SYMMETRIC GROUP CHARACTERS AS SYMMETRIC FUNCTIONS
(EXTENDED ABSTRACT)

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Abstract. The irreducible characters of the symmetric group are a symmetric polynomial in the
eigenvalues of a permutation matrix. They can therefore be realized as a symmetric function that
can be evaluated at a set of variables and form a basis of the symmetric functions. This basis of the
symmetric functions is of non-homogeneous degree and the (outer) product structure coefficients are
the stable Kronecker coefficients.

We introduce the irreducible character basis by defining it in terms of the induced trivial char-
tacters of the symmetric group which also form a basis of the symmetric functions. The irreducible
character basis is closely related to character polynomials and we obtain some of the change of basis
coefficients by making this connection explicit. Other change of basis coefficients come from a repre-
sentation theoretic connection with the partition algebra, and still others are derived by developing
combinatorial expressions.

This document is an extended abstract which can be used as a review reference so that this basis
can be implemented in Sage. A more full version of the results in this abstract can be found in
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1. Introduction

We begin with a very basic question in representation theory. Can the irreducible characters of the
symmetric group be realized as functions of the eigenvalues of the permutation matrices? In general,
characters of matrix groups are symmetric functions in the set of eigenvalues of the matrix and (as
we will show) the symmetric group characters define a basis of inhomogeneous degree for the space of
symmetric functions.

To arrive at an answer to this question and its role in representation theory we begin in the context
of Schur-Weyl duality. Consider an $n$-dimensional vector space $V = \{v_1, v_2, \ldots, v_n\}$ for which $Gl_n$ has
an action on $V$ and also acts diagonally on the tensor space $T^k(V)$. The symmetric group $S_k$ acts by
permuting the positions of the tensor space and the symmetric group algebra is the centralizer algebra
of the $Gl_n$ action. Schur-Weyl duality states that $T^k(V)$ decomposes as a direct sum of irreducible
$S_k$ and $Gl_n$ bimodules

\[ T^k(V) \simeq \bigoplus_{\lambda \vdash k} S^\lambda \otimes M^\lambda \]

where the $S^\lambda$ is an irreducible $S_k$ module and $M^\lambda$ is an irreducible $Gl_n$ module. The Schur functions
arise in two different contexts. The first as the Frobenius image of the character of $S^\lambda$,

\[ s_\lambda = \frac{1}{k!} \sum_{\sigma \in S_k} \text{char}_{S^\lambda}(\sigma) \text{type}(\sigma), \]

and the second as characters

\[ s_\lambda(x_1, x_2, \ldots, x_n) = \text{char}_{M^\lambda}(A) \]

where $A$ is a matrix whose eigenvalues are $x_1, x_2, \ldots, x_n$.

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partition algebra.

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It is this second context in which we propose to consider the irreducible characters of the symmetric group $S_n$ which sits inside of $GL_n$ as permutation matrices. We wish to consider the question that we posed in the first paragraph where we change the group $GL_n$ with another group. In the case that the group $GL_n$ is replaced with $SO_{2n}$, $SO_{2n+1}$ or $Sp_{2n}$, formulas for the characters were given by Weyl [Weyl] and explicit expressions of the characters as symmetric functions were worked out by Koike and Terada [KT]. The centralizer algebras of the group action in these cases are the Brauer algebras.

It is the case when group $GL_n$ is replaced by the symmetric group $S_n$ of permutation matrices that we are considering here. The centralizer algebra of this action was determined by Jones and Martin [Jon, Ma1, Ma2, Ma3, Ma4] to be the partition algebra. Halverson and Ram [Hal, HalRam] made the characters of the partition algebra explicit and we see these character values arise in the coefficients of the change of bases. We have not however seen the symmetric group characters treated in a manner similar to the characters of the classical groups.

To this end, we propose the introduction of the irreducible symmetric group characters as a basis (or, for short, irreducible character basis), $\tilde{s}$, similar to the characters of the classical groups.

$s$ is independent of $\nu$ for any partition $\lambda\vdash n$. We call this second basis the natural second basis is suggested by the algebra. We have not however seen the symmetric group characters treated in a manner similar to the characters of the classical groups.

The basis of $\tilde{s}$, $\tilde{\nu}$ and $\tilde{\nu}$ are characters, their products are the characters of the tensors of the representations. The structure coefficients of the irreducible character basis will be the reduced Kronecker coefficients [BOR]. That is, for an $n$ sufficiently large and $\lambda, \mu$ partitions of $n$, for any partition $\nu$ let $\underline{\nu} = (\nu_2, \nu_3, \ldots, \nu_{\ell(\nu)})$, then

$$\tilde{s}_{\underline{\alpha}}\tilde{s}_{\underline{\beta}} = \sum_{\nu \vdash n} k_{\lambda \mu \nu} \tilde{s}_{\nu}$$

$$\tilde{h}_{\underline{\alpha}}\tilde{h}_{\underline{\beta}} = \sum_{\nu \vdash n} K_{\lambda \mu \nu} \tilde{h}_{\nu}$$

where for partitions $\alpha, \beta, \tau$ of size $n$, and

$$k_{\alpha \beta \tau} = \sum_{\gamma \vdash n} \chi^{\alpha}(\gamma)\chi^{\beta}(\gamma)\chi^{\tau}(\gamma)$$

and

$$K_{\alpha \beta \tau} = \sum_{\gamma \vdash n} \langle h_{\alpha}, p_{\gamma} \rangle \langle h_{\beta}, p_{\gamma} \rangle \langle h_{\tau}, p_{\gamma} \rangle$$

The $k_{\alpha \beta \tau}$ and $K_{\alpha \beta \tau}$ are the coefficients that appear in the internal product of the Schur and complete symmetric function bases.

In addition, due to the Schur-Weyl duality with the partition algebra, the change of basis coefficients between the power sum basis $p_\mu$ and the elements $\tilde{s}_\lambda$ are the irreducible characters of the partition algebra and our elements must satisfy the Murnghana-Nakayma rule for those characters (see [Hal]). In addition, the product $\tilde{s}_{\underline{\alpha}}\tilde{s}_{\underline{\beta}}$ corresponds to the restriction of characters of the partition algebra $BDO$.

A recent paper by Church and Farb [CF] introduces a notion of representation stability. They describe a number of families of representations whose decomposition into irreducible representations is independent of $n$, if $n$ is sufficiently large. The bases that we introduce in this paper are likely to be a useful tool in finding expressions for their character.

There are a number of other results that we are not adequately able to summarize in this abstract. Some of these include product rules which come from known cases of Kronecker coefficients, change of basis coefficients, coproducts and connections with other questions in representation theory.

We believe that the relationship of the representation theory, combinatorics and symmetric functions already merits study of the induced character and irreducible character bases, however we are...
aware that these bases will only be seen as an important innovation in the theory of symmetric functions if they find proper applications.

Since inner and outer plethysm and the inner (Kronecker) product of Schur functions are all classes of symmetric function expressions which do not currently have a simple combinatorial formula, and yet they all three have the a stability property that is related to removing the first row, we expect that this basis will prove a useful tool for progress in explaining these operations.

2. Notation

The objects that arise in the following constructions are familiar building blocks of combinatorics: set, multi-set, partition, set partition, multi-set partition, composition, weak composition, set composition, multi-set composition, tableau, words, etc. We will use this section to establish notation conventions and use of language that we will need in this paper.

For non-negative integers $n$ and $\ell$, a partition of size $n$ and length $\ell$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_{i+1}$ for $1 \leq i < \ell$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. The size of the partition is denoted $|\lambda| = n$ and the length of the partition is denoted $\ell(\lambda) = \ell$. We will often use the shorthand notation $\lambda \vdash n$ to indicate that $\lambda$ is a partition of $n$. The symbols $\lambda$ and $\mu$ will be reserved exclusively for partitions. Let $m_i(\lambda)$ represent the number of times that $i$ appears in the partition $\lambda$. It will be convenient to sometimes represent our partitions in exponential notation where $m_i = m_i(\lambda)$ and $\lambda = (1^{m_1} 2^{m_2} \cdots k^{m_k})$. With this notation the number of permutations with cycle structure $\lambda \vdash n$ is $\frac{n!}{z_\lambda}$ where

$$z_\lambda = \prod_{i=1}^{\ell} m_i(\lambda)!i^{m_i(\lambda)}.$$  

(6)

The most common operation we will use will be adding a part of size $n - |\lambda|$ to the beginning of a partition. This will be denoted $(n - |\lambda|, \lambda)$. If $n < |\lambda| + \lambda_1$, this sequence will no longer be an integer partition.

The cells of a partition $\lambda$ are the set of points $\{(i, j) : 1 \leq i < j \leq \ell(\lambda)\}$. We will represent these cells as stacks of boxes in the first quadrant (following ‘French notation’ for a partition). A tableau is a mapping from the set of cells to a set of labels and a tableau will be represented by filling the boxes of the diagram for a partition with the labels. In our case, we will encounter tableaux where only a subset of the cells are mapped to a label.

Multi-sets will also be represented by exponential notation so that $\{1^{a_1}, 2^{a_2}, \ldots, \ell^{a_\ell}\}$ represents the multi-set where $i$ occurs $a_i$ times.

A set partition of a set $S$ is a set of subsets $\{S_1, S_2, \ldots, S_\ell\}$ with $S_i \cap S_j = \emptyset$ for $1 \leq i < j \leq \ell$ and $S_1 \cup S_2 \cup \cdots \cup S_\ell = S$. A multi-set partition $\pi = \{S_1, S_2, \ldots, S_\ell\}$ of a multi-set $S$ is a similar construction to a set partition, but now $S_i$ may be a multi-set, and it is possible that two sets $S_i$ and $S_j$ have non-empty intersection (and may even be equal). Let $\ell(\pi) := \ell$ and $\hat{m}(\pi)$ represent the partition of $\ell(\pi)$ consisting of the the multiplicities of the multi-sets which occur in $\pi$ (e.g. $\hat{m}(\{(1, 1, 2), (1, 1, 2), (1, 3)\}) = (2, 1)$ because $\{(1, 1, 2)\}$ occurs 2 times and $\{(1, 3)\}$ occurs 1 time). We will use the notation $\pi \vdash S$ to indicate that $\pi$ is a multi-set partition of the multi-set $S$.

For non-negative integers $n$ and $\ell$, a composition is an ordered sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of integers $\alpha_i > 0$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$. A weak composition is such a sequence with the condition that $\alpha_i \geq 0$. To indicate that $\alpha$ is a composition of $n$ we will use the notation $\alpha \models n$ and to indicate that $\alpha$ is a weak composition of $n$ we will use the notation $\alpha \models_w n$. For both compositions and weak compositions, $\ell(\alpha) := \ell$.

The ring of symmetric functions (for some modern references on this subject see for example [Mac, Sagan, Stanley, Lascoux]) will be denoted $\operatorname{Sym} = \mathbb{Q}[p_1, p_2, p_3, \ldots]$ and has the fundamental bases (each indexed by the set of partitions $\lambda$) power sum $\{p_\lambda\}_\lambda$, homogeneous/complete $\{h_\lambda\}_\lambda$, elementary $\{e_\lambda\}_\lambda$, Schur $\{s_\lambda\}_\lambda$ and monomial $\{m_\lambda\}_\lambda$. We will also refer to the irreducible character
of the symmetric group indexed by the partition $\lambda$ and evaluated at a permutation of cycle structure $\mu$ as the coefficient $(s_\lambda, p_\mu) = \chi^\lambda(\mu)$.

For $k > 0$, define
\[ \Xi_k := 1, e^{2\pi i/k}, e^{4\pi i/k}, \ldots, e^{2(k-1)\pi i/k} \]
as a symbol representing the eigenvalues of a permutation matrix of a $k$-cycle. Then for any partition $\mu$, let
\[ \Xi_\mu := \Xi_{\mu_1}, \Xi_{\mu_2}, \ldots, \Xi_{\mu(\mu)} \]
be the multi-set of eigenvalues of a permutation matrix with cycle structure $\mu$. We will evaluate symmetric functions at these eigenvalues. The notation $f[\Xi_\mu]$ represents taking the element $f \in \text{Sym}$ and replacing $p_k$ in $f$ with $x_1^k + x_2^k + \cdots + x_{|\mu|}^k$ and then replacing the variables $x_i$ with the values in $\Xi_\mu$.

3. SYMMETRIC GROUP CHARACTER BASES OF THE SYMMETRIC FUNCTIONS

To begin, we establish the following result connecting the evaluation of the homogeneous symmetric functions at roots of unity $\Xi_\mu$ with the values of the trivial character induced from $S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu(\mu)}$ to $S_{|\mu|}$. This is given by the scalar product $\langle h_\mu h_{|\mu|-|\lambda|}, p_\mu \rangle$.

**Theorem 1.** For all partitions $\lambda$,
\[ h_\lambda[\Xi_\mu] = \sum_{\pi \vdash \lambda, \nu \vdash |\lambda|} \langle h_{\tilde{\pi}(\pi)} h_{|\nu|-|\lambda|}, p_\mu \rangle. \tag{7} \]

We may also extend this further by computing a similar expression for the elementary basis. While we don’t include this result here as the result is auxiliary to what we are presenting, the expansion involves combinatorial objects which are set partitions of a multi-set (as opposed to multi-set partitions of a multi-set) since in each set of the set partition, repeated elements are not allowed.

We are now ready to define symmetric functions whose evaluations at roots of unity (the eigenvalues of a permutation matrix) are the values of characters. We base our definition on equation (7).

**Definition 2.** Let $\tilde{h}_\mu$ be the family of symmetric functions which satisfies:
\[ h_\lambda = \sum_{\pi \vdash \lambda, \nu \vdash |\lambda|} \tilde{h}_{\tilde{\pi}(\pi)} \tag{8} \]

We call this basis the induced trivial character basis of the symmetric functions.

This is a recursive definition for calculating this basis directly since there is precisely one multi-set partition of $\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}$ such that $\tilde{m}(\pi)$ is of size $|\lambda|$, hence
\[ \tilde{h}_\lambda = h_\lambda - \sum_{\nu \vdash |\lambda|, \tilde{m}(\pi) \neq \lambda} \tilde{h}_{\tilde{\pi}(\pi)}. \tag{9} \]

Now by equation (7) and an induction argument, we can conclude that for all partitions $\mu$,
\[ \tilde{h}_\lambda[\Xi_\mu] = \langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \rangle \tag{10} \]
and this is the value of the character of the trivial module which is induced from $S_{|\mu|-|\lambda|} \times S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda(\mu)}$ to the full symmetric group $S_{|\mu|}$.

**Example 3.** We list all of the multi-set partitions of $\{1, 1, 1, 2\}$. They are
\[
\{\{1\}, \{1\}, \{1\}, \{1\}\}, \{\{1\}, \{1\}, \{1, 2\}\}, \{\{1\}, \{1\}, \{1\}, \{2\}\}, \{\{1\}, \{1, 1\}, \{2\}\}, \{\{1, 1\}, \{1, 2\}\}, \{\{1, 1, 1\}, \{2\}\}, \{\{1, 1, 1\}, \{1, 2\}\}, \{\{1, 1\}, \{1, 2\}\}, \{\{1, 1\}, \{1, 2\}\}, \{\{1, 1, 1\}, \{1, 1, 2\}\}
\]

By Definition 2
\[ h_{31} = \tilde{h}_{31} + \tilde{h}_{21} + 3\tilde{h}_{111} + 3\tilde{h}_{111} + \tilde{h}_1. \]
Next we define the symmetric functions $\tilde{s}_\lambda$ by the change of basis with $\tilde{h}_\lambda$ basis and the formulas

\begin{equation}
\tilde{h}_\mu = \sum_{|\lambda| \leq |\mu|} K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)} \tilde{s}_\lambda
\end{equation}

and

\begin{equation}
\tilde{s}_\lambda = \sum_{|\mu| \leq |\lambda|} K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)}^{-1} \tilde{h}_\mu
\end{equation}

where $n$ is any positive integer greater than $\max(|\lambda| + \lambda_1, |\mu| + \mu_1)$ and $K_{\lambda\mu}$ are the Kostka coefficients (the change of basis coefficients between the complete symmetric functions and the Schur basis, or the number of column strict tableaux of shape $\lambda$ and content $\mu$). It should be clear from a number of perspectives that this definition is independent of the value of $n$ as long as $n$ is sufficiently large. In particular, if $n - |\lambda| \geq \lambda_1$ then the Kostka coefficients $K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)}$ do not change by increasing the value of $n$ since there is an isomorphism between the tableau with $n - |\mu|$ labels 1 in the first row and those that have $n - |\mu| + 1$ labels 1 in the first row where the length of the first row also differs by 1.

If $n$ is smaller than $|\lambda| + \lambda_1$, then the change of basis coefficients are the same as those between the complete symmetric functions and a Schur function indexed by a composition $\alpha = (|\mu| - |\lambda|, \lambda)$, namely the expression representing the Jacobi-Trudi matrix

\begin{equation}
\det [h_{\alpha_i + i-j}]_{1 \leq i,j \leq \ell(\lambda)+1}.
\end{equation}

We find then that $\tilde{s}_\lambda$ are the (unique) symmetric functions of inhomogeneous degree $|\lambda|$, such that

\begin{equation}
\tilde{s}_\lambda[\Xi_\gamma] = \sum_{|\mu| \leq |\lambda|} K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)}^{-1} \tilde{h}_\mu[\Xi_\gamma]
= \sum_{|\mu| \leq |\lambda|} K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)}^{-1} \langle h_{(n-|\mu|,\mu)}, p_\gamma \rangle
= \langle \tilde{s}_{(n-|\lambda|,\lambda)}, p_\gamma \rangle = \chi^{(n-|\lambda|,\lambda)}(\gamma).
\end{equation}

We conclude that the basis $\tilde{s}_\lambda$ are the characters of the symmetric group when the symmetric group is realized as permutation matrices acting on the representation.

The combinatorial interpretation of the coefficients for the $\tilde{s}$-expansion of a complete symmetric function comes from combining the notion of multi-set partition of a multi-set and column strict tableau.

**Proposition 4.** For a partition, choose an $n \geq \mu_1 + |\mu|$, then

\begin{equation}
\tilde{h}_\mu = \sum_{T} \tilde{s}_{\text{shape}(T)}
\end{equation}

where $\overline{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$ and the sum is over all column strict tableau with $n - |\mu|$ blank cells in the first row and the rest of the cells filled with multi-sets of labels such that the total content of the tableau is $\{1^\mu, 2^{\mu_2} \cdots \ell^{\mu_\ell}\}$.

**Example 5.** Consider the following 20 column strict tableaux whose entries are multi-sets with total content of the tableau $\{1^2, 2\}$ and with 5 blank cells.

```
1 1 2 2
1 1
1 1
1 1
1 1
```

```
1 1 2
1 1
1 1
1 1
1 1
```

```
1 2 2
1 1
1 1
1 1
1 1
```

```
1 2 2
1 1
1 1
1 1
1 1
```

```
1 2 2
1 1
1 1
1 1
1 1
```

```
1 2 2
1 1
1 1
1 1
1 1
```
Proposition 4 states then that

\[ h_{21} = 4\tilde{s}_1 + 7\tilde{s}_1 + 3\tilde{s}_{11} + 4\tilde{s}_2 + \tilde{s}_{21} + \tilde{s}_3. \]

**Example 6.** Say that we want to compute the decomposition of \( V \otimes S^4 \) where \( V = L\{x_1, x_2, x_3, \ldots, x_n\} \) as an \( S_n \)-module with the diagonal action. The module \( V \) has character equal to \( h_1 = h_1 \). Therefore to compute the decomposition of this character into \( S_n \) irreducibles we are looking for the expansion of \( h_1 \) into the \( \tilde{s} \)-basis.

Using Sage we compute that it is

\[ h_1 = 15\tilde{s}_1 + 37\tilde{s}_1 + 31\tilde{s}_{11} + 10\tilde{s}_{111} + 31\tilde{s}_2 + 20\tilde{s}_{21} + 3\tilde{s}_{211} + 2\tilde{s}_{22} + 10\tilde{s}_3 + 3\tilde{s}_{31} + \tilde{s}_4. \]

If \( n \geq 6 \) then the multiplicity of the irreducible \((n-3,3)\) will be 10. The combinatorial interpretation of this value is the number of column strict tableaux with entries that are multi-sets (or in this case sets) of \{1, 2, 3, 4\} of shape \((4,3)\) or \((5,3)\) and 4 blank entries in the first row. Those tableaux are

\[
\begin{array}{cccc}
1 & 2 & 3 & 0 \\
1 & 2 & 4 & 3 \\
1 & 3 & 4 & 0 \\
1 & 2 & 3 & 4
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 4 & 3 \\
1 & 2 & 3 & 4
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 4 & 3 \\
1 & 2 & 3 & 4
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 4 & 3 \\
1 & 2 & 3 & 4
\end{array}
\]

What is interesting about this example is that the usual combinatorial interpretation for the repeated Kronecker product \((\chi^{(n-1,1)})^k\) is stated in other places in the literature in terms of oscillating tableaux \([CG, Sund]\) so this more general formula is giving a slightly different way of combinatorially describing the multiplicities in terms of set valued tableaux.

4. CHARACTER POLYNOMIALS AND THE IRREDUCIBLE CHARACTER BASIS

Character polynomials were first used by Murnaghan \([Murg]\). Much later, Specht \([Sp]\) gave determinantal formulas and expressions in terms of binomial coefficients for these polynomials. They are treated as an example in Macdonald’s book \([Mac, ex. I.7.13 and I.7.14]\). More recently, Garsia and Goupil \([GG]\) gave an umbral formula for computing them. We will show in this section that character polynomials are a transformation of character symmetric functions and this will allow us to give an expression for character symmetric functions in the power sum basis.

To begin, we note that \( p_r[\Xi_k] = k \) if \( k \) divides \( r \) and it is equal to 0 otherwise. In general, we can express any partition \( \mu \) in exponential notation \( \mu = (1^{m_1}2^{m_2}\cdots r^{m_r}) \) where \( m_i \) are the number of parts of size \( i \) in \( \mu \). Therefore

\[ p_k[\Xi_\mu] = \sum_{d|k} dm_d \]

Hence any symmetric function \( f \) evaluated at some set of roots of unity is equal to a polynomial in variables \( m_1, m_2, \ldots, m_n \) where

\[ f[\Xi_\mu] = f\bigg|_{p_k \rightarrow \sum_{d|k} dm_d} = q(m_1, m_2, \ldots, m_n). \]
Moreover, if we know this polynomial \( q(m_1, m_2, \ldots, m_d) \) we can use Möbius inversion to recover the symmetric function since if \( p_k = \sum_{d|k} dm_d \), then \( km_k = \sum_{d|k} \mu(k/d)p_d \)

\[
(20) \quad \mu(r) = \begin{cases} 
(-1)^d & \text{if } r \text{ is a product of } d \text{ distinct primes} \\
0 & \text{if } r \text{ is not square free}
\end{cases}
\]

Therefore, we also have

\[
(21) \quad q \left( \frac{p_1}{2}, \frac{p_2 - p_1}{3}, \ldots, \frac{1}{n} \sum_{d|n} \mu(n/d)p_d \right) = f.
\]

Following the notation of [GG], a character polynomial is a multivariate polynomial \( q_\lambda(x_1, x_2, x_3, \ldots) \) in the variables \( x_i \) such that for specific integer values \( x_i = m_i \in \mathbb{Z} \),

\[
(22) \quad q_\lambda(m_1, m_2, m_3, \ldots) = \chi^{(n-|\lambda|, \lambda)}(1^{m_1}2^{m_2}3^{m_3} \ldots)
\]

where \( n = \sum_{i \geq 1} im_i \). As a consequence of this relationship we have the following relationship between the character polynomials \( q_\lambda(x_1, x_2, x_3, \ldots) \) and character basis \( \tilde{s}_\lambda \).

**Proposition 7.** For a partition \( \lambda \),

\[
q_\lambda(x_1, x_2, x_3, \ldots) = \tilde{s}_\lambda \bigg|_{p_k \to \sum_{d|k} dx_d}
\]

and

\[
\tilde{s}_\lambda = q_\lambda(x_1, x_2, x_3, \ldots) \bigg|_{x_k \to \frac{1}{k} \sum_{d|k} \mu(k/d)p_d}
\]

As an important intermediate result, we have a power sum expansion of the irreducible character basis. Consider the following example.

**Example 8.** In [GG], the formula that they use to compute the character polynomial is most simply stated algorithmically at the top of page 3. If we make an additional substitution in the last step of their formula with \( x_k \) by \( \frac{1}{k} \sum_{d|k} \mu(k/d)p_d \), then in order to compute \( \tilde{s}_\lambda \) we compute the following steps, plus a fifth step to relate the character polynomial to irreducible character symmetric function.

1. Expand the Schur function \( s_\lambda \) in the power sums basis \( s_\lambda = \sum_\gamma \frac{\chi^\gamma(x)}{x^\gamma}p_\gamma \).
2. Replace each power sum \( p_i \) by \( x_i - 1 \).
3. Expand each product \( \prod_i (x_i, x_i - 1)^{n_i} \) as a sum \( \sum_\gamma c_\gamma \prod_i x_i^{q_\gamma} \).
4. Replace each \( x_i^{q_\gamma} \) by \( (x_k)_{g_k} = x_k(x_k - 1) \cdots (x_k - g_k + 1) \).
5. Replace each \( x_k \) by \( \frac{1}{k} \sum_{d|k} \mu(k/d)p_d \).

For instance, to compute \( \tilde{s}_3 \) we follow the steps to obtain:

1. \( s_3 = \frac{1}{3}(p_1^3 + 2p_2 + 3p_3) \)
2. \( \frac{1}{6}(p_1 + 2p_2 + 3p_3) \to \frac{1}{6}((x_1 - 1)^3 + 3(2x_2 - 1)(x_1 - 1) + 2(3x_3 - 1)) \)
3. \( \frac{1}{6}((x_1 - 1)^3 + 3(2x_2 - 1)(x_1 - 1) + 2(3x_3 - 1)) = \frac{1}{6}x_1^3 - \frac{1}{2}x_1^2 + x_1x_2 - x_2 + x_3 \)
4. \( q_3 = \frac{1}{6}(x_1)_3 - \frac{1}{2}(x_1)_2 + x_1x_2 - x_2 + x_3 \)
5. \( \tilde{s}_3 = \frac{1}{3}(p_1)_3 - \frac{1}{2}(p_1)_2 + p_1 \frac{p_2 - p_1}{2} - \frac{p_2 - p_1}{2} + \frac{p_2 - p_1}{3} \)

As a consequence we state a slightly more explicit expression for the character basis that we derive by following the algorithm of [GG].

**Proposition 9.** For \( \lambda \vdash n \),

\[
(23) \quad \tilde{s}_\lambda = \sum_{\gamma | n} \chi^\gamma(\gamma) \frac{P_\gamma}{\gamma_{\gamma}}
\]


As a consequence of Proposition 10 we conclude

\[ p_r = \sum_{k=0}^{r} (-1)^{r-k} k^r \binom{r}{k} \left( \frac{1}{i} \sum_{d|i} \mu(i/d) p_d \right)_k \text{ and } p_\gamma := \prod_{i \geq 1} p_{\mu_i(\gamma)}. \]

5. A PRODUCT RULE FOR THE $\tilde{h}$-BASIS

The Kronecker product of two complete symmetric functions expanded in the complete basis is listed as Exercise 23 (c) in section 1.7 of [Mac]. It states that

\[ h_\lambda \cdot h_\mu = \sum_M \prod_{i} \prod_{j} h_{M_{i,j}} \]

summed over all matrices $M$ of non-negative integers with $\ell(\lambda)$ rows, $\ell(\mu)$ columns and row sums $\lambda_i$ and column sums $\mu_j$.

We propose a different (but equivalent) combinatorial interpretation for this product based on the $\tilde{h}$-basis. For two multi-set partitions $\pi = \{S_1, S_2, \ldots, S_{\ell(\pi)}\}$ and $\theta = \{T_1, T_2, \ldots, T_{\ell(\theta)}\}$ with $S_i \cap T_j = \emptyset$. Define a join operation (which we will denote as $\pi \circ \tau$) as the set of distinct multi-set partitions of the form

\[ \{S_1, S_2, \ldots, S_{\ell(\pi)-r}, T_1, T_2, \ldots, T_{\ell(\theta)-r}, S_{u_1} \cup T_{v_1}, S_{u_2} \cup T_{v_2}, \ldots, S_{u_r} \cup T_{v_r} \} \]

with $\{i_1, i_2, \ldots, \ell(\pi)-r, u_1, u_2, \ldots, u_r\} = \{1, 2, \ldots, \ell(\pi)\}$ and $\{j_1, j_2, \ldots, \ell(\theta)-r, v_1, v_2, \ldots, v_r\} = \{1, 2, \ldots, \ell(\theta)\}$.

**Proposition 10.** For disjoint multi-set partitions $\pi$ and $\theta$,

\[ \tilde{h}_{\tilde{m}(\pi)} \cdot \tilde{h}_{\tilde{m}(\theta)} = \sum_{\tau \in \pi \circ \theta} \tilde{h}_{\tilde{m}(\tau)}. \]

**Example 11.** Let $\pi = \{\{1\}, \{1\}, \{2\}\}$ and $\theta = \{\{3\}, \{3\}, \{4\}\}$

Below we list the multi-set partitions in the product along with the corresponding partition $\tilde{m}(\tau)$.

\[ \{\{1\}, \{1\}, \{2\}, \{3\}, \{3\}, \{4\}\} \rightarrow (2, 2, 1, 1) \quad \{\{1\}, \{1\}, \{2\}, \{3\}, \{3\}, \{4\}\} \rightarrow (1, 1, 1, 1, 1) \]

\[ \{\{1\}, \{1\}, \{2\}, \{3\}, \{3\}, \{4\}\} \rightarrow (2, 1, 1, 1) \quad \{\{1\}, \{1\}, \{2\}, \{3\}, \{3\}\} \rightarrow (2, 1, 1, 1) \]

\[ \{\{1\}, \{1\}, \{2\}, \{3\}, \{3\}\} \rightarrow (2, 2, 1) \quad \{\{1\}, \{1\}, \{2\}, \{3\}, \{4\}\} \rightarrow (2, 1, 1) \]

\[ \{\{1\}, \{1\}, \{2\}, \{3\}, \{3\}\} \rightarrow (1, 1, 1, 1) \quad \{\{1\}, \{1\}, \{2\}, \{3\}, \{4\}\} \rightarrow (1, 1, 1, 1) \]

\[ \{\{1\}, \{1\}, \{2\}, \{3\}, \{3\}\} \rightarrow (1, 1, 1, 1) \quad \{\{1\}, \{1\}, \{2\}, \{4\}, \{3\}\} \rightarrow (1, 1, 1, 1) \]

\[ \{\{1\}, \{1\}, \{2\}, \{4\}, \{3\}, \{3\}\} \rightarrow (2, 1) \quad \{\{1\}, \{1\}, \{2\}, \{4\}, \{3\}\} \rightarrow (1, 1, 1) \]

As a consequence of Proposition 10 we conclude

\[ \tilde{h}_{21} \cdot \tilde{h}_{21} = \tilde{h}_{11} + 4 \tilde{h}_{1111} + \tilde{h}_{11111} + \tilde{h}_{21} + \tilde{h}_{211} + 2 \tilde{h}_{2111} + \tilde{h}_{221} + \tilde{h}_{2211} \]

or in terms of Kronecker products

\[ h_{521} \cdot h_{521} = h_{511} + 4 h_{41111} + h_{311111} + h_{521} + h_{4211} + 2 h_{32111} + h_{3221} + h_{22211}. \]
6. Appendix: Using Sage to compute the character bases of symmetric functions

Sage is an open source symbolic calculation program based on the computer language Python. A large community of mathematicians participate in its support and add to its functionality [sage, sage-combinat]. In particular, the built-in library for symmetric functions includes a large extensible set of functions which makes it possible to do calculations within the ring following closely the mathematical notation that we use in this paper. The language itself has a learning curve, but the contributions made by the community towards the functionality make that a barrier worth overcoming.

In version 6.10 or later of Sage (released Jan 2016) these bases will be available as methods in the ring of symmetric functions.

We demonstrate examples of some of the definitions and results in this paper by sage calculations.

```sage
sage: Sym = SymmetricFunctions(QQ)
sage: Sym
Symmetric Functions over Rational Field
sage: st = Sym.irreducible_symmetric_group_character()
sage: st
Symmetric Functions over Rational Field in the irreducible character basis
sage: st[2]*st[2]
st[] + st[1] + st[1, 1] + st[1, 1, 1] + 2*st[2] + 2*st[2, 1] + st[2, 2]
+ st[3] + st[3, 1] + st[4]
sage: s = Sym.Schur(); s
Symmetric Functions over Rational Field in the Schur basis
sage: s[6,2].kronecker_product(s[6,2])
s[4, 2, 2] + s[4, 3, 1] + s[4, 4] + s[5, 1, 1, 1] + 2*s[5, 2, 1] + s[5, 3] + s[6, 1, 1] + 2*s[6, 2] + s[7, 1] + s[8]

We can express one basis in terms of another.

```sage
sage: h = Sym.complete(); h
Symmetric Functions over Rational Field in the homogeneous basis
sage: ht = Sym.induced_trivial_character(); ht
Symmetric Functions over Rational Field in the induced trivial character basis
sage: ht(h[2,2]) # express h_{2,2} in the ht-basis
ht[1] + 3*ht[1, 1] + ht[1, 1, 1] + ht[2] + 2*ht[2, 1] + ht[2, 2]
sage: st(ht[2,2])
st[] + 2*st[1] + st[1, 1] + 3*st[2] + 2*st[2, 2] + st[2, 2] + 2*st[3] + st[3, 1] + st[4]
sage: s(h[4,2,2])
s[4, 2, 2] + s[4, 3, 1] + s[4, 4] + 2*s[5, 2, 1] + 2*s[5, 3] + s[6, 1, 1] + 3*s[6, 2] + 2*s[7, 1] + s[8]
```

This ticket also introduces two operations that we use in this paper. The first is the evaluation of a symmetric function at the eigenvalues of a permutation matrix where the permutation has cycle structure $\mu$. We represented this operation in the paper as $f[\Xi_{\mu}]$. In Sage, elements of the symmetric functions have the method `eval_at_permutation_roots` which represents this operation.

```sage
sage: ht[3,1].eval_at_permutation_roots([3,3,2,2,1])
2
sage: st[3,1].eval_at_permutation_roots([3,3,2,2,1])
```
In Sage, these can be compared to the following coefficients in the power sum basis.

\[
sage: p = Sym.powersum(); p
\]
\[
\text{Symmetric Functions over Rational Field in the powersum basis}
\]
\[
sage: h[7,3,1].scalar(p[3,3,2,2,1])
\]
\[
2
\]
\[
sage: s[7,3,1].scalar(p[3,3,2,2,1])
\]
\[
-1
\]

The other operation that we define is one that interprets a symmetric function as a symmetric group character and then maps that character to the Frobenius image (or characteristic map) of the character. That is, for a symmetric function \(f\), the function computes

\[
(28) \quad \phi_n(f) = \sum_{\mu} f(\Xi_{\mu}) \frac{P_{\mu}}{z_{\mu}}.
\]

The elements of the symmetric functions in Sage will have the method `character_to_frobenius_image` which represents the map \(\phi_n\).

We have defined the \(\tilde{s}\) and \(\tilde{h}\) bases so that they have the property \(\phi_n(\tilde{s}_\lambda) = s(n-|\lambda|,\lambda)\) and \(\phi_n(\tilde{h}_\lambda) = h(n-|\lambda|,\lambda)\) respectively if \(n \geq |\lambda| + \lambda_1\). If \(n < |\lambda| + \lambda_1\), then the corresponding symmetric function will still be equivalent to a Schur function or complete symmetric function indexed by a list of integers.

\[
sage: s(st[3,2].character_to_frobenius_image(8))
\]
\[
s[3,3,2]
\]
\[
sage: h(ht[3,2].character_to_frobenius_image(6))
\]
\[
h[3,2,1]
\]

It is this operation that is the origin of our definitions since in the beginning of our investigations the \(\tilde{h}\) and \(\tilde{s}\) basis for us were the pre-images of the Schur and complete symmetric functions in the \(\phi_n\) map.

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