Cross-ownership as a structural explanation for rising correlations in crisis times

Nils Bertschinger\textsuperscript{1} and Axel A. Araneda\textsuperscript{2}

\textsuperscript{1}Frankfurt Institute for Advanced Studies; D-60438 Frankfurt am Main, Germany.
\textsuperscript{2}Institute of Financial Complex Systems, Faculty of Economics and Administration, Masaryk University; 602 00 Brno, Czech Republic.

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Abstract

In this paper, we examine the interlinkages among firms through a financial network where cross-holdings on both equity and debt are allowed. We relate mathematically the correlation among equities with the unconditional correlation of the assets, the values of their business assets and the sensitivity of the network, particularly the $\Delta$-Greek. We noticed also this relation is independent of the Equities level. Besides, for the two-firms case, we analytically demonstrate that the equities correlation is always higher than the correlation of the assets; showing this issue by numerical illustrations. Finally, we study the relation between equity correlations and asset prices, where the model arrives to an increase in the former due to a fall in the assets.

1 Introduction

The famous Merton (1974) model relates debt and equity of a firm with European put and call options respectively. Since then it has been developed into an industry standard for structural credit risk modeling and management. However, the increasingly complex interlinkages between financial institutions are at odds with an individual and separate valuation of risk (see De Bandt and Hartmann (2000) for an early survey of systemic risk). Especially, the latest financial crisis has painfully revealed the danger of contagion throughout
the financial system and spurred a wealth of interest in theoretical models of systemic risk. In this regard, the model of Eisenberg and Noe (2001) and its “clearing payment vector” insight have been identified as the seminal contribution in the field, forming the basis for numerous studies of financial contagion arising from cross-ownership of debt (Cifuentes, Ferrucci, & Shin, 2005; Gai & Kapadia, 2010; Elliott, Golub, & Jackson, 2014). See Caccioli, Barucca, and Kobayashi (2018) and Sasidevan and Bertschinger (2019) for recent surveys of the network approach into systemic risk.

An interesting and alternative viewpoint is provided by Suzuki (2002). While the model can be interpreted as an extension of the Eisenberg and Noe (2001) model, it is better seen as an extension of the Merton model allowing for multiple firms with cross-ownership of debt as well as equity. Thereby, considering financial contagion as a problem of firm valuation where debt and equity have to be assessed in a self-consistent fashion, e.g. solving a fixed point via Picard iteration (Hain & Fischer, 2015). Furthermore, Suzuki explicitly solves the valuation problem in case of two financial institutions, conditional on the values of the business assets where banks are solvent or in default (Suzuki areas). For three or more banks a formal solution can still be written down, but requires a case distinction between exponentially many solvency regions and can no longer be visualized in two dimensions. Further developments on the Suzuki model extend it to debts of multiple seniorities (Fischer, 2014), address the (joint) default probabilities under this model (Karl & Fischer, 2014) or compute analytic bounds assuming comonotonic asset endowments (Banerjee & Feinstein, 2021).

On the empirical side, it is an established “stylized fact” that correlations rise during bear markets and crises times (Longin & Solnik, 2001; Ang & Chen, 2002; Kalkbrener & Packham, 2015). Indeed, Baig and Goldfajn (1999) found that during the Asian crisis, correlations in stock markets, interest rates, exchange rates and sovereign spreads rose significantly as compared to tranquil times. Onnela, Chakraborti, Kaski, Kertesz, and Kanto (2003) investigated the empirical distribution of pairwise stock correlation coefficients of stocks traded at NYSE, estimating correlations in a rolling window fashion, finding a substantial increase of their mean value around the Black Monday of Oct 1987. Preis, Kenett, Stanley, Helbing, and Ben-Jacob (2012) analyzed, with the time-varying correlations among the 30 stocks composing the DJIA index and demonstrate a linear relationship with market stress. Instead, Adams, Füss, and Glück (2017) argue, based on econometric consideration, that correlations change in a step-like fashion due to particular financial events (structural breaks). Similar arguments are put forward in more recent works of Choi and Shin (2019) and Demetrescu and Wied (2019).

In terms of modeling, the works of Cizeau, Potters, and Bouchaud (2001); Lillo and Mantegna (2000); Kyle and Xiong (2001) have addressed the correlation issue from a structural perspective. Another interesting
approach in this line is provided by Cont and Wagalath (2013) who evaluate how fire sales lead to *endogeneous*
correlations in a simple multi-period model. However, to our knowledge, it has not been approached in
the context of cross-holding networks. Here, we show that network models readily explain the correlation
stylized fact. In particular, we depart from a simply specification of the Suzuki model, and proof that it
exhibits structural changes in the correlation between firm equity values depending on solvency conditions.
Furthermore, we seek to understand and quantify the precise influence of different model parameters on the
observed correlation structure.

The remainder of the paper is structured as follows: First, we lay out the general network valuation
model following Suzuki (2002). Then, we interpret values of firm equity and debt as derivative contracts,
i.e. extending the Merton model to multiple firms, and link the correlation between these derivatives with the
Δ sensitivities, the asset prices, and the leverage. In turn, concentrating on the two-firms case, we proof that
the correlation among the two firm equities never falls below their unconditional asset correlation. Finally,
we illustrate our results through numerical simulations, showing in particular that correlations tend to rise
when equity prices drop. Thereby, providing a novel structural explanation of this well-known stylized fact.

2 Model

2.1 Notation and mathematical preliminaries

Here we quickly summarize the mathematical notation employed in this paper. We write vectors \( x, y \in \mathbb{R}^n \)
with bold lower case and matrices \( A, B \in \mathbb{R}^{m \times n} \) with bold upper case letters. Individual entries of vectors
and matrices are written as \( x_i, A_{ij} \). \( \text{diag}(x) \) denotes the \( n \times n \) diagonal matrix \( D \) with entries \( D_{ii} = x_i \) along
its diagonal. The transpose of a matrix is denoted as \( A^T \). All products containing vectors and matrices are
understood as standard matrix products, e.g. \( AB \) denotes the matrix product of \( A \) and \( B \), \( xx \) is undefined
whereas \( x^T x \) is the scalar product of \( x \) with itself. Row- and column-wise stacking of vectors or matrices
is denoted by \( (x; y) \) and \( (x, y) \) respectively, i.e. \( (x; y) \) is a \( 2n \)-dimensional vector whereas \( (x, y) \) is a \( n \times 2 \)
matrix.

Random variables \( X, Y \) are written as upper case letters with individual outcomes \( x, y \) denoted in lower
case. Expectations are denoted as \( \mathbb{E}[f(X)] \) and understood with respect to the (joint) distribution of random
variables within the brackets. Sometimes we use \( \mathbb{E}_t^Q \) to denote that the expectation is taken over the risk-
neutral measure \( Q \), implicitly conditioned on the information filtration \( \mathcal{F}_t \) up to time \( t \).
2.2 Network valuation

Merton (1974) has shown that equity and firm debt can be considered as call and put options on the firm’s value respectively. In this model, a single firm is holding externally priced assets $a$ and zero-coupon debt with nominal amount $d$ due at a single, fixed maturity $T$. Then, at time $T$ the value of equity $s$ and the recovery value of debt $r$ are given as

\[
s = \max\{0, a - d\} = (a - d)^+ ,
\]

\[
r = \min\{d, a\} = d - (d - a)^+
\]

corresponding to an implicit call and put option respectively.

Suzuki (2002) and others (Elsinger, 2009; Fischer, 2014) have since generalized this model to multiple firms with equity and debt cross-holdings. In this paper we consider $n$ firms. Each firm $i = 1, \ldots, n$ holds an external asset $a_i > 0$ as well as a fraction $M^e_{ij}$ of firm $j$’s equity and debt $M^d_{ij}$. Here, the investment fractions $M^e_{ij}$ and $M^d_{ij}$ are bounded between 0 and 1, i.e. $0 \leq M^{s,d}_{ij} \leq 1$, and the actual value invested in the equity of counterparty $j$ is given as $M^e_{ij}s_j$. In the following we require:

**Assumption 1.** There are no self-holdings, i.e. $M^e_{ii} = M^d_{ii} = 0$ for all $i = 1, \ldots, n$, nor short positions, i.e. $M^e_{ij}, M^d_{ij} \geq 0$ for all $i, j = 1, \ldots, n$. Moreover, we require that the total fractions equity and debt held by any counterparty cannot exceed unity. In addition, we assume that some of each firms equity and debt are held externally, i.e. for all $j = 1, \ldots, n$ it holds that

\[
\sum_i M^e_{ij} < 1 \quad \text{and} \quad \sum_i M^d_{ij} < 1.
\]

That is, $M^e$ and $M^d$ are strictly (left) sub-stochastic matrices. Alternatively, we can express this as $\|M^d\|_1, \|M^e\|_1 < 1$.

Now, the value of all assets $v_i$ held by firm $i$ is given by

\[
v_i = a_i + \sum_{j=1}^n M^e_{ij}s_j + \sum_{j=1}^n M^d_{ij}r_j .
\]
Correspondingly, the firm’s equity and recovery value of debt are given by

\[
s_i = \max \left\{ 0, a_i + \sum_j M_{ij}^s s_j + \sum_j M_{ij}^d r_j - d_i \right\}, \quad (5)
\]

\[
r_i = \min \left\{ d_i, a_i + \sum_j M_{ij}^s s_j + M_{ij}^d r_j \right\}. \quad (6)
\]

In matrix notation, i.e. collecting equity and debt values into vectors \( s = (s_1, \ldots, s_n)^T \) and \( r = (r_1, \ldots, r_n)^T \) respectively, this can be rewritten as

\[
s = \max \left\{ 0, a + M^s s + M^d r - d \right\}, \quad (7)
\]

\[
r = \min \left\{ d, a + M^s s + M^d r \right\} \quad (8)
\]

Thus, the firms’ equity and debt values are endogenously defined as the solution of a fixed point. This is readily seen when collecting equity and debt row-wise into a single vector \( x = (s; r) \), i.e. \( s = x_{1:n} \) and \( r = x_{(n+1):2n} \), and writing

\[
x = g(a, x) \quad (9)
\]

with the vector valued function \( g = (g_1^s, \ldots, g_n^s, g_1^r, \ldots, g_n^r)^T \) where for \( i = 1, \ldots, n \)

\[
g_i^s(a, x) = \max \left\{ 0, a_i + \sum_j M_{ij}^s x_j + \sum_j M_{ij}^d x_{n+j} - d_i \right\}, \quad (10)
\]

\[
g_i^r(a, x) = \min \left\{ d_i, a_i + \sum_j M_{ij}^s x_j + \sum_j M_{ij}^d x_{n+j} \right\}. \quad (11)
\]

Each of the functions \( g_i^s \) and \( g_i^r \) is continuous and increasing in \( a \) and \( x \). Together with assumption 1 it follows that the fixed point of (9) is positive and unique.

**Theorem 1.** Suppose that assumption 1 holds. Then, for each value of external assets \( a > 0 \) there is a positive and unique \( x \) solving (9).

**Proof.** Our model is a special case of the one considered by (Fischer, 2014) with \( k = 1 \) and \( d_{r1, r0}^1 \equiv d \). Furthermore, Fischer’s assumption 3.1 holds by assumption 1 and assumptions 3.6 and 3.7 are trivial as our
nominal debt vector $d$ is constant. The result then follows by his theorem 3.8 (iv).

3 Risk-neutral valuation

The celebrated Merton model exploits the connection of Eq. 9 with option prices to obtain the ex-ante market prices at time $t < T$ as

$$s_t = E_Q^t[e^{-r	au} S_T] = E_Q^t[e^{-r	au} (A_T - d)^+] \quad r_t = E_Q^t[e^{-r	au} R_T] = E_Q^t[e^{-r	au} (d - (d - A_T)^+)]$$

respectively. Furthermore, assuming a geometric Brownian motion for the price of the external assets, i.e.

$$dA_t = rA_t dt + \sigma_a A_t dW_Q^t$$

the corresponding stochastic differential equation for $s_t$ can be obtained via Ito’s lemma as

$$dS_t = \left( \frac{\partial s_t}{\partial t} + \frac{\partial s_t}{\partial a_t} r A_t + \frac{1}{2} \frac{\partial^2 s_t}{\partial a_t^2} \sigma_a^2 A_t^2 \right) dt + \frac{\partial s_t}{\partial a_t} \sigma_a A_t dW_Q^t.$$  \hspace{1cm} (15)

Matching the volatility with $\sigma_s S_t$ one obtains the well known relation

$$\sigma_s = \frac{\partial s_t}{\partial a_t} \frac{a_t}{s_t} = \sigma_a \Delta$$

between equity and asset volatility. Here, $\Delta = \frac{\partial s_t}{\partial a_t}$ is the option Delta and $\lambda = \frac{A_t}{s_t}$ its leverage.

3.1 Network valuation

Denoting the unique solution of equation (9) by $x^*(a)$, we can consider the corresponding value of equity and debt claims as derivative contracts on the underlying $a$. Accordingly, the ex-ante market price at time $t < T$ is given as

$$x_t = E_Q^t[e^{-r\tau} x^*(A_T)]$$

with the risk-less interest rate $r$ and time to maturity $\tau = T - t$. The expectation is taken with respect to the risk-neutral measure $Q$ of external asset values $a$ at maturity $T$. In the following, we assume that the
risk-neutral asset values follow a multi-variate geometric Brownian motion, i.e.

\[ dA_t = rA_t \, dt + \text{diag}(\sigma) \text{diag}(A_t) \, dW^Q_t \]  

with possibly correlated Wiener processes \( W^Q_t \), i.e. \( \mathbb{E}[dW^Q_{t,t'} \, dW^Q_{j,t}] = \rho_{ij} \, dt \) with \( \rho_{ii} = 1 \).

The well-known solution of equation (18) is given by

\[ A_t = a_0 e^{(r - \frac{1}{2} \text{diag}(\sigma^2))t + \text{diag}(\sigma)W_t} \]  

where \( a_0 > 0 \) denotes the initial value and \( W_t \) is multivariate normal distributed with mean \( 0 \) and covariance matrix \( tC \) with entries \( C_{ij} = \rho_{ij} \).

As before, via the multi-variate Ito formula we obtain

\[
dX_{i,t} = \left( \frac{\partial x_{i,t}}{\partial t} + r \left( \frac{\partial x_{i,t}}{\partial a_t} \right)^T A_t + \frac{1}{2} \text{Tr} \left( \text{diag}(A_t) \text{diag}(\sigma)^T \frac{\partial^2 x_{i,t}}{\partial a_t \partial a_t} \text{diag}(\sigma) \text{diag}(A_t) \right) \right) dt \\
+ \left( \frac{\partial x_{i,t}}{\partial a_t} \right)^T \text{diag}(\sigma) \text{diag}(A_t) \, dW^Q_t .
\]  

Then, again matching the volatility with \( \sigma_{i,x} x_{i,t} \) we compute the equity and debt volatilities

\[ \sigma_{i,x} = \frac{1}{x_{i,t}} \left( \frac{\partial x_{i,t}}{\partial a_t} \right)^T \text{diag}(\sigma) \text{diag}(a_t) \]  

or collecting all terms into a volatility matrix

\[ L_x = \text{diag}(x_t)^{-1} \left( \frac{\partial x_t}{\partial a_t} \right)^T \text{diag}(\sigma) \text{diag}(a_t) . \]  

Note that the instantaneous covariance matrix of \( x_t \) at time \( t \) is then given by

\[ \Sigma_x = L_x tC L_x^T \]  

\[ = \text{diag}(x_t)^{-1} \left( \frac{\partial x_t}{\partial a_t} \right)^T \text{diag}(\sigma) \text{diag}(a_t) tC \text{diag}(a_t) \text{diag}(\sigma) \left( \frac{\partial x_t}{\partial a_t} \right) \text{diag}(x_t)^{-1} \]  

\[ = \text{diag}(x_t)^{-1} \left( \frac{\partial x_t}{\partial a_t} \right)^T \text{diag}(a_t) \Sigma_a \text{diag}(a_t) \left( \frac{\partial x_t}{\partial a_t} \right) \text{diag}(x_t)^{-1} \]  

where \( \Sigma_a = \text{diag}(\sigma) tC \text{diag}(\sigma)^T \) denotes the instantaneous asset covariance at time \( t \). This generalizes
equation (16) with the Delta matrix \( \Delta_x = \frac{\partial x}{\partial a_t} \) and \( \text{diag}(\mathbf{x}_t)^{-1}, \text{diag}(\mathbf{a}_t) \) acting as leverage. In contrast, to the uni-variate case these cannot be collected into a leverage matrix as \( \text{diag}(\mathbf{x}_t)^{-1} \) is multiplied from the left, i.e. acts on the rows, whereas \( \text{diag}(\mathbf{a}_t) \) is multiplied from the right, acts on the columns.

4 Two bank case

For two banks, i.e. \( i = 1, 2 \), the fixed point equations 10 for equity and debt can be solved explicitly. In particular, Suzuki (2002) has shown that the value of the equity and debt, at maturity, depends on the solvency conditions of the firms. Here, firm \( i \) is solvent (insolvent) if its total value \( v_i \) exceeds (falls short of) its nominal debt, i.e. \( v_{i,T} \geq (<) d_i \). Following Suzuki, we define four regions (Suzuki areas) which consider the combinatory of solvency or default condition at maturity Suzuki (2002); Karl and Fischer (2014):

\[
\Xi_{ss} = \left\{ (a_{1,T}, a_{2,T}) \in \mathbb{R}^2_+ : v_{1,T} \geq d_1 \land v_{2,T} \geq d_2 \right\}
\]

\[
\Xi_{sd} = \left\{ (a_{1,T}, a_{2,T}) \in \mathbb{R}^2_+ : v_{1,T} \geq d_1 \land v_{2,T} < d_2 \right\}
\]

\[
\Xi_{ds} = \left\{ (a_{1,T}, a_{2,T}) \in \mathbb{R}^2_+ : v_{1,T} < d_1 \land v_{2,T} \geq d_2 \right\}
\]

\[
\Xi_{dd} = \left\{ (a_{1,T}, a_{2,T}) \in \mathbb{R}^2_+ : v_{1,T} < d_1 \land v_{2,T} < d_2 \right\}
\]

After that, and using simply assumptions Suzuki (2002), we have a fix-point-solution for the system 5-6 conditional to each Suzuki area, given by:

\[
s_{1,T} = \begin{cases} 
\frac{a_{1,T} - d_1 + M_{12}d_2 + M_{11}(a_{2,T} - d_2 + M_{21}d_1)}{1 - M_{12}M_{21}}, & (a_{1,T}, a_{2,T}) \in \Xi_{ss} \\
\frac{a_{1,T} - d_1 + M_{12}d_2 + M_{11}(a_{2,T} - d_2 + M_{21}d_1)}{1 - M_{12}M_{21}}, & (a_{1,T}, a_{2,T}) \in \Xi_{sd} \\
0, & (a_{1,T}, a_{2,T}) \in \Xi_{ds} \\
0, & (a_{1,T}, a_{2,T}) \in \Xi_{dd}
\end{cases}
\]
\[
\begin{align*}
    s_{2,T} &= \begin{cases}
        a_2, T - d_2 + M_{21}^d (a_1, T - d_1 + M_{12}^d d_2) / (1 - M_{12}^d M_{21}^d), & (a_1, T, a_2, T) \in \Xi_{ss} \\
        0, & (a_1, T, a_2, T) \in \Xi_{sd} \\
        a_2, T - d_2 + M_{21}^d (a_1, T - d_1 + M_{12}^d d_2) / (1 - M_{12}^d M_{21}^d), & (a_1, T, a_2, T) \in \Xi_{ds} \\
        0, & (a_1, T, a_2, T) \in \Xi_{dd}
    \end{cases} \\
\end{align*}
\]

\[
\begin{align*}
r_{1,T} &= \begin{cases}
        d_1, & (a_1, T, a_2, T) \in \Xi_{ss} \\
        d_1, & (a_1, T, a_2, T) \in \Xi_{sd} \\
        a_1, T + M_{12}^d d_2 + M_{12}^s (a_2, T - d_2) / (1 - M_{12}^d M_{21}^d), & (a_1, T, a_2, T) \in \Xi_{ds} \\
        a_1, T + M_{12}^d a_2, T / (1 - M_{12}^d M_{21}^d), & (a_1, T, a_2, T) \in \Xi_{dd}
    \end{cases} \\
\end{align*}
\]

\[
\begin{align*}
r_{2,T} &= \begin{cases}
        d_2, & (a_1, T, a_2, T) \in \Xi_{ss} \\
        a_2, T + M_{21}^d d_2 + M_{21}^s (a_1, T - d_1) / (1 - M_{21}^d M_{12}^d), & (a_1, T, a_2, T) \in \Xi_{sd} \\
        a_2, T + M_{21}^d a_1, T / (1 - M_{21}^d M_{12}^d), & (a_1, T, a_2, T) \in \Xi_{dd}
    \end{cases} \\
\end{align*}
\]

4.1 Computing correlations

Consider two assets with covariance matrix

\[
\Sigma = \begin{pmatrix}
    \sigma_1^2 & \sigma_1 \sigma_2 \rho \\
    \sigma_1 \sigma_2 \rho & \sigma_2^2
\end{pmatrix},
\]

i.e. with volatilities \(\sigma_1, \sigma_2\) and correlation \(\rho\).

The Cholesky factor \(L\) with \(\Sigma = LL^T\) is then given by

\[
L = \begin{pmatrix}
    \sigma_1 & 0 \\
    \sigma_2 \rho & \sigma_2 \sqrt{1 - \rho^2}
\end{pmatrix} = \begin{pmatrix}
    l_{11} & 0 \\
    l_{21} & l_{22}
\end{pmatrix}.
\]

In particular, the correlation coefficient can be expressed directly in terms of the Cholesky coefficients as

\[
\rho = \frac{l_{21}}{\sqrt{l_{21}^2 + l_{22}^2}} = \frac{1}{\sqrt{1 + \left(\frac{l_{22}}{l_{21}}\right)^2}}.
\]

\[9\]
Furthermore, for the factor of the equity covariances we obtain

\[ L^s = \text{diag}(s)^{-1} \frac{\partial s}{\partial a} \text{diag}(a) L \]

or explicitly

\[ l^s_{ij} = \sum_k \frac{\Delta_{ik} a_k}{s_i} l_{kj} \] \hspace{1cm} (32)

where \( \Delta_{ij} = \frac{\partial a_i}{\partial a_j} \). Note that this is not a Cholesky factor, as it will not be lower triangular in general.

Nevertheless, we have \( \Sigma^s = L^s (L^s)^T \) or explicitly

\[ \sigma^s_{ij} = \sum_k l^s_{ik} l^s_{jk} \]

and therefore for the correlation coefficient

\[ \rho^s = \frac{\sigma^s_{12}}{\sqrt{\sigma^s_{11} \sigma^s_{22}}} \]

\[ = \frac{l^s_{11} l^s_{21} + l^s_{12} l^s_{22}}{\sqrt{((l^s_{11})^2 + (l^s_{12})^2)((l^s_{21})^2 + (l^s_{22})^2)}} \]

\[ = \text{sign}(l^s_{11} l^s_{21} + l^s_{12} l^s_{22}) \sqrt{\frac{(l^s_{11} l^s_{21})^2 + 2l^s_{11} l^s_{12} l^s_{21} l^s_{22} + (l^s_{12} l^s_{22})^2}{(l^s_{21})^2 + (l^s_{12} l^s_{22})^2}} \]

Assuming that \( l^s_{11} l^s_{21} + l^s_{12} l^s_{22} \) is nonzero, we can rewrite the above equation using a quadratic extension as

\[ \rho^s = \text{sign}(l^s_{11} l^s_{21} + l^s_{12} l^s_{22}) \sqrt{\frac{(l^s_{11} l^s_{21})^2 + 2l^s_{11} l^s_{12} l^s_{21} l^s_{22} + (l^s_{12} l^s_{22})^2}{(l^s_{21})^2 + (l^s_{12} l^s_{22})^2}} \]

\[ = \text{sign}(l^s_{11} l^s_{21} + l^s_{12} l^s_{22}) \sqrt{\frac{(l^s_{11} l^s_{21})^2 + 2l^s_{11} l^s_{12} l^s_{21} l^s_{22} + (l^s_{12} l^s_{22})^2}{(l^s_{12})^2 + (l^s_{11} l^s_{21})^2 + (l^s_{12} l^s_{22})^2}} \]

\[ = \text{sign}(l^s_{11} l^s_{21} + l^s_{12} l^s_{22}) \sqrt{\frac{1}{1 + \frac{(l^s_{11} l^s_{22} - l^s_{12} l^s_{21})^2}{(l^s_{11} l^s_{21})^2 + (l^s_{12} l^s_{22})^2}}} \] \hspace{1cm} (33)
To compute the correlation coefficient we start with equation (32) and obtain

$$l_{11}^s = \frac{1}{s_1}(\Delta_{11}a_1l_{11} + \Delta_{12}a_2l_{21}) = \frac{1}{s_1}(\Delta_{11}a_1\sigma_1 + \Delta_{12}a_2\sigma_2)$$

$$l_{12}^s = \frac{1}{s_1}(\Delta_{11}a_1l_{12} + \Delta_{12}a_2l_{22}) = \frac{1}{s_1}\Delta_{12}a_2\sigma_2\sqrt{1 - \rho^2}$$

$$l_{21}^s = \frac{1}{s_2}(\Delta_{21}a_1l_{11} + \Delta_{22}a_2l_{21}) = \frac{1}{s_2}(\Delta_{21}a_1\sigma_1 + \Delta_{22}a_2\sigma_2)$$

$$l_{22}^s = \frac{1}{s_2}(\Delta_{21}a_1l_{12} + \Delta_{22}a_2l_{22}) = \frac{1}{s_2}\Delta_{22}a_2\sigma_2\sqrt{1 - \rho^2}$$

where the simplified expressions follow from \(l_{12} = 0\).

Overall, we obtain for the relevant terms

$$l_{11}^s l_{22}^s - l_{12}^s l_{21}^s = \frac{1}{s_1 s_2}(\Delta_{11}a_1\Delta_{22}a_2 - \Delta_{12}a_2\Delta_{21}a_1)\sigma_1\sigma_2\sqrt{1 - \rho^2}$$

(34)

$$l_{11}^s l_{22}^s = \frac{1}{s_1 s_2}(\Delta_{11}a_1\sigma_1 + \Delta_{12}a_2\sigma_2)(\Delta_{21}a_1\sigma_1 + \Delta_{22}a_2\sigma_2)$$

$$l_{12}^s l_{21}^s = \frac{1}{s_1 s_2}(\Delta_{11}a_1\Delta_{21}a_1\sigma_1^2 + 2\Delta_{12}a_2\Delta_{21}a_1\sigma_2\rho + \Delta_{12}a_2\Delta_{22}a_1\sigma_1\sigma_2 + \Delta_{12}a_2\Delta_{22}a_2\sigma_2^2\rho^2)$$

(35)

$$l_{11}^s l_{22}^s + l_{12}^s l_{21}^s = \frac{1}{s_1 s_2}(\Delta_{11}a_1\Delta_{21}a_1\sigma_1^2 + (\Delta_{11}a_1\Delta_{22}a_2 + \Delta_{12}a_2\Delta_{21}a_1)\sigma_1\sigma_2 + \Delta_{12}a_2\Delta_{22}a_2\sigma_2^2)$$

and therefore

$$\frac{1}{(\rho^*)^2} = 1 + \frac{(\Delta_{11}a_1\Delta_{22}a_2 - \Delta_{12}a_2\Delta_{21}a_1)\sigma_1\sigma_2\sqrt{1 - \rho^2}}{(\Delta_{11}a_1\Delta_{21}a_1\sigma_1^2 + (\Delta_{11}a_1\Delta_{22}a_2 + \Delta_{12}a_2\Delta_{21}a_1)\sigma_1\sigma_2 + \Delta_{12}a_2\Delta_{22}a_2\sigma_2^2)^2}$$

$$= 1 + \frac{(1 - \Delta_{11}a_1\Delta_{22}a_2 + \Delta_{12}a_2\Delta_{21}a_1)\rho + \Delta_{12}a_2\Delta_{22}a_2\sigma_2^2\rho}{\Delta_{21}a_1\sigma_1 + (1 + \Delta_{12}a_2\Delta_{21}a_1)\rho + \Delta_{12}a_2\Delta_{22}a_2\sigma_2^2\rho^2}$$

(36)

where we have used that \(\Delta_{11}, \Delta_{22}, a_1, a_2, \sigma_1, \sigma_2 > 0\). As before, the sign of the correlation coefficient is given
by the sign of $l_{11}^s l_{21}^s + l_{12}^s l_{22}^s$.

Finally, the equity $\Delta$’s are given as (see appendix A for a detail computation):

\[
\begin{align*}
\Delta_{11} &= \mathbb{E}^Q \left[ \frac{1}{1 - M_{12}^s M_{21}^s} \frac{A_{1,T}}{a_1} \Xi_{ss} \right] + \frac{1}{1 - M_{12}^d M_{21}^d} \frac{A_{1,T}}{a_1} \Xi_{sd} \\
&= \pi_{ss} \frac{1}{1 - M_{12}^s M_{21}^s} \mathbb{E}^Q \left[ \frac{A_{1,T}}{a_1} \Xi_{ss} \right] + \pi_{sd} \frac{1}{1 - M_{12}^d M_{21}^d} M_{12}^d \mathbb{E}^Q \left[ \frac{A_{1,T}}{a_1} \Xi_{sd} \right] \\
\Delta_{12} &= \mathbb{E}^Q \left[ \frac{1}{1 - M_{12}^s M_{21}^s} M_{12}^s \frac{A_{2,T}}{a_2} \Xi_{ss} \right] + \frac{1}{1 - M_{12}^d M_{21}^d} M_{12}^d \frac{A_{2,T}}{a_2} \Xi_{sd} \\
&= \pi_{ss} \frac{1}{1 - M_{12}^s M_{21}^s} M_{12}^s \mathbb{E}^Q \left[ \frac{A_{2,T}}{a_2} \Xi_{ss} \right] + \pi_{sd} \frac{1}{1 - M_{12}^d M_{21}^d} M_{12}^d \mathbb{E}^Q \left[ \frac{A_{2,T}}{a_2} \Xi_{sd} \right] \\
\Delta_{21} &= \mathbb{E}^Q \left[ \frac{1}{1 - M_{12}^s M_{21}^s} \frac{A_{1,T}}{a_1} \Xi_{ss} \right] + \frac{1}{1 - M_{12}^d M_{21}^d} \frac{A_{1,T}}{a_1} \Xi_{ds} \\
&= \pi_{ss} \frac{1}{1 - M_{12}^s M_{21}^s} \mathbb{E}^Q \left[ \frac{A_{1,T}}{a_1} \Xi_{ss} \right] + \pi_{ds} \frac{1}{1 - M_{12}^d M_{21}^d} \mathbb{E}^Q \left[ \frac{A_{1,T}}{a_1} \Xi_{ds} \right] \\
\Delta_{22} &= \mathbb{E}^Q \left[ \frac{1}{1 - M_{12}^s M_{21}^s} \frac{A_{2,T}}{a_2} \Xi_{ss} \right] + \frac{1}{1 - M_{12}^d M_{21}^d} \frac{A_{2,T}}{a_2} \Xi_{ds} \\
&= \pi_{ss} \frac{1}{1 - M_{12}^s M_{21}^s} \mathbb{E}^Q \left[ \frac{A_{2,T}}{a_2} \Xi_{ss} \right] + \pi_{ds} \frac{1}{1 - M_{12}^d M_{21}^d} \mathbb{E}^Q \left[ \frac{A_{2,T}}{a_2} \Xi_{ds} \right].
\end{align*}
\]

4.2 Special cases and theorems

**Theorem 2.** The equity correlation $\rho^s$ exceeds the asset correlation $\rho$, i.e. $\rho^s \geq \rho$.

**Proof.** Here, we consider several cases.

\(\rho = 0\): Then, by equation (35)

\[
l_{11}^s l_{21}^s + l_{12}^s l_{22}^s = \frac{1}{s_1 s_2} (\Delta_{11} a_1 \Delta_{21} a_1 \sigma_1^2 + \Delta_{12} a_2 \Delta_{22} a_2 \sigma_2^2) \geq 0
\]

and therefore $\rho^s \geq 0$ as well.

\(\rho > 0\): Then, by equation (35) we find that $l_{11}^s l_{21}^s + l_{12}^s l_{22}^s > 0$ as well and therefore from 33 $\rho^s > 0$. 

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Furthermore, we compute

\[
\rho_s \geq \rho \iff \frac{1}{(\rho^*)^2} \leq \frac{1}{\rho^2}
\]

\[
1 + \left(\frac{x}{y}\right)^2 \leq \frac{1}{\rho^2}
\]

\[
\left(\frac{x}{y}\right)^2 \leq \frac{1}{\rho^2 - 1}
\]

\[
\left(\frac{x}{y}\right)^2 \leq \frac{1 - \rho^2}{\rho^2}
\]

where \(x = (1 - \frac{\Delta_{12}\Delta_{21}}{\Delta_{11}\Delta_{22}})\sqrt{1 - \rho^2}\) and \(y = \frac{\Delta_{21}\sigma_1}{\Delta_{22}\sigma_2} + (1 + \frac{\Delta_{12}\Delta_{21}}{\Delta_{11}\Delta_{22}})\rho + \frac{\Delta_{12}\sigma_2}{\Delta_{11}\sigma_1}\). Continuing, we reason

\[
\frac{x^2}{y^2} = \frac{(1 - \frac{\Delta_{12}\Delta_{21}}{\Delta_{11}\Delta_{22}})^2(1 - \rho^2)}{\left(\frac{\Delta_{21}\sigma_1}{\Delta_{22}\sigma_2} + (1 + \frac{\Delta_{12}\Delta_{21}}{\Delta_{11}\Delta_{22}})\rho + \frac{\Delta_{12}\sigma_2}{\Delta_{11}\sigma_1}\right)^2}
\]

\[
\leq \frac{(1 - \frac{\Delta_{12}\Delta_{21}}{\Delta_{11}\Delta_{22}})^2}{(1 + \frac{\Delta_{12}\Delta_{21}}{\Delta_{11}\Delta_{22}})^2} \cdot \frac{1 - \rho^2}{\rho^2}
\]

\[
\leq \frac{1 - \rho^2}{\rho^2}
\]

as required.

\[\rho < 0: \text{In this case, whenever } l_{11}^s l_{21}^s + l_{12}^s l_{22}^s \geq 0 \text{ we clearly have } \rho_s > \rho > 0. \text{ Thus, we assume that}
\]

\[l_{11}^s l_{21}^s + l_{12}^s l_{22}^s < 0
\]

\[\iff \Delta_{11} a_1 \Delta_{21} a_1\sigma_1^2 + (\Delta_{11} a_1 \Delta_{22} a_2 + \Delta_{12} a_2 \Delta_{21} a_1)\sigma_1 \sigma_2 \rho + \Delta_{12} a_2 \Delta_{22} a_2 \sigma_2^2 < 0
\]

\[\iff \frac{\Delta_{11} a_1 \Delta_{21} a_1 \sigma_1^2 + \Delta_{12} a_2 \Delta_{22} a_2 \sigma_2^2}{(\Delta_{11} a_1 \Delta_{22} a_2 + \Delta_{12} a_2 \Delta_{21} a_1)\sigma_1 \sigma_2} < -\rho
\]

\[\iff \frac{\Delta_{11} a_1 \sigma_1^2 + \Delta_{12} a_2 \sigma_2^2}{1 + \Delta_{12} a_2 \Delta_{21} a_1} < -\rho.
\]

Now, defining \(x = \frac{\Delta_{12} a_1}{\Delta_{11} \Delta_{22}}\) and \(y = \frac{\Delta_{21} a_1 \sigma_1}{\Delta_{22} a_2 \sigma_2} + \frac{\Delta_{12} a_2 \sigma_2}{\Delta_{11} a_1 \sigma_1}\) the above equation reads

\[
\frac{y}{1 + x} < -\rho \iff y < -\rho(1 + x).
\]
The desired result $\rho^* \geq \rho$ than follows if we can show that

$$
\frac{1}{(\rho^*)^2} = 1 + \frac{(1-x)^2(1-\rho^2)}{(y+(1+x)\rho)^2} \geq \frac{1}{\rho^2}
$$

$$
\Rightarrow \frac{(1-x)^2(1-\rho^2)}{(y+(1+x)\rho)^2} \geq \frac{1-\rho^2}{\rho^2}
$$

$$
\Rightarrow (y+(1+x)\rho)^2 \leq (1-x)^2\rho^2
$$

$$
\Rightarrow y^2 + 2y(1+x)\rho + \rho^2 (1-(1-x)^2) \leq 0.
$$

From equation (45) we have that

$$
y^2 + 2y(1+x)\rho + \rho^2 (1-(1-x)^2) < \rho^2 (1+x)^2 - 2(1+x)\rho(1+x)\rho + \rho^2 (1-(1-x)^2)
$$

$$
= \rho^2 (1-(1-x)^2 - (1+x)^2)
$$

$$
= -\rho^2 (1+2x^2)
$$

which is obviously negative and thereby completes the proof.

\[\Box\]

**Higher values:** Consider the limit $a_1, a_2 \to \infty$. Then,

$$
\Delta_{12} = \frac{\Delta_{11}}{M_1} = \frac{1}{\pi_{ss} M_1^2 M_2^2} M_{12}^s \mathbb{E} Q \left[ \frac{A_{2,T}}{a_2} \mid \Xi_{ss} \right] + \frac{1}{\pi_{sd} M_1^2 M_2^2} M_{12}^d \mathbb{E} Q \left[ \frac{A_{2,T}}{a_2} \mid \Xi_{sd} \right]
$$

$$
\pi_{ss} \frac{1}{1-M_1^2 M_2^2} \mathbb{E} Q \left[ \frac{A_{1,T}}{a_1} \mid \Xi_{ss} \right] + \pi_{sd} \frac{1}{1-M_1^2 M_2^2} \mathbb{E} Q \left[ \frac{A_{1,T}}{a_1} \mid \Xi_{sd} \right]
$$

$$
= \frac{1}{1-M_1^2 M_2^2} M_{12}^s \mathbb{E} Q \left[ \frac{A_{2,T}}{a_2} \mid \Xi_{ss} \right] + \frac{1}{1-M_1^2 M_2^2} M_{12}^d \mathbb{E} Q \left[ \frac{A_{2,T}}{a_2} \mid \Xi_{sd} \right]
$$

$$
\pi_{ss} \frac{1}{1-M_1^2 M_2^2} \mathbb{E} Q \left[ \frac{A_{1,T}}{a_1} \mid \Xi_{ss} \right] + \pi_{sd} \frac{1}{1-M_1^2 M_2^2} \mathbb{E} Q \left[ \frac{A_{1,T}}{a_1} \mid \Xi_{sd} \right]
$$

$$
\mathbb{E} Q \left[ \frac{A_{2,T}}{a_2} \mid \Xi_{ss} \right]
$$

$$
\mathbb{E} Q \left[ \frac{A_{2,T}}{a_2} \mid \Xi_{ss} \right] \mathbb{E} Q \left[ \frac{A_{1,T}}{a_1} \mid \Xi_{ss} \right]
$$

(46)

as $\pi_{sd} \to 0, \pi_{ss} \to 1$ for $a_1, a_2 \to \infty$ and the conditional expectations $\mathbb{E} Q \left[ \frac{A_{1,T}}{a_1} \mid \Xi_{ss} \right], \mathbb{E} Q \left[ \frac{A_{2,T}}{a_2} \mid \Xi_{ss} \right]$ are constants which do not depend on $a_1, a_2$. 

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Similarly,

\[
\Delta_{21} \Delta_{22} = \frac{1}{\pi ss M_{1} M_{2} \frac{1}{M_{2}} M_2 \mathbb{E} Q \left[ \frac{A_{1} T_{a_{1}}}{\Xi} \right]} + \frac{1}{\pi ds \frac{1}{M_{2}} M_{2} \mathbb{E} Q \left[ \frac{A_{2} T_{a_{2}}}{\Xi} \right]}
\]

\[
\frac{\Delta_{11}^{21} \Delta_{11}^{22} \Delta_{21} \Delta_{22}^{2} \mathbb{E} Q \left[ \frac{A_{1} T_{a_{1}}}{\Xi} \right]}{\Xi_{ss}} + \frac{\Delta_{21}^{21} \Delta_{22} \mathbb{E} Q \left[ \frac{A_{2} T_{a_{2}}}{\Xi} \right]}{\Xi_{ss}}.
\]

(47)

Yet, in general, the limit of \( \rho^s \) does not exist as it depends explicitly on \( \frac{a_{1}}{a_{2}} \) and thereby on the specific path along which \( a_{1}, a_{2} \) are taken to infinity.

4.2.1 Debt cross-holdings only

Assuming debt cross-holdings only, i.e. \( M^s = 0 \), we obtain the following proposition.

**Proposition 1.** Assuming debt cross-holdings only, we have

\[
\rho^s \xrightarrow{a_{1}, a_{2} \to \infty} \rho.
\]

**Proof.** First we show that in the limit \( a_{1}, a_{2} \to \infty \) we have \( (\rho^s)^2 \xrightarrow{a_{1}, a_{2} \to \infty} \rho^2 \). For this note that from the considerations above (equation (46) and equation (47)),

\[
\Delta_{11}^{21} \xrightarrow{a_{1}, a_{2} \to \infty} 0
\]

\[
\Delta_{21} \xrightarrow{a_{1}, a_{2} \to \infty} 0
\]

and thus

\[
\frac{1}{(\rho^s)^2} = 1 + \left( 1 - \frac{\Delta_{11}^{21} \Delta_{12} \Delta_{22}}{\Delta_{21} \Delta_{22} \sigma_{1} \sigma_{2}} + (1 + \frac{\Delta_{12} \Delta_{21}}{\Delta_{11} \sigma_{2}}) \rho + \frac{\Delta_{12} \Delta_{21}}{\Delta_{11} \sigma_{1}} \rho \right)^2
\]

\[
\xrightarrow{a_{1}, a_{2} \to \infty} 1 + \left( \frac{\sqrt{1 - \rho^2}}{0 + \rho + 0} \right)^2
\]

\[
\rho^2 + 1 - \rho^2 = \frac{1}{\rho^2}
\]

along any path at which \( \frac{a_{1}}{a_{2}} \) and \( \frac{a_{2}}{a_{1}} \) stay bounded.
Furthermore, by the same argument \( \frac{\Delta_{11}a_1}{\Delta_{22}a_2\sigma_2} + (1 + \frac{\Delta_{12}a_2}{\Delta_{11}a_1\sigma_1})\rho + \frac{\Delta_{12}a_2\sigma_2}{\Delta_{11}a_1\sigma_1} \to \rho \) meaning that the signs of \( \rho^* \) and \( \rho \) eventually agree and we conclude that \( \rho^* \to \rho \).  

4.2.2 Merton model

Sanity check of special cases

- No network; i.e., \( M^* = M^d \equiv 0 \). Then,

\[
\begin{align*}
\frac{l_{22}^*}{l_{21}^*} &= \left( \frac{\pi_{ss}E^Q[A_{2,T}\mid \Xi_{ss}] + \pi_{ds}E^Q[A_{2,T}\mid \Xi_{ds}]}{\pi_{ss}E^Q[A_{2,T}\mid \Xi_{ss}] + \pi_{ds}E^Q[A_{2,T}\mid \Xi_{ds}]} \right) \sigma_2 \sqrt{1 - \rho^2} = \frac{\sqrt{1 - \rho^2}}{\rho} \\
\rho^* &= \sqrt{1 + \frac{\rho^2}{\sigma^2}} = \rho
\end{align*}
\]

- Volatility and the Merton model. According to equation (30) we have \( \sigma_1 = l_{11} \) and therefore using equation (32) again

\[
\begin{align*}
\sigma_1^* &= \frac{\Delta_{11}a_1}{s_1} l_{11} + \frac{\Delta_{12}a_2}{s_1} l_{21} \\
&= \frac{\pi_{ss}E^Q[A_{1,T}\mid \Xi_{ss}] + \pi_{sd}E^Q[A_{1,T}\mid \Xi_{sd}]}{\pi_{ss}E^Q[A_{1,T}\mid \Xi_{ss}] - d_1 + M_{12}^s d_2 + M_{12}^d d_1} \cdot \frac{\pi_{sd}E^Q[A_{1,T}\mid \Xi_{sd}]}{1 - M_{12}^s M_{21}} \\
&= \frac{\pi_{ss}E^Q[A_{1,T}\mid \Xi_{ss}] + \pi_{sd}E^Q[A_{1,T}\mid \Xi_{sd}] - d_1 + M_{12}^s d_2 + M_{12}^d d_1}{\pi_{ss}E^Q[A_{1,T}\mid \Xi_{ss}] - d_1 + M_{12}^s d_2 + M_{12}^d d_1} \\
&= \frac{\pi_{ss}E^Q[A_{1,T}\mid \Xi_{ss}] + \pi_{sd}E^Q[A_{1,T}\mid \Xi_{sd}]}{\pi_{ss}E^Q[A_{1,T}\mid \Xi_{ss}] - d_1 + M_{12}^s d_2 + M_{12}^d d_1} \\n&= \frac{\pi_{ss}E^Q[A_{1,T}\mid \Xi_{ss}] + \pi_{sd}E^Q[A_{1,T}\mid \Xi_{sd}]}{\pi_{ss}E^Q[A_{1,T}\mid \Xi_{ss}] - d_1 + M_{12}^s d_2 + M_{12}^d d_1} \\
\end{align*}
\]

which should reduce to the standard Merton model formulas without a network:

\[
\begin{align*}
\sigma_1^* &= \frac{\pi_{ss}E^Q[A_{1,T}\mid \Xi_{ss}] + \pi_{sd}E^Q[A_{1,T}\mid \Xi_{sd}]}{\pi_{ss}E^Q[A_{1,T}\mid \Xi_{ss}] - d_1 + \pi_{sd}E^Q[A_{1,T}\mid \Xi_{sd}]} \cdot \frac{\sigma_1}{d_1} \\
&= \frac{\sigma_1}{1 - \pi_{ss}E^Q[A_{1,T}\mid \Xi_{ss}] - d_1 + \pi_{sd}E^Q[A_{1,T}\mid \Xi_{sd}]}
\end{align*}
\]

Translating to more standard notation, i.e. \( A_T = A_{1,T} \) and \( K = d_1 \), and using that the conditional expectation \( E^Q[A_T\mid A_T \geq K] \) can be computed as

\[
E^Q[A_T\mid A_T \geq K] = e^{\mu B S + \frac{\sigma^2}{2}} \Phi(\frac{\mu B S + \sigma^2}{\sigma B S} \frac{\ln K}{\sigma B S}) - \Phi(\frac{\mu B S}{\sigma B S})
\]

which reduces to

\[
= a_1 e^{r(T-t)} \frac{\Phi(d_+)}{\Phi(d_-)}
\]

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where $\mu_{BS} = (r - \frac{\sigma^2}{2})(T - t) + \ln a_t$ and $\sigma_{BS} = \sigma\sqrt{T - t}$. Further, $d_{\pm}$ denotes the familiar terms

$$d_{\pm} = \frac{\ln \frac{a_t}{K} + (r \pm \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}$$

and $\Phi$ the cumulative distribution function of a standard normal. Thus, plugging everything together we obtain the standard result

$$\sigma^s = \frac{\sigma}{1 - e^{2\sigma^2 \rho A_t^1 A_t^2 \mid A_t^1 \mid}}$$

$$= \frac{\sigma A_t \Phi(d_+)}{1 - a_t e^{-r(T - t)} \Phi(d_+)}$$

$$= \frac{a_t \Phi(d_+)}{a_t \Phi(d_+) - e^{-r(T - t)} K \Phi(d_-) \sigma}$$

$$= \Delta_{BS} \frac{a_t}{c_{BS}} \sigma$$

with the Black-Scholes Delta $\Delta_{BS} = \Phi(d_+)$ and call price $c_{BS} = a_t \Phi(d_+) - e^{-r(T - t)} K \Phi(d_-)$.

### 4.3 Numerical illustrations

#### 4.3.1 Equity correlation as function of the network parameters

One of the central results of this paper is given by the Theorem 2; i.e., $\rho^s \geq \rho$ for two firms. Here, we illustrate it numerically by plotting the equity correlation for different values of asset correlations, initial prices, volatilities and cross-holding fractions. For the sake of simplicity, we consider cross-holdings of debt only ($M^s = 0$) and symmetric initial conditions for the assets\(^1\), i.e., $\sigma_1 = \sigma_2 = \sigma$ and $a_{1,0} = a_{2,0} = a_{1,2}$. Figure 1 show the resulting equity correlation as a function of firm 1’s equity\(^2\), for different debt cross-holding fractions, asset correlations and volatilities. Here, the subplots correspond to cross-holding fractions of $M^d_{21} = 0, 0.2, \ldots, 0.8$ (vertical) and $M^d_{12} = 0, 0.2, \ldots, 0.8$ (horizontal). Furthermore, solid lines represent an asset volatility value of 0.2 while dotted lines a volatility value of 0.4. The colors correspond to different values of asset correlations $\rho_a = \{-0.4, 0, 0.4, 0.8\}$. As proved above (page 16), in case of no cross-holdings (left and upper subplot) we find $\rho^s = \rho$. In contrast, with cross-holdings the equity correlation shows a marked increase above the asset correlations until the equity reaches essentially zero. Most notable, even

\(^1\)Thus, the both firms assets have the same spot value, but are still log-normally distributed at maturity. The case of comonotonic asset endowments as considered by (Banerjee & Feinstein, 2021) corresponds to the trivial case of fully correlated assets, i.e., $\rho = 1$, and thus $\rho^s = 1$ as well.

\(^2\)Note that by symmetry of the setup, the corresponding figure for firm 2’s equity is just the mirror image, i.e., obtained by exchanging $M^d_{21}$ and $M^d_{12}$.
for anti-correlated business asset \((\rho_a = -0.4)\) the firm’s equities exhibit positive correlations for sufficiently large cross-holding fractions and stressed firm equities, i.e., during crises times with correspondingly low asset values. Similar effects are also observed for asymmetric asset values and with additional equity cross-holdings (as shown in appendix B).

![Figure 1: Equity correlations as function of firm 1’s equity value \(s_{1,0}\), for different debt cross-holding fractions, asset correlations and volatilities.](image)

### 5 Summary

We have used a financial network with cross-holdings to model the complex interlinkages around the financial firms and we put the accent in the study of the correlation of their derivatives. In fact, we uncover the capabilities of the Suzuki model to address the non-constant behavior of the correlation observed in the equity market; even from constant values of the business asset correlations. We shown mathematically, that the correlation among equities depends on the structure of the financial networks. Furthermore, we demonstrate analytically for the two firms case, the equity correlation is never lower than the unconditional correlation of the asset returns. Besides, the numerical simulations shows the power of the network approach too explain structurally the increase in correlation under crisis.
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A Computing the Greeks

The Greeks quantify the sensitivities of derivative prices to changes in underlying parameters. Here, we consider first-order Greeks only. In particular, we compute the sensitivities of equity and debt prices accounting for cross-holdings with respect to current asset values \( \Delta = \frac{\partial x_t}{\partial a_t} \).

First, we recall the solution \( A_t \) of equation (18) as

\[
A_t = a_0 e^{\left(-\frac{1}{2} \text{diag}(\sigma^2)\right) t + \text{diag}(\sigma) W_t},
\]

where \( a_0 > 0 \) denotes the initial value and \( W_t \) is multivariate normal distributed with mean \( 0 \) and covariance matrix \( tC \). Note that \( W_t \) can be obtained from independent standard normal variates \( Z \sim N(0, I_{n \times n}) \) as \( W_t = \sqrt{t}LZ \) with \( L^T L = C \). We will use this representation in the next section to express the risk-neutral market value of equity and debt contracts as

\[
x_t = \mathbb{E}_t^Q \left[ e^{-rT} x^* (a_T(Z)) \right] = \mathbb{E}_t^Q \left[ e^{-rT} x^* \left( a_t e^{\left(-\frac{1}{2} \sigma^2\right) T + \sqrt{T} \text{diag}(\sigma) LZ \right) \right].
\]  

A.1 Formal solution

Denoting all parameters of interest by \( \theta = (a_t, \sigma, r, \tau)^T \) and considering that the asset value \( a_T(Z; \theta) \) depends on the random variate \( Z \) and these parameters, we need to compute the following derivatives

\[
\frac{\partial}{\partial \theta} x_t = \frac{\partial}{\partial \theta} \mathbb{E}_t^Q \left[ e^{-rT} x^* (a_T(Z; \theta)) \right]
= \mathbb{E}_t^Q \left[ \left( \frac{\partial}{\partial \theta} e^{-rT} \right) x^* (a_T(Z; \theta)) + e^{-rT} \left( \frac{\partial}{\partial \theta} x^* (a_T(Z; \theta)) \right) \right],
\]

where we have used pathwise differentiation. Exchanging integration and differentiation requires some continuity conditions on \( x^* \). In particular, Broadie and Glasserman (1996, proposition 1) prove that pathwise differentiation is applicable for Lipschitz continuous functions.

Lemma 1. The function \( x^*(a) \) is Lipschitz continuous with Lipschitz constant

\[
L^* = (1 - \max_\xi \| K_\xi \|_1)^{-1}
\]
where

\[ K_\xi = \begin{pmatrix} \text{diag}(\xi) M^s & \text{diag}(\xi) M^d \\ \text{diag}(1-\xi) M^s & \text{diag}(1-\xi) M^d \end{pmatrix}. \]

Proof. First, we observe that \( g \) is continuous and piecewise linear. In particular, we have

\[
\|g(a_1, x) - g(a_2, x)\|_1 \leq \|a_1 - a_2\|_1
\]

\[
\|g(a, x_1) - g(a, x_2)\|_1 \leq \max_\xi \|K_\xi\|_1 \|x_1 - x_2\|_1
\]

showing that \( g \) is Lipschitz continuous with respect to \( a \) and \( x \).

Note that by assumption 1, we have \( \|M^s\|_1, \|M^d\|_1 < 1 \). Furthermore, the solvency indicator is either \( \xi_i = 1 \) if bank \( i \) is solvent or \( \xi_i = 0 \) otherwise. Thus, it holds that \( \|K_\xi\|_1 < 1, \forall \xi \in \{0, 1\}^N \).

Then, we compute

\[
\|x^*(a_1) - x^*(a_2)\|_1 = \|g(a_1, x^*(a_1)) - g(a_2, x^*(a_2))\|_1
\]

\[
= \|g(a_1, x^*(a_1)) - g(a_1, x^*(a_2)) + g(a_1, x^*(a_2)) - g(a_2, x^*(a_2))\|_1
\]

\[
\leq \|g(a_1, x^*(a_1)) - g(a_1, x^*(a_2))\|_1 + \|g(a_1, x^*(a_2)) - g(a_2, x^*(a_2))\|_1
\]

\[
\leq \max_\xi \|K_\xi\|_1 \|x^*(a_1) - x^*(a_2)\|_1 + \|a_1 - a_2\|_1
\]

\[
\Rightarrow \|x^*(a_1) - x^*(a_2)\|_1 \leq (1 - \max_\xi \|K_\xi\|_1)^{-1} \|a_1 - a_2\|_1.
\]

By the chain rule of differentiation we obtain

\[
\frac{\partial}{\partial \theta} x^*(a_r(Z; \theta)) = \frac{\partial}{\partial a} x^*(a)|_{a=a_r(Z; \theta)} \frac{\partial}{\partial \theta} a_r(Z; \theta).
\]

Note that \( \frac{\partial}{\partial a} x^*(a) \) is the derivative of the fixed point solving (9). In order to compute it, we make use of the implicit function theorem. A version of the theorem by Halkin (1974) is adopted to our notation:

**Theorem 3.** Let \( U \subset \mathbb{R}^m, V \subset \mathbb{R}^n \) and \( f : U \times V \to \mathbb{R}^n \) a continuously differentiable function. Suppose
that

\[ f(x^*, y^*) = 0 \]  \hspace{1cm} (52)

at a point \((x^*, y^*) \in U \times V\) and that the Jacobian matrices \(J_{f,x}f(x, y), J_{f,y}f(x, y)\) of partial derivatives exist at \((x^*, y^*)\). Further, \(J_{f,y}\) is invertible at this point. Then, there exists a neighborhood \(U^* \subset U\) and a continuously differentiable function \(h : U^* \to \mathbb{R}^n\) with

\[ h(x^*) = y^* \]  \hspace{1cm} (53)

and

\[ f(x, h(x)) = 0 \quad \forall x \in U^*. \]  \hspace{1cm} (54)

Moreover, the partial derivatives of \(h\) with respect to \(x \in U^*\) are given as

\[
\frac{\partial}{\partial x} h(x) = - [J_{f,y}f(x, h(x))]^{-1} \left[ \frac{\partial}{\partial x} f(x, h(x)) \right] \hspace{1cm} (55)
\]

As the function \(g(a, x)\) defined in equation (10) is Lipschitz continuous, it is almost everywhere differ-
entiable. The partial derivatives are given by

$$\frac{\partial}{\partial s_j} g^s_i(a, x) = \begin{cases} M^s_{ij} & \text{if firm } i \text{ is solvent} \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (56)

$$\frac{\partial}{\partial s_j} g^r_i(a, x) = \begin{cases} 0 & \text{if firm } i \text{ is solvent} \\ M^s_{ij} & \text{otherwise} \end{cases}$$  \hspace{1cm} (57)

$$\frac{\partial}{\partial r_j} g^s_i(a, x) = \begin{cases} M^d_{ij} & \text{if firm } i \text{ is solvent} \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (58)

$$\frac{\partial}{\partial r_j} g^r_i(a, x) = \begin{cases} 0 & \text{if firm } i \text{ is solvent} \\ M^d_{ij} & \text{otherwise} \end{cases}$$  \hspace{1cm} (59)

$$\frac{\partial}{\partial a_j} g^s_i(a, x) = \begin{cases} 1 & \text{if } i = j \text{ and firm } i \text{ is solvent} \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (60)

$$\frac{\partial}{\partial a_j} g^r_i(a, x) = \begin{cases} 0 & \text{if } i = j \text{ and firm } i \text{ is solvent} \\ 1 & \text{otherwise} \end{cases}$$  \hspace{1cm} (61)

Here, a firm $i$ is solvent if its asset value $v_i$ is sufficient to repay its nominal debt $d_i$, i.e. $v_i = a_i + \sum_{j=1}^{n} M^s_{ij} s_j + \sum_{j=1}^{n} M^d_{ij} r_j > d_i$. The derivatives of $g$ exist everywhere except for the boundary case $v_i = d_i$. Defining the solvency vector $\xi = (\mathbb{1}_{v_1 > d_1}(v_1), \ldots, \mathbb{1}_{v_n > d_n}(v_n))$, the partial derivatives of $g$ with respect to $x$ can be collected in a matrix as follows

$$\frac{\partial}{\partial x} g(a, x) = \begin{bmatrix} \text{diag}(\xi) M^s & \text{diag}(\xi) M^d \\ \text{diag}(1_n - \xi) M^s & \text{diag}(1_n - \xi) M^d \end{bmatrix}$$  \hspace{1cm} (62)

$$= \text{diag} \left((\xi; 1_n - \xi)\right) \begin{bmatrix} M^s & M^d \\ M^s & M^d \end{bmatrix}$$  \hspace{1cm} (63)

Thus, defining $f(a, x) = x - g(a, x)$ we obtain by the implicit function theorem 3
Corollary 1. The partial derivatives of $x^*(a)$ are given by

$$
\frac{\partial}{\partial a} x^*(a) = \left[ I_{2n \times 2n} - \frac{\partial}{\partial x} g(a, x) \right]^{-1} \begin{bmatrix} \text{diag}(\xi) \\ \text{diag}(1_n - \xi) \end{bmatrix}
$$

(64)

Proof. Use that $\frac{\partial}{\partial x} f(a, x) = I_{2n \times 2n} - \frac{\partial}{\partial x} g(a, x)$ and $\frac{\partial}{\partial a} f(a, x) = -\frac{\partial}{\partial a} g(a, x)$. Then, the result follows from theorem 3 and $\frac{\partial}{\partial a} g(a, x) = \begin{bmatrix} \text{diag}(\xi) \\ \text{diag}(1_n - \xi) \end{bmatrix}$. As explained below, assumption 1 ensures that $\frac{\partial}{\partial x} f(a, x)$ is invertible as required. \qed

Finally, combining equation (50) and (51) with corollary 1 we formally compute the network Greeks as

$$
\frac{\partial}{\partial \theta} x_t = \mathbb{E}_t^Q \left[ \left( \frac{\partial}{\partial \theta} e^{-r \tau} \right) x^*(a_T(Z; \theta)) 
\right.
$$

$$
+ e^{-r \tau} \left[ I_{2n \times 2n} - \frac{\partial}{\partial x} g(a, x) \right]^{-1} \begin{bmatrix} \text{diag}(\xi) \\ \text{diag}(1_n - \xi) \end{bmatrix} \frac{\partial}{\partial \theta} a_T(Z; \theta)
$$

(65)

where the expectation is well-defined as the derivatives exist almost everywhere, i.e. except for a set of measure zero.

A.2 Two bank Delta

In case of two banks, the network Delta, i.e. $\Delta = \frac{\partial x_t}{\partial a}$, can be computed explicitly. First, we drop the time index $t$ to ease notation and denote firm values (of equity and debt) and asset prices at time $t$ as $x = (s_1, s_2, r_1, r_2)$ and $a = (a_1, a_2)$ respectively. Then, using that $\frac{\partial}{\partial a} e^{-r \tau} = 0$ and

$$
\frac{\partial}{\partial a} A_T = \frac{\partial}{\partial a} e^{\left(-\frac{1}{2} \text{diag}(\sigma^2)\right)(T-t) + \text{diag}(\sigma)W_T}
$$

$$
= I e^{\left(-\frac{1}{2} \text{diag}(\sigma^2)\right)(T-t) + \text{diag}(\sigma)W_T} \begin{pmatrix} A_{1,T} & 0 \\ \frac{A_{1,T}}{a_1} & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} \frac{A_{1,T}}{a_1} & 0 \\ 0 & \frac{A_{2,T}}{a_2} \end{pmatrix}
$$

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from equation (19) with suitably shifted time indices, equation (65) simplifies to

\[
\frac{\partial}{\partial \alpha} x_t = E^Q_t \left[ I_{4 \times 4} - \frac{\partial}{\partial x} g(a, x) \right]^{-1} \begin{bmatrix} \text{diag}(\xi) \\ \text{diag}(1 - \xi) \end{bmatrix} \begin{bmatrix} \frac{A_{1,T}}{a_1} & 0 \\ 0 & \frac{A_{2,T}}{a_2} \end{bmatrix}
\]

Furthermore, we find from equation (62) that

\[
I_{4 \times 4} - \frac{\partial}{\partial x} g(a, x) = \begin{pmatrix}
1 & -\xi_1 M_{12}^s & 0 & -\xi_1 M_{12}^d \\
-\xi_2 M_{21}^s & 1 & -\xi_2 M_{21}^d & 0 \\
0 & -(1 - \xi_1) M_{12}^s & 1 & -(1 - \xi_1) M_{12}^d \\
-(1 - \xi_2) M_{21}^d & 0 & -(1 - \xi_2) M_{21}^d & 1
\end{pmatrix}
\]

Thus, the matrix is piecewise constant on each solvency region and \( \Delta \) can be found in all four cases. For illustration, we detail the case \( \xi_1 = \xi_2 = 1 \), i.e. both banks solvent:

\[
\begin{bmatrix} I_{4 \times 4} - \frac{\partial}{\partial x} g(a, x) \end{bmatrix}^{-1} \begin{bmatrix} \text{diag}(\xi) \\ \text{diag}(1 - \xi) \end{bmatrix} \begin{bmatrix} \frac{A_{1,T}}{a_1} & 0 \\ 0 & \frac{A_{2,T}}{a_2} \end{bmatrix} = \begin{bmatrix} \frac{A_{1,T}}{a_1} & 0 \\ 0 & \frac{A_{2,T}}{a_2} \end{bmatrix}
\]

containing the \( \Delta \)'s for equity (top) and debt (bottom) respectively. Note that the equity \( \Delta \)'s correspond to the terms for solvency region \( \Xi_{ss} \) in equations (37) – (43).

Here, we have used that the inverse of a block matrix can be expressed via the Schur complement as

\[
\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -B_{22}^{-1} B_{21} & I \end{pmatrix} \begin{pmatrix} (B_{11} - B_{12} B_{22}^{-1} B_{21})^{-1} & 0 \\ 0 & B_{22}^{-1} \end{pmatrix} \begin{pmatrix} I & -B_{12} B_{22}^{-1} \\ 0 & I \end{pmatrix}.
\]
As in our case, $B_{11} = I - M^s$, $B_{12} = M^d$, $B_{21} = 0$ and $B_{22} = I$, we obtain

$$
\begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix}
\begin{pmatrix}
(I - M^s)^{-1} & 0 \\
0 & I
\end{pmatrix}
\begin{pmatrix}
I & -M^d I \\
0 & I
\end{pmatrix} =
\begin{pmatrix}
(I - M^s)^{-1} & 0 \\
0 & I
\end{pmatrix}
$$

and the above results follows from

$$
\begin{pmatrix}
1 & -M_{12}^s \\
-M_{21}^d & 1
\end{pmatrix}^{-1} = \frac{1}{1 - M_{12}^s M_{22}^d}
\begin{pmatrix}
1 & M_{12}^s \\
M_{21}^d & 1
\end{pmatrix}.
$$

The $\Delta$’s on the other three solvency regions can be found analogously and are ommitted for brevity.

B Additional figures

Markedly rising equity correlations at sufficiently low asset values are also observed with additional equity cross-holdings of 10% (figure 2) and asymmetric asset values (figure 3). As the figure is no longer symmetric with respect to the firms equity values, we show the initial values of $a_1$ and $a_2$ (in log-scale) instead. The Suzuki areas, i.e., default boundaries, are indicated by grey lines and the equity correlations are color coded. Again, at sufficiently large debt cross-holding fractions a marked increase in equity correlations is observed, especially in the $\Xi_{dd}$ Suzuki area, i.e., during crisis.

![Figure 2](image-url)

Figure 2: Same as figure 1, but with additional equity cross-holdings of $M_{12}^d = M_{21}^d = 0.1$.  

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Figure 3: Suzuki areas and equity correlations as a function of asset values $a_1$ and $a_2$. Here, the asset correlation is fixed at $\rho = 0$. 