Estimation Mean by the Bayesian Approach on the Exponentially Weighted Moving Average Control Chart

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Abstract. A control chart is one of the primary techniques of statistical quality control. Exponentially Weighted Moving Average (EWMA) control chart is one of the control charts used to detect a small shift in the mean process. In the quality control process, the control chart is often constructed by ignoring parameter uncertainty. This can affect the long-term performance of the control chart in a controlled or out-of-control state, so parameter estimation is necessary. The parameter estimation method used in this paper is the Bayesian approach. The Bayesian approach is used to develop an EWMA control chart in monitoring a process. The performance of the EWMA control chart with the Bayesian approach has been evaluated using Average Run Length (ARL). Based on the comparison of the ARL value from the EWMA control chart with the Bayesian approach and the Classical EWMA control chart, it shows that the EWMA control chart with the Bayesian approach is more sensitive in detecting the shift in the mean process.

1. Introduction
In statistics, there is a statistical quality control system that aims to provide a new tool that makes the inspection process more effective [1]. In statistical quality control, it is known that control chart is used to monitor process output and detect when changes in the inputs are required to bring the process back to an in-control state. The control chart was first developed by Dr. Walter A. Shewhart in 1930 was used to detect large shifts in a process. This control chart is also known as Shewhart's control chart. According to [2] a major disadvantage of a Shewhart control chart is that it is relatively insensitive to small process shifts. This is because the Shewhart control chart uses only information contained in the last sample observation and it ignores the information provided by previous samples. Based on these reasons, cumulative-sum (CUSUM) control chart and exponentially weighted moving average (EWMA) control chart are developed.

In 1959 Robert developed EWMA control chart which was then widely developed by Crowder (1989) and Lucas & Saccucci (1990). This control chart combines information from several samples, causing the EWMA control chart to be more sensitive in monitoring small process shifts when compared to Shewhart control chart. But in practice, the process parameters are unknown and the control chart is built by ignoring the parameter uncertainty using estimates as a substitute for parameters. This is because the problem of parameter estimation only receives relatively simple attention in terms of control chart so that it affects the long-term performance of the control chart in of control or out of control conditions.
Ignoring parameter uncertainty involved in the quality control process can lead to errors. Therefore, it is important to make precise estimates of the parameters related to make the statistical process performance of the control chart accurate. In parameter estimation, there are two methods of approach, namely the classical approach method and the Bayesian approach method. In the Bayesian approach, the parameters are considered as variables that describe the initial information about the parameters expressed in the prior distribution. This approach combines sample information with previous information expressed in a distribution to overcome the parameters uncertainty. This method has been widely applied in every scientific discipline including statistical quality control and is extensively used in the framework of a control chart.

Research on control chart in the Bayesian framework has been previously carried out. In 2013, [3] developed an EWMA control chart for the mean and variance of the normal distribution using the Bayesian approach. Also based on the Bayesian approach the EWMA control chart is designed for monitor variance and for combine monitoring of mean and variance of a process [4]. Next, [5] using the Bayesian approach based on the loss function in the EWMA control chart to detect small shifts in a process. In his research, the EWMA control chart was developed by using posterior predictive distribution to monitor the mean of a process considering different loss functions. Based on the review, this paper discusses the comparison of EWMA control chart with the Bayesian approach and Classical EWMA control chart for detecting process mean shift.

The paper is organized as follows. We explained methods in section 2 such as the Bayesian approach, Classical EWMA control chart, EWMA control chart with the Bayesian approach, and Average Run Length. Section 3, we present the results. Then, section 4 concludes this paper with a summary.

2. Methods

2.1. Bayesian Approach

In the parameter estimation theory, there are two methods of approach, namely the classical approach method and the Bayesian approach method. In the classical approach more dependent on inferential processes on sample data from a population [6]. This method of approach views the parameter $\mu$ as a parameter with a fixed quantity that is unknown in price [7]. While the Bayesian approach in addition to utilizing information from sample data, it also takes account of previous information about unknown parameter probability distributions known as prior distributions. [6]. The Bayesian approach views the $\mu$ parameter as a random variable that describes the previous information about the parameters stated in a prior distribution from which we can determine the posterior distribution to obtain the Bayesian estimator [7].

The posterior distribution $f(\mu|x)$ is a combination of the likelihood function $f(x|\mu)$ and the prior distribution $f(\mu)$, where the prior selected is the non-informative prior. One of the approaches used to define non-informative prior distributions is Jeffrey's method, so it is also called Jeffrey's prior. The Jeffrey’s prior is defined as follows [8]:

$$f(\mu) = \left[f(\mu)\right]^{\frac{1}{2}}$$

where $f(\mu)$ is Fisher’s information. The Jeffrey's prior distribution is defined as follows:

$$p(\mu) = \frac{1}{\sqrt{n\sigma^{2}}} = constant$$

where $n$ and $\sigma^{2}$ are constants, so Jeffrey's prior is constant.

Therefore, the posterior distribution for continuous random variables can be written as follows:

$$f(\mu|x) = \frac{f(x|\mu)f(\mu)}{\int f(x|\mu)f(\mu) d\mu} = \frac{[2\pi(\sigma^{2}/n)]^{-1/2} \exp \left[-\frac{1}{2\sigma^{2}/n}(\mu - \bar{x})^{2}\right]}{\int \exp \left[-\frac{1}{2\sigma^{2}/n}(\mu - \bar{x})^{2}\right] d\mu}$$

An estimator minimizing the risk of Bayes can be obtained by minimizing the posterior risk defined as $R(\hat{\mu}, \pi) = \int L(\mu, \hat{\mu}) \times p(\mu|x) d\mu$, where $L(\mu, \hat{\mu}) = (\mu - \hat{\mu})^{2}$ is squared error loss function (SELF) [9]. SELF is a loss function that is widely used because it is easy to calculate and is symmetric
loss function [5]. Therefore, the Bayes estimator value based on the posterior distribution \(\mu|x\) are \(\hat{\mu}_{SEL} = \bar{x}\) and \(\sigma^2_{SEL} = \sigma^2/n\) or can be written \(\mu|x\sim N(\hat{\mu}_{SEL},\sigma^2_{SEL})\).

In the Bayesian method, posterior predictive distribution is the main tool for handling prediction problems in statistics so that making it easier to make predictions [9]. Posterior predictive distribution is defined as follows:

\[
f(y|x) = \int f(y|\mu) \times f(\mu|x) d\mu
\]

Let \(y_1, y_2, \ldots, y_n\) are the next \(n\) observations with the posterior predictive distribution \(Y|X \sim N(\hat{\mu}_{SEL}, \sigma^2_{Y|X})\) where \(\sigma^2_{Y|X} = \sigma^2 + (\sigma^2/n)\). For \(\tilde{y}_i = (y_{i1} + y_{i2} + \cdots + y_{ik})/k\) then \(\tilde{Y}|X \sim N(\hat{\mu}_{SEL}, \psi^2)\) where \(\psi^2 = (\sigma^2/k) + (\sigma^2/n)\).

### 2.2. Classical EWMA Control Chart

EWMA control chart is one of an alternative to Shewhart's control chart to monitoring small shifts in a process. The Classical EWMA statistics is defined as [2]:

\[
z_i = \lambda \tilde{x}_i + (1 - \lambda)z_{i-1}
\]

where \(0 < \lambda < 1\) which is the determinant of the value of the weight given in the current sample. The initial value \(z_0\) is usually specified as the target value of a process so that \(z_0 = \mu_0\). The center line and control limits for the Classical EWMA control chart are defined as follows:

\[
UCL = \mu_0 + 3\left[\sigma_{x_i}/\sqrt{n}\sqrt{\lambda/(2-\lambda)}\right]\left[1 - (1-\lambda)^i\right]
\]

\[
CL = \mu_0
\]

\[
LCL = \mu_0 - 3\left[\sigma_{x_i}/\sqrt{n}\sqrt{\lambda/(2-\lambda)}\right]\left[1 - (1-\lambda)^i\right]
\]

The form \((1 - (1-\lambda)^i)\) in equations (6) and (8) will converge to 1 when the value of \(i\) gets larger. This means that the control limits will approach steady-state values given by

\[
UCL = \mu_0 + 3\left[\sigma_{x_i}/\sqrt{n}\sqrt{\lambda/(2-\lambda)}\right]
\]

\[
CL = \mu_0
\]

\[
LCL = \mu_0 - 3\left[\sigma_{x_i}/\sqrt{n}\sqrt{\lambda/(2-\lambda)}\right]
\]

### 2.3. EWMA Control Chart with the Bayesian Approach

EWMA control chart with the Bayesian approach is also called Bayesian EWMA control chart. The EWMA statistics based on Bayesian estimator \(y|x\) is defined as [5]:

\[
z_i = \lambda (y|x_i) + (1 - \lambda)z_{i-1}
\]

The center line and control limits for the Bayesian EWMA control chart on the steady-state is as follows:

\[
UCL = \hat{\mu}_{SEL} + 3\psi\sqrt{\lambda/(2-\lambda)}
\]

\[
CL = \hat{\mu}_{SEL}
\]

\[
LCL = \hat{\mu}_{SEL} - 3\psi\sqrt{\lambda/(2-\lambda)}
\]

### 2.4. Average Run Length

Average Run Length (ARL) is the average number of points that must be plotted before a point indicates an out-of-control condition. The control charts with a smaller ARL value indicate that the smaller the number of samples taken to detect a shift in the process [2].

One of the methods that are widely used in determining the ARL value on the CUSUM and EWMA control chart is the Markov chain method developed by Brook and Evans (1972). The procedure in the Markov chain method divides the interval between the UCL and LCL values into \(k = 2m + 1\) subinterval with width \(2\delta\), where the width of each interval is obtained by the formula \(w = (UCL - LCL)/k\). The midpoint of each interval is defined \(S_i = LCL + (i - 0.5)w\) for \(i = -m, -m + 1, \ldots, m - 1, m\). The transition probability matrix represented in partitioned matrix form is given by [11]:

\[
P = \begin{pmatrix} R & (I - R) \end{pmatrix} \begin{pmatrix} 1 \
0 \end{pmatrix}
\]

\[
R = \begin{pmatrix} 0.01 & 0.99 \
0 & 1 \end{pmatrix}
\]

\[
I = \begin{pmatrix} 1 & 0 \
0 & 1 \end{pmatrix}
\]

\[
1 = \begin{pmatrix} 1 \
0 \end{pmatrix}
\]
where the sub-matrix $R = [r_{ij}]$ of $k \times k$ size is a controlled transition probability expressed by

$$r_{ij} = \Phi \left( \frac{(S_j + \delta) - (1 - \lambda)S_i - \lambda \mu}{\lambda \sigma} \right) - \Phi \left( \frac{(S_j - \delta) - (1 - \lambda)S_i - \lambda \mu}{\lambda \sigma} \right)$$

for $i, j = -m, -m + 1, \ldots, m - 1, m$. Therefore, ARL on the EWMA control chart for the mean process is given by:

$$ARL = q^\top (I - R)^{-1} \mathbf{1}$$ (17)

where $I$ is the $k \times k$ identity matrix, and $\mathbf{1}$ is the $k \times 1$ column vector of ones. We substitute $q = e_i$ in equation (17), where $e_i$ is the $i$th unit vector with all elements zero except the $i$th element which is 1.

In equation (17), we use $i = (k + 1)/2$ (12).

3. Results

Here, we demonstrate the performance and implementation of the EWMA control chart based on the Bayesian approach and its comparison with the Classical EWMA control chart. Dataset is taken from European J. Industrial Engineering in the paper [13]. The data contained 5 results of the weight measurement of yogurt cups in each subgroup, where the average of the weight measurement of yogurt cups in each subgroup can be seen in Figure 1.

![Figure 1. The average weight of yogurt cups in 20 sample subgroup](image)

Classical EWMA control chart for data measuring the weight of yogurt cups shown in Figure 2 with the mean of data is $\mu_0 = \left( \sum_{i=1}^{20} \bar{x}_i \right)/20 = 124,2748$ and the estimated standard deviation is $\sigma = \bar{S}/c_4 = 0,7398$, where $c_4 = \left( 2/(k - 1) \right)^{1/2} \left( \Gamma(k/2) \right) \left( \Gamma((k - 1)/2) \right)^{-1}$. Figure 2 shows that there are 10 from 20 averages data of weight measurement yogurt that are outside the control limits so that the product can be said uncontrolled.
Bayesian EWMA control chart for data measuring the weight of yogurt cups shown in Figure 3 with the values $\hat{\mu}_{SELP} = 124.2748$ and $\psi = 0.3187$. Figure 3 shows that there are 11 from 20 averages data of weight measurement yogurt that are outside the control limits so that the product can be said uncontrolled.

Based on the value $0 < \lambda < 1$ and the number of data that is outside the control limit show that Bayesian EWMA control chart is faster to detect data out of control than the Classical EWMA control chart. However, to be more precise will be used the calculation of the ARL value to detect the accuracy of the control chart. The ARL value in the Classical EWMA and Bayesian EWMA control charts can be seen in Table 1, where the ARL Bayesian EWMA control chart is smaller than the ARL Classical EWMA control chart.
| $\lambda$ | ARL Classical EWMA | ARL Bayesian EWMA | $\lambda$ | ARL Classical EWMA | ARL Bayesian EWMA | $\lambda$ | ARL Classical EWMA | ARL Bayesian EWMA |
|---------|-------------------|-------------------|---------|-------------------|-------------------|---------|-------------------|-------------------|
| 0.01    | 57.0883           | 52.1042           | 0.34    | 3.2882            | 3.1888            | 0.67    | 2.1414            | 2.1134            |
| 0.02    | 28.4407           | 26.8096           | 0.35    | 3.2246            | 3.1291            | 0.68    | 2.1226            | 2.0958            |
| 0.03    | 20.6010           | 19.5125           | 0.36    | 3.1643            | 3.0726            | 0.69    | 2.1043            | 2.0785            |
| 0.04    | 16.4750           | 15.6214           | 0.37    | 3.1071            | 3.0189            | 0.70    | 2.0863            | 2.0617            |
| 0.05    | 13.8452           | 13.1382           | 0.38    | 3.0526            | 2.9678            | 0.71    | 2.0689            | 2.0453            |
| 0.06    | 12.0082           | 11.4045           | 0.39    | 3.0008            | 2.9192            | 0.72    | 2.0518            | 2.0293            |
| 0.07    | 10.6476           | 10.1209           | 0.40    | 2.9514            | 2.8729            | 0.73    | 2.0351            | 2.0136            |
| 0.08    | 9.5967            | 9.1296            | 0.41    | 2.9042            | 2.8287            | 0.74    | 2.0188            | 1.9982            |
| 0.09    | 8.7587            | 8.3394            | 0.42    | 2.8591            | 2.7864            | 0.75    | 2.0028            | 1.9832            |
| 0.10    | 8.0738            | 7.6935            | 0.43    | 2.8160            | 2.7460            | 0.76    | 1.9872            | 1.9685            |
| 0.11    | 7.5027            | 7.1551            | 0.44    | 2.7747            | 2.7073            | 0.77    | 1.9719            | 1.9541            |
| 0.12    | 7.0188            | 6.6989            | 0.45    | 2.7351            | 2.6701            | 0.78    | 1.9569            | 1.9400            |
| 0.13    | 6.6031            | 6.3070            | 0.46    | 2.6971            | 2.6345            | 0.79    | 1.9422            | 1.9262            |
| 0.14    | 6.2418            | 5.9667            | 0.47    | 2.6605            | 2.6003            | 0.80    | 1.9278            | 1.9126            |
| 0.15    | 5.9249            | 5.6681            | 0.48    | 2.6254            | 2.5673            | 0.81    | 1.9137            | 1.8993            |
| 0.16    | 5.6444            | 5.4039            | 0.49    | 2.5916            | 2.5356            | 0.82    | 1.8998            | 1.8862            |
| 0.17    | 5.3943            | 5.1685            | 0.50    | 2.5590            | 2.5051            | 0.83    | 1.8862            | 1.8734            |
| 0.18    | 5.1699            | 4.9572            | 0.51    | 2.5276            | 2.4757            | 0.84    | 1.8728            | 1.8608            |
| 0.19    | 4.9673            | 4.7666            | 0.52    | 2.4973            | 2.4473            | 0.85    | 1.8597            | 1.8484            |
| 0.20    | 4.7835            | 4.5936            | 0.53    | 2.4680            | 2.4198            | 0.86    | 1.8468            | 1.8363            |
| 0.21    | 4.6159            | 4.4360            | 0.54    | 2.4398            | 2.3933            | 0.87    | 1.8341            | 1.8243            |
| 0.22    | 4.4625            | 4.2917            | 0.55    | 2.4124            | 2.3677            | 0.88    | 1.8216            | 1.8125            |
| 0.23    | 4.3214            | 4.1591            | 0.56    | 2.3859            | 2.3428            | 0.89    | 1.8093            | 1.8009            |
| 0.24    | 4.1913            | 4.0368            | 0.57    | 2.3603            | 2.3188            | 0.90    | 1.7972            | 1.7895            |
| 0.25    | 4.0709            | 3.9236            | 0.58    | 2.3354            | 2.2955            | 0.91    | 1.7853            | 1.7782            |
| 0.26    | 3.9591            | 3.8186            | 0.59    | 2.3113            | 2.2729            | 0.92    | 1.7736            | 1.7671            |
| 0.27    | 3.8551            | 3.7209            | 0.60    | 2.2879            | 2.2509            | 0.93    | 1.7621            | 1.7562            |
| 0.28    | 3.7579            | 3.6297            | 0.61    | 2.2652            | 2.2296            | 0.94    | 1.7507            | 1.7454            |
| 0.29    | 3.6671            | 3.5444            | 0.62    | 2.2431            | 2.2089            | 0.95    | 1.7394            | 1.7348            |
| 0.30    | 3.5819            | 3.4644            | 0.63    | 2.2217            | 2.1888            | 0.96    | 1.7284            | 1.7243            |
| 0.31    | 3.5018            | 3.3893            | 0.64    | 2.2008            | 2.1692            | 0.97    | 1.7174            | 1.7140            |
| 0.32    | 3.4265            | 3.3185            | 0.65    | 2.1805            | 2.1501            | 0.98    | 1.7067            | 1.7037            |
| 0.33    | 3.3554            | 3.2518            | 0.66    | 2.1607            | 2.1315            | 0.99    | 1.6960            | 1.6936            |
4. Conclusion
In this paper, we have presented the EWMA control chart with the Bayesian approach and compared the ARL values obtained with the ARL values Classical EWMA control chart. The results obtained show that the EWMA control chart with the Bayesian approach is more sensitive in detecting the shift in the process mean compared to the Classical EWMA control chart.

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