Gâteaux and Fréchet derivatives of the operator of geometrically nonlinear bending problem of sandwich plate

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Abstract. The geometrically nonlinear problem of bending of sandwich plates with a transversally soft core in a one-dimensional statement is considered. Mathematically, the problem is formulated as an integral identity generating an operator equation in the Sobolev space. The Gâteaux derivative of the operator is calculated and it is proved that it coincides with the Fréchet derivative.

1. Introduction
Layered structures, in particular, sandwich plates and shells (Fig. 1), are used in various fields of technology (aircraft manufacturing, shipbuilding, etc.). Three-layer structures have many qualities that conventional structures made of metal alone do not have. They have high specific stiffness and can withstand high specific loads. Layered plates and shells have good heat and sound insulation qualities, damping vibration-absorbing properties. [1–8].

Figure 1. Sandwich plate with transversely soft core. The cores: 1 – foam; 2 – corrugation; 3 – honeycomb.

This paper is devoted to finding the Gâteaux derivative [9] of an operator of a geometrically nonlinear bending problem for a sandwich plate with a transversally soft core, formulated as an operator equation in Sobolev space. It is established that the Gâteaux derivative of an operator is a continuous operator, whence it follows that the Gâteaux derivative coincides with the Fréchet derivative [9, 10]. Generalized statements of physically nonlinear and geometrically linear problems are considered in [11–14]. The nonlinear problems of the shells and the theory of soft net shells were studied in [15–25]. The numerical solution of geometrically nonlinear problems was carried out in [26–29].
2. Problem statement

We study the problems of determining the stress-strain state of infinitely wide sandwich plates with transversally soft core. The plate length is equal \( a \), the thickness of the aggregate is \( 2t \), the thickness of the supporting layers is \( 2t(k) \), \( k \) is the layer number. To describe the stress-strain state in the carrier layers, the Kirchhoff-Love model equations are used; in the core, the equations of the elasticity theory, simplified within the accepted model of the transversally soft layer and integrated over the thickness with satisfaction of the conjugation conditions of the layers by displacements [30–32]. We introduce the following notation: (hereinafter we assume that \( k = 1, 2 \)), \( H(k) = t + t(k) \), \( X^1(k) \), \( X^2(k) \) are the components of the surface load reduced to the middle surface of the \( k \)-th layer, \( w^{(k)} \) and \( u^{(k)} \) are the deflections and axial displacements of the points of the middle surface of the \( k \)-th layer, \( T^{11}(k) \), \( M^{11}(k) \) are the membrane forces and internal bending moments in the \( k \)-th layer, respectively. We assume that the edges of the plate are fixed, i.e., \( u^{(k)}(x) = 0, \ w^{(k)}(x) = 0 \), \( d w^{(k)}/dx = 0 \) at \( x = 0, \ x = a \). We consider the geometrically nonlinear case, i.e., \( T^{11}(k) = B(k) \left( du^{(k)}/dx + 0.5(d^2w^{(k)}/dx^2) \right) \), \( M^{11}(k) = D(k) d^2 u^{(k)}/dx^2 \), where \( B(k) = 2t(k) E^{(k)}(1 - v^{12}_1 v^{21}_1) \) is the tensile–compressive stiffness of the \( k \)-th layer, \( E^{(k)} \) and \( v^{12}_1, v^{21}_1 \) are the first kind of elastic modulus and the Poisson’s coefficients of the \( k \)-th layer material, \( D(k) = B(k) h^{2}_{(k)}/3 \) is the flexural stiffness of the \( k \)-th layer. Let \( U = (w^{(1)}, w^{(2)}, u^{(1)}, u^{(2)}) \) be the vector of displacements of points of the middle surfaces of the bearing layers, \( q^1 \) be the tangential stresses in the core. For \( q^1 \), we assume that the boundary conditions \( q^1(0) = q^1(a) = 0 \) are satisfied. Let \( G_{13}, E_3 \) be the transverse shear moduli and the compression of the core \( c_1 = 2t/G_{13}, \ c_2 = t^3/(3E_3), \ c_3 = E_3/(2t) \), \( M(k) \) be the surface moment of external forces, reduced to the middle surface of the \( k \)-th layer.

3. Generalized statement of the problem in the form of an operator equation

Let \( V = W_2^1(0, a) \) be the Sobolev spaces [33] with inner products

\[
(u, \eta)_k = \int_0^a d^k u / dx^k \cdot d^k \eta / dx^k \, dx, \quad k = 1, 2, \quad V = V_2 \times V_2 \times V_1 \times V_1.
\]

We denote the inner product in \( V \) by \((\cdot, \cdot)\). By analogy with (1), by solution of the problem we mean the element \((U, q^1)\) that is a solution the variational equation

\[
b((U, q^1), (Z, y)) = f(Z) \quad \forall (Z, y) \in W = V \times V_1,
\]

where the form \( b(\cdot, \cdot) \) and functional \( f \) given on \( W \times W \) and \( V \) are determined by the formulas

\[
b((U, q^1), (Z, y)) = \sum_{k=1}^a B(k) \left[ \frac{d u^{(k)}}{dx} \right] \left[ \frac{d \eta^{(k)}}{dx} \right] \cdot dx + \sum_{k=1}^a B(k) \left[ \frac{d w^{(k)}}{dx} \right] \left[ \frac{d z^{(k)}}{dx} \right] \cdot dx + \sum_{k=1}^a D(k) \left[ \frac{d^2 w^{(k)}}{dx^2} \right] \left[ \frac{d z^{(k)}}{dx} \right] \cdot dx + c_1 \int_0^a (w^{(2)} - w^{(1)}) (z^{(2)} - z^{(1)}) \, dx + \int_0^a \left[ \frac{d z^{(k)}}{dx} \right] \left( \frac{d z^{(k)}}{dx} \right) \, dx + (\eta^{(2)} - \eta^{(1)}) \right] q^1 \, dx \right.
\]

(2)
+ \left\{ \int_0^a \left[ \sum_{k=1}^{2} H_{(k)} \frac{d w_{(k)}}{d x} \right] + (u^{(2)} - u^{(1)}) + c_1 q_1 \right\} y + c_2 dq' l \, dx \right\} \, dy \, dx = 0 \\
\forall Z = (z^{(1)}, z^{(2)}, \eta^{(1)}, \eta^{(2)}) \in V, \quad \forall y \in V_1,
\end{align*}

\begin{align*}
f(Z) &= \int_0^a \left[ \sum_{k=1}^{2} [X_{(k)} \eta^{(k)}] + M_{(k)} \frac{d z^{(k)}}{d x} \right] + X_{(k)} z^{(k)} \right\} \, dx \\
&\forall Z \in V. \quad (3)
\end{align*}

The form given by (2) is linear and continuous with respect to the second argument, which means it generates an operator $A: W \rightarrow W$ defined by the formula

\begin{align*}
b((U, q^1), (Z, y)) &= (A(U, q^1), (Z, y))_W \\
&\forall (Z, y) \in W,
\end{align*}

where $(\cdot, \cdot)_W$ is the inner product in $W$, and the functional $f$ defined by (3) generates an element defined by the formula $(F, Z)_V = f(Z)$ for all $Z \in V$.

Therefore, problem (1) can be written as an operator equation

\begin{align*}
A(U, q^1) = (F, 0). \quad (5)
\end{align*}

4. **Gâteaux derivative of an operator of an equation**

Recall that if for the operator $A: Y \rightarrow Y$ there exists a limit

\begin{align*}
\lim_{t \rightarrow 0} \| t^{-1} (A(u + t\eta) - A(u)) - VA(u, \eta) \| = 0
\end{align*}

at the point $u \in Y$ for any $\eta \in Y$, then $VA(u, \eta)$ is called the Gâteaux variation at the point $u$ from the operator $A(u)$ (see [9]). In the case when the Gâteaux variation $VA(u, \eta): Y \times Y \rightarrow Y$ is a linear operator with respect to $\eta$, then the Gâteaux variation is called the Gâteaux differential and is denoted by $DA(u, \eta) = A'(u)\eta$, and $A'(u)$ is called the Gâteaux derivative of the operator $A$ at the point $u$.

Let’s calculate the Gâteaux derivative of the operator $A$ defined by (2), (4). We denote $U = (w^{(1)}, w^{(2)}, u^{(1)}, u^{(2)})$, $\hat{U} = (\hat{w}^{(1)}, \hat{w}^{(2)}, \hat{u}^{(1)}, \hat{u}^{(2)})$, $Z = (z^{(1)}, z^{(2)}, \eta^{(1)}, \eta^{(2)})$. Then

\begin{align*}
\lim_{t \rightarrow 0} t^{-1} (A(U + t\hat{U}, q^1 + t\hat{q}^1) - A(U, q^1), (Z, y))_W &= \int_0^a \left[ \sum_{k=1}^{2} B_{(k)} \left( \frac{d u^{(k)}}{d x} + \frac{d w^{(k)}}{d x} \right) \right] \, d \eta^{(k)} \, dx + \\
+ \left[ \sum_{k=1}^{2} B_{(k)} \left( \frac{d u^{(k)}}{d x} + \frac{d w^{(k)}}{d x} \right) \right] \frac{d \eta^{(k)}}{d x} \, dx + \left[ \sum_{k=1}^{2} B_{(k)} \left( \frac{d w^{(k)}}{d x} + \frac{d \eta^{(k)}}{d x} \right) \right] \frac{d \eta^{(k)}}{d x} \, dx + \\
+ \left[ \sum_{k=1}^{2} H_{(k)} \frac{d z^{(k)}}{d x} \right] + \left( \eta^{(2)} - \eta^{(1)} \right) \hat{q}^1 \, dx + \left[ \sum_{k=1}^{2} H_{(k)} \frac{d \hat{w}^{(k)}}{d x} \right] + (\hat{u}^{(2)} - \hat{u}^{(1)}) \right\} \, dx + \\
&+ c_1 \hat{q}^1 \, dx + c_2 \frac{d \hat{q}^1}{d x} \, dx = (DA(U, q^1), (\hat{U}, \hat{q}^1), (Z, y))_W.
\end{align*}

Thus, for all $(U, q^1), (\hat{U}, \hat{q}^1), (Z, y)$ from $W$ we have

\begin{align*}
\lim_{t \rightarrow 0} t^{-1} \left( A(U + t\hat{U}, q^1 + t\hat{q}^1) - A(U, q^1) - DA(U, q^1), (\hat{U}, \hat{q}^1), (Z, y) \right)_W = 0. \quad (7)
\end{align*}

By the corollary of the Han–Banach theorem (see [34], Theorem 2.7.4), we can choose a unit vector $(Z, y)$ from $W$ such that
(t^{-1}\left((A(U + t\hat{U}, q^1 + t\hat{q}^1) - A(U, q^1)) - DA((U, q^1), (\hat{U}, \hat{q}^1)), (Z, y)\right))_w = 
= \left\| t^{-1}\left((A(U + t\hat{U}, q^1 + t\hat{q}^1) - A(U, q^1)) - DA((U, q^1), (\hat{U}, \hat{q}^1))\right)\right\|_w.

From here and from the relation (7) it follows that
\lim_{t \to 0} \left\| t^{-1}\left((A(U + t\hat{U}, q^1 + t\hat{q}^1) - A(U, q^1)) - DA((U, q^1), (\hat{U}, \hat{q}^1))\right)\right\|_w = 0.

It is easy to see that the operator \( DA((U, q^1), (\hat{U}, \hat{q}^1)) \) is linear in \( (\hat{U}, \hat{q}^1) \), and therefore \( DA((U, q^1), (\hat{U}, \hat{q}^1)) = A'(U, q^1)(\hat{U}, \hat{q}^1) \), where \( A'(U, q^1) \) is the Gâteaux derivative of the operator the point \( (U, q^1) \). Thus, the following theorem holds.

**Theorem 2.** Let an operator \( A \) be generated by relations (2), (4). Then it is differentiable everywhere according to Gâteaux, its Gâteaux derivative is defined by the relation (6).

**5. Fréchet derivative of an operator of an equation**

Recall [9, 10] that if at the point \( u \in Y \) for the operator \( A: Y \to Y \) the relation \( A(u + \eta) - A(u) = dA(u, \eta) + o(\eta) \) holds, where \( dA(u, \eta) \) is the linear operator of \( \eta \) and \( \lim_{\eta \to 0} o(\eta) / | \| \eta ||| = 0 \) then \( dA(u, \eta) \) is called the Fréchet differential of the operator \( A(u) : Y \to Y \) at the point. If \( dA(u, \eta) \) is a bounded operator of \( \eta \), then \( dA(u, \eta) = A'(u) \eta \), and \( A'(u) \) is called the Fréchet derivative of the operator \( A \) at the point \( u \). We prove that the following theorem is true.

**Theorem 3.** Let the operator \( A \) be generated by relations (2), (4). Then its Gâteaux derivative is a continuous operator.

**Proof.** For all \((U, q^1), (\hat{U}, \hat{q}^1), (Y, r)\) and any unit vector \((Z, y)\) from \( W \), using the Sobolev embedding theorem and the generalized Hölder inequality, we obtain
\[
| (A'(U, q^1)(\hat{U}, \hat{q}^1) - A'(Y, r)(\hat{U}, \hat{q}^1), (Z, y))_w | \leq C^R \| U - Y \|_V \| \hat{U} \|_V.
\] (8)

By the corollary of the Han–Banach theorem ([34, Theorem 2.7.4]), we can choose a unit vector \((Z, y)\) from \( W \) such that
\[
(A'(U, q^1)(\hat{U}, \hat{q}^1) - A'(Y, r)(\hat{U}, \hat{q}^1), (Z, y))_w = \| A'(U, q^1)(\hat{U}, \hat{q}^1) - A'(Y, r)(\hat{U}, \hat{q}^1), (Z, y) \|_w .
\]
\[
\| A'(U, q^1)(\hat{U}, \hat{q}^1) - A'(Y, r)(\hat{U}, \hat{q}^1), (Z, y) \|_w \leq C^R \| U - Y \|_V \| \hat{U} \|_V.
\]

From here and from the relation (8) it follows that
\[
\| A'(U, q^1)(\hat{U}, \hat{q}^1) - A'(Y, r)(\hat{U}, \hat{q}^1), (Z, y) \|_w \leq C^R \| U - Y \|_V \| \hat{U} \|_V.
\]

Thus, \( A'(U, q^1) \) is a continuous operator. The theorem is proved.

**6. Conclusion**

It is proved that the operator of a geometrically nonlinear problem of bending of a three-layer plate with a transversely soft filler is differentiable according to Gâteaux and its Gâteaux derivative is calculated. It is established that the Gâteaux derivative coincides with the Fréchet derivative. In the future, this property will be used in the study of buckling of the plate and finding the critical load at which a buckling occurs. Approximate methods for solving this problem will be developed based on the approaches developed in [35–46].

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