Symplectic quasi-states on the quadric surface
and Lagrangian submanifolds

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Abstract

The quantum homology of the monotone complex quadric surface splits into the sum of two fields. We outline a proof of the following statement: The unities of these fields give rise to distinct symplectic quasi-states defined by asymptotic spectral invariants. In fact, these quasi-states turn out to be “supported” on disjoint Lagrangian submanifolds. Our method involves a spectral sequence which starts at homology of the loop space of the 2-sphere and whose higher differentials are computed via symplectic field theory, in particular with the help of the Bourgeois-Oancea exact sequence.

1 Introduction and main result

The quantum homology of the monotone complex quadric surface $S^2 \times S^2$ splits into the sum of two fields. The unities of these fields give rise to symplectic quasi-states defined by asymptotic spectral invariants (see [11]). One of these quasi-states is “supported” on a Lagrangian sphere, the anti-diagonal [12]. Our main finding (see Theorem 1.1 below) is that the second quasi-state is “supported” on the exotic monotone Lagrangian torus described in [12] which is disjoint from the anti-diagonal. Thus these quasi-states are distinct, and the exotic torus has strong symplectic rigidity properties. Let us pass to precise formulations.

Let $(W, \omega)$ be the standard symplectic quadric surface $S^2 \times S^2$, where both $S^2$ factors have equal areas 1. Consider the field $\mathcal{K} = \mathbb{C}[[t^{-1}, t]]$ of Laurent
series and the quantum homology algebra $QH(W, \omega) = H_\ast(W, \mathbb{C}) \otimes \mathcal{K}$ which is graded by

$$\deg(at^N) = \deg(a) + 4N.$$ 

The graded component $QH_4$ is an algebra over $\mathbb{C}$ with respect to the quantum product. It splits into the sum of two fields generated by idempotents $e_\pm = (1 \pm Pt)/2$, where $1 = [W]$ is the fundamental class and $P$ stands for the class of the point. Define the functionals $\zeta_\pm : C^\infty(W) \to \mathbb{R}$ by

$$\zeta_\pm(H) = \lim_{E \to \infty} c(e_\pm, EH)/E,$$

where $c(\cdot, \cdot)$ is a spectral invariant [17, 16].

It was shown in [11] that these functionals are symplectic quasi-states, that is they are monotone, linear on all Poisson-commutative subspaces and normalized by $\zeta_\pm(1) = 1$. Recall that a quasi-measure associated to a quasi-state $\zeta$ is a set-function whose value on a closed subset $X$ equals, roughly speaking, $\zeta(\chi_X)$ where $\chi_X$ is the indicator function of $X$. We write $\tau_\pm$ for the quasi-measures associated to $\zeta_\pm$. The reader is referred to [10, 11] for further preliminaries.

We view $W$ as the symplectic cut [14] of the unit cotangent bundle $\{|p| \leq 1\} \subset T^*S^2$, where the zero section is identified with the Lagrangian anti-diagonal $L \subset W$, and the length $|p|$ is understood with respect to the standard spherical metric so that the length of the equator equals 1. The level $\{|p| = 1/2\}$ contains unique (up to a Hamiltonian isotopy) monotone Lagrangian torus denoted by $K$. Write $T \subset W$ for the Lagrangian Clifford torus (the product of equators).

It has been proved in [12, Example 1.20] that $\tau_-(L) = 1$ and $\tau_-(K) = 0$. Together with the equality $\tau_-(T) = \tau_+(T) = 1$ (see [11]) this yields that $K$ and $T$ are not Hamiltonian isotopic.

**Theorem 1.1.** $\tau_+(L) = 0, \tau_+(K) = 1$. In particular, $\zeta_- \neq \zeta_+$.

Equality $\tau_+(K) = 1$ yields that $K$ is $e_+$-superheavy in the terminology of [12]. In particular, it is non-displaceable and intersects every image of the Clifford torus under a symplectomorphism.

The rest of the note contains a detailed outline of the proof of Theorem 1.1. Our method involves a spectral sequence which starts at homology of the loop space of the 2-sphere and whose higher differentials are computed via
symplectic field theory. The main technical ingredient comes from a paper by Bourgeois and Oancea \[5\].

Non-displaceability of the exotic torus $K$ has been recently established via Lagrangian Floer homology by several independent groups of researchers: Fukaya, Oh, Ohta and Ono \[13\], Chekanov and Schlenk \[6\] and Wehrheim (unpublished). The paper \[10\] presents new constructions of exotic Lagrangian tori in the product of spheres. A related construction of exotic tori is given by Biran and Cornea in \[4\] in the context of their study of narrow Lagrangian submanifolds. It would be interesting to understand whether these tori can be distinguished by appropriate symplectic quasi-measures.

The paper \[13\] contains a more general version of Theorem 1.1. According to \[13\], the exotic torus $K$ lies in an infinite family of non-displaceable Lagrangian tori whose Liouville class varies. Though our approach is quite different from the one in \[13\], there are some similarities which deserve further exploration. Let us mention also that the tori of the above-mentioned family appear as invariant sets of a semitoric integrable system (see \[18\] and Section 2 below). In dimension four, semitoric means that one of the integrals generates a Hamiltonian circle action. The study of this class of integrable systems was initiated recently in \[18\]. Semitoric systems have some amusing properties and appear in meaningful physical models. It would be interesting to detect “symplectically rigid” invariant Lagrangian tori in other examples of semitoric systems.

2 Reduction to a Floer-homological calculation

We work with Floer homology with $\mathbb{C}$-coefficients. In our conventions on the Conley-Zehnder indices, the PSS-isomorphism identifies $FH_k$ with $QH_{k+2}$ (see \[16\] for preliminaries).

Throughout that paper, we denote by $\Sigma$ the diagonal in $W = S^2 \times S^2$. In our picture, $\Sigma$ is obtained from the hypersurface $\{|p| = 1\}$ by the symplectic cut \[14\].

Fix $r \in (0; 1/2)$, $E > 0$ large enough, and $\epsilon > 0$ small enough. We assume that the data is “non-resonant”: $1/r \notin \mathbb{Z}$ and $(E + \epsilon)/\epsilon \notin \mathbb{Z}$.

Consider a Hamiltonian $H_E(|p|)$ on $W$ given by a piece-wise linear function which equals $E$ on $U := \{|p| \leq r - \epsilon\}$ and equals $-\epsilon$ on $V := \{|p| \geq r\}$. 

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We refer e.g. to [9] for a discussion on Floer-homological calculations with piece-wise linear Hamiltonians.

Orbits of period 1 of $H_E$ form several critical submanifolds, which we are going to describe now. Each of these submanifolds is equipped with a Morse function which is used for a small perturbation of the action functional associated to $H_E$. In addition, we fix capping discs for orbits from these submanifolds. Critical points of these Morse functions together with the capping discs give rise to generators of the Floer complex. Let us pass to the precise description of this data.

**The maximum set $U$:** Here we have constant orbits capped with the constant discs. We choose an exhausting Morse function $f_U$ on $U$ with two critical points: a saddle point $x_0$ of Morse index 2 and a maximum point $x_2$. Their Conley-Zehnder indices are 0 and 2 respectively, and their actions equal $E$. We refer to the elements of the Floer complex of the form $\gamma t^{-N}$ for $\gamma \in \{x_0, x_2\}$ as $U$-generators, and call the number $N$ the $t$-degree.

**The minimum set $V$:** Here we have constant orbits capped by the constant discs. We choose an exhausting Morse function $f_V$ on $V$ with two critical points: a saddle point $y_0$ of Morse index 2 and a minimum point $y_{-2}$. Their Conley-Zehnder indices are 0 and $-2$ respectively, and their actions equal $-\epsilon$. We refer to the elements of the Floer complex of the form $\gamma t^{-N}$ for $\gamma \in \{y_0, y_{-2}\}$ as $V$-generators.

**Non-constant orbits:** They form two series of submanifolds diffeomorphic to $\mathbb{R}P^3$. We denote these submanifolds by $Z_k^\pm$. Here the lower index $k$ corresponds to $k$-times-covered simple closed geodesics on $L$, $Z^+$ stands for the orbits on the submanifold $\{|p| = r - \epsilon\}$ and $Z^-$ stands for the orbits on the submanifold $\{|p| = r\}$. Note that the multiplicity $k$ satisfies inequalities $k \geq 1$ and

$$k \leq \frac{E + \epsilon}{\epsilon}. \tag{1}$$

In the discussion below we assume that $k \geq 1$ is arbitrary, and that inequality (1) is an extra restriction which selects orbits relevant for the Floer complex corresponding to the fixed value of $\epsilon > 0$. Next, we fix a Morse function $f_k^\pm$ on $Z_k^\pm$ with critical points $\hat{m}_k^\pm$, $\hat{M}_k^\pm$ and $\hat{M}_k^\pm$ of Morse indices 0, 1, 2, 3 respectively. It will be convenient for us to choose $f_k$ in such a way that the orbits in each pair of orbits $(\hat{m}_k^\pm, \hat{m}_k^\pm)$ and $(\hat{M}_k^\pm, \hat{M}_k^\pm)$ represent the same unparameterized orbit. The orbits from $Z_k^\pm$ are capped by discs lying in $W \setminus \Sigma$. We refer to the elements of the Floer complex of the form $\gamma t^{-N}$ for
\( \gamma \in \tilde{m}_k^+, \tilde{m}_k^-, \tilde{M}_k^+, \tilde{M}_k^- \) as upper generators and for \( \gamma \in \tilde{m}_k^-, \tilde{m}_k^-, \tilde{M}_k^-, \tilde{M}_k^- \) as lower generators, and call the number \( N \) the \( t \)-degree.

The Conley-Zehnder indices of the generators corresponding to the non-constant orbits are given in Table 1 below.

| \( \tilde{m}_k \) | \( \tilde{m}_k \) | \( \tilde{M}_k \) | \( \tilde{M}_k \) |
|-----------|-----------|---------|---------|
| upper     | 2k − 1    | 2k      | 2k + 1  |
| lower     | 2k − 2    | 2k − 1  | 2k      | 2k + 1  |

The actions of the lower generators equal \(-\epsilon + kr\) and of the upper generators \(E + kr\).

In what follows we write \( CZ(\gamma) \) for the Conley-Zehnder index of a capped orbit \( \gamma \) and \( A(\gamma) \) for its action. We have

\[
CZ(\gamma t^{-N}) = CZ(\gamma) - 4N, \quad A(\gamma t^{-N}) = A(\gamma) - N.
\]

A direct calculation (crucially based on the fact that \( r < 1/2 \)) yields the following lemma which will be used throughout the paper.

**Lemma 2.1.**

(i) All lower generators and \( V \)-generators of the Conley-Zehnder indices \(1, 2, 3\) have action < 1.

(ii) All upper generators and \( U \)-generators of the Conley-Zehnder indices \(1, 2, 3\) have action < \( E + 1 \) and non-negative \( t \)-degree.

(iii) Let \( \gamma \) be an upper generator of the Conley-Zehnder index \(1, 2, 3\) and action \( A(\gamma) > 1 \). Then inequality (1) holds automatically: \( k < \frac{E + \epsilon}{\epsilon} \).

(iv) There exist numbers \( \mu_-(E) < \mu_+(E) \), \( \mu_\pm(E) \to \infty \) as \( E \to \infty \) with the following property: Let \( \gamma \) be an upper generator or a \( U \)-generator of the Conley-Zehnder index \(1, 2, 3\) and \( t \)-degree \( N \). Then \( A(\gamma) > 1 \) for \( N \leq \mu_- \) and \( A(\gamma) < 1 \) for \( N > \mu_+ \).

**Proof. Lower Generators:** Take \( \gamma_k \in \{\tilde{m}_k^-, \tilde{m}_k^-, \tilde{M}_k^-, \tilde{M}_k^-\} \) and put \( \gamma = \gamma_k t^{-N} \). We have that

\[
CZ(\gamma) = 2k + j - 4N \in [1; 3], \quad j = -2, -1, 0, 1.
\]
Thus $0 \leq 2k - 4N \leq 4$. Note that

$$A(\gamma) = kr - \epsilon - N.$$ 

Since $r < 1/2$ we have that

$$A(\gamma) < (k/2 - N) - \epsilon < 1.$$ 

**Upper generators:** Take $\gamma_k \in \{\hat{m}_k^+, \hat{m}_k^+, \hat{M}_k^+, \hat{M}_k^+\}$ and put $\gamma = \gamma_k t^{-N}$. We have that

$$CZ(\gamma) = 2k + j - 4N \in [1; 3], \quad j = -1, 0, 1, 2.$$ 

Thus

$$0 \leq 2k - 4N \leq 4. \quad (2)$$ 

In particular, $N \geq 0$ since $k \geq 1$. Note that

$$A(\gamma) = E + kr - k\epsilon - N. \quad (3)$$ 

Since $r < 1/2$ we have that

$$A(\gamma) < E + (k/2 - N) - k\epsilon \leq E + 1.$$ 

Put $\kappa = 1 - 2r + 2\epsilon$. Observe that by (2) and (3) $A(\gamma) > 1$ yields $k < 2E/\kappa < (E + \epsilon)/\epsilon$ which proves statement (iii) of the lemma.

Further, pick

$$\mu_-(E) < (E - 1)/\kappa, \mu_+(E) > (E - \kappa)/\kappa.$$ 

Statement (iv) of the lemma readily follows from (2) and (3).

Finally, the only $U$-generator of the index 1, 2, 3 is $x_2$, and its action equals $E$ and its $t$-degree equals 0. The only $V$-generator of index 2 is $y_{-2t}$ and its action equals $1 - \epsilon$. This completes the proof of statements (i) and (ii) of the lemma. \hfill \Box

**Lemma 2.2** (Main lemma). $HF_2^{(1; E+1)}(H_E) = \mathbb{C}$. 

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Proof of Theorem 1.1 modulo Main Lemma:

**Step 1:** Look at the diagram

\[
\begin{array}{c}
HF_3^{(E+1, +\infty)}(H_E) \\
\downarrow j \\
HF_2^{(-\infty, 1)}(H_E) \xrightarrow{k} HF_2^{(-\infty, E+1)}(H_E) \xrightarrow{l} HF_2^{(1, E+1)}(H_E) = \mathbb{C} \\
\downarrow m \\
HF_2^{(-\infty, \infty)} = QH_4(M) = \mathbb{C}^2
\end{array}
\]

Here the horizontal and the vertical lines are exact, and the triangle is commutative. Since \(e_\pm = (1 \pm Pt)/2\) and \(\max H_E = E\) the spectral invariants \(c(e_\pm, H_E)\) do not exceed \(E + 1\). Thus, since \(QH_4\) is generated by \(e_-, e_+\) the map \(i\) is onto. By Lemma 2.1(i),(ii) \(HF_3^{(E+1, +\infty)}(H_E) = 0\). This yields that \(j = 0\) so \(i\) is an isomorphism, and in particular \(l\) has a non-trivial kernel. Thus \(k \neq 0\) and we conclude that \(m \neq 0\).

Assume that some non-zero quantum homology class \(a = \alpha e_- + \beta e_+\), \(\alpha, \beta \in \mathbb{C}\), lies in the image of \(m\). This yields \(c(a, H_E) \leq 1\). Since \(\tau_-(L) = 1\) (see [12]), we have that \(c(e_-, H_E) \geq E\), and therefore \(\beta \neq 0\). Observe that the quantum product \(e_+ \ast e_-\) equals 0, while \(e_+ \ast e_+ = e_+, e_- \ast e_- = e_-\). Thus, by the triangle inequality for spectral invariants,

\[
c(e_+, H_E) = c(a \ast e_+, H_E) \leq c(a, H_E) + c(e_+, 0) \leq 2 .
\]

Since this holds for every \(E\) and \(r < 1/2\), we conclude that

\[
\tau_+ (\{|p| < 1/2\}) = 0 . \quad (4)
\]

**Step 2:** Observe now that the Hamiltonian \(|p|^2\) on \(W \setminus \Sigma\) extends to a smooth Hamiltonian on the whole \(W\). This Hamiltonian is integrable and yields a foliation of \(W \setminus (L \cup \Sigma)\) by Lagrangian tori. Look at these tori in the tube \(\{|p| \geq 1/2\}\). One readily checks by an argument in the spirit of [15], that all these tori besides the monotone exotic torus \(K\) are displaceable.

One can prove the displaceability directly in the following way. Write \(W = S^2 \times S^2\) as

\[
\{x_1^2 + y_1^2 + z_1^2 = 1\} \times \{x_2^2 + y_2^2 + z_2^2 = 1\} \subset \mathbb{R}^3 \times \mathbb{R}^3 .
\]
Introduce functions $F$ and $G$ on $W$ by
\[
F(x_1, y_1, z_1, x_2, y_2, z_2) = z_1 + z_2,
\]
\[
G(x_1, y_1, z_1, x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2,
\]
and a map
\[
\Phi = (F, G) : W \to \mathbb{R}^2.
\]
One can check directly that within this model the Hamiltonian $|p|$ corresponds to $\sqrt{(1 + G)/2}$. It defines an integrable Hamiltonian system with an integral $F$ (since $F$ generates a circle action, such an integrable system is semitoric [18]). The above-mentioned Lagrangian tori are given by
\[
N_{a,b} := \Phi^{-1}(a, b) = \{z_1 + z_2 = a, x_1 x_2 + y_1 y_2 + z_1 z_2 = b\}.
\]
The monotone torus $K$ is given by $N_{0,-1/2}$.

Note that for $a \neq 0$, $N_{a,b}$ is displaceable by
\[
(x_1, y_1, z_1, x_2, y_2, z_2) \mapsto (-x_1, y_1, -z_1, -x_2, y_2, -z_2).
\]
For $a = 0$, the torus $N_{0,b}$, $b \in (-1/2, 1)$ can be displaced inside the hypersurface $\Pi := \{z_1 + z_2 = 0\}$. Indeed, let $\phi_i$ be the polar angle in the $(x_i, y_i)$-plane. Consider a fibration $\tau : \Pi \to C := (-1; 1) \times S^1$ given by
\[
(x_1, y_1, z_1, x_2, y_2, z_2) \mapsto (z_1, \phi_1 - \phi_2).
\]
One readily checks that for every $(z, \theta) \in C$ the preimage $\tau^{-1}(z, \theta)$ consists of a closed orbit of the Hamiltonian $z_1 + z_2$. Thus for every simple closed curve $\alpha \subset C$, the preimage $\tau^{-1}(\alpha)$ is a Lagrangian torus in $\Pi$. Denote by $\sigma$ the push-forward to $C$ of the symplectic form restricted to $\Pi$. Since the symplectic form on $W$ is given by
\[
\frac{1}{4\pi} (dz_1 \wedge d\phi_1 + dz_2 \wedge d\phi_2),
\]
we have that $\sigma = (4\pi)^{-1} dz \wedge d\theta$. Furthermore, $N_{0,b} = \tau^{-1}(\alpha_b)$ with
\[
\alpha_b = \left\{ z^2 = \frac{\cos \theta - b}{\cos \theta + 1} \right\}.
\]
Observe that $\alpha_b$ is a contractible simple closed curve in $C$. Integration yields
\[
\frac{1}{4\pi} \int_{\alpha_b} zd\theta = \frac{1}{2} = \frac{1}{2} \text{Area}_\sigma(C).
\]
For \( b > -1/2 \) the curve \( \alpha_b \) lies inside the disc bounded by \( \alpha_{-1/2} \) in \( C \). Thus \( \alpha_b \) is displaceable in \( C \), and therefore, by lifting the displacing isotopy to \( \Pi \), we get that \( N_{0,b} \) is displaceable in \( \Pi \). This completes the proof of the displaceability.

**Step 3:** Consider the push-forward \( \Phi_* \tau_+ \) of the quasi-measure \( \tau_+ \) to the plane \( \mathbb{R}^2 \) by the map \( \Phi \) given by (5). Let \( (u, v) \) be Euclidean coordinates on \( \mathbb{R}^2 \). Since \( F \) and \( G \) Poisson commute, \( \Phi_* \tau_+ \) extends to a measure, say \( \sigma \) on \( \mathbb{R}^2 \). Recall that in our model of \( W \) the function \( |p| \) corresponds to \( \sqrt{(1 + G)/2} \), and hence the tube \( \{|p| < 1/2\} \) is given by \( \{G < -1/2\} \). Formula (4) above implies that the support of \( \sigma \) lies in \( \{v \geq -1/2\} \).

Further, by Step 2, every non-empty fiber \( \Phi^{-1}(a, b) \) with \( b \geq -1/2, (a, b) \neq (0, -1/2) \) is displaceable in \( W \). Recall that every Floer-homological symplectic quasi-measure vanishes on displaceable subsets, and hence a point \( (a, b) \in \mathbb{R}^2 \) cannot lie in the support of \( \sigma \) provided the set \( \Phi^{-1}(a, b) \) is displaceable. Therefore the support of \( \sigma \) consists of the single point \( (0, -1/2) \), so that \( \sigma \) is the Dirac \( \delta \)-measure concentrated in this point. Since the torus \( K \) is given by \( \Phi^{-1}(0, -1/2) \), we get that

\[
\tau_+(K) = \sigma((0, -1/2)) = 1.
\]

This completes the proof of the theorem. \( \square \)

### 3 Proof of the Main Lemma

**The strategy of calculation:**

By Lemma 2.1 The Floer complex \( CF_i^{(1; E+1)}(H_E) \), \( i = 1, 2, 3 \) is generated by upper generators and \( U \)-generators in the action window \( (1; E + 1) \) satisfying the multiplicity bound (1). The lower generators and \( V \)-generators leave the stage. *Thus we shall suppress the upper index \( + \) and set \( Z_k = Z_k^+, \) \( f_k := f_k^+ \), \( \hat{m}_k = \hat{m}_k^+ \), etc.*

Denote by \( B_n, n \geq 0 \) the span over \( \mathbb{C} \) of generators

\[
\gamma \in \{x_0, x_2, \hat{m}_k, \hat{m}_k, \hat{M}_k, \hat{M}_k, k \geq 1\}
\]

of the Conley-Zehnder index \( n \).
Put \( C_{i,s} = t^{-s}B_{i + 4s} \), where \( i \geq 0 \) and \( s \geq 0 \). Write \( C_i = \oplus_s C_{i,s} \) and \( C = C_1 \oplus C_2 \oplus C_3 \). Denote by \( D \subset C \) the subspace consisting of the elements of action \(< 1\) and set \( D_{i,s} = D \cap C_{i,s} \). By Lemma 2.1(i)-(iii) the Floer complex of \( H_E \) in the action window \((1; E + 1)\) and in degrees 1, 2, 3 is given by \((C/D, \delta)\), where \( \delta : C/D \to C/D \) is the Floer differential. The differential \( \delta \) has the form

\[
\delta = \delta_0 + t^{-1}\delta_1 + t^{-2}\delta_2 + ... \quad \text{mod } D
\]

with \( \delta_l : B_* \to B_{* + 4l - 1} \). Let us emphasize that only negative powers of \( t \) appear in this expression: this follows from the fact that Floer trajectories of \( H_E \) are holomorphic near \( \Sigma \) and intersect it positively. With this notation,

\[
HF_2^{(1;E+1)}(H_E) = H_2(C/D, \delta).
\]

To motivate the strategy of calculation of this homology group, identify

\[
CF^{(1;E+1)}_i(H_E) = CF^{(1-E;1)}_i(H_E - E)
\]

and look at \( C_{i,0}/D_{i,0} \) considered as a subspace of \( CF^{(1-E;1)}_i(H_E - E) \). With this identification homology of the complex \((\oplus C_{i,0}/D_{i,0}, \delta_0)\) converge to symplectic homology \( SH(U') \) of the domain \( U' = \{|p| < r\} \) in the action window \((-\infty; 1)\) provided \( E \to \infty \) and \( \epsilon \to 0 \). Indeed, functions \( H_E - E \) restricted to \( U' \) form an exhausting sequence used in the definition of symplectic homology, the complex \( \oplus C_{i,0}/D_{i,0} \) is generated by closed orbits of \( H_E - E \) capped inside \( U' \) and the differential \( \delta_0 \) counts the Floer trajectories lying inside \( U' \).

The contribution of the lower generators disappears in this limit.

Now we can formulate the intuitive idea behind our calculation: The complex \((C/D, \delta)\) can be considered as a properly understood deformation of \((\oplus C_{i,0}/D_{i,0}, \delta_0)\) which involves capping discs and Floer trajectories intersecting \( \Sigma \). Eventually, the required homology \( H(C/D, \delta) \) can be calculated by an appropriate spectral sequence which starts at \( SH(U') \).

To make this precise, we use the technology developed by Bourgeois and Oancea [5] (and extended further in [3]) who identified symplectic homology of the Liouville domain \( \{|p| < r\} \) with the homology of the complex \( B := \oplus B_i = \oplus_i C_{i,0} \) equipped with certain differential \( d_0 \) which will be described below. In fact we shall introduce an appropriate deformation of their construction which takes into account the fact that Floer cylinders can intersect \( \Sigma \) and which will enable us to calculate homology of the deformed complex \((C/D, \delta)\).
As a graded and filtered vector space the deformed Bourgeois-Oancea complex is described as follows. Introduce the ring $\Lambda$ consisting of all Laurent series of the form

$$\sum_{s=0}^{+\infty} \lambda_s t^{-s}$$

$\lambda_s \in \mathbb{C}$.

With this notation the deformed Bourgeois-Oancea complex is given by $QB := B \otimes_{\mathbb{C}} \Lambda$. As before, this complex is graded by $CZ(\gamma t^{-s}) = CZ(\gamma) - 4s$ and filtered by the symplectic action of $H_E$: $A(\gamma t^{-s}) = A(\gamma) - s$. Its differential $d$ is $\Lambda$-linear and has the form

$$d = d_0 + t^{-1}d_1 + t^{-2}d_2 + ...$$

with $d_l : B_* \to B_{*+l-1}$. It is instructive to view $(QB, d)$ as a “quantum” deformation of the complex $(B, d_0)$ where $t$ plays the role of a deformation parameter. By [5] the group $H(B, d_0)$ coincides with symplectic homology of the Liouville domain $\{|p| < r\} \subset T^*S^2$. The latter, according to [1, 20], equals homology of the free loop space of $S^2 = L$. Therefore

$$H(B, d_0) = H(LS^2).$$

In order to describe the differential $d$ we need some preliminaries.

**Stretching-the-neck:**
Denote by $\pi : \nu \to \Sigma$ the holomorphic normal line bundle to $\Sigma$ in $W = \mathbb{C}P^1 \times \mathbb{C}P^1$. Perform a stretching-the-neck [8, 2] of $W$ at the hypersurfaces $\{|p| = r - \epsilon\}$ and $\{|p| = r\}$. The manifold $W$ splits into three pieces which after gluing in (asymptotically) cylindrical ends at their boundaries will be identified with $W_{left} := W \setminus \Sigma$, $W_{middle} := \nu \setminus \Sigma$ and $W_{right} := \nu$.

It would be convenient to view orbits from $Z_k$ as $k$-times-covered unit circles of the bundle $\nu$. In particular, the projection $\pi : \nu \to \Sigma$ gives rise to the natural map $\pi^k : Z_k \to \Sigma$.

We shall assume that the exhausting Morse function $f_U$ is defined on the whole $W_{left}$.

**Matrix coefficient ($d_l(\gamma_+, \gamma_-)$ for upper generators $\gamma_+, \gamma_-$):**
Denote by $P_{a,b}$ the cylinder $\mathbb{C} \setminus 0$ with the set of negative interior punctures $a = \{a_1, ..., a_{l_+}\}$ and the set of positive interior punctures $b = \{b_1, ..., b_{l_+}\}$. Here we put $l_+ = l$ (recall that we are defining $d_l$).

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1 One should shift our filtration by $E$ to get the standard filtration on symplectic homology used in [5], see [8].
Let $\gamma$ be an orbit from $Z_k$. In our picture it is interpreted in two different ways. First, it is a point of the corresponding critical variety $Z_k$ (recall that upper index $+$ is omitted). We denote this point by $A_\gamma$. Second, $\gamma$ is a (parameterized, in general multiply covered) unit circle in the fiber of the bundle $\nu$ over the projection $\pi(A_\gamma)$.

In what follows we work with holomorphic maps

$$u : P_{a,b} \to W_{\text{middle}} = \nu \setminus \Sigma.$$  

We say that $u$ enters $\gamma$ at a puncture $\eta \in a \cup \infty$ if $u(z)/|u(z)| \to \gamma(\arg(z))$ and $|u(z)| \to \infty$ as $z \to \eta$. We say that $u$ exits $\gamma$ at a puncture $\eta \in b \cup 0$ if $u(z)/|u(z)| \to \gamma(\arg(z))$ and $|u(z)| \to 0$ as $z \to \eta$.

Suppose that $u$ exits $\alpha$ at $0$, enters $\beta$ at $\infty$ and in addition exits (resp. enters) some orbits at all positive (resp. negative) interior punctures. We shall denote this by

$$\alpha \overset{u}{\rightarrow} \beta.$$  

We shall also book-keep the quantity $l_-$ by putting

$$\text{weight}(u) = 2^{l_-}. \quad (10)$$  

Note that geometrically such $u$’s either are contained in a single fiber of $\nu$, or correspond to multi-sections of $\nu$ with zeroes at $\pi(A_\alpha)$ and the projections of the asymptotic images of the positive punctures, and with poles at $\pi(A_\beta)$ and the projections of the asymptotic images of the negative punctures.

Suppose now that $\alpha, \beta$ belong to the same critical manifold $Z_k$. We shall write

$$\alpha \sim_{\nu} \beta \quad (11)$$  

if $v$ is a parameterized piece of trajectory of the negative gradient flow of $f_k$ joining the points $A_\alpha$ and $A_\beta$.\footnote{We will assume that the gradient vector field for each function $f_k$ satisfies the following condition: the 1-dimensional stable manifold of $\hat{M}_k$ (resp. the unstable manifold of $\hat{M}_k$) consists of orbits which differ from $\hat{M}_k$ (resp. from $\hat{M}_k$) only by their parameterization.}

Let $\gamma_+, \gamma_-$ be two upper generators representing critical points of $f_{k_+}, f_{k_-}$ on $Z_{k_+}, Z_{k_-}$ respectively. Assume that $k_+ \neq k_-$. Consider all possible configurations of the form

$$\gamma_+ \sim_{\nu} \alpha \overset{u}{\rightarrow} \beta \sim_{\nu} \gamma_- \quad (12)$$
We call weight($u$) the weight of this configuration. Note that the right and/or the left arrow could be empty. In case $k_+ = k_- = k$, we consider configurations of the form
\[
\gamma_+ \xrightarrow{u} \gamma_- ,
\]
and its weight is put to be 1. Finally, we define $(d_l \gamma_+, \gamma_-)$ as the sum of weights taken over the 0-dimensional components of the moduli spaces of configurations of the form [12] and [13]. Note that in the definition of the moduli spaces [12] the markers $a, b$ are varying as well. In addition, each weight should be taken with a sign responsible for the orientation of the moduli space. The orientation issue will be ignored in this note.

MATRIX COEFFICIENTS $(d_l x, \gamma)$ AND $(d_l \gamma, x)$ FOR AN UPPER ORBIT $\gamma$ AND AN $U$-ORBIT $x$:

First, note that all the coefficients $(d_l \gamma, x)$ vanish for $l > 0$ by index reasons, so we do not need to describe here the algorithm for their computing. For the description of $(d_0 \gamma, x)$ we refer to [3].

Let us describe the algorithm for computing of coefficients $(d_l x, \gamma)$. Suppose that the multiplicity of the orbit $\gamma$ is equal to $k$. Then the coefficient $(d_l x, \gamma)$ counts rigid configuration
\[
(g, u, c) ,
\]
where

- $g$ is a minus gradient trajectory of the function $f_U$ which begins at $x$ and ends at a point $p \in \partial U$; note that $\partial U$ can be canonically identified with the $S^1$-bundle associated with the complex line bundle $\nu$, and thus $p$ determines a ray $l_p$ in one of the fibers of $\nu$;

- $u : P_{a, b} \rightarrow W_{middle}$ a holomorphic map such that $u$ exits (resp. enters) some orbits at all positive (resp. negative) punctures, enters an orbit $\tilde{\gamma}$ of multiplicity $k$ at $\infty$ and $\lim_{z \rightarrow 0} u(z) \in l_p$; note that the set of positive punctures must be non-empty due to the maximum principle;

- $c$ is a minus gradient trajectory of the function $f_k$ connecting $\tilde{\gamma}$ and $\gamma$.

A new feature of configurations $(g, u, c)$ considered above is the ray $l_p$ which connects the holomorphic curve $u$ with the gradient trajectory $g$. This requires a justification which will be given elsewhere.
This completes the description of the differential $d$ on $QB$.

Comparison of Floer and Bourgeois-Oancea homologies:

Recall that $D$ denotes the subspace of $C$ consisting of elements of action $< 1$. Denote by $QD \subset QB$ the subspace consisting of elements of action $< 1$. We shall use the following equality:

$$H_2(C/D, \delta) = H_2(QB/QD, d).$$

(15)

Note that for $i = 1, 2, 3$ we have $C_i = QB_i = \oplus s B_{i+4s} \otimes \mathbb{C} t^{-s}$, and $D_i = QD_i = QB_i \cap D$. Let us compare the differentials. The stretching-the-neck procedure described above has the following effect on the original Floer trajectories of our Hamiltonian $H_E$: every Floer trajectory joining a pair of upper generators $\gamma_+$ and $\gamma_-$ splits into three pieces. The piece lying in $W_{middle} = \nu \setminus \Sigma$ is the Floer trajectory joining $\gamma_+$ and $\gamma_-$ with positive punctures (corresponding to the intersection points with $\Sigma$) and possibly some negative punctures. The orbit $\gamma_+$ lies on the connected component of the ideal boundary of $W_{middle}$ adjacent to $W_{right}$, while the orbit $\gamma_-$ lies on the connected component of the ideal boundary of $W_{middle}$ adjacent to $W_{left}$. The positive punctures are capped by rigid holomorphic planes (the fibers of $\nu$) lying in $W_{right}$. The negative punctures are capped by rigid holomorphic planes lying in $W_{left}$ corresponding to $\mathbb{C}P^1 \times \text{point}$ and $\text{point} \times \mathbb{C}P^1$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$. Thus every negative puncture is capped by exactly 2 rigid planes, which yields the factor $2^{l_-}$ in the definition of the weight in (10).

Let’s focus on the piece lying in $W_{middle}$: one extends Bourgeois-Oancea theory [5, Prop. 4] and finds an isomorphism between homology whose differential is determined by such punctured Floer trajectories and the homology whose differential is described by configurations of the form (12) and (13). This explains equality (15). The formal proof will be given in a forthcoming paper.

Remark 3.1. From the viewpoint of the Hamiltonian Floer theory, the differential $\delta : C/D \to C/D$ does not lifts canonically to a differential $C \to C$: the square of the expression in the right hand side of formula (6) vanishes modulo the subcomplex $D$. Since $d^2 = 0$, the argument above shows that in the limit, “when the neck is stretched”, the square of this expression vanishes in $C$ itself.
Unperturbed differential:
The explicit form of the differential \( d_0 \) is folkloric (private communications of F.Bourgeois and T.Ekholm). We shall present the result right now.

**Lemma 3.2.** We have

\[
\begin{align*}
\dot{m}_k &= \dot{m}_{k-1} + \dot{M}_{k-1}, \quad k \geq 2, \\
\dot{M}_k &= 2\dot{m}_k, \quad k \geq 1, \\
\dot{m}_1 &= 0, \\
\dot{M}_1 &= 2\dot{m}_1, \\
\dot{m}_2 &= 2\dot{m}_1 + 2x_2, \\
\dot{M}_2 &= 2\dot{m}_2, \\
\dot{m}_3 &= 2\dot{m}_2 + 2x_2 + \dot{m}_1, \\
\dot{M}_3 &= 2\dot{m}_3, \\
\dot{m}_4 &= 2\dot{m}_3 + 2\dot{m}_2 + \dot{m}_1, \\
\dot{M}_4 &= 2\dot{m}_4, \\
\dot{m}_5 &= 2\dot{m}_4 + 2\dot{m}_3 + \dot{m}_2, \\
\dot{M}_5 &= 2\dot{m}_5, \\
\dot{m}_6 &= 2\dot{m}_5 + 2\dot{m}_4 + \dot{m}_3, \\
\dot{M}_6 &= 2\dot{m}_6, \\
\dot{m}_7 &= 2\dot{m}_6 + 2\dot{m}_5 + \dot{m}_4, \\
\dot{M}_7 &= 2\dot{m}_7, \\
\dot{m}_8 &= 2\dot{m}_7 + 2\dot{m}_6 + \dot{m}_5, \\
\dot{M}_8 &= 2\dot{m}_8, \\
\dot{m}_9 &= 2\dot{m}_8 + 2\dot{m}_7 + \dot{m}_6, \\
\dot{M}_9 &= 2\dot{m}_9, \\
\dot{m}_{10} &= 2\dot{m}_9 + 2\dot{m}_8 + \dot{m}_7, \\
\dot{M}_{10} &= 2\dot{m}_{10}, \\
\dot{m}_{11} &= 2\dot{m}_{10} + 2\dot{m}_9 + \dot{m}_8, \\
\dot{M}_{11} &= 2\dot{m}_{11}, \\
\dot{m}_{12} &= 2\dot{m}_{11} + 2\dot{m}_{10} + \dot{m}_9, \\
\dot{M}_{12} &= 2\dot{m}_{12},
\end{align*}
\]

It follows that the homology of the complex \((B, d_0)\) are given by

\[
\begin{align*}
H_0(B, d_0) &= \text{Span}_\mathbb{C}(\{x_0\}), \\
H_2(B, d_0) &= \text{Span}_\mathbb{C}(\{x_1\}), \\
H_2(k+2) &= \text{Span}_\mathbb{C}(\{|\dot{M}_k]\}, k \geq 1, \\
H_1(B, d_0) &= \text{Span}_\mathbb{C}(\{\dot{m}_1]\), \\
H_2(k+1) &= \text{Span}_\mathbb{C}(\{|\dot{M}_k - \dot{m}_{k+1}\}], k \geq 1.
\end{align*}
\]

It remains to describe the differential \( d_l \) for \( l \geq 1 \).

**Calculation of the quantum corrections:**

**Lemma 3.3.**

(i) For all \( l \geq 2 \) one has \( d_l = 0; \)

(ii) For all \( k \geq 1 \),

\[
\begin{align*}
\dot{m}_1 &= \dot{m}_{k+1}, \quad \dot{M}_1 = \dot{M}_{k+1}, \quad d_1 \dot{m}_k = 0, \quad d_1 \dot{M}_k = 0.
\end{align*}
\]
Proof of Main Lemma:

1) Define a decreasing filtration \( \cdots \supset QB(\mu) \supset QB(\mu+1) \supset \cdots \) on \( QB \) by

\[
QB^i_s(\mu) = \oplus_{s \geq \mu+1} C_{i,s} = \oplus_{s \geq \mu+1} t^{-s} B_{s+4s},
\]

and observe that the differential \( d \) preserves the filtration. Furthermore, by Lemma 2.1(iv) for \( E \) large enough the subspace \( QD_i \subset QB_i, \ i = 1,2,3 \) consisting of elements of the \( H_E \)-action < 1 is squeezed between \( QB^i_{\mu} \) and \( QB^i_{\mu-} \) with \( \mu_\pm(E) \to \infty \) as \( E \to \infty \):

\[
QB^i_{\mu-} \supset QD_i \supset QB^i_{\mu+}.
\]

We shall show in the next step that for \( \mu \) large enough \( H_2(QB/QB(\mu),d) \) is isomorphic to \( \mathbb{C} \) and is generated by \( x_2 \mod QB(\mu) \). The latter implies that the map \( H_2(QB/QB(\mu_+),d) \to H_2(QB/QB(\mu-),d) \) is an isomorphism. Since it factors through \( H_2(QB/QD,d) \) we shall conclude that \( H_2(QB/QD,d) = \mathbb{C} \).

2) Lemma 3.3 yields that \( d_1 : H_{2k}(B,d_0) \to H_{2k+3}(B,d_0) \) vanishes while \( d_1 : H_{2k+1}(B,d_0) \to H_{2k+4}(B,d_0) \) is onto for all \( k \geq 0 \). Fix \( \mu \) large enough and identify \( QB := QB/QB(\mu) \) with \( \oplus_{s \leq \mu} C_{s,s} \).

We claim that

\[
H_2(QB,d) = H_2(B,d_0) = \mathbb{C}.
\]

Indeed, consider a filtration \( F_p QB_s := \oplus_{\mu \geq s \geq -p} C_{s,s} \) on the complex \( QB \). Look at the homology spectral sequence corresponding to this filtration [19]: we have that

\[
E^1_{ij} = \oplus_{s \leq \mu} H_j(C_{s,s},d_0) = \oplus_{s \leq \mu} t^{-s} H_{j+4s}(B,d_0)
\]

and the differential \( d^1 : E^1_{ij} \to E^1_{i-1,j} \) can be written as

\[
d^1 = \oplus d^1_s, \ d^1_s = [t^{-1} d_1] : H_j(C_{s,s},d_0) \to H_{j-1}(C_{s,s+1},d_0), s \leq \mu - 1.
\]

Since \( d = d_0 + t^{-1} d_1 \), the spectral sequence degenerates at the second page and converges to \( E^2_0 := H_j(E^1_1,d^1) \), and in particular \( H_2(QB,d) = E^2_2 \). In order to calculate \( E^2_2 \) look at the piece

\[
H_3(C_{s,s-1},d_0) \to H_2(C_{s,s},d_0) \to H_1(C_{s,s+1},d_0)
\]

of the complex \( (E^1,d^1) \). If \( \mu - 1 \geq s \geq 1 \), the left arrow is onto, while the right arrow is zero, thus the homology vanishes. If \( s = \mu \), the sequence (16) degenerates to

\[
H_3(C_{s,\mu-1},d_0) \to H_2(C_{s,\mu},d_0) \to 0
\]

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and since the left arrow is onto, the homology vanishes. Finally, for \( s = 1 \)
the sequence \((16)\) degenerates to
\[
0 \to H_2(C_{*,0}, d_0) \to H_1(C_{*,1}, d_0) .
\]
Since the right arrow vanishes, the resulting homology is
\[
H_2(C_{*,0}, d_0) = H_2(B, d_0) = \mathbb{C} .
\]
We conclude that \( H_2(QB, d) = E_2^2 = \mathbb{C} \), and the generator of \( H_2(QB, d) \) is \( x_2 \mod QB^{(\mu)} \), as required.

It remains to put all the pieces together: The calculation above together with equalities \((7)\) and \((15)\) yield
\[
HF_2^{(1; E+1)}(H_E) = H_2(C/D, \delta) = H_2(QB/QD, d) = H_2(B, d_0) = \mathbb{C} .
\]
This completes the proof. \(\square\)

**Proof of Lemma 3.3:**

1) First we explore coefficients of the form \((d_l \gamma_+, \gamma_-)\) with \( l = l_+ \geq 1 \),
\[
\gamma_+ \in \{ \tilde{m}_{k_\pm}, \tilde{m}_{k_\pm}, \tilde{M}_{k_\pm}, \tilde{M}_{k_\pm} \} .
\]
The index formula reads
\[
CZ(\gamma_+) - CZ(\gamma_-) + 4l_+ = 1 .
\]
Recall that
\[
CZ(\gamma_\pm) = 2k_\pm + j_\pm, \ j_\pm = -1, 0, 1, 2 .
\]
Put \( h = j_+ - j_- \). Thus the index formula yields
\[
2(k_+ - k_-) + h + 4l_+ = 1 . \tag{17}
\]

Consider a configuration of the form
\[
\gamma_+ \sim^x \alpha \sim^u \beta \sim^w \gamma_- .
\]
Denote by \( \Delta \) the degree of the projection of \( u \) to \( \Sigma \). Since the Chern class of \( \nu \) equals 2 we have
\[
(k_+ + l_+) - (k_- + l_-) = 2\Delta . \tag{18}
\]
It follows from (17) and (18) that
\[ l_+ + l_- = (1 - h) / 2 - 2\Delta. \]
Since \( l_+ \geq 1, \Delta \geq 0 \) and \( h \) is an integer from \([-3; 3]\) we conclude that \( \Delta = 0 \).
Thus our holomorphic curve lies in the single fiber of the bundle \( \nu \).

We claim that \( h \neq -3 \). Indeed otherwise we have the connecting trajectory of the form
\[
\tilde{m}_{k+} \xrightarrow{\nu} \alpha \xrightarrow{u} \beta \xrightarrow{\omega} \hat{M}_{k-} .
\] (19)
Since \( \tilde{m}_{k+} \) is the point of minimum of \( f_{k+} \), \( \nu \) is the constant trajectory. Since \( \hat{M}_{k-} \) is the point of maximum of \( f_{k-} \), \( w \) is the constant trajectory. Since \( u \) lies in the single fiber, the set of limit points of \( u(z)/|u(z)| \) as \( z \to \infty \) lie on a circle which is the fiber of \( Z_{k-} = \mathbb{R}P^3 \) over \( \pi^{k+}(\tilde{m}_{k+}) \). Generically, as the dimension count shows, this circle does not pass through \( \hat{M}_{k-} \), and hence configuration (19) does not exist. The claim follows.

Since \( h \neq -3 \), we get that
\[ h = -1, l_+ = 1, l_- = 0. \]
This readily yields that \( d_1\gamma_+ = 0 \) for \( l \geq 2 \) and the only possible non-trivial matrix coefficients could be (with \( k_+ := k \)) \( (d_1\tilde{m}_k, \hat{M}_{k+1}) \) and \( (d_1\hat{M}_k, \hat{M}_{k+1}) \). We claim that
\[ (d_1\tilde{m}_k, \hat{M}_{k+1}) = (d_1\hat{M}_k, \hat{M}_{k+1}) = 1. \]

Let us present a calculation (modulo orientations) for the coefficient \( (d_1\tilde{m}_k, \hat{m}_{k+1}) \) (the calculation for \( (d_1\hat{M}_k, \hat{M}_{k+1}) \) is analogous). Since \( \Delta = 0 \) and \( l_+ = 1 \), we work with holomorphic maps \( u \) of the cylinder \( P(b) \) punctured at a point \( b \in \mathbb{C} \setminus 0 \) lying in the single fiber of \( \nu \). Choose functions \( f_k \) and \( f_{k+1} \) on \( Z_k \) and \( Z_{k+1} \) respectively so that
\[ \pi^k(\tilde{m}_k) = \pi^{k+1}(\hat{m}_{k+1}) := m \in \Sigma \]
and so that the circle \( (\pi^{k+1})^{-1}(m) \subset Z_{k+1} \) forms the unstable manifold \( U \) of \( \hat{m}_{k+1} \). Identify the fiber of the line bundle \( \nu \) over \( m \) with \( \mathbb{C} \). Recall that we identified each \( Z_j \) with the unit circle bundle of \( \nu \). Assume that \( \tilde{m}_k \) corresponds to the point \(-1 \in \mathbb{C} \) and \( \hat{m}_{k+1} \) corresponds to the point \( 1 \in \mathbb{C} \). Since \( \tilde{m}_k \) is the minimum point of \( f_k \), the only gradient trajectories exiting \( \tilde{m}_k \) are the constant ones. Thus we are counting configurations of the form
\[
\tilde{m}_k \xrightarrow{\nu} \beta \xrightarrow{\omega} \hat{m}_{k+1} .
\] (20)
Note that the point $A_{\beta}$ lies both on the circle $U = (\pi^{k+1})^{-1}(m)$ (since the image of $u$ is contained in the single fiber) and on the stable manifold of $\hat{m}_{k+1}$. These two submanifolds intersect transversally at a single point, $\hat{m}_{k+1}$. In particular, $w$ is constant. Therefore it suffices to show that the holomorphic map $u$ is unique up to a reparameterization and up to multiplication by constants (these symmetries should be taken into account when one passes to the moduli space).

The general form of $u$ is $u(z) = \lambda z^k(z - b)$ with the asymptotic conditions

$$\text{Arg}(u(t)) \to 0, \ t \in \mathbb{R}_+, t \to +\infty$$

and

$$\text{Arg}(u(t)) \to \pi, \ t \in \mathbb{R}_+, t \to 0.$$

This readily yields $\lambda, b \in \mathbb{R}_+$. The change of variables $z \to cz$ with $c \in \mathbb{R}_+$ takes $u$ to the form $u(z) = \lambda c^{k+1} z^k(z - b/c)$. Thus $u$, up to a reparameterization and up to multiplication by constants coincides with $z^k(z - 1)$. Thus we have the unique configuration of the form (20), which completes the calculation.

Finally, we claim that $(d_1\hat{m}_k, \hat{M}_{k+1}) = 0$. Indeed, assume that $d_1\hat{m}_k = n\hat{M}_{k+1}$. Observe that since $d^2 = 0$ we have that $d_1d_0 + d_0d_1 = 0$. Thus (in view of the explicit formulas for the unperturbed differential)

$$d_1d_0\hat{m}_k + d_0d_1\hat{m}_k = 0 + nd_0\hat{M}_{k+1} = 0.$$

Since $d_0\hat{M}_{k+1} \neq 0$ we conclude that $n = 0$ as claimed.

2) Now we turn to calculation of $d_1x$ where $x \in \{x_0, x_2\}$ and $l \geq 1$. Let us observe that if $(d_1x, \gamma) \neq 0$ then the orbit $\gamma$ must have an odd grading, and hence $\gamma = \hat{m}_k$, or $\gamma = \hat{M}_k$. But in this case if there exists a gradient trajectory $c$ connecting an orbit $\hat{\gamma}$ with $\gamma$, and if $\hat{\gamma}_\alpha$ differs from $\hat{\gamma}$ by a reparameterization $s \mapsto se^{i\alpha}$, then for almost all values $\alpha$ there exists a gradient trajectory $c_\alpha$ connecting $\hat{\gamma}_\alpha$ with $\gamma$. This implies that there are no rigid configurations $(g, u, c)$ (see (14) above) which may contribute to $(d_1x, \gamma)$, and hence $(d_1x, \gamma) = 0$. Indeed, any such configuration belongs to a family $(g, u \circ r_\alpha, c_\alpha)$, where $r_\alpha : \mathbb{C} \setminus 0 \to \mathbb{C} \setminus 0$ is the rotation $z \mapsto ze^{i\alpha}$.

This finishes off the proof of the lemma.

Outline of the proof of Lemma 3.2: First we discuss coefficients of the form $(d_0\gamma_+, \gamma_-)$ with $l_+ = 0$,

$$\gamma_\pm \in \{\hat{m}_{k\pm}, \hat{M}_{k\pm}, \hat{M}_{k\pm}\}.$$
We argue as in Step 1 of the proof of Lemma 3.3, keeping the same notations and taking into account that \( l_+ = 0 \) This yields three possibilities:

(i) \( h = -3, \Delta = 0, l_- = 2 \);
(ii) \( h = -3, \Delta = 1, l_- = 0 \);
(iii) \( h = -1, \Delta = 0, l_- = 1 \).

Case (i) is ruled out exactly as in the proof of Lemma 3.3. Case (ii) yields

\((d_0\hat{m}_k, \hat{M}_{k-2}) = 2\), case (iii) yields

\((d_0\hat{m}_k, \hat{m}_{k-1}) = (d_0\hat{M}_k, \hat{M}_{k-1}) = 2\),

while all other coefficients vanish.

Further, we analyze the matrix coefficients involving \( x_0 \) and \( x_2 \). Equality \((d_0\hat{m}_2, x_2) = 2\) follows from the count of degree 1 (properly parameterized) sections of the bundle \( \nu \) passing through a given generic point and having a single zero of order two at the point \( \pi^2(\hat{m}_2) \in \Sigma \). Equality \((d_0\hat{M}_1, x_2) = 2\) corresponds to the fact that there are exactly two rigid spheres \( S^2 \times \text{point} \) and \( \text{point} \times S^2 \) in \( W \) passing through \( x_2 \).

The only tricky remaining coefficient is \((d_0\hat{m}_1, x_0) = 0\): Seemingly, there are two rigid spheres in \( W \) asymptotic to \( \hat{m}_1 \) which may contribute to this coefficient. We claim that in fact a cancelation happens. To see this, recall that by (9) and (7) \( H_1(B, d_0) = \mathbb{Z} \). If \((d_0\hat{m}_1, x_0) \neq 0\), we would get that \( H_1(B, d_0) = 0 \), and thus arrive at a contradiction.

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