Permutation Invariant Gaussian Matrix Models

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ABSTRACT

Permutation invariant Gaussian matrix models were recently developed for applications in computational linguistics. A 5-parameter family of models was solved. In this paper, we use a representation theoretic approach to solve the general 13-parameter Gaussian model, which can be viewed as a zero-dimensional quantum field theory. We express the two linear and eleven quadratic terms in the action in terms of representation theoretic parameters. These parameters are coefficients of simple quadratic expressions in terms of appropriate linear combinations of the matrix variables transforming in specific irreducible representations of the symmetric group $S_D$ where $D$ is the size of the matrices. They allow the identification of constraints which ensure a convergent Gaussian measure and well-defined expectation values for polynomial functions of the random matrix at all orders. A graph-theoretic interpretation is known to allow the enumeration of permutation invariants of matrices at linear, quadratic and higher orders. We express the expectation values of all the quadratic graph-basis invariants and a selection of cubic and quartic invariants in terms of the representation theoretic parameters of the model.
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1 Introduction

In the context of distributional semantics \([1, 2]\), the meaning of words is represented by vectors which are constructed from the co-occurrences of a word of interest with a set of context words. In tensorial compositional distributional semantics \([3, 4, 5, 6, 7]\), different types of words, depending on their grammatical role, are associated with vectors, matrices or higher rank tensors. In \([8, 9]\) we initiated a study of the statistics of these tensors in the framework of matrix/tensor models. We focused on matrices associated with adjectives or verbs, constructed by a linear regression method, from the vectors for nouns and for adjective-noun composites or verb-noun composites.

We developed a 5-parameter Gaussian model,

\[
\mathcal{Z}(\Lambda, a, b, J^0, J^S) = \int dM e^{-\frac{1}{2} \sum_{i=1}^{D} M_{ii}^2 + \frac{1}{4}(a+b) \sum_{i<j} (M_{ij}^2 + M_{ji}^2)} e^{-\frac{1}{4}(a-b) \sum_{i<j} M_{ij} M_{ji} + J^0 \sum_i M_{ii} + J^S \sum_{i<j} (M_{ij} + M_{ji})}.
\] (1.1)

The parameters \(J^S, J^0, a, b, \Lambda\) are coefficients of five linearly independent linear and quadratic functions of the \(D^2\) random matrix variables \(M_{i,j}\) which are permutation invariant, i.e. obey the equation

\[
f(M_{i,j}) = f(M_{\sigma(i),\sigma(j)})
\] (1.2)

for \(\sigma \in S_D\), the symmetric group of all permutations of \(D\) distinct objects. This \(S_D\) invariance implements the notion that the meaning represented by the word-matrices is independent of the ordering of the \(D\) context words. General observables of the model are polynomials \(f(M)\) obeying the condition \([1,2]\). At quadratic order there are 11 linearly independent polynomials, which are listed in Appendix B of \([8]\). A three dimensional subspace of quadratic invariants was used in the model above. The most general Gaussian matrix model compatible with \(S_D\) symmetry considers all the eleven parameters and allows coefficients for each of them. What makes the 5-parameter model relatively easy to handle is that the diagonal variables \(M_{ii}\) are each decoupled from each other and from the off-diagonal elements, and there are \(D(D-1)/2\) pairs of off-diagonal elements. For each \(i < j\), \(M_{ij}\) and \(M_{ji}\) mix with each other so the solution of the model requires an inversion of a \(2 \times 2\) matrix.

Expectation values of \(f(M)\) are computed as

\[
\langle f(M) \rangle \equiv \frac{1}{\mathcal{Z}} \int dM f(M) \text{EXP}
\] (1.3)

where \(\text{EXP}\) is the product of exponentials in \((1.1)\).

Representation theory of \(S_D\) offers the techniques to solve the general permutation invariant Gaussian model. The \(D^2\) matrix elements \(M_{ij}\) transform as the tensor product
$V_D \otimes V_D$ of two copies of the natural representation $V_D$. We first decompose $V_D \otimes V_D$ into irreducible representations of the diagonal $S_D$.

$$V_D \otimes V_D = 2V_0 \oplus 3V_H \oplus V_2 \oplus V_3$$  \hspace{1cm} (1.4)

The trivial (one-dimensional) representation $V_0$ occurs with multiplicity 2. The $(D - 1)$-dimensional irreducible representation (irrep) $V_H$ occurs with multiplicity 3. $V_2$ is an irrep of dimension $\frac{(D-1)(D-3)}{2}$ which occurs with multiplicity 1. Likewise, $V_3$ of dimension $\frac{(D-1)(D-2)}{2}$ occurs with multiplicity 1. As a result of these multiplicities, the 11 parameters can be decomposed as

$$11 = 1 + 1 + 3 + 6$$  \hspace{1cm} (1.5)

3 is the size of a symmetric $2 \times 2$ matrix. 6 is the size of a symmetric $3 \times 3$ matrix. More precisely the parameters form

$$\mathcal{M} = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{M}_2^+ \times \mathcal{M}_3^+$$  \hspace{1cm} (1.6)

where $\mathbb{R}^+$ is the set of real numbers greater or equal to zero, $\mathcal{M}_r^+$ is the space of positive semi-definite matrices of size $r$. Calculating the correlators of this Gaussian model amounts to inverting a symmetric $2 \times 2$ matrix, inverting a symmetric $3 \times 3$ matrix, and applying Wick contraction rules, as in quantum field theory, for calculating correlators. There is a graph basis for permutation invariant functions of $M$. This is explained in Appendix B of [8] which gives examples of graph basis invariants and representation theoretic counting formulae which make contact with the sequence A052171 - directed multi-graphs with loops on any number of nodes - of the Online Encyclopaedia of Integer Sequences (OEIS) [10].

In this paper we show how all the linear and quadratic moments of the graph-basis invariants are expressed in terms of the representation theoretic parameters of (1.6). We also show how some cubic and quartic graph basis invariants are expressed in terms of these parameters. These results are analytic expressions valid for all $D$.

The paper is organised as follows. Section 2 introduces the relevant facts from the representation theory of $S_D$ we need in a fairly self-contained way, which can be read with little prior familiarity of rep theory, but only knowledge of linear algebra. This is used to define the 13-parameter family of Gaussian models (equations (2.71), (2.72), (2.73)). Section 3 calculates the expectation values of linear and quadratic graph-basis invariants in the Gaussian model. Sections 4 and 5 describe calculations of expectation values of a selection of cubic and quartic graph-basis invariants in the model.

## 2 General permutation invariant Gaussian Matrix models

We solved a permutation invariant Gaussian Matrix model with 2 linear and 3 quadratic parameters [8], obtaining analytic expressions for low order moments of permutation in-
variant polynomial functions of a matrix variable as a function of the 5 parameters (section 6 of [8]). The linear parameters are coefficients of linear permutation invariant functions of $M$ and the quadratic parameters (denoted $\Lambda, a, b$) are coefficients of quadratic functions. We explained the existence of a $2 + 11$ parameter family of models, based on the fact that there are 11 linearly independent quadratic permutation invariant functions of a matrix. The general $2 + 11$-parameter family of models can be solved with the help of techniques from the representation theory of $S_D$.

We give a brief informal description of the key concepts we will use here. Further information can be found in [12, 11, 13, 14], and we will give more precise references below. A representation of a finite group $G$ is a pair $(V, D^V)$ consisting of a vector space $V$ and a homomorphism $D^V$ from $G$ to the space of invertible linear operators acting on $V$. Physicists often speak of a representation $V$ of $G$, when the accompanying homomorphism is left implicit. The homomorphism associates to each $g \in G$ a linear operator $D^V(g)$. Distinguished among the representations of $G$ are the irreducible representations (irreps). It is known that any representation of $G$ is isomorphic to a direct sum of irreducible representations. For further explanations of these statements see Lecture 1 of [12]. When a representation $V$ is a direct sum of $V_1, V_2, \cdots, V_k$, we express this as

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k.$$  \hspace{1cm} (2.1)

This implies that the linear operators $D^V(g)$ corresponding to group elements $g \in G$ can, after an appropriate choice of basis in $V$, be put in a block diagonal form where the blocks are $D^{V_1}(g), D^{V_2}(g), \cdots, D^{V_k}(g)$. The problem of finding this change of basis is called “reducing the representation $V$ into a direct sum of irreducibles”.

Given two representations $(V_1, D^{V_1})$ and $(V_2, D^{V_2})$ of $G$, the tensor product space $V_1 \otimes V_2$ is a representation of the product group $G \times G$, which consists of pairs $(g_1, g_2)$ with $g_1, g_2 \in G$. The product group $G \times G$ has a subgroup of pairs $(g, g)$ which is called the diagonal subgroup of $G$, denoted Diag($G$). The tensor product space $V_1 \otimes V_2$ is also a representation of this diagonal subgroup (see for example Chapter 1 of [13]). The linear transformation which reduces $V_1 \otimes V_2$ into a direct sum of irreducibles of Diag($G$) is called the Clebsch-Gordan decomposition. The matrix elements of the transformation are called Clebsch-Gordan coefficients. More details on these can be found in Chapter 5 of [11]. These can be used to construct projection operators for the subspaces of the tensor product space corresponding to particular irreducible representations.

In section 2.1 we introduce the natural representation $V_D$ of $S_D$. We note that the space of linear combinations of the matrix variables $M_{ij}$ is isomorphic as a vector space to $V_D \otimes V_D$. We recall the known fact that $V_D$ is isomorphic to a direct sum of two irreducible representations

$$V_D = V_0 \oplus V_H$$

and give the explicit change of basis which demonstrates this isomorphism. The tensor
product is thus isomorphic to a direct sum

\[ V_D \otimes V_D = (V_0 \otimes V_0) \oplus (V_0 \otimes V_H) \oplus (V_H \otimes V_0) \oplus (V_H \otimes V_H) \tag{2.2} \]

This leads to the definition (Equation (2.23)) of \( S_D \times S_D \) covariant variables \( S^{s0}, S^{0h}, S_{a0}, S_{ab}^{HH} \), which correspond to the four terms in the expansion (2.2).

In section 2.2 we describe the space of linear combinations of \( M_{ij} \) as a representation of \( \text{Diag}(S_D) \):

\[
\text{Span} \{ M_{ij} : 1 \leq i, j \leq D \} = V_0 \oplus V_0 \oplus V_H \oplus V_H \oplus V_2 \oplus V_3
\]

\[ = \bigoplus_{\alpha=1}^2 V_0^{(\alpha)} \bigoplus_{\alpha=1}^3 V_H^{(\alpha)} \oplus V_2 \oplus V_3 \tag{2.3} \]

The irreps \( V_2, V_3 \) have dimensions \( (D)(D-3)/2 \) and \( (D-1)(D-2)/2 \). The multiplicity index \( \alpha \) keeps track of the fact the same irrep appears multiple times in the decomposition into irreducibles of \( \text{Span} (M_{ij}) \). The isomorphism of representations of \( S_D \) above implies the identity relating the dimensions

\[ D^2 = 2 + 3(D-1) + \frac{D(D-3)}{2} + \frac{(D-1)(D-2)}{2} \tag{2.4} \]

This decomposition leads to the definition, in equations (2.51), (2.52), (2.53), of variables \( S^{V;\alpha} \) transforming according to the decomposition (2.3).

The next key observation is to think about the vector space of quadratic polynomials in indeterminates \( \{ x_1, x_2, \cdots, x_N \} \) in a way which is amenable to the methods of representation theory. Consider a vector space \( V_N \) spanned by \( x_1, x_2, \cdots, x_N \). The quadratic polynomials are spanned by the set of monomials \( x_i x_j \) which contains \( N(N+1)/2 \) elements. The vector space can be identified with the subspace of the tensor product \( V_N \otimes V_N \) which is invariant under the exchange of the two factors using the map

\[ x_i x_j \rightarrow (x_i \otimes x_j + x_j \otimes x_i). \tag{2.5} \]

This subspace of \( V_N \otimes V_N \) is denoted by \( \text{Sym}^2(V_N) \). In section (2.2) we apply this observation to the space of quadratic polynomials in the matrix variables \( M_{ij} \). They form a vector space which is isomorphic to \( \text{Sym}^2(V_D \otimes V_D) \).

Using the decomposition (2.3), we are able to find the \( S_D \) invariants by using a general theorem about invariants in tensor products of irreducible representations. For two irreps \( V_R, V_S \), the tensor product \( V_R \otimes V_S \) contains the trivial representation of the diagonal \( S_D \) only if \( R = S \), i.e. \( V_R \) is isomorphic to \( V_S \), and further it is also known that this invariant appears in the symmetric subspace \( \text{Sym}^2(V_R) \subset (V_R \otimes V_R) \). For further information on this useful fact, the reader is referred to Chapter 5 of [11].

This culminates in section (2.3) in an elegant representation theoretic description of the quadratic invariants in the matrix variables, using the linear combinations \( S^{V;\alpha} \). With
this description in hand, we introduce a set of representation theoretic parameters for the
13-parameter Gaussian matrix models, see equations (2.71) and (2.72). In terms of these
parameters, the linear and quadratic expectation values of $S_{V;\alpha}$ are simple (see equations
(2.75), (2.76), (2.77)). The computation of the correlators of low order polynomial in-
variant functions of the matrices then follows using Wick’s theorem from quantum field
theory (see for example Appendix A of [26]).

2.1 Matrix variables $M_{ij}$ and the natural representation of $S_D \times S_D$
The matrix elements $M_{ij}$, where $i, j$ run over $\{1, 2, \cdots, D\}$ span a vector space of dimen-
sion $D^2$. It is isomorphic to the tensor product $V_D \otimes V_D$, where $V_D$ is a $D$-dimensional
space. Consider $V_D$ as a span of $D$ basis vectors $\{e_1, e_2, \cdots, e_D\}$. This vector space $V_D$ is
a representation of $S_D$. For every permutation $\sigma \in S_D$, there is a linear operator $\rho_{V_D}(\sigma)$
defined by

$$\rho_{V_D}(\sigma)e_i = e_{\sigma^{-1}(i)}$$

on the basis vectors and extended by linearity. With this definition, $\rho_{V_D}$ is a homomor-
phism from $S_D$ to linear operators acting on $V_D$

$$\rho_{V_D}(\sigma_1)\rho_{V_D}(\sigma_2) = \rho_{V_D}(\sigma_1\sigma_2).$$

We introduce an inner product $(., .)$ where the $e_i$ form an orthonormal basis

$$(e_i, e_j) = \delta_{ij}.$$  

We can form the following linear combinations

$$E_0 = \frac{1}{\sqrt{D}}(e_1 + e_2 + \cdots + e_D)$$

$$E_1 = \frac{1}{\sqrt{2}}(e_1 - e_2)$$

$$E_2 = \frac{1}{\sqrt{6}}(e_1 + e_2 - 2e_3)$$

$$\vdots$$

$$E_a = \frac{1}{\sqrt{a(a+1)}}(e_1 + e_2 + \cdots + e_a - ae_{a+1})$$

$$\vdots$$

$$E_{D-1} = \frac{1}{\sqrt{D(D+1)}}(e_1 + e_2 + \cdots + e_{D-1} - (D-1)e_D).$$

(Eq. 2.9)

$E_0$ is invariant under the action of $S_D$

$$\rho_{V_D}(\sigma)E_0 = E_0$$

(Eq. 2.10)
since, for any $\sigma$, we have
\[
e^{-\sigma^{-1}(1)} + e^{-\sigma^{-1}(2)} + \cdots + e^{-\sigma^{-1}(D)} = e_1 + e_2 + \cdots + e_D. \tag{2.11}
\]
Thus the one-dimensional vector space spanned by $E_0$ is an $S_D$ invariant vector subspace of $V_D$. We can call this vector space $V_0$. The vector space spanned by $E_a$, where $1 \leq a \leq (D - 1)$, which we call $V_H$, is also an $S_D$-invariant subspace
\[
\rho_{V_D}(\sigma) E_a \in V_H. \tag{2.12}
\]
We have a matrix $D^H(\sigma)$ with matrix elements $D^H_{ab}(\sigma)$ such that
\[
\rho_{V_D}(\sigma) E_a = \sum_{b=1}^{D-1} D^H_{ab}(\sigma) E_b. \tag{2.13}
\]
These matrices are obtained by using the action on the $e_i$ and the change of basis coefficients. The vectors $E_A$ for $0 \leq A \leq D - 1$ are orthonormal under the inner product $\langle 2.8 \rangle$
\[
(E_A, E_B) = \delta_{A,B}. \tag{2.14}
\]
All the above facts are summarised by saying that the natural representation $V_D$ of $S_D$ decomposes as an orthogonal direct sum of irreducible representations of $S_D$ as
\[
V_D = V_0 \oplus V_H. \tag{2.15}
\]
By reading off the coefficients in the expansion of the $E_0, E_a$ in $V_H$, we can define the coefficients
\[
C_{0,i} = \langle E_0, e_i \rangle \quad C_{a,i} = \langle E_a, e_i \rangle \tag{2.16}
\]
using the inner product $\langle 2.8 \rangle$. They are
\[
C_{0,i} = \frac{1}{\sqrt{D}}, \\
C_{a,i} = N_a \left( -a \delta_{i,a+1} + \sum_{j=1}^{a} \delta_{ji} \right), \\
N_a = \frac{1}{\sqrt{a(a+1)}}. \tag{2.17}
\]
The orthonormality means that
\[
\sum_{i=1}^{D} C_{0,i} C_{0,i} = 1
\]
\[ \sum_{i=1}^{D} C_{a,i} C_{b,i} = \delta_{a,b} \]
\[ \sum_{i=1}^{D} C_{0,i} C_{a,i} = 0. \]  
(2.18)

The last equation implies that
\[ \sum_{i=1}^{D} C_{a,i} = 0. \]  
(2.19)

From
\[ \sum_{A=0}^{D-1} C_{A,i} C_{A,j} = C_{0,i} C_{0,j} + \sum_{a=1}^{D-1} C_{a,i} C_{a,j} = \delta_{ij} \]  
(2.20)

we deduce
\[ \sum_{a=1}^{D-1} C_{a,i} C_{a,j} = (\delta_{ij} - \frac{1}{D}) \equiv F(i, j). \]  
(2.21)

As we will see, this function \( F(i, j) \) will play an important role in calculations of correlators in the Gaussian model. It is the projector in \( V_D \) for the subspace \( V_H \), obeying
\[ \sum_{j=1}^{D} F(i, j) F(j, k) = F(i, k) \]
\[ \sum_{i=1}^{D} F(i, i) = (D - 1). \]  
(2.22)

Now we will use these coefficients \( C_{A,i} \) to build linear combinations of the matrix elements \( M_{i,j} \) which have well-defined transformation properties under \( S_D \times S_D \). Define
\[
S_{00}^{00} = \sum_{i,j=1}^{D} C_{0,i} C_{0,j} M_{ij} = \frac{1}{D} \sum_{i,j=1}^{D} M_{ij}
\]
\[
S_{a}^{0H} = \sum_{i,j=1}^{D} C_{0,i} C_{a,j} M_{ij} = \frac{1}{\sqrt{D}} \sum_{i,j=1}^{D} C_{a,j} M_{ij}
\]
\[
S_{a}^{H0} = \sum_{i,j=1}^{D} C_{a,i} C_{0,j} M_{ij} = \frac{1}{\sqrt{D}} \sum_{i,j=1}^{D} C_{a,i} M_{ij}
\]
\[
S_{ab}^{HH} = \sum_{i,j=1}^{D} C_{a,i} C_{b,j} M_{i,j} \]  
(2.23)
The $a, b$ indices range over $1 \cdots (D - 1)$. These variables are irreducible under $S_D \times S_D$, transforming as $V_0 \otimes V_0, V_0 \otimes V_H, V_H \otimes V_0, V_H \otimes V_H$. Under the diagonal $S_D$, the first three transform as $V_0, V_H, V_H$ while $S_{ab}^{HH}$ form a reducible representation.

Conversely, we can write these $M$ variables in terms of the $S$ variables, using the orthogonality properties of the $C_{0,i}, C_{a,i}$,

$$M_{ij} = C_{0,i}C_{0,j}S^{00} + \sum_{a=1}^{D-1} C_{0,a}C_{a,j}S_a^{0H} + \sum_{a=1}^{D-1} C_{a,i}C_{0,j}S_a^{H0} + \sum_{a,b=1}^{D-1} C_{a,i}C_{b,j}S_{ab}^{HH}$$

$$= \frac{1}{D} S^{00} + \frac{1}{\sqrt{D}} \sum_{a=1}^{D-1} C_{a,j}S_a^{0H} + \frac{1}{\sqrt{D}} \sum_{a=1}^{D-1} C_{a,i}S_a^{H0} + \sum_{a,b=1}^{D-1} C_{a,i}C_{b,j}S_{ab}^{HH}. \quad (2.24)$$

The next step is to consider quadratic products of these $S$-variables, and identify the products which are invariant. In order to do this we need to understand the transformation properties of the above $S$ variables in terms of the diagonal action of $S_D$. It is easy to see that $S^{00}$ is invariant. $S_a^{0H}$ and $S_a^{H0}$ both have a single $a$ index running over \{1, 2, \ldots , (D - 1)\}, and they transform in the same way as $V_H$. The vector space spanned by $S_{ab}^{HH}$ form a space of dimension $(D - 1)^2$ which is

$$V_H \otimes V_H. \quad (2.25)$$

Permutations act on this as

$$\sigma(S_{ab}^{HH}) = \sum_{a_1,b_1} D_{a_1 a}^H(\sigma) D_{b_1 b}^H(\sigma) S_{a_1,b_1}^{HH}. \quad (2.26)$$

using the matrices $D^H(\sigma)$ introduced in (2.13).

### 2.2 Decomposition of matrix variables as irreducible representations of $\text{Diag}(S_D) \subset S_D \times S_D$

In this section we will perform a further change of variables to introduce variables $S_{V_i;\alpha}^\alpha$ which transform according to irreps of the diagonal subgroup $\text{Diag}(S_D) \subset S_D \times S_D$.

- The representation space $V_H \otimes V_H$ can be decomposed into irreducible representations (irreps) of the diagonal $S_D$ action as

$$V_H \otimes V_H = V_0 \oplus V_H \oplus V_2 \oplus V_3. \quad (2.27)$$

In Young diagram notation for irreps of $S_D$, listing the row lengths of the Young diagram, we have

- $V_0 \rightarrow [D]$
- $V_H \rightarrow [D - 1, 1]$
- $V_2 \rightarrow [D - 2, 2]$
- $V_3 \rightarrow [D - 2, 1, 1]. \quad (2.28)$
These irreps are known to have dimensions $1, (D - 1), \frac{D(D-3)}{2}, \frac{(D-1)(D-2)}{2}$. They add up to $(D - 1)^2$ which is the dimension of $V_H \otimes V_H$.

The vector $\sum_{a=1}^{D-1} E_a \otimes E_a$ is invariant under the diagonal action of $\sigma$ on $V_H \otimes V_H$. Using the fact that $V_H$ is a subspace of $V_D$ described by the coefficients $C_{a,i}$ defined in (2.10), the action of $\sigma$ on $V_H$ is given by

$$D_{ab}^H(\sigma) = (E_a, \sigma E_b) = \sum_{i=1}^{D} C_{a,i} C_{b,\sigma(i)}.$$  \hfill (2.29)

These can be verified to satisfy the homomorphism property

$$\sum_{b=1}^{D-1} D_{ab}^H(\sigma) D_{bc}^H(\tau) = D_{ac}^H(\sigma \tau).$$ \hfill (2.30)

We also have $D_{ab}^H(\sigma^{-1}) = D_{ba}^H(\sigma)$. Using these properties, we can show that $\sum_a E_a \otimes E_a$ is invariant under the diagonal action. The vector

$$\frac{1}{\sqrt{D-1}} \sum_{a=1}^{D-1} E_a \otimes E_a$$ \hfill (2.31)

has unit norm, using the inner product on $V_D \otimes V_D$ obtained from (2.8), and defines a normalized vector in the $V_0$ subspace of the direct sum decomposition of $V_H \otimes V_H$ given in (2.27). From this expression, we can read off the Clebsch-Gordan coefficients for the trivial representation $V_0$ in $V_H \otimes V_H$

$$C_{a,b}^{H,H \rightarrow V_0} = \frac{\delta_{ab}}{\sqrt{D-1}}.$$ \hfill (2.32)

Using these we define $S^{HH \rightarrow V_0}$ as

$$S^{HH \rightarrow V_0} = \sum_{a,b=1}^{D-1} C_{a,b}^{H,H \rightarrow V_0} S_{ab}^{HH}$$

$$= \frac{1}{\sqrt{D-1}} \sum_{a=1}^{D-1} S_{aa}^{HH}$$ \hfill (2.33)

The vectors in the $V_H$ subspace on the RHS of the direct sum decomposition (2.27) are some linear combinations

$$\sum_{b,c=1}^{D-1} C_{b,c}^{H,H \rightarrow H} S_{bc}^{HH} = S_{a}^{H,H \rightarrow H}.$$ \hfill (2.34)
The coefficients $C_{b, c ; a}^{H, H \to H}$ are some representation theoretic numbers (called Clebsch-Gordan coefficients) which satisfy the orthonormality condition

$$\sum_{b, c = 1}^{D-1} C_{b, c ; a}^{H, H \to H} C_{b, c ; d}^{H, H \to H} = \delta_{a,d}. \quad (2.35)$$

As shown in Appendix B, these Clebsch-Gordan coefficients are proportional to $C_{a,b,c} \equiv \sum_i C_{a,i} C_{b,i} C_{c,i}$

$$C_{a, b ; c}^{HH \to H} = \sqrt{\frac{D}{D - 2}} C_{a,b,c}. \quad (2.36)$$

It is a useful fact that the Clebsch-Gordan coefficients for $V_H \otimes V_H \to V_H$ can be usefully written in terms of the $C_{a,i}$ describing $V_H$ as a subspace of the natural representation. This has recently played a role in the explicit description of a ring structure on primary fields of free scalar conformal field theory [15]. It would be interesting to explore the more general construction of explicit Clebsch-Gordan coefficients and projectors in the representation theory of $S_D$ in terms of the $C_{a,i}$.

- Similarly for $V_2, V_3$ we have corresponding vectors and Clebsch-Gordan coefficients

$$\sum_{b, c = 1}^{D-1} C_{b, c ; a}^{H, H \to V_2} S_{b,c} \equiv S_a^{HH \to V_2} \quad (2.37)$$

where $a$ ranges from 1 to $\text{Dim}(V_2) = \frac{D(D-3)}{2}$. We have the orthogonality property

$$\sum_{b, c = 1}^{D-1} C_{b, c ; a_1}^{H, H \to V_2} C_{b, c ; a_2}^{H, H \to V_2} = \delta_{a_1,a_2}. \quad (2.38)$$

And for $V_3$

$$\sum_{b, c = 1}^{D-1} C_{b, c ; a}^{H, H \to V_3} S_{bc} \equiv S_a^{HH \to V_3}$$

$$\sum_{b, c = 1}^{D-1} C_{b, c ; a_1}^{H, H \to V_3} C_{b, c ; a_2}^{H, H \to V_3} = \delta_{a_1,a_2}. \quad (2.39)$$

Here the $a, a_1, a_2$ runs over 1 to $\frac{(D-1)(D-2)}{2}$. 
• The projector for the subspace of $V_H \otimes V_H$ transforming as $V_H$ under the diagonal $S_D$ is

\[
(P_{H,H\rightarrow H})_{a,b,c,d} = \sum_{e=1}^{D-1} C_{a,b,e}^{H,H\rightarrow H} C_{c,d,e}^{H,H\rightarrow H}
\]

\[
= \frac{D}{(D-2)} \sum_{e=1}^{D-1} C_{a,b,e} C_{c,d,e}.
\]

(2.40)

• The projector $P_{H,H \rightarrow V_0}$ for $V_0$ in $V_H \otimes V_H$ is

\[
(P_{H,H \rightarrow V_0})_{a,b,c,d} = \frac{1}{(D-1)} \delta_{a,b} \delta_{c,d}.
\]

(2.41)

$V_{[n-2,1,1]} = V_3$ is just the anti-symmetric of $V_H \otimes V_H$. It is the orthogonal complement to $V_H \oplus V_0$ inside the symmetric subspace of $V_H \otimes V_H$ which is invariant under the swop of the two factors (often denoted $\text{Sym}^2(V_H)$)

\[
(P_{H,H \rightarrow V_2}) = (1 - P_{H,H \rightarrow H} - P_{H,H \rightarrow V_0}) \left( \frac{1 + s}{2} \right)
\]

(2.42)

where the swop $s$ acting on $V_H \otimes V_H$ has matrix elements

\[
(s)_{a,b,c,d} = \delta_{a,d} \delta_{b,c}.
\]

(2.43)

The quadratic invariant corresponding to $V_2$ is

\[
S_{ab}^{HH} (P_{H,H \rightarrow V_2})_{a,b,c,d} S_{cd}^{HH}.
\]

(2.44)

The quadratic invariant corresponding to $V_3$ is similar. We just have to calculate

\[
P_{H,H \rightarrow V_3} = \frac{1}{2} (1 - s).
\]

(2.45)

• The inner product

\[
\langle M_{ij}, M_{kl} \rangle = \delta_{ik} \delta_{jl}
\]

(2.46)

is invariant under the action $\sigma(M_{ij}) = M_{\sigma^{-1}(i),\sigma^{-1}(j)}$.

(2.47)

• The following is an important fact about invariants. Every irreducible representation of $S_D$, let us denote it by $V_R$ has the property that

\[
\text{Sym}^2(V_R)
\]

(2.48)
contains the trivial irrep once. This invariant is formed by taking the sum over an orthonormal basis $\sum_A e_A^V \otimes e_A^V$. The invariance is proved as follows

\[
D^V \otimes V(\sigma) \sum_A e_A^V \otimes e_A^V = \sum_A D^V(\sigma) e_A^V \otimes D^V(\sigma) e_A^V \\
= \sum_A \sum_{B,C} D^V_{BA}(\sigma) D^V_{CA}(\sigma) e_B^V \otimes e_C^V \\
= \sum_A \delta_{B,C} e_B^V \otimes e_C^V \\
= \sum_A e_A^V \otimes e_A^V. \tag{2.49}
\]

In the first equality we have used the definition of the diagonal action of $\sigma$ on the tensor product space.

- To summarize the matrix variables $M_{ij}$ can be linearly transformed to the following variables, organised according to representations of the diagonal $S_D$

  Trivial rep: $S^0_0$, $S^{HH \to V_0}$
  Hook rep: $S^H_0$, $S^{H0}_0$, $S^{HH \to H}$
  The rep $V_2$: $S^{HH \to V_2}_a$
  The rep $V_3$: $S^{HH \to V_3}_a$. \tag{2.50}

- For convenience, we will also use simpler names

\[
S^{V_0;1}_0 = S^0_0 \\
S^{V_0;2}_0 = S^{HH \to V_0}
\]

where we introduced labels 1, 2 to distinguish two occurrences of the trivial irrep $V_0$ in the space spanned by the $M_{ij}$. The variables $S^0, S^{H,H \to V_0}$ were first introduced in (2.23) and (2.33) respectively. We will also use

\[
S^{H;1}_a = S^{0,H \to H}_a \equiv S^0_a \\
S^{H;2}_a = S^{H0,H \to H}_a \equiv S^1_a \\
S^{H;3}_a = S^{H,H \to H}_a
\]

where we introduced labels 1, 2, 3 to distinguish the three occurrences of $V_H$ in the space spanned by $M_{ij}$. The variables $S^{H,H \to V_0}_a, S^{H0}_a$ were introduced earlier in (2.23). For the multiplicity-free cases, we introduce

\[
S^{V_2}_a = S^{HH \to V_2}_a
\]
The \( M_{ij} \) variables can be written as linear combinations of the \( S \) variables. Rep-basis expansion of \( M_{ij} \) is

\[
M_{ij} = C_{0,i} C_{0,j} S_{00} + \sum_{a,b=1}^{D-1} C_{a,i} C_{b,j} S_{ab} + \sum_{a=1}^{D-1} C_{0,i} C_{a,j} S_{a0} + \sum_{a=1}^{D-1} C_{a,i} C_{0,j} S_{0a} \\
= \frac{1}{D} S_{00} + \frac{1}{\sqrt{D}} \sum_{a=1}^{D-1} C_{a,j} S_{0i} + \frac{1}{\sqrt{D}} \sum_{a=1}^{D-1} C_{a,i} S_{0j} + \sum_{a,b=1}^{D-1} C_{a,i} C_{b,j} S_{ab} \\
+ \sum_{a,b=1}^{D-1} C_{a,i} C_{b,j} \sum_{V \in \{V_0, V_H, V_2, V_3\}} \sum_{c=1}^{\text{Dim} V} C_{HH \rightarrow V}^{a, b ; c} S_{HH \rightarrow V}^{c} \\
= \frac{1}{D} S_{00} + \frac{1}{\sqrt{D}} \sum_{a=1}^{D-1} C_{a,j} S_{0i} + \frac{1}{\sqrt{D}} \sum_{a=1}^{D-1} C_{a,i} S_{0j} + \sum_{a=1}^{D-1} C_{a,i} C_{a,j} S_{HH \rightarrow V_0} \\
+ \sum_{a,b=1}^{D-1} C_{a,i} C_{b,j} \sum_{c=1}^{D-1} C_{HH \rightarrow V}^{a, b ; c} S_{HH \rightarrow V}^{c} + \sum_{a,b=1}^{D-1} C_{a,i} C_{b,j} \sum_{c=1}^{\text{Dim} V_2} C_{HH \rightarrow V_2}^{a, b ; c} S_{HH \rightarrow V_2}^{c} \\
+ \sum_{a,b=1}^{D-1} \sum_{c=1}^{\text{Dim} V_3} C_{a,i} C_{b,j} C_{HH \rightarrow V_3}^{a, b ; c} S_{HH \rightarrow V_3}^{c}.
\]

In going from first to second line, we have used the fact that the transition from the natural representation to the trivial representation is given by simple constant coefficients

\[
C_{0,j} = \frac{1}{\sqrt{D}}.
\]

In the third line, we have used the Clebsch-Gordan coefficients for \( V_H \otimes V_H \rightarrow V \), obeying the orthogonality

\[
\sum_{a,b=1}^{D-1} C_{ab}^{HH \rightarrow V} C_{ab'}^{HH \rightarrow V} = \delta_{cc'}.
\]

For \( V = V_0 \), which is one dimensional, we just have

\[
C_{ab}^{HH \rightarrow V_0} = \frac{\delta_{ab}}{\sqrt{D - 1}}.
\]
in accordance with (2.31). The index \(c\) ranges over a set of orthonormal basis vectors for the irrep \(V\), i.e. extends over a range equal to the dimension of \(V\), denoted \(\text{Dim}V\). It is now useful to collect together the terms corresponding to each irrep \(V_0, V_H, V_2, V_3\)

\[
M_{ij} = \left( \frac{1}{D} S^{00} + \frac{1}{\sqrt{D-1}} \sum_{a=1}^{D-1} C_{a,i} C_{a,j} S^{HH \to 0} \right) \\
+ \left( \frac{1}{\sqrt{D}} \sum_{a=1}^{D-1} C_{a,i} S^{0H}_a + \frac{1}{\sqrt{D}} \sum_{a=1}^{D-1} C_{a,i} S^{H0}_a + \sum_{a,b,c=1}^{D-1} C_{a,i} C_{b,j} C^{HH \to H} \right) \\
+ \sum_{a,b=1}^{D-1 \text{Dim}V_2} \sum_{c=1}^{D-1 \text{Dim}V_3} C_{a,i} C_{b,j} C^{HH \to V_2} \sum_{c=1}^{D-1 \text{Dim}V_3} C_{a,i} C_{b,j} C^{HH \to V_3} \\
(2.58)
\]

Using the notation of (2.51), (2.52), (2.53), we write this as

\[
M_{ij} = \left( \frac{1}{D} S^{V_0;1} + \frac{1}{\sqrt{D-1}} \sum_{a=1}^{D-1} C_{a,i} C_{a,j} S^{V_0;2} \right) \\
+ \left( \frac{1}{\sqrt{D}} \sum_{a=1}^{D-1} C_{a,i} S^{H;1}_a + \frac{1}{\sqrt{D}} \sum_{a=1}^{D-1} C_{a,i} S^{H;2}_a + \sum_{a,b,c=1}^{D-1 \text{Dim}V_2} C_{a,i} C_{b,j} C^{H;3} \right) \\
+ \sum_{a,b=1}^{D-1 \text{Dim}V_2} \sum_{c=1}^{D-1 \text{Dim}V_3} C_{a,i} C_{b,j} C^{HH \to V_2} \sum_{c=1}^{D-1 \text{Dim}V_3} C_{a,i} C_{b,j} C^{HH \to V_3} \\
(2.59)
\]

• The discussion so far has included explicit bases for \(V_H\) inside \(V_D\) which are easy to write down. A key object in the above discussion is the projector \(F(i,j)\) defined in \(2.21\). For the irreps \(V_2, V_3\) which appear in \(V_H \otimes V_H\), we will not need to write down explicit bases. Although Clebsch-Gordan coefficients for \(H, H \to V_2\) and \(H, H \to V_3\) appear in some of the above formulae, we will only need some of their orthogonality properties rather than their explicit forms. The projectors for \(V_2, V_3\) in \(V_D \otimes V_D\) can be written in terms of the \(F(i,j)\), and it is these projectors which play a role in the correlators we will be calculating.

### 2.3 Representation theoretic description of quadratic invariants

With the above background of facts from representation theory at hand, we can give a useful description of quadratic invariants. Quadratic invariant functions of \(M_{ij}\) form the invariant subspace of Sym\(^2\)(\(V_D \otimes V_D\)) since \(M_{ij}\) transform as \(V_D \otimes V_D\).

\[
(V_D \otimes V_D) = (V_0 \oplus V_H) \otimes (V_0 \oplus V_H)
\]
\[(V_0 \oplus V_{0,H}^H \oplus V_{1,H}^H \oplus V_{H,H}^H \oplus V_{V_2}^H \oplus V_{V_3}^H). \quad (2.60)\]

So there are two copies of \(V_0\), namely \(V_{0,0}^0, V_{H,H}^0\). Sym\(^2(V_{0,0}^0 \oplus V_{H,H}^0)\) contains three invariants:

\[
\begin{align*}
(S^{00})^2 &= \left(S^{V_0;1}\right)^2 \\
(S^{00} S^{H,H \rightarrow 0}) &= S_{V_0;2} S_{V_0;1} = S_{V_0;1} S_{V_0;2} \\
(S^{H,H \rightarrow 0})^2 &= \left(S^{V_0;2}\right)^2. \quad (2.61)
\end{align*}
\]

These are all easy to write in terms of the original matrix variables, using the formulae for \(S\)-variables in terms of \(M\) given earlier. The relevant equations are \([2.51], [2.52], [2.53]\) where \(S\)-variables for irreps of \(\text{Diag}(S_D) \subset S_D \times S_D\) along with multiplicity labels are introduced, and earlier equations \([2.23], [2.31], [2.34], [2.37]\) which introduce \(S\)-variables labelled by \(S_D \times S_D\) irreps. These latter are more directly related to the Matrix variables, but the former are needed to give an elegant description of the quadratic invariants.

The general invariant quadratic function of the \(S_{V_0;\alpha}\) variables is

\[
\sum_{\alpha, \beta = 1}^2 (A_{V_0})_{\alpha \beta} S_{V_0;\alpha} S_{V_0;\beta}. \quad (2.62)
\]

\(A_{V_0}\) is a \(2 \times 2\) symmetric matrix. As we will see later, in defining the Gaussian model, this matrix will be restricted to be positive semi-definite.

There are three copies of \(V_H\), namely \(V_{0,H}^H, V_{H,0}^H, V_{H,H}^H\). These lead to 6 invariants:

\[
\begin{align*}
\sum_a S_{a,0,H \rightarrow H} S_{a,0,H \rightarrow H} &= \sum_a S_{a,H:1} S_{a,H:1} \\
\sum_a S_{a,0,H \rightarrow H} S_{a,0,H \rightarrow H} &= \sum_a S_{a,H:2} S_{a,H:2} \\
\sum_a S_{a,H,H \rightarrow H} S_{a,H,H \rightarrow H} &= \sum_a S_{a,H:3} S_{a,H:3} \\
\sum_a S_{a,0,H \rightarrow H} S_{a,H,0 \rightarrow H} &= \sum_a S_{a,H:1} S_{a,H:2} = \sum_a S_{a,H:2} S_{a,H:1} \\
\sum_a S_{a,0,H \rightarrow H} S_{a,H,0 \rightarrow H} &= \sum_a S_{a,H:1} S_{a,H:3} = \sum_a S_{a,H:3} S_{a,H:1} \\
\sum_a S_{a,0,H \rightarrow H} S_{a,H,0 \rightarrow H} &= \sum_a S_{a,H:2} S_{a,H:3} = \sum_a S_{a,H:3} S_{a,H:2}. \quad (2.63)
\end{align*}
\]

The sum over \(a\) runs over \((D - 1) = \text{Dim}(V_H)\) elements of a basis for \(V_H\). Thus the general quadratic invariants arising from the \(H\) representation among the \(M_{ij}\) are

\[
\sum_{\alpha, \beta = 1}^3 (A_H)_{\alpha \beta} \sum_a S_{a,H:\alpha} S_{a,H:\beta}. \quad (2.64)
\]
We introduced parameters \((\Lambda_H)_{\alpha\beta}\) forming a symmetric \(3 \times 3\) matrix. When we define the general Gaussian measure, we will see that this matrix will be required to be a positive definite matrix.

The quadratic invariants constructed from the \(V_2, V_3\) variables are

\[
\begin{align*}
(\Lambda_{V_2}) & \sum_{a=1}^{\text{Dim}V_2} S_a^{V_2} S_a^{V_2}, \\
(\Lambda_{V_3}) & \sum_{a=1}^{\text{Dim}V_3} S_a^{V_3} S_a^{V_3}.
\end{align*}
\] (2.65)

When we define the general Gaussian measure, we will take the parameters \(\Lambda_{V_2}, \Lambda_{V_3}\) to obey \(\Lambda_{V_2}, \Lambda_{V_3} \geq 0\).

### 2.4 Definition of the Gaussian models

The measure \(dM\) for integration over the matrix variables \(M_{ij}\) is taken to be the Euclidean measure on \(\mathbb{R}^{D^2}\) parametrised by the \(D^2\) variables

\[
dM \equiv \prod_i dM_{i} \prod_{i \neq j} dM_{ij}.
\] (2.66)

Since the variables \(S_a^{V;\alpha}\) defined in (2.51), (2.52), (2.53) are given by an orthogonal change of basis, we can show that

\[
dM = dS_{V_6;1}^S dS_{V_6;2}^S \prod_{a=1}^{\text{Dim}V_H} dS_a^{H;1} dS_a^{H;2} dS_a^{H;3} \prod_{a=1}^{\text{Dim}V_2} dS_a^{V_2} \prod_{a=1}^{\text{Dim}V_3} dS_a^{V_3}.
\] (2.67)

Indeed writing \(M_A\) for the matrix variables, where \(A\) runs over the \(D^2\) pairs \((i, j)\) and \(S_B\) for \(S_D\)-covariant variables, where \(B\) runs over all the factors in (2.67), we have

\[
dM = \prod_A dM_A = |\det J| \prod_B dS_B
\] (2.68)

with

\[
J_{AB} = \frac{\partial M_A}{\partial S_B}.
\] (2.69)

Now the \(S_B\) variables are obtained from \(M_A\) by an orthogonal basis change, and symmetric group properties also allow the matrix to be chosen to be real. This implies that the matrix is orthogonal

\[
J J^T = 1.
\] (2.70)
Hence $\det J$ has magnitude 1, and we have the claimed identity (2.67).

The model is defined by integration. The partition function is

$$Z(\mu_1, \mu_2; \Lambda V_0, \Lambda H, A V_2, \Lambda V_3) = \int dM e^{-S}$$  \hspace{1cm} (2.71)

where the action is a combination of linear and quadratic functions.

$$S = -\sum_{\alpha=1}^{2} \mu_{\alpha}^2 S_{V_0;\alpha} + \frac{1}{2} \sum_{\alpha,\beta=1}^{2} S_{V_0;\alpha} (\Lambda V_0)_{\alpha\beta} S_{V_0;\beta} + \frac{1}{2} \sum_{a=1}^{D-1} \sum_{\alpha,\beta=1}^{3} S_{a}^{H;\alpha} (\Lambda H)_{\alpha\beta} S_{a}^{H;\beta}$$

$$+ \frac{1}{2} \Lambda V_2 \sum_{a=1}^{D(D-1)/2} S_{a}^{V_2} S_{a}^{V_2} + \frac{1}{2} \Lambda V_3 \sum_{a=1}^{D(D-3)/2} S_{a}^{V_3} S_{a}^{V_3}.$$  \hspace{1cm} (2.72)

The expectation values of permutation invariant polynomials $f(M)$ are defined by

$$\langle f(M) \rangle = \frac{1}{Z} \int dM e^{-S} f(M).$$  \hspace{1cm} (2.73)

These expectation values can be computed using standard techniques from quantum field theory, specialised to matrix fields in zero space-time dimensions (See Appendix A for some explanations). Textbook discussions of these techniques are given, for example in [25], [26]. For linear functions, the non-vanishing expectation values are those of the invariant variables, which transform as $V_0$ under the $S_D$ action

$$\langle S_{V_0;\alpha} \rangle = \sum_{\beta} (\Lambda^{-1})_{\alpha\beta} \mu_{\beta}.$$  \hspace{1cm} (2.74)

We introduce the definition

$$\tilde{\mu}_{\alpha} \equiv \sum_{\beta} (\Lambda^{-1})_{\alpha\beta} \mu_{\beta}.$$  \hspace{1cm} (2.75)

We have defined variables $\tilde{\mu}_1, \tilde{\mu}_2$ for convenience. The variables transforming according to $V_H, V_2, V_3$ have vanishing expectation values

$$\langle S_{a}^{H;\alpha} \rangle = 0$$

$$\langle S_{a}^{V_2} \rangle = 0$$

$$\langle S_{a}^{V_3} \rangle = 0.$$  \hspace{1cm} (2.76)

The quadratic expectation values are

$$\langle S_{V_1;\alpha} S_{V_1;\beta} \rangle = \langle S_{V_1;\alpha} S_{V_1;\beta} \rangle_{\text{conn}} + \langle S_{V_1;\alpha} \rangle \langle S_{V_1;\beta} \rangle.$$  \hspace{1cm} (2.77)

where

$$\langle S_{a}^{V_1;\alpha} S_{b}^{V_1;\beta} \rangle_{\text{conn}} = \delta(V_i, V_j)(\Lambda^{-1})_{\alpha\beta} \delta_{ab}.$$  \hspace{1cm} (2.78)
The notation $\langle \cdot \rangle_{\text{conn}}$ is explained in the Appendix A. The $V_i, V_j$ can be $V_0, V_H, V_2, V_3$. The delta function means that these expectation values vanish unless the two irreps $V_i, V_j$ are equal. While $\delta_{ab}$ is the identity in the state space for each $V_i$. The fact that the mixing matrix in the multiplicity indices $\alpha, \beta$ is the inverse of the coupling matrix $\Lambda_V$ is a special (zero-dimensional) case of a standard result in quantum field theory, where the propagator is the inverse of the operator defining the quadratic terms in the action. The decoupling between different irreps follows because of the factorised form of the measure $dMe^{-S}$ in (2.71).

The requirement of an $S_D$ invariant Gaussian measure has led us to define variables $S^{V, \alpha}$, transforming in irreducible representations of $S_D$. The action is simple in terms of these variables. This is reflected in the fact that the above one and two-point functions are simple in terms of the parameters of the model.

When $\Lambda_{V_2} > 0, \Lambda_{V_3} > 0$ and $\Lambda_H, \Lambda_{V_0}$ are positive-definite real symmetric matrices (i.e real symmetric matrices with positive eigenvalues), then the partition function $Z$ is well defined as well as the numerators in the definition of $\langle f(M) \rangle$. We can relax these conditions, allowing $\Lambda_{V_2}, \Lambda_{V_3} \geq 0$ and $\Lambda_H, \Lambda_{V_0}$ positive semi-definite, by appropriately restricting the $f(M)$ we consider. For example, if $\Lambda_{V_2} = 0$, we consider functions $f(M)$ which do not depend on $S^{V_2}$, which ensures that the ratios defining $\langle f(M) \rangle$ are well-defined.

Thus the complete set of constraints on the representation theoretic parameters are

$$\begin{align*}
\det(\Lambda_{V_0}) &\geq 0 \\
\det(\Lambda_H) &\geq 0 \\
\Lambda_{V_2} &\geq 0 \\
\Lambda_{V_3} &\geq 0.
\end{align*}$$

(2.79)

More explicitly

$$\begin{align*}
(\Lambda_{V_0})_{11}(\Lambda_{V_0})_{22} - ((\Lambda_{V_0})_{12})^2 &\geq 0 \\
(\Lambda_H)_{11}((\Lambda_H)_{22}(\Lambda_H)_{33} - ((\Lambda_H)_{23})^2) - (\Lambda_H)_{12}((\Lambda_H)_{12}(\Lambda_H)_{33} - (\Lambda_H)_{23}(\Lambda_H)_{13}) &\geq 0 \\
+ (\Lambda_H)_{13}((\Lambda_H)_{12}(\Lambda_H)_{23} - (\Lambda_H)_{13}(\Lambda_H)_{22}) &\geq 0 \\
\Lambda_{V_2} &\geq 0 \\
\Lambda_{V_3} &\geq 0.
\end{align*}$$

(2.80)

With these linear and quadratic expectation values of representation theoretic matrix variables $S$ available, the expectation value of a general polynomial function of $M_{ij}$ can be expressed in terms of finite sums of products involving these linear and quadratic expectation values. This is an application of Wick’s theorem in the context of QFT. We will explain this for the integrals at hand in Appendix A and describe the consequences of Wick’s theorem explicitly for expectation values of functions up to quartic in the matrix variables. We will be particularly interested in the expectation values of polynomial functions of the $M_{ij}$ which are invariant under $S_D$ action and can be parametrised by
graphs. While the mixing between the \( S \) variables in the quadratic action is simple, there are non-trivial couplings between the \( D^2 \) variables \( M_{ij} \) if we expand the action in terms of the \( M \) variables. This will lead to non-trivial expressions for the expectation values of the graph-basis polynomials.

These expectation values were computed for the 5-parameter Gaussian model in \[8\]. They were referred to as theoretical expectation values \( \langle f(M) \rangle \), which were compared with experimental expectation values \( \langle f(M) \rangle_{\text{EXP}} \). These experimental expectation values were calculated by considering a list of words labelled by an index \( A \) ranging from 1 to \( N \), and their corresponding matrices \( M^A \),

\[
\langle f(M) \rangle_{\text{EXP}} = \frac{1}{N} \sum_{A=1}^{N} f(M^A).
\] (2.81)

We will now proceed to explicitly apply Wick’s theorem to calculate the expectation values of permutation invariant functions labelled by graphs for the case of quadratic functions (2-edge graphs), cubic (3-edge graphs) and quartic functions (4-edge graphs). We will leave the comparison of the results of this 13-parameter Gaussian model to linguistic data for the future.

### 3 Graph basis invariants in terms of rep theory parameters

In the graph theoretic description of \( S_D \) invariants constructed from \( M_{ij} \), nodes in the graph correspond to indices, \( M \) corresponds to directed edges. At linear order we have the one-node invariant \( \sum_i M_{ii} \) and the two-node invariant \( \sum_{i,j} M_{ij} \). At quadratic order in \( M \) we have up to three nodes. In this section, we calculate the expectation values of the linear and quadratic invariants in the Gaussian model defined in Section 2. The quadratic expectation values show non-trivial mixing between the different \( M_{ij} \), unlike the 5-parameter model, where the \( M_{ii} \) are decoupled from the off-diagonal elements and from each other. In that simple model, \( M_{ij} \) only mixes with \( M_{ji} \). Here the mixings are more non-trivial, but controlled by \( S_D \) representation theory.

The quantity \( F(i, j) \) defined in (2.21)

\[
F(i, j) = \sum_a C_{a,i} C_{a,j} = \left( \delta_{ij} - \frac{1}{D} \right)
\] (3.1)

will play an important role in the following. Its meaning is that it is the projector for the hook representation in the natural representation. Deriving expressions for expectation values of permutation invariant polynomial functions of the matrix variable \( M \) amounts to doing appropriate sums of products of \( F \) factors, with the arguments of these \( F \) factors
being related to each other according to the nature of the polynomial under consideration. In terms of the variables

$$\tilde{\mu}_\alpha = \sum_{b=1}^{2} (\Lambda^{-1}_{V_0})_{\alpha\beta} \mu_\beta$$

defined in Section 2, repeated here for convenience,

$$\langle S^{V_0;1} \rangle = \tilde{\mu}_1$$

$$\langle S^{V_0;2} \rangle = \tilde{\mu}_2 .$$

(3.2)

Using the expansion (2.59) of the matrix variables in terms of the rep-theoretic $S$ variables and the 1-point functions of these in (2.74), (2.75) and (2.76), we have the 1-point function for the matrix variables

$$\langle M_{ij} \rangle = \frac{\tilde{\mu}_i}{D} + \frac{\tilde{\mu}_j}{\sqrt{D-1}} F(i, j).$$

(3.3)

Using (A.4), (A.5) along with the expansion of $M$ in terms of $S$ variables (equation 2.59), and the two-point functions of the $S$-variables, we have

$$\langle M_{ij} M_{kl} \rangle = \langle M_{ij} M_{kl} \rangle_{\text{conn}} + \langle M_{ij} \rangle \langle M_{kl} \rangle$$

(3.4)

where

$$\langle M_{ij} M_{kl} \rangle_{\text{conn}} = \frac{1}{D^2} \langle S^{V_0;1} S^{V_0;1} \rangle_{\text{conn}} + \frac{1}{D-1} \sum_{a_1, a_2=1}^{D-1} C_{a_1,j} C_{a_1,k} C_{a_2,l} \langle S^{V_0;2} S^{V_0;2} \rangle_{\text{conn}}$$

$$+ \frac{1}{D \sqrt{D-1}} \sum_{a_1, a_2=1}^{D-1} \langle S^{V_0;1} S^{V_0;2} \rangle_{\text{conn}} C_{a_1,k} C_{a_1,l} + \frac{1}{D \sqrt{D-1}} \sum_{a_1, a_2=1}^{D-1} \langle S^{V_0;2} S^{V_0;1} \rangle_{\text{conn}} C_{a_1,j} C_{a_1,k}$$

$$+ \frac{1}{D} \sum_{a_1, a_2=1}^{D-1} C_{a_1,j} C_{a_2,k} \langle S^{H;1} S^{H;1} \rangle_{\text{conn}} + \frac{1}{D} \sum_{a_1, a_2=1}^{D-1} C_{a_1,j} C_{a_2,k} \langle S^{H;2} S^{H;2} \rangle_{\text{conn}}$$

$$+ \sum_{a_1, b_1, c_1, a_2, b_2, c_2=1}^{D-1} C_{a_1,j} C_{b_1,j} C_{a_2,k} C_{b_2,l} \langle S^{H;1} S^{H;1} \rangle_{\text{conn}} + \sum_{a_1, b_1, c_1, a_2, b_2, c_2=1}^{D-1} C_{a_1,j} C_{b_2,l} \langle S^{H;1} S^{H;2} \rangle_{\text{conn}}$$

$$+ \frac{1}{D} \sum_{a_1, a_2=1}^{D-1} C_{a_1,j} C_{a_2,k} \langle S^{H;1} S^{H;2} \rangle_{\text{conn}} + \frac{1}{D} \sum_{a_1, a_2=1}^{D-1} C_{a_1,j} C_{a_2,l} \langle S^{H;2} S^{H;1} \rangle_{\text{conn}}$$

$$+ \frac{1}{\sqrt{D}} \sum_{a_1=1}^{D-1} \sum_{a_2=1}^{D-1} C_{a_1,j} C_{a_2,k} \langle S^{H;1} S^{H;1} \rangle_{\text{conn}} + \frac{1}{\sqrt{D}} \sum_{a_1=1}^{D-1} \sum_{a_2=1}^{D-1} C_{a_1,j} C_{a_2,l} \langle S^{H;2} S^{H;2} \rangle_{\text{conn}}$$

$$+ \sum_{a_1, b_1, c_1=1}^{D-1} C_{a_1,j} C_{b_1,j} C_{a_2,k} \langle S^{H;1} S^{H;1} \rangle_{\text{conn}} + \sum_{a_1, b_1, c_1=1}^{D-1} C_{a_1,j} C_{a_2,l} \langle S^{H;2} S^{H;1} \rangle_{\text{conn}}$$

$$+ \sum_{a_1, b_1, c_1=1}^{D-1} C_{a_1,j} C_{b_1,j} C_{a_2,k} \langle S^{H;1} S^{H;1} \rangle_{\text{conn}} + \sum_{a_1, b_1, c_1=1}^{D-1} C_{a_1,j} C_{a_2,l} \langle S^{H;2} S^{H;1} \rangle_{\text{conn}}$$

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\[ + \frac{1}{\sqrt{D}} \sum_{a_1,b_1,c_1=1}^{D-1} \sum_{a_2,b_2,c_2=1}^{D-1} C_{a_1,i} C_{a_2,k} C_{b_2,l} C_{a_2,b_2; c_2}^{HH \to H} \langle S_{a_1}^{H;2} S_{c_2}^{H;3} \rangle_{\text{conn}} \]

\[ + \frac{1}{\sqrt{D}} \sum_{a_1,b_1,c_1=1}^{D-1} \sum_{a_2,b_2,c_2=1}^{D-1} C_{a_1,i} C_{b_1,j} C_{a_1,b_1; c_1} C_{a_2,k} \langle S_{c_1}^{H;3} S_{a_2}^{H;2} \rangle_{\text{conn}} \]

\[ + \sum_{a_1,b_1=1}^{D-1} \sum_{a_2,b_2=1}^{D-1} \sum_{c_1,c_2=1}^{\text{Dim} V_2} C_{a_1,i} C_{b_1,j} C_{a_1,b_1; c_1} C_{a_2,k} C_{b_2,l} C_{a_2,b_2; c_2} \langle S_{c_1}^{V_2} S_{c_2}^{V_2} \rangle_{\text{conn}} \]

\[ + \sum_{a_1,b_1=1}^{D-1} \sum_{a_2,b_2=1}^{D-1} \sum_{c_1,c_2=1}^{\text{Dim} V_3} C_{a_1,i} C_{b_1,j} C_{a_1,b_1; c_1} C_{a_2,k} C_{b_2,l} C_{a_2,b_2; c_2} \langle S_{c_1}^{V_3} S_{c_2}^{V_3} \rangle_{\text{conn}} . \tag{3.5} \]

All the terms can be expressed in terms of the \( F \)-function defined in (3.1)

\[ \langle M_{ij}M_{kl} \rangle_{\text{conn}} = \frac{1}{D^2} (\Lambda_{V_0}^{-1})_{i_1} + \frac{D(\Lambda_{H}^{-1})_{i_2}}{D(D-1)} F(i,j) F(k,l) + \frac{D(\Lambda_{H}^{-1})_{i_3}}{D(D-2)} \sum_{p,q=1}^{D} F(i,p) F(j,q) F(l,q) F(p,q) \]

\[ + \frac{D(\Lambda_{H}^{-1})_{i_4}}{\sqrt{D-2}} \left( \sum_{p=1}^{D} F(i,p) F(k,p) F(l,p) + F(i,p) F(j,p) F(k,p) \right) \]

\[ + \frac{(\Lambda_{V_2}^{-1})_{i_5}}{2} \left( F(i,k) F(j,l) + \frac{1}{2} F(i,l) F(j,k) - \frac{D}{D-2} \sum_{p,q=1}^{D} F(i,p) F(j,q) F(k,q) F(l,q) F(p,q) \right) \]

\[ - \frac{1}{D-1} F(i,j) F(k,l) \] \[ \right. \nonumber \]

\[ + \frac{1}{2} \left( F(i,k) F(j,l) - F(i,l) F(j,k) \right) . \tag{3.6} \]

We will refer to the terms depending on \( \Lambda_{V_0} \) as \( \text{V}_0 \)-channel contributions to the 2-point functions, those on \( \Lambda_{H} \) as \( \text{H}_V \)-channel (or \( \text{H}_H \))-channel contributions, those on \( \Lambda_{V_2} \) as \( \text{V}_2 \)-channel and those on \( \Lambda_{V_3} \) as \( \text{V}_3 \)-channel contributions. It will be convenient to denote these different channel contributions as \( \langle M_{ij}M_{kl} \rangle_V \) where \( V \in \{ \text{V}_0, \text{V}_H, \text{V}_2, \text{V}_3 \} \), so that we have

\[ \langle M_{ij}M_{kl} \rangle_{\text{conn}} = \langle M_{ij}M_{kl} \rangle_{\text{V}_0} + \langle M_{ij}M_{kl} \rangle_{\text{V}_H} + \langle M_{ij}M_{kl} \rangle_{\text{V}_2} + \langle M_{ij}M_{kl} \rangle_{\text{V}_3} . \tag{3.7} \]

In arriving at the expressions for the last two terms in (3.6), we used the fact that these terms in (3.5) can be expressed as

\[ \Lambda_{V_2}^{-1} \left( e_i \otimes e_j, P_{V_0} \otimes V_2 \rightarrow V_2 (e_k \otimes e_l) \right) + \Lambda_{V_3}^{-1} \left( e_i \otimes e_j, P_{V_0} \otimes V_3 \rightarrow V_3 (e_k \otimes e_l) \right) . \tag{3.8} \]
Here $P_{V_D \otimes V_D \to V_2}$ is the projector from the tensor product of natural reps. to $V_2$, the irrep of $S_D$ associated with Young diagram $[D - 2, 2]$. Similarly for $V_3 = [D - 2, 1, 1]$. Now it is useful to observe that $V_3$ is just the anti-symmetric part of $V_H \otimes V_H$. The symmetric part decomposes as $V_0 \oplus V_H \oplus V_2$. The projector for $V_0$ is

$$ (P_{H \rightarrow V_0})_{a_1b_1;a_2b_2} = \frac{1}{D - 1} \delta_{a_1a_2} \delta_{b_1b_2}. \quad (3.9) $$

For $V_H$ it is

$$ (P_{H \rightarrow H})_{a_1b_1;a_2b_2} = \sum_{c=1}^{D-1} C_{a_1b_1; c}^{H,H \rightarrow H} C_{a_2b_2; c}^{H,H \rightarrow H} = \frac{D}{D - 2} \sum_{c=1}^{D-1} C_{a_1b_1;c} C_{a_2b_2;c}. \quad (3.10) $$

The factor $\frac{D}{D - 2}$ is explained in Appendix C.

### 3.1 Calculation of $\sum_{i,j} \langle M_{ij} M_{ij} \rangle$

Following (3.7) the expectation value $\langle M_{ij} M_{ij} \rangle$ can be written as a sum over $V$-channel contributions, where $V$ ranges over the four irreps.

#### 3.1.1 Contributions from $V_2, V_3$

From (3.8) we find

$$ \sum_{i,j} \langle M_{ij} M_{ij} \rangle_{V_2}^{V_2} = (\Lambda_{V_2})^{-1} \text{tr}_{V_D \otimes V_D} (P_{V_D \otimes V_D \rightarrow V_2}) $$

$$ = (\text{Dim } V_2)(\Lambda_{V_2})^{-1} = D(D - 3) \frac{(D - 1)}{2} (\Lambda_{V_2})^{-1}. \quad (3.11) $$

The projector has eigenvalue 1 on the subspace transforming in the irrep $V_2$ and zero elsewhere, hence the (Dim $V_2$). Similarly

$$ \sum_{i,j} \langle M_{ij} M_{ij} \rangle_{V_3}^{V_3} = (\text{Dim } V_3)(\Lambda_{V_3})^{-1} = \frac{(D - 1)(D - 2)}{2} (\Lambda_{V_3})^{-1}. \quad (3.12) $$

Since $V_2, V_3$ appear inside the $V_H \otimes V_H$ subspace of $V_{\text{nat}} \otimes V_{\text{nat}}$, we can also write the trace in $V_H \otimes V_H$ and express this in terms of irreducible characters

$$ \text{tr}_{V_H \otimes V_H} P_{V_2} = \frac{(\text{Dim } V_2)}{D!} \sum_{\sigma \in S_D} \chi_{V_2}(\sigma) \chi_{V_H}(\sigma) \chi_{V_H}(\sigma) $$

$$ = (\text{Dim } V_2). \quad (3.13) $$

In the last line we have used the fact that the Kronecker coefficient for $V_H \otimes V_H \to V_2$ is 1.
3.1.2 Contribution from $V_H$ channel

The $(\Lambda_{H}^{-1})_{11}$ contribution is

$$(\text{Dim } V_H)(\Lambda_{H}^{-1})_{11} = (D - 1)(\Lambda_{H}^{-1})_{11}$$

(3.14)

which can be obtained easily from (3.5) or (3.6). Similarly, the $(\Lambda_{H}^{-1})_{12}$ contribution is zero. And the $(\Lambda_{H}^{-1})_{22}$ contribution is

$$(\text{Dim } V_H)(\Lambda_{H}^{-1})_{22} = (D - 1)(\Lambda_{H}^{-1})_{22} .$$

(3.15)

From either (3.5) or (3.6) we easily conclude that the $(\Lambda_{H}^{-1})_{13}, (\Lambda_{H}^{-1})_{23}$ contributions vanish. Starting from (3.5), we make repeated use of (2.19).

The $(\Lambda_{H}^{-1})_{33}$ contribution is

$$\sum_{i,j} (\tilde{\mu}_i D + \frac{\tilde{\mu}_j}{\sqrt{D-1}} F(i,j))^2 = \tilde{\mu}_1^2 + \tilde{\mu}_2^2.$$  

(3.16)

We used the second equation in (2.18).

3.1.3 Contribution from $V_0$ channel

This is

$$\sum_{i,j} (M_{ij} M_{ij})_{V_0} = (\Lambda_{V_0}^{-1})_{11} + (\Lambda_{V_0}^{-1})_{22} (\text{Dim } V_H) = (\Lambda_{V_0}^{-1})_{11} + (\Lambda_{V_0}^{-1})_{22} .$$

(3.17)

3.1.4 Summing all channels

$$\sum_{i,j} (M_{ij} M_{ij})_{\text{conn}} = (\Lambda_{V_0}^{-1})_{11} + (\Lambda_{V_0}^{-1})_{22} + (D - 1)(\Lambda_{V_0}^{-1})_{22} + (D - 1)(\Lambda_{V_0}^{-1})_{33} + (D - 1)(\Lambda_{V_0}^{-1})_{11}$$

$$+ \frac{D(D - 3)}{2} (\Lambda_{V_2})^{-1} + \frac{(D - 1)(D - 2)}{2} (\Lambda_{V_3})^{-1} .$$

(3.18)

The disconnected piece is

$$\sum_{i,j} (M_{ij}) (M_{ij}) = \sum_{i,j} \left( \tilde{\mu}_i D + \frac{\tilde{\mu}_j}{\sqrt{D-1}} F(i,j) \right)^2$$

$$= \tilde{\mu}_1^2 + \tilde{\mu}_2^2.$$ 

(3.19)

and using (3.4)

$$\sum_{i,j} (M_{ij} M_{ij}) = \tilde{\mu}_1^2 + \tilde{\mu}_2^2 + \sum_{i,j} (M_{ij} M_{ij})_{\text{conn}}$$

$$= \tilde{\mu}_1^2 + \tilde{\mu}_2^2 + (\Lambda_{V_0}^{-1})_{11} + (\Lambda_{V_0}^{-1})_{22} + (D - 1)(\Lambda_{V_0}^{-1})_{22} + (D - 1)(\Lambda_{V_0}^{-1})_{33} + (D - 1)(\Lambda_{V_0}^{-1})_{11}$$

$$+ \frac{D(D - 3)}{2} (\Lambda_{V_2})^{-1} + \frac{(D - 1)(D - 2)}{2} (\Lambda_{V_3})^{-1} .$$

(3.20)
3.2 Calculation of $\sum_{i,j} \langle M_{ij} M_{ji} \rangle$

As in (3.7) the expectation value $\langle M_{ij} M_{ij} \rangle$ can be written as a sum over $V$-channel contributions, where $V$ ranges over the four irreps.

### 3.2.1 Contribution from multiplicity 1 channels $V_2, V_3$

The $(\Lambda^{-1}_{V_2})$ contribution is

$$
(\Lambda^{-1}_{V_2}) \text{tr}_{V_H \otimes V_H} (P^2 \tau) \\
= (\Lambda^{-1}_{V_2}) d_{V_2} = (\Lambda^{-1}_{V_2}) \frac{D(D - 3)}{2}.
$$

(3.21)

$\tau$ is the swap which acts on the two factors of $V_H$. We have used the fact that $V_2$ appears in the symmetric part of $V_H \otimes V_H$.

The $(\Lambda^{-1}_{V_3})$ contribution is

$$
-(\Lambda^{-1}_{V_3}) \frac{(D - 1)(D - 2)}{2}.
$$

(3.22)

We use the fact that $V_3$ is the antisymmetric part of $V_H \otimes V_H$.

### 3.2.2 Contribution from $V_H$ channel

The $(\Lambda^{-1}_{H})_{11}$ contribution From (3.5), we have

$$
\frac{1}{D} \sum_{a_1, a_2} \sum_{i,j} C_{a_1,i} C_{a_2,j} (\Lambda^{-1}_{H})_{11} \delta_{a_1a_2} = \frac{1}{D} (\Lambda^{-1}_{H})_{11} \sum_a C_{a,i} C_{a,j} = 0
$$

(3.23)

using (2.19). Similarly the $(\Lambda^{-1}_{H})_{22}$ contribution is zero.

The $(\Lambda^{-1}_{H})_{12}$ contribution is

$$
\frac{2}{D} \sum_{a,i,j} C_{a,j} C_{a,j} (\Lambda^{-1}_{H})_{12} = 2(D - 1)(\Lambda^{-1}_{H})_{12}.
$$

(3.24)

The $(\Lambda^{-1}_{H})_{33}$ contribution is

$$
\sum_{i,j} \sum_{a,b,c,d,e} (\Lambda^{-1}_{H})_{33} C_{a,i} C_{b,j} C_{c,j} C_{d,i} C_{H \rightarrow H} C_{H \rightarrow H} C_{H \rightarrow H} C_{H \rightarrow H}
\\
= (\Lambda^{-1}_{H})_{33} \sum_{a,b,c,d,e} C_{H \rightarrow H} C_{H \rightarrow H} C_{H \rightarrow H} C_{H \rightarrow H}
\\
= (\Lambda^{-1}_{H})_{33} \text{tr}_{H \otimes H} P_{H \otimes H} \rightarrow H S = (\Lambda^{-1}_{H})_{33} \text{tr}_{H \otimes H} P_{H \otimes H} \rightarrow H (1 + s^2 - 1 - \tau^2)
\\
= (\Lambda^{-1}_{H})_{33} d_H = (D - 1)(\Lambda^{-1}_{H})_{33}.
$$

(3.25)
In the penultimate line, we have introduced the swop $s$ which exchanges the two factors in $H \otimes H$. We know that $H$ appears in the symmetric part of $H \otimes H$, so the swop leaves it invariant.

Use of the equation (2.19) shows that the $(\Lambda_H^{-1})_{13}, (\Lambda_H^{-1})_{23}$ dependent terms vanish.

Collecting all the $V_H$-channel contributions, we have

$$\sum_{i,j} \langle M_{ij}M_{ji} \rangle_{\text{conn}}^{V_H} = 2\frac{(D-1)}{D} (\Lambda_H^{-1})_{12} + (D-1)(\Lambda_H^{-1})_{33}.$$  \hspace{1cm} (3.26)

### 3.2.3 Contribution from $V_0$ channel

The first term from (3.5) is

$$(\Lambda^{-1}_{V_0})_{11}.$$  \hspace{1cm} (3.27)

The second term is

$$\frac{(\Lambda_{V_0}^{-1})_{22}}{d_H} \sum_{i,j} \sum_{a_1,a_2} C_{a_1,i}C_{a_1,j}C_{a_2,j}C_{a_2,i} = \frac{(\Lambda_{V_0}^{-1})_{22}}{D-1} \sum_{a_1,a_2} \delta_{a_1 a_2} \delta_{a_1 a_2} = (\Lambda^{-1}_{V_0})_{22}.$$  \hspace{1cm} (3.28)

The third term is

$$\frac{1}{D\sqrt{D-1}} \sum_{i,j} \sum_{a} C_{a,j}C_{a,i}(\Lambda^{-1}_{V_0})_{12}.$$  \hspace{1cm} (3.29)

which vanishes using (2.19). The last term vanishes for the same reason.

So collecting the $V_0$-channel contributions to $\sum_{i,j} \langle M_{ij}M_{ji} \rangle_{\text{conn}}$, we have

$$\sum_{i,j} \langle M_{ij}M_{ji} \rangle_{\text{conn}} = (\Lambda^{-1}_{V_0})_{11} + (\Lambda^{-1}_{V_0})_{22}.$$  \hspace{1cm} (3.30)

### 3.2.4 Summing all channels

$$\sum_{i,j} \langle M_{ij}M_{ji} \rangle_{\text{conn}} = (\Lambda^{-1}_{V_2}) \frac{(D)(D-3)}{2} - (\Lambda^{-1}_{V_3}) \frac{(D-1)(D-2)}{2} + 2(D-1)(\Lambda^{-1}_H)_{12} + (D-1)(\Lambda^{-1}_H)_{33} + (\Lambda^{-1}_{V_0})_{11} + (\Lambda^{-1}_{V_0})_{22}.$$  \hspace{1cm} (3.31)

Since $F(i,j) = F(j,i)$, the disconnected piece is the same in (3.19)

$$\sum_{i,j} \langle M_{ij} \rangle \langle M_{ji} \rangle = \sum_{i,j} \langle M_{ij} \rangle \langle M_{ij} \rangle = \tilde{\mu}_1^2 + \tilde{\mu}_2^2.$$  \hspace{1cm} (3.32)

and $\sum_{i,j} \langle M_{ij}M_{ij} \rangle$ in terms of the $\mu, \Lambda$ parameters of the Gaussian model is the sum of the expressions in (3.31) and (3.32).
3.3 Calculation of $\sum_{i,j} \langle M_{ii}M_{ij} \rangle$

An important observation here is that the sum over $j$ projects the representation $V_D$ to the trivial irrep $V_0$, which follows from the formula for $C_0$ in (2.17). This means that when we expand $M_{ii}$ and $M_{ij}$ into $S$ variables as in the first line of (2.54), we only need to keep the term $S^H$ or $S^{00}$ from the expansion of $M_{ij}$.

3.3.1 Contribution from $V_2, V_3$ channels

From the above observation, and since $V_2, V_3$ appear only in $\langle S^{HH} S^{HH} \rangle$, we immediately see that

$$\sum_{i,j} \langle M_{ii}M_{ij} \rangle_{V_2}^{V_{conn}} = \sum_{i,j} \langle M_{ii}M_{ij} \rangle_{V_3}^{V_{conn}} = 0 . \quad (3.33)$$

3.3.2 Contribution from $V_H$ channel

From the above observation, the only non-zero contributions in the $V_H$ channel come from $\langle S^{HH} \rightarrow H S^{HH} \rangle$, $\langle S^{HH} S^{HH} \rangle$ and $\langle S^{0H} S^{HH} \rangle$. These are

$$\frac{1}{D} \sum_{a_1,a_2,i,j} C_{a_1,i} C_{a_2,i} (\Lambda_H^{-1})_{12} \delta_{a_1 a_2} = (D - 1)(\Lambda_H^{-1})_{12} \quad (3.34)$$

$$\frac{1}{D} \sum_{a_1,a_2,i,j} C_{a_1,i} C_{a_2,i} (\Lambda_H^{-1})_{22} \delta_{a_1 a_2} = (D - 1)(\Lambda_H^{-1})_{22} \quad (3.35)$$

$$\frac{1}{\sqrt{D}} \sum_{a_1,b_1,a_2,i,j} C_{a_1,i} C_{b_1,i} C^{H,H \rightarrow H}_{a_1,b_1; c_1} C_{a_2,i} \delta_{c_1 a_2} (\Lambda_H^{-1})_{32} = \frac{D}{\sqrt{D - 2}} \sum_{a_1,b_1,a_2,i} C_{a_1,i} C_{b_1,i} C_{a_2,i} \delta_{c_1 a_2} (\Lambda_H^{-1})_{32} = \frac{D}{\sqrt{D - 2}} (\Lambda_H^{-1})_{32} \sum_{a_1,b_1,c_1} C_{a_1,b_1,c_1} C_{a_1,b_1,c_1} = (D - 1) \sqrt{(D - 2)} (\Lambda_H^{-1})_{23} . \quad (3.36)$$

Note that $\Lambda_H$ is a symmetric $3 \times 3$ matrix and $\Lambda_H^{-1})_{23} = (\Lambda_H^{-1})_{32}$. In the penultimate step, we have used the normalization equation (C.9). These add up to

$$\sum_{i,j} \langle M_{ii}M_{ij} \rangle_{V_H}^{V_{conn}} = (D - 1)(\Lambda_H^{-1})_{12} + (D - 1)(\Lambda_H^{-1})_{22} + (D - 1) \sqrt{(D - 2)}(\Lambda_H^{-1})_{23} . \quad (3.37)$$
3.3.3 Contribution from $V_0$ channel

The non-zero contributions come from $\langle S^{00}S^{00} \rangle$ and $\langle S^{HH\rightarrow 0}S^{00} \rangle$. They are

$$\sum_{i,j} \langle M_{ii}M_{ij} \rangle_{V_0}^{\text{conn}} = \frac{1}{D^2} \sum_{i,j} (\Lambda_{V_0}^{-1})_{11} + \frac{1}{D\sqrt{d_H}} \sum_{i,j} \sum_a (\Lambda_{V_0}^{-1})_{12} C_{a,i} C_{a,i}$$

$$= (\Lambda_{V_0}^{-1})_{11} + \frac{1}{\sqrt{D-1}} \sum_{i,a} (\Lambda_{V_0}^{-1}) C_{a,i} C_{a,i}$$

$$= (\Lambda_{V_0}^{-1})_{11} + \sqrt{d_H} (\Lambda_{V_0}^{-1})_{12} = (\Lambda_{V_0}^{-1})_{11} + \sqrt{(D-1)} (\Lambda_{V_0}^{-1})_{12} \cdot (3.38)$$

3.3.4 Summing all channels

$$\sum_{i,j} \langle M_{ii}M_{ij} \rangle_{\text{conn}} = (\Lambda_{V_0}^{-1})_{11} + \sqrt{(D-1)} (\Lambda_{V_0}^{-1})_{12}$$

$$+ (D-1) (\Lambda_{H}^{-1})_{12} + (D-1) (\Lambda_{H}^{-1})_{22} + (D-1) \sqrt{(D-2)} (\Lambda_{H}^{-1})_{23} \cdot (3.39)$$

The disconnected piece is

$$\sum_{i,j} \langle M_{ii} \rangle \langle M_{ij} \rangle = \sum_{i,j} \left( \frac{\bar{\mu}_1}{D} + \frac{\bar{\mu}_2}{\sqrt{D-1}} F(i,i) \right) \left( \frac{\bar{\mu}_1}{D} + \frac{\bar{\mu}_2}{\sqrt{D-1}} F(i,j) \right)$$

$$= \bar{\mu}_1^2 + \frac{\bar{\mu}_1 \bar{\mu}_2}{D\sqrt{D-1}} \sum_{i,j} (1 - \frac{1}{D})$$

$$= \bar{\mu}_1^2 + \bar{\mu}_1 \bar{\mu}_2 \sqrt{(D-1)} \cdot (3.40)$$

so we have

$$\sum_{i,j} \langle M_{ii}M_{ij} \rangle = \sum_{i,j} \langle M_{ii}M_{ij} \rangle_{\text{conn}} + \bar{\mu}_1^2 + \bar{\mu}_1 \bar{\mu}_2 \sqrt{(D-1)} \cdot (3.41)$$

with the first term given by (3.39)

3.4 Calculation of $\sum_{i,j} \langle M_{ii}M_{ji} \rangle$

We can write down the answer from inspection of (3.39)

$$\sum_{i,j} \langle M_{ii}M_{ji} \rangle_{\text{conn}} = (\Lambda_{V_0}^{-1})_{11} + \sqrt{(D-1)} (\Lambda_{V_0}^{-1})_{12}$$

$$+ (D-1) (\Lambda_{H}^{-1})_{12} + (D-1) (\Lambda_{H}^{-1})_{11} + (D-1) \sqrt{(D-2)} (\Lambda_{H}^{-1})_{13} \cdot (3.42)$$

The reasoning is as follows. The sum over $j$ projects to $V_0$. This means that the only non-zero contributions are, from the $V_0$ channel,

$$\langle S^{00}S^{00} \rangle^\text{conn}$$
\[ \langle S^{HH\rightarrow 0S^{00}} \rangle_{\text{conn}} \]  

(3.43)

and from the \( V_H \) channel

\[ \begin{align*} 
\langle S^{0H}\bar{S}^{0H} \rangle_{\text{conn}} \\
\langle S^{0H\bar{S}^{0H}} \rangle_{\text{conn}} \\
\langle S^{HH\rightarrow H\bar{S}^{0H}} \rangle_{\text{conn}} \\
\end{align*} \]  

(3.44)

This identifies the contributing entries of \( \Lambda_{V_0}^{-1}, \Lambda_{H}^{-1} \) using the indexing in (2.51) and (2.52). Given the similarity between the expectation value in section 3.3, we have contributions of the same form, up to taking care of the right indices on \( \Lambda_{V_0}^{-1}, \Lambda_{H}^{-1} \).

Given the symmetry of \( F(i,j) \) under exchange of \( i, j \), the disconnected piece is the same as above

\[
\sum_{i,j} (M_{ii}M_{ji}) = (\Lambda_{V_0}^{-1})_{11} + \sqrt{(D-1)}(\Lambda_{V_0}^{-1})_{12} \\
+ (D-1)(\Lambda_{H}^{-1})_{12} + (D-1)(\Lambda_{H}^{-1})_{11} + (D-1)\sqrt{(D-2)}(\Lambda_{H}^{-1})_{13} \\
+ \tilde{\mu}_1^2 + \tilde{\mu}_1\tilde{\mu}_2 \sqrt{(D-1)}. 
\]  

(3.45)

3.5 Calculation of \( \sum_{i,j,k} \langle M_{ij}M_{ik} \rangle \)

The sums over \( j, k \) project to \( V_0 \). The non-vanishing contributions are \( \langle S^{00S^{00}} \rangle_{\text{conn}} \) and \( \langle S^{H0\bar{S}^{0H}} \rangle_{\text{conn}} \). They add up to

\[
\sum_{i,j,k} \langle M_{ij}M_{ik} \rangle_{\text{conn}} = D(\Lambda_{V_0}^{-1})_{11} + D(D-1)(\Lambda_{H}^{-1})_{22}. 
\]  

(3.46)

The disconnected part is

\[
\sum_{i,j,k} \langle M_{ij} \rangle \langle M_{ik} \rangle = \tilde{\mu}_1^2 D 
\]  

(3.47)

leading to

\[
\sum_{i,j,k} \langle M_{ij}M_{ik} \rangle = D(\Lambda_{V_0}^{-1})_{11} + D(D-1)(\Lambda_{H}^{-1})_{22} + \tilde{\mu}_1^2 D. 
\]  

(3.48)

3.6 Calculation of \( \sum_{i,j,k} \langle M_{ij}M_{kj} \rangle \)

Now we are projecting to \( V_0 \) on the first index of both \( M \)'s. This means that the contributing terms are \( \langle S^{00S^{00}} \rangle \) and \( \langle S^{H0\bar{S}^{0H}} \rangle \).

Repeat the same steps as above in (3.46) to get

\[
\sum_{i,j,k} \langle M_{ij}M_{kj} \rangle_{\text{conn}} = D(\Lambda_{V_0}^{-1})_{11} + D(D-1)(\Lambda_{H}^{-1})_{11}. 
\]  

(3.49)
The only difference is that we are picking up the \((1, 1)\) matrix element of \((\Lambda^{-1}_H)\) instead of the \((2, 2)\) element, since we defined \(S^{0H} = S^{V_0;1}\) and \(S^{H0} = S^{V_0;2}\).

Adding the disconnected piece, which is the same as (3.47), we have
\[
\sum_{i,j,k} \langle M_{ij} M_{kj} \rangle = D (\Lambda^{-1}_{V_0})_{11} + D (D - 1) (\Lambda^{-1}_H)_{11} + \tilde{\mu}_i^2 D.
\] (3.50)

### 3.7 Calculation of \(\sum_{i,j,k} \langle M_{ij} M_{jk} \rangle\)

We are now projecting to \(V_0\) on first index of one of the matrices and second index of the other. Hence the contributing terms are \(\langle S_{00} S_{00} \rangle\) and \(\langle S_{0H} S_{H0} \rangle\). The result is
\[
\sum_{i,j,k} \langle M_{ij} M_{jk} \rangle_{\text{conn}} = D (\Lambda^{-1}_{V_0})_{11} + D (D - 1) (\Lambda^{-1}_H)_{12}.\] (3.51)

and
\[
\sum_{i,j,k} \langle M_{ij} M_{jk} \rangle = D (\Lambda^{-1}_{V_0})_{11} + D (D - 1) (\Lambda^{-1}_H)_{12} + \tilde{\mu}_i^2 D.\] (3.52)

### 3.8 Calculation of \(\sum_{i,j,k,l} \langle M_{ij} M_{kl} \rangle\)

Here we project to \(V_0\) on all four indices, so
\[
\sum_{i,j,k,l} \langle M_{ij} M_{kl} \rangle_{\text{conn}} = D^2 \langle S^{00} S^{00} \rangle = D^2 (\Lambda^{-1}_{V_0})_{11}.\] (3.53)

Adding the disconnected piece we have
\[
\sum_{i,j,k,l} \langle M_{ij} M_{kl} \rangle = D^2 (\Lambda^{-1}_{V_0})_{11} + \tilde{\mu}_i^2 D^2.\] (3.54)

### 3.9 Calculation of \(\sum_i \langle M^2_{ii} \rangle\)

#### 3.9.1 The \(V_0\) channel

The contribution from the \(V_0\) channel is given by
\[
\sum_i \langle M^2_{ii} \rangle^{V_0}_{\text{conn}} = \frac{1}{D} \sum_i (\Lambda^{-1}_{V_0})_{11} + \frac{1}{D^2} \sum_i F(i,i)^2 (\Lambda^{-1}_{V_0})_{22} + \frac{2}{D^2} \sum_i F(i,i) (\Lambda^{-1}_{V_0})_{12}\]
\[
= \frac{1}{D} (\Lambda^{-1}_{V_0})_{11} + \frac{(D - 1)}{D} (\Lambda^{-1}_{V_0})_{22} + \frac{2D - 1}{D} (\Lambda^{-1}_{V_0})_{12}.
\] (3.55)
3.9.2 The $V_H$ channel

It is convenient to use (3.6) to arrive at

$$
\sum_i \langle M^2_{\text{int}} \rangle_{\text{conn}}^V = \frac{D-1}{D} (\Lambda_H^{-1})_{11} + (\Lambda_H^{-1})_{22} \frac{(D-1)}{D} + D^{-1}(D-1)(D-2)(\Lambda_H^{-1})_{33}
+ 2 \frac{(D-1)}{D} (\Lambda_H^{-1})_{12} + 2 \frac{(\Lambda_H^{-1})_{13}}{D} (D-1) \sqrt{(D-2)} + 2 \frac{(\Lambda_H^{-1})_{24}}{D} (D-1) \sqrt{(D-2)}.
$$

(3.56)

Useful equations in arriving at the above are the sums

$$
\sum_{i,p,q} F(i,p)F(i,p)F(i,q)F(p,q) = \frac{(D-2)^2(D-1)}{D^2}
$$

$$
\sum_{i,p}(F(i,p))^3 = \frac{(D-1)(D-2)}{D}.
$$

(3.57)

which can be obtained by hand or with the help of Mathematica. In the latter case, it is occasionally easier to evaluate for a range of integer $D$ and fit to a form $Polynomial(D)/D^{n}$ some power.

3.9.3 The $V_2, V_3$ channels

Now calculate the $HH \rightarrow V_2$ and $HH \rightarrow V_3$ channel.

$$
\sum_i \langle M^2_{\text{int}} \rangle_{\text{conn}}^V = \sum_i \sum_{a,b,c,d} C_{a,i} C_{b,i} C_{c,i} C_{d,i} C_{e} C_{f} \langle S^H \rightarrow V_2 S^H \rightarrow V_2 \rangle_{\text{conn}}
$$

$$
= (\Lambda_H^{-1}) \sum_{i,j} \sum_{a,b,c,d,e} C_{a,i} C_{b,i} C_{c,i} C_{d,i} C_{e} C_{f} \langle S^H \rightarrow V_2 S^H \rightarrow V_2 \rangle_{\text{conn}}
$$

$$
= (\Lambda_H^{-1}) \sum_{a,b,c,d,e} C_{a,i} C_{b,i} \left( C_{c,i} C_{e,j} + \frac{1}{D} \right) C_{c,j} C_{d,j} C_{e} C_{f} \langle S^H \rightarrow V_2 S^H \rightarrow V_2 \rangle_{\text{conn}}
$$

$$
= \frac{D-2}{D} (\Lambda_H^{-1}) tr_H \otimes_H (P^H P^V) + \frac{1}{D} (\Lambda_H^{-1}) \sum_{a,b,c,d,e} C_{c,i} C_{d,i} C_{e} C_{f} \langle S^H \rightarrow V_2 S^H \rightarrow V_2 \rangle_{\text{conn}}
$$

$$
= \frac{D-2}{D} (\Lambda_H^{-1}) tr_H \otimes_H (P^H \rightarrow H P^H \rightarrow V_2) + \frac{1}{D} (\Lambda_H^{-1}) \sum_{a,b} C_{c,i} C_{d,i} C_{e} C_{f} \langle S^H \rightarrow V_2 S^H \rightarrow V_2 \rangle_{\text{conn}}
$$

$$
= \frac{D-2}{D} (\Lambda_H^{-1}) tr_H \otimes_H (P^H \rightarrow H P^H \rightarrow V_2) + \frac{1}{D} (\Lambda_H^{-1}) \sum_{a,b} C_{c,i} C_{d,i} C_{e} C_{f} \langle S^H \rightarrow V_2 S^H \rightarrow V_2 \rangle_{\text{conn}}
$$

$$
= 0 + \frac{(\Lambda_H^{-1})}{D-2} \sum_{a,b,i,j} C_{a,i} C_{b,i} C_{c,i} C_{d,i} C_{e} C_{f} \langle S^H \rightarrow V_2 S^H \rightarrow V_2 \rangle_{\text{conn}}
$$

$$
= \frac{(\Lambda_H^{-1})}{D-2} \sum_{i,j} F(i,j) F(i,i) F(j,j) = (\Lambda_H^{-1}) \frac{(D-1)^2}{D^2(D-2)} \sum_{i,j} F(i,j) = 0.
$$

(3.58)

We used the fact that the projectors for $H, V_2$ are orthogonal.
Similarly, the contribution from $V_3$ is zero. Another to arrive at the same answer is to recognize that $V_3$ is the antisymmetric part, so

$$P_{ab,cd}^{H,H \rightarrow V_3} = \frac{1}{2} (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}).$$  \hfill (3.59)$$

### 3.9.4 Summing the channels

$$\sum_i \langle M_{ii}^2 \rangle_{\text{conn}} = \frac{1}{D}(\Lambda_{V_0}^{-1})_{11} + \frac{D-1}{D}(\Lambda_{V_0}^{-1})_{22} + 2\frac{\sqrt{D-1}}{D}(\Lambda_{V_0}^{-1})_{12}$$

$$+ \sum_i \left( \frac{D-1}{D}(\Lambda_{H}^{-1})_{11} + \frac{D-1}{D}(\Lambda_{H}^{-1})_{22} + D^{-1}(D-1)(D-2)(\Lambda_{H}^{-1})_{33} \right)$$

$$+ 2\frac{\sqrt{D-1}}{D}(\Lambda_{H}^{-1})_{12} + 2\frac{(\Lambda_{H}^{-1})_{13}}{D}(D-1)\sqrt{(D-2)} + 2\frac{(\Lambda_{H}^{-1})_{24}}{D}(D-1)\sqrt{(D-2)}.$$  \hfill (3.60)

The disconnected part is

$$\sum_{i,j} \langle M_{ii} \rangle \langle M_{jj} \rangle_{\text{conn}} = \sum_i \left( \frac{\bar{\mu}_1^2}{D} + \frac{\bar{\mu}_2^2}{D}\sqrt{D-1} F(i,i) \right)^2$$

$$= \frac{\bar{\mu}_1^2}{D} + 2\bar{\mu}_1\bar{\mu}_2 \frac{\sqrt{D-1}}{D}.$$  \hfill (3.61)

### 3.10 Calculation of $\sum_{i,j} \langle M_{ii} M_{jj} \rangle$

Since $\sum_i M_{ii}$ and $\sum_j M_{jj}$ are $S_D$ invariant, we only have contributions from the $V_0$ channel. Use the first four terms of (3.5) to get

$$\sum_{i,j} \langle M_{ii} M_{jj} \rangle_{\text{conn}} = (\Lambda_{V_0}^{-1})_{11} + (D-1)(\Lambda_{V_0}^{-1})_{22} + 2\sqrt{D-1}(\Lambda_{V_0}^{-1})_{12}.$$  \hfill (3.62)

Using

$$\sum_i \left( \frac{\bar{\mu}_1}{D} + \frac{\bar{\mu}_2}{\sqrt{D-1}} F(i,i) \right) = \bar{\mu}_1 + \bar{\mu}_2 \sqrt{D-1}$$  \hfill (3.63)$$

the disconnected part is

$$\sum_{i,j} \langle M_{ii} \rangle \langle M_{jj} \rangle = \bar{\mu}_1^2 + 2\bar{\mu}_1\bar{\mu}_2 \sqrt{D-1} + \bar{\mu}_2^2(D-1).$$  \hfill (3.64)$$

so that

$$\sum_{i,j} \langle M_{ii} M_{jj} \rangle = \sum_{i,j} \langle M_{ii} M_{jj} \rangle_{\text{conn}} + \bar{\mu}_1^2 + 2\bar{\mu}_1\bar{\mu}_2 \sqrt{D-1} + \bar{\mu}_2^2(D-1).$$  \hfill (3.65)$$

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3.11 Calculation of $\sum_{i,j,k} \langle M_{ii}M_{jk} \rangle$

Here we get contributions from $\langle S^{HH\rightarrow 0}S^{00} \rangle_{\text{conn}}$ and $\langle S^{00}S^{00} \rangle_{\text{conn}}$. Adding these up from (3.5)

$$\sum_{i,j,k} \langle M_{ii}M_{jk} \rangle_{\text{conn}} = D(\Lambda_{V_0}^{-1})_{11} + D\sqrt{D-1}(\Lambda_{V_0}^{-1})_{12}. \quad (3.66)$$

The disconnected part is

$$\sum_{i,j,k} \langle M_{ii} \rangle \langle M_{jk} \rangle = (\bar{\mu}_1 + \bar{\mu}_2 \sqrt{D-1}) \ D\bar{\mu}_1 \quad (3.67)$$

hence

$$\sum_{i,j,k} \langle M_{ii}M_{jk} \rangle = D(\Lambda_{V_0}^{-1})_{11} + D\sqrt{D-1}(\Lambda_{V_0}^{-1})_{12} + (\bar{\mu}_1 + \bar{\mu}_2 \sqrt{D-1}) \ D\bar{\mu}_1. \quad (3.68)$$

3.12 Summary of results for quadratic expectation values in a large $D$ limit

It is interesting to collect the results for the connected quadratic expectation values and consider the large $D$ limit. Let us assume that all the $\Lambda_{V_0}, \Lambda_H, \Lambda_{V_2}, \Lambda_{V_3}$ scale in the same way as $D \rightarrow \infty$ and consider the sums normalized by the appropriate factor of $D$

$$\frac{1}{D^2} \sum_{i,j} \langle M_{ij}M_{ij} \rangle_{\text{conn}} = \frac{1}{2} \left( (\Lambda_{V_2})^{-1} + (\Lambda_{V_3})^{-1} \right)$$

$$\frac{1}{D^2} \sum_{i,j} \langle M_{ij}M_{ji} \rangle_{\text{conn}} = \frac{1}{2} \left( (\Lambda_{V_2})^{-1} - (\Lambda_{V_3})^{-1} \right)$$

$$\frac{1}{D^2} \sum_{i,j} \langle M_{ii}M_{ij} \rangle_{\text{conn}} = \frac{1}{D^{3/2}}(\Lambda_{V_0}^{-1})_{12} + \frac{1}{D^{3/2}}(\Lambda_H^{-1})_{23}$$

$$\frac{1}{D^2} \sum_{i,j} \langle M_{ii}M_{ji} \rangle_{\text{conn}} = \frac{1}{D^{3/2}}(\Lambda_{V_0}^{-1})_{12} + \frac{1}{D^{3/2}}(\Lambda_H^{-1})_{13}$$

$$\frac{1}{D^3} \sum_{i,j,k} \langle M_{ij}M_{ik} \rangle_{\text{conn}} = \frac{1}{D}(\Lambda_H^{-1})_{22}$$

$$\frac{1}{D^3} \sum_{i,j,k} \langle M_{ij}M_{kj} \rangle_{\text{conn}} = \frac{1}{D}(\Lambda_H^{-1})_{11}$$

$$\frac{1}{D^3} \sum_{i,j,k} \langle M_{ij}M_{jk} \rangle_{\text{conn}} = \frac{1}{D}(\Lambda_H^{-1})_{12}$$
The dominant expectation values in this limit are the first, second and ninth. These are the quadratic expressions which enter the simplified 5-parameter model considered in [8] (see Equation (1.1)). It will be interesting to systematically explore the different large $D$ scalings of the parameters in real world data, e.g., the computational linguistics setting of [8] or in any other situation where permutation invariant matrix Gaussian matrix distributions can be argued to be appropriate.

4 A selection of cubic expectation values

In this section we use Wick’s theorem from Appendix A to express expectation values of cubic functions of matrix variables in terms of linear and quadratic expectation values. The permutation invariance condition requires sums of indices over the range $\{1, \ldots, D\}$. This leads to non-trivial sums of products of the natural-to-hook projector $F(i, j)$. The invariants at cubic order are 52 in number (Appendix B of [8]) and correspond to graphs with up to 6 nodes.

4.1 1-node case $\sum_i \langle M_{ii}^3 \rangle$

Using [A.6] we have

$$\sum_i \langle M_{ii}^3 \rangle = 3 \sum_i \langle M_{ii}^2 \rangle_{\text{conn}} \langle M_{ii} \rangle + \langle M_{ii} \rangle^3. \quad (4.1)$$

Specialising (3.3)

$$\langle M_{ii} \rangle = \frac{1}{D} \tilde{\mu}_1 + \sqrt{\frac{(D - 1)}{D}} \tilde{\mu}_2. \quad (4.2)$$

Since this is independent of $i$, we can use (3.60) to get

$$\sum_i \langle M_{ii}^3 \rangle = 3 \left( \frac{1}{D} \tilde{\mu}_1 + \sqrt{\frac{(D - 1)}{D}} \tilde{\mu}_2 \right) \times$$
\[
\left( \frac{1}{D}(\Lambda^{-1}_V)_{11} + \frac{(D-1)}{D}(\Lambda^{-1}_V)_{22} + 2\sqrt{D-1} \right)(\Lambda^{-1}_V)_{12} \\
+ \frac{D-1}{D}((\Lambda^{-1}_H)_{11} + (\Lambda^{-1}_H)_{22}(D-1)) + D^{-1}(D-1)(D-2)(\Lambda^{-1}_H)_{33} \\
+ 2(D-1)(\Lambda^{-1}_H)_{12} + 2(\Lambda^{-1}_H)_{12}(D-1)\sqrt{(D-2)} + 2(D-1)\sqrt{(D-2)} \\
+ \frac{1}{D^2} \left( \bar{\mu}_1 + \sqrt{(D-1)} \bar{\mu}_2 \right)^3.
\]

(4.3)

### 4.2 A 2-node case \( \sum_{i,j} \langle M_{ij}^3 \rangle \)

Using \( \text{A.6} \) we have

\[
\langle M_{ij}^3 \rangle = \sum_{i,j} \langle M_{ij}^2 \rangle_{\text{conn}} \langle M_{ij} \rangle + \sum_{i,j} \langle M_{ij} \rangle^3.
\]

(4.4)

Calculating this requires doing a few sums, which can be done by hand or with Mathematica (the function KoneckerDelta is handy).

\[
\sum_{i,j} F(i,j) = 0 \\
\sum_{i,j} (F(i,j))^2 = (D-1) \\
\sum_{i,j} (F(i,j))^3 = D^{-1}(D-1)(D-2).
\]

(4.5)

Using (3.3), we find for the second term in (4.4)

\[
\sum_{i,j} (M_{ij})^3 = \frac{\bar{\mu}_1^3}{D} + \frac{3}{D} \bar{\mu}_1 \bar{\mu}_2^2 + \frac{(D-2)}{D\sqrt{D-1}} \bar{\mu}_3^2.
\]

(4.6)

For the first term on the RHS of (4.4)

\[
\sum_{i,j} \langle M_{ij} \rangle_{\text{conn}} \langle M_{ij} \rangle = \frac{3\bar{\mu}_1}{D} \sum_{i,j} \langle M_{ij}^2 \rangle_{\text{conn}} + \frac{3\bar{\mu}_2}{\sqrt{D-1}} \sum_{i,j} \langle M_{ij}^2 \rangle_{\text{conn}} F(i,j).
\]

(4.7)

The first term in (4.7) can be expressed as a function of the parameters of the Gaussian model using (3.18). The second term is calculated by specialising the fundamental quadratic moments (3.6) and doing the resulting sums over the \( F \)-factors. Consider the \( V_0 \) contributions to the second term above. The term proportional to \( (\Lambda^{-1}_V)_{11} \) vanishes due to the first of (4.5). The \( (22) \) contribution, using the third of (4.5) is

\[
\frac{3\bar{\mu}_2}{\sqrt{D-1}} \times \frac{D-2}{D} (\Lambda^{-1}_V)_{22} = \frac{3\bar{\mu}_2 (\Lambda^{-1}_V)_{22} (D-2)}{D\sqrt{D-1}}.
\]

(4.8)
The (12) contributions, using the second of (4.5) is
\[
\frac{3\tilde{\mu}_2}{\sqrt{D-1}} \times \frac{2\sqrt{D-1}}{D} (\Lambda_{V_0}^{-1})_{12} = \frac{6\tilde{\mu}_2 (\Lambda_{V_0}^{-1})_{12}}{D} . \tag{4.9}
\]

Now consider the \(V_H\) contribution to the second term in (4.7). The \((\Lambda_{H}^{-1})_{11}\) term is
\[
\frac{1}{D} (\Lambda_{H}^{-1})_{11} \sum_{i,j} F(i, j) F(j, j) = 0 . \tag{4.10}
\]

The \((\Lambda_{H}^{-1})_{22}\) contribution is similarly zero. The \((\Lambda_{H}^{-1})_{33}\) contribution is
\[
\frac{3\tilde{\mu}_2}{\sqrt{D-1}} \times \frac{D (\Lambda_{H}^{-1})_{33}}{(D-2)} \times \sum_{i,j,p,q} F(i, p) F(j, q) F(i, p) F(j, q) F(i, j) F(p, q)
\]
\[
= 3\tilde{\mu}_2 (\Lambda_{H}^{-1})_{33} \sqrt{D-1} (D-3) \frac{1}{D} . \tag{4.11}
\]

The sum of products of six \(F\)'s is readily done with Mathematica to give \(D^{-2}(D-1)(D-2)(D-3)\).

Contributions from the (1, 2) matrix element of symmetric matrix \(\Lambda_H\) give
\[
\frac{6\tilde{\mu}_2 \sqrt{(D-1)}}{D} (\Lambda_{H}^{-1})_{12} . \tag{4.12}
\]

This uses the second of (4.5). From (1, 3) and (3, 1) we have
\[
\frac{3\tilde{\mu}_2}{\sqrt{D-1}} \times \frac{\Lambda_{H}^{-1}}{\sqrt{D-2}} \times 2 \sum_{i,j,p} F(i, p) (F(j, p))^2 F(i, j)
\]
\[
= \frac{3\tilde{\mu}_2}{\sqrt{D-1}} \times \frac{\Lambda_{H}^{-1}}{\sqrt{D-2}} \times 2D^{-1}(D-1)(D-2)
\]
\[
= \frac{6\tilde{\mu}_2}{D} (\Lambda_{H}^{-1})_{13} \sqrt{(D-1)(D-2)} . \tag{4.13}
\]

From (2, 3) and (3, 2), we have
\[
\frac{6\tilde{\mu}_2}{D} \sqrt{(D-1)(D-2)} (\Lambda_{H}^{-1})_{23} . \tag{4.14}
\]

Now consider the contribution from \(V_2\). It is convenient to use (3.5)
\[
(\Lambda_{V_2}^{-1}) \sum_{i,j} a_{1, b_{1}, c_{1}, a_{2}, b_{2}, d} C_{a_{1}b_{1}c_{1}}^{H_H \rightarrow V_2} C_{a_{2}b_{2}c_{2}}^{H_H \rightarrow V_2} C_{d_{1}d_{2}}^{H_H \rightarrow V_2}
\]
\[
= (\Lambda_{V_2}^{-1}) \sum_{i,j} \sum_{a_{1}, b_{1}, c_{1}, a_{2}, b_{2}, d} C_{a_{1}b_{1}c_{1}} C_{a_{2}b_{2}c_{2}} C_{d_{1}d_{2}} (F_{H,H \rightarrow V_2})_{a_{1}b_{1};a_{2}b_{2}} .
\]

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\[ (\Lambda_{V_2}^{-1}) \sum_{a_1, b_1, a_2, b_2} C_{a_1 a_2 d} C_{b_1 b_2 d} (P^{H, H \rightarrow V_2})_{a_1 b_1; a_2 b_2} \]
\[ = (\Lambda_{V_2}^{-1}) \frac{D - 2}{D} \sum_{a_1, a_2, b_1, b_2} (P^{H, H \rightarrow H})_{a_1, a_2; b_1, b_2} (P^{H, H \rightarrow V_2})_{a_1 b_1; a_2 b_2} . \]  

(4.15)

In the last line, we have used the second equation in C.10 which gives the relation between the invariant \( C_{abc} \) in \( V_\text{nat}^\otimes 3 \) and the normalized Clebsch-Gordan Gordan coefficients \( C_{abc}^{H \rightarrow H} \), and the formula for the projector in terms of the Clebsch-Gordan coefficients.

The symmetric part of \( V_H \otimes V_H \), i.e. the subspace invariant under the swap of the two factors, decomposes into irreducible representations of the diagonal \( S_D \) action as \( V_0 \oplus V_H \oplus V_2 \). This means that

\[ p^{H, H \rightarrow V_2} = (1 - p^{H, H \rightarrow H} - p^{H, H \rightarrow V_0}) \frac{(1 + \tau)}{2} \]
\[ p^{H, H \rightarrow H} \frac{(1 + \tau)}{2} = p^{H, H \rightarrow H} . \]  

(4.16)

This means that

\[ (P^{H, H \rightarrow V_2})_{a_1 b_1; a_2 b_2} = \frac{1}{2}(\delta_{a_1 a_2} \delta_{b_1 b_2} + \delta_{a_1 b_2} \delta_{a_2 b_1}) - P^{H, H \rightarrow H}_{a_1, b_1; a_2, b_2} - P^{H, H \rightarrow V_0}_{a_1, b_1; a_2, b_2} . \]  

(4.17)

A useful fact following from (2.21) and (2.19) is

\[ \sum_a C_{aab} = 0 . \]  

(4.18)

When the expression (4.17) is substituted in (4.15) the first term on the RHS of (4.17) does not contribute because of (4.18). The second term gives

\[ (\Lambda_{V_2}^{-1}) \frac{D - 2}{D} \sum_{a_1, a_2, b_1, b_2} (P^{H, H \rightarrow H})_{a_1, a_2; a_2, a_1} = (\Lambda_{V_2}^{-1}) \frac{(D - 1)(D - 2)}{2D} . \]  

(4.19)

The third term gives

\[ -(\Lambda_{V_2}^{-1}) \frac{D - 2}{D} \sum_{a_1, a_2, b_1, b_2} (P^{H, H \rightarrow H})_{a_1, a_2; b_1, b_2} (P^{H, H \rightarrow H})_{a_1 b_1; a_2 b_2} \]
\[ = -(\Lambda_{V_2}^{-1}) \frac{D}{(D - 2)} \sum_{a_1, a_2, b_1, b_2, c_1, c_2} C_{a_1 a_2, c_1} C_{b_1, b_2, c_1} C_{a_1, b_1, c_2} C_{a_2, b_2, c_2} \]
\[ = -(\Lambda_{V_2}^{-1}) \frac{D}{(D - 2)} \sum_{a_1, a_2, b_1, b_2, c_1, c_2} \sum_{i, j, p, q} F(i, p) F(i, q) F(j, p) F(j, q) F(p, q) F(p, q) \]
\[ = -(\Lambda_{V_2}^{-1}) \frac{D}{(D - 2)} \frac{(D - 1)(D - 2)(D - 3)}{D^2} . \]  

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\[ -\left( \Lambda_{V_2}^{-1} \right) \frac{(D - 1)(D - 3)}{D}. \] (4.20)

The fourth term gives
\[ -\left( \Lambda_{V_2}^{-1} \right) \frac{D - 2}{D} \sum_{a_1,a_2,b_1,b_2} \left( P_{H,H \to H} \right)_{a_1,a_2;b_1,b_2} \left( P_{H,H \to V_0} \right)_{a_1,b_1;a_2,b_2} \]
\[ -\left( \Lambda_{V_2}^{-1} \right) \frac{D - 2}{D} \frac{1}{(D - 1)} \sum_{a_1,a_2,b_1,b_2} \left( P_{H,H \to H} \right)_{a_1,a_2;b_1,b_2} \delta_{a_1b_1} \delta_{a_2b_2} \]
\[ = -\left( \Lambda_{V_2}^{-1} \right) \frac{D - 2}{D}. \] (4.21)

Collecting terms from (4.19), (4.20) and (4.21) we get
\[ \left( \Lambda_{V_2}^{-1} \right) \frac{(3 - D)}{2}. \] (4.22)

Multiplying the factor \( \frac{3\tilde{\mu}_2}{\sqrt{D - 1}} \) from (4.7) to get a contribution to \( \sum_{i,j} \left( M_{ij}^3 \right) \), we get
\[ \frac{3\tilde{\mu}_2}{\sqrt{D - 1}} \left( \Lambda_{V_2}^{-1} \right) \frac{(3 - D)}{2} = 3\tilde{\mu}_2 \Lambda_{V_2}^{-1} \frac{3(3 - D)}{2\sqrt{D - 1}}. \] (4.23)

The contribution from \( V_3 \) is
\[ \frac{3\tilde{\mu}_2}{\sqrt{D - 1}} \left( \Lambda_{V_3}^{-1} \right) \frac{D - 2}{D} \sum_{a_1,a_2,b_1,b_2} \left( P_{H,H \to H} \right)_{a_1,a_2;b_1,b_2} \left( P_{H,H \to V_3} \right)_{a_1,b_1;a_2,b_2}. \] (4.24)

Now use the fact that
\[ \left( P_{H,H \to V_3} \right)_{a_1,b_1;a_2,b_2} = \frac{1}{2} \left( \delta_{a_1,a_2} \delta_{b_1,b_2} - \delta_{a_1,b_2} \delta_{a_2,b_1} \right) \] (4.25)

along with
\[ \sum_{a,b} \left( P_{H,H \to H} \right)_{aa;bb} = 0 \]
\[ \sum_{a_1,a_2} \left( P_{H,H \to H} \right)_{a_1,a_2;a_2,a_1} = \sum_{a_1,a_2} \left( P_{H,H \to H} \right)_{a_1,a_2;a_1,a_2} = (D - 1) \] (4.26)
to find
\[ -\left( \Lambda_{V_3}^{-1} \right) \frac{3\tilde{\mu}_2}{\sqrt{D - 1}} \frac{(D - 2)}{D} (D - 1) = -3\tilde{\mu}_2 \left( \Lambda_{V_3}^{-1} \right) \frac{(D - 2)\sqrt{D - 1}}{2D}. \] (4.27)

Collecting all the contributions we have
\[ \sum_{i,j} \left( M_{ij}^3 \right) = \frac{\tilde{\mu}_3^2}{D} + \frac{3}{D} \tilde{\mu}_1 \tilde{\mu}_2^2 + \frac{(D - 2)}{D\sqrt{D - 1}} \tilde{\mu}_2^3 + \frac{(D - 2)}{D\sqrt{D - 1}} \tilde{\mu}_2^3 \]
\[+ \frac{3\bar{\mu}_1}{D} \left( (\Lambda_{V_0}^{-1})_{11} + (\Lambda_{V_0}^{-1})_{22} + (D-1)(\Lambda_{H}^{-1})_{22} + (D-1)(\Lambda_{H}^{-1})_{33} + (D-1)(\Lambda_{V_0}^{-1})_{11}\right) + \frac{D(D-3)}{2} (\Lambda_{V_2})^{-1} + \frac{(D-1)(D-2)}{2} (\Lambda_{V_3})^{-1}\] 

\[+ 3\bar{\mu}_2 (\Lambda_{V_0}^{-1})_{22} \frac{D-2}{D\sqrt{D-1}} + \frac{6\bar{\mu}_2 (\Lambda_{V_0}^{-1})_{12}}{D} + 3\bar{\mu}_2 (\Lambda_{H}^{-1})_{33} \sqrt{D-1} \frac{(D-3)}{D} + 6\bar{\mu}_2 (\Lambda_{H}^{-1})_{12} \sqrt{(D-1)(D-2)} + 6\bar{\mu}_2 (\Lambda_{V_0}^{-1})_{13} \frac{D}{D} + \bar{\mu}_2 \Lambda_{V_2}^{-1} \frac{3(3-D)}{2\sqrt{D-1}} - 3 \bar{\mu}_2 (\Lambda_{V_3}^{-1}) (D-2) \sqrt{D-1} \frac{2D}{D}.\]

(4.28)

### 4.3 A 3-node case \(\sum_{i,j,k} \langle M_{ij}M_{jk}M_{ki} \rangle\)

Using Wick’s theorem (A.6)

\[\sum_{i,j,k} \langle M_{ij}M_{jk}M_{ki} \rangle = \sum_{i,j,k} \langle M_{ij} \rangle \langle M_{jk} \rangle \langle M_{ki} \rangle + \langle M_{ij}M_{jk} \rangle \langle M_{ki} \rangle + \langle M_{ij}M_{ki} \rangle \langle M_{jk} \rangle + \langle M_{jk}M_{ki} \rangle \langle M_{ij} \rangle.\]

(4.29)

The first term is

\[\sum_{i,j,k} \left( \frac{\bar{\mu}_1}{D} + \frac{\bar{\mu}_2}{\sqrt{D-1}} F(i, j) \right) \left( \frac{\bar{\mu}_1}{D} + \frac{\bar{\mu}_2}{\sqrt{D-1}} F(j, k) \right) \left( \frac{\bar{\mu}_1}{D} + \frac{\bar{\mu}_2}{\sqrt{D-1}} F(k, i) \right).\]

(4.30)

Using

\[\sum_{j} F(i, j) F(j, k) = F(i, k)\]

(4.31)

along with the first and second of (4.5) we can show that

\[\sum_{i,j,k} \langle M_{ij} \rangle \langle M_{jk} \rangle \langle M_{ki} \rangle = \bar{\mu}_1^3 + \frac{\bar{\mu}_2^3}{\sqrt{D-1}}.\]

(4.32)

Consider the remaining three terms. Focus on the first of these :

\[\sum_{i,j,k} \langle M_{ij}M_{jk} \rangle_{\text{conn}} \langle M_{ki} \rangle\]

\[= \sum_{i,j,k} \langle M_{ij}M_{jk} \rangle_{\text{conn}} \left( \frac{\bar{\mu}_1}{D} + \frac{\bar{\mu}_2}{\sqrt{D-1}} F(k, i) \right)\]

\[= \frac{\bar{\mu}_1}{D} \sum_{i,j,k} \langle M_{ij}M_{jk} \rangle_{\text{conn}} + \frac{\bar{\mu}_2}{\sqrt{D-1}} \sum_{i,j,k} \langle M_{ij}M_{jk} \rangle F(k, i).\]

(4.33)
We already know the first term from (3.51). So let us consider the second. An easy calculation using (3.5) (or equivalently using (3.6)) shows that the contribution from the $V_0$ channel is

$$
\frac{\tilde{\mu}_2}{\sqrt{D-1}} (\Lambda^{-1}_{V_0})_{22}.
$$

From the $V_H$ channel, the contributions are

$$
\tilde{\mu}_2 \sqrt{D-1} (\Lambda^{-1}_{H})_{33} + \tilde{\mu}_2 \sqrt{D-1} (\Lambda^{-1}_{H})_{12}.
$$

From the $V_2$ channel, we get

$$
\tilde{\mu}_2 \Lambda^{-1}_{V_2} \frac{D(D-3)}{2\sqrt{D-1}} = \frac{3}{2} \tilde{\mu}_2 \Lambda^{-1}_{V_2} - \frac{3}{2} \tilde{\mu}_2 (\Lambda^{-1}_{V_2})_{12}.
$$

Collecting terms

$$
\sum_{i,j,k} \langle M_{ij} M_{jk} \rangle_{\text{conn}} \langle M_{ki} \rangle = \sum_{i,j,k} \langle M_{ij} M_{ki} \rangle_{\text{conn}} \langle M_{jk} \rangle = \sum_{i,j,k} \langle M_{jk} M_{ki} \rangle_{\text{conn}} \langle M_{ij} \rangle
$$

Hence, we have

$$
\sum_{i,j,k} \langle M_{ij} M_{jk} M_{ki} \rangle = \tilde{\mu}_1^3 + \frac{\tilde{\mu}_1^3}{\sqrt{D-1}} + 3 \tilde{\mu}_1 \left( (\Lambda^{-1}_{V_0})_{11} + (D-1)(\Lambda^{-1}_{H})_{12} \right)
$$

$$
+ \frac{3 \tilde{\mu}_1}{\sqrt{D-1}} (\Lambda^{-1}_{V_0})_{22} + 3 \tilde{\mu}_2 \sqrt{D-1} (\Lambda^{-1}_{H})_{33} + 3 \tilde{\mu}_2 \sqrt{D-1} (\Lambda^{-1}_{H})_{12}
$$

$$
+ 3 \tilde{\mu}_2 \Lambda^{-1}_{V_2} \frac{D(D-3)}{2\sqrt{D-1}} - 3 \tilde{\mu}_2 (\Lambda^{-1}_{V_2})_{12}.
$$

4.4 A 6-node case $\sum_{i_1, \ldots, i_6} \langle M_{i_1 i_2} M_{i_3 i_4} M_{i_5 i_6} \rangle$

The sums over $i_1, \ldots, i_6$ project to the $V_0$ representations. As a result, using (A.6), along with (2.59), we have

$$
\sum_{i_1, \ldots, i_6} \langle M_{i_1 i_2} M_{i_3 i_4} M_{i_5 i_6} \rangle = \frac{3}{D^3} \sum_{i_1, \ldots, i_6} \langle S^{V_0;1} S^{V_0;1} \rangle \langle S^{V_0;1} \rangle + \frac{1}{D^3} \sum_{i_1, \ldots, i_6} \langle (S^{V_0;1})^3 \rangle
$$

$$
= 3 \tilde{\mu}_1 D^3 (\Lambda^{-1}_{V_0})_{11} + \tilde{\mu}_1^3 D^3.
$$
5 A selection of quartic expectation values

The methods we have used to calculate the cubic expectation values, which were explained in detail above, extend straightforwardly to quartic expectation values. The first step is to use Wick’s theorem. Then we use the formulae for quadratic and linear expectation values from Sections 2 and 3. In order to arrive at the final result as a function of $D$, $\bar{\mu}_i$, $\tilde{\bar{\mu}}_q$, $\Lambda_{V_0}$, $\Lambda_H$, $\Lambda_{V_2}$, $\Lambda_{V_3}$ we have to do certain sums over products of the natural-to-Hook projector $F(i,j)$. We will give some formulae below to illustrate these steps for the quartic case, without producing detailed formulae as in previous sections.

5.1 A 2-node quartic expectation value $\sum_{i,j} \langle M_{ij}^4 \rangle$

$$\sum_{i,j} \langle M_{ij}^4 \rangle = \sum_{i,j} \langle M_{ij} \rangle^4 + 6 \sum_{i,j} \langle M_{ij}^2 \rangle \langle M_{ij} \rangle^2 + 3 \sum_{i,j} \langle M_{ij}^2 \rangle \langle M_{ij}^2 \rangle \langle M_{ij} \rangle.$$  \hspace{1cm} (5.1)

The quadratic average is

$$\langle M_{ij}^2 \rangle_{\text{conn}} = \frac{(\Lambda_{V_0}^{-1})_{11}}{D^2} + \frac{(\Lambda_{V_0}^{-1})_{22}}{(D-1)^2} F(i,j)^2 + \frac{2(\Lambda_{V_0}^{-1})_{12}}{D\sqrt{D-1}} F(i,j)$$

$$+ \frac{(\Lambda_{H}^{-1})_{11}}{D} F(j,j) + \frac{(\Lambda_{H}^{-1})_{22}}{D} F(i,i) + \frac{(\Lambda_{H}^{-1})_{22}}{D-2} \sum_{p,q} F(i,p)F(j,q)F(i,q)F(j,p)F(p,q)$$

$$+ \frac{2(\Lambda_{H}^{-1})_{12}}{D} F(i,j) + \frac{2(\Lambda_{H}^{-1})_{12}}{D-2} \sum_{p} F(i,p)F(j,p)F(j,p)$$

$$+ \frac{2(\Lambda_{H}^{-1})_{22}}{\sqrt{D-2}} \sum_{p} F(i,p)F(j,p)F(j,p)$$

$$+ (\Lambda_{V_2})^{-1} \left( \frac{1}{2} F(i,i)F(j,j) + \frac{1}{2} F(i,j)F(i,j) - \frac{D}{D-2} \sum_{p,q} F(i,p)F(i,q)F(j,p)F(j,q)F(p,q) \right)$$

$$+ \frac{(\Lambda_{V_3}^{-1})}{2} \left( \frac{1}{2} F(i,i)F(j,j) - F(i,j)F(i,j) \right). \hspace{1cm} (5.2)$$

Using this and

$$\langle M_{ij} \rangle = \left( \frac{\bar{\mu}_i}{D} + \frac{\tilde{\bar{\mu}}_q}{\sqrt{D-1}} F(i,j) \right). \hspace{1cm} (5.3)$$

we can work out the formula for $\sum_{i,j} \langle M_{ij}^4 \rangle$ as a function of the 13 Gaussian model parameters. Mathematica would be handy in doing the sums over products of $F(i,j)$ which arise.
5.2 A 5-node quartic expectation value \( \sum_{i,j,k,p,q} \langle M_{ij} M_{jk} M_{kp} M_{pq} \rangle \)

From A.7 we have

\[
\sum_{i,j,k,p,q} \langle M_{ij} M_{jk} M_{kp} M_{pq} \rangle = \sum_{i,j,k,p,q} \langle M_{ij} \rangle \langle M_{jk} \rangle \langle M_{kp} \rangle \langle M_{pq} \rangle \\
+ \sum_{i,j,k,p,q} \langle M_{ij} M_{jk} \rangle \langle M_{kp} \rangle \langle M_{pq} \rangle \\
+ \sum_{i,j,k,p,q} \langle M_{jk} M_{kp} \rangle \langle M_{ij} \rangle \langle M_{pq} \rangle \\
+ \sum_{i,j,k,p,q} \langle M_{ij} M_{jk} \rangle \langle M_{kp} \rangle \langle M_{pq} \rangle \\
+ \sum_{i,j,k,p,q} \langle M_{jk} M_{kp} \rangle \langle M_{ij} \rangle \langle M_{pq} \rangle \\
+ \sum_{i,j,k,p,q} \langle M_{ij} M_{jk} \rangle \langle M_{kp} \rangle \langle M_{pq} \rangle \
\]

All the summands on the RHS can be evaluated using 3.5 or 3.6 in terms of \( F(i,j) \).

The sums can be done with the help of Mathematica to obtain expressions in terms of \( D, \tilde{\mu}_1, \tilde{\mu}_2, \Lambda_V \).

6 Summary and Outlook

We have used the representation theory of symmetric groups \( S_D \) in order to define a 13-parameter permutation invariant Gaussian matrix model, to compute the expectation values of all the graph-basis permutation invariant quadratic functions of the random matrix, and a selection of cubic and quartic invariants. In [8] analogous computations with a 5-parameter model were compared with matrix data constructed from a corpus of the English language. A natural direction is to extend that discussion of the English language, or indeed other languages, to the present 13-parameter model. Combining the experimental methods employed in [8] with machine learning methods such those used in [19], in the investigation of the 13-parameter model, would also be interesting to explore.

As a theoretical extension of the present work, it will be useful to generalise the representation theoretic parametrisation of the Gaussian models to perturbations of the Gaussian model, where we add cubic and quartic terms to the Gaussian action. Identifying parameter spaces of these deformations which allow well-defined convergent partition functions and expectation values will be useful for eventual comparison to data. If we ignore the convergence constraints, the general perturbed model at cubic and quartic order has 348 parameters, since there are 52 cubic invariants and 296 quartic invariants (Appendix A of [8]). As in the Gaussian case, we can expect that representation theory methods will be useful in handling this more general problem. Further techniques involving partition algebras underlying the representation theory of tensor products of the natural representation will likely play a role (see e.g. [20] for recent work in these directions).

It is worth noting that permutation invariant random matrix distributions have been approached from a different perspective, based on non-commutative probability theory
The approach of the present paper and [8] is based on the connection between statistical physics and zero dimensional quantum field theory (QFT). It would seem that the approach of the present paper can complement the theory developed in these papers [21] [22] [23] by producing integral representations (Gaussians or perturbed Gaussians) of random matrix distributions having finite expectation values for permutation invariant polynomial functions of matrices. The results on the central limit theorem from the above references would be very interesting to interpret from the present QFT perspective.

The computation of expectation values in Gaussian matrix models admits generalization to higher tensors. Indeed the motivating framework in computational linguistics discussed in [8] involves matrices as well as higher tensors in a natural way. Generalizations of the present work on representation theoretic parametrization of Gaussian models and computation of graph-basis observables to the tensor case is an interesting avenue for future research.

In this paper, we have focused on the explicit computation of permutation invariant correlators for general \( D \). Some simplifications at large \( D \) were discussed in section 3.12. For traditional matrix models having \( U(D) \) (or \( SO(D)/Sp(D) \) symmetries), there is a rich geometry of two dimensional surfaces and maps in the large \( D \) expansions which allows these expansions of matrix quantum field theories to have deep connections to string theory [28] [29]. It will be interesting to explore the possibility of two dimensional geometrical interpretations of the large \( D \) expansion in permutation invariant matrix models.

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A Multi-dimensional Gaussian Integrals and Wick’s theorem

Consider the multi-variable integral with a Gaussian integrand
\[
Z = \int dx \exp \left( -\frac{1}{2} \sum_{i,j=1}^{N} x_i A_{ij} x_j + \sum_i s_i x_i \right) = \sqrt{\frac{(2\pi)^N}{\det A}} \exp \left( \frac{1}{2} s_i (A^{-1})_{ij} s_j \right). \tag{A.1}
\]

\(x \in \mathbb{R}^N\). \(A \in \mathbb{C}^{N \times N}\) is a real symmetric positive definite matrix. \(s \in \mathbb{R}^N\) is an arbitrary complex vector (see for example \cite{17}, \cite{18}, Appendix A, Equations (8) and (9) of \cite{26}). One can also consider \(A\) more generally to be complex with positive definite real part, but to keep a probabilistic interpretation we keep \(A\) real symmetric. Expectation values of functions \(f(x)\) are defined by
\[
\langle f(x) \rangle = \frac{1}{Z} \int dx \ f(x) \exp \left( -\frac{1}{2} \sum_{i,j=1}^{N} x_i A_{ij} x_j + \sum_i s_i x_i \right). \tag{A.2}
\]

These expectation values can be calculated by taking derivatives with respect to \(s_i\) on both sides of (A.1). For the \(x\) variables
\[
\langle x_i \rangle = \sum_j (A^{-1})_{ij} s_j = \sum_j s_j (A^{-1})_{ji}. \tag{A.3}
\]

Application of this equation, along with the formula for \(dM\) in terms of the representation theoretic \(S\)-variables \((2.67)\) leads to \((2.74),(2.76)\). For expectation values of quadratic monomials we have
\[
\langle x_i x_j \rangle = (A^{-1})_{ij} + \langle x_i \rangle \langle x_j \rangle. \tag{A.4}
\]

We define the connected part as
\[
\langle x_i x_j \rangle_{\text{conn}} \equiv \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle = (A^{-1})_{ij}. \tag{A.5}
\]

The expressions \((2.77)\) and \((2.4)\) follow from these.

For cubic expressions
\[
\langle x_i x_j x_k \rangle = \langle x_i x_j \rangle_{\text{conn}} \langle x_k \rangle + \langle x_i x_k \rangle_{\text{conn}} \langle x_j \rangle + \langle x_j x_k \rangle_{\text{conn}} \langle x_i \rangle + \langle x_i \rangle \langle x_j \rangle \langle x_k \rangle. \tag{A.6}
\]

For quartic expressions
\[
\langle x_i x_j x_k x_l \rangle = \langle x_i x_j \rangle_{\text{conn}} \langle x_k x_l \rangle_{\text{conn}} + \langle x_i x_k \rangle_{\text{conn}} \langle x_j x_l \rangle_{\text{conn}} + \langle x_j x_k \rangle_{\text{conn}} \langle x_i x_l \rangle_{\text{conn}} + \langle x_i x_l \rangle_{\text{conn}} \langle x_j x_k \rangle_{\text{conn}} + \langle x_i x_k \rangle_{\text{conn}} \langle x_j \rangle \langle x_l \rangle + \langle x_j x_k \rangle_{\text{conn}} \langle x_i \rangle \langle x_l \rangle + \langle x_j x_l \rangle_{\text{conn}} \langle x_i \rangle \langle x_k \rangle + \langle x_k x_l \rangle_{\text{conn}} \langle x_i \rangle \langle x_j \rangle.
\]
\[ + \langle x_i \rangle \langle x_j \rangle \langle x_k \rangle \langle x_l \rangle . \tag{A.7} \]

These illustrate a general fact (known as Wick’s theorem in the quantum field theory context and Isserlis’ theorem in probability theory [27]) about Gaussian expectation values. Higher order expectation values can be expressed in terms of linear and quadratic expectation values. When applied to permutation invariant matrix models, we still have non-trivial sums left to do, after Wick’s theorem has been applied. This is illustrated in the calculations of section 4 and section 5.

B Rep theory of \( V_H \) and its tensor products

Some basics of rep theory of \( V_H \) can be presented in a self-contained way, assuming only knowledge of linear algebra and index notation.

Alternatively, we can observe that the matrices in \( V_H \) are the same as Young’s orthogonal basis. If we just follow the self-contained route, we define

\[
D^H_{ab}(\sigma) = (E_a, \sigma E_b) = \sum_i \sum_j C_{a,i} C_{b,j} \langle e_i, e_j \rangle = \sum_{i,j} C_{a,i} C_{b,j} \langle e_i, e_{\sigma^{-1}(j)} \rangle = \sum_{i,j} C_{a,i} C_{b,j} \delta_{i,\sigma^{-1}(j)} = \sum_i C_{a,i} C_{b,\sigma(i)} . \tag{B.1} \]

We have the orthogonality property:

\[
D^H_{ab}(\sigma^{-1}) = \sum_i C_{a,i} C_{b,\sigma^{-1}(i)} = \sum_i C_{a,\sigma(i)} C_{b,i} = D^H_{ba}(\sigma) . \tag{B.2} \]

The homomorphism property

\[
D^H_{ab}(\sigma) D^H_{bc}(\tau) = \sum_i C_{a,i} C_{b,\sigma(i)} \sum_j C_{b,j} C_{c,\tau(j)} = \sum_{i,j} C_{a,i} C_{b,j} C_{b,\sigma(i)} C_{c,\tau(j)} = \sum_{i,j} C_{a,i} C_{b,j} C_{b,\sigma(i)} C_{c,\tau(j)} \]

45
\[\sum_{i,j} C_{a,i} \left( \delta_{j,\sigma(i)} - \frac{1}{D} \right) C_{c,\tau(j)} = \sum_{i,j} C_{a,\sigma^{-1}(j)} C_{c,\tau(j)} - \frac{1}{D} \sum_{i,j} C_{a,i} C_{c,\tau(j)} = \sum_{j} C_{a,j} C_{c,\tau(\sigma(j))} = \sum_{j} C_{a,j} C_{c,\sigma\tau(j)} = D_{ac}^{H}(\sigma \tau). \] 

We used
\[\sum_{a} C_{a,a} = \left( \delta_{ij} - \frac{1}{D} \right), \quad \sum_{a} C_{a,i} = 0. \] 

Using the definition (B.1), we prove
\[\sum_{b} D_{ba}^{H}(\sigma) C_{b,i} = C_{a,\sigma(i)}. \] 

Indeed
\[\sum_{b} D_{ba}^{H}(\sigma) C_{b,i} = \sum_{b,j} C_{b,j} C_{a,\sigma(j)} C_{b,i} = \sum_{j} C_{a,\sigma(j)} (\delta_{i,j} - \frac{1}{D}) = C_{a,\sigma(i)}. \] 

It is useful to define \( C_{\sigma(a),i} = \sum_{b} D_{ba}^{H}(\sigma) C_{b,i} \) so the above can be expressed as an equivariance property
\[C_{\sigma(a),i} = C_{a,\sigma(i)}. \] 

which is an equivariance condition for the map \( V_{H} \to V_{\text{nat}} \) given by the coefficients \( C_{a,i} \).

This map intertwines the \( S_{n} \) action on the \( V_{H} \) and \( V_{\text{nat}} \). Now define \( C_{a,b,c} \)
\[C_{a,b,c} = \sum_{i} C_{a,i} C_{b,i} C_{c,i}. \] 

We show that this is an invariant tensor
\[C_{\sigma(a),\sigma(b),\sigma(c)} = C_{a,b,c}. \]
Indeed

\[ C_{\sigma(a),\sigma(b),\sigma(c)} = \sum_{i=1}^{n} C_{\sigma(a),i} C_{\sigma(b),i} C_{\sigma(c),i} \]

\[ = \sum_i C_{a,\sigma(i)} C_{b,\sigma(i)} C_{c,\sigma(i)} \]

\[ = \sum_i C_{a,i} C_{b,i} C_{c,i} \]

\[ = C_{a,b,c} . \quad (B.10) \]

We used the equivariance of the \( C \)'s, then the relabelled the sum \( i \to \sigma(i) \).

Using vectors \( \{e_a\} \) spanning \( V_H \), we write a basis for the tensor product \( V_H \otimes V_H \):

\[ e_a \otimes e_b . \quad (B.11) \]

There is a subspace of \( V_H \otimes V_H \), which transforms as the irrep \( V_H \). This is constructed using the invariant 3-index tensor \( C_{a,b,c} \). The linear combinations

\[ E_a = \sum_{a,b,c} C_{a,b,c} e_b \otimes e_c \]

span the subspace \( V_H \) in the direct sum decomposition of \( V_H \otimes V_H \) (Equation (2.27)) under the diagonal action of \( S_D \). To see this, we can write the diagonal action of \( \sigma \in S_D \)

\[ \sigma E_a = \sum_{a,b,c} C_{a,b,c}(\sigma e_b) \otimes (\sigma e_c) \]

\[ = \sum_{a,b,c} \sum_{b',c'} C_{a,b,c} D_{b'b}^{H}(\sigma) D_{c'c}^{H}(\sigma)(e_{b'} \otimes e_{c'}) \]

\[ = \sum_{a,b,c} \sum_{d,b',c'} C_{d,b,c} D_{d'c}^{H}(\sigma^{-1}) D_{b'b}(\sigma) D_{c'c}^{H}(\sigma)(e_{b'} \otimes e_{c'}) \]

\[ = \sum_{a,b,c} \sum_{d,b',c'} C_{d,b,c} D_{d'c}^{H}(\sigma^{-1}) D_{b'b}(\sigma) D_{c'c}^{H}(\sigma)(e_{b'} \otimes e_{c'}) \]

\[ = \sum_{a,a',b',c'} C_{\sigma^{-1}(a'),\sigma^{-1}(b'),\sigma^{-1}(c')} D_{a'a}^{H}(\sigma)(e_{b'} \otimes e_{c'}) \]

\[ = \sum_{a'} D_{a'a}^{H}(\sigma) \sum_{a,b,c} C_{a',b',c'}(e_{b'} \otimes e_{c'}) \]

\[ = \sum_{a'} D_{a'a}^{H}(\sigma) E_{a'} . \quad (B.13) \]

showing that the transformation is indeed by the matrix \( D^H \).

These vectors \( E_a \) are orthogonal. It is useful to calculate the inner product

\[ (E_{a_1}, E_{a_2}) = \sum_{b,c} C_{a_1,b,c} C_{a_2,b,c} \]

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\[
= \sum_{i,j} \sum_{b,c} C_{a_1,i} C_{b,i} C_{a_2,j} C_{b,j} C_{c,j}
= \sum_{i,j} C_{a_1,i} C_{a_2,j} \left( \delta_{i,j} - \frac{1}{D} \right) \left( \delta_{i,j} - \frac{1}{D} \right)
= (1 - \frac{2}{D}) \sum_i C_{a_1,i} C_{a_2,i}
= \frac{(D - 2)}{D} \delta_{a_1,a_2}.
\] (B.14)

which will be useful in the next section.

C Clebsch-Gordan coefficients and normalizations

The normalized Clebsch-Gordan coefficients for an orthonormal basis of a subspace of \( V_H \otimes V_H \) transforming as an irrep \( V \) obey the condition
\[
\sum_{a,b,c} C^{H,H \rightarrow V}_{a,b,c} C^{H,H \rightarrow V}_{a,b,c'} = \delta_{cc'}.
\] (C.1)

This means that
\[
\sum_{a,b,c} C^{H,H \rightarrow V}_{a,b,c} C^{H,H \rightarrow V}_{a,b,c} = \text{Dim } V.
\] (C.2)

If instead we consider the invariant state in \( H \otimes H \otimes V \), normalized to one, then
\[
\sum_{a,b,c} \sum_{a',b',c'} (C^{H,H \rightarrow V}_{a,b,c} e_a \otimes e_b \otimes e_c, C^{H,H \rightarrow V}_{a',b',c'} e_{a'} \otimes e_{b'} \otimes e_{c'}) = \sum_{a,b,c} (C^{H,H \rightarrow V}_{a,b,c})^2 = 1.
\] (C.3)

The equivariance property of the map \( C^{H,H \rightarrow V} \) is
\[
D^H \otimes H(\sigma \otimes \sigma) \sum_{a,b} C^{H,H \rightarrow V}_{a,b,c} e_a \otimes e_b = \sum_{c' \sigma'} D^{H}_{a' a}(\sigma) D^{H}_{b' b}(\sigma) C^{H,H \rightarrow V}_{a,b,c} e_{a'} \otimes e_{b'}
= \sum_{c'} D^{H}_{c' c}(\sigma) \sum_{a',b'} C^{H,H \rightarrow V}_{a',b',c'} e_{a'} \otimes e_{b'}.
\] (C.4)

which means
\[
\sum_{c'} D^{H}_{a' a}(\sigma) D^{H}_{b' b}(\sigma) C^{H,H \rightarrow V}_{a,b,c} = D^{H}_{c' c}(\sigma) C^{H,H \rightarrow V}_{a,b,c'}.
\] (C.5)

Multiplying on both sides by \( D^{H}_{c c'}(\sigma^{-1}) \) and summing over \( c \), we have, after using \( D^{H}_{a b}(\sigma^{-1}) = D^{H}_{b a}(\sigma) \) and relabelling indices
\[
\sum_{c'} D^{H}_{a' a}(\sigma) D^{H}_{b' b}(\sigma) D^{H}_{c c}(\sigma) C^{H,H \rightarrow V}_{a,b,c} = C^{H,H \rightarrow V}_{a,b,c}.
\] (C.6)
This means that we can identify

\[ C_{a,b,c}^{H,H \rightarrow V} = \sqrt{\text{Dim} V} \, C_{a,b,c}^{H,H,V}. \]  

We also know that

\[ C_{a,b,c} = \sum_i C_{a,i} C_{b,i} C_{c,i}. \]  

has the invariance property of \( C_{a,b,c}^{H,H,H} \). Since there is a unique invariant state in \( V_H \otimes V_H \otimes V_H \), \( C_{a,b,c} \) must be proportional to \( C_{a,b,c}^{H,H,H} \). We calculate

\[ \sum_{a,b,c} C_{a,b,c} C_{a,b,c} = \sum_{i,j} \sum_{a,b,c} C_{a,i} C_{b,i} C_{c,i} C_{a,j} C_{b,j} C_{c,j} \]
\[ = \sum_{i,j} (\delta_{ij} - \frac{1}{D})(\delta_{ij} - \frac{1}{D})(\delta_{ij} - \frac{1}{D}) \]
\[ = 1 \times D - \frac{3}{D} \times D + \frac{3}{D^2} \times D - \frac{1}{D^3} \times D^2 \]
\[ = D - 3 + 2D^{-1} = D^{-1}(D - 1)(D - 2). \]  

We can therefore identify

\[ C_{a,b,c}^{H,H,H} = \sqrt{\frac{D}{(D - 1)(D - 2)}} C_{a,b,c}, \]
\[ C_{a,b,c}^{H,H \rightarrow H} = \sqrt{\frac{D}{(D - 2)}} C_{a,b,c}. \]  

The second equation is also consistent, as expected, with (B.14). The projector for \( V_H \) in \( V_H \otimes V_H \) is given in terms of the normalized Clebsch-Gordan coefficients \( C_{a,b,c}^{H,H \rightarrow H} \) as

\[ P_{abcd}^{H,H \rightarrow H} = \sum_d C_{a,b,c}^{H,H \rightarrow H} C_{c,d,e}^{H,H \rightarrow H}. \]  

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