Harmonic functions, central quadrics, and twistor theory

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Abstract

Solutions to the \(n\)-dimensional Laplace equation which are constant on a central quadric are found. The associated twistor description of the case \(n=3\) is used to characterise Gibbons-Hawking metrics with tri-holomorphic \(SL(2,\mathbb{C})\) symmetry.

1 Introduction

Let \(V(x_1,\ldots,x_n) \in \mathbb{C}\) be a solution to a PDE

\[
F(V, V_i, V_{ij}, \ldots, V_{ij\cdots k}, x_i) = 0,
\]

where \(V_i = \partial V/\partial x_i\), and \((x_1,\ldots,x_n) \in \mathbb{C}^n\). The usual way of reducing this PDE to an ODE is to determine a group of transformations acting on dependent and independent variables, such that \(V\) is transformed to a different solution of (1.1), and reduce the number of independent variables down to one. There are several algorithmic procedures of varying levels of sophistication (Noether’s theorem, Lie Point symmetries, Cartan-Kähler formalism, ...) to study such reductions. See [14] for a very good review of some of these methods.

It is interesting to seek non-symmetric ways of reducing PDEs to ODE. One such method is the hyper-surface ansatz. Let \(\Sigma \subset \mathbb{C}^n\) be an algebraic hyper-surface. The ansatz (which is motivated by the work of Darboux [1] on orthogonal curvilinear coordinates) is to seek solutions constant on \(\Sigma\), or equivalently to seek symmetric objects

\[
M(V), M^i(V), M^{ij}(V), \ldots, M^{ij\cdots k}(V),
\]

so that a solution of equation (1.1) is determined implicitly by

\[
Q(x_i, V) := M(V) + M^i(V)x_i + M^{ij}(V)x_ix_j + \ldots + M^{ij\cdots k}(V)x_ix_j\ldots x_k = C,
\]

where \(C\) is a constant. Here \(\Sigma\) should be regarded as the zero locus of a polynomial \(Q(x_i, V) - C\) in \(\mathbb{C}^n\).

If \(V\) satisfies (1.1) and the algebraic constraint (1.2), then so does \(g^i(V)\), there \(g^i\) is a flow generated by any section of \(T\Sigma\). Note however that vectors tangent to \(\Sigma\) do not generate symmetries of (1.1), as the choice of \(\Sigma\) depends on \(V\).
In this paper we shall look the quadric ansatz
\[ Q(x_i, V) := M^{ij}(V)x_ix_j = C, \] (1.3)
which is the simplest non-trivial case of (1.2). The ansatz can be made whenever we have a PDE of the form
\[ \frac{\partial}{\partial x_j}(\eta_{ij}(V)\frac{\partial V}{\partial x_i}) = 0, \] (1.4)
where \( \eta \) is a given symmetric matrix whose components depend on \( V \).

The quadric ansatz has been applied to two non-linear integrable PDEs: If \( V \) is a solution of \( SU(\infty) \) Toda equation
\[ \exp(V)V_{12} - V_{33} = 0 \]
then \( M \) can be determined in terms of the third Painlevé transcendent [16]. If \( V \) is a solution to dispersionless KP equation
\[ (V_1 - V V_2)^2 - V_{33} = 0 \]
then \( M \) can be determined by one of the first two Painlevé transcendent [5]. Both \( SU(\infty) \) Toda, and dKP equations are solvable by twistor transform, and it would be desirable to achieve a characterisation twistor spaces corresponding to solutions constant on central quadrics.

Motivated by this problem we shall apply the quadric ansatz to the Laplace equation. We shall work in the holomorphic category, and regard \( V \) as a holomorphic function of complex coordinates \( x_1, ..., x_n \). We shall abuse the terminology, and call \( V \) a harmonic function, whenever it satisfies a complexified Laplace equation. The reality conditions may be imposed if desired, to characterise real solutions in signatures \((n - r, r)\) when \( r = 1, ..., n \).

In the next section we shall find harmonic functions constant on central quadrics (Theorem 2.1). In the remaining sections we shall focus on the three-dimensional case. In section 3 harmonic functions will be related to solutions to \( \text{SDiff}(\Sigma) \) Nahm’s equations by means of a hodograph transformation. Here \( \text{SDiff}(\Sigma) \) is a group of holomorphic symplectomorphisms of a two-dimensional complex symplectic manifold \( \Sigma \). Harmonic functions constant on a central quadric will be characterised by a reduction form \( \text{SDiff}(\Sigma) \) to \( SL(2, \mathbb{C}) \) (Proposition 3.3). In section 4 we shall review a twistor construction of solutions to 3D Laplace equation, and characterise solutions constant on central quadrics in terms of \( SL(2, \mathbb{C}) \) invariant holomorphic line bundles over \( T\mathbb{CP}^1 \) (Theorem 4.1). In section 5 we shall characterise \( \mathbb{C}^* \) invariant complexified hyper-Kähler metrics in 4D which admit tri-holomorphic \( SL(2, \mathbb{C}) \) transitive action (Proposition 5.1). In section 6 we shall give an example illustrating the construction. The basic facts about bundles over \( \mathbb{CP}^1 \) used in the paper are collected in the Appendix A. Appendix B is devoted to \( \text{SDiff}(\Sigma) \) Nahm’s equations.

2 Quadric ansatz for Laplace equation

**Theorem 2.1** Solutions of the Laplace equation
\[ \sum_{i=1}^{n} \frac{\partial^2 V}{\partial x_i^2} = 0 \] (2.5)
constant on a central quadric are given by

\[ V = \int \frac{dH}{\sqrt{(H - \beta_1)(H - \beta_2) \cdots (H - \beta_n)}}, \tag{2.6} \]

where

\[ \sum_{i=1}^{n} \frac{x_i^2}{H - \beta_i} = C, \tag{2.7} \]

and \( C, \beta_1, \beta_2, \ldots, \beta_n \) are constants (which can be normalised so that \( \beta_1 + \beta_2 + \cdots + \beta_n = 1 \)).

**Proof.** Equation (2.5) is equivalent to (1.4) if \( \eta_{ij} \) is an identity matrix. We assume that the level sets of \( V \) are of the form (1.3), and we differentiate (1.3) implicitly to find

\[ \frac{\partial V}{\partial x_i} = -2 \dot{Q} M_{ij} x_j, \quad \text{where} \quad \dot{Q} = \frac{\partial Q}{\partial V}. \tag{2.8} \]

Now we substitute this into (1.4) and integrate once with respect to \( V \). Introducing \( g(V) \) by

\[ \dot{g} = \frac{1}{2} \eta_{ij} M_{ij} = \frac{1}{2} \text{trace} (\eta M) \tag{2.9} \]

we obtain

\[ (g \dot{M}^{ij} - M^{ik} \eta_{km} M^{mj}) x_i x_j = 0, \]

so that as a matrix ODE

\[ g \ddot{M} = M \eta M. \tag{2.10} \]

This equation simplifies if written in terms of another matrix \( N(V) \) where

\[ N = -M^{-1}, \tag{2.11} \]

for then

\[ g \dot{N} = \eta, \tag{2.12} \]

and \( g \) can be given in terms of \( \Delta = \det (N) \) by

\[ g^2 \Delta = \zeta = \text{constant}. \tag{2.13} \]

Equation (2.12) implies that \( N(V) \) can be written as

\[ N(u) = \begin{pmatrix} H_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & H_{22} & h_{23} & \cdots & h_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \cdots & \cdots & H_{nn} \end{pmatrix} \tag{2.14} \]

where \( h_{ij} \) are constants, while

\[ \dot{H}_{ii} = g^{-1}, \quad H_{11} - H_{ii} = \gamma_i = \text{constant}, \quad i = 1, \ldots, n. \tag{2.15} \]

The constants \( h_{ij} \) can be eliminated by \( SO(n, \mathbb{C}) \) transformations of (2.5). Let \( H_{11} = H(V) \). Equation (2.13) yields

\[ H(H + \gamma_2) \cdots (H + \gamma_n) = \zeta \dot{H}^2, \]

3
and $M = -\text{diag}[H^{-1}, (H + \gamma_2)^{-1}, \ldots, (H + \gamma_n)^{-1}]$ so the quadric (1.3) is diagonal. We can rescale $H$, shift it by a constant and define a new set of constants $\beta_1, \ldots, \beta_n$ so that (1.3) yields (2.7) where $H(V)$ satisfies

$$\prod_{i=1}^{n} (H - \beta_i) = H^2.$$  \hfill (2.16)

The last equation is readily solved giving (2.6).

\[ \square \]

**Remarks**

1. In general solutions to the Laplace equations obtained from the quadric ansatz with $C \neq 0$ do not admit any symmetries. Solutions which are constant on a central cone ($C = 0$) are invariant under scaling transformations $x_i \rightarrow sx_i$.

2. If $V$ is a harmonic function given by the quadric ansatz (1.3) then $\hat{V} = \partial V / \partial C$ is also harmonic (but not necessarily constant on a quadric). Repeating the process yields an infinite set

$$V, \quad \partial V / \partial C, \quad \partial^2 V / \partial C^2, \quad \ldots$$

of solutions associated to the quadric ansatz. They can all be found by implicit differentiation of (1.3)

$$\dot{Q} \frac{\partial V}{\partial C} = C, \quad \ddot{Q} \left( \frac{\partial V}{\partial C} \right)^2 + \dot{Q} \frac{\partial^2 V}{\partial C^2} = 0, \quad \ldots$$

where $\dot{Q} = \dot{M}^{ij} x_i x_j$. For example

$$\hat{V} = -\left( \prod_{i=1}^{n} (H - \beta_i) \right)^{-1/2} \left( \sum_{i=1}^{n} \frac{x_i^2}{(H - \beta_i)^2} \right)^{-1},$$  \hfill (2.17)

where $H$ is an algebraic root of (2.7), and there is no need for hyper-elliptic integrals! Implicit differentiation of (2.6), and setting $c = 1$ shows that $2\hat{V} = \Upsilon(V)$, where $\Upsilon = x_i (\partial / \partial x_i)$ is the Euler’s homogeneity operator.

3. We can impose the Euclidean reality conditions, and seek solutions constant on confocal ellipsoids.

For $n = 2$ the solutions can be written in terms of holomorphic functions. The solution (2.6) is $\Re f(z)$, where $z = x + \sqrt{-1} y$, and $f = \ln(z + \sqrt{z^2 + \alpha})^2$ for $\alpha = (2\beta_1 - 1)/C$.

The case $n = 3$ of (2.17) has been previously characterised [13] in elliptic coordinates, and revisited in [8] in a context of gravitational instantons. The description of the quadric ansatz in terms of arbitrary holomorphic functions is also possible for $n = 3$, by means of twistor theory. This will be done in the Section 4.
3 The $SL(2,\mathbb{C})$ Nahm equation and the quadric ansatz

In this section we shall present a hodograph transformation between the Laplace equation and a system of first order PDEs. In three dimensions (when the first order system is SDiff(Σ) Nahm’s equations) solutions constant on quadrics will be characterised by a choice of $SL(2,\mathbb{C}) \subset SDiff(\Sigma)$.

Equation (2.5) is equivalent to

$$d \ast dV = 0,$$

where the Hodge operator $\ast$ defined by the metric $dx_1^2 + \cdots + dx_n^2$, and the volume form $dx^1 \wedge \cdots \wedge dx^n$. We say that $V$ generic if $|dV|^2 = dV \wedge \ast dV \neq 0$. The next Proposition shows that in the generic case the Laplace equation is equivalent to a system of first order PDEs, which we propose to call the Nambu-Nahm equations

**Proposition 3.1** Let $V = V(x_i)$ be a generic solution to the Laplace equation (2.5). The functions $P_a(x_i), a = 1, \cdots, n-1$ can be found such that $x_i = x_i(V, P_a)$ satisfy the following set of first order PDEs:

$$\dot{x}_i = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}_{NB},$$

where the Nambu bracket appearing on the RHS of (3.19) is given in terms of a Jacobian determinant by $f^{-1}(x \neq x_i)/\partial (P)$, and $f(P_a)$ is a nowhere vanishing function which locally can be set to 1.

Conversely, if $x_i = x_i(V, P_a)$ solve (3.19) then $V(x_i)$ satisfies (2.5).

**Proof.** In the generic case $|dV| \neq 0$ the condition (3.18), and the Poincare Lemma imply the existence of a foliation $U = \mathbb{C} \times \Sigma$ of $U \subset \mathbb{C}^n$ by $(n-1)$-dimensional complex manifolds $\Sigma$ with a holomorphic volume form $\omega$ such that

$$\ast dV = \omega.$$  

(3.20)

Let $P_a = (P_1, \ldots, P_{n-1})$ be local holomorphic coordinates\(^1\) on $\Sigma$, and let

$$\omega = f(P)dP_1 \wedge \cdots \wedge dP_{n-1}.$$  

(3.21)

The representation (3.20) leads to a hodograph transformation between solutions to (2.5) and solutions to a system of $n$ first order PDEs: Define $n$ holomorphic $n$–forms in an open set of $\mathbb{C}^{2n}$ by

$$\omega_i = dx_i \wedge (\ast dV - \omega).$$

(3.21)

The Laplace equation with $|dV| \neq 0$ is equivalent to $\omega_i = 0$. Selecting an $n$-dimensional surface (an integral manifold) in $\mathbb{C}^{2n}$ with $x_i$ as the local coordinates, and eliminating $P_a$ by cross-differentiating would lead back to (2.5). We are however free to make another choice and use $(V, P_a)$ as local coordinates. This yields (3.19).

Conversely, if $x_i(V, P_a)$ satisfy (3.19) then transforming it back to (3.21), and closing the ideal we deduce that $V(x_i)$ is harmonic.

\[\square\]

\(^1\)In the null case $|dV| = 0$ the existence of $P_1, \ldots, P_{n-1}$ can not be deduced even locally.
In the three-dimensional case \((\Sigma, \omega)\) is a holomorphic symplectic manifold with local coordinates \(P_a = (P, Q)\). We have therefore given an alternative proof of the equivalence between solutions to the Laplace equation and the \(\text{SDiff}(\Sigma)\) Nahm equation.

**Corollary 3.2** [19] In the three dimensions the Laplace equation is generically equivalent to the \(\text{SDiff}(\Sigma)\) Nahm equations

\[
\dot{x}_1 = \{x_2, x_3\}, \quad \dot{x}_2 = \{x_3, x_1\}, \quad \dot{x}_3 = \{x_1, x_2\},
\]

(3.22)

where \(\{,\}\) is a Poisson structure determined by symplectic form \(\omega\).

The above result fits into a general scheme of integrable background geometries [3]. In our case the background geometry is flat, and this makes the hodograph transformation \(x_i(P, Q, V) \rightarrow V(x_i)\) so effective.

From now on we shall restrict to the case \(n = 3\). Equations (3.22) are invariant under an infinite-dimensional group of holomorphic symplectomorphisms of \(\Sigma\) acting on the leaves of the foliation \(\mathbb{C} \times \Sigma\) (the Lax formulation of these equations is given in Appendix B). The next result characterises the solutions for which this symmetry reduces to \(\text{SL}(2, \mathbb{C})\).

**Proposition 3.3** Let \(x_i(P, Q, V)\) be solutions to the Nahm’s system (3.22) with the gauge group \(\text{SL}(2, \mathbb{C}) \subset \text{SDiff}(\Sigma)\). Then \(V = V(x_i)\) is a harmonic function constant on a central quadric.

**Proof.** Consider a symplectic action of \(\text{SL}(2, \mathbb{C})\) on \(\Sigma\), generated by Hamiltonian vector fields \(L_i\), such that the Lie brackets satisfy \([L_i, L_j] = -(1/2)\varepsilon_{ijk}L_k\). Let \(h_i = h_i(P, Q)\) be the corresponding Hamiltonians which satisfy

\[
\{h_i, h_j\} = \frac{1}{2}\varepsilon_{ijk}h_k.
\]

We notice that

\[
\{h_1^2 + h_2^2 + h_3^2, h_i\} = 0 \quad i = 1, 2, 3.
\]

The Poisson structure \(\{,\}\) comes from a symplectic two-form, and so is non-degenerate. Therefore the Hamiltonians satisfy the algebraic constraint

\[
h_1^2 + h_2^2 + h_3^2 = C,
\]

(3.23)

where \(C\) is a constant (which can always be scaled to 0 or 1). The case \(C = 0\) corresponds to a linear action, and \(C \neq 0\) to a Mōbius action.

Now assume that \(x_i = x_i(V, P, Q)\), satisfy \(\text{SL}(2, \mathbb{C})\) Nahm’s equation. Therefore \(x_i = A_{ij}(V)h_j(P, Q)\). The matrix \(A_{ij}\) can be made diagonal and so

\[
x_1 = w_1(V)h_1(P, Q), \quad x_2 = w_2(V)h_2(P, Q), \quad x_3 = w_3(V)h_3(P, Q).
\]

The constraint (3.23) implies

\[
\frac{x_1^2}{w_1(V)^2} + \frac{x_2^2}{w_2(V)^2} + \frac{x_3^2}{w_3(V)^2} = C,
\]

and the level sets of \(V\) are quadrics.

\[\square\]
We conclude that for solutions constant on quadrics the Nahm’s equations (3.22) reduce to the Euler equations.

\[ \dot{w}_1 = w_2 w_3, \quad \dot{w}_2 = w_1 w_3, \quad \dot{w}_3 = w_1 w_2. \]  

These equations readily reduce to

\[ (\dot{w}_3)^2 = (w_3^2 + A)(w_3^2 + B), \]

where \( A = w_1^2 - w_3^2 \) and \( B = w_2^2 - w_3^2 \) are constants. Setting

\[ H(V) = w_3^2(V) + \beta_3, \]

where \( \beta_3 = (A + B)/3 \) yields

\[ \dot{H}^2 = (H - \beta_1)(H - \beta_2)(H - \beta_3). \]

which is (2.16) with \( n = 3 \), and \( \beta_1 = \beta_3 - A, \beta_2 = \beta_3 - B. \)

4 Twistor description

Let \((x_1, x_2, x_3)\) be coordinates on \( \mathbb{C}^3 \) in which \( \eta = dx_1^2 + dx_2^2 + dx_3^2 \). The twistor space \( Z \) of \( \mathbb{C}^3 \) is the space of all planes \( Z \subset \mathbb{C}^3 \) with are null with respect to \( \eta \). It is the two-dimensional complex manifold \( Z = T \mathbb{C}P^1 \), which can be seen as follows: null vectors in \( \mathbb{C}^3 \) can be parametrised by \( k = (1 - \lambda^2, \sqrt{-1}(1 + \lambda^2), 2\lambda) \), where \( \lambda \in \mathbb{C}P^1 \). Points of \( Z \) correspond to null 2-planes in \( \mathbb{C}^3 \) via the incidence relation

\[ \mu = (x_1 + \sqrt{-1}x_2) + 2x_3\lambda - (x_1 - \sqrt{-1}x_2)\lambda^2. \]

Fixing \((\mu, \lambda)\) defines a null plane in \( \mathbb{C}^3 \) with \( \mu \) as its normal. An alternate interpretation of (4.26) is to fix \( x_i \). This determines \( \mu \) as a function of \( \lambda \) i.e. a section \( L_x \) of \( \mathcal{O}(2) \to \mathbb{C}P^1 \).

The total space of \( T \mathbb{C}P^1 \) is equivalent to the total space of the line bundle \( \mathcal{O}(2) \), and so every holomorphic section of \( Z \to \mathbb{C}P^1 \) can be written as a polynomial quadratic in \( \lambda \) with complex coefficients. The outlined twistor correspondence can be summarised as follows

points \( \leftrightarrow \) holomorphic sections

null planes \( \leftrightarrow \) points.

Another way of defining \( Z \) is by the double fibration

\[ \mathbb{C}^3 \xleftarrow{p} \mathcal{F} \xrightarrow{q} Z. \]

The correspondence space \( \mathcal{F} = \mathbb{C}^3 \times \mathbb{C}P^1 \) has a natural fibration over \( \mathbb{C}^3 \), and the projection \( q : \mathcal{F} \to Z \) is a quotient of \( \mathcal{F} \) by the two-dimensional distribution of vectors tangent to a null plane. In concrete terms this distribution is spanned by

\[ L_0 = (\partial_1 + \sqrt{-1}\partial_2) - \lambda \partial_3, \quad L_1 = \partial_3 - \lambda (\partial_1 - \sqrt{-1}\partial_2). \]

This leads to an alternative definition \( \mathcal{F} = \{(x, Z) \in \mathbb{C}^3 \times Z | Z \in L_x \}. \)
The integral formula for solutions to the Laplace equation
\[
\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2} = 0
\]
may be elegantly expressed in the twistor terms. Given a holomorphic function \( F(\lambda, \mu) \) on \( Z \) (secretly an element of \( H^1(\mathbb{C}P^1, \mathcal{O}(-2)) \)) restrict it to a section \( (4.26) \), and pull it back to \( F \). The general harmonic function in \( \mathbb{C}^3 \) is then given by
\[
V(x) = \oint_{\Gamma} q^* (F(\lambda, \mu)) d\lambda,
\]
where \( \Gamma \subset L_x \cong \mathbb{C}P^1 \) is a real closed contour.

It is natural to ask for a characterisation of twistor functions which give rise to harmonic functions constant on a central quadric. We shall first note that the dilation vector field \( \Upsilon = r \cdot \nabla \) corresponds to a holomorphic vector field \( \mu/\partial \mu \) on \( Z \). Therefore if a twistor function \( F \) gives rise to \( V \) constant on a central quadric, then \( \hat{V} \) given by \( (2.17) \) with \( n = 3 \) can be written as
\[
\hat{V}(x) = \oint_{\Gamma} q^* \left( \mu \frac{\partial F(\lambda, \mu)}{\partial \mu} \right) d\lambda.
\]
A direct attempt to invert \( (4.28) \) with \( V \) as in \( (2.17) \) leads to a messy calculation with an inconclusive outcome. We shall therefore choose a different route based on holomorphic line bundles over \( Z = \mathcal{O}(2) \) with the vanishing first Chern class. These objects are classified by elements of \( H^1(\mathcal{O}(2), \mathcal{O}) \). Let \( L \) be such a line bundle corresponding to a patching function \( \exp(f) \), where \( f \in H^1(\mathcal{O}(2), \mathcal{O}) \). One can view \( L \) in two different ways
1. A pull-back of the cohomology class \( \partial f/\partial \mu \) to the correspondence space gives a solution to the Laplace equation.
2. Lifts of holomorphic sections of \( \mathcal{O}(2) \to \mathbb{C}P^1 \) to \( L \) are rational curves with normal bundle \( \mathcal{O}(1) \oplus \mathcal{O}(1) \). Moreover \( L \) is fibred over \( \mathbb{C}P^1 \), and the canonical bundle of \( L \) is isomorphic to a pull-back of \( \mathcal{O}(-4) \) form \( \mathbb{C}P^1 \). Therefore \( L \) is a twistor space of a (complexified) hyper-Kähler four-manifold \( M \). [15, 14].

**Theorem 4.1** Let \( f \in H^1(\mathcal{O}(2), \mathcal{O}) \) define a holomorphic line bundle \( L \to \mathcal{O}(2) \) with \( c_1(L) = 0 \), and let \( \mu = \mu(\lambda) \) be a section of \( \mathcal{O}(2) \) given by \( (4.26) \). The following statements are equivalent

1. There exists a homomorphism of vector bundles,
   \[ \alpha : L \otimes \mathfrak{sl}(2, \mathbb{C}) \to TL \] \[ \text{such that } \text{rank}(\alpha) = 2. \]
2. Let \( F \in H^1(\mathcal{O}(2), \mathcal{O}(-2)) \) satisfy
   \[ \mu \frac{\partial F}{\partial \mu} = \frac{\partial f}{\partial \mu}. \]

Then
\[
V = \oint_{\Gamma} q^* (F(\lambda, \mu)) d\lambda
\]
is a solution to the Laplace equation constant on a central quadric.

The proof of this result is based on a fact that a hyper-Kähler metric corresponding to the twistor space \( L \) can be explicitly given in terms of a harmonic function \( \hat{V} \). We shall postpone this proof to the next section, where we have characterised hyper-Kähler metrics corresponding to (Euler derivatives of) harmonic functions constant on quadrics.
5 \( SL(2, \mathbb{C}) \) invariant Gibbons-Hawking metrics

In this section we shall show that harmonic functions constant on central quadrics (acted on by the homogeneity operator) characterise \( \mathbb{C}^* \) invariant hyper-Kähler metrics \[3\] which belong to the BGPP class \[2\]. The construction is a consequence of a simple observation: If \( h \) is a left-invariant metrics on \( SL(2, \mathbb{C}) \), and \( \mathbb{C}^* \subset SL(2, \mathbb{C}) \) then the metric induced on \( \Sigma = SL(2, \mathbb{C})/\mathbb{C}^* \) is conformal to a metric on a complex quadric.

Recall that a four-dimensional manifold \( M \) is (complexified) hyper-Kähler if it admits Kähler structures \( \Omega_i, i = 1, 2, 3 \) compatible with a fixed (holomorphic) Riemannian metric \( g \) and such that the endomorphisms \( I_i \) given by \( g(I_iX, Y) = \Omega_i(X, Y) \) satisfy the algebraic relation of quaternions. The hyper-Kähler metrics on \( \mathbb{C} \times SL(2, \mathbb{C}) \) with a transitive tri-holomorphic action of \( SL(2, \mathbb{C}) \) can be put in the form \[2\]

\[
g = w_1 w_2 w_3 dV^2 + \frac{w_2 w_3}{w_1} (\sigma_1)^2 + \frac{w_3 w_1}{w_2} (\sigma_2)^2 + \frac{w_1 w_2}{w_3} (\sigma_3)^2, \quad (5.32)
\]

where \( w_1, w_2, w_3 \) are functions of \( V \), and \( \sigma_i \) are left invariant one-forms on \( SL(2, \mathbb{C}) \) which satisfy

\[
d\sigma_1 = \sigma_2 \wedge \sigma_3, \quad d\sigma_2 = \sigma_3 \wedge \sigma_1, \quad d\sigma_3 = \sigma_1 \wedge \sigma_2. \quad (5.33)
\]

The self-dual two-forms are

\[
\begin{align*}
\Omega_1 &= w_1 \sigma_2 \wedge \sigma_3 + w_2 w_3 \sigma_1 \wedge dV, \\
\Omega_2 &= w_2 \sigma_3 \wedge \sigma_1 + w_1 w_3 \sigma_2 \wedge dV, \\
\Omega_3 &= w_3 \sigma_1 \wedge \sigma_2 + w_1 w_2 \sigma_3 \wedge dV,
\end{align*}
\]

and the hyper-Kähler condition \( d\Omega_i = 0 \) is\(^2\) equivalent to the Euler equations \[3,24\]. Therefore \( A = w_1^2 - w_3^2 \) and \( B = w_2^2 - w_3^2 \) are constants. The BGPP metric \[5.32\] is flat if \( A = B = 0 \), and is never complete if \( AB(A - B) \neq 0 \). The remaining cases correspond to a complete metric known as the Eguchi–Hanson solution.

Now choose a one-dimensional subgroup \( \mathbb{C}^* \subset SL(2, \mathbb{C}) \). Hyper-Kähler metric with a tri-holomorphic \( \mathbb{C}^* \) action can be put in the Gibbons–Hawking form \[9\]

\[
g = \tilde{V} (dx_1^2 + dx_2^2 + dx_3^2) + \tilde{V}^{-1} (dT + A)^2, \quad (5.35)
\]

where \( *d\tilde{V} = dA \). Here \( K = \partial/\partial T \) generates the \( \mathbb{C}^* \) action, and \( x_i \) are defined up to addition of a constant by \( dx_i = K J \Omega_i \). The one-form \( A \) is defined on the orbits of \( K \).

To characterise the harmonic functions \( \tilde{V} \) for which \[5.35\] belongs to the BGPP class we could expand \( K \) in a left-invariant basis of \( SL(2, \mathbb{C}) \), eliminate the Euler angles and \( V \) in favour of \( (x_i, t) \) and observe that \( \tilde{V}^{-1} = g(K,K) \), where \( g \) is given by \[5.32\]. This is essentially done in \[7\] and more explicitly in \[8\], where the \( n = 3 \) case of \[2.17\] is obtained.

We shall adopt a variation of this approach, and use the Nahm equations to establish the following

**Proposition 5.1** The Gibbons–Hawking metric \[5.35\] belongs to the BGPP class \[5.32\] if and only if \( \tilde{V} = r \cdot \nabla V \), and \( V \) is a harmonic function constant on a central quadric.

**Proof.** Let \((M = \mathbb{C}^* \times SL(2, \mathbb{C}), g)\) be a hyper-Kähler four-manifold with a transitive and tri-holomorphic \( SL(2, \mathbb{C}) \) action, and let \( \gamma : SL(2, \mathbb{C}) \rightarrow \Sigma = SL(2, \mathbb{C})/\mathbb{C}^* \) be a complexified Hopf bundle.

\(^2\)The \( SL(2, \mathbb{C}) \) action fixes all complex structures, so the invariant frame is covariantly constant.
The Corollary 3.2 allows us to introduce \((P,Q,\hat{V},T)\) as local coordinates on \(M\) such that \((P,Q)\) are local coordinates on \(\Sigma\), \(\hat{V}\) is a coordinate on the fibres of \(\gamma\), and \(T\) parametrises the \(SL(2,\mathbb{C})\) orbits. We shall regard \(x_i = x_i(P,Q,\hat{V})\) as functions on \(\mathbb{C}^3\). The hyper–Kähler condition is then equivalent to \(SDiff(\Sigma)\) Nahm equation

\[
\frac{\partial x_i}{\partial \hat{V}} = \frac{1}{2} \varepsilon_{ijk} \{x_j, x_k\},
\]

where the Poisson brackets are taken with respect to the symplectic form \(\omega = dA\).

Consider the left action of \(SL(2,\mathbb{C})\) on itself, generated by left invariant vector fields \(L_i\), such that

\[
\mathcal{L}_{L_i} g = 0, \quad [L_i, L_j] = -\frac{1}{2} \varepsilon_{ijk} L_k, \quad L_i \cdot \sigma_j = \delta_{ij}.
\]

The push-forward vector fields \(\gamma_*(L_i)\) generate symplectomorphisms of \(\Sigma\), and so they correspond to the Hamiltonians \(h_i = h_i(P,Q)\) (these are the Hamiltonians used in the proof of Proposition 3.3), such that

\[
\gamma_*(L_i)(h_j) = \{h_i, h_j\} = \frac{1}{2} \varepsilon_{ijk} h_k, \quad (5.36)
\]

and the algebraic constraint \(3.28\) holds. Let \(K = \gamma^*(h_1)L_1 + \gamma^*(h_2)L_2 + \gamma^*(h_3)L_3\). Note that

\[
[K, L_i] = 0
\]

as a consequence of \(5.30\). The moment-maps \(x_i\) are given by

\[
dx_i = K \cdot \Omega_i = \gamma^*(h_i) \omega_i dV + \omega_i d(\gamma^*(h_i)) = \gamma^* d(\omega_i h_i), \quad \text{(no summation)},
\]

where \(\Omega_i\) are given by \(5.31\). The formula \(5.35\) implies that \(\hat{V} = (g(K,K))^{-1}\). Making a substitution \(w_i = \sqrt{H - \beta_i}\), where \(H(V)\) satisfies \(2.13\) with \(n = 3\) reveals that \(\hat{V}\) is given by formulae \(2.17\) with \(n = 3\). Therefore \(\hat{V} = \Upsilon(V)\) is an Euler derivative of a solution \(2.6\) constant on a central quadric.

\[\square\]

We are now ready to present a proof of Theorem 4.1 and give the characterisation of inverse twistor functions corresponding to \(\hat{V}\) constant on a central quadric.

**Proof of Theorem 4.1.** From \((A40)\) it follows that cocycles in \(H^1(\mathbb{CP}^1,\mathcal{O}(n))\) can be represented by coboundaries if \(n \geq -1\). The freedom one has is measured by \(H^0(\mathbb{CP}^1,\mathcal{O}(n))\). In particular \(H^1(\mathbb{CP}^1,\mathcal{O}(-1)) = 0\) and its cocycles can be uniquely represented as coboundaries. Let \(\pi = (\pi_0, \pi_1)\) be homogeneous coordinates on \(\mathbb{CP}^1\). Let \(U_0\) and \(U_1\) be a covering of \(\mathcal{O}(2)\) such that \(\pi_1 \neq 0\) on \(U_0\), and \(\pi_0 \neq 0\) on \(U_1\), and let \(q^*U_0\) denote the open sets on the correspondence space \(\mathbb{C}^3 \times \mathbb{CP}^1\) that are the pre-image of \(U_0\), for \(\alpha = 0,1\). Let \(f \in H^1(\mathcal{O}(2), \mathcal{O})\) be a logarithm of a patching function of \(L\). Consider

\[
\pi \frac{\partial f}{\partial \mu} \in \mathbb{C}^2 \otimes H^1(\mathcal{O}(2), \mathcal{O}(-1)),
\]

which is homogeneous of degree \(-1\). Then we restrict it to a section of \(\mathcal{O}(2) \rightarrow \mathbb{CP}^1\) and pull it back to \(\mathcal{F}\) by \(4.27\), where we can split as \(\pi \partial f / \partial \mu = h_0 - h_1\).

Here \(h_\alpha\) is holomorphic on \(q^*U_\alpha\), and is given by

\[
h_\alpha = \oint_{\Gamma_\alpha} \rho \frac{\partial f}{\partial \mu} \cdot d\rho, \quad (5.37)
\]
where \( \rho \) are homogeneous coordinates on \( \mathbb{CP}^1 \), and \( \rho \cdot d\rho = d\lambda \) in affine coordinates. The contours \( \Gamma_\alpha \) and are homologous to the equator of \( \mathbb{CP}^1 \) in \( U_0 \cap U_1 \) and are such that \( \Gamma_0 - \Gamma_1 \) surrounds the point \( \rho = \pi \). We see that \( \pi \partial f/\partial \mu = h_0 - h_1 \) follows from the Cauchy’s integral formula. Moreover \( \pi \cdot (\pi f) = \pi \cdot h_0 - \pi \cdot h_1 = 0. \) Therefore \( \hat{V} := \pi \cdot h_0 = \pi \cdot h_1 \) is a global holomorphic function homogeneous of degree 0, and so by the Liouville Theorem it is constant on \( \mathbb{CP}^1 \). The formula (5.37) implies that \( \hat{V} \) is a harmonic function explicitly given by

\[
\hat{V} = \int_{\Gamma} q^* \left( \frac{\partial f(\lambda, \mu)}{\partial \mu} \right) \, d\lambda,
\]

which is (4.29) with

\[
\frac{\partial F}{\mu \partial \mu} = \frac{\partial f}{\partial \mu}.
\]

Now we shall regard \( L \) as a twistor space of a hyper-Kähler manifold \( (M, g) \). The fibres of \( L \rightarrow \mathbb{CP}^1 \) are symplectic manifolds, where the symplectic two-form \( \Omega \) takes values in \( O(2) \). The Hamiltonian vector field of \( \mu \in \Gamma(O(2)) \) with respect to \( \Omega \) preserves the projection of \( L \) onto \( \mathbb{CP}^1 \), and therefore induces a tri-holomorphic Killing vector \( K = \partial/\partial T \) on \( M \). Metrics which admit such Killing vectors are locally given by (5.35). We shall show that \( \hat{V} \) appearing in (5.35) is indeed the same as (5.38).

Introduce local homogeneous coordinates \( (\pi, \mu, \zeta_\alpha) \) on each set \( U_\alpha \) of some Stein cover of the twistor space; here \( \zeta_\alpha \) is a fibre coordinate up the fibres of the affine line bundle \( L \rightarrow O(2) \) on \( U_\alpha \) with patching relations \( \zeta_0 = \zeta_1 + f \) on \( U_0 \cap U_1 \). In these coordinates

\[ \Omega = dh_\mu \wedge dh_\zeta_\alpha, \]

where \( dh_\alpha \) denotes the exterior derivative in which \( \pi \) is held constant. The two–form \( \Omega \) is globally defined on vector fields tangent to the fibres of \( L \rightarrow \mathbb{CP}^1 \) as \( f \) does not depend on \( \zeta \). In order to calculate the self-dual two-forms \( \Omega_i \) (and so the metric) associated to \( f \) we pullback \( \Omega \) to \( M \times \mathbb{CP}^1 \), and determine \( \zeta_\alpha \) using integral splitting formulae analogous to (5.37) (note however, that \( h_\alpha \) are not twistor functions, as they don’t descend from \( F \) to \( Z \). On the other hand \( \zeta_\alpha \) are twistor function for \( L \). This yields

\[
q^* (\Omega) = (\Omega_1 + \sqrt{-1}\Omega_2) + 2\Omega_3 \lambda - (\Omega_1 - \sqrt{-1}\Omega_2) \lambda^2,
\]

where

\[
\Omega_i = (dT + A) \wedge dx_i - \frac{1}{2} \hat{V} \varepsilon_{ijk} dx_j \wedge dx_k.
\]

The two-forms \( \Omega_i \) are the self-dual two forms of (5.35), and \( x_i \) are given by (4.12). The function \( \hat{V} \), and a one-form \( A = A_i dx^i \) are given in terms of a \( 2 \times 2 \) matrix

\[
\Phi_{BC} = \int_{\Gamma_\alpha} \frac{\rho B^{LC}}{\rho \cdot t} \frac{\partial f}{\partial \mu} \rho \cdot d\rho, \quad t = (t_0, t_1) \in \mathbb{CP}^1, \quad B, C = 0, 1
\]

by

\[
\Phi = \begin{pmatrix}
A_1 + \sqrt{-1}A_2 & A_3 + \hat{V} \\
A_3 - \hat{V} & -(A_1 - \sqrt{-1}A_2)
\end{pmatrix}.
\]

Therefore \( \hat{V} \) is as in (5.35), and the monopole equation \( *d\hat{V} = dA \) is automatically satisfied. The two-forms \( \Omega_i \) are closed, as a consequence of the monopole equation.

Proposition 5.4 implies that \( \hat{V} = r \cdot \nabla V \), where \( V \) is harmonic and constant on a central quadric iff the associated metric (5.35) belongs to the BGPP class. To complete the proof we need to characterise the structures on \( L \) induced by a tri-holomorphic \( SL(2, \mathbb{C}) \) action.
on $M$. The $SL(2, \mathbb{C})$ action on $M$ preserves $g$ therefore it induces a holomorphic $SL(2, \mathbb{C})$ action on $L$. Each Killing vector $L_i$ on $M$ induces a holomorphic vector field $\hat{L}_i$ on $L$, and this gives rise to a homomorphism (4.30) such that $2 \leq \text{rk}(\alpha) \leq 3$. The group action on $M$ is tri-holomorphic, therefore $Q_i = L_i \cdot d\lambda = 0$ (in general $Q_i \in \mathcal{O}(2)$ gives rise to a vector field Hamiltonian with respect to $d\pi_0 \wedge d\pi_1$ which rotates the self-dual two-forms). The orbits of the $SL(2, \mathbb{C})$ action on $L$ are contained in the two-dimensional fibres of $L \rightarrow \mathbb{CP}^1$. We conclude that rank$(\alpha) = 2$.

\[
\square
\]

6 Example

We shall illustrate Proposition (5.1) with the example of a harmonic function corresponding to the Eguchi–Hanson metric. We impose the Euclidean reality conditions, and work with real harmonic functions on $\mathbb{R}^3$ giving rise to real hyper-Kähler metrics.

The one-forms $\sigma_i$ can be explicitly given in terms of Euler angles

\[
\sigma_1 = \cos \psi \, d\theta + \sin \psi \sin \theta \, d\phi, \quad \sigma_2 = \sin \psi \, d\theta - \cos \psi \sin \theta \, d\phi, \quad \sigma_3 = d\psi + \cos \theta \, d\phi.
\]

The functions

\[
h_1 = \sin \theta \sin \psi, \quad h_2 = -\sin \theta \cos \psi, \quad h_3 = \cos \theta
\]

satisfy (5.36) with the symplectic form $\omega = d(\cos \theta) \wedge d\psi$.

Consider the $SU(2)$ invariant hyper–Kähler metric (5.32) with $w_1 = w_2 \neq w_3$. The Euler equations (3.24) yield $w_3 = \rho(V)$, $w_1 = w_2 = \sqrt{\rho^2 - a^2}$, and (using $\rho$ as a coordinate)

\[
g = \frac{\rho}{\rho^2 - a^2} d\rho^2 + \rho(\sigma_1^2 + \sigma_2^2) + \frac{\rho^2 - a^2}{\rho} \sigma_3^2.
\]

(6.39)

This is the metric of Eguchi and Hanson [6]. It is complete, as the apparent singularity at $\rho = a$ is removed by allowing

\[
a^2 < \rho^2, \quad 0 \leq \psi \leq 2\pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \theta \leq \pi.
\]

Identifying $\psi$ modulo $2\pi$ makes the surfaces of constant $\rho^2 > a^2$ into $\mathbb{R}P^3$. At large value of $\rho^2$ the metric is asymptotically locally Euclidean.

The metric (6.39) can be put in the Gibbons–Hawking form with $\hat{V} = \rho/(\rho^2 - a^2 \cos^2 \theta)$. To see this perform a coordinate transformation $x_i = \omega_i h_i$, $T = \phi + \psi$ which yields (5.35) with

\[
\hat{V} = |r + a|^{-1} + |r - a|^{-1}, \quad \text{where} \quad a = (0, 0, a).
\]

We verify that $r \cdot \nabla V = \hat{V}$, where the harmonic function

\[
V = -\frac{2}{a} \text{arcoth} \left( \frac{|r + a| + |r - a|}{2a} \right)
\]

is obtained from (2.6) with $n = 3$, and $\beta_1 = \beta_2 = a^2, \beta_3 = 0$. The potential $V$ is constant on the ellipsoid

\[
\frac{x^2 + y^2}{a^2((\coth(aV/2))^2 - 1)} + \frac{z^2}{a^2((\coth(aV/2))^2} = 1,
\]

in agreement with Proposition (5.1).
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Appendix A. Bundles over $\mathbb{CP}^1$

Let $\mathbb{C}^2$ be a symplectic vector space, with anti-symmetric product

$$\pi \cdot \rho = \pi_0 \rho_1 - \pi_1 \rho_0 = -\rho \cdot \pi,$$

where $\pi = (\pi_0, \pi_1), \rho = (\rho_0, \rho_1) \in \mathbb{C}^2$. Remove $\pi = (0, 0)$ and use $\pi$ as homogeneous coordinates on $\mathbb{CP}^1$. We shall also use the affine coordinate $\lambda = \pi_0/\pi_1$. Holomorphic functions on $\mathbb{C}^2 - 0$ extend to holomorphic functions on $\mathbb{C}^2$ (Hartog’s Theorem). Therefore homogeneous functions on $\mathbb{CP}^1$ are polynomials. In particular, holomorphic functions homogeneous of degree 0 are constant (Liouville theorem). Let us summarize some facts about holomorphic line bundles over $\mathbb{CP}^1$. First define a tautological line bundle

$$\mathcal{O}(-1) = \{(\lambda, (\pi_0, \pi_1)) \in \mathbb{CP}^1 \times \mathbb{C}^2 | \lambda = \pi_0/\pi_1\}.$$

Other line bundles can be obtained from $\mathcal{O}(-1)$ by algebraic operations:

$$\mathcal{O}(-n) = \mathcal{O}(-1)^\otimes n, \quad \mathcal{O}(n) = \mathcal{O}(-n)^*, \quad \mathcal{O} = \mathcal{O}(-1) \otimes \mathcal{O}(1), \quad n \in \mathbb{N}. $$

Equivalently $\mathcal{O}(n)$ denotes the line bundle over $\mathbb{CP}^1$ with transition functions $\lambda^{-n}$ from the set $\lambda \neq \infty$ to $\lambda \neq 0$ (i.e. Chern class $n$). Its sections are given by functions homogeneous of degree $n$ in a sense that $f(\xi \lambda) = \xi^n f(\lambda)$. These are polynomials in $\lambda$ of degree $n$ with complex coefficients. The theorem of Grothendick states that all holomorphic line bundles over a rational curve are equivalent to $\mathcal{O}(n)$ for some $n$. The spaces of global sections, and the first cohomology groups are

$$H^0(\mathbb{CP}^1, \mathcal{O}(n)) = \begin{cases} 0 & \text{for } n < 0 \\ \mathbb{C}^{n+1} & \text{for } n \geq 0. \end{cases}$$

$$H^1(\mathbb{CP}^1, \mathcal{O}(-n)) = \begin{cases} 0 & \text{for } n < 2 \\ \mathbb{C}^{n-1} & \text{for } n \geq 2. \end{cases}$$  \hfill (A40)

Appendix B. $\text{SDiff}(\Sigma)$ Nahm’s equations

Let $G$ be a Lie group and let $[\cdot, \cdot]$ be the Lie bracket in the corresponding Lie algebra $\mathfrak{g}$. The Nahm equations for three $\mathfrak{g}$-valued functions $X_i = X_i(V)$ are

$$\dot{X}_1 = [X_2, X_3], \quad \dot{X}_2 = [X_3, X_1], \quad \dot{X}_3 = [X_1, X_2].$$  \hfill (B41)

These equation admit a Lax representation. Let

$$A(\lambda) = (X_1 + \sqrt{-1}X_2) + 2X_3\lambda - (X_1 - \sqrt{-1}X_2)\lambda^2.$$

Then

$$\dot{A} = [X_2 - \sqrt{-1}X_1, X_3] + 2[X_1, X_2]\lambda - [X_2 + \sqrt{-1}X_1, X_3]\lambda^2$$

$$= [A, -\sqrt{-1}X_3 + \sqrt{-1}(X_1 - \sqrt{-1}X_2)\lambda]$$

$$= [A, B].$$  \hfill (B42)
where \( B = -2\sqrt{-1}X_3 + \sqrt{-1}(X_1 - \sqrt{-1}X_2)\lambda \). The matrix \( A(\lambda) \) should be thought of as an \( O(2) \) valued section of a two-dimensional complex vector bundle over \( \mathbb{CP}^1 \). Let \((\mu, \lambda)\) be the local coordinates on the total space of \( O(2) \). The zero locus of the characteristic equation
\[
S = \{ (\mu, \lambda) \in \mathbb{Z} | \det(1\mu - A(\lambda)) = 0 \}
\]
defines an algebraic curve \( S \), called the spectral curve of \( A(\lambda) \) which (as a consequence of \( \text{B42} \)) is preserved by the Nahm equations. In the \( SL(2, \mathbb{C}) \) case the Riemann surface \( S \) has genus one, and is a torus parametrised by the elliptic function \( \text{B25} \).

For each point \((\mu, \lambda)\) on \( S \) we have a one-dimensional space \( L(\mu, \lambda) = \ker (1\mu - A(\lambda)) \) and this varies with \( V \). It forms a line bundle over the spectral curve. Hitchin \( \text{B10} \) shows that this line bundle evolves along a straight line on the Jacobian.

Now assume that \( g \) is the infinite-dimensional Lie algebra \( \text{sdiff}(\Sigma) \) of holomorphic symplectomorphisms of a two-dimensional complex symplectic manifold \( \Sigma \) with local holomorphic coordinates \( P,Q \) and the holomorphic symplectic structure \( \omega \). Elements of \( \text{sdiff}(\Sigma) \) are represented by the Hamiltonian vector fields \( X_h \) such that \( X_h \omega = dh \) where \( H \) is a \( \mathbb{C} \)-valued function on \( \Sigma \). The Poisson algebra of functions which we are going to use is homomorphic to \( \text{sdiff}(\Sigma) \). We shall make the replacement \( [\ ,\ ] \rightarrow \{\ ,\ \} \) in formulae \( \text{B41, B42} \). Here \( \{\ ,\ \} \) is a Poisson structure defined by \( \omega \). The components of \( X_i \) are therefore replaced by Hamiltonians \( x_i(P,Q,V) \) generating the symplectomorphisms of \( \Sigma \).

The Lax representation for the \( \text{SDiff}(\Sigma) \) Nahm system is
\[
\frac{\partial \Psi}{\partial V} = \{ (\mu/\lambda)_+, \Psi \}, \quad \{ \mu, \Psi \} = 0,
\]
where \( \Psi = \Psi(P,Q,V,\lambda), \) and
\[
\mu(\lambda) = (x_1 + \sqrt{-1}x_2) + 2x_3\lambda - (x_1 - \sqrt{-1}x_2)\lambda^2, \quad (\mu/\lambda)_+ = -2\sqrt{-1}x_3 + \sqrt{-1}(x_1 - \sqrt{-1}x_2)\lambda.
\]
The compatibility conditions for this over-determined system yield \( \text{B322} \).

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