ALMOST-PRIME VALUES OF REDUCIBLE POLYNOMIALS
AT PRIME ARGUMENTS

C. S. FRANZE AND P. H. KAO

ABSTRACT. We adopt A. J. Irving’s sieve method to study the almost-prime values produced by products of irreducible polynomials evaluated at prime arguments. This generalizes the previous results of Irving and Kao, who separately examined the almost-prime values of a single irreducible polynomial evaluated at prime arguments.

1. INTRODUCTION

In this paper, we adopt a sieve method developed by A. J. Irving in [7] to prove

**Theorem 1.** Let \( H(n) = h_1(n) \cdot \ldots \cdot h_g(n) \), where \( h_i \) are distinct irreducible polynomials each with integer coefficients and \( \deg h_i = k \) for all \( i = 1, \ldots, g \). Suppose that
\[
\#\{ a \pmod{p} : (a, p) = 1 \text{ and } H(a) \equiv 0 \pmod{p} \} < p - 1.
\]
Then, for sufficiently large \( x \), there exists a natural number \( r \) such that
\[
\sum_{x < p \leq 2x} \frac{1}{\Omega(H(p))} \sim \frac{x}{\log^g x}.
\]

If \( g \geq 2 \) and \( k \) is sufficiently large, we may select an \( r \) of the form
\[
(2) \quad r = gk + c_1g^{3/2}k^{1/2} + c_2g^2 + O(g \log gk),
\]
where \( c_1 \) and \( c_2 \) are \( O(1) \). Explicit admissible values of \( r \) for small \( g \) and \( k \) are given below.

| \( g \) \( k \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | - | - | 15 | 18 | 21 | 23 | 26 | 29 | 31 | 33 | 36 | 38 | 40 | 43 |
| 3 | - | - | - | 30 | 35 | 39 | 43 | 47 | 51 | 55 | 59 | 62 | 66 | 70 |
| 4 | - | - | - | - | 43 | 50 | 56 | 63 | 68 | 74 | 79 | 85 | 90 | 95 |

**Table 1.** Admissible values for \( r \) using Irving’s sieve method

The case \( g = 1 \) was first investigated by H.-E. Richert in 1969 [9], who showed that for each \( k \geq 1 \), \( r = 2k + 1 \) is an admissible choice. Virtually no progress was made until Irving’s work in 2015 [7], which showed that one could take an \( r \) of the form \( r = k + O(\log k) \). Explicit bounds for the \( O \)-term, as well as explicit values for \( r \) when \( k \) is small, are available in [7] and [8].

The more general case where \( g \geq 2 \) is studied in the book by Halberstam and Richert in 1974 [6], who showed that one could select an \( r \) of the form
\[
(3) \quad r = 2gk + O(g \log gk).
\]
Their method was refined in the book by Diamond and Halberstam [4], which offers the admissible $r$ described below in Table 2 (see [4, pp.149-150]). However, their admissible $r$ exhibit the same asymptotic behavior described in (3). Therefore, the results of Theorem 1 represent an improvement when $k \gg g$.

| $g \setminus k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|----------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| 2              | 7 | 11| 16| 20| 24| 28| 32| 36| 40| 44  | 48  | 52  | 56  | 60  |
| 3              | 12| 19| 25| 32| 38| 44| 50| 56| 62| 69  | 75  | 81  | 87  | 93  |
| 4              | 17| 27| 35| 44| 52| 61| 69| 77| 86| 94  | 102 | 110 | 118 | 126 |

Table 2. Classical admissible values for $r$

Irving’s innovation was to combine a linear (one-dimensional) sieve with a two-dimensional sieve that permits a level of distribution beyond that which is available using the Bombieri-Vinogradov theorem. We adopt this novel idea to the relevant $g$- and $g+1$-dimensional sieves used for the more general polynomial sequence, $H$, considered here. The sifting functions $F_g$ and $f_g$ are, however, more difficult to work with for $g \geq 2$.

2. MAIN SIEVE SETUP

Here, we adopt some standard sieve notation. Setting $P(z) = \prod_{p < z} p$, we require bounds on

$$(4) \quad S(A, z) = \# \{ n \in A : (n, P(z)) = 1 \}.$$ 

The sequence that we are going to sieve is

$$A = \{ H(p) : x < p \leq 2x \}.$$

Using the prime number theorem, we note that the cardinality $|A| \sim X$, where

$$(5) \quad X = \text{li} x.$$

Letting $A_d = \{ n \in A : n \equiv 0 \ (d) \}$, it is straightforward (e.g. see [4 pp.131-132]) to show that

$$(6) \quad |A_d| = \frac{\rho_1(d)}{\phi(d)} X + r_A(d),$$

where

$$\rho_1(d) := \# \{ a \ (\text{mod } d) : 1 \leq a \leq d \text{ and } (a, d) = 1 \text{ and } H(a) \equiv 0 \ (\text{mod } d) \},$$

and the remainder term, $r_A(d)$, is bounded by

$$|r_A(d)| \leq \rho(d) E(x, d) + \rho(d),$$

where

$$\rho(d) := \# \{ a \ (\text{mod } d) : 1 \leq a \leq d \text{ and } H(a) \equiv 0 \ (\text{mod } d) \},$$

and

$$E(x, d) = \max_{1 \leq m \leq d \ (m, d) = 1} \left| \pi(x, d, m) - \frac{\text{li} x}{\phi(d)} \right|.$$
The sieve dimension is $g$ since the density function $\rho_1(d)/\varphi(d)$ appearing in (6) satisfies

$$\sum_{p \leq x} \frac{\rho_1(p)}{\varphi(p)} \log p = g \log x + O(1).$$

This follows from Proposition 10.1 of [4], which gives

$$\sum_{p \leq x} \frac{\rho_1(p)}{p} \log p = g \log x + O(1),$$

since

$$\sum_{p \leq x} \left( \frac{\rho_1(p)}{\varphi(p)} - \frac{\rho_1(p)}{p} \right) \log p \ll \sum_{p \leq x} \frac{\rho_1(p)}{p^2} \log p \ll \sum_{p \leq x} \frac{\log p}{p^2} \ll 1,$$

where we used $\rho_1(p) \leq \rho(p) \leq \deg(H)$.

As a consequence of (7), the product

$$V(z) := \prod_{p < z} \left( 1 - \frac{\rho_1(p)}{\varphi(p)} \right) \gg (\log z)^{-g}.$$

Finally, we note that the Bombieri-Vinogradov theorem implies that for any $\tau_1 \leq \frac{1}{2}$,

$$\sum_{d \text{ squarefree}} A \omega(d) |r_A(d)| \ll \frac{X}{(\log X)^{\tau_1+1}},$$

for a suitably large value of $B$ (e.g. see [6, Lemma 3.5 on p.115, and p. 288]). The parameter $\tau_1$ is called the level of distribution.

3. An Auxiliary Sieve

The main difference between Irving’s approach, adopted here, and the classical one is the introduction of an auxiliary upper bound sieve for the sequence $A_p$, where $p$ is a prime $z \leq p < y$. Recall from (4) that

$$S(A_p, z) = \sum_{x < q \leq 2x \atop p \text{ prime}} 1.$$

If $z < x$, then for any prime $q > x$ we plainly have $(q, P(z)) = 1$. Therefore,

$$S(A_p, z) = \sum_{x < q \leq 2x \atop p \text{ prime}} 1 \leq \sum_{x < n \leq 2x \atop H(n) \equiv 0(p)} 1 = S(A', z),$$

where

$$A' = \{ nH(n) : x < n \leq 2x, p | H(n) \}.$$ 

Although the upper bound available for $S(A', z)$ is worse than that for $S(A_p, z)$, a larger level of distribution is available to us for $A'$, which involves integer arguments rather than primes. In this case, the cardinality $|A'| \sim X'$, where

$$X' = \frac{\rho_1(p)}{p} x.$$
and, using the Chinese remainder theorem, we observe that

\[ |A'_d| = \frac{\rho_2(d)}{d} X' + r_{A'}(d), \]

where

\[ \rho_2(d) := \#\{a (\text{mod} d) : aH(a) \equiv 0 (\text{mod} d)\}, \]

and the remainder term, \( r_{A'}(d) \), is bounded by

\[ |r_{A'}(d)| \leq \rho_1(p) \rho_2(d), \]

for \( d \mid P(z) \) and \( p \geq z \) large enough to ensure that \( p \nmid H(0) \) (see proof of Lemma 4.2 in [7]). The sieve dimension is \( g + 1 \) in this case since the density function \( \rho_2(d)/d \) appearing in (12) satisfies

\[ \sum_{p \leq x} \frac{\rho_2(p)}{p} \log p = (g + 1) \log x + O(1), \]

owing to the fact that \( \rho_2(p) = \rho_1(p) + 1 \). As a consequence of (14), we have

\[ V'(z) := \prod_{p < z} \left(1 - \frac{\rho_2(p)}{p}\right) \gg (\log z)^{(g+1)}. \]

More precisely, using Mertens’ product formula,

\[ V''(z) = \prod_{p < z} \left(1 - \frac{1}{p}\right) \left(1 - \frac{\rho_1(p)}{\phi(p)}\right) \sim \frac{e^{-\gamma}}{\log z} V(z). \]

Using (5), we note that

\[ x \sim X \log X, \]

and therefore,

\[ X' \sim \frac{\rho_1(p)}{p} X \log X. \]

In contrast to (10), upon setting \( z = X^{1/\nu} \), a small power of \( X \), we see that for any \( \tau_2 \leq 1 \),

\[ \sum_{d \mid P(z) \atop pd < X^{\tau_2 (\log X)^{-B'}}} 4^{w(d)} |r_{A'}(d)| = o \left( X' V'(z) \right), \]

for a suitably large \( B' \). This is easily obtained using (13) and (15) so that

\[ \sum_{d \mid P(z) \atop pd < X^{\tau_2 (\log X)^{-B'}}} 4^{w(d)} |r_{A'}(d)| \leq \frac{\rho_1(p)}{p} X^{\tau_2} (\log X)^{-B'} \sum_{d \mid P(z)} \frac{4^{w(d)} \rho_2(d)}{d}. \]

Proceeding in the manner of the proof of Lemma 4.3 in [4], we conclude that this is

\[ X' \left( \frac{X^{\tau_2}}{x} \right) (\log X)^{-B'} \prod_{p < z} \left(1 + \frac{4\rho_2(p)}{p}\right) \ll X' \left( \frac{\text{li} x}{x} \right) (\log X)^{-B'} V'(z)^{-4} = o \left( X' V'(z) \right), \]

for a suitably large \( B' \).
4. DIAMOND-HALBERSTAM-RICHERT SIEVE

We will employ the Diamond-Halberstam-Richert (DHR) sieve to estimate the number of survivors, \( S(A, z) \), \( S(A_p, z) \), and \( S(A', z) \). Recall from Theorem 9.1 of [4] that for any \( 2 \leq z \leq y \),

\[
S(A, z) \leq XV(z) \left( F_g \left( \frac{\log y}{\log z} \right) + O \left( \frac{\log \log y^2}{\log (\log 1/(2g+2))} \right) \right) + 2 \sum_{m \mid P(z) \atop m < y} 4^{\omega(m)} |r_A(m)|,
\]

and,

\[
S(A, z) \geq XV(z) \left( f_g \left( \frac{\log y}{\log z} \right) - O \left( \frac{\log \log y^2}{\log (\log 1/(2g+2))} \right) \right) - 2 \sum_{m \mid P(z) \atop m < y} 4^{\omega(m)} |r_A(m)|.
\]

The functions \( F_g \) and \( f_g \) are defined by the unique solutions to the differential-delay equations

\[
(u^g F_g(u))' = gu^{g-1} f_g(u - 1), \quad u > \alpha_g
\]

\[
(u^g f_g(u))' = gu^{g-1} F_g(u - 1), \quad u > \beta_g,
\]

with initial conditions

\[
F_g(u) = \frac{1}{\sigma_g(u)}, \quad 0 < u \leq \alpha_g,
\]

\[
f_g(u) = 0, \quad 0 < u \leq \beta_g,
\]

where \( \sigma_g \) is the Ankeny-Onishi function, and

\[
\alpha_1 = \beta_1 = 2 \quad \text{and} \quad \alpha_g > \beta_g > 2 \quad \text{for } g > 1.
\]

We suppose here that \( g \) is a positive integer, and remark that Booker and Browning [2] have recently compiled a list of values for \( \alpha_g \) and \( \beta_g \) for \( g \leq 50 \). The sifting limit \( \beta_g \) satisfies \( \beta_g \leq cg \), where \( c \approx 2.445 \) (see [3] Theorem 2), and [11]. The functions \( F_g \) and \( f_g \) satisfy

\[
F_g(u) = 1 + O \left( e^{-u} \right), \quad f_g(u) = 1 + O \left( e^{-u} \right),
\]

and \( F_g \) decreases monotonically, while \( f_g \) increases monotonically on \((0, \infty)\). In fact, Diamond and Halberstam establish in [4] Lemma 6.2 that for \( 1 \leq u_1 < u_2 \),

\[
0 \leq F_g(u_1) - F_g(u_2) \leq \frac{u_2 - u_1}{u_1} \cdot \frac{g}{\sigma_g(1)},
\]

and

\[
0 \leq f_g(u_2) - f_g(u_1) \leq \frac{u_2 - u_1}{u_1} \cdot \frac{g}{\sigma_g(1)}.
\]

5. RICHERT WEIGHTS

The aforementioned DHR sieve is enhanced by incorporating certain weights introduced by Richert [9]. The arithmetic significance of these weights are summarized in the lemma below.

**Lemma 5.1.** Suppose \( y = X^{1/u} \), \( z = X^{1/v} \), and \( 0 < \frac{1}{u} < \frac{1}{v} < \tau_2 \leq 1 \). Let \( r \) be a natural number such that \( r + 1 > gku \), and define \( \eta := r + 1 - gku \). Then for \( x \) sufficiently large,

\[
\sum_{n \in A \atop \Omega(n) \leq r \atop (n,P(z)) = 1} 1 \geq \frac{1}{r + 1} W(A) - o(XV(z)),
\]
where

\[ W(A) := \sum_{n \in A} \left( \eta - \sum_{z \leq p < y \atop p \mid n} \left( 1 - \frac{\log p}{\log y} \right) \right). \] (27)

Thus, if we can show that the weighted sum \( W(A) \) remains large even as \( x \) grows large, say for example \( W(A) \gg XV(z) \), then we succeed in demonstrating the abundance of elements \( n \in A \) which contain at most \( r \) prime factors. The proof of this lemma is contained in \([4, pp.140-141]\). We briefly reproduce it here for completeness.

**Proof.** We begin by observing that the number of elements \( n \in A \) that are divisible by \( p^2 \) for a \( z \leq p < y \) is negligible. More specifically,

\[ \sum_{z \leq p < y} |A_{p^2}| \ll \sum_{z \leq p < y} \rho(p^2) \left( \frac{x}{p^2} + O(1) \right) \ll_H \frac{x}{z} + y = o(XV(z)), \]

since \( \rho(p^2) \leq \deg(H)D^2 \), where \( D \) is the discriminant of \( H \) \([6, p.260]\). Therefore, we have

\[ W(A) = W(A^*) + o(XV(z)), \] (28)

where \( A^* := A \setminus \bigcup_{z \leq p < y} A_{p^2} \).

If an \( n \in A^* \) contains a repeated prime factor \( p \), then \( p \geq y \), and so

\[ \sum_{z \leq p < y \atop p \mid n} \left( 1 - \frac{\log p}{\log y} \right) \geq \sum_{z \leq p < y \atop p \mid n} \left( 1 - \frac{\log p}{\log y} \right) = \Omega(n) - \frac{\log |n|}{\log y} \geq \Omega(n) - \frac{\log X^{gk}}{\log X^{1/u}}, \] (29)

where \( \sum^* \) denotes summation over the appropriate multiplicity. It follows from (27) and (29) that

\[ W(A^*) \leq \sum_{n \in A^* \atop (n,P(z))=1} (r + 1 - \Omega(n)) \leq \sum_{n \in A^* \atop \Omega(n) \leq r \atop (n,P(z))=1} (r + 1). \]

Combining this inequality with (28) finishes the proof of the lemma since

\[ \sum_{n \in A \atop \Omega(n) \leq r \atop (n,P(z))=1} 1 \geq \sum_{n \in A^* \atop \Omega(n) \leq r \atop (n,P(z))=1} 1 \geq \frac{1}{r + 1} W(A) - o(XV(z)). \]

The observant reader may note that \( gk \) should be replaced with \( gk + \varepsilon \) in (29) since

\[ \max_{n \in A^*} |n| \leq X^{gk+\varepsilon}, \]

for \( x \) sufficiently large. The presence of this \( \varepsilon \), however, makes little difference in the final analysis.
6. APPROXIMATING THE WEIGHTED SUM

In this section, we turn our attention to approximating the weighted sum, \( W(A) \), by integrals. Recall that \( z = X^{1/v} \) and \( y = X^{1/u} \). Letting \( s \in (z, y) \), say \( s = X^{1/w} \), we have

\[
W(A) = \eta S(A, z) - (S_1 + S_2),
\]

where

\[
S_1 := \sum_{z \leq p < s} \left( 1 - \frac{\log p}{\log y} \right) S(A_p, z),
\]

and,

\[
S_2 := \sum_{s \leq p < y} \left( 1 - \frac{\log p}{\log y} \right) S(A_p, z).
\]

For \( S(A, z) \) and \( S_1 \), we invoke the Bombieri-Vinogradov theorem in [10] for the underlying \( g \)-dimensional sieve. However, for \( S_2 \), we will swap \( S(A_p, z) \) for \( S(A', z) \), where we can instead make use of [18] for the underlying \((g + 1)\)-dimensional sieve. For readers who wish to skip ahead, we are ultimately lead to an integral form for \( W(A) \) stated below in Lemma 6.4. The following three lemmas provide the necessary bounds for \( S(A, z), S_1 \), and \( S_2 \).

**Lemma 6.1.** Let \( z = X^{1/v} \), and \( 0 < \frac{1}{v} < \tau_1 \leq \frac{1}{2} \). Then

\[
S(A, z) \geq XV(z) \left\{ f_g \left( \tau_1 v \right) - o \left( 1 \right) \right\}.
\]

**Proof.** Letting \( y = X^{\tau_1 (\log X)^{-B}} \), \( X = \max X \), we conclude at once from [9], [10], and [20] that

\[
S(A, z) \geq XV(z) \left\{ f_g \left( \tau_1 v - Bv \log \log X \log X \right) - O \left( \frac{(\log \log X)^2}{(\log X)^{1/(2g+2)}} \right) \right\} - o \left( XV(z) \right).
\]

Finally, equation (25) allows us to perturb the argument of \( f_g \) at a small expense, so that

\[
f_g \left( \tau_1 v - Bv \frac{\log \log X}{\log X} \right) \geq f_g \left( \tau_1 v \right) - O \left( \frac{\log \log X}{\log X} \right).
\]

**Lemma 6.2.** Let \( z = X^{1/v} \), \( s = X^{1/w} \), and \( y = X^{1/u} \) where \( 0 < \frac{1}{v} < \frac{1}{w} < \tau_1 \leq \frac{1}{2} < \frac{1}{u} \). Then

\[
S_1 \leq XV(z) g \left\{ \int_{w}^{v} \left( 1 - \frac{u}{t} \right) F_g \left( u \left( \tau_1 - \frac{1}{t} \right) \right) \frac{dt}{t} + o(1) \right\}.
\]

**Proof.** We apply the \( g \)-dimensional upper bound DHR sieve in [19] to \( S(A_p, z) \) with level of distribution \( X^{\tau_1 / p} \). Letting \( z = X^{1/v}, y = X^{\tau_1 (\log X)^{-B}} / p \) in [19], we have

\[
\frac{(\log \log y)^2}{(\log y)^{1/(2g+2)}} \ll \frac{(\log \log X)^2}{(\log(X^{\tau_1 (\log X)^{-B}} / p))^{1/(2g+2)}} \ll \frac{(\log \log X)^2}{(\log(X^{\tau_1 - \frac{1}{p}} (\log X)^{-B}))^{1/(2g+2)}},
\]

and so,

\[
S(A_p, z) \leq \frac{\rho_1(p)}{\phi(p)} XV(z) \left( F_g \left( \frac{\log(X^{\tau_1 (\log X)^{-B}} / p)}{\log X^{1/v}} \right) + o(1) \right) + 2 \sum_{m \in \mathcal{M}_p} 4^{\omega(m)} |r_{A_p}(m)|,
\]

where

\[
\mathcal{M}_p := \{ m | P(z) : m < X^{\tau_1 (\log X)^{-B}} / p \}.
\]
Applying (24) to perturb the argument of \( F_g \) at a small expense, we have

\[
S(A_p, z) \leq \frac{\rho_1(p)}{\phi(p)} XV(z) \left( F_g \left( \tau_1 v - v \frac{\log p}{\log X} \right) + o(1) \right) + 2 \sum_{m \in \mathcal{M}_p} 4^{\omega(m)} |r_A(pm)|.
\]

Now, summing over \( p \) in \( S_1 \), we have

\[
S_1 \leq XV(z) \sum_{z \leq p < s} \left( 1 - \frac{\log p}{\log y} \right) \frac{\rho_1(p)}{\phi(p)} \left( F_g \left( \tau_1 v - v \frac{\log p}{\log X} \right) + o(1) \right) + o(XV(z)),
\]

since, by the Bombieri-Vinogradov in (10),

\[
\sum_{z \leq p < s} \sum_{m \in \mathcal{M}_p} 4^{\omega(m)} |r_A(pm)| \ll \sum_{n < X^{\tau_1(\log X)^{-B}}} 4^{\omega(n)} |r_A(n)| = o(XV(z)).
\]

Using (7), and recalling that \( z = X^{1/v} \), and \( s = X^{1/w} \), we find that

\[
\sum_{z \leq p < s} \frac{\rho_1(p)}{\phi(p)} \ll g \log \left( \frac{\log s}{\log z} \right) \ll g \log \frac{v}{w} \ll 1.
\]

Therefore, distributing the sum in (30) gives

\[
S_1 \leq XV(z) \left( \sum_{z \leq p < s} \left( 1 - \frac{\log p}{\log y} \right) \frac{\rho_1(p)}{\phi(p)} F_g \left( \tau_1 v - v \frac{\log p}{\log X} \right) + o(1) \right).
\]

Passing from this sum to the stated integral is a standard exercise in Riemann-Stieltjes integration, or summation by parts. For example, we may write the sum as

\[
(31) \quad \int_{z}^{s} \left( 1 - \frac{\log T}{\log y} \right) F_g \left( \tau_1 v - v \frac{\log T}{\log X} \right) \frac{dS(T)}{\log X}.
\]

with

\[
S(T) = \sum_{p \leq T} \frac{\rho_1(p)}{\phi(p)} \log p.
\]

If \( z = X^{1/v}, s = X^{1/w}, y = X^{1/u} \), then (7) implies that the integral in (31) is asymptotic to

\[
g \int_{X^{1/v}}^{X^{1/w}} \left( 1 - \frac{\log T}{\log X^{1/u}} \right) F_g \left( v \left( \tau_2 - \frac{1}{t} \right) \right) \frac{d\log T}{\log X}.
\]

Performing the change of variables \( T = X^{1/t} \) finishes the proof. \( \square \)

**Lemma 6.3.** Let \( z = X^{1/v}, s = X^{1/w}, y = X^{1/u} \) where \( 0 < \frac{1}{v} < \frac{1}{w} < \tau_1 \leq \frac{1}{2} < \frac{1}{u} < \tau_2 \leq 1 \).

\[
S_2 \leq XV(z) \frac{g v}{e^\tau} \left\{ \int_{u}^{w} \left( 1 - \frac{u}{t} \right) F_{g+1} \left( v \left( \tau_2 - \frac{1}{t} \right) \right) \frac{dt}{t} + o(1) \right\}.
\]

**Proof.** Here we use (11) to swap \( S(A_p, z) \) for \( S(A', z) \), since

\[
S(A_p, z) \leq S(A', z),
\]
and then apply the \((g+1)\)-dimensional upper bound DHR sieve in \([19]\) to \(S(A', z)\), with \(X\) replaced by \(X'\), \(V(z)\) replaced by \(V'(z)\), \(z = X^{1/v}\), and \(y = X'^2 (\log X)^{-B'}/p\) for a suitably large \(B'\). Using \([18]\) to control the remainder term gives

\[
S(A', z) \leq X'V'(z) \left( F_{g+1} \left( \frac{\log \left( X'^2 (\log X)^{-B'}/p \right)}{\log X^{1/v}} \right) + o(1) \right).
\]

Appealing to \([24]\) to perturb the argument of \(F_{g+1}\) so that

\[
F_{g+1} \left( \frac{\log \left( X'^2 (\log X)^{-B'}/p \right)}{\log X^{1/v}} - B' \frac{\log \log X}{\log z} \right) \leq F_{g+1} \left( \frac{\log \left( X'^2 (\log X)^{-B'}/p \right)}{\log X^{1/v}} \right) + O \left( \frac{\log \log X}{\log X} \right),
\]

gives

\[
S(A', z) \leq X'V'(z) \left( F_{g+1} \left( \frac{\log \left( X'^2 (\log X)^{-B'}/p \right)}{\log X^{1/v}} \right) + o(1) \right).
\]

Replacing \(V'(z)\) and \(X'\) with their corresponding expressions in \([16]\) and \([17]\),

\[
S(A', z) \leq XV(z) \left( \frac{\rho_1(p)}{p} e^{-\gamma} \frac{\log X}{\log z} \left( F_{g+1} \left( v\tau_2 - \frac{\log p}{\log X} \right) + o(1) \right) \right),
\]

Summing over \(s \leq p < y\) in \(S_2\) then gives

\[
S_2 \leq XV(z) e^{-\gamma} v \left( \sum_{s \leq p < y} \left( 1 - \frac{\log p}{\log y} \right) \frac{\rho_1(p)}{p} F_{g+1} \left( v\tau_2 - \frac{\log p}{\log X} \right) + o(1) \right),
\]

since \([8]\) implies that

\[
\sum_{s \leq p < y} \frac{\rho_1(p)}{p} \ll g \log \frac{\log y}{\log s} \ll g \log \frac{w}{u} \ll 1.
\]

Passing from the sum in \((32)\) to the stated integral is a standard exercise. Note that this sum is

\[
\int_{s^-}^y \left( 1 - \frac{\log T}{\log y} \right) F_{g} \left( \frac{\tau_2 v - \frac{\log T}{\log X}}{\log T} \right) \frac{dS(T)}{\log T}.
\]

with

\[
S(T) = \sum_{p \leq T} \frac{\rho_1(p)}{p} \log p.
\]

Recalling that \(s = X^{1/w}\), \(y = X^{1/u}\), and using \([8]\), the integral in \((33)\) is asymptotic to

\[
g \int_{X^{1/u}}^{X^{1/w}} \left( 1 - \frac{\log T}{\log X^{1/u}} \right) F_{g+1} \left( v \left( \frac{\tau_2}{\log X} \right) \right) \frac{d \log T}{\log T},
\]

Performing the change of variables \(T = X^{1/t}\) finishes the proof.

Combining Lemma \([6.1]\) Lemma \([6.2]\) and Lemma \([6.3]\) gives

**Lemma 6.4.** Let \(0 < \frac{1}{v} < \frac{1}{w} < \tau_1 \leq \frac{1}{2} < \frac{1}{u} < \tau_2 \leq 1\). Then

\[
W(A) \geq \left( \eta f_g(\tau_1 v) - \left( I(u, w, v) + \frac{v}{e^\gamma} J(u, w, v) \right) + o(1) \right) XV(z),
\]

where

\[
I(u, w, v) := g \int_{w}^{v} \left( 1 - \frac{u}{s} \right) F_{g} \left( v \left( \tau_1 - \frac{1}{s} \right) \right) \frac{ds}{s},
\]

\(\frac{1}{v} < \frac{1}{w} < \tau_1 \leq \frac{1}{2} < \frac{1}{u} < \tau_2 \leq 1\). Then

\[
W(A) \geq \left( \eta f_g(\tau_1 v) - \left( I(u, w, v) + \frac{v}{e^\gamma} J(u, w, v) \right) + o(1) \right) XV(z),
\]

where

\[
I(u, w, v) := g \int_{w}^{v} \left( 1 - \frac{u}{s} \right) F_{g} \left( v \left( \tau_1 - \frac{1}{s} \right) \right) \frac{ds}{s},
\]
Lemma 7.1. Let \( \xi_1 := v\tau_1 + 1 - \frac{u}{w} \), and \( 0 < \frac{1}{v} < \frac{1}{w} < \tau_1 \leq \frac{1}{u} < \tau_2 \leq 1 \). If \( \xi_1 \geq \beta_g \), then
\[
\frac{1}{f_g(\tau_1 v)} I \leq \left( g + \frac{u}{v} \xi_1 \left( 1 - \frac{f_g(\xi_1)}{f_g(\tau_1 v)} \right) \right) \log \frac{w}{w} + \left( 1 - \frac{f_g(\xi_1)}{f_g(\tau_1 v)} \right) \xi_1 \frac{w}{v} \left( 1 - \frac{u}{w} \right) - g \left( \frac{u}{w} - \frac{v}{w} \right).
\]

Proof. Let \( t - 1 = v(\tau_1 - \frac{1}{s}) \), so \( s = v/(v\tau_1 + 1 - t) \). Under this change of variables,
\[
I = g \frac{u}{v} \int_{\xi_1}^{v\tau_1} F_g(t - 1) \frac{t - \xi_1 + \frac{w}{u} - \frac{v}{w}}{v\tau_1 + 1 - t} dt.
\]

We then separate the integral so that
\[
(36) \quad I = I_1 + g \frac{u}{v} \left( \frac{w}{u} - \frac{v}{u} \right) I_2,
\]
where
\[
I_1 := g \frac{u}{v} \int_{\xi_1}^{v\tau_1} F_g(t - 1) \frac{t - \xi_1}{v\tau_1 + 1 - t} dt,
\]
and
\[
I_2 := \int_{\xi_1}^{v\tau_1} F_g(t - 1) \frac{dt}{v\tau_1 + 1 - t}.
\]

Integrating by parts,
\[
I_1 = -g \frac{u}{v} \int_{\xi_1}^{v\tau_1} F_g(t - 1) f_g(t - \xi_1) d \log(v\tau_1 + 1 - t)
\]
\[
= g \frac{u}{v} \int_{\xi_1}^{v\tau_1} \left( F_g(t - 1) + F'_g(t - 1) (t - \xi_1) \right) \log(v\tau_1 + 1 - t) dt
\]
\[
< g \frac{u}{v} \int_{\xi_1}^{v\tau_1} F_g(t - 1) \log(v\tau_1 + 1 - t) dt,
\]

since \( F \) is decreasing. Next, if \( \xi_1 \geq \beta_g \), then \( t \geq \beta_g \), and we can use (22) to observe that
\[
I_1 < g \frac{u}{v} \int_{\xi_1}^{v\tau_1} \left( t^g f_g(t) \right)' t^{1-g} \log(v\tau_1 + 1 - t) dt.
\]

Integrating by parts, and using the fact that \( f \) is increasing, gives
\[
I_1 < g \frac{u}{v} \int_{\xi_1}^{v\tau_1} f_g(t) \left( (g - 1) \log(v\tau_1 + 1 - t) + \frac{t}{v\tau_1 + 1 - t} \right) dt - g \frac{u}{v} \xi_1 f_g(\xi_1) \log \frac{v}{w}
\]
\[
< g \frac{u}{v} f_g(v\tau_1) \int_{\xi_1}^{v\tau_1} \left( (g - 1) \log(v\tau_1 + 1 - t) + \frac{t}{v\tau_1 + 1 - t} \right) dt - g \frac{u}{v} \xi_1 f_g(\xi_1) \log \frac{v}{w}.
\]
The remaining integral is \( \frac{v}{w} \left( g + \frac{w}{v} \xi_1 \right) \log \frac{v}{w} - g \left( \frac{w}{v} - 1 \right) \), so that

\[
I_1 < \int_{vτ_1} f_g(vτ_1) \left( \frac{u}{w} \left( g + \frac{w}{v} \xi_1 \right) \log \frac{v}{w} - g \left( \frac{u}{w} - \frac{u}{v} \right) - \frac{u}{v} \xi_1 \frac{f_g(ξ_1)}{f_g(τ_1 v)} \log \frac{v}{w} \right).
\]

For \( I_2 \), we make use of (22) and integrate by parts to observe that

\[
I_2 = \int_{ξ_1} ^{vτ_1} \left( t^θ f_g(t) \right) \frac{dt}{g t^{g-1}(vτ_1 + 1 - t)} = \frac{f_g(vτ_1)}{g} vτ_1 \xi_1 w + \int_{ξ_1} ^{vτ_1} f_g(t) \left( \frac{g - 1}{g(vτ_1 + 1 - t)} - \frac{t}{g(vτ_1 + 1 - t)^2} \right) dt.
\]

Since \( f \) is increasing,

\[
I_2 \leq \frac{f_g(vτ_1)}{g} \left( vτ_1 - \frac{1}{f_g(τ_1 v)} \xi_1 w + g \int_{ξ_1} ^{vτ_1} \left( \frac{g - 1}{g(vτ_1 + 1 - t)} - \frac{t}{g(vτ_1 + 1 - t)^2} \right) dt \right).
\]

The remaining integral is \( \log \frac{v}{w} - \frac{vτ_1 + 1}{g} \left( 1 - \frac{v}{w} \right) \), and so

\[
I_2 < \int_{vτ_1} f_g(τ_1 v) \left( \frac{1 - \frac{f_g(ξ_1)}{f_g(τ_1 v)} \xi_1 w + g \log \frac{v}{w}}{vτ_1} \right) dt.
\]

Inserting the bounds (37) and (38) into (36) gives the stated lemma.

Lemma 7.2. Let \( ξ_2 := vτ_2 + 1 - \frac{1}{u} \), and 0 < \( \frac{1}{v} < \frac{1}{w} < \tau_1 \leq \frac{1}{2} < \frac{1}{u} < \tau_2 \leq 1 \). If \( ξ_2 \geq β_{g+1} \), then

\[
\frac{v}{e^v f_g(τ_1 v)} J \leq \frac{1}{f_g(τ_1 v)} v \left( \log \frac{w}{u} - 1 + \frac{u}{w} \right) + ξ_2 \frac{g}{g + 1} e^v \left( 1 - \frac{f_{g+1}(ξ_2)}{f_g(τ_1 v)} \right) \log \frac{v}{w}.
\]

Proof. Let \( t - 1 = v(τ_2 - 1/s) \), so \( s = v/(vτ_2 + 1 - t) \). Under this change of variables,

\[
J = \frac{u}{v} \int_{ξ_2} ^{vτ_2 + 1 - \frac{1}{u}} F_{g+1}(t - 1) \left( \frac{v}{u} - vτ_2 - 1 + t \right) \frac{dt}{vτ_2 + 1 - t}
\]

\[
= -\frac{u}{v} \int_{ξ_2} ^{vτ_2 + 1 - \frac{1}{u}} F_{g+1}(t - 1)(t - ξ_2) d \log (vτ_2 + 1 - t).
\]

Integrating by parts, and then using the fact that \( F > 1 \), we have

\[
J < \frac{u}{v} \left[ \int_{ξ_2} ^{vτ_2 + 1 - \frac{1}{u}} \left( F_{g+1}(t - 1) - F_{g+1}(t - 1)(t - ξ_2) \right) \log (vτ_2 + 1 - t) dt \right.
\]

\[
- \left. \left( \frac{v}{u} - \frac{v}{w} \right) \log \frac{v}{w} \right].
\]

Since \( F \) is decreasing, \( F' < 0 \), and

\[
J < \frac{u}{v} \left[ J_1 - \left( \frac{v}{u} - \frac{v}{w} \right) \log \frac{v}{w} \right],
\]

where

\[
J_1 = \int_{ξ_2} ^{vτ_2 + 1 - \frac{1}{u}} F_{g+1}(t - 1) \log (vτ_2 + 1 - t) dt.
\]

Next, using (22), and assuming that \( ξ_2 \geq β_{g+1} \), we rewrite

\[
J_1 = \int_{ξ_2} ^{vτ_2 + 1 - \frac{1}{u}} \left( t^{g+1} f_{g+1}(t) \right)' \log (vτ_2 + 1 - t) \frac{dt}{t^{g+1}}.
\]
Integrating by parts, we find that

\[
J_1 = \frac{(v\tau_2 + 1 - \frac{v}{w})f_{g+1}(v\tau_2 + 1 - \frac{v}{w}) \log \frac{v}{w}}{(g + 1)} - \frac{\xi_2 f_{g+1}(\xi_2) \log \frac{v}{w}}{(g + 1)} + J_2,
\]

where

\[
J_2 = - \int_{\xi_2}^{v\tau_2 + 1 - \frac{v}{w}} t^{g+1} f_{g+1}(t) d \left( \frac{\log(v\tau_2 + 1 - t)}{t^g(g + 1)} \right).
\]

Now, since \( f < 1 \),

\[
J_1 < \frac{(v\tau_2 + 1 - \frac{v}{w}) \log \frac{v}{w}}{(g + 1)} - \frac{\xi_2 f_{g+1}(\xi_2) \log \frac{v}{w}}{(g + 1)} + J_2,
\]

and

\[
J_2 = - \int_{\xi_2}^{v\tau_2 + 1 - \frac{v}{w}} t^{g+1} f_{g+1}(t) d \left( \frac{\log(v\tau_2 + 1 - t)}{t^g(g + 1)} \right)
\]

\[
= \frac{1}{(g + 1)} \int_{\xi_2}^{v\tau_2 + 1 - \frac{v}{w}} f_{g+1}(t) \left\{ g \log(v\tau_2 + 1 - t) + \frac{t}{v\tau_2 + 1 - t} \right\} dt
\]

\[
< \frac{1}{(g + 1)} \int_{\xi_2}^{v\tau_2 + 1 - \frac{v}{w}} \left\{ g \log(v\tau_2 + 1 - t) + \frac{t}{v\tau_2 + 1 - t} \right\} dt.
\]

Calculating the remaining integral, we conclude that

\[
J_2 < \frac{g}{g + 1}(v\tau_2 + 1 - \frac{v}{w}) \log \frac{v}{w} - \frac{g}{g + 1} \xi_2 \log \frac{v}{u} + (v\tau_2 + 1) \log \frac{w}{u} - \frac{v}{u} + \frac{v}{w}.
\]

Combining (41) and (40), we have

\[
J_1 < (v\tau_2 + 1 - \frac{v}{w}) \log \frac{v}{w} - \frac{\xi_2}{g + 1} f_{g+1}(\xi_2) \log \frac{v}{u} - \frac{g}{g + 1} \xi_2 \log \frac{v}{u} + (v\tau_2 + 1) \log \frac{w}{u} - \frac{v}{u} + \frac{v}{w}.
\]

Since \( v\tau_2 + 1 - \frac{v}{w} = (\frac{w}{u} - \frac{w}{w}) + (v\tau_2 + 1 - \frac{v}{w}) = (\frac{w}{u} - \frac{w}{w}) + \xi_2 \), we conclude from (39) that

\[
J < \frac{u}{v} \left( - \frac{\xi_2}{g + 1} f_{g+1}(\xi_2) \log \frac{v}{u} + \xi_2 \log \frac{v}{u} - \frac{g}{g + 1} \xi_2 \log \frac{v}{u} + (v\tau_2 + 1) \log \frac{w}{u} - \frac{v}{u} + \frac{v}{w} \right),
\]

or equivalently,

\[
J < \frac{u}{v} \left( - \frac{\xi_2}{g + 1} f_{g+1}(\xi_2) \log \frac{v}{u} + \xi_2 \log \frac{v}{u} - \frac{g}{g + 1} \xi_2 \log \frac{v}{u} + \frac{v}{u} \log \frac{w}{u} - \frac{v}{u} + \frac{v}{w} \right).
\]

Simplifying the right-hand side, this reads,

\[
J < \frac{g}{g + 1} \frac{u}{v} \xi_2 (1 - f_{g+1}(\xi_2)) \log \frac{v}{u} + g \left( \log \frac{w}{u} - 1 + \frac{u}{w} \right).
\]

Multiplying this inequality by \( \frac{v}{e^{f_{g+1}(\tau_1 v)}} \) gives the stated lemma. \( \blacksquare \)
8. PROOF OF THEOREM 1

Setting \( \xi_1 = \beta_g \) in Lemma 7.1 and \( \xi_2 = \beta_{g+1} \) in Lemma 7.2 gives

\[
\frac{1}{f_g(\tau_1 v)} I \leq \left( g + \frac{u}{v} \beta_g \right) \log v \frac{v}{w} + \frac{w}{v} \beta_g \left( 1 - \frac{u}{w} \right) - g \left( \frac{u}{w} - \frac{u}{v} \right),
\]

and

\[
\frac{v}{e^{\gamma} f_g(\tau_1 v)} J \leq \frac{1}{f_g(\tau_1 v)} \left( \frac{v}{e^{\gamma} g} \left( \log \frac{w}{u} - 1 + \frac{u}{w} \right) + \frac{\beta_{g+1} u g}{g + 1} e^{\gamma} \log \frac{v}{u} \right).
\]

Setting \( \tau_1 = 1/2 \) and \( \tau_2 = 1 \), this choice of \( \xi_1 \) and \( \xi_2 \) implies that

\[
u = 1 + \frac{\beta_{g+1} - 1}{v - (\beta_{g+1} - 1)},
\]

and

\[
w = 2 \left( 1 + \frac{2(\beta_g - 1)}{v - 2(\beta_g - 1)} \right).
\]

The parameters \( u \) and \( w \) will therefore be completely determined by our choice of \( v \).

To simplify the analysis, we bound the ratio \( \frac{w}{u} \), defined for \( v > \max\{\beta_{g+1} - 1, 2(\beta_g - 1)\} \).

Let \( N \geq 3 \) be chosen so that \( N(\beta_{g+1} - 1) > \max\{\beta_{g+1} - 1, 2(\beta_g - 1), 4(\beta_g - 1) - (\beta_{g+1} - 1)\} \).

Assuming that

\[
v \geq N(\beta_{g+1} - 1),
\]

then

\[
\frac{4}{3} \leq \frac{w}{u} \leq 4.
\]

The upper bound is easy to see since

\[
\frac{w}{u} = \frac{2(v - (\beta_{g+1} - 1))}{v - 2(\beta_g - 1)} \leq 4
\]

if \( v \geq 4(\beta_g - 1) - (\beta_{g+1} - 1) \), which holds for (46). Next, if \( \max\{\beta_{g+1} - 1, 2(\beta_g - 1)\} = \beta_{g+1} - 1 \),

\[
g(v) := \frac{2(v - (\beta_{g+1} - 1))}{v - 2(\beta_g - 1)}
\]

is an increasing function. Therefore, for \( v \) satisfying (46),

\[
g(v) \geq g \left( N(\beta_{g+1} - 1) \right) = \frac{2(N - 1)(\beta_{g+1} - 1)}{N(\beta_{g+1} - 1) - 2(\beta_g - 1)} \geq \frac{2(N - 1)}{N} \geq \frac{4}{3},
\]

since \( \beta_g \geq 1 \), and \( N \geq 3 \). If, on the other hand, \( \max\{\beta_{g+1} - 1, 2(\beta_g - 1)\} = 2(\beta_g - 1) \), then

\[
g(v) \geq 2,
\]

since this is equivalent to \( \beta_{g+1} - 1 \leq 2(\beta_g - 1) \). In either case, the lower bound for \( \frac{w}{u} = g(v) \) in (47) holds.

Using (47), the bound for \( J \) in (43) simplifies to

\[
\frac{v}{e^{\gamma} f_g(\tau_1 v)} J \leq \left( \frac{v}{e^{\gamma} g} \left( \log 4 - 1 + \frac{3}{4} \right) + \frac{\beta_{g+1} u g}{g + 1} e^{\gamma} \log \frac{v}{u} \right) (1 + O\left( e^{-v/2} \right)) \]

\[
\leq \frac{v g}{C_0} + \beta_{g+1} \frac{u g}{g + 1} e^{\gamma} \log \frac{v}{u} + O\left( \frac{v g}{e^{v/2}} + \frac{u g \log v}{e^{v/2}} \right),
\]
where we have used the boundary condition in (23), and defined

\[ C_0 := \frac{e^\gamma}{\log 4 - \frac{1}{4}}. \]

Ultimately, our choice of \( v \) in (53) will guarantee that the error term above is \( o(1) \), and that our assumption that \( v \geq N(\beta_{g+1} - 1) \) in (46) is valid provided \( k \) is sufficiently large, say

\[ k \geq \frac{(N - 1)^2(\beta_{g+1} - 1)}{C_0}. \]

Lemma 5.1, Lemma 6.4, and (9) guarantee (1) is satisfied provided we select an \( r \) such that

\[ r > gk u - 1 + \frac{1}{f_g(\tau_1 v)} I(u, w, v) + \frac{v}{e^\gamma f_g(\tau_1 v)} J(u, w, v). \]

Ignoring error terms, the bounds in (42) and (48) show that it is enough to select an \( r \) such that

\[ r > gk u - 1 + \left( g + \frac{u \beta_g}{v} \right) \log \frac{v}{w} + \frac{w}{v} \beta_g \left( 1 - \frac{u}{w} \right) - g \left( \frac{u}{w} - \frac{u}{v} \right) + \frac{v g}{C_0} + \frac{\beta_{g+1} u g}{g + 1 e^\gamma \log \frac{v}{u}}. \]

In search of the smallest such \( r \), we choose \( v \) to minimize the expression on the right. For the sake of simplicity, we focus on the most problematic terms in this expression, given by

\[ M(v) := gk u + \frac{v g}{C_0} = gk + \frac{(\beta_{g+1} - 1)gk}{v - (\beta_{g+1} - 1)} + \frac{v g}{C_0}, \]

where \( C_0 \) is defined in (49), also keeping in mind (44). The minimum is achieved at

\[ v = \beta_{g+1} - 1 + \sqrt{C_0(\beta_{g+1} - 1)k}, \]

at which

\[ M(v) = gk + gk \left( 2\sqrt{\frac{(\beta_{g+1} - 1)/(C_0 k)}{(\beta_{g+1} - 1)/(C_0 k)}} \right), \]

and the remaining terms in (52) are \( O(g \log gk) \). Therefore, the admissible \( r \) in (52) take the form

\[ r > gk + c_1 g^{3/2}k^{1/2} + c_2 g^2 + O(g \log gk), \]

where

\[ c_1 = 2 \sqrt{\frac{\beta_{g+1} - 1}{C_0 g}} \quad \text{and} \quad c_2 = \frac{\beta_{g+1} - 1}{C_0 g}. \]

Both \( c_1 \) and \( c_2 \) are \( O(1) \). Thus, our admissible \( r \) take the form stated in (2).

Before moving on, note that we have shown (2) for \( k \) satisfying (50), but that we may need an even larger \( k \) to guarantee that these admissible \( r \) are asymptotically better than those in (3). In fact, the main term in (54) satisfies \( M(v) < 2gk \) if

\[ \frac{\beta_{g+1} - 1}{C_0 k} < 3 - 2\sqrt{2}. \]

Therefore, we suppose that

\[ k > \max \left\{ \frac{(N - 1)^2(\beta_{g+1} - 1)}{C_0}, \frac{\beta_{g+1} - 1}{C_0(3 - 2\sqrt{2})} \right\}. \]

However, numerical data suggests that the improvements appear much earlier.

For the admissible \( r \)-values in Table 2, we briefly describe our choices of \( v, w, \) and \( u \), for each fixed \( g \) and \( k \). All numerical experiments were conducted using W. Galway’s Mathematica.
We chose the parameter $v$ to be of the form $v = \alpha + n$, where $n$ is a positive integer. Next, we choose $w$ to minimize the expression on the right in (51), which amounts to solving
\[ F_g \left( v \left( \frac{1}{2} - \frac{1}{w} \right) \right) - \frac{v}{e^{\gamma}} F_{g+1} \left( v \left( 1 - \frac{1}{w} \right) \right) = 0. \]

With these choices of $v$ and $w$, we then chose $u$ to minimize the expression in (51) by solving
\[ k f_g \left( \frac{v}{2} \right) - \int_{w}^{v} F_g \left( v \left( \frac{1}{2} - \frac{1}{s} \right) \right) \frac{ds}{s^2} - \frac{v}{e^{\gamma}} \int_{w}^{u} F_{g+1} \left( v \left( 1 - \frac{1}{s} \right) \right) \frac{ds}{s^2} = 0. \]

This process was repeated for many values of $n$ to arrive at the stated admissible $r$-values.

9. CONCLUDING REMARKS

More general results are readily available. For example, one could consider polynomials $H$ whose irreducible components have different degrees. In addition, the work of Booker and Browning [2] allows one to capture squarefree values, rather than almost-primes, if these irreducible components have degree 3 or less. The polynomial sequence considered here was chosen mainly for illustrative purposes.

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DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY
E-mail address: franze.3@osu.edu

DEPARTMENT OF MATHEMATICS AND TECHNOLOGY, FLAGLER COLLEGE
E-mail address: ckao@flagler.edu