Route to chaos in optomechanics

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We establish the emergence of chaotic motion in optomechanical systems. Chaos appears at negative detuning for experimentally accessible values of the pump power and other system parameters. We describe the sequence of period doubling bifurcations that leads to chaos, and state the experimentally observable signatures in the optical spectrum. In addition to the semi-classical dynamics we analyze the possibility of chaotic motion in the quantum regime. We find that quantum mechanics protects the optomechanical system against irregular dynamics, such that simple periodic orbits reappear and replace the classically chaotic motion. In this way observation of the dynamical signatures makes it possible to pin down the crossover from quantum to classical mechanics.

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The coupling between light and matter lies at the heart of modern physics. In recent years the fabrication of optomechanical systems using, e.g., microtoroid resonators [1–3], suspended micromirrors [4, 5], whispering gallery microdisks [6, 7] or microsphere resonators [8–10] has opened up new possibilities for fundamental research and technological applications [11–13]. Because the light-matter coupling and other system parameters can be adjusted over large scales optomechanics provides a genuine opportunity to access the classical and quantum dynamics of mesoscopic driven dissipative systems in a variety of different regimes. Optomechanical systems have been used—or proposed to be used—for the creation of non-classical light [14], preparation of Schrödinger cat states [15], generation of light-matter entanglement [16], ultra-precision measurements [17, 18], and radiative cooling to the ground state [19, 20].

The basic optomechanical system consists of a cantilever in a cavity. The cantilever motion is affected by the radiation pressure of the cavity field, and thus implements light-matter coupling at a truly fundamental level. The cavity is pumped with an external laser, which drives the system out of equilibrium. Experiments have successfully demonstrated the optical bistability of the cavity-cantilever dynamics that leads to self-induced cantilever oscillations [2, 21, 22]. With a few exceptions [1, 23], previous studies mainly addressed the regime of simple periodic cantilever motion, and took the prevalence of regular over irregular dynamics for granted.

In this paper, we consider the dynamics of the optomechanical system with a view towards chaotic motion. We demonstrate the appearance of chaos at negative detuning and explain how to detect it experimentally through characteristic signatures in the optical spectrum. Chaos emerges already for slightly increased pump power which makes it accessible with present experimental setups. We identify the period doubling bifurcations on the way to chaos, and provide the bifurcation diagrams for the first chaotic orbits.

Chaotic dynamics of the optomechanical system appears in the bad-cavity limit and is described by the semi-classical equations of motion. In the quantum regime we use a Monte Carlo propagation technique [24, 25] to solve the master equation for the density matrix, which allows us to track the deviations from the classical dynamics systematically. Surprisingly, chaotic motion can be suppressed in favor of regular oscillatory motion of the cantilever by pushing the system into the quantum regime. We can relate the reemergence of periodic cantilever oscillations to the localization of individual quantum trajectories on simple limit cycles that are not accessible in the classical dynamics.

Our theoretical analysis is based on the generic Hamilton operator of optomechanics [26]

\[
H = \left[ -\Delta + g_0 (b + b^\dagger) \right] a^\dagger a + \Omega b^\dagger b + \alpha_L (a^\dagger + a) .
\]

(1)

It describes, e.g., the vibrational mode of a cantilever \((b^{(1)})\), with frequency \(\Omega\) under the influence of the radiation pressure \((\propto g_0)\) of the cavity photon field \((a^{(1)})\). The effect of the pump laser, with amplitude \(\alpha_L\) and detuning \(\Delta = \omega_{\text{las}} - \omega_{\text{cav}}\) of the laser and cavity frequency, is included in the rotating wave approximation. To account for radiative cavity losses \((\propto \kappa)\) and cantilever damping \((\propto \Gamma)\) we have to study the time evolution of the density matrix \(\rho(t)\) with the quantum-optical master equation

\[
\frac{d\rho}{dt} = -i[H, \rho] + \Gamma D[b, \rho] + \kappa D[a, \rho] ,
\]

(2)

where

\[
D[L, \rho] = L\rho L^\dagger - \frac{1}{2} (L^\dagger L\rho + \rho L^\dagger L) \]

(3)

for \(L \in \{a^{(1)}, b^{(1)}\}\) denotes the dissipative terms in Lindblad form. Here and in the remainder we work at zero temperature.

It is convenient to switch to dimensionless parameters \(\Delta \mapsto \Delta/\kappa, \Omega \mapsto \Omega/\kappa, \Gamma \mapsto \Gamma/\kappa, \alpha_L \mapsto \alpha_L g_0/\kappa^2\), measure time as \(\tau = \kappa t\), and introduce the new parameters

\[
P = \frac{8\alpha_L^2}{\Omega^3}, \quad \sigma = \frac{g_0}{\kappa} .
\]

(4)
FIG. 1. (Color online) (a) and (c): Amplitudes of cantilever oscillation limit cycles given by the sinusoidal ansatz (black line) and from the numerical solution of the SC equations of motions (5), (6) (red diamonds). (b) and (d): Initial dynamics of the cantilever converging to a period-1 resp. period-2 limit cycle.

The pump parameter $P$ gives the strength of the laser pumping of the cavity. Our definition of $P$ is identical to that used in [26, 27]. The quantum-classical scaling parameter $\sigma$ relates the zero-point fluctuations of the cantilever to the resonance width of the cavity [26]. Variation of $\sigma$ allows us to track how the quantum dynamics of the optomechanical system evolves towards the classical dynamics in the bad-cavity limit $\sigma \ll 1$.

For the numerical results we will fix the damping parameters $\kappa = 1$, $\Gamma = 0.001$ and the natural cantilever frequency $\Omega = 1$, which are typical values realized in experiments [11]. This leaves us with the three parameters $\sigma, \Delta, P$.

We first establish the emergence of chaotic motion in the bad-cavity limit $\sigma \ll 1$. In this limit, the dynamics of the optomechanical systems follows the semi-classical (SC) equations of motion

$$\frac{d\alpha}{dt} = i [\Delta \alpha - (\beta + \beta^*) \alpha - \alpha_L] - \frac{1}{2} \alpha, \quad (5)$$

$$\frac{d\beta}{dt} = -i [|\alpha|^2 + \Omega \beta] - \frac{1}{2} \Gamma \beta, \quad (6)$$

for the rescaled cavity and cantilever amplitude $\alpha = \sigma(a)$, $\beta = \sigma(b)$. For the cantilever we also use the phase space variables $x = 1/\sqrt{2}(\beta + \beta^*)$ and $p = -i/\sqrt{2}(\beta^* - \beta)$.

The SC equations of motion are obtained from the Ehrenfest equations of motion for the photon $(a^{(1)})$ and phonon $(b^{(1)})$ mode, together with the SC approximation $\langle b^\dagger b \rangle a \approx \langle b^\dagger b \rangle a$ in which all photon-phonon correlations are neglected. For $\sigma > 0$ the SC equations are an approximation to the full quantum dynamics in Eq. (2), but become exact in the limit $\sigma \to 0$.

The SC equations of motion predict the optical bistability of the optomechanical system, where self-induced cantilever oscillations arise through a Hopf bifurcation [26, 27]. The stable attractors of self-induced oscillations are shown in Fig. 1 (a). The oscillations can be described with a simple sinusoidal ansatz $x(t) = \bar{x} + A \cos(\Omega t)$ for the cantilever position. Inserting the ansatz into the SC equations of motion (6) allows for an analytical solution in terms of a Fourier series [26, 27].

The predictions of the ansatz agree well with the amplitudes extracted from the numerical solution of the SC equations (5), (6) (see Fig. 1 (b) for a sample trajectory).

We now follow the route from regular self-induced cantilever oscillations into the chaotic regime by increasing the pump power $P$. For $P = 1.3$ a period doubling bifurcation (PDB) takes place, and a new limit cycle with twice the period of the original simple periodic cycle appears for negative detuning and small amplitude, as shown in Fig. 1 (c). A sample trajectory located on the period-2 limit cycle is shown in Fig. 1 (d). The single-frequency ansatz fails trivially predicting the PDB, the four possible “amplitudes” of the period-2 cycle are extracted from the numerical solution of the SC equations.

Increasing the driving further leads to additional PDBs and the appearance of period-$n$ limit cycles (not shown here). Eventually, for $P = 1.4$, chaotic motion emerges as shown in Fig. 2. We distinguish chaotic and regular trajectories through the maximal Lyapunov exponent (LE), which we calculate with the “standard” method from [28, 29]. The LE vanishes at every PDB, and separates regular motion with a negative LE from chaotic motion with a positive LE. As the LE in Fig. 2 shows, the chaotic region is bounded and contained within a small window $\Delta \in [-1.0, -0.91]$.

The bifurcation diagrams get more and more complex with increasing $P$, as shown in Fig. 3. The chaotic regions do not only expand and fill larger intervals of the detuning, but they also split and form a fairly complex
intertwined sequence of windows of regular and chaotic dynamics. Notably, the appearance of regular or chaotic motion is very susceptible to the value of $\Delta$. Changing the laser-cavity detuning one can easily tune the optomechanical system in and out of chaos.

The appearance of chaos in the optomechanical system is summarized in Fig. 4. Experimental evidence for chaotic motion can be obtained from the cavity intensity spectrum, as shown in Fig. 5. For period-1 oscillations, with $\Delta = -0.4$ to the right of the chaotic window in Fig. 2, peaks in the spectrum occur only at multiples of the cantilever frequency $\Omega$ (panel (a)). Moving further into the negative detuning regime $\Delta < 0$, additional peaks occur between the peaks of the preceding spectrum with each PDB (panels (b), (c)), until the chaotic regime is reached and the spectrum becomes continuous (Fig. 5 (d)).

We now turn to the quantum dynamics of the optomechanical system. The non-linear SC dynamics emerges from the quantum dynamics only in the bad-cavity limit $\sigma \ll 1$. In this regime the photon and phonon modes are occupied up to high boson numbers ($\langle n \rangle, (b) \sim 10^3$), which renders the direct solution of the master equation (2) impossible. Instead we use the Monte Carlo method of quantum state diffusion (QSD) [24, 25] in the implementation of Schack and Brun [30]. In QSD the density matrix $\rho(t)$ is represented as a classical mixture of individual quantum trajectories. The time evolution of the trajectories is governed by a stochastic differential equation that replaces the master equation (2). One advantage of QSD over other unraveling techniques, e.g., the quantum jump method [31], is the dynamical localization of the quantum trajectories on classical orbits [32–34]. This property gives direct access to the emergence of SC dynamics in the bad-cavity limit through comparison of individual quantum and classical trajectories.

Typical quantum trajectories for different values of the scaling parameter $\sigma$ and different classical attractors are shown as a stroboscopic plot in Fig. 6. It can be observed clearly how with decreasing $\sigma$ the trajectories localize on the classical limit cycles. For more complex classical orbits, localization requires smaller values of $\sigma$. The localization properties change completely in the quantum regime ($\sigma = 0.1$, panels (b), (c)). Now the quantum trajectory localizes on a simple periodic orbit, which differs from the classical limit cycle and is not accessible in the SC dynamics.

The individual quantum trajectories can not be measured in experiment. Experimentally accessible quantities are obtained from the ensemble average over all trajectories. Fig. 7 shows the resulting cantilever position $x(t)$ in comparison to the SC trajectories after the initial transient dynamics has faded out. As the optomechanical systems evolves out of the SC limit one observes that the classical (period-2 or chaotic) motion is replaced by simple periodic cantilever oscillations in the quantum regime. As could be anticipated by the different localization properties of the individual quantum trajectories the quantum dynamics favors simple periodic motion, which does not need to have a classical counterpart. As witnessed by the two curves for $\sigma = 0.01$ in Fig. 7 the position of the crossover from classical to period-1 motion depends on the complexity of the classical limit cycle.
FIG. 6. (Color online) Stroboscopic \((x, p)\)-phase space plot of a single quantum trajectory (red points) for decreasing quantum-classical scaling parameter \(\sigma\), approaching a simple periodic limit cycle (a), a period-2 limit cycle (b) and a chaotic limit cycle (c) (black curves) in the limit \(\sigma \to 0\).

For more complex or chaotic orbits it takes place closer to the SC limit, such that the \(\sigma = 0.01\) curve still agrees with the classical dynamics for a period-2 orbit (left panels), but already shows the simple periodic oscillations of the quantum regime for a classically chaotic orbit (right panels).

To conclude, we here analyze the route to chaos in the optomechanical system and track the appearance of PDBs in the cantilever oscillations as the first step to mixed regular-chaotic dynamics. From comparison of the SC dynamics with the quantum dynamics we find that the quantum dynamics is strongly protected against chaotic motion. The central result is that in the optomechanical system quantum mechanics counteracts the classical route to chaos and stabilizes simple periodic orbits. This behavior can be traced back to the different localization properties of individual quantum trajectories.

Besides being of interest in itself, the existence of chaos in the optomechanical system could be relevant for ultra-precision measurements or fundamental tests on the physical conditions for classical dynamics. Because the mixed regular-chaotic dynamics we depicted is susceptible to small variations of the systems parameters, e.g., the cantilever mass or the laser-cavity detuning, such variations can be detected through drastic changes in the cantilever dynamics. On the other, the fact that the quantum dynamics favors simple periodic over multi-periodic or irregular chaotic motion may help to explain why optomechanical systems can be used in a controlled way even deep in the quantum regime.

First experimental results on the observation of PDBs and chaotic motion in the intensity spectrum of an optomechanical system were reported in [1, 23]. In light of our results we suggest to continue experimental studies in this direction, systematically tracing out the boundaries of the regular and chaotic regimes in comparison to the theoretical predictions garnered from the SC equations of motion. In particular, one should try to locate the crossover from classically multi-periodic or chaotic motion to the simple periodic quantum dynamics by changing the quantum-classical scaling parameter \(\sigma\), e.g., through variation of \(g_0\).

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