Dynamic Connectivity in ALOHA Ad Hoc Networks

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Abstract

In a wireless network the set of transmitting nodes changes frequently because of the MAC scheduler and the traffic load. Previously, connectivity in wireless networks was analyzed using static geometric graphs, and as we show leads to an overly constrained design criterion. The dynamic nature of the transmitting set introduces additional randomness in a wireless system that improves the connectivity, and this additional randomness is not captured by a static connectivity graph. In this paper, we consider an ad hoc network with half-duplex radios that uses multihop routing and slotted ALOHA for the MAC contention and introduce a random dynamic multi-digraph to model its connectivity. We first provide analytical results about the degree distribution of the graph. Next, defining the path formation time as the minimum time required for a causal path to form between the source and destination on the dynamic graph, we derive the distributional properties of the connection delay using techniques from first-passage percolation and epidemic processes. We consider the giant component of the network formed when communication is noise-limited (by neglecting interference). Then, in the presence of interference, we prove that the delay scales linearly with the source-destination distance on this giant component. We also provide simulation results to support the theoretical results.

I. INTRODUCTION

In a multihop ad hoc network, bits, frames or packets are transferred from a source to a destination in a multihop fashion with the help of intermediate nodes. Decoding, storing, and relaying introduces a delay that, measured in time slots, generally exceeds the number of hops.
For example, a five-hop route does not guarantee a delay of only five time slots. In a general setting, each node can connect to multiple nodes. So a large number of paths may form between the source and the destination. Each path may have taken a different time to form with the help of different intermediate nodes. Consider a network in which each node wants to transmit to its destination in a multihop fashion. In general in such a network, a relay node queues the packets from other nodes and its own packets and transmits them according to some scheduling algorithm. If one introduces the concept of queues, the analysis of the system becomes extremely complicated because of the intricate spatial and temporal dependencies between various nodes. In this paper we take a different approach. We are concerned only with the physical connections between nodes, i.e., we do not care when a node $i$ transmits a particular packet to a node $j$ (which depends on the scheduler), but we analyze when a (physical) connection (maybe over multiple hops) is formed between the nodes $i$ and $j$. This delay is a lower bound on the delay with any queueing scheduler in place.

We assume that the nodes are distributed as a Poisson point process (PPP) on the plane. In each time slot, every node decides to transmit or receive using ALOHA. Any transmitting node can connect to a receiving node when a modified version of the protocol model criterion introduced in [1] is met. Since at each time instant, the transmitting and receiving nodes change, the connectivity graph changes dynamically. We analyze the time required for a causal path to form between a source and a destination node. The system model is made precise in Section II.

This problem is similar in flavor to the problem of First-Passage Percolation (FPP) [2], [3], [4], and the process of dynamic connectivity also resembles an epidemic process [5], [6], [7] on a Euclidean domain. In a spatial epidemic process, an infected individual infects a certain (maybe random) neighboring population, and this process continues until the complete population is infected or the spreading of the disease stops. In the literature cited above, the spreading time of the epidemic is analyzed for different models of disease spread. We draw many ideas from this theory of epidemic process and FPP. The main difference between an epidemic process and the process we consider is that the spreading (of packets) depends on a subset of the population (due to interference) and is not independent from node to node. In [8], the latency for a message to propagate in a sensor network is analyzed using similar tools. They consider a Boolean connectivity model with randomly weighted edges and derive the properties of first-passage paths on the weighted graph. Their model does not consider interference and thus allows the
use of Kingman’s subadditive ergodic theorem [9] while ours does not. Percolation in signal-to-interference ratio graphs was analyzed in [10] where the nodes are assumed to be full-duplex. In practice, radios do not transmit and receive at the same time (at the same frequency), and hence the instantaneous network graph is always disconnected. In [11], [12], we have introduced the concept of dynamic connectivity graphs, and we proved that the average delay scales linearly with source-destination distance but the temporal correlation between interference was neglected. Baccelli et al. introduced a similar concept of SINR-time graphs for ALOHA networks [13] wherein they proved that below a certain ALOHA parameter $p$, the average delay of connectivity between nodes scales linearly with the distance by considering the temporal correlation of the interference. In this paper we show a similar result for the protocol model of communication. We also show that for a positive fraction of nodes, the time of connectivity scales linearly with the source-destination irrespective of the ALOHA parameter. Connectivity between nodes far apart occurs because of the dynamic nature of the MAC protocol. We first introduce a dynamic graph process to model and analyze connectivity and then derive the properties of this graph process for ALOHA.

In Section II, we introduce the system model. In Section III, we study the connectivity properties of the random geometric graph formed at any time instant. In Section IV, we derive the properties of the delay and the average number of paths between a source and destination and show that the delay increases linearly with increasing source-destination distance or, equivalently, that the propagation speed is constant, i.e., the distance of the farthest nodes to which the origin can connect increases linearly with time.

II. SYSTEM MODEL

The location of the wireless nodes (transceivers) is assumed to be a Poisson point process (PPP) $\phi$ of intensity $\lambda$ on the plane. We assume that time is slotted and the MAC protocol used is slotted ALOHA. In every time slot each node transmits with probability $p$. Nodes are half-duplex, and they act as receivers if they are not transmitting. We use the protocol model [1] to decide if the communication between a transmitter and a receiver is successful in a given time slot: A transmitting node located at $x$ can connect to a receiver located at $y$ if two conditions are met:
1) **Interference:** The disk $B(y, \beta \|x - y\|), \beta > 0$, does not contain any other transmitting nodes.

2) **Noise:** $\|x - y\| < \eta$.

$B(x, r)$ denotes a disk of radius $r$ centered around $x$ and $B_c(x, r) = \mathbb{R}^2 \setminus B(x, r)$. $\beta$ is a system parameter and captures the resilience of the receiver against interference. The standard physical SINR model of communication can be related to the protocol model easily when there is no fading. A detailed discussion about the protocol model can be found in [14]. An interference-limited regime can be modeled by dropping condition 2. In a similar fashion, a noise-limited scenario can be modeled by dropping condition 1.

We shall use $1(x \to y, \Delta, \eta)$ to represent a random variable that is equal to one if a transmitter at $x$ is able to connect to a receiver $y$ when the transmitting set is $\Delta$, i.e., the interfering set is $\Delta \setminus \{x\}$. We will drop $\Delta$ if there is no ambiguity. At any time instant $k$, we denote the set of transmitters (decided by ALOHA) by $\phi_t(k)$ and the set of receivers by $\phi_r(k)$. So we have $\phi_t(k) \cup \phi_r(k) = \phi$ and $\phi_t(k) \cap \phi_r(k) = \emptyset$, where $\emptyset$ denotes the empty set.

The connectivity at time $k$ is captured by a directed and weighted random geometric graph $g(k) = (\phi, E_k)$ with vertex set $\phi$ and edge set

$$E_k = \{(x, y): \ 1(x \to y, \phi_t(k), \eta) = 1, x \in \phi_t(k), y \in \phi_r(k)\}.$$ (1)

See Figure 1 for illustration of $g(0)$ and $g(1)$. Each edge in this graph $g(k)$ is associated with a weight $k$ that represents the time slot in which the edge was formed. Let $G(m, n)$ denote the
weighted directed multigraph (multiple edges with different time stamps are allowed between two vertices) formed between times $m$ and $n > m$, i.e.,

$$G(m, n) = \left( \phi, \bigcup_{k=n}^{m} E_k \right).$$

So $G(m, n)$ is the edge-union of the graphs $g(k)$, $m \leq k \leq n$. See Figure 2.

**Definition 1:** A directed path $x_0, e_0, x_1, e_1, \ldots, e_{q-1}, x_q$ between the nodes $x_0, x_q \in \phi$ where $e_i = (x_i, x_{i+1})$ denotes an edge in the multigraph is said to be a causal path if the weights of the edges $e_i$ are strictly increasing with $i$.

This means that the edge $e_{i-1}$ was formed before $e_i$ for $0 < i < q$. For the rest of the paper, we always mean causal path when speaking about a path. We observe that the random graph $g(k)$ is a snapshot of the ALOHA network at time instant $k$. The random graph process $G(0, m)$ captures the entire connectivity history up to time $m$. In the graph $G(0, m)$ there is a notion of time and causality, i.e., packets can propagate only on a causal path.
III. Properties of the snapshot graph \( g(k) \)

In this section, we will analyze the properties of the random graph \( g(k) \). We first observe that the graphs \( g(k) \) are identically distributed for all \( k \). So for this section we will drop the time index unless otherwise indicated. \( g \) a planar Euclidean graph even with straight lines as edges [15, Lemma 2]. In Figure 1, realizations of \( g \) are shown for \( p = 0.2 \) and \( p = 0.3 \). We first characterize the distribution of the in-degree of a receiver node and the out-degree of a transmit node.

A. Node degree distributions

Let \( N_t(x) \) denote the number of receivers a transmitter located at \( x \) can connect to, i.e., the out-degree of a transmitting node. Similarly, let \( N_r(x) \) denote the number of transmitters that can connect to a receiver at \( x \), i.e., the in-degree of a receiving node. We first calculate the average out-degree of a transmitting node.

**Proposition 1:** \( \mathbb{E}[N_t(x)] = \frac{1-p}{p} \beta^2 (1 - \exp(-\lambda p \pi \beta^2 \eta^2)). \)

**Proof:** By stationarity of \( \phi \), we have \( N_t(x) \overset{d}{=} N_t(o) \) where \( \overset{d}{=} \) stands for equality in distribution. So it is sufficient to consider the out-degree of a transmitter placed at the origin, which is given by \( \sum_{x \in \phi_r} 1(o \rightarrow x, \phi_t, \eta) \). So the average degree is

\[
\mathbb{E}[N_t(o)] = \mathbb{E}\left[ \sum_{x \in \phi_t} 1(o \rightarrow x, \phi_t, \eta) \right]
\]

\[
\overset{(a)}{=} \lambda(1-p) \int_{\mathbb{R}^2} \mathbb{E}_{\phi_t}[1(o \rightarrow x, \phi_t, \eta)] \, dx
\]

\[
\overset{(b)}{=} \lambda(1-p) \int_{B(o,\eta)} \exp(-\lambda p \pi \beta^2 \|x\|^2) \, dx
\]

\[
= \frac{1-p}{p} \beta^2 \left(1 - \exp(-\lambda p \pi \beta^2 \eta^2)\right),
\]

where \((a)\) follows from Campbell’s theorem [16] and the independence of \( \phi_r \) and \( \phi_t \). \((b)\) follows from the fact that \( 1(o \rightarrow x, \phi_t) \) is equal to one if and only if the ball \( B(x, \beta\|x\|) \) does not contain any interferers.

The average out-degree in the interference-limited case is obtained by \( \lim_{\eta \to \infty} \mathbb{E}[N_t(x)] \) and is \( \frac{1-p}{p} \beta^2 \). Similarly, the average out-degree in the noise-limited case is obtained as \( \lim_{\beta \to 0} \mathbb{E}[N_t(x)] \) and is equal to \( \lambda(1-p)\pi \eta^2 \).
Proposition 2: The probability distribution of $N_t$ is given by
\[
P(N_t = m) = \sum_{k=m}^{\infty} \frac{(-1)^{k+m}}{k!} \left( \frac{1-p}{p} \right)^k V_k,
\]
where $V_k = \int_{B(o, \sqrt{\lambda p})} \cdots \int_{B(o, \sqrt{\lambda p})} \exp \left( - \text{vol} \left( \bigcup_{i=1}^{k} B(x_i, \beta \|x_i\|) \right) \right) dx_1 \cdots dx_k$.

Proof: We provide the complete characterization of $N_t$ using the Laplace transform, given by
\[
\mathcal{L}_{N_t}(s) = \mathbb{E} \left[ \exp \left( -s N_t \right) \right]
= \mathbb{E} \left[ \exp \left( -s \sum_{x \in \phi_t} 1(o \to x, \phi_t, \eta) \right) \right]
\stackrel{(a)}{=} \mathbb{E}_{\phi_t} \exp \left[ -\lambda (1-p) \int_{\mathbb{R}^2} 1 - \exp \left( -s 1(o \to x, \phi_t, \eta) \right) dx \right]
= \mathbb{E}_{\phi_t} \exp \left[ -\lambda (1-p) (1-\exp(-s)) \int_{\mathbb{R}^2} 1(o \to x, \phi_t, \eta) dx \right]
= \mathbb{E}_{\phi_t} \exp \left[ -\lambda (1-p) (1-\exp(-s)) \int_{B(o, \eta)} 1(o \to x, \phi_t, \infty) dx \right],
\]
where $(a)$ follows from the probability generating functional of a PPP. Let $\nu$ denote a two dimensional Poisson point process of density 1. We then have
\[
1(o \to x, \phi_t, \infty) \doteq 1(o \to x \sqrt{\lambda p}, \nu, \infty).
\]
Hence
\[
\mathcal{L}_{N_t}(s) = \mathbb{E}_{\nu} \exp \left[ -\frac{1-p}{p} (1-\exp(-s)) \int_{B(o, \sqrt{\lambda p})} 1(o \to x, \nu, \infty) dx \right].
\]
Let $a = \frac{1-p}{p} (1-\exp(-s))$. Then
\[
\mathcal{L}_{N_t}(s) = \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \mathbb{E}_{\nu} \left( \int_{B(o, \sqrt{\lambda p})} 1(o \to x, \nu) dx \right)^k
= \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \int_{B(o, \sqrt{\lambda p})} \cdots \int_{B(o, \sqrt{\lambda p})} \mathbb{E}_{\nu} \left( 1(o \to x_1, \nu) \cdots 1(o \to x_k, \nu) \right) dx_1 \cdots dx_k
= 1 + \sum_{k=1}^{\infty} \frac{(-a)^k}{k!} \int_{B(o, \sqrt{\lambda p})} \cdots \int_{B(o, \sqrt{\lambda p})} \exp \left( -\text{vol} \left( \bigcup_{i=1}^{k} B(x_i, \beta \|x_i\|) \right) \right) dx_1 \cdots dx_k.
\]
By comparison of coefficients (replace $e^{-s}$ with $z$), we obtain (2).
A lower bound on $L_{N_t}(s)$ from (5) is obtained by using Jensen’s inequality:

$$
L_{N_t}(s) \overset{(a)}{\geq} \exp \left[ -\frac{1-p}{p} (1 - e^{-s}) \int_{B(o,\sqrt{\lambda \eta})} \mathbb{E}_{\nu} 1(o \to x, \nu, \infty) dx \right]
$$

$$
\overset{(b)}{=} \exp \left[ -\frac{1-p}{p\beta^2} (1 - e^{-s}) (1 - e^{-\pi \beta^2 \lambda \eta^2}) \right]
$$

where \((a)\) follows from Jensen’s inequality and \((b)\) follows since $\mathbb{E}_{\nu} 1(o \to x, \nu, \infty) = \exp(-\beta^2 \pi \|x\|^2)$. This is the Laplace transform of a Poisson random variable with mean $\frac{1-p}{p\beta^2} (1 - e^{-\pi \beta^2 \lambda \eta^2})$, which implies the following lower bound on the probability of a transmit node being isolated:

$$
\mathbb{P}(N_t = 0) \geq \exp \left( -\frac{1-p}{p\beta^2} (1 - e^{-\pi \beta^2 \lambda \eta^2}) \right).
$$

We next evaluate the in-degree distribution of a receiving node. Since the point process is stationary, the distribution of $N_r(x)$ is the same for all receivers $x$.

**Proposition 3:** The average in-degree $\mathbb{E}[N_r(x)]$ of a node in $g$ is $\beta^{-2}(1 - e^{-\pi \beta^2 \lambda \eta^2})$. When $\beta > 1$, $N_r$ is distributed as a Bernoulli random variable with mean $\beta^{-2}(1 - e^{-\pi \beta^2 \lambda \eta^2})$.

**Proof:** We have $N_r(x) \overset{d}{=} N_r(o)$ and hence,

$$
\mathbb{E}[N_r(o)] = \mathbb{E} \left[ \sum_{y \in \phi} 1_{\phi_t}(y) 1(y \to o, \phi_t, \eta) \right] = \lambda p \int_{\mathbb{R}^2} \mathbb{E}_{\phi_t} [1(y \to o, \phi_t, \eta)] dy = \lambda p \int_{B(o,\eta)} \exp \left( -\lambda p \beta^2 \|y\| \right) dy = \beta^{-2} (1 - e^{-\pi \beta^2 \lambda \eta^2}).
$$

If $\beta > 1$, at most one transmitter can connect to any receiver, so $N_r$ is Bernoulli. Since $\mathbb{E}[N_r(x)] = \beta^{-2}(1 - e^{-\pi \beta^2 \lambda \eta^2})$, we have $N_r(x) \sim \text{Bernoulli}(\beta^{-2}(1 - e^{-\pi \beta^2 \lambda \eta^2}))$.

Observe that $\mathbb{E}[N_t(x)]$ and $\mathbb{E}[N_r(x)]$ are spatial averages and not time averages. We observe that $p \mathbb{E}[N_t(o)] = (1-p) \mathbb{E}[N_r(o)]$,

i.e., the time averages of the in-degree and the out-degree are equal.

**B. Average time for single-hop connectivity**

A node may require multiple attempts (time slots) before it is able to connect to any other node. In this subsection we will consider the time it takes for a node to (opportunistically)
connect to some other node. We add a virtual node at the origin and define the number of time slots required to connect to any node,

\[ T_O = \min_k \left[ \mathbf{1}(o \in \phi_t(k)) \prod_{x \in \phi_t(k)} \mathbf{1}(o \to x, \phi_t(k), \eta) \right]. \]

**Lemma 1**: The average single-hop connection time in a Poisson network is infinite:

\[ \mathbf{E}T_O = \infty. \]

**Proof**: In the point process \( \phi \) the probability that the ball \( B(o, \eta) \) is empty is equal to \( \exp(-\lambda \pi \eta^2) \). Hence a typical transmitter at the origin cannot connect to any node with probability \( \exp(-\lambda \pi \eta^2) \) regardless of the number of attempts. Hence \( \mathbf{E}T_O = \infty. \)

From the above lemma we observe that the presence of noise which implies a finite connectivity radius makes the average single-hop connectivity time infinite. In a Poisson network this happens because the nearest-neighbor distance is Rayleigh [16] and there exists a positive fraction of nodes with large nearest-neighbor distance. We now consider an interference-limited network, i.e., neglect the finite connectivity radius assumption. Let \( \tilde{T}_O \) denote the opportunistic connectivity time with the interference limited assumption. Let \( \tilde{T}_N \) denote the time required for a connection to form between the origin and its nearest neighbor. We then have

\[ \tilde{T}_O \leq \tilde{T}_N. \]

**Lemma 2**: The average time for nearest neighbor connectivity is equal to

\[ \mathbf{E}\tilde{T}_N = \begin{cases} (p(1 - p) - p^2 \nu(\beta))^{-1}, & p < \frac{1}{1 + \nu(\beta)} \\ \infty, & \text{otherwise.} \end{cases} \]

where

\[ \nu(\beta) = \begin{cases} \beta^2 - \pi^{-1} \left\{ \beta^2 \cos^{-1} \frac{\beta}{2} + \cos^{-1} \left(1 - \frac{\beta^2}{2}\right) - \frac{\beta}{2} \sqrt{4 - \beta^2}\right\}, & \beta < 2 \\ \beta^2 - 1, & \beta > 2. \end{cases} \]

**Proof**: Let \( z \) denote the nearest neighbor of the origin \( o \). We first condition on the fact that the node at the origin always transmits and the node at \( z \) always listens. We then have,

\[ \mathbf{1}(o \to z, \phi_t(k)) = \prod_{x \in \phi \cap B(o, \|z\|)} \mathbf{1}(x \in B(z, \beta\|z\|)) \mathbf{1}(x \in \phi_t(k)). \]
The probability that $\tilde{T}_N > k$ is equal to
\[
\mathbb{P}(\tilde{T}_N > k) = \mathbb{E} \prod_{k=1}^{k} 1 - 1(o \to z, \phi_i(k)).
\] (6)

Let $N(o)$ denote the nearest neighbor of the origin $o$. Conditioning on the point process we have,
\[
\mathbb{P}(\tilde{T}_N > k \mid \phi, N(o) = z) = \left[ 1 - \prod_{x \in \phi \cap B(o, ||z||)}^{c} 1 - 1(x \in B(z, \beta \|z\|))p \right]^k.
\] (7)

So we have
\[
\mathbb{E}[\tilde{T}_N \mid N(o) = z] = \mathbb{E} \sum_{k=0}^{\infty} \mathbb{P}(\tilde{T}_N > k \mid \phi)
= \mathbb{E} \left[ \prod_{x \in \phi \cap B(o, ||z||)}^{c} 1 - 1(x \in B(z, \beta \|z\|))p \right]^{-1}
= \exp \left( -\lambda \int_{B(o, ||z||)}^{c} 1 - 1(x \in B(z, \beta \|z\|))p \, dx \right)
= \exp \left( \frac{p}{1 - p} \lambda \pi \|z\|^2 \nu(\beta) \right).
\] (8)

Averaging with respect to the nearest-neighbor distribution we have
\[
\mathbb{E}\tilde{T}_N = 2\pi \lambda \int_{0}^{\infty} z \exp(-\lambda \pi z^2) \exp \left( \frac{p}{1 - p} \lambda \pi z^2 \nu(\beta) \right) \, dz
= \frac{1}{1 - p(1 - p)^{-1} \nu(\beta)}, \quad p < \frac{1}{1 + \nu(\beta)}.
\] (9)

Removing the conditioning on the node at $o$ transmitting and the nearest neighbor listening, the result follows.

From the above lemma we observe that there exists a cutoff value for the ALOHA contention parameter above which $\mathbb{E}\tilde{T}_O = \infty$. See Figure 4. We also observe that the minimum value of $\mathbb{E}\tilde{T}_N$ occurs at $p = 0.5/(1 + \nu(\beta))$ and is equal to $4(1 + \nu(\beta))$.

We now provide a lower bound to the average time required for opportunistic communication for $\beta > 1$.

**Lemma 3:** The average time for opportunistic communication is lower bounded by:

\[1 < \beta < 2: \]
\[
\mathbb{E}\tilde{T}_O > \frac{(\beta - 1)^2 [2 + p + (\beta - 1)^2]}{p(1 - p^2)}.
\]
Figure 3. The ALOHA parameter $p$ above which the average time for nearest-neighbor connectivity $E\tilde{T}_N$ is infinite as a function of $\beta$.

$\beta > 2$ :

$$E\tilde{T}_O > \begin{cases} (p - p^2(\beta - 1)^2)^{-1}, & p < (\beta - 1)^{-2} \\ \infty, & \text{otherwise} \end{cases}$$

Proof: We observe that

$$1(o \to x, \phi_t(k)) \leq 1(\phi_t(k) \cap B(o, (\beta - 1)\|x\|) = \{o\}).$$

So the opportunistic success probability is upper bounded as

$$1 - \prod_{x \in \phi_r(k)} 1(o \to x, \phi_t(k)) \leq 1 - \prod_{x \in \phi_r(k)} 1(\phi_t(k) \cap B(o, (\beta - 1)\|x\|) = \{o\}). \quad (11)$$

Case 1: $1 < \beta < 2$.

Let $z \in \phi_r$ be the nearest receiver to the origin. We then have

$$B(o, (\beta_2 - 1)\|z\|) \subset B(o, (\beta_2 - 1)\|x\|) \quad \forall x \in \phi_r \setminus \{z\}.$$

Hence the success probability at time instant $k$ is bounded by

$$\mathbb{P}(\text{success} \mid \phi) \leq \mathbb{P}(\phi_t(k) \cap B(o, (\beta - 1)\|z\|) = \{o\}),$$

where $z$ is the nearest node of $\phi_r(k)$ to the origin. Let $\eta$ denote the nearest point of the point process $\phi$. Then the right hand side of the above equation is equal to the probability that there
is at least one receiver among the nodes in the annulus $A$ centered around the origin and radius $\eta$ and $\eta/(\beta - 1)$. Let $m$ denote the number of nodes of $\phi$ in $A$. We then have
\[ P(\phi_t(k) \cap B(o, (\beta - 1)\|z\|) = \{o\} \mid \phi) = 1 - p^{m+1}. \]

Hence
\[ P(\tilde{T}_O > n \mid \phi) = p^{(m+1)n}. \]

So we have
\[ \mathbb{E}\tilde{T}_O \geq \mathbb{E}\left[ \frac{1}{1 - p^{m+1}} \right]. \]

Therefore,
\[ \mathbb{E}\tilde{T}_O > \mathbb{E}\left[ \frac{1}{1 - p^m} \mid m = 0 \right] + \mathbb{E}\left[ \frac{1}{1 - p^{m+1}} \mid m > 1 \right] \]
\[ = \frac{1}{(1 - p)(A(\beta) + 1)} + \sum_{n=0}^{\infty} p^n \mathbb{E}[p^{nm} \mid m > 0] \]
\[ = \frac{1}{(1 - p)(A(\beta) + 1)} + 2A(\beta) \sum_{n=0}^{\infty} \frac{p^{2k}}{(A(\beta) + 1)(A(\beta)(1 - p^k) + 1)} \]
\[ > \frac{1}{(1 - p)(A(\beta) + 1)} + \frac{A(\beta)}{(A(\beta) + 1)^2(1 - p^2)}, \]

where $A(\beta) = (\beta - 1)^{-2} - 1$. Multiplying with the average time for the origin at $o$ to be a transmitter, we have the result.

Case 2: $\beta > 2$. For $\beta > 2$, we observe that the right hand side of (11) is equal to 1 if and only if the closest point of $\phi$ to the origin $\eta$ is a receiver and $B(o, (\beta - 1)\eta)$ is devoid of any transmitters. So we have
\[ P(\text{Success}) < (1 - p)^{m+1}, \]

where $m$ are the number of points of $\phi$ in the annulus of radii $\|\eta\|$ and $(1 - \beta)\|\eta\|$. Hence we have
\[ \mathbb{E}\tilde{T}_O > \mathbb{E}(1 - p)^{-m-1} \]
\[ = (1 - p)^{-1} \mathbb{E} \exp(\lambda \pi((\beta - 1)^2 - 1)\eta^2 p(1 - p)^{-1}) \]
\[ = (1 - p)^{-1}2\pi \lambda \int_0^{\infty} x \exp(\lambda \pi((\beta - 1)^2 - 1)x^2 p(1 - p)^{-1} - \pi \lambda x^2)dx \]

When $p < (\beta - 1)^{-2}$ the last integral converges. Removing the conditioning on the origin being a transmitter we have the result.
Figure 4. The lower and upper bounds for $E\bar{T}_O$ as a function of $p$ for different values of $\beta$. The upper bound corresponds to the average connectivity delay for the nearest-neighbor connectivity $E\bar{T}_N$.

IV. THE TIME EVOLUTION GRAPH $G(0, n)$

In the previous section we analyzed the snapshot connectivity graph formed at a particular time instant. In this section we will consider the superposition of these snapshot graphs and study how the connectivity evolves over time.

A. Asymptotic analysis of $G(0, n)$

We first define the connection time between two nodes. For $x, y \in \phi$, we denote the path formation time between $x$ and $y$ as

$$T(x, y) = \min \{k : G(0, k) \text{ has a path from } x \text{ to } y\}.$$

For general $x, y \in \mathbb{R}^2$, define $T(x, y) = T(x^*, y^*)$ where $x^*$ (resp. $y^*$) is the point in $\phi$ closest to $x$ (resp. $y$), with some fixed deterministic rule for breaking ties (there are no ties almost surely). Since the point process is isotropic, it is sufficient for most cases to consider destinations along a given direction. For notational convenience we define for $y \in \mathbb{R}$, $T(x, y) = T(x, (y, 0))$.

This path formation time is the minimum time required for a packet to propagate from a source $x$ to its destination $y$ in an ALOHA network. In this section we show that this propagation delay increases linearly with the source-destination distance. Similar to $T(x, y)$ we define

$$T_n(x, y) = \min_{k>n} \{k - n : G(n, k) \text{ has a path from } x \text{ to } y\}.$$
The evolution of the graph $G(0, n)$ is similar to the growth of an epidemic on the plane, and one can relate the spread of information on the graph $G(0, n)$ to the theory of Markovian contact processes [7] which was used to analyze the growth of epidemics. We now provide bounds on the path formation time between two points.

In the following arguments we rely on the spatial subadditivity of $T(o, x)$ to analyze the asymptotic properties. Subadditivity of random variables is a powerful tool which is often used to prove results in percolation and geometric graph theory. The problem of finding the minimum-delay path is similar to the problem of first-passage percolation. From the definition of $T(o, y)$, we observe that

$$T(o, y) \leq T(o, x) + T_{T(o,x)}(x, y).$$

(19)

We also have that $T_{T(o,n)}(x, y) \overset{d}{=} T(x, y)$ from the way the graph process is defined. Observe that (19) resembles the triangle inequality (especially if $T_{T(o,y)}(x, y)$ was $T(x, y)$) and thus provides a pseudo-metric, which holds in FPP problems and is the reason that the shortest paths in FPP are called geodesics. In the next two lemmata we show that the average time for a path to form between two nodes scales linearly with the distance between them.

**Lemma 4:** The time constant defined by

$$\mu = \lim_{x \to \infty} \frac{\mathbb{E}T(o, x)}{x}$$

exists.

**Proof:** From (19), we have

$$T(o, y + x) \leq T(o, y) + T_{T(o,y)}(y, y + x).$$

(20)

From the definition of the graph, the edge set $E_k$ does not depend on $E_i, \ i < k$. Hence $T_{T(o,y)}(y, y + x)$ has the same distribution as $T(y, y + x)$. Also from the invariance of the point process $\phi$, we have $T(y, y + x) \overset{d}{=} T(o, x)$. Taking expectations of (20), we obtain

$$\mathbb{E}T(o, y + x) \leq \mathbb{E}T(o, y) + \mathbb{E}T(o, x),$$

and the result follows from the basic properties of subadditive functions.

Consistent with the FPP terminology we will call $\mu$ the time constant of the process.

**Lemma 5:** The time constant for the disc model is infinite,

$$\mu = \infty.$$
Proof: Follows from Lemma 1.

The time constant is infinite because of noise. Because of the finite connectivity radius a positive fraction of the nodes will not be able to connect to any other node and hence the time constant is infinite. But if $\eta > \sqrt{1.435/\lambda}$ [17] the disc graph with radius $\eta$ and node set $\phi$ percolates. Hence there is a giant connected component that corresponds to the disc graph formed by just considering the noise and not the interference. We denote this giant connected component by $\Psi_\eta$.

B. Finiteness and positivity of the time constant $\mu$

We now prove that the any two nodes in this giant component can communicate in a time that scales linearly with the distance in between. Similar to $G(0, n)$ we define $G(0, n, \eta)$ as the dynamic graph on $\Psi_\eta$. We can similarly define for $x, y \in \Psi_\eta$.

$$T(x, y, \eta) = \min\{k : G(0, k, \eta) \text{ has a path from } x \text{ to } y\},$$

and for $x, y \in \mathbb{R}^2$, $T(x, y, \eta) = T(x^*, y^*, \eta)$ where $x^*$ and $y^*$ are the points in $\Psi_\eta$ closest to $x$ and $y$. The following Lemma has been proven in [18].

Lemma 6: For $x, y \in \mathbb{R}^2$ and $\|x - y\| < \infty$, $\|x^* - y^*\| < \infty$ almost surely.

We also have the following lemma from [18] which deals with the lengths of the shortest path in terms of the number of hops.

Lemma 7: For $x, y \in \Psi_\eta$, let $L(x, y)$ denote the length (in terms of number of hops) of the shortest path of the disc graph. If $\|x - y\| < \infty$, then $L(x, y) < \infty$.

We now prove that the time constant is finite and positive on the giant connected component.

Lemma 8: For any two nodes in $\Psi_\eta$, the average path formation time scales linearly with the distance, i.e.,

$$0 < \mu < \infty,$$

if $0 < p < 1$.

Proof: Upper bound: Let $n$ denote the point $(n, 0)$. By subadditivity and homogeneity we have

$$\mathbb{E}T(o, n, \eta) \leq n\mathbb{E}T(o, 1, \eta),$$

and hence it is sufficient to show that $\mathbb{E}T(o, 1\eta) < \infty$ to prove $\mu < \infty$. By Lemmata 6 and 7 we have $L(o^*, 1^*) < \infty$ almost surely. Hence the shortest path that connects $0^*$ and $1^*$ in
the disc graph has a finite number of edges. Denote the edges by $e_i, 1 \leq i \leq L(o^*, 1^*)$ and its corresponding Euclidean length by $|e_i|$. By the protocol model $|e_i| < \eta$. Let $T_i$ denote the average time for a direct connection to form on the edge $e_i$. Since the transmitting set of the giant component at time instant $k$ is a subset of $\phi_t(k)$, the average time obtained in (8) with $z = \eta$ upper-bounds $T_i$. Hence we have

$$T_i \leq \exp \left( \frac{p}{1-p} \lambda \pi \eta^2 \nu(\beta) \right).$$

So

$$\mathbb{E}T(o, 1, \eta) < \sum_{i=1}^{L(o^*, 1^*)} T_i < L(o^*, 1^*) \exp \left( \frac{p}{1-p} \lambda \pi \eta^2 \nu(\beta) \right),$$

which is finite when $p < 1$, and hence $\mu < \infty$.

*Lower bound:* By the protocol model any path between $o$ and $n$ should have at least $n/\eta$ hops and hence the average time is always greater than $n/\eta$ and hence $\mu > 0$.

Hence the information propagation time on the giant component scales linearly with distance. The fraction of nodes in the giant component increases as the maximum connectivity distance $\eta$ increases, and hence the set of nodes for which $\mu < \infty$ increases with increasing $\eta$.

V. Simulation Results

In this section we illustrate the results using simulation results. For the purpose of simulation we consider a PPP of unit density in the square $[-50, 50]^2$. For most of the simulations, we use $\beta = 1.2$, and we average over 200 independent realizations of the point process. In Figure 5, $\mathbb{E}T(o, x)$ is plotted with respect to $x$ for different values of $p$. The time constant $\mu$ is plotted as a function of $p$ in Figure 6. We make the following observations:

1) The time constant increases with the ALOHA parameter $p$.

2) In Figure 5, we observe that $\mathbb{E}T(o, x) \approx \mu(p)x + C(p)$, where $C(p)$ is a decreasing function of $p$ and $\mu(p)$ is increasing. For smaller values of $p$, the time taken for a node to become a transmitter is large, but the probability of a successful transmission is also high because of the low density of transmitters. This results in a large $C(p)$ and smaller $\mu(p)$ for small $p$.

3) Figure 5 also implies that the presence of interfering transmitters causes the delay to increase when the packet has to be transmitted over longer distances. So when the packet
Figure 5. $ET(o,x)$ as a function of $x$, for $\beta = 1.2$. We first observe the linear scaling of $ET(o,x)$ with the distance $x$ and that the slope increases with $p$. Also for small values of $x$ we observe that $ET(o,x) \approx p^{-1}$ since for small $x$ the path delay time is dominated by the MAC contention time. For small values of $p$, once the source is a transmitter, long edges form due to the low interference.

transmission distance is large, it is beneficial to decrease the density of contending transmitters.

4) For each $x$, there is an optimal $p$ which minimizes the delay, and the optimum $p$ is a decreasing function of $x$.

For two nodes located at $o$ and $x$ and $\|x\|$ large, there will in general be many paths between $o$ and $x$ which form by time $\mu \|x\|$. From such an ensemble of delay-optimal paths, we will consider paths which have the minimum number of hops and call them fastest paths. In Figure 7, we show the average number of hops in these paths. We observe that for a given $p$, the average hop length decreases as the source-destination distance $x$ increases. This shows that for larger source-destination distance, it is beneficial to use shorter hops since they are more reliable and form faster than longer hops. Also from Figure 6, we observe that for larger $x$, it is beneficial to be less aggressive in terms of spatial reuse and use a smaller $p$. 
Figure 6. The time constant $\mu$ as a function of $p$, for $\beta = 1.2$

Figure 7. Average hop length in the fastest path versus the source-destination distance.
VI. CONCLUSIONS

Connectivity in a wireless network is dynamic and directed because of the MAC scheduler and the half-duplex radios. Since these properties are not captured in static graph models that are usually used, we have introduced a dynamic connectivity graph and analyzed its properties for ALOHA. We have shown that the time taken for a causal path to form between a source and a destination on this dynamic ALOHA graph scales linearly with the source-destination distance for large fraction of nodes. The fraction of nodes for which the time-constant is finite increases with increasing power. So we can state the following: Networks are inherently noise-limited (or power-limited) as given sufficient time, the MAC protocol can induce enough randomness to deal with the interference. By simulations we showed that it is beneficial to use higher value of the ALOHA contention parameter for smaller source-destination distances and lower value for large distances, and that the average hop length of the fastest paths first increases rapidly but then decreases slowly as a function of the source-destination distance. These observations provide some insight how to choose the hop length for efficient routing in ad hoc networks.

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Average delay of nearest neighbor connectivity $E_{T_N}$

- $\beta = 1.25$
- $\beta = 3$
- $\beta = 2$

$p$-axis range: 0 to 0.5

$E_{T_N}$-axis range: 0 to 200