The State of Cepheid Pulsation Theory

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Abstract. We review the current state of Cepheid modeling and discuss its dominant deficiency, namely the use of time dependent mixing length. Notwithstanding, Cepheid modeling has achieved some excellent successes, and we mention some of the most recent ones. Discrepancies between observations and modeling appear not so much in the gross properties of single mode Cepheids, but rather when more subtle nonlinear effects are important, such as in double mode or even triple mode pulsations. Finally we discuss what we consider the most important challenges for the next decade. These are, first, realistic multi dimensional modeling of convection in a pulsating environment, and second, the nonlinear modeling of the nonradial pulsations that have been observed, and, relatedly, of the Blazhko like phenomenon that has recently been observed in Cepheids.

Keywords: Cepheids, Variable stars, Pulsations, Opacities, Metallicities, Chaotic Pulsations

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INTRODUCTION

When looking back to the pioneering days of Cepheid modeling (e.g., [1, 2]) we have to modestly admit that in terms of numerical modeling methods we have only made some progress, except that we now have at our disposal enormous computing power compared to that available in those days.

There has however been quite a bit of progress in terms of the quality of the opacities that are now used (OPAL [3], OP [4], and Andersen-Ferguson [5]). In addition, some treatment of convection is necessary and is routinely included in the calculations in the form of time-dependent mixing length.

In parallel to direct numerical simulations, the amplitude equation formalism has been developed [6, 7, 8, 9, 10] (for a review see [11]). The physical conditions that prevail in Cepheids and RR Lyrae, namely that the growth rates of the dominant modes are small compared to their frequencies, form the basis for the applicability of these techniques. We note in passing that, in contrast, amplitude equations do not apply in Pop. II Cepheids, RV Tau and Mira variables. In these stars the growth rates are comparable to the frequencies and therefore the amplitudes can change substantially on the time scale of a period, hence the frequent occurrence of irregularity in the pulsations [12, 13, 14].

In many respects the application of amplitude equations complements brute force modeling. Amplitude equations, or normal forms as they are known in nonlinear dynamics [15], are very general and describe the underlying mathematical structure of the temporal behavior, i.e., the pulsations in our case. In addition, when combined with numerical simulations [17], they provide not only an excellent description of the modal selection problem, e.g., of where the region of double mode behavior occurs, or where hysteresis occurs (‘either or’ regime in the jargon of stellar pulsations), but they also yield a deeper understanding of these bifurcations in the pulsational behavior [17].

CRITIQUE OF THE HYDROCODES

A number of hydrocodes have been developed over the last dozen years or so (e.g., the Italian code [18], the Viennese code [19], the Florida-Budapest code [20] code. They are all essentially the same in that they include some similar form of time-dependent mixing length approximation. The first code is based on the formulation of [21], and the second on that of [22, 23], and the third a hybrid of the two formulations. Some of their features have been compared in [24]. More recent codes are the Australian [25] and the Polish ones [26]. Some of these codes include a moving adaptive mesh that resolves better the sharp temperature features, in particular those associated with the hydrogen partial ionization front [19, 27]. Some in addition improve on the equilibrium heat diffusion [28].

We briefly recall the ingredients of time dependent mixing length. The effects of turbulence and convection in 1-dimensional hydrodynamics description appear through the turbulent pressure and viscous stresses $p_t$ and $p_v$, the convective $F_c$ and the energy coupling term $\mathcal{E}$.

\begin{align*}
\frac{du}{dt} &= -\frac{1}{\rho} \frac{\partial}{\partial r} \left( p + p_t + p_v \right) - \frac{GM_r}{r^2} \quad (1) \\
\frac{de}{dt} + p \frac{dv}{dt} &= -\frac{1}{\rho r^2} \frac{\partial}{\partial r} \left[ r^2 \left( F_r + F_c \right) \right] + \mathcal{E} \quad (2)
\end{align*}
The turbulent and convective quantities are assumed to be functions of the ‘turbulent energy’ \( e_t \) that is assumed to satisfy

\[
\frac{d e_t}{d t} + (p_e + p_v) \frac{d v}{d t} = - \frac{1}{\rho r} \frac{\partial}{\partial r} \left( r^2 F_c \right) - \mathcal{C} \tag{3}
\]

The expressions for \( p_e, F_c, p_v \) and \( \mathcal{C} \) can be derived in analogy with gas kinetic theory, however without the same solid physical basis, or more simply just on dimensional grounds plus a few assumptions. All these terms contain \( (\alpha) \) parameters of \( O(1) \) for whose values physics provides little guidance. There is also some ambiguity, e.g., possible physically acceptable choices for \( F_c \) and the source term \( S_t \) (that goes into the energy coupling term \( \mathcal{C} \)):

\[
\begin{align*}
F_c &\sim \alpha_e e_t^{1/2} Y \\
S_t &\sim \alpha_t e_t^{1/2} Y
\end{align*}
\tag{4}
\]

where

\[
Y \equiv \left[ - \frac{H_p}{c_p} \frac{\partial s}{\partial r} \right] + \quad \text{or} \quad Y \equiv - \frac{H_p}{c_p} \frac{\partial s}{\partial r} \tag{6}
\]

More \( \alpha \) parameters appear if a Péclet correction for radiative losses and a flux limiter for \( F_c < \rho c_p T \) are included.

The literature is very vast, so we will just mention some of the more recent work.

One of the most striking properties of the Cepheids, both for fundamental (F) mode pulsations and for first overtone (O1) mode pulsations is the progression of the Fourier decomposition coefficients, of the light curves as well as of the radial velocity curves (e.g., [42]). For F mode Cepheids this progression is of course related to the well-known Hertzsprung progression in the bump Cepheids. Full amplitude Cepheid model sequences do a good job at reproducing the Fourier properties of the F Cepheids [33] and of the O1 Cepheids [34].

In parallel, the amplitude equation formalism that was developed for explaining the effect of internal resonances on the appearance of the light and radial velocity curves has given a clear demonstration that it is the 2:1 resonance between the self-excited F mode and the vibrationally stable, but resonant second overtone [35, 36] that is responsible for the structure of the Fourier coefficients for periods around 10 days, rather than a shockwave that reflects off the core. For the first overtone Cepheids (or s Cepheids) it is the 2:1 resonance of the stable fourth overtone with the self-excited first overtone that causes structure at period in the vicinity of 4 days. Modeling also reproduces well the observed light and radial velocity curves of individual stars, e.g., [57, 58].

A full amplitude model survey of F and O1 Cepheids [39] has found that the phase lag between light and radial velocity curves is in good agreement with observations. This study has furthermore demonstrated that the phase lag can also be used observationally as a discriminant between F and O1 mode pulsation.

The period ratio \( P_1/P_0 \) as a function of period \( P_0 \) of the beat Cepheids is a sensitive function of the metallicity Z. This property has recently been taken advantage of to determine the metallicities in the LMC and SMC [40] with the help of Cepheid modeling [41, 42]. The same method has been applied to the 5 known beat Cepheids in M33. Interestingly, this yields a galactic metallicity gradient for M33 that is in good agreement with totally
FIGURE 1. Evolution of the specific turbulent energy $e_t$ during the pulsational phase; the lightness of the grey reflects the strength of $e_t$.

A recent study shows that a comparison of the light curves of full amplitude bump Cepheid models with observed light curves can also be used as metallicity tracers and distance indicators [44].

Theoretical period-color-luminosity relations are in good agreement with observational ones [45], [46]. A summary of the recent Italian work on this and other topics in Cepheid modeling appears in [47].

Theory has been ahead of observations by predicting the existence of 'strange' Cepheids and RR Lyrae [48, 49]. These are Cepheids or RR Lyrae that pulsate in a high (7th to 12th) overtone in which the pulsation is confined to the outer region, more specifically, above the hydrogen ionization front. Typically, the predicted periods of these self excited modes are 4 to 5 times smaller than the fundamental period of the same object. The amplitudes are predicted to be in the millimag range. On the observational side some evidence for the existence of strange Cepheids has been uncovered. However, it is hard to distinguish between intrinsic pulsation and ellipsoidal binary motion at the millimag level [50, 51]). Furthermore, in crowded field conditions, it is possible that the objects could be regular, albeit low amplitude Cepheids that appears above the P-L relation because of contamination by another star. Follow-up observations would be useful to ascertain that the identified objects are indeed strange Cepheids.

In Fig. 2 we display the results of a recent search for ultra-low amplitude (ULA) variability in the combined MACHO and OGLE LMC data [51]. For reference and to guide the eye, the well known Cepheid P-L strips are indicated as black dots. All the ULA (Fourier amplitude $A < 0.01$) objects that were found in the plotted period–Wesenheit magnitude ($W = I - 1.55(V - I)$) are shown as orange filled circles. It is possible that some of the objects are ellipsoidal binaries that with the available data, are indistinguishable from ULA Cepheids. The clustering of the ULA objects with the LMC Cepheids suggests though that the latter are indeed pulsating variables. The objects below the fundamental P-L relation are probably Pop. II Cepheids. The single object way above the P-L relation is a strange Cepheid candidate. When one concentrates on the remaining objects, their position is seen to correlate strongly with the LMC Cepheid P-L relation. One expects any ULA Cepheids to lie at the edges of the instability strip, but Fig. 2 might suggest that they lie on a parallel P-L relation that is shifted slightly upwards. If that is indeed the case, it is an interesting challenge to explain the nature of these ULA Cepheids.

The analyses of the OGLE data base e.g., [32, 52, 53]), to which we refer the reader for the following discussion, have yielded a beautiful overview of the HR diagram. In addition to the F and O1 Cepheids, a whole zoo of other types of Cepheid pulsational behavior has now been observed, namely O2 single mode, double mode
FIGURE 2. Ultralow amplitude objects in the LMC, shown as open (red and orange) circles. The smaller (orange) circles possibly have proper motions. The classical Cepheids are superposed as dots for reference.

F/O1, double mode O1/O2, triple mode, and Cepheids with Blazhko like modulations.

For comparison we present the results of a recently computed HR diagram [54] in Fig. 3. The bounding curves on the right [in red, in the color figure of the electronic version] and the [blue] ones on the left represent respectively the red and blue edges of the F, O1, O2 and O3 pulsational instability strips. We recall in passing that these boundaries are obtained from linear vibrational stability analysis, and as such, they only tell us where at least one mode is linearly unstable, i.e., where pulsations occur, but they generally do not tell us what type of pulsation actually results. For example, for F/O1 double mode behavior it is necessary, but not sufficient that both modes be linearly unstable (in the figure the intersection of the F and the O1 instability strips). Nonlinear calculations are necessary to determine which pulsational behavior can actually occur. (We say ‘can occur’ rather than ‘occurs’ because there may also be hysteresis, i.e., the pulsational state depends on the evolutionary path.)

Each of the symbols in Fig. 3 represents the result of one or more hydrodynamic integrations to steady full amplitude. The numbers on the left denote the mass (in $M_\odot$) which is constant along horizontal lines. The fundamental periods (in days) are constant along the dotted lines. The open squares [black] denote the regime where only single mode F full amplitude pulsations (limit cycles) occur, the open upside-down triangles [red filled triangles] the regime of single mode O1, and the open circles [blue filled circles] that of single mode O2 pulsations.

The solid [lightblue] squares denote the regime where either F or O1 pulsations occur, i.e., where there is hysteresis. A star, say with mass 5$M_\odot$, evolving leftward enters the F instability strip (rightmost red line) and starts to pulsate in the F mode. It continues to do so to the leftmost edge of the (lightblue) hysteretic region where it switches to O1 pulsations until it exits through the O1 blue edge on the left. On its return, moving now rightward, it enters the O1 instability strip (leftmost blue line) and starts to pulsate in the O1 mode. It continues to do so to the rightmost edge of the (lightblue) hysteretic region where it switches to F pulsations until it exits through the F red edge on the right.

Below, and as a continuation of the hysteretic region
we typically find F/O1 double mode pulsations denoted by crosses [yellow filled circles with a concentric black dot]. Care has to be exercised in the modeling computations that the pulsations are truly double mode (i.e., with constant amplitudes and phases) rather than that the model may be switching from one mode to the other. Indeed, it has been shown that the amplitudes may level off relatively fast, and then vary extremely slowly, giving the erroneous impression of steady pulsations. To be certain that double mode is indeed obtained it is necessary to integrate the model with several judiciously chosen initial kicks and to watch the evolution of the individual amplitudes in an amplitude–amplitude portrait [17].

Interestingly, F/O1 double mode pulsations can also occur in a narrow region above the hysteresis regime. This double mode behavior is induced by the \( P_0 = 2P_2 \) resonance in the vicinity of the 10 d period. The locus of the center of this resonance is indicated as a dashed line in the figure.

The width of the hysteresis regime [light blue lines] is clearly nonmonotone with mass. This is not a numerical problem, but we have again traced this behavior to a resonance, this time \( P_1 = 2P_4 \), as seen by the locus of the resonance center (solid line).

Similar regimes of hysteresis and double mode occur on the right towards the lower masses, but now for O1 and O2. The hysteretic regime of O1 and O2 ('either O1 or O2') is marked by solid triangles [green filled circles]. Below this occurs a regime of O1/O2 double mode pulsations labeled with x’s [orange filled circles].

We note in passing, and as a curiosity, that we have also found an extremely narrow strip of hysteresis with either F single mode pulsations or F/O1 double mode pulsations.

It is likely that multimode and hysteresis also occur for the 2\( M_\odot \) models, for which O3 is linearly unstable, but we have not checked this.

Overall, the modeling is seen to reproduce the different types of single mode and double mode behavior with approximately the right topography. However, some
types of observed pulsational behavior, such as triple mode, have yet to be modeled. It is clear that double mode or triple mode behavior is much more sensitive to the details of the physical input than, say single mode, because it depends on more subtle effects, viz. the non-linear coupling between the modes.

Before going on we also want to mention some well known, but serious discrepancies, for which stellar evolution calculations are at fault, but on which Cepheid modeling unfortunately has to rely.

First, low mass, low Z evolution loops do not penetrate the instability strip where Cepheids are actually observed. It has been suggested that this could be a metallicity selection effect [55, 56]. There remains disagreement about the treatment of convection and convective overshoots which leads to uncertainties in the Cepheid M-L-Z relation. Unfortunately, this in turn causes uncertainties in Cepheid modeling which depends on an ML relation. However, we note that the modeling of LMC and SMC Cepheids [57] based on the tracks of Girardi et al. [58] is in good agreement with the resonance constraints imposed by observations. Finally, the occurrence of a bump mass discrepancy was largely removed with the OPAL opacities [59], some authors claim that there remains a small discrepancy [44] which is at variance with the above mentioned work [57].

BEYOND TIME-DEPENDENT MIXING LENGTH

Since dependent mixing length has such a long list of known problems, efforts have been made to go beyond the simple model described above, while keeping its 1D feature. This unfortunately leads to a proliferation of equations [29, 60, 61, 62] with a concomitant numerical cost and perhaps little guarantee of substantial improvement.

In parallel, plume models have been proposed [63]. Along similar lines Stoekl has developed a 2 column model [64]. However, none of these approaches have been used in Cepheid models, nor are they expected to provide a parameter free description of convection which after all is 3 dimensional phenomenon.

TWO GRAND NUMERICAL CHALLENGES

In our opinion there remain two grand challenges for the modeling of Cepheids, and for that matter of RR Lyrae which are quite similar in many ways. These are the modeling of convection in the highly structured envelopes of Cepheids, on the one hand, and of nonradial pulsations, on the other.

1. Multidimensional simulations of convection in Cepheids and RR Lyrae

There is a great deal of literature on the numerical difficulties that arise in the modeling of astrophysical convection. The literally astronomically large Rayleigh numbers imply that the turbulence is well developed. This in turn implies that many degrees of freedom are involved which sets extreme mesh requirements. However, the hope of all large eddy simulations (LES) is that once we have attained a sufficient spatial resolution, then the dominant, large scale features of convective transport are well reproduced. Another problem arises from the extremely small Prandtl number. (The Pr number is the ratio of gas viscosity to heat conduction.) In the numerical modeling the viscosity is however not set by the physical extremely small viscosity, but by that of the numerical scheme. Effective Pr numbers of order 0.01 or less are hard to achieve.

There has been a great deal of excellent multidimensional numerical hydrodynamics work in other astrophysical contexts, such as solar models, stellar core burning, dynamos [65, 66]. Additional difficulties arise that are specific to Cepheids and RR Lyrae modeling. First, it is out of question to compute convection over the whole sphere, nor is it probably necessary. If convection occurs over a vertical range $\Delta R$, it seems reasonable that we limit ourselves to a horizontal sector whose width is a few times $\Delta R$, at the vertical center $R_v$ of the convective region, i.e., an angular sector $\Delta \theta = \Delta R_c/R_v$, typically a degree or so, and impose periodic horizontal boundary conditions. But even this leaves extreme mesh requirements. The resolution of the rapidly varying temperature structure in the partial hydrogen ionization region requires tiny $\Delta R$. However, the hot inner region that needs to be included in a realistic stellar model, even though no convection is taking place there requires large $\Delta R$ because the sound speed is large, so that we are not killed by the Courant condition. The mesh aspect ratios can thus vary from 0.01 to 100.

Second, the surface boundary condition is delicate because the envelope is surrounded by an optically thin, approximately isothermal atmosphere with an exponential decrease of density.

Third, the pulsating environment in which the stellar radius can change by more than 10%, combined with the exponentially decreasing atmosphere, make an Eulerian code unsuitable. A moving mesh is needed to follow the average pulsational motion, and adaptive features are required to track the sharp temperature gradients (H
ionization front).

It should be clear that the modeling of convection in a pulsating star is a challenging problem that is at the limit of current modeling possibilities. However an intermediate goal could be to make a small but sufficient number of 3D simulations in realistic Cepheid or RR Lyrae models, and then use the results to calibrate the mixing length recipe parameters, or if necessary to improve or even replace mixing length theory.

2. The Modeling of Nonradial Pulsations in Cepheids

The analysis of the observations of the OGLE Cepheids strongly suggest the excitation of nonradial modes \(^{[52, 53]}\).

On the theoretical side, the linear stability analysis of nonradial modes in Cepheids and RR Lyrae is notoriously difficult \(^{[67, 68]}\). The problem arises from the fact that the nonradial modes have mixed p and g mode nature with unresolvably rapid spatial oscillations in interior. This means that most nonradial modes are very strongly damped. However, some surface-trapped modes were found to be unstable or only mildly damped, and they can therefore compete dynamically with the low order radial modes \(^{[69, 70]}\). A recent revisit of the problem with realistic stellar models \(^{[71]}\) however finds no nonradial instability among the low \(\ell\) modes. (Only low \(\ell\) are detectable with full disk observations.) This problem therefore remains a puzzle even at the linear level. Ultimately, of course, nonlinear 3D hydrodynamic simulations will be necessary to model the observed nonradial pulsations, which is again a very tough numerical problem, for many of the same reasons as for convection.

The analyses of the LMC variables \(^{[52, 53]}\) also show the existence of Blazhko like modulations of the pulsations which bear some similarities with the same effect in RR Lyrae. Just as in the latter the physical origins remain obscure. Similar modulations also seem to occur in in V473 Lyr and in Polaris \(^{[72]}\).

Among the viable models for the Blazhko effect are the likely involvement of one or more nonradial modes \(^{[73]}\). A specific mechanism involving resonant coupling has been examined by \(^{[74]}\).

One may also wonder whether the nonradial modes can be destabilized by convection? Could this be the origin of the observed Blazhko modulations \(^{[53]}\)? We are somewhat skeptical that a simple dependent mixing length description is sufficiently good to answer that question. Most likely a full 3D simulation, as described in the previous section, will be necessary to treat the convection – pulsation interaction. There is also the possibility the Blazhko modulations arise from the radial pulsation - convection interaction. In any case an explanation of the physical mechanism of the Blazhko cycle remains a theoretical challenge.

Despite these serious difficulties, we think that these two grand modeling challenges will be attacked and possibly will be met in the next decade.

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