NORMAL AND ANOMALOUS SCALING OF THE FOURTH-ORDER CORRELATION FUNCTION OF A RANDOMLY ADVECTED PASSIVE SCALAR

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For a delta-correlated velocity field, simultaneous correlation functions of a passive scalar satisfy closed equations. We analyze the equation for the four-point function. To describe a solution completely, one has to solve the matching problems at the scale of the source and at the diffusion scale. We solve both the matching problems and thus find the dependence of the four-point correlation function on the diffusion and pumping scale for large space dimensionality $d$. It is shown that anomalous scaling appears in the first order of $1/d$ perturbation theory. Anomalous dimensions are found analytically both for the scalar field and for its derivatives, in particular, for the dissipation field.

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INTRODUCTION

It is already a commonplace to talk about an anomalous scaling of the high-order correlation functions in developed turbulence. By this, the deviation of the scaling exponents from their “naive” values taken from dimensional estimate or perturbation theory is usually meant. Another meaning ascribed to that term is related to the cases where the exponent of the 2n-th correlation function is not $n$ times the exponent of the second one so that the degree of non-Gaussianity depends on scale. The experimental evidence in favor of anomalous scaling of a scalar field advected by turbulence exists for some time \cite{1,2,3,4} while the attempts of the consistent derivation (starting from the equations of fluid mechanics) of the correlation functions of the order higher than two started only recently \cite{5,6,7,8,9}. The problem of a passive scalar advection, being of physical importance by itself, may serve also as a starting point in studying anomalous scaling in turbulence \cite{10,11}. At a first step, we found \cite{12} the whole set of the simultaneous correlation functions for the Batchelor-Kraichnan problem of a scalar advected by a large-scale random velocity field \cite{13,14}. It has been shown that, whatever be the (finite) temporal correlations of the velocity field, all correlation functions of the scalar are integer powers of a logarithm for all the distances in the convective interval of scales and no anomalous scaling thus appears at the leading terms. The present paper is an account of the next step: we consider multi-scale velocity field with power spectrum. Following Kraichnan \cite{15}, we restrict ourselves by the simplest possible temporal behavior assuming both velocity field and scalar source to be white in time. That leads to a substantial simplification of the analytical description since any simultaneous correlation function of a scalar satisfies closed linear differential equation of the second order (see \cite{16} and below).

In an isotropic turbulence, n-point correlation function depends on $n(n-1)/2$ distances for a dimensionality of space $d > n - 2$. For the pair correlation function, the respective ordinary differential equation could be readily solved for any distance between the points — see \cite{17} and \cite{18} below. The solution is expressed via the flux of a squared scalar $P_2$ and molecular and eddy diffusivity, the scaling exponent $c_2$ in the inertial interval being fixed by the condition of the flux constancy. If the scaling exponent of the 2n-th correlation function is $n c_2$, that is called a normal scaling. An anomalous scaling would mean that the true answer has additional factors like $(l/r)^\Delta$ where $\Delta$ is anomalous dimension, $r$ is the distance between points and $l$ is some length parameter. One may imagine three reasons for anomalous scaling according to the three lengths that may be relevant: i) $l$ is the pumping scale $L$ so that the anomalous scaling appears due to infrared non-locality, ii) $l$ is the diffusion scale $r_d$ so that the anomalous scaling appears due to non-locality \cite{19,20,21} and iii) anomalous scaling appears due to the existence of the high-order integrals of motion \cite{22} so that the length parameter appears from the ratio of the different fluxes. First nontrivial object that may reveal anomalous scaling is the fourth-order correlation function. One may be interested in two-point objects like $(\partial_1 - \partial_2)^4$ or pair correlation function $\langle \epsilon_1 \epsilon_2 \rangle$ of the dissipation field $\epsilon(t, \mathbf{r}) = \kappa [\nabla \theta(t, \mathbf{r})]^2$ yet they do not satisfy any closed equation. To find such two-point-fourth-order objects, one should solve the complete equation for the four-point function and then fuse some points. Considering $d > 2$, one has to deal with the space of six variables which makes the direct solution at arbitrary parameters quite difficult. Gawedski and Kupiainen \cite{23} recently developed the perturbation theory that describes the effect of the weak advection on the fourth-order correlation function for a non-cascade diffusion-like regime and found the
respective anomalous exponents. Our main target here is the description of the cascade in the convective interval of scales, we solve the equation for the four-point correlation function assuming the space dimensionality d to be large. Non-gaussian part of the correlation function is small in parameter 1/d which makes it possible to develop a regular perturbation theory. In the present paper, we obtain the scaling exponents analytically in the first order in 1/d (assuming 1/d to be the smallest parameter, in particular, 1/d ≪ ζ2) and show that the main contribution into the general fourth-order correlation function has L-related anomalous scaling with

\[ \Delta = \frac{4(2 - \zeta_2)}{d}. \]  

(0.1)

Note that Δ turns into zero when ζ2 → 2 which contradicts Kraichnan’s closure [6,7] and is in a qualitative agreement with Gawedski and Kupiainen result [10] even though they considered a different limit.

For the correlation functions between the points separated by some distances from the convective interval and by others from a diffusion interval of scales, we show that the main term does not depend on r_d. We also show that the subleading terms in the case of some distances being much smaller than others necessarily have another anomalous exponents with respect to small distances. Such subleading terms determine, in particular, correlations of the spatial derivatives of the scalar field, which thus have an r_d-related anomalous scaling (in addition to a general L-related scaling). Note that the results on r_d-dependence are obtained without 1/d-expansion so they are exact at any dimension. For example, the dissipation field has an r_d-related anomalous scaling with the exponent that is equal to the scaling exponent of the eddy diffusivity so that the irreducible correlator \( \langle (\epsilon_1 \epsilon_2) \rangle \propto (L/r_{12})^{2} r_d^d \) does not depend on the diffusion scale r_d when the distance r_{12} is in the convective interval. Another consequence of our results is the statement that \( \langle \theta^2 \rangle / \langle \theta \rangle^2 \) tends to infinity with Peclet number increasing. Contrary, the one-point statistics of \( \theta \) (say, the flatness \( \langle \theta^4 \rangle / \langle \theta^2 \rangle^2 \)) is independent of the Peclet number \( P_e = L/r_d \) at the limit of large \( P_e \).

Besides, we find the nontrivial r_d-related anomalous scaling describing the traceless tensor \( \nabla \alpha \theta \nabla \beta \theta - d^{-1} \delta_{\alpha \beta} (\nabla \theta)^2 \). We also argue that high-order integrals of motion may influence scaling if the pumping is non-Gaussian and that the different kinds of L-related anomalous scaling may be the case starting from the sixth correlation function.

The anomalous L-related scaling of the fourth-order correlation function of the scalar field confronted with the diagrammatic analysis that shows no infrared divergences at any finite order of a Wyld diagram technique [11] evidently means that such an anomalous scaling is a non-perturbative phenomenon.

The structure of the article is as follows. We formulate the problem and find the pair correlation function in Section 1 the second section is devoted to L-dependence while the third section to r_d dependence of the fourth-order correlation functions. The correlation functions of the scalar derivatives are also considered in the third section. Section 2 describes possible generalizations and Conclusion summarizes the results.

1. FORMULATION OF THE PROBLEM

We consider the advection of a passive scalar field \( \theta(t, r) \) by an incompressible turbulent flow. The advection is governed by the following equation

\[ (\partial_t - \bar{P}) \theta = \phi, \]  

(1.1)

\[ \bar{P}(t) = -u^\alpha \nabla^\alpha + \kappa \Delta, \quad \nabla^\alpha u^\alpha = 0, \]  

(1.2)

where both convective and diffusive terms are included in the operator \( \bar{P}(t) \). \( \Delta \) designates Laplacian here. The external velocity \( u(t, r) \) and the external source \( \phi(t, r) \) are random functions of \( t \) and \( r \). We regard the statistics of the velocity and of the source to be independent. Therefore all correlation functions of \( \theta \) are to be treated as averages over both statistics. Averaging over pumping will be designated by over-line and averaging over velocity will be designated by angular brackets.

A. Basic Relations

We assume that the source \( \phi \) is \( \delta \)-correlated in time and spatially correlated on a scale \( L \). The latter means e.g. that the pair correlation function of the source

\[ \phi(t_1, r_1) \phi(t_2, r_2) = \delta(t_1 - t_2) \chi(r_{12}), \]  

(1.3)

as a function of the argument \( r_{12} \equiv |r_1 - r_2| \) decays on the scale \( L \). The value \( \chi(0) = P_2 \) is the production rate of \( \theta^2 \). Since the pumping is Gaussian then high-order correlation functions are determined by \( \chi(r) \). For example,

\[ \phi(t_1, r_1) \phi(t_2, r_2) \phi(t_3, r_3) \phi(t_4, r_4) = \chi(r_{12}) \chi(r_{34}) \delta(t_1 - t_2) \delta(t_3 - t_4) \]  

\[ + \chi(r_{13}) \chi(r_{24}) \delta(t_1 - t_3) \delta(t_2 - t_4) \]  

\[ + \chi(r_{14}) \chi(r_{23}) \delta(t_1 - t_4) \delta(t_2 - t_3). \]  

(1.4)

A formal solution of (1.1) is

\[ \theta(t, r) = \int_{-\infty}^{t} dt' \tilde{U}(t, t') \phi(t', r), \]  

(1.5)

\[ \tilde{U}(t, t') = T \exp \left( \int_{t'}^{t} \bar{P}(\tau) d\tau \right) \equiv \]  

(1.6)

\[ \sum_{n=0}^{\infty} \int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2 \ldots \int_{t'}^{\tau_{n-1}} d\tau_n \bar{P}(\tau_1) \ldots \bar{P}(\tau_n). \]
Since $\theta$ and $\phi$ are related linearly then the pair product of the passive scalar field averaged over the statistics of pumping is expressed via the pair correlation function of the pumping
\[
\theta(t, \mathbf{r}_1)\theta(t, \mathbf{r}_2) = \int_{-\infty}^{t} d\tau \hat{U}_1(t, \tau)\hat{U}_2(t, \tau)\chi(r_{12})
\]
\[
= \int_{-\infty}^{t} d\tau T\exp\left(\int_{\tau}^{t} [\hat{P}_1(\tau) + \hat{P}_2(\tau)]d\tau\right)\chi(r_{12}).
\] (1.7)

Here and below $\hat{U}_i, \hat{P}_i$ designate operators acting on variables $\mathbf{r}_i$. Time-ordered exponents in the r.h.s. of (1.7) commute with each other (because they have different space arguments), that allows to rewrite in (1.7) their product as a single $T$ exp.

The next step is to average $T\exp$ in (1.7) with respect to statistics of the velocity field $\mathbf{u}$. Following Kraichnan, we consider the case of a velocity delta-correlated in time but multi-scale in space. Velocity statistics is completely determined by the pair correlation function
\[
\langle u^\alpha(t_1, \mathbf{r}_1)u^\beta(t_2, \mathbf{r}_2) \rangle = \delta(t_1 - t_2)V^{\alpha\beta},
\]
\[
V^{\alpha\beta} = V_0 \delta^{\alpha\beta} - \kappa^{\alpha\beta}(r_{12}), \quad \kappa^{\alpha\beta}(0) = 0.
\] (1.8)

Here the so-called eddy diffusivity is as follows
\[
\kappa^{\alpha\beta} = \frac{D}{\gamma}r^2\delta^{\alpha\beta} - \frac{D(d-1)}{2-\gamma}r^{\alpha\beta}r^{2-\gamma}.
\] (1.9)

where $0 < \gamma < 2$ is supposed and $\langle \ldots \rangle$ stands for an average over the velocity field statistics, isotropy being assumed. The representation (1.7) is valid for the scales less than velocity infrared cut-off $L_u$, which is supposed to be the largest scale of the problem. The marginal “logarithmic” $\gamma = 0$ and “diffusive” $\gamma = 2$ regimes require a special care.

**B. Simultaneous pair correlator**

Let us calculate the simultaneous pair correlation function of the passive scalar
\[
f(r_{12}) = \langle \theta(\mathbf{r}_1)\theta(\mathbf{r}_2) \rangle. \quad (1.10)
\]

Since we assume the statistics of the velocity field to be delta-correlated and Gaussian, then the $n$-th order correlator of the velocity field [which appears in the expansion of the $T$ exp from the integrand of (1.7)] can be found explicitly by the Wick theorem which prescribes to reduce any average to a product of pair correlation functions. Those terms are summed up into a usual operator exponent
\[
\langle T\exp\left(\int_{\tau}^{t} [\hat{P}_1(\tau) + \hat{P}_2(\tau)]d\tau\right) \rangle = e^{(t-\tau)\hat{L}_{12}}, \quad (1.11)
\]
\[
\hat{L}_{ij} = V^{\alpha\beta}(r_{ij})\nabla^\alpha\nabla^\beta + \left(\frac{V_0}{2} + \kappa\right)(\Delta_i + \Delta_j), \quad (1.12)
\]

where $r_{ij} = r_i - r_j$ and $\Delta_i = \nabla_i^2$. We find from (1.7)
\[
f(r_{12}) = \int_{-\infty}^{t} d\tau \exp(\langle t-\tau \hat{L}_{12} \rangle)\chi(r_{12}). \quad (1.13)
\]

Integrating the r.h.s. of (1.13) with respect to time ($\hat{L}_{ij}$ is a well defined integral operator) we get $\hat{L}_{12}f(r_{12}) = -\chi(r_{12})$ which is an ordinary second-order differential equation
\[
\hat{L}^{(p)}f = -\chi, \quad \partial_r \equiv \partial / \partial r \quad (1.14a)
\]
\[
\hat{L}^{(p)} = \frac{(d-1)D}{2-\gamma}r^{1-d}\partial_r(r^{2-\gamma} + r^{2-\gamma}_d)\partial_r^{d-1}r, \quad (1.14b)
\]

previously derived by Kraichnan [13]. In (1.14b) the diffusion scale $r_d$ was introduced
\[
r^{2-\gamma}_d = \frac{2\kappa(2-\gamma)}{D(d-1)}. \quad (1.15)
\]

Equation (1.14) is integrated explicitly. The solution is completely determined by two physical boundary conditions: zero at $r = \infty$ and absence of a singularity at $r = 0$. We consider the pumping correlation function close to a step function $\chi(r) = P_2$ at $r < L$ and zero otherwise:
\[
f(r) = P_2 \int_{r}^{\infty} g(r')dr', \quad (1.16)
\]
\[
at \quad r > L \quad g(r) = \frac{L^d}{2\kappa d^d + (r/r_d)^{2-\gamma}},
\]
\[
at \quad r < L \quad g(r) = \frac{r}{2\kappa d^d + (r/r_d)^{2-\gamma}}.
\]

For a general $\gamma$, the pair correlation function is expressed via a transcendental function
\[
f(r) = \frac{P_2 r^d}{d(d-1)} \left\{ \frac{\pi}{\sin[2\pi/(2 - \gamma)]} \right. \left. - \Phi\left[-(r/r_d)^{2-\gamma}, 1, 2/(2 - \gamma)\right] \right\}, \quad (1.17)
\]

- see [18], p.29 and [19] p.27. In the Richardson-Kolmogorov case, $d = 3, \gamma = 2/3$ and one gets
\[
f(0) - f(r) = \frac{P_2}{3} \left[ r^{3/3} - r^{2/3}_d \arctg(r/r_d)^{2/3} \right]. \quad (1.18)
\]

Both (1.17) and (1.18) are valid at any $r < L$ for the step-like pumping. At $r < L$, those formulas describe the pair correlation function for the case of an arbitrary pumping $\chi$ with the correlation scale $L$. One can see
that the transition from convective to diffusive interval is described by a universal function.

If the Peclet number $Pe = L/r_d$ is large, the pumping and diffusion scales separate three intervals of scales with a different scaling behavior:

$$f(r) \approx P_2 \left( \frac{L^\gamma(2-\gamma)}{\gamma(d-\gamma)(d-1)D} - \frac{r^2}{4\kappa d} \right),$$

$$r < r_d \ll L$$  \hspace{1cm} (1.19a)

$$f(r) \approx P_2 \left( \frac{2 - \gamma}{\gamma(d-\gamma)} \right) \frac{L^\gamma}{d - \gamma} - \frac{r^\gamma}{d},$$

$$r_d \ll r < L$$  \hspace{1cm} (1.19b)

$$f(r) \approx P_2 \frac{L^d(2-\gamma)r^{\gamma-d}}{d(d-1)(d-\gamma) D},$$

$$r_d \ll L < r$$  \hspace{1cm} (1.19c)

Those expressions fully determine the behavior of the pair correlation function.

C. Different-time pair correlator

The dynamical analog of (1.7) looks as follows ($t_2 > t_1$)

$$\tilde{\theta}(t_1, r_1) \tilde{\theta}(t_2, r_2) = \tilde{U}_2(t_2, t_1) \int_{-\infty}^t \! \! dt \tilde{U}_1(t_1, \tau) \tilde{U}_2(t_1, \tau) \chi(r_{12})$$

$$= \tilde{U}_2(t_2, t_1) \tilde{\theta}(t_1, r_1) \tilde{\theta}(t_1, r_2).$$  \hspace{1cm} (1.20)

Since the velocity field is delta-correlated, then one may reduce the average of the product from the r.h.s. of (1.20) to the product of the averages

$$\left\langle \theta(t_1, r_1) \theta(t_2, r_2) \right\rangle = (\tilde{U}_2(t_2, t_1)) f(r_{12})$$

$$= \exp \left( |t_2 - t_1| + (V_0 + \kappa \Delta) \right) f(r_{12}).$$

Let us stress that if $r_d \ll L$ then $V_0 \gg \kappa$. The point is that $V_0$ can be estimated as $V_0 \sim DL_u^{2-\gamma}$ where $L_u$ is the scale of the largest vortices. Since $L_u > L$ is assumed then $V_0 \gg \kappa$. That means that the time dependence of correlation functions of the passive scalar $\theta$ is determined by the velocity of the largest vortices and is therefore fast. Note that in the comoving reference frame this dynamics is much more slow [1].

2. FOUR-POINT CORRELATION FUNCTION OF THE PASSIVE SCALAR

We begin the analysis of the fourth-order correlation function $F_{1234}$. It will be based upon the equation (2.2), previously derived independently by Kraichnan [20], Sinai and Yakhot [21] and Shraiman and Siggin [22]. It is a second-order partial differential equation in the space of 4d variables. Isotropy allows one to diminish the number of variables to five at $d = 2$ and to six at $d > 2$, which is still too many to hope to find $F$ explicitly at all possible distances $r_{ij}$ ranging from zero to infinity. Our aim is modest: we are looking for the scaling exponents only. First, we shall find the overall scaling exponent $\zeta_4$ which describes how $\Gamma$ (which is the irreducible part of $F_{1234}$) scales if all the distances $r_{ij}$ are multiplied by the same factor. This is the subject of this section where we employ 1/d perturbation theory assuming space dimensionality to be large. That will also allow us to find the anomalous exponents that describe the $L$-dependence of the correlation function. Second, we shall consider the case with one or two distances being much less than other ones and find the scaling exponents with respect to small and large distances separately, this could be done at an arbitrary $d$. That will allow us to fuse some points and to find the $r_d$-dependence of the cumulants of second powers of spatial derivatives. This is the subject of the next section.

A. The equation for the simultaneous four-point correlation function

Using the technique developed in the preceding section we derive the equation for the simultaneous four-point correlation function $F_{1234}$. Let us form first a four-point object averaged over the pumping only by analogy with (1.7)

$$\tilde{\theta}(t, r_1) \tilde{\theta}(t, r_2) \tilde{\theta}(t, r_3) \tilde{\theta}(t, r_4) =$$

$$= \int_{-\infty}^t \! \! dt_1 \int_{-\infty}^t \! \! dt_2 \int_{-\infty}^t \! \! dt_3 \int_{-\infty}^t \! \! dt_4 \tilde{U}_1(t, \tau_1) \tilde{U}_2(t, \tau_2)$$

$$\tilde{U}_3(t, \tau_3) \tilde{U}_4(t, \tau_4) \tilde{\phi}(\tau_1, r_1) \tilde{\phi}(\tau_2, r_2) \tilde{\phi}(\tau_3, r_3) \tilde{\phi}(\tau_4, r_4).$$  \hspace{1cm} (2.1)

Due to (1.4) the object separates onto three parts $Q_{12;34} + Q_{13;24} + Q_{14;23}$, where

$$Q_{ij;kl} = \int_{-\infty}^t \! \! d\tau d\tilde{\tau} \tilde{U}_i(t, \tau) \tilde{U}_j(t, \tilde{\tau}) \tilde{U}_k(t, \tilde{\tau}) \tilde{U}_l(t, \tau) \chi(r_{ij}) \chi(r_{kl})$$

$$= \int_{-\infty}^t \! \! d\tau T \exp \left[ \int_{-\infty}^t \! \! d\tilde{\tau} (\tilde{P}_i + \tilde{P}_j + \tilde{P}_k + \tilde{P}_l) \right] \int_{-\infty}^t \! \! d\tilde{\tau} T \exp \left[ \int_{-\infty}^t \! \! d\tau' (\tilde{P}_i + \tilde{P}_j) \right] \chi(r_{ij}) \chi(r_{kl}).$$  \hspace{1cm} (2.2)

The commutativity of $\tilde{P}_i$ and $\tilde{P}_j$ at $i \neq j$ has been taken into account at (2.2).

For the delta-correlated velocity field, one may average the time-ordered exponents in the r.h.s. of (2.2) explicitly, first, expanding them into series and, second, performing the Gaussian decomposition of the velocity correlators in the same manner as it was done for the pair
correlator of the passive scalar in \((1.11)\). That results in the following expressions

\[
(Q_{ij;kl}) = \int_{-\infty}^{t} d\tau e^{(t-\tau)\hat{c}} \times \\
\int_{-\infty}^{t} d\tau \left( e^{(t-\tau)\hat{L}_{12}} + e^{(t-\tau)\hat{L}_{34}} \right) \chi(r_{ij})\chi(r_{kl}), \tag{2.3}
\]

where \(\hat{L}_{ij}\) was defined in \((1.12)\) and \(\hat{L}\) has the following form:

\[
\hat{L} \equiv \frac{1}{2} \sum_{ij} V^{\alpha\beta}(r_{ij})\nabla^\alpha \nabla^\beta + \kappa \sum \triangle_i, \tag{2.4}
\]

where \(\nabla^\alpha \equiv \partial/\partial r^\alpha\). The integration over \(\hat{c}\) in \((2.3)\) gives

\[-(\hat{L}^{-1}_{ij} + \hat{L}^{-1}_{kl})(\chi(r_{ij})\chi(r_{kl}) = f(r_{ij})\chi(r_{kl}) + \chi(r_{ij})f(r_{kl})\]

where \(\hat{L}_{ij}f(r_{ij}) = -\chi(r_{ij})\) was used. Inserting the last expressions into \((2.3)\) and collecting all the terms \((Q_{ij;kl})\) we find for the full four-point object \(F_{1234} \equiv \langle \theta(r_1)\theta(r_2)\theta(r_3)\theta(r_4) \rangle:\n\]

\[
F_{1234} = \int_{-\infty}^{t} d\tau \exp[(t-\tau)\hat{L}] \sum_{(ij)\neq (kl)} f(r_{ij})\chi(r_{kl}),
\]

where the notation \((ij)\) stands for a pair of non-coinciding \((i \neq j)\) site indices. That leads to the following equation

\[-\hat{L}F_{1234} = f(r_{12})\chi(r_{34}) + f(r_{34})\chi(r_{12}) + f(r_{13})\chi(r_{24}) + f(r_{24})\chi(r_{13}) + f(r_{14})\chi(r_{23}) + f(r_{23})\chi(r_{14}), \tag{2.5}\]

with the r.h.s. expressed in terms of the functions \(f(r_{ij})\) already found: see \((1.16,1.13)\). Note that at a convergence of the points \(r_i \to r_j\) the r.h.s. of \((2.5)\) tends to a constant. Using \((1.14)\) one can obtain an equation for the irreducible four-point correlator \(\Gamma_{1234} = \langle \theta(r_1)\theta(r_2)\theta(r_3)\theta(r_4) \rangle \equiv F_{1234} - f_{12}f_{34} - f_{13}f_{24} - f_{14}f_{23}:\n\]

\[-\hat{\Gamma}_{1234} = \Phi_{12;34} + \Phi_{13;24} + \Phi_{14;23}, \tag{2.6a}\]

\[
\Phi_{ij;kl} = \left[ \kappa^{\alpha\beta}(r_{ij}) - \kappa^{\alpha\beta}(r_{ik}) + \kappa^{\alpha\beta}(r_{jk}) - \kappa^{\alpha\beta}(r_{jl}) \right] \times \nabla^\alpha \nabla^\beta f(r_{ij})f(r_{kl}), \tag{2.6b}
\]

where \(\nabla^\alpha \equiv \partial/\partial r^\alpha\).

The operator \(\hat{L}\) \((2.4)\) is negatively defined. To prove this we represent \(\hat{L} = \kappa\hat{L}_{dif} + \hat{L}_c\) where the convective part could be written as \(\hat{L}_c = \langle [\sum u(r_i)\nabla_i]^2 \rangle_u\). Here we designate \(\langle \ldots \rangle_u\) as an average over the velocity field \(u(r)\) random in space – from the viewpoint of initial \(u(r,t)\), the averaging is over the instant configuration. Incompressibility guarantees that \(\hat{L}_c\) is Hermitian and negatively defined as well as \(\hat{L}_{dif}\). Therefore, \(\hat{L}\) has a continuous non-positive spectrum and the density of states regular at zero. The last statement follows from the inequality \(\|\kappa\hat{L}_{dif} + \hat{L}_c\| \geq \|\kappa\hat{L}_{dif}\|\) and the absence of singularity in the density of states for \(\hat{L}_{dif}\). We can thus conclude that the equation \(-\hat{\Gamma} = \Phi\) is well defined for \(\Gamma\) and \(\Phi\) that do not grow at infinity.

The operator \(\hat{L}\) is scale invariant if all the distances \(r_{ij}\) are either much larger than \(r_d\) or much smaller. The right-hand side is scale invariant if all the distances are either larger or smaller than \(L\). We thus could divide our space of \(r_{ij}\) into three domains where the scale invariance of \(\Gamma(r_{ij})\) is to be expected. It is natural to ask now a simple question: what prevents us from making the statement that the scaling exponent of the solution is equal to the difference between the exponents of the right-hand side and of the operator \(\hat{L}\)? Of course, the solutions with such “naive” scalings exist in all three regions (we call them forced solutions \(F_{force}\)). The problem is to match those particular solutions at pumping and diffusion scale, it may be necessary to include into the full solution the zero modes of the operator:

\[
F = F_{force} + Z. \tag{2.7}
\]

Here the zero mode \(Z\) may have a scaling different from that of \(F_{force}\). To avoid misunderstanding, note that since the operator \(\hat{L}\) is non-positive then it cannot have a global zero mode that satisfies boundary conditions. The parts of the solution with an anomalous scaling may nevertheless be considered as zero modes within separate domains. At \(r_{ij} \ll L\), for instance, one should worry about zero modes that appear due to matching conditions at \(r_{ij} \approx L\), such modes are allowed to grow with \(r_{ij}\). At an oversimplified level, this is what happens when we obtain the constant \(L\)-dependent term in the expressions \((1.19a,1.19b)\) for the pair correlation function. In a multidimensional space, operators may have zero modes much more complicated than a constant and, indeed, the operator \(\hat{L}\) does have an infinite number of zero modes.

We cannot yet find the zero modes and solve the matching problem analytically at arbitrary \(d\). Fortunately, for \(d > 2\) (generally, for \(d > n - 2\) where \(n\) is the number of points in the correlation function) we found the representation of \(\hat{L}\) that allows one to represent \(\Gamma_{ij}\) (with all distances of the same order or the main term if some distances are smaller than others) as a power series with respect to some numerical parameter \((1/d)\) in the convective interval). The coefficients in the series are functions of \(r_{ij}\) all having the same scaling exponents and logarithms that appear from expanding anomalous exponents in powers of \(1/d\). By analyzing those functions at the next subsection, we establish the overall scaling properties of \(\Gamma_{ij}\) in the convective and diffusion intervals. The possibility to sum the logarithms into a power functions is provided by an explicit scale invariance of the equation that determines the zero mode in the convective interval. The consideration of the subleading terms
at the case of some distances being small in comparison with other ones requires a special approach that will be developed in Section 3.

B. Representation of Tetrahedron’s Lengths

To establish overall scaling properties of the irreducible fourth-order correlator $\Gamma$ it will be convenient for us to use a special representation of the equations (2.5,2.6). In an isotropic case, the four-point correlation function $\Gamma$ is a function of six distances $r_{ij}$ between the points, $d > 2$ being assumed. The case $d = 2$ where there are only five independent variables deserves special consideration. In the variables $r_{ij} = |r_{ij}|$, the operator $\hat{L}$ is a sum of two parts, $\hat{L} = \hat{L}_0 + \hat{L}_1$:

$$\hat{L}_0 = \frac{D(d-1)}{2-\gamma} \sum_{i<j} r_{ij}^{-1-\gamma} \partial_{r_{ij}} \left( r_{ij}^{-\gamma} + r_{di}^{-\gamma} \right) r_{ij}^{-d-1} \partial_{r_{ij}}$$

$$\hat{L}_1 = -\frac{D(d-1)}{2(2-\gamma)} \sum_{i<j} \left( r_{ij}^{2} - r_{ij}^{-2} - r_{jm}^{-2} \right) \frac{r_{ij}^{-1-\gamma}}{r_{jm}} \frac{\partial^2}{\partial r_{ij} \partial r_{jm}}$$

$$- \frac{D}{4} \sum_{i<j} r_{ij}^{-1} \left( \frac{d+1-\gamma}{2-\gamma} r_{ij}^{-2} \left( r_{im}^{2} + r_{jm}^{2} - r_{ij}^{-2} - r_{mn}^{-2} \right) \right) \frac{\partial^2}{\partial r_{ij} \partial r_{jm}}$$

$$+ \frac{1}{2} \left( r_{ij}^{-2} + r_{im}^{-2} - r_{jm}^{-2} \right) \frac{\partial^2}{\partial r_{ij} \partial r_{im}}$$

$$+ \kappa \sum_{i<j} r_{ij}^{-2} r_{im}^{-2} - r_{mj}^{-2} \frac{\partial^2}{\partial r_{ij} \partial r_{im}}. \tag{2.8}$$

Here the summation is performed over subscripts satisfying the conditions $i \neq j$ and $m \neq i, j$, $n \neq i, j$. What is nice about that representation is that if we omit $\hat{L}_1$ from $\hat{L}$ then the reducible part $F_{1234} = F_{12} + F_{34} + F_{13} F_{24} + F_{14} F_{23}$ of the fourth-order correlator appears to be a solution of (2.3). It is

$$F^{(0)} = \sum_{i,j} f(r_{ij}) = \left( \text{in the conv. int.} \right) \frac{3(2-\gamma)^2 L^{2\gamma}}{\gamma^2(d-1)^2(d-\gamma)^2} \sum Z_{ijkl} + F_{\text{cor}},$$

$$Z_{ijkl} = r_{ij}^\gamma r_{kl}^\gamma - \frac{d}{2(d+\gamma)} \left( r_{ij}^\gamma - r_{kl}^\gamma \right), \tag{2.9}$$

$$F_{\text{cor}} = \frac{(2-\gamma)^2}{\gamma^2(d+\gamma)(d-1)^2} \frac{(2-\gamma)^2 \gamma \sum r_{ij}^\gamma}{\gamma^2(d-1)^2 d(d-\gamma)}. \tag{2.10}$$

which is the zero mode of $\hat{L}_0$ while that is remarkable that a direct check shows $F_{\text{cor}}$ to be a partial solution in the convective interval of the full equation $-$ $\hat{L} F_{\text{cor}} = f(r_{ij}) \gamma(r_{ij}) + \cdots$. It means, particularly, that if we develop the iteration procedure in $\hat{L}_0^{-1} L_1$ then all the terms appeared at higher steps will enter zero modes of the full operator $\hat{L}$ but not the forced term found. Another remarkable feature of the zero step is an absence among the terms three-point zero modes of $\hat{L}_0$, like $Z_{ij,ik}$. They will appear on the next step only.

The basic fact is that $\hat{L}_0 \propto d^2$ while $\hat{L}_1 \propto d$ as $d \to \infty$. Assuming $1/d$ to be a formal small parameter, we shall implement the iteration procedure with respect to $\hat{L}_1$

$$\hat{L}_0 \Gamma^{(1)} = -\hat{L}_1 F^{(0)}, \quad \hat{L}_0 \Gamma^{(n)} = -\hat{L}_1 \Gamma^{(n-1)}. \tag{2.11}$$

This procedure leads to a representation of $\Gamma$ as a series over the powers of $1/d$:

$$\Gamma = \sum_{n=1}^{\infty} (-\hat{L}_0^{-1} \hat{L}_1)^n F^{(0)}. \tag{2.12}$$

which is actually the series for the non-gaussian part of the zero mode.

One may note that the four-point correlation function is defined not in the whole six-dimensional space of $r_{ij}$ but rather in the physical subspace restricted by triangle inequalities $r_{ij} + r_{jk} \geq r_{ik}$. That makes no additional difficulties since the solution we shall find satisfies all boundary conditions in the physical subspace. In addition, the answer is expressed in terms of powers of $\hat{L}_0^{-1} \hat{L}_1$ which do not have singularities at the boundary of the subspace.

At each step of the iteration procedure we should solve the equation of the (2.11) type. Such equation is much easier to analyze than e.g. (2.6a). The point is that the operator $\hat{L}_0$ is a sum of the six sub-operators $\hat{L}^{(p)}$ from (1.14) each depending on the single variable $r_{ij}$ only. This particular form of $\hat{L}_0$ enables us to analyze the zero modes and establish the necessary properties of its resolvent. The solution of the equation (2.11) is expressed via the corresponding integral kernel as follows

$$\Gamma^{(n)}(\vec{r}) = \int dt \prod_{i>j} \int d\vec{r}' R(t; \vec{r}, \vec{r}') \hat{L}_1 \Gamma^{(n-1)}(\vec{r}'), \tag{2.13}$$

where $\vec{r}$ designates the set of six variables $r_{ij}$ and the integration is performed over time $t$ and six separations $r_{ij}'$. The resolvent $R$ can be represented as a product

$$R(t; \vec{r}, \vec{r}') = \prod_{i>j} R^{(p)}(t; r_{ij}, r_{ij}'), \tag{2.14}$$

where the function $R^{(p)}(t; r, r')$ satisfies the equation

$$(\partial_t - \hat{L}^{(p)}(t; r, r')) R^{(p)}(t; r, r') = \delta(t) \delta(r - r'), \tag{2.15}$$

with the condition $R^{(p)}(t; r, r') = 0$ at $t < 0$. Note the essential property of $R$ simplifying the subsequent analysis: it is independent of the pumping scale $L$. That means that there is only one characteristic length in $R$: the diffusion scale introduced by (1.13).

Now we are going to establish the properties of the resolvent $R^{(p)}(t; r, r')$. Let us remind that there are two terms in the operator $\hat{L}^{(p)}$ (1.14) which are of the same
order at the diffusion scale \( r_d \). Solving (2.15) at the diffusion and convection intervals, we get for the resolvent

\[
R^{(p)}(r, r', (D\tau)^{1/\gamma}) \equiv \frac{1}{4t\kappa} \exp \left( -\frac{r^2 + r'^2}{8t\kappa} \right) \int_{d/2 - 1}^{d/2} \left( \frac{rr'}{4t\kappa} \right)^{1-d/2} \frac{r^d}{r'^d},
\]

at \( r, r', \sqrt{t\kappa} \ll r_d \) (2.16a)

\[
R^{(p)}(r, r', (D\tau)^{1/\gamma}) \gg r_d
\]

\[
R^{(p)}(r, r') \approx \frac{(2-\gamma)}{\gamma(d-1)tD} \exp \left( -\frac{(r^2 + r'^2)(2-\gamma)}{\gamma^2(d-1)tD} \right) (2.16b)
\]

We shall need also the asymptotics in the mixed limit \( r, r_d \ll r', (D\tau)^{1/\gamma} \), for \( r \) and \( r' \) lying in the different intervals (diffusion and convective ones respectively). This asymptotics has to match with one following from (2.16b) in the sub-interval \( r_d \ll r \ll r', (D\tau)^{1/\gamma} \). It gives an idea: to improve the expansion of (2.16b) with respect to \( r/r' \), introducing a function \( y(r) \):

\[
y(r) = y(0) + (d-1)D \int_0^r \frac{xdx}{D(d-1)x^{2-\gamma} + 2\kappa(2-\gamma)}. \tag{2.17a}
\]

In the limit \( r_d \ll r \), the function \( y(r) \) should pass to \( r^\gamma \) as it follows from (2.16b). Substituting (2.17a) into (2.13) and solving the resulting equation for \( y \) one gets an expression with one unknown parameter:

\[
y(r) = y(0) + (d-1)D \int_0^r \frac{xdx}{D(d-1)x^{2-\gamma} + 2\kappa(2-\gamma)}. \tag{2.17b}
\]

Generally, the integration of the second order differential equation produces two parameters, but here one of them has been already fixed by the condition of the finiteness of \( y(r) \) at \( r \to 0 \). The only scale which can determine the dimensional parameter \( y(0) \) is \( r_d \). It gives the following estimate \( y(0) \sim r_d^\gamma \) and, thus, closes our analysis of the mixed asymptotics of the resolvent at \( r, r_d \ll r', (D\tau)^{1/\gamma} \).

C. Overall Scaling of the Fourth-Order Correlator

The analysis of the series (2.12) enables us to establish the scaling behavior of the irreducible part of the fourth-order correlator \( \Gamma \) in different regions of scales. The crucial point is that all terms of the series (2.12) have the same scaling behavior as \( \Gamma^{(1)} \) up to some logarithmic functions that are our main concern. Indeed, we should worry if some logarithms could appear at each step of the iteration procedure (2.11), summation of powers of logarithms can produce anomalous exponents changing the index of the whole sum in comparison with the first term of (2.12).

We shall analyze in this Section only logarithms \( \ln(L/r_{ij}) \). Besides, the logarithms of the ratios of separations \( \ln(r_{ij}/r_{kl}) \) may arise at any step of the iteration procedure (2.11) which could change the behavior at small ratios. This happens only in subleading terms where the coefficient at logarithms are proportional to a power of the small ratio. One may worry, nevertheless, if the logarithms could be summed into the large exponent which compensates the small factors. That means that the case of small ratios needs a separate non-perturbative analysis which is done in Section 3. The results of the analysis show that after re-summation the subleading terms remain the subleading ones. Therefore, the logarithms of the ratios can be neglected at the general investigation. Below in this section, we will speak about the overall scaling of the zero modes that is about their exponents in terms of the ratio \( L/r \).

First, we show that the zero modes of \( \hat{\mathcal{L}}_0 \) cannot contain logarithms \( \ln(L/r_{ij}) \). It can be proved using an “angular” representation in \( r_{ij} \)-space introduced as

\[
r_{ij}^{\gamma/2} = R^{\gamma/2} n_1, \quad r_{ij}^{\gamma/2} = R^{\gamma/2} n_2, \ldots, \tag{2.18}
\]

where \( R \) is a “radial” variable and \( \vec{n} \) is a unit 6-component vector. In terms of those variables the operator \( \hat{\mathcal{L}}_0 \) at \( r \gg r_d \) is written as

\[
\hat{\mathcal{L}}_0 = \frac{(d-1)D}{(2-\gamma)R^2} \left( R^2 \partial^2_{\mathcal{R}} + (6d + 1 - \gamma)R \partial_{\mathcal{R}} + \hat{\Upsilon} \right),
\]

where \( \hat{\Upsilon} \) is an angular operator:

\[
\hat{\Upsilon} = \frac{\gamma(1-\gamma + d)}{2} \sum_{m=1}^6 \left( n_m^1 - 6n_m \right) \frac{\partial}{\partial n_m} + (\gamma/2)^2 \sum_{m=1}^6 n_m^{2-\gamma-1} \frac{\partial}{\partial n_m} n_m^{1-\gamma} \frac{\partial}{\partial n_m}, \tag{2.19}
\]

\[
\frac{\partial}{\partial n_m} = \frac{\partial}{\partial n_m} - \sum_{l=1}^6 n_m n_l \frac{\partial}{\partial n_l}.
\]

The consequence of (2.19) is a possibility to look for the zero modes in the following form:

\[
Z_n = R^{a_n} Y_n(\vec{n}), \tag{2.20}
\]

\[
\hat{\Upsilon} Y_n(\vec{n}) = \lambda_n Y_n(\vec{n}), \tag{2.21}
\]

\[
a_n^2 + (6d - \gamma) a_n + \lambda_n = 0. \tag{2.22}
\]

The roots of the equation (2.22) determine exponents in the expression (2.20) for the zero modes which appear to be power functions [like (2.9)]. Dangerous logarithms would occur if the roots of (2.22) coincide. That corresponds to the values \( a_{n,cr} = -(6d - \gamma)/2 \), \( \lambda_n = \frac{\gamma(1-\gamma + d)}{2} \sum_{m=1}^6 \left( n_m^1 - 6n_m \right) \frac{\partial}{\partial n_m} + (\gamma/2)^2 \sum_{m=1}^6 n_m^{2-\gamma-1} \frac{\partial}{\partial n_m} n_m^{1-\gamma} \frac{\partial}{\partial n_m}, \tag{2.19}
\]

\[
\frac{\partial}{\partial n_m} = \frac{\partial}{\partial n_m} - \sum_{l=1}^6 n_m n_l \frac{\partial}{\partial n_l}.
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\]

\[
\frac{\partial}{\partial n_m} = \frac{\partial}{\partial n_m} - \sum_{l=1}^6 n_m n_l \frac{\partial}{\partial n_l}.
\]
(6d − γ)²/4, which is impossible since all eigenvalues λₙ of the operator Ŷ are negative. The property follows from the fact that all eigenvalues of \( \mathcal{L}_0 \) are negative and \( \mathcal{L}_0 \) is reduced to \( \mathcal{Y} \) for \( R \)-independent functions. Non-degeneracy of the eigenvalues guarantees the absence of the logarithms \( \ln(L/r_{ij}) \) in the zero modes of \( \mathcal{L}_0 \). Note that the coincidence of \( a_n \) for two different \( \lambda_n \) (i.e. the existence of two zero modes with the same scaling yet different angular structures) apparently does not lead to the appearance of logarithms as well.

To establish the exponent of the overall scaling, let us analyze the integral (2.13). First, we show that at any step of the iteration the integral converges at \( r' \gg L \) so that this region gives negligible contribution if we consider separations \( r_{ij} \lesssim L \). To do that, one should find the behavior of \( \Gamma^{(n)} \) at large scales. The decay of \( \Phi \) from (2.6b) is different for different direction in the six-dimensional space \{ri,j\}. By analyzing the most slowly decaying case we get \( \Phi \sim r^{-2} \) where \( R \) is determined by (2.18). Now we can analyze (2.13) and using the resolvent which is determined for all scales one can directly show that the region \( R' \gg L \) gives a negligible contribution. Then the analysis of \( \Gamma^{(1)} \) in the region \( R \gg L \) shows that \( \Gamma^{(1)} \) decays like \( \Phi \). That means that the above scheme is reproduced at any step of the iteration procedure.

Consider the case where all \( r_{ij} \) are in the convective interval. Then we can use (2.16b) for \( \mathcal{R}^{(p)} \). The contribution to \( \Gamma^{(1)} \) associated with \( r' < r \) is \( L \)-independent. One can check using (2.16b) and (1.19b) that the integral in (2.13) converges at small \( r' \) already in the convective interval. That means that the contribution associated with \( r' \) from the convective interval is also \( r \)-independent and it is thus given by the simple dimensional estimate: \( P_2^{(p)} \) multiplied by a scale invariant function of \( r_{ij} \) with the exponent 2γ. That contribution could be neglected in comparison with the terms proportional to positive powers of \( L \) which will be found below. Using the expressions analogous to (2.17) for small \( r' \) one can also check that the contribution to the integral due to the diffusion region is finite and negligible. The main contribution to \( \Gamma^{(n)} \) at the integration over \( r'_{ij} \) in (2.13) is given by the region \( r'_{ij} \sim L \), the characteristic time being estimated as \( t \sim L^2/\gamma \). It follows then from (2.16b) that for \( r \ll L \) the corresponding expression for \( \mathcal{R}^{(p)} \) is a regular expansion in \( r^2 \), what means that the contribution to \( \Gamma^{(1)} \) associated with the considered region \( r'_{ij} \sim L \) is also a regular expansion in \( r'_{ij}/L^3 \). Only the first terms, proportional to positive powers of \( L \), are important, the subsequent terms proportional to negative powers of \( L \) can be neglected in the convective interval.

The zero term in \( r^2 \)-expansion of (2.13) is a constant which can be estimated as \( P_2^{(p)} L^{2\gamma}/D^2 \). The first term (that would be proportional to \( L^2/r^2 \)) is actually equal to zero. That could be understood as follows. As it is seen from (1.19b, 2.8) in the convective interval, the r.h.s of (2.11) does not depend on \( L \): (1.19b) contains only an \( L \)-dependent constant, these constants has to be killed in (2.13) since the operator (2.8) contains only cross derivatives. That \( L \)-independence of the r.h.s. means that an \( L \)-dependent term in \( \Gamma^{(1)} \) should be treated as a zero mode (up to a possible logarithmic factor) of the operator \( \mathcal{L}_0 \) (in the convective interval only!). Zero modes with the scaling \( r^2 \) exist (e.g. \( r_1^2 - r_3^24 \)) but one cannot construct such mode symmetric in permutations of the points 1, 2, 3, 4. Thus the presence of such zero modes would violate the intrinsic \( r_i \leftrightarrow r_j \) symmetry of the problem.

The next term behaves \( \propto r^{2\gamma} \). If \( \Gamma^{(1)} \) contains logarithmic factors multiplied by \( r^{2\gamma} \) then the normal scaling is renormalized into an anomalous one. Let us consider the general expression for \( \Gamma^{(1)} \)

\[
\Gamma^{(1)}(r) = \int_{-\infty}^{\infty} dt \int d\tilde{r}r \mathcal{R}(r, \tilde{r}) \tilde{\mathcal{L}}_1 F^{(0)}(\tilde{r}) ,
\]

expand it in the series in \( r_{ij}^2 \) and calculate (for instance) the terms proportional to \( r_{1234}^2, r_{12}^2 r_{13}^2 r_{14}^2 \) and \( r_{12}^2 \) respectively:

\[
\Delta \Gamma^{(1)}_{12,34} = r_{12}^2 r_{34}^2 \int \frac{dt}{t} \mathcal{A}_1(tD/L^\gamma), \quad \Delta \Gamma^{(1)}_{12,13} = r_{12}^2 r_{13}^2 \int \frac{dt}{t} \mathcal{A}_2(tD/L^\gamma), \quad \Delta \Gamma^{(1)}_{12,12} = r_{12}^2 \int \frac{dt}{t} \mathcal{A}_3(tD/L^\gamma),
\]

\[
\mathcal{A}_1(t/L^\gamma) = \int L dr_{12}' \mathcal{R}^{(p)}(t; 0, r_{12}') \int L dr_{34}' \mathcal{R}^{(p)}(t; 0, r_{34}') \int L dr_{13}' \mathcal{R}^{(p)}(t; 0, r_{13}') \int L dr_{14}' \mathcal{R}^{(p)}(t; 0, r_{14}') \times \int L dr_{23}' \mathcal{R}^{(p)}(t; 0, r_{23}') \int L dr_{24}' \mathcal{R}^{(p)}(t; 0, r_{24}') \tilde{\mathcal{L}}_1(\tilde{r}) F^{(0)}(\tilde{r}),
\]

\[
\mathcal{A}_2(t/L^\gamma) = \int L dr_{12}' \mathcal{R}^{(p)}(t; 0, r_{12}') \int L dr_{13}' \mathcal{R}^{(p)}(t; 0, r_{13}') \int L dr_{34}' \mathcal{R}^{(p)}(t; 0, r_{34}') \int L dr_{14}' \mathcal{R}^{(p)}(t; 0, r_{14}') \times \int L dr_{23}' \mathcal{R}^{(p)}(t; 0, r_{23}') \int L dr_{24}' \mathcal{R}^{(p)}(t; 0, r_{24}') \tilde{\mathcal{L}}_1(\tilde{r}) F^{(0)}(\tilde{r}),
\]

\[
\mathcal{A}_3(t/L^\gamma) = \int L dr_{12}' \mathcal{R}^{(p)}(t; 0, r_{12}') \int L dr_{34}' \mathcal{R}^{(p)}(t; 0, r_{34}') \int L dr_{13}' \mathcal{R}^{(p)}(t; 0, r_{13}') \int L dr_{14}' \mathcal{R}^{(p)}(t; 0, r_{14}') \times \int L dr_{23}' \mathcal{R}^{(p)}(t; 0, r_{23}') \int L dr_{24}' \mathcal{R}^{(p)}(t; 0, r_{24}') \tilde{\mathcal{L}}_1(\tilde{r}) F^{(0)}(\tilde{r}),
\]
While integrating over time at $tD \ll L^\gamma$ one can expand the functions $A_i(tD/L^\gamma)$ in a regular series in $tD/L^\gamma$:

$$A_i(tD/L^\gamma) = C_0^{(i)} + C_1^{(i)} tD/L^\gamma + \cdots. \quad (2.31)$$

The crucial point is the presence of the zero term in the expansion. Were $C_0^{(i)}$ be equal to zero we would immediately figure out the part of the well known zero-mode from the divergent integral over $t$ ($\sim C_1^{(i)}$ in (2.24), the respective term in (2.22) being $L$-independent. As we show below, $C_0^{(i)} \neq 0$ so that one gets the logarithm $\ln[L/r]$ in the respective ($\sim C_1^{(i)}$) contribution to $\Delta \Gamma^{(1)}$. This term stems from the forced solution of the equation $\dot{L}_0 \Gamma^{(1)} = -\dot{L}_1 F^{(0)}$ (it is proved already that there are no logarithms in zero modes of $\dot{L}_0$). From the viewpoint of the initial equation (2.5), those logarithms stem from the zero modes of the full operator $\mathcal{L}$ — see (2.10) and below.

Let us calculate $C_0^{(i)}$ directly. First of all one finds from (2.8)

$$-\dot{L}_1 F^{(0)} = -\left[ \frac{2 - \gamma}{\gamma(d-1)d} \right]^2 \dot{L}_1 \left[ r_{12}^\gamma r_{24}^\gamma + r_{13}^\gamma r_{34}^\gamma + r_{14}^\gamma r_{23}^\gamma \right]$$

$$= \frac{(2 - \gamma)^2}{8(d-1)^2 d^2} \sum p \left( \frac{2}{2 - \gamma} r_{ij}^\gamma - \frac{r_{in}^\gamma + r_{jm}^\gamma - r_{nj}^\gamma - r_{mn}^\gamma}{r_{ij}^\gamma - r_{mn}^\gamma} \right) r_{ij}^{-\gamma} r_{mn}^{-\gamma} r_{in}^{-\gamma} r_{jm}^{-\gamma}, \quad (2.32)$$

where the summation is performed over all the sets of different subscripts $\{i, j, m, n\} = \{1, 2, 3, 4\}$. $F^{(0)}(r)$ and respectively the integrands in (2.28)-(2.30) are sums of a huge number of very simple terms that have been calculated by Mathematica. We do not present here the cumbersome expressions obtained formally at arbitrary $\gamma$ and $d$ since the physical meaning could be ascribed to them only when $\dot{L}_1/L_0$-expansion is a small-parameter expansion. The below formulas show that there are two cases where this is so: $C_0^{(i)}$ go to zero at $\gamma \to 2$ and at $d \to \infty$:

at $d \to \infty$ \hspace{1cm} $C_0^{(1)} \to \frac{4(2 - \gamma)^3}{\gamma^3 d^5}, \quad (2.33)$

at $d \to \infty$ \hspace{1cm} $C_0^{(2)} \to \frac{2(2 - \gamma)^3}{\gamma^3 d^5}, \quad (2.34)$

at $d \to \infty$ \hspace{1cm} $C_0^{(3)} \to \frac{2(2 - \gamma)^3}{\gamma^3 d^5}, \quad (2.35)$

at $\gamma \to 2^-$ \hspace{1cm} $C_0^{(1)} \to -\frac{(5d - 2 - d^2)(2 - \gamma)^3}{2(2 - d)(d - 1)^2 d^3}, \quad (2.36)$

at $\gamma \to 2^-$ \hspace{1cm} $C_0^{(2)} \to -\frac{(2 + d^2 - 4d)(2 - \gamma)^3}{4(2 - d)(d - 1)^2 d^3}, \quad (2.37)$

at $\gamma \to 2^-$ \hspace{1cm} $C_0^{(3)} \to +\frac{(2 - \gamma)^3}{4d^2(d - 1)(d - 2)}. \quad (2.38)$

It is natural that our $\hat{\mathcal{L}}^{-1}_0 \hat{\mathcal{L}}_1$-expansion gives small correction in the limit $d \to \infty$ while we should admit that the same behavior in the limit $\gamma \to 2$ is surprising for us. That means that $\hat{\mathcal{L}}^{-1}_0 \hat{\mathcal{L}}_1$-expansion might be meaningful also when $d$ is arbitrary while $2 - \gamma$ is small. To establish that, one should check whether non-logarithmic terms in $\Gamma^{(1)}$ are proportional to $(2 - \gamma)^3$ as well, which is beyond the scope of the present approach. A nontrivial task to develop a consistent $(2 - \gamma)$-expansion will be the subject of further publications. As $\gamma \to 2$ velocity spectrum diverges at small scales, one should introduce ultraviolet cut-off $r_\eta$ in (2.9) so that $r^{2 - \gamma} / (2 - \gamma) \to \ln(r/r_\eta)$. The scalar correlation functions contains logarithms already at $\gamma = 2$ (in the paper [10], the limit $\gamma \to 2, D/(2 - \gamma) \to \text{const}$ has been considered where such logarithms are absent).

Contrary, the perturbation theory in $1/d$ which is the main subject of this Section is regular and uniform with respect to distances $r_{ij}$. Note that we assume $1/d$ to be the smallest parameter in the problem, in particular, $1/d \ll \gamma$ so that our results do not describe the limit of small $\gamma$ which will be the subject of further publications.

We have found all the logarithmic terms entering the fourth-order correlator on the first step of the iteration procedure:

$$F_{12, 34} = F_0^{(1)} + \gamma \left\{ C_0^{(1)} \sum r_{ij}^{\gamma} r_{kl}^{\gamma} \ln[L/r] \right\} + C_0^{(2)} \sum r_{ij} r_{ik} \ln[L/r] + C_0^{(3)} \sum r_{ij}^{2\gamma} \ln[L/r] \right\} + L\text{-independent or non-logarithmic terms} + \cdots, \quad (2.39)$$

where the summations are performed over all the possible combinations supposing all the indexes $i, j, k, l$ to be different. The expression in front of the logarithms in (2.39) is a sum of the well-known zero modes ($\sim r^{2\gamma}$), that is guaranteed by the identity that could be directly checked.
\[ C_0^{(3)} + \frac{d}{2(d + \gamma)} (C_0^{(1)} + 4C_0^{(2)}) = 0. \]  

(2.40)

It agrees with the absence of any logarithms among zero modes of \( \hat{L}_0 \).

We have thus found the first (in \( 1/d \)) term of the expansion of the zero-mode part of the solution of the full operator

\[ Z = \frac{(2 - \gamma)^2}{\gamma^2 d^4} \sum Z_{ij,kl} + \gamma \left( C_0^{(1)} \sum Z_{ij,kl} \ln[L/r] + C_0^{(2)} \sum Z_{ij,ik} \ln[L/r] \right) + \cdots. \]  

(2.41)

Note that there are two types (\( Z_{ij,kl} \) and \( Z_{ij,ik} \)) of zero modes of \( \hat{L}_0 \) possessing scaling \( 2\gamma \), only one of them (\( Z_{ij,kl} \)) is present on the zero step of the iteration procedure. It is clear that (2.41) cannot be described in terms of a single anomalous exponent \( \Delta(d) \) because the expression in the last brackets does not coincide with \( \sum Z_{ij,kl} \).

We thus come to the conclusion that \( \Delta \) should consist of at least two terms with different scaling. On another language it means that one gets a degenerate case. Indeed, the full operator \( \hat{L} \), the bare operator \( \hat{L}_0 \) and the perturbative one \( \hat{L}_1 \) are scale invariant. The bare zero modes scales similarly. And it is very essential that the first perturbation step should dismiss the degeneracy: on the first step of the iteration procedure it should appear two zero modes

\[ Z_1 = \sum \left[ (\alpha Z_{ij,kl} + \beta Z_{ij,ik}) \left( 1 - \Delta_1 \ln[r/L] \right) \right], \]  

(2.42)

\[ Z_2 = \sum \left[ ((1 - \alpha) Z_{ij,kl} - \beta Z_{ij,ik}) \left( 1 - \Delta_2 \ln[r/L] \right) \right], \]  

(2.43)

such that

\[ Z = \frac{(2 - \gamma)^2}{\gamma^2 d^4} (Z_1 + Z_2), \]  

(2.44)

where \( Z \) was found above in (2.41).

To extract an additional information that will fix all the numbers \( \alpha, \beta, \Delta_1 \), we should calculate first-order logarithmic correction \( \hat{L}_0^{-1} \hat{L}_1 \sum Z_{in,jn} \) to the second bare zero mode \( \sum Z_{in,jn} \). First of all one finds

\[ \hat{L}_1 \sum Z_{in,jn} = \frac{(d - 1) \gamma^2}{2(2 - \gamma)} \sum (r_{ij}^2 - r_{in}^2 - r_{jn}^2) r_{jn}^{-2} \]  

\[ + \gamma^2 \sum \frac{r_{jn}^{-2} r_{in}^{-2}}{r_{ij}} \left[ \frac{d + 1 - \gamma}{2 - \gamma} r_{ij}^2 (r_{in}^2 + r_{jn}^2 - r_{ij}^2) \right] \]  

\[ + \frac{1}{2} r_{ij}^2 + r_{in}^2 - r_{jn}^2 \left( r_{ij}^2 + r_{jn}^2 - r_{in}^2 \right). \]  

(2.45)

Performing calculations analogous to what was done at the calculations of the coefficients \( C_0 \) one finds that all the logarithmic terms appeared are proportional to the bare zero mode \( \sum Z_{in,jn} \) only. It is a manifestation of the fact that the functions on a triangle constitute an invariant subspace of the full operator \( \hat{L} \): \( \hat{L} \) acting on an arbitrary function of three distances from a triangle \( r_{in}, r_{ij}, r_{jn} \) produces a function of the same three distances again. Thus the cancelation of a coefficient before the logarithm proportional to another four-point zero mode will occur on all the higher steps of the iteration of \( \sum Z_{in,jn} \) too. Thus, the resulting zero mode (say \( Z_1 \) with \( \alpha = 0 \)) is scale invariant with the following asymptotics of the anomalous exponent at \( d \to \infty \)

\[ \Delta_1 \to -\frac{(2 - \gamma)(2 + \gamma)}{2d}. \]  

(2.46)

The asymptotics of all the rest coefficients defining the second (mixed) scale-invariant zero mode \( Z_2 \) is restored from (2.41, 2.42, 2.43, 2.44, 2.46)

\[ \Delta_2 \to \frac{4(2 - \gamma)}{d} \beta \to \frac{4}{10 + \gamma}. \]  

(2.47)

Note that the above \( \hat{L}_1^{-1} \hat{L}_0 \) iteration procedure was not, strictly speaking, a direct \( 1/d \)-expansion: not only major terms \( \sim d^2 \) but also subleading ones \( \sim d \) have been included into \( \hat{L}_0 \). That made it possible to have both unperturbed operator and perturbation of the same (second) order. To reinforce the results and obtain the explicit solution for \( \Gamma_1 \) (not only it’s logarithmic part as above), let us briefly describe the direct \( 1/d \)-procedure where the main part of \( \hat{L} \) in the convective interval is the differential operator of the first order

\[ \hat{L}_0' = \frac{d^2}{2 - \gamma} \sum \frac{1}{r_{ij}} \frac{\partial}{\partial r_{ij}}, \]  

(2.48)

yet we will directly account for the necessary boundary conditions while integrating over characteristics below.

The zero modes of (2.48) which we are going to iterate are \( d \to \infty \) limits of \( \sum Z_{ij,kl} \) and \( \sum Z_{ij,ik} \): \( \hat{L} \) is \( \hat{L}_0' = \sum [2r_{ij}^2 r_{ik}^2 - \{ r_{ij}^2 + r_{ik}^2 \}] \) and \( Z_1' = \sum [2r_{ij}^2 r_{ik}^2 - \{ r_{ij}^2 + r_{ik}^2 \}] \). To find the first-order corrections to the bare zero modes one has to solve the differential equation

\[ \hat{L}_0 Z_1' = -\hat{L}_1' Z_1', \quad \hat{L}_{0} Z_0' = -\hat{L}_1' \hat{Z}_0'. \]  

(2.49)

\[ \hat{L}_1 = \frac{d}{2 - \gamma} \left\{ \sum (r_{ij}^2 - \gamma r_{ij}^2) \frac{\partial}{\partial r_{ij}} - \sum (r_{in}^2 - r_{jn}^2) \frac{\partial}{\partial r_{in} \partial r_{jn}} \right\}. \]  

(2.50)

where \( \hat{L}_1' \) is the part of the full operator (in the convective interval) proportional to \( d \), note that it stems both
from \( \hat{\mathcal{L}}_0 \) and \( \hat{\mathcal{L}}_1 \). The equations (2.49) are integrated by
characteristics, for example
\[
Z_1' = - \int dt \hat{\mathcal{L}}_1 Z_0' \left[ \gamma r_{ij} \to d^2 \gamma D_{2-\gamma} t + B_{ij} \right]. \tag{2.51}
\]
\[
Z_1' (\gamma = 1) = \frac{4}{d} \sum r_{ij}^2 - \frac{1}{2d} \sum (r_{im} - r_{ij}) (r_{im} - r_{in})^2 + \frac{r_{im} - r_{jm}}{r_{im} - r_{jn}} \ln |L/r_{in}| + \frac{(r_{jn} - r_{ij}) - (r_{jn} - r_{in})^2 - (r_{jn} - r_{jm})^2 + (r_{jn} - r_{jm})^2}{r_{im} - r_{jn}} \ln |L/r_{jn}|.
\]
\[
Z_1' (\gamma = 1) = - \frac{8}{d} \sum r_{ij}^2 - \frac{1}{d} \sum \frac{(r_{ij} - r_{ik})(r_{ij} - r_{kj})}{r_{ik} - r_{kj}} \left( (r_{ij} + r_{kj} - 2r_{ik}) \ln |L/r_{ik}| - (r_{ij} + r_{ik} - 2r_{kj}) \ln |L/r_{kj}| \right).
\]

One can directly check that those are the solutions (2.3) at the first order in \( 1/d \). Being interested only in
finding the overall scaling we can just put all the separations under the logarithm to be the same. Generally, one can extract the logarithmic parts of \( Z_1 \), \( Z_1' \) directly
from the r.h.s. of (2.51) putting formally the upper limit of the integral to be very large and looking for the
logarithmically divergent terms
\[
Z_1' = \frac{2(2-\gamma)}{d} (2Z_0' - \tilde{Z}_0) \ln \left( \frac{L}{r} \right) + \text{non-log. terms}, \tag{2.52}
\]
\[
Z_1' = \frac{\gamma^2 - 4}{d} Z_0' \ln \left( \frac{L}{r} \right) + \text{non-log. terms}. \tag{2.53}
\]
This is the direct analog of (2.41). The formulas (2.52,2.53) immediately give the leading \( d \to \infty \) asymptotics (2.46,2.47) obtained above from the general iteration procedure.

The structure of the resolvent (2.16a) at small \( r \) determined by (2.17) shows that for \( \Gamma^{(1)} \),
there exists the limit \( \kappa \to 0 \). This property being reproduced at any step of the iteration procedure enables one
to construct a solution \( \Gamma^{(0)} \) which is the limit of \( \Gamma \) at \( \kappa \to 0 \). The function \( \Gamma^{(0)} \)
is close to \( \Gamma \) in the convective interval and remains finite at any \( r_{ij} \to 0 \). The behavior of \( \Gamma^{(0)} \)
is determined by the estimation (2.54) where \( r \) is the maximal value among \( r_{ij} \).

We thus come to the conclusion that for separations \( r \) from the convective interval \( r_d \ll r \ll L \)
\[
\Gamma(r) - \Gamma(0) \sim P_2^2 r^{2\gamma} (L/r) \Delta / D^2, \quad \Gamma(0) \sim P_2^2 L^{2\gamma} / D^2 \tag{2.54}
\]
up to dimensionless constants depending on \( d \) and \( \gamma \) and the anomalous dimension \( \Delta \) is given by the largest exponent \( (\Delta_2) \) which is asymptotically
\[
\Delta = \Delta_2 = \frac{2-\gamma}{d} + O(1/d^2). \tag{2.55}
\]

Finally, we treat the case where all separations are in the diffusion interval. The contribution to \( \Gamma^{(1)} \) associated
with \( r' \gg r_d \) gives the constant \( \sim L^{2\gamma} (L/r_d)^\Delta P_2^2 / D^2 \).
The contribution associated with scales \( r' \lesssim r_d \) has to be analyzed carefully since for all separations from the
diffusion interval the typical integration time \( t \) in (2.13) is also characteristic of the diffusion region. We use
(2.16a) for \( R^{(p)} \) and (1.19c) for \( f \). The estimate for the \( r \)-independent contribution gives \( \sim r'^{2\gamma} P_2^2 / D^2 \).
To analyze a \( r \)-dependent contribution associated with \( r' \lesssim r_d \) it is convenient to return to the initial form (2.1) of
the equation for \( \Gamma \). At the analysis we will not divide \( \hat{\mathcal{L}}_1 \) into \( \hat{\mathcal{L}}_0 \) and \( \hat{\mathcal{L}}_1 \),
the consequent contributions are consequently non-perturbative. In the diffusion region, \( \hat{\mathcal{L}} \) is reduced
to the sum of Laplacians. The r.h.s. of (2.6) is proportional to \( r^4 - \gamma \), that generates the forced part of the
solution proportional to \( r^{6-\gamma} \). To find the principal \( r \)-dependent term one has to compare the forced solution
with zero modes of the sum of Laplacians. All the modes are well known: they are constructed from even powers
of \( r_{ij} \). There are no modes with second powers of \( r_{ij} \) in \( \Gamma \) since they are not invariant under permutations of points \( r_i \).

The zero mode of \( \hat{\mathcal{L}} \) of the fourth power possessing the permutation symmetry:
\[
\sum_{\{ij\}} \left( r_{ij}^2 r_{nm}^2 - \frac{d}{2(d+2)} (r_{ij}^4 + r_{nm}^4) \right), \tag{2.56}
\]
turns out to produce larger contribution than the forced term. Thus, we conclude that the principal term in \( \Gamma \)
is of the forth power in \( r \). The parametric dependence of the coefficient at the power can be easily established
from a matching at \( r \sim r_d \):
\[
\Gamma - (C_1 L^{2\gamma} + C_1' r_d^{2\gamma}) P_2^2 / D^2 \sim r^4 r_d^{2\gamma-4} (L/r_d) \Delta P_2^2 / D^2. \tag{2.57}
\]

The consideration of this Section shows that the degree of non-Gaussianity increases as the distances decrease
in the convective interval and eventually comes
to a constant when the distances are in the diffusion interval. For example, the flatness factor is proportional to \((L/r)^2\) in the convective interval \(L \gg r \gg r_d\) and it is an \(r\)-independent constant of the order of \((L/r_d)^2\) in the diffusive interval \(r_d \gg r\).

3. FOUR-POINT OBJECTS WITH STRONGLY DIFFERENT SEPARATIONS

In Section 2 we established the overall scaling behavior of the four-point irreducible correlation function \(\Gamma_{1234}\). In particular, one may check that at any step of the iteration procedure \((2.11)\), the contributions to \(\Gamma_{1234}\) associated with possible small values of the separation ratios are negligible in comparison with the leading term. To show that the summation over \(n\) could not convert subleading terms into leading ones, here we check the self-consistency of our approach nonperturbatively in \(1/d\) using the full operator \(\hat{\mathcal{L}}\). In the next subsection, we show that if any separation \(\rho\) is much less than other ones then the \(\rho\)-dependent contribution to \(\Gamma_{1234}\) is much less than the leading term. This property enables one to find such two-point objects as \((\{\theta_1 - \theta_2\}^4)\) which can be extracted from \(\Gamma_{1234}\) by fusings some points. Besides, to establish the properties of the correlation functions \(\langle\langle \epsilon_1 \epsilon_2 \rangle\rangle\) and \(\langle\langle \epsilon_1 \theta_2 \rangle\rangle\) one should know not only the limit of \(\Gamma_{1234}\) for coinciding points but also the dependence of \(\Gamma_{1234}\) on distances between nearby points in groups strongly separated from each other. It is the aim of this section to obtain the dependence. Here, we shall use perturbation theory with respect to the ratio between small and large distances. Calculating the corresponding contributions to \(\Gamma_{1234}\) one can obtain the two-point correlation functions of products of different spatial derivatives. We shall demonstrate how an absence of an \(r_d\)-related anomalous scaling in \(\Gamma_{1234}\) prescribes it for \(\langle\langle \epsilon_1 \theta_2^2 \rangle\rangle\) and \(\langle\langle \epsilon_1 \epsilon_2 \rangle\rangle\).

A. Four-Point Correlator with a small Separation

Suppose that among separations \(r_{ij}\) there is the separation \(\rho\), say \(r_{34}\), which is much smaller than other separations. Then the main term in the operator \(\hat{\mathcal{L}}\) \((2.4)\) is associated with derivatives over the vector \(\rho = r_3 - r_4\). This term can be written as

\[
\hat{\mathcal{L}}_\rho = \kappa \rho^2(\rho) \nabla^2 \nabla^2 + 2\kappa \Delta \rho
\]

where \(\nabla^2 \equiv \partial / \partial \rho_{\alpha}\). To solve the equation \((2.6a)\) for \(\Gamma_{1234}\) in this case, we can formulate an iteration procedure where \(\hat{\mathcal{L}}_\rho\) is treated as the main contribution to \(\hat{\mathcal{L}}\). Namely a solution of \((2.6a)\) can be represented as

\[
\Gamma_{1234} = \Gamma_0(r_{13}, r_{23}) + \sum_{n=1}^\infty \Gamma_n(r_{13}, r_{23}, \rho) ,
\]

where \(r_{13}\) and \(r_{23}\) are separations much larger than \(\rho\) and \(\Gamma_0\) is \(\rho\)-independent part of \(\Gamma\). The procedure introduced by \((2.2)\) can be considered as a series over the small parameters \(\rho / r_{13}, \rho / r_{23}\). We impose two boundary conditions on \(\Gamma_n\): to be of order of \(\Gamma_0\) at \(\rho \sim r_{13}, r_{23}\) and to remain finite at \(\rho = 0\).

The r.h.s. of \((3.2)\) can be assumed to be known. A solution of \((3.2)\) satisfying the imposed boundary conditions has the following form

\[
\Gamma_n(r_1, r_2, \rho) = \int_0^\infty dt \int d\mathcal{R}(t; \rho, r) \Phi_n(r_1, r_2, r) ,
\]

where \(\mathcal{R}\) is the kernel of the resolvent of the operator \(\hat{\mathcal{L}}_\rho\):

\[
(\partial_t - \hat{\mathcal{L}}_\rho) \mathcal{R}(t; \rho, r) = \delta(t) \delta(r - \rho) .
\]

Since the operator \(\hat{\mathcal{L}}_\rho\) explicitly depends on \(\rho\) the resolvent \(\mathcal{R}\) is not a function of the difference \(|r - \rho|\) only as it would be in a homogeneous case. The symmetry admits to treat \(\mathcal{R}\) as a function of \(\rho, r, x = pr / \rho\). It is convenient to expand the function in the series over Jacobi polynomials \([23]\) which are the eigenfunctions of the \(x\)-dependent part of \(\hat{\mathcal{L}}_\rho\):

\[
\mathcal{R}(\rho, r, x) = \sum_{n=0}^{\infty} \mathcal{R}^{(2n)}(\rho, r) P_{2n}^{(\nu, \nu)}(x) ,
\]

where \(\nu = (d - 3) / 2\) and only the polynomials with even numbers participate since \(\mathcal{R}\) is an even function of \(x\) (at \(d = 3\), Jacobi polynomials turn into Legendre ones). One can obtain from \((3.4)\) separate equations for \(\mathcal{R}^{(2n)}\):

\[
(\partial_t - \hat{\mathcal{L}}_\rho^{(\rho)}) + \left( \frac{d + 1 - \gamma}{2 - \gamma} \partial_\rho^2 - \gamma + 2\kappa \right) \frac{2n(2n + d - 2)}{\rho^2} \times \mathcal{R}^{(2n)}(\rho, r) = \delta(\rho - r) \rho^{-(d-1)} \times \frac{(d - 2 + 4n)(d - 2 + 2n)}{2d - 1 + (d - 1)^2} \times \mathcal{R}^{(2n)}(\rho, r) ,
\]

where \(\hat{\mathcal{L}}_\rho^{(\rho)}\) was introduced in \((1.14)\).

Solving \((3.4)\) one gets the behavior of \(\mathcal{R}^{(2n)}\) in the diffusive and convective intervals explicitly:

\[
\mathcal{R}^{(2n)}(\rho, r) \sim \frac{(d - 2 + 4n)(d - 2 + 2n)}{2d - 1 + (d - 1)^2} \times \frac{(\gamma - 2\gamma)(2 - \gamma)(\gamma + \rho)^{(2 - \gamma)}(\gamma - 2\gamma)}{\gamma^2(d - 1)^2} \times \mathcal{I}_{2n} \left( \frac{2(2 - \gamma)(\gamma + \rho)^{(2 - \gamma)}(\gamma - 2\gamma)}{\gamma^2(d - 1)^2} \right) ,
\]
where

\[ \eta_n = \frac{1}{\gamma} \sqrt{\frac{(d - \gamma)^2 + 8(d + 1 - \gamma)n(2n + d - 2)}{d - 1}}, \quad (3.8) \]

and \( I_\eta \) designate modified Bessel functions. Note that \( R^{(0)} = R^{(p)} \Gamma(d/2)p^{1-d}/(2\pi^{d/2}) \) where \( R^{(p)} \) was introduced by (2.13) and (2.16). It is possible to formulate for \( R^{(2n)} \) interpolation formulas of the (2.17) type.

To establish the behavior of \( \Gamma_n \) determined by (3.2) we should analyze the integral (3.3) using (3.7). The main part of the integral is determined by the region \( r \sim r_{13}, \, r_{23} \). Thus the \( \rho \)-dependence is associated with the limit of (3.7) at \( \rho \ll r \). We see that the resolvents in this case tend to zero with decreasing \( \rho \). We can also assert based on (3.8) that the higher is the number \( n \) of the angular harmonic the faster is the decay of the corresponding resolvent. Since the procedure (3.2) produces a convergent series over a small parameter then the above statements can be extended to the sum of the series. Thus we proved that the \( \rho \)-dependent part of \( \Gamma \) is parametrically smaller than its \( \rho \)-independent part at \( \rho \ll r_{1}, \, r_{2} \) what was the purpose of this subsection.

**B. Four-Point Correlator with two separated pairs**

At the rest of this section, we consider \( \Gamma_{1234} \) in the case of the special geometry with two separations between points being much smaller than all other ones, namely \( R \gg \rho_{1,2} \), where

\[ \begin{align*}
    r_1 - r_2 &= \rho_1, \\
    r_3 - r_4 &= \rho_2, \\
    R &= (r_3 + r_1 - r_1 - r_2)/2. 
\end{align*} \quad (3.9) \]

Here, the mutual orientations of the vectors and the ratio \( \rho_1/\rho_2 \) are supposed to be arbitrary. We denote by \( \Gamma_0(R) \) the main contribution to \( \Gamma_{1234} \) which is equal to \( \Gamma_{1234} \) at \( \rho_1 = \rho_2 = 0 \). The estimate for \( \Gamma_0 \) can be extracted from the results that have been obtained in the preceding subsection. Indeed, the statements (2.54, 2.57) are valid for \( r \) being the largest length \( R \) of the tetrahedron. One thus gets for \( \Gamma_0 \):

\[ \begin{align*}
    r_d &< R \ll L \quad \Gamma_0 \approx (C_1 L^{2\gamma} - C_2 R^{2\gamma}(L/R)^\Delta)P_2^2/D^2, \\
    R &< r_d \quad \Gamma_0 \approx (C_1 L^{2\gamma} - C_3 R^4(L/R)^\Delta r_d^{-2\gamma - 4})P_2^2/D^2, 
\end{align*} \quad (3.10a, b) \]

where \( C_1, \, C_2, \, C_3 \) are dimensionless constants depending on \( d \) and \( \gamma \).

Generally, we do not assume here and below in this section that \( \Delta \) is small (or \( d \) is large) yet the condition \( \Delta < 2\gamma \) is assumed. At the limit \( d \to \infty \), the numerical coefficients \( C_1, \, C_2, \, C_3 \) could be found from the results of Sect. II. In particular, we can get the following asymptotic expressions:

\[ \langle (\theta_1 - \theta_2)^4 \rangle \approx \frac{24(2 - \gamma)^2}{\gamma^2 d^4} \cdot (1 - 2\beta) \left( \frac{L}{r_{12}} \right)^{\Delta_2} + 2\beta \left( \frac{L}{r_{12}} \right)^{\Delta_1} + \cdots, \quad (3.11) \]

at \( r_{12} \ll r_{13} \approx r_{23} \): \( \langle (\theta_1 - \theta_2)^2 \theta_2^2 \rangle - \langle (\theta_1 - \theta_2)^2 \rangle^2 \approx 4(2 - \gamma)^2 \beta r_1^2 r_2^2 \gamma^2 L^{d/4} \left[ - \frac{L}{r_{13}}^{\Delta_2} + \frac{L}{r_{13}}^{\Delta_1} + \cdots \right], \quad (3.12) \]

at \( r_{12}, \, r_{34} \ll r_{13} \approx r_{23} \approx r_{24} \approx r_{14} \): \( \langle (\theta_1 - \theta_2)^2 (\theta_3 - \theta_4)^2 \rangle - \langle (\theta_1 - \theta_2)^2 \rangle \langle (\theta_3 - \theta_4)^2 \rangle \approx 4(2 - \gamma)^2 \gamma^2 (d - 1)^{d/2} r_1^2 r_3^4 r_4 \left( \frac{L}{r_{13}} \right)^{\Delta_2} + \cdots, \quad (3.13) \]

where dots designate the terms with a normal scaling (\( r^2 \)). Note that if one expands (3.11) in small \( \Delta \)-s then the remarkable cancelation of the terms linear in logs \((3.13b)\) at the conditions \( \gamma \) and to turn into zero at \( \Delta \) could be readily found from (3.14b) and (3.15).

The procedure introduced by (3.14) can be considered as a series over the small parameters \( \rho_1/R, \, \rho_2/R \). We impose two conditions on \( \Gamma_0 \): to be of order of \( \Gamma_0 \) at \( \rho_1, \, \rho_2 \ll R \) and to turn into zero at \( \rho_{1,2} = 0 \).

Before consistent derivation, let us notice that the \( r_d \)-dependence could be readily found from (3.14a) and the additional anomalous scaling of the subleading (\( \rho \)-dependent) terms naturally appears already at the first step as a consequence of the given scaling of \( \Gamma_0 \). Let us consider \( \rho \ll r_d \), then the solution of (3.14) for \( \Gamma_1 \) will be \( \rho_1^2 + \rho_2^2 \) multiplied by the \( R \)-dependent terms \( \Phi \) and \( \Gamma_1 \) which both are proportional to \( R^{(2\gamma - 2)}(L/R)^\Delta \) — see (3.27) below. Assuming that no special cancelations happen, one may conclude, in particular, that \( \langle (\theta^2(0)c(R)) \rangle \approx r_d^{-2} R^{2(2\gamma - 2)}(L/R)^\Delta \) instead of \( R^{2\gamma - 2}(L/R)^\Delta \) that would be given by a naive counting of powers.

Let us start now a regular examination of (3.14). A solution of that equation has the following form
\[ \Gamma_n(R; \rho_1, \rho_2) = \int_0^\infty dt \]
\[ \times \int \int d\tau \, d\tau' \mathcal{R}(t; \rho_1, \tau') \mathcal{R}(t; \rho_2, \tau') \Phi_n(R; \tau', \tau), \]

where \( \mathcal{R} \) is the kernel of the resolvent of the operator \( \mathcal{L}_t \) introduced by (3.4, 3.5, 3.7). The main contribution to the integral (3.15) is determined by the region \( r' \sim R \), \( r'' \sim R \). The higher is the number of the angular harmonic \( n \) in (3.3, 3.7) the smaller is the respective contribution into \( \Gamma \) due to additional powers of the small parameter \( \rho/R \) in \( \mathcal{R}^{(2n)} \). Therefore besides the first harmonic, we consider further only the second one to show how a non-trivial anomalous exponent appears due to an angular dependence.

C. Zero modes

The asymptotic formulas (3.7) for the resolvent enable one to extract a scaling behavior of the general expression (3.13) for a solution of (3.14b). We consider the case where the largest scale \( R \) of the tetrahedron is much smaller than the pumping scale \( L \). That means that the only external scales for the integrations in (3.15) can be \( R \) or \( r_d \) but not \( L \). To be interested in the contribution to \( \Gamma_n \) stemmed from large enough scales \( r' \), \( r'' \sim R \) and time \( t \sim \tau_R \) (where one defines \( \tau_R \) as \( r'^2/\kappa \) at \( t \ll r_d \) but as \( r'^2/D \) at \( t \gg r_d \)), one can expand the resolvents in (3.15) in a series over the small parameters \( \tau_n/t \) and analyze the convergence of the temporal integral for different terms of the expansion. Of course, the integrals depend on the concrete form of the \( \Phi \). However, in any case only a finite number of the first terms of the expansion are determined by the time \( t \sim \tau_R \). The respective contributions into \( \Gamma \) are the zero modes of the operator \( \mathcal{L}_t \), that grow with \( \rho \) and are symmetric under the permutation \( \rho_1 \leftrightarrow \rho_2 \). All the rest terms (stemmed from \( r', r'' \ll R \)) are summed up into the term estimated as \( \tau_n \Phi \), we will name this part of the solution the forced one \( \Gamma_f \).

The \( r \)-dependence of the forced part of the term \( \Gamma_n \) depends on a concrete form of the function \( \Phi_n \) in (3.14b). Contrary, the form of the zero modes is universal: they depend on the source term \( \Phi_n \) only via coefficients. We thus start from finding out the zero modes that give the main contributions at small scales. To find the zero modes that contribute to the solution, we will follow an indirect (but, probably, the simplest) way. Since the zero modes are formed at the large times \( t \sim \tau_R \) in the integral (3.15) (which means that they appear due to matching at \( \rho \sim R \)), then we can extract their scaling behavior expanding the expressions (3.3) for the resolvents in the series over \( \tau_n/t \) (3.7) and keeping the first terms of the expansion. Thus we can extract the exponents of the first zero modes and their angular dependence but not coefficients at different terms of the same order. To find the coefficients we can construct a combination of a given order with arbitrary coefficients and then find relations between the coefficients demanding that the combination is a zero mode of \( \mathcal{L}_{\rho_1} + \mathcal{L}_{\rho_2} \), what can be established by direct applying the operator.

Let us realize the scheme using the expressions (3.7). We see that the \( \rho \)-expansion of the expressions produces powers of \( \rho^\gamma \) and \( \rho^\delta \) where as follows from (3.8)

\[ \delta = \frac{1}{2} \left( \gamma - d + \sqrt{(d - \gamma)^2 + \frac{8(d + 1 - \gamma)d}{d - 1}} \right). \]

The angular structure of the zero modes also can be established if to take into account the explicit form of two first Jacobi polynomials:

\[ P_0^{(\nu, \nu)}(x) = 1, \quad P_2^{(\nu, \nu)}(x) = \frac{d + 1}{8}(dx^2 - 1). \]

Then one can directly check that the zero modes of the first and of the second orders are as follows:

\[ Z_2^{(0,0)} \sim \frac{P_2}{D^2} \left( \frac{L}{R} \right)^\Delta \left\{ \begin{array}{c} \rho_1^\gamma \rho_2 - \frac{d}{2(d + 1)}(\rho_1^\gamma + \rho_2^\gamma), \quad \rho \gg r_d, \\
\rho_1^\gamma \rho_2^\delta - \frac{2}{d + 1}(\rho_1^\gamma + \rho_2^\delta) \end{array} \right\}, \quad \rho \ll r_d; \]

\[ Z_2^{(1,0)} \sim \frac{P_2}{D^2} R^{2\gamma - \delta} \left( \frac{L}{R} \right)^\Delta \left\{ \begin{array}{c} \left( \rho_1^\delta - 2(d(R\rho_1^2 - R^2\rho_1^2)) \right) \left( \rho_1 \leftrightarrow \rho_2 \right) R^{-2}, \quad \rho \gg r_d, \\
\left( \rho_1^\delta - 2(d(R\rho_1^2 - R^2\rho_1^2)) \right) R^{-2} \end{array} \right\}, \quad \rho \ll r_d; \]

\[ Z_2^{(0,2)} \sim \frac{P_2}{D^2} R^{2\gamma - \delta} \left( \frac{L}{R} \right)^\Delta \left\{ \begin{array}{c} \left( \rho_1^\delta - 2(d(R\rho_1^2 - R^2\rho_1^2)) \right) \left( \rho_1 \leftrightarrow \rho_2 \right) R^{-2}, \quad \rho \gg r_d, \\
\left( \rho_1^\delta - 2(d(R\rho_1^2 - R^2\rho_1^2)) \right) \left( \rho_1 \leftrightarrow \rho_2 \right) R^{-2} \end{array} \right\}, \quad \rho \gg r_d; \]

\[ Z_2^{(2,0)} \sim \frac{P_2}{D^2} R^{2\gamma - \delta} \left( \frac{L}{R} \right)^\Delta \left\{ \begin{array}{c} \left( \rho_1^\delta - 2(d(R\rho_1^2 - R^2\rho_1^2)) \right) \left( \rho_1 \leftrightarrow \rho_2 \right) R^{-2} + c_1(\rho_1\rho_2)^2 - 2(d(R\rho_1^2 - R^2\rho_1^2)) \left( \rho_1 \leftrightarrow \rho_2 \right) R^{-4} + \left( \rho_1^\gamma \rho_2^\delta \right) \left( \rho_1 \leftrightarrow \rho_2 \right) R^{-2}, \quad \rho \gg r_d, \\
\left( \rho_1^\delta - 2(d(R\rho_1^2 - R^2\rho_1^2)) \right) \left( \rho_1 \leftrightarrow \rho_2 \right) R^{-2} + c_2(\rho_1\rho_2)^2 - 2(d(R\rho_1^2 - R^2\rho_1^2)) \left( \rho_1 \leftrightarrow \rho_2 \right) R^{-4} \end{array} \right\}, \quad \rho \gg r_d; \]

\[ (3.17d) \]
where \( c_{1,2} \) are dimensionless constants and
\[
b_1 = \frac{(\delta + \gamma)(d + 2\gamma - 2d)}{d\gamma}, \quad b_2 = \frac{d + 4}{d}. \tag{3.18}
\]
The upper indices of \( \mathcal{Z} \), introduced in (3.17), characterize the type of a zero mode with respect to the angular structure. The lower index denotes the original order of the resolvent’s expansion over \( \tau_\rho/t \) producing the concrete term. We keep the terms of the second order (3.17a, 3.17c, 3.17d) besides the first order term (3.17b) since they possess qualitatively different \( \rho \)-dependence: the term (3.17b) is an additive one while the terms (3.17a, 3.17c, 3.17d) contain cross-contributions.

Note that higher-order angular harmonics will be characterized by the exponents \( \delta_n \) which can be extracted from the asymptotic behavior of the resolvents (3.7) characterized by (3.8). Let us write the explicit expressions for the exponents:
\[
\delta_n = \frac{1}{2} \left( \gamma - d + \sqrt{(d - \gamma)^2 + \frac{8n(d + 1 - \gamma)(2n + d - 2)}{d - 1}} \right) \tag{3.19}
\]
which at \( n = 1 \) gives (3.16). The exponents \( \delta_n \) will figure in the expressions of the (3.17) type for higher harmonics.

The leading in \( \rho/R \) contribution in r.h.s. of (3.14c) is \( \rho \)-independent (and consequently angular independent). The contribution originates

a) from \( \hat{\mathcal{L}}\Gamma_0 \):
\[
\hat{\mathcal{L}}\Gamma_0 \rightarrow (\hat{\mathcal{L}}\rho - \kappa\Delta\rho)\Gamma_0(\rho); \tag{3.20a}
\]
b) from \( \Phi_{13,24} + \Phi_{14,23} \):
\[
\Phi_{13,24} + \Phi_{14,23} \rightarrow \Phi_R \equiv 2K_{\alpha\beta}^\rho \frac{\rho^\alpha R^\beta}{R^2} \left( \frac{df(\rho)}{d\rho} \right)^2. \tag{3.20b}
\]

The expression for the respective forced solution is
\[
\Gamma_{1,add}^f \approx \left[ \frac{(2 - \gamma)}{(d - 1)dD} \Phi_R + (\hat{\mathcal{L}}\rho - \kappa\Delta\rho)\Gamma_0(\rho) \right] \times \left( \int_0^{\rho_1} \frac{rdr}{2r^2 - \gamma + \frac{1}{2}} + \int_0^{\rho_2} \frac{rdr}{2r^2 - \gamma + \frac{1}{2}} \right). \tag{3.21}
\]

The subsequent additive terms are produced by taking into account the next \( \rho^2 \)-terms at the r.h.s. of (3.14c). They produce the additive forced contributions to \( \Gamma_1 \) with the scaling \( \rho^{2+\gamma} \) in the convective interval. The leading cross-term originates from \( \Phi_{12,34} \), the main term of its expansion is
\[
\Phi_{12,34} \approx \nabla_\mu \nabla_\nu K^{\alpha\beta}_R \rho^{\mu_1} \rho^{\nu_1} \rho^{\rho_1} \rho^{\rho_2} \frac{df_1(\rho_1)}{d\rho_1} \frac{df_2(\rho_2)}{d\rho_2}. \tag{3.22}
\]

Apart from (3.22), we have to take into account the cross-terms originating from other terms in r.h.s. of (3.14c). The terms have a character of a regular expansion in \( \rho/R \) and are consequently proportional to \( \rho^n \rho^2 \). Such terms are much less than (3.22) in the convective interval and are of the same order in the diffusive interval. Since we are interested not in the factors but only in the scaling behavior we can restrict ourselves with the analysis of (3.22). The term is proportional to \( \rho^{2\gamma} \) in the convective interval and to \( \rho^4 \) in the diffusive one, that gives for the principal behavior of the corresponding forced term \( \rho^{2\gamma} \) and \( \rho^4 \) respectively.

The source (3.22) produces not only cross-terms in \( \Gamma \) of scalar and tensor structures but also additive and generally mixed ones possessing all the variety of angular and \( \rho_1 \leftrightarrow \rho_2 \) symmetries used in the classification of the zero modes (3.17). All those terms scale with \( R \) as \( R^{-\gamma} \) both in the convective and diffusive intervals by \( \rho \). To match those forced terms from the convective and diffusive intervals one has to introduce the additional zero modes in the diffusive interval by \( \rho \)
\[
Z_{2,2}^{(0,0)} \sim Z_{2,2}^{(0,0)}(R/L)^\Delta (r_d/R)^\gamma, \tag{3.23a}
\]
\[
Z_{1,2}^{(0,2)} \sim Z_{1,2}^{(0,2)}(R/L)^\Delta (r_d/R)^{2\gamma - \delta}, \tag{3.23b}
\]
\[
Z_{2,2}^{(0,2)} \sim Z_{2,2}^{(0,2)}(R/L)^\Delta (r_d/R)^{2\gamma - \delta}, \tag{3.23c}
\]
\[
Z_{2,2}^{(2,2)} \sim Z_{2,2}^{(2,2)}(R/L)^\Delta (r_d/R)^{2\gamma - 2\delta}. \tag{3.23d}
\]
Let us now consider the forced term $\Gamma_{2}$ arising at the second step of the iteration procedure. Analyzing the structure of the r.h.s. of (3.14a) one concludes that it decreases downscales with $\rho$ not slower than $(\Gamma_{1}^{f} + \text{zero modes})$. Consequently, the forced term $\Gamma_{2}^{f}$ decreases downscales with $\rho$ not slower than $\tau_{\rho}(\Gamma_{1}^{f} + \text{zero modes})$ that is faster than $\Gamma_{1}^{f}$. Thus, one can drop $\Gamma_{2}^{f}$ in comparison with $\Gamma_{1}^{f}$ and all the set of zero modes. The same arguments allows one to drop the higher forced terms $\Gamma_{n}^{f}$, $n > 2$ too.

E. Comparison between the zero modes and forced terms

To find the main contributions into $\Gamma$ one should compare the zero modes and forced terms of the same angular structure. The leading term in the expansion in $\rho/R$ is the isotropic additive forced term (3.21) inside of both the convective and diffusive intervals.

The zero modes are $L$-dependent but the non-additive forced terms are not. Thus keeping the parameter $L/R$ to be large enough one forces the zero modes to dominate. A possibility for the forced terms to prevail appears at the smallest $\rho$-separations, when the dominance of the zero modes could be compensated by different $\rho$-dependencies of the zero modes and the forced terms. Present Subsection is devoted to such a comparison.

We compare (3.17a) with the forced terms generated by (3.22). Both in the convective and diffusive interval we get the dominance of the angular independent zero mode (3.17a) containing the cross-contribution. To be sure that (3.17a) does give the main contribution into $\langle \theta_{1} \theta_{2} \rangle$ (and thus provides for its anomalous scaling), we have to check whether or not the respective dimensionless coefficient in front of that mode can turn into zero. We consider that mode in a special case $R \gg L \gg \rho$ when the coefficient can be found unambiguously contrary to other cases where the matching at $\rho \sim R$ is necessary. If $R \gg L \gg \rho$ then one have to consider the integral (3.15) with (3.22) and the respective term from (3.7) of the expansion of $R^{(0)}$ – see (2.16b, 2.17). One can directly check that the main contribution into the integral stems from the scales $L$ and time $\tau_{L}$ so that no matching at $R$ is necessary. We do not write here the bulky expression for the integral, it turns into zero only at a single value $\gamma/d \approx 0.086$ found numerically. That numerical value depends on the shape of $\chi$. At the rest values of $\gamma/d$ the integral is nonzero and provides the contribution of the structure (3.17a).

Next we consider the second angular harmonic presented in (3.17b, 3.23), (3.17c, 3.23) and the product of the second angular harmonics in (3.17d, 3.23d). The scaling of the respective harmonics of the forced terms coincides with that of the isotropic contribution: $\rho^{3\gamma}$ and $\rho^{6}$ in the convective and diffusive intervals respectively. The zero modes (3.17b, 3.17d) and (3.17c, 3.17d) scale as $\rho^{4}, \rho^{4+\gamma}$ and $\rho^{5}$ in the convective interval, and as $\rho^{2}, \rho^{2}$ and $\rho^{4}$ in the diffusive one respectively. One concludes:

a) in the convective interval, the zero mode $\Gamma_{1}^{(2)}$ (or $\Gamma_{2}^{(2)}$) is dominant if $3\gamma > \delta$ (respectively $2\gamma > \delta$ or $\gamma > 2\delta/3$), otherwise at the smallest $\rho$ the forced terms prevail;

b) in the diffusive interval, at $3\gamma > \delta$ (2$\gamma > \delta$ or $\gamma > 2\delta/3$) the zero modes (3.17b) (respectively (3.17c) or (3.17d)) become dominant at small enough $\rho$; otherwise the zero modes (3.23b) (respectively (3.23c) or (3.23d)), matching with the forced solution originating from (3.22), prevails.

4. POSSIBLE GENERALIZATIONS

Let us remind the reader that the above results on $r_{d}$-dependence are general while those on $L$-dependence are formally obtained only when the respective anomalous exponent $\Delta$ is much less than the normal exponent $2\gamma$. The whole approach of Sect.2 was developed for the fourth-order correlation function at $d > 2$. The generalization for the $n$-th correlation function at $d > n - 2$ is straightforward by means of the same representation of $\mathcal{L}$ in terms of $r_{ij}$. Such a generalization leads to the new predictions for $n > 4$: the scaling exponent $2\gamma - \Delta$ should appear at all high-order functions. The direct check shows that the respective zero modes appear in the partially reducible contributions only. For example, the six-point object $F_{123456} \equiv \langle \theta(r_{1})\theta(r_{2})\theta(r_{3})\theta(r_{4})\theta(r_{5})\theta(r_{6}) \rangle$ should necessarily contain terms like $f(r_{12})\Gamma_{456}$ i.e. $\rho^{2\gamma-\Delta}L^{1+\Delta}$. Such contributions should cancel out in structure functions like $\langle \theta \theta_{1} - \theta_{2} \rangle^{n}$). To find the scaling exponents of the $n$-th structure functions in the limit $d \rightarrow \infty$, one should iterate the zero mode with the scaling exponent $n\gamma$, which will be published elsewhere.

Quite a different picture may appear for $d \leq n - 2$. In this case, $n(n - 1)/2$ distances between points are no longer the independent variables. Additional constraints should be imposed on that set of (otherwise very convenient) variables, which may lead to the possibility of additional zero modes.

As a final remark, let us emphasize that the above results are valid for a Gaussian pumping only and discuss what might be the consequences of the pumping non-Gaussianity. Let us add the fourth-order pumping correlation function (4.4) the irreducible part $\chi_{4}(r_{1}, r_{2}, r_{3}, r_{4})\delta(t_{1} - t_{2})\delta(t_{1} - t_{3})\delta(t_{1} - t_{4})$. One may model the function $\chi_{4}$ by the step function:

$$
\chi_{4}^{i}(r_{1}, r_{2}, r_{3}, r_{4}) = P_{d} \quad \text{if all } r_{ij} < L \text{ and zero if any } r_{ij} > L.
$$

The production rate of $\theta^{4}$ is $3P_{d}f(0) + P_{4}$ (let us remind that the advection preserves the integral of any power of the scalar field). The ratio $\tau_{*} = P_{4}/P_{2}^{2}$ having the dimensionality of time is a correlation time of the
pumping, which is assumed to be the smallest time in the problem. As a result, the terms generated by \( \chi_4 \) will be small yet we keep them to show what qualitatively new may appear in the case of a non-Gaussian pumping with a finite correlation time. The equation (2.6a) acquires additional term:

\[-\hat{\mathcal{L}} \Gamma_{1234} = \chi_4 + F_{12;34} + F_{13;24} + F_{14;23}.\]

The presence of the function \( \chi_4 \) changes scaling in the convective interval, it causes the terms with anomalous scaling \( \tau_\rho r_{ij}^0 \) to appear in \( \Gamma \) which corresponds to the scaling \( \zeta_n = \zeta_2 \) discussed in (139). Also, the terms with \( r_{ij}^0 \) appear in \( \langle \langle \epsilon_1 \theta_2^2 \rangle \rangle \), while no additional terms appear in \( \langle \langle \epsilon_1 \theta_2 \rangle \rangle \). Note that within our approach the \( \tau_\rho \)-related terms are small corrections even at the diffusion scale despite the fact that they decrease slower than the main terms as distances decrease. Most probably, finite correlation time of the pumping will lead to a substantial contributions with another anomalous scaling yet this is beyond the scope of the present analysis. The development of the theory for the finite correlation times of the velocity and pumping is necessary for a meaningful comparison between theory and experiment. This will be the subject of future publications. On the other hand, the more detailed experimental results are desirable, in particular, direct measurements of the correlation function of the dissipation field.

5. CONCLUSION

We have shown that the fourth-order correlation function \( F_{1234} \) has the scaling exponent \( 2\gamma - \Delta \) in the convective interval at \( d > 2 \). The anomalous exponent \( \Delta \) is analytically found at \( d \to \infty \). Considering the behavior of \( F_{1234} \) in the convective interval where some separations \( \rho \) tend to zero we established that the \( \rho \)-dependent contribution to \( F_{1234} \) also tends to zero. Moreover, nothing terrible happens when \( \rho \) passes through the diffusion scale \( r_d \) and we find that

\[ \langle \langle \theta_1 - \theta_2 \rangle \rangle^2 \propto |r_1 - r_2|^{2\gamma - \Delta} L^\Delta, \]

for \( r_{12} \) from the convective interval.

We analyzed also the case where two separations are much smaller than the distance \( R \) between the pairs and found the contribution, mixing both small distances, which determines the correlator

\[ \langle \langle \theta_1 - \theta_2 \rangle \rangle^2 (\theta_3 - \theta_4)^2 - \langle \langle \theta_1 - \theta_2 \rangle \rangle \langle \langle \theta_3 - \theta_4 \rangle \rangle^2, \]

at \( r_{12}, r_{34} \ll R \). We established that the scaling behavior of that correlation function in the convective interval is characterized by the additional anomalous exponent \( \delta \) defined by (5.16). The corresponding contributions to the irreducible part \( \Gamma_{1234} \) of the correlator \( F_{1234} \) are presented in (3.17a). The isotropic contribution \( F_{12;34} \) behaves \( \propto r_{12}^{-2} r_{34}^{-2} (L/R)^\Delta \), while the anisotropic one \( F_{13;24} \) behaves \( \propto r_{13}^{-2} r_{24}^{-2} (2\gamma - 2\delta) (L/R)^\Delta \). The cross term \( F_{14;23} \) behaves \( \propto (r_{14}^{-2} r_{23}^{-2} + r_{13}^{-2} r_{24}^{-2}) (L/R)^\Delta \). The analysis of the continuation of the correlation function (5.7) for \( r_{12}, r_{34} \) passing to the diffusive interval gives the following scaling laws

\[ \langle \langle \epsilon_1 \theta_2 \rangle \rangle = \kappa^2 \langle \langle (\nabla \theta_1)^2 (\nabla \theta_3)^2 \rangle \rangle \propto r_d^2 (L/r_{13})^\Delta, \]

\[ \langle \langle \nabla_\alpha \theta_1 \nabla_\beta \theta_1 - d^{-1} \delta_\alpha \beta (\nabla \theta_1)^2 \rangle \rangle \propto r_d^{2\delta - 2} r_{13}^{-2\gamma - 2\delta} (L/r_{13})^\Delta, \]

\[ \langle \langle \epsilon_1 \nabla_\alpha \theta_3 \nabla_\beta \theta_3 - d^{-1} \delta_\alpha \beta (\nabla \theta_3)^2 \rangle \rangle \propto r_d^{\delta - 2} r_{13}^{-2\gamma - \delta} (L/r_{13})^\Delta, \]

where the separation \( r_{13} \) lies in the convective interval. The double angular brackets designate irreducible correlation functions. The analogous analysis can be performed for the correlation function

\[ \langle \langle \theta_1 - \theta_2 \rangle \rangle^2 \rangle \propto r_d^2 (L/r_{13})^\Delta. \]

The anisotropic part of the correlation function (5.6) is determined by the contribution (3.17b) what is proportional to \( r_{13}^{-2\gamma - \delta} r_{12}^\delta \), it leads to

\[ \langle \langle \theta_1^2 \nabla_\alpha \theta_2 \nabla_\beta \theta_2 - d^{-1} \delta_\alpha \beta (\nabla \theta_2)^2 \rangle \rangle \propto r_d^{\delta - 2} r_{13}^{-2\gamma - \delta} (L/r_{13})^\Delta. \]

The appearance of \( r_{13}-dependence \) in the correlation functions (5.3) may correspond to the presence of the ultraviolet divergences found in (146) in the diagrams for powers of gradients both of the velocity and of the passive scalar. That means that, contrary to \( L \)-related scaling, \( r_d \)-related anomalous scaling of the scalar derivatives could be caught perturbatively. Moreover, in the asymptotic limits of the normal overall scaling of \( \Gamma_{1234} \) (at \( d \to \infty \) or \( \gamma \to 2^- \)) it is tempting to describe the structure of the part \( \epsilon \) of (5.13) (5.8) using the language of the so-called operator algebra (26, 28) (developed in the context of second-order phase transition theory) the validity of which for turbulence was argued in (16). Namely, at \( d \to \infty \) or \( \gamma \to 2^- \) in accordance with those expressions, the passive scalar \( \theta \) has the dimensionality \(-\gamma/2, (\nabla \theta)^2 \) has the dimensionality 0 and \( (\nabla_\alpha \nabla_\beta \theta - d^{-1} \delta_\alpha \beta (\nabla \theta)^2 \) has the dimensionality \( \delta - \gamma \). In the general case of arbitrary \( d \) and \( \gamma \) when the \( L \)-related anomalous scaling does exist, the underlying algebraic structure (if it exists at all) is unclear at the moment.
All said above concerns the terms associated with zero modes of the operator $\mathcal{L}$ in the convective interval. There exist also a forced solution in the convective interval originating from (5.23). The tails in the diffusion intervals created by this forced solution can be interpreted as zero modes (in the diffusive interval only) which are determined by the estimates (3.23).Those zero modes produce the contributions to the correlation functions (5.4, 5.5, 5.8) which are $\propto r_{\delta\gamma} (L/r_{\delta\gamma}^4)$ and prevail if $\delta > 3\gamma / 2, \delta > \gamma / 2$ and $\delta > 3\gamma$ correspondingly. Those contributions are not governed by the operator algebra even in the asymptotic cases of $d \to \infty$ or $\gamma \to 2^-$. It is natural from the diagrammatic point of view since those contributions correspond to “one-bridge” diagrams which does not contain series producing anomalous scaling. Thus the factor $\propto r_{\delta\gamma}$ is determined simply by the “bridge” factor $\nabla \nabla \mathcal{K} \propto R^{-\gamma}$. Of course analogous contributions exist in all above correlation functions but only in (5.4, 5.5, 5.8) they can prevail at some $\delta$.

To conclude, $1/d$-expansion tells us that the anomalous scaling is present already in the oversimplified model of the delta-correlated velocity field. If one formally uses our formula (1.1) for the anomalous exponent at Richardson-Kolmogorov case $d = 3, \gamma = 2/3$ it gives $\Delta = 16/9$ which substantially larger than the experiments and numerics give ($\Delta \approx 0.4 \div 0.5$), the difference could be accounted by inaccuracy of both an asymptotic formula at finite $d$ and the model with delta-correlated velocity. Note that our results are valid not only for steady state but also for decaying turbulence of a passive scalar with a compact spectrum as an initial condition. In that case, the scalar cascade is accelerated as it goes towards small scales (with typical time $t \propto r^{(2-\gamma)/2}$) so that the small-scale part of the scalar distribution is quasi-steady.

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