We calculate the euclidean partition function of the type IIA NS fivebrane wrapped on an arbitrary Calabi-Yau space in a double-scaling decoupling limit and in the presence of a flat RR 3-form background field. The result is the product of a theta function, coming from the classical fluxes of the self-dual tensor field, and a factor representing the quantum contributions. The quantum factor turns out to be related to topological B-model string amplitudes, and both factors satisfy a holomorphic anomaly equation. The result can teach us more about little string theories and about instanton corrections to four-dimensional effective quantities.

The work reported here was done in collaboration with R. Dijkgraaf and E. Verlinde [1]. This text is intended to give a brief overview of this work. Only the main results are reported here; the calculations and many other details can be found in the original paper.

1. DESCRIPTION OF THE SYSTEM

We consider euclidean type IIA string theory, compactified on an arbitrary Calabi-Yau manifold $X$. Around $X$, we wrap one or more euclidean NS five-branes, so that these objects are pointlike in the remaining four directions. We turn on a background RR-field $C$ which satisfies $dC = 0$ along the directions of $X$. Finally, we take a double scaling limit where we send both the distance between the five-branes and the string coupling constant to zero in such a way that their ratio becomes infinite.

The reason to study this system is twofold. First of all, the worldvolume theory on an NS-five-brane is the so-called little string theory, which is still not completely understood. As shown by Giveon and Kutakosov [2], in the above double scaling limit this theory turns into a weakly
coupled field theory. Therefore, the calculation of the partition function of this system could teach us something about little string theory. Secondly, in Calabi-Yau compactifications there are instanton corrections to several four-dimensional quantities, coming from wrapped NS five-branes. Our computation could be helpful in determining these instanton corrections.

2. THE CLASSICAL FACTOR

The bosonic part of the low-energy field theory on the NS five-brane contains a self-dual two-form field and five scalar fields. For clarity, we will not take the fermions – which are completely determined by supersymmetry – into consideration here. Also, the scalar fields do not contribute to the classical partition function. Four of them, corresponding to the position of the NS five-brane, are fixed while the fifth one cannot have any winding numbers since the Calabi-Yau does not have any closed one-cycles. Therefore, we want to study the classical partition sum of a self-dual two-form $B$ coupled to a flat background three-form field $C$. Ignoring the subtleties of the self-duality, this system can be described by the action

$$S = \frac{1}{4\pi} \int_X \left( \frac{1}{2} (H - C) \wedge \ast (H - C) - iH \wedge C \right),$$

(1.1)

where $H = dB$ is the field strength of the two-form field.

The field $H$ has quantized fluxes around the conjugate three-cycles $A^I, B_I$ of the Calabi-Yau:

$$\int_{A^I} H = 2\pi n^I, \quad \int_{B_I} H = 2\pi m_I.$$  

(1.2)

Similarly, we denote the (fixed) periods of the $C$-field around these cycles by $x^I_A, x_{B,I}$, and the fluxes of the holomorphic $(3,0)$-form on $X$ around the $A$-cycles (defining its complex structure) are denoted by $z^I$. Note that the variables $x$ and $z$ are not quantized.

The partition sum is now a sum over the fluxes $m, n$, which can be evaluated using a Poisson resummation. It is well-known how to incorporate self-duality in such a calculation [3]: basically, one has to choose a spin structure $(\alpha^I, \beta_I)$ on the manifold and take a holomorphic root of the resulting expression. The result of this somewhat lengthy but straightforward calculation of the classical partition function is

$$Z_X^{cl} = \Theta_{\alpha,\beta}(x^I, z^I)$$

(1.3)

i.e. a modified theta-function depending on the background field fluxes $x^I$ and on the periods $z^I$. The sum over the fluxes $m$ and $n$ (only one
sum is left because of the holomorphic root) has turned into the usual sum inside the theta function. For the full expression for $\Theta$, the reader is referred to [1].

3. **T-DUALITY**

It is known that after a transversal T-duality, the NS five-branes (which are four-dimensional instantons for the $B_{\mu\nu}$ field), turn into gravitational instantons. More specifically, it has been shown in [4] that the T-dual system in four dimensions is a Taub-NUT space (without five-branes) which in the limit where we let the compactification circle grow to infinite size becomes an ALE-space of type $A_{k-1}$, where $k$ is the number of five-branes in the original picture.

Comparing the limits on both sides of the T-duality, we find that the double scaling limit corresponds to a weak coupling limit with small Planck length, i.e. we can use an $\mathcal{N} = 2$ supergravity approximation on the type IIA side. The three-form field on the Calabi-Yau obtains a leg in the four remaining directions after the T-duality, so for every three-cycle $A^I$ on the Calabi-Yau, we obtain a vector field with field strength $F^I$ in the supergravity limit. (The cycles $B_I$ correspond to the Hodge duals of these forms.)

4. **THE QUANTUM FACTOR**

In the $\mathcal{N} = 2$ supergravity theory, we can again calculate the classical partition function of a single five-brane by summing over the fluxes of the gauge fields $F^I$, and one finds exactly the result (1.3). However, the interesting thing is that on this side one can also calculate the quantum contributions to the partition function. In fact, it was shown in [5] that the supergravity action obtains quantum corrections of the form

$$S_{qu} = \int \sum_{g=1}^{\infty} R_- \wedge R_-(g_s T_-)^{2g-2} \mathcal{F}_g(z, \bar{z}),$$

(1.4)

where $R_-$ is the anti-self-dual part of the curvature, $T_-$ is the anti-self-dual part of the graviphoton field strength (which is one of the gauge field strengths in our description above), and the $\mathcal{F}_g$ are topological string amplitudes at genus $g$.

A straightforward calculation shows that the integral over the curvature gives a factor of $k - 1$ – corresponding to the number of five-branes – and $g_s$ and $T_-$ in our scaling limit combine into an effective coupling constant $\lambda$, so the final result for the partition function of a single five-brane
is

\[ Z_X = \Theta_{\alpha,\beta}(x^I; z^I) \exp \left( \sum_{g=1}^{\infty} \lambda^{2g-2} \mathcal{F}_g(z, \bar{z}) \right) . \]  \quad (1.5)

5. OPEN PROBLEMS

Two important open questions arise from our calculations. First of all, since \( k \) five-branes correspond to an \( A_{k-1} \)-singularity, it is not clear what happens in the case of a single five-brane. In fact, the result we get for \( k \) five-branes in the Coulomb phase on the supergravity side seems to be the \((k-1)^{th}\) power of the result (1.5). In our paper [1], we give some arguments for the fact that there is indeed a “missing five-brane” at infinity in the supergravity calculation. However, it would be nice to see this explicitly, for example by doing the calculation on the Taub-NUT space instead of the ALE space.

The second open problem stems from the fact that the classical and quantum parts of 1.5 satisfy conjugate “holomorphic anomaly equations” [6], i.e. they are not truly holomorphic quantities in \( z \), but they are holomorphic in a background independent sense as explained by Witten in [7]. Integrating over the C-field fluxes \( x \) would give a truly holomorphic result. It would be nice to understand these facts and the relation to topological theories better.

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