Dynamical Systems Method (DSM) for solving nonlinear operator equations in Banach spaces

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Abstract

Let \( F(u) = h \) be an operator equation in a Banach space \( X \), \( \|F'(u) - F'(v)\| \leq \omega(\|u - v\|) \), where \( \omega \in C([0, \infty)) \), \( \omega(0) = 0 \), \( \omega(r) > 0 \) if \( r > 0 \), \( \omega(r) \) is strictly growing on \([0, \infty)\). Denote \( A(u) := F'(u) \), where \( F'(u) \) is the Fréchet derivative of \( F \), and \( A_a := A + aI \). Assume that (*) \( \|A_a^{-1}(u)\| \leq \frac{c_0}{|a|^b} \), \( |a| > 0 \), \( b > 0 \), \( a \in L \). Here \( a \) may be a complex number, and \( L \) is a smooth path on the complex \( a \)-plane, joining the origin and some point on the complex \( a \)-plane, \( 0 < |a| < \epsilon_0 \), where \( \epsilon_0 > 0 \) is a small fixed number, such that for any \( a \in L \) estimate (*) holds. It is proved that the DSM (Dynamical Systems Method)

\[
\dot{u}(t) = -A_{a(t)}^{-1}(u(t))[F(u(t)) + a(t)u(t) - f], \quad u(0) = u_0, \quad \dot{u} = \frac{du}{dt},
\]

converges to \( y \) as \( t \to +\infty \), where \( a(t) \in L \), \( F(y) = f \), \( \tau(t) := |a(t)| \), and \( \tau(t) = c_4(t + c_2)^{-c_3} \), where \( c_j > 0 \) are some suitably chosen constants, \( j = 2, 3, 4 \). Existence of a solution \( y \) to the equation \( F(u) = f \) is assumed. It is also assumed that the equation \( F(w_a) + aw_a - f = 0 \) is uniquely solvable for any \( f \in X \), \( a \in L \), and \( \lim_{|a| \to 0, a \in L} \|w_a - y\| = 0 \).

MSC 2000, 47J05, 47J06, 47J35
Key words: Nonlinear operator equations; DSM (Dynamical Systems Method); Banach spaces

1 Introduction

Consider an operator equation

\[
F(u) = f,
\]

where \( F \) is an operator in a Banach space \( X \). By \( X^* \) denote the dual space of bounded linear functionals on \( X \).

Assume that \( F \) is continuously Fréchet differentiable, \( F'(u) := A(u) \), and

\[
\|A(u) - A(v)\| \leq \omega(\|u - v\|), \quad \omega(r) = c_0 r^\kappa, \quad \kappa \in (0, 1],
\]

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1 Introduction

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\]
that the norm of the resolvent $A^{-1}$ introduced in [10], Chapter 8. This spectral assumption says, that the set $A$ is satisfied for the class of linear operators $A|L$ on a complex plane $\mathbb{C}$, consisting of regular points of the operator $A(u)$, such that the norm of the resolvent $A^{-1}(u)$ grows, as $a \to 0$, not faster than a power $|a|^{-b}$. Thus, assumption (3) is a weak assumption. For example, assumption (3) is satisfied for the class of linear operators $A$, satisfying the spectral assumption, introduced in [10], Chapter 8. This spectral assumption says, that the set $\{a : |\arg a - \pi| \leq \phi_0, 0 < |a| < \epsilon_0\}$ consists of the regular points of the operator $A$. This assumption implies the estimate $||A^{-1}|| \leq \frac{c_1}{a}$, $0 < a < \epsilon_0$, similar to estimate (3).

Assume additionally that the equation

$$F(w_a) + aw_a - f = 0, \quad a \in L,$$

is uniquely solvable for any $f \in X$, and

$$\lim_{a \to 0, a \in L} ||w_a - y|| = 0, \quad F(y) = f.$$

All the above assumptions are standing and are not repeated in the formulation of Theorem 2.7 which is our main result.

These assumptions are satisfied, e.g., if $F$ is a monotone operator in a Hilbert space $H$ and $L$ is a segment $[0, \epsilon_0]$, in which case $c_1 = 1$ and $b = 1$ (see [10]).

Every equation (1) with a linear, closed, densely defined in a Hilbert space $H$ operator $F = A$ can be reduced to an equation with a monotone operator $A^*A$, where $A^*$ is the adjoint to $A$. The operator $T := A^*A$ is selfadjoint and densely defined in $H$. If $f \in D(A^*)$, where $D(A^*)$ is the domain of $A^*$, then the equation $Au = f$ is equivalent to $Tu = A^*f$, provided that $Au = f$ has a solution, i.e., $f \in R(A)$, where $R(A)$ is the range of $A$. Recall that $D(A^*)$ is dense in $H$ if $A$ is closed and densely defined in $H$. If $f \in R(A)$ but $f \notin D(A^*)$, then equation $Tu = A^*f$ still makes sense and its normal solution $y$, i.e., the solution with minimal norm, can be defined as

$$y = \lim_{a \to 0} T_a^{-1}A^*f.$$

One proves that $Ay = f$, and $y \perp N(A)$, where $N(A)$ is the null-space of $A$. These results are proved in [11], [13], [14].

Our aim is to prove convergence of the DSM (Dynamical Systems Method) for solving equation (1), which is of the form:

$$\dot{u} = -A_{a(t)}^{-1}[F(u(t)) + a(t)u(t) - f], \quad u(0) = u_0,$$

$c_0 > 0$ is a constant. The function $\omega(r)$, in general, is a continuous strictly growing function, $\omega(0) = 0$.

Assume that

$$\|A_a^{-1}(u)\| \leq \frac{c_1}{|a|}; \quad |a| > 0, \quad A_a := A + aI, \quad c_1 = \text{const} > 0, \quad b > 0.$$ 

Here $a$ may be a complex number, $|a| > 0$, and there exists a smooth path $L$ on the complex plane $\mathbb{C}$, such that for any $a \in L$, $|a| < \epsilon_0$, where $\epsilon_0 > 0$ is a small fixed number independent of $u$, estimate (3) holds, and $L$ joins the origin and some point $a_0, 0 < |a_0| < \epsilon_0$. Assumption (3) holds if there is a smooth path $L$ on a complex $a$-plane, consisting of regular points of the operator $A(u)$, such that the norm of the resolvent $A_a^{-1}(u)$ grows, as $a \to 0$, not faster than a power $|a|^{-b}$. Thus, assumption (3) is a weak assumption. For example, assumption (3) is satisfied for the class of linear operators $A$, satisfying the spectral assumption, introduced in [10], Chapter 8. This spectral assumption says, that the set $\{a : |\arg a - \pi| \leq \phi_0, 0 < |a| < \epsilon_0\}$ consists of the regular points of the operator $A$. This assumption implies the estimate $||A_a^{-1}|| \leq \frac{c_1}{a}$, $0 < a < \epsilon_0$, similar to estimate (3).
where \( u_0 \in X \) is an initial element, \( a(t) \in C^1[0,\infty) \), \( a(t) \in L \). Our main result is formulated in Theorem 2.1 in Section 2.

The DSM for solving operator equations has been developed in the monograph [10] and in a series of papers [11]-[27]. It was used as an efficient computational tool in [5]-[9]. One of the earliest papers on the continuous analog of Newton’s method for solving well-posed nonlinear operator equations was [3].

The novel points in the current paper include the larger class of the operator equations than earlier considered, and the weakened assumptions on the smoothness of the nonlinear operator \( F \). While in [10] it was often assumed that \( F''(u) \) is locally bounded, in the current paper a weaker assumption (2) is used.

Our proof of Theorem 2.1 uses the following result from [4].

**Lemma 1.** Assume that \( g(t) \geq 0 \) is continuously differentiable on any interval \([0,T]\) on which it is defined and satisfies the following inequality:

\[
\dot{g}(t) \leq -\gamma(t)g(t) + \alpha(t)g^p(t) + \beta(t), \quad t \in [0,T),
\]

where \( p > 1 \) is a constant, \( \alpha(t) > 0 \), \( \gamma(t) \) and \( \beta(t) \) are three continuous on \([0,\infty)\) functions. Suppose that there exists a \( \mu(t) > 0 \), \( \mu(t) \in C^1[0,\infty) \), such that

\[
\alpha(t)\mu^{-p}(t) + \beta(t) \leq \mu^{-1}(t)[\gamma(t) - \dot{\mu}(t)\mu^{-1}(t)], \quad t \geq 0,
\]

and

\[
\mu(0)g(0) < 1.
\]

Then \( T = \infty \), i.e., \( g \) exists on \([0,\infty)\), and

\[
0 \leq g(t) \leq \mu^{-1}(t), \quad t \geq 0.
\]

This lemma generalizes a similar result for \( p = 2 \) proved in [10].

In Section 2 a method is given for a proof of the following conclusions: there exists a unique solution \( u(t) \) to problem (7) for all \( t \geq 0 \), there exists \( u(\infty) := \lim_{t \to \infty} u(t) \), and \( F(u(\infty)) = f \):

\[
\exists u(t) \quad \forall t \geq 0; \quad \exists u(\infty); \quad F(u(\infty)) = f.
\]

The assumptions on \( u_0 \) and \( a(t) \) under which (12) holds for the solution to (7) are formulated in Theorem 2.1 in Section 2. Theorem Theorem 2.1 in Section 2 is our main result. Roughly speaking, this result says that conclusions (12) hold for the solution to problem (7), provided that \( a(t) \) is suitably chosen.

## 2 Proofs

Let \( |a(t)| := r(t) > 0 \). If \( a(t) = a_1(t) + a_2(t) \), where \( a_1(t) = \text{Re}a(t), \quad a_2(t) = \text{Im}a(t) \), then

\[
|\dot{r}(t)| \leq |\dot{a}(t)|.
\]
Indeed,
\[ |\dot{r}(t)| = \frac{|a_1\dot{a}_1 + a_2\dot{a}_2|}{r(t)} \leq \frac{r(t)|\dot{a}(t)|}{r(t)}, \tag{14} \]
and (14) implies (13).

Let \( h \in X^* \) be arbitrary with \( \|h\| = 1 \), and
\[ g(t) := (z(t), h); \quad z(t) := u(t) - w_a(t), \tag{15} \]
where \( u(t) \) solves (7) and \( w_0(t) \) solves (4) with \( a = a(t) \). By the assumption, \( w_a(t) \) exists for every \( t \geq 0 \). The local existence of \( u(t) \), the solution to (7), is the conclusion of Lemma 2.

**Lemma 2.** If (4)-(5) hold, then \( u(t) \), the solution to (7), exists locally.

**Proof.** Differentiate equation (5) with respect to \( t \). The result is
\[ A_{a(t)}(w_a(t))\dot{w}_a(t) = -\dot{a}(t)w_a(t), \tag{16} \]
or
\[ \dot{w}_a(t) = -\dot{a}(t)A_{a(t)}^{-1}(w_a(t))w_a(t). \tag{17} \]
Denote
\[ \psi(t) := F(u(t)) + a(t)u(t) - f. \tag{18} \]
For any \( \psi \in H \) equation (18) is uniquely solvable for \( u(t) \) by the inverse function theorem, because, by our assumption (3), the Fréchet derivative \( F'(u(t)) + a(t)I \) is boundedly invertible, and (2) implies that the solution \( u(t) \) to (18) is continuously differentiable with respect to \( t \) if \( \psi(t) \) is. One may solve (18) for \( u \) and write \( u = G(\psi) \), where the map \( G \) is continuously Fréchet differentiable because \( F \) is.

Differentiate (18) and get
\[ \dot{\psi}(t) = A_{a(t)}(u(t))\dot{u}(t) + \dot{a}(t)u. \tag{19} \]
If one wants the solution to (18) to be a solution to (7), then one has to require that
\[ A_{a(t)}(u(t))\dot{u} = -\psi(t). \tag{20} \]
If (20) holds, then (19) can be written as
\[ \dot{\psi}(t) = -\dot{\psi} + \dot{a}(t)G(\psi), \quad G(\psi) := u(t), \tag{21} \]
where \( G(\psi) \) is continuously Fréchet differentiable. Thus, equation (21) is equivalent to (7) at all \( t \geq 0 \) if
\[ \psi(0) = F(u_0) + a(0)u_0 - f. \tag{22} \]
Indeed, if \( u \) solves (7) then \( \psi \), defined in (18), solves (21)-(22). Conversely, if \( \psi \) solves (21)-(22), then \( u(t) \), defined as the unique solution to (18), solves (7). Since the right-hand side of (21) is Fréchet differentiable, it satisfies a local Lipschitz condition. Thus, problem (21)-(22) is locally, solvable. Therefore, problem (7) is locally solvable.
Lemma 2 is proved. \( \square \)
To prove that the solution \( u(t) \) to (7) exists globally, it is sufficient to prove the following estimate
\[
\sup_{t \geq 0} \| u(t) \| < \infty. \tag{23}
\]

**Lemma 3.** Estimate (23) holds.

**Proof.** Denote
\[
z(t) := u(t) - w(t), \tag{24}
\]
where \( u(t) \) solves (7) and \( w(t) \) solves (4) with \( a = a(t) \). If one proves that
\[
\lim_{t \to \infty} \| z(t) \| = 0, \tag{25}
\]
then (23) follows from (25) and (5):
\[
\sup_{t \geq 0} \| u(t) \| \leq \sup_{t \geq 0} \| z(t) \| + \sup_{t \geq 0} \| w(t) \| < \infty. \tag{26}
\]

To prove (25) we use Lemma 1.

Let
\[
g(t) := \| z(t) \|. \tag{27}
\]
Rewrite (7) as
\[
\dot{z} = -\dot{w} - A_{u(t)}^{-1}(u(t))[F(u(t)) - F(w(t)) + a(t)z(t)]. \tag{28}
\]
Note that:
\[
\sup_{h \in X^*, \|h\| = 1} (\langle \dot{w}(t), h \rangle) = \| \dot{w}(t) \|. \tag{29}
\]

**Lemma 4.** If the norm \( \| w(t) \| \) in \( X \) is differentiable, then
\[
\| w(t) \| \leq \| \dot{w}(t) \|. \tag{30}
\]

**Proof.** The triangle inequality implies:
\[
\frac{\| w(t + s) \| - \| w(t) \|}{s} \leq \frac{\| w(t + s) - w(t) \|}{s}, \quad s > 0. \tag{31}
\]
Passing to the limit \( s \searrow 0 \), one gets (30).

Lemma 4 is proved.

The norm is differentiable if \( X \) is strictly convex (see, e.g. [1]). A Banach space \( X \) is called strictly convex if \( \| u + v \| < 2 \) for any \( u \neq v \in X \) such that \( \| u \| = \| v \| = 1 \). A Banach space \( X \) is called uniformly convex if for any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that for all \( u, v \in B(0,1) \) with \( \| u - v \| = \epsilon \) one has \( \| u + v \| \leq 2(1 - \delta) \). Here \( B(0,1) \) is the closed ball in \( X \), centered at the origin and of radius one.
Various necessary and sufficient conditions for the Fréchet differentiability of the norm in Banach spaces are known in the literature (see [2], starting with Shmulian’s paper of 1940, see [28].

Hilbert spaces, $L^p(D)$ and $\ell^p$-spaces, $p \in (1, \infty)$, and Sobolev spaces $W^{l,p}(D)$, $p \in (1, \infty)$, $D \subset \mathbb{R}^n$ is a bounded domain, have Fréchet differentiable norms. These spaces are uniformly convex and they have the $E$–property, i.e., if $u_n \rightharpoonup u$ and $||u_n|| \to ||u||$ as $n \to \infty$, then $\lim_{n \to \infty} ||u_n - u|| = 0$.

From (17), (29) and (3) one gets

$$\|\dot{w}\| \leq c_1|\dot{a}(t)|r^{-b}(t)\|w(t)\|, \quad r(t) = |a(t)|,$$

where $w(t) = w_a(t)$. Since $\lim_{t \to \infty} |a(t)| = 0$, (32) and (5) imply

$$\|\dot{w}\| \leq c_2|\dot{a}(t)|r^{-b}(t), \quad c_2 = \text{const} > 0,$$

because (5) implies the following estimate:

$$c_1\|w(t)\| \leq c_2, \quad t \geq 0. \quad (34)$$

Inequality (33) implies that inequality (38) holds if

$$\|\dot{w}\| \leq c_2|\dot{r}(t)|r^{-b}(t), \quad t \geq 0. \quad (35)$$

Note that

$$F(u) - F(w) = \int_0^1 F'(w + sz)dsz = A(u)z + \int_0^1 [A(w + sz) - A(u)]dsz. \quad (36)$$

Apply $h$ to (28), take $\sup_{h \in X^*, \|h\| = 1}$, and use Lemma 4, relation (36), estimate (3), and inequality (2), to get:

$$\dot{g}(t) \leq \|\dot{z}(t)\| \leq c_2|\dot{r}(t)|r^{-b}(t) + c_3r^{-b}(t)g^p - g, \quad (37)$$

where $g(t)$ is defined in (27),

$$p = 1 + \kappa, \quad c_3 := c_0c_1. \quad (38)$$

Inequality (37) is of the form (8) with

$$\gamma(t) = 1, \quad \alpha(t) = c_2r^{-b}(t), \quad \beta(t) = c_2|\dot{r}(t)|r^{-b}(t). \quad (39)$$

Choose

$$\mu(t) = \lambda r^{-k}(t), \quad \lambda = \text{const} > 0, \quad k = \text{const} > 0. \quad (40)$$

Then

$$\dot{\mu}\mu^{-1} = -kr^{-1}. \quad (41)$$

Let us assume that

$$r(t) \searrow 0, \quad \dot{r} < 0, \quad |\dot{r}| \searrow 0. \quad (42)$$
Condition (10) implies
\[ g(0) \frac{\lambda}{r^k(0)} < 1, \quad (43) \]
and inequality (49) holds if
\[ \frac{c_3 r^{-b}(t) r^k}{\lambda} + c_2 |\dot{r}(t)| r^{-b}(t) \leq \frac{r^k(t)}{\lambda} (1 - k |\dot{r}(t)| r^{-1}(t)), \quad t \geq 0. \quad (44) \]
Inequality (44) can be written as
\[ \frac{c_3 r^{k(p-1)-b}(t)}{\lambda^{p-1}} + \frac{c_2 \lambda |\dot{r}(t)|}{r^{k+b}(t)} + k |\dot{r}(t)| \leq 1. \quad (45) \]

Let us choose \( k \) so that
\[ k(p - 1) - b = 1, \]
that is,
\[ k = \frac{b + 1}{p - 1}. \quad (46) \]
Choose \( \lambda \) as follows:
\[ \lambda = \frac{r^k(0)}{2g(0)}. \quad (47) \]
Then inequality (43) holds, and inequality (45) can be written as:
\[ c_3 \frac{r(t)[2g(0)]^{p-1}}{[r^k(0)]^{p-1}} + c_2 \frac{r^k(0)}{2g(0)} \frac{|\dot{r}(t)|}{r^{k+b}(t)} + k |\dot{r}(t)| \leq 1, \quad t \geq 0. \quad (48) \]
Note that (46) implies:
\[ k + b = kp - 1. \quad (49) \]
Choose \( r(t) \) so that relations (42) hold and
\[ k |\dot{r}(t)| \leq \frac{1}{2}, \quad t \geq 0. \quad (50) \]
Then inequality (48) holds if
\[ c_2 \frac{[2g(0)]^{p-1}}{r^b+1(0)} + c_2 \frac{r^k(0)}{2g(0)} \frac{|\dot{r}(t)|}{r^{kp-1}} \leq \frac{1}{2}, \quad t \geq 0. \quad (51) \]
Denote
\[ c_2 \frac{r^k(0)}{2g(0)} = c_2 \lambda := c_4. \quad (52) \]
Let
\[ c_4 \frac{|\dot{r}(t)|}{r^{kp-1}} = \frac{1}{4}, \quad t \geq 0, \quad (53) \]
and $kp > 2$. Then equation (53) implies

$$r(t) = \left[ t + \frac{4c_4}{kp - 2} \right]^{-\frac{1}{p-2}} \left( \frac{kp - 2}{4c_4} \right)^{-\frac{1}{p-2}}.$$  \hspace{1cm} (54)

This $r(t)$ satisfies conditions (42), and equation (53) implies:

$$k \frac{\dot{r}(t)}{r(t)} = kr_{kp-2}(t), \quad t \geq 0.$$  \hspace{1cm} (55)

Recall that $r(t)$ decays monotonically. Therefore, inequality (50) holds if

$$kr_{kp-2}(0) \leq \frac{1}{2}.$$  \hspace{1cm} (56)

Inequality (56) holds if

$$k \frac{r_{kp-2}(0)}{2} \leq \frac{2g(0)}{c_2} = \frac{kg(0)}{c_2} r^{(p-1)-2}(0) \leq 1.$$  \hspace{1cm} (57)

Note that (46) implies:

$$k(p-1) - 2 = b - 1.$$  \hspace{1cm} (58)

Condition (57) holds if $g(0)$ is sufficiently small or $r^{b-1}(0)$ is sufficiently large:

$$g(0) \leq \frac{c_2}{k} r^{b-1}(0).$$  \hspace{1cm} (59)

If $b > 1$, then condition (59) holds for any fixed $g(0)$ if $r(0)$ is sufficiently large. If $b = 1$, then (59) holds if $g(0) \leq \frac{c_2}{k}$. If $b \in (0, 1)$ then (59) holds either if $g(0)$ is sufficiently small or $r(0)$ is sufficiently small.

Consequently, if (54) and (59) hold, then (53) holds. Therefore, (51) holds if

$$c_3 \frac{[2g(0)]^{p-1}}{r^{b+1}(0)} \leq \frac{1}{4}.$$  \hspace{1cm} (60)

It follows from (59) that (60) holds if

$$c_3 2^{p-1} \left( \frac{c_2}{k} \right)^{p-1} \frac{1}{r^{p+2b-2}(0)} \leq \frac{1}{4}.$$  \hspace{1cm} (61)

If $b \in (0, 1]$ then

$$p - pb + 2b > 0.$$  \hspace{1cm} (62)

Thus, (61) always holds if $r(0)$ is sufficiently large, specifically, if

$$r(0) \geq \left[ 4c_3 (2c_2 k^{-1})^{p-1} \right]^\frac{1}{p-1-b+2b}.$$  \hspace{1cm} (63)

We have proved the following theorem.
Theorem 2.1. If \( r(t) = |a(t)| \) is defined in (54), and if (59) and (63) hold, then

\[
\|z(t)\| < r^k(t)\lambda^{-1}, \quad \lim_{t \to \infty} \|z(t)\| = 0.
\]  

Thus, problem (7) has a unique global solution \( u(t) \) and

\[
\lim_{t \to \infty} \|u(t) - y\| = 0,
\]

where

\[
F(y) = f.
\]
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