Ergoregion instabilities and vortex stability in dilute Bose–Einstein condensates

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Guided by the analogy with gravitational physics, we investigate the origin of the dynamical instability of multiply quantized vortices in spatially homogeneous atomic Bose–Einstein condensates. A careful analysis of the role of boundary conditions at large distance in the linearized Bogoliubov problem shows that this is a dispersive version of the ergoregion instability of rotating spacetimes with respect to scalar field perturbations. Different mechanisms that suppress the instability in specific regimes are identified, either via interference effects in finite-size condensates or because of the superluminal dispersion of Bogoliubov modes in high angular momentum channels.

Since the pioneering work by Penrose [1] it has been known that in the so-called ergoregion surrounding rotating black holes, particles can have negative energies with respect to an asymptotic observer. As a consequence, (positive) energy can be extracted from a rotating black hole by accumulating negative energy there. Similarly, radiation incident on a rotating black hole can be amplified during reflection at the expense of a negative energy partner being transmitted into the ergoregion: this phenomenon is known as rotational superradiance [2].

This is an energetic instability of the rotating black hole that can become a dynamical one under suitable boundary conditions. Amplified superradiant scattering from a dynamically stable system is found if perturbations are free to propagate away both in the asymptotic outward direction and in the inward one across the horizon. On the other hand, superradiance can get self-stimulated in the presence of sizable reflections on either side, which gives rise to radiation modes with complex frequencies, corresponding to exponentially growing perturbations. If reflection occurs on the inside, e.g. in a rotating spacetime displaying an ergoregion but no horizon, this dynamical instability is called ergoregion instability [3, 4]. If instead reflection occurs in the external space away from the black hole, a black hole bomb is found, e.g. for rotating black holes in a AdS background [5, 6].

Since these phenomena only depend on the propagation of fields in a curved background, they directly transfer to condensed matter analog models of gravity, where collective excitations modes propagate on top of a moving medium according to the same equations as a massless scalar field on a curved spacetime [7]. Superradiant scattering has been widely studied in these systems [8–12] and has been recently observed using surface gravity waves on top of a water flow configuration displaying a draining vortex [13]. Also ergoregion instabilities were theoretically studied in purely rotating vortex configurations within the hydrodynamic approximation [14, 15].

Because of their exceptional control and flexibility, atomic Bose–Einstein condensates (BECs) [16] are among the most promising systems to realize analogue models of gravity: thanks to the ultracold temperature, quantum features are visible, allowing the investigation of very quantum aspects of the physics of fields in curved spacetimes. As a most celebrated example, pioneering results on the observation of analog Hawking radiation were recently reported [17]. Conversely, the gravitational analogy can also shed new light on the basic physics of condensates: schemes to exploit analog Hawking processes to entangle collective phonon modes were proposed in [18] and a one-dimensional superfluid flow instability was reinterpreted in [19] as a black hole lasing effect.

In this Letter we adopt this latter perspective to revisit the problem of the stability of vortices in BECs, much studied in trapped geometries [20–27] and less understood in spatially homogeneous ones [27, 28], and link it to the gravitational physics of rotating spacetimes. Vortices in BECs display an irrotational flow pattern with an azimuthal velocity \( v_\theta \propto 1/r \), which becomes supersonic in the vicinity of the vortex core. While this suggests the possibility of superradiant instabilities, the sudden density drop in the vortex core and the superluminal dispersions of collective excitations prevent a straightforward use of the hydrodynamic approximation at the basis of the gravitational analogy.

Going beyond these limitations, here we carry out a microscopic study of the Bogoliubov collective excitations around vortices of different charges in infinite and spatially homogeneous two-dimensional BECs. In addition to the confirmation that singly quantized vortices are stable while doubly (and higher) quantized ones are dynamically unstable, our analysis finally settles the long-standing debate on the nature of the instability [23, 27, 28], showing that it is of the ergoregion rather than of the black hole bomb type, a distinction that cannot be clearly made in the presence of a trap. This also shows that these gravitational instabilities are robust against the superluminal shape of the Bogoliubov dispersion and offers a deeper physical understanding of the known results for non-uniform condensates.

Vortices and the linear problem – Atomic BECs are well described at the mean field level by the Gross–Pitaevskii equation (GPE) for a scalar classical field de-
scribing the order parameter $\Psi(r, t)$ [16],

$$i\hbar \partial_t \Psi = \left[ -\frac{\hbar^2 \nabla^2}{2M} + g |\Psi|^2 + V_{\text{ext}} \right] \Psi, \quad (1)$$

where $g$ is the interparticle interaction constant and $M$ is the atomic mass. We consider two spatial dimensions, that is a good approximation for a pancake-shaped condensate tightly confined in the third direction. If needed, $V_{\text{ext}}(r)$ is a cylindrically-symmetric external trapping potential. We focus on vortices located at the origin of the coordinates $r = 0$ and described by cylindrically-symmetric stationary solutions of (1) of the form $\Psi_\ell(r, t) = f(r) \exp(i\ell \theta) \exp(-i\mu t/\hbar)$ with an amplitude $f(r) \rightarrow f_\infty$ at large distances and the chemical potential $\mu = g n_\infty = g |f_\infty|^2$. For the order parameter to be single valued, the vortex circulation must be quantized to integer values $\ell$. The current of a vortex of charge $\ell$ becomes supersonic at a radius $r_E \sim \xi$, with $\xi = \hbar/(mg f_{\infty})^{1/2}$ being the so called healing length.

To study the stability of these solutions one can adopt the Bogoliubov approach and linearize the GPE around the stationary state $\Psi_\ell$ [29]. To this purpose, we consider the (small) perturbation $\delta \Psi(r, \ell)$ and its complex conjugate $\delta \Psi^*$ as independent variables and group them into the spinor $(\delta \Psi^\dagger, \delta \Psi)^T$. Given the cylindrical symmetry of the problem, we decompose the perturbation in its angular momentum $m$ components,

$$\left( \begin{array}{c} \delta \Psi \\ \delta \Psi^* \end{array} \right) (r, \theta, t) = e^{im\theta} \left( \begin{array}{c} e^{i\ell \theta} e^{-i\mu t/\hbar} u_\ell(r, t) \\ e^{-i\ell \theta} e^{i\mu t/\hbar} v_\ell(r, t) \end{array} \right). \quad (2)$$

The time evolution for the spinor $|\phi\rangle := (u_\phi, v_\phi)^T$ is given by the Bogoliubov–de Gennes (BdG) equation

$$i\hbar \partial_t |\phi\rangle = \mathcal{L}_{\ell, m} |\phi\rangle \quad (3a)$$

$$\mathcal{L}_{\ell, m} = \begin{bmatrix} D_+ + V_{\text{ext}} + 2g f^2 - \mu & g f^2 \\ -g f^2 & -(D_- + V_{\text{ext}} + 2g f^2 - \mu) \end{bmatrix}$$

with

$$D_{\pm} = \frac{\hbar^2}{2M} \left( -\partial_r^2 - \frac{\partial_r}{r} + \frac{(\ell \pm m)^2}{r^2} \right). \quad (3b)$$

The evolution (3) is pseudo-unitary, so that the conserved inner product and norm

$$\langle \psi | \sigma_3 | \psi \rangle = \int dr \left[ u_\psi^*(r) u_\psi(r) - u_\psi^*(r) v_\psi(r) \right]$$

are non-positive definite. The energy of an eigenmode $|\psi_i\rangle$ of the BdG matrix (3b) with frequency $\omega_i$ is given by $E_i = \langle \psi_i | \sigma_3 | \psi_i \rangle \hbar \omega_i$, so that, for example, negative-norm modes with a positive frequency have negative energy. Besides pairs of particle-hole symmetric pairs of positive and negative norm modes also zero-norm modes may exist. The symmetries of the BdG matrix impose that $(\omega_i - \omega_j^*) \langle \psi_j | \sigma_3 | \psi_i \rangle = 0$ holds for eigenmodes and thus unstable complex frequency modes have zero norm: in physical terms, the exponential growth of unstable modes corresponds to the simultaneous creation of particles and antiparticles with opposite energies, thus leaving the energy unchanged. While in the long-distance hydrodynamic limit the BdG equations recover a Klein–Gordon equation in curved spacetime [7] and share many of its spectral properties, they differ for the superluminal dispersion at high momenta.

Vortices in trapped BECs – In an infinite condensate, a charge $\ell$ vortex has a greater energy than an array of $\ell$ singly charged vortices [30]: this means that multiply quantized vortices are energetically unstable.

Extensive studies with the Bogoliubov approach [20, 21] have shown that vortices in harmonically trapped condensates – even singly quantized ones – are always energetically unstable since they possess a negative energy $m = 1$ mode localized around the vortex core, corresponding to precession around the trap center. However, actual spiraling of the vortex out of the condensate requires some energy dissipation mechanism, for example via interaction with thermal atoms [31]. As a result, the vortex remains dynamically stable under the purely conservative dynamics (3). In contrast, multiply quantized ones display alternate intervals of dynamical instability and stability as the nonlinear interparticle interaction is varied with respect to the trap frequency [22].

Even though the literature agrees on the occurrence of this instability and unambiguous experimental evidence is available [25], the situation is much less clear for what concerns the physical origin of the instability. Several authors [28, 32] have ascribed it to the trapping potential, making it the equivalent of a black hole bomb instability, in the sense that the energetic instability of the core is turned to a dynamical one by the reflecting boundary condition at the outer edge of the trapped BEC. In contrast to this statement, we are now going to show that a multiply charged vortex is inherently unstable via an analog of the ergoregion instability, but the instability may be suppressed by a subtle destructive interference effect in suitably small BECs.

A charge 2 vortex in an infinite BEC – As a first step, we follow the path of [27] and investigate the stability of a doubly quantized vortex in an infinite BEC by looking how the spectrum of a finite system of size $R$ evolves in the infinite size limit $R \rightarrow \infty$. To this purpose, we numerically find the radial profile $f(r)$ of the GPE ground state $\Psi_\ell$ with a given circulation $\ell$ on an interval $[0, R]$. In order to mimic a spatially homogeneous BEC, Neumann boundary conditions $\partial_r f|_{r=R} = 0$ are imposed. The BdG spectrum is then obtained imposing Dirichlet boundary conditions at $r = R$ onto the perturbation, $u_\phi(r = R) = v_\phi(r = R) = 0$. The calculation is repeated for growing values of the size $R$.

The resulting discrete spectra of modes are shown
tends to a.

FIG. 1. Real (a) and imaginary (b) parts of the Bogoliubov eigenfrequencies for modes of azimuthal number \( m = 2 \) on a charge \( \ell = 2 \) vortex in a BEC of size \( R \). Black (solid), red (dotted), green (thicker) lines correspond to positive-, negative-, and zero-norm modes. A wider view of the imaginary part is given in panel (d). Here, the \( 1/\sqrt{R} \) dashed line envelopes the instability maxima up to moderate \( R \). The horizontal line indicates the instability rate extracted from the time-dependent simulation with absorbing boundary conditions shown in Fig.2. Panel (c): spatial shape of the core mode for \( R = 50\xi \). The green and red lines respectively show the \( u_\phi \) and \( v_\phi \) components of the Bogoliubov spinor. The dashed line shows the (rescaled) density profile of the vortex. In the left panels of Fig.1 as a function of \( R \) for an \( m = 2 \) perturbation on a charge \( \ell = 2 \) vortex. An energetically unstable mode (with negative norm and positive frequency) is clearly visible in panel (a) at an (almost) \( R \)-independent frequency around \( 0.44 \mu /\hbar \). This \( R \)-independence is a strong indication that the mode is localized in the core region, which is further verified in the exponential decay of the envelope of \( u_\phi \) and \( v_\phi \) (panel (c)); the oscillations are due to the interference of out-going waves with their reflection at \( r = R \). Given the symplectic \([33]\) nature of the Bogoliubov problem, the crossing of the negative norm core mode with one of the positive norm collective modes of the condensate gives rise to dynamically unstable zero-norm modes \([26]\) with the characteristic instability bubble visible in Fig.1(b).

Looking at Fig.1(d), an important distinction between the moderate-\( R \) and large-\( R \) regimes jumps to the eyes. In the former case, the positive-norm collective modes of the condensate are well distinct in energy and the stability islands (instability peaks) are well separated and occur whenever the phase of the reflected waves at the \( r = R \) boundary destructively (constructively) interferes with the oscillation of the core mode. In contrast, for large \( R \) the energy spacing between collective modes (at a given energy) decreases as \( 1/R \), faster than the \( 1/\sqrt{R} \) one of the instability rate maxima for moderate-\( R \) (dashed line in Fig.1(d)). As a result, neighboring instability bubbles end up merging with each other and instability rate tends to a \( R \)-independent value. Since the energy spacing corresponds to the inverse round-trip time from the core to the \( r = R \) boundary and back, this means that the instability develops so quickly that it does not have the time to feel the presence of the boundary.

While this way of taking the infinite-size limit may seem a sound one, one must not forget that the eigenmodes of finite systems have a standing-wave-shape and necessarily involve a reflected in-going wave. As a consequence, the spectrum of a closed system is generally very different (even in the infinite-size limit) from the one in the asymptotic outgoing boundary conditions case. Because of this crucial difference, well highlighted for the Klein–Gordon case in \([34]\), it is thus essential to put any conclusion on the infinite system on solid grounds by implementing radiative boundary conditions where all reflected waves are removed from the outset.

In Fig.2 we summarize a numerical study of time-dependent BdG equations (3) where such radiative boundary conditions are implemented by adding an effective absorption at large distances. A series of snapshots of the evolution of a \( m = 2 \) Gaussian perturbation (upper-left panel) incident on a \( \ell = 2 \) vortex core are shown. After a transient in which the perturbation wavepacket is reflected by the vortex core and then propagates away from it (upper-right panel), the dynamics is characterized by the unstable core mode being amplified while maintaining a constant shape (bottom panels) with an excitation current propagating to infinity. Compared to Fig.1(c), the outgoing boundary conditions remove the interference-induced oscillations at large distance, leaving only the exponential spatial decay. The temporal growth of the core mode can be precisely fitted with an exponential law (not shown) of instability rate \( 3(\omega) \approx 0.00242 \mu /\hbar \), indicated by the horizontal line in Fig.1(d) and in perfect agreement with the one found for

FIG. 2. Snapshots of the time evolution of a \( m = 2 \) perturbation scattering on a \( \ell = 2 \) vortex. Green and red lines respectively show the \( u_\phi \) and \( v_\phi \) components of the Bogoliubov spinor. Outgoing boundary conditions are imposed by including a wide and smooth imaginary potential of Gaussian spatial shape centered at the edge of the integration box \( r = 700\xi \), of variance \( 120\xi \) and amplitude \( 0.15\mu \), so to effectively absorb the perturbation spinor \( \phi \) and suppress the reflected waves.
the closed system in the $R \to \infty$ limit. In analogy to the superradiant emission from population-inverted atoms, this instability can thus be understood in terms of the parametric emission of pairs of Bogoliubov quanta into a (propagating) collective BEC excitation and the (localized) core mode. Our rigorous way of directly dealing with an infinite system confirms the conclusions of [27] and, furthermore, offers a physical understanding of the validity of their infinite-size-limit procedure.

**Higher charge vortices** – Based on this important result, we are now entitled to apply the infinite-size-limit procedure to more general cases, starting from $\ell > 2$ charge vortices. As an example, we display in Fig.3 the $\ell$-dependence of the different-$m$ spectra for a given charge $\ell = 4$. Independently from $\ell$, the $m = 1$ spectrum always shows a negative-norm mode approaching zero frequency from below as $R \to \infty$. This core mode corresponds to the zero energy mode found in [35] and associated to the translation of the vortex core. Its energetic stability is to be contrasted to the energetic instability of vortex translation in harmonically trapped BEC [20], whose inverted-parabola-shaped density profile favours expulsion of the vortex. Since the energetically unstable negative-norm mode is always located below the lowest positive-norm collective mode, the instability does not become dynamical. Quite unexpectedly, if the density profile showed a bump surrounded by a wide region of lower density, collective modes satisfying the resonance condition may be available for any $\ell$, leading to the instability of even singly-charged vortices (not shown).

While the $m = 1$ mode is the only core mode for $\ell = 1$ vortices, for larger $\ell$ other negative norm core modes appear for increasing $m$ at both negative and positive BdG frequencies, corresponding thus to positive and negative energies. Interestingly, the energy of the (lowest energy) core mode decreases until $m = \ell$ and then starts increasing again until it becomes positive and dynamical stability is recovered. Since they result from negative norm modes crossing the positive norm ones, dynamical instabilities are found in a finite $2 \leq m \leq 2\ell - 2$ range and not only for $m \leq \ell$ as claimed in [22] or for all $m$ as claimed in [26]. Anyway, their rate is strongest for $m = 2$ and then decreases with $m$. Since the instability of multiply charged vortices is associated to their splitting into an assembly of $\ell$ singly charged vortices, one may have expected the most unstable mode to be at $m = \ell$. Actually, it turns out that the vortex decay begins with lower-$m$ deformations of the core and the splitting in $\ell$ parts may start dominating only during the nonlinear stages of the later dynamics.

Finally, it is interesting to compare our results to the recent work [14] carried out for a purely hydrodynamic system for which the gravitational analogy holds exactly. By imposing a reflecting boundary condition at a finite radius, it was found that for a given size of the ergoregion all the high-enough $m$ modes are dynamically unstable, but the instability is stronger for the lower $m$ unstable modes. While this hydrodynamic result is fully recovered by our Bogoliubov calculation, we also find an upper bound on the unstable $m$ values. Whereas the overall similarity confirms that the nature of the instability is the same in the two cases, the minor yet remarkable differences can be ascribed to the microscopic density profile of the vortex core and, even more importantly, to the superluminal Bogoliubov dispersion that suppresses the instability at large (angular) momenta. Further evidence in support of this latter statement is provided by calculations for a dispersive version of the Schiff–Snyder–Weinberg effect [36], where a similar instability is also found [37] for a spatially homogeneous density.

**Conclusions.** – In this Letter we have investigated the physical origin of the intrinsic instability of multiply ($\ell \geq 2$) charged vortices in spatially infinite atomic BECs. Similarly to rotating spacetimes displaying an ergoregion but no horizon in gravitational physics, this dynamical instability of BECs is due to the presence of negative energy modes localized in the vortex core and resonantly coupled to outward propagating collective modes.
of positive energy. In stark contrast to the black hole bomb effect determined by the properties of the external space, the only role of the trapping potential away from the vortex core is to modulate the instability rate via an interference mechanism and even completely suppress the instability for suitable trap sizes. As a corollary of our theory, exotic geometries where even singly quantized vortices are dynamically unstable were identified. Well beyond atomic BECs, our conclusions are directly applicable to generic quantum fluids showing vortex excitations, in particular quantum fluids of light [38]. From the analogue gravity point of view, our calculations show the robustness of superradiant phenomena against superluminal shape of the Bogoliubov dispersion and confirms that the gravitational analogy provides qualitatively correct results.

In addition to on-going work on the dispersive Schifff–Snyder–Weinberg effect that allows for additional insight on ergoregression instabilities in simple geometries [36], future studies will address ergoregion instabilities in two-component condensates [39] and the consequences of a radial flow, which may quench the instability by dragging the localized core mode away in the inward direction. Going beyond mean-field, one can anticipate that the superradiant scattering of quantum fluctuations off the ergoregion will result in a spontaneous pair production similar to the Hawking effect process and in the ergoregion will result in a spontaneous pair production [40] similar to the Hawking effect process and in peculiar geometric signatures on the correlation function [40] similar to the Hawking effect process and in the ergoregion will result in a spontaneous pair production [40] similar to the Hawking effect process and in peculiar geometric signatures on the correlation function [40] similar to the Hawking effect process and in peculiar geometric signatures on the correlation function [40] similar to the Hawking effect process and in peculiar geometric signatures on the correlation function [40] similar to the Hawking effect process and in peculiar geometric signatures on the correlation function [40] similar to the Hawking effect process and in peculiar geometric signatures on the correlation function. Going beyond mean-field, one can anticipate that the superradiant scattering of quantum fluctuations off the ergoregion will result in a spontaneous pair production.

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