MEAN CURVATURE ONE SURFACES IN HYPERBOLIC SPACE,
AND THEIR RELATIONSHIP TO MINIMAL SURFACES IN
EUCLIDEAN SPACE

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Abstract. We describe local similarities and global differences between minimal surfaces in Euclidean 3-space and constant mean curvature 1 surfaces in hyperbolic 3-space. We also describe how to solve global period problems for constant mean curvature 1 surfaces in hyperbolic 3-space, and we give an overview of recent results on these surfaces. We include computer graphics of a number of examples.

Introduction

In this article we explain some aspects of research on constant mean curvature 1 surfaces in hyperbolic 3-space. The first half of the article emphasizes the most elementary elements of the theory, and the last half attempts to explain less elementary elements as elementarily as possible. Comparison with minimal surfaces in Euclidean 3-space is used, and such a comparison is natural (in fact, almost unavoidable), as ideas in the theory have evolved directly from ideas in minimal surface theory. We begin with this comparison in this introduction.

Minimal surfaces in Euclidean 3-space $\mathbb{R}^3$ have been a subject of investigation since the 1800’s, as evidenced by the names of nineteenth century mathematicians associated with some of the most famous complete (i.e. without boundary) examples and with the subject’s primary theorem: Enneper’s surface, Riemann’s singly-periodic staircase surface, Scherk’s periodic surfaces, and the Weierstrass representation theorem. The first half of the twentieth century, however, saw the focus turn to compact minimal surfaces with boundary, in particular, to the Plateau problem of determining the compact minimal surfaces with given boundary in $\mathbb{R}^3$, as seen in the works of Douglas, Rado, and Courant. (Although, in the early 1900’s, Bernstein did prove that a complete minimal graph must be a plane.)

But recently there has been much productive research on complete minimal surfaces, with the work of Meeks, Hoffman, Karcher, Osserman, Ros, Rosenberg, and others. And the research has been taking advantage of a modern-day tool – computer graphics. A striking example of how computer graphics has helped in the study of minimal surfaces is the Costa surface, whose proof of embeddedness [HM1] came as a direct result of computer graphics experiments (providing a counterexample to a longstanding conjecture that the plane, catenoid, and helicoid are the only complete embedded minimal surfaces with finite topology). And the recent proofs of existence of numerous examples of minimal surfaces is due in part to the ability to first check on a computer if existence is plausible. (The beauty of the graphics, as well, cannot be ruled out as a motivating factor.) Then, the many examples have given us a better understanding of the nature of minimal surfaces.

Minimal surfaces, and constant mean curvature (CMC) surfaces as well, are natural objects of study, as they are critical for area with respect to volume preserving
Figure 1. CMC 1 horosphere and catenoid cousin in $H^3$. The catenoid cousin was first described in [B].

variations that fix their boundaries. And minimal surfaces are especially important in the class of CMC surfaces, as they are area critical for all variations, not just volume preserving ones. The particular interest in minimal surfaces comes also from the elegance of the Weierstrass representation, which describes minimal surfaces in terms of a pair of holomorphic functions. Actually, nonminimal CMC surfaces can be described with holomorphic functions as well [DPW], but the descriptions are more complicated, so minimal surfaces are unique amongst all CMC surfaces in that they have a "simple" holomorphic description.

Recently, the field has also grown in breadth, by influencing related areas. As one among many examples, it has influenced the study of CMC surfaces in other 3 dimensional space forms (such as the 3 dimensional hyperbolic and spherical spaces $H^3$ and $S^3$). Regarding this, we should first remark on an elementary correspondence between CMC surfaces in the different 3 dimensional space forms, described by Lawson [L] and often attributed to him. It gives, in particular, a local correspondence between minimal surfaces in $R^3$ and CMC 1 surfaces in $H^3$. So it is not surprising that there is a kind of "Weierstrass" representation in terms of a pair of holomorphic functions for CMC 1 surfaces in $H^3$ as well, which we call the Bryant representation here, after its discoverer [B]. Furthermore, among all CMC surfaces in $H^3$, CMC 1 surfaces are unique in having a relatively simple description in terms of a pair of holomorphic functions, as we expected, since minimal surfaces in $R^3$ are similarly endowed. (However, since CMC 1 surfaces in $H^3$ are not minimal, they are area critical only for volume preserving variations – not all of the variational properties of minimal surfaces in $R^3$ are preserved by the Lawson correspondence).

So locally minimal surfaces in $R^3$ are indeed equivalent to CMC 1 surfaces in $H^3$, and share many analogous properties. But interesting and essential differences arise when considering the two types of surfaces globally. Both the Weierstrass and Bryant representations require an integration to produce the actual immersions into $R^3$ and $H^3$, which locally can always be done, but when we attempt to do it globally on a non-simply-connected region, there are period problems, i.e. upon integration, we might find that the immersion is well defined only on the region's universal cover. Solvability of these period problems (adjusting free parameters to get the immersion well defined on the region itself) is a global question, and the answer can be different in the two cases. As a general (but inexact) rule of thumb,
solvability is more likely in the $H^3$ case, leading to a wider variety of surfaces in the $H^3$ case. For example, a genus 1 surface with finite total curvature and two embedded ends cannot exist as a minimal surface in $R^3$, but it does exist as a CMC 1 surface in $H^3$. And a genus 0 surface with finite total curvature and two embedded ends exists as a minimal surface in $R^3$ only if it is a surface of revolution, but it may exist as a CMC 1 surface in $H^3$ without being a surface of revolution.

Another difference between these two types of surfaces is that there are no relations regarding embeddedness of the surfaces in the two different cases. For example, the minimal Costa surface is embedded in $R^3$, but a corresponding CMC 1 surface in $H^3$ is not embedded. And the minimal trinoid is not embedded in $R^3$, but a corresponding CMC 1 surface in $H^3$ is embedded. But again, this is a
Figure 3. Two different CMC 1 trinoid cousin duals (proven to exist in [UY3]) and a CMC 1 genus 1 trinoid cousin dual in $H^3$ (for which existence follows from Theorem 5). The graphics for the genus 1 surface here were made by Katsunori Sato of Tokyo Institute of Technology. Although these surfaces are proven to exist, and numerical experiments show that some of them are embedded (as two of the pictures here are), none have yet been proven to be embedded.

global question. (Umehara and Yamada, Rosenberg, and others have been studying embeddedness of CMC 1 surfaces in $H^3$ [CHR1], [ET1], [LR], [UY1].)

The goals of this article are three-fold:

1. to demonstrate some essential global differences between minimal surfaces in $R^3$ and CMC 1 surfaces in $H^3$, for which graphics are helpful.
2. to explain how period problems can be solved, without hiding the main ideas behind too many technical details. Showing solvability invariably requires detailed arguments, but here we restrict ourselves to giving plausibility arguments, to make the main ideas more transparent.
3. to show computer graphics of CMC 1 surfaces in $H^3$. Such graphics are largely absent from an already significant amount of literature on the subject, but are useful for understanding the nature of these surfaces (in the same way that computer graphics have been useful in the study of minimal surfaces in $R^3$). Consider this example: the CMC 1 Enneper cousin in $H^3$ has an end at which the hyperbolic Gauss map (to be defined later) has an
essential singularity, so it is difficult to imagine what this surface looks like, and no picture is in the literature. Here we show its picture, and pictures of other fundamental examples as well.

In Section 1, we give an elementary description of CMC surfaces and the Lawson correspondence. In Section 2, we describe hyperbolic space. In Sections 3 and 4, we describe the Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ and the Bryant representation for CMC 1 surfaces in $\mathbb{H}^3$, respectively. In Section 5, we describe the duals of CMC 1 surfaces in $\mathbb{H}^3$, and then we describe a way to solve the period problems on dual surfaces in Sections 5 and 6. In the final section, we include an overview of known results about CMC 1 surfaces in $\mathbb{H}^3$.

1. Defining CMC surfaces, and Lawson’s correspondence

Let $\Phi : \Sigma \to \mathbb{R}^3$ be an immersion of a 2 dimensional surface $\Sigma$ into Euclidean 3-space $\mathbb{R}^3$. The standard metric $dx_1^2 + dx_2^2 + dx_3^2$ (also written $(\cdot, \cdot)$) on $\mathbb{R}^3$ induces a metric (the first fundamental form) $ds^2 : T_p(\Sigma) \times T_p(\Sigma) \to \mathbb{R}$ on $\Sigma$, which is a bilinear map, where $T_p(\Sigma)$ is the tangent space at $p \in \Sigma$. If $(u, v)$ is a local coordinate of $\Sigma$, and if the basis $\{ \Phi_u = (\frac{\partial \Phi}{\partial u})_p, \Phi_v = (\frac{\partial \Phi}{\partial v})_p \}$ is chosen for $T_p(\Sigma)$ (we can identify $T_p(\Sigma)$ with the plane in $\mathbb{R}^3$ tangent to $\Phi(\Sigma)$ at $\Phi(p)$), then the metric $ds^2$ is represented by the matrix

$$I_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

that is, $ds^2(a\Phi_u + b\Phi_v, c\Phi_u + d\Phi_v) = (a \, b) I_1(c \, d)^t$ for any $a, b, c, d \in \mathbb{R}$. Likewise, the second fundamental form of $\Phi$, a symmetric bilinear map from $T_p(\Sigma) \times T_p(\Sigma)$ to the normal space $\{ r \cdot \vec{N}_p \mid r \in \mathbb{R} \} \approx \mathbb{R}$, where $\vec{N}$ is a local unit normal vector field to $\Phi(\Sigma)$ near $\Phi(p)$, is similarly represented by a matrix

$$I_2 = \begin{pmatrix} l & m & n \\ m & l & -n \\ n & -m & l \end{pmatrix}.$$

The linear map $(a \, b)^t \to I_1^{-1} I_2(a \, b)^t$ represents the shape operator taking $\nu = a\Phi_u + b\Phi_v \in T_p(\Sigma)$ to $-D_p \nu \vec{N} \in T_p(\Sigma)$, where $D$ is the directional derivative of $\mathbb{R}^3$. The eigenvalues $k_1, k_2$ and corresponding eigenvectors of the shape operator are the principal curvatures and principal curvature directions of the surface $\Phi(\Sigma)$ at $\Phi(p)$. Although a bit imprecise, intuitively the principal curvatures tell us the maximum and minimum amounts of bending of the surface toward the normal $\vec{N}$ at $\Phi(p)$, and the principal curvature directions tell us the directions in $T_p(\Sigma)$ of those maximal and minimal bendings.

Definition 1. The determinant and the trace of the shape operator are the Gaussian curvature $K$ and the mean curvature $H$, respectively. Thus

$$K = k_1 k_2 = \frac{ln - m^2}{EG - F^2}, \quad \text{and} \quad H = \frac{k_1 + k_2}{2} = \frac{Gl - 2Fm + En}{2(EG - F^2)}.$$ 

A surface is CMC if $H$ is constant, and is minimal if $H$ is identically zero.

It is a fact that one can locally always choose $(u, v)$ so that $z = u + iv$ becomes a conformal coordinate on $\Sigma$ with respect to $ds^2$ (i.e. $F \equiv 0$, $E \equiv G$). Hence from now on we assume the coordinates are chosen this way and furthermore that $\Sigma$ is a Riemann surface.

The mean curvature $H$ at $\Phi(p)$ can be equivalently defined as the average of the normal curvatures $-\langle \nu, D_p \vec{N} \rangle$ in all tangent directions $\nu \in T_p(\Sigma)$. (Intuitively, the normal curvature measures the rate at which the surface bends toward $\vec{N}$, in the
Thus a minimal surface has average normal curvature zero at every point, and this suggests a physical interpretation:

[HM2]: “Loosely speaking, one imagines the surface as made up of very many rubber bands, stretched out in all directions; on a minimal surface the forces due to the rubber bands balance out, and the surface does not need to move to reduce tension.”

To say this more rigorously, suppose $U$ is a compact domain in $\Sigma$, and define a smooth boundary-fixing variation of the immersion $\Phi(U)$ to be a $C^\infty$ map $\Phi_t : (-1, 1) \times U \to \mathbb{R}^3$ with three properties:

1. $\Phi_t(\cdot) : U \to \mathbb{R}^3$ is an immersion for all $t \in (-1, 1)$,
2. $\Phi_0 = \Phi$ on $U$,
3. $\Phi_t|_{\partial U} = \Phi|_{\partial U}$ for all $t \in (-1, 1)$.

Note that $\text{Area}(\Phi_t(U)) = \int_U dA_t$, where $dA_t$ is the volume element (the area 2 form) of the metric induced by the immersion $\Phi_t$. It turns out that the first variation formula for smooth boundary-fixing variations is then

$$\frac{d}{dt} \left. \text{Area}(\Phi_t(U)) \right|_{t=0} = -\int_{\Sigma} \left\langle H\vec{N}, (\Phi_t)_{\ast} \frac{\partial}{\partial t} \right|_{t=0} \right\rangle dA_0 .$$

In particular, minimal surfaces (with $H \equiv 0$) are area critical for all compact domains $U$ — and we could have defined them this way. (Actually, when $U$ is small enough, not only is $\Phi(U)$ critical for area, it is also the unique least-area surface with boundary $\partial \Phi(U)$, hence a "minimal" surface.)

Similarly, a nonminimal CMC surface could be defined as an immersion $\Phi$ such that for every compact domain $U$, $\Phi(U)$ is critical for area amongst all smooth boundary-fixing variations that keep the volume on one side of the surface unchanged. (The derivative of volume is $\int_U \langle \vec{N}, (\Phi_t)_{\ast} \frac{\partial}{\partial t} |_{t=0} \rangle dA_t$, so if the volume is unchanging and so $\int_U \langle \vec{N}, (\Phi_t)_{\ast} \frac{\partial}{\partial t} |_{t=0} \rangle dA_t = 0$, and if $H$ is constant, then $\frac{d}{dt} \text{Area}(\Phi_t(U)) |_{t=0} = 0$ [BCE].)

This is why minimal and CMC surfaces model physical soap films, which always move to minimize area. Minimal surfaces model soap films not enclosing bounded pockets of air, as such films are area minimizing for all boundary-fixing variations. Nonminimal CMC surfaces model soap films enclosing bounded pockets of air, as such films are area minimizing only for variations that keep the air pockets’ volumes fixed.

To define CMC surfaces in $\mathbb{H}^3$, we can proceed in the same way as the Euclidean case. We only need to replace $\Phi_u$ and $\Phi_v$ by linear differentials $\left( \frac{\partial}{\partial u} \right)_p$ and $\left( \frac{\partial}{\partial v} \right)_p$, and $\vec{N}_u$ and $\vec{N}_v$ by $\nabla_{\frac{\partial}{\partial u}} \vec{N}$ and $\nabla_{\frac{\partial}{\partial v}} \vec{N}$, where $\nabla$ is the Riemannian connection for $\mathbb{H}^3$. The shape operator becomes $\nu \mapsto -\nabla_{\nu} \vec{N}$. All the variational properties in the Euclidean case also hold when the ambient space is $\mathbb{H}^3$. And we can choose conformal coordinates $z = u + iv$ in the $\mathbb{H}^3$ case as well, so we again assume $\Sigma$ is a Riemann surface.

Regarding the Lawson correspondence between minimal surfaces in $\mathbb{R}^3$ and CMC 1 surfaces in $\mathbb{H}^3$, the essential ingredient is the fundamental theorem of surface theory, telling us that on a simply connected region $U$ of $\Sigma$ there exists an immersion with fundamental forms $I_1$ and $I_2$ if and only if $I_1$ and $I_2$ satisfy the Gauss and Codazzi equations. (This is true regardless of whether the ambient space is $\mathbb{R}^3$ or $\mathbb{H}^3$, but the Gauss and Codazzi equations are not the same in the two cases.) So if $\Phi : U \to \mathbb{R}^3$ is a conformal minimal immersion, then its fundamental forms $I_1$ and $I_2$ satisfy the Gauss and Codazzi equations for surfaces in $\mathbb{R}^3$, and one can then check that $\tilde{I}_1 = I_1$ and $\tilde{I}_2 = I_1 + I_2$ satisfy the Gauss and Codazzi equations for surfaces in $\mathbb{H}^3$, so there exists an immersion $\tilde{\Phi} : U \to \mathbb{H}^3$ with fundamental forms...
Figure 4. CMC 1 Costa cousin dual in $H^3$, proven to exist by Costa and Sousa Neto when they established that the minimal Costa surface in $R^3$ is symmetric and nondegenerate [CN]. Rather than showing graphics of this surface, we show two vertical cross sections by which the surface is reflectionally symmetric (including the "circles" at infinity), and a schematic of the central portion of the surface.

Figure 5. 5 ended CMC 1 surface in $H^3$, found by Umehara and Yamada in [UY1]. This surface has the same symmetries and end behavior as the minimal surface on the left hand side of Figure 11. Here we show only one of six congruent disks that form the surface. The full surface is constructed by reflections across planes containing boundary curves of the disk shown here.

\[ \tilde{I}_1 \text{ and } \tilde{I}_2. \text{ Since } \text{tr}(I_1^{-1}I_2) = 0, \text{ we have } \text{tr}(\tilde{I}_1^{-1}\tilde{I}_2) = 1, \text{ so } \tilde{\Phi} \text{ is CMC 1. And since } I_1 = \tilde{I}_1, \tilde{\Phi} \text{ is also conformal, and } \Phi(U) \text{ and } \tilde{\Phi}(U) \text{ are isometric. This is Lawson’s correspondence.} \]

We digress here to explain the Lawson correspondence in more detail: Let $M^3(\bar{K})$ be the 3 dimensional space form with constant section curvature $\bar{K}$ (e.g. $M^3(0) = R^3$, $M^3(-1) = H^3$, $M^3(1) = S^3$). For an immersion $\Phi : U \to M^3(\bar{K})$ with induced metric $\langle \cdot, \cdot \rangle$ and connection $\nabla$ and Gaussian curvature $K$ and shape operator $A$, the Gauss and Codazzi
Figure 6. CMC 1 genus 0 surface in $H^3$ with two embedded ends, which is not a surface of revolution. By Schoen’s result, such a surface cannot exist as a minimal surface in $R^3$, so this surface is not the cousin nor the dual cousin of any minimal surface with finite total curvature in $R^3$. Its existence is proven in [UY1]. One of two congruent pieces of the surface is shown here. This example, and the examples in Figures 7 and 13, show that the converse of the [RUY1] result does not hold.

The equations are satisfied:

$$K - \bar{K} = \det(A), \quad \langle A([[X, Y]]), Z\rangle = \langle \nabla_X A(Y), Z \rangle - \langle \nabla_Y A(X), Z \rangle$$

for all smooth vector fields $X$, $Y$, and $Z$ in the tangent space of $U$.

Assume $\Phi$ is CMC $H$, so $H = \text{tr}(A)$ is constant. Now choose any $c \in R$ and define

$$\tilde{A} = A + c \cdot (id.), \quad \tilde{K} = \bar{K} - 2c\text{tr}(A) - c^2.$$

Then the Codazzi equation clearly still holds when $A$ is replaced by $\tilde{A}$:

$$\langle \tilde{A}([[X, Y]]), Z\rangle = \langle A([[X, Y]]), Z\rangle + c\langle [X, Y], Z\rangle = \langle \nabla_X A(Y), Z \rangle - \langle \nabla_Y A(X), Z \rangle - c\langle \nabla_X Y, Z \rangle + c\langle \nabla_Y X, Z \rangle.$$

The Gauss equation in $M^3(\bar{K})$ also holds with $\tilde{A}$ (note that $K$ is intrinsic and does not change):

$$K - \tilde{K} = K - (\bar{K} - 2c\text{tr}(A) - c^2) = \det(A) + 2c\text{tr}(A) + c^2$$

$$= \det(A + c(id.)) = \det(\tilde{A}).$$

Therefore there exists an immersion $\tilde{\Phi} : U \to M^3(\tilde{K})$ with metric $\langle \cdot, \cdot \rangle$ and shape operator $\tilde{A}$, and $\tilde{\Phi}(U)$ is isometric to $\Phi(U)$. As the mean curvature of $\tilde{\Phi}(U)$ is

$$\tilde{H} = \text{tr}(\tilde{A}) = \text{tr}(A) + c = H + c,$$

this demonstrates the Lawson correspondence between a CMC $H$ surface in $M^3(\bar{K})$ and a CMC $(H + c)$ surface in $M^3(\bar{K} - 2cH - c^2)$. In particular, when $H = \bar{K} = 0$ and $c = 1$, we have the correspondence between minimal surfaces in $R^3$ and CMC 1 surfaces in $H^3$. 
MINIMAL SURF ACES IN $\mathbb{R}^3$ VS. CMC 1 SURF ACES IN $H^3$

Figure 7. CMC 1 genus 0 surface in $H^3$ with two nonembedded ends (each of winding order two), proven to exist in [UY1]. This surface again (like the surface in Figure 6) has no corresponding minimal surface in $\mathbb{R}^3$. Three partial views of the surface are also shown here.

2. HYPERBOLIC SPACE

Hyperbolic 3-space $H^3$ is the unique simply-connected 3 dimensional complete Riemannian manifold with constant sectional curvature $-1$, but it can be described by a variety of models, each with their own advantages. Here we describe the two models we will need: the Poincare ball model, convenient for showing computer graphics; and the Hermitian matrix model, the one Bryant chose for his representation.

The Poincare model is the 3 dimensional Euclidean unit ball

$$B^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 < 1\}$$

with the metric

$$ds^2 = \frac{2}{1 - x_1^2 - x_2^2 - x_3^2} (dx_1^2 + dx_2^2 + dx_3^2).$$

This metric is conformal to the Euclidean metric and so angles are the same as Euclidean angles, that is, angles are just as our "Euclidean" eyes tell us they are. However, distances are clearly not Euclidean. With this metric the unit ball is complete, simply-connected, and has constant sectional curvature $-1$, so it is $H^3$. 
The geodesics in the Poincaré model are segments of Euclidean lines and circles that intersect the boundary "sphere at infinity" $\partial B^3$ at right angles. The CMC hyperbolic planes with $H = 0$ (resp. hyperspheres with $|H| < 1$, spheres with $|H| > 1$, horospheres with $|H| = 1$) are the intersections of $B^3$ with Euclidean spheres and planes in $R^3$ that meet $\partial B^3$ orthogonally (resp. have nonempty nontangential intersection with $\partial B^3$, are in the interior of $B^3$, are in $B^3$ and are tangent to $\partial B^3$ at one point).

Now we describe the Hermitian model, which we will see is convenient for describing the isometric motions of $H^3$. We first recall that the 6 dimensional Lie group $SL(2, C)$ is the set of all $2 \times 2$ matrices with complex entries and determinant 1, with matrix multiplication as the group operation. We also mention, as we use it later, that $SU(2)$ is the 3 dimensional subgroup of matrices $\mu \in SL(2, C)$ such that $\mu \mu^*$ is the identity, where $\mu^* = \bar{\mu}^t$, or equivalently, $\mu = \begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix}$, for some $p, q \in C$ with $|p|^2 + |q|^2 = 1$.

The set \{AA$^*$ | $A \in SL(2, C)$\} will form the Hermitian model for $H^3$, which consists of the Hermitian symmetric matrices with determinant 1, that is, matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ \bar{a}_{12} & a_{22} \end{pmatrix},$$

where $a_{12} \in C$ and $a_{11}, a_{22} \in R$ and $a_{11}a_{22} - a_{12}\bar{a}_{12} = 1$. Such matrices can be bijectively mapped to points

$$\left(\frac{a_{12} + \bar{a}_{12}}{2 + a_{11} + a_{22}}, \frac{i(a_{12} - \bar{a}_{12})}{2 + a_{11} + a_{22}}, \frac{a_{11} - a_{22}}{2 + a_{11} + a_{22}}\right)$$

in the Poincaré model, so if this set of matrices is given the metric so that this map is an isometry, then it becomes $H^3$. We can then use the group $SL(2, C)$ to describe the isometries of $H^3$. A matrix $h \in SL(2, C)$ acts isometrically on $H^3$ via, for each matrix $x$ in the Hermitian model,

$$x \rightarrow h x h^*.$$

3. Weierstrass Representation

In the Weierstrass representation for complete oriented minimal surfaces of finite total curvature (i.e. $\int_{\Sigma} \kappa dA < +\infty$) in $R^3$, we incorporate that these surfaces are conformally equivalent to compact Riemann surfaces with finitely many points removed [O]:

**Theorem 2.** (Weierstrass representation) Let $\Sigma$ be a compact Riemann surface, and $\{p_j\}_{j=1}^k \subset \Sigma$ be a finite number of points (representing the ends of the minimal surface defined in this theorem). Fix a point $z_0 \in \Sigma \setminus \{p_j\}$. Let $g$ be a meromorphic function on $\Sigma$, and $f$ a holomorphic function on $\Sigma \setminus \{p_j\}$. Assume that, for any point in $\Sigma \setminus \{p_j\}$, $f$ has a zero of order $2k$ if and only if $g$ has a pole of order $k$, and that $f$ has no other zeroes on $\Sigma \setminus \{p_j\}$. Then, in terms of local holomorphic coordinates $z$ on $\Sigma \setminus \{p_j\}$,

$$\Phi(z) = Re \int_{z_0}^z \begin{pmatrix} (1 - g^2)f d\zeta \\ i(1 + g^2)f d\zeta \\ 2gf d\zeta \end{pmatrix}$$
Figure 8. Minimal catenoid and helicoid in $\mathbb{R}^3$. (Note that the ends have been cut away, and will also be cut away in all subsequent figures.) Schoen [S] has shown that any complete connected finite total curvature minimal immersion (not necessarily embedded) with two embedded ends in $\mathbb{R}^3$ must be a catenoid.

Figure 9. Minimal Enneper surface and Costa surface in $\mathbb{R}^3$, the Costa surface was proven to exist by Costa and proven to be embedded in [HM1].

is a conformal minimal immersion of the universal cover $\widetilde{\Sigma \setminus \{p_j\}}$ of $\Sigma \setminus \{p_j\}$ into $\mathbb{R}^3$. If $\Phi$ is well-defined on $\Sigma \setminus \{p_j\}$, then it has finite total curvature. Furthermore, any complete minimal surface with finite total curvature in $\mathbb{R}^3$ can be represented this way.

The function $g$ has a geometric interpretation: it is the composition of the Gauss map with stereographic projection to the complex plane $\mathbb{C}$. And the fundamental
Figure 10. Minimal trinoid (proven to exist in [JM]) and genus 1 trinoid (proven to exist in [BR]) in $\mathbb{R}^3$. These graphics were made using [MESH] software.

Figure 11. Minimal 5 ended surface in $\mathbb{R}^3$ with same symmetry group as the trinoid (proven to exist in [Ka], [R]), and a minimal genus 1 Enneper surface (proven to exist in [CG]) in $\mathbb{R}^3$. The right hand side figure was made by Edward C. Thayer of ZymoGenetics Inc. (Seattle, U.S.A.), and both figures were made using [MESH] software.

Forms and Gaussian curvature can be described using $g$ and $f$:

$$I_1 = \begin{pmatrix} (1 + g\bar{g})^2 f\bar{f} & 0 \\ 0 & (1 + g\bar{g})^2 f\bar{f} \end{pmatrix}, \quad I_2 = -2 \begin{pmatrix} \text{Re}(\frac{dg}{df} f) & \text{Re}(\frac{dg}{dx} f) \\ \text{Re}(\frac{dg}{dx} f) & -\text{Re}(\frac{df}{dx} f) \end{pmatrix},$$

$$K = -4 \left( \frac{|dg|}{|f|(1 + |g|^2)^2} \right)^2.$$
Theorem 3. (Bryant representation) Let the Weierstrass representation. Choose the holomorphic immersion where \( F(z) \) is the image of the composition of a first map, from any CMC 1 is a conformal CMC immersed perpendicular geodesic ray starting from \( \Phi(z) \) in the sphere at infinity (in the Poincare model) at the opposite end of the origin. This geometrical property is strikingly similar to that of the map, stereographic projection of the sphere at infinity to the complex plane. Weierstrass representation, two roles are served by the same function. Following the terminology of Umehara and Yamada, the \( \Phi \) in (3) is well defined on \( \Sigma \setminus \{ p_j \} \).

4. BRYANT REPRESENTATION

Theorem 3. (Bryant representation) Let \( \Sigma, \Sigma \setminus \{ p_j \}, z_0, f, \) and \( g \) be as in the Weierstrass representation. Choose the holomorphic immersion \( F : \Sigma \setminus \{ p_j \} \to SL(2, \mathbb{C}) \) so that \( F(z_0) \) is the identity matrix and \( F \) satisfies

\[
dF = F \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} dz,
\]

then \( \Phi : \Sigma \setminus \{ p_j \} \to \mathbb{H}^3 \) defined by

\[
(3) \quad \Phi = F \cdot F^*,
\]

is a conformal CMC 1 immersion in the Hermitian model of \( \mathbb{H}^3 \). Furthermore, any CMC 1 surface with finite total curvature in \( \mathbb{H}^3 \) can be represented this way.

Following the terminology of Umehara and Yamada, \( g \) is the secondary Gauss map of \( \Phi \). The fundamental forms and Gaussian curvature are again described using \( g \) and \( f \), and \( I_1 \) and \( K \) are the same as in equation (2). The second fundamental form is \( I_2 + I_1 \) (where \( I_1 \) and \( I_2 \) are as in equation (2)), as we saw from the Lawson correspondence.

A significant difference from the Weierstrass representation is that, even when the \( \Phi \) in (3) is well defined on \( \Sigma \setminus \{ p_j \} \), \( g \) and \( F \) might not be. We will soon see that the catenoid cousin is an example of this.

Suppose \( \Phi \) is indeed well defined on \( \Sigma \setminus \{ p_j \} \). The hyperbolic Gauss map \( G \) is the meromorphic function

\[
G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}},
\]

where \( F = (F_{ij})_{i,j=1,2} \). \( G \) is single-valued by its geometric interpretation [B]: \( G \) is the image of the composition of a first map, from \( z \in \Sigma \setminus \{ p_j \} \) to the pioint in the sphere at infinity (in the Poincare model) at the opposite end of the oriented perpendicular geodesic ray starting from \( \Phi(z) \) on the surface, and a second map, stereographic projection of the sphere at infinity to the complex plane \( \mathbb{C} \). This geometrical property is strikingly similar to that of the \( g \) in the Weierstrass representation.

So the Bryant representation has two "Gauss" maps \( g \) and \( G \), whereas the Weierstrass representation had only one \( g \). One way to think about this is that in the Weierstrass representation, two roles are served by the same function \( g \): the first is to describe the fundamental forms, and the second is to describe stereographic projection of the unit normal vector field. In the Bryant representation, we need two different functions to serve these two roles: \( g \) for the first, and \( G \) for the second.

Here are three simple examples of data for the Bryant representation:

1. horosphere: \( \Sigma \setminus \{ p_j \} = \mathbb{C}, g = 0, f = 1 \), like the data for a plane in \( \mathbb{R}^3 \).
(2) Enneper cousins: $\Sigma \setminus \{p_j\} = C, g = z, f = k, k \in \mathbb{R}$, like the data for Enneper’s surface in $\mathbb{R}^3$. $k$ is a nontrivial parameter, and the Enneper cousins for different $k$ do not differ by only a dilation, as would happen in the Weierstrass representation.

(3) catenoid cousins: $\Sigma \setminus \{p_j\} = C \setminus \{0\}, g = z^\mu, f = \frac{1-\mu^2}{4\mu}z^{-\mu-1}, \mu \in \mathbb{R}^3 \setminus \{1\}$. $\Phi$ is well defined on $C \setminus \{0\}$, even though $g$ (and $F$) is not.

The term "cousin" is often understood to mean that the CMC 1 surface is locally isometric (via Lawson correspondence) to the corresponding minimal surface in $\mathbb{R}^3$. The horosphere (resp. Enneper’s cousin with $k = 1$) is isometric to the plane (resp. Enneper’s surface) in $\mathbb{R}^3$. So we could call the horosphere a "plane cousin", but it already has a name. The catenoid cousin is locally isometric to a minimal catenoid, if one makes the coordinate transformation $w = z^\mu$ and an appropriate homothety of the minimal catenoid.

However, this interpretation of the word "cousin" is not universal. For example, in the paper [RYU1], the term is used to mean what we will call "dual cousin" in the next section, and sometimes the term is also used to describe the Lawson correspondence combined with a conjugation by angle $\pi/2$.

5. Dual CMC 1 surfaces in $\mathbb{H}^3$

An interesting property of CMC 1 surfaces in $\mathbb{H}^3$ is that if we change $F$ to $F^{-1}$ in equation (3) we get the CMC 1 "dual" surface $\Phi^\# = F^{-1}(F^{-1})^*$ in $\mathbb{H}^3$. Since $g = -\frac{dF}{\alpha F^2}$ and $G = \frac{dF}{\alpha F^2}$, changing $\Phi$ to $\Phi^\#$, that is, changing

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \quad \rightarrow \quad F^{-1} = \begin{pmatrix} F_{22} & -F_{12} \\ -F_{21} & F_{11} \end{pmatrix}$$

switches the secondary and hyperbolic Gauss maps. (Note that one of $\Phi$ and $\Phi^\#$ being well defined on $\Sigma \setminus \{p_j\}$ does not imply the other is also.)

Dual surfaces have useful properties, initially exploited by Umehara and Yamada [UY5]. One useful property is that period problems are easier to study for cousin dual surfaces than for the cousin surfaces, as we now explain:

To produce CMC 1 surfaces in $\mathbb{H}^3$, we could start with a well defined minimal surface in $\mathbb{R}^3$ with data $\Sigma \setminus \{p_j\}$ and $g$ and $f$, and insert this data into the Bryant representation to produce the cousin CMC 1 surface in $\mathbb{H}^3$, and try to solve the period problems. About a nontrivial loop $\alpha$ in $\Sigma \setminus \{p_j\}$, $F$ changes to $B_\alpha F$, for some constant matrix $B_\alpha \in SL(2, \mathbb{C})$, so for the cousin surface to well defined on $\Sigma \setminus \{p_j\}$, we need

$$FF^* = B_\alpha F(B_\alpha F)^* = B_\alpha FF^*B_\alpha^*$$

for all loops $\alpha$. But this can be difficult to establish, given that $F$ depends on $z \in \Sigma \setminus \{p_j\}$, and given that $B_\alpha$ and $B_\alpha^*$ are on the "outer" part of the product on the right hand side. And even when we can establish this, we produce only those special CMC 1 surfaces where the secondary Gauss map is well defined on $\Sigma \setminus \{p_j\}$.

On the other hand, let us consider the period problems of the dual cousin surface. Solving the period problems requires that

$$F^{-1}(F^{-1})^* = (B_\alpha F)^{-1}((B_\alpha F)^{-1})^* = F^{-1}B_\alpha^{-1}(B_\alpha^{-1})^*(F^{-1})^*$$

for all loops $\alpha$. In this case $B_\alpha$ and $B_\alpha^*$ are conveniently located in the "inner" part of the product on the right hand side, so we have a simple condition for solvability:

**Lemma 4.** (SU(2) condition) The dual surface $\Phi^\#$ is well-defined on $\Sigma \setminus \{p_j\}$ if $B_\alpha \in SU(2)$ for every closed loop $\alpha \in \Sigma \setminus \{p_j\}$. 
Dual cousins have another advantage. As \( g \) has been chosen from the Weierstrass data of a minimal surface well defined on \( \Sigma \setminus \{ p_j \} \), \( g \) is also well defined on \( \Sigma \setminus \{ p_j \} \). So once the \( SU(2) \) condition is solved, the dual cousin’s hyperbolic Gauss map, which is \( g \), is well defined on \( \Sigma \setminus \{ p_j \} \), and it should be. But solving the \( SU(2) \) condition does not force the secondary Gauss map, which is now \( G \), to be well defined on \( \Sigma \setminus \{ p_j \} \), so the dual cousin’s secondary Gauss map might not be well defined on \( \Sigma \setminus \{ p_j \} \), allowing for many more possibilities.

We now give three examples of data for dual cousins, which are the same as the data for the corresponding minimal surfaces (unlike the data for the original cousins in non-simply-connected cases):

(1) Enneper cousin duals: \( \Sigma \setminus \{ p_j \} = C, g = z, f = k, k \in \mathbb{R} \), like the data for Enneper’s surface. The Enneper cousin duals, having infinite total curvature, are truly different from the Enneper cousins, having finite total curvature.

(2) catenoid cousin “duals”: \( \Sigma \setminus \{ p_j \} = C \setminus \{ 0 \}, g = \frac{4}{z}, f = k, k \in \mathbb{R} \), like the data for a minimal catenoid. The catenoid cousin “duals” are the same as the catenoid cousins of the previous section (hence “duals” is in quotes). Here we are describing the data in terms of the hyperbolic Gauss map, rather than the secondary Gauss map, as we did in the previous section. So we see that the secondary Gauss map is not well defined.

(3) trinoid cousin duals: \( \Sigma \setminus \{ p_j \} = (C \cup \{ \infty \}) \setminus \{ 1, e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}} \}, g = z^2, f = \frac{k}{(z^3 - 1)^2}, k \in \mathbb{R} \), like the data for a trinoid. The secondary Gauss map of the trinoid cousin dual is also not well defined, and is in fact unknown.

6. MAKING DUAL COUSINS

As many of the examples shown in the figures here have been proven to exist by a method in [RUY1], we briefly outline the method here. For each minimal surface with finite total curvature satisfying certain conditions, the method implies existence of a one parameter family of corresponding CMC 1 dual cousins.

We start with a minimal surface \( \Phi : \Sigma \setminus \{ p_j \} \to \mathbb{R}^3 \) of finite total curvature with Weierstrass data \( f \) and \( g \). We require the immersion to be symmetric in the following sense, a condition that generically eliminates virtually all minimal surfaces, but eliminates none of the better known surfaces, which all have symmetries:

**Symmetry condition:** There is a disk \( D \subset \Sigma \setminus \{ p_j \} \) so that \( \Phi(D) \) is bounded by non-straight planar geodesics.

If \( \Phi \) is symmetric with respect to a disk \( D \), then \( \Phi(D) \) generates the full surface by reflections across planes containing the boundary planar geodesics of \( \partial \Phi(D) \) (by the Schwartz reflection principle [O]). Since the surface has finite total curvature, it is not periodic, so if any two of these boundary planar geodesics lie in parallel planes, they must lie in the same plane. And in fact, it is shown in [RUY1] that the boundary \( \partial \Phi(D) \) is contained entirely in only either two intersecting planes \( P_1, P_2 \), or in three planes \( P_1, P_2, \) and \( P_3 \) in general position. Let the boundary planar geodesics of \( \Phi(D) \) contained in \( P_3 \) be called \( S_{j,1}, S_{j,2}, \ldots, S_{j,2s} \) \( j = 1, \ldots, s \), for \( s = 2 \) or 3. Two examples of symmetric surfaces are:

(1) the genus 1 Costa surface. This surface can be placed in \( \mathbb{R}^3 \) so that the central planar end is asymptotic to the plane \( \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0 \} \) and so that reflections through the planes \( P_1 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0 \} \) and \( P_2 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = 0 \} \) are symmetries of the surface. Then the piece of the surface lying in the region \( \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2 \geq 0 \} \) is one of four congruent disks comprising the surface. The Costa surface is
symmetric with respect to this disk. The boundary of this disk is contained in \( P_1 \cup P_2 \), with four planar geodesics \( S_{1,1}, S_{1,2}, S_{2,1}, \) and \( S_{2,2} \).

(2) the genus 1 trinoid. This surface can be separated into twelve congruent disks, any of which is bounded by four planar geodesics lying in three planes \( P_1, P_2, \) and \( P_3 \). So the list of boundary planar geodesics of this disk could be written \( S_{1,1}, S_{1,2}, S_{3,1}, \) and \( S_{3,2} \).

We now define non-degeneracy of the period problems. Let \( d \) be the number of \( S_{j,\ell} \) minus the number of planes (\( d = d_1 + d_2 + d_3 - 3 \) if \( s = 3 \), and \( d = d_1 + d_2 - 2 \) if \( s = 2 \)).

**Nondegeneracy condition:** There exists a continuous \( d \)-parameter family of minimal disks \( \Phi(D)_\lambda \), (\( \lambda = (\lambda_1, ..., \lambda_d), \lambda \approx \vec{0} \)) such that

1. \( \Phi(D)_{(0,0,...,0)} = \Phi(D) \).
2. \( \partial \Phi(D) \lambda = \bigcup_{j=1}^s \bigcup_{\ell=2}^{d_j} S_{j,\ell}(\lambda) \) and each \( S_{j,\ell}(\lambda) \) is a planar geodesic lying in a plane \( P_{j,\ell}(\lambda) \) parallel to \( P_j \).
3. letting \( \text{Per}_{j,\ell}(\lambda) \) (\( j = 1, ..., s, \ell = 2, ..., d_j \)) be the oriented distance between the plane \( P_{j,\ell}(\lambda) \) and \( P_{j,1}(\lambda) \), the map from \( \lambda \) in \( \mathbb{R}^d \) to \( (\text{Per}_{j,\ell}(\lambda)) \) in \( \mathbb{R}^d \) is an open map onto a neighborhood of \( \vec{0} \).

**Theorem 5.** [RUY1] If the minimal immersion \( \Phi \) is symmetric and nondegenerate, then there exists an associated one-parameter family of CMC 1 cousin duals in \( \mathbb{H}^3 \).

We outline the proof: We will not show that the period problems are solvable, but we will at least explain why we have enough free parameters available so that the period problems are not overdetermined.

First note that studying CMC 1 surfaces in the standard hyperbolic space is equivalent to studying CMC \( c \) surfaces in the hyperbolic space of constant sectional curvature \( -c^2 \). (One can see this by dilating \( \mathbb{R}^3 \) from the origin by a factor \( \frac{1}{c} \), taking the standard Poincare model to the model \( \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 < \frac{1}{c^2} \} \) with metric \( dx_i^2 = \frac{4 \sum dx_j^2}{(1-x_i^2-x_j^2)^2} \), which is hyperbolic space of constant sectional curvature \( -c^2 \), and taking CMC 1 surfaces to CMC \( c \) surfaces.) So we can instead study the problem with \( c \approx 0 \), where we are looking at only slight deformations of the original minimal surface, so that the nondegeneracy of the minimal surface’s periods can be used.
Figure 13. CMC 1 genus 1 catenoid cousin dual in $H^3$, proven to exist in [RS]. The graphics were made by Katsunori Sato of Tokyo Institute of Technology. No corresponding minimal surface can exist, by Schoen’s result (which is mentioned in the caption of Figure 8). Levitt and Rosenberg [LR] have proved that any complete properly embedded CMC 1 surface in $H^3$ with asymptotic boundary consisting of at most two points is a surface of revolution, which implies that this example, and the examples in Figures 6 and 7, cannot be embedded, and we see that they are not.

Now suppose $D \subset \Sigma \setminus \{p_j\}$ is a disk by which the minimal surface is symmetric. Consider a local coordinate $\{z \in \mathbb{C} \mid |z| < 1\}$ of $\Sigma$ in a neighborhood of a point in $\partial D$, which can be chosen so that $\{z \in \mathbb{R} \mid |z| < 1\}$ lies on $\partial D$ (hence $\Phi$ maps it to a portion of some $S_{j,\ell} \subset P_j \cap \partial \Phi(D)$), and so that $z \to \bar{z}$ represents reflection of the surface across the plane $P_j$. It turns out that the $F$ from the Bryant representation satisfies

$$F(z) = \rho_{j,\ell} F(z) \sigma_j,$$

where $(n_{1,j}, n_{2,j}, n_{3,j})$ is a unit normal vector to the plane $P_j$, and $\sigma_j, \rho_{j,\ell} \in SL(2, \mathbb{C})$, and where $p_{j,\ell} \in \mathbb{C}$ and $\gamma_{1,j,\ell}, \gamma_{2,j,\ell} \in \mathbb{R}$. One can now think of loops in $\Sigma \setminus \{p_j\}$ as compositions of reflections (a reflection for each $S_{j,\ell}$), and, with a bit of consideration regarding how $F$ would change under the compositions of such reflections that represent closed loops, we can see that the matrices $\sigma_j$ never create any period problems, and that the $SU(2)$ condition is satisfied if each $\rho_{j,\ell} \in SU(2)$, i.e. $\gamma_{1,j,\ell} = \gamma_{2,j,\ell}$ for all $S_{j,\ell}$. Hence the period problem is 1 dimensional for each $S_{j,\ell}$, and so in full it is a $d + s$ dimensional problem, as there are $d + s$ curves $S_{j,\ell}$.

But, as we saw in the definition of nondegeneracy of the original minimal surface, there we only had a $d$ real dimensional period problem, so the Weierstrass data of the original minimal surface only guarantees us $d$ free parameters $\lambda_1, \ldots, \lambda_d$. We need an extra $s$ free parameters to prevent the period problems from being overdetermined, and we get them by changing the initial condition $F(z_0)$ in the Bryant representation from the identity to other matrices in $SL(2, \mathbb{C})$. (Changing the initial condition nontrivially changes the resulting dual cousin in $H^3$, even though a change in the initial condition of the Weierstrass representation amounts
Figure 14. CMC 1 surface in $H^3$, proven to exist in [UY1]. This example is interesting because the hyperbolic Gauss map has an essential singularity at one of its two ends, like the end of the Enneper cousin. And the geometric behavior of the end here is strikingly similar to that of the Enneper cousin’s end. Here we show three pictures consecutively including more of this end.

to only a trivial translation in $R^3$, and a change in the initial condition of the Bryant representation for the original cousins amounts to only a trivial isometric motion of $H^3$.) The upshot is that we have $d + s$ free parameters available for each value of $c$, and, as shown in [RUY1], the $SU(2)$ condition is solvable for each $c \approx 0$, using the nondegeneracy condition of the minimal surface. So we have a 1 parameter family of dual cousins in $H^3$ with parameter $c \approx 0$.

Since many types of minimal surfaces in $R^3$ are symmetric and nondegenerate, this result implies the existence of many types of CMC 1 dual cousins, some of which are shown in the figures.

7. Other Aspects of CMC 1 Surfaces in $H^3$

Up to this point, we have been considering how to make examples of CMC 1 immersions in $H^3$, a primary interest of the author. But of course there are other approaches to the study of these surfaces, so we close with a brief list of other
research and viewpoints that the author is aware of. Some terms in this section we leave undefined, but the reader can refer to the original sources.

First we note that nice general introductions to the subject can be found in [UY6] and [Ro]. In [UY6], like here, comparison with minimal surfaces in $\mathbb{R}^3$ is used, but the presentation and emphasized points are quite different than in this paper. In [Ro], there is a detailed explanation of the Bryant representation and of some fundamental examples.

We begin with some of the results by Umehara and Yamada – who have been pursuing a fruitful long-term study of CMC 1 surfaces in $\mathbb{H}^3$ – as their results bear the closest relation with the previous sections.

- In [UY1], Umehara and Yamada use the Frobenius method to prove a number of results. Among the results in [UY1] are the following:
  1. reducible CMC 1 surfaces in $\mathbb{H}^3$ have a Perez-Ros type deformation [PR], i.e. changing $g, f$ to $\lambda g, f/\lambda$ for $\lambda \in \mathbb{R} \setminus \{0\}$ leaves $\Phi$ still well-defined on $\Sigma \setminus \{p_j\}$;
  2. for CMC 1 surfaces in $\mathbb{H}^3$, equality never holds in the Cohn-Vossen inequality
     \[
     \frac{1}{2\pi} \int_{\Sigma} |K| \ dA > -\chi(\Sigma \setminus \{p_j\});
     \]
  3. there is a simple algebraic way to check if a complete regular end is embedded, relying only on the orders and the leading coefficients of $f$ and $g$ at the end;
  4. all genus 0 immersions with 2 regular ends are classified;
  5. a general procedure for solving period problems at regular ends is given, which is applied to make CMC 1 genus 0 surfaces;
  6. any regular end is shown to be tangent to the sphere at infinity at a single limiting point.

- In [UY3], a correspondence between complete CMC 1 surfaces in $\mathbb{H}^3$ with finite total curvature and abstract surfaces with constant Gauss curvature 1 and isolated conical singularities (Met$_1$ surfaces) is established, and also CMC 1 surfaces with dihedral symmetry (similar to the Jorge-Meeks minimal surfaces in $\mathbb{R}^3$ [JM]) and with the same symmetry as the Platonic solids (similar to surfaces in [Xu] and [Ka]) are shown to exist.

- In [UY5] and [Y2], dual CMC 1 surfaces are explained (these are the sources for Section 5 of this paper). Using duality, these papers establish an Osserman type inequality

\[
\frac{1}{2\pi} \int_{\Sigma} |K^#| \ dA^# \geq -\chi(\Sigma \setminus \{p_j\}) + \text{number of ends of original surface}
\]

for the total curvature of the dual surfaces, analogous to the Osserman inequality for minimal surfaces in $\mathbb{R}^3$. ($K^#$ and $dA^#$ are the Gaussian curvature and area form of the dual surfaces.) A point of interest here is that the Osserman inequality does not hold for the total curvature $\int_{\Sigma} K dA$ of the original CMC 1 surfaces in $\mathbb{H}^3$ (the Cohn-Vossen inequality is the best possible result), and one really must go to the dual surfaces to establish the Osserman inequality.

- In [UY7], the correlation between CMC 1 surfaces in $\mathbb{H}^3$ and abstract Met$_1$ surfaces is used to classify Met$_1$ surfaces with 3 conical singularities. Also, genus 0 CMC 1 irreducible immersions with 3 embedded ends all of which are asymptotic to catenoid cousin ends ("trinoids") are classified. We remark that Collin and Rosenberg have recently been creating a more geometric method for classifying trinoids.
With the author, Umehara and Yamada have recently used residue conditions of the data in the dual surfaces’ Bryant representation to show nonexistence of certain types of genus 0 CMC 1 surfaces in $H^3$ [RUY2].

Umehara and Yamada and the author have been classifying CMC 1 surfaces with low total curvature, and surfaces whose dual surfaces have low total curvature [RUY3], [RUY4], [RUY5], [RUY6]. (The Osserman inequality gives one reason for the interest in surfaces whose dual surfaces have low total curvature.)

The flux of CMC surfaces in $H^3$ is defined and explored by Korevaar, Kusner, Meeks and Solomon in [KKMS], and in particular this gives a flux when $H \equiv 1$. The residue conditions in [RUY2] have similar properties to the flux in [KKMS] when $H \equiv 1$, making it believable that the residue conditions are equivalent to the flux, but this is still unproven.

An interesting result by Small [Sm], using algebraic techniques, shows that when $g$ and $G$ are known, the surface can be given explicitly in terms of $g$, $G$, $G'$, and $G''$. (Small’s result is stated in [UY6] in the notation of Umehara and Yamada.) McCune and Umehara [MU] have recently found an analogy for CMC 1 surfaces in $H^3$ of the UP iteration for making minimal surfaces in $R^3$ [M]. With it, they make new examples of CMC 1 immersions in $H^3$.

The results above are about immersions. Other research has, however, focused on properties that CMC 1 embeddings must have, and we now list some results about this. We begin with an important recent work [CHR1].

Recently Collin, Hauswirth, and Rosenberg have given an impressive proof that any properly embedded CMC 1 surface in $H^3$ of finite topology has finite total curvature, and all of its ends are regular [CHR1], [Ro]. (An end $p_j$ is regular when $G$ is meromorphic at $p_j$, or equivalently order$_{p_j}$ (Hopf differential $= g' f dz^2$) $\geq -2$ [B].) Then the result of Earp and Toubiana [ET1] (see below) implies that each end is asymptotic to a catenoid cousin end or a horosphere end. (Finite total curvature and regularity are precisely the needed conditions in [ET1]. Also, we note here that this asymptotic behavior is implicit in the computations in Section 5 of [UY1], as argued in the appendix of [LiRo].) Among the corollaries of the [CHR1] result are:

1. a properly embedded CMC 1 surface in $H^3$ of finite topology that is simply-connected (resp. annular) must be a horosphere (resp. catenoid cousin);
2. irregular CMC 1 ends cannot be embedded, confirming a conjecture in [UY5].

Also in [CHR1], it is shown that a properly embedded CMC 1 surface in $H^3$ of finite topology with an end asymptotic to a horosphere end must be an actual horosphere. (This shows that many types of CMC 1 immersions cannot be embedded, and contrasts with the fact that properly embedded nonplanar minimal surfaces of finite topology in $R^3$ can have planar ends.)

Given a CMC 1 embedding $\Phi$ into $H^3$ in the Poincaré model $B^3$, let $\partial_\infty \Phi$ denote the limit set of $\Phi$ in the sphere at infinity $\partial_\infty B^3$ of the Poincaré model. Using the maximum principle, Levitt and Rosenberg [LR] showed that if $\partial_\infty \Phi$ lies in a single great circle of $\partial_\infty B^3$, then the surface has a reflective symmetry with respect to the geodesic plane whose limit set at infinity is that great circle. In other words, in this case the surface inherits the symmetry of its boundary. (In [LR], the result is stated more generally for CMC $H$ hypersurfaces in $H^n$.)

Do Carmo and Lawson showed that a complete CMC hypersurface which is a proper embedding $\Phi$ into $H^n$ with exactly one point in $\partial_\infty \Phi$ must be a
horosphere. They assume only that $H$ is constant, and not necessarily that it is 1. They also give other results for the $H \neq 1$ case – for example, if the hypersurface $\Phi$ is compact, then $\Phi$ is a sphere (known by Alexandrov); and if $\partial_\infty \Phi$ is a hypersphere in $\partial_\infty H^n$ and $\Phi$ separates the two components of $\partial_\infty H^n \setminus \partial_\infty \Phi$, then $\Phi$ is a hypersphere.

- Rodriguez and Rosenberg [RR] showed that a properly embedded CMC 1 surface $\Phi$ in $H^3$ lying in the interior region bounded by a horosphere is itself a horosphere. Furthermore, if $\Phi$ lies in the exterior region bounded by a horosphere, with mean curvature vector pointing toward that horosphere, then $\Phi$ again is a horosphere.

- Earp and Toubiana [ET1] showed that an embedded CMC 1 end that is regular ($\mathcal{G}$ extends meromorphically to the end) and of finite total curvature must be asymptotic to either a catenoid cousin end or a horosphere end. They also include an explicit description of CMC 1 helicoids in $H^3$. (These helicoids are also described in [Ro].)

- Do Carmo, Gomes and Thorbergsson [CGT] considered complete properly embedded CMC hypersurfaces $\Phi$ in $H^n$. When $H \in [0, 1)$, they showed that $\partial_\infty \Phi$ has no isolated points. They also showed that for any $H$, if $\partial_\infty \Phi$ is $C^2$ regular at infinity, then $H > 1$ if and only if $\partial_\infty \Phi$ is empty. (Ends that are asymptotic to hyperbolic Delaunay surfaces, as in [KKMS], are not $C^2$ regular at infinity.) These results contrast with the case of $H \equiv 1$, where $\partial_\infty \Phi$ is never empty and can have isolated points. Another result is that if $H > 1$, $\partial_\infty \Phi$ does not contain any component of codimension 1 in $\partial_\infty H^n$.

- Karcher [K] has recently constructed several different types of new embedded CMC 1 surfaces in $H^3$ of infinite topology, using properties of conjugate minimal surfaces in $R^3$. These surfaces have the same symmetries as Platonic tessellations of $H^3$.

Another line of inquiry is the stability and Morse index of CMC 1 surfaces in $H^3$. We saw in Section 1 that area of a CMC immersion $\Phi : \Sigma \setminus \{p_j\} \rightarrow H^3$ is critical for any volume-preserving compactly supported variation $\Phi_t$ of $\Phi$. So to determine if $\Phi_t$ reduces area, we must use the second variation formula. By reparametrizing the surfaces of $\Phi_t$ if necessary, we may assume that

$$
(\Phi_t)_* \frac{\partial}{\partial t} \bigg|_{t=0} = u \vec{N}
$$

for some compactly supported smooth function $u$ on $\Sigma \setminus \{p_j\}$ and that $\int_\Sigma u dA = 0$, and then the second variation formula becomes

$$
\frac{d^2}{dt^2} \text{Area}(\Phi_t(U)) \bigg|_{t=0} = \int_\Sigma u \cdot (\triangle + 2K) u ~ dA .
$$

If this integral is nonnegative for all compactly supported $u$ with (resp. without) the condition $\int_\Sigma u dA = 0$, then the surface is called stable (resp. strongly stable). The index of the surface is the maximum dimension of a vector space of compactly supported smooth functions $u$ such that the second variation formula is negative and $\int_\Sigma u dA = 0$, for all nonzero functions $u$ in the space. (Hence the surface is stable if and only if the index is 0.)

- Silveira [Si] showed that for a complete noncompact CMC immersion in $H^3$ with $H \geq 1$, stability of the surface implies that $H = 1$ and the surface is a horosphere. He also showed that a compact CMC 1 disk with boundary and total curvature less than $2\pi$ must be strongly stable, analogous to the result of [BC] for minimal surfaces in $R^3$. 


Do Carmo and Silveira [CS] showed that a CMC 1 surface in $H^3$ has finite index if and only if it has finite total curvature, analogous to the same result for minimal surfaces in $\mathbb{R}^3$ by Fischer-Colbrie [FC] and Gulliver [G]. Lima and the author [LiRo] followed this with index estimates for specific examples, in particular the index of the catenoid cousins is computed.

There is also a concept of strong index for CMC 1 surfaces in $H^3$, which is the number of negative eigenvalues of the operator $\triangle + 2K$ in the second variation formula with an appropriate domain of functions (strong index is explained in [CS], [LiRo], and [BB]). Strong index is always at least as big as the index, but is at most one greater when the index is finite. So the strong index is a good estimate for the index, and has the advantage of being easier to compute, with its more analytic nature. So it is interesting to know when index and strong index agree, and in fact, it is shown by Barbosa, Berard, and Hauswirth [BB], [BH] that they always agree for complete CMC 1 surfaces in $H^3$.

We remark that Goes, Galvao and Nelli [GGN], and also Earp and Toubiana [ET2], have developed alternatives to the Bryant representation, used to find examples.

Finally we note that there are results on the omitted points of the image of the hyperbolic Gauss map $G$. Yu [Y1] has shown that for a complete CMC 1 surface which is not a horosphere, $G$ omits at most 4 points. And recently Collin, Hauswirth and Rosenberg [CHR2] have shown that a complete CMC 1 immersion with finite total curvature that is not a horosphere has $G$ missing at most 3 points. Also, in [CHR2] it is shown that a properly embedded CMC 1 surface of finite topology that is not a horosphere or catenoid cousin has surjective $G$.

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