Deformed Nahm Equation and a Noncommutative BPS Monopole

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A deformed Nahm equation for the BPS equation in the noncommutative N=4 supersymmetric U(2) Yang-Mills theory is obtained. Using this, we constructed explicitly a monopole solution of the noncommutative BPS equation to the linear order of the noncommutativity scale. We found that the leading order correction to the ordinary SU(2) monopole lies solely in the overall U(1) sector and that the overall U(1) magnetic field has an expected long range component of magnetic dipole moment.

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As shown recently, quantum field theories in noncommutative spacetime naturally arise as a decoupling limit of the worldvolume dynamics of D-branes in a constant NS-NS two-form background [1]. But detailed dynamical effects due to the noncommutative geometry are still partially understood [1, 2, 3, 4].

In this note, we shall exploit the N=4 supersymmetric gauge theories on noncommutative $\mathbb{R}^4$ that is the worldvolume theory of D3-branes in the NS-NS two-form background. This theory was recently investigated to study the nature of monopoles and dyons in the noncommutative space [5]. The energy and charge of monopoles satisfying the noncommutative BPS equations were identified and shown to agree those of ordinary monopoles. Below, we shall concentrate on the construction of a self-dual monopole solution of the theory by generalizing ADHMN methods [6, 7, 8, 9] to the noncommutative case.

We begin with the noncommutative N=4 supersymmetric Yang-Mills theory. We shall restrict our discussion to the case of $U(2)$ gauge group. Among the six Higgs fields, only a Higgs field $\phi$ plays a role in the following discussions of a monopole. The bosonic part of the action is given by

$$S = -\frac{1}{4g_{YM}^2} \int d^4 x \, \text{tr} \left( F_{\mu \nu} * F^{\mu \nu} - 2 D_\mu \phi * D^\mu \phi \right),$$

(1)

where the $*$-product is defined by

$$a(x) * b(x) \equiv \left( e^{\frac{2i}{\theta} \partial_{\mu} \partial_{\nu} a(x) b(x')} \right) \bigg|_{x=x'}$$

(2)

that respects the associativity of the product. We shall assume in the following that $\theta_{01} = -\theta_{02} = 0$. Then without loss of generality, one may take the only nonvanishing components to be $\theta_{12} = -\theta_{21} \equiv \theta$. $F_{\mu \nu}$ and $D_\mu \phi$ are defined respectively by

$$F_{\mu \nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i (A_\mu * A_\nu - A_\nu * A_\mu)$$

$$D_\mu \phi \equiv \partial_\mu \phi + i (A_\mu * \phi - \phi * A_\mu)$$

(3)

The four vector potential and $\phi$ belong to $U(2)$ Lie algebra given by $T_4 = \frac{1}{2} I_{2 \times 2}$ and $(T_1, T_2, T_3) = \frac{1}{2} (\sigma_1, \sigma_2, \sigma_3)$ normalized by $\text{tr} \, T_m T_n = \frac{1}{2} \delta_{mn}$. The vacuum expectation value of the Higgs field $\langle \phi \rangle$ is taken to be $T_3 U$ in the asymptotic region.

As shown in Ref. [3], the energy functional

$$M = \frac{1}{2g_{YM}^2} \int d^3 x \, \text{tr} \left( E_i * E_i + D_0 \phi * D_0 \phi + B_i * B_i + D_i \phi * D_i \phi \right) \geq \frac{1}{g_{YM}^2} \int_{r=\infty} dS_k \text{tr} \, B_k * \phi,$$

(4)

is bounded as in the case of the ordinary supersymmetric Yang-Mills theory. The saturation of the bound occurs when the BPS equation

$$B_i = D_i \phi$$

(5)

is satisfied. The mass for the solution is

$$M = \frac{2\pi Q_M}{g_{YM}^2} U$$

(6)

where we define the magnetic charge $Q_M$ by

$$Q_M = \frac{1}{2\pi U} \int_{r=\infty} dS_k \text{tr} \, B_k \phi.$$

(7)
As argued in Ref. [3], the charge is to be quantized at integer values even in the noncommutative case. This is because the fields in the asymptotic region are slowly varying and, hence, the standard argument of the topological quantization of the magnetic charge holds in the noncommutative theory. The main aim of this note is to investigate the detailed form of one self-dual monopole solution in the noncommutative case.

Before proceeding we show first the fact that a noncommutative monopole solution inevitably involves nonvanishing overall $U(1)$ parts. To prove this, let us note the solution for the monopole with $\theta = 0$ is given by

$$
\tilde{\phi} = r \cdot \sigma \frac{1}{2r} \left( \coth r - \frac{1}{r} \right)
$$

$$
\tilde{A}_i = \epsilon_{ijk} x^j \sigma^k \frac{1}{2r^2} \left( \frac{r}{\sinh r} - 1 \right).
$$

We then expand the solution of the noncommutative BPS equation with respect to $\theta$ by

$$
\phi = \tilde{\phi} + \theta \phi^{(1)} + \theta^2 \phi^{(2)} \cdots,
$$

$$
A = \tilde{A} + \theta A^{(1)} + \theta^2 A^{(2)} \cdots.
$$

The leading corrections to $D_i \phi$ and $B_i$ are, respectively,

$$
(B_i)_{(1)} = \epsilon_{ijk} \left( i(\partial \tilde{A}_j \tilde{\partial} \tilde{A}_k - \tilde{\partial} \tilde{A}_j \partial \tilde{A}_k) + \partial_j (A^{(1)}_k) + i[\tilde{A}_j, (A^{(1)}_k)] \right),
$$

$$
(D_i \phi)_{(1)} = i(\partial \tilde{A}_i \tilde{\partial} \tilde{\phi} - \tilde{\partial} \tilde{A}_i \partial \tilde{\phi} - \tilde{\partial} \tilde{\phi} \partial \tilde{A}_i + \partial_i \phi^{(1)} + i[(A^{(1)}_i), \tilde{\phi}] + i[\tilde{A}_i, \phi^{(1)}]),
$$

where $\partial$ and $\tilde{\partial}$ denote derivative with respect to $x + iy$ and $x - iy$. The terms in the first parentheses follow from the evaluation of the $*$-product of the ordinary monopole solution and can be explicitly computed. They are

$$
i \epsilon_{ijk} \left( (\partial \tilde{A}_j \tilde{\partial} \tilde{A}_k - \tilde{\partial} \tilde{A}_j \partial \tilde{A}_k) \right) = \frac{1}{2r} \left( -(r^2 w^2)' \delta_{3i} + (w^2)' x^3 x^i \right) I_{2 \times 2}
$$

$$
i(\partial \tilde{A}_i \tilde{\partial} \tilde{\phi} - \tilde{\partial} \tilde{A}_i \partial \tilde{\phi} - \tilde{\partial} \tilde{\phi} \partial \tilde{A}_i - \partial_i \phi^{(1)} + i[(A^{(1)}_i), \tilde{\phi}] + i[\tilde{A}_i, \phi^{(1)}]) = \frac{1}{r} \left( -(r^2 wh)' \delta_{3i} + (wh)' x^3 x^i \right) I_{2 \times 2},
$$

where $w = \frac{r}{2r} \left( \frac{r}{\sinh r} - 1 \right)$ and $h = \frac{1}{2r} \left( \coth r - \frac{1}{r} \right)$. We note that these have overall $U(1)$ components only and the two are different from each other. Hence to satisfy the Bogomol’nyi equation, there should be overall $U(1)$ contributions in $\phi^{(1)}$ or $A^{(1)}$ to cancel the difference because the $SU(2)$ parts of $\phi^{(1)}$ and $A^{(1)}$ do not produce $U(1)$ components of the field strengths in this leading order.

We now establish the Nahm’s formalism in the noncommutative case to solve the BPS equation. The basic idea is simply to replace all the ordinary product of the standard derivation by $*$-product. This fact was briefly argued already in Ref. [5], but the deformed Nahm equation below was not found. To obtain the self-dual monopole solution, Nahm adapted the ADHM construction of instanton to solve the self-dual Yang-Mills equation in Euclidean four space such that the gauge fields are translationally invariant in the $x_4$-direction. Here we shall begin with reviewing ADHMN construction of instanton solutions.

\[ \Delta^\dagger = -\frac{d}{dt} I_{k \times k} \otimes I_{2 \times 2} + I_{k \times k} \otimes \sigma_i x_i + T_i \otimes \sigma_i, \]
where $T_i$’s are $k \times k$ matrices. The following two conditions are required on $\Delta$;

$$[\Delta^\dagger \Delta, I_{k \times k} \otimes \sigma_i] = 0, \quad \Delta^\dagger \Delta: \text{invertible}.$$  \hfill (17)

We then solve the equation

$$\Delta^\dagger V = 0,$$  \hfill (18)

with a normalization condition

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau \ V^\dagger V = I_{2k \times 2k},$$  \hfill (19)

where $V$ is a $2k \times 2k$ matrix. The gauge fields are given by

$$\tilde{\phi} = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau \ \tau V^\dagger V, \quad \tilde{A}_i = -i \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau \ V^\dagger \partial_i V.$$  \hfill (20)

One may directly verify that

$$F_{mn} = 2\bar{\eta}^i_{mn} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau \ \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau' \ V^\dagger(\tau) I_{k \times k} \otimes \sigma_i (\Delta^\dagger \Delta)^{-1}(\tau, \tau') V(\tau')$$  \hfill (21)

where the indices run from 1 to 4, we identify $A_4$ with $\phi$, i.e. $F_{i4} = D_i \phi$, and $\bar{\eta}^i_{mn} = \epsilon_{imn4} - \delta_{im} \delta_{4n} + \delta_{in} \delta_{4m}$ is the self-dual ’t Hooft tensor.

In the noncommutative case, the derivation goes through once all the product operations are replaced by $\ast$-product operations. Namely, the matrix operator (16) remains the same while the conditions become

$$[\Delta^\dagger \ast \Delta, I_{k \times k} \otimes \sigma_i] = 0, \quad \Delta^\dagger \ast \Delta: \text{invertible}.$$  \hfill (22)

We have to solve

$$\Delta^\dagger \ast V = 0,$$  \hfill (23)

with a normalization condition

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau \ V^\dagger \ast V = I_{2k \times 2k}.$$  \hfill (24)

The gauge fields are now given by

$$\phi = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau \ \tau V^\dagger \ast V, \quad A_i = -i \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau \ V^\dagger \ast \partial_i V.$$  \hfill (25)

These potentials then satisfy the noncommutative BPS equation. Namely, the field strength is the one obtained from (21) by replacing all the ordinary product operations with $\ast$-product operations, so the expression is manifestly self-dual.

Let us now work out the conditions in (17). The operator $\Delta^\dagger \ast \Delta$ reads

$$\Delta^\dagger \ast \Delta = -\frac{d^2}{d\tau^2} I_{k \times k} \otimes I_{2 \times 2} + (T_i + x^i I_{k \times k})(T^\dagger_i + x^i I_{k \times k}) \otimes I_{2 \times 2}$$

$$+ (T_i - T^\dagger_i) \otimes \sigma_i \frac{d}{d\tau} + (-\frac{d}{d\tau} T^\dagger_i + i \epsilon_{ijk} T_j T^\dagger_k - \theta \delta_{3i} I_{k \times k}) \otimes \sigma_i.$$  \hfill (26)
The two conditions demand that \( T_i \)'s are Hermitian matrices and that

\[
\frac{d}{d\tau} T_i = i\epsilon_{ijk} T_j T_k - \theta \delta_{3i} I_{k\times k} .
\]

(27)

This equation is nothing but the Nahm equation for the ordinary Yang-Mills theory if \( \theta \) is set to zero. This deformed Nahm equation can be reduced to the ordinary Nahm equation

\[
\frac{d}{d\tau} \tilde{T}_i = i\epsilon_{ijk} \tilde{T}_j \tilde{T}_k .
\]

(28)

introducing \( \tilde{T}_i \) by \( T_i = \tilde{T}_i - \theta \tau \delta_{3i} I_{k\times k} \). The boundary conditions at \( \tau = \pm 1/2 \) for the Nahm equation (28) are not to be changed from those of ordinary monopoles even for the noncommutative case because they correspond to the long range effects and the noncommutativity plays no role. Hence the boundary conditions are that \( \tilde{T}_i \)'s have simple poles at the boundaries and the residues form an irreducible representation of \( SU(2) \). The ordinary Nahm equation is naturally interpreted as describing supersymmetric ground states for the worldvolume theory of suspended D-strings between D3-branes[12].

The emerging picture is quite consistent with the direct analysis of a D1-brane in the NS-NS two-form background where the same slope appears[5]. Although interesting, we won’t exploit the relationship further here.

We now turn to the problem of the construction of actual solutions of the BPS equation. We try the case of \( k = 1 \), i.e. one monopole. The deformed Nahm equation is solved trivially by \( T_i = -\tau \delta_{3i} - c_i \), where \( c_i \)'s are constants related to the monopole position. We shall set \( c_i \) to zero by invoking the translational invariance of the noncommutative Yang-Mills theory.

The equation (23) is explicitly

\[
-\frac{d}{d\tau} V + \sigma_i x_i * V - \theta \tau \delta_{3i} V = -\frac{d}{d\tau} V + \sigma_i x_i V + \theta \hat{O} V = 0 ,
\]

(29)

where we define \( \hat{O} \) by

\[
\hat{O} = \frac{i\theta}{2} (\sigma_1 \partial_2 - \sigma_2 \partial_1) - \theta \tau \sigma_3 .
\]

(30)

This equation is equivalent to 3-dimensional Dirac equation with time dependent mass in the background magnetic field \( \theta \). The solution with \( \theta = 0 \) is simply

\[
\tilde{V}(\tau, r) = e^{\sigma \cdot r} K(r) ,
\]

(31)

where \( K(r) = (\frac{r}{sinh r})^{1/2} \) is determined by the normalization condition, [19]. One may solve the equation perturbatively by the Dyson series; the solution reads

\[
V = \tilde{V} + \theta e^{\sigma \cdot r} \left( \int_0^\tau ds_1 \hat{O}_I(s_1) K(r) + W_1(r) \right) \\
+ \theta^2 e^{\sigma \cdot r} \left( \int_0^\tau ds_1 \hat{O}_I(s_1) \left( \int_0^{s_1} ds_2 \hat{O}_I(s_2) K(r) + W_1(r) \right) + W_2(r) \right) + \cdots ,
\]

(32)

where \( \hat{O}_I(\tau) = e^{\sigma \cdot r} \hat{O}(\tau) e^{-\sigma \cdot r} \). All the integration constants \( W_n(r) \) are determined by the normalization condition, [24].
Explicit evaluation of $W_1$ by imposing the normalization condition leads to

$$W_1(r) = \frac{K(r)}{4r^2} \left( r\varphi - (S - 1) \right) \sigma_3 + \frac{x_3}{r} \left( \frac{r^2}{2} - r\varphi + (S - 1) \right) \sigma \cdot \hat{r} ,$$

(33)

where $\varphi \equiv \coth r - \frac{1}{r}$ and $S \equiv \frac{r}{\sinh r}$. The solution, $V_{(1)}$ reads explicitly

$$\frac{V_{(1)}}{K} = -\frac{x_3}{4r^2} \left( 2 \tau \cosh(\tau r) - \frac{\sinh(\tau r)}{\tau}(r\varphi + 2) \right) - \sinh(\tau r)(2\tau + \varphi \sigma \cdot \hat{r}) \frac{\sigma_3}{4r}$$

$$- \frac{\tau^2 x_3}{2r} \sigma \cdot \hat{r} e^{\sigma \cdot \hat{r}} + \frac{1}{2r^3} \left( \tau e^{-\sigma \cdot \hat{r}} - \sinh(\tau r) \right) (\sigma \cdot \tau \sigma_3 - x_3)$$

$$+ \frac{e^{\sigma \cdot \hat{r}}}{4r^2} \left( r\varphi - (S - 1) \right) \sigma_3 + \frac{x_3}{r} \left( \frac{r^2}{2} - r\varphi + (S - 1) \right) \sigma \cdot \hat{r} .$$

(34)

It is then straightforward to evaluate $\phi_{(1)}$ and $A_{(1)}$ using (23); they are

$$\phi_{(1)} = 0 ,$$

(35)

$$(A_{(1)})_i = \frac{(S - 1)}{8r^4} (2r\varphi - (S - 1)) \epsilon_{3ij} x_j I_{2 \times 2} .$$

(36)

In this order, there is no correction to the Higgs field, which provides the higher dimensional geometric picture of monopole as a D-string suspended between two D3-branes. Finally, using (12)-(13), the leading corrections to the magnetic field and $D\phi$ are found to be

$$B_{(1)} = (D\phi)_{(1)} = \left( -\frac{1}{r} \left( r^2 G' \hat{\epsilon}_3 + G' x_3 \hat{r} \right) \right) I_{2 \times 2} ,$$

(37)

where $G = \frac{(S - 1)\varphi}{4r^2}$. A few comments are in order. First, the self duality to this order is manifest in the above as it should be. As proved earlier, there are nonvanishing overall $U(1)$ components in the gauge connection. Furthermore, all the leading order corrections lie in the overall $U(1)$ sector. The solution in (12)-(13) is rather unique up to zero-mode and gauge fluctuations that may be shown to agree with those of ordinary SU(2) monopole to $O(\theta)$. Note that the translational zero modes of a monopole are already fixed by setting $c_1$ to zero. The gauge fields, Higgs field and field strengths are nonsingular everywhere to this order. Especially, the magnetic field at the origin behaves

$$\delta B = \theta \left( \frac{1}{36} \hat{\epsilon}_3 + O(\tau^2) \right) I_{2 \times 2} .$$

(38)

We see here constant $U(1)$ magnetic field is induced at the origin. In the asymptotic region, the correction behaves as

$$\delta B = \theta \frac{\hat{\epsilon}_3 + 3(\hat{\epsilon}_3 \cdot \hat{r})\hat{r}}{4r^2} I_{2 \times 2} + \theta \frac{\hat{\epsilon}_3 - 2(\hat{\epsilon}_3 \cdot \hat{r})\hat{r}}{2r^2} I_{2 \times 2} + \text{exponential corrections} .$$

(39)

This long range field does not contribute to the magnetic charge in (5) and, thus, we explicitly see that the magnetic charge remains at integer values. The leading overall $U(1)$ correction is the expected magnetic dipole contribution of the form, $-\frac{p^2 \hat{r} \hat{r}}{r^3}$, with $p = \frac{\theta}{4} \hat{\epsilon}_3 I_{2 \times 2}$; As discussed below (28), $\pm\frac{\theta}{2} \hat{\epsilon}_3$ are the end-point displacements of the D-string from the positions of $\theta = 0$, and the $U(1)$ magnetic charges with a proper normalization are $\pm\frac{\theta}{2}$ on the branes, so $p$ agrees with

The dipole term in this expression was first obtained in Ref. [4].
the charges multiplied by the displacements. The subleading long range term is not in the form of quadrupole moment and its physical origin is not clear.

There are many directions to go further. In the large $\theta$ limit, the structure of the equation \[29\] seems simplified and this may be helpful in understanding the nature of large $\theta$ limit by finding details. As said earlier, one may equivalently describe the noncommutative supersymmetric Yang-Mills theory by the ordinary Dirac-Born-Infeld theory with a magnetic field background \[1.15, 16, 17, 18\]. Understanding this relation was originally one of the main purpose of this note, but no progress is made in this direction.

Our analysis is expected to be directly generalized to the cases of dyons and 1/4 BPS dyons. In addition, the leading effect of noncommutativity to the moduli dynamics of monopoles and 1/4 BPS dyons will be of interest.

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