Environment Identification in Flight using Sparse Approximation of Wing Strain

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Abstract

This paper addresses the problem of identifying different flow environments from sparse data collected by wing strain sensors. Insects regularly perform this feat using a sparse ensemble of noisy strain sensors on their wing. First, we obtain strain data from numerical simulation of a Manduca sexta hawkmoth wing undergoing different flow environments. Our data-driven method learns low-dimensional strain features originating from different aerodynamic environments using proper orthogonal decomposition (POD) modes in the frequency domain, and leverages sparse approximation to classify a set of strain frequency signatures using a dictionary of POD modes. This bio-inspired machine learning architecture for dictionary learning and sparse classification permits fewer costly physical strain sensors while being simultaneously robust to sensor noise. A measurement selection algorithm identifies frequencies that best discriminate the different aerodynamic environments in low-rank POD feature space. In this manner, sparse and noisy wing strain data can be exploited to robustly identify different aerodynamic environments encountered in flight, providing insight into the stereotyped placement of neurons that act as strain sensors on a Manduca sexta hawkmoth wing.

Keywords: insect flight; proper orthogonal decomposition; sparse approximation; classification; sensor selection; unsteady aerodynamics

1. Introduction

Winged flight remains one of the most successful forms of animal locomotion and one of mankind’s foremost accomplishments. The aircraft of today are a direct result of centuries of fascination, inquiry and experimentation inspired by bat, bird and insect locomotion. Today’s most prevalent rigid-wing aircraft perform maneuvers and functions strikingly different from that of these flexible-winged animals, and the success of these aircraft in military and transport continues to spur development of rigid-wing technology. More recently, advances in robotics, materials and high-performance computation have advanced bio-inspired flexible-wing technologies including miniaturized unmanned micro-aerial vehicles (MAVs), ornithopters, hovercraft, and drones. Suitable controllers for these small-scale autonomous technologies require extensive understanding of low-Reynolds number, unsteady aerodynamics that is often experienced by bats, insects and most birds. Of particular interest in this work is understanding the role that a limited number of sensors (e.g. neurons) play in accurately informing control decisions in this low-Reynolds number regime, thus potentially helping to reveal bio-inspired flight control principles.

While the comprehensive mechanism of wing actuation, fluid-structure coupling and response maneuvers in winged animal flight is not fully understood, biologists and engineers have nevertheless made significant progress in identifying the propulsive forces in flapping-wing flight (Ellington, 1999; Zbikowski, 2002; Tangorra et al., 2007; Dabiri, 2009). The preliminary study of these forces began with quasi-steady,
Chord Wing Deflection \( \mu z t + (E I z x x)_{x x} = p \)

Fluid Loads

\[ p(x, t) = g(k, zt) \]

Insect Wing

Figure 1: Strain sensors in insect wings directly encode elastic wing deformation from the leading edge dynamics and flow feedback loads. These fluid loads are driven by dominant flapping and external reduced frequencies \( k \) and weak coupling from wing deformation.

Inviscid assumptions from rigid-wing thin airfoil theory that proved to be inadequate at predicting the additional lift generated by insects in experiments. It was later found that the unsteady aerodynamics and added-mass of the surrounding fluid are crucial for characterizing the forces experienced by animal wings (Ellington, 1994; Sane, 2003; Wang, 2005). Although the fluid’s added-mass would seem to complicate the robust generation of lift, it was discovered that animal wings rotate and flap in a manner that harness aerodynamic added-masses and leading edge vortices for additional lift in insects (Dickinson et al., 1999; Birch and Dickinson, 2001; Combes and Daniel, 2001; Song et al., 2008; Eldredge et al., 2010; Faruque and Humbert, 2010a,b), birds (Spedding et al., 2003), and bats (Hedenström et al., 2007; Clark and Smits, 2006). There is evidence that wings harness low-dimensional flow structure such as leading edge vortices to maximize lift and improve stability in flight (Birch and Dickinson, 2001; Videler et al., 2004; Dabiri, 2009). This is consistent with the observation that wing motion is itself low-dimensional. Proper orthogonal decomposition analysis has revealed that fin motion (Tangorra et al., 2007), bat (Riskin et al., 2008) and avian wing motion are constrained to only a few degrees of freedom. The lack of complex musculature in insect wings results in even fewer degrees of freedom. Such dimensionality reduction suggests that insects, which are constrained to small ranges of motion, capitalize on low-dimensional feature spaces to inform low-dimensional control protocols.

Wing mechanosensors in insects detect and leverage these fluid forces and accelerations for changing flight environments. Indeed, tactile sensory mechanisms have been identified on the wings of most animals - bats sense with small hairs on their wings (Sterbing-D’Angelo et al., 2011), birds sense with wing feathers (Brown and Fedde, 1993), but insects possess a small number of strain sensors on their wings called campaniform sensilla. The sensilla are strongly implicated in neurosensory flight control (Dickerson et al., 2014; Sane et al., 2007), in part because insect wings react to disturbances faster than visual stimulus transmission to the central nervous system (Collett and Land, 1975). Biological evidence shows that sensilla are stereotyped across specimens of the same species. Indeed, Cole and Palka (1982) show that the spatial distribution of sensilla are encoded in the genes of the fruit fly Drosophila, and sensilla determine maneuvers in flight control in locusts and Manduca sexta (Gettrup, 1966; Dickerson et al., 2014; Dickerson, 2015). Aerodynamic feedback is encoded within the strain signals, registering fluidic loading frequencies into the signal that can be exploited for characterizing flow environment. Evidence shows that insects exploit innate or learned knowledge of fluid environment through the strain encodings to make split-second decisions in flight. The insect nervous system has evolved specifically for the decision task, but controllers in hovercraft and MAVs require other means of exploiting point sensor feedback. Strain sensors are too sparsely distributed to fully resolve spatial flow encoding over the wing, and equation-based flow identification or prediction is expensive and difficult to generalize to different flow regimes.

In contrast, data-enabled methods for flow characterization or parameter estimation have shown re-
remarkable promise in the analysis of complex flows. A variety of data decompositions are being applied to flow measurement data for spectral analysis, model reduction, and control of complex flows, for example, proper orthogonal decomposition (POD) (Lumley, 1970; Holmes et al., 1998), dynamic mode decomposition (DMD) (Schmid and Sesterhenn, 2008; Rowley et al., 2009; Schmid, 2010; Tu et al., 2014; Kutz et al., 2016), compressed sensing (Bright et al., 2013; Bai et al., 2014; Brunton et al., 2014) and sparse regression (Brunton et al., 2016b), network theoretic approaches (Nair and Taira, 2015; Taira et al., 2016), as well as many other machine learning methods for flow control, as surveyed in Brunton and Noack (2015). Broadly, these equation-free methods characterize measurements of a system to inform its state and subsequent control decisions. In addition to intriguing evidence that biological strain sensors inform reactive decisions in flight, sensors play a pivotal role in the feedback control of many complex flows.

1.1. Contributions of this work

The focus of our work is sensor-enabled, data-driven flow environment classification using supervised learning and sparse approximation of incoming sensor measurements. Our data consists of strain point measurements from a numerical fluid-structure interaction model of a hawkmoth wing undergoing different environments of flow feedback. First, POD modes of strain measurements from different aerodynamic environments are assembled into a dictionary of low-rank dynamical states labelled by environment. This supervised learning stage mimics experiential learning and trains the data-driven model specifically for the task at hand. Next, incoming strain measurements are classified in this learned POD library using L1 constrained sparse approximation and sparse representation for classification (SRC) (Wright et al., 2009). We propose a framework in which sparse approximation is used to classify flow environments. Although other classifiers such as discriminant-based analysis and neural networks abound in machine learning, sparse approximation is particularly suited for the problem at hand since it is naturally robust to noise in measurement and accommodates classification between subspaces of different dimension. Moreover, sparse approximation aligns with plausible neurobiological sparse encoding strategies (Olshausen and Field, 2004).

In this work, sparse approximation of wing strain for flow classification is shown to be accurate and robust to high levels of sensor noise. Our strain measurements consist of Fourier coefficients in the frequency domain at a single spatial location on the wing chord. These frequency measurements are then sparsely approximated in an overcomplete library of POD bases that characterize different aerodynamic environments. An L1 sparsity constraint is imposed on the solution so that the nonzero solution components identify the originating flow environment. This analysis of strain time histories in the frequency domain permits the use of limited spatial sensors and provides robustness to sensor noise. Frequencies are sampled from a single, randomly chosen spatial location at a time. We also perform a location study in which the classification accuracy is compared across different regions along the chord. Based on this comparison, trailing edge frequency content is amplified and more suitable for aerodynamic environment discrimination. This is because the more pliable trailing edge contains more energetic aerodynamic contributions far away from leading edge actuation dynamics. Sparse representation accuracy suggests sampling certain frequencies yields higher flow classification accuracy, and measurement selection algorithms are used to identify these discriminating frequencies. This further suggests that a principled measurement strategy on flexible wings is advantageous, and is consistent with the stereotyped placement of strain-sensing neurons in the hawkmoth.

2. Background

This section is a brief overview of existing literature on lift generation in insect flight, the role of insect wing mechanosensors, and data-driven sparse approximation methods. We also introduce important aerodynamic flow and actuation parameters that will be used throughout the paper.

2.1. Unsteady aerodynamics of insect flight

The uniqueness of insect flight can be attributed to simple oscillating wing actuation (ranging from 10 to 500 Hz) that can achieve a large variety of maneuvers in exceedingly short timescales. Indeed, a wider range of motion is accessible to insects from relatively simpler wing actuation compared to birds or fixed wing structures. Unlike birds, insects also possess little to no wing musculature to control wing shape and do not require the large forward velocities of the fixed wings in airplanes. The study of insect wings is extremely relevant to miniaturized flight applications for unmanned MAVs that are required to perform sophisticated
maneuvers within small spatial and temporal scales. The rapid miniaturization of these technologies in surveillance and robotics has in part fueled the study of lift generation in insects.

Modeling the forces and moments acting on an insect wing is challenging because the rapid small scale movement is highly unsteady, meaning wings maneuver rapidly enough that excited vortices do not have time to convect the entire chord length before affecting the wing. In contrast, steady or quasi-steady aerodynamics typical of rigid airfoils rest on the assumption that maneuvers occur slowly relative to the fluid convection time. A comprehensive treatment of unsteady aerodynamics can be found in Leishman (2006), including the extensively used classical models of Wagner (1925) and Theodorsen (1935) (which we subsequently use in modeling fluid-structure interaction). Unsteady flight is commonly parametrized by the Strouhal number \( St = fM/U_\infty \) and reduced frequency of oscillation, \( k = 2\pi fb/U_\infty \), where \( f \) and \( M \) are the frequency and amplitude of oscillation, \( b \) is the half-chord length, and \( U_\infty \) is the free-stream velocity. In particular, gusts, flow disturbances and rapid maneuvers in unsteady regimes excite higher frequencies resulting in large reduced frequencies \( k \) greater than 1. The fluid-structure interaction in our work models a wide range of reduced frequency regimes to be subsequently characterized by sparse classification based on limited frequency measurements.

The unsteady effects, forces and lift generation experienced by insect wings have been extensively studied by Ellington (1994), Wang (2005) and Dickinson et al. (1999), among others. These seminal works in the study of insect wings all argue that unsteady effects are crucial for the additional lift generation previously unaccounted for by quasi-steady assumptions in thin airfoil theory. Although thin airfoil theory presents a starting point in characterizing surrounding flows, the fluid-structure interactions in airfoils and insect wings are markedly different. Airfoils respond to resonant frequencies between actuation and vortex shedding, and this is the focus of aeroelasticity and wing flutter analysis (Horikawa and Dowell, 1979; Dowell, 1996; Dowell and Hall, 2001). Insect wings, however, are strained and bent in response to both inertial-elastic forces and fluid loads which can be weakly coupled as shown in Figure 1. This complex interaction between elastic bending and fluid forcing is used in our simulations of the wing and the subsequent strain data.

2.2. Theodorsen model: unsteady forces on wings

Theodorsen (1935) addresses airfoil aerodynamic instability and flutter on thin plates undergoing oscillatory pitching and plunging movement. The Theodorsen model yields lift forces, pressures and moments of a rigid airfoil characterized by oscillatory movement in the angle of attack \( \alpha(k) \) and vertical position \( h(k) \). Its predictions agree quite well with experimental results (Leishman, 2006; Brunton et al., 2013) compared to the predictions of quasi-steady thin airfoil theory. This improvement is because Theodorsen augments

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**Nomenclature**

| Symbol | Description |
|--------|-------------|
| \( a \) | Vector of POD coefficients |
| \( b \) | Half-chord length of wing \([.01 \text{ m}]\) |
| \( c \) | Categories or number of environments |
| \( C(k) \) | Theodorsen transfer function |
| \( EI(x) \) | Flexural stiffness of wing \([\text{Nm}^2]\) |
| \( f \) | Frequency of maneuver \([\text{Hz}]\) |
| \( f \) | Vector of frequencies |
| \( k \) | Reduced frequency \( k \equiv 2\pi fb/U_\infty \) |
| \( h_0 \) | Initial vertical position of wing |
| \( h(t) \) | Vertical position of plate |
| \( m \) | Spatial grid resolution |
| \( N \) | Number of timesteps |
| \( p(x,t) \) | Chordwise loading of fluid pressure |
| \( P \) | Measurement selection matrix |
| \( r \) | Reduced-order model order |
| \( s \) | Vector of transverse strain of wing |
| \( S \) | Matrix of transverse strains |
| \( S \) | Discrete Cosine Transform of \( S \) |
| \( t \) | Time \([\text{s}]\) |

| Symbol | Description |
|--------|-------------|
| \( t \) | Vector of time \( t \) in seconds |
| \( U_\infty \) | Free stream velocity \([10 \text{ m/s}]\) |
| \( V \) | Matrix of right singular vectors of \( S \) |
| \( x \) | Chordwise spatial coordinate \([\text{m}]\) |
| \( x \) | High-dimensional signal |
| \( z(x,t) \) | Transverse deflection of wing \([\text{m}]\) |
| \( \alpha_0 \) | Base angle of attack |
| \( \alpha(t) \) | Angle of attack of wing |
| \( \omega \) | Angular velocity of maneuver |
| \( \mu \) | Linear wing density \([0.002 \text{ kg/m}]\) |
| \( \xi \) | Zero-mean sensor noise |
| \( \eta \) | Sensor noise variance |
| \( \epsilon \) | Error tolerance parameter |
| \( \sigma \) | Singular values of POD decomposition |
| \( \Sigma \) | Diagonal Matrix of singular values |
| \( \Phi \) | Basis of POD modes |
| \( \Phi_r \) | Low-rank basis of \( r \) dominant POD modes |
| \( \Phi_r \) | \( r \) dominant POD modes of \( S \) |
| \( \Psi, \hat{\Psi} \) | Library of multiple low-rank POD bases |
| \( \Theta_r \) | Joint POD modes of all available data |
the quasi-steady lift term from thin airfoil theory (Leishman, 2006),

\[ C_L = \frac{2\pi}{U_\infty} \left[ U_\infty \alpha + \dot{h} + \dot{\alpha} \left( \frac{1}{2} - \frac{b}{2} \right) \right], \]  

(1)

with unsteady terms multiplied by the Theodorsen transfer function, \( C(k) \), which is parametrized by reduced frequency of plate oscillation \( k \). The transfer function \( C(k) \) is derived by integrating forces generated by planar wake vorticity beyond the foil in inviscid, incompressible flow. We compare the quasi-steady lift coefficient (1) to the Theodorsen lift coefficient and distinguish added-mass and circulatory terms (Leishman, 2006):

\[ C_L = \frac{b\pi}{U_\infty} \left[ U_\infty \dot{\alpha} + \ddot{h} - \frac{b^2}{2} \ddot{\alpha} \right] - \frac{2\pi}{U_\infty} C(k) \left[ U_\infty \alpha + \dot{h} + \dot{\alpha} \left( \frac{1}{2} - \frac{b}{2} \right) \right]. \]  

(2)

\( C(k) \) is a quotient of Hankel functions of the second kind

\[ C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)}, \]  

(3)

where the subscripts of the Hankel function denote its order.

The model is an invaluable tool for understanding fluid-structure coupling along insect wings because it is based on unsteady assumptions and incorporates added-mass of the fluid surrounding the wing. Although the model rests on inviscid assumptions, Brunton and Rowley (2013) empirically adjusted coefficients in the model to agree with low-Reynolds number regimes typical of insect flight. The Theodorsen model will be used in our analysis for generating libraries of POD modes characterizing different aerodynamic environments parametrized by different \( k \).

2.3. Wing strain

Strain sensors encode the unsteady fluid added-masses on the wing and capture elastic stretching due to inertial-elastic forces from the flapping movement. Surprisingly, 80% of the elastic wing deflection in insects is due to inertial elastic forces (Combes and Daniel, 2003; Eberle et al., 2014), and the direct deformations due to flow feedback are relatively small in magnitude. Insects nevertheless appear to characterize flow environments using limited sensory information resulting from fluid forces upon the wing. Our fluid-structure interaction model accordingly simulates the wing using Theodorsen’s fluid pressures as a forcing term to a wing undergoing inertial-elastic deformation. To obtain strain data, we numerically simulate the wing chord deformation \( z(x, t) \) using classical elastic beam theory (Guenther and Lee, 1988) with external aerodynamic forcing term \( p(x, t) \) from Theodorsen’s chordwise loads,

\[ \mu z_{tt} + \left( EI z_{xx} \right)_{xx} = p(x, t), \quad x \in (0, 2b), t \geq 0. \]  

(4)

Parameters such as the flexural stiffness \( EI(x) \) in both the beam equation and the Theodorsen model reflect the wing morphological traits and leading edge flapping frequency of the representative hawkmoth insect. The chordwise spatial wing strain can be easily calculated from the structural deformation \( z(x, t) \) as the change in chord length divided by the initial length. We emphasize, however, that strain measurements used for classification are Fourier coefficients of the chordwise strain temporal dynamics at a single spatial location. Ultimately sparse approximation classifies sampled frequencies based on existing strain knowledge forced by different \( p(x, t) \).

2.4. Exploiting sparsity for classification

Most organisms, including insects, exploit low-dimensional structure to quickly characterize sensory input and execute swift response. They rely on innate or learned knowledge of new regimes of sensory stimuli to characterize stimuli on the fly. A data-driven protocol requires a similar means of acquiring low-rank knowledge from wing strain measurements. Many classifiers, including our sparse approximation methods, leverage the low-rank representation of the system’s input states rather than the inputs themselves.

Input states from sensor arrays or numerical discretization manifest with large dimension \( N \) despite arising from a system of much lower rank \( r \), where \( r \ll N \). The proper orthogonal decomposition (POD)
compresses high-dimensional data into a basis expansion consisting of \( r \) basis POD modes. These POD modes \( \phi \) are the \( r \) degrees of freedom in the system that span the spatial dynamics of system states \( x \in \mathbb{R}^N \):

\[
x = \sum_{j=1}^{r} a_j \phi_j(x) = \Phi_r a.
\]  

(5)

In the latter matrix form, the columns of \( \Phi \) are the POD modes and the vector \( a \in \mathbb{R}^r \) consists of \( r \) POD coefficients that uniquely define any input state \( x \). Given \( m \) states of a dynamical system stacked row-wise into a \( N \times m \) data matrix \( x = [x_1, x_2, \ldots, x_m] \), the POD is efficiently computed from the singular value decomposition of \( X \):

\[
X = \Phi \Sigma V^T,
\]

(6)

from which the dominant \( r \) POD modes are retained in \( \Phi_r \). The POD coefficients for \( x_k \) are \( a = \Phi_r^T x_k \). The strictly decreasing singular values \( \sigma_j \) scale the POD coefficients \( \Phi_r^T X = \Sigma_r V_r^T \) and determine the truncation level \( r \). In practice, \( r \) is chosen so that the energy of the last dominant mode \( \sigma_r / \sum_j \sigma_j \) is greater than some small threshold. The truncated POD basis is particularly suitable for low-rank approximation since the SVD is the explicit minimizer to the following optimization problem for a given target rank \( r \),

\[
\min_X \| X - \hat{X} \|_F \text{ subject to rank}(\hat{X}) = r,
\]

or equivalently, \( \hat{X} = \Phi_r \Sigma_r V_r^T \). This is the well-known Eckart-Young Theorem (Eckart and Young, 1936). The columns of \( \Phi_r \) span the column space of \( X \), so they yield the best rank-\( r \) least-squares approximation to the system \( X \) in the Frobenius norm. For this reason the POD is a popular model reduction tool for complex flow dynamics and is broadly used across many fields in which it is alternatively known as principal component analysis (PCA) (Pearson, 1901), Karhunen-Loève decomposition (Loève, 1955), and Hotelling Transform (Hotelling, 1933). POD modes are ordered by decreasing spectral energy (singular values), so the first few modes are sufficient for characterizing an intrinsically low-rank dynamical system. Lower energy modes can often be ignored. This parsimonious representation of system states by a few dominant modes facilitates sparse classification within a library of POD mode sets from different classes, especially given significant variation between the different classes or in our case, flow environments.

Remark: In subsequent sections, the truncation threshold \( r \) is chosen so that the first \( r \) normalized singular values capture 99% of the energy in the system, or equivalently, sum to 0.99. This high energy threshold is suitable for noiseless data from physics simulations. A more principled truncation method developed by Gavish and Donoho (2014) chooses an optimal truncation threshold based on noise level, singular value dropoff and data dimensions, which is more suitable for noisy data.

2.4.1. Library learning

The supervised learning stage constructs a POD library from \( c \) distinct environments expected to be encountered in flight. Data matrices of strain dynamics are collected through simulation for each flow environment \( i \), which requires knowledge of current environment in the learning phase. The construction of truncated POD modes \( \Phi_r \) for each data matrix \( X_i \) concludes the offline knowledge acquisition. The \( c \)
sets of POD modes are stacked into the library

\[ \Psi = [\Phi_{r_1} | \Phi_{r_2} | \ldots | \Phi_{r_c}] . \]

New state measurements \( x \) are then classified using POD library coefficients,

\[ a = [a_{T r_1}^T \ a_{T r_2}^T \ldots \ a_{T r_c}^T]^T, \]

where its environment \( i \) is identified by the set of POD modes that best approximate \( x \) in a least-squares sense

\[ \text{environment}(x) = \arg \min_i \| \Phi_{r_i} a_{r_i} - x \|_2. \quad (7) \]

The classifier decision is therefore determined by the POD library coefficients, which cannot be obtained by the standard inner product with \( x \) since \( \Psi \) is no longer unitary. Sparse library coefficients are sought so only the components of \( a \) that correspond to the correct set of POD modes \( \Phi_{r_i} \) are nonzero. The library coefficients require a sparsity-promoting solution method that simultaneously solves for all \( a_{r_i} \) at once, and we outline two such sparse approximation methods in what follows.

Library learning of low-rank features from data is well established in the computer science community. More recently, the mathematical framework has migrated into the reduced order modeling community for characterizing parametrized PDEs (Amsallem et al., 2016; Choi et al., 2015; Peherstorfer and Willcox, 2015c,b,a; Sargsyan et al., 2015). Thus libraries of ROM models that can be selected and/or interpolated through measurement and classification. Alternatively, cluster-based reduced order models use a k-means clustering to build a Markov transition model between dynamical states (Kaiser et al., 2014). Before these prototypical machine learning methods were considered for ROMs, it was already realized that parameter domains could be decomposed into subdomains and a local ROM/POD computed in each subdomain. Eftang et al. (2010) used a partitioning based on a binary tree whereas Amsallem et al. (2009) used a Voronoi Tessellation of the domain. Such methods were closely related to the work of Du and Gunzburger (2002) where the data snapshots were partitioned into subsets and multiple reduced bases computed. The multiple bases were then recombined into a single basis, which is different than the library building techniques used more recently. For a review of these domain partitioning strategies, please see Ref. Amsallem et al. (2015).

2.4.2. \( L_1 \) constrained approximation

First we use the POD library and incoming full state measurements in signal \( s \) to approximate library coefficients from the solution of the overdetermined linear system (Figure 2)

\[ s \approx \Psi a. \quad (8) \]

The high-dimensional state \( s \) (previously denoted \( x \)) is a linear combination of only one set of POD modes within the library and not the others, \( i.e., \) only a few adjacent library coefficients should be nonzero. Minimizing the number of nonzero components in a linear system (the \( L_0 \) norm of \( a \)) is a computationally intractable combinatorial search. However, relaxing the objective to \( L_1 \) norm minimization, where \( \| a \|_1 = \sum_k |a_k| \), results in the sparsest solution in many cases. The sparse solution is given by the optimization

\[ a = \arg \min_\hat{a} \| \hat{a} \|_1 \text{ subject to } \| \Psi \hat{a} - s \|_2 < \epsilon, \quad (9) \]

where \( \epsilon \) is a tunable error tolerance parameter. The performance is highly sensitive to the tolerance parameter - small values of \( \epsilon \) risk approximating the least squares solution (10) and large \( \epsilon \) risk library coefficients that are not sparse enough to select only one set of POD modes. The identifier (7) relies on the subset selecting sparsity of \( a \) that results from \( L_1 \) norm minimization.

On the other hand, the \( L_2 \) error minimizer given by the pseudo-inverse of \( \Psi \)

\[ a = \Psi^* s = a = \arg \min_\hat{a} \| \Psi \hat{a} - s \|_2, \quad (10) \]

has the undesirable effect of weighing all available library vectors and distributing nonzero entries across all components of \( a \), which confuses the classifier in (7).
2.4.3. Sparse Representation for Classification (SRC)

A related idea, sparse representation for classification (SRC), was originally formulated by Wright et al. (2009) for facial image recognition, and computational solution methods include convex optimization packages such as MATLAB’s cvx (Grant and Boyd, 2014) and greedy algorithms such as CoSaMP (Needell and Tropp, 2010). SRC acts on fewer state measurements than $R = \sum_{i=1}^{p} r_i$, the number of columns in $\Psi$, resulting in an underdetermined system of equations with multiple solutions and no prescribed error tolerance. The measurement operator $P \in \mathbb{R}^{p \times N}$ discretely samples $s$ at $p < R$ measurement locations that consists of $p$ rows of the $N \times N$ identity. Upon using the new state measurements $y = Ps$ and corresponding rows in the library $P\Psi$, the system has multiple solutions $\hat{a}$. The sparsest one is uniquely given by an optimization similar to (9),

$$
\hat{a} = \arg\min_{\tilde{a}} \|\tilde{a}\|_1 \text{ subject to } y = P\Psi\tilde{a}. 
$$

SRC is closely related to compressed sensing (Donoho, 2006; Candès et al., 2006; Baraniuk, 2007). Compressed sensing is widely used for signal recovery from fewer random measurements than the signal’s rank in the approximating basis. Perfect reconstruction requires $p = O(r \log(N/r))$ measurements that randomly encode the signal in a way that is incoherent with the approximating basis. This is alternatively known as the Restricted Isometry Property (RIP) that must be satisfied by the measurement matrix $P$. Later we show that $P$ can be optimized so that measurement locations are specially chosen to increase sparse classification accuracy. Furthermore, this can be done with fewer measurements than are required for reconstruction since classification is a milder objective.

In the reduced order modeling community, sparse sampling for state space reconstruction is not new. Indeed, for nonlinear model reduction involving time-dependent PDEs, there are two critically enabling mathematical ideas: (i) the rank-$r$ Galerkin projection of the dynamics onto POD modes, and (ii) the approximation of the nonlinearity and its inner products using gappy (sparse) POD sampling. First introduced by Everson and Sirovich (Everson and Sirovich, 1995), gappy POD (Everson and Sirovich, 1995; Willcox, 2006; Yildirim et al., 2009; Carlberg et al., 2013), and variants such as missing point estimation (MPE) (Astrid, 2004), “best points” method (Nguyen et al., 2008), empirical interpolation method (EIM) (Barrault et al., 2004), and discrete empirical interpolation method (DEIM) (Chaturantabut and Sorensen, 2010), take advantage of sparse sampling strategies to project nonlinearities onto the low-rank POD basis. Specifically, the constraint $y = P\Psi\tilde{a}$ in (9) is identical with the gappy POD mathematical architecture where $P$ is prescribed by one of the above sampling methods. In the ROM context, the projection is only on a specific, rank-$r$ POD basis so that $\Psi$ is not a library of modes, thus allowing one to easily use standard $L_2$-based reconstruction. In contrast, our sparse, $L_1$-based sampling strategy is used to select one of the many POD basis sets available for reconstruction. Thus the objective of the sparse sampling in ROMs is quite different, yet relies on a similar mathematical framework.

3. Fluid-Structure Interaction

This section outlines our numerical approach for analyzing the fluid-structure coupling of the wing, and motivates sparse representation for classification in the frequency domain (SRCf). The wing is modeled by classical elasticity theory (Guenther and Lee, 1988). As noted previously, the governing equations model the deformation of the elastic body subject to a function of the time-dependent chord loading $p(x, t)$. We define the spatial domain $x$ to represent the length of the wing chord that deforms in response to inertial-elastic forces. The range of $x$ is the closed interval $[0, 2b]$ where $b$ is the half-chord length, $x = 0$ represents the leading edge undergoing oscillatory actuation, and $x = 2b$ is the stress and shear free trailing edge. These boundaries are consistent with biology - most insect wings are actuated by exoskeletal structure at their leading edge, while their trailing edge consisting primarily of thin membrane flutter at the behest of air loads without experiencing any inertial stress or shear forces. True wings deform in three dimensions, however we can approximate the deforming wing using two spatial dimensions $x$ and $z$ where $z(x)$ is the vertical displacement that depends on position along the chord. Assuming normal deflections and ignoring rotational effects, the elastic wing can be characterized by a forced linear Euler Bernoulli beam with spatially varying flexural stiffness $EI(x)$ and a zero deformation initial state that corresponds to a flat stationary wing. However, the strain data is collected after the deformation achieves a steady state,
Unforced, $f = 5$ Hz

Forced, $f = 5$ Hz

Forced, $f = 51$ Hz

Figure 3: The strain dynamics for three different regimes, from left to right, (1) No gust disturbance, (2) low frequency gust disturbance $f = 5$ Hz, (3) high frequency gust disturbance $f = 51$ Hz. All three cases experience leading edge wing actuation at 26 Hz. Direct visualization of the strain dynamics offers no insight into the changing gust feedback between environments since inertial-elastic forces dominate over aerodynamic feedback forces.

Unforced, $\phi(x)$

Forced, $f = 5$ Hz

Forced, $f = 51$ Hz

Figure 4: The similarity of dominant spatial POD modes of wing strain between regimes 1 to 3 (Figure 3) suggests that spatial POD modes are not ideal for the SRC classification task.

ignoring any initial transients. The governing equations and boundary conditions are thus given by:

\begin{align}
\mu z_{tt} + (EI z_{xx})_{xx} &= p(x, t), & x \in (0, 2b), t \geq 0 \\
z(x, 0) &= 0 \\
z(0, t) &= h(t) \\
z_x(0, t) &= -\sin(\alpha(t)) \\
(12a) & & (12b) & & (12c) & & (12d)
\end{align}

where $z(x, t)$ is the displacement of the elastic body along the length $x \in (0, 2b)$, $\alpha(t)$ and $h(t)$ specify the leading edge time dynamics for pitching and plunging respectively, and $p(x, t)$ is the applied pressure loading that is computed from the Theodorsen model. The trailing edge boundary conditions specified at $x = 2b$ model a shear-free and stress-free boundary.

Numerical simulations use wing morphological parameters of the *Manduca sexta* hawkmoth, a representative species capable of hovering and other small-scale maneuvers that typify insect flight. The numerical model consists of two components that are weakly coupled as shown in Figure 1: (1) elastic deformation strains and (2) unsteady aerodynamic loads. The deformation $z(x, t)$ is simulated with the hawkmoth pitching and plunging motion at 26 Hz at the leading edge. The *Manduca* wing’s morphological parameters (Combes, 2002; Combes and Daniel, 2003) facilitate analysis of a small parameter space of a simplified 2D domain - the deforming wing chord along the $xz$ plane. Accordingly, the flexural stiffness distribution $EI(x) = 10^{-5} e^{-150x}$ is determined by averaged experimental values from the actual hawkmoth in Combes (2002), in which wings are shown to exhibit exponentially decreasing chordwise wing stiffness away from the leading edge. An implicit finite-difference scheme is used to solve the dynamic 1D beam equation for the normal deformation $z(x, t)$ with spatially varying flexural stiffness to investigate the resulting unsteady fluid pressures and lift forces. Denoting the vector of transverse deflection at time $t_i$ as $z_{ti}$, the strain vector...
which expresses the local chordwise loads as external fluid disturbances. The first source of loads from the shed wake includes added masses and Abdo (2004) derived this expression explicitly as an intermediate step to Theodorsen’s result for the space-

\[ s(x, t_1) = \sqrt{\Delta z_i^2 + \Delta x^2} - \sqrt{\Delta z_{i-1}^2 + \Delta x^2}, \]  

where \( \Delta z_{t_1} \) denotes the spatial one-sided difference \( z(x + \Delta x) - z(x) \) at time \( t_1 \).

Meanwhile, the loads on the insect wing, \( p(x, t; k) \), result from two sources – the shed wake as well as external fluid disturbances. The first source of loads from the shed wake includes added masses and unsteady effects of the wing oscillation. These contributions are computed using the integral equation (14), which expresses the local chordwise loads \( p(x, t; k) \) as a function of \( x \) and normal wing velocity \( z_t(x, t) \). Abdo (2004) derived this expression explicitly as an intermediate step to Theodorsen’s result for the space-averaged global pressure coefficient. However, the resulting changes in wing strain are small in magnitude and are characterized by the same dominant reduced frequency of wing oscillation at 26 Hz.

\[
p(x, t; k) = \frac{2}{\pi} \mu U_\infty \sqrt{1 - x} \int_{-1}^{1} \frac{z_t(\tilde{x}, t)}{1 - \tilde{x}} d\tilde{x} + \frac{1}{\pi} \mu b \int_{-1}^{1} z_t(\tilde{x}, t) L(x, \tilde{x}) d\tilde{x},
\]

where

\[
L(x, \tilde{x}) = \log \left( \frac{(x - \tilde{x})^2 + (1 - x^2 - \sqrt{1 - x^2})^2}{(x - \tilde{x})^2 + (1 - x^2 + \sqrt{1 - x^2})^2} \right).
\]

The second source of chordwise loads are assumed to be external sinusoidal gusts characterized by different frequencies. We assume these gusts generate loads that resemble forces that would occur if the wing oscillates precisely at the characteristic frequencies of sinusoidal gust. Consequently, the loads resulting from external gusts are also computed using Theodorsen’s model. This assumption approximates the aerelastic effects of sinusoidal gust fields, since the forcing to the elastic beam includes these loads. This forced coupling between the elasticity model and the aerodynamic model addresses the absence of inertial effects in the Theodorsen framework. Specifically, we evaluate the following expressions for external loads, \( p(x, t; k) = p_o(x, t; k) + p_h(x, t; k) \),

\[
p_o(x, t; k) = \frac{U_\infty}{2b\mu_\omega} \sqrt{1 - x^2} + \frac{U_\infty}{2b\mu_\omega} \sqrt{1 - x^2} - \frac{C(k)}{k} \sqrt{\frac{1 - x}{1 + x}} - \frac{1}{2k} (1 + 2x + 2C(k)) \sqrt{\frac{1 - x}{1 + x}}.
\]

\[
p_h(x, t; k) = \frac{i C(k)}{k} \sqrt{\frac{1 - x}{1 + x}}.
\]

given by Postel and Leppert (1948) explicitly in their derivation of Theodorsen’s lift coefficient (2). The resulting loads from the gusts represent varying flow environments, hence, we generate gusts characterized by a wide range of reduced frequencies. Since the model wing and base flow remain unchanged, this is accomplished by varying \( f \), hence \( k \), and holding free-stream velocity and chord length fixed.
Figure 6: SRCf promotes sparsity of the coefficients $a$ to help identify the correct originating environment $i$ using sparse frequency measurements $Ps$. The left plot contains 3 actual solution vectors. The blue line is the solution $a$ for a test vector from regime (1), red from (2), and green from (3). Each line correctly identifies its originating regime with its largest magnitude nonzero components.

First, wing strain dynamics are visualized for three environments $i = 1, 2, 3$, all of which undergo the wing oscillation at 26 Hz. Environment 1 is unforced, and environments 2 and 3 undergo gust forcings of 5 Hz and 51 Hz, respectively. These frequencies correspond to reduced frequencies of $k \approx 0.03$ and $k \approx 0.3$, the first of which is quasi-steady, and the latter represents unsteady flow. Figure 3 displays the resulting strain dynamics which, by inspection, appear quite similar. The dominant spatial POD modes of strain, shown in fig. 4, confirm that spatial modes remain largely unaffected by the changing forcing term. Indeed, without additional inertial or added masses in the model, the various forced dynamics are still governed by the intrinsic spatial modes. This poses a difficulty for data-driven classifiers based on spatial POD modes. Indeed, preliminary runs of SRC on strain snapshots from the three environments yield classification accuracies of 30% on average, which is no better than random guesses. To remedy this, POD modes must incorporate the time dynamics of wing strain, as detailed in the following section.

4. Frequency domain representation of wing strain

We introduce a sparse classification strategy that incorporates the time history of the strain signal through its frequency content. Indeed, insects and winged animals are thought to discriminate aerodynamic environments with only a few physical wing sensors that sample certain bandwidths of the feedback frequencies. Since gust forcing terms do not induce distinctive spatial modes, it is important to consider temporal strain dynamics, which are characterized by the jump discontinuities seen in fig. 3. These discontinuities are distributed across many temporal POD modes, posing difficulties in obtaining a low-rank truncated POD. Hence, we transform the time dynamics into the frequency domain using the discrete cosine transform (DCT), and adapt the POD library and sparse classifier to classify the transformed strain dynamics. Thus the input to the sparse classifier, $\hat{s}$ (previously denoted $x$), can be expanded in terms of frequency domain POD modes,

$$s(x, t) \xrightarrow{DCT} \hat{s}(x, f) = \sum_{j=1}^{r} a_j \hat{\phi}_j(f) = \hat{\Phi} a.$$  

This is distinct from the spatial POD modal expansion in (5). The numerical simulation for environment $i$ yields strain data $S_i$ for $m$ spatial gridpoints and $n$ timesteps. Recall that the standard POD is performed on a data matrix whose columns are time snapshots, that is, its columns span the spatial direction. For frequency domain analysis, the strain data matrices $S_i \in \mathbb{R}^{n \times m}$ are adjusted so that its columns span time dynamics

$$S_i = [s(x_1, t) \mid s(x_2, t) \mid \ldots \mid s(x_m, t)],$$

and the DCT applied to each column yields strain frequency content at every spatial location

$$\hat{S}_i = [\hat{s}(x_1, f) \mid \hat{s}(x_2, f) \mid \ldots \mid \hat{s}(x_m, f)].$$
Thus the procedure for building the POD library of frequency content remains unchanged

\[ \hat{S}_i = \hat{\Phi}_r \Sigma_r V^T_r, \]

where hatted notation indicates that POD modes span frequency content. Figure 5 illustrates the dominant POD modes of the strain data’s temporal frequency content for the three regimes. The frequency content reveals substantial differences between the aerodynamic regimes that facilitate robust, accurate classification.

4.1. Classification in the frequency domain

Given the frequency adjusted POD library, the SRC and compressed sensing procedures for classification remain identical. As before the POD modes \( \hat{\Phi}_r \) are stacked into the frequency POD library \( \hat{\Psi} \), and a new state is classified from the POD library coefficients. The input to the classifier is the frequency content at one spatial location on the wing, \( \hat{s} \). The \( L_1 \) constrained approximation for the full state is then given by

\[
a = \arg \min_{\tilde{a}} \| \tilde{a} \|_1 \text{ subject to } \| \hat{\Psi} \tilde{a} - \hat{s} \|_2 < \epsilon.
\]

(18)

Note that the error tolerance \( \epsilon \) must be chosen carefully to balance the trade-off between approximation and classification. Sampling the frequency content at a few limited frequency locations with \( P \) yields the measurement vector \( \hat{P}s \) used in SRC, with sparse solution coefficients given by

\[
a = \arg \min_{\tilde{a}} \| \tilde{a} \|_1 \text{ subject to } (\hat{P} \hat{\Psi}) \tilde{a} = \hat{P}s.
\]

The SRC schemes are illustrated in Figure 6. As before, the strength of \( a \)’s components will determine the subset of regimes to which the strain frequency signal belongs. Because the classifier now operates in the frequency domain, SRC in the frequency domain is abbreviated as \( SRC_f \).

4.2. Optimized frequency selection for \( SRC_f \)

Engineering and biological applications typically benefit from optimized measurement strategies for decision-making and estimation tasks. Recall that \( L_1 \) regularized SRC permits sparse input sampling using, for instance, ROM methods like empirical interpolation methods (§ 2.4.3). EIM advocates measurement locations optimized for regression within the POD basis. Since locations correspond to the frequency bandwidths accessible for measurement, we refer to this optimized measurement selection as frequency selection. This section briefly overviews existing methods for measurement selection, followed by our application of EIM for optimized frequency selection within POD modes of strain dynamics.
4.2.1. Background on measurement selection

While empirical interpolation as measurement selection is a relatively new idea, measurement selection itself is a well-researched problem in signal processing, machine learning, design of experiment (DoE) and control. It is often designated as sensor selection since it typically informs sensor placement in spatial domains. Since sensor selection in large domains is a combinatorially hard search across possible sensors, a variety of greedy optimization strategies have been advocated that scale better with the search space. Geometric approaches in Hochbaum and Maass (1985); González-Baños (2001) treat sensors as disks and attempt to cover the measurement space. If strong spatial correlations exist, sensors can be placed to favor regions that experience greater entropy or variance (Cressie, 1991; Shewry and Wynn, 1987). Meanwhile, Caselton and Zidek (1984); Krause et al. (2008) greedily maximize mutual information of a sensor, which quantifies how much information a sensor contains about unused locations. More general uncertainty quantification techniques (Zhu and Stein, 2006; Zimmerman, 2006) advocate sampling specifically for parameter estimation. Joshi and Boyd (2009) tackle sensor selection with convex optimization and survey related optimal experiment design and Bayesian approaches. In related work, Brunton et al. (2016a) develop an $L_1$-based convex optimization that operates directly on POD modes for facial image classification called sparse sensor placement optimization for classification (SSPOC). In addition, an overview of sensor selection for Gaussian processes can be found in Krause et al. (2008).

4.2.2. Empirical Interpolation

Sparse sampling for state space reconstruction uses EIM/DEIM points (as discussed in §2.4.3) that are well-conditioned for least-squares reconstruction in a single low-rank POD basis. To generalize these methods for $L_1$ classification in a library of modes, we first obtain a single low-rank POD basis for all categories of data. For categorical data, a joint POD of all categories yields fewer modes than a concatenated library of POD modes from each consecutive category. Importantly, both are assumed to characterize the same active subspace or column space, since a joint POD compactly represents redundant features shared between categories in a POD library. Explicitly, we construct a joint POD using strain data from all flow categories,

$$[\hat{S}_1 | \hat{S}_2 | \ldots | \hat{S}_c] = \Theta \Sigma V^T,$$

and truncate the left singular vectors to retain only the first $r$ dominant POD modes. This is advantageous for engineering the fewest measurements possible for sparse classification, since

$$r < \sum_{i=1}^c r_i.$$

Subsequently, sparse library coefficients $a$ may be recovered from selected measurements of the full signal, where we denote the measurement selection operator by $P$

$$(P\hat{\Psi})a \approx Ps. \quad (20)$$

This row selection operator, $P \in \mathbb{R}^{r \times n}$, consists of rows of the $n \times n$ identity, so that

$$P_{j,I} = 1.$$  

For the moment, we proceed as though the unknown state $\hat{s}$ is reconstructed (not classified) in $\Theta_r$, the POD feature space spanning all classes. Since $\hat{s}$ is unknown, it cannot be recovered from its POD coefficients $\Theta_r^T \hat{s}$. Instead, the goal is to optimize $P$ for approximating $\hat{s}$ from $y = Ps$ as follows

$$P_* = \arg \min_P \| \hat{s} - \Theta_r (P\Theta_r)^{-1} y \|_2.$$  

DEIM (Chaturantabut and Sorensen, 2010) casts this optimization as a conditioning problem in the spectral norm,

$$P_* = \arg \min_P \|(P\Theta_r)^{-1}\|_2, \quad (21)$$

where the attained minimum is denoted $\gamma_* = \|(P, \Theta_r)^{-1}\|_2.$ Equation (21) also maximizes the matrix volume of the product $P\Theta_r$, which is the absolute value of its determinant. A naïve brute-force search across all combinatorial $\binom{n}{r}$ row selections (measurements) of $\Theta_r$ is computationally intractable and would involve evaluating determinants for each selection. Empirical interpolation methods use greedy procedures...
to bypass this combinatorial search. DEIM iteratively selects measurements one at a time by locating them at maxima of successive residuals from approximation with previously chosen measurements. It is extensively used since it is computationally cheap and has established upper bounds for the minimizer $\gamma$.

Drmac and Gugercin (2016) demonstrate an even better choice for $P$ given by the column pivoted QR factorization of $\Theta^T_r$ called Q-DEIM. QR column pivoting has been a pioneering workhorse for the solution of underdetermined linear systems ever since its introduction by Businger and Golub (1965). Its utility in least-squares polynomial approximation has led to related work in finding near-optimal Fekete interpolation points from polynomial Vandermonde matrices (Sommariva and Vianello, 2009) and interpolation points in weighted polynomials (Seshadri et al., 2016). This procedure is designated QR selection to distinguish sampling for sparse approximation in a POD library from Q-DEIM sampling for state reconstruction in ROMs.

The rest of this discussion closely follows Drmac and Gugercin (2016), in which the pivoted QR factorization of $\Theta^T_r$ is shown to be a near-optimal solution of (21). The QR factorization expresses $\Theta^T_r$ as the product of an orthonormal matrix $Q$ and an upper-triangular matrix $R$. The column pivoted QR rearranges $r$ columns of $\Theta^T_r$ so that its first $r$ columns can be expressed as a product of $Q$ and $R_1$, the leading square $r \times r$ submatrix of $R$, such that

$$\Theta^T_r \Pi = QR_1.$$  \hspace{1cm} (22)

QR selection essentially pivots measurement space to construct a special diagonally dominant $R_1$ with a lower condition number than would have been possible without column pivoting. The resulting row selection is then given by

$$P = \Pi^T.$$

Construction of a diagonally dominant $R_1$ is beneficial because the spectral norm depends entirely on $R_1$, 

$$\gamma_{qr} = \| (P\Theta_r)^{-1} \|_2 = \| R_1^{-1} Q^T \|_2 = \| R_1^{-1} \|_2 = \frac{1}{\sigma_{\text{min}}(R_1)}.$$ 

Finally, Drmac and Gugercin (2016) demonstrate the improvement of pivoted QR selection over the leading method, DEIM, in theory and in practice. They derive upper bounds for $\gamma_{qr}$ that are smaller than $\gamma_{\text{deim}}$. Moreover, in practice, QR selection values for $\gamma$ fall well below that of the DEIM procedure. Indeed, the authors demonstrate empirically that $\gamma_{qr}$ often satisfy the conjectured optimal upper bound $\gamma_\star$, while that of DEIM surpasses it. Thus, QR selection is near-optimal in the matrix volume maximizing criterion.

4.2.3. QR Software Implementation

Software implementations of QR are readily available in most scientific computing packages, including LAPACK, ScaLAPACK, and MATLAB. Most subroutines implement Businger-Golub pivoting using Householder projections as detailed in Businger and Golub (1965), adapting the procedure as necessary to deal with rank-deficiency. However, any of the procedures may be used in this setting since we only consider matrices of full rank. After each successive orthogonal projection step, this method successively selects the next column with maximal 2-norm as the next pivot, with the effect of increasing the condition number of the column pivoted target matrix. Below is MATLAB code for constructing the QR selection operator $P_{qr}$, given $\Theta_r \in \mathbb{R}^{n \times r}$ and truncation level $r$:

```matlab
%% MATLAB code for given Theta_r matrix
n = size(Theta,1); % dimension of state space
Theta_r = Theta(:,1:r); % first r POD modes

% Pivoted QR factorization of POD modes transposed
[Q,R,pivot] = qr(Theta_r','vector');

% QR row selection indices
ind = pivot(1:r);

% Form row selection operator
I = eye(n);
P_qr = I(ind,:);
```
Table 1: Summary of numerical experiments and sparse classifiers, where ∅ refers to the flow environment with no gust forcing. All flow environments exhibit the dominant 26 Hz frequency of wing oscillation. Furthermore, the number of gust frequencies equals the total number of classes considered.

More generally, QR measurement selection can easily be adapted to incorporate measurement location constraints as would be commonly encountered in engineering applications. Undesirable measurement locations from a practical standpoint can simply be omitted from the QR pivoting algorithm by omitting the appropriate rows (measurements) from the input \( \Theta_r \). More complex preferences can be implemented by multiplying the input modes by application-specific weight matrices; this is the focus of ongoing work.

5. Results

In this section sparse approximation simulations are designed to detect flow-induced strain patterns across various flow environments, frequency samples and spatial correlations. Importantly, the advantages of sparse over conventional least squares approximation and trained QR row selections over random samples in SRCf are demonstrated. These patterns are of particular importance in engineering applications that require collection of strain data optimized for environment detection. These tests also stress the importance of sensor noise in these applications by demonstrating robust sparse approximation performance over a range of increasing noise levels in the strain dynamics.

In this section, machine learning terminology is used to describe the data. Single location frequency content are called observations or signals, to disambiguate from system states that may refer to snapshots in time. Observations to be classified, \( \hat{s} \) or \( y = P\hat{s} \), are called inputs to the classifier. As a pre-processing step, noise is added to each column of the untransformed input observations in the following order.

1. Normalize each column so that \( \|s(x_j,t)\|_2 = 1 \) for all \( j = 1 \rightarrow m \) gridpoints,
2. Add zero-mean Gaussian noise so \( S = S + \xi \), \( \xi \sim N(0, \eta^2) \), and
3. Transform \( S \) into frequency domain \( \hat{S} \): \( \hat{s}(x_j,f) = DCT[s(x_j,t)] \).

Thus sensor noise is added to the untransformed normalized signal in the space-time dimension as would be the case in engineering applications, and classifier inputs are normalized to facilitate comparing the effect of noise across different environmental regimes. Note that we do not normalize or add noise to the transformed signal \( \hat{s} \) in the frequency domain since wing strain measurements are noisy when collected (sensor noise), not when processed via the frequency domain (process noise). Table 1 provides a summary of the numerical experiments performed and the figures produced for this section.

5.1. Feedback identification from full state

We demonstrate sparse flow feedback identification between the three learned flow environments in § 3, and test it on randomly selected subsets of strain dynamics from each flow environment. The environments are chosen to highlight the extremes of gust frequencies - low feedback of \( f = 5 \) Hz in environment 2, high feedback of \( f = 51 \) Hz in environment 3, and no feedback in regime 1. All three share the 26 Hz frequency signature expressed by wing oscillation. It is expected that some environments will be identified with higher accuracy than the others, and strain dynamics from each environment are sparsely classified separately to reveal this structure in the data.

For each flow environment \( i \), the tests are conducted using tenfold cross validation. Ten percent of vectors in \( \hat{S} \), are randomly selected to be test inputs of strain dynamics, and the remaining 90% of the columns are used to train the POD library and optimal sensors, meaning the POD modes from each environment are
Figure 8: The cross-validated performance of $L_1$ approximation across environments 1-3 is shown here for sensor noise of increasing variance $\eta$. Red, blue and green represent input observations from environments 1, 2 and 3, respectively. High-frequency gusts from regime 3 are more robustly distinguished from the other two regimes.

Figure 9: The comparison of $L_1$ approximation to least squares approximation of the library coefficients demonstrates the power of the sparsity promoting $L_1$ norm constraint for subset selection.

trained from 500 observations. The POD library $\Psi$ consists of the stacked dominant POD modes of environments 1, 2 and 3 shown in Figure 5. After training the POD library, test frequency dynamics $\hat{s} \in \mathbb{R}^{256}$ are classified using $L_1$ constrained approximation of the overdetermined linear system (18). The performance is investigated across increasing levels of sensor noise, and the classification accuracies across all the tests for one level of sensor noise are displayed as one box and whisker in the subsequent plots. The mean, 25th quartile, and 75th quartile of the classification performance distribution are displayed as circles, the bottom, and top edges of the boxes, respectively. Whiskers extend to $\pm 2.7$ standard deviations of the result distribution, and any data outside this range is displayed as a small outlier point.

Box and whisker plots of classification accuracy for test dynamics from environments 1-3 are shown in Figure 8. $L_1$ constrained approximation achieves perfect 100% classification accuracy in the noiseless case for all three environments, and even at levels of 20% sensor noise, the performance remains impressively higher than chance or 33.3%, the scenario in which the classifier chooses between the three environments uniformly at random. Interestingly, as sensor noise increases, the identification of high-frequencies in environment 3 is more successful than the identification of environments 1 and 2. Environment 3 in particular contains high frequency content and higher amplitude spatial modes (Figure 3) that are amplified in the frequency domain analysis.

These results are quite sensitive to the choice of $L_2$ error tolerance $\epsilon$. In these computations, $\epsilon$ is set to some multiple of the least squares solution error. That is, given the cheaply computed least squares solution $\hat{a}_\text{ls} = \arg \min_{a} \| \Psi a - \hat{s} \|_2$, $\epsilon_{\text{ls}} = \min_{a} \| \Psi a - \hat{s} \|_2$. In practice, setting $\epsilon = \epsilon_{\text{ls}}$ in the noiseless case and relaxing the tolerance to $\epsilon = 1.2\epsilon_{\text{ls}}$ in the noisy case works quite well. The tolerance appears to depend on the noise level $\eta$ and $\epsilon_{\text{ls}}$, and there may exist an optimal choice based on the two parameters requiring further investigation.
5.2. Sparse ($L_1$) vs. least squares ($L_2$) approximation

It is instructive to overlay in Figure 9 results from classification using the least squares solution $a_{ls}$ in (7). The comparison demonstrates the clear advantage of using $L_1$ norm minimization of library coefficients over $L_2$ minimization of the residual (least squares), particularly in the presence of sensor noise, although least-squares achieves $L_1$ accuracy in the noiseless case. The identification of environment 3 is once again an exception due to high-frequency fitting that is not present in the other two, however, $L_1$ constrained approximation always outperforms least squares. Therefore the sparsity promotion of solution library coefficients is essential for effective classification in the discriminating frequency domain.

5.3. SRCf with QR selection

The resolution of strain measurements in time is directly informed by the optimized sampling of frequency content for classification. The strain dynamics from different flow environments by construction are linearly separable by flow feedback frequency. The precise discriminating frequencies are unknown in the training stage but are easily determined from QR selection with the joint POD of all strain dynamics. The following simulation compares the performance of SRCf with QR row selection over random samples in the frequency domain, confirming the advantage of QR selection.
Figure 12: This plot compares classification accuracy for sensors distributed around (left to right) leading edge, 1/4 chord, mid-chord, 3/4-chord and trailing edge for identically increasing sensor noise. Each error bar represents cross-validated classification accuracy across observations randomly chosen from a Gaussian distribution centered around a certain area of the chord, sampled from a randomly chosen environmental regime out of the five total regimes. Classification accuracy increases across the chord with accuracy increasing near the trailing edge.

QR selection is obtained from a richer training dataset of strain dynamics consisting of forced disturbances at 25 Hz, 50 Hz, 75 Hz and 100 Hz, in addition to the no forcing scenario. There are now five environments considered by the classifier, a more difficult task than the former. As before, the tested dynamics do not consist of the same random 90% of the data used for training POD library modes and QR selection. The five QR selected measurements for the five environment case in Figure 10 appear to cluster around the discriminating disturbance frequencies and even appear to discriminate the flow forcing at 25 Hz from wing oscillation at 26 Hz. The measurements do not exactly lie at these numbers because the DCT does not span whole number frequencies.

The performance of SRCf shown in Figure 11 demonstrates the advantage of QR selection of strain dynamics over random sampling. Up to a 35% increase in classification accuracy is achieved with only five samples (1.95% of the 256 element vector of strain dynamics). In addition, QR selection is more robust to sensor noise than an even larger number of random samples, although the accuracy gain drops off near sensor noise of level $\eta = 0.2$ that comprises 20% of the untransformed signal’s magnitude. The strain dynamics are saturated by noise at this higher noise level where SRCf performance drops off to randomized classification accuracy.

Interestingly, doubling the number of QR selections does not improve the accuracy, although doubling the number of random samples improves random performance as expected. The results indicate that cross-environment variation is well characterized by the QR selected measurements which are optimized for library approximation with $\hat{\Psi}$. Furthermore, increasing the number of random samples to 35 (Figure 12) does not achieve the same robust accuracy drop-off seen with QR selections.

5.4. SRCf with spatial bias

A related inquiry concerns the optimal spatial locations along the wing to capture strain dynamics. These locations would correspond to the physical placement of each strain sensor and campaniform sensilla on insect wings. Our previous simulations ignored any spatial correlation by testing classification for frequency content from random locations along the chord (columns of $\hat{S}$). We now introduce a spatial bias that draws strain frequency dynamics at random from Gaussian distributions centered at different regions of the wing chord and training with the remainder, using the same data considered previously in § 5.3. This essentially tests columns $\hat{s}$ that are adjacent in the data matrix $\hat{S}$. The different regions of the chord considered - the leading edge, 1/4-chord, midchord, 3/4-chord, and trailing - are important in the study of both insect wings and airfoils. The SRCf classifier is used with 35 random frequency samples in the frequency domain.

The expected performance profile for increasing sensor noise is observed in the results of Figure 12.
however there is an obvious classification advantage towards the trailing edge of the wing. This trend is consistent with downstream amplification of flow-induced strain towards the trailing edge, where the wing is most prone to deflection from the free boundary (12) and flexural stiffness is lowest. However, this finding deviates from observed campaniform sensilla locations that tend to be distributed away from the trailing edge. There are many possible explanations - the trailing edge is dominated by membrane surface without venal or neural conduits for the sensilla, or the non-stiff trailing edge of the wing may be prone to amplified flow process noise. Alternatively, the ensemble of campaniform sensilla may be sufficient to aggregate strain dynamics to amplify flow features without resorting to trailing edge sensors. Both scenarios are well worth examining in future research.

6. Conclusions and Outlook

This work develops a data-driven framework for flow environment identification based on supervised learning from simulated wing strain dynamics in the frequency domain. This is done using sparse classification in an overcomplete library of POD modes, which is assembled by simulating strain dynamics from each expected flow environment. Then, the problem of identifying the flow environment reduces to a classification task between the sets of POD modes in the library. The subset selection is accomplished by $L_1$ regularized, sparse approximation of POD library coefficients from input frequency dynamics. This $L_1$ regularization is extremely accurate and mitigates contributions from noise when compared to least squares approximation – even with sensor noise at 20% of the input signal’s magnitude, sparse classification is slightly biased towards the correct originating environment. Furthermore, SRCf with strategically subsampled signals is shown to be effective, with discriminating frequency measurements selected by the pivoted QR factorization of POD modes. In this manner, sparse classification of strain frequencies facilitates environment identification from single wing locations, providing insight into possible sparse encoding strategies employed by strain sensing neurons on insect wings. Moreover, our approach may be applied to sensor-equipped feedback systems in flight control, which is an active area of research.

6.1. Control Implications

Downstream environment identification from sensor data is an important problem in closed-loop control where sensors are expensive and measurements are corrupted by noise. Spatial strain mode shapes do not vary significantly across environments and require many more physical strain sensors to complete spatial knowledge. Frequency domain analysis of wing strain can help move toward autonomous flight technologies. Furthermore, the design of frequency samples chosen specifically to discriminate certain environments or frequencies greatly reduces the dimension of decision space and helps mitigate uncertainty in sensor measurements. Strain sensors and gauges are already being used to characterize stresses and guide wing and fin design in experiment (Kahn and Tangorra, 2015). As an added functionality they may assist in flow characterization using similar data-driven dimension reduction concepts as presented here. Our frequency distinction framework is particularly attractive for feedback analysis for controller decisions. Given suitable libraries of candidate frequency dynamics, the sparse classification framework can be efficiently implemented in flight controllers, especially when the decision space consists of only a few strategically selected frequencies. For example, controllers may mitigate undesirable frequencies encountered in flight such as the erratic high frequency content of turbulence and low frequencies induced by strong winds. Although we have not built a controller in this work, we note that convex optimization strategies, such as the one underlying sparse classification, are commonly employed in control applications.

6.2. Biological Implications

There are a number of intriguing biological conjectures that the current results help address. Specifically, the machine learning architecture of learned dictionaries and sparse sampling both seem to have advantageous properties in the context of flight. The learned libraries encoded from our numerical simulations suggest that insects possess a similar, innate library for flight dynamics. Indeed, insects at birth simply begin flying without a lengthy training stage, suggesting that flight dynamics, or dictionaries, are stereotyped behavior in the mechanosensory flight system (Cole and Palka, 1982; Gettrup, 1966; Dickerson et al., 2014). Thus at birth, a low-dimensional representation of mechanosensory codes are genetically inherited. The sparse and stereotyped placement of campaniform sensilla on the wings of a hawkmoth also suggest that sparse sampling does indeed occur for helping guide flight dynamics and control protocols. Of course, we have made no connection to control, but we certainly have demonstrated the tremendous bio-inspired advantages of sparsity and learned libraries.
6.3. Outlook

Here we propose several avenues of investigation aimed towards engineering and biology applications that exploit low-rank data structure. One direction concerns detecting and adapting this supervised learning and sparse classification framework to new, unseen classes of strain dynamics. In particular, turbulent and mixed frequency flow environments may introduce nonlinear decision boundaries between classes that may require a boosted classifier that could, for example, aggregate decisions from multiple spatial locations on the wing. The supervised library learning stage may be generalized to assemble modal representations other than POD, such as dynamic mode decomposition (DMD). The second component of measurement selection, empirical interpolation methods, would benefit from incorporation of measurement constraints that typically arise in engineering applications.

Furthermore, both components of this framework, sparse classification and measurement selection, stand to benefit from the quantification of uncertainty in environment identification. The assumption of white Gaussian sensor noise in our work is quite simplistic; more realistic sources of uncertainty may arise from process noise such as uncertainty in training data or spatially varying amplification of noise. Since this is an equation-free, data-driven approach to environment identification, more sophisticated learning strategies such as neural networks, decision trees, etc. may be employed in the training stage to identify and mitigate these sources of uncertainty.

This work is a piece that fits into a broader effort to understand neurosensory encoding for robust insect flight. In the present effort, a coupled fluid-structure interaction model was used to simulate instead loads on an insect-scale wing. However, applying this analysis to measurements of the strain on an actual wing presents an exciting avenue of future research. With increasingly small and inexpensive strain sensors, it may be possible to explore optimal sensor location in a robotic platform. Perhaps of greater interest, it may be possible in future studies to incorporate data directly from individual campaniform sensilla in a flying insect. It would also be interesting to compare the distribution of campaniform sensilla across insect wings to attempt to understand underlying biological optimization principles and goals.

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