Mixed Integer Programming to Globally Minimize the Economic Load Dispatch Problem With Valve-Point Effect

Michael Azzam, S. Easter Selvan, Augustin Lefèvre, and P.-A. Absil

Abstract—Optimal distribution of power among generating units to meet a specific demand subject to system constraints is an ongoing research topic in the power system community. The problem, even in a static setting, turns out to be hard to solve with conventional optimization methods owing to the consideration of valve-point effects which make the cost function nonsmooth and nonconvex. This difficulty gave rise to the proliferation of population-based global heuristics in order to address the multi-extremal and nonsmooth problem. In this paper, we address the economic load dispatch problem (ELDP) with valve-point effect in its classic formulation where the cost function for each generator is expressed as the sum of a quadratic term and a rectified sine term. We propose two methods that resort to piecewise-quadratic surrogate cost functions, yielding surrogate problems that can be handled by mixed-integer quadratic programming (MIQP) solvers. The first method shows that the global solution of the ELDP can often be found by using a fixed and very limited number of quadratic pieces in the surrogate cost function. The second method adaptively builds piecewise-quadratic surrogate under-estimations of the ELDP cost function, yielding a sequence of surrogate MIQP problems. It is shown that any limit point of the sequence of MIQP solutions is a global solution of the ELDP. Moreover, numerical experiments indicate that the proposed methods outclass the state-of-the-art algorithms in terms of minimization value and computation time on practical instances.

Index Terms—Economic load dispatch, Global convergence, Mixed integer quadratic programming, Valve-point effect.

I. INTRODUCTION

ECONOMIC LOAD DISPATCH PROBLEM (ELDP) attempts to minimize the cost associated with the power generation by optimally scheduling the load across generating units to meet a certain demand subject to system constraints. It is not uncommon to notice that the cost function for a generator is approximated by a quadratic function for the sake of simplicity. Nevertheless, when the cost function also takes into account highly nonlinear input-output characteristics due to valve-point loadings, even the static ELDP problem that ignores the ramp-rate constraints turns out to be difficult to solve. The challenges faced by the solvers stem from (i) the nonsmooth cost function and (ii) the multi-extremal nature of the problem.

A popular strategy to address the ELDP is to rely on a (population-based) stochastic search algorithm. Indeed, a myriad of such algorithms have been proposed during recent years, including genetic algorithms, evolutionary programming, particle swarm optimization, ant swarm optimization, differential evolution, firefly algorithm, bacterial foraging algorithm, and biogeography-based optimization.

Since heuristics enable the global exploration, and a local method aids to converge to a local optimum, a focussed effort has been made by many to integrate a global heuristic with a local optimizer, resulting in hybrid algorithms. Interested readers may refer to for an exhaustive list of such algorithms in the ELDP context. A few well-known local optimizers integrated with a global scheme to tackle the ELDP are the Nelder-Mead method, generalized pattern search, sequential quadratic programming, and a Riemannian subgradient steepest descent. Note that the equality constraint must be handled by way of a slack variable or a barrier approach, and the inequality constraints with the help of a penalty approach in global heuristics. Furthermore, even though the global heuristics favor finding the global minimum, these hybrid algorithms are guaranteed at best to find a local minimum.

In this paper, we introduce new techniques to efficiently build surrogates of the ELDP cost function in order to take advantage of powerful modern mixed-integer programming (MIP) solvers. In particular, the adaptive technique introduced in Section builds a sequence of surrogate piecewise-quadratic cost functions that aims at keeping low the number of pieces for the sake of efficiency while nevertheless offering the guarantee that the sequence of surrogate solutions converge to the global solution of the ELDP. It is interesting to note that the MIP method vested with theoretical guarantees as applied to a static ELDP with the valve-point effect may be extended to a dynamic setting, provided the number of generating units is not too large. Finally, we demonstrate that the minimization results from a 3-, two instances of 13-, and a 40-generator settings are lower than the results reported thus far in the literature with the same datasets.

While the rectified-sine model of the valve-point effect was proposed more than 20 years ago, it is only in the past few months that MIP techniques appeared in the literature to handle this problem formulation. The methods introduced in this paper contribute beyond this recent literature in two ways. (i) The approach proposed in Section while sharing several aspects with the recent report introduces fewer breakpoints in the surrogate cost function, allowing for a reduced complexity. Moreover, it takes...
into account that it is only the rectified sine term that is piecewise concave in (1b); this leads a piecewise-quadratic
under-approximation of (1b) that is handled by mixed-integer quadratic programming (MIQP).

The remainder of the article is organized as follows. In Section II, the ELDP with the valve-point effect is briefly presented. The principle behind our approach is first expounded in Section III using a simple linear approximation of the term that accounts for the nonconvexity and nonsmoothness. Section IV reviews MIP formulations for general piecewise-linear objective functions and shows how this technique can be integrated in the ELDP. The adaptive algorithm is introduced and its convergence analysis carried out in Section V. Numerical results are reported in Section VI, and conclusions are drawn in Section VII.

II. PROBLEM STATEMENT

In this section, we recall the formulation of the widely investigated ELDP with valve-point effect, as described, e.g., in [9].

In the ELDP, the main component of the cost that needs to be taken into account, is the cost of the input, that is to say the fuel. The objective function used in the problem is thus defined by how we represent the input-output relationship of each generator. The total cost is then naturally the sum of each contribution. The objective function is thus written

$$f(p) = \min_{p \in \mathbb{R}^n} \sum_{i=1}^{n} f_i(p_i), \quad (1a)$$

where $f$ is the total cost function in $$/h$$, equal to the sum of the $n$, $f_i$ univariate functions that give the individual contribution in the total cost of the $i$th generator, depending on the $p_i$ amount of power, in MW, assigned to this unit.

A classical, simple and straightforward approach to construct the cost functions is to use a quadratic function for each generator, i.e.,

$$f_i(p_i) = a_i p_i^2 + b_i p_i + c_i,$$

where $a_i$, $b_i$ and $c_i$ are the scalar coefficients.

However, in reality, performance curves do not behave so smoothly. In the case of generating units with multi-valve steam turbines, ripples will typically be seen in the curve. Large steam turbine generators usually have a number of steam admission valves that are opened in sequence to meet an increasing demand from a unit. And as each steam admission valve starts to open, a sharp increase in losses due to wire-drawing effects occur [9], [16]. This is the so-called valve-point effect. To try to capture this effect in the model, a rectified sine term is usually added to the fuel cost functions, so that they become

$$f_i(p_i) = a_i p_i^2 + b_i p_i + c_i + d_i |\sin(e_i(p_i - p_i^{min}))| \quad (1b)$$

where $d_i$ and $e_i$ are additional positive coefficients needed to take the valve-point effect into account [2]. We can see how the new term affects the cost function in the example of Fig. 1.

Unfortunately, this addition brings two detrimental properties in the problem: non-convexity and non-differentiability. These are two major hindrances that prevent the direct application of usual optimization algorithms.

Fig. 1. Examples of cost functions for a generator, without (solid) or with (dashed) valve-point effect.

Naturally, the problem has also some constraints that must be satisfied and restrict the search space. First, the producer must meet the demand, even though some power will be lost in the network. This is the power balance constraint, formulated through an equality constraint

$$\sum_{i=1}^{n} p_i = p_D + p_L(p) \quad (2)$$

with scalar $p_D$ and function $p_L$ being respectively the demand on the system and the transmission loss in the network, both in MW. The transmission loss is computed using the so-called $B$-coefficients as

$$p_L(p) = p^T B p + p^T b^0 + b^{00} \quad (3)$$

where $B$ is a symmetric positive-semidefinite matrix, $b^0$ a vector of size $n$, and $b^{00}$ a scalar.

The other type of constraints are generator capacity constraints, which take the form of box constraints imposing that each unit has its own range of possible power generation, from $p_i^{min}$ to $p_i^{max}$. This is easily transcribed as inequality constraints

$$p_i^{min} \leq p_i \leq p_i^{max}, \quad (4)$$

where $p_i^{min}$ and $p_i^{max}$ are obviously the minimum and maximum power output of the $i$th generator, in MW.

The set of points that satisfy constraints (2) and (4) is termed the feasible set of the ELDP.

As is often done, we will ignore the transmission loss in the network, i.e., we set $p_L = 0$ in (2). The static ELDP without losses is then classically written as

$$\min_{\mathbb{R}^n} \sum_{i=1}^{n} a_i p_i^2 + b_i p_i + c_i + d_i |\sin(e_i(p_i - p_i^{min}))|$$

subject to

$$\sum_{i=1}^{n} p_i = p_D,$$

$$p_i^{min} \leq p_i \leq p_i^{max} \quad \forall i \in \{1, ..., n\} \quad (5)$$
However, as we point out in the concluding section, considering the loss does not add much complexity.

III. A FIRST SIMPLE MIQP APPROACH

We now introduce our first, crude but effective way of building a piecewise-quadratic surrogate of the ELDP cost function \( f \), and we show how to solve the resulting surrogate problem with an MIQP solver.

As discussed in the previous section, the term

\[
| \sin(e_i(p_i - p_i^{\text{min}})) |
\]

of the objective function makes the optimization problem challenging because it breaks its smoothness and convexity.

In this paper, we overcome the difficulty by approximating \( | \sin x | \) with functions that are more manageable, namely piecewise-linear functions. Even though they do not restore smoothness nor convexity of the problem, they are conveniently handled by mixed-integer programming.

A first simple piecewise-linear approximation consists of replacing \( | \sin x | \) by \(| x |\) over \([−π/2, π/2]\) and completing the approximation over the whole domain by periodic extension, observing that \( | \sin x | \) is periodic of period \( π \). The underlying motivation is to keep low the number of linear pieces while capturing accurately the behavior of \( | \sin x | \) around its kink points, as the optimum tends to be located at those kink points.

The resulting function can be compactly written as

\[ | \arcsin(\sin x) | = \min_{k \in \mathbb{Z}} | x - k\pi | \]

which can be reformulated to become linear, as follows:

\[
\min_{(k,t) \in \mathbb{Z} \times \mathbb{R}} t
\]

\[ s.t. \quad x - k\pi \leq t \]

\[ x - k\pi \geq -t. \]

The problem is thus very simple, even though we introduce an integer variable. By introducing this in the original problem, we get

\[
\min_{p,k,t} \sum_i a_ip_i^2 + b_ip_i + c_i + d_i t_i
\]

\[ s.t. \quad \sum_i p_i = D \]

\[ p_i^{\text{min}} \leq p_i \leq p_i^{\text{max}} \]

\[ -t_i \leq e_i(p_i - p_i^{\text{min}}) - \pi k_i \leq t_i \]

\[ p_i \in \mathbb{R} \]

\[ t_i \in \mathbb{R} \]

\[ k_i \in \mathbb{N} \]

which is an MIQP problem. This class of problems can be solved exactly by solvers, for instance, CPLEX, Gurobi or Mosel. In Section VI, we see that this model gives results that are competitive with other methods suggested in the literature.

IV. FINER APPROXIMATION WITH PIECEWISE LINEAR FUNCTIONS

In this section, we show how to handle general piecewise-linear surrogate objective functions, then we consider specifically an over-approximation obtained from tangents to the rectified sine function.

Let us say we want to minimize on an interval \([v,w]\), a piecewise linear function \( g \) described by the slopes \( \alpha_i \) and intercepts \( \beta_i \) of its \( m \) line segments components as well as its breakpoints \( X_1 = v, X_2, \ldots, X_{m+1} = w \). This objective function can be expressed as a mixed integer linear problem such as

\[
\min \sum_{i=1}^{m} \alpha_i \chi_i + \beta_i \eta_i
\]

\[ s.t. \quad \sum_{i=1}^{m} \chi_i = x \]

\[ \sum_{i=1}^{m} \eta_i = 1 \]

\[ X_i \eta_i \leq x_i \leq X_{i+1} \eta_i \quad \forall i \in \{1, \ldots, m\} \]

\[ \eta_i \in \{0,1\} \quad \forall i \in \{1, \ldots, m\}. \]

This system of constraints ensures that for a given \( x \), only one of the binary variables \( \eta_i \) will be equal to one, indicating which segment is active, while one of the real variables \( \chi_i \) will hold the value of \( x \) and the others will be equal to zero. Thus, only one term of the objective will be non zero. It will be equal to the value of the linear function of slope \( \alpha_i \) and intercept \( \beta_i \) at this abscissa.

We now integrate this technique in our model. Instead of replacing \( | \sin x | \) by \( \min_{k \in \mathbb{Z}} | x - k\pi | \), we use the output of this as the abscissa for finer approximation through piecewise linear functions. For example, with \( m_i \) segments for the cost function of unit \( i \),
minimize \( \sum_i p_i \) subject to
\[
\begin{aligned}
& p_i^\text{min} \leq p_i \leq p_i^\text{max} \\
& t_i \leq \epsilon_i (p_i - p_i^\text{min}) - \pi k_i \leq t_i \\
& X_j \eta_{i,j} \leq \chi_{i,j} \leq X_{j+1} \eta_{i,j} \\
& t_i = \sum_{j=1}^{m_i} \chi_{i,j} \\
& \sum_{j=1}^{m_i} \eta_{i,j} = 1 \\
& p_i \in \mathbb{R} \\
& t_i, \chi_{i,j} \in \mathbb{R}^+ \\
& k_i \in \mathbb{N} \\
& \eta_{i,j} \in \{0, 1\}
\end{aligned}
\]

where \( t_i \) are the distances to the nearest root, \( \chi_{i,j} \) and \( \eta_{i,j} \) are auxiliary variables that help us get the right value in the objective function depending on which segment we are located in and \( \alpha_{ij} \) and \( \beta_{ij} \) are constants that parametrize the line segments.

A course of action we could opt for is following the logic of the previous section and build an over-approximation with first-order Taylor approximation to the sine at different points.

With three segments, it would mean we want to approximate \( \sin x \) by \( \min(x, T_1(x), T_2(x)) \), where \( T_1(x) \) and \( T_2(x) \) are the tangents to \( \sin x \) at respectively \( \theta_1 \) and \( \theta_2 \). A bit of algebra gives us the parameters

\[
\alpha_j = \cos \theta_j \\
\beta_j = \sin \theta_j - \theta_j \cos \theta_j
\]

and the endpoints of the intervals
\[
X_1 = \frac{\sin \theta_1 - \theta_1 \cos \theta_1}{1 - \cos \theta_1} \\
X_2 = \frac{\theta_2 \cos \theta_2 - \theta_1 \cos \theta_1 - \sin \theta_2 + \sin \theta_1}{\cos \theta_2 - \cos \theta_1}
\]

This is illustrated in Fig. 3. Numerical results are reported in Section VI.

V. GLOBAL METHOD BASED ON ADAPTIVE UNDER-APPROXIMATION

In this section, we will describe an algorithm that provably converges toward a global minimum of the ELDP. The way to achieve this is to change our approach to use under-approximations instead of over-approximations.

Over-approximations with tangents seem natural because we can then infer properties from Taylor’s theorem, but under-approximations have the useful property that if the under-approximation and the true function coincide at a global minimizer \( x^* \) of the under-approximation, then \( x^* \) is also a global minimizer of the true function.

**Proposition 1**: Let \( f : X \mapsto \mathbb{R} \) and \( g : X \mapsto \mathbb{R} \) be two functions such that \( g(x) \leq f(x) \) \( \forall x \in X \) and \( x^* \in X \) a global minimizer of \( g \). If \( g(x^*) = f(x^*) \), then \( x^* \) is a global minimizer of \( f \).

**Proof**: Since \( x^* \) is a global minimizer of \( g \), \( g(x^*) \leq g(x) \) for every \( x \in X \). And because \( g \leq f \), we have \( g(x^*) \leq g(x) \leq f(x) \) for every \( x \in X \). Finally, since \( f(x^*) = g(x^*) \), we can conclude.

So if we were able to find such a under-approximation, then we would have found the global minimum. What we suggest is to build a sequence of under-approximations that achieve this goal at the limit.

As before, we will make use of piecewise linear functions to approach the sine part of the real cost function. We start off with a simple chord of \( \sin x \) that links its extreme points. We then solve problem (9). We want the next approximation in the sequence to be equal to the true objective function at the solution \( p^0 \) we found for the first approximation. Therefore, we use the values of \( t_i \) as breaking points for the new piecewise linear approximation and compute the coefficients of the line segments so that \( g^1(p^0) = f(p^0) \). We can now simply repeat this procedure again, find the new solution \( p^1 \), build a new approximation \( g^2 \), so on, till convergence is reached.

At each iteration, the approximation becomes closer to the true function. Of course, it is very possible that for a generator, its assigned power does not change from one iteration to another and so, the approximation will stay the same. This is actually desired because it means we add less complexity than we might have expected.

This algorithm can be formalized for our specific needs as follows, where \( X_i \) is the set of break points for cost function \( i \), \( m_i \) the number of segments for its approximation, \( X_{i,j} \) the \( j \)th element of \( X_i \) in ascending order, \( \alpha_{ij} \) and \( \beta_{ij} \) the coefficients of line segment \( j \) of cost function \( i \), \( \delta \) the change in optimal value as a proxy to measure convergence, and \( \epsilon \) a given positive parameter.

We now analyze the convergence of Algorithm 1.
Let \( f \) denote the optimal value of the cost function \( f \) of the ELDP (5). For \( m = 0, 1, \ldots \), let \( g_m \) denote the piecewise-quadric surrogate cost function used by Algorithm 1 at iteration \( m \), and let \( p_m \in \mathbb{R}^n \) denote the power outputs produced by Algorithm 1 at iteration \( m \) by solving problem (9). Recall that \( g_m(p_m) = \min g_m \) and denote it by \( g_m^* \). Then \( \lim_{m \to \infty} g_m^* = \lim_{m \to \infty} f(p_m) = f^* \), and every limit point of \( (p_m)_{m \in \mathbb{N}} \) is a global solution of the ELDP.

Proof: We first show that \( f \) and \( g_m, m = 0, 1, \ldots \), are Lipschitz continuous on the ELDP feasible set with a common Lipschitz constant \( K \). Indeed, for every \( i \), the cost function \( f_i \) satisfies the Lipschitz property \( |f_i(p_i + \Delta) - f_i(p_i)| \leq (2a_i p_i^{\text{max}} + b_i + d_i \epsilon) \Delta \) for all \( p_i \) and \( p_i + \Delta \) that satisfy the generator capacity constraints (4). The ELDP cost function \( f \), being the sum of Lipschitz continuous functions, is thus Lipschitz continuous on the ELDP feasible set, with a constant \( K = \sum_{i=1}^{n} 2a_i p_i^{\text{max}} + b_i + d_i \epsilon_i \). Since \( g_m \) is obtained by replacing the rectified sines of \( f \) by chords, it follows that \( 2a_i p_i^{\text{max}} + b_i + d_i \epsilon_i \) is still a Lipschitz constant for the contribution of generator \( i \) to \( g_m \), and hence that \( K \) is also a Lipschitz constant for \( g_m \).

Since building \( g_{m+1} \) from \( g_m \) consists of inserting, for each generator, one new breakpoint for the piecewise-linear under-approximation of the piecewise-concave rectified sine, it follows that, for every \( m \),

\[
g_m \leq g_{m+1} \leq f. \tag{10}
\]

Therefore \( (g_m^*)_{m \in \mathbb{N}} \) is a nondecreasing sequence bounded by \( f^* \). Thus \( (g_m^*)_{m \in \mathbb{N}} \) converges and \( \lim_{m \to \infty} g_m^* \leq f^* \).

We show that \( \lim_{m \to \infty} g_m^* = f^* \). By contradiction, assume that \( \lim_{m \to \infty} g_m^* = f^* - \varepsilon \) with \( \varepsilon > 0 \). Since \( (p_m)_{m \in \mathbb{N}} \) is bounded in view of the generator capacity constraints (4), there exists a subsequence \( (p_{m_k})_{k \in \mathbb{N}} \) that converges. We then have the following inequalities which we justify hereafter:

\[
\|p_{m_k} - p_{m_{k+1}}\| \geq \frac{1}{K}[g_{m_{k+1}}(p_{m_k}) - g_{m_{k+1}}(p_{m_{k+1}})] \tag{11}
\]

\[
\geq \frac{1}{K}[f^* - g_{m_{k+1}}(p_{m_{k+1}})] \tag{12}
\]

\[
\geq \frac{1}{K}[f^* - (f^* - \varepsilon)] \tag{13}
\]

\[
\geq \frac{\varepsilon}{K}.
\]

a contradiction since \( (p_{m_k})_{k \in \mathbb{N}} \) converges. Inequality (13) states Lipschitz continuity of \( g_{m_{k+1}} \) with constant \( K \). Inequality (12) follows from \( g_{m_{k+1}}(p_{m_k}) = f(p_{m_k}) \geq f^* \); indeed, by construction, \( p_{m_k} \) is a breakpoint of \( g_{m_{k+1}} \) and all subsequent surrogate cost functions. Finally, (11) follows from \( g_{m_{k+1}}(p_{m_{k+1}}) = g_{m_{k+1}} \leq f^* - \varepsilon \).

We now show that \( \lim_{m \to \infty} f(p_m) = f^* \). By contradiction, suppose not. Then there is an infinite subsequence \( (p_{m_k})_{k \in \mathbb{N}} \) and \( \varepsilon > 0 \) such that \( f(p_{m_k}) \geq f^* + \varepsilon \). We assume w.l.o.g. that \( (p_{m_k})_{k \in \mathbb{N}} \) converges; if not, we extract a sub-subsequence that does. By the triangle inequality, we have

\[
|f(p_{m_k}) - f^*| \leq |f(p_{m_k}) - g_{m_{k+1}}(p_{m_k})| + |g_{m_{k+1}}(p_{m_k}) - g_{m_{k+1}}(p_{m_{k+1}})| + |g_{m_{k+1}}(p_{m_{k+1}}) - f^*|.
\]

The first term of the bound is zero by construction of the surrogate functions. The second term goes to zero as \( k \to \infty \) in view of the common Lipschitz constant \( K \) and the convergence of \( (p_{m_k})_{k \in \mathbb{N}} \). The third term goes to zero as \( k \to \infty \) since \( g_{m_{k+1}}(p_{m_{k+1}}) = g_{m_{k+1}}^* \) and \( \lim_{m \to \infty} g_m = f^* \). Hence \( |f(p_{m_k}) - f^*| \) goes to zero, a contradiction.

Finally, let \( (p_{m_k})_{k \in \mathbb{N}} \) be a convergent subsequence and let \( p_{m_{\infty}} \) denote its limit. By continuity of \( f \), we have that \( f(p_{m_{\infty}}) = \lim_{k \to \infty} f(p_{m_k}) = f^* \). 

Note that, in view of the breakpoint insertion procedure (step 15 of Algorithm 1), it generally does not hold that \( g_m \)

...
converges to $f$ pointwise; otherwise the proof above could have been more direct.

VI. NUMERICAL RESULTS

After having built such a model, we can hand it over to a solver that can handle MIQP. Here we will use the Gurobi solver [17]. To study the efficiency of our method, we will test it on the most common test cases in the literature: a 3-units setting with a demand of 850 MW [2] (I), a 13-units setting with a demand of 1800 MW [1] (IIa) and 2520 MW [18] (IIb), and a 40-units setting with a demand of 10,500 MW [1] (III). Using model (7) and these datasets, and feeding them to Gurobi, we get the solutions given in tables I, II, IV.

The resulting optimal value is slightly better than what we found earlier. The real time needed is 0.493 s for a CPU time of 0.902 s.

We also tested the method of Section IV on the different study cases. It so happens that it finds exactly the same solutions (except for a few swaps in equivalent generators) as the method of Section IV for case I, IIa and III thereby proving their optimality. The exception is case IIb, for which a slightly better solution is found (table VI). Note that the algorithm of Section IV always needs more time since it takes at least two iterations to stop: one to get a solution and another to prove their optimality. The exception is case IIb, for which a slightly better solution is found (table VI). Note that the algorithm of Section IV always needs more time since it takes at least two iterations to stop: one to get a solution and another.

| Table I | Found solution for the 3-units study case (I) using the simple model |
|---------|-------------------------------------------------|
| Unit    | Power (MW)                                      |
| $p_1$   | 300.267                                         |
| $p_2$   | 400.000                                         |
| $p_3$   | 149.733                                         |
| Total cost ($/h) | 8234.07                                         |
| Best in lit. ($/h) | 8234.07                                         |
| Real time (s) | 0.013                                          |
| CPU time (s) | 0.012                                          |

| Table II | Found solution for the 13-units study case (IIa) using the simple model |
|----------|--------------------------------------------------------------------------------|
| Unit     | Power (MW)                      |
| $p_1$    | 628.319                         |
| $p_2$    | 222.749                         |
| $p_3$    | 149.600                         |
| $p_4$    | 109.867                         |
| $p_5$    | 60.000                          |
| $p_6$    | 109.867                         |
| $p_7$    | 109.867                         |
| $p_8$    | 109.867                         |
| $p_9$    | 109.867                         |
| $p_{10}$ | 40.000                          |
| $p_{11}$ | 40.000                          |
| $p_{12}$ | 55.000                          |
| $p_{13}$ | 55.000                          |
| Total cost ($/h) | 17963.83                         |
| Best in lit. ($/h) | 17963.83                         |
| Real time (s) | 0.381                          |
| CPU time (s) | 0.708                          |

| Table III | Found solution for the 13-units study case (IIb) using the simple model |
|----------------|--------------------------------------------------------------------------------|
| Unit    | Power (MW)                      |
| $p_1$   | 628.319                         |
| $p_2$   | 299.199                         |
| $p_3$   | 299.199                         |
| $p_4$   | 159.733                         |
| $p_5$   | 159.733                         |
| $p_6$   | 159.733                         |
| $p_7$   | 159.733                         |
| $p_8$   | 159.733                         |
| $p_9$   | 159.733                         |
| $p_{10}$ | 77.400                          |
| $p_{11}$ | 77.400                          |
| $p_{12}$ | 90.042                          |
| $p_{13}$ | 90.042                          |
| Total cost ($/h) | 24170.66                         |
| Best in lit. ($/h) | 24169.92                         |
| Real time (s) | 0.054                          |
| CPU time (s) | 0.084                          |

| Table IV  | Found solution for the 40-units study case (III) using the simple model |
|------------|--------------------------------------------------------------------------------|
| Unit    | Power (MW)                      |
| $p_1$   | 110.800                         |
| $p_2$   | 110.800                         |
| $p_3$   | 97.400                          |
| $p_4$   | 179.733                         |
| $p_5$   | 90.279                          |
| $p_6$   | 140.000                         |
| $p_7$   | 259.600                         |
| $p_8$   | 284.600                         |
| $p_9$   | 284.600                         |
| $p_{10}$ | 130.000                         |
| $p_{11}$ | 168.800                         |
| $p_{12}$ | 168.800                         |
| $p_{13}$ | 214.760                         |
| $p_{14}$ | 394.279                         |
| $p_{15}$ | 394.279                         |
| $p_{16}$ | 304.520                         |
| $p_{17}$ | 489.279                         |
| $p_{18}$ | 489.279                         |
| $p_{19}$ | 511.279                         |
| $p_{20}$ | 511.279                         |
| Total cost ($/h) | 121415.31                         |
| Best in lit. ($/h) | 121412.54                         |
| Real time (s) | 0.116                          |
| CPU time (s) | 0.192                          |
TABLE V
SOLUTION FOR 40-UNITS CASE STUDY (III) USING THE MODEL OF SECTION [V]

| Unit | Power (MW) | Unit | Power (MW) |
|------|------------|------|------------|
| p1   | 110.800    | p21  | 523.279    |
| p2   | 110.800    | p22  | 523.279    |
| p3   | 97.400     | p23  | 523.279    |
| p4   | 179.733    | p24  | 523.279    |
| p5   | 87.800     | p25  | 523.279    |
| p6   | 140.000    | p26  | 523.279    |
| p7   | 259.600    | p27  | 10.000     |
| p8   | 284.600    | p28  | 10.000     |
| p9   | 284.600    | p29  | 10.000     |
| p10  | 130.000    | p30  | 87.800     |
| p11  | 94.000     | p31  | 190.000    |
| p12  | 94.000     | p32  | 190.000    |
| p13  | 214.760    | p33  | 190.000    |
| p14  | 394.279    | p34  | 164.800    |
| p15  | 394.279    | p35  | 200.000    |
| p16  | 394.279    | p36  | 194.398    |
| p17  | 489.279    | p37  | 110.000    |
| p18  | 489.279    | p38  | 110.000    |
| p19  | 511.279    | p39  | 110.000    |
| p20  | 511.279    | p40  | 511.279    |

Total cost ($/h) | 121412.54
Best in lit. ($/h) | 121412.54
Real time (s) | 0.493
CPU time (s) | 0.902

TABLE VI
GLOBAL SOLUTION FOR CASE III FOUND WITH THE ALGORITHM OF SECTION [V]

| Unit | Power (MW) |
|------|------------|
| p1   | 628.319    |
| p2   | 299.199    |
| p3   | 299.199    |
| p4   | 159.733    |
| p5   | 159.733    |
| p6   | 159.733    |
| p7   | 159.733    |
| p8   | 159.733    |
| p9   | 159.733    |
| p10  | 77.400     |
| p11  | 77.400     |
| p12  | 87.684     |
| p13  | 92.400     |

Total cost ($/h) | 24169.92
Best in lit. ($/h) | 24169.92
Real time (s) | 0.589
CPU time (s) | 0.592

to prove the optimality.

VII. CONCLUSION

This article concerns a piecewise quadratic underapproximation of the ELDP cost function that takes into account the valve-point effect, thereby providing a means to solve it globally with the MIQP method, despite the nonsmoothness and the nonconvexity of the original cost function. Furthermore, a convergence analysis is presented to show that, under mild assumptions, this strategy guarantees the global minimizer in the static ELDP context. In order to support our claim, the minimization results are furnished for the datasets corresponding to a 3-, two instances of 13-, and a 40-generator setting, wherein the transmission losses are omitted. Interestingly enough, one the one hand, in accordance with the global convergence guarantee, the outcome of the cost minimization by the method of Section [V] is never surpassed by the hybrid methods, and on the other hand, the computational times are quite impressive.

While these widely used datasets in the ELDP literature enable us to investigate how our approach compares with the state-of-the-art methods, the same framework is applicable for scenarios that do not ignore losses. The only modification needed is to add the loss term in the constraint to take it into account and to relax the equality into an inequality. The reason behind the relaxation is to make the constraint convex and the model easy enough to solve. This does not change the solution of the problem as long as the objective function is monotonically increasing.

Moreover, these methods can be used for other cost functions, where the valve-point effect would be modeled differently. As long as the valve-point effect is represented by a periodic piecewise-concave function, only minor adaptation would be needed.

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