SOME NEW TWO-WEIGHT TERNARY AND QUINARY CODES OF LENGTHS SIX AND TWELVE

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Abstract. Let \([n,k]_q\) be a projective two-weight linear code over \(GF(q)^n\). In this correspondence, 9 codes are constructed in which \(k = 3\).

1. Introduction

As surveyed in [6], Delsarte connected three areas of mathematics, coding theory, graph theory, and finite geometry. This connection motivates the study of projective two-weight linear \([n,3]_q\) codes over \(GF(q)^n\) and of two intersection sets in \(PG(2,q)\).

2. Definitions and background results

Let \(GF(q)\) denote the Galois field of order \(q\). A linear \([n,k]_q\) code \(C\) is a \(k\)-dimensional subspace of \(GF(q)^n\). A code \(C\) can be described in terms of its generator matrix. The generator matrix is a \(k \times n\) matrix over \(GF(q)\) whose row-space is the subspace \(C\). We say that \(C\) is projective if the \(i^{th}\) and \(j^{th}\) column of the generating matrix are linearly independent for any pair \((i,j), i \neq j\). Furthermore, \(C\) is a two-weight code if all the non-zero codewords of \(C\) have one of two (non-zero) weights. That is, if \(c\) is a codeword of \(C\) in \(GF(q)^n\) then the number of non-zero entries in \(c\) must be one of two values, which we call weights. We will denote the weights as \(w_1\) and \(w_2\) (\(w_1 < w_2\)).

Let \(PG(2,q)\) denote the Desarguesian projective plane of order \(q\). A \(t\)-set of type \((m_1, m_2)\) (or two intersection set) in \(PG(2,q)\) is a proper nonempty set \(\Omega\) consisting of \(t\) points in \(PG(2,q)\) such that for any line \(l \in PG(2,q)\), \(|l \cap \Omega| = m_1\) or \(m_2\) (\(m_1 < m_2\)).

The equivalence of projective two-weight codes and two intersection sets can be described in terms of the generator matrix. Let \(C\) be a projective two-weight \([n,3]_q\) code, then the columns of the generator matrix are linearly independent. Thus, we can take the columns of the generator matrix as a set of points in \(PG(2,q)\). That is, the columns of the generator matrix make up the points of the two intersection set.
in PG(2, q), while the weights of the code become the intersection numbers between the projective point set and the lines of PG(2, q). The intersection numbers are then given by \( n - w_1 \) and \( n - w_2 \). This equivalence holds for any value of \( k \); see [6].

Projective two-weight linear \([n, 3]_q\) codes also give rise to strongly regular graphs. A graph \( \Gamma \), which is simple, undirected, and loopless, of order \( v \) is a strongly regular graph with parameters \((v, K, \lambda, \mu)\) whenever \( \Gamma \) is not complete or edgeless and (i) each vertex is adjacent to \( K \) vertices, (ii) for each pair of adjacent vertices, there are \( \lambda \) vertices adjacent to both, and (iii) for each pair of non-adjacent vertices, there are \( \mu \) vertices adjacent to both. The parameters arising from projective two-weight linear \([n, 3]_q\) codes are

\[
\begin{align*}
v &= q^3, \\
K &= n(q - 1), \\
\lambda &= q^2m_1m_2 - qnm_1 - qnm_2 + q + qm_1 + qm_2 + n^2 - 3n, \\
\mu &= \frac{(n-m_1)(n-m_2)}{q},
\end{align*}
\]

where \( m_1 \) and \( m_2 \) are the intersection numbers for the corresponding two intersection set.

2.1. Survey of two intersection sets. The history of two intersection sets in finite projective planes stretches back at least to two 1966 papers by Tallini Scafati [28,29], with precursors dealing with special cases [1,4,27]. Calderbank and Kantor surveyed these sets (and their higher-dimensional counterparts) in 1986 [6] in what has become the standard reference. Postdating their survey, a number of constructions have been given of two intersection sets in PG(2, q), and in turn, two-weight codes [2,3,7–15,17–26]. To these, we add examples with new parameters with \( q = 25 \) and 81. To the best of our knowledge, the only examples of two intersection sets in finite Desarguesian planes admitting insoluble groups known before this paper are the regular hyperovals, the classical unitals, and Baer subplanes (and their complements). Thus, it is of some interest that four of our examples (and their complements) have insoluble groups. We also contribute further examples admitting soluble groups (but some without new parameters).

3. New constructions

3.1. Construction procedure. In order to construct projective two-weight codes, we construct the corresponding point sets. We begin by considering possible automorphism groups for two intersection sets. That is, we consider subgroups of PTL(3, q), the group of PG(2, q). Since \((m_1+p)(p^2-p-1)\)-sets of type \((m_1, m_1+p)\) for some prime \( p \) such that \( q = p^r \) are feasible, unions of orbits on points are considered so that the union of points satisfies the \((m_1+p)(p^2-p-1)\) condition. Upon obtaining the correct number of points for a point set, the intersection numbers are verified.

3.2. Results. We will denote the projective two-weight \([n, 3]_q\) codes as \((n, r, w_1)\) over GF(q), where \( q^2 = p^r \) for some prime \( p \). It should be noted that \( w_2 = w_1 + p \). In Table 1, the first eight results are for the case when \( q = 5^2 \) - the quinary codes of length 6. The last result is for the case when \( q = 3^4 \) - the ternary code of length 12.

It should be noted that all of the codes in Table I, except for the \((155, 6, 145)\) code in GF(25), give rise to new strongly regular graphs with previously unknown parameters. These results can be verified using Magma [5] and GAP [16].
| Code      | \((m_1, m_2)\) | \(v\) | \(K\) | \(\lambda\) | \(\mu\) | Group                          |
|-----------|----------------|-------|-------|-------------|---------|--------------------------------|
| \((155, 6, 145)\) | \((5, 10)\) | 15625 | 3720  | 935         | 870     | \(\text{PGL}(2, 5)\)          |
| \((210, 6, 200)\) | \((5, 10)\) | 15625 | 5040  | 1595        | 1640    | \(\text{PSL}(2, 5) \times C_2\) |
| \((231, 6, 220)\) | \((6, 11)\) | 16525 | 5544  | 1943        | 1980    | \(\text{AGL}(1, 5) \times C_2\) |
| \((252, 6, 240)\) | \((7, 12)\) | 15625 | 6048  | 2323        | 2352    | \(\text{ASL}(2, 3)\)          |
| \((252, 6, 240)\) | \((7, 12)\) | 15625 | 6048  | 2323        | 2352    | \((C_3 \times C_3) \rtimes C_6\) |
| \((315, 6, 300)\) | \((10, 15)\) | 15625 | 7560  | 3655        | 3600    | \(\text{AGL}(1, 5) \times C_3\) |
| \((315, 6, 300)\) | \((10, 15)\) | 15625 | 7560  | 3655        | 3600    | \(\text{PSL}(2, 9)\)          |
| \((315, 6, 300)\) | \((10, 15)\) | 15625 | 7560  | 3655        | 3600    | \(\text{PSL}(2, 5) \times C_2\) |
| \((3285, 12, 3240)\) | \((36, 45)\) | 531441 | 4536  | 1279        | 1332    | \(\text{PSL}(2, 9)\)          |

Table 1.

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