Zero-sum constants related to the Jacobi symbol

Santanu Mondal, Krishnendu Paul, Shameek Paul *

Ramakrishna Mission Vivekananda Educational and Research Institute, Belur, Dist. Howrah, 711202, India

Abstract

For a weight-set $A \subseteq \mathbb{Z}_n$, the $A$-weighted Davenport constant $D_A(n)$ is defined to be the smallest natural number $k$ such that any sequence of $k$ elements in $\mathbb{Z}_n$ has an $A$-weighted zero-sum subsequence and the constant $C_A(n)$ is defined to be the smallest natural number $k$ such that any sequence of $k$ elements in $\mathbb{Z}_n$ has an $A$-weighted zero-sum subsequence of consecutive terms.

When $n$ is odd, for $x \in \mathbb{Z}_n$ let $(\frac{x}{n})$ be the Jacobi symbol and $S(n) = \{ x \in U(n) : (\frac{x}{n}) = 1 \}$. We compute these constants for the weight-set $S(n)$. For a prime divisor $p$ of $n$, we also compute these constants for the weight-set $L(n;p) = \{ x \in U(n) : (\frac{x}{n}) = (\frac{x}{p}) \}$. We show that even though these weight-sets may have half the size of $U(n)$, they can have the same constants as for $U(n)$.

Keywords: Davenport constant, Jacobi symbol, Zero-sum sequence

AMS Subject Classification: 11B50

1 Introduction

The following definition was given in [6].

Definition 1.1. For a weight-set $A \subseteq \mathbb{Z}_n$, the $A$-weighted Davenport constant $D_A(n)$ is defined to be the least positive integer $k$, such that any sequence in $\mathbb{Z}_n$ of length $k$ has an $A$-weighted zero-sum subsequence.

The following definition was given in [12].
Definition 1.2. For a weight-set $A \subseteq \mathbb{Z}_n$, the $A$-weighted constant $C_A(n)$ is defined to be the least positive integer $k$, such that any sequence in $\mathbb{Z}_n$ of length $k$ has an $A$-weighted zero-sum subsequence of consecutive terms.

Let $U(n)$ denote the multiplicative group of units in the ring $\mathbb{Z}_n$, and let $U(n)^2 = \{ x^2 : x \in U(n) \}$. For an odd prime $p$, let $Q_p$ denote the set $U(p)^2$. For $n$ squarefree, let $\Omega(n)$ denote the number of distinct prime divisors of $n$.

Let $m$ be a divisor of $n$. We refer to the ring homomorphism $f_{n|m} : \mathbb{Z}_n \to \mathbb{Z}_m$ given by $a + n\mathbb{Z} \mapsto a + m\mathbb{Z}$ as the natural map. As this map sends units to units, we get a group homomorphism $U(n) \to U(m)$ which we also refer to as the natural map.

The Jacobi symbol is defined in Section 2 for odd $n$. It is denoted by $\left( \frac{a}{n} \right)$. The following are some of the results in this paper. Except for the first result, we assume that $n$ is an odd, squarefree number whose every prime divisor is at least seven.

- Let $S(n) = \{ x \in U(n) : \left( \frac{x}{n} \right) = 1 \}$.
  - If $n$ is prime, then $D_{S(n)}(n) = 3$, and $D_{S(n)}(n) = \Omega(n) + 1$ otherwise.
  - If $n$ is prime, then $C_{S(n)}(n) = 3$, and $C_{S(n)}(n) = 2^{\Omega(n)}$ otherwise.

- Let $L(n;p) = \{ x \in U(n) : \left( \frac{x}{n} \right) = \left( \frac{x}{p} \right) \}$ where $p$ is a prime divisor of $n$.
  - If $\Omega(n) = 2$, then $D_{L(n;p)}(n) = 4$, and $D_{L(n;p)}(n) = \Omega(n) + 1$ otherwise.
  - If $\Omega(n) = 2$, then $C_{L(n;p)}(n) = 6$, and $C_{L(n;p)}(n) = 2^{\Omega(n)}$ otherwise.

Remark 1.3. In [11] it was shown that if $A = \mathbb{Z}_n \setminus \{0\}$ and $B = \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, we have $D_A(n) = D_B(n)$. We make a similar observation in this paper. In Proposition 2.2 we show that $S(n)$ is a subgroup of $U(n)$ having index two when $n$ is not a square. From [8] and [12] we see that when $n$ is odd, we have $D_{U(n)}(n) = \Omega(n) + 1$ and $C_{U(n)}(n) = 2^{\Omega(n)}$. So from Theorems 3.3 and 3.4 we see that if in addition $n$ is not a prime, we have $D_{S(n)}(n) = D_{U(n)}(n)$ and $C_{S(n)}(n) = C_{U(n)}(n)$. Thus, even though these weight-sets may have different sizes, they can have the same constants. From Theorems 4.2 and 5.4 we see that when $\Omega(n) \neq 2$, we have $D_{L(n;p)}(n) = D_{U(n)}(n)$ and $C_{L(n;p)}(n) = C_{U(n)}(n)$.

If $p$ is a prime divisor of $n$, we use the notation $v_p(n) = r$ to mean that $p^r \mid n$ and $p^{r+1} \nmid n$. Let $p$ be a prime divisor of $n$ and $v_p(n) = r$. We denote the image in $U(p^r)$ of $x \in U(n)$ under $f_{n|p^r}$ by $x^{(p)}$. For a sequence $S = (x_1, \ldots, x_l)$ in $\mathbb{Z}_n$, let $S^{(p)}$ denote the sequence $(x_1^{(p)}, \ldots, x_l^{(p)})$ in $\mathbb{Z}_{p^r}$ which is the image of $S$ under the $f_{n|p^r}$. The following statement is Observation 2.2 in [8].
Observation 1.4. Let $S$ be a sequence in $\mathbb{Z}_n$. Suppose for every prime divisor $p$ of $n$, the sequence $S^{(p)}$ in $\mathbb{Z}_{p^r}$ is a $U(p^r)$-weighted zero-sum sequence where $r = v_p(n)$. Then $S$ is a $U(n)$-weighted zero-sum sequence.

The next result follows from Theorem 1.2 of [14] along with Theorem 1 of [11], and from Corollary 4 of [12].

Theorem 1.5. Let $n$ be odd. Then $D_{U(n)}(n) = \Omega(n) + 1$ and $C_{U(n)}(n) = 2^{\Omega(n)}$.

We get the next result from Theorem 2 of [6] and Theorem 4 of [12].

Theorem 1.6. Let $p$ be an odd prime. Then $C_{Q_p}(p) = D_{Q_p}(p) = 3$.

The next result is Lemma 3 of [12] which will be used in Theorem 5.6.

Lemma 1.7. Let $n = mq$. Let $A, B, C$ be subsets of $\mathbb{Z}_n, \mathbb{Z}_m, \mathbb{Z}_q$ respectively. Suppose $f_{n|m}(A) \subseteq B$ and $f_{n|q}(A) \subseteq C$. Then we have $C_{A}(n) \geq C_{B}(m) C_{C}(q)$.

We now prove a similar result for the weighted Davenport constant which we will use in Theorem 5.3. A generalization of this result was proved in Lemma 3.1 of [9] for abelian groups.

Lemma 1.8. Let $n = mq$. Let $A, B, C$ be subsets of $\mathbb{Z}_n, \mathbb{Z}_m, \mathbb{Z}_q$ respectively. Suppose $f_{n|m}(A) \subseteq B$ and $f_{n|q}(A) \subseteq C$. Then $D_{A}(n) \geq D_{B}(m) + D_{C}(q) - 1$.

Proof. Let $D_{B}(m) = k$ and $D_{C}(q) = l$. Assume that $k, l \geq 2$. There exists a sequence $S_1' = (u_1, \ldots, u_{k-1})$ of length $k - 1$ in $\mathbb{Z}_m$ which has no $B$-weighted zero-sum subsequence, and there exists a sequence $S_2' = (v_1, \ldots, v_{l-1})$ of length $l - 1$ in $\mathbb{Z}_q$ which has no $C$-weighted zero-sum subsequence.

As $f_{n|m}$ is onto, for $1 \leq i \leq k - 1$ there exist $x_i \in \mathbb{Z}_n$ such that $f_{n|m}(x_i) = u_i$ and as $f_{n|q}$ is onto, for $1 \leq j \leq l - 1$ there exist $y_j \in \mathbb{Z}_n$ such that $f_{n|q}(y_j) = v_j$.

Let $S$ be the sequence $(x_1, \ldots, x_{k-1}, y_1, \ldots, y_{l-1})$ of length $k + l - 2$ in $\mathbb{Z}_n$.

Let $S_1 = (x_1, \ldots, x_{k-1})$ and $S_2 = (y_1, \ldots, y_{l-1})$. Suppose $S$ has an $A$-weighted zero-sum subsequence $T$. If the sequence $T$ contains some term of $S_2$, by taking the image of $T$ under $f_{n|q}$ we get the contradiction that $S_2'$ has a $C$-weighted zero-sum subsequence, as $f_{n|q}(x_i) = 0$ and as $f_{n|q}(A) \subseteq C$.

Thus, $T$ does not contain any term of $S_2$ and so $T$ is a subsequence of $S_1$. Let $T'$ be the subsequence of $S_1'$ such that $u_i$ is a term of $T'$ if and only if $x_i$ is a term of $T$. As $f_{n|m}(A) \subseteq B$, by dividing the $A$-weighted zero-sum which is obtained from $T$ by $q$ and by taking the image under $f_{n|m}$ we get the contradiction that $T'$ is a $B$-weighted zero-sum subsequence of $S_1'$.

Hence, we see that $S$ does not have any $A$-weighted zero-sum subsequence. As $S$ has length $k + l - 2$, it follows that $D_{A}(n) \geq k + l - 1$. 

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If \( k = l = 1 \), then we are done. Suppose exactly one of them is equal to one. We may assume that \( k > 1 \) and \( l = 1 \). Then we take \( S'_2 \) to be the empty sequence in the above proof and so \( S_1 = S \). \( \square \)

2 Some results about the weight-set \( S(n) \)

From this point onwards, we will always assume that \( n \) is odd.

**Definition 2.1.** For an odd prime \( p \) and for \( a \in U(p) \) the symbol \( \left( \frac{a}{p} \right) \) is the Legendre symbol which is defined as

\[
\left( \frac{a}{p} \right) = \begin{cases} 
1 & \text{if } a \in Q_p \\
-1 & \text{if } a \notin Q_p.
\end{cases}
\]

For a prime divisor \( p \) of \( n \), we use the notation \( \left( \frac{a}{p} \right) \) to denote \( \left( \frac{f_{n/p}(a)}{p} \right) \)
where \( a \in U(n) \). Let \( n = p_1^{r_1} \cdots p_k^{r_k} \) where the \( p_i \)'s are distinct primes.

For \( a \in U(n) \), we define the Jacobi symbol \( \left( \frac{a}{n} \right) \) to be \( \left( \frac{a}{p_1} \right)^{r_1} \cdots \left( \frac{a}{p_k} \right)^{r_k} \).

Observe that we have \( \left( \frac{a}{n} \right) = \left( \frac{a^{(p_1)}}{p_1^{r_1}} \right) \cdots \left( \frac{a^{(p_k)}}{p_k^{r_k}} \right) \). Let \( S(n) \) denote the kernel of the homomorphism \( U(n) \to \{1, -1\} \) given by \( a \mapsto \left( \frac{a}{n} \right) \).

In Section 3 of \([3]\), the set \( S(n) \) was considered as a weight-set.

**Proposition 2.2.** \( S(n) \) is a subgroup having index two in \( U(n) \) when \( n \) is a non-square, and \( S(n) = U(n) \) when \( n \) is a square.

**Proof.** Let \( n = p_1^{r_1} \cdots p_k^{r_k} \) where the \( p_i \)'s are distinct primes. If \( n \) is a square, then all the \( r_i \) are even and so \( S(n) = U(n) \). If \( n \) is not a square, there exists \( j \) such that \( r_j \) is odd. As for any \( p, r \in \mathbb{N} \) the map \( f_{p^r} \) is onto, by the Chinese Remainder theorem we see that there is a unit \( b \in U(n) \) such that \( \left( \frac{b}{p_i} \right) = 1 \) when \( i \neq j \), and \( \left( \frac{b}{p_j} \right) = -1 \). It follows that \( \left( \frac{b}{n} \right) = -1 \) and so the homomorphism \( U(n) \to \{1, -1\} \) given by \( a \mapsto \left( \frac{a}{n} \right) \) is onto. Hence, we see that \( S(n) \) has index two in \( U(n) \).  \( \square \)

**Remark 2.3.** In particular, if \( n \) is squarefree then \( S(n) \) has index two in \( U(n) \). It follows that when \( p \) is an odd prime we have \( S(p) = Q_p \).

**Observation 2.4.** Let \( n = p_1 \cdots p_k \) where the \( p_i \)'s are distinct prime numbers. For \( a \in U(n) \), let \( \mu(a) \) denote the cardinality of \( \{1 \leq j \leq k : f_{n/p_j}(a) \notin Q_{p_j} \} \). Then \( a \in S(n) \) if and only if \( \mu(a) \) is even.

**Lemma 2.5.** Let \( d \) be a proper divisor of \( n \) such that \( d \) is not a square. Suppose \( d \) is coprime with \( n' \) where \( n' = n/d \). Then \( U(n') \subseteq f_{n|n'}(S(n)) \).
Proof. Let \( a' \in U(n') \). By the Chinese remainder theorem, there is an isomorphism \( \psi : U(n) \rightarrow U(n') \times U(d) \). If \( a' \in S(n') \), let \( a \in U(n) \) such that \( \psi(a) = (a',1) \). If \( a' \notin S(n') \), let \( b \in U(d) \setminus S(d) \) and let \( a \in U(n) \) such that \( \psi(a) = (a',b) \). Such a \( b \) exists by Proposition 2.2 because \( d \) is not a square. Then \( a \in S(n) \) and \( f_{n|n'}(a) = a' \).

\[ \square \]

Lemma 2.6. Let \( S \) be a sequence in \( \mathbb{Z}_n \) and let \( d \) be a proper divisor of \( n \) which divides every element of \( S \). Let \( n' = n/d \) and let \( d \) be coprime with \( n' \). Let \( S' \) be the sequence in \( \mathbb{Z}_{n'} \) which is the image of the sequence \( S \) under \( f_{n|n'} \). Let \( A \subseteq \mathbb{Z}_n \) and let \( A' \subseteq \mathbb{Z}_{n'} \) such that \( A' \subseteq f_{n|n'}(A) \). Suppose \( S' \) is an \( A' \)-weighted zero-sum sequence. Then \( S \) is an \( A \)-weighted zero-sum sequence.

Proof. Let \( S = (x_1, \ldots, x_k) \) be a sequence in \( \mathbb{Z}_n \) and let \( S' = (x'_1, \ldots, x'_k) \) where \( x'_i = f_{n|n'}(x_i) \) for \( 1 \leq i \leq k \). Suppose \( S' \) is an \( A' \)-weighted zero-sum sequence. Then for any \( 1 \leq i \leq k \), there exist \( a_i' \in A' \) such that \( a_i'x'_1 + \cdots + a_i'x'_k = 0 \). Since \( A' \subseteq f_{n|n'}(A) \), for \( 1 \leq i \leq k \) there exist \( a_i \in A \) such that \( f_{n|n'}(a_i) = a_i' \). As \( a_i'x'_1 + \cdots + a_i'x'_k = 0 \) in \( \mathbb{Z}_{n'} \), it follows that \( f_{n|n'}(a_1x_1 + \cdots + a_kx_k) = 0 \). Let \( x = a_1x_1 + \cdots + a_kx_k \in \mathbb{Z}_n \). As \( f_{n|n'}(x) = 0 \), we see that \( n' \mid x \) and as every term of \( S \) is divisible by \( d \), we see that \( d \mid x \). Now as \( d \) is coprime with \( n' \), it follows that \( x \) is divisible by \( n = n'd \) and so \( x = 0 \). Thus, \( S \) is an \( A \)-weighted zero-sum sequence.

\[ \square \]

The next result is Lemma 2.1 (ii) of [8], which we restate here using our terminology.

Lemma 2.7. Let \( p \) be an odd prime. If a sequence \( S \) in \( \mathbb{Z}_{p^r} \) has at least two terms coprime to \( p \), then \( S \) is a \( U(p^r) \)-weighted zero-sum sequence.

The next result is Lemma 1 in [7].

Lemma 2.8. Let \( A = U(n)^2 \) where \( n = p^r \) and \( p \geq 7 \) is a prime. Let \( x_1, x_2, x_3 \in U(n) \). Then \( Ax_1 + Ax_2 + Ax_3 = \mathbb{Z}_n \).

For the theorem in the next section, we need the following lemma which is similar to Lemma 2.8. We observe that when \( n = p^r \) where \( p \) is an odd prime and \( r \in \mathbb{N} \), then \( U(n) \) is a cyclic group (see [10]) and so \(-1\) is the unique element in \( U(n) \) of order 2. Thus, the map \( U(n) \rightarrow U(n) \) given by \( x \mapsto x^2 \) has kernel \( \{1, -1\} \) and so \( U(n)^2 \) is a subgroup of \( U(n) \) having index 2. Hence, \( |A_1| = |A_2| \) in the next lemma and so its proof is similar to the proof of Lemma 1 of [7].

Lemma 2.9. Let \( A_1 = U(n)^2 \) and \( A_2 = U(n) \setminus U(n)^2 \), where \( n = p^r \) and \( p \geq 7 \) is a prime. Let \( x_1, x_2, x_3 \in U(n) \) and let \( f : \{1, 2, 3\} \rightarrow \{1, 2\} \) be any function. Then \( A_{f(1)}x_1 + A_{f(2)}x_2 + A_{f(3)}x_3 = \mathbb{Z}_n \).
Corollary 2.10. Let \( n = p^r \) and \( p \geq 7 \) be a prime. Let \( S \) be a sequence in \( \mathbb{Z}_n \) such that at least three terms of \( S \) are in \( U(n) \). Then \( S \) is a \( U(n)^2 \)-weighted zero-sum sequence.

Proof. Let \( S = (x_1, x_2, \ldots, x_k) \) be a sequence in \( \mathbb{Z}_n \) as in the statement of the corollary and \( A = U(n)^2 \). Without loss of generality, we may assume that \( x_1, x_2, x_3 \in U(n) \). If \( k = 3 \), let \( y = 0 \). If \( k \geq 4 \), let \( y = x_4 + \cdots + x_k \). By Lemma 2.8 we get \(-y \in Ax_1 + Ax_2 + Ax_3 \). So there exists \( a_1, a_2, a_3 \in A \) such that \( a_1x_1 + a_2x_2 + a_3x_3 + y = 0 \). Thus, \( S \) is an \( A \)-weighted zero-sum sequence. \( \square \)

Remark 2.11. The conclusion of Corollary 2.10 may not hold when \( p < 7 \). One can check that the sequence \((1, 1, 1)\) in \( \mathbb{Z}_n \) is not a \( U(n)^2 \)-weighted zero-sum sequence, when \( n = 2 \) or \( 5 \) and the sequence \((1, 2, 1)\) in \( \mathbb{Z}_3 \) is not a \( U(3)^2 \)-weighted zero-sum sequence.

3 The constants \( D_{S(n)}(n) \) and \( C_{S(n)}(n) \)

Lemma 3.1. Let \( n \) be squarefree and \( S = (x_1, \ldots, x_l) \) be a sequence in \( \mathbb{Z}_n \). Suppose given any prime divisor \( p \) of \( n \), at least two terms of \( S \) are coprime to \( p \). If at most one term of \( S \) is a unit, then \( S \) is an \( S(n) \)-weighted zero-sum sequence.

Proof. As we have assumed that \( n \) is odd and as for every prime divisor \( p \) of \( n \) at least two terms of \( S \) are coprime to \( p \), by Lemma 2.7 for every prime divisor \( p \) of \( n \) the sequence \( S(p) = (x_1^{(p)}, \ldots, x_l^{(p)}) \) is a \( U(p) \)-weighted zero-sum sequence. Let \( n = p_1 \cdots p_k \) where the \( p_i \)'s are distinct primes. For \( 1 \leq i \leq k \) there exist \( c_{i, 1}, \ldots, c_{i, l} \in U(p_i) \) such that \( c_{i, 1}x_1^{(p_i)} + \cdots + c_{i, l}x_l^{(p_i)} = 0 \).

By Observation 1.24 for \( 1 \leq j \leq l \) there exist \( a_j \in U(n) \) such that \( a_1x_1 + \cdots + a_lx_l = 0 \) and such that for \( 1 \leq i \leq k \) we have \( (a_1^{(p_i)}}, \ldots, a_l^{(p_i)}) = (c_{i, 1}, \ldots, c_{i, l}) \). Let \( X \) denote the \( k \times l \) matrix whose \( i \)-th row is \((x_1^{(p_i)}, \ldots, x_l^{(p_i)}) \) and let \( C \) denote the \( k \times l \) matrix whose \( j \)-th row is \((c_{i, 1}, \ldots, c_{i, l}) \). We want to modify the entries of the matrix \( C \) so that for \( 1 \leq j \leq l \) the corresponding \( a_j \in U(n) \) which we get by the Chinese remainder theorem are in \( S(n) \).

Suppose the \( j \)-th column of \( X \) has a zero. Then there exists \( 1 \leq i \leq k \) such that \( x_i^{(p_i)} = 0 \). By making a suitable choice for \( c_{i, j} \) we can ensure that the corresponding \( a_j \in U(n) \) is in \( S(n) \) as \( \left( \frac{a_j}{n} \right) = \left( \frac{c_{1, j}}{p_1} \right) \cdots \left( \frac{c_{k, j}}{p_k} \right) \). Thus, we can modify the \( j \)-th column of \( C \) so that the corresponding \( a_j \in U(n) \) is in \( S(n) \).

We observe that a term \( x_j \) of \( S \) is a unit if and only if the \( j \)-th column of \( X \) does not have a zero. Hence, if no term of \( S \) is a unit then each column of \( X \)
has a zero. So in this case $S$ is an $S(n)$-weighted zero-sum sequence.

Suppose exactly one term of $S$ is a unit, say $x_{j_0}$. Then the $j_0^{th}$ column of $X$ does not have a zero and there is a zero in all the other columns of $X$. By multiplying the 1st row of $C$ by a suitable element of $U(p_1)$, we can modify $c_{1,j_0}$ so that $a_{j_0} \in S(n)$. As the other columns of $X$ have a zero, we can modify those columns of $C$ suitably so that $a_j \in S(n)$ for $j \neq j_0$. Thus, $S$ is an $S(n)$-weighted zero-sum sequence. □

**Lemma 3.2.** Let $n$ be squarefree, every prime divisor of $n$ be at least seven and $S = (x_1, \ldots, x_l)$ be a sequence in $\mathbb{Z}_n$ such that for every prime divisor of $n$, at least two terms of $S$ are coprime to it. Suppose there is a prime divisor $p$ of $n$ such that at least three terms of $S$ are coprime to $p$. Then $S$ is an $S(n)$-weighted zero-sum sequence.

**Proof.** If $\Omega(n) = 1$, then $n$ is a prime say $p$. As at least three terms of $S$ are coprime to $p$, so by Corollary 2.10 we have $S$ is a $Q_p$-weighted zero-sum sequence.

Let $\Omega(n) \geq 2$. As there are at least three units in the sequence $S^{(p)}$, by Lemma 2.7 it is a $U(p)$-weighted zero-sum sequence. So for $1 \leq i \leq l$ there exist $b_i \in U(p)$ such that $b_1x_1^{(p)} + \cdots + b_lx_l^{(p)} = 0$. Let us assume that $x_1^{(p)}, x_2^{(p)}$ and $x_3^{(p)}$ are units. A similar argument will work in the general case. We want to choose the $b_i$’s so that the corresponding $U(n)$-weighted zero-sum for $S$ (which we get using Observation 2.4 as in Lemma 3.1) is an $S(n)$-weighted zero-sum.

Using Observation 2.4 we choose the units $\{b_i : 4 \leq i \leq l\}$ so that for $4 \leq i \leq l$ we have $a_i \in S(n)$. Let us denote the negative of $b_4x_4^{(p)} + \cdots + b_lx_l^{(p)}$ by $y$. By Lemma 2.9 and using Observation 2.4 we can choose $b_1, b_2, b_3 \in U(p)$ so that $a_1, a_2, a_3 \in S(n)$ and $b_1x_1^{(p)} + b_2x_2^{(p)} + b_3x_3^{(p)} = y$. Thus, $S$ is an $S(n)$-weighted zero-sum sequence. □

**Theorem 3.3.** Let $n$ be squarefree. If $n$ is prime, we have $D_{S(n)}(n) = 3$. If $n$ is not a prime and every prime divisor of $n$ is at least seven, we have $D_{S(n)}(n) = \Omega(n) + 1$.

**Proof.** From Theorem 1.5 we have $D_{U(n)}(n) = \Omega(n) + 1$. As $S(n) \subseteq U(n)$ it follows that $D_{S(n)}(n) \geq D_{U(n)}(n)$ and so $D_{S(n)}(n) \geq \Omega(n) + 1$. If $\Omega(n) = 1$ then $n = p$ where $p$ is a prime and $S(n) = Q_p$. So by Theorem 1.6 we have $D_{S(n)}(n) = 3$.

Let $n$ be squarefree and let $\Omega(n) \geq 2$. We claim that $D_{S(n)}(n) \leq \Omega(n) + 1$. Let $S = (x_1, \ldots, x_l)$ be a sequence in $\mathbb{Z}_n$ of length $l = k + 1$ where $k = \Omega(n)$. We have to show that $S$ has an $S(n)$-weighted zero-sum subsequence. If any term
of $S$ is zero, then that term will give us an $S(n)$-weighted zero-sum subsequence of length 1.

**Case 3.3.1.** There is a prime divisor $p$ of $n$ such that at most one term of $S$ is coprime to $p$.

Let us assume without loss of generality that $x_i$ is divisible by $p$ for $2 \leq i \leq l$ and let $T$ denote the subsequence $(x_2, \ldots, x_l)$ of $S$. Let $n' = n/p$ and let $T'$ be the sequence in $\mathbb{Z}_{n'}$ which is the image of $T$ under $f_{n|n'}$. From Theorem 1.3 we see that $D_{U(n')}(n') = \Omega(n') + 1$. As $T'$ has length $l - 1 = \Omega(n) = \Omega(n') + 1$, it follows that $T'$ has a $U(n')$-weighted zero-sum subsequence. As $n$ is squarefree, $p$ is coprime to $n'$. Thus, by Lemmas 2.3 and 2.6 we see that $S$ has an $S(n)$-weighted zero-sum subsequence.

**Case 3.3.2.** For each prime divisor $p$ of $n$, exactly two terms of $S$ are coprime to $p$.

Suppose $S$ has at most one unit. By Lemma 3.1 we see that $S$ is an $S(n)$-weighted zero-sum sequence. So we can assume that $S$ has at least two units. By the assumption in this subcase, we see that $S$ will have exactly two units and the other terms of $S$ will be zero. As $S$ has length $k + 1$ and as $k \geq 2$, some term of $S$ is zero.

**Case 3.3.3.** For every prime divisor $p$ of $n$ at least two terms of $S$ are coprime to $p$, and there is a prime divisor $p'$ of $n$ such that at least three terms of $S$ are coprime to $p'$.

In this case, we are done by Lemma 3.2.

**Theorem 3.4.** Let $n$ be squarefree. If $n$ is a prime, we have $C_{S(n)}(n) = 3$. If $n$ is not a prime and every prime divisor of $n$ is at least seven, we have $C_{S(n)}(n) = 2^{\Omega(n)}$.

**Proof.** If $n = p$ where $p$ is a prime then $S(n) = Q_p$. As $p$ is odd, from Theorem 1.4 we get that $C_{S(n)}(n) = 3$. Let $n = p_1 \ldots p_k$ where $k \geq 2$. As $S(n) \subseteq U(n)$, it follows that $C_{S(n)}(n) \geq C_{U(n)}(n)$. As $n$ is odd, from Theorem 1.3 we have $C_{S(n)}(n) \geq 2^k$.

Let $S = (x_1, \ldots, x_l)$ be a sequence in $\mathbb{Z}_n$ of length $l = 2^k$. If we show that $S$ has an $S(n)$-weighted zero-sum subsequence of consecutive terms, it will follow that $C_{S(n)}(n) \leq 2^k$. If any term of $S$ is zero, we get an $S(n)$-weighted zero-sum subsequence of $S$ of length 1.

**Case 3.4.1.** There is a prime divisor $p$ of $n$ such that at most one term of $S$ is coprime to $p$.
We will get a subsequence, say \( T \), of consecutive terms of \( S \) of length \( l/2 \) whose all terms are divisible by \( p \). Let \( n' = n/p \) and let \( T' \) be the image of \( T \) under \( f_{n|n'} \). From Theorem 1.5 we have \( C_{U(n')}(n') = 2^{\Omega(n')} \). As the length of \( T' \) is \( 2^{\Omega(n')} \), it follows that \( T' \) has a \( U(n') \)-weighted zero-sum subsequence of consecutive terms. As \( n' \) is coprime with \( p \), by Lemmas 2.5 and 2.6 we get that \( T \) (and hence \( S \)) has an \( S(n) \)-weighted zero-sum subsequence of consecutive terms.

Case 3.4.2. For each prime divisor \( p \) of \( n \), exactly two terms of \( S \) are coprime to \( p \).

In this case, as \( \Omega(n) = k \) there are at most \( 2^k \) non-zero terms in \( S \). Let \( k \geq 3 \). As \( S \) has length \( 2^k \) and as \( 2^k > 2^k \), some term of \( S \) is zero and we are done. If \( k = 2 \), then \( S \) has length 4. If \( S \) has at most one unit, by Lemma 3.1 this sequence is an \( S(n) \)-weighted zero-sum sequence. So we can assume that \( S \) has exactly two units and so the other two terms of \( S \) will be zero.

Case 3.4.3. For every prime divisor \( p \) of \( n \) at least two terms of \( S \) are coprime to \( p \), and there is a prime divisor \( p' \) of \( n \) such that at least three terms of \( S \) are coprime to \( p' \).

In this case, we are done by Lemma 3.2.

4 Some results about the weight-set \( L(n; p) \)

To determine the constant \( D_{S(n)}(n) \) for some non-squarefree \( n \), we consider the following subset of \( \mathbb{Z}_n \) as a weight-set.

**Definition 4.1.** Let \( p \) be a prime divisor of \( n \) where \( n \) is odd. We define

\[
L(n; p) = \left\{ a \in U(n) : \left( \frac{a}{n} \right) = \left( \frac{a}{p} \right) \right\}
\]

Consider the homomorphism \( \varphi : U(n) \to \{1, -1\} \) given by \( \varphi(a) = \left( \frac{a}{n} \right) \left( \frac{a}{p} \right) \).

Then the kernel of \( \varphi \) is \( L(n; p) \) and so it follows that \( L(n; p) \) is a subgroup having index at most two in \( U(n) \).

**Proposition 4.2.** Let \( p \) be a prime divisor of \( n \). Then \( L(n; p) \) has index two in \( U(n) \), unless \( p \) is the unique prime divisor of \( n \) such that \( v_p(n) \) is odd.

**Proof.** Let \( n = p^r m \) where \( m \) is coprime to \( p \). Let \( \psi : U(n) \to U(p^r) \times U(m) \) be the isomorphism which is given by the Chinese remainder theorem. If we show that \(-1\) is in the image of the homomorphism \( \varphi : U(n) \to \{1, -1\} \) which was defined above, then \( \ker \varphi \) will be a subgroup of index two in \( U(n) \).
Case 4.2.1. \( r \) is odd.

Suppose \( m \) is a square. For any \( a \in U(n) \), we have \( \varphi(a) = \left( \frac{a}{m} \right) \left( \frac{a}{p^r+1} \right) = 1 \).
Thus \( \varphi \) is the trivial map and so \( L(n;p) = U(n) \).

Suppose \( m \) is not a square. By Proposition 2.2 we see that \( S(m) \) has index two in \( U(m) \). For \( c \in U(m) \setminus S(m) \), there exists \( a \in U(n) \) such that \( \psi(a) = (1, c) \).
Thus \( \left( \frac{a}{p} \right) = \left( \frac{1}{p} \right) = 1 \) and so \( \varphi(a) = \left( \frac{a}{m} \right) = \left( \frac{a}{n} \right) = 1 \).

Case 4.2.2. \( r \) is even.

Let \( m = 1 \). Then \( \left( \frac{a}{n} \right) = \left( \frac{a}{p} \right)^r = 1 \) and so \( \varphi(a) = \left( \frac{a}{p} \right) \).
Let \( b \in U(p) \setminus Q_p \).
There exists \( a \in U(n) \) such that \( f_{n|p}(a) = b \). Thus \( \varphi(a) = \left( \frac{b}{p} \right) = -1 \).

Suppose \( m > 1 \). Let \( b \in U(p) \setminus Q_p \). There exists \( b' \in U(p^r) \) such that \( f_{p^r|p}(b') = b \). For \( c \in S(m) \), there exists \( a \in U(n) \) such that \( \psi(a) = (b', c) \).
Thus \( \left( \frac{a}{n} \right) = \left( \frac{b}{p} \right)^r \left( \frac{c}{m} \right) = 1 \) and so \( \varphi(a) = \left( \frac{a}{n} \right) = \left( \frac{b}{p} \right) = -1 \).

Remark 4.3. In particular if \( n \) is a prime \( p \), then \( L(n;p) = U(p) \).

Lemma 4.4. Let \( p' \) be a prime divisor of \( n \) and \( p \) be a prime divisor of \( n \) which is coprime with \( n' = n/p \). Then \( S(n') \subseteq f_{n|n'}(L(n;p')) \).

Proof. Let \( b \in S(n') \) where \( n' = n/p \). As \( p \) is coprime with \( n' \), by the Chinese remainder theorem we have an isomorphism \( \psi : U(n) \to U(n') \times U(p) \).

Suppose \( p = p' \). Let \( a \in U(n) \) such that \( \psi(a) = (b, 1) \). Thus \( f_{n|n'}(a) = b \) and \( a \in L(n; p') \) as

\[
\left( \frac{a}{n} \right) = \left( \frac{b}{n} \right) \left( \frac{1}{p} \right) = \left( \frac{1}{p} \right) = \left( \frac{a}{p} \right) = \left( \frac{a}{p'} \right).
\]

Suppose \( p \neq p' \). Then \( p' \) divides \( n' \). Let \( c \in U(p) \) such that \( \left( \frac{c}{p} \right) = \left( \frac{b}{p'} \right) \) and let \( a \in U(n) \) such that \( \psi(a) = (b, c) \). Thus \( f_{n|n'}(a) = b \) and \( a \in L(n; p') \) as

\[
\left( \frac{a}{n} \right) = \left( \frac{b}{n'} \right) \left( \frac{c}{p} \right) = \left( \frac{c}{p} \right) = \left( \frac{b}{p'} \right) = \left( \frac{a}{p'} \right).
\]

Lemma 4.5. Let \( p' \) be a prime divisor of \( n \) which is coprime to \( n' = n/p' \). Then \( U(p') \subseteq f_{n|p'}(L(n;p')) \).
Proof. Let \(b \in U(p')\). As \(n' = n/p'\) is coprime to \(p'\), by the Chinese remainder theorem we have an isomorphism \(\psi : U(n) \to U(n') \times U(p')\). There exists \(a \in U(n)\) such that \(\psi(a) = (1, b)\). Thus \(f_{n|p'}(a) = b\) and \(a \in L(n; p')\) as

\[
\left(\frac{a}{n}\right) = \left(\frac{1}{n'}\right) \left(\frac{b}{p'}\right) = \left(\frac{b}{p'}\right) = \left(\frac{a}{p'}\right). 
\]

\[\square\]

**Observation 4.6.** Let \(A \subseteq \mathbb{Z}_n\), \(S\) be a sequence in \(\mathbb{Z}_n\) and \(n = m_1m_2\) where \(m_1\) and \(m_2\) are coprime. For \(i = 1, 2\), let \(A_i \subseteq \mathbb{Z}_{m_i}\) be given and \(S_i\) denote the image of the sequence \(S\) under \(f_{n|m_i}\). Suppose \(A_1 \times A_2 \subseteq \psi(A)\) where \(\psi : U(n) \to U(m_1) \times U(m_2)\) is the isomorphism given by the Chinese remainder theorem. If \(S_1\) is an \(A_1\)-weighted zero-sum sequence in \(\mathbb{Z}_{m_1}\) and \(S_2\) is an \(A_2\)-weighted zero-sum sequence in \(\mathbb{Z}_{m_2}\), then \(S\) is an \(A\)-weighted zero-sum sequence in \(\mathbb{Z}_n\).

**Lemma 4.7.** Let \(n\) be squarefree and \(p'\) be a prime divisor of \(n\). Suppose \(n' = n/p'\) and \(\psi : U(n) \to U(n') \times U(p')\) is the isomorphism given by the Chinese remainder theorem. Then \(S(n') \times U(p') \subseteq \psi(L(n; p'))\).

**Proof.** Let \((b, c) \in S(n') \times U(p')\). There exists \(a \in U(n)\) such that \(\psi(a) = (b, c)\). Then \(a \in L(n; p')\) as

\[
\left(\frac{a}{n}\right) = \left(\frac{b}{n'}\right) \left(\frac{c}{p'}\right) = \left(\frac{c}{p'}\right) = \left(\frac{a}{p'}\right). 
\]

\[\square\]

5 The constants \(D_{L(n;p)}(n)\) and \(C_{L(n;p)}(n)\)

**Lemma 5.1.** Let \(n\) be squarefree, \(p'\) be a prime divisor of \(n\), \(S = (x_1, \ldots, x_l)\) be a sequence in \(\mathbb{Z}_n\) such that for every prime divisor \(p\) of \(n\) at least two terms of \(S\) are coprime to \(p\). Assume that \(S'\) denotes the image of \(S\) under \(f_{n|n'}\), where \(n' = n/p'\). Suppose at most one term of \(S'\) is a unit or suppose there is a prime divisor \(p\) of \(n'/p'\) such that at least three terms of \(S\) are coprime to \(p\). Then \(S\) is an \(L(n; p')\)-weighted zero-sum sequence.

**Proof.** Let \(n' = n/p'\) and \(S'\) denote the image of the sequence \(S\) under \(f_{n|n'}\). As at least two terms of \(S'(n')\) are coprime to \(p'\), by Lemma 2.7 we have \(S'(n')\) is a \(U(p')\)-weighted zero-sum sequence.

If at most one term of \(S'\) is a unit, by Lemma 3.3 we see that \(S'\) is an \(S(n')\)-weighted zero-sum sequence in \(\mathbb{Z}_{n'}\), as \(n'\) is squarefree and for every prime divisor \(p\) of \(n'\) at least two terms of \(S'\) are coprime to \(p\).
If there is a prime divisor $p$ of $n/p'$ such that at least three terms of $S$ are coprime to $p$, by Lemma 4.2 we get that $S'$ is an $S(n')$-weighted zero-sum sequence, since at least three terms of $S'$ are coprime to $p$.

As $n$ is squarefree, $n'$ is coprime to $p'$. Let $\psi : U(n) \to U(n') \times U(p')$ be the isomorphism given by the Chinese remainder theorem. By Lemma 4.7 we see that $S(n') \times U(p') \subseteq \psi(L(n;p'))$. Hence, by Observation 4.6 we see that $S$ is an $L(n; p')$-weighted zero-sum sequence.

**Theorem 5.2.** Let $n$ be a squarefree number whose every prime divisor is at least seven. Suppose that $p'$ is a prime divisor of $n$ and $\Omega(n) \neq 2$. Then $D_{L(n; p')}(n) = \Omega(n) + 1$.

**Proof.** Let $p'$ be a prime divisor of $n$. We have $D_{U(n)}(n) \leq D_{L(n; p')}(n)$, as $L(n; p') \subseteq U(n)$. From Theorem 1.5 we have $D_{U(n)}(n) = \Omega(n) + 1$ and so $D_{L(n; p')}(n) \geq \Omega(n) + 1$. If $\Omega(n) = 1$, then $L(n; p') = U(n)$ and so by Theorem 1.5 we have $D_{L(n; p')}(n) = 2$.

Let $n$ be a squarefree number whose every prime divisor is at least seven. Suppose $\Omega(n) \geq 3$ and $S = (x_1, \ldots, x_l)$ is a sequence in $\mathbb{Z}_n$ of length $\Omega(n) + 1$. To show that $D_{L(n; p')}(n) \leq \Omega(n) + 1$, it suffices to show that $S$ has an $L(n; p')$-weighted zero-sum subsequence.

**Case 5.2.1.** There is a prime divisor $p$ of $n$ such that at most one term of $S$ is coprime to $p$.

Let us assume without loss of generality that $x_i$ is divisible by $p$ for $i > 1$ and let $T$ denote the subsequence $(x_2, \ldots, x_l)$ of $S$. Let $n' = n/p$ and let $T'$ denote the sequence in $\mathbb{Z}_{n'}$ which is the image of $T$ under $f_{n|n'}$. We have $n'$ is squarefree, $\Omega(n') \geq 2$, every prime divisor of $n'$ is at least seven and $T'$ has length $\Omega(n') + 1$.

So it follows from Theorem 3.6 that $T'$ has an $S(n')$-weighted zero-sum subsequence. As $n$ is squarefree, $p$ is coprime to $n'$. Now by Lemmas 2.6 and 4.4 we see that $T$ has an $L(n; p')$-weighted zero-sum subsequence.

**Case 5.2.2.** For every prime divisor $p$ of $n/p'$ exactly two terms of $S$ are coprime to $p$, and at least two terms of $S$ are coprime to $p'$.

Let $n' = n/p'$ and $S' = (x_1', \ldots, x_l')$ be the image of $S$ under $f_{n|n'}$. Suppose at most one term of $S'$ is a unit. By Lemma 5.1 we see that $S$ is an $L(n; p')$-weighted zero-sum sequence. Suppose at least two terms of $S'$ are units. By the assumption in this case we see that exactly two terms of $S'$ are units, say $x_{i_1}'$ and $x_{j_2}'$, and the other terms of $S'$ are zero. It follows that all terms of $S$ are divisible by $n'$ except $x_{j_1}$ and $x_{j_2}$.
Hence, if some term \( f_{n|p'}(x_j) \) of \( S^{(p')} \) is zero for \( j \neq j_1, j_2 \), then \( x_j = 0 \). So we can assume that all the terms of \( S^{(p')} \) are non-zero except possibly two terms. As \( k \geq 3 \), the sequence \( S \) has length at least 4. Let \( T \) be a subsequence of \( S \) of length at least two which does not contain the terms \( x_{j_1} \) and \( x_{j_2} \).

As all the terms of \( T^{(p')} \) are non-zero and as \( T^{(p')} \) has length at least 2, by Lemma 2.7 we see that \( T^{(p')} \) is a \( U(p') \)-weighted zero-sum sequence. Also all the terms of \( T \) are divisible by \( n' \). Hence, by Lemmas 2.6 and 4.5 we see that \( T \) is an \( L(n; p') \)-weighted zero-sum subsequence of \( S \).

**Case 5.2.3.** Given any prime divisor \( p \) of \( n \) at least two terms of \( S \) are coprime to \( p \), and there is a prime divisor \( p \) of \( n/p' \) such that at least three terms of \( S \) are coprime to \( p \).

In this case, we are done by Lemma 5.1.

**Theorem 5.3.** Let \( n = p'q \) where \( p' \) and \( q \) are distinct primes which are at least seven. Then \( D_{L(n; p')(n)} = 4 \).

**Proof.** Let \( n \) be as in the statement of the theorem. As \( L(n; p') \subseteq U(n) \), we have \( f_{n|p'}(L(n; p')) \subseteq U(p') \). Also observe that \( f_{n|q}(L(n; p')) \subseteq Q_q \). As from Theorem 1.5 we have \( D_{U(p')(p')} = 2 \) and from Theorem 1.4 we have \( D_{Q_q(q)} = 3 \), by Lemma 1.8 it follows that \( D_{L(n; p')(n)} \geq 4 \).

Let \( S = (x_1, x_2, x_3, x_4) \) be a sequence in \( \mathbb{Z}_n \). We will show that \( S \) has an \( L(n; p') \)-weighted zero-sum subsequence. It will follow that \( D_{L(n; p')(n)} = 4 \). If some term of \( S \) is zero, then we are done. So we can assume that all the terms of \( S \) are non-zero. We continue with the notations and terminology which were used in the proof of Theorem 5.2.

**Case 5.3.1.** There is a prime divisor \( p \) of \( n \) such that at most one term of \( S \) is coprime to \( p \).

We can find a subsequence \( T \) of \( S \) of length 3 such that all the terms of \( T \) are divisible by \( p \). Let \( n' = n/p \) and let \( T' \) be the sequence in \( \mathbb{Z}_{n'} \) which is the image of \( T \) under \( f_{n|n'} \). As all the terms of \( S \) are non-zero, no term of \( T \) can be divisible by \( n' \). So \( T' \) is a sequence of non-zero terms of length 3. As \( n' \) is a prime, \( S(n') = Q_{n'} \) and by Corollary 2.10 we see that \( T' \) is a \( Q_{n'} \)-weighted zero-sum subsequence. Thus, by Lemmas 2.6 and 4.5 we see that \( T \) is an \( L(n; p') \)-weighted zero-sum subsequence of \( S \).

**Case 5.3.2.** Exactly two terms of \( S \) are coprime to \( q \).

Let us assume that \( x_1 \) and \( x_2 \) are coprime to \( q \) and let \( T : (x_3, x_4) \). The sequence \( T^{(q)} \) has both terms zero and hence it is an \( S(q) \)-weighted zero-sum sequence. As \( S \) has all terms non-zero, we see that both the terms of \( T^{(p')} \) are
non-zero, and so by Lemma 2.7 we get that $T^{(p')}$ is a $U(p')$-weighted zero-sum sequence. Let $\psi : U(n) \to U(q) \times U(p')$ be the isomorphism given by the Chinese remainder theorem. By Lemma 4.3 we have $S(q) \times U(p') \subseteq \psi(L(n;p'))$. Thus, by Observation 1.6 we see that $T$ is an $L(n;p')$-weighted zero-sum subsequence of $S$.

**Case 5.3.3.** At least three terms of $S$ are coprime to $q$, and at least two terms of $S$ are coprime to $p'$.

In this case, we are done by Lemma 5.1. 

**Theorem 5.4.** Let $n$ be squarefree whose every prime divisor is at least seven. Suppose $p'$ is a prime divisor of $n$ and $\Omega(n) \neq 2$. Then $C_{L(n;p')} (n) = 2^{\Omega(n)}$.

**Proof.** If $n$ is a prime, then $n = p'$ and $L(n;p') = U(p')$. So from Theorem 1.5 we have $C_{L(n;p')} (n) = 2$. Let $n = p_1 \ldots p_k$ where $k \geq 3$ and let $p' = p_k$. As $L(n;p') \subseteq U(n)$, we have $C_{L(n;p')} (n) \geq C_{U(n)} (n)$. So from Theorem 1.5 we have $C_{L(n;p')} (n) \geq 2^{\Omega(n)}$. Let $S = (x_1, \ldots, x_l)$ be a sequence in $\mathbb{Z}_n$ of length $l = 2^{\Omega(n)}$. If we show that $S$ has an $L(n;p')$-weighted zero-sum subsequence of consecutive terms, it will follow that $C_{L(n;p')} (n) \leq 2^{\Omega(n)}$. If any term of $S$ is zero, then we get an $L(n;p')$-weighted zero-sum subsequence of $S$ of length 1.

**Case 5.4.1.** There is a prime divisor $p$ of $n$ such that at most one term of $S$ is coprime to $p$.

We can find a subsequence say $T$ of consecutive terms of $S$ of length $l/2$ such that all the terms of $T$ are divisible by $p$. Let $n' = n/p$ and let $T'$ be the image of $T$ under $f_{n|n'}$. As $\Omega(n') = \Omega(n) - 1 \geq 2$ and as $T'$ has length $2^{\Omega(n')}$, by Theorem 5.4 we see that $T'$ has an $S(n')$-weighted zero-sum subsequence of consecutive terms. By Lemma 4.3 we get $S(n') \subseteq f_{n|n'} (L(n;p'))$ and so by Lemma 2.6 we get that $T$ (and hence $S$) has an $L(n;p')$-weighted zero-sum subsequence of consecutive terms.

**Case 5.4.2.** For every prime divisor $p$ of $n/p'$ exactly two terms of $S$ are coprime to $p$, and at least two terms of $S$ are coprime to $p'$.

In this case, we can use a slight modification of the argument which was used in the same case of the proof of Theorem 5.2. We just observe that in a sequence $S$ of length at least eight which has at most two terms which are not divisible by $n'$, we can find a subsequence $T$ of consecutive terms of length at least two such that all the terms of $T$ are divisible by $n'$.

**Case 5.4.3.** For every prime divisor $p$ of $n$ at least two terms of $S$ are coprime to $p$, and there is a prime divisor $p$ of $n/p'$ such that at least three terms of $S$ are coprime to $p$.
In this case, we are done by Lemma 5.1.

**Theorem 5.5.** Let \( n = p'q \) where \( p' \) and \( q \) are distinct primes which are at least seven. Then \( C_{L(n;p')} ( n ) = 6 \).

**Proof.** Let \( n \) be as in the statement of the theorem. By Theorems 1.5 and 1.6, we see that \( C_{U(p')}(p') = 2 \) and \( C_{Q_{n}}(q) = 3 \). Also as \( f_{n\mid p'} (L(n; p')) \subseteq U(p') \) and \( f_{n\mid q} (L(n; p')) \subseteq Q_{q} \), by Lemma 1.7 it follows that \( C_{L(n;p')} ( n ) \geq 6 \).

Let \( S = (x_1, \ldots, x_6) \) be a sequence in \( \mathbb{Z}_n \). It is enough to show that \( S \) has an \( L(n;p') \)-weighted zero-sum subsequence of consecutive terms. We can assume that all the terms of \( S \) are non-zero.

**Case 5.5.1.** There is a prime divisor \( p \) of \( n \) such that at most one term of \( S \) is coprime to \( p \).

In this case, we can find a subsequence \( T \) of \( S \) of consecutive terms of length three whose all terms are divisible by \( p \). As all the terms of \( S \) are non-zero, all the terms of \( T \) are coprime to \( n' \) where \( n' = n/p \). If \( T' \) is the image of \( T \) under \( f_{n\mid n'} \), then \( T' \) is a sequence of non-zero terms of length three in \( \mathbb{Z}_{n'} \). As \( n' \) is a prime, \( S(n') = Q_{n'} \) and by Corollary 2.10 we get that \( T' \) is a \( Q_{n'} \)-weighted zero-sum sequence. By using Lemmas 2.6 and 4.4 it follows that \( T \) is an \( L(n; p') \)-weighted zero-sum subsequence of consecutive terms of \( S \).

**Case 5.5.2.** Exactly two terms of \( S \) are coprime to \( p \).

Let the terms \( x_{j_1} \) and \( x_{j_2} \) be coprime to \( q \). As \( S \) has length six, we can find a subsequence \( T \) of consecutive terms of \( S \) of length two, which does not have any term from the positions \( j_1 \) and \( j_2 \). As \( x_j \) is divisible by \( q \) when \( j \neq j_1, j_2 \), all the terms of \( T \) are divisible by \( q \). As \( S \) has all terms non-zero, all the terms of \( T \) are coprime to \( p' \).

By Lemma 2.7 we get that \( T(n') \) is a \( U(p') \)-weighted zero-sum sequence. So by Lemmas 2.6 and 4.4 it follows that \( T \) is an \( L(n; p') \)-weighted zero-sum subsequence of consecutive terms of \( S \).

**Case 5.5.3.** At least three terms of \( S \) are coprime to \( q \), and at least two terms of \( S \) are coprime to \( p' \).

In this case, we are done by Lemma 5.1.

**6 Concluding remarks**

We have \( S(15) = \{1, 2, 4, 8\} \). We can check that the sequence \( S : (1, 1, 1) \) does not have any \( S(15) \)-weighted zero-sum subsequence. So \( D_{S(15)}(15) \geq 4 \) and hence \( D_{S(15)}(15) > \Omega(15) + 1 \). This shows that the statement of Theorem 3.3.
is not true in general if some prime divisor of \( n \) is smaller than seven. It will be interesting to find the Davenport constant \( D_{S(n)}(n) \) for non-squarefree \( n \).

In [2], it was proposed to characterize the weight-sets \( A \subseteq \mathbb{Z}_n \) which have the same value of \( D_A(n) \). In this paper we have seen that if \( A \subseteq \mathbb{Z}_n \) is such that \( S(n) \subseteq A \subseteq U(n) \) and if \( n \) is not a prime, then \( D_A(n) = D_{U(n)}(n) \). We have also seen that if \( A \subseteq \mathbb{Z}_n \) is such that \( L(n;p) \subseteq A \subseteq U(n) \) and if \( \Omega(n) \neq 2 \), then again \( D_A(n) = D_{U(n)}(n) \). We can try to see whether this can happen for some other weight-sets \( A \subseteq \mathbb{Z}_n \).

Acknowledgement. Santanu Mondal would like to acknowledge CSIR, Govt. of India, for a research fellowship.

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