Immersion with bounded second fundamental form

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Abstract
We first consider immersions on compact manifolds with uniform $L^p$-bounds on the second fundamental form and uniformly bounded volume. We show compactness in arbitrary dimension and codimension, generalizing a classical result of J. Langer. In the second part, this result is used to deduce a localized version, being more convenient for many applications, such as convergence proofs for geometric flows.

1 Introduction
In [16] J. Langer investigated compactness of immersed surfaces in $\mathbb{R}^3$ admitting uniform bounds on the second fundamental form and the area of the surfaces. For a given sequence $f^i : \Sigma^i \to \mathbb{R}^3$, there exist after passing to a subsequence a limit surface $f : \Sigma \to \mathbb{R}^3$ and diffeomorphisms $\phi^i : \Sigma \to \Sigma^i$, such that $f^i \circ \phi^i$ converges in the $C^1$-topology to $f$. In particular, up to diffeomorphism, there are only finitely many manifolds admitting such an immersion. The finiteness of topological types was generalized by K. Corlette in [9] to immersions of arbitrary dimension and codimension. Moreover, the compactness theorem was generalized by S. Delladio in [10] to hypersurfaces of arbitrary dimension.

The general case, that is compactness in arbitrary dimension and codimension, is the first main theorem of this paper:

Theorem 1.1 (Compactness theorem for immersions on compact manifolds)

Let $q$ be a point in $\mathbb{R}^n$, $m$ a positive integer, $p > m$, and $A, V > 0$ constants. Let $\mathcal{F}$ be the set all mappings $f : M \to \mathbb{R}^n$ with the following properties:

- $M$ is an $m$-dimensional, compact manifold (without boundary)
- $f$ is an immersion in $W^{2,p}(M, \mathbb{R}^n)$ with

$$\|A(f)\|_{L^p(M)} \leq A \quad \text{(1.1)}$$
$$\text{vol}(M) \leq V \quad \text{(1.2)}$$
$$q \in f(M). \quad \text{(1.3)}$$

Then for every sequence $f^i : M^i \to \mathbb{R}^n$ in $\mathcal{F}$ there exist a subsequence $f^j$, a mapping $f : M \to \mathbb{R}^n$ in $\mathcal{F}$, and a sequence of diffeomorphisms $\phi^j : M \to M^j$, such that $f^j \circ \phi^j$ converges in the $C^1$-topology to $f$.

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Introduction

Here, the $L^p$-norm for the second fundamental form and the volume is measured with respect to the volume measure induced by $f$. Having shown Theorem 1.1 we will use the Nash embedding to generalize the result to complete Riemannian manifolds as target. For a definition of the $C^1$-topology see [12], p. 34–35. The assumption $q \in f(M)$ ensures that the immersions $f^i$ do not diverge uniformly. This can be weakened to $f(M) \cap K \not= \emptyset$ for a fixed compact set $K \subset \mathbb{R}^n$. In the case of an $L^\infty$-bound on the second fundamental form, the assumption $\text{vol}(M) \leq \mathcal{V}$ is equivalent to a bound on the diameter $\text{diam}(M) \leq D$. The theorem can easily be generalized to higher order convergence, provided we assume uniform bounds $\|\nabla^k A\|_{L^\infty(M)} \leq A_k$ for all covariant derivatives of $A$ up to some specific order. We remark that in general Theorem 1.1 fails to be true in the case $p = m$; in [16] on p. 227, Langer constructs a counterexample in dimension 2 by considering suitable inversions of a Clifford torus. A similar result was shown by C. B. Ndiaye and R. Schätzle in [19], considering surfaces with $L^2$-bounded second fundamental form that satisfy some additional hypotheses. Furthermore, the author showed in [16] compactness of immersions with local Lipschitz representation.

To prove Theorem 1.1 we will first show a weak notion of convergence, the convergence in the sense of graph systems. However, this does not directly imply the existence of a limit immersion $f : M \to \mathbb{R}^n$. In [16], in the case of surfaces, one defines $M^j$ as limit manifold; here $j$ is a fixed large integer. Afterwards one constructs the mappings $\phi^j : M^j \to M^i$ and shows, after passing to a subsequence, convergence to an immersion $f : M^j \to \mathbb{R}^3$. Here we like to take a more systematic approach. We will construct the limit manifold and immersion directly after having shown convergence of graph systems. In order to do so, we shall take the limit graph system and define appropriate identifications; this will enable us to recover the limit immersion by its image. Only after that, we construct the diffeomorphisms $\phi^j$. This abstract construction of the limit $f$ might be of its own interest for other applications.

As a corollary of Theorem 1.1 we directly obtain:

**Corollary 1.2** Let $\mathcal{F}$ be defined as in Theorem 1.1. Then there are only finitely many manifolds in $\mathcal{F}$ up to diffeomorphism.

Next we prove a localized version for smooth proper immersions admitting uniform $L^\infty$-bounds for the second fundamental form $A$ and its covariant derivatives $\nabla^k A$. Here the manifolds on which the immersions are defined are not required to be compact. For a proper immersion $f : M \to \mathbb{R}^n$ with induced metric $g$ and volume measure $\mu_g$ on $M$, let $\mu = f(\mu_g)$ be the Radon measure on $\mathbb{R}^n$ defined by $\mu(E) = \mu_g(f^{-1}(E))$ for $E \subset \mathbb{R}^n$. Abbreviating we write $\| \cdot \|_{L^\infty(B_R)}$ for the $L^\infty$-norm on $f^{-1}(B_R)$, where $B_R \subset \mathbb{R}^n$ is the open ball of radius $R$ centered at the origin. We obtain the following theorem:

**Theorem 1.3** (Compactness theorem for proper immersions)

Let $f^i : M^i \to \mathbb{R}^n$ be a sequence of proper immersions, where $M^i$ is an $m$-manifold without boundary and $0 \in f^i(M^i)$. With $\mu^i = f^i(\mu_{g^i})$ assume

$$\mu^i(B_R) \leq C(R) \quad \text{for any } R > 0, \tag{1.4}$$

$$\|\nabla^k A^i\|_{L^\infty(B_R)} \leq C_k(R) \quad \text{for any } R > 0 \text{ and } k \in \mathbb{N}_0. \tag{1.5}$$

Then there exists a proper immersion $f : M \to \mathbb{R}^n$, where $M$ is again an $m$-manifold without boundary, such that after passing to a subsequence there are diffeomorphisms $\phi^i : U^i \to (f^i)^{-1}(B_i) \subset M^i$, where $U^i \subset M$ are open sets with $U^i \subset \subset U^{i+1}$ and $M = \bigcup_{i=1}^\infty U^i$, such that $\|f^i \circ \phi^i - f\|_{C^0(U^i)} \to 0$, and moreover $f^i \circ \phi^i \to f$ locally smoothly on $M$.

Moreover, the immersion $f$ also satisfies (1.4) and (1.5), that is $\mu(B_R) \leq C(R)$ and $\|\nabla^k A\|_{L^\infty(B_R)} \leq C_k(R)$.
Again, the assumption $0 \in f^i(M^n)$ can be weakened to $f^i(M^n) \cap K \neq \emptyset$ for a fixed compact set $K \subset \mathbb{R}^n$. In contrast to the compact case, here the bound $||A||_{L^\infty(B_R)} \leq C(R)$ depends on the radius of the image. This explains the need of some technical refinements which allow us to handle an increasing norm of the second fundamental form. We like to remark that a similar result is shown by A. Cooper in [8], however the construction of the diffeomorphisms $\phi^i$ is not carried out there (see Remark 7.12 in this paper). Theorem 7.3 has some important applications such as convergence proofs for geometric flows — for example for the mean curvature flow or the Willmore flow (see e.g. [2], [3], [13], [15], [17]).

As a corollary of Theorem 1.3 we prove convergence of the corresponding measures:

**Corollary 1.4** Let $f^i$ and $f$ be as in Theorem 1.3 and let $\mu^i = f^i(\mu_g)$, $\mu = f(\mu_g)$. Then $\mu^i \to \mu$ in $C^0_\sigma(\mathbb{R}^n)'$ as $i \to \infty$.

Finally, we will give some further generalizations of Theorem 1.3. In particular, in Corollary 7.13 we shall give a generalization to proper immersions $f^i : M^m \to \Omega$ into an open subset $\Omega \subset \mathbb{R}^n$. Along with this corollary, our theorems cover a wide range of situations one encounters in various applications.

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### 2 Local representation as a function graph

First, in Sections 2 to 6 we will show Theorem 1.1. After a rotation and a translation, every immersion $f : M^m \to \mathbb{R}^{m+k}$ can locally be written as the graph of a function $u : B_r \to \mathbb{R}^k$, where $B_r$ denotes an open ball in $\mathbb{R}^m$ of radius $r$. In this section, we like to work out the details of such graph representations. First we have to introduce some notation:

For $n = m + k$ let $G_{n,m}$ denote the Grassmannian of (non-oriented) $m$-dimensional subspaces of $\mathbb{R}^n$. Unless stated otherwise, let $B_q$ always denote the open ball in $\mathbb{R}^m$ of radius $q > 0$ centered at the origin.

Now let $M$ be an $m$-dimensional manifold without boundary and $f : M \to \mathbb{R}^n$ a $C^1$-immersion. Let $q \in M$ and let $T_qM$ be the tangent space at $q$. Identifying vectors $X \in T_qM$ with $f_*X \in T_{f(q)}\mathbb{R}^n$, we may consider $T_qM$ as an $m$-dimensional subspace of $\mathbb{R}^n$. Let $(T_qM)^\perp$ denote the orthogonal complement of $T_qM$ in $\mathbb{R}^n$, that is

$$\mathbb{R}^n = T_qM \oplus (T_qM)^\perp$$

and $(T_qM)^\perp$ is perpendicular to $T_qM$. In this manner we may define a tangent and a normal map

$$\tau_f : M \to G_{n,m},$$

$$q \mapsto T_qM,$$  \hspace{1cm} (2.1)

and

$$\nu_f : M \to G_{n,k},$$

$$q \mapsto (T_qM)^\perp.$$  \hspace{1cm} (2.2)

Moreover, let $\pi_q^\tau : \mathbb{R}^n \to T_qM$ and $\pi_q^\perp : \mathbb{R}^n \to (T_qM)^\perp$ be the orthogonal projections onto $T_qM$ and onto $(T_qM)^\perp$ respectively.

First we like to consider immersions, that are already given as a graph. We like to begin with the following trivial lemma:
Lemma 2.1 Let $u, v : V \to \mathbb{R}^k$ be two $C^1$-mappings, where $V \subset \mathbb{R}^m$ is open and convex with $0 \in V$. Moreover let $f, g : V \to \mathbb{R}^{m+k}$, $f(x) = (x, u(x)), g(x) = (x, v(x))$.

a) The tangent space $\tau_\xi$ is spanned by the vectors $(e_1, \partial_1 u(x)), \ldots, (e_m, \partial_m u(x))$.

b) If $u(0) = 0$, then $|u(x)| \leq \|Du\|_{C^0(V)} |x|$.

c) Let $\zeta = (y, 0) \in \mathbb{R}^m \times \mathbb{R}^k$. Then $|\pi^\perp_\xi(\zeta)| \geq (1 + \|Du(x)\|)^{-\frac{1}{2}} |y|$.

d) Let $\xi = (0, z) \in \mathbb{R}^m \times \mathbb{R}^k$. Then $|\pi^\perp_\xi(\xi)| \geq (1 + \|Du(x)\|^{-\frac{1}{2}} |z|).

The proof of the lemma is trivial and shall be omitted here. In the next lemma, we estimate the $L^p$-norm of the second derivatives of $u$ from above by the supremum norm of the first derivative and the $L^p$-norm of the second fundamental form:

Lemma 2.2 For $B_r \subset \mathbb{R}^m$ and $n = m + k$, let $f \in W^{2,p}(B_r, \mathbb{R}^n)$ be a mapping of the form $f(x) = (x, u(x)) \in \mathbb{R}^m \times \mathbb{R}^k$. If $\|Du\|_{C^0(B_r)} < \infty$, then we have the estimate

$$\|D^2u\|_{L^p(B_r)} \leq (1 + \|Du\|_{C^0(B_r)}^\frac{2}{p}) \|A(f)\|_{L^p(B_r)}. \quad (2.3)$$

Proof:
Let $q \in B_r$. With Lemma 2.1 we have

$$|A_q(e_i, e_j)| = |\pi^\perp_q(\partial_{ij} f(q))|$$

$$= |\pi^\perp_q(0, \partial_{ij} u(q))|$$

$$\geq (1 + \|Du(q)\|^{-\frac{1}{2}} |\partial_{ij} u(q)|$$

$$\geq (1 + \|Du\|_{C^0(B_r)}^{-\frac{1}{2}} |\partial_{ij} u(q)|.$$ 

It follows

$$|\partial_{ij} u(q)| \leq (1 + \|Du\|_{C^0(B_r)}^\frac{2}{p}) |A_q(e_i, e_j)|$$

$$\leq (1 + \|Du\|_{C^0(B_r)}^\frac{2}{p}) |(e_i, \partial_i u(q))||q^\perp(e_j, \partial_j u(q))||A(q)||$$

$$\leq (1 + \|Du\|_{C^0(B_r)}^\frac{2}{p}) \|A(q)\|.$$ 

Integration yields the desired inequality. \hfill \Box

The following inequality is due to C. B. Morrey:

Lemma 2.3 Let $p > m$, $B_r \subset \mathbb{R}^m$ and $v \in (W^{1,p} \cap C^0)(B_r)$. Then there is a universal constant $C = C(m, p)$, such that for all $x \in B_r$

$$|v(x) - v(0)| \leq Cr^{1-\frac{m}{p}} \|Dv\|_{L^p(B_r)}. \quad (2.4)$$

Proof:
For a proof see for instance [1], p. 315, Theorem 8.11. The special case pointed out on p. 317 in Remark 8.12 2) and 3) is exactly (2.4). \hfill \Box

With this lemma, we are able to estimate the supremum norm of the derivative from above by the $L^p$-norm of the second derivatives:

Lemma 2.4 Let $p > m$, $B_r \subset \mathbb{R}^m$ and $u \in (W^{2,p} \cap C^1)(B_r, \mathbb{R}^k)$. Let $u$ satisfy $Du(0) = 0$. Then there is a universal constant $C = C(m, k, p)$, such that

$$\|Du\|_{C^0(B_r)} \leq Cr^{1-\frac{m}{p}} \|D^2u\|_{L^p(B_r)}. \quad (2.5)$$
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Proof:
Using Lemma 2.3 with \(Du \in (W^{1,p} \cap C^0)(B_r, \mathbb{R}^{k \times m})\) the statement follows. \(\square\)

Next we like to explain, how an immersion can locally be written as a function graph. The existence of such a graph representation is clear by the implicit function theorem. However, for our purposes, we have to go more into detail. First we need to introduce some more notation.

We call a mapping \(A : \mathbb{R}^n \to \mathbb{R}^n\) a **Euclidean isometry**, if there is a rotation \(R \in SO(n)\) and a translation \(T \in \mathbb{R}^n\), such that \(A(x) = Rx + T\) for all \(x \in \mathbb{R}^n\).

For a given point \(q \in M\) let \(A_q : \mathbb{R}^n \to \mathbb{R}^n\) be a Euclidean isometry, which maps the origin to \(f(q)\), and the subspace \(\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^k\) onto \(f(q) + \tau f(q)\). Let \(\pi : \mathbb{R}^n \to \mathbb{R}^m\) be the standard projection onto the first \(m\) coordinates.

Finally let \(U_{r,q} \subset M\) be the \(q\)-component of the set \((\pi \circ A_q^{-1} \circ f)^{-1}(B_r)\). Although the isometry \(A_q\) is not uniquely determined, the set \(U_{r,q}\) does not depend on the choice of \(A_q\).

We come to the central definition (as first defined in [16]):

**Definition 2.5** An immersion \(f : M \to \mathbb{R}^n\) is called an \((r, \alpha)\)-immersion, if for each point \(q \in M\) the set \(A_q^{-1} \circ f(U_{r,q})\) is the graph of a differentiable function \(u : B_r \to \mathbb{R}^k\) with \(\|Du\|_{C^0(B_r)} \leq \alpha\).

Here, for any \(x \in B_r\) we have \(Du(x) \in \mathbb{R}^{k \times m}\). In order to define the \(C^0\)-norm for \(Du\), we have to fix a matrix norm for \(Du(x)\). Let us agree upon

\[
\|A\| = \left(\sum_{j=1}^{m} |a_j|^2 \right)^{\frac{1}{2}}
\]

for \(A = (a_1, \ldots, a_m) \in \mathbb{R}^{k \times m}\). For this norm we have \(\|A\|_{op} \leq \|A\|\) for any \(A \in \mathbb{R}^{k \times m}\) and the operator norm \(\|\cdot\|_{op}\). Hence the bound \(\|Du\|_{C^0(B_r)} \leq \alpha\) directly implies that \(u\) is \(\alpha\)-Lipschitz (and all estimates in the previous lemmas are true). Moreover the norm \(\|Du\|_{C^0(B_r)}\) does not depend on the choice of the isometry \(A_q\).

For given \(\alpha > 0\), we would like to give an estimate for the admissible size of the radius \(r\), such that a given immersion is an \((r, \alpha)\)-immersion. Here the admissible size of \(r\) only depends on the \(L^p\)-norm of the second fundamental form:

**Theorem 2.6** Let \(p > m\) and \(0 < \alpha \leq 1\). Then there exists a universal constant \(c = c(m, k, p) > 0\) such that every immersion \(f \in W^{2,p}(M, \mathbb{R}^n)\) on a compact m-manifold \(M\) is an \((r, \alpha)\)-immersion for all \(r > 0\) with

\[
r^{1 - \frac{m}{p}} \leq c\alpha \|A(f)\|_{L^p(M)}^{\frac{1}{p}}.
\]

**Proof:** The proof of the 2-dimensional case in [16] also applies to the higher dimensional case, where we use Lemmas 2.2 and 2.4. \(\square\)

In the previous lemma, \(k\) denotes the codimension. The following lemma (Lemma 3.1 in [16]) is crucial for the proof and will also be needed (in a variation) for the noncompact case. The proof of Langer carries over to our situation:

**Lemma 2.7** Let \(f : M \to \mathbb{R}^n\) be an \((r, \alpha)\)-immersion and \(p, q \in M\).

a) If \(p \in U_{r,q}\), then \(|f(p) - f(q)| \leq (1 + \alpha^2)r\).

b) If \(\alpha^2 < \frac{1}{r}\) and \(U_{r,q}^* \cap U_{r,p}^* \neq \emptyset\), then \(U_{r,p}^* \subset U_{r,q}^*\).
3 Convergence of graph systems

For \((r, \alpha)\)-immersions \(f : M \to \mathbb{R}^n\) we define the notion of a \(\delta\)-net:

**Definition 2.8** Let \(Q = \{q_1, \ldots, q_s\}\) be a finite set of points in \(M\) and let \(0 < \delta < r\). We say that \(Q\) is a \(\delta\)-net for \(f\) if \(M = \bigcup_{j=1}^{s} U_{\delta,q_j}\).

The number of elements of a \(\delta\)-net can be bounded from above:

**Lemma 2.9** Assume \(\alpha^2 < \frac{1}{r}\) and \(0 < \delta < r\). Then every \((r, \alpha)\)-immersion \(f : M \to \mathbb{R}^n\) admits a \(\delta\)-net with at most \(\left(\frac{2}{\delta}\right)^n \text{vol}(M)\) points.

**Proof:**
The proof is the same as in the 2-dimensional case, see Lemma 3.2 in [10]. Note that one could even derive the bound \(\left(\frac{4}{\delta}\right)^n \text{vol}(M)\). \(\square\)

3 Convergence of graph systems

In the previous section we have seen, how any immersion in \(\mathfrak{H}\) can be written locally on sets \(U_{r,q}\) as the graph of a function. The notion of a \(\delta\)-net yields a cover of each manifold with such kind of sets. This is the starting point for the notion of graph systems, and for convergence of such systems.

First we like to explain how to represent an immersion in \(\mathfrak{H}\) as a system of graphs. For that we define the space of graph systems with \(s\) elements by

\[
\mathfrak{G}^s = \{(A_j, u_j)_{j=1}^s : A_j : \mathbb{R}^n \to \mathbb{R}^n \text{ is a Euclidean isometry},
\quad u_j \in W^{2,p}(B_r, \mathbb{R}^k)\}.
\]

Every Euclidean isometry \(A : \mathbb{R}^n \to \mathbb{R}^n\) splits uniquely into a rotation \(R \in \mathcal{O}(n)\) and a translation \(T \in \mathbb{R}^n\), such that \(A(x) = R(x) + T\) for all \(x \in \mathbb{R}^n\). If \(\| \cdot \|_{\text{op}}\) denotes the operator norm and if \(\Gamma = (A_j, u_j)_{j=1}^s \in \mathfrak{G}^s\), we set

\[
\vartheta(\cdot, \cdot) : \mathfrak{G}^s \times \mathfrak{G}^s \to \mathbb{R},
\quad \vartheta(\Gamma, \tilde{\Gamma}) = \sum_{j=1}^s (\| R_j - \tilde{R}_j \|_{\text{op}} + | T_j - \tilde{T}_j | + \| u_j - \tilde{u}_j \|_{C^1(B_r)}).
\]

This makes \((\mathfrak{G}^s, \vartheta)\) a metric space.

Now let \(f : M \to \mathbb{R}^n\) be an \((r, \alpha)\)-immersion and \(Q = \{q_1, \ldots, q_s\}\) a \(\delta\)-net for \(f\) with \(s\) elements. To each \(q_j \in Q\) we may assign a neighborhood \(U_{r,q_j}\), a Euclidean isometry \(A_j\), and a \(C^1\)-function \(u_j : B_r \to \mathbb{R}^k\) as described above. Hence, to given \(f\), \(r\) and \(Q\), we may assign a graph system

\[\Gamma = (A_j, u_j)_{j=1}^s \in \mathfrak{G}^s.\]

The isometries \(A_j\) and functions \(u_j\) are not uniquely determined, but we always have \(u_j(0) = 0\) and \(Du_j(0) = 0\).

For any \(j \in \{1, \ldots, s\}\) we finally set \(Z(j) := \{1 \leq k \leq s : U_{\delta,q_j} \cap U_{\delta,q_k} \neq \emptyset\}\).

With the preceding notations we are able to define a notion of convergence for graph systems:

**Definition 3.1 (Convergence in the sense of graph systems)**

Let a sequence \(f^i : M^i \to \mathbb{R}^n\) of immersions be given. We say \(f^i\) is convergent in the sense of graph systems, if there are fixed \(\alpha, r, \delta > 0\) with \(r > \delta\), and \(s \in \mathbb{N}\), such that the following properties are satisfied:
Each $f^i$ is an $(r,\alpha)$-immersion.

For each $f^i$ there exists a $\delta$-net with $s$ points, for which the following holds:
- $Z^i(j) = Z(j)$ for fixed sets $Z(j)$ independent of $i$.
- There exists a system $\Gamma \in \mathcal{G}$, such that the graph systems $\Gamma^i$ corresponding to $f^i$ converge in $(\mathcal{G}, \delta)$ to $\Gamma$.

The following statement is true:

**Theorem 3.2** Every sequence in $\mathcal{F}$ admits a subsequence that converges in the sense of graph systems.

**Proof:** Using the results above, the proof of Theorem 3.3 on p. 228 in [16] carries over to the higher dimensional case. 

Here, we only require a graph system $\Gamma$ as limit, but not an immersion $f$. Actually we could say, that any sequence in $\mathcal{F}$ admits a subsequence that is Cauchy in the sense of graph systems. In the next section we will show completeness in the sense that there exists an immersion $f$ with $\Gamma = \Gamma(f)$.

### 4 Construction of the limit manifold and immersion

In Theorem 3.2 for a given sequence of immersions $f^i$ in $\mathcal{F}$ we have found a subsequence, that converges in the sense of graph systems to a limit system $\Gamma$. However, it is not clear whether $\Gamma$ is the graph system of an immersion $f : M \to \mathbb{R}^n$ on a compact manifold $M$. In this section we like to show, that this is the case.

First we would like to construct the limit manifold $M$. We start with a sequence of $(r,\alpha)$-immersions, convergent in the sense of graph systems, with $\alpha^2 < \frac{1}{12}$, $\delta = \frac{1}{10}$; $\delta$-nets $Q^i = \{q^i_1, \ldots, q^i_k\}$ with $s$ elements, intersection sets $Z(j) = \{1 \leq i \leq s : U^i_{\delta,j} \cap U^i_{\delta,q^i_k} \neq \emptyset\}$ which are independent of $i$, limit isometries $A_j$ and limit functions $u_j : B_r \to \mathbb{R}^k$. Here we have to use $\delta/10$-nets and not only $\delta$-nets; this is in particular needed in the proof of Lemma 4.7. To simplify the notation, for $0 < \theta \leq r$ we set $U^i_{\theta,j} := U^i_{\delta,j}$. For the open ball $B_\delta \subset \mathbb{R}^m$ we set $B^i_\delta = B_\delta \times \{j\}$. This makes $\bigcup_{j=1}^s B^i_\delta$ a disjoint union. We endow $\bigcup_{j=1}^s B^i_\delta$ with the topology of the disjoint union, which is defined as follows: A subset $U \subset \bigcup_{j=1}^s B^i_\delta$ is open if and only if $U \cap B^i_\delta \subset B_\delta$ is open for every $j$.

We define a relation $\sim$ on $\bigcup_{j=1}^s B^i_\delta$. For $(x,j), (y,k) \in \bigcup_{j=1}^s B^i_\delta$ we set
\[
(x,j) \sim (y,k) \iff [k \in Z(j) \text{ and } A_j(x, u_j(x)) = A_k(y, u_k(y))].
\]

**Lemma 4.1** The relation $\sim$ is an equivalence relation.

**Proof:** Obviously the relation $\sim$ is reflexive and symmetric. Now let $(x,j) \sim (y,k)$, $(y,k) \sim (z,l)$ for $(x,j), (y,k), (z,l) \in \bigcup_{j=1}^s B^i_\delta$. As $k \in Z(j)$, $l \in Z(k)$, we have $U^i_{\delta,j} \cap U^i_{\delta,k} \neq \emptyset$, $U^i_{\delta,k} \cap U^i_{\delta,l} \neq \emptyset$. Using Lemma 2.7 2b) twice yields $U^i_{\delta,l} \subset U^i_{\delta,j}$. Moreover there is exactly one $\xi^i \in U^i_{\delta,l}$ with $f^i(\xi^i) = A_j(z, u_j(z))$. By the definition of $\sim$ it follows $A_j(x, u_j(x)) = A_k(z, u_k(z))$, which means together with the graph convergence $f^i(\xi^i) \to A_j(x, u_j(x))$ as $i \to \infty$. We define $x^i := \pi \circ (A^i_j)^{-1} \circ f^i(\xi^i)$.

As $U^i_{\delta,j} \subset U^i_{\sigma,r}$, we have $\xi^i \in U^i_{\sigma,r,j}$ and hence $x^i \in B_r$ and $f^i(\xi^i) = A^i_j(x^i, u^i_j(x^i))$. With the convergence of $f^i(\xi^i)$ it follows $A^i_j(x^i, u^i_j(x^i)) \to A_j(x, u_j(x))$ as $i \to \infty$. As $A^i_j \to A_j$ for $i \to \infty$ (in
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the sense of (3.2) it follows $\pi \circ (A^i_j)^{-1} \circ A_j(x, u_j(x)) \to x$, and, by using the triangular inequality, $x^i \to x$ in the ball $B_r$. In particular we have $x^i \in B_\delta$ for $i$ sufficiently large, and hence $\xi^i \in U^i_{\delta,j}$. We deduce $U^i_{\delta,j} \cap U^i_{\delta,s} \neq \emptyset$ (which is then automatically satisfied for all $i$) and hence $l \in Z(j)$. This shows transitivity.

This enables us to define the limit manifold. As set, $M$ is defined to be the quotient space

$$M = \left( \bigcup_{j=1}^s B^j_{\delta} \right) / \sim, \tag{4.2}$$

resulting from the equivalence relation of above. Let $M$ be endowed with the quotient topology. For $(x, k) \in \bigcup_{j=1}^s B^j_{\delta}$, let $\{ (x, k) \}$ denote the corresponding equivalence class. Let $P$ denote the canonical projection from $\bigcup_{j=1}^s B^j_{\delta}$ onto $M$, and $P_j$ the restriction $P|B^j_{\delta} : B^j_{\delta} \to P(B^j_{\delta})$. We can consider $P_j$ as a mapping defined on $B_{\delta}$. We note that $P$ is injective on $B^j_{\delta}$, and in particular $P_j$ invertible. For $V \subset B_{\delta}$ we set $V^j = V \times \{j\}$. For $V \subset B_{\delta}$ open we define

$$\varphi^j_v : P(V^j) \to V, \quad [\{x, j\}] \mapsto P_j^{-1}(\{x, j\}) \in V, \tag{4.3}$$

which yields a well-defined mapping. Finally we denote the set of all such mappings by $\mathfrak{A}$, that is $\mathfrak{A} = \{ \varphi^k_w : 1 \leq k \leq s, W \subset B_{\delta} \text{ open} \}$. To simplify the notation, we will often identify sets $V^j$ with $V$, and elements $(x, j)$ with $x$ (as already done above).

Lemma 4.2 The quotient projection $P$ is open.

Proof:

Let $V \subset \bigcup_{j=1}^s B^j_{\delta}$ be open. We have to show, that $P^{-1}(P(V))$ is open. For that let $x \in P^{-1}(P(V))$.

Then $x \in B^j_{\delta}$ for a $j \in \{1, \ldots, s\}$. We show the existence of an open neighborhood $U \subset B^j_{\delta}$ of $x$ with $U \subset P^{-1}(P(V))$.

It holds $x \sim y$ for a $y \in V$ and moreover $y \in B^k_{\delta}$ for a $k \in Z(j)$. Now consider $\psi : B_{\delta} \to \mathbb{R}^m$, $z \mapsto \pi \circ A^i_j \circ A_j(z, u_j(z))$. As $x \sim y$, we have $\psi(x) = y$. As $V$ is open, there is an open neighborhood $W \subset V$ of $y$ with $W \subset B^k_{\delta}$. As $\psi$ is continuous, $\psi^{-1}(W)$ is an open neighborhood of $x$.

We show that every point $z \in \psi^{-1}(W)$ is equivalent to a point in $W$, which implies the statement. For every $i$ there is exactly one $\xi^i \in U^i_{\delta,j}$ with $f^i(\xi^i) = A^i_j(z, u^i_j(z))$. As $k \in Z(j)$, with Lemma 4.4

b) it holds $U^i_{\delta,j} \subset U^i_{4\delta,k}$. Hence for every $i$ there is a $w^i \in B_{4\delta}$ with $A^i_k(w^i, u^i_k(w^i)) = A^i_j(z, u^i_j(z))$. For $i \to \infty$ we have $A^i_j(z, u^i_j(z)) \to A_j(z, u_j(z))$, and $A^i_k \to A_k$, $u^i_k \to u_k$ (in the sense of (4.2)) and for a subsequence $w^i \to w$ for a $w \in B_{4\delta}$. Using the triangular inequality, we deduce $A_k(w, u_k(w)) = A_j(z, u_j(z))$. Hence $w = \psi(z) \in W \subset B_{\delta}$ and $(z, j) \sim (w, k)$, which proves the lemma.

Lemma 4.3 The space $M$ is a second countable Hausdorff space.

Proof:

We first show that $M$ is Hausdorff. Let $p, q \in M$ with $p \neq q$. Then there are $j, k \in \{1, \ldots, s\}$ with $p \in P(B^j_{\delta})$, $q \in P(B^k_{\delta})$. If $k \notin Z(j)$, then $P(B^j_{\delta})$ and $P(B^k_{\delta})$ are disjoint open neighborhoods.

Now let us assume $k \in Z(j)$. Then there are $x \in B^j_{\delta}$, $y \in B^k_{\delta}$ with $p = P(x)$, $q = P(y)$. It follows $A_j(x, u_j(x)) \neq A_k(y, u_k(y))$, otherwise $p = q$. We define a mapping $\gamma : B_3 \times B_3 \to \mathbb{R}$, $(v, w) \mapsto [A_j(v, u_j(v)) - A_k(w, u_k(w))]$. Hence $\gamma(x, y) > 0$. As $\gamma$ is continuous, there are open neighborhoods $V, W$ of $x, y$ with $\gamma(V \times W) \subset (0, \infty)$. Using that the projection $P$ is open, $P(V^j)$ and $P(W^k)$ are disjoint open neighborhoods of $p$ and $q$.  

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Next we like to show that $M$ is second countable. Let $\mathfrak{B}$ be a countable basis of $\bigcup_{j=1}^{s} B_{δ}^j$. As the projection $P$ is open, $\{P(B) : B \in \mathfrak{B}\}$ is a countable basis of $M$.

**Lemma 4.4** The set $\mathfrak{A}$ is a differentiable atlas on $M$.

**Proof:**
First we note that $M$ is covered by the sets $P(V^j)$. Furthermore, every $\varphi_{v}^j : P(V^j) \to V$ is a bijective mapping between open sets with inverse mapping $P_{j}$ (more precisely $(\varphi_{v}^j)^{-1} : V \to P(V^j)$ with $(\varphi_{v}^j)^{-1}(x) = P_{j}(x)$). The quotient projection $P$ is open by Lemma 4.2 and certainly continuous. But continuous, open, bijective mappings are homeomorphisms. Hence $M$ is locally Euclidean.

It remains to show differentiability of the coordinate changes. For the charts $\varphi_{v}^j, \varphi_{w}^k$, the coordinate change is given by

$$\varphi_{v}^j \circ (\varphi_{w}^k)^{-1} : \varphi_{v}^k(\varphi_{v}^j(P(V^j) \cap P(W^k))) \to \varphi_{v}^j(P(V^j) \cap P(W^k)),$$

$$x \mapsto \pi \circ A_{j}^{-1} \circ A_{k}(x, u_{k}(x)).$$

But this is a composition of smooth mappings; hence $\varphi_{v}^j \circ (\varphi_{w}^k)^{-1}$ is smooth. □

Let us summarize our results:

**Theorem 4.5** The topological space $M$ is Hausdorff with countable basis and $\mathfrak{A}$ is a differentiable atlas on $M$. Hence $(M, \mathfrak{A})$ induces uniquely the structure of a differentiable manifold.

Finally we show compactness of $M$:

**Lemma 4.6** The limit manifold $M$ is compact.

**Proof:**
For the proof we already use Lemma 4.4. By this we have $M = \bigcup_{j=1}^{s} P(B_{δ/2}^j)$. As the quotient projection is continuous, with the compactness of $B_{δ/2}^j$ the statement follows. □

Now we define the limit immersion:

$$f : M \to \mathbb{R}^n,$$

$$[(x, j)] \mapsto A_{j}(x, u_{j}(x)).$$

(4.4)

If $(x, j) \sim (y, k)$, by the definition of $\sim$ we have $A_{j}(x, u_{j}(x)) = A_{k}(y, u_{k}(y))$. Hence $f$ is well-defined. Moreover $f$ admits the local representation $x \mapsto A_{j}(x, u_{j}(x))$ for $x \in B_{δ}$, which implies that $f$ is an immersion. Finally we note that the limit system $(A_{j}, u_{j})_{j=1}^{s}$ of the graph convergence is the graph system of an immersion.

The following lemmas are associated with the construction of the limit manifold above. All statements are needed only for technical reasons and will be required for the construction of the mappings $\phi^j$ in the next section, and in particular for showing injectivity of these mappings. Additionally, Lemma 4.7 is required in the proof of Lemma 4.6 stating that $M$ is compact.

By the definition of $M$, we have $M = \bigcup_{j=1}^{s} P(B_{δ}^j)$. The following lemma says, that there even exists a much finer cover:

**Lemma 4.7** It holds $M = \bigcup_{j=1}^{s} P(B_{δ}^j)$. 

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The next statement is the analogue to Lemma 2.7 b) for the limit immersion:

**Lemma 4.9**

It follows

Moreover, the points passing to a subsequence, we may assume \( j^i = k \) independent of \( i \). Then there are \( y^i \in B_{\delta/10} \) with \( f^i(\xi^i) = A_k(y^i, u_k^i(y^i)) \). A subsequence of \( y^i \) converges to \( y \in \overline{B_{\delta/10}} \subset B_{\delta/6} \). As \( f^i(\xi^i) \to f(q) \), \( A_k^i \to A_k \) and \( u_k^i \to u_k \) for \( i \to \infty \), we have \( f(q) = A_k(y, u_k(y)) \). As \( \xi^i \in U_{\delta,j}^i \), \( \xi^i \in U_{\delta,j}^{i,10,k} \); we have \( U_{\delta,j}^i \cap U_{\delta,k}^{i,10} \neq \emptyset \), and all the more \( U_{\delta,j}^i \cap U_{\delta,k}^i \neq \emptyset \). This implies \( k \in Z(j) \), and moreover \( (x, j) \sim (y, k) \).

It follows \( q \in P(B_{\delta/6}^k) \).

The next statement is the analogue to Lemma 2.7 b) for the limit immersion:

**Lemma 4.8**

If \( P(B_{\delta}^j) \cap P(B_{\delta}^k) \neq \emptyset \), then \( P(B_{\delta}^j) \subset P(B_{\delta}^k) \).

**Proof:**

The proof of Lemma 2.7 carries over to the limit immersion.

Analogous to the sets \( Z(j) \), we define intersection sets for a finer cover of \( M^i \) by

\[
\tilde{Z}^i(j) = \{ 1 \leq k \leq s : U_{\delta,j}^i \cap U_{\delta,k}^i \neq \emptyset \}.
\]

Passing to a subsequence, again we may assume \( \tilde{Z}^i(j) = \tilde{Z}(j) \) independent of \( i \).

The relation \( P(B_{\delta}^j) \cap P(B_{\delta}^k) \neq \emptyset \) implies \( k \in Z(j) \); however, in general \( P(B_{\delta}^j) \cap P(B_{\delta}^k) = \emptyset \) does not imply \( k \notin Z(j) \). Instead the following statement holds (where the numbers are adapted to the situation in the next section):

**Lemma 4.9**

If \( P(B_{\delta}^j) \cap P(B_{\delta}^k) = \emptyset \), then \( k \notin \tilde{Z}(j) \).

**Proof:**

Let \( P(B_{\delta/4}^j) \cap P(B_{\delta/4}^k) = \emptyset \). Suppose \( k \in \tilde{Z}(j) \). Then for every \( i \) there is a \( \xi^i \in U_{\delta/5,j}^i \cap U_{\delta/5,k}^i \). Moreover, the points \( f^i(\xi^i) \) lie in a ball of fixed radius. Hence there is a subsequence and an \( x \in \mathbb{R}^n \) with \( f^i(\xi^i) \to x \) as \( i \to \infty \). With the graph convergence and by arguments as in Lemma 4.7, we have \( x = A_j(y, u_j(y)) = A_k(z, u_k(z)) \) with \( y, z \in \overline{B_{\delta/5}} \subset B_{\delta/4} \). As \( k \in \tilde{Z}(j) \), we surely have \( k \in Z(j) \). It follows \( P(B_{\delta/4}^j) \cap P(B_{\delta/4}^k) \neq \emptyset \), contrary to our assumption.

## 5 Reparametrization of the immersions

We like to construct the reparametrizations \( \varphi^i : M \to M^i \). This is done by a kind of projection from the limit surface onto each of the surfaces \( f^i \).

Our starting point is a sequence of \((r, \alpha)\)-immersions \( f^i : M^i \to \mathbb{R}^n \) in \( \mathcal{F} \), which converges in the sense of graph systems to a limit immersion \( f : M \to \mathbb{R}^n \). Here we require \( \alpha^2 \leq \frac{1}{10} \). We will define the projection locally, using charts \( \varphi_j : P(B_{\delta}^j) \to B_{\delta} \). By such a chart, we shall often tacitly identify the set \( P(B_{\delta}^j) \) with the ball \( B_{\delta} \).

Let \( A_j \) and \( A_j^i \) denote the isometries of the previous sections corresponding to \( f \) and \( f^i \) respectively. As the following constructions are invariant under translations and rotations, we may assume \( A_j = \text{Id}_{\mathbb{R}^n} \) and replace \( A_j^i \) by \( A_j^{-1} \circ A_j^i \).

Then \( f(P(B_{\delta}^j)) \) is the graph of a function \( u_j : B_{\delta} \to \mathbb{R}^k \) with \( u_j(0) = 0 \), \( Du_j(0) = 0 \). The set \( f^i(U_{r,j}^i) \) is the graph of a function \( u_j^i : B_r \to \mathbb{R}^k \), however translated and rotated relatively to the
limit immersion by \( A_j^{-1} \circ A_j^i \). But actually \( A_j^i \to A_j \) as \( i \to \infty \) in the sense of the metric \((\cdot,\cdot)\). Hence the translation and rotation \( A_j^{-1} \circ A_j^i \) gets arbitrarily small relative to \( f \) as \( i \to \infty \). Hence we may assume that also \( f^i(U_{r,j}) \) is the graph of a function on a subset of \( \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^k \), which shall be denoted in the following by \( \tilde{u}_j^i \).

Furthermore for all \( \varrho \) with \( 0 < \varrho < r \) there is an \( N \in \mathbb{N} \), such that for all \( i > N \)

\[
\{(x, \tilde{u}_j^i(x)) : x \in B_{r-\varrho} \} \subset f^i(U_{r,j}).
\]

This is the situation represented in Figure 1.

![Figure 1](image1.png)

**Figure 1** Position of the immersions \( f^i \) relative to the limit immersion. Note that the figure is not true to scale, because in the proof we have \( r = 16\delta \).

As \( \|Du_j^i\|_{C^0(B_{r})} \leq \alpha \), we surely may assume \( \|D\tilde{u}_j^i\|_{C^0(B_{r-\varrho})} \leq 2\alpha \) for \( i \) sufficiently large. Moreover, by the graph convergence, for any \( \varepsilon > 0 \) we have \( |\tilde{u}_j^i(0)| < \varepsilon \) for \( i \) large.

Finally we like to simplify notation. All the following considerations are performed locally on \( P(B_j^r) \). We will fix the index \( j \) and suppress it in the notation. Hence we shall write for example \( u \) instead of \( u_j \), and \( \tilde{u}^i \) instead of \( \tilde{u}_j^i \).

If the limit immersion is sufficiently smooth, it is possible to project into the normal direction. However, if \( f \) is not \( C^2 \), in general this is not possible. Without an \( L^{\infty} \)-bound for the second fundamental form we might have a local concentration of curvature. In this case, projecting into the normal direction will not lead to injective mappings \( \phi^i \) (see Figure 2).

![Figure 2](image2.png)

**Figure 2** Normal projection in the case of concentrated curvature.
where we assume that in addition to (5.1), \(f\) is \(C^2\) close to \(f^i\). Then we can project from \(f\) in the normal direction \(\nu\) of \(g\) onto \(f^i\). Similarly one could use one of the approximation theorems for immersions in \([12]\). Slightly different is the approach using an averaged normal projection. It is described in \([16]\) for codimension 1. A generalization to arbitrary codimension using the Riemannian center of mass is presented in \([5]\).

Here we like to assume that we have already found (by one of the preceding methods) a smooth mapping \(\nu : M \to G_{n,k}\), which is close to the normal of \(f\). Let us explain what that means: As explained above \(f(P(B^4))\) is the graph of a function \(u\) on \(B_\delta \subset \mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^k\) with \(\|Du\|_{C^0(B_\delta)} \leq \alpha\). For \(q \in P(B^2)\) consider the subspace \(\nu_f(q)\), where \(\nu_f : M \to G_{n,k}\) is the normal of \(f\). Then \(\nu_f(q)\) is a graph over \(\{0\} \times \mathbb{R}^k\); more precisely there is a linear map \(\tilde{N}_q : \mathbb{R}^k \to \mathbb{R}^m\) such that

\[
\nu_f(q) = \{(\tilde{N}_q(z), z) : z \in \mathbb{R}^k\} \subset \mathbb{R}^m \times \mathbb{R}^k = \mathbb{R}^n.
\]

Moreover, as \(\|Du\|_{C^0(B_\delta)} \leq \alpha\), for the operator norm \(\| \cdot \|\) we have

\[
\|\tilde{N}_q\| \leq \alpha.
\]

The property of \(\nu\) being close to \(\nu_f\) (which can be reached by any of the described methods) shall mean, that for all \(q \in P(B^2)\) also the subspace \(\nu(q)\) is the graph of a linear map \(N_q : \mathbb{R}^k \to \mathbb{R}^m\) over \(\{0\} \times \mathbb{R}^k\) and that

\[
\|N_q\| \leq 2\alpha. \tag{5.2}
\]

Identifying \(P(B^2)\) with the ball \(B_\delta\) as described above, we may similarly assign to each \(x \in B_\delta\) a linear map \(N_x : \mathbb{R}^k \to \mathbb{R}^m\).

We like to show, that for \(q \in P(B^2)\) the affine subspace \(f(q) + \nu(q)\) has exactly one point of intersection with the set \(f^i(U^i_{\delta,j})\).

For that, in addition to (5.1), assume

\[
\{(x, \tilde{u}(x)) : x \in B_{\delta-\varrho}\} \subset f^i(U^i_{\delta,j}), \tag{5.3}
\]

where \(\varrho\) is small, say \(\varrho = \frac{\delta}{2}\) (suppose (5.1) is satisfied with the same \(\varrho\)).

The mapping \(F\):

For \(x \in B_\delta\) we denote by \(F(x)\) the unique intersection point of the affine subspace

\[
h(x) := (x, u(x)) + \nu(x)
\]

with \(\mathbb{R}^m \times \{0\}\). In that way, we obtain the mapping

\[
F : B_\delta \to \mathbb{R}^m, \quad x \mapsto x - N_x(u(x)). \tag{5.4}
\]

The mappings \(G^i_x\):

For \(x \in B_\delta\) and \(y \in B_{\varrho-\varrho}\) we denote by \(G^i_x(y)\) the unique intersection point of the affine subspace \((y, \tilde{u}(y)) + \nu(x)\) with \(\mathbb{R}^n \times \{0\}\). In that way we obtain for each fixed \(x \in B_\delta\) a mapping

\[
G^i_x : B_{\varrho-\varrho} \to \mathbb{R}^m, \quad y \mapsto y - N_x(\tilde{u}(y)). \tag{5.5}
\]
By (5.3), the affine subspace \( F \) is parallel to \( B \). By the definitions of \( H \) and \( G \), the mappings \( H \) and \( G \) intersect each other in \( (x, u(x)) \) and \( (y, \hat{u}'(y)) \) for each fixed \( x \) and \( y \). This point lies in \( (x, u(x)) \) for all \( x \in B \). Using the definition of \( H \) and \( G \), we estimate

\[
|y - G_x^i(y) + F(x)| \leq (1 + 18\alpha^2)\delta \leq 3\delta = 4\delta - 2\theta.
\]

With that we define for each fixed \( x \in B \) a mapping

\[
H_x^i : \overline{B_{4\delta - 2\theta}} \to \overline{B_{4\delta - 2\theta}}, \quad y \mapsto y - G_x^i(y) + F(x).
\]

(5.6)

**Lemma 5.1** For \( q \in P(B_x) \) the affine subspace \( f(q) + \nu(q) \) has exactly one point of intersection with the set \( F(B_{4\delta}) \). This point lies in \( f(U_{4\delta}) \).

**Proof:**

We pass to the local representation and consider \( h(x) = (x, u(x)) + \nu(x) \) for \( x \in B_\delta \). Using the definition of \( H_x^i \) and \( \alpha^2 \leq \frac{1}{16} \), we estimate

\[
|H_x^i(\xi) - H_x^i(\zeta)| = |N_x(\hat{u}'(\xi) - \hat{u}'(\zeta))| \leq 2\alpha||N_x|||\xi - \zeta| \leq 4\alpha^2|\xi - \zeta| \leq \frac{1}{2}|\xi - \zeta|.
\]

Hence \( H_x^i \) is a contraction. By the Banach fixed point theorem there is exactly one \( y \in \overline{B_{4\delta - 2\theta}} \) with \( H_x^i(y) = y \), that is with \( G_x^i(y) = F(x) \).

By the definitions of \( F \) and \( G_x^i \), the affine subspaces \( h(x) = (x, u(x)) + \nu(x) \) and \( (y, \hat{u}'(y)) + \nu(x) \) intersect each other in \( F(x) = G_x^i(y) \) and are parallel, hence \( h(x) = (y, \hat{u}'(y)) + \nu(x) \) and \( (y, \hat{u}'(y)) \in h(x) \). By \( \text{(5.3)} \), the affine subspace \( h(x) \) intersects the set \( f(U_{4\delta}) \) in \( (y, \hat{u}'(y)) \).

![Figure 3](attachment:image.png)  
*Figure 3 The mappings \( F \) and \( G_x^i \). The part between the parentheses on the immersion \( f^i \) represents the set \( f^i(U_{4\delta}) \).*
Similarly, we show that there is only one point of intersection with $f^i(U_{r,j}^i)$: For that we assume that we have chosen $r$ slightly smaller in the beginning, such that also the set $f^i(U_{r+2\delta,j}^i)$ is the graph of a function $\tilde{u}^i$ on a subset of $\mathbb{R}^m$ with $||D\tilde{u}^i||_{C^0} \leq 2\alpha$, and such that

$$f^i(U_{r,j}^i) \subset \{(x,\tilde{u}^i(x)) : x \in \overline{B_{r+\epsilon}}\} \subset f^i(U_{r+2\delta,j}^i).$$

Now for each fixed $x \in B_\delta$ define a function

$$\tilde{H}_x^i : \overline{B}_{r+\epsilon} \to \overline{B}_{r+\epsilon},$$

$$y \mapsto y - G_x^i(y) + F(x),$$

where we also extend $G_x^i$ to the ball $\overline{B}_{r+\epsilon}$. Using $r = 16\delta$, $\varrho = \frac{\delta}{2}$, $\alpha^2 \leq \frac{1}{10}$ and assuming $\varepsilon$ to be small, one shows $|y - G_x^i(y) + F(x)| \leq \frac{\delta}{2}$. Hence $\tilde{H}_x^i$ is well-defined. Then also $\tilde{H}_x^i$ is a contraction and there is exactly one point $y \in \overline{B}_{r+\epsilon}$ with $G_x^i(y) = F(x)$. By the definitions of $G_x^i$ and $F$, this shows the statement.

Before we come to the definition of the mappings $\phi^i : M \to M^i$, we need the following lemma, which will assure that the $\phi^i$ are well-defined:

**Lemma 5.2** Let $x \in P(B_\delta^i) \cap P(B_\delta^k)$. Moreover let $S_1$ be the point of intersection of $h(x)$ with $f^i(U_{r,j}^i)$, $S_2$ the point of intersection of $h(x)$ with $f^j(U_{r,j}^j)$, and $\sigma_1 \in U_{r,j}^i$ with $f^i(\sigma_1) = S_1$, $\sigma_2 \in U_{r,j}^j$ with $f^j(\sigma_2) = S_2$. Then $\sigma_1 = \sigma_2$.

**Proof:** By Lemma 5.1 we have $S_2 \in f^j(U_{r,j}^j)$, that is $\sigma_2 \in U_{r,j}^j$. The assumption $x \in P(B_\delta^i) \cap P(B_\delta^k)$ implies $k \in Z(j)$, hence by Lemma 2.7(b) $U_{r,j}^{i,k} \subset U_{r,j}^j$. Using again Lemma 5.1 the statement follows. □

With the preceding lemmas we are able to give a definition of the mappings $\phi^i : M \to M^i$. For that let $x \in M$. Then $x \in P(B_\delta^i)$ for some $j$. The set $h(x)$ intersects $f^j(U_{r,j}^j)$ in exactly one point $S_x$. Furthermore there is exactly one point $\sigma_x \in U_{r,j}^j$ with $f^j(\sigma_x) = S_x$. We set $\phi^i(x) := \sigma_x$. The mappings $\phi^i$ are well-defined by Lemma 5.2. Now we like to show that the mappings $\phi^i$ (after passing to a subsequence, if necessary) are diffeomorphisms.

Let $\gamma_{n,k} = \{(E, x) : E \in G_{n,k}, x \in E\}$ and let $p : \gamma_{n,k} \to G_{n,k}$, $(E, x) \mapsto E$, be the universal bundle over $G_{n,k}$. The local trivializations for this bundle are defined as follows: Let $E \in G_{n,k}$ and let $\pi_E : \mathbb{R}^n \to E$ be the orthogonal projection; we set $U_E = \{G \in G_{n,k} : \pi_E(G)\text{ is of dimension }k\}$; a local trivialization is then given by $\Psi : p^{-1}(U_E) \to U_E \times E \cong U_E \times \mathbb{R}^k$, $\Psi((G, x)) = (G, \pi_E(x))$.

Let $\nu : M \to G_{n,k}$ be as above. We now consider the pullback bundle $\nu^*\gamma_{n,k}$, which is a vector bundle over $M$ with bundle projection $\pi$ and $n$-dimensional total space

$$E = \{(x, y) \in M \times \mathbb{R}^n : y \in \nu(x)\}.$$

We set $E_j = \{(x, y) \in E : x \in P(B_\delta^j)\}$. Hence $\nu^*\gamma_{n,k}|E_j$ is a bundle over $P(B_\delta^j)$. As $P(B_\delta^j)$ is diffeomorphic to $B_\delta \subset \mathbb{R}^m$, and as $B_\delta$ is diffeomorphic to $f(P(B_\delta^j)) = \{(x, u_j(x)) : x \in B_\delta\}$, we may consider $\nu^*\gamma_{n,k}|E_j$ also as a bundle over one of the last-named sets. In particular, $\nu^*\gamma_{n,k}|E_j$ is a trivial bundle.

We sometimes identify the zero section of $\nu^*\gamma_{n,k}|E_j$ with $P(B_\delta^j)$. Finally we define a mapping

$$F : E \to \mathbb{R}^n,$$

$$(x, y) \mapsto f(x) + y,$$  

(5.7)

where $y \in \nu(x)$.
Lemma 5.3 (Local tubular neighborhood around the limit immersion) There exists an open neighborhood $V \subset E$ of the zero section of $\nu^*\gamma_{n,k}$, such that for every $j$ with $1 \leq j \leq s$ the following holds:

- $F|E_j \cap V$ is a diffeomorphism onto an open neighborhood of $f(P(B_{\delta}^j))$,
- $F|P(B_{\delta}^j) = f|P(B_{\delta}^j)$,
- for every fibre $E_q = \pi^{-1}(q)$ we have $F(E_q) = h(q)$.

Proof:
We note that for every $q \in M$ the affine subspace $f(q) + \nu(q)$ intersects $f(q)$ transversally. Moreover $\nu$ is a smooth mapping. Now the statement is a simple fact from differential topology about the existence of tubular neighborhoods (see [7] and [12]). In this way we find tubular neighborhoods on $P(B_{\delta}^j)$ for every $j$. Appropriately composing these neighborhoods, we obtain the desired neighborhood $V \subset E$ of the zero section of $\nu^*\gamma_{n,k}$. □

Lemma 5.4 After passing to a subsequence, each mapping $\phi^i : M \to M^i$ is surjective.

Proof:
By Lemma 5.3 for each $j$ the set $F(E_j \cap V)$ is an open neighborhood of $f(P(B_{\delta}^j)) = \{(x,u_j(x)) : x \in B_{\delta}\}$. We define sets $M_j = \{(x,u_j(x)) : x \in B_{\frac{2}{\sqrt{\delta}}} \cap f(P(B_{\delta}^j))$. As $M_j$ is compact, there is an $\varepsilon_j > 0$ with

$$
M_j^\varepsilon := \{(x,y) : x \in B_{\frac{2}{\sqrt{\delta}}}, \ y \in \mathbb{R}^k \text{ mit } |y - u_j(x)| < \varepsilon_j \} \subset F(E_j \cap V).
$$

We set $\hat{\varepsilon} = \min\{\varepsilon_1, \ldots, \varepsilon_s\}$. By definition of $\delta(\cdot, \cdot)$ and by graph convergence, it follows that $f^i(U_{\delta/2,j}^i)$ is a subset of $M_j^\hat{\varepsilon}$ for $i$ sufficiently large (see Figure 4). Further, it follows $U_{\delta/2,j}^i \subset \phi^i(P(B_{\delta}^j))$ for $j = 1, \ldots, s$. Hence, for every $q \in U_{\delta/2,j}^i$ there is a $p \in P(B_{\delta}^j)$ with $f^i(q) \in f(p) + \nu(p)$. By the definition of $\phi^i$, this yields $\phi^i(p) = q$. As $\{q_1^i, \ldots, q_s^i\}$ is a $\frac{\hat{\varepsilon}}{T}$-net for $f^i$, for every $q \in M^i$ there is a $j \in \{1, \ldots, s\}$ with $q \in U_{\delta/2,j}^i$. Hence, by the considerations of above, $\phi^i$ is surjective. □

Figure 4 Surjectivity of the mappings $\phi^i$.  

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For showing injectivity, we need the following lemma:

**Lemma 5.5** For $i$ sufficiently large, we have the inclusions

a) $\phi^i(P(B^j_{\delta})) \subset U^1_{\delta,j}$,

b) $\phi^i(P(B^j_{\delta})) \subset U^1_{\delta,j}$.

**Proof:**

Follow the arguments of Lemma 5.1. A calculation with the numbers of above proves a) and b). □

We first show local injectivity:

**Lemma 5.6 (Local injectivity)** After passing to a subsequence, for each $j$ the mappings $\phi^i : M \to M^i$ restricted to $P(B^j_{\delta})$ are injective.

**Proof:**

Let $x, y \in P(B^j_{\delta})$ with $x \neq y$. By the definition of $\phi^i$ and by Lemma 5.5 we have $f^i \circ \phi^i(x) \in h(x) = F(E_x)$, $f^i \circ \phi^i(y) \in h(y) = F(E_y)$, and

$$F(E_x \cap V) \cap F(E_y \cap V) = \emptyset.$$  \hfill (5.8)

By Lemma 5.5 a) we have $\phi^i(P(B^k_{\delta/3})) \subset U^i_{\delta/2,k}$ for each $k$, which implies for $z \in P(B^k_{\delta/3})$ with the arguments in the proof of Lemma 5.4, that $f^i \circ \phi^i(z) \in F(E_z \cap V)$. As by Lemma 4.9 it holds $M \subset \bigcup_{j=1}^{\infty} P(B^k_{\delta/3})$, we actually have $f^i \circ \phi^i(z) \in F(E_z \cap V)$ for all $z \in M$. With (5.8) it follows $f^i \circ \phi^i(x) \neq f^i \circ \phi^i(y)$, and hence $\phi^i(x) \neq \phi^i(y)$. □

Now we like to show global injectivity:

**Lemma 5.7 (Injectivity of $\phi^i$ )** After passing to a subsequence, the mappings $\phi^i : M \to M^i$ are injective.

**Proof:**

Let $x, y \in M$ with $x \neq y$. By Lemma 4.3 there are $j, k$ with $x \in P(B^j_{\delta/5}) \subset P(B^j_{\delta/4})$, $y \in P(B^k_{\delta/5}) \subset P(B^k_{\delta/4})$.

**Case 1:** $P(B^j_{\delta/5}) \cap P(B^k_{\delta/5}) = \emptyset$

By Lemma 5.5 b) we have $\phi^i(x) \in U^j_{\delta/5,j}$, $\phi^i(y) \in U^k_{\delta/5,k}$ and Lemma 4.3 implies $k \notin \tilde{Z}(j)$, that is $U^j_{\delta/5,j} \cap U^k_{\delta/5,k} = \emptyset$. It follows $\phi^i(x) \neq \phi^i(y)$.

**Case 2:** $P(B^j_{\delta/5}) \cap P(B^k_{\delta/5}) \neq \emptyset$

By Lemma 4.3 we have $P(B^k_{\delta/4}) \subset P(B^j_{\delta/4})$. By Lemma 5.6 $\phi^i$ is injective on $P(B^j_{\delta/4})$, hence again $\phi^i(x) \neq \phi^i(y)$. □

For showing, that each mapping $\phi^i$ is a diffeomorphism, we first show that the composition $f^i \circ \phi^i$ is an immersion. For that, we use that $F(E_j \cap V)$ is a tubular neighborhood both of $f(P(B^j_{\delta}))$ and of $f^i \circ \phi^i(P(B^j_{\delta}))$.

**Lemma 5.8** The mapping $f^i \circ \phi^i : M \to \mathbb{R}^n$ is an immersion.

**Proof:**

We show the statement by considering the local representation of $f^i \circ \phi^i$ on the set $P(B^j_{\delta})$. We regard $E_j$ as a bundle over $f(P(B^j_{\delta}))$. As $\nu^i_{\gamma_{n,k}|E_j}$ is a trivial bundle, there exists a trivialization $\Psi_j : E_j \to B^k \times \mathbb{R}^k \subset \mathbb{R}^n$ with $\Psi_j(E_q) = \{q\} \times \mathbb{R}^k$. As $f(P(B^j_{\delta}))$ is diffeomorphic to $B^k$, we may assume that the zero section is mapped by $\Psi_j$ onto $B^k$. We define restrictions $\tilde{\Psi}_j = \Psi_j|E_j \cap V : \mathbb{R}^n \to B^k$. Now we show that $\tilde{\Psi}_j$ is a diffeomorphism. □
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$E_j \cap V \to \Psi_j(E_j \cap V)$, and $F_j = F|E_j \cap V : E_j \cap V \to F(E_j \cap V)$ with $F$ as in Lemma $5.3$. Also by Lemma $5.3$, $F_j$ and hence also $\tilde{\Psi}_j \circ F_j^{-1}$ is a diffeomorphism (see Figure $3$). We note, that $f^i(U_{r,j}^i)$ and hence also $W_j^i := f^i(U_{r,j}^i) \cap F(E_j \cap V)$ is a smooth submanifold of $\mathbb{R}^n$. With Lemma $5.3$, and by construction of the projection, for $q \in P(B_q^i)$ we have $f^i \circ \phi^i(q) \in f^i(U_{r,j}^i) \cap h(q) = f^i(U_{r,j}^i) \cap F(E_q)$, and $f^i \circ \phi^i(B_B) = W_j^i$. We obtain $F_j^{-1} \circ f^i \circ \phi^i(q) \in E_q$ and hence $\tilde{\Psi}_j \circ F_j^{-1} \circ f^i \circ \phi^i(q) = (h_j^i(q)) \in B_B \times \mathbb{R}^k$ with a mapping $h_j^i : B_B \to \mathbb{R}^k$. As $\tilde{\Psi}_j \circ F_j^{-1}$ is a diffeomorphism and $W_j^i$ a submanifold, also $\tilde{\Psi}_j \circ F_j^{-1}(W_j^i) = \tilde{\Psi}_j \circ F_j^{-1} \circ f^i \circ \phi^i(B_B)$ is a smooth submanifold and hence $h_j^i$ a differentiable mapping. It follows, that $\tilde{\Psi}_j \circ F_j^{-1} \circ f^i \circ \phi^i$ is an immersion. As $\tilde{\Psi}_j \circ F_j^{-1}$ is a diffeomorphism, the statement follows.

Figure 5 Straightening of the tubular neighborhood. Here the set $f(P(B_q^i))$ is mapped by $\tilde{\Psi}_j \circ F_j^{-1}$ onto $B_B \times \{0\}$. Note that for $\nu \in C^2$ the mapping $f^i \circ \phi^i$ is in $W^{2,p}$.

Theorem 5.9 The mappings $\phi^i : M \to M'$ are diffeomorphisms.

Proof: The mappings $f^i$ and $f^i \circ \phi^i$ are immersions. It follows, that also $\phi^i$ is an immersion. Moreover $\phi^i$ is surjective by Lemma $5.3$ and injective by Lemma $5.7$. Hence $\phi^i$ is a diffeomorphism.

6 Convergence of the immersions

In this section we would like to show convergence of the sequence $f^i \circ \phi^i$ to $f$ in the $C^1$-topology. This means, we show $C^1$-convergence of the local representations of $f^i \circ \phi^i$ to the local representations of the limit immersion $f$ with respect to the atlas $\mathcal{A}$.

As a generalization, we like to show higher order convergence for immersions with graph representations that are uniformly bounded in $W^{k,p}$ with $k > 2$. For that reason, let us assume that the mappings $f$ and $f^i$, and hence also $u_j$, $u_j^i$ and $\tilde{u}_j^i$ are in $W^{k,p}$ for a $k \geq 2$, and that $\tilde{u}_j^i$ is uniformly bounded in $W^{k,p}$. We will discuss in the end of this section under which assumptions we obtain these higher order bounds.

We shall use the same notation as in the previous section. All considerations are performed locally on $P(B_B^i)$. As in the previous section, we will fix the index $j$ and suppress it in the notation. Hence again we shall write $u$ instead of $u_j$, and $\tilde{u}$ instead of $\tilde{u}_j^i$.

Instead let a lower index $\nu$ now denote the $\nu$-th coordinate of a vector in $\mathbb{R}^n$ or in $\mathbb{R}^k$. Moreover, let $\pi^{\nu}$ be the projection from $\mathbb{R}^n$ onto the first $m$ coordinates, and $\pi^{\nu}$ the projection onto the last $k$ coordinates. Finally, we shall simply write $f(x)$ instead of $(x, u(x))$. 

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For proving convergence, we additionally assume that we have chosen

$$\alpha \leq \frac{1}{\sqrt{k}} \quad (6.1)$$

in the beginning, where \( k \) denotes the codimension of the immersions (here we denote by \( k \) the degree of differentiability, and by \( k \) the codimension).

By the previous section, we project into the direction \( \nu \). Moreover \( \nu^*\gamma_{n,k}|E_j \) is a trivial bundle over \( B_\delta \). The fibre of this bundle over each point in \( B_\delta \) is a \( k \)-dimensional subspace of \( \mathbb{R}^n \). Now let \( \sigma = (e^1, \ldots, e^k) \) be a smooth frame of this bundle, that is \( e^1, \ldots, e^k : B_\delta \rightarrow \mathbb{R}^n \) and \( (e^1(x), \ldots, e^k(x)) \) is a basis of \( \nu(x) \in G_{n,k} \) for all \( x \in B_\delta \).

We define a mapping

$$G^i(x, t_1, \ldots, t_k) = \tilde{u}^i(\pi^h(f(x) + \sum_{\nu=1}^{k} t_\nu e^\nu)) - \pi^h(f(x) + \sum_{\nu=1}^{k} t_\nu e^\nu), \quad (6.2)$$

where \( x \in B_\delta \) and \( t_1, \ldots, t_k \in \mathbb{R} \) is sufficiently small, such that \( \pi^h(f(x) + \sum_{\nu=1}^{k} t_\nu e^\nu) \in B_{r^{-\theta}} \).

By the construction of the reparametrization in the previous section, for every \( x \in B_\delta \) there is exactly one tuple \((T^i_1(x), \ldots, T^i_k(x))\) with

$$G^i(x, T^i_1(x), \ldots, T^i_k(x)) = 0. \quad (6.3)$$

In this manner we obtain mappings \( T^i_\nu : B_\delta \rightarrow \mathbb{R} \) (depending on the choice of frame). We like to choose a frame, such that all calculations get as simple as possible.

For that let \( \hat{e}^1, \ldots, \hat{e}^k \) denote the standard orthonormal basis of \( \{0\} \times \mathbb{R}^k \subset \mathbb{R}^m \times \mathbb{R}^k = \mathbb{R}^n \). For every \( x \in B_\delta \) the \( k \)-space \( \nu(x) \) is a graph over \( \{0\} \times \mathbb{R}^k \). Now we define a frame \( \sigma = (\hat{e}^1, \ldots, \hat{e}^k) \) for \( \nu^*\gamma_{n,k}|E_j \) by projecting the basis \( \hat{e}^1, \ldots, \hat{e}^k \) orthogonally with respect to \( \{0\} \times \mathbb{R}^k \) onto each fibre \( \nu(x) \in G_{n,k} \). If \( \nu \) is a \( C^k \)-mapping, the basis vectors \( e^j : B_\delta \rightarrow \mathbb{R}^n \) are easily seen to be mappings of the same class. Moreover (as the bundle \( \nu^*\gamma_{n,k}|E_j \) can be continued to a trivial bundle on a larger set), the mappings \( e^j \) are bounded in \( C^k \).

By construction, this frame has the property \( \pi^h(\sum_{\nu=1}^{k} t_\nu e^\nu) = (t_1, \ldots, t_k)^t \). For the corresponding mappings \( T^i_1, \ldots, T^i_k \), using \( \pi^h(f(x)) = x \), it follows

$$0 = \tilde{u}^i(Id_{B_\delta} + \pi^h(\sum_{\nu=1}^{k} T^i_\nu e^\nu)) - u - (T^i_1, \ldots, T^i_k)^t, \quad (6.4)$$

that is for each coordinate

$$0 = \tilde{u}^i(Id_{B_\delta} + \pi^h(\sum_{\nu=1}^{k} T^i_\nu e^\nu)) - u_i - T^i \quad (6.5)$$

for all \( \epsilon \) with \( 1 \leq \epsilon \leq k \).

Now \( \tilde{u}^i(Id_{B_\delta} + \pi^h(\sum_{\nu=1}^{k} T^i_\nu e^\nu)) \) is just the local representation of the mapping \( f^i \circ \phi^j \), which is in \( W^{2,p} \). It follows directly, that also the mappings \( T^i_\nu \) are in \( W^{2,p} \). As we also like to show higher order convergence, let us assume that each mapping \( T^i_\nu \) is in \( W^{k,p} \) with \( k \geq 2 \).
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Convergence

We like to show, that the mappings $T_i^j$ are uniformly bounded in $W^{k,p}$.

Beforehand we define

$$ X^i : B_\delta \to B_{r_0}, \quad X^i = Id_{B_i} + \pi^i (\sum_{\nu=1}^k T^\nu_i e^\nu) . \quad (6.6) $$

Here all balls are subsets of $\mathbb{R}^m$. Inserting $X^i$ into \((6.3)\) gives

$$ T^i_i = \bar{u}_i \circ X^i - u_i . \quad (6.7) $$

The following expressions $S^i_l$ and $U^i_l$ need not be calculated explicitly; we only need to estimated the order of derivatives involved. For that we shall use the multi-index notation. Expressions of the form $\partial_w$ will denote the usual partial derivative. Lower indices (as, for example, in the expressions $X^i_\alpha, T^\nu_i$, $e^\nu_\alpha, u_i, \bar{u}_i$) will denote the corresponding coordinate of a vector.

Lemma 6.1 Let $\gamma \in \mathbb{N}_0^m, |\gamma| = l$, be a multi-index with $1 \leq l \leq k$. Then

$$ \partial^\gamma X^i_\alpha = \sum_{\nu=1}^k \partial^\gamma T^\nu_i \cdot e^\nu_\alpha + S^i_l, \quad (6.8) $$

where $S^i_l$ is a finite sum of terms of the form $C + \partial^\lambda T^\nu_i \cdot \partial^\mu e^\nu_\alpha$ with multi-indices $\lambda, \mu$ with $0 \leq |\lambda| \leq l-1$, $1 \leq |\mu| \leq l$ and $C$ a constant.

Proof:
The statement is easily shown by induction over $l$ (the order of the multi-index $\gamma$).

Lemma 6.2 Let $\gamma \in \mathbb{N}_0^m, |\gamma| = l$, be a multi-index with $1 \leq l \leq k$. Then

$$ \partial^\gamma T^i_i = \sum_{\alpha=1}^m \partial^\gamma \bar{u}_i^j (X^i) \cdot \partial^\gamma X^i_\alpha - \partial^\gamma u_i + U^i_l, \quad (6.9) $$

where $U^i_l$ is a finite sum of terms of the form $\partial^\lambda \bar{u}_i^j (X^i) \cdot \partial^\mu_1 X^i_{\beta_1} \cdot \ldots \cdot \partial^\mu_\eta X^i_{\beta_\eta}$ with $1 \leq |\lambda| \leq l$, $1 \leq \eta \leq l, 1 \leq |\mu_1|, \ldots, |\mu_\eta| \leq l-1$ and $1 \leq \beta_1, \ldots, \beta_\eta \leq m$.

Proof:
Again the statement is shown by induction over $l$. For $l = 1$ and $1 \leq w \leq m$ one calculates the derivative $\partial_w$ of equation \((6.7)\). The induction step is shown by straightforward calculations.

Before showing convergence in $C^{k-1}$, we show pointwise convergence:

Lemma 6.3 It holds pointwisely $T_i \to 0$ as $i \to \infty$.

Proof:
Let $x \in B_\delta$ and $\varepsilon > 0$. By the graph convergence there is an $N \in \mathbb{N}$, such that

$$ \| \bar{u}_i - u \|_{C^0 (B_{r_0})} < \frac{\varepsilon}{2} \quad \text{for all} \quad i > N, \quad (6.10) $$

where $u$ is the corresponding function of the limit graph system $\Gamma$. Let $y^i$ be the local representation of the point $f^i \circ \phi^i (x)$, that is $y^i = f(x) + \sum_{\nu=1}^k T^\nu_i (x) e^\nu (x)$, $y^i = (y^i_b, y^i_v) \in \mathbb{R}^m \times \mathbb{R}^k$. By construction of the mappings $\phi^i$, we have $y^i = (y^i_b, \bar{u}_i^j (y^i_v))$. We set $\varepsilon^i = | f^i \circ \phi^i (x) - f(x)|$. 

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The slope of \( u_\nu(x) \), and \( \| Du \|_{C^0(B_r)} \leq \alpha \) imply \( |\tilde{u}^i(y_i^k) - u(y_i^k)| > \frac{\varepsilon}{2} \). With (6.10) it follows \( \varepsilon^i < \varepsilon \) for all \( i > N \).

Hence \( \varepsilon^i = | \sum_{i=1}^k T^i_{\nu}(x)e^{\nu}(x) | \to 0 \) as \( i \to \infty \). As the vectors \( e^{\nu}(x), \ldots, e^{k}(x) \) are linearly independent, we finally conclude \( T^i(x) \to 0 \) as \( i \to \infty \).

Now we are able to show convergence in \( C^{k-1} \):

**Theorem 6.4** Under the assumptions at the beginning of this section, a subsequence of \( f^i \circ \phi^i \) converges in the \( C^{k-1} \)-topology to \( f \). In particular, it follows \( C^1 \)-convergence in the situation of Theorem 7.1.

**Proof:**
Let \( \gamma \) be as in Lemmas 6.1 and 6.2. We insert (6.8) into (6.9) and obtain

\[
\partial^\nu T^i = \sum_{\alpha=1}^m \partial_\alpha \tilde{u}_i^\nu(X^i) \cdot \left( \sum_{\nu=1}^k \partial^\nu T^i_{\nu} \cdot e^{\nu} + S_i^\nu \right) - \partial^\gamma u_i + U_i^\nu \tag{6.11}
\]

For \( 1 \leq \nu, \iota \leq k \) we define functions

\[
A_{\nu,\iota}^i : B_\delta \to \mathbb{R}, \quad A_{\nu,\iota}^i := \sum_{\alpha=1}^m \partial_\alpha \tilde{u}_i^\nu(X^i) \cdot e^{\nu},
\]

and

\[
B_{\iota}^{i,\gamma} : B_\delta \to \mathbb{R}, \quad B_{\iota}^{i,\gamma} := \sum_{\alpha=1}^m \partial_\alpha \tilde{u}_i^\nu(X^i) \cdot S_i^\nu - \partial^\gamma u_i + U_i^{\nu}.
\]

Inserting these functions into (6.11) yields

\[
\partial^\nu T^i = \sum_{\nu=1}^k A_{\nu,\iota}^i \cdot \partial^\nu T^i_{\nu} + B_{\iota}^{i,\gamma} \quad \text{for } 1 \leq \iota \leq k. \tag{6.12}
\]

For each \( x \in B_\delta \) we have \( \partial^\nu T^i(x) \in \mathbb{R}^k \), \( B_{\iota}^{i,\gamma}(x) \in \mathbb{R}^k \), and \( A^i(x) = (A_{\nu,\iota}^i(x)) \in \mathbb{R}^{k \times k} \). We obtain

\[
\partial^\nu T^i = A^i \cdot \partial^\nu T^i + B_{\iota}^{i,\gamma},
\]

hence almost everywhere

\[
|\partial^\nu T^i| \leq |A^i| \cdot |\partial^\nu T^i| + |B_{\iota}^{i,\gamma}| \leq \| A^i \|_{\text{op}} \| \partial^\nu T^i \| + |B_{\iota}^{i,\gamma}|, \tag{6.13}
\]

where \( \| \cdot \|_{\text{op}} \) denotes the operator norm. We write \( e^{\nu} = (e^{\nu}_h, e^{\nu}_k) \in \mathbb{R}^m \times \mathbb{R}^k \). By construction of the frame we have \( |e^{\nu}_h| \leq 2\alpha \leq 1 \), and by (6.1) moreover \( \| D\tilde{u}^i \|_{C^0(B_\delta \times \mathbb{R}^k)} \leq \frac{1}{2\sqrt{\alpha}} \). We estimate

\[
\| A^i \|^2_{\text{op}} \leq \sum_{\nu,\iota=1}^k \left( \sum_{\alpha=1}^m \partial_\alpha \tilde{u}_i^\nu(X^i) \cdot e^{\nu}_h \right)^2 = \sum_{\nu,\iota=1}^k \left( D\tilde{u}^i(X^i), e^{\nu}_h \right)^2 \leq \| D\tilde{u}^i \| \| e^{\nu}_h \|^2 \leq \frac{1}{4} \cdot k = \frac{1}{4}.
\]

Hence with (6.13) we obtain

\[
|\partial^\nu T^i| \leq 2|B_{\iota}^{i,\gamma}|. \tag{6.14}
\]

By Lemmas 6.1 and 6.2 \( B_{\iota}^{i,\gamma} \) only depends on derivatives of \( e^{1}, \ldots, e^{k}, u, \tilde{u}^i \) up to the order \( |\gamma| \), and on derivatives of \( T^i \) up to the order \( |\gamma| - 1 \). Moreover \( e^{1}, \ldots, e^{k} \) are fixed mappings, which are bounded
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in $C^k$, and $\nu$ is a fixed mapping, which is bounded in $W^{k,p}$ and hence also in $C^{k-1}$; the mappings $\tilde{u}^i$ are uniformly bounded in $W^{k,p}$ and hence also in $C^{k-1}$. From \[\text{Lemma 6.5}\] it follows that $T^i$ is uniformly bounded in $\mathcal{C}^0$. Hence for $|\gamma| = 1$, independently of $i$, we have the estimate $\|B^i\gamma\|_{L^\infty(B_h)} \leq K^1$, and inductively we obtain for $|\gamma| = I$, $1 < I < k$, the estimate $\|B^i\gamma\|_{L^\infty(B_h)} \leq K^I$, and finally for $|\gamma| = k$ the estimate $\|B^i\gamma\|_{L^\infty(B_h)} \leq K^k$. Hence $T^i$ is uniformly bounded in $W^{k,p}$. As for $p > m$ the space $W^{k,p}$ is compactly embedded in $C^{k-1}$, there is a function $T : B_\delta \to \mathbb{R}^k$, and a subsequence of $T^i$, which converges in $C^{k-1}$ to $T$. By \[\text{Lemma 6.3}\] we have $T \equiv 0$.

Hence we have locally

$$f + \sum_{\nu=1}^k T^\nu e^\nu \to f \text{ in } C^{k-1} \text{ as } i \to \infty,$$

which we wanted to show. \hfill \Box

Next we like to show that the limit immersion satisfies also the bounds for the second fundamental form and the volume. For that let $F : \mathbb{R}^N \times \mathbb{R}^{mN} \to \mathbb{R}$ be a function, where $N = mn$. For a domain $\Omega \subset \mathbb{R}^m$ and for each $W^{2,1}$-function $v : \Omega \to \mathbb{R}^n$ we define

$$\mathcal{F}(v) = \int_\Omega F(Dv, D^2v) d\mathcal{L}^m.$$ 

We first need the following lemma:

**Lemma 6.5** Suppose $F : \mathbb{R}^N \times \mathbb{R}^{mN} \to \mathbb{R}$ is continuous, nonnegative, and $F(\zeta, \cdot)$ is convex for every fixed $\zeta \in \mathbb{R}^N$. Then, if $v_i, v \in W^{2,1}(\Omega)$ and $Dv_i \to Dv$ in $L^1(\Omega)$, $D^2v_i \to D^2v$ weakly in $L^1(\Omega)$, it follows that

$$\mathcal{F}(v) \leq \liminf_{i \to \infty} \mathcal{F}(v_i).$$

**Proof:**

This is a special case of Theorem 1.6 in \[\text{[22]}\]. \hfill \Box

Now we come to the bounds for the limit, which is the final step in the proof of \[\text{Theorem 1.1}\].

**Theorem 6.6** The limit immersion $f : M \to \mathbb{R}^n$ satisfies $\|A(f)\|_{L^p(M)} \leq A$, $\text{vol}(M) \leq \mathcal{V}$, and moreover $q \in f(M)$ with $q$ as in Theorem \[\text{1.1}\].

**Proof:**

We consider the local representations $f \circ \varphi^{-1}_j : B_\delta \to \mathbb{R}^n$, $f^i \circ \phi^i \circ \varphi^{-1}_j : B_\delta \to \mathbb{R}^n$ (where $\varphi_j : P(B^j_\delta) \to B_\delta$ is a chart of the atlas $A$) and simply write $f$, $f^i \circ \phi^i$ for that. We consider the tensorial norm of $A$; for an immersion $f : B_\delta \to \mathbb{R}^n$ it is pointwisely given by

$$\|A\|^2 = \sum_{i,j,k,l=1}^m A_{ij} \cdot A_{kl} g^{ik} g^{jl},$$

where $G^{-1} = (g^{ij}) \in \mathbb{R}^{m \times m}$ is the inverse of $G = Df^i \cdot Df$, $A_{ij} = (\partial_{ij} f)^{-1}$ and $A_{ij} \cdot A_{kl}$ the Euclidean standard scalar product. Note that the projection onto the normal space only depends on $Df$ (more precisely $\pi^\perp = \text{Id} - Df G^{-1} Df$, see for instance \[\text{[4]}\], p. 555).

First we consider the $L^2$-norm of the second fundamental form. We set

$$F = F(Df, D^2f) := \|A\|^2.$$
6 Convergence of the immersions

and show, that $F$ satisfies the assumptions of Lemma 6.5.

For that we note, that $F$ is a homogeneous polynomial in the variables $\partial_j f_\mu$ (with $1 \leq i,j \leq m$, $1 \leq \mu \leq n$) of degree two, that is a sum of terms of the form $2c \partial_j f_\mu \partial_\mu f_\nu$ and $c (\partial_j f_\mu)^2$, where $c$ only depends on $Df$.

Now we write $F = F(\zeta, \xi)$, fix $\xi$ and calculate the Hessian $D^2F$ by the variables $\xi$. We obtain

$$D^2F(\zeta, \xi)(v,v) = 2F(\zeta, v) \geq 0.$$ 

Hence $F$ is convex in $\xi$ for each fixed $\zeta$. The chain rule implies convexity for the case $p > 2$. It remains to show convexity in the case of dimension $m = 1$ and $1 < p < 2$. For an immersion $f : (-\delta, \delta) \to \mathbb{R}^n$, the pointwise norm of $A$ simplifies to

$$\|A\| = \frac{1}{|Df|^2} |\pi^+ f''|,$$

where $\pi^+ = \text{Id} - DfG^{-1}Df^t$ as above. This time we set $F = F(Df, D^2f) := \|A\|$. Again we write $F = F(\zeta, \xi)$ and fix $\xi \neq 0$. Let $c := \frac{1}{|\zeta|^2}$, $C := \text{Id} - \frac{1}{|\zeta|^2} \zeta \zeta^t$. Then $F(\zeta, \xi) = c|C| \zeta$. Now let $\xi_1, \xi_2 \in \mathbb{R}^n$. Then for any $t$ with $0 \leq t \leq 1$, we conclude by the linearity of $C$ that

$$F(\zeta, (1-t)\xi_1 + t\xi_2) \leq (1-t)F(\zeta, \xi_1) + tF(\zeta, \xi_2).$$

Again, this shows that $F$ is convex in $\xi$ for each fixed $\zeta$. Then it is easily seen, that also

$$\|A\|^p = \frac{1}{|Df|^{2/p}} |\pi^+ f''|^p$$

is convex in $\xi$ for $p > 1$. In both cases ($p \geq 2$ and $1 < p < 2$) this implies the desired $L^p$-bound as follows:

In the convergence proof it is shown that the local representations of $f^i \circ \phi^i$ are uniformly bounded in $W^{2,p}$. Moreover $f^i \circ \phi^i$ converges in $C^1$ to $f$. As $W^{2,p}$ is reflexive, there exists a subsistence which converges weakly in $W^{2,p}$ and therefore also weakly in $W^{2,1}$ to $f$ (see e.g. [11], p. 220, Example 6.10 3)). Lemma 6.5 gives

$$\|A(f)\|_{L^p(P(B_{\delta}^j))} \leq \liminf_{i \to \infty} \|A(f^i \circ \phi^i)\|_{L^p(P(B_{\delta}^j))}$$

Note that above we have defined $F = F(\zeta, \xi)$ only for $\zeta \in \mathbb{R}^{n \times m} \approx \mathbb{R}^{nm}$ with rank $\zeta = m$; however Lemma 6.5 is also true under this restriction. Using a partition of unity we deduce

$$\|A(f)\|_{L^p(M)} \leq \liminf_{i \to \infty} \|A(f^i \circ \phi^i)\|_{L^p(M)} = \liminf_{i \to \infty} \|A(f^i)\|_{L^p(M^i)} \leq A.$$

For the volume we note $\text{vol}(P(B_{\delta}^j)) = \int_{B_{\delta}} \sqrt{\det g_{ij}} \, d\mathcal{L}^m$, where $G = (g_{ij}) \in \mathbb{R}^{m \times m}$ with $G = Df^i \cdot Df$. Now the bound on the volume of the limit manifold $M$ follows directly from $C^1$-convergence of the local representations. Finally we note that for each $i$ there is a point $p^i \in M^i$ with $f^i(p^i) = q$. Then the relation $q \in f(M)$ is obvious. \hfill \Box

Remark 6.7 With the compactness theorem (together with the lower semicontinuity of the norm of the second fundamental form) it is possible to derive existence theorems for minimizers of the $L^p$-norm of the second fundamental form, see [10], p. 224. Analogous results in the setting of integral rectifiable $m$-varifolds have been attained by J. Hutchinson in [17] and A. Mondino in [15].
7 Compactness for immersions on noncompact manifolds

We would like to conclude this section with some generalizations of Theorem 1.3. First we would like to show how to obtain higher order convergence, that is convergence in $C^{k-1}$ for $k \geq 2$ (again $k =$ degree of differentiability, $k =$ codimension). For that we assume in addition to the bounds

$$
\|\nabla^l A(f)\|_{L^\infty(M)} \leq A_l \quad \text{for any } l \text{ with } 0 \leq l \leq k-2,
$$

where $k \geq 2$. Here we assume that each immersion is sufficiently smooth, that is at least of class $C^k$. Additionally assume that also the mapping $\nu : M \to G_{n,k}$ we used to construct the diffeomorphisms $\phi^i$ is at least $C^k$. For $\alpha > 0$ choose $r > 0$ such that each immersion is an $(r, \alpha)$-immersion. We need a bound for higher derivatives of the graph functions $u$:

**Lemma 6.8** For $B_r \subset \mathbb{R}^m$ and $n = m+k$ let $f \in C^k(B_r, \mathbb{R}^n)$ be a mapping of the type $f(x) = (x, u(x)) \in \mathbb{R}^m \times \mathbb{R}^k$ with $u(0) = 0$. Suppose $\|Du\|_{C^0(B_r)} \leq \alpha < \infty$ and $\|\nabla^l A(f)\|_{L^\infty(B_r)} \leq A_l < \infty$ for any $l$ with $0 \leq l \leq k-2$. Then

$$
\|u\|_{C^0(B_r)} \leq C(r, \alpha, A_0, \ldots, A_{k-2})
$$

for a universal constant $C(r, \alpha, A_0, \ldots, A_{k-2}) < \infty$.

**Proof:**
We have $\|Du\|_{C^0(B_r)} \leq \alpha$, hence $\|u\|_{C^0(B_r)} \leq \alpha r$. The higher derivatives of $u$ are easily estimated by induction, see for instance Lemma 8.2 in the diploma thesis [20] for the case of codimension 1. The proof of the general case is left to the reader. \( \square \)

Starting from the situation in Lemma 6.8 the calculations in this section show that a subsequence of $f^i \circ \phi^i$ converges locally in $C^{k-1}$ to $f$. We will make use of this when proving higher order convergence in Theorem 1.3.

Finally we like to explain how to prove the theorem for immersions $f^i : M^i \to N$ with values in a complete Riemannian manifold $N$ (without boundary). Note that $N$ has to be complete, as otherwise we could take an open ball $N = B_1 \subset \mathbb{R}^n$ and construct a sequence of immersions converging to the boundary $\partial B_1$. We shall use the Nash embedding: Any Riemannian manifold $(N^n, g)$ can be isometrically embedded into $\mathbb{R}^n$, where $\nu = \nu(n)$. Let $\phi : N \to \mathbb{R}^\nu$ be such an embedding. Here we additionally assume that the second fundamental form of $\phi$ is bounded in $L^\infty$.

Let $f^i : M^i \to N$ be a sequence of immersions with $\|A(f^i)\|_{L^p(M^i)} \leq A$, $\text{vol}(M^i) \leq V$ and $q \in f^i(M^i)$ for a $q \in N$. Now consider the sequence $\phi \circ f^i : M^i \to \mathbb{R}^n$. Then $\phi \circ f^i : M^i \to \mathbb{R}^n$ is a sequence with $\|A(\phi \circ f^i)\|_{L^p(M^i)} \leq A'$, $\text{vol}(M^i) \leq V$ and $\phi(q) \in \phi \circ f^i(M^i)$. Hence we can apply the compactness theorem for immersions into $\mathbb{R}^n$. We obtain a limit immersion $f : M \to \mathbb{R}^{\nu}$ with $f(M) \subset \phi(N)$, and a subsequence of $\phi \circ f^i$ converging to $f$. Applying $\phi^{-1}$ to these mappings, we finally obtain a version of our compactness theorem for immersions with values in $N$. Again, one can formulate similar statements involving higher order convergence.

7 Compactness for immersions on noncompact manifolds

In the final section we want to prove Theorem 1.3 the compactness of proper immersions on manifolds which are not necessarily compact.

One of the technical main difficulties in the proof lies in fact that the norm of $A_i$ depends on $R$, that is

$$
\|A^i\|_{L^\infty(B_R)} \leq C_0(R).
$$

For that reason we do not have uniform estimates for the size of the radius $r$ of the function graphs as in (2.6) — we may only estimate $r$ for each fixed $R > 0$. This leads to the problem that we cannot
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directly apply Lemma 2.1(b) any more, which was of great importance for the proof in the compact case. This explains the need for some technical refinements which are carried out in the following.

Preparations for the noncompact case

First of all we have to adjust some definitions to the new situation. Let \( \mathbb{N} \) denote the integers greater than 0 and let \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Any sequence \((a_i)_{i \in \mathbb{N}_0}\) shall be denoted by \( a \). If \( a, b : \mathbb{N} \to \mathbb{R} \) are two sequences, then \( a < b \) if and only if \( a_i < b_i \) for all \( i \in \mathbb{N} \).

Let us assume here that the image \( f(M^i) \) of any immersion is unbounded in \( \mathbb{R}^n \) (as \( f^i \) is proper, this is the case whenever \( M^i \) is noncompact). If the \( f^i(M^i) \) are bounded uniformly in \( i \), then the proof of Theorem \( \ref{thm:compactness} \) applies. If the \( f^i(M^i) \) are bounded, but not uniformly in \( i \), then the statement is proven as in the unbounded case but with minor adaptations of the notation (for a more general formulation see Corollary \( \ref{cor:general} \)).

We are dealing here with balls both in \( \mathbb{R}^n \) and in \( \mathbb{R}^m \). For \( q > 0 \) let \( B_q \) denote the open ball of radius \( q \) in \( \mathbb{R}^m \) centered at the origin. Let \( \hat{B}_q \) denote the corresponding ball in \( \mathbb{R}^n \). For \( q \leq 0 \) we define \( \hat{B}_q = \emptyset \). Note that all balls \( B_R \) and \( \hat{B}_q \) in Theorem \( \ref{thm:compactness} \) are in fact balls in \( \mathbb{R}^n \) and should be written as \( \hat{B}_q \) and \( B_r \) in our new notation.

For a given immersion \( f : M \to \mathbb{R}^n \) and for \( p \in M \) we set

\[
\hat{p} = \hat{p}(f) := \min\{ j \in \mathbb{N} : f(p) \in \hat{B}_j \} \in \mathbb{N}.
\]

The notion of the \((r, \alpha)\)-immersion has to be adapted by replacing the real number \( r \) by a sequence. First we adapt the definition of \( U_{r,q} \):

**Definition 7.1** Let \( f : M \to \mathbb{R}^n \) be an immersion and let \( q \in M \). Let \( A_q \) and \( \pi \) be as in Section 4. Let \( r : \mathbb{N} \to \mathbb{R}_{>0} \) be a sequence. We define \( U_{r,q} \) to be the \( q \)-component of the set \( (\pi \circ A_q^{-1} \circ f)^{-1}(B_r) \).

With the preceding definition we come to the notion of an \((r, \alpha)\)-immersion:

**Definition 7.2** Let \( f : M \to \mathbb{R}^n \) be an immersion. Let \( r : \mathbb{N} \to \mathbb{R}_{>0} \) be a sequence and let \( \alpha > 0 \). We say that \( f \) is an \((r, \alpha)\)-immersion, if for each \( q \in M \) the set \( A_q^{-1} \circ f(U_{r,q}) \) is the graph of a \( C^1 \)-function \( u : B_{r_{q}} \to \mathbb{R}^k \) with \( \|Du\|_{C^0(B_{r_{q}})} \leq \alpha \).

Under the assumption that each immersion is proper, the condition \( \|A^i\|_{L^\infty(B_{\bar{r}})} \leq C_0(R) \) obviously implies, that for every \( \alpha > 0 \) there is a sequence \( r \) (which does not depend on \( i \)) such that each immersion \( f^i \) is an \((r, \alpha)\)-immersion. From now on \( r \) will always be a sequence. All sequences \( r \) and \( q \) with \( q \leq r \) are assumed to be greater than 0.

**Definition 7.3**

a) Let \( \nu : \mathbb{N}_0 \to \mathbb{N}_0 \) be a sequence. We say \( \nu \) is a subdivision, if \( \nu_0 = 0 \) and \( \nu \) is strictly increasing.

b) Let \( f : M \to \mathbb{R}^n \) be an \((r, \alpha)\)-immersion, \( \nu \) a subdivision and \( \delta \) a sequence with \( \delta < r \). Let \( Q = \{q_1, q_2, \ldots\} \) be a countable set of points in \( M \). We say \( Q \) is a \( \delta \)-net for \( f \) with subdivision \( \nu \), if for all \( i \in \mathbb{N} \) the following holds:

- \( f(q_k) \in B_{\bar{B}_{j-1}} \) for all \( k \) with \( \nu_{j-1} < k \leq \nu_j \),
- \( f^{-1}(\bar{B}_j) \subset \bigcup_{k=1}^{\nu_j} U_{s,q_k} \).

Next we have to adapt the definition of \((\mathfrak{S}^n, \delta)\) in order to handle graphs with different radii:
Lemma 7.6 Let \( f : \mathbb{N} \to \mathbb{R}^n \) be a decreasing sequence, \( u : \mathbb{N} \to \mathbb{R}^n \) a sequence with \( u_i = r_k \) for all \( i, k \in \mathbb{N} \) with \( \nu_k - 1 < i \leq \nu_k \). For \( j \in \mathbb{N} \cup \{ \infty \} \) with \( \nu_\infty := \infty \) we set

\[
\mathcal{G} = \mathcal{G}(r, \nu) = \{(A_i, u_i)_{i=1}^\nu : A_i : \mathbb{R}^n \to \mathbb{R}^n \text{ is a Euclidean isometry, } u_i \in C^1(B_{\gamma_i}, \mathbb{R}^k)\}. \tag{7.1}
\]

If \( \tilde{\Gamma} = (A_i, u_i)_{i=1}^\nu \in \mathcal{G} \), we define \( \tilde{\Gamma}_i := (A_i, u_i)_{i=1}^\nu \in \mathcal{G} \). Moreover, splitting \( A_i \) in a rotation \( R_i \in SO(n) \) and a translation \( T_i \in \mathbb{R}^n \), we set for \( j \in \mathbb{N} \)

\[
\vartheta(\cdot, \cdot) : \mathcal{G} \times \mathcal{G} \to \mathbb{R},
\]

\[
\vartheta(\Gamma, \tilde{\Gamma}) = \sum_{i=1}^\nu_i \left( \| R_i - \tilde{R}_i \| + \| T_i - \tilde{T}_i \| + \| u_i - \tilde{u}_i \|_{C^1(B_{\gamma_i})} \right). \tag{7.2}
\]

More accurately we should write \( \vartheta^j \) instead of \( \vartheta \), but we will maintain the notation without index \( j \). Again, as in the compact case, \( (\mathcal{G}, \vartheta) \) is a metric space. The reader should take care not to confuse the isometries \( A_j \) (of a fixed immersion) with the second fundamental forms \( A^i \) of the sequence of immersions \( f^i \).

Definition 7.5 Let \( f : M \to \mathbb{R}^n \) be an \((r, \alpha)\)-immersion, \( \delta < r \) a sequence and \( \nu \) a subdivision. Let \( Q = \{q_1, q_2, \ldots \} \) be a \( \delta \)-net for \( f \) with subdivision \( \nu \). As in the compact case we may assign to each \( q_j \in Q \) a neighborhood \( U_{r,q_j} \), a Euclidean isometry \( A_j \) and a \( C^\infty \)-function \( u_j : B_{r,q_j} \to \mathbb{R}^k \). We define

\[
\Gamma = \Gamma(f) = (A_j, u_j)_{j=1}^\nu \in \mathcal{G}. \tag{7.3}
\]

Furthermore for each \( j \in \mathbb{N} \) we define

\[
Z(j) := \{ l \in \mathbb{N} : U_{r,q_j} \cap U_{r,q_l} \neq \emptyset \} \in \mathcal{P}(\mathbb{N}). \tag{7.4}
\]

Lemma 7.6 Let \( f : M \to \mathbb{R}^n \) be an \((r, \alpha)\)-immersion with \( \alpha^2 < \frac{1}{3} \) and \( r_i \leq \frac{3}{2} \) for all \( i \in \mathbb{N} \). Let \( \delta \) be a sequence with \( \delta < r \) and let \( j \in \mathbb{N} \).

a) If \( p \in f^{-1}(\tilde{B}_{j+\frac{1}{2}} \setminus \tilde{B}_{j-\frac{1}{2}}) \), then \( U_{\delta,p} \subset f^{-1}(\tilde{B}_{j+\frac{1}{2}} \setminus \tilde{B}_{j-\frac{1}{2}}) \).

b) If \( p \in f^{-1}(\tilde{B}_{j}) \) and \( q \in M \setminus f^{-1}(\tilde{B}_{j+1}) \), then \( U_{\delta,p} \cap U_{\delta,q} = \emptyset \).

Proof:

a) Let \( x \in U_{\delta,p} \). With Lemma 2.7 a) we calculate

\[
|f(x) - f(p)| \leq (1 + \alpha^2) \delta_j \leq \frac{1}{2}.
\]

As \( j - 1 \leq |f(p)| < j \) it follows \( j - \frac{3}{2} \leq |f(x)| < j + \frac{1}{2} \). Hence \( U_{\delta,p} \subset f^{-1}(\tilde{B}_{j+\frac{1}{2}} \setminus \tilde{B}_{j-\frac{1}{2}}) \).

b) By part a) we have \( U_{\delta,p} \subset f^{-1}(\tilde{B}_{j+\frac{1}{2}}), U_{\delta,q} \subset f^{-1}(\mathbb{R}^n \setminus \tilde{B}_{j+\frac{1}{2}}) \), hence \( U_{\delta,p} \cap U_{\delta,q} = \emptyset \). \( \square \)

We obtain the following version of Lemma 2.7 b):

Lemma 7.7 Let \( f : M \to \mathbb{R}^n \) be an \((r, \alpha)\)-immersion with \( \alpha^2 < \frac{1}{3} \), where \( r \) is a decreasing sequence with \( r_i \leq \frac{3}{2} \). Let \( \delta^i \) be a decreasing sequence with \( \delta^i_j \leq \delta^i_{j+1} \) for all \( i \in \mathbb{N} \) and let \( \delta \) be a sequence with \( \delta_i \leq \frac{\delta^i_j}{2} \) for all \( i \in \mathbb{N} \).

a) If \( p, q \in M \) and \( U_{\delta,p} \cap U_{\delta,q} \neq \emptyset \), then \( U_{\delta,p} \subset U_{\delta',q} \subset U_{r,q} \).

b) If \( p, q \in M \) and \( U_{\delta',p} \cap U_{\delta',q} \neq \emptyset \), then \( U_{\delta',p} \subset U_{r,q} \).

c) If \( x, y, z \in M \) and \( U_{\delta,z} \cap U_{\delta,y} \neq \emptyset \), \( U_{\delta,y} \cap U_{\delta,z} \neq \emptyset \), then \( U_{\delta,z} \subset U_{r,z} \).
Proof:

a) Let \( j := \bar{p}, k := \bar{q} \), in other words \( p \in f^{-1}(\hat{B}_j \setminus \hat{B}_{j-1}), q \in f^{-1}(\hat{B}_k \setminus \hat{B}_{k-1}) \). As \( U_{\delta,q} \cap U_{\delta,p} \neq \emptyset \), Lemma 7.6 b) implies \( |j - k| \leq 1 \). Let \( \epsilon = \min\{j, k\} \) and let \( \zeta \in U_{\delta,p}, \xi \in U_{\delta,q} \cap U_{\delta,p} \). With \( \varphi_q = \pi \circ A_q^{-1} \circ f \), using \( \delta_i \leq \frac{\epsilon_i}{4} \) and the fact that \( \delta' \) is decreasing, we estimate

\[
|\varphi_q(\zeta)| \leq |f(\zeta) - f(q)| \leq |f(\zeta) - f(p)| + |f(p) - f(q)| + |f(\xi) - f(q)| \leq 3(1 + \alpha^2) \frac{\delta_i}{4} < \delta_{i+1}.
\]

Hence \( U_{\delta,p} \subset \varphi_q^{-1}(B_{\delta_i+4}) \). As \( \bar{q} \in \{i, i+1\} \) and as \( \delta' \) is decreasing, we conclude \( U_{\delta,p} \subset \varphi_q^{-1}(B_{\delta_i}) \). But \( U_{\delta,p} \cup U_{\delta,q} \) is a connected set containing \( q \) and is hence included in the \( q \)-component of the set \( \varphi_q^{-1}(B_{\delta_i}) \), that is in \( U_{\delta,q} \). Hence \( U_{\delta,p} \subset U_{\delta,q} \). The relation \( U_{\delta,q} \subset U_{\delta,q} \) is obvious.

b) The proof of the second part runs as before.

c) As \( \delta < \delta' \), the relation \( U_{\delta,x} \cap U_{\delta,y} \neq \emptyset \) implies \( U_{\delta,x} \cap U_{\delta,y} \neq \emptyset \). By part a) we have \( U_{\delta,z} \subset U_{\delta,y} \) and by part b) \( U_{\delta,y} \subset U_{\delta,x} \).

Remark 7.8 Let \( f : M \to \mathbb{R}^n \) be an \((r,\alpha)\)-immersion with \( \alpha^2 < \frac{1}{4} \), \( \delta < r \) a sequence and \( p, q \in M \) with \( U_{\delta,q} \cap U_{\delta,p} \neq \emptyset \). Then, under the additional assumption \( p, q \in f^{-1}(\hat{B}_j \setminus \hat{B}_{j-1}) \) for \( j \in \mathbb{N} \), we may apply Lemma 7.6 b) and obtain

\[
U_{\delta,q} \subset U_{\delta,p}. \tag{7.5}
\]

This will be used in the proof of the following lemma.

Lemma 7.9 Let \( f^i : M^i \to \mathbb{R}^n \) be a sequence as in Theorem 7.3. Moreover let \( \alpha > 0 \) with \( \alpha^2 < \frac{1}{4} \) and \( r \) a sequence with \( r_i \leq \frac{3}{4} \) for all \( i \in \mathbb{N} \), such that each \( f^i \) is an \((r,\alpha)\)-immersion. Let \( \delta \) be a sequence with \( \delta < r \). Then there exists a fixed subdivision \( \nu \), such that the following holds:

a) Each immersion \( f^i \) admits a \( \delta \)-net \( Q^i \) with subdivision \( \nu \).

b) For each \( j \in \mathbb{N} \) let \( Z^i(j) \) be the set corresponding to \( Q^i \) as defined in (7.4). Then, after passing to a subsequence, for each \( j \in \mathbb{N} \) there exists a finite set \( Z(j) \subset \mathbb{N} \), such that

\[
Z^i(k) = Z(k) \text{ for all } k \leq \nu_i.
\]

Proof:

a) We fix \( i, j \in \mathbb{N} \). Now consider the immersion \( f^i \). Let \( q^i_1 \in (f^i)^{-1}(\hat{B}_j \setminus \hat{B}_{j-1}) \). Assume we have found points \( \{q^i_1, \ldots, q^i_l\} \) in \((f^i)^{-1}(\hat{B}_j \setminus \hat{B}_{j-1})\) with the property \( U_{\delta,q^i_1} \cap U_{\delta,q^i_l} = \emptyset \) for \( a \neq b \). Suppose \( U_{\delta,q^i_1} \cup \ldots \cup U_{\delta,q^i_l} \) does not cover \((f^i)^{-1}(\hat{B}_j \setminus \hat{B}_{j-1})\). Then choose a point \( q^i_{l+1} \in (f^i)^{-1}(\hat{B}_j \setminus \hat{B}_{j-1}) \) from the complement. Then \( U_{\delta,q^i_{l+1}} \cap U_{\delta,q^i_{l+1}} = \emptyset \) for \( k \leq \nu_i \), as otherwise \( U_{\delta,q^i_{l+1}} \subset U_{\delta,q^i_1} \) by (7.4). Using (7.4) in the first line and Lemma 7.6 a) in the second, we estimate

\[
C(j + 1) \geq \mu^i(\hat{B}_{j+\frac{1}{2}} \setminus \hat{B}_{j-\frac{1}{2}}) \geq \sum_{k=1}^{s} \mu^i(U^i_{\delta,q^i_k}) \geq \sum_{k=1}^{s} C^m(B_{\delta_k}) \geq s \left( \frac{\delta_k}{4} \right)^m.
\]
7 Compactness for immersions on noncompact manifolds

Therefore, with \(|x| := \max \{n \in \mathbb{N}_0 : n \leq x\}\) for \(x \geq 0\), this procedure yields after at most \([\left(\frac{8}{3}\right)^m C(j+1)]\) steps a cover of \((f^i)^{-1}(B_j \backslash B_{j-1})\). Now define the subdivision \(\nu\) recursively as follows:

\[
\nu_0 := 0, \quad \nu_j := \nu_{j-1} + \left[ \frac{4}{\delta_j} m C(j+1) \right] \quad \text{for } j \geq 1.
\]

By the considerations of above we may choose for all \(i, j \in \mathbb{N}\) exactly \(\nu_j - \nu_{j-1}\) points \(q_{\nu_{j-1}+1}, \ldots, q_{\nu_j}\) in \((f^i)^{-1}(B_j \backslash B_{j-1})\), such that \((f^i)^{-1}(B_j) \subset \bigcup_{k=1}^{\nu_j} U^{i}_{\delta, q_k}\).

b) Fix \(j \in \mathbb{N}\). Let \(k \leq \nu_j\). If \(l > \nu_j+1\) then \(U^{i}_{\delta, q_k} \cap U^{i}_{\delta, q_l} = \emptyset\) by Lemma 7.6 b), hence \(l \notin Z^j(k)\) for each \(i \in \mathbb{N}\). But this means \(Z^j(k) \subset \{1, \ldots, \nu_j+1\}\), hence \(\|Z^j(k) : i \in \mathbb{N}\| \leq \|\mathcal{P}(\{1, \ldots, \nu_j+1\})\| = 2^{\nu_j+1}\). Hence we may pass to a subsequence \((f^{i_k})_{i \in \mathbb{N}}\) of \((f^i)_{i \in \mathbb{N}}\) with

\[
Z^{i_k}(k) = Z(k) \quad \text{for all } k \leq \nu_j \text{ and all } i \in \mathbb{N}.
\]

Choosing successively subsequences for any \(j \in \mathbb{N}\) and passing to the diagonal sequence, we obtain a subsequence with the desired property. (Note that the sets \(Z(k)\) do not depend on \(j\), if passing successively to subsequences.)

Lemma 7.10 Let \(f^i : M^i \to \mathbb{R}^n\) be a sequence as in Theorem 4.3 and \(r\) a sequence, such that each immersion \(f^i\) is an \((r, \alpha)\)-immersion. Let \(\delta < r\) be another sequence and \(\nu\) a subdivision. Let \(Q^i\) be \(\delta\)-nets for \(f^i\) with subdivision \(\nu\) and let \(\Gamma^i \in \mathcal{G}_\infty\) be as in (7.3). Then, after passing to a subsequence, there exists a graph system \(\Gamma \in \mathcal{G}_\infty\), such that for all \(j \in \mathbb{N}\)

\[
\Gamma^i \to \Gamma_j \quad \text{in } (\mathcal{G}_j, \mathcal{B}) \quad \text{as } i \to \infty.
\]

Proof:
Fix \(j \in \mathbb{N}\). With the arguments of Theorem 3.3 in [16], there exists a graph system \(\tilde{\Gamma}^j \in \mathcal{G}_j\) and a subsequence \((\tilde{\Gamma}^i_j)_{i \in \mathbb{N}}\) of \((\Gamma^i)_{i \in \mathbb{N}}\), such that

\[
\tilde{\Gamma}^i_j \to \tilde{\Gamma}^j \quad \text{in } (\mathcal{G}_j, \mathcal{B}) \quad \text{as } i \to \infty.
\]

By successively choosing subsequences for any \(j \in \mathbb{N}\) and passing to the diagonal sequence, we obtain a sequence with \(\Gamma^i_j \to \Gamma^j\) in \((\mathcal{G}_j, \mathcal{B})\) as \(i \to \infty\) for all \(j \in \mathbb{N}\). Moreover, if \(\tilde{\Gamma}^k = (\tilde{A}^k_j, \tilde{u}^k_j)_{j=1}^{\nu_k}\) and \(\tilde{\Gamma}^l = (\tilde{A}^l_j, \tilde{u}^l_j)_{j=1}^{\nu_l}\) with \(k \leq l\), we observe \((\tilde{A}^k_j, \tilde{u}^k_j) = (\tilde{A}^l_j, \tilde{u}^l_j)\) for \(j \leq \nu_k\). Define \(\Gamma = (A_j, u_j)_{j=1}^{\infty} \in \mathcal{G}_\infty\) by setting \((A_j, u_j) := (\tilde{A}^k_j, \tilde{u}^k_j)\) for an arbitrary \(k\) with \(\nu_k \geq j\). Hence \(\Gamma^i_j \to \Gamma^j\) for each \(j \in \mathbb{N}\), which completes the proof.

Construction of the limit manifold and immersion
Let \(f^i : M^i \to \mathbb{R}^n\) be a sequence of immersions as in Theorem 4.3. All constants have to be chosen such that all arguments of the compact case can be used. Let \(a > 0\) with \(a \leq \frac{1}{\sqrt{4m}}\) as in (6.1), in particular \(a^2 \leq \frac{1}{16}\) as in Section 8. Let \(r\) be a decreasing sequence with \(r_1 = \frac{3}{\sqrt{4}}\), such that each \(f^i\) is an \((r, \alpha)\)-immersion. Let the sequence \(\delta^i\) be defined by \(\delta^i = \frac{r_{i+1}}{a}\) for all \(i \in \mathbb{N}\), and \(\delta\) be defined by \(\delta_i = \frac{a \alpha^2}{4}\) for all \(i \in \mathbb{N}\), that is \(\delta_i = \frac{\delta_{i+1}}{1}\). Finally let \(\delta\) be the sequence defined by \(\delta_i = \delta_{i+1}\) for all \(i \in \mathbb{N}\). By Lemma 7.6 a) there exist a fixed subdivision \(\nu\) and countable subsets \(Q^i = \{q_1^i, q_2^i, \ldots\} \subset M^i\), such that each \(Q^i\) is a \(\frac{4}{\delta}\)-net for \(f^i\) with subdivision \(\nu\). Similar to the compact case, we have to use \(\frac{4}{\delta}\)-nets and not only \(\delta\)-nets (in the compact case we used \(\frac{4}{\delta}\)-nets). Moreover, by Lemma 7.6 b), we may pass to a subsequence such that for each \(j \in \mathbb{N}\) there exists a finite set \(Z(j) \subset N\) with

\[
Z^j(k) = Z(k) \quad \text{for all } k \leq \nu.j.
\]

(7.6)
(Here \( Z^i(k) := \{ l \in \mathbb{N} : U^i_{\delta,\ell_k} \cap U^i_{\delta,\ell_l} \neq \emptyset \} \) as in (7.4), that is \( Z^i(k) \) is not defined as \( \{ l \in \mathbb{N} : U^i_{\delta/10,q_k} \cap U^i_{\delta/10,q_l} \neq \emptyset \} \). Nevertheless, as any \( \frac{\delta}{10} \)-net is also a \( \delta \)-net, (7.6) holds.)

By Lemma 7.10 after passing to another subsequence, there exists a graph system \( \Gamma = (A_i, u_i)_{i=1}^\infty \in \mathcal{G}_\infty \), such that for each \( j \in \mathbb{N} \)

\[
\Gamma^j_i \to \Gamma^j_j \text{ in } (\mathcal{G}^j, \mathcal{D}) \text{ as } i \to \infty.
\]

We come to the construction of the limit manifold and limit immersion:

Let \( \rho \) be a sequence with \( \rho_j = \delta_k \) for all \( j, k \in \mathbb{N} \) with \( \nu_k-1 < j \leq \nu_k \). We define \( B^j_k := B_{\rho_j} \times \{ j \} \). The set \( \bigcup_{j=1}^\infty B^j_k \), endowed with the disjoint union topology, is a second countable space. Again we define a relation \( \sim \) on \( \bigcup_{j=1}^\infty B^j \). For \( (x, j), (y, k) \in \bigcup_{j=1}^\infty B^j \) we set

\[
(x, j) \sim (y, k) \iff [k \in Z(j) \text{ and } A_j(x, u_j(x)) = A_k(y, u_k(y))].
\]

Here the sets \( Z(j) \) shall be the fixed sets from (4.8).

To simplify the notation, for any sequence \( \rho \) with \( 0 < \rho \leq r \) we set

\[
U^{i,\rho} := U^{i,\rho}_{\delta,\ell_k}.
\]

Now observe that the construction of the limit manifold \( M \) can be performed in exactly the same manner as in the compact case. For that we note that the sequence \( r \) in the present case corresponds to the number \( r \) in the compact case. Similarly, the sequence \( \delta' \) corresponds to the number \( \frac{\delta}{10} \), the sequence \( \delta \) to the number \( \delta = \frac{\delta}{10} \), the sequence \( \frac{\delta}{10} \) to the number \( \frac{\delta}{100} \).

Lemma 2.8 is replaced by Lemma 7.7. By part c) of the latter, even the iterated case works. Moreover, in the compact case it was crucial that the sets \( Z^i(k) \) do not depend on \( i \), that is \( Z^i(k) = Z(k) \). This is replaced by Lemma 7.7 b), which ensures for fixed \( k \in \mathbb{N} \) that \( Z^i(k) = Z(k) \) for \( i \) sufficiently large. In the compact case, all arguments involving \( Z^i(k) = Z(k) \) were either needed for the construction of the limit, for which it is sufficient to consider \( i \) large, or for the reparametrizations \( \phi^i : M \to M^i \), which are in the present case replaced by diffeomorphisms \( \phi^i \) : \( U^i \to (f^i)^{-1}(B_t) \subset M^i \) for which the property (4.8) suffices. For the same reasons the convergence of graph systems \( \Gamma^j_i \to \Gamma^j_j \) for any \( j \in \mathbb{N} \), replacing \( \Gamma^i \to \Gamma^j \), is sufficient for our proof.

Following step by step the arguments of Lemma 4.4, we see that \( \sim \) defines an equivalence relation on \( \bigcup_{j=1}^\infty B^j_k \). Again we set \( M = (\bigcup_{j=1}^\infty B^j_k)/\sim \). We construct an atlas \( \mathfrak{A} \) as in Section 4 with charts \( \varphi^j_{\delta} : P(V^j) \to V \) for \( V \subset B_{\rho_j} \).

Similarly, we may follow the arguments of Lemmas 4.2, 4.3 and 4.4 stating that the quotient projection \( P : \bigcup_{j=1}^\infty B^j_k \to M \) is open, that \( M \) is a second countable Hausdorff space and \( \mathfrak{A} \) a differentiable atlas on \( \hat{M} \). Hence \( (M, \mathfrak{A}) \) induces uniquely the structure of a differentiable manifold.

Finally we define a smooth immersion on \( M \) by

\[
f : M \to \mathbb{R}^n, \quad [(x, j)] \mapsto A_j(x, u_j(x)),
\]

where \([(x, j)]\) denotes the equivalence class of \((x, j)\).
We have the following versions of Lemmas 4.7 and 4.8:

- It holds $M = \bigcup_{j=1}^{\infty} P(B_j^{1/6}).$

- If $P(B_j^{1/4}) \cap P(B_k^{1/4}) \neq \emptyset$, then $P(B_j^{1/4}) \subset P(B_k^{1/4}).$

As in the compact case, let us define sets $\tilde{Z}_i(j) = \{ l \in \mathbb{N} : U_i^{\delta/5,j} \cap U_i^{\delta/5,l} \neq \emptyset \}$.

With the arguments of Lemma 7.9 b), we may pass to a subsequence, such that $\tilde{Z}_i(k) = \tilde{Z}(k)$ for all $k \leq \nu_i$ for fixed finite sets $\tilde{Z}(k) \subset \mathbb{N}$.

Finally, the following version of Lemma 4.9 holds:

- If $P(B_j^{1/4}) \cap P(B_k^{1/4}) = \emptyset$, then $k \notin \tilde{Z}(j)$.

As additional lemma we have

**Lemma 7.11** The immersion $f : M \rightarrow \mathbb{R}^n$ is proper.

**Proof:**

Let $K \subset \mathbb{R}^n$ be compact. Then there is a $j \in \mathbb{N}$ with $K \subset B_j$. Let $x \in f^{-1}(B_j)$. As $M = \bigcup_{j=1}^{\infty} P(B_j^{1/2})$, there is a $k \in \mathbb{N}$ with $x \in P(B_k^{1/2})$. It holds $f(P(B_k^{1/2})) = A_k(\{(y,u_k(y)) : y \in B_{\rho_k/2}\})$.

With the argument of Lemma 7.6 a) we conclude $f(P(B_k^{1/2})) \subset B_{j+1}$. As $f^i(q_{j+k}^i) = A_k(0,u_k^i(0)) \rightarrow A_k(0,u_k(0))$, we conclude $f^i(q_{j+k}^i) \subset B_{j+1}$ for $i$ sufficiently large. As $Q^i$ is a $\delta$-net for $f^i$ with subdivision $\nu$, we conclude $k \leq \nu_{j+1}$. Hence

$$f^{-1}(K) \subset f^{-1}(B_j) \subset \bigcup_{l=1}^{\nu_{j+1}} P(B_l^{1/2}) \subset \bigcup_{l=1}^{\nu_{j+1}} P(B_l^{1/2}),$$

that is $f^{-1}(K)$ is a subset of a compact set. As $f$ is an immersion, it is also continuous, hence $f^{-1}(K)$ closed. But closed subsets of compact sets are compact. \hfill $\square$

Note that the limit manifold $M$ does not need to be connected, even if all manifolds $M^i$ are connected. A simple counterexample is given in Figure 6.

![Figure 6](image-url)

**Figure 6** The limit manifold $M$ does not need to be connected, even if all $M^i$ are connected. The limit $f(M)$ in the example are two parallel lines.

Let $\nu_f : M \rightarrow G_{n,k}$ denote the Gauss normal map with respect to the immersion $f$. With Lemma 6.8 we may conclude (as in Step 6 below) that the limit $f$ is in $C^\infty$. Hence also $\nu_f$ is in $C^\infty$. We come to the proof of our second main theorem.
Proof of Theorem 7.3:

Step 1: Definition of maps $\varphi^i$

First we define maps $\varphi^i : \bigcup_{j=1}^{\nu_i} P(B_{5j}^i) \to \bigcup_{j=1}^{\nu_i} U_{r,j}^i$ as follows:

Let $i \in \mathbb{N}$ be fixed. With the arguments of the compact case, we may choose $a_i \in \mathbb{N}$ sufficiently large with $a_i \geq i$, such that for all $j \leq \nu_i$ and all $x \in P(B_{5j}^i)$ the affine space $h(x) := f(x) + \nu_f(x)$ intersects $f^a(U_{r,j}^i)$ in exactly one point $S_x$ and that this point lies in $f^a(U_{r,j}^i)$. Furthermore there is exactly one point $\sigma_x \in U_{r,j}^i$ with $f^a(\sigma_x) = S_x$. We define

$$\varphi^a_i : \bigcup_{j=1}^{\nu_i} P(B_{5j}^i) \to \bigcup_{j=1}^{\nu_i} U_{r,j}^i \subset M^a,$$

$$x \mapsto \sigma_x.$$

We like to show that $\varphi^a_i$ is well-defined: Suppose $x$ also lies in $P(B_{5j}^i)$ for a $k \leq \nu_i$. Then $h(x)$ intersects $f^a(U_{r,j}^i)$ in exactly one point $S'_x$ and there is exactly one $\sigma'_x \in U_{r,j}^i$ with $f^a(\sigma'_x) = S'_x$. Again we have $S'_x \in f^a(U_{r,j}^i)$ and $\sigma'_x \in U_{r,j}^i$. As $a_i \geq i$, by (7.6) we have $Z^a(j) = Z(j)$ for all $j \leq \nu_i$. The relation $P(B_{5j}^i) \cap P(B_{5k}^i) \neq \emptyset$ implies $k \in Z(j)$. Hence $U_{r,j}^i \cap U_{r,k}^i \neq \emptyset$ and by Lemma 7.7 b) $U_{r,k}^i \subset U_{r,j}^i$. As $h(x)$ intersects $f^a(U_{r,j}^i)$ in exactly one point, we conclude $S'_x = S_x$. Hence $\varphi^a_i$ is well-defined.

In that way we define for any $i \in \mathbb{N}$ a map $\varphi^a_i$. Moreover we may choose $(a_i)_{i \in \mathbb{N}}$ to be strictly increasing. Passing to the subsequence $f^a_i$ and simply writing $f^i$ for it, we obtain maps $\varphi^i : \bigcup_{j=1}^{\nu_i} P(B_{5j}^i) \to \bigcup_{j=1}^{\nu_i} U_{r,j}^i$.

Step 2: $\varphi^i(P(B_{5j}^i)) \subset U_{r,j}^i \subset \varphi^i(P(B_{5j}^i)) \subset \varphi^i(P(B_{5j}^i)) \subset U_{r,j}^i$

Fix $i \in \mathbb{N}$. As in Lemmas 5.4 and 5.5 using convergence of graph systems, we may choose $a_i \in \mathbb{N}$ sufficiently large such that for all $j \leq \nu_i$

- $\varphi^a_i(P(B_{5j}^i)) \subset U_{r,j}^i$,
- $\varphi^a_i(P(B_{5j}^i)) \subset U_{r,j}^i$,
- $U_{r,j}^i \subset \varphi^a_i(P(B_{5j}^i))$.

Doing so for all $i \in \mathbb{N}$, we may choose $(a_i)_{i \in \mathbb{N}}$ to be strictly increasing. Denoting the subsequences $f^a_i$ and $\varphi^a_i$ simply by $f^i$ and $\varphi^i$, we obtain a sequence with

$$\varphi^i(P(B_{5j}^i)) \subset U_{r,j}^i \subset \varphi^i(P(B_{5j}^i)) \subset \varphi^i(P(B_{5j}^i)) \subset U_{r,j}^i \quad \text{for all } j \leq \nu_i.$$

Step 3: Construction of diffeomorphisms $\phi^i$

Consider the maps $\varphi^i : \bigcup_{j=1}^{\nu_i} P(B_{5j}^i) \to \bigcup_{j=1}^{\nu_i} U_{r,j}^i$. By Step 2 we have $U_{r,j}^i \subset \varphi^i(P(B_{5j}^i))$ for any $j \leq \nu_i$. As $Q^i$ is also a $\frac{r}{r}$-net for $f^i$, we conclude

$$(f^i)^{-1}(B_i) \subset \bigcup_{j=1}^{\nu_i} U_{r,j}^i.$$
so in particular

\[(f^i)^{-1}(\hat{B}_i) \subset \varphi^i((\bigcup_{j=1}^{\nu_i} P(B^{j}_{\delta/3}))). \quad (7.7)\]

Define \(U^i := (\varphi^i)^{-1}((f^i)^{-1}(\hat{B}_i))\). As \(f^i\) and \(\varphi^i\) are continuous, we conclude that \(U^i\) is open. Restricting \(\varphi^i\) to \(U^i\) yields maps \(\varphi^i : U^i \to (f^i)^{-1}(\hat{B}_i)\).

By definition \(\varphi^i\) is surjective. By Step 2 we have \(\varphi^i(P(B^{j}_{\delta/3})) \subset U^j_{\delta/2,j}\) for \(j \leq \nu_i\). Moreover, we know by Step 2 that \(\varphi^i(P(B^{j}_{\delta/6})) \subset U^j_{\delta/5,j}\) for every \(j \leq \nu_i\), which enables us to follow the arguments of Lemma 5.7 in order to conclude that \(\varphi^i\) is injective on all of \(\bigcup_{j=1}^{\nu_i} P(B^{j}_{\delta/3})\). Hence also \(\varphi^i\) is injective on \(U^i\).

With the arguments of Lemma 5.8 we may show that \(f^i \circ \varphi^i\) are immersions and finally that \(\varphi^i : U^i \to (f^i)^{-1}(\hat{B}_i)\) are diffeomorphisms.

**Step 4:** \(U^i \subset U^{i+1}\) and \(\bigcup_{i=1}^{\infty} U^i = M\)

First observe that by Step 2 for \(1 \leq j \leq \nu_i\), as \(\nu_i \leq \nu_{i+1}\) we have

\[\varphi^{i+1}(x) \in U^i_{\delta/2,j} \quad \text{for} \quad x \in P(B^{j}_{\delta/3}).\]

Hence by Lemma 7.6(a)

\[f^{i+1} \circ \varphi^{i+1}(x) \in \hat{B}_{i+1} \quad \text{for} \quad x \in \bigcup_{j=1}^{\nu_i} P(B^{j}_{\delta/3}).\]

By construction of the sets \(U^i\) this means

\[\bigcup_{j=1}^{\nu_i} P(B^{j}_{\delta/3}) \subset U^{i+1}.\]

Moreover, by (7.7) and as \(\varphi^i\) is injective on \(\bigcup_{j=1}^{\nu_i} P(B^{j}_{\delta/3})\), we conclude

\[U^i \subset \bigcup_{j=1}^{\nu_i} P(B^{j}_{\delta/3}),\]

hence

\[\overline{U^i} \subset \bigcup_{j=1}^{\nu_i} P(B^{j}_{\delta/3}) \subset U^{i+1}.\]

Moreover we observe

\[\bigcup_{i=1}^{\infty} U^i = \bigcup_{i=0}^{\infty} U^{i+1} \supset \bigcup_{i=0}^{\nu_i} P(B^{j}_{\delta/3}) \supset \bigcup_{i=1}^{\infty} P(B^{j}_{\delta/3}) = M,\]

hence \(\bigcup_{i=1}^{\infty} U^i = M\).
A short technical remark: If we pass another time to subsequences \( f^{a_i}, \phi^{a_i} \) of \( f^i, \phi^i \) (as will be done in Step 5 and 6), we shall restrict \( \phi^{a_i} \) to \( V^{a_i} := (\varphi^{a_i})^{-1}(f^{a_i})^{-1}(\hat{B}_l) \subset U^{a_i} \). Simply writing \( f^i, \phi^i, V^i \) instead of \( f^{a_i}, \phi^{a_i}, V^{a_i} \) and after that \( U^i \) instead of \( V^i \), we again obtain a sequence of sets \( U^i \) with \( U^i \subset U^{i+1} \) and \( \bigcup_{i=1}^\infty U^i = M \). Conversely we could also first choose all subsequences and define the sets \( U^i \) afterwards.

**Step 5: A subsequence with \( f^i \circ \phi^i \to f \) in \( C^0 \)**

For \( i \geq k \) we set \( \Theta_k^i := \| f^i \circ \varphi^i - f \|_{C^0(U^{i+1}_l, P(B_l^i))} \). Now we fix \( k \in \mathbb{N} \). By the convergence argument of the compact case there is a subsequence \( b_i \) with

\[
\Theta_k^{b_i} \to 0 \quad \text{as} \quad i \to \infty.
\]

Hence we may choose a strictly increasing sequence \( a : \mathbb{N} \to \mathbb{N} \) (in particular \( a_i \geq i \) for all \( i \in \mathbb{N} \)) with

\[
\Theta_k^{a_i} \leq \frac{1}{i}.
\]

Passing to the subsequence \( f^{a_i} \circ \varphi^{a_i} \) and denoting this sequence simply by \( f^i \circ \phi^i \), we obtain a sequence with

\[
\Theta_i^{b_i} = \| f^i \circ \varphi^i - f \|_{C^0(U^{i+1}_l, P(B_l^i))} < \frac{1}{i} \quad \text{as} \quad i \to \infty.
\]

Restricting \( \varphi^i \) to \( U^i \) and using the definition of \( \phi^i \), we finally obtain

\[
\| f^i \circ \phi^i - f \|_{C^0(U^i)} \to 0 \quad \text{as} \quad i \to \infty.
\]

**Step 6: Higher order convergence**

We like to find another subsequence such that \( f^i \circ \phi^i \to f \) locally smoothly. This means convergence with respect to the weak topology \( C_o^\infty(M, \mathbb{R}^n) \) as defined in [12], p. 34–36, which in our case is the same as convergence of \( f^i \circ \phi^i \circ \varphi^{-1} \) to \( f \circ \varphi^{-1} \) in \( C^k(\varphi(U), \mathbb{R}^n) \) for any chart \((\varphi, U)\) of the atlas \( \mathfrak{A} \) constructed above and for any \( k \in \mathbb{N} \).

Let \( \bar{\rho} \) be a sequence with \( \bar{\rho}_j = r_l \) for all \( j, l \in \mathbb{N} \) with \( \nu_{l-1} < j \leq \nu_l \). Then for \( j \leq \nu_l \), using \( \| \nabla^k A^i \|_{L^\infty(\hat{B}_{l+1})} \leq C_k(l+1) \), Lemma [6.8] implies

\[
\| u^i_j \|_{C^{k+1}(\bar{B}_{\bar{\rho}_j})} \leq C_k^k, \tag{7.8}
\]

where \( C_k^k \) is a constant depending on \( r_1, \alpha, C_0(l+1), \ldots, C_k(l+1) \) (here we use \( \bar{\rho}_j \leq r_1 \) as \( r \) is assumed to be decreasing).

Again let \( \rho \) be a sequence with \( \rho_j = \delta_l \) for all \( j, l \in \mathbb{N} \) with \( \nu_{l-1} < j \leq \nu_l \). Using (7.8) and Theorem [6.4], we may choose successively subsequences for any \( l \) and pass to the diagonal sequence in order to obtain

\[
f^i \circ \phi^i \circ \varphi^{-1}_j \to f \circ \varphi^{-1}_j \quad \text{in} \quad C^1(B_{\rho_j}, \mathbb{R}^n)
\]

for all \( j \in \mathbb{N} \) and charts \( \varphi_j := \varphi_{B_{\rho_j}} : P(B_{\rho_j}^i) \to B_{\rho_j} \).

Starting from this sequence, we may choose again successively subsequences for any \( k \) and pass to the diagonal sequence in order to obtain

\[
f^i \circ \phi^i \circ \varphi^{-1}_j \to f \circ \varphi^{-1}_j \quad \text{in} \quad C^k(B_{\rho_j}, \mathbb{R}^n)
\]

for all \( j \in \mathbb{N} \) and all \( k \in \mathbb{N}_0 \).
Step 7: Bounds for the limit

We like to show that
\[ \mu(\tilde{B}_R) \leq C(R) \quad \text{for any } R > 0, \]
\[ \|\nabla^k A\|_{L^\infty(\tilde{B}_R)} \leq C_k(R) \quad \text{for any } R > 0 \text{ and } k \in \mathbb{N}_0, \]
where \( \mu = f(\mu_y) \) and \( A \) is the second fundamental form of \( f \).

For the first inequality let \( R > 0 \). Let \( \varepsilon > 0 \). Choose \( \tilde{R} \) with \( 0 < \tilde{R} < R \) and
\[ \mu(\tilde{B}_R) \leq \mu(\tilde{B}_{\tilde{R}}) + \varepsilon \]
(which is always possible). By Theorem 1.3 we have
\[ \|f^i \circ \phi^i - f\|_{C^0(U^i)} \to 0 \quad \text{as } i \to \infty, \]
where \( U^i = (f^i \circ \phi^i)^{-1}(\tilde{B}_R) \). This yields
\[ f^{-1}(\tilde{B}_R) \subset (f^i \circ \phi^i)^{-1}(\tilde{B}_R) \quad \text{for } i \text{ sufficiently large.} \]
(7.10)

Denoting by \( \tilde{g}^i \) the metric induced by \( f^i \circ \phi^i \), we have
\[ \mu_{\tilde{g}^i}(f^i \circ \phi^i)^{-1}(\tilde{B}_R)) = \mu_{g^i}(f^i)^{-1}(\tilde{B}_R)) = \mu^i(\tilde{B}_R) \leq C(R). \]
(7.11)

Moreover
\[ f^i \circ \phi^i \to f \quad \text{on } f^{-1}(\tilde{B}_R) \subset M \text{ in } C^1. \]
(7.12)

Using (7.12) in the first line, (7.10) in the second, and (7.11) in the third, we obtain
\[ \mu(\tilde{B}_R) = \mu_g(f^{-1}(\tilde{B}_R)) = \lim_{i \to \infty} \mu_{\tilde{g}^i}(f^{-1}(\tilde{B}_R))) \leq \limsup_{i \to \infty} \mu_{\tilde{g}^i}(f^i \circ \phi^i)^{-1}(\tilde{B}_R))) \leq C(R). \]

With (7.10) this implies \( \mu(\tilde{B}_R) \leq C(R) + \varepsilon \). As this is true for any \( \varepsilon > 0 \), we finally conclude \( \mu(\tilde{B}_R) \leq C(R) \).

The bound \( \|\nabla^k A\|_{L^\infty(\tilde{B}_R)} \leq C_k(R) \) is shown in the same way, using the locally smooth convergence. Note that the first bound would also follow from Corollary 1.4 which is shown below. This completes the proof of Theorem 1.3.

Remark 7.12 In [3], the projection for the construction of the diffeomorphisms \( \phi^i \) is not carried out. In the cited paper, one considers sets \( W_{i,j} = \bigcup_{j=1}^{K^i} U_{\delta(j)} \subset M^i \), where \( K^i \) is a constant. Passing to a subsequence, the corresponding Euclidean isometries \( A^j \) converge for \( 1 \leq j \leq K^i \). Choosing \( i, i' \) large enough, the corresponding graph systems of \( W_{i,j} \) and \( W_{i,j'} \) are close to each other. It is concluded, that \( f^i(W_{i,j}) \) is a graph over \( f^j(W_{i,j'}) \) and that, therefore, \( W_{i,j} \) and \( W_{i,j'} \) are diffeomorphic. However, this conclusion is false. In fact, one can easily construct two sets \( W_{i,j} \) and \( W_{i,j'} \) that are arbitrarily close in the sense of graph systems, but not diffeomorphic — for example \( W_{i,j} = S^1 \), and \( W_{i,j'} \) a spiral that is close to \( S^1 \) but diffeomorphic to an open interval. Similarly, there are counterexamples where both \( W_{i,j} \) and \( W_{i,j'} \) are noncompact; also assuming a property of the intersections of the graphs as in Lemma 1.4 (b) does not suffice for the conclusion. Instead, one has to construct projections between the immersions in order to obtain diffeomorphisms between appropriate subsets of \( W_{i,j} \) and \( W_{i,j'} \). The same is needed in order to obtain \( C^k \)-convergence. This was done by Langer in [10], and by the author in the present paper.
7 Compactness for immersions on noncompact manifolds

Proof of Corollary 1.4
The measures $\mu^i$ converge to $\mu$ in $C^0_c(\mathbb{R}^n)'$ if and only if
\[
\lim_{i \to \infty} \int_{\mathbb{R}^n} f \, d\mu^i = \int_{\mathbb{R}^n} f \, d\mu \quad \text{for all } f \in C^0_c(\mathbb{R}^n).
\]
By [11], p. 54, Theorem 1, this is equivalent to the inequalities
\[
\limsup_{i \to \infty} \mu^i(K) \leq \mu(K) \quad \text{for each compact set } K \subset \mathbb{R}^n \quad \text{and}
\mu(U) \leq \liminf_{i \to \infty} \mu^i(U) \quad \text{for each open set } U \subset \mathbb{R}^n.
\]
We now will show these two inequalities.

So let $K \subset \mathbb{R}^n$ be compact. Let $V \subset \mathbb{R}^n$ be open with $K \subset V$. By Theorem 1.3 we have
\[
\|f \circ \phi^i - f\|_{C^0(V)} \to 0 \quad \text{as } i \to \infty,
\]
where $U^i = (f^i \circ \phi^i)^{-1}(\hat{B})$. Hence
\[
(f^i \circ \phi^i)^{-1}(K) \subset f^{-1}(V) \quad \text{for } i \text{ sufficiently large.}
\]
Thus we get, denoting by $\tilde{g}^i$ the metric induced by $f^i \circ \phi^i$,
\[
\mu^i(K) = \mu_{\tilde{g}^i}((f^i)^{-1}(K)) = \mu_{\tilde{g}^i}((f^i \circ \phi^i)^{-1}(K)) \leq \mu_{\tilde{g}^i}(f^{-1}(V)).
\]
Letting $i \to \infty$ yields
\[
\limsup_{i \to \infty} \mu^i(K) \leq \mu_{\tilde{g}}(f^{-1}(V)) = \mu(V).
\]
As $\mu(K) = \inf \{\mu(W) : W \text{ open, } K \subset W\}$ by Theorem 1.3 in [21], we finally obtain $\limsup_{i \to \infty} \mu^i(K) \leq \mu(K)$.

Next let $U \subset \mathbb{R}^n$ be open. Let $C \subset \mathbb{R}^n$ be compact with $C \subset U$. Then
\[
f^{-1}(C) \subset (f^i \circ \phi^i)^{-1}(U) \quad \text{for } i \text{ sufficiently large.}
\]
This implies
\[
\mu_{\tilde{g}^i}(f^{-1}(C)) \leq \mu_{\tilde{g}^i}((f^i \circ \phi^i)^{-1}(U)) = \mu_{\tilde{g}^i}((f^i)^{-1}(U)) = \mu^i(U).
\]
Again letting $i \to \infty$ yields
\[
\mu(C) = \mu_{\tilde{g}}(f^{-1}(C)) \leq \liminf_{i \to \infty} \mu^i(U).
\]
As $\mu(U) = \sup \{\mu(E) : E \text{ compact, } E \subset U\}$ by Remark 1.4 in [21], we obtain $\mu(U) \leq \liminf_{i \to \infty} \mu^i(U)$, which proves Corollary 1.4.

Finally we would like to give some generalizations of Theorem 1.3 First we remark that the assumptions (1.4) and (1.5) can be weakened as follows: Let $f^i : M^i \to \mathbb{R}^n$ be as in Theorem 1.3 with $f^i(M^i) \cap K \neq \emptyset$ for a compact set $K \subset \mathbb{R}^n$. Let $(R_i)_{i \in \mathbb{N}}$ be a sequence in $\mathbb{R}_{>0}$ with $R_i \to \infty$ as $i \to \infty$, and assume
\[
\mu^i(B_R) \leq C(R) \quad \text{for any } R < R_i, \quad (i)
\]
\[
\|\nabla^k A^i\|_{L^\infty(B_R)} \leq C_k(R) \quad \text{for any } R < R_i \text{ and } k \in \mathbb{N}_0. \quad (ii)
\]
Then the same statement as is Theorem 1.3 holds. The bounds (i) and (ii) for the limit, that is \( \mu(B_R) \leq C(R) \) and \( \|\nabla^k A^i\|_{L^\infty(B_R)} \leq C_k(R) \), hold for any \( R > 0 \). This statement is needed in [17]. The proof is essentially the same as for Theorem 1.3.

A further reaching generalization is to consider proper immersions into open subsets \( \Omega \subset \mathbb{R}^n \):

**Corollary 7.13** Let \( f^i : M^i \to \Omega \) be a sequence of proper immersions, where \( M^i \) is an \( m \)-manifold without boundary, \( \Omega \subset \mathbb{R}^n \) open, and \( f^i(M^i) \cap C \neq \emptyset \) for a compact set \( C \subset \Omega \). Assume

\[
\mu^i(K) \leq C(K) \quad \text{for any } K \subset \Omega \text{ compact}, \quad (i')
\]
\[
\|\nabla^k A^i\|_{L^\infty(K)} \leq C_k(K) \quad \text{for any } K \subset \Omega \text{ compact and } k \in \mathbb{N}_0. \quad (ii')
\]

Then there exists a proper immersion \( f : M \to \Omega \), where \( M \) is again an \( m \)-manifold without boundary, such that after passing to a subsequence there are diffeomorphisms

\[
\phi^i : U^i \to (f^i)^{-1}(\Omega^i) \subset M^i,
\]

where \( \Omega^i \subset \Omega \), \( U^i \subset M \) are open sets with \( \Omega^i \subset \subset \Omega^{i+1} \), \( U^i \subset \subset U^{i+1} \) and \( \Omega = \bigcup_{i=1}^\infty \Omega^i \), \( M = \bigcup_{i=1}^\infty M^i \), such that \( \|f^i \circ \phi^i - f\|_{C^0(U^i)} \to 0 \), and moreover \( f^i \circ \phi^i \to f \) locally smoothly on \( M \).

Moreover, the immersion \( f \) also satisfies (i') and (ii'), that is \( \mu(K) \leq C(K) \) and \( \|\nabla^k A\|_{L^\infty(K)} \leq C_k(K) \).

**Proof:**
We set \( V^i : = B_i(0) \cap \Omega_{1/i} \), where \( \Omega_{\delta} : = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \} \). As \( C \) is compact, there is an \( i_0 \in \mathbb{N} \) with \( C \subset V^{i_0} \) for all \( i \geq i_0 \). We set \( \Omega^i : = V^{i_0+i} \). As \( \overline{\Omega^i} \subset \Omega \) is compact, on \( \Omega^i \) we have uniform bounds for the volume and the second fundamental form. Now we can proceed as in the proof of Theorem 1.3. For a given immersion \( f : M \to \Omega \) and \( p \in M \) define \( p = \bar{p}(f) : = \min\{ j \in \mathbb{N} : f(p) \in \Omega^j \} \). Now always replace the balls \( B_i \) by the sets \( \Omega^i \) (for example in Definition 7.3). Following step by step the arguments of Theorem 1.3 the corollary follows. \( \square \)

Although in the corollary above the target \( \Omega \) is not necessarily complete, the limit does not only lie in \( \overline{\Omega^i} \), but even in \( \Omega \). This is possible as here we only desire local convergence. Theorem 1.3 and Corollary 7.13 can be generalized further. First one could formulate a version of Corollary 7.13 as in the paragraph preceding this statement. Moreover there are versions for proper \( C^k \)-immersions with

\[
\|\nabla^i A^i\|_{L^\infty(B_R)} \leq C_i(R) \quad \text{for any } R > 0 \text{ and } l \text{ with } 0 \leq l \leq k - 2
\]

(or with uniform bounds on compact sets \( K \subset \Omega \)) with convergence in \( C^{k-1} \). Finally also for the noncompact case there are versions for proper immersions into Riemannian manifolds \( N \). For that we again use an isometric embedding \( N \hookrightarrow \mathbb{R}^\nu \).

We would like to give an example how Corollary 7.13 can be used. Let us consider the graph situation where \( g^i : \mathbb{R}^m \to \mathbb{R}^n \), \( g^i(x) = (x, u^i(x)) \) with \( u^i : \mathbb{R}^m \to \mathbb{R}^k \). Now assume that there is exactly one point where curvature concentrates, say in \( 0 \in \mathbb{R}^m \). We cut out a ball \( B_0(0) \), and obtain proper immersions \( f^i : \mathbb{R}^m \setminus B_0(0) \to \Omega \), \( f^i : = g^i|_{\mathbb{R}^m \setminus B_0(0)} \) with \( \Omega := (\mathbb{R}^m \setminus B_0(0)) \times \mathbb{R}^k \). Note that \( f^i \) is not proper as a mapping into \( \mathbb{R}^n \). Let us assume that \( f^i \) admits uniform bounds on \( \nabla^k A^i \). Then we are in a situation to apply Corollary 7.13 and to conclude convergence of a subsequence. Similarly, we can also apply the corollary in the case of graphs defined on annuli \( B_R \setminus B_r \).

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