On random stable partitions

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Abstract
It is well known that the one-sided stable matching problem (“stable roommates problem”) does not necessarily have a solution. We had found that, for the independent, uniformly random preference lists, the expected number of solutions converges to $e^{1/2}$ as $n$, the number of members, grows, and with Rob Irving we proved that the limiting probability of solvability is below $e^{1/2}/2$, at most. Stephan Mertens’s extensive numerics compelled him to conjecture that this probability is of order $n^{-1/4}$. Jimmy Tan introduced a notion of a stable cyclic partition, and proved existence of such a partition for every system of members’ preferences, discovering that presence of odd cycles in a stable partition is equivalent to absence of a stable matching. In this paper we show that the expected number of stable partitions with odd cycles grows as $n^{1/4}$. However the standard deviation of that number is of order $n^{3/8} \gg n^{1/4}$, i.e. too large to conclude that the odd cycles exist with probability $1-o(1)$. Still, as a byproduct, we show that with probability $1-o(1)$ the fraction of members with more than one stable “predecessor” is of order $n^{-1/2+o(1)}$. Furthermore, with probability $1-o(1)$ the average rank of a predecessor in every stable partition is of order $n^{1/2}$. The likely size of the largest stable matching is $n/2 - O(n^{1/4+o(1)})$, and the likely number of pairs of unmatched members blocking the optimal complete matching is $O(n^{3/4+o(1)})$.

Keywords Stable matchings · Partitions · Random preferences · Asymptotics

Mathematics Subject Classification 05C30 · 05C80 · 05C05 · 34E05 · 60C05

1 Introduction and main results

A roommates problem instance is specified by an even integer $n$, which is the number of members, and for each $i$ ($1 \leq i \leq n$) a permutation $\sigma_i$ of the set $[n] = \{1, 2, \ldots, n\}$ in which $i$ itself occupies position $n$, $(\sigma_i(n) = i)$. The permutation $\sigma_i$ forms the
preference list of person $i$: $\sigma_i(k) = j$ if person $j$ occupies position $k$ in the preference list of person $i$, and each person $i$ is at the end of their own preference list. Equivalently, the instance can be specified by the ranking list $R_i$ of each person $i$: $R_i(j) = k$ if the person $j$ is the $k$-th best for person $i$.

For a given problem instance, a complete matching $M$ on the set $[n]$ is called stable if no pair of members $(i, j)$ unmatched in $M$ prefer each other to their partners under $M$. It is well known that for every even $n \geq 4$ there exist instances of preference lists without any stable matching. Tan (1991a) introduced a less restrictive notion of a stable permutation on the members set $[n]$. A generic permutation $\Pi$ on $[n]$ is naturally viewed as a partition of $[n]$ into disjoint union of directed cycles. So, for each $i$, $\Pi(i)$ and $\Pi^{-1}(i)$ are respectively the successor and the predecessor of $i$ on the directed cycle that contains $i$. A permutation $\Pi$ is stable if

\[
\begin{align*}
(1) \forall i \in [n]: & \quad R_i(\Pi(i)) \leq R_i(\Pi^{-1}(i)) ; \\
(2) \forall 1 \leq i \neq j \leq n : & \quad R_i(j) < R_i(\Pi^{-1}(i)) \implies R_j(i) > R_j(\Pi^{-1}(j)).
\end{align*}
\]

Interpretation: the condition (1) states that no person prefers his predecessor to his successor, and the condition (2) states that no two members $i$ and $j \neq \Pi(i), \Pi^{-1}(i)$ prefer each other to their predecessors. Clearly a stable permutation $\Pi$ with all cycles of length 2 corresponds to a stable matching with pairs of members forming those cycles, i.e. transpositions.

Note that equality in condition (1) is possible iff $\Pi^2(i) = i$, i.e. either $i$ is a fixed point of $\Pi(i = \Pi(i))$, in which case $i$ is its own predecessor and successor, or $(i, \Pi(i))$ is a transposition in $\Pi$, i.e. $\Pi(i)$ is both successor and predecessor of $i$, and we say that $(i, \Pi(i))$ is a matched pair in the partition $\Pi$.

Thus inequality (1) is not vacuous iff $i$ is in a cycle of length 3 or more, in which case it is strict. Also if $i$ is a fixed point, i.e. $i = \Pi(i)$, then $R_i(\Pi^{-1}(i)) = R_i(i) = n$; so condition (2) implies that there are no other fixed points, and every $j \neq i$ prefers his own predecessor to $i$. Intuitively, each member $i$ proposes to $\Pi(i)$ and holds a proposal from $\Pi^{-1}(i)$.

In a remarkable development Tan (1991a) proved that, for every instance of preferences, (1) there is at least one stable permutation; (2) all stable permutations have the same odd cycles (“parties”); (3) replacing each even cycle $(i_1, i_2, \ldots, i_{2m})$ of a stable permutation by the transpositions $(i_1, i_2), \ldots, (i_{2m-1}, i_{2m})$, or by the transpositions $(i_2, i_3), \ldots, (i_{2m}, i_1)$ we get another stable, reduced, permutation; (4) a stable matching exists iff there are no odd cycles.

Interestingly, years before Tan (1991a), Shapley and Scarf 1974 studied a commodity trading market with $n$ traders bringing to the market their own individual good, each trader having his own preferences for the goods on the market. An allocation of $n$ goods among $n$ traders can be viewed as a permutation $\Pi$ on $[n]$: trader $i$ gets the good of trader $j = \Pi(i)$. They proved existence, and uniqueness, of a core allocation $\Pi$, the one that cannot be blocked by a coalition of traders, and described a simple (top trading cycles) algorithm for computing this allocation.

Suppose that the random problem instance, call it $I_n$, is chosen uniformly at random among all $[(n - 1)!]^n$ instances. We showed Pittel (1993b) that the expected number

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of stable matchings is \(e^{1/2}\) in the limit, implying that the number of stable matchings, if any exist, is bounded in probability. With Irving and Pittel (1994) we proved that the probability that a stable matching exists is at most \(e^{1/2}/2 < 1\) in the limit. In a pleasing contrast, the stable partitions do not have a fixed point (odd party of size 1) with surprisingly high probability \(\geq 1 - O(e^{-c\sqrt{n}})\). So while a stable matching may not exist, stable partitions (that exist always) with high probability have no “pariah”: every member holds a proposal from another member, while his own proposal is held by possibly a different member.

Our goal is to analyze asymptotic behavior of a series of leading parameters of the full family of stable (reduced) partitions for \(I_n\), and we will focus on those that have no pariah.

**Theorem 1.1** Let \(S_n\) and \(O_n\) denote the total number of stable (reduced) partitions and the total number of odd cycles. (If present, those cycles are common to all stable partitions.) Then we have

\[
E[S_n] = (1 + o(1)) \frac{\Gamma(1/4)}{\sqrt{\pi} e^{21/4}} n^{1/4}, \tag{1.2}
\]

\[
E[O_n] \leq (1 + o(1)) \frac{\Gamma(1/4)}{4\sqrt{\pi} e^{21/4}} n^{1/4} \log n; \tag{1.3}
\]

\((\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \, dt, x > 0.)\)

The fact that \(E[S_n] \to \infty\), but at a moderate rate, can be charitably viewed as supporting the conjecture that, in probability, \(S_n \to \infty\). Since the expected number of stable matchings approaches a finite limit \(e^{1/2}\), we would then be able to claim that with probability \(1 - o(1)\) there are odd cycles, meaning that \(P_n := P(\exists\) a stable matching) \(\to 0\). We had proved, Pittel (1993b), that \(P_n \geq \sqrt{\frac{4e^3}{\pi}} n^{-1/2}\); so if \(P_n\) does approach zero, convergence rate is quite moderate. Numerical experiments conducted by Mertens (2005) made him conjecture that \(P_n\) is of order \(n^{-1/4}\).

To prove rigorously that \(\lim P_n = 0\) one would normally want to show that \(\text{Var}(S_n) \ll E^2[S_n]\), with Chebyshev’s inequality completing the proof. It turns out, however, that \(\text{Var}(S_n)\) is of order \(n^{3/4}\), thus exceeding \(E^2[S_n]\) by the factor \(n^{1/4}\), which invalidates this naive two-moment approach. Can this device be gainfully modified by narrowing the pool of stable partitions?

A key tool for estimating \(\text{Var}(S_n)\) is an asymptotic formula for the probability that each of two generic (reduced) partitions \(\Pi_1\) and \(\Pi_2\) (with the same odd parties, of course) are stable. The symmetric difference of the set of matched pairs in \(\Pi_1\) and the set of matched pairs in \(\Pi_2\) (i.e. the set of pairs of members matched in only one of \(\Pi_1\) and \(\Pi_2\), is the edge set of the disjoint even cycles of length \(\geq 4\). In each such cycle every other edge is a matched pair in \(\Pi_1\), and every edge in between is a matched pair in \(\Pi_2\). Each such cycle can be viewed as an even rotation in both partitions, so that the pair \((\Pi_1, \Pi_2)\) gives rise to \(2^\mu\) of stable, reduced, partitions, with \(\mu\) being the total number of those even cycles, Tan (1991a). Define a random graph \(G_n = (\mathcal{V}_n, \mathcal{E}_n)\), where \(\mathcal{V}_n\) is the set of all stable partitions \(\Pi\), and \(\mathcal{E}_n\) is the set of pairs \((\Pi_1, \Pi_2)\), each giving rise to a single even cycle. By \((5.23)\), \(E[\mathcal{V}_n] = E[S_n]\) is of order \(n^{1/4}\). It turns
out that \( \mathbb{E}[\mathcal{E}_n] \) is of order \( n^{1/4} \) as well. What, if anything, does this fact tell about the likely range of \( S_n \)?

There are three positive results that stem from (1.2)–(1.3), and the analysis of \( \text{Var}(S_n) \). First, Tan (1991a, b) defined a maximum stable matching for an instance \( I \) as a maximum-size matching \( M = M(I) \) which is internally stable, i.e. not blocked by any two members from the agent (vertex) set of \( M \). He proved that \( |M(I)| = (n - O(I))/2 \). Combining Tan’s result with (1.3) we assert

**Theorem 1.2** Let \( M_n \) denote the largest number of matched pairs in an internally stable matching for the random instance \( I_n \). If \( \omega(n) \to \infty \) however slowly, then

\[
P\left( M_n \geq \frac{n - \omega(n)n^{1/4}\log n}{2} \right) \to 1.
\]

In short, with probability \( 1 - o(1) \), the number of members not in the maximum-size stable matching is of order \( n^{1/4} \log n \), at most.

Second, Abraham et al. (2006) introduced the alternative notion of a “maximally stable” matching, i.e. a matching on \([n]\) that is blocked by the smallest number of pairs, call it \( B(I) \), of agents unmatched in \( M \). They obtained a two-sided bound for \( B(I) \) in terms of preference lists lengths and the odd cycles. A cruder version of the ABM upper bound states that \( B(I) \leq d(I)O(I) \), where \( d(I) \) is the length of the longest preference list. Clearly \( d(I) \) can be replaced by \( d_{\text{max}}(I) \), which is the largest rank of a member’s predecessor among all stable partitions. Combining the ABM bound \( B(I) \leq d_{\text{max}}(I)O(I) \) and (1.3) we will prove

**Theorem 1.3** Let \( D_n = d_{\text{max}}(I_n) \) and \( B_n = B(I_n) \), i.e. the number of pairs of members that block a maximally stable matching on \([n]\). Then, for every \( \delta > 0 \), \( D_n \leq n^{1/2}(\log n)^{1+\delta} \) with probability \( \geq 1 - \exp(-c(\log n)^{2(1+\delta)}) \). Consequently, with probability \( 1 - o(1) \), there exists a complete matching which is blocked only by \( O(n^{3/4+o(1)}) \) pairs.

Third, we use the notion of a rotation exposed in a stable partition and the techniques for computing \( \text{Var}(S_n) \) to prove

**Theorem 1.4** The expected fraction of members with multiple stable predecessors is of order \( n^{-1/2+o(1)} \).

Finally

**Theorem 1.5** With probability \( 1 - o(1) \), the sum of the ranks of predecessors in every stable partition is asymptotic to \( n^{3/2} \). Consequently the worst predecessor’s rank in every stable partition is \( n^{1/2}(1 - o(1)) \) at least, which nearly matches the likely upper bound \( n^{1/2}(\log n)^{1+\delta} = n^{1/2+o(1)} \).

Here is an application. Suppose we shrink every member’s preference list to their own best \( d \) choices. If the constrained instance has no fixed point then neither does the full-lists instance. Consider an instance \( I_{n,d} \) of the stable partition
problem chosen uniformly at random among all instances with some $d$ acceptable choices for every member. Randomly, and independently, ordering the remaining $n - 1 - d$ members for every member, we will get the uniformly random (full-lists) instance $I_n$. It follows then that if $d \leq (1 - \varepsilon)n^{1/2}$ ($d \geq n^{1/2}/(\log n)^{1+\delta}$ resp.) then with probability $1 - o(1)$ stable partitions for $I_{n,d}$ have (do not have resp.) a fixed point.

1.1 Organization of the proofs; guiding comments

(1) In Sect. 2 we derive the (multidimensional) integral formulas for $P(\Pi$ is stable), $P(\Pi_1, \Pi_2$ are stable) and $P(\Pi$ is stable and the total predecessors rank is $k$). Derivation is based on a refined probability space, with the random preferences induced by a random $n \times n$ matrix with the entries $X_{i,j}$ being independent, $[0, 1]$-uniform variables. Intuitively $X_{i,j}$ is a quantitative measure by member $i$ of member $j$ as a potential predecessor. (The less $X_{i,j}$ is, the better $j$ is for $i$.) Each integrand is the probability of the attendant event conditioned on the event $\{X_{i,\Pi^{-1}(i)} = x_i, \ i \in [n]\}$, which turns out to be of product type, since the stability conditions are conditionally independent. Expanding those products and integrating would lead to hopelessly monstrous formulas. We review informally the exponential bounds for those products, which will allow us to go directly to asymptotic results for $n \to \infty$.

(2) In Sect. 3 we formulate/prove the basic analytical facts necessary for asymptotic estimation/evaluation of the integrals in question: an identity/bound for the joint density of $S := \sum_{i \in [v]} X_i, X_i$ being independent $[0, 1]$-Uniforms, and the fraction vector $\{X_i/S\}$. It turns out that there is an important connection between these fractions and the intervals lengths $L_i$ in the partition of $[0, 1]$ by $v - 1$ random points. We cite/prove the probabilistic inequalities for $\{L_i\}$. This connection and the bound mentioned in the item (1) will enable us to show, for instance, that the dominant contribution to the value of $P(\Pi) := P(\Pi$ is stable) comes from $x = \{x_i\}$ with $s := \sum_{i \in [n]} x_i$ relatively close to $n^{1/2}$, and $\{x_i/s\}$ close to the likely $\{L_i\}$.

(3) We start Sect. 4 with the proof of the above-cited exponential bounds for the products in the integrands representing the probabilities listed in the item (1). In Sect. 4.1 we use these bounds to bound $P(\Pi)$, $P(\Pi$ is stable and has a fixed point), and to show that it is very unlikely that the total membership of all odd parties (cycles) exceeds $n^{1/2} \log n$.

In Sect. 4.2 we use insight gained in Sect. 4.1 and the connection with $L = \{L_i\}_{i \in [n]}$ to prove a rather sharp asymptotic formula $P(\Pi)$. In Sect. 4.3 we obtain asymptotic estimates/bounds for the expected number of stable partitions and the total number of odd cycles, and apply the latter to show that, with probability $1 - o(1)$, the maximum stable matching contains all but $n^{1/2+o(1)}$ members. In Sect. 4.4 we show that, with probability $1 - o(1)$, there exists a perfect matching blocked by at most $n^{3/4+o(1)}$ pairs.

In Sect. 4.5 we show that, with probability $1 - o(1)$, the total rank of the predecessors in every stable $\Pi$ is sharply concentrated around $n^{3/2}$. We think that the Sects. 3 and 4 already provide an ample evidence of how efficient the integral formulas might be for the probabilistic analysis of the stable partitions and matchings.
(4) In Sect. 5 we derive an asymptotic bound for the variance of the number of stable partitions. This time the center stage belongs to the integral formula for \( P(\Pi_1, \Pi_2) \), which is the probability that two generic partitions are both stable. The proof has certain semblance with the argument for the expected number of stable partitions. It is considerably more technical, because the exponent in the bound for the double-indexed product in the corresponding integrand (Lemma 4.1) is a combination of three squared sums, one term having plus sign. Still this analysis provides a substantial insight into how two stable partitions may likely have common matched pairs, in addition to the common odd cycles.

We would not rule out a possibility that a similar technique can be used for a properly “filtered” set of stable partitions, with the attendant variance comparable to the squared expectation. If so, one could possibly improve the existing upper bound \( 1 - e^{1/2} / 2 \) for the limiting solvability probability, Pittel (1993b), if not show that this limit is zero. For now, we apply this approach to prove possibly the deepest result in this paper, namely that the fraction of members with multiple predecessors is \( n^{-1/2 + o(1)} \) at most.

2 Integral formulas for stability probabilities

At the core of our proofs are the integral formulas, including one for the probability that a generic cyclic partition is stable, and another for the probability that two generic cyclic partitions are stable.

**Lemma 2.1** Let \( \Pi \) be a permutation of \([n]\) with even cycles of length 2 only, and possibly a single fixed point \( h^* \), i.e. \( \Pi(h^*) = h^* \). Let Odd (\( \Pi \)) be the set of all elements from the odd cycles of \( \Pi \) with an exception of the fixed point if it is present. Let \( D(\Pi) \) be the set of unordered pairs \( (i \neq j) \) such that \( i = \Pi(j) \) or, not exclusively, \( j = \Pi(i) \). Then

\[
P(\Pi) := P(\Pi \text{ is stable}) = \int \cdots \int F(x) \, dx,
\]

where

\[
F(x) := \prod_{h \in \text{Odd}(\Pi)} x_h \cdot \prod_{(i,j) \notin D(\Pi)} (1 - x_i x_j) \cdot \prod_{k \neq h^*} (1 - x_k);
\]

(2.1)

if there is no fixed point \( h^* \), then the third product is replaced by 1, and \([0, 1]^{n-1}\) by \([0, 1]^n\).

If \( \Pi \) is a matching, we get (Pittel 1993b)

\[
P(\Pi \text{ is stable}) = \int \cdots \int \prod_{(i,j) \notin D(\Pi)} (1 - x_i x_j) \, dx.
\]

(2.2)

**Proof** To generate the random instance \( I_n \), introduce an array of the independent random variables \( X_{i,j} \) (\( 1 \leq i \neq j \leq n \)), each distributed uniformly on \([0, 1]\). Assume that each member \( i \in [n] \) ranks the members \( j \neq i \) in increasing order of the variables.
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Such an ordering is uniform for every $i$, and the orderings by different members are independent. Then

$$
P(\Pi \text{ is stable} \mid X_{i, \Pi^{-1}(i)} = x_i, \ i \in [n])
\quad = \prod_{h \in \text{Odd}(\Pi)} x_h \cdot \prod_{(i, j) \notin D(\Pi)} (1 - x_i x_j) \prod_{k \neq h^*} (1 - x_k) \tag{2.3}
$$

Indeed, by (1.1), $\Pi$ is stable iff

1. for every $h \in \text{Odd}(\Pi)$: $X_{h, \Pi(h)} < X_{h, \Pi^{-1}(h)}$,
2. for every $(i, j) \notin D(\Pi)$, $i, j \neq h^*$: $X_{i, j} < X_{i, \Pi^{-1}(i)} \Rightarrow X_{j, i} > X_{j, \Pi^{-1}(j)}$,
3. for every $i \neq h$: $X_{i, \Pi^{-1}(i)} < X_{i, h^*}$.

And, conditioned on the event $\{X_{i, \Pi^{-1}(i)} = x_i, \ i \in [n]\}$, the events above are independent, with (conditional) probabilities $x_h$, $1 - x_i x_j$, and $1 - x_k$ respectively. Integrating both sides of the identity over $[0, 1]^{n-1}$, we have (2.1). $\square$

Like analogous formulas in Pittel (1993a, b) and Irving and Pittel (1994), this is a non-bipartite counterpart of Knuth’s formula for stable bipartite matchings, Knuth (1996). His derivation was based on the inclusion-exclusion method, applied directly to the random preference lists and coupled with ingenious observation that the resulting sum equals the multidimensional integral of a product-type integrand resembling our $F(x)$. Of course, we could get a sum-type formula for $P(\Pi \text{ is stable})$ by expanding the product in (2.3) and integrating the resulting sum term-wise. Moving in the opposite direction, i.e. starting with an inclusion-exclusion formula for $P(\Pi \text{ is stable})$, finding an integral-type representation of the generic summand, and discerning that the sum of the attendant integrands happens to be an expansion of the “out-of-the-blue” product in (2.3), would have been very problematic.

The identity (2.1) is indispensable for asymptotic estimates, thanks to a bound

$$
\prod_{(i, j) \notin D(\Pi)} (1 - x_i x_j) \leq \exp\left(-\frac{s^2}{2} + 4.5\right), \quad s := \sum_{i \in [n]} x_i. \tag{2.4}
$$

Its proof will be our first step toward an asymptotic analysis of the multidimensional integrals like the one in Lemma 2.1. We will see, or instance, that this bound and the inequality $\prod_k (1 - x_k) \leq e^{-s}$ will almost immediately yield that the stable partitions have no fixed point with probability $\geq 1 - e^{\Theta(n^{1/2})}$. Here and below we use the notation $\Theta(f(n))$ for the quantity of order $f(n)$ exactly.

We will prove a surprisingly simple, yet qualitatively sharp estimate: uniformly for a fixed-point free partitions $\Pi$,

$$
P(\Pi \text{ is stable}) = O\left(\frac{1}{(n + m - 1)!!}\right), \quad m := |\text{Odd}(\Pi)|. \tag{2.5}
$$
We note that Alcalde (1995) defined an *exchange stable* matching as a matching $M$ that, to quote from Manlove (2013), “admits no *exchange-blocking pair*, which is a pair of members each of whom prefers the other’s partner in $M$ to their own”. Cechlárová and Manlove (2005) proved that, in stark contrast with the classic stable roommates model, the problem of determining whether a given instance admits an exchange-stable matching is NP-complete. The interested reader may wish to check that the formula (2.2) continues to hold for $P(a$ matching $M$ is exchange-stable $)$.

Consequently the expected number of exchange-stable matchings and the expected number of the classic stable matchings are exactly the same, implying that the former is also asymptotic to $e^{1/2}$. Let us call a (fixed-point free) partition $\Pi$ exchange stable if no two members prefer each other predecessors to their own predecessors under $\Pi$. What about the partitions that are “doubly-stable”, i.e. stable and exchange stable? It turns out that

$$P(\Pi) := P(\Pi \text{ is doubly stable}) = \int \cdots \int F_2(x) \, dx,$$

$$F_2(x) := \prod_{h \in \operatorname{Odd}(\Pi)} x_h \cdot \prod_{(i, j) \not\in D(\Pi)} (1 - x_i x_j)^2.$$ 

The counterpart of (2.5) is $P(\Pi) = O\left(2^{-\frac{n+m}{2}}/(n + m - 1)!\right)$, implying that the expected number of the doubly stable partitions is of order $2^{-n/2}$, way down from $n^{1/4}$ for the stable partitions.

Continuing, introduce $R(\Pi)$, the sum of the ranks of all predecessors in the preference lists of their successors in a partition $\Pi$. Let $P_k(\Pi) := P(\Pi \text{ is stable and } R(\Pi) = k)$. Let $[z^k \sum_{j \geq 0} a_j z^j]$ stand for $a_j$, i.e. the coefficient by $z^j$ in the series.

**Lemma 2.2** Suppose $\Pi$ is fixed-point free. Then, letting $m := |\operatorname{Odd}(\Pi)|$, and $\bar{x} = 1 - x$, we have

$$P_k(\Pi) = \int \cdots \int \left[ z^{k-n-m} \right] F(x, z) \, dx,$$

$$F(x, z) := \prod_{h \in \operatorname{Odd}(\Pi)} x_h \cdot \prod_{(i, j) \not\in D(\Pi)} \left( \bar{x}_i \bar{x}_j + z x_i \bar{x}_j + z \bar{x}_i x_j \right).$$

**Proof** First of all, using $\chi(A)$ to denote the indicator of an even $A$, we have

$$P_k(\Pi) = [z^k] E\left[ z^{R(\Pi)} \chi(\Pi \text{ is stable}) \right].$$

Here

$$\chi(\Pi \text{ is stable}) = \prod_{(i, j) \not\in D(\Pi)} \chi(X_{i, j} > X_{i, \pi^{-1}(i)} \text{ or } X_{j, i} > X_{j, \pi^{-1}(j)})$$
Furthermore

\[
R(\Pi) = \sum_{(i,j) \notin D(\Pi)} \left[ \chi(X_{i,j} < X_{i,\Pi^{-1}(i)}) + \chi(X_{j,i} < X_{j,\Pi^{-1}(j)}) \right] + \sum_{i \in [n]} 1 + \sum_{h \in \text{Odd}(\Pi)} \chi(X_h, \Pi(h) < X_h, \Pi^{-1}(h)),
\]

where the second sum accounts for the pairs \((i, \Pi^{-1}(i)), i \in [n]\). So

\[
E\left[z^{R(\Pi)} \chi(\Pi \text{ is stable}) \middle| X_{i,\Pi^{-1}(i)} = x_i, i \in [n] \right] = z^{n+|\text{Odd}(\Pi)|} \prod_{h \in \text{Odd}(\Pi)} x_h \times \prod_{(i,j) \notin D(\Pi)} E[z^{\chi(X_{i,j} < x_i) + \chi(X_{j,i} < x_j)} \chi(X_{i,j} > x_i \text{ or } X_{j,i} > x_j)] = z^{n+m} \prod_{h \in \text{Odd}(\Pi)} x_h \cdot \prod_{(i,j) \notin D(\Pi)} (\bar{x}_i \bar{x}_j + zx_i \bar{x}_j + z\bar{x}_i x_j).
\]

So

\[
E\left[z^{R(\Pi)} \chi(\Pi \text{ is stable}) \right] = z^{n+m} \int \cdots \int F(x, z) \, dx,
\]

which proves (2.6).

Finally, suppose we have a pair of distinct cyclic partitions, \(\Pi_1\) and \(\Pi_2\). Let \(P(\Pi_1, \Pi_2)\) denote the probability that both \(\Pi_1\) and \(\Pi_2\) are stable. We assume the two partitions have the same odd cycles, since otherwise the probability is zero. Suppose also there is no fixed point. Let Odd_{1,2} stand for the vertex set of the family of odd cycles, common to both partitions; so \(\Pi_1(h) = \Pi_2(h)\) for all \(h \in \text{Odd}_{1,2}\). The cardinality \(|\text{Odd}_{1,2}|\) is even, and \(\Pi_1\) and \(\Pi_2\) induce a pair of perfect matchings \((M_1, M_2)\) on Even_{1,2} := \([n] \setminus \text{Odd}_{1,2}\). Together, \(M_1\) and \(M_2\) determine a graph \(G(M_1, M_2) = \left(\text{Even}_{1,2}, E\right)\), with the edge set \(E\) formed by the pairs \((i, j) \in M_1 \cup M_2\). Each component of \(G(M_1, M_2)\) is either an edge \(e \in M_1 \cap M_2\), or a circuit of even length at least 4, in which the edges from \(M_1\) and \(M_2\) alternate. The edge set for all these (alternating) circuits is the symmetric difference \(M_1 \Delta M_2\).

Note. A stable partition \(\Pi\) with at least one even cycle of length \(\geq 4\), thus \textit{not reduced}, can be considered as a superposition of two \textit{reduced} stable partitions \(\Pi_1\) and \(\Pi_2\). Indeed, using every other pair of neighbors in each even cycle as a matched pair we obtain \(\Pi_1\), and using all other “in-between” edges we get the matched pairs for the accompanying \(\Pi_2\).
Lemma 2.3  Let $P(\Pi_1, \Pi_2)$ denote the probability that both $\Pi_1$ and $\Pi_2$ are stable. For $r = 1, 2$, let $D_r$ be the set of unordered pairs $(i \neq j)$ such $i = \Pi_r(j)$ or, not exclusively, $j = \Pi_r(i)$. Then, denoting $a \wedge b := \min\{a, b\}$, we have

$$P(\Pi_1, \Pi_2) = \int \cdots \int F(\mathbf{x}, \mathbf{y}) \, d\mathbf{x}d\mathbf{y},$$

$$F(\mathbf{x}, \mathbf{y}) = \prod_{h \in \text{Odd}_{1,2}} x_h \cdot \prod_{(i, j) \in D_1^c \cup D_2^c} \left[1 - x_i x_j - y_i y_j + (x_i \wedge y_i)(x_j \wedge y_j)\right];$$

where $\mathbf{x}, \mathbf{y} \in [0, 1]^n$, $y_i = x_i$ for $\Pi_1(i) = \Pi_2(i)$.

$$d\mathbf{x} = \prod_{i \in [n]} d\mathbf{x}_i, \quad d\mathbf{y} = \prod_{i : \Pi_1(i) \neq \Pi_2(i)} d\mathbf{y}_i,$$

and for every circuit $\{i_1, \ldots, i_\ell\}$ of $G(M_1, M_2)$:

either $x_{i_1} > y_{i_1}, x_{i_2} < y_{i_2}, \ldots, x_{i_\ell} < y_{i_\ell}$,

or $x_{i_1} < y_{i_1}, x_{i_2} > y_{i_2}, \ldots, x_{i_\ell} > y_{i_\ell}$.

Proof  As in the proof of Lemma 2.1, we use the array $\{X_{i, j} : 1 \leq i \neq j \leq n\}$. By the definition of stability, we have

$$\{\Pi_1, \Pi_2 \text{ are both stable}\} = \bigcap_{h \in \text{Odd}(\Pi_{1,2})} A_h \cap \bigcap_{(i, j) \in D_1^c \cup D_2^c} (B(i, j))^c.$$  

Here $A_h = \{X_{h, \Pi_{1,2}(h)} < X_{h, \Pi_{1,2}^{-1}(h)}\}$. Furthermore: (1) if $(i, j) \in D_1^c \cap D_2^c$, then

$$B(i, j) = \left\{X_{i, j} < X_{i, \Pi_{1}^{-1}(i)}; X_{j, i} < X_{j, \Pi_{1}^{-1}(j)}\right\} \cup \left\{X_{i, j} < X_{i, \Pi_{2}^{-1}(i)}; X_{j, i} < X_{j, \Pi_{2}^{-1}(j)}\right\};$$

(2) if $(i, j) \in D_1^c \cap D_2$, then necessarily $(i, j) \in M_1^c \cap M_2$, and, by stability of $\Pi_1$,

$$B(i, j) = \left\{X_{i, \Pi_{1}^{-1}(i)} < X_{i, \Pi_{1}^{-1}(i)}; X_{j, \Pi_{1}^{-1}(j)} < X_{j, \Pi_{1}^{-1}(j)}\right\};$$

(3) if $(i, j) \in D_1 \cap D_2^c$, then necessarily $(i, j) \in M_1 \cap M_2^c$ and, by stability of $\Pi_2$,

$$B(i, j) = \left\{X_{i, \Pi_{1}^{-1}(i)} < X_{i, \Pi_{1}^{-1}(i)}; X_{j, \Pi_{1}^{-1}(j)} < X_{j, \Pi_{1}^{-1}(j)}\right\}.$$  

Conditioned on the values

$$X_{i, \Pi_{1}^{-1}(i)} = x_i, \quad (i \in [n]), \quad X_{i, \Pi_{2}^{-1}(i)} = y_i, \quad (i \in [n] : \Pi_1(i) \neq \Pi_2(i)).$$
the events $A_h, B_{(i,j)}$ are all independent. And, denoting the characteristic function of a set $U \subset [0, 1)^{2n}$ by $\chi(U)$, we have $P(A_h|\cdot) = x_h$,

$$P((B_{(i,j)})^C|\cdot) = \begin{cases} 1 - x_i x_j - y_i y_j + (x_i \land y_i)(x_j \land y_j), & \text{Case (1)}, \\ \chi(y_i \geq x_i \text{ or } y_j \geq x_j), & \text{Case (2)}, \\ \chi(x_i \geq y_i \text{ or } x_j \geq y_j), & \text{Case (3)}. \end{cases}$$

Therefore

$$P(\Pi_1, \Pi_2 \text{ are both stable}|\cdot)$$

$$\prod_{h \in \text{Odd}1,2} x_h \cdot \prod_{(i,j) \in D_1^c \cup D_2^c} [1 - x_i x_j - y_i y_j + (x_i \land y_i)(x_j \land y_j)],$$

provided that $\forall (i, j) \in M_1^c \cap M_2$, we have $y_i \geq x_i$ or $y_j \geq x_j$ and $\forall (i, j) \in M_1 \cap M_2^c$, we have $x_i \geq y_i$ or $x_j \geq y_j$. (The conditional probability is zero otherwise.) Since the edges from $M_1 \Delta M_2$ form the disjoint alternating circuits of length $\geq 4$, the condition means that for every such circuit $\{i_1, i_2, \ldots, i_\ell\}$ [with $(i_1, i_2) \in M_1, (i_2, i_3) \in M_2, \ldots, (i_\ell, i_1) \in M_2, \text{say}$], we have

$$y_{i_1} \leq x_{i_1} \text{ or } y_{i_2} \leq x_{i_2},$$
$$y_{i_2} \geq x_{i_2} \text{ or } y_{i_3} \geq x_{i_3},$$
$$\vdots$$
$$y_{i_{\ell-1}} \leq x_{i_{\ell-1}} \text{ or } y_{i_\ell} \leq x_{i_\ell},$$
$$y_{i_\ell} \geq x_{i_\ell} \text{ or } y_{i_1} \geq x_{i_1}.$$

We may, of course, assume that all these inequalities are strict. Thus there are only two options on the circuit: either $x_{i_1} > y_{i_1}, x_{i_2} < y_{i_2}, \ldots, x_{i_\ell} < y_{i_\ell}$, or $x_{i_1} < y_{i_1}, x_{i_2} > y_{i_2}, \ldots, x_{i_\ell} > y_{i_\ell}$. In both options, the inequalities alternate. Application of Fubini’s theorem completes the proof. \hfill \Box

A counterpart of the bound (2.4) is

$$\prod_{(i,j) \in D_1^c \cup D_2^c} [1 - x_i x_j - y_i y_j + (x_i \land y_i)(x_j \land y_j)] \leq e^{28} \exp \left( -\frac{s_1^2}{2} - \frac{s_2^2}{2} + \frac{s_{1,2}^2}{2} \right); \quad (2.7)$$

here $s_1 = \sum_i x_i, s_2 = \sum_i y_i, s_{1,2} = \sum_i (x_i \land y_i)$ and $i$ runs over $[n]$. Never mind enormity of $e^{28}$. Like (2.4), the bound (2.7) is both simple and instrumental in identifying a relatively small, eminently tractable, part of the integration domain which is “in charge” of the asymptotic behavior of $P(\Pi_1, \Pi_2)$.

Note. The reader interested in our prior work on stable roommates problem (Pittel 1993a, b; Irving and Pittel 1994) will not find the inequalities (2.4) and (2.7) there.
Working on this project, we detected a technical, estimational, glitch (see the next Section for details) in Pittel (1993a), equally consequential for analysis in Pittel (1993b) and Irving and Pittel (1994). Luckily the new bounds (2.4)–(2.7) allow to repair this error and, as an unexpected bonus, to simplify the arguments as well. The analysis in this paper can be viewed, in part, as a close template for the correction of that embarrassing oversight. We emphasize that, fortunately, this correction leaves the ultimate asymptotic results in those references intact.

Note. Transforming each such circuit \( \{i_1, i_2, \ldots, i_\ell, \} \) into the cycle \( \{i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_\ell \} \) (\( \{i_1 \leftarrow i_2 \leftarrow \cdots \leftarrow i_\ell \} \) resp.) for the second (first resp.) option, we could have obtained a stable, non-reduced, partition.

3 Estimation tools

To estimate the integrals in Lemmas 2.1 and 2.3 for \( n \to \infty \), we will need the following claim, see Pittel (1989, 1993b):

**Lemma 3.1** Let \( X_1, \ldots, X_v \) be independent \([0, 1]\)-Uniforms. Let \( S = \sum_{i \in [v]} X_i \) and \( V = \{V_i = X_i / S ; i \in [v]\} \), so that \( \sum_{i \in [v]} V_i = 1 \). Let \( L = \{L_i ; i \in [v]\} \) be the set of lengths of the \( v \) consecutive subintervals of \([0, 1]\) obtained by selecting, independently and uniformly at random, \( v - 1 \) points in \([0, 1]\). Then (with \( \chi(A) \) standing for the indicator of an event \( A \)) the joint density \( f_{S, V}(s, v) \), \( (v = (v_1, \ldots, v_{v-1})) \), of \((S, V)\) is given by

\[
f_{S, V}(s, v) = s^{v-1} \chi(\max_{i \in [v]} v_i \leq s^{-1}) \chi(s^{-1} + v_{v-1} \leq 1)
\leq \frac{s^{v-1}}{(v - 1)!} f_L(v);
\]

where \( v_v := 1 - \sum_{i=1}^{v-1} v_i \), and \( f_L(v) = (v - 1)! \chi(v_1 + \cdots + v_{v-1} \leq 1) \) is the density of \((L_1, \ldots, L_{v-1})\).

We will also use the classic identities, Andrews et al. (1999), Section 1.8:

\[
\int_{x \geq 0} \cdots \int_{x_1 + \cdots + x_v = 1} \prod_{i \in [v]} x_i^{\alpha_i} \, dx_1 \cdots dx_{v-1} = \frac{\prod_{i \in [v]} \alpha_i!}{(v - 1 + \alpha)!}, \quad \alpha := \sum_{i \in [v]} \alpha_i,
\]

(3.2)
The identity/bound (3.1) is useful since the random vector $L$ had been well studied. It is known, for instance, that

$$L \equiv \left\{ \frac{w_i}{\sum_{j \in [\nu]} w_j} \right\}_{i \in [\nu]},$$

where $w_j$ are independent, exponentially distributed, with the same parameter, 1 say; see Karlin and Taylor (1981), Section 5.4. Here is this property at work. (Recall that, for $g > 0$, $\Theta(g)$ stand for a quantity of order $g$, i.e. sandwiched between $k_1 g$ and $k_2 g$ for two absolute positive constants $0 < k_1 \leq k_2$.)

**Lemma 3.2** Let $s \geq 2$, $\delta < \frac{1}{s+1}$ and $\varepsilon_v = v^{-\delta}$. Then we have (1)

$$P\left( \left| \sum_{j \in [\nu]} L_j^s - 1 \right| \geq \varepsilon_v \right) = O\left( \exp\left( -\Theta(\varepsilon_v v^{s+1}) \right) \right).$$

(2) For $\nu$ even, we have

$$P\left( \left| \sum_{j \in [\nu]/2} L_j L_j + 1 - 1 \right| \geq \varepsilon_2 \right) = O\left( \exp\left( -\Theta(\varepsilon_v v^{s+1}) \right) \right).$$

**Proof** Observe that $E[W] = 1$, $E[W^s] = s!$. Choose

$$a = \left( 1 + \frac{\varepsilon_v}{3} \right) s!, \quad b = 1 - \frac{\varepsilon_v}{3s},$$

so that $a/b^s < (1 + \varepsilon)s!$, for $v$ sufficiently large. Then, denoting $W^{(1)} = \sum_j W_j^1,$

$$P\left( \left| \sum_{j \in [\nu]} L_j^s \right| \geq (1 + \varepsilon_v)s! \right) = P\left( \frac{W^{(s)}}{W^{(1)}} \geq \left( 1 + \varepsilon_v \right) s! \right)$$

$$\leq P\left( W^{(s)} \geq av \text{ or } W^{(1)} < bv \right) \leq P\left( W^{(s)} \geq av \right) + P\left( W^{(1)} < bv \right).$$

Since $E[e^{-zW}] < \infty$ for every $z \geq 0$, the standard application of Chernoff’s method (Chernoff 1952) yields

$$P(W^{(1)} < bv) \leq \exp(-vc(b)), \quad c(b) = b - 1 - \log b = \Theta(\varepsilon_v^2).$$

Bounding $P\left( W^{(s)} \geq av \right)$ is more problematic since $E[e^{zW^2}] = \infty$ for $z > 0$. Truncation to the rescue! Introduce $V = \min\{W, n^\alpha\}$, ($\alpha < 1$); then

$$P(W_j \neq V_j, j \in [\nu]) \leq \nu P(W \geq v^\alpha) = \nu e^{-v^\alpha} = e^{-\Theta(\varepsilon_v^2)}.$$
Further

\[
E \left[ e^{-n^{-\alpha} V_s} \right] = \int_0^n e^{(n^{-\alpha} w)^s} e^{-w} \, dw + e^{1-n^{-\alpha}} \\
\leq 1 + n^{-\alpha} \int_0^\infty w^s e^{-w} \, dw + O(n^{-2\alpha}) \\
= 1 + n^{-\alpha} s! + O(n^{-2\alpha}).
\]

So

\[
P \left( \sum_{j \in [v]} V_j^s \geq a \nu \right) \leq \left( 1 + n^{-\alpha} s! + O(n^{-2\alpha}) \right)^\nu \exp \left( (\nu n^{-\alpha}) \right) \\
= \exp \left( -\nu^{-1-s\alpha} (a - s!) + O(\nu^{1-2\alpha}) \right).
\]

Combining this bound with (3.7), we select the best $\alpha = 1/(s + 1)$ and obtain

\[
P(W(s) \geq a \nu) \leq \exp \left( -\hat{c} \varepsilon_v \nu^{-s+1} \right), \quad (\hat{c} > 0).
\] (3.8)

This bound combined with (3.6) prove that

\[
P \left( \nu^{s-1} \sum_{j \in [v]} L_j^s \geq (1 + \varepsilon_v) s! \right) = O \left( \exp \left( \nu \varepsilon_v \right) \right).
\]

Since $E[e^{-z W_s}] < \infty$ for all $z > 0$, there is no need for truncation, and we get

\[
P \left( \nu^{s-1} \sum_{j \in [v]} L_j^s \leq (1 - \varepsilon_v) s! \right) \leq e^{-\Theta(\nu^{s+1})},
\]

So (3.4) follows. The proof of (3.5) is similar, and we omit it. 

\[\square\]

**Note.** In Pittel (1993a) we claimed that the probabilities in Lemma 3.2, for $\varepsilon$ fixed, could be shown to be exponentially small, and used this claim also in Pittel (1993b) and Irving and Pittel (1994). Having returned to these bounds, we realized though that for the right tail of the sums’ distributions we could get only a sub-exponential bound, see (3.8). Fortunately, with the new inequalities (2.4)–(2.7) put to use, the sub-exponential bounds (3.4) and (3.5) are all we need. The interested reader may see for themselves that the resulting proof provides a clear recipe for local (swap-step type) correctives in Pittel (1993a,b) and Irving and Pittel (1994), which make the thorny issue of exponential bounds go away completely.
In addition to the bounds (3.4), we will need

\[ P \left( \max_{j \in [\nu]} L_j^{(\nu)} \geq \frac{1.01 \log^2 \nu}{\nu} \right) \leq e^{-\log^2 \nu}, \]  

which directly follows from

\[ P \left( \max_{j \in [\nu]} L_j^{(\nu)} \geq x \right) \leq \nu P(L_1 \geq x) = \nu (1 - x)^{\nu - 1}, \]

as \( L_1, \ldots, L_\nu \) are equidistributed.

### 4 Estimates of \( E[S_n] \) and \( E[O_n] \), ramifications

We need to identify a part of the cube \([0, 1]^n\) that provides the dominant contribution to the integral in (2.1). This will allow us to estimate, sharply, the expected total length of the odd cycles in the random instance \( I_n \). Many of the intermediate estimates can be traced back to Pittel (1993a, b) and Irving and Pittel (1994). We begin with the pair of two new, instrumental, bounds for the products in the integrands expressing \( P(\Pi) := P(\Pi \text{ is stable}) \) and \( P(\Pi_1, \Pi_2) := P(\Pi_1 \text{ and } \Pi_2 \text{ are both stable}) \). The second bound will be used in Sect. 5 to handle \( E[S_n^2] \).

In the statement below and elsewhere we will write \( A_n \leq_b B_n \) as a shorthand for \( A_n = O(B_n) \), uniformly over parameters that determine \( A_n, B_n \), when the expression for \( B_n \) is uncomfortably bulky for an argument of the big O notation. We will also write \( A_n \lesssim B_n \) if \( \limsup \frac{A_n}{B_n} \leq 1 \).

**Lemma 4.1**

\[
\prod_{(i, j) \notin D(Pi)} (1 - x_i x_j) \leq_b \exp \left( -\frac{s_1^2}{2} \right), \quad s := \sum_{i \in [n]} x_i,
\]

\[
\prod_{(i, j) \in D_1 \cup D_2} \left[ 1 - x_i x_j - y_i y_j + (x_i \wedge y_i)(x_j \wedge y_j) \right] \leq_b \exp \left( -\frac{s_1^2}{2} - \frac{s_2^2}{2} + \frac{s_{1, 2}^2}{2} \right);
\]

here \( s_1 = \sum_i x_i, s_2 = \sum_i y_i, s_{1, 2} = \sum_i (x_i \wedge y_i) \) and \( i \) runs through \([n]\).

**Proof** (1) Using \( 1 - z \leq e^{-z - z^2/2} \), we have

\[
\prod_{(i, j) \notin D(Pi)} (1 - x_i x_j) \leq \exp \left( -\sum_{(i, j) \notin D(Pi)} \left( x_i x_j + \frac{x_i^2 x_j^2}{2} \right) \right), \quad (4.1)
\]
Here, using $2ab \leq a^2 + b^2$,

$$\sum_{(i,j) \notin D(\Pi)} x_i x_j = \frac{s^2}{2} - \frac{1}{2} \sum_{i \in [n]} x_i^2 - \sum_{i \in [n/2]} x_i x_{i+n/2} - \sum_{h \in \text{Odd}(\Pi)} x_h x_{\Pi(h)} \geq \frac{s^2}{2} - \frac{3}{2} \sum_{i \in [n]} x_i^2. \quad (4.2)$$

Analogously, and using $\max_i x_i \leq 1$,

$$\sum_{(i,j) \notin D(\Pi)} x_i^2 x_j^2 \geq \frac{1}{2} \left( \sum_{i \in [n]} x_i^2 \right)^2 - \frac{3}{2} \sum_{i \in [n]} x_i^4 \quad (4.3)$$

$$\geq \frac{1}{2} \left( \sum_{i \in [n]} x_i^2 \right)^2 - \frac{3}{2} \sum_{i \in [n]} x_i^2. \quad (4.4)$$

Therefore

$$\sum_{(i,j) \notin D(\Pi)} \left( x_i x_j + \frac{x_i^2 x_j^2}{2} \right) \geq \frac{s^2}{2} + \frac{1}{2} \left( \sum_{i \in [n]} x_i^2 \right)^2 - 3 \sum_{i \in [n]} x_i^2 \quad (4.5)$$

$$\geq \frac{s^2}{2} - 4.5, \quad (4.6)$$

so that

$$\prod_{(i,j) \notin D(\Pi)} (1 - x_i x_j) \leq \exp \left( -\frac{s^2}{2} + 4.5 \right).$$

(2) Let $M_1$ be the perfect matching on $\text{Even}_{1,2} = [n] \setminus \text{Odd}_{1,2}$, induced by $\Pi_1$. Then $M_1 \cap M_2$ is the set of matched pairs common to $\Pi_1$ and $\Pi_2$, and $M_1 \Delta M_2$ is the edge set of the even circuits, of length 4 at least, formed (in alternating fashion) by the matched pairs in $M_1$ and $M_2$. So $D_1 \cup D_2$ is the disjoint union of $M_1 \cap M_2$, $M_1 \Delta M_2$ the set of pairs $(i, u_{\Pi_{1,2}})$.

So, given $u_i, i \in [n]$,

$$\sum_{(i,j) \notin D_1 \cap D_2} u_i u_j = \sum_{(i,j) \notin D_1 \cup D_2} u_i u_j - \sum_{(i,j) \in D_1 \cup D_2} u_i u_j$$

$$= \sum_{(i,j) \notin D_1 \cap D_2} u_i u_j - \sum_{(i,j) \in M_1 \cap M_2} u_i u_j - \sum_{(i,j) \in M_1 \Delta M_2} u_i u_j - \sum_{i \in \text{Odd}_{1,2}} u_i u_{\Pi_{1,2}(i)} \quad (4.7)$$

$$= \frac{1}{2} \left( \sum_{i \in [n]} u_i \right)^2 - \frac{1}{2} \sum_{i \in [n]} u_i^2 - \sum_{(i,j) \in M_1 \cap M_2} u_i u_j - \sum_{(i,j) \in E_{1,2}} u_i u_j.$$
here $E_{1,2}$ is the edge set of the odd cycles and the even circuits, formed by $\Pi_1$ and $\Pi_2$. This exact formula certainly implies that

$$\frac{1}{2} \left( \sum_{i \in [n]} u_i \right)^2 - 3 \sum_{i \in [n]} u_i^2 \leq \sum_{(i \neq j) \in D_1^c \cap D_2^c} u_i u_j \leq \frac{1}{2} \left( \sum_{i \in [n]} u_i \right)^2.$$  

Therefore we bound

$$\sum_{(i \neq j) \in D_1^c \cap D_2^c} \left[ x_i x_j + y_i y_j - (x_i \land y_i)(x_j \land y_j) \right]$$

$$\geq \sum_{(i \neq j)} \left[ x_i x_j + y_i y_j - (x_i \land y_i)(x_j \land y_j) \right] - 3 \sum_{i \in [n]} (x_i^2 + y_i^2)$$

$$\geq \frac{s_1^2}{2} + \frac{s_2^2}{2} - \frac{s_{1,2}^2}{2} - 3 \sum_{i \in [n]} (x_i^2 + y_i^2).$$

Furthermore

$$\left[ x_i x_j + y_i y_j - (x_i \land y_i)(x_j \land y_j) \right]^2 \geq \left[ x_i x_j + y_i y_j - (x_i x_j \land y_i y_j) \right]^2$$

$$\geq \left( \frac{x_i x_j + y_i y_j}{2} \right)^2 \geq \frac{1}{8} (x_i^2 x_j^2 + y_i^2 y_j^2).$$

So

$$\sum_{(i \neq j) \in D_1^c \cap D_2^c} \left[ x_i x_j + y_i y_j - (x_i \land y_i)(x_j \land y_j) \right]^2$$

$$\geq \frac{1}{8} \sum_{(i \neq j) \in D_1^c \cap D_2^c} (x_i^2 x_j^2 + y_i^2 y_j^2)$$

$$\geq \frac{1}{16} \left( \sum_{i \in [n]} x_i^2 \right)^2 + \frac{1}{16} \left( \sum_{i \in [n]} y_i^2 \right)^2 - \sum_{i \in [n]} (x_i^4 + y_i^4).$$

As

$$\sum_{i \in [n]} (x_i^4 + y_i^4) \leq \sum_{i \in [n]} (x_i^2 + y_i^2),$$

we obtain

$$\prod_{(i \neq j) \in D_1^c \cap D_2^c} \left[ 1 - x_i x_j - y_i y_j + (x_i \land y_i)(x_j \land y_j) \right]$$

$$\leq \exp \left( -\frac{s_1^2}{2} - \frac{s_2^2}{2} + \frac{s_{1,2}^2}{2} \right).$$
\[ \times \exp \left[ -\frac{1}{32} \left( \sum_{i \in [n]} x_i^2 \right)^2 - \frac{1}{32} \left( \sum_{i \in [n]} y_i^2 \right)^2 + 4 \sum_{i \in [n]} (x_i^2 + y_i^2) \right]. \]

It remains to observe that \(-\frac{z^2}{32} + 4z \leq 128. \]

Note. The gist of the top bound in Lemma 4.1 is that it depends on \(s = \sum_i x_i\) only. This will allow us to show that the dominant contribution to the value of \(P(\Pi)\) comes from the \(x\) with \(s\) very close to \(n^{1/2}\). Once we focus on those \(x\), we will be able to reduce further the integration domain, keeping only the \(x\)'s such that the \(n\) fractions \(\frac{x_i}{n}\) are close to the likely set \(\{L_i : i \in [n]\}\) of the interval lengths in the partition of \([0, 1]\) by \((n - 1)\) random points in \([0, 1]\). The bottom bound will play the similar role in the analysis of \(P(\Pi_1, \Pi_2)\).

4.1 Bounds for \(P(\Pi)\), the probability of a fixed point in a stable \(\Pi\), and the likely \(|\text{Odd} (\Pi)|\)

Here are our first applications of Lemma 4.1.

**Lemma 4.2** Denoting \(m = |\text{Odd} (\Pi)|\),

\[ P(\Pi) \leq \begin{cases} 
\frac{e^{-n^{1/2}}}{(n + m - 2)!!}, & \text{if } \Pi \text{ has a fixed point}, \\
\frac{1}{(n + m - 1)!!}, & \text{if } \Pi \text{ has no fixed point}.
\end{cases} \]

**Proof** For the second case, by (2.1) and Lemma 4.1,

\[ P(\Pi) \leq b \text{ Int}(m) := \int_{x \in [0, 1]^n} \cdots \int_{x \in [0, 1]^n} e^{-\frac{x^2}{2}} \prod_{h=n-m+1}^n x_h \, dx. \]

Disregarding the constraint \(\max_i x_i \leq 1\), and using (3.2), we obtain

\[ \text{Int}(m) \leq \frac{1}{(n + m - 1)!!} \int_0^\infty e^{-\frac{x^2}{2}} x^{n+m-1} \, dx = \frac{(n + m - 2)!!}{(n + m - 1)!!} = \frac{1}{(n + m - 1)!!}. \quad (4.8) \]

If \(\Pi\) has a fixed point \(h^*\), then using

\[ \prod_{k \neq h^*} (1 - x_k) \leq e^{-s}, \quad s := \sum_{k \neq h^*} x_k, \]

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we obtain that
\[
P(\Pi) \leq b \frac{1}{(n+m-2)!} \int_0^{\infty} e^{-s - \frac{s^2}{2}} s^{n+m-2} ds.
\]

A quick glance at the integrand shows that the dominant contribution to the integral comes from \( s \) within, say, \( \log n \) distance from the integrand’s maximum point
\[
s^* = (n + m - 2)^{1/2} - \frac{1}{2} + O(n^{-1/2}),
\]
\((n + m - 2)^{1/2}\) being the maximum point of \( e^{-s^2/2} s^{n+m-2} \). So the above integral is of order
\[
e^{-n^{1/2}} \int_0^{\infty} e^{-\frac{s^2}{2}} s^{n+m-2} ds = e^{-n^{1/2}} (n + m - 3)!!,
\]
whence
\[
P(\Pi \text{ is stable}) \leq b \frac{e^{-n^{1/2}}}{(n + m - 2)!!}.
\]

Now the total number of permutations \( \Pi \) of \([n]\) with a fixed point and \( |\text{Odd}(\Pi)| = m \) is at most
\[
n \begin{pmatrix} n - 1 \\ m \end{pmatrix} m!(n - m - 2)!! = \frac{n!}{(n - m - 1)!!}.
\]

**Corollary 4.3**

\[
P(\text{stable \( \Pi \)'s have a fixed point}) = O(n^2 e^{-\sqrt{n}}) \to 0.
\]

**Proof** By Lemma 4.1 and the union bound, the probability in question is of order
\[
e^{-n^{1/2}} \sum_{m \geq 3} \frac{n!}{(n - m - 1)!! (n + m - 2)!!} \leq e^{-n^{1/2}} \frac{n!}{(n - m - 1)!! (n + m - 2)!!} \bigg|_{m=3} = O(n^2 e^{-n^{1/2}}).
\]

Our original proof in Pittel (1993b) was considerably more involved, and reliant on the problematic existence of the exponential bounds, the issue we touched upon in the previous sections, and will stop bringing up in the sequel.

From now on we focus on stable partitions without a fixed point. Here is another low hanging fruit.
Corollary 4.4 Denoting by $\text{Odd}(\Pi)$ the set of members in the odd cycles of stable partitions,

$$P(|\text{Odd}(\Pi)| \geq n^{1/2} \log n) \leq b \exp(-\log^2 n/3),$$

i.e. with super-polynomially high probability (quite surely in terminology of Knuth et al. (1991)) the total length of all odd cycles is below $n^{1/2} \log n$.

Proof Denote $m_n = \lceil n^{1/2} \log n \rceil$. The total number of potential stable partitions with an even $|\text{Odd}(\Pi)| = m \geq 4$ is at most

$$\binom{n}{m} m!(n-m-1)!! = \frac{n!}{(n-m)!!}.$$

So, by Lemma 4.1, Stirling formula, and the inequality

$$(1+x) \log(1+x) + (1-x) \log(1-x) \geq x^2,$$

the probability in question is of order

$$\leq \sum_{m=m_n}^{n} \exp\left(-\frac{m^2}{2n}\right) \leq b n^{1/2} \int_{x \geq m_n/n^{1/2}} e^{-\frac{x^2}{2}} \, dx \ll e^{-\log^2 n/3}.$$

\hfill \Box

Focusing on the likely stable partitions, we may and will consider only the permutations $\Pi$ without a fixed point and with $|\text{Odd}(\Pi)| \leq m_n$.

4.2 Sharp estimate of $P(\Pi)$

In steps, we will chop off the peripheral parts of the integration cube $[0, 1]^n$ till we get to its part narrow enough to allow us to approximate the integrand in the formula (2.1) within $1 + o(1)$ factor, so that the accumulative error cost is of order $e^{-\Theta(\log^2 n)}$.

Step 1. For the first reduction, we set $s_n = n^{1/2} + 3 \log n$, and define

$$P_1(\Pi) := \int_{\mathbf{x} \in C_1} \cdots \int F(\mathbf{x}) \, d\mathbf{x}, \quad C_1 := \{\mathbf{x} \in [0, 1]^n : s \leq s_n\}, \quad (4.9)$$

Lemma 4.5

$$P(\Pi) - P_1(\Pi) \leq b \frac{e^{-3 \log^2 n}}{(n + m - 1)!!}. \quad (4.10)$$
Proof By Lemma 4.1,

\[ P(\Pi) - P_1(\Pi) \leq_b \frac{1}{(n + m - 1)!} \int_{s \geq s_n} \exp \left( -\frac{s^2}{2} \right) s^{n+m-1} \, ds. \]  

(4.11)

The integrand, write it as \( e^{h(s)} \), attains its maximum at \( s_{n,m} = (n + m - 1)^{1/2} \), and

\[ e^{h(s(n,m))} = \exp \left( -\frac{n + m - 1}{2} \right) (n + m - 1)^{\frac{n+m-1}{2}} \leq_b n^{1/2}(n + m - 2)!! \]

Further

\[ h(s_n) = h(s(n,m)) + (1 + o(1)) \frac{h''(s(n,m))}{2} (s_n - s(n,m))^2 \leq h(s(n,m)) - 4 \log^2 n, \]

\[ h'(s_n) = -s_n + \frac{n + m - 1}{s_n} \leq -5 \log n. \]

Now, since \( h(s) \) is concave, we have

\[ \int_{s \geq s_n} e^{h(s)} \, ds \leq e^{h(s_n)} \int_{s \geq s_n} \exp(h'(s_n)(s - s_n)) = \frac{e^{h(s_n)}}{-h'(s_n)}. \]

Therefore

\[ P(\Pi) - P_1(\Pi) \leq_b \frac{n^{1/2}e^{-4\log^2 n} (n + m - 2)!!}{\log n} \leq \frac{e^{-3\log^2 n}}{(n + m - 1)!!}. \]

Next, motivated by the inequalities (4.2) and (4.3), we will derive sharp asymptotics, on progressively smaller \( C_j \subseteq C_1 \), for the leading sums \( \sum_{i \in [n]} x_i^2 \), \( \sum_{i \in [n]/2} x_i x_{i+n/2}, (n_1 := n - m) \), and obtain sufficiently strong upper bounds for the secondary sums \( \sum_{h \in \text{Odd}(\Pi)} x_h^2 \), \( \sum_{i \in [n]} x_i^4 \) and \( \sum_{i \in [n]} x_i^6 \). We will end up with a rather sharp asymptotic formula for \( \prod (1 - x_i x_j) \) on the terminal dominant subset of \( C_1 \).

**Step 2.** With \( s := \sum_{i \in [n]} x_i \), define \( u = \{ u_i = x_i/s : i \in [n] \} \). Introduce \( t_1(u) = \max_{i \in [n]} u_i \). Define \( C_2 = \{ x \in C_1 : t_1(u) \leq 1.01 \log^2 n \} \), and let \( P_j(\Pi) \) be the integral of \( F(x) \) over \( C_j \). Introduce \( L_1, \ldots, L_n \), the lengths of the \( n \) consecutive subintervals of \([0, 1]\) obtained by choosing, at random, \( n - 1 \) points in \([0, 1]\). Applying Lemma 3.1, the identity (3.2) and Lemma 3.2 (part (1)) with \( v = n \), we have
\[ P_1(\Pi) - P_2(\Pi) \leq \int_{x \geq 0} \prod_{h \in \text{Odd}(\Pi)} u_h \, dx \]

\[
\leq \frac{\mathbb{E}\left[ \chi \left\{ \max_{i \in [n]} L_i \geq 1.01 \log^2 n \right\} \prod_{h=1}^m L_h \right]}{(n-1)!} \int_0^\infty e^{-\frac{s^2}{2}} s^{m+n-1} \, ds.
\]

By the union bound, the expected value is below

\[
n \mathbb{E}\left[ \chi \left\{ L_n \geq 1.01 \frac{\log^2 n}{n} \right\} \prod_{h=1}^{m-1} L_h \right] \leq \int \cdots \int z_h \, dz \leq n! e^{-1.01 \log^2 n} (n+m-2)!
\]

Lemma 4.6

\[
P_1(\Pi) - P_2(\Pi) \leq \frac{e^{-\log^2 n}}{(n+m-1)!!}.
\]

In addition, since \( m \leq m_n = \lceil n^{1/2} \log n \rceil \) and \( s \leq s_n = n^{1/2} + 3 \log n \), it follows from (4.13) that on \( C_2 \)

\[
\sum_{h \in \text{Odd}(\Pi)} x_h \leq b s n^{-2} \log^3 n, \quad \sum_{h \in \text{Odd}(\Pi)} x_h^2 \leq b n^{-1/2} \log^5 n,
\]

\[
\sum_{i=1}^n x_i^4 \leq b n^{-1} \log^8 n, \quad \sum_{i=1}^n x_i^6 \leq b n^{-2} \log^{12} n.
\]

Step 3. With \( \xi := \sum_{i \in [n_1]} x_i \), define \( v = \{ v_i = x_i / \xi : i \in [n_1] \} \). Introduce

\[
t_2(v) = \sum_{i \in [n_1]} v_i^2.
\]

Define

\[
C_3 = \left\{ x \in C_2 : \left| \frac{n}{2} t_2(v) - 1 \right| \leq n^{-\delta} \right\}, \quad \delta < 1/3.
\]
Introduce $L_1, \ldots, L_{n_1}$, the lengths of the $n_1$ consecutive subintervals of $[0, 1]$ in the partition of $[0, 1]$ by the random $n_1 - 1$ points. Analogously to Step 2, we have

$$P_2(\Pi) - P_3(\Pi) \leq \int \cdots \int_{x \geq 0} e^{-\frac{x^2}{2}} \left\{ \frac{\sum_{i \in [n_1]} L_i} {n_1} \leq n^{-\sigma} \right\} \prod_{h \in \text{Odd}(\Pi)} u_h \, dx$$

$$\leq \frac{P \left( \left\{ \frac{\sum_{i \in [n_1]} L_i} {n_1} \leq n^{-\sigma} \right\} \right)} {(n_1 - 1)!(2m - 1)!} \int \int_{\eta, \xi \geq 0} e^{-\frac{\eta^2}{2}} \xi^{n_1 - 1} \eta^{2m - 1} \, d\xi \, d\eta$$

$$\leq \frac{e^{-\Theta(n^{1/3 - \sigma})}} {(n + m - 1)!!} \int_0^\infty e^{-\frac{s^2}{2}} s^{n + m - 1} \, ds = \frac{e^{-\Theta(n^{1/3 - \sigma})}} {(n + m - 1)!!}.$$ 

Lemma 4.7

$$P_2(\Pi) - P_3(\Pi) \leq \frac{e^{-\Theta(n^{1/3 - \sigma})}} {(n + m - 1)!!}. \quad (4.15)$$

Similarly, with $t_3(\nu) := \sum_{i \in [n_1/2]} v_{i-1} v_{i+1/2}$ and 

$$C_4 := \{ x \in C_3 : 2n_1 t_3(\nu) - 1 \leq \sigma \},$$

Lemma 4.8

$$P_3(\Pi) - P_4(\Pi) \leq \frac{e^{-\Theta(n^{1/3 - \sigma})}} {(n + m - 1)!!}. \quad (4.16)$$

Combining the estimates $(4.10)$, $(4.13)$, $(4.15)$, we have

Lemma 4.9 Let $\Pi$ be such that $m = |\text{Odd}(\Pi)| \leq m_n$. Then

$$P(\Pi) - P_4(\Pi) \leq \frac{e^{-\Theta(\log^2 n)}} {(n + m - 1)!!},$$

where $P_4(\Pi)$ is the integral of $F(x)$ over $C_4 \subset [0, 1]^n$ defined by the additional constraints: with $s := \sum_{i \in [n]} x_i$, $s_n = n^{1/2} + 3 \log n$, $\xi = \sum_{i \in [n_1]} x_i$,

$$s \leq s_n, \quad \max_{i \in [n]} x_i \leq 1.02 \frac{s \log^2 n}{n}; \quad (4.17)$$

$$\left| \frac{n_1 \sum_{i \in [n_1]} x_i^2}{2\xi^2} - 1 \right| \leq n^{-\sigma}, \quad \left| \frac{2n_1 \sum_{i \in [n_1/2]} x_i x_{i+n_1/2}}{\xi^2} - 1 \right| \leq n^{-\sigma}. \quad (4.18)$$

The constraint (sumusps involves only $\{x_i\}_{i \in [n_1]}$. Furthermore, given $s$, the constraint (4.17) imposes the uniform upper bound for the individual components $x_i$, $i \in [n]$: no
mixing the components \( x_i, \ i \in [n_1] \), and \( x_h, \ h \in \text{Odd}(\Pi) \), either. Also, this constraint implies that

\[
\max_{i \in [n]} x_i \leq 4n^{-1/2} \log^2 n = o(1),
\]  

(4.19)

meaning that the constraint \( \max_i x_i \leq 1 \) is superfluous. Moreover, the inequality (4.19) yields the equality

\[
\prod_{(i,j) \not\in D(\Pi)} (1 - x_ix_j) = \exp \left( - \sum_{(i,j) \not\in D(\Pi)} \left( x_ix_j + \frac{x_i^2x_j^2}{2} \right) + O \left( \sum_{i \in [n]} x_i^6 \right) \right),
\]

that holds uniformly for \( \mathbf{x} \in C_4 \), with the remainder term \( \ll n^{-1} \), see (4.14). It is the matter of simple algebra to obtain from the constraints on \( C_4 \):

**Lemma 4.10** Uniformly for \( m \leq mn \) and \( \mathbf{x} \in C_4 \),

\[
F(\mathbf{x}) = \exp \left( - \frac{s^2}{2} \left( 1 - \frac{3}{n} \right) - \frac{s^4}{n^2} + O(n^{-\sigma}) \right) \prod_{h \in \text{Odd}(\Pi)} x_h.
\]

(4.20)

Thus, introducing \( \eta = \sum_{h \in \text{Odd}(\Pi)} x_h \), so that as \( s = \xi + \eta \), within the factor \( 1 + O(n^{-\sigma}) \) the integrand depends on \( (\xi, \eta) \) and \( \prod_h x_h \). Observe also that, on \( C_4 \),

\[
\max_{i \in [n_1]} x_i^s/\xi \sim \max_{i \in [n_1]} x_i/s \leq 1.02 \frac{\log^2 n}{n} \ll \frac{1}{\xi}.
\]

So denoting \( \psi_n(s) = \frac{s^2}{2} \left( 1 - \frac{3}{n} \right) + \frac{s^4}{n^2} \), and applying Lemma 3.1, (3.1), we have: \( P_4(\Pi) \), the integral of \( F(\mathbf{x}) \) over \( C_4 \), is given by

\[
P_4(\Pi) = \left( 1 + O(n^{-\sigma}) \right) \int_{\mathbf{x} \in C_4} \ldots \int e^{-\psi_n(\xi + \eta)} \prod_{h \in \text{Odd}(\Pi)} x_h \, d\mathbf{x}
\]

\[
= \left( 1 + O(n^{-\sigma}) \right) \int_{\xi + \eta \leq s_n} \int e^{-\psi_n(\eta + \xi)} \left( \frac{\xi^{n_1 - 1}}{(n_1 - 1)!} \cdot \frac{\eta^{2m - 1}}{(m - 1)!} \right) d\eta \, d\xi
\]

\[
\times P \left( \left| \sum_{i \in [n_1]} L_i^2 - 1 \right| \leq n^{-\sigma}, \ \left| 2n_1 \sum_{i \in [n_1/2]} L_i L_{i+n_1/2} - 1 \right| \leq n^{-\sigma} \right).
\]

(4.21)

From the step (4) we know that the probability factor is at least \( 1 - e^{-\log^2 n} \). The double integral, denote it \( I_{n,m} \), is given by

\[
I_{n,m} = \frac{(2m - 1)!}{(m - 1)! (n + m - 1)!} \int_{s \leq s_n} e^{-\psi_n(s)} s^{n+m-1} \, ds.
\]
The integrand attains its maximum at $\hat{s} = (n + m - 1)^{1/2} - \Theta(n^{-1/2})$, so that $s_n - \hat{s} \geq 2 \log n$. and it is easy to show that

$$\int_{|s - \hat{s}| \geq \log n} e^{-\psi_n(s)} s^{n+m-1} ds \leq e^{-\Theta(\log^2 n)} \int_{|s - s^*| \geq \log n} e^{-\psi_n(a)} s^{n+m-1} ds.$$ 

Besides, $s^4/n^2 = 1 + O(m/n)$ for $|s - \hat{s}| \leq \log n$. Therefore

$$I_{n,m} = e^{-1 + O(m/n)} \frac{(2m - 1)!}{(m - 1)! (n + m - 1)!} \int_{s \geq 0} e^{-\frac{2(1-3/n)}{z}} s^{n+m-1} ds$$

$$= e^{-1 + O(m/n)} \frac{(2m - 1)!}{(m - 1)! (n + m - 1)!} \cdot \frac{(n + m - 2)!!}{(1 - 3/n)(n+m)/2}$$

$$= e^{1/2 + O(m/n)} \frac{(2m - 1)!}{(m - 1)! (n + m - 1)!}.$$ 

Since $m/n = O(n^{-1/2} \log n)$, and $\sigma < 1/3$ in (4.21), we have proved

**Lemma 4.11** Uniformly for even $m \leq m_n$ and $\Pi$ with $\text{Odd}(\Pi) = m$, 

$$P_4(\Pi) = (1 + O(n^{-\sigma})) \frac{e^{1/2}}{(n + m - 1)!!}.$$ 

Consequently, by Lemma 4.9,

$$P(\Pi) = (1 + O(n^{-\sigma})) \frac{e^{1/2}}{(n + m - 1)!!}.$$ (4.22)

**Note.** This sharp formula is qualitatively very close to the upper bound for $P(\Pi)$ in Lemma 4.1.

**Note.** The formula (4.22) works for $m = 0$ as well, meaning that

$$P(\text{matching } \Pi \text{ is stable}) = (1 + O(n^{-\sigma})) \frac{e^{1/2}}{(n - 1)!!}.$$ 

So the expected number of stable matchings tends to $e^{1/2}$ as $n \to \infty$, Pittel (1993a).

### 4.3 The expectations of the numbers of stable partitions and odd parties

**Theorem 4.12** Let $S_n$ and $O_n$ denote the total number of odd stable partitions $\Pi$, and the total number of odd cycles. Then

$$\mathbb{E}[S_n] = (1 + O(n^{-1/4})) \frac{\Gamma(1/4)}{\sqrt{\pi} e^{21/4}} n^{1/4},$$ (4.23)

$$\mathbb{E}[O_n] \lesssim \frac{\Gamma(1/4)}{4 \sqrt{\pi} e^{21/4}} n^{1/4} \log n.$$ (4.24)
**Proof** For even \( m \), let \( f(m) \) denote the total number of permutations of \( m \), having only odd cycles, each of length 3 at least. For even \( k \), let \( f(m, k) \) denote the total number of permutations of \( m \) having only \( k \) odd cycles, each of length 3, at least; so \( f(m) = \sum_k f(m, k) \). Then the total number of permutations of \( [n] \) with \( k \) odd cycles, each of length 3 at least, with \( m \) elements overall, and even cycles of length 2 only, is \((n \choose m) f(m, k)(n - m - 1)!!\). So, by Lemma 4.11, we have

\[
E[S_n] = (e^{1/2} + O(n^{-\sigma})) \sum_{m \leq mn} \binom{n}{m} \frac{f(m)(n - m - 1)!!}{(n + m - 1)!!}.
\] (4.25)

A standard argument from permutation enumeration (Bóna 2004, Section 3.4) shows that

\[
\sum_{m \geq 4} \frac{f(m)}{m!} x^m = \exp\left( \sum_{\text{odd } j \geq 3} \frac{x^j}{j} \right) = e^{-x} \sqrt{\frac{1+x}{1-x}}. \quad (|x| < 1).
\] (4.26)

So, using the saddle-point method (Flajolet and Sedgewick 2009),

\[
f(m) = (e^{-1/2} + O(m^{-1})) \frac{(2m - 1)!!}{2^m}.
\] (4.27)

With a bit of work, based on Stirling formula, it follows that

\[
(e^{1/2} + O(n^{-\sigma})) \binom{n}{m} \frac{f(m)(n - m - 1)!!}{(n + m - 1)!!} = (1 + O(n^{-\sigma} + m^{-1})) \sqrt{\frac{2}{\pi e}} \cdot m^{-1/2} \exp\left( -\frac{m^2}{2n} \right).
\]

Combining this formula with (4.25), and choosing \( \sigma = 1/4 \), we complete the proof of (4.23).

A bivariate extension of (4.26) is

\[
\sum_{m \geq 4} \frac{x^m}{m!} \sum_{k \geq 2} y^k f(m, k) = \exp\left( y \sum_{\text{odd } j \geq 3} \frac{x^j}{j} \right).
\]

Differentiating this identity at \( y = 1 \), we obtain

\[
\sum_{m \geq 4} \frac{x^m}{m!} \sum_{k \geq 2} k f(m, k) = \sum_{\text{odd } j \geq 3} \frac{x^j}{j} \exp\left( \sum_{\text{odd } j \geq 3} \frac{x^j}{j} \right)
= \left( \frac{1}{2} \log \frac{1}{1-x} + \frac{1}{2} \log(1+x) - 1 \right) e^{-x} \sqrt{\frac{1+x}{1-x}}.
\] (4.28)
So, analogously to (4.27), we obtain
\[\sum_{k \geq 2} kf(m, k) = (1 + O(m^{-1})) \frac{e^{-1/2} \log m}{2} \cdot \frac{(2m - 1)!!}{2^m}.\]

Combining this formula with the counterpart of (4.25), i.e. with
\[E[O_n] \leq (e^{1/2} + O(n^{-\sigma})) \sum_{m \leq m_n} \left(\frac{n}{m}\right)^{(n-m)!! \sum_{k \geq 2} kf(m, k)},\]
we have (4.24).

Tan (1991a, b) defined a maximum stable matching for an instance \(I\) as a matching \(M\) of maximum size (number of matched pairs) such that no pair of members, both having a partner in \(M\), prefer each other to their partners. In short, no two members assigned in \(M\), but not to each other, block \(M\). He proved that a maximum stable matching has size \((n - O)/2\), (see also Manlove 2013).

**Corollary 4.13** Let \(M_n\) denote the size of the maximum stable matching for the random instance \(I_n\). Then
\[E[M_n] \geq \frac{n - cn^{1/4} \log n}{2}, \quad c = \frac{\Gamma(1/4)}{3\sqrt{\pi} e^{1/4}},
\]
\[P(M_n \geq \frac{n - n^{1/4} \log^2 n}{2}) \geq 1 - O(\log^{-1} n),\]
so that the number of members unassigned in the maximum stable matching is likely to be of order \(O(n^{1/4} \log^2 n)\).

### 4.4 A “maximally stable” matching in the random instance \(I_n\)

For a given set of preferences, Abraham et al. (2006) (see also Manlove 2013) defined a “maximally stable” matching as a perfect matching \(M\) on \([n]\) that is blocked by the smallest number of pairs, \(B(I_n)\), of members not matched with each other in \(M\). (Two members block \(M\) if they prefer each other to their partners in \(M\).) A weaker corollary of the bound in Abraham et al. (2006) states that \(B(I_n) \leq d(I_n)O(I_n)\), where \(O(I_n)\) is the number of odd parties (common to all stable partitions for \(I_n\)) and \(d(I_n)\) is the length of the longest preference list.

Once we estimate \(R_{\text{max}}\), defined as the largest rank of a predecessor in the uniformly random instance \(I_n\), we will be able to apply the ABM inequality via replacing \(d(I_n)\) with \(R_{\text{max}}\).

For a stable \(\Pi\) (without a fixed point), introduce \(X(\Pi) := \max_{\{i\}} X_i, \Pi^{-1}(i)\). Intuitively, \(\max_{\Pi} X(\Pi)\) controls the worst predecessor’s rank. From Lemma 4.9, and the proof of Theorem 4.12, it follows that
\[
P \left( \max_{\Pi} X(\Pi) \geq \frac{\log^2 n}{n^{1/2}} \right) \leq e^{-\Theta(\log^2 n)}.
\]

A bit more generally, for every \( \delta > 0 \),
\[
P_\delta := P \left( \max_{\Pi} X(\Pi) \geq \frac{\log^{1+\delta} n}{n^{1/2}} \right) \leq e^{-\Theta(\log^{1+\delta} n)}.
\]

(4.29)

Denoting \( x_n = \frac{\log^{1+\delta} n}{n^{1/2}} \), let \( R_i := |\{ j \neq i : X_{i,j} \leq x_n \}|. \) Since \( X_{i,j} \) are independent \([0, 1]-\)Uniforms, we have \( R_i \overset{D}{=} \text{Bin}(n - 1, p = x_n) \). Let \( c > 1 \); by the classic (Chernoff) bound for the tails of the binomial distribution (Chernoff 1952),
\[
P( R_i \geq c(n - 1)x_n ) \leq \exp\left( -f(c)(n - 1)x_n \right), \quad f(c) := 1 + c(\log c - 1).
\]

So, \( P(R_i \geq 2nx_n) \leq e^{-n^{1/2}/3} \) if \( n \) is large enough. Invoking (4.29), we have then
\[
P\left( R_{\max} \geq 2n^{1/2} \log^{1+\delta} n \right) \leq P_\delta + \sum_{i \in [n]} P \left( R_i \geq 2n^{1/2} \log^{1+\delta} n \right)
\leq e^{-\Theta(\log^{1+\delta} n)} + ne^{-n^{1/2}/3}.
\]

Thus

**Lemma 4.14** For \( \delta > 0 \) arbitrarily small, quite surely \( R_{\max} \) is of order \( n^{1/2} \log^{1+\delta} n \).

Combining Lemma 4.14 with (4.24) in Theorem 4.12, we have proved

**Corollary 4.15** With high probability, there exists a perfect matching which is blocked by at most \( n^{3/4}(\log n)^{2+\delta} \) unmatched pairs.

### 4.5 Likely range of \( R(\Pi) \) in a stable, fixed-point free, partition \( \Pi \)

In Lemma 2.2 we proved that \( P_k(\Pi) \) the probability that \( \Pi \) is stable and the total rank of all \( n \) predecessors \( R(\Pi) \) equals \( k \), necessarily exceeding \( n + |\text{Odd}(\Pi)| \), is given by

\[
P_k(\Pi) = \int_{x \in [0, 1]^n} \cdots \int_{x \in [0, 1]^n} \left[ z^k \right] F(x, z) \, dx,
\]

(4.30)

\[
F(x, z) := \prod_{h \in \text{Odd}(\Pi)} x_h \cdot \prod_{(i,j) \notin D(\Pi)} \left( \bar{x}_i \bar{x}_j + z x_i \bar{x}_j + z \bar{x}_i x_j \right),
\]

where \( m := |\text{Odd}(\Pi)| \) and \( \bar{k} := k - (n + m) \).

**Theorem 4.16** For \( \varepsilon \in (0, 1) \),
\[
P \left( \max_{\Pi} \left| \frac{R(\Pi)}{n^{3/2}} - 1 \right| \geq \varepsilon \right) \leq e^{-\Theta(\log^2 n)}.
\]
Proof Predictably, we will prove the claim via the union bound, i.e. summing the bounds of the respective probabilities for the individual partitions. It suffices then to consider the partitions $\Pi$ with $m \leq m_n = \lceil n^{1/2} \log n \rceil$.

First of all, since $F(\mathbf{x}, z)$ in (4.30) is a polynomial of $z$ with non-negative coefficients, we have a Chernoff-type bound: for $k := \lceil (1 + \varepsilon)n^{3/2} \rceil$,

$$\Pr(\mathcal{R}(\Pi) \geq k) \leq I(\Pi, k) := \int_{\mathbf{x} \in [0, 1]^n} \cdots \int_{z \geq 1} \inf_{z \geq 1} \left[ z^{-\bar{k}} F(\mathbf{x}, z) \right] d\mathbf{x}.$$  

The integrand is, at most,

$$F(\mathbf{x}, 1) = F(\mathbf{x}) \leq b e^{-\frac{z^2}{2}} \prod_{h \in \text{Odd}(\Pi)} x_h,$$

($s = \sum_{i \in [n]} x_i$), see Lemma 4.1. Therefore the proof of Lemma 4.9 delivers, with only notational modification, that

$$I(\Pi, k) - I_4(\Pi, k) \leq \frac{e^{-\Theta(n^{1/2} \log n)}}{(n + m - 1)!!}.$$  \hfill (4.31)

Here $I_4(\Pi, k)$ is the integral of $\inf_{z \geq 1} \left[ z^{-\bar{k}} F(\mathbf{x}, z) \right]$ over $C_4 \subset [0, 1]^n$, defined by the additional constraints: with $\xi = \sum_{i \in [n_1]} x_i$, $(n_1 := n - m)$,

$$s \leq s_n := n^{1/2} + 3 \log n, \quad \max_{i \in [n]} x_i \leq 1.02 \frac{s \log^2 n}{n};$$ \hfill (4.32)

$$\left| \frac{n_1 \sum_{i \in [n_1]} x_i^2}{2 \xi^2} - 1 \right| \leq n^{-\sigma}, \quad \frac{2n_1 \sum_{i \in [n_1/2]} x_i x_i + n_1/2}{\xi^2} - 1 \leq n^{-\sigma}. \hfill (4.33)$$

Instead of looking for the best $\mathbf{z} = \mathbf{z}(\mathbf{x}) \geq 1$ where $z^{-\bar{k}} F(\mathbf{x}, z)$ attains, or is close to, its infimum, we confine ourselves to a sub-optimal $\mathbf{z} = \mathbf{z}(s) \geq 1$ (i.e. dependent on $s$ only), which makes $z^{-\bar{k}} F(\mathbf{x}, z)$ suitably small for all $\mathbf{x} \in C_4$. Consider $z \leq \frac{2\bar{k}}{sn}$; as we shall see shortly, the minimum point of an auxiliary bound for the integrand does satisfy this constraint.

Using $1 + x \leq e^x$, the constraints (4.32), (4.33) and $z \leq \frac{2\bar{k}}{sn}$, we have

$$\prod_{(i, j) \notin D(\Pi)} \left( \tilde{x}_i \tilde{x}_j + z x_i x_j + z \tilde{x}_i \tilde{x}_j \right) = \prod_{(i, j) \notin D(\Pi)} \left( 1 + (1 - 2z)x_i x_j + (z - 1)(x_i + x_j) \right)$$

$$\leq \exp \left( \sum_{(i, j) \notin D(\Pi)} \left[ (1 - 2z)x_i x_j + (z - 1)(x_i + x_j) \right] \right)$$

$$\leq b \exp \left( (1 - 2z) \frac{s^2}{2} + n(z - 1)s \right);$$
therefore
\[
z^{-\bar{k}} F(x, z) \leq_b \exp \left( (1 - 2z) \frac{s^2}{2} + n(z - 1)s - \bar{k} \log z \right) \prod_{h \in \text{Odd}(\Pi)} x_h.
\]

So, applying the identity (3.2),
\[
I_4(\Pi, k) \leq_b \frac{1}{(n + m - 1)!} \int_0^{s_n} \exp(H(z, s)) \, ds,
\]
\[
H(z, s) := (1 - 2z) \frac{s^2}{2} + n(z - 1)s - \bar{k} \log z + (n + m - 1) \log s. \tag{4.34}
\]

Let us use (4.34) to prove that
\[
I_4(\Pi, k) \leq e^{-\Theta(\varepsilon^2 n)} \frac{(n + m - 1)!!}{(n + m - 1)!}. \tag{4.35}
\]

As a function of \( z \), \( H(z, s) \) is convex, and has its absolute minimum at
\[
\bar{z} = \bar{z}(s) := \frac{\bar{k}}{(n - s)s} \sim \frac{\bar{k}}{ns} < \frac{2\bar{k}}{ns}.
\]

This decreasing function of \( s \) is “Chernoff-admissible” when \( s \) is such that \( z(s) \geq 1 \).

Let \( s_1 \) be the smaller root of the (quadratic) equation \( \bar{z}(s) = 1 \):
\[
s_1 > \frac{\bar{k}}{n}, \quad s_1 = \frac{\bar{k}}{n} + O(1) = (1 + \varepsilon)n^{1/2} + O(1).
\]

Thus our best hope is a function
\[
z(s) = \begin{cases} \bar{z}(s), & \text{if } s \leq s_1, \\ 1, & \text{if } s > s_1. \end{cases}
\]

(i) \( s > s_1 \). Here \( s_1 \sim \frac{\bar{k}}{n} > (n + m - 1)^{1/2} \), the maximum point of
\[
h(s) := H(1, s) = -\frac{s^2}{2} + (n + m - 1) \log s.
\]

So, arguing as in the proof of Lemma 4.5,
\[
\frac{1}{(n + m - 1)!} \int_{s_1}^{s_n} \exp(H(1, s)) \, ds \leq \frac{e^{h(s_1)}}{(-h'(s_1))(n + m - 1)!} \leq e^{-\Theta(\varepsilon^2 n)(n + m - 2)!!} \leq e^{-\Theta(\varepsilon^2 n)} \frac{(n + m - 1)!!}{(n + m - 1)!}. \tag{4.36}
\]
(ii) $s < s_1$. Let $\bar{h}(s) := H(z(s), s)$. Since $H_z(z(s), s) = 0$, we have

$$\bar{h}'(s) = H_z(z(s), s) = \left(s - \frac{\bar{k}}{n-s} \right) + \left(\frac{\bar{k}}{s} - n\right) + \frac{n + m - 1}{s},$$

$$\bar{h}''(s) = 1 - \frac{\bar{k}}{(n-s)^2} - \frac{k - 1}{s^2}.$$  

By the second formula, we have $\bar{h}''(s) < 0$ for $s \leq s_n$, i.e. $\bar{h}(s)$ is concave. By the first formula, we have

$$\bar{h}'(\bar{k}/n) \geq -(1 + o(1)) \frac{k^2}{n^3} + \frac{n^2}{k} \sim \frac{n^{1/2}}{1 + \varepsilon} \to \infty,$$

$$\bar{h}'(s_1) = \left(\frac{n + m - 1}{s_1} - s_1^2\right) \sim -\frac{(2\varepsilon + \varepsilon^2)n^{1/2}}{1 + \varepsilon} \to -\infty.$$  

Thus $\max \{\bar{h}(s) : s \leq s_n\}$ is attained at a unique point $s_2 \in [\bar{k}/n, s_1]$; in particular, $s_1 - s_2 = O(1)$. Since $|\bar{h}''(s)| = O(n^{1/2})$, it follows via Taylor’s approximation of $\bar{h}(s_1)(= h(s_1))$ at $s_2$—that $\bar{h}(s_2) = h(s_1) + O(n^{1/2})$. Therefore, similarly to (4.36), we obtain

$$\frac{1}{(n + m - 1)!} \int_0^{s_1} \exp(H(z(s), s)) \, ds \leq \frac{e^{-\Theta(\varepsilon^2 n)}}{(n + m - 1)!!}.$$  

This bound together with (4.36) imply (4.35), which in combination with (4.31) deliver

$$\text{P}(\mathcal{R}(\Pi) \geq k) \leq \frac{e^{-\Theta(\log^2 n)}}{(n + m - 1)!!}.$$  

As in the proof of Corollary 4.4, it follows that

$$\text{P}(\exists \, \Pi : \mathcal{R}(\Pi) \geq (1 + \varepsilon)n^{3/2}) \leq e^{-\Theta(\log^2 n)}.$$  

Similarly

$$\text{P}(\exists \, \Pi : \mathcal{R}(\Pi) \leq (1 - \varepsilon)n^{3/2}) \leq e^{-\Theta(\log^2 n)}.$$  

\[\square\]

Note. In the introduction we mentioned the Shapley-Scarf model of the market in indivisible goods, a precursor of Tan’s model of stable partitions. One of the parameters of the core allocation is the total rank $\mathcal{R}$ of the goods assigned to their ultimate owners. Suppose that the individual preferences of the traders are independent uniform, just like in the stable partitions we study. Jointly with Frieze and Pittel (1996) we proved that $\text{E}[\mathcal{R}]$ is sandwiched between $0.5n \log n$ and $n \log n$, and later Knuth (1996) proved an exact formula $\text{E}[\mathcal{R}] = (n + 1)H_n - n$ (Knuth 1996).
5 \(E[\mathcal{S}_n^2]\) and the expected number of members with multiple stable predecessors

Introduce \((\mathcal{S}_n)_2 := \mathcal{S}_n(\mathcal{S}_n - 1)\), which is the total number of ordered pairs of stable partitions \(\{\Pi_1, \Pi_2\}\). Then

\[
E[(\mathcal{S}_n)_2] = \sum_{\Pi_1 \neq \Pi_2} P(\Pi_1, \Pi_2),
\]

where \(P(\Pi_1, \Pi_2)\) is the probability that \(\Pi_1\) and \(\Pi_2\) are both stable. By Lemma 2.3,

\[
P(\Pi_1, \Pi_2) = \int_{x,y \in [0,1]^n} \cdots \int F(x, y) \, dx \, dy,
\]

\[
F(x, y) := \prod_h x_h \cdot \prod_{(i \neq j)} [1 - x_i x_j - y_i y_j + (x_i \land y_i) (x_j \land y_j)],
\]

\[
dx = \prod_{i \in [n]} dx_i, \quad dy = \prod_{i \in [n]:\Pi_1(i) \neq \Pi_2(i)} dy_i,
\]

where \(h \in \text{Odd}(\Pi_{1,2}), (i \neq j) \in D_1^c \cap D_2^c, (D_t = D(\Pi_t)), y_i = x_i\) if \(\Pi_1(i) = \Pi_2(i)\), and for every circuit \(\{i_1, \ldots, i_\ell\}, (\ell \geq 4)\), formed by alternating pairs matched either in \(\Pi_1\) or \(\Pi_2\), we have:

\[
either \quad \text{either } x_{i_1} > y_{i_1}, \, x_{i_2} < y_{i_2}, \ldots, \, x_{i_\ell} < y_{i_\ell}, & \text{ or } \quad x_{i_1} < y_{i_1}, \, x_{i_2} > y_{i_2}, \ldots, \, x_{i_\ell} > y_{i_\ell}.
\]

Let \(\mu = \mu(\Pi_1, \Pi_2)\) be the total number of these circuits, and \(2\nu = 2\nu(\Pi_1, \Pi_2)\), be their total length. Obviously, there are \(2^\mu\) ways to select one of two “alteration” sequences described in (5.2) for each of the \(\mu\) circuits. Whatever the choice, there are exactly \(\nu\) vertices determined by these inequalities, on those circuits, where \(y_i > x_i\) and \(\nu\) vertices where \(y_i < x_i\).

Let \(A\) and \(B\) denote the correspondent subsets, \(|A| = |B| = \nu\). So

\[
y_i > x_i, \quad \text{if } i \in A; \quad y_i < x_i, \quad \text{if } i \in B,
\]

\[
y_i = x_i, \quad \text{if } i \in [n] \setminus (A \cup B) = \text{Odd}_{1,2} \cup (\text{Even}_{1,2} \setminus (A \cup B)),
\]

\(\text{Even}_{1,2} := [n] \setminus \text{Odd}_{1,2}\). The individual contributions of these \(2^\mu\) choices (of the inequalities along the circuits) to the integral in (5.1) are all the same. This means that \(P(\Pi_1, \Pi_2)\) equals the RHS integral in (5.1), with inequalities (5.3) instead of (5.2), times \(2^\mu\).

As in the previous section, we need first to identify the subrange of \((x, y)\) that provides an asymptotically dominant contribution to the integral, and second to find a sharp approximation for that contribution. Like Theorem 4.12, the key instrument is the bound for the double-indexed product in the definition of \(F(x, y)\) proved in Lemma 4.1:
\[ F(x, y) \leq b \exp\left(-\frac{s_1^2}{2} - \frac{s_2^2}{2} + \frac{s_{1,2}^2}{2}\right) \prod_h x_h, \]  
\[ s_1 := \sum_{i \in [n]} x_i, \quad s_2 := \sum_{i \in [n]} y_i, \quad s_{1,2} := \sum_{i \in [n]} (x_i \wedge y_i). \]  

Here \((x, y)\) are subject to the constraints (5.3). To make use of this bound, we change the variables of integration:

\[ x_i' = \begin{cases} x_i - y_i, & i \in B, \\ x_i, & i \notin B, \end{cases} \quad y_i' = \begin{cases} y_i - x_i, & i \in A, \\ y_i, & i \notin A. \end{cases} \]  

Here \(x', y' \in [0, 1]^n\), such that \(x_i' = y_i'\) if \(\Pi_1(i) = \Pi_2(i)\), and the Jacobian \(\partial(x, y)/\partial(x', y')\) equals 1. Furthermore, switching to \((x', y')\) and introducing

\[ \xi_1 = \sum_{i \in [n] \setminus B} x_i' + \sum_{i \in B} y_i', \quad \xi_2 = \sum_{i \in B} x_i', \quad \xi_3 = \sum_{i \in A} y_i', \]  
we obtain

\[ -\frac{s_1^2}{2} - \frac{s_2^2}{2} + \frac{s_{1,2}^2}{2} = -\frac{1}{2}\left(\sum_{i \in [n] \setminus B} x_i' + \sum_{i \in B} y_i' + \sum_{i \in B} x_i'\right)^2 \]  
\[ -\frac{1}{2}\left(\sum_{i \in [n] \setminus B} x_i' + \sum_{i \in B} y_i' + \sum_{i \in A} y_i'\right)^2 + \frac{1}{2}\left(\sum_{i \in [n] \setminus B} x_i' + \sum_{i \in B} y_i'\right)^2 \]  
\[ = -\frac{1}{2}(\xi_1 + \xi_2)^2 - \frac{1}{2}(\xi_1 + \xi_3)^2 + \frac{1}{2}\xi_1^2 = -\frac{1}{2}(\xi_1 + \xi_2 + \xi_3)^2 + \xi_2\xi_3. \]  

Notice that

\[ \xi_1 + \xi_2 + \xi_3 = \sum_{i \in [n]} x_i' + \sum_{i \in A \cup B} y_i' = \sum_{i \in [n]} (x_i \vee y_i), \]  
\[ \xi_2 + \xi_3 = \sum_{i \in B} x_i' + \sum_{i \in A} y_i' = \sum_{i \in [n]} |x_i - y_i|. \]  

In full analogy with the case of \(E[O_n]\), the bound (5.4) and the identity (5.7) will allow us to shrink, in several steps, the range of \((x, y)\) to a core range, on which the integrand \(F(x, y)\) can be sharply approximated.

(1) Recall that we consider the partitions \(\Pi\) with the total length of all odd cycles \(m = m(\Pi) \leq m_n = [n^{1/2} \log n]\). Our first step is to dispense with the pairs \((\Pi_1, \Pi_2)\) of the partitions such that \(2\nu = 2\nu(\Pi_1, \Pi_2) \geq 2m_n\).

**Lemma 5.1**

\[ E[(S_n)_2] - E_1[(S_n)_2] \leq e^{-\Theta(\log^2 n)}, \]
\[ E_1[(S_n)_2] := \sum_{\nu(\Pi_1, \Pi_2) \leq m_n} P(\Pi_1, \Pi_2). \]

**Proof** By the Eqs. (5.4) and (5.7), and the identity (3.2), we have

\[
P(\Pi_1, \Pi_2) \leq b \cdot 2^\mu \int_{x', y' \geq 0} \exp \left( -\frac{1}{2} \left( \sum_j \xi_j \right)^2 + \xi_2 \xi_3 \right) \times \left( \prod_{h \in \text{Odd}_1, 2} x'_h \right) \prod_{i \in [n]} dx'_i \prod_{j \in A \cup B} dy'_j = 2^\mu \iint \exp \left( -\frac{1}{2} \left( \sum_j \xi_j \right)^2 + \xi_2 \xi_3 \right) \frac{\xi_1^{n+m-1}}{(n+m-1)!} \cdot \frac{(\xi_2 \xi_3)^{v-1}}{[(v-1)!]^2} \, d\xi.
\]

Expanding \( \exp(\xi_2 \xi_3) = \sum_{k \geq 0} \frac{k^k k^k}{k!} \) and using again, term-wise, (3.2), we obtain

\[
P(\Pi_1, \Pi_2) \leq b \cdot 2^\mu \sum_{k \geq 0} s(n+m, v, k),
\]

\[
s(n+m, v, k) := \frac{[(v-1+k)!]^2}{[(v-1)!]^2 k!(n+m+2(v+k)-1)!!}.
\]

For \( m = 0 \) this sum was estimated in Pittel (1993b). For our case the estimate from Pittel (1993b) becomes

\[
\sum_{k \geq 0} s(n+m, v, k) \leq b \cdot n \left( \frac{e}{n+m} \right)^{\frac{n+m}{2}} (n+m)^{-v}.
\]

Furthermore, the number of ordered pairs \((\Pi_1, \Pi_2)\) with parameters \(m, v\) and \(\mu\) is

\[
\binom{n}{m} f(m) \binom{n-m}{2v} (n-m-2v)!! \cdot 2^\mu f(2v, \mu);
\]

here, as we recall, \(f(m)\) is the total number of permutations of \([m]\) with only odd cycles of length 3 or more, and \(f(2v, \mu)\) is the total number of circuit partitions of \([2v]\) with \(\mu\) circuits, each of even length 4 at least. The factor \(2^\mu\) counts the total number of ways to assign, in the alternating fashion, the edges of the circuits to the matching sets of \(\Pi_1\) and \(\Pi_2\). Clearly then, \(2^\mu f(2v, \mu)\) is the total number of permutations of \([2v]\) with only even cycles, of length 4 at least. We add that \((n-m-2v)!!\) is the total number of ways to form the \((n-m-2v)/2\) matched pairs out of \(n-m-2v\) elements outside the circuits, i.e. the pairs common to \(\Pi_1\) and \(\Pi_2\).
So, by (5.10), (5.11) and (5.12), we obtain

\[ \sum_{\nu(P_1, P_2) \geq m_n} P(P_1, P_2) \leq b n \sum_{m \leq m_n} \left( \frac{n}{m} \right) f(m) \left( \frac{e}{n + m} \right)^{n + m} \]

\[ \times \sum_{\nu \geq m_n} \left( \frac{n - m}{2} \right) (n - m - 2\nu - 1)!!(n + m)^{-\nu} \sum_\mu 2^{\mu} f(2\nu, \mu). \] (5.13)

Now \( f(m) \leq m! \), and from Pittel (1993b) (Appendix) it follows that

\[ \sum_\mu 2^{\mu} f(2\nu, \mu) = e^{-1 + O(\nu^{-1})(2\nu)!} = O((2\nu)!). \] (5.14)

Also

\[ \frac{(n - m - 2\nu - 1)!!}{(n - m - 2\nu)!} = \left[ 2^{\frac{n-m-2\nu}{2}} \left( \frac{n - m - 2\nu}{2} \right)! \right]^{-1}. \]

So the bound (5.13) yields

\[ \sum_{\nu(P_1, P_2) \geq m_n} P(P_1, P_2) \leq b n! \cdot \sum_{m \leq m_n} \left( \frac{e}{n + m} \right)^{n + m} \]

\[ \times \sum_{\nu \geq m_n} \left( \frac{n + m}{2} \right)^{n + m} \left( \frac{n - m - 2\mu}{2} \right)! \]

\[ \leq n! n^2 \sum_{m \leq m_n} \left( \frac{e}{n + m} \right)^{n + m} \left( \frac{n + m}{2} \right)^{n + m} \left( \frac{n - m - 2\mu}{2} \right)! \]

since in the \( \nu \)-sum the terms decrease with \( \nu \). Applying Stirling formula for the two factorials and using \( m \leq m_n \ll n^{2/3} \) in the expansions of \( \log(1 + z) \), \( z = m/n \), \( z = -\frac{m + 2m_n}{n} \), we transform this bound into

\[ \sum_{\nu(P_1, P_2) \geq m_n} P(P_1, P_2) \leq n^2 \cdot \sum_{m \leq m_n} \exp \left( -\frac{(m + 2m_n)^2}{4n} \right) \leq e^{-0.99 \log^2 n}. \] (5.15)

\[ \Box \]

From now on we will consider only admissible pairs \((P_1, P_2)\) of reduced partitions, i.e. those satisfying \( m(P_1, P_2) \leq m_n \) and \( \nu(P_1, P_2) \leq m_n \). In light of the note preceding Lemma 2.3, a byproduct of the proof above is

**Corollary 5.2** With probability \( \geq 1 - e^{-0.99 \log^2 n} \), for every stable partition \( P \) the total length of the odd cycles and of the cycles of even length \( \geq 4 \) is below \( 2m_n \).
(2) For the admissible pairs \((\Pi_1, \Pi_2)\), we can discard large parts of the \((x, y)\)’s range, like we did for individual partitions \(\Pi\) in the case of \(E[S_n]\). For a generic set \(C\) of \((x, y)\) with \(x, y \in [0, 1]^n\), we define

\[
P_C(\Pi_1, \Pi_2) = \int \cdots \int F(x, y) \, dxdy, \quad E_C[(S_n)_2] = \sum_{\Pi_1, \Pi_2} P_C(\Pi_1, \Pi_2).
\]

**Lemma 5.3** Introducing \(s_n = n^{1/2} + 6 \log n\), and \(C_1 = \{x, y : \sum_{i \in [n]} (x_i \lor y_i) \leq s_n\}\), we have

\[
E_1[(S_n)_2] - E_{C_1}[(S_n)_2] \leq e^{-\Theta(\log^2 n)}.
\]

**Proof** We already observed, (5.8), that \(\sum_i (x_i \lor y_i) = \sum_j \xi_j\). So, similarly to (5.9)-(5.10) we have:

\[
P(\Pi_1, \Pi_2) - P_{C_1}(\Pi_1, \Pi_2)
\leq_b 2^{\mu} \int \cdots \int \exp\left(-\frac{1}{2} \left(\sum_j \xi_j\right)^2 + \xi_2 \xi_3\right) \frac{s_n^{n+m-1}}{(n+m-1)!} \frac{(\xi_2 \xi_3)^{v-1}}{[(v-1)!]^2} \, d\xi
\]

\[
= 2^{\mu} \sum_{k \geq 0} \frac{[(v-1+k)!]^2}{[(v-1)!]^2 k!(n+m+2(v+k)-1)!} \times \int_{s \geq s_n} \exp\left(-\frac{s^2}{2}\right) s^{n+m+2(v+k)-1} \, ds.
\]

(5.16)

(Relaxing the constraint on \(s\) to \(s \geq 0\) we get back to (5.10).) The last integrand attains its maximum at

\[
s_{\text{max}} = (n + m + 2(v + k) - 1)^{1/2},
\]

which is below \(s_n - 3 \log n\) if \(k \leq m_n\). Let \(S_{\leq m_n}\) and \(S_{> m_n}\) denote the sub-sums of the sum above, for \(k \leq m_n\) and \(k > m_n\) respectively. Then, expanding integration to \([0, \infty)\), we obtain

\[
S_{> m_n} \leq \sum_{k > m_n} \frac{[(v-1+k)!]^2}{[(v-1)!]^2 k!(n+m+2(v+k)-1)!}
\]

\[
\leq_b \frac{[(v+m_n)!]^2}{[(v-1)!]^2 m_n!(n+m+2(v+m_n)+1)!};
\]

since \(v \leq m_n\), the ratio of the consecutive terms in the sum is below 2/3. Dropping \([(v-1)!]^2\) in the denominator and using the Stirling formula for the other factorials, we simplify the bound to
\[
S_{> m_n} \leq b \left( \frac{e}{n + m} \right)^{\frac{n+m}{2}} (n + m)^{-(\nu + m_n)}.
\] (5.17)

The bound is smaller than the bound (5.11) for the full sum of \( s(n + m, \nu, k) \) by the factor \((n + m)^{m_n} \). Turn to \( S_{\leq m_n} \). This time the bottom integral over \( s \geq s_n \) in (5.16) is small, compared to the integral over all \( s \geq 0 \), because for \( k \leq m_n \) the maximum point of the integrand is at distance \( 3 \log n \), at least, from the interval \([s_n, \infty) \). More precisely, using the argument in the proof of Lemma 4.5, we have

\[
\int_{s \geq s_n} \exp \left( -\frac{s^2}{2} \right) s^{n+m+2(v+k)-1} \, ds \leq b \, e^{-8 \log^2 n} (n + m + 2(v + k) - 2)!!.
\]

Therefore

\[
S_{\leq m_n} \leq b \, e^{-8 \log^2 n} \sum_{k \leq m_n} \frac{[(\nu - 1 + k)!]^2}{[(\nu - 1)!]^2 k! (n + m + 2(v + k) - 1)!!}.
\]

Combining (5.17), (5.18) and (5.11) we transform the inequality (5.16) into

\[
P(\Pi_1, \Pi_2) - P_{C_1}(\Pi_1, \Pi_2) \leq e^{-\Theta(\log^2 n)} \sum_{\Pi_1, \Pi_2} \left( \frac{e}{n + m} \right)^{\frac{n+m}{2}} (n + m)^{-\nu}.
\] (5.19)

So, like the part (1) in the proof of Lemma 5.1,

\[
\sum_{\Pi_1, \Pi_2} \left[ P(\Pi_1, \Pi_2) - P_{C_1}(\Pi_1, \Pi_2) \right] \leq e^{-\Theta(\log^2 n)} n! m_n
\]

\[
\times \sum_{m \leq m_n} \left( \frac{e}{n + m} \right)^{\frac{n+m}{2}} [ (n + m)^{0} 2^{\frac{n-m-2 \cdot 0}{2}} \left( \frac{n - m - 2 \cdot 0}{2} \right) ]^{-1}.
\] (5.20)

\[
= e^{-\Theta(\log^2 n)}.
\]

We need some additional reduction of the last range \( C_2 \). The bound (5.4) will continue to be the key tool, until the resulting range is narrow enough to permit a sufficiently sharp bound of the double product

\[
G(x, y) = \prod_{(i \neq j) \in D_1 \cap D_2} \left[ 1 - x_i x_j - y_i y_j + (x_i \wedge y_i)(x_j \wedge y_j) \right]
\]
Define $\mathcal{N} = \mathcal{N}(\Pi_1, \Pi_2)$ and $\mathcal{M} = \mathcal{M}(\Pi_1, \Pi_2)$ as the vertex set of all odd cycles and even cycles, of length 4 or more, and the vertex set of the edges common to both partitions, respectively. So $|\mathcal{N}| = m + 2\nu$, and $|\mathcal{M}| = n - (m + 2\nu)$. Arguing as in the proof of Lemma 4.1, but retaining more terms, we have

$$G(x, y) \leq \exp\left( -\frac{s_1^2}{2} - \frac{s_2^2}{2} + \frac{s_{1,2}^2}{2} + \frac{1}{2} \sum_{i \in \mathcal{M}} x_i^2 + \sum_{(i \neq j) \in M_1 \cap M_2} x_i x_j \right)$$

$$- \frac{1}{4} \left( \sum_{i \in [n]} (x_i \wedge y_i)^2 \right)^2 + O\left( \sum_{i \in \mathcal{N}} (x_i^2 + y_i^2) \right) + O\left( \sum_{i \in [n]} x_i^4 \right).$$

(5.21)

Thus we have to find sharp approximations of the three explicit sums and to establish the $o(1)$ bounds of the remainders for almost all $(x, y) \in \mathcal{C}_1$. With those approximations at hand we will obtain an explicit upper bound for $E\left[ (S_n)_2 \right]$. For brevity will not present a proof of a matching lower bound.

(3) By (5.5) and (5.6), $s := \xi_1 + \xi_2 + \xi_3 = \sum_{i \in [n]} x_i^2 + \sum_{i \in A \cup B} y_i^2$.

**Lemma 5.4** Define $u' = \{u'_i\}_{i \in [n]}$, where $u'_i = x_i^2/s$, for $i \in [n]$, and $u'_i = y_i^2/s$ for $i \in A \cup B$. Define $T_1(u') = \max_i u'_i$. For

$$\mathcal{C}_2 := \{ (x, y) \in \mathcal{C}_1 : T_1(u') \leq 1.01 \log^2 n \},$$

we have

$$P_{\mathcal{C}_1}(\Pi_1, \Pi_2) - P_{\mathcal{C}_2}(\Pi_1, \Pi_2) \leq 2^\mu e^{-\Theta(\log^2 n)} \left( \frac{e}{n + m} \right)^{\frac{n + m}{2} n^{-\nu}}.$$

**Proof** Introduce $L'_1, \ldots, L'_{n+2\nu}$, the intervals lengths in the random partition of $[0, 1]$ by the $n + 2\nu - 1$ random points. Analogously to (5.9), but using the sharper inequality in Lemma 3.1, (3.1), we have: with $s := \xi_1 + \xi_2 + \xi_3$.

$$P_{\mathcal{C}_1}(\Pi_1, \Pi_2) - P_{\mathcal{C}_2}(\Pi_1, \Pi_2)$$

$$\leq b \int \cdots \int e^{-\frac{x^2}{2} + \frac{y^2}{2} + \xi_3} \prod_{h \in \text{Odd}_{1,2}} x_h \prod_{i \in [n]} dx'_i \prod_{j \in A \cup B} dy'_j$$

$$\begin{array}{c}
\text{subject to: } T_1(u') > 1.01 \log^2 n \\
\int \cdots \int e^{-\frac{x^2}{2} + \frac{y^2}{2} + \xi_3} s^m \prod_{i \in [n]} dx'_i \prod_{j \in A \cup B} dy'_j,
\end{array}$$

$$\leq \frac{2^\mu}{(n + 2\nu - 1)!} E\left[ \chi\left( T_1(L') \geq 1.01 \log^2 n \right) \prod_{h \in \text{Odd}_{1,2}} L'_h \right]$$

$$\times \int \cdots \int e^{-\frac{x^2}{2} + \frac{y^2}{2} + \xi_3} s^m \prod_{i \in [n]} dx'_i \prod_{j \in A \cup B} dy'_j.$$
Arguing as in (4.12), the expectation factor is less than
\[
e^{-1.01 \log^2 n} \frac{(n + 2\nu)!}{(n + m + 2\nu - 2)!}.
\]
The integral is less than
\[
\int \int \int e^{-\frac{\xi^2}{2} + \xi_3 s \xi_1^{n-1}} \frac{(\xi_2 \xi_3)^{\nu-1}}{((n-1)!)^2} d\xi
\]
\[
= \sum_{k \geq 0} \frac{((v - 1 + k)!)^2}{((v - 1)!)^2 k! (2(v + k) - 1)!} \int \int e^{-\frac{(\xi_3 + \xi_4)^2}{2}} (\xi_1 + \xi_4)^m \xi_1^{n-1} \xi_4^{2v+k-1} d\xi_1 d\xi_4.
\]
Here the double integral equals
\[
\frac{(n - 1)! (2(v + k) - 1)! (n + m + 2(v + k) - 2)!!}{(n + 2(v + k) - 1)!}.
\]
So
\[
I_n(m, \nu) = \sum_{k \geq 0} \frac{((v - 1 + k)!)^2 (n + m + 2(v + k) - 2)!!}{((v - 1)!)^2 k! (n + 2(v + k) - 1)!}
\]
Therefore
\[
P_{\mathcal{C}_1}(\Pi_1, \Pi_2) - P_{\mathcal{C}_2}(\Pi_1, \Pi_2) \leq_b 2^\mu e^{-1.01 \log^2 n} \sum_{k \geq 0} s'(n, m, \nu, k),
\]
\[
s'(n, m, \nu, k) := \frac{((v - 1 + k)!)^2 (n + m + 2(v + k) - 2)!! (n + 2\nu)!}{(n + m + 2\nu - 2)!! ((v - 1)!)^2 k! (n + 2(v + k) - 1)!}.
\]
The summand \(s'(n, m, \nu, k)\) is similar to the summand \(s(n + m, \nu, k)\) defined in (5.10). Closely following the derivation of the bound for \(\sum_{k \geq 0} s(n, \nu, k)\) in Pittel (1993b), we obtain
\[
\sum_{k \geq 0} s'(n, m, \nu, k) \leq_b n^2 \left( \frac{e}{n + m} \right)^{n+m} n^{-\nu},
\]
compare to (5.11). The last two bounds complete the proof.

\[ \square \]
On $C_1 \supset C_2$ we have

$$s = \sum_{i \in [n]} x_i' + \sum_{i \in A \cup B} y_i' = \sum_{i \in [n]} (x_i \lor y_i) \leq s_n = n^{1/2} + 6 \log n.$$ 

and on $C_2$

$$\max \left\{ \max_i \frac{x_i'}{s}, \max_{j \in A \cup B} \frac{y_j'}{s} \leq 1.01 \frac{\log^2 n}{n} \right\}.$$ 

Since $m, \nu \leq n^{1/2} \log n$, we have then the counterparts of the bounds in (4.14). Namely, on $C_2$,

$$\sum_{i \in \mathcal{N}} (x_i' + y_i') \leq b s_n^{-1} \log^3 n, \quad \sum_{i \in \mathcal{N}} (x_i' + y_i')^2 \leq b n^{-1/2} \log^5 n,$$

$$\sum_{i \in \mathcal{N}} (x_i' + y_i')^4 \leq b n^{-1} \log^8 n.$$  

(5.22)

(4) With $\xi := \sum_{i \in \mathcal{M}} x_i' = \sum_{i \in \mathcal{M}} x_i$, define $v_i' = x_i' / \xi$ for $i \in \mathcal{M}$, $|\mathcal{M}| = n - m - 2\nu$. Introduce $T_2(v') = \sum_{i \in \mathcal{I}} (v_i')^2$, and $V'$ the set of all $v'$ such that

$$\left| \frac{n - m - 2\nu}{2} T_2(v') - 1 \right| \leq n^{-\sigma}.$$ 

Lemma 5.5 For $\sigma < 1/3$, let $C_3 = \{ (x, y) \in C_2 : v' \in V \}$. Then

$$P_{C_2}(\Pi_1, \Pi_2) - P_{C_3}(\Pi_1, \Pi_2) \leq b 2^\mu e^{-\Theta(n^{1/3-\sigma})} \left( \frac{e}{n + m} \right)^{\frac{n + m}{2}} n^{-v}.$$ 

Proof Introduce

$$\xi_4 := \xi_1 - \xi = \sum_{i \in B^c \cap \mathcal{M}} x_i' + \sum_{i \in B} y_i' = \sum_{i \in \text{Odd}_1, 2} x_i' + \sum_{i \in A} x_i' + \sum_{i \in B} y_i'.$$

Then with $s := \xi + \xi_4 + \xi_2 + \xi_3$,

$$P_{C_2}(\Pi_1, \Pi_2) - P_{C_3}(\Pi_1, \Pi_2) \leq b 2^\mu \int_{x', y'} \ldots \int_{v' \in V} e^{-\frac{x'^2}{2} + \xi_4 + \xi_2 + \xi_3} \prod_{i \in \mathcal{M}} x_i' \prod_{j \in \text{Odd}_1, 2} x_j dx_j$$

$$\times \prod_{k \in A} dx_k \prod_{\ell \in B} dy_\ell \prod_{b \in B} dx'_b \prod_{a \in A} dy'_a.$$
Now the integrand depends on \( \{x_i\}_{i \in \mathcal{M}} \) only through \( \xi = \sum_{i \in \mathcal{M}} x_i' \). So, introducing the random intervals \( \mathcal{L}'_1, \ldots, \mathcal{L}'_{n-m-2v} \) forming the partition of \([0, 1]\), we obtain

\[
P_{C_2}(\Pi_1, \Pi_2) - P_{C_3}(\Pi_1, \Pi_2) \\
\leq b \ 2^\mu \frac{e^{(n-m-2v)2\xi_2^{\sigma}}}{|\mathcal{M}|-1} d\xi \\
\times \frac{\xi_4^{2m+2v-1} d\xi_4}{(2m+2v-1)!} \frac{\xi_2^{v-1} d\xi_2}{(v-1)!} \frac{\xi_3^{v-1} d\xi_3}{(v-1)!}.
\]

The probability is of order \( e^{-\Theta(n^{1/3-\sigma})} \), and the integral equals the bottom 3-dimensional integral in (5.9). Jointly with (5.10) and (5.11) this proves the claim.

Finally, introduce \( T_3(v') = \sum_{(i,j) \in M_1 \cap M_2} v'_i v'_j \); (here, of course, \( i, j \in \mathcal{M} \)).

**Lemma 5.6** *For \( \sigma < 1/3 \), let

\[
C_4 = \{(x, y) \in C_3 : |2(n-m-2v)T_3(v') - 1| \leq n^{-\sigma}\}
\]

Then

\[
P_{C_3}(\Pi_1, \Pi_2) - P_{C_4}(\Pi_1, \Pi_2) \leq 2^\mu e^{-\Theta(n^{1/3-\sigma})} \left( \frac{e}{n+m} \right) \frac{n^m}{2} n^{-\sigma}.
\]

The proof is a copy of the previous argument.

The Lemmas 5.3, 5.4, 5.5 and 5.6 imply

**Lemma 5.7** *For every admissible pair \( \Pi_1, \Pi_2 \),

\[
P(\Pi_1, \Pi_2) - P_{C_4}(\Pi_1, \Pi_2) \leq b \ 2^\mu e^{-\Theta(\log^2 n)} 2^\mu \left( \frac{e}{n+m} \right) \frac{n^m}{2} n^{-\sigma}.
\]

Here \( P_{C_4}(\Pi_1, \Pi_2) \) is the integral of \( F(x, y) \) over

\[
C_4 \subset \{(x, y) \in [0, 1]^n : x_i = y_i \text{ if } \Pi_1(i) = \Pi_2(i)\},
\]

defined by the additional constraints: denoting \( \xi := \sum_{i \in \mathcal{M}} x_i' \) (\( = \sum_{i \in \mathcal{M}} x_i \)),

\[
s := \sum_{i \in [n]} x_i' + \sum_{j \in A \cup B} y_j' (= \xi_1 + \xi_2 + \xi_3) \leq s_n (= n^{1/2} + 6 \log n),
\]

\[
\max \left\{ \max_{i \in [n]} x_i', \max_{j \in A \cup B} y_j' \right\} \leq 1.01 \frac{s \log^2 n}{n},
\]
\[
\frac{|M|}{2\xi^2} \sum_{i \in M} x_i^2 - 1 \leq n^{-\sigma}, \quad \frac{2|M|}{\xi^2} \sum_{(i, j) \in M_1 \cap M_2} x_i x_j - 1 \leq n^{-\sigma}.
\] (5.25)

The constraint (5.25) involves only \( \{x_i\}_{i \in M} \), and the constraint (5.24) imposes the bound for the individual components \( x_i' \) and \( y_j' \). Since \( s_n \leq 2n^{1/2} \), the latter implies that

\[
\max \left\{ \max_{i \in [n]} x_i', \max_{j \in A \cup B} y_j' \right\} \leq 3n^{-1/2} \log^2 n,
\] (5.26)

obviating the constraint \( x_i' \leq 1, y_j' \leq 1 \). On \( C_4 \) the inequality (5.21) can be drastically simplified. First of all, the bottom part of the bound (5.21) is

\[
-\frac{1}{4} \left( \sum_{i \in M} x_i^2 \right)^2 + O(n^{-1/2} \log^5 n).
\]

Second,

\[
\sum_{i \in M} x_i^2 = (1 + O(n^{-\sigma})) \frac{2\xi^2}{|M|} = \frac{2\xi^2}{|M|} + O(n^{-\sigma}),
\]

\[
\sum_{(i, j) \in M_1 \cap M_2} x_i x_j = (1 + O(n^{-\sigma})) \frac{\xi^2}{2|M|} = \frac{\xi^2}{2|M|} + O(n^{-\sigma}),
\]

and \( \xi = \xi_1 (1 + O(n^{-1/2} \log^2 n)) \). In addition, \( |M| = n(1 + O(n^{-1/2} \log n)) \). Therefore (5.21) becomes

\[
G(x, y) \leq (1 + O(n^{-\sigma})) \exp[H(\xi)], \quad H(\xi) = -\frac{s^2}{2} + \xi_2 \xi_3 + \frac{3\xi_1^2}{2n} - \frac{\xi_1^4}{n^2}.
\]

**Lemma 5.8**

\[
P_{C_4}(\Pi_1, \Pi_2) \leq \frac{2^{1 + O(n^{-\sigma})}}{(n + m - 1) \left[ (\nu - 1) \right]^{3/2}} \cdot \mathcal{I}(n + m, \nu),
\]

\[
\mathcal{I}(n + m, \nu) := \iiint_{(\xi_1, \xi_2, \xi_3) \in \mathbf{R}} \exp[H(\xi)] \xi_1^{n+m-1} \cdot (\xi_2 \xi_3)^{\nu-1} \, d\xi,
\]

\[
\mathbf{R} := \{ \xi \geq 0 : \xi_1 \leq n^{1/2} + 6 \log n; \xi_2, \xi_3 \leq 2 \log^3 n \}.
\]

The proof, of course, is based on the description of \( C_4 \), and it runs along the familiar lines of our preceding proofs; in particular, see the proof of Lemma 5.5. We omit the details. Furthermore, by the asymptotic formula for \( \mathcal{I}(n, \nu) \) from Pittel (1993b) (3.60),
we have
\[ I(n + m, \nu) = (1 + o(1)) \left( \frac{\pi e}{n + m} \right)^{1/2} \left( \frac{n + m}{e} \right)^{\frac{n+m}{2}} (n + m)^{-\nu} [(\nu - 1)!]^2. \]

So, by Lemma 5.8,
\[ P_{\mathcal{C}_4}(\Pi_1, \Pi_2) \leq \frac{2^\mu \left( 1 + O(n^{-\sigma}) \right)}{(n + m - 1)!} \left( \frac{\pi e}{n + m} \right)^{1/2} \left( \frac{n + m}{e} \right)^{\frac{n+m}{2}} (n + m)^{-\nu}. \]

Therefore, within the factor \( 1 + O(n^{-\sigma}) \),
\[
\sum_{\Pi_1, \Pi_2} P_{\mathcal{C}_4}(\Pi_1, \Pi_2)
\leq \sum_{m \leq m_n} \left( \frac{n}{m} \right) \frac{f(m)}{(n + m - 1)!} \left( \frac{\pi e}{n + m} \right)^{1/2} \left( \frac{n + m}{e} \right)^{\frac{n+m}{2}} (n + m)^{-\nu}
\times \sum_{\nu \leq m_n} \frac{(n - m)}{2\nu} (n - m - 2\nu - 1)! (n + m)^{-\nu} \cdot \sum_\mu 2^{2\mu} f(2\nu, \mu)
= n! \sum_{m \leq m_n} \frac{f(m)}{m! (n + m - 1)!} \left( \frac{\pi e}{n + m} \right)^{1/2} \left( \frac{n + m}{e} \right)^{\frac{n+m}{2}}
\times \sum_{\nu \leq m_n} \frac{(n - m - 2\nu - 1)!}{(n - m - 2\nu)! (2\nu)!} (n + m)^{-\nu} \cdot \sum_\mu 2^{2\mu} f(2\nu, \mu).
\]
cf. (5.13). The sum over \( \mu \) is \( e^{-1}(2\nu)! (1 + O(1/\nu)) \), see (5.14). So the sum over \( \nu \) is asymptotic to
\[
\frac{e^{-1}}{(n - m)!} \sum_{\nu \leq m_n} (n + m)^{-\nu} \prod_{j=0}^{\nu-1} (n - m - 2j) \sim \frac{e^{-1}}{(n - m)!} \sum_{\nu \leq m_n} e^{\frac{\nu^2}{2} - \frac{2\nu m}{n}}.
\]

Thus, since \( f(m)/m! = e^{-1} \sqrt{\frac{2}{\pi m}} (1 + O(m^{-1})) \), the \( m \)-term in the resulting sum is (within a factor \( 1 + O(m^{-1}) \))
\[
e^{-2} \sqrt{\frac{2}{\pi m}} \cdot \frac{n!}{(n - m)! (n + m - 1)!} \left( \frac{e}{n + m} \right)^{1/2} \left( \frac{n + m}{e} \right)^{\frac{n+m}{2}}
\times \sum_{\nu \leq m_n} e^{\frac{\nu^2}{2} - \frac{2\nu m}{n}}
\sim e^{-3/2} \sqrt{\frac{2}{\pi^2 m}} \cdot e^{-\frac{m^2}{2n}} \sum_{\nu \leq m_n} e^{\frac{\nu^2}{2} - \frac{2\nu m}{n}}.
\]
So

\[ \sum_{\Pi_1, \Pi_2} P_{C_4}(\Pi_1, \Pi_2) \lesssim e^{-3/2} \sqrt{\frac{2}{\pi^2}} \sum_{m, n} \frac{1 + O(m^{-1})}{m^{1/2}} e^{-\frac{m^2}{2n} - \frac{2mn}{n} - \frac{\nu^2}{n}} \sim cn^{3/4}, \]

\[ c := e^{-3/2} \sqrt{\frac{2}{\pi^2}} \int \int_{x \geq y \geq 0} x^{-1/2} e^{-\frac{x^2}{2} - 2xy - y^2} \, dx \, dy \approx 0.617. \]

Thus, since \( E[S_n] \) is of order \( n^{1/4} \), we have

**Theorem 5.9** \( E[S_n^2] \lesssim cn^{3/4} \).

With extra work, we could have proved that \( E[S_n^2] \gtrsim cn^{3/4} \), as well. Since \( E[S_n^2] \gg E^2[S_n] \), we cannot deduce that \( S_n \to \infty \) in probability, even though \( E[S_n] \to \infty \). We firmly believe that the argument itself may help to define a subset of stable partitions for which the two-moments approach will work just fine. For now we are content to use the techniques above to prove a result that would have been out of reach if not for the analysis of \( E[S_n^2] \).

**Theorem 5.10** Let \( q_n \) denote the fraction of members that have more than one stable predecessor. Then \( E[q_n] = O(n^{-1/2} \log n) \), so that with high probability the fraction of members with multiple predecessors is \( O(n^{-1/2} + o(1)) \).

**Proof** It suffices to consider the members outside the odd cycles. Here is a simple observation. Given a reduced stable partition \( \Pi \), let us split the set of pairs matched in \( \Pi \) into two disjoint subsets, \( M_1(\Pi) \) and \( M_2(\Pi) \): \( M_1(\Pi) \) consists of the pairs that belong to the rotations exposed in \( \Pi \). Then all pairs in \( M_2(\Pi) \) are pairs in every other reduced stable partition \( \Pi' \). If not, then a pair \((u, v) \in M_2(\Pi)\) is not a pair in some reduced \( \Pi' \), and so \((u, v)\) must be in a circuit of even length \( \geq 4 \), formed by matched pairs from \( \Pi \) (\((u, v)\) among them) and \( \Pi' \), in alternating fashion, meaning that \((u, v)\) is in a rotation: contradiction. It remains to observe that, by Corollary 5.2, quite surely for every stable, non-reduced, partition the number of members in the even cycles of length \( \geq 4 \) is \( O(n^{1/2} \log n) \). \( \square \)

### 6 Conclusion

We hope the reader shares our view that the stable matchings/partitions with many agents are surprisingly amenable to asymptotic analysis in the case of independent uniform preferences. In combination with local nature of stability conditions, a series of leading probabilities/expectations are expressed through the values of multidimensional integrals, that in turn are sharply estimated because of the product form of the integrands. Is there a possibility to enhance these techniques with a deeper combinatorial (game-theoretical) insight in order to confirm rigorously the conjecture that with probability \( 1 - o(1) \) no stable matching exists? It is probably telling that a single, non-trivial, upper bound \( e^{1/2}/2 \) for solvability probability was obtained in Irving
and Pittel (1994) by analysis of the Tan’s proposal algorithm based on the integral representations of the relevant probabilities/expectations.

Can the method of the integral representations be extended to the stable matchings/partitions with ties/incomparability in the preference lists?

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Appendix

Proof of Lemma 5.3 We already observed, (5.8), that \( \sum_i (x_i \vee y_i) = \sum_j \xi_j \). So, similarly to (5.9)–(5.10) we have:

\[
P(\Pi_1, \Pi_2) - P_{C_1}(\Pi_1, \Pi_2) \leq b 2^{\mu} \int \int \int \frac{\sum_j (\xi_j)^2 + \xi_2 \xi_3}{(n + m - 1)!} \cdot \frac{\xi_1^{n+m-1}}{(v - 1)!} d\xi
\]

\[
= 2^{\mu} \sum_{k \geq 0} \frac{[(v - 1 + k)!]^2}{[(v - 1)!]^2 k! (n + m + 2(v + k) - 1)!} \times \int_{s \geq s_n} \exp \left( -\frac{s^2}{2} \right) s^{n+m+2(v+k)-1} ds. \tag{6.1}
\]

(Relaxing the constraint on \( s \) to \( s \geq 0 \) we get back to (5.10)). The last integrand attains its maximum at

\[ s_{\text{max}} = (n + m + 2(v + k) - 1)^{1/2}, \]

which is below \( s_n - 3 \log n \) if \( k \leq m_n \). Let \( S_{\leq m_n} \) and \( S_{> m_n} \) denote the sub-sums of the sum above, for \( k \leq m_n \) and \( k > m_n \) respectively. Then, expanding integration to \([0, \infty)\), we obtain

\[
S_{> m_n} \leq \sum_{k > m_n} \frac{[(v - 1 + k)!]^2}{[(v - 1)!]^2 k! (n + m + 2(v + k) - 1)!!} \leq b \frac{[(v + m_n)!]^2}{[(v - 1)!]^2 m_n! (n + m + 2(v + m_n) + 1)!!};
\]
since \( \nu \leq m_n \), the ratio of the consecutive terms in the sum is below \( 2/3 \). Dropping \( [(\nu - 1)!]^2 \) in the denominator and using the Stirling formula for the other factorials, we simplify the bound to

\[
S_{>m_n} \leq b \left( \frac{e}{n + m} \right)^{\frac{n + m}{2}} (n + m)^{-v(n + m)}.
\]

(6.2)

The bound is smaller than the bound (5.11) for the full sum of \( s(n + m, \nu, k) \) by the factor \( (n + m)^{mn} \). Turn to \( S_{\leq m_n} \). This time the bottom integral over \( s \geq s_n \) in (6.1) is small, compared to the integral over all \( s \geq 0 \), because for \( k \leq m_n \) the maximum point of the integrand is at distance \( 3 \log n \), at least, from the interval \([s_n, \infty)\). More precisely, using the argument in the proof of Lemma 4.5, we have

\[
\int_{s \geq s_n} \exp \left( -\frac{s^2}{2} \right) s^{n + m + 2(v + k) - 1} ds \leq b e^{-8 \log^2 n} (n + m + 2(v + k) - 2)!!.
\]

Therefore

\[
S_{\leq m_n} \leq b e^{-8 \log^2 n} \sum_{k \leq m_n} \frac{[(\nu - 1 + k)!]^2}{[(\nu - 1)!]^2 k!(n + m + 2(v + k) - 1)!}.
\]

\[
\leq e^{-8 \log^2 n} \sum_{k \geq 0} \frac{[(\nu - 1 + k)!]^2}{[(\nu - 1)!]^2 k!(n + m + 2(v + k) - 1)!}.
\]

(6.3)

Combining (6.2), (6.3) and (5.11) we transform the inequality (6.1) into

\[
P(\Pi_1, \Pi_2) - P_{C_1}(\Pi_1, \Pi_2) \leq e^{-\Theta(\log^2 n)} 2^{\mu} \left( \frac{e}{n + m} \right)^{\frac{n + m}{2}} (n + m)^{-v}.
\]

(6.4)

So, like the part (1) in the proof of Lemma 5.1, we obtain

\[
\sum_{\Pi_1, \Pi_2} \left[ P(\Pi_1, \Pi_2) - P_{C_1}(\Pi_1, \Pi_2) \right] \leq e^{-\Theta(\log^2 n)} n! m_n
\]

\[
\times \sum_{m \leq m_n} \left( \frac{e}{n + m} \right)^{\frac{n + m}{2}} \left[(n + m)^0 2^{\frac{n - m - 2}{2}} \left( \frac{n - m - 2 \cdot 0}{2} \right)! \right]^{-1}
\]

(6.5)

\[
= e^{-\Theta(\log^2 n)}.
\]

Proof of Lemma 5.4 Introduce \( L'_1, \ldots, L'_{n+2\nu} \), the intervals lengths in the random partition of \([0, 1]\) by the \( n + 2\nu - 1 \) random points. Analogously to (5.9), but using
the sharper inequality in Lemma 3.1, (3.1), we have: with \( s := \xi_1 + \xi_2 + \xi_3 = \sum_{i \in [n]} y_i' + \sum_{i \in A \cup B} y_i' \) (see (5.5) and (5.6)),

\[
\begin{align*}
\mathbb{P}_{C_1}(\Pi_1, \Pi_2) - \mathbb{P}_{C_2}(\Pi_1, \Pi_2) & \leq b \ 2^\mu \int_{x', y' \geq 0} \cdots \int_{T_1(u') > 1.01 \log^2 n \over n} e^{-\frac{s^2}{2} + \xi_2 \xi_3} \prod_{i \in [n]} x_i \prod_{i \in A \cup B} y_i' d\xi_1 \cdots d\xi_3 \\
& \leq \frac{2^\mu}{(n + 2v - 1)!} \mathbb{E} \left[ \chi \left( T_1(L') \geq \frac{\log^2 n}{n} \prod_{h \in \text{Odd}_1} L'_h \right) \right] \\
& \times \int_{x', y' \geq 0} e^{-\frac{s^2}{2} + \xi_2 \xi_3} \sum_{i \in [n]} \prod_{j \in A \cup B} y_i' d\xi_1 \cdots d\xi_3
\end{align*}
\]

Arguing as in (4.12), the expectation factor is less than

\[
e^{-\Theta((\log^2 n) \over n)} \frac{(n + 2v)!}{(n + m + 2v - 2)!}.
\]

The integral is less than

\[
I_n(m, v) := \int_{\xi_1, \xi_4 \geq 0} e^{-\frac{s^2}{2} + \xi_2 \xi_3} \sum_{k \geq 0} \frac{[\log^2 n \over n]^{v-1}}{(v-1)!^2} \sum_{k \geq 0} \frac{[(v-1+k)!]^2}{(v-1)!^2 k! (2(v+k)-1)!} \\
\times \int_{\xi_1, \xi_4 \geq 0} e^{-\frac{(\xi_1 + \xi_4)^2}{2}} (\xi_1 + \xi_4)^m \xi_1^{n-1} \xi_4^{2(v+k)-1} d\xi_1 d\xi_4.
\]

Here the double integral equals

\[
\frac{(n-1)! (2(v+k) - 1)! (n + m + 2(v+k) - 2)!!}{(n + 2(v+k) - 1)!}.
\]

So

\[
I_n(m, v) = \sum_{k \geq 0} \frac{[(v-1+k)!]^2}{[(v-1)!^2 k! (n + 2(v+k) - 1)!}.
\]
Therefore

\[ P_{C_1}(\Pi_1, \Pi_2) - P_{C_2}(\Pi_1, \Pi_2) \leq b 2^\mu e^{-\Theta(\log^2 n)} \sum_{k \geq 0} s'(n, m, v, k), \]

\[ s'(n, m, v, k) := \frac{[(v - 1 + k)!]^2 (n + m + 2(v + k) - 2)! (n + 2v)!}{(n + m + 2v - 2)! [(v - 1)!]^2 k! (n + 2(v + k) - 1)!}. \]

The summand \( s'(n, m, v, k) \) is similar to the summand \( s(n + m, v, k) \) defined in (5.10). Closely following the derivation of the bound for \( \sum_{k \geq 0} s(n, v, k) \) in Pittel (1993b), we obtain

\[ \sum_{k \geq 0} s'(n, m, v, k) \leq b n^2 \left( \frac{e}{n + m} \right)^{n+m} n^{-v}, \]

compare to (5.11). The last two bounds complete the proof. \( \square \)

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