The Černy conjecture

A.N. Trahtman

Abstract. A word $w$ of letters on edges of underlying graph $Γ$ of deterministic finite automaton (DFA) is called synchronizing if $w$ sends all states of the automaton to a unique state. J. Černy discovered in 1964 a sequence of $n$-state complete DFA possessing a minimal synchronizing word of length $(n-1)^2$. The hypothesis, well known today as the Černy conjecture, claims that it is also precise upper bound on the length of such a word for a complete DFA. The hypothesis was formulated in 1966 by Starke. The problem has motivated great and constantly growing number of investigations and generalizations.

To prove the conjecture, we use algebra $w$ on a special class of row monomial matrices (one unit and rest zeros in every row), induced by words in the alphabet of labels on edges. These matrices generate a space with respect to the mentioned operation.

The proof is based on connection between length of words $u$ and dimension of the space generated by solutions $L_x$ of matrix equation $M_u L_x = M_s$ for synchronizing word $s$, as well as on the relation between ranks of $M_u$ and $L_x$.

Keywords: deterministic finite automaton, synchronizing word, Černy conjecture.

Introduction

The problem of synchronization of DFA is a natural one and various aspects of this problem have been touched in the literature. Prehistory of the topic, the emergence of the term, the connections with the early coding theory, first efforts to estimate the length of synchronizing word [21], [22], different problems of synchronization one can find in surveys [14], [17].

Synchronization makes the behavior of an automaton resistant against input errors since, after detection of an error, a synchronizing word can reset the automaton back to its original state, as if no error had occurred. The synchronizing word limits the propagation of errors for a prefix code. Deterministic finite automaton is a tool that helps to recognized language in a set of DNA strings.

A problem with a long story is the estimation of the minimal length of synchronizing word.

J. Černy in 1964 [6] found the sequence of $n$-state complete DFA with shortest synchronizing word of length $(n-1)^2$ for an alphabet of size two. The hypothesis,
well known today as the Černý’s conjecture, claims that this lower bound on the length of the synchronizing word of aforementioned automaton is also the upper bound for the shortest synchronizing word of any \( n \)-state complete DFA:

**Conjecture 1** The deterministic complete \( n \)-state synchronizing automaton over alphabet \( \Sigma \) has synchronizing word in \( \Sigma \) of length at most \( (n - 1)^2 \) [28] (Starke, 1966).

The problem can be reduced to automata with a strongly connected graph [6]. An attempt to prove this hypothesis is proposed below.

This famous conjecture is true for a lot of automata, but in general the problem still remains open although several hundreds of articles consider this problem from different points of view [35].

Moreover, two conferences ”Workshop on Synchronizing Automata” (Turku, 2004) and ”Around the Černý conjecture” (Wrocław, 2008) were dedicated to this longstanding conjecture. The problem is discussed in ”Wikipedia” - the popular Internet Encyclopedia and on many other sites.

As well as the Road Coloring problem [1], [12], [33], this simple-looking conjecture was arguably the most longstanding and famous open combinatorial problems in the theory of finite automata [17], [24], [25], [28], [29].

We consider a class of matrices \( M_u \) of mapping induced by words \( u \) in the alphabet of letters on edges of the underlying graph \( \Gamma \). The matrix \( M_u \) of word \( u \) belongs to the class of matrices with one unit in every row and rest zeros (row monomial). We call them also matrices of word.

There are no examples of automata such that the length of the shortest synchronizing word is greater than \( (n - 1)^2 \). Moreover, the examples of automata with shortest synchronizing word of length \( (n - 1)^2 \) are infrequent. After the sequence of Černý and the example of Černý, Piricka and Rosenauerova [9] of 1971 for \( |\Sigma| = 2 \), the next such examples were found by Kari [15] in 2001 for \( n = 6 \) and \( |\Sigma| = 2 \) and by Roman [27] for \( n = 5 \) and \( |\Sigma| = 3 \) in 2004.

The package TESTAS [33], [36] studied all automata with strongly connected underlying graph of size \( n \leq 11 \) for \( |\Sigma| = 2 \), of size \( n \leq 8 \) for \( |\Sigma| \leq 3 \) and of size \( n \leq 7 \) for \( |\Sigma| \leq 4 \) and found five new examples of DFA with shortest synchronizing word of length \( (n - 1)^2 \) with \( n \leq 4 \). Don and Zantema present in [10] an ingenious method of designing new automata from existing examples of size three and four and proved that for \( n \geq 5 \) the method does not work. So there are up to isomorphism exactly 15 DFA for \( n = 3 \) and exactly 12 DFA for \( n = 4 \) with shortest synchronizing word of length \( (n - 1)^2 \).

The authors of [10] support the hypothesis from [31] that all automata with shortest synchronizing word of length \( (n - 1)^2 \) are known, of course, with essential correction found by themselves for \( n = 3, 4 \).

There are several reasons [2], [4], [5], [10], [31] to believe that the length of the shortest synchronizing word for remaining automata with \( n > 4 \) (except the sequence of Černý and examples for \( n = 5, 6 \)) is essentially less and the gap grows with \( n \). For several classes of automata, one can find some estimations on the length in [2], [8], [16], [18], [32].
Initially found upper bound for the minimal length of synchronizing word was big and has been consistently improved over the years by different authors. The upper bound found by Frankl in 1982 [11] is equal to \((n^3 - n)/6\). The result was reformulated in terms of synchronization in [26] and repeated independently in [19]. The cubic estimation of the bound exists since 1982. Attempts to improve Frankl’s result were unsuccessful.

The considered deterministic automaton \(A\) can be presented by a complete underlying graph with edges labelled by letters of an alphabet.

Our work uses the class of row monomial matrices \(M_u\) of mapping induced by words \(u\) in the alphabet of letters on edges of the underlying graph and properties of corresponding space.

The matrix approach for synchronizing automata supposed first by Béal [3] proved to be fruitful [4], [5], [7].

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We consider the equation \(M_u L_x = M_s (2)\) for synchronizing word \(s\) and the space generated by row monomial solutions \(L_x\). A connection between the set of nonzero columns of matrix of word, subsets of states of automaton and our kind \(L_x\) of solutions of (2) is revealed in Remarks.

Theorems [11] [2] finish our attempt to prove the Černý conjecture. Theorem [3] and some corollaries contain certain consequences. The ideas of the proof are illustrated on example of automata with a maximal length of synchronizing word from [15].

Preliminaries

We consider a complete \(n\)-state DFA with strongly connected underlying graph \(\Gamma\) and transition semigroup \(S\) over a fixed finite alphabet \(\Sigma\) of labels on edges of \(\Gamma\) of an automaton \(A\). The trivial cases \(n \leq 2\), \(|\Sigma| = 1\) and \(|A\sigma| = 1\) for \(\sigma \in \Sigma\) are excluded.

The restriction on strongly connected graphs is based on [6]. The states of the automaton \(A\) are considered also as vertices of the graph \(\Gamma\).

If there exists a path in an automaton from the state \(p\) to the state \(q\) and the edges of the path are consecutively labelled by \(\sigma_1, \ldots, \sigma_k\), then for \(s = \sigma_1 \ldots \sigma_k \in \Sigma^+\) let us write \(q = ps\).

Let \(Px\) be the set of states \(q = px\) for all \(p\) from the subset \(P\) of states and \(x \in \Sigma^+\). Let \(Ax\) denote the set \(Px\) for the set \(P\) of all states of the automaton.
A word \( s \in \Sigma^+ \) is called a *synchronizing* (reset, magic, recurrent, homing, directable) word of an automaton \( A \) with underlying graph \( \Gamma \) if \( |As| = 1 \). The word \( s \) below denotes minimal synchronizing word such that for a state \( q.As = q \).

The states of the automaton are enumerated, the state \( q \) has number one.

An automaton (and its underlying graph) possessing a synchronizing word is called *synchronizing*.

Let us consider a linear space generated by row monomial (one unit and rest of zeros in every row) \( n \times n \)-matrices.

We connect a mapping of the set of states of the automaton made by a word \( u \) with an \( n \times n \)-matrix \( M_u \) such that for an element \( m_{i,j} \in M_u \) takes place

\[
    m_{i,j} = \begin{cases} 
        1, & q_i u = q_j; \\
        0, & \text{otherwise.}
    \end{cases}
\]

Any mapping of the set of states of the automaton \( A \) can be presented by some row monomial word \( u \) and by a corresponding matrix \( M_u \). For instance,

\[
    M_u = \begin{pmatrix}
        0 & 0 & 1 & \ldots & 0 \\
        1 & 0 & 0 & \ldots & 0 \\
        0 & 0 & 0 & \ldots & 1 \\
        \ldots & \ldots & \ldots & \ldots & \ldots \\
        0 & 1 & 0 & \ldots & 0 \\
        1 & 0 & 0 & \ldots & 0
    \end{pmatrix}
\]

Let us call the matrix \( M_u \) of the mapping induced by the word \( u \), for brevity, the matrix of word \( u \).

\[ M_u M_a = M_{ua} \] 3.

The set of nonzero columns of \( M_u \) (set of second indexes of its elements) of \( M_u \) is denoted as \( R(u) \).

Zero matrix is considered as a matrix of empty word.

The subset of states \( A_u \) is denoted as \( c_u \) with number of states \( |c_u| \). In \( n \)-vector \( c_u \) the coordinate \( j \) has unit if the state \( j \in c_u \) and zero in opposite case.

For linear algebra terminology and definitions, see [20], [23].

1 Mappings induced by a word and subword

**Remark 1** The invertible matrix \( M_a \) does not change the number of units of every column of \( M_u \) in its image of the product \( M_a M_u \).

Every unit in the product \( M_u M_a \) is the product of two units, first unit from nonzero column of \( M_u \) and second unit from a row with one unit of \( M_a \).

**Remark 2** The columns of the matrix \( M_u M_a \) are obtained by permutation of columns \( M_u \). Some columns can be merged (units of columns are moved along row to a common column) with \( |R(ua)| < |R(u)| \).

The rows of the matrix \( M_u M_a \) are obtained by permutation of rows of the matrix \( M_u \). Some of these rows may disappear and replaced by another rows of \( M_u \).
Lemma 1 The number of nonzero columns \(|R(b)|\) is equal to the rank of \(M_b\).

\(|R(ua)| \leq |R(u)|\) and \(R(au) \subseteq R(u)\).

For invertible matrix \(M_a\) \(R(au) = R(u)\) and \(|R(ua)| = |R(ua)|\).

For the set of states of deterministic finite automaton \(A\) and any words \(u\) and a \(Aua \subseteq Aa\).

Nonzero columns of \(M_{ua}\) have units also in \(M_a\).

Proof. The matrix \(M_b\) has submatrix with nonzero determinant having only one unit in every row and in every nonzero column. Therefore \(|R(b)|\) is equal to the rank of \(M_b\).

The matrix \(M_a\) in the product \(M_aM_u\) shifts column of \(M_u\) to columns of \(M_aM_u\) without changing the column itself by Remark 2 or merging some columns of \(M_a\).

In view of possible merged columns, \(|R(ua)| \leq |R(u)|\).

Some rows of \(M_u\) can be replaced in \(M_aM_u\) by another row and therefore some rows from \(M_u\) may be changed, but zero columns of \(M_u\) remain in \(M_aM_u\) (Remark 1).

Hence \(R(au) \subseteq R(u)\) and \(|R(ua)| \leq |R(ua)|\).

For invertible matrix \(M_a\) in view of existence \(M_a^{-1}\) we have \(|R(ua)| = |R(u)|\) and \(R(au) = R(u)\).

From \(R(ua) \subseteq R(a)\) follows \(Aua \subseteq Aa\).

Nonzero columns of \(M_{ua}\) have units also in \(M_a\) in view of \(R(ua) \subseteq R(ua)\).

Corollary 1 The invertible matrix \(M_a\) keeps the number of units of any column of \(M_u\) in corresponding column of the product \(M_aM_u\).

Corollary 2 The matrix \(M_s\) of word \(s\) is synchronizing if and only if \(M_s\) has zeros in all columns except one and units in the residual column.

All matrices of right subwords of \(s\) also have at least one unit in this column.

2 Necessary conditions of the operation of summation in the class of row monomial matrices

Lemma 2 Suppose that for row monomial matrices \(M_i\) and \(M\)

\[ M = \sum_{i=1}^{k} \lambda_i M_i. \]  \hspace{1cm} (1)

with coefficients \(\lambda\) from \(Q\).

Then the sum \(\sum_{i=1}^{k} \lambda_i = 1\) and the sum \(S_j\) of values in every row \(j\) of the sum in (1) also is equal to one.

If \(\sum_{i=1}^{k} \lambda_i M_i = 0\) then \(\sum_{i=1}^{k} \lambda_i = 0\) and \(S_j = 0\) for every \(j\) with \(M_u = 0\).

If the sum \(\sum_{i=1}^{k} \lambda_i\) in every row is not unit [zero] then \(\sum_{i=1}^{k} \lambda_i M_i\) is not a row monomial matrix.
Proof. The nonzero matrices $M_i$ have $n$ cells with unit in the cell. Therefore, the sum of values in all cells of the matrix $\lambda_i M_i$ is $n \lambda_i$.

For nonzero $M$ the sum is $n$. So one has in view of $M = \sum_{i=1}^{k} \lambda_i M_i$ 

\[ n = n \sum_{i=1}^{k} \lambda_i, \text{ whence } 1 = \sum_{i=1}^{k} \lambda_i. \]

Let us consider the row $j$ of matrix $M_j$ in $\mathbb{1}$ and let $1_j$ be unit in the row $j$. The sum of values in a row of the sum $\mathbb{1}$ is equal to unit in the row of $M$. So $1 = \sum_{i=1}^{k} \lambda_i 1_i = \sum_{i=1}^{k} \lambda_i$.

$\sum_{i=1}^{k} \lambda_i M_i = 0$ implies $S_j = \sum_{i=1}^{k} \lambda_i 1_i = \sum_{i=1}^{k} \lambda_i = 0$ for every row $j$.

If the matrix $M = \sum_{i=1}^{k} \lambda_i M_i$ is a matrix of word or zero matrix then $\sum_{i=1}^{k} \lambda_i \in \{0, 1\}$. If $\sum_{i=1}^{k} \lambda_i \notin \{0, 1\}$ or the sum in $0, 1$ is not the same in every row then we have opposite case and the matrix does not belong to the set of row monomial matrix.

The set of row monomial matrices is closed with respect to the considered operation and together with zero matrix generates a space.

3 Useful lemmas

Lemma 3 The set $V$ of all $n \times k$-matrices of words (or $n \times n$-matrices with zeros in fixed $n - k$ columns for $k < n$) has $n(k - 1) + 1$ linear independent matrices.

Proof. Let us consider distinct $n \times k$-matrices of word with at most one nonzero cell outside the last nonzero column $k$.

Let us begin from the matrices $V_{i,j}$ with unit in $(i, j)$ cell ($j < k$) and units in $(m, k)$ cells for all $m$ except $i$. The remaining cells contain zeros. So we have $n - 1$ units in the $k$-th column and only one unit in remaining $k - 1$ columns of the matrix $V_{i,j}$. Let the matrix $K$ have units in the $k$-th column and zeros in the other columns. There are $n(k - 1)$ matrices $V_{i,j}$. Together with $K$ they belong to the set $V$. So we have $n(k - 1) + 1$ matrices. For instance, 

\[
V_{1,1} = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 1 \\
\vdots \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad V_{3,2} = \begin{pmatrix}
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 1 \\
0 & 1 & 0 & \ldots & 0 \\
\ldots \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad K = \begin{pmatrix}
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 1 \\
\ldots \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

The first step is to prove that the matrices $V_{i,j}$ and $K$ generate the space with the set $V$. For arbitrary matrix $T$ of word from $V$ for every $t_{i,j} \neq 0$ and $j < k$, let us consider the matrices $V_{i,j}$ with unit in the cell $(i, j)$ and the sum of them $\sum V_{i,j} = Z$.

The first $k - 1$ columns of $T$ and $Z$ coincide. Hence in the first $k - 1$ columns of the matrix $Z$ there is at most only one unit in any row. Therefore in the cell of $k$-th column of $Z$ one can find only value of $m$ or $m - 1$. The value of $m$ appears if there are only zeros in other cells of the considered row. Therefore $\sum V_{i,j} - (m - 1)K = T$. Thus every matrix from the set $V$ is a span of $(k - 1)n + 1$
matrices from $V$. It remains now to prove that the set of matrices $V_{i,j}$ and $K$ is a set of linear independent matrices.

If one excludes a certain matrix $V_{i,j}$ from the set of these matrices, then it is impossible to obtain a nonzero value in the cell $(i, j)$ and therefore to obtain the matrix $V_{i,j}$. So the set of matrices $V_{i,j}$ is linear independent. Every non-trivial linear combination of the matrices $V_{i,j}$ equal to a matrix of word has at least one nonzero element in the first $k - 1$ columns. Therefore, the matrix $K$ could not be obtained as a linear combination of the matrices $V_{i,j}$. Consequently the set of matrices $V_{i,j}$ and $K$ forms a basis of the set $V$.

**Corollary 3** The set of all row monomial $n \times (n - 1)$-matrices of words (or $n \times n$-matrices with zeros in a fixed column) has $(n - 1)^2$ linear independent matrices.

The set of row monomial $n \times 2$-matrices of words has at most $n + 1$ linear independent matrices.

The set of row monomial matrices of words with one column has at most $n$ linear independent matrices.

**Corollary 4** There are at most $n(n - 1) + 1$ linear independent matrices of words in the set of $n \times n$-matrices.

**Lemma 4** Distributivity

For every words $b$ and $x_i$

$$M_b \sum \tau_i M_{x_i} = \sum \tau_i (M_b M_{x_i}).$$

$$(\sum \tau_i M_{x_i}) M_b = \sum \tau_i (M_{x_i} M_b).$$

**Proof.** The matrix $M_b$ from left shifts rows of every $M_{x_i}$ and of the sum of them in the same way according to Remark 2. $M_b$ removes common row of them and replace also by common row (Remark 2).

Therefore the matrices $M_b M_{x_i}$ and the sum $\sum \tau_i M_b M_{x_i}$ has the origin rows with one unit from $M_{x_i}$ and maybe in another order than in its linear combination $\sum \tau_i M_{x_i}$. The matrix $M_b$ from right shifts column $m$ of every $M_{x_i}$ and of the sum of them in the same way (to the same column $k$) according to Remark 2. $M_b$ merges columns of the sum and of terms in the same way too. (Remark 2).

Therefore the matrices $M_{x_i} M_b$ and the sum $(\sum \tau_i M_{x_i}) M_b$ has the origin columns (sometimes merged) of $M_{x_i}$ and from its linear combination $(\sum \tau_i M_{x_i}) M_b$, with the same merged columns.

### 4 Linear independent matrices $M_u$

**Lemma 5** Let the space $W$ be generated by linear independent $n \times n$-matrices $M_u$ of words $u$ of restricted length $1 \leq |u| \leq j$ and units only in first $k < n$ columns of $M_u$.

Then some matrix $M_{u\beta} \notin W$ for generator $M_u$ of $W$ and some letter $\beta$, sometimes with $|R(v)| < |R(u)|$. 


Proof. Assume the contrary: for every word $u$ with $1 \leq |u| \leq j$ of generator $M_{u_i}$ of $W$ and every letter $\beta$ with the length $(|u_i\beta| \leq j + 1)$ the matrix $M_{u_i\beta}$ in $W$.

For every matrix $M \in W$

$$M = \sum \tau_i M_{u_i}$$

with generators $M_{u_i}$ in $W$ such that $|u_i| \leq j$ with units in first $k < n$ columns.

Therefore by distributivity from Lemma 4

$$M \beta = \sum \tau_i M_{u_i} \beta$$

for matrices $M_{u_i} \beta = M_{u_i\beta}$ with its units only in first $k < n$ columns.

Therefore also $M_{u\beta} = M \beta = \sum \tau_i M_{u_i\beta}$ belongs to $W$ and by induction for every word $t$ the matrix $M_{u,t}$ belongs to $W$ with its units only in first $k < n$ columns.

By induction, for every word $t$ the matrix $M_{vt} \in W$.

Contradiction for considered synchronizing automaton because for some word $t$ the matrix $M_{vt}$ has nonzero column $n$.

Corollary 5 Let the sequence of spaces $W_j$ of dimension $j$ be ordered by inclusion by grow of $j$. The basis of $W_1$ contains one letter that maps a pair of states into one and the basis of $W_j \supset W_{j-1}$. The space $W_j$ is extended by matrix $M_{u\beta}$ for a letter $\beta$ and some $M_u$ from basis of $W_{j-1}$.

Then the length of the word $u$ of every generator $M_u$ of $W_j$ is not greater than $j$.

Matrices of left subword of every generator of $W_j$ are linear independent because all generators and its right subwords were obtained by adding a letter from right.

5 The equation with unknown $L_x$

Definition 1 We denote

$$M_u \sim_q M_v$$

if the columns of the state $q$ of both matrices are equal.

If the set of cells with units in the column $q$ of the matrix $M_v$ is a subset of the analogous set of the matrix $M_u$ then we write

$$M_v \subseteq_q M_u$$

As $= q$ for synchronizing word $s$.

The solution $L_x$ of the equation

$$M_u L_x = M_s$$

for synchronizing matrix $M_s$ and arbitrary $M_u$ must have units in the column of the state $q$.

Lemma 6 Every equation $M_u L_x = M_s$ has a solutions $L_x$ with at least $n \geq |R(u)| > 0$ units in column $q$. Every nonzero column $j$ of of $M_u$ corresponds a unit in the cell $j$ of column $q$. 


For solution $L_x$ with only $|R(u)|$ units in column $q$ (a minimal solution) $L_x \subseteq L_y$ for any other solution $L_y$ of (2).

There exists one-to-one correspondence between units in the column $q$ of minimal solution $L_x$ and the set $c_u$ of states.

Proof. The matrix $M_s$ of rank one has nonzero column of the state $q$.

For every nonzero column $j$ of $M_u$ with elements $u_{i,j} = 1$ and $s_{i,q} = 1$ in the matrix $M_s$ the cell $(j,q)$ must have unit in the matrix $L_x$. So the unit in the column $q$ of matrix $M_s$ is a product of every unit from the column $j$ of $M_u$ and unit in the sell $j$ of column $q$ of $L_x$.

The set $R(u)$ of nonzero columns of $M_u$ corresponds the set of cells of the column $q$ with unit of $L_x$.

Therefore the minimal solution $L_x$ has in the column $q$ $|R(u)|$ units.

So to the column $q$ of every solution belong at least $|R(u)|$ units. The remaining units of the solution $L_x$ belong to the next columns, one unit in a row. The remaining cells obtain zero.

Lastly every solution $L_x$ is a row monomial matrix of word.

Zeros in the column $q$ of minimal $L_x$ correspond zero columns of $M_u$. Therefore for matrix $L_y$ such that $L_x \subseteq L_y$ we have $M_u L_y = M_s$. On the other hand, every solution $L_y$ must have units in cells of column $q$ that correspond nonzero columns of $M_u$.

Thus minimal $L_x$ has $|R(u)|$ units in column $q$ and the equality $M_u L_x = M_u L_y = M_s$ is equivalent to $L_x \subseteq L_y$.

The matrix $M_u$ has set $R(u)$ of nonzero columns and maps the automaton on the set $c_u$ of states and on the set of units in the column $q$ of minimal $L_x$.

Lemma 6 explains the following

Remark 3 Every permutation and shift of $m$ nonzero columns $M_u$ induces corresponding permutation of the set of $m$ units in the column $q$ of minimal solution $L_x$ of (2), and vice versa.

6 Allocation of linear independent matrices $L_x$

Lemma 7 There exists allocation of matrix $M_s$ and $k n$ linear independent solutions $L_x$ in first $k + 1 < n$ columns of $n \times n$-matrix $T$.

Proof. The dimension of the space of matrices with $k + 1$ nonzero columns is at most $k n + 1$ cells by Lemma 6.

$M_s$ is placed at the beginning in column $q$ because $A s = q$.

Cells without unit in $T$ let us call empty. First all cells in last $n - 1$ columns are empty after allocation of $M_s$.

We continue extend set of linear independent $L_x$ using consistent allocation of its units. Only one unit of $L_x$ will be placed in empty cell, the remaining units we place in non-empty cells of allocation of former $L_x$. The goal is a sequence
of linear independent matrices $L_x$. The unit in empty cell guarantees linear independence matrix $L_x$ from previously allocated $L_x$.

There are at most $n$ linear independent matrices $M_u$ with $n - 1$ units in the column $q$, last units of them belong to distinct rows outside $q$ (Corollary 9). One can shift every such unit along row to column 2 without change linear independence.

Then let us allocate linear independent matrices $L_x$ with $|R(x)| = n - k$ units in column $q$ and continue by growth of $k$ from two to $n - 2$. There are for every $k$ at most $nk + 1$ linear independent matrices $L_x$ with $|R(x)| = n - k$ units in column $q$ is a considered part of them. So one has at most $nk + 1$ linear independent matrices $L_x$ in at most $nk + n$ cells of first $k + 1$ columns of $T$. ($M_s$ was allocated first in $n$ cells.) The allocation for fixed $k$ has three steps for the set of $L_x$ and its $n$ units.

At the beginning for given $k$ and considered $L_x$, $|R(u)|$ units of $L_x$ are allocated in cells of column $q$ corresponding nonzero columns of matrix $M_u$ in equation $M_u L_x = M_s$. So $L_x$ satisfies the equation (2) as minimal solution. Then we choose from remaining $n - |R(u)|$ units of $L_x$ only one unit for empty cell in possible minimal column $i \leq (k + 1)$. We reduce on such way the set of empty cells in $k + 1$ first columns and number of $L_x$ of given $k$. The remaining units are allocated arbitrarily from column two in non-empty cells of previously allocated $L_x$.

Thus one can place at most $n(n - 2)$ linear independent matrices $L_x$ together with $M_s$ in at most $n((n - 2) + 1)$ cells of first $n - 1$ columns with $|R(u)|$ units in column $q$ of every $L_x$ corresponding $M_u$ from (2).

7 Theorems

**Theorem 1** The deterministic complete $n$-state synchronizing automaton $A$ with strongly connected underlying graph over alphabet $\Sigma$ has synchronizing word in $\Sigma$ of length at most $(n - 1)^2$.

The matrices of left subword of some synchronizing word of the automaton are linear independent.

**Proof.** We consider solutions $L_x$ of (2) for linear independent words $u$ from Lemma 5. By Lemma 7 at most $kn$ linear independent matrices $L_x$ together with $M_s$ can be allocated in first nonzero $k$ columns of some matrix, say $W$.

For $k = n - 2$, one has a space of matrices $L_x$ and $M_s$ of dimension at most $n(n - 2) + 1$. All matrices $L_x$ with at least two units in column $q$ have allocation in first $n - 1$ columns. Therefore for remaining words $u$ there are only matrices $L_x$ with one unit in column $q$.

By Lemma 7 one unit of such $L_x$ of next $u$ belongs to column $q$, another unit is in last column $n$ and remaining units are in columns from 2 to $n - 1$.

We replace the matrix $M_u$ from the basis of $W$ by this $L_x$ with one unit in column $q$. The dimension of new space is $n(n - 2) + 1$ as before.
By Lemma 6 corresponding word $u$ of such $L_x$ has $|R(u)| = 1$, whence the matrix $M_u$ has one nonzero column of synchronizing word $u$ with $Au = q$. The length of $u$ is restricted by $n(n - 2) + 1$ by Corollary 5.

Moreover, by Corollary 5 the length of the word $u$ of every generator of considered space is not greater than $n(n - 2) + 1$ and all these generators have linear independent matrices of its left subword. Hence the obtained synchronizing word $u$ has linear independent matrices of left subwords of $M_u$.

**Corollary 6** For every integer $k < n$ of deterministic complete $n$-state synchronizing automaton $A$ with strongly connected underlying graph over alphabet $\Sigma$ there exists a word $v$ of length at most $n(k - 1) + 1$ such that $|Av| \leq n - k$.

**Theorem 2** The deterministic complete $n$-state synchronizing automaton $A$ with underlying graph over alphabet $\Sigma$ has synchronizing word in $\Sigma$ of length at most $(n - 1)^2$.

Follows from Theorem 1 because the restriction for strongly connected graphs can be omitted due to [6].

**Theorem 3** Suppose that $|\Gamma \alpha| < |\Gamma| - 1$ for a letter $\alpha \in \Sigma$ in deterministic complete $n$-state synchronizing automaton $A$ with underlying graph $\Gamma$ over alphabet $\Sigma$.

Then the minimal length of synchronizing word of the automaton is less than $(n - 1)^2$.

Proof. We follow the proof of Theorem 1.

The difference is that at the beginning of the proof the equation (2) has at least two linear independent nontrivial solutions for the matrix $M_\alpha$ of a letter $\alpha$ equal to the first word $u = \alpha$ of length one. Number of linear independent matrices $L_x$ is greater then corresponding $|u|$. The difference remains on every step.

Hence we obtain finally synchronizing word of length less than $(n - 1)^2$ with strongly connected ideal $I$.

Let us go to the case of not strongly connected underlying graph with $n - |I| > 0$ states outside $I$.

This ideal has synchronizing word of length at most $(|I| - 1)^2$ (Theorem 1). There is a word $p$ of length at most $(n - |I|)(n - |I| + 1)/2$ such that $Ap \subset I$.

$(|I| - 1)^2 + (n - |I|)(n - |I| + 1)/2 < (n - 1)^2$. Thus, the restriction for strongly connected automata can be omitted.

**8 Examples**

J. Kari [15] discovered the following example of $n$-state automaton with minimal synchronizing word of length $(n - 1)^2$ for $n = 6$. 
The minimal synchronizing word

\[ s = ba^2 bababa^2 b^2 aba^2 ba^2 baba^2 b \]

has the length at the Černy border.

Every line below presents a pair (word \( u \), column \( q \) of \( L_x \)) of linear independent \( L_x \) from the sequence with evidently non-linear picture.

(\( b, 111110 \)) \( R(u) = 5 \)
(\( ba, 111011 \))
(\( ba^2, 111101 \))
(\( ba^2 b, 111100 \)) \( R(u) = 4 \)
(\( ba^2 ba, 111010 \))
(\( ba^2 bab, 011110 \))
(\( ba^2 babab, 101111 \)) \( R(v) = 5 \) (\( 101011 \) of \( L_x \))
(\( ba^2 babab, 011110 \)) \( R(u) = 4 \)
(\( ba^2 bababa, 110101 \))
(\( ba^2 bababa^2 b, 011101 \))
(\( ba^2 bababa^2 b, 111000 \)) \( R(u) = 3 \)
(\( ba^2 bababa^2 b^2, 011100 \))
(\( ba^2 bababa^2 b^2 a, 110111 \)) \( R(v) = 5 \) (\( 101010 \) of \( L_x \))
(\( ba^2 bababa^2 b^2 ab, 001110 \)) \( R(u) = 3 \)
(\( ba^2 bababa^2 b^2 aba, 100011 \))
(\( ba^2 bababa^2 b^2 aba^2 b, 011111 \)) \( R(v) = 5 \) (\( 010101 \) of \( L_x \))
(\( ba^2 bababa^2 b^2 aba^2 b, 011000 \)) \( R(u) = 2 \)
(\( ba^2 bababa^2 b^2 aba^2 ba^2 b, 101000 \))
(\( ba^2 bababa^2 b^2 aba^2 ba^2 b, 001101 \)) \( R(v) = 3 \) (\( 001100 \) of \( L_u \))
(\( ba^2 bababa^2 b^2 aba^2 ba^2 b, 100010 \)) \( R(u) = 2 \)
(\( ba^2 bababa^2 b^2 aba^2 ba^2 bab, 001010 \))
(\( ba^2 bababa^2 b^2 aba^2 ba^2 bab, 001011 \)) \( R(v) = 3 \) (\( 000011 \) of \( L_u \))
(\( ba^2 bababa^2 b^2 aba^2 ba^2 bab, 000101 \)) \( R(u) = 2 \)
(\( ba^2 bababa^2 b^2 aba^2 ba^2 babab^2 b = s, 100000 \)) \( R(s) = 1 \)

By the bye, the matrices of left subwords of \( s \) are simply linear independent.

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