OPTIMAL APPROXIMATION RATES FOR DEEP RELU NEURAL NETWORKS ON SOBOLEV SPACES

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January 9, 2023

ABSTRACT

Let \( \Omega = [0, 1]^d \) be the unit cube in \( \mathbb{R}^d \). We study the problem of how efficiently, in terms of the number of parameters, deep neural networks with the ReLU activation function can approximate functions in the Sobolev space \( W^s(L_q(\Omega)) \) with error measured in \( L_p(\Omega) \). This problem is important when studying the application of neural networks in scientific computing and has previously been completely solved only in the case \( p = q = \infty \). Our contribution is to provide a complete solution for all \( 1 \leq p, q \leq \infty \) and \( s > 0 \), including asymptotically matching upper and lower bounds. The key technical tool is a novel bit-extraction technique which gives an optimal encoding of sparse vectors. This enables us to obtain sharp upper bounds in the non-linear regime where \( p > q \). We also provide a novel method for deriving \( L_p \)-approximation lower bounds based upon VC-dimension when \( p < \infty \). Our results show that very deep ReLU networks significantly outperform classical methods of approximation in terms of the number of parameters, but that this comes at the cost of parameters which are not encodable.

1 Introduction

Deep neural networks have achieved remarkable success in both machine learning [27] and scientific computing [22, 35]. However, a precise theoretical understanding of why deep neural networks are so powerful has not been attained and is an active area of research. An important part of this theory is the study of the approximation properties of deep neural networks, i.e. to understand how efficiently a given class of functions can be approximated using deep neural networks. In this work, we solve this problem for the class of deep ReLU neural networks [32] when approximating functions lying in a Sobolev space with error measured in the \( L_p \)-norm. We remark that the ReLU activation functions is very widely used and is a major driver of many recent breakthroughs in deep learning [20, 27, 32].

Let us begin by giving a description of the Sobolev function class, which is widely used in the theory of solutions to partial differential equations (PDEs) [18]. Suppose that \( \Omega \subset \mathbb{R}^d \) is a bounded domain which satisfies an appropriate Sobolev extension theorem (see for instance [17, 18]), so that without loss of generality we may set \( \Omega = [0, 1]^d \) to be the unit cube in \( \mathbb{R}^d \) in the following. We denote by \( L_p(\Omega) \) the set of functions \( f \) for which the \( L_p \)-norm on \( \Omega \) is finite, i.e.

\[
\| f \|_{L_p(\Omega)} = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p} < \infty.
\] (1.1)

When \( p = \infty \), this becomes \( \| f \|_{L_\infty(\Omega)} = \text{ess sup}_{x \in \Omega} |f(x)| \). Suppose that \( s > 0 \) is a positive integer. Then \( f \in W^s(L_q(\Omega)) \) is in the Sobolev space (see [10], Chapter 2 for instance) with \( s \) derivatives in \( L_q \) if \( f \) has weak derivatives of order \( s \) and

\[
\| f \|_{W^s(L_q(\Omega))} := \| f \|_{L_q(\Omega)} + \sum_{|\alpha| = k} \| D^\alpha f \|_{L_q(\Omega)} < \infty.
\] (1.2)
When the weight matrix \( A \) we use the notation

\[
|f|_{W^{s}(L_{q}(\Omega))} := \left( \sum_{|\alpha|=k} \|D^{\alpha}f\|_{L_{q}(\Omega)}^{q} \right)^{1/q},
\]

and the standard modifications are made when \( q = \infty \).

When \( s > 0 \) is not an integer, we write \( s = k + \theta \) with \( k \geq 0 \) an integer and \( \theta \in (0, 1) \). The Sobolev semi-norm is defined by (see [13] Chapter 4 or [17] Chapter 1 for instance)

\[
|f|_{W^{s}(L_{q}(\Omega))} := \int_{\Omega \times \Omega} \frac{|D^{\alpha}f(x) - D^{\alpha}f(y)|^{q}}{|x - y|^{d+\theta q}} dxdy
\]

when \( 1 \leq q < \infty \) and

\[
|f|_{W^{s}(L_{\infty}(\Omega))} := \sup_{x,y \in \Omega} \sup_{|\alpha|=k} \frac{|D^{\alpha}f(x) - D^{\alpha}f(y)|}{|x - y|^{\theta}}.
\]

We define the Sobolev norm by

\[
\|f\|_{W^{s}(L_{q}(\Omega))} := |f|_{W^{s}(L_{q}(\Omega))} + |f|_{L_{q}(\Omega)}^{q},
\]

with the usual modification when \( q = \infty \). We remark that in the case of non-integral \( s \) these spaces are also called Sobolev-Slobodeckij spaces.

Sobolev spaces are widely used in PDE theory and a priori estimates for PDE solutions are often given in terms of Sobolev norms [18]. For applications of neural networks to scientific computing it is thus important to understand how efficiently neural networks can approximate functions from \( W^{s}(L_{q}(\Omega)) \).

One of the most important classes of neural networks are deep ReLU neural networks, which we define as follows. We use the notation \( A_{W,b} \) to denote the affine map with weight matrix \( W \) and offset, or bias, \( b \), i.e.

\[
A_{W,b}(x) = Wx + b.
\]

When the weight matrix \( W \) is an \( k \times n \) and the bias \( b \in \mathbb{R}^{k} \), the function \( A_{W,b} : \mathbb{R}^{n} \to \mathbb{R}^{k} \) maps \( \mathbb{R}^{n} \) to \( \mathbb{R}^{k} \). Let \( \sigma \) denote the ReLU activation function [32], specifically

\[
\sigma(x) = \begin{cases} 
0 & x < 0 \\
|_x & x \geq 0.
\end{cases}
\]

The ReLU activation function \( \sigma \) has become ubiquitous in deep learning in the last decade and is used in most state-of-the-art architectures. Since \( \sigma \) is continuous and piecewise linear, it also has the nice theoretical property that neural networks with ReLU activation function represent continuous piecewise linear functions. This property has been extensively studied in the computer science literature [3, 24, 37, 46] and has been connected with traditional linear finite element methods [25].

When \( x \in \mathbb{R}^{n} \), we write \( \sigma(x) \) to denote the application of the activation function \( \sigma \) to each component of \( x \) separately, i.e. \( \sigma(x)_{i} = \sigma(x_{i}) \). The set of deep ReLU neural networks with width \( W \) and depth \( L \) mapping \( \mathbb{R}^{d} \) to \( \mathbb{R}^{k} \) is given by

\[
Y^{W,L}(\mathbb{R}^{d}, \mathbb{R}^{k}) := \{ A_{W_{L},b_{L}} \circ \sigma \circ A_{W_{L-1},b_{L-1}} \circ \sigma \circ \cdots \sigma \circ A_{W_{1},b_{1}} \circ \sigma \circ A_{W_{0},b_{0}} \},
\]

where the weight matrices satisfy \( W_{L} \in \mathbb{R}^{k \times d} \), \( W_{0} \in \mathbb{R}^{W \times d} \), and \( W_{1}, \ldots, W_{L-1} \in \mathbb{R}^{W \times \mathbb{R}} \) and \( b_{0}, \ldots, b_{L-1} \in \mathbb{R}^{W} \) and \( b_{L} \in \mathbb{R}^{k} \). Notice that our definition of width does not include the input and output dimensions and only includes the intermediate layers. When the depth \( L = 0 \), i.e. when the network is an affine function, there are no intermediate layers and the width is undefined, in this case we write \( Y^{0}(\mathbb{R}^{d}, \mathbb{R}^{k}) \). We also use the notation

\[
Y^{W,L}(\mathbb{R}^{d}, \mathbb{R}) := Y^{W,L}(\mathbb{R}^{d}, \mathbb{R})
\]

to denote the set of ReLU deep neural networks with width \( W \), depth \( L \) which represent scalar functions. We note that our notation only allows neural networks with fixed width. We do this to avoid excessively cumbersome notation. We remark that the dimension of any hidden layer can naturally be expanded and thus any fully connected network can be made to have a fixed width.

The problem we study in this work is to determine optimal \( L_{p} \)-approximation rates

\[
\sup_{\|f\|_{W^{s}(L_{q}(\Omega))} \leq 1} \left( \inf_{f_{L} \in Y^{W,L}(\mathbb{R}^{d})} \|f - f_{L}\|_{L_{p}(\Omega)} \right)
\]
Then we have that
\[ W \equiv \text{determine the optimal rates in (1.11) under this condition. Specifically, we prove the following two Theorems. The necessary precondition that we have any approximation rate in (1.11) at all is for the Sobolev space } W^s(L^p(\Omega)) \text{ works } [12, 14, 39, 49]. \]
\[ \text{The super-convergence has a limit, however, and the rate (1.12) is shown to be optimal using } \]
\[ \text{the VC-dimension of deep ReLU neural networks } [6, 39, 49]. \]

The key technical difficulty in proving Theorem 1 is to deal with the case when the strict Sobolev embedding condition (1.13) holds with equality it is not a priori clear whether one has an embedding or not (this depends on the precise values of p, q, and d). Consequently this boundary case is much more subtle and we leave this for future work.

Theorem 1. Let \( \Omega = [0,1]^d \) be the unit cube in \( \mathbb{R}^d \) and let \( 0 < s < \infty \) and \( 1 \leq q \leq \infty \) and \( 1 \leq p \leq \infty \). Assume that \( \frac{1}{q} - \frac{1}{p} > \frac{s}{d} \), which guarantees that we have the compact Sobolev embedding
\[ W^s(L^q(\Omega)) \subset \subset L^p(\Omega). \]

Then we have that
\[ \inf_{f \in Y^{k,d}(\mathbb{R}^d)} \| f - f_L \|_{L^p(\Omega)} \leq C \| f \|_{W^s(L^q(\Omega))} L^{-2s/d}, \]
for a constant \( C := C(s,q,p,d) < \infty \).

Note that the width \( W = 25d + 31 \) of our networks are fixed as \( L \to \infty \), but scale linearly with the input dimension \( d \). We remark that a linear scaling with the input dimension is necessary since if \( d \geq W \), then the set of deep ReLU networks is known to not be dense in \( C(\Omega) \). The next Theorem gives a lower bound which shows that the rates in Theorem 1 are sharp in terms of the number of parameters.

Theorem 2. Let \( p, q \geq 1 \) and \( \Omega = [0,1]^d \) be the unit cube and \( W, L \geq 1 \) be integers. Then there exists an \( f \) with
\[ \| f \|_{W^s(L^q(\Omega))} \leq 1 \] such that
\[ \inf_{f \in Y^{k,d}(\mathbb{R}^d)} \| f - f_{W,L} \|_{L^p(\Omega)} \geq C(p) \min \{ W^2 L^2 \log(WL), W^3 L^2 \}^{-s/d}. \]

We remark that if the Sobolev embedding condition (1.13) strictly fails, then a simply scaling argument shows that \( W^s(L^q(\Omega)) \not\subset L^p(\Omega) \) and we cannot get any approximation rate. On the boundary where the Sobolev embedding condition (1.13) holds with equality it is not a priori clear whether one has an embedding or not (this depends on the precise values of s, p, and q). Consequently this boundary case is much more subtle and we leave this for future work.

The key technical difficulty in proving Theorem 1 is to deal with the case when \( p > q \), i.e. when the target function’s (weak) derivatives are in a weaker norm than the error. Classical methods of approximation using piecewise polynomials or wavelets can attain an approximation rate of \( CN^{-s/d} \) with \( N \) wavelet coefficients or piecewise polynomials with
\(N\) pieces. When \(p \leq q\) this rate can be achieved by linear methods, while for \(p > q\) nonlinear, i.e. adaptive, methods are required. For the precise details of this theory, see for instance [15, 16, 29].

Thus, in the linear regime where \(p \leq q\) we can use piecewise polynomials on a fixed uniform grid to approximate \(f\), while in the non-linear regime we need to use piecewise polynomials on an adaptive (i.e. depending upon \(f\)) non-uniform grid. This greatly complicates the bit-extraction technique used to obtain super-convergence, since the methods in [38, 40, 49] are only applicable to regular grids. The tool that we develop to overcome this difficulty is a novel bit-extraction technique, presented in Theorem 3 which optimally encodes sparse vectors using deep ReLU networks. Specifically, suppose that \(x \in \mathbb{Z}^N\) is an \(N\)-dimensional integer vector with \(\ell^1\)-norm bounded by

\[ ||x||_1 \leq M. \]  

(1.17)

In Theorem 5 we give (depending upon \(N\) and \(M\)) a deep ReLU neural network construction which optimally encodes \(x\).

We remark, however, that super-convergence comes at the cost of parameters which are non-encodable, i.e. cannot be encoded using a fixed number of bits, and this makes the numerical realization of this approximation rate inherently unstable. In order to better understand this, we introduce the notion of metric entropy first introduced by Kolmogorov. The metric entropy numbers \(\epsilon_N(A)\) of a set \(A \subset X\) in a Banach space \(X\) are given by (see for instance [29], Chapter 15)

\[ \epsilon_N(A)_H = \inf \{ \varepsilon > 0 : A \text{ is covered by } 2^N \text{ balls of radius } \varepsilon \}. \]  

(1.18)

An encodable approximation method consists of two maps, an encoding map \(E : A \rightarrow \{0, 1\}^N\) mapping the class \(A\) to a bit-string of length \(N\), and a decoding map \(D : \{0, 1\}^N \rightarrow X\) which maps each bit-string to an element of \(X\). This reflects the fact that any method which is implemented on a classical computer must ultimately encode all parameters using some number of bits. The metric entropy numbers give the minimal reconstruction error of the best possible encoding scheme.

Let \(U^{s}(L_q(\Omega)) := \{ f : \|f\|_{W^{s}(L_q(\Omega))}\} \) denote the unit ball of the Sobolev space \(W^{s}(L_q(\Omega))\). The metric entropy of this function class is given by

\[ \epsilon_N(U^{s}(L_q(\Omega)))_{L_p(\Omega)} \approx N^{-s/d} \]  

(1.19)

whenever the Sobolev embedding condition (1.13) is strictly satisfied. This is known as the Birman-Solomyak Theorem 8. So the approximation rate in Theorem 1 is significantly smaller than the metric entropy of the Sobolev class. This manifests itself in the fact that in the construction of the approximation rate in Theorem 1 the parameters of the neural network cannot be specified using a fixed number of bits, but rather need to be specified to higher and higher accuracy as the network grows [50], which is a direct consequence of the bit-extraction technique.

Concerning the lower bounds, the key difficulty in proving Theorem 3 is to extend the VC-dimension arguments used to obtain lower bounds when the error is measured in \(L_{\infty}\) to the case when the error is measured in the weaker norm \(L_p\) for \(p < \infty\). We do this by proving Theorem 4 which gives a general lower bound for \(L_p\)-approximation of Sobolev spaces by classes with bounded VC dimension. We have recently learned of a different approach to obtaining \(L_p\) lower bounds using VC-dimension [11], which is more generally applicable but introduces additional logarithmic factors in the lower bound.

Finally, we remark that there are other results in the literature which obtain approximation rates for deep ReLU networks on Sobolev spaces, but which do not achieve superconvergence, i.e. for which the approximation rate is only \(CN^{-s/d}\) (up to logarithmic factors), where \(N\) is the number of parameters [21, 48]. In addition, the approximation of other novel function classes (other than Sobolev spaces, which suffer the curse of dimensionality) by neural networks has been extensively studied recently, see for instance [4, 11, 12, 26, 34, 41–43].

The rest of the paper is organized as follows. First, in Section 2 we describe a variety of deep ReLU neural network constructions which will be used to prove Theorem 1. Many of these constructions are trivial or well-known, but we collect them for use in the following Sections. Then, in Section 3 we prove Theorem 3 which gives an optimal representation of sparse vectors using deep ReLU networks and will be key to proving superconvergence in the nonlinear regime \(p > q\). In Section 4 we give the proof of the upper bound Theorem 1. Finally, in Section 5 we prove the lower bound Theorem 2 and also prove the optimality of Theorem 3. We remark that throughout the paper, unless otherwise specified, \(C\) will represent a constant which may change from line to line, as is standard in analysis. The constant \(C\) may depend upon some parameters and this dependence will be made clear in the presentation.

## 2 Basic Neural Network Constructions

In this section, we collect some important deep ReLU neural network constructions which will be fundamental in our construction of approximations to Sobolev functions. Many of these constructions are well-known and will be used repeatedly to construct more complex networks later on, so we collect them here for the reader’s convenience.
2.1 Elementary Facts

We begin by making some fundamental observations and constructing some basic networks. Much of these are trivial consequences of the definitions, but we collect them here for future reference. We begin by noting that we can always widen a ReLU neural network without changing the function it represents.

**Lemma 1** (Widening Networks). If \( W_1 \leq W_2 \), then \( Y^{W_1,L}(\mathbb{R}^d,\mathbb{R}^k) \subset Y^{W_2,L}(\mathbb{R}^d,\mathbb{R}^k) \).

*Proof.* We simply add \( W_2 - W_1 \) rows and columns to the inner weight matrices \( W_i \) for \( i = 1,...,L-1 \), and \( W_2 - W_1 \) zeros to \( b_l \) to get
\[
W'_i := \begin{pmatrix} W_i & 0 \\ 0 & 0 \end{pmatrix}, \quad b'_l := \begin{pmatrix} b_l \\ 0 \end{pmatrix}.
\]

The first and last weight matrices are modified in an analogous manner. \( \square \)

Next we give an elementary Lemma on compositions which follows immediately from the definitions.

**Lemma 2** (Composing Networks). Suppose \( L_1, L_2 \geq 1 \) and that \( f \in Y^{W,L_1}(\mathbb{R}^d,\mathbb{R}^k) \) and \( g \in Y^{W,L_2}(\mathbb{R}^k,\mathbb{R}^l) \). Then the composition satisfies
\[
g(f(x)) \in Y^{W,L_1+L_2}(\mathbb{R}^d,\mathbb{R}^l).
\]

Further, if \( f \) is affine, i.e. \( f \in Y^0(\mathbb{R}^d,\mathbb{R}^k) \), then
\[
g(f(x)) \in Y^{W,L_2}(\mathbb{R}^d,\mathbb{R}^l).
\]

Finally, if instead \( g \) is affine, i.e. \( f \in Y^0(\mathbb{R}^k,\mathbb{R}^l) \) then
\[
g(f(x)) \in Y^{W,L_1}(\mathbb{R}^d,\mathbb{R}^l).
\]

*Proof.* These facts follow immediately from the definitions and the fact that compositions of affine maps are affine. \( \square \)

We remark that combining this with Lemma 1 we can apply Lemma 2 to networks with different widths and the width of the resulting network will be the maximum of the two widths. We will use this extension without comment in the following.

Next, we give a simple construction allowing us to apply two networks networks in parallel.

**Lemma 3** (Concatenating Networks). Let \( d = d_1 + d_2 \) and \( k = k_1 + k_2 \) with \( d_i, k_i \geq 1 \). Suppose that \( f_1 \in Y^{W_1,L}(\mathbb{R}^{d_1},\mathbb{R}^{k_1}) \) and \( f_2 \in Y^{W_2,L}(\mathbb{R}^{d_2},\mathbb{R}^{k_2}) \). We view \( \mathbb{R}^d = \mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2} \) and \( \mathbb{R}^k = \mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2} \). Then the function \( f = f_1 \oplus f_2 : \mathbb{R}^d \to \mathbb{R}^k \) defined by
\[
(f_1 \oplus f_2)(x_1 + x_2) = f_1(x_1) + f_2(x_2)
\]

satisfies \( f_1 \oplus f_2 \in Y^{W_1+W_2,L}(\mathbb{R}^d,\mathbb{R}^k) \).

*Proof.* This follows by setting the weight matrices \( W_j = W_j^1 \oplus W_j^2 \) and \( b_i = b_i^1 \oplus b_i^2 \), where \( W_j^1, b_i^1 \) and \( W_j^2, b_i^2 \) represent the parameters defining \( f_1 \) and \( f_2 \) respectively. Recally that the direct sum of matrices is simply given by
\[
A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.
\]

\( \square \)

Note that this result can be applied recursively to concatenate multiple networks. The next Lemma shows that the identity function can be implemented using ReLU networks.

**Lemma 4** (Identity Networks). The identity map on \( \mathbb{R} \) is in \( Y^{2,1}(\mathbb{R},\mathbb{R}) \).

*Proof.* Consider the identity \( x = \sigma(x) - \sigma(-x) \). This can be implemented by setting
\[
W_0 = W_1^T = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

and \( b_0 = b_1 = 0 \). \( \square \)
We remark that if the input to the network can be bounded, then the width in Lemma 4 can be reduced to 1 by taking $\sigma(x + b) - b$ for a sufficiently large bias $b$, although we will not use this in what follows.

Combining Lemmas 3 and 4, we obtain the following basic fact, which will be used extensively in the following construction. It states that we can apply a given network to only a few components of the input.

**Lemma 5.** Let $m \geq 0$ and suppose that $f \in Y^{W,L}([\mathbb{R}^{d_i}, \mathbb{R}^k])$. Then the function $f \oplus I$ on $\mathbb{R}^{d+m}$ defined by

$$ (f \oplus I)(x_1 \oplus x_2) = f(x_1) \oplus x_2 \tag{2.7} $$

satisfies $f \oplus I \in Y^{W+2m,L}([\mathbb{R}^{d+m}, \mathbb{R}^{k+m}])$.

**Proof.** Simply apply Lemma 3 to $f$ and the identity network constructed in Lemma 4.

Using these basic lemmas we show how to construct a deep network which represents the sum of a collection of smaller networks. This will serve as a simple example of how the previous observations can be combined to construct more sophisticated networks.

**Proposition 1 (Summing Networks).** Let $f_i \in Y^{W,L_i}(\mathbb{R}^{d_i}, \mathbb{R}^k)$ for $i = 1, \ldots, n$. Then we have

$$ \sum_{i=1}^n f_i \in Y^{W+2d+2k,L}(\mathbb{R}^d, \mathbb{R}^k), \tag{2.8} $$

where $L = \sum_{i=1}^n L_i$.

**Proof.** We will show by induction on $j$ that

$$ \begin{pmatrix} x \\ 0 \end{pmatrix} \to \begin{pmatrix} \sum_{i=1}^j x_i \\ \sum_{i=1}^j f_i(x) \end{pmatrix} \in Y^{W+2d+2k,L}([\mathbb{R}^{d+k}, \mathbb{R}^{d+k}]) \tag{2.9} $$

for $L = \sum_{i=1}^j L_i$. The base case $j = 0$ is trivial since the identity map is affine. Suppose we have shown this for $j-1$, i.e.

$$ \begin{pmatrix} x \\ 0 \end{pmatrix} \to \begin{pmatrix} \sum_{i=1}^{j-1} x_i \\ \sum_{i=1}^{j-1} f_i(x) \end{pmatrix} \in Y^{W+2d+2k,L}([\mathbb{R}^{d+k}, \mathbb{R}^{d+k}]), \tag{2.10} $$

where $L = \sum_{i=1}^{j-1} L_i$. Compose this map with an affine map which duplicates the first entry to get

$$ \begin{pmatrix} x \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ \sum_{i=1}^{j-1} f_i(x) \end{pmatrix} \in Y^{W+2d+2k,L}([\mathbb{R}^{d+k}, \mathbb{R}^{d+k}]), \tag{2.11} $$

Now, we use Lemma 5 to apply $f_j$ to the middle entry. This gives

$$ \begin{pmatrix} x \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ f_j(x) + \sum_{i=1}^{j-1} f_i(x) \end{pmatrix} \in Y^{W+2d+2k,L_j+L}([\mathbb{R}^{d+k}, \mathbb{R}^{d+k}]), \tag{2.12} $$

We finally compose with the affine map

$$ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y+z \end{pmatrix} \in Y^d([\mathbb{R}^{d+2k}, \mathbb{R}^{d+k}]), \tag{2.13} $$

and apply Lemma 2 to obtain

$$ \begin{pmatrix} x \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ \sum_{i=1}^j f_i(x) \end{pmatrix} \in Y^{W+2d+2k,L_j+L}([\mathbb{R}^{d+k}, \mathbb{R}^{d+k}]), \tag{2.14} $$

which completes the inductive step.

Applying the induction up to $j = n$, we have that

$$ x \to \begin{pmatrix} x \\ 0 \end{pmatrix} \to \begin{pmatrix} x \\ \sum_{i=1}^n f_i(x) \end{pmatrix} \to \sum_{i=1}^n f_i(x) \in Y^{W+2d+2k,L}([\mathbb{R}^d, \mathbb{R}^k]), \tag{2.15} $$

where $L = \sum_{i=1}^n L_i$, since the first and last maps above are affine (applying Lemma 3).
A key application of this is the following result showing how to represent piecewise linear functions using deep networks.

**Proposition 2.** Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a continuous piecewise linear function with \( k \) pieces. Then \( f \in Y^{5,k-1}(\mathbb{R}) \).

*Proof.* First observe that any piecewise linear function \( f \) with \( k \) pieces can be written as
\[
f(x) = a_0 x + c + \sum_{i=1}^{k-1} a_i \sigma(x - b_i)
\]
for appropriate weights \( a_0, \ldots, a_{k-1} \) and \( b_1, \ldots, b_{k-1} \). Specifically, the \( b_i \) are simply equal to the breakpoints points at which the derivative of \( f \) is discontinuous, while the \( a_i \) give the jump in derivative at those points. \( a_0 \) is set equal to the derivative in the left-most component and \( c \) is set to match the value at 0.

Now we apply Proposition 1 to the sum (2.16) to get the desired result, since we easily see that
\[
x \to a_0 x + c \in Y^{0}(\mathbb{R})
\]
and
\[
x \to a_i \sigma(x - b_i) \in Y^{1,1}(\mathbb{R}).
\]

\[ \Box \]

### 2.2 Approximating Products

Next we describe how to approximate products using deep ReLU networks. This will be necessary in the following to approximate piecewise polynomial functions. The method for doing this is based upon a construction of Telgarsky [44] and was first applied to approximating smooth functions using neural networks by Yarotsky [48]. This construction has since become an important tool in the analysis of deep ReLU networks and has been used by many different authors [14, 30, 34]. For the readers convenience, we reproduce a complete description of the construction.

**Proposition 3** (Product Network, Proposition 3 in [48]). Let \( k \geq 1 \). Then there exists a network \( f_k \in Y^{13,6k+3}([\mathbb{R}^2]) \) such that for all \( x, y \in [-1,1] \) we have
\[
|f_k(x,y) - xy| \leq 6 \cdot 4^{-k}.
\]

*Proof.* Observe that the piecewise linear hat function
\[
f(x) = \begin{cases} 
2x & x \leq 1/2 \\
2(1-x) & x > 1/2 
\end{cases}
\]
satisfies \( f \in Y^{5,1}(\mathbb{R}) \) by Proposition 2. On the interval \([0,1]\), \( f \) composed with itself \( n \) times is the sawtooth function
\[
f^{\circ n}(x) := (f \circ \cdots \circ f)(x) = f(2^{n-1}x - [2^{n-1}x]),
\]
and one can calculate that (see [48])
\[
x^2 = x - \sum_{n=1}^{\infty} 4^{-n} f^{\circ n}(x)
\]
for \( x \in [0,1] \).

Using this, we construct a network \( g_k \in Y^{7,k}(\mathbb{R}) \) such that
\[
\sup_{x \in [0,1]} |x^2 - g_k(x)| \leq 4^{-k}.
\]

To do this, we first apply the affine map which duplicates the input
\[
x \to \begin{pmatrix} x \\ x \end{pmatrix} \in Y^0([\mathbb{R}, \mathbb{R}^2]).
\]

Next, we show by induction on \( k \) that the map
\[
x \to \left( x - \sum_{n=1}^{k} 4^{-n} f^{\circ n}(x) \right) \in Y^{7,k}([\mathbb{R}, \mathbb{R}^2]).
\]

The base case \( k = 0 \) is simply (2.22).
For the inductive step suppose that (2.23) holds for \( k \geq 0 \). We use Lemma 5 to apply \( f \in \mathcal{Y}^{5,1}(\mathbb{R}) \) to the second coordinate, showing that
\[
\left( \frac{x}{y} \right) \rightarrow \left( \frac{x}{f(y)} \right) \in \mathcal{Y}^{7,1}(\mathbb{R}^2, \mathbb{R}^2).
\] (2.24)

Using the inductive assumption and Lemma 2, we compose this with the map in (2.23) to get
\[
x \rightarrow \left( x - \sum_{n=1}^{k} 4^{-n} f^{\circ n}(x) \right) \in \mathcal{Y}^{7,k+1}(\mathbb{R}, \mathbb{R}^2).
\] (2.25)

We again use Lemma 2 and compose with the affine map
\[
\left( \frac{x}{y} \right) \rightarrow \left( \frac{x + y}{y} \right) \in \mathcal{Y}^{0}(\mathbb{R}^2, \mathbb{R}^2)
\] (2.26)
to complete the inductive step.

To construct \( g_k \) we then simply compose the map in (2.23) with the map in (2.27) to get
\[
\left( \frac{x}{y} \right) \rightarrow x \in \mathcal{Y}^{0}(\mathbb{R}^2, \mathbb{R})
\] (2.27)
which forgets the second coordinate. Then for \( x \in [0, 1] \) we have
\[
g_k(x) = x - \sum_{n=1}^{k} 4^{-n} f^{\circ n}(x)
\] (2.28)
and by (2.20) we get the bound (2.21). This gives us a network which approximates \( x^2 \) on the interval \([0, 1]\).

In order to obtain a network which approximates \( x^2 \) on \([-1, 1]\) we observe that if \( x \in [-1, 1] \), then \( \sigma(x), \sigma(-x) \in [0, 1] \), and
\[
x^2 = \sigma(x)^2 + \sigma(-x)^2.
\] (2.29)

We begin with the single layer network
\[
x \rightarrow \left( \frac{\sigma(x)}{\sigma(-x)} \right) \in \mathcal{Y}^{2,1}(\mathbb{R}, \mathbb{R}^2).
\] (2.30)

Further, applying Lemma 5 we see that
\[
\left( \frac{x}{y} \right) \rightarrow \left( \frac{g_k(x)}{y} \right) \in \mathcal{Y}^{0,k}(\mathbb{R}^2, \mathbb{R}^2)
\] (2.31)
and also
\[
\left( \frac{x}{y} \right) \rightarrow \left( \frac{x}{g_k(y)} \right) \in \mathcal{Y}^{0,k}(\mathbb{R}^2, \mathbb{R}^2)
\] (2.32)
Finally, composing all of these and then applying the affine summation map
\[
\left( \frac{x}{y} \right) \rightarrow x + y \in \mathcal{Y}^{0}(\mathbb{R}^2),
\] (2.33)
we get, using Lemma 2 (note that we can expand the width of the network in (2.30), a function \( h_k \in \mathcal{Y}^{0,2k+1}(\mathbb{R}) \) such that on \([-1, 1]\), we have
\[
|x^2 - h_k(x)| \leq |\sigma(x)^2 - h_k(\sigma(x))| + |\sigma(-x)^2 - h_k(\sigma(-x))| \leq 4^{-k}.
\] (2.34)
(Since one of \( \sigma(x) \) and \( \sigma(-x) \) is 0.)

Finally, to construct a network which approximates products, we use the formula
\[
xy = 2 \left( \left( \frac{x+y}{2} \right)^2 - \left( \frac{x}{2} \right)^2 - \left( \frac{y}{2} \right)^2 \right)
\] (2.35)
If \( x, y \in [-1, 1] \), then all of the terms which are squared in the previous equation are also in \([-1, 1]\), so that we can approximate these squares using the network \( h_k \). Applying the affine map
\[
\left( \frac{x}{y} \right) \rightarrow \left( \frac{(x+y)/2}{x/2}, \frac{x/2}{y/2} \right) \in \mathcal{Y}^{0}(\mathbb{R}^2, \mathbb{R}^3),
\] (2.36)
then successively applying $h_k$ to the first, second, and third coordinates using Lemmas 5 and 2 and finally applying the affine map
\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to 2(x - y - z) \in \mathcal{Y}^2(\mathbb{R}^3),
\] (2.37)
we obtain a network $f_k \in \mathcal{Y}^{13,6\epsilon+3}(\mathbb{R}^2)$ such that for $x, y \in [-1, 1]$ we have
\[
|f_k(x, y) - xy| \leq 6 \cdot 4^{-k},
\] (2.38)
as desired.

\section{2.3 Bit Extraction Networks}

The key to obtaining superconvergence for deep ReLU networks is the bit extraction technique, which was first introduced in [5] with the goal of lower bounding the VC dimension of the class of neural networks with polynomial activation function. This technique as also been used to obtain sharp approximation results for deep ReLU networks [39, 49]. In the following Proposition, which is a minor modification of Lemma 11 in [6], we construct the bit extraction networks that we will need in our approximation of Sobolev functions.

\begin{prop}[Bit Extraction Network]
Let $n \geq m \geq 0$ be an integer. Then there exists a network $f_{n,m} \in \mathcal{Y}^{9,4m}(\mathbb{R}, \mathbb{R}^2)$ such that for any input $x \in [0, 1]$ with at most $n$ non-zero bits, i.e.
\[
\begin{align*}
x &= 0, x_1 x_2 \cdots x_n
\end{align*}
\] (2.39)
with bits $x_i \in \{0, 1\}$, we have
\[
\begin{align*}
f_{n,m}(x) &= \begin{pmatrix} 0, x_{m+1} \cdots x_n \\ x_1 x_2 \cdots x_m, 0 \end{pmatrix}.
\end{align*}
\] (2.40)
\end{prop}

\begin{proof}
We begin by noting that for any $\epsilon > 0$ the piecewise linear maps
\[
\begin{align*}
b_\epsilon(x) &= \begin{cases} 
0, & x \leq 1/2 - \epsilon \\
\epsilon^{-1}(x - 1/2 + \epsilon), & 1/2 - \epsilon < x \leq 1/2 \\
1, & x > 1/2
\end{cases}
\end{align*}
\] (2.41)
and
\[
\begin{align*}
g_\epsilon(x) &= \begin{cases} 
x, & x \leq 1 - \epsilon \\
\frac{1-\epsilon}{\epsilon}(1-x), & 1 - \epsilon < x \leq 1 \\
x - 1, & x > 1
\end{cases}
\end{align*}
\] (2.42)
satisfy $b_\epsilon, g_\epsilon \in \mathcal{Y}^{5,2}(\mathbb{R})$ by Proposition 2. In addition, these functions have been designed so that if $\epsilon < 2^{-n}$, we have for any $x$ of the form (2.39) that
\[
b_\epsilon(x) = x_1, \ g_\epsilon(2x) = 0, x_2 x_3 \cdots x_n.
\] (2.43)

We now construct the network $f_{n,m}$ by induction on $m$. In what follows, we assume that all of our inputs $x$ are of the form (2.39). The base case when $m = 0$ is simply the affine map
\[
x \to \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{Y}^0(\mathbb{R}, \mathbb{R}^2).
\] (2.44)

For the inductive step, we suppose that we have constructed a map
\[
\begin{align*}
f_{n,m-1}(x) &= \begin{pmatrix} 0, x_m x_{m+1} \cdots x_n \\ x_1 x_2 \cdots x_{m-1}, 0 \end{pmatrix} \in \mathcal{Y}^{9,4(m-1)}(\mathbb{R}, \mathbb{R}^2)
\end{align*}
\] (2.45)
We then compose this network with an affine map which doubles and duplicates the first component
\[
\begin{align*}
\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2x \\ y \end{pmatrix} \in \mathcal{Y}^0(\mathbb{R}^2, \mathbb{R}^3)
\end{align*}
\] (2.46)
to get the map
\[
x \to \begin{pmatrix} x_{m-1} x_m x_{m+1} \cdots x_n \\ 0, x_m x_{m+1} \cdots x_{n-1} \\ x_1 x_2 \cdots x_{m-1}, 0 \end{pmatrix} \in \mathcal{Y}^{9,4(m-1)}(\mathbb{R}, \mathbb{R}^3).
\] (2.47)
Next we choose \( \varepsilon < 2^{-n} \) and use Lemmas 2 and 5 to apply \( g_\varepsilon \) to the first component and then \( b_\varepsilon \) to the second component. This gives a map
\[
x \rightarrow \begin{pmatrix} 0, x_{m+1}, \ldots, x_n \\ x_m \\ x_1, x_2, \ldots, x_{m-1} \end{pmatrix} \in \mathcal{Y}^{2,4m}(\mathbb{R}, \mathbb{R}^3)
\] (2.48)

Finally, we complete the inductive step by composing with the affine map
\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x \\ 2z + y \end{pmatrix} \in \mathcal{Y}^0(\mathbb{R}^3, \mathbb{R}^2).
\] (2.49)

2.4 Order Statistic Networks

In order to deal with the case when the error is measured in \( L_m \), we will need the following technical construction. We construct a ReLU network which takes an input in \( \mathbb{R}^d \) and returns the \( k \)-th largest entry. The first step is the following simple Lemma.

**Lemma 6 (Max-Min Networks).** There exists a network \( p \in \mathcal{Y}^{1,1}(\mathbb{R}^2, \mathbb{R}^2) \) such that
\[
p \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \max(x, y) \\ \min(x, y) \end{pmatrix}.
\] (2.50)

**Proof.** We observe the basic formulas:
\[
\max(x, y) = x + \sigma(y - x), \quad \min(x, y) = x - \sigma(x - y).
\] (2.51)

We begin with the affine map
\[
\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} y - x \\ x - y \end{pmatrix} \in \mathcal{Y}^0(\mathbb{R}^2, \mathbb{R}^2).
\] (2.52)

Next, we use the fact that \( \sigma \in \mathcal{Y}^{1,1}(\mathbb{R}) \) and Lemmas 2 and 5 to apply \( \sigma \) to the last two coordinates. We get
\[
\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \sigma(y - x) \\ \sigma(x - y) \end{pmatrix} \in \mathcal{Y}^{1,1}(\mathbb{R}^2, \mathbb{R}^2).
\] (2.53)

Finally, we use Lemma 2 to compose with the affine map
\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x + y \\ x - z \end{pmatrix}.
\] (2.54)

Using these networks as building blocks, we can implement a sorting network using deep ReLU neural networks.

**Proposition 5.** Let \( k \geq 1 \) and \( d = 2^k \) be a power of 2. Then there exists a network \( s \in \mathcal{Y}^{d,d}(\mathbb{R}^d, \mathbb{R}^d) \) where \( L = \binom{k+1}{2} \) which sorts the input components.

Note that the power of 2 assumption is for simplicity and is not really necessary. It is also known that the depth \( \binom{k+1}{2} \) can be replaced by a multiple \( Ck \) where \( C \) is a very large constant [2, 33], but this will not be important in our argument.

**Proof.** Suppose that \((i_1, j_1), \ldots, (i_{2^k-1}, j_{2^k-1})\) is a pairing of the indices of \( \mathbb{R}^d \). By Lemma 6 and Lemma 3 there exists a network \( g \in \mathcal{Y}^{d,d}(\mathbb{R}^d, \mathbb{R}^d) \) which satisfies for all \( l = 1, \ldots, k - 1 \)
\[
g(x)_{i_l} = \max(x_{i_l}, x_{j_l}), \quad g(x)_{j_l} = \min(x_{i_l}, x_{j_l}),
\] (2.55)
i.e. which sorts the entries in each pair. By a well-known construction of sorting networks (for instance bitonic sort [7]), composing \( \binom{k+1}{2} \) such functions can be used to sort the input.

Finally, we note that by selecting a single output (which is an affine map), we can obtain a network which outputs any order statistic.

**Corollary 1.** Let \( 1 \leq k \leq d \) and \( d \) is a power of 2. Then there exists a network \( g_k \in \mathcal{Y}^{d,d}(\mathbb{R}^d) \) with \( L = \binom{k+1}{2} \) such that
\[
g_k(x) = x_{(k)},
\] (2.56)
where \( x_{(k)} \) is the \( k \)-th largest entry of \( x \).
3 Optimal Representation of Sparse Vectors using Deep ReLU Networks

In this section, we prove the main technical result which enables the efficient \(L_p\)-approximation of Sobolev functions \(f \in W^p(L_q(\Omega))\) in the non-linear regime when \(q < p\). Specifically, we have the following Theorem showing how to optimally represent sparse integer vectors using deep ReLU neural networks.

**Theorem 3.** Let \(M \geq 1\) and \(N \geq 1\) and \(x \in \mathbb{Z}^N\) be an \(N\)-dimensional vector satisfying
\[
\|x\|_1 \leq M. \tag{3.1}
\]
Then if \(N \geq M\), there exists a neural network \(g \in \mathcal{Y}^{17, L}(\mathbb{R}, \mathbb{R})\) with depth \(L \leq C \sqrt{M(1 + \log(N/M))}\) which satisfies \(g(n) = x_n\) for \(n = 1, \ldots, N\).

Further, if \(N < M\), then there exists a neural network \(g \in \mathcal{Y}^{17, L}(\mathbb{R}, \mathbb{R})\) with depth \(L \leq C \sqrt{N(1 + \log(M/N))}\) which satisfies \(g(n) = x_n\) for \(n = 1, \ldots, N\).

In Theorem 3 from Section 5 we prove that this result is optimal as long as \(M\) is not exponentially small or exponentially large relative to \(N\).

**Proof of Theorem 3.** Let \(M \geq 1\) and \(N \geq 1\) be fixed. There are two cases to consider, when \(N \geq M\) and when \(N < M\).

The key to the construction in both cases will be a length \(k\) binary encoding of the set
\[
S_{N,M} = \{x \in \mathbb{Z}^N, \|x\|_1 \leq M\}. \tag{3.2}
\]
By a length \(k\) binary encoding we mean a pair of maps:
- \(E : S_{N,M} \to \{0, 1\}^{\leq k}\) (an encoding map which maps \(S_{N,M}\) to a bit-string of length at most \(k\))
- \(D : \{0, 1\}^{\leq k} \to S_{N,M}\) (a decoding map which recovers \(x \in S_{N,M}\) from a bit-string of length at most \(k\))

which satisfy
\[
D(E(x)) = x. \tag{3.3}
\]
Note that this construction proves that \(E\) is injective and \(D\) is surjective, which means that \(|S_{N,M}| \leq 2^{k+1} - 1\). We will give explicit algorithms which implement both maps, and these will be used to construct the neural network \(g\).

Let us begin with the first case, when \(N \geq M\). In this case, we set \(k = 2M(3 + \lfloor \log(N/M) \rfloor)\) (note that all logarithms are taken with base 2). The encoding map \(E\) is defined as
\[
E(x) = f_1 t_1 f_2 t_2 \cdots f_{Rt} t_R, \tag{3.4}
\]
the concatenation of \(R \leq 2M\) blocks consisting of \(f_i \in \{0, 1\}^{1 + \lfloor \log(N/M) \rfloor}\) and \(t_i \in \{0, 1\}^2\). The \(f_i\)-bits encode an offset in \(\{0, 1, \ldots, \lfloor N/M \rfloor\}\) (via binary expansion), and the \(t_i\)-bits encode a value in \(\{0, \pm 1\}\) (via \(0 = 00, 1 = 10, \text{and} -1 = 01\)). The \(f_i\) and \(t_i\) are determined from the input \(x \in S_{N,M}\) by Algorithm 1.

The decoding map \(D\), which reconstructs \(x\) from the sequence \(f_1 t_1 \cdots f_{Rt} t_R\) is given in Algorithm 2. It is clear that the number of blocks \(R\) produced by Algorithm 1 is at most \(2M\) since in each round of the while loop either \(f_i = \lfloor N/M \rfloor\) (which can happen at most \(M\) times before the index \(j > N\)) or the entry \(r_j\) is decremented (which can happen at most \(M\) times since \(|x\|_1 \leq M\)). We also easily verify that Algorithm 2 reconstructs the vector \(x\) from the output of Algorithm 1.

Next, we show how to use these algorithms to construct an appropriate deep ReLU neural network \(g\). We first choose a threshold parameter
\[
S := \left\lfloor \sqrt{M(4 + \log(N/M))}^{-1} \right\rfloor. \tag{3.5}
\]
Given a vector \(x \in \mathbb{Z}^N\) we divide it into two pieces \(x = x^B + x^s\) (here \(x^B\) represents the ‘big’ part and \(x^s\) the ‘small’ part). We define
\[
x^B_i = \begin{cases} x_i & x_i \geq S \\ 0 & x_i < S \end{cases} \tag{3.6}
\]
and
\[
x^s_i = \begin{cases} 0 & x_i \geq S \\ |x_i| & x_i < S \end{cases} \tag{3.7}
\]
We claim that since this means that there is a piecewise linear function with at most 3
Algorithm 2 run with this input and let
\[ p \]
x
Algorithm 1 are executed at least \( \ell \) times at index \( j \) at the end of step 0 is the beginning of the algorithm. Then
\[ 1 = j_0 \leq j_1 \leq \cdots \leq j_R \leq N, \]
and \( j_{i+1} = j_i + f_i \). Consider the index \( i \) at evenly spaced steps with spacing determined by the parameter \( S \),
\[ 1 = j_0 \leq j_S \leq j_{2S} \leq \cdots \leq j_{pS}, \]
where \( p = \lfloor R/S \rfloor \leq 2\sqrt{M(4 + \log(N/M))} \) is the largest integer such that \( pS \leq R \).
We claim that since \( \|x^i\|_{\ell \infty} < S \), we have that
\[ j_0 < j_S < j_{2S} < \cdots < j_{pS}. \]
Note that if \( j = j_{kS} = \cdots = j_{(k+1)S} \), then \( f_i = 0 \) for all \( i = 1 + kS, \ldots, (k+1)S \). This implies that lines 13 and 14 in Algorithm 1 are executed at least \( S \) times at index \( j \), so that \( |x^i_j| \geq S \). This contradicts the bound \( \|x^i\|_{\ell \infty} < S \), and so (3.12) holds.

**Algorithm 1** Small \( \ell^1 \)-norm Encoding Algorithm

**Input:** \( x \in \mathbb{Z}^N \), \( \|x\|_{\ell^1} \leq M \)
1. Set \( j = 1 \), \( r = x \) (Set pointer to beginning of vector and residual to input)
2. Set \( i = 1 \)
3. while \( r \neq 0 \) do
4. \( \ell = \min \{ i : r_i \neq 0 \} \) (Find the first non-zero index in the residual)
5. if \( i - j \leq \lfloor N/M \rfloor \) then \( \{ \) If we can make it to the next non-zero index, do so \( \} \)
6. \( f_i = 1 - j \)
7. \( j = \ell \)
8. else \{ Otherwise go as far as we can \}
9. \( f_i = \lfloor N/M \rfloor \)
10. \( j = j + \lfloor N/M \rfloor \)
11. end if
12. if \( j = \ell \) then \( \{ \) If we are at the next non-zero index, \( t_i \) captures its sign \( \} \)
13. \( t_i = \text{sgn}(r_i) \)
14. \( r_j = r_j - t_i \) (This decrements \( \|r\|_{\ell^1} \) which can happen at most \( M \) times)
15. else \{This can only happen if \( f_i = \lfloor N/M \rfloor \), which can occur at most \( M \) times\}
16. \( t_i = 0 \)
17. end if
18. \( i = i + 1 \)
19. end while

**Algorithm 2** Small \( \ell^1 \)-norm Decoding Algorithm

**Input:** A bit string \( f_1f_2\cdots f_R \)
1. Set \( x = 0 \) and \( j = 1 \) \{Start with the 0 vector\}
2. for \( i = 1, \ldots, R \) do
3. \( j = j + f_i \) \{Shift index by \( f_i \}\}
4. \( x_j = x_r + t_i \) \{Increment value by \( t_i \}\}
5. end for

The \( \ell^1 \)-norm bound \( \|x\|_{\ell^1} \leq M \) on \( x \) implies that the support of \( x^B \) is at most of size
\[ |\{ n : x^B_n \neq 0 \}| \leq \frac{\|x^B\|_{\ell^1}}{S} \leq \frac{\|x\|_{\ell^1}}{S} \leq \frac{M}{S} \leq \sqrt{M(4 + \log(N/M))}. \tag{3.8} \]
This means that there is a piecewise linear function with at most \( 3 \sqrt{M(4 + \log(N/M))} \) pieces which matches the values of \( x^B \), so by Proposition 2 there exists a network
\[ g_B \in Y^5L(\mathbb{R}) \]
with depth bounded by \( L \leq 3 \sqrt{M(4 + \log(N/M))} \) such that \( g_B(n) = x^B_n \) for \( n = 1, \ldots, N \).

To efficiently represent the small values \( x^i \) using a deep ReLU neural network, we use the encoding and decoding Algorithms 1 and 2. Specifically, let \( f_1f_2\cdots f_R \) be the output of the encoding Algorithm 1 run on input \( x^i \). Consider Algorithm 2 run with this input and let \( j_i \) denote the value of the position variable \( j \) at the end of step \( i \) (note that the end of step 0 is the beginning of the algorithm). Then
\[ 1 = j_0 \leq j_1 \leq \cdots \leq j_R \leq N, \tag{3.9} \]
and \( j_{i+1} = j_i + f_i \). Consider the index \( i \) at evenly spaced steps with spacing determined by the parameter \( S \),
\[ 1 = j_0 \leq j_S \leq j_{2S} \leq \cdots \leq j_{pS}, \tag{3.10} \]
where \( p = \lfloor R/S \rfloor \leq 2\sqrt{M(4 + \log(N/M))} \) is the largest integer such that \( pS \leq R \).
We claim that since \( \|x^i\|_{\ell \infty} < S \), we have that
\[ j_0 < j_S < j_{2S} < \cdots < j_{pS}. \tag{3.11} \]
Note that if \( j = j_{kS} = \cdots = j_{(k+1)S} \), then \( f_i = 0 \) for all \( i = 1 + kS, \ldots, (k+1)S \). This implies that lines 13 and 14 in Algorithm 1 are executed at least \( S \) times at index \( j \), so that \( |x^i_j| \geq S \). This contradicts the bound \( \|x^i\|_{\ell \infty} < S \), and so (3.12) holds.
For \( q = 0, \ldots, p \), set
\[
T_q = \min\{(q + 2)S, R\} = \begin{cases} (q + 2)S & q \leq p - 2 \\ R & q = p - 1 \text{ or } p \end{cases}
\]  
(3.12)
and let \( r_q \in [0, 1] \) denote the real number
\[
r_q = 0.f_qS+1f_{qS+1}f_{qS+2} \cdots f_{qS+T_q}t_q
\]  
(3.13)
whose binary expansion contains the blocks of the encoding of \( x^i \) from index \( i = qS + 1, \ldots, T_q \) (followed by zeros). Note that the number of non-zero blocks in each number \( r_q \) is at most \( 4S \).

Given an index \( n = 1, \ldots, N \), we define
\[
q(n) = \begin{cases} 0 & n = 1 \\ \max\{q : j_qS < n\} & n > 1 \end{cases}
\]  
(3.14)
and construct piecewise linear functions \( J \) and \( R \) which satisfy
\[
J(n) = j_qS, \quad R(n) = r_q(n)
\]  
(3.15)
for integers \( n = 1, \ldots, N \). Since the functions \( J \) and \( R \) each take at most \( p + 1 \) distinct values at the integers \( n = 1, \ldots, N \), they can be constructed to have at most \( 2p + 1 \) pieces (\( p + 1 \) constant pieces for each different value and \( p \) linear pieces where the values change). Proposition 2 thus implies that \( J, R \in \mathcal{C}^{2p}([R]) \).

We construct the following network. We begin with the affine map
\[
x \rightarrow \begin{pmatrix} x \\ x \end{pmatrix} \in \mathcal{Y}_0([R, \mathbb{R}^3]).
\]  
(3.16)
We use Lemmas 2 and 5 to apply \( K \) to the first component and then apply \( R \) to the second component to get
\[
x \rightarrow \begin{pmatrix} J(x) \\ x \end{pmatrix} \rightarrow \begin{pmatrix} J(x) \\ R(x) \end{pmatrix} \in \mathcal{Y}^{9,4p+2}([R, \mathbb{R}^3]).
\]  
(3.17)
Composing with the affine map
\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} z - x \\ y \\ 0 \end{pmatrix} \in \mathcal{Y}_0([\mathbb{R}^3, \mathbb{R}^3]),
\]  
(3.18)
and using Lemma 2 we get that
\[
x \rightarrow \begin{pmatrix} x - J(x) \\ R(x) \\ 0 \end{pmatrix} \in \mathcal{Y}^{9,4p+2}([R, \mathbb{R}^3]).
\]  
(3.19)
When applied to an integer input \( n = 1, \ldots, N \), this network takes the value
\[
n \rightarrow \begin{pmatrix} n - J(n) \\ R(n) \\ 0 \end{pmatrix} = \begin{pmatrix} 0.f_{i+1}t_{i+1} \cdots f_{q(n)}t_{q(n)}x_n \end{pmatrix}
\]  
(3.20)
for \( i = q(n)S \). Here \( x_n^i \) is the value of the \( n \)-th entry \( x_n \) at the end of the \( i \)-th step of Algorithm 2 run on the encoding \( f_{i+1}t_{i+1} \cdots f_{q(n)}t_{q(n)} \) of \( x^i \).

The equality of the first two components in (3.20) follows from the definitions of \( J \) and \( R \). For the last component we need to check that \( x_n^i = 0 \) when \( i = q(n)S \). This follows since by definition when \( n \geq 1 \) we have \( j_qS < n \), so that at the end of step \( i = q(n)S \), the position variable \( j \) has not yet reached \( n \). This means that \( x_n \) cannot have been changed yet in Algorithm 2. In addition, when \( n = 1 \) we set \( q(n) = 0 \) so that we are at the beginning of Algorithm 2. In either case we have that \( x_n^i = 0 \) when \( i = q(n)S \).

Next, we will construct a deep ReLU network which maps
\[
\begin{pmatrix} x \\ 0.f_1t_1 \cdots f_{4S}t_{4S} \end{pmatrix} \rightarrow \begin{pmatrix} x - f_1 \\ 0.f_2t_2 \cdots f_{4S}t_{4S} \\ z + \delta_{x-f_1t_1} \end{pmatrix}
\]  
(3.21)
when $x$ is an integer and the second component takes the form in (3.21). Here $\delta_x$ for $x \in \mathbb{Z}$ is the delta function, i.e.

$$
\delta_x = \begin{cases} 
1 & x = 0 \\
0 & x \neq 0.
\end{cases}
$$

This network implements a single step of Algorithm 2 where the first component $x$ tracks the difference $n - j$ between the input $n$ and the position variable $j$, the second component contains a real number $r = 0.f_1t_1 \cdots f_4s_4S$ whose binary expansion contains at most $4S$ blocks. Here we choose $n \geq 4S(4 + \log(N/M))$, which is guaranteed to be larger than the length of a bit-string containing at most $4S$ blocks. To do this, we first implement a network which extracts the two bits corresponding to $t$ and then adds $t$ to the third component if the first component is 0. Let $h(x)$ denote the piecewise linear function

$$
h(x) = \begin{cases} 
0 & x \leq -1 \\
1 & -1 < x \leq 0 \\
0 & 0 < x \leq 1 \\
1 & x > 1.
\end{cases}
$$

For integer inputs, $h$ is simply the delta function, i.e. $h(x) = \delta_x$ for $x \in \mathbb{Z}$, and by Proposition 4 we have $h \in \mathcal{Y}^{5,3}(\mathbb{R})$. We first apply an affine map which duplicates the first coordinate,

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{Y}^{0}(\mathbb{R}^3, \mathbb{R}^4).
$$

Then, we use Lemma 5 to apply $h$ to the second coordinate and then apply the bit extractor network $f_{n,m}$ from Proposition 4 to the third component. As before, we choose $n \geq 4S(4 + \log(N/M))$ which is guaranteed to be larger than the length of a bit-string containing at most $4M$ blocks. This gives (note that we write $b_1b_2$ for the two bits corresponding to $t_1$)

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ h(x) \\ \delta_1 \\ 0.b_1b_2f_2\cdots f_4s_4S \\ z \end{pmatrix} \in \mathcal{Y}^{15,7}(\mathbb{R}^3, \mathbb{R}^5),
$$

Now we apply the map

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 + x_3 - 1 \\ x_4 \\ x_5 \\ x_5 + \sigma(x_2 + x_3 - 1) \end{pmatrix} \in \mathcal{Y}^{7,1}(\mathbb{R}^5, \mathbb{R}^3).
$$
We note that when applied to integer inputs this does not cause a problem since the remaining blocks are simply taken to be 0 and in this case the network we compose the network (3.19) with 4 copies of $h$ gives

$$g : \mathbb{Z} \rightarrow \mathbb{Z}.$$

Composing the networks in (3.29) and (3.30) will extract the block $g_{q(n)}$ for $q(n) = p - 1$ or $q(n) = p$, in which case $T_{q(n)} < (q(n) + 2)S$, then the initial bit-string $r_q$ contains fewer than 4S blocks. This does not cause a problem since the remaining blocks are simply taken to be 0 and in this case the network $h$, i.e. applying one step of Algorithm 2, leaves its input unchanged.

Finally, to check the correctness of $g$, we must verify that $x_{j^i} = x_{j^i}$ when $i = T_{q(n)}$. When $q(n) = p$ or $q(n) = p - 1$ this is immediate, since $T_{q(n)} = R$ is the end of Algorithm 2 which reconstructs $x^q$. Otherwise, when $q(n) \leq p - 2$, we have

$$T_{q(n)} = (q(n) + 2)S.$$

We note that

$$j_{q(n)} + 2S > j_{q(n) + 1}S \geq n.$$

Here the first inequality follows from (3.24) while the second follows from the definition of $q(n)$, since $q(n)$ is the maximal value of $q$ for which $j_{qS} \leq n$. Thus, at step $T_{q(n)}$ in Algorithm 2 the position index $j$ has passed $n$ and so the entry $x_n$ will no longer be modified. Since Algorithm 2 reconstructs $x^q$, this proves the correctness of the network $g$.

We construct the final network $g$ as follows. We begin by duplicating the input using an affine map. Then we use Lemmas 5 and 2 to apply $g_b$ to the first copy, and then $g_s$ to the second copy. Finally, we add both coordinates using an affine map. This gives a $g \in \mathcal{Y}^{17L}(\mathbb{R})$ which maps

$$g : x \rightarrow \begin{pmatrix} x \\ g_B(x) \\ g_s(x) \end{pmatrix} \rightarrow g_s(x) + g_B(x).$$

When applied to integer inputs $n = 1, \ldots, N$, this network satisfies

$$g(n) = g_s(n) + g_B(n) = x^B_n + x^B_n = x_n.$$

Further, the depth of $g$ satisfies

$$L \leq 3\sqrt{M(4 + \log(N/M))} + 4p + 2 + 4S(16 + 4m).$$

Recalling that $p \leq 2\sqrt{M(4 + \log(N/M))}$, $m = 1 + \log(N/M)$, and the value of the parameter $S$ from (3.3), we finally see that

$$L \leq C\sqrt{M(1 + \log(N/M))}.$$

This completes the proof when $N \geq M$. 

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Next, we consider the case where $N < M$. In this case the construction is similar, but we consider a different encoding of $S_{N,M}$. Specifically, we set $k = 2N(3 + \lceil \log(M/N) \rceil)$, and define the encoding map $E$ is via

$$E(x) = t_1 f_1 t_2 f_2 \cdots t_R f_R,$$

i.e. $E(x)$ is the concatenation of $R \leq 2N$ blocks consisting of $t_i \in \{0,1\}^{2 + \lceil \log(M/N) \rceil}$ and $f_i \in \{0,1\}$. The $f_i$-bits encode an offset in $\{0,1\}$, and the $t_i$-bits encode a value in $\{-[M/N],\ldots,[M/N]\}$. Here the first bit of each $t_i$ determines its sign, while the remaining $1 + \lceil \log(M/N) \rceil$ bits consist of the binary expansion of its magnitude (which lies in $\{0,\ldots,[M/N]\}$). The $t_i$ and $f_i$ are determined from the input $x \in S_{N,M}$ by Algorithm 3.

The decoding map $D$, which reconstructs $x$ from the sequence $t_1 f_1 \cdots t_R f_R$ is given in Algorithm 4. It is clear that the number of blocks $R$ produced by Algorithm 3 is at most $2N$ since in each round of the while loop either $t_i = 1$ and is incremented (which can happen at most $N$ times before the index $j > N$) or the entry $r_j$ is decremented by at least $\lceil M/N \rceil$ (which can happen at most $N$ times since $\|x\|_{\ell^1} \leq M$). We also easily verify that Algorithm 4 reconstructs the vector $x$ from the output of Algorithm 3.

Algorithm 3 Large $\ell^1$-norm Encoding Algorithm

**Input:** $x \in \mathbb{Z}^N$, $\|x\|_{\ell^1} \leq M$

1. Set $j = 1$, $r = x$ {Set pointer to beginning of vector and residual to input}
2. Set $i = 1$
3. while $r \neq 0$ do
4. if $|r| \leq [M/N]$ then {If we can fully capture the current value, do so}
5. $t_i = r$
6. $f_i = 0$
7. $j = j + 1$ {This can happen at most $N$ times before $j > n$}
9. else {Otherwise capture as much as we can}
10. $t_i = \text{sgn}(r_j)[M/N]$
11. $f_i = 0$
13. end if
14. $i = i + 1$
15. end while

Algorithm 4 Large $\ell^1$-norm Decoding Algorithm

**Input:** A bit string $t_1 f_1 \cdots t_R f_R$

1. Set $x = 0$ and $j = 1$ {Start with the 0 vector}
2. for $i = 1,\ldots,N$ do
3. $x_j = x_j + t_i$ {Increment value by $t_i$}
4. $j = j + f_i$ {Shift index by $f_i$}
5. end for

The next step is to use these algorithms to construct a deep ReLU neural network $g$ which interpolates the values of $x$. As in the previous case, the first step is to divide $x$ into a small and large part. For this, we choose a threshold parameter

$$T := \left\lfloor \frac{M}{\sqrt{N(4 + \log(M/N))}} \right\rfloor,$$  \hspace{1cm} (3.37)

and decompose $x = x^g + x^l$ where

$$x^g = \begin{cases} |x_i| & x_i \geq T \\ 0 & x_i < T \end{cases}$$  \hspace{1cm} (3.38)

and

$$x^l = \begin{cases} 0 & x_i \geq T \\ |x_i| & x_i < T \end{cases}$$  \hspace{1cm} (3.39)

The $\ell^1$-norm bound $(3.1)$ on $x$ implies that the support of $x^g$ is at most of size

$$|\{n : x^g_{n} \neq 0\}| \leq \frac{\|x^g\|_{\ell^1}}{T} \leq \frac{\|x\|_{\ell^1}}{T} \leq \frac{M}{T} \leq \sqrt{N(4 + \log(M/N))}.  \hspace{1cm} (3.40)$$
We claim that since

we now employ an almost identical strategy as in the previous case when

This means that there is a piecewise linear function with at most \(3\sqrt{N(4 + \log(M/N))}\) pieces which matches the

values of \(x^n\), so by Proposition\(^2\) there exists a network

\[ g_B \in Y^{5L}(\mathbb{R}) \]

with depth bounded by \(L \leq 3\sqrt{N(4 + \log(M/N))}\) such that \(g_B(n) = x^n_n\) for \(n = 1, \ldots, N\).

Next, to represent \(x^n\) efficiently using a deep ReLU neural network we use Algorithms\(^5\) and \(^4\). Specifically, let \(t_1 f_1 \cdots t_R f_R\) be encoding of \(x^n\) using Algorithm\(^5\). Consider Algorithm\(^2\) run on this input and let \(j_i\) denote the value of the position variable \(j\) at the end of step \(i\). Then

\[ 1 = j_0 \leq j_1 \leq \cdots \leq j_K \leq N, \quad (3.41) \]

and \(j_{i+1} = j_i + f_{i+1}\). We introduce the spacing parameter

\[ S := \left\lceil \sqrt{N(4 + \log(M/N))}^{-1} \right\rceil \quad (3.42) \]

and consider an even spacing of the index \(i\) with difference \(S\),

\[ 1 = j_0 \leq j_S \leq j_{2S} \leq \cdots \leq j_{pS}, \quad (3.43) \]

where \(p = \lfloor R/S \rfloor \leq 2\sqrt{N(4 + \log(M/N))}\) is the largest integer such that \(pS \leq R\).

We claim that since \(\|x^n\|_{\infty} < T\), we have that

\[ j_0 < j_S < j_{2S} < \cdots < j_{pS}. \quad (3.44) \]

This holds since if \(j = j_{kS} = \cdots = j_{(k+1)S}\), then \(f_i = 0\) for all \(i = 1 + kS, \ldots, (k+1)S\). This implies that lines 10 and 11 in Algorithm\(^3\) are executed at least \(S\) times at index \(j\), so that

\[ |x^n_j| \geq |M/N|S \geq M/\sqrt{N(4 + \log(M/N))} \]

As \(|x^n_j|\) is an integer, this contradicts the bound \(\|x^n\|_{\infty} < T\), which implies that (3.44) holds.

We now employ an almost identical strategy as in the previous case when \(M \leq N\). Specifically, for \(q = 0, \ldots, p\) we define \(T_q\) as in (3.12) and define the real number \(r_q \in [0, 1]\),

\[ r_q = 0.t_qs + 1.t_{q+1}s + \cdots + t_{q+1}s, \quad (3.45) \]

analogously to (3.13). Given an input \(n \in \{1, \ldots, N\}\) we define \(q(n)\) as in (3.14) and construct piecewise linear functions \(J, R \in Y^{52p}(\mathbb{R})\) with the same property (3.15).

The remainder of the construction and argument is exactly the same as in the case when \(M \leq N\) except that the network in (3.21) is replaced by a network

\[
\begin{pmatrix}
0.t_1 f_1 \cdots t_{4S} f_{4S} \\
0.t_2 f_2 \cdots t_{4S} f_{4S} \\
0.t_3 f_3 \cdots t_{4S} f_{4S} \\
0.t_4 f_4 \cdots t_{4S} f_{4S}
\end{pmatrix} \rightarrow \begin{pmatrix}
x - f_1 \\
x - f_2 \\
x - f_3 \\
x - f_4
\end{pmatrix},
\]

which implements a single step of Algorithm\(^4\). As before, the first component \(z\) tracks the difference \(n - j\) between the input \(n\) and the position variable \(j\), the second component contains a real number

\[ r = 0.t_1 f_1 \cdots t_{4S} f_{4S} \quad (3.47) \]

whose binary expansion contains at most \(4S\) remaining blocks \((t, F)\) (note that we can pad with zero blocks), and the last component tracks the value of \(x_a\).

Let us describe how to implement this using a deep ReLU network. Set \(m = 2 + \lceil \log(M/N) \rceil\) to be the length of the \(t\)-block. First, we first extract a single bit (containing the sign of \(t\)) from the second component, using Lemma\(^8\) to apply the bit extractor network \(f_{n,1}\) from Proposition\(^4\) to the second component. We choose \(n \geq 4S(4 + \log(M/N))\), which is guaranteed to be larger than the length of a bit-string containing at most \(4S\) blocks. This gives the map (as before, we have written out the bits \(b_1 \cdots b_m\) of \(t_i\))

\[
\begin{pmatrix}
0.b_1 \cdots b_m f_1 t_2 f_2 \cdots t_{4S} f_{4S} \\
0.b_2 \cdots b_m f_1 t_2 f_2 \cdots t_{4S} f_{4S}
\end{pmatrix} \rightarrow \begin{pmatrix}
x \\
0.b_1 \cdots b_m f_1 t_2 f_2 \cdots t_{4S} f_{4S}
\end{pmatrix} \in Y^{13,4}(\mathbb{R}^3, \mathbb{R}^4),
\]

(3.48)
Next, we extract the remaining \(m-1\) bits corresponding to the first \(t\)-block to obtain the magnitude of \(t\). We do this using Lemma \[5\] and the bit extractor network \(f_{n,m-1}\) from Proposition \[4\]. This gives a map

\[
\begin{pmatrix}
  x \\
  z
\end{pmatrix}
\mapsto
\begin{pmatrix}
  x \\
  \frac{b_1}{b_2} \cdot b_3 \cdot \ldots \cdot b_m \\
  0 \cdot f_1 f_2 f_3 \cdot \ldots \cdot t_4 s f_{4S}
\end{pmatrix}
\in \mathbb{R}^{15,4m} \left( \mathbb{R}^3, \mathbb{R}^5 \right). \tag{3.49}
\]

We compose this map with the following two networks, where \(h\) represents the piecewise continuous function in \([3.25]\).

The first network is given by

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{pmatrix}
\mapsto
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{pmatrix}
\mapsto
\begin{pmatrix}
  x_3 - 2^m(1 - h(x_1) + x_2) \\
  x_4 \\
  x_5
\end{pmatrix}
\mapsto
\begin{pmatrix}
  \sigma(x_3 - 2^m(1 - h(x_1) + x_2)) \\
  x_4 \\
  x_5
\end{pmatrix}
\mapsto
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{pmatrix}
\in \mathbb{R}^{15,4} \left( \mathbb{R}^3, \mathbb{R}^5 \right), \tag{3.50}
\]

while the second is given by

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{pmatrix}
\mapsto
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{pmatrix}
\mapsto
\begin{pmatrix}
  x_3 - 2^m(2 - h(x_1) - x_2) \\
  x_4 \\
  x_5
\end{pmatrix}
\mapsto
\begin{pmatrix}
  \sigma(x_3 - 2^m(2 - h(x_1) - x_2)) \\
  x_4 \\
  x_5
\end{pmatrix}
\mapsto
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{pmatrix}
\in \mathbb{R}^{15,4} \left( \mathbb{R}^3, \mathbb{R}^5 \right), \tag{3.51}
\]

In the above two networks, the first, third, and last maps are affine, while the second map is in \(\mathbb{R}^{15,4} \left( \mathbb{R}^6, \mathbb{R}^6 \right)\) by Lemma \[5\] since \(h \in \mathbb{R}^{5,3} \left( \mathbb{R}^6 \right)\), and the fourth map is in \(\mathbb{R}^{11,1} \left( \mathbb{R}^6, \mathbb{R}^6 \right)\) by Lemma \[5\].

Note that when \(x_1 \in \mathbb{Z}\) and \(x_2 \in \{0, 1\}\), we have that (recall that \(h(x) = \delta_1\) for integer \(x\))

\[
(1 - h(x_1) + x_2) = \begin{cases} 
0 & x_1 = 0 \text{ and } x_2 = 0 \\
1 & x_1 \neq 0 \text{ and } x_2 = 0 \\
1 & x_1 = 0 \text{ and } x_2 = 1 \\
2 & x_1 \neq 0 \text{ and } x_2 = 1.
\end{cases} \tag{3.52}
\]

If we also have that \(x_3 \in \{0, \ldots, 2^m\}\), then it follows that

\[
\sigma(x_3 - 2^m(1 - h(x_1) + x_2)) = \begin{cases} 
x_3 & x_1 = 0 \text{ and } x_2 = 0 \\
0 & \text{otherwise.}
\end{cases} \tag{3.53}
\]

Thus, when applied to an input for which \(x_1 \in \mathbb{Z}\), \(x_2 \in \{0, 1\}\) and \(x_3 \in \{0, \ldots, 2^m\}\), the first network in \(\text{(3.50)}\) has the effect of adding \(x_3\) to \(x_3\) iff \(x_2 = 0\) and \(x_1 = 0\). In an analogous manner, the second network in \(\text{(3.51)}\) has the effect of subtracting \(x_3\) from \(x_3\) iff \(x_2 = 1\) and \(x_1 = 0\). Thus, composing both of these with the network in \(\text{(3.49)}\) and applying an affine map which drops the second and third components gives

\[
\begin{pmatrix}
  x \\
  z
\end{pmatrix}
\mapsto
\begin{pmatrix}
  x \\
  \frac{0, b_1 \cdot b_2 \cdot b_3 \cdot \ldots \cdot b_m f_1 f_2 f_3 \cdot \ldots \cdot t_4 s f_{4S}}{z + \delta_1(-1)^{b_1 b_2 b_3 \cdots b_m}}
\end{pmatrix}
\in \mathbb{R}^{15,4m+8} \left( \mathbb{R}^3, \mathbb{R}^3 \right). \tag{3.54}
\]
Recalling the encoding of \( t_1 \) (using the first bit to determine the sign), we thus get

\[
\begin{bmatrix}
0.t_1t_2t_3f_3s_4s_4 \\
0.f_1t_2t_3s_4s_4
\end{bmatrix} \to \begin{bmatrix}
0.f_1t_2t_3f_3s_4s_4 \\
0.f_1t_2t_3s_4s_4
\end{bmatrix} \in \mathbb{Y}_{15,4m+8}^4(\mathbb{R}^3, \mathbb{R}^3).
\]  
(3.55)

Finally, we extract the \( f_1 \) using the bit extractor network \( f_{a_1} \) from Proposition 4 and subtract this from \( x \) to get

\[
\begin{bmatrix}
x \\
x \\
x
\end{bmatrix} \to \begin{bmatrix}
x \\
x \\
x
\end{bmatrix} \to \begin{bmatrix}
x \\
x \\
x
\end{bmatrix} \in \mathbb{Y}_{13,4}^4(\mathbb{R}^3, \mathbb{R}^3)
\]  
(3.56)

Composing this with the network in (3.55) we get a network \( d \in \mathbb{Y}_{15,4m+12}^4(\mathbb{R}^3, \mathbb{R}^3) \). We now complete the construction as in the case where \( M \leq N \) to get a network \( g \in \mathbb{Y}_{17,4}^{17}(\mathbb{R}, \mathbb{R}) \) with depth

\[
L = 3\sqrt{N(4 + \log(M/N))} + 4p + 2 + 4S(4m + 12)
\]  
(3.57)

such that \( g(n) = x_n \) for \( n = 1, ..., N \). Recalling that in this case \( p \leq 2\sqrt{N(4 + \log(M/N))} \) and \( m = 2 + \lfloor \log(M/N) \rfloor \) was the length of the \( t \)-block, we get

\[
L \leq C\sqrt{N(1 + \log(M/N))}.
\]  
(3.58)

This completes the proof when \( N < M \).

\section{Optimal Approximation of Sobolev Functions Using Deep ReLU Networks}

In this section, we give the main construction and proof of Theorem 1. A key component of the proof is the approximation of piecewise polynomial functions using deep ReLU neural networks. To describe this, we first introduce the some notation.

Throughout this section, unless otherwise specified, let \( b \geq 2 \) be a fixed integer. To avoid excessively cumbersome notation, we suppress the dependence on \( b \) in the following notation. Let \( l \geq 0 \) be an integer and consider the \( b \)-adic decomposition of the cube \( \Omega = [0, 1]^d \) (note that by removing a zero-measure set it suffices to consider this half-open cube in the proof) at level \( l \) given by

\[
\Omega = \bigcup_{i \in I} \Omega_i^l,
\]  
(4.1)

where the index \( i \) lies in the index set \( I := \{0, ..., b^l - 1\}^d \), and \( \Omega_i^l \) is defined by

\[
\Omega_i^l = \prod_{j=1}^d [b^{-l}i_j, b^{-l}(i_j + 1)].
\]  
(4.2)

Note that for each \( l \), the \( b^l \) subcubes \( \Omega_i^l \) form a partition of the original cube \( \Omega \). For an integer \( k \geq 0 \), we let \( P_k \) denote the space of polynomials of degree at most \( k \) and consider the space

\[
P_k^l = \{ f : \Omega \to \mathbb{R}, f_{|\Omega_i^l} \in P_k \text{ for all } i \in I \}
\]  
(4.3)

of (non-conforming) piecewise polynomials subordinate to the partition \( [4.1] \). The space \( P_k^l \) has dimension \( (d + k)b^l \) and a natural \( (L^\infty \text{-normalized}) \) basis

\[
\rho_{\alpha}^l(x) = \prod_{j=1}^d (b^l x_j - i_j)^{\alpha_j} \quad x \in \Omega_i^l
\]  
(4.4)

indexed by \( i \in I \) and \( \alpha \) a \( d \)-dimensional multi-index with \( |\alpha| \leq k \).

In our construction, we will approximate piecewise polynomial functions from \( P_k^l \) by deep ReLU neural networks. However, since a deep ReLU network can only represent a piecewise continuous function, this approximation will not be over the full cube \( \Omega \). Rather, we will need to remove an arbitrarily small region from \( \Omega \). This idea is from the method in [39], where this region was called the trifling region. Given \( \varepsilon > 0 \) we define sets

\[
\Omega_i^{l, \varepsilon} = \prod_{j=1}^d \begin{cases} [b^{-l}i_j, b^{-l}(i_j + 1)] - \varepsilon & i_j < b^l - 1 \\ [b^{-l}i_j, b^{-l}(i_j + 1)) & i_j = b^l - 1 \end{cases}
\]  
(4.5)
which are slightly shrunk sub-cubes (except at one edge) from (4.2). We then define the good region to be
\[ \Omega_{l,x} := \bigcup_{i \in [l]} \Omega_{l,x}^i. \] (4.6)
Next, we will show how to approximate piecewise polynomials from \( \mathcal{P}_d \) on the set \( \Omega_{l,x} \). For this, we begin with the following Lemma, which first appears in [39]. We give a detailed proof for the reader’s convenience.

**Lemma 7.** Let \( l \geq 0 \) be an integer and \( 0 < \varepsilon < b^{-l} \). Then there exists a deep ReLU neural network \( q_d \in \mathcal{Y}^d,2(b^{-l})l(\mathbb{R}^d) \) such that
\[ q_d(\Omega_{l,x}) = \text{ind}(i) := \sum_{j=1}^b b^{(j-1)}i_j. \] (4.7)
Note that here \( \text{ind}(i) \in \{0, \ldots, b^d - 1\} \) is just an integer index corresponding to the sub-cube position \( i \).

**Proof.** Start with the following piecewise linear function
\[ g_\varepsilon(x) := \begin{cases} 0 & x \leq 0 \\ (j + \varepsilon^{-1}(x - j/b)) & j/b - \varepsilon < x \leq j/b, \text{ for } j = 1, \ldots, b - 1 \\ j & j/b < x \leq (j + 1)/b - \varepsilon, \text{ for } j = 0, \ldots, b - 1 \\ b - 1 & x > 1. \end{cases} \] (4.8)
Note that this function has \( 2b - 1 \) pieces and so by Proposition[2] we have \( g_\varepsilon \in \mathcal{Y}^5,2(b^{-l})(\mathbb{R}) \).

We set \( x_0 = x \) and \( q_0 = 0 \) and consider the following recursion
\[ x_{n+1} = b(x_n - g_\varepsilon(x_n)), \quad q_{n+1} = bq_n + g_\varepsilon(x_n). \] (4.9)
It is easy to verify that if \( x_0 = x \in [jb^{-l}, (j + 1)b^{-l} - \varepsilon) \), then \( q_l = j \), since in this case all iterates \( x_n \in \cup_{j=1}^b (j/b - \varepsilon, j/b) \) so that \( g_\varepsilon \) extracts the first bit in the \( b \)-ary expansion of \( x_n \)
\[ g_\varepsilon(x_n) = j \quad \text{if} \quad j/b \leq x_n < (j + 1)/b. \]
In addition, when \( x_0 = x \in [1 - b^{-l}, 1] \), then \( q_l = b^l - 1 \) since \( g_\varepsilon(x_n) = b - 1 \) for all \( n \).

We implement this recursion using a deep ReLU network as follows. We begin with the affine map
\[ x \rightarrow \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \in \mathcal{Y}(\mathbb{R}, \mathbb{R}^3). \] (4.10)
Now, we use induction. Suppose that the map
\[ x \rightarrow \begin{pmatrix} x_n \\ g_\varepsilon(x_n) \\ q_n \end{pmatrix} \in \mathcal{Y}^d,2(b^{-l})n(\mathbb{R}, \mathbb{R}^3) \] (4.11)
has already been implemented. Then we use Lemmas[2] and[5] to apply \( g_\varepsilon \) to only the second coordinate. This gives the map
\[ x \rightarrow \begin{pmatrix} x_n \\ g_\varepsilon(x_n) \\ q_n \end{pmatrix} \in \mathcal{Y}^d,2(b^{-l})(\mathbb{R}, \mathbb{R}^3). \] (4.12)
Finally, we compose with the affine map
\[ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} 2(x - y) \\ 2(x - y) \\ 2z + y \end{pmatrix} \in \mathcal{Y}(\mathbb{R}^3, \mathbb{R}^3) \] (4.13)
to complete the inductive step. After \( l \) steps of induction, we then compose with the affine map which selects the last coordinate to get the network \( q_1 \in \mathcal{Y}^d,2(b^{-l})l(\mathbb{R}). \)

For higher dimensional cubes \( \Omega = [0,1]^d \), we construct an indexing network \( q_d \in \mathcal{Y}^d,2(b^{-l})l(\mathbb{R}^d) \). We use Lemma[3] to apply \( q_1 \) to each coordinate of the input. Then, we compose with the affine map
\[ x \rightarrow \sum_{j=1}^d b^{(j-1)}x_j \] (4.14)
We now compose with the affine map
\[
q_d(\Omega_{i,x}) = \text{ind}(i) := \sum_{j=1}^{d} b^{(j-1)i} x_j.
\] (4.15)

Using this Lemma we prove the following key technical Proposition, which shows how to efficiently approximate piecewise polynomial functions on the good set \(\Omega_{i,x}\).

**Proposition 6.** Let \(l \geq 0\) be an integer and \(\varepsilon > 0\). Suppose that \(f \in P^l_k\) is expanded in terms of the bases \(\rho_{l,i}^\alpha\)
\[
f(x) = \sum_{|\alpha| \leq k, i \in I_l} a^\alpha_i \rho_{l,i}^\alpha(x).
\] (4.16)

Let \(1 \leq q \leq p \leq \infty\) and choose a parameter \(\delta > 0\) and an integer \(m \geq 1\). Then there exists a deep ReLU network \(f_{\delta,m} \in \mathcal{Y}^{2d+18L}(\mathbb{R}^d)\) such that
\[
\|f - f_{\delta,m}\|_{L^p(\Omega_{i,x})} \leq C \left( \delta \min\left\{ 1, b^{-dl} \delta^{-q} \right\}^{1/p} + 4^{-m} \right) \left( \sum_{|\alpha| \leq k} |a^\alpha_i|^q \right)^{1/q}
\] (4.17)

(with the standard modification when \(q = \infty\)), and whose depth satisfies
\[
L \leq C \left\{ m + l + \delta^{-q/2} \sqrt{1 + dl \log(b) + q \log(\delta)} \right\} \delta^{-q} \leq b^{dl} \delta^{-q} > b^{dl}.
\] (4.18)

Here the constants \(C := C(p,q,d,k,b)\) only depend upon \(p,q,d,k\) and the base \(b\), but not on \(f, \delta, l, \varepsilon, \) or \(m\).

**Proof.** We begin by decomposing \(f = \sum_{|\alpha| \leq k} f_\alpha\) where
\[
f_\alpha(x) = \sum_{i \in I_l} a^\alpha_i \rho_{l,i}^\alpha(x).
\] (4.19)

By Proposition 1 and the triangle inequality, it suffices to prove the result for each \(f_\alpha\) individually with width \(W = 20d + 17\) (at the expense of larger constants). So in the following we assume that \(f = f_\alpha\) and write \(a_\alpha := a^\alpha_i\). By normalizing \(f\) we may assume also without loss of generality that
\[
\left( \sum_{i \in I_l} |a_\alpha|^q \right)^{1/q} \leq 1.
\] (4.20)

We construct the following network. First, duplicate the input \(x \in \mathbb{R}^d\) three times using an affine map
\[
x \rightarrow \begin{pmatrix} x \\ x \\ x \end{pmatrix} \in \mathcal{Y}^0(\mathbb{R}^d, \mathbb{R}^{3d}).
\] (4.21)

Next, we use Lemmas 5 and 2 to apply the network \(q_d\) from Lemma 7 to the last coordinate and apply \(q_1\) from Lemma 7 to each entry of the first coordinate to get
\[
x \rightarrow \begin{pmatrix} q_1(x_1) \\ \vdots \\ q_1(x_d) \\ x \\ q_d(x) \end{pmatrix} \in \mathcal{Y}^{20d,2d}(\mathbb{R}^d, \mathbb{R}^{2d+1}).
\] (4.22)

We now compose with the affine map
\[
\begin{pmatrix} x \\ y \\ r \end{pmatrix} \rightarrow \begin{pmatrix} b^{l} y - x \\ r \end{pmatrix} \in \mathcal{Y}^0(\mathbb{R}^{2d+1}, \mathbb{R}^{d+1}),
\] (4.23)
where \( x,y \in \mathbb{R}^d \) and \( r \in \mathbb{R} \), to get
\[
x \rightarrow \begin{pmatrix} b^t x_1 - q(x_1) \\ \vdots \\ b^t x_d - q(x_d) \\ q_d(x) \end{pmatrix} \in \mathcal{Y}^{20d,2(b-1)l}(\mathbb{R}^d,\mathbb{R}^{d+1}).
\] (4.24)

On the set \( \Omega_{1,\delta} \) from (4.5) this map becomes
\[
x \rightarrow \begin{pmatrix} b^t x_1 - i_1 \\ \vdots \\ b^t x_d - i_d \\ \text{ind}(i) \end{pmatrix}.
\] (4.25)

The next step in the construction will be to approximate the coefficients \( a_i \). To do this we round the \( a_i \) down to the nearest multiple of \( \delta \) (in absolute value) to get approximate coefficients
\[
\tilde{a}_i := \delta \text{sgn}(a_i) \left\lfloor \frac{|a_i|}{\delta} \right\rfloor.
\] (4.26)

We estimate the \( \ell^p \)-norm of the error this incurs as follows. Write
\[
||a - \tilde{a}||_{\ell^p} = \left( \sum_{i \in I} |a_i - \tilde{a}_i|^p \right)^{1/p}
\] (4.27)
with the standard modification when \( p = \infty \). Note that
\[
||a - \tilde{a}||_{\ell^p} \leq ||a||_{\ell^p} \leq 1
\] (4.28)
by (4.20). In addition, it is clear from the rounding procedure that \( ||a - \tilde{a}||_{\ell^\infty} \leq \delta \). Hölder’s inequality thus implies that (since \( p \geq q \))
\[
||a - \tilde{a}||_p \leq ||a - \tilde{a}||_{\ell^q}^{q/p} ||a - \tilde{a}||_{\ell^\infty}^{1-q/p} \leq \delta^{1-q/p}.
\] (4.29)

On the other hand, using that |\( I \big| = b^d l \), we can use the bound \( ||a - \tilde{a}||_{\ell^\infty} \leq \delta \) to get
\[
||a - \tilde{a}||_p \leq b^dl^p \delta.
\] (4.30)

Putting these together, we get
\[
||a - \tilde{a}||_{\ell^p} \leq \delta \min\{b^dl^p, \delta^{-q}\}^{1/p}.
\] (4.31)

Next we construct a ReLU neural network which maps the index \( \text{ind}(i) \) to the rounded coefficients \( \tilde{a}_i \). For this Theorem 3 will be key. We set \( N = b^dl^p \) and write \( \tilde{a}_i = \delta x_{\text{ind}(i)} \) for a vector \( x \in \mathbb{Z}^N \) defined by
\[
x_{\text{ind}(i)} = \text{sgn}(a_i) \left\lfloor \frac{|a_i|}{\delta} \right\rfloor.
\] (4.32)

We proceed to estimate \( ||x||_{\ell^1} \). We observe that by (4.20)
\[
\sum_{i=1}^N |x_i|^q \leq \sum_{i \in I} \left( \frac{|a_i|}{\delta} \right)^q \leq \delta^{-q}.
\] (4.33)

Thus \( ||x||_{\ell^q} \leq \delta^{-1} \). Moreover, since \( x \in \mathbb{Z}^N \), (4.33) implies that the number of non-zero entries in \( x \) satisfies
\[
||i : x_i \neq 0|| \leq \min\{\delta^{-q},N\}.
\] (4.34)

We can thus use Hölder’s inequality to get the bound
\[
||x||_{\ell^1} \leq ||i : x_i \neq 0||^{1-1/q} ||x||_{\ell^q} \leq \delta^{-1} \min\{\delta^{-q},N\}^{1-1/q}.
\] (4.35)

Using this we apply Theorem 3 with \( M = \delta^{-1} \min\{\delta^{-q},N\}^{1-1/q} \) to the vector \( x \). We calculate that if \( \delta^{-q} \leq N \), then
\[
M = \delta^{-1} \delta^{-q(1-1/q)} = \delta^{-q} \leq N,
\] (4.36)
while if \( \delta^{-q} > N \), then
\[
M = \delta^{-1} N^{(1-1/q)} = N(\delta^{-q}N^{-1})^{1/q} \geq N.
\] (4.37)
Thus, Theorem 3 (combined with a scaling by $\delta$) gives a network $g \in \mathcal{Y}^{17L}(\mathbb{R})$ such that $g(\text{ind}(i)) = \bar{a}_i$, whose depth is bounded by

$$L \leq C \left\{ \frac{\delta^{-q/2} \sqrt{1 + dL \log(b) + q \log(\delta)}}{b^{dL/2} \sqrt{1 - \log \delta - (dL/q) \log(b)}} \right\} \frac{\delta^{-q}}{\delta^{-q} > b^{dL}}. \quad (4.38)$$

Using Lemma 5 to apply $g$ to the last coordinate of the output in (4.24) gives a network $\tilde{f}_\delta \in \mathcal{Y}^{2d+17L}$ with depth bounded by

$$L \leq 2(b-1)l + C \left\{ \frac{\delta^{-q/2} \sqrt{1 + dL \log(b) + q \log(\delta)}}{b^{dL/2} \sqrt{1 - \log \delta - (dL/q) \log(b)}} \right\} \frac{\delta^{-q}}{\delta^{-q} > b^{dL}}, \quad (4.39)$$

such that for $x \in \Omega_{L}$ we have

$$\tilde{f}_\delta(x) = \begin{pmatrix} b'x_1 - i_1 \\ \vdots \\ b'x_d - i_d \\ \bar{a}_i \end{pmatrix}. \quad (4.40)$$

Finally, to obtain the network $f_{\delta,m}$ we use Lemma 2 to compose $\tilde{f}_\delta$ with a network $P_m$ which approximates the product

$$\begin{pmatrix} z_1 \\ \vdots \\ z_d \\ z_{d+1} \end{pmatrix} \rightarrow z_{d+1} \prod_{j=1}^{d} z_j^{\alpha_j} \quad (4.41)$$

on the set where $|z_j| \leq 1$ for all $j = 1, \ldots, d + 1$. Note from the bound (4.20) we see that $|\bar{a}_i| \leq |a_i| \leq ||a||_\infty \leq 1$. In addition, it is easy to see that for $x \in \Omega_{L}$ we have $|b'x_j - i_j| \leq 1$ for $j = 1, \ldots, d$. Thus the output of $\tilde{f}_\delta$ satisfies these assumptions for any $x \in \Omega_{L}$.

We construct the network $P_m$ using Proposition 3 as follows. Choose a parameter $m \geq 1$. We first approximate a function which multiplies the last entry $z_{d+1}$ by the $i$-th entry $z_i$. We do this by duplicating the $i$-th entry using an affine map and then applying Lemma 5 to apply the network $f_m$ from Proposition 3 to the $i$-th and last entries

$$\begin{pmatrix} z_1 \\ \vdots \\ z_d \\ z_{d+1} \end{pmatrix} \rightarrow \begin{pmatrix} z_1 \\ \vdots \\ z_d \\ z_{d+1} \end{pmatrix} \rightarrow \begin{pmatrix} z_1 \\ \vdots \\ z_d \\ f_m(z_{d+1}, z_i) \end{pmatrix} \in \mathcal{Y}^{2d+13.6m+3}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1}). \quad (4.42)$$

In order to ensure that the resulting approximate product is still bounded in magnitude by 1 (so that we can recursively apply these products), we apply the map $z \rightarrow \max(\min(z, -1), 1) \in \mathcal{Y}^{5.2}(\mathbb{R})$ to the last component. This gives a network $P_m^m \in \mathcal{Y}^{2d+13.6m+5}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1})$, which maps

$$P_m : \begin{pmatrix} z_1 \\ \vdots \\ z_d \\ z_{d+1} \end{pmatrix} \rightarrow \begin{pmatrix} z_1 \\ \vdots \\ z_d \\ f_m(z_{d+1}, z_i) \end{pmatrix}, \quad (4.43)$$

where $f_m(z_{d+1}, z_i) = \max(\min(f_m(z_{d+1}, z_i), -1), 1)$. Observe that since the true product $z_{d+1}z_i \in [-1, 1]$ the truncation cannot increase the error, so that Proposition 3 implies

$$|f_m(z_{d+1}, z_i) - z_{d+1}z_i| \leq |f_m(z_{d+1}, z_i) - z_{d+1}z_i| \leq 6 \cdot 4^{-m}. \quad (4.44)$$

We construct $P_m$ by composing (using Lemma 2) $\alpha_j$ copies of $P_m$ and then applying an affine map which selects the last coordinate. Thus $P_m \in \mathcal{Y}^{2d+13L}(\mathbb{R}^{d+1})$ with $L \leq k(6m + 5)$. Moreover, since all entries $z_i$ are bounded by 1, we calculate that

$$\left| P_m(x) - z_{d+1} \prod_{j=1}^{d} z_j^{\alpha_j} \right| \leq \sum_{j=1}^{d} \alpha_j |f_m(z_{d+1}, z_j) - z_{d+1}z_i| \leq 6k \cdot 4^{-m}. \quad (4.45)$$
We obtain the network \( f_{\delta,m} \in Y^{2d+17,L}(\mathbb{R}^d, \mathbb{R}) \) by composing \( f_\delta \) and \( P_m \) using Lemma 2. Its depth is bounded by
\[
L \leq 2(b - 1)l + k(6m + 5) + C \left\{ \delta^{-q/2} \sqrt{1 + dl \log(b) + q \log(\delta)} \right\},
\]
(4.46)
and we note that \( k(6m + 5) \leq Cm \) for integers \( m \geq 1 \) and a constant \( C := C(k) \) which depends upon \( k \).

We bound the error using equations (4.31), (4.40), (4.45), and the fact that the basis \( \rho_{\alpha}^q \) is normalized in \( L_\infty \) and has disjoint support for fixed \( \alpha \) to get
\[
\| f - f_{\delta,m} \|_{L_p(\Omega_{\epsilon}, \epsilon)}^p \leq 2^{-ld} \sum_{I \in \mathcal{L}} |a_i - \tilde{a}_i|^p + (6k \cdot 4^{-m})p,
\]
(4.47)
so that
\[
\| f - f_{\delta,m} \|_{L_p(\Omega_{\epsilon}, \epsilon)} \leq 2^{-ld/p} \| a - \tilde{a} \|_{L_p} + 6k \cdot 4^{-m} \leq C \left( \delta \min \left\{ 1, 2^{-di \delta^{-q}} \right\} \right)^{1/p} + 4^{-m},
\]
(4.48)
which completes the proof.

Next, we use the construction in Proposition 6 to approximate a target function \( f \in W^s(L_q(\Omega)) \) in \( L_p(\Omega) \) using deep ReLU neural networks, again removing an arbitrarily small trifling set in the spirit of [39].

**Proposition 7.** Let \( \Omega = [0,1]^d \), \( 1 < q \leq p \leq \infty \) and \( f \in W^s(L_q(\Omega)) \) with \( \| f \|_{W^s(L_q(\Omega))} \leq 1 \) for \( s > 0 \). Suppose that the Sobolev embedding condition is strictly satisfied, i.e.
\[
\frac{1}{q} - \frac{1}{p} - \frac{s}{d} < 0,
\]
(4.49)
which guarantees that the the compact embedding \( W^s(L_q(\Omega)) \subset L_p(\Omega) \) holds. Let \( \epsilon > 0 \) and \( l_0 \geq 1 \) be an integer and set \( l^* = \lceil \kappa l_0 \rceil \) with
\[
\kappa := \frac{s}{s + d/p - d/q}.
\]
(4.50)
Note that \( 1 \leq \kappa < \infty \) by the Sobolev embedding condition. Then there exists a network \( f_{l_0,\epsilon} \in Y^{24d+20,L}(\mathbb{R}^d) \) such that
\[
\| f - f_{l_0,\epsilon} \|_{L_p(\Omega_{\epsilon}, \epsilon)} \leq Cb^{-l_0}
\]
(4.51)
and whose depth is bounded by
\[
L \leq Cb^{l_0/2}.
\]
(4.52)
Here the constants \( C := C(s,p,q,d,b) \) do not depend upon \( l_0, f \) or \( \epsilon \).

**Proof.** For a function \( f \in L_q(\Omega) \), we write
\[
\Pi_k^l(f) = \arg \min_{p \in \mathbb{P}_k^l} \| f - p \|_{L_p(\Omega)}
\]
for the \( L_q \)-projection of \( f \) onto the space of piecewise polynomials of degree \( k \). We will utilize the following well-known multiscale dyadic decomposition of the function \( f \), which is a common tool in harmonic analysis [8, 28, 31] and the analysis of multigrid methods [10],
\[
f = \sum_{l=0}^{m} f_l,
\]
(4.54)
where the components at level \( l \) are defined by \( f_0 = \Pi_k^0(f) \) and \( f_l = \Pi_k^l(f) - \Pi_k^{l-1}(f) \) for \( l \geq 1 \). Expanding the components \( f_l \) in the basis \( \rho_{\alpha}^q \), we write
\[
f_l(x) = \sum_{|\alpha| = k, \xi = b} a^q_{\alpha,l} \rho_{\alpha,l}^q(x).
\]
(4.55)
Utilizing the Bramble-Hilbert lemma [9] and a well-known scaling argument, we will prove the coefficient bound
\[
|a^q_{\alpha,l}| \leq Cb^{d/q - s} \| f \|_{W^s(L_q(\Omega_{l-1}^{l-1})))},
\]
(4.56)
where \( \Omega_{l-1}^{l-1} \supset \Omega_l^l \) is the parent domain of \( \Omega_l^l \) when \( l \geq 1 \). When \( l = 0 \), we have the simple modification
\[
|a^q_{\alpha,0}| \leq C \| f \|_{W^s(L_q(\Omega))}.
\]
Indeed, for \( l \geq 1 \) consider the scaling map \( S_{l, j} \) which scales the small domain \( \Omega_{l+1}^{j-1} \) up to the large domain \( \Omega_j \), defined by
\[
S_{l, j}(f)(x) = f(b^{l-1}x - i^j) \in L^q(\Omega_j) \tag{4.57}
\]
for \( f \in L^q(\Omega_{l+1}^{j-1}) \). We easily verify that
\[
\|S_{l, j}(f)|_{W^s(\Omega_j(\Omega_j))} = b^{(d/q-s)(l-1)}f|_{W^s(\Omega_j(\Omega_j))}
\]
\[
S_{l, j}(\Pi_k^j(f) - \Pi_k^{j-1}(f)) = \Pi_k^j(S_{l, j}(f)) - \Pi_k^{j-1}(S_{l, j}(f))
\tag{4.58}
\]
\[
S_{l, j}(\rho_{k}^j) = \rho_{k}^{j-1},
\]
where \( j \in \{0, 1\}^d \) is the index of \( \Omega_j \) in \( \Omega_{l+1}^{j-1} \), i.e. \( j \equiv i \) (mod \( p \)). From these facts we deduce that it suffices to prove (4.56) when \( l = 1 \).

To prove (4.56) when \( l = 1 \), we use the Bramble-Hilbert lemma [9]. We calculate using the Bramble-Hilbert lemma that
\[
\|\Pi_k^j(f) - f\|_{L^q(\Omega_j^1)} \leq \|\Pi_k^j(f) - f\|_{L^q(\Omega_j)} \leq C|f|_{W^s(\Omega_j(\Omega_j))} \tag{4.59}
\]
Combining these two estimates, we get
\[
\|\Pi_k^j(f) - f\|_{L^q(\Omega_j^1)} \leq C|f|_{W^s(\Omega_j(\Omega_j))} \tag{4.60}
\]
When \( l = 0 \) we make the modification
\[
\|\Pi_k^j(f)\|_{L^q(\Omega_j^1)} \leq \|f\|_{L^q(\Omega_j)} + \|\Pi_k^j(f) - f\|_{L^q(\Omega_j)} \leq C\|f\|_{W^s(\Omega_j(\Omega_j))} \tag{4.61}
\]
Now we use the fact that all norms on the finite dimensional space of polynomials of degree at most \( k \) are equivalent, which implies (4.56).

From (4.56) we deduce the following bound on the coefficients of \( f_i \).
\[
\left( \sum_{|\alpha| \leq k, i \in I} |d_{\alpha}^{q, j}|^q \right)^{1/q} \leq C b^{(d/q-s)l} \left( \sum_{|\alpha| \leq k, i \in I} |f|^q_{W^s(\Omega_{l+1}^{j-1})} \right)^{1/q} \leq C b^{(d/q-s)l}, \tag{4.62}
\]
since \( \|f\|_{W^s(\Omega_j(\Omega_j))} \leq 1 \). This follows from the sub-additivity of the Sobolev norm,
\[
\sum_{i \in I \setminus I} |f|^q_{W^s(\Omega_{l+1}^{j-1})} \leq |f|^q_{W^s(\Omega_j(\Omega_j))},
\]
since each \( \Omega_{l+1}^{j-1} \) appears a finite number of times (namely \( (k+1)b^l \)) in the sum (4.62). Note that it also holds when the standard modifications are made for \( q = \infty \). We also note the following bound which follows from (4.62), the \( L^\infty \)-normalization of the basis functions \( \rho_{k+1}^j \), the fact that for fixed \( \alpha \) the functions \( \rho_{k+1}^j \) have disjoint support, and the assumption that \( p \geq q \)
\[
\|f_i\|_{L^p(\Omega_j^i)} \leq \sum_{|\alpha| \leq k} b^{-d/|\alpha|} \left( \sum_{i \in I} |d_{\alpha}^{q, j}|^p \right)^{1/p} \leq b^{-d/|\alpha|} \left( k + d \right)^{1-1/p} \left( \sum_{|\alpha| \leq k, i \in I} |d_{\alpha}^{q, j}|^p \right)^{1/p} \leq C b^{(d/q-d/p-s)l}. \tag{4.63}
\]
To complete the proof, we choose \( \tau > 0 \) such that
\[
\frac{d}{q} - \frac{d}{p} - \tau + \left( 1 - \frac{q}{p} \right) \tau < 0.
\]
Note that this condition is can be satisfied since \( q \leq p \) and the Sobolev embedding condition (4.49) holds. For each level \( l \) we choose
\[
\delta = \delta(l) = \begin{cases} 
b^{-dl/q + \tau(l-\delta)} & l \geq \delta \\
b^{-dl/q + (s+1)(l-\delta)} & l < \delta \end{cases}
\tag{4.64}
\]
and
\[
m = m(l) = \left\lfloor \frac{s \log(b)}{2} \right\rfloor l_0 + \left\lfloor \frac{d \log(b)}{2q} \right\rfloor l. \tag{4.65}
\]
Note that $\delta(l)^{-q} \leq b^{\ell l}$ when $l \geq l_0$ and $\delta(l)^{-q} < b^{\ell l}$ when $l < l_0$.

Define the network $f_{l_0,\varepsilon}$ using Proposition 1 to be

$$f_{l_0,\varepsilon} = \sum_{i=0}^{\ell l} f_{\delta(l),m(l)},$$

where $f_{\delta(l),m(l)}$ is constructed using Proposition 6 with parameters $\delta = \delta(l)$ and $m = m(l)$. Propositions 1 and 6 together imply that $f_{l_0,\varepsilon} \in \mathcal{Y}^{2d+20L}(\mathbb{R}^d)$ with

$$L \leq C \left( \sum_{j=0}^{\ell l} \frac{s \log(b)}{2} \bigg| l_0 + \left( \frac{d \log(b)}{2q} \right) + 1 \right) + b^{d l_0} \sum_{j=0}^{\ell l-1} b^{d(l-l_0)/2} \sqrt{1 + \log(b)(s+1)(l-l_0)} + b^{d l_0} \sum_{j=0}^{\ell l-1} b^{-\epsilon(l-l_0)/2} \sqrt{1 + \log(b)(d/q + \tau)(l-l_0)}.$$  (4.67)

Summing the series above (and noting that the latter two are bounded by convergent geometric series), we get

$$L \leq C((\ell l)^2 + b^{d l_0}/2).$$  (4.68)

Since $\ell l \leq \kappa l$ is a linear function of $l$, the quadratic term $(\ell l)^2 \leq (\kappa l)^2$ is dominated by the exponential second term. Thus we get $L \leq Cb^{d l_0}/2$.

Finally, we use Proposition 6 together with the bounds (4.62) and (4.63) and the observation that $\Omega_{l_0,\varepsilon} \supset \Omega_{l,\varepsilon}$ if $l \leq l'$ to get

$$\|f - f_{l_0,\varepsilon}\|_{L^p(\Omega_{l,\varepsilon})} \leq C \left( b^{-s l_0} \sum_{j=0}^{\ell l} b^{l(l-l_0)} + b^{-s l_0} \sum_{j=0}^{\ell l} b^{d/q - p - s + (1-q/p)\tau}(l-l_0) + b^{-s l_0} \sum_{j=0}^{\ell l} b^{-\epsilon l} + \sum_{j=0}^{\ell l} b^{d/q - p - s l} \right).$$  (4.69)

Our condition on $\tau$ implies that all of the geometric series above are summable, so we get

$$\|f - f_{l_0,\varepsilon}\|_{L^p(\Omega_{l,\varepsilon})} \leq C\left( b^{-s l_0} + b^{d/q - p - s l}\right).$$  (4.70)

Finally, we use the definition of $l'$ and $\kappa$ to see that

$$b^{d/q - p - s l} \leq Cb^{-s l_0}.$$  (4.71)

\[\square\]

Finally, we show how to remove the trifling region to give a proof of Theorem 1. This is a technical construction similar to the method in [30, 39, 40], but we significantly reduce the size of the required network (in particular the width no longer depends exponentially on the input dimension) by using the sorting network construction from Corollary 1.

**Proof of Theorem 1** We assume without loss of generality that $f \in W^s(L_q(\Omega))$ has been normalized, i.e. so that $\|f\|_{W^s(L_q(\Omega))} \leq 1$.

In order to remove the trifling region from the preceding construction we will make use of different bases $b$. Let $r$ be the smallest integer such that $2r \geq 2d + 2$ (so that $2r \leq 4d + 4$), set $m = 2r$, and set $b_i = p_i$ (the $i$-th prime number) for $i = 1, \ldots, m$.

Let $n \geq b_m$ be an integer. We will construct a network $f_L \in \mathcal{Y}^{30d+24L}(\mathbb{R}^d)$ such that

$$\|f - f_L\|_{L^p(\Omega)} \leq Cn^{-s}$$  (4.72)

with depth $L \leq Cn^{d/2}$, which will complete the proof.

For $i = 1, \ldots, m$, set $l_i = \lceil \log(n)/\log(b_i) \rceil$ to be the largest power of $b_i$ which is at most $n$, and write $l_i' = \lceil \kappa l_i \rceil$ where $\kappa$ is defined as in Proposition 1. Note that since the $p_i$ are all pairwise relatively prime, the numbers

$$S := \left\{ \frac{1}{p_1^{l_1'}}, \ldots, \frac{1}{p_1^{l_1'}}, \frac{1}{p_2^{l_2'}}, \ldots, \frac{1}{p_2^{l_2'}}, \frac{1}{p_3^{l_3'}}, \ldots, \frac{1}{p_3^{l_3'}}, \ldots, \frac{1}{p_m^{l_m'}}, \ldots, \frac{1}{p_m^{l_m'}} \right\}$$  (4.73)
are all distinct. Choose an \( \varepsilon > 0 \) which satisfies
\[
\varepsilon < \min_{x,y \in S} |x - y|,
\]
\( (4.74) \)
i.e. which is smaller than the distance between the two closest elements of \( S \). This \( \varepsilon \) has the property that any \( x \in [0, 1] \)
is contained in at most one of the sets
\[
[jp_i - l_i, jp_i - l_i - \varepsilon) \) for \( i = 1, \ldots, m \) and \( j = 1, \ldots, p_i - 1 \).
\]
\( (4.75) \)
This means that for any \( x \in \Omega \), we have \( x \notin \Omega_{l_i, \varepsilon}^j \) for at most \( d \) different values \( i \). Here \( \Omega_{l_i, \varepsilon}^j \) is the good region at level \( l_i^j \) with base \( b_i \). This holds since \( x \) has \( d \) coordinates and each coordinate can be contained in at most one bad set from \( \Omega_{l_i, \varepsilon}^j \).

We now use Proposition \( 7 \) setting \( l_0 = l_i \) and using an \( \varepsilon \) satisfying \( (4.74) \), to construct \( f_i \in \mathbb{Y}^{24d + 20L} (\mathbb{R}^d) \) which satisfies
\[
||f - f_i||_{L^p(\Omega_{l_i, \varepsilon}^j)} \leq C p_i^{-sl_i} \leq C n^{-s}
\]
and has depth bounded by
\[
L \leq C p_i^{dl_i/2} \leq C n^{d/2}.
\]
\( (4.76) \)
\( (4.77) \)
Finally, we construct the following network. We sequentially duplicate the input and apply the network \( f_i \) to the new copy using Lemma \( 5 \) to get
\[
x \rightarrow \begin{pmatrix} x \cr x \cr f_1 (x) \end{pmatrix} \rightarrow \begin{pmatrix} x \cr x \cr f_2 (x) \cr f_1 (x) \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x \cr x \cr \vdots \cr f_m (x) \cr f_1 (x) \end{pmatrix} \in \mathbb{Y}^{30d + 24L} (\mathbb{R}^d, \mathbb{R}^m)
\]
\( (4.78) \)
with \( L \leq C \Sigma_{i=1}^m p_i^{dl_i/2} \leq C n^{d/2} \).

We construct the network \( f_L \in \mathbb{Y}^{30d + 24L} (\mathbb{R}^d) \) by composing the network from \( (4.78) \) with the order statistic network which selects the median, i.e. the \( m/2 \)-largest value. By construction the network depth of \( f_L \) satisfies
\[
L \leq C n^{d/2} + \left( \frac{m + 1}{2} \right) \leq C n^{d/2},
\]
\( (4.79) \)
since \( \left( \frac{m + 1}{2} \right) \) is a constant independent of \( n \).

To bound the approximation error of \( f_L \) we introduce the following notation. Given \( x \in [0, 1]^d \), we write
\[
\mathcal{K}(x) = \{ i : x \in \Omega_{l_i, \varepsilon}^j \}
\]
\( (4.80) \)
for the set of indices such that \( x \) is contained in the good region for the base \( b_i \) decomposition. Since \( x \) fails to be in \( \Omega_{l_i, \varepsilon}^j \) for at most \( d \) values of \( i \), we get
\[
|\mathcal{K}(x)| \geq m - d \geq m/2 + 1
\]
\( (4.81) \)
since \( m \geq 2d + 2 \). Thus the \( m/2 \)-largest element among the \( f_1 (x), \ldots, f_m (x) \) is both smaller and larger than some element of \( \{ f_i (x), i \in \mathcal{K}(x) \} \), which implies
\[
\min_{i \in \mathcal{K}(x)} f_i (x) \leq f_L (x) \leq \max_{i \in \mathcal{K}(x)} f_i (x),
\]
\( (4.82) \)
so that
\[
|f_L (x) - f(x)| \leq \max_{i \in \mathcal{K}(x)} |f_i (x) - f(x)|.
\]
\( (4.83) \)
This completes the proof when \( p = \infty \), since if \( i \in \mathcal{K}(x) \) then \( |f_i (x) - f(x)| \leq C n^{-s} \) by Proposition \( 7 \) and the definition of \( \mathcal{K}(x) \).

For \( p < \infty \), we note that
\[
\int_{\Omega} |f_L (x) - f(x)|^p dx \leq \int_{\Omega} \max_{i \in \mathcal{K}(x)} |f_i (x) - f(x)|^p dx \leq \int_{\Omega} \sum_{i \in \mathcal{K}(x)} |f_i (x) - f(x)|^p dx
\]
\[
\leq \sum_{i=1}^m ||f_i - f||_{L^p(\Omega_{l_i, \varepsilon}^j)}^p \leq C n^{-sp}.
\]
\( (4.84) \)
Taking \( p \)-th roots completes the proof. \( \square \)
5 Lower Bounds

In this section, we study lower bounds on the approximation rates that deep ReLU neural networks can achieve on Sobolev spaces. Our main result is to prove Theorem 2 which shows that the construction of Theorem 1 is optimal in terms of the number of parameters. In addition, we show that the representation of sparse vectors proved in Theorem 3 is optimal.

The key concept is the notion of VC dimension, which was used in [39, 49] to prove lower bounds for approximation in the \( L_\infty \)-norm. We generalize these results to obtain sharp lower bounds on the approximation in \( L_p \) as well. Let \( K \) be a class of functions defined on \( \mathbb{R}^d \). The VC-dimension [45] of \( K \) is defined to be the largest number \( n \) such that there exists a set of points \( x_1, ..., x_n \in \Omega \) such that

\[
|\{(\sgn(g(x_1)),...,\sgn(g(x_n))), \ g \in K\}| = 2^n, \tag{5.1}
\]

i.e. such that every sign pattern at the points \( x_1, ..., x_n \) can be matched by a function from \( K \). Such a set of points is said to be shattered by \( K \).

The VC dimension of classes of functions defined by neural networks has been extensively studied and the most precise results are available for piecewise polynomial activation functions. We will discuss two main results concerning the VC dimension of \( \mathcal{Y}^{W,L}(\mathbb{R}^d) \). The first bound is most useful when the depth \( L \) is fixed and the width \( W \) is large and is given by

\[
\text{VC-dim}(\mathcal{Y}^{W,L}(\mathbb{R}^d)) \leq C(W^2L^2 \log(WL)). \tag{5.2}
\]

This was proved in Theorem 6 of [6]. The second bound, which is most informative when the width \( W \) is fixed and the depth \( L \) is large is

\[
\text{VC-dim}(\mathcal{Y}^{W,L}(\mathbb{R}^d)) \leq C(W^3L^2). \tag{5.3}
\]

This was proved in Theorem 8 of [6] using a technique developed in [19]. In either case, the VC-dimension of a deep ReLU neural network with \( P = O(W^2L) \) parameters is bounded by \( CP^2 \), with this bound achieved up to a constant only in the case where the width \( W \) is fixed and the depth \( L \) grows. This bound on the VC-dimension was used in [39, 49] to prove Theorem 2 in the case \( p = \infty \). However, in order to extend the lower bound to \( p < \infty \) a more sophisticated analysis is required. The key argument is captured in the following Proposition.

**Theorem 4.** Let \( p > 0, \Omega = [0, 1]^d \) and suppose that \( K \) is a translation invariant class of functions whose VC-dimension is at most \( n \). By translation invariant we mean that \( f \in K \) implies that \( f(\cdot - v) \in K \) for any fixed vector \( v \in \mathbb{R}^d \). Then there exists an \( f \in W^q(\mathcal{L}_\infty(\Omega)) \) such that

\[
\inf_{g \in K} \| f - g \|_{L^p(\Omega)} \geq C(d, p)n^{-\frac{n}{2q}} \| f \|_{W^q(\mathcal{L}_\infty(\Omega))}. \tag{5.4}
\]

Although the translation invariance holds for many function classes of interest, it is an interesting problem whether it can be removed. Before proving this result, we first show how Theorem 2 follows from this.

**Proof of Theorem 2.** Note that the class of deep ReLU networks \( \mathcal{Y}^{W,L}(\mathbb{R}^d) \) is translation invariant. Combining this with the VC-dimension bounds (5.2) and (5.3), Theorem 4 implies Theorem 2 in the case \( q = \infty \). The case \( q < \infty \) follows trivially since \( W^q(\mathcal{L}_\infty(\Omega)) \subseteq W^q(\mathcal{L}_q(\Omega)) \) for any \( q < \infty \). \( \square \)

Let us turn to the proof of Theorem 4. A key ingredient is the well-known Sauer-Shelah lemma [36, 38].

**Lemma 8** (Sauer-Shelah Lemma). Suppose that \( K \) has VC-dimension at most \( n \). Given any collection of \( N \) points \( x_1, ..., x_N \in \Omega \), we have

\[
|\{(\sgn(g(x_1)),...,\sgn(g(x_N))), \ g \in K\}| \leq \sum_{i=0}^{n} \binom{N}{i}. \tag{5.5}
\]

We will also utilize the following elementary bound on the size of a Hamming ball.

**Lemma 9.** Suppose that \( N > 2n \), then

\[
\sum_{i=0}^{n} \binom{N}{i} \leq 2^{NH(n/N)}, \tag{5.6}
\]

where \( H(p) \) is the entropy function

\[
H(p) = -p \log(p) - (1-p) \log(1-p). \tag{5.7}
\]

(Note that all logarithms here are taken base 2.)
Proof. Observe that since $N - n > n$, we have
\[
\left(\frac{N - n}{N}\right)^{N-n} \left(\frac{n}{N}\right)^n \sum_{i=0}^{n} \binom{N}{i} \leq \sum_{i=0}^{n} \binom{N}{i} \left(\frac{N - n}{N}\right)^{N-i} \left(\frac{n}{N}\right)^i < \sum_{i=0}^{N} \binom{N}{i} \left(\frac{N - n}{N}\right)^{N-i} \left(\frac{n}{N}\right)^i = 1. \tag{5.8}
\]
This means that
\[
\sum_{i=0}^{n} \binom{N}{i} \leq \left[ \left(\frac{N - n}{N}\right)^{N-n} \left(\frac{n}{N}\right)^n \right]^{-1}. \tag{5.9}
\]
Taking logarithms, we obtain
\[
\log \left( \sum_{i=0}^{n} \binom{N}{i} \right) \leq -N \left( \left(\frac{N - n}{N}\right) \log \left(\frac{N - n}{N}\right) + n \log \left(\frac{n}{N}\right) \right) = NH(n/N) \tag{5.10}
\]
as desired.

Utilizing these lemmas, we give the proof of Theorem 4.

Proof of Theorem 4 Let $c < 1/2$ be chosen so that $H(c) < 1/2$ (for instance $c = 0.1$ will work) and fix
\[
k := \lfloor \sqrt{n/c} \rfloor \leq C(d)n^{1/d},
\]
and $\epsilon := k^{-1}$. Next, we consider shifts of an equally spaced grid with side length $\epsilon$. Specifically, for each $\lambda \in [0, \epsilon]^d$, define the point set
\[
X_\lambda = \left\{ \lambda + \epsilon z, z \in \lfloor k \rfloor^d \right\}, \tag{5.11}
\]
where we have written $\lfloor k \rfloor := \{0, \ldots, k - 1\}$ for the set of integers from 0 to $k - 1$.

Let us now investigate the set of sign patterns which the class $K$ can match on $X_\lambda$. To do this, we will introduce some notation. For a function $g \in K$, we write
\[
\text{sgn}(g|_{X_\lambda}) \in \{\pm 1\}^{\lfloor k \rfloor^d}, \quad \text{sgn}(g|_{X_\lambda})(z) = \text{sgn}(g(\lambda + \epsilon z)) \tag{5.12}
\]
for the set of signs which $g$ takes at the (shifted) grid points $X_\lambda$. Here the vector $\text{sgn}(g|_{X_\lambda})$ is indexed by the coordinate $z \in \lfloor k \rfloor^d$ which specifies the location of a point in the shifted grid $X_\lambda$.

We write
\[
\text{sgn}(K|_{X_\lambda}) := \left\{ \text{sgn}(g|_{X_\lambda}) : g \in K \right\} \subset \{\pm 1\}^{\lfloor k \rfloor^d} \tag{5.13}
\]
for the set of sign patterns attained by the class $K$ on $X_\lambda$. Observe that since $K$ is assumed to be translation invariant, the set $\text{sgn}(K|_{X_\lambda})$ is independent of the shift $\lambda$. To see this, let $\lambda, \mu \in [0, \epsilon]^d$ be two different shifts and let $g \in K$. By the translation invariance, we find that the function $g'$ defined by
\[
g'(x) = g(x + \lambda - \mu)
\]
is also in $K$. We easily calculate that
\[
\text{sgn}(g|_{X_\lambda}) = \text{sgn}(g'|_{X_\mu}), \tag{5.14}
\]
which implies that $\text{sgn}(K|_{X_\lambda}) = \text{sgn}(K|_{X_\mu})$. In the following we simplify notation and write $\text{sgn}(K) \subset \{\pm 1\}^{\lfloor k \rfloor^d}$ for this set.

Next, we will show that there exists a choice of signs $\alpha \in \{\pm 1\}^{\lfloor k \rfloor^d}$ which differs from every element of $\text{sgn}(K)$ in a constant fraction of its entries. To do this, it is convenient to use the notion of the Hamming distance between two sign patterns, which is defined as the number of indices in which they differ, i.e.
\[
d_H(\alpha, \beta) := |\{ z \in \lfloor k \rfloor^d : \alpha(z) \neq \beta(z) \}|. \tag{5.15}
\]
We also use the notion of the Hamming ball of radius $m$ around a sign pattern $\alpha \in \{\pm 1\}^{\lfloor k \rfloor^d}$, which is defined to be the set of sign patterns which differ from $\alpha$ by at most $m$ entries, i.e.
\[
B_H(\alpha, m) = \{ \beta \in \{\pm 1\}^{\lfloor k \rfloor^d} : d_H(\alpha, \beta) \leq m \}. \tag{5.16}
\]
We note that Lemma 9 implies the following estimate on the size of $B_H(\alpha, m)$ when $2m < k^d$:

\[ |B_H(\alpha, m)| = \sum_{i=0}^{m} \binom{k^d}{i} \leq 2^{k^d H(m/k^d)}. \]  

(5.17)

Further, our assumption on the VC-dimension of $K$ combined with Lemmas 8 and 9 implies that

\[ |\text{sgn}(K)| \leq 2^{k^d H(n/k^d)} \leq 2^{k^d H(c)} < 2^{k^d/2} \]  

from our choice of $c$. If we choose $m := \lfloor c k^d \rfloor \leq c k^d$, it follows that

\[ \left| \bigcup_{\beta \in \text{sgn}(K)} B_H(\beta, m) \right| < 2^{k^d/2} 2^{k^d H(m/k^d)} < 2^{k^d/2} 2^{k^d H(c)} < 2^{k^d}, \]  

(5.19)

so that there must exist an $\alpha \in \{ \pm 1 \}^{k^d}$ such that

\[ \alpha \notin \bigcup_{\beta \in \text{sgn}(K)} B_H(\beta, m), \]

and hence

\[ \inf_{\beta \in \text{sgn}(K)} d_H(\alpha, \beta) \geq m + 1 \geq c k^d. \]  

(5.20)

Finally, we choose a compactly supported smooth positive bump function $\phi$ which vanishes outside of the unit cube $\Omega$ and consider the function

\[ f(x) = \sum_{z \in [k]^d} \alpha(z) \phi(kx - z). \]  

(5.21)

Since the supports of the functions $\phi(kx - z)$ are all disjoint, we calculate

\[ \|f\|_{W^s(L^\infty(\Omega))} = \|\phi(kx - z)\|_{W^s(L^\infty(\Omega))} = k^s \|\phi\|_{W^s(L^\infty(\Omega))}. \]  

(5.22)

Next, let $g \in K$ be arbitrary. We calculate

\[ \int_{\Omega} |f(x) - g(x)|^p dx = \int_{[0,e]^d} \sum_{z \in [k]^d} |f(\lambda + \varepsilon z) - g(\lambda + \varepsilon z)|^p d\lambda \]

\[ = \int_{[0,e]^d} \sum_{z \in [k]^d} |\alpha(z) \phi(k\lambda) - g(\lambda + \varepsilon z)|^p d\lambda. \]  

(5.23)

From equation (5.20) and the fact that $\text{sgn}(g|_{X_k}) \in \text{sgn}(K)$, we see that

\[ |\{ z \in [k]^d, \alpha(z) \neq \text{sgn}(g(\lambda + \varepsilon z)) \}| \geq c k^d. \]  

(5.24)

Further, if $\alpha(z) \neq \text{sgn}(g(\lambda + \varepsilon z))$, then we have the lower bound

\[ |\alpha(z) \phi(k\lambda) - g(\lambda + \varepsilon z)| \geq \phi(k\lambda) \]

since $\phi \geq 0$. This implies that for every $\lambda \in [0,e]^d$ we have the lower bound

\[ \sum_{z \in [k]^d} |\alpha(z) \phi(k\lambda) - g(\lambda + \varepsilon z)|^p \geq c k^d \phi(k\lambda)^p. \]  

(5.25)

We thus obtain

\[ \int_{\Omega} |f(x) - g(x)|^p dx \geq c k^d \int_{[0,e]^d} \phi(k\lambda)^p d\lambda = c \int_{\Omega} \phi(x)^p dx, \]  

(5.26)

from which we deduce

\[ \|f - g\|_{L^p(\Omega)} \geq c \phi^{\frac{1}{p}} \|\phi\|_{L^p(\Omega)}. \]  

(5.27)

Combining this with the bound (5.22), using that $k \leq C(d)n^{1/d}$, $\phi$ is a fixed function, and $g \in K$ was arbitrary, we get

\[ \inf_{g \in K} \|f - g\|_{L^p(\Omega)} \geq C(d, p) k^{-s} \|f\|_{W^{s,L^\infty(\Omega)}} \geq C(d, k) n^{-s/d} \|f\|_{W^{s,L^\infty(\Omega)}}, \]  

(5.28)

as desired. \qed
We conclude this section by proving that Theorem 3 is optimal up to a constant as long as the $\ell^1$-norm $M$ is not too large and not too small. Specifically, we have the following.

**Theorem 5.** Let $M,N \geq 1$ be integers and define

$$S_{N,M} = \{x \in \mathbb{Z}^N, \|x\|_1 \leq M\} \quad (5.29)$$

as in the proof of Theorem 3. Suppose that $W,L \geq 1$ are integers and that for any $x \in S_{N,M}$ there exists an $f \in \mathcal{Y}^{W,L}(\mathbb{R})$ such that $f(i) = x_i$ for $i = 1, \ldots, N$. Then there exists a constant $C < \infty$ such that if $C \log(N) < M \leq N$, then

$$W^3L^2 \geq C^{-1}M(1 + \log(M/N)), \quad (5.30)$$

and if $N \leq M < \exp(N/C)$, then

$$W^3L^2 \geq C^{-1}N(1 + \log(N/M)). \quad (5.31)$$

This result implies that if $\mathcal{Y}^{W,L}(\mathbb{R})$ can match the values of any vector in $S_{N,M}$ for $M$ in the range $(C \log(N), \exp(N/C))$, then the number of parameters must be larger than a constant multiple of the upper bound proved in Theorem 3. Thus Theorem 3 is sharp in this range. If $M < C \log(N)$ then piecewise linear functions with $O(M)$ pieces can fit $S_{N,M}$, and if $M > \exp(N/C)$ then piecewise linear functions with $O(N)$ pieces can fit $S_{N,M}$. This implies that Theorem 3 is no longer sharp outside this range.

**Proof.** Suppose first that $N/2 < M \leq 2N$, i.e. that $M$ is of the same order as $N$. For any subset $S \subset \{1, \ldots, N/2\}$ it is easy to construct an $x \in S_{N,M}$ such that $x_i > 0$ if $i \in S$. Thus the class $\mathcal{Y}^{W,L}(\mathbb{R})$ must shatter a set of size at least $N/2$ and the VC-dimension bound (5.3) implies the result.

In the case where $M << N$ or $M >> N$, the proof proceeds in a similar manner as the VC-dimension bounds from [6,19] although the VC-dimension cannot directly be used.

We begin with the case where $M \leq N/2$. We will bound the total number of sign patterns that $\mathcal{Y}^{W,L}(\mathbb{R})$ can match on the input set $X = \{1, \ldots, N\}$. For $i = 1, \ldots, L$, let $\epsilon_i \in \{0, 1\}^W$ be a sign pattern. Given an input $x \in X$ and a neural network with parameters $W_i$ and $b_i$, consider the signs of the following quantities

$$(AW_{0,b_0}(x))_j, j = 1, \ldots, W$$

$$(AW_{1,b_1} \circ \epsilon_1 \circ AW_{0,b_0}(x))_j, j = 1, \ldots, W$$

$$(AW_{2,b_2} \circ \epsilon_2 \circ AW_{1,b_1} \circ \epsilon_1 \circ AW_{0,b_0}(x))_j, j = 1, \ldots, W$$

$$\vdots$$

$$(AW_{L-1,b_{L-1}} \circ \epsilon_{L-1} \circ \cdots \circ \epsilon_2 \circ AW_{1,b_1} \circ \epsilon_1 \circ AW_{0,b_0}(x))_j, j = 1, \ldots, W$$

Here $\epsilon_i$ represents pointwise multiplication by the sign vector $\epsilon_i$. For any input $x \in \mathbb{R}$ the definition of the ReLU activation function implies that if we recursively set

$$\epsilon_i = \text{sgn}(AW_{i-1,b_{i-1}} \circ \epsilon_{i-1} \circ \cdots \circ \epsilon_2 \circ AW_{1,b_1} \circ \epsilon_1 \circ AW_{0,b_0}(x)),$$

then we will have

$$AW_{L,b_L} \circ \epsilon_L \circ \cdots \circ \epsilon_2 \circ AW_{1,b_1} \circ \epsilon_1 \circ AW_{0,b_0}(x) = AW_{L,b_L} \circ \sigma \cdots \circ \sigma \circ AW_{1,b_1} \circ \sigma \circ AW_{0,b_0}(x). \quad (5.34)$$

This implies that the signs of the quantities in (5.32) ranging over all sign vectors $\epsilon_1, \ldots, \epsilon_L \in \{0, 1\}^W$ uniquely determine the value of the network at $x$. Thus the number of sign patterns achieved on the set $X$ is bounded by the number of sign patterns achieved in (5.32) as $x$ ranges over the input set $X$, the $\epsilon_i$ range over the sign vectors $\{0, 1\}^W$, and the parameters $W_i, b_i$ range of the set of all real numbers. As the $\epsilon_i$ range over the sign vectors $\{0, 1\}^W$ and $x$ ranges over $X$, the quantities in (5.32) range over $N(WL+1)^{2W^2}$ polynomials in the $P \leq CW^2L$ parameter variables $W_i, b_i$ of degree at most $L$. We can thus use Warren’s Theorem ([47], Theorem 3) to bound the total number of sign patterns by

$$\left(\frac{4eLN(WL+1)^{2WL}}{P}\right)^P \leq (4eLN(WL+1)^{2WL})^{CW^2L}. \quad (5.35)$$

Suppose that $\mathcal{Y}^{W,L}(\mathbb{R})$ can match the values of any element in $S_{N,M}$. Since the set $S_{N,M}$ contains the indicator function of every subset of $\{1, \ldots, N\}$ of size $M$, we get that

$$\binom{N}{M} \leq (4eLN(WL+1)^{2WL})^{CW^2L}. \quad (5.36)$$
Taking logarithms, we get

\[
M \log(N/M) \leq CW^3L^2 + CW^2L \log(N) + CW^2L \log(4eLWL + 1)) \\
\leq CW^3L^2 + CW^2L \log(N).
\] (5.37)

Since \( M \leq N/2 \), we conclude that

\[
M(1 + \log(N/M)) \leq CM \log(N/M) \leq C \max\{W^3L^2, W^2L \log(N)\}.
\] (5.38)

In the next few equations, let \( C \) denote the constant in (5.38). Suppose that \( W^3L^2 < C^{-1}M(1 + \log(N/M)) \). Then equation (5.38) implies that

\[
W^2L \geq M \frac{1 + \log(N/M)}{C \log(N)}.
\] (5.39)

But this would mean that

\[
M \frac{1 + \log(N/M)}{C \log(N)} \leq W^2L = \sqrt{W^4L^2} \leq \sqrt{W^2L^2} < \sqrt{C^{-1}M(1 + \log(N/M))}.
\] (5.40)

Rearranging this, we get the inequality

\[
\sqrt{M} < \frac{\sqrt{C \log(N)}}{\sqrt{1 + \log(N/M)}}.
\] (5.41)

from which we deduce that \( M \leq C \log(N) \) for a (potentially larger) new constant \( C \).

Next, we consider the case where \( M > 2N \). In this case we consider the following modification of (5.32)

\[
(A_{W_{i,j}}(x))_{j=1,...,W} \\
(A_{W_{1,j}} \circ \varepsilon_1 \circ A_{W_{0,j}}(x))_{j=1,...,W} \\
(A_{W_{2,j}} \circ \varepsilon_2 \circ A_{W_{1,j}} \circ \varepsilon_1 \circ A_{W_{0,j}}(x))_{j=1,...,W} \\
\vdots \\
(A_{W_{L-1,j}} \circ \varepsilon_{L-1} \circ \cdots \circ \varepsilon_2 \circ A_{W_{1,j}} \circ \varepsilon_1 \circ A_{W_{0,j}}(x))_{j=1,...,W} \\
A_{W_{L,j}} \circ \varepsilon_L \circ A_{W_{L-1,j}} \circ \varepsilon_{L-1} \circ \cdots \circ \varepsilon_2 \circ A_{W_{1,j}} \circ \varepsilon_1 \circ A_{W_{0,j}}(x) - k, k = 0, \ldots, \lceil M/N \rceil.
\] (5.42)

The number of sign patterns that can be obtained as \( x \) ranges over \( X \), the \( \varepsilon_i \) range over \( \{0,1\}^W \), and the parameters range over the set of real numbers is bounded (using Warren’s Theorem [47]) by

\[
\left( \frac{4eLN(WL + M)2^{WL}}{P} \right)^P \leq (4eLN(WL + M)2^{WL})^{CW^2L}.
\] (5.43)

However, the set \( S_{N,M} \) contains all \((\lceil M/N \rceil + 1)^{N-1}\) vectors whose first \( N - 1 \) coordinates are arbitrary integers in \( \{0,1,\ldots,\lceil M/N \rceil\} \) and whose last coordinate is chosen to make the \( \ell^1 \)-norm equal to \( M \). Thus, setting \( \varepsilon_i \) recursively according to (5.33), we see that if every vector in \( S_{N,M} \) can be represented by an element of \( Y^{WL}(\mathbb{R}) \), then

\[
(\lceil M/N \rceil + 1)^{N-1} \leq (4eLN(WL + M)2^{WL})^{CW^2L}.
\] (5.44)

Taking logarithms and calculating in a similar manner as before, we get

\[
N(1 + \log(M/N)) \leq CW^3L^2 + CW^2L \log(M).
\] (5.45)

As before we now deduce that if \( W^3L^2 < C^{-1}N(1 + \log(M/N)) \), then \( N \leq C \log(M) \).

\[\square\]

### 6 Acknowledgements

We would like to thank Professor Ron DeVore for suggesting this problem, and Professors Andrea Bonito, Geurgana Petrova, Zuowei Shen, George Karniadakis, and Jinchao Xu for helpful comments while preparing this manuscript. This work was supported by the National Science Foundation (DMS-2111387 and CCF-2205004).
References

[1] El Mehdi Achour, Armand Foucault, Sébastien Gerchinovitz, and François Malgouyres, *A general approximation lower bound in $L^p$ norm, with applications to feed-forward neural networks*, arXiv preprint arXiv:2206.04360 (2022).

[2] Miklós Ajtai, János Komlós, and Endre Szemerédi, *An $O(n \log n)$ sorting network*, Proceedings of the fifteenth annual ACM symposium on Theory of computing, 1983, pp. 1–9.

[3] Raman Arora, Amitabh Basu, Poorya Mianjy, and Anirbit Mukherjee, *Understanding deep neural networks with rectified linear units*, International Conference on Learning Representations, 2018.

[4] Francis Bach, *Breaking the curse of dimensionality with convex neural networks*, The Journal of Machine Learning Research 18 (2017), no. 1, 629–681.

[5] Peter Bartlett, Vitaly Maiorov, and Ron Meir, *Almost linear $vc$ dimension bounds for piecewise polynomial networks*, Advances in neural information processing systems 11 (1998).

[6] Peter L. Bartlett, Nick Harvey, Christopher Liaw, and Abbas Mehrabian, *Nearly-tight $vc$-dimension and pseudodimension bounds for piecewise linear neural networks*, The Journal of Machine Learning Research 20 (2019), no. 1, 2285–2301.

[7] Kenneth E Batcher, *Sorting networks and their applications*, Proceedings of the April 30–May 2, 1968, spring joint computer conference, 1968, pp. 307–314.

[8] Mikhail Shlemovich Birman and Mikhail Zakharovich Solomyak, *Piecewise-polynomial approximations of functions of the classes $W^p_a$*, Matematicheskii Sbornik 115 (1967), no. 3, 331–355.

[9] James H Bramble and SR Hilbert, *Estimation of linear functionals on sobolev spaces with application to fourier transforms and spline interpolation*, SIAM Journal on Numerical Analysis 7 (1970), no. 1, 112–124.

[10] James H Bramble, Joseph E Pasciak, and Jinchao Xu, *Parallel multilevel preconditioners*, Mathematics of computation 55 (1990), no. 191, 1–22.

[11] Ingrid Daubechies, Ronald DeVore, Nadav Dym, Shira Faigenbaum-Golovin, Shahar Z Kovalsky, Kung-Chin Lin, Josiah Park, Guergana Petrova, and Barak Sober, *Neural network approximation of refinable functions*, IEEE Transactions on Information Theory (2022).

[12] Ingrid Daubechies, Ronald DeVore, Simon Foucart, Boris Hanin, and Guergana Petrova, *Nonlinear approximation and (deep) relu networks*, Constructive Approximation 55 (2022), no. 1, 127–172.

[13] Françoise Demengel, Gilbert Demengel, and Reinie Erné, *Functional spaces for the theory of elliptic partial differential equations*, Springer, 2012.

[14] Ronald DeVore, Boris Hanin, and Guergana Petrova, *Neural network approximation*, Acta Numerica 30 (2021), 327–444.

[15] Ronald A DeVore, *Nonlinear approximation*, Acta numerica 7 (1998), 51–150.

[16] Ronald A DeVore and George G Lorentz, *Constructive approximation*, vol. 303, Springer Science & Business Media, 1993.

[17] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci, *Hitchhikers guide to the fractional sobolev spaces*, Bulletin des sciences mathématiques 136 (2012), no. 5, 521–573.

[18] Lawrence C Evans, *Partial differential equations*, vol. 19, American Mathematical Soc., 2010.

[19] Paul Goldberg and Mark Jerrum, *Bounding the vapnik-chervonenkis dimension of concept classes parameterized by real numbers*, Proceedings of the sixth annual conference on Computational learning theory, 1993, pp. 361–369.

[20] Ian Goodfellow, Yoshua Bengio, and Aaron Courville, *Deep learning*, MIT press, 2016.

[21] Ingo Gühring, Gitta Kutyniok, and Philipp Petersen, *Error bounds for approximations with deep relu neural networks in $p$ norms*, Analysis and Applications 18 (2020), no. 05, 803–859.

[22] Jiequn Han, Arnulf Jentzen, and Weinan E, *Solving high-dimensional partial differential equations using deep learning*, Proceedings of the National Academy of Sciences 115 (2018), no. 34, 8505–8510.

[23] Boris Hanin, *Universal function approximation by deep neural nets with bounded width and relu activations*, Mathematics 7 (2019), no. 10, 992.

[24] Boris Hanin and David Rolnick, *Complexity of linear regions in deep networks*, International Conference on Machine Learning, PMLR, 2019, pp. 2596–2604.
[25] Juncai He, Lin Li, Jinchao Xu, and Chunyue Zheng, *Relu deep neural networks and linear finite elements*, Journal of Computational Mathematics 38 (2020), no. 3, 502–527.

[26] Jason M Klusowski and Andrew R Barron, *Approximation by combinations of relu and squared relu ridge functions with \( \ell^1 \) and \( \ell^0 \) controls*, IEEE Transactions on Information Theory 64 (2018), no. 12, 7649–7656.

[27] Yann LeCun, Yoshua Bengio, and Geoffrey Hinton, *Deep learning*, nature 521 (2015), no. 7553, 436–444.

[28] John E Littlewood and Raymond EAC Paley, *Theorems on fourier series and power series*, Journal of the London Mathematical Society 1 (1931), no. 3, 230–233.

[29] George G Lorentz, Manfred v Golitschek, and Yuly Makozov, *Constructive approximation: advanced problems*, vol. 304, Springer, 1996.

[30] Jianfeng Lu, Zuowei Shen, Haizhao Yang, and Shijun Zhang, *Deep network approximation for smooth functions*, SIAM Journal on Mathematical Analysis 53 (2021), no. 5, 5465–5506.

[31] Stéphane Mallat, *A wavelet tour of signal processing*, Elsevier, 1999.

[32] Vinod Nair and Geoffrey E Hinton, *Rectified linear units improve restricted boltzmann machines*, Icml, 2010.

[33] Michael S Paterson, *Improved sorting networks witho (logn) depth*, Algorithmica 5 (1990), no. 1, 75–92.

[34] Philipp Petersen and Felix Voigtlaender, *Optimal approximation of piecewise smooth functions using deep relu neural networks*, Neural Networks 108 (2018), 296–330.

[35] Maziar Raissi, Paris Perdikaris, and George E Karniadakis, *Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations*, Journal of Computational physics 378 (2019), 686–707.

[36] Norbert Sauer, *On the density of families of sets*, Journal of Combinatorial Theory, Series A 13 (1972), no. 1, 145–147.

[37] Thiago Serra, Christian Tjandraatmadja, and Srikumar Ramalingam, *Bounding and counting linear regions of deep neural networks*, International Conference on Machine Learning, PMLR, 2018, pp. 4558–4566.

[38] Saharon Shelah, *A combinatorial problem; stability and order for models and theories in infinitary languages*, Pacific Journal of Mathematics 41 (1972), no. 1, 247–261.

[39] Zuowei Shen, Haizhao Yang, and Shijun Zhang, *Optimal approximation rate of relu networks in terms of width and depth*, Journal de Mathématiques Pures et Appliquées 157 (2022), 101–135.

[40] Zhang Shijun, *Deep neural network approximation via function compositions*, Ph.D. thesis, National University of Singapore (Singapore), 2021.

[41] Jonathan W Siegel and Jinchao Xu, *Approximation rates for neural networks with general activation functions*, Neural Networks 128 (2020), 313–321.

[42] , *High-order approximation rates for shallow neural networks with cosine and reluk activation functions*, Applied and Computational Harmonic Analysis 58 (2022), 1–26.

[43] , *Sharp bounds on the approximation rates, metric entropy, and n-widths of shallow neural networks*, Foundations of Computational Mathematics (2022), 1–57.

[44] Matus Telgarsky, *Benefits of depth in neural networks*, Conference on learning theory, PMLR, 2016, pp. 1517–1539.

[45] Vladimir N Vapnik and A Ya Chervonenkis, *On the uniform convergence of relative frequencies of events to their probabilities*, Measures of complexity, Springer, 2015, pp. 11–30.

[46] Shuning Wang and Xusheng Sun, *Generalization of hinging hyperplanes*, IEEE Transactions on Information Theory 51 (2005), no. 12, 4425–4431.

[47] Hugh E Warren, *Lower bounds for approximation by nonlinear manifolds*, Transactions of the American Mathematical Society 133 (1968), no. 1, 167–178.

[48] Dmitry Yarotsky, *Error bounds for approximations with deep relu networks*, Neural Networks 94 (2017), 103–114.

[49] , *Optimal approximation of continuous functions by very deep relu networks*, Conference on learning theory, PMLR, 2018, pp. 639–649.

[50] Dmitry Yarotsky and Anton Zhevnerchuk, *The phase diagram of approximation rates for deep neural networks*, Advances in neural information processing systems 33 (2020), 13005–13015.