Isogeny orbits in a family of abelian varieties

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Abstract

We prove that if a curve of a non-isotrivial family of abelian varieties over a curve contains infinitely many isogeny orbits of a finitely generated subgroup of a simple abelian variety then it is special. This result fits into the context of Zilber-Pink conjecture and partially generalizes a result of Faltings. Moreover by using the polyhedral reduction theory we give a new proof of a result of Bertrand.

1 Introduction

In this paper we are interested in a curve of an abelian scheme that contains infinitely many isogeny orbits of a finitely generated group of a simple abelian variety. We prove that it is special. Zilber-Pink have conjectured, roughly speaking, that a subvariety containing many special points must be special. This generalizes many well-known problems including conjectures of Mordell-Lang, Manin-Mumford and André-Oort. Special points considered in this paper are isogeny orbits which are closely related to the generalized Hecke orbits considered by Zilber-Pink. Therefore our result fits into the context of their conjectures.

Let $S$ be a smooth irreducible abstract algebraic curve over $\overline{\mathbb{Q}}$, and $\pi : A \to S$ be an abelian scheme. An abelian scheme $A \to S$ refers to a smooth proper group scheme with geometrically connected fibers. Then $A$ can be regarded as a smooth family of abelian varieties over $S$. Take an abelian variety $A'$ defined over $\overline{\mathbb{Q}}$ and a finitely generated group $\Gamma \subset A'(\overline{\mathbb{Q}})$. We call a point $q \in A_t(\overline{\mathbb{Q}})$, where $t \in S(\overline{\mathbb{Q}})$, special if there exist an isogeny $\phi : A' \to A_t$ and $\gamma \in \Gamma$ with $\phi(\gamma) = q$. In this paper we prove

**Theorem 1.1.** Assume that $A'$ is simple and $A$ is non-isotrivial. If an irreducible Zariski closed algebraic curve $X$ of $A$, over $\overline{\mathbb{Q}}$, contains infinitely many special points, then either $X$ is a subtorus of $A_q$ for some $q \in S(\overline{\mathbb{Q}})$ or there exists $n \in \mathbb{N}$ such that $[n]X(\overline{\mathbb{Q}}) = 0$.

In the next section we prove a partially stronger result in case of a family of elliptic curves, and indicate two major obstructions preventing that argument from
working in case of families of abelian varieties. In Section 3 we use the polyhedral
reduction theory to give a new proof of a result of Bertrand, which is crucial for this
paper. In Section 4 we present the proof of the main theorem. The basic strategy
of this paper is to compare different heights in number theory, including geometric
Faltings height, Neron-Tate height and Weil height.

Throughout this paper \( h_F(A) \) refers to the geometric Faltings height of an
abelian variety \( A \) over \( \mathbb{Q} \), \( h_{X,D} : X(\overline{\mathbb{Q}}) \to \mathbb{R} \) refers to a Weil height function of
a variety over \( \overline{\mathbb{Q}} \) with respect to the divisor \( D \) and \( \hat{h}_{A,D} : A(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0} \) refers to the
canonical height function of an abelian variety over \( \overline{\mathbb{Q}} \) with respect to a symmetric
divisor \( D \). Two linearly equivalent divisors respectively isomorphic line bundles ar e
connected by \( \sim \) respectively \( \cong \). The set of linear equivalence of divisors or line
bundles of \( X \) is \( \text{Pic}(X) \). For a complete nonsingular curve \( X \) over \( \mathbb{Q} \), we have a
canonical surjective homomorphism \( \text{deg} : \text{Pic}(X) \to \mathbb{Z} \).

Let \( L \) be an invertible sheaf
of an abelian variety \( A \), \( \chi(L) \) be its Euler characteristic and \( \lambda_L \) be the morphism
from \( A(\overline{\mathbb{Q}}) \) to \( \text{Pic}(A) \). The subgroup \( \text{Pic}^0(A) \) of \( \text{Pic}(A) \) consists of invertible sheaves
\( L \) for which \( \lambda_L \equiv 0 \). The Neron-Severi group of \( A \) is denoted by \( \text{NS}(A) \). Points of
the dual abelian or Picard variety \( A^\vee \) of \( A \) parametrize the elements of \( \text{Pic}^0(A) \).

A homomorphism of abelian varieties \( \phi : A \to B \) gives rise to a dual \( \phi^\vee : B^\vee \to A^\vee \).
Write \( \text{End}(A) = \text{End}(A) \otimes \mathbb{Q} \), on which we have another
deg function. The set of isomorphism classes of pairs \( (A,\lambda) \) with \( A \) an abelian
variety of dimension \( g \) and \( \lambda \) a polarization of \( A \) of degree \( d = \chi(\lambda) \) is parameterized
by the Siegel modular variety \( M_{g,d} \), in particular \( M_{1,1} = \mathbb{A}^1 \). If \( \Gamma \) is a finitely
generated abelian group then we let \( \Gamma_t \) respectively \( \Gamma_{nt} \) to be the torsion subgroup
respectively its complement.

2 Isogeny orbits in a family of elliptic curves

In this section we let \( S \) be a smooth irreducible abstract algebraic curve over \( \overline{\mathbb{Q}} \),
and \( \pi : A \to S \) an abelian scheme of dimension one. Then \( A \) can be regarded as
a smooth family of elliptic curves over \( S \). Take an elliptic curve \( A' \) defined over \( \overline{\mathbb{Q}} \)
and a \( p \in A'(\overline{\mathbb{Q}}) \). We call \( q \in A_t(\overline{\mathbb{Q}}) \), where \( t \in S(\overline{\mathbb{Q}}) \), special if there exists either
an isogeny \( \phi : A' \to A_t \) with \( \phi(p) = q \), or an isogeny \( \phi : A_t \to A' \) with \( \phi(q) = p \).

We prove

**Proposition 2.1.** Assume that \( A \) is non-isotrivial. If an irreducible Zariski closed
algebraic curve \( X \) of \( A \), over \( \overline{\mathbb{Q}} \), contains infinitely many special points, then either
\( X \) is some special fiber \( A_t \) that is isogenous to \( A' \) or there exists \( n \in \mathbb{N} \) such that
\[ |n|X(\overline{\mathbb{Q}}) = 0. \]

**Proof.** Firstly we assume that \( X \) is not any fiber \( A_t \), otherwise there is nothing to prove.
Secondly we notice that it suffices to prove the result under the assumption that \( X \) is a section \( s : S \to A \) of \( \pi : A \to S \). Indeed in the general case if let \( X' \)
be a smooth resolution of $X$, then $A \times_S X' \to X'$ is also a smooth family of elliptic curves over $X'$ and we write $f : X' \to A$ to be the natural morphism.

The above commutative diagram provides a section $s : X' \to A \times_S X'$ of $A \times_S X'/X'$. Moreover it is easy to check that $s(X') \subset A \times_S X'$ contains infinitely special points if and only if $X \subset A$ does and that $[n|X = 0$ if and only if $(n|\circ s)X = 0$. Therefore the general case reduces to a special case that $X$ comes from a section.

If $p$ is a torsion point and $A'$ has complex multiplication, then our assertion is a special case of a result of André [2]. If $p$ is a torsion point and $A'$ has no complex multiplication, then by a lemma of Habegger [6, Lemma 5.8] there are only finitely many elliptic curves isogenous to $A'$ with bounded height (The proof of Habegger’s statement relies heavily on the work of Szpiro and Ullmo). This makes André’s argument valid line by line after replacing only Poonen’s lemma [8] by Habegger’s lemma. From here on we assume that $p$ is not torsion.

Let $D$ be the divisor given by the zero section of the abelian scheme $A/S$, then the canonical height function on $A_t(\mathbb{Q})$ with respect to $\deg D_t$ is simply $\hat{h}_{A_t}$ for any $t \in S(\mathbb{Q})$. According to the non-isotriviality of $A$, the modular map $j : S(\mathbb{Q}) \to \mathbb{A}^1(\mathbb{Q})$ is non-constant. Without ambiguity we write $h : \mathbb{A}^1(\mathbb{Q}) \to \mathbb{R}$ for the standard Weil height function on $\mathbb{A}^1$ and $h : s \in S(\mathbb{Q}) \to h(j(s)) \in \mathbb{R}$ for the Weil height function on $S$ with respect to $j^*((0))$.

There are two types of special points. The back orbit of $p$ and the forward orbit of $p$. We first assume that $X$ contains infinitely many back orbits $q_i(i = 1, 2, \ldots)$ of
p. If \( q_i \in A_{t_i} \), where \( t_i \in S(Q) \), then \( q_i = s(t_i) \). Let \( \phi_i : A_{t_i} \to A' \) be the isogeny that satisfies \( \phi_i(q_i) = p \) then we have

\[
\hat{h}_{A_{t_i}}(q_i) = \frac{\hat{h}_{A'}(p)}{\deg \phi_i}.
\] (1)

Using the lemma of Habegger [6, Lemma 5.8], there are only finitely many elliptic curves over \( \mathbb{Q} \) within the isogeny class of \( E' \) with bounded Weil height. Using the non-isotriviality of \( A \), given any elliptic curve \( E_1 \) there are only finitely many \( i \) such that \( E_{t_i} \) is isomorphic to \( E_1 \) over \( \mathbb{Q} \). These two facts clearly lead to

\[
\lim_{i \to \infty} h(t_i) \to \infty.
\] (2)

By Silverman’s specialization theorem [10] there is a constant \( C \) such that

\[
\lim_{h(t) \to \infty} \frac{\hat{h}_{A_{t}}(s(t))}{\hat{h}(t)} = C.
\] (3)

Because of (2) we can apply (3) to \( t_i \) and obtain

\[
\lim_{i \to \infty} \frac{\hat{h}_{A_{t}}(q_i)}{\hat{h}(t_i)} = C.
\]

Using (1) we have

\[
\lim_{i \to \infty} \frac{\hat{h}_{A_{t}}(q_i)}{\deg \phi_i \cdot \hat{h}(t_i)} = C.
\]

As \( p \) is not torsion, we have \( h_{A'}(p) > 0 \) and therefore the above identity gives \( C = 0 \). Recall in Silverman’s specialization theorem [10], the constant \( C = 0 \) means that the canonical height of \( X \) regarded as a point in the abelian variety \( A_\eta \) over the generic point is zero. According to the non-isotriviality of \( A \), the \( \mathbb{Q}(S)/\mathbb{Q} \) trace of \( A \) is trivial. This implies that \( X \) is a torsion of the abelian variety over the generic fiber. There exists \( n \in \mathbb{N} \) such that \( [n]X(\mathbb{Q}) = 0 \), a contradiction to our assumption that \( p \) is not torsion.

Now we assume that \( X \) contains infinitely many forward orbits \( \{q_i\}_{i=1}^{\infty} \) of \( p \). Let \( \phi_i : A' \to A_{t_i} \) be the isogeny that satisfies \( \phi_i(p) = q_i \), then

\[
\hat{h}_{A_{t_i}}(q_i) = \deg \phi_i \cdot \hat{h}_{A'}(p).
\] (4)

The inequality of Faltings [4] implies that

\[
h_F(A_{t_i}) \leq h_F(A') + \log \deg \phi_i/2.
\] (5)

We claim that \( \lim_{i \to \infty} \deg \phi_i = \infty \). Otherwise there are infinitely many \( i \) such that \( A_{t_i} \) are isomorphic to each other over \( \mathbb{Q} \), a contradiction to the fact that \( A \) is non-isotrivial. Therefore [4], [3] and \( \hat{h}_{A'}(p) > 0 \) lead to

\[
\lim_{i \to \infty} \frac{\hat{h}_{A_{t}}(s(t_i) = q_i)}{\hat{h}(t_i)} = \infty.
\]
The identity (2) is still valid. This makes the above equality a contradiction to Silverman’s specialization theorem (3).

The above argument does not work in the context of families of abelian varieties for the following reasons. Firstly the relation between canonical heights of points on isogenous abelian varieties is not as simple as in (1). Secondly the lemma of Habegger is not known in higher dimensional case. More precisely we do not know whether within an isogeny class of abelian varieties there are only finitely many ones with bounded height. Without this result we have no validity of (2) in general yet, which is essential if we want to directly apply Silverman’s specialization theorem.

3 Polyhedral reduction theory and canonical heights

In this section we give a proof of Lemma 3.1 below, based on the polyhedral reduction theory [1]. Actually this lemma is not new. G. Rémont pointed out to us that it is equivalent to the main theorem of Bertrand [3] in case of simple abelian varieties, linked by the theorem of Mordell-Weil. We still present the proof here, as our approach is rather distinct from Bertrand’s original one.

Throughout this section $A$ is an abelian variety over $\mathbb{Q}$. Any symmetric line bundle $L$ of $A$ defines a canonical height function $\hat{h}_{A,L}$ that is quadratic on $A(\mathbb{Q})$. We remark that $\hat{h}_{A,L}$ depends only on the class of $L$ in $\text{NS}(A)$. Indeed if a symmetric line bundle $L$ maps to zero in the short exact sequence

$$0 \rightarrow \text{Pic}^0(A) \rightarrow \text{Pic}(A) \rightarrow \text{NS}(A) \rightarrow 0,$$

then we have $L \in \text{Pic}^0(A)$. This leads to $[-1]^*L = L^{-1}$. Together with $[-1]^*L = L$ we have $L^2 = 0$ and therefore $\hat{h}_{A,L} \equiv 0$. Hence there will be no confusion when we label the canonical height function of a symmetric line bundle to its image in the Neron-Severi group.

When $A$ is a simple abelian variety, a recent result of Kawaguchi and Silverman [5] tells us that for any nonzero nef symmetric $L \in \text{Pic}(A) \otimes \mathbb{R}$ the canonical height $\hat{h}_{A,L} = 0$ if and only if $x \in A(\mathbb{Q})$.

The endomorphism algebra $\text{End}^0(A)$ is semi-simple and contains $\text{End}(A)$ as a lattice. The unit $(\text{End}^0(A) \otimes \mathbb{R})^\times$ is reductive, and $\text{Aut}(A)$ is an arithmetic group. The function $\text{deg}$ extends to a homogeneous function of degree $2g$ on $\text{End}^0(A) \otimes \mathbb{R}$.

For any line bundle $L$ the theorem of square leads to a group homomorphism

$$\lambda_L : A(\overline{\mathbb{Q}}) \rightarrow A^\vee(\overline{\mathbb{Q}})$$

which takes $x$ to $T^*_xL \otimes L^{-1}$. It is an isogeny if and only if $L$ is ample.

From here on we fix an ample line bundle $N$, which defines a Rosati involution $\dagger$ of $\phi \in \text{End}^0(A)$ by $\phi^\dagger = \lambda_N^{-1} \circ \phi^\vee \circ \lambda_N$. The map

$$\text{NS}(A)_\mathbb{Q} \rightarrow \text{End}^0(A)$$
defined by $L \mapsto \lambda_N^{-1} \circ \lambda_L$ identifies $\text{NS}(A)_Q$ with the subset of $\text{End}^0(A)$ of elements fixed by $\hat{\gamma}$. Given $\phi \in \text{Aut}(A)$ and $L \in \text{Pic}(A)$ it is straightforward to check that $\lambda_{\phi^*(L)} = \phi^* \circ \lambda_L \circ \phi$. Which extends to be an action of $\text{End}^0(A)$ on $\text{NS}(A)_Q \subset \text{End}^0(A)$ by $\alpha^* = \phi^* \circ \alpha \circ \phi$. The bilinear form

$$\langle \phi, \psi \rangle \mapsto \text{Tr}(\phi \circ \psi^*)$$

on $\text{End}^0(A) \times \text{End}^0(A)$ is positive definite.

As a finite dimensional algebra over $\mathbb{R}$ with a positive involution, $\text{End}^0(A) \otimes \mathbb{R}$ is isomorphic to $\prod H_{r_i}(\mathbb{R}) \times \prod H_{s_j}(\mathbb{C}) \times \prod P_{t_k}(\mathbb{H})$ where the involution is given by conjugations. Under this identification $N^1(A) = \text{NS}(A) \otimes \mathbb{R}$ respectively the ample cone $\text{Amp}(A)$ is isomorphic to $\prod P_{r_i}(\mathbb{R}) \times \prod H_{s_j}(\mathbb{C}) \times \prod P_{t_k}(\mathbb{H})$ respectively $\prod P_{r_i}(\mathbb{R}) \times \prod P_{s_j}(\mathbb{C}) \times \prod P_{t_k}(\mathbb{H})$, where $H_r$ is the space of symmetric or Hermitian symmetric matrices and $P_t$ consists of positive ones. Let $G(\text{Amp}(A))$ be the automorphism group of the cone $\text{Amp}(A)$ and $G(\text{Amp}(A))^0$ its identity component. The homomorphism $(\text{End}^0(A) \otimes \mathbb{R})^\times \to G(\text{Amp}(A))^0$ is surjective. According to Ash’s main result of polyhedral reduction theory, there exists a rational polyhedral cone $F \subset \overline{\text{Amp}(A)}$ such that $(\text{Aut}(A) \cdot F) \cap \text{Amp}(A) = \text{Amp}(A)$. For more details of this paragraph we refer [1] and [3].

Using the theorem of Ash we give a new proof of

**Lemma 3.1 (Bertrand).** Let $A$ be a simple abelian variety defined over $\overline{\mathbb{Q}}$, and let $\Gamma \subset A(\mathbb{Q})$ be a finitely generated group. Then there exists a constant $C > 0$ depending on $\Gamma$ and $A$ such that for any symmetric ample divisor $M \in \text{Pic}(A)$ and nontorsion $x \in \Gamma$ we have

$$\hat{h}_{A,M}(x) \geq C(\chi(M))^{1/\rho}.$$

**Proof.** It is well-known that $\text{End}(A)$ is of finite rank, hence $\text{End}(A)(\Gamma)$ is also a finitely generated group. Therefore it suffices to prove our lemma under the assumption that $\Gamma$ is invariant under $\text{Aut}(A)$.

By the polyhedral reduction theory [1], there is a rational polyhedral fundamental domain $F \subset \overline{\text{Amp}(A)}$ under the action of $\text{Aut}(A)$. The rationality of $F$ guarantees that there is a basis $\{v_1, \ldots, v_t\} \subset \text{NS}(A) \cap \overline{\text{Amp}(A)}$ such that if $w \in F \cap \text{NS}(A)$ then there are nonnegative real numbers $r_i$ with $w = \sum_{i=1}^t r_i v_i$.

We have $\Gamma = \Gamma_t + \Gamma_{nt}$. Because $v_i$ are nef, a result of Kawaguchi-Silverman [2] tells us that $\hat{h}_{A,v_i}$ is a positive bilinear function on the finitely generated abelian $\Gamma_{nt}$, therefore there exists a constant $c_1$ such that

$$\hat{h}_{A,v_i}(\gamma) \geq c_1$$

for all $1 \leq i \leq t$ and all non-zero $\gamma \in \Gamma_{nt}$. Furthermore for any $M \in \text{Amp}(A)$ and $x = x_1 + x_2 \in \Gamma$ we have $\hat{h}_{A,M}(x) = \hat{h}_{A,M}(x_2)$. Consequently the inequality (6) is valid for all $1 \leq i \leq t$ and non-torsion $\gamma \in \Gamma$. 

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Take a symmetric ample divisor with image \( M \subset F \), then we have nonnegative real numbers \( r_i \) such that \( M = \sum_{i=1}^{t} r_i v_i \). In particular for any non-torsion \( \gamma \in \Gamma \) we have

\[
\hat{h}_{A,M}(\gamma) \geq c_1 \max_{i=1}^{t} \{r_i\}
\] (7)

The degree function \( \text{deg} : \text{End}^0(A) \otimes \mathbb{R} \to \mathbb{R} \) is homogeneous of degree \( g \). Take \( \text{deg} M \) to be the degree of the image of \( M \) in \( \text{End}^0(A) \otimes \mathbb{R} \). Then we shall have

\[
\text{deg} M = \text{deg}(\lambda_M) / \text{deg}(\lambda_N) = c_2 \chi(M)^2
\] (8)

where \( c_2 = 1 / \text{deg}(\lambda_N) \). The homogeneity of \( \text{deg} : \text{End}^0(A) \otimes \mathbb{R} \to \mathbb{R} \) implies that there is a positive constant \( c_3 \) which depends only on \( v_i \) but not on \( M \) such that

\[
\text{deg} M \leq c_3 \sum_{i=1}^{t} r_i^{2g}.
\] (9)

The above (7), (8) and (9) obviously lead to a constant \( C > 0 \) such that

\[
\hat{h}_{A,M}(\gamma) \geq C(\chi(M))^{1/g}
\]

for all \( M \subset F \cap \text{Amp}(A) \) and non-torsion \( \gamma \in \Gamma \).

For general \( M \in \text{Amp}(A) \), there exists \( \sigma \in \text{Aut}(A) \) such that \( \sigma^*(M) \in F \). We have assumed that \( \sigma^{-1}(x) \in \Gamma \) and it is clear that \( \chi(\sigma^*(M)) = \chi(M) \). Therefore for any non-torsion \( \gamma \in \Gamma \)

\[
\hat{h}_{A,M}(\gamma) = \hat{h}_{A,\sigma^*(M)}(\sigma^{-1}(\gamma)) \geq C(\chi(M))^{1/g},
\]

which proves what has been claimed in the lemma. \( \square \)

4 Proof of the main theorem

It is unknown to us whether in an isogeny class of abelian varieties there are only finitely many ones with bounded height. Therefore we can not directly use Silverman’s specialization theorem as before. Instead we shall use some arguments of [10] to prove our main theorem.

**Proof of Theorem [10]** Firstly we may assume that \( \Gamma \) is invariant under \( \text{Aut}(A') \). Secondly by the same trick used in the proof of Proposition [2.1] we assume that \( X \) is a section \( s : S \to A \) of \( \pi : A \to S \). We write \( \epsilon : S \to A \) for the zero section.

In the decomposition \( \Gamma = \Gamma_t + \Gamma_{nt} \), \( \Gamma_t \) is finite, there exists a positive integer \( n \) such that \( [n] \Gamma_t = 0 \). If there are infinitely many \( t \in S(\mathbb{Q}) \) such that there exists \( \gamma_t \in \Gamma_t \) and isogeny \( \phi_t : A' \to A_t \) with \( \phi_t(x_t) = s(t) \), then we also have \( [n]s(t) = 0 \). This implies that \( [n]X(\overline{\mathbb{Q}}) \) intersects with the zero section infinitely many times. This leads to \( [n]X(\overline{\mathbb{Q}}) = 0 \).
Now we assume that there are infinitely many distinct $t_i \in S(\mathbb{Q})(i = 1, 2, 3, \ldots)$ such that exists $\gamma_i \in \Gamma \setminus \Gamma_t$ and isogenies $\phi_i : A' \to A_{t_i}$ with $\phi_i(\gamma_i) = s(t_i)$.

By theory of Theta functions, there is a smooth semiabelian scheme $\mathcal{A}/\mathcal{S}$ extending the family $A/S$ (where $\mathcal{S}$ is the smooth compactification of $S$) together with a symmetric very ample line bundle $\mathcal{L}'$ of $A/S$ that extends to a very ample one $\mathcal{L}$ of $\mathcal{A}/\mathcal{S}$. Moreover $\mathcal{L}$ is indeed very ample on $\mathcal{A}/\mathbb{Q}$ and makes the total space a projective variety. For details of this paragraph we refer [7].

The Euler characteristic $\chi(\mathcal{L}_t)$ is a constant function of $t \in S(\mathbb{Q})$, and we denote it by $d$. Because $A$ is non-isotrivial, the modular map $j : S \to M_{g,d}$ is not constant. We claim that

$$\lim_{i \to \infty} \deg \phi_i = \infty.$$ 

Otherwise there are infinitely many $t \in S(\mathbb{Q})$ such that $A_t$ are all isomorphic to each other over $\mathbb{Q}$, According to a well-known geometric finiteness theorem, given any abelian variety $A^0$ and $d \in \mathbb{N}$ there are only finitely many isomorphism classes of polarized abelian varieties $(A^0, \lambda)$ with $\lambda$ of degree $d$. These two facts together force the modular map $j : S \to M_{g,d}$ to be constant. This contradicts to the non-isotriviality of $A$ and proves the claim.

Concerning isogenous $\phi_i$ of abelian varieties Faltings’ inequality (11) gives

$$h_F(A_{t_i}) \leq h_F(A') + \log \deg \phi_i/2.$$ 

Under the isogeny canonical heights satisfy $\hat{h}_{\mathcal{A}_{t_i}, \mathcal{L}_{t_i}}(s(t_i)) = \hat{h}_{\mathcal{A}', \phi_i^*(\mathcal{L}_{t_i})}(\gamma_i)$. The Euler characteristics satisfy $\chi(\phi_i^*(\mathcal{L}_{t_i})) = \deg \phi_i \cdot \chi(\mathcal{L}_{t_i}) = d \deg \phi_i$. By the lemma in the last section there exists a positive constant $c_1$ such that

$$\hat{h}_{\mathcal{A}_{t_i}, \mathcal{L}_{t_i}}(s(t_i)) \geq c_1(\deg \phi_i)^{1/g}.$$ 

The group morphism $[2] : A/S \to A/S$ extends to a morphism on the semiabelian scheme $\mathcal{A}/\mathcal{S}$. As mentioned before $\mathcal{L}$ gives a projective embedding of $\mathcal{A}$. By this condition a theorem of Silverman-Tate [10] Theorem A applies, and consequently there exists positive constants $c_2$ and $c_3$ such that

$$\left| \hat{h}_{\mathcal{A}_{t_i}, \mathcal{L}_{t_i}}(s(t_i)) - h_{\mathcal{A}', \mathcal{L}}(s(t_i)) \right| < c_2 h_{\mathcal{S}', \varepsilon^*(\mathcal{L})}(t_i) + c_3.$$ 

Because $h_{\mathcal{A}', \mathcal{L}}(s(t_i)) = h_{\mathcal{S}', \varepsilon^*(\mathcal{L})}(t_i)$ and because both $\varepsilon^*(\mathcal{L})$ and $s^*(\mathcal{L})$ are ample, there exists positive constants $c_4$ and $c_5$ such that

$$h_{\mathcal{A}', \mathcal{L}}(s(t_i)) \leq c_4 h_{\mathcal{S}', \varepsilon^*(\mathcal{L})}(t_i) + c_5.$$ 

As indicated in [11], Zarhin’s trick works for families and therefore $B = (A \times A')^4$ is an abelian scheme over $S$ with principal polarization. Because a constant family of abelian varieties contains no nonconstant subfamily, $B$ is also non-isotrivial. The modular map $J : S \to M_{g,1}$ attached to $B$ with respect to this principle polarization
is nonconstant. Let $\mathcal{N}$ be an ample line bundle of the Baily-Borel compactification of $M_{g,1}$. By another inequality of Faltings [[1]], by $h_F((A_t \times A_t'))^4 = 8h_F(A_t)$ and by the fact that $h_{\mathcal{M}}^{\ast}(\mathcal{L})$ is almost proportional to $h_{\mathcal{M}}^{\ast}(\mathcal{N})$ there exist positive constants $c_6$, $c_7$ and $c_8$ such that

$$|h_F(A_t) - c_6 h_{\mathcal{M}}^{\ast}(\mathcal{L})(t_i)| \leq c_7 + c_8 \log(\max(1, h_{\mathcal{M}}^{\ast}(\mathcal{L})(t_i))). \quad (14)$$

Notice that although not explicitly mentioned in [[1]], one can check carefully (or see the lemma below) that $c_8$ is independent of the number field $K$. Combining (11), (12) and (13) there exist positive constants $c_9$ and $c_{10}$ such that

$$c_9(\deg \phi_i)^{1/g} \leq h_{\mathcal{M}}^{\ast}(\mathcal{L})(t_i) + c_{10}. \quad (15)$$

Combining (10) and (14) there exist positive constants $c_{11}$, $c_{12}$ and $c_{13}$ such that

$$h_{\mathcal{M}}^{\ast}(\mathcal{L})(t_i) \leq c_{11} + c_{12} \log(\max(1, h_{\mathcal{M}}^{\ast}(\mathcal{L})(t_i))) + c_{13} \log(\deg \phi_i). \quad (16)$$

It is clear that (15) contradicts to (10) as the degree of the isogenies $\deg \phi_i$ goes to the infinity.

Lastly we sketch a calculation to make sure that positive constants $c_3, c_4$ obtained in [[3], p.356] are independent of number fields.

**Lemma 4.1.** Let $X \subset \mathbb{P}^n_Z$ be Zariski-closed, $Y \subset X$ closed, $\|\|$ a hermitian metric on $\mathcal{O}(1)|X(\mathbb{C}) - Y(\mathbb{C})|$ with logarithmic singularities along $Y$ and $\|\|_1$ a hermitian metric on $\mathcal{O}(1)|X(\mathbb{C})$. For $x \in X(\overline{\mathbb{Q}}) - Y(\overline{\mathbb{Q}})$ one defines $h(x)$ and $h_1(x)$ as in [[7]]. There exists positive constant $c_3$ and $c_4$ such that for all $x \in X(\overline{\mathbb{Q}}) - Y(\overline{\mathbb{Q}})$ we have

$$|h(x) - h_1(x)| \leq c_3 + c_4 \log(\max(1, h_1(x))).$$

**Proof.** We may assume that $Y$ is the intersection of $X$ with a linear subspace and the set of common zeros of $f_1, \ldots, f_r \in \Gamma(X/\mathbb{Z}, \mathcal{O}(1))$ is exactly $Y$. By multiplying the metric we assume $\|f_i\|_1 \leq 1$. A rational point $x \in X(K) - Y(K)$ corresponds to $\rho : \text{Spec}(R) \to X$, where $R$ is the integer ring of the number field $K$. We assume $f_1(x) \neq 0$. By definition we have

$$[K : \mathbb{Q}]h_1(x) \geq \sum_\sigma - \log(\|f_1\|_1(\sigma(x)))$$

and $[K : \mathbb{Q}]|h(x) - h_1(x)| = \sum_\sigma \log(\|f_1\|/\|f_1\|_1(\sigma(x)))$, where $\sigma$ runs through all embeddings of $K \hookrightarrow \mathbb{C}$. According to logarithmic singularities of the metric there exist positive constants $c_1, c_2$ such that for all number field $K$ and for $K$ rational point $x$

$$[K : \mathbb{Q}]|h(x) - h_1(x)| \leq c_1[K : \mathbb{Q}] + c_2 \sum_\sigma \log(-\log(\|f_1\|_1(\sigma(x)))).$$
Furthermore we have
\[
\sum_{\sigma} \log \left( -\log \|f_1\|_1(\sigma(x)) \right) \leq \log \left( \sum_{\sigma} -\log \|f_1\|_1(\sigma(x))/|K : Q| \right)^{|K : Q|}
\leq [K : Q] \log h_1(x).
\]

The above inequalities prove the desired claim. □

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