Fréchet algebraic deformation quantization of the Poincaré disk

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Abstract

Starting from formal deformation quantization we use an explicit formula for a star product on the Poincaré disk $\mathbb{D}_n$ to introduce a Fréchet topology making the star product continuous. To this end a general construction of locally convex topologies on algebras with countable vector space basis is introduced and applied. Several examples of independent interest are investigated as e.g. group algebras over finitely generated groups and infinite matrices. In the case of the star product on $\mathbb{D}_n$ the resulting Fréchet algebra is shown to have many nice features: it is a strongly nuclear Köthe space, the symmetry group SU$(1,n)$ acts smoothly by continuous automorphisms with an inner infinitesimal action, and evaluation functionals at all points of $\mathbb{D}_n$ are continuous positive functionals.

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1 Introduction

Deformation quantization as introduced in [1] aims at the construction of a quantum observable algebra out of the observable algebra of a classical mechanical system, usually modeled by the smooth functions on a Poisson manifold. The idea is to implement the necessary noncommutativity by means of a deformation of the commutative product in direction of the Poisson bracket with deformation parameter $\hbar$ resulting in a star product. Here several different settings are possible: in formal deformation quantization one follows the algebraic deformation program of Gerstenhaber [17] and uses formal power series in $\hbar$, ignoring questions of convergence completely. In this setting very strong existence and classification results are available, cumulating in Kontsevich’s formality theorem which guarantees the existence and provides the full classification of formal star products on Poisson manifolds in general [20]. A gentle introduction with a large bibliography can be found in the textbook [26].

For many reasons a formal deformation quantization is not believed to be the end of the story: for applications in quantum theory $\hbar$ has to be treated as a positive number. But also applications of deformation quantization in noncommutative geometry [15], where it provides important examples,
require a more analytic framework. Several notions of “convergent” deformations are available. 
Rieffel’s notion of strict deformation quantization \[24, 25\], see also Landsman’s monograph \[21\],
are located in the realm of \(C^*\)-algebras. Here one tries to construct continuous fields of \(C^*\)-algebras deforming a given commutative \(C^*\)-algebra sitting at a particular value of the underlying space of parameters. Many examples are known and studied in great detail. Unfortunately, a general existence and classification as in the formal case is not yet available and perhaps not even possible: here things usually start to depend strongly on the example and many completely different approaches and constructions compete.

While a \(C^*\)-algebraic formulation has clear motivations from quantum physics and noncommutative geometry, it also poses several technical difficulties. Ultimately, they originate in the fact that the passage from smooth (or even algebraic) differential geometry to the continuous world of \(C^*\)-algebras is quite a long way. Thus it might be reasonable to find intermediate types of deformations where on one hand the formal character of formal star products is already overcome but one still has closer contact to the smooth structures present in differential geometry.

In this work we want to advocate a Fréchet algebraic deformation quantization where the formal series in \(\hbar\) are investigated by means of a suitable Fréchet topology on a certain class of functions.

To achieve this we first provide a general construction of a locally convex topology for an algebra with a countable vector space basis, the completion of it will then be a Fréchet algebra. Whether our general construction works will depend on some technical details of the underlying algebra which seem hard to pin down in general. However, we discuss and illustrate our program in several examples like polynomial algebras, infinite matrices and group algebras over finitely generated groups. These examples seem to be interesting enough to pursue a further investigation. Nevertheless, we will turn to deformation quantization after a short glance at them.

In principle, there are other constructions of Fréchet topologies for algebras with countable vector space basis: the construction in \[16, \text{Prop. 2.1}\] can be interpreted as such, see Remark \[2.11\]. However, the resulting highly recursive construction of seminorms seems to be very difficult to handle. Nevertheless, this suggest that one should not expect any reasonable uniqueness theorems for such Fréchet topologies unless one specifies further properties.

In deformation quantization the first example to look at is of course the Weyl product for the flat phase space \(\mathbb{R}^{2n}\). We show that our general construction coincides with a previous construction \[3\] and yields a very explicit completion of the Weyl algebra to a nuclear Fréchet algebra. The underlying Fréchet space is a certain Köthe space and hence enjoys many nice properties.

The main task of this work is devoted to a more nontrivial example from deformation quantization: the Poincaré disk and its higher dimensional cousins \(\mathbb{D}_n\). This is a first example of a symplectic manifold with a curved Kähler structure: it is a Hermitian symmetric space of noncompact type with symmetry group SU(1, \(n\)). Moreover, the Poincaré disk is the covering space of the Riemann surfaces of higher genus. Thus an invariant quantization should ultimately also induce quantizations of these Riemann surfaces. Finally, the Poincaré disk is a first nontrivial example of a reduced phase space, allowing to test ideas of the quantization of phase space reduction also in a convergent framework.

The principal result is that our general construction applies to all stages of the phase space reduction as introduced in \[6,7\]: first we start with a star product \(\tilde{\ast}_{\text{Wick}}\) obtained from a particular equivalence transformation of the Wick star product on \(\mathbb{C}^{n+1}\). The U(1)-invariant functions are a subalgebra in which we find a subalgebra with countable vector space basis, essentially given by polynomials. We obtain a completion to a nuclear Fréchet algebra which consists of real-analytic functions of a particular type. The former vector space basis becomes an absolute Schauder basis after completion. The passage to the Poincaré disk is a quotient by a two-sided closed ideal. The resulting nuclear Fréchet algebra \(\hat{\mathcal{A}}_{\hbar}(\mathbb{D}_n)\) of functions on the disk can again be characterized.
very explicitly as a certain Köthe space with absolute Schauder basis consisting of representative functions with respect to the SU(1, n)-action on the disk.

After the construction of $\hat{A}_h(D_n)$ we determine several further properties: first we show that the symmetry group SU(1, n) acts by continuous automorphisms. The action turns out to be smooth and the corresponding Lie algebra action is inner via the classical momentum map. Second, the dependence on the parameter $h$ is shown to be holomorphic for $h \in \mathbb{C} \setminus \{-\frac{1}{2}, -\frac{1}{4}, \ldots, 0\}$. Third, we consider real $h$ in which case the pointwise complex conjugation is a continuous $^\ast$-involution of $\hat{A}_h(D_n)$ for which the evaluation functionals at points of $D_n$ are (continuous) positive functionals. The corresponding GNS construction is determined explicitly.

This last result will hopefully provide the bridge to compare our construction to other deformation quantizations of the Poincaré disk. A well-studied approach uses a Berezin-Toeplitz like quantization based on coherent states. The star product is then obtained as asymptotic expansion of a symbol calculus using the Toeplitz operators on the (anti-)holomorphic sections of higher and higher tensor powers of certain holomorphic line bundles. Here one should consult in particular [11–14] and [8] for a general background on deformation quantization of Kähler manifolds as well as [10]. Here our GNS representation should be compared to the (pseudo-differential) operator approaches. Alternatively, the algebra $\hat{A}_h(D_n)$ should be related directly to the algebras constructed and studied in [4,5]. There, explicit integral formulas for a star product on the Poincaré disk where given which allow for a $C^\ast$-algebraic quantization. The precise relations will be subject to a future project.

The paper is organized as follows: in Section 2 we establish the general construction in two versions. First we construct a countable system of seminorms for an algebra with countable vector space basis in terms of the structure constants. The difficulty is that the “seminorms” may diverge. To avoid this a closer look at the example will be necessary. Then we can describe the Fréchet algebra completion and discuss some general properties of it in Theorem 2.6. The second version will be a finer topology making a given linear functional continuous as well. Again we have to show by hand that the recursive definition will produce finite seminorms. General properties of this second version are then discussed in Theorem 2.10.

In Section 3 we discuss several basic examples to illustrate our general construction: first we consider polynomials as well as Laurent polynomials and study some of the properties of the resulting Fréchet algebras. As a second and now noncommutative example we consider infinite matrices with finitely many nonzero entries. Here we discuss the dependence of our general construction on the chosen vector space basis. The third example is the group algebra of an infinite but finitely generated group. The canonical basis given by the group elements has to be rescaled by an appropriate prefactor in order to make the general construction work: here the condition of finitely many generators becomes crucial. We take the factorial of the word length of a group element as rescaling. Clearly, this example would be interesting to investigate further. A last example is the Wick star product on flat $\mathbb{C}^n$. We show that the general construction reproduces an earlier approach [3] which was actually the main motivation for the present work. Moreover, we determine the resulting Fréchet algebra explicitly as a particular Köthe space. Thereby we show in Theorem 3.19 that the Weyl algebra has a completion which is nuclear and has the monomials as absolute Schauder basis.

Section 4 contains the main example: first we recall some basic properties of the Poincaré disk to fix our notation. Next, the construction of the (formal) star product according to [6,7] is recalled. We consider U(1)-invariant representative functions on $\mathbb{C}^{n+1}$ with respect to the linear SU(1, n)-action. A particular vector space basis is chosen and we compute the structure constants explicitly. The first version of our general construction is shown to produce the Cartesian product topology on the span of the basis vectors. Hence the completion is not interesting as we cannot interpret the elements of it as functions anymore. Thus we evoke the second version to make the evaluation
functionals at all points continuous as well. In a last step we have to divide by the (classical)
vanishing ideal of a level set of the Hamiltonian of the U(1)-action in order to get functions on the
Poincaré disk. As we can show the vanishing ideal to be a closed ideal also with respect to the
star product upstairs, we obtain a Fréchet algebra $\tilde{A}_h(\mathbb{D}_n)$ of functions on $\mathbb{D}_n$. The main result
in Theorem 4.21 determines the algebra $\tilde{A}_h(\mathbb{D}_n)$ as a certain Köthe space with absolute Schauder
basis together with some further properties.

Section 5 contains some further results on the algebra $\tilde{A}_h(\mathbb{D}_n)$. We show that the canonical
action of SU(1, n) on $\mathbb{D}_n$ induces a smooth action by automorphisms of $\tilde{A}_h(\mathbb{D}_n)$ with inner Lie
algebra action, see Theorem 5.8. Moreover, for the allowed values of $h$ we show that $\tilde{A}_h(\mathbb{D}_n)$ is
a holomorphic deformation in the sense of [23] in Theorem 5.18. Finally, we show the positivity
of the evaluation functionals at all points of $\mathbb{D}_n$ and start investigating the corresponding GNS
representation.

Finally, Appendix A contains some well-known facts about Köthe spaces which we will need at
several places.

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2 The general construction

The purpose of this section is to establish a general construction of a locally convex topology
consisting of countably many seminorms on an algebra $A$ with a countable vector space basis such
that the product becomes continuous. In general, the construction will require some additional
estimates in order to succeed which we will discuss in detail.

2.1 First version

To fix our notation we consider a complex algebra $A$ with a vector space basis $\{e_\alpha\}_{\alpha \in I}$ where $I$ is
a countably infinite index set. The algebra multiplication will be denoted by $\star$ and gives rise to
structure constants $C_{\alpha\beta}^\gamma$ defined by

$$e_\alpha \star e_\beta = \sum_{\gamma \in I} C_{\alpha\beta}^\gamma e_\gamma. \quad (2.1)$$

Note that for given $\alpha$ and $\beta$ we have only finitely many $\gamma$ with $C_{\alpha\beta}^\gamma \neq 0$. An element $a \in A$ can thus be written as

$$a = \sum_{\gamma \in I} a_\gamma e_\gamma \quad (2.2)$$

with only finitely many $a_\gamma$ different from zero. In particular, the choice of the basis gives the
coefficient functionals

$$e_\gamma : A \ni a \mapsto a_\gamma \in \mathbb{C}. \quad (2.3)$$

Remark 2.1 For the following we will be interested in associative algebras over the complex
numbers or even in $*$-algebras. However, the construction will work equally well over the real
numbers and also for other types of algebras like Lie algebras. The associativity will not be used
in the construction at all. The Lie case will clearly be of independent interest.
Based on the structure constants $C^\gamma_{\alpha\beta}$ we can now state the following (recursive) definition:

**Definition 2.2** For $m \in \mathbb{N}_0$, $\ell = 0, \ldots, 2^m - 1$, and $\gamma \in I$ one defines recursively the maps $h_{m,\ell,\gamma}: \mathcal{A} \to [0, +\infty]$ by setting $h_{0,0,\gamma}(a) = |a_\gamma|$ and

$$h_{m+1,\ell,\gamma}(a) = \sum_{\alpha \in I} (h_{m,\ell,\alpha}(a))^2 \sum_{\beta \in I} |C^\gamma_{\alpha\beta}| \quad \text{and} \quad h_{m+1,\ell+1,\gamma}(a) = \sum_{\beta \in I} (h_{m,\ell,\beta}(a))^2 \sum_{\alpha \in I} |C^\gamma_{\alpha\beta}|. \tag{2.4}$$

Moreover, we define $\| \cdot \|_{m,\ell,\gamma}: \mathcal{A} \to [0, +\infty]$ by

$$\|a\|_{m,\ell,\gamma} = 2^m \sqrt{h_{m,\ell,\gamma}(a)}. \tag{2.5}$$

Before we proceed let us state a few simple remarks. Clearly, the convergence (divergence) of the series is absolute as they consist of nonnegative terms only. Moreover, the resulting values depend strongly on the choice of the basis and, as we shall see later, even the convergence itself does. This is an unpleasant feature but the generality of the construction seems to have this price.

If the algebra is commutative, then the index $\ell$ does not play any role: a simple induction shows that for given $m$ and $\gamma$, the quantities $h_{m,\ell,\gamma}(a)$ coincide for all $\ell = 0, \ldots, 2^m - 1$ in this case.

Finally, it will be useful to assume that the series

$$C^\gamma_{\alpha\beta} = \sum_{\beta \in I} |C^\gamma_{\alpha\beta}| \quad \text{and} \quad C^\gamma_{\alpha\beta} = \sum_{\alpha \in I} |C^\gamma_{\alpha\beta}| \tag{2.6}$$

are finite for all $\alpha, \beta, \gamma \in I$. Otherwise the corresponding $h_{1,\ell,\gamma}$ will be infinite for $a \neq 0$ and thus all the following ones as well. Note that this is already a nontrivial convergence condition for the structure constants. Using these constants (whether finite or not) we get the following simple estimate

$$2^{m+1} \sqrt{C^\gamma_{\alpha\cdot}} \|a\|_{m,\ell,\alpha} \leq \|a\|_{m+1,2\ell,\gamma} \tag{2.7}$$

for all $a \in \mathcal{A}$ and all $\alpha, \gamma \in I$. Analogously we have

$$2^{m+1} \sqrt{C^\gamma_{\alpha\beta}} \|a\|_{m,\ell,\beta} \leq \|a\|_{m+1,2\ell+1,\gamma}. \tag{2.8}$$

The following lemma explains now our interest in the quantities $h_{m,\ell,\gamma}$.

**Lemma 2.3** Let $m \in \mathbb{N}_0$, $\ell = 0, \ldots, 2^m - 1$ and $\gamma \in I$. Then the maps $\| \cdot \|_{m,\ell,\gamma}$ have the following properties:

i.) For all $z \in \mathbb{C}$ and $a \in \mathcal{A}$ one has $\|za\|_{m,\ell,\gamma} = |z| \|a\|_{m,\ell,\gamma}$.

ii.) For all $a, b \in \mathcal{A}$ one has $\|a + b\|_{m,\ell,\gamma} \leq \|a\|_{m,\ell,\gamma} + \|b\|_{m,\ell,\gamma}$.

iii.) For all $a, b \in \mathcal{A}$ one has

$$\|a * b\|_{m,\ell,\gamma} \leq \|a\|_{m+1,\ell,\gamma} \|b\|_{m+1,2^m+\ell,\gamma}. \tag{2.9}$$

Proof. The first part is clear. For the second, the statement is clear for $m = 0$ so we can proceed by induction on $m$. First we get

$$\|a + b\|_{m+1,2\ell,\gamma} = \left(\sum_{\alpha \in I} \left(\|a + b\|_{m,\ell,\alpha}^2 C^\gamma_{\alpha\cdot}\right)^{2^{m+1}}\right)^{\frac{1}{2^{m+1}}}. \tag{}$$
\[
\leq \left( \sum_{\alpha \in I} \left( \|a\|_{m,\ell,\alpha} + \|b\|_{m,\ell,\alpha} \right)^{2m+1} C_{\alpha,\gamma}^{\gamma} \right)^{\frac{1}{2m+1}}
\]

\[
\leq \left( \sum_{\alpha \in I} \left( \|a\|_{m,\ell,\alpha} \right)^{2m+1} C_{\alpha,\gamma}^{\gamma} \right)^{\frac{1}{2m+1}} + \left( \sum_{\alpha \in I} \left( \|b\|_{m,\ell,\alpha} \right)^{2m+1} C_{\alpha,\gamma}^{\gamma} \right)^{\frac{1}{2m+1}}
\]

\[
= \|a\|_{m+1,2\ell,\gamma} + \|b\|_{m+1,2\ell,\gamma},
\]

where we used first the induction hypothesis and second the Minkowski inequality. The case of \( \| \cdot \|_{m+1,2\ell+1,\gamma} \) is analogous. For the third part we first note that the case \( m = 0 \) is a simple application of the Cauchy-Schwarz inequality. Then we proceed again by induction on \( m \) and get for even \( \ell \)

\[
\|a * b\|_{m,\ell,\gamma} = \left( \sum_{\alpha \in I} \left( \|a * b\|_{m-1,\ell/2,\alpha} \right)^{2m} C_{\alpha,\gamma}^{\gamma} \right)^{\frac{1}{2m}}
\]

\[
\leq \left( \sum_{\alpha \in I} \left( \|a\|_{m,\ell/2,\alpha} \|b\|_{m,\ell/2+2m-1,\alpha} \right)^{2m} C_{\alpha,\gamma}^{\gamma} \right)^{\frac{1}{2m}}
\]

\[
\leq \left( \left( \sum_{\alpha \in I} \left( \|a\|_{m,\ell/2,\alpha} \right)^{2m+1} C_{\alpha,\gamma}^{\gamma} \right) \left( \sum_{\alpha \in I} \left( \|b\|_{m,\ell/2+2m-1,\alpha} \right)^{2m+1} C_{\alpha,\gamma}^{\gamma} \right) \right)^{\frac{1}{2m+1}}
\]

\[
= \|a\|_{m+1,\ell,\gamma} \|b\|_{m+1,2m+\ell,\gamma},
\]

where we first used the induction hypothesis and then the Cauchy-Schwarz inequality. The case with \( \ell \) odd is analogous.

Remark 2.4 The last part of this lemma was the original motivation for considering this recursion scheme: when trying to make the product continuous for the zeroth seminorms \( \| \cdot \|_{0,0,\gamma} \), i.e. for the topology of pointwise convergence with respect to the coefficient functionals \( e^{\gamma} \), then one is lead rather directly to \( (2.4) \).

Up to now it may well happen that all the higher “seminorms” \( \|a\|_{m,\ell,\gamma} \) will produce \(+\infty\) for \( a \neq 0 \). In this case, our construction stops and we have not succeeded in finding a nice locally convex topology making the product continuous. This motivates to consider the following subset

\[
A_{\text{nice}} = \left\{ a \in A \mid \|a\|_{m,\ell,\gamma} (a) < +\infty \text{ for all } m \in \mathbb{N}_0, \ell = 0, \ldots, 2^m - 1, \gamma \in I \right\} \subseteq A.
\]

(2.10)

From Proposition 2.3 one immediately obtains the following corollary. Note that already the seminorms \( \| \cdot \|_{0,0,\gamma} \) are enough to guarantee a Hausdorff topology.

Corollary 2.5 The subset \( A_{\text{nice}} \) is a subalgebra of \( A \) and the maps \( \| \cdot \|_{m,\ell,\gamma} \) are seminorms on \( A_{\text{nice}} \) such that \( A_{\text{nice}} \) becomes a Hausdorff locally convex algebra.

Thus it is reasonable to restrict our attention to \( A_{\text{nice}} \) and replace \( A \) by this subalgebra. Note however, that we do not have any a priori guarantee that \( A_{\text{nice}} \) is different from \( \{0\} \).

Having a locally convex algebra we can form its completion. Here we will get a rather explicit description which will constitute the first main theorem of this section:
Theorem 2.6 Let \( \mathcal{A} = \mathbb{C} \cdot \text{span}\{e_\alpha\}_{\alpha \in I} \) be an algebra with countable vector space basis and assume \( \mathcal{A}_{\text{nice}} = \mathcal{A} \).

i.) The completion \( \hat{\mathcal{A}} \) of \( \mathcal{A} \) becomes a Fréchet algebra. The underlying Fréchet space can be described explicitly as

\[
\hat{\mathcal{A}} = \left\{ a = \sum_{\alpha \in I} a_\alpha e_\alpha \in \prod_{\alpha \in I} \mathbb{C} e_\alpha \mid \|a\|_{m,\ell,\gamma} = +\infty \text{ for all } m \in \mathbb{N}_0, \ell = 0, \ldots, 2^m - 1, \gamma \in I \right\},
\]

viewed as subspace of the vector space \( \prod_{\alpha \in I} \mathbb{C} e_\alpha \). The topology of \( \hat{\mathcal{A}} \) is finer than the Cartesian product topology of \( \prod_{\alpha \in I} \mathbb{C} e_\alpha \).

ii.) The evaluation functionals \( e^\alpha \) extend to continuous linear functionals

\[
e^\alpha : \hat{\mathcal{A}} \longrightarrow \mathbb{C}.
\]

iii.) The vectors \( \{e_\alpha\}_{\alpha \in I} \) form an unconditional Schauder basis of \( \hat{\mathcal{A}} \), i.e.

\[
a = \sum_{\alpha \in I} e^\alpha(a)e_\alpha
\]

converges unconditionally for all \( a \in \hat{\mathcal{A}} \).

Proof. The second part is clear from the continuity of the \( e^\alpha \) on \( \mathcal{A} \): they extend continuously to the completion. In fact, the continuity of the \( e^\alpha \) on \( \mathcal{A} \) can be seen directly from the estimate (equality)

\[
|e^\alpha(a)| = |a_\alpha| = \|a\|_{0,0,\alpha},
\]

which also shows that the topology of \( \hat{\mathcal{A}} \) will be finer than the Cartesian product topology which is determined by the seminorms \( \|\cdot\|_{0,0,\alpha} \) alone. For the remaining statements we can essentially follow the arguments from [3, Thm. 3.9 and Thm. 3.10]. From the recursive definition it is clear that there are constants \( \mu_{m,\ell,\gamma,\alpha_1,\ldots,\alpha_s} \geq 0 \) with \( s = 2^m \) such that

\[
h_{m,\ell,\gamma}(a) = \sum_{\alpha_1,\ldots,\alpha_s} \mu_{m,\ell,\gamma,\alpha_1,\ldots,\alpha_s}|a_{\alpha_1}| \cdots |a_{\alpha_s}|.
\]

The precise form of the \( \mu_{m,\ell,\gamma,\alpha_1,\ldots,\alpha_s} \) is complicated but irrelevant for the following. We view \( a \) as a function \( f_a : I^s \longrightarrow \mathbb{C} \) assigning the \( s \)-tuple of indices \( (\alpha_1,\ldots,\alpha_s) \) the value \( a_{\alpha_1} \cdots a_{\alpha_s} \). The coefficients \( \mu_{m,\ell,\gamma,\alpha_1,\ldots,\alpha_s} \) define a weighted counting measure \( \mu_{m,\ell,\gamma} \) on \( I^s \) such that (*) simply becomes

\[
h_{m,\ell,\gamma}(a) = \|f_a\|_{\ell^1} \quad \text{and thus} \quad \|a\|_{m,\ell,\gamma} = \sqrt[2^m]{\|f_a\|_{\ell^1}},
\]

where the \( \ell^1 \)-seminorm is defined with respect to the measure \( \mu_{m,\ell,\gamma} \). For fixed \( m, \ell, \gamma \) we consider the subset \( \mathcal{A}_{m,\ell,\gamma} \) of those \( a \in \prod_{\alpha \in I} \mathbb{C} e_\alpha \) with \( h_{m,\ell,\gamma}(a) < \infty \). These correspond precisely to the integrable functions \( f_a \). Thus the map \( a \mapsto f_a \) is topological homeomorphism of metric spaces (and even isometric up to the root \( \sqrt[2^m]{\gamma} \)) though clearly not a linear homeomorphism. Now let \( (a_i)_{i \in \mathbb{N}} \) be a Cauchy sequence in \( \hat{\mathcal{A}} \). The seminorms \( \|\cdot\|_{0,0,\gamma} \) guarantee the Hausdorff property and thus there is at most one limit. Moreover, \( (a_i)_{i \in \mathbb{N}} \) is a Cauchy sequence in \( \mathcal{A}_{m,\ell,\gamma} \) and hence convergent by the usual completeness of the \( \ell^1 \)-spaces. Note that it might happen that the weighted counting measures \( \mu_{m,\ell,\gamma} \) lead to nonempty subsets of \( I^s \) of measure zero. Hence the \( \ell^1 \)-spaces \( \mathcal{A}_{m,\ell,\gamma} \) are not yet Hausdorff but nevertheless complete. Thus \( a_i \longrightarrow a \) converges to some \( a \in \mathcal{A}_{m,\ell,\gamma} \) with
a being uniquely determined on those indices of nonzero measure. But for those the evaluation functionals \( e^\alpha \) are continuous and hence the limits \( a \) obtained for different \( m, \ell, \gamma \) coincide on those \( \alpha \) with nonzero measure. From \( \mathcal{A} = \bigcap_{m, \ell, \gamma} A_{m, \ell, \gamma} \) and the Hausdorff property obtained from \( m = 0 \) we see that \( a_i \to a \in \mathcal{A} \) proving the completeness. Now let \( a \in \mathcal{A} \) be given and let \( K \subseteq I \) be a finite subset of indices. Let \( a_K = \sum_{\alpha \in K} a_\alpha e_\alpha \). We claim that \( a_K \to a \) for any exhausting sequence of finite subsets of \( I \). Indeed, in our measure-theoretic picture \( a_K \) corresponds to the function \( f_{a_K} = f_a \chi_{K^s} \) where \( \chi_{K^s} \) is the characteristic function of \( K^s \subseteq I^s \). Thus we can apply the dominated convergence theorem to conclude \( f_{a_K} \to f_a \) in the \( \ell^1 \)-seminorm of \( A_{m, \ell, \gamma} \) which implies \( a_K \to a \) in the seminorm \( \| \cdot \|_{m, \ell, \gamma} \). This shows the unconditional convergence of (2.13). Since the evaluation functionals are continuous, we have an unconditional Schauder basis for \( \mathcal{A} \). But this finally shows that \( \mathcal{A} \subseteq \mathcal{A} \) is dense, completing also the proof of the first part.

Remark 2.7 In general, the topology will not be multiplicatively convex and hence \( \mathcal{A} \) will not be a locally \( m \)-convex Fréchet algebra. The relevant index for this failure of \( m \)-convexity is the index \( m \) in \( \| \cdot \|_{m, \ell, \gamma} \). We also see that the index \( \ell \) plays a minor role: we can easily take the maximum over the finitely many values of \( \ell \) for a given \( m \) and \( \gamma \). In particular, the seminorms defined by

\[
\| a \|_{m, \gamma} = \max_{0 \leq \ell \leq 2^m-1} \| a \|_{m, \ell, \gamma}
\]  

will yield the same topology as the original ones.

2.2 Second version

While the above construction will already produce interesting examples by its own, we will need yet a refinement. The idea is that while we have now a continuous product we also want certain linear functionals on \( A \) to be continuous. Thus let \( \omega : A \to \mathbb{C} \) be a linear functional. Since we have a vector space basis this is entirely determined by the coefficients \( \omega_\alpha = \omega(e_\alpha) \). In general, \( \omega \) will not be continuous with respect to the seminorms \( \| \cdot \|_{m, \ell, \gamma} \). If we insist on a topology making \( \omega \) continuous as well then we have to add at least the seminorm \( \| a \|_{0, 0, \omega} = |\omega(a)| \). However, adding just this single seminorm will spoil the continuity of the product in general. Thus we have to start again a recursion enforcing the continuity of \( * \). This will lead to the following definition:

Definition 2.8 Let \( \omega : A \to \mathbb{C} \) be a linear functional. Then we define the maps \( \| \cdot \|_{m, \ell, \omega} : A \to [0, +\infty] \) for \( a \in A \) by

\[
\| a \|_{m, \ell, \omega} = \sqrt[m]{\sum_{\gamma \in I} |\omega_\gamma| h_{m, \ell, \gamma}(a)},
\]  

where \( m \in \mathbb{N}_0 \) and \( \ell = 0, \ldots, 2^m - 1 \).

Lemma 2.9 Let \( \omega : A \to \mathbb{C} \) be a linear functional. Then for all \( m \in \mathbb{N}_0 \) and \( \ell = 0, \ldots, 2^m - 1 \) the following statements hold:

i.) For all \( z \in \mathbb{C} \) and \( a \in A \) one has \( \| za \|_{m, \ell, \omega} = |z| \| a \|_{m, \ell, \omega} \).

ii.) For all \( a, b \in A \) one has \( \| a + b \|_{m, \ell, \omega} \leq \| a \|_{m, \ell, \omega} + \| b \|_{m, \ell, \omega} \).

iii.) For all \( a, b \in A \) one has

\[
\| a \ast b \|_{m, \ell, \omega} \leq \| a \|_{m+1, \ell, \omega} \| b \|_{m+1, 2^m + \ell, \omega}.
\]  

9
Proof. Using the properties of the maps \( h_{m,\ell,\gamma} \) according to Lemma 2.3 all the statements are a simple verification.

Of course, it may again well happen that the quantities \( \|a\|_{m,\ell,\omega} \) are all \(+\infty\) for \( a \neq 0 \) even though the \( \|a\|_{m,\ell,\gamma} \) are always finite. Thus we typically add nontrivial conditions when we require \( \|a\|_{m,\ell,\omega} < \infty \). As before, we consider only those algebra elements where this is finite and obtain a subalgebra \( A_{\omega-\text{nice}} \) of \( A \) inside the previous \( A_{\text{nice}} \). Moreover, we can even select an arbitrary family \( \Omega = \{ \omega: A \to \mathbb{C} \} \) of linear functionals and consider the seminorms \( \| \cdot \|_{m,\ell,\omega} \) for all \( \omega \in \Omega \). We shall refer to this topology as the \( \Omega \)-nice topology. This will lead to subalgebras

\[
A_{\Omega-\text{nice}} = \bigcap_{\omega \in \Omega} A_{\omega-\text{nice}} \subseteq A_{\text{nice}} \subseteq A, \tag{2.17}
\]

which of course might be trivial, depending on the choice of \( \Omega \). Clearly, the inclusions are continuous and the topologies become coarser when moving to the right in (2.17). If \( \Omega \) is at most countably infinite then we still get a countable set of seminorms. Note that taking \( \Omega = \{ e^\alpha, \alpha \in I \} \) just reproduces the seminorms \( \| \cdot \|_{m,\ell,\gamma} \) we had before since \( e^\alpha(e_\gamma) = \delta^\alpha_\gamma \). Putting things together properly, we get the analogous statement to Theorem 2.6 also in this more general situation:

**Theorem 2.10** Let \( A = \mathbb{C} \)-span\{\( e_\alpha \)\}_{\alpha \in I} be an algebra with countable vector space basis. Moreover, let \( \Omega \) be a family of linear functionals on \( A \) and assume that \( A_{\Omega-\text{nice}} = A \).

i.) The completion \( \widehat{A} \) of \( A \) becomes a complete locally convex algebra which is Fréchet if \( \Omega \) is countable. The underlying locally convex space can be described explicitly as

\[
\widehat{A} = \left\{ a \in \prod_{\alpha \in I} \mathbb{C}e_\alpha \left| \|a\|_{m,\ell,\gamma}, \|a\|_{m,\ell,\omega} < +\infty \text{ for all } m \in \mathbb{N}_0, \ell = 0, \ldots, 2^m - 1, \gamma \in I, \omega \in \Omega \right. \right\}, \tag{2.18}
\]

viewed as subspace of \( \prod_{\alpha \in I} \mathbb{C}e_\alpha \) as before.

ii.) The evaluation functionals \( e^\alpha \) as well as the functionals \( \omega \in \Omega \) extend to continuous linear functionals

\[
e^\alpha, \omega: \widehat{A} \to \mathbb{C}. \tag{2.19}
\]

iii.) The vectors \( \{ e_\alpha \}_{\alpha \in I} \) form an unconditional Schauder basis of \( \widehat{A} \), i.e.

\[
a = \sum_{\alpha \in I} e^\alpha(a)e_\alpha \quad (2.20)
\]

converges unconditionally for all \( a \in \widehat{A} \).

iv.) If \( \Omega' \subseteq \Omega \) then \( A_{\Omega'-\text{nice}} = A \), too, and the \( \Omega \)-nice topology is finer than the \( \Omega' \)-nice topology.

After this presentation of the general features of the subalgebra \( A_{\text{nice}} \) as well as \( A_{\Omega-\text{nice}} \) the remaining but also quite nontrivial question is how we can guarantee that these subalgebras are different from \{0\} at all. Unfortunately, we have no general argument for this: it will boil down to a case by case study in the examples. This clearly limits the above method, however, many seemingly unrelated examples turn out to be just the above construction.

Even if one succeeds in showing that \( A_{\text{nice}} = A \) the construction will depend strongly on the chosen basis: this is really not avoidable as taking the absolute values of the structure constants in the definition of the seminorms is not at all well-behaved under a change of the basis. We can hardly expect any reasonable way to compare the results for different bases. Even worse, as we
shall see in the examples, a simple rescaling of the basis may change the convergence scheme and also the topology. This can be taken as advantage in order to cure bad behavior by rescaling.

Since the basis enters in such a crucial way, one should take this construction only for algebras where one has a quite distinguished choice of a basis. In fact, this is more often the case than one might first think.

Remark 2.11 There are other possibilities of constructing a countable system of seminorms on $\mathcal{A}$ for which the product becomes continuous: In fact, elaborating on the construction in the proof of [16, Prop. 2.1] one can again start with the evaluation functionals $e^\alpha$ and the corresponding seminorms $\| \cdot \|_\alpha = |e^\alpha(\cdot)|$ making them continuous. Then the recursion in [16, Prop. 2.1] will produce a countable system of seminorms build on top of the $\| \cdot \|_\alpha$ such that the product becomes continuous. The completion will be a Fréchet algebra, too, with a topology depending on the choice of the basis in a similarly “obscure” way than our construction. However, for the examples we shall study, our construction will be manageable to yield quite explicit properties of the Fréchet algebras in question. We leave it as a future task to examine further relations between the two approaches.

3 First examples: polynomials, matrices, and group algebras

In this section we collect some simple examples to illustrate the general method developed in the previous section.

3.1 Polynomials

We first consider the algebra of polynomials in one variable $\mathcal{A} = \mathbb{C}[z]$ with the basis given by the monomials. For the structure constants we see

$$z^n z^m = \sum_{k=0}^{\infty} C_{nm}^k z^k \quad \text{with} \quad C_{nm}^k = \delta_{k,m+n}, \quad (3.1)$$

where $n, m, k \geq 0$. Since $\mathbb{C}[z]$ is commutative we can ignore the index $\ell$ in this case. We first compute the constants (2.6) giving

$$C_n^k = \sum_{m=0}^{\infty} C_{nm}^k = \begin{cases} 0 & \text{for } n > k \\ 1 & \text{for } n \leq k. \end{cases} \quad (3.2)$$

In particular, $C_n^k < \infty$. From the recursion we see that

$$h_{m,k}(a) = \sum_{n=0}^{\infty} h_{m-1,n}(a)^2 C_n^k = \sum_{n=0}^{k} h_{m-1,n}(a)^2, \quad (3.3)$$

starting with $h_{0,k}(a) = |a_k|$ for $a = \sum_n a_n z^n$. Thus for $h_{m,k}(a)$ only finitely many $h_{m-1,n}(a)$ contribute. By a simple induction we get the following result:

**Proposition 3.1** For the polynomials $\mathcal{A} = \mathbb{C}[z]$ the topology of the seminorms $\| \cdot \|_{n,k}$ is the topology of the Cartesian product. The completion gives the locally multiplicatively convex Fréchet algebra $\mathbb{C}[[z]]$. 


Let us now also implement the second version for the polynomial algebra. Since the first version gives the formal power series \( \mathbb{C}[[z]] \) the evaluation functionals \( \delta_p \) for \( p \in \mathbb{C} \) cannot be continuous unless \( p \neq 0 \). Thus we want to enforce their continuity. Since the absolute value of the evaluation of the monomials at \( p \) only depends on \( |p| \) we can restrict ourselves to some positive radius \( R > 0 \) on which we want to evaluate. Then the additional seminorms are

\[
\|a\|_{m,R} = \sum_{k=0}^{\infty} R^k h_{m,k}(a). \tag{3.4}
\]

Now \( h_{1,k}(a) \geq |a_n|^2 \) for all \( k \geq n \) and hence \( \|a\|_{1,R} \) diverges for \( R \geq 1 \) unless \( a = 0 \). In this case, the second version fails. Consider now the case \( R < 1 \) and let \( a \in \mathbb{C}[z] \). Then a simple induction shows that \( h_{m,k}(a) \leq c_m k^{a_m} \) with some constant \( c_m > 0 \) depending on \( a \). Thus we see that in this case the series needed for \( \|a\|_{m,R} \) converges and hence the second version works. Moreover, for \( |p| \leq R \) we have \( |\delta_p(a)| \leq \|a\|_{0,R} \) and hence the topology we get is finer than the uniform topology on the disk of radius \( R \). After completion, we get a subalgebra of those functions which are holomorphic on at least the closed disk of radius \( R < 1 \). In general, not every such function will be in our completion: take \( R < r < 1 \) with \( r^2 < R \) and consider the coefficients \( a_n = \frac{1}{n!} \) which define a holomorphic function \( a(z) = \sum_{n=0}^{\infty} a_n z^n \) on the closed disk of radius \( R \). Then \( h_{1,k}(a) \geq \frac{1}{k^2} \) and hence the series needed for \( \|a\|_{1,R} \) will no longer converge. Hence we have a proper subalgebra and the topology will be strictly finer than the (Banach) topology of uniform convergence on the closed disk of radius \( R \).

**Proposition 3.2** For \( R \geq 1 \) the second version of our construction with respect to the basis of the monomials fails for \( \mathbb{C}[z] \) while for \( R < 1 \) we get a proper subalgebra of the Banach space of holomorphic functions on the closed disk of radius \( R \) with a strictly finer Fréchet topology. All evaluation functionals for points \( p \) with \( |p| \leq R \) are continuous.

We illustrate now the dependence on the basis: motivated by the usual Taylor formula we can also rewrite a polynomial as \( a = \sum_n \tilde{a}_n \frac{z^n}{n!} \). Now the rescaled monomials \( \frac{z^n}{n!} \) will serve as basis. The new structure constants will be rescaled as well and this will result in the new recursion

\[
\tilde{h}_{m+1,k}(a) = \sum_{n=0}^{k} \tilde{h}_{m,n}(a)^2 {k \choose n}, \tag{3.5}
\]

with \( \tilde{h}_{0,k}(a) = |\tilde{a}_k| \). For the first version, the same argument as for Proposition 3.1 still applies and hence we get \( \mathbb{C}[z] \) as completion. But for the second version things will change: the evaluation of the basis vectors on \( p \in \mathbb{C} \setminus \{0\} \) are now \( \frac{p^n}{n!} \) and hence the additional seminorms are

\[
\|a\|_{m,R} = \sum_{k=0}^{\infty} \frac{R^k}{k!} \tilde{h}_{m,k}(a), \tag{3.6}
\]

where again \( R = |p| > 0 \). We denote the nice part of \( \mathbb{C}[z] \) with respect to the given \( R \) by \( A_R \) and its completion by \( \tilde{A}_R \). The following argument shows in particular that \( A_R = \mathbb{C}[z] \) as vector spaces.

Suppose that \( a \in \mathbb{C}[[z]] \) has Taylor coefficients which satisfy a sub-factorial growth\(^1\), i.e. for all \( \varepsilon > 0 \) there is a constant \( c_0 > 0 \) with \( |\tilde{a}_k| \leq c_0 |k|! \varepsilon \). Then a simple induction shows that for \( |z| \leq R \) and \( |p| > R \) has a convergent Taylor expansion for all \( |z| \leq R \) then \( f \) has in fact a holomorphic extension to some slightly larger open disk.\(^2\)

\(^1\)Here “sub-factorial” is not to be confused with the sub-factorial \(|n|\) which we shall never need in this work.
all $\epsilon > 0$ also $h_{m,k}(a)$ can be bounded by $c_m(k!)^\epsilon$. It will be important to have this bound not only for one $\epsilon$. Taking now $\epsilon$ sufficiently small shows that (3.6) converges and thus such a formal series $a$ belongs to the completion $\hat{A}_R$, no matter what $R > 0$ is. Now fix $R > 0$ and suppose that $a \in \mathbb{C}[[z]]$ belongs to $\hat{A}_R$. Then we have the estimate

$$\tilde{h}_{m,k}(a) \leq k^m \left( \|a\|_{m,R} \right)^2$$

for all $m$ and $k$. By taking always only the term with $n = k$ in the recursion (3.6) it is clear that $|\tilde{a}_k|^2 \leq \tilde{h}_{m,k}(a)$ and hence

$$|\tilde{a}_k| \leq \frac{2^{n-k}}{R^{2m}} \|a\|_{m,R}.$$  

(3.8)

Thus $a$ has a sub-factorial growth as above. This results in the following Proposition:

**Proposition 3.3** Let $R > 0$.

i.) $\hat{A}_R$ consists of those entire functions $a = \sum_{n=0}^{\infty} \tilde{a}_n \frac{z^n}{n!}$ having Taylor coefficients with sub-factorial growth, i.e. for all $\epsilon > 0$ there exists a constant $c > 0$ with $|\tilde{a}_n| \leq c(n!)^\epsilon$.

ii.) The Fréchet topology of $\hat{A}_R$ does not depend on $R$ and is strictly finer than the usual Fréchet topology of the entire functions $O(\mathbb{C})$.

iii.) The evaluation functionals are continuous for all $p \in \mathbb{C}$.

iv.) The algebra $\hat{A}_R$ is not locally multiplicatively convex.

v.) An equivalent defining system of seminorms is given by

$$\|a\|_\epsilon = \sup_{n \in \mathbb{N}_0} \frac{|\tilde{a}_n|}{(n!)^\epsilon},$$

where $0 < \epsilon < 1$. As a Fréchet space, $\hat{A}_R$ is isomorphic to the Köthe space $\Lambda$ of sequences with sub-factorial growth. It is strongly nuclear and the Schauder basis is even absolute.

Proof. We have already shown the first statement. For two different $R, R'$ we have the same vector space for the completions and clearly if $R \leq R'$ then the topology of $\hat{A}_R$ is coarser than that topology of $\hat{A}_{R'}$, which one sees directly from the seminorms (3.6). But then the two Fréchet spaces $\hat{A}_R$ and $\hat{A}_{R'}$ already coincide by the open mapping theorem. In particular, all the seminorms $\| \cdot \|_{m,R'}$ will be continuous on $\hat{A}_R$. The seminorms $\{\| \cdot \|_{1,R}\}_{R>0}$ constitute a defining set of seminorms for the topology of locally uniform convergence in $O(\mathbb{C})$ and hence the second part is shown as well. The third part is clear and the fourth part follows from the fact that $\hat{A}_R$ does not have an entire calculus: otherwise it would be equal to $O(\mathbb{C})$. For the last part we first observe that (3.8) implies that the seminorms $\| \cdot \|_\epsilon$ can be estimated by the seminorms $\| \cdot \|_{m,R}$ for appropriate $m$. A careful examination of the bound $h_{m,k}(a) \leq c_m(k!)^\epsilon$ shows also the reverse estimate: however, this is also clear by general arguments as $\hat{A}_R$ is clearly a Fréchet space for both topologies and one is finer than the other. Hence by the open mapping theorem they coincide. Then the remaining statements follow from general properties of the Köthe space $\Lambda$, see Appendix A.

We see that already for this simple example of polynomials we get a rather rich structure and interesting completions. Moreover, the dependence on the chosen basis is manifest in this example.
### 3.2 Laurent polynomials

As a second example we consider the Laurent polynomials $A = \mathbb{C}[z, z^{-1}]$. Here the structure constants are similar to those of $\mathbb{C}[z]$, explicitly given by

\[
z^n z^m = \sum_{k=-\infty}^{\infty} C_{nm}^k z^k \quad \text{with} \quad C_{nm}^k = \delta_{n+m,k}
\]

with $n, m, k \in \mathbb{Z}$ instead of $\mathbb{N}_0$ as above. This has a nontrivial impact. We have

\[
C_n^k = \sum_{m=-\infty}^{\infty} C_{nm}^k = 1,
\]

since now there is always precisely one $m$ matching the condition $n + m = k$. Hence the recursion is

\[
h_{m,k}(a) = \sum_{n=-\infty}^{\infty} h_{m-1,n}(a)^2,
\]

with $h_{0,k}(a) = a_k$ as before. In particular, we have

\[
h_{1,k}(a) = \sum_{n=-\infty}^{\infty} |a_n|^2,
\]

which is finite for every $a \in \mathbb{C}[z, z^{-1}]$ but independent of $k$. Thus the next iteration gives $h_{2,k}(a) = +\infty$ unless $a = 0$. Our constructions fail in this case.

In order to fix this divergence we will now rescale the basis first. Instead of taking the monomials $z^n$ as basis we consider $e_n = \frac{1}{|n|!} z^n$, where $n \in \mathbb{Z}$ as before. Of course this is sort of arbitrary but here it is again motivated by the prefactors in the usual Laurent expansion around $z = 0$, i.e. we write now

\[
a = \sum_{n \in \mathbb{Z}} \frac{a_n}{|n|!} z^n
\]

for $a \in \mathbb{C}[z, z^{-1}]$ with only finitely many $a_n$ different from zero. The structure constants (again denoted by $C_{nm}^k$) are now given by

\[
e_n e_m = \sum_{k=-\infty}^{\infty} C_{nm}^k e_k \quad \text{with} \quad C_{nm}^k = \frac{|k|!}{|n|!|k-n|!} \delta_{n+m,k}.
\]

The corresponding constants (3.6) are given by

\[
C_n^k = \sum_{m=-\infty}^{\infty} C_{nm}^k = \frac{|k|!}{|n|!|k-n|!} < \infty,
\]

now depending on both indices $k$ and $n$. The recursion for the seminorms is changed to

\[
h_{m,k}(a) = \sum_{n=-\infty}^{\infty} h_{m-1,n}(a)^2 \frac{|k|!}{|n|!|k-n|!}
\]

with $h_{0,k}(a) = a_k$ according to (3.14). Now we want to show that for $a \in \mathbb{C}[z, z^{-1}]$ all the quantities $h_{m,k}(a)$ are finite. In fact, we shall determine the completion directly:
Proposition 3.4 We have $A_{\text{nice}} = A$. Moreover, a formal series $a \in \mathbb{C}[[z, z^{-1}]]$ belongs to the completion $\hat{A}$ iff its Laurent coefficients $a_n$ have sub-factorial growth, i.e. for all $\epsilon > 0$ there is a constant $c > 0$ depending on $a$ with $|a_n| \leq c(|n|!)^\epsilon$.

Proof. As an inequality in $[0, +\infty]$ we first prove that for all $a \in \mathbb{C}[[z, z^{-1}]]$ we have

$$|a_n| \leq 2^m \frac{|n|!|n-k|!}{|k|!} \|a\|_{m,k},$$

where $m \geq 1$ and $k \in \mathbb{Z}$. Indeed, for $m = 1$ we estimate

$$h_{1,k}(a) = \sum_{\ell \in \mathbb{Z}} |a_\ell|^2 \frac{|k|!}{|\ell|!|k-\ell|!} \geq |a_n|^2 \frac{|k|!}{|n|!|k-n|!},$$

which gives (*). Now we proceed by induction: assuming (*) for $m$ gives

$$h_{m+1,k}(a) \geq \sum_{\ell \in \mathbb{Z}} |a_\ell|^{2m+1} \left( \frac{|\ell|!}{|n|!|n-\ell|!} \right)^2 \frac{|k|!}{|\ell|!|k-\ell|!} \geq |a_n|^{2m+1} \frac{|k|!}{|n|!|k-n|!},$$

which shows (*) also for $m+1$. Now taking $k = 0$ gives the estimate

$$|a_n| \leq 2^m \sqrt{|n|!} \|a\|_{m+1,0}.$$

Hence, if $a \in \hat{A}_{\text{nice}}$, i.e. $\|a\|_{m,k} < \infty$ for all $m$ and $k$, then $a$ has sub-factorial growth. Conversely, assume that $a$ has sub-factorial growth. We have to show that all the quantities $h_{m,k}(a)$ are finite. In fact, we claim that for each $m$ also $h_{m,n}(a)$ behaves in a sub-factorial way. For $m = 0$ this is the assumption about $a$ itself. Now assume that for all $\epsilon > 0$ we have $h_{m,n}(a) \leq c_{m}(|n|!)^\epsilon$. Then we estimate for $k \geq 0$ and $\epsilon$ small enough

$$h_{m+1,k}(a) = \sum_{n=0}^{\infty} h_{m,-n}(a) \frac{k!}{n!(k+n)!} + \sum_{n=0}^{k} h_{m,n}(a) \frac{k!}{n!(k-n)!} + \sum_{n=k+1}^{\infty} h_{m,n}(a) \frac{k!}{n!(n-k)!}$$

$$\leq \sum_{n=0}^{\infty} c_m(n!) 2^\epsilon \frac{1}{n!} + \sum_{n=0}^{k} c_m(n!) 2^\epsilon \binom{k}{n} + \sum_{n=k+1}^{\infty} c_m(n!) 2^\epsilon \frac{k!}{n!(n-k)!}$$

$$\leq c + c_m(k!) 2^\epsilon k^2 + c_m(k!) 2^\epsilon \sum_{n=k+1}^{\infty} \frac{(k!)^{1-2\epsilon}}{(n!)^{1-2\epsilon}} \frac{1}{(n-k)!}$$

$$\leq c + c' c_m(k!) 2^\epsilon + c_m(k!) 2^\epsilon e,$$

where $c$ is the finite value of the first sum and $c'$ is a constant such that $2^k \leq c'(k!)^\epsilon$. We see that this can be estimated by $h_{m+1,k}(a) \leq c_{m+1}(k!)^\epsilon$ with an appropriate $c_{m+1}$. Since $\epsilon > 0$ was arbitrary we get again a sub-factorial growth, proving our claim. The case $k < 0$ is analogous. Hence all the quantities $h_{m,k}(a)$ are finite and $a$ belongs to the completion.

We shall now show that the elements of $\hat{A}$ can still be evaluated at points $p \in \mathbb{C} \setminus \{0\}$. Hence we can still interpret them as functions. We are able to show the following result:

Proposition 3.5 Let $a \in \hat{A}$.
i.) For \( p \in \mathbb{C} \setminus \{0\} \) the evaluation \( a(p) \) is well-defined and yields a continuous character
\[
\delta_p \colon \hat{A} \ni a \mapsto a(p) \in \mathbb{C}.
\] (3.18)

ii.) Any \( a \in \hat{A} \) can be viewed as holomorphic function \( a \in \mathcal{O}(\mathbb{C} \setminus \{0\}) \) via \((3.18)\).

iii.) The map \( \hat{A} \to \mathcal{O}(\mathbb{C} \setminus \{0\}) \) is a continuous injective algebra homomorphism with dense image.

iv.) The topology of \( \hat{A} \) is strictly finer than the locally uniform topology of \( \mathcal{O}(\mathbb{C} \setminus \{0\}) \) and it is not locally multiplicatively convex.

Proof. Let \( a = \sum_{n \in \mathbb{Z}} a_n z^n \in \hat{A} \) be given and \( p \in \mathbb{C} \setminus \{0\} \). Then we consider the convergence of the series \( a(p) = \sum_{n \in \mathbb{Z}} a_n p^n \). Using the estimate for the Laurent coefficients \( a_n \) as in \((**)*\) from the proof of Proposition 3.3 for \( m = 1 \) gives
\[
|a(p)| \leq \sum_{n \in \mathbb{Z}} \frac{|a_n| |p|^n}{|n|!} \leq \|a\|_{2,0} \sum_{n \in \mathbb{Z}} \frac{|p|^n}{|n|!} = c_{|p|} \|a\|_{2,0},
\]
which is the desired continuity estimate for \( \delta_p \) since the remaining series \( c_{|p|} \) converges for all \( p \neq 0 \). Clearly, the evaluation is a character of the algebra \( \hat{A} \) and hence also for \( \hat{A} \) by continuity. This shows the first part and the second is clear as we have convergence for all \( p \in \mathbb{C} \setminus \{0\} \). In fact, this is also clear from Proposition 3.4. For the next part, recall that the locally uniform topology of \( \mathcal{O}(\mathbb{C} \setminus \{0\}) \) can be obtained e.g. from the seminorms \( \| \cdot \|_R \) with \( R > 1 \) where
\[
\|f\|_R = \sum_{n \in \mathbb{Z}} \frac{|f_n|}{n!!} R^n,
\]
where \( f(z) = \sum_{n \in \mathbb{Z}} f_n z^n \) is the convergent Laurent expansion of \( f \). Then the above computation shows \( \|a\|_R \leq c_R \|a\|_{2,0} \) which is the continuity of the inclusion. The image is dense as already \( \mathbb{C}[z, z^{-1}] \) is dense. The last part is clear as \( \hat{A} \) does not coincide with \( \mathcal{O}(\mathbb{C} \setminus \{0\}) \) and it has no entire calculus.

\[\square\]

**Remark 3.6** Again, a simple verification shows that the seminorms \( \|a\|_\epsilon = \sup_n \frac{|a_n|}{|n|!} \) for \( \epsilon > 0 \) will produce an equivalent system of seminorms. Hence, as a Fréchet space, also here \( \hat{A} \) is just the (strongly nuclear) Köthe space \( \Lambda \) of sequences with sub-factorial growth and the Schauder basis is absolute. Note that indexing sequences by \( n \in \mathbb{Z} \) does not cause any difficulties here.

### 3.3 Infinite matrices

The third example is the noncommutative and nonunital algebra \( A = M_\infty(\mathbb{C}) \) of infinite matrices with only finitely many nonzero entries. Let \( E_{ij} \) be the matrix with 1 in the \((i, j)\)-th position and zeros everywhere else. Then \( \hat{A} = \mathbb{C}\text{-span}\{E_{ij}\}_{i, j \in \mathbb{N}} \) and the matrix multiplication gives the structure constants
\[
E_{ij} E_{kl} = \sum_{r, s=1}^{\infty} C_{(i, j), (k, l)}^{(r, s)} E_{rs} \quad \text{with} \quad C_{(i, j), (k, l)}^{(r, s)} = \delta_{ir} \delta_{jk} \delta_{ls}
\] (3.19)
as usual. The corresponding constants from \((2.6)\) are now given by
\[
C_{(i, j), (k, l)}^{(r, s)} = \sum_{k, l=1}^{\infty} C_{(i, j), (k, l)}^{(r, s)} = \sum_{k, l=1}^{\infty} \delta_{ir} \delta_{jk} \delta_{ls} = \delta_{ir}
\] (3.20)
as well as

\[ C^{(r,s)}_{r,0,(k,l)} = \sum_{i,j=1}^{\infty} C^{(r,s)}_{(i,j),(k,l)} = \sum_{i,j=1}^{\infty} \delta_{ir} \delta_{jk} \delta_{ls} = \delta_{ls}. \]  

(3.21)

Note that now we indeed have a noncommutative algebra and hence two types of such constants. We get the recursion

\[ h_{m+1,2\ell,(r,s)}(A) = \sum_{i,j=1}^{\infty} h_{m,\ell,(i,j)}(A)^2 C^{(r,s)}_{(i,j),r} = \sum_{j=1}^{\infty} h_{m,\ell,(r,j)}(A)^2 \]  

(3.22)

and analogously for the odd case

\[ h_{m+1,2\ell+1,(r,s)}(A) = \sum_{k,l=1}^{\infty} h_{m,\ell,(k,l)}(A)^2 C^{(m,n)}_{(k,l),r} = \sum_{k=1}^{\infty} h_{m,\ell,(k,s)}(A)^2. \]  

(3.23)

We see that the even case of \( 2 \ell \) does not depend on the index \( s \) while the odd case \( 2 \ell + 1 \) is independent on \( r \). After the second iteration we get infinite sums over constants and hence \( h_{2,\ell,(r,s)}(A) = +\infty \) for all \( A \neq 0 \). Again, we conclude that the method fails for this choice of a basis.

Thus we rescale the basis as we did already in the Laurent case: we shall discuss two different options here. First we consider the new basis

\[ \hat{E}_{ij} = \frac{1}{\sqrt{i! j!}} E_{ij}. \]  

(3.24)

Consequently, the new structure constants with respect to this basis are given by

\[ \hat{C}^{(r,s)}_{(i,j),(k,l)} = \frac{\sqrt{r! s!}}{\sqrt{i! j! k! l!}} C^{(r,s)}_{(i,j),(k,l)} = \frac{1}{j!} \delta_{ir} \delta_{jk} \delta_{ls}, \]  

(3.25)

and hence

\[ \hat{C}^{(r,s)}_{(i,j),r} = \frac{1}{j!} \delta_{ir} \quad \text{and} \quad \hat{C}^{(r,s)}_{(i,j),(k,l)} = \frac{1}{k!} \delta_{ls}. \]  

(3.26)

The recursive definition of the seminorms is now changed into

\[ \hat{h}_{m+1,2\ell,(r,s)}(A) = \sum_{j=1}^{\infty} \frac{1}{j!} \hat{h}_{m,\ell,(r,j)}(A)^2 \quad \text{and} \quad \hat{h}_{m+1,2\ell+1,(r,s)}(A) = \sum_{k=1}^{\infty} \frac{1}{k!} \hat{h}_{m,\ell,(k,s)}(A)^2. \]  

(3.27)

with \( \hat{h}_{0,0,(i,j)}(A) = \hat{A}_{ij} = \sqrt{i! j!} |A_{ij}| \) as starting point. As usual \( \|A\|_{m,\ell,(r,s)} = 2^m \sqrt{\hat{h}_{m,\ell,(r,s)}(A)} \).

Note that for even \( \ell \), the seminorms do not depend on \( s \) while for odd \( \ell \) they do not depend on \( r \). Thus there is a certain redundancy. One can show that the recursion will work for this choice of the basis.

**Proposition 3.7** Let \( M_{\infty}(\mathbb{C}) \) be endowed with the seminorms \( \| \cdot \|_{m,\ell,(r,s)} \) for \( r, s \in \mathbb{N} \).

i.) For all \( A \in M_{\infty}(\mathbb{C}) \) we have \( \|A\|_{m,\ell,(r,s)} < \infty \).

ii.) The completion of \( M_{\infty}(\mathbb{C}) \) to a Fréchet algebra contains at least those \( A \) with coefficients \( \hat{A}_{rs} \) having sub-factorial growth with respect to \( r + s \).

iii.) The trace functional \( \text{tr}: M_{\infty}(\mathbb{C}) \to \mathbb{C} \) extends to a continuous linear trace functional on the completion.
Proof. Let $\hat{A}_{rs}$ have sub-factorial growth, i.e. for all $\epsilon > 0$ there is a constant $c > 0$ with $|\hat{A}_{rs}| \leq c((r + s)!)^\epsilon$. Equivalently, we can replace $(r + s)!$ also by $\max(r, s)!$ or by $r!s!$. We claim that the recursion yields $\hat{h}_{m,\ell,(r,s)}(A)$ still having sub-factorial growth for all $m, \ell$. In particular, $\|A\|_{m,\ell,(r,s)} < \infty$. Indeed, this is a simple induction following directly from (3.27). This proves the first and second part. For the third, we note that a straightforward application of Hölder’s inequality gives

$$|\text{tr}(A)| \leq \sum_{r=1}^{\infty} \frac{|\hat{A}_{rr}|}{r!}$$

$$\leq \left( \sum_{r=1}^{\infty} \frac{|\hat{A}_{rr}|^4}{(r!)^2} \right)^{\frac{1}{4}} \left( \sum_{r=1}^{\infty} \frac{1}{(r!)^{\frac{1}{2}}} \right)^{\frac{1}{4}}$$

$$\leq c \left( \sum_{s,r=1}^{\infty} \frac{|\hat{A}_{sr}|^4}{s!(r!)^2} \right)^{\frac{1}{4}}$$

$$\leq c \left( \sum_{s=1}^{\infty} \frac{1}{s!} \left( \sum_{r=1}^{\infty} \frac{|\hat{A}_{sr}|^2}{r!} \right)^2 \right)^{\frac{1}{4}}$$

$$= c \|A\|_{2,1,(i,j)}^2,$$

where $c$ is the numerical constant coming from the second series in the second step and $(i, j)$ are arbitrary. This proves the last part. □

Alternatively, we can make use of a polynomial rescaling. We consider the basis of $M_\infty(\mathbb{C})$ defined by

$$\tilde{E}_{ij} = \frac{1}{ij} E_{ij}$$ (3.28)

leading to the structure constants $\tilde{C}^{(r,s)}_{(i,j),(k,l)} = \frac{1}{j} \delta_{ir} \delta_{jk} \delta_{ls}$ and hence

$$\tilde{C}^{(r,s)}_{(i,j),\cdot} = \frac{1}{j^2} \delta_{ir} \quad \text{and} \quad \tilde{C}^{(r,s)}_{\cdot,(k,l)} = \frac{1}{k^2} \delta_{ls}. \quad (3.29)$$

Accordingly, the recursion for the seminorms is now given by

$$\tilde{h}_{m+1,2\ell,(r,s)}(A) = \sum_{j=1}^{\infty} \frac{1}{j^2} \tilde{h}_{m,\ell,(r,j)}(A)^2 \quad \text{and} \quad \tilde{h}_{m+1,2\ell+1,(r,s)}(A) = \sum_{k=1}^{\infty} \frac{1}{k^2} \tilde{h}_{m,\ell,(k,s)}(A)^2 \quad (3.30)$$

with starting point $\tilde{h}_{0,0,(i,j)}(A) = |\hat{A}_{ij}| = ij|A_{ij}|$.

Proposition 3.8 Let $M_\infty(\mathbb{C})$ be endowed with the seminorms $\| \cdot \|_{m,\ell,(r,s)}$ for $r, s \in \mathbb{N}$.

i.) For all $A \in M_\infty(\mathbb{C})$ we have $\|A\|_{m,\ell,(r,s)} < \infty$.

ii.) The completion of $M_\infty(\mathbb{C})$ to a Fréchet algebra contains at least those $A$ with coefficients $\hat{A}_{rs}$ being bounded.

iii.) The trace functional is not continuous and the completion contains matrices not being trace-class.
Proof. The convergence of the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) allows to show inductively that for bounded \( |\hat{A}_{rs}| \) also all the quantities \( h_{m,\ell,(r,s)}(A) \) stay bounded with respect to \( (r,s) \). This proves the first and second part. For the third, we consider the matrix \( A \) with coefficients \( \hat{A}_{rs} = r\delta_{rs} \). We claim that also this matrix is in the completion (though it does not have bounded coefficients). Indeed, we have \( \tilde{h}_{1,0,(r,s)}(A) = \frac{1}{r} \) and \( \tilde{h}_{1,1,(r,s)}(A) = \frac{1}{s} \) which is now bounded with respect to \( (r,s) \). Hence by the induction from the second part, all higher \( \tilde{h}_{m,\ell,(r,s)}(A) \) will be bounded as well. Clearly, \( A \) is not trace-class. However, the series \( A = \sum_{r,s=1}^{\infty} \delta_{rs} \tilde{E}_{rs} = \sum_{r=1}^{\infty} \frac{1}{r} \tilde{E}_{rr} \) converges according to the general fact that the \( \tilde{E}_{rs} \) form a Schauder basis. Thus \( \text{tr} \) cannot be continuous. \( \Box \)

Remark 3.9 Here we see that the two rescalings yield two different Fréchet algebras. Furthermore, one can show that the topology of the first version is strictly finer than the well-known topology of fast decreasing matrices (the Fréchet space being the Schwarz space), which then is strictly finer than the topology of the second version. More details on this example and a further discussion can be found in \[2\, \text{Sect. 4.3} \].

3.4 Group algebras of finitely generated groups

Generalizing the example of the Laurent polynomials, the fourth example is based on a countable but infinite group \( G \) and the corresponding group algebra \( \mathcal{A} = \mathbb{C}[G] \). For \( G = \mathbb{Z} \) we are back at the second example.

Since we have already seen in the case \( G = \mathbb{Z} \) that a rescaling of the canonical basis might be necessary, we immediately start by considering the basis \( e_g = \frac{1}{c(g)} g \) for \( g \in G \) where \( c(g) > 0 \) will be a rescaling. We will require \( c(g) = c(g^{-1}) \) for all \( g \in G \) as well as \( c(e) = 1 \). The structure constants are now given by

\[
e_g e_h = \sum_{k \in G} C^k_{g,h} e_k \quad \text{with} \quad C^k_{g,h} = \frac{c(k)}{c(g)c(h^{-1}k)} \delta_{gh,k}.
\]

(3.31)

The constants \((2.6)\) needed for the construction of the topology on \( \mathbb{C}[G] \) are given by

\[
C^k_{g,\cdot} = \frac{c(k)}{c(g)c(g^{-1}k)} \quad \text{and} \quad C^k_{\cdot,h} = \frac{c(k)}{c(gh^{-1})c(h^{-1})} = \frac{c(k^{-1})}{c(hk^{-1})c(h^{-1})} = C^{k^{-1}}_{\cdot,h^{-1}}.
\]

(3.32)

according to our convention \( c(g) = c(g^{-1}) \). Writing \( a = \sum_{g \in G} a_g e_g \) we have \( \|a\|_{0,0,g} = |a_g| \) as usual. For the higher seminorms we have the recursion

\[
h_{m+1,2\ell,k}(a) = \sum_{g \in G} h_{m,\ell,g}(a)^2 \frac{c(k)}{c(g)c(g^{-1}k)}
\]

(3.33)

and by \( (3.32) \)

\[
h_{m+1,2\ell+1,k-1}(a) = \sum_{g \in G} h_{m,\ell,g}(a)^2 C^{k-1}_{\cdot,g} = \sum_{g \in G} h_{m,\ell,g}(a)^2 C^k_{g,\cdot} = h_{m+1,2\ell,k}(a).
\]

(3.34)

While it is tempting to choose \( c(g) = 1 \) for all \( g \), the following lemma shows that this choice will not lead to \( \mathcal{A}_{\text{nice}} = \mathcal{A} \).

Lemma 3.10 A necessary condition for \( \mathcal{A}_{\text{nice}} = \mathcal{A} \) is \( \frac{1}{c} \in \ell^2(G) \).
Proof. First we note that $h_{1,0,k}(a) \geq |a_g|^2 \frac{c(k)}{c(g)c(g^{-1}k)}$ for all $g$. Hence for $m = 2$ we get

$$h_{2,0,h}(a) \geq \sum_{k \in G} |a_g|^4 \frac{c(k)^2}{c(g)c(g^{-1}k)^2} \frac{c(h)}{c(k)c(k^{-1}h)} = |a_g|^4 \frac{c(h)}{c(g)^2} \sum_{k \in G} \frac{c(k)}{c(g^{-1}k)^2c(k^{-1}h)}.$$  

Thus a necessary condition for $h_{2,0,h}(a) < \infty$ for all $a \in A$ is the convergence

$$\sum_{k \in G} \frac{c(k)}{c(g^{-1}k)^2c(k^{-1}h)} < \infty$$

for all $g, h \in G$. Taking $g = h = e$ gives the summability of $\frac{1}{c}$ as claimed. \hfill \qed

However, this will be just a necessary condition, the higher seminorms will yield more conditions in general. While it is not yet clear whether we can actually find a suitable $c$, the next lemma shows that the growth of the coefficients will be limited, in the same spirit as we have seen that for the polynomials and the Laurent polynomials:

**Lemma 3.11** For every $m \geq 0$ and $a \in A$ one has

$$|a_g| \leq 2^m \sqrt{c(g)} \|a\|_{m+1,0,e}. \quad (3.35)$$

Proof. We claim that $h_{m,0,k}(a) \geq |a_g|^2 c\left(\frac{c(k)}{c(g)c(g^{-1}k)}\right)$ which gives (3.35) for $k = e$. Indeed, the case $m = 0$ was already obtained in the proof of Lemma 3.10 By induction on $m$ we have

$$h_{m+1,0,k}(a) \geq \sum_{h \in G} |a_g|^{2m+1} \frac{c(h)^2}{c(g)^2c(g^{-1}h)^2} \frac{c(k)}{c(h)c(h^{-1}k)} \geq |a_g|^{2m+1} \frac{c(k)}{c(g)c(g^{-1}k)}.$$  

\hfill \qed

**Lemma 3.12** For every rescaling with $A_{\text{nice}} = A$ the canonical $^*$-involution and the canonical antipode $S$ of $C[G]$ are continuous.

Proof. Recall that by definition $e_g^* = e_{g^{-1}}$ since we assume $c(g) = c(g^{-1})$. Hence $(a^*)_g = \overline{a}_{g^{-1}}$. Thus $\|a^*\|_{0,g} = \|a\|_{0,g^{-1}}$ for all $g \in G$. For the higher seminorms we have to take care of even and odd $\ell$: the effect of $^*$ is to flip between even and odd $\ell$ as well as between $k$ and $k^{-1}$. The details require some notational effort using the binary expansion of $\ell$ which we leave out. The argument for the antipode $S(a) = \sum_{g \in G} a_ge_{g^{-1}}$ is identical. \hfill \qed

After these generalities we are now looking for a rescaling $c$ such that we can guarantee $A_{\text{nice}} = A$. The idea is to look also for a class of functions $\mathcal{F} \subseteq \text{Fun}(G, [0, \infty))$ subject to the following conditions:

i.) For every $\chi \in \mathcal{F}$ there is another $\psi \in \mathcal{F}$ and $c > 0$ such that for all $h \in G$

$$\sum_{g \in G} \frac{\chi(g)^2}{c(g)c(g^{-1}h)} \leq c \frac{\psi(h)}{c(h)}.$$  

ii.) There is a $\chi \in \mathcal{F}$ with $\chi(g) > 0$ for all $g \in G$.

iii.) For $\chi \in \mathcal{F}$ also $\chi^{-1} \in \mathcal{F}$ where $\chi^{-1}(g) = \chi(g^{-1})$. 

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Suppose we have such a matching pair \((c,F)\) then one has the following statement:

**Lemma 3.13** Suppose \(c\) is a rescaling and \(F\) is a class of functions such that the above compatibility holds. Then for \(A = \mathbb{C}[G]\) we have:

i.) \(A_{\text{nice}} = A\).

ii.) If \(a \in \mathbb{C}[\langle G \rangle]\) is a formal series \(a = \sum_{g \in G} a_g e_g\) such that there is a \(\chi \in F\) with \(|a_g| \leq c\chi(g)\) for some \(c > 0\) then \(a \in \hat{A}\).

Proof. Suppose that \(a \in \mathbb{C}[\langle G \rangle]\) satisfies the estimate from the second part. We claim that for all \(m\) and \(\ell\) there exists a \(c > 0\) and a \(\chi \in F\) (depending on \(m\) and \(\ell\)) such that \(h_{m,\ell,k}(a) \leq c\chi(k)\) for all \(k \in G\). Indeed, for \(m = 0\) this is the assumption about \(a\). Then by induction

\[
h_{m+1,2\ell,k}(a) = \sum_{g \in G} h_{m,\ell,g}(a)^2 \frac{c(k)}{c(g)c(g^{-1}k)} \leq \sum_{g \in G} c^2 \chi(g)^2 \frac{c(k)}{c(g)c(g^{-1}k)} \leq c^2 c' \psi(k),
\]

where \(\psi\) and \(c'\) are chosen according to (3.36). For \(2\ell + 1\) instead of \(2\ell\) we can proceed analogously using the property \(\chi^{-1} \in F\) whenever \(\chi \in F\). This establishes the claim. In particular, all the \(h_{m,\ell,k}(a)\) are finite for such a \(a \in \mathbb{C}[\langle G \rangle]\). Since we have at least one nontrivial \(\chi \in F\) with \(\chi(g) > 0\) for all \(g \in G\), it follows that the finite sums \(a \in \mathbb{C}[G]\) can clearly be estimated by such a \(\chi\). Hence \(A_{\text{nice}} = A\) follows, proving the first part. The second was already shown on the way.

In general it is not clear whether we can find such a rescaling \(c\) and a sufficiently interesting class of functions \(F\) meeting the above criteria. However, if we make the additional assumption that \(G\) is finitely generated then we can construct such a matching pair. We choose a set of generators \(g_1, \ldots, g_N \in G\). This allows to define the word length functional

\[
L: G \rightarrow \mathbb{N}_0
\]

in the usual way, i.e. \(L(g)\) is the minimum of the number of generators and their inverses needed to obtain \(g\) as a product of generators. By convention we set \(L(e) = 0\). The crucial properties of this functional are now

\[
L(gh) \leq L(g) + L(h), \quad L(e) = 0, \quad \text{and} \quad L(g^{-1}) = L(g)
\]

for all \(g, h \in G\). As usual, \(L\) depends of course on the choice of the generators.

**Proposition 3.14** Assume \(G\) is a finitely generated group with word length functional as above and fix \(\varepsilon > 0\). Let \(c_\varepsilon(g) = (L(g)!^\varepsilon\) and

\[
F = \left\{ \chi_R: g \mapsto \chi_R(g) = R^{L(g)} \mid R > 0 \right\}.
\]

Then the pair \(c_\varepsilon\) and \(F\) satisfies the conditions [i.), [ii.), and [iii.)] needed for Lemma 3.13.

Proof. The second and third condition is trivial. It remains to check the first. Let \(R > 0\) be given. For a fixed \(\ell\) we have at most \((2N)\ell^\varepsilon\) group elements \(g\) with \(L(g) = \ell\), where \(2N\) is the number of generators and their inverses. Indeed, the bound is exceeded for the free group in \(N\) letters and any other finitely generated group has possibly less thanks to possible relations. From this we see that

\[
\sum_{g \in G} \frac{R^{L(g)}}{(L(g)!^\varepsilon} \leq \sum_{\ell=0}^\infty \frac{R^\ell}{(\ell)!^\varepsilon} \leq \frac{(2N)\ell^\varepsilon}{(\ell)!^\varepsilon}
\]
converges. Hence \( \text{Lexp}_\epsilon(z) = \sum_{g \in G} \frac{z^L(g)}{(L(g))!^\epsilon} \) defines an entire function for \( z \in \mathbb{C} \). For \( h \in G \) and \( R > 0 \) we have

\[
\sum_{g \in G} \left( \frac{R^{L(g)}}{(L(g)! L(g^{-1}h)!)} \right)^\epsilon
\]

\[
= \left( \sum_{L(g) \geq L(h)} + \sum_{L(g) < L(h) \land L(g^{-1}h) \geq L(h)} + \sum_{L(g) < L(h) \land L(g^{-1}h) < L(h)} \right) R^{2L(g)} \left( \frac{L(h)!}{L(g)! L(g^{-1}h)!} \right)^\epsilon
\]

\[
\leq \sum_{L(g) \geq L(h)} \frac{R^{2L(g)}}{(L(g^{-1}h)!)} + \sum_{L(g) < L(h) \land L(g^{-1}h) \geq L(h)} \frac{R^{2L(g)}}{(L(g)!)^\epsilon} + \sum_{L(g) < L(h) \land L(g^{-1}h) < L(h)} R^{2L(g)} \left( \frac{(L(g) + L(g^{-1}h))!}{L(g)! L(g^{-1}h)!} \right)^\epsilon
\]

\[
\leq \sum_{k \in \mathbb{N}} \frac{(1 + R)^{2L(k^{-1}h)}}{(L(k)!)} + \text{Lexp}_\epsilon(R^2) + \sum_{L(g) < L(h) \land L(g^{-1}h) < L(h)} (1 + R)^{2L(h)} 2^{2L(h)} L(h)
\]

\[
\leq (1 + R)^{2L(h)} \text{Lexp}_\epsilon((1 + R)^2) + \text{Lexp}_\epsilon(R^2) + (2N)^L(h)(1 + R)^{2L(h)} 2^{2L(h)},
\]

where in (a) we use \( L(h) \leq L(g) + L(g^{-1}h) \), see \[3.38\], and in (b) we use that (rough) estimate that \( \binom{n+m}{m} \leq 2^{n+m} \) for \( n = L(g) \) and \( m = L(g^{-1}h) \) both being \( \leq L(h) \). Thus the binomial coefficient is \( \leq 2^{2L(h)} \). But the last estimate shows that we have again an at most exponential growth in \( L(h) \), which establishes the second property.

Thanks to the proposition, there always exists a good choice (in fact many) of a rescaling \( c \) for a finitely generated group. For the fixed choice of generators we write \( \hat{A}_\epsilon \) for \( \mathbb{C}[G] \) equipped with the above topology. The corresponding Fréchet *-algebra \( \hat{A}_\epsilon \) has several nice properties.

**Proposition 3.15** Let \( G \) be a finitely generated infinite group with chosen word length functional \( L \) and let \( \epsilon > 0 \). Then for the corresponding Fréchet algebra \( \hat{A}_\epsilon \) one has:

i.) \( \hat{A}_\epsilon \) is continuously included into \( l^1(G) \).

ii.) Every complex-valued group character \( \chi \) of \( G \) extends continuously to an algebra character \( \chi \) on \( \hat{A}_\epsilon \).

Proof. Let \( a \in \hat{A}_\epsilon \). Then Lemma 3.11 yields

\[
||a||_{l^1(G)} = \sum_{g \in G} \frac{|a_g|}{(L(g)!^\epsilon)} \leq ||a||_{m+1,0,\epsilon} \sum_{g \in G} \frac{2^m}{(L(g)!^\epsilon)} = c_m ||a||_{m+1,0,\epsilon}
\]

with a finite \( c_m > 0 \) for \( m \geq 1 \). This shows the first part. For the second, we note that for a character \( \chi \) we have the estimate \( |\chi(g)| \leq R^{L(g)} \) where \( R \) is the maximum of \( |\chi| \) on the generators.
But then

$$|\chi(a)| \leq \sum_{g \in G} \frac{|a_g|}{(L(g))!^\epsilon} |\chi(g)| \leq \|a\|_{m+1,0,\epsilon} \sum_{g \in G} \frac{(L(g)!^\epsilon) R^L(g)}{(L(g))!} = c_{m,R} \|a\|_{m+1,0,\epsilon}$$

with a finite constant $c_{m,R} > 0$ for $m \geq 1$. This shows the continuity of $\chi$. \hfill \Box

The image of $\hat{A}_{\epsilon}$ in $\ell^1(G)$ is a dense subalgebra which is different from $\ell^1(G)$ since we have the nontrivial growth conditions (3.35) not necessary for an arbitrary element of $\ell^1(G)$.

Hence the Fréchet topology of $\hat{A}_{\epsilon}$ is strictly finer than the Banach topology of $\ell^1(G)$.

**Remark 3.16** As our primary interest is a different one, we leave now the group algebra case with several open questions: first, one would like to know how sensitive the topology of $\hat{A}_{\epsilon}$ is on the choice of the generators as well as on the parameter $\epsilon$. At the present state it seems to be rather difficult to approach this question. In particular, it would be nice to see if one can prove a similar statement as in the Laurent series case: the completion consists of those formal series with sub-factorial growth with respect to the length functional. Second, in view of Lemma 3.12 it would be interesting to find a reasonable topology on $A_{\epsilon} \otimes A_{\epsilon}$ such that also the canonical coproduct determined by $\Delta(g) = g \otimes g$ is continuous, leading to a locally convex Hopf algebra structure.

### 3.5 The Wick star product on $\mathbb{C}^n$

Recall that the Wick star product on $\mathbb{C}^n$ is explicitly given by

$$f \star_{\text{Wick}} g = \sum_{N=0}^{\infty} \frac{(2\hbar)^{|N|}}{N!} \frac{\partial^{|N|} f}{\partial z^N} \frac{\partial^{|N|} g}{\partial \bar{z}^N},$$

where $z^1, \ldots, z^n$ are the usual complex coordinates on $\mathbb{C}^n$ and we make use of the standard multiindex notation: we use a multiindex $N = (N_1, \ldots, N_n) \in \mathbb{N}^n_0$ and set $N! = N_1! \cdots N_n!$ as well as $|N| = N_1 + \cdots + N_n$. Sums, min and max, and partial derivatives and powers of $z$ and $\bar{z}$ for multiindices are defined componentwise as usual.

Usually, the Wick star product is treated as a formal star product with $h$ being a formal parameter and $f, g \in C^\infty(\mathbb{C}^n)[[\hbar]]$, see e.g. [20, Sect. 5.2.4] or [6] for a detailed exposition. However, it is clear from the definition that the Wick star product makes perfect sense for polynomials $f$ and $g$ and $h$ an arbitrary complex number. For physical reasons, we choose $\hbar > 0$. Then $\mathbb{C}[z, \bar{z}]$ endowed with the Wick star product is just the usual Weyl algebra since we have the canonical commutation relations $[z^k, \bar{z}^\ell]_{\text{Wick}} = 2\hbar \delta_{k\ell}$. In fact, using the complex conjugation as $^*$-involution we get the more familiar Hermitian generators $q^k = \frac{1}{2}(z^k + \bar{z}^k)$ and $p_k = \frac{1}{2i}(z^k - \bar{z}^k)$.

The monomials in $z$ and $\bar{z}$ constitute a vector space basis of all the polynomials which we shall use now to implement our general construction. In fact, we use the rescaled monomials

$$e_{I,J} = \frac{z^I \bar{z}^J}{I! J!(2\hbar)^{|I|+|J|}}.$$  \hfill (3.41)

We will not carry out the details of the computation but the procedure is clear: one has to compute the structure constants of $\ast_{\text{Wick}}$ with respect to this basis and set up the recursion for the seminorms $\| \cdot \|_{m,\ell,(I,J)}$ as defined in Section 2.1. One can check that the seminorms are indeed all finite on polynomials and hence we get a locally convex algebra structure on $\mathbb{C}[z, \bar{z}]$ which we shall denote by $A_{\text{Wick}}$.  

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Moreover, we can require that an evaluation functional \( \delta_p \) for some \( p \in \mathbb{C}^n \) is continuous and hence implement the second version of our general construction as in Section 2.2. Again, one shows that this works for any given \( p \) and yields a locally convex algebra structure on \( \mathbb{C}[z, \bar{z}] \) which we shall denote by \( A_p \).

Finally, in [3] a more ad-hoc recursive construction of a locally convex topology for the Weyl algebra \((\mathbb{C}[z, \bar{z}], \star_{\text{Wick}})\) was given, depending also on the choice of a point \( p \). We denote the resulting locally convex algebra by \( A_{h,p} \). In fact, this work was the starting point for our more general construction. Even though the precise recursion is not the same as the one from our general construction, one can show the following result [2, Sect. 4.5]:

**Theorem 3.17** For all \( p \in \mathbb{C}^n \) the locally convex algebras \( A_{\text{Wick}}, A_p \) and \( A_{h,p} \) are the same. The completion yields a Fréchet algebra with unconditional Schauder basis given by the \( e_{I,J} \).

In [3] many additional properties of the resulting Fréchet algebra have been studied. In particular, the completion contains also the exponentials of \( z \) and \( \bar{z} \) and hence the generators of the \( C^* \)-algebraic version of the Weyl algebra. Moreover, the Heisenberg group acts on \( \hat{A}_{\text{Wick}} \) by inner \( \ast \)-automorphisms. Finally, every evaluation functional is a continuous positive functional and the corresponding GNS representation gives the usual representation on the Bargmann-Fock space.

**Remark 3.18 (Weyl algebra)** The Weyl algebra has of course been studied by many people for many different aspects, e.g. in the work of Cuntz [16] the finest locally convex topology on the Weyl algebra was used to define and study its bivariant \( K \)-theory. Note however that this topology is of course very far from being Fréchet.

We can even determine the Fréchet space \( \hat{A}_{\text{Wick}} \) explicitly. Let \( a \in \hat{A}_{\text{Wick}} \) be given as Taylor series

\[
a = \sum_{I,J} a_{I,J} e_{I,J} = \sum_{I,J} \frac{a_{I,J}}{I!J!(2h)^{|I|+|J|}} z^I \bar{z}^J.
\]

In the notation of [3] we have an estimate of the Taylor coefficient \( a_{I,J} \) in terms of the zeroth seminorms of the form

\[
|a_{I,J}| \leq \frac{1}{\sqrt{(2h)^{|I|+|J|}}} \|a\|_{0,0,I,J}^h,
\]

see [3, Def. 3.1]. From that recursive definition of the quantities \( h_{m,\ell,0,1,0,1,0}^{0,h} \) one immediately sees that we have an estimate of the form \( h_{m,\ell,1,0}^{0,h}(a) \geq h_{m-1,\ell/2,0,1,0}^{0,h}(a)^2 \) for even \( \ell \) and \( h_{m,\ell,1,0}^{0,h}(a) \geq h_{m-1,\ell/2,0,1,0}^{0,h}(a)^2 \) for odd \( \ell \). Thus, taking \( \ell = 0 \) for simplicity, we arrive at the estimate

\[
|a_{I,J}| \leq \frac{1}{h_{m,0,0,1,0}^{0,h}(a)} = \|a\|_{0,0,0,1,0}^{0,h} \leq \sqrt{(2h)^{|I|+|J|}} \|a\|_{m+2,1,0,0}^{0,h},
\]

where for the last we used [3, Prop. 3.3, 6.]). Thus the Taylor coefficients have sub-factorial growth with respect to \( |I| + |J| \) and \( \max(|I|, |J|) \), respectively. Conversely, and more simple, one can use the explicit recursive definition of the seminorms to see that sub-factorial growth with respect to \( |I| + |J| \) is also sufficient for a formal Taylor series to belong to the completion. This is done analogously to [3, Thm. 3.6, 4.]), where an exponential growth was shown to be sufficient. More details on the algebra \( \hat{A}_{\text{Wick}} \) can be found in [2, Sect. 4.5]. We summarize this as follows:
Theorem 3.19 The completion $\hat{A}_{\text{wick}}$ is a strongly nuclear Fréchet algebra with absolute Schauder basis given by the Taylor monomials $\{e_{IJ}\}_{I,J \in \mathbb{N}_0^n}$ and it consists of real-analytic functions with Taylor coefficients having sub-factorial growth with respect to $|I| + |J|$. The underlying Fréchet space is isomorphic to a Kôthe space of sub-factorial growth.

4 The star product on the Poincaré disk

We come now to our main example, the star product on the Poincaré disk and on its higher dimensional cousins. The star product we are interested in appeared in the literature in the work [22], see also [13, Sect. 9] for an explicit formula for a particular class of functions. We shall rely on the construction from [6, 7], which works both for $\mathbb{C}P^n$ and the disk. It is inspired by classical phase space reduction and yields very explicit and manageable formulas.

4.1 The geometry of the disk

In this subsection we recall some basic features of the geometry of the disk $\mathbb{D}_n$ to fix our notation.

On $\mathbb{C}^{n+1}$ we consider a modified symplectic structure by means of a different pseudo-Kähler metric: let $g = \text{diag}(-1, +1, \ldots, +1)$ and consider the function

$$y(z) = -g(z, \overline{z}) = |z^0|^2 - |z^1|^2 - \cdots - |z^n|^2,$$

where $z^0, \ldots, z^n$ are the usual holomorphic coordinate functions on $\mathbb{C}^{n+1}$. Everything we will do will take place in the open cone $C_{n+1}^+ = \{ z \in \mathbb{C}^{n+1} \mid y(z) > 0 \}$. (4.2)

The disk is now obtained as a phase space reduction with respect to the U(1)-action induced by the Hamiltonian flow of $y$. Indeed, the flow of $y$ with respect to the Poisson bracket corresponding to $g$ is easily shown to be the U(1)-action $z \mapsto e^{i\phi} z$ where the point $z$ becomes simply multiplied by the phase. Clearly, $C_{n+1}^+$ is invariant under this action. More precisely, every level set of constant $y$ is invariant. Thus first restricting to such a “mass shell”, say for $y = 1$, and then dividing by the remaining U(1)-symmetry gives again a symplectic manifold, the disk $\mathbb{D}_n$. To handle the U(1)-symmetry, we shall use the holomorphic and anti-holomorphic Euler operators $E = \sum_{i=0}^n z^i \frac{\partial}{\partial z^i}$ and $\overline{E}$. Then a function is U(1)-invariant iff $Ef = \overline{E}f$.

The disk $\mathbb{D}_n$ can alternatively be viewed as an open subset of $\mathbb{C}P^n$ where we just take those complex lines in $\mathbb{C}^{n+1}$ which lie in $C_{n+1}^+$, i.e.

$$\mathbb{D}_n = \{ [z] \in \mathbb{C}P^n \mid z \in C_{n+1}^+ \}. \quad (4.3)$$

This makes $\mathbb{D}_n$ a complex manifold. Since for every $[z] \in \mathbb{D}_n$ we have $z^0 \neq 0$, the (local) holomorphic coordinates $v^i = \frac{z^i}{z^0}$ of $\mathbb{C}P^n$ are globally defined holomorphic coordinates on $\mathbb{D}_n$. This shows that $\mathbb{D}_n$ is diffeomorphic to $\mathbb{R}^{2n}$ and holomorphic to the open unit ball in $\mathbb{C}^n$ determined by the inequality $|v^1|^2 + \cdots + |v^n|^2 < 1$ as $y(z) > 0$. The particular case of $n = 1$ gives the usual unit disk $\mathbb{D}_1 = \{ v \in \mathbb{C} \mid |v| < 1 \}$. Moreover, it can be shown that the reduced symplectic form and the complex structure fit together nicely and yield a (now actually positive definite) Kähler manifold of constant negative holomorphic curvature. The canonical projection

$$\pi: C_{n+1}^+ \longrightarrow \mathbb{D}_n \quad (4.4)$$

is a holomorphic surjection. It allows to pull back functions $u \in C^\infty(\mathbb{D}_n)$ to $C^\infty(C_{n+1}^+)$ which will be called homogeneous functions on $C_{n+1}^+$. Indeed, the functions of the form $f = \pi^* u$ with
\( u \in C^\infty(\mathbb{D}_n) \) are precisely the functions invariant under the multiplicative action of \( \mathbb{C} \setminus \{0\} \). With other words, they are U(1)-invariant and constant in direction \( y \). Equivalently, they satisfy \( Ef = 0 = \overline{Ef} \).

Finally, \( \mathbb{D} \) is the noncompact dual of \( \mathbb{C}P^n \) as a Hermitian symmetric space: the function \( y \) is clearly invariant under the usual linear group action of SU\((1, n)\) on \( \mathbb{C}^{n+1} \). Hence it also acts on the complex lines in \( \mathbb{C}^{n+1} \), even in a transitive manner. The stabilizer of the standard point \((1, 0, \ldots, 0)\) in the level set \( y = 1 \) is now \( \text{S(U}(1) \times \text{U}(n)) \) and hence the last version of the disk is

\[
\mathbb{D}_n = \frac{\text{SU}(1, n)}{\text{S(U}(1) \times \text{U}(n))}.
\]

The symmetry respects both the complex structure and the reduced symplectic structure. The symmetry is even Hamiltonian with an equivariant momentum map given by

\[
J_\xi([z]) = \frac{i}{2} \sum_{i,j,k=0}^{n} g^{ij} \xi_k z^k \frac{\partial f}{\partial z^i} \frac{\partial g}{\partial z^j},
\]

for \( \xi \in \text{su}(1, n) \),

where we use \( \text{su}(1, n) \subseteq M_{n+1}(\mathbb{C}) \). For the case \( n = 1 \) the group \( \text{SU}(1, 1) = \text{SL}_2(\mathbb{R}) \) acts by Möbius transformations on \( \mathbb{D}_1 \). Moreover, in this particular case \((4.5)\) simplifies to \( \mathbb{D}_1 = \text{SU}(1, 1)/\text{U}(1) \) and the level set \( y = 1 \) can be identified with \( \text{SU}(1, 1) \): for a given point \( p \) in the mass shell \( y = 1 \) there is precisely one group element \( U \in \text{SU}(1, 1) \) which moves the point \((1, 0)\) to \( p \). In higher dimensions this uniqueness fails. These statements are of course all very much standard and can be found e.g. in textbooks like [18, Chap. VIII and Chap. IX].

### 4.2 Construction of the star product

In a next step we recall the basic features of the star product on the disk and review its explicit construction from [6, 7].

Matching to the modified symplectic structure on \( \mathbb{C}^{n+1} \) we consider the corresponding Wick star product

\[
f \ast_{\text{Wick}} g = \sum_{r=0}^{\infty} \frac{(2\lambda)^r}{r!} \sum_{i_1, \ldots, i_r = 0}^{n} g^{i_1 j_1} \cdots g^{i_r j_r} \frac{\partial^r f}{\partial z^{i_1} \cdots \partial z^{i_r}} \frac{\partial^r g}{\partial \overline{z}^{j_1} \cdots \partial \overline{z}^{j_r}},
\]

again first in the formal power series setting, i.e. for \( f, g \in C^\infty(\mathbb{C}^{n+1})[[\lambda]] \). We note that the polynomials in \( z \) and \( \overline{z} \) form a subalgebra where we can substitute \( \lambda \) by \( \hbar \) without difficulties.

**Remark 4.1** The Wick star product is clearly invariant under \( \text{SU}(1, n) \): for \( U \in \text{SU}(1, n) \) we have for all \( f, g \in C^\infty(\mathbb{C}^{n+1})[[\lambda]] \)

\[
(U^* f) \ast_{\text{Wick}} (U^* g) = U^*(f \ast_{\text{Wick}} g).
\]

It is even strongly invariant with respect to the usual momentum map, i.e. for

\[
J_\xi(z) = \frac{i}{2} \sum_{i,j,k} g^{ij} \xi_k z^k
\]

with \( \xi \in \text{su}(1, n) \) we have

\[
J_\xi \ast_{\text{Wick}} f - f \ast_{\text{Wick}} J_\xi = i\lambda \{J_\xi, f\} \quad \text{and} \quad J_\xi \ast_{\text{Wick}} J_\eta - J_\eta \ast_{\text{Wick}} J_\xi = i\lambda J_{[\xi, \eta]},
\]

where \( f \in C^\infty(\mathbb{C}^{n+1})[[\lambda]] \) and \( \xi, \eta \in \text{su}(1, n) \). It is this invariance which we will need later.
We call a function \( f \in C^\infty(C_{n+1}^+) \) radial if it depends only on the “radius” \( y \), i.e. if there is a smooth function \( g : (0, \infty) \rightarrow \mathbb{C} \) with \( f = g \circ y \). It will be useful to consider the following global vector field \( \frac{\partial}{\partial y} = \frac{1}{2y}(E + iE) \) on \( C_{n+1}^+ \). On radial functions it is indeed just differentiation with respect to \( y \). With this vector field we have the following formula

\[
R \ast_{\text{Wick}} F = \sum_{r=0}^{\infty} \frac{(2\lambda)^r}{r!} y^r \frac{\partial^r R \partial^r F}{\partial y^r \partial y^r} = F \ast_{\text{Wick}} R,
\]

(4.11)

for the Wick star product of a radial function \( R \) and a U(1)-invariant function \( F \). In particular, the radial functions constitute a commutative subalgebra.

In a next step one makes use of an equivalence transformation \( S \) from the Wick star product to a new one, denoted by \( \tilde{\ast}_{\text{Wick}} \), i.e.

\[
f \ast_{\text{Wick}} g = S(S^{-1} f \ast_{\text{Wick}} S^{-1} g).
\]

(4.12)

The aim is that for \( \tilde{\ast}_{\text{Wick}} \) the product of a radial function and an arbitrary U(1)-invariant function becomes not only commutative but pointwise. As usual the equivalence \( S \) is a formal series of differential operators starting with the identity. We require that it contains only powers of \( \frac{\partial}{\partial y} \).

Hence it is completely determined by its symbol \( \tilde{S}(\alpha, y) = e^{-\alpha y} S e^{\alpha y} \). The required property gives a functional equation for the symbol which can be solved in an essentially unique way. The resulting formal differential operator has the following properties which we recall from [6]. For all \( r \in \mathbb{N} \) we have

\[
S1 = 1, \quad Sy^r = y^r \prod_{k=0}^{r-1} \left( 1 + k \frac{2\lambda}{y} \right), \quad \text{and} \quad Sy^{-r} = y^{-r} \prod_{k=1}^{r} \left( 1 - k \frac{2\lambda}{y} \right)^{-1}.
\]

(4.13)

We see that we can substitute \( \lambda \) by \( h \). Moreover, we can rewrite the action of \( S \) on \( y^r \) for \( r \geq 0 \) by means of the Pochhammer symbols (or raising factorials) in the following way

\[
Sy^r = (2h)^r \left( \frac{y}{2h} \right)^r,
\]

(4.14)

where as usual \( (\alpha)_r = \alpha(\alpha + 1) \cdots (\alpha + (r - 1)) \).

Since we are interested in evaluating this on \( y = 1 \) later on, we have to take care of the zeros of the Pochhammer symbols. We will call \( h \) an allowed value if

\[
2h \in \mathbb{C} \setminus \left\{ 0, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots \right\}.
\]

(4.15)

Equivalently, \( \frac{1}{2\pi} e^{\lambda \alpha} \neq 0 \) for all \( \alpha \in \mathbb{N}_0 \). In [6,7] the case of \( \mathbb{CP}^n \) was considered. Here the critical values of \( h \) were \( +\frac{1}{2} \) on the positive half axis. This is the reason why we will deal with the disk \( \mathbb{D}_n \) instead: all positive \( h \) will be allowed.

According to [6,7], the explicit formula for \( \tilde{\ast}_{\text{Wick}} \) is now

\[
F \ast_{\text{Wick}} G = \sum_{r=0}^{\infty} \frac{1}{r!} \left( \frac{2\lambda}{y} \right)^r \prod_{k=1}^{r} \left( 1 - k \frac{2\lambda}{y} \right)^{-1} y^r \sum_{j_1, \ldots, j_r = 0}^{n} g^{i_1 j_1} \cdots g^{i_r j_r} \frac{\partial^r F}{\partial z^{i_1} \cdots \partial z^{i_r}} \frac{\partial^r G}{\partial \bar{z}^{j_1} \cdots \partial \bar{z}^{j_r}}.
\]

(4.16)

for U(1)-invariant functions \( F, G \in C^\infty(C_{n+1}^+)[[\lambda]] \). Moreover, since the radial functions behave like scalars by the very design of \( \tilde{\ast}_{\text{Wick}} \), one can show that the ideal generated with respect to \( y - 1 \) coincides with the classical vanishing ideal of \( y = 1 \). This allows to set \( y = 1 \) in (4.16) for U(1)-invariant functions to obtain an explicit star product \( \ast_{\mathbb{D}_n} \) on the disk \( \mathbb{D}_n \).
Remark 4.2 The equivalence transformation $S$ is clearly SU(1, $n$)-invariant. Thus the new star product $\tilde{\star}_{\text{Wick}}$ is again a SU(1, $n$)-invariant star product. Moreover, since $Sy = y$ and $\frac{1}{y}J_{\xi}$ is homogeneous we conclude that $SJ_{\xi} = J_{\xi}$ for all $\xi \in \mathfrak{su}(1, n)$. It follows easily that $\tilde{\star}_{\text{Wick}}$ is again strongly invariant under SU(1, $n$) with the same (quantum) momentum map $J_{\xi}$ as already $\star_{\text{Wick}}$, see also [7, Lem. 5] for details.

4.3 The basis and the structure constants

In a next step we shall consider a subalgebra of the formal star product where we have convergence for $\hbar$ in the set (4.15) and a countable vector space basis. Let $P, Q \in \mathbb{C}^6_n$ be multiindices of length $n$ and let $\alpha \in \mathbb{C}^0_6$ with $|P|, |Q| \leq \alpha$. Then we consider the functions

$$ e_{P,Q,\alpha}(z) = \left( z^0 \right)^{\alpha - |P|} z^P \left( z^0 \right)^{\alpha - |Q|} z^Q, $$

where we use the multiindex notation $z^P = (z_0^1)^P_1 \cdots (z_0^n)^P_n$ as usual. Clearly, $e_{P,Q,\alpha}$ is U(1)-invariant. In fact, such a monomial is homogeneous of degree $\alpha$ for both Euler operators $E$ and $E_\alpha$, i.e. we have $E e_{P,Q,\alpha} = \alpha e_{P,Q,\alpha} = E_\alpha e_{P,Q,\alpha}$. Note that we view the monomials as functions on $\mathbb{C}^n_+$. Note also that every U(1)-invariant polynomial is a linear combination of these monomials which explains our choice. In the following, we call $(P, Q, \alpha)$ an index triple if $P$ and $Q$ are multiindices with $|P|, |Q| \leq \alpha$.

Lemma 4.3 For all index triples $(P, Q, \alpha)$ and $(R, S, \beta)$ we have

$$ e_{P,Q,\alpha} \star_{\text{Wick}} e_{R,S,\beta} = \sum_{k=0}^{\min(\alpha - |P|, \beta - |S|)} \sum_{K=0}^{\min(P, S)} \left( \frac{2\hbar}{k!K!} \right)^k (-1)^k \frac{(\alpha - |P|)!}{(\alpha - |P| - k)!} \frac{P!}{(P - K)!} \frac{(\beta - |S|)!}{(\beta - |S| - k)!} \frac{S!}{(S - K)!} e_{P+R-K,Q+S-K,\alpha+\beta-k-|K|}, $$

where the minimum of multiindices is taken componentwise as usual.

Proof. This is just an exercise in differentiation. \qed

We take now the monomials and apply the equivalence $S$ to them. A priori, this will give a formal power series in $\lambda$ but from the explicit formula we see that the expression will make sense for all $\hbar \neq 0$. The following result is clear from (4.14):

Lemma 4.4 For all index triples $(P, Q, \alpha)$ and $\hbar \neq 0$ we have

$$ Se_{P,Q,\alpha} = (2\hbar)^{\alpha} \left( \frac{y}{2\hbar} \right)^{\alpha} e_{P,Q,\alpha}. $$

Proof. Indeed, since $S$ is the identity on pullbacks $f = \pi^* u$ of functions on $\mathbb{D}_n$, we have by (4.14)

$$ Se_{P,Q,\alpha} = S \left( y^\alpha \frac{e_{P,Q,\alpha}}{y^\alpha} \right) = S(y^\alpha) \frac{e_{P,Q,\alpha}}{y^\alpha} = (2\hbar)^{\alpha} \left( \frac{y}{2\hbar} \right)^{\alpha} e_{P,Q,\alpha}. $$

\qed

For $\hbar \neq 0$ the functions obtained from the monomials after applying $S$ are still linearly independent.
Lemma 4.5 Let $h \neq 0$. Then the functions
\[
\left\{ \left( \frac{y}{2h} \right)^{\epsilon_{P,Q,\alpha}} y^\alpha \right\}_{\alpha \in \mathbb{N}_0, P,Q \in \mathbb{N}_0^n, \alpha \geq \max(|P|,|Q|)}
\] (4.20)
are linearly independent in $C^\infty(\mathbb{C}^+)_{n+1}$.

Proof. First it is clear that the monomials $e_{P,Q,\alpha}$ are linearly independent. Second, the Pochhammer symbol $\left( \frac{y}{2h} \right)^{\alpha}$ has leading term $(y)^{\alpha}$ for $y \to +\infty$ while $\frac{\epsilon_{P,Q,\alpha}}{y^\alpha}$ stays bounded. Thus we can recover $\alpha$ from the asymptotics of $\left( \frac{y}{2h} \right)^{\epsilon_{P,Q,\alpha}} y^\alpha$. From these two facts the statement follows easily. $\Box$

We anticipate here that evaluating at $y = 1$ will give additional linear dependencies if $h$ is not an allowed value.

In view of the product formula it will be advantageous to rescale the functions slightly. From a more physical point of view, we can make them dimensionless in order to get dimensionless structure constants, i.e. independent of $h$. This gives the final definition: for an index triple $(P,Q,\alpha)$ and $h \neq 0$ we consider the functions
\[
f_{P,Q,\alpha} = \frac{1}{(2h)^\alpha} \frac{1}{P!(\alpha - |P|)!Q!(\alpha - |Q|)!} S e_{P,Q,\alpha} = \frac{1}{P!(\alpha - |P|)!Q!(\alpha - |Q|)!} \left( \frac{y}{2h} \right)^{\epsilon_{P,Q,\alpha}} y^\alpha.
\] (4.21)

They are still linearly independent and hence we consider their span inside $C^\infty(\mathbb{C}^+)_{n+1}$ which we will denote by
\[
A_h(\mathbb{C}^+)_{n+1} = \mathbb{C}\text{-span} \left\{ f_{P,Q,\alpha} \mid \alpha \in \mathbb{N}_0, P,Q \in \mathbb{N}_0^n, \alpha \geq \max(|P|,|Q|) \right\},
\] (4.22)
for which the $f_{P,Q,\alpha}$ will be a countable vector space basis. We write $a = \sum (P,Q,\alpha) a_{P,Q,\alpha} f_{P,Q,\alpha}$ for $a \in A_h(\mathbb{C}^+)_{n+1}$ as usual. In fact, $A_h(\mathbb{C}^+)_{n+1}$ will be a subalgebra with respect to $\bar{\tau}_{\text{Wick}}$.

Proposition 4.6 Let $h \neq 0$.

i.) $A_h(\mathbb{C}^+)_{n+1}$ is a unital subalgebra with respect to $\bar{\tau}_{\text{Wick}}$.

ii.) For real $h$ the algebra $A_h(\mathbb{C}^+)_{n+1}$ is a $^*$-algebra with respect to the pointwise complex conjugation and one has
\[
\overline{f_{P,Q,\alpha}} = f_{\bar{P},\bar{Q},\bar{\alpha}}.
\] (4.23)

iii.) The structure constants of $A_h(\mathbb{C}^+)_{n+1}$ with respect to the basis (4.21) are explicitly given by
\[
C_{(P,Q,\alpha),(R,S,\beta)}^{(I,J,\gamma)}(\mathbb{C}^+)_{n+1} = \frac{(-1)^{\alpha+\beta-\gamma-|P|-|R|+|I|}}{\epsilon(P,Q,\alpha,R,S,\beta,I,J,\gamma)} \frac{\alpha + \beta - \gamma - |P| - |R| + |I|}{(P+R-I)!} \bigg( \binom{I}{R} \binom{J}{Q} \bigg( \frac{\gamma - |I|}{\beta - |R|} \bigg) \bigg( \frac{\gamma - |J|}{\alpha - |Q|} \bigg) \delta_{P+R-I,Q+S-J}.
\] (4.24)

where the quantity
\[
\epsilon(P,Q,\alpha,R,S,\beta,I,J,\gamma) = \sum_{k=0}^{\min(\alpha-|P|,\beta-|S|)} \sum_{K=0}^{\min(P,S)} \delta_{I,P+R-K} \delta_{\gamma,\alpha+\beta-k-|K|}.
\] (4.25)

takes only the values 0 or 1. In particular, the structure constant will be zero if one of the conditions
\[
R \leq I, \quad Q \leq J, \quad \max(\alpha, \beta) \leq \gamma \leq \alpha + \beta
\] (4.26)
is violated.
Proof. First we note that \( f_{0,0,0} = 1 \). If \( h = \hbar \) then \( S \) commutes with the pointwise complex conjugation and hence \( \tilde{\star}_{\text{Wick}} \) is a Hermitian star product since \( \star_{\text{Wick}} \) is obviously a Hermitian star product. To show that \( \mathcal{A}_h(C_{n+1}^+) \) is a subalgebra, we have to compute the product of two elements of the basis. Using (4.21) as well as Lemma 4.3 we have

\[
\begin{align*}
&f_{P,Q,\alpha} \tilde{\star}_{\text{Wick}} f_{R,S,\beta} \\
&= S \left( S^{-1} f_{P,Q,\alpha} \star_{\text{Wick}} S^{-1} f_{R,S,\beta} \right) \\
&= S \left( \frac{1}{(2\hbar)^{\alpha}} P!(|\alpha|) Q!(|\alpha|) R!(|\beta|) S!(|\beta|) \epsilon_{P,Q,\alpha} \star_{\text{Wick}} \epsilon_{R,S,\beta} \right) \\
&= S \left( \frac{1}{Q!(\alpha-|Q|)!} \frac{1}{R!(\beta-|R|)!} \sum_{k=0}^{\min(|\alpha-|P|,|\beta-|S|)} \sum_{K=0}^{\min(|P|,|S|)} \frac{(2\hbar)^k}{k!K!} (-1)^k \right) \\
&\times \frac{1}{(\alpha-|P|-k)!} (P-K)! \frac{1}{(\alpha-|Q|-k)!} (Q-S-K)! \left( \frac{1}{(\alpha-|P|-k)!} (P-K)! \frac{1}{(\alpha-|Q|-k)!} (Q-S-K)! \right) \\
&\times f_{P+R-K,Q+S-K,\alpha+\beta-k-|K|} \sum_{k=0}^{\min(|\alpha-|P|,|\beta-|S|)} \sum_{K=0}^{\min(|P|,|S|)} \frac{(-1)^k}{k!K!} f_{P+R-K,Q+S-K,\alpha+\beta-k-|K|} \\
&\times \left( \frac{P+R-K}{R} \right) \left( \frac{Q+S-K}{Q} \right) \left( \frac{\alpha+\beta-k-|P|-|R|}{\beta-|R|} \right) \left( \frac{\alpha+\beta-k-|Q|-|S|}{\alpha-|Q|} \right).
\end{align*}
\]

This shows that \( \mathcal{A}_h(C_{n+1}^+) \) is indeed a subalgebra, completing the proof of the first part. Moreover, we can read off the structure constants of \( \tilde{\star}_{\text{Wick}} \) from this formula and get

\[
\begin{align*}
C_{(P,Q,\alpha), (R,S,\beta)}^{(I,J,\gamma)} \\
&= \sum_{k=0}^{\min(|\alpha-|P|,|\beta-|S|)} \sum_{K=0}^{\min(|P|,|S|)} \frac{(-1)^k}{k!K!} \delta_{I,P+R-K} \delta_{J,Q+S-K} \delta_{\gamma,\alpha+\beta-k-|K|} \\
&\times \left( \frac{P+R-K}{R} \right) \left( \frac{Q+S-K}{Q} \right) \left( \frac{\alpha+\beta-k-|P|-|R|}{\beta-|R|} \right) \left( \frac{\alpha+\beta-k-|Q|-|S|}{\alpha-|Q|} \right) \\
&\times \sum_{k=0}^{\min(|\alpha-|P|,|\beta-|S|)} \sum_{K=0}^{\min(|P|,|S|)} \delta_{I,P+R-K} \delta_{\gamma,\alpha+\beta-k-|K|}.
\end{align*}
\]
We have three $\delta$'s and only two summations. Hence one $\delta$ will survive. Implementing the conditions from the $\delta$'s we have a remaining summation over $k$ and $K$ with two of the $\delta$'s. This sum will either give 0 or 1 depending on whether in the allowed ranges we find the correct from the $\delta$.

In view of the $\delta$ in front of the sum we can replace $\delta_{I,P+R-K}$ also by $\delta_{I,Q+S-K}$. From this we get the two necessary conditions $Q \leq J$ and $R \leq I$ in order to get a nonzero structure constant. The condition on $\gamma$ is more delicate to evaluate. The reason is that the minimum $\min(min(P,S))$ might have strictly smaller length than the minimum of the lengths $min(|P|,|S|)$. Nevertheless, we get an estimate that the structure constant is certainly zero unless

$$\max(\alpha,\beta) \leq \gamma \leq \alpha + \beta.$$  

Note however, that there will be situations where the structure constants still vanish, even though (*) is satisfied. □

The important feature is that for a given index triple $(I,J,\gamma)$ we only have finitely many $(P,Q,\alpha)$ and $(R,S,\beta)$ such that the corresponding structure constant is nonzero. This follows from the estimate $\gamma \geq \max(\alpha,\beta)$ and the conditions $\gamma \geq |I|,|J|$.

Note also that the rescaling resulted in structure constants not depending on the deformation parameter $h$ anymore. Thus the recursion for the seminorms will not contain $h$. However, for a given function $a \in A_h(C_{n+1}^+)$ the seminorms do depend on $h$ and so does the topology, as the basis vectors $f_{P,Q,\alpha}$ do. We will have to come back to this $h$-dependence in Subsection 5.2.

The Wick star product $\ast_{\text{wick}}$ and also $\wedge_{\text{wick}}$ are Hermitian star products if one treats the deformation parameter as a real quantity. Since we have absorbed $h$ into the definition of the basis, we get a symmetry of the structure constants originating from the complex conjugation being a $*$-involution for real $h$.

**Proposition 4.7** For all index triples the structure constants $C_{(P,Q,\alpha),(R,S,\beta)}^{(I,J,\gamma)}$ are real and one has

$$C_{(S,R,\beta),(Q,P,\alpha)}^{(I,J,\gamma)} = C_{(P,Q,\alpha),(R,S,\beta)}^{(J,I,\gamma)}.$$  

(4.27)

We conclude this subsection by the following observation. The restrictions (4.26) on $\gamma$ lead to a filtration of the algebra $A_h(C_{n+1}^+)$. We set

$$A_h^\gamma(C_{n+1}^+) = \bigoplus_{I,J} \mathbb{C}f_{I,J,\gamma} \quad \text{and} \quad A_h^{(\gamma)}(C_{n+1}^+) = \bigoplus_{\alpha=0}^\gamma A_h^{\gamma}(C_{n+1}^+),$$  

(4.28)

which are finite-dimensional subspaces with $A_h^{(\gamma)}(C_{n+1}^+) \subseteq A_h^{(\gamma+1)}(C_{n+1}^+)$. The following is then an immediate consequence of Proposition 4.6.

**Corollary 4.8** Let $h \neq 0$. The algebra $A_h(C_{n+1}^+)$ is filtered via the subspaces $A_h^{(\gamma)}(C_{n+1}^+)$, i.e.

$$A_h(C_{n+1}^+) = \bigcup_{\gamma=0}^\infty A_h^{(\gamma)}(C_{n+1}^+) \quad \text{and} \quad A_h^{(\alpha)}(C_{n+1}^+) \ast_{\text{wick}} A_h^{(\beta)}(C_{n+1}^+) \subseteq A_h^{(\alpha+\beta)}(C_{n+1}^+).$$  

(4.29)
4.4 First version

To proceed we need to compute the constants \([26]\) from the structure constants as usual. Even though one can compute them explicitly, the following properties and rough estimates will be all we need:

**Lemma 4.9** Let \((I, J, \gamma)\) be an index triple.

i.) For all index triples \((P, Q, \alpha)\) the constant \(C^{(I, J, \gamma)}_{(P, Q, \alpha)}\) is finite and zero for \(\alpha > \gamma\). Analogously, for all index triples \((Q, P, \alpha)\) the constant \(C^{(I, J, \gamma)}_{(Q, P, \alpha)}\) is finite and zero for \(\beta > \gamma\).

ii.) For all index triples \((P, Q, \alpha)\) and \((R, S, \beta)\) one has

\[
\sum_{I,J} C^{(I, J, \gamma)}_{(P, Q, \alpha)} \leq (\gamma + 1)^{4n+1} 4^\gamma \quad \text{and} \quad \sum_{I,J} C^{(I, J, \gamma)}_{(R, S, \beta)} \leq (\gamma + 1)^{4n+1} 4^\gamma. \tag{4.30}
\]

iii.) For all index triples \((P, Q, \alpha)\) and all \(\gamma \geq \alpha\) one has

\[
C^{(P, Q, \gamma)}_{(P, Q, \alpha)} \geq 1 \quad \text{and} \quad C^{(P, Q, \gamma)}_{(R, S, \beta)} \geq 1. \tag{4.31}
\]

Proof. With the explicit formula from Proposition \([16]\) this is now a simple argument. First recall that in an index triple \((I, J, \gamma)\) there are only finitely many allowed \(I\) and \(J\) for a fixed \(\gamma\) since \(|I|, |J| \leq \gamma\). Thus we only have to take care of the indices \(\alpha, \beta,\) and \(\gamma\). For a given \(\gamma\) there are only finitely many \(\alpha\) and \(\beta\) such that \(\max(\alpha, \beta) \leq \gamma \leq \alpha + \beta\) can hold. In particular, \(\alpha, \beta \leq \gamma\) is a necessary condition. But then the summation over \(\beta\) gives a finite constant \(C^{(I, J, \gamma)}_{(P, Q, \alpha)}\), while the summation over \(\alpha\) gives a finite \(C^{(\gamma, J, \gamma)}_{(R, S, \beta)}\). In addition, the conditions \(\alpha \leq \gamma\) and \(\beta \leq \gamma\), respectively, still persist giving the first statement. For the second we have

\[
\sum_{I,J} C^{(I, J, \gamma)}_{(P, Q, \alpha)} = \sum_{I,J} \sum_{(R, S, \beta)} C^{(I, J, \gamma)}_{(P, Q, \alpha), (R, S, \beta)}
\]

\[
\leq \sum_{I,J} \sum_{(R, S, \beta)} \frac{(1_\gamma) (\gamma - |I|) (\gamma - |J|)}{(\alpha + \beta - \gamma - |P| - |R| + |I|)! (P + R - I)!}
\]

\[
\leq \sum_{I,J} \sum_{(R, S, \beta)} 4^\gamma \leq (\gamma + 1)^{4n+1} 4^\gamma,
\]

since each index \(I_\ell\) in \(I\) with \(\ell = 1, \ldots, n\) runs at most from 0 to \(\gamma\) and analogously for \(J, R,\) and \(S\). Also \(\beta\) runs at most from 0 to \(\gamma\). The other estimate is analogous. For the third estimate we note that the product of \(f_{P, Q, \alpha}\) with \(f_{0, 0, \beta}\) gives a contribution for \(f_{P, Q, \alpha + \beta}\) with a prefactor given by \((\alpha + \beta - |P|)(\alpha + \beta - |Q|) \geq 1\). This corresponds to the summation indices \(k = 0\) and \(K = 0\) in the computations in the proof of Proposition \([16]\). Hence \(C^{(P, Q, \alpha + \beta)}_{(P, Q, \alpha), (0, 0, \beta)} = C^{(P, Q, \alpha + \beta)}_{(0, 0, \beta), (P, Q, \alpha)} \geq 1\). But then the third claim is clear. \(\Box\)

In principle, we can even compute the constants explicitly, using the explicit formulas obtained in Proposition \([16]\). However, to determine the topology of \(A_\hbar(C_{n+1}^+)\), the above simple counting is already sufficient, a situation very similar to the case of the polynomials in Subsection 3.1.

**Proposition 4.10** Let \(\hbar \neq 0\). The topology of \(A_\hbar(C_{n+1}^+)\) according to the construction as in Theorem \([2, 6]\) is the Cartesian product topology inherited from \(\prod_{(P, Q, \alpha)} C f_{P, Q, \alpha}\).
Proof. Thanks to Lemma 4.9 we have for the recursive definition of the \( h_{m,\ell,(I,J,\gamma)} \) the finite sums
\[
h_{m+1,2\ell,(I,J,\gamma)}(a) = \sum_{(P,Q,\alpha)} \text{finite } h_{m,\ell,(P,Q,\alpha)}(a) C_{(P,Q,\alpha)}^{(I,J,\gamma)},
\]
and
\[
h_{m+1,2\ell+1,(I,J,\gamma)}(a) = \sum_{(R,S,\beta)} \text{finite } h_{m,\ell,(R,S,\beta)}(a) C_{(R,S,\beta)}^{(I,J,\gamma)}.
\]
A simple induction shows that in \( h_{m,\ell,(I,J,\gamma)}(a) \) only finitely many coefficients \( |a_{P,Q,\alpha}| \) of \( a \) contribute. Hence, up to a constant, we can estimate \( \|a\|_{m,\ell,(I,J,\gamma)} \) by the maximum of those finitely many \( |a_{P,Q,\alpha}| \) which constitutes a continuous seminorm of the Cartesian product. Thus the Cartesian product topology is finer in this case. But it is coarser in general according to Theorem 2.6.

To conclude this subsection we collect a few more technical properties of the seminorms which we shall need later on. Since for a given \( \gamma \) we have only finitely many \( I \) and \( J \) with \( |I|, |J| \leq \gamma \) we can consider the new combination
\[
h_{m,\ell,\gamma}(a) = \sum_{I,J} h_{m,\ell,(I,J,\gamma)}(a). \tag{4.32}
\]
We have corresponding seminorms \( \|a\|_{m,\ell,\gamma} = \sqrt[2m]{h_{m,\ell,\gamma}(a)} \). Clearly, they will produce the Cartesian product topology as well.

**Lemma 4.11** Let \( m \in \mathbb{N}_0, \ell = 0, \ldots, 2^m - 1, \) and \( \gamma \in \mathbb{N}_0. \)

i.)
\[
\sum_{\alpha=0}^{\gamma} h_{m,\ell,\alpha}(a)^2 \leq (\gamma + 1)^{2n} h_{m+1,2\ell,\gamma}(a) \quad \text{and} \quad \sum_{\alpha=0}^{\gamma} h_{m,\ell,\alpha}(a)^2 \leq (\gamma + 1)^{2n} h_{m+1,2\ell+1,\gamma}(a). \tag{4.33}
\]

ii.)
\[
\sum_{I,J} h_{m,\ell,(I,J,\gamma)}(a)^2 \leq h_{m,\ell,\gamma}(a)^2. \tag{4.34}
\]

iii.) If \( m \geq 1 \) then \( \| \cdot \|_{m,\ell,\gamma} \) is a norm on \( A_h^{(\gamma)}(C_{n+1}^+) \) and identically zero on the complement \( \bigoplus_{(P,Q,\alpha),\alpha > \gamma} C_{(P,Q,\alpha)} \).

Proof. For the first part we use \( C_{(P,Q,\alpha),\gamma} \geq 1 \) according to Lemma 4.9. Thus we get
\[
\sum_{\alpha=0}^{\gamma} h_{m,\ell,\alpha}(a)^2 = \left( \sum_{|P|,|Q| \leq \alpha} h_{m,\ell,(P,Q,\alpha)}(a) \right)^2 \leq (\gamma + 1)^{2n} \sum_{\alpha=0}^{\gamma} \sum_{|P|,|Q| \leq \alpha} h_{m,\ell,(P,Q,\alpha)}(a)^2 \leq (\gamma + 1)^{2n} \sum_{\alpha=0}^{\gamma} \sum_{|P|,|Q| \leq \alpha} h_{m,\ell,(P,Q,\alpha)}(a)^2 C_{(P,Q,\alpha)}(P,Q,\gamma),
\]

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\[
\leq (\gamma + 1)^{2n} \sum_{|I|, |J| \leq \gamma} \sum_{\alpha = 0}^{\gamma} \sum_{|P|, |Q| \leq \alpha} h_{m,\ell,(P,Q,\alpha)}(a)^2 C_{(P,Q,\alpha)}^{(I,J,\gamma)}.
\]

\[
= (\gamma + 1)^{2n} h_{m+1,2\ell,\gamma}(a),
\]

where in the first estimate we use Hölder’s inequality for finite sums and the rough estimate that all indices run at most from 0 to \(\gamma\). In the second we used Lemma 4.9. Analogously, we can take \(C_{(P,Q,\alpha)}^{(I,J,\gamma)}\) in the second estimate, proving the first part. The second part is trivial since all terms in (4.32) are nonnegative. For the last we note that \(\|a\|_{0,\alpha} = \sum_{P,Q} |a_{P,Q,\alpha}|\) is a norm on the span of the \(f_{P,Q,\alpha}\) for fixed \(\alpha\). But then the first part shows that \(h_{1,0,\gamma}(a)\) being zero implies \(h_{0,0,\alpha}(a) = 0\) for all \(\alpha \leq \gamma\). Hence it follows that \(\|\cdot\|_{1,0,\gamma}\) is a norm on the span of all \(f_{P,Q,\alpha}\) with \(\alpha \leq \gamma\) and analogously for \(\|\cdot\|_{1,1,\gamma}\). For the higher \(m\) the norm property follows again by the first part by induction. Moreover, a simple induction using Lemma 4.9, shows that for \(h_{m,\ell,\gamma}(a)\) one never uses coefficients \(|a_{P,Q,\alpha}|\) with \(\alpha > \gamma\).

\[
(4.35)
\]

\[
(4.36)
\]

\[
(4.37)
\]

#### 4.5 Making the evaluation functionals continuous

We want to refine the topology now in such a way that the evaluation functionals at an arbitrary point in \(C^+_{n+1}\) become continuous. This will allow us to interpret the elements of the completion still as functions on \(C^+_{n+1}\), a feature which we do not want to loose.

The following technical lemma will help us to show that the completion will not be too small. The situation is very similar to the polynomial algebra from Section 3.1: again a sub-factorial behavior with respect to the index \(\gamma\) will reproduce in the recursion of the \(h_{m,\ell,(I,J,\gamma)}\).

**Lemma 4.12** Let \(h \neq 0\) and \(a = \sum_{I,J,\gamma} a_{I,J,\gamma} f_{I,J,\gamma} \in \prod_{I,J,\gamma} C_f(I,J,\gamma)\) be an element in the Cartesian product with sub-factorial growth with respect to \(\gamma\), i.e. for all \(\epsilon > 0\) there is a constant \(c_0 > 0\) such that

\[
|a_{I,J,\gamma}| \leq c_0 (\gamma!)^\epsilon
\]

for all index triples \((I,J,\gamma)\). Then for all \(m \in \mathbb{N}_0\) and \(\ell = 0, \ldots, 2^m - 1\) and all \(\epsilon > 0\) we have a constant \(c_m > 0\) with

\[
h_{m,\ell,(I,J,\gamma)}(a) \leq c_m (\gamma!)^\epsilon.
\]

Proof. Again, we prove this by induction on \(m\). For \(m = 0\) this is precisely the assumption on \(a\). From Lemma 4.9, we know that the constants \(C_{(I,J,\gamma)}^{(P,Q,\alpha)}\), as well as \(C_{(P,Q,\alpha)}^{(I,J,\gamma)}\), can be estimated by some \(c^\gamma\). Hence for all \(\epsilon > 0\)

\[
h_{m+1,2\ell,(I,J,\gamma)}(a) = \sum_{(P,Q,\alpha)} h_{m,\ell,(P,Q,\alpha)}(a)^2 C_{(P,Q,\alpha)}^{(I,J,\gamma)} \leq \sum_{(P,Q,\alpha)} c_m(a!)^2 c^\gamma \leq c_{m+1}(\gamma!)^{2\ell},
\]

since the sum is finite and \(\alpha \leq \gamma\). Thus also \(h_{m+1,2\ell,(I,J,\gamma)}(a)\) has sub-factorial growth. The case \(2\ell + 1\) is analogous. \(\square\)

Note that even though the topology induced by the \(h_{m,\ell,\gamma}\) is the (rather trivial) Cartesian product topology, the above statement is nontrivial and would immediately fail if the \(h_{m,\ell,(I,J,\gamma)}\) are multiplied by a suitable \(\gamma\)-dependent factor. Moreover, since for a given \(\gamma\) we have only finitely many \(I\) and \(J\) also the quantities \(h_{m,\ell,\gamma}(a)\) have sub-factorial growth for \(a\) having sub-factorial growth, i.e. for all \(\epsilon > 0\) we have a (different) constant \(c_m > 0\) with

\[
h_{m,\ell,\gamma}(a) \leq c_m (\gamma!)^\epsilon.
\]

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Let $w \in C_{n+1}^+$. Then we consider the evaluation functional

$$\delta_w : \mathcal{A}_h(C_{n+1}^+) \ni a \mapsto a(w) \in \mathbb{C}. \quad (4.38)$$

This gives us new seminorms $\| \cdot \|_{m,\ell,\delta_w}$ on $\mathcal{A}_h(C_{n+1}^+)$ which have shown to be finite by hand. According to the general construction from Subsection 2.2 they are given by

$$\|a\|_{m,\ell,\delta_w} = 2^m \sqrt{\sum_{(I,J,\gamma)} |f_{I,J,\gamma}(w)| h_{m,\ell,(I,J,\gamma)}(a)}. \quad (4.39)$$

From the explicit form of the functions $f_{I,J,\gamma}$ the evaluation at $w$ gives

$$|\delta_w (f_{I,J,\gamma})| = \frac{1}{I!(\gamma - |I|)!J!(\gamma - |J|)!} \left( \frac{y(w)}{2h} \right)^2 \frac{|w|^{2\gamma - |I| - |J|} |w|^{f+J}}{y(w)^{\gamma}}. \quad (4.40)$$

To elaborate further on the new seminorms we recall some basic estimate on the Pochhammer symbols. Let $z \in \mathbb{C}$ then

$$ab^\gamma \leq \frac{1}{\gamma!} \left( |z| \right)^\gamma \leq c^\gamma \quad (4.41)$$

for constants $b, c > 0$ and $a \geq 0$ depending on $z$. Note that $c$ can be chosen locally uniformly in $z$. Moreover, if in addition $-z$ is not in $\mathbb{N}_0$ then also $a > 0$ and $a$ and $b$ can be chosen locally uniformly in $z$ as well. Clearly, for $-z \in \mathbb{N}_0$ the Pochhammer symbol is 0 for large enough $\gamma$.

**Lemma 4.13** Let $h \neq 0$. The system of seminorms $\{\| \cdot \|_{m,\ell,\delta_w}\}_{w \in C_{n+1}^+}$ is equivalent to the system of seminorms $\{\| \cdot \|_{m,\ell,R}\}_{R > 0}$ where

$$\|a\|_{m,\ell,R} = 2^m \sqrt{\sum_{\gamma=0}^{\infty} \frac{R^\gamma}{\gamma!} h_{m,\ell,\gamma}(a)}. \quad (4.42)$$

On those $a \in \prod_{I,J,\gamma} \mathbb{C} f_{I,J,\gamma}$ with sub-factorial growth as in Lemma 4.12 they take finite values. Moreover, it is equivalent to the system $\{\| \cdot \|_{m,\ell,\delta_w}\}$ where we only take $w \in C_{n+1}^+$ with $y(w) = 1$, provided $h$ is an allowed value.

Proof. Of course, we prove the mutual estimates as inequalities in $[0, +\infty]$ first and then deduce their finiteness. In fact, $\|a\|_{m,\ell,R} < \infty$ is trivial for those $a$ with sub-factorial growth according to (4.37). To prove the equivalence of the two systems we first consider

$$\left( \|a\|_{m,\ell,\delta_w} \right)^{2^m} \leq \sum_{(I,J,\gamma)} \frac{1}{\gamma!^{12}} \left( \frac{1}{I!} \left( \gamma - |I| \right) \right) \left( \frac{1}{J!} \left( \gamma - |J| \right) \right) \left( \frac{y(w)}{2h} \right)^2 \frac{|w|^{2\gamma - |I| - |J|} |w|^{f+J}}{y(w)^{\gamma}} h_{m,\ell,(I,J,\gamma)}(a)$$

$$\leq \sum_{\gamma=0}^{\infty} \frac{R^\gamma}{\gamma!} c^\gamma (n+1)^2 \frac{\|w\|_\infty^{2\gamma}}{y(w)^{\gamma}} \sum_{|I|,|J| \leq \gamma} h_{m,\ell,(I,J,\gamma)}(a)$$

$$= \sum_{\gamma=0}^{\infty} \frac{R^\gamma}{\gamma!} h_{m,\ell,\gamma}(a),$$

where we have used the standard estimate for the $(n+1)$-multinomial coefficients and where

$$R = c(n+1)^2 \frac{\|w\|_\infty^{2}}{y(w)} \quad (*)$$

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with $c$ begin the constant in (4.31) for $z = \frac{y(w)}{2n}$. For the reverse estimate let $R > 0$ be given. Then we have for an arbitrary point $w$ with $-\frac{y(w)}{2n} \not\in N_0$

\[
\sum_{\gamma=0}^{\infty} \frac{R^\gamma}{\gamma!} h_{m,\ell,\gamma}(a) \leq \frac{1}{a} \sum_{\gamma=0}^{\infty} \frac{R^\gamma}{\gamma!} \frac{1}{b^\gamma} \left| \frac{1}{\gamma!} \left( \frac{y(w)}{2h} \right)^\gamma \right| \sum_{|I|,|J| \leq \gamma} h_{m,\ell,(I,J,\gamma)}(a)
\]

\[
\leq \frac{1}{a} \sum_{\gamma=0}^{\infty} \frac{R^\gamma}{\gamma!} \frac{1}{b^\gamma} \left| \frac{1}{\gamma!} \left( \frac{y(w)}{2h} \right)^\gamma \right| \sum_{|I|,|J| \leq \gamma} \left( \frac{\gamma}{|I|} \right) J! \left( \frac{\gamma}{|J|} \right) J! h_{m,\ell,(I,J,\gamma)}(a)
\]

\[
= \frac{1}{a} \sum_{(I,J,\gamma)} \frac{1}{I!(\gamma - |I|)!J!(\gamma - |J|)!} \left| \frac{y(w)}{2h} \right| \frac{R^\gamma}{b^\gamma} h_{m,\ell,(I,J,\gamma)}(a).
\]

We choose the point $w$ such that in addition to the requirement $-\frac{y(w)}{2n} \not\in N_0$ we have $|w^i| \geq \sqrt{\frac{R}{9y(w)}}$ for all $i = 0, \ldots, n$. Note that we always can fulfill this additional requirement since we have points $w$ with arbitrarily large positive coefficients $w^i$ but fixed $y(w)$. Then we can estimate further and get

\[
\sum_{\gamma=0}^{\infty} \frac{R^\gamma}{\gamma!} h_{m,\ell,\gamma}(a) \leq \frac{1}{a} \sum_{(I,J,\gamma)} \frac{1}{I!(\gamma - |I|)!J!(\gamma - |J|)!} \left| \frac{y(w)}{2h} \right| \frac{|w|^{|\gamma - |I||} |w|^{|\gamma - |J||} |w|^J}{|y(w)^\gamma|} h_{m,\ell,(I,J,\gamma)}(a)
\]

\[
= \frac{1}{a} \|a\|_{m,\ell,\delta_w}^{2n}.
\]

If $h$ is an allowed value this shows that even the points with $y(w) = 1$ will suffice. \hfill \Box

**Corollary 4.14** Let $h \neq 0$. The estimate of $\|a\|_{m,\ell,\gamma}$ by some $\|a\|_{m,\ell,R}$ is locally uniform in $w$, i.e.
for every compact subset $K \subseteq C_{n+1}^+$ there is a $R > 1$ with

\[
\sup_{w \in K} \|a\|_{m,\ell,\delta_w} \leq \|a\|_{m,\ell,R}.
\]

(4.43)

**Proof.** Indeed, from (*) in the above proof this is clear by taking $R$ to be the maximum value of $c(n+1)^2 \|w\|_\infty^2$ for $w \in K$. \hfill \Box

From now on we endow the subalgebra $A_h(C_{n+1}^+)$ of $\prod_{I,J,\gamma} C f_{I,J,\gamma}$ with the topology arising from the additional seminorms $\| \cdot \|_{m,\ell,\delta_w}$ for $w \in C_{n+1}^+$ which we know to be finite according to the lemma. Equivalently, we can add all the seminorms $\| \cdot \|_{m,\ell,R}$ with $R > 0$. The completion $\hat{A}_h(C_{n+1}^+)$ is the locally convex algebra according to the general construction from Theorem 2.10 where we choose the set of all evaluation functionals for the construction of the additional seminorms.

We collect these results and the explicit characterization of the completion in the following theorem:

**Theorem 4.15** Let $h \neq 0$. Then $A_h(C_{n+1}^+)$ has the following properties:

i.) $A_h(C_{n+1}^+)$ is a Hausdorff locally convex algebra.

ii.) Every evaluation functional $\delta_w$ for all $w \in C_{n+1}^+$ is continuous.

iii.) The completion $\hat{A}_h(C_{n+1}^+)$ is a Fréchet algebra.
iv.) The completion $\hat{A}_h(C^+_{n+1})$ coincides with those $a \in \prod_{(I,J,\gamma)} \mathcal{C}f_{I,J,\gamma}$ having sub-factorial growth\footnote{iv.} with respect to $\gamma$. An equivalent system of seminorms is given by $\{\| \cdot \|_\epsilon\}_{0<\epsilon<1}$ where

$$\|a\|_\epsilon = \sup_{(I,J,\gamma)} \frac{|a_{I,J,\gamma}|}{(\gamma!)^\epsilon}.$$ (4.44)

v.) As Fréchet space $\hat{A}_h(C^+_{n+1})$ is isomorphic to a Kőthe space of sub-factorial growth. It is strongly nuclear and the Schauder basis $\{f_{I,J,\gamma}\}$ is absolute.

vi.) The completion $\hat{A}_h(C^+_{n+1})$ can be continuously included into the continuous functions $C^0(C^+_{n+1})$.

Proof. The first and second part is clear by construction and Lemma \ref{lem:properties}. By the same lemma we note that the topology is determined by countably many seminorms $\| \cdot \|_{m,\ell,R}$ with an increasing sequence $R_n \to \infty$. Hence the completion is Fréchet. For the fourth part we already know that the elements with sub-factorial growth belong to the completion, this was also mentioned in Lemma \ref{lem:properties}. To show the converse, let $a \in \hat{A}_h(C^+_{n+1})$ be given. From the estimate (4.33) we get by induction that

$$|a_{I,J,\gamma}|^{2m} \leq (\gamma + 1)^{k_m} h_{m,0,\gamma}(a)$$

with some universal exponent $k_m > 0$ depending only on the dimension and $m$. Thus the definition of the seminorm $\| \cdot \|_{m,\ell,R}$ as in (4.42) gives the estimate

$$|a_{I,J,\gamma}|^{2m} \leq (\gamma + 1)^{k_m} h_{m,0,\gamma}(a) \leq (\gamma + 1)^{k_m} \frac{\gamma!}{R^\gamma} \|a\|_{m,0,R}^{2m},$$ (*)

for all $R > 0$ and all $m \in \mathbb{N}_0$. Since $(\gamma + 1)^{k_m}$ and $R^{-\gamma}$ clearly grow sub-factorially we can take the $2m$-th root of this inequality to see that $|a_{I,J,\gamma}|$ grows sub-factorially, too. Now we note that $\|a\|_\epsilon$ is clearly a well-defined seminorm for those $a$ with sub-factorial growth. Moreover, (*) shows that the seminorm $\|a\|_\epsilon$ with $\epsilon = \frac{1}{2m}$ can be estimated by $\|a\|_{m,0,R}$ for $R > 1$. Clearly, those $\epsilon$ are sufficient to conclude that the original topology is finer than the one induced by the system of all the $\| \cdot \|_\epsilon$. For the converse estimates one carefully examines the proof of Lemma \ref{lem:properties} to conclude that the constant $c_m$ in (4.36) can be chosen to be a numerical factor times the $2m$-th power of $\|a\|_\epsilon$ for an appropriate $\epsilon' > 0$. From this we get the reverse estimate of $\|a\|_{m,\ell,R}$ by some $\|a\|_\epsilon$. It is also clear from general arguments as on a Fréchet space any coarser Fréchet topology is necessarily the same by the open mapping theorem. This proves the fourth part. Then the fifth part is a simple consequence of (4.44) and the general facts from Appendix A. The last part is clear by Corollary \ref{cor:properties}.

\[ \blacksquare \]

Remark 4.16 While the first version gave us just the uninteresting Cartesian product topology, the second is now much more subtle. In particular, the precise form of the $h_{m,\ell,\gamma}$ enter the game as we still have to guarantee the continuity of the product via Theorem \ref{thm:properties}.

The functions in $\hat{A}_h(C^+_{n+1})$ can be characterized further: we consider the diagonal map

$$\Delta: C^+_{n+1} \ni z \mapsto (z,z) \in C^+_{n+1} \times C^+_{n+1}. $$ (4.45)

Moreover, denote by $\mathcal{O}(C^+_{n+1} \times C^+_{n+1})$ the functions which are holomorphic in the first and antiholomorphic in the second variables. Then we have for every index triple $(P,Q,\alpha)$ the function $\hat{e}_{P,Q,\alpha} \in \mathcal{O}(C^+_{n+1} \times C^+_{n+1})$ given by $\hat{e}_{P,Q,\alpha}(u,v) = (u^0)^{\alpha - \gamma} |P|_u |\mathcal{Q}|_v$ such that $\Delta^* \hat{e}_{P,Q,\alpha} = e_{P,Q,\alpha}$. Similarly, $\hat{y}(u,v) = u^0 v^0 - u^1 v^1 - \cdots - u^{\mathcal{Q}} v^{\mathcal{Q}}$ satisfies $\Delta^* \hat{y} = y$. Since $\hat{y}$ is still different from 0 on $C^+_{n+1} \times C^+_{n+1}$ we have also $\hat{f}_{P,Q,\alpha} \in \mathcal{O}(C^+_{n+1} \times C^+_{n+1})$ by the analogous formulas. Thus every element $a \in A_h(C^+_{n+1})$ has a (necessarily unique) extension $\hat{a}$ to a function in $\mathcal{O}(C^+_{n+1} \times C^+_{n+1})$, i.e. $\Delta^* \hat{a} = a$. This still holds for the completion:
Proposition 4.17 Under the identification of \( \tilde{A}_h(C_{n+1}^+) \) with functions on \( C_{n+1}^+ \), any function \( a \in \tilde{A}_h(C_{n+1}^+) \) has an extension \( \hat{a} \in \mathcal{O}(C_{n+1}^+ \times C_{n+1}^+) \). In particular, \( a \) is real-analytic on \( C_{n+1}^+ \).

Proof. First we note that for a basis vector we get the estimate

\[
|\hat{f}_{P,Q,\alpha}(u,v)| \leq \frac{1}{\alpha!} (n+1)^{2\alpha} c(u,v)^\alpha \frac{\|u\|_\infty \|v\|_\infty}{|\hat{y}(u,v)|^\alpha},
\]

where \( c(u,v) \) is the constant in the estimate (4.41) of the Pochhammer symbol \( \left( \frac{\hat{y}(u,v)}{2\hbar} \right)^\gamma \). Since this constant can be chosen to be locally uniform, for every compact subset \( K \subseteq C_{n+1}^+ \times C_{n+1}^+ \) we can define

\[
R = \max_{(u,v) \in K} (n+1)^2 c(u,v) \frac{\|u\|_\infty \|v\|_\infty}{|\hat{y}(u,v)|}.
\]

On \( \tilde{A}_h(C_{n+1}^+) \) we can define the evaluation functional \( \delta(u,v) \) of \( a \) as \( \delta(u,v)(a) = \hat{a}(u,v) \) for \( u,v \in C_{n+1}^+ \). We claim that this is continuous. Indeed, writing \( a = \sum_{(P,Q,\alpha)} a_{P,Q,\alpha} \hat{f}_{P,Q,\alpha} \) we estimate

\[
|\delta(u,v)(a)| \leq \sum_{(P,Q,\alpha)} |a_{P,Q,\alpha}| |\hat{f}_{P,Q,\alpha}(u,v)| \leq \sum_{\alpha=0}^\infty h_{0,0,\alpha} R^{\alpha} \frac{\|a\|_{0,0,R}}{\alpha!} = \|a\|_{0,0,R}
\]

for all \( (u,v) \in K \). Thus \( \delta(u,v) \) is continuous and extends continuously to \( \tilde{A}_h(C_{n+1}^+) \) and hence we can define \( \hat{a} \) pointwise as \( \hat{a}(u,v) = \delta(u,v)(a) \). Since \( a = \sum_{(P,Q,\alpha)} a_{P,Q,\alpha} \hat{f}_{P,Q,\alpha} \) converges for every \( a \in \tilde{A}_h(C_{n+1}^+) \) we have, thanks to \((*)\), a locally uniform approximation of \( \hat{a} \) by elements of \( \mathcal{O}(C_{n+1}^+ \times C_{n+1}^+) \). Hence also \( \hat{a} \in \mathcal{O}(C_{n+1}^+ \times C_{n+1}^+) \).

Note that we get a rather strong regularity property for the functions \( a \in \tilde{A}_h(C_{n+1}^+) \): not every real-analytic function on \( C_{n+1}^+ \) has such an extension. In general, only an extension in a neighborhood of the diagonal is possible. Moreover, from the last estimate \((*)\) in the proof we note that the topology of \( \tilde{A}_h(C_{n+1}^+) \) is finer than the canonical topology of \( \mathcal{O}(C_{n+1}^+ \times C_{n+1}^+) \), i.e. the locally uniform one. Thus we get a nontrivial refinement of Theorem 4.15 (43). Finally, the characterization of the growth as in Theorem 4.15 (43), gives another (and more direct) way to prove the proposition, however, the continuity statement of the evaluation functionals \( \delta(u,v) \) requires the above argument.

4.6 The algebra on the disk

In the last step we have to pass from \( C_{n+1}^+ \) to the disk \( \mathbb{D}_n \). In the formal power series setting the idea of \[67\] was to show that the classical vanishing ideal of the hypersurface \( y = 1 \) coincides with the two-sided ideal generated by \( y - 1 \) with respect to \( \hat{\psi}_{\text{Wick}} \), simply because \( \hat{\psi}_{\text{Wick}} \) with a radial function is just the pointwise product.

Before we can set \( y = 1 \) in our situation, we have to check a few compatibilities: since we have now only a subalgebra of \( C^{\infty}(C_{n+1}^+)[[\lambda]] \) it is not clear whether the ideal generated by \( y - 1 \) is still the vanishing ideal or not. Here we will need that \( h \) is an allowed value:

Lemma 4.18 Let \( h \) be an allowed value. In \( \tilde{A}_h(C_{n+1}^+) \) the vanishing ideal of the hypersurface \( y = 1 \) coincides with the ideal generated by \( y - 1 \).

Proof. First we note that since \( y - 1 \) is a radial function, the ideal generated by \( y - 1 \) with respect to \( \hat{\psi}_{\text{Wick}} \) is just the classical ideal generated by \( y - 1 \). Hence we only have to care about the pointwise
product. Clearly, the ideal generated by $y - 1$ is contained in the vanishing ideal. To show the reverse inclusion we argue as follows: Let $\gamma$ be given and let $a \in \mathcal{A}_h^{(\gamma)}(C_{n+1}^+)$ be in the vanishing ideal. Then $a = \sum_{(P,Q,\alpha)} a_{P,Q,\alpha} f_{P,Q,\alpha}$ satisfies

$$\sum_{(P,Q,\alpha)} a_{P,Q,\alpha} \frac{1}{P!|m|!Q!|n|!} \left(\frac{1}{2\hbar}\right)_\alpha e_{P,Q,\alpha}(w) = 0$$

for every $w \in C_{n+1}^+$ with $y(w) = 1$. By assumption $\left(\frac{1}{2\hbar}\right)_\alpha \neq 0$. This shows that the polynomial

$$\tilde{a} = \sum_{(P,Q,\alpha)} a_{P,Q,\alpha} \frac{1}{P!|m|!Q!|n|!} \left(\frac{1}{2\hbar}\right)_\alpha e_{P,Q,\alpha}$$

vanishes on the hypersurface $y = 1$. Since $\tilde{a}$ is a U(1)-invariant polynomial of degree at most $2\gamma$, we find a polynomial $\tilde{b}$ of degree at most $2(\gamma - 1)$ with $\tilde{a} = (y - 1)\tilde{b}$. Moreover, $\tilde{b}$ is unique and still U(1)-invariant. Denote the dimension of the space of U(1)-invariant polynomials of degree at most $2\gamma$ by $n_\gamma$. Then this shows that the vanishing ideal intersected with $\mathcal{A}_h^{(\gamma)}(C_{n+1}^+)$ is $n_{\gamma - 1}$-dimensional. But also $\mathcal{A}_h^{(\gamma)}(C_{n+1}^+)$ is $n_{\gamma}$-dimensional and the map $b \mapsto (y - 1)b$ is still injective for $b \in A_h^{(\gamma - 1)}(C_{n+1}^+)$. Hence the space of elements of the form $(y - 1)b$ with $b \in A_h^{(\gamma - 1)}(C_{n+1}^+)$ is $n_{\gamma - 1}$-dimensional, too. Since the latter is contained in the former, both have to coincide. Note however, that the map $a \mapsto \tilde{a}$ is not at all an algebra morphism, we only use it to compare the sizes of the ideals.

This shows now that the ideal generated by $y - 1$ is already a closed ideal in $A_h(C_{n+1}^+)$ since all the evaluation functionals $\delta_w$ with $w$ in the hypersurface $y = 1$ are continuous. Hence the intersection of their kernels is closed in the (noncomplete) algebra $\mathcal{A}_h(C_{n+1}^+)$. Moreover, after completion to $\overline{\mathcal{A}_h}(C_{n+1}^+)$ the closure of the vanishing ideal is still just the vanishing ideal, i.e.

$$\mathcal{I}_{y = 1} = ((y - 1) \tilde{e}_{\text{Wick}} A_h(C_{n+1}^+))^\text{cl} = \left(\bigcap_w \ker \delta_w\right)^\text{cl} = \bigcap_w \ker \delta_w \subseteq \overline{\mathcal{A}_h}(C_{n+1}^+),$$

where $w$ runs through the points of the $y = 1$ hypersurface and $\delta_w$ denotes the extension of $\delta_w$ to the completion. Thus the closure is still an ideal with respect to $\tilde{e}_{\text{Wick}}$ and we can now divide by this ideal. Since $\overline{\mathcal{A}_h}(C_{n+1}^+)$ has the interpretation of a space of functions on $C_{n+1}^+$, the quotient by the vanishing ideal has the interpretation of a space of functions on $\mathbb{D}_n$. This will now be the motivation for the following definition:

**Definition 4.19 (The algebra $\overline{\mathcal{A}_h}(\mathbb{D}_n)$)** Let $h$ be an allowed value. The quotient Fréchet algebra $\overline{\mathcal{A}_h}(C_{n+1}^+)/\mathcal{I}_{y = 1}$ is denoted by $\overline{\mathcal{A}_h}(\mathbb{D}_n)$ and its multiplication will be denoted by $*_{\mathbb{D}_n}$.

Analogously, we set $A_h(\mathbb{D}_n) = A_h(C_{n+1}^+)/(|y - 1| \tilde{e}_{\text{Wick}} A_h(C_{n+1}^+))$. Then the completion of $A_h(\mathbb{D}_n)$ is indeed $\overline{\mathcal{A}_h}(\mathbb{D}_n)$ justifying our notation.

Since the ideal generated by $y - 1$ is precisely the vanishing ideal, we can just set $y$ equal to 1 in all our above formulas to get the corresponding formulas for the algebra on the disk. In particular, the basis vectors $f_{P,Q,\alpha}$ behave as follows: recall that the canonical coordinates on the disk $\mathbb{D}_n$ are given by $v^i = \frac{z^i}{2}$ for $i = 1, \ldots, n$. Then the function on $\mathbb{D}_n$ corresponding to the equivalence class of $f_{P,Q,\alpha}$ is explicitly given by

$$[f_{P,Q,\alpha}](v) = \frac{1}{P!(\alpha - |m|)!Q!(\alpha - |n|)!} \left(\frac{1}{2\hbar}\right)_\alpha \frac{v^P \overline{v}^Q}{(1 - |v|^2)\alpha}.$$  \hfill (4.47)
and their product is given by the very same formula (4.24). However, now these functions are no longer linearly independent as we have used the relation $y = 1$. It was this class of functions for which the star product on the disk was given in [13, Sect. 4].

**Lemma 4.20** Let $h$ be an allowed value. Then the functions

$$f_{P,Q}(v) = |f_{P,Q,\alpha}(v)| = \frac{1}{P!|\alpha - |P||Q!(|\alpha - |Q||)} \left( \frac{1}{2h} \right)_{\alpha} \frac{v^P \overline{\pi}^Q}{(1 - |v|^2)^\alpha}$$

(4.48) for $P, Q \in \mathbb{N}_0^n$ and $\alpha = \max(|P|, |Q|)$ form a vector space basis of $\hat{A}_h(\mathbb{D}_n)$.

Proof. First we note the obvious relation

$$\frac{1}{1 - |v|^2} = 1 + \sum_{\beta=1}^{n} \frac{|v|^{2\beta}}{1 - |v|^2}$$

which gives for all $\alpha \geq 1$ the relation

$$\frac{1}{(1 - |v|^2)^\beta} (1 - |v|^2)^{\max(|R|, |S|)} = \sum_{|I| = 0}^{\beta} \binom{\beta}{|I|} \frac{|I|!}{|I|!} \frac{v^{R+I} \overline{\pi}^S I}{(1 - |v|^2)^{|I| + \max(|R|, |S|)}}$$

(∗)

Clearly, each term on the right hand side is in the linear span of the functions $f_{P,Q}$ showing that the linear span of the $f_{P,Q}$ is all of $\hat{A}_h(\mathbb{D}_n)$. To show their linear independence we have to use that $h$ is an allowed value: we note that the asymptotic behavior of $f_{P,Q}$ for $|v| \to 1$ allows to recover $\alpha = \max(|P|, |Q|)$. Then the linear independence of the monomials $v^P \overline{\pi}^Q$ for fixed $\max(|P|, |Q|)$ gives immediately the linear independence of the $f_{P,Q}$. 

Note finally that the non-allowed values of $h$ would yield a vanishing Pochhammer symbol in (4.47) for all large enough $\alpha$ and hence a finite-dimensional quotient, a case which was discussed in detail in [7].

We summarize now some first properties of the algebra on the disk:

**Theorem 4.21 (The algebra on the disk)** Let $h$ be an allowed value.

.i.) The algebra $\hat{A}_h(\mathbb{D}_n)$ on the disk is a unital Fréchet algebra with respect to $\ast_{\mathbb{D}_n}$.

ii.) $\hat{A}_h(\mathbb{D}_n)$ is a dense subalgebra of $\hat{A}_h(\mathbb{D}_n)$.

iii.) The evaluation functionals $\delta_v : \hat{A}_h(\mathbb{D}_n) \to \mathbb{C}$ defined by

$$\delta_v([a]) = \delta_w(a),$$

(4.49)

where $w \in C_{n+1}^+$ is a preimage of $v \in \mathbb{D}_n$ with $y(w) = 1$, are continuous linear functionals.

iv.) The elements $[a] \in \hat{A}_h(\mathbb{D}_n)$ can be identified with certain real-analytic functions on $\mathbb{D}_n$ having an extension to $\mathbb{D}_n \times \mathbb{D}_n$ begin holomorphic in the first and anti-holomorphic in the second variables. The topology of $\hat{A}_h(\mathbb{D}_n)$ is finer than the locally uniform topology of $\mathcal{O}(\mathbb{D}_n \times \mathbb{D}_n)$.

v.) $\hat{A}_h(\mathbb{D}_n)$ inherits the filtration of $\hat{A}_h(C_{n+1}^+)$ from Corollary 4.8.

vi.) The functions $f_{P,Q} \in \hat{A}_h(\mathbb{D}_n)$ form an unconditional Schauder basis.

vii.) A formal series $\sum_{P,Q} a_{P,Q} f_{P,Q}$ belongs to $\hat{A}_h(\mathbb{D}_n)$ iff the coefficients $a_{P,Q}$ have sub-factorial growth with respect to $\max(|P|, |Q|)$.

viii.) An equivalent system of seminorms for $\hat{A}_h(\mathbb{D}_n)$ is given by

$$\|[a]\|_{\epsilon} = \sup_{P,Q} \frac{|a_{P,Q}|}{(\max(|R|, |S|)!\epsilon)^\epsilon},$$

(4.50)

with $0 < \epsilon < 1$. It follows that $\hat{A}_h(\mathbb{D}_n)$ is isomorphic to a Köthe space of sub-factorial growth. It is strongly nuclear and the Schauder basis $\{f_{R,S}\}$ is absolute.
Proof. The first part is clear by abstract arguments: a Fréchet algebra modulo a closed two-sided ideal is again a Fréchet algebra. Note that it is important that we are in a Fréchet situation, otherwise quotients by closed subspaces might not be complete again. The second is also clear from general arguments on the compatibility of quotients and closures. For \( w \in C^+_{n+1} \) with \( y(w) = 1 \) the evaluation functional \( \delta_w \) is well-defined on the quotient since we divide by the intersection of all the kernels of these functionals. By the universal property of the locally convex quotient topology the resulting functional on \( \tilde{A}_h(\mathbb{D}_n) \) is still continuous. For the fourth part we note that the elements of \( \tilde{A}_h(\mathbb{D}_n) \) have this property as they are linear combinations of the functions \( \{ \mathcal{R}, \mathcal{S} \} \) for which we have the extension, explicitly given by

\[
[f_{P,Q,\alpha}](v, u) = \frac{1}{P!(\alpha - |P|)Q!(\alpha - |Q|)!} \left( \frac{1}{2\pi} \right)^{\alpha} \frac{v^P w^Q}{(1 - v\pi)^\alpha}.
\]

First let \((z, w) \in C^+_{n+1} \times C^+_{n+1}\) satisfy \( \hat{y}(z, w) = 1 \). We claim that \([a] \mapsto \hat{a}(z, w)\) is well-defined for \([a] \in \tilde{A}_h(\mathbb{D}_n)\). Indeed, let \(a \in A_h(C^+_{n+1})\) be given. Then \((y - 1)b(z, w) = (\hat{y}(z, w) - 1)b(z, w) = 0\). Since we divide by the ideal generated by \(y - 1\), the evaluation is well-defined. We have a section \(\varphi\) of the projection \(\pi \times \pi : C^+_{n+1} \times C^+_{n+1} \rightarrow \mathbb{D}_n \times \mathbb{D}_n\) whose image lies in the hypersurface of \(\hat{y} = 1\): there are many choices, one convenient possibility is

\[
\varphi(v, u) = \left( \frac{1}{1 - \overline{v}w}, \frac{v}{1 - \overline{v}w}, (1, u) \right).
\]

Moreover, we have the following consistency \([f_{P,Q,\alpha}](v, u) = f_{P,Q,\alpha}(\varphi(v, u))\), which is checked immediately from \((*)\) and \((**)*\). Thus we get for \(a \in A_h(C^+_{n+1})\) the relation

\[
[a](v, u) = \hat{a}(\varphi(v, u)).
\]

Now let \(K \subseteq \mathbb{D}_n \times \mathbb{D}_n\) be a compact subset. Then \(\varphi(K)\) is compact and contained in the hypersurface of \(\hat{y} = 1\). For \((v, u) \in K\) we get the estimate

\[
||[a](v, u)|| = |a(\varphi(v, u))| \leq ||a||_{0,0,R},
\]

where \(R > 0\) is the parameter corresponding to the compact subset \(\varphi(K)\) as in the proof of Proposition 4.17. Since this holds for all representatives \(a\) of \([a]\), we get the estimate also for the corresponding seminorm on the quotient \(\tilde{A}_h(\mathbb{D}_n)\). This shows that the evaluation at \((v, u)\) is continuous with a locally uniform estimate concerning the point \((v, u)\). Using the second part, the fourth part follows. The filtration property is clear. Now consider the vector space basis \(f_{P,Q}\) from Lemma 4.20. We have to show that the evaluation functionals with respect to this basis are continuous. Thus let \(a = \sum_{(P,Q,\alpha)} a_{P,Q,\alpha} \tilde{f}_{P,Q,\alpha} \in \tilde{A}_h(C^+_{n+1})\) be given. For its equivalence class \([a] \in \tilde{A}_h(\mathbb{D}_n)\) we get by a straightforward computation

\[
[a] = \sum_{R,S} \sum_{(P,Q,\alpha)} a_{P,Q,\alpha} \sum_{|I| = 0} \frac{(P + I)!(\max(0, |Q| - |P|))!(Q + I)!(\max(0, |P| - |Q|))!}{P!(\alpha - |P|)!Q!(\alpha - |Q|)!} \frac{1}{(2\pi)^\alpha} \frac{1}{(2\pi)^{|I| + \max(|P|, |Q|)}} \frac{\alpha - \max(|P|, |Q|)!}{\alpha - \max(|P|, |Q|) - |I|!} \delta_{P+I,R} \delta_{Q+I,S} f_{R,S},
\]

using \((*)\) from the proof of Lemma 4.20. For a given pair \((R, S)\), only those \(P, Q\) can contribute where we have an \(I\) with \(P + I = R\) and \(Q + I = S\). Moreover, \(\alpha\) has to satisfy \(\alpha \geq \max(|R|, |S|)\).
Finally, $I \leq \min(R, S)$ has to hold as well. For abbreviation we set $M = \max(|R|, |S|)$ and $m = \min(|R|, |S|)$. This allows to compute the coefficient $a_{R,S}$ of the basis vector $f_{R,S}$ as follows

$$a_{R,S} = \sum_{\alpha \geq M} \sum_{I \leq \min(R, S)} a_{R-I, S-I, \alpha}(R) (S) \frac{I!(M-m)!}{I!(\alpha-M+|I|)!(\alpha-m+|I|)!} \frac{(\frac{1}{m})}{M} \alpha (\alpha-M)!$$

$$= \sum_{\alpha \geq M} \sum_{I \leq \min(R, S)} a_{R-I, S-I, \alpha}(R) (S) \frac{I!(M-m)!}{I!(\alpha-M+|I|)!(\alpha-m+|I|)!} \frac{1}{M} \alpha (\alpha-M)$$

We claim that this converges for all $a \in \widehat{A}_h(C^{+}_{n+1})$. Indeed, we know that $a_{P,Q,\alpha}$ has sub-factorial growth in $\alpha$. Hence let $\epsilon > 0$ and thus $|a_{R-I, S-I, \alpha}| \leq c_1 (\alpha!)^\epsilon$ for some appropriate $c_1$. Using the estimates for the Pochhammer symbols according to (4.11) we get

$$|a_{R,S}| \leq \sum_{\alpha=M}^{\infty} \sum_{I \leq \min(R, S)} c_1 (\alpha!)^\epsilon 2^{M+m} \frac{I!(M-m)!}{I!(\alpha-m+|I|)!M!(\alpha-M)!} \frac{c_3 M}{2}$$

$$= \sum_{\alpha=M}^{\infty} \sum_{I \leq \min(R, S)} c_1 (\alpha!)^\epsilon 2^{M+m} \frac{I!}{|I|!} \frac{1}{(\alpha-m+|I|)!M!(\alpha-M)!} \frac{1}{2} \frac{c_3 M}{2}$$

$$\leq \sum_{\alpha=M}^{\infty} \sum_{I \leq \min(R, S)} c_1 (\alpha!)^\epsilon c_4 c_5 \frac{1}{M!(\alpha-M)!}$$

$$\leq c_1 c_6^M (M!)^\epsilon \sum_{\alpha=M}^{\infty} \left( \frac{\alpha}{M} \right)^\epsilon \frac{c_7 M}{(\alpha-M)!^{1-\epsilon}}$$

$$\leq c_7 c_8^M (M!)^\epsilon,$$

with corresponding appropriate constants $c_1, \ldots, c_8$. This shows two things: first the series converges (absolutely) and hence the linear functional $a \mapsto a_{R,S}$ is a weakly* convergent series of continuous linear functionals on $\widehat{A}_h(C^{+}_{n+1})$. Second, the resulting values $a_{R,S}$ have a sub-factorial growth in $M = \max(|R|, |S|)$. Since we are in a Fréchet situation, the resulting linear functionals are continuous. By the very definition of the quotient topology, also the induced functionals $[a] \mapsto a_{R,S}$ are continuous on $\widehat{A}_h(D_n)$. Now let $a_{P,Q}$ have sub-factorial growth with respect to $\max(|P|, |Q|)$ then set $a_{P,Q,\alpha} = a_{P,Q}$ for $\alpha = \max(|P|, |Q|)$ and 0 else. By Theorem 4.15 (1) and Theorem 2.10 (1) we have an unconditionally convergent $a = \sum_{P,Q,\alpha} a_{P, Q, \alpha} f_{P, Q, \alpha}$ in $\widehat{A}_h(C^{+}_{n+1})$. But this shows that also $[a] = \sum_{P,Q} a_{P, Q} f_{P, Q}$ converges unconditionally, proving the sixth part. The next part was already shown. Since we can take for $[a] = \sum_{P,Q} a_{P, Q} f_{P, Q}$ a representative $a = \sum_{P, Q, \alpha} a_{P, Q, \alpha} f_{P, Q, \alpha}$ with $a_{P, Q, \alpha} = a_{P, Q}$ for $\alpha = \max(|P|, |Q|)$ and zero elsewhere, the estimate $\|a\|_{P, Q} \leq \|a\|_{P, Q}$ is immediate. This shows that the seminorms of (4.50) are continuous seminorms on $\widehat{A}_h(D_n)$. Thanks to the seventh part, they make $\widehat{A}_h(D_n)$ a Fréchet space as well implying that the two topologies have to coincide. Then the last part is an easy consequence as we have argued already several times.

Note that even though we have a vector space basis also for $A_h(D_n)$ the corresponding structure constants are not as easily described as the ones of $A_h(C^{+}_{n+1})$. The reason is the quotient procedure by $\delta_{y=1}$, in particular, using the explicit formula for the product of $f_{P, Q}$ with $f_{R, S}$ according to (4.24) by evaluating $f_{P, Q, \max(|P|, |Q|)} f_{R, S, \max(|R|, |S|)}$ for $y = 1$ gives a redundant description.
which has first to be expressed again in terms of the linearly independent $f_{I,J}$ alone and not in terms of the $[f_{I,J,\gamma}]$.

Note also that parts of the last statement can be obtained more directly: in particular, $\hat{A}_h(\mathbb{D}_n)$ is a quotient of a nuclear space by a closed subspace and hence nuclear itself. However, the above Schauder basis allows to give an explicit description of the Fréchet space structure of $\hat{A}_h(\mathbb{D}_n)$ as a Köthe space of sub-factorial growth.

5 Further properties of $\hat{A}_h(\mathbb{D}_n)$

In this concluding section we collect some further properties of the algebra on the Poincaré disk $\mathbb{D}_n$: we discuss the symmetry inherited from the classical symmetry (4.5), the dependence on the deformation parameter $\hbar$, and the $^*$-involution.

5.1 The $SU(1,n)$-symmetry

Recall that on the level of formal star products $\tilde{\star}$ Wick was invariant under the canonical $SU(1,n)$-action, allowing even an equivariant quantum momentum map, see Remark 4.2. In fact, the momentum map is part of $\hat{A}_h(C^+_n)$:

**Lemma 5.1** For all $\xi \in su(1,n)$ we have $J_{\xi} \in \hat{A}_h^1(C^+_n)$.

Proof. Since $Sy = y$ this follows immediately from the definition of $J$ in (4.9).

Thus on $\hat{A}_h(C^+_n)$ we have an inner action of $su(1,n)$ via the inner derivations $\frac{1}{i\hbar}[J_{\xi}, \cdot]_{\tilde{\star}}$ Wick. From the filtration property (4.29) we cannot expect the spaces $\hat{A}_h^\gamma(C^+_n)$ to be invariant. Nevertheless, the strong invariance of $\tilde{\star}$ Wick together with the fact that $L_{\xi}y = 0$ for all $\xi \in su(1,n)$ shows that even the spaces $\hat{A}_h^\gamma(C^+_n)$ are invariant. We have

$$\frac{1}{i\hbar}[J_{\xi}, \cdot]_{\tilde{\star}} \hat{A}_h^\gamma(C^+_n) = L_{\xi} \hat{A}_h^\gamma(C^+_n) \subseteq \hat{A}_h^\gamma(C^+_n)$$

(5.1)

for all $\gamma$ and all $\xi \in su(1,n)$. This observation is one of the motivations for our choice of the basis of the $f_{I,J,\gamma}$. The next lemma gives an integrated version of this:

**Lemma 5.2** Let $\hbar$ be an allowed value and $\gamma \in \mathbb{N}_0$.

i.) $\hat{A}_h^\gamma(C^+_n)$ is invariant under $SU(1,n)$ and thus defines a finite-dimensional representation.

ii.) For $U \in SU(1,n)$ the representation on $\hat{A}_h^\gamma(C^+_n)$ has matrix coefficients

$$U^*f_{I,J,\gamma} = \sum_{K,L} M_{I,J}^{KL}(U,\gamma)f_{K,L,\gamma},$$

(5.2)

which satisfy an estimate of the form

$$|M_{I,J}^{KL}(U,\gamma)| \leq (n + 1)^{2\gamma} \|U\|_\infty^{2\gamma}$$

(5.3)

for all multiindices and $U \in SU(1,n)$ where $\|\cdot\|_{\infty}$ denotes the supremum norm of the matrix elements.

Proof. The first statement is an easy consequence of the fact that the monomial $e_{I,J,\gamma}$ is transformed into a linear combination of $e_{K,L,\gamma}$ for all allowed $K,L$ but the same $\gamma$. Since the function $y$ is invariant, the claim follows at once. For the second part, we consider again the monomials $e_{I,J,\gamma}$.
Expanding the summations in $U^*e_{I,J,\gamma}$ by means of the multinomial theorem gives immediately the result, since the additional prefactor in $f_{I,J,\gamma}$ only depends on $y$ and is not changed by the invariance of $y$. 

With other words, the functions in $A_h(C^1_{n+1})$ consist of representative functions, i.e. functions which have a finite-dimensional orbit under the group action of SU(1, n). Since we already know, see Remark 4.2, that SU(1, n) acts by automorphisms of $\tilde{\mathcal{A}}_{\text{Wick}}$ on the whole space $C^\infty(C^1_{n+1})[[\lambda]]$ we get an action by automorphisms on $A_h(C^1_{n+1})$, too.

In a next step we show that the action of SU(1, n) is continuous. The key observation is contained in the following lemma:

**Lemma 5.3** Let $m \in \mathbb{N}_0$. Then there exists a constant $c_m > 1$ such that for all $U \in U(1, n)$, all $a \in A_h(C^1_{n+1})$, all $\gamma \in \mathbb{N}_0$, and all $\ell = 0,\ldots,2^m-1$ we have

$$\|U^*a\|_{m,\ell,\gamma} \leq c_m^\gamma \|U\|_\infty 2^{2\gamma} \|a\|_{m,\ell,\gamma}.$$  \hfill (5.4)

Proof. As usual we prove this by induction on $m$. For $m = 0$ we have

$$h_{0,0,\gamma}(U^*a) = \sum_{I,J} h_{0,0,(I,J,\gamma)}(U^*a) \leq \sum_{I,J} \left| \sum_{K,L} a_{K,L,\gamma} M^{K,L}_{I,J}(U,\gamma) \right|_{\|\cdot\|_{m,\ell,\gamma}} \leq c_0 \sum_{I,J} (n+1)^{2\gamma} \|U\|_\infty^{2\gamma}.$$  \hfill (5.3)

Since the remaining sum has at most $(n+1)^{2\gamma}$ terms, we get (5.2) by taking $c_0 = (n+1)^4$. By induction, we estimate

$$h_{m+1,2\ell,\gamma}(U^*a) = \sum_{I,J} \sum_{(P,Q,\alpha)} h_{m+1,2\ell,(P,Q,\alpha)}(U^*a)^2 C_{(P,Q,\alpha)}^{(I,J,\gamma)}.$$  \hfill (5.4)

$$\leq (\gamma + 1)^4 m+1 4^{2\gamma} \sum_{(P,Q,\alpha)} h_{m+1,2\ell,(P,Q,\alpha)}(U^*a)^2.$$  \hfill (5.5)

$$\leq (\gamma + 1)^4 m+1 4^{2\gamma} \sum_{\alpha=0}^\gamma h_{m+1,2\ell,\alpha}(U^*a)^2.$$  \hfill (5.6)

$$\leq (\gamma + 1)^4 m+1 4^{2\gamma} \sum_{\alpha=0}^\gamma c_m^{m+1,\alpha} \|U\|_\infty^{2m+2\gamma} h_{m+1,2\ell,\alpha}(a)^2.$$  \hfill (5.7)

where we use $\|U\|_\infty \geq 1$ for $U \in SU(1,n)$ and $c_m > 1$ in the last step. Defining $c_{m+1}$ appropriately and taking now the $2^{m+1}$-th root gives the claim. The case $2\ell + 1$ is analogous. Since there are only finitely many $\ell$, we can take $c_{m+1}$ independently of $\ell$. 

Since the action of SU(1, n) preserves the filtration by $\gamma$ and since each $A_h^\gamma(C^1_{n+1})$ is finite-dimensional, it is rather obvious that the action is via continuous maps with respect to the Cartesian product topology, i.e. the one determined by the seminorms $\|\cdot\|_{m,\ell,\gamma}$. From this point of view, Lemma 5.3 is not surprising. However, the precise estimate (5.4) becomes crucial for the following result:
Lemma 5.4 For every $U \in SU(1, n)$ the automorphism $U^* : A_h(C^+_{n+1}) \rightarrow A_h(C^+_{n+1})$ is continuous. More precisely, for all $R > 0$ we have

$$\|U^* a\|_{m, \ell, R} \leq \|a\|_{m, \ell, R'}$$

\( (5.5) \)

with $R' = c_m^m \|U\|_{\infty}^{2m+1} R$.

Proof. Since the seminorms $\| \cdot \|_{m, \ell, R}$ specify the topology of $A_h(C^+_{n+1})$ by Lemma 4.13 we only have to check the estimate. We have

$$\sum_{\gamma=0}^{\infty} R^\gamma h_{m, \ell, \gamma}(U^* a) \leq \sum_{\gamma=0}^{\infty} \frac{R c_m^{2m} \|U\|_{\infty}^{2m+1}}{\gamma!} h_{m, \ell, \gamma}(a).$$

Then taking the $2m$-th root gives \( (5.5) \).

Thus the action extends uniquely to an action by automorphisms on the completion $\hat{A}_h(C^+_{n+1})$. It turns out to be a continuous action:

Proposition 5.5 The action of $SU(1, n)$ on $\hat{A}_h(C^+_{n+1})$ yields a continuous map

$$SU(1, n) \times \hat{A}_h(C^+_{n+1}) \rightarrow \hat{A}_h(C^+_{n+1}).$$

\( (5.6) \)

Proof. First we note that a sequence $U_j \in SU(1, n)$ converges to $U \in SU(1, n)$ iff $\|U_j - U\|_{\infty} \rightarrow 0$, even though $U_j - U$ is of course not in $SU(1, n)$ in general but just in $M_{n+1}(\mathbb{C})$. Moreover, since $\hat{A}_h(C^+_{n+1})$ is a Fréchet space it suffices to consider sequences instead of nets. Thus let $U_j \rightarrow U$ and $a_j \rightarrow a$ be convergent sequences in $SU(1, n)$ and $\hat{A}_h(C^+_{n+1})$, respectively. Since $\| \cdot \|_{\infty}$ is continuous we have a constant $c > 0$ with $\|U_j\|_{\infty}, \|U\|_{\infty} \leq c$ for all $j$. Now let $m, \ell$, and $R$ be given and define $R' = c_m^m c^{2m+1} R$ such that we can apply Lemma 5.4 for all $U_j$ and $U$ simultaneously. Then we first note the estimate

$$\|U_j^* a_j - U^* a\|_{m, \ell, R} \leq \|U_j^* a_j - U_j^* a\|_{m, \ell, R} + \|U_j^* a - U^* a\|_{m, \ell, R} \leq \|a_j - a\|_{m, \ell, R} + \|U_j^* a - U^* a\|_{m, \ell, R}. $$

Now for $j \rightarrow \infty$, the first contribution converges to 0. Hence we only have to take care of the second. First we note that from $(\alpha + \beta)^k \leq 2^{k-1}(\alpha^k + \beta^k)$ for $\alpha, \beta \geq 0$ and all positive $k$ we get

$$h_{m, \ell, \gamma}(U_j^* a - U^* a) = \frac{2m}{m, \ell, \gamma} \leq 2^{m-1} \left( \|U_j^* a\|_{m, \ell, \gamma}^{2m} + \|U^* a\|_{m, \ell, \gamma}^{2m} \right) \leq \bar{c} \|a\|_{m, \ell, \gamma}^{2m},$$

with $\bar{c} > 0$ build out of the constants in \( (5.4) \) and the above $c$ estimating the sup-norms of $U_j$ and $U$. Thus the series $\sum_{\gamma=0}^{\infty} \frac{R^\gamma}{\gamma!} h_{m, \ell, \gamma}(U_j^* a - U^* a)$ can be dominated by the series $\sum_{\gamma=0}^{\infty} \frac{R^\gamma}{\gamma!} h_{m, \ell, \gamma}(a)$. Since clearly the group action is strongly continuous with respect to the single (semi-) norm $\| \cdot \|_{m, \ell, \gamma}$, we have $h_{m, \ell, \gamma}(U_j^* a - U^* a) \rightarrow 0$ for $j \rightarrow \infty$. Both arguments together allow us to exchange the limit $j \rightarrow \infty$ with the summation over $\gamma$ and we get $\|U_j^* a - U^* a\|_{m, \ell, \gamma} \rightarrow 0$ as wanted. This establishes the sequential continuity of \( (5.6) \) at the point $(U, a)$ which is all we have to prove.

Note that though the seminorms $\| \cdot \|_{m, \ell, \gamma}$ just specify the Cartesian product topology for which the group action is continuous for rather obvious reasons, the continuity with respect to the topology of $\hat{A}_h(C^+_{n+1})$ uses the specific form of the $h_{m, \ell, \gamma}$ and is therefore nontrivial. Having the continuity it is now fairly easy to see that the action is even smooth: we can rely on some general arguments for this. The following lemma should be well-known.
Lemma 5.6 Let $G$ be a connected Lie group acting on a sequentially complete Hausdorff locally convex space $V$ by continuous endomorphism such that the action map $G \times V \to V$ is continuous. Suppose that for every $\xi \in \mathfrak{g}$ we are given a continuous operator $L_\xi : V \to V$ such that $\xi \mapsto L_\xi$ is linear. Finally, suppose $V_0 \subseteq V$ is a dense subspace invariant under $G$ and under all the operators $L_\xi$ such that for $v \in V_0$ one has

$$L_\xi v = \lim_{t \to 0} \frac{\exp(t\xi)v - v}{t}$$

in the topology of $V$. Then the action is smooth and the smooth topology of $V$ coincides with the original one.

Proof. The crucial point is that we assume that $L_\xi$ is continuous. Let $p$ be a continuous seminorm on $V$ and let $\xi \in \mathfrak{g}$. Since $L_\xi$ is continuous we get a continuous seminorm $q$ such that $p(L_\xi v) \leq q(v)$ for all $v \in V$. Moreover, we have a continuous function $t \mapsto \exp(t\xi)v$. For $v \in V_0$ we get for $t \neq 0$

$$p\left(\frac{\exp(t\xi)v - v}{t} - L_\xi v\right) = p\left(\int_0^1 \frac{d}{ds}\exp(ts\xi)v \, ds - L_\xi v\right)$$

$$= p\left(\int_0^1 \frac{d}{d\epsilon}_{\epsilon=0} \exp(t\epsilon \xi) \exp(ts\xi)v \, ds - L_\xi v\right)$$

$$= p\left(\int_0^1 L_\xi \exp(t\epsilon \xi) \exp(ts\xi)v \, ds - L_\xi v\right)$$

$$= p\left(L_\xi \int_0^1 (\exp(ts\xi)v - v) \, ds\right)$$

$$\leq \int_0^1 q(\exp(ts\xi)v - v) \, ds,$$

where we have used the invariance of $V_0$ and the existence of the derivative on $V_0$. Note that a naive Riemann integral will do the job for the above estimates, hence sequential completeness is all we have to require here. Since now $L_\xi$ is continuous and hence both sides are a continuous expression in $v$, the estimate is still true for all vectors $v \in V$ by $V_0$ being dense, i.e. we get

$$p\left(\frac{\exp(t\xi)v - v}{t} - L_\xi v\right) \leq \int_0^1 q(\exp(ts\xi)v - v) \, ds.$$

The continuity of the integrand allows to take the limit $t \to 0$ yielding 0 on the right hand side. Thus (5.7) holds for all $v \in V$. Using the action property we see that $\frac{d}{dt} \exp(t\xi)v = L_\xi \exp(t\xi)v = \exp(t\xi)L_\xi v$ holds for all $t$. Hence all first directional derivatives exist and are continuous in a neighborhood of $e \in G$ which implies that the action is $C^1$ everywhere. Since the operators $L_\xi$ are defined on all of $V$ we can iterate this argument now to conclude that the action is $C^\infty$. Finally, the additional seminorms of the $C^\infty$-topology on $V$ are given by $v \mapsto p(L_{\xi_1} \cdots L_{\xi_n} v)$ where $n \in \mathbb{N}_0$ and $\xi_1, \ldots, \xi_n \in \mathfrak{g}$. Since all the operators $L_\xi$ are continuous, the resulting system of seminorms consists of continuous seminorms with respect to the original topology. Hence they coincide. \hfill \Box

The following lemma shows that for the $SU(1,n)$-action on $\hat{A}_h(C^+_{n+1})$ we are indeed in this situation:

Lemma 5.7 The dense subspace $A_h(C^+_{n+1})$ of $\hat{A}_h(C^+_{n+1})$ satisfies the conditions of Lemma 5.6 where $L_\xi = \frac{i}{\hbar}[J_\xi, \cdot]_{\text{Wick}}$ for $\xi \in \mathfrak{su}(1,n)$.  

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We have already seen that the structure constants with respect to the basis ℏ \( a \in \hat{A}_h(C^+_{n+1}) \) find a γ with \( a \in \hat{A}_h(C^+_{n+1}) \). But this is a finite-dimensional subspace and hence there is only one Hausdorff locally convex topology. For this topology it is clear that the action is differentiable with derivatives given by (5.7) and \( L_ξ = \frac{1}{\hbar}[J_ξ, \cdot]_{\text{Wick}} \). Indeed, this is just a trivial computation. The claim follows since the product \( \tilde{\gamma} \) is continuous and hence also the commutators with \( J_ξ \) are continuous.

Proof. First we note that \( \hat{A}_h(C^+_{n+1}) \) is invariant under the action of SU(1, n) as well as under taking commutators with \( J_ξ \) as we have seen in (5.1). For \( a \in \hat{A}_h(C^+_{n+1}) \) we find a γ with \( a \in \hat{A}_h(C^+_{n+1}) \). We leave it as an open problem whether one can actually exponentiate \( \tilde{\gamma} \) would indeed yield an inner action and hence the continuity and smoothness statements would follow trivially: the action would be even analytic with an entire extension. However, we expect that the algebra is not locally multiplicatively convex and hence the existence of an exponential is far from being obvious.

Remark 5.9 It is tempting to try to use the inner Lie algebra action via \( \frac{1}{\hbar}[J_ξ, \cdot]_{\text{Wick}} \) to obtain also an inner action of SU(1, n) by exponentiating \( J_ξ \). This \( \tilde{\gamma} \)-exponential would indeed yield an inner action and hence the continuity and smoothness statements would follow trivially: the action would be even analytic with an entire extension. However, we expect that the algebra is not locally multiplicatively convex and hence the existence of an exponential is far from being obvious. We leave it as an open problem whether one can actually exponentiate \( J_ξ \).

5.2 The dependence on \( h \)

We have already seen that the structure constants with respect to the basis \( f_{P,Q,α} \) are independent of \( h \). However, since we have the concrete interpretation of \( a \in \hat{A}_h(C^+_{n+1}) \) as functions on \( C^+_{n+1} \), the coefficients \( a_{P,Q,α} \) depend on the choice of \( h \) since the functions \( f_{P,Q,α} \) do so, see (4.21). Hence the algebra \( \hat{A}_h(C^+_{n+1}) \) including its topology depend on \( h \), a priori. To emphasize the dependence on \( h \) we denote the coefficients of \( a \in \hat{A}_h(C^+_{n+1}) \) by \( a_{P,Q,α}(h) \) and write \( f_{P,Q,α}(h) \) in this subsection. We also decorate the seminorms \( \| \cdot \|_{m,ℓ,(I,J,γ),h} \) with an additional \( h \).

In order to understand this dependence we introduce the following auxiliary algebra \( \mathfrak{A}(C^+_{n+1}) \) being the span of basis vectors \( F_{P,Q,α} \) for all index triples \((P, Q, α)\) endowed with the product \( \star \) specified by the structure constants \( C_{(P,Q,α),(R,S,β)} \) as in (4.21). Thanks to the precise form of the structure constants we have no dependence on \( h \) in the definition of \( \mathfrak{A}(C^+_{n+1}) \). Repeating our general construction we get a definition of a locally convex topology on \( \mathfrak{A}(C^+_{n+1}) \) first based on the \( h_{m,ℓ,(I,J,γ)} \), yielding the Cartesian product topology. In a second step we use the definition of the
semimnorms $\| \cdot \|_{m,\ell,R}$ as in (4.42) also for $\mathfrak{A}(C_{n+1}^+)$ for which the completion $\hat{\mathfrak{A}}(C_{n+1}^+)$ becomes a Fréchet algebra, still independent on any choice of $h$. Equivalently, we can use the seminorms $\| \cdot \|_\epsilon$ build as in (4.41), making $\hat{\mathfrak{A}}(C_{n+1}^+)$ isomorphic to a Köthe space of sub-factorial growth.

Lemma 5.10 Let $h, h' \neq 0$.

i.) The map

$$\Phi_h: \hat{A}_h(C_{n+1}^+) \ni \sum_{P,Q,\alpha} a_{P,Q,\alpha}(h) f_{P,Q,\alpha}(h) \mapsto \sum_{(P,Q,\alpha)} a_{P,Q,\alpha}(h) F_{P,Q,\alpha} \in \hat{\mathfrak{A}}(C_{n+1}^+)$$

(5.8)

is a well-defined isomorphism of Fréchet algebras mapping $A_h(C_{n+1}^+)$ to $\mathfrak{A}(C_{n+1}^+)$. 

ii.) Suppose $t = \frac{h}{h'} > 0$. Then

$$\Phi^{-1}_h \circ \Phi_h = \sqrt{t} R$$

(5.9)

where $R_t: C_{n+1}^+ \ni z \mapsto tz \in C_{n+1}^+$ is the rescaling by $t$.

Proof. First we consider $a \in A_h(C_{n+1}^+)$ for which the above map is clearly a well-defined linear map as all involved summations are finite. It clearly gives a bijection onto $\mathfrak{A}(C_{n+1}^+)$. Since the structure constants of the $f_{P,Q,\alpha}(h)$ do not depend on $h$, $\Phi_h$ is an algebra homomorphism. Finally, by the very construction of the seminorms we have $\|a\|_{m,\ell,R,h} = \|\Phi_h(a)\|_{m,\ell,R,h}$. From this it is clear that also the seminorms $\| \cdot \|_{m,\ell,R,h}$ of $A_h(C_{n+1}^+)$ correspond to the seminorms $\| \cdot \|_{m,\ell,R}$ of $\mathfrak{A}(C_{n+1}^+)$ under $\Phi_h$, showing that $\Phi_h$ as well as its inverse are continuous. The first part follows. For the second part we note that $(\Phi^{-1}_h \circ \Phi_h)(f_{P,Q,\alpha}(h)) = f_{P,Q,\alpha}(h')$ by the very definition. For positive $t = \frac{h}{h'}$ we see immediately that $R^{*}\sqrt{t} = ty$. Since the functions $e_{P,Q,\alpha}^{t}$ are constant along the complex rays in $C_{n+1}^+$ this gives

$$R^{*}\sqrt{t} f_{P,Q,\alpha}(h) = f_{P,Q,\alpha}(h').$$

Since $\Phi^{-1}_h \circ \Phi_h: \hat{A}_h(C_{n+1}^+) \longrightarrow \hat{A}_{h'}(C_{n+1}^+)$ is an isomorphism of Fréchet algebras by the first part, we get

$$\left(\Phi^{-1}_h \circ \Phi_h\right)(a) = \left(\Phi^{-1}_h \circ \Phi_h\right) \left(\sum_{(P,Q,\alpha)} a_{P,Q,\alpha}(h) f_{P,Q,\alpha}(h)\right)$$

$$= \sum_{(P,Q,\alpha)} a_{P,Q,\alpha}(h) \left(\Phi^{-1}_h \circ \Phi_h\right)\left(f_{P,Q,\alpha}(h)\right)$$

$$= \sum_{(P,Q,\alpha)} a_{P,Q,\alpha}(h) R^{*}\sqrt{t} f_{P,Q,\alpha}(h)$$

for $a \in \hat{A}_h(C_{n+1}^+)$. Now for finite sums, i.e. $a \in A_h(C_{n+1}^+)$ this clearly coincides with $R^{*}\sqrt{a}$. Since in general the series converges in the topology of $\hat{A}_h(C_{n+1}^+)$ and since all evaluation functionals at points in $C_{n+1}^+$ are continuous, the last series also coincides with $R^{*}\sqrt{a}$ defined pointwise. Hence we have shown two things: first $R^{*}\sqrt{t}$ maps $\hat{A}_h(C_{n+1}^+)$ continuously into $\hat{A}_{h'}(C_{n+1}^+)$, and, second, (5.9) holds.

Corollary 5.11 For $h \neq 0$ all the Fréchet algebras $\hat{A}_h(C_{n+1}^+)$ are isomorphic.
However, the isomorphism is nontrivial in the following sense: since we know that \( \hat{A}_h(C^+_{n+1}) \subseteq C^\omega(C^+_{n+1}) \) is also a subspace of the real-analytic functions on \( C^+_{n+1} \), we can directly compare the elements as functions. Here we have the following result:

**Lemma 5.12** For \( h \neq h' \) the subspaces \( \hat{A}_h(C^+_{n+1}) \) and \( \hat{A}_{h'}(C^+_{n+1}) \) of \( C^\omega(C^+_{n+1}) \) are different.

Proof. Here the argument is similar as for the proof of Lemma 4.18. First we note that we can evaluate \( a \in \hat{A}_h(C^+_{n+1}) \) at every point \( z \in C^{n+1} \) in a continuous way: indeed, the sub-factorial growth of the coefficients \( a_{I,J,\gamma} \) with respect to the Schauder basis \( f_{I,J,\gamma} \) together with the factorial decrease of \( f_{I,J,\gamma}(z) \) according to (4.40) shows the continuity of all evaluation functionals. Thus the vanishing ideal of the hypersurface \( y = -2h \) is a closed subspace in \( \hat{A}_{h'}(C^+_{n+1}) \) for all nonzero \( h, h' \). Now it is easy to see that for \( h = h' \) this subspace has finite co-dimension. In fact, the quotient is spanned by the classes of \( f_{0,0,0} = 1 \) and the \( f_{i,j,1} \) with \( i, j = 1, \ldots, n \). For all other \( h' \) the co-dimension is strictly bigger: either infinite in the generic case, or, for particular values of \( h' \), finite but larger. Hence we conclude that the two subspaces \( \hat{A}_h(C^+_{n+1}) \) and \( \hat{A}_{h'}(C^+_{n+1}) \) are different for \( h \neq h' \).

\( \square \)

Things change when we pass to the disk: here the Pochhammer symbol becomes just a numerical constant (depending on \( h \)) but the span of the basis vectors \( f_{P,Q} \) is independent of \( h \):}

**Lemma 5.13** Let \( h, h' \) be allowed values. Then the subspaces \( \hat{A}_h(D_n) \) and \( \hat{A}_{h'}(D_n) \) of \( C^\omega(D_n) \) coincide. Moreover, the Fréchet topology of \( \hat{A}_h(D_n) \) is independent of \( h \).

Proof. First we note that the only difference between \( f_{P,Q}(h) \) and \( f_{P,Q}(h') \) are the prefactors given by the Pochhammer symbols \( \left( \frac{1}{2h} \right)^{\max(|P|,|Q|)} \) and \( \left( \frac{1}{2h'} \right)^{\max(|P|,|Q|)} \). Since both are non-vanishing their quotient can be estimated with exponential bounds. But then it is clear that for \( a = \sum_{P,Q} a_{P,Q}(h)f_{P,Q}(h) \) with coefficients \( a_{P,Q}(h) \) having sub-factorial growth, the coefficients

\[ a_{P,Q}(h') = \left( \frac{1}{2h'} \right)^{\max(|P|,|Q|)} a_{P,Q}(h) \] (\(*\))

have still sub-factorial growth. More precisely, for \( \epsilon' > 0 \) and every \( \epsilon' > \epsilon > 0 \) we have

\[
\|a\|_{C^\omega(D_n)}^h = \sup_{P,Q} \frac{|a_{P,Q}(h')|}{\max(|P|,|Q|)!^{\epsilon'}} \leq \sup_{P,Q} \left( \frac{1}{h} \right)^{\max(|P|,|Q|)} \left( \frac{c(h)}{a(h')b(h')^{\max(|P|,|Q|)}} \sup_{P,Q} \frac{|a_{P,Q}(h)|}{\max(|P|,|Q|)!^{\epsilon'}} \right) \leq \sup_{P,Q} \frac{1}{\max(|P|,|Q|)!^{\epsilon' - \epsilon}} \left( \frac{c(h)^{\max(|P|,|Q|)}}{a(h')b(h')^{\max(|P|,|Q|)}} \sup_{P,Q} \frac{|a_{P,Q}(h)|}{\max(|P|,|Q|)!^{\epsilon'}} \right) \leq c(h, h') \|a\|_\epsilon^h,
\] (\(**\))

where \( a(h'), b(h') \), and \( c(h) \) are the constants from the standard estimate of the Pochhammer symbols (4.41). Since in (4.41) we can chose the parameters to be locally uniform, we can arrange
defines a holomorphic map $A$ locally uniform in $c$ and a constant $\epsilon > 0$ all R, S for all $\epsilon > 0$ a constant, the inverse of the Pochhammer symbol $(\frac{1}{2n})_{\max(|P|,|Q|)}$.

Moreover, (** in the proof of Lemma 5.13 shows that the sub-factorial growth of $a_{P,Q}(h)$ is locally uniform in $h$, i.e. for all $h$ in some compact subset $K$ within the allowed values we get for all $\epsilon > 0$ a constant $c_K$ with

$$|a_{P,Q}(h)| \leq c_K(\max(|P|,|Q|))!^\epsilon. \tag{5.10}$$

For the next lemma it is again crucial that the topology of $\tilde{A}_h(D_n)$ is independent of $h$. An analogous statement for the Schauder basis upstairs does not even make sense according to Lemma 5.12.

Lemma 5.15 For all $P, Q$ the map $C \ni h \mapsto f_{P,Q}(h) \in \tilde{A}_h(D_n)$ is holomorphic on $C \setminus \{0\}$.

Proof. This is trivial as $f_{P,Q}(h)$ is the holomorphic Pochhammer symbol $(\frac{1}{2n})_{\max(|P|,|Q|)}$ times a fixed vector in $\tilde{A}_h(D_n)$.

Lemma 5.16 Let $a \in \tilde{A}_h(D_n)$. Then

$$A(h) = \sum_{P,Q} a_{P,Q}(h)F_{P,Q,\max(|P|,|Q|)} \tag{5.11}$$

defines a holomorphic map $A$ from the allowed values of $h$ to $\mathcal{A}(\mathbb{C}^{n+1})$.

Proof. Indeed, each $h \mapsto a_{P,Q}(h)F_{P,Q,\max(|P|,|Q|)}$ is holomorphic by Lemma 5.14. Moreover, $\|F_{P,Q,\max(|P|,|Q|)}\|_\epsilon = \frac{1}{\max(|P|,|Q|)}!$. But then the locally uniform estimate (5.10) shows that the series (5.11) converges absolutely and locally uniformly with respect to each seminorm $\|\cdot\|_\epsilon$. Thus the resulting map is again holomorphic.

We will need a refinement of our consideration in Theorem 4.21. Let $A = \sum_{(P,Q,\alpha)} A_{P,Q,\alpha} F_{P,Q,\alpha} \in \mathcal{A}(\mathbb{C}^{n+1})$ be given and consider $a(h) = [\Phi_h^{-1}(A)] \in \tilde{A}_h(D_n)$. The $h$-dependence of $a$ comes from the map $\Phi_h$ and from the quotient procedure. We are then interested in the growth properties of the coefficients $a_{R,S}$ of $a$.

Lemma 5.17 For all $\epsilon > 0$ and all compact subsets $K$ of allowed values of $h$ there exists an $\epsilon > 0$ and a constant $c_K$ such that

$$|a_{R,S}(h)| \leq c_K \|A\|_\epsilon (\max(|R|,|S|)!)^\epsilon \tag{5.12}$$

for all $R, S \in \mathbb{N}_0^n$ and all $h \in K$.
Proof. The main point is that $c_K$ can be chosen locally uniform in $h$. In the proof of Theorem 4.21 in (⋆) we had an estimate for $a_{R,S}$ involving several constants: fixing $\epsilon' > 0$, we chose a $0 < \epsilon < \epsilon'$ and set $c_1 = \|A\|_\epsilon$ yielding the estimate $|A_{P,Q,\alpha}| \leq c_1(\alpha!)^\epsilon$ needed. Moreover, we note that in (⋆) we can choose $c_2$ and $c_3$ locally uniformly in $h$ thanks to (4.11). Then (⋆) will yield an estimate $|a_{R,S}(h)| \leq c_7 c_8^\epsilon (M!)^\epsilon$ with $c_7 = \|A\|_\epsilon$ times a numerical constant being locally uniform in $h$ and $c_8$ being locally uniform in $h$ as well. From this we deduce the claim. \hfill \Box

With other words, the continuity of the map $A \mapsto a(h)$, which was clear before, can be sharpened to the estimate

$$\|a(h)\|_{\epsilon'} \leq c \|A\|_{\epsilon} \tag{5.13}$$

with a locally uniform constant $c$ concerning the dependence on $h$.

Finally, assume that $h \mapsto A(h)$ is itself a holomorphic function with values in $A(C_{n+1}^+)$. Then clearly all the coefficients $A_{P,Q,\alpha}(h)$ are scalar holomorphic functions since we have a Schauder basis. Moreover, $h \mapsto \|A(h)\|_{\epsilon}$ is a continuous function for all $\epsilon$ and hence locally bounded. From the construction of the $a_{R,S}$ we see that we have absolute and locally uniform convergence of holomorphic functions, implying that also the scalar function $h \mapsto a_{R,S}(h)$ is holomorphic. This will eventually lead to the following result:

**Theorem 5.18 (Holomorphic deformation)** Let $a,b \in \hat{A}_h(D_n)$ be given. Then the product $a \star_{D_n} b$ depends holomorphically on $h$ for all allowed values of $h$.

Proof. First we note that by Lemma 5.16 the maps $h \mapsto A(h), B(h) \in A(C_{n+1}^+)$ are holomorphic where $A(h) = \sum_{P,Q} a_{P,Q}(h)F_{P,Q,\max(|P|,|Q|)}$ and analogously for $B(h)$. Since the product $\star$ of $A(C_{n+1}^+)$ is continuous (and independent of $h$), also $A(h) \star B(h)$ is holomorphic and we have $a \star_{D_n} b = [\Phi_h^{-1}(A(h) \star B(h))]$. Thus the coefficients $(a \star_{D_n} b)_{R,S}(h)$ of

$$a \star_{D_n} b = \sum_{R,S} (a \star_{D_n} b)_{R,S}(h) f_{R,S}(h) \tag{⋆}$$

are scalar holomorphic functions as we just argued. By Lemma 5.15 also $f_{R,S}(h)$ depends holomorphically on $h$. Finally, Lemma 5.17 together with $\|f_{R,S}\|_{\epsilon} = \frac{1}{(\max(|R|,|S|))}$ shows that the convergence of the series (⋆) is not only absolute with respect to the seminorms $\|\cdot\|_{\epsilon}$ but even locally uniform in $h$. Hence the result is again holomorphic, proving the claim. \hfill \Box

**Remark 5.19** We are thus in the situation of a holomorphic deformation in the sense of [23] with the crucial difference that $h = 0$ is not in the domain of definition. In fact, the singularities at the non-allowed values on the negative axis, i.e. the zeros of the Pochhammer symbols, accumulate at 0. Hence there is no chance to extend the domain such that $h = 0$ is included. As an alternative to the above argument, a more direct estimate of $a \star_{D_n} b$ can be used to show the locally uniform convergence in $h$ by expanding everything in terms of the coefficients with respect to the Schauder basis.

5.3 The $\ast$-involution for real $h$

We consider now the particular case of real $h$. From Proposition 4.6 [23]), we know that $A_h(C_{n+1}^+)$ is a $\ast$-algebra with respect to the pointwise complex conjugation as $\ast$-involution. In fact, the complex conjugation is continuous and extends therefore to a $\ast$-involution of $A_h(C_{n+1}^+)$. We arrive at the following statement:
Proposition 5.20 Let \( h \) be real and an allowed value.

i.) The complex conjugation extends to a continuous \(*\)-involution of \( \hat{A}_h(C_{n+1}^+) \) and for all \( 0 < \epsilon < 1 \) and \( a \in \hat{A}_h(C_{n+1}^+) \) we have

\[
\|\overline{a}\|_{\epsilon} = \|a\|_{\epsilon}. \tag{5.14}
\]

ii.) The ideal \( \mathcal{J}_{y=1} \subseteq \hat{A}_h(C_{n+1}^+) \) is a \(*\)-ideal.

iii.) The induced continuous \(*\)-involution on \( \hat{A}_h(D_n) = \hat{A}_h(C_{n+1}^+)/\mathcal{J}_{y=1} \) is the pointwise complex conjugation once we identify elements of \( \hat{A}_h(D_n) \) with functions on \( D_n \). For all \( 0 < \epsilon < 1 \) and \( [a] \in \hat{A}_h(D_n) \) we have

\[
\|\hat{a}\|_{\epsilon} = \|[a]\|_{\epsilon}. \tag{5.15}
\]

Proof. For the first part it suffices to show (5.14). From (1.23) we conclude \((\hat{a})_{P,Q,\alpha} = \hat{a}_{Q,P,\alpha}\) and hence (5.14) is obvious. For any \( w \in C_{n+1}^+ \) the evaluation functional \( \delta_w \) is real in the sense that \( \overline{\delta_w(a)} = \delta_w(\overline{a}) \). Hence \( \ker \delta_w \) is stable under complex conjugation and thus also \( \mathcal{J}_{y=1} \), proving the second part. The third part is then clear by construction. \( \square \)

5.4 Positivity of \( \delta \)-functionals and their GNS construction

Let \( h > 0 \) be positive. Then we consider the evaluation functionals on the disk for the algebra \( \hat{A}_h(D_n) \). Since the group SU\((1,n)\) acts transitively on \( D_n \), it will be sufficient to consider one point only, all others will be obtained by pulling back the evaluation functional via the \(*\)-automorphisms \( U^* \) of \( \hat{A}_h(D_n) \). Hence we can concentrate on the point \( v = 0 \in D_n \).

Lemma 5.21 Let \( h > 0 \).

i.) The evaluation functional \( \delta_0 : \hat{A}_h(D_n) \rightarrow \mathbb{C} \) is positive.

ii.) The Gel’fand ideal of \( \delta_0 \) is given by

\[
\mathcal{J}_0 = \left\{ a = \sum_{P,Q} a_{P,Q} f_{P,Q} \in \hat{A}_h(D_n) \mid a_{0,Q} = 0 \text{ for all } Q \in \mathbb{N}_0^n \right\}. \tag{5.16}
\]

Proof. First we note that \( \delta_0(a) \) can be obtained by evaluating a representative of \( a \) in \( \hat{A}_h(C_{n+1}^+) \) at the point \( w = (1,0,\ldots,0) = \pi^{-1}(0) \in C_{n+1}^+ \). For a general index triple \( (I,J,\gamma) \) we have

\[
\delta_w(f_{I,J,\gamma}) = \frac{1}{I!J!(\gamma - |I|!)(\gamma - |J|)!} \left( \frac{1}{2h} \right)_{\gamma} \delta_{I,0}\delta_{J,0} = \frac{1}{\gamma!}\left( \frac{1}{2h} \right)_{\gamma} \delta_{I,0}\delta_{J,0}. \tag{*}
\]

Now let \( P, Q, R, \) and \( S \) be given and set \( \alpha = \max(|Q|,|P|) \) and \( \beta = \max(|R|,|S|) \). Then we have

\[
\delta_0(f_{P,Q}*_{D_n} f_{R,S})
\]

\[
= \delta_0(f_{Q,P}*_{D_n} f_{R,S})
\]

\[
= \delta_w(f_{Q,P,\alpha}^*_{Wick} f_{R,S,\beta})
\]

\[
\min(\alpha - |Q|, \beta - |S|) \min(Q,S)
\]

\[
= \sum_{k=0}^{\min(\alpha - |Q|, \beta - |S|)} \sum_{K=0}^{\min(Q,S)} \frac{(-1)^k}{k!K!} \delta_w(f_{Q+R-K,P+S-K,\alpha+\beta-k-|K|})
\]

\[
\left( Q + R - K \right) \left( P + S - K \right) \left( \alpha + \beta - k - |Q| - |R| \right) \left( \alpha + \beta - k - |P| - |S| \right).
\]

\( \tag{**} \)
In order to get a nonzero contribution in (**) we need $Q + R - K = 0 = P + S - K$ according to (*). Since $K$ has range $0, \ldots, \min(Q, S)$ this implies $K = Q = S$ and $P = R = 0$. Hence $\alpha = |Q|$ and $\beta = |S|$ which implies that in (**) only $k = 0$ occurs. For these values, all the binomial coefficients in (**) are 1. Then remaining evaluation is $\delta_w(f_{0,0,|Q|}) = \frac{1}{|Q||Q|!} \left( \frac{1}{2h} \right)^{|Q|}$. This allows to compute

$$\delta_0(\pi \star_{\mathbb{D}_n} b) = \sum_{P,Q,R,S} \frac{a_{P,Q} b_{R,S}}{|Q||Q|!} \delta_0(\pi_{P,Q} \star_{\mathbb{D}_n} f_{R,S}) = \sum_{Q} \frac{a_{0,Q} b_{0,Q}}{|Q||Q|!} \left( \frac{1}{2h} \right)^{|Q|}.$$ 

Since for $h > 0$ the Pochhammer symbol $\left( \frac{1}{2h} \right)^{|Q|}$ is also positive, we see that $\delta_0$ is a positive functional by setting $a = b$. Moreover, (5.16) is clear from the last formula. $\square$

**Corollary 5.22** Let $h > 0$. For all points $v \in \mathbb{D}_n$, the evaluation functional $\delta_v : \mathcal{A}_h(\mathbb{D}_n) \to \mathbb{C}$ is positive.

**Lemma 5.23** Let $h > 0$.

i.) The Gelfand ideal $\mathcal{J}_0$ of $\delta_0$ is a closed subspace.

ii.) The Gelfand ideal $\mathcal{J}_0$ has a closed complementary subspace $\mathcal{J}_0 \oplus \mathbb{D}_h = \mathcal{A}_h(\mathbb{D}_n)$ explicitly given by

$$\mathbb{D}_h = \left\{ \psi = \sum_Q \psi_Q f_{0,Q} \left| (\psi_Q)_{Q \in \mathbb{N}_0^n} \text{ has sub-factorial growth} \right\} \subseteq \mathcal{A}_h(\mathbb{D}_n). \right.$$ (5.17)

iii.) The GNS pre Hilbert space is canonically isomorphic to $\mathbb{D}_h$ with the inner product

$$\langle \psi, \phi \rangle_h = \sum_Q \overline{\psi_Q} \phi_Q \left( \frac{1}{2h} \right)^{|Q|}.$$ (5.18)

iv.) The nuclear Fréchet topology of $\mathbb{D}_h$ is finer than the pre Hilbert space topology induced by $\langle \cdot, \cdot \rangle_h$.

Proof. The first part is clear since $\star_{\mathbb{D}_n}$, the complex conjugation, and the evaluation functional $\delta_0$ are continuous. Having an absolute basis it is clear that we can split the total space into a direct sum of two closed subspaces by splitting the set of basis vectors and taking their closures of their spans, respectively. Thus the second statement follows from the characterization of the Gelfand ideal according to Lemma 5.21 (ii). The third part is clear from the computation in the proof of the previous lemma. For the last part, we first recall that $\mathbb{D}_h$ is necessarily nuclear, being a closed subspace of a nuclear space. In fact, it is again isomorphic to a Köthe space of sub-factorial growth. Since $\star_{\mathbb{D}_n}$, the complex conjugation, and the evaluation functional $\delta_0$ is continuous, also the inner product is continuous, proving that the pre Hilbert space topology is coarser. $\square$

Clearly, the Hilbert space completion $\mathcal{H}_h$ of $\mathbb{D}_h$ can be described as the $\ell^2$-space of sequences indexed by multiindices $Q \in \mathbb{N}_0^n$ with a weighted counting measure with weights given by $\frac{1}{|Q||Q|!} \left( \frac{1}{2h} \right)^{|Q|}$. This is clear from (5.18). More surprising is the fact that we can view elements in the Hilbert space $\mathcal{H}_h$ still as *functions* on $\mathbb{D}_n$. We collect this with some other first properties of $\mathcal{H}_h$ in the following lemma:

**Lemma 5.24** Let $h > 0$. 

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i.) The vectors \( f_0, Q \in \mathfrak{D}_h \) are pairwise orthogonal and yield a Hilbert basis

\[
E_Q = \frac{|Q|!}{\sqrt{\left(\frac{1}{2\pi}\right)|Q|}} f_{0,Q}
\]  

(5.19)

after normalization, where \( Q \in \mathbb{N}_0^n \).

ii.) The evaluation functional at \( w \in \mathfrak{D}_n \) extends to a continuous linear functional on \( \mathfrak{H}_h \). More precisely, for every compact \( K \subseteq \mathfrak{D}_n \) there is a constant \( c_K > 0 \) with

\[
|\psi(w)| \leq c_K \|\psi\|_{\mathfrak{H}_h}
\]  

(5.20)

for all \( \psi \in \mathfrak{H}_h \) and \( w \in K \).

iii.) For \( w \in \mathfrak{D}_n \) the evaluation functional \( \delta_w \) can be written as \( \psi(w) = \langle E_w, \psi \rangle_{\mathfrak{H}_h} \) with

\[
E_w = \sum_Q \frac{w^Q}{(1 - |w|^2)^{|Q|}} \frac{|Q|!}{Q!} f_{0,Q} \in \mathfrak{D}_h.
\]  

(5.21)

The series converges absolutely in the nuclear Fréchet topology of \( \mathfrak{D}_h \) and in the Hilbert space topology of \( \mathfrak{H}_h \).

Proof. The first statement is clear from the explicit form of the inner product as in (5.18). For the second part we use the Cauchy-Schwarz inequality to obtain

\[
|\psi(w)|^2 = \left| \sum_Q \psi_Q f_{0,Q}(w) \right|^2
\]

\[
= \left| \sum_Q \psi_Q \frac{1}{|Q|!Q!} \left( \frac{1}{2\pi} \right)^{|Q|} (1 - |w|^2)^{|Q|} \right|^2
\]

\[
\leq \left( \sum_Q \frac{|\psi_Q|^2}{|Q|!Q!} \left( \frac{1}{2\pi} \right)^{|Q|} (1 - |w|^2)^{|Q|} \right) \left( \sum_Q \frac{1}{Q!Q!} \left( \frac{1}{2\pi} \right)^{|Q|} (1 - |w|^2)^{|Q|} \right)
\]

\[
= \|\psi\|_{\mathfrak{H}_h}^2 f(w),
\]

with a function \( f \) determined by the second series. Now clearly the series \( f \) converges for all \( w \in \mathfrak{D}_n \) by the standard estimate for the Pochhammer symbols (4.41). It even converges locally uniformly and thus defines a continuous function \( f \). Taking \( c_K^2 = \max_{w \in K} f(w) \) proves the second part. From this continuity statement it is clear that there exists a uniquely determined vector \( E_w \in \mathfrak{H}_h \) with \( \psi(w) = \langle E_w, \psi \rangle_{\mathfrak{H}_h} \). Using the explicit formula (5.18) for the inner product as well as the explicit formula for \( \psi(w) \) in terms of the series expansion using the functions \( f_{0,Q} \) gives immediately (5.21).

Since the coefficients with respect to the \( f_{0,Q} \) have exponential and thus sub-factorial growth, we get \( E_w \in \mathfrak{D}_h \) as claimed.

\[\square\]

From the general GNS construction we know that \( \tilde{\mathcal{A}}_h(\mathfrak{D}_n) \) acts on the GNS pre Hilbert space \( \tilde{\mathcal{A}}_h(\mathfrak{D}_n)/\mathfrak{J}_0 \) in the canonical way. The projection onto \( \mathfrak{D}_h \) and the injection of \( \mathfrak{D}_h \) into \( \tilde{\mathcal{A}}_h(\mathfrak{D}_n) \) provide isomorphisms of the GNS pre Hilbert space with \( \mathfrak{D}_h \). Using those, we can compute the GNS representation explicitly:
Proposition 5.25 Let $h > 0$. Let $a = \sum_{P,Q} a_{P,Q} f_{P,Q} \in \hat{A}_h(\mathbb{D}_n)$ and $\psi = \sum_{Q} \psi_Q f_{0,Q} \in \mathcal{D}_h$ be given. Then the GNS representation is determined by
\begin{equation}
\pi_0(a)\psi = \text{pr}_{\mathcal{D}_h}(a \ast_{\mathcal{D}_n} \iota(\psi)) = \sum_{Q,S} \sum_{P \leq S} a_{P,Q} \psi_S \frac{1}{P!(\alpha - |Q|)!} \left(\frac{Q + S - P}{Q}\right) \left(\alpha + |S| - |P|\right) \left(\frac{1}{|\alpha + |S| - |P|!}\right) e^{\frac{i}{2h} |\alpha + |S| - |P|!}} \cdot \frac{1}{|Q + |S| - |P|!}\left(\frac{2\pi}{|Q + |S| - |P|!}\right) f_{0,Q+S-P},
\end{equation}

where $\alpha = \max(|P|, |Q|)$ as usual. The series converges in the Fréchet topology of $\mathcal{D}_h$.

Proof. First we note that the projection $\text{pr}_{\mathcal{D}_h} : \hat{A}_h(\mathbb{D}_n) \rightarrow \mathcal{D}_h$ is continuous, this is clear from the explicit description of the topology by means of the seminorms $\| \cdot \|_{\epsilon}$. Similarly, the injection $\iota$ is continuous. Together with the continuity of the product $\ast_{\mathcal{D}_n}$ and the (absolute) convergence of the series $a = \sum_{P,Q} a_{P,Q} f_{P,Q} \psi_S$ and $\psi = \sum_{Q} \psi_S f_{0,Q}$ it is clear that we can take all the summations in from of the algebraic manipulations and still have convergence in the Fréchet topology of $\mathcal{D}_h$. Thus the following computations are justified. We compute things upstairs on $C^+_{n+1}$ with $\alpha = \max(|P|, |Q|)$ and $\beta = \max(0, |S|) = |S|$ as usual and get from Proposition 4.6

\begin{equation}
a \ast_{\mathcal{D}_n} \iota(\psi) = \sum_{P,Q} \sum_{S} a_{P,Q} \psi_S \sum_{k=0}^{\min(\alpha - |P|, \beta - |S|)} \sum_{K=0}^{\min(P,S)} \frac{(-1)^k}{k! K!} \left[ f_{P+0-K,Q+S-K,\alpha+\beta-k-|K|} \right] \left(\frac{P + 0 - K}{0}\right) \left(\frac{Q + S - K}{Q}\right) \left(\alpha + \beta - k - |P| - 0\right) \left(\alpha + \beta - k - |S|\right) \left(\beta - 0\right) \left(\alpha - |Q|\right).
\end{equation}

Next we have to project back to $\mathbb{D}_h$. Hence we have to compute $\text{pr}_{\mathcal{D}_h} \left[ f_{P-K,Q+S-K,\alpha+|S|-|K|} \right]$. In general, $[f_{I,J,\gamma}]$ is a nontrivial linear combination of the basis vectors $f_{R,S}$ which we have computed in the proof of Theorem 4.21. However, since we are interested in the $\mathbb{D}_h$-component only, the computation simplifies drastically. In fact, from (*) in the proof of Lemma 4.20 we see that the only nontrivial contribution in $\text{pr}_{\mathcal{D}_h} \left[ f_{I,J,\gamma} \right]$ arises for $I = 0$ and is given by
\begin{equation}
\text{pr}_{\mathcal{D}_h} \left[ f_{I,J,\gamma} \right] = \frac{1}{\gamma! J! (\gamma - |J|)!} \left(\frac{1}{2\pi}\right) \left(\frac{\pi!}{(\frac{2\pi}{|J|})^{|J|}}\right) \delta_{I,0} = \frac{1}{\gamma! |J|} \left(\frac{1}{2\pi}\right)^{|J|} \left(\frac{2\pi}{|J|}\right)^{|J|} |J|! f_{0,J} \delta_{I,0}.
\end{equation}

Inserting this in the above computation gives us first the condition $P - K = 0$ and hence $P \leq S$. Then it results in (5.23).

Remark 5.26 The above characterization of the GNS representation is of course not yet very illuminating but shows that one can efficiently compute the representation. It will be a future project to relate it to the more familiar construction of a Berezin-Toeplitz like quantization as this has been used by various people. In particular, a direct comparison with the results on [13 Sect. 4] should be within reach. Moreover, the vector states $\mathcal{E}_n$ should directly correspond to the coherent states used in [13]. Finally, we believe that the above GNS representation will provide the bridge in order to compare the algebra $\hat{A}_h(\mathbb{D}_n)$ to other (deformation) quantizations of the Poincaré disk which are based on more operator-algebraic approaches. Here in particular the approaches of [4,5,8,10] should be mentioned. As this certainly will require some more effort we postpone a detailed study to some future projects.
A Köthe spaces of sub-factorial growth

In this appendix we collect some well-known and basic features of the Köthe space of those sequences which have sub-factorial growth. For details on Köthe spaces we refer e.g. to [19], in particular to the Sections 1.7.E, 3.6.D, and 21.6. First, let

$$\Lambda = \left\{ a = (a_n)_{n \in \mathbb{N}} \right\} \quad \text{with} \quad p_\epsilon(a) = \left\| \left( \frac{a_n}{(n!)^\epsilon} \right)_{n \in \mathbb{N}} \right\|_{\ell^1} < \infty \text{ for all } 0 < \epsilon < 1 \quad (A.1)$$

be the Köthe sequence space of sequences with sub-factorial growth, endowed with its topology induced by the seminorms $p_\epsilon$ as usual. Equivalently, we can use the system of seminorms

$$\|a\|_\epsilon = \sup_{n \in \mathbb{N}} \frac{|a_n|}{(n!)^\epsilon}, \quad (A.2)$$

where again $0 < \epsilon < 1$. Indeed, the estimate $\|a\|_\epsilon \leq p_\epsilon(a)$ is obvious. For the reverse we note that

$$p_\epsilon(a) = \sum_{n \in \mathbb{N}} \frac{|a_n|}{(n!)^\epsilon} \leq \sup_{n \in \mathbb{N}} \frac{|a_n|}{(n!)^{\epsilon/2}} \sum_{n \in \mathbb{N}} \frac{1}{(n!)^{\epsilon/2}}. \quad (A.3)$$

Since the last series converges, this gives the estimate $p_\epsilon(a) \leq c \|a\|_{\epsilon/2}$ with $c$ being the value of the above series. Since clearly a countable subset of the parameters $\epsilon$ will suffice to specify the topology, $\Lambda$ is a Fréchet space.

As for all Köthe spaces, the sequences $e_k = (\delta_{nk})_{n \in \mathbb{N}}$ constitute an absolute Schauder basis, see e.g. [19, Theorem 24.8.8].

Consider now $\epsilon > 0$ and the corresponding sequence $\left( \frac{1}{(n!)^\epsilon} \right)_{n \in \mathbb{N}}$. Then the sequence of quotients

$$\left( \frac{(n!)^{\epsilon/2}}{(n!)^\epsilon} \right)_{n \in \mathbb{N}} = \left( \frac{1}{(n!)^{\epsilon/2}} \right)_{n \in \mathbb{N}}$$

is $p$-summable for all $p > 0$. Thus, by the Grothendieck-Pietch criterion, this implies that $\Lambda$ is nuclear and, in fact, even strongly nuclear by [19, Prop. 21.8.2].

Sometimes we will meet “sequences” not indexed by $n \in \mathbb{N}$ but by multiindices. Thus consider

$$\Lambda_d = \left\{ a = (a_N)_{N \in \mathbb{N}^d} \right\} \quad \text{with} \quad p_\epsilon(a) = \left\| \left( \frac{a_N}{(|N|!)^\epsilon} \right)_{N \in \mathbb{N}^d} \right\|_{\ell^1} < \infty \text{ for all } 0 < \epsilon < 1 \quad (A.4)$$

Since the number of points $N$ with given $|N|$ grows polynomially we see that the same arguments as for $\Lambda = \Lambda_1$ apply. We have an equivalent system of seminorms

$$\|a\|_\epsilon = \sup_{N \in \mathbb{N}^d} \frac{|a_n|}{(|N|!)^\epsilon}. \quad (A.5)$$

By choosing an enumeration, $\Lambda_d$ can be viewed as a Köthe space as well. However, it will be easier to use multiindices in many situations. The vectors $e_K = (\delta_{KN})_{N \in \mathbb{N}^d}$ yield an absolute Schauder basis. Finally, the summability properties of the functions $\frac{1}{(|N|!)^\epsilon}$ over $\mathbb{N}^d$ are the same and thus $\Lambda_d$ is strongly nuclear, too.

There are several further generalizations. In particular, the index triples $(I,J,\gamma)$ can be used to construct a Köthe space by imposing sub-factorial growth with respect to $\gamma$. Since there are only finitely many $|I|, |J| \leq \gamma$ in an index triple and their number grows polynomially in $\gamma$, also the index triples give rise to a Köthe space as above.

We shall speak of Köthe spaces of sub-factorial growth whenever the reference to the index set and the relevant factorial is clear from the context.
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