The classification of rigid hyperelliptic fourfolds

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Abstract
We provide a fine classification of rigid hyperelliptic manifolds in dimension four up to biholomorphism and diffeomorphism. These manifolds are explicitly described as finite étale quotients of a product of four Fermat elliptic curves.

Keywords Rigid complex manifold · Deformation theory · Flat Kähler manifold · Hyperelliptic variety · Bieberbach group

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1 Introduction
It is classically known that any compact flat Riemannian manifold $X$ is a quotient of the affine space $\mathbb{R}^n$ by a torsion-free, discrete and cocompact group $\Gamma \leq \mathbb{E}(n)$ of Euclidean motions. Such groups $\Gamma$ are called Bieberbach groups. By Bieberbach’s first structure theorem, there is a short exact sequence

$$0 \to \Lambda \to \Gamma \to G \to 1,$$

where the subgroup $\Lambda$ of translations is a lattice of full rank and $G$ is a finite group isomorphic to the holonomy group of $X$. In particular, $X$ is the quotient of the torus $T = \mathbb{R}^n / \Lambda$ by the induced action of $G$. Obviously, not every flat manifold $X = \mathbb{R}^{2n} / \Gamma$ has a complex Kähler structure, which means that the holonomy representation is unitary. In case of existence, the manifold $X$ is called a flat Kähler manifold or a (generalized) hyperelliptic manifold. They have been classified in the surface case, i.e., in complex dimension 2 by Bagnera-de Franchis [3] as well as Enriques-Severi [13]. In complex dimension 3, these manifolds have been investigated in works of Uchida-Yoshihara [22], Lange [19] and Catanese-Demleitner [11].
In [12], Demleitner derived a complete list of holonomy groups of generalized hyperelliptic fourfolds. At the moment, a fine classification up to biholomorphism is not yet established. As a first step toward such a classification, we restrict our attention to those examples having a rigid complex structure.

From the $C^\infty$-point of view, compact flat Riemannian manifolds behave very nicely. Indeed, as a consequence of Bieberbach’s second theorem, the diffeomorphism type of a compact flat Riemannian manifold is uniquely determined by the fundamental group of the underlying topological space. The example of complex tori shows that we cannot expect the analogous property in the holomorphic category without any additional assumptions. In fact, in this paper we present diffeomorphic but non-biholomorphic rigid hyperelliptic fourfolds; hence, this property fails even under the strong assumption of rigidity. We want to point out that Bieberbach’s third theorem shows that in any dimension, there are only finitely many compact flat Riemannian manifolds up to diffeomorphism. Our first main result is:

**Theorem 1.1** Let $X = T/G$ be a rigid hyperelliptic manifold with holonomy $G$, then:

(a) $\dim(X) \geq 4$.

(b) If $\dim(X) = 4$, then $G \cong \mathbb{Z}_3^2$ or $G \cong \text{He}(3)$, the Heisenberg group of order 27:

$$
\text{He}(3) := \langle g, h, k \mid g^3 = h^3 = k^3 = [g, k] = [h, k] = 1, [g, h] = k \rangle. \quad (1.1)
$$

Our second main result is the full classification of rigid hyperelliptic fourfolds, up to biholomorphism and diffeomorphism:

**Theorem 1.2** Let $E := \mathbb{C}/\mathbb{Z}[\zeta_3]$ and $t := (1 + 2\zeta_3)/3 \in E[3]$. Then:

(1) There are exactly twelve biholomorphism classes $X_i$ of rigid hyperelliptic fourfolds with holonomy $\mathbb{Z}_3^2$. They are realized as quotients of $E^4/K_i$ by the actions

$$
\phi_i(a, b)(z) := \text{diag}(\zeta_3^a, \zeta_3^b, \zeta_3^{2a+b}, \zeta_3^{a+b}) \cdot z + \tau_i(a, b),
$$

where $K_i$ and $\tau_i$ are according to the table below: These twelve complex manifolds form

| $i$ | $K_i$ | $\tau_i(1, 0)$ | $\tau_i(0, 1)$ |
|-----|--------|----------------|----------------|
| 1   | $0$    | $(0, t, t, t)$ | $(t, 0, 0, 0)$ |
| 2   | $(0, 0, t, t)$ | $(0, t, t, t)$ | $(t, 0, 0, 0)$ |
| 3   | $(0, t, t, t)$ | $(0, 1/3, 1/3, 1/3)$ | $(t, 0, 0, 0)$ |
| 4   | $(0, 0, 0, t)$ | $(0, t, t, 1/3)$ | $(2/3, 0, 0, 0)$ |
| 5   | $(t, 0, t, t)$ | $(0, 1/3, 1/3, 1/3)$ | $(2/3, 0, 0, 0)$ |
| 6   | $(t, t, t, t)$ | $(0, 1/3, 1/3, 1/3)$ | $(2/3, 0, 0, 0)$ |
| 7   | $(0, t, t, t), (0, t, t, 0)$ | $(0, 1/3, 1/3, 1/3)$ | $(t, 0, 0, 0)$ |
| 8   | $(0, 0, t, t), (0, t, 0, t)$ | $(0, t, 1/3, 2/3)$ | $(2/3, 0, 0, 0)$ |
| 9   | $(t, 0, 0, t), (0, 0, t, 0)$ | $(0, t, 1/3, 1/3)$ | $(2/3, 0, 0, 0)$ |
| 10  | $(0, 0, t, t), (t, 0, 0, t)$ | $(0, 1/3, 1/3, 2/3)$ | $(2/3, 0, 0, 0)$ |
| 11  | $(t, 0, 0, t), (t, t, t, t)$ | $(0, 1/3, 1/3, 2/3)$ | $(2/3, 0, 0, 0)$ |
| 12  | $(-t, t, 0, 0), (t, 0, t, t), (t, t, t, 0)$ | $(0, 1/3, 1/3, 2/3)$ | $(1/3, 0, 0, 0)$ |

eight diffeomorphism classes:

$X_1, X_2 \cong_{\text{diff}} X_4, X_3 \cong_{\text{diff}} X_5, X_6, X_7 \cong_{\text{diff}} X_8, X_9, X_{10} \cong_{\text{diff}} X_{11}, X_{12}$.

Manifolds belonging to different $C^\infty$-classes have non-isomorphic fundamental groups. A hyperelliptic fourfold $X$ whose fundamental group is isomorphic to the fundamental
group of a rigid hyperelliptic fourfold with holonomy $\mathbb{Z}_3^2$ is rigid and therefore biholomorphic to some $X_j$.

(2) There are exactly four biholomorphism classes $X_{i,j}$ of hyperelliptic fourfolds with holonomy $\text{He}(3)$. They are realized as $E^4/K_i$ by the actions

$$\phi_{i,j}(g)(z) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot z + \tau_j(g) \quad \text{and} \quad \phi_{i,j}(h)(z) := \begin{pmatrix} \zeta_3 \\ 1 \\ \zeta_3^2 \\ \zeta_3 \end{pmatrix} \cdot z + \tau_j(h),$$

where $K_1 := \langle (0, t, t, t) \rangle$, $K_2 := \langle (0, t, t, t), (0, t, -t, 0) \rangle$ and

$\tau_1(g) := (1/3, 0, 0, 0)$, $\tau_1(h) := (0, 1/3, 1/3, 1/3)$,

$\tau_2(g) := (1/3, 0, 0, -t)$, $\tau_2(h) := (0, 1/3, 1/3, 1/3)$.

The manifolds $X_{i,j}$ have pairwise distinct fundamental groups. Each hyperelliptic manifold with holonomy $\text{He}(3)$ is rigid and therefore biholomorphic to one of the $X_{i,j}$.

Observe that all manifolds in the theorem are finite quotients of a product of elliptic curves and therefore projective. The projectivity is not a coincidence, in fact, rigid hyperelliptic fourfolds cannot have global nonzero holomorphic 2-forms. Thus, $H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^{1,1}(X)$, and consequently, there exist Kähler classes represented by positive line bundles. More generally, in [10], it is shown that a hyperelliptic manifold has arbitrary small algebraic approximations.

Some of the manifolds of Theorem 1.2 were already discussed in the literature. In [12, Section 10.1.4], the author already presented the rigid hyperelliptic fourfold $X_{1,2}$ with holonomy $\text{He}(3)$. In [1, Theorem 3.4], the reader can find for each $n \geq 4$ an example of a rigid hyperelliptic manifold with holonomy $\mathbb{Z}_3^2$. For $n = 4$, their example is $X_1$ from our Theorem 1.2. The same fourfold was found and discussed in [14, Section 5.1]. In [5, Theorem 5.4], two rigid hyperelliptic fourfolds with holonomy $\mathbb{Z}_3^2$ and different fundamental groups are constructed. In our classification, these examples are $X_1$ and $X_3$. We point out that our manifolds have $b_1 = 0$: such manifolds are of independent interest and were studied in the papers [15] and [16], where the authors give $X_1$ as an example.

We will now sketch the outline of the paper. In Sect. 2, we collect some preliminaries concerning hyperelliptic manifolds and explain the necessary tools from deformation theory. In particular, we show that the rigidity of the hyperelliptic manifold is encoded in the complex holonomy representation. Section 3 is devoted to prove the first main result of our paper, Theorem 1.1. Going through the list of complex holonomy groups [22] in dimension 3 and analyzing their representation theory, we prove that none of these groups allow a rigid and free action. In dimension four, we use Demleitner’s list of 79 complex holonomy groups to show that only the two groups $\mathbb{Z}_3^2$ and $\text{He}(3)$ allow a rigid action. Moreover, for both of these groups, there is, up to equivalence and automorphism, a unique candidate for the complex holonomy representation $\rho$. In the third section, we recall Bieberbach’s structure theorems of crystallographic and Bieberbach groups and explain their geometric consequences in our setting. In the fourth section, we determine all lattices $\Lambda$ which have a $\text{He}(3)$ or $\mathbb{Z}_3^2$ module structure via the holonomy representation $\rho$. Moreover, we determine all free and rigid actions of our holonomy groups on the tori $\mathbb{C}^4/\Lambda$. The linear parts of these actions are given by $\rho$, while the translation parts are so called special cohomology classes in the group cohomology $H^1(G, \mathbb{C}^4/\Lambda)$. In Section 6, we decide, using the developed theory about Bieberbach groups, which fourfolds found in the previous chapter are biholomorphic or diffeomorphic, respectively. This amounts to determine the orbits of the action of a certain group on the special cohomology classes of $H^1(G, \mathbb{C}^4/\Lambda)$. To list the actions and determine

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these orbits, we use the computer algebra system MAGMA [2], which allows for an efficient computation. The interested reader can find our code on the website:

http://www.staff.uni-bayreuth.de/~bt300503/publi.html.

Finally, in the last section, we summarize the proof of Theorem 1.2.

**Notation.** We use the standard notation from complex geometry and representation theory of finite groups. The group of affine linear transformations of $\mathbb{R}^n$ is denoted by $\text{AGL}(n, \mathbb{R})$. We write $\text{Aut}(T)$ for the group of biholomorphic automorphisms of a complex torus $T$, whereas $\text{Aut}_0(T)$ is the subgroup of group automorphisms. Similarly, $\text{Aff}(T)$ is the group of affine diffeomorphisms of $T$ and $\text{Aff}_0(T)$ the subgroup of diffeomorphisms fixing the origin.

## 2 Basic definitions and preliminaries

In this section, we collect preliminaries concerning rigidity and hyperelliptic manifolds that we will use in our paper.

Let $T = V/\Lambda$ be a complex torus. Since holomorphic maps between complex tori are affine, an automorphism $g \in \text{Aut}(T)$ can be decomposed into its linear part $\rho(g)$ and its translation part $\tau(g)$, i.e., $g(z) = \rho(g)z + \tau(g)$. A subgroup $G \leq \text{Aut}(T)$ therefore defines two representations

$$\rho : G \to \text{GL}(V) \quad \text{and} \quad \rho_\Lambda : G \to \text{GL}(\Lambda),$$

which both map an element of $G$ to its linear part, viewed as automorphisms of $V$ and $\Lambda$, respectively. These representations are called the *analytic or complex holonomy* and the *integral holonomy* representation. The relation between the two representations comes from the Hodge decomposition $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = V \oplus \overline{V}$, which implies that $\rho_\Lambda \otimes \mathbb{C}$ is equivalent to $\rho \oplus \overline{\rho}$. In particular, the characteristic polynomial of $\rho(g) \oplus \overline{\rho(g)}$ has integral coefficients for all $g \in G$.

**Remark 2.1** Let $G$ be a finite group of automorphisms of the complex torus $T$. Denote by $H$ the normal subgroup of $G$ consisting of the translations. Then, $T/H$ is again a complex torus, and the canonical map $(T/H)/(G/H) \to T/G$ is biholomorphic. For this reason, it suffices to study actions of finite groups on complex tori which do not contain any translations.

**Definition 2.2** A *hyperelliptic manifold* is a quotient $X = T/G$ of a complex torus $T$ by a finite, non-trivial group $G \leq \text{Aut}(T)$ which acts freely on $T$ and does not contain any translations. The group $G$ is called the *holonomy group* of $X$.

**Remark 2.3** Let $X = T/G$ be a hyperelliptic manifold with holonomy $G$.

1. The associated analytic representation $\rho : G \to \text{GL}(V)$ is faithful because $G$ does not contain translations.
2. Since $g \in G \setminus \{\text{id}_T\}$ acts freely on $T = V/\Lambda$, the lift of the fixed point equation

   $$(\rho(g) - \text{id}_V)z = \lambda - \tau(g)$$

   has no solution in $z \in V$ and $\lambda \in \Lambda$. In particular, 1 is an eigenvalue of $\rho(g)$.

There are several notions of rigidity, see [1, Definition 2.1]. We recall only the parts which are relevant to us.

**Definition 2.4** Let $X$ be a compact complex manifold.
(1) A deformation of $X$ consists of the following data:

- a flat and proper holomorphic map $\pi : X \rightarrow B$ of connected complex spaces,
- a point $0 \in B$,
- an isomorphism $\pi^{-1}(\{0\}) \simeq X$.

(2) We call $X$ (locally) rigid if for every deformation $\pi : X \rightarrow B$ of $X$, there is an open neighborhood $U \subset B$ of $0$ such that $\pi^{-1}(U) \simeq X \times U$ and $\pi|_{\pi^{-1}(U)} : X \times U \rightarrow U$ is the projection onto the second factor.

(3) We call $X$ infinitesimally rigid if $H^1(X, \Theta_X) = 0$, where $\Theta_X$ is the holomorphic tangent bundle of $X$.

Kodaira–Spencer–Kuranishi theory shows that an infinitesimally rigid complex manifold is also rigid (see [9] for an account on deformation theory). It was a question of Kodaira and Morrow [18, Problem on p. 45] whether the converse implication holds. In general, this is false. The first counterexamples were given by Bauer and Pignatelli [7], see also the paper of Böhning, Bothmer and Pignatelli [6]. However, for hyperelliptic manifolds, the two notions coincide:

**Proposition 2.5** A hyperelliptic manifold $X = T/G$ is rigid if and only if it is infinitesimally rigid.

**Proof** According to [10, Section 5 and Proposition 3], $X$ is rigid if and only if the pair $(T, G)$ is rigid, i.e., Def$(T)^G := \text{Def}(T) \cap H^1(T, \Theta_T)^G$ is a point. Since Def$(T)$ is smooth of dimension $h^1(T, \Theta_T)$ (cf. [17, p. 230 ff.]), the set Def$(T)^G$ consists of a single point if and only if $H^1(T, \Theta_T)^G = 0$. This precisely means that $H^1(X, \Theta_X) \simeq H^1(T, \Theta_T)^G = 0$ because the quotient map $\pi : T \rightarrow T/G$ is unramified. □

The rigidity of a hyperelliptic manifold $X = T/G$ depends only on the associated analytic representation:

**Corollary 2.6** A hyperelliptic manifold $X = T/G$ is rigid if and only if the analytic representation $\rho : G \rightarrow \text{GL}(V)$ and its conjugate $\overline{\rho}$ have no common irreducible subrepresentation.

**Proof** Using Dolbeault’s interpretation of cohomology, we write

$$H^1(T, \Theta_T) \simeq H^{0,1}(\Theta_T) = \langle d\overline{z}_1, \ldots, d\overline{z}_n \rangle \otimes \langle \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \rangle.$$  

By definition, the $G$-action on $V = \langle \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \rangle$ is the analytic representation, while the action on $\langle d\overline{z}_1, \ldots, d\overline{z}_n \rangle$ is the dual of the complex conjugate of the analytic representation. This gives an isomorphism of representations $H^1(T, \Theta_T) \simeq \overline{V} \otimes V \simeq \text{Hom}(\overline{V}, V)$ and the claim follows from Schur’s lemma and Proposition 2.5. □

**Remark 2.7** (1) The corollary tells us in particular that the analytic representation of a rigid hyperelliptic manifold contains no self-conjugate irreducible representations, i.e., no representations of real or quaternionic type [20, p. 108].

(2) A rigid hyperelliptic manifold $X$ has no nonzero global holomorphic 2-forms because

$$0 = \langle \overline{\rho}, \chi_{\rho} \rangle = \langle \overline{\rho}^2, \chi_{\text{triv}} \rangle = \langle \wedge^2(\overline{\rho}^2), \chi_{\text{triv}} \rangle + \langle \text{Sym}^2(\overline{\rho}), \chi_{\text{triv}} \rangle$$  

implies $h^0(X, \Omega^2_X) = 0$.  

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3 Proof of the first main theorem

In order to prove Theorem 1.1, we use the classification of hyperelliptic groups according to Bagnera-de Franchis [3] in dimension 2, Uchida-Yoshihara [22] in dimension 3 (later completed by Catanese and Demleitner [11]) and Demleitner [12] in dimension 4. It would be interesting to find a proof which does not depend on these classification results.

Proof of Theorem 1.1 (a) Hyperelliptic surfaces are never rigid because their holonomy groups are cyclic: Remark 2.3 (2) then implies that the associated analytic representation contains the trivial representation. To prove that there are no rigid hyperelliptic threefolds, we use the list of groups of Uchida-Yoshihara [22]: the group is either the dihedral group $D_4$ of order 8 or an Abelian group of the form $\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2}$, where $d_1 | d_2$. For the latter, we can restrict to the ones with $d_1 > 1$, i.e., the non-cyclic Abelian groups. Since the representation theories of $D_4$ and $\mathbb{Z}_2^2$ are real, no hyperelliptic manifold with holonomy group $D_4$ or $\mathbb{Z}_2^2$ is rigid. For $(d_1, d_2) \neq (2, 2), d_1 \neq 1$, the assertion follows from [19, Lemma 6.5] or [12, Theorem 6.1.8], which tells us that the associated analytic representation contains the trivial representation.

Proof of Theorem 1.1 (b) In analogy to part (a), we use the classification of hyperelliptic groups in dimension 4 achieved in [12]. For each of the 79 holonomy groups $G$, we check the existence of a representation $\rho : G \to \text{GL}(4, \mathbb{C})$ with the following properties:

1. $\rho$ is faithful (Remark 2.3 (1)),
2. Each representation matrix $\rho(g)$ contains 1 as an eigenvalue (Remark 2.3 (2)),
3. The characteristic polynomial of $\rho(g) \oplus \overline{\rho}(g)$ is in $\mathbb{Z}[x]$ for all $g \in G$ (integral representation),
4. The character $\chi$ of $\rho$ and its conjugate $\overline{\chi}$ do not contain common irreducible characters (Corollary 2.6).

To verify the existence of such a representation, only the character table of $G$ is needed. Clearly, the kernel of a representation $\rho$ is equal to the kernel of its character:

$$\ker(\chi) := \{ g \in G \mid \chi(g) = \chi(1) \}.$$  

Moreover, it is well known that the characteristic polynomial $\rho(g)$ can be determined from the character values $\chi(g^i)$ thanks to the Newton identities. We use MAGMA to run through the 79 groups and check if there is a representation $\rho$ satisfying conditions (1)–(4). We find that the only groups admitting such a representation are $\mathbb{Z}_2^3$ and He(3).

Remark 3.1 We recall that He(3) has exactly two non-equivalent irreducible representations of dimension 3: the representation $\rho_3$, given by

$$\rho_3(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho_3(h) = \begin{pmatrix} 1 \\ \zeta_3^2 \\ \zeta_3 \end{pmatrix}, \quad \rho_3(k) = \begin{pmatrix} \zeta_3 \\ \zeta_3 \\ \zeta_3 \end{pmatrix}$$  

and its complex conjugate $\overline{\rho_3}$. Moreover, there are nine 1-dimensional representations obtained from the central quotient $\mathbb{Z}_3^2$ by inflation.

Remark 3.2 Analyzing the output of the MAGMA computation in the proof of Theorem 1.1 (b), we see that after application of a suitable automorphism of $G$, the analytic representation of a rigid hyperelliptic fourfold $X = T/G$ is equivalent to:

1. $\rho(a, b) = \text{diag}(\zeta_3^a, \zeta_3^b, \zeta_3^{2a+b}, \zeta_3^{a+b})$ if $G = \mathbb{Z}_3^2$ and
• $\rho = \rho_1 \oplus \rho_3$ if $G = \text{He}(3)$. Here, $\rho_1$ is defined by $\rho_1(g) := 1$ and $\rho_1(h) := \zeta_3$.

As a consequence of Remark 3.2, all rigid hyperelliptic fourfolds with a fixed holonomy group have the same Hodge numbers:

**Corollary 3.3** The Hodge numbers of a rigid hyperelliptic fourfold $X = T/G$ are:

| $p$ | $q$ |
|-----|-----|
| 1   | 0   |
| 0   | 2   |
| 0   | 1   |
| 1   | 1   |
| 0   | 0   |

and

| $p$ | $q$ |
|-----|-----|
| 1   | 3   |
| 0   | 6   |
| 0   | 0   |

$G = \text{He}(3)$

$G = \mathbb{Z}_3^2$

**Proof** We consider the case $G = \text{He}(3)$ first. Since the action of $\text{He}(3)$ on $T$ is free, the Hodge numbers of $X$ are given as

$$h^{p,q}(X) = \dim_{\mathbb{C}}(H^{p,q}(T)^{\text{He}(3)}).$$

They are the multiplicities of the trivial representation in

$$\psi_{p,q}: \text{He}(3) \rightarrow \text{GL}(H^{p,q}(T)), \quad u \mapsto [\omega \mapsto \rho(u^{-1})^* \omega].$$

According to Remark 3.2, we may assume that $\rho = \rho_1 \oplus \rho_3$. Then, the character of $\psi_{p,q}$ is given by:

$$\chi_{p,q} = \sum_{s_1 + s_2 = p \atop t_1 + t_2 = q} \wedge^{s_1}(\chi_1) \wedge^{s_2}(\chi_3) \wedge^{t_1}(\chi_1) \wedge^{t_2}(\chi_3), \quad \text{where } \chi_i := \text{character of } \rho_i.$$

Now, the Hodge numbers $h^{p,q}(X) = \langle \chi_{p,q}, \chi_{\text{triv}} \rangle$ can be easily computed using the formulae for $\chi_{p,q}$ and the identities

$$\wedge^3(\chi_3) = \det(\rho_3) = \chi_{\text{triv}} \quad \text{and} \quad \wedge^2(\chi_3) = \overline{\chi_3}.$$

To determine the Hodge numbers $h^{p,q}$ in the $G = \mathbb{Z}_3^2$ case, we restrict the characters $\chi_{p,q}$ to the subgroup $(h, k) \leq \text{He}(3)$, which we identify with $\mathbb{Z}_3^2$ by the isomorphism $h \mapsto (1, 0), k \mapsto (0, 1)$. We then compute the inner product of these characters with the trivial character of $\mathbb{Z}_3^2$.

**Proposition 3.4** Any hyperelliptic fourfold with holonomy group $\text{He}(3)$ is rigid.

**Proof** Since the associated analytic representation $\rho: \text{He}(3) \rightarrow \text{GL}(4, \mathbb{C})$ is faithful, it decomposes in a one dimensional and an irreducible 3-dimensional representation. The freeness of the action implies that the 1-dimensional summand is not trivial, [12, Proposition 10.1.21]. Corollary 2.6 implies the rigidity.

4 Bieberbach’s structure theorems and their geometric consequences

In this section, we consider hyperelliptic manifolds from the differential geometric point of view as compact flat Riemannian manifolds or, equivalently, as quotients of Euclidean spaces by Bieberbach groups. Using Bieberbach’s structure theorems, we translate the problem of biholomorphic, diffeomorphic and homeomorphic classification of hyperelliptic manifolds into a group theoretical problem. This will play a crucial role in the proof of our main theorem because in our setting, the latter can be solved algorithmically.
Remark 4.1

(1) In the sequel, we often identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via

$$(x_1, y_1, \ldots, x_n, y_n) \mapsto (z_1, \ldots, z_n), \quad \text{where} \quad z_i = x_i + \sqrt{-1} y_i.$$ 

In this way, we can view any complex representation $\rho : G \to \text{GL}(n, \mathbb{C})$ as a real representation $\rho_{\mathbb{R}} : G \to \text{GL}(2n, \mathbb{R})$. Over $\mathbb{C}$, the representation $\rho_{\mathbb{R}}$ decomposes into $\rho \oplus \overline{\rho}$.

(2) A matrix $A \in \text{GL}(2n, \mathbb{R})$ induces, under the identification in (1), a bijection $f_A : \mathbb{C}^n \to \mathbb{C}^n$ which is in general only $\mathbb{R}$-linear. We say that the matrix $A$ is $\mathbb{C}$-linear, if $f_A$ is $\mathbb{C}$-linear, and $\mathbb{C}$-antilinear if $f_A$ is $\mathbb{C}$-antilinear, i.e., $f_A(\lambda v) = \overline{\lambda} f_A(v)$ for all $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$.

(3) The fundamental group of a hyperelliptic manifold $X = T/G$ is isomorphic to the group of deck transformations $\Gamma$ of the universal cover

$$\mathbb{C}^n \to T \to X.$$ 

It consists of all lifts of the elements of $G$ to $\mathbb{C}^n$ and is therefore a cocompact, free and discrete group of affine transformations. More precisely, since $G$ is finite, we may assume that the analytic representation $\rho$ is unitary. Then, we can consider $\Gamma$ as a subgroup of $\mathbb{C}^n \rtimes U(n)$. The identification in (1) allows us to view $\rho$ as a real representation $\rho_{\mathbb{R}} : G \to O(2n)$ and $\Gamma$ as a subgroup of the group of Euclidean motions $\mathbb{E}(2n)$. As the action of $G$ on $T$ does not contain translations, the lattice $\Lambda$ of the torus $T$ is equal to the intersection $\Gamma \cap \mathbb{C}^n$, which is the translation subgroup of $\Gamma$.

Definition 4.2 A discrete cocompact subgroup of $\mathbb{E}(n)$ is called a crystallographic group. A Bieberbach group is a torsion-free crystallographic group.

The above remark tells us that the fundamental group of a rigid hyperelliptic manifold with holonomy group $G$ is a Bieberbach group and, with the above notation, $\Gamma / \Lambda \simeq G$.

Remark 4.3 Given a Bieberbach group $\Gamma \leq \mathbb{E}(n)$, the quotient $X = \mathbb{R}^n / \Gamma$ is a differentiable manifold. It inherits a natural Riemannian metric, induced by the Euclidean inner product, and is therefore flat, i.e., locally isometric to $\mathbb{R}^n$. Conversely, any compact flat Riemannian manifold is isometric to a quotient of $\mathbb{R}^n$ by a Bieberbach group.

In the sequel, we will use the classical structure theorems of crystallographic and Bieberbach groups, see [8, Chapter I] and [21, Theorem 2.1]. We would like to remark that the classical reference [8] applies the theory to compact flat Riemannian manifolds only, whereas the book [21] studies, in addition to other recent developments, also hyperelliptic and flat Kähler manifolds, respectively.

Theorem 4.4 [Bieberbach’s structure theorems]

(1) The translation subgroup $\Lambda := \Gamma \cap \mathbb{R}^n$ of a crystallographic group $\Gamma \leq \mathbb{E}(n)$ is a lattice of rank $n$ and $\Gamma / \Lambda$ is finite. All other normal Abelian subgroups of $\Gamma$ are contained in $\Lambda$.

(2) Let $\Gamma, \Gamma' \leq \mathbb{E}(n)$ be two crystallographic groups and $\gamma : \Gamma \to \Gamma'$ be an isomorphism. Then, there exists an affine transformation $\alpha \in \text{AGL}(n, \mathbb{R})$ such that $\gamma(g) = \alpha \circ g \circ \alpha^{-1}$ for all $g \in \Gamma$.

(3) In each dimension, there are only finitely many isomorphism classes of crystallographic groups.
In our setting, these theorems have the following consequences (cf. [14, Section 3]):

**Corollary 4.5** Let $\Phi : G \to \text{Aut}(T)$ and $\Phi' : G' \to \text{Aut}(T')$ be free and translation-free holomorphic actions of finite groups $G$ and $G'$. Assume there is an isomorphism $\gamma : \Gamma \to \Gamma'$ between the fundamental groups of $X = T/G$ and $X' = T'/G'$.

1. We view the groups $\Gamma$ and $\Gamma'$ as Bieberbach groups in $\mathbb{E}(2n)$. Thanks to Bieberbach’s first theorem, $\gamma$ restricts to an isomorphism $\gamma : \Lambda \to \Lambda'$, and it follows that $G$ and $G'$ are isomorphic.

2. According to Bieberbach’s second theorem, $\gamma$ is the conjugation by an affinity
   
   $$\alpha(x) = Ax + d, \quad A \in \text{GL}(2n, \mathbb{R}), \quad d \in \mathbb{R}^{2n}.$$ 

   In particular, $\Lambda' = A \cdot \Lambda$. The affinity $\alpha$ induces diffeomorphisms $\tilde{\alpha}$ and $\tilde{\alpha}'$, such that the following diagram commutes:

   $\begin{array}{ccc}
   T & \xrightarrow{\tilde{\alpha}} & T' \\
   \downarrow & & \downarrow \\
   X & \xrightarrow{\tilde{\alpha}} & X'.
   \end{array}$

   This holds in particular for $\gamma = f_*$, where $f : X_1 \to X_2$ is a homeomorphism.

3. There is an isomorphism $\varphi : G \to G'$ such that
   
   $$A \Phi(u)(x) + d = \Phi'(\varphi(u))(Ax + d) \quad \text{for all } u \in G \text{ and } x \in T.$$ 

   This comes from the fact that $\Phi(G)$ and $\Phi'(G')$ are conjugated by $\tilde{\alpha}$.

4. If $f : X \to X'$ is biholomorphic, then it lifts to a biholomorphism of the tori, i.e., it is induced by an affinity
   
   $$\alpha(x) = Ax + d, \quad \text{where } A \in \text{GL}(n, \mathbb{C}).$$

As a consequence, in the category of hyperelliptic manifolds, or more generally compact flat Riemannian manifolds, the following equivalence relations coincide:

- isomorphic fundamental groups, homeomorphic, diffeomorphic and affine diffeomorphic.

In view of the classification problem, we may restrict our attention to affine diffeomorphisms and biholomorphic maps. According to Corollary 4.5 (1), it suffices to consider hyperelliptic manifolds $X, X'$ with the same holonomy group $G$.

**Remark 4.6** Let $f : X \to X'$ be a diffeomorphism induced by an affinity $\alpha(x) = Ax + d$. Writing $\phi(u)(x) = \rho_R(u)(x) + \tau(u)$ and similarly $\phi'(u)(x) = \rho'_R(u)(x) + \tau'(u)$, the equation in Corollary 4.5 (3) is equivalent to

1. $A \rho_R(u)A^{-1} = \rho'_R(\varphi(u))$ and
2. $(\rho'_R(\varphi(u)) - I_{2n})d = A \tau(u) - \tau'(\varphi(u))$

for all $u \in G$. Item (1) means that $\rho_R$ and $\rho'_R \circ \varphi$ are equivalent as real representations. Item (2) is an equation holding in the group $T'$. If $f$ is holomorphic, i.e., $A$ is $\mathbb{C}$-linear, then $\rho$ and $\rho' \circ \varphi$ are equivalent as complex representations.

**Remark 4.7** Let $f : X \to X'$ be a diffeomorphism induced by $\alpha(x) = Ax + d$.

1. Since $A \cdot \Lambda = \Lambda'$, condition (1) of Remark 4.6 precisely means that $A$ is contained in
   
   $$\mathcal{N}_R(\Lambda, \Lambda') := \{ A \in \text{GL}(2n, \mathbb{R}) \mid A \cdot \Lambda = \Lambda', \ A \cdot \text{im}(\rho_R) = \text{im}(\rho'_R) \cdot A \}.$$ 

   If $f$ is biholomorphic, then $A$ is even contained in the subset
   
   $$\mathcal{N}_C(\Lambda, \Lambda') := \mathcal{N}_R(\Lambda, \Lambda') \cap \text{GL}(n, \mathbb{C}).$$
(2) In contrast with the linear parts, the translation parts $\tau : G \to T$ and $\tau' : G \to T'$ of the actions $\Phi$ and $\Phi'$ are not homomorphisms, but 1-cocycles:

$$\tau(u_1u_2) = \tau(u_1) + \rho(u_1)\tau(u_2), \quad \tau'(u_1u_2) = \tau'(u_1) + \rho'(u_1)\tau'(u_2)$$

for all $u_1, u_2 \in G$. Their classes define elements in the cohomology groups $H^1(G, T)$ and $H^1(G, T')$. Conversely, a class in $H^1(G, T)$ together with $\rho$ defines an action on $T$, which is unique up to conjugation. The class is called special if the action is free. Condition (2) in Remark 4.6 has the following cohomological interpretation: replacing $u$ with $\varphi^{-1}(u)$, it reads

$$\rho'(u)d - d = A\tau(\varphi^{-1}(u)) - \tau'(u),$$

which just means that the cocycles $A\tau \circ \varphi^{-1}$ and $\tau'$ belong to the same cohomology class.

(3) In the special case where $\rho = \rho'$ and $T = T'$, the sets $N_{R}(\Lambda, \Lambda)$ and $N_{C}(\Lambda, \Lambda)$ are the normalizers of $\text{im}(\rho_R)$ in $\text{Aff}_0(T)$ and in $\text{Auto}_0(T)$. For simplicity, we denote them by $N_R(\Lambda)$ and $N_C(\Lambda)$. By (2), they act on $H^1(G, T)$ by

$$A \ast \tau(u) := A \cdot \tau(\varphi^{-1}_A(u)),$$

where $\varphi_A$ is the unique automorphism of $G$ with $A\rho_RA^{-1} = \rho_R \circ \varphi_A$. It follows that $A$ is the linear part of an affine diffeomorphism (or biholomorphism) $X \to X'$ if and only if $A \ast \tau$ and $\tau'$ represent the same class in $H^1(G, T)$.

The strategy to derive of our main theorem is now clear:

**Scheme for the Classification**

(1) In the first step, we determine all lattices $\Lambda$ which have a $G = \text{He}(3)$ (or $\mathbb{Z}_3^2$) module structure via the representation $\rho$ from Remark 3.2.

(2) In the second step, we determine for each $T = \mathbb{C}^4/\Lambda$ and for all special cohomology classes in $H^1(G, T)$ a representative $\tau$. Taking the quotients of $T$ by the actions

$$\Phi(u)(z) = \rho(u)z + \tau(u),$$

we obtain all possible rigid hyperelliptic fourfolds.

(3) In the final step, we decide which fourfolds found in the previous step are biholomorphic or diffeomorphic, respectively.

**Remark 4.8** Let $X = T/G$ be a hyperelliptic manifold, where $T = \mathbb{C}^n/\Lambda$ is a complex torus. The short exact sequence of $G$-modules

$$0 \to \Lambda \to \mathbb{C}^n \to T \to 0$$

induces the long exact sequence

$$\ldots \to H^1(G, \mathbb{C}^n) \to H^1(G, T) \to H^2(G, \Lambda) \to H^2(G, \mathbb{C}^n) \to \ldots$$

in group cohomology. By the vanishing of $H^i(G, \mathbb{C}^n)$ for $i \geq 1$, we obtain an isomorphism $H^1(G, T) \simeq H^2(G, \Lambda)$. Hence, the 1-cocycle $\tau$ defining the translation part of the $G$-action on $T$ yields a class in $H^2(G, \Lambda)$, which corresponds to the extension

$$0 \to \Lambda \to \Gamma \to G \to 1,$$

where $\Gamma = \pi_1(X)$.

The biholomorphism or diffeomorphism problem of hyperelliptic fourfolds can therefore be reinterpreted in terms of group extensions: Let $X$ and $X'$ be two hyperelliptic manifolds...
with the same holonomy group $G$, and let $\tau$ and $\tau'$ be the 2-cocycles corresponding to the extensions

$$0 \to \Lambda \to \Gamma \to G \to 1 \quad \text{and} \quad 0 \to \Lambda' \to \Gamma' \to G \to 1.$$ 

Then, $X$ and $X'$ are biholomorphic (or diffeomorphic) if and only if there exists a matrix $A \in \mathcal{N}_{\mathbb{C}}(\Lambda, \Lambda')$ (or $\mathcal{N}_{\mathbb{R}}(\Lambda, \Lambda')$) such that $\tau'$ and $A \cdot \tau$ belong to the same cohomology class of $H^2(G, \Lambda')$. Here, the 2-cocycle $A \cdot \tau$ is defined in the obvious way:

$$A \cdot \tau(u_1, u_2) := A \tau(\varphi^{-1}_A(u_1), \varphi^{-1}_A(u_2)).$$

This amounts to say that $\Gamma$ and $\Gamma'$ are conjugated by an affinity $\alpha(x) = Ax + d$, for a suitable $d$. The conjugation isomorphism induces an isomorphism of short exact sequences:

$$0 \to \Lambda \to \Gamma \to G \to 1$$

$$\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$$

$$0 \to \Lambda' \to \Gamma' \to G \to 1$$

In view of the discussion above, our strategy parallels the classification scheme for Bieberbach groups outlined in [8, Chapter III, Section 2] and [21, Section 3.1, p. 30], except for the last step. Here, for sake of the biholomorphic classification, we consider the refined equivalence relation of conjugation of Bieberbach groups with holomorphic affinities $\alpha(x) = Ax + d$, i.e., those where $A$ is $\mathbb{C}$-linear.

## 5 Lattices and cocycles corresponding to rigid hyperelliptic fourfolds

We follow our classification strategy and first determine the possible lattices $\Lambda$. As a first step, we show that each candidate for $\Lambda$ contains $\mathbb{Z}[\zeta_3]^4$ as a sublattice of finite index.

**Notation 5.1** In the sequel, we will often consider $\mathbb{Z}_3^2$ as a subgroup of $\text{He}(3)$ by identifying $(1, 0)$ with $h$ and $(0, 1)$ with $k$. Observe that this is compatible with the analytic representations given in Remark 3.2.

**Proposition 5.2** Let $X = T/G$ be a rigid hyperelliptic fourfold, then the torus $T$ is equivariantly isogenous to a product of four elliptic curves $E_i \subset T$, each of which is a copy of the Fermat elliptic curve $\mathbb{C}/\mathbb{Z}[\zeta_3]$.

**Proof** According to Theorem 1.1, the group is $G = \mathbb{Z}_3^2$ or He(3). Thanks to Remark 3.2, we can assume

$$\rho(h) = \text{diag}(\zeta_3, 1, \zeta_3^2, \zeta_3) \quad \text{and} \quad \rho(k) = \text{diag}(1, \zeta_3, \zeta_3, \zeta_3),$$

in both cases $G = \mathbb{Z}_3^2$ and He(3). Consider the following subtori

- $E_1 := \ker(\rho(k) - \text{id}_T)^0$,
- $E_2 := \ker(\rho(h) - \text{id}_T)^0$,
- $E_3 := \ker(\rho(hk) - \text{id}_T)^0$ and
- $E_4 := \ker(\rho(hk^2) - \text{id}_T)^0$.

Here, the superscript 0 denotes the connected component of the identity. By construction, $\dim(E_i) = 1$ and $\zeta_3 \in \text{Aut}(E_i)$; therefore, $E_i \simeq \mathbb{C}/\mathbb{Z}[\zeta_3]$. We conclude the proof, because the addition map

$$\mu : E_1 \times \ldots \times E_4 \to T$$
is a surjective holomorphic group homomorphism between tori of the same dimension, i.e.,
an isogeny (cf. [4, p. 12]). The equivariance of $\mu$ holds by construction.

Proposition 5.2 allows us to write $T \simeq (E_1 \times E_2 \times E_3 \times E_4)/K$, where $K$ is a finite group of translations. We will usually identify all $E_i$ with the Fermat elliptic curve $E := \mathbb{C}/\mathbb{Z}[\xi_3]$ and only keep the indices if they are relevant for our arguments.

Let $T/G$ be a rigid hyperelliptic fourfold. From now on, we assume that $T = E^4/K$ (where $E = \mathbb{C}/\mathbb{Z}[\xi_3]$ is the Fermat elliptic curve) and that the associated analytic representation $\rho$ is given as in Remark 3.2.

**Remark 5.3** Let $\Phi : G \hookrightarrow \text{Aut}(T)$ be a faithful rigid holomorphic action.

- If $G = \text{He}(3)$, then, up to a change of origin in the elliptic curves, the translation part
  $\tau : G \to T$ of $\Phi$ can be written in the form
  $$\tau(g) = (a_1, a_2, a_3, a_4), \quad \tau(h) = (0, b_2, b_3, b_4), \quad \tau(k) = (c_1, 0, 0, 0).$$

- If $G = \mathbb{Z}_3^2 = \langle h, k \rangle$, then the translation part of $\Phi$ can be written as
  $$\tau(h) = (0, b_2, b_3, b_4), \quad \tau(k) = (c_1, 0, 0, 0).$$

We call an action with such a translation part an action in *standard form*. Sometimes, we may write $u(z)$ instead of $\Phi(u)(z)$ for $u \in G$ and $z \in T$, by a slight abuse of notation.

Recall the presentation 1.1 of $\text{He}(3)$ given in Theorem 1.1.

**Lemma 5.4** Let $\Phi : G \hookrightarrow \text{Aut}(T)$ be a faithful rigid holomorphic action in standard form. Then:

1. If $G = \mathbb{Z}_3^2$, the following three elements are zero in $T$:
   $$v_1 := ((\xi_3 - 1)c_1, (1 - \xi_3)b_2, (1 - \xi_3)b_3, (1 - \xi_3)b_4),$$
   $$v_2 := (0, 3b_2, 0, 0),$$
   $$v_3 := (3c_1, 0, 0, 0).$$

   Conversely, given $b$ and $c$ such that $v_1, v_2, v_3$ are zero in $T$, we obtain a faithful holomorphic action of $\mathbb{Z}_3^2$ on $T$ in standard form.

2. If $G = \text{He}(3)$, the elements $v_1, \ldots, v_6$ are zero in $T$, where
   $$v_4 := ((1 - \xi_3)a_1 - c_1, b_2 - \xi_3b_3, b_2 - \xi_3b_3, b_2 - \xi_3b_3 + (1 - \xi_3^2)a_4),$$
   $$v_5 := (3a_1, a_2 + a_3 + a_4, a_2 + a_3 + a_4, a_2 + a_3 + a_4),$$
   $$v_6 := (0, (1 - \xi_3)a_2, (1 - \xi_3)a_3, (1 - \xi_3)a_4)$$

   and $v_4, v_5, v_6$ are as above.

   Conversely, given $a$, $b$, and $c$ such that $v_1, \ldots, v_6$ are zero in $T$, we obtain a faithful holomorphic action of $\text{He}(3)$ on $T$ in standard form.

**Proof** We will only sketch the proof of (2). Since $\Phi$ is a homomorphism, the images $\Phi(g)$, $\Phi(h)$ and $\Phi(k)$ of the generators fulfill the six defining relations of $\text{He}(3)$. As an example, the relation $\Phi(g) \circ \Phi(k) = \Phi(k) \circ \Phi(g)$ precisely means that the difference $v_6$ of

$$gk(z) = (z_1 + a_1 + c_1, \xi_3z_4 + a_2, \xi_3z_2 + a_3, \xi_3z_3 + a_4), \quad \text{and} \quad kg(z) = (z_1 + a_1 + c_1, \xi_3z_4 + a_2, \xi_3z_2 + a_3, \xi_3z_3 + a_4)$$

is zero, where $z = (z_1, \ldots, z_4)$. The remaining five relations yield the vanishing of the other $v_j$'s.

Conversely, if the elements $v_1, \ldots, v_6$ are zero, then it is clear that the formulae in Remark 5.3 define a faithful homomorphism $\Phi$.  \(\square\)
Corollary 5.5 Let $\Phi : G \hookrightarrow \text{Aut}(T)$ be a rigid and free action in standard form, then:

(a) The elements $c_1$ and $b_2, b_3, b_4$ have order 3 in $E_1$ and $E_2, E_3, E_4$, respectively.

(b) Let $p_i : K \to E_i$ be the projection on the $i$-th coordinate. Then $p_i(K)$ does not contain $E_i[3] := \{z_i \in E_i \mid 3z_i = 0\}$. 

Proof (a) We use the important property that the maps $E_i \hookrightarrow E_1 \times \ldots \times E_4 \to T = (E_1 \times \ldots \times E_4)/K$ are injective: it implies that if three of the four coordinates of an element of $K$ are zero, then the fourth coordinate is zero as well. It then follows from $v_2, v_3 \in K$ that $3c_1 = 0, 3b_2 = 0$. Moreover, $c_1$ and $b_2$ are clearly non-trivial, since else $h$ resp. $k$ do not act freely. In order to show that $\text{ord}(b_3) = \text{ord}(b_4) = 3$, a similar argument works: we observe that 

$$hk(z) = (\zeta_3 z_1 + \zeta_3 c_1, \zeta_3 z_2 + b_2, z_3 + b_3, \zeta_3^2 z_4 + b_4),$$

$$hk^2(z) = (\zeta_3 z_1 + 2\zeta_3 c_1, \zeta_3^2 z_2 + b_2, \zeta_3 z_3 + b_3, z_4 + b_4)$$

have order 3, and so, the elements $(0, 0, 3b_3, 0), (0, 0, 0, 3b_4)$ are zero in $T$, which in turn implies that $3b_3 = 3b_4 = 0$ in $E$. Moreover, $b_3$ and $b_4$ cannot be zero, because the elements $hk$ and $hk^2$ act freely by assumption.

(b) If $p_i(K)$ contains $E_i[3]$, we can find an element in $K$ whose first coordinate is $c_1$ (if $i = 1$) or $b_i$ (if $i \in \{2, 3, 4\}$). Thus, we obtain a fixed point of $k$ (if $i = 1$), $h$ (if $i = 2$), $hk$ (if $i = 3$) or $hk^2$ (if $i = 4$).

□

Assumption 5.6 Since we are only interested in free actions, we can and will assume from now on that $\text{ord}(c_1) = 3$ and $\text{ord}(b_j) = 3$ for all $j$.

We will denote the fixed locus of $\zeta_3 \in \text{Aut}(E)$ by 

$$\text{Fix}_E(\zeta_3) := \{x \in E \mid \zeta_3 x = x\} = \{0, t, -t\}, \quad \text{where} \quad t := \frac{1 + 2\zeta_3}{3}.$$ 

Besides knowing the fixed locus of $\zeta_3$, we also need a description of the kernel of $3(\zeta_3 - 1)$.

Lemma 5.7 Let $E = \mathbb{C}/\mathbb{Z}[\zeta_3]$ be the Fermat elliptic curve.

(a) The kernel of the multiplication map $3(\zeta_3 - 1) : E \to E$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_9$. More precisely, it is spanned by 1/3 and $t/3 = (1 + 2\zeta_3)/9$.

(b) Let $x \in E$ be an element of order 9. If $3x$ is fixed by $\zeta_3$, then $x$ and $\zeta_3 x$ span $\text{ker}(3(\zeta_3 - 1))$.

Proof (a) This is just a calculation.

(b) By (a), it suffices to exclude that $(x, \zeta_3 x) \simeq \mathbb{Z}_9$. Assume for a contradiction that $\zeta_3 x \in \langle x \rangle$. Since $x$ has order 9, $\zeta_3$ is an automorphism of $\langle x \rangle$ of order 3 so that we are left with the possibilities $\zeta_3 x \in \{4x, 7x\}$. In these cases, $(\zeta_3 - 1)x = \pm 3x$. Multiplying both sides with $(\zeta_3 - 1)$ yields

$$-3\zeta_3 x = (\zeta_3 - 1)^2 x = \pm (\zeta_3 - 1)3x = 0,$$

which implies $3x = 0$, in contradiction to the assumption $\text{ord}(x) = 9$. □

Proposition 5.8 Let $\Phi : G \hookrightarrow \text{Aut}(T)$ be a rigid and free action in standard form, where $T = E^4/K$. Then, $K \leq \text{Fix}_E(\zeta_3)^4 \simeq \mathbb{Z}_3^4$. In particular, every non-trivial element of $K$ has order 3.
Proof We preliminarily remark that for all \( u \in G \) we may view \( \rho(u) \) as an automorphism of \( E^4 \) that maps \( K \) to \( K \). Let \((t_1, t_2, t_3, t_4) \in K \). By the initial remark, the element
\[
(\rho(h) - \text{id}_{E^4})(\rho(hk) - \text{id}_{E^4})(\rho(hk^2) - \text{id}_{E^4})(t_1, t_2, t_3, t_4) = ((\xi_3 - 1)^3t_1, 0, 0, 0).
\]
is contained in \( K \) as well. Since \((\xi_3 - 1)^2 = -3\xi_3 \) and \( E_1 \hookrightarrow T \) is injective, we infer that \( 3t_1 \in \text{Fix}_E(\xi_3) \). Analogous arguments yield that \( 3t_i \in \text{Fix}_E(\xi_3) \). It follows in particular that \( t_i \in E_i[9] \) for all \( i \). Suppose that there is \( i \) such that \( \text{ord}(t_i) = 9 \). Then, we arrive at a contradiction as follows: by Lemma 5.7 (b), the two elements \( t_i, \xi_3 t_i \) span a subgroup of \( E_i[9] \) isomorphic to \( \mathbb{Z}_3 \times \mathbb{Z}_3 \). This shows that \( 3t_1 = 3t_2 = 3t_3 = 3t_4 = 0 \). If some \( t_i \) is not fixed by \( \xi_3 \), then \( t_i \) and \( \xi_3 t_i \) span \( E_i[3] \), and we again arrive at the contradiction \( E_i[3] \leq p_i(K) \).

\( \square \)

Remark 5.9 Let \( X = T/G \) be a rigid hyperelliptic fourfold, where \( T = E^4/K \). By construction, \( K \) is the kernel of the addition map \( E^4 \to T \). By Proposition 5.8, \( K \leq \text{Fix}_E(\xi_3) \). Moreover, \( K \) does not contain any nonzero multiples of unit vectors. There are 129 subgroups of \( \mathbb{Z}_3^4 \) with these two properties, the set of which we denote by \( K \).

5.1 Kernels and actions for \( \text{He}(3) \)

In this subsection, we finish part (1) and (2) of the classification scheme for \( G = \text{He}(3) \). Here, only those elements in \( K \) stabilized by \( \rho(g) \) can occur as a kernel. Out of the 129 possibilities found in Remark 5.9, just the following nine survive:

| \( \dim_{\mathbb{Z}_3}(K) \) | \( K \) |
|-----------------|------------------|
| 0               | \[[0] \]          |
| 1               | \((0, t, t, t), (0, t, t, t), (0, t, t, t)\) |
| 2               | \((0, t, t, t), (0, t, t, t), (0, t, t, t)\), \((0, t, t, t), (0, t, t, t), (0, t, t, t)\) |
| 3               | \((0, t, t, t), (0, t, t, t), (0, t, t, t), (0, t, t, t)\) |

As a first step, we exclude the last four possibilities for \( K \) in the table.

Proposition 5.10 Let \( T/\text{He}(3) \) be a rigid hyperelliptic fourfold, where \( T = E^4/K \). Then, \( K \) is either generated by an element \((t', t'', t''', t''') \) where \( t', t'' \in \text{Fix}_E(\xi_3) \), or \( K = \langle (0, t, t, t), (0, t, t, t), (0, t, t, t) \rangle \).

Proof The cyclic case follows from Remark 5.9. If \( K \) is non-cyclic, investigating the five subgroups introduced in Remark 5.9 shows that \((0, t, t, t) \in K \) in each case. From this and since the last three coordinates of
\[
v_5 = (3a_1, a_2 + a_3 + a_4, a_2 + a_3 + a_4, a_2 + a_3 + a_4) \in K \quad (v_5 \text{ as in Lemma 5.4})
\]
coincide, we conclude that \( 3a_1 = 0 \). This shows that \((1 - \xi_3)a_1 \) is fixed by \( \xi_3 \). Since \((1 - \xi_3)a_1 - c_1 \) is the first coordinate of \( v_4 \in K \), it is fixed by \( \xi_3 \) as well. We obtain that \( c_1 \in \text{Fix}_E(\xi_3) \), and hence, \( K \) cannot contain an element whose first coordinate is
nonzero: otherwise, \( k \) does not act freely. Out of the five non-cyclic possibilities, only \( K = \langle (0, t, t, t), (0, t, -t, 0) \rangle \) remains. \( \Box \)

We observe in particular that the above proposition shows that every element \((t_1, t_2, t_3, t_4) \in K\) has the property that \( t_2 + t_3 + t_4 = 0 \) in \( E \). This observation is useful to prove a simple criterion for the freeness of the action of \( \text{He}(3) \) on \( T = E^4 / K \).

**Lemma 5.11** An action \( \Phi: \text{He}(3) \leftrightarrow \text{Aut}(T) \) is free if and only if none of the elements \( k, g, h, gh \) and \( gh^2 \) have a fixed point. If \( \Phi \) is in standard form, the freeness of these elements can be characterized as follows:

1. \( k: \) \( c_1 \) is never the first coordinate of an element in \( K \),
2. \( g: \) \( (\xi_3 - 1) a_1 \neq 0 \) in \( E \),
3. \( h: \) \( b_2 \) is never the second coordinate of an element in \( K \),
4. \( gh: \) \( \xi_3^2(a_2 + a_3) + a_4 + \xi_3^2(b_2 + b_4) + b_3 \neq 0 \) in \( E \),
5. \( gh^2: \) \( \xi_3(a_2 + a_3) + a_4 - \xi_3(b_2 + b_3) - b_4 \neq 0 \) in \( E \).

**Remark 5.12** Later we will show that \( K = \langle (0, t, t, t) \rangle \) or \( \langle (0, t, -t, 0), (0, t, t, t) \rangle \). In particular condition (1) will be obsolete, since \( c_1 \neq 0 \) in \( E_1 \) (Assumption 5.6). Furthermore, condition (3) then translates into the condition that \( (\xi_3 - 1)b_2 \neq 0 \).

**Proof of Lemma 5.11** Note that an element \( u \in \text{He}(3) \) acts freely if and only if \( u^2 \) acts freely, because \( u^3 = 1 \) for all \( u \in \text{He}(3) \). Observe that \( (gh)^2 \) is conjugate to \( g^2h^2 \) and \( (g^2h)^2 \) is conjugate to \( gh^2 \). Thus, the representatives \( g, g^2, h, h^2, k, k^2, gh, g^2h, gh^2 \) and \( g^2h^2 \) of the ten non-trivial conjugacy classes act freely if and only if \( g, h, k, gh \) and \( gh^2 \) act freely.

Clearly, \( k \) and \( h \) act freely if and only if (1) and (3) hold, respectively. The element \( g \) has a fixed point on \( T \) if and only if there is \( z = (z_1, \ldots, z_4) \in T \) such that

\[
g(z) = (z_1 + a_1, z_4 + a_2, z_2 + a_3, z_3 + a_4) = (z_1, \ldots, z_4) \in T
\]

\[\iff (a_1, z_4 - z_2 + a_2, z_2 - z_3 + a_3, z_3 - z_4 + a_4) = 0 \text{ in } T.\]

Reading the second line of the above equivalence as an equation in \( E^4 \) and setting \( w_2 := z_4 - z_2, w_3 := z_2 - z_3, w_4 := z_3 - z_4 \), we infer that the above is satisfied if and only if there is \( (t_1, t_2, t_3, t_4) \in K \) such that

\[
(a_1, w_2 + a_2, w_3 + a_3, -w_2 - w_3 + a_4) = (t_1, t_2, t_3, t_4) \text{ in } E^4.
\]

Thus, equation (5.1) can only be satisfied if \( a_1 = t_1, w_2 = t_2 - a_2, w_3 = t_3 - a_3, w_4 = t_4 \). Calculating the fourth coordinate, we obtain that

\[
-w_2 - w_3 + a_4 = -t_2 - t_3 + a_2 + a_3 + a_4 = t_4.
\]

Since \( t_2 + t_3 + t_4 = 0 \) (Proposition 5.10), we obtain that \( a_2 + a_3 + a_4 = 0 \).

In total, this shows that \( g \) has a fixed point on \( T \) if and only if \( (\xi_3 - 1)a_1 = 0 \) and \( a_2 + a_3 + a_4 = 0 \). It follows that \( (\xi_3 - 1)a_1 \neq 0 \) is a sufficient condition for the freeness of \( g \). Moreover, this condition is necessary for the freeness of \( k \), since otherwise the first coordinate of the vector \( -v_4 = 0 \in T \) defined in Lemma 5.4 is \( c_1 \), which is impossible by (1).

Conditions (4) and (5) mean that \( gh \) and \( gh^2 \) act freely, respectively: this is proved as above.

We are now in the situation to complete step (1) of our classification scheme (see p. 8) in the Heisenberg case. In fact, the upcoming proposition shows that only the two kernels \( K_1 \) and \( K_2 \) from Theorem 1.2 (2) allow a free \( \text{He}(3) \)-action.
Proposition 5.13 Let \( \phi : \text{He}(3) \to \text{Aut}(E^4/K) \) be a free action in standard form. Then, \( K \neq \{0\} \) and \( K \neq \langle (t', t, t, t) \rangle \) for \( t' \neq 0 \).

Proof Let \( v := \zeta_3^2(a_2 + a_3) + a_4 + \zeta_3^2(b_2 + b_4) + b_3 \) be the element of Lemma 5.11 (4). A direct calculation yields

\[
(gh)^3(z) = (z_1, z_2 + \zeta_3 v, z_3 + \zeta_3 v, z_4 + v).
\]

Since \((gh)^3\) is the identity on \( E^4/K \), the element

\[
(0, \zeta_3 v, \zeta_3 v, v)
\]

is contained in the kernel \( K \). Now, if \( v \neq 0 \), then \( K \neq \langle (t', t, t, t) \rangle \) for \( t' \neq 0 \). On the other hand, if \( K = \{0\} \), then \( v = 0 \), and hence, Lemma 5.11 (4) shows that the action of \( \text{He}(3) \) is not free. \( \square \)

Let \( T = \mathbb{C}^4/\Lambda \), where \( \Lambda \) is one of the lattices

\[
\Lambda_1 := \mathbb{Z}[\zeta_3]^4 + \langle (0, t, t, t) \rangle \quad \text{or} \quad \Lambda_2 := \Lambda_1 + \langle (0, t, -t, 0) \rangle
\]

Corresponding to the two remaining kernels \( K_1 \) and \( K_2 \). The results from above allow us to determine all free actions \( \Phi : \text{He}(3) \to \text{Aut}(T) \) in standard form. For a more efficient computation, we prove a refinement of Corollary 5.5:

Lemma 5.14 Let \( T \) be one of the two complex tori above, and let \( \Phi : \text{He}(3) \to \text{Aut}(T) \) be a free action in standard form. Then, the following statements hold in \( E \):

(a) \( (1 - \zeta_3) a_1 = c_1 \) and \( a_1 \) is an element of order 3 which is not fixed by \( \zeta_3 \).

(b) The elements \( b_2, b_3 \) and \( b_4 \) of order 3 are not fixed by \( \zeta_3 \). In particular, the element \( v_1 \) defined in Lemma 5.4 is equal to \( \pm (0, t, t, t) \) in \( E^4 \).

(c) There are \( j_3, j_4 \) such that \( b_3 = \zeta_3^{j_3} b_2 \) and \( b_4 = \zeta_3^{j_4} b_2 \).

(d) \( \text{ord}(a_2), \text{ord}(a_3), \text{ord}(a_4) \in \{1, 3\} \).

Proof (a) The first coordinate of every element of \( K \) is zero, and thus, the property that \( v_4 \) and \( v_5 \) are zero in \( T \) shows that \( 3a_1 = 0 \) and \( (1 - \zeta_3) a_1 = c_1 \). Now, if \( a_1 \) is fixed by \( \zeta_3 \), the first coordinate of \( v_4 \) is \(-c_1 \). Thus, \( c_1 = 0 \) and \( k \) does not act freely, a contradiction.

(b) If \( b_2 \) (resp. \( b_3, b_4 \)) is fixed by \( \zeta_3 \), then the torsion group \( K \) contains an element whose second (resp. third, fourth) coordinate is \( b_2 \) (resp. \( b_3, b_4 \)); in this case, the element \( h \) (resp. \( hk, hk^2 \)) does not act freely on \( T \). The last statement follows from the description of the possible kernels \( K \) since \( (1 - \zeta_3) b_1 \neq 0 \).

(c) It was shown in (b) that \( b_2, b_3 \) and \( b_4 \) are elements of order 3 that are not fixed by \( \zeta_3 \). Hence

\[
b_3, b_4 \in E[3] \setminus \text{Fix}_E(\zeta_3) = \{b_2, \zeta_3 b_2, \zeta_3^2 b_2, -b_2, -\zeta_3 b_2, -\zeta_3^2 b_2 \}. \tag{5.2}
\]

Furthermore, the second assertion of part (b) yields

\[
(1 - \zeta_3) b_2 = (1 - \zeta_3) b_3 = (1 - \zeta_3) b_4 \in \langle t, -t \rangle.
\]

In particular, \( b_2 - b_3 \) and \( b_2 - b_4 \) are contained in \( \text{Fix}_E(\zeta_3) \). Combined with 5.2, this easily implies that \( b_3, b_4 \in \{b_2, \zeta_3 b_2, \zeta_3^2 b_2 \} \).

(d) Since \( v_6 \) is contained in \( K \), the elements \( (1 - \zeta_3) a_i \) are fixed by \( \zeta_3 \) for \( i \in \{2, 3, 4\} \). This is the case if and only if \( 3a_i = 0 \). \( \square \)
Remark 5.15 Let \( \Phi : \text{He}(3) \hookrightarrow \text{Aut}(T) \) be an action in standard form. In the case where \( b_2 \) is not fixed by \( \zeta_3 \), it holds \( b_2 = (\zeta_3^i/3, 1) \), and conjugation by \((-1)^i \zeta_3^{-i} \text{id}_T \) yields an action in standard form with \( b_2 = 1/3 \).

Remark 5.16 According to part (b) and Remark 5.15, we may assume that \( b_2 = 1/3 \). By part (a), we have \( a_1 = \pm \zeta_3^i/3 \), and, up to conjugation of the action in the first coordinate, \( a_1 = 1/3 \) (cf. Remark 5.15). Thus, \( c_1 = (1 - \zeta_3)/3 \).

It is now easy to construct all free actions \( \Phi \):

- Run over all \( j_3 \), \( j_4 \in \{0, 1, 2\} \) and define \( b_3 := \zeta_3^{j_3}/3, b_4 := \zeta_3^{j_4}/3 \).
- Run over all \( a_2, a_3, a_4 \in \mathbb{E}[3] \) and define the elements \( v_1, \ldots, v_6 \) of Lemma 5.4,
- Check if \( v_1, \ldots, v_6 \) are contained in \( \Lambda_i \) and if the freeness conditions of Lemma 5.11 are satisfied.

Running a MAGMA implementation, we find 108 actions for the lattice \( \Lambda_1 \) and 324 for the lattice \( \Lambda_2 \), some of which may coincide on \( \mathbb{E}^4/\Lambda_i \). For both lattices \( \Lambda_i \), there are four special cohomology classes in \( H^1(\text{He}(3), \mathbb{E}^4/\Lambda_i) \).

### 5.2 Kernels and actions for \( \mathbb{Z}_3^2 \)

In the \( \mathbb{Z}_3^2 \) case, kernels \( K \in \mathcal{K} \) are clearly invariant under the action of \( \mathbb{Z}_3^2 \) via \( \rho \). However, different kernels might yield isomorphic hyperelliptic manifolds. In order to take this into account, we define a group action on \( \mathcal{K} \) such that kernels belonging to different orbits cannot correspond to isomorphic hyperelliptic manifolds.

**Proposition 5.17** Any biholomorphism \( f : X \to X' \) between two hyperelliptic manifolds with holonomy group \( \mathbb{Z}_3^2 \) is induced by a biholomorphic map \( \hat{f} : E^4 \to E^4, z \mapsto Az + d \). This means that \( A \) is contained in the normalizer \( N_{\text{Aut}(E^4)}(\mathbb{Z}_3^2) \).

**Proof** Write \( X = T/\mathbb{Z}_3^2 \) and \( X' = T'/\mathbb{Z}_3^2 \), where \( T = E^4/K \) and \( T' = E^4/K' \). By Corollary 4.5 (4), the isomorphism \( f : X \to X' \) is induced by an affinity \( \alpha(z) = Az + d \) with \( A \in \text{GL}(4, \mathbb{C}) \). Moreover, there exists an automorphism \( \varphi \in \text{Aut}(\mathbb{Z}_3^2) \) such that \( A \rho A^{-1} = \rho \circ \varphi \). Note that the only matrix in the image of \( \rho \) having the eigenvalue \( \zeta_3 \) with multiplicity 3 is \( \rho(k) \); thus, we infer that \( \varphi(k) = k \). On the other hand, \( \varphi(h) \in \{h, hk, hk^2\} \) because the matrices \( \rho(h), \rho(hk) \) and \( \rho(hk^2) \) have the same eigenvalues and all other matrices do not. It follows that \( A \) takes one of the following forms:

\[
\begin{pmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
\end{pmatrix},
\begin{pmatrix}
  a_1 & 0 & 0 & 0 \\
  0 & a_2 & 0 & 0 \\
  0 & 0 & a_3 & 0 \\
  0 & a_4 & 0 & 0 \\
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
  a_1 & 0 & 0 & 0 \\
  0 & 0 & a_2 & 0 \\
  0 & a_3 & 0 & 0 \\
  0 & 0 & a_4 & 0 \\
\end{pmatrix}.
\]

Note that for the above matrices, \( A \cdot e_i \) is a vector with only one nonzero entry. Denoting by \( \Lambda \) and \( \Lambda' \) the lattices of \( T \) and \( T' \), respectively, this observation, together with the condition \( A \cdot \Lambda = \Lambda' \) shows that the entries of \( A \) are Eisenstein integers. The same holds for \( A^{-1} \), and it follows that the complex numbers \( a_i \) are contained in \( \mathbb{Z}[\zeta_3]^4 = \langle -\zeta_3 \rangle \), hence \( A \cdot \mathbb{Z}[\zeta_3]^4 = \mathbb{Z}[\zeta_3]^4 \). \( \square \)

As a by-product of the proof we obtain:
Corollary 5.18 The normalizer $N_{\text{Aut}(E^4)}(\mathbb{Z}_3^2)$ is a finite group with $6^4 \cdot 3 = 3888$ elements generated by the matrices

$$\begin{pmatrix} -\zeta_3 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\zeta_3 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

As an abstract group, $N_{\text{Aut}(E^4)}(\mathbb{Z}_3^2)$ is isomorphic to $\mathbb{Z}_6^4 \rtimes \mathcal{A}_3$.

Similarly as in the Heisenberg case, we formulate a simple criterion for the freeness of an action $\Phi: \mathbb{Z}_3^2 \to \text{Aut}(T)$ in standard form.

Lemma 5.19 An action $\Phi: \mathbb{Z}_3^2 \to \text{Aut}(T)$ is free if and only if none of the elements $h, k, hk$ and $hk^2$ have a fixed point. If $\Phi$ is in standard form, the latter holds precisely when

1. $c_1$ is never the first coordinate of an element in $K$,
2. $b_j$ is never the $j$-th coordinate of an element in $K$ for all $j$.

Observe that Corollary 5.5 shows that the elements $c_1, b_2, b_3$ and $b_4$ have order 3, respectively. In particular, there are only finitely many free actions in standard form. Our MAGMA code determines all of these actions. The number of these actions is displayed in the middle column of the table from below.

Remark 5.20 Note that the normalizer $N_{\text{Aut}(E^4)}(\mathbb{Z}_3^2)$ acts on $K$. Proposition 5.17 tells us that two hyperelliptic fourfolds $X$ and $X'$ cannot be biholomorphic if the corresponding kernels $K, K' \in \mathcal{K}$ belong to different orbits of this action. Performing a MAGMA computation, we find exactly 13 orbits: For all kernels $K_i$, except for $K_{\text{exc}}$, there are rigid and free actions

| $i$ | Representative $K_i$ of the orbit | # of free actions | # of special classes in $H^1(\mathbb{Z}_3^2, E^4/K_i)$ |
|-----|----------------------------------|-------------------|---------------------------------------------|
| 1   | $\langle 0 \rangle$             | 16                | 16                                          |
| 2   | $\langle (0, 0, t, t) \rangle$  | 72                | 8                                           |
| 3   | $\langle (0, t, t, t) \rangle$  | 108               | 12                                          |
| 4   | $\langle (t, 0, 0, t) \rangle$  | 72                | 8                                           |
| 5   | $\langle (t, 0, t, t) \rangle$  | 108               | 12                                          |
| 6   | $\langle (t, t, t, t) \rangle$  | 162               | 18                                          |
| 7   | $\langle (0, 0, t, t), (0, t, -t, 0) \rangle$ | 108 | 4 |
| 8   | $\langle (0, 0, t, t), (t, 0, -t, 0) \rangle$ | 108 | 4 |
| 9   | $\langle (t, 0, 0, t), (0, 0, t, t) \rangle$ | 324 | 4 |
| 10  | $\langle (0, 0, t, t), (t, t, 0, t) \rangle$ | 162 | 2 |
| 11  | $\langle (t, 0, 0, t), (t, t, t, -t) \rangle$ | 162 | 2 |
| exc | $\langle (-t, t, 0, 0), (t, t, t, 0) \rangle$ | 486 | 6 |

on $T_i = E^4/K_i$. Thus, there are at least 12 biholomorphism classes of rigid fourfolds with holonomy $\mathbb{Z}_3^2$. Moreover, every hyperelliptic fourfold with holonomy $\mathbb{Z}_3^2$ is obtained as a quotient of $T = E^4/K_i$.
6 Biholomorphism and diffeomorphism classes of rigid hyperelliptic fourfolds

In this section, we perform the final step of our classification scheme, outlined at the end of Sect. 4. The main part is devoted to determine for each group \( G = \text{He}(3) \) and \( \mathbb{Z}_3^2 \) and for each kernel \( K \) the normalizers \( \mathcal{N}_\mathbb{C}(\Lambda_K) \) and \( \mathcal{N}_\mathbb{R}(\Lambda_K) \) explicitly, where \( \Lambda_K = \mathbb{Z} \langle \xi_3 \rangle^4 + K \). With the help of MAGMA, we then determine the orbits of the action of these groups on the special cohomology classes in \( H^1(G, E^4/K) \). Propositions 6.10 and 6.13 impose conditions on the coboundaries and therefore allow an efficient computer search. Thereby, we distinguish all isomorphism and diffeomorphism classes of rigid hyperelliptic fourfolds and prove our main theorems. In the code, the special cohomology classes are represented by the translation parts of rigid and free \( G \)-actions in standard form.

6.1 The case \( \text{He}(3) \)

To compute the sets \( \mathcal{N}_\mathbb{R}(\Lambda, \Lambda') \) and \( \mathcal{N}_\mathbb{C}(\Lambda, \Lambda') \), where \( \Lambda \) and \( \Lambda' \) are one of
\[
\Lambda_1 = \mathbb{Z} \langle \xi_3 \rangle^4 + \langle (0, t, t, t) \rangle, \quad \Lambda_2 = \Lambda_1 + \langle (0, t, -t, 0) \rangle,
\]
respectively, we will use the real representation theory of \( \text{He}(3) \), which we briefly recall. Throughout the subsection, we will use the notation \( \Lambda_i^\oplus := \Lambda_i \cap (\{0\} \times \mathbb{C}^3) \) and exploit the fact that
\[
\Lambda_i = \mathbb{Z} \langle \xi_3 \rangle \oplus \Lambda_i^\oplus.
\]

Remark 6.1 The Heisenberg group \( \text{He}(3) \) has 5 irreducible non-trivial real representations:

- Four 2-dimensional representations induced from the central quotient \( \mathbb{Z}_3^2 \) by inflation. They map \((a, b) \in \mathbb{Z}_3^2 \) to \( b^a \), \( b^{a+b} \), \( b^a \) and \( b^{2a+b} \), respectively, where \( B := -\frac{1}{2} \left( \begin{smallmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 1 \end{smallmatrix} \right) \).
- One irreducible 6-dimensional representation is the decomplexification \( \rho_{3\mathbb{R}} \) of the representation \( \rho_3 \) defined in 3.1.

The endomorphism algebra of each of these representations is isomorphic to \( \mathbb{C} \).

Remark 6.2 A matrix \( A \in \text{GL}(8, \mathbb{R}) \) is contained in \( \mathcal{N}_\mathbb{R}(\Lambda, \Lambda') \) if and only if
\[
A \rho_\mathbb{R} A^{-1} = \rho_\mathbb{R} \circ \varphi \quad \text{for some} \quad \varphi \in \text{Aut}(\text{He}(3)) \tag{6.1}
\]
and \( A \cdot \Lambda = \Lambda' \). Since \( \rho_\mathbb{R} = \rho_{1\mathbb{R}} \oplus \rho_{3\mathbb{R}} \) is the sum of an irreducible real 2-dimensional and an irreducible 6-dimensional representation, Schur’s Lemma implies that \( A \) is of block diagonal form
\[
A = \text{diag}(C, D), \quad \text{where} \quad C \in \text{GL}(2, \mathbb{R}) \quad \text{and} \quad D \in \text{GL}(6, \mathbb{R}).
\]

In particular, the characters \( \chi_{3\mathbb{R}} \) of \( \rho_{3\mathbb{R}} \) are stabilized by \( \varphi \), i.e., \( \chi_{3\mathbb{R}} = \chi_{1\mathbb{R}} \circ \varphi \). We write \( \varphi \in \text{Stab}(\chi_{1\mathbb{R}}, \chi_{3\mathbb{R}}) \). Note that \( \text{Stab}(\chi_{1\mathbb{R}}, \chi_{3\mathbb{R}}) = \text{Stab}(\chi_{1\mathbb{R}}) \) because \( \rho_{3\mathbb{R}} \) is the unique irreducible 6-dimensional real representation of \( \text{He}(3) \).

On the other hand, given \( \varphi \in \text{Stab}(\chi_{1\mathbb{R}}) \), there exist invertible matrices \( C_\varphi \in \text{GL}(2, \mathbb{R}) \) and \( D_\varphi \in \text{GL}(6, \mathbb{R}) \) such that \( A_\varphi := \text{diag}(C_\varphi, D_\varphi) \) fulfills Eq. 6.1.
Clearly, the matrices $C_\phi$ and $D_\phi$ are unique only up to a nonzero element in the endomorphism algebras $\text{End}_{\text{He}(3)}(\rho_{1\mathbb{R}})$ and $\text{End}_{\text{He}(3)}(\rho_{3\mathbb{R}})$, respectively. Both of these algebras are isomorphic to $\mathbb{C}$. Any $\phi \in \text{Stab}(\chi_{1\mathbb{R}})$ either stabilizes $\chi_i$ or maps $\chi_i$ to $\overline{\chi_i}$ because $\chi_{1\mathbb{R}}$ is equal to the sum of the complex characters $\chi_i$ and $\overline{\chi_i}$. Moreover, a matrix defining an equivalence between $\rho_{1\mathbb{R}}$ and $\rho_{1\mathbb{R}} \circ \phi$ is $\mathbb{C}$-linear if and only if $\chi_i$ is stabilized and $\mathbb{C}$-antilinear if and only if $\chi_i$ is mapped to $\overline{\chi_i}$, cf. Remark 4.1. In particular, $A_\phi = \text{diag}(C_\phi, D_\phi)$ is $\mathbb{C}$-linear if and only if $\phi \in \text{Stab}(\chi_1, \chi_3)$.

**Remark 6.3** Suppose that $C \in \text{GL}(2, \mathbb{R})$ and $\phi \in \text{Stab}(\chi_{1\mathbb{R}})$ such that $C \rho_{1\mathbb{R}}(u)C^{-1} = \rho_{1\mathbb{R}}(\phi(u))$. If $\phi$ belongs to $\text{Stab}(\chi_1)$, then the matrix $C$ commutes with $B$, otherwise it holds $CB = B^2C$. We see that the matrix $C$ must have the form

$$C = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \quad \text{or} \quad C = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}.$$ 

In complex coordinates, $C(z) = cz$ or $C(z) = cz^\ast$, where $c := \lambda + \sqrt{-1}\mu$. If there exists $D \in \text{GL}(6, \mathbb{R})$ such that $A = \text{diag}(C, D) \in \mathcal{N}_{\mathbb{R}}(\Lambda, \Lambda')$, then $CZ[\zeta_3] = \mathbb{Z}[\zeta_3]$, because $\Lambda_i = \mathbb{Z}[\zeta_3] \oplus \Lambda_i^\ast$. Hence, the scalar $c$ must be a unit in $\mathbb{Z}[\zeta_3]$. In total, there are 12 such matrices, they form a dihedral group $D_6$.

To determine the possibilities for $D$ is more involved. According to Remark 6.2, we obtain for each $\phi \in \text{Stab}(\chi_{1\mathbb{R}})$ a matrix $D_\phi$, which is either $\mathbb{C}$-linear or $\mathbb{C}$-antilinear and unique up to a nonzero constant. Hence, $\phi$ determines a class $[D_\phi] \in \text{PGL}(3, \mathbb{C}) \rtimes \mathbb{Z}_2$. It is clear that $[D_\phi] \cdot [D_\phi'] = [D_{\phi \circ \phi'}]$. In other words, we obtain a semi-projective representation

$$\Xi : \text{Stab}(\chi_{1\mathbb{R}}) \to \text{PGL}(3, \mathbb{C}) \rtimes \mathbb{Z}_2, \quad \phi \mapsto [D_\phi].$$

It is faithful because $\rho_{\mathbb{R}}$ is faithful, i.e., any class in the image of $\Xi$ corresponds to a unique automorphism $\phi$. The normalizer $\mathcal{N}_{\mathbb{R}}(\Lambda)$ has therefore a fairly simple description:

**Corollary 6.4** The normalizer $\mathcal{N}_{\mathbb{R}}(\Lambda)$ is equal to the fiber product

$$\mathcal{N}_{\mathbb{R}}(\Lambda) = D_6 \times_{\mathbb{Z}_2} \{ D \in \text{GL}(3, \mathbb{C}) \rtimes \mathbb{Z}_2 \mid [D] \in \text{im}(\Xi), \ D \Lambda^\natural = \Lambda^\natural \}$$

defined via the natural homomorphisms $D_6 \to \mathbb{D}_6/(-\zeta_3) \simeq \mathbb{Z}_2$ and

$$\{ D \in \text{GL}(3, \mathbb{C}) \rtimes \mathbb{Z}_2 \mid [D] \in \text{im}(\Xi), \ D \Lambda^\natural = \Lambda^\natural \} \to \text{Stab}(\chi_{1\mathbb{R}})/\text{Stab}(\chi_1) \simeq \mathbb{Z}_2.$$

Our next goal is to explicitly describe the sets $\mathcal{N}_{\mathbb{R}}(\Lambda, \Lambda')$ and $\mathcal{N}_{\mathbb{C}}(\Lambda, \Lambda')$. We begin with the case $\Lambda = \Lambda'$ and determine $\mathcal{N}_{\mathbb{R}}(\Lambda)$ by finding generators of the group

$$\{ D \in \text{GL}(3, \mathbb{C}) \rtimes \mathbb{Z}_2 \mid [D] \in \text{im}(\Xi), \ D \Lambda^\natural = \Lambda^\natural \} \leq \text{GL}(3, \mathbb{C}) \rtimes \mathbb{Z}_2.$$

The starting point is the following lemma:

**Lemma 6.5** The group $\text{Stab}(\chi_{1\mathbb{R}}) \leq \text{Aut}(\text{He}(3))$ is generated by four automorphisms $\varphi_1, \ldots, \varphi_4$. These automorphisms and representatives $D_{\varphi_i}$ of the classes $\Xi(\varphi_i)$ are listed in the table below:

The representatives $D_{\varphi_i}$ are chosen such that $D_{\varphi_i} \cdot \Lambda^\natural = \Lambda^\natural$, independent of $\Lambda = \Lambda_1$ or $\Lambda_2$.

**Proof** We first determine the normal subgroup $\text{Stab}(\chi_1, \chi_3) \leq \text{Stab}(\chi_{1\mathbb{R}})$. Since $\text{Aut}(\text{He}(3)) \simeq \text{AGL}(2, \mathbb{F}_3)$ acts transitively on the eight non-trivial complex $\text{He}(3)$-characters of degree one, we have $|\text{Stab}(\chi_1)| = 432/8 = 54$. The automorphism $\varphi$ defined by $\varphi(g) = g^2$ and $\varphi(h) = h$ belongs to $\text{Stab}(\chi_1)$. It maps $\chi_3$ to $\overline{\chi_3}$, which is the second of the two irreducible complex
Lemma 6.6. On the other hand, a MAGMA computation shows that μ/Lambda1♮. The semi-linear transformations Dϕ generate the group all other representatives with this property differ by a unit of Z.

Proof. Let Dϕ solves the equation Dρ3(u) = (ρ3 ◦ ϕ)(u)D, u = g, h.

Here, ρ3 is the representation defined in 3.1. On the other hand, ϕ4 exchanges χ and Χ, which implies that Dϕ4 is antilinear. It is easy to check that Dϕ4(z2, z3, z4) = (z2, z3, z4) solves the equation Dρ3(u) = (ρ3 ◦ ϕ4)(u)D for u = g and h. □

Up to now, it is not even clear that NIR(Λ) is finite. The next lemma assures this property.

Lemma 6.6 Suppose that a representative Dϕ of Ξ(ϕ) is chosen such that DϕΛ2 = Λ5, then all other representatives with this property differ by a unit of Z[ζ3].

Proof. Let μ ∈ C∗ with μDϕΛ2 = Λ2. Since DϕΛ2 = Λ5, we have μΛ5 = μDϕΛ2 = Λ5. In particular, μ · (1, 0, 0) ∈ Λ2, which implies μ ∈ Z[ζ3]. On the other hand, the equation μΛ5 = Λ5 is equivalent to μ−1Λ5 = Λ5, and we conclude that μ−1 ∈ Z[ζ3]. □

Proposition 6.7 The semi-linear transformations Dϕ1, . . . , Dϕ4 in the table of Lemma 6.5 generate the group

{D ∈ GL(3, C) × Z2 | [D] ∈ im(Ξ), DΛ5 = Λ5} ≤ GL(3, C) × Z2.

Proof. By construction, {Dϕ1, . . . , Dϕ4} is contained in {D ∈ GL(3, C) × Z2 | [D] ∈ im(Ξ)}, DΛ5 = Λ5). The latter has at most |Stab(χ1IR)| · 6 = 648 elements, thanks to Lemma 6.6. On the other hand, a MAGMA computation shows that {Dϕ1, . . . , Dϕ4} has also 648 elements. Hence, both groups are equal. □

Corollary 6.8 For both Λ = Λ1 and Λ2, the normalizers NIR(Λ) and NC(Λ) are

NIR(Λ) = D6 × Z2 ∨ {Dϕ1, . . . , Dϕ4} and NC(Λ) = (−ζ3) ∨ {Dϕ1, Dϕ2}.

Proof. Only the claim for NC(Λ) = NIR(Λ) ∩ GL(4, C) needs to be justified. According to Remark 6.2, a matrix A = Aϕ belongs to NC(Λ) if and only if ϕ ∈ Stab(χ1, χ3). For this reason,

NC(Λ) = (−ζ3) ∨ {D ∈ GL(3, C) | [D] ∈ Ξ(Stab(χ1, χ3)), DΛ5 = Λ5}.
Since $\text{Stab}(\chi_1, \chi_3)$ is generated by $\varphi_1$ and $\varphi_2$, the image $\Xi(\text{Stab}(\chi_1, \chi_3))$ is generated by the projective transformations $[D_{\varphi_1}]$ and $[D_{\varphi_2}]$. Thanks to Lemma 6.6, we conclude that

$$\{D \in \text{GL}(3, \mathbb{C}) \mid [D] \in \Xi(\text{Stab}(\chi_1, \chi_3)) \}, \quad D \Lambda^2 = \Lambda^2 \} = \langle D_{\varphi_1}, D_{\varphi_2} \rangle,$$

in analogy to Proposition 6.7. \hfill \Box

**Proposition 6.9** The set $N_{\mathbb{R}}(\Lambda_1, \Lambda_2)$ is empty. In particular, hyperelliptic fourfolds with holonomy $\text{He}(3)$ corresponding to different lattices are topologically distinct.

**Proof** If $N_{\mathbb{R}}(\Lambda_1, \Lambda_2)$ is not empty, let $A = \text{diag}(C, D) \in N_{\mathbb{R}}(\Lambda_1, \Lambda_2)$, so that $A \cdot \Lambda_1 = \Lambda_2$.

It holds $D \cdot \Lambda^2_1 = \Lambda^2_2$. Since the class of $D$ belongs to $\text{im}(\Xi) \leq \text{PGL}(3, \mathbb{C}) \times \mathbb{Z}_2$, there is a constant $\lambda \in \mathbb{C}^*$ such that $\lambda D \in \langle D_{\varphi_1}, \ldots, D_{\varphi_4} \rangle$. This implies $\lambda \Lambda^2_1 = \lambda D \Lambda^2_1 = \Lambda^2_2$ and in particular, $\lambda e_1 \in \Lambda^2_1$, which shows that $\lambda \in \mathbb{Z}[\zeta_3]$. On the other hand, we have $\lambda^{-1} \Lambda^2_1 = \Lambda^2_2$, which implies $\lambda^{-1} \in \mathbb{Z}[\zeta_3]$, and we conclude that $\lambda$ is a unit of $\mathbb{Z}[\zeta_3]$. Hence, $D$ itself is an element in $\langle D_{\varphi_1}, \ldots, D_{\varphi_4} \rangle$. This leads to the contradiction $D \Lambda^2_1 = \Lambda^2_2 \subseteq \Lambda^2_2$. \hfill \Box

In order to decide whether two given translation parts corresponding to free actions are in the same $N_{\mathbb{C}}(\Lambda)$- or $N_{\mathbb{R}}(\Lambda)$-orbit, we determine the possible $d$’s, which define the coboundaries in Condition (2) of Remark 4.6 (see also Remark 4.7 (2)). Here, we use the notation $d = (d_1, \ldots, d_4) \in \mathbb{C}^4 \cong \mathbb{R}^8$, as in Remark 4.1 (1).

**Proposition 6.10** Let $X$ and $X'$ be quotients of $T = \mathbb{C}^4/\Lambda$ with respect to the rigid and free actions $\Phi, \Phi' : \text{He}(3) \to \text{Aut}(T)$, where $\Lambda = \Lambda_1$ or $\Lambda_2$ and $\Lambda^2 := \Lambda \cap (0 \times \mathbb{C}^3)$. Suppose that $f : X \to X'$ is a diffeomorphism induced by the affinity $\alpha(x) = Ax + d$. Then:

1. The element $(d_2, d_3, d_4)$ is contained in the kernel of $(\zeta_3 - 1) : \mathbb{C}^3/\Lambda^2 \to \mathbb{C}^3/\Lambda^2$.

2. The first coordinate $d_1$ of $d$ is contained in the kernel of the map $3(\zeta_3 - 1) : E \to E$.

**Proof** (1) Since the center $Z(\text{He}(3)) = \langle k \rangle$ is characteristic, we have $\varphi(k) = k$ or $k^2$ for all automorphisms $\varphi$ of $\text{He}(3)$. By definition, $\Phi'(k^j)$ acts on the last three coordinates by multiplication by $\zeta_3^j$. Spelling out condition (2) of Remark 4.6 for $u = k$ thus yields

$$(\zeta_3^j - 1)(d_2, d_3, d_4) = 0 \in \mathbb{C}^3/\Lambda^2.$$ 

The statement follows, because $\ker(\zeta_3 - 1) = \ker(\zeta_3^2 - 1)$.

(2) We use Remark 4.6 (2) for $u = \varphi^{-1}(h)$. Since the right-hand side is 3-torsion by Lemma 5.14, we conclude $3(\zeta_3 - 1)d_1 \in \mathbb{Z}[\zeta_3]$. \hfill \Box

**Proposition 6.11** There are precisely four biholomorphism classes of rigid hyperelliptic fourfolds with holonomy $\text{He}(3)$. They are topologically distinct.

**Proof** We use MAGMA to determine for each lattice $\Lambda = \Lambda_1$ and $\Lambda_2$ the set of all possible translation parts $\tau$ of free actions in standard form, as described in Remark 5.16. They represent the special cohomology classes in $H^1(\text{He}(3), \mathbb{C}^4/\Lambda)$. For every pair of special cocycles $\tau, \tau'$, we check whether there is a matrix $A \in N_{\mathbb{C}}(\Lambda)$ and a vector $d$ according to Proposition 6.10 such that

$$(\rho(u) - I_4)d = A \ast \tau(u) - \tau'(u)$$

for $u = g, h$. Our code finds two orbits for each of the two lattices $\Lambda_1$ and $\Lambda_2$. This computation and Proposition 6.9 imply that there are exactly four biholomorphism classes and at least two different topological types. We claim that the two biholomorphism classes obtained for each lattice are topologically distinct. To verify the claim, we check with MAGMA that the two respective cohomology classes in $H^1(\text{He}(3), \mathbb{C}^4/\Lambda_i)$ belong to different $N_{\mathbb{R}}(\Lambda_i)$-orbits. \hfill \Box
6.2 The case \( \mathbb{Z}_3^2 \)

**Proposition 6.12** Let \( K \) be one of the twelve kernels from Remark 5.20, then
\[
\mathcal{N}_C(\Lambda_K) = \{ A \in N_{\text{Auto}}(E^4)(\mathbb{Z}_3^2) \mid AK = K \}.
\]

**Proof** As we have seen in the proof of Proposition 5.17, each \( A \in N_{\text{Auto}}(E^4)(\mathbb{Z}_3^2) \) also defines an automorphism of \( E^4 \) and therefore stabilizes \( K \). The other inclusion is clear. \( \square \)

We determine the possible translation parts of potential biholomorphisms, mimicking the statement in the Heisenberg case (Proposition 6.10).

**Proposition 6.13** Let \( X \) and \( X' \) be quotients of \( T = E^4/K \) with respect to the rigid and free actions \( \Phi, \Phi' : \mathbb{Z}_3^2 \to \text{Aut}(T) \) in standard form. Suppose that \( f : X \to X' \) is a biholomorphism induced by the affinity \( \alpha(x) = Ax + d \). Then:

1. The element \((d_2, d_3, d_4)\) is contained in the kernel of \((\xi_3 - 1) : \mathbb{C}^3/p(\Lambda_K) \to \mathbb{C}^3/p(\Lambda_K)\), where \( p : \mathbb{C}^4 \to \mathbb{C}^3 \) is the projection onto the last three coordinates.
2. The first coordinate \( d_1 \) of \( d \) is contained in the kernel of the map \((\xi_3 - 1) : E \to E\).

**Proof** (1) The proof of Proposition 5.17 shows that \( \phi(k) = k \), where \( \phi \) is the unique automorphism such that \( A\rho A^{-1} = \rho \circ \phi \) (see Remark 4.7 (3)). We can thus argue as in the proof of Proposition 6.10.

(2) This is similar to the proof of Proposition 6.10 (2). \( \square \)

**Proposition 6.14** For each of the twelve kernels \( K_i \) from Remark 5.20, there exists one and only one biholomorphism class of a rigid hyperelliptic fourfold \( X_i \).

**Proof** In analogy to Proposition 6.11, we use MAGMA to verify that \( \mathcal{N}_R(\Lambda_{K_i}) \) acts transitively on the special cohomology classes in \( H^1(\mathbb{Z}_3^2, E^4/K_i) \) for each kernel \( K_i \). Recall that there are no rigid and free \( \mathbb{Z}_3^2 \)-actions on \( E^4/K_{\text{exc}} \) (i.e., no special cohomology classes in \( H^1(\mathbb{Z}_3^2, E^4/K_{\text{exc}}) \)), see Remark 5.20. \( \square \)

Using Proposition 6.14, the diffeomorphism problem can be reformulated in the following way:

**Corollary 6.15** The fourfolds \( X_i \) and \( X_j \) corresponding to kernels \( K_i \) and \( K_j \) are diffeomorphic if and only if \( \mathcal{N}_R(\Lambda_{K_i}, \Lambda_{K_j}) \) is not empty.

**Proof** Assume that \( A \in \mathcal{N}_R(\Lambda_{K_i}, \Lambda_{K_j}) \) and let \( \Phi_i \) be a free holomorphic action giving \( X_i \), then, \( \psi := A\Phi_i A^{-1} \) is a free action on \( E^4/K_j \) with linear part \( A\rho A^{-1} = \rho \circ \varphi_A \). In particular, \( \psi \) is holomorphic and rigid. The matrix \( A \) induces a diffeomorphism between \( X_i \) and the quotient \( Z \) with respect to \( \psi \). However, \( Z \) and \( X_j \) are biholomorphic since by Proposition 6.14, there is precisely one biholomorphism class corresponding to the kernel \( K_j \). The converse is obvious. \( \square \)

We therefore need to determine the \( i \neq j \) for which \( \mathcal{N}_R(\Lambda_{K_i}, \Lambda_{K_j}) \) is non-empty.

**Remark 6.16** Let \( A \) be a matrix in \( \mathcal{N}_R(\Lambda_{K_i}, \Lambda_{K_j}) \), then there exists an automorphism \( \varphi \in \text{Aut}(\mathbb{Z}_3^2) \) such that
\[
A\rho R A^{-1} = \rho \circ \varphi.
\]
Note that the representation $\rho_{\mathcal{R}}$ is the direct sum of all four irreducible 2-dimensional real representations of $\mathbb{Z}_3^2$:

$$
\rho_{\mathcal{R}}(a, b) = \text{diag}(B^a, B^b, B^{2a+b}, B^{a+b}), \quad \text{where} \quad B = -\frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.
$$

By Schur’s lemma, $A$ is, up to a permutation, a block diagonal matrix of $2 \times 2$ blocks. In analogy to Remark 6.3, they are $\mathbb{C}$-linear or antilinear, i.e., of the form $z \mapsto cz$ or $z \mapsto c\bar{z}$.

Since $AK_i = K_j$, we conclude as in the proof of Proposition 5.17 that $A \cdot \mathbb{Z}[\zeta_3] = \mathbb{Z}[\zeta_3]$. This in turn implies that the scalars $c$ are units in $\mathbb{Z}[\zeta_3]$. In other words,

$$
\mathcal{N}_\mathcal{R}(\Lambda_{K_i}, \Lambda_{K_j}) = \{ A \in N_{\text{Aff}_0(E_4)}(\mathbb{Z}_3^2) \mid AK_i = K_j \}.
$$

The group $N_{\text{Aff}_0(E_4)}(\mathbb{Z}_3^2)$ has $|\text{Aut}(\mathbb{Z}_3^2)| \cdot 6^4 = 62208$ elements and is generated by the maps

- $M_1(z_1, z_2, z_3, z_4) := (\bar{z}_1, z_2, z_4, z_3)$,
- $M_2(z_1, z_2, z_3, z_4) := (z_2, \bar{z}_3, z_4, z_1)$ and
- $M_3(z_1, z_2, z_3, z_4) := (-\zeta_3 z_1, z_2, z_3, z_4)$.

Note that $N_{\text{Aff}_0(E_4)}(\mathbb{Z}_3^2)$ is a subgroup of $O(8)$. In particular, the determinant of any of its elements is $\pm 1$.

**Corollary 6.17** Let $\mu_m(K)$ be the number of elements in $K$ with exactly $m$ nonzero entries. Then, $\mathcal{N}_\mathcal{R}(\Lambda_{K_i}, \Lambda_{K_j})$ is empty if $\mu_m(K_i) \neq \mu_m(K_j)$ for some $m$.

**Proof** According to the previous discussion, any $A \in \mathcal{N}_\mathcal{R}(\Lambda_{K_i}, \Lambda_{K_j})$ can be regarded as an isomorphism $A: K_i \rightarrow K_j$ of finite groups. The functions $\mu_m$ are invariant because $A$ is the direct sum of four maps of the form $z \mapsto cz$ or $z \mapsto c\bar{z}$, up to a permutation. \(\square\)

Going through the list of all kernels $K_i$ from Remark 5.20, we obtain:

**Proposition 6.18** The set $\mathcal{N}_\mathcal{R}(\Lambda_{K_i}, \Lambda_{K_j})$ is empty for all $1 \leq i < j \leq 12$, except in the following cases: Furthermore, in each set $\mathcal{N}_\mathcal{R}(\Lambda_{K_i}, \Lambda_{K_j})$ contained in the table, half of the elements are orientation-preserving, the other half orientation-reversing.

| $i$ | $j$ | Size of $\mathcal{N}_\mathcal{R}(\Lambda_{K_i}, \Lambda_{K_j})$ |
|-----|-----|----------------------------------|
| 2   | 4   | 5184                             |
| 3   | 5   | 3888                             |
| 7   | 8   | 3888                             |
| 10  | 11  | 1296                             |

**Proposition 6.19** A hyperelliptic fourfold $X$ whose fundamental group is isomorphic to the fundamental group of a rigid hyperelliptic fourfold with holonomy $\mathbb{Z}_3^2$ is rigid.

**Proof** According to Corollary 4.5 (2), the isomorphism of fundamental groups induces an affine diffeomorphism of the manifolds. Note that the complex holonomy representation $\rho': \mathbb{Z}_3^2 \rightarrow \text{GL}(4, \mathbb{C})$ of $X$ cannot contain complex conjugate sub-representations,
otherwise the decomplexification $\rho'_R$ consists of at most three distinct non-trivial irreducible real representations—a contradiction because $\rho'_R$ is equivalent to $\rho_R(a, b) = \text{diag}(B^a, B^b, B^{2a+b}, B^{a+b})$ up to an automorphism of $\mathbb{Z}_2^3$ (see Remark 4.6). The latter, however, contains four non-trivial distinct irreducible real representations. According to Corollary 2.6, the manifold $X$ is rigid.

7 Proof of the second main theorem

In this section, we summarize the proof of our second main result.

Proof of Theorem 1.2 In Remark 3.2, the analytic representations for both $G = \mathbb{Z}_2^3$ and $\text{He}(3)$ are described. Proposition 5.2 shows that a hyperelliptic fourfold with such a holonomy representation is finitely covered by a product of four Fermat elliptic curves.

(1) It was proved in Proposition 6.14 that there are exactly twelve biholomorphism classes of rigid hyperelliptic fourfolds with holonomy $\mathbb{Z}_2^3$. We obtain the translation parts $\tau_i$ listed in the theorem using our MAGMA code. The statement regarding the diffeomorphism types follows from Corollary 6.15 and the computation in Proposition 6.18. Proposition 6.19 shows that a hyperelliptic fourfold whose fundamental group is isomorphic to the fundamental group of a rigid hyperelliptic fourfold with holonomy $\mathbb{Z}_2^3$ is rigid and therefore biholomorphic to one of the $X_i$.

(2) The classification follows from Proposition 6.9 and Proposition 6.11. The listed cocycles are a by-product of our MAGMA computation. In Proposition 3.4, we explained that any hyperelliptic fourfold with holonomy $\text{He}(3)$ is rigid.

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