Mathematical modeling of Russian northern coastal waters in the shallow water approximation

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Abstract. In this paper, the authors present a model for computing a coastal waters flow in the shallow seas. The software implementation of the model consists in coupling the global software and the local hydrodynamic computations. The authors describe the use of the joint two computational packages: the global model for the ocean dynamics NEMO (Nucleus for European Modelling of the Ocean) which provides the initial and boundary conditions and the code WRF (Weather Research and Forecasting) for predicting meteorological conditions is used to calculate the wind loads. A hydrodynamic model based on the shallow water (SW) approximation is used to compute the coastal waters flow within a local domain, typically, 1000 km in size. The software implementation of the algorithm is based on regularized SW equations. As an example, the authors have computed the flow of the coastal waters in the vicinity of the Kara Strait. The study carried out demonstrate that all three algorithms successfully combined in the framework of the local computation and give the consistent results.

1. Introduction

Modeling natural waters is a complex problem whose solution requires taking into account many different parameters, such as meteorological conditions, ice situation, tides, bathymetry features. For this reason, practical applications need a complex of codes and models interacting with one another.

One possible approach to compute the sea flows dynamics consists in using a system of hydrodynamic equations as a basis and additional packages as independent modules that provide boundary and initial conditions as well as external conditions such as external forces. This approach results in the fact that external models do not respond to changes in the state of the sea. This probably reduces the accuracy of the results but simplifies the model used and leads to more time-efficient computations. However, even with module independency and without feedback, combining software packages in the framework of hydrodynamic equations is a rather labor-consuming task.

Generally, hydrodynamic problems are treated with 3D models [1, 2, 3]. However, even when a high accuracy is not required, a full description of the problem, including the treatment of vertical flows presents significant drawbacks: the complexity of its implementation and large computational times. Two-dimensional models are a useful alternative, such as models...
that average 3D equations over the water depth: the shallow water (SW) approximation. This approach can be justified in the case of a small depth and a weak stratification of the water volume. The advantage of reducing the dimensionality of the problem is a simplified implementation and results are obtained rapidly. As an example, a 2D model has already been used to predict stormy surges in the Okhotsk Sea [4].

In the present study, the sea dynamics is described in the 2D SW formulation as:

\begin{equation}
\frac{\partial h}{\partial t} + \text{div} \left( h \mathbf{u} \right) = 0,
\end{equation}

\begin{equation}
\frac{\partial (h \mathbf{u})}{\partial t} + \text{div} \left( h \mathbf{u} \otimes \mathbf{u} \right) + \nabla gh^2 = h (f^v - g \nabla b) + f^s.
\end{equation}

Referring to Figure 1, the following notations are used: \( \mathbf{x} = (x, y) \), \( h(\mathbf{x}, t) \) is the thickness of the water layer measured from the bottom, \( b(\mathbf{x}, t) \) is the bottom level, measured from the deepest point, \( \mathbf{u} = \{ u_x(\mathbf{x}, t), u_y(\mathbf{x}, t) \} \) – is the horizontal velocity vector, \( g \) is the gravity acceleration directed downwards along the \( z \)-axis, \( \mathbf{f}^v = \{ f^v_x(\mathbf{x}, t), f^v_y(\mathbf{x}, t) \} \) is the vector of the volume external forces acting on the whole water depth, \( \mathbf{f}^s = \{ f^s_x(\mathbf{x}, t), f^s_y(\mathbf{x}, t) \} \) is the vector of the surface external forces, \( \xi(\mathbf{x}, t) = h(\mathbf{x}, t) + b(\mathbf{x}) \) is the water surface level.

It is convenient to introduce \( \eta = h_0 - h(\mathbf{x}, t) \), i.e. the departure of the water level from its equilibrium state value \( h_0 \).

2. Regularized shallow water equations

To solve the hydro-and gas-dynamic equations using the finite difference explicit algorithm developed by B. N. Chetverushkin, T. G. Elizarova and their students, the regularization of those equations has been introduced [5, 6] and has received further developments [7].

The regularization of equations (1)–(2) resulted in a system of Regularized Shallow Water Equations (RSWE) [8] which was solved by the finite difference explicit algorithm that involved two time steps at each time layer. This approach was validated for a number of test cases and practical problems [8, 9, 10].

The RSWE system is written down as

\begin{equation}
\frac{\partial h}{\partial t} + \text{div} \mathbf{j}_m = 0,
\end{equation}

\begin{equation}
\frac{\partial (h \mathbf{u})}{\partial t} + \text{div} \left( \mathbf{j}_m \otimes \mathbf{u} \right) + \nabla gh^2 = h^* (f^v - g \nabla b) + f^s + \text{div} \mathbf{\Pi},
\end{equation}

\begin{equation}
h^* = h - \tau \text{div} (h \mathbf{u}),
\end{equation}

\begin{equation}\mathbf{j}_m = h (\mathbf{u} - \mathbf{w}),\end{equation}
where \( W, M_2, N_2, O_1, P_1, Q_1, S_2 \): associated with the sea, the water level is given as the sum of the eight tide harmonics: \( K_1, K_2, \ldots \).

This process, local changes of the sea level are less influential than the global tide waves.

An important factor that governs a flow as well as the sea level is the occurrence of tides. In this paper, the Coriolis force and the tide were taken into account in addition to the wind friction and the bottom friction.

3. Accounting for external factors

To improve the description of the problem, a number of external factors must be regarded. In the present paper, the Coriolis force and the tide were taken into account in addition to the wind friction and the bottom friction.

3.1. The Coriolis force and friction forces

The Coriolis force is a volume force equal to

\[
\tau^c = f^c u_y, \quad f^c_y = -f^c u_x,
\]

where \( f^c = 2\Omega \sin \varphi \), and \( \Omega = 7.2921 \cdot 10^{-5} \text{ c}^{-1} \) is the angular velocity of the Earth.

The friction force due to the wind is denoted as \( \tau^w = \{\tau^w_x, \tau^w_y\} \) and estimated by an empirical formula [11]:

\[
\tau^w = \gamma |W| W, \quad \gamma = 0.001 \frac{\rho_a}{\rho_w} (1.1 + 0.04 |W|),
\]

where \( W = \{W_x(x, y, t), W_y(x, y, t)\} \) is the wind velocity \( |W| = \sqrt{W_x^2 + W_y^2} \); \( \gamma \) is the wind friction coefficient on a free water surface; \( \rho_a = 1.3 \cdot 10^{-3} \text{ g/cm}^3 \) and \( \rho_w = 1.025 \text{ g/cm}^3 \) are the air and water densities, respectively. The coefficient 0.04 in (10) has the dimension in \( (\text{m/s})^{-1} \).

The friction force along the bottom is described as a classical quadratic friction. It is denoted as \( \tau^b = \{\tau^b_x, \tau^b_y\} \) and estimated in the same way as the wind force:

\[
\tau^b = -\mu^b |u| u,
\]

where \( |u| = \sqrt{u_x^2 + u_y^2} \) and \( \mu^b \) is the bottom friction coefficient.

Thus, the volume and surface forces in the RSWE are written down as

\[
\begin{align*}
\tau^v_x &= f^c u_y, \quad \tau^v_y = -f^c u_x; \\
\tau^s_x &= \gamma |W| W_x - \mu^b |u| u_x, \\
\tau^s_y &= \gamma |W| W_y - \mu^b |u| u_y.
\end{align*}
\]

3.2. The tide

An important factor that governs a flow as well as the sea level is the occurrence of tides. In this process, local changes of the sea level are less influential than the global tide waves.

In the present study, similar to [12], at the boundaries of the computational domain that are associated with the sea, the water level is given as the sum of the eight tide harmonics: \( K_1, K_2, M_2, N_2, O_1, P_1, Q_1, S_2 \):

\[
\eta(x, y, t) = \eta_0 + \sum_{i=1}^{8} f_i A_i(x, y) \cos(q_i t + \psi_0i - g_i(x, y)),
\]
where \( \eta_0 \) is the average sea level in the computational domain; \( t \) is the time (UTC) expressed in hours with origin at midnight; \( A_i(x,y), g_i(x,y) \) and \( q_i \) are the amplitude, phase and angular velocity, respectively for the \( i \)-th tidal harmonic at each point of the liquid boundary; \( \psi_0 i \) and \( f_i \) are the initial phase and an attenuation factor determined by astronomical data. The formulas for \( \psi_0 i \) and \( f_i \) are well described in [13].

4. The finite difference approximation

The properties of equation system (3)–(8) have been extensively studied and are well known. As was done for the Quasi Gaso-Dynamic (QGD) equations, their discretization is based on a rectangular uniform grid. The basic variables \( h(x,y,t) \), \( b(x,y) \) and \( u(x,y,t) \) affect the central nodes of the space grid \((i,j)\). Their values at the intermediate points \((i \pm 1/2,j)\) and \((i,j \pm 1/2)\) are obtained as the half-sum of their values at the neighboring nodes, i.e., \( h_{i \pm 1/2,j} = 0.5(h_{i+1,j} + h_{i,j}) \). Their values at the centers of cells are obtained as the arithmetical average of their values at the neighboring nodes, i.e., for instance, \( h_{i+1/2,j+1/2} = 0.25(h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}) \).

The quantities \( u_x, u_y, b \) are approximated in the same way. The approximation of flow quantities is done at half-points. The finite difference scheme for the RSWE system is given in [8, 9].

The stability of the method described is governed by the term that includes the coefficient \( \tau \) given by:

\[
\tau = \alpha \frac{l}{c},
\]

(15)

where \( l \) is a characteristic length of the space cell, e.g. \( l = \sqrt{s} \), where \( s \) is the cell area; \( c = \sqrt{gh} \) is the velocity of long wave propagation; \( \alpha \) is the stability coefficient chosen to ensure both the accuracy and stability of the computation. Usually \( 0 < \alpha < 1 \) and a typical value \( \alpha = 0.5 \) can be retained as a first guess. For a correct choice of \( \alpha \), the parameter \( \tau \) characterizes the time needed for a small perturbation to travel through the area of a computational cell.

The stability condition has the form of the Courant condition [7], where the time step is chosen as

\[
\Delta t = \beta \left( \frac{\Delta x + \Delta y}{2c} \right)_{\text{min}}.
\]

(16)

The Courant factor \( 0 < \beta < 1 \) depends on the regularization parameter \( \tau \) as \( \beta = \beta(\alpha) \) and is chosen during the computation to ensure a monotonic numerical solution.

When modeling natural waters, it is often necessary to account for changes in the location of the bank boundary, i.e. the limit between a water and a dry domain. The peculiarity of the dry domain is a zero water depth \( h = 0 \). This requires prescribing special conditions in the algorithm to avoid the formation of numerical instabilities.

The description of the dry domain used in the present study, has been presented by the authors (e.g. in [9]), as well as in many other papers [14, 15].

Accounting for the boundary of the dry domain is done through the clipping parameter \( \varepsilon \) that defines the minimal value of the water depth \( h \) that can be calculated using the finite difference scheme.

As applied to the RSWE, the dry bottom condition is written down as

\[
u_x = u_y = 0 \text{ if } h \leq \varepsilon.
\]

(17)

Furthermore, if a dry cell is higher than its neighbor, \( \xi_{\text{dry}} > \xi_{\text{neighbor}} \), to both of them, an additional condition to ensure the absence of mass-flow through the boundary is given. As applied to the RSWE, this condition is as follows:

\[
\tau = 0.
\]

(18)
5. Computational domain
Modeling the seas, as well as predicting the sea level and studying its variability is an urgent
task for the Northern regions of Russia. One of the most important Russian transport arteries,
namely, the Northern Sea Route (NSR) in fact runs there, and most important harbors are
located along that coast. For this reason, the present research addresses the region of the Kara
Gate Strait.

A rectangular domain that includes the Kara Gate Strait, the Pechora Sea and a part of the
Kara Sea is defined by longitudes between 51.425 and 69.275 degrees East and latitudes between
68 and 71.8 degrees North (Figure 2). In the strait region, a rectangular grid is built with the
Cartesian coordinates \( x \) along the latitude circles and \( y \) along the meridians.

The data relative to the sea bottom were taken from the General Bathymetric Chart of the
Oceans (GEBCO), whose spatial resolution is \( 0.0083^\circ \). The bathymetry of the simulated region
is shown in Figure 2.

![Figure 2. Bathymetry of the simulated region.](image)

The simulated region has the size of \( 2290 \times 456 \) points, but to ensure the grid uniformity
a limitation was applied in the horizontal direction: \( 715 \times 456 \) points, which corresponds to
\( \Delta x = 956 \) m, \( \Delta y = 926 \) m. The size of the computational domain is approximately \( 682 \times 421 \)
\( \text{km}^2 \).

The location of the computational domain was chosen to ensure that the vertical (East) and
the horizontal (South) boundaries were fully in a dry zone. This choice does not affect the
physical problem and simplifies the expression of boundary conditions.

6. Initial and boundary conditions
Initial and boundary conditions were obtained by using the model NEMO (Nucleus for European
Modelling of the Ocean). This model of the global oceanic circulation was developed at the
Institute of Pierre-Simon Laplace (Paris) and is described in numerous papers, e.g. [16, 17].
Computational parameters required by the NEMO and used in the present study are described
in [18].

For the present study the data from NEMO were obtained as an average of a day.
The volume of the available NEMO output data is very large. In view of using the RSWE,
the following data were considered:

- \( ssh \) distribution of the sea surface altitude, further denoted as \( \hat{h}_{\text{NEMO}} \).
• $\hat{u}, \hat{v}$ distribution of the flow velocity dependence on the depth from West to East and from North to South, respectively.

Because the shallow water model considers the horizontal velocity to be averaged over the depth, the velocities $u_x, u_y$ were taken as the mean values of $\hat{u}$ and $\hat{v}$, respectively (further denoted as $\hat{u}_{\text{NEMO}}$ and $\hat{v}_{\text{NEMO}}$).

The NEMO data have a space resolution of $0.25^\circ$. Therefore, a bilinear interpolation was used to obtain the velocity and the sea surface altitude. The computational domain in the NEMO has $77 \times 17$ resolution, with 13 points along the West boundary and $\approx 55$ along the North boundary, resulting in more than 650 points within the water domain.

Along the West and North boundaries, the sea level is given as the sum of eight tide harmonics. The levels obtained by the NEMO were used as the mean sea level in the domain under consideration.

$$h|_r = \hat{h}_{\text{NEMO}} + \sum_{i=1}^{8} f_i A_i(x, y) \cos (q_i t + \psi_{0i} - g_i(x, y)),$$

where $\hat{h}_{\text{NEMO}}$ is first time-interpolated at given points of the initial NEMO grid, then space-interpolated to the nodes of the RSWE grid. The tide waves are computed using the constants $A_i$ and $g_i$, obtained by the OSU TPXO Tidal Model [19], a model for the global tides on a spherical grid with $10 \times 10^3(1/6^\circ)$ resolution.

As was done in [12], the derivatives of both components of velocity are equal to zero along the West and North liquid boundaries:

$$\frac{\partial u_x}{\partial n} = \frac{\partial u_y}{\partial n} = 0,$$

where $n$ is the direction perpendicular to the boundary oriented outwards.

7. Computation of the wind speed using the package WRF

The computational package Weather Research and Forecasting (WRF) [20] is widely used for the prediction of meteorological conditions and includes the computation of the wind loads. It is also used in conjunction with hydrodynamic circulation models, e.g. [3].

The WRF is based on non-hydrostatic equations for compressible fluids, written in the horizontal Cartesian coordinates and vertical orographic coordinates.

The WRF computational grid consists of a parallelepiped that is horizontally rectangular and divided by meridians and parallels with $75 \times 45$ resolution, $\Delta x = \Delta y = 10$ km. Its center is located at the coordinates $70^\circ N, 60^\circ E$. In the vertical direction, 41 levels are considered. The global data were taken from the database [21]. The physical model parameters are described in [22].

A comparison of the WRF outputs with historical data is shown in Figure 3. Historical data were observed at the meteorological station (E. K. Fedorov, WMO index – 20946), whose coordinates are $70^\circ 26' N, 59^\circ 5' E$, and the altitude is 12 m above the sea level. As is seen in Figure 3, both the speed and direction of the wind are in reasonable agreement between the model and observations.

8. Computational results

As an example, the computation was carried out for a 96-hour period from 12:00, June 1, 2013, up to 12:00 June 5, 2013. The shallow water model does not account for the ice cover. This explains the choice of a period of the year when the ice cover of the northern seas is at a minimum.
Figure 3. Comparison of the WRF computations with observations of the meteorological station of E. K. Fedorov (WMO index – 20946). The origin of time corresponds to the start of the computation; (a) the wind speed, (b) azimuth of the direction from which the wind is blowing.

The following parameters were taken for the computation:

\[ \alpha = 0.5, \quad \beta = 0.2, \quad \varepsilon = 0.1 \text{ m}. \] (21)

The coefficient of the bottom friction \( \mu^b \) was taken from [12]: \( \mu^b = 2.5 \cdot 10^{-3} \).

The origin of the time in the figures corresponds to the start of the computation.

The sea level evolution, as well as that of the streamlines during a period of 24 hours are visible in Figure 4. The tide flows formation is visible with the speeds \( |u| \sim 15 - 40 \text{ cm/s} \) (Figure 5).

A change in the sea level at a given point is even more conclusive. This point was chosen as the location of the platform "Prirazlomnaya" 69°15'57"N, 57°17'09"E. The sea level evolution during 84 hours is plotted in Figure 6, as it is obtained from the model RSWE and from the tide formula. The RSWE results are in reasonable agreement with those of (14). A small difference (6–10 cm) can be explained by the fact that equation (14) describes the only tide, whereas the RSWE also accounts for the wind, bottom friction and Coriolis forces. Anyway, the comparison in Figure 6 demonstrates that the model RSWE reproduces tide oscillations inside the domain when the sea level at boundaries is given by formula (19).

9. Conclusion
In the present paper, three models were successfully combined: a global model for the oceanic dynamics NEMO, a predictive model for meteorological conditions WRF and the Regularized Shallow Water Equations (RSWE). The algorithm for dry regions used in the present study appears to be rather a simple description of the complex coastline even on a rectangular space grid, which is often complicated in the global models as well as in many local models.

The computations demonstrate that the model RSWE well describes the tide flows and even oscillations of the sea level even for the formulation of the problem. At the present time, the full combined model is available to solve specific problems, which is already being planned.

Acknowledgments
The authors gratefully acknowledge that the present research is supported by the Russian Science Foundation (project 19-11-00076).
Figure 4. The calculated departure of the sea level from its equilibrium value and a plot of streamlines; (a) $t = 6$ h; (b) $t = 12$ h; (c) $t = 18$ h; (d) $t = 24$ h.

Figure 5. The wind speed $|u|$ and streamlines at the time $t = 40$ h from the start of the calculation.

The authors are also grateful to J.-C. Lengrand for fruitful discussions about the RSWE applications and assistance in the manuscript preparation.
Figure 6. The sea level evolution at the location of the station "Prirazlomnaya", 69°15′57″N 57°17′09″E for 84 hours.

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