THE BUSINESS OF HEIGHT PAIRINGS

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Abstract. In algebraic geometry there is the notion of a height pairing of algebraic cycles, which lies at the confluence of arithmetic, Hodge theory and topology. After explaining a motivating example situation, we introduce new directions in this subject.

In celebration of Steven M. Zucker’s 65th birthday.
A true pioneer in Hodge theory!

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1. Introduction

From topology one has the notion of the local linking number (or index) of two curves in 3-space. Basically this determines locally how many times a given curve winds around another (with orientation). If one of curves bounds a membrane (we think of that membrane as a pre-cycle, in the sense that it’s boundary is not zero), then the sum of these local links can be interpreted as an intersection pairing. Paragraph 2.1. in [Be3] comes to mind about this. The height pairing of two algebraic cycles is an algebraic generalization of this. Here is an example (see [C-L]) of how we view a classical algebraic cycle as bounding a pre-cycle. Let \( X \) be a projective algebraic manifold of dimension \( d \) and \( \{ Z_\alpha \} \) a finite collection of irreducible subvarieties of codimension \( r-1 \) in \( X \). Let \( f_\alpha \in \mathbb{C}(Z_\alpha)^\times \), and consider the pre-cycle

\[
\xi_1' := \sum_\alpha (f_\alpha, Z_\alpha).
\]

Put

\[
\xi_1 := \sum_\alpha \text{div}_{Z_\alpha}(f_\alpha) \in z^r(X),
\]

where \( z^r(X) \) are the cycles of codimension \( r \) in \( X \). Note that by definition \( \xi_1 \in z^r_{\text{rat}}(X) \), the subgroup of cycles in \( z^r(X) \) rationally equivalent to zero. Alternate take: Let \( \Box := \mathbb{P}^1 \setminus \{1\} \). Then one can interpret

\[
\xi_1' = \sum_\alpha \text{graph}_{Z_\alpha \times \Box}(f_\alpha) \in z^r(X \times \Box),
\]

with

\[
\partial(\xi_1') = \partial_0(\xi_1') - \partial_\infty(\xi_1') = \xi_1.
\]

If \( \xi_2 \in z^{d-r+1}(X) \) is in general position with respect to \( \xi_1 \) (and \( \xi_1' \)), then \( |\xi_1| \cap |\xi_2| = \emptyset \); moreover

\[
\sum_\alpha \int_{Z_\alpha \cap \xi_2} \log |f_\alpha| \in \mathbb{R},
\]

becomes the analog of the total linking index of \( \xi_1 \) and \( \xi_2 \). Now suppose that \( \xi_1 = 0 \), i.e. \( \partial\xi_1' = 0 \). Then the real regulator of the “\( K_1^{(r)}(X) \)” cycle \( \xi_1' \), given by the formula,

\[
R_{r,1}(\xi_1') \in H^{r-1,r-1}(X, \mathbb{R}) \simeq H^{d-r+1,d-r+1}(X, \mathbb{R})^\vee,
\]

\[
\omega \in H^{d-r+1,d-r+1}(X, \mathbb{R}) \mapsto \sum_\alpha \int_{Z_\alpha} \log |f_\alpha| \omega \in \mathbb{R},
\]
is well-defined (see [Ja1], or [KLM] and the references cited there). If 
\( \omega = [\xi_2] \) is algebraic, then

\[
R_{r,1}(\xi'_1)(\omega) = \sum_\alpha \int_{Z_\alpha \cap \xi_2} \log |f_\alpha|.
\]

Finally, if \( \omega = 0 \), e.g. \( \omega = [\xi_2] \) where \( \xi_2 \in z_{\text{rat}}^{d-r+1}(X) \), then \( R_{r,1}(\xi'_1)(\omega) = 0 \). We deduce:

**Proposition 1.1.** We have a pairing

\[
\langle , \rangle : z_{\text{rat}}^r(X) \times z_{\text{rat}}^{d+1-r}(X) \rightarrow \mathbb{R}
\]

given by

\[
\langle \xi_1, \xi_2 \rangle = R_{r,1}(\xi'_1)(\xi_2) = \sum_\alpha \int_{Z_\alpha \cap \xi_2} \log |f_\alpha| \in \mathbb{R},
\]

where \( \xi_1 \in z_{\text{rat}}^r(X) \), \( \xi_2 \in z_{\text{rat}}^{d+1-r}(X) \) and \( \xi'_1 = \sum(f_\alpha, Z_\alpha) \) is a higher Chow precycle whose divisor (boundary) is \( \xi_1 \). It is easy to see that the pairing is well-defined, i.e., it is independent of the exact choice of \( \xi'_1 \), since if \( \text{div}(\xi'_1 - \xi'_2) = 0 \), then

\[
R_{r,1}(\xi'_1 - \xi'_2)(\xi_2) = 0
\]

as \( \xi_2 \sim_{\text{rat}} 0 \).

The projection formula holds trivially from the definition. That is, we have

**Proposition 1.2.** Let \( \pi : X \rightarrow Y \) be a flat surjective morphism between two smooth projective varieties \( X \) and \( Y \). Then \( \langle \xi_1, \pi^*\xi_2 \rangle = \langle \pi_*\xi_1, \xi_2 \rangle \) for all \( \xi_1 \in z_{\text{rat}}^r(X) \) and \( \xi_2 \in z_{\text{rat}}^{d+1-r}(Y) \) with \( |\pi_*\xi_1| \cap |\xi_2| = \emptyset \).

A little less obvious fact is that this pairing is symmetric. That is, it has the following property which we will call the *reciprocity property* of the pairing.

**Proposition 1.3.** For all \( \xi_1 \in z_{\text{rat}}^r(X) \), \( \xi_2 \in z_{\text{rat}}^{d-r+1}(X) \) with \( |\xi_1| \cap |\xi_2| = \emptyset \), \( \langle \xi_1, \xi_2 \rangle = \langle \xi_2, \xi_1 \rangle \).

**Proof.** Let \( (f, D) \) and \( (g, E) \) be the higher Chow precycles such that \( \xi_1 = \text{div}(f) \) and \( \xi_2 = \text{div}(g) \). We can assume, using some additional machinery ([Blo1], (Lemma 4.2)), that with regard to the pairs \( (f, D) \), \( (g, E) \), everything is in “general” position. For notational simplicity, let us assume that \( D \) and \( E \) are irreducible and meet properly along an irreducible curve \( C \). Let

\[
f_C := f|_C \in \mathbb{C}(C)^\times, \quad g_C := g|_C \in \mathbb{C}(C)^\times.
\]
For every point $p \in C$, put

$$T_p \{f_c, g_c\} = (-1)^{\nu_p(f_c)\nu_p(g_c)} \left( \frac{f_c^{\nu_p(g_c)}}{g_c^{\nu_p(f_c)}} \right)_p.$$

where $\nu_p(h)$ is the vanishing order of a function $h$ at $p$. Since $|\xi_1| \cap |\xi_2| = \emptyset$, it follows that

$$T_p \{f_c, g_c\} = \begin{cases} 
 f_c^{\nu_p(g_c)}(p) & \text{if } \nu_p(g_c) \neq 0 \\
 g_c^{-\nu_p(f_c)}(p) & \text{if } \nu_p(f_c) \neq 0 \\
 1 & \text{otherwise.}
\end{cases}$$

Then it is a consequence of Weil reciprocity:

$$\prod_{p \in C} T_p \{f_c, g_c\} = 1,$$

that

$$\int_{D \cap \text{div}(g)} \log |f| = \int_{E \cap \text{div}(f)} \log |g|.$$

Obviously, this is equivalent to $\langle \xi_1, \xi_2 \rangle = \langle \xi_2, \xi_1 \rangle$. The reader can consult chapters 2.2.2 and 2.2.3 of [GS] for a generalization of this. □

In addition, this pairing is also non-degenerate in the sense of detecting rational equivalence (see [C-L] for details). This pairing is a special case of the complex Archimedean height pairing, well presented in [MS1], and plays a role at “infinity” in §4. We will return to a generalization of this Archimedean height pairing in §9.

2. Notation and a breezy review of background material

- Unless otherwise specified, $X$ is a smooth projective variety of dimension $d$ defined over a subfield $k \subseteq \mathbb{C}$, and $H^\bullet(X(\mathbb{C}))$ is singular cohomology, treating $X$ as a complex analytic space.
- For a quasi-projective variety $W$ over a field (or more generally a noetherian and separated scheme $W$), $z^r(W)$ is the free abelian group generated by subvarieties of codimension $r$ in $W$. The Chow group of $W$ is defined as $\text{CH}^r(W) = z^r(W)/z_r^{\text{rat}}(W)$, where $z_r^{\text{rat}}(W)$ is the subgroup of cycles rationally equivalent to zero. The rational Chow groups will be denoted by $\text{CH}^r(W; \mathbb{Q}) := \text{CH}^r(W) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- Let $A \subseteq \mathbb{R}$ be a subring. The reader is assumed to have some familiarity with the abelian category of $A$-MHS (mixed Hodge structures). Two excellent reference sources are [B-Z] and [Ja1]. If $r \in \mathbb{Z}$, then the Tate twist $A(r) := (2\pi i)^r A$ is the (pure) Hodge structure with weight $-2r$ and Hodge type $(-r, -r)$. It is customary to make the further assumption that $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a field, and we will assume this. The reasons
have to do with Deligne's observation (his $I^{p,q}$ decomposition theorem - a user-friendly explanation provided in [St]) that the weight functor $W_*$ is exact (same for the Hodge filtration functor $F^\bullet$). Let $V_1, V_2$ be $\mathbb{A}$-MHS. Carlson [Ca] was the first to give an explicit description of $\text{Ext}^1_{\mathbb{A}}(V_1, V_2)$ in terms of a “torus”, with the consequence that $\text{Ext}^n_{\mathbb{A}}(V_1, -)$ is a right exact functor. If we assume for the moment that $\mathbb{A}$-MHS has enough injectives, then it is clear from formal homological algebra arguments that $\text{Ext}^n_{\mathbb{A}}(V_1, V_2) = 0$ form $n \geq 2$. In general, one uses an Yoneda-Ext argument. The vanishing of the higher Ext’s was first proven by Beilinson [Be1].

- Let’s fix $\mathbb{A}$ as per the above paragraph, and put, for $V$ a $\mathbb{A}$-MHS, $\Gamma(V) = \text{hom}_{\mathbb{A}-\text{MHS}}(A(0), V)$, $J(V) = \text{Ext}^1_{\mathbb{A}-\text{MHS}}(A(0), V)$. For instance, if $\mathbb{A} = \mathbb{Q}$, then the classical Hodge conjecture asserts that $\Gamma(H^{2r}(X(\mathbb{C}), \mathbb{Q}(r)))$ is generated by the fundamental classes of cycles $\gamma^r(X; \mathbb{Q}) := \gamma^r(X) \otimes \mathbb{Q}$. The space $\Gamma(H^{2r}(X(\mathbb{C}), \mathbb{Q}(r)))$, of dimension $M$ say, in untwisted form is precisely $F^*H^{2r}(X(\mathbb{C}), \mathbb{C}) \cap H^{2r}(X(\mathbb{C}), \mathbb{Q}) \cong \bigoplus_{i=0}^{M} \mathbb{Q}(-r)$. In general, $F^*H^i(X(\mathbb{C}), \mathbb{C}) \cap H^i(X(\mathbb{C}), \mathbb{Q})$ need not be a Hodge structure, as first observed by Grothendieck (see [Lew1, §7]). The (unique) largest Hodge structure in $F^*H^i(X(\mathbb{C}), \mathbb{C}) \cap H^i(X(\mathbb{C}), \mathbb{Q})$ is denoted by $N^r_H H^i(X(\mathbb{C}), \mathbb{Q})$. There is also a filtration by coniveau, denoted by

$$N^r_H H^i(X, \mathbb{Q}) \subseteq N^r_H H^i(X(\mathbb{C}), \mathbb{Q}).$$

The (Grothendieck amended) general Hodge conjecture (GHC) asserts that the aforementioned inclusion is an equality (the reader can again consult [Lew1](§7) for details).

- If $V$ is a $\mathbb{A}$-MHS, then by ([Ca], [Ja2]),

$$J(V) \simeq \frac{W_0V_{\mathbb{C}}}{F^0W_0V_{\mathbb{C}} + W_0V'}.$$

As an example for $X/\mathbb{C}$, $J(H^{2r-1}(X(\mathbb{C}), \mathbb{Z}(r)))$ denotes the $r$-th Griffiths jacobian, and $J(H^{2r-2}(X(\mathbb{C}), \mathbb{R}(r))) \simeq H^{r-1,r-1}(X(\mathbb{C}), \mathbb{R}(r-1))$, the target space (after incorporating twists, viz., after multiplication

1Exactness is implied by strict compatibility which means that $h(F^rV_{1,C}) = h(V_{1,\mathbb{C}}) \cap F^rV_{2,\mathbb{C}}$ and $h(W_0V_{1,\mathbb{A} \otimes \mathbb{Q}}) = h(V_{1,\mathbb{A} \otimes \mathbb{Q}}) \cap W_0V_{2,\mathbb{A} \otimes \mathbb{Q}}$ for all $r$ and $\ell$. The idea is this. For any $\mathbb{A}$-MHS $V$, $V_C$ has a $\mathbb{C}$-splitting into a bigraded direct sum of complex vector spaces $I^{p,q} := F^p \cap W_{p+q} \cap [F^q \cap W_{p+q} + \sum_{i \geq 2} F^{q-i+1} \cap W_{p+q-i}]$, where one shows that $F^rV_C = \oplus_{p \geq r} \oplus_q I^{p,q}$ and $W_0V_C = \oplus_{p+q \leq \ell} I^{p,q}$. Then by construction of $I^{p,q}$, one has $h(I^{p,q}(V_{1,C})) \subseteq I^{p,q}(V_{2,C})$. Hence $h$ preserves both the Hodge and complexified weight filtrations. Now use the fact that $\mathbb{A} \otimes \mathbb{Q}$ is a field to deduce that $h$ preserves the weight filtration over $\mathbb{A} \otimes \mathbb{Q}$.}
by \((2\pi i)^{r-1}\) of \(R_{r,1}\) in \([\Pi]\). Indeed, \(J\left(H^{2r-2}(X(\mathbb{C}),\mathbb{R}(r-1))\right)\) is a version of real Deligne cohomology \(H_{\mathbb{R}}^{2r-1}(X(\mathbb{C}),\mathbb{R}(r))\), where we consider \(X/\mathbb{C}\) as a real variety via \(X \to \text{Spec}(\mathbb{C}) \to \text{Spec}(\mathbb{R})\) (see \([\text{Ja1}]\)).

3. INTERMEZZO I

Steven Zucker’s seminal work \([Z]\), the \(L_2\)-cohomology in the Poincaré metric associated to a polarizable variation of Hodge structure over a base curve, turned out to provide one instance of a \(L_2\)-cohomology interpretation of a corresponding intersection cohomology, the coincidence in the general situation over an arbitrary base manifold \(S\) with \(S\text{-Kähler},\) conjectured by Deligne, and settled by the works of W. Schmid, A. Kaplan, and E. Cattani, following the development of Schmid’s \(sl_2\) orbit theorem to several variables. In this part, we are interested in a lesser known result of Zucker’s work, as it relates to a global function field height pairing due to Beilinson \([\text{Be3}]\), albeit in characteristic zero. We wish to make it clear that the construction here is simply an interpretation of section 1 in \([\text{Be3}]\), from the point of view of the \(L_2\)-cohomology in \([Z]\). Start off with a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\rho} & \overline{X} \\
\downarrow & & \downarrow \overline{\rho} \\
C & \xrightarrow{j} & \overline{C}
\end{array}
\]

where \(\overline{C}\) is a smooth projective curve, \(C\) affine, \(\overline{\rho}\) is proper, \(\rho\) is smooth and proper, and all varieties are smooth, defined over a field \(k \subseteq \mathbb{C}\). Let \(K = k(C) = k(\eta), \eta \in C/k\) the generic point, and set \(X_K = X_\eta\), the generic fiber. Note that

\[
\text{CH}^r(X_\eta) = \lim_{U \subseteq \overline{C}/k} \text{CH}^r(\rho^{-1}(U/k)),
\]

and that the cycle class map

\[
\text{CH}^r(X_\eta; \mathbb{Q}) \to H^{2r}(X_\eta(\mathbb{C}),\mathbb{Q}(r)) := \lim_{U \subseteq \overline{C}/k} H^{2r}(\rho^{-1}(U(\mathbb{C})),\mathbb{Q}(r))
\]

is induced by

\[
\lim_{U \subseteq \overline{C}/k} \left( \text{CH}^r(\rho^{-1}(U/k); \mathbb{Q}) \to H^{2r}(\rho^{-1}(U(\mathbb{C})),\mathbb{Q}(r)) \right).
\]

\(^2\)The reader is encouraged to consult \([\text{C-K-S}]\) for more precise details concerning this discussion.
Warning. The definition of $H^{2r}(X_K; \mathbb{Q}(r))$, which is commonly interpreted as $H^{2r}(X_K(C), \mathbb{Q}(r))$, should not be misconstrued as the same object as $H^{2r}(\mathcal{X}_\eta(C), \mathbb{Q}(r))$, the latter defined by a limit process.

The affine Lefschetz theorem, the fact that $C$ is a curve, together with the (known degeneration of the) Leray spectral sequence (Deligne, but see [Z] 15 and the references cited there), tells us that the Leray filtration

$$H^{2r}(\mathcal{X}(C), \mathbb{Q}(r)) = L_0 \supset L_1 \supset L_2 \supset \{0\},$$

satisfies

$$L_0/L_1 = H^0(C, R^{2r}\rho_\ast\mathbb{Q}(r)),$$

$$L_1/L_2 = H^1(C, R^{2r-1}\rho_\ast\mathbb{Q}(r)),$$

$$L_2 H^2(C, R^{2r-2}\rho_\ast\mathbb{Q}(r)) = 0,$$

with same story for $H^{2r}(\mathcal{X}_\eta(C), \mathbb{Q}(r))$, where we replace $C$ by $\eta$. It is clear then that $\xi \in CH^r_{\text{hom}}(X_K; \mathbb{Q}) = CH^r_{\text{hom}}(\mathcal{X}_\eta; \mathbb{Q})$ maps to zero in $H^0(\eta, R^{2r}\rho_\ast\mathbb{Q}(r))$ by the Leray spectral sequence associated to $\rho$. Indeed, $\xi$ will have a spread cycle $\tilde{\xi} \in CH^r(\overline{\mathcal{X}}; \mathbb{Q})$, with $\tilde{\xi}|_X \mapsto 0 \in H^0(C, R^{2r}\rho_\ast\mathbb{Q}(r))$. Thus $\tilde{\xi}|_X \in H^1(C, R^{2r-1}\rho_\ast\mathbb{Q}(r))$. Let $d = \dim X_K$, which is the relative dimension of the flat morphism $\overline{\rho}$. Observe that the product $H^1(C, R^{2r-1}\rho_\ast\mathbb{Q}(r)) \otimes H^1(C, R^{2d-2r+1}\rho_\ast\mathbb{Q}(d-r+1)) \cong H^2(C, R^{2d}\rho_\ast\mathbb{Q}(d)) = 0$, is zero. Indeed, to re-iterate, this is due to the affine Lefschetz theorem applied to a smooth affine $C$ with cohomological degree $2 > 1 = \dim C$. Notice however that $\overline{\mathcal{X}}$ is complete (and smooth), and hence $H^{2r}(\overline{\mathcal{X}}, \mathbb{Q}(r))$ is a pure Hodge structure of weight zero, viz., $H^{2r}(\overline{\mathcal{X}}, \mathbb{Q}(r)) = W_0H^{2r}(\overline{\mathcal{X}}, \mathbb{Q}(r))$. Furthermore $[\xi]$ is the restriction of $[\tilde{\xi}]$ in $H^{2r}(\overline{\mathcal{X}}, \mathbb{Q}(r))$. Thus it is well known (rather implicit after reading [Z] 14 and more to the point in [PM]), that for $\xi \in CH^r_{\text{hom}}(X_K; \mathbb{Q})$,

$$[\xi] \in W_0H^1(\eta, R^{2r-1}\rho_\ast\mathbb{Q}(r)) = W_0 H^1(C, R^{2r-1}\rho_\ast\mathbb{Q}(r))$$

$$= H^1(\overline{C}, j_\ast R^{2r-1}\rho_\ast\mathbb{Q}(r)),$$

the latter being the object of interest in [Z]. Note that the pairing (also being a pairing of intersection cohomologies)

$$H^1(\overline{C}, j_\ast R^{2r-1}\rho_\ast\mathbb{Q}(r)) \otimes H^1(\overline{C}, j_\ast R^{2d-2r+1}\rho_\ast\mathbb{Q}(d-r+1))$$

$$\cong H^2(\overline{C}, R^{2d}\rho_\ast\mathbb{Q}(d)) \simeq \mathbb{Q},$$

is non-degenerate [Z]. Now for $\xi_1 \in CH^r_{\text{hom}}(X_K; \mathbb{Q})$, $\xi_2 \in CH^{d-r+1}_{\text{hom}}(X_K; \mathbb{Q})$, we arrive at Beilinson’s (global case) height pairing over the function field of a curve [Be3], viz., $\langle \cdot, \cdot \rangle : CH^r_{\text{hom}}(X_K; \mathbb{Q}) \otimes CH^{d-r+1}_{\text{hom}}(X_K; \mathbb{Q})$ → $H^1(\overline{C}, j_\ast R^{2r-1}\rho_\ast\mathbb{Q}(r)) \otimes H^1(\overline{C}, j_\ast R^{2d-2r+1}\rho_\ast\mathbb{Q}(d-r+1)) \simeq \mathbb{Q}$. 


Remark 3.1. For each closed point $v \in C$, Beilinson attaches a local linking number $\langle \cdot, \cdot \rangle_v$, and shows that the global height pairing is the sum of local ones.

4. The arithmetic scenario

Now let $X$ be a smooth projective variety of dimension $d$, defined over a number field $k$ (i.e., $[k : \mathbb{Q}] < \infty$). Denote by $z_{\text{hom}}^r(X)$ the nullhomologous cycles. Under some assumptions, Beilinson in [Be3] defined a height pairing

$$z_{\text{hom}}^r(X; \mathbb{Q}) \times z_{\text{hom}}^{d-r+1}(X; \mathbb{Q}) \to \mathbb{R},$$

which factors through $z_{\text{rat}}^r(X; \mathbb{Q})$, viz., an induced height pairing

$$\langle \cdot, \cdot \rangle_{\text{HT}} : \text{CH}_r^\text{hom}(X; \mathbb{Q}) \times \text{CH}_{d-r+1}^\text{hom}(X; \mathbb{Q}) \to \mathbb{R}.\quad (3)$$

This pairing should have a number of conjectural properties, for example

Conjecture 4.1. (Conjecture 5.4 (a) of [Be3]) The height pairing is non-degenerate.

Conjecture 4.2. (Hodge-index conjecture 5.5 of [Be3]) Assume that a hard Lefschetz conjecture holds on null-homologous cycles (conjecture 5.3 of [Be3]) and consider the primitive cycle decomposition. Let $L_X \in \text{CH}^1(X)$ be the class of a hyperplane section. Then the form $\langle \cdot, L^{d-2r+1}\cdot \rangle_{\text{HT}}$ is definite of sign $(-1)^r$ on the primitive $r$-cycles for $r \leq \frac{d+1}{2}$.

These conjectures seem to mimic the nondegeneracy and the polarizing properties of the cohomology of $X$. For example, Conjecture 4.1 is an analog of the non-degeneracy of

$$H^{2r-1}(X(\mathbb{C}), \mathbb{Q}) \times H^{2d-2r+1}(X(\mathbb{C}), \mathbb{Q}) \to \mathbb{Q}.$$

It is instructive to explain the idea behind the pairing: The ingredient comes from

4.3. Arithmetic Chow groups. References for this section are [GS], [BGS], and [Ku]. We will only provide a brief glimpse into this fascinating theory. Interested readers may consult the references cited above for details (especially [BGS]). We begin with a motivating example (Chapter III of [Neu]):
Example 4.4. Consider a number field \( k \) with the number ring \( \mathcal{O}_k \). A prime \( \wp \) of \( k \) is a class of equivalent valuations of \( k \). The non-Archimedean equivalence classes are called \textit{finite} primes and accordingly the Archimedean ones are \textit{infinite} primes. The infinite primes \( \wp \) are obtained from the embeddings \( \tau : k \rightarrow \mathbb{C} \). There are two sorts of these: the real primes, corresponding to the real embeddings, and the complex primes corresponding to the pairs of complex conjugate non-real embeddings. The finite primes will be denoted formally by \( \wp \) and the infinite primes by \( \wp_\infty \).

To each prime \( \wp \in k \) (finite or infinite), we associate a canonical homomorphism
\[
v_\wp : k^* \rightarrow \mathbb{R}
\]
from the multiplicative group \( k^* \) of \( k \). If \( \wp \) is a finite prime, then \( v_\wp \) is a \( \wp \)-adic exponential valuation which is normalized by the condition \( v_\wp(k^*) = \mathbb{Z} \). If \( \wp \) is infinite, then \( v_\wp \) is given by \( v_\wp(a) = -\ln |\tau a| \), where \( \tau \) is an embedding which defines \( \wp \).

\textbf{Arakelov class group of} \( \mathcal{O}_k \): The group \( \hat{\text{Div}}(\mathcal{O}_k) \) of (Arakelov) divisors is defined by elements of the form
\[
D := \sum \wp m_\wp \wp + \sum \wp_\infty \lambda_\wp_\infty \wp_\infty,
\]
where \( m_\wp \in \mathbb{Z} \) and \( \lambda_\wp_\infty \in \mathbb{R} \), respectively. The principal divisors \( \hat{\mathcal{P}}(\mathcal{O}_k) \) are of the form
\[
\sum \wp v_\wp(\alpha) \wp + \sum \wp_\infty (-\log |\alpha|_{\wp_\infty}),
\]
where \( |\alpha|_{\wp_\infty} = |\tau \alpha| \) if \( \wp_\infty \) is real, and \( |\alpha|_{\wp_\infty} = |\tau \alpha|^2 \) if \( \wp_\infty \) is complex. We define the Arakelov class group of \( \mathcal{O}_k \) as the quotient
\[
\hat{\text{Cl}}(\mathcal{O}_k) := \hat{\text{Div}}(\mathcal{O}_k)/\hat{\mathcal{P}}(\mathcal{O}_k).
\]
One can define a real number, called the degree of a divisor \( D \) as
\[
\hat{\text{deg}}(D) := \sum \wp m_\wp \log N_\wp + \sum \wp_\infty \lambda_\wp_\infty,
\]
with \( N_\wp = |\mathcal{O}_k/\wp| \). The degree of a principal divisor is zero by the product formula. Hence we get a well-defined (continuous) homomorphism
\[
\hat{\text{deg}} : \hat{\text{Cl}}(\mathcal{O}_k) \rightarrow \mathbb{R}.
\]
More generally, consider a regular, projective and flat scheme \( \tilde{X} \rightarrow S = \text{Spec}(\mathcal{O}_k) \) of absolute dimension \( d + 1 \). Such a scheme will be referred to as a regular arithmetic variety. Note that, from the definition
\(\tilde{X}\) can be seen as a projective and flat scheme over \(\text{Spec}(\mathbb{Z})\).

For a regular arithmetic variety \(\tilde{X}\), and any integer \(r \geq 0\) we let \(Z^r(\tilde{X})\) be the free abelian group of cycles of codimension \(p\) over \(\tilde{X}\). The set of complex points \(\tilde{X}(\mathbb{C})\) of \(\tilde{X}\) can be identified with the disjoint union \(\coprod_{\sigma:k \to \mathbb{C}} \tilde{X}_\sigma(\mathbb{C})\). Let \(F_\infty : \tilde{X}(\mathbb{C}) \to \tilde{X}(\mathbb{C})\) be the antiholomorphic involution coming from complex conjugation. We denote by \(D^r,\sigma(\tilde{X}_\mathbb{R})\) the set of real currents in \(D^r,\sigma(\tilde{X}(\mathbb{C}))\) (with respect to a suitable action \(F_\infty^*\) of \(F_\infty\) on \(D^r,\sigma(\tilde{X}(\mathbb{C}))\)).

Now, any cycle \(Z \in Z^r(\tilde{X})\) defines a current \(\delta_Z \in D^r,\sigma(\tilde{X}_\mathbb{R})\) by integration on its set of complex points. A Green current for \(Z\) is any current \(g \in D^{r-1,\sigma-1}(\tilde{X}_\mathbb{R})\) such that \(dd^c g + \delta_Z\) is smooth (see §1 of \([BGS]\) for details and notations). Denote by \(\hat{Z}^r(\tilde{X})\) the group of pairs \((Z, g_Z)\) where \(Z \in Z^r(\tilde{X})\) and \(g_Z\) is a Green current for \(Z\), with addition defined component-wise. Let \(\hat{R}^r(\tilde{X}) \subset \hat{Z}^r(\tilde{X})\) be the subgroup generated by pairs of the form \((0, \partial u + \overline{\partial} v)\), where \(u\) and \(v\) are currents of type \((r - 2, \sigma - 1)\) and \((r - 1, \sigma - 2)\) respectively (\(\partial\) and \(\overline{\partial}\) being suitable operations on the space of currents), and \((\text{div}(f), - \log |f|^2)\), where \(f\) is a rational function on an integral subscheme \(\tilde{Y} \subset \tilde{X}\) of codimension \(r - 1\), and \(- \log |f|^2\) is the current on \(\tilde{X}(\mathbb{C})\) obtained by restricting forms to the smooth part of \(\tilde{Y}(\mathbb{C})\) and integrating against the \(L^1\) function \(- \log |f|^2\). Now define the arithmetic Chow group of codimension \(r\) as

\[\hat{\text{CH}}^r(\tilde{X}) = \hat{Z}^r(\tilde{X})/\hat{R}^r(\tilde{X}).\]

**Remark 4.5.** Arithmetic Chow groups can be defined for more general types of arithmetic varieties assuming only that the generic fibre is smooth (refer to §3.2 of \([GS]\) for details). In case our arithmetic variety is \(S = \text{Spec}(\mathcal{O}_k)\) for the number ring \(\mathcal{O}_k\) of \(k\), the arithmetic Chow group \(\hat{\text{CH}}^1_1(S) \cong \hat{\text{Cl}}(\mathcal{O}_k)^3\)

We note down some crucial properties of arithmetic Chow groups:

- (Theorem 4.2.3 of \([GS]\)) There is a cup product of arithmetic Chow groups

\[\hat{\text{CH}}^r(\tilde{X}) \otimes \hat{\text{CH}}^s(\tilde{X}) \to \hat{\text{CH}}^{r+s}(\tilde{X}; \mathbb{Q}),\]

\(3\)A comment is in order here: In Arakelov setting, a current \(\alpha \in D^r,\sigma(\tilde{X}(\mathbb{C}))\) is called real if \(F_\infty^*(\alpha) = (-1)^r \alpha\) (see either section 3.2 of \([GS]\) or 2.1 of \([BGS]\)).

4One uses a special case of theorem 3.3.5, exact sequence (i) of \([GS]\), setting \(\tilde{X} = S = \text{Spec}(\mathcal{O}_k)\) (see section 3.4 of \([GS]\) for details).
formally defined by the formula \[ ([Z_1, g_{Z_1}]) \cdot ([Z_2, g_{Z_2}]) = ([Z_1 \cdot Z_2, g_{Z_1 \ast g_{Z_2}}]) \], where \(*\) denotes the star product of Green currents (§1 of [GS]).

- (Theorem 3.6.1 and 4.2.3 of [GS]) Let \( f : \tilde{X} \to \tilde{Y} \) be a morphism of regular arithmetic varieties. Then there is a pull-back homomorphism \( f^* : \widehat{CH}^r(\tilde{Y}) \to \widehat{CH}^r(\tilde{X}; \mathbb{Q}) \). It is multiplicative, i.e., given \( \alpha \in \widehat{CH}^r(\tilde{Y}) \) and \( \beta \in \widehat{CH}^s(\tilde{Y}) \), we have
\[
f^*(\alpha \cdot \beta) = f^*(\alpha) \cdot f^*(\beta).
\]

Further if \( f \) is proper, \( f_k : \tilde{X}_k \to \tilde{Y}_k \) is smooth and \( \tilde{X}, \tilde{Y} \) are equidimensional, then there is a push-forward homomorphism
\[
f_* : \widehat{CH}^r(X) \to \widehat{CH}^{r-\delta}(Y), \ (\delta := \dim(X) - \dim(Y)),
\]
satisfying the projection formula
\[
f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta).
\]

### 4.6. Arithmetic height pairing.

Let \( \widehat{CH}^*(\tilde{X}) \) be the arithmetic Chow theory as defined above. One can define an arithmetic degree map as a push-forward
\[
\widehat{deg}_X : \widehat{CH}^{d+1}(\tilde{X}) \to \widehat{CH}^1(S) \cong \widehat{Cl}(\mathcal{O}_k) \to \mathbb{R},
\]
where \( \widehat{Cl}(\mathcal{O}_k) \to \mathbb{R} \) is the \( \widehat{deg} \) map of Example 4.4. Together with the arithmetic intersection, it defines a pairing
\[
\widehat{CH}^r(\tilde{X}; \mathbb{Q}) \otimes \widehat{CH}^{d-r+1}(\tilde{X}; \mathbb{Q}) \to \widehat{CH}^{d+1}(\tilde{X}; \mathbb{Q}) \to \mathbb{R}.
\]

For a smooth projective variety \( X/k \), assume that it has a regular model \( \tilde{X} \), i.e., a regular arithmetic variety which is projective and flat over \( S := \text{Spec}(\mathcal{O}_k) \), together with an isomorphism \( \tilde{X}_k \cong X/k \). Let’s explore this a little bit further. We are considering a family \( \tilde{X} \to S \), where the generic fibre is isomorphic to \( X/k \). This resembles the situation \( \rho : X \to C \) of §3. Now for each finite prime \( \varphi \in S \), we get a finite fibre \( \tilde{X}_\varphi := \tilde{X} \times_S \text{Spec}(\mathcal{O}_k/\varphi\mathcal{O}_k) \), and for each embedding \( \sigma : k \hookrightarrow \mathbb{C} \), we get a “fibre at infinity” \( \tilde{X}_\sigma := \tilde{X} \times_\sigma \mathbb{C} \). We think of the whole family (fibres over finite and infinite primes/embeddings) as a completion of

\footnote{We remark here that this \( \widehat{deg} \) is really the composition of the push-forward morphism \( \widehat{CH}^*(S) \to \widehat{CH}^1(\text{Spec}(\mathbb{Z})) \) attached to the unique morphism \( S \to \text{Spec}(\mathbb{Z}) \), and the isomorphism \( \widehat{CH}^1(\text{Spec}(\mathbb{Z})) \cong \mathbb{R} \) (see 2.1.3 of [BGS] for a detailed discussion of arithmetic degree maps).}
\[ \tilde{X} \] over \( \mathcal{S} := S \cup \{ \sigma : k \hookrightarrow \mathbb{C} \} \), resembling the situation \( \rho \) in \( \S 3 \).

**Remark 4.7.** The existence of a regular model for \( X \) is a highly non-trivial problem. As a basic example, smooth projective curves have regular models (after possibly extending the base field). Apart from that, triple product of curves (Gross-Schoen) and abelian varieties (Künemann) provides us with a large class of examples.

With this set up and under a further assumption ((17) of \([Ku]\)), Beilinson’s height pairing can be interpreted in light of arithmetic intersection

\[ \hat{\text{CH}}^r(\tilde{X}; \mathbb{Q}) \otimes \hat{\text{CH}}^{d-r+1}(\tilde{X}; \mathbb{Q}) \to \hat{\text{CH}}^{d+1}(\tilde{X}; \mathbb{Q}) \to \mathbb{R}. \]

This pairing may a priori depend on the choice of \( \tilde{X} \). Since our primary aim is to detect non-trivial cycles, this choice is not a hinderance, once we have one. To get a more earthly description, we stretch the analogy with Example 4.4 even further. Think of \( \tilde{X} \) as a projective scheme \( \text{Proj}(\mathcal{O}_k[X_0, \cdots, X_N]/I) \to S, \) for some homogeneous ideal \( I \). The height pairing then is a sum of non-Archimedean and Archimedean parts

\[ \langle \xi_1, \xi_2 \rangle_{HT} := \sum_{\wp} \langle \xi_1, \xi_2 \rangle_{\wp} + \sum_{\wp_{\infty}} \langle \xi_1, \xi_2 \rangle_{\wp_{\infty}}, \]

where (at least for finite primes \( \wp \) of good reduction) \( \langle \xi_1, \xi_2 \rangle_{\wp} \) is given by

\[ \log N_{\wp} \times (\#|\xi_1 \cap \xi_2|_{\wp}, \text{counting multiplicities}), \text{ albeit heuristically!} \]

On the subgroup of cycles algebraically equivalent to zero, the height pairing is given by the Néron-Tate pairing

\[ J_{\text{alg}}^r(X)(\overline{\mathbb{Q}}) \times J_{\text{alg}}^{d-r+1}(X)(\overline{\mathbb{Q}}) \to \mathbb{R}, \]

where \( J_{\text{alg}}^r(X)(\overline{\mathbb{Q}}) := \Phi_r(\text{CH}_{\text{alg}}^r(X/\overline{\mathbb{Q}})) \), and \( \Phi_r : \text{CH}_{\text{hom}}^r(X/k) \to J(H^{2r-1}(X(\mathbb{C}), \mathbb{Z}(r))) \) is the Griffiths Abel-Jacobi map.

**Remark 4.8.** Much like the notion of a height function, one can extend the height pairing for a smooth and projective \( X \) defined over \( \mathbb{Q} \) (see 4.0.6 of \([Be3]\)).

**Remark 4.9.** The Archimedean part of the height pairing \( \sum_{\wp_{\infty}} \langle \xi_1, \xi_2 \rangle_{\wp_{\infty}} \) is given by the star product of green currents \( g_{\xi_1} \) and \( g_{\xi_2} \) (see §1 of \([MS1]\) for details). Restricting further to the subgroups \( \tau_{\text{rat}}(X) \) and \( \tau_{\text{rat}}^{d-r+1}(X) \), this Archimedean part of the height pairing is the one defined in \( \S 7 \) (up to factors).
5. Bloch-Beilinson filtration

It was first indicated in [Blo2], and later fortified by Beilinson, that for $X$ smooth and projective over a field $k$, there should be a descending filtration

$$F^0 := CH^r(X; \mathbb{Q}) \supset F^1 = CH^r_{\text{hom}}(X; \mathbb{Q}) \supset \cdots \supset F^r \supset \{0\},$$

satisfying

$$Gr^r CH^r(X; \mathbb{Q}) := \frac{F^r CH^r(X; \mathbb{Q})}{F^{r+1} CH^r(X; \mathbb{Q})} \cong \text{Ext}^r_{\mathcal{MM}}(\text{Spec}(k), h^{2r-r}(X)(r)),$$

where $\mathcal{MM}$ is the conjectural category of mixed motives over $k$. A number of candidate filtrations have been proposed (names will suffice) by Jannsen [Ja3], S. Saito [SSa], M. Saito/M. Asakura [A], Murre [Mu], Griffiths-Green [G-G], Lewis [Lew2], Lewis/Kerr [K-L], Raskind [Ra], and so forth.... In the case $k = \mathbb{C}$ we have seen that in the category of MHS, $\text{Ext}^r_{\text{MHS}} = 0$, and yet $F^2 CH^r(X; \mathbb{Q})$ need not be zero (Mumford [M], Bloch [Blo2], Lewis (op. cit. and [Lew4]), Schoen (see [Ja2]), Roitman [Ro], Griffiths-Green [G-G-P]...). Even in the case where $\text{trdeg}_Q k = 1$, there are examples from some of the references (op. cit.) that $F^2 CH^r(X_k; \mathbb{Q}) \neq 0$. Indeed Beilinson and Bloch have independently conjectured the following:

**Conjecture 5.1.** Let $X/\mathbb{Q}$ be smooth and projective. Then the Griffiths Abel-Jacobi map

$$\Phi_r : CH^r_{\text{hom}}(X/\overline{\mathbb{Q}}; \mathbb{Q}) \to J(H^{2r-1}(X(\mathbb{C}), \mathbb{Q}(r))),$$

is injective.

**Remark 5.2.** Assuming the classical Hodge conjecture, one can argue that $X$ in the conjecture can be replaced by a smooth quasi-projective variety over $\overline{\mathbb{Q}}$. This follows from a weight filtered spectral sequence argument [K-L] (p. 371).

For $k = \mathbb{C}$, the following theorem best summarizes one’s expectations: First consider fields $\overline{\mathbb{Q}} \subset K \subset \mathbb{C}$, where $K/\mathbb{Q}$ is finitely generated. One first constructs a filtration on $CH^r(X_K; \mathbb{Q})$. The “lift” from $K$ to $\mathbb{C}$ follows from:

$$F^\nu CH^r(X_\mathbb{C}; \mathbb{Q}) = \lim_{K \subset \mathbb{C}} F^\nu CH^r(X_K; \mathbb{Q})$$

**Theorem 5.3** ([Lew2]). Let $X/K$ be smooth projective of dimension $d$. Then for all $r$, there is a filtration,

$$CH^r(X_K; \mathbb{Q}) = F^0 \supset F^1 \supset \cdots \supset F^\nu \supset F^{\nu+1} \supset \cdots$$
which satisfies the following

(i) \( F^1 = CH^r_{\text{hom}}(X_K; \mathbb{Q}) \).

(ii) \( F^2 \subset \left( \ker \Phi_r : CH^r_{\text{hom}}(X_K; \mathbb{Q}) \to J(H^{2r-1}(X_K(\mathbb{C}), \mathbb{Q}(r))) \right) \).

(iii) \( F^{\nu_1} CH^1(X_K; \mathbb{Q}) \cdot F^{\nu_2} CH^2(X_K; \mathbb{Q}) \subset F^{\nu_1+\nu_2} CH^{1+r_2}(X_K; \mathbb{Q}) \), where \( \cdot \) is the intersection product.

(iv) \( F^\nu \) is preserved under the action of correspondences between smooth projective varieties over \( K \).

(v) Assume that the K"unneth components of the diagonal class \( [\Delta_X] = \bigoplus_{p+q=2d}[\Delta_X(p, q)] \in H^{2d}(X \times X, \mathbb{Q}(d)) \) are algebraic and defined over \( K \). Then

\[
\Delta_X(2d - 2r + \ell + m, 2r - \ell - m) |_{Gr^\nu CH^r(X; \mathbb{Q})} = \delta_{\ell, \nu} \cdot \text{Identity}.
\]

[If we assume the conjecture that homological and numerical equivalence coincide, then (v) says that \( Gr^\nu \) factors through the Grothendieck motive.]

(vi) Let \( D^r(X_K) := \bigcap \nu F^\nu \). If Conjecture 5.1 holds for smooth quasi-projective varieties defined over \( \overline{\mathbb{Q}} \) (vis-à-vis Remark 5.2), then \( D^r(X_K) = 0 \) (hence \( D^r(X_{\mathbb{C}}) = 0 \)).

It is instructive to briefly explain how this filtration comes about. For \( X/K \) smooth projective, one can find a smooth quasi-projective \( S/\mathbb{Q} \) such that \( \mathbb{Q}(S) \) is identified with \( K \). One can then spread out \( X/K \) to a family \( \rho : X \to S \), where \( \rho \) is a smooth and proper morphism of smooth quasi-projective varieties over \( \overline{\mathbb{Q}} \), and \( X/K \) is the generic fiber. As a momentary digression, we offer the reader an illuminating illustration of the notion of spreads:

**Example 5.4.** Let

\[
Y/\mathbb{C} = \text{Spec} \left\{ \frac{\mathbb{C}[x, y]}{(\pi y^2 + (\sqrt{\pi} + 4)x^3 + ex)} \right\}.
\]

\[
S/\mathbb{Q} = \text{Spec} \left\{ \frac{\mathbb{Q}[u, v, w]}{(u - v^2)} \right\};
\]

Set:

\[
\mathcal{Y}_S = \text{Spec} \left\{ \frac{\mathbb{Q}[x, y, u, v, w]}{(uy^2 + (v + 4)x^3 + wx, u - v^2)} \right\}
\]

The inclusion

\[
\frac{\mathbb{Q}[u, v, w]}{(u - v^2)} \subset \frac{\mathbb{Q}[x, y, u, v, w]}{(uy^2 + (v + 4)x^3 + wx, u - v^2)}
\]
defines a morphism $\mathcal{Y}_S \to S$, as varieties over $\mathbb{Q}$. Let $\eta \in S$, be the generic point. Then
\[
\mathbb{Q}(\eta) = \text{Quot}\left(\frac{\mathbb{Q}[u, v, w]}{(u - v^2)}\right).
\]

Note that the embedding
\[
\mathbb{Q}(\eta) \hookrightarrow \mathbb{C}, \quad (u, v, w) \mapsto (\pi, \sqrt{\pi}, e), \quad \Rightarrow \mathcal{Y}_{S, \eta} \times \mathbb{C} = Y/\mathbb{C}.
\]

We will have more to say about this in the next section.

Now here is the key point. Beilinson’s absolute Hodge cohomology $H^*_\mathcal{H} [\text{Be1}]$, is a highly sophisticated cohomology theory with a number of similar properties to Deligne-Beilinson cohomology, with the advantage of incorporating weights. For our purposes here, we need the short exact sequence (p.2 of [Be1]):
\[
J(H^{2r-1}(\mathcal{X}(\mathbb{C}); \mathbb{Q}(r))) \hookrightarrow H^{2r}_\mathcal{H}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r)) \twoheadrightarrow \Gamma(H^{2r}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r))).
\]

There is a cycle class map $\text{cl}_r : CH^r(\mathcal{X}/\mathbb{Q}; \mathbb{Q}) \to H^{2r}_\mathcal{H}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r))$, and according to Conjecture [A.1] and Remark [B.2] one anticipates that $\text{cl}_r$ is injective. The lowest weight part, $H^{2r}_\mathcal{H}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r)) \subset H^{2r}_\mathcal{H}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r))$ is given by the image $H^{2r}_\mathcal{H}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r)) \to H^{2r}_\mathcal{H}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r))$, where $\mathcal{X}/\mathbb{Q}$ is a smooth compactification of $\mathcal{X}/\mathbb{Q}$. Note that $CH^r(\mathcal{X}/\mathbb{Q}) \to CH^r(\mathcal{X}/\mathbb{Q})$, is surjective; likewise there is a cycle class map $CH^r(\mathcal{X}; \mathbb{Q}) \to H^{2r}_\mathcal{H}(\mathcal{X}, \mathbb{Q}(r))$. Thus we conjecturally have an injection
\[
\text{cl}_r : CH^r(\mathcal{X}/\mathbb{Q}; \mathbb{Q}) \to H^{2r}_\mathcal{H}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r)).
\]

The filtration $F^\nu CH^r(\mathcal{X}/\mathbb{Q}; \mathbb{Q})$ is given by the pullback of the $\nu$-th Leray filtration of $\rho$ on $H^{2r}_\mathcal{H}(\mathcal{X}(\mathbb{C}), \mathbb{Q}(r))$, to $CH^r(\mathcal{X}/\mathbb{Q}; \mathbb{Q})$. (For an excellent motivic description of the Leray filtration, the reader should consult [A.1].) Let $\eta_S$ be the generic point of $S$, and put $K := \mathcal{X}(\eta_S)$, and note that the sequence in (4) remains exact at the generic point, by properties of direct limits. Write $X_K := \mathcal{X}(\eta_S)$. The injectivity of $\text{cl}_r$ passes to the generic point of $S$, viz., $\text{cl}_r : CH^r(\mathcal{X}(\eta_S); \mathbb{Q}) \hookrightarrow H^{2r}_\mathcal{H}(\mathcal{X}(\eta_S); (\mathbb{C}, \mathbb{Q}(r))),$ leading to a filtration \(\{F^\nu CH^r(X_K; \mathbb{Q})\}_{\nu \geq 0}\). Thus following [Lew2] we introduced a decreasing filtration $F^\nu CH^r(X_K; \mathbb{Q})$, with the property that $Gr^\nu F^r CH^r(X_K; \mathbb{Q}) \hookrightarrow E^\nu_{\infty, 2r-\nu}(\mathbb{Q}(r))$, where $E^\nu_{\infty, 2r-\nu}(\mathbb{Q}(r))$ is the $\nu$-th graded piece of the Leray filtration associated to $\rho$ on $H^{2r}_\mathcal{H}(\mathcal{X}(\eta_S), \mathbb{Q}(r))$. It is proven in [Lew2] that the term $E^\nu_{\infty, 2r-\nu}(\mathbb{Q}(r))$ fits in a short exact sequence:
\[
0 \to E^\nu_{\infty, 2r-\nu}(\eta_S) \to E^\nu_{\infty, 2r-\nu}(\eta_S) \to E^\nu_{\infty, 2r-\nu}(\eta_S) \to 0,
\]
where
\[ E_{\nu,2r-\nu}^{\nu,2r-\nu}(\eta_S) = \Gamma(H^\nu(\eta_S, R^{2r-\nu}\rho_*\mathbb{Q}(r))), \]
\[ J(W_{-1}H^{\nu-1}(\eta_S, R^{2r-\nu}\rho_*\mathbb{Q}(r))) \subset J(H^{\nu-1}(\eta_S, R^{2r-\nu}\rho_*\mathbb{Q}(r))). \]

Here the latter inclusion is a result of the short exact sequence:
\[ 0 \to W_{-1}H^{\nu-1}(\eta_S, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \to W_0H^{\nu-1}(\eta_S, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \to Gr_0W H^{\nu-1}(\eta_S, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \to 0. \]

We attend to (vi). The idea comes from [Ra]; however, as noted in [Ra], it goes back to a hard Lefschetz argument due to Beauville (see [Lew2]). This will imply that \( D_r(X) := D_r(X_K) = 0 \) under Conjecture 5.1 for smooth quasi-projective varieties. It is instructive to the reader to explain this argument. It suffices to show that
\[ \lim_{U \subset S/\mathbb{Q}} F^{r+1}CH^r(\rho^{-1}(U); \mathbb{Q}) =: F^{r+1}CH^r(X_K; \mathbb{Q}) \]
\[ = \lim_{U \subset S/\mathbb{Q}} F^{r+j}CH^r(\rho^{-1}(U); \mathbb{Q}) =: F^{r+j}CH^r(X_K; \mathbb{Q}), \text{ for all } j \geq 1. \]

Let \( L_X \) be the operation of cupping with the hyperplane class of the fibers of \( \rho \), and \( \psi_{r+j} : F^{r+j}CH^r(\rho^{-1}(U); \mathbb{Q}) \to E^{r+j,r-j}_\infty \), the natural map. There is a commutative diagram
\[
\begin{array}{ccc}
F^{r+j}CH^r(\rho^{-1}(U); \mathbb{Q}) & \xrightarrow{\psi_{r+j}} & E^{r+j,r-j}_\infty \\
\downarrow L_X^{d-r+j} & & \downarrow L_X^{d-r+j} \\
F^{r+j}CH^{d+j}(\rho^{-1}(U); \mathbb{Q}) & \xrightarrow{} & E^{r+j,2d-r+j}_\infty
\end{array}
\]
Since \( F^{r+j+1}CH^{r+j}(\rho^{-1}(U); \mathbb{Q}) = \ker \psi_{r+j} \), and that
\[ \lim_{U \subset S/\mathbb{Q}} CH^{d+j}(\rho^{-1}(U); \mathbb{Q}) = CH^{d+j}(X_K; \mathbb{Q}) = 0, \]
for \( j \geq 1 \), as \( \dim X_K = d \), it follows that (5) holds, and we’re done.
6. The Business of Spreads

Consider the smooth elliptic curve

$$E_C := \text{Proj}\left( \frac{\mathbb{C}[z_0, z_1, z_2]}{(2e z_0 z_2^2 - \pi z_1^3 + \sqrt{\pi} z_1 z_0^2 + \sqrt{3} i z_0^3)} \right) \subset \mathbb{P}^2(\mathbb{C})$$

An analytic geometer may view this as a compact Riemann surface endowed with the analytic topology. If for the moment we view $E$ as a prototypical projective algebraic manifold, then one key distinguishing feature that $E$ (or for that matter any complex algebraic variety) has over general complex manifolds, is that it can be arrived at via base extension from a smaller subfield $K \subset \mathbb{C}$. There are lots of choices for $K$, but it is customary to think of it as finitely generated over $\mathbb{Q}$. To muddy the water a bit, let’s consider $\xi = \{\sqrt{3}z_1^2 z_1 + iz_2^2 = 0\} \cap E_C \in \text{CH}^1(E_C)$. In this case, let’s choose $K = \mathbb{Q}(\sqrt{\pi}, e, i, \sqrt{5})$, and define

$$E_K := \text{Proj}\left( \frac{K[z_0, z_1, z_2]}{(2^{-1}e z_0 z_2^2 - \pi z_1^3 + \sqrt{\pi} z_1 z_0^2 + \sqrt{3} i z_0^3)} \right) \subset \mathbb{P}^2(K).$$

By base change, we have $E_C = E_K \times_K \mathbb{C} := E_K \times_{\text{Spec}(K)} \text{Spec}(\mathbb{C})$. Now $K$ itself can be representative of the process of evaluation of a general point over $\mathbb{Q}$. Let

$$\mathcal{S} = \text{Spec}\left( \frac{\mathbb{Q}[u, v, w, t, s]}{(w^2 + 1, t^2 - 3, s^2 - 5)} \right).$$

Let $\eta_S \in \mathcal{S}/\mathbb{Q}$ be the generic point. Note that by definition $\mathbb{Q}(\eta_S) = \mathbb{Q}(\mathcal{S})$ and that the evaluation map

$$(6) \quad \mathbb{Q}(\mathcal{S}) \hookrightarrow \mathbb{C}, \ (u, v, w, t, s) \mapsto p := (e, \sqrt{\pi}, i, \sqrt{3}, \sqrt{5}) \in \mathcal{S}(\mathbb{C}),$$

identifies $\mathbb{Q}(\mathcal{S})$ with $K$. Any other point $q \in \mathcal{S}(\mathbb{C})$ for which evaluation defines an embedding $K \hookrightarrow \mathbb{C}$ is called a general point of $\mathcal{S}$. Now consider the quasi-projective variety $\mathcal{E}_Q$ defined by

$$\begin{cases} 2^{-1}u z_0 z_2^2 - v z_1^3 + v z_1 z_0^2 + w t z_0^3 = 0 \\ w^2 + 1 = t^2 - 3 = s^2 - 5 = 0 \end{cases} \subset \mathbb{P}_Q^2 \times Q \text{Spec}(\mathbb{Q}[u, v, w, t, s]).$$

Likewise, $\xi$ has an obvious spread $\tilde{\xi}$ given by

$$\begin{cases} 2^{-1} u z_0 z_2^2 - v z_1^3 + v z_1 z_0^2 + w t z_0^3 = 0 \\ s z_0^4 z_1 + w z_2^5 = 0 \\ w^2 + 1 = t^2 - 3 = s^2 - 5 = 0 \end{cases} \subset \mathbb{P}_Q^2 \times Q \text{Spec}(\mathbb{Q}[u, v, w, t, s]).$$

As in Example 5.4, there is a morphism $\rho : \mathcal{E}_Q \to \mathcal{S}_Q$. Then $\tilde{\xi} \in \text{CH}^1(\mathcal{E}_Q)$. Indeed $\tilde{\xi}_{\eta_S} = \xi \in \text{CH}^1(E_K; \mathbb{Q})$, where $\mathcal{E}_{\eta_S}$ is identified with $E_K$, under the embedding given in (6). Let us also view $\rho : \mathcal{E}(\mathbb{C}) \to \mathcal{S}(\mathbb{C})$ as the induced map of complex spaces. The datum associated to
the Leray sheaf $R^* \rho_* \mathbb{Q}$ amounts to an arithmetic variation of Hodge structure, and these ideas have played a big role in constructing algebraic invariants associated to Chow groups of algebraic cycles, as for example seen in the previous section. The reader should also consult [A], [G-G] and [Lew4] as further exploitation of these ideas. A different line of enquiry involving spreads can be found in [V]. Finally one can also spread $E$ over $\mathbb{Z}$, by including the equation $2x - 1 = 0$. This leads to an arithmetic scheme over $\mathbb{Z}$ where the business of height pairings can be addressed.

7. Intermezzo II

At this point, it should be reasonably clear to the reader that the notion of a height pairing of the form

$$F^\nu \text{CH}^r(X_K; \mathbb{Q}) \times F^\nu \text{CH}^{d-r+\nu}(X_K; \mathbb{Q}) \to \mathbb{R},$$

generalizing (3), and providing a “polarization” on “primitive” pieces of $Gr^\nu_F \text{CH}^r(X; \mathbb{Q})$, much the same way as with the Hodge-Riemann bilinear relations on the primitive cohomology of a projective algebraic manifold, should exist. Here $K$ is finitely generated over $\mathbb{Q}$. As we will see below, there is the technical requirement that $K$ have transcendence degree $\nu - 1$ over $\mathbb{Q}$, $\nu \geq 1$. Unfortunately, a proof of such a pairing seems elusive at this given time, and so we were forced to make further concessions ($\S 8$).

The relevance of these ideas should be clear. The idea of attaching a conjecturally non-degenerate pairing on graded pieces of the Bloch-Beilinson filtration is a unique new idea that is at the cross roads of arithmetic, Arakelov geometry and Hodge theory. At the heart of the notion of a height pairing of two cycles, is the idea of “spreading” a cycle out so as to form an intersection pairing, very similar to the aforementioned idea of defining a linking number of two disjoint curves in 3-space, where one curve bounds a membrane, thus creating an intersection number with the other curve. In (co-)homology theory, it is often the case that to determine whether a (co-)cycle is non-zero, is via an intersection/cup product with a complementary dimensional cycle. The “definite” properties of the Néron-Tate pairing (and conjecturally that of the Beilinson pairing) should convince one that this technology may lead to similar role in detecting the non-triviality of a specific “interesting” algebraic cycle.

8. A new pairing

Throughout this section, we will assume Conjecture [5,4] and the GHC.
If \( \overline{Q} \) of the previous section is replaced by \( K \), a field of finite transcendence degree over \( \overline{Q} \), then as alluded to earlier, Conjecture 5.1 is false. However as indicated in §5, the notion of a conjectural Bloch-Beilinson filtration involves spreads, which is key to a generalized pairing. We will continue with the notation of §5, with \( K = \overline{Q}(S) \) finitely generated over \( \overline{Q} \). Let \( \text{Gr}_\nu \text{CH}^r(X_K; \mathbb{Q}) \) denote the graded pieces of the filtration, we have a non-canonical motivic decomposition (albeit \( \text{Gr}_\nu \text{CH}^r(X_K; \mathbb{Q}) \) is unique)

\[
\text{CH}^r(X_K; \mathbb{Q}) = \bigoplus_{\nu \geq 0} \Delta_{X_K}(2d - 2r + \nu, 2r - \nu)_* \text{CH}^r(X_K; \mathbb{Q})
\]

\[
\simeq \bigoplus_{\nu \geq 0} \text{Gr}_\nu \text{CH}^r(X_K; \mathbb{Q}),
\]

much like the Hodge decomposition of the de Rham cohomology of \( X \). Now if \( X/\overline{Q} \) is smooth projective, in light of Conjecture 5.1, Beilinson’s height pairing could be interpreted as a pairing on \( \text{Gr}_1 \text{CH}^r(X_K; \mathbb{Q}) \). In \([S-G]\), we obtained the following extension of Beilinson’s pairing for higher graded pieces:

**Theorem 8.1.** Let \( X/\overline{Q} \) be a smooth projective variety of dimension \( d \) and let \( K/\overline{Q} \) be a finitely generated overfield of transcendence degree \( \nu - 1 \), where \( \nu \geq 1 \) is an integer. Then there exists a pairing

\[
\langle \cdot, \cdot \rangle^{\nu}_{\text{HT}} : \text{Gr}^{\nu}_F \text{CH}^r(X_K; \mathbb{Q}) \times \text{Gr}^{\nu}_F \text{CH}^{d-r+\nu}(X_K; \mathbb{Q}) \to \mathbb{R},
\]

extending Beilinson’s height pairing.

**Proof.** (Sketch only.) First note that \( K \cong \overline{Q}(S) \) where \( S/\overline{Q} \) is a smooth projective variety of dimension \( \nu - 1 \) and let \( \eta_S \) be the generic point of \( S \). In this case \( \rho : X \to S \) is given by \( \text{Pr}_S : S \times X \to S \). We have the short exact sequence at the generic point

\[
0 \to E_{\infty}^{\nu,2r-\nu}(\eta_S) \to E_{\infty}^{\nu,2r-\nu}(\eta_S) \to E_{\infty}^{\nu,2r-\nu}(\eta_S) \to 0,
\]

where

\[
E_{\infty}^{\nu,2r-\nu}(\eta_S) = \Gamma \left( H^{\nu}(\eta_S, R^{2r-\nu}_{\eta_S} \rho_* \mathbb{Q}(r)) \right) = 0
\]

by the affine Lefschetz theorem and

\[
E_{\infty}^{\nu,2r-\nu}(\eta_S) = \frac{J (W_{-1} (H^{\nu-1}(\eta_S, \mathbb{Q}) \otimes H^{2r-\nu}(X, \mathbb{Q}) (r)) \otimes \text{Gr}_W^{\eta} (H^{\nu-1}(\eta_S, \mathbb{Q}) \otimes H^{2r-\nu}(X, \mathbb{Q}) (r)) (r))}{\Gamma (\text{Gr}_W^{\eta} (H^{\nu-1}(\eta_S, \mathbb{Q}) \otimes H^{2r-\nu}(X, \mathbb{Q}) (r)) (r))}.
\]

We also have \( \text{Gr}^{\nu}_F \text{CH}^r(X_K; \mathbb{Q}) \hookrightarrow E_{\infty}^{\nu,2r-\nu}(\eta_S) \).

The following two propositions are key to the proof.
Proposition 8.2 (Lewis). There is an injective map
$$Gr_F^\nu CH^r(X_K; \mathbb{Q}) \hookrightarrow J(H_0).$$
Here $J(H_0)$ denotes the jacobian of the pure Hodge structure $H_0$ defined by
$$H_0 := \left( \frac{H^{\nu-1}(S, \mathbb{Q})}{N^\nu_H H^{\nu-1}(S, \mathbb{Q})} \otimes \frac{H^{2r-\nu}(X, \mathbb{Q})}{N^\nu_H H^{2r-\nu}(X, \mathbb{Q})} \right)(r).$$

Rather than explain the details of the proof of Proposition 8.2, the main philosophical point is the expectation$^6$ that
$$\text{Ext}_{\mathcal{M}_K}(\text{Spec}(K), h^{2r-\nu}(X_K)(r)) \simeq \text{Ext}_{\mathcal{M}_K}(\text{Spec}(K), \Delta^{2r-\nu}(X_K)(r)).$$

Unfortunately, any attempt to extend Proposition 8.2 beyond $X = X_{\mathbb{Q}}$, viz., to $X_K$, involving a twisted spread $X \rightarrow S$, $\overline{\mathbf{Q}}(S) = K$, seems highly non-trivial. Next,

Proposition 8.3 (Lewis). There is a surjective map
$$CH^r_{\text{hom}}((S \times X)_{\mathbb{Q}}; \mathbb{Q}) \twoheadrightarrow Gr_F^\nu CH^r(X_K; \mathbb{Q})$$
given by the projector $\Delta_S \otimes \Delta_X(2d - 2r + \nu, 2r - \nu)$.

Proof. First of all we observe that $CH^r(S \times \mathbb{Q} X) \twoheadrightarrow CH^r(X_K)$ is surjective. Therefore by Theorem 5.3(v), the composite involving the full Chow group:
$$\Delta_{X_K}(2d - 2r + \nu, 2r - \nu)_* CH^r(X_K; \mathbb{Q}) \simeq Gr_F^\nu CH^r(X_K; \mathbb{Q}),$$
is surjective. Now for any smooth affine subvariety $U \subset S/\mathbb{Q}$ of dimension $< \nu$, the affine Lefschetz theorem implies that $H^\nu(U, \mathbb{Q}) = 0$. Applying the Künneth formula to $H^{2r}(U \times X, \mathbb{Q}(r))$, it follows that
$$(\text{Id} \otimes \Delta_{X_K}(2d - 2r + \nu, 2r - \nu))_* CH^r(U \times \mathbb{Q} X; \mathbb{Q}) \hookrightarrow 0 \in H^{2r}(U \times X, \mathbb{Q}(r)),$$
and hence accordingly,
$$\Delta_{X_K}(2d - 2r + \nu, 2r - \nu)_* CH^r(X_K; \mathbb{Q}) \simeq Gr_F^\nu CH^r(X_K; \mathbb{Q}),$$
is surjective. Finally
$$CH^r_{\text{hom}}(S \times \mathbb{Q} X; \mathbb{Q}) \twoheadrightarrow CH^r_{\text{hom}}(U \times \mathbb{Q} X; \mathbb{Q}),$$
is surjective by the Hodge conjecture, and the proposition follows. □

$^6$This is also apparent in the work of Shuji Saito [SSa].
By our assumptions,
\[ \Phi_r : \text{CH}_{\text{hom}}( (S \times X)_{\overline{Q}} ; \mathbb{Q}) \hookrightarrow J( H^{2r-1}(S \times X, \mathbb{Q}(r)) ) . \]

We have the following decomposition at the level of jacobians.
\[ J( H^{2r-1}(S \times X, \mathbb{Q}(r)) ) \cong J( H_0 ) \oplus J( H_0^+ ) , \]
where \( H_0^+ \) arises due to polarization. Let \( P_1 \) be the projector
\[ H^{2r-1}(S \times X, \mathbb{Q}(r)) \twoheadrightarrow H_0 , \]
and \( w_1 \) be an algebraic cycle lying in the K"unneth component
\[ H^{2(d+r-1)}(S \times X, \mathbb{Q}(d+r)) \otimes H^{2r-1}(S \times X, \mathbb{Q}(r)) \]
corresponding to it. Let
\[ \Xi_1 := w_1,_* ( \text{CH}_{\text{hom}}( (S \times X)_{\overline{Q}} ; \mathbb{Q}) ) . \]

Since we are assuming Conjecture 5.1,
\[ F^2 \text{CH}^r( (S \times X)_{\overline{Q}} ; \mathbb{Q}) = 0 \]
and \( \Xi_1 \) is independent of the choice of algebraic cycle representative corresponding to \( P_1 \). Viewing everything inside the jacobian, we get
\[ P_1,_* : \text{CH}_{\text{hom}}( (S \times X)_{\overline{Q}} ; \mathbb{Q}) \twoheadrightarrow \text{Gr}^r_F \text{CH}^r( X_K ; \mathbb{Q}) \]
and
\[ \Phi_r|_{\Xi_1} : \Xi_1 \cong \text{Gr}^r_F \text{CH}^r( X_K ; \mathbb{Q}) . \]

By a similar procedure, we get \( \Xi_2 := w_2,_* ( \text{CH}_{\text{hom}}^{d-r+r}( (S \times X)_{\overline{Q}} ; \mathbb{Q}) ) \), for an algebraic cycle \( w_2 \) (similar to \( w_1 \)), and an isomorphism
\[ \Phi_{d-r+r}|_{\Xi_2} : \Xi_2 \cong \text{Gr}^r_F \text{CH}^{d-r+r}( X_K ; \mathbb{Q}) . \]

Note that \( d-r+r = (d+r-1)-r+1 \) and we have Beilinson’s height pairing
\[ \text{CH}_{\text{hom}}^r( (S \times X)_{\overline{Q}} ; \mathbb{Q}) \times \text{CH}_{\text{hom}}^{d-r+r}( (S \times X)_{\overline{Q}} ; \mathbb{Q}) \rightarrow \mathbb{R} , \]
and hence between \( \Xi_1 \) and \( \Xi_2 \). The desired pairing \( \langle , \rangle_{\text{HT}}^r \) between the spaces \( \text{Gr}^r_F CH^r( X_K ; \mathbb{Q}) \) and \( \text{Gr}^r_F CH^{d-r+r}( X_K ; \mathbb{Q}) \) is now obtained through the isomorphisms above. \( \square \)

**Remark 8.4.** One can show that the height pairing above is independent of the choice of smooth projective variety \( S/\overline{Q} \) with \( \overline{Q}(S) \cong K \).

Since our height pairing \( \langle , \rangle_{\text{HT}}^r \) is given by the one developed by Beilinson, it is only natural that the conjectures in [Be3] have a natural extension for graded pieces. As an example:
Assuming Conjecture 4.2, we conclude and Proposition 8.5 follows immediately. □

Definition 8.6. Let $L_{X_K}$ denote the operation of intersecting with a hyperplane section. Then for $x \neq 0 \in Gr_F \CH^r(X_K; \mathbb{Q})$ such that $L_{X_K}^{d-2r+\nu+1}(x) = 0$, the height pairing

$$(-1)^r \langle x, L_{X_K}^{d-2r+\nu}(x) \rangle_{HT} > 0,$$

when $r \leq (d + \nu)/2$.

Proof. First note that the filtration developed in [Lew2] already has the property that $L_{X_K}^{d-2r+\nu}$ defines an isomorphism between $Gr_F \CH^r(X_K; \mathbb{Q})$ and $Gr_F \CH^{d-r+\nu}(X_K; \mathbb{Q})$, so Proposition 8.5 makes sense. Now for any $x \in \Xi_1$

$$\Phi_{d-r+\nu}(L_{S \times X}^{d-2r+\nu}(x) - w_{2,*} \circ L_{S \times X}^{d-2r+\nu}(x))$$

$$= [L_{S \times X}]^{d-2r+\nu}(\Phi_r(x)) - [w_2] * [L_{S \times X}]^{d-2r+\nu}(\Phi_r(x))$$

$$= [L_{S \times X}]^{d-2r+\nu}((\Phi_r(x)) - [L_{S \times X}]^{d-2r+\nu}(\Phi_r(x)) = 0.$$

Since we are assuming Conjecture 5.1, we get

$$L_{S \times X}^{d-2r+\nu}(x) = w_{2,*} \circ L_{S \times X}^{d-2r+\nu}(x),$$

which shows that $L_{S \times X}^{d-2r+\nu}$ maps $\Xi_1$ to $\Xi_2$, isomorphically. Further, let $\Xi_2' \subset CH_{d-r+\nu+1}^{\text{hom}}(S \times X; \mathbb{Q})$ be such that $\Xi_2' \cong Gr_F \CH^{d-r+\nu+1}(X_K; \mathbb{Q})$. For $x' \in \Xi_1$,

$$\Phi_r(x') = x \in Gr_F \CH^r(X_K; \mathbb{Q}) \implies \Phi_{d-r+\nu+1}(L_{S \times X}^{d-2r+\nu+1}(x')) = L_{X_K}^{d-2r+\nu+1}(x).$$

So, $L_{X_K}^{d-2r+\nu+1}(x) = 0 \implies L_{S \times X}^{d-2r+\nu+1}(x') = 0$. We also have

$$(-1)^r \langle x, L_{X_K}^{d-2r+\nu}(x) \rangle_{HT} = (-1)^r \langle x', L_{S \times X}^{d-2r+\nu}(x') \rangle_{HT}.$$

Note that $x' \in \Xi_1 \subset CH_{d-r+\nu+1}^{\text{hom}}(S \times X; \mathbb{Q})$ and $L_{S \times X}^{d-2r+\nu+1}(x') = 0$. Now assuming Conjecture 4.2, we conclude

$$(-1)^r \langle x', L_{S \times X}^{d-2r+\nu}(x') \rangle_{HT} > 0,$$

and Proposition 8.5 follows immediately. □

We study the following subspace of $Gr_F \CH^r(X_K; \mathbb{Q})$:

Definition 8.6. Let $F^\nu CH_{\text{alg}}(X_K; \mathbb{Q}) :=$

$$F^\nu \CH^r(X_K; \mathbb{Q}) \bigcap \left[ \text{Im}(CH_{\text{alg}}^r((S \times X)_Q; \mathbb{Q}) \to \CH^r(X_K; \mathbb{Q})) \right].$$

Then we define

$$Gr_F CH_{\text{alg}}^r(X_K; \mathbb{Q}) := \text{Im} \left( F^\nu CH_{\text{alg}}^r(X_K; \mathbb{Q}) \to Gr_F \CH^r(X_K; \mathbb{Q}) \right).$$
There is one remark in order: If $S'$ is another such variety, then we can dominate both $S$ and $S'$ by a desingularization of a third $S'' \to S \times S'$. From this, and the fact that the rational Chow group of cycles algebraically equivalent to zero being a $\mathbb{Q}$ vector space, one can show
\[
\text{Im} \left( \text{CH}^r_{\text{alg}}((S \times X)_{\mathbb{Q}}; \mathbb{Q}) \to \text{CH}^r(X_K; \mathbb{Q}) \right)
\]
and
\[
\text{Im} \left( \text{CH}^r_{\text{alg}}((S' \times X)_{\mathbb{Q}}; \mathbb{Q}) \to \text{CH}^r(X_K; \mathbb{Q}) \right),
\]
are the same. Thus the definition of $\text{Gr}_\nu^{\text{FCH}_{\text{alg}}}(X_K; \mathbb{Q})$ is independent of the choice of $S$. Now we have the following

**Theorem 8.7.** Under the same set up as in Theorem 8.1, we have the height pairing
\[
\langle \cdot, \cdot \rangle_{\text{HT, alg}} : \text{Gr}^{\nu}_{\text{F}} \text{CH}^r_{\text{alg}}(X_K; \mathbb{Q}) \times \text{Gr}^{\nu}_{\text{F}} \text{CH}^{d-r+\nu}_{\text{alg}}(X_K; \mathbb{Q}) \to \mathbb{R},
\]
extending the Néron-Tate pairing.

**Proof.** Assuming Conjecture 5.1 we get that
\[
\text{CH}^r_{\text{alg}}((S \times X)_{\mathbb{Q}}; \mathbb{Q}) \hookrightarrow J^r_{\text{alg}}(S \times X)_{\mathbb{Q}},
\]
and
\[
\text{CH}^{d-r+\nu}_{\text{alg}}((S \times X)_{\mathbb{Q}}; \mathbb{Q}) \hookrightarrow J^{d-r+\nu}_{\text{alg}}(S \times X)_{\mathbb{Q}}.
\]

The proof now goes exactly in the same way as Theorem 8.1, if we replace $\text{CH}^r_{\text{hom}}((S \times X)_{\mathbb{Q}}; \mathbb{Q})$ (resp. $\text{CH}^{d-r+\nu}_{\text{hom}}((S \times X)_{\mathbb{Q}}; \mathbb{Q})$) with $\text{CH}^r_{\text{alg}}((S \times X)_{\mathbb{Q}}; \mathbb{Q})$ (resp. $\text{CH}^{d-r+\nu}_{\text{alg}}((S \times X)_{\mathbb{Q}}; \mathbb{Q})$). We obtain $\Xi_{1, \text{alg}} \subset \text{CH}^r_{\text{alg}}((S \times X)_{\mathbb{Q}}; \mathbb{Q})$ (respectively $\Xi_{2, \text{alg}} \subset \text{CH}^{d-r+\nu}_{\text{alg}}((S \times X)_{\mathbb{Q}}; \mathbb{Q})$), such that
\[
\Xi_{1, \text{alg}} \cong \text{Gr}^{\nu}_{\text{F}} \text{CH}^r_{\text{alg}}(X_K; \mathbb{Q}), \quad \Xi_{2, \text{alg}} \cong \text{Gr}^{\nu}_{\text{F}} \text{CH}^{d-r+\nu}_{\text{alg}}(X_K; \mathbb{Q}).
\]
The height pairing $\langle \cdot, \cdot \rangle_{\text{HT, alg}}$ is now given as the pairing between $\Xi_{1, \text{alg}}$ and $\Xi_{2, \text{alg}}$.

**Remark 8.8.** The reasons for restricting to this particular subspace are the following:

1. Since the height pairing for cycles algebraically equivalent to zero is given by the Néron-Tate pairing ([Be3], Remark 4.0.8), one can work without the assumption of Conjecture 5.1

2. Further for cycles algebraically equivalent to zero, assumption (17) of [Ku] is no longer necessary ([Ku], §8).

So in effect, one can freely use the machineries available from arithmetic intersection theory to compute the height pairing, albeit (GHC). We will illustrate this with an example.
8.9. An example computation. Using the formalism of arithmetic intersection theory discussed in §1, we present here a computation related to the theory developed so far.

**Example 8.10.** Let \( X = C_1 \times C_2 \) be the product of smooth projective curves \( C_1 \) and \( C_2 \), defined over \( \overline{\mathbb{Q}} \). For \( \nu = 2 \), we fix an embedding \( K = \overline{\mathbb{Q}}(C_2) \hookrightarrow \mathbb{C} \) (so naturally \( S = C_2 \) following the set up of Theorem 8.1), and let \( p = \eta_2 \in C_2(\mathbb{C}) \) be a very general point corresponding to this embedding. To be more precise, \( \eta_2 \) is regarded as the generic point of \( \overline{\mathbb{Q}}(C_2) \) (so \( \overline{\mathbb{Q}}(C_2) = \overline{\mathbb{Q}}(\eta_2) \)), and recall that any point \( p \in S(\mathbb{C}) \) for which evaluation at \( p \) defines an embedding \( \overline{\mathbb{Q}}(\eta_2) \hookrightarrow \mathbb{C} \) is defined to be a very general point. Although this notation is a bit slang, we write \( p = \eta_2 \). We fix \( e_2 \in C_2(\overline{\mathbb{Q}}) \). For distinct points \( p_1, q_1, p_2, q_2 \in C_1(\overline{\mathbb{Q}}) \), let

\[
\xi_1 := (p_1 - q_1) \times (\eta_2 - e_2) \in \text{Gr}_2^2 \text{CH}^2_{\text{alg}}(X_K; \mathbb{Q}), \\
\xi_2 := (p_2 - q_2) \times (\eta_2 - e_2) \in \text{Gr}_2^2 \text{CH}^2_{\text{alg}}(X_K; \mathbb{Q}).
\]

Assume also

\[
N^1_H(H^1(C_1, \mathbb{Q}) \otimes H^1(C_2, \mathbb{Q})) = 0. 
\]

Then

\[
\langle \xi_1, \xi_2 \rangle_{\text{HT,alg}} = \text{deg} \left( \Delta_{C_2}^2(1, 1) \right) \langle p_1 - q_1, p_2 - q_2 \rangle_{\text{NT}},
\]

where on the RHS we have Néron-Tate pairing.

We add a remark before we begin the proof:

**Remark 8.11.** The assumption in (3) holds for example if we take \( X = E_1 \times E_2 \), a product of two non-isogenous elliptic curves; for here \( N^1_H(H^1(E_1, \mathbb{Q}) \otimes H^1(E_2, \mathbb{Q})) = H^2_{\text{alg}}(E_1 \times E_2, \mathbb{Q}) \cap (H^1(E_1, \mathbb{Q}) \otimes H^1(E_2, \mathbb{Q})) = 0 \) follows from the fact that any non-zero element

\[
[\xi] \in H^2_{\text{alg}}(E_1 \times E_2, \mathbb{Q}) \cap (H^1(E_1, \mathbb{Q}) \otimes H^1(E_2, \mathbb{Q}))
\]

will in turn define an isogeny between \( E_1 \) and \( E_2 \).

Since Example 8.10 illustrates the potential of our theory so effectively, we will provide a more detailed proof.

**Proof.** From the assumptions of Example 8.10 we get that \( S \times X = C_2 \times (C_1 \times C_2) \cong C_1 \times (C_2 \times C_2) \). We have the Chow-Kühneth decomposition for smooth curves

\[
\Delta_{C_2}(1, 1) = \Delta_{C_2} - e_2 \times C_2 - C_2 \times e_2.
\]

Now put

\[
\xi_1 := (p_1 - q_1) \times \Delta_{C_2}(1, 1) \in \text{CH}^2_{\text{alg}}(C_1 \times (C_2 \times C_2); \mathbb{Q}),
\]
Lemma 8.12. Let \( \xi_i \in \Xi_{alg} \mapsto \xi_i, i = 1, 2 \) under the isomorphism \( \Xi_{alg} \cong G^2_{\lambda} \text{CH}^2_{alg}(X_k; \mathbb{Q}) \).
Here \( \Xi_{alg} \subset \text{CH}^2_{alg}(C_1 \times (C_2 \times C_2); \mathbb{Q}) \) is the suitable subspace (see Theorem 8.7 for details). Thus, the height pairing relation,
\[
\langle \xi_1, \xi_2 \rangle_{HT, \text{alg}} \text{ is given by } \langle \xi_1, \xi_2 \rangle_{HT}.
\]
We provide a general computation in the formalism of arithmetic intersection theory, for which \( \langle \tilde{\xi}_1, \tilde{\xi}_2 \rangle_{HT} \) is a particular case.

**Lemma 8.12.** Let \( C \) be smooth projective curve and \( X \) be a smooth projective variety of dimension \( d - 1 \), both defined over a number field \( k \). Let \( \alpha_1, \alpha_2 \in \text{CH}^d_{\text{alg}}(C; \mathbb{Q}) \) and \( \pi_1 : C \times X \to C \) and \( \pi_2 : C \times X \to X \) are the projections. Given \( w_1 \in \text{CH}^{d-1}(X; \mathbb{Q}) \) and \( w_2 \in \text{CH}^{d-r-1}(X; \mathbb{Q}) \) and the cycles
\[
a_1 := \pi_1^*(\alpha_1) \cdot \pi_2^*(w_1) \in \text{CH}^d_{\text{alg}}(C \times X; \mathbb{Q})
\]
\[
a_2 := \pi_1^*(\alpha_2) \cdot \pi_2^*(w_2) \in \text{CH}^{d-r+1}_{\text{alg}}(C \times X; \mathbb{Q})
\]
We get the following height pairing relation:
\[
\langle a_1, a_2 \rangle_{HT} = (\deg(w_1 \cdot w_2)_X)\langle \alpha_1, \alpha_2 \rangle_{\text{NT}} ,
\]
where \( (w_1 \cdot w_2)_X \) is the usual intersection pairing on \( X \).

**Proof.** Let \( \tilde{C} \) be a regular semi-stable model for \( C \) over \( \text{Spec}(\mathcal{O}_{k'}) \) (after a finite extension \( k' \) of the ground field \( k \)). Choose \( Z_i, i = 1, 2 \) cycles on \( \tilde{C} \) of codimension 1 such that
\[
\begin{align*}
(1) & \quad \left[ Z_i \right]_{C} = \alpha_i, \\
(2) & \quad Z_i \cdot V = 0 \text{ for any vertical cycle } V.
\end{align*}
\]
One can arrange the above situation by Th. 1.3 of [Hi]. Choose \( g_i, i = 1, 2 \). Green’s functions for \( Z_i \) such that \( dd^c g_i + \delta_{Z_i} = 0 \) (since \( \alpha_i \) is null-homologous, the cohomology class \( [\omega_{Z_i}] = 0 \)). We have
\[
[(Z_i, g_i)] \in \widehat{\text{CH}}^1(\tilde{C}), \ i = 1, 2.
\]
Then,
\[
\langle \alpha_1, \alpha_2 \rangle_{\text{NT}} = \deg_{\tilde{C}}((Z_1, g_1) \cdot (Z_2, g_2)) \in \widehat{\text{CH}}^1(\text{Spec}(\mathbb{Z})) \otimes \mathbb{Q} \cong \mathbb{R},
\]
is independent of the choices of \( Z_i, g_i \).

Now, for any projective and flat model \( \tilde{X}' \) over \( \text{Spec}(\mathcal{O}_{k'}) \) of \( X \), by de Jong’s alteration ([Hd], Theorem 8.2) we get a projective, flat and regular scheme \( \tilde{X} \) over a finite extension of \( k' \) (in turn a finite extension of \( k \)), with a finite and surjective morphism to \( \tilde{X}' \). In particular
dim(\(\tilde{X}'\)) = dim(\(\tilde{X}\)). Let \(W_i, i = 1, 2\) be cycles on \(\tilde{X}\) of codimensions \(r - 1\) and \(d - r\) respectively such that
\[W_i|_X = w_i, \ i = 1, 2.\]
Let \(g_{W_1}\) (resp. \(g_{W_2}\)) be a Green current for \(W_1\) (resp. \(W_2\)). Then
\[
[(W_1, g_{W_1})] \in \widehat{\text{CH}}^{r-1} (\tilde{X})
\]
\[
[(W_2, g_{W_2})] \in \widehat{\text{CH}}^{d-r} (\tilde{X}).
\]
For the scheme \(\tilde{C} \times_{\text{Spec}(\mathcal{O}_{k'})} \tilde{X}\), we can use the alteration trick once more to obtain a regular flat and projective scheme \(Z\) over \(\text{Spec}(\mathcal{O}_{k''})\), where \(k''\) is a finite extension of \(k\) and a dominant and finite morphism \(f : Z \to \tilde{C} \times_{\text{Spec}(\mathcal{O}_{k'})} \tilde{X}\). In particular \(\dim(Z) = \dim(\tilde{C} \times_{\text{Spec}(\mathcal{O}_{k'})} \tilde{X}) = d + 1\). For the projections
\[
\pi_{\tilde{C}} : \tilde{C} \times_{\text{Spec}(\mathcal{O}_{k'})} \tilde{X} \to \tilde{C},
\]
\[
\pi_{\tilde{X}} : \tilde{C} \times_{\text{Spec}(\mathcal{O}_{k'})} \tilde{X} \to \tilde{X},
\]
consider
\[
f_{\tilde{C}} := \pi_{\tilde{C}} \circ f
\]
\[
f_{\tilde{X}} := \pi_{\tilde{X}} \circ f,
\]
and the cycles
\[
\tilde{a}_1 := f^*_{\tilde{C}}((Z_1, g_{Z_1})) \cdot f^*_\tilde{X}([W_1, g_{W_1}])
\]
\[
\tilde{a}_2 := f^*_{\tilde{C}}((Z_2, g_{Z_2})) \cdot f^*_\tilde{X}([W_2, g_{W_2}]).
\]
Then (up to rational multiples, which will arise since we are using alterations and extensions of the base field \(k\))
\[
\langle a_1, a_2 \rangle_{HT} = \widehat{\deg}_Z (\tilde{a}_1 \cdot \tilde{a}_2) \in \widehat{\text{CH}}^1 (\text{Spec}(\mathbb{Z})) \otimes \mathbb{Q} \cong \mathbb{R}.
\]
Since \(f^*_{\tilde{C}}\) and \(f^*_\tilde{X}\) are morphisms of rings (\([\text{GS}], 4.4.3 (5)\)),
\[
\tilde{a}_1 \cdot \tilde{a}_2 = f^*_{\tilde{C}}((Z_1, g_{Z_1}) \cdot ([Z_2, g_{Z_2}]) \cdot f^*_\tilde{X}([W_1, g_{W_1}] \cdot ([W_2, g_{W_2}])
\]
By the projection formula for arithmetic intersection pairing (\([\text{GS}], 4.4.3 (7)\))
\[
f_{\tilde{C},*}(\tilde{a}_1 \cdot \tilde{a}_2) = \left[\langle Z_1, g_{Z_1} \rangle \cdot [Z_2, g_{Z_2}] \cdot \left[ f_{\tilde{C},*}(\tilde{a}_1) \cdot f_{\tilde{X},*}(\tilde{a}_2) \right] \right]_{\text{CH}^r(\tilde{C}, \mathbb{Q})}
\]
Since
\[
\widehat{\deg}_Z (\tilde{a}_1 \cdot \tilde{a}_2) = \widehat{\deg}_{\tilde{C}} \left( f_{\tilde{C},*}(\tilde{a}_1 \cdot \tilde{a}_2) \right)
\]
and
\[
f_{\tilde{C},*}(f^*_\tilde{X}([W_1, g_{W_1}] \cdot ([W_2, g_{W_2}])) = \deg(w_1 \cdot w_2)_X,
\]
we obtain our desired result. We note here that since we are using $\mathbb{Q}$-valued intersection pairing, the relations among various height-pairings won’t change.

Quite generally, one can also prove the following:

**Theorem 8.13 (S-G).** Given smooth projective curves $C_1, \ldots, C_d$ over $\overline{\mathbb{Q}}$, let $X = C_1 \times \cdots \times C_d$. For $\nu \geq 2$, we fix an embedding $K = \mathbb{Q}(C_2 \times \cdots \times C_\nu) \hookrightarrow \mathbb{C}$, and let $p = (\eta_2, \ldots, \eta_\nu) \in C_2(\mathbb{C}) \times \cdots \times C_\nu(\mathbb{C})$ be a very general point corresponding to this embedding (see Example 8.10 for a clarification of “general”). We fix $e_2 \in C_2(\mathbb{Q})$, $e_3 \in C_3(\mathbb{Q}), \ldots, e_d \in C_d(\mathbb{Q})$. For distinct points $p_1, q_1, p_2, q_2 \in C_1(\overline{\mathbb{Q}})$ and $\nu \geq r \leq d$, let

$$
\xi_1 := Pr^*_{1,\ldots,\nu}((p_1 - q_1) \times (\eta_2 - e_2) \times \cdots \times (\eta_\nu - e_\nu)) \bigcap Pr^*_{\nu+1,\ldots,(\nu+r)}(e_{\nu+1}, \ldots, e_r) \in Gr^\nu_F CH^d_{\text{alg}}(X_K; \mathbb{Q}),
$$

$$
\xi_2 := Pr^*_{1,\ldots,\nu}((p_2 - q_2) \times (\eta_2 - e_2) \times \cdots \times (\eta_\nu - e_\nu)) \bigcap Pr^*_{r+1,\ldots,d}(e_{r+1}, \ldots, e_d) \in Gr^\nu_F CH^{d-r+\nu}_{\text{alg}}(X_K; \mathbb{Q}).
$$

Assume also

$$
N_1^H (H^1(C_1, \mathbb{Q}) \otimes \cdots \otimes H^1(C_\nu, \mathbb{Q})) = 0,
$$

and

$$
N_2^Q (H^1(C_2, \mathbb{Q}) \otimes \cdots \otimes H^1(C_d, \mathbb{Q})) = 0.
$$

Then, $\langle \xi_1, \xi_2 \rangle_{HT, \text{alg}} = 0$.

**Remark 8.14.** For a self-product of a CM-elliptic curve (S-G, §8.2), we were able to eliminate the assumption in (8) altogether.

9. **An Archimedean pairing involving the equivalence relation defining higher Chow groups**

In this section, and for each $m \geq 0$, we construct a pairing on the cycle level, involving the equivalence relation in the definition of Bloch’s higher Chow groups $CH^r(X, m)$ defined below. The case when $m = 0$ has already been defined in [P] and the nature of this pairing is more akin to the Archimedean height pairing defined in the literature. It was first discussed in [C-L]; however presentation here is intended to be more user friendly. A general construction of this pairing for all $m$ is in order. We first recall that two subvarieties $V_1, V_2$ of a given
variety intersect properly if \( \text{codim}\{V_1 \cap V_2\} \geq \text{codim} \, V_1 + \text{codim} \, V_2 \). This notion naturally extends to algebraic cycles.

(i) **Higher Chow groups.** Let \( W/k \) be a quasi-projective variety over a field \( k \). Put \( z^r(W) = \text{free abelian group generated by subvarieties of codimension } r \) in \( W \),

\[
\Delta^m := \text{Spec} \left( \frac{k[t_0, \ldots, t_m]}{1 - \sum_{j=0}^m t_j} \right)
\]

the standard \( m \)-simplex, and \( z^r(W,m) = \{ \xi \in z^k(W \times \Delta^m) \mid \xi \text{ meets all faces } \{ j_1 = \cdots = j_\ell = 0 \mid \ell = 1, \ldots, m \} \text{ properly} \} \).

**Definition 9.1** ([Blo1]). \( CH^\bullet(W, \bullet) = \text{homology of } \{ z^\bullet(W, \bullet), \partial \} \). We put \( CH_k(W) := CH_k(W, 0) \).

(ii) **Cubical version.** Let \( \square^m := (\mathbb{P}^1 \setminus \{1\})^m \) with coordinates \( z_i \) and \( 2^m \) codimension one faces obtained by setting \( z_i = 0, \infty \), and boundary maps \( \partial = \sum (-1)^{i-1}(\partial_i^0 - \partial_i^\infty) \), where \( \partial_i^0, \partial_i^\infty \) denote the restriction maps to the faces \( z_i = 0, z_i = \infty \) respectively. The rest of the definition is completely analogous for \( z^r(W,m) \subset z^r(W \times \square^m) \), except that one has to quotient out by the subgroup \( z^r_{\text{dgt}}(W,m) \subset z^r(W,m) \) of degenerate cycles obtained via pullbacks \( \sum_{j=1}^m pr_j^* : z^r(W,m-1) \to z^r(W,m), \) \( pr_j : W \times \square^m \to W \times \square^{m-1} \) the \( j \)-th canonical projection. It is known that both complexes are quasi-isomorphic (Bloch (unpublished)/Levine [Lv]; independently).

9.2. **A quick detour via Milnor K-theory.** An excellent reference for this part is [B-T]. Let \( F \) be a field with multiplicative group \( F^\times \subset F \). Consider the graded tensor algebra

\[
T(F) := \bigoplus_{r=0}^{\infty} (F^\times)^\otimes z^m = \mathbb{Z} \oplus F^\times \oplus \cdots,
\]

and let \( R(F) \) be the graded 2-sided ideal generated by

\[
\{ \tau \otimes (1 - \tau) \mid \tau \in F^\times \setminus \{1\} \}.
\]

Recall that the Milnor K-theory of \( F \) is given by

\[
K^M_\bullet(F) := T(F)/R(F) = \bigoplus_{r=0}^{\infty} K^M_r(F).
\]

Further, recall that \( K^M_r(F) \simeq CH^r(\text{Spec}(F), r) \), (Nesterenko/Suslin (1990), Totaro (1992)). Now let \( W/k \) be a smooth scheme over a field \( k \). If one replaces \( F \) by \( O^\times_W \), then we arrive at the sheaf \( K^M_{r,W} \) of Milnor
$K$-groups. To be more precise, let $O_W$ be the sheaf of regular functions on $X$, with sheaf of units $O_W^\times$. As in [Ka], we put

$$K^M_{r,W} := \left(\underset{r \text{ times}}{O_W^\times \otimes \cdots \otimes O_W^\times}\right)/\mathcal{J},$$

(Milnor sheaf),

where $\mathcal{J}$ is the subsheaf of the tensor product generated by sections of the form:

$$\{\tau_1 \otimes \cdots \otimes \tau_r \mid \tau_i + \tau_j = 1, \text{ for some } i \neq j\}.$$

For example, $K^M_{1,W} = O_X^\times$. The higher Chow groups $\text{CH}^r(C(m))$ come naturally equipped with a coniveau filtration involving codimension of cycles when projected to $W$, whose graded pieces can be computed via a local-to-global spectral sequence ([BO], [Blo1]), involving flasque resolutions of certain sheaves. Via the works of Elbaz-Vincent/Müller-Stach (1998), and Gabber (1992), (see [MS2], together with [Ke]), one of those sheaves is $K^M_{r,W}$. This, together with partial degeneration of the aforementioned spectral sequence leads to:

**Theorem 9.3** (See [MS2]). For $0 \leq m \leq 2$, there is an isomorphism

$$H_{Z,\text{Zar}}^{r-m}(W, K^M_{r,W}) \simeq \text{CH}^r(W, m).$$

In the context of Milnor $K$-theory, the last 3 terms of the flasque resolution of $K^M_{r,W}$ are

$$\bigoplus_{\text{cd}_W Z = r-2} K^M_2(\mathbb{C}(Z)) \xrightarrow{T} \bigoplus_{\text{cd}_W Z = r-1} K^M_1(\mathbb{C}(Z)) \xrightarrow{\text{div}} \bigoplus_{\text{cd}_W Z = r} K^M_0(\mathbb{C}(Z)).$$

If we interpret this in terms of global sections, this leads to a complex whose last three terms and corresponding homologies (norm/graph maps, indicated at $\uparrow$) for $0 \leq m \leq 2$ are:

$$\bigoplus_{\text{cd}_W Z = r-2} K^M_2(\mathbb{C}(Z)) \xrightarrow{T} \bigoplus_{\text{cd}_W Z = r-1} \mathbb{C}(Z)^\times \xrightarrow{\text{div}} \bigoplus_{\text{cd}_W Z = r} \mathbb{Z}$$

(9)

where as a reminder, $\text{div}$ is the divisor map of zeros minus poles of a rational function, and $T$ is the Tame symbol map. Again as a reminder, the Tame symbol map

$$T : \bigoplus_{\text{cd}_X Z = r-2} K^M_2(\mathbb{C}(Z)) \to \bigoplus_{\text{cd}_D Z = r-1} K^M_1(\mathbb{C}(D)),$$

is defined as follows. First $K^M_2(\mathbb{C}(Z))$ is generated by symbols $\{f, g\}$, $f, g \in \mathbb{C}(Z)^\times$, under $f \otimes g \mapsto \{f, g\}$. 

For $f, g \in \mathbb{C}(Z)^\times$,
\[
T(\{f, g\}) = \sum_D (-1)^{\nu_D(f)\nu_D(g)} \left( \frac{f^{\nu_D(g)}}{g^{\nu_D(f)}} \right)_D,
\]
where $(\cdots)_D$ means restriction to the generic point of $D$, and $\nu_D$ represents order of a zero or pole along an irreducible divisor $D \subset Z$.

**Example 9.4.** Taking cohomologies of the complex in (9), we have:

(i) $\text{CH}^r(W, 0) = z^r(W)/z^r_{\text{rat}}(W) =: \text{CH}^r(W)$.

(ii) $\text{CH}^r(W, 1)$ is represented by classes of the form $\xi = \sum_j (f_j, D_j)$, where $\text{codim}_X D_j = r - 1$, $f_j \in \mathbb{C}(D_j)^\times$, and $\sum \text{div}(f_j) = 0$; modulo the image of the Tame symbol.

(iii) $\text{CH}^r(W, 2)$ is represented by classes in the kernel of the Tame symbol; modulo the image of a higher Tame symbol.

In this section we will adopt the cubical version of $\text{CH}^\bullet(W, \bullet)$, albeit a simplicial version can also be arranged [KLL]. The intersection product for cycles in the cubical version, is easy to define. On the level of cycles, and in $W \times W \times \square^{m+n}$, one has
\[
z^r(W, m) \times z^k(W, n) \to z^{r+k}(W \times W, m + n);
\]
however the pullback along the diagonal
\[
z^{r+k}(W \times W, m + n) \to z^{r+k}(W, m + n),
\]
is not well-defined, even for smooth $W$. In particular, for smooth $W$, the issue of when an intersection product is defined, which is a general position statement involving proper intersections, has to be addressed since we will be working on the level of cycles. On the level of Chow groups, a moving lemma of Bloch (adapted to the cubical situation) guarantees a pullback for smooth $W$:
\[
\text{CH}^\bullet(W \times W, \bullet) \to \text{CH}^\bullet(W, \bullet),
\]
and hence an intersection product for smooth $W$.

Let us return to the situation where $X$ be a projective algebraic manifold of dimension $d$, and let $z^r_{\text{rat}}(X, m) := \partial(z^r(X, m + 1)) \subset z^r(X, m)$ be the equivalence relation subgroup defining the higher Chow groups $\text{CH}^r(X, m)$. As in [KLM], we will need to restrict ourselves to those precycles $\xi \in z^r(X, \bullet)$ that are in general position with respect to

---

As a reminder, for $\xi \in z^r(X, \bullet)$ to be a cycle, we require $\partial \xi = 0.$
the real subsets $X \times [-\infty, \infty]$, and we will denote this by $z^*_\infty(X, \bullet)$. Now introduce

$$\Lambda^0(r, m, X) = \left\{ (\xi_1, \xi_2) \in z^r_{\Re, \rat}(X, m) \times z^{d-r+m+1}_{\Re, \rat}(X, m) \mid \xi_1 \cap |\xi_2| = \emptyset \right\}.$$ 

$$\Lambda^+(r, m, X) = \left\{ (\xi_1, \xi_2) \in \Lambda^0(r, m, X) \mid \xi_1 = \partial \xi'_1, \text{ where } \xi'_1 \cap \xi_2 \text{ is defined } \right\},$$

$$\Lambda^-(r, m, X) = \left\{ (\xi_1, \xi_2) \in \Lambda^0(r, m, X) \mid \xi_2 = \partial \xi'_2, \text{ where } \xi_1 \cap \xi'_2 \text{ is defined } \right\},$$

$$\Lambda(r, m, X) \subset \Lambda^+(r, m, X) \cap \Lambda^-(r, m, X),$$

is characterized by the requirement that $\xi'_1 \cap \xi'_2$ is defined, viz,

$$\xi'_1 \cap \xi'_2 \in z^{d+m+1}_{\Re}(X, 2m + 2).$$

**Theorem 9.5.** There are natural pairings

$$\langle \cdot, \cdot \rangle_+^m : \Lambda^+(r, m, X) \to \mathbb{C}/\mathbb{Z}(1),$$

$$\langle \cdot, \cdot \rangle_-^m : \Lambda^-(r, m, X) \to \mathbb{C}/\mathbb{Z}(1),$$

which satisfy the following:

(i) **(Reciprocity)** On $\Lambda(r, m, X)$, $\langle \cdot, \cdot \rangle_+^m = (-1)^m \langle \cdot, \cdot \rangle_-^m$.

(ii) **(Bilinearity)** If $(\xi_1^{(1)}, \xi_2), (\xi_1^{(2)}, \xi_2) \in \Lambda^+(r, m, X)$, then

$$\langle \xi_1^{(1)} + \xi_2^{(2)}, \xi_2 \rangle^+_m = \langle \xi_1^{(1)}, \xi_2 \rangle^+_m + \langle \xi_1^{(2)}, \xi_2 \rangle^+_m.$$ 

If $(\xi_1, \xi_2^{(1)}), (\xi_1, \xi_2^{(2)}) \in \Lambda^-(r, m, X)$, then

$$\langle \xi_1^{(1)} + \xi_2^{(2)}, \xi_2 \rangle^-_m = \langle \xi_1^{(1)}, \xi_2 \rangle^-_m + \langle \xi_1, \xi_2^{(2)} \rangle^-_m.$$ 

(iii) **(Projection formula)** Let $\pi : X \to Y$ be a flat surjective morphism between two smooth projective varieties, with $\dim X = d$. Then

$$\langle \xi_1, \pi^* \xi_2 \rangle^\pm_m = \langle \pi_* \xi_1, \xi_2 \rangle^\pm_m \text{ for all } \xi_1 \in z^r_{\rat}(X, m) \text{ and } \xi_2 \in z^{d-r+m+1}_{\rat}(Y, m),$$

with $\pi_* \xi_1, \xi_2 \in \Lambda^\pm(r + s - d, m, Y)$, where $s := \dim Y$.

**Proof.** We first recall the definition of Deligne cohomology. Good sources for this are [Kl], [Ja1] and [KLM]. Let $\mathcal{D}^*_X$ be the (fine) sheaf of complex-valued currents acting on $C^\infty$ complex-valued compactly supported $(2d - \bullet)$-forms, where we recall $\dim X = d$. One has a decomposition into Hodge type:

$$\mathcal{D}^*_X = \bigoplus_{p+q=\bullet} \mathcal{D}^{p,q}_X,$$
where $\mathcal{D}_X^{p,q}$ acts on $(d-p, d-q)$ forms, with Hodge filtration,

$$F^r \mathcal{D}_X^• = \bigoplus_{p+q=•, p \geq r} \mathcal{D}_X^{p,q}.$$  

Likewise, for a subring $\mathbb{A} \subseteq \mathbb{C}$, there is the (soft) sheaf subcomplex $\mathcal{C}_X^•(\mathbb{A}) \subset \mathcal{D}_X^•$ of $\mathbb{A}$-coefficient Borel-Moore chains on $X$. The global sections of a given sheaf $\mathcal{S}$ over $X$ will be denoted by $\mathcal{S}(X)$. Next, for a morphism of complexes $\lambda : \mathcal{A}^• \to \mathcal{C}^•$, we recall the cone complex:

$$\text{Cone}(A^• \xrightarrow{\lambda} B^•) = A^•[1] \oplus B^•,$$

with differential

$$\delta_D : A^{q+1} \oplus B^q \to A^{q+2} \oplus B^{q+1}, \quad (a, b) \xrightarrow{\delta_D} (-da, \lambda(a) + db).$$

**Definition 9.6.** Fix a subring $\mathbb{A} \subseteq \mathbb{R}$. The Deligne cohomology of $X$ is given by

$$H'^i_D(X, \mathbb{A}(j)) := H^i(\text{Cone}(\mathcal{C}_X^•(X, \mathbb{A}(j)) \bigoplus F^j \mathcal{D}_X^•(X) \xrightarrow{\varepsilon-l} \mathcal{D}_X^•(X))[-1]).$$

It is customary of thinking of currents as associated to homology. Note that by simply regarding $\mathcal{C}_X^•(\mathbb{A}(j))$, $F^j \mathcal{D}_X^•$, as acyclic resolutions of the respective sheaves $\mathbb{A}(r)$ and $\Omega^j_{X, d=\text{closed}}$, with quasi-isomorphisms, $\{ \mathbb{A}(j) \to 0 \to \cdots \} \approx \mathcal{C}_X^•(\mathbb{A}(j))$, $\Omega^j_{X, d=\text{closed}} \approx F^j \mathcal{D}_X^•$, the above definition, when compared with the one in [EV], already incorporates Poincaré duality.

**Remark 9.7.** Generally speaking, one thinks of currents as well behaved under proper push-forwards, albeit with no defined pull-back. However, the rules can be broken here if one replaces the sheaf complex of currents on a given manifold with another which is quasi-isomorphic and having better properties with respect to pull-backs and multiplication. The situation is well documented in [K] (§4) and [K-L] (§8). The reader should keep this in mind in the discussion below. To simplify our notation, we will use the notation “$.$” to refer to multiplication of currents. Also we use the principal branch of the log function below.

Continuing with the proof of Theorem 9.5 we now recall the description of the regulator on the level of complexes [KLM],

$$\text{cl}_{r,m,X} : \text{CH}^r(X, m) \to H^D_{2r-m}(X, \mathbb{Z}(r)), \text{ viz., } \mathbb{A} = \mathbb{Z}.$$  

\footnote{In the end, acyclicity is all that matters here.}
Consider $\square^m$ with affine coordinates $(z_1, \ldots, z_m)$ and introduce the currents: ($\delta_t \cdot$ means integration over $V$)

$$\Omega_m := \left( \bigwedge_{j=1}^m d \log z_j \right) \cdot \delta_{\square^m},$$

$$T_{z_1} = \delta_{[-\infty,0] \times \square^{m-1}}, \ldots, T_m := T_{z_1} \cap \cdots \cap T_{z_m} = \delta_{[-\infty,0]^m} := \int_{[-\infty,0]^m} (-),$$

$$R_m := \log z_1 d \log z_2 \wedge \cdots \wedge d \log z_m \cdot \delta_{\square^m} - (2\pi i) \log z_2 d \log z_3 \wedge \cdots \wedge d \log z_m \cdot T_{z_1} + \cdots + (-1)^{m-1} (2\pi i)^{m-1} \log z_m \cdot T_{z_1} \cap \cdots \cap T_{z_{m-1}}.$$

For $\xi \in z^r(X,m)$, and if we let $\text{pr}_\square : X^m \to \square^m$, $\text{pr}_X : X \times \square^m \to X$ be the obvious projections, we consider the currents on $X$:

$$T_m(\xi) := \int_{\xi} \text{pr}_\square^*(T_m) \wedge \text{pr}_X^*(-),$$

$$\Omega_m(\xi) := \int_{\xi} \text{pr}_\square^*(\Omega_m) \wedge \text{pr}_X^*(-),$$

$$R_m(\xi) := \int_{\xi} \text{pr}_\square^*(R_m) \wedge \text{pr}_X^*(-).$$

One has the following identities [KLM]:

\begin{align}
\partial T_m(\xi) &= T_{m-1}(\partial \xi), \quad d[\Omega_m(\xi)] = 2\pi i \Omega_{m-1}(\partial \xi), \\
\quad d[R_m(\xi)] &= \Omega_{m}(\xi) - (2\pi i)^m T_{m}(\xi) - 2\pi i R_{m-1}(\partial \xi).
\end{align}

The map $\text{cl}_{r,m,X}$ is induced (up to the normalizing twist $(2\pi i)^{-r-m}$) by

\begin{align}
\xi \in z^r(X,m) \mapsto ((2\pi i)^m T_m(\xi), \Omega_m(\xi), R_m(\xi)),
\end{align}

with the following caveat. One expects a quasi-isomorphism $z^r(X,\bullet) \approx z^r(X,\bullet)$, which certainly holds after tensoring with $\mathbb{Q}$ [K-L]. Having said this, by the very definition of $\Lambda^\pm$, we can drop the $\mathbb{Q}$-coefficients from this discussion without compromising the theorem. It is easy to check that

$$((2\pi i)^m T_m(\xi), \Omega_m(\xi), R_m(\xi)) = (0,0,0)$$

for $\xi \in z^r_{d\Omega}(X,m)$.

For $m = 0$, note that $(T_0(\xi), \Omega_0(\xi), (2\pi i)^m R_0(\xi)) = (\xi, \delta_0, 0)$. First an observation. For precycles $\alpha \in z^p(X,\ell)$ and $\beta \in z^q(X,n)$ (in general position), one has the relation [KLM]:

\begin{align}
R_{\ell+n}(\alpha \cup \beta) = (-2\pi i)^q T_\ell(\alpha) \cdot R_n(\beta) + R_\ell(\alpha) \cdot \Omega_n(\beta).
\end{align}
9.8. The pairings. For \((\xi_1, \xi_2) \in \Lambda^+(r, m, X)\), we put
\[
\langle \xi_1, \xi_2 \rangle_m^+ := (-2\pi i)^{m+1} T_{m+1}(\xi_1') \cdot R_m(\xi_2) + R_{m+1}(\xi_1) \cdot \Omega_m(\xi_2)
\]
\[
\in \mathbb{C}/\mathbb{Z}(2m + 1) \cong \mathbb{C}/\mathbb{Z}(1),
\]
where the latter \(\simeq\) is given by multiplication by \((-4\pi^2)^{-m}\), and for \((\xi_1, \xi_2) \in \Lambda^-(r, m, X)\), we put (under \(\mathbb{C}/\mathbb{Z}(2m + 1) \cong \mathbb{C}/\mathbb{Z}(1)\)),
\[
\langle \xi_1, \xi_2 \rangle_m^- := (-2\pi i)^{-m} T_m(\xi_1) \cdot R_{m+1}(\xi_1') + R_m(\xi_1) \cdot \Omega_{m+1}(\xi_2').
\]

Note that for dimension and general position reasons alone,
\[
T_{m+1}(\xi_1') \cdot T_m(\xi_2) = 0 = T_m(\xi_1) \cdot T_{m+1}(\xi_2') \in z_{2R}^{d+m+1}(X, 2m + 1),
\]
and likewise over \(|\xi_1 \cap \xi_2|\) or \(|\xi_1' \cap \xi_2|\),
\[
\Omega_{m+1}(\xi_1') = 0 = \Omega_{m+1}(\xi_2'),
\]
using the fact that \(\dim |\xi_1 \cap \xi_2|\), \(\dim |\xi_1' \cap \xi_2|\) \leq m and that \(\Omega_{m+1}(\xi_1')\), \(\Omega_{m+1}(\xi_2')\) are meromorphic currents involving \(m + 1\) holomorphic differentials. This, together with
\[
R_m(\xi_1) \wedge \Omega_{m+1}(\xi_2') = \text{pr}_2^*(R_m \wedge \Omega_{m+1}) \cdot \delta_{\xi_1 \cap \xi_2},
\]
(by fiberwise Fubini), implies (using (12)), the simpler expression:
\[
\langle \xi_1, \xi_2 \rangle_m^- := (-2\pi i)^m T_m(\xi_1) \cdot R_{m+1}(\xi_2').
\]
Furthermore, the vanishing relations in (14) and (15) imply that the pairings \(\langle \xi_1, \xi_2 \rangle_m^+\) correspond (up to twist) to (*) in a Deligne complex triple of the form \((0, 0, *)\), (see the RHS of (11)). Note that if either \(\partial \xi_1' = 0\) or \(\partial \xi_2' = 0\), then the pairings \(\langle \xi_1, \xi_2 \rangle_m^+\) amount to a cup product in Deligne cohomology of the regulator of a higher Chow cycle, together with one which is nullhomologous (in Deligne cohomology), which is zero in:
\[
H^{2d+1}_D(X, \mathbb{Z}(d + m + 1)) \cong \mathbb{C}/\mathbb{Z}(1),
\]
where firstly after incorporating the normalizing twist (just preceding (11)), and in our setting, we arrive at the isomorphisms:
\[
H^{2d+1}_D(X, \mathbb{Z}(d + m + 1)) \cong \mathbb{C}/\mathbb{Z}(d + m + 1) \times (2\pi i)^{d+m} \cong \mathbb{C}/\mathbb{Z}(1).
\]
Hence the pairings \(\langle \xi_1, \xi_2 \rangle_m^\pm\) do not depend on the choices of the \(\xi_j'\)'s. For simplicity, we will assume given \((\xi_1, \xi_2) \in \Lambda(r, m, X)\). By definition, this implies that
\[
\delta_{\xi_1 \cap \xi_2}, \xi_1 \cap \xi_2 \in z_{2R}^{d+m+1}(X, 2m + 1), \xi_1' \cap \xi_2' \in z_{2R}^{d+m+1}(X, 2m + 2),
\]
which is important in ensuring that the currents above are defined. Next, the relations
\[
R_{2m+1}(\partial(\xi_1' \cup \xi_2')) = R_{2m+1}(\xi_1 \cup \xi_2') + (-1)^m R_{2m+1}(\xi_1' \cup \xi_2),
\]
imply that
\[(16) \quad \langle \xi_1, \xi_2 \rangle^+_m = (-1)^m \langle \xi_1, \xi_2 \rangle^-_m.\]
We remark in passing that in the case \(m = 0\), and after taking real parts, equation \((16)\) implies the reciprocity result in Proposition \(1.3\).

The remaining claims in Theorem \(9.5\) are left to the reader.

\[\square\]

**Remark 9.9.** We can pass to a real-valued height pairing using the composite \(\mathbb{C}/\mathbb{Z}(1) \to \mathbb{C}/\mathbb{R}(1) \simeq \mathbb{R}\).

We put
\[\langle \cdot, \cdot \rangle^0_m := \langle \cdot, \cdot \rangle^+_m|_{\Lambda(r,m,X)} = (-1)^m \langle \cdot, \cdot \rangle^-_m|_{\Lambda(r,m,X)},\]

and thus we have pairings
\[\langle \cdot, \cdot \rangle^0 : \Lambda^0(r,0,X) \to \mathbb{C}/\mathbb{Z}(1),\]
\[\langle \cdot, \cdot \rangle^R := \langle \cdot, \cdot \rangle^0 \circ \Lambda \to \mathbb{R},\]

Let \(\xi_1 := \text{div}(f,D) \in z^r_{\text{rat}}(X,0), \xi_2 := \text{div}(g,E) \in z^{d-r+1}_{\text{rat}}(X,0)\) be given. In this case \(D\) and \(E\) are irreducible subvarieties of \(X\) of codim\(X\) \(D = r - 1\) and codim\(X\) \(E = d - r\), and \(f \in \mathbb{C}(D)^{\times}, g \in \mathbb{C}(E)^{\times}\). Then
\[\langle \xi_1, \xi_2 \rangle^0 = \int_{D \setminus \{f^{-1}(-\infty,0)\} \cap \xi_2} \log f \in \mathbb{C}/\mathbb{Z}(1).\]

Similarly,
\[\langle \xi_1, \xi_2 \rangle^R = \int_{D \cap \xi_2} \log |f|.

**Remark 9.10.** It is instructive to work out the case \(m = 1\). Let
\[\xi_1 := \sum_j (g_j, D_j) \in z^r_{\text{rat}}(X,1),\]
and
\[\xi_2 := T(\{f_1, f_2\}, E) \in z^{d-r+2}_{\text{rat}}(X,1),\]
be given, where \(T\) is the Tame symbol. (We will also be working under the assumption that \((\xi_1, \xi_2) \in \Lambda(r,1,X)\).) In this case \(E\) and \(D_j\) are
irreducible subvarieties of $X$ of $\text{codim}_X D_j = r - 1$ and $\text{codim}_X E_j = d - r$, and $f_i \in \mathbb{C}(E)^\times$, $i = 1, 2$, and $g_j \in \mathbb{C}(D)^\times$. Set $C_j = D_j \cap E$, which is a curve in $X$, and put

$$g_j|_{C_j} := g|_{C_j}, \quad f_i|_{C_j} = f_i|_{C_j}.$$  

Then using the identification $\langle \xi_1, \xi_2 \rangle_1 = -\langle \xi_1, \xi_2 \rangle_1$, a simple computation yields:

$$\langle \xi_1, \xi_2 \rangle_1 = \sum_j \left[ (2\pi i) \int_{g_j^{-1}C_j[-\infty,0]} \log f_1 d \log f_2 ight] - (2\pi i)^2 \int_{(f_1C_j \times g_jC_j)^{-1}[-\infty,0]^2} \log f_2 \right] \in \mathbb{C}/\mathbb{Z}(3) \simeq \mathbb{C}/\mathbb{Z}(1).$$  

(Recall in equation (13) the identification $\mathbb{C}/\mathbb{Z}(2m + 1) \simeq \mathbb{C}/\mathbb{Z}(1)$, which explains the need for the identification $\mathbb{C}/\mathbb{Z}(3) \simeq \mathbb{C}/\mathbb{Z}(1)$ in the case $m = 1$.) Let $\gamma$ be the (closed) curve given by

$$\gamma := \sum_j g_j^{-1}C_j[-\infty,0].$$  

Taking the real part of $\langle \xi_1, \xi_2 \rangle_1$ and applying a Stokes’ theorem argument, one can show that:

$$\langle \xi_1, \xi_2 \rangle_1^R = -2\pi \int_\gamma \left[ \log |f_1| d \arg f_2 - \log |f_2| d \arg f_1 \right].$$  

Equation (17) is easily seen to be non-trivial. [Take for example $X := E = \mathbb{P}^2 \ni [z_0, z_1, z_2]$, and consider $\mathbb{P}^1 = \ell_j := D_j := V(z_j)$, $g_j = -z_j/\ell_j$, $g_1 = -z_1/z_2$, $g_2 = -z_0/z_1$. Note that $g_j \in \mathbb{C}(\ell_j)^\times$ and that $\sum_{j=0}^2 \text{div}_{\ell_j}(g_j) = 0$. Put $L := z_0 + z_1 + z_2$, and $t_j := z_j/L$, hence $t_0 + t_1 + t_2 = 1$. Thus in affine coordinates, $g_0 = -t_1/t_2$, $g_1 = -t_2/t_0$, $g_2 = -t_0/t_1$, and $\ell_j = V(t_j)$. Now let $\gamma$ be the corresponding 1-cycle, which is the boundary of the real simplex $\{t_0 + t_1 + t_2 = 1 \mid t_j \in [0,1]\}$. Let $\nu : \mathbb{P}^2 \to \mathbb{P}^1$ be the projection from $[0,0,1]$ (explicit: $\nu([z_0, z_1, z_2]) = [z_0, z_1]$). Then $\nu(t_0, t_1, t_2) = (t_0, t_1)$ and the aforementioned real simplex becomes $\{t_0 + t_1 \leq 1 \mid t_j \in [0,1]\}$, and $\nu_*(\gamma)$ the obvious boundary. Consider $p$ in the interior of $\nu_*(\gamma)$, and in terms of the coordinate $w = t_0 + it_1$, $t_j \in \mathbb{R}$, set $h(w) = w - p$. Observe that $\int_{\nu_*(\gamma)} d \log h \neq 0$. Now put

$$f_2 := \nu^*(h(w)) = \frac{z_0 + iz_1 + p \cdot L}{L},$$  

and choose $f_1 \in \mathbb{R}^\times \setminus \{\pm 1\}.$]
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