ON FOURIER-LAPLACE TRANSFORM OF A CLASS OF GENERALIZED FUNCTIONS

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Abstract. We consider a subspace of Schwartz space of fast decaying infinitely differentiable functions on an unbounded closed convex set in a multidimensional real space with a topology defined by a countable family of norms constructed by means of a family $M$ of a logarithmically convex sequences of positive numbers. Owing to the mentioned conditions for these sequence, the considered space is a Fréchet-Schwartz one. We study the problem on describing the strong dual space for this space in terms of the Fourier-Laplace transforms of functionals. Particular cases of this problem were considered by J.W. De Roever in studying problems of mathematical physics, complex analysis in the framework of a developed by him theory of ultradistributions with supports in an unbounded closed convex set; similar studies were also made by P.V. Fedotova and by the author of the present paper. Our main result, presented in Theorem 1, states that the Fourier-Laplace transforms of the functionals establishes an isomorphism between the strong dual space of the considered space and some space of holomorphic functions in a tubular domain of the form $\mathbb{R}^n + iC$, where $C$ is an open convex acute cone in $\mathbb{R}^n$ with the vertex at the origin; the mentioned holomorphic functions possess a prescribed growth majorants at infinity and at the boundary of the tubular domain. The work is close to the researches by V.S. Vladimirov devoted to the theory of the Fourier-Laplace transformation of tempered distributions and spaces of holomorphic functions in tubular domains. In the proof of Theorem 1 we apply the scheme proposed by M. Neymark and B.A. Taylor as well as some results by P.V. Yakovleva (Fedotova) and the author devoted to Paley-Wiener type theorems for ultradistributions.

Keywords: Fourier-Laplace transform of functionals, holomorphic functions.

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1. Introduction

1.1. Problem. Let $C$ be an open convex acute cone in $\mathbb{R}^n$ with the vertex at the origin [1, Ch. 1, Sect. 4], $b$ be a convex continuous positive homogeneous of degree 1 function on $C$, which the closure of $C$. A pair $(b, C)$ defines a closed convex unbounded domain

$$U(b, C) = \{\xi \in \mathbb{R}^n : -\langle \xi, y \rangle \leq b(y), \forall y \in C\},$$

containing no entire straight line. We note that the interior $U(b, C)$ is non-empty and coincides with the set

$$V(b, C) = \{\xi \in \mathbb{R}^n : -\langle \xi, y \rangle < b(y), \forall y \in C\},$$

and the closure of $V(b, C)$ is $U(b, C)$.

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Let $\mathfrak{M} = \{M^{(m)}\}_{m \in \mathbb{N}}$ be the family of logarithmically convex sequences $M^{(m)} = (M_k^{(m)})_{k=0}^\infty$ with $M_0^{(m)} = 1$ such that for each $m \in \mathbb{N}$

\begin{itemize}
  \item[i_1]\quad \sup_{k \in \mathbb{Z}_+} \frac{M_k^{(m+1)}}{M_k^{(m)}} < +\infty,
  \item[i_2]\quad \lim_{k \to \infty} M_k^{(m+1)} = 0,
  \item[i_3]\quad \lim_{k \to \infty} \left( \frac{M_k^{(m)}}{e^{k/2}} \right)^{1/2} > 0.
\end{itemize}

To each sequence $M^{(m)}$ we associate a function $\omega_m : [0, \infty) \to [0, \infty)$ by the rule:

$$\omega_m(r) = \sup_{k \in \mathbb{Z}_+} \ln \frac{r^k}{M_k^{(m)}}, \quad r > 0; \quad \omega_m(0) = 0.$$ 

For the sake of brevity we denote the set $U(b, C)$ by $U$ and the set $V(b, C)$ is denoted by $V$. Then we define a space $G_{\mathfrak{M}}(U)$ as follows. For each $m \in \mathbb{N}$ we introduce the space $G_m(U)$ consisting of the functions $f$ in the class $C^\infty$ on $U$ with finite norms

$$p_m(f) = \sup_{x \in V, a \in \mathbb{Z}_+^n} \frac{|(D^\alpha f)(x)|(1 + ||x||)^m}{M^{(m)}_a}.$$ 

By condition $i_2$, the space $G_{m+1}(U)$ is continuously embedded into $G_m(U)$ for each $m \in \mathbb{N}$. We let $G_{\mathfrak{M}}(U) = \bigcap_{m=1}^\infty G_m(U)$. Being equipped with usual summing and multiplication by complex numbers, the set $G_{\mathfrak{M}}(U)$ becomes a linear space. We also introduce the topology of inductive limit of the spaces $G_m(U)$ in $G_{\mathfrak{M}}(U)$. It is obvious that $G_{\mathfrak{M}}(U)$ is the Fréchet space continuously embedded into the Schwartz space $S(U)$ of fast decaying functions in the class $C^\infty$ on $U$.

It is well-known that for each $z \in T_C = \mathbb{R}^n + iC$, the function $f_z(\xi) = e^{i(z, \xi)}$ belongs to the space $S(U)$ [1], [2]. We also have $f_z \in G_{\mathfrak{M}}(U)$ (Lemma 4). This is why for each linear continuous functional $\Phi$ on $S(U)$ ($G_{\mathfrak{M}}(U)$), in the domain $T_C$, a function $\Phi$ is well-defined being the Fourier-Laplace transform of the functional $\Phi$ defined by the formula $\Phi(z) = (\Phi, e^{i(z, \xi)})$, $z \in T_C$.

Under additional assumptions on the family $\mathfrak{M}$, the space $G_{\mathfrak{M}}(U)$ and its strongly dual space $G^*_{\mathfrak{M}}(U)$ were studied by J.W. de Roever [2] in relation with problems in mathematical physics (quantum field theory), in complex analysis (solvability of convolution equations and systems of convolution equations, interpolation theory, Palemmodov-Ehrenpreis fundamental principle) in the framework of theory of ultradistributions supported in an unbounded closed convex set. In particular, we obtained the description of the space $G^*_{\mathfrak{M}}(U)$ in terms of the Fourier-Laplace transform of the functionals in the case, when the family $\mathfrak{M}$ consists in sequences $M^{(m)}$ of form $(\varepsilon_m M_k^k)_{k=0}^\infty$, where $(\varepsilon_m)_{m=1}^\infty$ is an arbitrary decaying to zero sequence of positive numbers $\varepsilon_m$, and $M = (M_k)_{k=0}^\infty$ is a non-decaying logarithmically convex sequence of positive numbers $M_0 = 1$ satisfying, for some $h > 0$ and $K > 0$, the following conditions:

\begin{itemize}
  \item[i_4]\quad M_{p+q} \leq h^{p+q} M_p M_q, \quad p, q \in \mathbb{Z}_+;
  \item[i_5]\quad \sum_{q=p+1}^{\infty} M_{q-1} M_q \leq K p M_p M_{p+1}, \quad p \in \mathbb{N}.
\end{itemize}

The mentioned description was given as a some subspace in the space $H(T_C)$ of holomorphic in a tubular domain $T_C$ functions with certain growth estimates at infinity and near the boundary of the domain. More precisely, it follows from his results [2] Thms. 1.21.i, 2.24.ii that $G^*_{\mathfrak{M}}(U)$ is isomorphic to the projective limit of the spaces $H_{C_1, \varepsilon}$, where $\varepsilon > 0$, $C_1$ is a cone compact in the cone $C$, and $H_{C_1, \varepsilon}$ is the inductive limit of the spaces

$$H^{(m)}_{C_1, \varepsilon} = \left\{ f \in H(T_{C_1}) : \| f \|^{(m)}_{C_1, \varepsilon} = \sup_{z \in T_{C_1}, ||y|| \geq \varepsilon} \frac{|f(z)|}{e^{b(y)+\omega_m(||z||)}} < \infty \right\}, \quad m \in \mathbb{N}.$$ 

We note that Condition $i_4$ implies that the sequence $M$ satisfies the condition $i_6$. There exist numbers $H_1 > 1$, $H_2 > 1$ such that $M_{k+1} \leq H_1 H_2^k M_k$, $\forall k \in \mathbb{Z}_+$, while Condition $i_5$ and the logarithmic convexity imply that $M$ satisfies the condition
for some $Q_1 > 0$ and $Q_2 > 0$ the inequalities $M_k \geq Q_1 Q_2^k k!, k \in \mathbb{Z}_+$ hold.

Under the same assumptions on the structure of the family $\mathcal{M}$, a theorem of Paley-Wiener-Schwartz type was obtained for the space $G_{2\mathcal{M}}(U)$ in $[3]$ under weaker restrictions for $M$. Namely, Conditions $i_4$ and $i_5$ were replaced by Conditions $i_6$ and $i_7$. Thus, in $[3]$, the sequence $M$ could be quasi-analytic. Moreover, taking into consideration that the space $G_{2\mathcal{M}}(U)$ is independent of the choice of the sequence $(\varepsilon_m)_{m=1}^\infty$, we can assume that $\varepsilon_m = H_2^{-m}$ $(m \in \mathbb{N})$. Then the family the sequences \{(\varepsilon_m M_k)_{k=0}^\infty \}_{m \in \mathbb{N}} satisfies Condition $i_1$). On the other hand, if $(\varepsilon_m)_{m=1}^\infty$ is an arbitrary decaying to zero scalar sequence, $M = (M_k)_{k=0}^\infty$ is an arbitrary sequence of positive numbers and the family of sequences \{(\varepsilon_m M_k)_{k=0}^\infty \}_{m \in \mathbb{N}} satisfies Condition $i_1$), then the sequence $M$ satisfies Condition $i_6$) for some $H_1 > 1, H_2 > 1$.

The aim of the present work is to describe the space $G_{2\mathcal{M}}(U)$ in terms of the Fourier-Laplace transform of the functionals under the assumption that the family $\mathcal{M}$ consists in non-decreasing logarithmically convex sequences $M^{(m)} = (M_k^{(m)})_{k=0}^\infty$ with $M_0^{(m)} = 1$, which, apart of Conditions $i_1), i_2), i_3), i_8$), satisfy also the following condition:

\[ \sum_{|\alpha| \geq 0} M_{|\alpha|}^{m+|\alpha|} < \infty. \]

We note that the family \{(\varepsilon_m M_k)_{k=0}^\infty \}_{m \in \mathbb{N}} in works $[2], [3]$ satisfies Conditions $i_1), i_2), i_3), i_8$).

1.2. Notations. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

\[ |\alpha| = \alpha_1 + \ldots + \alpha_n, \quad \alpha! = \alpha_1! \ldots \alpha_n!, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}. \]

For $u = (u_1, \ldots, u_m) \in \mathbb{R}^m(C^m), v = (v_1, \ldots, v_m) \in \mathbb{R}^m(C^m)$ we let

\[ (u, v) = u_1 v_1 + \ldots + u_m v_m, \quad \|u\| = \sqrt{|u_1|^2 + \ldots + |u_m|^2}, \quad |u|_m = \max_{1 \leq j \leq m} |u_j|. \]

A polydisk $\{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1| \leq 1, \ldots, |z_n| \leq 1\}$ is denoted by $\Pi$. For $r > 0, z \in \mathbb{C}^m$ we let $B(z, r) = \{\zeta \in \mathbb{C}^m : |\zeta - z| \leq r\}$.

The symbol $\lambda_m$ denotes the Lebesgue measure in $\mathbb{C}^m$, $T_C = \mathbb{R}^n + iC, \Delta_C(y)$ is the distance from a point $y \in C$ to the boundary of $C, d(z)$ is the distance from $z = x + iy \in T_C$ to the boundary of $T_C$.

For a locally convex space $X$, by $X'$ we denote the space of linear continuous functionals on $X$, while the symbol $X^*$ stands for the strongly dual space.

Hereafter $\mathcal{M}$ is a family of non-decreasing logarithmically convex sequences $M^{(m)} = (M_k^{(m)})_{k=0}^\infty$ with $M_0^{(m)} = 1, m \in \mathbb{Z}_+$, satisfying Conditions $i_1) - i_3), i_8$).

By $S(U)$ we denote the Schwartz space of $C^\infty(U)$-functions $f$ such that for each $p \in \mathbb{Z}_+$ we have

\[ \|f\|_{p,U} = \sup_{x \in V,|\alpha| \leq p} |(D^\alpha f)(x)|(1 + \|x\|)^p < \infty, \]

and $S_p(U)$ is the completion of $S(U)$ by the norm $\| \cdot \|_{p,U}$.

By $C(K)$ we denote the space of functions continuous on a compact set $K \subset \mathbb{R}^n$ with a usual topology, $H(O)$ is the space of functions holomorphic in the domain $O \subset \mathbb{C}^n$ equipped with the topology of uniform convergence on compact subsets $O$.

1.3. Main result and structure of work. For each $m \in \mathbb{N}$ we define normed spaces

\[ H_{b,m}(T_C) = \left\{ f \in H(T_C) : \|f\|_m = \sup_{z \in T_C} |f(z)| e^{b|y| + \omega_m(|z|_m)}(1 + \frac{1}{\Delta_C(y)})^m < \infty \right\}, \]

where $z = x + iy, x \in \mathbb{R}^n, y \in C$. Let $H_{b,\mathbb{R}}(T_C) = \bigcup_{m=0}^\infty H_{b,m}(T_C)$. The set $H_{b,\mathbb{R}}(T_C)$ with the summing and multiplication by complex numbers is a linear space. We equip $H_{b,\mathbb{R}}(T_C)$ with the topology of inductive limit of the spaces $H_{b,m}(T_C)$.

The main result of the present work is the following theorem.
Theorem 1. The Laplace-Fourier transform establishes an isomorphism between the spaces $G^*_m(U)$ and $H_{b,2R}(T_C)$.

The proof of Theorem 1 is based on ideas by M. Neymark [4] and B.A. Taylor [5] and employs a series of results from [6]; this proof is given in Section 3. It is presented in a rather brief form since it follows the same lines as the proof of Theorem 2 in [3]. We also observe that in the proof of Theorem 1, we show how to cover a series of gaps in the proof of Theorem 2 in [3]. Section 2 is devoted to auxiliary results.

2. Auxiliary results

We recall that the space represented as the projective limit of a sequence of normed spaces $S_n$, $n \in \mathbb{N}$, with respect to linear continuous mappings $g_{mn} : S_n \rightarrow S_m$, $m < n$, such that $g_{n,n+1}$ is completely continuous for each $n$, is called space $(M^*)$ [7]. Employing Arzelà-Ascoli and Condition $i_2$), it is easy to prove the following statement.

Lemma 1. The space $G^*_m(U)$ is the space $(M^*)$.

Thus, $G^*_m(U)$ is a Fréchet-Schwartz space [8].

In what follows a general form of a functional in $G^*_m(U)$. Because of this, we introduce the space $C^m(U)$ as the projective limit of the spaces $C^m(U) = \{ f \in C(U) : \hat{p}_m(f) = \sup_{x \in U} |f(x)|(1 + \|x\|)^m < \infty \}, \quad m \in \mathbb{N}$.

By a known scheme, cf. [5, Props. 2.10, 2.11, Cor. 2.12], with employing Condition $i_2$), one can prove the following statement.

Lemma 2. Let a functional $T \in G^*_m(U)$, numbers $c > 0$ and $m \in \mathbb{N}$ be such that

$$|(T, f)| \leq cp_m(f), \quad f \in G^*_m(U).$$

Then there exist functionals $T_\alpha \in C^\alpha_m(U)$, $\alpha \in \mathbb{Z}_+^n$, such that

$$|(T_\alpha, f)| \leq \frac{c\hat{p}_m(f)}{M^{(m)}(\alpha)}, \quad f \in C^\alpha_m(U),$$

and

$$(T, f) = \sum_{\alpha \geq 0} (T_\alpha, D^\alpha f), \quad f \in G^*_m(U).$$

Lemma 3. For each $m \in \mathbb{N}$ there exists a constant $q \geq 0$ such that

$$w_m(r) + \ln(1 + r) \leq w_{m+1}(r) + q, \quad r \geq 0.$$

Proof. Let $m \in \mathbb{N}$. For each $r > 0$ we have:

$$w_m(r) + \ln r = \sup_{k \in \mathbb{Z}_+} \ln \frac{r^{k+1}}{M^{(m)}_k} = \sup_{k \in \mathbb{Z}_+} \ln \frac{r^{k+1}}{M^{(m+1)}_{k+1}M^{(m)}_k} \leq \sup_{k \in \mathbb{Z}_+} \ln \frac{r^{k+1}}{M^{(m+1)}_{k+1}} + \sup_{k \in \mathbb{Z}_+} \ln \frac{M^{(m+1)}_{k+1}M^{(m)}_k}{M^{(m)}_k} \leq w_{m+1}(r) + \sup_{k \in \mathbb{Z}_+} \ln \frac{M^{(m+1)}_{k+1}}{M^{(m)}_k}.$$ 

Now in view of Condition $i_1$), we arrive easily to the statement of the lemma. The proof is complete.

Lemma 4. Let $S \in G^*_m(U)$. Then $\hat{S} \in H_{b,2R}(T_C)$. 


Proof. We first mention that if $z = x + iy \in T_C$ ($x \in \mathbb{R}^n, y \in C$), then the function $f_z(\xi) = e^{i\langle \xi, z \rangle}$ belongs to the space $G_{2\Omega}(U)$. Indeed, for each $m \in \mathbb{N}$,

$$p_m(f_z) = \sup_{\xi \in \mathcal{V}, \alpha \in \mathbb{Z}_+^n} \frac{|(iz)^n e^{i\langle \xi, z \rangle}||1 + ||\xi|||^m}{M^{(m)}_{|\alpha|}} \leq \sup_{\alpha \in \mathbb{Z}_+^n} \frac{|z|^n}{M^{(m)}_{|\alpha|}} \sup_{\xi \in \mathcal{V}} \exp(-\langle \xi, y \rangle + m(1 + ||\xi||))$$

$$= \exp(\omega_m(|z|^n) + \sup_{\xi \in \mathcal{V}}(-\langle \xi, y \rangle + m(1 + ||\xi||))).$$

It is known [3, Lm. 1] that there exists a number $\delta > 0$ independent of $y$ such that

$$\sup_{\xi \in \mathcal{V}}(-\langle \xi, y \rangle + m(1 + ||\xi||)) \leq b(y) + dm + 3m \ln(1 + \frac{1}{\Delta C(y)}) + 2m \ln(1 + ||y||).$$

Employing this inequality and Lemma 3, we obtain a final estimate:

$$p_m(f_z) \leq A e^{b(y) + \omega_m(|z|^n)} \left(1 + \frac{1}{\Delta C(y)}\right)^{3m},$$

where $A$ is some positive constant independent of $z \in T_C$. Thus, $f_z \in G_{2\Omega}(U)$ and if $S \in G_{2\Omega}'(U)$, then on $T_C$, the following function is well-defined: $\hat{S}(z) = (S, e^{i\langle \xi, z \rangle})$. Employing Lemma 2 and Condition $(i)_k$, it is easy to show that $\hat{S} \in H(T_C)$. There exist numbers $m \in \mathbb{N}$ and $c > 0$ such that

$$|(S, f)| \leq cp_m(f), \quad f \in G_{2\Omega}(U).$$

By (2) this implies:

$$|\hat{S}(z)| \leq c A e^{b(y) + \omega_m(|z|^n)} \left(1 + \frac{1}{\Delta C(y)}\right)^{3m}.$$ 

Therefore, $\hat{S} \in H_{b,2\Omega}(T_C)$. The proof is complete. \[ \square \]

Standard arguing with applying Montel theorem and Lemma 3 show that for each $m \in \mathbb{N}$ the embeddings $j_m : H_{b,m}(T_C) \to H_{b,m+1}(T_C)$ are completely continuous. This means that $H_{b,2\Omega}(T_C)$ is a space $(LN^\ast)$ or, following a more modern terminology, a space $DFS$. \[ \square \]

In the proof of Theorem 1, while passing from integral weighted estimates for holomorphic functions in a tubular domain $T_C$ to uniform estimates, we shall make use of the following lemma [6, Lm. 9].

**Lemma 5.** Let $K$ be an open convex cone in $\mathbb{R}^n$ with the vertex at the origin. Let $h$ be a convex continuous positive homogeneous of degree 1 function on the closure on the cone $K$. Then for each $\varepsilon > 0$ there exists a constant $A_\varepsilon > 0$ such that

$$|h(y_2) - h(y_1)| \leq \varepsilon \|y_1\| + \varepsilon \|y_2\| + A_\varepsilon$$

for all $y_1, y_2 \in K$ such that $\|y_2 - y_1\| \leq 1$.

3. Description of space $G_{2\Omega}^\ast(U)$

3.1. Three important results. We first provide three important results playing a key role in the proof of Theorem 1. The first result is the Paley-Wiener-Schwartz theorem for the space $S(U)$ obtained in [3] by a scheme from [1]. It will be applied in the proving that the Fourier-Laplace transform is bijective. To formulate it, we define a space $V_b(T_C)$ as follows. For each $m \in \mathbb{N}$ we define normed spaces

$$V_{b,m}(T_C) = \left\{ f \in H(T_C) : N_m(f) = \sup_{z \in T_C} \frac{|f(z)|e^{-b(y)}}{(1 + \|z\|)^m(1 + \frac{1}{\Delta C(y)})^m} < \infty \right\},$$

where $z = x + iy, x \in \mathbb{R}^n, y \in C$. Let $V_b(T_C) = \bigcup_{m=1}^{\infty} V_{b,m}(T_C)$. The set $V_b(T_C)$ with summing and multiplication by complex numbers is a linear space. We equip $V_b(T_C)$ by the topology of the inductive limit of the spaces $V_{b,m}(T_C)$. 


Theorem 2. The Laplace-Fourier transform $\mathcal{F}: S^*(U) \to V_b(T_C)$ defined by the rule $\mathcal{F}(T) = \hat{T}$ is an isomorphism.

For $b(y) = a\|y\|$ ($a > 0$), Theorem 2 was proved by V.S. Vladimirov [1].

The second result we shall need is established by J.W. de Roever [2, Thm. 3.1]. It will be employed in the proof that the Fourier-Laplace transform is surjective.

**Theorem 3.** Let a linear subspace in $\mathbb{C}^n$ of dimension $n - k$ be defined by linear functions $\theta_1 = s_1(\theta_{k+1}, \ldots, \theta_n), \ldots, \theta_k = s_k(\theta_{k+1}, \ldots, \theta_n)$, or, briefly, $w = s(z), w \in \mathbb{C}^k, z \in \mathbb{C}^{n-k}$. Let $\Omega_1 \subset \Omega_2 \subset \Omega$ be the holomorphy domains in $\mathbb{C}^n$ such that for some $\varepsilon > 0$, the $\varepsilon$-neighbourhood of $\Omega_1$ in the first $k$ coordinates in the semi-circle metrics is contained in $\Omega_2$, that is,

$$
\{(\theta_1, \ldots, \theta_n) : |\theta_j - \theta_j^0| < \varepsilon, j = 1, \ldots, k; \theta_j = \theta_j^0, j = k + 1, \ldots, n; \theta^0 = (\theta_0^1, \ldots, \theta_0^n) \in \Omega_1\} \subset \Omega_2.
$$

Let $\varphi$ be a plurisubharmonic function on $\Omega$ and, for $\theta = (\theta_1, \ldots, \theta_n) \in \Omega_1$,

$$
\varphi_\varepsilon(\theta) = \max\{\varphi(\theta_1 + \xi_1, \ldots, \theta_n + \xi_n : |\xi_j| < \varepsilon, j = 1, \ldots, k\}.
$$

Let

$$
\Omega' = \{z \in \mathbb{C}^{n-k} : (s(z), z) \in \Omega\}, \quad \Omega'_j = \{z \in \mathbb{C}^{n-k} : (s(z), z) \in \Omega_j\}, \quad j = 1, 2,
$$

and $\tilde{\varphi}(z) = \varphi(s(z), z), z \in \Omega'$.

Then given a function $f$ analytic in $\Omega'$, there exists a function $F$ analytic in $\Omega_1$ such that $F(s(z), z) = f(z), z \in \Omega'$, and for some $K > 0$ depending only on $k$ and $s_1, \ldots, s_k$, the inequality

$$
\int_{\Omega_1} \frac{F(\theta)\exp(-\varphi_\varepsilon(\theta))}{(1 + |\theta|^2)^{4k}} \, d\lambda_n(\theta) \leq K\varepsilon^{-2k} \int_{\Omega'_2} |f(z)|^2 e^{-\tilde{\varphi}(z)} \, d\lambda_{n-k}(\theta),
$$

holds, where $\lambda_n$ and $\lambda_{n-k}$ denote the Lebesgue measure in $\mathbb{C}^n$ and $\mathbb{C}^{n-k}$, respectively. If the right hand side of the latter inequality is finite, then $F$ depends on $f, \Omega_1, \varepsilon, \varphi$.

**Theorem 4.** Let $\mathcal{O}$ be a holomorphy domain in $\mathbb{C}^n$, and $h$ be a plurisubharmonic function in $\mathcal{O}$ and $\varphi$ be plurisubharmonic function in $\mathbb{C}^n$ such that

$$
|\varphi(z) - \varphi(t)| \leq c_h \quad \text{if} \quad \|z - t\| \leq \frac{1}{(1 + \|t\|)^\nu},
$$

for some $c_h > 0$ and $\nu > 0$. Let a function $S \in H(\mathbb{C}^n \times \mathcal{O})$ satisfies the inequality

$$
|S(z, \zeta)| \leq e^{\varphi(z)+h(\zeta)}, \quad z \in \mathbb{C}^n, \quad \zeta \in \mathcal{O},
$$

and $S(\zeta, \zeta) = 0$ for $\zeta \in \mathcal{O}$.

Then there exist functions $S_1, \ldots, S_n \in H(\mathbb{C}^n \times \mathcal{O})$ such that

a) $S(z, \zeta) = \sum_{j=1}^n S_j(z, \zeta)(z_j - \zeta_j), \quad (z, \zeta) \in \mathbb{C}^n \times \mathcal{O}$;

b) for some $m \in \mathbb{N}$ independent of $S$, we have

$$
\int_{\mathbb{C}^n \times \mathcal{O}} \frac{|S_j(z, \zeta)|^2}{e^{2(\varphi(z)+h(\zeta)+m\ln(1+(|z|+|\zeta|))})} \, d\lambda_{2n}(z, \zeta) < \infty, \quad j = 1, \ldots, n.
$$

Theorem 4 was proved in [6, Lm. 11]. It will be employed in the proof that the Laplace-Fourier transform is injective.

**3.2. Proof of Theorem 1.** We first observe that by Lemma 4, the linear mapping $L : S \in G_{2\mathbb{R}}^*(U) \to \tilde{S}$ acts from $G_{2\mathbb{R}}^*(U)$ into $H_{b,2\mathbb{R}}(T_C)$. The continuity of $L$ is established in the same way as in the proof of Theorem 2 in [3].

We prove that the mapping $L$ is bijective by following a scheme in [4, 5]. Let us show that $L$ is a surjection. Given $F \in H_{b,2\mathbb{R}}(T_C)$, we have $F \in H_{b,m}(T_C)$ for some $m \in \mathbb{N}$. Taking into consideration that $d(z) = \Delta_C(y)$, we get:

$$
\int_{T_C} |F(z)|^2 \exp(-2(b(\Im z) + \omega_m(|z|) + m\ln(1+\frac{1}{\pi M})+(n+1)\ln(1+|z|^2))) \, d\lambda_n(z) < \infty.
$$

(3)
We let $\mathcal{K} = \mathbb{R}^n \times C$. In Theorem 3 we replace $n$ by $2n$ and choose

$$\Omega = \Omega_1 = \Omega_2 = \mathbb{R}^{2n} + i\mathcal{K}.$$  

It is obvious that

$$\Omega = \Omega_1 = \Omega_2 = \mathbb{C}^n \times T_C.$$  

As a linear subspace in this theorem we consider the subspace

$$W = \{ (z, \xi) \in \mathbb{C}^{2n} : z_1 = \xi_1, \ldots, z_n = \xi_n \}$$  

of complex dimension $n$. Then

$$\Omega' = \Omega'_1 = \Omega'_2 = \{ z \in \mathbb{C}^n : (z, z) \in \Omega = \mathbb{C}^n \times T_C \} = T_C.$$  

Then in Theorem 3 as $\varepsilon$ we take 1, while as $\varphi$, we choose the function

$$\varphi(z, \zeta) = 2(b(\text{Im} z) + \omega_m(|z|_n) + m \ln(1 + \frac{1}{d(\zeta)}) + (n + 1) \ln(1 + \|z, \zeta\|^2)),$$

where $z = x + iy \in \mathbb{C}^n$, $\zeta \in T_C$. We note that $\varphi(z, \zeta)$ is plurisubharmonic in $\mathbb{C}^n \times T_C$ and

$$\bar{\varphi}(z) = 2(b(\text{Im} z) + \omega_m(|z|_n) + m \ln(1 + \frac{1}{d(z)}) + (n + 1) \ln(1 + 2\|z\|^2)), \quad z \in T_C.$$  

In view of (3),

$$\int_{T_C} |F(z)|^2 e^{-\bar{\varphi}(z)} d\lambda_n(z) < \infty.$$  

Hence, by Theorem 3, there exists a function $\Phi \in H(\mathbb{C}^n \times T_C)$ such that $\Phi(z, z) = F(z)$ for $z \in T_C$ and for some $B > 0$ the estimate

$$\int_{\mathbb{C}^n \times T_C} \frac{|\Phi(z, \zeta)|^2 e^{-\varphi_1(z, \zeta)}}{(1 + \|z, \zeta\|^2)^{3n}} d\lambda_{2n}(z, \zeta) \leq B \int_{T_C} |F(z)|^2 e^{-\bar{\varphi}(z)} d\lambda_n(z)$$

holds. Here $\varphi_1(z, \zeta) = \max_{t \in \Pi} \varphi(z + t, \zeta)$. Since

$$|\ln(1 + x_2^2) - \ln(1 + x_1^2)| \leq |x_2 - x_1|, \quad x_1, x_2 \in \mathbb{R},$$

and for some $b_m > 0$ and for $r_1, r_2 \geq 0$ such that $|r_2 - r_1| \leq 1$ we have [10] Lm. 1

$$|w_m(r_2) - w_m(r_1)| \leq b_m,$$  

then

$$|\varphi_1(z, \zeta) - \varphi(z, \zeta)| \leq c_0, \quad (z, \zeta) \in \mathbb{C}^n \times T_C,$$

where $c_0 = 2(n + 1) \sqrt{n} + 2b_m$. Hence,

$$\int_{\mathbb{C}^n \times T_C} \frac{|\Phi(z, \zeta)|^2 e^{-\varphi_1(z, \zeta)}}{(1 + \|z, \zeta\|^2)^{3n}} d\lambda_{2n}(z, \zeta) \leq Be^{c_0} \int_{T_C} |F(z)|^2 e^{-\bar{\varphi}(z)} d\lambda_n(z).$$  

(5)

We denote the right hand side of this inequality by $B_F$. Letting

$$h_m(z, \zeta) = 2 \left( b(\text{Im} \zeta) + \omega_m(|z|_n) + m \ln \left( 1 + \frac{1}{d(\zeta)} \right) \right), \quad z \in \mathbb{C}^n, \quad \zeta \in T_C,$$

for brevity, we obtain uniform estimates for $\Phi(z, \zeta)$. Let $(z, \zeta) \in \mathbb{C}^n \times T_C$ and $R = \min(1, \frac{d(\zeta)}{4})$. We note that if $(t, u) \in \mathbb{C}^n \times T_C$ belongs to the ball $B((z, \zeta), R)$, then by Lemma 5, for some $A_\varepsilon > 0$,

$$|b(\text{Im} u) - b(\text{Im} \zeta)| \leq 2\varepsilon \|\text{Im} \zeta\| + A_\varepsilon + \varepsilon;$$

and by inequality (4)

$$|\omega_m(|t|_n) - \omega_m(|z|_n)| \leq b_m.$$
It is obvious that
\[
|\ln(1 + \| (t, u) \|^2) - \ln(1 + \| (z, \zeta) \|^2)| \leq 1;
\]
\[
|\ln \left( 1 + \frac{1}{d(u)} \right) - \ln \left( 1 + \frac{1}{d(\zeta)} \right)| \leq \frac{5}{4}.
\]

Employing these inequalities, for each \( \varepsilon > 0 \) we have:
\[
|h_m(t, u) + (5n + 2) \ln(1 + \| (t, u) \|^2) - (h_m(z, \zeta) + (5n + 2) \ln(1 + \| (z, \zeta) \|^2))| \leq 4\varepsilon \| \Im \zeta \| + B_{\varepsilon, m},
\]
where
\[
B_{\varepsilon, m} = 2A_\varepsilon + 2\varepsilon + \frac{5m}{2} + 2b_m.
\]

Employing the latter inequality, the plurisubharmonicity of the function \(|\Phi(t, u)|^2\) in \( \mathbb{C}^n \times T_C \) and inequality (5), we obtain that
\[
|\Phi(z, \zeta)|^2 \leq \frac{B_F}{\lambda_{2n}(R)} e^{h_m(z, \zeta) + (5n + 2) \ln(1 + \| (z, \zeta) \|^2) + 4\varepsilon \| \Im \zeta \| + B_{\varepsilon, m}}.
\]

Hence, for each \( \varepsilon > 0 \) there exists a number \( c_1 > 0 \) such that
\[
|\Phi(z, \zeta)| \leq c_1 \exp \left( b(\Im \zeta) + \omega_m(|z|_n) + (m + 2n) \ln \left( 1 + \frac{1}{d(\zeta)} \right) + (5n + 2) \ln(1 + \| (z, \zeta) \|) + 2\varepsilon \| \Im \zeta \| \right)
\]
for \( (z, \zeta) \in \mathbb{C}^n \times T_C \). Since \( \Phi(z, \zeta) \) is entire in \( z \), then, expanding \( \Phi(z, \zeta) \) in powers of \( z \), we find:
\[
\Phi(z, \zeta) = \sum_{|\alpha| > 0} C_\alpha(\zeta) z^\alpha, \quad \zeta \in T_C, \quad z \in \mathbb{C}^n.
\]

By the Cauchy formula, the identity
\[
C_\alpha(\zeta) = \frac{1}{(2\pi i)^n} \int_{|z_i| = R} \ldots \int_{|z_n| = R} \frac{\Phi(z, \zeta)}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}} dz_1 \ldots dz_n
\]
holds, where \( \alpha \in \mathbb{Z}_+^n \) and \( R > 0 \) is arbitrary. This implies that \( C_\alpha \in H(T_C) \). Employing (6), we obtain:
\[
|C_\alpha(\zeta)| \leq \frac{c_1((1 + \sqrt{n}R)(1 + \| \zeta \|))^{5n + 2} e^{b(\Im \zeta) + 2\varepsilon \| \Im \zeta \| + \omega_m(R) \left( 1 + \frac{1}{d(\zeta)} \right)^m + 2n}}{R^{|\alpha|}}.
\]

By Lemma 3 we find a constant \( c_2 > 0 \) depending on \( \varepsilon \) such that
\[
|C_\alpha(\zeta)| \leq c_2 e^{\omega_m + (5n + 2)(R)} \frac{e^{b(\Im \zeta) + 2\varepsilon \| \Im \zeta \| + \omega_m(R) \left( 1 + \frac{1}{d(\zeta)} \right)^m + 2n}}{R^{|\alpha|}}, \quad \zeta \in T_C.
\]

Therefore, for \( \zeta \in T_C \),
\[
|C_\alpha(\zeta)| \leq c_2 \left( \inf_{R > 0} \frac{e^{\omega_m + (5n + 2)(R)}}{R^{|\alpha|}} \right) e^{b(\Im \zeta) + 2\varepsilon \| \Im \zeta \| + \omega_m(R) \left( 1 + \frac{1}{d(\zeta)} \right)^m + 2n}.
\]

Hence, taking into consideration the identity
\[
\inf_{r > 0} \frac{e^{\omega_m + 5n + 2(r)}}{r^k} = \frac{1}{M_k^{(m + 5n + 2)}}, \quad k = 0, 1, \ldots,
\]
for each \( \varepsilon > 0 \) and all \( \alpha \in \mathbb{Z}_+^n \), \( \zeta \in T_C \) we have:
\[
|C_\alpha(\zeta)| \leq c_2 \frac{e^{b(\Im \zeta) + 2\varepsilon \| \Im \zeta \| + \omega_m(R) \left( 1 + \frac{1}{d(\zeta)} \right)^m + 2n}}{M_k^{(m + 5n + 2)}}, \quad k = 0, 1, \ldots.
\]
For each $\varepsilon > 0$ we define a function $b_\varepsilon$ on $\mathcal{C}$ by the rule $b_\varepsilon(y) = b(y) + \varepsilon\|y\|$ and the space $V_{b_\varepsilon}(T_C)$ being the inductive limit of normed spaces

$$V_{b_\varepsilon,m}(T_C) = \left\{ f \in H(T_C) : N_{k,\varepsilon}(f) = \sup_{z \in T_C} \frac{|f(z)|e^{-b_\varepsilon(y)}}{(1 + |z|)^k(1 + \frac{1}{2\varepsilon_0(y)})^k} < \infty \right\},$$

where $k \in \mathbb{Z}_+$, $z = x + iy$, $x \in \mathbb{R}^n$, $y \in C$. By the latter inequality we see that for each $\alpha \in \mathbb{Z}_n^+$ the function $C_\alpha$ belongs to the space $H_b(T_C)$, which the projective limit of the spaces $V_{b_\varepsilon}(T_C)$. By estimate (7), the set $\{M^{(m+5n+2)}C_\alpha\}_{\alpha \in \mathbb{Z}_n^+}$ is bounded in each space $V_{b_\varepsilon}(T_C)$. Hence, it is bounded also in $H_b(T_C)$. The space $H_b(T_C)$ coincides with the space $V_(T_C)$ [3 Thm. D]; for the case $b(y) = a\|y\|$ with $a \geq 0$ this fact was established by V.S. Vladimirov [1 Ch. 2]. Hence, the set $\{M^{(m+5n+2)}C_\alpha\}_{\alpha \in \mathbb{Z}_n^+}$ is bounded in $V_{b_\varepsilon}(T_C)$. Since the spaces $S^*(U)$ and $V_b(T_C)$ are isomorphic (Theorem 1), there exist functionals $S_\alpha \in S^*(U)$ such that $\hat{S}_\alpha = C_\alpha$, the set $\mathcal{A} = \{M^{(m+5n+2)}S_\alpha\}_{\alpha \in \mathbb{Z}_n^+}$ is bounded in $S^*(U)$ and hence, is weakly bounded. By the Schwartz theorem [1 Ch. 1, Sect. 5], there exist numbers $c_3 > 0$ and $p \in \mathbb{N}$ such that

$$|(F, f)| \leq c_3\|f\|_{p,U}, \quad F \in \mathcal{A}, \quad f \in S(U).$$

Thus,

$$|(S_\alpha, D^\alpha f)| \leq \frac{c_3}{M^{(m+5n+2)}_{|\alpha|}}\|f\|_{p,U}, \quad f \in S(U), \quad \alpha \in \mathbb{Z}_n^+.$$ (8)

We define a functional $T$ on $G_{2\Omega}(U)$ by the rule:

$$(T, f) = \sum_{|\alpha| \geq 0} (S_\alpha, (-i)^{|\alpha|}D^\alpha f), \quad f \in G_{2\Omega}(U).$$ (9)

Let us show that $T \in G'_{2\Omega}(U)$. Employing inequality (8), for $f \in G_{2\Omega}(U)$, $\alpha \in \mathbb{Z}_n^+$, natural numbers $s \geq p$ we have:

$$|(S_\alpha, D^\alpha f)| \leq \frac{c_3}{M^{(m+5n+2)}_{|\alpha|}} \sup_{x \in U, |\beta| \leq p} |(D^{\alpha+\beta} f)(x)|(1 + \|x\|)^p$$

$$\leq \frac{c_3}{M^{(m+5n+2)}_{|\alpha|}} \sup_{x \in U, |\beta| \leq p} \frac{p_{m+s}(f)M^{(m+s)}_{|\alpha|+|\beta|}(1 + \|x\|)^p}{(1 + \|x\|)^{m+s}}$$

$$\leq c_3p_{m+s}(f) \frac{M^{(m+s)}_{|\alpha|}}{M^{(m+5n+2)}_{|\alpha|}}.$$ (9a)

Employing Condition $i_8$, we choose $s \geq p$ so that the series $\sum_{|\alpha| \geq 0} \frac{M^{(m+s)}_{|\alpha|+p}}{M^{(m+5n+2)}_{|\alpha|}}$ converges. Therefore, for each $f \in G_{2\Omega}(U)$, the series in the right hand side in (9) converges absolutely and for some $c_4 > 0$ independent of $f \in G_{2\Omega}(U)$ we have

$$|(T, f)| \leq c_4p_{m+s}(f).$$ (9b)

Therefore, the linear functional $T$ is well-defined and continuous. It is easy to see that $\hat{T} = F$. Thus, the mapping $L$ is surjective.

We are going to show that the mapping $L$ is injective. Let $T \in G'_{2\Omega}(U)$ $\hat{T} \equiv 0$. There exist numbers $m \in \mathbb{N}$ and $c_5 > 0$ such that

$$|(T, f)| \leq c_5p_{m}(f), \quad f \in G_{2\Omega}(U).$$

By Lemma 2, there exist functionals $T_\alpha \in C'_m(U)$, $\alpha \in \mathbb{Z}_n^+$, such that

$$(T, f) = \sum_{\alpha \in \mathbb{Z}_n^+} (T_\alpha, D^\alpha f), \quad f \in G_{2\Omega}(U).$$
and
\[ |(T_\alpha, g)| \leq \frac{c_5}{M_{|\alpha|}} \bar{P}_m(g), \quad g \in C_m(U). \] (10)

Therefore, for each \( z \in T_C \) we have:
\[ \hat{T}(z) = \sum_{\alpha \in \mathbb{Z}^n_+} i^{\alpha}(T_\alpha, e^{i(\xi, \zeta)}) z^\alpha. \]

Let \( V_\alpha(z) = i^{\alpha}(T_\alpha, e^{i(\xi, \zeta)}) \). It is obvious that \( V_\alpha \in H(T_C) \). Employing inequalities (10) and (1), we get:
\[ |V_\alpha(z)| \leq \frac{c_6}{M_{|\alpha|}} (1 + \|z\|)^{2m} (1 + \frac{1}{\Delta_C(y)})^{3m} e^{b(y)}, \] (11)
where \( c_6 > 0 \) is a constant independent of \( z = x + iy \in T_C \) and \( \alpha \). We consider the function
\[ S(u, z) = \sum_{|\alpha| \geq 0} V_\alpha(z) u^\alpha, \quad z \in T_C, \quad u \in \mathbb{C}^n. \]

Employing estimate (11), we obtain:
\[ |S(u, z)| \leq c_6 \left( 1 + \frac{1}{\Delta_C(y)} \right)^{3m} (1 + \|z\|)^{2m} e^{b(y)} \sum_{|\alpha| \geq 0} |u|^{|\alpha|}_{M_{|\alpha|}}. \]

Employing Condition \( i_8 \), we choose \( \nu \in \mathbb{N} \) so that the series \( \sum_{|\alpha| \geq 0} \frac{M_{|\alpha|}^{(m+\nu)}}{M_{|\alpha|}^{(m)}} \) converges. Then
\[ |S(u, z)| \leq c_7 e^{b(y)} \left( 1 + \frac{1}{\Delta_C(y)} \right)^{3m} (1 + \|z\|)^{2m} e^{\nu m + \nu(|\alpha|)}, \]
where
\[ c_7 = c_6 \sum_{|\alpha| \geq 0} \frac{M_{|\alpha|}^{(m+\nu)}}{M_{|\alpha|}^{(m)}}. \]

We note that
\[ S(z, z) = \sum_{|\alpha| \geq 0} V_\alpha(z) z^\alpha = 0 \]
for each \( z \in T_C \). Then by Theorem 4 there exist functions \( S_1, \ldots, S_n \in H(\mathbb{C}^n \times T_C) \) such that
\[ S(z, \zeta) = \sum_{j=1}^n S_j(z, \zeta)(z_j - \zeta_j), \quad z \in \mathbb{C}^n, \quad \zeta \in T_C, \]
and for some \( c_8 > 0 \) and \( k \in \mathbb{N} \) and all \( j = 1, 2, \ldots, n, z \in \mathbb{C}^n, \zeta \in T_C \) we have
\[ |S_j(z, \zeta)| \leq c_8 \exp \left( \frac{\omega_k(|z|_n) + b(\Im \zeta) + k \ln \left( 1 + \frac{1}{d(\zeta)} \right)}{k} \right). \] (12)

We expand \( S_j \) into the Taylor series in powers of \( z \):
\[ S_j(z, \zeta) = \sum_{|\alpha| \geq 0} S_{j, \alpha}(\zeta) z^\alpha, \quad z \in \mathbb{C}^n, \quad \zeta \in T_C. \]

Proceeding as estimating the functions \( C_\alpha \), by (12) we obtain:
\[ |S_{j, \alpha}(\zeta)| \leq c_8 \exp \left( b(\Im \zeta) + k \ln \left( 1 + \frac{1}{d(\zeta)} \right) \right) \frac{1}{M_{|\alpha|}^{(k)}}. \]

Since the Fourier-Laplace transform establishes a topological isomorphism between \( S^*(U) \) and \( V(T_C) \), there exist functionals \( \psi_{j, \alpha} \in S^*(U) \) such that \( \hat{\psi}_{j, \alpha} = S_{j, \alpha} \). It follows from the latter estimate that the set \( \{ S_{j, \alpha} M_{|\alpha|}^{(k)} \}_{\alpha \in \mathbb{Z}^n_+} \) is bounded in \( V(T_C) \). Then the set \( \Psi = \{ M_{|\alpha|}^{(k)} \psi_{j, \alpha} \}_{\alpha \in \mathbb{Z}^n_+, j=1, \ldots, n} \) is bounded
in $S^*(U)$. Hence, it is weakly bounded. By the Schwartz theorem [11 Ch. 1, Sect. 5], there exist numbers $c_0 > 0$ and $p \in \mathbb{N}$ such that

$$|\langle F, \varphi \rangle| \leq c_0 \|\varphi\|_{p,U}, \quad F \in \Psi, \quad \varphi \in S(U).$$

Thus,

$$|\langle \Psi_{j,\alpha}, f \rangle| \leq \frac{c_0}{M(\alpha)} \|f\|_{p,U}, \quad f \in S(U), \quad \alpha \in \mathbb{Z}_+^n, \quad j = 1, \ldots, n. \quad (13)$$

For $j = 1, \ldots, n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with at least one negative component, let $\Psi_{j,\alpha}$ be a zero functional in $S^*(U)$ and $S_{j,\alpha}(z) = 0$ for all $z \in \mathbb{C}^n$. Then

$$S(z, \zeta) = \sum_{j=1}^n \sum_{|\alpha| \geq 0} (S_{j,(\alpha_1, \ldots, \alpha_j-1, \ldots, \alpha_n)}(\zeta) - S_{j,\alpha}(\zeta)\zeta^\alpha), \quad z \in \mathbb{C}^n, \quad \zeta \in T_C.$$

Therefore,

$$V_{\alpha}(\zeta) = \sum_{j=1}^n (S_{j,(\alpha_1, \ldots, \alpha_j-1, \ldots, \alpha_n)}(\zeta) - S_{j,\alpha}(\zeta)\zeta^\alpha), \quad \alpha \in \mathbb{Z}_+^n.$$

That is,

$$V_{\alpha}(\zeta) = \sum_{j=1}^n \left( \Psi_{j,(\alpha_1, \ldots, \alpha_j-1, \ldots, \alpha_n)}(\zeta) + i \left( \Psi_{j,\alpha}, \frac{\partial}{\partial \zeta_j} e^{i\zeta_j} \right) \right).$$

This means that the right hand side in the latter identity is the Fourier-Laplace transform of the functional acting by the rule:

$$f \in S(U) \rightarrow \sum_{j=1}^n \left( \Psi_{j,(\alpha_1, \ldots, \alpha_j-1, \ldots, \alpha_n)}, f \right) + i \left( \Psi_{j,\alpha}, \frac{\partial}{\partial \zeta_j} f \right).$$

Hence,

$$(T_{\alpha}, f) = (-i)^{|\alpha|} \sum_{j=1}^n \left( i \left( \Psi_{j,\alpha}, \frac{\partial}{\partial \zeta_j} f \right) + (\Psi_{j,(\alpha_1, \ldots, \alpha_j-1, \ldots, \alpha_n)}, f) \right).$$

Thus,

$$(T_{\alpha}, f) = \sum_{|\alpha| > 0} (-i)^{|\alpha|} \sum_{j=1}^n \left( i \left( \Psi_{j,\alpha}, \frac{\partial}{\partial \zeta_j} D^\alpha f \right) + (\Psi_{j,(\alpha_1, \ldots, \alpha_j-1, \ldots, \alpha_n)}, D^\alpha f) \right), \quad f \in G_{2n}(U).$$

For an arbitrary $N \in \mathbb{N}$ we define sets

$$B_N = \{ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n : \alpha_1 \leq N, \ldots, \alpha_n \leq N \},$$

$$R_{N,j} = \{ \alpha_1 \leq N, \ldots, \alpha_j = N, \ldots, \alpha_n \leq N, \alpha \in \mathbb{Z}_+^n \}, \quad j = 1, \ldots, n,$$

and a functional $T_N$ on $G_{2n}(U)$ by the rule:

$$(T_N, f) = \sum_{\alpha \in B_N} (-i)^{|\alpha|} \sum_{j=1}^n \left( i \left( \Psi_{j,\alpha}, \frac{\partial}{\partial \zeta_j} D^\alpha f \right) + (\Psi_{j,(\alpha_1, \ldots, \alpha_j-1, \ldots, \alpha_n)}, D^\alpha f) \right).$$

Then

$$(T, f) = \lim_{N \to \infty} (T_N, f), \quad f \in G_{2n}(U).$$

As in [5], we confirm that

$$(T_N, f) = \sum_{j=1}^n \sum_{\alpha \in R_{N,j}} (-i)^{|\alpha|} i(\Psi_{j,\alpha}, \frac{\partial}{\partial \zeta_j} D^\alpha f), \quad f \in G_{2n}(U).$$

Hence, in view of (13), we have

$$|(T_N, f)| \leq \sum_{j=1}^n \sum_{\alpha \in R_{N,j}} \frac{c_0}{M(\alpha)} \sup_{\xi \in U, |\xi| \leq p} \left( (D^{(\alpha_1+\gamma_1, \ldots, \alpha_j+\gamma_j, \ldots, \alpha_n+\gamma_n)}) f(\xi) \left( 1 + \|\xi\|^p \right) \right)$$
for all \( f \in G_{2R}(U) \). For each natural number \( \nu \geq p \) and for all \( f \in G_{2R}(U) \) we have:

\[
|\langle T_N, f \rangle| \leq \sum_{j=1}^{n} \sum_{\alpha \in R_{N,j}} c_9 \frac{M^{(\nu)}_{|\alpha|+\gamma} + 1}{M_{|\alpha|}} \sup_{\xi \in U, |\gamma| \leq p} \frac{M^{(\nu)}_{|\alpha|+p+1}}{M_{|\alpha|}} \leq c_9 p^\nu(f) \sum_{j=1}^{n} \sum_{\alpha \in R_{N,j}} \frac{M^{(\nu)}_{|\alpha|+p+1}}{M_{|\alpha|}}.
\]

Employing Condition \( i_8 \), we choose \( \nu \in \mathbb{N} \) so that the series \( \sum_{|\alpha| \geq 0} \frac{M^{(\nu)}_{|\alpha|+p+1}}{M_{|\alpha|}} \) converges. Then

\[
|\langle T_N, f \rangle| \leq n c_9 p^\nu(f) \sum_{|\alpha| \geq N} \frac{M^{(\nu)}_{|\alpha|+p+1}}{M_{|\alpha|}}.
\]

This implies that \( \langle T_N, f \rangle \to 0 \) as \( N \to \infty \). Hence, \( \langle T, f \rangle = 0 \) for all \( f \in G_{2R}(U) \). Thus, the mapping \( L \) is injective.

By the theorem on open mapping \( [11, \text{Append. } 1] \), the mapping \( L^{-1} \) is continuous. Thus, \( L \) is a topological isomorphism. The proof is complete.

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