mathbb{G}_a, (x, t) \mapsto (x, t + \frac{1}{2}) \). Then \( \hat{P} \) is obtained by glueing \( M_1 \times \mathbb{G}_a \) and \( M_2 \times \mathbb{G}_a \) along \( U_1 \times \mathbb{G}_a \) and \( U_2 \times \mathbb{G}_a \) via the morphism \( U_1 \times \mathbb{G}_a \to U_2 \times \mathbb{G}_a, (x, y, s) \mapsto (x, xy, s + \frac{1}{x}) \).

We define a morphism \( \eta : \hat{P} \to \mathbb{A}^5 \) by

\[
\eta : (x, y, s) \mapsto \begin{cases} 
(x, xy, xs + 1, yxs + y, xs^2 + s) & \text{if } (x, y, s) \in M_1 \times \mathbb{G}_a \\
(x, y, xs, yxs, xs^2 - s) & \text{if } (x, y, s) \in M_2 \times \mathbb{G}_a.
\end{cases}
\]

This is in fact a closed immersion whose image is the irreducible smooth subvariety \( Z \hookrightarrow \mathbb{A}^5 \) that is given by the three equations \( T_1 T_4 - T_2 T_3 = T_2 T_5 + T_4 - T_3 T_4 = T_1 T_5 - T_3^2 + T_3 = 0 \). The inverse morphism \( \eta^{-1} : Z \to \hat{P} \) is given by

\[
\eta^{-1} : (a, b, c, d, e) \mapsto \begin{cases} 
(a, \frac{d}{c}, \frac{e}{c}) & \text{if } c \neq 0 \\
(a, b, \frac{e}{c - 1}) & \text{if } c \neq 1.
\end{cases}
\]

It follows that in this example we have \( B = \mathbb{C}[x, y, xs, yxs, xs^2 - s] \subset \mathbb{C}[x, y, s] \). Note that the vertical \( \mathbb{G}_m \)-action defined on \( \mathbb{A}^2 \times \mathbb{G}_a \subset M_2 \times \mathbb{G}_a \) does extend to \( M_2 \times \mathbb{G}_a \), but not to \( \hat{P} \).
We now pass on to the second kind of extensions of the trivial $\mathbb{G}_a$-bundle, i.e. extensions such that the natural morphism $j: \mathbb{A}^2 \times \mathbb{G}_a \to \hat{P}$ contracts $0 \times \mathbb{G}_a$ to a singular point.

**Lemma 2.2.** An extension $\hat{P}$ of the trivial $\mathbb{G}_a$-bundle is of the second kind if and only if $B$ is a subalgebra of $\mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} (x, y)s^\nu$.

**Proof.** Note that the (non finitely generated) algebra $\mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} (x, y)s^\nu$ consists exactly of those functions on $\mathbb{A}^2 \times \mathbb{G}_a$ that are constant along $0 \times \mathbb{G}_a$. Suppose that $j(0 \times \mathbb{G}_a) = \{x_0\}$ for some $x_0 \in \hat{P}$. Then $j^*(f)(0 \times \mathbb{G}_a) = \{f(x_0)\}$ for all $f \in \mathcal{O}(\hat{P})$, so $B \subset \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} (x, y)s^\nu$. Conversely, if $j^*(f)$ is constant along $0 \times \mathbb{G}_a$ for all $f \in \mathcal{O}(\hat{P})$, it follows that $j$ contracts $0 \times \mathbb{G}_a$ since $\mathcal{O}(\hat{P})$ separates points in $\hat{P}$. \[\square\]

**Remark 2.3.** It follows from Lemma 2.2 that $j: \mathbb{A}^2 \times \mathbb{G}_a \to \hat{P}$ contracts $0 \times \mathbb{G}_a$ if the $\mathbb{G}_m$-action extends to $\hat{P}$, except in the case where $\hat{P} = \mathbb{A}^2 \times \mathbb{G}_a$.

The vertical $\mathbb{G}_m$-action on $\mathbb{A}^2 \times \mathbb{G}_a$ corresponds to the $s$-grading of $\mathbb{C}[x, y, s]$, and the subalgebra $B$ inherits this $s$-grading if and only if the vertical $\mathbb{G}_m$-action on $\mathbb{A}^2 \times \mathbb{G}_a$ extends to $\hat{P}$. It follows from Lemma 2.2 and Remark 2.3 that if the $\mathbb{G}_m$-action extends to $\hat{P}$, then $B$ is of the form $B = \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} m_\nu s^\nu$, where $(m_\nu)_{\nu \in \mathbb{N}_{>0}}$ is a sequence of $(x, y)$-primary ideals $m_\nu \subset \mathbb{C}[x, y]$ such that $m_\nu m_\lambda \subset m_{\nu+\lambda}$ for all $\nu, \lambda \in \mathbb{N}_{>0}$. Moreover, by the fact that the $\mathbb{G}_m$-action on $\mathbb{A}^2 \times \mathbb{G}_a$ extends to $\hat{P}$, the algebra $B$ is $\partial_s$-invariant and thus the sequence $(m_\nu)_{\nu \in \mathbb{N}_{>0}}$ is decreasing.

**Example 2.4.** If $b \subset \mathbb{C}[x, y]$ is any $(x, y)$-primary ideal, and $m_\nu = b^\nu$, then $\hat{P}$ is the relative affine cone belonging to the projective spectrum $\text{Proj}(\hat{B})$, where $\hat{B}$ is endowed with its $s$-grading. In this situation, $\text{Proj}(B)$ is the blowup of $\mathbb{A}^2$ at the ideal $b$.

In particular, if $b = (g, h)$ with two polynomials $g, h \in \mathbb{C}[x, y]$ with their origin as support, we have $B = \mathbb{C}[x, y, gs, hs]$, $$\hat{P} \cong \{(x, y, u, v) \in \mathbb{A}^4; h(x, y)u - g(x, y)v = 0\},$$ and the natural morphism is given by with $j(x, y, s) = (x, y, g(x, y)s, h(x, y)s)$. The case $g = x, h = y$ gives the Segre cone.

We recall the notion of a $\mathbb{G}_m$-bundle, used in the formulation of Theorem 2.

**Definition 2.5.** An affine morphism $q: X \to Y$ from a $\mathbb{G}_m$-variety $X$ to a variety $Y$ is called a $\mathbb{G}_m$-bundle if the following two conditions hold.

(1) The $\mathbb{G}_m$-orbits are the $q$-fibres.

(2) We have $V \cong q^{-1}(V)/\mathbb{G}_m$ for any affine open subset $V \subset Y$.

**Remark 2.6.** A $\mathbb{G}_m$-bundle $q: X \to Y$ is principal if and only if the $\mathbb{G}_m$-action on $X$ is free.

**Example 2.7.** For a finitely generated algebra $B = \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} m_\nu s^\nu$ endowed with its $s$-grading and ideals $m_\nu, \nu \in \mathbb{N}$, as above, there is a natural morphism $p: \text{Proj}(B) \to \mathbb{A}^2$;

it is a proper modification of $\mathbb{A}^2$ centered at the origin and

$$q: \hat{P}_s := \text{Sp}(B) \setminus V(B_{>0}) \to \text{Proj}(B)$$

is a $\mathbb{G}_m$-bundle. The singular point $x_0 \in \hat{P} = \text{Sp}(B)$ (c.f. Theorem 1) is the unique $\mathbb{G}_m$-fixed point in $E \hookrightarrow \hat{P}$. If $m_\nu = b^\nu$, where $b \subset \mathbb{C}[x, y]$ is a proper ideal with the origin as support, then $p: \text{Proj}(B) \to \mathbb{A}^2$ is the blowup of $\mathbb{A}^2$ along the ideal $b$ and $q: \hat{P}_s \to \text{Proj}(B)$ is a principal $\mathbb{G}_m$-bundle.
Proof of Theorem 2. We have seen already in Remark 2.3 that $j: \mathbb{A}^2 \times \mathbb{G}_a \to \hat{P}$ contracts $0 \times \mathbb{G}_a$. Denote $Z \to \text{Proj}(B)$ a resolution of the singularities. Then $Z \to \mathbb{A}^2$ is a composition of blowups at points, hence the zero fiber has irreducible components isomorphic to $\mathbb{P}^1$. Thus the irreducible components of the zero fiber of $\text{Proj}(B) \to \mathbb{A}^2$ are dominated by $\mathbb{P}^1$, hence are rational curves. It remains to show that $\mathbb{G}_a$ acts trivially on $E$. The standard action of the affine group $\mathbb{G}_a \times \mathbb{G}_m$ on $\mathbb{G}_a \cong \mathbb{A}^1$ yields an action on $P \cong \mathbb{A}^2 \times \mathbb{G}_a$ extending to $\hat{P}$. Its orbits have at most dimension 1, since that holds on $P$ and $P$ is dense in $\hat{P}$. Assume that there is a nontrivial $\mathbb{G}_a$-orbit $\mathbb{G}_a \ast x \to E$. Then we have even

$$(\mathbb{G}_a \times \mathbb{G}_m)x = \mathbb{G}_a \ast x,$$

the left hand side being irreducible and one-dimensional, hence it is the union of the singular point $x_0 \in E$ and a $\mathbb{G}_m$-orbit. But since $E$ is purely two dimensional, there are infinitely many such orbits - a contradiction, since different orbits are disjoint! \hfill \Box

3. Principal $\mathbb{G}_a$-bundles over the punctured plane

Let $g, h \in \mathbb{C}[x, y]$ be polynomials with the origin as their only common zero, and consider the variety

$$P_{g,h} := \{(x, y, u, v) \in \mathbb{A}^4; h(x, y)u - g(x, y)v = 1\}.$$  

It is a principal $\mathbb{G}_a$-bundle over $\mathbb{A}^2$, whose $\mathbb{G}_a$-action is given by the locally nilpotent derivation which is defined by $D(x) = D(y) = 0, D(u) = g(x, y)$ and $D(v) = h(x, y)$. The projection $\pi: P_{g,h} \to \mathbb{A}^2, (x, y, u, v) \mapsto (x, y)$ has equivariant local trivializations with respect to the covering $U = \{\mathbb{A}^2_y, \mathbb{A}^2_h\}$, given respectively by

$$\pi^{-1}(\mathbb{A}^2_y) \to \mathbb{A}^2_y \times \mathbb{G}_a \quad \text{and} \quad \pi^{-1}(\mathbb{A}^2_h) \to \mathbb{A}^2_h \times \mathbb{G}_a,$$

$$(x, y, u, v) \mapsto ((x, y), u/g(x, y)) \quad \text{and} \quad (x, y, u, v) \mapsto ((x, y), v/h(x, y)).$$

Example 3.1. If we take $g(x, y) = x$ and $h(x, y) = y$, we get $\text{SL}_2 \cong P_{x,y}$.

The main goal of this section is to show Proposition 3.2 from which it follows that the total space of any nontrivial principal $\mathbb{G}_a$-bundle is affine. The latter is a fact which was mentioned already in [DuFil11 \S1.2].

Proposition 3.2. For any nontrivial $\mathbb{G}_a$-principal bundle $P \to \mathbb{A}^2$ there are polynomials $g, h \in \mathbb{C}[x, y]$ with the origin as their only common zero, such that

$$P \cong P_{g,h}.$$  

For the proof of Proposition 3.2 we study the algebra $A := \mathcal{O}(P)$ of regular functions on $P$; it has a locally nilpotent derivation, corresponding to its $\mathbb{G}_a$-action.

Definition 3.3. If $A$ is a $\mathbb{C}$-algebra with a locally nilpotent derivation $D: A \to A$, we define the $D$-filtration $(A_{\leq \nu})_{\nu \in \mathbb{N}}$ of $A$ by $A_{\leq \nu} := \ker D^{\nu+1}$.

In our case, where $A = \mathcal{O}(P)$ is a principal $\mathbb{G}_a$-bundle $P$, the $\mathbb{C}[x, y]$-submodule $A_{\leq \nu} \subset A$ consists of the functions whose restriction to any fiber is a polynomial of degree $\leq \nu$, and we have the following very explicit description.

Proposition 3.4. If $A = \mathcal{O}(P)$, where $\pi: P \to \mathbb{A}^2$ is a nontrivial principal $\mathbb{G}_a$-bundle, there are functions $f_0, f_1 \in A_{\leq 1}$, whose restrictions to any fiber $\pi^{-1}(x, y)$ are linearly independent affine functions. In particular the $\mathbb{C}[x, y]$-modules $A_{\leq \nu}$ are free:

$$A_{\leq \nu} = \bigoplus_{\alpha \in \mathbb{N}^2; |\alpha| = \nu} \mathbb{C}[x, y] f^\alpha$$

with $f^\alpha := f_0^\alpha f_1^{\alpha_1}$ for $\alpha = (\alpha_0, \alpha_1) \in \mathbb{N}^2$. 


Remark 3.5. The statement of Proposition 3.4 is, of course, also true in the case of the trivial $\mathbb{G}_a$-bundle. The essential difference between the trivial and the nontrivial bundle case is that $\mathbb{C}[x, y]$ is a direct summand of $\mathcal{O}(P)_{\leq 1}$ if $P$ is trivial, otherwise it is not.

Proof of Proposition 3.4. Consider the locally nilpotent derivation $D : A \to A$ induced by the $\mathbb{G}_a$-action on $P$. We sheafify the filtration $(A_{\leq \nu})_{\nu \in \mathbb{N}}$ as follows: for any open subset $U \subset \mathbb{A}_2^*$ we define
\[ \mathcal{F}_\nu(U) := \mathcal{O}(\pi^{-1}(U)_{\leq \nu} \subset \mathcal{O}(\pi^{-1}(U)). \]
Indeed $\mathcal{F}_\nu$ is a locally free sheaf of rank $\nu + 1$. But a locally free $\mathcal{O}_{\mathbb{A}_2}$-module is free according to Lemma 3.6 below, in particular
\[ \mathcal{F}_1 \cong (\mathcal{O}_{\mathbb{A}_2})^2. \]
Denote $f_0, f_1 \in \mathcal{F}_1(\mathbb{A}_2^*) = A_{\leq 1}$ the functions corresponding to $(1, 0), (0, 1) \in (\mathcal{O}(\mathbb{A}_2^*)^2 \cong \mathcal{F}_1(\mathbb{A}_2^*)$. Since the latter isomorphism is induced by a sheaf isomorphism, the sections $f_0, f_1 \in \mathcal{F}_1(\mathbb{A}_2^*)$ generate over any fiber the space of affine linear functions. It follows from Lemma 3.7 that
\[ A_{\leq \nu} = \mathcal{F}_\nu(\mathbb{A}_2^*) = \bigoplus_{\alpha \in \mathbb{N}^2; |\alpha| = \nu} \mathcal{O}(\mathbb{A}_2^*) f^\alpha. \]
\[ \square \]

Lemma 3.6. Any locally free $\mathcal{O}_{\mathbb{A}_2}$-module $\mathcal{F}$ of finite rank is free.

Proof. According to [Hor64] it is trivial in a punctured Zariski-neighbourhood of the origin, hence extends to a locally free module $\mathcal{F}$ on $\mathbb{A}_2^*$, but such a module is free, see [TLam06]. \[ \square \]

Lemma 3.7. If $p_0, p_1 \in \mathbb{C}[x]$ are linearly independent polynomials of degree $\leq 1$, we have
\[ \mathbb{C}[x]_{\leq \nu} = \bigoplus_{\alpha \in \mathbb{N}^2; |\alpha| = \nu} \mathbb{C}p^\alpha. \]

Proof. It suffices to show that the monomials are generators, and this can be proved by induction using the case $\nu = 1$ and $\mathbb{C}[x]_{\leq \nu + 1} = \mathbb{C}[x]_{\leq 1} \cdot \mathbb{C}[x]_{\leq \nu}$. \[ \square \]

Proof of Proposition 3.2. Let $P$ be a nontrivial principal $\mathbb{G}_a$-bundle, and take $g := Df_0, h := Df_1 \in \mathbb{C}[x, y]$, with the notation from Proposition 3.4. These functions have no common zero in the punctured plane $\mathbb{A}_2^*$, and we claim that $P \cong P_{g, h}$. It follows from the fact $D(hf_0 - gf_1) = 0$ that $hf_0 - gf_1 \in \mathcal{F}_0(\mathbb{A}_2^*) \subset \mathcal{F}_1(\mathbb{A}_2^*)$ is a nowhere vanishing function ($f_0, f_1$ being fiberwise linearly independent and $g, h$ having no common zero in $\mathbb{A}_2^*$) which is constant on $\pi$-fibers. Thus it is a pullback of a regular function on the punctured plane $\mathbb{A}_2^*$. But on the other hand $\mathcal{O}(\mathbb{A}_2^*)^* = \mathbb{C}^*$. It follows that $hf_0 - gf_1$ is a nonzero constant, which we may and will assume to be 1, i.e. $hf_0 - gf_1 = 1$. It follows that
\[ P \to P_{g, h}, z \mapsto (\pi(z), f_0(z), f_1(z)) \]
is an isomorphism. \[ \square \]

Corollary 3.8. Any nontrivial principal $\mathbb{G}_a$-bundle $\pi : P \to \mathbb{A}_2^*$ is isomorphic to a pullback $\varphi^*(\text{SL}_2)$ with a morphism $\varphi := (g, h) : \mathbb{A}_2^* \to \mathbb{A}_2^*$.

We now explain a homogeneity property that $\text{SL}_2$ have; see also [DuF11 §1.3].
Definition 3.9. Let $p, q \in \mathbb{N}$ be relatively prime natural numbers. A principal $\mathbb{G}_a$-bundle $P \to \mathbb{A}^2$ is called $(p, q)$-homogeneous, if there is a $\mathbb{G}_m$-action $P \times \mathbb{G}_m \to P$ such that the bundle projection $P \to \mathbb{A}^2$ is equivariant with respect to the action

$$\mathbb{A}^2 \times \mathbb{G}_m \to \mathbb{A}^2, ((x, y), \lambda) \mapsto (\lambda^p x, \lambda^q y).$$

With a homogeneous bundle we mean a $(p, q)$-homogeneous bundle for some pair $(p, q) \in \mathbb{N}^2$, and a $\mathbb{G}_m$-action of the above type will be called a horizontal action.

Since the translations form a normal subgroup of the affine linear group, the action of $\mathbb{G}_m$ on a $(p, q)$-homogeneous bundle $P$ normalizes the $\mathbb{G}_a$-action, i.e. there is a $d \in \mathbb{Z}$, such that $(z \ast \tau) \lambda = (z\lambda) \ast \lambda^{-d}\tau$ for all $x \in P$. In other words, the push forward of the $\mathbb{G}_a$-action with respect to the action of $\lambda \in \mathbb{G}_m$ results in a multiplication with $\lambda^{-d}$ of the acting "additive" parameter $\tau \in \mathbb{G}_a$. This normalization property is equivalent to the fact that the corresponding nilpotent derivation $D: A \to A$ is homogeneous of degree $d$ with respect to the grading which corresponds to the $\mathbb{G}_m$-action.

Example 3.10. If $g, h \in \mathbb{C}[x, y]$ are $(p, q)$-homogeneous of degree $m, n > 0$ respectively (i.e. $g(\lambda^p x, \lambda^q y) = \lambda^m g(x, y)$ and $h(\lambda^p x, \lambda^q y) = \lambda^n h(x, y)$), then $P \cong P_{p, h}$ admits a $\mathbb{G}_m$-action lifting the $(p, q)$-action on $\mathbb{A}^2$, namely

$$P \times \mathbb{G}_m \to P, ((x, y, u, v), \lambda) \mapsto (\lambda^p x, \lambda^q y, \lambda^{-n}u, \lambda^{-m}v).$$

Here we have $d = m + n$. In particular $\text{SL}_2$ is $(p, q)$-homogeneous for any $(p, q)$ with the $\mathbb{G}_m$-action where $\lambda \in \mathbb{G}_m$ acts via

$$\begin{pmatrix} x & u \\ y & v \end{pmatrix} \lambda := \begin{pmatrix} \lambda^p x & \lambda^{-q}u \\ \lambda^q y & \lambda^{-p}v \end{pmatrix}.$$ 

In this situation $d = p + q$. The $\mathbb{G}_m$-variety $\text{SL}_2$ with this action is denoted $P(p,q)$.

In order to obtain invariants for affine extensions we introduce the associated graded algebra of a $\mathbb{C}$-algebra endowed with a locally nilpotent derivation:

Definition 3.11. Given a $\mathbb{C}$-algebra $A$ together with a locally nilpotent derivation $D: A \to A$, we define the associated graded algebra $\text{gr}_D(A)$ as

$$\text{gr}_D(A) := \bigoplus_{\nu=0}^\infty A_{\leq \nu}/A_{\leq \nu-1}.$$ 

The "leading term" $\text{gr}(f) \in \text{gr}_D(A)$ of $f \in A \setminus \{0\}$ is defined as

$$\text{gr}(f) := f + A_{\leq \nu-1} \in \text{gr}_D(A)_\nu,$$

where $\nu \in \mathbb{N}$ is the unique natural number such that $f \in \ker D^{\nu+1} \setminus \ker D^\nu$.

We can always regard $\text{gr}_D(A)$ as a subalgebra of the polynomial algebra $A_{\leq 0}[S]$ in one indeterminate $S$ over $A_{\leq 0}$.

Proposition 3.12. Let $D: A \to A$ be a locally nilpotent derivation of the $\mathbb{C}$-algebra $A$. Then the sequence of ideals

$$a_{\nu} := D^\nu(A_{\leq \nu}) \hookrightarrow A_{\leq 0}$$

is decreasing and satisfies $a_0 = A_{\leq 0}$, and $a_\nu a_\mu \subset a_{\nu+\mu}$. Furthermore we have

$$\text{gr}_D(A) \cong \bigoplus_{\nu=0}^\infty a_\nu S^\nu \hookrightarrow A_{\leq 0}[S].$$

Proof. The isomorphism is induced by

$$\text{gr}_D(A)_\nu \to a_\nu S^\nu, a + A_{\leq \nu-1} \mapsto D^\nu a S^\nu.$$ 

$\square$
Corollary 3.13. Let \( A = \mathcal{O}(P) \) with a principal \( \mathbb{G}_a \)-bundle \( P \), and let \( f_0, f_1 \in A_{\leq 1} \) be as in Proposition 3.14. Then with \( g := D(f_0), h := D(f_1) \in \mathbb{C}[x, y] \) we have
\[
gr_D(A) = \bigoplus_{\nu=0}^{\infty} (g, h)^\nu S^\nu \hookrightarrow A_{\leq 0}[S].
\]

Proof. For \( \alpha = (\alpha_0, \alpha_1) \) with \( \alpha_0 + \alpha_1 = \nu \) we have \( D(f^\alpha) = \alpha! g^{\alpha_0} h^{\alpha_1}. \)

4. AFFINE EXTENSIONS

Let \( P \) be a principal \( \mathbb{G}_a \)-bundle, and let \( \hat{P} \) be an affine extension of \( P \). With our definition of affine extensions, \( \hat{P} = P \) is allowed if \( P \) is affine, which according to Proposition 3.12 is the case if and only if it is a nontrivial principal \( \mathbb{G}_a \)-bundle. We will call an extension ”proper” if \( P \subseteq \hat{P} \).

It is a general fact that the complement of an open affine subvariety of a variety is purely one-codimensional. In particular the exceptional fiber \( E = \pi^{-1}(0) \) of a proper affine extension \( \hat{P} \) of a nontrivial principal \( \mathbb{G}_a \)-bundle \( P \to \mathbb{A}^*_2 \) is purely two-dimensional.

Example 4.1. Let \( g, h, p \in \mathbb{C}[x, y] \) be polynomials, where \( g, h \) have the origin as their only common zero. Take \( \hat{P} := Y^{(2)} \) to be the normalization of
\[
Y := \{(x, y, u, v) \in \mathbb{A}^4; h(x, y)u - g(x, y)v = p(x, y)\}
\]
with the \( \mathbb{G}_a \)-action \((x, y, u, v) * \tau = (x(x, y), u + \tau g(x, y), v + \tau h(x, y)) \) and \( P := (\mathbb{A}^2_2 \times \mathbb{A}^2_2) \cap Y \).

Then \( P \) is a nontrivial principal \( \mathbb{G}_a \)-bundle if and only if \( p \notin (g, h) \), and \( \hat{P} \) is a proper affine extension of \( P \) if and only if \( p(0, 0) = 0 \). In that case the exceptional fibre \( E \) consists of fixed points only.

Remark 4.2. If \( A = \mathcal{O}(P) \) is the algebra of regular functions on \( P \), then affine extensions \( \hat{P} \) correspond to finitely generated normal subalgebras \( B \subset A \), such that
\[
\begin{align*}
(1) & \quad \mathbb{C}[x, y] \subset B, \\
(2) & \quad B_x = A_x \text{ and } B_y = A_y, \text{ and} \\
(3) & \quad D(B) \subset B, \text{ where } D \text{ is the locally nilpotent derivation which corresponds to the } \\
& \quad \mathbb{G}_a \text{-action on } P.
\end{align*}
\]

In order to obtain a rough classification of affine extensions of a given principal \( \mathbb{G}_a \)-bundle, we consider the associated graded subalgebras
\[
gr_D(B) \hookrightarrow gr_D(A) \hookrightarrow \mathbb{C}[x, y][S].
\]
More precisely we have
\[
gr_D(B) = \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} m_\nu S^\nu,
\]
where \( m_\nu = D^\nu(B_{\leq \nu}) \) is a decreasing sequence of ideals which satisfy \( m_\nu m_\mu \subset m_{\nu + \mu} \) for all \( \nu, \mu \geq 1 \) and, by Corollary 3.13, \( m_\nu \subset (g, h)^\nu \) if \( P = P_{g, h} \). Moreover, the \( m_\nu \) are \((x, y)\)-primary; this follows from the localization condition (2) of Remark 4.2 above, since \( gr_D(A)_f = gr_D(A_f) = gr_D(B_f) = gr_D(B)_f \) for \( f = x, y \).

Remark 4.3. If \( gr_D(B) \subset gr_D(A) \) is finitely generated, then it defines an (not necessarily normal) affine extension of the trivial bundle \( \mathbb{A}^*_2 \times \mathbb{G}_a \), where even the vertical \( \mathbb{G}_m \)-action extends. The latter follows from the discussion after Remark 2.3.

Remark 4.4. If \( B = \mathbb{C}[x, y][f_1, \ldots, f_r] \subset A \), it follows that
\[
\mathbb{C}[x, y][gr(f_1), \ldots, gr(f_r)] \subset gr_D(B),
\]
but it is not clear that the last inclusion always is an equality; actually, it is not even clear that \( gr_D(B) \) has to be finitely generated.
Remark 4.5. We have just seen that the graded subalgebra $\text{gr}_D(B) \hookrightarrow \text{gr}_D(A)$, where $B \subset A$ is the algebra of an affine extension, is of the form
\[
C = \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} c_{\nu} S^{\nu}
\]
with some $(x, y)$-primary ideals satisfying $c_{\nu} \subset (g, h)^{\nu}$. It would be interesting to know if all subalgebras of $\text{gr}_D(A)$ of the above form arise from subalgebras $B \hookrightarrow A$ belonging to an affine extension $\hat{P} = \text{Sp}(B)$.

Proposition 4.6. The algebra $B = \psi^* (\mathcal{O}(\hat{P})) \subset A$ of regular functions of an affine extension $\hat{P}$ is uniquely determined by $\text{gr}_D(B) \subset \text{gr}_D(A)$ if $\text{gr}_D(B)$ is generated by $\text{gr}_D(B)_1$ as a $\mathbb{C}[x, y]$-algebra.

Proof. If $\text{gr}_D(B)$ is finitely generated, we can take the generators to be homogeneous, i.e.
\[
\text{gr}_D(B) = \mathbb{C}[x, y][\text{gr}(f_1), \ldots, \text{gr}(f_r)]
\]
for some $f_1, \ldots, f_r \in B$. Then it follows that $B = \mathbb{C}[x, y][f_1, \ldots, f_r]$, since $B_{\leq n} \subset \mathbb{C}[x, y][f_1, \ldots, f_r]$ holds by induction for all $n \in \mathbb{N}$. Now, if $\text{gr}_D(B)$ is generated by $\text{gr}_D(B)_1$, we may use that for $f \in A_{\leq 1}$ we have: $f \in B \iff \text{gr}(f) \in \text{gr}_D(B)_1$ because of $B_{\leq 0} = A_{\leq 0} = \mathbb{C}[x, y]$. \qed

Remark 4.7. If $B \subset \hat{B} \subset A$, we still have $B = \hat{B} \iff \text{gr}_D(B) = \text{gr}_D(\hat{B})$. Namely, if $\hat{b} \in \hat{B}_{\leq \nu}$, there exists $b \in B_{\leq \nu}$ such that $\text{gr}(\hat{b}) = \text{gr}(b)$, and hence $b - \hat{b} \in B_{\leq \nu - 1} = B_{\leq \nu - 1}$. It follows by induction that $b \in B_{\nu}$.

Proposition 4.8. If $\hat{P} \not\cong \mathbb{A}^2 \times \mathbb{G}_a$ and $\text{gr}_D(B)$ is generated in degree 1, then the exceptional fiber $E \hookrightarrow \hat{P}$ consists of fixed points only.

Proof. Choose generators $\text{gr}(f_1), \ldots, \text{gr}(f_r) \in \text{gr}_D(B)_1$. Then
\[
\psi : \hat{P} \to \mathbb{A}^2 \times \mathbb{A}^r, z \mapsto (\pi(z), f_1(z), \ldots, f_r(z))
\]
is an equivariant embedding, when we endow the right hand side with the $\mathbb{G}_a$-action
\[
(x, y, u_1, \ldots, u_r) \ast \tau = (x, y, u_1 + \tau g_1(x, y), \ldots, u_r + \tau g_r(x, y)),
\]
where $g_i := Df_i$ regarded as function on $\mathbb{A}^2$. Assume that $g_i(0, 0) \neq 0$ for some $i$. Then $\hat{P} \to \mathbb{A}^2$ admits a trivialization over some neighbourhood of the origin, hence is a principal $\mathbb{G}_a$-bundle and thus $\hat{P} \cong \mathbb{A}^2 \times \mathbb{G}_a$.

We remark that $\psi(\pi^{-1}(x, y))$ is an affine line for $(x, y) \in \mathbb{A}^2$ and that the exceptional fiber consists of all lines in $\mathbb{A}^2 \times \mathbb{A}^r$, which are limits of such lines. \qed

5. Extensions of $\text{SL}_2$

If we take $P = \text{SL}_2$, it follows from Corollary 3.13 that
\[
\text{gr}_D(A) = \bigoplus_{\nu=0}^{\infty} (x, y)^\nu S^{\nu}.
\]
We have seen that any affine extension of $P$ induces a graded subalgebra of $\text{gr}_D(A)$; in this section we investigate this correspondence between affine extensions and graded subalgebras closer, and construct examples. We start with different graded subalgebras $C \hookrightarrow \text{gr}_D(A)$, and find affine extensions $\hat{P} = \text{Sp}(B)$ such that $C = \text{gr}_D(B)$. We always take $C$ of the form
\[
C = \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} m_{\nu} S^{\nu}
\]
where \( m_\nu \subset (x, y)^\nu \) are monomial ideals

\[
m_\nu = m(G_\nu) = \bigoplus_{(\alpha, \beta) \in G_\nu \cap \mathbb{Z}^2} x^\alpha y^\beta,
\]
and where \((G_\nu)_{\nu \geq 1}\) denotes a decreasing sequence of regions in the first quadrant of \(\mathbb{R}^2\) which satisfy \(G_\nu + G_\mu \subset G_{\nu + \mu}\). Note that \( C \) is generated by \( C_1 \) if and only if \( G_\nu = \sum_{\nu} G_1 \) holds for all \( \nu \in \mathbb{N}_{>0} \).

**Example 5.1.** (1) For \( n \in \mathbb{N} \), we take the sequence

\[
G_\nu = \{ (\alpha, \beta) \in \mathbb{R}^2; \alpha + \beta \geq (n + 2)\nu \}.
\]

We will shortly, for each \( n \), construct an affine extension \( \hat{P}_n \) of \( SL_2 \) corresponding to this sequence, and it follows from Proposition 4.6 that it is unique.

(2) For relatively prime \( p, q \in \mathbb{N}_{>0} \) and \( d = p + q \), we take the sequence

\[
G_\nu = \{ (\alpha, \beta) \in \mathbb{R}^2; p\alpha + q\beta \geq d\nu \},
\]
and construct corresponding affine extensions \( \hat{P}(p, q) \) of \( SL_2 \).

In both cases, \( \mathbb{G}_a \) acts trivially on the exceptional fiber \( E \). For the first family that follows from Proposition 4.8 and for the second family this is explained in Remark 5.4.

**Remark 5.2.** For \((p, q) = (1, 1)\) we get \( m_\nu = (x, y)^{2\nu} \), so \( C \) is generated by \( C_1 \) in this case and for the corresponding affine extensions we have \( \hat{P}(1, 1) \cong \hat{P}_0 \).

**Example 5.3.** If \((p, q) = (2, 1)\) we get

\[
G_1 = \{ (\alpha, \beta); 2\alpha + \beta \geq 3 \} \quad \text{and} \quad G_2 = \{ (\alpha, \beta); 2\alpha + \beta \geq 6 \}.
\]
It follows that \((3, 0) \in G_2\), but \((3, 0) \notin G_1 + G_1\), so \( C \) is not generated by \( C_1 \).

1.) **The family \( \hat{P}(p, q) \):** Denote \( P(p, q) \) the \( \mathbb{G}_m \)-variety \( SL_2 \) equipped with the action of Example 3.10. The \( \mathbb{G}_m \)-action on \( P(p, q) \) corresponds to a \( \mathbb{Z} \)-gradings

\[
A = \bigoplus_{\mu = -\infty}^\infty A(\mu),
\]
with respect to which the locally nilpotent derivation \( D: A \to A \) is homogeneous. Writing a matrix in \( SL_2 \) as in Example 3.10 and viewing its components as regular functions \( x, y, u, v \in A \) we have

\[
A_{\leq \nu} = \bigoplus_{k+\ell=\nu} \mathbb{C}[x, y]u^kv^\ell \quad \text{and} \quad A(\mu) = \sum_{p(i-\ell)+q(j-k)=\mu} \mathbb{C}x^iy^ju^kv^\ell.
\]

The locally nilpotent derivation on \( A \) is given by \( D(x) = D(y) = 0, D(u) = x, \) and \( D(v) = y \); in particular it is homogeneous of degree \( d := p + q \). Furthermore the horizontal \( \mathbb{G}_m \)-action on \( P \) has possibly nontrivial stabilizers only along two orbits, namely \( \mathbb{G}_m \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \) and \( \mathbb{G}_m \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \) with stabilizers the groups \( C_p \) and \( C_q \) of \( p \)-th and \( q \)-th roots unity respectively.

The quotient morphism with respect to the \( \mathbb{G}_m \)-action is given by

\[
P(p, q) \to P(p, q)/\mathbb{G}_m \cong \{(a, b, c) \in \mathbb{A}^3 \mid ac = b^q(b-1)^p\},
\]

\[
\left( \begin{array}{cc} x & u \\ y & v \end{array} \right) \mapsto (x^ay^p, xv, y^pv^q),
\]
and it is a \( \mathbb{G}_m \)-bundle in the sense of Definition 2.5 principal above the regular part of \( P(p, q)/\mathbb{G}_m \). Note that \( P(p, q)/\mathbb{G}_m \) has at most two singular points; it is smooth if and only if \( p = q = 1 \).
Now we take the above $G_m$-bundle and add a zero section:

$$\hat{P}(p, q) = P(p, q) \cup E,$$

i.e. we form the quotient

$$\hat{P}(p, q) := P(p, q) \times_{G_m} \mathbb{A}^1$$

with respect to the action $\lambda(z, x) := (z\lambda, \lambda^{-1}x)$. Here $E \cong P(p, q)/G_m$ is the image of $P(p, q) \times \{0\}$. Then we have $\hat{P} = \text{Sp}(B)$ with the non-negative part

$$B = \bigoplus_{\mu=0}^{\infty} A(\mu)$$

of the $\mathbb{Z}$-graded algebra

$$A = \bigoplus_{\mu=-\infty}^{\infty} A(\mu).$$

In particular, the $G_a$-action on $P$ extends to an action on $\hat{P}$, since $B$ is $D$-invariant, $D$ being homogeneous of degree $d \geq 2$. Let us now determine $\text{gr}_D(B)$ using the $G_m$-decomposition of $A$. Since the locally nilpotent derivation $D : A \to A$ is homogeneous, the $G_m$-grading descends to the associated graded algebra

$$\text{gr}_D(A) = \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} (x, y)^\nu S^\nu,$$

where the $(p, q)$-grading on $\mathbb{C}[x, y]$ satisfies $\deg(x) = p, \deg(y) = q$ and associates the degree $-d$ to the variable $S$ (though $S \not\in \text{gr}_D(A)$). Indeed $xS = \text{gr}(u)$ and $yS = \text{gr}(v)$.

It follows that

$$\text{gr}_D(B) = \text{gr}_D(A)_{\geq 0} \hookrightarrow \text{gr}_D(A),$$

where the subscript refers to the above grading. In other words,

$$\text{gr}_D(B) = \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} m_\nu S^\nu$$

with

$$m_\nu = ((x, y)\nu)_{\geq \nu d} = \bigoplus_{\alpha p + \beta q \geq \nu d} \mathbb{C}x^\alpha y^\beta.$$

Remark 5.4. For the above affine extensions $\hat{P}(p, q) \to \mathbb{A}^2$, the exceptional fiber $E \hookrightarrow \hat{P}$ consists of fixed points of the $G_a$-action. This follows since any function $D(f), f \in B$, vanishes on $E$ because of

$$D(B) \subset \bigoplus_{\mu=d} A(\mu).$$

2.) The family $\hat{P}_n, n \in \mathbb{N}$, consists of $\text{SL}_2$-embeddings $\hat{P}_n = \text{Sp}(B_n) \to \mathbb{A}^2, n \geq 0$, satisfying $m_\nu = (x, y)^{(n+2)\nu}$.

We saw in Remark 5.2 that $\hat{P}_0 \cong \hat{P}(1, 1)$. From now on we consider $P = \text{SL}_2 = \text{Sp}(A)$ with its structure of $(1, 1)$-homogeneous principal $G_a$-bundle: $P = P(1, 1)$. In order to define $\hat{P}_n$ for $n > 0$, we remark that the composition

$$P = \text{SL}_2 \to \mathbb{A}_+^2 \to \mathbb{P}^1$$

can be identified with

$$\text{SL}_2 \to \text{SL}_2/T_2,$$
where $T_2 \subset \text{SL}_2$ is the subgroup of upper triangular matrices of determinant 1. Indeed it is a locally trivial principal $T_2$-bundle, and the group homomorphism
\[ G_a \hookrightarrow T_2, s \mapsto \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \]
defines via matrix multiplication from the right the $G_a$-action, while the horizontal $G_m$-action corresponds in the same way to
\[ G_m \hookrightarrow T_2, t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}. \]
Assume now we have a simultaneous right and left embedding $F$ of $T_2$, i.e. a variety
\[ F \supset T_2 \cong \text{Sp}(\mathbb{C}[s, t, t^{-1}]) = \mathbb{A}^1 \times \mathbb{A}^1_*, \]
containing
\[ \mathbb{A}^1 \times \mathbb{A}^1_* = G_a \times_\sigma G_m \cong T_2, (s, t) \mapsto \begin{pmatrix} t & t^{-1}s \\ 0 & t^{-1} \end{pmatrix} \]
as dense open subset (the bijection defining a group isomorphism when regarding the left hand side as the semidirect product of $G_a$ and $G_m$ with respect to $\sigma: G_m \rightarrow \text{Aut}(G_a), t \mapsto \sigma_t$ and $\sigma_t(s) = t^2s$), such that both the right and left action of $T_2$ on itself extend respectively to a right and left action
\[ F \times T_2 \rightarrow F, \quad T_2 \times F \rightarrow F, \]
and the projection $T_2 = \mathbb{A}^1 \times \mathbb{A}^1_* \rightarrow \mathbb{A}^1_*$ extends to
\[ F \rightarrow \mathbb{A}^1 \]
with $F \setminus T_2$ as zero fiber. In order to check the extension property for one of the two actions, it suffices to consider the $G_m$- and $G_a$-action obtained by the respective above inclusions of $G_a$ and $G_m$ into $T_2$. Then
\[ Q := \text{SL}_2 \times_{T_2} F \]
is a (not necessarily affine) extension of $P = \text{SL}_2$. Here the construction of the right hand side should not be thought of as an algebraic quotient, the group $T_2$ not being reductive; instead we replace in the locally trivial principal $T_2$-bundle
\[ \text{SL}_2 \rightarrow \mathbb{P}^1 \cong \text{SL}_2/T_2 \]
the fiber $T_2$ with $F$.

We shall discuss several possible choices $F_n, n \in \mathbb{N}$, for the new fiber $F$. In order to do so, we regard $T_2$ as the embedding of the torus $\text{Sp}(\mathbb{C}[s, s^{-1}, t, t^{-1}]) = \mathbb{A}^1_\times \times \mathbb{A}^1_*$ corresponding to the cone
\[ \sigma := \mathbb{R}_{\geq 0}(1, 0) \subset \mathbb{R}^2. \]
Then we take $F_n$ to be the torus embedding defined by the cone
\[ \sigma_n := \mathbb{R}_{\geq 0}(1, 0) + \mathbb{R}_{\geq 0}(-n, 1). \]
Now the left additive action of $G_a \hookrightarrow T_2$ corresponds to the derivation
\[ D_\ell = \frac{\partial}{\partial s}: \mathbb{C}[s, t, t^{-1}] \rightarrow \mathbb{C}[s, t, t^{-1}], \]
while
\[ D_r: \mathbb{C}[s, t, t^{-1}] \rightarrow \mathbb{C}[s, t, t^{-1}], s \mapsto t^2, t \mapsto 0, \]
corresponds to the right additive action. It follows that the algebra of regular functions $O(F_n) = \mathbb{C}[st^n, t] \hookrightarrow \mathbb{C}[s, t, t^{-1}]$ is both $D_r$- and $D_\ell$-invariant, hence the $G_a$-actions extend to $F_n$. The right- and left $G_m$-actions given by $G_m \hookrightarrow T_2$ are induced by lattice
vectors, namely \((0, 1)\) and \((1, 1)\) respectively, thus extend as well. Altogether it follows that we have an extended left and right \(T_2\)-action on \(F_n\) and can form
\[
Q_n := \text{SL}_2 \times_{T_2} F_n.
\]
For \(n > 0\), \(Q_n\) is a non-affine extension: the extended horizontal \(G_m\)-action is fiber preserving (corresponding to the lattice vector \((0, 1)\)), and it induces an action on \(F_n \cong \mathbb{C}^2\) which is elliptic with exponents \((1, n)\) and which has a unique fixed point. It follows that its global fixed point set \(Q_n^{\text{ext}} \cong \mathbb{P}^1\) is a projective line.

**Proposition 5.5.** The algebra \(B_n := \mathcal{O}(Q_n) \subset A = \mathcal{O}(\text{SL}_2)\) is \(D\)-invariant and satisfies
\[
\text{gr}_D(B_n) = \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} (x, y)^{(n+2)\nu} S^\nu.
\]

In particular \(B_n\) is a finitely generated \(\mathbb{C}[x, y]\)-algebra and for \(n > 0\) the natural morphism \(\psi: Q_n \to \tilde{P}_n := \text{Sp}(B_n)\) is a small resolution of the unique singular point of \(\tilde{P}_n\).

**Proof.** Since \(\text{SL}_2\) acts on both \(Q_n\) and \(\mathbb{A}^2\), the ideals
\[
m_\nu := D\nu(\mathcal{O}(Q_n))_{\leq \nu}
\]
are \(\text{SL}_2\)-invariant and thus \(m_\nu = (x, y)^{k_\nu}, k_\nu \in \mathbb{N}\). Indeed they are graded ideals and their homogeneous summands irreducible \(\text{SL}_2\)-modules. Thus we have achieved our goal if we can show

1. \(m_\nu \subset (x, y)^{(n+2)\nu}\) for all \(\nu \geq 1\), and
2. \(m_1 \not\subset (x, y)^{n+3}\).

In order to obtain the first inclusion we fix a point \((a, b) \in \mathbb{A}_2^2\) and consider the following "fiber embedding diagram"
\[
\begin{array}{ccc}
F_n & \hookrightarrow & Q_n \\
\downarrow & & \downarrow \\
\mathbb{A}^1 & \hookrightarrow & \mathbb{A}^2
\end{array}
\]
where the first vertical arrow is induced by the inclusion \(\mathbb{C}[t] \hookrightarrow \mathbb{C}[st^n, t]\) and the second one is the extension of the bundle projection \(\text{SL}_2 \rightarrow \mathbb{A}_2^2\). The lower horizontal arrow identifies \(\mathbb{A}^1\) with the line \(\mathbb{C}(a, b) \hookrightarrow \mathbb{A}^2\), it is given by the algebra homomorphism \(\mathbb{C}[x, y] \hookrightarrow \mathbb{C}[t], x \mapsto at, y \mapsto bt\). Finally the morphism \(\iota\) is the extension of a coset embedding
\[
\mathbb{A}^1 \times \mathbb{A}_s^2 \equiv \left(\begin{array}{cc}a & c \\ b & d\end{array}\right) \ realize \ \text{SL}_2 \hookrightarrow \text{SL}_2, (s, t) \mapsto \left(\begin{array}{cc}a & c \\ b & d\end{array}\right) \cdot \left(\begin{array}{cc}t & t^{-1}s \\ 0 & t^{-1}\end{array}\right)
\]
where \(c, d \in \mathbb{C}\) with \(ad - bc = 1\). Now look at some \(f \in (B_n)_{\leq \nu}\). Its pullback \(\circ \iota\) to \(F_n\) is of the form
\[
f \circ \iota = \sum_{k \leq \nu, \ell} a_{k \ell} (st^n)^k t^\ell.
\]

Since \(D\) acts on both \(B_n = \mathcal{O}(Q_n)\) and \(\mathcal{O}(F_n) = \mathcal{C}[st^n, t]\) (with \(D = D_c\)), we see that
\[
D\nu(f)(at, bt) = \sum_{\ell \geq 0} a_{\nu \ell} t^{(n+2)\nu + \ell}.
\]

Since that holds for all \((a, b) \in \mathbb{A}_s^2\), it follows that \(D\nu(f) \in (x, y)^{(n+2)\nu}\).

To get the second point, we look at trivializations of \(Q_n \to \mathbb{P}^1\) over \(U_0 = \mathbb{P}^1 \setminus \{[0 : 1]\}\) and \(U_1 = \mathbb{P}^1 \setminus \{[1 : 0]\}\) and see that, as a subset of \(A = \mathcal{O}(\text{SL}_2)\), we have
\[
(x, y)^{n+1} u + (x, y)^{n+1} v \subset (B_n)_{\leq 1},
\]
but, for instance, \(D(x^{n+1}u) = x^{n+2} \not\subset (x, y)^{n+3}\).
As for the geometry of the morphism \( \psi: Q_n \to \hat{P}_n = \text{Sp}(B_n) \), we look first at the horizontal \( \mathbb{G}_m \)-actions on \( P = \text{SL}_2 \) and \( Q_n \supset P \) and the corresponding decompositions
\[
B := B_n = \bigoplus_{\mu = -\infty}^{\infty} B(\mu) \hookrightarrow A = \bigoplus_{\mu = -\infty}^{\infty} A(\mu).
\]
Recall that \( x, y \in A(1), u, v \in A(-1) \). Now \( B \) is generated as a \( \mathbb{C} \)-algebra by \( B_{\leq 1} \), i.e. by functions

\[
f(x, y)u + g(x, y)v, \ f(x, y)x + g(x, y)y \in (x, y)^{n+2},
\]
and it follows that
\[
B = \mathbb{C} \bigoplus_{\mu = 1}^{\infty} B(\mu).
\]
Hence the \( \mathbb{G}_m \)-action on \( \hat{P}_n \) has exactly one fixed point \( z_0 \in \hat{P}_n \) and \( \lim_{\lambda \to 0} z\lambda = z_0 \) for all \( z \in \hat{P}_n \). Note that in \( Q_n \) that limit exists as well, it is a point in the compact fixed point set \( F := Q_n^{\mathbb{G}_m} \). From that we can conclude that \( \psi \) is a proper morphism, hence onto, being dominant.

On the other hand, the functions in \((B_n)_{\leq 1}\) separate pairs of points in \( Q_n \), which do not simultaneously lie in the fixed point set \( F \), so \( \psi \) is injective outside the fixed point and \( \psi|_F \equiv z_0 \); indeed \( \psi^{-1}(z_0) = F \). Since \( B_n \) is normal, the variety \( Q_n \) being smooth, we may conclude that
\[
\psi|_{Q_n \setminus F}: Q_n \setminus F \to \hat{P}_n \setminus \{z_0\}
\]
is an isomorphism. Finally \( z_0 \in \hat{P} \) is a singular point: this follows with the same argument as in the proof of Theorem \([\text{I}]\). □

Proof of Theorem \([\text{II}]\). Let \( B_n := \iota^*(\mathcal{O}(\hat{P}_n)) \). If \( \hat{P} = \text{Sp}(B) \) is any extension and
\[
gr_D(B) = \mathbb{C}[x, y] \bigoplus_{\nu = 1}^{\infty} \mathfrak{m}_n S^\nu,
\]
choose \( n \in \mathbb{N} \) with \((x, y)^{n+2} \subset \mathfrak{m}_1 \). Then we have \((B_n)_{\leq 1} \subset B_{\leq 1}\) and thus \( B_n \subset B \), since \( B_n \) is, as a \( \mathbb{C}[x, y] \)-algebra, generated by \((B_n)_{\leq 1}\). □

Finally we construct some extensions with empty fixed point set and with one dimensional fixed point set.

**Proposition 5.6.** Assume that the exceptional fiber \( E \hookrightarrow Y = \text{Sp}(B) \) of an affine extension \( Y \to \mathbb{A}^2 \) of \( P \to \mathbb{A}^2 \) is a local hypersurface and coincides with the fixed point set \( E = Y^{\mathbb{G}_a} \). Let \( C \hookrightarrow E \) be a closed subset. Denote \( \text{Bl}_C(Y) \to Y \) the blowup of \( Y \) with center \( C \hookrightarrow E \) and \( \hat{E} \hookrightarrow \text{Bl}_C(Y) \) the strict transform of \( E \). Then
\[
Y_1 := \text{Bl}_C(Y) \setminus \hat{E}
\]
is an affine extension.

Proof. It is clear that \( Y_1 \) inherits a \( \mathbb{G}_a \)-action, since the center of the blowup is fixed by the \( \mathbb{G}_a \)-action on \( Y \). It remains to show \( Y_1 \) is affine. It is enough to check that the morphism \( Y_1 \to Y \) is affine, and for this we may assume that \( E \hookrightarrow Y \) is a hypersurface, i.e. \( I(E) = (f) \subset B = \mathcal{O}(Y) \). But then, if \( I(C) = (g_1, \ldots, g_s) \ni f \), we have
\[
Y_1 = \text{Sp}(B[\frac{g_1}{f}, \ldots, \frac{g_s}{f}]).
\]
□
Now we start with $Y = \tilde{P}_0 \supset \text{SL}_2$ and take $Y_1 := \text{Bl}_a(Y) \setminus \tilde{E}$ with the exceptional fiber $E \hookrightarrow \tilde{P}_0$ and some point $a \in E$. Using the realization of $Y$ as locally trivial bundle over $\mathbb{P}^1$, we see that we may think of $a \in Y$ as the origin in

$$a = (0, 0, 0) \in \mathbb{A}^3 = \text{Sp}(\mathbb{C}[x, y, z]) =: U$$

where $(x, y, z) \mapsto [1 : z]$ is the bundle projection, while $\mathbb{A}^1 \times \mathbb{A}_z^1 \times \mathbb{A}^1$ is $\mathbb{G}_a \times_{\alpha} \mathbb{G}_m \times \mathbb{A}^1$. Furthermore the $\mathbb{G}_a$-action is given by

$$D: \mathbb{C}[x, y, z] \rightarrow \mathbb{C}[x, y, z], \ x \mapsto y^2, \ y \mapsto 0, \ z \mapsto 0$$

and $E \cap U = \mathbb{A}_x^1 \times 0 \times \mathbb{A}^1$. Then above $U$, in the blowup, we have

$$U_1 := \text{Sp}(\mathbb{C}[x/y, z/y]),$$

with $D(x/y) = y$, and $D(y) = 0 = D(z/y)$. If we take $\xi = x/y, \eta = y, \zeta = z/y$, we see that $\mathbb{G}_a$ acts linearly on $U_1 = \mathbb{A}_x^3 = \text{Sp}(\mathbb{C}[\xi, \eta, \zeta])$, namely

$$D = \eta \frac{\partial}{\partial \xi},$$

or, equivalently,

$$t * (\xi, \eta, \zeta) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}.$$

Note that

$$Y_1 \supset U_1 \supset E_1 = \mathbb{A}_x^1 \times 0 \times \mathbb{A}^1.$$

Now let us apply the recipe of Proposition $5.6$ with some subvariety $C \hookrightarrow E_1$. We obtain an affine extension $Y_2$. We discuss several choices of $C$:

1. If $C = \{(0, 0, 0)\}$, the exceptional fiber is naturally isomorphic to

$$E_2 \cong \mathbb{P}(\mathbb{A}^3) \setminus \mathbb{P}(\mathbb{A}_x^1 \times 0 \times \mathbb{A}^1)$$

with the restriction of the induced linear $\mathbb{G}_a$-action on $\mathbb{P}(\mathbb{A}^3) \cong \mathbb{P}(T_0(\mathbb{A}^3))$. Hence the $\mathbb{G}_a$-action on $Y_2$ is free.

2. Now let $C \hookrightarrow E_1 = \mathbb{A}_x^1 \times 0 \times \mathbb{A}^1$ be a smooth curve. For the fiber $F_b$ over a point $b \in C$ there is a natural isomorphism

$$F_b \cong \mathbb{P}^1(\mathbb{A}^3/T_0(C)) \setminus \{(\mathbb{A}_x^1 \times 0 \times \mathbb{A}^1)/T_0(C)\}.$$

We distinguish two cases:

1. If $T_0(C) = \mathbb{C}(1, 0, 0)$, the $\mathbb{G}_a$-action on $F_b$ is trivial.

2. If $T_0(C) = \mathbb{C}(\alpha, 0, \beta)$ with $\beta \neq 0$, the $\mathbb{G}_a$-action on $F_b$ is free. So if $C$ is not a line parallel to $\mathbb{C}(1, 0, 0)$, the fixed point set has at least dimension one.

6. Extensions of nontrivial principal $\mathbb{G}_a$-bundles

We have seen that any nontrivial principal $\mathbb{G}_a$-bundle over the punctured plane is of the form $P = \varphi^*(\text{SL}_2)$, with a morphism $\varphi = (g, h): \mathbb{A}_x^3 \rightarrow \mathbb{A}_x^2$, where the polynomials $g, h$ have the origin as their only common zero. We write $\varphi = (g, h): \mathbb{A}_x^2 \rightarrow \mathbb{A}_x^2$ for the continuation of $\varphi$.

**Lemma 6.1.** The image of $\varphi: \mathbb{A}_x^2 \rightarrow \mathbb{A}_x^2$ contains $0 \in \mathbb{A}_x^2$ as an interior point.

**Proof.** If $0 \in \varphi(\mathbb{A}_x^2)$ is not an interior point, there is an irreducible curve $C \subset \mathbb{A}_x^2$ which passes through the origin such that $C \cap \varphi(\mathbb{A}_x^2)$ is finite. With the embedding $\mathbb{A}_x^2 \hookrightarrow \mathbb{P}^2, (x, y) \mapsto [x : y : 1]$, $\varphi$ induces a rational map $\varphi_P: \mathbb{P}^2 \rightarrow \mathbb{P}^2$, and by Noether-Castelnuovo’s classical theorem, there is a blowup $\xi: X \rightarrow \mathbb{P}^2$ and a projective morphism $\eta: X \rightarrow \mathbb{P}^2$ such that the following diagram commutes. Note that $\xi: X \rightarrow \mathbb{P}^2$ is the
blowup of a finite number of points in $\mathbb{P}^1 \times \{0\} \subset \mathbb{P}^2$, so in particular we may think of $\mathbb{A}^2$ as a subset of $X$.

Since projective morphisms pull back divisors to divisors, $\eta^*(C)$ is a curve in $X$. At least one of its irreducible components, $D$, passes through the point $\xi^{-1}(0:0:1)$, and then the curve $D \cap \mathbb{A}^2$ is sent by $\hat{\varphi}$ onto $0 \in \mathbb{A}^2$. This is a contradiction since $\hat{\varphi}^{-1}(0)$ is finite. □

**Proposition 6.2.** If $\hat{\mathbb{S}}$ is an affine extension of $\text{SL}_2$, then the normalized reduction $\hat{\mathbb{P}}$ of the pull back $\hat{\varphi}^*(\hat{\mathbb{S}})$ is an affine extension of $\mathbb{P}$ and $\hat{\varphi}$ induces an injection

$$\text{gr}_D(\mathcal{O}(\hat{\mathbb{S}})) \hookrightarrow \text{gr}_D(\mathcal{O}(\hat{\mathbb{P}})).$$

**Proof.** An extension $\hat{\mathbb{P}}$ of $\mathbb{P}$ is defined by completing the pullback diagram for $\mathbb{P}$ into a cartesian diagram as follows. Here $\psi := \varphi^*(\pi)$ and $\hat{\psi} := \hat{\varphi}^*(\hat{\pi})$.

$$\begin{array}{ccc}
P & \longrightarrow & \text{SL}_2 \\
\downarrow & & \downarrow \\
\mathbb{A}_2^* & \xrightarrow{\varphi} & \mathbb{A}_2^* \\
\downarrow & & \downarrow \\
\Omega^{\mathbb{A}_2^*} & \xrightarrow{\psi} & \Omega^{\mathbb{A}_2^*} \\
\downarrow & & \downarrow \\
\mathbb{P} & \longrightarrow & \hat{\mathbb{S}} \\
\downarrow & & \downarrow \\
\hat{\mathbb{P}} & \longrightarrow & \hat{\mathbb{S}} \\
\downarrow & & \downarrow \\
\mathbb{A}_2^* & \xrightarrow{\hat{\psi}} & \mathbb{A}_2^* \\
\downarrow & & \downarrow \\
\Omega^{\mathbb{A}_2^*} & \xrightarrow{\hat{\psi}} & \Omega^{\mathbb{A}_2^*} \\
\end{array}$$

By Lemma 6.1, the image $\hat{\varphi}(\mathbb{A}_2^*)$ contains $0 \in \mathbb{A}_2^*$ as an interior point, and it follows that $P \subset \hat{\mathbb{P}}$ is dense. In particular we obtain an injection $\mathcal{O}(\hat{\mathbb{S}}) \hookrightarrow \mathcal{O}(\hat{\mathbb{P}})$ such that

$$\mathcal{O}(\hat{\mathbb{S}})^{\leq n} = \mathcal{O}(\hat{\mathbb{P}})^{\leq n} \cap \mathcal{O}(\hat{\mathbb{S}}).$$

In order to see that $\hat{\mathbb{P}} \sslash \mathbb{G}_a \cong \mathbb{A}_2^*$, we consider the following diagram where the vertical arrows are the restrictions, the left one is an isomorphism and the right one injective. It follows that the the upper horizontal arrow is an isomorphism as well.

$$\begin{array}{ccc}
\mathcal{O}(\mathbb{A}_2^*) & \xrightarrow{\psi^*} & \mathcal{O}(\hat{\mathbb{P}})^{\mathbb{G}_a} \\
\downarrow & & \downarrow \\
\mathcal{O}(\mathbb{A}_2^*) & \cong & \mathcal{O}(\hat{\mathbb{P}})^{\mathbb{G}_a} \\
\end{array}$$

**Remark 6.3.** Unfortunately we don’t know if all extensions of $P$ can be obtained in this way, and second, if different extensions of $\text{SL}_2$ induce different extensions of $P$.

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EXTENSIONS OF PRINCIPAL \(\mathbb{G}_a\)-BUNDLES OVER \(\mathbb{A}^2\)

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