Small Solutions of Quadratic Congruences, and Character Sums with Binary Quadratic Forms

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1 Introduction

Let $Q(x) = Q(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ be a quadratic form. This paper, which may be seen as a continuation of the author’s earlier work [10], [11] seeks to understand the smallest solution of the congruence $Q(x) \equiv 0 \pmod{q}$ in non-zero integers $x$. Thus we shall set

$$m(Q; q) := \min\{|x| : x \in \mathbb{Z}^n - \{0\}, Q(x) \equiv 0 \pmod{q}\}$$

where $|x|$ denotes the Euclidean norm, and ask (in the first instance) about

$$B_n(q) := \max_Q m(Q; q),$$

where the maximum is taken over all integral quadratic forms in $n$ variables. (This definition differs slightly from that used in [10] and [11].) The interested reader may refer to Baker [1, Chapter 9] for an account of this problem and its applications.

It is trivial that $B_n(q)$. When $q$ is square-free it is easy to see that $B_n(q) \geq q$ for $n = 1$ or $2$. Moreover the form $Q(x) = x_1^2 + \ldots + x_n^2$ has $m(Q; q) \geq q^{1/2}$ so that $B_n(q) \geq q^{1/2}$ for every $q$ and every $n$. When $n = 3$ and $q$ is square-free one has

$$B_3(q) \geq m(Q; q) \geq q^{2/3} + O(q^{1/3})$$

for a suitable singular form

$$Q(x_1, x_2, x_3) = (x_1 - bx_2)^2 - a(x_2 - bx_3)^2.$$  (1.1)

(Details for the case in which $q$ is prime are given in [10] Theorem 3] but the argument readily extends to any square-free $q$.) It is reasonable to conjecture that such lower bounds represent the true order of magnitude for $B_n(q)$ in general, so that one would have

$$B_n(q) \ll \epsilon \left\{\begin{array}{ll}
q^{2/3+\epsilon}, & n = 3 \\
q^{1/2+\epsilon}, & n \geq 4
\end{array}\right.$$  

for any fixed $\epsilon > 0$ (uniformly in $n$, by the non-increasing property).

A basic upper bound for $B_n(q)$ was provided by Schinzel, Schlickewei and Schmidt [16], who showed that

$$B_n(q) \ll \left\{\begin{array}{ll}
q^{1/2+1/(2n)}, & n \text{ odd,} \\
q^{1/2+1/(2n-2)}, & n \text{ even.}
\end{array}\right.$$  

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In particular one sees that $q^{2/3}$ is the true order of magnitude of $B_3(q)$, at least when $q$ is square-free.

For $n \geq 4$ and any $\varepsilon > 0$ one has

$$B_n(q) \ll \varepsilon q^{1/2+\varepsilon}$$

if $q$ has at most 2 prime factors, (see the author [11, Theorem 1]); that

$$B_4(q) \ll \varepsilon q^{5/8+\varepsilon}$$

(see [11, Theorem 2]); and that

$$B_n(q) \ll \varepsilon, n \, q^{1/2+3/n^2+\varepsilon}$$

for every even $n \geq 2$ (see [11, Theorem 3]). Indeed a number of other such bounds are possible.

It might appear from the above discussion that our question is completely resolved for $n = 3$, but wide open for $n \geq 4$. None the less, the main goal of this paper is a further exploration of the situation for $n = 3$ (!) It will be observed that the example (1.1) is a singular form. It turns out that one can do better if one restricts attention to ternary forms which are nonsingular modulo $q$. Before stating our result we should make two simple observations. Firstly, if $q = q_0^2 q_1$ and $q_1 | Q(x)$, then $q | Q(q_0 x)$. It follows that $m(Q; q) \leq q_0 m(Q; q_1)$. In particular, if we have proved that $B_n(q) \ll q^\theta$ for all square-free $q$, for some exponent $\theta \geq \frac{1}{2}$, then we may deduce that $B_n(q) \ll q^\theta$ for every $q$. Secondly, if $q = 2 q_1$ and $q_1 | Q(x)$, then $q | Q(2x)$. It follows in this case that $m(Q; q) \leq 2 m(Q; q_1)$. Once again, if we have proved that $B_n(q) \ll q^\theta$ for all odd square-free $q$, for some exponent $\theta$, then we may deduce that $B_n(q) \ll q^\theta$ for every square-free $q$. These observations allow us to focus on odd square-free $q$. Indeed we shall assume without further comment throughout the remainder of this paper that $q$ is odd and square-free. In this situation we can represent $Q(x)$ modulo $q$ via a symmetric integer matrix, which we also denote by $Q$, by abuse of notation.

We now define

$$B^*_3(q) := \max_Q m(Q; q),$$

where the maximum is taken over all integral ternary quadratic forms $Q$ with $(\det(Q), q) = 1$. This notation allows us to state our principal result.

**Theorem 1** Let $q \in \mathbb{N}$ be odd and square-free, and let $\varepsilon > 0$ be given. Then

$$B^*_3(q) \ll \varepsilon q^{5/8+\varepsilon}.$$

So we see that we can go below the exponent $2/3$ which is the limiting exponent for $B_3(q)$. We now have the same exponent $5/8$ for (non-singular) forms in 3 variables as we previously had for 4 variables. (However it is explained in [11] that one can reduce the exponent to $13/21$ with more work, in the 4 variable case.)

It now seems that one should conjecture a bound

$$B^*_3(q) \ll \varepsilon q^{1/2+\varepsilon}. \quad (1.2)$$
The proof of Theorem 1 proceeds by reducing the problem to a second question, which we now explain. If $Q$ is a quadratic form in $n \geq 2$ variables we write
\[
\hat{m}(Q; q) := \min \{ ||x|| : x \in \mathbb{Z}^n - \{0\}, \exists t \in \mathbb{Z}, Q(x) \equiv t^2 \pmod{q} \}
\]
and
\[
\hat{B}_n(q) := \max_Q \hat{m}(Q; q),
\]
where the maximum is taken over all integral quadratic forms in $n$ variables such that $(\det(Q), q) = 1$.

We then have the following result.

**Lemma 1** Let $q \in \mathbb{N}$ be odd and square-free. Then if $Q$ is a ternary quadratic form with $(\det(Q), q) = 1$ we have
\[
m(Q; q) \ll q^{1/2} \hat{m}(-Q^{\text{adj}}; q)^{1/2},
\]
where $Q^{\text{adj}}$ is the adjoint matrix for $Q$. In particular one has
\[
B^*_3(q) \ll q^{1/2} \hat{B}_3(q)^{1/2}.
\]

This naturally leads us to speculate about the size of $\hat{B}_3(q)$, and the natural conjecture is that
\[
\hat{B}_3(q) \ll \varepsilon q^\varepsilon
\]
for any fixed $\varepsilon > 0$. Of course Lemma 1 immediately shows that this latter conjecture implies (1.2).

If $q$ is odd and square-free there is a real character
\[
\chi_d(m) = \left( \frac{m}{d} \right)
\]
for each divisor $d$ of $m$, and the congruence $Q(x) \equiv t^2 \pmod{q}$ will have a solution $t$ if and only if
\[
\sum_{d|q} \chi_d(Q(x)) > 0.
\]

We can therefore attempt to show that $\hat{B}_3(q)$ is small by investigating the character sums
\[
S(\chi, B, Q) := \sum_{||x|| \leq B} \chi(Q(x))
\]
for primitive characters $\chi$ to modulus $d > 1$. If we can show that
\[
S(\chi, B, Q) \ll B^{3-\delta}
\]
for some fixed $\delta > 0$, for every primitive $\chi$ to modulus $d \geq 2$, then we will be able to deduce that $\hat{m}_3(Q; q) \leq B$, since one has $S(1, B, Q) \gg B^3$ for the trivial character.

It seems plausible that (1.4) should hold for $B \geq q^\eta$, for any fixed $\eta > 0$, and with $\delta = \delta(\eta) > 0$. This would suffice for (1.3), and hence also for (1.2).
One standard procedure to estimate sums such as $S(\chi, B, Q)$ is to complete the sum and use bounds of Weil–Deligne type. It is very instructive to carry this out in detail. What one finds, if $d = q$ for example, is essentially that

$$S(\chi, B, Q) \ll \frac{B^3}{q^3} \sum_{\gamma \in \mathbb{Z}^3} W\left(\frac{q}{B}\gamma\right) S(\gamma),$$

where $W \ll 1$ is a suitable weight function and

$$S(\gamma) := \sum_{x \pmod{q}} e_q(yx)\chi(Q(x)).$$

These complete sums can be computed explicitly. Taking $q$ to be prime for simplicity, and assuming that $q \nmid \det(Q)$, one finds that $S(\gamma)$ is of order $q^2/2$ when $q \mid Q^{\text{adj}}(y)$, and of order $q$ otherwise. This may be something of a surprise, since one typically expects complete sums in $n$ variables to have size around $q^{n/2}$. As a result this analysis leads to a bound which one may think of as

$$S(\chi, B, Q) \ll \varepsilon q^{1/3} B^{3/2} q^{-1/2} \# \left\{ y \ll q/B : q \mid Q^{\text{adj}}(y) \right\}. $$

Since we have estimated $m(Q; q)$ in terms of sums $S(\chi, B, Q^{\text{adj}})$ it is apparent that the above analysis ultimately connects small solutions of $q \mid Q(y)$ with small solutions of $q \mid Q^{\text{adj}}(y)$. In fact the argument is not completely circular, and one can show in this way that $B_3^2(q) \ll \varepsilon q^{2/3 + \varepsilon}$, at least when $q$ is prime. Alternatively one can provide an upper bound for

$$\# \left\{ y \ll q/B : q \mid Q^{\text{adj}}(y), y \text{ primitive} \right\} \ll (q/B)^{3/2},$$

by using $O((q/B)^{1/2})$ plane slices of the type $a \gamma = 0$. Each such slice produces a binary quadratic form of rank 1 or 2, which will have $O(1)$ primitive zeros modulo $q$, under the assumption that $q$ is prime. In this way one finds that

$$\# \left\{ y \ll q/B : q \mid Q^{\text{adj}}(y) \right\} \ll (q/B)^{3/2},$$

and hence

$$S(\chi, B, Q) \ll \varepsilon q^2 (q + B^{3/2} q^{1/2}).$$

We therefore have a non-trivial bound for $B \geq q^{1/3 + \varepsilon}$. Unfortunately this merely yields $B_3^2(q) \ll \varepsilon q^{1/3 + \varepsilon}$ and hence $B_3^3(q) \ll q^{2/3 + \varepsilon}$.

We have been unable to obtain a non-trivial bound for $S(\chi, B, Q)$ when $B \leq q^{1/3}$. However, if one replaces $Q$ by a binary form one can do better. Indeed the following result of Chang [7, Theorem 11] is the main inspiration for this paper.

**Theorem 2 (Chang).** For any $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

$$\left| \sum_{X' < x \leq X + X'} \sum_{y' \leq y \leq Y + Y'} \chi(x^2 + axy + by^2) \right| < p^{-\delta} XY$$

for any non-trivial character $\chi$ modulo $p$, any $X, Y > p^{1/4 + \varepsilon}$, and any integers $a, b$ with $a^2 \neq 4b \pmod{p}$. 


This improves on the corresponding results of Burgess \cite{5} and \cite{6}, which were non-trivial only for \(X, Y > p^{1/3+\varepsilon}\).

The proof of Chang’s result crucially uses the fact that a binary quadratic form over \(\mathbb{F}_p\) factors over \(\mathbb{F}_{p^2}\), and of course this limits the approach to the case \(n = 2\). Since we are interested in composite \(q\) we will require a variant of Theorem \(3\). The argument in \cite{7} splits into two rather different cases, one in which the form factors over \(\mathbb{F}_p\), and one in which it does not. In order to handle composite \(q\) we need to devise a treatment which handles both cases in the same way. Our result is the following.

**Theorem 3** Let \(\varepsilon > 0\) and an integer \(r \geq 3\) be given, and suppose that \(C \subset \mathbb{R}^2\) is a convex set contained in a disc \(\{x \in \mathbb{R}^2 : ||x - x_0|| \leq R\}\). Let \(q \geq 2\) be odd and square-free, and let \(\chi\) be a primitive character to modulus \(q\). Then if \(Q(x, y)\) is a binary quadratic form with \((\det(Q), q) = 1\) we have

\[
\sum_{(x, y) \in C} \chi(Q(x, y)) \ll \varepsilon, r R^{2-1/r} q^{(r+2)/(4r^2)+\varepsilon}
\]

for \(q^{1/4+1/2r} \leq R \leq q^{5/12+1/2r}\).

For comparison we observe that the standard Burgess bound [\cite{4}, Theorem 2] yields

\[
\sum_{x, y \leq R} \chi(xy) \ll \varepsilon, r R^{2-2/r} q^{(r+1)/(2r^2)+\varepsilon},
\]

relative to which our theorem has a loss of \((Rq^{-1/4})^{1/r}\). In Section \(3\) we will apply Theorem \(3\) with \(C = \{x \in \mathbb{R}^2 : ||x|| \leq R\}\) and \(R = q^{1/4+\delta}\). Taking \(r > (2\delta)^{-1}\) we will be able to deduce that \(B_2(q) \ll \varepsilon q^{1/4+\delta}\). We then go on to conclude that \(B_3(q) \ll \varepsilon q^{1/4+\varepsilon}\) and hence, via Lemma \(\mathbf{1}\) that \(B^*_3(q) \ll \varepsilon q^{5/8+\varepsilon}\).

Before embarking on the proofs we need to mention one point of notation. We shall follow the common convention that the small positive number \(\varepsilon\) will be allowed to change between appearances, allowing us to write \(\varepsilon q^{\log q} \ll \varepsilon q^r\), for example.

### 2 Proof of Lemma \(\mathbf{1}\)

Suppose that \(\overline{-Q\text{adj}}(a) \equiv t^2 \pmod q\) with \(||a|| = \overline{m}(-Q\text{adj}; q)\) and \(a \neq 0\). Write \(a = \alpha a_0\) with \(\alpha \in \mathbb{N}\) and \(a_0\) primitive, and let

\[
\Lambda := \{x \in \mathbb{Z}^2 : a_0 x = 0\}
\]

This will be a 2-dimensional lattice of determinant \(||a_0||\). Let \(x_1\) be the shortest non-zero vector in \(\Lambda\), and \(x_2\) the shortest vector non-proportional to \(x_1\). Then \(x_1\) and \(x_2\) form a basis for \(\Lambda\), and we have

\[
||x_1||, ||x_2|| \ll ||a_0||
\]

and

\[x_1 \wedge x_2 = \pm a_0.\]
We proceed to write $R(u, v) := Q(u\mathbf{x}_1 + v\mathbf{x}_2)$, so that $R$ is a binary quadratic form. We then have

$$
\det(R) = \det(Q(u\mathbf{x}_1 + v\mathbf{x}_2)) = Q^{\text{adj}}(\mathbf{x}_1 \wedge \mathbf{x}_2) = Q^{\text{adj}}(\mathbf{a}_0)
$$
as an identity, so that

$$
-\alpha^2 \det(R) = -Q^{\text{adj}}(\mathbf{a}) \equiv \epsilon^2 (\text{mod } q).
$$

Let $(q, \alpha) = q_0$ and $q = q_0q_1$. It follows that $R$ factors over $\mathbb{F}_p$ for every prime factor $p$ of $q_1$. We may then use the Chinese Remainder Theorem to write $R(u, v) \equiv L_1(u, v)L_2(u, v)$ (mod $q_1$) for certain integral linear forms $L_1$ and $L_2$.

Our strategy is now to find a short vector $\mathbf{x} \in \Lambda$ such that $q_1 \mid L_1(u, v)$. We will then automatically have $q_1 \mid R(u, v)$ and hence $q_1 \mid Q(u\mathbf{x}_1 + v\mathbf{x}_2)$. This will produce $q \mid Q(\mathbf{x})$ with $\mathbf{x} = q_0(u\mathbf{x}_1 + v\mathbf{x}_2)$.

Let

$$
U := \left(q_1 \frac{||\mathbf{x}_2||}{||\mathbf{x}_1||}\right)^{1/2} \quad \text{and} \quad V := \left(q_1 \frac{||\mathbf{x}_1||}{||\mathbf{x}_2||}\right)^{1/2},
$$

so that $UV = q_1$. Then an easy application of the pigeon-hole principle shows that one can find $(u, v) \in \mathbb{Z}^2 - \{(0, 0)\}$ with $q_1 \mid L_1(u, v)$ and satisfying $|u| \leq U$ and $|v| \leq V$. We then deduce that

$$
||\mathbf{x}|| = q_0||u\mathbf{x}_1 + v\mathbf{x}_2|| \\
\leq q_0(U||\mathbf{x}_1|| + V||\mathbf{x}_2||) \\
= 2q_0 \left(q_1 ||\mathbf{x}_1|| \cdot ||\mathbf{x}_2||\right)^{1/2} \\
\ll q_0(q_1 ||\mathbf{a}_0||)^{1/2} \\
= q^{1/2} \left(q_0 ||\mathbf{a}_0||\right)^{1/2} \\
\leq q^{1/2} (\alpha ||\mathbf{a}_0||)^{1/2} \\
= q^{1/2} \left|\mathbf{a}\right|^{1/2}
$$
by (2.1). Since $\mathbf{x}$ must be non-zero we deduce that

$$
m(Q; q) \ll (q||\mathbf{a}||)^{1/2} = q^{1/2} \tilde{m}(-Q^{\text{adj}}; q)^{1/2},
$$
as required.

3 Deduction of Theorem 1

In this section we will show how Theorem 1 follows from Theorem 3. Clearly it suffices to prove that $\tilde{m}(Q; q) \ll q^{1/4+\epsilon}$ uniformly for any ternary form $Q$ with $(\det(Q), q) = 1$.

Our first task is to establish the following corollary to Theorem 1.

Lemma 2 For any $\delta > 0$ there is a corresponding $\eta > 0$ such that, if $q > 1$ and $C \subset \mathbb{R}^2$ is a convex set contained in a disc $\{\mathbf{x} \in \mathbb{R}^2 : ||\mathbf{x} - \mathbf{x}_0|| \leq R\}$, then

$$
\sum_{(x, y) \in C} \chi(Q(x, y)) \ll_{\delta} R^{2-\eta}
$$
for $R \geq q^{1/4+\delta}$, uniformly for every primitive character $\chi$ modulo $q$ and for every binary quadratic form $Q$ with $(\det(Q), q) = 1$. 
We prove this in three steps, beginning with the case in which \( q^{1/4+\delta} \leq R \leq q^{5/12} \). In this range we choose

\[
r = 3 + \left\lfloor \frac{1}{\delta} \right\rfloor, \quad \eta = \frac{1}{(r^2 + 4r)} \quad \text{and} \quad \varepsilon = \eta/4.
\]

Then \( r \geq 3 \) and \( 1/4 + \delta \geq 1/4 + 1/r \), so that

\[
q \leq R^{1/(1/4+\delta)} \leq R^{4r/(r+4)}.
\]

Thus Theorem 3 produces

\[
\sum_{(x,y) \in \mathcal{C}} \chi(Q(x,y)) \ll_{\varepsilon,r} R^2 q^{2(r+2)/r(r+4)+\varepsilon}
\]

\[
= R^{2-2/r(r+4)+\eta}
\]

\[
= R^{2-\eta}.
\]

Next, when \( q^{5/12} \leq R \ll q \), we cover \( \mathbb{R}^2 \) with disjoint squares of side \( q^{5/12} \) to obtain a partition of \( \mathcal{C} \) into \( O(R^2 q^{-5/6}) \) convex subsets, each with diameter at most \( q^{5/12} \). On applying the result above with \( \delta = 1/6 \) we find that

\[
\sum_{(x,y) \in \mathcal{C}} \chi(Q(x,y)) \ll R^2 q^{-5/6} (q^{5/12})^{2-\eta} \ll R^{2-5\eta/12} \tag{3.1}
\]

for some absolute constant \( \eta > 0 \).

Finally we examine the case \( R \gg q \). This time we cover \( \mathcal{C} \) with squares of side \( q \), and observe that

\[
\sum_{x,y \,(\text{mod } q)} \chi(Q(x,y)) = 0.
\]

(By multiplicativity it suffices to prove this when \( q \) is prime, in which case it is an easy exercise, relying on the fact that \( Q \) is nonsingular modulo \( q \).) Since \( \mathcal{C} \) will be partitioned into \( O(R^2 q^{-2}) \) complete squares and \( O(Rq^{-1}) \) partial squares we may use the result (3.1) to conclude that

\[
\sum_{(x,y) \in \mathcal{C}} \chi(Q(x,y)) \ll R q^{-1} q^{2-5\eta/12} \leq R^{2-5\eta/12},
\]

and the lemma follows.

We next estimate \( \hat{m}(Q; q) \) for binary forms \( Q \).

**Lemma 3** For any fixed \( \delta > 0 \) we have

\[
\hat{m}(Q; q) \ll_{\delta} q^{1/4+\delta}
\]

uniformly over odd square-free moduli \( q \), and over binary forms \( Q \) subject to \((\text{det}(Q), q) = 1\).

As already noted in the introduction, if we let \( d \) run over all divisors of \( q \) then if \( \sum_d \chi_d(m) > 0 \) we must have \( m \equiv t^2 \,(\text{mod } q) \) for some integer \( t \). It follows that \( \hat{m}(Q; q) \leq R \) provided that

\[
\sum_{d|q} \sum_{\|(x,y)\| \leq R} \chi_d(Q(x,y)) > 0.
\]
The number of divisors of $q$ is $O(q^{1/4+\delta})$, for any $\varepsilon > 0$. Choosing $\varepsilon = \eta/8$ it follows from Lemma 2 that if $R \geq q^{1/4+\delta}$ then

$$\sum_{d|q, d>1} \sum_{||x,y||\leq R} \chi_d(Q(x,y)) \ll R^{2-\eta} q^{1/2} \ll R^{2-\eta/2}.$$

On the other hand

$$\sum_{||x,y||\leq R} 1 \gg R^2,$$

and Lemma 3 follows.

Finally we need to estimate $\hat{B}_3(q)$ in terms of $\hat{B}_2(q)$. Lemma 4

We have

$$\hat{B}_3(q) \ll \varepsilon q^{1/4} \hat{B}_2(q)$$

for any fixed $\varepsilon > 0$.

Once this is proved we may deduce from Lemma 3 that $\hat{B}_3(q) \ll q^{1/4+\varepsilon}$ for any $\varepsilon > 0$, whence Lemma 4 yields $B^*_3(q) \ll q^{8/8+\varepsilon}$. This is the result required for Theorem 1.

To establish Lemma 4 we will find short vectors

$$(a_1, a_2), (a_3, a_4), (a_5, a_6) \in \mathbb{Z}^2$$

such that the form

$$R(u, v) := Q(a_1 u + a_2 v, a_3 u + a_4 v, a_5 u + a_6 v) \quad (3.3)$$

has $(\det(R), q) = 1$. We can then choose $u, v \ll \hat{B}_2(q)$, not both zero, such that $R(u, v)$ is a square modulo $q$, which will produce a corresponding vector

$$x = (a_1 u + a_2 v, a_3 u + a_4 v, a_5 u + a_6 v)$$

for which $Q(x)$ is a square modulo $q$, and with

$$||x|| \ll ||(u, v)|| \max(a_1, \ldots, a_6).$$

If $x$ were to vanish the three vectors (3.2) would all have to be proportional. But then the form (3.3) would have rank at most 1, so that $\det(R) = 0$. This would contradict our assumption that $(\det(R), q) = 1$. It follows that we must have $x \neq 0$. Thus to complete the proof of Lemma 4 it will suffice to show that we can choose the coefficients $a_1, \ldots, a_6$ to be of size $O(q^{\varepsilon})$.

Define

$$\Delta(a_1, \ldots, a_6) := \det(Q(a_1 u + a_2 v, a_3 u + a_4 v, a_5 u + a_6 v)).$$

This will be a sextic form in the 6 variables $a_1, \ldots, a_6$. We claim that for each prime factor $p$ of $q$ there is at least one choice of $a \in \mathbb{Z}^6$ such that $p \nmid \Delta(a)$. Since we can diagonalize $Q$ by a unimodular transformation over $\mathbb{F}_p$, a moment’s reflection shows that it is enough to verify the claim when $Q$ is a diagonal form. However the result is trivial in this case since $p \nmid \det(Q)$.

We can now call on the following lemma, which we will prove in a moment.
Lemma 5 Let $\varepsilon, \delta > 0$ be given. Suppose that $F(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ is a form of degree $d$, and let $q \in \mathbb{N}$. Assume that for every prime divisor $p$ of $q$ there is at least one $a \in \mathbb{Z}^n$ such that $p \nmid F(a)$. Then

$$\# \{a \in \mathbb{N}^n : \max a_i \leq A, \langle F(a), q \rangle = 1 \} \gg_{d,n,\varepsilon} A^n q^{-\varepsilon}$$

as soon as $A \geq q^\delta$ and $q \gg_{n,d} 1$.

This result shows that we have at least one vector $a$ of size $\|a\| \ll q^\varepsilon$ such that $\Delta(a)$ is coprime to $q$, which suffices to complete the proof.

It remains to prove Lemma 5. Define

$$N(e) := \# \{a \pmod{e} : e \mid F(a) \}$$

for each $e \in \mathbb{N}$. Then $N(e)$ is multiplicative, and $N(p) \ll p^n$ for $p \mid q$ by the hypothesis of the lemma. Moreover, when $N(p) < p^n$ the form $F$ cannot vanish identically modulo $p$, whence $N(p) \ll_{d,n} p^{n-1}$. It follows that

$$N(e) \ll_{d,n,\eta} e^{n-1+\eta}$$

for $e \mid q$, for any fixed $\eta > 0$.

We now consider

$$N(e, A) := \# \{a \in \mathbb{N}^n : \max a_i \leq A, e \mid F(a) \}.$$

The set $(0, A]^n$ contains $[A/e]^n$ disjoint cubes of side-length $e$, and is included in a union of $(1 + [A/e])^n$ such cubes. It follows that

$$N(e, A) = \frac{A^n}{e^n} N(e) + O_n(A^{n-1} e^{1-n} N(e)) = \frac{A^n}{e^n} N(e) + O_{d,n,\eta}(A^{n-1} e^n) \quad (3.4)$$

when $e \mid q$ and $e \leq A$. To handle larger values of $e$ we use a rather general result of Browning and Heath-Brown [2]. For each $p_i \mid q$ let $V_i$ be the affine variety over $\mathbb{F}_{p_i}$ given by $F = 0$. Since $F$ does not vanish identically modulo $p_i$ this has dimension $n - 1$. We now apply [2, Lemma 4] with $W = \mathbb{A}^n$, and $k_i = n - 1$ for every index $i$. Taking $e \mid q$ with $e \geq A$ we find that there is a constant $C = C(d,n)$ such that

$$N(e, A) \ll C^{\omega(e)}(A^n e^{-1} + \omega(e) A^{n-1}) \ll_{d,n,\eta} e^{\eta} A^{n-1}$$

for any fixed $\eta > 0$. It follows that if $e \geq A$ we will have

$$N(e, A) = \frac{A^n}{e^n} N(e) + O(A^n N(e) e^{-n}) + O_{d,n,\eta}(A^{n-1} e^n)$$

$$= \frac{A^n}{e^n} N(e) + O_{d,n,\eta}(A^n e^{-1+\eta}) + O_{d,n,\eta}(A^{n-1} e^n)$$

$$= \frac{A^n}{e^n} N(e) + O_{d,n,\eta}(A^{n-1} e^{\eta}),$$

so that (3.4) holds whether $e \leq A$ or not.
We now examine
\[
\# \{ a \in \mathbb{N}^n : \max a_i \leq A, (F(a), q) = 1 \} = \sum_{e | q} \mu(e) N(e, A) = \sum_{e | q} \mu(e) \frac{A^n}{e^n} N(e) + O_{d,n,q} \left( \sum_{e | q} A^{n-1} e^n \right) = A^n \prod_{p | q} (1 - N(p)p^{-n}) + O_{d,n,q} (A^{n-1} q^{2n}).
\]

Since \( N(p) < p^n \) and \( N(p) \leq c_0 p^{n-1} \) for some constant \( c_0 \) depending only on \( d \) and \( n \), we may deduce that
\[
\prod_{p | q} (1 - N(p)p^{-n}) \geq \prod_{p \leq 2c_0} (1 - (p^n - 1)p^{-n}) \prod_{p > 2c_0} (1 - c_0 p^{-1}) \geq \prod_{p \leq 2c_0} p^{-n} \prod_{p > 2c_0} (1 - p^{-1})^{2c_0} \gg_{d,n} \left( \frac{\phi(q)}{q} \right)^{2c_0} q^{-n}.
\]

It follows that
\[
\# \{ a \in \mathbb{N}^n : \max a_i \leq A, (F(a), q) = 1 \} \geq c_2 A^n q^{-\eta} - c_3 A^{n-1} q^{2\eta}
\]
for suitable positive constants \( c_2 \) and \( c_3 \) depending on \( d \) and \( n \). The lemma then follows on taking \( \eta = \min(\epsilon, \delta/4) \).

### 4 Proof of Theorem 3

For the proof we will write
\[
\Sigma = \sum_{(x,y) \in C} \chi(Q(x,y))
\]
for convenience. Let \( N \in \mathbb{N} \) be a parameter to be chosen, satisfying \( N \leq Rq^{-1/100} \), say, and set \( S = [R/N] \). We need to specify a “good” set of vectors \( s \in \mathbb{N}^2 \), and this will require a further definition. The form \( Q(X,Y) \) should be thought of as lying in \( (\mathbb{Z}/q\mathbb{Z})[X,Y] \), and we need an appropriate lift to \( \mathbb{Z}[X,Y] \). To achieve this we write \( Q(x_1, x_2) = Ax_1^2 + Bx_1x_2 + Cx_2^2 \) and
\[
\Lambda = \{ y \in \mathbb{Z}^3 : y \equiv \lambda(A, B, C) \pmod{q} \text{ for some } \lambda \in \mathbb{Z} \}, \tag{4.1}
\]
and we let \( (A^*, B^*, C^*) \) be a non-zero vector in \( \Lambda \) of minimal length. As there is a non-zero vector \( (A', B', C') \equiv (A, B, C) \pmod{q} \) in \( \Lambda \) with \( |A'|, |B'|, |C'| \leq q/2 \) we see that \( q \) cannot divide \( (A^*, B^*, C^*) \). We now define
\[
Q^*(X,Y) = A^* X^2 + B^* XY + C^* Y^2.
\]
Note that \( \det(Q^*) \equiv \lambda^2 \det(Q) \pmod{q} \) for an appropriate \( \lambda \). Since \( q \) cannot divide \( \lambda \), and is square-free and coprime to \( \det(Q) \), we will have \( q \nmid \det(Q^*) \). In particular \( Q^* \) is nonsingular, but there is no guarantee that \( (\det(Q^*), q) = 1 \).

We can now take our set of “good” vectors \( \mathcal{S} \) to be

\[
\mathcal{S} = \{(s_1, s_2) \in \mathbb{N}^2 : ||s|| \leq S, (Q(s), q) = 1, Q^*(s) \neq 0\}.
\]

There are \( O(S) \) vectors for which \( Q^*(s) = 0 \), uniformly over all non-zero forms \( Q^* \). Thus, according to Lemma 5 we have

\[
\# \mathcal{S} \gg S^2 q^{-\varepsilon}, \tag{4.2}
\]

for \( S \gg q^\varepsilon \), for any fixed \( \varepsilon > 0 \).

For any positive integer \( n \leq N \) we proceed to write

\[
(#\mathcal{S})\Sigma = \sum_{(s_1, s_2) \in \mathcal{S}} \sum_{(x, x_0) \in \mathbb{Z}^2} \chi(Q(x_1 + ns_1, x_2 + ns_2)) \mathbbm{1}_C(x_1 + ns_1, x_2 + ns_2),
\]

where \( \mathbbm{1}_C \) is the characteristic function for the set \( C \). It follows that

\[
N(#\mathcal{S})\Sigma = \sum_{(s_1, s_2) \in \mathcal{S}} \sum_{(x, x_0) \in \mathbb{Z}^2} \sum_{n \in I} \chi(Q(x_1 + ns_1, x_2 + ns_2)),
\]

where

\[
I = \{n \leq N : (x_1 + ns_1, x_2 + ns_2) \in C\}.
\]

Since \( C \) is convex, \( I \) is an interval. Moreover if \( I \) is nonempty, containing \( x + ns \), then \( ||x + ns - x_0|| \leq R \) and \( ||ns|| \leq NS \leq R \), whence \( ||x - x_0|| \leq 2R \). We therefore deduce, via (4.2), that

\[
\Sigma \ll \varepsilon N^{-1} S^{-2} q^\varepsilon \sum_{s \in \mathcal{S}} \sum_{x \in \mathbb{Z}^2} \max_{I \subseteq (0, N)} \left| \sum_{n \in I} \chi(Q(x_1 + ns_1, x_2 + ns_2)) \right|.
\]

If the reader compares this with the corresponding stage in the argument of Chang [7], see [7, (4.3)] for example, then it will be observed that Chang has a product \( st \) in place of our variable \( n \). Indeed our method is slightly different from Chang’s, requires one variable fewer, and does not use an argument corresponding to [7, Lemma 3].

To proceed further we use the readily verified identity

\[
Q(x_1 + ns_1, x_2 + ns_2) = Q(s) \bar{Q}(n + a(s, x), b(s, x)),
\]

where \( \bar{Q}(x_1, x_2) := x_1^2 + B x_1 x_2 + AC x_2^2 \), and

\[
a(s, x) = \frac{Ax_1 s_1 + Bx_1 s_2 + Cx_2 s_2}{Q(s_1, s_2)}, \quad b(s, x) = \frac{x_2 s_1 - x_1 s_2}{Q(s_1, s_2)}. \tag{4.3}
\]

Here the fractions are to be interpreted in the ring \( \mathbb{Z}/q\mathbb{Z} \), the denominators \( Q(s_1, s_2) \) being units by our choice of the set \( \mathcal{S} \). We now write

\[
N(a, b) = \# \{(s, x) \in \mathcal{S} \times \mathbb{Z}^2 : ||x - x_0|| \leq 2R, a(s, x) = a, b(s, x) = b\},
\]

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whence

\[ \Sigma \ll \varepsilon \, N^{-1} S^{-2} q^2 \sum_{a, b \ (\text{mod } q)} N(a, b) \max_{I \subseteq (0, N)} \left| \sum_{n \in I} \chi(\tilde{Q}(n + a, b)) \right|. \]

We must now consider the mean square of \( N(a, b) \), for which we will prove the following bound.

**Lemma 6** For any fixed \( \varepsilon > 0 \) we have

\[ \sum_{a, b \ (\text{mod } q)} N(a, b)^2 \ll \varepsilon \, q^2 R^2 S^2 (1 + RSq^{-1/2} + R^2 S^2 q^{-4/3}). \]

This will be established in the next section.

We also have the trivial bound

\[ \sum_{a, b \ (\text{mod } q)} N(a, b) \leq \# \{(s, x) \in S \times \mathbb{Z}^2 : ||x - x_0|| \leq 2R\} \ll R^2 S^2, \]

whence Hölder’s inequality yields

\[ \Sigma^{2r} \ll_{\varepsilon, r} (N^{-1} S^{-2} q^2)^{2r} \left( \sum_{a, b \ (\text{mod } q)} N(a, b)^2 \right)^{2r-2} \left( \sum_{a, b \ (\text{mod } q)} N(a, b) \right)^{2r} \]

\[ \ll_{\varepsilon, r} N^{-2r} R^{4r-2} S^{-2} q^2 (1 + RSq^{-1/2} + R^2 S^2 q^{-4/3}) \]

\[ \times \sum_{a, b \ (\text{mod } q)} \max_{I \subseteq (0, N)} \left| \sum_{n \in I} \chi(\tilde{Q}(n + a, b)) \right|^{2r} \]

\[ \ll_{\varepsilon, r} N^{-2r} R^{4r-4} q^4 (1 + R^2 N^{-1} q^{-1/2} + R^2 N^{-2} q^{-4/3}) \]

\[ \times \sum_{a, b \ (\text{mod } q)} \max_{I \subseteq (0, N)} \left| \sum_{n \in I} \chi(\tilde{Q}(n + a, b)) \right|^{2r}, \quad (4.4) \]

on employing our convention concerning the values taken by \( \varepsilon \).

We are therefore led to consider sums of the form

\[ S(q; H) := \sum_{a, b \ (\text{mod } q)} \left| \sum_{n \leq H} \chi(\tilde{Q}(n + a, b)) \right|^{2r}. \]

To estimate these we expand to obtain

\[ S(q; H) = \sum_{n_1, \ldots, n_{2r} \leq H} \Sigma(q; \underline{u}) \]

with

\[ \Sigma(q; \underline{u}) = \sum_{a, b \ (\text{mod } q)} \chi(F_+(a, b; \underline{u})) \chi(F_-(a, b; \underline{u})) \]
and
\[ F_+(X, Y; \eta) = \prod_{i=1}^{r} \tilde{Q}(a_i + X, Y), \quad F_-(X, Y; \eta) = \prod_{i=r+1}^{2r} \tilde{Q}(a_i + X, Y). \]

The sums \( \Sigma(q; \eta) \) have a standard multiplicative property. If \( q = uv \), say, then \( u \) and \( v \) will be coprime and square-free, and we can write \( \chi = \chi_u \chi_v \) for suitable primitive characters to moduli \( u \) and \( v \) respectively. We will then have
\[
\Sigma(q; \eta) = \Sigma(u; \eta) \Sigma(v; \eta) \quad (4.5)
\]
It therefore suffices to understand \( \Sigma(q; \eta) \) when \( q \) is prime, for which we have the following result.

**Lemma 7** Let \( p \) be an odd prime not dividing \( \det(\tilde{Q}) \), and let \( \chi \) be a non-principal character to modulus \( p \). Write
\[
\Delta_i = \prod_{1 \leq j \leq 2r, j \neq i} (n_j - n_i)
\]
and
\[
\Delta = \text{h.c.f.}(\Delta_1, \ldots, \Delta_{2r}).
\]
Then
\[
|\Sigma(p; \eta)| \leq 4r^2 p(p, \Delta).
\]

We will prove this in Section 6. By summing over the \((2r)\)-tuples \( \eta \) we are then able to establish the following bound for \( S(q; H) \).

**Lemma 8** For any \( \varepsilon > 0 \) and \( r \in \mathbb{N} \) we have
\[
S(q, H) \ll_{\varepsilon, r} (qH^\varepsilon(qH^{2r} + q^2H^r)).
\]
This will be proved in Section 7.

Having established this there is a standard procedure to insert a maximum over subintervals of \((0, N]\), which goes back to Rademacher [15] and Menchov [13]. We do not repeat the details, but instead refer the reader to Gallagher and Montgomery [9, Section 3] or Heath-Brown [13, Section 2]. The outcome is the following result.

**Lemma 9** For any \( \varepsilon > 0 \) and \( r \in \mathbb{N} \) we have
\[
\sum_{a,b \ (\text{mod} \ q)} \max_{I \subseteq [0, N]} \left| \sum_{n \in I} \chi(\tilde{Q}(n + a, b)) \right|^{2r} \ll_{\varepsilon, r} (qN)^{\varepsilon}(qN^{2r} + q^2N^r).
\]

We are now ready to complete the proof of Theorem 3. We insert the bound of Lemma 9 into (4.3), to give
\[
\Sigma^{2r} \ll_{\varepsilon, r} N^{2r-2}R^{4r-4}(1 + R^2 N^{-1} q^{-1/2} + R^4 N^{-2} q^{-4/3})(qN)^{\varepsilon}(qN^{2r} + q^2N^r).
\]
In order to balance the final two terms we choose \( N = \lfloor q^{1/r} \rfloor \), which satisfies our constraint \( N \leq Rq^{-1/100} \) provided that \( R \geq q^{1/4 + 1/2r} \) and \( r \geq 3 \). On re-defining \( \varepsilon \) we then find that
\[
\Sigma^{2r} \ll_{\varepsilon, r} q^{1/2 + 1/r + \varepsilon} R^{4r-2}(R^{-2} q^{1/2 + 1/r} + 1 + R^2 q^{-5/6 - 1/r}),
\]
and the theorem follows.
5 Proof of Lemma 6

We now prove Lemma 6. In view of the definitions (4.3), we have the identity

\[(Ab(s, x)X - a(s, x)Y)(s_2X - s_1Y) = s_2b(s, x)Q(X, Y) - (x_2X - x_1Y)Y\]

in \((\mathbb{Z}/q\mathbb{Z})[X, Y]\). Thus if \(a(s, x) = a(s', x') = a\) and \(b(s, x) = b(s', x') = b\) then

\[(AbX - aY)(s_2X - s_1Y)(s'_2X - s'_1Y) = (s_2bQ(X, Y) - (x_2X - x_1Y)Y)(s'_2X - s'_1Y),\]

and also

\[(AbX - aY)(s'_2X - s'_1Y)(s_2X - s_1Y) = (s'_2bQ(X, Y) - (x'_2X - x'_1Y)Y)(s_2X - s_1Y).\]

Thus by subtraction we deduce that

\[Y((x_2X - x_1Y)(s'_2X - s'_1Y) - (x'_2X - x'_1Y)(s_2X - s_1Y))\]

\[= b(s'_2s_1 - s'_1s_2)YQ(X, Y),\]

still in \((\mathbb{Z}/q\mathbb{Z})[X, Y]\).

We then deduce that

\[(x_2s'_2 - x'_2s_2, x'_2s_1 + x'_1s_2 - x_2s'_1 - x_1s'_2, x_1s'_1 - x'_1s_1) \in \Lambda, \quad (5.1)\]

with \(\Lambda\) given by (4.1). It follows that

\[
\sum_{a, b} N(a, b)^2 \leq \sum_{s, s' \in S} N_1(s, s'),
\]

where \(N_1(s, s')\) counts pairs of vectors \((x, x')\) each lying in the disc \(||x - x_0|| \leq 2R\), such that (5.1) holds. Now suppose that \((x_1, x'_1)\) is a pair counted by \(N_1(s, s')\). For any other such pair we write \(x = x_1 + u\) and \(x' = x'_1 + u'\) whence, by subtraction, we find firstly that \(||u||, ||u'|| \leq 4R\), and secondly that

\[(u_2s'_2 - u'_2s_2, u'_2s_1 + u'_1s_2 - u_2s'_1 - u_1s'_2, u_1s'_1 - u'_1s_1) \in \Lambda. \quad (5.2)\]

Thus \(N_1(s, s') \leq N_2(s, s')\), where \(N_2(s, s')\) counts pairs of vectors \(u, u'\) satisfying (5.2), and having length at most \(4R\).

We have already chosen \((A^*, B^*, C^*) = v_1\), say, as the shortest vector in \(\Lambda\). As in the proof of Davenport [3] Lemma 5], we can then construct a basis \(y_1, y_2, y_3\) for \(\Lambda\), such that if \(y = \lambda_1y_1 + \lambda_2y_2 + \lambda_3y_3\), then \(\lambda_i \ll ||y||/||y_i||\) for \(i = 1, 2, 3\). Moreover we will have

\[||y_1|| \leq ||y_2|| \leq ||y_3||\]

and

\[||y_1||/||y_2||/||y_3|| \geq \det(\Lambda).\]

In our case we have \(\det(\Lambda) = q^2\), whence

\[||y_2||/||y_3|| \geq q^{4/3}.
\]
Moreover one sees from the definition of $\Lambda$ that $q \mid v_1 \wedge v_2$, and since the vectors $v_1$ and $v_2$ are not proportional it follows that

$$q \leq ||v_1 \wedge v_2|| \leq ||v_1|| ||v_2|| \leq ||v_2||^2.$$  

The vector

$$\chi = (u_2 s_2' - u_2' s_2, u_2' s_1 + u_1' s_2 - u_2 s_1' - u_1 s_2', u_1 s_1' - u_1' s_1)$$

has length at most $32RS$ so that the corresponding coefficients satisfy

$$\lambda_2 \ll \frac{RS}{||v_2||}$$ and

$$\lambda_3 \ll \frac{RS}{||v_3||}.$$  

If we break the available vectors counted by $N_2(s, s')$ into subsets according to the values of $\lambda_2$ and $\lambda_3$, then the number of such subsets will be

$$\ll \left(1 + \frac{RS}{||v_2||}\right) \left(1 + \frac{RS}{||v_3||}\right) \ll 1 + \frac{RS}{||v_2||} + \frac{R^2 S^2}{||v_2|| ||v_3||} \ll 1 + \frac{RS}{q^{4/3}} + \frac{R^2 S^2}{q^{4/3}}.$$  

If $(u_1, u_1')$ and $(u_2, u_2')$ are two pairs belonging to the same subset, and we write $u = u_1 - u_2$ and $u' = u_1' - u_2'$, then

$$\chi = (u_2 s_2' - u_2' s_2, u_2' s_1 + u_1' s_2 - u_2 s_1' - u_1 s_2', u_1 s_1' - u_1' s_1) \quad (5.3)$$

will be a multiple of $v_1 = (A^*, B^*, C^*)$, and we will have $||u||, ||u'|| \leq 8R$.

We therefore conclude that

$$N_2(s, s') \ll (1 + RSq^{-1/2} + R^2 S^2 q^{-4/3}) N_3(s, s')$$

where $N_3(s, s')$ counts pairs of vectors $u, u'$ having length at most $8R$, and for which the vector $(5.3)$ is an integer multiple of $(A^*, B^*, C^*)$. The quadratic form corresponding to $\chi$ is

$$(u_2 s_2' - u_2' s_2) X^2 + (u_2' s_1 + u_1' s_2 - u_2 s_1' - u_1 s_2') Y X + (u_1 s_1' - u_1' s_1) Y^2$$

$$= (u_2 X - u_1 Y)(s_2' X - s_1' Y) - (u_2' X - u_1' Y)(s_2 X - s_1 Y).$$

We therefore conclude that

$$Q^*(X, Y) \mid (u_2 X - u_1 Y)(s_2' X - s_1' Y) - (u_2' X - u_1' Y)(s_2 X - s_1 Y). \quad (5.4)$$

Thus, to complete the proof of Lemma 6 it suffices to show that

$$\#\{ (u, u', s, s') \in \mathbb{Z}^2 \times \mathbb{Z}^2 \times S \times S : ||u||, ||u'|| \leq 8R, \quad (5.4) \text{ holds} \} \ll \varepsilon q^2 R^2 S^2. \quad (5.5)$$

Given two binary quadratic forms $Q_1$ and $Q_2$ one may define a covariant $C(Q_1, Q_2)$ as the discriminant of the binary form $D(\alpha, \beta) = \det(\alpha Q_1 + \beta Q_2)$. One readily confirms that $C(Q_1, Q_2) = C(Q_1 + \lambda Q_2, Q_2)$ for any constant $\lambda$, and moreover that

$$C((u_2 X - u_1 Y)(v_2 X - v_1 Y), Q) = Q(u_1, u_2)Q(v_1, v_2).$$
Taking $Q_1 = (u_2X - u_1Y)(s_2'X - s_1'Y)$ and $Q_2 = Q^*$ we deduce that

$$Q^*(u_1, u_2)Q^*(s_1', s_2') = Q^*(u_1', u_2')Q^*(s_1, s_2).$$

In defining the set $S$ we arranged that $Q^*(s) \neq 0$. If $Q^*(u) \neq 0$ then $Q^*(u_1, u_2)Q^*(s_1', s_2')$ has $O_\varepsilon(q^\varepsilon)$ divisors, since

$$|Q^*(u_1, u_2)Q^*(s_1', s_2')| \ll \max(|A^*|, |B^*|, |C^*|)^2||u||^2|s||^2 \ll q^2R^2S^2 \ll q^6.\) Moreover, when $d \neq 0$ the equation $Q^*(u_1', u_2') = d$ will have $\ll_{\varepsilon}(qR)^\varepsilon \ll_{\varepsilon} q^\varepsilon$ solutions $u'$ with $||u'|| \ll 8R$, by Theorem 13 of Heath-Brown [12] for example. (Here we use crucially the fact that $Q^*$ is nonsingular.) Similarly $Q^*(s_1, s_2) = d'$ will have $O_\varepsilon(q^\varepsilon)$ solutions for any $d' \neq 0$. It then follows that the contribution arising from 4-tuples $(u, u', s, s')$ in which $Q^*(u) \neq 0$ will be $O_\varepsilon(q^2R^2S^2)$, which is satisfactory for (5.5).

It remains to deal with the case in which $Q^*(u) = 0$. In view of (5.6) we will then have $Q^*(u') = 0$, since $Q^*(s)$ and $Q^*(s')$ are non-zero. We now claim that either $u = u' = 0$, or $Q^*(X, Y)$ factors over $\mathbb{Z}$ into linear factors $L_1(X, Y)$ and $L_2(X, Y)$ such that $L_1(X, Y)$ divides both $u_2X - u_1Y$ and $u_2'X - u_1'Y$. To see this, suppose that $u \neq 0$, say. Then we must have $L_1(X, Y) | u_2X - u_1Y$ for some integral linear factor of $Q^*$. It would then follow from (5.6) that $L_1(X, Y) | u_2'X - u_1'Y$, since $Q^*(s) \neq 0$. The claim then follows.

Clearly the contribution to (5.5) arising from the case $u = u' = 0$ is $O(S^4) = O(R^2S^2)$, which is satisfactory, so it remains to consider the case in which

$$u_2X - u_1Y = kL_1(X, Y) \quad \text{and} \quad u_2'X - u_1'Y = k'L_1(X, Y)$$

with integers $k, k'$ such that $|k|, |k'| \leq 8R$. We then have

$$k(s_2'X - s_1'Y) \equiv k'(s_2X - s_1Y) \pmod{L_2(X, Y)},$$

by (5.6). If $L_2(X, Y) = aX - bY$, say then we must have

$$k(s_2'b - s_1'a) = k'(s_2b - s_1a).$$

Moreover $s_2'b - s_1'a$ and $s_2b - s_1a$ are non-zero, since $Q^*(s)$ and $Q^*(s')$ do not vanish. If $k' = 0$ then $k = 0$, which would put us in the case $u = u' = 0$ which has already been dealt with. Since at most one of $a$ or $b$ can vanish we may suppose that $b$ say, is non-zero. There are then $O(RS^3)$ possibilities for $s_1', s_1, s_2$ and $k'$, and the number of divisors $k$ of $k'(s_2b - s_1a)$ will be $O_\varepsilon((qS)^\varepsilon)$, since $|a|, |b| \ll (|A^*|, |B^*|, |C^*|)^2 \ll q$. The complementary divisor to $k$ is then $s_2'b - s_1'a$, which determines $s_2'$. We therefore conclude that the corresponding contribution to (5.5) is $O_\varepsilon(q^2R^2S^2)$, since $S \leq R \leq q$. This completes the proof of (5.6), and hence of Lemma 7.

## 6 Proof of Lemma 7

Our proof of Lemma 7 is inspired by the viewpoint taken by Chang [7]. We first consider the case in which $Q(X, Y) = X^2 + BXY + ACY^2$ factors modulo $p$. In this case we may replace $Q(X, Y)$ by $(X + \lambda Y)(X + \mu Y)$ say, where $p \nmid \lambda - \mu$ since $p \nmid \det(Q)$. Then $Q(n + a, b) = (n + a')(n + b')$ where $a' = a + \lambda b$ and $b' = b + \mu b$. We proceed as in the case $Q(X, Y) = X^2 + BXY + ACY^2$.
\[ b' = a + \mu b \text{ are independent of } n. \] Moreover \((a', b')\) runs over \(\mathbb{F}_p^2\) as \((a, b)\) does. It follows that
\[
\Sigma(p; u) = \sum_{a, b \pmod{p}} \chi(G_+(a, b; u)) \overline{\chi(G_-(a, b; u))}
\]
with
\[
G_+(X, Y; u) = \prod_{i=1}^{r} (n_i + X)(n_i + Y), \quad G_-(X, Y; u) = \prod_{i=r+1}^{2r} (n_i + X)(n_i + Y).
\]
We then see that
\[
\Sigma(p; u) = \Sigma_1(p; u)^2
\]
with
\[
\Sigma_1(p; u) = \sum_{a \pmod{p}} \chi(H_+(a; u)) \overline{\chi(H_-(a; u))}
\]
and
\[
H_+(X; u) = \prod_{i=1}^{r} (n_i + X), \quad H_-(X; u) = \prod_{i=r+1}^{2r} (n_i + X).
\]
The sum \(\Sigma_1(p; u)\) occurs in the work of Burgess [3, Lemma 1], from which one readily sees that
\[
|\Sigma_1(p; u)| \leq 2r\sqrt{\#F} \tag{6.1}
\]
unless every linear factor of the polynomial \(H_+(X; u)H_-(X; u)\) has multiplicity two or more, modulo \(p\). In the exceptional case we have \(p \mid \Delta_i\) for every \(i\), whence \(p \mid \Delta\). We deduce that (6.1) holds whenever \(p \nmid \Delta\). In the remaining case we have a trivial bound \(|\Sigma_1(p; u)| \leq p\), so that
\[
|\Sigma_1(p; u)| \leq 2r\sqrt{p(p, \Delta)}^{1/2}
\]
whether or not \(p \nmid \Delta\). We therefore conclude that
\[
|\Sigma(p; u)| \leq 4r^2p(p, \Delta)
\]
whenever \(\tilde{Q}\) factors modulo \(p\). This is satisfactory for Lemma 7.

We turn now to the case in which \(\tilde{Q}\) is irreducible over \(\mathbb{F}_p\). It will be typographically convenient to write \(F\) for the field \(\mathbb{F}_p\). In the case under consideration, there is a factorization \(\tilde{Q}(X, Y) = (X + \lambda Y)(X + \lambda' Y)\) say, over \(F\) with \(\lambda\) and \(\lambda'\) being conjugates in \(F/\mathbb{F}_p\). We may now define a function \(\psi\) from \(F\) to \(\mathbb{C}\) by setting
\[
\psi(a + \lambda b) = \chi((a + \lambda b)(a + \lambda'b)) = \chi(\tilde{Q}(a, b)).
\]
One easily sees that this is a non-trivial multiplicative character on \(F\), and that
\[
\Sigma(p; u) = \sum_{\alpha \in F} \psi(H_+(\alpha; u)) \overline{\psi(H_-(\alpha; u))}.
\]
Burgess’ proof of (6.1), based on Weil’s “Riemann Hypothesis” for curves over arbitrary finite fields, immediately extends to \(\Sigma(p; u)\), and shows that
\[
|\Sigma(p; u)| \leq 2r\sqrt{\#F} = 2rp
\]
17
It follows that the set 

\[ \{ H \} \]

are at most \( i \) there is any index so that it suffices to establish the estimate 

\[ \text{Indeed, we have} \]

\[ \nu \]

the 

thus to prove Lemma 8 it will be enough to show that 

\[ \text{However} \]

\[ \| \]

in view of the trivial bound \( |\Sigma(p; \nu)| \leq p^2 \). As above, these bounds are satisfactory for Lemma 7.

### 7 Proof of Lemma 8

It follows from Lemma 7, along with the multiplicative relation (4.5) that 

\[ \Sigma(q; \nu) \leq (4r^2)\omega(q)q(\epsilon, \nu) \ll \epsilon, r q^{1+\epsilon}(q, \nu). \]

Thus to prove Lemma 8 it will be enough to show that 

\[ \sum_{n_1, \ldots, n_{2r} \leq H} (q, \nu) \ll \epsilon, r (qH)^\epsilon(H^{2r} + qH^r). \]

Indeed, we have 

\[ \sum_{n_1, \ldots, n_{2r} \leq H} (q, \nu) \leq \sum_{k|q} k\#\{ \nu \in \mathbb{N}^{2r} \cap (0, H]^{2r} : k \mid \nu \}, \]

so that it suffices to establish the estimate 

\[ \#\{ \nu \in \mathbb{N}^{2r} \cap (0, H]^{2r} : k \mid \nu \} \ll \epsilon, r (kH)^\epsilon(H^{2r}k^{-1} + H^r). \quad (7.1) \]

We first consider vectors \( \nu \) for \( \Delta_1 = \ldots = \Delta_{2r} = 0 \). Then if \( \nu \in \mathbb{N} \) and there is any index \( i \) such that \( n_i = \nu \), there must be at least two such indices. It follows that the set \( \{n_1, \ldots, n_{2r}\} \) contains at most \( r \) distinct elements. There are at most \( H^r \) choices for these elements, \( \nu_1, \ldots, \nu_s \) say, with \( 1 \leq s \leq r \). Once the \( \nu_j \) have been chosen there are (at most) \( s \) choices for each \( n_i \). It follows that there are \( O_r(H^r) \) vectors \( \nu \) for which \( \Delta_1 = \ldots = \Delta_{2r} = 0 \). This is satisfactory for (7.1).

In the remaining case we have \( \Delta_j \neq 0 \) for some index \( j \), and 

\[ \#\{ \nu \in \mathbb{N}^{2r} \cap (0, H]^{2r} : k \mid \nu, \Delta \neq 0 \} \]

\[ \leq \sum_{j=1}^{2r} \#\{ \nu \in \mathbb{N}^{2r} \cap (0, H]^{2r} : k \mid \Delta_j, \Delta_j \neq 0 \}. \]

However \( |\Delta_j| \leq H^{2r-1} \), so that there are at most \( 2H^{2r-1}k^{-1} \) possibilities for \( \Delta_j \). For each such choice of \( \Delta_j \) there are at most \( 2d(|\Delta_j|) \ll \epsilon, r H^r \) possibilities for each of its divisors \( n_i \) \( n_j \). Thus, taking account of the \( O(H) \) possibilities for \( \Delta_j \) itself, we find that 

\[ \#\{ \nu \in \mathbb{N}^{2r} \cap (0, H]^{2r} : k \mid \Delta_j, \Delta_j \neq 0 \} \ll \epsilon, r H^{2r-1}k^{-1}(H^r)^{2r-1}H. \]

After replacing \( \epsilon \) by \( \epsilon/(2r-1) \) we see that this is \( O_{\epsilon, r}(H^{2r+1}k^{-1}) \). Since this is satisfactory for (7.1) the proof of Lemma 8 is complete.
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