RESTRICTED LAZARSFELD-MUKAI BUNDLES AND CANONICAL CURVES

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Dedicated to Professor Shigeru Mukai on his sixtieth birthday, with admiration

For a $K3$ surface $S$, a smooth curve $C \subset S$ and a globally generated linear series $A \in W^d_r(C)$ with $h^0(C, A) = r + 1$, the Lazarsfeld-Mukai vector bundle $E_{C,A}$ is defined via the following elementary modification on $S$

$$0 \to E_{C,A}^\vee \to H^0(C, A) \otimes \mathcal{O}_S \to A \to 0.$$  

The bundles $E_{C,A}$ have been introduced more or less simultaneously in the 80’s by Lazarsfeld [L1] and Mukai [M1] and have acquired quite some prominence in algebraic geometry. On one hand, they have been used to show that curves on general $K3$ surfaces verify the Brill-Noether theorem [L1], and this is still the only class of smooth curves known to be general in the sense of Brill-Noether theory in every genus. When $\rho(g, r, d) = 0$, the vector bundle $E_{C,A}$ is rigid and plays a key role in the classification of Fano varieties of coindex 3. For $g = 7, 8, 9$, the corresponding Lazarsfeld-Mukai bundle has been used to coordinatize the moduli space of curves of genus $g$, thus giving rise to a new and more concrete model of $\mathcal{M}_g$, see [M2], [M3], [M4]. Furthermore, Lazarsfeld-Mukai bundles of rank two have led to a characterization of the locus in $\mathcal{M}_g$ of curves lying on $K3$ surfaces in terms of existence of linear series with unexpected syzygies [F], [V]. For a recent survey on this circle of ideas, see [A].

Recently, Lazarsfeld-Mukai bundles have proven to be effective in shedding some light on an interesting conjecture of Mercat in Brill-Noether theory, see [FO1], [FO2], [LMN]. Recall that the Clifford index of a semistable vector bundle $E \in \mathcal{U}_C(n, d)$ on a smooth curve $C$ of genus $g$ is defined in [LN1] as

$$\gamma(E) := \mu(E) - \frac{2}{n} h^0(C, E) + 2 \geq 0.$$  

Then the higher Clifford indices of the curve $C$ are defined as the quantities

$$\text{Cliff}_n(C) := \min \left\{ \gamma(E) : E \in \mathcal{U}_C(n, d), \ d \leq n(g - 1), \ h^0(C, E) \geq 2n \right\}.$$  

For any line bundle $L$ on $C$ such that $h^i(C, L) \geq 2$ for $i = 0, 1$, that is, contributing to the classical Clifford index $\text{Cliff}(C)$, by computing the invariants of the strictly semistable vector bundle $E := L^{\otimes n}$, one finds that $\text{Cliff}_n(C) \leq \text{Cliff}(C)$. Mercat [Me1] predicted that for any smooth curve $C$ of genus $g$, the following equality

$$(M_n) : \quad \text{Cliff}_n(C) = \text{Cliff}(C).$$

should hold. Counterexamples to $(M_2)$ have been found on curves lying on $K3$ surfaces that are special in Noether-Lefschetz sense, see [FO1], [FO2] and [LN2]. However, $(M_2)$ is expected to hold for a general curve of genus $g$, and in fact even for a curve $C$ lying
on a $K3$ surface $S$ such that $\text{Pic}(S) = \mathbb{Z} \cdot C$. For instance, it is known that statement $(M_2)$ holds on $M_{11}$ outside a certain Koszul divisor (which also admits a Noether-Lefschetz realization), see [FO2] Theorem 1.3. It has also been shown that $(M_2)$ holds generically on $M_{10}$ for $g \leq 16$, see [FO1].

It has been proved in [LMN] that rank three restricted Lazarsfeld-Mukai bundles invalidate statement $(M_3)$ in genus 9 and 11 respectively, that is, Mercat’s conjecture in rank three fails generically on $M_9$ and $M_{11}$ respectively. This was then extended in [FO2] Theorem 1.4, to show that on a $K3$ surface $S$ with $\text{Pic}(S) = \mathbb{Z} \cdot C$, where $C^2 = 2g - 2$, if $A \in W_d^2(C)$ is a linear system where $d := \lfloor \frac{2g+8}{3} \rfloor$, the restriction to $C$ of the Lazarsfeld-Mukai bundle $E_{C,A}$ is stable and has Clifford index strictly less than $\lfloor \frac{g-1}{2} \rfloor$, in particular, statement $(M_3)$ fails for the curve $C$. For further background on this problem, we refer to the papers [Mc], [LN1] and [CMN].

The restricted Lazarsfeld-Mukai bundle $E|_C := E_{C,A} \otimes O_C$ sits in the following exact sequence on the curve $C \subset S$

\[
0 \longrightarrow Q_A \longrightarrow E|_C \longrightarrow K_C \otimes A^\vee \longrightarrow 0,
\]

where $Q_A = M_A^\vee$ is the dual of the kernel bundle defined by the sequence

\[
0 \longrightarrow M_A \longrightarrow H^0(C, A) \otimes O_C \longrightarrow A \longrightarrow 0.
\]

One then easily shows [VI], [FO2] that the sequence (2) is exact on global sections, that is,

\[
h^0(C, E|_C) = h^0(C, K_C \otimes A^\vee) + h^0(C, Q_A) = g - d + 2r + 1.
\]

By choosing the degree $d$ minimal such that $W_d^r(C) \neq \emptyset$, precisely $d = r + \lfloor \frac{r(g+1)}{r+1} \rfloor$, it becomes clear that, for sufficiently high $g$, one has

\[
\gamma(E|_C) < \text{Cliff}(C),
\]

that is, $E|_C$, when semistable, provides a counterexample to Mercat’s conjecture $(M_{r+1})$. We prove the following result, extending to rank 4 a picture studied in smaller ranks in the papers [M1], [VI], respectively [FO2].

**Theorem 0.1.** Let $S$ be a $K3$ surface with $\text{Pic}(S) = \mathbb{Z} \cdot L$, where $L^2 = 2g - 2$ and write

\[
g = 4i - 4 + \rho \quad \text{and} \quad d = 3i + \rho,
\]

with $\rho \geq 0$ and $i \geq 6$. Then for a general curve $C \subset |L|$ and a globally generated linear series $A \in W_d^3(C)$ with $h^0(C, A) = 4$, the restriction to $C$ of the Lazarsfeld-Mukai bundle $E_{C,A}$ is stable.

Note that in Theorem 0.1, $\dim W_d^3(C) = \rho$. The rank 3 version of this result was proved in [FO2]. We record the following consequence of Theorem 0.1.

**Corollary 0.2.** For $C \subset S$ with $g \geq 20$ and $\text{Pic}(S) = \mathbb{Z} \cdot C$, we set $d := \lfloor \frac{4g+14}{3} \rfloor$ and $A \in W_d^3(C)$ with $h^0(C, A) = 4$. Then $E|_C$ is a stable rank 4 bundle with $\gamma(E|_C) < \lfloor \frac{g-1}{2} \rfloor$. It follows that the statement $(M_4)$ fails for $C$.

The curves $C$ appearing in Corollary 0.2 are Brill-Noether general, that is, they satisfy $\text{Cliff}(C) = \lfloor \frac{g-1}{2} \rfloor$, see [LL].
Theorem 0.1 and Corollary 0.2 fit into a more general set of results that are independent from the structure of Pic(S). For example, we show that under mild restrictions, on a very general K3 surface, the extension \( E \) is non-trivial and the restricted Lazarsfeld-Mukai bundle \( E|_C \) is simple (see Theorem 1.3). We expect that the bundle \( E|_C \) remains stable also for higher rank \( r + 1 = h^0(C, A) \), at least when \( \text{Pic}(S) = \mathbb{Z} \cdot C \). However, our method of proof based on the Bogomolov inequality, seems not to extend easily for \( r \geq 4 \).

The second topic we discuss in this paper concerns a connection between normal bundles of canonical curves and Mercat’s conjecture. The question we pose is however fundamental and interesting irrespective of Mercat’s conjecture.

For a smooth non-hyperelliptic canonically embedded curve \( C \subset \mathbb{P}^{g-1} \) of genus \( g \), we consider the normal bundle \( N_C := N_{C/\mathbb{P}^{g-1}} \); we then define the twist of the conormal bundle \( E := N_C^* \otimes K_C^{\otimes 2} \). By direct calculation

\[
\det(E) = K_C^{\otimes (g-5)} \quad \text{and} \quad \text{rk}(E) = g - 2.
\]

In particular, the vector bundle \( E \) contributes to \( \text{Cliff}_{g-2}(C) \) if and only if \( g \leq 8 \). Since \( M_{K_C}(-1) = \Omega_{\mathbb{P}^{g-1},C} \), the bundle \( E \) sits in the following exact sequence

\[
0 \to E \
\to M_{K_C} \otimes K_C \xrightarrow{\gamma_{K_C}} K_C^{\otimes 3} \to 0,
\]

where \( \gamma_{K_C} : H^0(C, M_{K_C} \otimes K_C) \to H^0(C, K_C^{\otimes 3}) \) is the Gaussian map of \( C \), see [W]. The map \( \gamma_{K_C} \) vanishes on symmetric tensors, hence \( \ker(\gamma_{K_C}) = I_2(K_C) \oplus \ker(\psi_{K_C}) \), where

\[
\psi_{K_C} := \gamma_{K_C}|_{\text{H}^2H^0(C, K_C)} : \bigwedge^2 H^0(C, K_C) \to H^0(C, K_C^{\otimes 3}),
\]

and \( I_2(K_C) = K_{1,1}(C, K_C) \) is the space of quadrics containing the canonical curve \( C \). The map \( \psi_{K_C} \) has been studied intensely in the context of deformations in \( \mathbb{P}^g \) of the cone over the canonical curve \( C \subset \mathbb{P}^{g-1} \), see [W]. It is in particular known [CHM], [V] that \( \psi_{K_C} \) is surjective for a general curve \( C \) of genus \( g \geq 12 \).

We now specialize to the case \( g = 7 \), when \( E \) contributes to \( \text{Cliff}_{5}(C) \). Then

\[
\text{rk}(E) = 5 \quad \text{and} \quad \det(E) = K_C^{\otimes 2},
\]

therefore \( \mu(E) = \frac{44}{5} \). It is easy to show that the Gaussian map \( \psi_{K_C} \) is injective for every smooth curve \( C \) of genus 7 having maximal Clifford index \( \text{Cliff}(C) = 3 \). In particular, the space

\[
H^0(C, E) = I_2(K_C)
\]

is 10-dimensional and \( \gamma(E) = 2 + \frac{4}{5} < \text{Cliff}(C) \). We establish the following result:

**Theorem 0.3.** The normal bundle \( N_{C/\mathbb{P}^6} \) of every canonical curve \( C \) of genus 7 with maximal Clifford index is stable. In particular, the Mercat conjecture (\( M_5 \)) fails for a general curve of genus 7.

The proof of Theorem 0.3 uses in an essential way Mukai’s realisation [M3] of a canonical curve \( C \) of genus 7 with \( \text{Cliff}(C) = 3 \) as a linear section of the 10-dimensional spinorial variety \( OG(5, 10) \subset \mathbb{P}^{15} \). In particular, the vector bundle \( E \) is the restriction
to $C$ of the rank 5 spinorial bundle on $OG(5, 10)$, which endows $E$ with an extra structure that only exists in genus 7. Note that the normal bundle of every canonical curve of genus at most 6 is unstable, and more generally, the normal bundle of a tetragonal canonical curve of any genus is unstable (see also Section 3). In particular, we have the following identification of cycles on $M_7$

$$\{ \left[ C \right] \in M_7 : N_C \text{ is unstable} \} = M^1_{7,4},$$

where the right hand side denotes the divisor of tetragonal curves of genus 7. We make the following conjecture:

**Conjecture 0.4.** The normal bundle $N_{C/p_{g-1}}$ of a general canonical curve $C$ of genus $g \geq 7$ is stable.

Note that the stability of the normal bundle $N_{C/p_r}$ of a curve of genus $g$ is not known even in the case of a non-special embedding $C \hookrightarrow \mathbb{P}^r$ given by a line bundle $L \in \text{Pic}(C)$ of large degree. This is in stark contrast with the case the kernel bundle $M_L = \Omega_{\mathbb{P}^r|C}(1)$, whose stability easily follows by a filtration argument due to Lazarsfeld [L2]. For some very partial results in this direction, see [EL]. In general, one can show by degenerating a canonical curve $C \subset \mathbb{P}^{g-1}$ to the transversal union of two rational normal curves in $\mathbb{P}^{g-1}$ meeting transversally in $g + 1$ points, that $N_{C/p_{g-1}}$ is not too unstable. Due to the fact that the slope $\mu(N_{C/p_{g-1}})$ is not an integer, this simple minded technique does not seem to lead to a full solution, because one cannot expect to find a specialization in which the corresponding limit of the normal bundle is a direct sum of line bundles of the same degree. It is of course, natural to ask whether a generalization of the equality (4) exists for higher genus.

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1. Simplicity of restricted Lazarsfeld-Mukai bundles

We fix a $K3$ surface $S$, a smooth curve $C \subset S$ of genus $g$ and a globally generated linear series $A \in W^r_d(C)$, with $h^0(C, A) = r + 1$. Using the evaluation sequence (1), we form the vector bundle $F = F_{C, A}$; by dualizing, we obtain an exact sequence for the dual bundle $E = E_{C, A} := F^\vee_{C, A}$.

$$0 \to H^0(C, A)^\vee \otimes \mathcal{O}_S \to E_{C, A} \to K_C \otimes A^\vee \to 0.$$  

It is well-known [M1], [L1] that $c_1(E) = [C]$ and $c_2(E) = d$; moreover $h^0(S, F) = 0$ and $h^1(S, E) = h^1(S, F) = 0$. Finally, one also has that $\chi(S, E \otimes F) = 2 - 2\rho(g, r, d)$;
in particular, if $E$ is a simple bundle, then $\rho(g, r, d) \geq 0$. Assuming furthermore that $\text{Pic}(S) = \mathbb{Z} \cdot C$, it is also well-known that both $E$ and $F$ are $C$-stable bundles on $S$.

### 1.1. The rank 2 case.

We begin by showing that in rank 2, irrespective of the structure of $\text{Pic}(S)$, a splitting of the restriction $E|_C$ can only be induced by an elliptic pencil on the $K3$ surface.

**Theorem 1.1.** Let $C \subset S$ be as above and $A \in W^1_d(C)$ a base point free pencil of degree $2 < d < g - 1$ with $K_C \otimes A^\vee$ globally generated. The following conditions are equivalent:

(i) $E|_C \cong A \oplus (K_C \otimes A^\vee)$;

(ii) There exists an elliptic pencil $N \in \text{Pic}(S)$ such that $N|_C = A$.

(iii) $h^0(S, E \otimes F) < h^0(C, E \otimes F|_C)$.

**Corollary 1.2.** With notation as above, if $g \leq 2d - 2$ and $A$ is not induced by an elliptic pencil on $S$, then $E|_C$ is simple if and only if $E$ is simple.

Note that it is easy to see that if $E|_C$ is simple, then $E$ is also simple. It is also known that if $E$ is simple, then automatically $g \leq 2d - 2$.

**Proof.** (of Theorem 1.1) (ii)$\Rightarrow$(i). Let $N$ be an elliptic pencil with $N|_C = A$. Consider the exact sequence

$$0 \longrightarrow N^\vee \longrightarrow F \longrightarrow N(-C) \longrightarrow 0.$$  

Its restriction to $C$ gives a splitting of the dual of the sequence (2) characterizing $E|_C$. Observe that since $d < g - 1$, there is no morphism from $A^\vee$ to $K_C^\vee \otimes A$.

(i)$\Rightarrow$(ii). Conversely, suppose that $E|_C = A \oplus (K_C \otimes A^\vee)$. Applying $\text{Hom}(K_C \otimes A^\vee, -)$ to the sequence (1), we obtain an exact sequence

$$0 \longrightarrow \text{Ext}^1(K_C \otimes A^\vee, F) \longrightarrow \text{Ext}^1(K_C \otimes A^\vee, H^0(C, A) \otimes O_S) \longrightarrow \text{Ext}^1(K_C \otimes A^\vee, A).$$

Since the extension class $[E] \in \text{Ext}^1(K_C \otimes A^\vee, H^0(C, A) \otimes O_S)$ maps to the trivial extension in $\text{Ext}^1(K_C \otimes A^\vee, A)$, it follows that there exists a rank 2 bundle $G$ on $S$ which fits into a commutative diagram:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & & & & & & & & \\
0 & F & H^0(A) \otimes O_S & A & 0 & 0 & 0 & 0 & 0 \\
\downarrow & & & & & & & & \\
0 & G & E & A & 0 & 0 & 0 & 0 & 0 \\
\downarrow & & & & & & & & \\
K_C \otimes A^\vee & K_C \otimes A^\vee & K_C \otimes A^\vee & K_C \otimes A^\vee & K_C \otimes A^\vee & K_C \otimes A^\vee & K_C \otimes A^\vee & K_C \otimes A^\vee & K_C \otimes A^\vee
\end{array}
\]

Using that $H^0(S, F) = H^1(S, F) = 0$, we obtain $H^0(S, G) \cong H^0(C, K_C \otimes A^\vee)$. Since $h^0(S, E) = h^0(C, A) + h^1(C, A) = h^0(C, A) + h^0(S, G)$, and $h^1(S, E) = 0$, it follows that $H^1(S, G) = 0$. From the second row of (5), we find that $H^0(S, G(-C)) = 0$.  


Furthermore, we compute \( c_1(G) = 0 \) and \( c_2(G) = 2d - 2g + 2 \). So \( c_2(G) < 0 = c_1^2(G) \), that is, \( G \) violates Bogomolov’s inequality, and then it sits in an extension
\[
0 \longrightarrow M \longrightarrow G \longrightarrow M^\vee \otimes \mathcal{I}_{\Gamma/S} \longrightarrow 0,
\]
where \( \Gamma \) is a zero-dimensional subscheme of \( S \), and \( M \in \text{Pic}(S) \) is such that \( M^2 > 0 \) and \( M \cdot H > 0 \) for any ample line bundle \( H \) on \( S \). In particular, \( H^0(S, M^\vee) = 0 \), and hence \( H^0(S, M) \cong H^0(S, G) \cong H^0(C, K_C \otimes A^\vee) \neq 0 \). Moreover, since
\[
h^0(S, M^\vee \otimes \mathcal{I}_{\Gamma/S}) = h^1(S, G) = 0,
\]
it also follows that \( H^1(S, M) = 0 \).

On the other hand \( H^0(S, F) = 0 \), which implies that the composed map
\[
M \longrightarrow G \longrightarrow K_C \otimes A^\vee
\]
is non-zero; in fact, we claim that it is surjective, that is, \( M|_C = K_C \otimes A^\vee \). Suppose that \( M|_C = K_C \otimes A^\vee(-D') \), with \( D' \neq 0 \) an effective divisor on \( C \). Since \( h^0(S, G(-C)) = 0 \), we have \( h^0(S, M(-C)) = 0 \), which implies \( h^0(S, M) \leq h^0(C, M|_C) \). Since we assumed \( K_C \otimes A^\vee \) to be globally generated, we have that
\[
h^0(S, M) \leq h^0(C, K_C \otimes A^\vee(-D')) < h^0(C, K_C \otimes A^\vee) = h^0(S, M),
\]
a contradiction.

Setting \( N := M^\vee(C) \), we have shown that \( N|_C = A \) and there is an exact sequence
\[
0 \longrightarrow M^\vee \longrightarrow N \longrightarrow A \longrightarrow 0.
\]
Since \( h^0(S, M^\vee) = h^1(S, M) = 0 \), it follows that \( h^0(N) = h^0(C, A) = 2 \) and hence \( N \) defines an elliptic pencil.

\( (iii) \Rightarrow (i) \). From the sequence (1) twisted by \( E(-C) \cong F \), we obtain that
\[
H^0(S, E \otimes F(-C)) \subset H^0(C, A) \otimes H^0(S, E(-C)),
\]
and, since \( F \) has no sections, it follows that \( H^0(S, E \otimes F(-C)) = 0 \). We have an exact sequence
\[
0 \longrightarrow H^0(S, E \otimes F) \longrightarrow H^0(S, E \otimes F|_C) \longrightarrow H^1(S, E \otimes F(-C)).
\]
The hypothesis implies that \( H^1(S, E \otimes F(-C)) \neq 0 \). From (1) twisted by \( E(-C) \cong F \), we obtain the exact sequence in cohomology
\[
0 \longrightarrow H^0(C, E|_C \otimes K_C^\vee \otimes A) \longrightarrow H^1(S, E \otimes F(-C)) \longrightarrow H^0(C, A) \otimes H^1(S, E(-C)) = 0,
\]
therefore \( h^0(C, E|_C \otimes K_C^\vee \otimes A) \neq 0 \). The sequence (2) yields to an exact sequence
\[
0 = H^0(C, K_C^\vee \otimes A \otimes \mathcal{O}_C) \longrightarrow H^0(C, E|_C \otimes K_C^\vee \otimes A) \longrightarrow H^0(C, E \otimes \mathcal{O}_C) \rightarrow H^1(C, K_C^\vee \otimes A \otimes \mathcal{O}_C).
\]
Then \( H^0(C, E|_C \otimes K_C^\vee \otimes A) \rightarrow H^0(C, \mathcal{O}_C) \) is an isomorphism and, under the coboundary map
\[
H^0(C, \mathcal{O}_C) \ni 1 \mapsto 0 \in H^1(C, K_C^\vee \otimes A \otimes \mathcal{O}_C),
\]
that is, the sequence (2) is split.

Note that we also have \( h^1(S, E \otimes F(-C)) = 1 \) and \( h^0(C, E \otimes F|_C) = h^0(S, E \otimes F) + 1 \).

\( (i) \Rightarrow (iii) \). From the hypothesis and from the sequence (2), we find
\[
h^0(C, E|_C \otimes A^\vee) = h^0(C, K_C \otimes A \otimes (-2)) + 1.
\]
Furthermore, \( h^0(S, E \otimes F) = h^0(C, E|_C \otimes A^\vee) \); twist \(^5\) by \( F \) and use the vanishing of \( h^0(F) \) and that of \( h^1(F) \).

On the other hand, since \( E|_C \cong A \otimes K_C \otimes A^\vee \), we have
\[
h^0(C, E \otimes F|_C) = 2 + h^0(C, K_C \otimes A^\otimes(-2)),
\]
hence \( h^0(C, E \otimes F|_C) = h^0(S, E \otimes F) + 1 \).

\[\square\]

1.2. Lazarsfeld-Mukai bundles of higher rank. We study when the restriction \( E|_C \) is a simple vector bundle. Our main tool is a variant of the Bogomolov instability theorem.

**Theorem 1.3.** Let \( S \) be a K3 surface and \( C \subset S \) a smooth curve of genus \( g \geq 4 \) such that \( \text{Pic}(S) = \mathbb{Z} \cdot C \). We fix positive integers \( r \) and \( d \) such that
\[
\rho(g, r, d) \geq 0, \quad g \geq 2r + 4 \text{ and } d \leq \frac{3r(g - 1)}{2r + 2}.
\]
Then for any linear series \( A \in W^r_d(C) \) such that \( h^0(C, A) = r + 1 \) and \( K_C \otimes A^\vee \) is globally generated, the restricted Lazarsfeld-Mukai bundle \( E|_C \) is simple.

Note that in the special case \( \rho(g, r, d) = 0 \), the constraints from the previous statement give rise to the bound \( g > 2r + 5 \).

**Proof. Step 1.** We first establish that the natural extension \(^2\), that is,
\[
0 \rightarrow Q_A \rightarrow E|_C \rightarrow K_C \otimes A^\vee \rightarrow 0
\]
is non-trivial. Assuming that \(^2\) is trivial. Then there is an injective morphism from \( K_C \otimes A^\vee \) to \( E|_C \) and hence a surjective map \( F(C) \rightarrow A \). Then
\[
G := \ker\{ F(C) \rightarrow A \}
\]
is a vector bundle of rank \( r + 1 \) with Chern classes \( c_1(G) = (r - 1)|C| \) and
\[
c_2(G) = c_2(F(C)) - c_1(F(C)) \cdot C + \deg(A) = 2d + r(r - 3)(g - 1).
\]
We compute the discriminant of \( G \)
\[
\Delta(G) = 2rk(G)c_2(G) - (rk(G) - 1)c_1^2(G) = 4d(r + 1) - 8r(g - 1) < 0,
\]
hence \( G \) is unstable. Applying \(^{[HL]}\) Theorem 7.3.4, there exists a subsheaf \( M \subset G \) with
\[
\xi_{M,G}^2 \geq -\frac{\Delta(G)}{r(r + 1)^2},
\]
where \( \xi_{M,G} = c_1(M)/rk(M) - c_1(G)/rk(G) \). Setting \( c_1(M) = k \cdot |C| \) and \( s := rk(M) \), the previous inequality becomes
\[
\left( \frac{k}{s} - \frac{r - 1}{r + 1} \right)^2 (2g - 2) \geq \frac{8r(g - 1) - 4d(r + 1)}{r(r + 1)^2}.
\]
Note that \( M \) destabilizes \( G \), which coupled with the stability of \( F(C) \) yields
\[
\frac{r - 1}{r + 1} \leq \frac{k}{s} < \frac{r}{r + 1},
\]
implying after manipulations \( 2d(r + 1) > 3(g - 1)r \), thus contradicting the hypothesis.
Step 2. Assuming that $E|_C$ is non-simple, we deduce that the extension (2) splits. We consider the exact sequence

$$H^0(S, E \otimes F) \rightarrow H^0(C, E \otimes F|_C) \rightarrow H^1(S, E \otimes F(-C)).$$

and it suffices to show that $H^1(S, E \otimes F(-C)) = 0$. Assuming this not to be the case, twisting (1) by $E(-C)$ induces the exact sequence

$$H^0(C, A \otimes E|_C \otimes K_C^\vee) \rightarrow H^1(S, E \otimes F(-C)) \rightarrow H^0(C, A) \otimes H^1(S, E(-C)).$$

Since $H^1(S, E(-C)) = 0$, we obtain that $H^0(C, A \otimes E|_C \otimes K_C^\vee) \neq 0$. Furthermore, $Q_A$ is a stable bundle and since $\mu(Q_A \otimes A \otimes K_C^\vee) < 0$, we find that

$$H^0(C, Q_A \otimes A \otimes K_C^\vee) = 0,$$

hence we also have the sequence induced from (2) after twisting with $A \otimes K_C^\vee$

$$0 \rightarrow H^0(C, E|_C \otimes K_C^\vee \otimes A) \rightarrow H^0(C, O_C) \rightarrow H^1(C, K_C^\vee \otimes A \otimes Q_A).$$

We conclude that the coboundary map $H^0(C, O_C) \rightarrow H^1(C, Q_A \otimes A \otimes K_C^\vee)$ is trivial, that is, $E|_C \cong Q_A \oplus (K_C \otimes A^\vee)$, which completes the proof. \hfill $\Box$

2. Stability of restricted Lazarsfeld-Mukai bundles

2.1. The rank 2 case. If $C \subset S$ is an ample curve, then with one exception ($g = 10$ and $C$ a smooth plane sextic), Cliff$(C)$ is computed by a pencil, see [CP] Proposition 3.3. We show that in rank 2 the semistability of the LM bundle is preserved under restriction.

**Theorem 2.1.** Let $S$ be a $K3$ surface, $C \subset S$ an ample curve of genus $g \geq 4$ and $A \in W^1_d(C)$ a pencil computing Cliff$(C)$. If $E_{C,A}$ is $C$-semistable on $S$, then $E|_C$ is also semistable on $C$. Moreover, if $E_{C,A}$ is $C$-stable on $S$, then $E|_C$ is stable on $C$.

**Proof.** The proof of the stability is similar, and hence we discuss the semistability part only. We write $A = O_C(D)$, where $D$ is an effective divisor on $C$. Suppose $E|_C$ is unstable and consider an exact sequence

$$0 \rightarrow L_1 \rightarrow E|_C \rightarrow K_C \otimes L_1^\vee \rightarrow 0,$$

with $\deg(L_1) \geq g$. Since $L_1 \not\subset A$, the composed map $L_1 \rightarrow E|_C \rightarrow K_C \otimes A^\vee$ must be non-zero, that is, $L_1 = K_C(-D - D_1)$, where $D_1$ is an effective divisor on $C$. Set $d_1 := \deg(D_1)$. Consider the elementary modification

$$0 \rightarrow V \rightarrow E \rightarrow A(D_1) \rightarrow 0$$

induced by the composition $E \rightarrow E|_C \rightarrow A(D_1)$. Then

$$c_1(V) = 0 \text{ and } c_2(V) = 2d + d_1 - 2g + 2 < 0,$$

hence $V$ is unstable with respect to any polarization and fits in an exact sequence

$$0 \rightarrow M \rightarrow V \rightarrow M^\vee \otimes T_{V/S} \rightarrow 0,$$

where $\Gamma \subset S$ is a 0-dimensional subscheme and $M$ is a divisor class that intersects positively any ample class on $S$ and with $M^2 > 0$. From (8) and (9) we find that $H^0(S, M) \cong H^0(S, V)$ and $H^0(S, M(-C)) = 0$. Dualizing (8), we obtain the sequence

$$0 \rightarrow F \rightarrow V^\vee \rightarrow K_C(-D - D_1) \rightarrow 0,$$

from which, using that $V \cong V^\vee$, we obtain $H^0(S, V) = H^0(C, K_C(-D - D_1)).$
We claim that \( \text{Cliff}(A(D_1)) = \text{Cliff}(C) \). Recall that \( h^0(S, E) = h^0(C, A) + h^1(C, A) \), and, from the sequence (8) we write
\[
h^0(S, E) \leq h^0(C, A(D_1)) + h^1(C, A(D_1)).
\]
By assumption, the pencil \( A \) computes \( \text{Cliff}(C) \), which implies
\[
\text{Cliff}(C) = g + 1 - h^0(A) - h^1(A) \geq g + 1 - h^0(A(D_1)) - h^1(A(D_1)) = \text{Cliff}(A(D_1)).
\]
It follows that \( \text{Cliff}(A(D_1)) = \text{Cliff}(C) \), in particular \( K_C(-D-D_1) \) is globally generated.

Clearly, \( M \not\subseteq F \), hence the composition \( \varphi: M \to V \to K_C(-D-D_1) \) is non-zero and one writes \( \text{Im}(\varphi) = K_C(-D-D_1-D_2) \), where \( D_2 \) is an effective divisor on \( C \). If \( D_2 \neq 0 \), then one has the sequence of inequalities
\[
h^0(S, M) \leq h^0(C, K_C(-D-D_1-D_2)) < h^0(C, K_C(-D-D_1)) = h^0(S, M),
\]
a contradiction. Therefore \( M|_C = K_C(-D-D_1) \). Viewing \( M \) as a subsheaf of \( E \), we find \( \mu(M) = M \cdot C = \deg(L_1) > \mu(E) \), thus bringing the proof to an end. \( \square \)

**Remark 2.2.** If \( E_{C,A} \) is stable, then it is simple and hence \( d = \left\lfloor \frac{g+3}{2} \right\rfloor \), see [LI]. Conversely, if \( C' \subset S \) is an ample curve of genus \( g \) and gonality \( \left\lfloor \frac{2g+3}{2} \right\rfloor \), then it was shown in [LC] that the LM bundle \( E_{C,A} \) corresponding to a general curve \( C \in |O_S(C')| \) and a pencil \( A \in W^1_{\left\lfloor \frac{2g+3}{2} \right\rfloor}(C) \) is \( C \)-semistable (even stable when \( g \) is odd).

### 2.2. Stability of Lazarsfeld-Mukai bundles of rank four

We show that restrictions of \( \text{LM} \) bundles of rank 4 on very general \( K3 \) surfaces of genus \( g \geq 20 \) are stable. Similar results were established in [V] and [FO2] for rank 2 and 3 respectively. We fix integers \( i \geq 6 \) and \( \rho \geq 0 \) and write
\[
g := 4i - 4 + \rho \quad \text{and} \quad d := 3i + \rho,
\]
so that \( \rho(g,3,d) = \rho \). Let \( S \) be a \( K3 \) surface and \( C \subset S \) a curve of genus \( g \) such that \( \text{Pic}(S) = \mathbb{Z} \cdot C \), and pick a globally generated linear series \( A \in W^1_d(C) \) with \( h^0(C, A) = 4 \).

**Proof of Theorem 0.7.** Our previous results show that \( E|_C \) is simple, hence indecomposable. Suppose \( E|_C \) is not stable and fix a maximal destabilizing sequence
\[
0 \longrightarrow M \longrightarrow E|_C \longrightarrow N \longrightarrow 0.
\]
Put \( d_N := \deg(N) \) and \( d_M := \deg(M) = 2g - 2 - d_N \). Since \( M \) is destabilizing,
\[
\frac{d_M}{\text{rk}(M)} \geq \frac{g-1}{2}, \quad \frac{d_N}{\text{rk}(N)} \leq \frac{g-1}{2}.
\]

The bundle \( N \), being a quotient of \( E \), is globally generated. Since \( h^0(C, E|_C) = 0 \), clearly \( N \neq O_C \), therefore \( h^0(C, N) \geq 2 \). From the inequalities (10) it follows that \( \text{rk}(N) > 1 \), because \( C \) has maximal gonality.

**Step 1.** We prove that \( M \) is a line bundle. Assume that, on the contrary,
\[
\text{rk}(M) = \text{rk}(N) = 2
\]
and consider the elementary modification \( G := \text{Ker}(E \to N) \). Its Chern classes are given as follows:
\[
c_1(G) = -[C], \quad c_2(G) = d + d_N - 2(g-1),
\]
and its discriminant equals $\Delta(G) = -64i + 110 + 8d_N - 14p < 0$, because of (10). In particular, there exists a saturated subsheaf $F \subset G$ which verifies the inequalities

(11) \hspace{1cm} \mu(G) \leq \mu(F) < \mu(E), \quad \text{and}

(12) \hspace{1cm} \xi_{F,G}^2 \geq -\frac{\Delta(G)}{48}.

Write $c_1(F) = \alpha \cdot [C]$ and $\text{rk}(F) = \beta \leq 3$. The above inequality (12) becomes

\[
\left(\frac{\alpha}{\beta} + \frac{1}{4}\right)^2 (2g - 2) \geq -\frac{\Delta(G)}{48}.
\]

We apply (11) for $\mu(F) = \alpha(2g - 2)/\beta$ and obtain

\[
-\frac{1}{4} \leq \frac{\alpha}{\beta} \leq \frac{1}{4},
\]

hence $\alpha = 0$, and the inequality (12) reads in this case $d_N \geq 5i - 10 + p$. Recalling that $d_N \geq g - 1 = 4i - 5 + p$, we obtain a contradiction whenever $i \geq 6$.

\text{Step 2. We construct an elementary modification, in order to reach a contradiction.}

From (10), we have $d_M \geq \frac{9g}{4}$. The composite map $M \rightarrow E|_C \rightarrow K_C \otimes A'$ is not zero, for else $M \rightarrow Q_A$ and since $\mu(Q_A \otimes M') < 0$, one contradicts the semistability of $Q_A$. We set $A_1 := K_C \otimes A' \otimes M'$ and obtain a surjection $F(C)|_C \rightarrow A \otimes A_1$ inducing, as before, an elementary modification

$V := \text{Ker}\{F(C) \rightarrow A \otimes A_1\}$.

By direct computation we show that $\Delta(V) < 0$. Indeed, we compute

$c_1(V) = 2 \cdot [C], \quad c_2(V) = d + 2g - 2 - d_M, \quad \text{hence}$

$\Delta(V) = 8c_2(V) - 3c_1^2(V) = 8(d - d_M - g + 1) = 8(5 - d_M - i) < 0$.

We obtain a destabilizing sheaf $P \subset V$, with $\text{rk}(P) = b \leq 3$ and $c_1(P) := a \cdot [C]$, such that the following inequalities are both satisfied

(13) \hspace{1cm} \left(\frac{a}{b} - \frac{1}{2}\right)^2 (2g - 2) \geq -\frac{\Delta(V)}{48} \quad \text{and} \quad \mu(V) \leq \mu(P) < \mu(F(C)).

The second inequality gives $\frac{1}{2} \leq \frac{a}{b} < \frac{3}{4}$, which leaves two possibilities: either $a = 1$ and $b = 2$, when via (13) one finds that $\Delta(V) \geq 0$, a contradiction, or else $a = 2$ and $b = 3$, when inequalities (13) and (10) clash. \hfill \Box

3. Normal bundle of canonical curves of genus 7

The aim of this section is to prove Theorem 13 and we begin by recalling Mukai’s results [M1] on canonical curves of genus 7. We choose a vector space $U := \mathbb{C}^9$ and a non-degenerate quadratic form $q : U \rightarrow \mathbb{C}$, defining a smooth 8-dimensional quadric $Q \subset \mathbb{P}(U) = \mathbb{P}^9$.

The algebraic group $\text{Spin}(U)$ corresponding to the Dynkin diagram $D_5$ admits two 16-dimensional half-spin representations $S^+$ and $S^-$, which correspond to maximal weights $\alpha^+$ and $\alpha^-$ respectively. The homogeneous spaces $V^\pm := \text{Spin}(U)/P(\alpha^\pm)$ are both 10-dimensional and can be realized as the two irreducible components of the
Grassmannian $G_q(5, U)$ of projective 4-planes inside $\mathbb{P}(U)$ which are isotropic with respect to the quadratic form $q$. From now on, we set

$$V := V^+ \subset \mathbb{P}(S^+) = \mathbb{P}^{15}.$$  

Note that $\text{Aut}(V) = SO(10)$. If $E$ is the restriction to $V$ of the tautological bundle on $G(5, 10)$, one has an exact sequence of vector bundles on $V$:

$$0 \to E^\vee \to U \otimes O_V \to E \to 0. \tag{14}$$

By the adjunction formula, smooth curvilinear sections of $V$ are canonical curves of genus 7 and Mukai [M3] showed that each curve $[C] \in M_7$ with $\text{Cliff}(C) = 3$ appears in this way. Precisely, there is a birational map

$$\alpha : G(7, 16)/SO(10) \dashrightarrow \overline{M}_7, \quad \alpha(\lambda) := [\lambda \cap V],$$

where $\lambda \cong \mathbb{P}^6$. Given a curve $[C] \in M_7$, the inverse $\alpha^{-1}([C])$ is constructed precisely via the twist of the conormal bundle on $C$ mentioned in the introduction.

Let $C \subset \mathbb{P}^6$ be a smooth canonical curve with $\text{Cliff}(C) = 3$, and set $E := N^\vee_C \mathbb{P}^6(2)$. One has an identification $H^0(C, E) = I_2(K_C)$ and $E$ is a globally generated bundle. The tautological map

$$\phi_E : C \to G(5, H^0(C, E))$$

is easily shown to be injective and its image lies on $V$. In particular, the vector bundle $E$ is the restricted spinorial bundle, that is, $E = E|_C$ and one has an exact sequence:

$$0 \to E^\vee \to H^0(C, E) \otimes O_C \to E \to 0. \tag{15}$$

Note that $W^1_4(C) = \emptyset$, while $W^1_5(C)$ is a curve. We are going to make essential use of the following fact:

**Lemma 3.1.** Let $C$ as above and $A \in W^1_5(C)$. Then there are no surjections $E \twoheadrightarrow A$.

**Proof.** We proceed by contradiction. Assume that there is such a pencil $A \in W^1_5(C)$, then use the base point free pencil trick to write the following diagram:

$$\begin{array}{cccccc}
0 & \to & E^\vee & \to & H^0(C, E) \otimes O_C & \to & E \to 0 \\
& & & & & & \\
0 & \to & A^\vee & \to & H^0(C, A) \otimes O_C & \to & A \to 0 \\
& & & & & & \\
0 & & & & & & \\
\end{array} \tag{16}$$

In particular, $H^0(C, E \otimes A^\vee) \neq 0$. Via the identification $H^0(C, E) = I_2(K_C)$, this implies that if $L := K_C \otimes A^\vee \in W^2_7(C)$, then the multiplication map

$$\text{Sym}^2 H^0(C, L) \to H^0(C, L \otimes 2)$$

is not injective. This is possible only if $L$ is not birationally very ample, in particular, $C$ must be trigonal, which is not the case. \qed
We are now in a position to prove that the twist $E$ of the conormal bundle of a canonical curve of genus 7 is stable.

**Proof of Theorem 0.3** Suppose that $0 \to F \to E \to M \to 0$ is a destabilizing sequence for the vector bundle $E$, that is, with $\mu(F) \geq \mu(E) = \frac{24}{7}$. Since $E$ is globally generated, so is any of its quotient, in particular $M$. We distinguish several possibilities, depending on the ranks that appear:

(i) $\text{rk}(F) = 4$ and $M$ is line bundle. Then $\deg(F) \geq 20$, hence $\deg(M) \leq 4$. Since $C$ is not tetragonal, $h^0(C, M) \leq 1$. Note that $M \not\cong \mathcal{O}_C$, for $H^0(C, E^\vee) = 0$. It follows that $M$ is not globally generated, a contradiction.

(ii) $\text{rk}(F) = 3$, and we may assume that $\deg(F) = 5$. Suppose first that $h^0(C, F) = 0$, therefore $h^0(C, K_C \otimes F^\vee) = 1$, and hence $K_C \otimes F^\vee$ is not globally generated. Since one has a surjection $E^\vee(1) \to K_C \otimes F^\vee$, we reach a contradiction by observing that $E^\vee(1)$ is globally generated. Indeed, via Serre duality, this last statement is equivalent to the equality $h^0(C, E(p)) = h^0(C, E) = 10$, for every point $p \in C$. From the exact sequence

$$0 \to E(p) \to M_{KC} \otimes K_C(p) \to K_C^{\otimes 3}(p) \to 0,$$

we obtain that $H^0(C, E(p)) = \text{Ker}\left\{H^0(C, M_{KC} \otimes K_C(p)) \to H^0(C, K_C^{\otimes 3}(p))\right\}$. The conclusion follows, since $H^0(C, M_{KC} \otimes K_C) = H^0(C, M_{KC} \otimes K_C(p))$.

Suppose now that $h^0(C, F) \geq 1$. The case $h^0(C, F) \geq 2$ having been discarded in the course of proving Lemma 3, we assume that $h^0(C, F) = 1$, hence $h^0(C, K_C \otimes F^\vee) = 2$. We obtain that the multiplication map

$$\text{Sym}^2 H^0(C, K_C \otimes F^\vee) \to H^0(C, K_C^{\otimes 2} \otimes F^{\otimes (-2)})$$

is not injective, which contradicts the base point free pencil trick.

(iii) $\text{rk}(F) = 3$, and then $\deg(F) \geq 15$, hence $\deg(M) \leq 9$. This time we may assume that $F$ is stable. If $M$ is not stable, we choose a line subbundle $A \subset M$ of maximal degree, which we pull-back under the surjection $E \to M$, to obtain the exact sequence

$$0 \to G \to E \to M/A \to 0.$$

We obtain that $\deg(M/A) \leq \deg(M)/2 \leq 9/2$, that is, $\deg(M/A) \leq 4$. In particular, $M/A$ is not globally generated, which is again a contradiction, so we can assume that both $F$ and $M$ are stable vector bundles. Since $h^0(C, M) + h^0(C, F) \geq h^0(C, E) = 10$, the strategy is to use the fact that the Mercat statements $(M_2)$ and $(M_3)$ have been established for curves $C$ of genus 7 with maximal Clifford index, that is,

$$\text{Cliff}_2(C) = \text{Cliff}_3(C) = 3,$$

see [LN3] Theorem 4.5. In particular, if both $F$ and $M$ contribute to their respective Clifford indices, that is, $h^0(C, F) \geq 6$ and $h^0(C, M) \geq 4$ respectively, then we write

$$\frac{9}{2} + 3 \leq \frac{3}{2} \gamma(F) + \gamma(M) = \frac{1}{2} \left(\deg(F) + \deg(M)\right) - h^0(C, F) - h^0(C, M) + 5,$$

that is, $h^0(C, F) + h^0(C, M) \leq \frac{10}{2}$, a contradiction.

Assume now that one of the bundles $F$ or $M$ does not contribute to its Clifford index. Since $M$ is globally generated, $h^0(C, M) \geq 2$. We can have $h^0(C, M) = 2$, only when
$M = \mathcal{O}_C^{\oplus 2}$, which is impossible, for $\mathcal{O}_C^{\oplus 2}$ is not a direct summand of $E$. If $h^0(C, M) = 3$, then $\deg(M) \geq 7$, and one has equality if and only if $M = Q_L$, where $L \in W_7^2(C)$. Assuming this to be the case, we choose two points $p, q \in C$ that correspond to a node in the plane model $\phi_L : C \to \mathbb{P}^2$, that is, $A := L(p - q) \in W_5^1(C)$. Then there is a surjection $Q_L \to A$, which by composition gives rise to a surjective morphism $E \to A$. This contradicts Lemma 3.

Thus we may assume that $\deg(M) \geq 8$, and accordingly, $\deg(F) \leq 16$. We may assume this time that $M$ is stable. If $F$ is not stable, then it has a line subbundle $A \hookrightarrow F$ with $\deg(A) \geq 5$, and we are back to case (ii). Thus both $M$ and $F$ are stable bundles, and we proceed precisely like in case (iii).

It is instructive to remark that the normal bundle of a canonical curve of genus $g < 7$ is never stable. More generally we have the following:

**Proposition 3.2.** The normal bundle of a tetragonal canonical curve of genus $g$ is unstable.

**Proof.** More generally, we begin with a $k : 1$ covering $f : C \to \mathbb{P}^1$, and consider the rank $(k - 1)$-vector bundle $\mathcal{F}^\vee := f_*\mathcal{O}_C/\mathcal{O}_p$ on the projective line. Then $\pi : X = \mathbb{P}(\mathcal{F}) \to \mathbb{P}^1$ is a scroll of dimension $k - 1$, which contains the canonical curve $C$ and which can be embedded by the tautological bundle $\mathcal{O}_X(1)$ in $\mathbb{P}^{g-1}$ as a variety of degree $g - k + 1$. Denoting by $H, R \in \text{Pic}(X)$ the class of the hyperplane section and that of the ruling respectively, we have

$$K_X \equiv -(k - 1)H + (g - k - 1)R,$$

whereas obviously $C \cdot H = 2g - 2$ and $C \cdot R = k$. We compute the degree of the normal bundle $N_{C/X}$ and find:

$$\deg(N_{C/X}) = \deg(T_{X|C}) + \deg(K_C) = k(g + k - 1).$$

We write the usual exact sequence relating normal bundles

$$0 \to N_{C/X} \to N_{C/\mathbb{P}^{g-1}} \to N_{X/\mathbb{P}^{g-1}} \otimes \mathcal{O}_C \to 0,$$

and compare the slopes

$$\mu(N_{C/X}) = \frac{k(g + k - 1)}{k - 2} \quad \text{and} \quad \mu(N_{C/\mathbb{P}^{g-1}}) = \frac{2(g - 1)(g + 1)}{g - 2}.$$

We conclude that for $k = 4$ and $g \geq 6$, the normal bundle $N_{C/X}$ is a destabilizing subbundle of $N_{C/\mathbb{P}^{g-1}}$. For $g$ at most 5, every canonical curve of genus $g$ is a complete intersection which obviously produces a destabilizing line subbundle. □
REFERENCES

[A] M. Aprodu, Lazarsfeld-Mukai bundles and applications, in: Commutative Algebra, Expository Papers Dedicated to David Eisenbud on the occasion of his 65th birthday (I. Peeva-ed.), Springer Verlag 2013, 1-23.

[AF] M. Aprodu and G. Farkas, The Green Conjecture for smooth curves lying on arbitrary K3 surfaces, Compositio Math. 147 (2011), 839-851.

[CHM] C. Ciliberto, J. Harris and R. Miranda, On the surjectivity of the Wahl map, Duke Mathematical Journal 57 (1988), 829-958.

[CP] C. Ciliberto and G. Pareschi, Pencils of minimal degree on curves on a K3 surface, Journal für die reine und angewandte Mathematik 460 (1995), 15-36.

[EL] L. Ein and R. Lazarsfeld, Stability and restrictions of Picard bundles with an application to the normal bundles of elliptic curves, in: Complex Projective Geometry, London Mathematical Society Lecture Notes Series 179, Cambridge University Press 1992, 149-156.

[F] G. Farkas, Aspects of the birational geometry of $\overline{M}_g$, Surveys in Differential Geometry, Vol. 14, 2010, 57-110.

[FO1] G. Farkas and A. Ortega, The maximal rank conjecture and rank two Brill-Noether theory, Pure Applied Math. Quarterly, 7 (2011), 1265-1296.

[FO2] G. Farkas and A. Ortega, Higher rank Brill-Noether on sections of K3 surfaces, International Journal of Mathematics 23 (2012), 1250075.

[GMN] I. Grzegorczyk, V. Mercat and P.E. Newstead, Stable bundles of rank 2 with 4 sections, International Journal of Mathematics 22 (2011), 1743-1762.

[HL] D. Huybrechts and M. Lehn, The geometry of the moduli space of sheaves, Second Edition, Cambridge University Press 2010.

[L1] R. Lazarsfeld, Brill-Noether-Petri without degenerations, Journal of Differential Geometry, 23 (1986), 299-307.

[L2] R. Lazarsfeld, A sampling of vector bundle techniques in the study of linear series, in: Lectures on Riemann Surfaces, World Scientific Press, Singapore, 1989, 500-559.

[LMN] H. Lange, V. Mercat and P.E. Newstead, On an example of Mukai, Glasgow Mathematical Journal, 54 (2012), 261-271.

[LN1] H. Lange and P.E. Newstead, Clifford indices for vector bundles on curves, arXiv:0811.4680, in: Affine Flag Manifolds and Principal Bundles (A. Schmitt-ed.), Trends in Mathematics, 165-202, Birkhäuser 2010.

[LN2] H. Lange and P.E. Newstead, Further examples of stable bundles of rank 2 with 4 sections, Pure Applied Math. Quarterly 7 (2011), 1517-1528.

[LN3] H. Lange and P.E. Newstead, Bundles of rank 3 on curves of Clifford index 3, Journal of Symbolic Computation 57 (2013), 3-18.

[LC] M. Lelli-Chiesa, Stability of rank 3 Lazarsfeld-Mukai bundles on K3 surfaces, Proceedings of the London Mathematical Society 107 (2013) 451-479.

[Me1] V. Mercat, Clifford’s theorem and higher rank vector bundles, International Journal of Mathematics 13 (2002), 785-796.

[M1] S. Mukai, Biregular classification of Fano 3-folds and Fano manifolds of coindex 3, Proceedings of the National Academy of Sciences USA, Vol. 86, 3000-3002, 1989.

[M2] S. Mukai, Curves and Grassmannians, in: Algebraic Geometry and Related Topics (1992), eds. J.-H. Yang, Y. Namikawa, K. Ueno, 19-40.

[M3] S. Mukai, Curves and symmetric spaces I, American Journal of Mathematics 117 (1995), 1627-1644.

[M4] S. Mukai, Curves and symmetric spaces II, Annals of Mathematics 172 (2010), 1539-1558.

[V] C. Voisin, Sur l’application de Wahl des courbes satisfaisant la condition de Brill-Noether-Petri, Acta Mathematica 168 (1992), 249-272.

[W] J. Wahl, Gaussian maps on algebraic curves, Journal of Differential Geometry 32 (1990), 77-98.
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