WEIERSTRASS SECTIONS FOR SOME TRUNCATED PARABOLIC SUBALGEBRAS.

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Abstract. In this paper we construct explicitly Weierstrass sections for some truncated parabolic subalgebras. A Weierstrass section $W$ of a Lie algebra $\mathfrak{a}$ consists in linearizing some symmetric invariants of $\mathfrak{a}$. Indeed $W$ is an affine subspace of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ such that restriction of functions of the symmetric algebra of $\mathfrak{a}$ to $W$ induces an algebra isomorphism between the algebra of symmetric invariants $Y(\mathfrak{a})$ of $\mathfrak{a}$ and the algebra of polynomial functions on $W$. The Weierstrass section requires here the construction of an adapted pair, which is the analogue of a principal $\mathfrak{sl}_2$-triple in the non-reductive case. Moreover Weierstrass sections and adapted pairs provide a certain number of nice properties, as the polynomiality of $Y(\mathfrak{a})$ but also the existence of an affine slice or the nonsingularity of $\mathfrak{a}$.

Mathematics Subject Classification: 16 W 22, 17 B 22, 17 B 35.

Key words: Weierstrass section, adapted pair, slice, parabolic subalgebra, polynomiality, symmetric invariants.

1. Introduction.

The base field $k$ is algebraically closed of characteristic zero.

1.1. The present work is a continuation of our work in [12], [13] and [1], where we studied maximal parabolic subalgebras in a simple Lie algebra, that is, proper parabolic subalgebras in a simple Lie algebra whose Levi factor is the product of at most two simple Lie algebras, and constructed an adapted pair for their canonical truncation, which provided a Weierstrass section for them and all derived properties explained below (for definitions, see 2.3, 2.4 and 2.5).

Here we focus on other particular parabolic subalgebras, whose Levi factor is the product $l_1 \times l_2 \times \ldots \times l_{k+1}$ of simple Lie algebras $l_i$, $1 \leq i \leq k+1$, with $l_2, \ldots, l_k$ (if $k \geq 2$) isomorphic to $\mathfrak{sl}_2$. If $k = 1$, such a parabolic subalgebra is maximal. If $k = 2$ that is, when its Levi factor is isomorphic to $l_1 \times \mathfrak{sl}_2 \times l_3$, such a parabolic subalgebra will be called submaximal.

Here we construct adapted pairs (which then provide Weierstrass sections) for two cases of parabolic subalgebras as described above, in a simple Lie algebra of type B or D. The first one concerns truncated submaximal parabolic subalgebras $\mathfrak{a}$, in the case when polynomiality of the algebra $Y(\mathfrak{a})$ of symmetric invariants was not yet known (for definition see 2.4). One may observe that our construction of an adapted pair does no more work when
To be more precise, the first case we consider concerns truncated submaximal parabolic subalgebras $\mathfrak{a}$ in a simple Lie algebra of type $B$ or $D$, for which the criterion in [16, Thm. 6.7] does not apply and then for which the polynomiality of the algebra $Y(\mathfrak{a})$ of symmetric invariants was not yet known. More precisely the Levi factor of $\mathfrak{a}$ is isomorphic to $\mathfrak{sl}_2^k \times \mathfrak{sl}_2 \times \mathfrak{so}_m$ with $k, m \in \mathbb{N}^*$, additionally $m \geq 4$ in type $D$. We adopt the convention that $\mathfrak{so}_1 = \{0\}$, $\mathfrak{so}_3 = \mathfrak{sl}_2$, $\mathfrak{so}_4 = \mathfrak{sl}_2 \times \mathfrak{sl}_2$ and $\mathfrak{so}_6 = \mathfrak{sl}_4$.

We show that $Y(\mathfrak{a})$ is a polynomial algebra, by constructing a Weierstrass section for $\mathfrak{a}$ which is also an affine slice to the coadjoint action (see 2.6 for a definition).

For this purpose we use the powerful method consisting of constructing an adapted pair for $\mathfrak{a}$ and then of computing a so-called improved upper bound such that, if it is equal to some lower bound, provides a Weierstrass section for $\mathfrak{a}$. This method is based on [18, Thm. 8.6] (slightly modified) and on [19, Lem. 6.11] and was already used in [13] and [7].

The second case we consider concerns truncated parabolic subalgebras $\mathfrak{a}$ in a simple Lie algebra of type $B$ or $D$, whose Levi factor is isomorphic to $\mathfrak{sl}_{2k+1} \times \mathfrak{sl}_2^\ell \times \mathfrak{so}_m$ with $k, \ell \in \mathbb{N}$ and $m \in \mathbb{N}^*$, additionally $m \geq 4$ in type $D$, and with the convention that $\mathfrak{sl}_1 = \{0\}$. When $\ell = 0$ the Levi factor of $\mathfrak{a}$ is simply isomorphic to $\mathfrak{sl}_{2k+1} \times \mathfrak{so}_m$, hence $\mathfrak{a}$ is maximal and this case has been treated in [12]. If $\ell = 1$, note that $\mathfrak{a}$ is submaximal. For all these parabolic subalgebras $\mathfrak{a}$ the criterion in [16, Thm. 6.7] applies, then we already know that the algebra $Y(\mathfrak{a})$ of symmetric invariants is a polynomial algebra over $k$ but slices were not yet exhibited, except for truncated maximal parabolic subalgebras, which corresponds to the case $\ell = 0$. Again adapted pairs (which generalize those constructed in [12]) are very efficient and provide Weierstrass sections for all these truncated parabolic subalgebras described above.

As in [12], [13] and [7] the Weierstrass sections here are provided by adapted pairs, as defined for instance in [13, Def 2.1]. An adapted pair is the analogue of a principal $\mathfrak{sl}_2$-triple in the non-reductive case. Such adapted pairs were introduced in [17] but do not always exist and are quite hard to construct. However when they exist, adapted pairs are a powerful tool to prove some properties, notably the polynomiality for the algebra of symmetric invariants (although this latter needs some other extra conditions), or some geometric properties (as the codimension two property or the existence of an affine slice, as defined below).

To be more precise, consider $\mathfrak{a}$ an algebraic finite dimensional Lie algebra over the field $k$, which has no proper semi-invariants (as it holds for
a truncated parabolic subalgebra \( p \)). In particular, by a result of Chevalley-Dixmier ([3]) it implies that the Gelfand-Kirillov dimension of the algebra \( Y(\mathfrak{a}) \) is equal to the index \( \text{ind } \mathfrak{a} \) of \( \mathfrak{a} \), which is the minimal codimension of a coadjoint orbit in the dual space \( \mathfrak{a}^* \) of \( \mathfrak{a} \) : elements of \( \mathfrak{a}^* \) whose coadjoint orbit is of minimal codimension are called regular.

If \( \mathfrak{a} \) admits an adapted pair, then \( \mathfrak{a} \) does not automatically have a Weierstrass section. However when the algebra of symmetric invariants \( Y(\mathfrak{a}) \) for \( \mathfrak{a} \) is polynomial, then it does ([21, 2.3]). To insure the polynomiality of \( Y(\mathfrak{a}) \), two efficient criterions ([16, Thm. 6.7] or [19, lem. 6.11]) may be used.

Note however that \( Y(\mathfrak{a}) \) may be a polynomial algebra without having any Weierstrass section, as it occurs for the truncated Borel \( \mathfrak{a} \) of \( \mathfrak{g} \) simple of type \( C_2 \) (see [20, 11.4 Example 2]).

For \( \mathfrak{a} = \mathfrak{g} \) a semisimple Lie algebra, it is well known that \( Y(\mathfrak{a}) \) is a polynomial algebra by a result of Chevalley and admits a Weierstrass section by a result of Kostant (see [22]).

1.6. By [11] a Weierstrass section for \( \mathfrak{a} \) implies the existence of an affine slice (defined in [11] or [20]) to the coadjoint action of \( \mathfrak{a} \) on \( \mathfrak{a}^* \), which extends the notion of the Kostant slice (see [22]) given by a principal \( \mathfrak{sl}_2 \)-triple in the case when \( \mathfrak{a} = \mathfrak{g} \) is a semisimple Lie algebra.

1.7. The existence of an adapted pair for \( \mathfrak{a} \) implies by [21, 1.7] that the algebra \( \mathfrak{a} \) is nonsingular, that is, the codimension of the set \( \mathfrak{a}^*_{\text{sing}} \) of singular (that is, non regular) elements in \( \mathfrak{a}^* \) is bigger or equal to two : the nonsingularity property is also called in [23, Def. 1.1] the “codimension two property” and is an important property.

For example, when polynomiality of the algebra of symmetric invariants \( Y(\mathfrak{a}) \) for \( \mathfrak{a} \) holds and when \( \mathfrak{a} \) is nonsingular, then by [21, 5.6] the sum of the degrees of a set of homogeneous algebraically independent generators \( f_1, \ldots, f_l \) of \( Y(\mathfrak{a}) \) is equal to the so-called “magic number” of \( \mathfrak{a} \), which is the integer equal to the half sum of the dimension of \( \mathfrak{a} \) and the index of \( \mathfrak{a} \).

Finally by [25] with the above hypotheses, the differentials of \( f_1, \ldots, f_l \) are linearly independent exactly at regular points of \( \mathfrak{a}^* \) (this property is called in [28] the Kostant regularity criterion, since it is true by [22] for a semisimple Lie algebra).

1.8. Observe that even when the field of invariant fractions of \( S(\mathfrak{a}) \) is a pure transcendental extension of the base field \( k \), it can happen that the algebra of symmetric invariants \( Y(\mathfrak{a}) \) is not a polynomial algebra and even not of finite type, as it occurs for a nilpotent Lie algebra \( \mathfrak{a} \) of dimension 45 by [4, 4.9.20].

Then the polynomiality of the algebra \( Y(\mathfrak{a}) \) of symmetric invariants is a much more stronger property than the question on whether the field of invariant fractions is a pure transcendental extension of the base field \( k \).
Several authors have studied the polynomiality of $Y(a)$ for some non-reductive Lie algebras $a$ and have exhibited (algebraic or affine) slices for $a$.

When $a$ is a truncated parabolic or biparabolic (seaweed) subalgebra of a simple Lie algebra $g$, the algebra $Y(a)$ of invariants was shown in a lot of cases, notably when $g$ is of type $A$ or $C$, but also in other cases, to be a polynomial algebra, whose number of generators, their weight and degree have been computed (see [9], [15], [16], [13], [7]). Moreover an affine (and/or algebraic) slice to the coadjoint action has also been constructed for some of them (see [18], [12], [13], [7], [30]).

When $a$ is the centraliser $g_x$ of a nilpotent element $x$ in a simple Lie algebra $g$, it was also shown in [26] that the algebra $Y(g_x)$ is a polynomial algebra, notably when $g$ is of type $A$ or $C$. Note that, when $x$ corresponds to the vector of highest root in $g$, the centraliser $g_x$ is equal to a truncated parabolic subalgebra of $g$.

For truncated parabolic subalgebras $p$, only one counter-example to the polynomiality of $Y(p)$ was shown in [31] (with $p$ also equal to the centraliser of the vector of highest root in a simple Lie algebra of type $E_8$). For centralisers in general, another counter-example was found in [23], in type $D_7$.

When $a$ is a semi-direct product, it was also shown for some particular cases that $Y(a)$ is a polynomial algebra (see [27], [28], [29], [32], [33]).

2. Some definitions.

In what follows, we precise the notions mentioned in Sect. 1.

Let $a$ be an algebraic finite dimensional Lie algebra over $k$, which acts on its symmetric algebra $S(a)$ by the adjoint action (denoted by $\text{ad}$) which extends by derivation the adjoint action of $a$ on itself given by the Lie bracket. We may regard $S(a)$ as the algebra of polynomial functions on the dual space $a^*$ of $a$, that is, $S(a) \simeq k[a^*]$ as an algebra.

2.1. Algebra of symmetric invariants. An invariant of $S(a)$ (symmetric invariant of $a$ for short) is an element $s \in S(a)$ such that, for all $x \in a$, $(\text{ad } x)(s) = 0$.

We denote by $Y(a)$ the set of symmetric invariants of $a$ : it is a subalgebra of $S(a)$. We may notice that the algebra $Y(a)$ also coincides with the centre of $S(a)$ for its natural Poisson structure (and that is why we will call it sometimes the Poisson centre of $S(a)$ or of $a$ for short). Moreover if $A$ is the adjoint group of $a$ then $Y(a)$ also coincides with the algebra $S(a)^A$ of invariants of $S(a)$ under the action of $A$.

2.2. Algebra of semi-invariants. An element $s \in S(a)$ is called a semi-invariant of $a$, if there exists $\lambda \in a^*$ verifying that, for all $x \in a$, $(\text{ad } x)(s) = \lambda(x) \cdot s$. We denote by $S(a)_\lambda \subset S(a)$ the space of such semi-invariants. The vector-space generated by all semi-invariants of $a$ will be denoted by $Sy(a) :$ it is a subalgebra of $S(a)$. One has that $Sy(a) = \bigoplus_{\lambda \in a^*} S(a)_{\lambda}$. A linear
2.3. Canonical truncation. By [1] there exists a canonically defined subalgebra of \( \mathfrak{a} \), called the canonical truncation of \( \mathfrak{a} \) and denoted by \( \mathfrak{a}_\Lambda \), such that \( Y(\mathfrak{a}_\Lambda) = S\mathfrak{y}(\mathfrak{a}_\Lambda) = S\mathfrak{y}(\mathfrak{a}) \). We say that \( \mathfrak{a}_\Lambda \) is the truncated subalgebra of \( \mathfrak{a} \) : it is the larger subalgebra of \( \mathfrak{a} \) which vanishes on the weights of \( S\mathfrak{y}(\mathfrak{a}) \).

In particular, the canonical truncation of \( \mathfrak{a} \) admits no proper semi-invariants. By [15, 7.9] (see also [6, Chap. I, Sec. B, 8.2]) the algebra of symmetric invariants of a proper parabolic subalgebra in a simple Lie algebra is reduced to scalars, while by [4], its algebra of semi-invariants is never. That is why we consider the algebra of semi-invariants \( S\mathfrak{y}(\mathfrak{a}) \) of a parabolic subalgebra \( \mathfrak{a} \) rather than its algebra of invariants, knowing moreover that \( S\mathfrak{y}(\mathfrak{a}) \) is the algebra of symmetric invariants of the truncated parabolic \( \mathfrak{a}_\Lambda \) associated to \( \mathfrak{a} \).

2.4. Adapted pairs. An adapted pair for \( \mathfrak{a} \) is a pair \((h, y) \in \mathfrak{a} \times \mathfrak{a}^*\) such that \((\text{ad} h)(y) = -y\), where \( \text{ad} \) denotes here the coadjoint action of \( \mathfrak{a} \) on \( \mathfrak{a}^* \), \( h \) is a semisimple element of \( \mathfrak{a} \) and \( y \) is a regular element in \( \mathfrak{a}^* \), that is, its coadjoint orbit is of minimal codimension, called the index of \( \mathfrak{a} \) (ind \( \mathfrak{a} \)). Call an element of \( \mathfrak{a}^* \) singular if it is not regular and denote by \( \mathfrak{a}_{\text{sing}}^* \) the set of singular elements in \( \mathfrak{a}^* \).

The set of regular elements in \( \mathfrak{a}^* \) is open dense in \( \mathfrak{a}^* \) and the codimension of \( \mathfrak{a}_{\text{sing}}^* \) is always bigger or equal to one and when equality holds the algebra \( \mathfrak{a} \) is said to be singular (nonsingular otherwise). Hence \( \mathfrak{a} \) is nonsingular if the set of regular elements in \( \mathfrak{a}^* \) is big.

Note that, if \((h, y)\) is an adapted pair for \( \mathfrak{a} \), then \( y \) belongs to the zero set of the ideal of \( S(\mathfrak{a}) \) generated by the homogeneous elements of \( Y(\mathfrak{a}) \) with positive degree.

It follows by [21, 1.7] that, in the case when \( \mathfrak{a} \) admits an adapted pair and has no proper semi-invariants, the algebra \( \mathfrak{a} \) is nonsingular.

If moreover \( \mathfrak{a} \) is a truncated parabolic subalgebra of a simple Lie algebra \( \mathfrak{g} \) and admits an adapted pair \((h, y)\), then by identifying elements of \( \mathfrak{a}^* \) via the Killing form on \( \mathfrak{g} \) with elements of \( \mathfrak{g} \), then \( y \) is necessarily a nilpotent element of \( \mathfrak{g} \).

2.5. Weierstrass sections. A Weierstrass section of \( \mathfrak{a} \) is an affine subspace \( y + V \) of \( \mathfrak{a}^* \) (with \( y \in \mathfrak{a}^* \) and \( V \) a vector subspace of \( \mathfrak{a}^* \)) such that restriction of functions of \( S(\mathfrak{a}) = \mathbb{k}[\mathfrak{a}^*] \) to \( y + V \) induces an algebra isomorphism between \( Y(\mathfrak{a}) \) and the algebra of polynomial functions \( \mathbb{k}[y + V] \) on \( y + V \). Of course, since \( \mathbb{k}[y + V] \) is isomorphic to \( S(V^*) \), the existence of a Weierstrass section
of \( a \) implies that the algebra \( Y(a) \) is isomorphic to \( S(V^*) \) and then that \( Y(a) \) is a polynomial algebra (on \( \dim V \) generators). Moreover, under this isomorphism, a set of homogeneous algebraically independent generators of \( Y(a) \) is sent to a basis of \( V^* \), hence each element of this set is linearized.

2.6. **Affine slice.** An affine slice to the coadjoint action of \( a \) on \( a^* \) is an affine subspace \( y + V \) of \( a^* \) such that \( A.(y + V) \) is dense in \( a^* \) and \( y + V \) meets every coadjoint orbit in \( A.(y + V) \) at exactly one point and transversally.

2.7. **The reductive case.** Take for example \( a = g \) semisimple. Then there exists a principal \( \mathfrak{sl}_2 \)-triple \((x, h, y)\) of \( g \) with \( h \) a semisimple element and \( x \) and \( y \) regular in \( g \) such that \([x, y] = h \) and \([h, y] = -y\). The pair \((h, y)\) is an adapted pair for \( g \) and by \([22]\) \( y + g^x \) is a Weierstrass section and also an affine slice to the coadjoint action of \( g \) on \( g^* \), where \( g^x \) is the centraliser of \( x \) in \( g \).

2.8. **Magic number.** The magic number \( c(a) \) of \( a \) is \( c(a) = \frac{1}{2}(\dim a + \text{ind } a) \). It is always an integer.

By \([24]\) Prop. 3.1] one always has that \( c(a_A) = c(a) \).

When \( a \) has no proper semi-invariants and is nonsingular (which is the case when \( a \) admits an adapted pair for instance) and when \( Y(a) \) is a polynomial algebra in necessarily \( \text{ind } a \) homogeneous algebraically independent generators \( f_1, \ldots, f_l \) \((l = \text{ind } a)\) then one has

\[
\sum_{i=1}^{l} \deg(f_i) = c(a)
\]

by \([21] 5.6\) for example.

When \( a = g \) is semisimple, the above equality holds and \( c(g) = \dim b \) where \( b \) is a Borel subalgebra of \( g \).

3. **Notation.**

Let \( g \) be a semisimple Lie algebra over \( k \) and \( h \) a fixed Cartan subalgebra of \( g \). Let \( \Delta \) be the set of roots of \( g \) with respect to \( h \) and \( \pi \) a chosen set of simple roots. Denote by \( \Delta^\pm \) the subset of \( \Delta \) formed by the positive, resp. negative, roots of \( \Delta \), with respect to \( \pi \).

We use Bourbaki’s labelling for the roots, as in \([22]\) Planches II, resp. IV] when \( g \) is simple of type \( B_n \), resp. \( D_n \).

To each root \( \alpha \in \Delta \) we associate a root vector space \( g_\alpha \) and a nonzero root vector \( x_\alpha \in g_\alpha \). For all \( A \subset \Delta \), set \( g_A = \bigoplus_{\alpha \in A} g_\alpha \) and \( -A = \{ \gamma \in \Delta \mid -\gamma \in A \} \). We denote by \( \alpha^\vee \) the coroot associated to the root \( \alpha \in \Delta \) and by \( n, \text{ resp. } n^- \), the subalgebra of \( g \) such that \( n = g_{\Delta^+} \), resp. \( n^- = g_{\Delta^-} \). We have the following triangular decomposition

\[
g = n \oplus h \oplus n^-.
\]

Recall that \((\alpha^\vee)_{\alpha \in \pi}\) is a basis for the \( k \)-vector space \( h \).
A standard parabolic subalgebra of $\mathfrak{g}$ is given by the choice of a subset $\pi'$ of $\pi$. That is why we will denote it by $\mathfrak{p}_{\pi'}$. Let $\Delta_{\pi'}^\pm$ denote the subset of $\Delta^\pm$ associated to $\pi'$, namely $\Delta_{\pi'}^\pm = \pm \mathbb{N}\pi' \cap \Delta^\pm$. Set $\mathfrak{n}_{\pi'}^\pm = \mathfrak{g}_{\Delta_{\pi'}^\pm}$. Then

$$\mathfrak{p}_{\pi'} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_{\pi'}^-.$$  

The opposite algebra of $\mathfrak{p}_{\pi'}$ will be denoted by $\mathfrak{p}_{\pi'}^-$. One has

$$\mathfrak{p}_{\pi'}^- = \mathfrak{n}_{\pi'}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}_{\pi'}^-.$$  

Via the Killing form $K$ on $\mathfrak{g}$, the dual space $\mathfrak{p}_{\pi'}^*$ of $\mathfrak{p}_{\pi'}$ is isomorphic to $\mathfrak{p}_{\pi'}^-$ which is then endowed with the coadjoint action of $\mathfrak{p}_{\pi'}$. 

We denote by $(, )$ the non-degenerate symmetric bilinear form on $\mathfrak{h}^* \times \mathfrak{h}^*$ invariant under the action of the Weyl group of $(\mathfrak{g}, \mathfrak{h})$, induced by the Killing form on $\mathfrak{h} \times \mathfrak{h}$. 

If $\pi = \{\alpha_1, \ldots, \alpha_n\}$, we denote by $\varpi_i$, $1 \leq i \leq n$, the fundamental weight associated to $\alpha_i$. Similarly, if $\pi' = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ we denote by $\varpi_{i_j}$ the fundamental weight associated to $\alpha_{i_j}$. 

Recall the definition of the canonical truncation given in Sect.2 and denote by $\mathfrak{p}_{\pi', \Lambda}$ the canonical truncation of $\mathfrak{p}_{\pi'}$. Then one has that

$$\mathfrak{p}_{\pi', \Lambda} = \mathfrak{n} \oplus \mathfrak{h}_{\Lambda} \oplus \mathfrak{n}_{\pi'}^-,$$

where $\mathfrak{h}_{\Lambda} \subset \mathfrak{h}$ is the largest subalgebra of $\mathfrak{h}$ which vanishes on $\Lambda(\mathfrak{p}_{\pi'})$, the set of weights of $S\mathfrak{g}(\mathfrak{p}_{\pi'})$ which may be identified with a subset of $\mathfrak{h}^*$. 

Denote by $\mathfrak{p}'_{\pi'}$ the derived subalgebra of $\mathfrak{p}_{\pi'}$ and set $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{p}'_{\pi'}$. Then $\mathfrak{h}' \subset \mathfrak{h}_{\Lambda}$. 

Let $w_0$ be the longest element of the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. If $w_0 = -Id$ then $\mathfrak{h}_{\Lambda} = \mathfrak{h}'$. In particular if $\mathfrak{g}$ is of type $B$, then we have that $\mathfrak{h}_{\Lambda} = \mathfrak{h}'$. 

For convenience we will replace $\mathfrak{p}_{\pi'}$ by its opposite algebra $\mathfrak{p}_{\pi'}^-$ (simply denoted by $\mathfrak{p}$) and we will consider the canonical truncation $\mathfrak{p}_{\Lambda} = \mathfrak{p}_{\pi', \Lambda}^-$ of $\mathfrak{p} = \mathfrak{p}_{\pi'}^-$. We have that

$$\mathfrak{p}_{\Lambda} = \mathfrak{p}_{\pi', \Lambda}^- = \mathfrak{n}^- \oplus \mathfrak{h}_{\Lambda} \oplus \mathfrak{n}_{\pi'}^+,$$

and its dual space $\mathfrak{p}_{\Lambda}^*$ may be identified via the Killing form on $\mathfrak{g}$ with $\mathfrak{p}_{\pi', \Lambda}^-$. 

We will denote by $\mathfrak{g}'$ the Levi factor of $\mathfrak{p}$ (and of $\mathfrak{p}_{\Lambda}$), namely :

$$\mathfrak{g}' = \mathfrak{n}_{\pi'}^+ \oplus \mathfrak{h}' \oplus \mathfrak{n}_{\pi'}^-.$$  

4. Construction of an adapted pair.

As we already said in the previous sections, our Weierstrass sections require the construction of an adapted pair. This construction uses the notions we already introduced in [7], [12] and [13]. For convenience we recall some of them, notably the Heisenberg sets and the Kostant cascades.
4.1. Heisenberg sets and Kostant cascades. A Heisenberg set with centre \( \gamma \in \Delta \) ([7](Def. 7)) is a subset \( \Gamma_\gamma \) of \( \Delta \) such that \( \gamma \in \Gamma_\gamma \) and for all \( \alpha \in \Gamma_\gamma \setminus \{ \gamma \} \), there exists a (unique) \( \alpha' \in \Gamma_\gamma \setminus \{ \gamma \} \) such that \( \alpha + \alpha' = \gamma \).

A typical example of Heisenberg set is given by the Kostant cascade ([7](Example 8)). More precisely assume that the semisimple Lie algebra \( \mathfrak{g} \) admits a set of roots \( \Delta = \cup \Delta_i \), each \( \Delta_i \) being a maximal irreducible root system with highest root \( \beta_i \). Then take \( (\Delta_i)_{\beta_i} = \{ \alpha \in \Delta_i \mid (\alpha, \beta_i) = 0 \} \) and decompose it into maximal irreducible root systems \( \Delta_{ij} \) with highest roots \( \beta_{ij} \). Continuing we obtain a set \( \beta_\pi \) of strongly orthogonal positive roots \( \beta_K \), called the \( \gamma \) set with centre \( \Gamma \). For all \( \alpha \in \Delta_i \), each \( \alpha \in \Delta_i \), there exists a (unique) \( \alpha' \in \Delta_i \setminus \{ \gamma \} \) such that \( \alpha + \alpha' = \gamma \).

Finally let \( \Gamma = \nabla \Delta \) be a Heisenberg set with centre \( \Gamma \). We also need to recall the following proposition of regularity (see [7](Prop. 9)), which is a generalization of [18](Thm. 8.6).

Proposition 4.2.1. Let \( S \) be a subset of \( \Delta^+ \cup \Delta^- \). For each \( \gamma \in S \), let \( \Gamma_\gamma \subset \Delta^+ \cup \Delta^- \) be a Heisenberg set with centre \( \gamma \) such that all the sets \( \Gamma_\gamma \) are disjoint. Set \( \Gamma = \cup_{\gamma \in S} \Gamma_\gamma \). Decompose \( S \) into \( S^+ \cup \Delta^- \), resp. \( S^- \), the subset of \( S \) containing those \( \gamma \in S \) with \( \Gamma_\gamma \subset \Delta^+ \), resp. \( \Gamma_\gamma \subset \Delta^- \). Finally let \( T, T^* \) be disjoint subsets of \( \Delta^+ \cup \Delta^- \), also disjoint from \( \Gamma \). For all \( \gamma \in S \), denote by \( \Gamma_\gamma^0 = \Gamma_\gamma \setminus \{ \gamma \} \) and set \( O = \cup_{\gamma \in S} \Gamma_\gamma^0 \) and \( O^\pm = \cup_{\gamma \in S \pm} \Gamma_\gamma^0 \).

Let \( y = \sum_{\gamma \in S} x_\gamma \). We assume that:

1. \( S_{\mathfrak{h}_\Lambda} \) is a basis for \( \mathfrak{h}_\Lambda \).
2. If \( \alpha \in \Gamma_\gamma^0 \), with \( \gamma \in S^+ \), is such that there exists \( \beta \in O^+ \), with \( \alpha + \beta \in S \), then \( \beta \in \Gamma_\gamma^0 \) and \( \alpha + \beta = \gamma \).
3. If \( \alpha \in \Gamma_\gamma^0 \), with \( \gamma \in S^- \), is such that there exists \( \beta \in O^- \), with \( \alpha + \beta \in S \), then \( \beta \in \Gamma_\gamma^0 \) and \( \alpha + \beta = \gamma \).
4. \( \Delta^+ \cup \Delta^- = \cup_{\gamma \in S} \Gamma_\gamma \cup T \cup T^* \).
5. For all \( \alpha \in T^* \), \( \mathfrak{g}_\alpha \subset (\text{ad } \mathfrak{p}_\Lambda)(y) + \mathfrak{g}_T \).
6. \( |T| = \text{ind } \mathfrak{p}_\Lambda \).

Then \( y \) is regular in \( \mathfrak{p}_\Lambda \) and \( (\text{ad } \mathfrak{p}_\Lambda)(y) \oplus \mathfrak{g}_T = \mathfrak{p}_\Lambda^* \). Moreover we can uniquely define \( h \in \mathfrak{h}_\Lambda \) by \( \gamma(h) = -1 \) for all \( \gamma \in S \), and then \( (h, y) \) is an adapted pair for \( \mathfrak{p}_\Lambda \).

Notice that [18](Thm. 8.6) is a special case of the above Proposition, with \( T^* = \emptyset \). Here we need to take a set \( T^* \neq \emptyset \) as in [7]. In [13] we also used a similar proposition as above (see [13](Lem. 3.2 and Lem. 6.1)) but with
different Heisenberg sets than in [7]. Actually the proof was considerably simplified in [7] (for type B) where a set \( T^* \neq \emptyset \) was introduced.

5. Some examples.

Before giving the general statement, we give some examples which will enlighten our construction of an adapted pair.

The first two examples illustrate the first case described in Sect. 1.1 (one in type B and the other in type D) and the third example illustrates the second case of Sect. 1.1.

5.1. First example. We assume that the Lie algebra \( \mathfrak{g} \) is simple of type \( B_6 \) and we set \( \pi' = \pi \setminus \{ \alpha_2, \alpha_4 \} \). Then we consider the submaximal parabolic subalgebra \( \mathfrak{p} = \mathfrak{p}_{\pi'} \) as defined in Sect. [3] We take \( S = S^+ \sqcup S^- \) with
\[
S^+ = \{ \epsilon_1 + \epsilon_3, \epsilon_2 + \epsilon_4 + \epsilon_5 \}
\]
\[
S^- = \{ -\epsilon_5 - \epsilon_6 \}
\]
\[
T = \{ \epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_3, \epsilon_2 + \epsilon_4, \epsilon_4 - \epsilon_5, \epsilon_3 - \epsilon_6, \epsilon_5 \}
\]
\[
T^* = \{ \epsilon_2 + \epsilon_6, \epsilon_2 + \epsilon_5, \epsilon_2 - \epsilon_1, \epsilon_2 - \epsilon_4, \epsilon_2 - \epsilon_3, \epsilon_6, \epsilon_2 - \epsilon_6 \}.
\]

We set
\[
\Gamma_{\epsilon_1 + \epsilon_3} = \{ \epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_4, \epsilon_3 - \epsilon_4, \epsilon_1 + \epsilon_5, \epsilon_3 - \epsilon_5, \epsilon_1 + \epsilon_6, \epsilon_3 - \epsilon_6, \epsilon_1, \epsilon_3, \epsilon_1 - \epsilon_6, \epsilon_3 + \epsilon_6, \epsilon_1 - \epsilon_5, \epsilon_3 + \epsilon_5, \epsilon_1 - \epsilon_4, \epsilon_3 + \epsilon_4, \epsilon_1 - \epsilon_2, \epsilon_2 + \epsilon_3 \}
\]
\[
\Gamma_{\epsilon_2} = \{ \epsilon_2 \}
\]
\[
\Gamma_{\epsilon_4 + \epsilon_5} = \{ \epsilon_4 + \epsilon_5, \epsilon_4 + \epsilon_6, \epsilon_5 - \epsilon_6, \epsilon_4, \epsilon_4 - \epsilon_6, \epsilon_5 + \epsilon_6 \}
\]
\[
\Gamma_{\epsilon_5 - \epsilon_6} = \{ -\epsilon_5 - \epsilon_6, -\epsilon_5, -\epsilon_6 \}.
\]

By setting \( y = \sum_{\gamma \in S} x_\gamma \) and
\[
h = \epsilon_1 - \epsilon_2 - 2\epsilon_3 + 2\epsilon_4 - 3\epsilon_5 + 4\epsilon_6 = \alpha_1^\vee - 2\alpha_3^\vee - 3\alpha_4^\vee + 1/2\alpha_6^\vee
\]
one verifies that Proposition 4.2.1 is satisfied. Hence the pair \((h, y)\) is an adapted pair for \( p_\Lambda \).

Let \( \lambda \in \mathfrak{k} \) and set \( q = h_\Lambda \oplus \mathfrak{g}_O \oplus \mathfrak{g}_S \oplus \mathfrak{g}_T^* \subset p_\Lambda^* \) (one has that \( q \oplus \mathfrak{g}_T = p_\Lambda^* \)). Recall that the endomorphism \( \text{ad} h \) of \( p_\Lambda^* \), resp. of \( p_\Lambda \) (with \( \text{ad} \) the co-adjoint action, resp. the adjoint action) is semisimple. Then \( \lambda \) is an eigenvalue of \( \text{ad} h \) on \( p_\Lambda^* \) if and only if \( -\lambda \) is an eigenvalue of \( \text{ad} h \) on \( p_\Lambda \). Write \( m_\lambda' \) for the multiplicity of \( \lambda \) in \( q \), \( m_\lambda \) for the multiplicity of \( \lambda \) in \( p_\Lambda \) and \( m_\lambda^* \) for the multiplicity of \( \lambda \) in \( p_\Lambda^* \). Then by the above \( m_{-\lambda} = m_\lambda \) and obviously \( m_\lambda' \leq m_{-\lambda} \). Moreover since \( (\text{ad} h)(y) = -y \) and that \( p_\Lambda^* = (\text{ad} p_\Lambda)(y) \oplus \mathfrak{g}_T \), we must have that
\[
m_\lambda' \leq m_{\lambda+1}
\]
(see also [7, 7.1]).

In the table below we give the multiplicities \( m_\lambda' \) and \( m_\lambda^* = m_{-\lambda} \) for all eigenvalue \( \lambda \in \mathfrak{k} \) of \( \text{ad} h \) in \( p_\Lambda^* \) and one easily checks that equality \((*)\) above holds.
5.2. Second example. We assume that the Lie algebra $\mathfrak{g}$ is simple of type $D_8$ and we set $\pi' = \pi \setminus \{\alpha_4, \alpha_6\}$. We consider the submaximal parabolic subalgebra $p = p_{\pi'}^-\Lambda$ associated to $\pi'$. We take $S = S^+ \cup S^-$ with
\[
S^+ = \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_5, \varepsilon_4 + \varepsilon_8, \varepsilon_4 - \varepsilon_8, \varepsilon_6 + \varepsilon_7\}
\]
\[
S^- = \{\varepsilon_3 - \varepsilon_1\}
\]
\[
T = \{\varepsilon_1 - \varepsilon_2, \varepsilon_3 + \varepsilon_4, \varepsilon_3 - \varepsilon_5, \varepsilon_4 + \varepsilon_6, \varepsilon_6 - \varepsilon_7, \varepsilon_6 - \varepsilon_5, \varepsilon_8 - \varepsilon_7\}
\]
\[
T^* = \{\varepsilon_4 + \varepsilon_7, \varepsilon_4 - \varepsilon_7, \varepsilon_4 - \varepsilon_6, \varepsilon_4 - \varepsilon_5, \varepsilon_4 - \varepsilon_1, \varepsilon_4 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, -\varepsilon_7 - \varepsilon_8\}.
\]
We set $\Gamma_{\varepsilon_1 + \varepsilon_2} = H_{\varepsilon_1 + \varepsilon_2}$ where $H_\gamma$ is the maximal Heisenberg set with centre $\gamma$ defined in subsection 4.1 for $\gamma$ a root in the Kostant cascade of $\mathfrak{g}$. We set
\[
\Gamma_{\varepsilon_3 + \varepsilon_5} = \{\varepsilon_3 + \varepsilon_5, \varepsilon_3 + \varepsilon_6, \varepsilon_5 - \varepsilon_6, \varepsilon_3 + \varepsilon_7, \varepsilon_5 - \varepsilon_7, \varepsilon_3 + \varepsilon_8, \varepsilon_5 - \varepsilon_8, \varepsilon_3 - \varepsilon_8, \varepsilon_5 + \varepsilon_8, \varepsilon_3 - \varepsilon_7, \varepsilon_5 + \varepsilon_7, \varepsilon_3 - \varepsilon_6, \varepsilon_5 + \varepsilon_6, \varepsilon_3 - \varepsilon_4, \varepsilon_4 + \varepsilon_5\}
\]
\[
\Gamma_{\varepsilon_6 + \varepsilon_7} = \{\varepsilon_6 + \varepsilon_7, \varepsilon_6 + \varepsilon_8, \varepsilon_7 - \varepsilon_8, \varepsilon_6 - \varepsilon_8, \varepsilon_7 + \varepsilon_8\}
\]
\[
\Gamma_{\varepsilon_4 + \varepsilon_8} = \{\varepsilon_4 + \varepsilon_8\}
\]
\[
\Gamma_{\varepsilon_4 - \varepsilon_8} = \{\varepsilon_4 - \varepsilon_8\}
\]
\[
\Gamma_{\varepsilon_3 - \varepsilon_1} = \{\varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \varepsilon_2 - \varepsilon_1\}.
\]
By setting $y = \sum_{\gamma \in S} x_\gamma$ and
\[
h = 3\varepsilon_1 - 4\varepsilon_2 + 2\varepsilon_3 - \varepsilon_4 - 3\varepsilon_5 + 3\varepsilon_6 - 4\varepsilon_7 = 3\alpha_1^\vee - \alpha_2^\vee + \alpha_3^\vee - 3\alpha_5^\vee - 2\alpha_7^\vee - 2\alpha_8^\vee
\]
one verifies that Proposition 4.2.1 is satisfied. Hence the pair $(h, y)$ is an adapted pair for $p_\Lambda$.

In the table below we give the multiplicities $m'_\lambda$ and $m^*_\lambda = m_{-\lambda}$ for all eigenvalue $\lambda \in \mathbb{k}$ of $\text{ad} h$ in $p^*_\Lambda$ and one easily checks that equality $(*)$ above holds.

| $\lambda$ | -8 | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 |
|-----------|----|----|----|----|----|----|----|----|---|
| $m'_\lambda$ | 1  | 4  | 2  | 2  | 7  | 4  | 2  | 10 | 10|
| $m_{-\lambda}$ | 1  | 4  | 2  | 2  | 7  | 4  | 2  | 10 | 10|

| $\lambda$ | -8 | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 |
|-----------|----|----|----|----|----|----|----|----|---|
| $m'_\lambda$ | 2  | 4  | 7  | 2  | 2  | 4  | 1  |    |   |
| $m_{-\lambda}$ | 3  | 5  | 7  | 3  | 3  | 5  | 3  |    |   |
5.3. Third example. We assume that the Lie algebra $\mathfrak{g}$ is simple of type $B_7$ and set $\pi' = \pi \setminus \{\alpha_3, \alpha_5, \alpha_7\}$. We consider the parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_{\pi'}$ associated to $\pi'$ as defined in Sect. 3. The Levi factor $\mathfrak{g}'$ of $\mathfrak{p}$ is isomorphic to $\mathfrak{sl}_3 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ (we are then in the second case described in Sect. 1.1).

Then for $S^+$, resp. $S^-$, it suffices to take the positive roots $\beta_i = \varepsilon_{2i-1} + \varepsilon_{2i}$, $1 \leq i \leq 3$, of the Kostant cascade of $\mathfrak{g}$, resp. the negative root $-\beta_i' = \varepsilon_3 - \varepsilon_1$ whose opposite $\beta_i'$ lies in the Kostant cascade of the Levi factor $\mathfrak{g}'$ of $\mathfrak{p}$.

For each Heisenberg set $\Gamma_{\gamma}$, $\gamma \in S$, we take $\Gamma_{\gamma} = H_{\gamma}$ if $\gamma \in S^+$, resp. $H_{\gamma} = -H_{-\gamma}$ if $\gamma \in S^-$, where the Heisenberg sets $H_{\gamma}$ are the maximal Heisenberg sets defined in subsection 4.1.

Then we take $T = \{\varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4, \varepsilon_5 - \varepsilon_6, \varepsilon_7, \varepsilon_5 - \varepsilon_4, \varepsilon_7 - \varepsilon_6\}$ and $T^* = \emptyset$.

By setting $y = \sum_{\gamma \in S} x_{\gamma}$ and

$$h = 2\varepsilon_1 - 3\varepsilon_2 + 2\varepsilon_4 + 2\varepsilon_5 - 3\varepsilon_6 + 3\varepsilon_7 = 2\alpha_1^\vee - \alpha_2^\vee - 2\alpha_4^\vee - 3\alpha_6^\vee$$

one verifies that Proposition 4.2.1 is satisfied. Hence the pair $(h, y)$ is an adapted pair for $\mathfrak{p}_A$.

In the table below we give the multiplicities $m'_\lambda$ and $m^*_\lambda = m_{-\lambda}$ for all eigenvalue $\lambda \in \mathbb{k}$ of $\text{ad} h$ in $\mathfrak{p}_A^*$ and one easily checks that equality $(*)$ above holds.

| $\lambda$ | -6 | -5 | -4 | -3 | -2 | -1 | 0 |
|-----------|----|----|----|----|----|----|---|
| $m'_\lambda$ | 3  | 5  | 2  | 2  | 4  | 10 | 10|
| $m^*_\lambda$ | 3  | 5  | 2  | 2  | 4  | 10 | 10|

6. First Case.

We keep the same conventions as in Sect. 1.2.

Indeed in this Section the Lie algebra $\mathfrak{g}$ is simple of type $B_n$, $n \geq 4$, resp. $D_n$, $n \geq 6$, and we consider as in Sect. 3 the parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_{\pi'}$ of $\mathfrak{g}$ associated to the subset $\pi' = \pi \setminus \{\alpha_s, \alpha_{s+2}\}$ of simple roots, with $s$ an even integer, $2 \leq s \leq n - 2$, resp. $2 \leq s \leq n - 4$. Then the Levi factor $\mathfrak{g}'$ of $\mathfrak{p}$ is isomorphic to the product $\mathfrak{sl}_s \times \mathfrak{sl}_2 \times \mathfrak{so}_m$ of at most three simple Lie algebras, with $m \in \mathbb{N}^*$, and $m \geq 4$ if $\mathfrak{g}$ is of type $D_n$. More precisely if $\mathfrak{g}$ is of type $B_n$ one has that $m = 2n - 2s - 3$, and when $\mathfrak{g}$ is of type $D_n$ one has that $m = 2n - 2s - 4$. In our terminology such a parabolic subalgebra of $\mathfrak{g}$ is called a submaximal parabolic subalgebra.

In this case the classical bounds given by [16, Thm. 6.7] do not coincide, hence we cannot conclude that the algebra of semi-invariants $S_y(\mathfrak{p})$ is or not polynomial. However we will construct an adapted pair for the truncated parabolic subalgebra $\mathfrak{p}_A$ associated to $\mathfrak{p}$ and then use the criterion in [19, lem. 6.11] to conclude that the algebra of symmetric invariants $Y(\mathfrak{p}_A) = S_y(\mathfrak{p})$...
is a polynomial algebra over $k$ for which the weights and degrees of a set of homogeneous generators will be computed.

### 6.1. An adapted pair for $\mathfrak{p}_\Lambda$

For this purpose, we will use Proposition [4.2.1]. Denote by $\varepsilon_i$, $1 \leq i \leq n$, the elements of an orthonormal basis of $\mathbb{R}^n$ according to which the simple roots $\alpha_i$ are expanded as in \cite{2} Planches II, resp. IV] for type $B_n$, resp. $D_n$.

For type $B_n$, we set

$$ S^+ = \{ \varepsilon_s, \varepsilon_{2i-1} + \varepsilon_{2i}, \varepsilon_{s-1} + \varepsilon_{s+1}, \varepsilon_{2j} + \varepsilon_{2j+1}; \\
1 \leq i \leq s/2 - 1, s/2 + 1 \leq j \leq [(n-1)/2] \} $$

and

$$ S^- = \{ \varepsilon_{s-i} - \varepsilon_i, -\varepsilon_{2j-1} - \varepsilon_{2j}; \\
1 \leq i \leq s/2 - 1, s/2 + 2 \leq j \leq [n/2] \} $$

For type $D_n$, we set

$$ S^+ = \{ \varepsilon_s - \varepsilon_n, \varepsilon_s + \varepsilon_n, \varepsilon_{2i-1} + \varepsilon_{2i}, \varepsilon_{s-1} + \varepsilon_{s+1}, \varepsilon_{2j} + \varepsilon_{2j+1}; \\
1 \leq i \leq s/2 - 1, s/2 + 1 \leq j \leq [(n-2)/2] \} $$

and

$$ S^- = \{ \varepsilon_{s-i} - \varepsilon_i, -\varepsilon_{2j-1} - \varepsilon_{2j}; \\
1 \leq i \leq s/2 - 1, s/2 + 2 \leq j \leq [(n-1)/2] \} $$

Remark that the above sets $S^\pm$ contain the same elements as those defined in \cite{13} or in \cite{7} for maximal parabolic subalgebras, except for one of them which is missing, namely the element $-\varepsilon_{s+1} - \varepsilon_{s+2}$, since it does no more belong to $\Delta^-$. As we already noticed in Sect. \ref{3} for type $B_n$, we have that $\mathfrak{h}_\Lambda = \mathfrak{h}'$. It is also true in type $D_n$ with our hypotheses ($s \leq n - 4$) by the description of $\mathfrak{h}_\Lambda$ given in \cite{10} Cor. 5.2.9 and 5.2.10] (see also \cite{12} 2.2).

As in \cite{13} Lem. 7.1], we prove the following lemma.

**Lemma 6.1.1.** Set $S = S^+ \cup S^-$ as above. Then $S|_{\mathfrak{h}_\Lambda}$ is a basis for $\mathfrak{h}_\Lambda^\ast$.

**Proof.** The proof is quite similar to that of \cite{13} Lem. 7.1]. We give it below for the reader’s convenience. First observe that $|S| = n - 2$. The elements of $S$ will be denoted by $s_i$, with $1 \leq i \leq n - 2$. When $\mathfrak{g}$ is of type $B_n$, we set $s_{n-3} = \varepsilon_s$ and $s_{n-2} = \varepsilon_{n-1} + \varepsilon_n$ if $n$ is odd, resp. $s_{n-2} = -\varepsilon_{n-1} - \varepsilon_n$ if $n$ is even. When $\mathfrak{g}$ is of type $D_n$, we set $s_{n-3} = \varepsilon_s$ and $s_{n-2} = \varepsilon_s + \varepsilon_n$.

Then we set $s'_i = s_i$ for all $1 \leq i \leq n - 2$ if $\mathfrak{g}$ is of type $B_n$. If $\mathfrak{g}$ is of type $D_n$, we set $s'_i = s_i$ for all $1 \leq i \leq n - 4$, $s'_{n-3} = \varepsilon_s$ and $s'_{n-2} = \varepsilon_n$.

It suffices to verify that, if $\{h_j\}_{1 \leq j \leq n-2}$ is a basis of $\mathfrak{h}_\Lambda = \mathfrak{h}'$, then $\det(s'(h_j))_{1 \leq i, j \leq n-2} \neq 0$.

To prove this, we order the basis $\{h_j\}_{1 \leq j \leq n-2}$ of $\mathfrak{h}_\Lambda$ as

$$ \{ \alpha_{2i}^\vee, \alpha_{s-1}^\vee, \alpha_{2j-1}^\vee, \alpha_{s-2j-1}^\vee, \alpha_k^\vee; 1 \leq i \leq s/2 - 1, \\
1 \leq j \leq [s/4], s + 1 \leq k \leq n, k \neq s + 2 \} $$
without repetitions. The elements $s'_{ij}, 1 \leq i \leq n - 2$, are ordered as
\[
\{ \varepsilon_{2i-1} + \varepsilon_{2i}, \varepsilon_s, \varepsilon_{s-j} - \varepsilon_j, \varepsilon_s-1 + \varepsilon_s+1, \varepsilon_{2k} + \varepsilon_{2k+1}, -\varepsilon_{2k+1} - \varepsilon_{2k+2},
\]
\[(-1)^n(\varepsilon_{n-2} + \varepsilon_{n-1}), s'_{n-2}; 1 \leq i, j \leq s/2 - 1, s/2 + 1 \leq k \leq [(n - 3)/2]\]
without repetitions.

Then one verifies that $(s'_{ij}(h))_{1 \leq i, j \leq n - 2} = \begin{pmatrix} A & 0 & 0 & 0 \\ * & B & 0 & 0 \\ * & * & C & 0 \\ * & * & * & D \end{pmatrix}$ with $A$, resp. $B$, a $(s/2 - 1) \times (s/2 - 1)$, resp. $s/2 \times s/2$, lower triangular matrix with 1, resp. $-1$, on its diagonal.

Moreover $C = (1)$ and $D = \begin{pmatrix} D' & 0 \\ * & D'' \end{pmatrix}$ with $D'$ a $(n - s - 4) \times (n - s - 4)$ lower triangular matrix with alternating 1 and $-1$ on its diagonal, and $D''$ an invertible $2 \times 2$ matrix.

To each $\gamma \in S$, we need now to associate a Heisenberg set $\Gamma_{\gamma}$ with centre $\gamma$.

Recall that $\beta_i := \varepsilon_{2i-1} + \varepsilon_{2i},$ for all $1 \leq i \leq s/2 - 1,$ is a positive root which belongs to the Kostant cascade of $g$. We then set, for all $1 \leq i \leq s/2 - 1,$ $\Gamma_{\beta_i} = H_{\beta_i}$ where $H_{\beta_i}$ is the maximal Heisenberg set with centre $\beta_i$ defined in subsection 4.1.

We set
\[
\Gamma_{\varepsilon_{s-1} + \varepsilon_{s+1}} = \{ \varepsilon_{s-1} + \varepsilon_{s+1}, \varepsilon_{s-1} \pm \varepsilon_s, \varepsilon_{s+1} \pm \varepsilon_s, \varepsilon_s + \varepsilon_{s+1}; s + 2 \leq i \leq n \}
\]
for $g$ of type $B_n$ and for $g$ of type $D_n$, $\Gamma_{\varepsilon_{s-1} + \varepsilon_{s+1}}$ is taken to be the same set but without $\varepsilon_{s-1}$ and $\varepsilon_{s+1}$ which are not roots in this type.

For $s/2 + 1 \leq j \leq [(n - 1)/2]$ for type $B_n$, resp. $s/2 + 1 \leq j \leq [(n - 2)/2]$ for type $D_n$, we set $\Gamma_{\varepsilon_{2j} + \varepsilon_{2j+1}} = \{ \varepsilon_{2j} + \varepsilon_{2j+1}, \varepsilon_{2j}, \varepsilon_{2j+1} \pm \varepsilon_s, \varepsilon_{2j} + \varepsilon_{2j+1} \pm \varepsilon_s, 2j + 2 \leq k \leq n \}$, resp. the same set but without $\varepsilon_{2j}$ and $\varepsilon_{2j+1}$.

For all $1 \leq i \leq s/2 - 1$, we set $\Gamma_{\varepsilon_{s-i} - \varepsilon_i} = \{ \varepsilon_{s-i} - \varepsilon_i, \varepsilon_{s-i} - \varepsilon_j, \varepsilon_j - \varepsilon_i \mid i + 1 \leq j \leq s - i - 1 \}$.

For all $s/2 + 1 \leq j \leq [n/2]$ for type $B_n$, resp. $s/2 + 1 \leq j \leq [(n - 1)/2]$ for type $D_n$, we set $\Gamma_{-\varepsilon_{2j-1} - \varepsilon_{2j}} = -H_{\varepsilon_{2j-1} + \varepsilon_{2j}}$ where $H_{\varepsilon_{2j-1} + \varepsilon_{2j}}$ is the maximal Heisenberg set with centre $\beta'_j := \varepsilon_{2j-1} + \varepsilon_{2j}$ in the Kostant cascade of $g'$ defined in subsection 4.1.

Finally for $g$ of type $B_n$, we set $\Gamma_{\varepsilon_s} = \{ \varepsilon_s \}$ and for $g$ of type $D_n$, we set $\Gamma_{\varepsilon_s + \varepsilon_n} = \{ \varepsilon_s + \varepsilon_n \}$ and $\Gamma_{\varepsilon_s - \varepsilon_n} = \{ \varepsilon_s - \varepsilon_n \}$.

By construction all the above sets $\Gamma_{\gamma}, \gamma \in S$, are Heisenberg sets with centre $\gamma$ and they are pairwise disjoint.

Moreover the above sets $\Gamma_{\gamma}, \gamma \in S$, are chosen to be the same as in [7] (for type $B_n$), except for $\Gamma_{\varepsilon_{s-1} + \varepsilon_{s+1}}$ where here the roots $\varepsilon_{s-1} - \varepsilon_s$ and $\varepsilon_s + \varepsilon_{s+1}$ are added. However the proofs of [7] Lem. 14 and 15] (themselves based on
Lem. 2.2]) can still be applied to show that conditions (2) and (3) of Proposition 4.2.1 are satisfied.

Now for the set $T$ we take

$$
T = \{\varepsilon_{s-1} + \varepsilon_s, \varepsilon_{s-1} - \varepsilon_{s+1}, \varepsilon_s + \varepsilon_{s+2}, \varepsilon_{2i-1} - \varepsilon_{2i}, \varepsilon_{s+2j} - \varepsilon_{s+2j+1}, -\varepsilon_{s+2k-1} + \varepsilon_{s+2k} : \varepsilon_i \leq s/2 - 1, 1 \leq j \leq [(n - s - 1)/2], 1 \leq k \leq [(n - s)/2]\}.
$$

One checks that $T \subset \Delta^+ \cup \Delta^-$ and that $T$ is disjoint from $\Gamma = \bigcup_{\gamma \in S} \Gamma_\gamma$.

Note also that this set $T$ has the same elements as the set $T$ in [7], except that $\alpha_{s-1} = \varepsilon_{s-1} - \varepsilon_s$ now belongs to $\Gamma_{\varepsilon_{s-1} + \varepsilon_s}$, and is replaced by $\varepsilon_s + \varepsilon_{s+2}$.

**Lemma 6.1.2.** We have that $|T| = \text{ind } p_\Lambda$.

**Proof.** One checks that $|T| = n - s/2 + 1$.

On the other hand by [8, 3.2] we know that $\text{ind } p_\Lambda = |E(\pi')|$ where $E(\pi')$ is the set of $\langle ij \rangle$-orbits in $\pi$, with $i$ and $j$ being involutions of $\pi$ defined in [12, 2.2] for example.

Denote by $\pi'_1$, $\pi'_2$, $\pi'_3$ the three irreducible components of $\pi'$. Then $\pi'_1$ is of type $A_{n-1}$, $\pi'_2$ is of type $A_1$ and $\pi'_3$ is of type $B_{n-2}$, resp. $D_{n-2}$ if $g$ is of type $B_n$, resp. $D_n$.

Then $i|\pi'_i$ exchanges $\alpha_t$ and $\alpha_{s-t}$ for all $1 \leq t \leq s/2 - 1$ and fixes $\alpha_{s/2}$, $i|\pi'_2 = Id_{\pi'_2}$ and $(ij)|\pi'_3 = Id_{\pi'_3}$ since $n$ and $n-s-2$ are of the same parity (and $n-s-2 \geq 2$ if $g$ is of type $D_n$). Moreover for all $\alpha \in \pi \setminus \pi'$, $i(\alpha) = j(\alpha) = \alpha$.

Then the set $E(\pi')$ of $\langle ij \rangle$-orbits in $\pi$ is

$$
E(\pi') = \{\{\alpha_t, \alpha_{s-t}\}, \{\alpha_{s/2}\}, \{\alpha_u\} : 1 \leq t \leq s/2 - 1, s \leq u \leq n\}.
$$

They are in $n - s/2 + 1$ in number. Hence the lemma.

Finally for the set $T^*$ we take

$$
T^* = \{\varepsilon_s - \varepsilon_i, \varepsilon_s + \varepsilon_j, (-1)^n s \varepsilon_n : 1 \leq i \leq n, i \neq s, s + 3 \leq j \leq n\}
$$

if $g$ is of type $B_n$, and

$$
T^* = \{\varepsilon_s - \varepsilon_i, \varepsilon_s + \varepsilon_j, (-1)^{n-1} s \varepsilon_n : 1 \leq i \leq n-1, i \neq s, s + 3 \leq j \leq n-1\}
$$

if $g$ is of type $D_n$.

In type $B_n$, note that this set $T^*$ is the same as $T^*$ in [7], except that two elements here are missing : $\varepsilon_s + \varepsilon_{s+1}$ which now belongs to $\Gamma_{\varepsilon_{s-1} + \varepsilon_{s+1}}$ and $\varepsilon_s + \varepsilon_{s+2}$ which now belongs to $T$. By construction $T^*$ is disjoint from $\Gamma \cup T$.

Actually if we denote by $\Delta^-_1$ the set of negative roots in the case when $\pi' = \pi \setminus \{\alpha_s\}$ (like in [7]) and by $\Delta^-_2$ the set of negative roots in our present case when $\pi' = \pi \setminus \{\alpha_s, \alpha_{s+2}\}$, then one has that $\Delta^+_1 = \Delta^-_2 \cup -H_{\varepsilon_{s+1} + \varepsilon_{s+2}}$.

By a similar proof as in [7, Lem. 13] we have that $\Delta^+ \cup \Delta^-_1 = \Gamma \cup T \cup T^*$. Hence condition (4) of Proposition 4.2.1 is satisfied. It remains to verify condition (5) of Proposition 4.2.1.
The proofs of [7] Lem. 16, 17, 18, 19 can still be applied in type $B_n$. In type $D_n$ they have to be adapted. For completeness, we give a proof below.

Set $y = \sum_{\gamma \in S} x_\gamma$.

**Lemma 6.1.3.** Let $\gamma \in T^\ast$. Then $g_\gamma \subset (\text{ad } p_\Lambda)(y) + g_T$.

**Proof.** Recall (Sect. 3) that $p_\Lambda = n^- \oplus h' \oplus n_\pi^+$ and that we have chosen, for each $\alpha \in \Delta$, a nonzero root vector $x_\alpha \in g_\alpha$.

Given $\gamma, \delta \in \Delta_\pm$ such that $\gamma + \delta \in \Delta_\pm$, one has that $(\text{ad } x_\gamma)(x_\delta) = [x_\gamma, x_\delta] \in g_{\gamma + \delta} \setminus \{0\}$ by say [5] 1.10.7, then it is a nonzero multiple of $x_{\gamma + \delta}$.

Assume that $g$ is of type $B_n$ and rescale if necessary the nonzero root vectors $x_\gamma, \gamma \in \Delta \setminus S$.

Let $s + 3 \leq j \leq n - 1$ and $j$ odd. Then one has that

$$\begin{cases} (\text{ad } x_{\epsilon_j})(y) = x_{\epsilon_j + \epsilon_j} + x_{-\epsilon_j + 1} \\ (\text{ad } x_{-\epsilon_j - \epsilon_j + 1})(y) = x_{-\epsilon_j + 1} \end{cases}$$

Hence $x_{\epsilon_j + \epsilon_j} = (\text{ad } x_{\epsilon_j} - x_{-\epsilon_j - \epsilon_j + 1})(y) \in (\text{ad } p_\Lambda)(y)$.

If $j = n$ is odd, then $x_{\epsilon_n + \epsilon_n} = (\text{ad } x_{\epsilon_n})(y) \in (\text{ad } p_\Lambda)(y)$.

Let $s + 4 \leq j \leq n$ and $j$ even. Then $x_{\epsilon_j + \epsilon_j} = (\text{ad } x_{\epsilon_j} - x_{-\epsilon_j - \epsilon_j - 1})(y) \in (\text{ad } p_\Lambda)(y)$.

Let $1 \leq i \leq s - 3$ and $i$ odd, or $s + 2 \leq i \leq n - 1$ and $i$ even. Then

$$\begin{cases} x_{\epsilon_i - \epsilon_i} = (\text{ad } x_{-\epsilon_i} - x_{\epsilon_i + 1 - \epsilon_i} + x_{-\epsilon_i - i - 2 - \epsilon_i})(y) & \text{if } i \leq s/2 - 2 \\ x_{\epsilon_i - \epsilon_i} = (\text{ad } x_{-\epsilon_i} - x_{\epsilon_i + 1 - \epsilon_i})(y) & \text{otherwise} \end{cases}$$

Hence $x_{\epsilon_i - \epsilon_i} \in (\text{ad } p_\Lambda)(y)$.

Let $2 \leq i \leq s - 2$ and $i$ even, or $s + 3 \leq i \leq n$ and $i$ odd. Then

$$\begin{cases} x_{\epsilon_i - \epsilon_i} = (\text{ad } x_{-\epsilon_i} - x_{\epsilon_i + 1 - \epsilon_i} + x_{-\epsilon_i - i + 2 - \epsilon_i})(y) & \text{if } 4 \leq i \leq s/2 \\ x_{\epsilon_i - \epsilon_i} = (\text{ad } x_{-\epsilon_i} - x_{\epsilon_i + 1 - \epsilon_i} + x_{-\epsilon_i - \epsilon_i + 1})(y) & \text{if } i = 2 \\ x_{\epsilon_i - \epsilon_i} = (\text{ad } x_{-\epsilon_i} - x_{\epsilon_i + 1 - \epsilon_i})(y) & \text{otherwise} \end{cases}$$

Hence $x_{\epsilon_i - \epsilon_i} \in (\text{ad } p_\Lambda)(y)$.

If $i = s - 1$ then

$$x_{\epsilon_{s - \epsilon_{s - 1}}} = (\text{ad } x_{-\epsilon_{s - 1} - \epsilon_{s - 1}})(y) \in (\text{ad } p_\Lambda)(y).$$

If $i = s + 1$ then

$$x_{\epsilon_{s + \epsilon_{s + 1}}} = (\text{ad } x_{-\epsilon_{s + 1} - \epsilon_{s - 1}})(y) \in (\text{ad } p_\Lambda)(y).$$

If $i = n$ is even, then

$$x_{\epsilon_{s - \epsilon_n}} = (\text{ad } x_{-\epsilon_n})(y) \in (\text{ad } p_\Lambda)(y).$$

Finally, if $n$ is odd, then $x_{-\epsilon_n} = (\text{ad } x_{-\epsilon_n})(y) \in (\text{ad } p_\Lambda)(y)$ and if $n$ is even, then $x_{\epsilon_n} = (\text{ad } x_{-\epsilon_n})(y) \in (\text{ad } p_\Lambda)(y)$.

Hence the lemma for $g$ of type $B_n$. 

WEIERSTRASS SECTIONS FOR SOME TRUNCATED PARABOLIC SUBALGEBRAS.
Assume that $\mathfrak{g}$ is of type $D_n$.
Let $s + 3 \leq j \leq n - 2$ and $j$ odd.
Up to rescaling the nonzero root vectors $x_{\varepsilon_j - \varepsilon_n}$, $x_{-\varepsilon_s - \varepsilon_{j+1}}$, $x_{\varepsilon_j + \varepsilon_n}$ in $\mathfrak{p}_\Lambda$ and $x_{-\varepsilon_{j+1} - \varepsilon_n}$, $x_{\varepsilon_n - \varepsilon_{j+1}}$ in $\mathfrak{p}_\Lambda^*$, one has that
\[
[x_{\varepsilon_j - \varepsilon_n}, x_{\varepsilon_s + \varepsilon_n}] = x_{\varepsilon_s + \varepsilon_j},
\]
\[
[x_{\varepsilon_j - \varepsilon_n}, x_{-\varepsilon_j - \varepsilon_{j+1}}] = x_{-\varepsilon_j - \varepsilon_{j+1} - \varepsilon_n} = [x_{-\varepsilon_s - \varepsilon_{j+1}}, x_{\varepsilon_s - \varepsilon_n}],
\]
\[
[x_{-\varepsilon_{j+1} - \varepsilon_n}, x_{\varepsilon_s + \varepsilon_n}] = x_{\varepsilon_n - \varepsilon_{j+1}} = [x_{\varepsilon_j + \varepsilon_n}, x_{-\varepsilon_j - \varepsilon_{j+1}}].
\]
Then by applying Jacobi identity several times and using the above equalities, one obtains that
\[
[x_{\varepsilon_j - \varepsilon_n}, x_{\varepsilon_s + \varepsilon_n}, x_{-\varepsilon_j - \varepsilon_{j+1}}] = [x_{\varepsilon_j + \varepsilon_n}, x_{\varepsilon_s - \varepsilon_n}, x_{-\varepsilon_j - \varepsilon_{j+1}}] (\neq 0).
\]
Hence $[x_{\varepsilon_j + \varepsilon_n}, x_{\varepsilon_s - \varepsilon_n}] = x_{\varepsilon_s + \varepsilon_j}$.
It follows that
\[
\begin{align*}
x_{\varepsilon_s + \varepsilon_j} + x_{-\varepsilon_{j+1} - \varepsilon_n} &= (\text{ad} x_{\varepsilon_j - \varepsilon_n})(y) \in (\text{ad} \mathfrak{p}_\Lambda)(y) \\
x_{-\varepsilon_{j+1} - \varepsilon_n} + x_{\varepsilon_n - \varepsilon_{j+1}} &= (\text{ad} x_{-\varepsilon_s - \varepsilon_{j+1}})(y) \in (\text{ad} \mathfrak{p}_\Lambda)(y) \\
x_{\varepsilon_n - \varepsilon_{j+1}} + x_{\varepsilon_s + \varepsilon_j} &= (\text{ad} x_{\varepsilon_j + \varepsilon_n})(y) \in (\text{ad} \mathfrak{p}_\Lambda)(y)
\end{align*}
\]
Hence $x_{\varepsilon_s + \varepsilon_j} \in (\text{ad} \mathfrak{p}_\Lambda)(y)$.
Now if $j = n - 1$ is odd, then $x_{\varepsilon_s + \varepsilon_{n-1}} = (\text{ad} x_{\varepsilon_{n-1} - \varepsilon_n})(y) \in (\text{ad} \mathfrak{p}_\Lambda)(y)$.
Let $s + 4 \leq j \leq n - 1$ and $j$ even. Then similarly as above, one has that
\[
\begin{align*}
x_{\varepsilon_s + \varepsilon_j} + x_{-\varepsilon_{j-1} - \varepsilon_n} &= (\text{ad} x_{\varepsilon_j - \varepsilon_n})(y) \in (\text{ad} \mathfrak{p}_\Lambda)(y) \\
x_{-\varepsilon_{j-1} - \varepsilon_n} + x_{\varepsilon_n - \varepsilon_{j-1}} &= (\text{ad} x_{-\varepsilon_s - \varepsilon_{j-1}})(y) \in (\text{ad} \mathfrak{p}_\Lambda)(y) \\
x_{\varepsilon_n - \varepsilon_{j-1}} + x_{\varepsilon_s + \varepsilon_j} &= (\text{ad} x_{\varepsilon_j + \varepsilon_n})(y) \in (\text{ad} \mathfrak{p}_\Lambda)(y)
\end{align*}
\]
Hence $x_{\varepsilon_s + \varepsilon_j} \in (\text{ad} \mathfrak{p}_\Lambda)(y)$.
Let $1 \leq i \leq s - 3$ and $i$ odd, or $s + 2 \leq i \leq n - 2$ and $i$ even.
Then similarly as above, by rescaling some nonzero root vectors and using Jacobi identity, one has that
\[
\begin{align*}
x_{\varepsilon_s - \varepsilon_i} + x_{\varepsilon_{i+1} - \varepsilon_n} &= (\text{ad} x_{-\varepsilon_i - \varepsilon_n})(y) \in (\text{ad} \mathfrak{p}_\Lambda)(y) \\
x_{\varepsilon_{i+1} + \varepsilon_s} + x_{\varepsilon_s - \varepsilon_i} &= (\text{ad} x_{\varepsilon_n - \varepsilon_i})(y) \in (\text{ad} \mathfrak{p}_\Lambda)(y) \\
x_{\varepsilon_{i+1} - \varepsilon_n} + x_{\varepsilon_{i+1} + \varepsilon_n} &\in (\text{ad} \mathfrak{p}_\Lambda)(y)
\end{align*}
\]
since
\[
\begin{align*}
x_{\varepsilon_{i+1} - \varepsilon_n} + x_{\varepsilon_{i+1} + \varepsilon_n} &= (\text{ad} (x_{\varepsilon_{i+1} - \varepsilon_s} - x_{-\varepsilon_s - i - 2 - \varepsilon_s}))(y) \quad \text{if } i \leq s/2 - 2 \\
x_{\varepsilon_{i+1} - \varepsilon_n} + x_{\varepsilon_{i+1} + \varepsilon_n} &= (\text{ad} x_{\varepsilon_{i+1} - \varepsilon_s})(y) \quad \text{otherwise}
\end{align*}
\]
Hence $x_{\varepsilon_s - \varepsilon_i} \in (\text{ad} \mathfrak{p}_\Lambda)(y)$.
For $i = n - 1$ even, one has that $x_{\varepsilon_s - \varepsilon_{n-1}} = (\text{ad} x_{\varepsilon_{n-1} - \varepsilon_n})(y) \in (\text{ad} \mathfrak{p}_\Lambda)(y)$.
For $2 \leq i \leq s - 2$ and $i$ even, or $s + 3 \leq i \leq n - 1$ and $i$ odd, a similar computation shows that one also has that $x_{\varepsilon_s - \varepsilon_i} \in (\text{ad} \mathfrak{p}_\Lambda)(y)$ in these cases.
Let $i = s - 1$. Then
\[
\begin{align*}
    x_{\varepsilon_{s} - \varepsilon_{s-1}} + x_{\varepsilon_{s+1} - \varepsilon_{s-1}} &= (\text{ad} \ x_{-\varepsilon_{s-1}})(y) \\
    x_{\varepsilon_{s+1} - \varepsilon_{s}} + x_{\varepsilon_{s+1} + \varepsilon_{s}} &= (\text{ad} \ x_{\varepsilon_{s+1} - \varepsilon_{s}})(y) \\
    x_{\varepsilon_{s+1} + \varepsilon_{s}} + x_{\varepsilon_{s+1} - \varepsilon_{s}} &= (\text{ad} \ x_{\varepsilon_{n} - \varepsilon_{s-1}})(y)
\end{align*}
\]

Hence \(x_{\varepsilon_{s} - \varepsilon_{s-1}} \in (\text{ad} \ p_{\Lambda})(y)\).

A similar computation shows that \(x_{\varepsilon_{s} - \varepsilon_{s+1}} \in (\text{ad} \ p_{\Lambda})(y)\).

Finally assume that \(n\) is even. Then \((\text{ad} \ x_{-\varepsilon_{s} - \varepsilon_{n-1}})(y) = x_{-\alpha_{n}} + x_{-\alpha_{n-1}} \in (\text{ad} \ p_{\Lambda})(y)\) and \(x_{-\alpha_{n-1}} = \mathfrak{g}_{T}\). Thus \(x_{-\alpha_{n}} \in (\text{ad} \ p_{\Lambda})(y) + \mathfrak{g}_{T}\).

If \(n\) is odd, then \((\text{ad} \ x_{\varepsilon_{n} - \varepsilon_{s}})(y) = x_{\alpha_{n}} + x_{\alpha_{n-1}} \in (\text{ad} \ p_{\Lambda})(y)\) and \(x_{\alpha_{n-1}} \in \mathfrak{g}_{T}\). Thus \(x_{\alpha_{n}} \in (\text{ad} \ p_{\Lambda})(y) + \mathfrak{g}_{T}\).

The proof is complete.  \(\square\)

All conditions of Proposition 4.2.1 are satisfied. Thus one has that

\[
(\text{ad} \ p_{\Lambda})(y) \oplus \mathfrak{g}_{T} = p_{\Lambda}^{*}
\]

that is, \(y\) is regular in \(p_{\Lambda}^{*}\).

Moreover, by Lemma 6.1.1 there exists a uniquely defined element \(h \in \mathfrak{h}_{\Lambda}\) such that \(\gamma(h) = -1\) for all \(\gamma \in \Sigma\). Then \((h, y)\) is an adapted pair for \(p_{\Lambda}\).

### 6.2. Computation of bounds for the formal character of \(Y(p_{\Lambda})\).

For \(\mathfrak{h}\)-modules \(M = \bigoplus_{\nu \in \mathfrak{h}^*} M_{\nu}\), \(M' = \bigoplus_{\nu \in \mathfrak{h}^*} M'_{\nu}\), with finite dimensional weight subspaces \(M_{\nu}\), resp. \(M'_{\nu}\), one writes \(\text{ch} \ M = \sum_{\nu \in \mathfrak{h}^*} \dim M_{\nu} e^{\nu}\), resp. \(\text{ch} \ M' = \sum_{\nu \in \mathfrak{h}^*} \dim M'_{\nu} e^{\nu}\) for the formal character of \(M\), resp. \(M'\) and \(\text{ch} \ M \leq \text{ch} \ M'\) if \(\dim M_{\nu} \leq \dim M'_{\nu}\) for all \(\nu \in \mathfrak{h}^*\).

Recall that \(y = \sum_{\gamma \in \Sigma} x_{\gamma}\) is regular in \(p_{\Lambda}^{*}\) and that \((\text{ad} \ p_{\Lambda})(y) \oplus \mathfrak{g}_{T} = p_{\Lambda}^{*}\).

Since \(S_{\mathfrak{h}_{\Lambda}}\) is a basis for \(\mathfrak{h}_{\Lambda}^{*}\), one may observe that, for each \(\gamma \in \mathfrak{h}^*\) and in particular for each \(\gamma \in T\), there exists a unique element \(s(\gamma) \in \mathbb{Q} S\) such that \(\gamma + s(\gamma)\) vanishes on \(\mathfrak{h}_{\Lambda}\).

Then by [19] Lem. 6.11], one has that

\[
\text{ch} \ (Y(p_{\Lambda})) \leq \prod_{\gamma \in T} (1 - e^{-\gamma + s(\gamma)})^{-1}
\]

Moreover by [16] Thm. 6.7] one has also that

\[
\prod_{\Gamma \in E(\pi')} (1 - e^{\delta_{\Gamma}})^{-1} \leq \text{ch} \ (Y(p_{\Lambda})) \leq \prod_{\Gamma \in E(\pi')} (1 - e^{\varepsilon_{\Gamma} \delta_{\Gamma}})^{-1}
\]

with \(E(\pi')\) defined in the proof of Lemma 6.1.2 and for each \(\Gamma \in E(\pi')\),

\[
\delta_{\Gamma} = -\sum_{\gamma \in \Gamma} \omega_{\gamma} - \sum_{\gamma \in j(\Gamma)} \omega_{\gamma} + \sum_{\gamma \in \Gamma \cap \pi'} \omega'_{\gamma} + \sum_{\gamma \in i(\Gamma \cap \pi')} \omega'_{\gamma}
\]

where \(\omega_{\alpha}\), resp. \(\omega'_{\alpha}\), is the fundamental weight associated to \(\alpha \in \pi\), resp. \(\alpha \in \pi'\).

Moreover for each \(\Gamma \in E(\pi')\) the number \(\varepsilon_{\Gamma}\) is the following :
where $B_\pi$, resp. $B_{\pi'}$, is the set of weights of the algebra of semi-invariants $Sy(n \oplus h)$, resp. $Sy(n_\pi^+ \oplus h')$, which is always a polynomial algebra over $k$ by [14], in $|\pi|$, resp. $|\pi'|$, generators whose weight and degree is given in [14] Tables I and II] (see also [9, Table] for an erratum).

If both bounds in (**) coincide, then one deduces that $Sy(p) = Y(p_\Lambda)$ is a polynomial algebra over $k$ on $|E(\pi')|$ generators, whose weight is $\delta_\pi$, $\Gamma \in E(\pi')$. Moreover in this case, by [9 7.1], $Sy(p)$ is equal (up to graduations) to a polynomial algebra for which the degree of each homogeneous generator may be computed by [9 5.4.2].

Actually in our present case, one may check that there exists $\Gamma \in E(\pi')$ such that $\varepsilon_\Gamma = 1/2$, thus both bounds in (**) do not coincide.

Assume now that $\prod_{\Gamma \in E(\pi')} (1 - e^{\delta_\pi})^{-1} = \prod_{\gamma \in T} (1 - e^{-(\gamma + s(\gamma))})^{-1}$. Then equality holds in (**) and by [19 Lem. 6.11] the restriction map gives an isomorphism $Y(p_\Lambda) \cong k[y + g_T]$. Then $y + g_T$ is a Weierstrass section for $p_A$ as defined in [23] This implies that $Y(p_\Lambda)$ is a polynomial algebra over $k$ on $|E(\pi')| = |T|$ algebraically independent homogeneous generators whose weight is $\delta_\pi$ for all $\Gamma \in E(\pi')$ (this weight is also equal to $-(\gamma + s(\gamma))$, for some $\gamma \in T$). Moreover the degree of each of these generators is equal to $1 + |s(\gamma)|$, $\gamma \in T$, where $|s(\gamma)| = \sum_{a \in S} m_{a,\gamma}$ if $s(\gamma) = \sum_{a \in S} m_{a,\gamma} \alpha$ $(m_{a,\gamma} \in \mathbb{N})$. For all $\gamma \in T$, the integer $|s(\gamma)|$ is also equal to the eigenvalue of $x_\gamma$ with respect to ad $h$. (For more details, see [19, 6.11].)

**Lemma 6.2.1.** If $g$ is of type $B_n$, resp. $D_n$, then

$$\prod_{\Gamma \in E(\pi')} (1 - e^{\delta_\pi})^{-1} = \prod_{\gamma \in T} (1 - e^{-(\gamma + s(\gamma))})^{-1}$$

Thus $y + g_T$ is a Weierstrass section for $p_A$. More precisely one has the following.

(i) If $g$ is of type $B_n$ and $s + 2 < n$, then

$$\text{ch} (Y(p_\Lambda)) = (1 - e^{-2\omega_2})^{-s/2}(1 - e^{-\omega_s})^{-1}(1 - e^{-2\omega_{s+2}})^{-1}(n - s - 2)$$

$$(1 - e^{-\omega_s + 2})^{-1}(1 - e^{-(\omega_s + \omega_{s+2})})^{-1}$$

(ii) If $g$ is of type $B_n$ and $n = s + 2$, then

$$\text{ch} (Y(p_\Lambda)) = (1 - e^{-2\omega_s})^{-s/2}(1 - e^{-\omega_s})^{-1}(1 - e^{-2\omega_{s+2}})^{-1}(1 - e^{-(\omega_s + 2\omega_{s+2})})^{-1}$$

(iii) If $g$ is of type $D_n$, then

$$\text{ch} (Y(p_\Lambda)) = (1 - e^{-2\omega_s})^{-s/2}(1 - e^{-\omega_s})^{-1}(1 - e^{-2\omega_{s+2}})^{-1}(n - s - 3)$$

$$(1 - e^{-\omega_s + 2})^{-2}(1 - e^{-(\omega_s + \omega_{s+2})})^{-1}$$
Proof. Recall the set $E(\pi')$ given in the proof of Lemma \[6.1.2\] and set for all $1 \leq t \leq s/2 - 1$, $\Gamma_t = \{\alpha_t, \alpha_{s-t}\}$, $\Gamma_{s/2} = \{\alpha_{s/2}\}$ and $\Gamma_u = \{\alpha_u\}$ for all $s \leq u \leq n$.

Observe that $j(\Gamma) = \Gamma$ (and then $i(\Gamma \cap \pi') = j(\Gamma) \cap \pi' = \Gamma \cap \pi'$) for all $\Gamma \in E(\pi')$, except in type $D_n$, with $n$ odd, for $\Gamma = \{\alpha_{n-1}\}$ or $\Gamma = \{\alpha_n\}$.

One checks that:

\[
\begin{align*}
\delta_{\Gamma_t} &= 2(\omega'_t - \omega_t + \omega'_{s-t} - \omega_{s-t}) = -2\omega_s, \text{ for all } 1 \leq t \leq s/2 - 1, \\
\delta_{\Gamma_{s/2}} &= 2(\omega'_{s/2} - \omega_{s/2}) = -\omega_s, \\
\delta_{\Gamma_s} &= -2\omega_s, \\
\delta_{\Gamma_{s+1}} &= -2\omega_{s+2}, \\
\text{and } \delta_{\Gamma_{s+1}} &= 2(\omega'_{s+1} - \omega_{s+1}) = -(\omega_s + \omega_{s+2}), \text{ for type } D_n \text{ and for type } B_n \text{ if } s + 2 < n.
\end{align*}
\]

For type $B_n$ with $s + 2 = n$ one checks that $\delta_{\Gamma_{s+1}} = -2\omega_{s+2}$.

If $s + 3 \leq u \leq n - 1$ for type $B_n$, resp. $s + 3 \leq u \leq n - 2$ for type $D_n$, then one checks that $\delta_{\Gamma_u} = -2\omega_{s+2}$.

If $u = n$ for $g$ of type $B_n$ (and $s + 2 < n$), resp. $u = n - 1$ or $u = n$ for $g$ of type $D_n$, then one checks that $\delta_{\Gamma_u} = -2\omega_{s+2}$.

Thus $\prod_{\Gamma \in E(\pi')}(1 - e^{\delta\Gamma})^{-1}$ is equal to the right hand side of (i), (ii) or (iii).

It remains to check that $\prod_{\Gamma \in E(\pi')}(1 - e^{\delta\Gamma})^{-1} = \prod_{\gamma \in T}(1 - e^{-(\gamma + s(\gamma))})^{-1}$.

Recall the set $T$ given before Lemma \[6.1.2\]

For $\gamma = \epsilon_{s-1} + \epsilon_s$, one checks that $s(\gamma) = (\epsilon_1 + \epsilon_2) + \ldots + (\epsilon_{s-3} + \epsilon_{s-2})$ so that $\gamma + s(\gamma) = \omega_s$.

For $\gamma = \epsilon_{s-1} - \epsilon_{s+1}$, one checks that $s(\gamma) = 2(\epsilon_1 + \epsilon_2) + \ldots + \epsilon_{s-3} + \epsilon_{s-2}) + (\epsilon_{s-1} + \epsilon_{s+1}) + 2\epsilon_s$ if $g$ of type $B_n$, $s(\gamma) = 2(\epsilon_1 + \epsilon_2) + \ldots + (\epsilon_{s-3} + \epsilon_{s-2}) + (\epsilon_{s-1} + \epsilon_{s+1}) + (\epsilon_s + \epsilon_n) + (\epsilon_s - \epsilon_n)$ if $g$ of type $D_n$ and for both types that $\gamma + s(\gamma) = 2\omega_s$.

For $\gamma = \epsilon_s + \epsilon_{s+2}$, one checks that $s(\gamma) = (\epsilon_1 + \epsilon_2) + \ldots + (\epsilon_{s-3} + \epsilon_{s-2}) + (\epsilon_{s-1} + \epsilon_{s+1})$ and that $\gamma + s(\gamma) = \omega_{s+2}$ for $g$ of type $D_n$ or $g$ of type $B_n$ with $s + 2 < n$, and that $\gamma + s(\gamma) = 2\omega_{s+2}$ for $g$ of type $B_n$ with $s + 2 = n$.

Let $1 \leq i \leq s/2 - 1$ and set $\gamma = \epsilon_{2i-1} - \epsilon_{2i}$. As in \[13\] Proof of Lem. 7.9, one checks that:

- If $s \leq 4i - 2$, then

\[
s(\epsilon_{2i-1} - \epsilon_{2i}) = 2 \sum_{j=1}^{s-2i} (\epsilon_{s-j} - \epsilon_j) + 4 \sum_{j=1}^{s/2-i} (\epsilon_{2j-1} + \epsilon_{2j}) + 2 \sum_{j=s/2-i+1}^{s-2i+1} (\epsilon_{2j-1} + \epsilon_{2j}) + 2 \epsilon_s
\]

in type $B_n$ and the same as above in type $D_n$ but with $2\epsilon_s$ replaced by $(\epsilon_s + \epsilon_n) + (\epsilon_s - \epsilon_n)$.

- If $s > 4i - 2$, then
\[ s(\varepsilon_{2i-1} - \varepsilon_{2i}) = 2 \sum_{j=1}^{2i-1} (\varepsilon_{s-j} - \varepsilon_{j}) + 4 \sum_{j=1}^{i-1} (\varepsilon_{2j-1} + \varepsilon_{2j}) + 2 \sum_{j=i+1}^{s/2-i} (\varepsilon_{2j-1} + \varepsilon_{2j}) + 3\varepsilon_{2i-1} + \varepsilon_{2i} + 2\varepsilon_s \]

in type \(B_n\) and the same as above in type \(D_n\) but with \(2\varepsilon_s\) replaced by \((\varepsilon_s + \varepsilon_n) + (\varepsilon_s - \varepsilon_n)\).

In both cases one obtains that \(\gamma + s(\gamma) = 2\varpi_s\).

Let \(1 \leq j \leq [(n-s-1)/2]\) and set \(\gamma = \varepsilon_{s+2j} - \varepsilon_{s+2j+1}\). One checks that:

\[ s(\gamma) = 2((\varepsilon_1 + \varepsilon_2) + \ldots + (\varepsilon_{s-3} + \varepsilon_{s-2})) + 2(\varepsilon_{s-1} + \varepsilon_{s+1}) \]

\[ -2 \sum_{k=2}^{j} (\varepsilon_{s+2k-1} + \varepsilon_{s+2k}) + 2 \sum_{k=2}^{j} ((\varepsilon_{s+2k-2} + \varepsilon_{s+2k-1}) + (\varepsilon_{s+2k} + \varepsilon_{s+2k+1}) + 2\varepsilon_s \]

in type \(B_n\), resp. in type \(D_n\) with \(n\) even (with \(2\varepsilon_s\) replaced by \((\varepsilon_s - \varepsilon_n) + (\varepsilon_s - \varepsilon_n))\), so that \(\gamma + s(\gamma) = 2\varpi_{s+2}\). In type \(D_n\) with \(n\) odd, for all \(1 \leq j \leq [(n-s-1)/2] - 1\), one also obtains that \(\gamma + s(\gamma) = 2\varpi_{s+2}\). If \(\varphi\) is of type \(D_n\), with \(n\) odd, then for \(\gamma = \varepsilon_{s+2j} - \varepsilon_{s+2j+1}\) with \(j = [(n - s - 1)/2] = (n - s - 1)/2\), one has that:

\[ s(\gamma) = (\varepsilon_1 + \varepsilon_2) + \ldots + (\varepsilon_{s-3} + \varepsilon_{s-2}) + (\varepsilon_{s-1} + \varepsilon_{s+1}) \]

\[ + (\varepsilon_{s+2} + \varepsilon_{s+3}) + \ldots (\varepsilon_{n-3} + \varepsilon_{n-2}) \]

\[ - ((\varepsilon_{s+3} + \varepsilon_{s+4}) + \ldots + (\varepsilon_{n-2} + \varepsilon_{n-1})) + (\varepsilon_s + \varepsilon_n) \]

so that \(\gamma + s(\gamma) = \varpi_{s+2}\).

Let \(2 \leq k \leq [(n-s)/2]\) and set \(\gamma = -\varepsilon_{s+2k-1} + \varepsilon_{s+2k}\).

One checks that:

\[ s(\gamma) = 2((\varepsilon_1 + \varepsilon_2) + \ldots + (\varepsilon_{s-3} + \varepsilon_{s-2})) + 2(\varepsilon_{s-1} + \varepsilon_{s+1}) \]

\[ -2 \sum_{k=3}^{k} (\varepsilon_{s+2k-3} + \varepsilon_{s+2k}) + 2 \sum_{k=2}^{k} ((\varepsilon_{s+2k-2} + \varepsilon_{s+2k-1}) + (\varepsilon_{s+2k} + \varepsilon_{s+2k+1}) + 2\varepsilon_s \]

in type \(B_n\), resp. in type \(D_n\) with \(n\) odd (with \(2\varepsilon_s\) replaced by \((\varepsilon_s - \varepsilon_n) + (\varepsilon_s - \varepsilon_n))\), so that \(\gamma + s(\gamma) = 2\varpi_{s+2}\). If \(\varphi\) is of type \(D_n\), with \(n\) even, then for all \(2 \leq k \leq [(n - s)/2] - 1\), one also obtains that \(\gamma + s(\gamma) = 2\varpi_{s+2}\).

Now for \(\varphi\) of type \(D_n\), with \(n\) even and for \(\gamma = -\varepsilon_{s+2k-1} + \varepsilon_{s+2k}\), with \(k = [(n - s)/2] = (n - s)/2\), one has that:

\[ s(\gamma) = (\varepsilon_1 + \varepsilon_2) + \ldots + (\varepsilon_{s-3} + \varepsilon_{s-2}) + (\varepsilon_{s-1} + \varepsilon_{s+1}) \]

\[ + (\varepsilon_{s+2} + \varepsilon_{s+3}) + \ldots (\varepsilon_{n-2} + \varepsilon_{n-1}) \]

\[ - ((\varepsilon_{s+3} + \varepsilon_{s+4}) + \ldots + (\varepsilon_{n-3} + \varepsilon_{n-2})) + (\varepsilon_s - \varepsilon_n) \]

so that \(\gamma + s(\gamma) = \varpi_{s+2}\).

Finally set \(\gamma = \varepsilon_{s+2} - \varepsilon_{s+1} = -\alpha_{s+1} \in T\). Then one has that \(s(\gamma) = 2((\varepsilon_1 + \varepsilon_2) + \ldots + (\varepsilon_{s-3} + \varepsilon_{s-2})) + 2(\varepsilon_{s-1} + \varepsilon_{s+1}) + 2\varepsilon_s\) so that \(\gamma + s(\gamma) = \varpi_s + \varpi_{s+2}\) if \(s + 2 < n\) and if \(s + 2 = n\) (necessarily in type \(B_n\)) then
\[ \gamma + s(\gamma) = \varpi_s + 2\varpi_{s+2}. \]

One can conclude that both bounds coincide, then equalities in (i), (ii) and (iii) hold and \( y + g_T \) is a Weierstrass section for \( p \).

\[ \square \]

6.3. **Weights and Degrees of homogeneous generators for \( Y(p) \).**

Thanks to Lemma 6.2.1 and by what we said before this Lemma, one can deduce the following Corollary.

**Corollary 6.3.1.** The algebra of symmetric invariants \( Y(p) = Sy(p) \) is a polynomial algebra over \( k \) on \( n - s/2 + 1 \) algebraically independent homogeneous generators. The weights and degrees of these generators are the following:

- \( s/2 \) generators have weight \(-2\varpi_s\) and degree \( s + 2 \), or \( s + 4i \) for all \( 1 \leq i \leq \lfloor s/4 \rfloor \), or \( 3s - 4i + 2 \) for all \( \lfloor s/4 \rfloor + 1 \leq i \leq s/2 - 1 \).
- One generator has weight \(-\varpi_s\) and degree \( s/2 \).
- One generator has weight \(-2\varpi_{s+2}\) if \( n = s + 2 \) (necessarily in type \( B_n \)), \(-\varpi_{s+2}\) otherwise, and degree \( s/2 + 1 \).
- If \( g \) is of type \( D_n \), then one more generator has weight \(-\varpi_{s+2}\) and degree \( n - s/2 - 1 \).
- If \( g \) is of type \( B_n \) and \( s + 2 < n \), or if \( g \) is of type \( D_n \), then \( n - s - 3 \) generators have weight \(-2\varpi_{s+2}\) and degree \( s + 4j \) for all \( 1 \leq j \leq \lfloor (n - s - 2)/2 \rfloor \), or \( s + 4j - 2 \) for all \( 2 \leq j \leq \lfloor (n - s - 1)/2 \rfloor \).
- If \( g \) is of type \( B_n \) and \( s + 2 < n \), then one more generator has weight \(-2\varpi_{s+2}\) and degree \( 2n - s - 2 \).
- One generator has weight \(-\varpi_s - 2\varpi_{s+2}\) if \( n = s + 2 \) (necessarily in type \( B_n \)), \(-\varpi_s - \varpi_{s+2}\) otherwise, and degree \( s + 3 \).

6.4. **The semisimple element \( h \) of the adapted pair for \( p \).**

By direct computation, one may obtain the expansion of the semisimple element \( h \) of the adapted pair for \( p \) constructed in subsection 6.1

**Lemma 6.4.1.** In terms of the elements \( \varepsilon_i \), \( 1 \leq i \leq n \), the semisimple element \( h \in h \) of the adapted pair \((h, y)\) constructed in subsection 6.1 has the following expansion. Set \( u = 0 \) in type \( D_n \), resp. \( u = 1 \) in type \( B_n \).

\[
\begin{align*}
    h &= \sum_{k=1}^{[s/4]}(s/2 + 2k - 1)\varepsilon_{2k-1} + \sum_{k=\lfloor s/4 \rfloor + 1}^{s/2 - 1}(3s/2 - 2k)\varepsilon_{2k-1} \\
    &- \sum_{k=1}^{s/4}(s/2 + 2k)\varepsilon_{2k} - \sum_{k=\lfloor s/4 \rfloor + 1}^{s/2 - 1}(3s/2 + 1 - 2k)\varepsilon_{2k} \\
    &+ (s/2)\varepsilon_s - \varepsilon_s - (s/2 + 1)\varepsilon_{s+1} \\
    &+ \sum_{k=1}^{\lfloor (n-s-1+w)/2 \rfloor}(2k - 1 + s/2)\varepsilon_{s+2k} \\
    &- \sum_{k=2}^{\lfloor (n-s+w)/2 \rfloor}(2k - 2 + s/2)\varepsilon_{s+2k-1} \\
\end{align*}
\]

In terms of the coroots \( \alpha_k^\vee \), \( 1 \leq k \leq n \), \( k \not\in \{s, s+2\} \), the element \( h \) has the following expansion.
Remark 6.4.2. By what we said before Lemma 6.2.1, one may check that
\[ \text{ad} Y \text{homogeneous generators of} \]
\[ \mathfrak{g} \]
Assume that
\[ \{ \alpha_n \} \]
\[ \text{chosen for the case} \]
\[ \pi \text{ of Lemma 6.1.2} \]
\[ \text{that} \]
\[ \text{of Lemma 4.2.1} \]
\[ \text{no more satisfied since} \]
\[ h \text{ to} \]
\[ \text{more belong to} \]
\[ \text{and} \]
\[ \text{Set} \]
\[ H = - \sum_{k=1}^{s/2-1} k \alpha_{2k} + \sum_{k=1}^{[s/4]} (s/2 + k) \alpha_{2k-1} \]
\[ + \sum_{k=[s/4]+1}^{[s/2]} (3s/2 + 1 - 3k) \alpha_{2k-1} \]
\[ + \sum_{k=s/2+1}^{(n-2+u)/2} (k - 1 - s/2) \alpha_{2k} \]
\[ - \sum_{k=s/2+1}^{(n-1+u)/2} k \alpha_{2k-1} \]
Then
\[ \begin{cases} 
  h = H + (n - s - 2)/4 \alpha_n' & \text{if} \ \mathfrak{g} \ \text{is of type} \ B_n \ \text{with} \ n \ \text{even} \\
  h = H - (n + 1)/4 \alpha_n' & \text{if} \ \mathfrak{g} \ \text{is of type} \ B_n \ \text{with} \ n \ \text{odd} \\
  h = H - n/4(\alpha_n' - \alpha_n) & \text{if} \ \mathfrak{g} \ \text{is of type} \ D_n \ \text{with} \ n \ \text{even} \\
  h = H + (n - s - 3)/4(\alpha_{n-1}' + \alpha_n') & \text{if} \ \mathfrak{g} \ \text{is of type} \ D_n \ \text{with} \ n \ \text{odd} 
\end{cases} \]

Remark 6.4.3. Assume that \( \mathfrak{g} \) is simple of type \( B_n \) \((n \geq 6)\), that \( \pi' = \pi \setminus \{ \alpha_s, \alpha_{s+2}, \alpha_{s+4} \} \) with \( s \) an even integer and consider the parabolic subalgebra \( \mathfrak{p} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_+^\mathfrak{p} \) associated to \( \pi' \). Then one may easily check as in the proof of Lemma 6.1.2 that \( \text{ind} \mathfrak{p}_\Lambda = n - s/2 + 1 \). Take the same set \( S^+ \) as this chosen for the case \( \pi' = \pi \setminus \{ \alpha_s, \alpha_{s+2} \} \) and the same set \( S^- \) but without the element \(-\varepsilon_{s+3} - \varepsilon_{s+4}\) which does no more belong to \( \Delta^-_\pi \). Then restriction to \( \mathfrak{h}' = \mathfrak{h}_\Lambda \) of \( S = S^+ \sqcup S^- \) is still a basis for \( \mathfrak{h}_\Lambda^* \). Take also the same sets \( T \) and \( T^* \) as before, which still lie in \( \Delta^+ \sqcup \Delta^-_\pi \). Unfortunately condition (5) of Lemma 4.2.1 is no more satisfied since \( x_{\varepsilon_{s+3}} \) and \( x_{\varepsilon_{s+4}} \) in \( T^* \) do no more belong to \( (\text{ad} \mathfrak{p}_\Lambda)(y) + \mathfrak{g}_\pi \). Thus our construction cannot be generalized to the more general case described in Sect. 6.4.

7. Second Case.

We keep the same conventions as in Sect. 6.3. Indeed in this Section we consider a parabolic subalgebra \( \mathfrak{p} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_+^\mathfrak{p} \) associated to the subset \( \pi' = \pi \setminus \{ \alpha_s, \alpha_{s+2}, \ldots, \alpha_{s+2\ell} \} \) with \( \ell \in \mathbb{N} \) and \( s \) an odd integer, \( 1 \leq s \leq n \), in a simple Lie algebra \( \mathfrak{g} \) of type \( B_n \), resp. \( D_n \), with \( n \geq 2 \), resp. \( n \geq 4 \).

If \( \ell = 0 \), then the parabolic subalgebra \( \mathfrak{p} \) is maximal and this case was already treated in 6.2. Thus we will assume from now on that \( \ell \geq 1 \) and then that \( s \leq n - 2 \). Observe also that our work contains the special case of submaximal parabolic subalgebras (namely when \( \ell = 1 \)).

Here the bounds of 6.7 coincides, then the algebra of symmetric invariants \( Y(\mathfrak{p}_\Lambda) \) is known to be polynomial. However Weierstrass sections were not yet constructed for such parabolic subalgebras.

We will still use Proposition 4.2.1 which is easier to apply than in the first case (see Sect. 6.3). Indeed it suffices to take as Heisenberg sets the maximal Heisenberg sets (or their opposite) associated to the Kostant cascade of \( \mathfrak{g} \) or
$\mathfrak{g}'$, defined in subsection 4.1 (and then to take in $S$ and in $T$ elements, or their opposite, in the Kostant cascade of $\mathfrak{g}$ or $\mathfrak{g}'$). Notice that, in Sect. 6 (with $s$ even and $\ell = 1$), such Kostant cascades could not have been taken in $S$, since otherwise the semisimple element $h$ of the adapted pair should verify both: $\varpi_s(h) = ((\epsilon_1 + \epsilon_2) + \ldots + (\epsilon_{s-1} + \epsilon_s))((h) = (-1) \times s/2$ and $\varpi_s(h) = 0$ since $h \in \mathfrak{h}_\Lambda = \mathfrak{h}'$.

7.1. An adapted pair for $p_\Lambda$. Assume first that $\mathfrak{g}$ is of type $B_n$ with $s + 2\ell \leq n$, resp. $\mathfrak{g}$ is of type $D_n$ with $s + 2\ell \leq n - 2$. Recall that $s$ is an odd integer and that $\ell \geq 1$. We set:

$$S^+ = \{\beta_i = \epsilon_{2i-1} + \epsilon_{2i} \mid 1 \leq i \leq [n/2]\}$$

$$S^- = \{-\beta'_i = \epsilon_{s+1-i} - \epsilon_i, -\beta''_j = -(\epsilon_{s+2\ell+2j-1} + \epsilon_{s+2\ell+2j}); 1 \leq i \leq (s-1)/2, 1 \leq j \leq [(n-s-2\ell)/2]\}$$

For $\mathfrak{g}$ of type $B_n$ we set:

\[
\begin{align*}
T^+ &= \{\alpha_i = \epsilon_{2i-1} - \epsilon_{2i} \mid 1 \leq i \leq [n/2]\} \\
T^- &= \{-\alpha_{s+2i-1} = \epsilon_{s+2i} - \epsilon_{s+2i-1}, -\alpha_{s+2\ell+2j-1} = \epsilon_{s+2\ell+2j} - \epsilon_{s+2\ell+2j-1}, \alpha_n = -\epsilon_n \mid 1 \leq i \leq \ell, 1 \leq j \leq [(n-s-2\ell)/2]\}
\end{align*}
\]

and if $n$ is odd

\[
\begin{align*}
T^+ &= \{\alpha_i = \epsilon_{2i-1} - \epsilon_{2i}, \alpha_n = \epsilon_n \mid 1 \leq i \leq [n/2]\} \\
T^- &= \{-\alpha_{s+2i-1} = \epsilon_{s+2i} - \epsilon_{s+2i-1}, -\alpha_{s+2\ell+2j-1} = \epsilon_{s+2\ell+2j} - \epsilon_{s+2\ell+2j-1}; 1 \leq i \leq \ell, 1 \leq j \leq [(n-s-2\ell)/2]\}
\end{align*}
\]

For $\mathfrak{g}$ of type $D_n$ we set:

\[
\begin{align*}
T^+ &= \{\alpha_i = \epsilon_{2i-1} - \epsilon_{2i} \mid 1 \leq i \leq [n/2]\} \\
T^- &= \{-\alpha_{s+2i-1} = \epsilon_{s+2i} - \epsilon_{s+2i-1}, -\alpha_{s+2\ell+2j-1} = \epsilon_{s+2\ell+2j} - \epsilon_{s+2\ell+2j-1}; 1 \leq i \leq \ell, 1 \leq j \leq [(n-s-2\ell)/2]\}
\end{align*}
\]

Now assume that $\mathfrak{g}$ is of type $D_n$ and that $s + 2\ell \in \{n-1, n\}$. Since the case $\pi' = \pi \setminus \{\alpha_s, \alpha_{s+2}, \ldots, \alpha_{s+2\ell-2}, \alpha_{n-1}\}$ and the case $\pi' = \pi \setminus \{\alpha_s, \alpha_{s+2}, \ldots, \alpha_{s+2\ell-2}, \alpha_n\}$ are symmetric, one may suppose that we are in the latter case. Then we set:

$$S^+ = \{\beta_i = \epsilon_{2i-1} + \epsilon_{2i} \mid 1 \leq i \leq [(n-1)/2]\}$$

$$S^- = \{-\beta'_i = \epsilon_{s+1-i} - \epsilon_i, -\beta''_i = \epsilon_n - \epsilon_{n-2} \mid 1 \leq i \leq (s-1)/2\}$$

if $n$ is even
$$S^- = \{-\beta'_i = \varepsilon_{s+1-i} - \varepsilon_i ; 1 \leq i \leq (s-1)/2\}$$

if $n$ is odd.

Moreover if $n$ is even, we set :

$$\begin{cases} 
T^+ & = \{ \alpha_i = \varepsilon_{2i-1} - \varepsilon_{2i}, \alpha_n = \varepsilon_{n-1} + \varepsilon_n ; 1 \leq i \leq [n/2]\} \\
T^- & = \{-\alpha_{s+2i-1} = \varepsilon_{s+2i} - \varepsilon_{s+2i-1} ; 1 \leq i \leq \ell - 1\} 
\end{cases}$$

and if $n$ is odd, we set :

$$\begin{cases} 
T^+ & = \{ \alpha_i = \varepsilon_{2i-1} - \varepsilon_{2i} ; 1 \leq i \leq [n/2]\} \\
T^- & = \{-\alpha_{s+2i-1} = \varepsilon_{s+2i} - \varepsilon_{s+2i-1} ; 1 \leq i \leq \ell\} 
\end{cases}$$

Set $S = S^+ \cup S^-$, $T = T^+ \cup T^-$ and $T^- = \emptyset$.

Then $S^+ , T^+ \subset \Delta^+$ and $S^- , T^- \subset \Delta^-$. Recall the Kostant cascade $\beta_\pi$ of $\mathfrak{g}$, resp. the Kostant cascade $\beta_\pi'$ of $\mathfrak{g}'$, explained in subsection 4.1. Set

$$\beta_\pi^0 = \beta_\pi \setminus (\beta_\pi \cap \pi), \quad \beta_\pi'^0 = \beta_\pi' \setminus (\beta_\pi' \cap \pi').$$

Then one may observe that $S^+ = \beta_\pi^0$ (except for $\mathfrak{g}$ of type $D_n$ with $n$ even and $s + 2\ell \leq n - 2$, since in this case $S^+ = \beta_\pi^0 \cup \{\alpha_n = \varepsilon_{n-1} + \varepsilon_n\}$). Similarly $S^- = -\beta_\pi'^0$, (except for $\mathfrak{g}$ of type $D_n$ with $n$ odd and $s + 2\ell \leq n - 2$, since in this case $S^- = -\beta_\pi'^0 \cup \{-\alpha_n = -(\varepsilon_{n-1} + \varepsilon_n)\}$).

Then $T^+ = \beta_\pi \cap \pi$ and $T^- = -\beta_\pi' \cap \pi'$, except for $\mathfrak{g}$ of type $D_n$ with $n$ even and $s + 2\ell \leq n - 2$, since in this case $T^+ = (\beta_\pi \cap \pi) \setminus \{\alpha_n\}$, and for $\mathfrak{g}$ of type $D_n$ with $n$ odd and $s + 2\ell \leq n - 2$, since in this case $T^- = -(\beta_\pi' \cap \pi') \setminus \{-\alpha_n\}$.

In all cases we have then that $\beta_\pi = S^+ \cup T^+$ and $-\beta_\pi' = S^- \cup -T^-$. See also [13, Sect. 7] or [12, Sect. 4 and 5] for further details: the method here being very similar to that used in [12].

Then for all $\gamma \in S^+$, resp. $\gamma \in S^-$, we choose $\Gamma_\gamma = H_\gamma$, resp. $\Gamma_\gamma = -H_\gamma$, as defined in subsection 4.1.

Recall also that, if $\alpha \in \beta_\pi \cap \pi$, then $H_\alpha = \{\alpha\}$.

Hence by [14, Lem. 2.2] (see also [12, Lem. 3]), conditions (2), (3), (4) of Proposition 4.2.1 are satisfied. Moreover condition (5) is empty since $T^+ = \emptyset$. It remains to verify conditions (1) and (6).

**Lemma 7.1.1.** $S_{|\Lambda}$ is a basis for $\mathfrak{h}_\Lambda^*$.

**Proof.** First remark that $\mathfrak{h}_\Lambda = \mathfrak{h}'$ and that $|S| = \dim \mathfrak{h}' = n - \ell - 1$.

Assume first that $\mathfrak{g}$ is of type $D_n$ with $n$ odd and that $s + 2\ell = n$. Then $|S| = (n - 1)/2 + (s - 1)/2 = n - \ell - 1$ and one may order the elements $s_u$ of $S$ as

$$\beta_1, \beta_2, \ldots, \beta_{(n-1)/2}, -\beta'_1, -\beta'_2, \ldots, -\beta'_{(s-1)/2}$$

and choose the following basis $h_u$ of $\mathfrak{h}'$:

$$\alpha'_{2i}, \alpha'_{2j-1}, \alpha'_{s-2j}, 1 \leq i \leq (n - 1)/2, 1 \leq j \leq [(s + 1)/4]$$
without repetition.

Then observe that, for all $1 \leq i \leq (n-3)/2$, one has $\beta_i = \varepsilon_{2i-1} + \varepsilon_{2i} = \omega_{2i} - \omega_{2i-2}$ if we set $\omega_0 = 0$ and $\beta_{(n-1)/2} = \varepsilon_{n-2} + \varepsilon_{n-1} = \omega_{n-1} + \omega_n - \omega_{n-3}$.

Moreover consider the connected component $\pi'_1$ of $\pi'$ of type $A_{s-1}$ and recall the construction of the Kostant cascade explained in [18] in this case. Set $\Delta^+_1 = \Delta^+_{\pi'_1}$, then set $\Delta^+_2 = \{ \alpha \in \Delta^+_1; (\alpha, \beta'_1) = 0 \}$ (recall that $\beta'_1 = \omega'_1 + \omega'_{s-1}$). Continuing we set $\Delta^+_{i+1} = \{ \alpha \in \Delta^+_i; (\alpha, \beta'_j) = 0 \}$. We then have that $\Delta^+_{i+1} \subset \Delta^+_i \subset \cdots \subset \Delta^+_1$ and then $(\beta'_i, \alpha)$ = 0 for all $\alpha \in \Delta^+_j$ with $j > i$.

Finally observe that, for all $1 \leq j \leq [(s+1)/4]$, $\alpha_{2j-1} \in \Delta^+_{2j-1}$, $\alpha_{s-2j} \in \Delta^+_{2j}$, $\beta'_{2j-1}(\alpha'_{2j-1}) = (\beta'_{2j-1}(\alpha'_{2j-1})$, $\alpha_{2j-1}) = 1$ and that $\beta'_{2j}(\alpha'_{s-2j}) = (\beta'_{2j}, \alpha_{s-2j}) = 1$ while $2j \leq (s-1)/2$.

It follows that the matrix $(s_u(h_v))_{1 \leq u, v \leq (n-\ell-1)}$ has the form

$$
\begin{pmatrix}
A & 0 \\
* & B
\end{pmatrix}
$$

where $A$ is a $(n-1)/2 \times (n-1)/2$ lower triangular matrix with 1 on the diagonal, and $B$ is a $(s-1)/2 \times (s-1)/2$ lower triangular matrix with 1 on the diagonal. Hence $\det(s_u(h_v))_{1 \leq u, v \leq (n-\ell-1)} \neq 0$ and we are done in this case.

Assume now that $\mathfrak{g}$ is of type $D_n$ with $n$ even and that $s + 2\ell = n - 1$. Then consider the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ associated to $\pi' = \pi \setminus \{ \alpha_s, \alpha_{s+2}, \ldots, \alpha_{n-3}, \alpha_n \}$ (1 $\leq s \leq n-3$ is still an odd integer). Then $|S| = (n-2)/2 + (s-1)/2 + 1 = n - \ell - 1$ and one may order the elements $s_u$ of $S$ as

$$
\beta_1, \beta_2, \ldots, \beta_{(n-2)/2}, -\beta'_1, -\beta'_2, \ldots, -\beta'_{(s-1)/2}, \varepsilon_n - \varepsilon_{n-2}
$$

and choose the following basis $h_u$ of $\mathfrak{h}'$:

$$
\alpha'^{\vee}_{2i}, \alpha'^{\vee}_{2j-1}, \alpha'^{\vee}_{s-2j}, \alpha'^{\vee}_{n-1}, 1 \leq i \leq (n-2)/2, 1 \leq j \leq [(s+1)/4]
$$

without repetition.

Similarly as above one obtains that the matrix $(s_u(h_v))_{1 \leq u, v \leq (n-\ell-1)}$ has the form

$$
\begin{pmatrix}
A & 0 \\
* & B
\end{pmatrix}
$$

where $A$ is a $(n-2)/2 \times (n-2)/2$ lower triangular matrix with 1 on the diagonal, and $B$ is a $(s+1)/2 \times (s+1)/2$ lower triangular matrix with 1 on the diagonal. Hence $\det(s_u(h_v))_{1 \leq u, v \leq (n-\ell-1)} \neq 0$ and we are done in this case.

Assume that $\mathfrak{g}$ is of type $B_n$.

Then one may order the elements $s_u$ of $S$ as follows:
\[ \beta_i, -\beta'_j, -\beta''_k; \]
\[ 1 \leq i \leq [n/2], 1 \leq j \leq (s - 1)/2, 1 \leq k \leq [(n - s - 2\ell)/2] \]
and choose the following basis \( h_v \) of \( \h' \):
\[ \alpha^\vee_{2i}, \alpha^\vee_{2j-1}, \alpha^\vee_{s-2j}, \alpha^\vee_{2k+1}; \]
\[ 1 \leq i \leq [n/2], 1 \leq j \leq [(s + 1)/4], (s + 2\ell + 1)/2 \leq k \leq [(n - 1)/2] \]
without repetition.

Now if \( g \) is of type \( D_n \) with \( s + 2\ell \leq n - 2 \), we take the same set \( S \) ordered as above and the same basis of \( \h' \), up to replacing \( \alpha^\vee_n \) by \( 2\varepsilon_n \).

Then by what we explained before, the matrix \( (s_u(h_v))_{1 \leq u, v \leq (n-\ell-1)} \) has the form
\[
\begin{pmatrix}
A & 0 & 0 \\
* & B & 0 \\
* & * & C
\end{pmatrix}
\]
where \( A \) is a \([n/2] \times [n/2]\) lower triangular matrix with one on the diagonal (except for the case \( n \) even where 2 is the last entry of the diagonal), \( B \) is a \((s-1)/2 \times (s-1)/2\) lower triangular matrix with -1 on the diagonal and \( C \) is a \([(n - s - 2\ell)/2] \times [(n - s - 2\ell)/2]\) lower triangular matrix with -1 on the diagonal (except for the case \( n \) odd where -2 is the last entry of the diagonal). Hence \( \det(s_u(h_v))_{1 \leq u, v \leq (n-\ell-1)} \neq 0 \) and the proof is complete.

Now we check that the set \( T = T^+ \sqcup T^- \) has cardinality equal to \( \text{ind} \, p_A \).

**Lemma 7.1.2.** One has that \( |T| = \text{ind} \, p_A \).

**Proof.** Recall the proof of Lemma 6.1.2 that \( \text{ind} \, p_A = |E(\pi')| \) where \( E(\pi') \) is the set of \( (ij)\)-orbits in \( \pi \).

Assume first that \( g \) is of type \( D_n \) and that \( s + 2\ell \in \{n - 1, n\} \).

If \( n \) is odd, then
\[ E(\pi') = \{ \Gamma_u = \{\alpha_u, \alpha_{s-u}\}, \Gamma_v = \{\alpha_v\}, \Gamma_n = \{\alpha_n, \alpha_{n-1}\}; 1 \leq u \leq (s-1)/2, s \leq v \leq n - 2 \} \]

If \( n \) is even, then (with the hypothesis that \( \pi' = \pi \setminus \{\alpha_s, \alpha_{s+2}, \ldots, \alpha_{s+2\ell-2}, \alpha_n\} \))
\[ E(\pi') = \{ \Gamma_u = \{\alpha_u, \alpha_{s-u}\}, \Gamma_v = \{\alpha_v\}, \Gamma_n = \{\alpha_n-2, \alpha_{n-1}\}, \Gamma_{n-1} = \{\alpha_{n-1}, \alpha_n\}, 1 \leq u \leq (s-1)/2, s \leq v \leq n - 3 \} \]

Hence \( \text{ind} \, p_A = |E(\pi')| = n - (s + 1)/2 \).

On the other hand, one has that for \( n \) even, \( |T^+| = n/2 + 1 \) and \( |T^-| = \ell - 1 \) and \( s + 2\ell = n - 1 \) and if \( n \) is odd, \( |T^+| = (n - 1)/2, |T^-| = \ell \) and \( s + 2\ell = n \).

Then \( |T| = \text{ind} \, p_A \) in both cases.

Now assume that \( g \) is of type \( B_n \). Then
\[ E(\pi') = \{ \Gamma_u = \{ \alpha_u, \alpha_{s-u} \}, \Gamma_v = \{ \alpha_v \}; 1 \leq u \leq (s-1)/2, s \leq v \leq n \} \]

Hence \( \text{ind} p_A = |E(\pi')| = n - (s-1)/2 \) and one checks that this is also equal to \( |T| \).

Finally assume that \( g \) is of type \( D_n \) and that \( s + 2\ell \leq n - 2 \). Then

\[ E(\pi') = \{ \Gamma_u = \{ \alpha_u, \alpha_{s-u} \}, \Gamma_v = \{ \alpha_v \}, \Gamma_{n-1} = \{ \alpha_{n-1}, \alpha_n \}; 1 \leq u \leq (s-1)/2, s \leq v \leq n - 2 \} \]

Hence \( \text{ind} p_A = |E(\pi')| = n - (s+1)/2 \) and one checks that this is also equal to \( |T| \).

\[ \square \]

All conditions of Proposition 4.2.1 thus one can deduce the following corollary.

**Corollary 7.1.3.** Set \( y = \sum_{\alpha \in S} x_\alpha \). Then \( y \) is regular in \( p_A^* \) and more precisely \((\text{ad} p_A)(y) \oplus g_Y = p_A^*\). Moreover since \( S|_{\mathfrak{h}_A} \) is a basis for \( \mathfrak{h}_A^* \), there exists a uniquely defined element \( h \in \mathfrak{h}_A \) such that \( \alpha(h) = -1 \) for all \( \alpha \in S \). Thus the pair \((h, y)\) is an adapted pair for \( p_A \).

### 7.2. A Weierstrass section for \( p_A \)

As we already said in Sect. 11 the existence of an adapted pair does not imply in general the existence of a Weierstrass section. However if \( Y(p_A) \) is a polynomial, then it does by [21, 2.3]. Actually in our present case, both bounds in (**) of subsection 6.2 coincide and then \( Y(p_A) \) is polynomial.

This is the following lemma.

**Lemma 7.2.1.** For all \( \Gamma \in E(\pi') \), one has that \( \varepsilon_\Gamma = 1 \). Then \( Y(p_A) \) is a polynomial algebra over \( \mathbb{k} \).

**Proof.** Recall the set \( E(\pi') \) given in the proof of Lemma 7.1.2 and set \( d_\Gamma = \sum_{\gamma \in \Gamma} w_\gamma \) and \( d_\Gamma^* = \sum_{\gamma \in \Gamma} w_\gamma^* \).

In [14 Tables I, II] and [9 Table] for an erratum, one can find the generators of the semigroup \( \mathcal{B}_\pi \).

Then for all \( 1 \leq u \leq (s-1)/2 \), one has that \( d_{\Gamma_u} = w_u + w_{s-u} \notin \mathcal{B}_\pi \).

Hence \( \varepsilon_{\Gamma_u} = 1 \).

Let \( s \leq v \leq n-2 \), \( (v \leq n-3 \text{ if } n \text{ even and } g \text{ of type } D_n \text{ with } s+2\ell = n-1) \), one has that \( d_{\Gamma_v} = w_v \notin \mathcal{B}_\pi \) if \( v \) is odd and \( d_{\Gamma_v}^* = w_v^* \notin \mathcal{B}_{\pi'} \) if \( v \) is even.

Hence \( \varepsilon_{\Gamma_v} = 1 \).

Now if \( g \) is of type \( D_n \), \( s + 2\ell = n - 1 \) and \( n \) even, one has that \( d_{\Gamma_{n-1}} = w_{n-1} + w_n \notin \mathcal{B}_\pi \) and \( d_{\Gamma_{n-1}}^* = w_{n} \notin \mathcal{B}_{\pi'} \). Hence \( \varepsilon_{\Gamma_{n-1}} = \varepsilon_{\Gamma_{n-1}}^* = 1 \).

If \( g \) is of type \( D_n \), \( s + 2\ell = n \) and \( n \) odd, then \( d_{\Gamma_{n-1}} = w_{n-1} + w_n \in \mathcal{B}_\pi \), but \( d_{\Gamma_{n-1}}^* = w_{n} \notin \mathcal{B}_{\pi'} \). Hence \( \varepsilon_{\Gamma_{n-1}} = 1 \).

If \( g \) is of type \( D_n \), \( s + 2\ell \leq n - 2 \), then \( d_{\Gamma_{n-1}} = w_{n-1} + w_n \), and \( d_{\Gamma_{n-1}}^* = w_{n-1}^* + w_n^*. \) One of them does not belong to \( \mathcal{B}_\pi \), resp. \( \mathcal{B}_{\pi'} \), since the simple
part of \( g' \) that we consider here is of type \( D_{n-s-2\ell} \), and since \( n \) and \( n-s-2\ell \) are of different parity. Hence \( \epsilon_{\Gamma_{n-1}} = 1 \).

Finally assume that \( g \) is of type \( B_n \) and take \( v \in \{n-1, n\} \). Then \( d_{\Gamma_n} = \varpi_n \not\in B_\pi \) if \( n \) is even. If now \( n \) is odd then \( d_{\Gamma_n} = \varpi_n \not\in B_\pi \) while \( d_{\Gamma_{n-1}} = \varpi_{n-1} \in B_\pi \). But since \( n \) is odd, \( \alpha_{n-1} \in \pi' \) and \( d_{\Gamma_{n-1}}' = \varpi_{n-1}' \not\in B_\pi' \).

Hence \( \epsilon_{\Gamma_n} = 1 \). This completes the proof. \( \square \)

**Corollary 7.2.2.** Let \( y = \sum_{\gamma \in S} x_\gamma \). Then \( y + g_T \) is a Weierstrass section for \( p_\Lambda \).

### 7.3 Weights and Degrees of homogeneous generators for \( Y(p_\Lambda) \)

As we said in subsection 6.2 when both bounds in (**) coincide, then \( Y(p_\Lambda) \) is (up to graduations) equal to a polynomial algebra for which each generator has \( \delta_\pi \) as a weight (given by (***) of 6.2 and a degree \( \delta_\Gamma \) (for each \( \Gamma \in E(\pi') \)) which may be computed by [9, 5.4.2].

Note, since \( y + g_T \) is a Weierstrass section for \( p_\Lambda \), that one may also compute weights and degrees of generators of \( Y(p_\Lambda) \), using that (**) of 6.2 is an equality and then that weights of generators are \( -(\gamma + s(\gamma)) \), and degrees are \( 1 + |s(\gamma)| \), for all \( \gamma \in T \).

Below are weights and degrees of a set of homogeneous algebraically independent generators of \( Y(p_\Lambda) \), each of them corresponding to an \( \langle ij \rangle \)-orbit \( \Gamma_i \) in \( E(\pi') \).

Assume that \( g \) is of type \( B_n \) and that \( s + 2\ell < n \):

| \( \langle ij \rangle \)-orbit in \( E(\pi') \) | Weight | Degree |
|---------------------------------|--------|-------|
| \( \Gamma_u = \{\alpha_u, \alpha_{s-u}\}, 1 \leq u \leq (s-1)/2 \) | \(-2\varpi_s \) | \( s + 1 + 2u \) |
| \( \Gamma_v = \{\alpha_v\}, v = s + 2k, 0 \leq k \leq \ell \) | \(-2\varpi_v \) | \( v + 1 \) |
| \( \Gamma_w = \{\alpha_w\}, v = s + 2k - 1, 1 \leq k \leq \ell \) | \(-\varpi_{v-1} - \varpi_{v+1} \) | \( v + 1 \) |
| \( \Gamma_v = \{\alpha_v\}, s + 2\ell + 1 \leq v \leq n - 1 \) | \(-2\varpi_s + 2\ell \) | \( 2v + 1 - s - 2\ell \) |
| \( \Gamma_n = \{\alpha_n\} \) | \(-\varpi_s + 2\ell \) | \( n - \ell + (1 - s)/2 \) |

Assume that \( g \) is of type \( B_n \) and that \( s + 2\ell = n \) (hence \( n \) is odd):

| \( \langle ij \rangle \)-orbit in \( E(\pi') \) | Weight | Degree |
|---------------------------------|--------|-------|
| \( \Gamma_u = \{\alpha_u, \alpha_{s-u}\}, 1 \leq u \leq (s-1)/2 \) | \(-2\varpi_s \) | \( s + 1 + 2u \) |
| \( \Gamma_v = \{\alpha_v\}, v = s + 2k, 0 \leq k \leq \ell - 1 \) | \(-2\varpi_v \) | \( v + 1 \) |
| \( \Gamma_w = \{\alpha_w\}, v = s + 2k - 1, 1 \leq k \leq \ell - 1 \) | \(-\varpi_{v-1} - \varpi_{v+1} \) | \( v + 1 \) |
| \( \Gamma_n = \{\alpha_{n-1}\} \) | \(-\varpi_{n-2} - 2\varpi_n \) | \( n \) |
| \( \Gamma_n = \{\alpha_n\} \) | \(-\varpi_n \) | \((n + 1)/2 \) |

Assume that \( g \) is of type \( D_n \) and that \( s + 2\ell \leq n - 2 \):

| \( \langle ij \rangle \)-orbit in \( E(\pi') \) | Weight | Degree |
|---------------------------------|--------|-------|
| \( \Gamma_u = \{\alpha_u, \alpha_{s-u}\}, 1 \leq u \leq (s-1)/2 \) | \(-2\varpi_s \) | \( s + 1 + 2u \) |
| \( \Gamma_v = \{\alpha_v\}, v = s + 2k, 0 \leq k \leq \ell \) | \(-2\varpi_v \) | \( v + 1 \) |
| \( \Gamma_w = \{\alpha_w\}, v = s + 2k - 1, 1 \leq k \leq \ell \) | \(-\varpi_{v-1} - \varpi_{v+1} \) | \( v + 1 \) |
| \( \Gamma_v = \{\alpha_v\}, s + 2\ell + 1 \leq v \leq n - 2 \) | \(-2\varpi_s + 2\ell \) | \( 2v + 1 - s - 2\ell \) |
| \( \Gamma_{n-1} = \{\alpha_{n-1}, \alpha_n\} \) | \(-2\varpi_s + 2\ell \) | \( 2n - s - 2\ell - 1 \) |
Assume that \( g \) is of type \( D_n \) and that \( s + 2\ell = n \) (hence \( n \) is odd):

\[
\begin{array}{|c|c|c|}
\hline
\langle ij \rangle\text{-orbit in } E(\pi') & \text{Weight} & \text{Degree} \\
\hline
\Gamma_u = \{\alpha_u, \alpha_{s-u}\}, 1 \leq u \leq (s-1)/2 & -2\varpi_s & s + 1 + 2u \\
\Gamma_v = \{\alpha_v\}, v = s + 2k, 0 \leq k \leq \ell - 1 & -2\varpi_v & v + 1 \\
\Gamma_v = \{\alpha_v\}, v = s + 2k - 1, 1 \leq k \leq \ell - 1 & -\varpi_{v-1} - \varpi_{v+1} & v + 1 \\
\Gamma_{n-1} = \{\alpha_{n-1}, \alpha_n\} & -2\varpi_{n-2} - 2\varpi_n & n \\
\Gamma_n = \{\alpha_n\} & -2\varpi_n & n/2 \\
\hline
\end{array}
\]

Assume that \( g \) is of type \( D_n \) and that \( s + 2\ell = n - 1 \) (hence \( n \) is even and recall that \( \pi' = \pi \setminus \{\alpha_s, \alpha_{s+2}, \ldots, \alpha_{n-3}, \alpha_n\} \)):

\[
\begin{array}{|c|c|c|}
\hline
\langle ij \rangle\text{-orbit in } E(\pi') & \text{Weight} & \text{Degree} \\
\hline
\Gamma_u = \{\alpha_u, \alpha_{s-u}\}, 1 \leq u \leq (s-1)/2 & -2\varpi_s & s + 1 + 2u \\
\Gamma_v = \{\alpha_v\}, v = s + 2k, 0 \leq k \leq \ell - 1 & -2\varpi_v & v + 1 \\
\Gamma_v = \{\alpha_v\}, v = s + 2k - 1, 1 \leq k \leq \ell - 1 & -\varpi_{v-1} - \varpi_{v+1} & v + 1 \\
\Gamma_{n-1} = \{\alpha_{n-1}, \alpha_{n-2}\} & -2(\varpi_{n-3} + \varpi_n) & 3n/2 \\
\Gamma_n = \{\alpha_n\} & -2\varpi_n & n/2 \\
\hline
\end{array}
\]

Remark 7.3.1. It would be interesting to find an adapted pair for \( g \) of type \( B \) or \( D \) with \( \pi' = \pi \setminus \{\alpha_s, \alpha_{s+4}\} \) with \( s \) odd, since in this case the criterion of [1] Thm. 6.7 does not apply and then polynomiality of \( Y(\mathfrak{p}_A) = Sy(\mathfrak{p}) \) is not yet known. However the adapted pair that we have constructed in subsection 7.1 for submaximal parabolic subalgebras (with the set \( S = \beta_0^\pi \cup (-\beta_0^\pi) \)) does no more work in this case. Indeed one may notice that for all \( \beta \in \beta_0^\pi \), and for all \( \beta' \in \beta_0^\pi \), one has \( \beta(\alpha_{s+2}) = \beta'(\alpha_{s+2}) = 0 \) while \( \alpha_{s+2} \in \mathfrak{h}_A \). It follows that the restriction of \( \beta_0^\pi \cup (-\beta_0^\pi) \) to \( \mathfrak{h}_A \) cannot give a basis for \( \mathfrak{h}_A^\ast \).

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