Geometric rigidity of torsion of thin-walled pipes: theory and experiment

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Abstract. Thin-walled rods as structural elements of buildings, structures and machines that absorb torsion deformations are widely used in construction and engineering. Development of analytical methods for solving the problem of torsion of rods of non-circular cross section is one of the most important modern problems of construction mechanics. When solving torsion problems, the geometric rigidity of the section $I_k$ is primarily determined. The analysis of the considered literature sources showed that the study of the problem in question using the isoperimetric properties of cross-sections of rods is very effective. In the present paper, for thin-walled pipes with an arbitrary convex contour and constant wall thickness, the authors obtain calculation formulae that associate the reduced geometric rigidity of the cross-sections of the pipes $I_k$ with their form factor. It allows formulating the isoperimetric properties of the parameter under study depending on the shape of the pipe sections, and using geometric methods to determine the geometric rigidity of the cross-sections of pipes of arbitrary shape and a convex contour. The article also contains the results of experimental studies of triangular pipes confirming this isoperimetric relationship of the reduced geometric rigidity of pipes with the form factor.

1. Introduction

Thin-walled rods as structural elements that perceive deformation of bending and torsion are widely used in construction and engineering. First of all, these are elements of metal structures of trusses and structural coverings, beams and columns of box section, etc. Development of analytical methods for solving the problem of torsion of rods of non-circular cross section is one of the most important problems of construction mechanics. At the root of the solution to this problem were the famous researchers: Saint-Venant [1], G. Polya [2,3], S.P. Tymoshenko [4], V.Z. Vlasov [5], L.S. Leibenson [6], N.I. Muskhelishvili [7], N.Kh. Arutyunyan, B.L. Abramyan [8] and many others. A large number of papers have been devoted to the study of torsion problems by experimental methods using membrane analogy, for example, the work of G. Trayer and H. March [9]. The problem of torsion of rods with doubly connected cross-section and, in particular, pipes of non-circular cross section occupies a special place. G.Polya and G. Szegő [3] dealt with this problem.

When solving torsion problems, the geometric rigidity of the cross-section $I_k$ is primarily determined. For some sections in the form of a regular triangle, a square and an ellipse, the geometric rigidity is determined exactly by elementary formulae. For sections of a more complex type, numerical methods are used, in particular, the finite element method, and more complex analytical methods [10].

In the last two decades, a very effective engineering method, the method of interpolation by the form factor [11-14], has been actively developing to solve two-dimensional problems in the theory of
elasticity, which makes it possible to visualize the desired results among the entire set of possible solutions of the problem under consideration. This method was also used to solve certain problems in the theory of torsion of elastic rods [11,13,15-17]. However, there were no experimental studies confirming the results of the application of geometric methods in the study of torsion problems. In the present article, the problem of torsion of elastic prismatic pipes is being studied with the use of isoperimetric properties of the cross sections, herewith the formulae are derived that relate the geometric rigidity of the pipes to their form factor, which made it possible to formulate the isoperimetric properties of the reduced geometric rigidity of the pipe sections. The results of experimental studies of triangular pipes confirming this physico-geometric relationship are also presented.

2. Determination of the geometric rigidity of the section in the torsion of thin-walled pipes

In the theory of elasticity [18], a solution is given to the problem of torsion of pipes – elastic prismatic rods with a thin-walled doubly connected contour, which reduces to determining the geometric rigidity of the section $I_k$ and tangential stresses $\tau$:

$$I_k = 4A_0^2 \int_0^{\delta} \frac{ds}{L_0}; \quad \int_0^{\delta} \tau ds = 2G\theta A_0,$$  \hspace{1cm} (1)

Here $A_0$ is the area of the figure bounded by the closed line $L_0$ drawn in the middle of the cross-section wall; $\delta$ is the thickness of the pipe wall (figure 1); $G$ is the material shear modulus; $\theta$ is the relative angle of torsion. For pipes with the same wall thickness ($\delta = \text{const}$)

$$I_k = 4A_0^2 \delta / L_0; \quad \tau = M/(2A_0\delta) = 2G\theta A_0 / L_0.$$  \hspace{1cm} (2)

We transform expression (2) as follows:

$$I_k = \frac{4A_0^2 \delta}{L_0} = \frac{4A_0^2 \delta}{L_0^2} \frac{L_0^2 \delta}{L_0^2} = \frac{L_0}{\delta} \frac{4A_0^2}{L_0^2} \frac{L_0^2 \delta^2}{L_0^2} = \frac{L_0}{\delta} \frac{4A_0^2}{L_0^2} \bar{A} ,$$

where $\bar{A} = L_0 \delta$ is the cross-sectional area of the pipe. Taking this ratio into account, the reduced geometric rigidity of torsion $i_k = I_k / \bar{A}^2$ will be determined by formula

$$i_k = \frac{L_0}{\delta} \frac{4A_0^2}{L_0^2}.$$  \hspace{1cm} (4)

We introduce a new symbol $K_f = L_0^2/(2A_0)$ . This geometric parameter is well known in building mechanics and is called the form factor of figures in the form of polygons, all sides of which touch the inscribed circle [11]. Since this parameter is dimensionless, it can be expressed in terms of the outer dimensions of the tubular section: Substituting this equation in (4), we get:

$$i_k = L_0 \frac{1}{\delta} K_f^2.$$  \hspace{1cm} (5)

For pipes of arbitrary section with a convex contour, we should use inequality

$$i_k \geq L_0 \frac{1}{\delta} K_f^2,$$  \hspace{1cm} (6)
since it is known from [11] that for such figures $K_f \geq L^2/(2A)$, where equality is achieved for polygons, all sides of which touch the inscribed circle. Detailed studies of isoperimetric properties of the form factor are given in the monograph [19].

![Figure 1. The section of a thin-walled pipe of a triangular shape.](image)

Dependence (6), represented graphically in the coordinate axes $i_k \delta/L_0 - 4A_0^2/L_0^4$, will be depicted by a straight line passing through the origin at an angle of 45° (figure 2). In the coordinate axes $10^3i_k \delta/L - 1/K_f^2$, the dependence (6) is shown in figure 3, where the straight line 0-1-2-3 corresponds to pipes with section in the form of polygons, all sides of which touch the inscribed circle; p. 3 corresponds to a circular pipe, p. 2 – a square pipe, p. 1 is a pipe in the form of an equilateral triangle; the average curve corresponds to rectangular pipes, and the upper one corresponds to elliptical ones.

![Figure 2. Dependence $i_k \delta/L_0 - 4A_0^2/L_0^4$.](image)

![Figure 3. Dependence $10^3i_k \delta/L_0 - 1/K_f^2$.](image)

By analogy with the properties proved in paper [11], we can formulate the following isoperimetric properties of the reduced geometric torsion rigidity of sections in the form of thin-walled pipes.

Property 1. The entire set of values of the reduced geometric torsion rigidity for sections in the form of pipes with an arbitrary convex contour, represented up to a dimensional factor $L_0/\delta$, is
bounded below by values \( i_k \) corresponding to sections in the form of polygons, all sides touching the inscribed circle, and from above – the values \( i_k \) corresponding to sections in the form of ellipses.

Property 2. The entire set of values of the reduced geometric torsion rigidity for sections in the form of pipes of an arbitrary quadrangle, represented to within a dimensional factor \( L_0/\delta \), is bounded below by values \( i_k \), corresponding to sections in the form of polygons, all sides touching the inscribed circle, and above – the values \( i_k \) corresponding to sections \( i_k \) in the form of rectangles.

For piping sections of a triangular shape at \( L_0/\delta = \text{const} \), the graph of the change in the geometric torsion rigidity will be displayed by the straight line 0-1 in figures 2 and 3.

Such a graphical representation of the boundaries of the change in the reduced geometric torsion rigidity of thin-walled pipes of triangular cross-section is of great theoretical interest, since it allows presenting regularities of variation of integral characteristics in two-dimensional problems of the theory of elasticity uniformly, as for simply-connected sections [15].

Since dependence (5) is an exact solution of the problem under consideration, it can be used in future experiments as a test problem, which makes it possible to reliably estimate the accuracy of the experimental results, as well as the accuracy of the solutions obtained by the method of interpolation with respect to the shape coefficient.

In concluding this section, we express the tangential stresses through the form factor of the region, using formula (3):

\[
\tau = \frac{2G\theta A_t}{L_0} = G\theta \frac{2A_0 L_0}{L_0^2} \geq G\theta \frac{L_0}{K_f}.
\]  

(7)

3. Tests of thin-walled triangular pipes on torsion

For testing, a set of standard equipment of the resistance laboratory was used. The tests are carried out on the SM-1P installation (figure 4). The test sample 1 is a cantilever rod made of steel sheet steel. The left end of the sample is rigidly clamped in the frame 2. The double-armed lever 3 is attached to the free end of the sample. The ratio of the arms of the lever is 1:1, the length of the arms is \( z = 300 \) mm. Loading of the sample was carried out by imposing loads on the weights that were connected to the levers (not shown in the diagram). The deformation was measured with the aid of an indicator of the watch type 4, which rested at the end of the console 5, fixedly connected to the free cross-section of the rod.

The angle of rotation \( \phi \) was calculated according to the indications of the indicator according to the formula \( \tan \phi = a/\ell \), where \( a \) is the deviation of the console, measured by the indicator (the indicator divides 0.01mm); \( \ell \) is the length of the console in mm. When performing the experiments and processing their results, the dependences known from the course of the theory of elasticity and the resistance of materials were used [18]:

\[
\phi = \frac{M_k \ell}{G I_k}; \quad I_k = \frac{M_k \ell}{G}; \quad I_k = \frac{I_k}{A^2} = \frac{M_k \ell}{G A^2};
\]  

(8)

where \( M_k \) is the rotational moment. The shear modulus \( G \) was determined according to the formula

\[
G = \frac{E}{2(1 + \nu)}.
\]  

(9)

where \( \nu \) is Poisson’s ratio.
The tests were carried out in the following sequence.
1. The initial data were calculated: the cross-sectional area $A$, the shear modulus of the material $G$.
2. The test rod was fixed with one end in the clamping device of the installation and the length of its free end $\ell$ was measured.
3. The first load stage was applied by setting the weights to the load device and the rotational moment $M_k$ was determined.
4. The angle of twisting of the section $\beta_1$ from the load of the first stage was calculated.
5. The value $i_{k1}$ was calculated according to the last formula from (8).
6. The next step of the load was applied and the operations were repeated in steps 4 and 5. In this sequence, five loading stages were carried out.
7. To assess the accuracy of the measurement results, operations in 3 ... 6 were performed five times. Thus, if this sequence of experiments is observed, a sample of the results is obtained in the amount of 25.

For the testing of torsion, four samples of steel thin-walled rods 650 mm long with a closed contour and section in the form of isosceles triangles were made. The thickness of the steel plate is $\delta = 2$ mm. The rods were manufactured using special equipment in the repair shop of the Orel steel rolling plant using plastic bending along the vertices of the triangular section and welding the joined edges of the steel sheet along one of the edges of the prismatic rod. Welding was performed on the entire thickness of the sheet, followed by facile treatment of the faces at the welding points.

The transverse dimensions of the test thin-walled prismatic rods in the form of equilateral isosceles triangles are shown in Table 1.

### Table 1. Geometric characteristics of sections of thin-walled rods.

| Name of a sample | The angle at the vertex $\gamma$, grades | The length of foundation $a$, mm | Cross-sectional area $A$, mm$^2$ | Conventional cross-sectional area $A_0$, mm$^2$ | The length of conditional perimeter $L_0$, mm |
|------------------|-----------------------------------------|---------------------------------|-------------------------------|---------------------------------|------------------|
| Sample N 1       | 60                                      | 80                              | 2771.28                       | 2536.48                         | 229.61           |
| Sample N 2       | 90                                      | 80                              | 1412.69                       | 1412.69                         | 181.48           |
| Sample N 3       | 120                                     | 100                             | 1235.95                       | 1235.95                         | 199.39           |
| Sample N 4       | 45                                      | 60                              | 1961.79                       | 1961.21                         | 205.97           |

The material of the samples in accordance with the certificate for steel sheets had the following physico-mechanical characteristics: elastic modulus $E = 2.1 \times 10^5$ MPa, Poisson’s ratio $\nu = 0.25$. Taking into account these values
When planning the experiment to evaluate the reliability of measurement results, the methods of mathematical statistics were used. The results of measurements of the geometric torsion rigidity for all samples were subjected to a detailed statistical study. The following data were accepted: the number of measurements \( n = 25 \), the confidence probability \( p_d = 0.95 \).

The final results after the statistical processing are presented in Table 2.

### Table 2. Results of statistical processing of experimental data.

| Parameters of plates | \( x \cdot 10^3 \) | \( D \cdot 10^6 \) | \( \sigma \cdot 10^3 \) | \( \sigma_b \cdot 10^3 \) | \( \mu \cdot 10^3 \) | \( \chi_d \cdot 10^3 \) | % |
|----------------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|----|
| Sample N 1           | 8.799              | 0.0194             | 0.1394             | 0.0279             | 0.0576             | 8.799±0.058         | 0.65 |
| Sample N 2           | 7.623              | 0.0149             | 0.1221             | 0.0244             | 0.0504             | 7.623±0.050         | 0.66 |
| Sample N 3           | 3.667              | 0.0050             | 0.0704             | 0.0141             | 0.0291             | 3.667±0.029         | 0.79 |
| Sample N 4           | 8.033              | 0.0218             | 0.1478             | 0.0295             | 0.0609             | 8.033±0.061         | 0.76 |

In the process of statistical analysis, it was found that improving the accuracy of measurements by reducing systematic errors could only be achieved by improving the measuring equipment. An increase in the number of measurements does not lead to a noticeable improvement in their results.

When analyzing the results of rod testing, it is not necessary to select an approximating function, since an exact theoretical solution of the problem under consideration (5) is known. It is only necessary to find the deviations of the mean value of the reduced geometric torsion rigidity of each section, obtained during the experiments, from the straight line. Such a comparison of the theoretical and experimental data is given in Table 3.

### Table 3. Comparison of theoretical and experimental results.

| Name of a sample | The angle at the vertex \( \beta \) | The form factor \( K_r \) | Theoretical result | Experimental result | Difference, % |
|------------------|---------------------------------|--------------------------|--------------------|---------------------|---------------|
| Sample N 1       | 60                              | 10.392                   | 9.259·10^{-3}      | 8.799·10^{-3}      | 4.97          |
| Sample N 2       | 90                              | 11.657                   | 7.359·10^{-3}      | 7.623·10^{-3}      | 3.63          |
| Sample N 3       | 120                             | 16.083                   | 3.866·10^{-3}      | 3.667·10^{-3}      | 5.15          |
| Sample N 4       | 45                              | 10.819                   | 8.543·10^{-3}      | 8.033·10^{-3}      | 5.97          |

Note – All results for \( h_k \) are reduced to one dimension \( L_0/\delta \) corresponding to the right triangle.

It can be seen from the table that the experimental results differ from the theoretical ones with an error of up to 6.0%.

### 4. Conclusions

Summarizing the results of the analysis of physico-geometric analogies, we can draw the following conclusions:

1. In the paper the calculation formulae connecting the reduced geometric cross-section in the form of thin-walled pipes with their form factor were obtained. This made it possible to describe the region of the distribution of the entire set of such sections with a convex contour and indicate its characteristic boundaries. The constructed graphs \( 10^3 h_k/\delta L_0 - 1/K_f^2 \) make it possible to solve the problem of determining the reduced geometric rigidity of a section in the form of thin-walled pipes using the isoperimetric method [20] and the method of interpolation by the form factor [11].

2. Graphical representation of the boundaries of the change in the entire set of values of the reduced geometric rigidity of the torsion cross sections in the form of thin-walled pipes allows visualizing the desired result in the entire set of required solutions.
3. To confirm the theoretical prerequisites of the considered methods for solving the problem of the torsion of thin-walled pipes, experimental studies were carried out to test triangular pipes for which there is an exact solution. The experiments showed that the deviation of the experimental results from the theoretical ones does not exceed 6.0%.

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