Geometric Satake equivalence in mixed characteristic and Springer correspondence

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Abstract

The geometric Satake equivalence and the Springer correspondence are closely related when restricting to small representations of the Langlands dual group. We prove this result for étale sheaves, including the case of the mixed characteristic affine Grassmannian, assuming a sufficient ramification. In this process, we construct a monoidal structure on the restriction functor of Satake categories. We construct also a canonical isomorphism between a mixed characteristic affine Grassmannian under a sufficient ramification and an equal characteristic one.

1 Introduction

For a complex reductive group $G$, the close relationship between the geometric Satake equivalence and the Springer correspondence are studied in [AHR15]. We will prove this result for étale sheaves, including the case of the mixed characteristic affine Grassmannian defined in [Zhu17a], assuming a sufficiently large ramification.

Let $k$ be an algebraically closed field of characteristic $p > 2$ and let $G$ be a reductive group over $\mathcal{O}$, where $\mathcal{O}$ is $k[[t]]$ or a totally ramified finite extension of the ring of Witt vectors $W(k)$. We denote the affine Grassmannian by $\text{Gr}_G$. Let $N_G$ be the nilpotent cone in the Lie algebra of $\tilde{G}$ (the reduction of $G$). Let $\ell$ be a prime different from $p$.

Consider the following four functors (see §2 for precise definitions):

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(1) The restriction of the geometric Satake equivalence $\mathcal{S}: \text{Perv}_{L^+G}(\text{Gr}, \bar{\mathbb{Q}}_\ell) \to \text{Rep}(\hat{G}, \bar{\mathbb{Q}}_\ell)$ to small representations of the Langlands dual group:

$$\mathcal{S}^\text{sm}_G: \text{Perv}_{L^+G}(\text{Gr}^\text{sm}_G, \bar{\mathbb{Q}}_\ell) \to \text{Rep}(\hat{G}, \bar{\mathbb{Q}}_\ell)^\text{sm}.$$

(2) The Springer correspondence

$$S_G = \text{Hom}(\text{Spr}, -): \text{Perv}_G(\mathcal{N}_G^\mathbf{p}^{-\infty}, \bar{\mathbb{Q}}_\ell) \cong \text{Perv}_G(\mathcal{N}_G, \bar{\mathbb{Q}}_\ell) \to \text{Rep}(W_G, \bar{\mathbb{Q}}_\ell)$$

where $\text{Spr} \in \text{Perv}_G(\mathcal{N}_G, \bar{\mathbb{Q}}_\ell)$ is the Springer sheaf and $W_G$ is the (finite) Weyl group of $G$.

(3) By taking the zero weight space of a representation of $\hat{G}$ and tensoring it with the sign character $\varepsilon: W_G \to \mathbb{Q}_\ell^\times$, we obtain a functor

$$\Phi_G: \text{Rep}(\hat{G}, \bar{\mathbb{Q}}_\ell)^\text{sm} \to \text{Rep}(W_G, \bar{\mathbb{Q}}_\ell).$$

(4) Assuming a sufficient ramification of $O$, there is an open dense subspace $\mathcal{M}_G \subset \text{Gr}^\text{sm}_G$ and $\pi_G: \mathcal{M}_G \to \mathcal{N}_G^\mathbf{p}^{-\infty}$ which is the perfection of a finite map. We obtain a functor

$$\Psi_G := (\pi_G)_\ast \circ (j_G)^!: \text{Perv}_{L^+G}(\text{Gr}^\text{sm}_G, \bar{\mathbb{Q}}_\ell) \to \text{Perv}_G(\mathcal{N}_G^\mathbf{p}^{-\infty}, \bar{\mathbb{Q}}_\ell)$$

where $j_G: \mathcal{M}_G \hookrightarrow \text{Gr}^\text{sm}_G$ is the inclusion.

We have the diagram

$$\begin{array}{ccc}
\text{Perv}_{L^+G}(\text{Gr}^\text{sm}_G, \bar{\mathbb{Q}}_\ell) & \xrightarrow{\mathcal{S}^\text{sm}_G} & \text{Rep}(\hat{G}, \bar{\mathbb{Q}}_\ell)^\text{sm} \\
\Psi_G \downarrow & & \Phi_G \downarrow \\
\text{Perv}_G(\mathcal{N}_G^\mathbf{p}^{-\infty}, \bar{\mathbb{Q}}_\ell) & \xrightarrow{S_G} & \text{Rep}(W_G, \bar{\mathbb{Q}}_\ell).
\end{array}$$

The main theorem is the following:

**Theorem 1.1.** Assuming a sufficient ramification of $O$, there is a canonical isomorphism of functors:

$$\Phi_G \circ \mathcal{S}^\text{sm}_G \leftrightarrow S_G \circ \Psi_G.$$

The meaning of “a sufficient ramification” is explained in the proof of Theorem 5.12.

For many parts of the proof, the same method as [AHR15] can be used. However, some parts require new methods. First, the construction of the open subset $\mathcal{M}_G \subset \text{Gr}^\text{sm}_G$...
in [AHR15 §2.6] does not work in the mixed characteristic case since the “big open cell” \( \text{Gr}_{G,0} \subset \text{Gr}_G \) is not known to exist in mixed characteristic. For this, we construct a canonical isomorphism between a mixed characteristic affine Grassmannian (under a sufficient ramification) and an equal characteristic one in §5 and define \( \mathcal{M}_G \) in the mixed characteristic case as the pullback of \( \mathcal{M}_G \) in the equal characteristic case by this isomorphism.

Furthermore, in §4.3 we need the monoidal structure of the restriction functor \( \mathcal{R}_L : \text{Perv}_{L^+G}(\text{Gr}^{\text{sm}}_G, \overline{Q}_\ell) \to \text{Perv}_{L^+L}(\text{Gr}^{\text{sm}}_L, \overline{Q}_\ell) \) to a Levi subgroup \( L \). The method of constructing this monoidal structure in the equal characteristic case as in [MV07, Proposition 6.4] does not work for the mixed characteristic affine Grassmannians since there is no analogue of the fusion product in the setting of [Zhu17a]. If \( L = T \), the method in [Zhu17a, Proposition 2.36] can be used, but this is not applicable to general \( L \). So we prove this in another way, which is a new method even in the equal characteristic situation.

**Outline of the paper.** In §2 we explain our notations and define four functors in Theorem 1.1, leaving the definition of \( \mathcal{M}_G \) until §5. In §3 we explain the plan of proof of Theorem 1.1. This plan is almost the same as the plan in [AHR15, §3]. Namely, we explain the method to reduce to the case where \( G \) is semisimple rank 1. In §4 we define restriction functors to a Levi subgroup and their transitivities. In this process, we construct a monoidal structure on the restriction functor of Satake categories. In §5 we define the open subset \( \mathcal{M}_G \subset \text{Gr}^{\text{sm}} \), assuming a sufficiently large ramification. In this process, we construct a canonical isomorphism between a mixed characteristic affine Grassmannian (under a sufficient ramification) and an equal characteristic one. In §6 we prove the commutativity of the four functors in Theorem 1.1 with the restriction functors. In §7 we complete the proof by considering the case where \( G \) is semisimple rank 1.

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2 Preliminaries

2.1 Notations

Let $k$ be an algebraically closed field of characteristic $p > 0$. Fix a prime number $\ell \neq p$. In this paper, all sheaves are $\mathbb{Q}_\ell$-sheaves for the étale topology. If $X$ is an (ind-) $k$-variety or the perfection of it, we write $D^b(X, \overline{\mathbb{Q}}_\ell)$ or simply $D^b(X)$ for the $\ell$-adic bounded constructible derived category of $X$. If $H$ is a connected (pro-)algebraic group over $k$ or the perfection of it, we write $\text{Perv}_H(X, \mathbb{Q}_\ell)$ for the full abelian subcategory (of $D^b(X, \mathbb{Q}_\ell)$) consisting of $H$-equivariant perverse sheaves. We write $D^b_H(X, \mathbb{Q}_\ell)$ for the $H$-equivariant derived category defined in [BL94], and $\text{Perv}'_H(X, \mathbb{Q}_\ell)$ for its core with respect to the perverse $t$-structure. Note that the forgetful functor $\text{For}: D^b_H(X, \mathbb{Q}_\ell) \to D^b(X, \mathbb{Q}_\ell)$ induces the equivalence $\text{For}: \text{Perv}'_H(X, \mathbb{Q}_\ell) \to \text{Perv}_H(X, \mathbb{Q}_\ell)$ (see [BL94, Proposition 2.5.3]).

For a $k$-algebra $R$, let $W(R)$ denote the ring of Witt vectors, and $W_{h}(R)$ the truncated Witt vectors of length $h$. For $a \in R$, write $[a]$ for the Teichmüller lift of $a$. Let $F$ be $k((t))$ or a totally ramified finite extension of $W(k)[1/p]$. Let $\mathcal{O}$ be the ring of integers of $F$. We take a uniformizer $\varpi$ of $\mathcal{O}$ (If $F = k((t))$, set $\varpi = t$).

For a $k$-algebra $R$, we write $W_{\mathcal{O}}(R) = W(R) \otimes W(k) \mathcal{O}$,

$$\mathcal{O}_R = \begin{cases} W_{\mathcal{O}}(R) & \text{if } F \text{ is a totally ramified finite extension of } W(k)[1/p], \\ R[[t]] & \text{if } F = k((t)) \end{cases}$$

and

$$W_{\mathcal{O},h}(R) = W_{\mathcal{O}}(R)/\varpi^h,$$

$$\mathcal{O}_{R,h} = \mathcal{O}_R/\varpi^h.$$  

Throughout this paper, $G$ is a connected reductive group over $\mathcal{O}$. We choose a Borel subgroup $B$ of $G$ and a maximal torus $T$ of $B$. Let $U$ be the unipotent radical of $B$. We write $\overline{G}, \overline{B}, \overline{U}, \overline{T}$ for the reduction modulo $\varpi$ of $G, B, U, T$, respectively and $\mathfrak{g}, \mathfrak{b}, \mathfrak{u}, \mathfrak{t}$ for the Lie algebra of $\overline{G}, \overline{B}, \overline{U}, \overline{T}$, respectively. Let $W_G$ be the Weyl group $N_G(T)/T$.

If $P$ is a parabolic subgroup of $G$ containing $B$, then it has Levi decomposition $P = LU_P$ where $L$ is a Levi subgroup containing $T$ and $U_P$ is the unipotent radical of $P$. Sometimes $B_L$ denotes $B \cap L$. The group $B_L$ is a Borel subgroup of $L$ containing $T$.

Let $X^\bullet(T)$ and $X_\bullet(T)$ be the character and cocharacter lattice of $T$. Let $X^\bullet_{+}(T) \subset X_\bullet(T)$ be the subset of dominant cocharacters with respect to $B$. 

4
2.2 Affine Grassmannians

Let $H$ be a smooth affine group scheme over $O$. We consider the following presheaves on the category of perfect affine $k$-schemes

\[
L^+ H(R) = H(O_R), \\
L^h H(R) = H(O_{R,h}), \\
LH(R) = H(O_R[1/\omega]).
\]

By [Zhu17b, Proposition 1.3.2] or [Zhu17a, 1.1], $L^+ H$ and $L^h H$ are represented by pfpp (perfectly of finite presentation) perfect group schemes over $k$, and $LH$ is represented by an ind perfect scheme over $k$. We also write $L^+ H^{(h)}$ for the $h$-th congruence group (i.e. the kernel of the natural map $L^+ H \to L^h H$).

Affine Grassmannian of $H$ is the perfect space $\text{Gr}_H$ defined by

\[
\text{Gr}_H := [LH/L^+ H].
\]

where $[\cdot/\cdot]$ means the quotient stack. By [BS17, Theorem 1.1], $\text{Gr}_H$ is representable by the inductive limit of perfections of quasi-projective varieties over $k$, along closed immersions. If $H$ is a reductive group scheme, then $\text{Gr}_H$ is represented by the inductive limit of perfections of projective varieties over $k$, along closed immersions.

**Remark 2.1.** If $F = k((t))$, there are non-perfect versions of the above spaces (see [Zhu17b]). We write $\text{Gr}'_H$ for this canonical deperfection of $\text{Gr}_H$. All the results of this paper hold for this version by the same arguments.

For $\lambda \in X_\bullet(T)$, let $\varpi^\lambda \in \text{Gr}_G(k)$ be the image of $\varpi \in F^\times = LG_m(k)$ under the map

\[
L G_m \xrightarrow{\lambda} LT \to LG \to \text{Gr}_G.
\]

For $\mu \in X_\bullet(T)$, define $\text{Gr}_{G,\mu}$ as the $L^+ G$-orbit of $\varpi^\mu$, and let

\[
\text{Gr}_{G,\leq \mu} = \bigcup_{\mu' \leq \mu} \text{Gr}_{G,\mu'}.
\]

The space $\text{Gr}_{G,\leq \mu}$ is a closed subspace of $\text{Gr}_G$, and $\text{Gr}_{G,\mu}$ is an open subspace of $\text{Gr}_{G,\leq \mu}$. We call $\text{Gr}_{G,\leq \mu}$ the Schubert variety corresponding to $\mu$, and $\text{Gr}_{G,\mu}$ the Schubert cell corresponding to $\mu$.

For $\lambda \in X_\bullet(T)$ defined $S_{G,\lambda} = S_\lambda$ as the $LU$-orbit of $\varpi^\lambda$, and let

\[
S_{\leq \lambda} = \bigcup_{\lambda' \leq \lambda} S_{\lambda'}.
\]
The space $S_{\leq \lambda}$ is a closed subspace of $\text{Gr}_G$, and $S_\lambda$ is an open subspace of $S_{\leq \lambda}$. We can regard $S_\lambda$'s as semi-infinite orbits as the attractor locus of certain torus-action on $\text{Gr}_G$ (see [Zhu17a] §2.2 for details). The natural map

$$\text{Gr}_B \rightarrow \text{Gr}_G$$

can be identified with

$$\prod_{\lambda \in X^+ (T)} S_\lambda \rightarrow \text{Gr}_G.$$

By the Iwasawa decomposition, this map is bijective.

Let $\Phi$ be the set of roots of $(\tilde{G}, \tilde{T})$. The Weyl group of $\tilde{G}$ is identified with the Weyl group of $G$, denoted by $W_G$. The group $W_G$ acts on $X^+ (T) = X^* (T)$.

**Definition 2.2.** A character $\mu \in X^* (\tilde{T})$ is said to be small for $\tilde{G}$ if

$$\begin{cases} 
\mu \in \mathbb{Z} \Phi, \\
\text{convex hull of } W_G \cdot \mu \text{ does not contain any element of } \{2 \tilde{\alpha} \mid \tilde{\alpha} \in \Phi\}.
\end{cases}$$

The closed subspace $Gr_{\text{sm}} \subset \text{Gr}$ is defined as the union of $Gr_\mu$ for small $\mu \in X^+ (T)$.

### 2.3 Geometric Satake equivalence

By [MV07] or [Zhu17a], $\text{Perv}_{L^+ G}(\text{Gr}_G, \overline{\mathbb{Q}}_\ell)$ has a monoidal structure under the convolution product $\star$, and the functor

$$F_G := H^* (\text{Gr}_G, -) : \text{Perv}_{L^+ G}(\text{Gr}_G, \overline{\mathbb{Q}}_\ell) \rightarrow \text{Vect}_{\overline{\mathbb{Q}}_\ell}$$

is monoidal. Let $\tilde{G}$ be the Langlands dual group (over a field of characteristic zero) of $G$. Then there is an identification

$$\tilde{G} \cong \text{Aut}^* (F_G).$$

Therefore $F_G$ gives rise to an equivalence of monoidal categories

$$\mathcal{I}_G : \text{Perv}_{L^+ G}(\text{Gr}_G, \overline{\mathbb{Q}}_\ell) \rightarrow \text{Rep}(\tilde{G}, \overline{\mathbb{Q}}_\ell).$$

This is called geometric Satake equivalence. More explicitly, the intersection cohomology sheaf $\text{IC}_\mu$ on $\text{Gr}_{\leq \mu}$ corresponds to an irreducible representation with highest weight $\mu$. 
Let \( z_G : \text{Gr}^*_G \hookrightarrow \text{Gr}_G \) be the inclusion where \( \text{Gr}^*_G \) is the connected component of \( \text{Gr} \) containing \( \varpi^0 \). The functor

\[
(z_G)_* : \text{Perv}^L(\text{Gr}_G^*, \overline{\mathbb{Q}}_\ell) \rightarrow \text{Perv}^L(\text{Gr}_G, \overline{\mathbb{Q}}_\ell)
\]

is fully faithful. The essential image of \( \mathcal{J}_G \circ (z_G)_* \) is the subcategory \( \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell)^{Z(\hat{G})} \) of \( \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell) \) consisting of representations on which \( Z(\hat{G}) \) acts trivially. Let \( I_G : \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell)^{Z(\hat{G})} \rightarrow \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell) \) be the inclusion. There is a unique equivalence of categories

\[
\mathcal{J}_G^0 : \text{Perv}^L(\text{Gr}_G^*, \overline{\mathbb{Q}}_\ell) \rightarrow \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell)^{Z(\hat{G})}
\]

such that

\[
I_G \circ \mathcal{J}_G^0 = \mathcal{J}_G \circ (z_G)_*.
\]

Since \((z_G)_*\) is left adjoint to \((z_G)^!\) and \( I_G \) is left adjoint to \((-)^{Z(\hat{G})} : \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell) \rightarrow \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell)^{Z(\hat{G})}\), there is a canonical isomorphism

\[
(-)^{Z(\hat{G})} \circ \mathcal{J}_G \leftrightarrow \mathcal{J}_G^0 \circ (z_G)^!.
\]  

(2.1)

Let \( f_G : \text{Gr}^{\text{sm}}_G \hookrightarrow \text{Gr}_G \) be the inclusion. The functor

\[
(f_G)_* : \text{Perv}^L(\text{Gr}_G^{\text{sm}}, \overline{\mathbb{Q}}_\ell) \rightarrow \text{Perv}^L(\text{Gr}_G, \overline{\mathbb{Q}}_\ell)
\]

is fully faithful. The essential image of \( \mathcal{J}_G \circ (f_G)_* \) is the subcategory \( \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell)^{\text{sm}} \) of \( \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell) \) consisting of representations whose \( T \)-weights are small. Let \( I_G : \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell)^{\text{sm}} \rightarrow \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell) \) be the inclusion. There is a unique equivalence of categories

\[
\mathcal{J}_G^{\text{sm}} : \text{Perv}^L(\text{Gr}_G^{\text{sm}}, \overline{\mathbb{Q}}_\ell) \rightarrow \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell)^{\text{sm}}
\]

such that

\[
I_G \circ \mathcal{J}_G^{\text{sm}} = \mathcal{J}_G \circ (f_G)_*.
\]

Let \( f_G^\circ : \text{Gr}^{\text{sm}}_G \hookrightarrow \text{Gr}_G^\circ \) and \( I_G^\circ : \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell)^{\text{sm}} \rightarrow \text{Rep}(\hat{G}, \overline{\mathbb{Q}}_\ell)^{Z(\hat{G})} \) be the inclusion. Since \( f_G = z_G \circ f_G^\circ \) and \( I_G = I_G^\circ \circ I_G^\circ \) holds, we obtain a canonical isomorphism

\[
I_G^\circ \circ \mathcal{J}_G^{\text{sm}} \leftrightarrow \mathcal{J}_G^0 \circ (f_G^\circ)_*.
\]  

(2.2)

### 2.4 Springer correspondence

Let \( \mathcal{N}_G \subset \mathfrak{g} \) be the nilpotent cone, and let \( \mathfrak{g}_{\text{rs}} \subset \mathfrak{g} \) be the subset consisting of regular semisimple elements. Recall there is a diagram

\[
\begin{array}{ccccccc}
    G \times \mathfrak{g} & \xrightarrow{i_{\mathfrak{h}}} & G \times \mathfrak{b} & \xleftarrow{\mathcal{N}_G} & G \times \mathfrak{b} & \leftarrow & (\mathfrak{g}_{\text{rs}} \cap \mathfrak{b}) \\
    \mu_N \downarrow & \square & \mu_0 & \square & \mu_0^\ast & \downarrow & \mu_0^\ast \\
    \mathcal{N}_G & \xrightarrow{i_\mathfrak{h}} & \mathfrak{g} & \leftarrow & \mathfrak{g}_{\text{rs}}
\end{array}
\]
where all the horizontal maps are the inclusions, and all the vertical maps send \((g, x)\) to \(g \cdot x\). Since \(\mu_g\) is proper and small,

\[
\text{Groth} := (\mu_g)^* \overline{\mathbb{Q}}_{\ell} \otimes \mathfrak{g}[\dim \mathfrak{g}]
\]

is a \(\mathcal{G}\)-equivariant perverse sheaf on \(\mathfrak{g}\). There is a canonical isomorphism

\[
\text{Groth} \cong (j_g)_* ((\mu_{N_{\mathcal{G}}}^s) \overline{\mathbb{Q}}_{\ell} \otimes \dim N_{\mathcal{G}})
\]

and \(\mu_{N_{\mathcal{G}}}^s\) is a Galois covering with Galois group \(\mathcal{W}_{\mathcal{G}}\). Hence we obtain a \(\mathcal{W}_{\mathcal{G}}\)-action on \(\text{Groth}\).

Moreover, since \(\mu_N\) is proper and small,

\[
\text{Spr} := (\mu_N)^* \overline{\mathbb{Q}}_{\ell} \otimes \mathfrak{g}[\dim \mathfrak{g}]
\]

is a \(\mathcal{G}\)-equivariant perverse sheaf on \(\mathcal{N}_{\mathcal{G}}\). There is a canonical isomorphism

\[
\text{Spr} \cong (i_g)^* \text{Groth}[-r].
\]

where \(r = \text{rk}(\mathfrak{g}) = \dim \mathfrak{g} - \dim \mathcal{N}_{\mathcal{G}}\). Hence we obtain a \(\mathcal{W}_{\mathcal{G}}\)-action on \(\text{Spr}\). This induces a functor

\[
S_G: \text{Perv}(\mathcal{N}_{\mathcal{G}}^{-\infty}, \overline{\mathbb{Q}}_{\ell}) \cong \text{Perv}(\mathcal{N}_{\mathcal{G}}, \overline{\mathbb{Q}}_{\ell}) \to \text{Rep}(\mathcal{W}_{\mathcal{G}}, \overline{\mathbb{Q}}_{\ell}),
\]

\[
M \mapsto \text{Hom}_{\text{Perv}(\mathcal{N}_{\mathcal{G}}, \overline{\mathbb{Q}}_{\ell})} (\text{Spr}, M).
\]

### 2.5 Functor \(\Phi_{\mathcal{G}}\)

For \(V \in \text{Rep}(\mathcal{G}, \overline{\mathbb{Q}}_{\ell})\), the Weyl group \(\mathcal{W}_{\mathcal{G}} = \mathcal{N}_{\mathcal{G}}(\check{T})/\check{T}\) naturally act on the zero weight space \(V^{\check{T}}\). Let \(\varepsilon_{\mathcal{G}}\) be the sign character of the coxeter group \(\mathcal{W}_{\mathcal{G}}\). Then we obtain the functor

\[
\Phi_{\mathcal{G}}: \text{Rep}(\mathcal{G}, \overline{\mathbb{Q}}_{\ell})_{\text{sm}} \to \text{Rep}(\mathcal{W}_{\mathcal{G}}, \overline{\mathbb{Q}}_{\ell}),
\]

\[
V \mapsto \check{V}^T \otimes \varepsilon_{\mathcal{G}}.
\]

### 2.6 Functor \(\Psi_{\mathcal{G}}\)

Let \(v_F\) be the normalized valuation on \(F\). The functor \(\Psi_{\mathcal{G}}\) will not be defined unless \(p > 2\) and \(v_F(p) \geq C\) where \(C = C_{\mathcal{G}} \in \mathbb{N}\) is a constant depending only on the isomorphism class of \(\mathcal{G}\).
We will prove later that if \( p > 2 \) and \( v_F(p) \geq C \), there is an open subspace \( j_G: \mathcal{M}_G \hookrightarrow \text{Gr}^\text{sm}_G \) and \( \pi_G: \mathcal{M}_G \rightarrow \mathfrak{g} \) which is the perfection of a finite morphism. These induce an exact functor

\[
\Psi_G := (\pi_G)_* \circ (j_G)^! \colon \text{Perv}_{L+G}(\text{Gr}^\text{sm}_G, \overline{\mathbb{Q}}_\ell) \rightarrow \text{Perv}_G(\mathcal{N}_G^{-\infty}, \overline{\mathbb{Q}}_\ell). \tag{2.3}
\]

If \( \mathcal{O} = \mathbb{k}[[t]] \), then we can define a non-perfect version of \( \mathcal{M}_G \) by \( \mathcal{M}_G' := \text{Gr}^\text{sm}_G \cap \text{Gr}^-_{G,0} \). The space \( \text{Gr}^-_{G,0} \) is isomorphic to the kernel \( G = \text{Ker}(\bar{G}(\mathbb{k}[t^{-1}])/t^2) \rightarrow \bar{G}(\mathbb{k}) \) by the homomorphism \( G \rightarrow \text{Gr}^-_{G,0}, g \mapsto g \cdot t^0 \), and \( \mathfrak{g} \) is a kernel of the map \( \bar{G}(\mathbb{k}[t^{-1}]/t^{-2}) \rightarrow \bar{G}(\mathbb{k}) \). Hence there is a canonical map \( \pi_G^*: \text{Gr}^-_{G,0} \rightarrow \mathfrak{g} \).

By the same argument as [AH13, Theorem 1.1] (using the assumption that \( p > 2 \)), one can show that \( \pi_G^! \mathcal{M}_G' \subset \mathcal{N}_G \) and \( \pi_G := \pi_G^! |_{\mathcal{M}_G'} : \mathcal{M}_G' \rightarrow \mathcal{N}_G \) is a finite morphism. Put \( \mathcal{M}_G := (\mathcal{M}_G')^{-\infty} \) and \( \pi_G := (\pi_G')^{-\infty} \). By the \( G(\mathbb{k}) \)-equivariance of \( j_G: \mathcal{M}_G \hookrightarrow \text{Gr}^\text{sm}_G \) and \( \pi_G \), we obtain an exact functor \( \Psi_G \) as in (2.3).

In the mixed characteristic case, we will show later the following:

**Lemma 2.3.** Assume that \( p > 2 \). Let \( G^0 := \bar{G} \otimes_k \mathbb{k}[[t]] \). If \( v_F(p) \geq C \), then \( \text{Gr}^\text{sm}_G \) has a natural \( \bar{G} \)-action (defined later) and there is a \( \bar{G} \)-equivariant isomorphism

\[ \text{Gr}^\text{sm}_G \cong \text{Gr}^\text{sm}_{G^0}. \]

So we can construct \( \mathcal{M}_G \) and \( \pi_G \) as pullbacks of \( \mathcal{M}_{G^0} \) and \( \pi_{G^0} \) along this isomorphism.

### 2.7 Statement of main theorem

We have the diagram

\[
\begin{array}{ccc}
\text{Perv}_{L+G}(\text{Gr}^\text{sm}_G, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\varphi_G} & \text{Rep}(\bar{G}, \overline{\mathbb{Q}}_\ell)_\text{sm} \\
\Psi_G \downarrow & & \Phi_G \downarrow \\
\text{Perv}_G(\mathcal{N}_G^{-\infty}, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\varphi_G} & \text{Rep}(W_G, \overline{\mathbb{Q}}_\ell).
\end{array}
\]

The main theorem is the following:

**Theorem 2.4.** Assume that \( p > 2 \). There exists a constant \( C_G \) depending only on \( \bar{G} \) such that the following holds: If \( v_F(p) > C_G, C_L \), there is a canonical isomorphism of functors:

\[ \Phi_G \circ \mathcal{F}_G^\text{sm} \iff \mathcal{S}_G \circ \Psi_G. \]
3 Plan of proof

Now we have to construct the isomorphism

$$\alpha_G: \Phi_G \circ \mathcal{I}_G^\text{sm} \iff \mathcal{S}_G \circ \Psi_G,$$

as in the main theorem.

First, we will construct certain restriction functors to a Levi subgroup $L$:

- $\mathcal{R}_L^G$:
  $$\text{Perv}_{L^+}(\text{Gr}^\text{sm}_G, \overline{\mathbb{Q}}_\ell) \to \text{Perv}_{L^+}(\text{Gr}^\text{sm}_L, \overline{\mathbb{Q}}_\ell),$$

- $\mathcal{R}_L^\mathcal{G}$:
  $$\text{Rep}(\mathcal{G}, \overline{\mathbb{Q}}_\ell)^\text{sm} \to \text{Rep}(\mathcal{L}, \overline{\mathbb{Q}}_\ell)^\text{sm},$$

- $\mathcal{R}_L^\mathcal{G}$:
  $$\text{Perv}_G(\mathcal{N}_G^{\sigma^\infty}, \overline{\mathbb{Q}}_\ell) \to \text{Perv}_L(\mathcal{N}_L^{\sigma^\infty}, \overline{\mathbb{Q}}_\ell),$$

- $\mathcal{R}_W^G$:
  $$\text{Rep}(W_G, \overline{\mathbb{Q}}_\ell) \to \text{Rep}(W_L, \overline{\mathbb{Q}}_\ell).$$

The functor $\mathcal{R}_W^G$ is the obvious restriction functor. The other functors will be defined later.

Next, we will define natural isomorphisms called transitivity isomorphisms

- $\mathcal{R}_T^G \iff \mathcal{R}_T^L \circ \mathcal{R}_L^G$,
- $\mathcal{R}_T^G \iff \mathcal{R}_T^L \circ \mathcal{R}_L^G$,
- $\mathcal{R}_T^G \iff \mathcal{R}_T^L \circ \mathcal{R}_L^G$,
- $\mathcal{R}_W^G \iff \mathcal{R}_W^L \circ \mathcal{R}_W^G$,

and intertwining isomorphisms

- $\mathcal{R}_W^G \circ \Phi_G \iff \Phi_L \circ \mathcal{R}_L^G$,
- $\mathcal{R}_L^G \circ \Psi_G \iff \Psi_L \circ \mathcal{R}_L^G$,
- $\mathcal{R}_L^G \circ \mathcal{I}_G^\text{sm} \iff \mathcal{I}_L^\text{sm} \circ \mathcal{R}_L^G$,
- $\mathcal{R}_W^G \circ \mathcal{S}_G \iff \mathcal{S}_L \circ \mathcal{R}_L^G$.
such that the following prisms are commutative:

\[
\begin{array}{c}
\text{Rep}(\mathcal{G}, \mathcal{Q}_\ell)_{\text{sm}} \xrightarrow{\Phi_{\mathcal{G}}} \text{Rep}(\mathcal{W} \mathcal{G}, \mathcal{Q}_\ell) \\
\text{Perv}_{\mathcal{L} + \mathcal{G}}(\text{Gr}^{\text{sm}}_{\mathcal{L}}, \mathcal{Q}_\ell) \xrightarrow{\Psi_{\mathcal{G}}} \text{Perv}_{\mathcal{L}}(\mathcal{N}^{p-\infty}_{\mathcal{L}}, \mathcal{Q}_\ell) \\
\text{Perv}_{\mathcal{L} + \mathcal{T}}(\text{Gr}^{\text{sm}}_{\mathcal{T}}, \mathcal{Q}_\ell) \xrightarrow{\Psi_{\mathcal{T}}} \text{Perv}_{\mathcal{T}}(\mathcal{N}^{p-\infty}_{\mathcal{T}}, \mathcal{Q}_\ell) \\
\text{Perv}_{\mathcal{T}}(\mathcal{N}^{p-\infty}_{\mathcal{T}}, \mathcal{Q}_\ell) \xrightarrow{S_{\mathcal{T}}} \text{Rep}(\mathcal{W} \mathcal{T}, \mathcal{Q}_\ell)
\end{array}
\]

where (Tr) and (Intw) mean the transitivity functors and the intertwining functors, respectively.

And then, we will construct the isomorphism in the case where \( G = T \) or \( G \) is
semisimple rank 1. If $G$ is general, then we have the isomorphism
\[
\phi_{G,T}: R_{W_T}^{W_G} \circ \Phi_{G} \circ \mathcal{S}_{G}^{\text{sm}} \xrightarrow{(\text{Intw})} 
\Phi_{T} \circ R_{T}^{G} \circ \mathcal{S}_{G}^{\text{sm}},
\]
for a maximal torus $T$, and similarly,
\[
\phi_{G,L}: R_{W_L}^{W_G} \circ \Phi_{G} \circ \mathcal{S}_{G}^{\text{sm}} \xrightarrow{\text{Intw}} R_{W_L}^{W_G} \circ \mathcal{S}_{G} \circ \Psi_{G}
\]
for any Levi subgroup $L$ which is semisimple rank 1. The commutativity of prism implies the equality
\[
R_{W_T}^{W_L} \phi_{G,L} = \phi_{G,T}.
\]
It means that for any $M \in \text{Perv}_{L+G}^{G}(\text{Gr}_{G}^{\text{sm}}, \overline{G}_{\ell})$, there is an isomorphism
\[
R_{W_T}^{W_G} \circ \Phi_{G} \circ \mathcal{S}_{G}^{\text{sm}}(M) \xrightarrow{\phi_{G,T}} R_{W_T}^{W_G} \circ \mathcal{S}_{G} \circ \Psi_{G}(M) \tag{3.1}
\]
which is $W_L$-equivariant for any Levi subgroup $L$ which is semisimple rank 1. It follows that (3.1) is $W_G$-equivariant since $W_L$’s generate $W_G$.

### 4 Definition of restriction functors and transitivity isomorphisms

For a Levi subgroup $L$ in $G$, we want to define the restriction functors
\[
\mathfrak{N}_{L}^{G}: \text{Perv}_{L+G}^{G}(\text{Gr}_{G}^{\text{sm}}, \overline{G}_{\ell}) \rightarrow \text{Perv}_{L+G}^{G}(\text{Gr}_{L}^{\text{sm}}, \overline{G}_{\ell}), \\
R_{L}^{G}: \text{Rep}(G, \overline{G}_{\ell}) \rightarrow \text{Rep}(L, \overline{G}_{\ell}), \\
\mathcal{R}_{L}^{G}: \text{Perv}_{G}(N_{L}^{p^{-\infty}}, \overline{G}_{\ell}) \rightarrow \text{Perv}_{L}(N_{L}^{p^{-\infty}}, \overline{G}_{\ell}), \\
R_{W_L}^{W_G}: \text{Rep}(W_G, \overline{G}_{\ell}) \rightarrow \text{Rep}(W_L, \overline{G}_{\ell})
\]
and the natural isomorphisms called transitivity isomorphisms
\[
\mathfrak{N}_{T}^{G} \leftrightarrow \mathfrak{N}_{T}^{L} \circ \mathfrak{N}_{L}^{G}, \\
\mathcal{R}_{T}^{G} \leftrightarrow \mathcal{R}_{T}^{L} \circ \mathcal{R}_{L}^{G}, \\
R_{T}^{G} \leftrightarrow R_{T}^{L} \circ R_{L}^{G}, \\
R_{W_T}^{W_G} \leftrightarrow R_{W_T}^{W_L} \circ R_{W_L}^{W_G}.
\]
4.1 Category $\text{Rep}(W_G, \overline{Q}_\ell)$

As $W_L$ can be regarded as a subgroup of $W_G$, we can define a restriction functor as the usual restriction of group action. Furthermore, the transitivity isomorphism is simply the identity morphism.

4.2 Category $\text{Perv}_{L+G}(\text{Gr}^{sm}_G, \overline{Q}_\ell)$

There is a diagram of algebraic groups

\[
L \leftarrow P \rightarrow G
\]  

where the first morphism is the natural projection, and the second is the inclusion. It induces the diagram of affine Grassmannians

\[
\text{Gr}_L \xrightarrow{q_P} \text{Gr}_P \xrightarrow{i_P} \text{Gr}_G
\]

where $q_P$ induces a bijection between the set of connected component. First, define a functor $\tilde{R}^G_L : D^b(\text{Gr}_G, \overline{Q}_\ell) \rightarrow D^b(\text{Gr}_L, \overline{Q}_\ell)$ as the composition

\[
D^b(\text{Gr}_G, \overline{Q}_\ell) \xrightarrow{(i_P)^!} D^b(\text{Gr}_P, \overline{Q}_\ell) \xrightarrow{(q_P)_*} D^b(\text{Gr}_L, \overline{Q}_\ell).
\]

We also define its equivariant version $\tilde{R}^{L+G}_L : D^b_{L+G}(\text{Gr}_G, \overline{Q}_\ell) \rightarrow D^b_{L+L}(\text{Gr}_L, \overline{Q}_\ell)$ as the composition

\[
D^b_{L+G}(\text{Gr}_G) \xrightarrow{\text{For}^{L+G}_{L+P}} D^b_{L+P}(\text{Gr}_G) \xrightarrow{(i_P)^!} D^b_{L+P}(\text{Gr}_P) \xrightarrow{(q_P)_*} D^b_{L+P}(\text{Gr}_L) \xrightarrow{\text{For}^{L+P}_{L+L}} D^b_{L+L}(\text{Gr}_L)
\]

There is an isomorphism

\[
\tilde{R}^G_L \circ \text{For}^{L+G} \iff \text{For}^{L+L} \circ \tilde{R}^G_L
\]  

(4.2)

defined by

\[
D^b_{L+G}(\text{Gr}_G) \xrightarrow{\text{For}^{L+G}_{L+P}} D^b_{L+P}(\text{Gr}_G) \xrightarrow{(i_P)^!} D^b_{L+P}(\text{Gr}_P) \xrightarrow{(q_P)_*} D^b_{L+P}(\text{Gr}_L) \xrightarrow{\text{For}^{L+P}_{L+L}} D^b_{L+L}(\text{Gr}_L)
\]  

We want to consider the transitivity for the functors $\tilde{R}^G_L$ and $\tilde{R}^G_L$. For that, notice the following cartesian squares:
Lemma 4.1. The commutative squares

\[
\begin{array}{ccc}
\text{Gr}_B & \longrightarrow & \text{Gr}_P \\
\downarrow & & \downarrow \\
\text{Gr}_{BL} & \longrightarrow & \text{Gr}_L, \\
\downarrow & & \downarrow \\
\text{Gr}_B^o & \longrightarrow & \text{Gr}_P^o \\
\downarrow & & \downarrow \\
\text{Gr}_{BL}^o & \longrightarrow & \text{Gr}_L^o
\end{array}
\]
(4.3)

are cartesian.

Proof. It suffices to show that

\[
\begin{array}{ccc}
\text{Gr}_B^o & \longrightarrow & \text{Gr}_P \\
\downarrow & & \downarrow \\
\text{Gr}_{BL}^o & \longrightarrow & \text{Gr}_L
\end{array}
\]

is cartesian because of the equivariance of the morphisms. Recall that the morphisms

\(a, b\) are locally closed embeddings and that \(\text{Gr}_B^o = \text{Gr}_U\) and \(\text{Gr}_{BL}^o = \text{Gr}_{U_{BL}}\) holds. Hence it remains to prove

\[a(\text{Gr}_U) = \{x \in \text{Gr}_P \mid q_P(x) \in b(\text{Gr}_{U_{BL}})\},\]

but it can be easily checked. \(\square\)

Thus we obtain the following diagram:

\[
\begin{array}{ccc}
\text{Gr}_G & \longrightarrow & \text{Gr}_P \\
\downarrow & & \downarrow \\
\text{Gr}_B & \longrightarrow & \text{Gr}_{BL}
\end{array}
\]

Corollary 4.2. There is a natural isomorphism

\[
\begin{align*}
\mathcal{R}_T^G & \iff \mathcal{R}_T^L \circ \mathcal{R}_L^G : D^b(\text{Gr}_G, \overline{\mathbb{Q}}_p) \to D^b(\text{Gr}_T, \overline{\mathbb{Q}}_p), \\
\mathcal{R}_T^G & \iff \mathcal{R}_T^L \circ \mathcal{R}_L^G : D^b_{L+G}(\text{Gr}_G, \overline{\mathbb{Q}}_p) \to D^b_{L+T}(\text{Gr}_T, \overline{\mathbb{Q}}_p).
\end{align*}
\]
(4.6)
(4.7)
Proof. We can define these isomorphisms by the following pasting diagrams:

\[
\begin{array}{c}
D^b(\text{Gr}_G) \xrightarrow{(\cdot)^*} D^b(\text{Gr}_P) \xrightarrow{(\cdot)^*} D^b(\text{Gr}_L) \\
\uparrow^{(\cdot)^*} \quad \uparrow^{(\cdot)^*} \quad \uparrow^{(\cdot)^*} \\
D^b(\text{Gr}_B) = D^b(\text{Gr}_{B_L}) \xrightarrow{(\cdot)^*} D^b(\text{Gr}_{B_L}) \xrightarrow{(\cdot)^*} D^b(\text{Gr}_T),
\end{array}
\]

where the isomorphisms (Co), (BC), (For), (Tr) are defined in §A. □

The connected components of $\text{Gr}_L$ correspond to the characters of $Z(\bar{L})$, where $\bar{L} \subset \bar{G}$ is the Levi subgroup containing $\bar{T}$ whose roots are dual to those of $L$. Let us write $(\text{Gr}_L)_\chi$ for the connected component of $\text{Gr}_L$ corresponding to $\chi \in \mathbb{X}^\bullet(Z(\bar{L}))$. Set $\rho_{GL} := \rho_G - \rho_L$, where $\rho_G, \rho_L$ are the half sum of positive roots of $G, L$, respectively. We define a functor $\widehat{\mathcal{R}}_L^G : D^b(\text{Gr}_G, \mathbb{Q}_\ell) \to D^b(\text{Gr}_L, \mathbb{Q}_\ell)$ by

\[
\widehat{\mathcal{R}}_L^G(M) = \bigoplus_{\chi \in \mathbb{X}^\bullet(Z(\bar{L}))} \widehat{\mathcal{R}}_L^G(M)|_{(\text{Gr}_L)_\chi}[\langle \chi, 2\rho_{GL} \rangle]
\]

and its equivariant version $\widetilde{\mathcal{R}}_L^G : D^b_{L+G}(\text{Gr}_G, \mathbb{Q}_\ell) \to D^b_{L+L}(\text{Gr}_L, \mathbb{Q}_\ell)$ by

\[
\widetilde{\mathcal{R}}_L^G(M) = \bigoplus_{\chi \in \mathbb{X}^\bullet(Z(\bar{L}))} \widetilde{\mathcal{R}}_L^G(M)|_{(\text{Gr}_L)_\chi}[\langle \chi, 2\rho_{GL} \rangle].
\]
Then we have an isomorphism
\[ \mathcal{R}_L^G \circ \text{For}^{L+G} \iff \text{For}^{L+L} \circ \mathcal{R}_L^G \] (4.8)
by shifting (4.4). Set
\[ F_G := H^*(\text{Gr}_G, -) : \text{Perv}_{L+G}(\text{Gr}_G) \to \text{Vect}_{\mathbb{Q}_\ell} \]
where \( \text{Vect}_{\mathbb{Q}_\ell} \) is the category of finite dimensional \( \mathbb{Q}_\ell \)-vector spaces.

**Lemma 4.3.**

(i) \( \mathcal{R}_L^G \) is exact with respect to the perverse \( t \)-structure.

(ii) The functor \( \mathcal{R}_L^G \) induces a functor
\[ \mathcal{R}_L^G : \text{Perv}_{L+G}(\text{Gr}_G, \mathbb{Q}_\ell) \to \text{Perv}_{L+L}(\text{Gr}_L, \mathbb{Q}_\ell). \]

(iii) There is a natural isomorphism
\[ \mathcal{R}_T^G \iff \mathcal{R}_T^L \circ \mathcal{R}_L^G : \text{Perv}_{L+G}(\text{Gr}_G, \mathbb{Q}_\ell) \to \text{Perv}_{L+T}(\text{Gr}_T, \mathbb{Q}_\ell). \] (4.9)

**Proof.** First, we can construct the transitivity isomorphisms between functors on derived categories by shifting the results in Lemma 4.2:

\[ \mathcal{R}_T^G \iff \mathcal{R}_T^L \circ \mathcal{R}_L^G : D^b(\text{Gr}_G, \mathbb{Q}_\ell) \to D^b(\text{Gr}_T, \mathbb{Q}_\ell), \] (4.10)

\[ \tilde{\mathcal{R}}_T^G \iff \tilde{\mathcal{R}}_T^L \circ \mathcal{R}_L^G : D^b_{L+G}(\text{Gr}_G, \mathbb{Q}_\ell) \to D^b_{L+T}(\text{Gr}_T, \mathbb{Q}_\ell). \] (4.11)

Therefore, if we show (ii), it implies (iii) by restricting the isomorphism (4.10) to the category of equivariant perverse sheaves.

Also, as the functor \( \text{For} : D^b_{L+G}(\text{Gr}_G) \to D^b(\text{Gr}_G) \) induces the equivalence
\[ \text{For} : \text{Perv}_{L+G}(\text{Gr}_G) \overset{\sim}{\to} \text{Perv}_{L+G}(\text{Gr}_G), \]
it follows from (4.8) that (i) implies (ii).

We want to prove (i). For this, we can use the argument in [BD, Proposition 5.3.29]. Namely, as follows:

For \( \mathcal{A} \in \text{Perv}_{L+G}(\text{Gr}_G) \), we know \( \mathcal{R}_T^G(\mathcal{A}) \in \text{Perv}(\text{Gr}_T) \) by [MV07, Theorem 3.5] or [Zhu17a, Proposition 2.7]. Also, \( \mathcal{R}_T^G \) is faithful on \( \text{Perv}_{L+G}(\text{Gr}_G) \) (by [Zhu17a, Corollary 2.10]). Moreover, the forgetful functor \( \text{For}^{L+G} : D^b_{L+G}(\text{Gr}_G) \to D^b(\text{Gr}_G) \) is faithful and exact with respect to the perverse \( t \)-structure. Hence by (4.8), it follows that \( \tilde{\mathcal{R}}_T^G \) is faithful and exact with respect to the perverse \( t \)-structure. Thus for \( \mathcal{A}' \in D^b_{L+G}(\text{Gr}_G) \),
the object $A'$ is a perverse sheaf if and only if the object $\mathcal{R}_T(A')$ is a perverse sheaf. Therefore, if $A'$ is an object in $\text{Perv}'_{L+G}(\text{Gr}_G)$, then $\mathcal{R}_T(A')$ is a perverse sheaf, and from the isomorphism

$$\mathcal{R}_T(A') \cong \mathcal{R}_T(\mathcal{R}_L(A')),$$

obtained by shifting (1.7), it follows that $\mathcal{R}_L(A')$ is a perverse sheaf. This proves the lemma.

Now we construct a functor $\mathcal{R}_L^G$. Set

$$\text{Gr}_P^{\text{sm}} := \text{Gr}_P^{\circ} \cap (i_P)^{-1}(\text{Gr}_G^{\text{sm}}).$$

**Lemma 4.4.** There is an inclusion $q_P(\text{Gr}_P^{\text{sm}}) \subset \text{Gr}_L^{\text{sm}}$.

**Proof.** Assume the contrary. Then there exists $\lambda \in \mathcal{X}^\circ(T)$ such that $\lambda$ is not small for $\tilde{L}$ and $q_P(\text{Gr}_P^{\text{sm}})^{-1}(S_{L,\lambda}) \neq \emptyset$, where $S_{L,\lambda}$ is the semi-infinite orbit defined in §2.2.

From the cartesian square (4.3), it follows that $i_P(q_P^{-1}(S_{L,\lambda})) = S_{G,\lambda}$. Hence we have an inclusion

$$(\emptyset \neq) i_P(q_P^{-1}(q_P(\text{Gr}_P^{\text{sm}}) \cap S_{L,\lambda})) \subset \text{Gr}_G^{\text{sm}} \cap S_{G,\lambda}.$$  

This implies that there exists a $\mu \in \mathcal{X}^\circ(T)$ small for $\bar{G}$ such that $\text{Gr}_{G,\mu} \cap S_{G,\lambda} \neq \emptyset$.

By [Zhu17a, Corollary 2.8] or [MV07, Theorem 3.2], it means that $\lambda$ is in the convex hull of $W_G \cdot \mu$. Therefore the convex hull of $W_L \cdot \lambda$ is contained in the convex hull of $W_G \cdot \mu$, which contradicts the fact that $\lambda$ is not small for $\tilde{L}$.

Then we have the following commutative diagram

$\begin{array}{cccc}
\text{Gr}_L^{\text{sm}} & \xrightarrow{q_P^{\text{sm}}} & \text{Gr}_P^{\text{sm}} & \xrightarrow{i_P^{\text{sm}}} & \text{Gr}_G^{\text{sm}} \\
\downarrow f_L^\circ & & \downarrow f_P^\circ & & \downarrow f_G^\circ \\
\text{Gr}_L^\circ & \xrightarrow{q_P^\circ} & \text{Gr}_P^\circ & \xrightarrow{i_P^\circ} & \text{Gr}_G^\circ \\
\downarrow z_L & & \downarrow z_P & & \downarrow z_G \\
\text{Gr}_L & \xrightarrow{q_P} & \text{Gr}_P & \xrightarrow{i_P} & \text{Gr}_G
\end{array}$

(4.12)

where all vertical arrows are inclusions. The top right square is cartesian by definition of $\text{Gr}_P^{\text{sm}}$, and the bottom left square is cartesian since $q_P$ induces a bijection between the set of connected component. Put

$$\mathcal{M}_L^G := (q_P^\circ)^* \circ (i_P^\circ)^!: D^b(\text{Gr}_G^{\circ}, \mathcal{O}_L) \to D^b(\text{Gr}_L^{\circ}, \mathcal{O}_L),$$

$$\mathcal{M}_L^{G^{\text{sm}}} := (q_P^{\text{sm}})^* \circ (i_P^{\text{sm}})^!: D^b(\text{Gr}_G^{\text{sm}}, \mathcal{O}_L) \to D^b(\text{Gr}_L^{\text{sm}}, \mathcal{O}_L).$$
Then we have the following lemma:

**Lemma 4.5.**

(i) There is a natural isomorphism

\[(z_L)^! \circ \mathcal{R}_L^G \iff \mathcal{R}_L^G \circ (z_G)^! .\]  

In particular, \( \mathcal{R}_L^G \) induces a functor

\[ \mathcal{R}_L^G : \text{Perv}_{L^+G}(\text{Gr}_G^\circ) \to \text{Perv}_{L^+L}(\text{Gr}_L^\circ). \]

(ii) There is a natural isomorphism

\[(f_L^*)_* \circ \mathcal{R}_L^G \iff \mathcal{R}_L^G \circ (f_G^*)_* .\]  

In particular, combining with (i), \( \mathcal{R}_L^G \) induces a functor

\[ \mathcal{R}_L^G : \text{Perv}_{L^+G}(\text{Gr}_{\text{sm}}^G) \to \text{Perv}_{L^+L}(\text{Gr}_{\text{sm}}^L). \]

**Proof.** (i) Since the connected component \( \text{Gr}_G^\circ \) corresponds to \( 0 \in X^*(Z(\tilde{L})) \), we obtain \( (z_L)^! \circ \mathcal{R}_L^G = (z_L)^! \circ \mathcal{R}_G^L \). Hence it suffices to show

\[(z_L)^! \circ \mathcal{R}_L^G \iff \mathcal{R}_L^G \circ (z_G)^! .\]

However, this isomorphism can be defined by the following pasting diagram:

\[
\begin{array}{ccc}
D^b(\text{Gr}_L) & \xrightarrow{(i_P)^!} & D^b(\text{Gr}_P) & \xrightarrow{(q_P)_*} & D^b(\text{Gr}_G) \\
(z_G)^! & \downarrow & (z_P)^! & \downarrow & (z_L)^! \\
D^b(\text{Gr}_L^g) & \xrightarrow{(i_P)^!} & D^b(\text{Gr}_P^g) & \xrightarrow{(q_P)_*} & D^b(\text{Gr}_G^g) \\
\end{array}
\]

where the isomorphism \( (\text{Co}), (\text{BC}) \) is obtained by the diagram \( \ref{112} \).

In particular, we have \( (z_L)^! \circ \mathcal{R}_L^G \circ (z_L)_* \iff \mathcal{R}_L^G \), hence \( \mathcal{R}_L^G \) induces a functor from \( \text{Perv}_{L^+G}(\text{Gr}_G^\circ) \) to \( \text{Perv}_{L^+L}(\text{Gr}_L^\circ) \).

(ii) Similarly, the desired isomorphism can be defined by the following pasting diagram:

\[
\begin{array}{ccc}
D^b(\text{Gr}_{\text{sm}}^L) & \xrightarrow{(i_P)^!} & D^b(\text{Gr}_{\text{sm}}^P) & \xrightarrow{(q_P^\text{sm})_*} & D^b(\text{Gr}_{\text{sm}}^G) \\
(f_L^*)_* & \downarrow & (f_P^*)_* & \downarrow & (f_G^*)_* \\
D^b(\text{Gr}_L^\circ) & \xrightarrow{(i_P)^!} & D^b(\text{Gr}_P^\circ) & \xrightarrow{(q_P)_*} & D^b(\text{Gr}_G^\circ) \\
\end{array}
\]
where the isomorphism $(\text{Co}), (\text{BC})$ is obtained by the diagram (4.12).

In particular, $\mathcal{R}_G^L$ induces a functor $\text{Perv}_{L+G}(\text{Gr}_G^\text{sm})$ to $\text{Perv}_{L+L}(\text{Gr}_L^\text{sm})$ since the functor $(f_\ell)_*$ is fully faithful and exact with respect to the perverse $\ell$-structure.

In order to construct the transitivity isomorphism for $\mathcal{R}_G^L$, we want to find cartesian diagrams:

**Lemma 4.6.** The commutative square

$$
\begin{array}{ccc}
\text{Gr}_B^\text{sm} & \longrightarrow & \text{Gr}_P^\text{sm} \\
\downarrow & & \downarrow \\
\text{Gr}_B^\circ & \longrightarrow & \text{Gr}_P^\circ
\end{array}
$$

is cartesian.

**Proof.** We want to prove that

$$
\begin{array}{ccc}
\text{Gr}_B^\text{sm} & \longrightarrow & \text{Gr}_P^\text{sm} \\
\downarrow & & \downarrow \\
\text{Gr}_B^\circ & \longrightarrow & \text{Gr}_P^\circ
\end{array}
$$

is cartesian. For that, it suffices to show that the following two squares are cartesian:

$$
\begin{array}{ccc}
\text{Gr}_B^\text{sm} & \longrightarrow & \text{Gr}_P^\text{sm} \\
\downarrow & & \downarrow \\
\text{Gr}_B^\circ & \longrightarrow & \text{Gr}_P^\circ
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\text{Gr}_B^\circ & \longrightarrow & \text{Gr}_P^\circ \\
\downarrow & & \downarrow \\
\text{Gr}_B^\circ & \longrightarrow & \text{Gr}_P^\circ
\end{array}
$$

The left square is cartesian by the definition of $\text{Gr}_B^\text{sm}, \text{Gr}_P^\text{sm}$, and the right square is cartesian by Lemma 4.1.

By the same argument as Corollary 4.2, we obtain the transitivity isomorphism for $\mathcal{R}_L^G$ from Lemma 4.6.

**Corollary 4.7.** There is a natural isomorphism

$$
\mathcal{R}_L^G \iff \mathcal{R}_L^T \circ \mathcal{R}_L^G : \text{Perv}_{L+G}(\text{Gr}_G, \overline{\mathbb{Q}}_\ell) \rightarrow \text{Perv}_{L+T}(\text{Gr}_T, \overline{\mathbb{Q}}_\ell).
$$

### 4.3 Category $\text{Rep}(\tilde{G}, \overline{\mathbb{Q}}_\ell)_{\text{sm}}$

We want to construct a restriction functor for $\text{Rep}(\tilde{G}, \overline{\mathbb{Q}}_\ell)_{\text{sm}}$ using the geometric Satake correspondence.
4.3.1 Monoidal structure on $\mathcal{R}_L^G$

We want to construct a canonical monoidal structure on $\mathcal{R}_L^G$. Recall that for an algebraic group $H$, the convolution affine Grassmannian $\text{Gr}_H \times \text{Gr}_H$ is defined by $\text{Gr}_H \times \text{Gr}_H := L H \times L^+ H \text{Gr}_H$. The convolution map $m_H: \text{Gr}_H \times \text{Gr}_H \to \text{Gr}_H$ is induced by the multiplication map $L H \times L H \to L H$. For perverse sheaves $A, B \in \text{Perv}_{L^+G}(\text{Gr}_G)$, there is a perverse sheaf $A \boxtimes B \in \text{Perv}_{L^+G}(\text{Gr}_G)$ called the “external twisted product” such that the pullback of $A \boxtimes B$ along $LG \times \text{Gr}_G$ is equal to the pullback of $A \widetilde{\otimes} B$ along $LG \times \text{Gr}_G \to \text{Gr}_G \sim \text{Gr}_G$, see §A.21. The convolution product $A \otimes B \in \text{Perv}_{L^+G}(\text{Gr}_G)$ is defined by

$$A \otimes B = (m_G)_!(A \widetilde{\otimes} B).$$

Consider the diagrams

$$\text{Gr}_L \sim \text{Gr}_L \underbrace{q^P \times q_P}_{\text{Gr}_P \times \text{Gr}_P} \underbrace{i_P \times i_P}_{\text{Gr}_G \times \text{Gr}_G}$$

and define a functor $\mathcal{R}_L^G$ by

$$\mathcal{R}_L^G := (q_P \times q_P)_* \circ (i_P \times i_P)^! : D^b(\text{Gr}_G \sim \text{Gr}_G) \to D^b(\text{Gr}_L \sim \text{Gr}_L).$$

Also, define a functor $\mathcal{R}_L^G$ by

$$\mathcal{R}_L^G(M) = \bigoplus_{(\chi_1, \chi_2) \in X^*(Z(L)) \times X^*(Z(L))} (\mathcal{R}_L^G(M))|_{(\text{Gr}_L)_{\chi_1 \sim (\text{Gr}_L)_{\chi_2}}[(2\rho_{GL}, \chi_1 + \chi_2)]},$$

see \(\text{A.22}\) for the notations.

**Lemma 4.8.** For any $A, B \in \text{Perv}_{L^+G}(\text{Gr}_G)$, there is a canonical isomorphism

$$\mathcal{R}_L^G(A \boxtimes B) \cong \mathcal{R}_L^G(A) \boxtimes \mathcal{R}_L^G(B).$$

**(Proof.** By Braden’s theorem \([\text{Bra03}]\), there is a canonical isomorphism

$$q_P \sim q_P)_* \circ (i_P \sim i_P)^! \cong (q_{P^-} \sim q_{P^-})_* \circ (i_{P^-} \sim i_{P^-})^*$$ \hspace{1cm} (4.16)

$$q_P)_* \circ (i_P)^! \cong (q_{P^-})_* \circ (i_{P^-})^*$$ \hspace{1cm} (4.17)

where $P^-$ is the parabolic subgroup opposite to $P$. Thus it suffices to show the lemma under the replacement $(\cdot)_*$ by $(\cdot)_!$ and $(\cdot)^!$ by $(\cdot)^*$. This follows from §A.21.1, §A.21.2 with some shift.

**Theorem 4.9.** The functor $\mathcal{R}_L^G$ has a canonical monoidal structure.

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Proof. By (4.16) and (4.17), it suffices to show the theorem under the replacement $(\cdot)_*$ by $(\cdot)$, and $(\cdot)^!$ by $(\cdot)^*$ in the definition of the restriction functors $\mathcal{R}^G_L$, $\mathcal{R}^G_L$, $\mathcal{R}^G_L$ and $\mathcal{R}^G_L$. Thus in this proof, we temporarily redefine $\mathcal{R}^G_L$ as $(q_P)_! \circ (i_P)^*$, and so on.

Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Gr}_L \times \text{Gr}_L & \xrightarrow{q_P \times q_P} & \text{Gr}_P \times \text{Gr}_P \\
\downarrow m_L & & \downarrow m_P \\
\text{Gr}_L & \xrightarrow{q_P} & \text{Gr}_P \\
\end{array}
\]

where $m_G, m_P, m_L$ are the convolution maps. Consider the space $\text{Gr}_P \sim \text{Gr}_G := LP \times \text{Gr}_G$. Then we have the following diagram:

\[
\begin{array}{ccc}
\text{Gr}_L \times \text{Gr}_L & \xrightarrow{q_P \times q_P} & \text{Gr}_P \times \text{Gr}_P \\
\downarrow m_L & & \downarrow m_P \\
\text{Gr}_L & \xrightarrow{q_P} & \text{Gr}_P \\
\end{array}
\]

where the right square is cartesian. Then we have

\[
(m_L)_! \left( \mathcal{R}^G_L(A) \boxtimes \mathcal{R}^G_L(B) \right) \overset{\text{(4.16)}}{=} (m_L)_! \left( \mathcal{R}^G_L(A \boxtimes B) \right)
\]

\[
= (m_L)_!(q_P \times q_P)_!(i_P \times i_P)^*(A \boxtimes B)
\]

\[
\overset{\text{(Co)}}{=} (q_P)_!(m_P)_!(1 \times i_P)^*(i_P \times 1)^*(A \boxtimes B)
\]

\[
\overset{\text{(BC)}}{=} (q_P)_!(i_P)^*(m'_!)(i_P \times 1)^*(A \boxtimes B)
\]

\[
= \mathcal{R}^G_L \left( (m'_!)(i_P \times 1)^*(A \boxtimes B) \right) \quad (4.18)
\]

for any $A, B \in \text{Perv}_{L^+G}(\text{Gr}_G)$. Thus it suffices to show that

\[
(m'_!)(i_P \times 1)^*(A \boxtimes B) \overset{\text{def}}{=} (m_G)_!(A \boxtimes B).
\]

It can be proved by an argument similar to [MV07, Theorem 3.6]. Namely, it is as follows:

Recall that there is an isomorphism

\[
(pr_1, m_G)_! : \text{Gr}_G \times \text{Gr}_G \to \text{Gr}_G \times \text{Gr}_G.
\]

We have a commutative diagram

\[
\begin{array}{ccc}
\text{Gr}_P \times \text{Gr}_G & \xrightarrow{i_P \times 1} & \text{Gr}_G \times \text{Gr}_G \\
(pr_1, m')_! \downarrow & & \downarrow (pr_1, m_G)_! \\
\text{Gr}_P \times \text{Gr}_G & \xrightarrow{i_P \times 1} & \text{Gr}_G \times \text{Gr}_G.
\end{array}
\]
Set $$\mathcal{C} := (\text{pr}_1, m_G)_!(A \boxtimes B).$$

Then it suffices to show

$$(\text{pr}_2)_!(i_P \times 1)^*(\mathcal{C}) \cong (\text{pr}_2)_!(\mathcal{C}).$$

Here the maps are as in the following diagram:

\[
\begin{array}{ccc}
\text{Gr}_P \times \text{Gr}_G & \xrightarrow{i_P \times 1} & \text{Gr}_G \times \text{Gr}_G \\
\downarrow \text{pr}_2 & & \downarrow \text{pr}_2 \\
\text{Gr}_G. & & \\
\end{array}
\]

Let $$S_{\chi}$$ be the connected component of $$\text{Gr}_P$$ corresponding to $$\chi \in \mathbb{X}^\bullet(Z(\tilde{L}))$$. Since the map

$$\text{Gr}_P = \bigsqcup_{\chi \in \mathbb{X}^\bullet(Z(\tilde{L}))} S_{\chi} \rightarrow \text{Gr}_G$$

is bijective, there are $$\lambda_1, \ldots, \lambda_m \in \mathbb{X}^\bullet(Z(\tilde{L}))$$ such that

$$\text{Supp}(\mathcal{C}) \subset \bigcup_{i=1}^m S_{\lambda_i} \times \text{Gr}_P.$$ 

Rearranging their indexes if necessary, we may assume that

$$\text{Supp}(\mathcal{C}) \cap \left( \bigcup_{i=1}^{j-1} S_{\lambda_i} \times \text{Gr}_P \cup S_{\lambda_j} \times \text{Gr}_P \right)$$

$$= \text{Supp}(\mathcal{C}) \cap \left( \bigcup_{i=1}^j S_{\lambda_i} \times \text{Gr}_P \right).$$

for all $$j \in \{1, \ldots, m\}$$. Let

$$S_{\leq j} := \bigcup_{i=1}^j S_{\lambda_i} \subset \text{Gr}_G$$

and let $$i_{\leq j} : S_{\leq j} \hookrightarrow \text{Gr}_G$$, $$i_j : S_{\lambda_j} \hookrightarrow \text{Gr}_G$$ be the inclusions. Put

$$\mathcal{F}_j := (\text{pr}_2)_!(i_j \times 1)^*(\mathcal{C}),$$

$$\mathcal{F}_{\leq j} := (\text{pr}_2)_!(i_{\leq j} \times 1)^*(\mathcal{C}),$$

$$\mathcal{F}_\infty := (\text{pr}_2)_!(\mathcal{C}).$$
Then \( F_{\leq m} = F_{\infty} \) by definition. We want to construct a canonical isomorphism

\[
F_{\infty} = \bigoplus_{j=1}^{m} F_{j}
\]
since

\[
\bigoplus_{j=1}^{m} F_{j} = (pr_2)_! (i_P \times 1)^!(C).
\]

By (4.18), we have

\[
\mathcal{R}^G_L \left( \bigoplus_{j=1}^{m} F_{j} \right) \cong \mathcal{R}^G_L A \ast \mathcal{R}^G_L B,
\]

so this is a perverse sheaf. From the fact that \( \mathcal{R}^G_L \) is faithful and exact with respect to the perverse \( t \)-structure, it follows that \( \bigoplus_{j=1}^{m} F_{j} \) is a perverse sheaf.

By induction on \( j \) using the distinguished triangle

\[
F_{j} \to F_{\leq j} \to F_{\leq j-1} \to,
\]

the complex \( F_{\leq j} \) is also a perverse sheaf. Thus the above distinguished triangle can be regarded as an exact sequence of perverse sheaves.

On the other hand, consider the orbits \( \{ T_{\lambda} \mid \lambda \in \mathbb{X}^\bullet(Z(\tilde{L})) \} \) opposite to \( \{ S_{\lambda} \mid \lambda \in \mathbb{X}^\bullet(Z(\tilde{L})) \} \). It holds that

\[
\text{Gr}_{P^\perp} = \coprod_{\lambda \in \mathbb{X}^\bullet(Z(\tilde{L}))} T_{\lambda}.
\]

We have the diagram

\[
\begin{array}{ccc}
\text{Gr}_{P^\perp} \times \text{Gr}_G & \xrightarrow{i_{P^\perp,1} \times 1} & \text{Gr}_G \times \text{Gr}_G \\
pr_2 \downarrow & & \downarrow \text{pr}_2 \\
\text{Gr}_G & &
\end{array}
\]

Let

\[
T_{\leq j} := \bigcup_{i=1}^{j} T_{\lambda_i} \subset \text{Gr}_G
\]

and let \( i'_{\leq j} : T_{\leq j} \hookrightarrow \text{Gr}_G, i'_j : T_{\lambda_j} \hookrightarrow \text{Gr}_G \) be the inclusion. Put

\[
\begin{align*}
F'_{j} &:= (pr_2)_! (i'_{j} \times 1)^!(C), \\
F'_{\leq j} &:= (pr_2)_! (i'_{\leq j} \times 1)^!(C), \\
F'_{\infty} &:= (pr_2)_! (C).
\end{align*}
\]
Then $\mathcal{F}'_{\leq m} = \mathcal{F}'_{\infty}$ by definition. Since the restriction of $\text{pr}_2: \text{Gr}_G \times \text{Gr}_G \to \text{Gr}_G$ to $\text{Supp} C$ is perfectly proper, $\mathcal{F}'_{\leq m} = \mathcal{F}'_{\infty}$. Also, there is an isomorphism

\[
\mathcal{F}'_j \cong (\text{pr}_2)_*(q_{P-} \times 1)_*(i'_j \times 1)^!(C)
\]

\[
\cong (\text{pr}_2)_!(q_{P-} \times 1)_!(i'_j \times 1)^!(C) \quad \text{pr}_2|_{\text{Supp} C \text{ is properly proper}}
\]

\[
\cong (\text{pr}_2)_!(i_j \times 1)^!(C)
\]

\[
\cong \mathcal{F}_j.
\] (4.20)

By the same argument above, there is a distinguished triangle

\[
\mathcal{F}'_{\leq j-1} \to \mathcal{F}'_{\leq j} \to \mathcal{F}'_j \to
\] (4.21)

and this can be regarded as an exact sequence of perverse sheaves.

We claim that the composition of the natural morphisms

\[
\varphi_j: \mathcal{F}'_{\leq j} \to \mathcal{F}'_{\infty} = \mathcal{F}_{\infty} \to \mathcal{F}_{\leq j}
\]

is an isomorphism.

We will show this by descending induction on $j$. If $j = m$, then the claim is clear since $\mathcal{F}'_{\leq j} \to \mathcal{F}'_{\infty}$ and $\mathcal{F}_{\infty} \to \mathcal{F}_{\leq j}$ are isomorphisms. Assume $j < m$. The homomorphism $\varphi_{j+1}$ is an isomorphism by induction, and one can show that the morphism

\[
\mathcal{F}_{j+1} \to \mathcal{F}'_{\leq j+1} \xrightarrow{\varphi_j^{-1}} \mathcal{F}'_{j+1} \to \mathcal{F}_{j+1}
\]

is the same as the morphism in (4.20) (cf. [Bra03, Theorem 3.6]). Thus the exact sequence (4.19) and (4.21) prove the claim.

From this claim, it follows that the two filtration on $\mathcal{F}_{\infty}$

\[
\text{Fil}^j = \ker(\mathcal{F}_{\infty} \to \mathcal{F}_{\leq j}),
\]

\[
\text{Fil}^{i,j} = \text{im}(\mathcal{F}_{\leq j} \to \mathcal{F}_{\infty})
\]

are complementary. This shows $\mathcal{F}_{\infty} = \bigoplus_{j=1}^m \mathcal{F}_j$. \hfill $\square$

### 4.3.2 Restriction for category $\text{Rep}(\overline{G}, \overline{\mathbb{Q}_\ell})_{\text{sm}}$

By [Zhu17a, Corollary 2.10] and [MV07, Theorem 3.6], there is an isomorphism

\[
\mathcal{F}_G \leftrightarrow \mathcal{F}_T \circ \overline{\mathbb{R}}^G_T.
\] (4.22)
Then we have an isomorphism

\[ F_G \iff F_T \circ \tilde{R}_R \]

\[ \iff F_T \circ \tilde{R}_L \circ \tilde{R}_L^G \]

\[ \iff F_L \circ \tilde{R}_L^G. \quad (4.23) \]

**Lemma 4.10.** The isomorphism \((4.23)\) is monoidal with respect to the monoidal structure of \(G \circ \tilde{R}_L^G\) defined in §4.3.1 and the monoidal structure of \(F_G, F_L\) defined in [Zhu17a, §2.3] or [Zhu17b, §5.2].

**Proof.**

(i) There is an isomorphism

\[ F_G \iff F_L \circ \tilde{R}_L^G \quad (4.24) \]

obtained by applying Braden’s theorem to the diagram

\[ \text{Gr}_L \xrightarrow{q_P} \text{Gr}_{P} \xrightarrow{i_P} \text{Gr}_G \]

and discussing in the same way as [MV07, Theorem 3.6]. This isomorphism can be written as follows:

\[ \begin{array}{c}
\text{Perv}_{L+G}(\text{Gr}_G) \\
\downarrow \text{(Br)} \end{array} \xrightarrow{\sim} \begin{array}{c}
\tilde{R}_L^G \\
\downarrow \text{(Br)} \end{array} \xrightarrow{\sim} \begin{array}{c}
\text{D}^b(\text{Gr}_L) \\
\downarrow \text{(Br)} \end{array} \xrightarrow{\sim} \begin{array}{c}
\text{D}^b(\{\ast\}). \\
\end{array} \quad (4.25) \]

One can show that this isomorphism coincides with the isomorphism \((4.23)\) using the diagram \((4.5)\) and proving the commutativity of the following prism:

\[ \begin{array}{ccc}
\text{Perv}_{L+G}(\text{Gr}_G) & \xrightarrow{\sim} & \text{D}^b(\{\ast\}) \\
\downarrow \text{(Br)} & & \downarrow \text{(Br)} \\
\tilde{R}_L^G & \xrightarrow{\sim} & \text{D}^b(\{\ast\}) \\
\downarrow \text{(Br)} & & \downarrow \text{(Br)} \\
\text{D}^b(\text{Gr}_L) & \xrightarrow{\sim} & \text{D}^b(\{\ast\}) \\
\downarrow \text{(Br)} & & \downarrow \text{(Br)} \\
\text{D}^b(\text{Gr}_T) & \xrightarrow{\sim} & \text{D}^b(\{\ast\}). \\
\end{array} \]
(ii) From (4.2), it follows that the isomorphism (4.25) lifts to the following isomorphism also denoted by (Bra):

$$
\begin{array}{ccc}
Perv'_{L+G}(Gr_G) & \xrightarrow{\mathcal{G}^G_{L+G}} & D^b_{L+G}(Gr_L) \\
\rightarrow & \mathcal{R}^G_{L+L}(Gr_L) \rightarrow & D^b_{L+L}(Gr_L) \\
For^{L+G_G(\cdot)} & \rightarrow & For^{L+L_G(\cdot)} \rightarrow \\
\end{array}
$$

(4.26)

Define the functor $\mathcal{R}^G_L : D^b_{L+G}(Gr_G \times Gr_G) \rightarrow D^b_{L+L}(Gr_L \times Gr_L)$ as the composition

$$
\begin{array}{ccc}
D^b_{L+G}(Gr_G \times Gr_G) & \xrightarrow{\mathcal{R}^G_{L+G}} & D^b_{L+G}(Gr_G \times Gr_G) \\
\rightarrow & D^b_{L+G}(Gr_G \times Gr_G) \rightarrow & D^b_{L+L}(Gr_L \times Gr_L) \\
\langle ip \rangle \rightarrow & D^b_{L+G}(Gr_G \times Gr_G) \rightarrow & D^b_{L+L}(Gr_L \times Gr_L) \\
\end{array}
$$

There is a transitivity isomorphism for this functor just as $\bar{\mathcal{R}}^G_L$. Write $Perv''_{L+G}(Gr_G \times Gr_G)$ for the essential image of the functor

$$
\cdot \mathcal{R} : Perv'_{L+G}(Gr_G) \times Perv'_{L+G}(Gr_G) \rightarrow D^b_{L+G}(Gr_G \times Gr_G) \rightarrow D^b_{L+G}(Gr_G \times Gr_G)
$$

where the second functor is an isomorphism obtained from the isomorphism $(pr_1, m_G) : Gr_G \times Gr_G \rightarrow Gr_G \times Gr_G$. As well as (4.26), we have the following isomorphisms:

$$
\begin{array}{ccc}
Perv''_{L+G}(Gr_G \times Gr_G) & \xrightarrow{\mathcal{R}^G_{L+G \times L+G}} & D^b_{L+L}(Gr_L \times Gr_L) \\
\rightarrow & D^b_{L+L}(Gr_L \times Gr_L) \rightarrow & D^b_{L+L}(Gr_L \times Gr_L) \\
For^{L+G_G(\cdot)} & \rightarrow & For^{L+L_G(\cdot)} \rightarrow \\
\end{array}
$$

(4.27)

Also, by [Zhu17a §2.3] or [Zhu17b §5.2], there is an isomorphism

$$
\begin{array}{ccc}
D^b_{L+G}(Gr_G) \times D^b_{L+G}(Gr_G) & \xrightarrow{\mathcal{R}^G_{L+G \times L+G}} & D^b_{L+G}(Gr_G \times Gr_G) \\
\rightarrow & D^b_{L+G}(Gr_G \times Gr_G) \rightarrow & D^b_{L+G}(Gr_G \times Gr_G) \\
(For^{L+G_G(\cdot)} \times For^{L+G_G(\cdot)}) & \rightarrow & For^{L+G_G(\cdot)} \rightarrow \\
D^b(\{\ast\}) \times D^b(\{\ast\}) & \rightarrow & D^b(\{\ast\}) \\
\end{array}
$$

(4.27)
By an argument similar to (i) using the diagram

\[
\begin{array}{c}
\text{Gr}_G \times \text{Gr}_G \\
\downarrow \\
\text{Gr}_L \times \text{Gr}_G \\
\downarrow \\
\text{Gr}_L \times \text{Gr}_L \\
\end{array}
\]

instead of \((4.25)\), and using the isomorphism

\[(\text{pr}_1, m_G) : \text{Gr}_G \times \text{Gr}_G \to \text{Gr}_G \times \text{Gr}_G,\]

one can show that the commutativity of the following prism proves the lemma:

\[
\begin{array}{c}
\text{Perv}'(\text{Gr}_G \times \text{Gr}_G) \\
\downarrow \\
\text{D}^b((-1)) \times \text{D}^b((+1)) \\
\end{array}
\]

(iii) The commutativity of the prism \((4.28)\) follows from some straightforward calculations as follows:

From the construction of the usual Künneth formula and the construction of the isomorphisms obtained from Braden’s theorem, one can show that the prism \((4.28)\) would be commutative under the replacement of \(\circ\) by \(\otimes\). Thus it suffices to show that the commutativity of \((4.28)\) is reduced to this replaced version.

Put \(\text{Gr}_G^{(m)} := LG/L^+G(m)\) where \(L^+G(m) \subset L^+G\) is the \(m\)-th congruence subgroup. Consider the \(L^+G \times L^+G \times L^+G\)-action on \(\text{Gr}_G^{(m)} \times \text{Gr}_G\) defined by \((g, h, k) \cdot (x, y) \mapsto (gxh^{-1}, ky)\). On the other hand, consider \(L^+G \times L^+G\)-action on \(\text{Gr}_G^{(m)} \times \text{Gr}_G\) defined by \((g, h) \cdot (x, y) \mapsto (gxh^{-1}, hy)\). Then by definition the isomorphism \((4.27)\) is obtained from the isomorphism

\[
\begin{array}{c}
\text{D}^b_{L^+G \times L^+G \times L^+G}(\text{Gr}_G^{(m)} \times \text{Gr}_G) \\
\downarrow \\
\text{D}^b_{L^+G \times L^+G}(\text{Gr}_G^{(m)} \times \text{Gr}_G) \\
\end{array}
\]

(4.28)
together with the usual K"unneth formula and some calculations. Here we re-
gard $L^+G \times L^+G$ as a subgroup of $L^+G \times L^+G \times L^+G$ by the embedding
$L^+G \times L^+G \to L^+G \times L^+G \times L^+G$, $(g,h) \mapsto (g,h,h)$. (Note that the cate-
gory $D^b_{L^+G \times L^+G \times L^+G}(Gr^m_G \times Gr_G)$ is equivalent to $D^b_{L^+G \times L^+G \times L^+G}(Gr_G \times Gr_G)$ and
that $D^b_{L^+G \times L^+G}(Gr^m_G \times Gr_G)$ is equivalent to $D^b_{L^+G}(Gr_G \times Gr_G)$.)
One can show that the isomorphism (4.29) is compatible with the isomorphism (Bra).

Then one can show that this problem is reduced to the commutativity of (4.28) under the
replacement of $\tilde{\boxtimes}$ by $\boxtimes$, as desired.

\textbf{Corollary 4.11.} The monoidal structure of $\mathcal{R}_T^G$ defined in Theorem 4.9 coincides with
the one defined in [Zhu17a, Proposition 2.36].

\textbf{Proof.} By [Zhu17a, Proposition 2.36], a monoidal structure which makes the isomor-
phism (4.22) monoidal is unique. Hence the coincidence follows from Lemma 4.10.

Then we have a morphism of algebraic groups

$\iota_L^G: L = \text{Aut}^*(F_L) \to \text{Aut}^*(F_L \circ \mathcal{R}_L^G) \cong \text{Aut}^*(F_G) = \hat{G}$.

Then define

$$\mathcal{R}_L^G := (\iota_L^G)_*: \text{Rep}(\hat{G}, \underline{Q}_\ell) \to \text{Rep}(\hat{L}, \underline{Q}_\ell).$$

Since the following diagram of natural isomorphisms

$$\begin{array}{ccc}
F_G & \cong & F_T \circ \mathcal{R}_T^G \\
\downarrow & & \downarrow \\
F_L \circ \mathcal{R}_L^G & \cong & F_T \circ \mathcal{R}_T^L \circ \mathcal{R}_L^G
\end{array}$$

is commutative, we have $\iota_T^G = \iota_L^G \circ \iota_T^L$. It implies that

$$\mathcal{R}_T^G = \mathcal{R}_T^L \circ \mathcal{R}_L^G.$$

Put

$$\mathcal{R}_L^G := (-)^{Z(L)} \circ \mathcal{R}_L^G \circ \mathcal{I}_G: \text{Rep}(\hat{G})^{Z(\hat{G})} \to \text{Rep}(\hat{L})^{Z(\hat{L})}.$$

From the fact that $Z(\hat{G}) \subset Z(\hat{L})$, it follows that

$$\mathcal{R}_L^G \circ (-)^{Z(\hat{G})} = (-)^{Z(\hat{L})} \circ \mathcal{R}_L^G.$$
and that
\[ \mathcal{R}_T^G = \mathcal{R}_T^L \circ \mathcal{R}_L^G : \text{Rep}(\check{G})^{Z(\check{G})} \to \text{Rep}(\check{T})^{Z(\check{T})}. \] (4.30)

**Lemma 4.12.** If \( V \in \text{Rep}(\check{G})_{\text{sm}} \), then \( V' := (\mathcal{R}_L^G V)^{Z(\check{L})} \in \text{Rep}(\check{L})_{\text{sm}} \).

**Proof.** Since \( Z(\check{L}) \) acts trivially on \( V' \), all \( \check{T} \)-weights of \( V' \) are roots of \( \check{L} \). The convex hull of weights of \( V' \) is included in the convex hull of weights of \( V \), hence no weight is of the form \( 2\check{\alpha} \) for a root \( \check{\alpha} \) of \( \check{L} \). \( \square \)

This lemma implies that there is a unique functor \( \mathcal{R}_L^G : \text{Rep}(\check{G})_{\text{sm}} \to \text{Rep}(\check{L})_{\text{sm}} \) such that
\[ \mathcal{R}_L^G \circ \mathcal{I}_G^0 = \mathcal{I}_L^0 \circ \mathcal{R}_L^G. \]
Then by (4.30), we have
\[ \mathcal{R}_T^G = \mathcal{R}_T^L \circ \mathcal{R}_L^G. \]

### 4.4 Category \( \text{Perv}_G(\mathcal{N}_G^{\check{p} \rightarrow \infty}) \)

Let \( \mathcal{N}_P \subset \mathfrak{p} \) denote the nilpotent cone of \( \check{P} \). Then the diagram (4.11) induces the diagram
\[ \mathcal{N}_L \xrightarrow{p_P} \mathcal{N}_P \xrightarrow{m_P} \mathcal{N}_G. \]
Put
\[ \mathcal{R}_L^G := (p_P)_* \circ (m_P)^! : D^b(\mathcal{N}_G, \overline{\mathbb{Q}}_{\ell}) \to D^b(\mathcal{N}_L, \overline{\mathbb{Q}}_{\ell}). \]

First, we need to prove the following:

**Proposition 4.13.** The functor \( \mathcal{R}_L^G \) restrict to an exact functor
\[ \mathcal{R}_L^G : \text{Perv}_G(\mathcal{N}_G, \overline{\mathbb{Q}}_{\ell}) \to \text{Perv}_G(\mathcal{N}_L, \overline{\mathbb{Q}}_{\ell}). \]

To prove this, consider the functor \( \mathcal{R}_L^G \) for equivariant derived categories defined by the composition
\[ D^b_G(\mathcal{N}_G) \xrightarrow{\text{For}_P^G} D^b_P(\mathcal{N}_G) \xrightarrow{(m_P)^!} D^b_P(\mathcal{N}_P) \xrightarrow{(p_P)_*} D^b_P(\mathcal{N}_L) \xrightarrow{\text{For}_L^P} D^b_L(\mathcal{N}_L) \]
where \( \text{For} \) denotes a forgetful functor defined in §A.6.1. There is an isomorphism
\[ \mathcal{R}_L^G \circ \text{For} \iff \text{For} \circ \mathcal{R}_L^G \] (4.31)
defined by the following diagram

\[
\begin{array}{c}
D^b_G(N_G) \xrightarrow{\text{For}_G^b} D^b_P(N_G) \xrightarrow{(m_P)_!} D^b_P(N_P) \xrightarrow{(p_P)_*} D^b_P(N_L) \\
\text{For} \quad \text{For} \quad \text{For} \quad \text{For} \quad \text{For} \\
D^b_G(N_G) \xrightarrow{\text{For}_G^b} D^b_P(N_G) \xrightarrow{(m_P)_!} D^b_P(N_P) \xrightarrow{(p_P)_*} D^b_P(N_L) \\
\end{array}
\]

where the isomorphism (For) is defined in [A.6.1]. The forgetful functor \(\text{For}_G^b\) has a left adjoint \(\gamma_H^G\) defined in [A.6.1] so \(\mathcal{R}^G_L\) has a left adjoint \(\mathcal{T}^G_L: D^b_L(N_L) \to D^b_G(N_G)\) defined by

\[
D^b_G(N_G) \xrightarrow{\gamma^G_L} D^b_P(N_G) \xrightarrow{(m_P)_!} D^b_P(N_P) \xrightarrow{(p_P)_*} D^b_P(N_L) \xrightarrow{\gamma^G_L} D^b_L(N_L).
\]

**Lemma 4.14.** The functor \(\mathcal{T}^G_L\) is right exact for perverse \(t\)-structure.

**Proof.** For an \(L\)-equivariant smooth sheaf \(\mathcal{E}\) on an \(L\)-orbit \(\mathcal{O}\) in \(N_L\), we write

\[
\Delta(\mathcal{O}, \mathcal{E}) := (j_\mathcal{O})_! \mathcal{E}[\dim \mathcal{O}] \in \mathcal{P}D_{\mathcal{L}}^{\leq 0}(N_L)
\]

where \(j_\mathcal{O}: \mathcal{O} \hookrightarrow N_L\) is the inclusion. Then \(\mathcal{P}D_{\mathcal{L}}^{\leq 0}(N_L)\) is a thick subcategory of \(D_{\mathcal{L}}(N_L)\) generated by all \(\Delta(\mathcal{O}, \mathcal{E})[n]\) with \(n \geq 0\). Hence it suffices to show

\[
\mathcal{T}^G_L \Delta(\mathcal{O}, \mathcal{E}) \in \mathcal{P}D_{\mathcal{G}}^{\leq 0}(N_G)
\]

for all \(\mathcal{O}\) and \(\mathcal{E}\). Let \(j'_\mathcal{O}: \mathcal{O} + u_P \hookrightarrow N_P\) and \(i_\mathcal{O}: \mathcal{O} + u_P \hookrightarrow \tilde{G} \times \mathcal{P} (\mathcal{O} + u_P)\) be the inclusion, and let

\[
n_\mathcal{O}: \tilde{G} \times \mathcal{P} (\mathcal{O} + u_P) \to N_G
\]

be the natural map induced by the \(\tilde{G}\)-action on \(N_G\). Then we have the following diagram

\[
\begin{array}{c}
D^b_G(\tilde{G} \times \mathcal{P} (\mathcal{O} + u_P)) \xrightarrow{\gamma^G_L} D^b_P(\tilde{G} \times \mathcal{P} (\mathcal{O} + u_P)) \xrightarrow{(m_P)_!} D^b_P(\mathcal{O} + u_P) \xrightarrow{(p_P)_*} D^b_P(\mathcal{O}) \\
\text{(Int)} \quad \text{(Co)} \quad \text{(BC)} \quad \text{(Int)} \quad \text{(Int)} \\
D^b_G(N_G) \xrightarrow{\gamma^G_L} D^b_P(N_G) \xrightarrow{(m_P)_!} D^b_P(N_P) \xrightarrow{(p_P)_*} D^b_P(N_L) \\
\end{array}
\]

where (Int) is defined in [A.6.1] Since \(\gamma^G_L(\mathcal{E}) = \mathcal{E}\), we have

\[
\mathcal{T}^G_L \Delta(\mathcal{O}, \mathcal{E}) \cong (n_\mathcal{O})_! \gamma^G_L(i_\mathcal{O})_!(p_P)_* \gamma^G_L(\mathcal{E})[\dim \mathcal{O}] \\
\cong (n_\mathcal{O})_! \gamma^G_L(i_\mathcal{O})_!(\mathcal{E} \boxtimes \mathcal{O} \ell_{u_P})[\dim \mathcal{O}].
\]

By [A.12], \(\gamma^G_L(i_\mathcal{O})_!\) is a categorical equivalence and the quasi-inverse is the functor \((i_\mathcal{O})^* \text{For}_G^b\), it implies that \(\gamma^G_L(i_\mathcal{O})_!(\mathcal{E} \boxtimes \mathcal{O} \ell_{u_P})[\dim \mathcal{O}]\) is concentrate in degree \(- \dim(\mathcal{O}) - \dim \tilde{G} + \dim \tilde{L}\). It implies that \(\gamma^G_L(i_\mathcal{O})_!(\mathcal{E} \boxtimes \mathcal{O} \ell_{u_P})[\dim \mathcal{O}]\) is concentrate in degree \(- \dim(\mathcal{O}) - \dim \tilde{G} + \dim \tilde{L}\).
On the other hand, by [Lus84, Proposition 1.2(b)]
\[ \dim(n_{\theta}^{-1}(x)) \leq \frac{1}{2}(\dim \tilde{G} - \dim(G \cdot x) - \dim \tilde{L} + \dim \theta) \]
for any \( x \in \mathcal{N}_G \).

It follows that
\[ H^i((\tilde{T}^G_L \Delta(\theta, \mathcal{E}))|_x) \cong H^i_c(\mathcal{N}_{\theta}^{-1}(x), (\gamma^G_G(i_\theta)!)(\mathcal{E} \boxtimes \mathbb{C}_{\ell_{\mu P}})[\dim \theta]|_{n_{\theta}^{-1}(x)}) \]
vanishes unless \( i \leq -\dim(G \cdot x) \). This proves the claim. \( \square \)

**Proof of Proposition 4.13.** Since \( \mathcal{R}^G_L \) is right adjoint to \( \tilde{T}^G_L \), Lemma 4.14 implies that \( \mathcal{R}^G_L \) is left exact. By the equivalence \( \text{For}: \text{Perv}^\ast_G(\mathcal{N}_G) \cong \text{Perv}_G(\mathcal{N}_G) \) and the isomorphism in (4.31), it follows that \( \mathcal{R}^G_L \) sends \( \text{Perv}_G(\mathcal{N}_G) \) into \( pD^{\geq 0}(\mathcal{N}_L) \).

Similarly, consider the functor \( \mathcal{R}^G_L \) defined by
\[ \mathcal{R}^G_L := (p_{P^{-}})! \circ (m_{P^{-}})^*: D^b(\mathcal{N}_G) \to D^b(\mathcal{N}_L) \]
where \( P^{-} \) is the parabolic subgroup opposite to \( P \). A lift \( \mathcal{R}^G_L \) to equivariant derived categories can be defined as
\[
\begin{array}{ccccccc}
D^b_G(\mathcal{N}_G) & \xrightarrow{\text{For}^G_{P^{-}}} & D^b_{P^{-}}(\mathcal{N}_G) & \xrightarrow{(m_{P^{-}})^*} & D^b_{P^{-}}(\mathcal{N}_{P^{-}}) & \xrightarrow{(p_{P^{-}})!} & D^b_{P^{-}}(\mathcal{N}_L) \\
\text{For}^G_{P^{-}} & \text{For}^G_{P^{-}} & \text{For}^G_{P^{-}} & \text{For}^G_{P^{-}} & \text{For}^G_{P^{-}} & \text{For}^G_{P^{-}} & \text{For}^G_{P^{-}}
\end{array}
\]
and it has a right adjoint \( \tilde{T}^G_L \) defined as
\[
\begin{array}{ccccccc}
D^b_G(\mathcal{N}_G) & \xrightarrow{\Gamma^G_{P^{-}}} & D^b_{P^{-}}(\mathcal{N}_G) & \xrightarrow{(m_{P^{-}})^*} & D^b_{P^{-}}(\mathcal{N}_{P^{-}}) & \xrightarrow{(p_{P^{-}})!} & D^b_{P^{-}}(\mathcal{N}_L) \\
\Gamma^G_{P^{-}} & \Gamma^G_{P^{-}} & \Gamma^G_{P^{-}} & \Gamma^G_{P^{-}} & \Gamma^G_{P^{-}} & \Gamma^G_{P^{-}} & \Gamma^G_{P^{-}}
\end{array}
\]
where \( \Gamma^H_K \) is the right adjoint of \( \text{For}^H_K \).

The similar argument using \( \mathcal{R}^G_L \) and \( \tilde{T}^G_L \) proves that \( \mathcal{R}^G_L \) sends \( \text{Perv}_G(\mathcal{N}_G) \) into \( pD^{\geq 0}(\mathcal{N}_L) \) (see [AHR15] Lemma 4.11, Lemma 4.12 for details).

Finally, for any \( M \in \text{Perv}_G(\mathcal{N}_G) \), there is a canonical isomorphism \( \mathcal{R}^G_L(M) \cong \mathcal{R}^G_L(M) \) by the Braden’s theorem [Bra03, Theorem 1]. This proves the Proposition 4.13. \( \square \)

We now get the restriction functor
\[ \mathcal{R}^G_L: \text{Perv}_G(\mathcal{N}_G^p_{-\infty}) \to \text{Perv}_L(\mathcal{N}_L^p_{-\infty}) \]
(note that \( \text{Perv}_G(\mathcal{N}_G) \) is canonically equivalent to \( \text{Perv}_G(\mathcal{N}_G^p_{-\infty}) \)). One can easily show that the square
\[
\begin{array}{ccc}
\mathcal{N}_B & \to & \mathcal{N}_P \\
\downarrow & & \downarrow \\
\mathcal{N}_{BL} & \to & \mathcal{N}_L
\end{array}
\]

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is cartesian. Hence we have the restriction isomorphism

\[ \mathcal{R}_T^G \leftrightarrow \mathcal{R}_T^L \circ \mathcal{R}_L^G : \text{Perv}_G(\mathcal{N}_G^{p-\infty}) \to \text{Perv}_T(\mathcal{N}_T^{p-\infty}) \]

defined by the following pasting diagram:

\[
\begin{array}{c}
D^b(\mathcal{N}_G) \xrightarrow{(\cdot)^!} D^b(\mathcal{N}_P) \xrightarrow{(\cdot)_!} D^b(\mathcal{N}_L) \\
\downarrow (\cdot)^! \quad \downarrow (\cdot)_! \quad \downarrow (\cdot)^! \\
D^b(\mathcal{N}_B) \xrightarrow{(\cdot)^!} D^b(\mathcal{N}_{BL}) \xrightarrow{(\cdot)_!} D^b(\mathcal{N}_T).
\end{array}
\]

## 5 Definition of open subset \( \mathcal{M}_G \)

In this section, we will vary \( \mathcal{O} \), so we write \( \mathcal{O}_1, \mathcal{O}_2, \ldots \) to distinguish them. We want to define an open subset \( \mathcal{M}_G \subset \text{Gr}_G^{\text{sm}} \) as in \( \S 2.6 \).

### 5.1 Preliminaries on affine Grassmannian for \( \text{GL}_n \)

In this subsection, we will introduce some definitions and fundamental results about the affine Grassmannian for \( \text{GL}_n \).

Recall there is another interpretation of affine Grassmannian. Namely,

\[ \text{Gr}_G(R) = \left\{ (\mathcal{E}, \beta) \mid \mathcal{E} \text{ is a } G\text{-torsor on } \text{Spec } \mathcal{O}_R, \right. \]

\[ \left. \quad \beta : \mathcal{E}|_{\text{Spec } \mathcal{O}_R[1/\varpi]} \to \mathcal{E}_0|_{\text{Spec } \mathcal{O}_R[1/\varpi]} \text{ is an isomorphism.} \right\} \]

where \( \mathcal{E}_0 \) is the trivial \( G \)-torsor on \( \text{Spec } \mathcal{O}_R \). In particular, if \( G = \text{GL}_n \), we have

\[ \text{Gr}_G(R) = \left\{ (\mathcal{E}, \beta) \mid \mathcal{E} \text{ is a projective } \mathcal{O}_R\text{-module of rank } n, \right. \]

\[ \left. \quad \beta : \mathcal{E}[1/\varpi] \to \mathcal{E}_0[1/\varpi] \text{ is an isomorphism} \right\} \quad (5.1) \]

where \( \mathcal{E}_0 = \mathcal{O}_R^n \).

For finite projective \( \mathcal{O}_R\)-modules \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \), an isomorphism \( \beta : \mathcal{E}_2[1/\varpi] \to \mathcal{E}_1[1/\varpi] \) is called a quasi-isogeny. We write this as \( \beta : \mathcal{E}_2 \dashrightarrow \mathcal{E}_1 \). It is called isogeny if it is induced by a genuine map \( \mathcal{E}_2 \to \mathcal{E}_1 \).

Recall that when \( G = \text{GL}_n \), we can make the identifications

\[ \mathbb{X}^\bullet(T) = \mathbb{Z}^n, \]

\[ \mathbb{X}^{\pm}(T) = \{(m_1, \ldots, m_n) \in \mathbb{Z}^n \mid m_1 \geq \cdots \geq m_n\} \]

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and the partial order on $X_\bullet(T)$ can be described as follows:

$(m_1, \ldots, m_n) \leq (l_1, \ldots, l_n)$ if and only if

\[
\begin{align*}
    m_1 + \cdots + m_j &\leq l_1 + \cdots + l_j \quad (1 \leq j \leq n), \\
    m_1 + \cdots + m_n &\leq l_1 + \cdots + l_n
\end{align*}
\]

Let $R$ be a perfect field over $k$. Then for a quasi-isogeny $\beta : \mathcal{E}_1 \to \mathcal{E}_2$, there exists a basis $(e_1, \ldots, e_n)$ of $\mathcal{E}_1$ and a basis $(f_1, \ldots, f_n)$ of $\mathcal{E}_2$ such that

$\beta(e_i) = p^{m_i}f_i$

and $(m_1, m_2, \ldots, m_n) \in X_\bullet(T)$. Then we write

$\text{Inv}(\beta) = (m_1, \ldots, m_n)$

and call it the relative position of $\beta$.

Let $R$ be a general perfect $k$-algebra, and let $\beta : \mathcal{E}_1 \to \mathcal{E}_2$ be a quasi-isogeny. For $x \in \text{Spec } R$, we write $\beta_x := \beta \otimes_{O_R} O_{k(x)}$.

Then for $\mu \in X_\bullet(T)$, the Schubert variety and the Schubert cell can be described as

$\text{Gr}_{G, \leq \mu} = \{(E, \beta) \in \text{Gr}_G \mid \text{Inv}(\beta_x) \leq \mu \text{ for all } x \in \text{Spec } R\}$,

$\text{Gr}_{G, \mu} = \{(E, \beta) \in \text{Gr}_G \mid \text{Inv}(\beta_x) = \mu \text{ for all } x \in \text{Spec } R\}$.

Let $N$ be a nonnegative integer. Define the closed subspace $\overline{\text{Gr}}_{G,N} \subset \text{Gr}_G$ by

$\overline{\text{Gr}}_{G,N} = \text{Gr}_{G, \leq (N,0,\ldots,0)}$.

There is a fundamental result as follows:

**Lemma 5.1.** Let $R$ be a perfect $k$-algebra, and let $\beta : \mathcal{E}_1 \to \mathcal{E}_2$ be a quasi-isogeny. For $x \in \text{Spec } R$, write

$\text{Inv}(\beta_x) = (m_{x1}, \ldots, m_{xn})$.

Then $\beta$ is an isogeny if and only if

$m_{xn} \geq 0$ for any $x \in \text{Spec } R$.

This can be prove by the same argument as [Zhu17a, Lemma 1.5], for example. By Lemma 5.1, $\overline{\text{Gr}}_{G,N}$ can be also described as

$\overline{\text{Gr}}_{G,N}(R) = \left\{ \mathcal{E} \subset \mathcal{E}_0 \mid \mathcal{E} \text{ is a projective } O_R\text{-submodule of rank } n, \text{Inv}(\beta_x) \leq (N,0,\ldots,0) \text{ for all } x \in \text{Spec } R. \right\}$.
5.2 The case \( G = \text{GL}_n \)

In this subsection, we want to prove

**Theorem 5.2.** Set \( G_1 := \text{GL}_n \otimes \mathcal{O}_1 \) and \( G_2 := \text{GL}_n \otimes \mathcal{O}_2 \). If \( v_{F_1}(p) \geq N \) and \( v_{F_2}(p) \geq N \), then there is a canonical isomorphism

\[
\overline{\text{Gr}}_{G_1,N} \cong \overline{\text{Gr}}_{G_2,N}.
\]

We only have to consider the case that \( \mathcal{O}_1 \) is of mixed characteristic and \( \mathcal{O}_2 \) is of equal characteristic. Namely, we may assume that

\[
\begin{cases}
\mathcal{O}_1 \text{ is a totally ramified extension of } W(k) \text{ with } [\mathcal{O}_1 : W(k)] \geq N, \\
\mathcal{O}_2 = k[[t]].
\end{cases}
\]

(5.2)

Then let us denote

\[
\begin{align*}
\mathcal{O} &:= \mathcal{O}_1, \\
\text{Gr} &:= \text{Gr}_{G_1}, \text{Gr}^b := \text{Gr}_{G_2}
\end{align*}
\]

for simplicity. Let \( \mu \in X^+_*(T) \) be such that \( \mu \leq (N,0,\ldots,0) \). Then we can write

\[
\mu = \omega_{N_1} + \omega_{N_2} + \cdots + \omega_{N_r}, \quad (N_1 \geq N_2 \geq \cdots \geq N_r)
\]

where \( \omega_i := (1,\ldots,1,0,\ldots,0) \in \mathbb{Z}^n \). Define \( \tilde{\text{Gr}}_\mu \) by

\[
\tilde{\text{Gr}}_\mu(R) = \left\{ \mathcal{E}_r \supset \cdots \supset \mathcal{E}_1 \supset \mathcal{E}_0 \left| \begin{array}{c}
\mathcal{E}_i \text{'s are projective } \mathcal{O}(R)-\text{modules of rank } n, \\
\beta_i \text{ is an isogeny of relative position } \omega_{N_i}.
\end{array} \right. \right\}.
\]

First we want to prove the following:

**Lemma 5.3.** Assume \((\ref{5.2})\). If \( \mu \leq (N,0,\ldots,0) = N \omega_1 \), then

\[
\tilde{\text{Gr}}_\mu \cong \text{Gr}^b_\mu.
\]

To prove this lemma, we need some preparation.

**Lemma 5.4.** If \( (\mathcal{E}_i, \beta_i) \in \tilde{\text{Gr}}_\mu(R) \), then \( \beta_i \) induces a chain of inclusions \( \varpi \mathcal{E}_{i-1} \subset \mathcal{E}_i \subset \mathcal{E}_{i-1} \), and \( \mathcal{E}_{i-1}/\mathcal{E}_i \) is a projective \( R \)-module of rank \( N_i \).

In particular, \( \varpi^N \mathcal{E}_0 \subset \mathcal{E}_i \subset \mathcal{E}_0 \) for all \( i \).

**Proof.** One can use the same argument as in [Zhu17a, Lemma 1.5]. \( \square \)
Lemma 5.5. Consider the sequence of \( W_0(R) \)-submodules

\[ \varpi^n \mathcal{E}_0 \subset \mathcal{E}_n \subset \mathcal{E}_{n-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0. \]

Assume that \( \mathcal{E}_{i-1}/\mathcal{E}_i \) is a projective \( R \)-module annihilated by \( \varpi \) for any \( i \). Then \( \mathcal{E}_i \) is a finite projective \( W_0(R) \)-module for any \( i \).

Proof. Induction on \( i \). If \( i = 0 \), it is clear from definition. Assume \( i > 0 \). Since \( \mathcal{E}_{i-1}/\mathcal{E}_i \) is \( R \)-projective, it is a direct summand of \( \mathcal{E}_{i-1}/\varpi \mathcal{E}_{i-1} \) as \( R \)-module, hence also as \( W_0(R) \)-module.

By induction hypothesis, \( \mathcal{E}_{i-1}/\mathcal{E}_i \) is finite projective \( W_0(R) \)-module. Therefore \( \mathcal{E}_{i-1}/\varpi \mathcal{E}_{i-1} \) is finitely presented, and so is \( \mathcal{E}_{i-1}/\mathcal{E}_i \). Furthermore, it follows that

\[ \text{pd}_{W_0(R)}(\mathcal{E}_{i-1}/\mathcal{E}_i) \leq \text{pd}_{W_0(R)}(\mathcal{E}_{i-1}/\varpi \mathcal{E}_{i-1}) = 1 \]

where \( \text{pd}_{W_0(R)}(\mathcal{E}) \) means its projective dimension over \( W_0(R) \). It implies that \( \mathcal{E}_i \) is finite projective \( W_0(R) \)-module.

Lemma 5.6. In the situation of Lemma 5.5, \( \text{Inv}(\mathcal{E}_i \rightarrow \mathcal{E}_{i-1}) = \omega_{N_i} \) if and only if the projective module \( \mathcal{E}_{i-1}/\mathcal{E}_i \) has the constant rank \( N_i \).

Proof. Let \( \beta \) denote the map \( \mathcal{E}_i \rightarrow \mathcal{E}_{i-1} \). If \( \text{Inv}(\beta) = \omega_{N_i} \), then the projective module \( \mathcal{E}_{i-1}/\mathcal{E}_i \) has the constant rank \( N_i \) by Lemma 5.4.

Conversely, suppose \( \mathcal{E}_{i-1}/\mathcal{E}_i \) has the constant rank \( N_i \). As in the proof of [Zhu17a, Lemma 1.5], one can show

\[ (\mathcal{E}_{i-1}/\mathcal{E}_i) \otimes_R k(x) \cong \text{Coker}(\beta \otimes_{W_0(R)} W_0(k(x))) \]

for all \( x \in \text{Spec} R \). Since \( \mathcal{E}_{i-1}/\mathcal{E}_i \) is annihilated by \( \varpi \), and \( \dim_{k(x)}((\mathcal{E}_{i-1}/\mathcal{E}_i) \otimes_R k(x)) = N_i \), we obtain

\[ \text{Coker}(\beta \otimes_{W_0(R)} W_0(k(x))) \cong (W_0(k(x))/\varpi)^{N_i} \]

as \( W_0(k(x)) \)-module. It means \( \text{Inv}(\beta \otimes_{W_0(R)} W_0(k(x))) = \omega_{N_i} \) for all \( x \in \text{Spec} R \), that is, \( \text{Inv}(\beta) = \omega_{N_i} \).

Now we can prove Lemma 5.3.

Proof of Lemma 5.3. Define a functor \( \widetilde{\text{Gr}}_\mu \) by

\[ \widetilde{\text{Gr}}_\mu(R) \cong \left\{ \mathcal{E}_r \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 \bigg| \begin{array}{l} \text{\mathcal{E}_r’s are } W_0(R)/\varpi^r \text{-submodules of } \mathcal{E}_0, \\ \mathcal{E}_{i-1}/\mathcal{E}_i \text{ is annihilated by } \varpi, \\ \text{and } \mathcal{E}_{i-1}/\mathcal{E}_i \text{ is a finite projective } R \text{-module} \end{array} \right\} \]
for a perfect $k$-algebra $R$, where $E_0 := (W_0/\omega^n)^n$. By Lemma 5.4, 5.5, 5.6 we obtain the following bijection:

$$\tilde{\text{Gr}}_\mu(R) \rightarrow \tilde{\text{Gr}}^b_\mu(R),$$

$$(E_i)_{i=1}^n \mapsto (E_i/\omega^N E_0)_{i=1}^n.$$

This is a natural isomorphism, so $\tilde{\text{Gr}}_\mu \cong \tilde{\text{Gr}}^b_\mu$ follows. Similarly, define a functor $\tilde{\text{Gr}}^b_\mu$ by

$$\tilde{\text{Gr}}^b_\mu(R) \cong \left\{ E_r \subset \cdots \subset E_1 \subset E_0 \mid \begin{aligned} &E_i's \text{ are } R[[t]]/t^N\text{-submodules of } E_0, \\
&\frac{E_i}{E_{i-1}}/E_0 \text{ is annihilated by } t, \\
&\text{and } \frac{E_i}{E_{i-1}} \text{ is a finite projective } R\text{-module.} \end{aligned} \right\}$$

where $E_0 := (k[[t]]/t^N)^n$. Then we have an isomorphism $\tilde{\text{Gr}}^b_\mu \cong \tilde{\text{Gr}}^b_\mu$ defined by

$$\tilde{\text{Gr}}^b_\mu(R) \rightarrow \tilde{\text{Gr}}^b_\mu(R),$$

$$(E_i)_{i=1}^n \mapsto (E_i/t^N E_0)_{i=1}^n.$$

For $a, b \in R$, the value $[a + b] - [a] - [b] \in pW_0(R)$ vanishes in $W_0(R)/\omega^N$, since $[O : W(k)] \geq N$. Therefore, we obtain a ring isomorphism

$$W_0(R)/\omega^N \cong R[[t]]/t^N, \quad (5.3)$$

$$\sum_{k=0}^{N-1} a_k t^k \rightarrow \sum_{k=0}^{N-1} a_k t^k.$$

Through this isomorphism, we have

$$\tilde{\text{Gr}}^b_\mu \cong \tilde{\text{Gr}}^b_\mu.$$

It implies

$$\tilde{\text{Gr}}_\mu \cong \tilde{\text{Gr}}^b_\mu.$$

\[\square\]

**Lemma 5.7.** There is an isomorphism

$$\text{Gr}_\mu \cong \text{Gr}^b_\mu. \quad (5.4)$$

**Proof.** By [BS17 Lemma 7.13], the natural map

$$\pi: \tilde{\text{Gr}}_\mu \rightarrow \text{Gr}_{\leq \mu},$$

$$(E_r \subset \cdots \subset E_1 \subset E_0) \mapsto (E_r \subset E_0)$$

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restricts to an isomorphism
\[ \pi^{-1}\text{Gr}_\mu \cong \text{Gr}_\mu. \]

An element \((E_\bullet) \in \widetilde{\text{Gr}}_\mu(k)\) is an element of \(\pi^{-1}\text{Gr}_\mu(k)\) if and only if
\[ E_0/E_r \cong \bigoplus_{i=1}^{n} W_\mathcal{O}(k)/\varpi^{\mu_i} \]
where \(\mu = (\mu_1, \ldots, \mu_n)\). Similarly, the natural map
\[ \pi^b : \widetilde{\text{Gr}}_\mu^b \to \text{Gr}_\leq^b, \]
\[ (E^b_r \subset \cdots \subset E^b_1 \subset E^b_0) \mapsto (E^b_r \subset E^b_0) \]
restricts to an isomorphism
\[ (\pi^b)^{-1}\text{Gr}_\mu^b \to \text{Gr}_\mu^b. \]

An element \((E^b_\bullet) \in \widetilde{\text{Gr}}_\mu^b(k)\) is an element of \(\pi^{-1}\text{Gr}_\mu^b(k)\) if and only if
\[ E^b_0/E^b_r \cong \bigoplus_{i=1}^{n} k[[t]]/t^{\mu_i}. \]

Therefore the open subspace \(\pi^{-1}\text{Gr}_\mu \subset \widetilde{\text{Gr}}_\mu^b\) and \(\pi^{-1}\text{Gr}_\mu^b \subset \widetilde{\text{Gr}}_\mu\) correspond each other by the isomorphism in Lemma 5.3. It implies \(\text{Gr}_\mu \cong \text{Gr}_\mu^b\). \(\square\)

If \(\beta : E_2 \to E_1\) is a quasi isogeny satisfying \(\text{Inv}(\beta_x) \leq N\omega_1\) for all \(x \in \text{Spec } R\), then we have
\[ \varpi^N E_1 \subset \beta(E_2) \subset E_1 \]
by applying Lemma 5.1 to \(\beta\) and \(1/\varpi^N \beta^{-1}\). Hence if \((E_1 \subset E_0) \in \text{Gr}_N\), then \(E_0/E_1\) is a \(W_\mathcal{O}(R)/\varpi^N\)-module. By the isomorphism in (5.3), \(E_0/E_1\) is also a \(R[[t]]/t^N\)-module, in particular \(R\)-module.

**Lemma 5.8.** If \((E_1 \subset E_0) \in \text{Gr}_N(R)\), then \(E_0/E_1\) is a finite projective \(R\)-module.

**Proof.** The proof is almost the same as in [Zhu17a, Lemma 1.5]. Namely, for perfect \(R\)-algebra \(R'\), there is an isomorphism
\[ W_\mathcal{O,N}(R) \otimes_R R' \cong R[[t]]/t^N \otimes_R R' \]
\[ \cong R'[[[t]]]/t^N \]
\[ \cong W_\mathcal{O,N}(R'). \]
Therefore, for \( x \in \text{Spec} \, R \), we obtain
\[
(E_0 / E_1) \otimes_R k(x) \cong (E_0 / E_1) \otimes_{W_{O,N}(R)} W_{O,N}(k(x)) \\
\cong (E_0 / E_1) \otimes_{W_{O}(R)} W_{O}(k(x)) \\
\cong (E_0 \otimes_{W_{O}(R)} W_{O}(k(x)))/ (E_1 \otimes_{W_{O}(R)} W_{O}(k(x))) \\
= \text{Coker}(\beta_x).
\]
Since Inv(\( \beta_x \)) \( \leq N \omega_1 \), we have dim Coker(\( \beta_x \)) = \( N \) for all \( x \in \text{Spec} \, R \). Hence \( \text{dim}(E_0/E_1) \otimes_R k(x) \) is constant on \( \text{Spec} \, R \).

On the other hand, \( E_0/E_1 \) is the cokernel of \( E_1/\mathfrak{p}^N \to E_0/\mathfrak{p}^N \). Also, \( E_i/\mathfrak{p}^N \) (\( i = 0,1 \)) is a finite projective \( W_{O}(R) \)-module, and hence a finite projective \( R \)-module. Therefore, \( E_0/E_1 \) is finitely presented as \( R \)-module.

Over a reduced ring, a finitely presented module whose fiber dimensions are constant is locally free (see [Sta18, Tag0FWG]). The lemma follows.

Fix an isomorphism \( W_{O,N}(k)^n \cong k^{nN} \) of vector spaces. By Lemma 5.8, we obtain a morphism \( i_{\overline{\text{Gr}}_N} : \overline{\text{Gr}}_N \to \text{Gr}(nN)^{p^{-\infty}} \) defined by
\[
\overline{\text{Gr}}_N(R) \ni (E_1 \subset E_0) \mapsto (E_0/E_1) \in \text{Gr}(nN)^{p^{-\infty}}(R)
\]
(5.6)
where \( \text{Gr}(nN) \) is a usual Grassmannian, classifying finite dimensional subspaces in \( k^{nN} \).

**Lemma 5.9.** The morphism \( i_{\overline{\text{Gr}}_N} \) is a closed immersion.

**Proof.** We know that \( \overline{\text{Gr}}_N \) is perfectly proper (i.e. separated and universally closed) over \( k \) and \( \text{Gr}(nN)^{p^{-\infty}} \) is separated (see [BS17] Lemma 3.4). Therefore \( i_{\overline{\text{Gr}}_N} \) is perfectly proper.

Furthermore, the map between the sets of \( R \)-valued points
\[
i_{\overline{\text{Gr}}_N}(R) : \overline{\text{Gr}}_N(R) \to \text{Gr}(nN)^{p^{-\infty}}(R)
\]
is injective for any perfect \( k \)-algebra \( R \). In particular, \( i_{\overline{\text{Gr}}_N}(K) \) is injective for any algebraic closed field \( K \). It implies that \( i_{\overline{\text{Gr}}_N} \) is universally injective.

By [BS17] Lemma 3.8, a universally homeomorphism between perfect schemes is an isomorphism. It follows that a perfectly proper and universally injective morphism between perfect schemes is a closed immersion. This proves the claim.

Now we can prove Theorem 5.2.
Proof of Theorem 5.2. From Lemma 5.9, we obtain a closed immersion $i_{\Gr N} : \overline{\Gr N} \hookrightarrow \Gr(nN)^p$ by fixing an isomorphism

$$W_{\O,N}(k)^n \cong k^{nN} \quad (5.7)$$

of vector spaces. Consider the isomorphism

$$(k[[t]])^{\pm N} \cong W_{\O,N}(k)^n \cong k^{nN} \quad (5.7)$$

Then similarly we obtain a closed immersion

$$i : \overline{\Gr N}^{\pm} \hookrightarrow \Gr(nN)^p.$$ 

Let $\mu \in \mathfrak{X}^{\pm}(T)$ be such that $\mu \leq N\omega_1$. Then by construction, the following diagram is commutative:

$$\begin{array}{ccc}
\Gr_{\mu} & \hookrightarrow & \overline{\Gr N}^{\pm} \\
\downarrow & & \downarrow \\
\Gr_{\mu}^{\pm} & \hookrightarrow & \overline{\Gr N}^{\pm} \\
\end{array}$$

Since $\overline{\Gr N} = \bigcup_{\mu \leq N\omega_1} \Gr_{\mu}$ and $\overline{\Gr N}^{\pm} = \bigcup_{\mu \leq N\omega_1} \Gr_{\mu}^{\pm}$, it follows that $\overline{\Gr N}$ and $\overline{\Gr N}^{\pm}$ coincides as perfect closed subschemes of $\Gr(nN)^p$.

More precisely, the isomorphism $\overline{\Gr N} \cong \overline{\Gr N}^{\pm}$ has some equivariance. To explain this, consider the following lemma:

Lemma 5.10. The $L^+GL_n^{(N)}$-action on $\overline{\Gr N}$ is trivial. Similarly, $L^+GL_\mu^{(N)}$-action on $\overline{\Gr N}$ is trivial.

Here, $L^+GL_n^{(N)}$, $L^+GL_\mu^{(N)}$ is the $N$-th congruence groups.

Proof. By the $L^+GL_n$-action on $\overline{\Gr N}$, an element $A \in L^+GL_n(R)$ sends $(\mathcal{E}_1 \subset \mathcal{E}_0) \in \overline{\Gr N}(R)$ to $(\mathcal{E}_1 \subset \mathcal{E}_0$) $\in \overline{\Gr N}(R)$.

By Lemma 5.9 and (5.5), a point $(\mathcal{E}_1 \subset \mathcal{E}_0) \in \overline{\Gr N}(R)$ is completely determined by its quotient $\mathcal{E}_1/\mathcal{E}_0$.

Since $\mathcal{E}_1/\mathcal{E}_0$ does not change by the $L^+GL_n^{(N)}$-action, the lemma follows.

By Lemma 5.10, $L^NGL_n$ acts on $\overline{\Gr N}$. Similarly, $L^NGL_\mu$ acts on $\overline{\Gr N}$. But there is an isomorphism

$$L^NGL_n \cong L^NGL_\mu \quad (5.8)$$

by the isomorphism (5.3).
Proposition 5.11. The isomorphism $\overline{G}_N \cong \overline{G}_N$ in Theorem 5.2 is $L_N\text{GL}_n$-equivariant through the isomorphism (5.8).

Proof. This follows from the construction of the isomorphism $\overline{G}_N \cong \overline{G}_N$. Write $GL_n$ for the perfection of the general linear group over $k$. Then by Proposition 5.11, the isomorphism $\overline{G}_N \cong \overline{G}_N$ is in particular $GL_n$-equivariant through the map $GL_n \to L_NGL_n$ coming from the natural map $k \to k[[t]]/t^N$.

5.3 Isomorphism for general $G$

If $N$ is a positive integer, and if $v_{F_1}(p) \geq N$ and $v_{F_2}(p) \geq N$ hold, then by the same argument as (5.3), there is a ring isomorphism

$$O_1/\varpi_1^N \cong O_2/\varpi_2^N,$$

$$\sum_{i=0}^{N-1} [a_i] \varpi_1 \mapsto \sum_{i=0}^{N-1} [a_i] \varpi_2$$

(5.10)

where in the equal characteristic case, we define $[\cdot]$ by $[a] := a$ for any $a \in k$.

Theorem 5.12. Let $(G_1, T_1, B_1)$ be a reductive group over $O_1$ and let $(G_2, T_2, B_2)$ be a reductive group over $O_2$. Assume that the reductions $(\overline{G}_1, \overline{T}_1, \overline{B}_1)$ and $(\overline{G}_2, \overline{T}_2, \overline{B}_2)$ are isomorphic as algebraic groups over $k$, and identify them. Let $\mu$ be an element in $X^+_1(\overline{T}_1) = X^+_2(\overline{T}_2)$. Then there exists a constant $N = N_{\overline{G}_1, \mu} \in \mathbb{Z}_{>0}$ depending only on the isomorphism class of $\overline{G}_1$ and $\mu$ such that if $v_{F_1}(p) \geq N$ and $v_{F_2}(p) \geq N$, then for each isomorphism

$$\phi: (G_1 \mod \varpi_1^N) \sim (G_2 \mod \varpi_2^N)$$

via the isomorphism (5.2) such that $(\phi \mod \varpi_1)$ is equal to the above identification between $\overline{G}_1$ and $\overline{G}_2$, there is a “canonical” isomorphism

$$\alpha_{\mu, \phi}: \text{Gr}_{G_1, \leq \mu} \cong \text{Gr}_{G_2, \leq \mu}.$$ (5.11)

(An isomorphism $\phi$ that satisfies the above conditions always exists.) Moreover, the Schubert cells $\text{Gr}_{G_1, \lambda}$ and $\text{Gr}_{G_2, \lambda}$ correspond under $\alpha_{\mu, \phi}$ for any $\lambda \leq \mu$.

Here the term “canonical” means the following: Let $(G_1', T_1', B_1')$ and $(G_2', T_2', B_2')$ be another pair of reductive groups satisfying the same
condition as \((G_1, T_1, B_1)\) and \((G_2, T_2, B_2)\). Let \(\mu'\) be an element in \(\mathbb{X}_+^\ast(T_1') = \mathbb{X}_+^\ast(T_2')\). Put \(\tilde{N} := \max\{N_{G_1, \mu}, N_{G_1', \mu'}\}\). Consider two isomorphisms

\[
\tilde{\phi}: (G_1 \mod \varpi_1^{\tilde{N}}) \sim \to (G_2 \mod \varpi_2^{\tilde{N}}),
\]

\[
\tilde{\phi}': (G_1' \mod \varpi_1^{\tilde{N}}) \sim \to (G_2' \mod \varpi_2^{\tilde{N}})
\]

via the isomorphism \((5.9)\) such that \((\tilde{\phi} \mod \varpi_1)\) and \((\tilde{\phi}' \mod \varpi_1)\) are equal to the identity. Put \(\phi := (\tilde{\phi} \mod \varpi_1^{N_{G_1, \mu}})\) and \(\phi' := (\tilde{\phi}' \mod \varpi_1^{N_{G_1', \mu'}})\). Let \(f_1: G_1 \to G_1'\) and \(f_2: G_2 \to G_2'\) be homomorphisms of algebraic groups such that the following square is commutative:

\[
\begin{array}{ccc}
G_1 \mod \varpi_1^{\tilde{N}} & \xrightarrow{f_1} & G_1' \mod \varpi_1^{\tilde{N}} \\
\tilde{\phi} \downarrow & & \tilde{\phi}' \downarrow \\
G_2 \mod \varpi_2^{\tilde{N}} & \xrightarrow{f_2} & G_2' \mod \varpi_2^{\tilde{N}}
\end{array}
\]

Suppose that the image of \(\text{Gr}_{G_1, \leq \mu}\) by the induced map \(\text{Gr}_{G_1} \to \text{Gr}_{G_1'}\) is contained in \(\text{Gr}_{G_1', \leq \mu'}\). Then the image of \(\text{Gr}_{G_2, \leq \mu}\) by the induced map \(\text{Gr}_{G_2} \to \text{Gr}_{G_2'}\) is contained in \(\text{Gr}_{G_2', \leq \mu'}\), and the following square is commutative:

\[
\begin{array}{ccc}
\text{Gr}_{G_1, \leq \mu} & \xrightarrow{\alpha_{\mu, \phi}} & \text{Gr}_{G_1', \leq \mu'} \\
\downarrow \alpha_{\mu', \phi'} & & \downarrow \alpha_{\mu', \phi'} \\
\text{Gr}_{G_2, \leq \mu} & \xrightarrow{\alpha_{\mu, \phi'}} & \text{Gr}_{G_2', \leq \mu'}
\end{array}
\]

**Proof of Theorem 5.12.** Again, we only have to consider the case that \(O_1\) is of mixed characteristic and \(O_2\) is of equal characteristic. Put

\[
F := F_1,
\]

\[
G := G_1, \ G^0 := G_2,
\]

\[
\text{Gr}_G := \text{Gr}_{G_1}, \ \text{Gr}^0_G := \text{Gr}_{G_2}
\]

and so on.

Choose a closed embedding \(i: G \hookrightarrow \text{GL}_n\) of algebraic groups for some \(n\). Then the induced map

\[
i: \text{Gr}_G \hookrightarrow \text{Gr}_{\text{GL}_n}
\]

is a closed immersion (see [Zhu17a, Proposition 1.20] and [Alp14, Corollary 9.7.7]). Since \(\text{Gr}_{G, \leq \mu}\) is a connected scheme, we obtain

\[
i(\text{Gr}_{G, \leq \mu}) \subset \varpi^m \overline{\text{Gr}_N}
\]
for some $m$ and $N$. Choose $\widetilde{i}$ and $N$ so that $N$ is the smallest, and define $N_{G,\mu}$ as this minimum $N$. Assume $v_{F}(p) \geq N_{G,\mu}$. Then by Theorem 5.2 there is a canonical isomorphism

$$\varpi^m \overline{\text{Gr}_N} \cong t^m \overline{\text{Gr}_N}. \quad (5.12)$$

Furthermore, this isomorphism is $L^NGL_n(\cong L^NGL^n_{\mu})$-equivariant by Proposition 5.11.

Let $\phi: (G \text{ mod } \varpi^N) \xrightarrow{\sim} (G^\flat \text{ mod } t^N)$ be an isomorphism via the isomorphism (5.3). Let $\overline{\pi}: G^\flat \hookrightarrow GL^n_\mu$ be an embedding which makes the following square commutative:

$$
\begin{array}{ccc}
(G \text{ mod } \varpi^N) & \xrightarrow{\overline{\pi}} & (GL_n \text{ mod } \varpi^N) \\
\phi \downarrow & & \psi \downarrow \\
(G^\flat \text{ mod } t^N) & \xrightarrow{\overline{\pi}} & (GL^\flat_n \text{ mod } t^N)
\end{array}
$$

where $\psi: (GL_n \text{ mod } \varpi^N) \xrightarrow{\sim} (GL^\flat_n \text{ mod } t^N)$ is a canonical isomorphism using (5.3).

The embedding $\overline{\pi}$ induces a closed immersion

$$\overline{\pi}: Gr^\flat_G \hookrightarrow Gr^\flat_{GL_n}.$$

Its image $\overline{\pi}(Gr^\flat_{G,\leq \mu})$ is the smallest $L^+G^\flat$-stable subspace of $Gr^\flat_{GL_n}$ containing $\overline{\pi}(t^\lambda)$ for all $\lambda \leq \mu$. From the fact that $i(\varpi^\lambda)$ and $\overline{\pi}(t^\lambda)$ correspond under the isomorphism (5.12), it follows that $\overline{\pi}(t^\lambda) \in t^m \overline{\text{Gr}_N}$ for all $\lambda \leq \mu$, and that

$$\overline{\pi}(Gr^\flat_{G,\leq \mu}) \subset t^m \overline{\text{Gr}_N}.$$

We want to show that $i(Gr_{G,\lambda})$ and $\overline{\pi}(Gr^\flat_{G,\lambda})$ correspond under the isomorphism (5.12) for any $\lambda \leq \mu$. First, $i(Gr_{G,\lambda})$ is the $L^+G$-orbit of $i(\varpi^\lambda)$. Since $L^+(N)(\subset L^+GL^n_{\mu})$ acts trivially on $\varpi^m \overline{\text{Gr}_N}$ by Lemma 5.10, $i(Gr_{G,\lambda})$ is the $L^N G$-orbit of $i(\varpi^\lambda)$. Similarly, $\overline{\pi}(Gr^\flat_{G,\lambda})$ is the $L^N G^\flat$-orbit of $\overline{\pi}(t^\lambda)$.

From the facts that $i(\varpi^\lambda)$ and $\overline{\pi}(t^\lambda)$ correspond under the isomorphism (5.12), and that the isomorphism (5.12) is $L^NGL_n(\cong L^NGL^n_{\mu})$-equivariant, it follows that $i(Gr_{G,\lambda})$ and $\overline{\pi}(Gr^\flat_{G,\lambda})$ correspond under the isomorphism (5.12).

Hence we obtain the isomorphism

$$\alpha_{\mu,\phi}: Gr_{G,\leq \mu} \cong Gr^\flat_{G,\leq \mu}.$$

Finally, we want to prove the canonicity. (The canonicity implies that the isomorphism $\alpha_{\mu,\phi}$ does not depend on the choice of $\widetilde{i}$, as we will mention later.) Let $(G', T', B')$ and $(G'^\flat, T'^\flat, B'^\flat)$ be another pair of reductive groups satisfying the same condition
as \( (G, T, B) \) and \( (G', T', B') \). Let \( \mu' \) be an element in \( X_+^a(T') = X_+(\bar{T}') \). Choose embeddings \( \tilde{i} : G \rightarrow \text{GL}_n \) and \( \tilde{\varphi} : G' \rightarrow \text{GL}_{n'} \) so that these embeddings induce closed immersions \( i : \text{Gr}_{G, \mu} \rightarrow \bar{\omega}^m \text{Gr}_{\text{GL}_n, \leq N_{G, \mu}} \) and \( i : \text{Gr}_{G', \mu'} \rightarrow \bar{\omega}^{m'} \text{Gr}_{\text{GL}_{n'}, \leq N_{G', \mu'}} \) for some \( m, m' \). Put \( N := N_{G, \mu}, N' := N_{G', \mu'} \) and \( \bar{N} := \max\{N, N'\} \). Consider two embeddings

\[
\tilde{i} : G \rightarrow \text{GL}_n \quad \text{and} \quad \tilde{i} : G' \rightarrow \text{GL}_{n'}
\]

so that these embeddings induce closed immersions \( i : \text{Gr}_{G, \mu} \rightarrow \text{Gr}_{\text{GL}_n, \leq N_{G, \mu}}, \omega_1 \) and \( i : \text{Gr}_{G', \mu'} \rightarrow \text{Gr}_{\text{GL}_{n'}, \leq N_{G', \mu'}}, \omega_1 \) for some \( m, m' \). Put \( N := N_{G, \mu}, N' := N_{G', \mu'} \) and \( \bar{N} := \max\{N, N'\} \).

Let \( \tilde{\phi} : (G \mod \bar{\omega}^N) \rightarrow (G' \mod \bar{\omega}^N) \) and \( \tilde{\phi}' : (G' \mod \bar{\omega}^N) \rightarrow (G' \mod \bar{\omega}^N) \) via the isomorphism (5.3) such that \( \tilde{\phi} \) and \( \tilde{\phi}' \) are equal to the identity.

Put \( \phi := (\tilde{\phi} \mod \bar{\omega}^N) \) and \( \phi' := (\tilde{\phi}' \mod \bar{\omega}^N) \). Let \( f : G \rightarrow G' \) and \( f' : G' \rightarrow G' \) be homomorphisms of algebraic groups such that the following square is commutative:

\[
\begin{array}{ccc}
(G \mod \bar{\omega}^N) & \xrightarrow{f} & (G' \mod \bar{\omega}^N) \\
\downarrow \tilde{\phi} & & \downarrow \tilde{\phi}' \\
(G' \mod \bar{\omega}^N) & \xrightarrow{f'} & (G' \mod \bar{\omega}^N)
\end{array}
\]

Suppose that the image of \( \text{Gr}_{G, \leq \mu} \) by the induced map \( \text{Gr}_{G}^b \rightarrow \text{Gr}_{G'}^b \) is contained in \( \text{Gr}_{G', \leq \mu'} \).

Let \( \tilde{\varphi} : G \rightarrow \text{GL}_n^b \) and \( \tilde{\varphi}' : G' \rightarrow \text{GL}_{n'}^b \) be embeddings which make the following squares commutative:

\[
\begin{array}{ccc}
(G \mod \bar{\omega}^N) & \xrightarrow{\tilde{\varphi}} & (\text{GL}_n \mod \bar{\omega}^N) \\
\downarrow \phi & & \downarrow \psi \\
(G' \mod \bar{\omega}^N) & \xrightarrow{\tilde{\varphi}'} & (\text{GL}_{n'} \mod \bar{\omega}^N)
\end{array}
\]

where \( \psi, \psi' \) are canonical isomorphisms using (5.3).

By the same argument as above, the image of the morphism

\[
(i' \circ f' : \text{Gr}_{G, \leq \mu} \rightarrow \text{Gr}_{\text{GL}_n, \leq N_{\omega_1}} \times \text{Gr}_{\text{GL}_{n'}, \leq N'_{\omega_1}})
\]

is contained in \( (t^m \text{Gr}_{\text{GL}_n, \leq N_{\omega_1}}) \times (t^{m'} \text{Gr}_{\text{GL}_{n'}, \leq N'_{\omega_1}}) \) and there is an isomorphism which
makes the following diagram commute, assuming only that $v_F(p) \geq \tilde{N}$:

\[
\begin{array}{ccc}
\text{Gr}_{G, \leq \mu} & \xrightarrow{(i, i' \circ f)} & (\varpi^m \text{Gr}_{\text{GL}_n, \leq N \omega_1}) \times (\varpi^{m'} \text{Gr}_{\text{GL}_{n'}, \leq N' \omega_1}) \\
\downarrow & & \downarrow \\
\text{Gr}_{G, \leq \mu} & \xrightarrow{(\tilde{i}, i' \circ f')} & (t^m \text{Gr}_{\text{GL}_n, \leq N \omega_1}) \times (t^{m'} \text{Gr}_{\text{GL}_{n'}, \leq N' \omega_1}).
\end{array}
\]

Then we obtain the diagram

\[
\begin{array}{ccc}
\text{Gr}_{G, \leq \mu} & \xrightarrow{(i, i' \circ f)} & (\varpi^m \text{Gr}_{\text{GL}_n, \leq N \omega_1}) \times (\varpi^{m'} \text{Gr}_{\text{GL}_{n'}, \leq N' \omega_1}) \\
\downarrow & & \downarrow \\
\text{Gr}_{G, \leq \mu} & \xrightarrow{(\tilde{i}, i' \circ f')} & (t^m \text{Gr}_{\text{GL}_n, \leq N \omega_1}) \times (t^{m'} \text{Gr}_{\text{GL}_{n'}, \leq N' \omega_1}).
\end{array}
\]

where all the squares are commutative.

We have the map $\text{Gr}_{G, \leq \mu} \to \text{Gr}_{G', \leq \mu'}$ which makes the cube commutative. This map coincides with the map induced by $f^\circ$. This proves the canonicity. The canonicity shows that $\alpha_{\mu, \phi}$ does not depend on the choice of $\tilde{i}$, by considering the case where $G = G'$, $G^\circ = G'^\circ$, $f = \text{id}_G$, $f^\circ = \text{id}_{G'}$, $\tilde{\phi} = \tilde{\phi}'$ and $\tilde{i} \neq \tilde{i}'$ in the above argument.

**Corollary 5.13** (restatement of Lemma 2.3). Let $G$ be a reductive group over $O$. Put $G^\circ := \bar{G} \otimes_k k[[t]]$. There is a constant $C = C_G \in \mathbb{N}$ depending only on $\bar{G}$ such that if $v_F(p) \geq C$, then there is a canonical isomorphism

\[\text{Gr}_{G}^{\text{sm}} \cong \text{Gr}_{G^\circ}^{\text{sm}}.\]

which is $L^C G(\cong L^C G^\circ)$-equivariant.

**Proof.** Since the number of coweights which is small is finite and $\text{Gr}_{G}^{\text{sm}}$ is closed in $\text{Gr}_{G}$, we can write

\[\text{Gr}_{G}^{\text{sm}} = \text{Gr}_{G, \leq \mu_1} \cup \cdots \cup \text{Gr}_{G, \leq \mu_r},\]

for some $\mu_1, \ldots, \mu_r$. Then Theorem 5.12 proves the claim.
Now we can define an open subspace $\mathcal{M}_G \subset \text{Gr}^{\text{sm}}_G$ as explained in §2.6, assuming that $p > 2$. Therefore, we can define a functor $\Psi_G$ as

$$\Psi_G := (\pi_G)^* \circ (j_G)^! : \text{Perv}_{L_G}^+(\text{Gr}^{\text{sm}}_G, \underline{\mathbb{Q}}_\ell) \to \text{Perv}_G^p_{\mathcal{N}^{p-\infty}_G, \underline{\mathbb{Q}}_\ell}$$

where $j_G : \mathcal{M}_G \hookrightarrow \text{Gr}^{\text{sm}}_G$ and $\pi_G : \mathcal{M}_G \to \mathcal{N}^{p-\infty}_G$ is the perfection of a finite morphism as in §2.6.

6 Intertwining isomorphisms and commutativity of prism

In this section, we will construct the intertwining isomorphisms for each functor, and prove the commutativity of prisms.

6.1 Functor $\Phi_\hat{G}$

For $V \in \text{Rep}(\hat{G})$, we have an equality as $W_L$-module

$$\Phi_L \circ R^\hat{G}_L(V) = \Phi_L(V^{Z(L)})$$

$$= (V^{Z(L)})^T \otimes \varepsilon_{W_L}$$

$$= V^T \otimes \varepsilon_{W_L}$$

$$= R^W_{W_L}(V^T \otimes \varepsilon_{W_G})$$

$$= R^W_{W_L} \circ \Phi_G(V)$$

where $\varepsilon_{W_L}$ is the sign character of the Coxeter group $W_L$. Hence we obtain the intertwining isomorphism

$$\text{id} : R^W_{W_L} \circ \Phi_\hat{G} = \Phi_L \circ R^\hat{G}_L.$$

And the commutativity of the following prism is clear (all the natural isomorphism is the identity):
6.2 Functor $\Psi_G$

In this subsection, we assume that $p > 2$ and that $v_F(p)$ is sufficiently large so that $\Psi_L$ can be defined for all Levi subgroup $L \subset G$.

**Lemma 6.1.** We have an inclusion $\mathcal{M}_G \cap Gr_L \subset \mathcal{M}_L$.

**Proof.** By the definition of $\mathcal{M}_G$, this lemma only needs to be shown in the case where $F = k((t))$ before the perfections. We write $\mathcal{M}_G'$ for a deperfection of $\mathcal{M}_G$ defined in §2.6, and so on.

From the fact that $Gr_{G,0} = \{x \in Gr_G' \mid \lim_{s \to \infty} s \cdot x = t^0\}$ for some $\mathbb{G}_m$-action, we have $Gr_{G,0} \cap Gr_L' = Gr_{L,0}'$.

On the other hand, by the definition of smallness, we have an inclusion $Gr_{t^{sm}} \cap Gr_{L,0} \subset Gr_{L,0}^{t^{sm}}$. Therefore we have

$$\mathcal{M}_G' \cap Gr_L' = Gr_{G,0} \cap Gr_{G}^{t^{sm}} \cap Gr_L' = Gr_{L,0} \cap Gr_L' \subset Gr_{L,0} \cap Gr_L^{t^{sm}} = \mathcal{M}_L'.$$

\[\square\]

**Lemma 6.2.** The following square is cartesian:

$$\begin{array}{ccc}
\mathcal{M}_G \cap Gr_L & \longrightarrow & \mathcal{M}_G \\
\pi_L|_{\mathcal{M}_G \cap Gr_L} & \downarrow & \pi_G \\
\mathcal{N}_{G}^{p^{-\infty}} & \longrightarrow & \mathcal{N}_{G}^{p^{-\infty}}.
\end{array}$$

**Proof.** By the definition of $\mathcal{M}_G$, this lemma only needs to be shown in the case where $F = k((t))$ before the perfections.

For the commutativity of the square, see [AH13 Lemma 2.5]. Let $x \in \mathcal{N}_L$. Then $x$ is fixed by $Z^0(L)$, where $Z^0(L)$ is the identity component of the center of $L$. From the fact that $(\pi_G)^{-1}(x)$ is a finite set and $\pi_G'$ is $\bar{G}$-equivariant, it follow that each point $y \in (\pi_G')^{-1}(x)$ is also fixed by $Z^0(L)$. It is known that the fixed-point set of $Z^0(L)$ on $Gr_G'$ is $Gr_L'$. Hence we have $(\pi_G')^{-1}(x) \subset \mathcal{M}_G' \cap Gr_L'$. This proves the lemma. \[\square\]

Put

$$\mathcal{M}_P := (q_{P}^{sm})^{-1}(\mathcal{M}_L)$$

and denote by $j_P : \mathcal{M}_P \hookrightarrow Gr_P^{sm}$.
Proposition 6.3. It holds that $i_P(M_P) \subset M_G$, and there is a morphism $\pi_P: M_P \to N_P^{p^{-\infty}}$ making the following square cartesian:

\[
\begin{array}{ccc}
M_P & \xrightarrow{i_P} & M_G \\
\downarrow{\pi_P} & & \downarrow{\pi_G} \\
N_P^{p^{-\infty}} & \xrightarrow{m_P} & N_G^{p^{-\infty}}.
\end{array}
\] (6.1)

Proof. By the definition of $M_G$, this lemma only needs to be shown in the case where $F = k((t))$ before the perfections.

First, we can see that $i'(P)(M'_P) \subset i'(P)(q'_P)^{-1}(\text{Gr}_{0,L})$

\[
= L(k[t^{-1}]) \cdot U_P(k((t))) \cdot t^0 \\
= L(k[t^{-1}]) \cdot U_P(k[t]) \cdot U_P(k[[t]]) \cdot t^0 \\
= L(k[t^{-1}]) \cdot U_P(k[t]) \cdot t^0 \\
= P(k[t^{-1}]) \cdot t^0 \\
\subset \text{Gr}_{G,0}.
\] (6.2)

Also we have

\[
i'(P)(M'_P) \subset i'(P)(\text{Gr}''_{P,\text{sm}}) \subset \text{Gr}''_{G,\text{sm}}.
\]

Hence $i'(P)(M'_P) \subset M'_G$. Furthermore, (6.2) implies that

\[
\pi'_G(i'(P)(M'_P)) \subset p \cap N_G = N_P.
\]

So we can set $\pi'_G: M'_P \to N_P$ as the induced map by $\pi'_G \circ i'_P$. This makes the square (6.1) commutative.

To prove that the square (6.1) is cartesian, let $x \in M'_G$ and $\pi'_G(x) \in N_P$. We have to show $x \in M'_P$. If $\pi'_G(x) \in N_L$, then we have $x \in M'_G \cap \text{Gr}'_{L}$ by Lemma 6.2. Hence by Lemma 6.1 we have

\[
x \in M'_G \cap \text{Gr}'_{L} \subset M'_L \subset \text{Gr}'_{L} \subset \text{Gr}'_{P}
\]

and

\[
x \in M'_G \subset \text{Gr}'_{G,\text{sm}},
\]

which implies that $x \in \text{Gr}'_{G,\text{sm}}$. Again by the fact that $x \in M'_L$, we obtain $x \in M'_P$.  

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Assume that \( \pi'_G(x) \in N_P \setminus N_L \). Let \( \lambda = (2\check{\rho}_G - 2\check{\rho}_L) \in X_\bullet(T) \), where \( \check{\rho}_G \) (resp. \( \check{\rho}_L \)) is the half sum of positive coroots of \( G \) (resp. \( L \)). The group \( \mathbb{G}_m \) acts on \( \text{Gr}'_G \) and \( N_G \) through \( \lambda \).

Let \( y = \lim_{s \to 0} \lambda(s) \cdot x \) and \( z = \lim_{s \to \infty} \lambda(s) \cdot x \). Then \( y, z \) are elements of \( \text{Gr}_L \). If \( y \in \mathcal{M}'_G \), then by Lemma 6.1, \( y \in \mathcal{M}'_L \). Hence we have

\[
\lambda \in (q'_P)^{-1}(\mathcal{M}'_L) \subset (q'_P)^{-1}(\text{Gr}'_L) = \text{Gr}'_P
\]

and

\[
x \in \mathcal{M}'_G \subset \text{Gr}'_G
\]

which implies that \( x \in \text{Gr}'_P \). Since \( q'_P(x) = y \in \mathcal{M}'_L \), we have \( x \in \mathcal{M}'_P \).

If \( z \in \mathcal{M}'_G \), then by Lemma 6.1, \( z \in \mathcal{M}'_L \). Hence we have \( x \in \mathcal{M}'_{P^-} \) as above, where \( P^- \) is the parabolic subgroup opposite to \( P \). It implies that \( \pi'_G(x) \in N_{P^-} \), which contradicts to \( \pi'_G(x) \in N_P \setminus N_L \).

If \( y \notin \mathcal{M}'_G \) and \( z \notin \mathcal{M}'_G \), then the \( \mathbb{G}_m \)-orbit of \( x \) is closed in \( \mathcal{M}'_G \). As \( \pi'_G \) is a finite morphism, we have that the \( \mathbb{G}_m \)-orbit of \( \pi'_G(x) \) is closed in \( N_G \). This contradicts to \( \pi'_G(x) \in N_P \setminus N_L \), since \( N_P = \{ x \in N_G \mid \lim_{s \to 0} \lambda(s) \cdot x \in N_L \} \).

We now have the following diagram

![Diagram](image)

where the top right square is cartesian by Proposition 6.3 and the bottom left square is cartesian by the definition of \( \mathcal{M}_P \).

From this diagram, we obtain the intertwining isomorphism

\[
\mathcal{R}^G_L \circ \Psi_G \iff \Psi_L \circ \mathcal{R}^G_L : D^b(\text{Gr}_G^\text{sm}) \to D^b(N_L^\text{p-\infty})
\]
defined by

\[
\begin{array}{cccccc}
D^b(\text{Gr}_{\text{sm}}^G) & \longrightarrow & D^b(M_G) & \longrightarrow & D^b(N_G^{p-\infty}) \\
\downarrow & & \downarrow & & \downarrow \\
D^b(\text{Gr}_{\text{sm}}^P) & \longrightarrow & D^b(M_P) & \longrightarrow & D^b(N_P^{p-\infty}) \\
\downarrow & & \downarrow & & \downarrow \\
D^b(\text{Gr}_{\text{sm}}^L) & \longrightarrow & D^b(M_L) & \longrightarrow & D^b(N_L^{p-\infty}).
\end{array}
\]

We also get the prism

\[
\begin{array}{cccccc}
D^b(\text{Gr}_{\text{sm}}^G) & \longrightarrow & D^b(N_G^{p-\infty}) \\
\downarrow & & \downarrow & & \downarrow \\
D^b(\text{Gr}_{\text{sm}}^L) & \longrightarrow & D^b(N_L^{p-\infty}) \\
\downarrow & & \downarrow & & \downarrow \\
D^b(\text{Gr}_{\text{sm}}^T) & \longrightarrow & D^b(N_T^{p-\infty}).
\end{array}
\]

which is commutative by gluing the prisms §A.2(a)(d), §A.3(c)(d), §A.4(b)(d). Here we use the fact that the following squares are cartesian:

\[
\begin{array}{cccc}
\mathcal{M}_B & \longrightarrow & \mathcal{M}_B & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_P & \longrightarrow & \mathcal{M}_P & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{N}_B^{p-\infty} & \longrightarrow & \mathcal{N}_B^{p-\infty}.
\end{array}
\]

The first square is cartesian by the definitions of $\mathcal{M}_B$ and $\mathcal{M}_B$. The second one is cartesian by the fact that the square

\[
\begin{array}{cccc}
\mathcal{M}_B & \longrightarrow & \mathcal{M}_B & \\
\downarrow & & \downarrow & \\
\text{Gr}_B^{\text{sm}} & \longrightarrow & \text{Gr}_B^{\text{sm}}
\end{array}
\]
is cartesian from the first square and Lemma 4.6. The third one is cartesian by Proposition 6.3 and its analogue with $B$ in place of $P$.

### 6.3 Functor $\mathcal{I}^{\text{sm}}_G$

#### 6.3.1 Functor $\mathcal{I}_G$

The geometric Satake equivalence $\mathcal{I}_G$ was induced by the monoidal functor

$$F_G : \text{Perv}_{L+G}(\text{Gr}_G) \to \text{Vect}_{\overline{\mathbb{Q}}_l},$$

and the restriction functor $\mathfrak{R}_L^G$ was induced by the homomorphism $\text{Aut}^*(F_L) \to \text{Aut}^*(F_L \circ \mathfrak{R}_L^G) \cong \text{Aut}^*(F_G)$. Therefore existence of the intertwining isomorphism

$$\text{Perv}_{L+G}(\text{Gr}_G) \xrightarrow{\mathcal{I}_G} \text{Rep}(\hat{G})$$

and the commutativity of the prism

follow from the construction. See [AHR15, Lemma 4.1] for details.

#### 6.3.2 Functor $\mathcal{I}^o_G$

Recall the restriction functors

$$\mathfrak{R}^G_L : \text{Perv}_{L+G}(\text{Gr}_G, \overline{\mathbb{Q}}_l) \to \text{Perv}_{L+L}(\text{Gr}_L, \overline{\mathbb{Q}}_l), \quad (6.4)$$

$$\mathfrak{R}^o_L : \text{Perv}_{L+G}(\text{Gr}^o_L, \overline{\mathbb{Q}}_l) \to \text{Perv}_{L+L}(\text{Gr}^o_L, \overline{\mathbb{Q}}_l) \quad (6.5)$$

defined in §4.2. We constructed an isomorphism

$$(z_L)^! \circ \mathfrak{R}^G_L \iff \mathfrak{R}^G_L \circ (z_G)^!$$
in (4.13) and a transitivity isomorphism
\[
\mathcal{R}_T^G \iff \mathcal{R}_T^L \circ \mathcal{R}_L^G
\]
in (4.9). Also, there is a transitivity isomorphism
\[
\mathcal{R}_T^G \iff \mathcal{R}_T^L \circ \mathcal{R}_L^G
\] (6.6)
defined by
\[
D^b(Gr_G) \xrightarrow{(z_G)^!} D^b(Gr_G^0) \xrightarrow{(z_G)^*} D^b(Gr_L^G) \\
D^b(Gr_B) \xrightarrow{(z_B)^!} D^b(Gr_B^L) \xrightarrow{(z_B)^*} D^b(Gr_T^L)
\]
Here we used the fact that the square (4.14) is cartesian.

**Lemma 6.4.** The following prism is commutative:
\[
Perv_{L+G}(Gr_G) \xrightarrow{(z_G)^!} Perv_{L+G}(Gr_G^0) \xrightarrow{(z_G)^*} Perv_{L+G}(Gr_L^G) \\
Perv_{L+T}(Gr_T) \xrightarrow{(z_T)^!} Perv_{L+T}(Gr_T^0) \xrightarrow{(z_T)^*} Perv_{L+T}(Gr_T^L)
\] (6.7)

**Proof.** It suffices to prove that the prism
is commutative. But it follows by gluing the diagrams §\ref{A.2}(d), §\ref{A.3}(d), §\ref{A.4}(d). Here it is required that the square
\[
\begin{array}{ccc}
\text{Gr}_B^o & \rightarrow & \text{Gr}_{B_L}^o \\
\downarrow & & \downarrow \\
\text{Gr}_B & \rightarrow & \text{Gr}_{B_L}
\end{array}
\]
is cartesian, which follows from the fact that the squares
\[
\begin{array}{ccc}
\text{Gr}_B^o & \rightarrow & \text{Gr}_T^o \\
\downarrow & & \downarrow \\
\text{Gr}_B & \rightarrow & \text{Gr}_T
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\text{Gr}_{B_L}^o & \rightarrow & \text{Gr}_T^o \\
\downarrow & & \downarrow \\
\text{Gr}_{B_L} & \rightarrow & \text{Gr}_T
\end{array}
\]
are cartesian by the bottom left square in (4.12). \hfill \Box

Let us define an intertwining isomorphism for $\mathcal{S}^o_G$.

Recall the isomorphism
\[
(-)^{Z(\hat{G})} \circ \mathcal{S}^o_G \iff \mathcal{S}^o_G \circ (z_G)^! \tag{2.1}
\]
in (2.1). Since $(z_G)^!: \text{Perv}_{L+G}(\text{Gr}_G) \to \text{Perv}_{L+G}(\text{Gr}_G^o)$ is full and essentially surjective, there is a unique isomorphism
\[
\mathcal{R}_L^\hat{G} \circ \mathcal{S}^o_G \iff \mathcal{S}^o_L \circ \mathcal{R}_L^G \tag{6.8}
\]
such that the following cube is commutative:

\[
\begin{array}{cccc}
\text{Perv}_{L+G}(\text{Gr}_G) & \overset{\mathcal{J}_G}{\longrightarrow} & \text{Rep}(\hat{G}) \\
\downarrow & & \downarrow \\
\mathcal{R}_L^G & \overset{(z_G)^!}{\longrightarrow} & \text{Perv}_{L+G}(\text{Gr}_G^o) \\
\downarrow & & \downarrow \\
\text{Perv}_{L+L}(\text{Gr}_L) & \overset{(z_L)^!}{\longrightarrow} & \text{Rep}(\hat{L}) \\
\downarrow & & \downarrow \\
\text{Perv}_{L+L}(\text{Gr}_L^o) & \overset{\mathcal{J}_L}{\longrightarrow} & \text{Rep}(\hat{L}) \overset{(z(\hat{L}))}{\longrightarrow} & \text{Rep}(\hat{L})^Z(\hat{L}).
\end{array}
\]
Lemma 6.5. The following prism is commutative:

\[
\begin{array}{c}
\text{Perv}_{L+G}(\text{Gr}_G^\circ) \\
\downarrow \gamma^G_T \\
\text{Perv}_{L+T}(\text{Gr}_T^\circ)
\end{array}
\xrightarrow{\mathcal{F}_G}
\begin{array}{c}
\text{Rep}(\hat{G})Z(\hat{G}) \\
\downarrow R^G_T \\
\text{Rep}(T)Z(T).
\end{array}
\] (6.10)

**Proof.** By the essential surjectivity of \((z_G)^1\), it suffices to show the commutativity of a diagram obtained by gluing the prisms (6.7), (6.10) and the cube (6.9) for \((G, T)\). But this prism can be also obtained by gluing the prism (6.3), the prism

\[
\begin{array}{c}
\text{Rep}(\hat{G}) \\
\downarrow R^G_T \\
\text{Rep}(T)
\end{array}
\xrightarrow{(-)Z(T)}
\begin{array}{c}
\text{Rep}(\hat{T})Z(\hat{T}) \\
\downarrow R^T_T \\
\text{Rep}(T).
\end{array}
\] (4.14)

and the cube (6.9) for \((G, L), (L, T)\). This proves the lemma. \(\square\)

6.3.3 Functor \(\mathcal{F}_G^{\text{sm}}\)

Recall that there is an isomorphism

\[(f_L^\circ)_* \circ \mathfrak{R}^G_L \Longleftrightarrow \mathfrak{R}^G_L \circ (f_G^\circ)_*,\]

as in (4.14).

**Lemma 6.6.** The following prism is commutative:

\[
\begin{array}{c}
\text{Perv}_{L+G}(\text{Gr}_G^{\text{sm}}) \\
\downarrow \gamma^G_T \\
\text{Perv}_{L+T}(\text{Gr}_T^{\text{sm}})
\end{array}
\xrightarrow{(f_G^\circ)_*}
\begin{array}{c}
\text{Perv}_{L+G}(\text{Gr}_G^\circ) \\
\downarrow \gamma^G_T \\
\text{Perv}_{L+T}(\text{Gr}_T^\circ)
\end{array}
\] (6.11)

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Proof. It suffices to prove that the prism

\[
\begin{array}{c}
D^b(\text{Gr}_G^{\text{sm}}) \xrightarrow{(f_G^0)^*} \text{Perv}_{L+G}(\text{Gr}_G^0) \\
\downarrow g^G_F \quad \downarrow g^G_L \quad \downarrow g^{G}_T \\
D^b(\text{Gr}_T^{\text{sm}}) \xrightarrow{(f_T^0)^*} \text{Perv}_{L+T}(\text{Gr}_T^0)
\end{array}
\]

is commutative. But it follows by gluing the diagrams $A.2(a)$, $A.3(c)$, $A.4(b)$. \qed

Let us define an intertwining isomorphism for $\mathcal{S}_G^{\text{sm}}$. Recall the isomorphism

\[
\mathcal{I}_G \circ \mathcal{S}_G^{\text{sm}} \iff \mathcal{S}_G \circ (f_G^0)_* \tag{2.2}
\]

in (2.2). Since $\mathcal{I}_G : \text{Perv}_{L+G}(\text{Gr}_G^{\text{sm}}) \to \text{Perv}_{L+G}(\text{Gr}_G^0)$ is fully faithful, there is a unique isomorphism

\[
\mathcal{R}_L^G \circ \mathcal{S}_G \iff \mathcal{S}_L^{\text{sm}} \circ g_L^G \tag{6.12}
\]

such that the following cube is commutative:

\[
\begin{array}{c}
\text{Perv}_{L+G}(\text{Gr}_G^{\text{sm}}) \xrightarrow{(f_G^0)^*} \text{Rep}(\tilde{G})^{\text{sm}} \\
\downarrow g^G_F \quad \downarrow g^G_L \quad \downarrow g^{G}_T \\
\text{Perv}_{L+L}(\text{Gr}_L^{\text{sm}}) \xrightarrow{(f_L^0)^!} \text{Rep}(L)^{\text{sm}} \\
\downarrow g^L_F \quad \downarrow g^L_L \quad \downarrow g^{L}_T \\
\text{Perv}_{L+L}(\text{Gr}_L^0) \xrightarrow{(f_L^0)^!} \text{Rep}(L)^{Z(L)}
\end{array}
\]

\[
\begin{array}{c}
\text{Rep}(\tilde{G})^{\text{sm}} \xrightarrow{\mathcal{I}_G} \text{Rep}(\tilde{G})^Z(\tilde{G}) \\
\downarrow \mathcal{I}_G \quad \downarrow \mathcal{I}_G \\
\text{Rep}(L)^{\text{sm}} \xrightarrow{\mathcal{I}_L} \text{Rep}(L)^{Z(L)}
\end{array}
\]

\[
\begin{array}{c}
\text{Perv}_{L+G}(\text{Gr}_G^0) \xrightarrow{\mathcal{S}_G} \text{Rep}(\tilde{G})^{Z(\tilde{G})} \\
\downarrow \mathcal{S}_G \\
\text{Perv}_{L+L}(\text{Gr}_L^0) \xrightarrow{\mathcal{S}_L} \text{Rep}(L)^{Z(L)}
\end{array}
\]
Lemma 6.7. The following prism is commutative:

\[
\begin{array}{c}
Perv_{L+G}(Gr^\text{sm}_G) \xrightarrow{\mathcal{F}^\text{sm}} \text{Rep}(\hat{G})_{\text{sm}} \\
\downarrow \quad \Downarrow \quad \Downarrow \\
Perv_{L+T}(Gr^\text{sm}_T) \xrightarrow{\mathcal{F}^\text{sm}} \text{Rep}(\hat{T})_{\text{sm}}
\end{array}
\]

(6.14)

Proof. By the fully faithfulness of \(\mathcal{F}_G\), it suffices to show the commutativity of a diagram obtained by gluing the prism (6.14), the prism

\[
\begin{array}{c}
\text{Rep}(\hat{G})_{\text{sm}} \xrightarrow{\mathcal{F}_G} \text{Rep}(\hat{G})^G(Z(G)) \\
\downarrow \quad \Downarrow \quad \Downarrow \\
\text{Rep}(\hat{L})_{\text{sm}} \xrightarrow{\mathcal{F}_L} \text{Rep}(\hat{L})^L(Z(L))
\end{array}
\]

and the cube (6.13) for \((G,T)\). But this prism can be also obtained by gluing the prisms (6.10), (6.11) and the cube (6.13) for \((G,L)\), \((L,T)\). This proves the lemma.

6.4 Functor \(S_G\)

6.4.1 Intertwining isomorphism for \(S_G\)

In this subsection, we want to show the following lemma:

Lemma 6.8. There is an isomorphism

\[
R^G_{WL} \circ S_G \iff S_G \circ R^G_L
\]

such that the following prism is commutative:

\[
\begin{array}{c}
Perv_G(N^\infty_G) \xrightarrow{S_G} \text{Rep}(W_G) \\
\downarrow \quad \Downarrow \quad \Downarrow \\
Perv_T(N^\infty_T) \xrightarrow{S_T} \text{Rep}(W_T)
\end{array}
\]

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It suffices to show the lemma under the replacement $\mathcal{N}_G^{P^{\to \infty}}$ by $\mathcal{N}_G$. Recall the functor

$$\widetilde{\mathcal{R}}^G_L := \text{For}_L^P \circ (p_P)^* \circ (m_P)^! \circ \text{For}_L^G : D^b_G(\mathcal{N}_G) \to D^b_L(\mathcal{N}_L)$$

and the isomorphism

$$\mathcal{R}^G_L \circ \text{For} \iff \text{For} \circ \widetilde{\mathcal{R}}^G_L. \quad (6.15)$$

Applying base change for the cartesian square

\[ \begin{array}{ccc}
\mathcal{N}_B & \to & \mathcal{N}_P \\
\downarrow & & \downarrow \\
\mathcal{N}_B & \to & \mathcal{N}_L
\end{array} \]

we obtain an isomorphism

$$\widetilde{\mathcal{R}}^G_T \iff \widetilde{\mathcal{R}}^G_L \circ D^b_G(\mathcal{N}_G) \to D^b_T(\mathcal{N}_T) \quad (6.16)$$

just as (4.7). Then the prism

\[ \begin{array}{ccc}
D^b_G(\mathcal{N}_G) & \xrightarrow{\text{For}} & D^b(\mathcal{N}_G) \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{R}}^G_L & \xrightarrow{\text{For}} & D^b(\mathcal{N}_L) \\
\downarrow & & \downarrow \\
D^b_T(\mathcal{N}_T) & \xrightarrow{\text{For}} & D^b(\mathcal{N}_T)
\end{array} \]

is commutative by §A.7(a)(c)(f), §A.8(a)(d), §A.9(a).

Now recall that the diagram

\[ \begin{array}{ccc}
Perv'_G(\mathcal{N}_G) & \xrightarrow{\text{For}} & Perv_G(\mathcal{N}_G) \\
\downarrow & & \downarrow \\
D^b_G(\mathcal{N}_G) & \xrightarrow{\text{For}} & D^b(\mathcal{N}_G)
\end{array} \]

is commutative and that $\text{For}$ is compatible with restriction in the sense that the diagram

\[ \begin{array}{ccc}
Perv'_G(\mathcal{N}_G) & \xrightarrow{\text{For}} & Perv_G(\mathcal{N}_G) \\
\downarrow & & \downarrow \\
Perv'_L(\mathcal{N}_L) & \xrightarrow{\text{For}} & Perv_L(\mathcal{N}_L)
\end{array} \]

is commutative. Therefore, to prove Lemma 6.8 it suffices to show the following lemma:
Lemma 6.9. There is an isomorphism

\[ R^W_{GL} \circ S_G \circ \text{For} \iff S_L \circ \tilde{R}^G_L : \text{Perv}^G(N_G) \to \text{Rep}(W_L) \]

such that the following prism is commutative:

The functor \( S_G \circ \text{For} \) extends to a functor \( S'_G : D^b_G(N_G) \to \text{Rep}(W_G) \) defined by

\[ M \mapsto \text{Hom}_{D^b_G(N_G)}(\text{Spr}_G, M). \]

Hence it suffices to show the following lemma:

Lemma 6.10. There is an isomorphism

\[ R^W_{WL} \circ S'_G \iff S'_L \circ \tilde{R}^G_L : D^b_G(N_G) \to \text{Rep}(W_L) \]

such that the following prism is commutative:

\[ \text{D}^b_G(N_G) \xrightarrow{S'_G} \text{Rep}(W_G) \]

\[ \tilde{R}^G_L \]

\[ S'_L \]

\[ D^b_L(N_L) \xrightarrow{S'_L} \text{Rep}(W_L) \]

\[ 6.17 \]
First, recall the functor \( \mathcal{T}_L^G \) left adjoint to \( \mathcal{R}_L^G \). By the adjointness, there is a transitivity isomorphism

\[
\mathcal{T}_L^G \iff \mathcal{T}_L^G \circ \mathcal{T}_T^L
\]

(6.19)
such that the following diagram is commutative for any \( M \in D^b_G(\mathcal{N}_G) \):

\[
\begin{array}{ccc}
\text{Hom}(\text{Spr}_T, \mathcal{R}_T^G M) & \xrightarrow{\text{adj}} & \text{Hom}(\mathcal{T}_T^G \text{Spr}_T, M) \\
\downarrow^{(6.16)} & & \downarrow^{(6.19)} \\
\text{Hom}(\text{Spr}_T, \mathcal{R}_T^L \mathcal{R}_L^G M) & \xrightarrow{\text{adj}} & \text{Hom}(\mathcal{T}_T^L \text{Spr}_T, \mathcal{R}_L^G M) & \xrightarrow{\text{adj}} & \text{Hom}(\mathcal{T}_T^L \mathcal{T}_T^L \text{Spr}_T, M).
\end{array}
\]

Suppose that we could construct a \( W_L \)-equivariant isomorphism

\[
\mathcal{T}_L^L(\text{Spr}_L) \cong \text{Spr}_G
\]

(6.20)
such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{T}_L^L(\text{Spr}_L) & \xrightarrow{\sim} & \text{Spr}_G \\
\uparrow & & \uparrow \\
\mathcal{T}_L^L(\mathcal{T}_T^L(\text{Spr}_T)) & \xrightarrow{\sim} & \mathcal{T}_T^L \text{Spr}_T.
\end{array}
\]  

(6.21)

Then we have an intertwining isomorphism

\[
D^b_G(\mathcal{N}_G) \xrightarrow{\mathcal{S}_G} \text{Rep}(W_G) \xrightarrow{\text{(Intw)}} \mathcal{R}_G^L \xrightarrow{\mathcal{S}_L} D^b_L(\mathcal{N}_L) \xrightarrow{\text{Rep}(W_L)} \text{Rep}(W_L)
\]

defined by an isomorphism

\[
\text{Hom}(\text{Spr}_L, \mathcal{R}_L^G M) \xrightarrow{\text{adj}} \text{Hom}(\mathcal{T}_L^G \text{Spr}_L, M) \cong \text{Hom}(\text{Spr}_G, M).
\]  

(6.20)

The commutativity of the prism in (6.18) is exactly the commutativity of the outer square in the following diagram:

\[
\begin{array}{ccc}
\text{Hom}(\text{Spr}_T, \mathcal{R}_T^G M) & \xrightarrow{\text{adj}} & \text{Hom}(\mathcal{T}_T^G \text{Spr}_T, M) & \xrightarrow{\text{adj}} & \text{Hom}(\text{Spr}_G, M) \\
\downarrow^{(6.16)} & & \downarrow^{(6.19)} & & \downarrow^{(6.20)} \\
\text{Hom}(\text{Spr}_T, \mathcal{R}_T^L \mathcal{R}_L^G M) & \xrightarrow{\text{adj}} & \text{Hom}(\mathcal{T}_T^L \text{Spr}_T, \mathcal{R}_L^G M) & \xrightarrow{\text{adj}} & \text{Hom}(\mathcal{T}_T^L \mathcal{T}_T^L \text{Spr}_T, M) & \xrightarrow{\text{adj}} & \text{Hom}(\mathcal{T}_T^L \mathcal{T}_T^L \text{Spr}_T, M).
\end{array}
\]

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So it suffices to construct an isomorphism (6.20) which makes the square (6.21) commute.

In other words, it suffices to prove the following lemma:

**Lemma 6.11.** There is an isomorphism

\[
\begin{array}{ccc}
W_G & \xrightarrow{\text{Spr}_G} & D^b_G(N_G) \\
\downarrow & & \uparrow \\
W_L & \xrightarrow{\text{Spr}_L} & D^b_L(N_L)
\end{array}
\]

such that the following prism is commutative:

\[
\begin{array}{ccc}
W_G & \xrightarrow{\text{Spr}_G} & D^b_G(N_G) \\
\downarrow & & \uparrow \\
W_L & \xrightarrow{\text{Spr}_L} & D^b_L(N_L) \\
\downarrow & & \uparrow \\
W_T & \xrightarrow{\text{Spr}_T} & D^b_T(N_T)
\end{array}
\]

Here we think of $W_G, W_L$ and $W_T$ as one-object categories.

### 6.4.2 In regular part

Consider the diagram

\[
\begin{array}{ccc}
L \times L (b_L \cap g^{rs}) & \xrightarrow{\gamma_L} & G \times L (b \cap g^{rs}) \\
\downarrow & & \downarrow \\
I \cap g^{rs} & \xrightarrow{\nu} & p \cap g^{rs} \\
\downarrow & & \downarrow \\
G \times L (g^{rs}) & \xrightarrow{\gamma_G} & D^b_G(g^{rs})
\end{array}
\]

where $\nu$, $\gamma$, $\mu_I$, $\mu_P$, respectively. Note that $\mu_I$ and $\mu_P$ is a Galois covering with Galois group $W_L$, and $\nu$ is a Galois covering with Galois group $W_G$.

Define a functor $\text{Spr}_G^\mathcal{L}: D^b_L(I \cap g^{rs}) \to D^b_G(g^{rs})$ as the composition

\[
D^b_L(I \cap g^{rs}) \xrightarrow{\gamma_L} D^b_P(I \cap g^{rs}) \xrightarrow{(\cdot)^*} D^b_P(p \cap g^{rs}) \xrightarrow{(\cdot)} D^b_P(g^{rs}) \xrightarrow{\gamma_P} D^b_G(g^{rs}).
\]

We will show the following lemma:
Lemma 6.12. There is a $W_L$-equivariant isomorphism

$$rs\mathcal{I}_L^G((\mu_1^{rs}, \mu_2^{rs})) \cong (\mu_1^{rs}, \mu_2^{rs})_L[\dim g]. \quad (6.22)$$

Define a functor $rs\mathcal{I}_L^G: D^b_\ell(\bar{L} \times B_L (b_L \cap g^{rs})) \to D^b_\ell(G \times B (b \cap g^{rs}))$ as the composition

$$D^b_\ell(\bar{L} \times B_L (b_L \cap g^{rs})) \to D^b_\ell(\bar{L} \times B_L (b_L \cap g^{rs})) \xrightarrow{\gamma_L^F} D^b_\ell(G \times B (b \cap g^{rs})). \quad (6.23)$$

It follows from definition that for any $w \in W_L$, there is an isomorphism

$$w^* \circ rs\mathcal{I}_L^G \iff rs\mathcal{I}_L^G \circ w^* \quad (6.23)$$

defined by

There is an isomorphism

$$rs\mathcal{I}_L^G([\ell_\dim g]) \cong \mathcal{I}_L[\dim g] \quad (6.24)$$

defined by

where (CII), (CI), (CIE) is defined in \&A.1.3, \&A.6.4, \&A.16. This isomorphism is compatible with the $W_L$-action in a sense that the prism

is commutative for all $w \in W_L$ (by \&A.11, \&A.19(c)).
Finally, there is an isomorphism

\[(\mu^G_\mathcal{T})! \circ \tau^G_{\mathcal{T}} \Longleftrightarrow \tau^G_{\mathcal{T}} \circ (\mu_\mathcal{T}^l)! \quad \text{(6.25)}\]

defined by

\[
\begin{array}{cccc}
D^b_L(\bar{L} \times B, (b_L \cap \mathfrak{g}^\alpha)) & \overset{(\mu^l_\mathcal{T})!}{\longrightarrow} & D^b_G(\bar{G} \times B, (b \cap \mathfrak{g}^\alpha)) \\
\gamma^L_{\mathcal{T}} & \downarrow & \gamma^G_{\mathcal{T}} \\
D^b_L(\bar{L} \times B, (b_L \cap \mathfrak{g}^\alpha)) & \overset{(\mu^l_\mathcal{T})!}{\longrightarrow} & D^b_G(\bar{G} \times B, (b \cap \mathfrak{g}^\alpha))
\end{array}
\]

This isomorphism is also compatible with the $W_L$-action in a sense that the prism

\[
\begin{array}{cccc}
D^b_L(\bar{L} \times B, (b_L \cap \mathfrak{g}^\alpha)) & \overset{w^*}{\longrightarrow} & D^b_G(\bar{G} \times B, (b \cap \mathfrak{g}^\alpha)) \\
(\mu^l_\mathcal{T})! & \downarrow & (\mu^G_\mathcal{T})! \\
D^b_L(\bar{L} \times B, (b_L \cap \mathfrak{g}^\alpha)) & \overset{w^*}{\longrightarrow} & D^b_G(\bar{G} \times B, (b \cap \mathfrak{g}^\alpha))
\end{array}
\]

is commutative for any $w \in W_L$ (by [A.4 (a), A.9 (b)]).

Gluing these two prisms implies Lemma [6.12].

6.4.3 Isomorphism for Groth

Consider the diagram

\[
\begin{array}{ccc}
\bar{L} \times B & \overset{\mu}{\longrightarrow} & \bar{P} \times B \\
& \mu_p \downarrow & \mu \downarrow \\
\mathfrak{g} & \longrightarrow & \mathfrak{g}.
\end{array}
\]

Define the functor $\mathcal{T}^G_L$ by

\[
D^b_L(\bar{L} \times B, b_L) \overset{\gamma^L_{\mathcal{T}}}{\longrightarrow} D^b_{\bar{P}}(\bar{L} \times B, b_L) \overset{\text{(1)}}{\longrightarrow} D^b_{\bar{P}}(\bar{P} \times B, b) \overset{\text{(1)}}{\longrightarrow} D^b_{\bar{P}}(\bar{G} \times B, b) \overset{\gamma^G_{\mathcal{T}}}{\longrightarrow} D^b_{\bar{G}}(\bar{G} \times B, b)
\]

and define $\mathcal{T}^G_{\mathcal{T}}$ by

\[
D^b_L(\mathfrak{l}) \overset{\gamma^L_{\mathcal{T}}}{\longrightarrow} D^b_{\mathfrak{p}}(\mathfrak{l}) \overset{\text{(1)}}{\longrightarrow} D^b_{\mathfrak{p}}(\mathfrak{p}) \overset{\text{(1)}}{\longrightarrow} D^b_{\mathfrak{p}}(\mathfrak{g}) \overset{\gamma^G_{\mathcal{T}}}{\longrightarrow} D^b_{\mathfrak{g}}(\mathfrak{g}).
\]
Since the square
\[ \bar{B} \times \bar{B} \bar{b} \rightarrow \bar{B}_L \times \bar{B}_L \bar{b}_L \]
\[ \bar{P} \times \bar{B} \bar{b} \rightarrow \bar{L} \times \bar{B}_L \bar{b}_L \]
is cartesian, there is a transitivity isomorphism
\[ \mathcal{T}_T^G \iff \mathcal{T}_L^G \circ \mathcal{T}_T^L. \]
by the same argument as (6.19). Similarly, since the square
\[ b \rightarrow \bar{b}_L \]
\[ p \rightarrow l \]
is cartesian, there is a transitivity isomorphism
\[ \mathcal{T}_T^G \iff \mathcal{T}_L^G \circ \mathcal{T}_T^L. \] (6.26)

We will show the following lemma:

**Lemma 6.13.** There is a $W_L$-equivariant isomorphism
\[ \mathcal{T}_L^G(\text{Groth}_L) \cong \text{Groth}_G \]
such that the following square commutes:
\[ \begin{array}{ccc}
\mathcal{T}_L^G(\text{Groth}_L) & \sim & \text{Groth}_G \\
\Downarrow \quad \gamma & & \Downarrow \quad \gamma \\
\mathcal{T}_L^G(\mathcal{T}_T^G(\text{Groth}_T)) & \cong & \mathcal{T}_T^G \text{Groth}_T.
\end{array} \] (6.27)

By the same argument as (6.24) and (6.25), we have isomorphisms
\[ \mathcal{T}_L^G(\overline{Q}_{\ell}[\text{dim } l]) \cong \overline{Q}_{\ell}[\text{dim } g] \] (6.28)
\[ (\mu_{\overline{g}})_! \circ \mathcal{T}_L^G \iff \mathcal{T}_L^G \circ (\mu_{\overline{g}})_!. \] (6.29)

Hence we obtain
\[ \mathcal{T}_L^G(\text{Groth}_L) = \mathcal{T}_L^G((\mu_{\overline{g}})_!(\overline{Q}_{\ell}[\text{dim } l])) \]
\[ \cong (\mu_{\overline{g}})_!(\mathcal{T}_L^G(\overline{Q}_{\ell}[\text{dim } l])) \]
\[ \cong (\mu_{\overline{g}})_!(\overline{Q}_{\ell}[\text{dim } g]) \]
\[ = \text{Groth}_G. \] (6.30)
It suffice to show that this isomorphism is $W_L$-equivariant and it makes the square \([6.27]\) commute.

Proof of the $W_L$-equivariance. Let us write $j_\theta : g^{rs} \hookrightarrow g$, $j_{p : p \cap g^{rs}} \hookrightarrow p$ and $j_{l : l \cap g^{rs}} \hookrightarrow l$ for the inclusions.

It suffices to prove that the isomorphism $j_\theta^* G_{\overline{L}}(\text{Groth}_L) \cong j_\theta^* \text{Groth}_L$ induced by \([6.30]\) is naturally identified with the $W_L$-equivariant isomorphism in Lemma \([6.12]\).

There is an isomorphism

\[(j_\theta)^* \circ \overline{I_L} \iff \overline{I_L} \circ (j'_l)^* \quad (6.31)\]

defined by

\[
\begin{array}{c}
D_L^b(l) \xrightarrow{\gamma_L^b} D_P(b) \xrightarrow{(\cdot)^*} D_P(p) \xrightarrow{(\cdot)} D_P(g) \xrightarrow{\gamma_P^b} D_G^b(g^{rs}) \\
D_L^b(l \cap g^{rs}) \xrightarrow{\gamma_L^b} D_P(b \cap g^{rs}) \xrightarrow{(\cdot)^*} D_P(p \cap g^{rs}) \xrightarrow{(\cdot)} D_P(g) \xrightarrow{\gamma_P^b} D_G^b(g^{rs}).
\end{array}
\]

Let us write $k_\theta : G \times B (b \cap g^{rs}) \hookrightarrow G \times B b$ and $k_{l'} : \overline{L} \times \overline{B}_L (b_L \cap g^{rs}) \hookrightarrow \overline{L} \times \overline{B}_L b_L$ for the inclusions. Similarly, there is an isomorphism

\[(k_\theta)^* \circ \overline{I_L} \iff \overline{I_L} \circ (k_{l'}^*)^* \quad (6.32)\]

We obtain the following cube

\[
\begin{array}{c}
D_G^b(G \times B b) \xrightarrow{(\mu_\theta)^\dagger} D_G^b(g) \xrightarrow{(j_\theta)^*} D_G^b(g^{rs}) \\
D_G^b(\overline{L} \times \overline{B}_L b_L) \xrightarrow{(\mu_l)^\dagger} D_L^b(b_L \cap g^{rs}) \xrightarrow{(j'_l)^*} D_L^b(l \cap g^{rs})
\end{array}
\]
which is commutative by §A.4(a), §A.9(b). Moreover, we obtain the following cube

which is commutative by §A.5, §A.11(b), §A.19(c).

Gluing these two cubes, we have a commutative diagram

Since the vertical isomorphisms are \( W_L \)-equivariant by definition, the \( W_L \)-equivariance of \( (6.30) \) follows from Lemma 6.12.

\[ \begin{array}{c}
\text{Proof of the commutativity of the square } (6.27). \text{ We obtain a prism}
\end{array} \]

which is commutative by §A.2(b), §A.3(a), §A.4(a), §A.7(a), §A.8(f), §A.9(b). More-
over, we obtain a prism

\[
\begin{array}{c}
\mathbb{Q}[\dim g] \\
\downarrow \id \\
\mathbb{Q}[\dim t] \\
\end{array}
\quad \rightarrow
\begin{array}{c}
D^b_G (\bar{G} \times^B b) \\
\downarrow \mathcal{I}_G \\
D^b_L (\bar{L} \times^{B_L} b_L) \\
\end{array}
\]

which is commutative by §A.5, §A.10(b), §A.11(b), §A.17(c), §A.18(c), §A.19(c).

Gluing these two prisms, we have a commutative diagram (6.27).

\[
\begin{array}{c}
\mathbb{Q}[\dim g] \\
\downarrow \id \\
\mathbb{Q}[\dim t] \\
\end{array}
\quad \rightarrow
\begin{array}{c}
D^b_G (\bar{G} \times^B b) \\
\downarrow \mathcal{I}_G \\
D^b_L (\bar{L} \times^{B_L} b_L) \\
\end{array}
\]

6.4.4 Isomorphism for \( \text{Spr} \)

Finally, we will show Lemma 6.11. Note that there is a \( W_G \)-equivariant isomorphism

\[ \text{Spr}_G \approx (i_g)^\Diamond \text{Groth}_G, \]

where \( (\cdot)^\Diamond = (\cdot)^*[-\dim g + \dim N_G] \).

By Lemma 6.13 we have an isomorphism

\[
\begin{array}{c}
W_G \text{Groth}_G \quad D^b_G (g) \\
\downarrow \mathcal{I}_G \\
W_L \text{Groth}_L \quad D^b_L (l) \\
\end{array}
\]

such that the following prism is commutative:

\[
\begin{array}{c}
W_G \text{Groth}_G \quad D^b_G (g) \\
\downarrow \mathcal{I}_G \\
W_L \text{Groth}_L \quad D^b_L (l) \\
\end{array}
\]

On the other hand, there is an isomorphism

\[
(i_l)^\Diamond \circ \mathcal{I}_L \iff \mathcal{I}_L^G \circ (i_g)^\Diamond
\]

(6.33)
defined just as (6.31). We have a prism

\[
\begin{array}{c}
D^b_G(\mathfrak{g}) \xrightarrow{(i_g)\circ} D^b_G(\mathcal{N}_G) \\
\xrightarrow{\mathscr{I}^G_L} D^b_T(t) \xrightarrow{(i_t)\circ} D^b_T(\mathcal{N}_T)
\end{array}
\]

which is commutative by §A.2(c), §A.3(b), §A.4(c), §A.7(g), §A.8(e), §A.9(b).

Gluing these two prisms proves Lemma 6.11.

7 The case $G$ is a torus or semisimple rank 1

As explained in §3, all that remains to prove the main theorem is the case $G = T$ or $G$ is semisimple rank 1. Assume that $p > 2$ and that $v_F(p)$ is sufficiently large so that $\Psi_G$ can be defined.

7.1 The case $G = T$

If $G = T$, then $\text{Gr}_T^{\text{sm}}$ and $\mathcal{N}_T^{p, -\infty}$ are both single points, and $\Psi_T$ is the canonical identification. Moreover, $\Phi_T \circ \mathscr{J}_T^{\text{sm}}$ is the equivalence $H^0: \text{Perv}_{L^+}(\text{Gr}_T^{\text{sm}}, \mathbb{Q}_\ell) \to \text{Vect}_{\mathbb{Q}_\ell}$ and $\mathcal{S}_T$ is canonically isomorphic to $H^0: \text{Perv}_T(\mathcal{N}_T^{p, -\infty}, \mathbb{Q}_\ell) \to \text{Vect}_{\mathbb{Q}_\ell}$. Hence we have a canonical isomorphism

\[
\alpha_T: \Phi_T \circ \mathscr{J}_T^{\text{sm}} \leftrightarrow \mathcal{S}_T \circ \Psi_T.
\]

7.2 The case $G$ is semisimple rank 1

We may assume $G = \text{PGL}_2$ since all the functors are invariant under the replacement of $G$ by $G/Z(G)$. Using $\alpha_T$, there is an isomorphism

\[
\phi_{G,T}: R^W_G \circ \Phi_G \circ \mathscr{J}_G^{\text{sm}} \leftrightarrow R^W_G \circ \mathcal{S}_T \circ \Psi_G
\]

as in §3. It suffices to show that this isomorphism is $W_G$-equivariant.

Identify $X_\bullet(T)$ with $\mathbb{Z}$. Then

\[
\text{Gr}_T^{\text{sm}} = \text{Gr}_0 \amalg \text{Gr}_2.
\]
Let $j_i : \text{Gr}_i \hookrightarrow \text{Gr}$ ($i = 0, 2$) be the inclusion map. Set

$$\text{IC}_i := (j_i)_!(\overline{Q}_\ell[i]).$$

Recall that $\text{Perv}_{L^+G}(\text{Gr}^\text{sm}_G, \overline{Q}_\ell)$ is semisimple and simple objects are $\text{IC}_0$ and $\text{IC}_2$. Therefore, it suffices to show that $\phi_{G,T}$ is $W_G$-equivariant for the objects $\text{IC}_0, \text{IC}_2$.

Since $G = \text{PGL}_2$, the map $\pi : \mathcal{M}_G \to \mathcal{N}_{G}^{p-\infty}$ is an isomorphism (see [AH13]). Hence $\Phi_G(\text{IC}_i)$ is canonically isomorphic to the intermediate extension in $\mathcal{N}_{G}^{p-\infty}$ of $\overline{Q}_\ell \mathcal{N}_{p-\infty} \cap \text{Gr}_i$.

Note that $\mathcal{N}_{p-\infty} \cap \text{Gr}_0$ and $\mathcal{N}_{p-\infty} \cap \text{Gr}_2$ are the $\bar{G}$-orbits in $\mathcal{N}_{G}^{p-\infty}$. From the theory of the Springer correspondence, it follows that $S_{G}(\Phi_G(\text{IC}_0))$ is isomorphic to the trivial representation of $W = S_2$, and $S_{G}(\Phi_G(\text{IC}_2))$ is isomorphic to the sign representation of $W = S_2$.

On the other hand, from the theory of the geometric Satake, $\mathcal{I}^\text{sm}_{G}^{G}(\text{IC}_i)$ is the irreducible representation of $\bar{G}$ with highest weight $i$. By the Schur-Weyl duality, $\Phi_G(\mathcal{I}^\text{sm}_{G}^{G}(\text{IC}_0))$ is isomorphic to the trivial representation of $W = S_2$, and $\Phi_G(\mathcal{I}^\text{sm}_{G}^{G}(\text{IC}_2))$ is isomorphic to the sign representation of $W = S_2$. As a result, $S_{G}(\Phi_G(\text{IC}_i))$ and $\Phi_G(\mathcal{I}^\text{sm}_{G}^{G}(\text{IC}_i))$ are irreducible as $W$-modules and their isomorphism classes coincide.

But $W = S_2$ acts on an irreducible representation by the scalar multiplication, it follows that the isomorphism of vector spaces

$$\Phi_G(\mathcal{I}^\text{sm}_{G}^{G}(\text{IC}_i)) \overset{\phi_{G,T}}{\cong} S_{G}(\Phi_G(\text{IC}_i))$$

is automatically $W$-equivariant.

### A Commutativity lemmas

#### A.1 Composition, Base change and Constant sheaf

##### A.1.1 Composition

Consider a diagram of scheme morphisms

$$X \overset{a}{\longrightarrow} Y \overset{b}{\longrightarrow} Z$$
and let $c = ba$. Then we obtain the following isomorphisms of functors denoted by (Co):

As a result, given a commutative square

$$
\begin{array}{c}
W \\ c \\
Y \\
\end{array} \quad \begin{array}{c}
\xrightarrow{a}
\end{array} \quad \begin{array}{c}
X \\
\xrightarrow{b}
\end{array} \quad \begin{array}{c}
\xrightarrow{d}
\end{array} \quad \begin{array}{c}
Z
\end{array}
$$
we have the following isomorphisms of functors also denoted by (Co):

\[ D^b(W) \xrightarrow{a_*} D^b(X) \]
\[ c_* \downarrow \quad \downarrow c_! \]
\[ D^b(Y) \xrightarrow{b_*} D^b(Z) \]

\[ D^b(W) \xrightarrow{a_t} D^b(X) \]
\[ c_t \downarrow \quad \downarrow c_t \]
\[ D^b(Y) \xrightarrow{b_t} D^b(Z) \]

\[ D^b(W) \xleftarrow{a^*} D^b(X) \]
\[ c^* \downarrow \quad \downarrow c_t \]
\[ D^b(Y) \xleftarrow{b^*} D^b(Z) \]

\[ D^b(W) \xleftarrow{a^t} D^b(X) \]
\[ c^t \downarrow \quad \downarrow c^t \]
\[ D^b(Y) \xleftarrow{b^t} D^b(Z). \]

### A.1.2 Base change

Consider a cartesian square of scheme morphisms

\[
\begin{array}{ccc}
W & \xrightarrow{a} & X \\
\downarrow{c} & \square & \downarrow{b} \\
Y & \xrightarrow{d} & Z.
\end{array}
\]

Then we obtain base change isomorphisms denoted by (BC) as follows:

\[ D^b(X) \xrightarrow{a_*} D^b(W) \]
\[ b_* \downarrow \quad \downarrow b_* \]
\[ D^b(Z) \xrightarrow{c_*} D^b(Y) \]

\[ D^b(X) \xrightarrow{a_t} D^b(W) \]
\[ b_* \downarrow \quad \downarrow b_* \]
\[ D^b(Z) \xrightarrow{c_*} D^b(Y). \]
A.1.3 Constant sheaf under inverse image

The constant sheaf $\mathbb{Q}_{\ell X}$ on a scheme $X$ can be seen as a functor

$$\mathbb{Q}_{\ell X} : 1 \to D^b(X)$$

where 1 is the trivial group regarded as a one-object category. For a morphism $f : X \to Y$, we have an isomorphism of functors denoted by (CII):

$$\begin{array}{ccc}
1 & \xrightarrow{1_{\mathbb{Q}_{\ell Y}}} & \mathbb{Q}_{\ell Y} \\
\mathbb{Q}_{\ell X} & \xrightarrow{f^*} & D^b(Y) \\
D^b(X) & \xrightarrow{f^*} & D^b(Y).
\end{array}$$

A.2 Iterated composition

Given a prism of morphisms

$$\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow{a} & & \downarrow{a'} \\
Y & \xrightarrow{c} & Y' \\
\downarrow{b} & & \downarrow{c'} \\
Z & \xrightarrow{h} & Z'
\end{array}$$

then following prisms are commutative:

(a) $D^b(X) \xrightarrow{f_*} D^b(X')$

(b) $D^b(X) \xrightarrow{f^*} D^b(X')$

(c) $D^b(X) \xrightarrow{f_*} D^b(X')$

(d) $D^b(X) \xrightarrow{f^*} D^b(X')$
A.3 Base change and composition

Consider a prism of morphisms

\[
\begin{array}{c}
X \xrightarrow{f} X' \\
\downarrow c \quad \downarrow a' \\
Y \xrightarrow{i} Y' \\
\downarrow b \quad \downarrow h \\
Z \xrightarrow{Y} Z'
\end{array}
\]

and suppose that all the squares are cartesian. Then following prisms are commutative:

(a) \[
\begin{array}{c}
D^b(X) \xrightarrow{f_i} D^b(X') \\
\downarrow a^* \quad \downarrow (a')^* \\
D^b(Z) \xrightarrow{h_i} D^b(Z'),
\end{array}
\]

(b) \[
\begin{array}{c}
D^b(X) \xrightarrow{f_i} D^b(X') \\
\downarrow c_i \quad \downarrow (c')^* \\
D^b(Z) \xrightarrow{h_i} D^b(Z'),
\end{array}
\]

(c) \[
\begin{array}{c}
D^b(X) \xrightarrow{f_i} D^b(X') \\
\downarrow b^* \quad \downarrow (b')^* \\
D^b(Z) \xrightarrow{h_i} D^b(Z'),
\end{array}
\]

(d) \[
\begin{array}{c}
D^b(X) \xrightarrow{f_i} D^b(X') \\
\downarrow c_i \quad \downarrow (c')^* \\
D^b(Z) \xrightarrow{h_i} D^b(Z'),
\end{array}
\]

A.4 Base change and composition

As a result of (A.3), given a cube of morphisms

\[
\begin{array}{c}
W \xrightarrow{a} X \\
\downarrow f_w \quad \downarrow c \\
Y \xrightarrow{d} b \\
\downarrow f_y \quad \downarrow f_x \\
Z \\
\downarrow f_z \\
W' \xrightarrow{a'} X' \\
\downarrow f_y' \quad \downarrow f_x' \\
Y' \xrightarrow{a'} Z' \\
\end{array}
\]
then the following cubes are commutative:

\[
\begin{array}{c}
(a)\ D^b(W) \xrightarrow{a^*} D^b(X) \xrightarrow{b_*} D^b(Z) \\
\end{array}
\]

\[
\begin{array}{c}
(b)\ D^b(W) \xrightarrow{a^*} D^b(X) \xrightarrow{b_*} D^b(Z) \\
\end{array}
\]

\[
\begin{array}{c}
(c)\ D^b(W) \xrightarrow{a^*} D^b(X) \xrightarrow{b_*} D^b(Z) \\
\end{array}
\]

\[
\begin{array}{c}
(d)\ D^b(W) \xrightarrow{a^*} D^b(X) \xrightarrow{b_*} D^b(Z) \\
\end{array}
\]

### A.5 Constant sheaf and composition

Consider a diagram of morphisms

\[
X \xrightarrow{a} Y \xrightarrow{b} Z
\]

and set \( c = ba \). Then the following prism is commutative:

\[
\begin{array}{c}
1 \xrightarrow{\text{id}} D^b(X) \xrightarrow{a^*} D^b(Y) \xrightarrow{b^*} D^b(Z) \\
\end{array}
\]

### A.6 Equivariant version

If an algebraic group \( H \) acts on a scheme \( X \), then we can consider the equivariant derived category \( D^b_H(X) \) defined in \cite{BL94}. We remark that \( D^b_H(X) \) can be seen as “a
derived category of a quotient stack $[H \backslash X]$” (cf. [LO08]).

We can define an equivariant version of the constant sheaf, denoted by $\underline{Q}_X^H$. If $f$ is $H$-equivariant, $f_*, f^!, f^*, f_!$ can be also defined. Moreover, we can construct the equivariant version of the above isomorphisms, also denoted by (Co), (BC) and (CII).

A.6.1 Forgetting and integration

Let $K$ be a closed subgroup of $H$, and let $H$ act on $X$.

Then there is a forgetful functor

$$\text{For}_K^H : \mathcal{D}_b^H(X) \to \mathcal{D}_b^K(X)$$

defined in [BL94] §2.6.1 (where it is denoted by $Res_{K,H}$). This functor has a left adjoint

$$\gamma_K^H : \mathcal{D}_b^K(X) \to \mathcal{D}_b^H(X)$$

defined in [BL94] §3.7.1 (where it is denoted by $Ind_K^H$). If $K$ is the trivial group, we write $\text{For}_K^H$ for $\text{For}_K^H$, and $\gamma_K^H$ for $\gamma_K^H$. If we write $q : [K \backslash X] \to [H \backslash X]$ for the natural projection of quotient stacks, then we can see $\text{For}_K^H$ as $q^* = q^! [-2 \dim(H/K)]$, and $\gamma_K^H$ as $q_! [2 \dim(H/K)]$.

For a $H$-morphism $f : X \to Y$ of $H$-schemes, we have the following isomorphisms of functors denoted by (For):

$$
\begin{array}{cccccccc}
D_b^H(X) & \xrightarrow{\text{For}_K^H} & D_b^K(X) & \xrightarrow{f_*} & D_b^H(X) & \xrightarrow{\text{For}_K^H} & D_b^K(X) & \xrightarrow{f_*} & D_b^H(X) & \xrightarrow{\text{For}_K^H} & D_b^K(X) \\
D_b^H(Y) & \xrightarrow{\text{For}_K^H} & D_b^K(Y) & \xrightarrow{f_*} & D_b^H(Y) & \xrightarrow{\text{For}_K^H} & D_b^K(Y) & \xrightarrow{f_*} & D_b^H(Y) & \xrightarrow{\text{For}_K^H} & D_b^K(Y)
\end{array}
$$

and the followings denoted by (Int):

$$
\begin{array}{cccccccc}
D_b^H(X) & \xrightarrow{\gamma_K^H} & D_b^K(X) & \xrightarrow{f_*} & D_b^H(Y) & \xrightarrow{\gamma_K^H} & D_b^K(Y) & \xrightarrow{f_*} & D_b^H(X) & \xrightarrow{\gamma_K^H} & D_b^K(X) & \xrightarrow{f_*} & D_b^H(Y) & \xrightarrow{\gamma_K^H} & D_b^K(Y)
\end{array}
$$

and

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Considering \( \text{For}_K^H = q^* = q^![-2 \dim(H/K)] \) and \( \gamma_K^H = q_![-2 \dim(H/K)] \), these can be seen as analogues of (Co) and (BC) in A.1.1 and A.1.2.

### A.6.2 Transitivity isomorphisms of forgetting and integration

Also, for a chain of closed subgroups \( K \subset J \subset H \), we have *transitivity isomorphisms* denoted by (Tr)

\[
\begin{array}{ccc}
D^b_H(X) & \xrightarrow{\text{For}^H_J} & D^b_J(X) \\
\downarrow{\text{For}^H_K} & & \downarrow{\text{For}^J_K} \\
D^b_K(X) & & D^b_K(X)
\end{array}
\]

\[
\begin{array}{ccc}
D^b_H(X) & \xrightarrow{\text{For}^J_K} & D^b_J(X) \\
\downarrow{\gamma^H_K} & & \downarrow{\gamma^J_K} \\
D^b_K(X) & & D^b_K(X)
\end{array}
\]

As a result, for chains of closed subgroups \( K \subset J \subset H \) and \( K \subset I \subset H \), we have the following isomorphisms also denoted by (Tr):

\[
\begin{array}{ccc}
D^b_I(X) & \xrightarrow{\text{For}^I_J} & D^b_J(X) \\
\downarrow{\gamma^I_K} & & \downarrow{\gamma^I_K} \\
D^b_K(X) & & D^b_K(X)
\end{array}
\]

These can be seen as analogues of (Co) in A.1.1.

### A.6.3 Forgetting and integration when \( H/K \) is acyclic

Let \( K \) be a closed subgroup of \( H \), and let \( H \) act on \( X \). Assume that \( H/K \) is \( \infty \)-acyclic (for example, isomorphic to an affine space. See [BL94, Definition 1.9.1] for the definition of the \( \infty \)-acyclicity). Then by [BL94, Theorem 3.7.3],

\[
\text{For}_K^H : D^b_H(X) \xrightarrow{\sim} D^b_K(X)
\]

is equivalent, and its left inverse is

\[
\gamma_K^H : D^b_K(X) \xrightarrow{\sim} D^b_H(X).
\]

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A.6.4 Constant sheaf under forgetting and integration

Let $K$ be a closed subgroup of $H$, and let $H$ act on $X$. Then we have the following isomorphisms denoted by (CF):

$$\mathbb{Q}_\ell^K \xrightarrow{\text{(CF)}} D^b_H(X)$$

This can be seen as analogues of (CII) in A.1.3. If $H/K$ is $\infty$-acyclic, then by A.6.3 we have the following isomorphisms denoted by (CI):

$$\mathbb{Q}_\ell^K \xrightarrow{\text{(CI)}} D^b_H(X)$$
A.7 Forgetting/integration and transitivity

Let \( f: X \to Y \) be a \( H \)-morphism of \( H \)-schemes, and let \( K \subset J \subset H \) be a chain of closed subgroups. Then the following prisms are commutative.

These can be seen as analogues of the prisms in [A.2] and [A.3].
A.8 Forgetting/integration and composition

Consider a diagram

\[ X \xrightarrow{a} Y \xrightarrow{b} Z \]

of \( H \)-morphisms, and let \( c = ba \). Let \( K \subset H \) be a closed subgroup. Then the following prisms are commutative:

These can be seen as analogues of the prisms in A.2 and A.3.

A.9 Forgetting/integration and base change

Consider a cartesian square of \( H \)-morphisms

\[ W \xrightarrow{a} X \]
\[ \xrightarrow{c} \]
\[ Y \xrightarrow{d} Z. \]
Then the following cubes are commutative:

These can be seen as analogues of the cubes in A.3.

A.10 Transitivity and constant sheaf

Let $K \subset J \subset H$ be a chain of closed subgroup. Then the following prism is commutative:
This can be seen as an analogue of the prism in \textcolor{red}{A.5}. If $J/K$ and $H/J$ are $\infty$-acyclic, then the following prism is commutative:

![Prism Diagram]

\textcolor{red}{A.11} Forgetting/integration and constant sheaf

Let $f : X \rightarrow Y$ be an $H$-morphism and let $K \subset H$ be a closed subgroup. Then the following cube is commutative:

![Cube Diagram]

If $H/K$ is $\infty$-acyclic, the following cube is commutative:

![Cube Diagram]
A.12 Induction equivalence

Let \( K \subset H \) be a closed subgroup, and let \( K \) act on \( X \). Consider the induced \( H \)-scheme \( \widetilde{X} = H \times^K X \) and let \( i: X \to \widetilde{X} \) be the inclusion. Then there is an equivalence

\[
i^* \circ \text{For}^H_K: D^b_H(\widetilde{X}) \cong D^b_K(X)
\]

defined in [BL94, §2.6.3] (where it is denoted by \( \nu^* \)). This can be understood from the fact that a composition \( [K \setminus X] \xrightarrow{i} [K \setminus \widetilde{X}] \to [H \setminus \widetilde{X}] \) of quotient stacks is an isomorphism. It follows that

\[
\gamma^H_K \circ \iota_i[-2 \dim(H/K)]: D^b_K(X) \to D^b_H(\widetilde{X})
\]

is an inverse of \( i^* \circ \text{For}^H_K \).

A.13 Transitivity isomorphism for induction equivalence

Let \( K \subset J \subset H \) be a chain of closed subgroups, and let \( n = 2 \dim(J/K), m = 2 \dim(H/K) \). Let \( \widetilde{X} = H \times^K X, X_J = J \times^K X \) and let \( i_1: X \to X_J, i_2: X_J \to \widetilde{X} \) and \( i: X \to \widetilde{X} \) be the inclusions. Then we have the following transitivity isomorphism:

\[
\begin{align*}
D^b_K(X) & \xrightarrow{\gamma^j_K(i_1)_!} D^b_j(X_J) \\
D^b_K(\widetilde{X}) & \xrightarrow{\gamma^j_H(i_2)_!} D^b_J(\widetilde{X})
\end{align*}
\]

defined by

\[
\begin{align*}
D^b_K(X) & \xrightarrow{(i_1)_!} D^b_K(X_J) \xrightarrow{\gamma^j_K} D^b_j(X_J) \\
D^b_K(\widetilde{X}) & \xrightarrow{\gamma^j_H} D^b_H(\widetilde{X}) \\
D^b_K(X) & \xrightarrow{(i_2)_!} D^b_K(\widetilde{X}) \xrightarrow{\gamma^j_H} D^b_H(\widetilde{X})
\end{align*}
\]
A.14 Integration and induction equivalence

Let $K, I$ be closed subgroups of $H$ such that $H = IK$. Then we can identify $I \times i_{\cap K} X$ with $\tilde{X}$. We have the following isomorphism denoted by (IEI):

\[
\begin{array}{ccc}
D^b_H(\tilde{X}) & \xrightarrow[\gamma^H_{K}i]{} & D^b_K(X) \\
\downarrow[\gamma^H_I] & & \downarrow[\gamma^K_{i\cap K}] \\
D^b_I(\tilde{X}) & \xrightarrow[\gamma^I_{i\cap K}i]{} & D^b_{i\cap K}(X)
\end{array}
\]

defined by

\[
\begin{array}{ccc}
D^b_H(\tilde{X}) & \xrightarrow[\gamma^H_K]{} & D^b_K(\tilde{X}) & \xrightarrow[i]{} & D^b_K(X) \\
\downarrow[\gamma^H_I] & & \downarrow[\gamma^K_{i\cap K}] & & \downarrow[\gamma^K_{i\cap K}] \\
D^b_I(\tilde{X}) & \xrightarrow[\gamma^I_{i\cap K}]{} & D^b_{i\cap K}(\tilde{X}) & \xrightarrow[i]{} & D^b_{i\cap K}(X).
\end{array}
\]

A.15 Inverse image and induction equivalence

Let $f: X \to Y$ be a $K$-morphism and let $\tilde{f}: X \to Y$ be the induced $H$-morphism. Then the square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
i & & \downarrow j \\
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y}
\end{array}
\]

is cartesian. Let $m = 2 \dim(H/K)$. We have the following isomorphism denoted by (IBC):

\[
\begin{array}{ccc}
D^b_H(\tilde{X}) & \xrightarrow[\gamma^H_Ki]{} & D^b_K(X) \\
\downarrow[\gamma^H_I] & & \downarrow[f^*] & & \downarrow[f^*] \\
D^b_H(\tilde{Y}) & \xrightarrow[\gamma^H_Kj]{} & D^b_K(Y)
\end{array}
\]

defined by

\[
\begin{array}{ccc}
D^b_H(\tilde{X}) & \xrightarrow[\gamma^H_K]{} & D^b_K(\tilde{X}) & \xrightarrow[i]{} & D^b_K(X) \\
\downarrow[\gamma^H_I] & & \downarrow[\tilde{f}^*] & & \downarrow[\tilde{f}^*] \\
D^b_H(\tilde{Y}) & \xrightarrow[\gamma^H_K]{} & D^b_K(\tilde{Y}) & \xrightarrow[j]{} & D^b_K(Y).
\end{array}
\]

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A.16 Constant sheaf and induction equivalence

Let \( f: X \to Y \) be a \( K \)-morphism. From the fact that \( i^\ast \circ \text{For}_K^H(\mathcal{Q}_X^H) \overset{(CIE)}{\cong} \mathcal{Q}_X^K \), we have the following isomorphism denoted by (CIE):

\[
\begin{array}{c}
\mathcal{Q}_X^H[2 \dim(H/K)] \\
\downarrow \text{(CIE)} \\
D^b_K(X)
\end{array}
\]

where \( m = 2 \dim(H/K) \).

A.17 Transitivity and induction equivalence

Let \( K \subset J \subset H \) be a chain of closed subgroups, and let \( n = 2 \dim(J/K), m = 2 \dim(H/K) \). Let \( \tilde{X} = H \times^K X, X_J = J \times^K X \) and let \( i_1: X \to X_J, i_2: X_J \to \tilde{X} \) and \( i: X \to \tilde{X} \) be the inclusions. The following prism is commutative:

A.18 Integration and induction equivalence

Let \( K, I \) be closed subgroups of \( H \) such that \( H = IK \). Then we can identify \( I \times^\iota^{} K X \) with \( \tilde{X} \). Let \( m = 2 \dim(H/K) \). Assume that \( H/I \) is \( \infty \)-acyclic. Then the
A.19 Inverse image and induction equivalence

Let $f: X \to Y$ be a $K$-morphism and let $m = 2\dim(H/K)$. Assume the square

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow i & \square & \downarrow j \\
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y}
\end{array}
$$

is cartesian. Then the following cube is commutative:

A.20 Free action

Let $N$ be a closed normal subgroup of $H$. Assume that $N$ acts on $X$ freely. Let $q: X \to N\backslash X$ be the quotient map. Then there is a natural equivalence of categories (see [BL94, Theorem 2.6.2])

$$q^*: D^b_{H/N}(N\backslash X) \simto D^b_H(X)$$

(A.1)
such that the following diagram is commutative:

\[
\begin{array}{ccc}
D^b_{H/N}(N\backslash X) & \xrightarrow{q^*} & D^b_H(X) \\
\downarrow_{For^{H/N}} & & \downarrow_{For^H} \\
D^b(N\backslash X) & \xrightarrow{q^*} & D^b(X).
\end{array}
\]

A.20.1 composition/base change and Free action

The functor (A.1) satisfies analogues of the isomorphism (Co), (BC). For example, let \( f: X \to Y \) be the \( H \)-equivariant morphisms, and assume that \( K \) acts on \( X, Y \) freely. Let \( f: K\backslash X \to K\backslash Y \) be the induced map. Then there are isomorphisms

\[
\begin{align*}
D^b_H(X) & \xrightarrow{f^*} D^b_H(Y) & D^b_H(X) & \xrightarrow{f^*} D^b_H(Y) \\
q_x^* \downarrow & & q_y^* \downarrow & \downarrow \mathrm{(Co)} & \downarrow \mathrm{(BC)}
\end{align*}
\]

where \( q_X: X \to K\backslash X \) and \( q_Y: Y \to K\backslash Y \) are the quotient maps.

A.20.2 Forgetting and free action

Let \( K \subset H \) be a closed subgroup. Assume \( H \) acts on \( X \) freely. Let \( q: X \to H\backslash X \), \( q_1: X \to K\backslash X \) and \( q_2: K\backslash X \to H\backslash X \) be the quotient maps. Then from the definition we have the following isomorphism denoted by (FF):

\[
\begin{array}{ccc}
D^b_K(N\backslash X) & \xrightarrow{q^*} & D^b_K(X) \\
\downarrow_{For^H_K} & & \downarrow_{(q_1)^*} \\
D^b(K\backslash X) & \xrightarrow{f^*} & D^b(K\backslash X).
\end{array}
\]

A.20.3 Free action and trivial action

Let \( K, N \subset H \) be subgroups such that \( H = N \rtimes K \). Assume that \( N \) is \( \infty \)-acyclic and that \( N \) acts on \( X \) freely and acts on \( Y \) trivially. Let \( f: X \to Y \) be an \( H \)-equivariant map and let \( f: N\backslash X \to Y \) be the induced map. Then we have the following isomorphism denoted by (FT):

\[
\begin{array}{ccc}
D^b_K(N\backslash X) & \xrightarrow{q^*} & D^b_K(X) \\
\downarrow_{(q_2)^*} & & \downarrow_{(q_1)^*} \\
D^b(K\backslash X) & \xrightarrow{f^*} & D^b(K\backslash X).
\end{array}
\]
where \( q: X \to N \setminus X \) be the quotient map. This is because the two functors \( q^* \circ f^* \circ \text{For}_K^H \) and \( f^* \) are isomorphic after composing the isomorphism \( \text{For}_K^H: D_K^H(X) \to D_K^b(X) \).

### A.21 Twisted exterior product

Assume that \( H \) acts on \( \bar{X} \) freely from the right and acts on \( Y \) from the left. Let \( X = \bar{X}/H \) be the quotient. Consider the \( H \)-action on \( \bar{X} \times Y \) defined by \( h \cdot (x, y) = (x \cdot h^{-1}, h \cdot y) \). Put

\[
X \times^H Y := \bar{X} \times^H Y := H \backslash (\bar{X} \times Y)
\]

and let \( q: \bar{X} \to X, \pi: \bar{X} \times Y \to X \times Y \) be the quotient map. Consider the \( H \)-action on the second factor of \( X \times Y \). Then \( \pi' := (q \times 1): \bar{X} \times Y \to X \times Y \) is \( H \)-equivariant. We obtain a functor \( D^b(X) \times D^b_H(Y) \to D^b(X \times^H Y), (A, B) \mapsto A \boxtimes B \) defined by the composition

\[
D^b(X) \times D^b_H(Y) \xrightarrow{\boxtimes} D^b_H(X \times Y) \xrightarrow{(\pi')^*} D^b_H(\bar{X} \times Y) \xrightarrow{(\pi)^*} D^b(X \times^H Y).
\]

#### A.21.1 Inverse image of twisted exterior product

Let \( K \subset H \) be a closed subgroup. Assume that \( K \) acts on \( \bar{X}_1 \) freely from the right and acts on \( Y_1 \) from the left and that \( H \) acts on \( \bar{X}_2 \) freely from the right and acts on \( Y_2 \) from the left. Let \( X_1 = \bar{X}_1/K, X_2 = \bar{X}_2/H \) be quotients. Let \( \tilde{f}: \bar{X}_1 \to \bar{X}_2, g: Y_1 \to Y_2 \) be \( K \)-equivariant maps. Let \( f: X_1 \to X_2 \) and \( f \times g: X_1 \times Y_1 \to X_2 \times Y_2 \) be the induced maps.

From the pasting diagram

\[
\begin{array}{ccccccccc}
D^b_K(X_1 \times Y_1) & \xrightarrow{(\pi)^*} & D^b_K(X_2 \times Y_2) & \xrightarrow{(\pi')^*} & D^b_K(X_1 \times Y_1) \\
\downarrow{(\text{FF})} & & \downarrow{(\text{FF})} & & \downarrow{(\text{FF})} \\
D^b_K(\bar{X}_1 \times Y_1) & \xrightarrow{(\pi)^*} & D^b_K(\bar{X}_2 \times Y_2) & \xrightarrow{(\pi')^*} & D^b_K(\bar{X}_1 \times Y_1) \\
\downarrow{(\text{Co})} & & \downarrow{(\text{Co})} & & \downarrow{(\text{Co})} \\
D^b(\bar{X}_1 \times Y_1) & \xrightarrow{(\pi)^*} & D^b(\bar{X}_2 \times Y_2) & \xrightarrow{(\pi')^*} & D^b(\bar{X}_1 \times Y_1) \\
\end{array}
\]

and the canonical \( K \)-equivariant isomorphism

\[
(f \times g)^*(A \boxtimes B) \cong f^*A \boxtimes g^*B,
\]

it follows that

\[
(f \times g)^*(A \boxtimes B) \cong f^*A \boxtimes g^*B
\]

for any \( A \in D^b(X_2) \) and \( B \in D^b_H(Y_2) \).
A.21.2 Küneth formula for twisted exterior product

Let $K, N \subset H$ be subgroups such that $H = N \rtimes K$ and that $N$ is $\infty$-acyclic. Assume that $H$ acts on $\tilde{X}_1$ freely from the right and acts on $Y_1$ from the left and that $K$ acts on $\tilde{X}_2$ freely from the right and acts on $Y_2$ from the left. Let $X_1 = \tilde{X}_1/H, X_2 = \tilde{X}_2/H$ be quotients. Let $\tilde{f}: \tilde{X}_1 \to \tilde{X}_2, g: Y_1 \to Y_2$ be $H$-equivariant maps. Let $f: X_1 \to X_2$ and $f \times g: X_1 \times Y_1 \to X_2 \times Y_2$ be the induced maps.

From the pasting diagram

$$
\begin{array}{ccccccccc}
D^b_H(X_1 \times Y_1) & \xrightarrow{(\cdot)_!} & D^b_H(X_1 \times Y_2) & \xrightarrow{\text{For}_K^H} & D^b_K(X_1 \times Y_2) & \xrightarrow{(\cdot)_!} & D^b_K(X_2 \times Y_2) \\
\downarrow{(\cdot)^*} & \text{(BC)} & \downarrow{(\cdot)^*} & \text{(FT)} & \downarrow{(\cdot)^*} & \text{(BC)} & \downarrow{(\cdot)^*} \\
D^b_H(\tilde{X}_1 \times Y_1) & \xrightarrow{(\cdot)_!} & D^b_H(\tilde{X}_1 \times Y_2) & \xrightarrow{(\cdot)_!} & D^b_K(\tilde{X}_1 \times Y_2) & \xrightarrow{(\cdot)_!} & D^b_K(\tilde{X}_2 \times Y_2) \\
\downarrow{(\cdot)^*} & \text{(BC)} & \downarrow{(\cdot)^*} & \text{(BC)} & \downarrow{(\cdot)^*} & \text{(BC)} & \downarrow{(\cdot)^*} \\
D^b(\tilde{X}_1 \times^H Y_1) & \xrightarrow{(\cdot)_!} & D^b(\tilde{X}_1 \times^H Y_2) & \xrightarrow{(\cdot)_!} & D^b(\tilde{X}_2 \times^K Y_2) 
\end{array}
$$

and the canonical $H$-equivariant isomorphism by the Küneth formula

$$(f \times g)! (\mathcal{A} \boxtimes \mathcal{B}) \cong f_! \mathcal{A} \boxtimes g_! \mathcal{B},$$

it follows that

$$(f \times g)! (\mathcal{A} \boxtimes \mathcal{B}) \cong f_! \mathcal{A} \boxtimes g_! \mathcal{B}.$$

for any $\mathcal{A} \in D^b(X_1)$ and $\mathcal{B} \in D^b_H(Y_1)$.

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