Research Article

Relative Gottlieb Groups of Embeddings between Complex Grassmannians

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Let $\text{Gr}(k, n)$ be the complex Grassmann manifold of $k$-linear subspaces in $\mathbb{C}^n$. We compute rational relative Gottlieb groups of the embedding $i: \text{Gr}(k, n) \rightarrow \text{Gr}(k, n + r)$ and show that the $G$-sequence is exact if $r \geq k(n - k)$.

1. Introduction

We work in the category of spaces having the homotopy type of simply connected CW complexes of finite type. We denote by $h: X \rightarrow X_\mathbb{Q}$ the rationalization of $X$ [1, 2]. Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a pointed continuous mapping and map $(X, Y; f)$ be the component of $f$ in the space of all continuous maps $g: X \rightarrow Y$. Consider the evaluation map $\text{ev}: (X, Y; f) \rightarrow Y$ at the base point $x_0$, that is, $\text{ev}(g) = g(x_0)$. The $n$th evaluation subgroup of $f$, $G_n(Y, X; f)$, is the image of $\pi_n(\text{ev})$ in $\pi_n(Y)$ [3]. In the special case where $X = Y$ and $f = 1_X$, one obtains the Gottlieb group $G_n(X)$ of $X$ [4]. Gottlieb groups play an important role in topology. For instance, if $G_n(X) = 0$, then any fibration $X \rightarrow E \rightarrow S^{n+1}$ admits a section (Corollary 2–7 in [4]).

In [2], Lee and Woo introduce relative evaluation groups $G_n^\text{rel}(Y, X; f)$ and obtain a long sequence,

\begin{equation}
\cdots \rightarrow G_{n+1}^\text{rel}(Y, X; f) \rightarrow G_n(X) \rightarrow G_n(Y, X; f) \rightarrow G_n^\text{rel}(Y, X; f) \rightarrow \cdots,
\end{equation}

called $G$-sequence [5]. This sequence is exact in some cases, for instance, if $f$ is a homotopy monomorphism [6].

2. Rational Relative Gottlieb Groups

The rationalization $h: Y \rightarrow Y_\mathbb{Q}$ induces a rationalization $h_*: \text{map}(X, Y; f) \rightarrow \text{map}(X, Y; h \circ f)$ [7]. Therefore,

\[ \text{ev}_*(\pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}) \equiv \text{ev}_*(\pi_n(\text{map}(X, Y_\mathbb{Q}; h \circ f))). \]

(2)

In this paper, we study the $G$-sequence of the natural inclusion $\text{Gr}(k, n) \rightarrow \text{Gr}(k, n + r)$ using models of function spaces in rational homotopy [8, 9]. In particular, we show that the $G$-sequence is exact if $r \geq k(n - k)$. We work with algebraic models in rational homotopy theory introduced by Sullivan and Quillen [10, 11]. In this section, we give relevant definitions and fix notation. Details can be found in [1]. All vector spaces and algebras are over the field of rational numbers $\mathbb{Q}$.

Let $(A, d)$ be a cochain algebra. The degree of an homogeneous element $a \in A^p$ is written $|a|$. We assume that
(A, d) is 1-connected, that is, $H^0(A, d) = \mathbb{Q}$ and $H^1(A, d) = 0$. The algebra A is called commutative if $ab = (-1)^{|a||b|}ba$ for homogeneous elements $a, b \in A$.

**Definition 1.** A commutative differential graded algebra (cdga, for short) $(A, d)$ is called a Sullivan algebra if $A = S(V^{\text{even}}) \otimes E(V^{\text{odd}})$, where $V = \oplus_{k \geq 2} V^k$. It will be denoted by $(\Lambda V, d)$.

Moreover, a Sullivan algebra $(\Lambda V, d)$ is called minimal if $dV \subset \Lambda^2 V$. A Sullivan model of $(A, d)$ is given by a Sullivan algebra $(\Lambda V, d)$ together with a quasi-isomorphism $f: (\Lambda V, d) \rightarrow (A, d)$. It is unique up to isomorphism.

**Definition 2.** If $X$ is a simply connected space of finite type, then the (minimal) Sullivan model of $X$ is the (minimal) Sullivan model of cdga $A_{pt}(X)$ of polynomial differential forms on $X$ [1, 5]. A simply connected topological space $X$ is called formal if there exists a quasi-isomorphism $(\Lambda V, d) \rightarrow H^*(X, \mathbb{Q})$, where $(\Lambda V, d)$ is a Sullivan model of $X$. Formal spaces include homogeneous spaces $G/H$, where $G$ and $H$ have the same rank.

The complex Grassmann manifold $Gr(k, n)$ is the space of $k$-dimensional subspaces of $\mathbb{C}^n$. Moreover, $G(k, n) = U(n)/(U(k) \times U(n-k))$, where $U(n)$ is the unitary group. Hence, $G(k, n)$ is formal (see also [11, 12]). As $G(k, n) \equiv G(n-k, n)$, we will assume that $k \leq n/2$. As $G(k, n)$ is a formal, its Sullivan model can be computed from its cohomology algebra. Precisely,$$
H^*(Gr(k, n)) = \frac{\Lambda(h_{k-1}, \ldots, h_{n-k+1}, x_{2k})}{(h_{n-k+1}, \ldots, h_{n-k+1})}, \quad (3)
$$
where $h_i$ is the polynomial of degree $2j$ in the Taylor expansion of the expression $1/(1 + x_2 + \cdots + x_{2k})$ [13]. A Sullivan model is given by
$$
(\Lambda V, d) = (\Lambda(x_2, \ldots, x_{2k}, x_{2n-2k+1}, \ldots, x_{2n-1}), d), \quad (4)
$$
where $dx_{2i} = 0$ and $dx_{2n-2k+2i} = h_{n-k+i}$, $i = 1, \ldots, k$. Moreover, this model is minimal.

Let
$$
(\Lambda V, d) = (\Lambda(x_2, \ldots, x_{2k}, x_{2n-2k+2i-1}, \ldots, x_{2n-1}), d), \quad (\Lambda W, d) = (\Lambda(y_2, \ldots, y_{2k}, y_{2n-2k+1}, \ldots, y_{2n-1}), d),
$$
be respective minimal Sullivan models of $Gr(k, n + r)$ and $Gr(k, n)$. A Sullivan model of the inclusion $i: Gr(k, n) \rightarrow Gr(k, n + r)$ is then
$$
\phi: (\Lambda V, d) \rightarrow (\Lambda W, d), \quad (6)
$$
which is defined by
$$
\phi(x_2) = y_2, \ldots, \phi(x_{2k}) = y_{2k}, \quad \phi(x_{2n-2k+2i-1}) = \sum_{j=0}^{k-1} p_{ij} y_{2n-2k+2j+1},
$$
where $p_{ij}$ is a polynomial of degree $2(r+i-j)$ in $y_{2j}, \ldots, y_{2kj}$, for $i, j = 0, 1, 2, \ldots, k-1$, provided that $r+i-j \geq 0$.

The polynomials $p_{ij}$ encode the relationships between $h_i$'s. They can be explicitly expressed from the equality:
$$
(1 + x_2 + \cdots + x_{2k})(1 + h_1 + h_2 + \cdots) = 1. \quad (8)
$$
For instance, for $k = 2$,
$$
h_1 = -4, \quad h_2 = x_4 - x_4, \quad h_3 = -x_2^2 + 2x_2x_4, \quad h_4 = x_4^2 - 3x_2^2x_4 + x_4^2, \quad h_5 = -x_2^2h_2 - x_4, \quad h_6 = (x_4^2 - x_2^2)h_4 + x_2x_4h_3, \quad h_7 = h_4h_1 + (x_2^2x_4 + x_4^3)h_3.
$$

**Example 1.** The inclusion $Gr(2, 4) \rightarrow Gr(2, 7)$ has a Sullivan model:
$$
\phi: (\Lambda(x_2, x_4, x_{11}, x_{13}), d) \rightarrow (\Lambda(y_2, y_4, y_5, y_7), d), \quad (10)
$$
where
$$
dx_2 = dx_4 = 0, \quad dx_{11} = (x_2^2 - x_4)h_1 + x_2x_4h_3, \quad dx_{13} = h_4h_3 + (-x_2^2x_4 + x_4^2)h_3, \quad dy_2 = dy_4 = 0, \quad dy_5 = -y_2^3 + 2y_2y_4, \quad dy_7 = y_4^3 - 3y_2^2y_4 + y_4^3, \quad (11)
$$
$$
\phi(x_2) = y_2, \quad \phi(x_4) = y_4, \quad \phi(x_{11}) = y_2y_4y_5 + (y_2^3 - y_4)y_7, \quad \phi(x_{13}) = (-y_2^2y_4 + y_4^2)y_5 + (-y_2^3 + 2y_2y_4)y_7.
$$
We note that $-y_2y_4 + y_4^2 = d(y_2y_4 + y_7)$; therefore,
$$
\phi(x_{13}) = d(y_2y_4 + y_7) + d(y_5)y_7. \quad (12)
$$
Recall that if $\phi: (A, d_A) \rightarrow (B, d_B)$ is a map of chain complexes; the mapping cone of $\phi$, denoted by $\text{Rel}(\phi)$, is defined by
$$
\text{Rel}(\phi)_* = \langle sA_{-1} \oplus B_*, D \rangle, \quad (13)
$$
where the differential is defined by $D(sa, b) = (-sd_A(a), \phi(a) + d_B(b))$ [9] or p. 46 in [14]. Define chain maps $J: B_n \rightarrow \text{Rel}_n(\phi)$ and $P: \text{Rel}_n(\phi) \rightarrow A_{n-1}$ by $J(b) = (0, b)$ and $P(sa, b) = a$. There is an exact sequence of chain complexes:
0 \longrightarrow B_* \xrightarrow{\partial} \text{Rel}_* (\phi) \xrightarrow{p} A_{*-1} \longrightarrow 0, \quad (14)

which induces a long exact sequence:

\[ H_n(B) \xrightarrow{H_n(f)} H_n(\text{Rel}(\phi)) \xrightarrow{H_n(p)} H_{n-1}(A) \xrightarrow{\partial} H_{n-1}(B) \rightarrow \cdots \tag{15} \]

(see Proposition 4.3 in [14]).

**Definition 3.** Let \( \phi: (A, d) \longrightarrow (B, d) \) be a morphism of cdga’s. A \( \phi \)-derivation of degree \( k \) is a linear mapping \( \theta: A^* \longrightarrow B^{* - k} \) such that \( \theta(ab) = \theta(a)\phi(b) + (-1)^{|a|}\phi(a)\theta(b) \). We denote by \( \text{Der}_n(A, B; \phi) \) the vector space of \( \phi \)-derivations of degree \( n \) and by \( \text{Der}(A, B; \phi) = \oplus_n \text{Der}_n(A, B; \phi) \) the \( \mathbb{Z} \)-graded vector space of all \( \phi \)-derivations. The differential on \( \text{Der}(A, B; \phi) \) is defined by \( \partial \theta = d\theta - (-1)^{|\theta|} \theta d \). We will restrict to derivations of positive degree; however, in degree one, we only consider those derivations which are cyclic.

If \( \phi: A \longrightarrow A \) is the identity mapping, we simply write \( \text{Der}A \) for \( \text{Der}(A, A; 1_A) \). Moreover, if \( A = \wedge V \), where \( \{v_1, v_2, \ldots \} \) is a basis of \( V \) and \( \phi: (\wedge V, d) \longrightarrow (B, d) \) is a morphism of cdga’s, we denote by \( (v_i, b) \) the unique \( \phi \)-derivation \( \theta \) such that \( \theta(v_i) = b \) and zero on other elements of the basis.

Define the Gottlieb group of \( (\wedge V, d) \):

\[ \text{Der}(\wedge V, \wedge W; \phi) \xrightarrow{\phi^*} \text{Der}(\wedge V; \phi) \xrightarrow{f} \text{Rel}(\phi^*) \xrightarrow{\varepsilon_*} \text{Rel}(\phi^*; \varepsilon_*) \xrightarrow{\partial} \cdots \tag{17} \]

Then, rational evaluation subgroups are corresponding images in the lower ladder induced in homology by vertical maps. Therefore, there is a long sequence:

\[ \cdots \longrightarrow G_n(\wedge V) \xrightarrow{H(\phi)} G_n(\wedge V, \wedge W; \phi) \xrightarrow{H(\tilde{\phi})} G_n(\wedge V, \wedge W; \phi) \xrightarrow{H(\tilde{\phi})} \cdots \tag{18} \]

We will use the following result for our computations (Theorem 2.1 in [9] or Corollary 1 in [15]).

**Theorem 1** (see [9]). Let \( f: X \longrightarrow Y \) be a map between simply connected CW complexes, where \( X \) is of finite type and \( \phi: (\wedge V, d) \longrightarrow (\wedge W, d) \) its Sullivan model. The long exact sequence induced by the map \( f: \text{map}(X, X; 1_X) \longrightarrow \text{map}(X, Y; f) \) on rational homotopy groups is equivalent to the long exact sequence of \( \phi^*: \text{Der}(\wedge W, d) \longrightarrow \text{Der}(\wedge V, \wedge W; \phi) \). \( \tag{19} \)

We consider the particular case, where \( f \) is the inclusion \( i: \text{Gr}(k, n) \longrightarrow \text{Gr}(k, n + r) \), where \( r \geq 1 \) and its Sullivan model \( \phi: (\wedge V, d) \longrightarrow (\wedge W, d) \) as given in equation (6).

**Theorem 2.** Let \( \phi: (\wedge V, d) \longrightarrow (\wedge W, d) \) be a Sullivan model of the inclusion \( i: \text{Gr}(k, n) \longrightarrow \text{Gr}(k, n + r) \), where \( r \geq k(n - k) \):

1. \( G_* (\wedge V, \wedge W; \phi) \equiv V^\# \), the dual of \( V \)
2. \( G_* (\wedge V, \wedge W; \phi) \equiv sG_* (\wedge V, \wedge W; \phi) \equiv sG_* (\wedge V)^{BV^\#} \)
Proof

(1) Recall that \( \Lambda V = \Lambda (x_2, \ldots, x_{2k}, x_{2n+2r-2k+1}, \ldots, x_{2n+1-2r}) \), \( \Lambda W = \Lambda (y_2, \ldots, y_{2k}, y_{2n+2k+1}, \ldots, y_{2n-1}) \), and \( \phi: \Lambda V \to \Lambda W \) are defined by \( \phi(x_2) = y_2, \ldots, \phi(x_{2k}) = y_{2k} \), \( \phi(x_{2n+2r-2k+1}) = \sum_{i=1}^{k} p_{ij} y_{2n-2k+2j-1} \), and \( p_{ij} \) is a polynomial of degree 2 \((r + i - j)\) in \( y_{2n-2k+2j-1} \) and \( i \in \{1, \ldots, k\} \).

We consider the composition \( \phi: \Lambda V, d \to (\Lambda W, d) \), we say that \( \Lambda V \) is a quasi-isomorphism, then the \( G \)-sequence of the inclusion is computed from the long exact sequence induced by the cone of the map:

\[
\phi^* : \text{Der}(\Lambda V, H^*(\Lambda W); p) \to \text{Der}(\Lambda V, H^*(\Lambda W); \phi).
\]

Each of the derivations \( x_{2n+2r-2k+1}^* \in (\Lambda V, H^*(\Lambda W); \phi) \) is a cycle of degree at least \( 2k + 2r + 2i - 1 > 2k + 2r \) and cannot be boundary as all even degree derivations in \( \text{Der}(\Lambda V, H^*(\Lambda W); \phi) \) are of degree at most 2k. Hence, \( \{x_{2n+2r-2k+1}^*\} \) is nonzero in \( G_*(\Lambda V, H^*(\Lambda W); \phi) \).

Consider the derivations \( x_i^* = (x_{2i}, 1) \in \text{Der}(\Lambda V, H^*(\Lambda W), d), i \in \{1, \ldots, k\} \). Then,

\[
(\delta x_i^*)(x_{2n+2r-2k+2j-1}) \in H^{2(n+r-k)+j}(\Lambda W, d).
\]

Moreover, as \( 1 \leq i, j \leq k \), then \( j - i \geq k \). Therefore,

\[
2(n + r - k + j - i) \geq 2(n + r - k + 1)
\]

\[
\geq 2(r + 1), \quad \text{as } n \geq 2k.
\]

Therefore, \( (\delta x_i^*)(x_{2n+2r-2k+2j-1}) \in H^{2k(n-k)+j} = 0 \).

Hence, \( x_i^* \) is a cycle for \( i = 1, \ldots, k \). Moreover, \( x_{2i}^* \)

cannot be a boundary as all odd degree derivations are of degree at least \( 2n + 2r - 2k + 1 - 2k(n - k) > 2(n - k) + 1 \).

Therefore, \( x_{2n+2r-2k+2j-1}^* \) are cycles which cannot be boundaries for degree reasons. Hence, \( G_*(\Lambda V, H^*(\Lambda W, d), \phi) \equiv V^\phi \).

(2) First, we note that \( H_{\text{even}}((\Lambda V, H^*(\Lambda W, d); p)) = 0 \), and consequently, \( G_{\text{even}}((\Lambda V, H^*(\Lambda W, d); p) = 0 \) [1, 16]. Moreover, a straightforward calculation shows that

\[
G_{\text{odd}}((\Lambda V, H^*(\Lambda W, d), p) \equiv (y_{2n-2k+1}^*, \ldots, y_{2n-1}^*).
\]

(23)

We consider the vector space:

\[
\text{Rel}(\phi^*) = s \text{Der}(\Lambda V, H^*(\Lambda W); p) \oplus \text{Der}(\Lambda V, \Lambda W; \phi),
\]

(24)

where the differential is defined by \( D(\alpha, \beta) = (s \delta \alpha, \phi^*(\alpha) + \delta \beta) \).

Consider \( W_1^* = \langle y_{2n-2k+1}^*, \ldots, y_{2n-1}^* \rangle \) in \( \text{Der}(\Lambda V, H^*(\Lambda W); p) \). For degree reasons, \( \phi^*(W_1^*) = 0 \). Therefore, \( D(y^*, 0) = 0 \), for \( y^* \in W_1^* \). Hence, \( s y_{2n-2k+1}^*, \ldots, s y_{2n-1}^* \) represent nonzero homology classes in \( G_{*}^d(\Lambda V, H^*(\Lambda W); \phi) \). We conclude that \( G_{*}^d(\Lambda V, H^*(\Lambda W); \phi) = s G_*(\Lambda V, H^*(\Lambda W); p) \oplus G_*(\Lambda V, H^*(\Lambda W); \phi) \).

\[
\square
\]

Corollary 1. If \( r \geq k(n - k) \), then the rational G-sequence of the inclusion \( i: Gr(k, n) \to Gr(k, n + r) \) is exact.

Proof. It comes from the previous lemma that the G-sequence is

\[
0 \to G_*(\Lambda V, H^*(\Lambda W); \phi) \to G_*(\Lambda V, H^*(\Lambda W); \phi) \oplus s G_*(\Lambda V, H^*(\Lambda W); p)
\]

\[
\to G_*(\Lambda V, H^*(\Lambda W); p) \to 0,
\]

(25)

which is exact.

3. Inclusion \( Gr(k, n) \to Gr(k, n + 1) \)

In the range \( 1 \leq r < k(n - k) \), the G-sequence of the inclusion \( Gr(k, n) \to Gr(k, n + r) \) is more challenging to characterize, as shown in the following example.
\( \phi^* : \text{Der} (B, H^* (B); p) \longrightarrow \text{Der} (A, H^* (B); \varphi) \),

where \( \varphi = p \circ \phi \). Moreover, \( G_\ast (B, H^* (B); p) = \langle y_1^2, y_2^2 \rangle \),

where \( y_1^2 = (y_1, 1) \) and similarly \( y_2^2 = (y_2, 1) \). Furthermore, \( \delta x_1^2 = 0 \); hence, \( [x_1^2] \) represents a nonzero homology class in \( \text{Der} (A, H^* (B); \varphi) \). A simple calculation shows that

\[
\delta x_1^2 = (x_{11}, \omega / 2), \quad \text{where } \omega = [x_1^2].
\]

Hence,

\[
G_\ast (A, H^* (B); \varphi) = \langle [x_1^2], [x_1^3], [x_1^6] \rangle.
\]

(27)

Consider

\[
\text{Rel}_\ast (\varphi^*) = (s \text{Der} (B, H^* (B); p) \circ \text{Der} (A, H^* (B); \varphi), D).
\]

(28)

\[
G_\ast^{rel} (A, H^* (B); \varphi) \xrightarrow{H_\ast (P)} G_\ast (B, H^* (B); p) \xrightarrow{H_\ast (\varphi^*)} G_\ast (A, H^* (B); \varphi),
\]

which is not exact.

In the same way,

\[
G_\ast^{rel} (A, H^* (B); \varphi) \xrightarrow{H_\ast (P)} G_\ast (B, H^* (B); p) \xrightarrow{H_\ast (\varphi^*)} G_\ast (A, H^* (B); \varphi),
\]

is not exact. Moreover, \( H_\ast (J) : G_\ast (A, H^* (B); \varphi) \longrightarrow G_\ast^{rel} (A, H^* (B); \varphi) \) is an isomorphism.

Although the G-sequence of the inclusion \( \text{Gr} (k, n) \longrightarrow \text{Gr} (k, n + r) \) might not be exact for some values of \( 1 \leq r < k(n-k) \), we have the following result for \( r = 1 \).

**Theorem 3.** Let \( \varphi : (\land V, d) \longrightarrow (\land W, d) \) be a Sullivan model of the inclusion \( \text{Gr} (k, n) \longrightarrow \text{Gr} (k, n + 1) \):

1. \( G^{rel} (\land V, \land W; \varphi) \) has dimension 1
2. The G-sequence of the inclusion \( \text{Gr} (k, n) \longrightarrow \text{Gr} (k, n + 1) \) is not exact.

**Proof.** Recall from Section 2 that the minimal Sullivan model of \( \text{Gr} (k, n) \) is \( (\land W, d) \), where

\[
W = \langle y_2, y_3, \ldots, y_{2k}, y_{2k+1}, \ldots, y_{2n-1} \rangle,
\]

\[
dy_2 = \cdots = dy_{2k} = 0,
\]

\[
dy_{2(n-k+1)} = h_{n-k+1}, \quad \text{for } i = 1, \ldots, k.
\]

Similarly, a model of \( G(k, n + 1) \) is \( (\land V, d) \), where

\[
V = \langle x_2, \ldots, x_{2k}, x_{2k+1}, x_{2k+2}, \ldots, x_{2n+1} \rangle,
\]

\[
dx_2 = \cdots = dx_{2k} = 0,
\]

\[
dx_{2(n-k+1)} = h_{n-k+1}, \quad \text{for } i = 1, \ldots, k.
\]

We have the following relations:

\[
\phi^*(y_2^i) = x_{2i}, \quad \text{for } i = 1, \ldots, n
\]

\[
\phi^*(y_{2n-1}) = x_{2n-1} - (x_{2n+1}, y_2)
\]

\[
\phi^*(y_{2n-3}) = x_{2n-3} - (x_{2n+1}, y_4),
\]

\[
\phi^*(y_2(i-k+2)) = x_{2(i-k+2)} - (x_{2n+1}, y_{2k-2})
\]

\[
\phi^*(y_2(i-k+1)) = -x_{2n+1}, y_{2k}
\]

\[
\phi^*(y_2(n-k+1)) = -x_{2n+1}, y_{2k}
\]

(39)
As a result in
\[ \text{Rel}(\phi^*) = \text{sDer}(\Lambda W, H^* (\Lambda W); p) \oplus \text{Der}(\Lambda V, H^* (\Lambda W); \varphi), \]
we have the following relations:
\[ D(0, x_{2(n-k)+2i+1}^*) = 0, \quad \text{for } i = 1, \ldots, k \]
\[ D(s y_{2n-1}^*, 0) = (0, x_{2n-1}^* - (x_{2n+1}, y_4)) \]
\[ D(s y_{2n-3}^*, 0) = (0, x_{2n-3}^* - (x_{2n+1}, y_4)) \]
\[ \vdots \]
\[ D(s y_{2n-2k+3}^*, 0) = (0, x_{2n-2k+3}^* - (x_{2n+1}, y_{2k-2})) \]
\[ D(s y_{2n-2k+1}^*, 0) = (0, -x_{2n+1}, y_{2k}). \]

We consider the commutative diagram:
\[ \begin{array}{ccc}
\text{Der}(\Lambda W, H^* (\Lambda W); p) & \xrightarrow{\phi^*} & \text{Der}(\Lambda V, H^* (\Lambda W); \varphi) \\
\downarrow \epsilon_* & & \downarrow \epsilon_* \\
\text{Der}(\Lambda W, Q; \epsilon) & \xrightarrow{\tilde{\phi}^*} & \text{Der}(\Lambda V, Q; \epsilon)
\end{array} \]
\[ \text{for } i = 1, \ldots, k - 1, \quad \text{and} \]
\[ G_{2m+1}(\Lambda V, H^* (\Lambda W); p) \xrightarrow{\eta(\tilde{\phi})} G_{2m+1}^{\text{rel}}(\Lambda V, H^* (\Lambda W); \varphi), \]
and a nonexact part,
\[ 0 \to G_{2n-2k+1}(\Lambda W, H^* (\Lambda W); p) \to 0. \]

Example 3. We consider a model of the inclusion \( \text{Gr}(2, 4) \to \text{Gr}(2, 5) \) which is of the form
\[ \phi: (\Lambda V, d) = (\Lambda (x_2, x_4, x_5, y_2), d) \to (\Lambda (y_2, y_4, y_5, y_7), d) = (\Lambda W, d), \]
\[ \text{where } dx_2 = dx_4 = 0, \quad dx_7 = x_2^4 - 3x_2^2x_4 + x_4^3, \]
\[ dx_4 = -x_2(x_2^4 - 3x_2^2x_4 + y_7^3) - x_4(x_2^4 + 2x_2x_4), \quad dy_2 = dy_4 = 0, \quad dy_5 = y_2^3 + 2y_2y_4, \]
\[ dy_7 = y_2^4 - 3y_2y_4 + y_7^4, \quad \phi(x_2) = y_2, \quad \phi(x_4) = y_4, \quad \phi(x_7) = y_7, \quad \text{and} \]
\[ \phi(y_2) = -y_2y_5 - y_4y_7. \]

We compose which the quasi-isomorphism \( p: (\Lambda W, d) \to H^* (\Lambda W, d) \) to get \( \varphi: (\Lambda V, d) \to H^* (\Lambda W, d). \)
\[ \text{Rel}(\phi)_* = s\text{Der}(\Lambda W, H^* (\Lambda W); p) \oplus \text{Der}(\Lambda V, H^* (\Lambda W); \varphi), \]
we have the following relations:
\[
D((sy^e_0,0)) = (0, (x_9, -y_9)),
\]
\[
D((sy^e_7,0)) = (0, x^*_7 + (x_9, -y_9)).
\]

Consider
\[
\text{Rel}_\pi = (\text{sDer}(\Lambda W, Q; \epsilon) @ \text{Der}(\Lambda V, Q; \epsilon), D) \equiv \left( sW^\# @ V^\#, D \right),
\]
where
\[
D(s\tilde{y}^*_0, 0) = (0, 0),
\]
\[
D(s\tilde{y}^*_7, 0) = (0, \tilde{x}^*_7),
\]
\[
D(0, \tilde{x}^*_7) = D(0, \tilde{x}_9) = (0, 0).
\]

Hence,
\[
H_*(\text{Rel}_\pi^*) \equiv \langle [(s\tilde{y}^*_0, 0)], [(0, \tilde{x}^*_9)] \rangle.
\]

However, \(\text{im} \epsilon_* \subseteq \langle [(0, \tilde{x}^*_9)] \rangle\). Therefore, \(G^s_* (\Lambda V, H^*(\Lambda W, d); \phi) \equiv \langle \langle [x^*_7], [\tilde{x}^*_9] \rangle \rangle\) and \(G_* (\Lambda V, H^*(\Lambda W, d), p) = \langle \langle [\tilde{y}^*_7], [\tilde{y}^*_9] \rangle \rangle\), then the G-sequence reduces to exact nonzero fragments:
\[
0 \longrightarrow G_5(\Lambda V, H^*(\Lambda W, d); p) \longrightarrow 0, \quad 0 \longrightarrow G_2(\Lambda V, H^*(\Lambda W, d); p) \longrightarrow 0, \quad 0 \longrightarrow 0.
\]

and a nonexact sequence,
\[
0 \longrightarrow G_5(\Lambda W, H^*(\Lambda W, d); p) \longrightarrow 0.
\]

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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