Abstract—The literature on Bessel beams is both fragmented and incomplete. We present here a unified and completed perspective of such beams, irrespective to their orbital angular momentum (OAM) – zero, integer or noninteger – and mode – scalar or vectorial, and LSE/LSM or TE/TM in the latter case. The unification is based on the integral superposition of constituent waves along the angular-spectrum cone of the beam, and allows to describe, compute, relate, and implement all the Bessel beams in a universal fashion. The paper first establishes the integral superposition theory. Then, it demonstrates the previously unreported existence of noninteger-OAM TE/TM Bessel beams, compares the LSE/LSM and TE/TM modes, and establishes useful mathematical relations between them. It also provides an original description of the position of the noninteger-OAM singularity in terms of the initial phase of the constituent waves. Finally, it introduces a general technique to generate Bessel beams by an adequate superposition of properly tuned waves. This global perspective and theoretical complement may open up new avenues in applications such as spectroscopy, sources. This global perspective and theoretical complement may open up new avenues in applications such as spectroscopy, microscopy, and optical/quantum force manipulations.

Electromagnetic Bessel beams represent a fundamental form of structured light. They are localized waves that carry orbital angular momentum (OAM) along their propagation axis. Localized waves are waves that propagate without spatial dispersive (or diffraction) and without temporal (or chromatic) dispersion. Their energy is thus uniformly confined and invariant perpendicular to and along, respectively, their direction of propagation. The OAM is a beam property whose macroscopic manifestation is an isophase surface that has the form of a vortex along the axis of the beam. It may be integer or noninteger. In the former case, the wave has the transverse phase dependence \(e^{in\phi} (n \in \mathbb{Z})\), corresponding to an OAM of \(n\hbar\) per photon, while in the latter case the wave is made of a superposition of integer-OAM waves that combine so as to produce a noninteger-OAM per photon. This property association of localization and OAM confers to Bessel beams specific capabilities for manipulating light that may be exploited in diverse applications, such as nanoparticle guiding, spectroscopy, microscopy, and quantum key distribution. Figure 1 shows a general classification of Bessel beams that pertain to the sequel of the paper.

Bessel beams are the simplest form of light OAM after the Laguerre-Gauss beams and the most studied localized waves. They are monochromatic beams with a transverse amplitude pattern that follows Bessel functions of the first kind, \(J_n(\alpha \rho)\), multiplied by the phase function \(e^{in\phi} (n \in \mathbb{Z})\).

\(^{1}\)OAM waves, or optical vortices, behave to some extent as charged particles: They may repel and attract each other, and mutually annihilate upon colliding \(^{1}\). For this reason, \(n\) is also called the topological charge.

\(^{2}\)The terminology LSE/LSM (longitudinal section electric/magnetic) is borrowed from the theory of waveguides \(^{11}\). Such modes are often referred to a linearly polarized beams in free-space optics \(^{12}\), but the terminology LS is more rigorous since such fields also include a longitudinal component.
by a metasurface [18].

Unfortunately, the literature on Bessel beams is very fragmented. It is composed of different theoretical perspectives that are often restricted to a specific type of Bessel beam or deprived of physical insight. Moreover, the circular cylindrical Bessel modes have received surprisingly little attention to date, with their noninteger version still remaining open question to date, despite their fundamental character. Finally, the aforementioned implementation techniques are as specific of the theoretical approaches, cumbersome or/and inefficient.

This paper resolves this double issue of fragmentation and incompleteness by presenting a unified and completed perspective of Bessel beams. The unification is based on the integral construction of the beam along its angular-spectrum cone, and allows to describe, compute, relate, and implement all the Bessel beams (see Fig. 1) in an equally simple fashion. The paper first establishes the theory of this approach. Then, it shows, as an example, that Eq. (1) in this case [Eq. (2d)] represents a continuous set of waves that propagate at the angle $\nu$ of a Bessel function, as illustrated in Fig. 2, and where apex, so as to form a transverse interference pattern in the form with the oblique wave vector (see Supp. Mat. A). These waves have the mathematical form

$$U_{\nu}(x,y,z,t) = \int_0^{2\pi} \psi_{\nu}(\phi_G) d\phi_G,$$

where $\psi_{\nu}(\phi_G)$ represents a continuous set of waves that propagate at the angle $\phi_G$ along a circular cone towards its apex, so as to form a transverse interference pattern in the form of a Bessel function, as illustrated in Fig. 2 and where $\nu$ is equal to the OAM of the beam when it is integer or half-integer (see Supp. Mat. A). These waves have the mathematical form

$$\psi_{\nu}(\phi_G) = e^{i(k(\phi_G) \cdot \mathbf{r} + \gamma_{\nu}(\phi_G) - \omega t)} U(\xi),$$

with the oblique wave vector

$$k(\phi_G) = -\alpha \cos(\phi_G) \mathbf{x} + \sin(\phi_G) \mathbf{y} + \beta \mathbf{z},$$

where

$$\alpha = k_0 \sin(\delta), \quad \beta = k_0 \cos(\delta),$$

so that $\alpha^2 + \beta^2 = k_0^2 = (\omega/c)^2$, and $\delta$ is called the axicon angle, and with the linear phase

$$\gamma_{\nu}(\phi_G) = 2\pi \nu \frac{\phi_G}{2\pi} \left(1 - \frac{\phi_G}{2\pi}\right), \quad \nu \in \mathbb{R}.$$  \hspace{1cm} (2d)

In these relations, $U(\xi)$ is the transverse apodization of each wave with respect to its propagation direction, $k(\phi_G)$, with $\xi$ being the radial variable of the corresponding local coordinate system, $\text{frac}(\cdot)$ is the fractional part function [19], and $\phi_{G,0}$ is related to the position of the phase singularity for $\nu \in \mathbb{R}, \mathbb{Z}$, as will be seen later. Note that Eq. (2d) represents a sawtooth function of $\phi_G$ that reduces to $\gamma_{\nu}(\phi_G) = \nu \phi_G$ for $\phi_G \in [0,2\pi]$ and $\phi_{G,0} = 0$.

Practically, the waves $\psi_{\nu}(\phi_G)$ must be spatially limited (e.g. Gaussian cross-sectional apodization $U(\xi)$), but they will be initially considered as plane waves ($U(\xi) = \text{const}$) because, as we shall see, an analytical solution can be derived for Eq. (1) in this case. Moreover, their azimuthal separation (\(\Delta \phi_G\)) is infinitesimally small, so that they effectively merge into the azimuthal continuum corresponding to the integral [1]. Finally, these waves may be temporally considered as a scalar, but they will later be seen to represent any of the components of fully vectorial electromagnetic waves.

The Bessel beam superposition in Eq. (1) is plotted in Fig. 3 using 20 equi-spaced plane waves for $\psi_{\nu}(\phi_G)$ in Eq. (2), with the top row showing the superposition of the maxima and minima, and the bottom row plotting the superposition of the actual waves with continuous sinusoidal gradients. This figure shows a perfectly smooth Bessel pattern is obtained around the axis of the beam with a restricted number of constituent waves and illustrates the increasing structural complexity of the Bessel beam without OAM ($\nu = 0$), with integer OAM ($\nu = 1$) and with non-integer OAM ($\nu = 1.5$).

Substituting Eq. (2) with $U(\xi) = A^{\text{PW}}(\text{const})$ into Eq. (1), simplifying the resulting integral (see Supp. Mat. C), and decomposing the field into its transverse-dependence and longitudinal/temporal-dependence parts as $U_{\nu}(\rho, \phi, z, t) = \psi_{\nu}(\phi_G)$.

\textsuperscript{3}Note that the beam is still localized in this case due to the decreasing envelope of the Bessel interference pattern.\textsuperscript{4}Eq. (1) may then be alternatively expressed as the inverse Fourier transform $U_{\nu}(\rho, \phi, z) = \mathcal{F}^{-1}\{\hat{U}_{\nu}(k_{\rho}, k_\phi, k_z)\} = \int_0^\infty \int_0^{2\pi} \int_0^\infty \hat{U}_{\nu}(k_{\rho}, k_\phi, k_z) e^{ik_{\rho} r - \omega t} k_{\rho} dk_{\rho} dk_\phi dk_z$ with the spectrum $\hat{U}_{\nu}(k_{\rho}, k_\phi, k_z) = \frac{\delta(k_{\rho} - \nu)}{2\pi} e^{ik_{\rho} r + (2\pi - (\phi_{G,0} \pm \pi))} \delta(k_z - \beta)$ if $k_{\rho} < \phi_{G,0} \pm \pi$ and $\hat{U}_{\nu}(k_{\rho}, k_\phi, k_z) = \frac{\delta(k_{\rho} - \nu)}{2\pi} e^{ik_{\rho} r - (\phi_{G,0} \pm \pi)} \delta(k_z - \beta)$. If $k_{\rho} \geq \phi_{G,0} \pm \pi$ (see Supp. Mat. B).

\textsuperscript{5}The top row of Fig. 3 shows that this radial effect is due to the radially decreasing density of the constituent waves.

![Fig. 2. General description of a Bessel beam [Eq. (1)] as a superposition of waves [Eq. (2a)] with different phases [Eq. (2d)] propagating along a cone of opening angle $\delta = \tan^{-1}(\alpha/\beta)$ [Eq. (2b)] towards its apex. The field panel shows, as an example, $|U_1(x,y,0,t)| = |J_1(\alpha \rho)|$.](image-url)
with the complex weighting distribution
\[ A_m^{BB}(\nu, \phi_G, 0) = 2\pi i^{-m}e^{-i\nu\phi_G, 0}A^P W\sin((m-\nu)\pi)e^{-i(m-\nu)(\phi_{G,0}+\pi)}. \] (6b)

Equation (6) contains the integer-OAM Bessel solution, since \( \nu = n \in \mathbb{Z} \) transforms Eq. (6b) into \( A_m^{BB} = 2\pi i^{-m}e^{-i\nu\phi_G, 0}A^P W\delta_m \), which reduces the sum in Eq. (6a) to the single term \( U_{m,\perp} = 2\pi i^{-n}e^{-i\nu\phi_G, 0}A^P WJ_n(\alpha\rho)e^{im\phi} \). But it also contains noninteger-OAM (\( \nu \in \mathbb{R} \setminus \mathbb{Z} \)) solutions, where the satisfaction of the circular periodic boundary condition is realized by a superposition of integer-OAM Bessel waves with proper phase \( (e^{im\phi}) \) and weighting coefficients \( (A_m^{BB}(\nu, \phi_G, 0)) \). In this noninteger-OAM case, the sum in Eq. (6a) must be practically truncated to an integer \( m = \pm M \) that is large enough to provide a satisfactory approximation of the Bessel beam.

Equation (6a) reveals that the parameter \( \alpha \) of the cone in Fig. 2 corresponds to the spatial frequency of the Bessel pattern. Since this parameter is proportional to the axicon angle, \( \delta \), according to Eq. (2c), we find that increasing the aperture of the cone in the integral construction of Eq. (1) compresses the Bessel ring pattern towards the axis of the beam.

Figure 4 plots the magnitude and phase of the Bessel beam given by Eq. (6) for different integer and noninteger OAMs. The cases \( \nu = 0, 1 \) and 1.5 correspond to the instantaneous field plots in Fig. 3. The OAM-less beam \( \nu = 0 \) has simultaneously azimuthally symmetric magnitude and phase. The integer beams \( \nu = n \in \mathbb{N} \) have azimuthally symmetric magnitudes but asymmetric phase (OAM). The beams \( \nu \in \mathbb{R} \setminus \mathbb{Z} \) have simultaneously asymmetric magnitude and phase. The parameter \( \phi_{G,0} \), which is here 0, corresponds to a dummy initial phase of the integer-OAM and the position of the discontinuity of the noninteger-OAM in the individual waves, with increasing \( \phi_{G,0} \) clockwise rotating the asymmetric magnitude of the noninteger-OAM pattern (see Supp. Mat. E).
waves under special conditions such as the paraxial condition ($\delta \ll \pi/2$) in the electromagnetic case. On the other hand, they fail to describe Bessel beams with a large axicon aperture (angle $\delta$ in Fig. 2), which are relevant to applications such as microscopy and optical force manipulations. Therefore, we extend here the previous scalar generalization to the vectorial case.

For the scalar case, we have established two alternative solutions, the integral solution of Eq. (1) and the analytical solution of Eq. (6). In the present extension to the vectorial case, we shall restrict our treatment to the integral approach because it provides more insight into the physical nature of the beam and because it will constitute the basis for the practical implementation to be discussed later. We shall still assume a plane wave construction ($\omega(\xi) = A^{\text{PW}} \, (\text{const.})$ in Eq. (2b)), for simplicity.

As indicated in Fig. 1, the vectorial Bessel beams can be LSE/LSM, with $\iota = \{x, y\}$ or TE/TM, where the subscript denotes the field component that is zero. In the former case, the electric/magnetic transverse field component that is nonzero is set as the scalar Bessel beam solution $U_\nu$ in Eq. (1), while in the latter case, it is the magnetic/electric longitudinal component of the field that is set as $U_\nu$, and the other components are found from Eq. (2) via Maxwell equations (see Supp. Mat. F).

The LSE$_y$ ($E_y = 0, E_x = U_\nu$) and TM$_z$ ($H_z = 0, E_z = U_\nu$) solutions are respectively given by (see Supp. Mat. F)

$$E(\rho, \phi, z) = \int_0^{2\pi} d\phi_G \, \psi_\nu(\phi_G) \times (\hat{x} + \tan(\delta) \cos(\phi_G) \hat{z}),$$

(7a)

$$\eta H(\rho, \phi, z) = \int_0^{2\pi} d\phi_G \, \psi_\nu(\phi_G) \times \left( -\sin(\delta) \tan(\delta) \sin(\phi_G) \cos(\phi_G) \hat{x} + \left( \sin(\delta) \tan(\delta) \cos^2(\phi_G) + \cos(\delta) \right) \hat{y} \right.$$

(7b)

$$+ \sin(\delta) \sin(\phi_G) \hat{z}),$$

and

$$E(\rho, \phi, z) = \int_0^{2\pi} d\phi_G \, \psi_\nu(\phi_G) \times \left( \cot(\delta) \cos(\phi_G) \hat{x} + \cot(\delta) \sin(\phi_G) \hat{y} + \hat{z} \right),$$

(8a)

$$\eta H(\rho, \phi, z) = \int_0^{2\pi} d\phi_G \, \psi_\nu(\phi_G) \times \left( -\csc(\delta) \sin(\phi_G) \hat{x} + \csc(\delta) \cos(\phi_G) \hat{y} \right.$$

(8b)

where $\eta$ is the impedance of free-space. The most striking difference between the LSE/LSM and TE/TM beams resides in the simplest feature of their respective constituent waves.

The former have a linear transverse polarization, while the latter have of a constant transverse magnitude.

A detailed investigation of these solutions reveals that the axicon angle ($\delta$) distinctly affects the LSE/LSM and TE/TM modes (see Supp. Mat. G). In both cases, increasing $\delta$ compresses the Bessel ring pattern towards the axis of the beam; however, this variation also breaks the symmetry of the transverse LSE/LSM patterns, even for $\nu = 0$, whereas it leaves the TE/TM pattern azimuthally symmetric. It is also interesting to note that, for a small axicon angle, i.e., $\delta \ll \pi/2$, the LSE$_y$ mode can essentially reduces to its $E_x$ and $H_y$ components, similarly to the scalar form.

Figures 3 and 6 depict the LSE$_y$ and TM$_z$ Bessel beams of global order $\nu = 1.5$ corresponding to the solutions of Eqs. (7) and (8), respectively. The vectorial field distributions plotted in the panels (a) and (b) of the two figures represent samples of the constituting waves of the integral construction of the beam (Fig. 2). Their strong vectorial nature starkly contrasts with the configuration of the scalar solution, except for the LSE$_y$ case in the aforementioned axicon limit ($\delta \ll \pi/2$). Note that the electric field of the constituent waves of the LSE$_y$ mode is linearly polarized in the $x$-direction, while that of the TM$_z$ modes is radially polarized. Nonzero and non-integer $\nu$ vectorial modes are obtained from their fundamental counterpart by simply setting the $\nu$ parameter in the initial phase of the constituent waves, i.e., $\gamma_\nu(\phi_G)$ in Eq. (2f), to the desired OAM.

Fig. 5. Noninteger global order ($\nu = 1.5$) LSE$_y$ Bessel beam, computed by Eq. (7), (a) Transverse vectorial fields at $z = z_G$ for 12 plane wave samples. (b) Corresponding complete vectorial fields in 4 different cut planes (dotted circles) with axicon angle $\delta = 25^\circ$. (c) Magnitude and phase of the fields over the cross-sectional area of $8\lambda \times 8\lambda$.

Figure 7 plots time-average Poynting vectors of integer and non-integer LSE$_y$ and TM$_y$ Bessel beams. Interestingly, whereas the maxima of the LSE$_y$ transverse Poynting vector components are superimposed with those of the longitudinal Poynting vector component, the TM$_z$ transverse maxima are not overlapping the longitudinal maxima. Also notice that the
Fig. 6. Noninteger global order ($\nu = 1.5$) TM$_z$ Bessel beam, computed by Eq. (8), with the same parameters and panels as in Fig. 5.

The LSE/LSM and TM/TE electromagnetic vectorial Bessel beams are related by the following relations, which may be easily verified upon comparing Eqs. (7) and (8):

\[
\frac{1}{2} \left\{ \begin{array}{l}
\text{LSE}_y,\nu - 1 + \text{LSE}_y,\nu + 1 \\
\text{LSM}_y,\nu - 1 + \text{LSM}_y,\nu + 1
\end{array} \right\}
\]
\[\pm \sin(\delta) \frac{1}{2} \left\{ \begin{array}{l}
\text{TE}_{z,\nu - 1} - \text{TE}_{z,\nu + 1} \\
\text{TM}_{z,\nu - 1} - \text{TM}_{z,\nu + 1}
\end{array} \right\} = \left\{ \begin{array}{l}
\text{LSE}_{y,\nu} \\
\text{LSM}_{y,\nu}
\end{array} \right\},
\]
\[\pm \tan(\delta) \frac{1}{2} \left\{ \begin{array}{l}
\text{TE}_{z,\nu - 1} + \text{TE}_{z,\nu + 1} \\
\text{TM}_{z,\nu - 1} + \text{TM}_{z,\nu + 1}
\end{array} \right\} = \left\{ \begin{array}{l}
\text{LSE}_{x,\nu} \\
\text{LSM}_{x,\nu}
\end{array} \right\},
\]

where the second subscripts ($\nu$ and $\nu \pm 1$) correspond, as usual, to the global OAM $\nu$ in Eq. (2a). Note the OAM conservation between the LSE/LSM and TM/TE modes in each of these relations, and the interesting mediation of the axicon angle. Note also that, in the paraxial approximation ($\delta \ll \pi/2$), the TM/TE beams, with their complex cylindrical (radial/azimuthal) polarizations, be realized superposing two transversally linearly-polarized LSE/LSM beams, according to Eq. (2a).

Several techniques have been proposed for generating Bessel beams experimentally. The main ones are axicon lenses illuminated by a Laguerre-Gauss beam [15], spatial light modulators [17], open circular waveguides with selectively excited modes [24], antenna arrays with proper phase feeding network [25], metasurfaces illuminated by plane waves [18], and 2D circular leaky-wave antennas [26]. Unfortunately, these techniques are restricted to simple beams, excessively complex to implement, bulky and expensive, or suffering from poor efficiency.

The unified integral formulation presented in this paper (Fig. 2 with Eq. (1) for the scalar case, and Eqs. (7) and (8) for the vectorial case) naturally points to a generation technique that is immune of these issues and that offers in addition a universal implementation framework. Indeed, circularly distributing a set of sources with the phases, amplitudes and polarizations of the derived modal field solutions (e.g., top panels of Figs. (5) and (6)) would exactly and efficiently produce the corresponding Bessel beams, irrespective to their order or complexity.

Specifically, the integral-formulation generation technique consists in the following design steps: 1) select a sufficient number of sources ($N$) to properly sample the desired OAM according to the Nyquist criterion, 2) determine an appropriate beam apodization ($w(\xi)$) for each of the constituent waves to be radiated by these sources, and 3) adequately set the phase, magnitude and polarization of each of the sources, and orient them so as to launch the constituent waves along a cone with
the selected axicon angle (δ). This is mathematically expressed by the formula

$$\mathbf{E} = \sum_{\phi_G \in [0, 2\pi]} \tilde{w}(\phi_G) \mathbf{E}^\text{PW}(\phi_G) \Delta \phi_G,$$

(10a)

where \(w(\phi_G)\) is the apodization of the constituent waves, \(\mathbf{E}^\text{PW}(\phi_G)\) is their plane-wave modal field solution (e.g., Eqs. (7) or (8), and \(\Delta \phi_G = 2\pi/N\). In the case of a (typical) Gaussian apodization, we have

$$w(\phi_G) = e^{-(\alpha^2 G_x^2 + y^2 G_y^2)} / w_0^2,$$

(10b)

where \(w_0\) is the waist of the beam, and \((x_0(\phi_G), y_0(\phi_G))\) represents the local conical coordinates

$$x_0(\phi_G) = (x \cos(\phi_G) + y \sin(\phi_G)) \cos(\delta) - z \sin(\delta),$$

(10c)

$$y_0(\phi_G) = -x \sin(\phi_G) + y \cos(\phi_G),$$

(10d)

which are related to the radial conical coordinate \(\sqrt{x_0(\phi_G)^2 + y_0(\phi_G)^2} = \xi(\phi_G)\) in \(w(\xi)\) (Eq. (2a)).

Note that apodization of the plane wave \(\mathbf{E}^\text{PW}\) by the function \(w(\xi)\) results in a localization of the beam in a restricted of extent \(L = w_0 / \sin(\delta)\) about the center of the cone at \((z = 0)\). Moreover, the discretization of the integral induces a distortion of the Bessel pattern, which grows with the distance from the axis of the beam, as previously explained, so that \(N\) may have to be increased to provide a satisfactory beam approximation across the transverse area of interest.

Figure 8 depicts the experimental implementation of the integral-formulation Bessel beam generation. Figure 8(a) represents a direct incarnation of this formulation, which consists of a circular array of laser beams with proper magnitudes, phases and polarizations, as illustrated in Fig. 8(b). Such an implementation, involving \(N\) independent lasers with respective magnitude, phase and polarization controls, is quite complex and cumbersome. Fortunately, recent advances in metasurface technology suggests the much more practical implementation shown in Fig. 8(c). Indeed, this metasurface-based Bessel beam generator requires only one laser source, while being ideally compact and inexpensive.

The metasurface required in the implementation of the Bessel beam generator depicted in Fig. 8(c) can be easily realized using latest metasurface synthesis techniques [27]. The simplest implementation strategy would consist in cascading metasurfaces that separately tailor the amplitude, the phase, the inclination, and the polarization of the incident wave in the transverse plane of the system. Specifically, assuming a linearly polarized incident wave, such a design would then consist in three cascaded metasurfaces. Two of these metasurfaces would be common to the LSE/LSM and TE/TM cases, with one metasurface providing the required azimuthal phase distribution via azimuthal sectors made of particles inducing progressive transmission delays, and the other providing the required conical inclination via a constant radial phase gradient. In contrast, third metasurface would be different for the LSE/LSM and TE/TM cases. In the former case, given the linear transverse polarization (Eq. (7) and Fig. 5), there is

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Supplementary Material A. OAM per Photon for a Bessel Beam of General Order

The total time-averaged angular momentum (AM) per photon for a monochromatic vectorial beam with propagation axis \( \hat{z} \) is

\[
J_z = \hbar \omega \int_V \langle \mathbf{r} \times (\mathbf{g}) \rangle \cdot \hat{z} \, dV / \int_V \langle \mathbf{u} \rangle \, dV, \tag{A1}
\]

where \( \langle \mathbf{g} \rangle = \text{Re} \{ \mathbf{D}^* \times \mathbf{B} \} / 2 \) is the time-averaged linear electromagnetic momentum density, \( \langle \mathbf{u} \rangle = \text{Re} \{ \mathbf{E}^* \cdot \mathbf{D} + \mathbf{B}^* \cdot \mathbf{H} \} / 4 \) is the time-averaged energy density, and \( V \) is a cylindrical volume enclosing the beam.

Wave beams, including Bessel beams, are typically satisfactorily described in terms of their paraxial approximation. In this regime, the fields \( \mathbf{D}, \mathbf{B}, \mathbf{H} \) can be expressed, from the Maxwell equations and from the constitutive equations, in terms of the field \( \mathbf{E} \), and the longitudinal part of \( \mathbf{E}, E_z \), can be expressed in terms of its transverse part, \( \mathbf{E}_T \). Expressing then \( \langle \mathbf{g} \rangle \) and \( \langle \mathbf{u} \rangle \) in Eq. (A1) in terms of \( \mathbf{E}_T \), neglecting the terms in \( 1/k^2 \) and \( 1/k^4 \) \cite{28}, and integrating the remaining terms by part yields

\[
J_z = L_z + S_z, \tag{A2a}
\]

which splits into

\[
L_z = \int_V \text{Re} \{ -i \hbar \mathbf{E}_z^* \cdot \left( \mathbf{r} \times \nabla \right) \mathbf{E}_T \} / \int_V \text{Re} \{ \mathbf{E}_T^* \cdot \mathbf{E}_T \} \, dV \tag{A2b}
\]

and

\[
S_z = \int_V \text{Re} \{ -i \hbar (\mathbf{E}_z^* + \mathbf{E}_T) \cdot \hat{z} \} \, dV / \int_V \text{Re} \{ \mathbf{E}_T^* \cdot \mathbf{E}_T \} \, dV \tag{A2c}
\]

where \( L_z \) is the orbital AM (OAM) and \( S_z \) is the spin AM (SAM).

In the case of a Bessel beam with transverse linear polarization (case of LSE and LSM modes, but not TE and TM modes, in the paper), i.e., \( \mathbf{E}_T = U \hat{x} \), Eq. (A2b) reduces to

\[
L_z = \hbar \int_0^{2\pi} \int_0^R U^* \rho \rho \left[ G_{\nu}(k_{\rho}, k_{\phi}, k_z) \right] \, d\rho \, d\phi , \tag{A3}
\]

where the radial integrals have been truncated to a finite radius, \( R \), to account for the finite energy of a practical beam.

Figure S1 plots, using Eq. (A3), the OAM per photon for a Bessel beam \( [U = U_\nu \text{ in Eq. (1)}] \) with \( \psi_\nu(\phi_G) \) in Eq. (2) for \( R = 25 \text{ m} \). As can be seen, the OAM is a continuous and wiggly function of the parameter \( \nu \), which coincides with the linear function \( L = \hbar \nu \) only at integer and half-integer values of \( \nu \).

Fig. S1. OAM per photon carried by a Bessel beam of order \( \nu \) (solid curve) computed by Eq. (A3) with \( k = 2\pi, \delta = 25^\circ, \phi_G,0 = 0 \) and \( R = 25 \text{ m} \), and corresponding linear relation (dashed curve), showing the (nontrivial) relation between the parameter \( \nu \) and the OAM \( (L_z / \hbar) \). The OAM for \( \nu \in \mathbb{Z} \) is called the topological charge.

Supplementary Material B. Verification of the Conical Spatial Angular Spectrum

We shall show here that the conical spatial angular spectrum

\[
\hat{U}_\nu(k_{\rho}, k_{\phi}, k_z) = \left\{ \begin{array}{ll}
\frac{\delta(k_{\rho} - \alpha)}{k_{\rho}} e^{i\nu (k_\phi + (2\pi - (\phi_G,0,0,\pm\pi)))} \delta(k_z - \beta) & \text{if } k_\phi < \phi_G,0,0,\pm\pi, \\
\frac{\delta(k_{\rho} - \alpha)}{k_{\rho}} e^{i\nu (k_\phi - (\phi_G,0,0,\pm\pi))} \delta(k_z - \beta) & \text{if } k_\phi > \phi_G,0,0,\pm\pi,
\end{array} \right.
\]

(B1)

where the ± solutions in each of the two expressions correspond to the ranges \( 0 \leq \phi_G,0,0,\pm\pi \leq \pi \) (± sign) and \( \pi \leq \phi_G,0,0,\pm\pi < 2\pi \) (− sign), which appears in the the spatial inverse Fourier transform

\[
U_\nu(\rho, \phi, z) = \mathcal{F}\{U_\nu(k_{\rho}, k_{\phi}, k_z)\} = \int_0^\infty \int_0^{2\pi} \int_0^\infty \hat{U}_\nu(k_{\rho}, k_{\phi}, k_z) \times e^{i(k_{\rho} - \omega)\rho} dk_{\rho} dk_{\phi} dk_z , \tag{B2a}
\]

where

\[
k(k_{\phi}) = k_{\rho} \cos(k_{\phi}) \hat{x} + k_{\rho} \sin(k_{\phi}) \hat{y} + k_z \hat{z} \tag{B2b}
\]

and

\[
\mathbf{r}(\rho, \phi, z) = \rho \cos(\phi) \hat{x} + \rho \sin(\phi) \hat{y} + z \hat{z}, \tag{B2c}
\]

(B2c)

corresponds to the integral representation of Eq. (1) with Eq. (2), or Eq. (3).
Substituting Eq. (B1) into Eq. (B2) and solving the simple integrals in \( k_\rho \) and \( k_z \) yields

\[
U_\nu(\rho, \phi, z) = \int_{0}^{\phi_{0,0} \pm \pi} e^{i(\alpha \cos(k_\rho \phi + \gamma_\nu(\phi_\rho) - \omega t)\rho \phi + \beta z - \omega t)} dk_\phi e^{iw_1 h_1} dh_1 \\
+ \int_{\phi_{0,0} \pm \pi}^{2\pi} e^{i(\alpha \cos(k_\rho \phi + \gamma_\nu(\phi_\rho) - \omega t)\rho \phi + \beta z - \omega t)} dk_\phi.
\] (B3)

The remaining integrals, in \( k_\phi \), can then be solved via the change of variables \( h_1 = k_\rho + \left( 2\pi - (\phi_{0,0} \pm \pi) \right) \) and \( h_2 = k_\rho - (\phi_{0,0} \pm \pi) \), which leads to

\[
U_\nu(\rho, \phi, z) = \int_{\phi_{0,0} \pm \pi}^{2\pi} e^{i\left(-\alpha \cos(h_1 + \phi_{0,0} - \phi) + \beta z - \omega t\right)} e^{iw_1 h_1} dh_1 \\
+ \int_{\phi_{0,0} \pm \pi}^{2\pi} e^{i\left(-\alpha \cos(h_2 + \phi_{0,0} - \phi) + \beta z - \omega t\right)} dh_2.
\] (B4)

Then, defining \( h_1 := h_0 \) and \( h_2 := h_0 \) transforms this expression into

\[
U_\nu(\rho, \phi, z) = \int_{0}^{2\pi} e^{i\left(-\alpha \cos(h + \phi_{0,0} - \phi) + \beta z - \omega t\right)} e^{iw_0 h_0} dh_0.
\] (B5)

Finally, using the change of variables \( \phi' = h_0 + \phi_{0,0} \) transforms this expression into

\[
\tilde{U}_\nu(\rho, \phi, z) = \int_{\phi_{0,0}}^{\phi_{0,0} + 2\pi} e^{i\left(-\alpha \cos(\phi' + \phi_{0,0} - \phi) + \beta z - \omega t\right)} e^{iw_0 h_0} d\phi_0',
\] (B6)

which is identical to Eq. (3). This proves that \( \tilde{U}_\nu(\rho, \phi, z) \) in Eq. (B1) is indeed the angular spectrum of \( U_\nu(\rho, \phi, z) \) in Eq. (2), i.e., that a Bessel beam has an angular spectrum given by Eq. (B1).

**Supplementary Material C. Derivation of Eq. (3)**

Substituting Eq. (2) with \( w(\xi) = A_{PW} \) (const.) into Eq. (1) yields

\[
U_\nu(x, y, z, t) = \int_{0}^{2\pi} A_{PW} e^{i(k(\phi_0) + \gamma(\phi_0) - \omega t)} d\phi_0,
\] (C1)

where, assuming \( 0 < \phi_0 < 2\pi \),

\[
\gamma(\phi_0) = \begin{cases} 
\nu(\phi_0 + (2\pi - \phi_{0,0})) & 0 \leq \phi_0 < \phi_{0,0}, \\
\nu(\phi_0 - \phi_{0,0}) & \phi_{0,0} \leq \phi_0 < 2\pi.
\end{cases}
\] (C2)

Inserting the Eq. (C2) into Eq. (C1) splits the integral of the former as

\[
U_\nu(x, y, z, t) = A_{PW} \int_{0}^{\phi_{0,0}} e^{i(k(\phi_0) + \gamma(\phi_0) - \omega t)} d\phi_0 \\
+ A_{PW} \int_{\phi_{0,0}}^{2\pi} e^{i(k(\phi_0) + \gamma(\phi_0) - \omega t)} d\phi_0.
\] (C3)

Operating next the change of variable \( \phi_1 = \phi_0 + 2\pi \) in the first integral transforms this relation into

\[
U_\nu(x, y, z, t) = A_{PW} \int_{0}^{2\pi} e^{i(k(\phi_1) + \gamma(\phi_1) - \omega t)} d\phi_1 \\
+ A_{PW} \int_{\phi_{0,0}}^{2\pi} e^{i(k(\phi_0) + \gamma(\phi_0) - \omega t)} d\phi_0.
\] (C4)

Finally, the two integrals in this relation can be grouped into a single one via the definitions \( \phi_1 := \phi' \) and \( \phi_2 := \phi'' \), which results into

\[
U_\nu(x, y, z, t) = A_{PW} \int_{\phi_{0,0}}^{\phi_0 + 2\pi} e^{i(k(\phi_1) + \gamma(\phi_1) - \omega t)} d\phi_1.
\] (C5)

**Supplementary Material D. Derivation of Eq. (4)**

As any function, the (nonperiodic) complex exponential function \( e^{im\phi'} \) with \( \nu \in \mathbb{R} \) and domain \( [\phi_{0,0}, \phi_{0,0} + 2\pi] \) in Eq. (3) can be decomposed over a complete orthogonal basis. Choosing for this basis the periodic complex exponential function \( e^{im\phi'} (m \in \mathbb{Z}) \) leads then to the expansion

\[
e^{i\nu \phi'} = \sum_{m=-\infty}^{+\infty} c_m e^{im\phi'}.
\] (D1)

Multiplying both sides of this relation by \( e^{-im'\phi'} \) and integrating over one period of the function \( e^{im\phi'} \) yields

\[
\int_{\phi_{0,0}}^{\phi_{0,0} + 2\pi} e^{i\nu \phi'} e^{-im' \phi'} d\phi' = \int_{\phi_{0,0}}^{\phi_{0,0} + 2\pi} \sum_{m=-\infty}^{+\infty} c_m e^{-i(m'-m)\phi'} d\phi'.
\] (D2)

Inverting the order of the integral and the sum in the right-handside term and using the orthogonality property of the integer-order complex integral function gives then

\[
\int_{\phi_{0,0}}^{\phi_{0,0} + 2\pi} e^{i\nu \phi'} d\phi' = 2\pi c_m,
\] (D3)

which integrates into

\[
c_m = \frac{\sin \left(\frac{(m - \nu)\pi}{\nu}\right)}{\sin \left(\frac{(m - \nu)\pi}{\nu}\right)} e^{-i(m-\nu)(\phi_{0,0} + \pi)},
\] (D4)

where the dummy prime has been dropped.

Substituting Eq. (D4) into Eq. (D1) finally yields the explicit expansion

\[
e^{i\nu \phi'} = \sum_{m=-\infty}^{+\infty} \frac{\sin \left(\frac{(m - \nu)\pi}{\nu}\right)}{\sin \left(\frac{(m - \nu)\pi}{\nu}\right)} e^{-i(m-\nu)(\phi_{0,0} + \pi)} e^{im\phi'}.
\] (D5)

\[\text{The description of the nonperiodic function } e^{im\phi'} \text{ over the domain } \phi' \in [\phi_{0,0}, \phi_{0,0} + 2\pi] \text{ is equivalent to describing this nonperiodic function as a piecewise function in terms of } \phi_0 \in [0, 2\pi] \text{ where conditions depend on } \phi_{0,0} \text{.}\]
**SUPPLEMENTARY MATERIAL E. PHYSICAL INTERPRETATION OF THE PHASE PARAMETER $\phi_{G,0}$**

The physical role of the phase term $\phi_{G,0}$ in Eq. (2d) and Eq. (S6) is not trivial. This section clarifies this issue, using two distinct and complementary strategies: 1) analyzing the problem in the spatial-frequency spectral ($k$) domain, where it is simpler, and 2) inspecting a beam made of truncated constituent waves in a cross-sectional plane where these waves weakly overlap and therefore only locally (i.e., only with nearest neighbors, see Fig. 2) interfere, which might be considered as a spatial alternative to 1).

The spatial-frequency spectrum solution of the general solution in Eq. (S6) is (see Supp. Mat. B)

$$\tilde{U}_\nu(k_\rho, k_\phi, k_z) =
\begin{cases}
\frac{\delta(k_\rho - \alpha)}{k_\rho - \alpha} e^{i\nu(k_\rho + 2\pi - (\phi_{G,0} \pm \pi))} \delta(k_z - \beta) & \text{if } k_\rho < \phi_{G,0} \pm \pi, \\
\frac{\delta(k_\rho - \alpha)}{k_\rho - \alpha} e^{i\nu(k_\rho - (\phi_{G,0} \pm \pi))} \delta(k_z - \beta) & \text{if } k_\rho \geq \phi_{G,0} \pm \pi.
\end{cases}$$

(E1)

Equation (E1) describes a cone with different initial phases in the $k$-space. If $\nu = n \in \mathbb{Z}$, the term $e^{i\nu 2\pi} = 1$ in the top expression disappears so that this expression reduces to the second; as a result, the initial phase $\phi_{G,0} \pm \pi$ does not depend any more on $k_\rho$, and represents therefore an dummy initial phase that can be set to zero, which reduces the spectrum to

$$\tilde{U}_n(k_\rho, k_\phi, k_z) = \delta(k_\rho - \alpha) e^{i\nu n k_\rho} \delta(k_z - \beta).$$

If $n \in \mathbb{R} \setminus \mathbb{Z}$, the situation is obviously more complex, with a phase ($k_\phi$) discontinuity of $2\pi \nu$ appearing in Eq. (E1), specifically

$$\lim_{\varepsilon \to 0} \angle \tilde{U}_\nu(k_\phi = \phi_{G,0} \pm \pi) - \angle \tilde{U}_\nu(k_\phi = \phi_{G} \pm \pi - \varepsilon) = 2\pi \nu.$$

(E2)

Thus, the parameter $\phi_{G,0}$ corresponds then the position of the spectral phase discontinuity as $k_\phi = \phi_{G,0} + \pi$. Figure S2 illustrates these results.

Fig. S2. Effect of $\phi_{G,0}$ on the spectrum of a Bessel beam [Eq. (E1)] formed by plane constituent waves with $\nu = 1.5$ and $\delta = 25^\circ$. In the first two rows, we have introduced a non-zero width to circular section of the cone for visualization. The bottom row shows, for comparison, the amplitude of the direct field, obtained by inverse-Fourier transforming Eq. (E1) and represented at the apex of the cone ($z = 0$).

Moving now on to the second strategy, Fig. S3 plots the effect of $\phi_{G,0}$ on the direct fields associated with Bessel beam with circular-cylindrically truncated constituting waves. Here, the parameter $\phi_{G,0}$, has also an impact of the phase, with the neighboring waves canceling each other at the angle $\phi = \phi_{G,0} (z < 0)$.

Fig. S3. Complex amplitude and phase of the direct fields associated with a Bessel beam with Gaussian circular-cylindrically truncated constituting in a plane of overlap only the nearest neighbours ($z_G < z < 0$) and in the overlapping plane ($z = 0$) (same parameters as in Fig. S2).

Figures S4 and S5 show the effect of the phase discontinuity, as the Figures S2 and S3 respectively, for a given $\phi_{G,0} = 0$ and different global order $\nu$. Figure S5 shows how the amplitude (of neighboring waves) opening is affected by the phase discontinuity $2\pi \nu$.

Fig. S4. Same parameters as in Fig. S2 with different global order $\nu$. 
SUPPLEMENTARY MATERIAL F. CONSTRUCTION OF THE ELECTROMAGNETIC VECTORIAL BESSEL WAVES

A. Maxwell Equations for a Plane Wave

The free-space Maxwell equations for a harmonic plane wave propagating in the direction \( \mathbf{k} \) read:

\[
\begin{align*}
\eta \mathbf{H}^{\text{PW}} &= \frac{1}{k} \left( (k_y E_y^{\text{PW}} - k_z E_z^{\text{PW}}) \mathbf{x} + (-k_x E_x^{\text{PW}} + k_z E_z^{\text{PW}}) \mathbf{y} + (k_x E_y^{\text{PW}} - k_y E_y^{\text{PW}}) \mathbf{z} \right), \\
\mathbf{E}^{\text{PW}} &= \frac{\eta}{k} \left( (-k_y H_y^{\text{PW}} + k_z H_z^{\text{PW}}) \mathbf{x} + (k_x H_x^{\text{PW}} - k_y H_y^{\text{PW}}) \mathbf{y} + (-k_x H_x^{\text{PW}} + k_y H_y^{\text{PW}}) \mathbf{z} \right), \\
k_z E_x^{\text{PW}} + k_y E_y^{\text{PW}} + k_x E_z^{\text{PW}} &= 0, \\
k_x H_x^{\text{PW}} + k_y H_y^{\text{PW}} + k_z H_z^{\text{PW}} &= 0,
\end{align*}
\]

where \( \eta = \sqrt{\mu_0/\varepsilon_0} \) is the impedance of free-space and \( k = \omega \sqrt{\mu_0/\varepsilon_0} \) is the wavenumber of free-space, with \( \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m} \) and \( \mu_0 = 1.257 \times 10^{-6} \text{ N/A}^2 \).

B. Construction of the LSE/LSM Modes

Let us start with the LSE\( _y \) (\( E_y = 0 \)) modes. In this case, we have:

\[
E_y^{\text{PW}}(\phi_G) = 0,
\]

from which we find, upon solving Eq. (F1c) for \( E_z^{\text{PW}} \),

\[
E_z^{\text{PW}}(\phi_G) = -\frac{k_x(\phi_G)}{k_z(\phi_G)} E_x^{\text{PW}}(\phi_G), \tag{F2a}
\]

which becomes, upon setting \( E_z^{\text{PW}}(\phi_G) = \psi_x(\phi_G) \) (transverse field in the scalar case) in Eq. (2a) and using Eq. (2b),

\[
E_x^{\text{PW}}(\phi_G) = \tan(\delta) \cos(\phi_G) \psi_x(\phi_G). \tag{F2b}
\]

The corresponding components of \( \mathbf{H}^{\text{PW}} \) are found by inserting Eqs. (F1) and (F2b) into Eq. (F1a), and successively solving for the different components, which results into:

\[
\eta H_z^{\text{PW}}(\phi_G) = \left(-\sin(\delta) \tan(\delta) \sin(\phi_G) \cos(\phi_G)\right) \psi_x(\phi_G), \tag{F3a}
\]

\[
\eta H_y^{\text{PW}}(\phi_G) = \left(\sin(\delta) \tan(\delta) \cos^2(\phi_G) + \cos(\delta)\right) \psi_x(\phi_G), \tag{F3b}
\]

\[
\eta H_z^{\text{PW}}(\phi_G) = \sin(\phi_G) \psi_x(\phi_G). \tag{F3c}
\]

The LSE\( _y \) Bessel beam corresponding to the integral form of Eq. (1) is then formed, upon grouping the previous results, by the field components

\[
\begin{align*}
E_x &= \int_0^{2\pi} d\phi_G \psi_x(\phi_G), \\
E_y &= 0, \\
E_z &= \int_0^{2\pi} d\phi_G \psi_x(\phi_G) \tan(\delta) \sin(\phi_G), \\
\eta H_x &= \int_0^{2\pi} d\phi_G \psi_x(\phi_G) \left(-\sin(\delta) \tan(\delta) \sin(\phi_G) \cos(\phi_G)\right), \\
\eta H_y &= \int_0^{2\pi} d\phi_G \psi_x(\phi_G) \left(\sin(\delta) \tan(\delta) \cos^2(\phi_G) + \cos(\delta)\right), \\
\eta H_z &= \int_0^{2\pi} d\phi_G \psi_x(\phi_G) \sin(\phi_G).
\end{align*}
\]

The LSE\( _x \) Bessel beam field components are similarly found as:

\[
\begin{align*}
E_x &= 0, \tag{F5a} \\
E_y &= \int_0^{2\pi} d\phi_G \psi_x(\phi_G), \tag{F5b} \\
E_z &= \int_0^{2\pi} d\phi_G \psi_x(\phi_G) \tan(\delta) \sin(\phi_G), \tag{F5c} \\
\eta H_x &= \int_0^{2\pi} d\phi_G \psi_x(\phi_G) \left(-\sin(\delta) \tan(\delta) \sin^2(\phi_G) - \cos(\delta)\right), \tag{F5d} \\
\eta H_y &= \int_0^{2\pi} d\phi_G \psi_x(\phi_G) \left(\sin(\delta) \tan(\delta) \sin(\phi_G) \cos(\phi_G)\right), \tag{F5e} \\
\eta H_z &= \int_0^{2\pi} d\phi_G \psi_x(\phi_G) \left(-\sin(\delta) \cos(\phi_G)\right). \tag{F5f}
\end{align*}
\]
LSM\textsubscript{y} modes, which are respectively
\begin{align}
E_x &= \int_{0}^{2\pi} d\phi_G \, \psi_x(\phi_G) (\sin(\delta) \tan(\delta) \sin^2(\phi_G) + \cos(\delta)), \\
E_y &= \int_{0}^{2\pi} d\phi_G \, \psi_y(\phi_G) (- \sin(\delta) \tan(\delta) \sin(\phi_G) \cos(\phi_G)), \\
E_z &= \int_{0}^{2\pi} d\phi_G \, \psi_z(\phi_G) \sin(\delta) \cos(\phi_G),
\end{align}
\begin{align}
\eta H_x &= 0, \\
\eta H_y &= \int_{0}^{2\pi} d\phi_G \, \psi_y(\phi_G), \\
\eta H_z &= \int_{0}^{2\pi} d\phi_G \, \psi_z(\phi_G) \tan(\delta) \sin(\phi_G).
\end{align}
and
\begin{align}
E_x &= \int_{0}^{2\pi} d\phi_G \, \psi_x(\phi_G) \sin(\delta) \tan(\delta) \sin(\phi_G) \cos(\phi_G) \\
E_y &= \int_{0}^{2\pi} d\phi_G \, \psi_y(\phi_G) (- \sin(\delta) \tan(\delta) \cos^2(\phi_G) - \cos(\delta)) \\
E_z &= \int_{0}^{2\pi} d\phi_G \, \psi_z(\phi_G) (- \sin(\delta) \sin(\phi_G))
\end{align}
\begin{align}
\eta H_x &= \int_{0}^{2\pi} d\phi_G \, \psi_x(\phi_G), \\
\eta H_y &= 0, \\
\eta H_z &= \int_{0}^{2\pi} d\phi_G \, \psi_z(\phi_G) \tan(\delta) \cos(\phi_G).
\end{align}

\section{Construction of the TM/TE Modes}

Let us start with the TM\textsubscript{z} \((H_z = 0)\) modes. In this case, we have
\begin{equation}
H_{z}^{\text{PW}}(\phi_G) = 0.
\end{equation}
Additionally, we find from the \(z\)-component of Eq. (F1a) the following relation between the transverse components of the electric field:
\begin{align}
E_{x}^{\text{PW}}(\phi_G) &= \frac{k_y(\phi_G)}{k_y(\phi_G)} E_{y}^{\text{PW}}(\phi_G) = \cot(\delta) E_{y}^{\text{PW}}(\phi_G), \\
or
E_{y}^{\text{PW}}(\phi_G) &= \frac{k_y(\phi_G)}{k_x(\phi_G)} E_{x}^{\text{PW}}(\phi_G) = \tan(\delta) E_{x}^{\text{PW}}(\phi_G),
\end{align}
where Eq. (2b) has been used in the last equalities. We find then, upon successively solving Eq. (F1c) for \(E_{x}^{\text{PW}}\) and \(E_{y}^{\text{PW}}\), and respectively using Eq. (F2b) and Eq. (F2a),
\begin{align}
E_{x}^{\text{PW}}(\phi_G) &= - \frac{k_y(\phi_G) k_z(\phi_G)}{k_x(\phi_G) + k_y(\phi_G)^2} E_{z}^{\text{PW}}(\phi_G) \\
= \cot(\delta) \cos(\phi_G) E_{z}^{\text{PW}}(\phi_G), \\
or
E_{y}^{\text{PW}}(\phi_G) &= \frac{-k_y(\phi_G) k_z(\phi_G)}{k_x(\phi_G) + k_y(\phi_G)^2} E_{z}^{\text{PW}}(\phi_G) \\
= \cot(\delta) \sin(\phi_G) E_{z}^{\text{PW}}(\phi_G),
\end{align}
which becomes, upon setting \(E_{z}^{\text{PW}}(\phi_G) = \psi_x(\phi_G)\) (transverse field in the scalar case) in Eq. (2a) and using Eq. (2b),
\begin{equation}
E_{x}^{\text{PW}}(\phi_G) = \cot(\delta) \cos(\phi_G) \psi_x(\phi_G),
\end{equation}
or
\begin{equation}
E_{y}^{\text{PW}}(\phi_G) = \cot(\delta) \sin(\phi_G) \psi_x(\phi_G).
\end{equation}
The corresponding components of \(H^{\text{PW}}\) are found by inserting Eqs. (F3c) and (F3d) into Eq. (F1a), and successively solving for the different components, which results into
\begin{align}
\eta H_{x}^{\text{PW}}(\phi_G) &= (- \csc(\delta) \sin(\phi_G)) \psi_y(\phi_G), \\
\eta H_{y}^{\text{PW}}(\phi_G) &= \csc(\delta) \cos(\phi_G) \psi_y(\phi_G), \\
\eta H_{z}^{\text{PW}}(\phi_G) &= 0.
\end{align}
The TM\textsubscript{z} Bessel beam corresponding to the integral form of Eq. (1) is then formed, upon grouping the previous results, by the field components
\begin{align}
E_x &= \int_{0}^{2\pi} d\phi_G \, \psi_x(\phi_G) \cot(\delta) \cos(\phi_G), \\
E_y &= \int_{0}^{2\pi} d\phi_G \, \psi_y(\phi_G) \cot(\delta) \sin(\phi_G), \\
E_z &= \int_{0}^{2\pi} d\phi_G \, \psi_z(\phi_G),
\end{align}
\begin{align}
\eta H_x &= \int_{0}^{2\pi} d\phi_G \, \psi_x(\phi_G) (- \csc(\delta) \sin(\phi_G)), \\
\eta H_y &= \int_{0}^{2\pi} d\phi_G \, \psi_y(\phi_G) \csc(\delta) \cos(\phi_G), \\
\eta H_z &= 0.
\end{align}
The TE\textsubscript{z} Bessel beam integral field components are similarly found as
\begin{align}
E_x &= \int_{0}^{2\pi} d\phi_G \, \psi_x(\phi_G) \csc(\delta) \sin(\phi_G) \\
E_y &= \int_{0}^{2\pi} d\phi_G \, \psi_y(\phi_G) (- \csc(\delta) \cos(\phi_G)) \\
E_z &= 0 \\
\eta H_x &= \int_{0}^{2\pi} d\phi_G \, \psi_x(\phi_G) \cot(\delta) \cos(\phi_G) \\
\eta H_y &= \int_{0}^{2\pi} d\phi_G \, \psi_y(\phi_G) \cot(\delta) \sin(\phi_G) \\
\eta H_z &= \int_{0}^{2\pi} d\phi_G \, \psi_z(\phi_G).
**Supplementary Material G. Effect of the Axicon Angle on the Transverse Pattern**

Figures S6 and S7 show the effect the axicon angle (δ) on the amplitude of the transverse fields.

Fig. S6. Cross-sectional view of the complex fields amplitude of the \( \text{LSE}_{y,0} \) Bessel beam, computed by Eq. (1) with (2a) with 500 constituent plane waves and for different δ angles, with \( \nu = 0, \omega(\xi) = 1, \phi_{G,0} = 0 \), and over the cross-sectional area of \( 4\lambda \times 4\lambda \) (\( \lambda = 2\pi c/\omega \)) co-centered with the axis of the beam.

Fig. S7. Same as in Fig. S6 but for the mode \( \text{TM}_{z,0} \).

Essentially, increasing δ increases the transverse spatial frequency \( \alpha \) (see Eqs. (2b) and (6a)) of the Bessel waveform of each of the field components and hence compresses the Bessel ring pattern toward the beam axis.

Figure S8 shows the effect of δ on the norm of the time-averaged Poynting vector and time-averaged energy density \( \langle \mu \rangle \): Increasing δ breaks the azimuthal symmetry of the Poynting vector and the energy density of the \( \text{LSE}_{y} \) mode, but not that of the \( \text{TM}_{z} \) mode, as could be expected from the previous results.