A unified framework for non-classicality: emergence of non-locality and entanglement

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Non-classical probability (along with its underlying logic) is a defining feature of quantum mechanics. A formulation that incorporates them, inherently and directly, would promise a unified description of seemingly different prescriptions of non-classicality of states that have been proposed so far. This paper sets up such a formalism. It is based on elementary considerations, free of ad-hoc definitions, and is completely operational. It permits a systematic construction of non-classicality conditions on states. As important applications, we show the emergence of two important criteria, non-locality and entanglement, as consequences of breakdown of classically valid elementary logical propositions.

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I. INTRODUCTION

Quantum mechanics is a theory that has altered the very way we comprehend laws of nature. Equally so, it has altered the way we formulate laws of probability. Thus, the concept of non-classicality of states in quantum physics is as much a reflection of non-classical probability as it is of physics. This was recognised quite early, in a rather formal manner, by Birkhoff and Von Neumann [1] (see also [2, 3]). Developments in quantum information have brought the further realization that non-classical states are, in fact, resources for information processes which would be impossible within the classical framework [4, 5]. This, in turn, has spawned a large number of definitions/criteria for non-classicality of states [6–17]. In parallel, there has been a vigorous experimental activity, both for probing the foundations of quantum mechanics[18–21] and for eminently practical applications[4, 22–24].

Spectacular though these developments are, the present understanding of non-classicality is not entirely satisfactory. Each definition or criterion is pinned to a specific context, and the underlying logical basis, or its interrelationship with other criteria is not always clear. Since rules of classical probability are based on laws governing valid logical propositions and rules governing their compositions involving operations such as AND, and OR, it ought to be possible to trace, in principle, all expressions of non-classicality to a breakdown of the validity of one or more such rules in the quantum domain. This would require setting up of a formalism which has (i) non-classical logic (in the sense of [1]) and probability inbuilt in it, (ii) is completely operational, and (iii) which allows for a systematic determination of non-classicality conditions starting from basic underlying propositions. Finally, it should be free of ad-hoc constructions. This task, if accomplished, would provide a unified framework to describe non-classicality of states and allow a systematic way for devising tests for verifying non-classicality of any given state.

This paper accomplishes this task. Starting from well established rules of quantum mechanics which are simple but equally general, we introduce what we call pseudo-projections, and show how they may be harnessed to obtain an infinitely large number of tests of non-classicality of states. For concrete illustration, we devote our attention to bipartite systems, and study two important criteria of non-classicality in this paper, the Bell-CHSH non-locality and entanglement. Explicitly, we show how they emerge as violations of Boolean rules that govern logical operations resulting in classically valid propositions. For reasons that will become clear, the study of entanglement is restricted to two qubit systems.

II. THE FORMALISM

It is convenient to start with a query, articulated clearly, by Fine[25, 26]: are there circumstances under which a given quantum state permits assignments of joint probabilities for the outcomes of a given set of observables? This important question, which forms the basis of our analysis, will be formulated mathematically, by introducing what we call pseudo-projections.

A. Joint probabilities, conjunctions and pseudo-projections

Consider a set of observables, \(A^1, A^2, \ldots, A^r\), defined over a phase space \(\Phi\). Let \(A^k\) take values belonging to a set \(\{a^k_1, a^k_2, \ldots\}\). For a classical system in a state \(f\), the joint probability for a conjunction of events \(\{A^1 = a^1_1, A^2 = a^2_2, \ldots, A^r = a^r_r\}\) always exists and admits a straightforward construction. Let \(S \subset \Phi\) be the support for the joint outcomes, and \(1_S\), its indicator function. Then, the joint probability is given by the overlap of \(f\) with \(1_S\). \(1_S\) is, by itself, a Boolean observable: it takes value 1 in \(S\) and vanishes outside the support. If \(S_{i_0}\) are
The unique operator, the pseudo-projection \( \Pi \), joint outcome, is given by the symmetrised product: 

\[ \rho \Pi \alpha \equiv \frac{1}{2} \{ \rho \alpha \Pi, \rho \alpha \Pi \}. \]

It is hermitian, but not necessarily idempotent (or for that matter, non-negative), unless the corresponding projections commute. The vi-

The eigen-projections \( \pi \) are the quantum representa-

tives of indicator functions. We call them pseudo-projections. Pseudo-projections hold the key to non-classicality.

1. Construction of Pseudo-projections

Consider, first, two observables \( A, B \). Let \( \pi^A_{a_i}, \pi^B_{b_j} \) be the projection operators representing the respective indicator functions for the outcomes \( A = a_i \) and \( B = b_j \).

The unique operator, the pseudo-projection, that represents the indicator function, \( 1_{S \cap S'} \), for the classical joint outcome, is given by the symmetrised product: 

\[ \Pi_{a_i b_j}^{AB} = \frac{1}{2} \{ \pi^A_{a_i}, \pi^B_{b_j} \}, \]

which is but the simplest example of Weyl ordering [27]. It is hermitian, but not necessarily idempotent (or for that matter, non-negative), unless 

\[ [\pi^A_{a_i}, \pi^B_{b_j}] = 0. \]

Pseudo-projections that represent an indicator function for joint outcomes of more than two observables are not unique. Of all the choices, we shall, for the present, employ the one obtained from Weyl ordering. This choice, which is completely symmetric in all the projections has a close relationship with Moyal brackets [28] and Wigner distribution functions [29] which are central to semi classical descriptions. Any residual ambiguity may be avoided by adhering to the stipulation that commutativity, and idempotency of each indicator function be fully employed before Weyl ordering is implemented.

B. Pseudo-projections and pseudo-probabilities

Pseudo-projections generate pseudo-probabilities. Let 

\[ \Pi^N(\{A^\alpha = a^\alpha_i\}) \equiv \frac{1}{2} \{ \rho \Pi^N(\{A^\alpha = a^\alpha_i\}) \}. \]

Pseudo-probabilities can take negative values. A complete set of pseudo-probabilities, corresponding to all possible outcomes of a given set of observables, is the quantum analogue of the classical joint probability scheme. We call this the pseudo-probability scheme, or in short, the scheme. The entries in the scheme do add to unity. Their marginals, corresponding to sets of mutually commuting observables, are always true probability schemes, and are identical to the the quantum expectation values.

For comparison, we note that the formalism developed in [1] treats joint outcomes as absurd propositions, unless the corresponding projections commute. The vi-

The violations of Boolean logic that they observe arise from this identification. The present work makes the notion sharper, and quantitative.

C. Definition of non-classicality

Pseudo-probability schemes serve to define non-classicality of states in a very broad sense.

**Definition:** Let \( S_n(\{A^n\}) \) be the scheme for a set of \( N \) observables when a system is in a state \( \rho \). The state is deemed to be classical with respect to these observables if, and only if, all pseudo-probabilities in the scheme are non-negative.

It follows from the definition that the classicality of a state is not absolute. It has a meaning only relative to the sets of observables considered. Non-classicality is signalled by the appearance of negative entries in the scheme. At times, it may also get reflected in a pseudo-probability which exceeds unity. The many methods of designating states to be non-classical have consisted, so far, of important, but specific considerations, as for example, in [6, 7, 14]. But, as we demonstrate in Sect. 4 for non-locality and entanglement, it ultimately reduces
to identifying sets of observables for which the corresponding scheme produces unphysical entries. On the other hand, if no restriction is placed on the observables, there is always a scheme which has negative entries. In that sense, all states are non-classical though, in importance and as resources, not all of them would be on the same footing.

A scheme which has only non-negative entries, even when the underlying projections do not commute, automatically yields an explicit construction of contextual hidden parameters for the non-commuting observables[30]. Also, along the sidelines, the definition brings out the real import of the idea of negative probability advocated by Dirac,[31], Bartlett [32], and most forcefully, by Feynman [33].

1. Disjunction and negation

Conjunction (AND) is but one logical operation. The pseudo-projections that represent the disjunction (OR), and the negation (NOT) may be obtained by employing standard rules. Thus, e.g., the disjunction, $A = a_i$ OR $B = b_j$, is represented by the pseudo-projection

$$\pi_{a_i b_j} = \pi_{a_i} + \pi_{b_j} - \pi_{a_i b_j},$$

(2)

which is simply the quantum representative of the indicator function for the union: $1_{S_i∪S_j} = 1_{S_i} + 1_{S_j} - 1_{S_i∩S_j}$ of the corresponding indicator functions. Consistency is guaranteed in the sense that this operation is identical to the extraction of the pseudo-probability from the parent scheme for $A, B$. In a similar manner, the negation of a pseudo-projection is represented by subtracting it from identity. As ever, the indicator function for the null set is represented by the operator zero, and for the full set, by identity.

For illustrating the results obtained above, and for preparing the ground for studying non-locality and entanglement, the single qubit case is discussed first. The single qubit case is interesting in itself since it affords a large number of non-classicality tests. But that is not our focus here.

III. SINGLE QUBITS

We employ the notation $A_i \equiv \vec{\sigma} \cdot \hat{m}_i$. Let $A_1, A_2$ be two observables, each having two outcomes: $a_{1,2} = \pm 1$. The corresponding complete set of pseudo-projections for their joint occurrence is given, in terms of respective projections, by

$$\Pi_{a_1 a_2} = \frac{1}{2} \{ \pi_{a_1}, \pi_{a_2} \}; \quad a_{1,2} = \pm 1.$$  

(3)

The four pseudo-probabilities for a qubit in a state $\rho = \frac{1}{2}(1 + \vec{\sigma} \cdot \vec{P})$ can be read off as

$$\mathcal{P}_{a_1 a_2} = \frac{1}{4} \{ 1 + a_1 a_2 \hat{m}_1 \cdot \hat{m}_2 + \vec{P} \cdot (a_1 \hat{m}_1 + a_2 \hat{m}_2) \}.$$  

(4)

As averred, the marginal scheme obtained by summing over $a_1$ yields the quantum mechanical probabilities for $A_2$ and vice versa.

Several quick conclusions may be drawn from Eq. (4). If no condition on the two directions is imposed, all states except the completely mixed turn out to be non-classical. More explicitly, so long as $|\vec{P}| \neq 0$, one can always find two observables for which the scheme becomes negative. On the other hand, consider the restriction $\hat{m}_1 \cdot \hat{m}_2 = 0$, which is equivalent to simultaneous specification of two orthogonal components of the spin vector. Modulo this restriction, the state would be deemed to be classical if $|\vec{P}| < \frac{1}{2}$. This conclusion, together with extension to three orthogonal observables, is dual to the POVM for joint measurement of observables[34].

Remarkably, Eq. (4) coincides with the expression obtained by Cohen and Scully [35] who, starting with quantum analog of characteristic functions for a pair of observables in a two level system, computed the probability for their joint outcomes. This exact agreement merits further investigation. But moving further on, it may be checked that if three incompatible observables are considered, even the completely mixed state will also exhibit non-classicality. More pertinently, the present work, apart from being much simpler, clarifies the precise meaning of what Cohen and Scully [35] mean by joint probability, and goes beyond, by developing a general framework with applicability to any number of observables.

IV. NON-LOCALITY AND ENTANGLEMENT

This section presents the main applications of the formalism. We identify the elementary proposition that underlies Bell - CHSH non-locality, and construct a chain of more involved propositions that describe entanglement. Bell-CHSH criterion is dimension independent, and so is our analysis. In contrast, analytic tools for testing entanglement in higher dimensional systems do not yet exist. The associated complications and richness are illustrated through the chain of propositions which have been worked out for two qubit systems.

First, we establish some notations for the sake of compactness. All observables, considered henceforth, are dichotomic, with eigenvalues, say $\pm 1$. Thus, the two outcomes are negations of each other. The proposition $\mathcal{L}(A = +1)$ is denoted by $\mathcal{L}(A)$, and its negation, $\mathcal{L}(A = -1)$, by $\mathcal{L}(\bar{A})$. Conjunctions are written as simple juxtapositions, by omitting the sign $\wedge$. We agree to separate observables belonging to different subsystems by a semicolon. The following example illustrates the notation.

$$\mathcal{L}(A_1 = +1) \wedge \mathcal{L}(B_1 = -1) \wedge \mathcal{L}(B_2 = +1) \equiv \mathcal{L}(A_1; B_1 B_2)$$

Here, the observables $A_1$ and $B_{1,2}$ belong to the first and the second subsystems respectively. However, the OR operation (disjunction) will be explicitly denoted by the standard symbol $\lor$.

We also introduce a pictorial representation along-
Accordingly, its indicator function for the support $S$ is given by the sum of the corresponding indicator functions, as emphasized by the double bond.

The supports $S_i$ for the classical outcomes for each conjunction $L^i$ in fig. 3 are mutually disjoint. Therefore, the support $S$ for the proposition is their union. Accordingly, its indicator function $1_S$ is given by the sum of the corresponding indicator functions,

$$1_S = \sum_{i=1}^{4} 1_{S_i}. \quad (5)$$

The quantum representative of $1_S$ is the corresponding sum, $\Pi_N = \sum_{i} \Pi_i$ of the pseudo projections representing each conjunction. Note that none of the pseudo-projections in the summand is a projection. A state $\rho$ would be classical with respect to this proposition if the corresponding pseudo-probability respects the bounds

$$0 \leq Tr[\rho \Pi_N] \leq 1 \quad (6)$$

The identity, $\pi_{A1}^A = \frac{1}{2}(1 \pm A)$, immediately leads to the classic Bell-CHSH inequality

$$\left| \langle A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 \rangle \right| \leq 2. \quad (7)$$

This derivation essentially identifies the inadmissibility of standard Boolean rules to operations on a set of valid logical propositions. Thereby, it throws further light on the violation of the corresponding rule of classical probability in the form expressed in [7]. It brings to the fore the fact that violation of Bell-CHSH inequality is autonomous of the kinematics of inertial frames, though, by no means does this diminish the deep physical and philosophical consequences that follow from combining non-locality with special relativity. It also shows that the so called jointly measurable observables cannot lead to the violation of the Bell-CHSH inequality [25, 26, 36, 37]. It will be seen that this last conclusion continues to hold true for entanglement also.

Finally, for future comparison, we recast Eq. (7) for the two-qubit case. We employ the forms $A_{1,2} = \hat{\sigma} \cdot \hat{a}_{1,2}$ and $B_{1,2} = \sum \cdot \hat{b}_{1,2}$ in terms of the Pauli bases in the respective subspaces. We further choose the special geometry $a_1 \cdot a_2 = b_1 \cdot b_2 = 0$ and define $\hat{b} = (\hat{b}_1 + \hat{b}_2)/\sqrt{2}$ and $\hat{b}' = (\hat{b}_1 - \hat{b}_2)/\sqrt{2}$ to get:

$$\left| \langle \hat{\sigma} \cdot \hat{a}_1 \sum \cdot \hat{b} + \hat{\sigma} \cdot \hat{a}_2 \sum \cdot \hat{b}' \rangle \right| > \sqrt{2}. \quad (8)$$

### B. Entanglement

Though all non-local states are entangled, the converse statement is not necessarily true[10], suggesting that entanglement admits further refinements. Further, settling whether a state is entangled or not is a hard problem, which has led to several more modest approaches such as majorisation relations, conditions based on correlation tensors and studies involving concurrence [38–40]. Rather than hunt for a single proposition that would deliver a witness which is capable of detecting all entangled states, we take up two-qubit systems, and systematically construct four inequivalent propositions of increasing complexity, and show that each of them yields an entanglement witness, emphasising the breakdown of the validity of an underlying classical proposition.

#### 1. Proposition 1

As usual, $M$ stands for the observable $\hat{\sigma} \cdot \hat{m}$. Henceforth, we denote the observables in the first and the second subsystems respectively by Latin and Greek symbols. The respective Pauli operators will be denoted by $\sigma_i$ and $\Sigma_i$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Pictorial representation of the proposition $L^1\{A_1; B_1; B_2\} \lor L^2\{A_1; B_1; B_2\}$. Inset: The pictorial representation of the outputs $\pm 1$ of an observable.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Figure showing the proposition underlying Bell-CHSH non-locality and its pictorial representation.}
\end{figure}
Consider, then, a pair of mutually orthogonal observables \( A_1, A_2 \) for the first qubit and two triplets of mutually orthogonal observables \( \{\Phi_1, \Phi_2, \Phi_3\}, \{\Theta_1, \Theta_2, \Theta_3\} \) for the second. To specify the detector geometry completely, the normalised sums of the vectors, in the respective triplets \( \{\Phi_i\} \) and \( \{\Theta_i\} \), given by \( \phi = \sum_{i=1}^3 \hat{\phi}_i / \sqrt{3} \) and \( \hat{\theta} = \sum_{i=1}^3 \theta_i / \sqrt{3} \), will be chosen such that \( \phi \cdot \hat{\theta} = 0 \). The full form of the first logical proposition is displayed in fig. 4.

### Proposition 2

The inequality Eq. (9) covers only a subset of entangled states. To explore further improvement, we now consider two doublets of mutually orthogonal observables \( \{A_1, A_2\}, \{B_1, B_2\} \) for the first qubit, and two doublets of orthogonal observables \( \{\Phi_1, \Phi_2\}, \{\Theta_1, \Theta_2\} \) for the second. As before, the normalised sum of the vectors in the doublet \( \{A_i\} \) is denoted by \( \hat{a}_i \), and similarly so for all other doublets. We further choose the normalised sums to be mutually orthogonal for each qubit. Explicitly, \( \hat{a}_1 \cdot \hat{b}_2 = \hat{\theta}_3 \cdot \hat{\phi} = 0 \). The proposition is displayed in fig. 5.

The new proposition leads to the same correlations as in proposition 1, but with an improved bound, and hence a better witness \( W_2 \). We arrive at a better sufficiency condition for non-separability, given by

\[
W_2 : \left< \sigma \cdot \hat{a} \Sigma \cdot \phi + \sigma \cdot \hat{b} \Sigma \cdot \theta \right> < -1.
\]  

This improvement raises the possibility that inclusion of more observables (classical joint observations) may lead to even better witnesses. Two possibilities remain if we impose the same orthogonality conditions that have been employed so far.

### Proposition 3

The proposition is shown shown in fig. 6. Basically, for the first subsystem, we have three sets of doublets \( \{A_i\}, \{B_i\} \{C_i\} \) where observables within each set are mutually orthogonal. For the second system, we choose three sets of triplets \( \{\Phi_i\}, \{\Theta_i\}, \{\Psi_i\} \) where again, each triplet consists of mutually orthonormal observables. As in the earlier cases, the detector geometry is specified by requiring that both the sets of normalised sums \( \{\hat{a}, \hat{b}, \hat{c}\} \) and \( \{\hat{\phi}, \hat{\theta}, \hat{\psi}\} \) be orthonormal sets.

### Proposition 4

Finally, we consider three triplets of mutually orthogonal observables \( \{A_i\}, \{B_i\}, \{C_i\} \) for the first qubit, and another three triplet of orthogonal observables
\( \{\Phi_i\}, \{\Theta_i\}, \{\Psi_i\} \) for the second. As before, the normalised sums are chosen to be mutually orthogonal for each qubit. The logical proposition which is more involved is displayed in fig. 7.

The resulting inequality is no more complicated than in (11), but gives an improved sufficiency bound, the witness \( W_4 \), for the same operator

\[
W_4 : \left( \hat{\sigma} \cdot \hat{a} \hat{\Sigma} \cdot \hat{\phi} + \hat{\sigma} \cdot \hat{b} \hat{\Sigma} \cdot \hat{\theta} + \hat{\sigma} \cdot \hat{c} \hat{\Sigma} \cdot \hat{\psi} \right) < -1 \tag{12}
\]

The last of the conditions, \( W_4 \), has been derived earlier by Gühne et al. [41] and Horodecki et al. [42], who express their results with the choice \( \hat{a} = \hat{\phi} = \hat{x}; \hat{b} = \hat{\theta} = \hat{y}; \hat{c} = \hat{\psi} = \hat{z} \). It should be borne in mind that those derivations are driven by purely algebraic considerations, in contrast to our approach which is motivated by violations of classical rules of probability associated with operations involving classically valid logical propositions.

V. EXAMPLES AND DISCUSSION

The set of four inequalities, arising from violations of classical rules for different propositions, may be seen to induce a further refinement in the quantification of entanglement. For illustration, and further discussion, we shall begin with the special class of states, obtained by the addition of a local term to the Werner states,

\[
\rho = \frac{1}{4}(1 + \sigma \cdot \Sigma + \alpha \phi \cdot \phi + \beta \theta \cdot \theta + \gamma \psi \cdot \psi).
\]

The main results are captured in fig. 8. The region bounded by the points \( A, B, C \) and \( D \), represents the space of all allowed states. The line \( AC \) represents the Werner states, and the region contained between the arcs \( BED \) and \( BCD \) corresponds to separable states. The vertex \( A \) represents the fully entangled singlet state, and the point \( O \), the completely mixed state.

The five vertical lines mark the boundaries of sets of entangled states (triangles with \( A \) as their common vertex) detected by respective propositions. Of them, the first line \( L_0 \) marks the boundary between non-local and local states. The subsequent lines represent, in order, the sets encompassed by the set of four sufficiency conditions, \( W_1 \), in the same order. The last line, \( L_4 \), encompasses the largest region, which includes all the entangled Werner states. But it is still not exhaustive since it fails to detect entangled states in the region shaded in pink. The existence of a logical proposition that would lead to a witness that detects all the entangled state is yet to be demonstrated.

It is possible to draw several strong conclusions for more general states. Consider an arbitrary two-qubit state in the SVD basis,

\[
\rho = \frac{1}{4}(1 + \sigma \cdot \Sigma + \alpha \phi \cdot \phi + \beta \theta \cdot \theta + \gamma \psi \cdot \psi + \sum_{i=1}^{3} t_i \sigma_i \Sigma_i).
\]

The witnesses, comprising entirely of correlations, are sensitive only to the singular values \( t_i \). Each witness, therefore detects entangled states in regions, determined by the corresponding set of inequalities imposed on the singular values. The region of the correlation space that represents a state is a tetrahedron, defined by the four inequalities [43],

\[
\begin{align*}
1 - t_1 - t_2 - t_3 & \geq 0 \\
1 - t_1 + t_2 + t_3 & \geq 0 \\
1 + t_1 - t_2 + t_3 & \geq 0 \\
1 + t_1 + t_2 - t_3 & \geq 0.
\end{align*}
\]

First consider the witness \( W_4 \). It partitions the correlation space into two parts. States that are separable and those that are entangled but evade detection by \( W_4 \),...
satisfy the conditions:
\[
\begin{align*}
1 + t_1 + t_2 + t_3 & \geq 0 \\
1 + t_1 - t_2 - t_3 & \geq 0 \\
1 - t_1 + t_2 - t_3 & \geq 0 \\
1 - t_1 - t_2 + t_3 & \geq 0.
\end{align*}
\] (16)

Together with conditions in Eq. (15), they constitute the interior (and surface) of an octahedron.

The complementary region lying outside the octahedron corresponds to the entangled states detected by \( W_4 \). This sufficiency condition also becomes necessary for the Bell diagonal states, as may be seen by employing partial transpose criterion, and as also displayed in fig. 8.

Thus, for the Bell diagonal states \( W_4 \) is the strongest. But it still leaves the relative strengths of the four witnesses undetermined. To settle that we shall consider the other three witnesses. Starting with \( W_3 \) we arrive at a new set of conditions:
\[
\begin{align*}
\frac{\sqrt{3}}{2} - t_1 + t_2 + t_3 & \geq 0 \\
\frac{\sqrt{3}}{2} - t_1 - t_2 - t_3 & \geq 0 \\
\frac{\sqrt{3}}{2} - t_1 + t_2 - t_3 & \geq 0 \\
\frac{\sqrt{3}}{2} - t_1 - t_2 + t_3 & \geq 0.
\end{align*}
\] (17)

which, together with Eq. (15) define a larger octahedron, which contains the separable and (undetected) entangled states. As before, \( (t_1, t_2, t_3) \) corresponding to the entangled states detected by \( W_3 \) must lie outside the octahedron. Between \( W_4 \) and \( W_3 \), the former is, of course, stronger. Of real interest, however, is to compare them with the conditions obtained by \( W_{1,2} \). These yield two sets of 12 bounds (that define dodecahedrons),
\[
\begin{align*}
c + t_1 \pm t_2 & \geq 0 \\
c + t_2 \pm t_3 & \geq 0 \\
c + t_3 \pm t_1 & \geq 0
\end{align*}
\] (18)

with \( c = \frac{2}{\sqrt{3}}(1) \) for \( W_1(W_2) \). These 12 conditions, in conjunction with Eq. (15), yield the required region in the correlation space, that contains all the separable states and also some (undetected) entangled states. The states lying outside the respective regions are all entangled and get detected.

With the regions thus identified, we may immediately conclude that \( W_4 \) is the strongest and that \( W_3 \) is stronger than \( W_1 \). We already know that \( W_2 \) is stronger than \( W_1 \). However, as may be seen in fig. 9, \( W_2 \) and \( W_3 \) are mutually independent, since the entangled states that are detected have only partial overlaps. In fig. 9 we show projection of the correlation space, with \( t_3 = 0.5 \). The hexagon \( QDEMHA \) and the rectangle \( PRLN \) constitute the set of separable and undetected entangled states vis-a-vis \( W_2 \) and \( W_3 \), respectively. The overlap is the octagon \( ABCDEFGH \). The triangles \( QBC \) and \( MGF \) are detected only by \( W_3 \). Likewise the triangles \( CDR, EFL, GHN, ABP \) are detected only by \( W_2 \). These results, together with the examples discussed, reinforce the statement that different witnesses reflect substructures in the space of entangled states that arise from violations of different classical rules.

VI. CONCLUSION

In conclusion, this paper proposes and develops an alternative way of studying non-classical attributes of states. The basic blocks are classically meaningful propositions involving outcomes of measurements with multiple observables, and the quantum representatives of the corresponding indicator functions — which are constructed from standard rules of quantum mechanics. We have shown that any state affords an infinite number of non-classicality tests, and that the current classification rests upon imposing reasonable restrictions on the so called joint observables. As an illustration, we have recovered the classic Bell-CHSH inequality, purely from logical propositions, without reference to non-locality [6], or a violation of a specific rule of classical probability [7]. We have also demonstrated the emergence a chain of entanglement witnesses for two qubit systems from basic propositions which indicate, essentially, further refinements in that overarching concept. Thus, this study
brings out the unity underlying the works involving the logic of quantum mechanics [1–3], negative probability [31–33], and criteria for non-classicality [6–17]. More pertinently, it opens up a fertile field for a systematic theoretical and experimental investigation of many more tests of non-classicality of states.

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