Scalar gravity: Post-Newtonian corrections via an effective field theory approach

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The problem of motion in General Relativity has lost its academic status and become an active research area since the next generation of gravity wave detectors will rely upon its solution. Here we will show within scalar gravity, how ideas borrowed from Quantum Field Theory can be used to solve the problem of motion in a systematic fashion. We will concentrate in Post-Newtonian corrections. We will calculate the Einstein-Infeld-Hoffmann action and show how a systematic perturbative expansion puts strong constraints on the couplings of non-derivative interactions in the theory.

I. INTRODUCTION

Consider the apparently simple problem of the earth motion around the sun. The Newtonian solution is an excellent approximation but suppose we wish to be more accurate. A closer look reveals that there are many sources of complication. Einstein theory teaches us how to correct for relativistic effects. However, the earth is clearly not a point particle, and will thus deform under the influence of tidal forces. In addition the whole sun-earth system will radiate energy away in the form of gravitational waves. The inclusion of all these effects can make the problem of solving for the trajectory intractable. In the past, solving this problem was only of academic interest, but the next generation of gravity wave detectors will rely upon its solution [1]. The construction of accurate templates for gravity wave interferometers is a daunting task. After more than ten years of work the templates have been completed up to third Post-Newtonian (PN) order for non-spinning compact binaries [2]. However, it was not clear how to: proceed to higher orders in a systematic fashion, include finite size effects due to spin or spin-spin corrections. During the last years a new framework has emerged, coined NRGR (Non-Relativistic General Relativity) due to its similarities with Effective Field Theory (EFT) ideas in particle physics, where all of the apparent obstacles of the traditional approach can been successfully overcome [3, 4]. NRGR naturally allows for a systematic account of the internal structure of the binary constitutes and permits us to calculate back reaction as well as dissipative effects [5]. Moreover, new results for spinning compact binaries have been recently reported [6]. In this short contribution we will show within scalar gravity, how an EFT approach can be used to solve the problem of motion in a systematic fashion [7]. In particular we will calculate the Einstein-Infeld-Hoffmann (EIH) action for the case of two scalar-gravitating bodies, accurate up to 1PN. The purpose of this contribution is pedagogical, allowing us to concentrate on the conceptual aspects. As we shall see a systematic perturbative expansion puts strong constraints on the couplings of non-derivative interactions in the theory.

II. SCALAR GRAVITY

The starting point of the EFT approach consists of a theory of point particles coupled to a real scalar field $\phi$ we shall call the “s-graviton”. For simplicity we will consider here a massive $\phi^3$ theory in a Minkowski background, though we will discuss other type of models later on. The action will be given by $S = S_g + S_{pp}$, with

$$
S_g = \int d^4x \left( \partial_\mu \phi \partial^\mu \phi - \mu^2 \phi^2 - \lambda \phi^3 \right), \quad S_{pp} = -\sum_a m_a \int d\tau_a \sqrt{1 + \frac{\phi}{M}}
$$

(1)

describing the s-graviton dynamics and motion of the binary system ($a = 1, 2$). In this equation $M$ sets the coupling to matter, and $\lambda, \mu$, the self-interaction and s-graviton mass respectively. Also $d\tau = \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu}$ represents the proper time along the $a$-th particle and $\eta_{\mu\nu} \equiv \text{diag}(+,-,-,-)$, we work in $\hbar = c = 1$ units. The choice of matter coupling is meant to resemble Einstein case, at least for the $h_{00}$ mode, with $M$ playing the role of the Planck Mass. The normalization is also chosen to mimic the graviton propagator. We could in principle add a set of higher order operators in the worldline action to account for finite size effects. However, $\phi^3$ theory is super-renormalizable and it is...
possible to show that the $n$-point function is UV finite and no higher order operators are needed. One other aspect of the super-renormalizability is the fact that a $\phi^3$ self-interaction in four spacetime dimensions has a dimensionful coupling and the perturbative approach breaks down at distances of order $1/\lambda$. This is connected to IR divergences (in the massless limit) which appear in the perturbative expansion due to factors of $\lambda/E$, with $E$ the energy of the s-graviton. These IR divergences must cancel in any physical observable, such as the binding energy of the binary system. However, a resummation procedure is in general needed in order to achieve a finite result. There are a few ways to overcome this. We could work in six dimensions where the self-interaction coupling of the theory. We will discuss later on under which circumstances this is a more generic phenomena.

III. NRGR

The power of the EFT formalism resides in a manifest power counting in the expansion parameter of the theory, in this case the relative velocity $v$. Here we will pinpoint the necessary steps and refer to Goldberger’s contribution for further details. The expansion of the worldline Lagrangian leads to

$$L_{pp} = \sum_a \frac{m_a}{2} \left[ v^2_a - \left(1 - \frac{v^2_a}{2}\right) \frac{\phi^2}{M} + \frac{1}{4} \frac{\phi^4}{M^4} + \frac{1}{4} \frac{\phi^2}{M^2} \right] + \ldots,$$

(2)

where we have chosen $x^0$ as the worldline parameter. The propagator for the field $\phi$ appearing in $L_{pp}$ is still fully relativistic, and therefore a small velocity expansion has yet to be performed. To deal with this problem it is convenient to decompose the s-graviton field into potential modes ($\phi$) with momentum scaling $k^\mu \sim (v/r, 1/r)$ (notice they can never go on shell), and radiation modes ($\Phi$) whose momentum scale as $k^\mu \sim (v/r, v/r)$. In the EFT spirit potential modes do not propagate and can be thus integrated out at each order in perturbation theory. Radiation s-gravitons on the other hand can appear on shell and must be kept as propagating degrees of freedom in order to reproduce the correct long distance physics.

A. Power counting

In the EFT approach one computes the effective action perturbatively, in our case in $v$, based on systematic power counting rules. In order to obtain the latter one starts with the scaling laws for the ($\phi, \Phi$) fields. For convenience one first introduces $\Phi_k$, where the large momentum piece of the potential s-graviton is factored out. By expanding we get

$$\langle \Phi_k(x^0)\Phi_q(0) \rangle = (2\pi)^3 \delta^3(k + q) \delta(x^0) \frac{-i}{2(k^2 + \mu^2)} \langle \phi(x)\phi(0) \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{2(k^2 + \mu^2)} e^{ikx},$$

(3)

for the propagators. Notice that we have decided to keep the mass “non-perturbatively” to cure IR divergences, although we will assume $\mu r < v$ in what follows, and treat it as a perturbation when allowed, in order to resemble the massless power counting rules and a $1/r$ leading order potential. If we assign the scaling $x^0 \sim v/r$ we obtain the following leading order power counting rules

$$\phi \sim v/r, \Phi_k \sim \sqrt{v/r} \rightarrow \Phi \sim \frac{v^2}{\sqrt{L}}.$$

(4)

where $L = mvr$. The last arrow follows from the assumption that the leading order potential is given by $1/r$ and hence the virial theorem, $v^2 \sim \frac{m}{Mr^2}$, applies. This assumption is true in the case of $\lambda = 0, \mu r < v$, however $\lambda \neq 0$ can easily jeopardize our power counting due to the introduction of a dimensionful coupling. In order to tackle this

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1. Notice also that using field re-definitions ($\sim$ equations of motion) we can always trade $\partial^2 \phi$ by a polynomial, so higher dimensional operator are always of the form $\phi^a$ and could be absorbed into the worldline couplings.

2. Notice it also implies $m/M \sim \sqrt{Lv}$.
problem let us exercise our scaling rules and power count the first correction due to \( \lambda \). The diagram is shown in fig. 2a and it scales as

\[
\text{fig. 2a} \sim \left[ \frac{dx^0 m}{M} \phi \right]^3 \left[ \frac{dx^0 \delta^3(\mathbf{k}) \lambda \phi^3}{M} \right] \sim \left[ \frac{r m}{v} \frac{\lambda}{\sqrt{L}} \right] \left[ \frac{M \phi^2}{\sqrt{L}} \right]^3 \sim L v^2(\lambda M r^2),
\]

\( v^2(\lambda M r^2) \) times the leading order term which scales as \( L^3 \). It is easy to see higher order terms in \( \lambda \) follow the same pattern. For \( \lambda = g^2 M \), with \( g \) a dimensionless coupling, we end up with \( r \sim \frac{1}{\sqrt{M}} \) for the validity of the perturbative approximation and power counting. In order to make sense of the perturbative approach we had to cure this IR singularities before expecting any power counting to work, and that is what the s-graviton mass is doing. If we demand our leading order potential to match the Newtonian case we will set \( M \equiv m_{pl} \) and therefore the perturbative expansion is valid for \( r < l_{pl}/g \), with \( l_{pl} = 10^{-33} \) cm, the Planck length. To avoid entering the quantum realm\(^3\) we will have to fine tune \( g \) to an extremely small number of the order of \( 10^{-40} \) for typical binary systems in the solar mass range in the inspiral regime. This obviously defies naturalness arguments and puts a flag on the phenomenological viability of such theory since it implies a ridiculously small self-coupling, \( \lambda \sim 10^{-80} m_{pl} \sim 10^{-60} \text{GeV}! \) Notice that the problem does not lie in the self-coupling itself but in the strength of the worldline coupling which determines the leading order scaling laws. In Einstein theory this is taken care of by the three graviton coupling, \( g_3 \sim k^2/M \rightarrow g_3 M r^2 \sim 1 \).

The condition \( \mu \leq v/r \) also implies a stunningly tiny s-graviton mass of the order of \( 10^{-30} \text{eV} \). These are consistent, and somehow equivalent, to solar system constraints\(^4\), whereas by naturalness arguments \( \mu \sim \lambda \) would produce an even smaller value. We will hereon assume \( \mu r \sim v, \lambda r \ll 1 \), and proceed with this theory as a playground.

IV. EINSTEIN-INFELD-HOFFMANN

Let us concentrate now in the calculation of the 1PN correction to the gravitational potential. The leading order one s-graviton exchange can be easily seen to reproduce Newtonian gravity \(^3\). We also need to take into account diagrams with one single s-graviton exchange which are down by \( v^2 \) shown in figures 1a and 1b plus the non linear terms depicted in figures 2a and 2b. We will treat the s-graviton mass as perturbation in the one s-graviton exchange (\( \mu r \sim v \)) and that is shown in diagram 1c. The computation proceeds systematically by using the Feynman rules of the EFT order by order. We will concentrate in detail in fig. 2a, we will display the full result later on. For the three s-graviton diagram we will have

\[
\text{fig. 2a} = \frac{1}{2} \left( \frac{-i m_2}{2M} \right)^2 \frac{-i m_1}{2M} \int dt_1 dt_2 dt_3 \langle T(\Phi(x_1)\Phi(x_2)\Phi(x_3)) \rangle.
\]

Our task now is to compute the three-point function. For a \( \phi^3 \) theory one obtains

\[
\langle T(\Phi(x_1)\Phi(x_2)\Phi(x_3)) \rangle = 3!(-i\lambda)\delta(t_1 - t_2)\delta(t_1 - t_3)\int \prod_r \frac{d^3k_r}{(2\pi)^3} e^{-i\sum_r k_r \cdot \Phi(x_r)} \prod_i \frac{1}{2(\mathbf{k}_j^2 + \mu^2)}.
\]

The next step would be to plug this expression back into \(^1\) , get a finite result which we will have to further expand in powers of \( \mu r \sim v \) and keep the leading order piece, already at 1PN for \( \lambda M r^2 \sim 1 \). In the EFT spirit a better way to proceed is to treat \( \mu \) as a perturbation in the same way time derivatives are treated, by expanding the propagators in powers of \( \mu/|\mathbf{k}| \). For the one s-graviton exchange this represents no harm. In general one faces the problem that IR divergences will only cancel out after all the terms are included. If we are willing to accept that is the case one can calculate the 1PN correction by taking the massless limit of \(^7\) and keep the (non-constant) finite piece. Therefore, introducing \( d = 3 + \epsilon \) and taking the limit \( \epsilon \rightarrow 0 \) one gets

\[
\text{fig. 2a} = i \lambda \frac{3m_1 m_2^2}{64 M^3} \int dt d^3k_2 d^3k_1 \frac{1}{(2\pi)^3} e^{i\mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2)} = i \lambda \frac{3\pi G N m_1 m_2^2}{16 M} \int dt d^3k_1 \frac{1}{(2\pi)^3} e^{-i\mathbf{k}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_2)}
\]

\[
= i 3\lambda G N m_1 m_2^2 \Gamma(\epsilon/2) \int dt \left( \frac{|x_1 - x_2|}{4} \right)^{-\epsilon/2} \rightarrow -i 3\lambda M G N^2 m_2^2 m_1 \int dt \log(|x_1 - x_2|) + \text{constant}.
\]

\(^3\) Recall that loop effects in NRGR for gravitons are suppressed by \( 1/L \) and can be thus ignored in the classical scenario \(^4\).
FIG. 1: One s-graviton exchange contribution at 1PN. The \( \times \) represents a correction to the propagator, and \( \times \) a mass insertion.

FIG. 2: Non linear contributions at 1PN.

with \( G_N = \frac{1}{24\pi M^2} \), and the \( \text{“constant”} \) piece also contains the \( \frac{1}{r} \) IR pole\(^4\).

Our final task consists in collecting the other few pieces. We refer to \( \text{[3, 7]} \) for details since the calculations are almost identical. Let us compute the result for the new term in fig. 1c due to the s-graviton mass insertion,

\[
\text{fig. 1c} = -\frac{i m_1 m_2}{8 M^2} \int dt_1 dt_2 \delta(t_1 - t_2) \int \frac{d^3 k}{(2\pi)^3} \frac{\mu^2}{k^4} e^{-ik \cdot (x_1 - x_2)} = \frac{i}{2} \int dt \ G_N m_1 m_2 \mu^2 |x_1 - x_2|, \tag{9}
\]

which is nothing but the \( \mathcal{O}(v^2) \) piece in the expansion of the Yukawa potential, \( -\frac{\mu r}{r} \sim \mu - \frac{1}{r}(1 + \frac{v^2}{r^2} + ...) \).

Putting everything together, including mirror images, we finally obtain

\[
L_{EIH} = \frac{1}{8} \sum_a m_a v_a^4 + \frac{G_N m_1 m_2}{2|x_1 - x_2|} \left[ v_1^2 + v_2^2 + (v_1 \cdot v_2) - \frac{(v_1 \cdot x_{12})(v_2 \cdot x_{12})}{|x_1 - x_2|^2} \right] + \frac{G_N^2 m_1 m_2 (m_1 + m_2)}{2|x_1 - x_2|^2} + \frac{1}{2} G_N m_1 m_2 \mu^2 |x_1 - x_2| - 3\lambda G_N^2 M m_2 m_1 (m_1 + m_2) \log(\mu |x_1 - x_2|) \tag{10}
\]

where we have also included the relativistic corrections to the kinetic energy of the point particles. The logarithmic potential introduces a very interesting feature, namely a \( \frac{1}{r} \) force into the equations of motion and therefore \( v^2 \sim a/r + b + \ldots \), which implies a \textit{dark matter} type of effect for the galaxy rotation curves. This is however by no means a serious candidate and we mention this only as a curiosity.

\[\text{V. DISCUSSION - CONCLUSIONS}\]

The EFT approach is a powerful tool within the PN framework. Using no more than dimensional analysis many conclusions can be already drawn before dwelling into the details of the calculations. We applied the techniques in the case of a massive \( \phi^3 \) theory as a playground but the ideas can be easily extended to more complicated scenarios. From the NRGR power counting rules we learned that in order to produce a well defined perturbative expansion, \( \lambda \) had to be fine tuned to a ridiculously small (compared with \( m_{Pl} \)) scale. One might however ask whether this is a feature of a \( \phi^3 \) theory or will this be faced in other scenarios. Let us consider a more general case,

\[
S_\phi = \int d^4 x \ (\gamma(\phi)\partial_\mu \phi \partial^\mu \phi + B(\phi)) \tag{11}
\]

In what follows we will consider two distinct case.

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\(^4\) In the massive case this is translated into a \( \log \mu \) factor. The full result is a Bessel function, \( K_0(\mu r) \), whose leading order piece in \( \mu r \) reproduces the logarithmic potential.
A. \( B(\phi) = 0 \)

If we expand \( \gamma(\phi) \sim 1 + \phi/M + \ldots \), we will get a kinetic piece plus a potential \( V(\phi) \) with terms like \( (\frac{\phi}{M})^n \phi \partial^2 \phi \) \( n \geq 1 \). This theory is not renormalizable, and it is easy to show it resembles Einstein case. We can also show that the perturbative approach is under control by power counting the contribution from a generic term in \( V(\phi) \). For an \((n + 2)\) s-graviton diagram we will get,

\[
\sqrt{L}^{n+2} \left( \frac{v^2}{\sqrt{L}} \right)^{n+2} M^2 v^2 / v \sim L v^{2n}.
\] (12)

For instance the first term in the expansion, \( \phi^2 \partial^2 \phi \), resembles the three graviton coupling in Einstein theory (up to tensor structure). Had we chosen this interaction we would have ended up with a similar 1PN correction as in the original Einstein-Infeld-Hoffmann action [3,4].

B. \( B(\phi) \neq 0 \)

This case is substantially different. Let us study a generic term, \( gM^4(\phi/M)^n \), with \( g \) a dimensionless coupling and \( n \geq 4 \). The \( n \) s-graviton diagram will scale as

\[
gM^4v^{2n-1} \sim gLv^{2n-7} \sim L g(m/M)^2v^{2n-8}.
\] (13)

To have a controlled perturbative expansion we would have to impose \( v^{2n-8}(m/M)^2 < 1 \). For the marginal case \( n = 4 \), setting \( M = m_{Pl} \) one needs \( g < \left( \frac{m_{Pl}}{m} \right)^2 \sim 10^{-70} \) for solar mass binary constitutes. We can improve this number by considering higher dimensional terms, namely larger \( n \), but the enhancement is really minute. Notice that this problem arises at the \textit{classical} level since the coupling to elementary particles is already too small to represent any trouble. Is only in the superposition of terms, which build up the massive lump of the star, that the perturbative expansion breaks down. From this analysis we conclude that in pure scalar gravity non-derivative self-interactions are extremely constrained.

One could then wonder about more general models including scalar fields, like tensor-scalar gravity [10]. In the latter in addition to the graviton field a scalar interaction is added with an action similar to [11] in a curved spacetime background. Within this type of scenarios the problems we encountered here can be cured by modifying the power counting. For instance, a large mass can be added to the scalar field (larger than the inverse of the solar system distance), which will render the field a negligible short range interaction. Another possibility would be to keep it nearly massless but weaken the coupling to matter to a much feeble strength \( M \gg m_{Pl} \). In this case the 3-scalar diagram (fig 2a) will now scale as \( L \frac{\Lambda}{m_{Pl}} \left( \frac{m}{M} \right)^3 \). For \( \Lambda \sim gM \) one needs \( g(m/M)^2 < v^2 \) in order to be treat as a perturbation. By naturalness argument one would expect \( g \sim 1 \), and we will then have a very tiny coupling to elementary particles. For instance, the coupling to a proton will be of the order of \( m_{proton}/M \sim 10^{-60} \) for a \( \phi^4 \) theory the 4-scalar diagram would now scale as \( L^4 v (m/M)^4 \). Compared to the leading Newtonian potential we get a suppression of order \( \hat{\lambda} (m/M)^3 (Mr^2)^{-1} \), which can be seen to be effectively small for \( \hat{\lambda} \sim 1 \), \( M \sim m \). Again the coupling to elementary particles is highly suppressed. Both solutions will keep the theory consistent with experimental data, both rely however in the introduction of a high mass scale into the theory, much higher than the Planck scale or the scale of particle physics. Perhaps this is an indication that non-derivative self-interactions are not present in nature [5).

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[5] The possibility that the dilaton, the scalar Goldstone mode of the SSB of conformal invariance, could provide an analog of Einstein theory was recently raised in [11]. Contrary to the traditional lore shift invariance does not forbid a non-derivative interaction as in the case of internal symmetries. A similar fine tuning, which was argued in [11] to be of the same nature as the cosmological constant, was invoked to avoid unstable configurations.
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