\textbf{A$^1$-HOMOTOPY INVARIANTS OF CORNER SKEW LAURENT POLYNOMIAL ALGEBRAS}

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Abstract. In this note we prove some structural properties of all the A$^1$-homotopy invariants of corner skew Laurent polynomial algebras. As an application, we compute the mod-1 algebraic K-theory of Leavitt path algebras using solely the kernel/cokernel of the incidence matrix. This leads naturally to some vanishing and divisibility properties of the K-theory of these algebras.

1. Corner skew Laurent polynomial algebras

Let $k$ be a field, $A$ a unital $k$-algebra, $e$ an idempotent of $A$, and $\phi: A \sim eAe$ a “corner” isomorphism. Following Ara-Barroso-Goodearl-Pardo [3, §2], the associated \textit{corner skew Laurent polynomial algebra} $A[t_+, t_-; \phi]$ is defined as follows: the elements are formal expressions $t^m a_{-m} + \cdots + t a_0 + a t_+ \cdots + a_n t^n_+$ with $a_{-i} \in \phi'(1)A$ and $a_i \in A\phi'(1)$ for every $i \geq 0$; the addition is defined componentwise; the multiplication is determined by the distributive law and by the relations $t_- t_+ = 1$, $t_+ t_- = e$, $a t_- = t_- \phi(a)$ for every $a \in A$, and $t_+ a = \phi(a) t_+$ for every $a \in A$. Note that $A[t_+, t_-; \phi]$ admits a canonical $\mathbb{Z}$-grading with $\deg(t_+) = \pm 1$.

As proved in [3, Lem. 2.4], the corner skew Laurent polynomial algebras can be characterized as those $\mathbb{Z}$-graded algebras $C = \bigoplus_{n \in \mathbb{Z}} C_n$ containing elements $t_+ \in C_1$ and $t_- \in C_{-1}$ such that $t_- t_+ = 1$. Concretely, we have $C = A[t_+, t_-; \phi]$ with $A := C_0$, $e := t_+ t_-$, and $\phi: C_0 \to t_+ t_- C_0 t_+ t_-$ given by $c_0 \mapsto t_0 c_0 t_-$.

Example 1.1 (Skew Laurent polynomial algebras). When $e = 1$, $A[t_+, t_-; \phi]$ reduces to the classical skew Laurent polynomial algebra $A \rtimes_\phi \mathbb{Z}$. In the particular case where $\phi$ is the identity, $A \rtimes_\phi \mathbb{Z}$ reduces furthermore to $A[t, t^{-1}]$.

Example 1.2 (Jacobson algebras). Following [8], the \textit{Jacobson algebra} $J_n$, $n \geq 0$, is the $k$-algebra generated by elements $x_0, \ldots, x_n, y_0, \ldots, y_n$ subject to the relations $y_i x_j = \delta_{ij}$. Note that the canonical $\mathbb{Z}$-grading, with $\deg(x_i) = 1$ and $\deg(y_i) = -1$, makes $J_n$ into a corner skew Laurent polynomial algebra. The algebras $J_n$ are also usually called Cohn algebras (see [1]), and $J_0$ the (algebraic) Toeplitz algebra.

Example 1.3 (Leavitt algebras). Following [12], the \textit{Leavitt algebra} $L_n$, $n \geq 0$, is the $k$-algebra generated by elements $x_0, \ldots, x_n, y_0, \ldots, y_n$ subject to the relations $y_i x_j = \delta_{ij}$ and $\sum_{i=0}^n x_i y_i = 1$. Note the canonical $\mathbb{Z}$-grading, with $\deg(x_i) = 1$ and $\deg(y_i) = -1$, makes $L_n$ into a corner skew Laurent polynomial algebra. Note also that $L_0 \cong k[t, t^{-1}]$. In the remaining case $n \geq 1$, $L_n$ is the universal example of a $k$-algebra of module type $(1, n+1)$, i.e. $L_n \cong L_n^{\oplus (n+1)}$ as right $L_n$-modules.

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Example 1.4 (Leavitt path algebras). Let $Q = (Q_0, Q_1, s, r)$ be a finite quiver; $Q_0$ and $Q_1$ stand for the sets of vertex and arrows, respectively, and $s$ and $r$ for the source and target maps, respectively. We assume that $Q$ has no sources, i.e. vertices $i \in Q_0$ such that $\{ \alpha \mid r(\alpha) = i \} = \emptyset$. Consider the double quiver $\overline{Q} = (Q_0, Q_1 \cup Q_1^\ast, s, r)$ obtained from $Q$ by adding an arrow $\alpha^\ast$, in the converse direction, for each arrow $\alpha \in Q_1$. Following Abrams-Pino [2] and Ara-Moreno-Pardo [5], the Leavitt path algebra $L_Q$ of $Q$ is the quotient of the quiver algebra $k\overline{Q}$ (which is generated by elements $\alpha \in Q_1 \cup Q_1^\ast$ and $e_i$ with $i \in Q_0$) by the Cuntz-Krieger’s relations: $\alpha^\ast\beta = \delta_{\alpha\beta}e_{r(\alpha)}$ for every $\alpha, \beta \in Q_1$; $\sum_{\alpha \in Q_1, s(\alpha) = i} \alpha\alpha^\ast = e_i$ for every non-sink $i \in Q_0$. Note that $L_Q$ admits a canonical $\mathbb{Z}$-grading with $\deg(\alpha) = 1$ and $\deg(\alpha^\ast) = -1$. For every vertex $i \in Q_0$ choose an arrow $\alpha_i$ such that $r(\alpha_i) = i$ and consider the associated elements $t_+ := \sum_{i \in Q_0} \alpha_i$ and $t_- := t_+^\ast$. Since $\deg(t_\pm) = \pm 1$ and $t_-t_+ = 1$, $L_Q$ is also an example of a corner skew Laurent polynomial algebra.

In the particular case where $Q$ is the quiver with one vertex and $n + 1$ arrows, $L_Q$ is isomorphic to $L_n$. Similarly, when $Q$ is the quiver with two vertices $\{1, 2\}$ and $2(n + 1)$ arrows ($n + 1$ from $1$ to $1$ and $n + 1$ from $1$ to $2$), we have $L_Q \simeq J_n$.

2. $\mathbb{A}^1$-HOMOTOPY INVARIANTS

A dg category $\mathcal{A}$, over a base field $k$, is a category enriched over cochain complexes of $k$-vector spaces; see §4.1. Every (dg) $k$-algebra $A$ gives rise to a dg category with a single object. Another source of examples is provided by schemes since the category of perfect complexes $\text{perf}(X)$ of every quasi-compact quasi-separated $k$-scheme $X$ admits a canonical $\mathbb{Z}$-grading with $\deg(\alpha) = 1$ and $\deg(\alpha^\ast) = -1$. For every vertex $i \in Q_0$ choose an arrow $\alpha_i$ such that $r(\alpha_i) = i$ and consider the associated elements $t_+ := \sum_{i \in Q_0} \alpha_i$ and $t_- := t_+^\ast$. Since $\deg(t_\pm) = \pm 1$ and $t_-t_+ = 1$, $L_Q$ is also an example of a corner skew Laurent polynomial algebra.

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Example 2.1 (Noncommutative mixed motives). Let $\text{Mot}_{loc}(k)$, resp. $\text{Mot}^{\mathbb{A}^1}_{loc}(k)$, be the category of noncommutative mixed motives constructed in [14, §8.2], resp.
in [14, §8.5.1]. By construction, this triangulated category comes equipped with a localizing invariant $U_{\text{loc}}: \text{dgcat}(k) \to \text{Mot}_{\text{loc}}(k)$, resp. with an $A^1$-homotopy invariant $U_{\text{loc}}^{A^1}: \text{dgcat}(k) \to \text{Mot}_{\text{loc}}^{A^1}(k)$, which is initial among all localizing invariants, resp. among all $A^1$-homotopy invariants; consult [14, §8-9] for further details.

3. Statement of results and applications

Let $A$ be a unital $k$-algebra, $e$ an idempotent of $A$, and $\phi: A \to eAe$ a "corner" isomorphism. Out of this data, we can construct the associated corner skew Laurent polynomial algebra $A[t_+, t_-; \phi]$. In the same vein, let us write $A[t_+; \phi]$ for the algebra defined by the formal expressions $a_0 + a_1 t_+ + a_2 t_+^2 + \cdots + a_n t_+^n$ and by the relations $t_+ a = \phi(a) t_+$. Consider the $A$-$A$-bimodule $A$ associated to $\phi$ (see Notation 4.1) and the square-zero extension $A[e] := A \otimes (\phi A[1])$ of $A$ by the suspension $\phi A[1]$ of $\phi A$. Concretely, $A[e] = A \oplus (eA)[1]$ with multiplication law $(a, eb) \cdot (a', eb') := (aa', e(ab' + ba'))$. Consider also the dg $A[e]$-$A[t_+; \phi]$-bimodule $B$ whose restriction to $A[t_+; \phi]$ is the projective resolution $t_+ \cdots : A[t_+; \phi] \to eA[t_+; \phi]$ of the trivial right $A[t_+; \phi]$-$A$-module $eA$. The left dg action is given by the canonical identification of $A[e]$ with the Ext-algebra $\text{Ext}_{A[t_+; \phi]}^*(eA)$. As explained in §4.1, the dg bimodule $B$ corresponds to a morphism from $A[e]$ to $A[t_+; \phi]$ in the localization of dgcat($k$) with respect to derived Morita equivalences, we obtain an induced morphism $E(B): E(A[e]) \to E(A[t_+; \phi])$.

**Theorem 3.1.** For every localizing invariant $E$, we have the homotopy colimit

\[(3.2) \quad \text{hocolim} E(B) \to \text{hocolim} E(B) \to \cdots \to E(A[t_+, t_-; \phi]),\]

where the transition morphism(s) is induced by the corner isomorphism $\phi$.

When $E$ is an $A^1$-homotopy invariant, $E(B)$ reduces to id $- E(eA): E(A) \to E(A)$ and the composition (3.2) is an isomorphism. Consequently, we obtain a triangle

\[(3.3) \quad E(A) \xrightarrow{\text{id} - E(eA)} E(A) \to E(A[t_+, t_-; \phi]) \xrightarrow{\partial} \Sigma E(A).\]

**Remark 3.4 (Generalization).** Given a dg category $A$, Theorem 3.1 holds more generally with $A$ and $A[t_+, t_-; \phi]$ replaced by $A \otimes A$ and $A \otimes A[t_+, t_-; \phi]$; see §7.

**Corollary 3.5.** Given a dg category $A$, we have a distinguished triangle of spectra:

\[KH(A \otimes A) \xrightarrow{\text{id} - KH(eA \otimes A)} KH(A \otimes A) \to KH(A \otimes A[t_+, t_-; \phi]) \xrightarrow{\partial} \Sigma KH(A \otimes A).\]

**Remark 3.6 (Related work).** Given a (not necessarily unital) $k$-algebra $C$, Ara-Brustenga-Cortiñas proved in [4, Thms. 3.6 and 8.4] the analogue of Corollary 3.5 with $A$ replaced by $C$. Their proof, which is inspired from operator $K$-theory, makes essential use of non-unital algebras. Since these latter objects don’t belong to the realm of dg categories, our proof of Corollary 3.5 is necessarily different. Among other ideas used in the proof, we show that every corner skew Laurent polynomial algebra is derived Morita equivalent to a dg orbit category in the sense of Keller; see Proposition 6.6. Finally, note that in contrast with Ara-Brustenga-Cortiñas’ result, Corollary 3.5 (and more generally Theorem 3.1) holds for dg algebras and schemes.

Let $Q = (Q_0, Q_1, s, r)$ be a finite quiver, without sources, with $v$ vertices and $v'$ sinks. We assume that the set $Q_0$ is ordered with the first $v'$ elements corresponding to the sinks. Let $I_Q$ be the incidence matrix of $Q$, $I_Q^t$ the matrix obtained from $I_Q$ by removing the first $v'$ rows (which are zero), and $I_Q^t$ the transpose of $I_Q$. 


Theorem 3.7. Let $\mathcal{A}$ be a dg category and $Q$ a finite quiver without sources. For every $k$-homotopy invariant $E$, we have a distinguished triangle

$$
\oplus_{i=1}^{n-\nu'} E(\mathcal{A}) \xrightarrow{\left(\begin{array}{c}
0 \\
1 \\
0
\end{array}\right)} \oplus_{i=1}^{\nu} E(\mathcal{A}) \to E(\mathcal{A} \otimes L_Q) \xrightarrow{\partial} \oplus_{i=1}^{n-\nu'} \Sigma E(\mathcal{A}).
$$

Theorem 3.7 shows that all the information concerning $k$-homotopy invariants of Leavitt path algebras $L_Q$ is encoded in the incidence matrix of the quiver $Q$.

Example 3.8 (Jacobson algebras). Let $Q$ be the quiver with two vertices $\{1, 2\}$ and $2(n+1)$ arrows ($n+1$ from 1 to 1 and $n+1$ from 1 to 2). In this particular case, the distinguished triangle of Theorem 3.7 reduces to

$$
E(\mathcal{A}) \xrightarrow{n \cdot \text{id}} E(\mathcal{A}) \to E(\mathcal{A} \otimes J_n) \xrightarrow{\partial} \Sigma E(\mathcal{A}).
$$

Since $(n, n+1) = 1$, we conclude that $E(\mathcal{A} \otimes J_n) \simeq E(\mathcal{A})$. This shows that the $k$-homotopy invariants don’t distinguish the Jacobson algebras $J_n$ from $k$. Note that $J_n$ is much bigger than $k$; for instance, it contains the path algebra $kQ$.

Example 3.9 (Leavitt algebras). Let $Q$ be the quiver with one vertex and $n+1$ arrows. In this particular case, the distinguished triangle of Theorem 3.7 reduces to

$$
E(\mathcal{A}) \xrightarrow{n \cdot \text{id}} E(\mathcal{A}) \to E(\mathcal{A} \otimes L_n) \xrightarrow{\partial} \Sigma E(\mathcal{A}).
$$

When $n = 0$, the distinguished triangle (3.10) splits and gives rise to the “fundamental” isomorphism $E(\mathcal{A} \otimes L_0) \simeq E(\mathcal{A}) \oplus \Sigma E(\mathcal{A})$. When $n = 1$, we have $E(\mathcal{A} \otimes L_1) = 0$. In the remaining case $n \geq 2$, $E(\mathcal{A} \otimes L_n)$ identifies with the mod-$n$ Moore object of $E(\mathcal{A})$. Intuitively speaking, this shows that the functor $\mathcal{A} \to \mathcal{A} \otimes L_n$, with $n \geq 2$, is a model of the mod-$n$ Moore construction.

Proposition 3.11. Let $l_1^{e_1} \times \cdots \times l_r^{e_r}$ be the prime decomposition of an integer $n \geq 2$. For every dg category $\mathcal{A}$ and $k$-homotopy invariant $E$, we have a direct sum decomposition $E(\mathcal{A} \otimes L_{l_1^{e_1}}) \oplus \cdots \oplus E(\mathcal{A} \otimes L_{l_r^{e_r}})$.

Roughly speaking, Proposition 3.11 shows that all the $k$-homotopy invariants of Leavitt algebras are “$l$-local”. Note that $L_n \neq L_{l_1^{e_1}} \times \cdots \times L_{l_r^{e_r}}$.

Remark 3.12 (Homotopy $K$-theory). By taking $E = KH$ in Theorem 3.7, we obtain a distinguished triangle of spectra

$$
\oplus_{i=1}^{n-\nu'} KH(\mathcal{A}) \xrightarrow{\left(\begin{array}{c}
0 \\
1 \\
0
\end{array}\right)} \oplus_{i=1}^{\nu} KH(\mathcal{A}) \to KH(\mathcal{A} \otimes L_Q) \xrightarrow{\partial} \oplus_{i=1}^{n-\nu'} \Sigma KH(\mathcal{A}).
$$

Given a (not necessarily unital) $k$-algebra $C$, Ara-Brustenga-Cortiñas constructed in [4, Thm. 8.6] the analogue of the preceding distinguished triangle with $\mathcal{A}$ replaced by $C$. Our construction is different and applies also to dg categories and schemes.

Remark 3.13. By taking $E = KH$ and $n = l^r$ in Example 3.9, we obtain an isomorphism between $KH(\mathcal{A} \otimes L_{l^r})$ and the mod-$l^r$ homotopy $K$-theory spectrum $KH(\mathcal{A}; \mathbb{Z}/l^r)$. When $l \nmid \text{char}(k)$, the latter spectrum is isomorphic to $Kh(\mathcal{A}; \mathbb{Z}/l^r)$.

Mod-$l^r$ algebraic $K$-theory of Leavitt path algebras. Let $l^r$ be a prime power such that $l \neq \text{char}(k)$ and $Q$ a finite quiver without sources. By taking $\mathcal{A} = k$ and $E = Kh(\mathcal{A}; \mathbb{Z}/l^r)$ in Theorem 3.7, we obtain a distinguished triangle of spectra

$$
\oplus_{i=1}^{n-\nu'} Kh(k; \mathbb{Z}/l^r) \xrightarrow{\left(\begin{array}{c}
0 \\
1 \\
0
\end{array}\right)} \oplus_{i=1}^{\nu} Kh(k; \mathbb{Z}/l^r) \to Kh(L_Q; \mathbb{Z}/l^r) \xrightarrow{\partial} \oplus_{i=1}^{n-\nu'} \Sigma Kh(k; \mathbb{Z}/l^r).
$$
Remark 3.14. The preceding triangle follows also from the work of Ara-Brustenga-Cortiñas [4]. Indeed, since by hypothesis \( l \nmid \text{char}(k) \), the functors \( \mathcal{K}(\_; \mathbb{Z}/l^\nu) \) and \( KH(\_; \mathbb{Z}/l^\nu) \) are isomorphic. Moreover, as explained in Remark 3.13, the latter functor identifies with \( KH(\_ \otimes \mathcal{L}_{l^\nu}) \). Therefore, if in Remark 3.12 we take for \( C \) the \( k \)-algebra \( \mathcal{L}_{l^\nu} \), we obtain the preceding distinguished triangle of spectra.

Assume that \( k \) is algebraically closed. As proved by Suslin\(^2\) in [13, Cor. 3.13], we have \( \mathcal{K}_n(k; \mathbb{Z}/l^\nu) \simeq \mathbb{Z}/l^\nu \) if \( n > 0 \) is even and \( \mathcal{K}_n(k; \mathbb{Z}/l^\nu) = 0 \) otherwise. Consequently, making use of the long exact sequence of algebraic \( K \)-theory groups associated to the preceding triangle of spectra, we obtain the following result:

**Corollary 3.15.** We have the following computation

\[
\mathcal{K}_n(L_Q; \mathbb{Z}/l^\nu) \simeq \begin{cases} 
\text{kernel of } \text{M} & \text{if } n \geq 0 \text{ even} \\
\text{cokernel of } \text{M} & \text{if } n \geq 0 \text{ odd} \\
0 & \text{if } n < 0,
\end{cases}
\]

where \( \text{M} \) stands for the homomorphism \( \bigoplus_{\nu = 1}^{v - 1} \mathbb{Z}/l^\nu (\nu) - I_Q^0 \bigoplus_{\nu = 1}^v \mathbb{Z}/l^\nu. \)

Corollary 3.15 provides a complete and explicit computation of the mod-\( l^\nu \) (non-connective) algebraic \( K \)-theory of Leavitt path algebras. To the best of the author’s knowledge, these computations are new in the literature. In particular, they yield a complete answer to the “mod-\( l^\nu \) version” of Question 2 raised by Gabe-Ruiz-Tomforde-Whalen in [7, page 38]. These computations lead also naturally to the following vanishing and divisibility properties of algebraic \( K \)-theory:

**Proposition 3.16.** (i) If there exists a prime power \( l^\nu \) and an even (resp. odd) integer \( n' \geq 0 \) such that \( \mathcal{K}_{n'}(L_Q; \mathbb{Z}/l^\nu) \neq 0 \), then for every even (resp. odd) integer \( n \geq 0 \) at least one of the groups \( \mathcal{K}_n(L_Q), \mathcal{K}_{n - 1}(L_Q) \) is non-zero.

(ii) If there exists a prime power \( l^\nu \) such that \( \mathcal{K}_n(L_Q; \mathbb{Z}/l^\nu) = 0 \) for every \( n \geq 0 \), then the groups \( \mathcal{K}_n(L_Q), n \geq 0, \) are uniquely \( l^\nu \)-divisible, i.e. \( \mathbb{Z}[1/l^\nu] \)-modules.

**Proof.** Combine the universal coefficients sequence (see [14, §2.2.2])

\[
0 \rightarrow \mathcal{K}_n(L_Q) \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \rightarrow \mathcal{K}_n(L_Q; \mathbb{Z}/l^\nu) \rightarrow \mathcal{K}_{n - 1}(L_Q) \rightarrow 0
\]

with the computation of Corollary 3.15. \( \square \)

**Example 3.17** (Quivers without sinks). Let \( Q \) be a quiver without sinks. In this case, \( (I_Q^0 - I_Q^1) \) is a square matrix. If \( l \) is a prime such that \( l \nmid \det((I_Q^0 - I_Q^1)) \), then the homomorphism \( \text{M} \) of Corollary 3.15 is invertible. Consequently, \( \mathcal{K}_n(L_Q; \mathbb{Z}/l^\nu) \neq 0 \) for every \( n \geq 0 \). Making use of Proposition 3.16(ii), we then conclude that the algebraic \( K \)-theory groups \( \mathcal{K}_n(L_Q), n \geq 0, \) are uniquely \( l^\nu \)-divisible.

**Schemes and stacks.** Let \( X \) be a quasi-compact quasi-separated \( k \)-scheme. By applying the results/examples/remarks of §3 to the dg category \( \mathcal{A} = \text{perf}_{dg}(X) \), we obtain corresponding results/examples/remarks concerning the scheme \( X \). For instance, Remark 3.13 yields an isomorphism between \( KH(\text{perf}_{dg}(X) \otimes \mathcal{L}_{l^\nu}) \) and \( KH(X; \mathbb{Z}/l^\nu) \). When \( l \nmid \text{char}(k) \), the latter spectrum is isomorphic to \( \mathcal{K}(X; \mathbb{Z}/l^\nu) \).

Roughly speaking, the dg category \( \text{perf}_{dg}(X) \otimes \mathcal{L}_{l^\nu} \) may be understood as the “noncommutative mod-\( l^\nu \) Moore object of \( X \)”. More generally, we can consider the

\(^2\)Given a quiver \( Q \), let \( C_0^\ast(Q) \) be the associated Cuntz-Krieger \( C^\ast \)-algebra. Cortiñas kindly informed the author that the work of Suslin was also used in [4, Thm. 9.4] in order to prove that \( \mathcal{K}_n(C \otimes L_Q) \simeq \mathcal{K}_n^{\text{top}}(C_0^\ast(Q)), n \geq 0, \) for every quiver \( Q \) without sinks such that \( \det((Q^0 - I_Q^0)) \neq 0 \).
A dg category \( \mathcal{A} = \text{perf}_{dg}(\mathcal{X}) \) of perfect complexes of an algebraic stack \( \mathcal{X} \). In the particular case of a quotient stack \( \mathcal{X} = [X/G] \), with \( G \) an algebraic group scheme acting on \( X \), Remark 3.13 yields an isomorphism between \( K\text{H}(\text{perf}_{dg}([X/G]) \otimes \mathbb{L}^{l'\nu}) \) and the mod-\( l'\nu \) \( G \)-equivariant homotopy \( K \)-theory spectrum \( K\text{H}^G(X; \mathbb{Z}/l'\nu) \). When \( l \nmid \text{char}(k) \), the latter spectrum is isomorphic to \( K^G(X; \mathbb{Z}/l'\nu) \).

4. Preliminaries

4.1. Dg categories. Let \( \mathcal{C}(k, \otimes, k) \) be the category of cochain complexes of \( k \)-vector spaces. A dg category \( \mathcal{A} \) is a category enriched over \( \mathcal{C}(k) \) and a dg functor \( F : \mathcal{A} \to \mathcal{B} \) is a functor enriched over \( \mathcal{C}(k) \); consult Keller’s ICM survey [9].

Let \( \mathcal{A} \) be a dg category. The opposite dg category \( \mathcal{A}^{op} \) has the same objects as \( \mathcal{A} \) and \( \mathcal{A}^{op}(x, y) := \mathcal{A}(y, x) \). A right dg \( \mathcal{A} \)-module is a dg functor \( \mathcal{A}^{op} \to \mathcal{C}_{dg}(k) \) with values in the dg category \( \mathcal{C}_{dg}(k) \) of complexes of \( k \)-vector spaces. Let us denote by \( \mathcal{C}(\mathcal{A}) \) the category of right dg \( \mathcal{A} \)-modules. Following [9, §3.2], the derived category \( \mathcal{D}(\mathcal{A}) \) of \( \mathcal{A} \) is defined as the localization of \( \mathcal{C}(\mathcal{A}) \) with respect to the objectwise quasi-isomorphisms. We write \( \mathcal{D}_c(\mathcal{A}) \) for the subcategory of compact objects.

A dg functor \( F : \mathcal{A} \to \mathcal{B} \) called a derived Morita equivalence if it induces an equivalence of categories \( \mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{B}) \); see [9, §4.6]. As proved in [14, Thm. 1.37], \( \text{dgcat}(k) \) admits a Quillen model structure whose weak equivalences are the derived Morita equivalences. Let \( \text{Hmo}(k) \) be the associated homotopy category.

The tensor product \( \mathcal{A} \otimes \mathcal{B} \) of (small) dg categories is defined as follows: the set of objects is the cartesian product and \((\mathcal{A} \otimes \mathcal{B})((x, w), (y, z)) := \mathcal{A}(x, y) \otimes \mathcal{B}(w, z)\). As explained in [9, §2.3], this construction gives rise to a symmetric monoidal structure on \( \text{dgcat}(k) \), which descends to the homotopy category \( \text{Hmo}(k) \).

A dg \( \mathcal{A}-\mathcal{B} \)-bimodule \( \mathcal{B} \) is a dg functor \( \mathcal{A} \otimes \mathcal{B}^{op} \to \mathcal{C}_{dg}(k) \) or equivalently a right \( \mathcal{A} \)-\( \mathcal{B} \)-bimodule \( \mathcal{B} : \mathcal{A} \otimes \mathcal{B}^{op} \to \mathcal{C}_{dg}(k) \), \((x, z) \mapsto \mathcal{B}(z, F(x))\). Let us write \( \text{rep}(\mathcal{A}, \mathcal{B}) \) for the full triangulated subcategory of \( \mathcal{D}(\mathcal{A}^{op} \otimes \mathcal{B}) \) consisting of those dg \( \mathcal{A}-\mathcal{B} \)-bimodules \( \mathcal{B} \) such that for every object \( x \in \mathcal{A} \) the associated right \( \mathcal{B} \)-module \( \mathcal{B}(x, \cdot) \) belongs to \( \mathcal{D}_c(\mathcal{B}) \). Clearly, the dg \( \mathcal{A}-\mathcal{B} \)-bimodules \( \mathcal{B} \) belongs to \( \text{rep}(\mathcal{A}, \mathcal{B}) \).

As explained in [14, §1.6.3], there is a natural bijection between \( \text{Hom} \text{Hmo}(k)(\mathcal{A}, \mathcal{B}) \) and the set of isomorphism classes of \( \text{rep}(\mathcal{A}, \mathcal{B}) \). Under this bijection, the composition law corresponds to the tensor product of dg bimodules.

Notation 4.1. Given a non-unital homomorphism \( \phi : \mathcal{A} \to \mathcal{B} \) between unital \( k \)-algebras, let us denote by \( \phi \mathcal{B} \) the \( \mathcal{A} \)-\( \mathcal{B} \)-bimodule \( \phi(1)\mathcal{B} \) equipped with the \( \mathcal{A}-\mathcal{B} \)-action \( a \cdot \phi(1)\mathcal{B} \cdot b := \phi(a)\mathcal{B}b \). Note that \( \phi \mathcal{B} \) belongs to \( \text{rep}(\mathcal{A}, \mathcal{B}) \).

Square-zero extensions. Let \( \mathcal{A} \) be a dg category and \( \mathcal{B} \) a dg \( \mathcal{A} \)-\( \mathcal{B} \)-bimodule. The square-zero extension \( \mathcal{A} \ltimes \mathcal{B} \) of \( \mathcal{A} \) by \( \mathcal{B} \) is the dg category with the same objects as \( \mathcal{A} \) and complexes of \( k \)-vector spaces \((\mathcal{A} \ltimes \mathcal{B})(x, y) := \mathcal{A}(x, y) \oplus \mathcal{B}(y, x)\). Given morphisms \((f, f') \in (\mathcal{A} \ltimes \mathcal{B})(x, y)\) and \((g, g') \in (\mathcal{A} \ltimes \mathcal{B})(y, z)\), the composition \((g, g') \circ (f, f') \) is defined as \((g \circ f, g'f + g'f')\).

Dg orbit categories. Let \( F : \mathcal{A} \to \mathcal{A} \) be an equivalence of dg categories. Following Keller [10, §5.1], the associated dg orbit category \( \mathcal{A}/F^2 \) has the same objects as \( \mathcal{A} \) and complexes of \( k \)-vector spaces \((\mathcal{A}/F^2)(x, y) := \bigoplus_{n \in \mathbb{Z}} \mathcal{A}(x, F^n(y))\). Given objects \( x, y, z \) and morphisms \( f = \{f_n\}_{n \in \mathbb{Z}} \in \bigoplus_{n \in \mathbb{Z}} \mathcal{A}(x, F^n(y)) \) and \( g = \{g_n\}_{n \in \mathbb{Z}} \in \bigoplus_{n \in \mathbb{Z}} \mathcal{A}(y, F^n(z)) \), the \( m \)-th component of \( g \circ f \) is defined as \( \sum_n F^n(g_{m-n}) \circ f_n \).
When $\mathcal{A}$ is a $k$-algebra $A$, the dg functor $F$ reduces to an isomorphism $\phi: A \rightarrow A$ and the dg orbit category $\mathcal{A}/F^Z$ to the skew Laurent polynomial algebra $A \rtimes \phi Z$.

Let us write $\mathcal{A}/F^N$ for the dg category with the same objects as $\mathcal{A}$ and complexes of $k$-vector spaces $(\mathcal{A}/F^N)(x, y) := \bigoplus_{n \geq 0} \mathcal{A}(x, F^n(y))$. The composition law is defined as above. By construction, we have a canonical dg functor $\mathcal{A}/F^N \rightarrow \mathcal{A}/F^Z$.

5. DG categories of idempotents

Let $\mathcal{A}$ be a (not necessarily unital) $k$-algebra.

**Definition 5.1.** The dg category of idempotents of $\mathcal{A}$, denoted by $\mathcal{A}_e$, is defined as follows: the objects are the symbols $e$ with $e$ an idempotent of $\mathcal{A}$; the (complexes of) $k$-vector spaces $\mathcal{A}(e, e')$ are given by $e\mathcal{A}e'$; the composition law is induced by the multiplication in $\mathcal{A}$; the identity of the object $e$ is the idempotent $e$.

**Notation 5.2.** Let $\text{alg}(k)$ be the category of (not necessarily unital) $k$-algebras and (not necessarily unital) $k$-algebra homomorphisms.

Note that the preceding construction gives rise to the following functor:

$$
\text{alg}(k) \rightarrow \text{dgcat}(k) \quad A \mapsto \mathcal{A}_e \quad \phi \mapsto \mathcal{A}_e
$$

(5.3)

**Lemma 5.4.** The functor (5.3) preserves filtered colimits.

**Proof.** Consider a filtered diagram $\{A_i\}_{i \in I}$ in $\text{alg}(k)$ with colimit $A$. Given an idempotent element $e$ of $A$, there exists an index $i' \in I$ and an idempotent $e_{i'} \in A_{i'}$ such that $e$ is the image of $e_{i'}$ under $A_{i'} \rightarrow A$. This implies that the induced dg functor $\text{colim}_i A_i \rightarrow A$ is not only (essentially) surjective but also fully-faithful. □

Given a unital $k$-algebra $A$ with unit 1, let us write $\iota: A \rightarrow A$ for the (unique) dg functor sending the single object of $A$ to the symbol 1.

**Lemma 5.5.** The dg functor $\iota$ is a derived Morita equivalence.

**Proof.** Note first that the dg functor $\iota$ is fully-faithful. Given an idempotent element $e$ of $A$, the morphisms $1 \overset{\iota}{\rightarrow} e$ and $e \overset{\iota}{\rightarrow} 1$ present the object $e$ as a direct summand of 1. This allows us to conclude that $\iota$ is a derived Morita equivalence. □

**Remark 5.6.** Given a non-unital homomorphism $\phi: A \rightarrow B$ between unital $k$-algebras, note that $\iota B \circ \phi B = \iota B \circ \iota A$ in the homotopy category $\text{Hmo}(k)$.

Let $A$ be a unital $k$-algebra and $M_2(A)$ the associated $k$-algebra of $2 \times 2$ matrices.

Consider the following non-unital homomorphisms

$$
j_1, j_2: A \rightarrow M_2(A) \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad a \mapsto \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.
$$

Note that if there exist elements $t_+, t_-$ of $A$ such that $t_- t_+ = 1$, then we can also consider the non-unital homomorphism $\phi^{\pm}: A \rightarrow A, a \mapsto t_+ a t_-^{-1}$.

**Proposition 5.7.** (i) The dg functors $j_1$ and $j_2$ are derived Morita equivalences. Moreover, their images in the homotopy category $\text{Hmo}(k)$ are the same.

(ii) The dg functor $\phi^{\pm}$ is a derived Morita equivalence. Moreover, its image in the homotopy category $\text{Hmo}(k)$ is the identity morphism.
Proof. (i) Recall first that a dg functor $F: A \to B$ is a derived Morita equivalence if and only if its image $F \mathcal{B}$ in the homotopy category $\text{Hmo}(k)$ is invertible. Thanks to Lemma 5.5 and Remark 5.6, it suffices then to show that the $A$-$M_2(A)$-bimodules $j_1M_2(A)$ and $j_2M_2(A)$ are invertible in $\text{Hmo}(k)$. Note that their inverses are given by the $M_2(A)$-$A$-bimodules $M_2(A)j_1(1)$ and $M_2(A)j_2(1)$, respectively. This shows the first claim. The second claim follows from the isomorphism $j_1M_2(A) \xrightarrow{\sim} j_2M_2(A)$ of $A$-$M_2(A)$-bimodules given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} t_+ & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t_- & 0 \\ 0 & 1 \end{pmatrix}.

(ii) Consider the following non-unital homomorphism

$\varphi^\pm: M_2(A) \to M_2(A)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} t_+ & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t_- & 0 \\ 0 & 1 \end{pmatrix}.$

Note that $j_1 \circ \varphi^\pm = \varphi^\pm \circ j_1$ and $\varphi^\pm \circ j_2 = j_2$ in the category $\text{alg}(k)$. By applying the functor (5.3), we hence conclude from item (i) that the dg functor $\varphi^\pm$ is not only a derived Morita equivalence but moreover that its image in the homotopy category $\text{Hmo}(k)$ is the identity morphism. □

6. PROOF OF THEOREM 3.1

Consider the sequential colimit diagram $A \xrightarrow{\sigma} A \xrightarrow{\sigma} \cdots \to C$ in the category $\text{alg}(k)$. Note that the $k$-algebra $C$ is non-unital and that the homomorphism $\sigma$ gives rise to an isomorphism $\sigma: C \xrightarrow{\sim} C$. Let us denote by $C \rtimes_{\sigma} N$, resp. $C \rtimes_{\sigma} Z$, the associated skew polynomial algebra, resp. skew Laurent polynomial algebra. Note also that $\sigma$ extends to (non-unital) homomorphisms $\sigma: A[t_+; \sigma] \to A[t_+; \sigma], a \mapsto t_+ at_-$, and $\sigma^\pm: A[t_+, t_-; \sigma] \to A[t_+, t_-; \sigma], a \mapsto t_+ at_-$. Under these notations, we have the following sequential colimit diagrams:

(6.1) $A[t_+; \sigma] \xrightarrow{\sigma} A[t_+; \sigma] \xrightarrow{\sigma} \cdots \to C \rtimes_{\sigma} N$

(6.2) $A[t_+, t_-; \sigma] \xrightarrow{\sigma^\pm} A[t_+, t_-; \sigma] \xrightarrow{\sigma^{\pm}} \cdots \to C \rtimes_{\sigma} Z$.

Lemma 6.3. The dg category of idempotents of $C \rtimes_{\sigma} N$, resp. $C \rtimes_{\sigma} Z$, is derived Morita equivalent to the dg category $C/\hat{\sigma}^N$, resp. $C/\hat{\sigma}^Z$.

Proof. We focus ourselves in the algebra $C \rtimes_{\sigma} Z$ and in the dg orbit category $C/\hat{\sigma}^Z$; the proof of the other case is similar. Let us denote by $1_n$ the unit of the $n^{th}$ copy of $A[t_+, t_-; \sigma]$ and by $e_n$ the image of $1_n$ under the induced homomorphism $A[t_+, t_-; \sigma] \to C \rtimes_{\sigma} Z$. Given an idempotent element $e$ of $C \rtimes_{\sigma} Z$, there exists an integer $n \gg 0$ such that $e$ is a direct summand of $e_n$. Since $e_n$ belongs to $C \subset C \rtimes_{\sigma} Z$, this allows us to conclude, in particular, that the dg category of idempotents of $C \rtimes_{\sigma} Z$ is derived Morita equivalent to its full dg category of symbols $e$ with $e$ an idempotent element of $C \subset C \rtimes_{\sigma} Z$. Given any two such symbols $e$ and $e'$, note that we have the following equalities of (complexes of) $k$-vector spaces:

$$(C \rtimes_{\sigma} Z)(e, e') := e(C \rtimes_{\sigma} Z)e' = \bigoplus_{n \in \mathbb{Z}} eC\hat{\sigma}^n(e') = \bigoplus_{n \in \mathbb{Z}} C(e, \hat{\sigma}^n(e')).$$
Under these equalities, the composition law of the dg category of idempotents of $C \times_{\hat{\phi}} \mathbb{Z}$ corresponds to the composition law of the dg orbit category $C/\hat{\phi}$. □

By combining Lemmas 5.4-5.5 with Remark 5.6, we conclude from Lemma 6.3 that (6.1)-(6.2) give rise to the following sequential (homotopy) colimit diagrams

\begin{align*}
(6.4) & \quad A[t_+; \phi] \xrightarrow{\phi A[t_+; \phi]} A[t_+; \phi] \xrightarrow{\phi A[t_+; \phi]} \cdots \longrightarrow C/\hat{\phi}^N \\
(6.5) & \quad A[t_+, t_-; \phi] \xrightarrow{\phi A[t_+, t_-; \phi]} A[t_+, t_-; \phi] \xrightarrow{\phi A[t_+, t_-; \phi]} \cdots \longrightarrow C/\hat{\phi}^Z
\end{align*}

in the homotopy category $\text{Hmo}(k)$. Proposition 5.7(ii) (with $A[t_+, t_-; \phi]$ instead of $A$) leads then automatically to the following result:

**Proposition 6.6.** The (transfinite) composition (6.5) is an isomorphism. Consequently, the dg categories $A[t_+, t_-; \phi]$ and $C/\hat{\phi}^Z$ are derived Morita equivalent.

Now, consider the square-zero extension $C[e] := C \times (\hat{C}[1])$ of $C$ by the suspension of $\hat{C}[1]$ of the dg $C$-$C$-bimodule $\hat{\phi}C$ associated to $\phi$. Consider also the dg $C[e]$-$(C/\hat{\phi}^Z)$-bimodule $B$ introduced in [10, §4]; denoted by $B'$ in loc. cit.

**Lemma 6.7.** We have the following sequential (homotopy) colimit diagram

\[ A[e] \xrightarrow{\phi A[e]} A[e] \xrightarrow{\phi A[e]} \cdots \longrightarrow C[e] \]

in the homotopy category $\text{Hmo}(k)$.

**Proof.** Thanks to Lemma 5.4, we have $A \xrightarrow{\phi} A \xrightarrow{\phi} \cdots \longrightarrow C$ in the category $\text{dgcat}(k)$. Consider the square-zero extension $A[e] := A \times (\hat{A}[1])$ of $A$ by the suspension of $\hat{A}[1]$ of the dg $A$-$A$-bimodule $\hat{\phi}A$ associated to $\phi$. Similarly to the proof of Lemma 5.4, we have the following induced sequential colimit diagram

\[ A[e] \xrightarrow{\phi} A[e] \xrightarrow{\phi} \cdots \longrightarrow C[e]. \]

Note that the dg functor $A[e] \rightarrow A[e]$ sending the single object $A[e]$ to the symbol $1$, where $1$ is the unit of $A$, is a derived Morita equivalence. Under such derived Morita equivalence the dg functor $\phi; A[e] \rightarrow A[e]$ corresponds to the morphism $\phi A[e] : A[e] \rightarrow A[e]$ in the homotopy category $\text{Hmo}(k)$. This concludes the proof. □

By combining Lemma 6.7 with the sequential (homotopy) colimit diagram (6.4), we obtain the following sequential (homotopy) colimit diagram

\begin{align*}
(6.8) & \quad C[e] \xrightarrow{\hat{\phi}} C/\hat{\phi}^N \\
& \quad \vdots \\
& \quad A[e] \xrightarrow{\phi A[e]} A[t_+; \phi] \\
& \quad A[e] \xrightarrow{\phi A[e]} A[t_+; \phi]
\end{align*}
in the homotopy category $H_{mo}(k)$. As explained in [10, §4], given any localizing invariant $E$, we have a distinguished triangle

$$E(C[ε]) \rightarrow E(C/\hat{\phi}^N) \rightarrow E(C/\hat{\phi}^Z) \rightarrow \hat{\phi} \Sigma E(C[ε]).$$

Therefore, by combining the diagram (6.8) with Proposition 6.6, we obtain the searched homotopy colimit diagram (3.2). This concludes the proof of the first claim.

We now prove the second claim. Let $E$ be an $\mathbb{A}^1$-homotopy invariant. As explained in [10, Prop. 4.6], $E(B)$ reduces to $\text{id} - E(\hat{\phi}A): E(A) \rightarrow E(A)$. Similarly, the morphism $E(\hat{B})$ reduces to $\text{id} - E(\hat{\phi}C): E(C) \rightarrow E(C)$. Therefore, making use of Lemma 5.5 and of Remark 5.6, we observe that by applying the functor $E$ to (6.8) we obtain (up to isomorphism) the following sequential colimit diagram:

$$E(C) \rightarrow E(C) \rightarrow \cdots \rightarrow E(C).$$

We now claim that the induced (transfinite) composition

$$\text{hocofib}(\text{id} - E(\hat{\phi})) \xrightarrow{E(\hat{\phi})} \text{hocofib}(\text{id} - E(\hat{\phi})) \xrightarrow{E(\hat{\phi})} \cdots \xrightarrow{E(\hat{\phi})}$$

is an isomorphism. As explained in [14, Thm. 8.25], the functor $U_{\text{loc}}^{\mathbb{A}^1}$ is the initial $\mathbb{A}^1$-homotopy invariant. Therefore, it suffices to prove the latter claim in the particular case where $E = U_{\text{loc}}^{\mathbb{A}^1}$. By construction, we have a factorization

$$U_{\text{loc}}^{\mathbb{A}^1}: \text{dgcat}(k) \xrightarrow{U_{\text{add}}} \text{Mot}_{\text{add}}(k) \xrightarrow{\gamma} \text{Mot}_{\text{loc}}^{\mathbb{A}^1}(k),$$

where $\text{Mot}_{\text{add}}(k)$ is a certain compactly generated triangulated category of noncommutative mixed motives, $U_{\text{add}}$ is a certain functor sending sequential (homotopy) colimits to sequential homotopy colimits, and $\gamma$ is a certain homotopy colimit preserving functor; consult [14, §8.4.2] for details. The triangulated category $\text{Mot}_{\text{add}}(k)$ is moreover enriched over spectra; we write $\text{Hom}_{\text{Spt}}(-, -)$ for this enrichment. Let $NM$ be a compact object of $\text{Mot}_{\text{add}}(k)$. In order to prove our claim, it is then enough to show that the (transfinite) composition obtained by applying the functor $\text{Hom}_{\text{Spt}}(NM, -)$ to (6.11) (with $E = U_{\text{add}}$) is an isomorphism. Since the spectrum $\text{Hom}_{\text{Spt}}(NM, U_{\text{add}}(C))$ is the sequential homotopy colimit of $\text{Hom}_{\text{Spt}}(NM, U_{\text{add}}(A))$, with respect to the transition morphism(s) $\text{Hom}_{\text{Spt}}(NM, U_{\text{add}}(\hat{\phi}))$, the proof follows now automatically from the general result [4, Lem. 3.3] concerning spectra. This finishes the proof of Theorem 3.1.
7. Proof of the Generalization of Theorem 3.1

The triangulated category $\text{Mot}_{\text{loc}}(k)$ carries a symmetric monoidal structure making the functor $U_{\text{loc}}$ symmetric monoidal; see [14, §8.3.1]. Therefore, the distinguished triangle (6.9) (with $E = U_{\text{loc}}$) gives rise to the distinguished triangle:

$$U_{\text{loc}}(A \otimes C[k]) \xrightarrow{U_{\text{loc}}(\text{id} \otimes \text{id})} U_{\text{loc}}(A \otimes C/k) \xrightarrow{\partial} \Sigma U_{\text{loc}}(A \otimes C[k]).$$

Since the functor $A \otimes -$ preserves (sequential) homotopy colimits, the combination of the preceding triangle with the commutative diagram (6.8) and with Proposition 6.6 leads then to the following sequential homotopy colimit diagram

$$\text{hocifib} U_{\text{loc}}(\text{id} \otimes B) \rightarrow \text{hocifib} U_{\text{loc}}(\text{id} \otimes A) \rightarrow \cdots \rightarrow U_{\text{loc}}(A \otimes A[t_+, t_-; \phi]),$$

where the transition morphism(s) is induced by the corner isomorphism $\phi$. The proof of the first claim follows now automatically from the fact that $U_{\text{loc}}$ is the initial localizing invariant; see [14, Thm. 8.5].

The triangulated category $\text{Mot}^{A^1}_{\text{loc}}(k)$ carries a symmetric monoidal structure making the functor $U_{\text{loc}}^{A^1}$ symmetric monoidal; see [14, §8.5.2]. Therefore, the distinguished triangle (3.3) (with $E = U_{\text{loc}}^{A^1}$) gives rise to the distinguished triangle:

$$U_{\text{loc}}^{A^1}(A \otimes A) \xrightarrow{\text{id} - U_{\text{loc}}^{A^1}(\text{id} \otimes \phi A)} U_{\text{loc}}^{A^1}(A \otimes A) \xrightarrow{U_{\text{loc}}^{A^1}(A \otimes A[t_+, t_-; \phi])} \Sigma U_{\text{loc}}^{A^1}(A \otimes A).$$

The proof of the second claim follows now automatically from the fact that $U_{\text{loc}}^{A^1}$ is the initial $A^1$-homotopy invariant; see [14, Thm. 8.25].

8. Proof of Theorem 3.7

Similarly to the arguments used in §7, it suffices to prove Theorem 3.7 in the particular case where $E = U_{\text{loc}}^{A^1}$ and $A = k$. As mentioned in Example 1.4, the Leavitt path algebra $L := L_Q$ is a corner skew Laurent polynomial algebra. Let $L_0$ be the homogeneous component of degree 0 and $\phi$: $L_0 \xrightarrow{\sim} eL_0e$ the “corner” isomorphism. Thanks to Theorem 3.1 (with $E = U_{\text{loc}}^{A^1}$), we have a triangle

$$U_{\text{loc}}^{A^1}(L_0) \xrightarrow{\text{id} - U_{\text{loc}}^{A^1}(\text{id} \otimes \phi A)} U_{\text{loc}}^{A^1}(L_0) \xrightarrow{U_{\text{loc}}^{A^1}(L_0)} \Sigma U_{\text{loc}}^{A^1}(L_0)$$

in the category $\text{Mot}^{A^1}_{\text{loc}}(k)$. Following Ara-Brustenga-Cortiñas [4, §5], the $k$-algebra $L_0$ admits a “length” filtration $L_0 = \bigcup_{n=0}^{\infty} L_{0,n}$. Concretely, $L_{0,n}$ is the $k$-linear span of the elements of the form $\sigma \zeta^*$, where $\sigma$ and $\zeta$ are paths such that $r(\sigma) = r(\zeta)$ and $\deg(\sigma) = \deg(\zeta) = n$. It turns out that the $k$-algebra $L_{0,n}$ is isomorphic to the product of $(n+1)v' + (v-v')$ matrix algebras with $k$-coefficients. Making use of the (derived) Morita equivariance between a matrix algebra with $k$-coefficients and $k$, we hence conclude that $U_{\text{loc}}^{A^1}(L_{0,n})$ is isomorphic to the direct sum of $(n+1)v' + (v-v')$ copies of $U_{\text{loc}}^{A^1}(k)$. Recall from [14, Thm. 8.28] that we have an isomorphism $\text{Hom}_{\text{Mot}^{A^1}_{\text{loc}}(k)}(U_{\text{loc}}^{A^1}(k), U_{\text{loc}}^{A^1}(k)) \simeq K_0(k) \simeq \mathbb{Z}$. Under this identification, the inclusion $L_{0,n} \subset L_{0,n+1}$ corresponds to the matrix morphism (see [4, §5]):

$$\begin{pmatrix} \text{id} & 0 \\ 0 & I_Q^{(n+1)v' + (v-v')} \end{pmatrix} : \bigoplus_{i=1}^{(n+1)v' + v} U_{\text{loc}}^{A^1}(k) \rightarrow \bigoplus_{i=1}^{(n+1)v' + v} U_{\text{loc}}^{A^1}(k).$$
In the same vein, the homomorphism \( \phi: L_{0,n} \rightarrow L_{0,n+1} \), which increases the degree of the filtration by 1, corresponds to the matrix morphism

\[
\begin{pmatrix} 0 \\
\text{id} \end{pmatrix} : \bigoplus_{i=1}^{n' + v} U^k_{\text{loc}}(k) \rightarrow \bigoplus_{i=1}^{(n+1)'} U^k_{\text{loc}}(k).
\]

Since the functor \( U^k_{\text{loc}} \) sends sequential (homotopy) colimits to sequential homotopy colimits, we hence obtain the following sequential homotopy colimit diagram

\[
\begin{array}{ccc}
U^k_{\text{loc}}(L_{0,0}) & \xrightarrow{(8.2)} & U^k_{\text{loc}}(L_{0,1}) & \xrightarrow{(8.2)} & \cdots & \xrightarrow{(8.2)} & U^k_{\text{loc}}(L_0) \\
\downarrow \text{(8.2)-(8.3)} & & \downarrow \text{(8.2)-(8.3)} & & \cdots & & \downarrow \text{id} - U^k_{\text{loc}}(a L_0) \\
U^k_{\text{loc}}(L_{0,1}) & \xrightarrow{(8.2)} & U^k_{\text{loc}}(L_{0,2}) & \xrightarrow{(8.2)} & \cdots & \xrightarrow{(8.2)} & U^k_{\text{loc}}(L_0).
\end{array}
\]

Simple matrix manipulations show that the homotopy cofibers of the vertical morphisms of the diagram (8.4) are all equal to the homotopy cofiber of the morphism \( (a)_0 - I_Q : \bigoplus_{i=1}^{n' - v'} U^k_{\text{loc}}(k) \rightarrow \bigoplus_{i=1}^{n'} U^k_{\text{loc}}(k) \). This allows us then to conclude that distinguished triangle (8.1) yields the following distinguished triangle

\[
\bigoplus_{i=1}^{n' - v'} U^k_{\text{loc}}(k) \xrightarrow{(a)_0 - I_Q} \bigoplus_{i=1}^{n'} U^k_{\text{loc}}(k) \rightarrow U^k_{\text{loc}}(L_Q) \xrightarrow{\partial} \bigoplus_{i=1}^{n' - v'} \Sigma U^k_{\text{loc}}(k).
\]

Consequently, the proof is finished.

9. PROOF OF PROPOSITION 3.11

By construction, the triangulated category Mot\(^k_{\text{loc}}(k)\) comes equipped with an action of the homotopy category of spectra (see [14, §A.3]):

\[
\text{Spt} \times \text{Mot}^k_{\text{loc}}(k) \rightarrow \text{Mot}^k_{\text{loc}}(k) \quad (S, NM) \mapsto S \otimes NM.
\]

Consider the distinguished triangle of spectra \( S \xrightarrow{n} S \rightarrow S/n \rightarrow \Sigma S \), where \( S \) stands for the sphere spectrum. Since \( S/n \otimes U^k_{\text{loc}}(A) \) identifies with the mod-\( n \) Moore object of \( U^k_{\text{loc}}(A) \) and we have an isomorphism \( S/n \simeq S/l_1^{n'} \oplus \cdots \oplus S/l_r^{n'} \) in Spt, we then conclude from Example 3.9 (with \( E = U^k_{\text{loc}} \)) that

\[
U^k_{\text{loc}}(A \otimes L_n) \simeq U^k_{\text{loc}}(A \otimes L_1^{n'}) \oplus \cdots \oplus U^k_{\text{loc}}(A \otimes L_r^{n'})
\]

The proof follows now automatically from the fact that \( U^k_{\text{loc}} \) is the initial \( k \)-homotopy invariant; see [14, Thm. 8.25].

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