Random Walk in Changing Environment

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Abstract

In this paper we introduce the notion of Random Walk in Changing Environment - a random walk in which each step is performed in a different graph on the same set of vertices, or more generally, a weighted random walk on the same vertex and edge sets but with different (possibly 0) weights in each step. This is a very wide class of RW, which includes some well known types of RW as special cases (e.g. reinforced RW, true SAW). We define and explore various possible properties of such walks, and provide criteria for recurrence and transience when the underlying graph is \( \mathbb{N} \) or a tree. We provide an example of such a process on \( \mathbb{Z}^2 \) where conductances can only change from 1 to 2 (once for each edge) but nevertheless the walk is transient, and conjecture that such behaviour cannot happen when the weights are chosen in advance, that is, do not depend on the location of the RW.

1 Introduction

Theseus is thrown into Daedalus’ labyrinth, this time without a ball of thread. Noticing that the labyrinth is a subgraph of \( \mathbb{Z}^2 \), Theseus decides to simply random walk his way out - he knows that he will almost surely reach the exit eventually. What Theseus doesn’t know is that Daedalus, aware of the recurrence of his labyrinth, is working relentlessly to amend this vulnerability. He is continually digging new passages throughout the labyrinth, following a carefully laid plan. He cannot, however, block existing passages, only create new ones and only between adjacent rooms, such that the labyrinth is a subgraph of \( \mathbb{Z}^2 \) at any point. Will Theseus find his way to the exit or is it possible that Daedalus’ cunning plan will deceive him forever (with positive probability)?

It turns out that if Daedalus is aware of Theseus whereabouts he can devise a plan to lure poor Theseus further and further into the labyrinth with positive probability (see Thm 6.10). We conjecture that this is not the case if Daedalus is not aware of Theseus whereabouts.

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Conjecture 1.1. If Daedalus is oblivious of Theseus location, then Theseus will almost surely reach the exit (infinitely many times, if he chooses to stay in the labyrinth). In other words, Theseus’ Random Walk is recurrent.

See Section 7 for formal statement and further open problems.

In this paper we introduce the notion of Random Walk in Changing Environment or RWCE. Generally speaking, a RWCE is a random walk in which each step is performed in a different graph on the same set of vertices. By different we may mean that the set of edges is different, but it is easier and more general to assume that the underlying graph is the same throughout the walk and what changes are the conductances of the edges. This is a very wide class of RW, which includes some well known types of RW as special cases, most notably the reinforced RW. We define and explore various possible properties of RWCE with the conclusion that the interesting case is when the walk is monotone (Daedalus can only create passages) and bounded (all the passages are edges of $\mathbb{Z}^2$). Under these assumptions we give criteria for recurrence and transience when the underlying graph is $\mathbb{N}$ or a tree, even when the sequence of graphs may depend on the history of the walk (the latter we call "adaptive", see section 2 for exact definitions).

We show that the above criteria cannot hold for general graphs: we provide an example where the underlying graph is $\mathbb{Z}^2$, but the RWCE is transient. This example is an RWCE on $\mathbb{Z}^2$ where each edge is started with weight 1 and at each stage we change the weight of the edge to the right of the walker to 2 if this was not already done. The idea behind this construction is to try and mimic the behaviour of excited random walk on two dimensions (see [2]) in which the walker gets a bias to the right whenever it visits a point for the first time, and was shown in [2] to be transient. However, it turns out that the proof carried out in [2] depends quite delicately on the model, and one must take care when working out the details. In particular, a similar attempt to mimic multi-excited RW in dimension 1 (see [17]) cannot succeed, as follows from our results on RWCEs on trees. Note that the above example was an adaptive RWCE, and we conjecture that such behaviour cannot occur in non-adaptive RWCE.

Related works: In recent years there have been a number of papers that studied related models. These works have some overlap with our model and some of the examples, but not with the results of this paper, and generally speaking the emphasis of these works are in different directions. Avin, Kouck and Lotker [1] studied RWCE on a sequence of finite unweighted graphs (which they called "evolving graphs") They were interested mainly in the problem of the cover time of the walk, showing , in particular, that contrary to a regular random walk on the graph, the RWCE may have exponential cover time. Dembo, Huang and Sidoravicius studied models on random walks on "monotone domains" - that is they assumed that the sequence if graphs in the RWCE is an increasing sequence of subgraphs of a pre-given graph, with a focus on $\mathbb{Z}^d$. They proved criteria for recurrence and transience of such walks, with one paper [5] focusing on the non-adaptive case (where they also consider a continuous analog for brownian motion), and the other [4] focusing on the adaptive case - that is when there is an interaction between the walk and the graph sequence.
As will be seen in section 3, the generality of these models implies that some further assumptions must be taken in order to get meaningful criteria, and Dembo, Huang and Sidoravicius focus on several interaction mechanisms (such as, e.g. the walker uncovering new edges when it approaches them) and give criteria for transience and recurrence as well as some conjectures, some of which carry a similar flavour to the ones in this paper.

Structure of the paper: In Section 2 we give the basic definitions and examples of known random walk models which falls into our framework. In Section 3 we give simple examples of Random Walks in Changing Environment which illustrate that if the environment is unbounded or nonmonotone then the random walk can have (almost) any behavior. Sections 4 and 5 give the main results about bounded monotone RWCE on $\mathbb{N}$ and on trees, respectively. Section 6 gives an example of a bounded monotone (adaptive) RWCE on $\mathbb{Z}^2$ which is transient, thus showing that the results on recurrent trees cannot be extended to general recurrent graphs. We conclude with a conjecture and some open problems.

2 Definitions

We begin by giving a rigorous definition of what a random walk in changing environment is, in the broadest sense.

**Definition 1.** A Random Walk in Changing Environment (RWCE), on a graph $G = (V, E)$ is a stochastic process $\{\langle X_t, G_t \rangle\}_{t=0}^{\infty}$, where $G_t = (V, E, C_t)$ are graphs with a conductances function $C_t : E \to [0, \infty)$ over a fixed vertex set $V$ and edge set $E$, and for all $t$, $X_t \in V$ and

$$
\mathbb{P}(X_{t+1} = v \mid \langle X_0, G_0 \rangle, \ldots, \langle X_t, G_t \rangle) = \frac{C_t(X_t, v)}{\sum_{e \in E \mid X_t \in e} C_t(e)}.
$$

We call the sequence $\{X_t\}_{t=0}^{\infty}$ the Random Walk and the sequence $\{G_t\}_{t=0}^{\infty}$ the Environment.

In other words, the law of the process governs the changes in the environment, while the distribution of $X_t$ is determined by $G_t$ in the sense that each step is like a network random walk step in $G_t$.

In our Labyrinth example, Daedalus was creating new edges, not changing conductances. It is easy to see, however, that the definition using conductances is a generalization of this scenario.

**Definition 2.** A RWCE is called proper if $0 < C_t(e) < \infty$ for all $t \in \mathbb{N}$ and $e \in E$. It is called improper otherwise.

**Definition 3.** A RWCE is said to be bounded from above (below) by $G = (V, E, C)$ if $C_t(e) \leq C(e)$ ($C_t(e) \geq C$) for all $e \in E$ and all $t \in \mathbb{N}$, almost surely.

Note that a RWCE bounded from above and below is necessarily proper. All the RWCE in this paper are proper unless otherwise noted. Also note the requirement $C_t(e) < \infty$ in the definition.
is formally redundant as the conductances were defined to be real numbers. However, in the
more naive approach of a changing graph, a conductance of infinity would correspond to merging
(shorting) two vertices together.

**Definition 4.** A RWCE is called nonadaptive if the distribution of $G_{t+1}$ given $G_0,..,G_t$ is inde-
pendent of $X_0,..,X_t$. It is called adaptive otherwise.

The Labyrinth example is nonadaptive if Daedalus is oblivious of Theseus whereabouts or
adaptive if Daedalus responds to it.

**Definition 5.** A RWCE is called monotone increasing (decreasing) if $C_{t+1} \geq C_t$ ($C_{t+1} \leq C_t$)
almost surely.

The Labyrinth example is monotone increasing, since Daedalus only adds new edges, i.e. raises
the conductance by 1.

Note that the definition of a general RWCE is very broad. Actually, it is too broad, as an
adaptive, improper, non-monotone RWCE on the full graph can implement any behavior at any
stage. But even with some restrictions, many interesting walks can be implemented as RWCE in a
natural way. We next give several examples of well-known random walks and how they fit into our
definition:

**Example 2.1.** The 1-reinforced random walk (see [6]) on $\mathbb{Z}^2$, is a proper, adaptive monotone
increasing RWCE. At the beginning the conductance of each edge is 1, and at each stage, if the RW
traversed an edge with conductance 1, replace it with an edge of conductance $c$ (for a fixed constant
$c$). This RWCE is bounded between 1 and $c$. Other reinforced random walks also fit similarly into
the RWCE framework. See [12] for a survey of such models.

**Example 2.2.** The Bridge Burning Random Walk (where the conductance of each edge the walk
traverses is reduced to 0) on $\mathbb{Z}^2$, is an improper, adaptive monotone decreasing RWCE. It is as the
1-reinforced RW with $c = 0$.

**Example 2.3.** The Laplacian random walk from between $v_0$ and $u$ (which is equivalent to the loop
erased random walk from $v_0$ to $u$, see [10]), which starts at $X_0 = v_0$ and chooses which neighbour
to move to at each step with probabilities proportional to the value of the harmonic function which
is 0 on the path of the RW up to this time and 1 on $u$, can be described as a monotone bounded
improper adaptive RWCE.

**Example 2.4.** The ”true” self-avoiding walk with bond repulsion (see e.g. [14]) is a nearest neigh-
bor random walk, for which the probability of jumping along an edge $e$ is proportional to $e^{-ck(e)}$,
where $k(e)$ is the number of times $e$ has been traversed. This is an adaptive, monotone, proper
RWCE.

The main question we will be interested in, is whether a given RWCE is recurrent. Note that for
RWCE, the dichotomy between recurrence and transience is not always as clear cut as for simple
RW. There might be a difference between a.s. returning to the origin, a.s. visiting every vertex, a.s. returning to the origin infinitely many times and a.s. visiting every vertex infinitely many times. Also, since no 0-1 law holds in general for RWCE, we can have a RWCE which return to the origin infinitely many times with probability which is positive but less then 1.

In most natural cases, however, the various possible definitions of recurrence and transience for RWCE coincide. We will therefore use the strictest definitions.

**Definition 6.** A RWCE on $G = (V, E)$ is called recurrent if it visits every vertex in $V$ infinitely many times almost surely. A RWCE is called transient if it visits every vertex a finite number of times almost surely. The RWCE is said to be of mixed type otherwise.

### 3 Simple examples

The aim of this section is to demonstrate the myriad possible behaviors of unrestricted RWCE. We begin with a simple example on general graphs. Let $G$ be any graph and $X_0$ be a vertex in $G$.

**Example 3.1.** For any distribution on paths in $G$ (starting with $X_0$), there is an improper, adaptive, nonmonotone RWCE inducing this distribution on $X$.

Since we have complete control over the conductances of the edges emerging from $X_t$, we can arbitrarily determine the distribution of the next step, and therefore the distribution of the sequence.

A distribution on paths in $G$ is called elliptic if for every finite path in $G$, $v_0, \ldots, v_n$, with $v_0 = X_0$, we have $\mathbb{P}(X_0 = v_0, \ldots, X_n = v_n) > 0$.

**Example 3.2.** For any elliptic distribution on paths in $G$ (starting with $X_0$), there is a proper, adaptive, nonmonotone RWCE inducing this distribution on $X$.

This example is the same as the previous one except you can’t have probability 0 for any transition. Next, note that since multiplying the conductances by some constant does not change the next step distribution, the previous example can be made monotone, either increasing or decreasing. Also, the starting set of conductances $C_0$ can be arbitrary (except for conductances of edges emerging from $X_0$) and by monotonicity the RWCE is bounded (from above or below) by $C_0$. Put together we have:

**Example 3.3.** For any elliptic distribution on paths in $G = (V, E)$ (starting with $X_0$), and any (proper) choice of conductances $C$ there is a proper, adaptive, monotone (increasing or decreasing) RWCE, bounded (from below or above, resp.) by $(V, E, C)$, inducing this distribution on $X$.

If we drop monotonicity, but require boundedness instead then we can still produce any distribution that has bounded conditional probabilities, i.e. the probability for traversing a given edge
is uniformly bounded away from zero. In particular, we have the following example on \( \mathbb{N} \) \((C_t(j)\) denotes the conductance of the edges \((j, j + 1)\) at time \(t)\):

**Example 3.4.** The RWCE with conductances \(C_t(X_t, X_t + 1) = 2\) and \(C_t(j, j + 1) = 1\) for \(j \neq X_t\) is bounded from above and below by a recurrent graph, adaptive, non-monotone and transient.

Indeed, \(X_t\) is simply a biased RW and is therefore transient.

We have thus seen that neither boundedness nor monotonicity are enough to draw any significant conclusions about the RWCE, at least in the adaptive setting.

The next example shows that even in the nonadaptive setting, boundedness does not imply recurrence or transience.

**Example 3.5.** The RWCE with conductances \(C_t(j, j + 1) = 100\) for \(t \equiv j \mod 100\) and \(C_t(j, j + 1) = 1\) otherwise is bounded from above and below by a recurrent graph, nonadaptive, nonmonotone and transient.

Proof. When \(X_t \equiv t \mod 100\) the conductance to the right of \(X_t\) is 100 while to the left it is only 1. Therefore, with probability 100/101, \(X_{t+1} = X_t + 1\), in which case \(X_{t+1} = t + 1 \mod 100\). This happens for an expected number of 101 times, after which the walk is simple until the next 100 conductance "catches up". This takes about 100 steps in which the expected displacement is 0. All in all, the RW gets a strong bias to the right about half the time and so it is transient. \[\blacksquare\]

Note that the same conductances would work even if the RW would have some probability of staying at the same vertex, thus nullifying the bipartiteness of the graph. The calculation would be slightly more involved, however, since the RW would sometime get a bias to the left.

Similarly, we can make the RW recurrent, even if it is bounded by a transient graph.

**Example 3.6.** The RWCE with conductances \(C_t(j, j + 1) = 100 \cdot 2^t\) for \(t \equiv -j \mod 100\) and \(C_t(j, j + 1) = 2^t\) otherwise is bounded from above and below by a transient graph, nonadaptive, nonmonotone and recurrent.

The argument is the same except that when \(X_t \not\equiv -t \mod 100\) the RW gets a bias to the right instead of being balanced. However, simple calculation shows that this bias is not enough to counter the bias to the left when \(X_t \equiv -t \mod 100\) (and we could always change the constants, anyway).

4 RWCE on \(\mathbb{N}\)

In this section we study RWCE whose underlying graph is \(\mathbb{N}\) (with edges between consecutive integers). All the theorems here apply equally to RWCE on \(\mathbb{Z}\), but the proofs are slightly simpler
for \( \mathbb{N} \) since there’s only one way to infinity. For such graphs we can prove quite general conditions which ensure the RWCE is recurrent (or transient).

The main idea of the proofs in this section and the next is as follows. We will define a potential sequence - an adaptive sequence of functions \( F_t : V \to \mathbb{R}^+ \) satisfying:

1. **Harmonicity:** \( F_t \) is harmonic on \((V, E, C_t)\) except at 0.

2. **Monotonicity:** \( F_t(v) \) is either monotone increasing for all \( v \in V \) or monotone decreasing for all \( v \in V \).

Note that \( F_t \) may depend on \( H_t \), the history of the RWCE up to time \( t \), even if the RWCE itself is nonadaptive. The two properties above imply that \( F_t(X_t) \) is either a supermartingale or a submartingale as long as \( X_t \neq 0 \). This is because \( E(F_t(X_{t+1})|H_t) = F_t(X_t) \) by harmonicity of \( F_t \) and because \( F_{t+1}(X_{t+1}) \geq F_t(X_{t+1}) \) (or \( \leq \)) by monotonicity. We will then use the optional stopping theorem to deduce bounds on the probability of return to 0.

The following theorems all require the RWCE be bounded from below and above by some graph. When this condition holds, the walk is elliptic (uniformly in time), that is, the probability of traversing each edge when the walk is at one of its endpoints is bounded away from 0. On \( \mathbb{N} \) this implies that such a walk cannot stay on a finite segment indefinitely - it will a.s. visit every vertex to the right of its current location. Therefore, when trying to determine whether the process is recurrent or transient, we can assume that the walk starts at any vertex of \( \mathbb{N} \), as long as the conditions of the theorem still hold for the RWCE at that time. Ellipticity also means that the walk is recurrent (by our definition) exactly when it visits some vertex infinitely many times almost surely and transient exactly when it visits some vertex only finitely many times almost surely.

**Theorem 4.1.** If \( \{(X_t, G_t)\} \) is a monotone increasing adaptive RWCE on \( \mathbb{N} \), bounded above by some recurrent connected graph \( G_\infty = (\mathbb{N}, C_\infty) \) then the walk is recurrent.

**Proof.** First notice that since the RWCE may be adaptive, \( G_\infty \) is just a bound on \( G_i \) and not necessarily its limit. Second, since \( G_\infty \) is recurrent, so is every possible \( G_t \).

Assume that the walk starts at some \( X_0 > 0 \). We will show that the walk almost surely hits 0. Since the conditions of the Theorem continue to hold at this hitting time, this implies the walk will a.s. hit 0 infinitely often and is therefore recurrent. The potential sequence we use in this case is

\[
F_t(v) = \sum_{j=0}^{v-1} \frac{1}{C_t(j)}
\]

i.e. the resistance between 0 and \( v \) on the graph \( G_t \). That \( F_t \) is harmonic on \( G_t \) is well known (and easily verified). Monotonicity follows from the monotonicity of the RWCE. Therefore, \( F_t(X_t) \) is a super-martingale until the first time \( X_t = 0 \).
Since $G_\infty$ is recurrent, we know that $\sum_{j=0}^\infty 1/C_\infty(j) = \infty$. Therefore, given any $A > 0$ there is a $v \in \mathbb{N}$ such that $F_\infty(v) \geq A$. Let $\tau$ to be the first time the walk hits either $v$ or 0. By ellipticity, $\tau$ is finite almost surely. By the optional stopping theorem $F_0(X_0) \geq \mathbb{E}(F_\tau(X_\tau))$. Denote by $p_v$ the probability that $X_\tau = v$, i.e. that the RW hits $v$ before 0. Noting that $F_t(0) = 0$ for all $t$ and that $F_t(v) \geq F_\infty(v) \geq A$ we have

$$F_0(X_0) \geq \mathbb{E}(F_\tau(X_\tau)) \geq A p_v$$

and therefore

$$p_v \leq \frac{F_0(X_0)}{A}.$$

Since $A$ was arbitrary, the proof is complete.

**Theorem 4.2.** If $\{(X_t, G_t)\}$ is a monotone increasing adaptive RWCE on $\mathbb{N}$, with $G_0$ transient, and bounded above by some transient graph $G_\infty = (\mathbb{N}, C_\infty)$ then the walk is transient.

**Proof.** Note that $G_0$ bounds the sequence $G_t$ from below. The potential sequence is

$$F_t(v) = \sum_{j=0}^\infty \frac{1}{C_t(j)},$$

i.e. the resistance between $v$ and infinity. Harmonicity and monotonicity hold as above and $F_t(X_t)$ is therefore a super-martingale. $G_0$ is transient, thus, given $\varepsilon > 0$ there is $v \in \mathbb{N}$ such that $F_0(v) < \varepsilon$. By ellipticity, we may assume that $X_0 = v$ and since $F_t(v)$ is decreasing we have $F_0(X_0) < \varepsilon$.

Let $\tau$ be the first time $X_t = 0$, or infinity if the walk never reaches 0. Let $p_v$ be the probability that $\tau < \infty$. Since $F_t(X_t)$ is positive and using the optional stopping theorem we have

$$p_v F_\infty(0) \leq \mathbb{E}(F_\tau(X_\tau)) \leq F_0(X_0)$$

which implies

$$p_v \leq \frac{\varepsilon}{F_\infty(0)}.$$

Since $\varepsilon$ was arbitrary, there exists a vertex $v \in \mathbb{N}$ such that $p_v < \frac{1}{2}$. Ellipticity implies that whenever the walk is at 0 it will almost surely visit $v$ at some later time and thereafter it would never visit 0 again with probability $\frac{1}{2}$. Therefore, 0 would be visited only a finite number of times, almost surely.

**Theorem 4.3.** If $\{(X_n, G_n)\}$ is a monotone decreasing adaptive RWCE on $\mathbb{N}$, bounded below by some transient graph $G_\infty = (\mathbb{N}, C_\infty)$, then the walk is transient.

**Proof.** Note that in this case every possible $G_t$ in the sequence is also transient. The potential sequence is

$$F_t(v) = \sum_{j=0}^{v-1} \frac{1}{C_t(j)}$$
i.e. the resistance between 0 and v. Harmonicity and monotonicity hold as above and $F_t(X_t)$ is therefore a sub-martingale. Obviously, this sub-martingale is bounded by 0 and $F_\infty(\infty) = \sum_{j=0}^{\infty} 1/C_\infty(j)$ which is finite.

Assume that the walk starts at $X_0 > 0$ and fix some $v > X_0$. Let $\tau$ be the first time the walk hits 0 or $v$, which, by ellipticity, happens almost surely. Let $p_v$ be the probability that the walk hits $v$ first. By the optional stopping theorem we have

$$F_0(X_0) \leq E(F_\tau(X_\tau)) \leq (1 - p_v)F_\infty(0) + p_vF_\infty(v) \leq p_vF_\infty(\infty)$$

and therefore

$$p_v \geq \frac{F_0(X_0)}{F_\infty(\infty)}.$$ 

This holds for all $v > X_0$ and thus the probability that the walk never visits 0 is at least $F_0(X_0)/F_\infty(\infty)$. Since $F_i$ is increasing, this bound holds every time the walk returns to $X_0$ and therefore the walk will visit 0 only finitely many times, almost surely. ■

**Theorem 4.4.** If $\{(X_t,G_t)\}$ is a monotone decreasing adaptive RWCE on $\mathbb{N}$, with $G_0$ recurrent and bounded below by $G_\infty = (\mathbb{N},C_\infty)$ with $C_\infty = c C_0$ for some $0 < c$, then the walk is recurrent.

**Proof.** Let $X_0$ be arbitrary. Given $X_0$, let $n$ be such that

$$\frac{1}{2} \sum_{j=0}^{n-1} \frac{1}{C_0(j)} \leq \sum_{j=X_0}^{n-1} \frac{1}{C_0(j)}.$$ 

(1)

This is possible since $G_0$ is recurrent. The potential sequence will be

$$F_t(v) = \sum_{j=v}^{n-1} \frac{1}{C_t(j)}$$

i.e. the resistance between $v$ and $n$. Then $F_t(X_t)$ is a sub-martingale until the first time that the RW reaches either 0 or $n$. Let $\tau$ be that time and let $p_0$ be the probability that $X_\tau = 0$. By the optional stopping theorem we have

$$F_0(X_0) \leq E(F_\tau(X_\tau)) = p_0E(F_\tau(0)|X_\tau = 0) + (1 - p_0)0 \leq p_0F_\infty(0) = \frac{p_0F_0(0)}{c}$$

Combining the above with 1 we conclude that $p \geq c/2$. This bound holds for any $X_0$, i.e. regardless of the current state of the RWCE, the probability of reaching 0 in the future is at least $c/2$. A standard argument now shows that this probability must actually be 1. ■

Unlike the other theorems in this section, the last theorem requires the RWCE to have bounded ratio between $G_0$ and $G_\infty$. As the example below shows, this requirement is essential.
Example 4.5. The RWCE with conductances \( C_t(j) = 2^{-j} \) for \( j < t \) and \( C_t(j) = 1 \) otherwise is monotone decreasing, nonadaptive, bounded from above and below by a recurrent graph and is of mixed type.

Proof. Indeed, with probability \( \prod_{t=0}^{\infty} 1/(1 + 2^{-t}) > 0 \) the RW will always go to the right and otherwise it will eventually perform a simple random walk on the graph with conductances \( 2^{-t} \), which is recurrent. 

It is not too difficult to make this example transient. Let \( D^n_t(j) = 2^{-j} \) for \( n \leq j < t \) and 1 otherwise. So the conductances of the last example are \( D^n_0(j) \).

Example 4.6. There exist an increasing sequence \( t_n \) such that the RWCE with conductances \( C_t(j) = \prod_{\{n|t_n < t\}} D^{t-t_n}_n(j) \) is monotone decreasing, nonadaptive, bounded from above and below by a recurrent graph and transient.

Proof. In our first example we had, in essence, a "wave" of conductances \( 2^{-j} \) threatening to carry the RW away. In this example there’s a multitude of such waves, each starting one edge further, so that the final conductance of each edge is finite, and each have some fixed positive probability of carrying the RW away. The sequence \( t_n \) has to be increasing fast enough, so that the RW would have a fixed positive probability of being to the right of the \( n \)-th "wave" when it starts.

5 RWCE on trees

Theorems 4.1 and 4.3 can be extended to the case where the underlying graph is a tree. In order to do that first notice that both proofs use the same potential sequence. Second, notice that these functions can be described as follows: Consider the trivial unit flow (on \( \mathbb{N} \)) from 0 to infinity and fix the potential at 0 to be 0. Then \( F_t(v) \) is the potential of \( v \) in \( G_t \). If the underlying graph is a tree, there are many possible choices of flows, each determining a potential. Harmonicity and monotonicity are true for any of these potential sequences, but some care in choosing the right flow is still needed. For general graphs, however, this method fails, since not every flow determines a potential. More precisely, if the graph contains cycles, then there are 2 distinct flows from the source to some vertex and the potential is well defined only when when Kirchoff’s cycle law is satisfied, which is not necessarily the case.

Theorem 5.1. If \( \{(X_t,G_t)\} \) is a monotone increasing adaptive RWCE on a tree \( T \), bounded above by some recurrent tree \( G_\infty = (T,C_\infty) \) then the walk is recurrent.

Proof. Fix \( A > 0 \). Since \( G_\infty \) is recurrent, there is some \( n \in \mathbb{N} \) such that the effective resistance (in \( G_\infty \)) between the root of the tree (denoted 0) and the outside of the ball of radius \( n \) around 0 is at least \( A \), i.e.

\[ R = R_{\text{eff}}(0 \leftrightarrow \partial B_n(0); G_\infty) \geq A \].
Fix such an \( n \) and let \( i \) be the unit current flow induced by putting a voltage difference of \( R \) between 0 and \( \partial B_n(0) \) in \( G_\infty \). Let
\[
F_t(v) = \sum_e \frac{i(e)}{C_t(e)}
\]
where the sum is over all edges \( e \) on the (unique) path connecting 0 and \( v \). In words, \( F_t(v) \) is the voltage which is induced by the flow \( i \) on \( G_t \). Harmonicity follows, as usual, from Kirchhoff’s law and Monotonicity is trivial since \( F_t(v) \) is a fixed positive linear combination of \( C_t(e) \)’s. Therefore, \( F_t(X_t) \) is a super-martingale until the first time \( X_t = 0 \) or \( X_t \in \partial B_n(0) \). From the definition of the flow \( F_\infty(v) = R \) for any \( v \in \partial B_n(0) \). Since \( C_t \leq C_\infty \) we have \( F_t(v) \geq R \geq A \) for all \( v \in \partial B_n(0) \).

The rest of the proof is the same as in theorem 4.1. Let \( \tau \) be the first time the walk hits either 0 or \( \partial B_n(0) \). Denote by \( p \) the probability that the RW hits \( \partial B_n(0) \) first. Since \( F_t(0) = 0 \) for all \( t \), by the optional stopping theorem we have
\[
F_0(X_0) \geq \mathbb{E}(F_\tau(X_\tau)) \geq Ap
\]
and therefore
\[
p \leq \frac{F_0(X_0)}{A}.
\]

Since \( A \) was arbitrary, the proof is complete. \( \square \)

**Theorem 5.2.** If \( \{(X_t,G_t)\} \) is a monotone decreasing adaptive RWCE on a tree \( T \), bounded below by some transient tree \( G_\infty = (T,C_\infty) \), then the walk is transient.

**Proof.** To prove transience, it is enough to show that under these conditions there is a vertex \( u \) such that such the RWCE, starting from \( X_0 = u \), has at least some fixed probability of never returning to 0. Indeed, by ellipticity, every time the walk returns to 0 it visits \( u \) with some fixed probability and will therefore return to 0 only finitely many times, almost surely.

Since \( G_\infty \) transient, the effective resistance, \( R \) between between 0 and infinity is finite, that is, if \( i \) is the unit current flow from 0 to infinity then the corresponding potential is bounded by \( R \). Let
\[
F_t(v) = \sum_e \frac{i(e)}{C_t(e)}
\]
where the sum is over all edges \( e \) on the (unique) path connecting 0 and \( v \). This is the same as the previous proof except now \( F_t(X_t) \) is a sub-martingale since \( C_t \) is decreasing.

Let \( u \) be a neighbor of 0 such that the flow from 0 to \( u \) is positive and assume that \( X_0 = u \). By definition, \( F_0(u) = i(0,u)/C_0(0,u) \) is positive too.

The rest of the proof is the same as in theorem 4.3. Let \( \tau \) be the first time the walk hits either 0 or \( \partial B_n(0) \). Denote by \( p_n \) the probability that the RW hits \( \partial B_n(0) \) first. By the optional stopping theorem we have
\[
F_0(X_0) \leq \mathbb{E}(F_\tau(X_\tau)) \leq p_n \mathbb{E}(F_\tau(X_\tau)|X_\tau \neq 0) \leq p_n R
\]

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and therefore
\[ p_n \geq \frac{F_0(X_0)}{R}. \]

This holds for all \( n \) and thus the probability that the walk never visits 0 is at least \( F_0(X_0)/R \). Since \( F_t \) is increasing, this bound holds every time the walk returns to \( u \) and therefore the walk will visit 0 only finitely many times, almost surely.

\[ \Box \]

### 6 RWCE on \( \mathbb{Z}^2 \)

One could hope that theorem 5.1 would hold for any monotone RWCE. But, unfortunately, this is not true, as the following example of an adaptive RWCE in 2 dimensions shows.

The example we build is a monotone increasing adaptive RWCE on \( \mathbb{Z}^2 \), with \( C_0 \equiv 1 \) and bounded above by 2.

We shall try to mimic the behavior of excited random walk in our model as follows. When the walk reaches a vertex, we will try to give it a push to the right by increasing the conductance of the right edge to 2. If its left neighbor was never visited, this will make the probabilities for the next step equal to \( \frac{1}{7} \) for the left and \( \frac{2}{7} \) for all the other directions which will give a drift to the right. In the other case, all probabilities will be \( \frac{1}{4} \) so the drift would be zero. Call this walk MAW for “Monotone Adaptive Walk” (the MAW is a specific example of a RWCE). This is obviously quite similar to excited random walk so one is tempted to assume we will get transience, as in [2]. One should be careful, though, because in one dimension a similar attempt to mimic the results of [16] would fail, as Theorem 4.1 shows. So this kind of result is quite sensitive to specific details of the model.

From a “calculatory” point of view, [2] reduces to the fact that simple random walk starting from \( (0,0) \) has a probability of \( n^{-1/4} \) to avoid hitting the right half line for the first \( n \) steps (Kesten’s lemma: see [8]). This \( n^{-1/4} \) factor manifests itself in the fact that, finally, they prove a \( n^{3/4} \) drift (up to logarithmic factors). In their settings the probabilities for going up or down never change — the effect of the drift is only to move weight around between the left and right probabilities. We do not know how to mimic this particular detail in the settings of monotone adaptive conductances so we will need to work without it, and this would complicate the geometric settings somewhat.

Below we work out our replacement for Kesten’s lemma. Hence for a while we will only develop properties of simple random walk. The impatient can jump to lemma 6.9 to see how this is used.

Below \( \varepsilon \in (0, \frac{1}{2}) \) is some parameter that will be kept fixed throughout.

**Lemma 6.1.** Let \( E_1 \) be the event that a random walk starting from 0 will avoid hitting the point \((-1,0)\) for the first \( \lceil n^\varepsilon \rceil \) steps. Then
\[
\mathbb{P}(E_1) \approx \frac{C}{\log n}.
\]

This is a well known fact. See e.g. [13].
Lemma 6.2. Let \( W \) be Brownian motion starting from 0. Let \( K \) be an infinite cone with opening \( \theta \in [0, \pi] \) and tip \( v \), and assume \( |v| \leq 2d(0,K) \). Let \( t > 4|v|^2 \) and denote \( \mu = \sqrt{t/|v|} \). Then

\[
\mathbb{P}(W[0,t] \cap K = \emptyset) \geq \frac{c\mu^{-\pi/(2\pi-\theta)}}{\sqrt{\log \mu}} \quad (2)
\]

\[
\mathbb{P}(\{W[0,t] \cap K = \emptyset\} \cap |W(t)_2| < \delta \sqrt{t}) \leq C\delta \mu^{-\pi/(2\pi-\theta)} \sqrt{\log \mu}. \quad (3)
\]

for any \( \delta < 1 \).

We remark that both \( \sqrt{\log} \) factors above can be removed without much difficulty. See some additional blurbs on this in the remark on page 17 below.

Proof. Denote \( \xi = \pi/(2\pi-\theta) \). Let \( T_r \) be the stopping time of \( W \) on \( \partial B(0,r) \). Conformal invariance (and a few calculations) show that

\[
\mathbb{P}(W[0,T_r] \cap K = \emptyset) \approx (|v|/r)^\xi.
\]

On the other hand, \( \mathbb{P}(t > T_{c\sqrt{t/\log \mu}}) \leq \mu^{-1} \) for some \( C \) sufficiently large, so

\[
\mathbb{P}(W[0,t] \cap K = \emptyset) \geq \mathbb{P}(W[0,T_{c\sqrt{t/\log \mu}}] \cap K = \emptyset) - \mu^{-1} \geq c \left( \frac{1}{\mu \sqrt{\log \mu}} \right)^\xi.
\]

For the other part, first notice that \( \mathbb{P}(\frac{1}{2}t < T_{c\sqrt{t/\log \mu}}) \leq \mu^{-1} \) which gives similarly that \( \mathbb{P}(W[0,\frac{1}{2}t] \cap K = \emptyset) \leq c(\sqrt{\log \mu}/\mu)^\xi \). After \( \frac{1}{2}t \) we have that for any \( x \in \mathbb{R}^2 \) that Brownian motion \( W \) starting from \( x \) has probability \( \leq C\delta \) to be in the strip \( \{(x,y) : |y| < \delta \sqrt{t}\} \). These two facts prove (3). \( \blacksquare \)

Lemma 6.3. For \( m \in (n^{2\varepsilon}, n] \) let \( E_2(m,n) \) be the event that a random walk \( R \) starting from 0, satisfies

1. \( E_1(m,n) \)
2. \( R[|n^{\varepsilon}|,m] \cap F = \emptyset \) where \( F \) is the funnel

\[
F = \{(x,y) : x \geq -1, |y| \leq \log^3 n \sqrt{x+2}\}. \quad (4)
\]

Then

\[
\mathbb{P}(E_2(m,n)) \geq \left( \frac{n^{\varepsilon}}{m} \right)^{1/4+o(1)},
\]

\[
\mathbb{P}(E_2(m,n) \cap \{|R(m)| < \delta \sqrt{m}\}) \leq \delta \left( \frac{n^{\varepsilon}}{m} \right)^{1/4+o(1)}
\]

if only \( \delta \sqrt{m} \geq 1 \).

Here and below \( o(1) \) stands for an entry that goes to 0 as \( n \to \infty \) uniformly in \( m > n^{2\varepsilon} \).
Proof. Denote \( v = R([n^\varepsilon]) \). The first ingredient is Hungarian coupling [7, Theorem 4], see also [3, 15, 9], which gives that we can couple random walk starting from \( v \) to Brownian motion \( W \) also starting from \( v \) such that with probability \( \geq 1 - n^{-10} \) we have \( |R(t) - W(t)| \leq C_1 \log^2 t \). We therefore find two cones \( K^\pm \) satisfying \( K^- + B(0, C_1 \log^2 n) \subset F \) and \( F + B(0, C_1 \log^2 n) \subset K^+ \). Specifically we choose

\[
K^- = \{(x, 0) : x \geq C_1 \log^2 n\}
\]

\[
K^+ = \{(x, y) : x \geq -n^\varepsilon/4, |y| \leq \frac{\log^3 n}{n^{\varepsilon/8}}(x + n^\varepsilon/4)\}
\]

and the inclusion conditions will be satisfied for \( n \) sufficiently large.

Next we want to estimate the distance of \( v \) from \( K^+ \). With probability \( > 1 - C \log^{-2} n \) we have that \( d(v, K^+) > n^\varepsilon/2 \log^{-2} n \). To see this fix some \( \lambda = 1, 2, \ldots \) and examine the annulus \( A := n^\varepsilon/2 (B(0, \lambda) \setminus B(0, \lambda - 1)) \). For every \( w \in A \) one has that \( \mathbb{P}(v = w) \leq C n^{-\varepsilon} e^{-\lambda^2} \) while the inflated cone \( (K^+ + B(0, n^\varepsilon/2 \log^{-2} n)) \cap A \) contains \( \leq C n^\varepsilon/2 (\lambda n^{(3/8)\varepsilon} \log^2 n + n^\varepsilon/2 \log^{-2} n) \leq C \lambda n^\varepsilon \log^{-2} n \) points. Summing over \( \lambda \) we get the estimate for \( d(v, K^+) \). Comparing to the probability of \( E_1 \) we get for \( n \) sufficiently large

\[
\mathbb{P}(E_1 \cap \{d(v, K^+) > n^\varepsilon/2 \log^{-2} n\}) \approx \frac{C}{\log n}. \tag{5}
\]

Now we may invoke lemma 6.2 and get that, assuming \( d(v, K^+) > n^\varepsilon/2 \log^{-2} n \),

\[
\mathbb{P}(R([n^\varepsilon], m) \cap F = \emptyset) \geq \mathbb{P}(W[0, m - [n^\varepsilon]] \cap K^- = \emptyset) \geq \left( \frac{C}{\sqrt{\log n}} \left( \frac{n^\varepsilon/2 \log^{-2} n}{\sqrt{m - |n^\varepsilon|}} \right)^{1/4} \right) \geq \left( \frac{n^\varepsilon}{m} \right)^{1/4 + o(1)}
\]

and

\[
\mathbb{P}\left( \{R([n^\varepsilon], m) \cap F = \emptyset\} \cap \{|R(m)| \leq \delta \sqrt{m}\} \right) \leq \mathbb{P}\left( \{W[0, m - [n^\varepsilon]] \cap K^+ = \emptyset\} \cap \{|W(m - [n^\varepsilon])| \leq \delta \sqrt{m} + C_1 \log^2 n\} \right) \leq C(\delta + \frac{C_1 \log^2 n}{\sqrt{m}}) \sqrt{\log n} \left( \frac{n^\varepsilon/2 \log^{-2} n}{\sqrt{m - |n^\varepsilon|}} \right)^{\pi/(2n^{\varepsilon/8} \log^3 n)} \leq \delta \left( \frac{n^\varepsilon}{m} \right)^{1/2 + o(1)}
\]

Where in the last inequality we used \( \delta \sqrt{m} \geq 1 \) to bound \( C_1 m^{-1/2} \log^2 n \leq \delta \log^2 n \) and then this log factors can be folded into the \( o(1) \) in the exponent like all the other log-s (including the one from \( E_1 \)). Notice also that we didn’t write the negligible probability for the coupling to fail, but it does not affect the result for \( n \) sufficiently large.

\[\blacksquare\]
Lemma 6.4. Let $E_3(m, n)$, $m \geq n^{2\varepsilon}$ be the event that a random walk $R$ starting from 0 satisfies that

1. $R[m - \lceil n^{\varepsilon} \rceil, m]$ avoids $(-1, 0) + R(m)$; and
2. $R[0, m - \lceil n^{\varepsilon} \rceil]$ avoids $F + R(m)$ where $F$ is the funnel defined in (4).

Then

$$P(E_3) \geq \left( \frac{n^{\varepsilon}}{m} \right)^{1/4 + o(1)},$$

$$P(E_3 \cap \{|R(m)| < \delta \sqrt{m}\}) \leq \delta \left( \frac{n^{\varepsilon}}{m} \right)^{1/4 + o(1)}.$$

for any $m \in [n^{2\varepsilon}, n]$.

Proof. This follows immediately from lemma 6.3 and time reversal symmetry.

Following [2] we will call $m$ satisfying $E_3(m, n)$ “tan points” (imagine the sun being at the right infinity, then $R(m)$ gets a tan without (almost) any previous point blocking a whole “tanning funnel”).

Lemma 6.5. Let $m_1 < m_2$ and $m_2 - m_1 > n^{2\varepsilon}$. Then

$$P(E_3(m_1, n) \cap E_3(m_2, n)) \leq P(E_3(m_1, n))P(E_3(m_2 - m_1, n)).$$

Proof. One only needs to notice notices that it is easier for $R(m_2)$ is a tan point with respect to the walk starting from $R(m_1)$ then to be a regular tan point. In other words, if $S(i) := R(m_1 + i) - R(m_1)$ then $S$ is a random walk starting from 0; and if $E^*$ is the event that $m_2 - m_1$ is a tan point for $S$; then $E_3(m_2, n) \subset E^*$.

Lemma 6.6. With probability $> 1 - Cn^{-2}$ there are at least $n^{3/4-(7/4)\varepsilon+o(1)}$ $n^{\varepsilon}$-separated tan points up to time $n$.

Proof. Let $h = \lfloor n^{1/2} \log^{-2} n \rfloor$ and $l = \lfloor h^2 \log^{-1} n / \lceil n^{2\varepsilon} \rceil \rfloor$. Let $T_i$ be the stopping times on the double line $\{(x, \pm ih) : x \in \mathbb{R} \}$. For all $i \in \mathbb{N}$ and $j = l, l + 1, \ldots, 2l$ let $Y_{i,j}$ be the event that $T_i + j \lceil n^{2\varepsilon} \rceil$ is a tan point with respect to $T_i$. Define $X_i := \# \{ j : Y_{i,j} \}$. The first step is to show that

$$P(X_i > n^{3/4-(7/4)\varepsilon+o(1)}) > c.$$

We use second moment methods. First by (6) we have

$$E(X_i) \geq (l + 1) \cdot \left( \frac{n^{\varepsilon}}{2l \lceil n^{2\varepsilon} \rceil} \right)^{1/4 + o(1)} \geq n^{3/4-(7/4)\varepsilon+o(1)}.$$
For the second moment write
\[ E(X_i^2) = \sum_j \mathbb{P}(Y_j) + \sum_{j<k} 2\mathbb{P}(Y_j \cap Y_k) \]
and by lemma 6.5
\[ \leq \mathbb{E}X_i + \sum_{j<k} 2\mathbb{P}(Y_j)\mathbb{P}(Y_{k-j}) \leq \mathbb{E}X_i + 2(\mathbb{E}X_i)^2. \]
By the well known inequality \( \mathbb{P}(X \geq \frac{1}{4}\mathbb{E}X) \geq (\mathbb{E}X)^2/4\mathbb{E}(X^2) \) we get for \( n \) sufficiently large
\[ \mathbb{P}(X_i > n^{3/4-(7/4)\varepsilon+o(1)}) \geq \frac{1}{12}. \]
Next we define \( Y_{i,j}^* \) to be the event
\[ Y_{i,j} \cap \{|R(T_i + j \lceil n^{2\varepsilon} \rceil)_2| > ih + n^{1/4} \log^4 n\}. \]
And \( X_i^* = \#\{j : Y_{i,j}^*\} \). We shall now estimate \( X_i^* \) under the assumption that \( R(T_i)_2 = ih \) (rather than \(-ih\) — the other case is symmetric). Examine the event
\[ B_{i,j} = Y_{i,j} \cap \{|R(T_i + j \lceil n^{2\varepsilon} \rceil)_2 - ih| \leq n^{1/4} \log^4 n\}. \]
By (7) we have (remember the definitions of \( l \) and \( h \)) that
\[ \mathbb{P}(B_{i,j}) \leq \frac{n^{1/4} \log^4 n}{\sqrt{jn^{2\varepsilon}}} \left( \frac{n^{\varepsilon}}{j \lceil n^{2\varepsilon} \rceil} \right)^{1/4+o(1)} \leq n^{-1/2+\varepsilon/4+o(1)} \]
and summing over \( j \) we get \( \leq n^{1/2-(7/4)\varepsilon+o(1)} \). Estimating with Markov’s inequality we see that the \( B_{i,j} \) are negligible and then
\[ \mathbb{P}(\#\{Y_{i,j} \setminus B_{i,j}\} > n^{3/4-(7/4)\varepsilon+o(1)}) > c. \]
Now \( Y_{i,j} \setminus B_{i,j} \) is equal to \( Y_{i,j} \setminus \{its \ symmetric \ image\} \). Therefore we get
\[ \mathbb{P}(X_i^* > N) = \mathbb{P}(\#\{j : Y_{i,j} \setminus (B_{i,j} \cup Y_{i,j}^*)\} > N) \quad \forall N \]
And hence \( \mathbb{P}(X_i^* > N) \geq \frac{1}{2}\mathbb{P}(\#\{Y_{i,j} \setminus B_{i,j}\} > 2N) \).
Finally we define the event
\[ G_i = \{X_i^* > n^{3/4-(7/4)\varepsilon+o(1)}\} \cap \{T_{i+1} - T_i > h^2 \log^{-1} n\}. \]
Then since \( \mathbb{P}(T_{i+1} - T_i \leq h^2 \log^{-1} n) < e^{-c \log^2 n} \) we get that \( \mathbb{P}(G_i) > c. \)
However, \( G_i \) is independent of \( R(T_i) \), including of whether it is in the line \( \mathbb{R} \times \{ih\} \) or \( \mathbb{R} \times \{-ih\} \), since \( Y_{i,j}^* \) and the rest of the conditions are invariant with respect to translations in the \( x \) direction and reflections through the \( x \) axis. Therefore (since \( G_i \) does not examine the walk beyond \( T_{i+1} \))
the $G_i$ are independent events. Further, with probability $> 1 - n^{-2}$ we have $\max_{m \leq n} |R(m)_1| \geq c\sqrt{n/\log n}$ and then there are at least $c \log^{3/2} n$ different $i$-s for which $T_{i+1} < n$. This shows that with probability $> 1 - Cn^{-2}$ at least one of the $G_i$-s occurred. Further, with the same probability we may also assume

$$\max_{m \leq n} |R(m)_1| \leq \sqrt{n \log n}. \quad (9)$$

This finishes the lemma. Indeed, $(9) \cap Y^*_{i,j} \Rightarrow T_i + j \lceil n^{2\varepsilon} \rceil$ is a tan point since the funnel $F + R(T_i + j \lceil n^{2\varepsilon} \rceil)$ intersects the band $\{(x, y) : |y| \leq ih\}$ only for $|x| > c\sqrt{n \log^2 n}$ and $R$ does not go so far. Hence the $T_i + j \lceil n^{2\varepsilon} \rceil$-s for which $Y^*_{i,j}$ occurred are $n^{-\varepsilon}$-separated tan points and the lemma is proved.

**Remark 6.7:** Lemma 6.5 alleviates most of the agony usually associated with second moment methods. However it is by no means necessary. There are at least two additional paths one might take to prove the result i.e. lemma 6.6:

- It is not very difficult to get rid of all the log factors we have so lavishly neglected and show explicitly that $\mathbb{P}(E_3) \approx \frac{1}{\log n} \left( \frac{n^\varepsilon}{m} \right)^{1/4}$ — the $1/\log n$ comes from $E_1$ and is the only log that represents a real phenomenon. Further one can show that $\mathbb{P}(E_3(m, n) \cap \{ R(m)_2 > \sqrt{m} \}) \approx \frac{1}{\log n} \left( \frac{n^\varepsilon}{m} \right)^{1/4}$ which would allow to estimate $X^*_i$ without going through the symmetry argument.

- Alternatively, if the second moment methods only show that $\mathbb{P}(G_i) > c \log^{-10} n$ one can simply take $h = \sqrt{n \log^{-12} n}$. This will give $\log^{11+1/2} n$ possible $i$-s and one of them would satisfy $E_i$.

This concludes what we need to know about simple random walk. The next step is to couple MAW and SRW. We shall do so in the natural way: if the MAW is in a vertex whose left neighbor was visited in the past (NV-vertex), make the MAW and the SRW walk together. Otherwise, do as follows:

- With probability $\frac{1}{6}$ they both walk to the left
- With probability $\frac{1}{6}$ they both walk up, another $\frac{1}{6}$ for right, and another for down.
- With probability $\frac{1}{28}$ the SRW walks left and the MAW walk up, etc.

Denoting by $R$ the SRW and by $E$ the MAW we get that $E(n) - R(n)$ changes only when $E$ is in a non-NV vertex. $(E(n) - R(n))_1$ only increases while $(E(n) - R(n))_2$ performs a random walk at these times.
Lemma 6.8. Let $R$ and $E$ be an SRW and an MAW coupled as above. Let $D(k, l) = E(l) - R(l) - E(k) + R(k)$. Let $n$ be some number. Then

$$\mathbb{P}\left( \exists k < l < n : |D(k, l)| \geq C \log n \sqrt{D(k, l)_{1+1}} \right) \leq n^{-1}$$

for some $C$ sufficiently large.

Proof. Fix some $k < l$. Let $K$ be the number of non-NV vertices visited by $E$ between $k$ and $l$. It is easy to see that

$$\mathbb{P}(D(k, l) + 1 < \frac{1}{\lambda}K) \leq Ce^{-\lambda} \quad \forall \lambda > 0, \forall K$$

so for some $C_1$ sufficiently large, setting $\lambda = C_1 \log(nK)$ will ensure that the probability is $\leq \frac{1}{n^2}(nK)^{-3}$. Denote this event by $B_1(K)$. Next we note that

$$\mathbb{P}(D(k, l) > \lambda \sqrt{K}) \leq Ce^{-\lambda^2} \quad \forall \lambda > 0$$

and setting $\lambda = C_2 \log(nK)$ for some $C_2$ sufficiently large will ensure that the probability is $\leq \frac{1}{n^2}(nK)^{-3}$. Denote this event by $B_2(K)$. We get that

$$\mathbb{P}\left( \bigcup_{K=1}^{\infty} B_1(K) \cup B_2(K) \right) \leq \frac{1}{n^2}.$$  

However, if this event did not happen then (10) gives that $K \leq C \log n(D(k, l) + 1)$ and with (11) we get $|D(k, l)| \leq C \log n(D(k, l) + 1)$. Summing over $k$ and $l$ proves the lemma. \hfill \blacksquare

Lemma 6.9. Let $R$ and $E$ be an SRW and an MAW coupled as above. Assume the event of lemma 6.8 did not happen. Let $m < n$ be a tan point of $R$. Then at least one of $[m - n^\epsilon, m]$ is a non-NV point of $E$.

Proof. If all of $[m - n^\epsilon, m]$ were NV points of $E$, then $E(l) - R(l)$ did not change throughout this time. Hence the first condition in the definition of a tan point, that $R(l)$ did not visit the left neighborhood of $R(m)$, ensures that $E(l)$ did not visit the left neighbor of $E(m)$. So this period is secured. Examine now the time $[0, m - n^\epsilon]$. If for some $l \in [0, m - n^\epsilon]$ we have that $E(l)$ is the left neighbor of $E(m)$, then $R(l) = \text{left neighbor of } R(m) + D(l, m)$. But we assumed (this is the event of lemma 6.8) that $|D(l, m)| \leq C \log n \sqrt{D(l, m)}$. Hence (if $n$ is sufficiently large), $R(l) \in R(m) + F$, in contradiction to the second condition in the definition of a tan point. \hfill \blacksquare

Theorem 6.10. MAW is transient.

Proof. Couple the MAW $E$ to a SRW $R$. Examine the first $n$ steps of both. Couple it to an SRW $R$ as above. By lemma 6.6 there are (with probability $> 1 - C/n$) $n^{3/4+o(1)}$ tan points $m_i < n$ which are separated i.e. $|m_i - m_j| > n^\epsilon$ for all $i \neq j$. By lemma 6.9 this shows that there are at least so many visits of $E$ to non-NV vertices. This shows that with probability $> 1 - C/n$ that $(E(n) - R(n))_1 > n^{3/4+o(1)}$. This shows that with probability 1, $E(2^n) - R(2^n) > 2n^{3/4+o(n)}$. Since $(E - R)_1$ is monotone we get $E(n) - R(n) > n^{3/4+o(1)}$ with probability 1. Hence $E$ is transient. \hfill \blacksquare
7 A conjecture and some open problems

The following conjecture seems the most interesting to us:

**Conjecture 7.1.** Theorems 4.1, 4.2, 4.3 and 4.4 are true for nonadaptive RWCE on any graph.

**Remark 7.2:** Note that these conjectures claim that there is an essential difference between adaptive and non-adaptive walks. The Theorems in this paper do not provide proof of any such difference. However, as pointed to us by Ben Morris, it seems that using the methods of evolving sets ([11]) it is possible to show that when the graph satisfies an isoperimetric inequality that implies transience (e.g. \( \mathbb{Z}^d \) for \( d \geq 3 \)) and the environment is monotone, bounded between two constant multiples of the same conductance function and nonadaptive the RWCE is also transient. On the other hand, an adaptive example similar that in Section 6 can likely be constructed on \( \mathbb{Z}^d \) by mimicking the behaviour of excited random walk towards the middle - a walk which gets a bias towards the origin every time it visits a new vertex, for which there is a sketch of proof for recurrence. It seems, however, that proving recurrence of the "excited towards the middle" RWCE is more technically involved then Theorem 6.10.

One could also consider a continuous time version of the RWCE, where the edges are equipped with Poisson clocks with rates equal to their conductance, and the walk jumps over whatever edge rings first. There seems to be an essential difference between the continuous time and discrete time models, which is that for a continuous time random walk on a graph, the stationary measure is always uniform. This seems to affect greatly the amount of control one can get using adaptivity. We therefore ask:

**Question 7.3.** Do Theorems 4.1, 4.2, 4.3 and 4.4 hold for continuous time RWCE on any graph?

In fact, one could ask the same thing in the discrete time RWCE, by simply demanding that the stationary measure on all the graphs \( G_i \) are equal. (e.g. by keeping the sum of the conductances from each vertex fixed).

**Question 7.4.** Do Theorems 4.1, 4.2, 4.3 and 4.4 hold for RWCE on any graph as long as we demand all \( G_i \) share the same stationary measure?

The latter question seems closely related to the results of [1] on RWCE’s on finite graphs, where it is shown that contrary to the general case where the cover time may be exponential, for a sequence of graphs with common stationary measures, the cover time is only polynomial.

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