MAGNETIC ENERGY BUILDUP FOR RELATIVISTIC MAGNETAR GIANT FLARES

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Received 2011 April 13; accepted 2011 June 17; published 2011 August 12

ABSTRACT

Motivated by coronal mass ejection studies, we construct general relativistic models of a magnetar magnetosphere endowed with strong magnetic fields. The equilibrium states of the stationary, axisymmetric magnetic fields in the magnetar magnetosphere are obtained as solutions of the Grad–Shafranov equation in a Schwarzschild spacetime. To understand the magnetic energy buildup in the magnetar magnetosphere, a generalized magnetic virial theorem in the Schwarzshild metric is newly derived. We carefully address the question whether the magnetar magnetospheric magnetic field can build up sufficient magnetic energy to account for the work required to open up the magnetic field during magnetar giant flares. We point out the importance of the Aly–Sturrock constraint, which has been widely studied in solar corona mass ejections, as a reference state in understanding magnetar energy storage processes. We examine how the magnetic field can possess enough energy to overcome the Aly–Sturrock energy constraint and open up. In particular, general relativistic (GR) effects on the Aly–Sturrock energy constraint in the Schwarzschild spacetime are carefully investigated. It is found that, for magnetar outbursts, the Aly–Sturrock constraint is more stringent, i.e., the Aly–Sturrock energy threshold is enhanced due to the GR effects. In addition, neutron stars with greater mass have a higher Aly–Sturrock energy threshold and are more difficult to erupt. This indicates that magnetars are probably not neutron stars with extreme mass. For a typical neutron star with mass of 1–2 $M_\odot$, we further explore the cross-field current effects, caused by the mass loading, on the possibility of stored magnetic field energy exceeding the Aly–Sturrock threshold.

Key words: pulsars: general – stars: magnetic field – stars: neutron

1. INTRODUCTION

After the discovery of soft gamma repeaters and anomalous X-ray pulsars (Mazets et al. 1979; Mereghetti & Stella 1995), magnetar models of these sources are proposed to explain the relevant phenomena (Duncan & Thompson 1992; Thompson et al. 2002). Magnetars are believed to be neutron stars with strong magnetic field, $\sim 10^{14}$–$10^{15}$ G (Duncan & Thompson 1992). The magnetar outbursts, such as giant flares, occur with huge release of magnetic energy $\sim 10^{44}$–$10^{46}$ erg. The energy for magnetar outbursts is widely accepted to be supplied by the star’s magnetic field. However the physical process by which the energy is stored and released is one of the great puzzles in high-energy astrophysics. Two possibilities exist for the location where the magnetic energy is stored prior to an eruption: in the magnetar crust or in the magnetosphere. For the former possibility, a giant flare may be caused by a sudden untwisting of the magnetar interior magnetic field (Thompson & Duncan 2001). Subsequently, a sudden and brittle fracture of the crust leads to the giant flare. In this crust scenario, the energy stored in the external twist is limited by the tensile strength of the crust, but instead by the total external magnetic field energy.

The magnetospheric storage model of magnetar giant flare shares similar magnetic energy buildup process to solar eruptions, such as coronal mass ejections (CMEs). In this model, the energy released during an eruption is stored in the magnetospheric magnetic field before the eruption. Large-scale eruptive CMEs often give rise to the opening up of magnetic field lines that were originally closed. The processes of magnetic fields opening up have been extensively investigated in the CME studies (Barnes & Sturrock 1972; Aly 1984; Mikic & Linker 1994). It is physically reasonable to assume that the pre-eruption closed state must possess more magnetic energy than the post-eruption open state. As will be discussed in detail below, requiring the magnetic field to open imposes an extreme energy constraint on theories for CMEs. This energy requirement on solar CMEs has been under extensive theoretical studies in the past decades (Aly 1984; Sturrock 1991; Wolfson & Dlamini 1997; Zhang & Low 2005). The energy storage processes take place quasi-statically on a long timescale. When the magnetic field reaches a threshold, due to the instability or loss of confinement, the field erupts suddenly on a much shorter dynamical timescale. Analogous processes of magnetic field opening up are believed to occur in magnetar giant flares (Woods et al. 2001; Thompson et al. 2002; Beloborodov 2009). All these features of the storage model are in good agreement with the observations of magnetar giant flares (Lyutikov 2006).

The similarity between solar eruptions and magnetar giant flares (Lyutikov 2003) motivates this study on the energy buildup process in the magnetar magnetosphere. We note that there are important differences between solar eruptions and magnetar outbursts. For situations in the magnetar magnetosphere...
energy above the Aly–Sturrock constraint. But no theoretical calculations were performed to corroborate this idea. In this work we will provide such a demonstration. Another possibility for the magnetic energy to exceed the Aly–Sturrock constraint is the formation of detached field lines from the magnetar surface (magnetic bubble or magnetic flux rope, e.g., Low & Smith 1993; Flyer et al. 2004), which will be further discussed by C. Yu (2011, in preparation).

This paper is organized as follows: in Section 2 we introduce the generalized magnetic virial theorem in the Schwarzschild metric. In Section 3 we will discuss how the Aly–Sturrock field energy is affected by GR effects. We will explore the cross-field effects caused by the mass loading on the magnetic energy storage in Section 4. Conclusions and discussions are given in Section 5.

2. GENERALIZED VIRIAL THEOREM IN SCHWARZSCHILD SPACETIME

The virial theorem is of vital importance for understanding the magnetic energy storage in the magnetar magnetosphere. In the flat spacetime, it was proposed by Chandrasekhar (1961) and has been used widely in solar physics researches (e.g., Low & Smith 1993). We focus in this paper on the physical behavior near the magnetar surface, GR effects should be incorporated. Because observed magnetars have a very slow rotation rate, we ignore the rotation effects and adopt the Schwarzschild metric as the background spacetime. In this section, we establish the virial theorem in the Schwarzschild metric including effects of magnetic fields. We consider a steady-state magnetosphere around magnetars. The metric $g_{\mu\nu}$ of Schwarzschild geometry reads (Misner et al. 1973)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \alpha^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

where $\alpha(r) = \sqrt{1 - \frac{2g}{r}}$.

The factor of $\alpha$ is defined as

$$\alpha(r) = \sqrt{1 - \frac{2g}{r}}.$$

A plasma containing only a perfect fluid and an electromagnetic field is described by the energy–momentum tensor (Anile 1989)

$$T^{\mu\nu} = T^{\mu\nu}_{\text{fluid}} + T^{\mu\nu}_{\text{EM}} = (p + \rho + b^2)u^\mu u^\nu + \left( p + \frac{b^2}{2} \right) g^{\mu\nu} - b^\mu b^\nu,$$

where $p$ is the isotropic pressure, $\rho = \rho_0 + \frac{\dot{r}^2}{2}$ is the energy density (including that due to the rest mass $\rho_0$), and $b^2 = h_i b^i$. A polytropic equation of state is adopted and we take $\gamma = 4/3$ throughout this paper. Here the Einstein summation rule is assumed and Greek letters take on the values $i, j, \theta, \phi$. The magnetic field four-vector is

$$b^\mu = F^{\mu\nu} u_\nu,$$

where $F^{\mu\nu}$ is the Maxwell tensor and $u_\mu$ is the four-velocity of the comoving observer (Antón et al. 2006). The plasma is assumed to be in magnetostatic equilibrium, thus the four-velocity is $u^\mu = ((-g_{\mu\nu})^{-1/2}, 0, 0, 0)$. Under such circumstances, the condition

$$\nabla_\nu T^{\mu\nu} = 0$$

1. Here simple geometries mean that the two ends of all field lines are anchored onto the neutron star surface.
where becomes the usual virial theorem in the flat spacetime as

\[ E + (3\gamma - 4)U = \int_{\partial V} d\mathbf{S} \cdot \mathbf{r} \]

where \( E = M + U + W \),

\[ M = \int \frac{B_r^2}{2} dV, \]

\[ U = \int \frac{p}{\gamma - 1} dV, \]

\[ W = -\int \frac{\partial G M_{\text{ls}}}{r} dV, \]

are the magnetic, internal and gravitational potential energy, respectively. Here we have absorbed a \( 4\pi \) factor into the definition of the magnetic fields throughout this paper. In the above equations, the magnetic field \( \mathbf{B} \) in the “ordinary” orthogonal basis (defined in Section 4) is used. The relation between \( \mathbf{B} \) and the magnetic field four-vector \( b^\mu \) is given explicitly in Appendix A. Note that \( B_r \) is the radial component of \( \mathbf{B} \) and \( B_r^2 = B_\theta^2 + B_\phi^2 + B_r^2 \). Throughout this paper, we mainly work with the magnetic field \( \mathbf{B} \). This choice is made mainly for the convenience of comparison between the results in the curved spacetime and the flat spacetime.

The last integral on the right-hand side in Equation (7) appears owing to GR effects. This term disappears when taking the flat spacetime limit, i.e., \( \alpha^2 \rightarrow 1 \). Note also that this equation becomes the usual virial theorem in the flat spacetime as \( \alpha^2 \rightarrow 1 \) (Chandrasekhar 1961). In particular, for the magnetically dominated force-free field, we arrive at

\[ M = \int_{\partial V} \frac{\alpha^2 B_r^2}{2} (\mathbf{r} \cdot d\mathbf{S}) - \int_{\partial V} \alpha^2 (\mathbf{B} \cdot \mathbf{r}) (\mathbf{B} \cdot d\mathbf{S}) + \int_{V} \frac{(1 - \alpha^2)}{2} (B_r^2 + B_\theta^2) dV. \]

Assuming that the magnetic field vanishes sufficiently rapidly at large distances, we find that the energy of the force-free fields in the exterior \( r > r_0 \) of the neutron star is

\[ M = \pi r_0^3 \int (\mathbf{B}^2 - B_\theta^2) |_{r=r_0} \sin \theta d\theta + \int_{V} \frac{r}{\gamma - 1} (2B_r^2 + B_\theta^2) dV, \]

where \( r_0 \) is the radius of the neutron star.

We note that, in a flat spacetime, the second term on the right-hand side of the above equation disappears and the total magnetic energy of a force-free magnetic field in the exterior region \( r > r_0 \) of a sphere is uniquely determined by the field values at the boundary \( r = r_0 \). However, this is no longer the case for the curved spacetime, since additional terms proportional to \( r \) appear on the right-hand side of this equation. Close observation of Equation (13) shows that, when GR effects are ignored, no force-free field that is completely detached from the solar surface (i.e., \( B_r = 0 \) at \( r = r_0 \)) in the exterior region \( r \geq r_0 \) can exist (Low 2001). However, such completely detached field configurations in the GR magnetar magnetosphere, due to the spacetime curvature, may be in the equilibrium state. This suggests that, besides the normal flux at the magnetar surface, the GR spacetime curvature provides additional self-confining effects. As a result, it needs more work to be done to open the magnetic field in the curved spacetime than in the flat spacetime. It is conceivable that when the magnetar mass increases, this effect becomes more evident (see Figure 2). Such GR effects have important implications for the magnetic energy storage process in the magnetar magnetosphere. In the next section, we will quantitatively calculate their influences on the Aly–Sturrock constraint.

3. ALY–STURROCK CONSTRAINT FOR MAGNETIC FIELD ENERGY

To discuss the magnetic energy in the magnetar magnetosphere, it is beneficial to introduce the potential field \( \mathbf{B}_{\text{pot}} \) in the Schwarzschild metric which satisfies (Uzdensky 2004)

\[ \nabla \times (a \mathbf{B}) = 0 \]

and the boundary condition

\[ r = r_0, \quad B_r = F(\theta). \]

In this paper, we mainly discuss the dipole field and its relevant open state. The explicit expression of the dipole field can be found in Appendix B. In this case the above boundary becomes \( B_r = C \cos \theta \), where \( C \) is a constant. Note that the potential field now involves the spacetime curvature term \( \alpha \) in Equation (14). This is quite different from the flat spacetime definition (Komissarov 2004). Note that, as \( \alpha \rightarrow 1 \), this potential field definition reduces to its flat spacetime form. The associated magnetic energy of the potential field is designated as \( M_{\text{pot}} \). For the force-free field in the magnetosphere, there exists one interesting energy reference state, the Aly–Sturrock state (Aly 1984, 1991; Sturrock 1991). Imagine all force-free magnetic fields complying with the boundary condition (15), with one end of each line of force anchored to the star’s surface

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2 Although we will be treating non-force-free magnetosphere in which cross-field effects (caused by mass loading) are important, the discussion here is restricted to magnetically dominated force-free fields. The relevance will become clear as we proceed.

3 See the magnetic field configuration in Figure 8(b) of Low (2001), which cannot maintain equilibrium in flat spacetime. But such configurations can be self-confined by the spacetime curvature effects.
Strictly speaking, this condition is not fulfilled since the field lines open at the equator (Aly 1984, 1991; Sturrock 1991). and the other out to infinity. Among all these fields, the one with the lowest energy is potential everywhere except for a current sheet at the equator (Aly 1984, 1991; Sturrock 1991). This lowest energy state is the Aly–Sturrock state. Call this magnetic field configuration $B_{\text{open}}$. The total energy of this state is designated as $M_{\text{open}}$. The well-known Aly–Sturrock conjecture claims that for any fully closed force-free field with the boundary condition (15), its total energy $M_{\text{FF}}$ satisfies the following relation:

$$M_{\text{pot}} < M_{\text{FF}} < M_{\text{open}}.$$  

(16)

The first half of this inequality means a current-free potential field is the lowest energy state. And the second half suggests that the opening process of an initial closed force-free magnetic field requires considerable amount of work to be done on the magnetic field. Of particular interest is whether the pre-eruption magnetic energy $M$ can exceed the threshold set by the Aly–Sturrock field. This is crucial for the magnetically driven outbursts.

Some numerical experiments have recently demonstrated the validity of this conjecture (Antiochos et al. 1999; Hu 2004). Due to the importance of the Aly–Sturrock constraint for the magnetic eruption, it is worthwhile to reconsider this problem when GR effects are important. Note that this Aly–Sturrock state is unique (Aly 1984; Sturrock 1991) and can be constructed by the following technique. Modify the boundary condition (15) to

$$r = r_0, \quad B_r = |F(\theta)|.$$  

(17)

After getting the field with this boundary condition and reversing the directions of those lines at the boundary $r = r_0$ where $B_r < 0$ of this field, we could get the Aly–Sturrock state (see also Low & Smith 1993). We have calculated the fully open field $B_{\text{open}}$ and the relevant energy $M_{\text{open}}$ numerically. The details to obtain the Aly–Sturrock field and the magnetic energy $M_{\text{open}}$ are discussed in Appendix C. In Figure 1, an illustrative example of the fully open Aly–Sturrock field is shown. The current sheet at the equator is shown by a thick solid line.

![Figure 1](image1.png)

**Figure 1.** Fully open Aly–Sturrock field with an initial dipole field boundary condition. The thick solid line at the equator denotes the current sheet in the field.

![Figure 2](image2.png)

**Figure 2.** Variation of the $M_{\text{open}}$ (in units of $M_{\text{pot}}$) with $r_g$ (equivalently, the magnetar mass). Note that $r_g$ ranges from 0.15 to 0.45, which corresponds to a mass range of 1–3 $M_\odot$. Note that in flat spacetime the fully open field energy $M_{\text{open}} = 1.662 M_{\text{pot}}$ for the dipole field boundary condition, which is denoted by dot-dashed line in this figure.

3.1. Dependence of $M_{\text{open}}$ on Neutron Star Masses

To investigate the spacetime curvature effects on the Aly–Sturrock constraint, we calculate the Aly–Sturrock threshold $M_{\text{open}}$ for different magnetar masses. Throughout this paper we take the neutron star radius $r_0 = 1$, so for a neutron star mass of 1–3 $M_\odot$, $r_g$ ranges from 0.15 to 0.45 (for simplicity, we keep the neutron star radius fixed at 10 km, though this is not the case in reality). In Figure 2, we show the variation of $M_{\text{open}}$ (in units of $M_{\text{pot}}$) with the neutron star mass. This figure shows that the more massive the magnetar, the higher the threshold is. For instance, for the dipole field with $r_g = 0.15$ (1 $M_\odot$), the energy of the fully open Aly–Sturrock field is $M_{\text{open}} = 1.80 M_{\text{pot}}$; when $r_g = 0.21$ (1.4 $M_\odot$), the energy becomes $M_{\text{open}} = 1.88 M_{\text{pot}}$. Consequently, it is more difficult for more massive neutron stars to surpass the Aly–Sturrock energy threshold. From this figure, we also note that as $r_g \to 0$ the Aly–Sturrock threshold approaches the flat spacetime limit $M_{\text{open}} = 1.662 M_{\text{pot}}$. This is consistent with our physical expectation.

This increase with mass of the Aly–Sturrock energy threshold stems entirely from the spacetime curvature self-confining effects mentioned in Section 2 and this behavior is quite

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4 Strictly speaking, this condition is not fulfilled since the field lines open at the light cylinder. Fortunately, magnetars are slow rotators, so the light cylinder is quite far away from the neutron star surface and this effects can be negligible. For this reason, we focus in this paper on the non-rotating neutron stars.
different from the solar eruption in flat spacetime, in which the Aly–Sturrock field energy (∼1.662 $M_{\odot}$) is independent of the star mass. It should be emphasized that in the magnetar outbursts, the Aly–Sturrock field energy constraint is more stringent than for the solar CME-type eruptions. From Figure 2, we can infer that magnetars are probably not neutron stars with extreme mass $\sim 3M_{\odot}$, as the Aly–Sturrock threshold could hardly be reached.

For typical neutron star masses (∼1–2 $M_{\odot}$, $r_g \sim 0.15–0.3$), it is necessary to seek initial magnetic fields which possess magnetic energy in excess of the threshold set by the Aly–Sturrock energy $M_{\text{open}}$. One possibility is due to the mass loading effects. The estimated ejected mass loading is about $10^{22}$ g (Lyutikov 2006). This mass loading can be balanced by pressure forces in the vertical direction. The pressure gradient in the horizontal direction, however, requires magnetic forces associated with cross-field currents, i.e., $\mathbf{J} \times \mathbf{B} \neq 0$, to maintain the equilibrium state. The deviations from a strictly force-free magnetic field, i.e., the cross-field contribution, are worth further investigations (Low & Smith 1993; Wolfson & Diamini 1997). Physically speaking, the mass loading would act as a lid over the magnetic field. The field can be compressed globally by a sufficient amount of plasma. As a result, the energy of the compressed magnetic field increases as the total load increases, and eventually the magnetic energy exceeds $M_{\text{open}}$. In other words, the cross-field current densities provide additional sources of magnetic free energy which may be enough to enable the magnetic field to clear the threshold $M_{\text{open}}$.

4. AXISYMMETRIC MAGNETOSTATIC MAGNETOSPHERE WITH CROSS-FIELD CURRENTS

In this section, we explore the cross-field effects on the magnetic energy storage properties in the magnetar magnetosphere. Similar investigations in solar CMEs have been carried out by Zhang & Low (2004). Specifically, we will focus on the question whether the magnetic energy in the magnetosphere can exceed the Aly–Sturrock threshold. In what follows, we consider that the magnetar magnetosphere evolves quasi-statically on sufficiently slow timescale that we can treat the magnetosphere as being essentially in magnetostatic equilibrium. A steady-state axisymmetric, purely poloidal magnetic field in the Schwarzschild metric can be written as

$$\mathbf{B} = \mathbf{B}_{\text{pol}} = \nabla \Psi \times \nabla \phi,$$

where $\Psi(r, \theta)$ is the poloidal magnetic stream function. The "ordinary" orthogonal basis is used, where $\mathbf{e}_r = g_{\mu\nu}^{1/2} \partial_\mu$, (Weinberg 1972, no summation rule over $\mu$ is used in this equation), namely,

$$\mathbf{e}_r = \alpha \partial_\alpha, \quad \mathbf{e}_\theta = \frac{1}{r} \partial_\theta, \quad \mathbf{e}_\phi = \frac{1}{r \sin \theta} \partial_\phi.$$  

The poloidal magnetic field components are (Uzdensky 2004)

$$\mathbf{B} = \frac{1}{r \sin \theta} \left( \frac{\partial \Psi}{\partial \theta} - \alpha \frac{\partial \Psi}{\partial r} \right).$$

To account for the cross-field current effects induced by the mass loading, we must go beyond the force-free approximations (Yu 2011) and turn to the full magnetohydrodynamic (MHD) equations (6). This equation decomposes into the following two equations:

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \theta} \right) + \frac{r^2 \sin^2 \theta}{\partial \Psi} \frac{\partial \rho}{\partial \Psi} = 0,$$

$$g_r' \frac{\partial \rho}{\partial r} + \left( \frac{p + \rho}{r^2} \right) G M_{\text{ns}} = 0,$$

for balance across and along the magnetic field (Low & Smith 1993). A simple solution to Equation (22) reads

$$p = \frac{P(\Psi)}{r^{m+1}}.$$  

where $P(\Psi)$ is a free function of the magnetic stream function $\Psi$ and $m$ is a constant.

To keep the problem mathematically tractable, we take the free function $P$ to be linear in $\Psi$. Subsequently, Equations (23) and (24) become

$$p = \frac{\lambda(\Psi + \Psi_0)}{r^{m+1}},$$

$$\rho_0 = \frac{1}{G M_{\text{ns}}} \frac{\lambda(\Psi + \Psi_0)}{r^m} \left[ m + 1 - \left( 2m + 2 + \frac{\gamma}{\gamma - 1} \right) \frac{r_g}{r} \right],$$

where $\Psi_0$ and $\lambda$ are constants. Substitute Equation (25) into Equation (21), we obtain the following linear Grad–Shafranov equation:

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \theta} \right) + \frac{\lambda}{r^{m-1}} \frac{\sin^2 \theta}{\partial \Psi} = 0.$$  

The general solution to the above equation can be written as

$$\Psi = f_m(r) \sin^2 \theta + \Psi_{\text{pol}},$$

where $\Psi_{\text{pol}}$ is an arbitrary potential stream function satisfying (Ghosh 2000)

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi_{\text{pol}}}{\partial r} \right) + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Psi_{\text{pol}}}{\partial \theta} \right) = 0.$$  

This equation can be readily solved by the variable separation method (see Appendix B). The function $f_m(r)$ satisfies the following second-order ordinary differential equation (ODE):

$$\left( 1 - \frac{2r_g}{r} \right) f'' + 2 \frac{r_g}{r^2} f' - \frac{r_g^2}{r^2} f + \frac{\lambda}{r^{m-1}} = 0.$$  

where prime denotes derivatives with respect to $r$. The particular solutions can be readily obtained analytically. For $m = 3, 4, 5, 6, 7$, and 8, the radial function $f_m$ are given explicitly in Appendix D. The simple linear solutions given by the above equations cannot be expected to describe the magnetar magnetosphere in realistic details. However, the solution given by Equation (28) with $\Psi_{\text{pol}} = 0$ can be used to obtain a physical estimate of how much energy can be stored in the magnetosphere prior to eruptions.

The magnetic energies for different values of $m$ and $r_g$ are listed in Table 1. We find that, for $r_g = 0.15$ and 0.21, the field

Yu

Equation (26)}
configurations are able to sustain magnetic energy higher than the Aly–Sturrock threshold as \(m \geq 8\). Simple estimation shows that the total magnetic energy of a magnetar with magnetic field \(\sim 10^{14}–10^{15} \text{G}\) is approximately \(10^{44}–10^{46} \text{erg}\). Given the actual giant flare energy release \(\sim 10^{44}–10^{46} \text{erg}\), we know that a few percent of the magnetic energy in excess of the Aly–Sturrock threshold is needed to release during a giant flare. This energy requirement can be fulfilled as \(m\) reaches a critical value. For instance, we find that for \(r_g = 0.15\) the magnetic energy \(M\) with \(m = 8\) is approximately 15% above the Aly–Sturrock threshold \(M_{\text{open}}\), which is enough to drive magnetar giant flares.

In Figure 3, we show the \(m = 8\) solution with \(r_g = 0.15\)

\[
\Psi = \frac{f_3(r)}{f_3(r_0)} \sin^2 \theta, \\
\rho_0 = \frac{1}{G \mathcal{M}_{\text{ns}}} \frac{\lambda \Psi}{r^m} \left[ m + 1 - \left( 2m + 2 + \frac{\gamma}{\gamma - 1} \right) \frac{r_g}{r} \right],
\]

where the stream flux is normalized to unity at \(r = r_0\) and \(\theta = \pi/2\), and we have set the constant \(\Psi_0 = 0\) in Equation (26).

The left panel in this figure shows the magnetic field lines and the right one shows the contour of the density departure5 from an arbitrary, spherically symmetric distribution. In this particular state, the magnetic energy \(M\) is 2.12 \(M_{\text{pot}}\), which is greater than the Aly–Sturrock state \(M_{\text{open}} = 1.80 M_{\text{pot}}\).

The solution with \(m = 3\) and \(\Psi_{\text{pot}} = 0\) is a purely radial magnetic field,

\[
B_r = \frac{\lambda \cos \theta}{r^2}, \quad B_\theta = 0.
\]

This solution has been extensively discussed in the Blandford–Znajek process (e.g., Blandford & Znajek 1977) related to relativistic astrophysical jets. But in our discussion this state itself is of no particular interest as we are more concerned with the initial closed state. To introduce the closed field structures, we add a dipole field to the \(m = 3\) purely radial magnetic field, i.e.,

\[
\Psi = \lambda \frac{f_3(r)}{f_3(r_0)} \sin^2 \theta \pm \Psi_{\text{dipole}}, \quad (31)
\]

5 Note that an arbitrary, spherically symmetric density distribution corresponds to the term that is proportional to the constant \(\Psi_0\) in Equation (26), which is ignored in this figure.

### Table 1

| \(m\) | \(r_g = 0.15\) | \(r_g = 0.21\) | \(r_g = 0.3\) |
|-------|--------------|--------------|--------------|
| 3     | 2.19         | 2.30         | 2.55         |
| 4     | 1.00         | 1.00         | 1.00         |
| 5     | 1.15         | 1.13         | 1.10         |
| 6     | 1.44         | 1.38         | 1.30         |
| 7     | 1.77         | 1.67         | 1.52         |
| 8     | 2.12         | 1.98         | 1.76         |
| 9     | 2.48         | 2.29         | 2.00         |
| 10    | 2.84         | 2.61         | 2.25         |

Notes. The gravitational radius \(r_g\) is taken as 0.15, 0.21, and 0.3, which correspond to the magnetar mass of 1.0, 1.4, and 2.0 \(M_\odot\), respectively. The Aly–Sturrock energy thresholds, shown in Figure 2 for the three values of the magnetar mass, are \(M_{\text{open}} = 1.80 M_{\text{pot}}\), \(M_{\text{open}} = 1.88 M_{\text{pot}}\), and \(M_{\text{open}} = 2.06 M_{\text{pot}}\), respectively. According to this table, we note that, when \(m \geq 8\) for \(r_g = 0.15, 0.21\) and \(m \geq 10\) for \(r_g = 0.3\), the magnetic energy in the magnetosphere could be higher than the Aly–Sturrock threshold.
where the stream function is also normalized. The magnetic fields with $r_{g} = 0.15$ are shown in Figure 4. The left panel in this figure corresponds to the “+” sign, which approximately models the effects of the neutron star wind (Bucciantini et al. 2006). Such configurations are also discussed by Low & Tsinganos (1986) and applied to model the effects of solar wind. Note that when $\lambda$ increases to 5.0, the magnetic energy in the left panel is about $1.83 M_{\odot}$, exceeding the corresponding Aly–Sturrock energy $M_{\text{pot}}$ by about 2%, which suggests this state may support a giant flare. If $\lambda$ is even increased, more magnetic energy can be obtained. The right panel takes the “−” sign in the above equation. Though the right panel shows a state that is physically unacceptable, it is worth pointing out that the energy of the state with detached field lines ($\sim 2.86 M_{\text{pot}}$) is much higher than the energy in the left panel. This also suggests that, when there are detached fields in the magnetosphere, the stored magnetic energy can be much larger than those configurations whose field lines are all anchored to the magnetar surface. This possibility to bypass the Aly–Sturrock constraint has been discussed by Flyer et al. (2004) for solar CMEs and will be further discussed for magnetar giant flares (C. Yu 2011, in preparation).

5. CONCLUSIONS AND DISCUSSIONS

We construct GR models of non-rotating neutron stars endowed with strong magnetic fields. The equilibrium states of axisymmetric force-free magnetic fields in magnetar magnetospheres are found as solutions of the Grad–Shafranov equations in the Schwarzschild geometry. A newly derived GR magnetic virial theorem is presented in this work. Based on this magnetic virial theorem, we carefully examine the GR effects on the well-known Aly–Sturrock energy threshold. We found that this energy threshold increases with the magnetar mass. As a result, it is more difficult for massive magnetars to erupt. By this observation, we conclude that magnetars are probably not neutron stars with extreme mass. The non-force-free magnetic field induced by the mass loading is further investigated as a possibility to bypass the Aly–Sturrock constraint for typical magnetar mass around $\sim 1.4 M_{\odot}$.

We mainly discuss dipolar surface boundary conditions in this paper. This is the case for magnetar’s large scale fields. But observations show a striking feature that the emergence of a strong four-peaked pattern in the light curve of the 1998 August 27 event from SGR 1900+14, which was shown in data from the Ulysses and BeppoSAX gamma-ray detectors (Feroci et al. 2001). These remarkable data may imply that the geometry of the magnetic field was quite complicated in regions close to the star where GR effects are important. As a result, complex boundary conditions should be important for the outburst of magnetars. Effects of different boundary conditions on the energy buildup in magnetars are worth further investigations (Antiochos et al. 1999).

For simplicity, we have neglected the relativistic wind from the neutron star surface. Actually, the wind from the neutron star (e.g., Bucciantini et al. 2006) may cause part of the magnetic field lines to be in the open states before eruption. Similar effects have been explored in solar CMEs (Low & Smith 1993; Wolfson 1993). It is interesting to investigate the effects of neutron star wind on the magnetic energy storage properties.

Helicity has been discussed extensively in solar physics (Zhang & Low 2005). CMEs are believed to be the unavoidable products of the coronal evolution as a result of magnetic helicity accumulation (Zhang et al. 2006). But helicity in the GR regime is not a well-explored issue. Finding a self-consistent definition of helicity in the curved spacetime and investigating the relevant helicity properties are interesting topics for further explorations.

The field topology change from a closed state to an open state must be accompanied by the magnetic reconnection. After a certain threshold is reached, the dynamical instability sets in. The gradual quasi-static evolution of the magnetar’s
magnetosphere will be replaced by the dynamical evolution of the field. This naturally explains the problem as to how a very slow buildup of the external shear (over an interval of \(\sim 100 \text{ yr}\)) could lead to the sudden release of external magnetic energy on a much shorter timescale (Lyutikov 2006). The magnetic energy dissipation in the strongly magnetized plasma is caused by the tearing mode instability (Lyutikov 2003; Komissarov et al. 2007). Relativistic tearing instability induced reconnections in the nonlinear regime need further studies to better understand the magnetar outburst behaviors.

Our theoretical models cannot address the nonlinear dissipation processes that occur during giant flares. However, current GRMHD simulations provide a unique opportunity to study the dynamical outburst physics. The models constructed in this work are likely to be useful as initial states in GRMHD numerical simulations to explore the dynamics of magnetic eruptions (Gammie et al. 2003; C. Yu 2011).

We thank the anonymous referee for important comments and suggestions that improve this paper greatly. The research is supported by the Natural Science Foundation of China (grants 10873033, 10703012, 10778702, and 10973034), the Western Light Young Scholar Program, and the 973 Program (grant 2009CB824800). The computation is performed at HPC Center, Kunming Institute of Botany, CAS, China.

APPENDIX A

DERIVATION OF VIRIAL THEOREM IN SCHWARZSCHILD METRIC

The four equations expressing conservation of energy momentum are

\[ \nabla_c T^{\mu \nu} = 0, \quad (A1) \]

where the Einstein summation rule is assumed and Greek letters take on the values \(t, r, \theta, \phi\). The four-velocity for a plasma in magnetostatic equilibrium is

\[ u^\nu = (-g_{tt})^{-1/2}, \quad u^t = u^\theta = u^\phi = 0. \quad (A2) \]

Given the energy–momentum tensor in Equation (3), the covariant derivative can be expanded as follows:

\[ \nabla_c T^{\mu \nu} = g^{\mu \nu} \frac{\partial}{\partial x^\nu} \left( p + \frac{b^2}{2} \right) + \Gamma^\nu_{\rho \sigma} (p + \rho + b^2) u^\rho u^\sigma - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left( \sqrt{-g} b^\mu b^\nu \right) - \Gamma^\nu_{\beta \sigma} b^\beta b^\sigma. \quad (A3) \]

The radial component of the above equation becomes (note that the connection coefficients \(\Gamma^\nu_{\mu \sigma} = -\frac{1}{2g} g^{\mu \nu} \frac{\partial g_{\tau \sigma}}{\partial x^\tau}\))

\[ g^{\nu \tau} \frac{\partial}{\partial r} \left( p + \frac{b^2}{2} \right) + g^{\nu \tau} \frac{1}{2g_{tt}} \left( p + \rho + b^2 \right) \frac{\partial g_{tt}}{\partial r} - \left( \frac{1}{\sqrt{-g}} \frac{\partial}{\partial r} \left( \sqrt{-g} b^\nu b^\tau \right) + \Gamma^\nu_{\tau \rho} b^\rho b^\tau + \Gamma^\nu_{\rho \sigma} b^\rho b^\sigma + \Gamma^\nu_{\mu \phi} b^\mu b^\phi \right) = 0, \quad (A4) \]

where

\[ \Gamma^\nu_{\tau \rho} = -\frac{r_g}{r(r-2r_g)}, \quad \Gamma^\nu_{\rho \phi} = -(r-2r_g), \quad \Gamma^\nu_{\mu \phi} = -(r-2r_g) \sin^2 \theta. \]

The “ordinary” component of the magnetic field \(B\) in the orthogonal basis (Weinberg 1972) is related to the magnetic field four-vector \(b^\mu\) by

\[ B_r = \sqrt{g_{rr}} b^r = \sqrt{g^{rr}} b_r, \quad B_\theta = \sqrt{g_{\theta \theta}} b^\theta = \sqrt{g^{\theta \theta}} b_\theta, \quad B_\phi = \sqrt{g_{\phi \phi}} b^\phi = \sqrt{g^{\phi \phi}} b_\phi. \quad (A5) \]

and

\[ b^2 = b_\mu b^\mu = B_r^2 + B_\theta^2 + B_\phi^2. \quad (A6) \]

Multiplying the Equation (A4) by \(r\) and expressing the magnetic field four-vector \(b^\mu\) by the “ordinary” magnetic fields \(B\) in Equation (A5), we may arrive at

\[ \alpha^2 r \frac{\partial}{\partial r} \left( p + \frac{B^2}{2} \right) + \frac{r_g}{r} (p + \rho + B^2) - \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^3 \alpha^2 B_r^2 \right) - \frac{\alpha}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta B_r B_\theta \right) + \frac{r_g}{r} B_r^2 + \frac{r_g}{r} B_\theta^2 + \alpha^2 (B_r^2 + B_\theta^2 + B_\phi^2) = 0. \quad (A7) \]

Performing the volume integral with the usage of Gauss’s theorem, the above equation can be re-arranged to give the generalized virial theorem, Equation (7) in the main text.

APPENDIX B

DIPOLE FIELD BOUNDARY CONDITIONS AND SEPARABLE SOLUTIONS FOR POTENTIAL FIELDS

To get the dipole field boundary conditions, we need to obtain the current-free potential field. To be self-contained, we describe the separable solutions of the homogenous Grad–Shafranov equation, which are also the building blocks for the Aly–Sturrock fully
open field. The homogenous Grad–Shafranov equation reads

\[ \frac{\partial}{\partial r} \left[ \left( 1 - \frac{2r_g}{r} \right) \frac{\partial \Psi}{\partial r} \right] + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \theta} \right) = 0. \]  
\( \text{(B1)} \)

Separable solutions of the above equation are of the form

\[ \Psi(r, \theta) = R(r) \Theta(\theta). \]  
\( \text{(B2)} \)

Substitute the above equation into Equation (B1), we obtain

\[ \frac{d}{d\theta} \left( \frac{1}{\sin \theta} \frac{d\Theta}{d\theta} \right) = -\lambda \frac{\Theta}{\sin \theta}, \]  
\( \text{(B3)} \)

\[ \frac{d}{dr} \left[ \left( 1 - \frac{2r_g}{r} \right) \frac{dR}{dr} \right] = \lambda \frac{R}{r^2}, \]  
\( \text{(B4)} \)

where \( \lambda \) is the separation constant. The lowest order of solution is the special case with \( \lambda = 0 \), which can be obtained by setting \( \lambda = 0 \) in the above two equations. The solutions are then

\[ \Theta(\theta) = a \cos \theta + b, \]  
\( \text{(B5)} \)

\[ R(r) = c, \]  
\( \text{(B6)} \)

where \( a, b, \) and \( c \) are constants. This solution is the Schwarzschild monopole.

The order of the solution is denoted by the ordinal number \( m \) ( \( m = 1 \) corresponds to the dipole field), related to the constant \( \lambda \) by \( \lambda = m(m + 1) \). Equations (B3) and (B4) become (Ghosh 2000)

\[ (1 - \mu^2) \frac{d^2 \Theta}{d\mu^2} + m(m + 1) \Theta = 0, \]  
\( \text{(B7)} \)

\[ (1 - z^2) \frac{d^2 R}{dz^2} - 2 \frac{dR}{dz} + m(m + 1) R = 0, \]  
\( \text{(B8)} \)

where \( \mu = \cos \theta \) and \( z = r/r_g - 1 \).

The solution of Equation (B7) is

\[ \Theta(\mu) = (1 - \mu^2) \frac{dP_m(\mu)}{d\mu}, \]  
\( \text{(B9)} \)

where \( P_m(\mu) \) is the Legendre polynomial. The solutions of Equation (B8) are

\[ R(r) = r^2 \left\{ \begin{array}{ll} P_{m-1}^{(0,2)}(z) \\
Q_{m-1}^{(0,2)}(z) \end{array} \right\}, \]  
\( \text{(B10)} \)

where \( P_{m-1}^{(0,2)}(z) \) and \( Q_{m-1}^{(0,2)}(z) \) are Jacobi polynomial and Jacobi functions of the second kind, respectively. For \( r \gg r_g \), the Jacobi polynomial and Jacobi function’s asymptotic behaviors are (Szegő 1939)

\[ P_{m-1}^{(0,2)}(z) \sim r^{m-1}, \quad Q_{m-1}^{(0,2)}(z) \sim r^{-m-2}. \]  
\( \text{(B11)} \)

The superscripts in the Jacobi polynomial and Jacobi function will be suppressed hereafter, as the values remain the same throughout this study. The explicit expressions for the Jacobi polynomials and Jacobi functions can be found in Gradshteyn & Ryzhik (1980).

Of particular interest is the dipole configuration (\( m = 1 \)) determined by

\[ \Psi = \left[ (1 - \mu^2) \frac{P_1(\mu)}{d\mu} \right] r^2 Q_0(z) = \left[ \frac{r^2}{2} \ln \left( \frac{r}{r - 2r_g} \right) - rr_g - r_g^2 \right] \sin^2 \theta. \]  
\( \text{(B12)} \)

This solution can be used as boundary conditions.
APPENDIX C
DETERMINATION OF THE ALY–STURROCK FIELD

To appreciate the Aly–Sturrock constraint on the availability of free magnetic energy, we need to determine the Aly–Sturrock state numerically. Following Low & Smith (1993), the boundary conditions of Aly–Sturrock fully opened field can be obtained by flipping the flux function according to the boundary condition (17):

\[
\Psi_{\text{modify}} = \begin{cases} 
\Psi(r_0, \theta) & 0 \leq \theta \leq \pi/2 \\
2\Psi(r_0, \pi/2) - \Psi(r_0, \theta) & \pi/2 \leq \theta \leq \pi. 
\end{cases}
\]

(C1)

Specifically, for the original dipole boundary condition, the modified boundary condition becomes

\[
\Psi_{\text{modify}}(r_0, \theta) = B_0 A_1 \times \begin{cases} 
\sin^2 \theta & 0 \leq \theta \leq \pi/2 \\
2 - \sin^2 \theta & \pi/2 \leq \theta \leq \pi,
\end{cases}
\]

(C2)

where

\[
A_1 = \frac{r_0^2}{2} \ln \left( \frac{r_0}{r_0 - 2 r_g} \right) - r_0 r_g - r_g^2
\]

and \(r_0\) is the magnetar radius. The solutions to the homogeneous Grad–Shafranov equation are of the form

\[
\Psi(r, \theta) = \sum_{n=1}^{\infty} a_n (r^2 Q_{n-1}(r)) \left( \sin^2 \theta \frac{d P_n(\mu)}{d \mu} \right) + \alpha_0 + \alpha_1 \cos \theta,
\]

(C3)

where \(\mu = \cos \theta\) and \(Q_{n-1}(r)\) is the Jacobi function of the second kind. It is clear that

\[
\alpha_0 = B_0 A_1, \quad \alpha_1 = -B_0 A_1.
\]

We define the following flux function as

\[
\Psi^*(r, \theta) = \Psi(r, \theta) - \alpha_0 - \alpha_1 \cos \theta.
\]

The problem becomes to determine the coefficient \(a_n\)

\[
\Psi^*(r, \theta) = \sum_{n=1}^{\infty} a_n (r^2 Q_{n-1}(r)) \left( \sin^2 \theta \frac{d P_n(\mu)}{d \mu} \right),
\]

(C5)

subject to the modified boundary condition (C2)

\[
\Psi^*(r_0, \theta) = \Psi(r_0, \theta) - \alpha_0 - \alpha_1 \cos \theta = B_0 A_1 \times \begin{cases} 
\sin^2 \theta - 1 + \cos \theta & 0 \leq \theta \leq \pi/2 \\
1 - \sin^2 \theta + \cos \theta & \pi/2 \leq \theta \leq \pi.
\end{cases}
\]

(C6)

According to the orthogonality of associated Legendre polynomials \(P_n^1(\mu)\), we have that

\[
a_n = -\frac{1}{r_0^2 Q_{n-1}(r_0)} \frac{2n + 1}{2n(n + 1)} \int_0^\pi \Psi^*(r_0, \theta) P_n(\mu) d\theta.
\]

(C7)

Note that \(\Psi^*(r_0, \theta)\) is an odd function of \(\theta\) in the integration range. When \(n\) is an odd integer, the coefficients \(a_n\)'s vanish. The non-zero coefficients \(a_n\)'s \((n = 2m)\) can be written as

\[
a_{2m} = -\frac{B_0 A_1}{r_0^2 Q_{2m-1}(r_0)} \frac{4m + 1}{2m(2m + 1)} \int_0^{\pi/2} (\sin^2 \theta - 1 + \cos \theta) P_{2m}^1(\cos \theta) d\theta,
\]

(C8)

where \(P_{2m}^1(\cos \theta)\) is the associated Legendre polynomial. After some manipulations, we arrive at

\[
a_{2m} = \frac{B_0 A_1}{r_0^2 Q_{2m-1}(r_0)} \frac{4m + 1}{m(2m + 1)} \frac{(-1)^{m-1}(2m - 2)!}{2^{2m}(m - 1)!(m + 1)!} \frac{2^{2m}(m - 1)!}{Q_{2m-1}(r_0)}.
\]

(C9)

The radial and \(\theta\) components of the magnetic field, according to Equation (20), are

\[
B_r = \sum_{m=1}^{\infty} a_{2m} Q_{2m-1}(r) [2m(2m + 1) P_{2m}] - \frac{\alpha_1}{r^2},
\]

(C10)
The potential dipole field energy

where prime denotes derivative with respect to \( r \). The magnetic energy of the open field, according to Equation (13), is

\[
M_{\text{open}} = \frac{\pi r_0^3}{\sin \theta_0} \left\{ 1 - \frac{2r_g}{r_0} \right\} \sqrt{1 - \frac{2r_g}{r}} \sum_{m=1}^{\infty} c_{2m} \frac{4m(2m+1)}{4m+1} \times \left[ 2m(2m+1) - \left( 1 - \frac{2r_g}{r_0} \right) \left( \frac{2 + r_0 Q_{2m-1}(r_0)}{Q_{2m-1}(r_0)} \right)^2 \right] \]

\[
+ 2 \left( 4\pi r_g B_0^2 A_1^2 \right) \int_{r_0}^{\infty} \frac{1}{r^3} dr + 4\pi r_g \sum_{m=1}^{\infty} \frac{c_{2m}^2}{4m+1} \int_{r_0}^{\infty} \left( \frac{Q_{2m-1}(r)}{Q_{2m-1}(r_0)} \right)^2 r dr \]

\[
+ 2\pi r_g \sum_{m=1}^{\infty} \frac{c_{2m}^2}{4m+1} \int_{r_0}^{\infty} \left( r - 2r_g \right) \left( \frac{Q_{2m-1}(r) + r Q_{2m-1}(r)}{Q_{2m-1}(r_0)} \right)^2 dr. \quad (C12)
\]

The potential dipole field energy \( M_{\text{pot}} \) can be calculated as follows:

\[
B_r = 2B_0 g_r(r) \cos \theta, \quad (C13)
\]

\[
B_\theta = B_0 g_\theta(r) \sin \theta, \quad (C14)
\]

where

\[
g_r(r) = \frac{1}{2} \ln \left( \frac{r}{r - 2r_g} \right) - \frac{r_g}{r} - \frac{r_g^2}{r^2}, \quad (C15)
\]

\[
g_\theta(r) = \sqrt{1 - \frac{2r_g}{r}} \left[ \frac{2r_g(r - r_g)}{r(r - 2r_g)} - \ln \left( \frac{r}{r - 2r_g} \right) \right]. \quad (C16)
\]

The potential dipole field energy is

\[
M_{\text{pot}} = \frac{1}{2} \int B_0^2 \left( 4g_r^2(r) \cos^2 \theta + g_\theta^2(r) \sin^2 \theta \right) dV. \quad (C17)
\]

**APPENDIX D**

**SOLUTIONS FOR THE ORDINARY DIFFERENTIAL EQUATION (30)**

For \( m = 3, 4, 5, 6, 7, \) and \( 8, \) the functions \( f_m(r) \) are

\[
f_3 = \frac{\lambda}{2}, \quad (D1)
\]

\[
f_4 = \frac{2rr_g + r^2 + r \ln(r - 2r_g) - r^2 \ln r}{8r_g^3}, \quad (D2)
\]

\[
f_5 = \frac{6r^2r_g + 6rr_g^2 + 8r_g^3 + 3r^3 \ln(r - 2r_g) - 3r^3 \ln r}{48r_g^4}, \quad (D3)
\]

\[
f_6 = \frac{6r^3r_g + 6r^2r_g^2 + 8rr_g^3 + 12r_g^4 + 3r^4 \ln(r - 2r_g) - 3r^4 \ln r}{192r^2g^5}, \quad (D4)
\]

\[
f_7 = \frac{30r^4r_g + 30r^4r_g^2 + 40r^2r_g^3 + 60rr_g^4 + 96r_g^5 + 15r^5 \ln \frac{r - 2r_g}{r}}{2880r^3g^6}, \quad (D5)
\]

\[
f_8 = \frac{30r^5r_g + 30r^4r_g^2 + 40r^3r_g^3 + 60r^2r_g^4 + 96rr_g^5 + 160r_g^6 + 15r^6 \ln \frac{r - 2r_g}{r}}{7680r^4g^7}, \quad (D6)
\]

respectively. In the calculation of the magnetic energy in the exterior of the neutron star, the stream functions are normalized as

\[
\Psi = \frac{f_m(r)}{f_m(r_0)} \sin^2 \theta. \quad (D7)
\]
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