EXTENSIONS OF THEOREMS OF RATTRAY AND MAKEEV

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Abstract. We consider extensions of the Rattray theorem and two Makeev’s theorems, showing that they hold for several maps, measures, or functions simultaneously, when we consider orthonormal \( k \)-frames in \( \mathbb{R}^n \) instead of orthonormal basis (full frames).

We also present new results on simultaneous partition of several measures into parts by \( k \) mutually orthogonal hyperplanes.

In the case \( k = 2 \) we relate the Rattray and Makeev type results with the well known embedding problem for projective spaces.

1. Introduction

In this paper we consider extensions of the following results of Rattray and Makeev:

- any odd continuous map \( S^{n-1} \to S^{n-1} \) maps some orthonormal basis to an orthonormal basis, the Rattray theorem \[20\];
- for any absolutely continuous probabilistic measure \( \mu \) in \( \mathbb{R}^n \) there exist \( n \) mutually orthogonal hyperplanes \( H_1, \ldots, H_n \) such that any two of them partition \( \mu \) into 4 equal parts, the Makeev theorem \[17, Theorem 4\].

These results share a common family of possible solutions, the manifold of all orthonormal basis \( O(n) \) in \( \mathbb{R}^n \). Moreover, they can be seen as a consequence of a single result, Theorem \[1\] proved implicitly already in \[20\].

A continuous function \( f : S^{n-1} \times S^{n-1} \to \mathbb{R} \) will be called:

- (a) odd, if for any \( x, y \in S^{n-1} \)
  \[ f(-x, y) = -f(x, y), \quad f(x, -y) = -f(x, y); \]

- (b) symmetric, if for any \( x, y \in S^{n-1} \)
  \[ f(x, y) = f(y, x). \]

**Theorem 1.** Suppose \( f : S^{n-1} \times S^{n-1} \to \mathbb{R} \) is an odd and symmetric function. Then there exists an orthonormal basis \( (e_1, \ldots, e_n) \in O(n) \) such that for any \( i < j \)

\[ f(e_i, e_j) = 0. \]

**Proof.** Consider a particular case when \( f(x, y) \) is a generic symmetric bilinear form. It follows from the diagonalization theorem in linear algebra that the required orthonormal basis \( e_1, \ldots, e_n \) exists and is unique modulo the action of the group \( W_n = (\mathbb{Z}_2)^n \rtimes \Sigma_n \subset O(n) \). Here the group \( W_n \) acts on basis \( (e_1, \ldots, e_n) \in O(n) \) by

\[ \varepsilon_i \cdot (e_1, \ldots, e_n) = (e'_1, \ldots, e'_n) \text{ where } e'_j = \begin{cases} -e_j & \text{for } j = i \\ e_j & \text{for } j \neq i \end{cases} \]

for the generators \( e_1, \ldots, e_n \) of the component \( (\mathbb{Z}_2)^n \) and by

\[ \pi \cdot (e_1, \ldots, e_n) = (e_{\pi(1)}, \ldots, e_{\pi(n)}) \]

for the permutation \( \pi \in \Sigma_n \) from the symmetric group component of the group \( W_n \).

Let us show that:

- the differential of the corresponding system of equations evaluated at the solution \( e_1, \ldots, e_n \) is nonzero, and

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the solution set represents a nonzero element of the 0-homology $H_0(O(n)/W_n;\mathbb{F}_2)$.

Suppose the base vector $e_i$ has coordinates $b_{ij}$, and

$$f(x, y) = \sum_i \lambda_i x_i y_i$$

in the coordinate representation. Since $f$ is a generic symmetric bilinear form we can assume that $\lambda_1, \ldots, \lambda_n$ are distinct real numbers. The solution is $b_{ij} = \delta_{ij}$, and its first order deformation is $b_{ij} = \delta_{ij} + s_{ij}$, where $s_{ij}$ is a skew symmetric $n \times n$ matrix. Consider

$$f(e_k, e_l) = \sum_i \lambda_i b_{ik} b_{li}.$$ 

The linear part, with respect to $s_{ij}$, is

$$df(e_k, e_l) = \sum_i \lambda_i \delta_{ik} s_{il} + \sum_i \lambda_i s_{ik} \delta_{il} = \lambda_k s_{kl} + \lambda_l s_{lk} = (\lambda_k - \lambda_l)s_{kl}.$$ 

Since all values $\lambda_k - \lambda_l$ are nonzero, that the differentials $df(e_k, e_l)$ give together a bijective map from the space of skew symmetric matrices to the space of all symmetric expressions of the form $s_{kl}$ for $k \neq l$.

Since any $f$ can be $W_n$-deformed (by a convex combination) to this particular case, it follows that for generic $f$ the solution set represents the generator of $H_0(O(n)/W_n;\mathbb{F}_2)$ (and is nonempty). Therefore, the solution set must be nonempty for all other $f$ by compactness considerations. □

In this paper we consider the following generalized problems of Rattray and Makeev type.

**Generalized Rattray problem.** Determine the set $R_{odd}^{orth} \subset \mathbb{N}^3$ $[R_{odd,sym}^{orth} \subset \mathbb{N}^3]$ of all triples $(n, m, k)$ with the property that for any collection $f_1, \ldots, f_m$ of $m$ odd [and symmetric] functions $S^{n-1} \times S^{n-1} \to \mathbb{R}$ there exists an orthonormal $k$-frame $(e_1, \ldots, e_k) \in V_{n,k}$ such that for any $1 \leq l \leq m$ and $1 \leq i < j \leq k$

$$f_l(e_i, e_j) = 0.$$ 

Here $V_{n,k}$ stands for the Stiefel manifold of all orthonormal $k$-frames in $\mathbb{R}^n$.

This problem has a natural variation when the requirement for the vectors $e_1, \ldots, e_k$ to be orthonormal is dropped. Determine the set $R_{odd} \subset \mathbb{N}^3$ $[R_{odd,sym} \subset \mathbb{N}^3]$ off all triples $(n, m, k)$ with the property that for any collection $f_1, \ldots, f_m$ of $m$ odd [and symmetric] functions $S^{n-1} \times S^{n-1} \to \mathbb{R}$ there exist $k$ unit vectors $e_1, \ldots, e_k$ such that for any $1 \leq l \leq m$ and $1 \leq i < j \leq k$

$$f_l(e_i, e_j) = 0.$$ 

An elementary observation is that $R_{odd}^{orth} \subset R_{odd}^{orth}$ $[R_{odd,sym}^{orth} \subset R_{odd,sym}]$ and

$$(n, m, k) \in R_{odd} \Rightarrow (n, m - 1, k) \in R_{odd}^{orth}$$

$$(n, m, k) \in R_{odd,sym} \Rightarrow (n, m - 1, k) \in R_{odd,sym}^{orth}$$

by putting inner product on $\mathbb{R}^n$ for $f_m$.

**Generalized Makeev problem.** Let $H = \{x \in \mathbb{R}^n \mid \langle x, v \rangle = \alpha\}$ be an affine hyperplane in $\mathbb{R}^n$. Here $v$ is a vector in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$ some constant. The affine hyperplane $H$ determines two open halfspaces

$$H^- = \{x \in \mathbb{R}^n \mid \langle x, v \rangle < \alpha\}$$

and $H^+ = \{x \in \mathbb{R}^n \mid \langle x, v \rangle > \alpha\}$.

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ be an arrangement of affine hyperplanes in $\mathbb{R}^d$. An orhant of the arrangement $\mathcal{H}$ is an intersection of halfspaces $\mathcal{O} = H_{1,\alpha_1}^+ \cap \cdots \cap H_{k,\alpha_k}^+$, for some $\alpha_j \in \mathbb{Z}_2$. For convenience we assume that $\mathbb{Z}_2 = \{\{+1, -1\}, \}$ with obvious abbreviation $H_{+1}^+ = H^+$ and $H_{-1}^- = H^-$. There are $2^k$ orthants determined by $\mathcal{H}$. The orthants are not necessary non-empty. They can be indexed by elements of the group $(\mathbb{Z}_2)^k$ in a natural way.

Let $\mu$ be an absolutely continuous probabilistic measure on $\mathbb{R}^n$. The arrangement $\mathcal{H}$ equiparts the measure $\mu$ if for each orthant $\mathcal{O}$ determined by the arrangement $\mu(\mathcal{O}) = \frac{1}{2^k} \mu(\mathbb{R}^n)$.

Generalized Makeev problem is to determine the set $M \subset \mathbb{N}^4$ $[M^{orth} \subset \mathbb{N}^4]$ of all quadruples $(n, m, k, l)$, where $1 \leq l \leq k$, with the property that for every collection of $m$ absolutely continuous probabilistic measures $\mu_1, \ldots, \mu_m$ on $\mathbb{R}^n$ there exist $k$ [mutually orthogonal] hyperplanes $H_1, \ldots, H_k$ such that any $l$ of them equipart all the measures.

It is obvious that $M^{orth} \subset M$. Moreover, by taking $\mu_m$ to be the uniform probability measure on the unit ball in $\mathbb{R}^n$ we can derive that

$$(n, m, k, l) \in M \Rightarrow (n, m - 1, k, l) \in M^{orth}.$$
The generalized Makeev problem for \( l = k \) is known as a generalized Grünbaum mass partition problem as introduced by Grünbaum in [12, 4, Remarks (v)] and further studied by Ramos in [19] and Mani-Levitska, S. Vrećica, R. Živaljević in [16].

2. Statement of main results

Let \( A = F_2[t_1, \ldots, t_k] \) denote the polynomial algebra with variables \( t_1, \ldots, t_k \) of degree 1. Then

\[
w_1 = t_1 + \cdots + t_k, \quad w_k = t_1 t_2 \cdots t_k
\]

are elementary symmetric polynomials in \( A \) with respect to permutation of variables. Set for \( l \geq 1, \)

\[
\bar{w}_l = \sum_{i_1, i_2, \ldots, i_k \geq 0 \atop 1 + 2 i_2 + \cdots + k i_k = l} \binom{i_1 + \cdots + i_k}{i_1, i_2, \ldots, i_k} w_1^{i_1} \cdots w_k^{i_k},
\]

where \( \binom{i_1 + \cdots + i_k}{i_1, i_2, \ldots, i_k} \) stands for \( \frac{(i_1 + \cdots + i_k)!}{i_1! \cdots i_k!} \) modulo 2.

2.1. Rattray type results. These results give sufficient conditions for a triple \((n, m, k)\) to be in \( R_\ast \) and can be formulated in the following way.

**Theorem 2.** Let \((n, m, k) \in \mathbb{N}^3\). Then

(a) \( \prod_{1 \leq i < j \leq k} (t_i + t_j)^{2m} \notin \langle t_1^n, \ldots, t_k^n \rangle \) \( \implies (n, m, k) \in R_{\text{odd}} \),

(b) \( \prod_{1 \leq i < j \leq k} (t_i + t_j)^m \notin \langle t_1^n, \ldots, t_k^n \rangle \) \( \implies (n, m, k) \in R_{\text{odd, sym}} \),

(c) \( \prod_{1 \leq i < j \leq k} (t_i + t_j)^{2m} \notin \langle \bar{w}_{n-k+1}, \ldots, \bar{w}_n \rangle \) \( \implies (n, m, k) \in R_{\text{odd}} \),

(d) \( \prod_{1 \leq i < j \leq k} (t_i + t_j)^m \notin \langle \bar{w}_{n-k+1}, \ldots, \bar{w}_n \rangle \) \( \implies (n, m, k) \in R_{\text{odd, sym}} \).

**Remark.** The degree of the polynomial

\[
\prod_{1 \leq i < j \leq k} (t_i + t_j) = \det \left( t_{i,j}^{j-1} \right)_{i,j=1}^k
\]

is at most \( \frac{1}{2} k(k - 1) \) and degree of each variable is at most \( k - 1 \). Therefore,

\[
(1) \quad (k - 1)m + n \implies \prod_{1 \leq i < j \leq k} (t_i + t_j)^m \notin \langle t_1^n, \ldots, t_k^n \rangle \implies (n, m, k) \in R_{\text{odd, sym}}.
\]

Similarly, \( 2(k - 1)m + n \) implies \((n, m, k) \in R_{\text{odd}}\).

**Remark.** Direct application of the criterion (d) of the theorem, for example, implies that \((3, 2, 2), (4, 1, 2), (4, 2, 2), (5, m, 2)\) for \(1 \leq m \leq 6\) and \((5, 1, 3)\) are elements of \( R_{\text{odd, sym}} \). The most striking example is that \((5, 6, 2) \in R_{\text{odd, sym}} \) since the triple does not fulfill even the inequality bound from the previous remark for being element of \( R_{\text{odd, sym}} \). The fact \((5, 6, 2) \in R_{\text{odd, sym}} \) is the consequence of

\[
(t_1 + t_2)^6 = t_0^6 + t_1^2 t_2^4 + t_1^4 t_2^2 + t_0^2 \notin \langle \bar{w}_4, \bar{w}_5 \rangle
\]

where

\[
\bar{w}_4 = w_1^4 + w_2^4 + w_2^2 = t_1^4 + t_1^2 t_2^2 + t_1^2 t_2 + t_2^4,
\]

\[
\bar{w}_5 = w_1^5 + w_1 w_2^2 = t_1^5 + t_1 t_2^4 + t_1^4 t_2 + t_1^2 t_2 + t_2^5,
\]

and \( w_1 = t_1 + t_2, w_2 = t_1 t_2 \).

Let us present some immediate consequences of Theorem 2 that generalize results from [13].

**Corollary 3.** Let \((n, k, m) \in R_{\text{odd, sym}} \).

(a) For every collection \( \phi_1, \ldots, \phi_m \) of \( m \) odd maps \( S^{n-1} \to S^{n-1} \) there exists an orthonormal \( k \)-frame \( (e_1, \ldots, e_k) \in V_n^k \) such that for any \( 1 \leq l \leq m \) the set \( \{\phi_l(e_1), \ldots, \phi_l(e_k)\} \) is an orthonormal frame too.

(b) For every collection \( g_1, \ldots, g_m \) of \( m \) continuous even functions \( \mathbb{R}^n \to \mathbb{R} \) there exists an orthonormal \( k \)-frame \( (e_1, \ldots, e_k) \in V_n^k \) such that for any \( 1 \leq l \leq m \) and \( 1 \leq i < j \leq k \)

\[
g_l(e_i + e_j) = g_l(e_i - e_j).
\]

**Proof.** For the first claim take \( f_i(x, y) = (\phi_l(x), \phi_l(y)) \) and apply Theorem 2 while for the second one take \( f_i(x, y) = g_l(x + y) - g_l(x - y) \).

□

In some particular cases the obvious inequality bound [1] can be substantially improved by more precise cohomology computations.
Theorem 4. Let \( n \in \mathbb{N} \) and \( P(n) = \min \{ 2^s \mid s \in \mathbb{N}, 2^s \geq n \} \). Then
\[
P(n) \geq m + 2 \iff n \geq \frac{1}{2}P(m + 2) + 1 \implies (n, m, 2) \in \mathcal{R}^\text{orth}_{\text{odd, symm}}.
\]

A further improvement of this result is possible, relating the Rattray problem for 2-frames to the famous problem of embedding of projective spaces into a Euclidean space.

Theorem 5. If \( \mathbb{R}^{m-1} \) cannot be embedded into \( \mathbb{R}^m \) because of the “deleted square obstruction”, then
\[
(n, m, 2) \in \mathcal{R}^\text{orth}_{\text{odd, symm}}.
\]

Remark. The deleted square obstruction for an embedding \( M \to \mathbb{R}^m \) is the obstruction to the existence of a \( \mathbb{Z}_2 \)-equivariant map \( (M \times M) \setminus \Delta(M) \to S^{m-1} \). Here \( \mathbb{Z}_2 \) acts on the deleted square \( (M \times M) \setminus \Delta(M) \) by interchanging coordinates and on \( S^{m-1} \) antipodally. The Haefliger theorem [13] states that in the range \( m \geq 2n/3 \) (the metastable range) this is the only obstruction for embedding. The results in [9] (see also the table [6]) for some low-dimensional cases show that asymptotically the required inequality for embedding of the projective space has the form \( m \geq 2n - O(\log n) \), i.e., falls into the metastable range. It follows that for large enough \( n \) the condition \( (n, m, 2) \in \mathcal{R}^\text{orth}_{\text{odd, symm}} \) also has the asymptotic form \( m \leq 2n - O(\log n) \).

Let us state more results in case \( k = 3 \). If we want to calculate in mod 2 equivariant cohomology, we may consider the Sylow subgroup \( W_3^{(2)} = D_8 \times \mathbb{Z}_2 \) (\( D_8 \) is the square group). We obtain the following algebraic criterion.

Theorem 6. Consider the graded algebra \( \mathbb{F}_2[x, y, w, t] \) with \( \dim x = \dim y = \dim t = 1, \dim w = 2 \), and relation \( xy = t. \) Put
\[
\begin{align*}
(1) & \quad w_* = (1 + x + y + w)(1 + t); \\
(2) & \quad \bar{w}_* = (w_* - 1).
\end{align*}
\]

In the above notation, if \( y^m(t^2 + t(x + y) + w)^m \not\in \langle \bar{w}_{n-2}, \bar{w}_{n-1}, \bar{w}_n \rangle \) then \( (n, m, 3) \in \mathcal{R}^\text{orth}_{\text{odd, symm}} \).

Remark. It can be checked “by hand” than \( (3, 1, 3) \in \mathcal{R}^\text{orth}_{\text{odd, symm}} \), i.e., the Rattray theorem for \( n = 3 \) follows from this theorem.

The results of Rattray type can be extended also in the following direction. It can be asked in addition for the “diagonal” values \( f_1(e_i, e_i) \) to be equal.

Theorem 7. Let \( k \) and \( m \) be positive integers. There exists a function \( n : \mathbb{N}^2 \to \mathbb{N} \) such that for every \( n \geq (n, m(k), m) \) and any collection \( f_1, \ldots, f_m \) of \( m \) odd functions \( S^{n-1} \times S^{n-1} \to \mathbb{R} \) there exists an orthonormal \( k \)-frame \( (e_1, \ldots, e_k) \) in \( V^n_m \) such that for any \( 1 \leq t \leq m \) and \( 1 \leq i < j \leq k \)
\[
f_i(e_i, e_j) = 0 \quad \text{and} \quad f_i(e_1, e_1) = \ldots = f_i(e_k, e_k).
\]

Remark. Description of the function \( n(k, m) \) remains a challenging open problem.

The final result of Rattray type we present is the following theorem.

Theorem 8. Let \( \psi : S^{n-1} \to S^{m-1} \) be an odd continuous map and \( 1 \leq k \leq n \). For any linear subspace \( L \subseteq \mathbb{R}^m \) of codimension \( n - k \) there exists an orthonormal \( k \)-frame \( (e_1, \ldots, e_k) \) in \( \mathbb{R}^n \) such that \( (\psi(e_1), \ldots, \psi(e_k)) \) is an orthonormal \( k \)-frame in \( L \).

Remark. This theorem implies that \( m \) must be at least \( n \) (when considered \( k = n \)), i.e. it implies the Borsuk–Ulam theorem.

2.2. Makeev type results. The following theorem gives sufficient conditions for \( (n, m, k, l) \) to be in \( \mathcal{M}^* \).

Theorem 9. Let \( (n, m, k, l) \in \mathbb{N}^4 \). Then
\[
\begin{align*}
\text{a)} & \quad \prod_{1 \leq s_1, \ldots, s_k \leq l} (s_1t_1 + s_2t_2 + \cdots + s_k t_k)^m \not\in \langle t_{1}^{n+1}, \ldots, t_{k}^{n+1} \rangle \implies (n, m, k, l) \in \mathcal{M}, \\
\text{b)} & \quad \prod_{1 \leq s_1, \ldots, s_k \leq l} (s_1t_1 + s_2t_2 + \cdots + s_k t_k)^m \not\in \langle w_{n-k+1}, \ldots, w_n \rangle \implies (n, m, k, l) \in \mathcal{M}^\text{orth}.
\end{align*}
\]

Remark. By considering maximal degree of the test polynomial in every variable we can get a rough bound
\[
n \geq m \left( \sum_{i=0}^{l} \binom{k-1}{i} \right) \implies (n, m, k, l) \in \mathcal{M}.
\]

Remark. Notice that for \( m = 1 \) and \( l = 2 \) algebraic conditions of Theorem 9 (b) and Theorem 2 (d) coincide.
Remark. For $l = k$, the case (a) is equivalent to the main result of the paper by Mani-Levitska, S. Vrečica, R. Živaljević [16, Theorem 39]. They obtained that

$$n \geq 2^{q+k-1} + r \implies (n, 2^q + r, k, k) \in \mathcal{M}$$

where $m = 2^q + r$ and $0 \leq r \leq 2^q - 1$.

Similar to Theorem 5, we prove another particular result on partitioning measures by pairs of hyperplanes. This result is a projective analogue of the “ham sandwich” theorem [22 [21], the concept of “projective measure partitions” is due to Benjamin Matschke (private communication).

Theorem 10. Suppose $\mathbb{R}^{Pn-1}$ cannot be embedded into $\mathbb{R}^m$ because of the “deleted square obstruction”. Let $\mu_0, \ldots, \mu_m$ be $m + 1$ absolutely continuous probabilistic measures on $\mathbb{R}^{Pn-1}$. Then there exists a pair of hyperplanes $H_1, H_2 \subseteq \mathbb{R}^{Pn-1}$, partitioning every measure $\mu_i$ into two equal parts.

Remark. A single hyperplane does not partition a projective space, but two hyperplanes partition it into two parts.

Remark. The condition is asymptotically $m \leq 2n - O(\log n)$, as in Theorem 5

3. Equivariant cohomology of the Stiefel manifold

Let $V_n^k$ denote the Stiefel manifold of all orthonormal $k$-frames in $\mathbb{R}^n$. Any subgroup $G \subseteq O(k)$ acts naturally on $k$-frames by

$$(e_1, \ldots, e_k) \cdot g = \left( \sum_j e_j s_{j1}, \ldots, \sum_j e_j s_{jk} \right)$$

where $(e_1, \ldots, e_k) \in V_n^k$ and $g = (s_{ij})_{j=1}^k \in O(k)$. The action is right, but it transforms in a left action in the usual way $g \cdot (e_1, \ldots, e_k) := (e_1, \ldots, e_k) \cdot g^{-1}$.

In this section we compute the Fadell–Husseini index of the Stiefel manifold $V_n^k$ with the respect to the action of any subgroup $G \subseteq O(k)$ and coefficients $\mathbb{F}_2$, i.e., we determine the generators of the following ideal

$$\text{Index}_{G, \mathbb{F}_2} V_n^k = \ker \left( H^* (G; \mathbb{F}_2) \to H^* (EG \times_G V_n^k; \mathbb{F}_2) \right).$$

In particular, we determine explicitly the index with respect to the subgroup $\mathbb{Z}_2^n$ of diagonal matrices with $\{−1, 1\}$ entries on diagonal. One description of the index $\text{Index}_{\mathbb{Z}_2^n, \mathbb{F}_2} V_n^k$ is given in the initial paper of Fadell and Husseini [10, Theorem 3.16, page 78].

3.1. The cohomology of the Stiefel manifold $V_n^k$ with $\mathbb{F}_2$ coefficients is the quotient algebra (consult [6])

$$H^* \left( V_n^k; \mathbb{F}_2 \right) = \mathbb{F}_2[\epsilon_{n-k}, \ldots, \epsilon_{n-1}] / \mathcal{J}_n^k$$

where $\deg \epsilon_i = i$ and $\mathcal{J}_n^k$ is the ideal generated by the relations

$$\epsilon_i^2 = \epsilon_{2i} \quad \text{for} \ 2i \leq n - 1$$

$$\epsilon_i^2 = 0 \quad \text{for} \ 2i \geq n.$$

In what follows, for a vector bundle $F \to \xi \to B$ we denote by $w_i(\xi) \in H^i(B; \mathbb{F}_2)$ the associated Stiefel–Whitney classes, by $\bar{w}_i(\xi) \in H^i(B; \mathbb{F}_2)$ its dual Stiefel–Whitney classes, $i \geq 0$. There is a relation between these classes expressed via the total class by $w \cdot \bar{w} = 1$ or particularly for $l \geq 1$ by

$$\bar{w}_l(\xi) = \sum_{i_1 + 2i_2 + \cdots + ki_k = l} \left( \begin{array}{c} i_1 + \cdots + i_k \\ i_1 \ i_2 \ \ldots \ i_k \end{array} \right) w_1^{i_1}(\xi) \cdots w_k^{i_k}(\xi).$$

Let us recall that:

(a) the Grassmann manifold $G^k(\mathbb{R}^\infty)$ of all $k$-flats in $\mathbb{R}^\infty$ is the classifying space of the group $O(k)$ and we denote $G^k(\mathbb{R}^\infty)$ also by $BO(k)$,

(b) the Stiefel manifold $V^\infty_k$ of all $k$-frames in $\mathbb{R}^\infty$ as a contractible free $O(k)$ space serves as a model for $EO(k)$,
(c) the associated canonical bundle:

\[ \mathbb{R}^k \rightarrow \gamma^k \rightarrow \text{Grassmanian} \]

can be seen as a Borel construction of the O(k)-space \( \mathbb{R}^k \) [where the action is given by the matrix multiplication from the left]:

\[ \mathbb{R}^k \rightarrow \text{EO}(k) \times_{O(k)} \mathbb{R}^k \rightarrow \text{BO}(k) \]

(d) the cohomology of the Grassmanian \( \text{Grassmanian} \) \( \approx \text{BO}(k) \) with coefficients in \( \mathbb{F}_2 \) is the polynomial algebra generated by the Stiefel–Whithey classes \( w_1, \ldots, w_k \) of the canonical vector bundle \( \gamma^k \):

\[ H^*(\text{BO}(k); \mathbb{F}_2) = \mathbb{F}_2 [w_1, \ldots, w_k] \]

Now we state a very useful result from [6] (see also [15, Theorem 3.3]).

**Proposition 11.** Let \( (E^r_{i,*}, d_i) \geq 2 \) denote the Leray–Serre spectral sequence associated with the Borel construction

\[ \mathbb{R}^k \rightarrow \text{EO}(k) \times_{O(k)} \mathbb{R}^k \rightarrow \text{BO}(k) . \]

Then

\[ \text{Index}_{O(k), \mathbb{F}_2} V^k_n = (\bar{w}_{n-k+1}, \ldots, \bar{w}_n) \subset \mathbb{F}_2 [w_1, \ldots, w_k] \]

where \( \bar{w}_i = \bar{w}_i (\gamma^k) = d_{i-1} (e_{i-1}) \).

3.2. The Borel construction is a functorial construction and therefore there is a morphism of fiber bundles induced by the inclusion \( i : G \subseteq O(k) \):

\[ \text{EO}(k) \times_G V^k_n \rightarrow \text{EO}(k) \times_{O(k)} V^k_n \]

\[ \pi \quad \mu \]

\[ \text{BG} \quad \text{B} \]

\[ \text{BO}(k) \]

In the bundle on the left, \( \text{EO}(k) \) is used as a model for \( EG \). The action of \( O(k) \) on the Stiefel manifold \( V^k_n \) is free. Therefore, the \( E^{p,q}_r \)-term of the Leray–Serre spectral sequence for the fibration \( \text{EO}(k) \times_{O(k)} V^k_n \rightarrow \text{BO}(k) \) has to vanish for \( p + q > \dim V^k_n \). Furthermore, \( O(k) \) acts trivially on the cohomology \( H^*(V^k_n; \mathbb{F}_2) \) and so by Proposition 11 we have that \( d_i (e_i) = \bar{w}_{i+1} \) for \( n - k \leq i \leq n - 1 \). Here \( d_i \) denotes the \( i \)-th differential of the Leray–Serre spectral sequence. The morphism of the bundles we considered induces a morphism of the associated Leray–Serre spectral sequences as well. The morphism in the \( E_2 \)-term on the 0-column is the identity and on the 0-row determines the restriction morphism \( \ast = \text{res}_G(k) \). Thus,

\[ \text{Index}_{G, \mathbb{F}_2} V^k_n = \ker \pi^* = \text{res}_G^{O(k)} (\ker \mu^*) = \text{res}_G^{O(k)} ([\bar{w}_{n-k+1}, \ldots, \bar{w}_n]) \]

\[ = [\text{res}_G^{O(k)} (\bar{w}_{n-k+1})], \ldots, \text{res}_G^{O(k)} (\bar{w}_n) \].

We have proved the following claim.

**Proposition 12.** \( \text{Index}_{G, \mathbb{F}_2} V^k_n = [\text{res}_G^{O(k)} (\bar{w}_{n-k+1}), \ldots, \text{res}_G^{O(k)} (\bar{w}_n)] \).

3.3. In the final step we identify the restriction morphism \( \text{res}_G^{O(k)} \). Consider \( \mathbb{R}^k \) as an \( O(k) \)-space where the action is given by the left matrix multiplication. The inclusion \( i : G \subseteq O(k) \) gives to \( \mathbb{R}^k \) the structure of \( G \)-space. Again, there is a morphism of associated Borel constructions, which in this case is also a morphism of vector bundles:

\[ \text{EO}(k) \times_G \mathbb{R}^k \rightarrow \text{EO}(k) \times_{O(k)} \mathbb{R}^k \]

\[ \phi \quad \psi \]

\[ \text{BG} \quad \text{B} \]

\[ \text{BO}(k) \]

The naturality of the Stiefel–Whitney classes implies that

\[ w_i (\text{EO}(k) \times_G \mathbb{R}^k) = \ast (w_i) = \text{res}_G^{O(k)} (w_i) \]
and consequently
\[ \bar{w}_i(EO(k) \times_G \mathbb{R}^k) = \text{res}^{O(k)}_G(\bar{w}_i). \]

Thus we have proved the following fact.

**Proposition 13.** Index\(_{G,F_2} V_n^k = \langle \bar{w}_{n-k+1}(EO(k) \times_G \mathbb{R}^k), \ldots, \bar{w}_n(EO(k) \times_G \mathbb{R}^k) \rangle.\]

3.4. Let \( G = \mathbb{Z}_2^k \) be the subgroup of diagonal matrices with \( \{ -1, 1 \} \) entries. Let \( A := H^* (\mathbb{Z}_2^k; \mathbb{F}_2) = \mathbb{F}_2[t_1, \ldots, t_k] \) be the polynomial algebra with variables \( t_1, \ldots, t_k \) of degree 1.

It is well known that the \( k \)-dimensional real \( \mathbb{Z}_2^k \)-representation \( \mathbb{R}^k \) can be decomposed into the sum of \( 1 \)-dimensional irreducible real \( \mathbb{Z}_2^k \)-representation. The total Stiefel–Whitey class of \( EO(k) \times_{\mathbb{Z}_2^k} \mathbb{R}^k \) is given by

\[ w\left( EO(k) \times_{\mathbb{Z}_2^k} \mathbb{R}^k \right) = \prod_{i=1}^{k} (1 + t_i) = 1 + \omega_1 + \cdots + \omega_k \]

where \( \omega_i \) denotes both: the elementary symmetric polynomial of degree \( i \) in variables \( t_1, \ldots, t_k \) and the \( i \)-th Stiefel–Whitey class of \( w_i\left( EO(k) \times_{\mathbb{Z}_2^k} \mathbb{R}^k \right) \). For example, \( \omega_1 = t_1 + t_2 + \cdots + t_k \) while \( \omega_k = t_1 t_2 \cdots t_k \).

Finally, we obtain the following result.

**Proposition 14.** Let \( \bar{\omega}_1 = \sum_{i_1+i_2+\cdots+i_k \geq 0} \left( \begin{array}{c} i_1+i_2+\cdots+i_k \\ i_1, i_2, \ldots, i_k \end{array} \right) \omega_1^{i_1} \cdots \omega_k^{i_k}, \text{ for } l \geq 1, \text{ and then} \]

\[ \text{Index}_{\mathbb{Z}_2^k} V_n^k = \langle \bar{\omega}_{n-k+1}, \ldots, \bar{\omega}_n \rangle \subset A. \]

4. PROOF OF RATTRAY TYPE RESULTS

4.1. The proofs of these results will be done via the configuration space / test map method. There are two different natural configuration spaces of interest:

\[ X = \left( S^{n-1} \right)^k = \text{the space of all collections of } k \text{ vectors on the sphere } S^{n-1}, \]
\[ Y = V_n^k = \text{the space of all orthogonal } k \text{-frames in } \mathbb{R}^n. \]

The group \( W_k = (\mathbb{Z}_2)^k \rtimes \Sigma_k \subset O(k) \) acts naturally on both configurations spaces. For the generators \( \varepsilon_1, \ldots, \varepsilon_n \) of the component \( (\mathbb{Z}_2)^n \) and \( (e_1, \ldots, e_k) \in X \) or \( Y \) the action is given by

\[ \varepsilon_i \cdot (e_1, \ldots, e_k) = (e_1', \ldots, e_k') \text{ where } e_i' = -e_i \text{ and } e_j' = e_j \text{ for } j \neq i, \]

and for the permutation \( \pi \in \Sigma_k \) by

\[ \pi \cdot (e_1, \ldots, e_k) = (e_{\pi(1)}, \ldots, e_{\pi(k)}). \]

Let us consider the space \( M_k \) of all real \( k \times k \)-matrices as a real \( O(k) \)-representation with respect to the action

\[ m \mapsto gmg^{-1} \]

where \( m \in M_k \) and \( g \) is \( k \times k \)-matrix representing an element of \( O(k) \). Then \( M_k \) has a structure of a real \( W_k \)-representation via the inclusion map \( W_k \hookrightarrow O(k) \). Consider following real vector subspaces of \( M_k \):

\[ R_k \text{ of all } k \times k \text{ symmetric matrices with zeros on the diagonal,} \]
\[ U_k \text{ of all } k \times k \text{ matrices with zeros on the diagonal, and} \]
\[ I_k \text{ of all } k \times k \text{ matrices with zeros outside the diagonal and trace zero.} \]

These are all real \( W_k \)-subrepresentations of \( M_k \). Moreover, when we consider only the subgroup \( (\mathbb{Z}_2)^k \) there is a decomposition \( U_k \cong R_k \oplus R_k \) of \( (\mathbb{Z}_2)^k \)-representation.

For an odd [and symmetric] function \( f : S^{n-1} \times S^{n-1} \rightarrow \mathbb{R} \) and \( k \)-vectors [\( k \)-frame] \( (e_1, \ldots, e_k) \), we denote by:

- \( \mu f (e_1, \ldots, e_k) \in U_k [\mu f (e_1, \ldots, e_k) \in R_k] \) the matrix given by entries

\[ (\mu f (e_1, \ldots, e_k))_{ij} = \begin{cases} f(e_i, e_j), & i \neq j \\ 0, & i = j. \end{cases} \]

- \( \eta f (e_1, \ldots, e_k) \in I_k \) the matrix given by entries

\[ (\eta f (e_1, \ldots, e_k))_{ij} = \begin{cases} f(e_i, e_j) - c, & i = j \\ 0, & i \neq j, \end{cases} \]

where \( c = \frac{1}{k} (f(e_1, e_1) + \cdots + f(e_k, e_k)) \).
4.2. Proof of Theorem 2. Let \((n, m, k) \in \mathbb{N}^3\) and \(f_1, \ldots, f_m\) be a collection of \(m\) odd [and symmetric] functions \(S^{n-1} \times S^{n-1} \to \mathbb{R}\). Let us introduce the test maps for the Rattray problems:

\[
\tau_{\text{odd}} : X \to U_k^{\oplus m}, \quad \tau_{\text{odd,sym}} : X \to R_k^{\oplus m}, \quad \tau_{\text{orth}} : Y \to U_k^{\oplus m}, \quad \tau_{\text{orth,sym}} : Y \to R_k^{\oplus m}.
\]

All four test maps are defined by the same formula

\[
(e_1, \ldots, e_k) \mapsto (\mu_{f_1}(e_1, \ldots, e_k))^{m}_{f=1}
\]

assuming appropriate domains and codomains. Have in mind that the test maps are functions of the \(f_i\), \(i = 1, \ldots, m\), even we abbreviate this from notation. The test maps are all \(W_k\)-equivariant maps and moreover have the following obvious but very important properties: If for every collection \(f_1, \ldots, f_m\) of \(m\) odd [and symmetric] functions \(S^{n-1} \times S^{n-1} \to \mathbb{R}\)

- \(0 \in U_k^{\oplus m} \in \tau_{\text{odd}}(X)\), then \((n, m, k) \in \mathcal{R}_{\text{odd}}\),
- \(0 \in U_k^{\oplus m} \in \tau_{\text{odd,sym}}(X)\), then \((n, m, k) \in \mathcal{R}_{\text{odd,sym}}\),
- \(0 \in U_k^{\oplus m} \in \tau_{\text{orth}}(Y)\), then \((n, m, k) \in \mathcal{R}_{\text{orth}}\),
- \(0 \in U_k^{\oplus m} \in \tau_{\text{orth,sym}}(Y)\), then \((n, m, k) \in \mathcal{R}_{\text{orth,sym}}\).

Let us assume that Theorem 2 fails in each case. This means that for a specific collection \(f_1, \ldots, f_m\) of \(m\) odd [and symmetric] functions \(0 \in U_k^{\oplus m}\) or \(0 \in R_k^{\oplus m}\) is not in the image of any of the test maps. Therefore, we have constructed the following \(W_k\)-equivariant maps

\[
X \to U_k^{\oplus m} \setminus \{0\}, \quad X \to R_k^{\oplus m} \setminus \{0\}, \quad Y \to U_k^{\oplus m} \setminus \{0\}, \quad Y \to R_k^{\oplus m} \setminus \{0\},
\]

i.e., after \(W_k\)-equivariant homotopy, the \(W_k\)-equivariant maps

\[
X \to S(U_k^{\oplus m}), \quad X \to S(R_k^{\oplus m}), \quad Y \to S(U_k^{\oplus m}), \quad Y \to S(R_k^{\oplus m}).
\]

Obviously all these maps are \(\mathbb{Z}_2^k\)-equivariant maps, where \(\mathbb{Z}_2^k\) is the diagonal subgroup of \(W_k\).

The basic monotonicity property of the Fadell–Husseini index theory [10] states that when there is a \(G\) map \(A \to B\) between \(G\)-spaces \(A\) and \(B\) there has to be an inclusion of associated indexes \(\text{Index}_G(A) \supseteq \text{Index}_G(B)\). Using the subgroup \(\mathbb{Z}_2^k\) of \(W_k\) the maps (4) induce following inclusions

\[
\text{Index}_{\mathbb{Z}_2^k} X \supseteq \text{Index}_{\mathbb{Z}_2^k} S(U_k^{\oplus m}), \quad \text{Index}_{\mathbb{Z}_2^k} X \supseteq \text{Index}_{\mathbb{Z}_2^k} S(R_k^{\oplus m}), \quad \text{Index}_{\mathbb{Z}_2^k} Y \supseteq \text{Index}_{\mathbb{Z}_2^k} S(U_k^{\oplus m}), \quad \text{Index}_{\mathbb{Z}_2^k} Y \supseteq \text{Index}_{\mathbb{Z}_2^k} S(R_k^{\oplus m}).
\]

We determine all Fadell–Husseini indexes appearing in (5).

Claim. With notation already introduced:

- (a) \(\text{Index}_{\mathbb{Z}_2^k} X = \langle t_1^1, \ldots, t_k^1 \rangle\),
- (b) \(\text{Index}_{\mathbb{Z}_2^k} Y = \langle \omega_{n-k+1}, \ldots, \omega_n \rangle\),
- (c) \(\text{Index}_{\mathbb{Z}_2^k} S(U_k^{\oplus m}) = \langle \prod_{1 \leq a < b \leq k} (t_a + t_b)^m \rangle\),
- (d) \(\text{Index}_{\mathbb{Z}_2^k} S(R_k^{\oplus m}) = \langle \prod_{1 \leq a < b \leq k} (t_a + t_b)^{2m} \rangle\).

Proof. (a) Since the \(\mathbb{Z}_2^k\) action on \(X\) is component-wise antipodal the index of \(X\) is computed in the paper of Fadell and Husseini [10] Example 3.3, p. 76).

(b) This fact is established in Proposition 4.

(c) Let us denote by \(R_{ab}\), for \(1 \leq a < b \leq k\), the 1-dimension real vector subspace of \(R_k\) described by

\[
R_{ab} = \{m \in R_k \mid m_{ij} = 0 \text{ for } (i, j) \notin \{(a, b), (b, a)\} \text{ and } m_{ab} = m_{ba} \in \mathbb{R}\}.
\]

The subspace \(R_{ab}\) is \(\mathbb{Z}_2^k\)-invariant and

\[
\varepsilon_i \cdot m = \begin{cases} 
-m & \text{for } i \in \{a, b\} \\
 m & \text{for } i \in \{1, \ldots, k\} \setminus \{a, b\}
\end{cases}.
\]

Moreover, \(R_k \cong \bigoplus_{1 \leq a < b \leq k} R_{ab}\) as a \(\mathbb{Z}_2^k\)-module. Since the Fadell–Husseini index of a sphere in this case is a principal ideal generated by the Euler class \(=\) the top Stiefel–Whitney class of the vector bundle

\[
R_k \to \mathbb{E}\mathbb{Z}_2^k \times \mathbb{Z}_2^k R_k \to B\mathbb{Z}_2^k
\]

and

\[
\varepsilon(\mathbb{E}\mathbb{Z}_2^k \times \mathbb{Z}_2^k R_k) = \prod_{1 \leq a < b \leq k} \varepsilon(\mathbb{E}\mathbb{Z}_2^k \times \mathbb{Z}_2^k R_{ab}) = \prod_{1 \leq a < b \leq k} (t_a + t_b).
\]
For details consult [5]. Proof of Proposition 3.11. It follows directly that
\[ \text{c}(E \mathbb{Z}_2^k \times \mathbb{Z}_2^l R_k^{\oplus m}) = \prod_{1 \leq a < b \leq k} (t_a + t_b)^m \]
and consequently Index_{\mathbb{Z}_2^k \times \mathbb{Z}_2^l} S (R_k^{\oplus m}) = \langle \prod_{1 \leq a < b \leq k} (t_a + t_b)^m \rangle.

(d) Follows from the decomposition \(U_k \cong R_k \oplus R_k\) of \(\mathbb{Z}_2^k\)-module. \(\square\)

Now, the inclusions (d) with just determined indexes imply that:
\[ \prod_{1 \leq a < b \leq k} (t_a + t_b)^m \in \langle t_1, \ldots, t_k \rangle, \quad \prod_{1 \leq a < b \leq k} (t_a + t_b)^m \in \langle t_1, \ldots, t_k \rangle, \quad \prod_{1 \leq a < b \leq k} (t_a + t_b)^m \in \langle \bar{\omega}_{n-k+1}, \ldots, \bar{\omega}_n \rangle \]
This gives a contradiction with the assumptions of Theorem 2. Therefore, all claims of Theorem 2 hold.

4.3. Proof of Theorem 3. Before starting the proof let us once more isolate an important property of Stiefel–Whitney classes already used in the proof of Theorem 2. Let \(H\) be a subgroup of a group \(G\) and \(V\) a real \(G\)-representation. Then the following equality between the total Stiefel–Whitney classes holds:
\[ w(EH \times_H V) = \text{res}^G_H (w(EG \times_G V)) \iff w_1(EH \times_H V) = \text{res}^G_H (w_1(EG \times_G V)) \] for all \(i \geq 1\) where \(V\) inherits the \(H\)-representation structure from the inclusion map \(H \hookrightarrow G\).

In the proof we use the complete group of symmetries \(W_2 = (\mathbb{Z}_2)^2 \times \mathbb{Z}_2 = ((\varepsilon_1) \times (\varepsilon_2) \times (\sigma)\) which is isomorphic to the dihedral group \(D_8\). The cohomology of the dihedral group \(D_8\) with \(\mathbb{F}_2\) coefficients is given by
\[ H^*(D_8; \mathbb{F}_2) = \mathbb{F}_2[x,y,w]/(xy). \]
where \(\text{deg} x = \text{deg} y = 1\) and \(\text{deg} w = 2\). Consult [1] Section IV.1, page 116 or [5] Section 4.2. In what follows we use the notations introduced in the paper [5] Section 4.3.2. For example subgroup \((\mathbb{Z}_2)^2\) is denoted by \(H_1\), while subgroup \((\sigma)\) is either \(K_4\) or \(K_5\). Let us assume for clarity that \(K_5 = (\sigma)\).

Let us consider \(W_2 = D_8\) and its already introduced representations \(R_2\) and \(\mathbb{R}^2\). Computation of the total Stiefel–Whitney class \(w(E(\mathbb{Z}_2)^2 \times (\mathbb{Z}_2)^2)\) conducted in Section 4.2 when translated into the notation of [5] Section 4.3.2, gives us that
\[ w(EH_1 \times_{H_1} R_2) = 1 + (a + a + b) = 1 + b \]
Moreover, since \(EK_5 \times K_5 R_2\) is a trivial vector bundle
\[ w(EK_5 \times K_5 R_2) = 1 \]
Thus, the restriction diagram presented in [5] Section 4.3.2, equations (26) and (27)] implies that
\[ w(ED_8 \times_{D_8} R_2) = 1 + y. \]
On the other hand, presented in the new notation
\[ w(EH_1 \times_{H_1} \mathbb{R}^2) = (1 + a)(1 + a + b) = 1 + b + a(a + b). \]
The 2-dimensional real \(K_5\)-representation \(\mathbb{R}^2\) can be decomposed into the direct sum \(\mathbb{R}^2 \cong V_0 \oplus V_1\) of the trivial 1-dimensional real \(K_5\)-representation \(V_0\) and the 1-dimensional real \(K_5\)-representation \(V_1\) where the action of generator \(\sigma \in K_5\) is given by \(\sigma \cdot v = -v\), for \(v \in V_1\). Then the total Stiefel–Whitney class is
\[ w(EK_5 \times_{K_5} \mathbb{R}^2) = 1 + t_5. \]
Again the restriction diagram [5] Section 4.3.2, equations (26) and (27)] implies that
\[ w(ED_8 \times_{D_8} \mathbb{R}^2) = 1 + (y + x) + w. \]

Proposition 15. With notation already introduced:
(a) Index_{\mathbb{Z}_2^k} V_n = \langle w_{n-1}(EO(2) \times_{D_8} \mathbb{R}^2), w_n (EO(2) \times_{D_8} \mathbb{R}^2) \rangle \subseteq H^*(D_8, \mathbb{F}_2) where
\[ (1 + w_1(EO(2) \times_{D_8} \mathbb{R}^2) + w_2(EO(2) \times_{D_8} \mathbb{R}^2) + \cdots ) (1 + (y + x) + w) = 1. \]
(b) Index_{\mathbb{Z}_2^k} S (R_2^{\oplus m}) = \langle y^{m} \rangle.
(c) \( y^{m} \notin \langle w_{n-1}(EO(2) \times_{D_8} \mathbb{R}^2), w_n (EO(2) \times_{D_8} \mathbb{R}^2) \rangle \) \(\implies\) \((n, m, 2) \in \mathbb{R}^{\text{orth, odd, sym}}\).
(d) \( y^{m} \notin \langle w_{n-1}(EO(2) \times_{D_8} \mathbb{R}^2), w_n (EO(2) \times_{D_8} \mathbb{R}^2), x \rangle \) \(\implies\) \((n, m, 2) \in \mathbb{R}^{\text{orth, odd, sym}}\).
Proof. (a) Proposition [13] together with the evaluated total Stiefel–Whitney class [7] imply the claim.
(b) From [5] it follows that $\epsilon(ED_8 \times D_2 \times R_2) = y$ and consequently $\epsilon(ED_8 \times D_2 R_2^{\infty m}) = y^m$. Since the Fadell–Husseini index of a sphere in this case is a principal ideal generated by the Euler class [5]. Proof of Proposition 3.11] the claim is proved.
(c) This is a direct consequence of the configuration test map construction presented at the beginning of Section 4.2.
(d) If $y^m$ is not an element of the bigger ideal
$$\langle \bar{w}_{n-1}(EO(2) \times D_n \mathbb{R}^2), w_n (EO(2) \times D_n \mathbb{R}^2), x \rangle$$
it certainly cannot belong to the smaller ideal
$$\langle \bar{w}_{n-1}(EO(2) \times D_n \mathbb{R}^2), w_n (EO(2) \times D_n \mathbb{R}^2) \rangle.$$
The statement follows from (c).

Hence, the final effort is to determine a condition on the integer $m$ such that
$$y^m \notin \langle \bar{w}_{n-1}(EO(2) \times D_n \mathbb{R}^2), w_n (EO(2) \times D_n \mathbb{R}^2), x \rangle$$
or
$$0 \neq y^m \in F_2[y, w]/\langle \bar{w}_{n-1}, \bar{w}_n \rangle$$
where $(1 + y + w)(1 + \bar{w}_1 + \bar{w}_2 + \cdots) = 1$.

If $y$ and $w$ are interpreted as the first and second Stiefel–Whitney class in the cohomology of the Grassmannian $G^2(\mathbb{R}^n)$ we can identify $F_2[y, w]/\langle \bar{w}_{n-1}, \bar{w}_n \rangle$ with $H^*(G^2(\mathbb{R}^n) ; F_2)$. Then our final step coincides with the well known problem of determining the height (maximal nonzero power) of the first Stiefel–Whitney class in the cohomology of the Grassmannian $G^2(\mathbb{R}^n)$. In [14] Proposition 2.6, page 525) the following statement is proved:

Lemma. Let $n \geq 2$, and let $P(n) := 2^s$ be the minimal power of two, satisfying $2^s \geq n$. For the first Stiefel-Whitney class $w_1$ of the Grassmannian $G^2(\mathbb{R}^n)$ holds
$$w_1^{2^s-2} \neq 0 \quad \text{and} \quad w_1^{2^s-1} = 0.$$

Therefore,
$$P(n) = m + 2 \iff n \geq 2 P(m, 2) + 1 \implies (n, m, 2) \in R_{odd,sym}^{orth}.$$

4.4. Proof of Theorem [5]. Consider the Stiefel manifold $V^2_n$ with $D_8$ action on it. We want to know whether $V^2_n$ can be mapped $D_8$-equivariantly to $(R_2)^m \setminus \{0\}$.

Denote by $\sigma_1, \sigma_2, \tau$ the generators of $D_8$, where $\sigma_1$ and $\sigma_2$ reflect the base vectors in $\mathbb{R}^2$, and $\tau$ transposes the base vectors. $R_2$ is the one-dimensional real $D_8$-representation on which $\sigma_1$ and $\sigma_2$ act antipodally, and $\tau$ acts trivially.

Now consider an automorphism of $D_8$, defined by
$$\begin{array}{ccc}
\sigma_1' & = & \sigma_1 \sigma_2 \tau \\
\sigma_2' & = & \tau \\
\tau' & = & \sigma_1.
\end{array}$$

Under this automorphism the representation of $D_8$ on $\mathbb{R}^2$ remains the same (it is sufficient to change the base $e_1' = e_1 + e_2, e_2' = -e_1 + e_2$). The representation $R_2$ is now given by trivial action of $\sigma_1'$ and $\sigma_2'$ and by antipodal action of $\tau'$. Thus, we pass to the space $X_n = V^2_n/(\sigma_1', \sigma_2')$ of all ordered pairs of orthogonal lines through the origin in $\mathbb{R}^n$. This space has the action of $Z_2 = (\tau')$ which permutes the lines. We want to know whether $X$ can be mapped $Z_2$-equivariantly to $\gamma^m \setminus \{0\}$, where $\gamma$ is the unique non-trivial one-dimensional representation of $Z_2$. It is well known that $X$ is homotopy equivalent to the deleted square of the projective space $\mathbb{R}P^{n-1}$, i.e.,
$$X \simeq (\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}) \setminus \Delta(\mathbb{R}P^{n-1}).$$
The existence of a $Z_2$-equivariant map $X \to S(\gamma^m)$ is exactly the “deleted square obstruction” for the embedding of $\mathbb{R}P^{n-1}$ to $\mathbb{R}^m$.

The idea of considering the same automorphism of $D_8$ was used by González and Landweber in [11], where the deleted square obstruction is related to another problem of finding the symmetric topological complexity of the projective space.
4.5. Proof of Theorem\[\] We consider the group $G := W_3^{(2)} = D_8 \times \mathbb{Z}_2$. We already know that

$$H^*(D_8, \mathbb{F}_2) = \mathbb{F}_2[x, y, w]/\langle xy \rangle, \quad H^*(\mathbb{Z}_2, \mathbb{F}_2) = \mathbb{F}_2[t],$$

and therefore $H^*(G, \mathbb{F}_2) = \mathbb{F}_2[x, y, w, t]/\langle xy \rangle$ by the Künneth formula. The Stiefel–Whitney class of the standard $G$-representation on $\mathbb{R}^3$ is

$$w(\mathbb{R}^3) = (1 + x + y + w)(1 + t),$$

and the Euler class of the representation $R_3$ is

$$c(R_3) = y(t^2 + t(x + y) + w),$$

because $\mathbb{R}^3(G) = \mathbb{R}^2(D_8) \oplus \mathbb{R}^1(\mathbb{Z}_2)$ and $R_3(G) = R_2(D_8) \oplus \mathbb{R}^2(\mathbb{Z}_2)$ in the obvious notation. The rest of the proof proceeds in the footsteps of the proof of Theorem\[.\]

4.6. Proof of Theorem\[.\] Before proving Theorem\[.\] we recall some basic facts and results on the following Borsuk–Ulam type problem (consult the book [4]).

**Problem.** Let $G$ be a finite group and $V$ its real representation such that $V^G = \{0\}$. Determine the conditions for the vector bundle

$$EG \times V \to EG$$

to have a $G$-equivariant nonzero section.

The following result for $p$-groups will be used, consult [2,3,4,7].

**Lemma.** Let $G$ be a $p$-group and $V$ its real representation such that $V^G = \{0\}$. Then the image of an equivariant map $f : EG \to V$ intersects $V^G = \{0\}$. Moreover, there exists an integer $n(G, V)$ such that for every free $G$-space $X$ is $(n-1)$-connected where $n \geq n(G, V)$, the image of an equivariant map $f : X \to V$ meets $V^G = \{0\}$.

In order to prove Theorem\[.\] we slightly change the configuration test map construction given at the beginning of this chapter. Let us fix positive integers $k$ and $m$, and consider a collection of $m$ odd functions $f_1, \ldots, f_m$. The test map in this case is the $W_k$-equivariant map $v : Y \to R_k^{\oplus m} \oplus I_k^{\oplus m}$ defined by

$$(e_1, \ldots, e_k) \mapsto (\mu_{f_r}(e_1, \ldots, e_k))_{r=1}^m \oplus (\eta_{f_r}(e_1, \ldots, e_k))_{r=1}^m$$

where $Y$ stands for the Stiefel manifold $V^k_n$ as before. If there exists a positive integer $n = n(k, m)$ such that there is no $W_k$-equivariant map

$$Y \to (R_k^{\oplus m} \oplus I_k^{\oplus m}) \setminus \{0\} \to S (R_k^{\oplus m} \oplus I_k^{\oplus m})$$

then Theorem\[.\] is proved.

Without loss of generality we may increase $n$ and $k$ in such a way that $k$ becomes power of 2. This can be done since we do not need an optimal $n$ and moreover proving the theorem for bigger $k$ and fixed $n$ and $m$ yields the same result for smaller $k$. Now consider the 2-Sylow subgroup $W_k^{(2)}$ of $W_k$. Since the $W_k^{(2)}$-fixed point set of the representation $R_k^{\oplus m} \oplus I_k^{\oplus m}$ is trivial, i.e., $(R_k^{\oplus m} \oplus I_k^{\oplus m})^{W_k^{(2)}} = \{0\}$ the previously presented lemma implies that every map $Y \to R_k^{\oplus m} \oplus I_k^{\oplus m}$ must meet origin. Thus there cannot be any $W_k^{(2)}$-equivariant (and consequently $W_k$-equivariant) map $Y \to S (R_k^{\oplus m} \oplus I_k^{\oplus m})$. This completes the proof of the theorem.

4.7. Proof of Theorem\[.\] Let $\lambda_1, \ldots, \lambda_n$ be independent linear forms defining the subspace $L$ in $\mathbb{R}^m$. In this proof we take $R_k$ to be an $O(k)$-representation where the action is given by the left matrix multiplication. The inclusion $W_k \subseteq O(k)$ gives to $R_k$ also the structure of a $W_k$-representation. Let us denote this $W_k$-representation by $P_k$. Consider the following $W_k$-equivariant maps

- $\phi_0 : V_n^k \to R_k$ given by

$$\phi_0(e_1, \ldots, e_k) = (\psi(e_1), \psi(e_j))_{1 \leq i < j \leq k},$$

- $\phi_r : V_n^k \to P_k$, for $1 \leq r \leq n - k$, given by

$$\phi_r(e_1, \ldots, e_k) = (\lambda_r(\psi(e_1)), \ldots, \lambda_r(\psi(e_k)))$$

for $1 \leq i \leq k$.\[\]
The sum of these maps, the \(W_k\)-equivariant map, \(\phi = \phi_0 \oplus \phi_1 \oplus \cdots \oplus \phi_{n-k} : V^k_n \rightarrow R_k \oplus (P_k)^{n-k}\) has the property that if the image of \(\phi\) meets zero in \(R_k \oplus (P_k)^{n-k}\) then the theorem follows. It is sufficient to show that the Euler class
\[
\varepsilon(R_k \oplus P_k^{n-k}) \in H^*(BW_k; \mathbb{F}_2)
\]
has nonzero image in \(H^*_k(V^k_n; \mathbb{F}_2)\), i.e.,
\[
\varepsilon(R_k \oplus P_k^{n-k}) \not\in \text{Index}_{W_k, X} V^k_n.
\]

Let us prove non-vanishing of the Euler class by counting zeroes of a generic map. We construct another \(W_k\)-equivariant map:
\[
\tau : V^k_n \rightarrow R_k \oplus P_k^{n-k}
\]
with the unique (up to \(W_k\)-action) non-degenerated zero. This will imply that \(\varepsilon(R_k \oplus P_k^{n-k}) \neq 0\) as an element of \(H^*_k(V^k_n; \mathbb{F}_2)\).

Let \(M = R^k \subseteq R^n\) be a standard inclusion, and let \(f(x, y)\) be a symmetric quadratic form, such that \(f|_{M \times M}\) is generic. Put
\[
\tau_0(e_1, \ldots, e_k) = (f(e_i, e_j))_{1 \leq i < j \leq k},
\]
and for \(1 \leq r \leq n-k\)
\[
\tau_r(e_1, \ldots, e_k) = (x_{k+r}(e_1), \ldots, x_{k+r}(e_k)),
\]
where \(x_{k+r}\) are coordinate functions in \(R^n\). Then a unique (up to \(W_k\)-action) basis in \(M\) is mapped by \(\tau\) to zero; because the conditions \(\tau_r(e_1, \ldots, e_k) = 0\) (for \(1 \leq r \leq n-k\)) imply \(e_1, \ldots, e_k \in M\) and condition \(\tau_0(e_1, \ldots, e_k) = 0\) implies that \(f|_{M \times M}\) is diagonal in the basis \((e_1, \ldots, e_k)\) of \(M\). This zero is non-degenerate, because the image of the differential \(d\tau\) at \((e_1, \ldots, e_k)\)
- contains \(R_k\), similar to the proof of the Rattray theorem;
- surjects onto \(P_k^{n-k}\), because in the first order approximation the frame \((e_1 + \delta_1, \ldots, e_k + \delta_k)\) is orthonormal for any \(\delta_1, \ldots, \delta_k \in M^\perp\).

Thus \(0 \neq \varepsilon(R_k \oplus P_k^{n-k}) \in H^*_k(V^k_n; \mathbb{F}_2)\) and the proof is complete.

5. Proof of Makeev type results

5.1. Proof of Theorem \ref{thm:makeev}

Makeev type results will be considered via the classical configuration space / test map scheme used for mass partition problems by hyperplanes, consult \cite{16} or \cite{5} for more details. We consider two different configuration spaces depending whether we require configuration of orthogonal hyperplanes or not.

Let \(R^n\) be embedded in \(R^{n+1}\) by \((x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 1)\). Every oriented affine hyperplane \(H\) in \(R^n\) determines a unique oriented hyperplane \(H^\perp\) through the origin in \(R^{n+1}\) by \(H^\perp \cap R^n = H\). Converse is also true if the hyperplane \(x_{n+1} = 0\) is excluded. Any oriented hyperplane \(H\) in \(R^{n+1}\) passing through the origin is uniquely determined by the unit vector \(v \in S^d\) pointing inside the halfspace \(H^\perp\).

Such a hyperplane we denote also by \(H_v\). Notice that \(H^{\perp}_{-v} = H_v^\perp\). Thus, the space of all oriented affine hyperplanes in \(R^n\) (including two hyperplanes at “infinity”) can be considered to be the sphere \(S^n\). The first configuration space we consider is
\[
X = (S^n)^k = \text{the space of all collections of } k \text{ oriented affine hyperplanes in } R^n.
\]

Let \(\mu\) be an absolutely continuous probabilistic measure on \(R^n\) with connected support. Then the second configuration space \(Y^k = V^k_n\) is shaped by \(\mu\) in the following way: every orthonormal \(k\)-frame \((e_1, \ldots, e_k) \in V^k_n\) determines a unique collection of \(k\) oriented affine hyperplanes \((H_1, \ldots, H_k)\) in \(R^n\) with the property that \(e_i \perp H_i\) and \(\mu(H^+_i) = \mu(H^-_i)\) for all \(1 \leq i \leq k\). This is because for every given direction \(e_i\) there is a unique hyperplane orthogonal to \(e_i\) that partitions \(\mu\) into equal halves. In case \(\mu\) has disconnected support, we may approximate \(\mu\) by a sequence of measures with connected support, prove the theorem in this case, and then go to the limit using the compactness of the following space: for a given \(0 < \varepsilon < 1\) consider the space of hyperplanes \(H\) that partition \(\mu\) into parts \(H^+, H^-\) with difference \(|\mu(H^+) - \mu(H^-)| \leq \varepsilon\).

The group \(W_k = (\mathbb{Z}_2)^k \rtimes \Sigma_k \subset O(k)\) acts on both configuration spaces \(X\) and \(Y\) in the same way as in Section \ref{sec:w-action}.
Before defining the test maps let us introduce a particular \( W_k \) and \((\mathbb{Z}_2)^k\)-representation on the vector space \( \mathbb{R}^2 \) and study its structure. If we assume that the coordinate functions \( x(a_1, \ldots, a_k) \) on \( \mathbb{R}^k \) are indexed by the elements \((a_1, \ldots, a_k)\) of the group \((\mathbb{Z}_2)^k\), then the \( W_k \)-action we consider is given by

\[
((b_1, \ldots, b_k) \cdot \pi) \cdot x(a_1, \ldots, a_k) = x(b_{a_k+1}, \ldots, b_{a_k-1}, b_{a_k})
\]

where \((b_1, \ldots, b_k) \in (\mathbb{Z}_2)^k\) and \(\pi \in \Sigma_k\). The inclusion \((\mathbb{Z}_2)^k \subset W_k\) induces also the structure of \((\mathbb{Z}_2)^k\)-representation on \(\mathbb{R}^2\).

All real irreducible representations of the group \((\mathbb{Z}_2)^k\) are all 1-dimensional. They are completely determined by characters \(\chi : (\mathbb{Z}_2)^k \rightarrow \mathbb{Z}_2\). For \((a_1, \ldots, a_k) \in (\mathbb{Z}_2)^k = \{+1, -1\}^k\), let

\[
V_{a_1, \ldots, a_k} = \text{span}\{v_{a_1, \ldots, a_k}\} \subset \mathbb{R}^2
\]
denotes the 1-dimensional representation given by

\[
\xi_i \cdot v_{a_1, \ldots, a_k} = a_i \cdot v_{a_1, \ldots, a_k}.
\]

Then there is a decomposition of the real \((\mathbb{Z}_2)^k\)-representation

\[
\mathbb{R}^2 \cong \sum_{a_1, \ldots, a_k \in (\mathbb{Z}_2)^k} V_{a_1, \ldots, a_k} \cong V_{+} \oplus \sum_{a_1, \ldots, a_k \in (\mathbb{Z}_2)^k \setminus \{+\ldots+\}} V_{a_1, \ldots, a_k}.
\]

Observe that \(V_{+} \oplus \ldots \oplus V_{+}\) is the trivial 1-dimensional real \((\mathbb{Z}_2)^k\)-representation. In order to simplify further notation let us define for \(1 \leq i \leq j \leq k\) the following \((\mathbb{Z}_2)^k\)-representation

\[
S_{ij} = \sum_{a_1, \ldots, a_k \in (\mathbb{Z}_2)^k \setminus \{+\ldots+\} \atop i \leq (a_1, \ldots, a_k) \leq j} V_{a_1, \ldots, a_k}
\]

where \(s(a_1, \ldots, a_k)\) denotes the number of \(-1\) in the sequence \((a_1, \ldots, a_k)\).

Let \(\mu_1, \ldots, \mu_m\) be a collection of \(m\) absolutely continuous probabilistic measures on \(\mathbb{R}^n\). The test maps we consider

\[
\tau : X \rightarrow S_{1l}^\otimes m \quad \text{and} \quad \tau^{\text{orth}} : Y_{\mu_1} \rightarrow S_{1l}^\otimes m
\]

are defined by

\[
(v_1, \ldots, v_k) \mapsto \left( (\mu_1(H_{a_1}^{(n)} \cap \cdots \cap H_{a_k}^{(n)}) - \frac{1}{\rho} \mu_1(R^d)) \right)_{(a_1, \ldots, a_k) \in (\mathbb{Z}_2)^k} \in \{1, \ldots, m\}
\]

\[
(e_1, \ldots, e_k) \mapsto \left( (\mu_1(H_{a_1}^{(n)} \cap \cdots \cap H_{a_k}^{(n)}) - \frac{1}{\rho} \mu_1(R^d)) \right)_{(a_1, \ldots, a_k) \in (\mathbb{Z}_2)^k} \in \{1, \ldots, m\}
\]

for \((v_1, \ldots, v_k) \in X\) and \((e_1, \ldots, e_k) \in Y_{\mu_1}\). Since the configuration space \(Y_{\mu_1}\) is chosen in such a way that each hyperplane equipartitions the measure \(\mu_1\) the test map \(\tau^{\text{orth}}\) factors

\[
Y_{\mu_1} \xrightarrow{\rho} S_{2l} \oplus S_{1l}^\otimes (m-1) \xrightarrow{\tau^{\text{orth}}} S_{1l}^\otimes m
\]

so that \(\tau^{\text{orth}} = \iota \circ \rho\) and \(\iota\) is induced by the inclusion \(S_{2l} \rightarrow S_{1l}\).

All test maps \(\tau\), \(\tau^{\text{orth}}\) and \(\rho\) are \(W_k\)-equivariant maps, when the introduced actions on the spaces are assumed. The key property of these test maps is that: If for every collection \(\mu_1, \ldots, \mu_m\) of \(m\) absolutely continuous probabilistic measures on \(\mathbb{R}^n\)

- if \(\emptyset \in S_{1l}^\otimes m\) \(\notin \tau(X)\), then \((n, m, k, l) \in \mathcal{M}\),
- if \(\emptyset \in S_{2l} \oplus S_{1l}^\otimes (m-1)\) \(\notin \rho(Y_{\mu_1})\), then \((n, m, k, l) \in \mathcal{M}^{\text{orth}}\).

Using the contraposition we get that

- \((n, m, k, l) \notin \mathcal{M}\) \(\Rightarrow\) there exists a collection of \(m\) absolutely continuous probabilistic measures on \(\mathbb{R}^n\) such that \(\emptyset \in S_{1l}^\otimes m\) \(\notin \tau(X)\)
- \((n, m, k, l) \notin \mathcal{M}^{\text{orth}}\) \(\Rightarrow\) there exists a \(W_k\)-equivariant map

\[
X = (S^n)^k \rightarrow S_{1l}^\otimes m \setminus \{\emptyset\} \rightarrow S(S_{1l}^\otimes m),
\]

- \((n, m, k, l) \in \mathcal{M}^{\text{orth}}\) \(\Rightarrow\) there exists a collection of \(m\) absolutely continuous probabilistic measures on \(\mathbb{R}^n\) such that \(\emptyset \in S_{2l} \oplus S_{1l}^\otimes (m-1)\) \(\notin \rho(Y_{\mu_1})\)
- \((n, m, k, l) \in \mathcal{M}^{\text{orth}}\) \(\Rightarrow\) there exists a \(W_k\)-equivariant map

\[
Y_{\mu_1} = V_n^k \rightarrow S_{2l} \oplus S_{1l}^\otimes (m-1) \setminus \{\emptyset\} \rightarrow S(S_{2l} \oplus S_{1l}^\otimes (m-1)).
\]

This implies that

- if there is no \(W_k\)-equivariant map \(X = (S^n)^k \rightarrow S(S_{1l}^\otimes m)\), then \((n, m, k, l) \in \mathcal{M}\),
- if there is no \(W_k\)-equivariant map \(Y_{\mu_1} = V_n^k \rightarrow S(S_{2l} \oplus S_{1l}^\otimes (m-1))\), then \((n, m, k, l) \in \mathcal{M}^{\text{orth}}\).
Therefore, by proving the next proposition we conclude the proof of Theorem 10.

**Proposition 16.**

(a) If
\[
\prod_{s_1,\ldots,s_k \in \mathbb{Z}_2, 1 \leq s_1 + \cdots + s_k \leq l} (s_1 t_1 + s_2 t_2 + \cdots + s_k t_k)^m \notin \langle t_1^{n+1}, \ldots, t_k^{n+1} \rangle
\]
then there is no $W_k$-equivariant map $X = (S^n)^k \to S(S^m_l)$.

(b) If
\[
\prod_{s_1,\ldots,s_k \in \mathbb{Z}_2, 1 \leq s_1 + \cdots + s_k \leq l} \frac{1}{t_1 \cdots t_k} (s_1 t_1 + s_2 t_2 + \cdots + s_k t_k)^m \notin \langle \bar{w}_n, \ldots, \bar{w}_n \rangle
\]
then there is no $W_k$-equivariant map $Y_{\mu_i} = V^k_{n_i} \to S(S^m_l)$. 

**Proof.** Both statements follow from the Fadell–Husseini index computations:

\[
\begin{align*}
\text{Index}_{\mathbb{Z}_2^k \times \mathbb{F}_2} (S^n)^k & = \langle t_1^{n+1}, \ldots, t_k^{n+1} \rangle, \\
\text{Index}_{\mathbb{Z}_2^k \times \mathbb{F}_2} S^m_l & = \langle \prod_{s_1,\ldots,s_k \in \mathbb{Z}_2, 1 \leq s_1 + \cdots + s_k \leq l} (s_1 t_1 + s_2 t_2 + \cdots + s_k t_k)^m \rangle, \\
\text{Index}_{\mathbb{Z}_2^k \times \mathbb{F}_2} V^k_{n_i} & = \langle \bar{w}_n, \ldots, \bar{w}_n \rangle, \\
\text{Index}_{\mathbb{Z}_2^k \times \mathbb{F}_2} S^m_l \oplus S^m_{l_1} & = \langle \frac{1}{\prod_{s_1,\ldots,s_k \in \mathbb{Z}_2, 1 \leq s_1 + \cdots + s_k \leq l} (s_1 t_1 + s_2 t_2 + \cdots + s_k t_k)^m \rangle, \\
\end{align*}
\]
and its basic property that if there is a $G$-equivariant map $X \to Y$ then $\text{Index}_{G \times X} \supseteq \text{Index}_{G \times Y}$. \(\square\)

### 5.2. Proof of Theorem 10

Let us lift the measures to $S^{n-1} \subseteq \mathbb{R}^n$: we obtain $m + 1$ centrally symmetric measures on the sphere. It is sufficient to find a pair of oriented hyperplanes through the origin $H_1, H_2$ such that for every $i = 0, 1, \ldots, m$
\[
\mu_i(H_1^+ \cap H_2^+) = \mu_i(H_1^- \cap H_2^-) = \mu_i(H_1^+ \cap H_2^-) = \mu_i(H_1^- \cap H_2^+).
\]
Since the conditions $\mu_i(H_1^+ \cap H_2^+) = \mu_i(H_1^- \cap H_2^-)$ and $\mu_i(H_1^+ \cap H_2^-) = \mu_i(H_1^- \cap H_2^+)$ hold always (because of the central symmetry), we may select the components of the test map to be
\[
f_i(H_1, H_2) = \mu_i(H_1^+ \cap H_2^+) - \mu_i(H_1^- \cap H_2^-) - \mu_i(H_1^+ \cap H_2^-) + \mu_i(H_1^- \cap H_2^+).
\]
The rest of the proof would follow directly from the proof of Theorem 9 (see Section 4.3), if we had $m$ measures. We are going to provide an additional argument to partition $m + 1$ measures. Take the measure $\mu_0$ and assume that its support equals $S^{n-1}$. Any measure can be approximated by such a measure, and the standard compactness argument (the configuration space of all pairs $(H_1, H_2)$ is compact) extends the solution to arbitrary measures. We are going to show the following:

**Proposition 17.** If the support of $\mu_0$ is the whole $S^{n-1}$, then the configuration space $X$ of pairs $(H_1, H_2)$ that equipartition $\mu_0$ (i.e. $f_0(H_1, H_2) = 0$) is $D^8$-equivariantly homeomorphic to $V^2_n$.

**Proof.** Take an orthogonal 2-frame $(e_1, e_2)$. Denote the orthogonal complement of $(e_1, e_2)$ by $L^\perp(e_1, e_2)$, and denote the reflections
\[
\sigma_1 : x \mapsto x - 2(x, e_1)e_1, \quad \sigma_2 : x \mapsto x - 2(x, e_2)e_2
\]
Note that the hyperplane $H_1$ is uniquely defined by the following conditions:

- $H_1 \supseteq L^\perp(e_1, e_2)$,
- $e_1, e_2 \in H_1^+$,
- $H_2 = \sigma_1(H_1) = -\sigma_2(H_1)$,
- $f_0(H_1, H_2) = 0$.

The dependence of $H_1$ on $(e_1, e_2) \in V^2_n$ is continuous, and therefore we obtain a homeomorphism between $X$ and $V^2_n$, if the action of $D_8$ on $V^2_n$ is chosen properly. \(\square\)

Now we continue the proof of Theorem 10. The functions $f_1, \ldots, f_m$ may be considered as functions on $V^2_n$. If we consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset D_8$, generated by $\sigma_1, \sigma_2$, then the functions $f_i$ are invariant under this group action. Therefore they define the map $\mathbb{Z}_2 = D_8/(\mathbb{Z}_2 \times \mathbb{Z}_2)$-equivariant map
\[
\tilde{f} : V^2_n/(\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq (\mathbb{R}^{P^n-1} \times \mathbb{R}^{P^n-1}) \setminus \Delta(\mathbb{R}^{P^n-1}) \to \mathbb{R}^m,
\]
where the action on $(\mathbb{R}^{P^n-1} \times \mathbb{R}^{P^n-1}) \setminus \Delta(\mathbb{R}^{P^n-1})$ is given by interchanging factors in the product while the action on $\mathbb{R}^m$ is antipodal. This map must have a zero, because the “deleted square obstruction” guarantees the existence of a zero.
REFERENCES

[1] A. Adem, R.J. Milgram. Cohomology of Finite Groups, Second Edition, Grundlehren der Mathematischen Wissenschaften 300, Springer-Verlag, Berlin, 2004.

[2] T. Bartsch, M. Clapp, D. Puppe. A mountain pass theorem for actions of compact Lie groups. // J. reine angew. Math., 419, 1991, 55–66.

[3] T. Bartsch. On the existence of Borsuk-Ulam theorems. // Topology, 31, 1992, 533–543.

[4] T. Bartsch. Topological methods for variational problems with symmetries. Berlin-Heidelberg: Springer-Verlag, 1993.

[5] P. Blagojević, G. Ziegler. The ideal-valued index for a dihedral group action, and mass partition by two hyperplanes. // Topology and its Applications, 158, 1326-1351, 2011, http://dx.doi.org/10.1016/j.topol.2011.05.008.

[6] A. Borel. Sur la cohomologie des espaces fibrés principaux et des espace homogènes de groupes de Lie compact. // Ann. Math., 57, 1953, 115–207.

[7] M. Clapp, W. Marzantowicz. Essential equivariant maps and Borsuk–Ulam theorems. // J. London Math. Soc., 61(2), 2000, 950–960.

[8] D. Davis. Table of immersions and embeddings of real projective spaces, http://www.lehigh.edu/ dmd1/immtable.

[9] D. M. Davis. A strong nonimmersion theorem for real projective spaces. // Ann. of Math, 120, 1984, 517–528.

[10] E. Fadell, S. Husseini. An ideal-valued cohomological index, theory with applications to Borsuk–Ulam and Bourgin–Yang theorems. // Ergod. Th. and Dynam. Sys., 8, 1988, 73–85.

[11] J. González, P. Landweber. Symmetric topological complexity of projective and lens spaces. // arXiv:0809.0816, 2008.

[12] B. Grünbaum. Partition of mass-distributions and convex bodies by hyperplanes. // Pacific J. Math., 10, 1960, 1257–1261.

[13] A. Haefliger. Plongements differentiables dans le domaine stable. // Comment. Math. Helv., 36, 1962–63, 155–176.

[14] H.L. Hiller. On the cohomology of real grassmannians. // Trans. Amer. Math. Soc., 257(2), 1980, 521–533.

[15] J. Jaworowski. Maps of Stiefel manifolds and a Borsuk–Ulam theorem. // Proceedings of the Edinburgh Mathematical Society (Series 2), 32, 1989, 271–279.

[16] P. Mani-Levitska, S. Vrečica, R. Zivaljević. Topology and combinatorics of partitions of masses by hyperplanes. // Advances in Mathematics, 207(1), 2006, 266–296.

[17] V.V. Makeev. Equipartition of a continuous mass distribution. // Journal of Mathematical Sciences, 140(4), 2007, 551–557.

[18] V.V. Makeev. Geometry of finite-dimensional normed spaces, and continuous functions on the Euclidean sphere. // Journal of Mathematical Sciences, 140(4), 2007, 558–563.

[19] E.A. Ramos. Equipartition of mass distributions by hyperplanes. // Discrete and Computational Geometry, 15, 1996, 147–167.

[20] B.A. Rattray. An antipodal-point, orthogonal-point theorem. // Ann. of Math., 60(3), 1954, 502–512.

[21] V.V. Makeev. Equipartition of a continuous mass distribution. // Journal of Mathematical Sciences, 140(4), 2007, 551–557.

[22] A.H. Stone, J.W. Tukey. Generalized 'Sandwich' Theorems. // Duke Math. J., 9, 1942, 356–359.

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