On delayed choice and quantum erasure in two-slit experiment for testing complementarity

Guang-Liang Li©

Department of Electrical and Electronic Engineering, University of Hong Kong, Hong Kong, People’s Republic of China
E-mail: glli@eee.hku.hk

Keywords: Young’s double-slit experiment, wave–particle duality, complementarity, delayed choice, quantum eraser

Abstract
The principle of complementarity is one of the cornerstones of quantum theory. The aim of this study was to advance our understanding of complementarity by analyzing the role of delayed choice and quantum erasers in two-slit experiments, and by proposing experiments for verifying the analysis. The analysis is based on models consisting of measurable spaces and probability measures involved in the experiments. The main findings are as follows: (a) The complementarity principle manifests itself in such a way that wave and particle behaviors cannot be simultaneously observed almost surely with respect to any single, fixed measure. (b) Described by different measures, complementary properties can coexist in the same experimental setup. (c) Which-way information will not preclude or erase interference fringes. (d) Delayed choice and quantum erasers are irrelevant to testing complementarity. (e) It is possible for us to know through which slit each quantum object passed almost surely with respect to the measure corresponding to the slit while the interference pattern is intact. Based on the experiments analyzed, realizable experiments are proposed for verifying the above results.

1. Introduction

Is it possible for us to know, with probability one, through which slit each quantum object (particle or photon) passed without disturbing interference fringes? This question concerns complementarity illustrated by the famous Young’s double-slit experiment, which embodies the counterintuitive features of quantum physics [1]. It is well known from the literature that the standard answer to this question is no; an affirmative answer is considered inconsistent with the complementarity principle. The negative answer is usually explained by Heisenberg’s uncertainty relation: simultaneous observation of wave and particle behaviors in the same experimental setup violates the uncertainty relation and hence is prohibited [2, 3].

The uncertainty relation is not the only explanation, however. For instance, researchers have proposed an alternative explanation based on two-slit quantum-eraser experiments for testing complementarity [4–6]. According to the alternative explanation, which-way (particle-like) information precludes or erases interference (wave-like) fringes, but the precluded or erased interference pattern can be restored by erasing which-way information after a particle or a signal photon has been detected. For the experiment introduced in [4], the choice of erasing or not erasing which-way information is made randomly by an idler photon, which is the entangled twin of the signal photon.

The alternative explanation was not accepted by some researchers; according to their argument [7], the uncertainty relation is relevant to the double-slit experiment, for any which-way measurement causes a momentum transfer, which is large enough to violate the uncertainty relation and destroy interference fringes. In the literature, this issue was addressed by observing a weak-valued momentum-transfer distribution [8]. This distribution is obtained by weak measurements [9–13]. According to the experiment reported in [8], interference fringes are destroyed by which-way information measured in the experiment; this is in agreement with the alternative explanation, because weak-valued probabilities can be negative [14], and the momentum-
transfer distribution has a variance consistent with zero. However, this momentum-transfer distribution also supports the argument against the alternative explanation; as a weak-valued distribution [8], it exhibits features characteristic of both the alternative explanation given in [4–6] and the argument in [7].

The striking feature of the alternative explanation is quantum erasure. The notion of ‘quantum eraser’ as interpreted in [4–6] implies a stronger form of ‘delayed choice’ in the sense illustrated by Wheeler [15], and has stimulated a spirited debate about what ‘delayed choice’ really means [16–25]. Various delayed-choice experiments have been proposed to test complementarity. In particular, the two-slit quantum-eraser experiment proposed in [5] has generated considerable research interest. Several quantum-eraser experiments have been reported, see, for example, [6, 26–29].

The present paper concentrates solely on two experiments. One is the two-slit atom interference gedanken experiment introduced in [4]; the other is the real experiment with pairs of entangled signal-idler photons reported in [6]. The former illustrates the notion of ‘delayed-choice quantum eraser’ straightforwardly. The latter is a full demonstration of the original scheme of the delayed-choice quantum eraser proposed in [5]. Each of the two experiments involves several probability measures on their common measurable space. However, the measures involved in both experiments are not specified, but the corresponding probability densities are used to explain the experimental results, which makes the explanation questionable.

The aim of this study was to advance our understanding of complementarity by analyzing the role of delayed choice and quantum erasers in the above experiments, and by proposing realizable experiments for verifying the analysis. The analysis is based on models consisting of measurable spaces and probability measures involved in the experiments. The findings reported in this paper are as follows. Which-way information can be used to specify two exclusive subsets of spots produced by quantum objects on a screen; each of the subsets serves as the domain of a random vector (or variable) representing coordinates of spots only in the subset. A probability conditional on this subset determines the density of the corresponding random vector (or variable). The density describes a pattern formed by spots of the subset, and characterizes the particle-like behavior. Addition of such densities is invalid, because the sum of the densities implies a false assumption, and violates the total probability theorem. The well-known interference pattern is described by the density of a random vector (or variable) representing coordinates of spots in a set different from those specified by which-way information. This density characterizes the wave-like behavior. The complementarity principle manifests itself in such a way that wave and particle behaviors cannot be simultaneously observed almost surely with respect to any single, fixed measure. Nevertheless, based on the total probability theorem, it is shown that the wave-like aspect and the particle-like aspect are described by different measures, and can coexist in the same experimental setup. Which-way information will not preclude or erase interference fringes observed in the experiments. Delayed choice and quantum erasers are irrelevant to testing complementarity. Without violating the complementarity principle, the experiments reveal a possibility for us to know through which slit each quantum object passed almost surely with respect to the measure corresponding to the slit.

The gedanken experiment [4] and the real experiment [6] are analyzed in section 2 and section 3, respectively. Based on the experiments analyzed, realizable experiments are proposed in section 4 for verifying the results reported in this paper. The paper is concluded in section 5.

2. On delayed choice in two-slit quantum-eraser experiment with atoms

The gedanken experiment [4] has demonstrated convincingly a way to bypass Heisenberg’s position-momentum uncertainty obstacle. A brief description of this experiment is as follows [3, 4]. A quantum-eraser detector sits between two micro-maser cavities, one in front of each slit. Two shutters shield the cavities from each other (figure 1).

The cavities act as which-way detectors. Initially, both cavities are empty and the shutters are closed. Atoms are sent to the apparatus one at a time, and collimated by a series of wider slits before they arrive at the narrow slits where the interference pattern originates. Immediately preceding the cavities, a laser beam is introduced to excite collimated atoms. After an atom passes through the laser beam, it absorbs a short-wavelength photon from the laser and becomes excited. The excited atom emits a longer-wavelength photon in one of the cavities, and goes through the corresponding slit with its motion unaffected. Hence the longer-wavelength photon carries potential which-way information. The quantum-eraser detector is used to detect the ‘tell-tale’ photon emitted by the atom. Before the shutters are opened, the wave associated with the photon consists of two partial waves, one in each cavity. After the atom has hit the screen and produced a spot there, the potential which-way information becomes actual, and experimenters can erase the which-way information by opening the shutters. Once the shutters are opened, the two partial waves become a single one. But the tell-tale photon may or may not be detected, depending on whether the partial waves reinforce or extinguish each other at the site of the
quantum-eraser detector. The above procedure is then repeated for the next atom. As this process continues, a sequence of spots will be produced. The spots are elementary outcomes of the experiment.

2.1. Model of experimental outcomes

To analyze the role of delayed choice and the quantum eraser in the above experiment, it is necessary to specify the involved probability measures on their common measurable space to describe the experimental outcomes. Let \( (S, \mathcal{A}) \) be the measurable space, where \( S \) is the sample space consisting of all the spots produced by atoms of an ensemble prepared for the experiment, and \( \mathcal{A} \) the \( \sigma \)-algebra of subsets of \( S \). Let \( i \) label the slits, \( i = 1, 2 \). For a specific \( i \), denote by \( H_i(s) \) a proposition, which characterizes a property of a spot \( s \), where \( H_i(s) \) means that \( s \) is produced by an atom that went through slit \( i \). This proposition concerns the particle-like behavior of an atom. According to whether \( H_i(s) \) is true or false for \( s \), all the spots fall into two subsets

\[
S_i = \{ s \in S : H_i(s) \}, \quad S_\neg i = \{ s \in S : H_\neg i(s) \},
\]

which constitute a partition of \( S \). Spots in \( S_i \) carry (actual) which-way information, because they imply a ‘slit-spot’ correlation between slit \( i \) and a spot \( s \) produced by an atom that passed through slit \( i \). Denote this correlation by \( C_i(s) \) if \( H_i(s) \) is true for given \( i \) and \( s \). Besides the property characterized by \( H_i(s) \), each spot \( s \) also has a property characterized by one of two propositions, \( H_+ (s) \) or \( H_- (s) \), where \( H_+ (s) \) means that \( s \) is produced by an atom with its tell-tale photon detected when the shutters are opened, and \( H_- (s) \) is the negation of \( H_+ (s) \). The two propositions concern wave-like behaviors of atoms. Write

\[
S_+ = \{ s \in S : H_+(s) \}, \quad S_- = \{ s \in S : H_-(s) \},
\]

which constitute a partition of \( S \) different from that formed by \( S_1 \) and \( S_2 \). Spots in \( S_\pm \) imply a ‘spot-photon’ correlation between \( s \) and the tell-tale photon of an atom that produced \( s \). Similarly, if \( H_+(s) \) or \( H_-(s) \) is true, the correlation is represented by \( C_+(s) \) or \( C_-(s) \).

Let \( \mathbb{P} \) be a probability measure on \( (S, \mathcal{A}) \). This measure characterizes uniquely the distribution of spots on the screen, and does not depend on specific coordinate systems. By definition, probabilities are values in the interval \([0, 1]\) assigned by a probability measure to events of a random experiment. Although probabilities are often expressed by numerical values, they are not pure numbers, because probabilities are calculated according to the rules different from those for calculating pure numbers. To calculate probabilities, it is necessary to distinguish different events, especially when their probabilities are numerically identical. If probabilities are expressed by numerical values without specifying associated events, the distinction is impossible. This can cause confusion. Fortunately, probabilities can also be expressed by symbols. When it is necessary to avoid confusion, it is helpful to express probabilities by symbols, indicating associated events explicitly. In this paper, the probabilities of \( S_1 \) and \( S_\pm \) are expressed by symbols \( \mathbb{P}(S_1) \) and \( \mathbb{P}(S_\pm) \). Their numerical values satisfy the following conditions.

\[
\mathbb{P}(S_1) + \mathbb{P}(S_2) = \mathbb{P}(S_+) + \mathbb{P}(S_-) = 1,
\]
and

\[
0 < P(S_1) < 1, \quad 0 < P(S_2) < 1, \\
0 < P(S_+) < 1, \quad 0 < P(S_-) < 1.
\]  
(1)

Write \( V = (X, Y, Z) \), which is a random vector defined for each \( s \in S \) with \( V(s) \) representing the coordinate of \( s \). Let \( P_V \) and \( f_V \) be the distribution and density of \( V \) on the measurable space \((\mathbb{R}^3, \mathcal{B})\), where \( \mathbb{R}^3 \) is the set of all ordered triples \( r = (x, y, z) \), and \( \mathcal{B} \) the \( \sigma \)-algebra of subsets of \( \mathbb{R}^3 \). Write \( I_x = I_y \times I_x \times I_y \), where \( I_x = (x, x + dx) \) is an interval of an infinitesimal length \( dx \) on the real line. Similarly \( I_x = (y, y + dy) \) and \( I_x = (z, z + dz) \). Write \( G(r) = \{ s \in S: V(s) \in I_x \} \).

The probability measure \( P \) on \((S, \mathcal{A})\) determines \( f_V \) and \( P_V \) uniquely. By definition,

\[
f_V(r) \, dr = P_V(I_x) = P[G(r)].
\]  
(2)

If \( s \) belongs to \( G(r) \) for some \( r \), the only information contained in \( G(r) \) is that \( V(s) \) belongs to \( I_x \). To use the correlation \( C(s) \), let \( V_r \) be the restriction of \( V \) to \( S_r \). To use the correlation \( C_r(s) \), denote by \( V_r \) and \( V_r \) the restrictions of \( V \) to \( S_r \) and \( S_r \). By definition, if \( U \) is a random vector on \( S_r \) and \( S_r \) a subset of \( S \) with \( 0 < P(S') < 1 \), the restriction of \( U \) to \( S' \) is a new random vector \( U' \) given by \( U'(s) = U(s) \) for \( s \in S' \). Let \( P_V \) and \( f_V \) be the distribution and density of \( V \). Let \( P_V \) and \( f_V \) be the distributions, and \( f_V \) and \( f_V \) the densities of \( V \) and \( V \). Define

\[
G_r(r) = \{ s \in S: V(s) \in I_x \}, \\
G_r(r) = \{ s \in S: V(s) \in I_x \}.
\]

Spots in \( G_r(r) \) imply the correlation \( C_r(s) \), and spots in \( G_r(r) \) imply the correlation \( C_r(s) \), as shown by the following simple properties of \( G_r(r) \) and \( G_r(r) \).

\[
\begin{align*}
G_r(r) &= G_r(r) \cap S_r, \\
G_r(r) &= G_r(r) \cap S_r, \\
G_r(r) \cup G_r(r) &= G_r(r) \cup G_r(r) = G(r), \\
G_r(r) \cap G_r(r) &= G_r(r) \cap G_r(r) = \emptyset.
\end{align*}
\]  
(3)

Similar to (2),

\[
f_V(r) \, dr = P_V(I_x) = P[G_r(r)],
\]  
(4)

\[
f_V(r) \, dr = P_V(I_x) = P[G_r(r)],
\]  
(5)

where \( P[G_r(r)] \) and \( P[G_r(r)] \) are given by conditional probabilities determined by \( P \) on \((S, \mathcal{A})\). Similar to \( P \), the measures \( P_r \) and \( P_r \) are independent of specific coordinate systems. By definition, for any event \( E \) in \( \mathcal{A} \),

\[
\begin{align*}
P_r(E) &= P(E|S_r) = \frac{P(E \cap S_r)}{P(S_r)}, \\
P_r(E) &= P(E|S_r) = \frac{P(E \cap S_r)}{P(S_r)}.
\end{align*}
\]  
(6)

Accordingly,

\[
\begin{align*}
P[G_r(r)] &= P[G_r(r)|S_r] = \frac{P[G_r(r) \cap S_r]}{P(S_r)} = \frac{P[G_r(r)]}{P(S_r)}, \\
P[G_r(r)] &= P[G_r(r)|S_r] = \frac{P[G_r(r) \cap S_r]}{P(S_r)} = \frac{P[G_r(r)]}{P(S_r)}.
\end{align*}
\]  
(8)

(9)

Note \( G_r(r) \subset S_r \), \( G_r(r) \subset S_r \), see (3). From (6), \( P(S_r) = \delta_0 \) follows immediately, where \( \delta_0 \) is the Kronecker delta. From (7), a similar result follows immediately.

The distributions and densities of \( V, V_r \) and \( V_r \) are defined on \((\mathbb{R}^3, \mathcal{B})\) with \( r \) serving as coordinates of spots. Properties of a spot \( s \), such as those characterized by \( H_1(s) \) and \( H_2(s) \), cannot be described by its coordinate. To describe the properties of \( s \), it is necessary to use \( P_r \) and \( P_r \) determined by \( P \). However, because these measures are all defined on \((S, \mathcal{A})\), the meaning of ‘with probability one’ or ‘almost surely’ becomes ambiguous. A convention in measure theory adopted here is helpful to prevent confusion and make the exposition easier. If, for each \( s \in S \), \( Q(s) \) is a proposition concerning \( s \), and if \( P \) is a probability measure on \((S, \mathcal{A})\), then the symbol \( Q(s)[P] \) represents the statement that \( Q(s) \) is true almost surely with respect to \( P \). In other words, if \( Q(s) \) is false for some \( s \), then this \( s \) belongs to a set \( \cdot N \) such that \( P(\cdot N) = 0 \). The definitions of \( P_r \) and \( P_r \) imply immediately that \( H_1(s)[P_r] \) and \( H_2(s)[P_r] \) are true.

Now consider again the question raised at the beginning of section 1: Is it possible for us to know, with probability one, through which slit each quantum object (particle or photon) passed without disturbing...
interference fringes? If ‘probability’ in the phrase ‘with probability one’ is $P$ (or any single, fixed measure), it is impossible to tell through which slit each atom passed almost surely with respect to $P$. By (1), $H_i(s)[P]$ and $H_2(s)[P]$ are both false. However, if ‘probability’ refers to $P_i$, then $H_i(s)[P]$ is true, where $i$ is determined by the correlation $C_i(s).$ For a specific $s$, there are two cases:

(1) $i = 1$, i.e., $s \in S_1$. Because $s$ must be somewhere on the screen, there is some $r$ such that $s \in G_1(r) = G_1(r) \cup G_2(r)$, see (3). Moreover, $P_1[G_1(r)] = 0$, since $G_1(r) \subset S_2$ and $P_1(S_2) = 0$. Therefore,

$$P_1[s \in G_1(r) \text{ for some } r] = P_1(S_1) = 1.$$

(2) $i = 2$, i.e., $s \in S_2$. Hence

$$P_2[s \in G_2(r) \text{ for some } r] = P_2(S_2) = 1.$$  

Similarly, for each $s \in S$, one and only one of two equations below must hold.

$$P_s[s \in G_s(r) \text{ for some } r] = P_s(S_s) = 1,$$

$$P_s[s \in G_s(r) \text{ for some } r] = P_s(S_s) = 1.$$

Spots in different subsets of $S$ may constitute different patterns. To avoid saying clumsily ‘a pattern formed by all spots in a set’ very often in the following analysis, let $\Pi(E)$ stand for a pattern formed by all spots in a set $E \subset S$. This pattern is described by the density of a random vector, which represents coordinates of $s \in E$. For instance, $\Pi(S)$ is formed by all $s \in S$ and described by $s^2$; $\Pi(S_i)$ is formed by all $s \in S_i$ and described by $f_{S_i}$. Similarly, $\Pi(S_{s_1})$ and $\Pi(S_{s_2})$ are described by $f_{S_{s_1}}$ and $f_{S_{s_2}}$.

### 2.2. Analysis and discussion

In the gedanken experiment [4], the role of the quantum eraser is to manipulate which-way information. Experimenters use a laser beam to excite an atom before it enters one of the cavities (figure 1). The excited atom emits a photon in the cavity, and passes through the corresponding slit with its motion undisturbed. Hence the photon carries potential which-way information. Once the atom hits the screen, the potential which-way information becomes actual and available for experimenters to exploit the correlations $C_i(s)$ or $C_s(s)$ for sorting the data. The cavities are separated by a shutter-detector combination. Experimenters can erase the which-way information by opening the shutters after the atom has produced a spot. When the laser is turned off, no atom becomes excited, and experimenters do not have which-way information. In the absence of which-way information, the motion of the center-of-mass of the atom (in the interference region) is described by the following wave function, which is the sum of two terms referring to the two slits

$$\Psi(r') = \frac{1}{\sqrt{2}} [\psi_1(r') + \psi_2(r')] |i\rangle,$$

where $r'$ is the coordinate of the center-of-mass of the atom before it hits the screen, and $|i\rangle$ represents the internal state of the atom. Corresponding to (10), the density of coordinates of spots on the screen is

$$\frac{1}{2} |\psi_1(r) + \psi_2(r)|^2$$

$$= \frac{1}{2} [|\psi_1(r)|^2 + |\psi_2(r)|^2] + \Re[\psi_1^*(r)\psi_2(r)],$$

which describes the well-known interference pattern, where $\Re$ represents the real part of a complex number, and $\psi_1^*(r)$ the complex conjugate of $\psi_1(r)$.

According to [4], if the laser is turned on, after an atom passes through one of the cavities and makes the transition from the excited state to the unexcited state $|b\rangle$, the system of the atom, cavities, and quantum-eraser detector is described by the state

$$\Psi(r') = \frac{1}{\sqrt{2}} [\psi_1(r')|1_10_2\rangle + \psi_2(r')|0_11_2\rangle] |b\rangle |d\rangle,$$

where $|d\rangle$ is the ground state of the quantum-eraser detector before the shutters are opened, $|1_10_2\rangle$ and $|0_11_2\rangle$ denote the cavity states. For example, $|1_10_2\rangle$ means that there is one photon in cavity 1 and none in cavity 2.

The following is a brief description of how the quantum eraser works [4]. In terms of symmetric and antisymmetric atomic states $\psi_+$ and $\psi_-$ and symmetric and antisymmetric states $\pm$ of the radiation fields in the cavities, the state given by (12) appears as
where
\[
\psi_+(\mathbf{r}') = \frac{1}{\sqrt{2}}[\psi_1(\mathbf{r}') + \psi_2(\mathbf{r}')] ,
\]

(13)

Once the atom has arrived at the screen and produced a spot there, experimenters open the shutters. The action of the quantum eraser is to change (13) into
\[
\Psi(\mathbf{r}) = \frac{1}{\sqrt{2}}[\psi_+(\mathbf{r})|0_10_2\rangle|e\rangle + \psi_-(\mathbf{r})|d\rangle] ,
\]

(14)

where $|e\rangle$ is the excited state of the quantum-eraser detector after it absorbs a photon. According to (4), (14) results in a density
\[
\frac{1}{2}[f_V(\mathbf{r}) + f_V'(\mathbf{r})] = \frac{1}{2}[|\psi_1(\mathbf{r})|^2 + |\psi_2(\mathbf{r})|^2] ,
\]

(15)

where
\[
f_V(\mathbf{r}) = |\psi_+(\mathbf{r})|^2 = \frac{1}{2}[|\psi_1(\mathbf{r}) + \psi_2(\mathbf{r})|^2 = \frac{1}{2}[|\psi_1(\mathbf{r})|^2 + |\psi_2(\mathbf{r})|^2 + 2\Re[\psi_1^*(\mathbf{r})\psi_2(\mathbf{r})] ,
\]

(16)

and
\[
f_V'(\mathbf{r}) = |\psi_-(\mathbf{r})|^2 = \frac{1}{2}[|\psi_1(\mathbf{r}) - \psi_2(\mathbf{r})|^2 = \frac{1}{2}[|\psi_1(\mathbf{r})|^2 + |\psi_2(\mathbf{r})|^2 - 2\Re[\psi_1^*(\mathbf{r})\psi_2(\mathbf{r})] .
\]

(17)

(16) and (17) describe wave-like properties of spots in $S_+$ and $S_-$; (16) is identical to (11), and (17) corresponds to so-called ‘anti-fringes’.

In [4], it is claimed that $\Pi(S)$ is described by (15) when the laser is turned on. The left-hand side of (15) is due to the orthogonality of $|e\rangle$ and $|d\rangle$, see (14). If the final state of the quantum-eraser detector is unknown, then the interference terms in $f_V$ and $f_V'$ cancel each other, and the cancellation is due to addition of these two densities; if experimenters observe the state of the quantum-eraser detector and correlate the observed state to the corresponding spot on the screen, then $f_V$ describes the fringes retrieved by erasing which-way information, and $f_V'$ describes the anti-fringes. The right-hand side of (15) is also a consequence of the orthogonality of $|1_10_2\rangle$ and $|0_11_2\rangle$, which makes the interference terms disappear, see (12).

(15) implies an assumption: in (11), (16), and (17), the terms $|\psi|^2$ serve as $f_V$. By this assumption, $|1_10_2\rangle$ and $|0_11_2\rangle$ are correlated with $\psi_1$ and $\psi_2$, respectively, which leads to an assertion that interference fringes are precluded simply by knowing which-way information or even by having the ability to acquire which-way information [4, 19].

However, the assumption underlying (15) is false, and the assertion is merely a consequence of explaining the role played by which-way information based on this false assumption. The assumption is false, for it results in illegitimate addition of densities. As shown by (4), (5), (8), and (9), $f_V(\mathbf{r})$ and $f_V'(\mathbf{r})$ on the left-hand side of (15) are determined by probabilities conditional on mutually exclusive events; similarly, $f_V$ and $f_V'$ are also determined by conditional probabilities; addition of such densities is illegitimate. The falsity of the assumption indicates that (15) is invalid, and hence $\Pi(S)$ is not described by (15).

No doubt, the state of the quantum-eraser detector implies information about wave-like properties of spots produced by atoms, and tell-tale photons carry potential which-way information, which becomes actual after the corresponding atoms hit the screen. But such information does not preclude interference fringes on the screen, and cannot change the experimental results. Experimenters can use acquired information to construct the restrictions of $V$ to $S_+$ or $S_-$. Such restrictions only serve to describe the spots on the screen from different aspects, but cannot modify the wave function given by (10). Therefore, (10) and the corresponding experimental results, which are represented by the interference pattern $\Pi(S)$, are intact. In other words, the density that describes $\Pi(S)$ is (11), whether or not which-way information is available. Hence experimenters need not use the quantum eraser to restore the fringes.

In the following, $f_V$ refers to (11). As components of $f_V$, $f_V'$, and $f_V$, the terms $|\psi|^2$ do not serve as $f_V$, even if $|\psi|^2$ and $f_V$ have the same functional form. By (16), $f_V$ is identical to $f_V'$. The anti-fringes are due to $G_-(\mathbf{r}) \cup G_+(\mathbf{r}) = G(\mathbf{r})$, see (3).
contains most spots of G(r), then G₁(r) consists of few spots remained in G(r), and vice versa. All spots in G₋(r) and G₆(r) constitute G(r), a portion of Π(S).

After the above clarification, the experiment can now be further analyzed based on the total probability theorem. This theorem holds for all physically meaningful probability densities, including those obtained by Born’s probabilistic interpretation of wave functions. By the total probability theorem, the wave-like aspects (interference fringes) described by \( P[G(r)] \) and \( P[G₆(r)] \) can coexist with the particle-like aspect (which-way information) described by \( P[G₁(r)] \), as shown below.

\[
\begin{align*}
\mathbb{P}[G(r)] &= \mathbb{P}_1[G(r)]\mathbb{P}(S₁) + \mathbb{P}_6[G(r)]\mathbb{P}(S₆) \\
&= \mathbb{P}_1[G(r)]\mathbb{P}(S₁) + \mathbb{P}_6[G(r)]\mathbb{P}(S₆).
\end{align*}
\]

Note that \( \mathbb{P}_1[G(r)] \) and \( \mathbb{P}_6[G(r)] \) are conditional probabilities, see (6) and (7). By (3),

\[
G(r) \cap S₁ = G₁(r), \quad G(r) \cap S₆ = G₆(r),
\]

and hence

\[
\begin{align*}
\mathbb{P}_1[G(r)] &= \mathbb{P}_1[G₁(r)], & \mathbb{P}_6[G(r)] &= \mathbb{P}_6[G₆(r)],
\end{align*}
\]

where \( \mathbb{P}_1[G₁(r)] \) and \( \mathbb{P}_6[G₆(r)] \) are given by (8) and (9), respectively. Consequently, (18) reduces to

\[
\begin{align*}
\mathbb{P}[G(r)] &= \mathbb{P}[G₁(r)]\mathbb{P}(S₁) + \mathbb{P}[G₆(r)]\mathbb{P}(S₆) \\
&= \mathbb{P}[G₁(r)]\mathbb{P}(S₁) + \mathbb{P}[G₆(r)]\mathbb{P}(S₆).
\end{align*}
\]

By (2), (4), and (5), an equivalent expression of (19) is

\[
\begin{align*}
\mathbb{P}[G(r)] &= \int f_V(r)dr \mathbb{P}(S₁) + \int f_V(r)dr \mathbb{P}(S₆) \\
&= \int f_V(r)dr \mathbb{P}(S₁) + \int f_V(r)dr \mathbb{P}(S₆).
\end{align*}
\]

To ensure the correctness of (19) and (20), the total probability theorem implies two requirements: (i) In (19), each of the expressions below

\[
\begin{align*}
\mathbb{P}_1[G₁(r)]\mathbb{P}(S₁), \quad \mathbb{P}_6[G₆(r)]\mathbb{P}(S₆), \quad \mathbb{P}_1[G₁(r)]\mathbb{P}(S₆), \quad \mathbb{P}_6[G₆(r)]\mathbb{P}(S₁),
\end{align*}
\]

and each of their counterparts in (20) represent a probability assigned by \( \mathbb{P} \) to an event in \( \mathcal{A} \). (ii) Both sides of each equality sign in (19) and in (20) represent the probability assigned by \( \mathbb{P} \) to the same event \( G(r) \) in \( \mathcal{A} \). By the requirements above, each of the following expressions in (19)

\[
\begin{align*}
\mathbb{P}_1[G₁(r)], \quad \mathbb{P}_6[G₆(r)], \quad \mathbb{P}_1[G₁(r)]\mathbb{P}(S₁), \quad \mathbb{P}_6[G₆(r)]\mathbb{P}(S₆), \quad \mathbb{P}_1[G₁(r)]\mathbb{P}(S₆), \quad \mathbb{P}_6[G₆(r)]\mathbb{P}(S₁),
\end{align*}
\]

and each of their counterparts in (20) must be treated or calculated as a whole. By (4), (5), (8), and (9),

\[
\begin{align*}
\{\int f_V(r)dr \mathbb{P}(S₁)\} + \{\int f_V(r)dr \mathbb{P}(S₆)\} &= \{\mathbb{P}_1[G₁(r)]\mathbb{P}(S₁)\} + \{\mathbb{P}[G₆(r)]\mathbb{P}(S₆)\} \\
&= \mathbb{P}[G₁(r)] + \mathbb{P}[G₆(r)] = \mathbb{P}[G(r)].
\end{align*}
\]

By (3), the last step in the above calculation yields \( \mathbb{P}[G(r)] \). Similarly,

\[
\begin{align*}
\{\int f_V(r)dr \mathbb{P}(S₁)\} + \{\int f_V(r)dr \mathbb{P}(S₆)\} &= \{\mathbb{P}_1[G₁(r)]\mathbb{P}(S₁)\} + \{\mathbb{P}[G₆(r)]\mathbb{P}(S₆)\} \\
&= \mathbb{P}[G₁(r)] + \mathbb{P}[G₆(r)] = \mathbb{P}[G(r)].
\end{align*}
\]

Because \( G₁(r), \ G₋(r), \ G₆(r), \) and \( G₆(r) \) are characterized by random vectors different from \( V \), their probabilities cannot be calculated with \( f_V \). By definition, each event characterized by \( V \) can be represented by an event in \( \mathcal{A} \), and hence \( \mathbb{P} \) can be used to calculate the probability of any event characterized by \( V \). However, there are events in \( \mathcal{A} \) that describe complementary phenomena but cannot be characterized by \( V \). In other words, the complementary nature of the experimental results cannot be captured by using \( f_V \). Nevertheless, as shown above based on the total probability theorem, complementary properties of atoms can be described by several probability densities in the same experimental setup, where the densities are determined by different measures.

The requirements of the total probability theorem are necessary to prevent illegitimate addition of probabilities conditional on different events. Probabilities must be calculated according to the rules for operations with probabilities. (15) not only fails to meet the requirements but also breaks the rules, because it implies the false assumption that the terms \( |\psi|^2 \) in the expressions of \( f_V, G₁, \) and \( f_V \) serve as \( f_V \). Although this assumption allows (15) to be expressed as
The two-slit quantum-eraser experiment combined with delayed choice is legitimate for calculating probabilities of any events relevant to the experiment. As shown in section 2.1, $f_V(r)$, $f_V(r)\, dr$, and $f_V(r)\, dr$ are probabilities conditional on $S_1$, $S_2$, and $S_3$ respectively. Moreover, $f_V$ and $f_V$ and $f_V$ are even not probability densities; they are invalid and have nothing to do with the experiment. Probabilities are not pure numbers. Even though $\mathbb{P}(S_1)$, $\mathbb{P}(S_2)$, $\mathbb{P}(S_3)$, and $\mathbb{P}(S_4)$ are numerically identical, these probabilities do not represent and hence cannot be treated as the same pure number. The distributive law for calculating pure numbers is not applicable to probabilities. Hence $\mathbb{P}[G(r)]$ is invalid.

Because the interference pattern $\Pi(S)$ is described by $\mathbb{P}$, and because neither $H_V(s) [\mathbb{P}^1]$ nor $H_V(s) [\mathbb{P}^2]$ is true, the complementarity principle manifests itself in such a way that wave and particle behaviors of atoms cannot be simultaneously observed almost surely with respect to $\mathbb{P}$ or any single, fixed measure. Nevertheless, as shown by (19) and (20), wave and particle aspects of atoms are described by different measures, and can peacefully coexist in the same experimental setup. Moreover, by (3), the correlation $C_V(s)$ implied by $G_V(r)$ and the correlation $C_V(s)$ implied by $G_V(r)$ cannot change $G_V(r)$, and hence $\Pi(S)$ remains intact in the presence of the correlations. In particular, ascertaining which–way information, i.e., $H_V(s) [\mathbb{P}^2]$ and $H_V(s) [\mathbb{P}^2]$, will not preclude $\Pi(S)$.

Experimenters can use the correlations to construct graphs of desired densities without changing anything on the screen.

Let $f$ be a continuous function of a three-dimensional vector with real-valued components, where the range of $f$ is $\mathbb{R}^3$. This shows again that (15) is invalid.

In a conceivable realization of the experiment [4], spots produced by atoms of the ensemble constitute a sequence $s_n$. The $n$-th spot $s_n$ is produced by the $n$-th atom sent to the apparatus. For an ordered triple $r$ of real numbers, the terms of $(s_n)_{n \geq 1}$ in $G(r)$ form a subsequence, denoted by $(s_n)_{n \geq 1} \cap G(r)$. Without the correlations $C_V(s)$ and $C_V(s)$, experimenters only know $V(s_n) \in I_r$ for any term in $(s_n)_{n \geq 1} \cap G(r)$. Using the correlation $C_V(s)$, experimenters can divide $(s_n)_{n \geq 1} \cap G(r)$ into two subsequences, $(s_n)_{n \geq 1} \cap G(r)$ and $(s_n)_{n \geq 1} \cap G(r)$ and $(s_n)_{n \geq 1} \cap G(r)$ and $(s_n)_{n \geq 1} \cap G(r)$ if experimenters exploit the correlation $C_V(s)$. As the number of the terms of $(s_n)_{n \geq 1}$ increases, there are more and more points of $\mathbb{R}^3$ representing coordinates of the terms, and gradually, the terms in $(s_n)_{n \geq 1}$ and in its subsequences above for different $r$ constitute $\Pi(S)$, $\Pi(S)$ or $\Pi(S)$, described by $\Gamma(s_n)$, $\Gamma(s_n)$ or $\Gamma(s_n)$, respectively. But only the interference pattern $\Pi(S)$ will emerge on the screen; it will not be precluded by which–way information. Delayed choice and the quantum eraser are irrelevant to testing complementarity; they cannot change $\Pi(S)$ or $f_V$, because all $s_n$ and their coordinates $V(s_n)$ are already fixed once the atoms hit the screen.

As shown above, the mathematical description of $\Pi(S)$ given in [4] is incorrect; it involves illegitimate addition of probability densities and violates the total probability theorem. The sorting of the full ensemble into sub-ensembles that have specific properties, such as known paths or interference patterns with full fringe visibility, is not sufficient for us to capture quantitative aspects of wave-particle duality.

3. On delayed ‘choice’ in two-slit quantum-eraser experiment with photons

The two-slit quantum-eraser experiment combined with delayed ‘choice’ [6] is a full demonstration of the original scheme of the delayed-choice quantum eraser introduced in [5]. In the experimental demonstration [6], spontaneous parametric down conversion is used to prepare an ensemble of pairs of entangled signal-idler photons.

When a pump laser beam passes through the two slits (labeled by A and B), it illuminates two tiny regions of a nonlinear optical crystal behind the two slits (figure 2). Each of the slits allows one and only one of the regions to be illuminated. One of the regions (labeled by A) corresponds to slit A, and the other region (labeled by B)
corresponds to slit B. Thus the regions play the role of the slits. Pairs of entangled signal-idler photons are generated from either region A or region B. In the apparatus, there is at most one pair of entangled signal-idler photons. Signal photons from both regions are sent through a lens to a screen. The lens is used to achieve the ‘far field’ condition. Detector $D_0$, movable along the x-axis, is used to detect signal photons. Idler photons from both regions are sent to an interferometer, consisting of one prism, three 50-50 beam splitters (BSA, BSB, and BS), and two reflecting mirrors ($M_A$ and $M_B$), see figure 3. The prism separates idler photons into different paths corresponding to various measurement options. Detectors $D_i$ ($i = 1, 2, 3, 4$) are used to detect idler photons; coincidences between $D_0$ and $D_i$ are recorded by a coincidence circuit, see figure 2.

The delayed ‘choice’ for experimenters to observe either wave or particle behavior of a signal photon is not made by experimenters; the ‘choice’ is made randomly by the idler photon in the same pair. Once the signal photon hits the screen, the delayed ‘choice’ made by the idler photon cannot change the position of $D_0$ where the signal photon is detected. Because photons in the same pair are entangled and hence have the same properties, experimenters can observe the path taken by the idler photon, and infer the behavior of the signal photon.

Figure 2. Schematic of the experimental setup in [6]. Reprinted figure 2 with permission from [6], Copyright (2000) by the American Physical Society.

Figure 3. Schematic of the interferometer in [6]. Reprinted figure 2 with permission from [6], Copyright (2000) by the American Physical Society.
without disturbing it in any way. Each idler photon travels along one of the six paths listed below, and is detected at least 7.7 ns later than the detection of its twin, the corresponding signal photon.

(1) region A $\rightarrow$ BSA $\rightarrow$ D$_4$,
(2) region B $\rightarrow$ BSB $\rightarrow$ D$_3$,
(3) region A $\rightarrow$ BSA $\rightarrow$ M$_A$ $\rightarrow$ BS $\rightarrow$ D$_1$,
(4) region B $\rightarrow$ BSB $\rightarrow$ M$_B$ $\rightarrow$ BS $\rightarrow$ D$_1$,
(5) region A $\rightarrow$ BSA $\rightarrow$ M$_A$ $\rightarrow$ BS $\rightarrow$ D$_2$,
(6) region B $\rightarrow$ BSB $\rightarrow$ M$_B$ $\rightarrow$ BS $\rightarrow$ D$_2$.

3.1. Model of experimental outcomes

To describe outcomes of the above experiment, it is necessary to specify the involved probability measures on their common measurable space. The notations used here are the same as (or similar to) those used in section 2, but their meanings may be redefined.

Let $(S, \mathcal{A})$ be the measurable space. The sample space $S$ consists of all the positions of detector $D_0$ where it ‘clicks’, and $\mathcal{A}$ is the $\sigma$-algebra of subsets of $S$. Let $\mathbb{P}$ be a probability measure on $(S, \mathcal{A})$. This measure characterizes uniquely the distribution of all $s \in S$, and does not depend on specific coordinate systems. For a specific $i > 0$, denote by $H_i(s)$ a proposition, which characterizes a property of an $s \in S$, where $H_i(s)$ means that $D_i$ also ‘clicks’ when $D_0$ ‘clicks’ at $s$. According to whether $H_i(s)$ is true or false for $s$, the sample space $S$ can be divided into four exclusive subsets

$$S_i = \{s \in S: H_i(s)\}, \quad i = 1, 2, 3, 4,$$

which constitute a partition of $S$. The probabilities of $S_i$ satisfy the following conditions.

$$\sum_{i=1}^{4} \mathbb{P}(S_i) = 1,$$

and

$$0 < \mathbb{P}(S_i) < 1, \quad i = 1, 2, 3, 4. \quad (22)$$

For a signal photon detected by $D_0$ at $s \in S$, if $H_i(s)$ is true, and if $i = 1$ or $i = 2$, the corresponding idler photon took one of the last four paths of the interferometer in [6] (listed before section 3.1), and hence the signal photon came from either region A or region B. In contrast, if $H_i(s)$ is true, the idler photon took path 2), and hence the signal photon came from region B. Similarly, if $H_i(s)$ is true, the idler photon took path 1) and the signal photon came from region A. As shown above, no which-way information is carried by sample points in $S_1 \cup S_2$. Only those in $S_3 \cup S_4$ have which-way information.

Denote by $X$ a random variable defined for each $s \in S$ with $X(s)$ representing the x-coordinate of $s$. Let $P_X$ and $f_X$ be the distribution and density of $X$ on $(\mathbb{R}, \mathbb{#})$. Write

$$G(x) = \{s \in S: X(s) \in I_x\}.$$

The probability measure $\mathbb{P}$ on $(S, \mathcal{A})$ determines $f_X$ and $P_X$ uniquely. If $s$ belongs to $G(x)$ for some $x$, the only information contained in $G(x)$ is that $X(s)$ belongs to $I_x$. Let $X_i$ be the restriction of $X$ to $S_i$. Let $P_{X_i}$ and $f_{X_i}$ be the distribution and density of $X_i$. Define

$$G_i(x) = \{s \in S: X_i(s) \in I_x\}.$$

It is easy to see that $G_i(x)$ have the following properties.

$$G_i(x) = G(x) \cap S_i, \quad \bigcup_{i=1}^{4} G_i(x) = G(x), \quad G_i(x) \cap G_j(x) = \emptyset, \quad i \neq j. \quad (23)$$

The events $G_i(x)$ and $G_j(x)$ describe wave-like behaviors; $G_i(x)$ and $G_d(x)$ describe the particle-like behavior. By definition,

$$f_{X_i}(x)dx = P_{X_i}(I_x) = \mathbb{P}[G_i(x)],$$
$$f_{X_i}(x)dx = P_{X_i}(I_x) = \mathbb{P}[G_i(x)],$$

where $\mathbb{P}[G_i(x)]$ are given by conditional probabilities determined by $\mathbb{P}$ on $(S, \mathcal{A})$. Similar to $\mathbb{P}$, the measures $\mathbb{P}_i$ are independent of specific coordinate systems.
However, by interference pattern, just like what we see in an ordinary Young's simpli-
ed experiment, which-way information is absent. According to quantum mechanics, this interpretation is incorrect,
because it violates the total probability theorem as we shall see below.

As shown in the experiment 3.2. Analysis and discussion

As a real number x, G(x) is a portion of the interference pattern II(S) characterized by X, and G(x), i = 1, 2, 3, 4 are subsets of G(x), see (23). By the total probability theorem,

\[ P[G(x)] = \sum_{i=1}^{4} P[G_i(x)]P(S_i) = \sum_{i=1}^{4} \int f_{X}^{i}(x)dxP(S_i) = f_{X}(x)dx. \]
For a specific $i$, $G(x)$ is a portion of $\Pi(S_i)$, characterized by $X_i$, which is the restriction of $X$ to $S_i$. The observed interference fringes and anti-fringes are due to $G_1(x) \cup G_2(x) = G'(x)$, where $G'(x) = G(x) \setminus [G_2(x) \cup G_1(x)]$, see also section 2.2. Experimenters can simultaneously construct $\Gamma(f) = \Gamma_3(f) \otimes \mathbb{R} \otimes \mathbb{R}^+$ to describe the corresponding patterns formed by points of $(S, \mathcal{A})$. However, such descriptions can by no means alter the patterns so formed, because $\Gamma(f_i)$ and $\Gamma(f_{i'})$ are constructed according to sample points obtained in the experiment, and because experimenters must first obtain these sample points, which are elementary experimental outcomes, before using the outcomes to construct $\Gamma(f_3)$ and $\Gamma(f_4)$.

In addition, idler photons registered at $D_i$ are irrelevant to $\Pi(S_i)$ if $i \neq i$, because $S_i$, $i = 1, 2, 3, 4$ constitute a partition of $S$, and hence $S_i \cap S_j = \emptyset$ when $i \neq j$, see section 3.1. In particular, ‘clicks’ at $D_i$ or $D_4$ will not erase interference fringes in $\Pi(S_1)$ or $\Pi(S_4)$, as idler photons registered at $D_i$ or $D_4$ are irrelevant to $\Pi(S_1)$ and $\Pi(S_4)$. Hence $\Pi(S_1)$ and $\Pi(S_4)$ are intact when idler photons are detected by $D_1$ or $D_2$. Similarly, ‘clicks’ at $D_1$ or $D_2$ will not erase which–way information carried by $s \in S_2 \cup S_3$, for idler photons registered at $D_1$ or $D_2$ are irrelevant to $\Pi(S_2)$ and $\Pi(S_3)$. Hence $\Pi(S_2)$ and $\Pi(S_3)$ remain unchanged when idler photons are detected by $D_1$ or $D_2$.

Furthermore, it is not meaningful to compare $\Gamma(f_{i1} + f_{i1})$ with $\Gamma(f_{i1})$ or $\Gamma(f_{i1})$, because $f_{i1} + f_{i1}$ violates the total probability theorem; it is neither a probability density nor relevant to the experiment, see also section 2.2.

As shown above, the analysis here leads to essentially the same findings as those presented in section 2: Because the interference pattern $\Pi(S)$ is described by $f_S$, which is determined by $\mathbb{P}$, and because $H_i(s)[\mathbb{P}]$, $i = 1, 2, 3, 4$ are all false, the complementarity principle manifests itself in such a way that wave and particle behaviors of photons cannot be simultaneously observed almost surely with respect to $\mathbb{P}$ or any single, fixed measure. Nevertheless, $H_i(s)[\mathbb{P}]$ are true. Based on the total probability theorem, the analysis also shows that complementary aspects of photons are described by different measures, and can coexist in the same experimental setup. Undoubtedly, each idler photon is detected at least 7.7 ns later than the detection of the corresponding signal photon. But the delayed ‘choice’ made by idler photons can neither alter the patterns on the screen, nor change the densities describing the patterns, because all $s \in S$ and their coordinates $X(s)$ and $X(s)$ for $s \in S$ are already fixed once signal photons are detected by $D_0$ at $s$. The quantum eraser is irrelevant to testing complementarity.

4. Toward advancing our understanding of complementarity

Besides the experiment proposed for observing the interference pattern $\Pi(S)$ (section 3.2), which is a simplified version of the realized experiment [6], the following experiments are proposed for verifying the results reported in this paper; the results and proposed experiments may be helpful to advance our understanding of complementarity. The proposed experiments are also simplified versions of the experiment [6], and hence realizable.

Proposed experiment 2. The purpose of this experiment is to show that which–way information (such as ‘clicks’ at $D_3$ or $D_4$ in the original experiment [6]) will not erase interference fringes. To this end, the original interferometer is modified accordingly, such that the beam splitters BSA and BSB are replaced by the mirrors $M_A$ and $M_B$, respectively; detectors $D_1$, $D_2$, and the beam splitter BS are removed (figure 5). The other parts of the original setup remain essentially the same. In this simplified experiment, only two paths that reveal which–way information are available for idler photons:

- region A $\rightarrow$ $M_A \rightarrow D_0$.
- region B $\rightarrow$ $M_B \rightarrow D_4$.

Compare paths 1) and 2) of the original interferometer in [6], listed before section 3.1. Consequently, $S_3$ and $S_4$ form a partition of $S$, and $X_3$, $X_4$ are the restrictions of $X$ to $S_3$, $S_4$. By the total probability theorem,

$$\mathbb{P}[G(x)] = \mathbb{P}[G_3(x)]\mathbb{P}(S_3) + \mathbb{P}[G_4(x)]\mathbb{P}(S_4) = [f_{X_3}(x)dx]\mathbb{P}(S_3) + [f_{X_4}(x)dx]\mathbb{P}(S_4) = f_{X}(x)dx.$$

The interference pattern $\Pi(S)$ is described by $\Gamma(f_{i0})$, while $\Pi(S_3)$ and $\Pi(S_4)$, the patterns corresponding to two exclusive subsets of $S$, are described by $\Gamma(f_{i0})$ and $\Gamma(f_{i0})$. Although neither $\Pi(S_3)$ nor $\Pi(S_4)$ exhibits interference fringes, the absence of interference fringes in $\Pi(S_3)$ and $\Pi(S_4)$ will not change the interference pattern $\Pi(S)$, which is formed by all $s \in S$. Moreover, the sum of $f_{X_3}$ and $f_{X_4}$ violates the total probability theorem; $f_{X_3} + f_{X_4}$ is even not a probability density and has nothing to do with the experiment, see also section 2.2.

By the total probability theorem, the three patterns, $\Pi(S)$, $\Pi(S_3)$, and $\Pi(S_4)$, can coexist in the same experimental setup. In other words, interference fringes exhibited in $\Pi(S)$ are intact in the presence of which–way information provided by ‘clicks’ at $D_3$ or $D_4$. Because $S_3 \cup S_4 = S$, we can tell with probability one from which region (slit) each idler photon and its entangled twin came without disturbing $\Pi(S)$ in any way, where
probability in the phrase ‘with probability one’ refers to the measure corresponding to the region (slit). That is, we can ascertain $H_i(s)[\mathcal{P}], i = 3, 4$ while interference fringes remain intact.

Proposed experiment 3. In this experiment, the interferometer used in [6] is modified, such that BSA, BSB, $D_3$, and $D_4$ are removed, but the other components of the original experimental setup remain unchanged (figure 6). With this simplified setup, we can readily see the relation between interference fringes and anti-fringes. No sample point in this experiment carries which-way information, because each idler photon can only take one of the paths below:

- region A $\rightarrow M_A \rightarrow$ BS $\rightarrow D_1$,
- region B $\rightarrow M_B \rightarrow$ BS $\rightarrow D_1$,
- region A $\rightarrow M_A \rightarrow$ BS $\rightarrow D_2$,
- region B $\rightarrow M_B \rightarrow$ BS $\rightarrow D_2$.

Figure 5. Schematic of the modified interferometer for proposed experiment 2.

Figure 6. Schematic of the modified interferometer for proposed experiment 3.
Compare the last four paths of the original interferometer in [6], listed before section 3.1. The sets \( S_1 \) and \( S_2 \) form a partition of \( S \), and \( X_i \) is the restriction of \( X \) to \( S_i \), where \( i = 1, 2 \). Again, by the total probability theorem,

\[
\mathbb{P}[G(x)] = \mathbb{P}[G_1(x)] \mathbb{P}(S_1) + \mathbb{P}[G_2(x)] \mathbb{P}(S_2) = \int f_{X_1}(x) \mathbb{P}(S_1) + \int f_{X_2}(x) \mathbb{P}(S_2) = f_X(x) \mathbb{P}(S).
\]

As a result, we can simultaneously construct \( \Gamma(f_{X_1}), \Gamma(f_{X_2}), \) and \( \Gamma(f_X) \) to describe \( \Pi(S), \Pi(S_1), \) and \( \Pi(S_2) \), respectively. One of the two patterns, \( \Pi(S_1) \) or \( \Pi(S_2) \), exhibits interference fringes just like those in \( \Pi(S) \), and the other exhibits anti-fringes. Similar to \( f_{X_1} + f_{X_2} \), the sum \( f_X \) is invalid and cannot make interference fringes in \( \Pi(S) \) disappear. See also section 2.2.

5. Conclusion

Recall the question concerning complementarity illustrated by the Young’s double-slit experiment: Is it possible for us to know, with probability one, through which slit each quantum object (particle or photon) passed without disturbing interference fringes? So far, in the literature, the standard answer to this question has been no. Traditionally, the negative answer is explained by Heisenberg’s uncertainty relation. In contrast, the explanation given in [4, 6] claims that interference fringes are precluded or erased by which-way information, but the precluded or erased fringes can be restored by erasing which-way information after an atom or a signal photon has been detected.

The explanation given in [4, 6] relies on densities determined by probabilities conditional on mutually exclusive events. These events are sets of spots. Such sets serve as domains of different random vectors (or variables). However, the probability measures that determine these densities are not specified in [4, 6], but the densities are used to explain the experimental results, which leads to violation of the total probability theorem and makes the explanation questionable.

The present paper has shown that the complementarity principle manifests itself in such a way that wave and particle behaviors cannot be simultaneously observed almost surely with respect to any single, fixed measure. Nevertheless, based on the total probability theorem, it is shown that complementary aspects of quantum objects are described by different measures, and can peacefully coexist in the same experimental setup. Which-way information will not preclude or erase interference fringes observed in the experiments. Delayed choice and quantum erasers are irrelevant to testing complementarity.

In conclusion, for the question about complementarity illustrated by the double-slit experiment, an affirmative answer is not only conceivable but also reasonable: Without violating the complementarity principle, we may tell through which slit each quantum object passed almost surely with respect to the measure corresponding to the slit. This answer cannot be found in the literature, but it can be tested by experiment. Based on the experiments analyzed, realizable experiments are proposed for verifying the results reported in this paper. The results and proposed experiments may be helpful to advance our understanding of complementarity.

Acknowledgments

The author would like to thank anonymous referees for their helpful comments and suggestions.

Data availability statement

No new data were created or analysed in this study.

ORCID iDs

Guang-Liang Li @ https://orcid.org/0000-0001-7210-1239

References

[1] Feynman R P, Leighton R B and Sands M 1965 The Feynman Lectures on Physics 3 (Reading: Addison-Wesley)
[2] Wootters W K and Zurek W H 1979 Phys. Rev. D 19 473–84
[3] Englert B G, Scully M O and Walther H 1994 Sci. Am. 271 86–92
[4] Scully M O, Englert B G and Walther H 1991 Nature (London) 351 111–6
[5] Scully M O and Duhé L 1982 Phys. Rev. A 25 2208–13
[6] Kim Y H, Yu R, Kulik S P, Shih Y and Scully M O 2000 Phys. Rev. Lett. 84 1–5
[7] Storey E P, Tan S M, Collett M J and Walls D F 1994 Nature 367 626–8
[8] Mir R, Landeen J S, Mitchell M W, Steinberg A M, Garretson J L and Wiseman H M 2007 New J. Phys. 9 287
[9] Aharonov Y, Albert D Z and Vaidman L 1988 Phys. Rev. Lett. 60 1351
[10] Ritchie N W M, Story J G and Hulet R G 1991 Phys. Rev. Lett. 66 1107–10
[11] Flack R and Hiley B J 2014 Journal of Physics Conference Series 504 012016
[12] Kocsis S, Braverman B, Ravets S, Stevens M J, Mirin R P, Shalm L K and Steinberg A M 2011 Science 332 1170–3
[13] Steinberg A, Feizpour A, Rozema L, Mahler D and Hayat A 2013 Physics World 26 35–40
[14] Wiseman H M 2003 Phys. Lett. A 311 285–91
[15] Wheeler J A 1978 The ‘past’ and the ‘delayed-choice’ Double-Slit Experiment (New York, NY: Academic) pp 9–48
[16] Mohrhoff U 1996 Am. J. Phys. 64 1468–75
[17] Englert B G, Scully M O and Walther H 1999 Am. J. Phys. 67 325–9
[18] Mohrhoff U 1999 Am. J. Phys. 67 330
[19] Scully M O and Walther H 1998 Found. Phys. 28 399–413
[20] Hiley B J and Callaghan R E 2006 Found. Phys. 36 1869–83
[21] Fearn H 2016 Found. Phys. 46 44–69
[22] Kastner R E 2019 Found. Phys. 49 717–27
[23] Ingraham R L 1994 Phys. Rev. A 50 4502–5
[24] Ingraham R L 1995 Phys. Rev. A 51 4295
[25] Aharonov Y, Popescu S and Vaidman L 1995 Phys. Rev. A 52 4984–5
[26] Herzog T J, Kwiat P G, Weinfurter H and Zeilinger A 1995 Phys. Rev. Lett. 75 3034–7
[27] Peng T, Chen H, Shih Y and Scully M O 2014 Phys. Rev. Lett. 112 180401.1–5
[28] Walborn S P, Terra Cunha M O, Pádua S and Monken C H 2002 Phys. Rev. A 65 033818
[29] Walborn S P, Terra Cunha M O, Pádua S and Monken C H 2003 Amer. Sci. 91 336–43