THE LEVEL OF DISTRIBUTION OF THE THUE–MORSE SEQUENCE

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Abstract. The level of distribution of a complex valued sequence $b$ measures “how well $b$ behaves” on arithmetic progressions $nd + a$. Determining whether $\theta$ is a level of distribution for $b$ involves summing a certain error over $d \leq D$, where $D$ depends on $\theta$; this error is given by comparing a finite sum of $b$ along $nd + a$ and the expected value of the sum. We prove that the Thue–Morse sequence has level of distribution 1, which is essentially best possible. More precisely, this sequence gives one of the first nontrivial examples of a sequence satisfying a Bombieri–Vinogradov type theorem for each exponent $\theta < 1$. In particular, this result improves on the level of distribution $2/3$ obtained by M"ullner and the author.

As an application of our method, we show that the subsequence of the Thue–Morse sequence indexed by $\lfloor nc \rfloor$, where $1 < c < 2$, is simply normal. That is, each of the two symbols appears with asymptotic frequency $1/2$ in this subsequence. This result improves on the range $1 < c < 3/2$ obtained by M"ullner and the author and closes the gap that appeared when Mauduit and Rivat proved (in particular) that the Thue–Morse sequence along the squares is simply normal. In the proofs, we reduce both problems to an estimate of a certain Gowers uniformity norm of the Thue–Morse sequence similar to that given by Konieczny (2017).

1. Introduction

The Thue–Morse sequence $t$ is one of the most easily defined automatic sequences. Like any automatic sequence, it can be defined using a constant-length substitution over a finite alphabet: $t$ is the unique fixed point of the substitution $0 \mapsto 01$, $1 \mapsto 10$ that starts with 0. Therefore $t = 011010011010110 \ldots$. Alternatively, this sequence can be defined using the binary sum-of-digits function $s$, which counts the number of 1s in the binary expansion of a nonnegative integer $n$: we have $t(n) = 0$ if and only if $s(n) \equiv 0 \mod 2$. The equivalence of these two definitions can be proved via a third description: start with the one-element sequence $t(0) \equiv \{0\}$ and define $t(n+1)$ by concatenating $t(n)$ and the Boolean complement $\neg t(n)$. Then $t$ is the pointwise limit of these finite sequences. In this work, we will adapt the second viewpoint. In fact, in the proofs we will work with the sequence $(-1)^{s(n)}$ instead of $t$. For an overview on the Thue–Morse sequence, we refer the reader to the article by Allouche and Shallit [1], which points out occurrences of this sequence in different fields of mathematics and offers a good bibliography. Moreover, we wish to mention the paper [24] by Mauduit. For a comprehensive account of automatic and morphic sequences, see the book [2] by Allouche and Shallit.

The main topic of this article is the study of $t$ along arithmetic progressions and, more generally, along Beatty sequences $\lfloor n\alpha + \beta \rfloor$. This topic can be traced back at least to Gelfond [15], who proved the following theorem on the base-$q$ sum-of-digits function $s_q$.

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Theorem A (Gelfond). Let \( q, m, d, b, a \) be integers and \( q, m, d \geq 2 \). Suppose that \( \gcd(m, q-1) = 1 \). Then
\[
|\{1 \leq n \leq x : n \equiv a \mod d, s_q(n) \equiv b \mod m\}| = \frac{x}{dm} + \mathcal{O}(x^\lambda)
\]
for some \( \lambda < 1 \) not depending on \( x, d, a, \) and \( b \).

We are particularly interested in the error term for sparse arithmetic progressions, having large common difference \( d \). This leads us directly to the other main concept of this paper, the notion of level of distribution. Very roughly speaking, the level of distribution is a measure of how well a given sequence behaves on arithmetic progressions. A formal definition is given by Fouvy and Mauduit [14], for example, which we adapt here.

Definition 1. Let \( c = (c_n)_{n \geq 0} \) be a sequence of complex numbers, and for each integer \( d \geq 1 \) let \( Q(d) \) and \( R(d) \neq \emptyset \) be subsets of \( \mathbb{Z}/d\mathbb{Z} \) such that \( Q(d) \subseteq R(d) \). The sequence \( c \) has level of distribution \( \theta \) with respect to \( Q \) and \( R \) if for all \( \varepsilon > 0 \) and \( A > 0 \) we have for all \( x \geq 1 \)
\[
\sum_{1 \leq d \leq x^{\theta-\varepsilon}} \max_{0 \leq k \leq x} \max_{0 \leq a < d} \left| \sum_{0 \leq n < y \atop n \equiv a \mod d \atop n+dZ \in Q(d)} c_n - \frac{1}{|R(d)|} \sum_{0 \leq n < y \atop n+dZ \in R(d)} c_n \right| = \mathcal{O}\left(\sum_{0 \leq n < x} |c_n| \right) \left(\log 2x\right)^{-A}.
\]
The implied constant may depend on \( A \) and \( \varepsilon \). Moreover, in this definition the maximum over the empty index set is defined as 0.

The level of distribution (also called exponent of distribution by some authors) is an important concept in sieve theory. As a striking application, a variant of this concept was used in the “bounded gaps between primes” paper by Zhang [36]. For more information on this subject, we refer the reader to the survey by Kontorovich [22]. Moreover, we wish to draw the attention of the reader to the book [16] on sieve theory by Friedlander and Iwaniec, in particular Chapter 22 on the level of distribution.

We are ready to present our main result.

Theorem 1.1. The Thue–Morse sequence has level of distribution 1 with respect to \( Q \) and \( R \) given by \( Q(d) = R(d) = \mathbb{Z}/d\mathbb{Z} \). More precisely, for all \( \varepsilon > 0 \) we have
\[
\sum_{1 \leq d \leq x^{\theta-\varepsilon}} \max_{0 \leq y \leq x} \max_{0 \leq a < d} \left| \sum_{0 \leq n < y \atop n \equiv a \mod d} (-1)^{s(n)} \right| = \mathcal{O}(x^{1-\eta})
\]
for some \( \eta > 0 \) depending on \( \varepsilon \), where \( D = x^{1-\varepsilon} \).

Before presenting some history, we wish to say a word about the proof: we are going to reduce the problem to the estimation of a certain Gowers uniformity norm of the Thue–Morse sequence. These expressions appear by repeated application of van der Corput’s inequality and have the form
\[
\sum_{0 \leq n < 2^x} \prod_{0 \leq r_1, \ldots, r_m < 2^x} (-1)^{s_p(n+\varepsilon r)},
\]
where \( \varepsilon \cdot r = \sum_{1 \leq i \leq m} \varepsilon_i r_i \) and \( s_p \) is the truncated sum-of-digits function in base 2 defined by \( s_p(n) = s(n \mod 2^p) \). The proof of a very similar statement was given recently by Konieczny [21], and we use ideas from that paper to prove our estimate.

In order to put Theorem 1.1 into context, we present some related theorems. The well-known Bombieri–Vinogradov theorem concerns the level of distribution of the von Mangoldt function...
\( \Lambda \), which is defined by \( \Lambda(n) = \log p \) if \( n = p^k \) for some prime \( p \) and some \( k \geq 1 \) and \( \Lambda(n) = 0 \) otherwise. This theorem states that \( \Lambda \) has level of distribution 1/2 with respect to \( \mathcal{Q} \) and \( \mathcal{R} \) given by \( \mathcal{Q}(d) = \mathcal{R}(d) \equiv (\mathbb{Z}/d\mathbb{Z})^* \).

**Theorem B** (Bombieri–Vinogradov). Let \( d \geq 1 \) and \( a \) be integers and define

\[
\psi(x; d, a) = \sum_{\substack{1 \leq n \leq x \mod d \leq D \leq \max \max_{0 \leq a < d}} 1 \leq y \leq x} \Lambda(n).
\]

For all real numbers \( A > 0 \) there exist \( B > 0 \) and a constant \( C \) such that setting \( D = x^{1/2}(\log x)^{-\frac{1}{2}} \) we have for all \( x \geq 2 \)

\[
\sum_{1 \leq d \leq D} \max_{1 \leq y \leq x} \max_{0 \leq a < d} \frac{\psi(y; d, a) - y}{\varphi(d)} \leq Cx(\log x)^{-A}.
\]

Here \( \varphi \) denotes Euler’s totient function.

No improvement on the level of distribution 1/2 in this theorem is currently known; meanwhile the Elliott–Halberstam conjecture \( [10] \) states that we can choose \( D = x^{1-\varepsilon} \) for any \( \varepsilon > 0 \). That is, it is conjectured that the primes have level of distribution 1. Improvements on the exponent 1/2 exist for certain sequences of integers; we refer to the articles \( [11, 12] \) by Fouvry, by Fouvry and Iwaniec \( [13] \) and by Friedlander and Iwaniec \( [17] \). Moreover, we note the series \( [3, 4, 5] \) by Bombieri, Friedlander and Iwaniec concerning these questions. In this context, we also note the result of Goldston, Pintz, and Yıldırım \( [19] \), who showed in particular the following conditional result: if the primes have level of distribution \( \theta \) for some \( \theta > 1/2 \), there is a constant \( C \) such that \( p_{n+1} - p_n < C \) infinitely often, where \( p_n \) is the \( n \)-th prime. In a groundbreaking paper we mentioned before, Zhang \( [36] \) used the Goldston–Pintz–Yıldırım method and a variant of the Bombieri–Vinogradov theorem to prove the above result unconditionally. Maynard \( [29] \) later proved the bounded gaps result using only the classical Bombieri–Vinogradov theorem.

Improvements on the level 1/2 are also known for the sum-of-digits function modulo \( m \). Fouvry and Mauduit \( [15] \) established 0.5924 as level of distribution of the Thue–Morse sequence, with respect to \( \mathcal{Q} \) and \( \mathcal{R} \), where \( \mathcal{Q}(d) = \mathcal{R}(d) = \mathbb{Z}/d\mathbb{Z} \).

**Theorem C** (Fouvry–Mauduit). Set

\[
A(x; d, a) = \left\lfloor \left\{ 0 \leq n < x : t(n) = 0, n \equiv a \mod d \right\} \right\rfloor.
\]

Then

\[
(1.1) \quad \sum_{1 \leq d \leq D} \max_{1 \leq y \leq x} \max_{0 \leq a < d} \left| A(y; d, a) - \frac{y}{2d} \right| \leq Cx(\log 2x)^{-A}
\]

for all real \( A \) and \( D = x^{0.5924} \).

More generally, for \( m \geq 2 \) they also study the sum-of-digits function in base 2 modulo \( m \), obtaining the weaker level of distribution 0.55711. Using sieve theory, they apply this result to the study of the sum of digits modulo \( m \) of numbers having at most two prime factors. Later, Mauduit and Rivat \( [28] \), in an important paper, managed to treat the sum of digits modulo \( m \) of prime numbers, thereby answering one of the questions posed by Gelfond \( [13] \).

Müllner and the author \( [31] \) improved the exponent 0.5924 to \( \frac{2}{3} - \varepsilon \), thereby establishing 2/3 as an admissible level of distribution of the Thue–Morse sequence.

Fouvry and Mauduit \( [14] \) also considered, more generally, the sum-of-digits function \( s_q \) in base \( q \) modulo an integer \( m \) such that \( \gcd(m, q - 1) = 1 \). They obtain the remarkable result that the level of distribution approaches 1 as the base \( q \) gets larger.
Theorem D (Fouvry–Mauduit). Let \( q \geq 2, m \geq 1 \) and \( b \) be integers such that \( \gcd(m, q-1) = 1 \). Then for all \( A \) and \( \varepsilon > 0 \) we have for all \( x \geq 1 \)

\[
\sum_{1 \leq d \leq x^{\theta_q - \varepsilon}} \max_{0 \leq y \leq x} \max_{0 \leq a < d} \left| \sum_{n < y, s_q(n) \equiv b \mod m} \frac{1}{n} - \sum_{n \equiv a \mod d} \frac{1}{n} - \frac{1}{d} \right| = O(x(\log 2x)^{-A}),
\]

where \( \theta_q \rightarrow 1 \) as \( q \rightarrow \infty \). The implied constant depends at most on \( m, q, A \) and \( \varepsilon \).

As an application of this theorem, they study the sum \( \sum_{n < x, s_q(n) \equiv b \mod m} \Lambda_k(n) \), where \( \Lambda_k(n) \) is the generalized von Mangoldt function of order \( k \geq 1 \) ([14 Corollaire 2]).

Theorem D motivates us to ask which sequences have level of distribution equal to 1. In the above-cited paper by Fouvry and Mauduit [14], for example, a list of sequences having this property is given. Moreover, we note [16 Chapter 22.3], which studies the level of distribution for additive convolutions, giving further examples. However, in these examples, other than the trivial example \( c_n = 1 \) for all \( n \), the maximum over \( a \) does not play a rôle: the set \( Q(d) \) consists of at most one element.

We are interested in sequences \( c \) having level of distribution 1 and such that the set \( Q(d) \) contains “many” residue classes. In other words, we want to find analogues of the Elliott–Halberstam conjecture. Requiring monotonicity of \( c \), examples can be constructed easily: \( c(n) = n \) is such an example, and more generally, increasing sequences \( c \) satisfying certain growth conditions have this property. Apart from such “trivial” sequences, no other examples seem to be known. Our Theorem 1.1 giving such an example, is therefore of considerable significance.

Moreover, we note that our method can certainly be adapted to \( s_q(n) \) mod \( m \) for general bases \( q \geq 2 \) and \( m \geq 1 \), which yields \( \theta_q = 1 \) in Theorem D.

The second focus of this paper concerns Piatetski-Shapiro sequences, which are sequences of the form \( ([n^c])_{n \geq 0} \) for some \( c \geq 1 \). For stating the second main theorem, we do not need additional preparation.

**Theorem 1.2.** Let \( 1 < c < 2 \). The Thue–Morse sequence along \( [n^c] \) is simply normal. That is, each of the letters 0 and 1 appears with asymptotic frequency 1/2 in \( n \rightarrow t([n^c]) \).

By the argument given in our earlier paper [31] with Müllein, this theorem is proved via a Beatty sequence variant of Theorem 1.1. That theorem in turn is proved by arguments analogous to the arguments in the proof of Theorem 1.1 and reduces to the same estimate of the Gowers uniformity norm of Thue–Morse. Theorem 1.2 is therefore an application of the method of proof of Theorem 1.1.

Again, we present some historical background. Studying Piatetski–Shapiro subsequences of a given sequence can be seen as a step towards proving theorems on polynomial subsequences. For example, it is unknown whether there are infinitely many primes of the form \( n^2 + 1 \); therefore it is of interest to consider primes of the form \( [n^c] \) for \( 1 < c < 2 \) and prove an asymptotic formula for the number of such primes. Piatetski-Shapiro [32] proved such a formula for \( 1 < c < 12/11 \), and the currently best known bound is \( 1 < c < 2817/2426 \) due to Rivat and Sargos [33]. In an analogous way, the study of the sum-of-digits function along \( [n^c] \) was motivated. It is another problem posed by Gelfond [18] to study the distribution of the sum of digits of polynomial sequences in residue classes. Since this problem could not be solved at first, Mauduit and Rivat [25, 26] considered \( q \)-multiplicative functions along \( [n^c] \) (where a \( q \)-multiplicative function \( f : \mathbb{N} \rightarrow \{z \in \mathbb{C} : |z| = 1\} \) satisfies \( f(aq^k + b) = f(aq^k)f(b) \) for nonnegative integers \( a, b, k \) such that \( b < q^k \)) and they obtained an asymptotic formula for \( c < 7/5 \).

**Theorem E** (Mauduit–Rivat). Let \( 1 < c < 7/5 \) and set \( \gamma = 1/c \). For all \( \delta \in (0, (7-5c)/9) \) there exists a constant \( C > 0 \) such that for all \( q \)-multiplicative functions \( f : \mathbb{N} \rightarrow \{z \in \mathbb{C} : |z| = 1\} \)
and all \( x \geq 1 \) we have

\[
\left| \sum_{1 \leq n \leq x} f(\lfloor n^c \rfloor) - \sum_{1 \leq m \leq x^c} \gamma m^{\gamma-1} f(m) \right| \leq Cx^{1-\delta}.
\]

Since the Thue–Morse sequence is 2-multiplicative, it follows in particular that the subsequence indexed by \([n^c]\) assumes each of the two values 0, 1 with asymptotic frequency 1/2, as long as \(1 < c < 7/5\). This means that this subsequence is simply normal. In the paper [7] by Deshouillers, Drmota, and Morgenbesser, a statement as in Theorem E for arbitrary automatic sequences and \(1 < c < 7/5\) is proved.

Some progress on Gelfond’s question on polynomials was made by Drmota and Rivat [9] and by Dartyge and Tenenbaum [6]; finally, Mauduit and Rivat [27] managed to answer Gelfond’s question for the polynomial \(n^2\). This latter paper was generalized by Drmota, Mauduit and Rivat [8], who showed that in fact \(t(n^2)\) defines a normal sequence, by which we understand an infinite sequence on \(\{0, 1\}\) such that every finite sequence of length \(k\) occurs as a factor (contiguous finite subsequence) with asymptotic frequency \(2^{-k}\). This result also generalizes a paper by Moshe [30] who showed that every finite word on \(\{0, 1\}\) occurs as a factor of \(n \mapsto t(n^2)\) at least once.

However, the distribution of the sum of digits of \([n^c]\) in residue classes, for \(c \in [1, 4, 2]\), remained an open problem. Progress in this direction was made by the author [34], who improved the bound on \(c\) to \(1 < c \leq 1.42\) for the Thue–Morse sequence. The key idea in that paper is to approximate \([n^c]\) by a Beatty sequence \([na + \beta]\) and thus reduce the problem to a linear one.

Müllner and the author [31], using the same linearization argument and a Bombieri–Vinogradov type theorem for the Thue–Morse sequence on Beatty sequences, were able to extend this range to \(1 < c < 3/2\). Moreover, we could handle occurrences of factors in Piatetski-Shapiro subsequences of \(t\), thus showing that \(t([n^c])\) defines a normal sequence for \(1 < c < 3/2\).

**Theorem F** (Müllner–Spiegelhofer). Let \(1 < c < 3/2\). Then the sequence \(u = (t([n^c]))_{n \geq 0}\) is normal. More precisely, for any \(L \geq 1\) there exists an exponent \(\eta > 0\) and a constant \(C\) such that

\[
\left| \{ n < N : u(n+i) = \omega_i \text{ for } 0 \leq i < L \} - N/2^L \right| \leq CN^{1-\eta}
\]

for all \((\omega_0, \ldots, \omega_{L-1}) \in \{0, 1\}^L\).

This theorem also improved on an earlier result by the author [35], who obtained normality for \(1 < c < 4/3\), using an estimate for Fourier coefficients related to the Thue–Morse sequence provided by Drmota, Mauduit and Rivat [8].

Our Theorem 1.2 finally closes the gap in the set of exponents \(c\) such that we have an asymptotic formula for Thue–Morse on \([n^c]\). This gap appeared with the Mauduit–Rivat result on squares; at that time, the gap was \([1.4, 2]\), now it was only left to close the smaller gap \([1.5, 2]\).

However, the case \(c > 2\) remains open for now, for \(c \in \mathbb{Z}\) (which is contained in Gelfond’s problem on polynomial subsequences) as well as for Piatetski-Shapiro sequences. For example, it is a notorious open question to prove that 0 occurs with frequency 1/2 in \(n \mapsto t(n^3)\). (If this result is proved some day, there will be a new gap to be closed.)

Mauduit [24] Conjecture 1] conjectures that

\[
\lim_{N \to \infty} \frac{1}{N} \{ 1 \leq n \leq N : s_q([n^c]) \equiv b \mod m \} = \frac{1}{m}
\]

for almost all \(c > 1\), where \(q \geq 2, m \geq 1\) and \(b\) are integers. While this almost-all result is known for \(1 < c < 2\), as he notes just before this conjecture, we believe (as we noted before) that our method can be adapted to generalize our results to general sequences \(s_q(n) \mod m\) and
thus to prove the asymptotic identity for all $c \in (1, 2)$. However, while we are confident that the asymptotic identity in Mauduit’s conjecture holds for all non-integer $c > 1$, the case $c > 2$ cannot yet be handled by our methods.

Moreover, we note that it would be interesting to generalize the normality result from Theorem F to all exponents $1 < c < 2$.

**Notation.** For a real number $x$, we write $e(x) = \exp(2\pi i x)$, $\{x\} = x - \lfloor x \rfloor$, $\|x\| = \min_{n \in \mathbb{Z}}|x - n|$ and $\langle \cdot \rangle = \lfloor x + 1/2 \rfloor$ (the “nearest integer” to $x$). For a prime number $p$ let $\nu_p(n)$ be the exponent of $p$ in the prime factorization of $n$. We define the truncated binary sum-of-digits function $s_\lambda(n) := s(n')$, where $0 \leq n' < 2^\lambda$ and $n' \equiv n \mod 2^\lambda$, which is the $2^\lambda$-periodic extension of the restriction of $s$ to $\{0, \ldots, 2^\lambda - 1\}$. For $\mu \leq \lambda$ we define the two-fold restricted binary sum-of-digits function $s_{\mu,\lambda}(n) = s_\lambda(n) - s_\mu(n)$.

For a real number $x \geq 0$, we set $\log^+ x = \max \{1, \log x\}$.

The symbol $\mathbb{N}$ denotes the set of nonnegative integers.

2. Results

In order to (re)state our main theorem, we introduce some notation. Let $\alpha, \beta, y$ and $z$ be nonnegative real numbers such that $\alpha \geq 1$. We define

$$A(y, z; \alpha, \beta) = \left\lfloor 1 \leq d \leq D \right\rfloor \max_{0 \leq z'y \leq z, 0 \leq s < d, s \equiv a \mod d} \left| A(y, z; d, a) - \frac{y}{2d} \right| \leq Cx^{1-\eta}$$

for $x \geq 1$ and $D = x^{1-\varepsilon}$.

Note that this theorem allows intervals $[y, z)$ for arbitrary $y \geq 0$, which is more general than our definition of a level of distribution.

Our second result concerns Piatetski-Shapiro subsequences of the Thue–Morse sequence.

**Theorem 2.1.** The Thue–Morse sequence has level of distribution 1. More precisely, for all $\varepsilon > 0$ there exist $\eta > 0$ and $C$ such that

$$\sum_{1 \leq d \leq D} \max_{0 \leq z'y \leq z, 0 \leq s < d, s \equiv a \mod d} \left| A(y, z; d, a) - \frac{y}{2d} \right| \leq Cx^{1-\eta}$$

for $x \geq 1$ and $D = x^{1-\varepsilon}$.

For proving this theorem, we follow the general argument presented in Section 4.2 of [31]. This argument uses linear approximation of $\lfloor n^c \rfloor$ by $\lfloor n\alpha + \beta \rfloor$ and thus reduces the problem to Beatty sequences. Therefore Theorem 2.1 is a corollary of the following Beatty sequence version of a statement on the level of distribution.

**Theorem 2.2.** Let $1 < c < 2$. Then the sequence $n \mapsto t(\lfloor n^c \rfloor)$ is simply normal. More precisely, there exists an exponent $\eta > 0$ and a constant $C$ such that

$$\frac{1}{N} \left| \{0 \leq n < N : t(\lfloor n^c \rfloor) = 0\} \right| - \frac{1}{2} \leq CN^{-\eta}.$$
Theorem 2.3. Let $0 < \theta_1 \leq \theta_2 < 1$. There exist $\eta > 0$ and $C$ such that
\[
\int_0^x \max_{y \geq 0} \max_{\beta \geq 0} \left| A(y, z; \alpha, \beta) - \frac{z - y}{2\alpha} \right| \, \text{d}a \leq C x^{1-\eta}
\]
for all $x$ and $D$ such that $x \geq 1$ and $x^{\theta_1} \leq D \leq x^{\theta_2}$.

In order to derive Theorem 2.2 from this result, it is essential that we have the maximum over $\beta$ inside the integral over $\alpha$, since we need to approximate $\lfloor n \rceil$ by inhomogeneous (shifted) Beatty sequences $\lfloor n \alpha + \beta \rceil$.

Concerning Theorem 2.1, a version of this result without the maximum over $a$ follows from work of Martin, Mauduit and Rivat, as we show now.

Remark. Martin, Mauduit and Rivat [23, Proposition 3] proved an estimate of a sum of type II containing the following special case: let $a_m$ and $b_n$ be complex numbers satisfying $|a_m| \leq 1$ and $|b_n| \leq 1$. Assume that $x \geq 2$, $0 < \varepsilon \leq 1/2$, $x^\varepsilon \leq M$, $N \leq x$ and $MN \leq x$. Then
\[
S_0 = \sum_{M < m \leq 2M} \sum_{N < n \leq 2N} \sum_{0 \leq \varepsilon \leq x} a_m b_n (\varepsilon^{\lfloor mn \rfloor}) \ll x^{1-\eta}
\]
for an absolute implied constant and some $\eta > 0$ only depending on $\varepsilon$. By dyadic decomposition and using the trivial estimate for $n < x^\varepsilon$, we obtain
\[
\sum_{M < m \leq 2M} \left| \sum_{0 \leq \varepsilon \leq x} a_m b_n (\varepsilon^{\lfloor mn \rfloor}) \right| \ll x^{1-\eta} \log N + M x^\varepsilon
\]
for $M$ and $N$ satisfying the same restrictions, and with an implied constant that may depend on $\varepsilon$. Let $x$ be given and assume that $x^\varepsilon \leq M \leq x^{\theta_1}$ for some $\theta \in (1/2, 1)$. Set $\varepsilon = \frac{1-\theta}{2} \leq 1/2$ and $N = x/M$. Then $N \geq x^\varepsilon$ and the condition $mn \leq x$ implies $n \leq 2N$. We obtain
\[
\sum_{M < m \leq 2M} \left| \sum_{0 \leq \varepsilon \leq x} a_m b_n (\varepsilon^{\lfloor mn \rfloor}) \right| \ll x^{1-\eta} \log x + M x^\varepsilon.
\]

We use dyadic decomposition again (in $m$), moreover Fouvry and Mauduit [15] in order to handle residue classes having small modulus $m$, that is, $m \leq x^\varepsilon$. Moreover, we note (as we did in [31]) that the error term in their estimate [15] (1.6)) is in fact $x^{1-\eta}$ for some $\eta > 0$, which follows from their Théorème 2. We obtain
\[
\sum_{1 \leq d \leq D} \left| \sum_{0 \leq n \leq x} \sum_{n \equiv 0 \mod d} (-1)^{s(n)} \right| \leq C x^{1-\eta}
\]
for $D = x^\theta$ and some $\eta > 0$ and $C$ depending on $\theta$. This is a weak version of a statement of the type “the Thue–Morse sequence has level of distribution 1”, where $Q(d)$ has only one element. (We note that we could also handle the maximum over $y \leq x$, using the factor $c(\beta mn)$ that appears in [23, Proposition 3].) The additional value of our paper lies in the maximum over the residue classes modulo $d$.

Finally, we note the following open questions concerning Theorems 2.1 and 2.2:

1. In Theorem 2.1 can we choose $D = x (\log x)^{-B}$ for some $B > 0$ (using $x (\log x)^{-A}$ as error term)?
2. Does Theorem 2.2 hold for $|x^2 (\log x)^{-C}|$ (and similar sequences, possibly with a worse error term)?
Plan of the paper. In Section 3 we state two results (Propositions 3.1 and 3.2) from which Theorems 2.1 and 2.3 follow, moreover an important Gowers uniformity norm estimate of the Thue–Morse sequence, Proposition 3.3. We also give an idea of the proof of Proposition 3.1. In Section 4 we state lemmas needed for proving the results from Section 3. Section 5 is devoted to proving Propositions 3.1 and 3.2. Finally, we prove Proposition 3.3 and a technical lemma appearing in the proof of Propositions 3.1 and 3.2.

3. Auxiliary results

As in our earlier paper with Müllner ([31, Section 4.1], and using Fouvry and Mauduit [15, Théorème 2] for handling small $d$), it is sufficient to prove the following two results in order to obtain our main theorems.

**Proposition 3.1.** For real numbers $N, D \geq 1$ and $\xi$ set

\[
S_0 = S_0(N, D, \xi) = \sum_{D \leq d < 2D} \max_{a \geq 0} \left| \sum_{0 \leq n < N} e\left(\frac{1}{2} s(nd + a)\right) e(n\xi) \right|.
\]

Let $\rho_2 \geq \rho_1 > 0$. There exists an $\eta > 0$ and a constant $C$ such that

\[
\frac{S_0}{ND} \leq CN^{-\eta}
\]

holds for all $\xi \in \mathbb{R}$ and all real numbers $N, D \geq 1$ satisfying $N^{\rho_1} \leq D \leq N^{\rho_2}$.

**Proposition 3.2.** For real numbers $D, N \geq 1$ and $\xi$ set

\[
S_0 = S_0(N, D, \xi) = \int_D^{2D} \max_{\beta \geq 0} \left| \sum_{0 \leq n < N} e\left(\frac{1}{2} s(\lfloor n\alpha + \beta \rfloor)\right) e(n\xi) \right| \, d\alpha.
\]

Let $\rho_2 \geq \rho_1 > 0$. There exist $\eta > 0$ and a constant $C$ such that

\[
\frac{S_0}{ND} \leq CN^{-\eta}
\]

holds for all real numbers $D, N \geq 1$ satisfying $N^{\rho_1} \leq D \leq N^{\rho_2}$ and for all $\xi \in \mathbb{R}$.

In the proof of these results, we will use the following essential Gowers uniformity norm estimate of the Thue–Morse sequence (see Konieczny [21]).

**Proposition 3.3.** Let $m \geq 2$ be an integer. There exists some $\eta > 0$ and some $C$ such that

\[
\frac{1}{2^{(m+1)\rho}} \sum_{0 \leq n < 2^\rho} e\left(\frac{1}{2} \sum_{\varepsilon \in \{0,1\}^m} s_\rho(n + \varepsilon \cdot r)\right) \leq C2^{-\rho\eta}
\]

for all $\rho > 0$, where $\varepsilon \cdot r = \sum_{1 \leq i \leq m} \varepsilon_i r_i$.

We wish to give a rough idea of the proof of Proposition 3.1 (Proposition 3.2 being proved essentially in the same way.)

**Idea of the proof of Proposition 3.1.** The key idea is to reduce the number of digits that have to be taken into account, and thus to replace the sum-of-digits function $s$ by its truncated version $s_\rho$. Here $2^\rho$ will be significantly smaller than $N$, so that (we simplify things a bit to convey the idea) we may replace the sum over $s(nd + a)$ by a full sum over the periodic function $s_\rho(n)$. This reducing of the digits is achieved by a refinement of the method used by Müllner and the author [31], which in turn builds on the ideas from the papers [27, 28] by Mauduit and Rivat.

First, we apply van der Corput’s inequality and use a “carry propagation lemma” in order to replace $s$ by $s_\lambda$. In general, $2^\lambda$ will be much larger than $N$, so that we have to reduce $\lambda$
further. The next step is to apply the generalized van der Corput inequality repeatedly. With each application, we remove $\mu$ many digits. This is achieved by appealing to the Dirichlet approximation theorem, by which we can find a multiple of $\alpha = d/2^{\mu}$ that is close to a multiple of $2^\mu$. This property can be used to discard the $\mu$ lowest digits.

By this repeated application the estimate is reduced to an estimate of a so-called Gowers uniformity norm of the Thue–Morse sequence; a related estimate was recently given by Konieczny [21].

4. Lemmas

We have the following series of lemmas that can also be found in our earlier paper with Müller [31].

The first lemma can be proved by elementary considerations.

Lemma 4.1. Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$.

(4.1) If $\|a\| < \varepsilon$ and $\|b\| \geq \varepsilon$, then $\lfloor a + b \rfloor = \langle a \rangle + \lfloor b \rfloor$.

(4.2) $\|na\| \leq n\|a\|$.

(4.3) If $\|a\| < \varepsilon$ and $2n\varepsilon < 1$, then $\langle na \rangle = n\langle a \rangle$.

As an essential tool, we will use repeatedly the following generalized van der Corput inequality [27, Lemme 17].

Lemma 4.2. Let $I$ be a finite interval containing $N$ integers and let $z_n$ be a complex number for $n \in I$. For all integers $K \geq 1$ and $R \geq 1$ we have

$$\left| \sum_{n \in I} z_n \right|^2 \leq \frac{N + K(R - 1)}{R} \sum_{0 \leq |r| < R} \left( 1 - \frac{|r|}{R} \right) \sum_{\substack{n \in I \atop n + Kr \in I}} z_{n + Kr} \overline{z_n}.$$  

Assume that $\alpha$ is a real number and $N$ is a nonnegative integer. We define the discrepancy of the sequence $n\alpha$ modulo 1:

$$D_N(\alpha) = \sup_{0 \leq x \leq 1} \left| \frac{1}{N} \sum_{n < N} c_{[0,x]} + y + \alpha(n\alpha) - x \right|.$$  

Applying this definition, using $x = 1/(KT)$ and $\alpha/K$ instead of $\alpha$, we obtain the following lemma.

Lemma 4.3. Let $J$ be an interval in $\mathbb{R}$ containing $N$ integers and let $\alpha$ and $\beta$ be real numbers. Assume that $t, T, k$ and $K$ are integers such that $0 \leq t < T$ and $0 \leq k < K$. Then

$$\left| \{ n \in J : \frac{t}{T} \leq \{ n\alpha + \beta \} < \frac{t + 1}{T}, \{ n\alpha + \beta \} \equiv k \mod K \} \right| = \frac{N}{KT} + O \left( N D_N \left( \frac{\alpha}{K} \right) \right)$$  

with an absolute implied constant.

In the estimation of our error terms, we will use the following mean discrepancy results.

Lemma 4.4. For integers $\mu \geq 0$ and $N \geq 1$ we have

$$\sum_{0 \leq d < 2^\mu} D_N \left( \frac{d}{2^\mu} \right) \ll \frac{N + 2^\mu}{N} \left( \log^+ N \right)^2.$$  

Moreover, the estimate

$$\int_0^1 D_N(\alpha) \, d\alpha \ll \frac{\left( \log^+ N \right)^2}{N}$$  

holds. The implied constants in these estimates are absolute.
The following “carry propagation lemma” will allow us to replace the sum-of-digits function $s$ by its truncated version $s_\lambda$. Statements of this type were used by Mauduit and Rivat in their papers on the sum of digits of primes and squares \[27, 28\].

**Lemma 4.5.** Let $r, N, \lambda$ be nonnegative integers and $\alpha > 0, \beta \geq 0$ real numbers. Assume that $I$ is an interval containing $N$ integers. Then

$$
\left| \{ n \in I : s(\lfloor (n+r)\alpha + \beta \rfloor) - s(\lfloor n\alpha + \beta \rfloor) \neq s_\lambda(\lfloor (n+r)\alpha + \beta \rfloor) - s_\lambda(\lfloor n\alpha + \beta \rfloor) \} \right| \\
\leq r(N\alpha/2^\lambda + 2).
$$

Let $F_n$ the set of rational numbers $p/q$ such that $1 \leq q \leq n$, the Farey series of order $n$. Each $a \in F_n$ has two neighbours $a_L, a_R \in F_n$, satisfying $a_L < a < a_R$ and $(a_L, a) \cap F_n = (a, a_R) \cap F_n = \emptyset$. We have the following elementary lemma concerning this set (see \[20\] chapter 3).

**Lemma 4.6.** Assume that $a/b$, $c/d$ are reduced fractions such that $b, d > 0$ and $a/b < c/d$. Then $a/b < (a+c)/(b+d) < c/d$. If $a/b$ and $c/d$ are neighbours in the Farey series $F_n$, then $bc - ad = 1$ and $b + d > n$, moreover

$$(a + c)/(b + d) - a/b < \frac{1}{bn} \quad \text{and} \quad c/d - (a + c)/(b + d) < \frac{1}{dn}. $$

Let $\alpha \in \mathbb{R}$ and $Q$ a positive integer. We assign a fraction $pQ(\alpha)/qQ(\alpha)$ to $\alpha$ according to the Farey dissection of the reals: consider reduced fractions $a/b < c/d$ that are neighbours in the Farey series $F_Q$, such that $a/b \leq \alpha < c/d$. If $\alpha < (a+c)/(b+d)$, then set $pQ(\alpha) = a$ and $qQ(\alpha) = b$, otherwise set $pQ(\alpha) = a$ and $qQ(\alpha) = d$. Lemma 4.6 implies

$$\left| \frac{qQ(\alpha)\alpha - pQ(\alpha)}{Q} \right| < Q^{-1}. $$

We will call an interval of the form $\{ \alpha \in \mathbb{R} : pQ(\alpha) = p, qQ(\alpha) = q \}$ a Farey interval around $p/q$.

5. **Proof of Propositions 3.1 and 3.2**

As in the proof of Proposition 2.5 in \[31\], it is sufficient to prove that there exists $\eta > 0$ and a constant $C$ such that

$$
\frac{S_0(N, 2^\nu, \xi)}{N 2^\nu} \leq CN^{-\eta}
$$

for all real numbers $\xi$ and for all positive integers $N$ and $\nu$ such that there exists a real number $D \geq 1$ satisfying $N^{p_1} \leq D \leq N^{p_2}$ and $D < 2^\nu \leq 2D$, where $S_0$ is defined according to \[3.1\] and \[3.2\].

In order to treat the two propositions to some extent in parallel, we will work with two measures $\mu$: for Proposition 3.1 we take the measure defined by $\mu(A) = |A \cap \mathbb{Z}|$, while for Proposition 3.2 $\mu$ is the Lebesgue measure. Moreover, we note that in this proof, implied constants in estimates depend only on $m$.

By Cauchy–Schwarz, followed by van der Corput’s inequality \[14\] ($R_0$ will be specified later), we obtain

$$
\left| S_0(N, 2^\nu, \xi) \right|^2 \leq 2^\nu \frac{N + R_0}{R_0} \int_{2^\nu}^{2^\nu+1} \sup_{\beta \geq 0} \sum_{0 \leq |r_0| < R_0} \left( 1 - \frac{|r_0|}{R_0} \right) e(r_0\xi) \times \sum_{0 \leq n < N} e\left( \frac{1}{2} s(\lfloor (n+r_0)\alpha + \beta \rfloor) - \frac{1}{2} s(\lfloor n\alpha + \beta \rfloor) \right) d\mu(\alpha)
$$
We apply the “carry propagation lemma” (Lemma 4.5), treat the summand \( r_0 = 0 \) separately, and omit the condition \( 0 \leq n + r_0 < N \). Moreover, we consider \( r_0 \) and \(-r_0\) synchronously. In this way we obtain for all \( \lambda \geq 0 \)

\[
|S_0(N, 2^\nu, \xi)|^2 \ll \frac{2^\nu N^2}{R_0} \sum_{1 \leq r_0 < R_0} \left( \sum_{\nu = 0}^{2^{\nu+1}} \sup_{\beta \geq 0} \left| \sum_{0 \leq n < N} e \left( \frac{1}{2} s_{\lambda\xi} \left( \lfloor n + (n + r_0)\alpha + \beta \rfloor \right) - \frac{1}{2} s_{\lambda\xi} \left( \lfloor n\alpha + \beta \rfloor \right) \right) \right| \right) d\mu(\alpha),
\]

where

\[
E_0 = \frac{1}{R_0} + \frac{R_0}{2R} + \frac{R_0}{R N}.
\]

We apply Cauchy–Schwarz on the sum over \( r_0 \) and the integral over \( \alpha \) in order to prepare our expression for another application of van der Corput’s inequality. It follows that

\[
|S_0(N, 2^\nu, \xi)|^4 \ll \frac{2^{3\nu} N^2}{R_0} \sum_{1 \leq r_0 < R_0} \left( \sum_{\nu = 0}^{2^{\nu+1}} \sup_{\beta \geq 0} |S_1|^2 \right) d\mu(\alpha) + (2^\nu N)^4 E_0,
\]

where

\[
S_1 = \sum_{0 \leq n < N} e \left( \frac{1}{2} s_{\lambda\xi} \left( \lfloor n + (n + r_0)\alpha + \beta \rfloor \right) - \frac{1}{2} s_{\lambda\xi} \left( \lfloor n\alpha + \beta \rfloor \right) \right).
\]

(Note that the error term is also squared, but if it is larger or equal to 1, the estimate is trivial anyway. We will use this argument again in a moment.) We apply van der Corput’s inequality \([4.4]\) with \( R = R_1 \) and \( K = K_1 \) to be chosen later:

\[
|S_1|^2 \leq \frac{N + K_1(R_1 - 1)}{R_1} \sum_{0 \leq |r_1| < R_1} \left( 1 - \frac{|r_1|}{R_1} \right) \times \sum_{0 \leq n < N \atop 0 \leq n + r_1 K_1 < N} \left( \frac{1}{2} \right) \sum_{\nu_0, \nu_1 \in \{0, 1\}} s_{\lambda\xi} \left( \lfloor n + \nu_0 r_0 + \nu_1 r_1 K_1 \alpha + \beta \rfloor \right),
\]

therefore, taking together the summands for \( r_1 \) and \(-r_1\) and omitting the condition \( 0 \leq n + r_1 K_1 < N \),

\[
|S_0(N, 2^\nu, \xi)|^4 \ll \frac{2^{3\nu} N^3}{R_0 R_1} \sum_{1 \leq r_0 < R_0} \left( \sum_{\nu = 0}^{2^{\nu+1}} \sup_{\beta \geq 0} |S_2|^2 \right) d\mu(\alpha) + (2^\nu N)^4 (E_0 + E_1),
\]

where

\[
S_2 = \sum_{0 \leq n < N} e \left( \frac{1}{2} \sum_{\nu_0, \nu_1 \in \{0, 1\}} s_{\lambda\xi} \left( \lfloor n + \nu_0 r_0 + \nu_1 r_1 K_1 \alpha + \beta \rfloor \right) \right)
\]

and

\[
E_1 = \frac{R_1 K_1}{N}.
\]

Cauchy–Schwarz over \( r_0, r_1 \) and \( \alpha \) yields

\[
|S_0(N, \nu, \xi)|^8 \ll \frac{2^6 N^6}{R_0 R_1} \sum_{1 \leq r_0 < R_0} \left( \sum_{\nu = 0}^{2^{\nu+1}} \sup_{\beta \geq 0} |S_2|^2 \right) d\mu(\alpha) + (2^\nu N)^8 (E_0 + E_1).
\]
We apply van der Corput’s inequality with $R = R_2$ and $K = K_2$ to be chosen later:

$$\frac{|S_0(N, 2^\nu, \xi)|^8}{(2^\nu N)^8} \ll (E_0 + E_1 + E_2) + \frac{1}{R_0 R_1 R_2 2^{\nu N}} \sum_{1 \leq r_0 < R_0 \atop 0 \leq r_1 < R_1 \atop 0 \leq r_2 < R_2} \int_{2^\nu}^{2^{\nu+1}} \sup_{\beta \geq 0} |S_3| \, d\mu(\alpha),$$

where

$$S_3 = \sum_{0 \leq n < N} e \left( \frac{1}{2} n \alpha + \sum_{\epsilon_0, \epsilon_1, \epsilon_2 \in \{0, 1\}} s_\lambda \left( [n\alpha + \beta + \epsilon_0 r_0 \alpha + \epsilon_1 r_1 K_1 \alpha + \epsilon_2 r_2 K_2 \alpha] \right) \right)$$

and $E_2 = R_2 K_2 / N$. Continuing in this manner and replacing the range of integration (we note that we are going to choose $\lambda > \nu$ later), we obtain

$$(5.1) \quad \frac{|S_0(N, 2^\nu, \xi)|^{2^{\nu+1}}}{2^{\nu N}} \ll (E_0 + E_1 + \cdots + E_m) + \frac{1}{R_0 R_1 \cdots R_m 2^{\nu N}} \sum_{1 \leq r_0 < R_0 \atop 0 \leq r_1 < R_1 \atop 0 \leq r_2 < R_2} \int_{2^\nu}^{2^{\nu+1}} \sup_{\beta \geq 0} |S_4| \, d\mu(\alpha),$$

where

$$S_4 = \sum_{0 \leq n < N} e \left( \frac{1}{2} n \alpha + \sum_{\epsilon_0, \ldots, \epsilon_m \in \{0, 1\}} s_\lambda \left( [n\alpha + \beta + \sum_{i=0}^m \epsilon_i r_i K_i \alpha] \right) \right)$$

and

$$E_0 = \frac{1}{R_0} + \frac{R_0}{2^\lambda} + \frac{R_0}{N}, \quad E_i = \frac{R_i K_i}{N} \quad \text{for } 1 \leq i \leq m.$$

Now we choose the multiples $K_1, \ldots, K_m$ in such a way that the number of digits to be taken into account is reduced from $\lambda$ to $\rho := \lambda - (m + 1)\mu$, where $\mu$ is chosen later. For this we use Farey series, see (4.3). Let

$$K_1 = q_{2^{2m+2}\sigma} \left( \frac{\alpha}{2^\mu} \right) q_{2^\nu} \left( \frac{p_{2^{2m+2}\sigma} \left( \frac{\alpha}{2^\mu} \right)}{2^{(m-1)\mu}} \right);$$

$$K_i = q_{2^{2}\sigma} \left( \frac{\alpha}{2^{(i+1)\mu}} \right) q_{2^\nu} \left( \frac{p_{2^{2}\sigma} \left( \frac{\alpha}{2^{(i+1)\mu}} \right)}{2^{(m-1)\mu}} \right) \quad \text{for } 2 \leq i \leq m;$$

$$K_m = q_{2^\nu} \left( \frac{\alpha}{2^{(m+1)\mu}} \right),$$

where $\sigma$ is chosen later. Moreover, we set

$$M_1 = p_{2^{2}\sigma} \left( \frac{\alpha}{2^\mu} \right) q_{2^\nu} \left( \frac{p_{2^{2}\sigma} \left( \frac{\alpha}{2^\mu} \right)}{2^{(m-1)\mu}} \right);$$

$$M_i = p_{2^{2}\sigma} \left( \frac{\alpha}{2^{(i+1)\mu}} \right) q_{2^\nu} \left( \frac{p_{2^{2}\sigma} \left( \frac{\alpha}{2^{(i+1)\mu}} \right)}{2^{(m-1)\mu}} \right) \quad \text{for } 2 \leq i \leq m;$$

$$M_m = p_{2^{2}\sigma} \left( \frac{\alpha}{2^{(m+1)\mu}} \right).$$

By Lemma 4.6 estimating the second factor in the definition of $K_i$ and $M_i$ by $2^\sigma$, we have
Applying (4.1), setting \(
\epsilon = (4.3) \) in Lemma 4.1 with \( \epsilon \) in the argument by 2\( \mu \) digits and thus reduce the number of digits to be taken into account from \( \lambda \) to \( \lambda - 2\mu \).

\[
S_4 = \sum_{0 \leq n < N} e\left(\frac{1}{2} \sum_{\epsilon_0, \ldots, \epsilon_m \in \{0,1\}} s_{2\mu, \lambda}(|n\alpha + \beta + \epsilon_0 r_0 \alpha + \epsilon_1 r_1 M_1 2^{2\mu} + \epsilon_2 r_2 K_2 \alpha + \cdots + \epsilon_m r_m K_m \alpha|)\right)
= \sum_{0 \leq n < N} e\left(\frac{1}{2} \sum_{\epsilon_0, \ldots, \epsilon_m \in \{0,1\}} s_{\lambda - 2\mu}(\left|\frac{n\alpha + \beta + \epsilon_0 r_0 \alpha + \epsilon_1 r_1 M_1 2^{2\mu} + \epsilon_2 r_2 K_2 \alpha + \cdots + \epsilon_m r_m K_m \alpha}{2^{2\mu}}\right|)\right).
\]

In the case \( \alpha \notin \mathbb{Z} \), we use the inequalities (5.2) and the argument that \( n\alpha \)-sequences are usually not close to an integer. This can be made precise as follows. Assume that

\[
\|n\alpha + \beta\| \geq R_1 / 2^\sigma,
\]

where \( \beta' = \beta + \epsilon_0 r_0 \alpha + \epsilon_2 r_2 K_2 \alpha + \cdots + \epsilon_m r_m K_m \alpha \), and that \( 2R_1 < 2^\sigma \). Using the inequality (4.3) in Lemma 4.1 with \( \epsilon = 1/2^\sigma \), where \( \sigma \geq 1 \) is chosen later, and (4.3), we obtain

\[
\langle r_1 K_1 \alpha \rangle = r_1 \langle K_1 \alpha \rangle = r_1 2^{2\mu} M_1.
\]

Applying (4.1), setting \( \epsilon = R_1 / 2^\sigma \), we see that (5.3) together with (5.2) implies

\[
|n\alpha + r_1 K_1 \alpha + \beta'| = |n\alpha + r_1 2^{2\mu} M_1 + \beta'|.
\]

The number of \( n \) where hypothesis (5.3) fails for some \( \epsilon_0, \epsilon_2, \ldots, \epsilon_m \) can be estimated by discrepancy estimates for \( \{n\alpha\} \)-sequences: for all positive integers \( N \) and \( 2R_1 < 2^\sigma \) we have

\[
\left|\left\{ n \in [0, N - 1] : |n\alpha + \beta'| \leq R_1 / 2^\sigma \right\}\right| = \left|\left\{ n \in [0, N - 1] : n\alpha + \beta' \in \left[-R_1 / 2^\sigma, R_1 / 2^\sigma\right] + \mathbb{Z}\right\}\right| = \left|\left\{ n \in [0, N - 1] : n\alpha \in [0, 2R_1 / 2^\sigma] - \beta' - R_1 / 2^\sigma + \mathbb{Z}\right\}\right| \leq N D_N(\alpha) + 2R_1 N / 2^\sigma.
\]

Therefore, the number of \( n \in [0, N - 1] \) such that \( |n\alpha + \beta'| \leq R_1 / 2^\sigma \) for some \( \epsilon_0, \epsilon_2, \ldots, \epsilon_m \in \{0,1\} \) is bounded by \( 2^n N (D_N(\alpha)+2R_1 / 2^\sigma) \), which is \( \ll N (D_N(\alpha)+2R_1 / 2^\sigma) \) by our convention that implied constants may depend on \( m \).

We replace \( K_1 \alpha \) by \( 2^{2\mu} M_1 \) and subsequently shift the digits by \( 2\mu \) and obtain

\[
S_4 = \mathcal{O}(ND_N(\alpha) + NR_1 / 2^\sigma) + \sum_{0 \leq n < N} e\left(\frac{1}{2} \sum_{\epsilon_0, \ldots, \epsilon_m \in \{0,1\}} s_{\lambda - 2\mu}\left(\left|\frac{n\alpha + \beta}{2^{2\mu}}\right|\right)\right).
\]
\[ + \varepsilon_0 r_0 \alpha + \varepsilon_1 r_1 M_1 + \varepsilon_2 r_2 K_2 \alpha + 2 \mu \cdot \varepsilon + \cdots + \varepsilon_m r_m K_m \alpha \cdot 2 \mu \cdot \varepsilon \right) \].

Repeating this argument for all \( i \in \{2, \ldots, m\} \), we obtain
\[
S_4 = NO \left( \tilde{D}_N(\alpha) + D_N \left( \frac{\alpha}{2^\mu} \right) + \cdots + D_N \left( \frac{\alpha}{2^{(m+1)\mu}} \right) + \frac{R_1 + \cdots + R_m}{2^\sigma} \right) + S_5,
\]
where \( \tilde{D}_N(\alpha) = D_N(\alpha) \) if \( \alpha \notin \mathbb{Z} \) and \( \tilde{D}_N(\alpha) = 0 \) otherwise.

Now the second factor in the definition of \( K_i \) comes into play. We use the definition of \( M_i \) together with the approximation property \((4.5)\), and apply the discrepancy estimate for \( na \)-sequences again to obtain
\[
(5.4) \quad S_4 = NO \left( \tilde{D}_N(\alpha) + D_N \left( \frac{\alpha}{2^\mu} \right) + \cdots + D_N \left( \frac{\alpha}{2^{(m+1)\mu}} \right) + \frac{R_1 + \cdots + R_m}{2^\sigma} \right) + S_5,
\]
where
\[
S_5 = \sum_{0 \leq n < N} e \left( \frac{1}{2} \sum_{\varepsilon_0, \ldots, \varepsilon_m \in \{0,1\}} s_{\lambda-(m+1)\mu} \left( \left[ \frac{n \alpha + \beta + \varepsilon_0 r_0 \alpha}{2^{(m+1)\mu}} \right] + \sum_{1 \leq i \leq m} \varepsilon_i r_i p_i \right) \right),
\]
and
\[
p_1 = p_{2^\sigma} \left( \frac{p_{2^{\mu+2^\sigma}} \left( \frac{\alpha}{2^\mu} \right)}{2^{(m-1)\mu}} \right),
\]
\[
p_i = p_{2^\sigma} \left( \frac{p_{2^{\mu+2^\sigma}} \left( \frac{\alpha}{2^\mu} \right)}{2^{(m-1)\mu}} \right) \quad \text{for } 2 \leq i < m;
\]
\[
p_m = p_{2^{\mu+2^\sigma}} \left( \frac{\alpha}{2^{(m+1)\mu}} \right).
\]

Our next goal is to remove the Beatty sequence occurring in \( S_5 \), and also to remove the integers \( p_i \). The resulting expression can be handled by the Gowers norm estimate given in Proposition \ref{prop1}, which will finish the proof.

We start by splitting the Beatty sequence into two summands. Let \( t, T \) be integers such that \( 0 \leq t < T \) and define
\[
S_6 = \sum_{0 \leq n < N} e \left( \frac{1}{2} \sum_{\varepsilon_0, \ldots, \varepsilon_m \in \{0,1\}} s_{\lambda-(m+1)\mu} \left( \left[ \frac{n \alpha + \beta + \varepsilon_0 r_0 \alpha}{2^{(m+1)\mu}} \right] + \sum_{1 \leq i \leq m} \varepsilon_i r_i p_i \right) \right),
\]

We define
\[
G = \left\{ 1 \leq t < T : \frac{t}{T} + \frac{\varepsilon_0 r_0 \alpha}{2^{(m+1)\mu}} + \frac{t + 1}{T} + \frac{\varepsilon_0 r_0 \alpha}{2^{(m+1)\mu}} \cap \mathbb{Z} = \emptyset \right\}.
\]

Clearly we have \(|G| \geq T - 2\), since we have to exclude at most one \( t \). For \( t \in \{0, \ldots, T - 1\} \setminus G \) we estimate \( S_6 \) trivially, using Lemma \ref{lem2} we obtain
\[
(5.6) \quad S_6 \leq \frac{N}{T} + ND_N \left( \frac{\alpha}{2^{(m+1)\mu}} \right).
\]
Assume that \( t \in G \) and that \( t/T \leq \{ (n \alpha + \beta)/2^{(m+1)\mu} \} < (t + 1)/T \). Then
\[
\left[ \frac{n \alpha + \beta}{2^{(m+1)\mu}} \right] + \frac{\varepsilon_0 r_0 \alpha}{2^{(m+1)\mu}} \leq \frac{n \alpha + \beta + \varepsilon_0 r_0 \alpha}{2^{(m+1)\mu}} < \left[ \frac{n \alpha + \beta}{2^{(m+1)\mu}} \right] + \frac{t + 1}{T} + \frac{\varepsilon_0 r_0 \alpha}{2^{(m+1)\mu}}.
\]
and the assumption $t \in G$ gives
\[
\left\lceil \frac{n\alpha + \beta + \varepsilon_0 r_0 \alpha}{2^{(m+1)\mu}} \right\rceil = \left\lceil \frac{n\alpha + \beta}{2^{(m+1)\mu}} \right\rceil + \left\lceil \frac{t}{T} + \frac{\varepsilon_0 r_0 \alpha}{2^{(m+1)\mu}} \right\rceil
\]
for $\varepsilon_0 \in \{0,1\}$. From these observations we obtain for $t \in G$:
\[
S_6 = \sum_{0 \leq k < 2^\omega} \sum_{0 \leq n < N} e\left(\frac{1}{2} \sum_{\varepsilon_0, \ldots, \varepsilon_m \in \{0,1\}} s_\rho \left( k + \frac{t}{T} + \frac{\varepsilon_0 r_0 \alpha}{2^{(m+1)\mu}} + \sum_{1 \leq i \leq m} \varepsilon_i r_i p_i \right) \right).
\]

Note that the Beatty sequence $\lfloor (n\alpha + \beta)/2^{(m+1)\mu} \rfloor$ does not occur in the summand any more. We may therefore remove the second summation by estimating the number of times the three conditions under the summation sign are satisfied. At this point we want to stress the fact that $N$ is going to be significantly larger than $2^n = 2^{\lambda-(m+1)\mu}$. Using Lemma 4.3 and the usually very small discrepancy of $n\alpha$-sequences, this fact will enable us to remove the summation over $n$, while introducing only a negligible error term for most $\alpha$. This is the point in the proof where the successive “cutting away” of binary digits with the help of Farey series pays off.

By Lemma 4.3 applied with $K = 2^\omega$, and noting that $\lambda = (m+1)\mu + \rho$, we obtain for $t \in G$
\[
S_6 = \frac{N}{2^\omega T} S_7 + O\left(2^\rho N D_N \left(\frac{\alpha}{2^\omega}\right)\right),
\]
(5.7)
where
\[
S_7 = \sum_{0 \leq k < 2^\omega} e\left(\frac{1}{2} \sum_{\varepsilon_0, \ldots, \varepsilon_m \in \{0,1\}} s_\rho \left( k + \frac{t}{T} + \frac{\varepsilon_0 r_0 \alpha}{2^{(m+1)\mu}} + \sum_{1 \leq i \leq m} \varepsilon_i r_i p_i \right) \right).
\]

We note the important fact that this expression is independent of $\beta$. This will allow us to remove the maximum over $\beta$ inside the integral over $\alpha$, and thus prove the strong statement on the level of distribution.

We wish to simplify this expression in such a way that Proposition 3.3 is applicable. To this end, we use the summation over $r_i$ and the integral over $\alpha$. We define
\[
S_8 = \int_0^{2^\lambda} \sum_{0 \leq r_1, \ldots, r_m < 2^\omega} |S_7| \, d\mu(\alpha),
\]
which is an expression that will appear when we expand the original sum $S_6$.

We are going to apply the argument that for most $\alpha < 2^\lambda$ (with respect to $\mu$) the 2-valuation of $p_1, \ldots, p_m$ is small. For these $\alpha$, the term $r_i p_i \mod 2^\omega$ attains each $k \in \{0, \ldots, 2^\lambda - 1\}$ not too often, as $r_i$ runs. We may therefore replace $r_i p_i$ by $r_i$ and thus obtain full sums over $r_i$ (we note that we will set $R_i = 2^\rho$ for $1 \leq i \leq m$). In order to make this argument work, we are going to utilize the following technical result, the proof of which we give in section 5.2.

Lemma 5.1. Let $\mu, \lambda, \sigma, \gamma, m$ be nonnegative integers such that $m \geq 2$ and
\[
\lambda \geq (m+1)\mu, \quad \gamma \leq \lambda - (m+1)\mu, \quad \mu \geq 4\sigma, \quad \sigma \geq \gamma \geq 1.
\]
(5.8)
Let $p_1, \ldots, p_m$ be defined by (5.6) and set
\[
A = \{\alpha \in \{0, \ldots, 2^\lambda - 1\} : 2^\lambda \gamma - 1 \leq 2^\lambda - 1 \}.
\]
Then
\[
|A| = O(2^{\lambda-\gamma}).
\]
Analogously, if

$$A = \{ \alpha \in [0, 2^\lambda] : 2^{3^\lambda} \mid p_i \text{ for some } i = 1, \ldots, m \}.$$  

Then

$$\lambda(A) = O(2^{\lambda - \gamma}),$$

where $\lambda$ is the Lebesgue measure. The implied constants are independent of $\mu, \lambda, \sigma,$ and $\gamma$.

Let $A$ be defined as in this lemma. We choose $R_i = 2^\rho$ for $1 \leq i \leq m$.

Assume that $\alpha \notin A$. Then by an elementary argument, $r_i p_i \mod 2^\rho$ attains each value not more than $2^{3^\lambda}$ times, as $r_i$ runs through $\{0, \ldots, 2^\rho - 1\}$. The contribution for $\alpha \in A$ will be estimated trivially by the lemma. We obtain

$$S_8 \leq O(2^{\lambda + (m + 1)\rho - \gamma}) + 2^{3^\gamma m} \int_0^{2^\lambda} \sum_{0 \leq r_1, \ldots, r_m < 2^\rho} |S_9| \, d\mu(\alpha),$$

where

$$S_9 = \sum_{0 \leq n < 2^\rho} e \left( \frac{1}{2} \sum_{\varepsilon_0, \ldots, \varepsilon_m \in \{0, 1\}} s_\rho \left( n + \left\lfloor \frac{n}{\rho} \right\rfloor + \sum_{1 \leq i \leq k} \varepsilon_i r_i \right) \right).$$

The next step is removing the remaining floor function, using the integral over $\alpha$. In the continuous case, the expression $\left\lfloor t/T + r_0 K_0/2^{(m + 1)\mu} \right\rfloor$ mod $2^\rho$ runs through $\{0, \ldots, 2^\rho - 1\}$ in a completely uniform manner. That is, for $r_0 \neq 0$ we have

$$\lambda \left( \left\{ \alpha \in [0, 2^\lambda] : \left\lfloor t/T + r_0 \alpha/2^{(m + 1)\mu} \right\rfloor \equiv k \mod 2^\rho \right\} \right) = 2^{\lambda - \rho},$$

where $\lambda$ is the Lebesgue measure. We consider the discrete case. Assume that $r_0 \leq 2^{(m + 1)\mu}$ (we will choose $R_0$ very small at the end of the proof, so that this will be satisfied). Then the set of $\alpha \in \{0, \ldots, 2^\lambda - 1\}$ such that $\left\lfloor t/T + r_0 \alpha/2^{(m + 1)\mu} \right\rfloor \equiv k \mod 2^\rho$ decomposes into at most $r_0 + 1$ many intervals (note that $\lambda = (m + 1)\mu + \rho$), each having $\leq 2^{(m + 1)\mu}/r_0 + 1$ elements. In total we have $\ll 2^{\lambda - \rho}$ elements, where the implied constant is absolute. It follows that

$$S_8 \ll 2^{\lambda + (m + 1)\rho - \gamma} + 2^{\lambda - \rho + 3^\gamma m} \sum_{0 \leq r_0, \ldots, r_m < 2^\rho} |S_{10}(r_0, \ldots, r_m)|,$$

where

$$S_{10}(r_0, \ldots, r_m) = \sum_{0 \leq n < 2^\rho} e \left( \frac{1}{2} \sum_{\varepsilon_0, \ldots, \varepsilon_m \in \{0, 1\}} s_\rho \left( n + \sum_{0 \leq i \leq m} \varepsilon_i r_i \right) \right).$$

As a final step in the procedure of reducing the main theorems to Proposition 3.3, we are going to remove the absolute value around $S_{10}$. For brevity, we set

$$g(n) = \sum_{\varepsilon_0, \ldots, \varepsilon_m \in \{0, 1\}} s_\rho \left( n + \sum_{0 \leq i \leq m} \varepsilon_i r_i \right)$$

By the $2^\rho$-periodicity of $g$ we have

$$\sum_{0 \leq r_0, \ldots, r_m < 2^\rho} |S_{10}(r_0, \ldots, r_m)|^2 = \sum_{0 \leq r_0, \ldots, r_m < 2^\rho} \sum_{0 \leq n_1, n_2 < 2^\rho} e \left( \frac{1}{2} g(n_1) + \frac{1}{2} g(n_2) \right)$$

$$= \sum_{0 \leq r_0, \ldots, r_m < 2^\rho} \sum_{0 \leq n_1 < 2^\rho} \sum_{0 \leq r_{m+1} < 2^\rho} e \left( \frac{1}{2} g(n_1) + \frac{1}{2} g(n_1 + r_{m+1}) \right)$$

$$= \sum_{0 \leq r_0, \ldots, r_{m+1} < 2^\rho} \sum_{0 \leq n_1 < 2^\rho} e \left( \frac{1}{2} g(n_1) + \frac{1}{2} g(n_1 + r_{m+1}) \right)$$
We have therefore removed the absolute value around $S_{10}$ for the price an additional variable $r_{m+1}$. This means that we have reduced our main theorems to Proposition 3.3.

By this Proposition and Cauchy-Schwarz we obtain
\begin{equation}
S_{8} \ll 2^{\lambda+(m+1)\rho} \left(2^{-\gamma} + 2^{\alpha m - \eta \rho}\right)
\end{equation}
for some $\eta > 0$.

It remains to collect the error terms and to choose values for the free variables. Using (5.7) and (5.6), we obtain
\[
S_5 \ll \sum_{t \notin G} \mathcal{O}\left(\frac{N}{T} + N D_N \left(\frac{\alpha}{2(m+1)\mu}\right)\right) + \sum_{t \notin G} \mathcal{O}\left(2^\rho N D_N \left(\frac{\alpha}{2\lambda}\right)\right)
\]
\[
= \frac{N}{2^\rho T} \sum_{t \notin G} S_t + \mathcal{O}\left(\frac{N}{T} + N D_N \left(\frac{\alpha}{2(m+1)\mu}\right) + 2^\rho N T D_N \left(\frac{\alpha}{2\lambda}\right)\right)
\]
and by (5.4) and (5.1) we obtain
\begin{equation}
\left|\frac{S_0(N, \nu, \xi)}{2^\nu N}\right|^{2m+1} \ll \mathcal{O}\left(\frac{1}{R_0} + \frac{R_0 2^\nu}{2^\lambda} + \frac{R_0}{N} + \frac{R_1 K_1}{N} + \cdots + \frac{R_m K_m}{N}\right)
\end{equation}
\[
+ \frac{1}{2^\nu N} \int_0^{2^\lambda} N \mathcal{O}\left(\tilde{D}_N(\alpha) + D_N \left(\frac{\alpha}{2^{2\mu}}\right) + \cdots + D_N \left(\frac{\alpha}{2(m+1)\mu}\right) + \frac{R_1 + \cdots + R_m}{2^\sigma}\right) d\mu(\alpha),
\]
\[
+ \frac{1}{2^\nu N} \int_0^{2^\lambda} \mathcal{O}\left(\frac{N}{T} + N D_N \left(\frac{\alpha}{2(m+1)\mu}\right) + 2^\rho N T D_N \left(\frac{\alpha}{2\lambda}\right)\right) d\mu(\alpha),
\]
\[
+ \frac{1}{R_0 \cdots R_m 2^\nu N^2 T} \sum_{t \notin G} \sum_{1 \leq r_0 < R_0} \int_0^{2^\lambda} \sum_{0 \leq r_0, \ldots, r_m < 2^\nu} |S_t| d\mu(\alpha).
\]

We employ the mean discrepancy estimates from Lemma 4.4. Assume that $\delta \leq \lambda$. In the continuous case we have
\[
\frac{1}{2^\nu} \int_0^{2^\lambda} D_N \left(\frac{\alpha}{2^\delta}\right) d\alpha \ll 2^{\lambda - \nu - \delta} \int_0^{2^\delta} D_N \left(\frac{\alpha}{2^\delta}\right) d\alpha \ll 2^{\lambda - \nu} \left(\log^+ N\right)^2 \frac{1}{N},
\]
while the discrete case gives
\[
\frac{1}{2^\nu} \sum_{0 \leq d < 2^\lambda} D_N \left(\frac{d}{2^\delta}\right) \ll 2^{\lambda - \delta - \nu} N \frac{2^\delta}{N} \left(\log^+ N\right)^2 = 2^{\lambda - \nu} \left(\log^+ N\right)^2 \left(\frac{1}{N} + \frac{1}{2^\delta}\right)
\]
In total, noting that $\lambda \geq (m+1)\mu$, the discrepancy terms can be estimated by
\[
\ll 2^{\lambda - \nu} \left(\log^+ N\right)^2 2^\rho T \left(\frac{1}{N} + \frac{1}{2^\nu}\right).
\]
By (5.9), the last summand in (5.10) can be estimated by
\[
\ll 2^{\lambda - \nu} \left(2^{-\gamma} + 2^{\alpha m - \eta \rho}\right).
\]
Moreover, using the facts $R_1 = \cdots = R_m = 2^\rho$ and $K_i \leq 2^{2\rho + 3\sigma}$ for $1 \leq i \leq m$, we obtain
that, for large $N$

\[ 2 \leq \rho N \]

a way that

\[ 2^\rho N \]

with some implied constant only depending on $m$. Collecting also the requirements on the variables we assumed in the course of our calculation, we see that this estimate is valid as long as

\[ R_0, T \geq 1, m \geq 2, \gamma, \nu, \lambda, \rho, \mu \geq 0, \quad R_1 = \cdots = R_m = 2^\sigma, \]

\[ \lambda > \nu, \quad \rho = \lambda - (m + 1)\mu, \]

\[ \gamma \leq \rho < \sigma - 1, \quad \mu \geq 4\sigma, \]

\[ R_0 \leq 2^{(m+1)\mu}. \]

It remains to choose the variables within these constraints. Choose the integer $j \geq 1$ in such a way that $N^{j-1} \leq 2^\rho < N^j$ and set $m = 3j - 1$. Clearly, $m \geq 2$. We define

\[ \mu = \left\lfloor \frac{\nu}{m + 1 + 1/8} \right\rfloor, \quad \sigma = \lfloor \mu/4 \rfloor, \quad \tilde{\rho} = \nu - (m + 1)\mu. \]

We obtain the inequalities $N \geq 2^m\mu$, $\mu \geq 4\sigma$, $\tilde{\rho} \geq 0$. Moreover, for large $\nu$ we obtain $\tilde{\rho} \sim \mu/8$.

Choose $\gamma = \lceil \tilde{\rho} n/(6m) \rceil$ and $R_0 = \lceil 2^\gamma/\mu \rceil$. Then the last summand in (5.11) is $\ll 2^{\lambda - \nu} (2^{-\gamma} + 2^{(-(\gamma/2)}) \ll 2^{\lambda - \nu - \gamma}$. Finally, set $\lambda = \nu + \lceil \gamma/2 \rceil$, $T = 2^\gamma$ and $\rho = \lambda - (m + 1)\mu$. It follows that

\[ \rho = \tilde{\rho} + \lceil \gamma/2 \rceil \sim \frac{\mu}{8} \left(1 + \eta/(12m)\right) \leq \mu/8 + \mu/192. \]

Using these definitions, it is not hard to see that, for large $N$ and $\nu$, the requirements (5.12) are met.

Moreover, using the statements $N^{2\rho} \leq D \leq N^{\rho^2}$ and $D < 2^\rho \leq 2D$ we can easily estimate (5.11) term by term and conclude that $S_0(N, \nu, \xi)/(2^\nu N) \leq C N^{-\eta}'$ for some $\eta' > 0$ and some constant $C$. This finishes the proof of Propositions 3.1 and 3.2 and therefore of our main theorems. It remains to prove our auxiliary results.

5.1. Proof of Proposition 3.3. We utilize ideas from the paper [21] by Konieczny. Set

\[ A_\rho(a) = \frac{1}{2^{(m+1)\rho}} \sum_{0 \leq n < 2^\rho \leq r_1, \ldots, r_m < 2^\rho} e \left( \frac{1}{2} \sum_{\epsilon \in \{0,1\}^m} s_\rho(n + \epsilon \cdot r + a_\epsilon) \right). \]

Then in analogy to equation (16) of [21], we get after a similar calculation (using $m \geq 2$)

\[ A_{\rho+1}(a) = \frac{(-1)^{|a|}}{2^{m+1}} \sum_{e_0, \ldots, e_m \in \{0,1\}} A_\rho(\delta(a, e)), \]

where $|a| = \sum_{\epsilon \in \{0,1\}^m} a_\epsilon$ and

\[ \delta(a, e) = \left[ \frac{a_\epsilon + e_0 + \sum_{1 \leq i \leq m} \epsilon_i e_i}{2} \right]. \]

We define a directed graph with weighted edges according to (5.13). The set of vertices is given by the set of families $a \in \mathbb{Z}^{\{0,1\}^m}$. There is an edge from $a$ to $b$ if and only if there is an $e = (e_0, \ldots, e_m) \in \{0,1\}^{m+1}$ such that $\delta(a, e) = b$ and this edge has the weight

\[ w(a, b) = \frac{(-1)^{|a|}}{2^{m+1}} \left| \{ e \in \{0,1\}^{m+1} : \delta(a, e) = b \} \right|. \]
Note that
\begin{equation}
\sum_{\mathbf{b} \in \mathbb{Z}^{(0,1)^m}} |w(\mathbf{a}, \mathbf{b})| = 1,
\end{equation}
which we will need later. We are interested in the subgraph \((V, E, w)\) induced by the set of vertices reachable from \(0\). This graph is finite: we have
\[
\max_{x \in (0,1)^m} |\delta(\mathbf{a}, e)_x| \leq \frac{1}{2} \left( \max_{x \in (0,1)^m} |a_x| + m + 1 \right)
\]
and by induction, it follows that \(\max_{x \in (0,1)^m} |a_x| < m + 1\) for all \(a \in V\), which implies the finiteness of \(V\).

Moreover, this subgraph is strongly connected. We prove this by showing that \(0\) is reachable from each \(a \in V\). This follows immediately by considering the path \((a = a^{(0)}, a^{(1)}, \ldots, a^{(k)}, a^{(k+1)})\) defined by \(a^{(j+1)} = \delta(a^{(k)},(0,\ldots,0))\). It is clear from the definition of \(\delta\) that such a path reaches \(0\) if \(k\) is large enough.

We wish to apply (5.13) recursively. We therefore define, for two vertices \(a, b \in V\) and a positive integer \(k\), the weight \(w_k(a,b)\) as the sum of all weights of paths of length \(k\) from \(a\) to \(b\). (Here the weight of a path is the product of the weights of the edges.)

In order to prove Proposition 5.3 it is sufficient to prove that there is a \(k\) such that
\[
\sum_{b \in V} |w_k(a,b)| < 1
\]
for all \(a \in V\). In order to prove this, it is sufficient, by the strong connectedness of the graph and (5.14), to prove that there are two paths of the same length from \(0\) to \(0\) such that their respective weights have different sign. One of this paths is the trivial one, choosing \(e_0 = \cdots = e_m = 0\) in each step. This path has positive weight.

For the second path, we follow Konieczny [21, proof of Proposition 2.3]. As in that paper, we define \(a^{(0)} = a^{(m+1)} = 0\) and for \(1 \leq j \leq m\),
\[
a^{(j)}_x = \begin{cases} 1, & \text{if } \varepsilon_1 = \cdots = \varepsilon_j = 1; \\ 0, & \text{otherwise}. \end{cases}
\]
Assuming for a moment that there is an edge from \(a^{(j)}\) to \(a^{(j+1)}\) for all \(j \in \{0, \ldots, m\}\), it is easy to see that each edge \((a^{(j)}, a^{(j+1)})\) has positive weight for \(0 \leq j < m\), while \((a^{(m)}, a^{(m+1)})\) has negative weight. Proving that these vertices indeed define a path is contained completely in the argument given by Konieczny. This finishes the proof of Lemma 5.3.

5.2. Proof of Lemma 5.1. We choose an integer \(\gamma > 0\) and bound the size of the set of \(a < 2^\lambda\) such that \(2^\gamma \mid p_i\) for some \(i \in \{1, \ldots, m\}\). We will need the following two lemmas.

**Lemma 5.2.** Let \(\lambda\) be the Lebesgue measure. Assume that \(K \geq 1\) and \(\gamma \geq 0\) are integers. Then
\[
\lambda(\{x \in [0,1] : 2^\gamma \mid q_K(x)\}) \ll \frac{1}{2^\gamma} + \frac{1}{K}.
\]
The constant in this estimate is absolute.

**Proof.** We have to sum up the lengths of the Farey intervals around \(p/q\) such that \(2^\gamma \mid q\). By Lemma 4.6, each such fraction contributes at most \(2/(Kq)\). By summing over \(p \in \{1, \ldots, q\}\), this gives a contribution \(2/K\) for each multiple \(q\) of \(2^\gamma\), and we obtain a total contribution
\[
\ll \sum_{1 \leq q \leq K} \frac{1}{K} \leq \frac{1}{2^\gamma} + \frac{1}{K}.
\]
\(\square\)
Lemma 5.3. Let $x_0, \ldots, x_{M-1} \in [0,1]$ and $\delta > 0$. Assume that $\|x_i - x_j\| \geq \delta$ for $i \neq j$. Then
\[
\{ n \in \{0, \ldots, M - 1\} : 2^\gamma | q_K(x_i) \} \ll \frac{K^2}{2^\gamma} + \frac{1}{\delta} \left( \frac{1}{2^\gamma} + \frac{1}{K} \right).
\]
The implied constant is absolute.

Proof. In each Farey interval around $p/q$ such that $q$ is divisible by $2^\gamma$ there are at most $2/(Kq\delta) + 1$ many points $x_i$. By summing over $p$ and $q$, we can bound the number of points in such intervals by
\[
\ll \sum_{1 \leq q \leq 2^{1/2}|q|} \sum_{1 \leq p \leq q} \left( \frac{1}{qK\delta} + 1 \right) = \sum_{1 \leq q \leq K/2^\gamma|q|} \frac{1}{K\delta} + q = (K2^{-\gamma} + 1) \frac{1}{K\delta} + \sum_{1 \leq q \leq K/2^\gamma|q|} q
\leq \frac{1}{2^\gamma \delta} + \frac{1}{K\delta} + 2^\gamma \sum_{1 \leq q' \leq [K2^{-\gamma}]} q' \ll \frac{K^2}{2^\gamma} + \frac{1}{2^\gamma \delta} + \frac{1}{K\delta}.
\]

We proceed to the proof of Lemma 5.1. Consider $p_1$ and the case “$\alpha$ discrete”. In this case, we have $p_{2^{j+2} \alpha} (\alpha/2^\mu) = \alpha$. Assume therefore that $\alpha = \alpha_0 + 2^{(m-1)\mu} \alpha_1$, where $\alpha_0 \in \{0, \ldots, 2^{(m-1)\mu} - 1\}$ and $\alpha_1 \in \{0, \ldots, 2^{\lambda-(m-1)\mu} - 1\}$.

Then
\[
p_1 = p_{2^\sigma} (\alpha/2^{(m-1)\mu}) = p_{2^\sigma} (\alpha_0/2^{(m-1)\mu}) + q_{2^\sigma} (\alpha_0/2^{(m-1)\mu}) \alpha_1.
\]
By Lemma 5.3, using also 5.8, it follows that the number of $\alpha_0 \in \{0, \ldots, 2^{(m-1)\mu} - 1\}$ such that $2^\gamma \nmid q_{2^\sigma} (\alpha_0/2^{(m-1)\mu})$ is $2^{(m-1)\mu} (1 - O(2^{-\gamma}))$. For each such $\alpha_0$, we let $\alpha_1$ run through $\{0, \ldots, 2^{\lambda-(m-1)\mu} - 1\}$. Then two occurrences $\alpha_1, \alpha_1'$ such that $2^{2\gamma} \nmid p_1$ are separated by at least $2^\gamma$ steps; it follows that the number of such $\alpha_1$ is bounded by $2^{\lambda-(m-1)\mu-\gamma}$. Putting these errors together, we see that the number of $\alpha \in \{0, \ldots, 2^\lambda - 1\}$ such that $2^{2\gamma} \nmid p_1$ is given by $2^{(m-1)\mu} (1 - O(2^{-\gamma})) 2^{(m-1)\mu} (1 - O(2^{-\gamma})) = 2^{\lambda} (1 - O(2^{-\gamma}))$.

Next, we consider the continuous case. We write $\alpha = \alpha_0 + 2^{2\mu} \alpha_1 + 2^{(m-1)\mu} \alpha_2$ , where $\alpha_0 \in [0,2^{2\mu})$ is real and $\alpha_1 < 2^{(m-1)\mu}$ and $\alpha_2 < 2^{\lambda-(m-1)\mu}$ are nonnegative integers. Set $p = p_{2^{j+2}\alpha} (\alpha_0/2^{2\mu})$ and $q = q_{2^{j+2}} (\alpha_0/2^{2\mu})$. Then
\[
p_{2^{j+2}\alpha} (\alpha/2^{(m-1)\mu}) = p + (\alpha_1 + 2^{(m-1)\mu} \alpha_2) q.
\]
By the approximation property 4.5 (note that $\sigma \geq 1$) we have
\[
p_1 = \left\lfloor \frac{p + \alpha_1 q}{2^{(m-1)\mu}} \right\rceil \alpha_2 q \left( \frac{p + \alpha_1 q}{2^{(m-1)\mu}} \right) = \left\lfloor \frac{p + \alpha_1 q}{2^{(m-1)\mu}} \right\rceil \alpha_2 q \left( \frac{p + \alpha_1 q}{2^{(m-1)\mu}} \right) + \alpha_2 q_2 \left( \frac{p + \alpha_1 q}{2^{(m-1)\mu}} \right)
\]
and we note that the first summand does not depend on $\alpha_2$.

As $\alpha_0$ runs through $[0,2^{2\mu})$, we have by Lemma 5.2 $2^{2\gamma} \nmid q$ in a set of measure $2^{2\mu} (1 - O(2^{-\gamma} + 2^{-2\mu-2\gamma}))$. By 5.5, this is $2^{2\mu} (1 - O(2^{-\gamma}))$. Assume that $\alpha_0$ is such that $2^\gamma \nmid q$ and set $\gamma' = \nu_2(q) < \gamma$. Next, we let $\alpha_1$ run. We choose $x_j = \{(p+jq)/2^{(m-1)\mu}\}$ for $0 \leq j < 2^{(m-1)\mu-\gamma'}$ and we note that these points satisfy $\|x_i - x_j\| \geq 1/2^{(m-1)\mu-\gamma'}$ for $i \neq j$. By Lemma 5.3 it follows that
\[
\left\{ \alpha_1 \in \{0, \ldots, 2^{(m-1)\mu-\gamma'} - 1\} : 2^{2\gamma} | q_{2^\sigma} (\alpha_1/2^{(m-1)\mu}) \right\} \ll \frac{2^{2\gamma}}{2^\gamma} + 2^{(m-1)\mu-\gamma'} \left( \frac{1}{2^\gamma} + \frac{1}{2^\sigma} \right).
\]
By \((5.8)\), this is \(< 2^{(m-1)\mu - \lambda + \gamma}\). Performing this also for the other intervals of length \(2^{(m-1)\mu - \lambda + \gamma}\), we obtain
\[
\left\{ \alpha_1 \in \{0, \ldots, 2^{(m-1)\mu} - 1\} : 2\gamma \mid q_2^\sigma \left( \frac{p + \alpha_1 q}{2^{(m-1)\mu}} \right) \right\} \ll 2^{(m-1)\mu - \lambda + \gamma}.
\]

Finally, \(\alpha_2\) runs through \(\{0, \ldots, 2^{2\lambda-(m+1)\mu} - 1\}\) and we consider \(p_1\). For given good \(\alpha_1\) and \(\alpha_0\) (such that \(2\gamma \mid q\) and \(2\gamma \mid q_2^\sigma((p + \alpha_1 q)/2^{(m-1)\mu}))\), \(p_1\) is an arithmetic progression in \(\alpha_2\) whose common difference is not divisible by \(2^{2\gamma}\). Similarly to the discrete case, it follows that \(p_1\) is divisible by \(2^{2\gamma}\) for at most \(2^{2\lambda-(m+1)\mu - \gamma}\) many \(\alpha_2\). It follows that there is a set of measure
\[
2^{2\lambda}(1 - O(2^{-\gamma}))2^{(m-1)\mu}(1 - O(2^{-\gamma}))2^{2\lambda-(m+1)\mu}(1 - O(2^{-\gamma})) = 2^{\lambda}(1 - O(2^{-\gamma}))
\]
of \(\alpha < 2^\lambda\) such that \(2^{2\gamma} \mid p_1\).

The cases \(2 \leq i \leq m\) do not require any new ideas; we only give a sketch of a proof. Let \(2 \leq i \leq m\). We treat the discrete and continuous cases in parallel. We write \(\alpha = \alpha_0 + 2^{(i+1)\mu}\alpha_1 + 2^{(m+1)\mu}\alpha_2\), where \(\alpha_0 < 2^{(i+1)\mu}\), and \(\alpha_1 < 2^{(m-1)\mu}\) and \(\alpha_2 < 2^{2\lambda-(m+1)\mu}\) are nonnegative integers. Set \(p = p_{2^{i+2}\sigma}(\alpha_0/2^{(i+1)\mu})\) and \(q = q_{2^{i+2}\sigma}(\alpha_0/2^{(i+1)\mu})\). Then
\[
p_{i} = \left( \frac{p + \alpha_1 q}{2^{(m-1)\mu}} \right) q_2^\sigma \left( \frac{p + \alpha_1 q}{2^{(m-1)\mu}} \right) + \alpha_2 q_2^\sigma \left( \frac{p + \alpha_1 q}{2^{(m-1)\mu}} \right),
\]
as before. By Lemmas \(5.2\) and \(5.3\) we have \(2\gamma \mid q\) for \(\alpha_0\) in a set of measure \(2^{(i+1)\mu}(1 - O(2^{-\gamma}))\), where we used \(2\mu + 4\sigma \leq (i + 1)\mu\) in the discrete case. (We note that this last inequality is the reason for defining \(p_1\) separately, using \(2^{2\gamma}\) instead of \(2^\gamma\).) The remaining steps are as before, and this case is finished.

Finally, in the case \(i = m\) we write \(\alpha = \alpha_0 + 2^{(m+1)\mu}\alpha_1\), where \(\alpha_0 < (m + 1)\mu\) and \(\alpha_1 \in \{0, \ldots, 2^{2\lambda-(m+1)\mu - 1}\}\). Then
\[
p_m = p_{2^{i+2}\sigma}(\alpha_0/2^{(m+1)\mu}) + q_{2^{i+2}\sigma}(\alpha_0/2^{(m+1)\mu})\alpha_1.
\]

By Lemmas \(5.2\) and \(5.3\) and \(5.8\) we have \(2\gamma \mid q_{2^{i+2}\sigma}(\alpha_0/2^{(m+1)\mu})\) for \(\alpha_0\) in a set of measure \(O(2^{(m+1)\mu - \gamma})\) and the statement follows as before.

In total, we have a set of measure \(2^{\lambda}(1 - O(2^{-\gamma}))\) of \(\alpha < 2^\lambda\) such that \(2^{2\gamma} \mid p_i\) for all \(i\).

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