Gram’s Law Fails a Positive Proportion of the Time

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Abstract
It is straightforward to show that all the non-trivial zeroes of the Riemann zeta-function \( \zeta(s) \) are confined to the critical strip: \( 0 \leq \Re(s) \leq 1 \). It is another matter to seek out their precise location. Extending the work of Riemann, Gram introduced a procedure for detecting these zeroes: Gram’s Law. It is known that Gram’s Law fails infinitely often, and that a weaker formulation of Gram’s Law is true infinitely often. This paper extends these results by showing that there is a positive proportion of both failures and (weak) successes.

1 Gram’s Law
Connected to \( \zeta(s) \) are two functions known as the Riemann-Siegel functions,

\[
Z(t) = e^{i\theta(t)} \zeta \left( \frac{1}{2} + it \right),
\]

where

\[
\theta(t) \sim \frac{t}{2} \log \frac{t}{2\pi} + O(t),
\]

is a steadily increasing function of \( t \) and \( Z(t) \) is real-valued whenever \( t \) itself is real-valued. Hence the zeroes of \( \zeta \left( \frac{1}{2} + it \right) \) coincide precisely with those of \( Z(t) \). Therefore the search for zeroes of the zeta-function is equivalent to a search for changes in sign of \( Z(t) \). From (1) it follows that

\[
\zeta \left( \frac{1}{2} + it \right) = Z(t) \cos \{ \theta(t) \} - iZ(t) \sin \{ \theta(t) \},
\]

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and so at the **Gram points** \( \theta(g_n) = n\pi \)

\[
\zeta \left( \frac{1}{2} + ig_n \right) = (-1)^n Z(g_n).
\]

(4)

Gram [4] found that \( \Re\{\zeta(\frac{1}{2} + it)\} > 0 \) frequently so that, in particular \( Z(g_n) \) and \( Z(g_{n+1}) \) were often of opposite sign, whence a zero of \( \zeta(\frac{1}{2} + it) \) must occur in this interval. Gram noted that this pattern can reasonably be expected to continue: one zero of \( \zeta(\frac{1}{2} + it) \) between successive Gram points. Gram located the first 15 non-trivial zeroes of \( \zeta(s) \) using this method and found each one to lie on the line. Hutchinson extended these results in [5] to show that the first 138 zeroes are on the line and later Titchmarsh [9] improved this to the first 1041 zeroes. In all cases it was shown that there were no other complex zeroes up to these heights. For a comprehensive historical account of the applications of Gram’s Law to finding zeroes of the zeta-function, see [1, pp.171-182].

**Gram’s Law**, as defined by Hutchinson (ibid) is that statement that there is exactly one zero of \( \zeta(\frac{1}{2} + it) \) for \( t \in (g_n, g_{n+1}] \) and that this is the only zero of \( \zeta(\frac{1}{2} + it) \) for \( t \in (g_n, g_{n+1}] \). A weakened version of Gram’s Law (hereafter referred to as the **Weak Gram Law**) is the statement that there is at least one zero of \( \zeta(\frac{1}{2} + it) \) for \( t \in (g_n, g_{n+1}] \). The successes of Gram’s Law were first investigated by analysing the discrete properties of the function \( Z(t) \), but later the argument function \( S(t) \) was used to obtain improvements.

### 1.1 The function \( S(t) \) and Gram’s Law

As is standard let \( S(T) = \pi^{-1} \arg\{\zeta(\frac{1}{2} + iT)\} \) (for more properties on the function \( S(T) \) see, for example [10, pp. 212-223]) and so by the Riemann-von Mangoldt formula

\[
S(T) = N(T) - \pi^{-1} \theta(T) - 1,
\]

(5)

where \( N(T) \) is the number of zeroes of \( \zeta(\sigma + it) \) for \( 0 \leq t \leq T \). Since \( \theta(g_n) = n\pi \) and \( \theta(t) \) is a steadily increasing function of \( t \), it is seen at once that \( S(T) \) is integral precisely at the Gram points. Furthermore if \( S(g_n) = \lambda \) and there is exactly one zero in the Gram interval \( (g_n, g_{n+1}] \) then \( S(g_{n+1}) = \lambda \) and \( |S(t) - \lambda| \leq 1 \) throughout the interval. So intervals in which Gram’s Law is valid induce some constancy in the function \( S(t) \) and it is this constancy which forms the basis of the following analysis.

### 2 General Failures

Titchmarsh showed in [8] that Gram’s Law fails infinitely often. This section will show that Gram’s Law and the Weak Gram Law fail a positive proportion of the time (given in **Theorem 1** on p.8 and **Theorem 2** on p.11 respectively).
The following result due to Fujii [2] concerns the ‘shifted moments’ of $S(t)$, viz.

$$I(T) = \int_T^{2T} |S(t + h) - S(t)|^2 \, dt$$

$$= \pi^{-2} T \log (3 + h \log T) + O \left( T \log (3 + h \log T)^{\frac{1}{2}} \right), \quad (6)$$

which is valid for $0 \leq h \leq \frac{1}{4}T$. This becomes an asymptotic relationship, i.e.

$$I(T) \sim \pi^{-2} T \log (3 + h \log T), \quad (7)$$

if $h \log T$ is sufficiently large. Henceforth $h = C_0 (\log T)^{-1}$, where $C_0$ is a constant that is chosen to be sufficiently large to ensure the dominance of the main term in (6) over the error term.

If $t$ and $t + h$ are in a connected union of Gram intervals in which Gram’s Law is valid, then $|S(t + h) - S(t)| \leq 2$. Thence $I(T) \leq 4T$ which is ‘too small’, in that this is not asymptotic to $\pi^{-2} T \log (3 + h \log T)$. This lends credence to what has already been shown using the work of Ghosh [3]: that for sufficiently large $T$ there must be at least one failure of Gram’s Law between heights $T$ and $2T$. In some loose sense, if $S$ is the set on which Gram’s Law is valid and $\overline{S}$ is the complement of $S$ in $[T, 2T]$, then (6) can be rewritten

$$I(T) \sim \pi^{-2} T \log (3 + h \log T)$$

$$= \int_S |S(t + h) - S(t)|^2 \, dt + \int_{\overline{S}} |S(t + h) - S(t)|^2 \, dt$$

$$\leq 4|S| + \int_{\overline{S}} |S(t + h) - S(t)|^2 \, dt, \quad (8)$$

whence an estimate on $|\overline{S}|$ can be made.

To this end, let the sequences \{\(i_n\}\} and \{\(j_n\)\} index the Gram points such that Gram’s Law holds on the collection of intervals \([g_{i_n}, g_{j_n}]\) and Gram’s Law fails on the collection of intervals \([g_{j_n}, g_{i_{n+1}}]\). Also let \(k_n = i_{n+1} - j_n\), that is, the number of consecutive Gram points between which Gram’s Law fails. So then \(\sum_n k_n = N_F(2T)\): the number of failures between heights $T$ and $2T$. It is now appropriate to introduce the following elementary result concerning $N_F$; the number of Gram points between heights $T$ and $2T$,

**Lemma 1** $N_F \sim (2\pi)^{-1} T \log T$. Furthermore if $g_n$ and $g_m$ are Gram points in the interval $[T, 2T]$ then $g_n - g_m = O(\frac{\log T}{\log T})$.

The first statement follows from both (2) and the definition of the Gram points. Since it can be shown (see [10, p 263]) that $\theta(t) \sim \frac{1}{2} \log t$, then the mean value theorem gives

$$\frac{\theta(g_n) - \theta(g_m)}{g_n - g_m} = \frac{(n - m)\pi}{g_n - g_m} = \frac{1}{2} \log \xi, \quad (9)$$

Footnote 3: This has been shown by the author and is being prepared for publication.
for some \( \xi \in (g_n, g_m) \), whence the result follows.

It is clear that the relative locations of \( t \) and \( t + h \) will determine the bound on \( |S(t+h) - S(t)| \): namely if \( g_n \leq t \leq t + h \leq g_{j_n} \) then \( |S(t+h) - S(t)| \leq 2 \). This leads to the definition

\[
S := \{ t \in [T, 2T] : \exists n \quad : g_n \leq t \leq t + h \leq g_{j_n} \},
\]

whence \( \int_{S} |S(t+h) - S(t)|^2 \, dt \leq 4T \), as claimed.

Now let \( \overline{S} \) be the complement of \( S \) in \([T, 2T]\). Then, if \( t \) belongs to \( \overline{S} \) either \( t \in (g_n, g_{j_n}] \) and \( t + h \geq g_{j_n} \); or \( t \in (g_{j_n}, g_{j_{n+1}}] \). The former condition is equivalent to \( g_{j_n} \geq t \geq g_{j_n} - h \) and so in any case \( g_{j_n} - h \leq t \leq g_{j_{n+1}} \). These intervals may overlap in \([T, 2T]\) and indeed

\[
\overline{S} \subset \bigcup_n (g_{j_n} - h, g_{j_{n+1}}].
\]

Whether or not these intervals are disjoint is of no consequence for Lemma 1 gives

\[
|\overline{S}| \ll \sum_n h + \frac{k_n}{\log T} \ll \left( h + \frac{1}{\log T} \right) N_F(2T).
\]

Ultimately an estimate on this number \( N_F(2T) \) is sought and hence the imposition of a lower bound of (12) would be useful. Returning to (8) it is seen that

\[
\pi^{-2}T \log (3 + h \log T) \leq 4|S| + \int_{\overline{S}} |S(t+h) - S(t)|^2 \, dt
\]

\[
\leq 4T + \int_{\overline{S}} |S(t+h) - S(t)|^2 \, dt. \tag{13}
\]

Currently \( h = C_0(\log T)^{-1} \) and \( C_0 \) is chosen to be sufficiently large such that the main term in (6) dominates the error term. If, in addition to this, \( C_0 \) is taken large enough to make the quantity \( \pi^{-2}T \log(3 + h \log T) \) larger than \( 5T \), then (13) gives

\[
T \ll \int_{\overline{S}} |S(t+h) - S(t)|^2 \, dt. \tag{14}
\]

Results on higher moments of the function \( S(t) \) have been developed by Fujii following the work of Selberg, as detailed in [10, pp. 245-246]. These may be employed after an application of Cauchy’s inequality to give

\[
\int_{S} |S(t+h) - S(t)|^2 \, dt \leq \left( \int_{S} |S(t+h) - S(t)|^4 \, dt \right)^{\frac{1}{2}} \times \left( \int_{S} dt \right)^{\frac{1}{2}}
\]

\[
= |S|^\frac{1}{2} \times \left( \int_{S} |S(t+h) - S(t)|^4 \, dt \right)^{\frac{1}{2}}, \tag{15}
\]

whence via (14) it follows that

\[
T \ll |S|^\frac{1}{2} \times \left( \int_{S} |S(t+h) - S(t)|^4 \, dt \right)^{\frac{1}{2}}. \tag{16}
\]
The particular result needed here is

\[ \int_T^{2T} |S(t + h) - S(t)|^4 \, dt \ll T \log^2 (3 + h \log T). \]  

(17)

Since \( \int_S | \cdot | \leq \int_T^{2T} | \cdot | \), the inequality in (16) can be combined with (17) to give

\[ T \ll |S|^{\frac{1}{2}} \log (3 + h \log T), \]  

and so a lower bound on \( |S| \) is attained, viz.

\[ |S| \gg \frac{T}{\log^2 (3 + h \log T)}. \]  

(19)

Now the upper bound for \( |S| \) in (12) can be combined with the lower bound in (19) to give,

\[ \frac{T}{{\log}^2 (3 + h \log T)} \ll \left( \frac{1}{\log T} + h \right) N_F(2T), \]  

(20)

or

\[ N_F(2T) \gg \frac{T \log T}{(1 + C_0) \log^2 (3 + C_0)} \gg AT \log T, \]  

(21)

for some positive constant \( A \). By Lemma 1 the total number of Gram points between heights \( T \) and \( 2T \) is \( O(T \log T) \) which proves the following

**Theorem 1** For sufficiently large \( T \) there is a positive proportion of failures of Gram’s Law between \( T \) and \( 2T \).

### 2.1 Further failures

A small point to note is that it is now possible to deduce that there is a positive proportion of Gram intervals which do not contain a zero of \( \zeta(\frac{1}{2} + it) \). For \( i = 0, 1, 2, \ldots \) let \( F_i \) denote a Gram interval in which \( i \) zeroes are located. Furthermore, let \( N_{F_i} \) denote the number of Gram intervals between heights \( T \) and \( 2T \) which contain exactly \( i \) zeroes: so that \( N_{F_0} \) is the number of \( F_0 \) intervals, \( N_{F_1} \) the number of intervals in which Gram’s Law is valid, and so on. Then

\[ N_{F_0} + N_{F_1} + \ldots + N_{F_k} + \ldots = N_g = \frac{T}{2\pi} \log T + O(T), \]  

(22)

where \( N_g \) is the total number of Gram intervals between heights \( T \) and \( 2T \). That there is a positive proportion of failures is represented by the following equation

\[ N_{F_0} + N_{F_2} + \ldots + N_{F_k} + \ldots \geq AN_g, \]  

(23)

where, as before, the number \( N_{F_1} \) is absent since this does not represent any failures. Lastly since all the zeroes on the critical line between heights \( T \) and \( 2T \),
denoted by $N_0(T)$, fall within Gram intervals a third relation may be written, viz.

$$N_{F_1} + 2N_{F_2} + \ldots + kN_{F_k} + \ldots = N_0(T) \leq N(T) = \frac{T}{2\pi} \log T + O(T), \quad (24)$$

where $N(T)$ is the number of complex zeroes of $\zeta(s + it)$ with $0 \leq t \leq T$. The subtraction of equation (22) from (24) gives

$$O(T) \geq -N_{F_0} + N_{F_2} + 2N_{F_3} \ldots + (k-1)N_{F_k} + \ldots \quad (25)$$

$$\geq -N_{F_0} + N_{F_2} + N_{F_3} \ldots + N_{F_k} + \ldots \quad (26)$$

whence, upon an addition of $2N_{F_0}$ and an invocation of (23) it is seen that

$$2N_{F_0} + O(T) \geq N_{F_0} + N_{F_2} + N_{F_3} \ldots N_{F_k} + \ldots \geq AN_g, \quad (27)$$

so that

$$\frac{N_{F_0}}{N_g} \geq \frac{A}{2} + O \left( \frac{1}{\log T} \right). \quad (28)$$

Thus the following has been now been proved

**Theorem 2** *For sufficiently large $T$ there is a positive proportion of failures of the Weak Gram Law between $T$ and $2T$.*

Since the number of $F_0$ intervals is certainly less than the total number of violations of Gram’s Law, the order of $N_{F_0}$ is exactly determined, viz. $AT \log T \leq N_{F_0} \leq AT \log T$. There is little else to be said about the nature of $F_0$ intervals, so it is natural to now turn to the remaining cases: those Gram intervals which contain at least one zero of $\zeta(\frac{1}{2} + it)$.

## 3 $F_k$ intervals

Titchmarsh showed in [7] that the Weak Gram Law is true infinitely often. What is actually shown in his proof is that there is an infinite number of Gram intervals which contain an odd number of zeroes. His proof concludes that the proportion of Gram intervals between $T$ and $2T$ which contain an odd number of zeroes of $\zeta(\frac{1}{2} + it)$ is greater than $A(T^{1/3} \log^2 T)^{-1}$, with $A$ a positive constant. This section will show (in **Theorem 3** on p.9) that the Weak Gram Law is true a positive proportion of the time.

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4One possibility is to calculate these constants, but this is not achievable via the methods in this paper.

5One remark to be made is that it is not yet known whether Gram’s Law is true infinitely often.
3.1 Outline

It is difficult to investigate the quantities $N_{F_k}$ for ‘small’ $k$, since the induced behaviour in $S(t)$ is virtually undetectable. Indeed the methods used in §1 viz. shifted moments of $S(t)$ are unable to distinguish a collection of $F_1$ intervals from a sequence of alternating $F_0$ and $F_2$ intervals. Investigations into the frequency of successes of Gram’s Law (or the quantity $N_{F_1}$) must be made through some other route. What can be said is a measure of the success of Gram’s Law in its weak sense: that is the number of intervals which contain at least one zero of $\zeta(\frac{1}{2} + it)$.

By Selberg’s result, a positive proportion of zeroes lie on the critical line; clearly each zero is contained within a Gram interval. There is a possibility that when $k$ is arbitrarily large there are many $F_k$ intervals which contain the bulk of these zeroes. To rule out this possibility it is necessary to place a bound on the growth of $N_{F_k}$ as $k \to \infty$. Once this has been established it will be shown that there is a $K$ such that the Gram intervals containing fewer than $K$ zeroes together contain the positive proportion of zeroes.

3.2 Improvements in the function $S(t)$

Much work has been done concerning the number of zeroes of $\zeta(\frac{1}{2} + it)$ of multiplicity greater than one. Extending this work to short intervals, particularly Gram intervals is natural since a zero of order $m$ will induce similar behaviour in $S(t)$ as will $m$ simple zeroes. The following result is due to Korolev [6]

$$I_m = \int_T^{T + H} |S(t + h) - S(t)|^{2m} dt \leq (Cm^2)^m H, \quad (29)$$

where

$$H = T^{\frac{27}{28}} + \epsilon; \quad 0 < \epsilon < 0.001; \quad h = \frac{2\pi}{3 \log \frac{T}{2\pi}}, \quad (30)$$

and $C$ is given as an explicit positive constant. This formula has $h = c (\log T)^{-1}$, where $c$ is small relative to the length of Gram intervals. In order to easily detect the contribution of an $F_k$ interval to the integrand in (29), the $h$ must be replaced with $N h$ (with $N$ to be chosen later) such that $N h$ is longer than a Gram interval. The following, which is easily deduced from Lemma 1, will prove useful

**Lemma 2** Denote the length of the longest Gram interval in $[T, T + H]$ by $L^+$ and the length of the shortest by $L^-$. Then

$$L^+ = \frac{2\pi}{\log \frac{T}{2\pi}}, \quad (31)$$

and

$$L^- = \frac{2\pi}{\log \frac{T + H}{2\pi}} = \frac{2\pi}{\log \frac{T}{2\pi} + \log(1 + \frac{H}{T})} = \frac{2\pi}{\log \frac{T}{2\pi}} \{1 + o(T)\}. \quad (32)$$
Now suppose the interval \((g_n, g_{n+1}]\) is an \(F_k\) interval and that \(S(g_n) = \lambda\). Then \(S(g_{n+1}) = \lambda + k - 1\) and thenceforth \(S(t)\) can decrease by at most one on the interval \((g_{n+1}, g_{n+2}].\) Furthermore for \(t \in (g_{n-2}, g_{n-1}]\) it follows that \(S(t) < \lambda + 2\). The choice of \(N\) must be made such that \(t + Nh \geq g_{n+1}\), which is satisfied if
\[
Nh \geq (g_{n+1} - g_{n-2}) \geq 3L^+ = 9h, \tag{33}
\]
so that \(N = 9\) will suffice. When \(T\) is sufficiently large, Lemma 2 shows \(t+Nh < g_{n+2}\) and so over an interval of length \(L^-\) the difference \(|S(t+h) - S(t)|\) is now bounded below by \(|k - 4|\). By an application of the Hölder inequality \((29)\) then becomes
\[
\int_T^{T+H} |S(t+h) - S(t)|^{2m} dt \leq 9^{2m-1} \sum_{i=0}^{8} \int_{T+ih}^{T+i+h} |S(t+h) - S(t)|^{2m} dt. \tag{34}
\]
Applying \((29)\) with \(T + ih\) in place of \(T\) it is seen that,
\[
\int_T^{T+H} |S(t+Nh) - S(t)|^{2m} dt \leq (Am^2)^m H(1 + o(1)), \tag{35}
\]
where \(A\) is a positive constant. Now suppose there are \(N_{F_k}\) intervals between heights \(T\) and \(T + H\). Each one will contribute at least \(|k - 4|^{2m}\) in the above integrand over a length at least \(L^-\). Thus
\[
(Am^2)^m H(1 + o(1)) \geq \frac{\pi N_{F_k}(T)(k - 4)^{2m}}{\log \frac{T}{2^m}}, \tag{36}
\]
or, expressed more succinctly,
\[
\frac{N_{F_k}(T)}{H \log T} \ll \left( \frac{Am^2}{(k - 4)^2} \right)^m. \tag{37}
\]
Now the task is to find the value of \(m\) depending on \(k\) that minimises the right hand inequality. Let
\[
F(m) = \left( \frac{Am^2}{(k - 4)^2} \right)^m, \tag{38}
\]
then
\[
\frac{dF(m)}{dm} = \left( \frac{Am^2}{(k - 4)^2} \right)^m \left\{ 2 + \log \left( \frac{Am^2}{(k - 4)^2} \right) \right\}, \tag{39}
\]
and clearly this stationary point \(m^* = \frac{(k-4)}{e \sqrt{A}}\) is indeed a minimum. Since \(m\) is an integer the value to be taken is whichever of \([m^*]\) or \([m^*] + 1\) is the nearer to \(m\). The error of such an assignment of value is \(O(1)\) in the exponent and can be absorbed into the ultimate \(O\)-constant. Thus, given a value of \(k\) the value \(m = m^*\) gives the bound
\[
\frac{N_{F_k}(T)}{H \log T} \ll \exp(-Ak), \tag{40}
\]
where $A$ is a positive constant, and clearly this result remains valid if $H$ is replaced with $T$. So $F_k$ intervals, for large $k$ are ‘exponentially rare’.

Since a positive proportion of zeroes lie on the critical line, the inequality in (40) can be used to show that a positive proportion of Gram intervals contain at least one zero. For, there is a constant $A'$ such that

$$0 < A' < \frac{N_{F_1}(T) + 2N_{F_2}(T) + \ldots + kN_{F_k}(T) + \ldots}{T \log T},$$

(41)

and by (40) this series on the right hand side is convergent. So, if $\delta$ is any small positive number, choose $K$ so large that the sum $(T \log T)^{-1} \sum_{k=K+1}^{\infty} kN_{F_k}(T)$ is not greater than $A' - \delta$. Then

$$0 < \delta < \frac{\sum_{k=1}^{K} kN_{F_k}(T)}{T \log T} < K \frac{\sum_{k=1}^{K} N_{F_k}(T)}{T \log T},$$

(42)

whence the number of Gram intervals which contain at least one zero is at least $AT \log T$, with $A$ a positive constant. Thus the following has been proved

**Theorem 3** For sufficiently large $T$ there is a positive proportion of successes of the Weak Gram Law between $T$ and $2T$.

### 4 Concluding Remarks

From (42) it follows that there must be a positive proportion of at least one of the $N_{F_k}$’s. Intuitively one might expect $N_{F_k}$ to be steadily decreasing with $k$ (which would be an improvement to the estimate in (40)). If such a relation could be shown it would therefore follow that there is a positive proportion of intervals in which Gram’s Law is valid.

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