KIDs prefer special cones

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Abstract

As complement to Chruściel and Paetz (2013 Class. Quantum Grav. 30 235036) we analyse Killing Initial Data (KID) on characteristic Cauchy surfaces in conformally rescaled vacuum space-times satisfying Friedrich’s conformal field equations. As an application, we derive and discuss the KID equations on a light-cone with vertex at past timelike infinity.

Keywords: characteristic initial value problem, Killing initial data, light-cone with vertex at past timelike infinity, conformal field equations
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1. Introduction

Gaining a better insight into properties and peculiarities of space-times which represent (physically meaningful) solutions to Einstein’s field equations belongs to the core of the analysis of general relativity. One question of interest concerns the existence of space-times which possess certain symmetry groups, mathematically expressed via a Lie algebra of Killing vector fields on that space-time. A fundamental issue in this context is to systematically construct such space-times in terms of an initial value problem. By that it is meant to supplement the usual constraint equations, which need to be satisfied by a suitably specified set of initial data, by some further equations which make sure that the emerging space-time contains one or several Killing vector fields. In vacuum, such Killing Initial Data (KIDs) are well-understood in the spacelike case as well as in the characteristic case, cf [1, 3, 10] and references therein. In this article we would like to complement the analysis of the characteristic case given in [3] to space-times satisfying Friedrich’s conformal field equations, and in particular to analyse the case where the initial surface is a light-cone with vertex at past timelike infinity.

In a first step, section 3, we translate the Killing equation into the unphysical, conformally rescaled space-time. The so-obtained ‘unphysical Killing equations’ constitute the main focus of our subsequent analysis. Assuming the validity of the conformal field equations, recalled in section 2, we will derive necessary-and-sufficient conditions on a characteristic initial surface...
which guarantee the existence of a vector field satisfying the unphysical Killing equations, cf theorem 3.4. In section 4 we then restrict attention to four space-time dimensions (it will be indicated that the higher dimensional case is more intricate). As in [3] we shall see that many of the hypotheses appearing in theorem 3.4 are automatically satisfied. The remaining ‘KID equations’ are collected in theorem 4.4 (cf proposition 4.9) for a light-cone, and in theorem 4.13 for two characteristic hypersurfaces intersecting transversally.

In section 5 we then apply theorem 4.4 to the ‘special cone’ $C_r$ whose vertex is located at past timelike infinity (assuming the cosmological constant to be zero). As for ‘ordinary cones’ treated in [3] it turns out that some of the KID equations determine a class of candidate fields on the initial surface while the remaining ‘reduced KID equations’ provide restrictions on the initial data to make sure that one of these candidate fields does indeed extend to a space-time vector field satisfying the unphysical Killing equations. However, contrary to the ‘ordinary case’, and this explains our title, on $C_r$ the candidate fields can be explicitly computed, and, besides, the reduced KID equations can be given in terms of explicitly known quantities. The main result for the $C_r$-cone is the contents of theorem 5.1.

Finally, in appendix A we recall a result on Fuchsian ODEs which will be of importance in the main part, in appendix B we review conformal Killing vector fields on the round 2-sphere.

2. Setting

Our analysis will be carried out in the so-called unphysical space-time $(\mathcal{M}, g, \Theta)$, related to the physical space-time $(\mathcal{M}, \tilde{g})$, $\tilde{g}$ being a solution to Einstein’s field equations, via a conformal rescaling,

$$\tilde{g} \mapsto g := \Theta^2 \tilde{g}, \quad \mathcal{M} \xrightarrow{\phi} \mathcal{M}, \quad \Theta_{\phi(\mathcal{M})} > 0.$$  

The part of $\partial \phi(\mathcal{M})$ on which the conformal factor $\Theta$ vanishes represents ‘infinity’ in the physical space-time.

In $(\mathcal{M}, g, \Theta)$ Einstein’s vacuum field equations with cosmological constant $\lambda$ are replaced by Friedrich’s conformal field equations (cf e.g. [7]), which read in $d \geq 4$ space-time dimensions

$$\nabla_{\mu}d_{\mu\nu\sigma\rho} = 0, \quad \nabla_{\mu}L_{\nu\sigma} - \nabla_{\nu}L_{\mu\sigma} = \Theta^{d-4}\nabla_{\mu}\Theta d_{\nu\sigma\rho}, \quad (2.1)$$

$$\nabla_{\mu}\nabla_{\nu}\Theta = -\Theta L_{\mu\nu} + sg_{\mu\nu}, \quad \nabla_{\nu}s = -L_{\mu\nu}\nabla^{\mu}\Theta, \quad (2.2)$$

$$(d - 1)(2\Theta s - \nabla_{\mu}(\Theta \nabla^{\mu}\Theta)) = \lambda, \quad (2.3)$$

$$R_{\mu\nu\sigma}^{\kappa}[g] = \Theta^{d-3}d_{\mu\nu\sigma\rho}^{\kappa} + 2(g_{\sigma[\mu}L_{\nu]}^{\kappa} - \delta_{[\mu}^{\kappa}L_{\nu]\rho]), \quad (2.4)$$

with $g_{\mu\nu}$, $\Theta$, $s$, $L_{\mu\nu}$ and $d_{\mu\nu\sigma\rho}$ regarded as unknowns. The trace of (2.3) can be read as the definition of the function $s$,

$$s := \frac{1}{d}g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\Theta + \frac{1}{2d(d - 1)}R_{\Theta}. \quad (2.7)$$

This issue has already been analysed in [8]. However, it is claimed there that regularity of the principal part of a wave equation suffices to guarantee uniqueness of solutions, and counter-examples of this assertion can be easily constructed. For instance, let $\Theta$ be the unique solution of the wave-equation $\Box_{\Theta} \Theta = 1$ which vanishes on the initial surface which we assume to be a light-cone $C_0$, i.e. $\Theta\big|_{C_0} = 0$. Then $\Theta\big|_{\partial C_0} > 0$, at least sufficiently close to $O$. Consider the non-regular wave-equation $\Box_{\Theta} f - \frac{1}{1 + \Theta}f = 0$. For given initial data $f\big|_{C_0} = 0$ there exist at least three solutions: $f = 0$, $\pm \Theta$.
The tensor field $L_{\mu\nu}$ is the Schouten tensor
\[ L_{\mu\nu} := \frac{1}{d-2} R_{\mu\nu} - \frac{1}{2(d-1)(d-2)} R g_{\mu\nu}, \tag{2.8} \]
while
\[ d_{\mu\nu\sigma}^\rho := \Theta^{3-d} C_{\mu\nu\sigma}^\rho \tag{2.9} \]
is a rescaling of the conformal Weyl tensor $C_{\mu\nu\sigma}^\rho$.

The conformal field equations are equivalent to the vacuum Einstein equations where $\Theta$ is positive, but remain regular even where $\Theta$ vanishes. The Ricci scalar $R$ turns out to be a conformal gauge source function which reflects the freedom to choose the conformal factor $\Theta$. It can be prescribed arbitrarily.

In (2.1)–(2.6) the fields $s$, $L_{\mu\nu}$ and $d_{\mu\nu\sigma}^\rho$ are treated as independent of $g_{\mu\nu}$ and $\Theta$. However, once a solution has been constructed they are related to them via (2.7)–(2.9). When talking about a solution of the conformal field equations we therefore just need to specify the pair $(g_{\mu\nu}, \Theta)$.

The conformal field equations imply a wave equation for the Schouten tensor, which will be of importance later on: One starts by taking the divergence of (2.2). Using then (2.1), (2.3), (2.6) and the tracelessness of the rescaled Weyl tensor one finds (cf [11] where the four-dimensional case is treated in detail)
\[
\Box_{L} L_{\mu\nu} = -2R_{\mu}^\alpha \nu_\beta L_{\alpha\beta} + g_{\mu\nu} |L|^2 + \frac{1}{2(d-1)} \nabla_\mu \nabla_\nu R + \frac{1}{d-1} R L_{\mu\nu} + (d-4) \left[ L_{\mu}^\alpha L_{\alpha\nu} + \Theta^{d-3} \nabla_\alpha \Theta \nabla_\beta d_{\mu\nu}^\beta \right],
\]
with $|L|^2 := L_{\mu}^\alpha L_{\mu}^\alpha$. Supposing that $\Theta$ has no zeros, or that we are in the four-dimensional case, we can use (2.2) to rewrite this as
\[
\Box_{L} L_{\mu\nu} = -2R_{\mu}^\alpha \nu_\beta L_{\alpha\beta} + g_{\mu\nu} |L|^2 + \frac{1}{2(d-1)} \nabla_\mu \nabla_\nu R + \frac{1}{d-1} R L_{\mu\nu} + (d-4) \left[ L_{\mu}^\alpha L_{\alpha\nu} + 2 \Theta^{-1} \nabla_\alpha \Theta \nabla_\mu L_{\mu}^\nu \right]. \tag{2.10} \]

3. KID equations in the unphysical space-time

3.1. The Killing equation in terms of a conformally rescaled metric

**Lemma 3.1.** A vector field $\tilde{X}$ is a Killing vector field in the physical space-time $(\mathcal{M}, \tilde{g})$ if and only if its push-forward $X := \phi_* \tilde{X}$ is a conformal Killing vector field in the unphysical space-time $(\mathcal{M}, g, \Theta)$ and satisfies there the equation $X^\mu \nabla_\mu \Theta = \frac{1}{d} \Theta \nabla_\mu X^\mu$.

**Proof.** By definition $\tilde{X}$ is a Killing field if and only if (set $\tilde{X}_\mu := \tilde{g}_{\mu\nu} \tilde{X}^\nu$ and $X_\mu := g_{\mu\nu} X^\nu$)
\[
\nabla_\mu (\Theta^{-2} \tilde{X}_\mu) = 0 \\
\iff \tilde{\nabla}_\mu (\Theta^{-2} \tilde{X}_\mu) = 0 \\
\iff \nabla_\mu (\Theta^{-2} X_\mu) + 2 \Theta^{-2} X_\mu \nabla_\nu \log \Theta = g_{\mu\nu} \Theta^{-2} X_\nu \nabla_\mu \Theta \log \Theta \\
\iff \nabla_\mu (X_\mu) = g_{\mu\nu} \Theta^{-1} X_\nu \nabla_\mu \Theta \\
\iff \nabla_\mu X_\mu = \frac{1}{d} \nabla_\mu X^\mu g_{\mu\nu} \quad \& \quad X^\mu \nabla_\mu \Theta = \frac{1}{d} \Theta \nabla_\mu X^\mu \tag{3.1} \]

(note that $\Theta (\phi, \tilde{\mathcal{M}}) > 0$). \qed

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Remark 3.2. The conditions (3.1), which replace the Killing equation in the unphysical space-time, make sense also where $\Theta$ is vanishing, supposing that $g$ can be smoothly extended across $\{\Theta = 0\}$ (note that the conformal Killing equation induces a linear symmetric hyperbolic system of propagation equations for $\phi_\ast \tilde{X}$ which then implies that $\phi_\ast \tilde{X}$ is smoothly extendable across the conformal boundary [6]).

Remark 3.3. We will refer to (3.1) as the unphysical Killing equations.

The main object of this work is to extract necessary-and-sufficient conditions on a characteristic initial surface which ensure the existence of some vector field $X$ which fulfils the unphysical Killing equations, so that its pull-back is a Killing vector field of the physical space-time.

3.2. Necessary conditions for the existence of Killing vector fields

Let us first derive some implications of the unphysical Killing equations (3.1) under the hypothesis that the conformal field equations (2.1)–(2.6) are satisfied.

From the conformal Killing equation we first derive a system of wave equations for $X$ and for the function $Y := \frac{1}{d} \nabla_\nu X^\nu$ (set $\Box_g := g^{\mu\nu} \nabla_\mu \nabla_\nu$):

\[ \Box_g X_\mu + R_\mu X + (d - 2) \nabla_\mu Y = 0, \]  
(3.3)

\[ \Box_g Y + \frac{1}{d - 1} \left( \frac{1}{2} X_\mu \nabla_\mu R + RY \right) = 0. \]  
(3.4)

With (3.1) and (2.7) we find

\[ 0 = \Box_g (X_\mu \nabla_\mu \Theta - \Theta Y) \]
\[ = \Box_g X_\mu \nabla_\mu \Theta + X_\mu \nabla_\mu \Box_g \Theta + 2 \nabla_\mu X_\mu \nabla_\nu \nabla_\nu \Theta - Y \Box_g \Theta - \Theta \Box_g Y - 2 \nabla_\mu \Theta \nabla_\nu Y \]
\[ = d (X_\mu \nabla_\mu s + s Y - \nabla_\nu \Theta \nabla_\nu Y). \]  
(3.5)

We set

\[ A_{\mu\nu} := 2 \nabla_\mu X_\nu - 2Y g_{\mu\nu}. \]  
(3.6)

Using the second Bianchi identity, (2.8), (3.3) and (3.4) we obtain

\[ \Box_g A_{\mu\nu} \equiv 2 \nabla_\mu \Box_g X_\nu + 2 \nabla_\nu \Box_g X_\mu - 2 R_\mu \nabla_\alpha \beta A_{\alpha\beta} - 4 R_{\mu\nu} Y \]
\[ + 2 X_\kappa \nabla_\mu (R_{\nu\kappa}) - 2 X_\kappa \nabla_\nu R_{\mu\kappa} - 2 \Box_g Y g_{\mu\nu} \]
\[ = 2 R_\mu \nabla_\nu A_{\alpha\beta} - 2 R_\mu \nabla_\alpha \nabla_\beta A_{\nu\nu} - 2 (d - 2) (\mathcal{L}_X L_{\mu\nu} + \nabla_\mu \nabla_\nu Y). \]  
(3.7)

(Remember that $\mathcal{L}_X L_{\mu\nu} \equiv X_\nu \nabla_\mu L_{\mu\nu} + 2 L_\nu (\nabla_\nu X^\kappa$.)

Hence the conformal Killing equation for $X$, which is $A_{\mu\nu} = 0$, implies

\[ B_{\mu\nu} := \mathcal{L}_X L_{\mu\nu} + \nabla_\mu \nabla_\nu Y = 0. \]  
(3.8)
3.3. KID equations on a characteristic initial surface

3.3.1. First main result. We are now in a position to formulate our first main result. Here and in the following we use an overbar to denote restriction to the initial surface.

**Theorem 3.4.** Assume we have been given, in dimension \( d \geq 4 \), an ‘unphysical’ space-time \( (\mathcal{M}, g, \Theta) \), with \((g, \Theta)\) a smooth solution of the conformal field equations (2.1)–(2.6). Assume further that \( \Theta \) is bounded away from zero if \( d \geq 5 \). Consider some characteristic initial surface \( N \subset \mathcal{M} \) (for definiteness we think of a light-cone or two transversally intersecting null hypersurfaces). Then there exists a vector field \( \hat{X} \) satisfying the unphysical Killing equations (3.1) on \( D^+(N) \) (i.e. representing a Killing field of the physical space-time) if and only if there exists a vector field \( X \) and a function \( Y \) which fulfil the following equations (recalling the definitions (2.7) and (2.8) for \( s \) and \( L_{\mu \nu} \), respectively)

\[
\begin{align*}
(i) \quad & \Box g_{\mu \nu} + R_{\mu \rho} \, g^{\rho \nu} X_\mu + (d - 2) \nabla_\rho Y = 0, \\
(ii) \quad & \Box g + \frac{1}{2(d - 1)} (X^\mu \nabla_\mu R + R Y) = 0, \\
(iii) \quad & \overline{\phi} = 0 \text{ with } \phi = X^\mu \nabla_\mu \Theta - \Theta Y, \\
(iv) \quad & \overline{\psi} = 0 \text{ with } \psi = X^\mu \nabla_\mu s + s Y - \nabla_\mu \Theta \nabla^\mu Y, \\
(v) \quad & \overline{A}_{\mu \nu} = 0 \text{ with } A_{\mu \nu} = 2 \nabla_{(\mu} X_{\nu)} - 2 Y g_{\mu \nu}, \\
(vi) \quad & \overline{B}_{\mu \nu} := B_{\mu \nu} - \frac{1}{2} \overline{B}_{\mu \rho} \overline{B}^\rho_{\nu} = 0 \text{ with } B_{\mu \nu} = \mathcal{L}_X L_{\mu \nu} + \nabla_\mu \nabla_\nu Y.
\end{align*}
\]

**Proof.** ‘\( \Rightarrow \)’: Follows from the considerations above if one takes \( X = \hat{X} \) and \( Y = \frac{1}{2} \nabla_\mu \hat{X}^\mu \).

‘\( \Leftarrow \)’; We will derive a homogeneous system of wave equations from which we conclude the vanishing of \( A_{\mu \nu} \) and \( \phi \) as well as the relation \( Y = \frac{1}{2} \nabla_\mu X^\mu \), which imply that \( X \) satisfies (3.1). Since by assumption (3.3) and (3.4) hold we can repeat the steps which led us to (3.7),

\[
\Box A_{\mu \nu} = 2 R_{\mu \sigma} A_{\sigma \nu} - 2 R_{\mu \nu} A_{\sigma \sigma} - 2 (d - 2) B_{\mu \nu}.
\]  

With (i), (ii) and the definition (2.7) of \( s \) we find

\[
\begin{align*}
\Box \phi &= \Box (X_\mu \nabla^\mu \Theta + X^\mu \nabla_\mu \Box + X^\mu R_{\mu \nu} \nabla_\nu \Theta + 2 \nabla_\nu X_\mu \nabla^\nu \nabla^\mu \Theta) \\
&\quad - Y \Box \Theta - \Box Y - 2 \nabla_\nu \Theta \nabla^\nu Y \\
&= d \psi - \frac{1}{2(d - 1)} R \phi + A_{\mu \nu} \nabla^\mu \nabla^\nu \Theta.
\end{align*}
\]

We use (i), (ii), (2.7), (2.8) and the conformal field equations (2.3) & (2.4) (which imply \( \Box s = 0 \) and \( \Box (s R + \nabla^\mu \Theta \nabla_\mu R) \)) to obtain

\[
\Box \psi = \Box (X_\mu \nabla^\mu s + X^\mu \nabla_\mu \Box + X^\mu R_{\mu \nu} \nabla_\nu s + A_{\mu \nu} \nabla^\mu \nabla^\nu s) \\
\quad + 3 Y \Box s + s \Box Y + 2 \nabla_\nu s \nabla^\nu Y - \nabla_\nu \Box \nabla^\mu Y - 2 R_{\mu \nu} \nabla_\nu \Theta \nabla^\mu Y \\
\quad - \nabla^\mu \Theta \nabla_\nu \Box Y - 2 \nabla_\nu \nabla_\nu \Theta \nabla^\mu Y \\
\quad = L^2 \phi + A_{\mu \nu} (\nabla^\mu \nabla^\nu s - 2 \Theta L_{\mu \nu} \phi) + 2 \Theta L^{\mu \nu} B_{\mu \nu} \\
\quad + \frac{1}{2(d - 1)} (A_{\mu \nu} \nabla^\mu R \nabla^\nu \Theta - \nabla^\mu R \nabla_\nu \phi - R \psi).
\]

As an immediate consequence of the first Bianchi identity we observe the identity

\[
\frac{1}{2} \nabla_\mu A_{\nu \lambda} + \nabla_\nu A_{\lambda \mu} = \nabla_\mu X_{\nu \lambda} + R_{\mu \nu \lambda} A_{\lambda \sigma} + 2 \nabla_\mu Y g_{\nu \lambda} + \nabla_\nu Y g_{\mu \lambda}.
\]

Another useful relation which follows from the Bianchi identities is

\[
2 L^{\alpha \beta} (\nabla_\beta \nabla_\alpha A_{\mu \nu} - \nabla_\mu \nabla_\alpha A_{\beta \nu}) = 2 L^{\alpha \beta} (X^\mu \nabla_\nu R_{\mu \alpha \beta}) \\
+ L^{\mu \beta} R_{\mu \nu \rho \sigma} A_{\rho \sigma} - R_{\mu \nu} \alpha \beta A_{\lambda \sigma} + 4 \nabla_\beta (R_{(\mu \nu \rho \sigma)} \nabla_\rho X_\sigma) + R_{(\mu \nu \rho \sigma)} \nabla_\rho X_\sigma \\
- 2 g_{\mu \lambda} L^{\alpha \beta} \nabla_\mu \nabla_\beta Y + 4 L_{\mu \nu \rho \sigma} \nabla_\rho Y - \frac{1}{d - 1} R \nabla_\mu \nabla_\nu Y.
\]
Employing (i), (ii), the conformal field equations, the wave equation for the Schouten tensor (2.10) as well as the identities (3.12) and (3.13) a tedious computation reveals that

$$
\Box_{\mu}B_{\nu} \equiv 2\left(\Sigma_{\mu}^\nu - R_{\mu}^\nu \right)B_{\sigma} + 2R_{\mu}^\nu B_{\nu \sigma} + \frac{2}{d-1} R g_{\mu \nu}
+ 2L_{\mu}^\nu \left(\nabla_{[\nu} A_{\sigma]} \nu \right) - \nabla_{\mu} \left(\nabla_{[\nu} A_{\sigma]} \nu \right)
+ (\nabla_{[\mu} A_{\sigma]} \nu) + 2\nabla_{[\nu} A_{\sigma]} \nu \right) \left(2\nabla^\mu L_{\nu} - \frac{1}{4(d-1)} \delta_{\mu}^\nu \nabla^\rho R \right)
+ A^a_{\mu} \left[ \nabla^\alpha L_{\nu \alpha} - 2R_{\ell}^\alpha \nabla^\ell \alpha + 2L_{\mu}^\alpha R_{\nu \ell}^\alpha + L_{\nu}^\alpha \left(2R_{\ell \rho \ell \ell} + R_{\ell \rho \ell \ell} \right)
- 2g_{\mu \nu} L_{\ell \alpha} \nu \right] + |L|^2 A_{\mu v} + L_{\mu}^a R_{\nu \ell}^a A_{\nu \ell} - \frac{1}{d-1} R L_{[\mu} A_{\nu] \ell}
+ (d-4)\left(2L_{\ell}^\nu B_{\nu \beta} - L_{\nu}^\alpha L_{\nu}^\beta A_{\alpha \beta} - \Theta^{\ell 5} \nabla_{\alpha} \Theta d_{\mu \nu} a \left(\nabla^\nu \phi - \nabla^\rho \Theta A^a \right) \right)
- (d-4)\Theta^{\ell 5} \nabla^\nu \Theta \left[2\nabla_{[\nu} B_{\mu]} + \left(\nabla_{[\nu} A_{\mu] \ell} + 2\nabla_{[\nu} A_{\ell] \mu} \right) L_{\mu \ell} \right].
\tag{3.14}
$$

While the before-last line contains negative powers of $\Theta$ merely in five dimensions, the last line contains such powers in any dimension $d \geq 5$. This is the point where our assumption enters that $\Theta$ is bounded away from zero for $d \geq 5$, since this ensures that (3.14) is a regular equation also in higher dimensions.

Note that the right-hand side of (3.14) involves second-order derivatives of $A_{\mu \nu}$, which is why we regard $\nabla_{\sigma} A_{\mu \nu}$ as another unknown for which we derive a wave equation. However, since the right-hand side of (3.9) does not involve derivatives, such a wave equation is easily obtained by differentiation (and, once again, the second Bianchi identity),

$$
\Box_{\mu} \nabla_{\nu} A_{\mu \nu} = 2\nabla_{\sigma} \left( R_{\ell}^\nu A_{\sigma] \nu} - R_{\ell}^\nu \nabla^\ell \nu A_{\sigma] \nu} \right) + 2A_{\sigma] \nu} \left( \nabla_{\rho} R_{\sigma}^\rho - \nabla^{\sigma} R_{\nu \rho} \right)
- 4R_{\ell \sigma] \nu} \nabla^\ell \nu A_{\rho \ell} + R_{\sigma] \nu} \nabla^{\sigma} A_{\rho \ell} - 2(d-2)\nabla_{\nu} R_{\mu \rho}. \tag{3.15}
$$

In the current setting the equations (3.9)–(3.11), (3.14) and (3.15) form a closed homogeneous system of regular wave equations for $A_{\mu \nu}$, $\phi$, $\psi$, $B_{\mu \nu}$ and $\nabla_{\sigma} A_{\mu \nu}$. The assumptions (iii)–(v) assure that the first three fields vanish initially. By (ii) and (v) we have

$$
\bar{R}_{\mu}^\nu = \frac{1}{2(d-1)} \Xi^\mu \nabla^\nu \bar{R} + 2L_{\mu \nu} \nabla^\nu \bar{X} + \Box_{\mu} \bar{Y} = L_{\mu \nu} A_{\mu \nu} = 0,
\tag{3.16}
$$

which, together with (vi), implies

$$
\bar{R}_{\mu \nu} = 0. \tag{3.17}
$$

It remains to verify the vanishing of $\nabla_{\sigma} A_{\mu \nu}$. This follows from lemma 3.5 below, together with (i), (ii), (v) and (3.16). We thus have vanishing initial data for the homogeneous system of wave equations (3.9)–(3.11), (3.14) and (3.15). It follows from [5] in the light-cone-case and from [12] in the case of two characteristic hypersurfaces intersecting transversally that there exists a unique solution, whence all the fields involved need to vanish identically.

It is important to note that we have treated $X$ and $Y$ as independent so far. The vanishing of $A_{\mu \nu}$ and $\phi$ implies that the unphysical Killing equations (3.1) hold for $X$ only once we have shown that $Y = \frac{1}{d} \nabla_{\nu} X^\nu$. Fortunately we have

$$
0 = A_{\mu}^\nu = 2\nabla_{\nu} X^\nu - 2dY, \tag{3.17}
$$

and the theorem is proved.
3.3.2. Adapted null coordinates. Before we state and prove lemma 3.5, which is needed to complete our proof of theorem 3.4, it is useful, also with regard to later purposes, to introduce \textit{adapted null coordinates} on light-cones and on transversally intersecting null hypersurfaces. We will be rather sketchy here, the details can be found e.g. in [2, 12].

First we consider a light-cone \( C_{\mathcal{O}} \subset \mathcal{M} \) with vertex \( O \in \mathcal{M} \) in a \( d \)-dimensional space-time \((\mathcal{M}, g)\). We use coordinates \((x^0, x^1 = r, x^A), A = 2, \ldots, d - 1\), adapted to \( C_{\mathcal{O}} \) in the sense that \( C_{\mathcal{O}} \setminus \{ O \} = \{ x^0 = 0 \}, \) \( r \) parameterizes the null geodesics generating the cone, and the \( x^A \)'s are local coordinates on the level sets \( \{ x^0 = 0, r = \text{const} \} \equiv S^{d - 2} \). On \( C_{\mathcal{O}} \) the metric then reads

\[
\begin{align*}
\bar{g} &= \bar{g}_{00}(dx^0)^2 + 2v_0 dx^0 dx^1 + v_0 (dx^0)^2 + \bar{g}_{AB} dx^A dx^B.
\end{align*}
\]

Note that these coordinates are singular at the vertex of the cone. Moreover, we stress that we do not impose any gauge condition off the cone. The inverse metric takes the form

\[
\bar{g}^\nu = 2v_0^0 \delta_0^0 + \bar{g}^{11} \delta_1^2 + 2\bar{g}^{AB} \delta_A \delta_B,
\]

with

\[
v^0 = (v_0)^{-1}, \quad \bar{g}^{1A} = -v_0^0 \bar{g}^{AB} v_B, \quad \bar{g}^{11} = (v^0)^2 (\bar{g}^{AB} v_A v_B - \bar{g}_{00}).
\]

It is customary to introduce the following quantities:

\[
\chi^B_A := \frac{1}{2} \bar{g}^{BC} \partial_C \bar{g}_{AC} \quad \text{null second fundamental form},
\]

\[
\tau := \chi^B_A \quad \text{expansion},
\]

\[
\sigma^B_A := \chi^B_A - \frac{\tau}{d - 2} \bar{g}^B_A \quad \text{shear tensor}.
\]

Next, let us consider two smooth hypersurfaces \( N_a, a = 1, 2, \) with transverse intersection along a smooth submanifold \( S \). Near the \( N_a \)'s one can introduce coordinates \((x^1, x^2, x^A), A = 3, \ldots, d\), such that \( N_a = \{ x^a = 0 \} \). On \( N_1 \) the coordinate \( x^2 \) parameterizes the null geodesics \( \{ x^1 = 0, x^A = \text{const} \} \) generating \( N_1 \) and vice versa. Since the hypersurfaces are required to be characteristic the metric takes there the specific form, on \( N_1 \) say,

\[
g|_{N_1} = \bar{g}_{11}(dx^1)^2 + 2\bar{g}_{12} dx^1 dx^2 + 2\bar{g}_{1A} dx^1 dx^A + \bar{g}_{AB} dx^A dx^B,
\]

similarly on \( N_2 \). The quantities \( \tau, \sigma^B_A, \chi_A^B \) are defined on \( N_1 \) and \( N_2 \) analogous to the light-cone-case.

3.3.3. Some useful relations. In this section we consider a light-cone. However, we note that exactly the same relations hold in the case of two intersecting null hypersurfaces.

Recall that the wave equations for \( X \) and \( Y \), (3.3)–(3.4), imply the wave equation (3.9) for \( A_{\mu\nu} \). One straightforwardly verifies that in adapted null coordinates

\[
\square_{\mu
u} A_{\mu\nu} = 2v_0 (\nabla_1 \nabla_\nu A_{\mu\nu} + \bar{R}_0(\mu, \alpha) a_\alpha A_{\nu}\nu) + \bar{g}^{11} \nabla_1 \nabla_\nu A_{\mu\nu} + 2\bar{g}^{1A} (\nabla_1 \nabla_A A_{\mu\nu} + \bar{R}_{1A}(\mu, \alpha) a_\alpha A_{\nu}\nu) + \bar{g}^{AB} \nabla_A \nabla_B A_{\mu\nu}.
\]

We equate the trace of (3.9) on the initial surface with (3.18). Making use of the formulae for the Christoffel symbols in adapted null coordinates in [2, appendix A], an elementary calculation yields the following set of equations where \( f, f_\alpha \) and \( f_{AB} \) denote generic (multi-linear) functions which vanish whenever their arguments vanish:

\[
(\mu\nu) = (11):
\]

\[
(\partial_1 + \frac{\tau}{2} - \bar{\Gamma}_0^0 - 2\bar{\Gamma}_{11}^1)\nabla_0 A_{11} = (\bar{R}_{11} + |\chi|^2)\bar{A}_{01} - (d - 2)v_0 \bar{B}_{11} + f(\bar{A}_{ij})
\]

(3.19)
Lemma 3.5. Assume that the wave equations for \(X\) and \(Y\), (3.3) and (3.4), are fulfilled. Assume further that \(\bar{\alpha}_{\mu\nu} = 0 = \bar{B}_{\mu\nu}\) on either a light-cone or two transversally intersecting null hypersurfaces. Then \(\bar{\nabla}_\mu \bar{A}_{\mu\nu} = 0\).

**Proof.** We start with the light-cone case. By assumption the equations (3.19)–(3.24) hold. Invoking \(\bar{\alpha}_{\mu\nu} = 0 = \bar{B}_{\mu\nu}\) they become

\[
\left( \partial_1 + \frac{\tau}{2} - \Gamma_{01} - 2\Gamma_{11} \right) \bar{\nabla}_0 \bar{A}_{11} = 0, \tag{3.26}
\]

\[
\left( \partial_1 + \frac{d - 4}{2(d - 2)} \tau - \nu^0 \partial_1 v_0 \right) \bar{\nabla}_0 \bar{A}_{1A} - \sigma_A \bar{\nabla}^0 \bar{A}_{AB} = f_A (\bar{\nabla}_0 \bar{A}_{11}), \tag{3.27}
\]

\[
\left( \partial_1 + \frac{d - 6}{2(d - 2)} \tau - \nu^0 \partial_1 v_0 \right) \bar{\nabla}_0 \bar{A}_{AB} - 2\sigma_A \bar{\nabla}^0 \bar{A}_{B\sigma} = f_A (\bar{\nabla}_0 \bar{A}_{11}). \tag{3.28}
\]

\[
\left( \partial_1 + \frac{\tau}{2} + \Gamma_{11} - 2\nu^0 \partial_1 v_0 \right) \bar{\nabla}_0 \bar{A}_{11} = f (\bar{\nabla}_0 \bar{A}_{ij}), \tag{3.29}
\]

\[
\left( \partial_1 + \frac{d - 4}{2(d - 2)} \tau - 2\nu^0 \partial_1 v_0 \right) \bar{\nabla}_0 \bar{A}_0 - \sigma_A \bar{\nabla}^0 \bar{A}_{0B} = f_A (\bar{\nabla}_0 \bar{A}_{ij}, \bar{\nabla}_0 \bar{A}_{01}). \tag{3.30}
\]
\[
\left( \partial_1 + \frac{\tau}{2} - 3\Gamma^0_{01} \right) \nabla_0 A_{00} = f(\nabla_0 A_{ij}, \nabla_0 A_{0i}).
\]

(3.31)

Taking the behaviour of the metric components at the tip of the cone into account, cf the formulae (4.41)–(4.51) in [2], we have
\[
\tau = \frac{d - 2}{r} + O(r), \quad \sigma^A = O(r), \quad v_0 = 1 + O(r^2), \quad \partial_1 v_0 = O(r),
\]

(3.32)

and it follows from [2, appendix A] that
\[
\Gamma^0_{01} = O(r), \quad \Gamma^1_{11} = O(r).
\]

(3.33)

With (3.32)–(3.33) we observe that the equations (3.26)–(3.31) form a hierarchical system of Fuchsian ODEs which can be solved step-by-step. Existence of a regular conformal Killing field \(X\) requires the tensor field \(A_{\mu\nu}\) to be regular, as well. Then \(\nabla_0 A_{\mu\nu}\) needs to show the following behaviour near the vertex:
\[
\nabla_0 A_{11} = O(1), \quad \nabla_0 A_{1A} = O(r), \quad \nabla_0 A_{AB} = O(r^2),
\]

(3.34)

\[
\nabla_0 A_{01} = O(1), \quad \nabla_0 A_{0A} = O(r), \quad \nabla_0 A_{00} = O(1).
\]

(3.35)

Standard results on Fuchsian ODEs (cf e.g. [3, appendix A]) imply that the only solution of (3.26)–(3.35) is provided by \(\nabla_0 A_{\mu\nu} = 0\).

In the case of two transversally intersecting null hypersurfaces one can derive the same hierarchical system of ODEs on \(N_1\) and \(N_2\), respectively, which now is a system of regular ODEs. The assumption \(A_{\mu\nu} = 0\) implies \(\nabla_\sigma A_{\mu\nu} \mid_S = 0\). We thus have vanishing initial data for the ODEs and the unique solutions are \(\nabla_1 A_{\mu\nu} \mid_{N_1} = 0\) and \(\nabla_2 A_{\mu\nu} \mid_{N_2} = 0\).

\(\square\)

3.4. A special case: \(\Theta = 1\)

Let us briefly analyse the implications of theorem 3.4 in the special case where the conformal factor \(\Theta\) is identical to one,
\[
\Theta = 1
\]

(note that thereby the gauge freedom to prescribe the Ricci scalar is lost). Then the unphysical space-time can be identified with the physical space-time. The conformal field equations (2.1)–(2.6) imply the equations
\[
s = \frac{1}{2(d - 1)} \lambda,
\]
\[
L_{\mu\nu} = s g_{\mu\nu} \quad \iff \quad R_{\mu\nu} = \lambda g_{\mu\nu},
\]

i.e. in particular the vacuum Einstein equations hold.

Let us analyse the conditions (i)–(vi) of theorem 3.4 in this setting: condition (iii) gives \(Y = 0\), which we take as initial data for the wave equation (ii) which then implies \(Y = 0\), i.e. \(X\) needs to be divergence-free, as desired. We observe that (iv) is automatically satisfied. Moreover,
\[
B_{\mu\nu} = \mathcal{L}_X L_{\mu\nu} = s \mathcal{L}_X g_{\mu\nu} = 2 s \nabla_{(\mu} X_{\nu)}.
\]

so (vi) follows from (v). To sum it up, the hypotheses of theorem 3.4 are satisfied if and only if there is a vector field \(X\) which satisfies
\[
\Box g_{\mu\nu} + \lambda X_{\mu} = 0,
\]
\[
\nabla_{(\mu} X_{\nu)} = 0.
\]

This was the starting point of the analysis in [3].
4. KID equations in four dimensions

Theorem 3.4 can be applied to dimensions \( d \geq 5 \) only when the conformal factor \( \Theta \) is bounded away from zero. In fact, this situation is rather uninteresting since then there is no need to pass to a conformally rescaled space-time (or to put it differently, it is just a matter of gauge to set \( \Theta = 1 \)). One reason why we included this case, though, was to emphasize that there arise difficulties when one tries to go from four to higher dimensions (which is in line with the observation that the conformal field equations provide a good evolution system only in four space-time dimensions). Another reason was to be able to consider the limiting case \( \Theta = 1 \) in any dimension \( d \geq 4 \) where the unphysical space-time can be identified with the physical space-time, and to compare the resulting equations with those in [3]. This is also a reason why we avoided to make the common gauge choice \( R = 0; \Theta = 1 \) is compatible with \( R = 0 \) solely when the cosmological constant vanishes. Henceforth we restrict attention to \( d = 4 \) space-time dimensions.

4.1. A stronger version of theorem 3.4 for light-cones

A more careful analysis of the computations made in the proof of lemma 3.5 will lead us to a refinement of theorem 3.4. We first treat the light-cone case. We will assume the vanishing of \( A_{ij} \) and \( B_{ij} \) together with the validity of the wave equations (3.3) for \( X \) and \( Y \), and explore the consequences concerning the vanishing of other components of these tensors, including certain transverse derivatives thereof.

Indeed as a straightforward consequence of theorem 3.4 and lemmas 4.2 and 4.3 below we establish the following result.

**Theorem 4.1.** Assume that we have been given a \( 3+1 \)-dimensional space-time \((M, g, \Theta)\), with \((g, \Theta)\) a smooth solution of the conformal field equations (2.1)–(2.6). Let \( CO \subset M \) be a light-cone. Then there exists a vector field \( \hat{X} \) satisfying the unphysical Killing equations (3.1) on \( D^+(CO) \) if and only if there exists a pair \((X, Y)\), \( X \) a vector field and \( Y \) a function, which fulfils the following conditions:

\[
\begin{align*}
(i) \quad & \Box_g X_\mu + R_{\mu\nu} X_\nu + 2\nabla_\mu Y = 0, \\
(ii) \quad & \Box_g Y + \frac{1}{2} X^\mu \nabla_\mu R + \frac{1}{2} R Y = 0, \\
(iii) \quad & \phi = 0 \text{ with } \phi = X^\mu \nabla_\mu \Theta - \Theta Y, \\
(iv) \quad & \psi = 0 \text{ with } \psi = X^\mu \nabla_\mu s + s Y - \nabla_\mu \Theta \nabla^\mu Y, \\
(v) \quad & A_{ij} = 0 \text{ with } A_{\mu\nu} = 2\nabla_{(\mu} X_{\nu)} - 2Y g_{\mu\nu}, \\
(vi) \quad & A_{01} = 0, \\
(vii) \quad & B_{ij} = 0 \text{ with } B_{\mu\nu} = \mathcal{L}_X L_{\mu\nu} + \nabla_\mu \nabla_\nu Y.
\end{align*}
\]

In that case one may take \( \hat{X} = X \) and \( \nabla_\tau \hat{X}^\tau = 4Y \). The condition (vi) is not needed on the closure of those sets where \( \tau \) is non-zero.

4.1.1. Vanishing of \( A_{0i} \). We take the trace of (3.12) which together with the wave equation (3.3) for \( X \) implies the relation

\[
\nabla_\nu A_\mu^\nu - \frac{1}{2} \nabla_\mu A_v^v = 0. \tag{4.1}
\]

On the initial surface that yields in adapted null coordinates,

\[
0 = \nabla^\rho (2\nabla_{(0} A_{1)}) + g_{11}(\nabla_1 A_{11} - \frac{1}{2} \nabla_1 A_{11}) + g_{0} A_{2\rho}(2\nabla_{(0} A_{1)}) - \frac{1}{2} \nabla_{(0} A_{11}). \tag{4.2}
\]
In the following we shall always assume that (3.3) and (3.4) hold, and thereby in particular (4.2) and (3.19)–(3.21).

With the assumptions \( \overline{A}_{ij} = 0 \) and \( \overline{B}_{ij} = 0 \) equation (3.19) becomes

\[
(\partial_1 + \frac{1}{\tau} \tau - \Gamma_{11}^1 - \nu^0 \partial_1 v_0) \nabla_{\nu} \overline{A}_{11} = -
\overline{A}_{01}(\partial_1 - \Gamma_{11}^1) \tau,
\]

where we have fallen back on the identity [2]

\[
\overline{K}_{11} \equiv - (\partial_1 - \Gamma_{11}^1) \tau - |x|^2.
\] (4.3)

Moreover, we deduce from the \( \mu = 1 \)-component of (4.2) that

\[
\tau \overline{A}_{01} + \nabla_{\nu} \overline{A}_{11} = 0,
\] (4.4)

which leads us to an ODE satisfied by \( \overline{A}_{01} \),

\[
\tau (\partial_1 + \frac{1}{\tau} \tau - \nu^0 \partial_1 v_0) \overline{A}_{01} = 0.
\] (4.5)

Since \( \tau \) has no zeros sufficiently close to the vertex it follows from regularity, which requires \( \overline{A}_{01} = 0 \) to be bounded near the vertex, that \( \overline{A}_{01} = 0 \) (and thus \( \nabla_{\nu} \overline{A}_{11} = 0 \) in that region. Even more, \( \overline{A}_{01} = 0 \) will automatically vanish on the closure of those sets on which \( \tau \) is non-zero.

Next we assume \( \overline{A}_{ij} = 0, \overline{A}_{01} = 0, \nabla_{\nu} \overline{A}_{11} = 0, \overline{B}_{IA} = 0 \). Then, due to (3.20), (4.3) and the identity

\[
\overline{K}_{IA} \equiv - (\partial_1 - \Gamma_{11}^1) \chi_{IA} - \chi_{A} \chi_{C} \chi_{B} \]

we have

\[
(\partial_1 + \frac{1}{\tau} \tau - \nu^0 \partial_1 v_0) \nabla_{\nu} \overline{A}_{IA} - \sigma_A \nu^0 \nabla_{\nu} \overline{A}_{AB} = -\overline{A}_{0A} (\partial_1 - \Gamma_{11}^1) \tau - \overline{A}_{0B} (\partial_1 - \Gamma_{11}^1) \sigma_A \nu^0 \nu^A.
\] (4.6)

With the current assumptions the \( \mu = A \)-component of (4.2) can be written as

\[
\nabla_{\nu} \overline{A}_{IA} + (\partial_1 + \tau - \nu^0 \partial_1 v_0) \overline{A}_{0A} = 0,
\]

whence

\[
(\partial_1 + \tau - \nu^0 \partial_1 v_0) [(\partial_1 - \Gamma_{01}^0) \overline{A}_{0A} - \sigma_A \nu^0 \overline{A}_{0B}] = 0.
\] (4.7)

Regularity requires \( \overline{A}_{0A} = O(\tau) \), and \( \overline{A}_{0A} = 0 \) is the only solution of (4.8) with this property. Of course, \( \nabla_{\nu} \overline{A}_{IA} v \) will then vanish as well.

In the final step we assume (in addition to (3.3) and (3.4)) \( \overline{A}_{ij} = 0, \overline{A}_{01} = 0, \nabla_{\nu} \overline{A}_{11} = 0 \), and \( g^{AB} \overline{B}_{AB} = 0 \). Taking the \( g_{AB} \)-trace of (3.21) and using again (4.3) we find

\[
(\partial_1 + \frac{1}{\tau} \tau + \Gamma_{11}^1 - \nu^0 \partial_1 v_0) (g^{AB} \nabla_{\nu} \overline{A}_{AB}) = \nu^0 \overline{A}_{00} (\partial_1 - \Gamma_{11}^1) \tau.
\]

From the \( \mu = 0 \)-component of (4.2) we derive the equation

\[
\nu^0 (\partial_1 + \tau + 2 \Gamma_{11}^1 - 2 \nu^0 \partial_1 v_0) \overline{A}_{00} - \frac{1}{2} g^{AB} \nabla_{\nu} \overline{A}_{AB} = 0,
\]

and end up with an ODE satisfied by \( \overline{A}_{00} \),

\[
(\partial_1 + \tau + \Gamma_{11}^1 - \nu^0 \partial_1 v_0) [\nu^0 (\partial_1 + \frac{1}{\tau} \tau + 2 \Gamma_{11}^1 - 2 \nu^0 \partial_1 v_0) \overline{A}_{00}] = 0.
\]

Regularity requires \( \overline{A}_{00} = O(1) \), which leads to \( \overline{A}_{00} = 0 \), which in turn implies \( g^{AB} \nabla_{\nu} \overline{A}_{AB} = 0 \). Assuming \( \overline{B}_{AB} = 0 \) we further have \( \nabla_{\nu} \overline{A}_{AB} = 0 \) due to (3.21).

Altogether we have proved the lemma.

**Lemma 4.2.** Assume that (3.3) and (3.4) hold, and that \( \overline{A}_{ij} = \overline{A}_{01} = 0 = \overline{B}_{ij} \). Then \( \overline{A}_{01} = 0 \) and \( \nabla_{\nu} \overline{A}_{ij} = 0 \). On the closure of those sets where \( \tau \) is non-zero, in particular sufficiently close to the vertex of the cone, the assumption \( \overline{A}_{01} = 0 \) is not needed, but follows from the remaining hypotheses.

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4.1.2. Vanishing of $\bar{B}_{0ij}$. By the second Bianchi identity we have
\[ \nabla_{\mu}B_{\nu}^{\alpha} - \frac{1}{2}\nabla_{\nu}B_{\mu}^{\alpha} = A_{\rho\sigma}(\nabla_{\mu}L_{\rho}^{\sigma} - \frac{1}{2}\nabla_{\nu}L_{\rho}^{\sigma}) + L_{\mu}^{\alpha}(\Box_{\sigma}X_{\nu} + R_{\sigma}^{\alpha}X_{\nu} + 2\nabla_{\sigma}X_{\nu}) + \frac{1}{2}\nabla_{\nu}\left(\Box_{\sigma}Y + \frac{1}{6}X_{\sigma}\nabla_{\nu}R + \frac{1}{3}\nabla_{\nu}R\right). \]
Assuming the wave equations (3.3) and (3.4) for $X$ and $Y$ as well as $\bar{A}_{\mu\nu} = 0$ this induces on the initial surface the relation
\[ \nabla_{\mu}B_{\nu}^{\alpha} - \frac{1}{2}\nabla_{\nu}B_{\mu}^{\alpha} = 0. \] (4.10)
As for $A_{\mu\nu}$, equation (4.2), we deduce that in adapted coordinates we have
\[ 0 = v^0(2\nabla_{\mu}B_{\nu}^{\alpha} - \nabla_{\nu}B_{\mu}^{\alpha}) + g^{11}(\nabla_{\mu}B_{1\nu}^{\alpha} - \frac{1}{2}\nabla_{\nu}B_{11}^{\alpha}) + g^{0\alpha}(2\nabla_{\mu}B_{0\nu}^{\alpha} - \frac{1}{2}\nabla_{\nu}B_{00}^{\alpha}) + g^{0\mu}(\nabla_{\nu}B_{0\nu}^{\alpha} - \frac{1}{2}\nabla_{\nu}B_{00}^{\alpha}). \] (4.11)
Recall that (3.3), (3.4) and the conformal field equations imply a wave equation (4.14) which is satisfied by $B_{\mu\nu}$. Assuming $\bar{A}_{\mu\nu} = 0$, $\nabla_{\mu}A_{\nu\alpha} = 0$ and $\bar{B}_{ij} = 0$, evaluation on the initial surface yields
\[ \Box_{\nu}B_{ij} = 2(g_{\mu}L_{\nu}^{\mu} - R_{\nu}^{\alpha\beta})B_{\alpha\beta} = 2R_{ij}^{\nu\lambda}B_{\mu\lambda} + 2\nabla_{\nu}(\nabla_{\mu}B_{\lambda\lambda} - \frac{1}{2}\nabla_{\lambda}B_{\mu\lambda}). \] (4.12)
In adapted null coordinates we have, as for the corresponding expression (3.18) for $A_{\mu\nu}$,
\[ \Box_{\nu}B_{ij} = 2v_{\nu}(\nabla_{\lambda}B_{\mu\lambda} + R_{\mu\lambda}^{\nu\sigma}B_{\sigma\lambda}) + g^{11}\nabla_{\mu}B_{1j} + 2g^{11}\nabla_{\lambda}B_{1\lambda} + \nabla_{\nu}(\nabla_{\mu}B_{1\lambda} + R_{1\lambda}^{\nu\sigma}B_{\sigma\lambda}). \] (4.13)
Moreover, we have seen that (3.3) & (3.4) imply the wave equation (3.15) is satisfied by $\nabla_{\nu}A_{\mu\nu}$. Assuming $\bar{A}_{\mu\nu} = 0$ and $\nabla_{\mu}A_{\nu\alpha} = 0$ we compute its trace on the initial surface,
\[ \Box_{\nu}\nabla_{\mu}A_{\nu\alpha} = 2v_{\nu}(\nabla_{\lambda}A_{\mu\lambda} + R_{\mu\lambda}^{\nu\sigma}A_{\sigma\lambda}) + 2g^{11}\nabla_{\mu}A_{1\alpha} + 2g^{0\alpha}\nabla_{\mu}A_{0\alpha} - 4\nabla_{\nu}B_{ij}. \] (4.14)
In adapted null coordinates and with the current assumptions the left-hand side becomes
\[ \Box_{\nu}\nabla_{\mu}A_{\nu\alpha} = 2v_{\nu}(\nabla_{\lambda}A_{\mu\lambda} + R_{\mu\lambda}^{\nu\sigma}A_{\sigma\lambda}) + 2g^{11}\nabla_{\mu}A_{1\alpha} + 2g^{0\alpha}\nabla_{\mu}A_{0\alpha} - 4\nabla_{\nu}B_{ij}. \] (4.15)
Recall that $\bar{A}_{\mu\nu} = 0$ suffices to establish $B_{\alpha\beta} = 0$. In that case $\bar{B}_{ij} = 0$ implies
\[ \nabla_{\mu}A_{\nu\alpha} = 0, \] (4.16)
and, by (3.22),
\[ \nabla_{\nu}A_{01} = 0. \]
As for $A_{\mu\nu}$, the $\mu = 1$-component of (4.11) yields
\[ \tau B_{01} + \nabla_{\nu}B_{11} = 0 \implies \nabla_{\nu}B_{11} = 0. \] (4.17)
The $(ij)$-component of (4.14) reads
\[ \Box_{\nu}\nabla_{\mu}A_{11} = 0 \implies (\partial_{1} + \frac{1}{2}\tau - 2\nu\partial_{1}v_{0})\nabla_{\nu}\nabla_{0}A_{11} = 0 \implies \nabla_{\nu}\nabla_{0}A_{11} = 0 \]
by regularity.
At this stage we can and will assume $\bar{A}_{\mu\nu} = \nabla_{\nu}A_{\mu\sigma} = \nabla_{0}A_{01} = \nabla_{0}A_{01} = B_{ij} = \bar{B}_{01} = \nabla_{0}B_{11} = 0$. With (3.23) we then find for the $(ij)$ = (1A)-components of (4.12)
\[ \Box_{\nu}B_{1A} = 2v_{\nu}(\nabla_{1\lambda}B_{1\lambda} + \frac{1}{2}(\nu\partial_{1})^{2}R_{1\lambda}\nabla_{\nu}\nabla_{0}A_{1\lambda} + \frac{1}{4}(\nu\partial_{1})^{2}R_{1\lambda}\nabla_{0}A_{0\lambda} + \frac{1}{2}(\nu\partial_{1})^{2}R_{1\lambda}^{\nu\sigma}A_{\sigma\lambda} - \nabla_{\nu}(\chi^{2}B_{0\lambda} + 2\chi^{2}A_{0\lambda}B_{0B})). \]
The $(ij)$ = (1A)-components of (4.13) read
\[ \Box_{\nu}B_{1A} = 2v_{\nu}(\nabla_{1\lambda}B_{1\lambda} - \nu\partial_{1}v_{0})\nabla_{\nu}B_{1A} - 2\nu\sigma^{\mu}_{A\mu}\nabla_{\nu}B_{1B} - v_{0}(\chi^{2}B_{0A} + 2\chi^{2}A_{0A}B_{0B}). \]
Evaluating both expressions for $\Box_y B_{1A}$ and using (4.6) we deduce that
\[ 2(\partial_1 - v^0\partial_1 v_0)\Box_0 B_{1A} - 2\sigma^\beta_0 \Box_0 B_{1B} = -2R_{0B}(\partial_1 - \Gamma^1_{1A})\chi_B + |\chi|^2 R_{0A} + \frac{1}{2} v^0 R_{11} \Box_0 \Box_0 A_{1A} + \frac{1}{2} R_{00}^1 \Box_0 A_{00} + \frac{1}{2} v^0 \sigma^\alpha_0 \Box_0 A_{0B}. \] (4.18)

Evaluation of (4.14) for $(i,j) = (1A)$ leads to
\[ \Box_y \Box_0 A_{1A} = v^0 R_{11} \Box_0 A_{1A} + 2v^0 R_{1A} \Box_0 A_{0B} - 4\Box_0 B_{1A}, \]
while (4.15) becomes
\[ \Box_y \Box_0 A_{1A} = v^0(\partial_1 + \Gamma^1_{11} - 2v^0\partial_1 v_0)\Box_0 \Box_0 A_{1A} - 2v^0\sigma^\beta_0 \Box_0 \Box_0 A_{1B} - v^0|\chi|^2 \Box_0 A_{00} - 2v^0\chi_B \Box_0 A_{0B}. \]

Using (4.3) and (4.6) we end up with
\[ 2(\partial_1 + \Gamma^1_{1A} - 2v^0 v_0)\Box_0 \Box_0 A_{1A} - 2\sigma^\beta_0 \Box_0 \Box_0 A_{1B} = -2\Box_0 A_{00}(\partial_1 - \Gamma^1_{11})\tau - 2\Box_0 A_{0B}(\partial_1 - \Gamma^1_{1A})\sigma_B - 4v_0 \Box_0 B_{1A}. \] (4.19)

The $\mu = A$-components of (4.11) give, again in close analogy to the corresponding equations for $A_{\mu\nu}$,
\[ (\partial_1 + \tau - \Gamma^0_{01}) B_{0A} + \Box_0 B_{1A} = 0. \] (4.20)

By (3.23) we have
\[ (\partial_1 - 2\Gamma^0_{01}) \Box_0 A_{0A} - \sigma^\beta_0 \Box_0 A_{0B} = -2v_0 B_{0A}. \] (4.21)

Taking the behaviour of the metric components at the vertex into account, cf [2, section 4.5], we observe that the ODE-system (4.18)–(4.21) for $B_{0A}$, $\Box_0 B_{1A}$, $\Box_0 A_{0A}$ and $\Box_0 \Box_0 A_{1A}$ is of the form
\[ \begin{bmatrix} \partial_1 + \begin{pmatrix} 2r^{-1} + O(r) & 1 & 0 & 0 \\ -2r^{-2} + O(1) & O(1) & O(r^{-1}) & O(1) \\ 2 + O(r^2) & 0 & O(r) & 0 \\ 0 & 2 + O(r^2) & -2r^{-2} + O(1) & O(r) \end{pmatrix} & 0 \\ 0 & \Box_0 B_{1A} & \Box_0 A_{0A} & \Box_0 A_{1A} \end{bmatrix} = 0, \]
where each matrix entry is actually a $2 \times 2$-matrix. Regularity requires
\[ B_{0A}, \Box_0 B_{1A}, \Box_0 A_{0A}, \Box_0 \Box_0 A_{1A} = O(r). \]

But then a necessary condition for (4.20) to be satisfied is
\[ B_{0A} = O(r^2), \] (4.22)
whence it follows from (4.21) and (4.19) that
\[ \Box_0 A_{0A} = O(r^3), \quad \Box_0 \Box_0 A_{1A} = O(r^3). \] (4.23)

Setting $\tilde{B}_{0A} := r^{-2}B_{0A}$, $\tilde{\Box}_0 B_{1A} = r^{-1}\Box_0 B_{1A}$, $\tilde{\Box}_0 A_{0A} := r^{-3} \Box_0 A_{0A}$ and $\tilde{\Box}_0 \Box_0 A_{1A} = r^{-2} \Box_0 \Box_0 A_{1A}$
the ODE-system adopts the form
\[ \begin{bmatrix} \partial_1 + r^{-1} \begin{pmatrix} 4 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 2 & -2 & 2 \end{pmatrix} + M & 0 \\ 0 & \tilde{B}_{0A} & \tilde{\Box}_0 A_{0A} & \tilde{\Box}_0 \Box_0 A_{1A} \end{bmatrix} = O(1), \]
where $M = O(r)$ is some matrix. Setting
\[ \tilde{v} := T^{-1} v = O(1), \] (4.24)

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where
\[ T := \begin{pmatrix} 0 & -1/2 & -1/3 & 0 \\ 0 & 1/2 & 2/3 & 0 \\ -1 & -1/2 & 2/3 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \]
is the change of basis matrix which transforms the leading order matrix to Jordan normal form, we end up with the Fuchsian ODE-system
\[ \partial_1 \tilde{v} + r^{-1} \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tilde{v} + \tilde{M} \tilde{v} = 0, \quad \tilde{M} := T^{-1}MT = O(r). \quad (4.25) \]
In appendix A it is shown that any solution of (4.25) which is $O(1)$ needs to vanish identically (take, in the notation used there, $\lambda = -1$). Hence $\overline{B}_{0A} = \nabla_0 \overline{B}_{1A} = \nabla_0 \overline{A}_{0A} = \nabla_0 \nabla_0 \overline{A}_{1A} = 0$, which we can and will assume in the subsequent computations.

The $\mathcal{C}_{AB}$-trace of the $(ij) = (AB)$-component of (4.12) reads
\[ \mathcal{C}_{AB}^{ij} \overline{B}_{AB} = \frac{1}{2} (\nu^0)^2 \nabla_0 \nabla_0 \nabla_0 \nabla_0 \nabla_0 - \frac{1}{2} (\nu^0)^2 \nabla_1 \nabla_0 \nabla_0. \]
For the corresponding component of (4.13) we find
\[ \mathcal{C}_{AB}^{ij} \overline{B}_{AB} \equiv 2 \nu^0 (\partial_1 + \frac{1}{2} r - \nabla_0^0) (\mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0) + 2 (\nu^0)^2 \chi^2 \overline{B}_{00}, \]
and thus
\[ (\partial_1 + \frac{1}{2} r - \nabla_0^0) (\mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0) + \chi^2 \overline{B}_{00} = \frac{1}{2} (\nu^0)^2 \nabla_1 \nabla_0 \nabla_0. \quad (4.26) \]
From (4.14) we deduce
\[ \mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0 \nabla_0 = -2 (\nu^0)^2 \nabla_1 \nabla_0 \nabla_0 - 4 \nu^0 \nabla_0 \nabla_0. \]
while from (4.15) we obtain
\[ \mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0 \nabla_0 = 2 \nu^0 (\partial_1 + \frac{1}{2} r - 2 \nabla_0^0) (\mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0) + 2 (\nu^0)^2 \chi^2 \nabla_0 \nabla_0. \]
Invoking (4.3) we are led to the equation
\[ (\partial_1 + \frac{1}{2} r - 2 \nabla_0^0) (\mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0 \nabla_0) = \nu^0 \nabla_0 \nabla_0 (\partial_1 - \nabla_1^1) r - 2 \nu^0 \mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0. \quad (4.27) \]
The $\mu = 0$-component of (4.11) reads
\[ (\partial_1 + \frac{1}{2} r - 3 \nabla_0^0) \overline{B}_{00} - \nu^0 \mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0 = 0. \quad (4.28) \]
Recall that by (3.24) we have
\[ (\partial_1 + \frac{1}{2} r - 3 \nabla_0^0) \nabla_0 \nabla_0 + 2 \nu^0 \overline{B}_{00} = 0. \quad (4.29) \]
Using again the results of [2, section 4.5] we find that the ODE-system (4.26)–(4.29) for $\overline{B}_{00}$, $\mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0 \nabla_0$ and $\mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0$ is of the form
\[ \begin{pmatrix} 2r^{-1} + O(r) & -\frac{1}{2} r + O(r^2) & 0 & 0 \\ 2r^{-2} + O(1) & r^{-1} + O(r) & O(r^{-1}) & O(1) \\ 2 + O(r^2) & 0 & r^{-1} + O(r) & 0 \\ 0 & 2 + O(r^2) & 2r^{-2} + O(1) & r^{-1} + O(r) \end{pmatrix} \begin{pmatrix} \overline{B}_{00} \\ \mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0 \nabla_0 \\ \mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0 \nabla_0 \end{pmatrix} = 0. \]
Due to regularity we have
\[ \overline{B}_{00}, \mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0 \nabla_0, \mathcal{C}_{AB}^{ij} \nabla_0 \nabla_0 \nabla_0 = O(1). \]
Where \( \tilde{\nu} \) is the rescaled fields \( g^{AB} \tilde{\nabla}_0 \tilde{\nabla}_0 A_{AB} = r^{-1} g^{AB} \tilde{\nabla}_0 \tilde{\nabla}_0 A_{AB} \) and \( \tilde{B}_{00} := r^{-1} \tilde{B}_{00} \).

In terms of the rescaled fields \( \tilde{g}^{AB} \tilde{\nabla}_0 \tilde{\nabla}_0 A_{AB} = \tilde{r}^{-1} \tilde{g}^{AB} \tilde{\nabla}_0 \tilde{\nabla}_0 A_{AB} \) and \( \tilde{B}_{00} := \tilde{r}^{-1} \tilde{B}_{00} \), the ODE-system takes the form

\[
\left[ \partial_t + r^{-1} \begin{pmatrix} 3 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix} + M \right] v = 0, \quad v := \begin{pmatrix} \tilde{B}_{00} \\ \tilde{g}^{AB} \tilde{\nabla}_0 \tilde{\nabla}_0 A_{AB} \end{pmatrix} = O(1),
\]

with \( M = O(r) \) being some matrix. The change of basis matrix

\[
T := \begin{pmatrix} 0 & -1/3 & -1/3 & 0 \\ 0 & -2/3 & 0 & 0 \\ 1/\sqrt{3} & 2/3 & 4/3 & 0 \\ 2/\sqrt{3} & 8/3 & 0 & 1 \end{pmatrix},
\]

transforms the indicial matrix to Jordan normal form, and we end up with another Fuchsian ODE-system,

\[
\partial_t \tilde{v} + r^{-1} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tilde{v} + \tilde{M} \tilde{v} = 0,
\]

where \( \tilde{v} := T^{-1} v = O(1) \) and \( \tilde{M} := T^{-1} M T = O(r) \). Again, lemma A.1 in appendix A (with \( \lambda = -1 \)) implies \( \tilde{v} = 0 \), and thus \( \tilde{B}_{00} = \tilde{g}^{AB} \tilde{\nabla}_0 \tilde{\nabla}_0 B_{AB} = \tilde{\nabla}_0 A_{00} = \tilde{g}^{AB} \tilde{\nabla}_0 \tilde{\nabla}_0 A_{AB} = 0 \).

In this section we have proved:

**Lemma 4.3.** Assume that (3.3) and (3.4) hold, and that \( \tilde{A}_{\mu\nu} = \tilde{B}_{ij} = \tilde{\nabla}_0 A_{ij} \). Then \( \tilde{B}_{00} = 0, \tilde{g}^{AB} \tilde{\nabla}_0 \tilde{\nabla}_0 B_{AB} = 0, \tilde{\nabla}_0 A_{00} = 0 \) and \( \tilde{g}^{AB} \tilde{\nabla}_0 \tilde{\nabla}_0 A_{AB} = 0 \).

### 4.1.3. The (proper) KID equations.

The conditions (iv), (vi) and (vii) in theorem 4.1 are not intrinsic in the sense that they involve transverse derivatives of \( X \) and \( Y \) which are not part of the initial data for the wave equations (i) and (ii). However, they can be eliminated via these wave equations. In fact, this is crucial if one wants to check for a certain candidate field defined only on the initial surface whether it extends to a vector field satisfying the unphysical Killing equations or not. In essence this is what we will do next.

We have

\[
\Box Y := 2 v^0 (\nabla_i + \frac{1}{2} \tau) \nabla_i Y + g^{ij} D_i D_j Y,
\]

where \( D_i \) is the derivative operator introduced in [3],

\[
D_i Y := Y_i, \quad D_i \tilde{X}_\mu := \tilde{X}_\mu, \quad D_i \tilde{D}_j Y := \partial_i \tilde{D}_j Y - \Gamma^k_{ij} D_k Y, \quad D_i \tilde{D}_j \tilde{X}_\mu := \partial_i \tilde{D}_j \tilde{X}_\mu - \Gamma^k_{ij} \tilde{D}_k \tilde{X}_\mu = \Gamma^k_{ij} \tilde{D}_k Y.
\]
i.e. one simply removes the transverse derivatives which would appear in the corresponding expressions with covariant derivatives. Since the action of \( \nabla_i \) and \( D_i \) coincides in many cases relevant to us one may often use them interchangeably. Nevertheless, we shall use \( D_i \) consistently whenever derivatives of \( X \) or \( Y \) appear to stress that no transverse derivatives of these fields are involved.

By (ii) and (4.33) the function \( \Upsilon := \partial_0 Y \) (note that \( \Upsilon \) is not a scalar) satisfies the ODE
\[
(\partial_1 + \frac{\tau}{2} - \Gamma_{01}^0) \Upsilon - \Gamma_{01}^0 \nabla_1 Y + \frac{1}{2} \partial_0 (g^\nu (D_i D_j) Y + \frac{1}{6} X^\mu \nabla_\mu R + \frac{1}{3} \nabla Y) = 0.
\]
Regularity requires \( \Upsilon = O(1) \).

It is useful to make the following definition
\[
S_{\mu\nu} := \nabla_\mu Y_{,\nu} - \nabla_\nu Y_{,\mu} - 2Y_{,\mu} g_{\nu\sigma} + \nabla_\nu Y g_{\mu\sigma}.
\]
It follows from the identity (3.12) that
\[
S_{\mu\nu} = \nabla_{(\mu} A_{\nu)} - \frac{1}{2} \nabla_\sigma A_{\mu\nu}.
\]
Note that this implies the useful relations
\[
2S_{\mu(\nu;\sigma)} = \nabla_\mu A_{\nu;\sigma},
\]
\[
S_{(\mu;\nu)} = 0.
\]
Recall that (4.4) is a consequence of (i) and (v). Hence
\[
\tilde{S}_{110} = \nabla_1 A_{10} - \frac{1}{2} \nabla_0 A_{11} = (\partial_1 + \frac{\tau}{2} - \nu^0 \partial_0) A_{01} = 0.
\]
Taking regularity into account we conclude that
\[
A_{01} = 0 \iff \tilde{S}_{110} = 0.
\]
This leads us to the following stronger version of theorem 4.1:

**Theorem 4.4.** Assume that we have been given a 3 + 1-dimensional space-time \( (\mathcal{M}, g, \Theta) \), with \( (g, \Theta) \) being a smooth solution of the conformal field equations (2.1)–(2.6). Let \( \tilde{X} \) be a vector field and \( \tilde{Y} \) a function defined on a light-cone \( C_D \subset \mathcal{M} \). Then there exists a smooth vector field \( X \) with \( \tilde{X} = X \) and \( \nabla_i \tilde{X} = 4Y \) satisfying the unphysical Killing equations (3.1) on \( D^+ (C_D) \) (i.e. representing a Killing field of the physical space-time) if and only if

(a) the conditions (iii) and (v) in theorem 4.1 hold,

(b) \( \tilde{Y} := \tilde{X}^\mu \nabla_\mu \tilde{Y} + 3\tilde{Y} - \nabla^\mu \Theta D_\mu \tilde{Y} - \nu^0 \tilde{Y} \nabla_1 \Theta = 0, \)

(c) \( \tilde{S}_{110} = D_1 D_0 X_0 = -\tilde{R}_{011} \tilde{X}_0 - 2\nu^0 D_1 \tilde{Y} = 0, \)

(d) \( \tilde{B}_{ij} = \tilde{X}^\mu \nabla_\mu \tilde{L}_{ij} + 2\tilde{L}_{a(i} D_{j)} \tilde{X}_a + D_i D_j \tilde{Y} = 0, \)

(e) \( \tilde{B}^a_{ij} := \tilde{X}^\mu \nabla_\mu \tilde{L}_{AB} + 2\tilde{L}_{a(i} D_{j)} \tilde{X}_a^B + D_i D_j \tilde{Y} + \nu^0 Y \tilde{X}_{AB} = 0, \)

(f) \( \tilde{X} \) and \( \tilde{Y} \) are restrictions to the light-cone of smooth space-time fields.

The function \( \Upsilon \) is the unique solution of
\[
(\partial_1 + \frac{\tau}{2} - \Gamma_{01}^0) \Upsilon - \Gamma_{01}^0 \nabla_1 \tilde{Y} + \frac{1}{2} \partial_0 (g^\nu (D_i D_j) \tilde{Y} + \frac{1}{6} \tilde{X}^\mu \nabla_\mu R + \frac{1}{3} \nabla \tilde{Y}) = 0
\]
which is bounded near the tip of the cone. The condition (c) is not needed on the closure of those sets on which the expansion \( \tau \) is non-zero.

**Proof.** It needs to be shown that \( (\tilde{X}, \tilde{Y}) \) extends to a pair \( (X, Y) \) satisfying (i)–(vii) in theorem 4.1. From the considerations above it becomes clear that (a)–(e) do imply (i)–(vii) in theorem 4.1 if \( \tilde{X} \) and \( \tilde{Y} \) can be extended to smooth solutions of the wave equations (3.3) and (3.4) for \( X \) and \( Y \). However, this follows from [5] and (f). □
**Remark 4.5.** The conditions (a)–(e) will be called *(proper)* \(^2\) KID equations *(cf proposition 4.9)* below which shows that condition (f) is not needed.

**Remark 4.6.** Theorem 4.4 can e.g. be applied to a light-cone with vertex at past timelike infinity for vanishing cosmological constant (this is done in section 5), or to light-cones with vertex on past null infinity for vanishing or positive cosmological constant.

### 4.1.4. Extendability of the candidate fields

A drawback of theorem 4.4 is the condition (f): it is a non-trivial issue to make sure that the candidate fields \(\tilde{X}\) and \(\tilde{Y}\) which are constructed from (a subset of) (a)–(e) are restrictions to the light-cone of smooth space-time fields. (Nonetheless we shall see in section 5 that (f) becomes trivial on the \(C_\tau\)-cone.) We therefore aim to prove that (f) follows directly and without any restrictions from the KID equations (a)–(e).

Since the validity of (f) is non-trivial only in some neighbourhood of the vertex of the cone, we can and will assume in this section that the expansion \(\tau\) has no zeros.

The procedure will be in close analogy to [3, section 2.5]. First of all we shall compute the divergence \(\nabla^\sigma S_{\mu\nu\sigma}\) which contains certain transverse derivatives of \(\tilde{X}\) and \(\tilde{Y}\) (which eventually drop out from the relevant formulae). For these expressions to make sense let \(X\) and \(Y\) be smooth extensions of \(\tilde{X}\) and \(\tilde{Y}\) from the cone \(C_\tau\) to a punctured neighbourhood of \(O\). We stress that no assumptions are made concerning the behaviour of \(X\) and \(Y\) as the tip of the cone is approached.

By (4.35) and the second Bianchi identity we have
\[
\nabla^\sigma S_{\mu\nu\sigma} = \frac{1}{2} \nabla_\mu \nabla_\nu A_\sigma - 2B_{\mu\nu} + \frac{1}{2} R A_{\mu\nu} + (B_\sigma^\sigma - L^{\alpha\beta} A_{\alpha\beta}) g_{\mu\nu} .
\]  
(4.41)

In adapted null coordinates the trace of the left-hand side on the cone reads,
\[
\nabla^\sigma S_{\mu\nu\sigma} = \nu^0 (\nabla_0 S_{\mu\nu 0} + \nabla_1 S_{\mu\nu 1}) + \tilde{g}^{\alpha\beta} (\nabla_1 S_{\mu\nu \alpha} + \nabla_\alpha S_{\mu\nu 1}) + \tilde{g}^{\mu\nu} \nabla_1 S_{\mu\nu 1} + \tilde{g}^{\mu\nu} \nabla_A S_{\mu\nu A} .
\]  
(4.42)

The undesirable transverse derivatives which appear in \(\nabla_0 S_{\mu\nu 1}\) can be eliminated via
\[
\nabla_0 \nabla_\mu \nabla_\nu X_\sigma = \nabla_\mu \nabla_\nu A_\sigma - \nabla_\mu \nabla_\nu \nabla_\sigma X_0 + 2g_{0\nu} \nabla_\mu \nabla_\sigma Y + \nabla_\mu (R_{0\nu\sigma} X_\epsilon) + R_{0\mu\nu\sigma} \nabla_\sigma X_0 + R_{0\mu\nu\sigma} \nabla_\sigma X_\epsilon .
\]  
(4.43)

**Lemma 4.7.** Assume \(\overline{A}_{ij} = 0\). Then
\[
2\overline{B}_{11} = \tau \nu^0 \overline{S}_{110} .
\]

**Proof.** Equation (4.41) with \((\mu\nu) = (11)\) yields
\[
\nabla^\sigma S_{11\sigma} = \nu^0 \nabla_1 \nabla_1 \overline{A}_{01} + 2\overline{B}_{11} .
\]  
(4.44)

Note that it follows from (4.36) that the vanishing of \(\overline{A}_{ij}\) implies the vanishing of \(\overline{S}_{11}\), and all permutations thereof. Due to (4.42) we further have \(\overline{S}_{i\alpha\beta} = \overline{S}_{\alpha\beta i} = 0\). From (4.42) we then obtain with \((\mu\nu) = (11)\)
\[
\nabla^\sigma S_{11\alpha} = \nu^0 (\overline{V}_0 \overline{S}_{111} + \nu^0 \nabla_1 \overline{S}_{110}) - 2\chi^{\alpha\beta} \overline{S}_{11\beta} + \tau \nu^0 \overline{S}_{110} .
\]  
(4.45)

while (4.43) gives
\[
\overline{V}_0 \overline{S}_{111} = \overline{V}_0 \nabla_1 \nabla_1 X_1 = \nabla_1 \nabla_1 \overline{A}_{01} - \nabla_1 \overline{S}_{110} .
\]

Equating (4.44) with (4.45) yields the desired result. \(\square\)

\(^2\) In the sense that they do not involve transverse derivatives of \(X\) or \(Y\).
Lemma 4.8. Assume $\vec{A}_{ij} = 0$ and $\vec{S}_{110} = 0$. Then

$$2\nu_0 \vec{B}_{iA} = (\partial_i + \tau - \nu_0 \partial_i \nu_0) \vec{S}_{1A0}.$$ 

Proof. From the $(\mu \nu) = (A1)$-components of (4.41) we deduce

$$\Delta^\nu \vec{S}_{1A\nu} = \frac{1}{2} \nabla_A \nabla_i \vec{A}_{\alpha \nu} + 2\vec{B}_{iA}. \quad (4.46)$$

It follows from (4.43) that

$$\nabla_0 \vec{S}_{1A0} = \nabla_0 \nabla_A \nabla_i X_i = \nabla_A \nabla_i \vec{A}_{01} - \nabla_A \vec{S}_{110} = \nabla_A \nabla_i \vec{A}_{01} + 2\chi^B \vec{S}_{B10}.$$

Recall that $\vec{S}_{110}$ as well as all permutations thereof vanish, and that $\vec{S}_{1A1} = \vec{S}_{IA} = 0$. Equation (4.42) with $(\mu \nu) = (A1)$ and (4.37) then yield

$$\Delta^\nu \vec{S}_{1A\nu} = \nu_0 \nabla_0 \vec{S}_{1A1} + \nu_0 \nabla_i \vec{S}_{1A0} + \frac{2}{\mu} \nabla_B \vec{S}_{A1} + \frac{\mu}{\nu} \nabla_C \vec{S}_{A1B}$$

$$= \nu_0 \nabla_A \nabla_i \vec{A}_{01} + 2\nu_0 \chi^B \vec{S}_{B10} + \nu_0 \nabla_i \vec{S}_{1A0} + \frac{2}{\mu} \nabla_B \nabla_A \vec{A}_{11} + \frac{\mu}{\nu} \nabla_{C} \vec{S}_{A1B}. \quad (4.47)$$

Combining (4.46) and (4.47) and invoking again (4.37) we obtain

$$2\vec{B}_{iA} = \nu_0 \nabla_i \vec{S}_{1A0} + 2\nu_0 \chi^B \vec{S}_{B10} + 2\frac{\mu}{\nu} \nabla_C \vec{S}_{A1B} + \frac{\nu}{\mu} \nu_0 \nabla_B \nabla_A \vec{A}_{11} - \nabla_A \nabla_i \vec{A}_{1B}. \quad (4.48)$$

Since

$$\frac{1}{2} \nabla_B \nabla_A \vec{A}_{11} - \nabla_A \nabla_i \vec{A}_{1B} = -\nabla_A \vec{S}_{11B} = 0,$$

and

$$2\frac{\mu}{\nu} \nabla_C \vec{S}_{A1B} = \tau \nu_0 \vec{S}_{1A0} - \nu_0 \chi^B \vec{S}_{B10} + 2\frac{\mu}{\nu} \chi^C \vec{S}_{C1B} = 0$$

by (4.36)

the lemma is proved. $\square$

As in [3] one checks via the formulae in [2; section 4.5] and assuming

$$\dot{X}_1, \partial_i \dot{X}_1 = O(1), \quad \dot{X}_0, \partial_i \dot{X}_0, \partial_i \partial_i \dot{X}_0 = O(1), \quad (4.49)$$

$$\dot{X}_A, \partial_i \dot{X}_A = O(\tau), \quad \partial_i \dot{X}_A = O(1), \quad \dot{Y}, \partial_i \dot{Y} = O(1), \quad (4.50)$$

that $\vec{S}_{1A0}$ needs to exhibit the following behaviour near the tip of the cone:

$$\vec{S}_{1A0} = O(r^{-N}). \quad (4.51)$$

Note that (4.49)–(4.50) are necessarily satisfied by any pair $(\dot{X}, \dot{Y}) = (X, \frac{1}{2} \nabla X)$ with $X$ a smooth vector field.

It now follows immediately from lemmas 4.7 and 4.8 that for any vector field $\dot{X}$ and any function $\dot{Y}$ which satisfy $\vec{A}_{ij} = 0$ and $\vec{B}_{ii} = 0$ the equation

$$\vec{S}_{ii0} = 0 \quad (4.52)$$

holds sufficiently close to the vertex of the cone where $\tau$ has no zeros.

Let us define an antisymmetric tensor field $\vec{F}_{i\alpha}$ via

$$\vec{F}_{ij} := \nabla_i \dot{X}_j, \quad (4.53)$$

$$\vec{F}_{i0} := \nabla_i \dot{X}_0 - \vec{S}_{0i}. \quad (4.54)$$

We also define a covector field $\vec{H}_i$,

$$\vec{H}_i := \nabla_i \dot{Y}, \quad (4.55)$$

$$\vec{H}_0 := 0. \quad (4.56)$$
In the following computations we assume
\[ S_{i0} = 0 = \bar{A}_{ij} = \bar{B}_{ij}. \]  
(4.57)

Then, due to the first Bianchi identity,
\[ \dot{F}_{ij} = \nabla_i \dot{X}_j - \frac{1}{2} \bar{A}_{ij} - \dot{\bar{g}}_{ij} = \nabla_i \dot{X}_j, \]
\[ \nabla_i \dot{F}_{ij} = \nabla_i [\bar{A}_{ij}] - 2 \bar{g}_{ij} [\dot{H}_k] - \bar{g}_{ij} \alpha \dot{X}_a = -\bar{R}_{ij \alpha} \dot{X}_a, \]
\[ \nabla_i F_{\alpha \beta} = \bar{S}_{i0} - \bar{R}_{i0} \alpha \dot{X}_a + \nu_0 \dot{H}_i - \bar{g}_{i0} \nabla_0 Y = \nu_0 \dot{H}_i - \bar{R}_{i0} \alpha \dot{X}_a. \]

Moreover,
\[ \nabla_i H_i = \bar{B}_{i0} - \bar{g}_{i0} \dot{Y} = \bar{B}_{i0} - \bar{L}(1) / \bar{L}_{i0} - \dot{X}_a \bar{g}_{a0} \dot{Y} - 2 \bar{L}_1 / \bar{L}_{i1} - \dot{Y}, \]
\[ \nabla_i \dot{H}_0 = - \bar{L}_0 / \bar{L}_1 \dot{H}_1. \]

Therefore the candidate fields \( \dot{X} \) and \( \dot{Y} \) solving (a)–(c) in theorem 4.4 form a solution of the following problem on \( C_o \),
\[ \nabla_i \dot{X}_a = \bar{F}_{i\mu} + \bar{g}_{i0} \dot{Y}, \]
\[ \nabla_i \dot{F}_{\mu \nu} = 2 \bar{g}_{i0} [\bar{H}_j] - \bar{R}_{\mu \nu \alpha} \dot{X}_a, \]
\[ \nabla_i \dot{Y} = H_i, \]
\[ \nabla_i \dot{H}_0 = - \bar{L}_0 / \bar{L}_1 \dot{H}_1, \]
\[ \nabla_i \dot{H}_j = - \dot{X}_a \bar{g}_{a0} \dot{Y} - 2 \bar{L}_1 / \bar{L}_{i1} ([\bar{F}_{i\alpha} / \bar{F}_{\mu \alpha} + \bar{g}_{i0} \dot{Y}]), \]

which is uniquely defined by the values of \( \dot{X}_a, \bar{F}_{\mu \nu}, \dot{Y} \) and \( \dot{H}_\mu \) at the vertex of the cone.

We want to show that the fields which solve (4.58) are restrictions to the cone of smooth space-time fields: Given any vector \( \ell^\mu \) in the tangent space at \( O \) define \((x^\mu(s), X^\mu(s), \bar{F}_{\nu \mu}(s), Y(s), H_\mu(s))\) as the unique solution of the problem
\[ \frac{d^2 x^\mu}{ds'} + \Gamma_\mu^\alpha \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0, \]
\[ \frac{dx^\alpha}{ds} + \Gamma_\mu^\alpha \frac{dx^\mu}{ds} = F_{\mu \alpha} \frac{dx^\alpha}{ds} + g_{\mu \nu} Y \frac{dx^\nu}{ds}, \]
\[ \frac{d\bar{F}_{\alpha \beta}}{ds} - \Gamma_\nu^\alpha \bar{F}_{\nu \mu} \frac{dx^\nu}{ds} - \Gamma_\nu^\nu \bar{F}_{\mu \alpha} \frac{dx^\nu}{ds} = 2 g_{\nu \mu} [\bar{H}_j] \frac{dx^\nu}{ds} - \bar{R}_{\mu \nu \alpha} \dot{X}_a \frac{dx^\nu}{ds}, \]
\[ \frac{dY}{ds} = H_0 \frac{dx^\nu}{ds}, \]
\[ \frac{dH_0}{ds} = - \Gamma_\mu^\mu H_\mu \frac{dx^\mu}{ds} = \left\{ - \dot{X}_a \bar{g}_{a0} \bar{L}_{\mu \nu} - 2 \bar{L}_1 / \bar{L}_{i1} ([\bar{F}_{i\alpha} / \bar{F}_{\mu \alpha} + \bar{g}_{i0} \dot{Y}]) \right\} \frac{dx^\nu}{ds}, \]
\[ \frac{dx^\mu}{ds}(0) = \ell^\mu, \]

\[ \ell^\mu(0) = 0, \]

for given initial data \((X^\mu(0), \bar{F}_{\nu \mu}(0), Y(0), H_\mu(0))\). As in [3, section 2.4] the system (4.59), together with the property that solutions of ODEs depend smoothly upon initial data, and that the trace of solutions of (4.59) on \( C_o \) solve (4.58), shows that the fields solving (4.58) are restrictions to the cone of smooth space-time fields. We have proved:

**Proposition 4.9.** The condition (f) in theorem 4.4 can be removed.
4.2. A stronger version of theorem 3.4 for two transversally intersecting null hypersurfaces

4.2.1. Stronger version. We want to establish the analogues of lemmas 4.2 and 4.3 for two transversally intersecting null hypersurfaces.

Lemma 4.10. Assume that the wave equations (3.3) and (3.4) for X and Y hold, and that, on \( N_1 \), \( \overline{A}_{2\mu} = \overline{A}_{AB} = 0 = \overline{B}_{22} = \overline{B}_{2A} = \overline{B}_{AB} \), similarly on \( N_2 \). Furthermore, we assume that \( \nabla_1(A_{2A}|_S) = 0 \). Then, on \( N_1 \), \( A_{11} = \overline{A}_{1A} = 0 \) and \( \nabla_1 A_{2A} = \nabla_1 A_{1A} = 0 \), and a corresponding statement holds on \( N_2 \). On the closure of those sets where \( \tau \) is non-zero the assumption \( \overline{A}_{12} = 0 \) is not needed but follows from the remaining assumptions, supposing that \( A_{12}|_S = 0 \).

Proof. We can repeat most of the steps which were necessary to prove lemma 4.2. The only difference is that the ODEs are not of Fuchsian type anymore, but regular ones. To make sure that all the fields involved vanish on \( N_1 \cup N_2 \) we therefore need to make sure that we have vanishing initial data on \( S \). This is the case if, on \( S \),

\[
A_{11} = A_{22} = A_{1A} = A_{2A} = \nabla_1 A_{2A} = \nabla_2 A_{1A} = g^{\mu\nu} \nabla_1 A_{AB} = g^{\mu\nu} \nabla_2 A_{AB} = 0.
\]

Observing that the analogue of (4.7) for light-cones holds, i.e.

\[
\nabla_1(A_{2A}|_S) = 0
\]

this is an obvious consequence of the hypotheses made above. \( \square \)

In analogy to lemma 4.3 we have

Lemma 4.11. Assume that (3.3) and (3.4) hold, and that \( \overline{A}_{\mu\nu} = 0 \). Moreover, assume that, on \( N_1 \), \( \overline{B}_{22} = \overline{B}_{2A} = \overline{B}_{AB} = 0 \) and \( \nabla_1 A_{22} = \nabla_1 A_{2A} = \nabla_1 A_{AB} = 0 \), similarly on \( N_2 \). Then, on \( N_1 \), \( \overline{B}_{1\mu} = 0 \), \( \nabla_1 B_{22} = \nabla_1 B_{2A} = \nabla_1 B_{AB} = 0 \), \( \nabla_1 A_{1\mu} = 0 \) and \( \nabla_1 \nabla_1 A_{22} = \nabla_1 \nabla_1 A_{2A} = \nabla_1 \nabla_1 A_{AB} = 0 \), and similar conclusions can be drawn on \( N_2 \).

Proof. Again, we just need to make sure that all the initial data for the ODEs vanish on \( S \). For all the field components involving covariant derivatives of \( A_{\mu\nu} \), this follows directly from the vanishing of \( \overline{A}_{\mu\nu} \). The vanishing of those field components involving (covariant derivatives of) \( B_{\mu\nu} \) follows from the same fact, since, by (3.9), they can be expressed in terms of \( A_{\mu\nu} \) and covariant derivatives thereof. \( \square \)

Altogether we have proved

Theorem 4.12. Assume that we have been given a \( 3 + 1 \) dimensional space-time (\( \mathcal{M}, g, \Theta \)), with \( (g, \Theta) \) a smooth solution of the conformal field equations. Let \( N_c \subset \mathcal{M} \), \( a = 1, 2 \), be two transversally intersecting null hypersurfaces with transverse intersection along a smooth two-dimensional submanifold \( S \). Then there exists a vector field \( \hat{X} \) satisfying the unphysical Killing equations (3.1) on \( D^* (N_1 \cup N_2) \) if and only if there exists a pair \((X, Y)\), \( X \) a vector field and \( Y \) a function, which fulfils the following conditions:

(a) the conditions (i)–(iv) in theorem 3.4 hold,

(b) \( \overline{A}_{12} = 0 \) with \( A_{\mu\nu} \equiv 2 \nabla_1 X_{\mu} - 2 Y_{\mu\nu} \),

(c) \( \overline{A}_{AB} = 0 \) with \( A_{AB} |_{N_1} = \overline{A}_{2A} |_{N_1} = \overline{A}_{11} |_{N_1} = \overline{A}_{1A} |_{N_1} \),

(d) \( \nabla_1 |_{A_{2A}|_S} = 0 \),

(e) \( \overline{B}_{AB} = 0 \) with \( B_{AB} |_{N_1} = \overline{B}_{2A} |_{N_1} = \overline{B}_{11} |_{N_1} = \overline{B}_{1A} |_{N_1} \) with \( B_{\mu\nu} \equiv \mathcal{L}_X L_{\mu\nu} + \nabla_1 Y_{\mu\nu} \).

In that case one may take \( \hat{X} = X \) and \( \nabla_c \hat{X}^c = 4Y \). The condition (b) is not needed on the closure of those sets where \( \tau \) is non-zero, supposing that \( A_{12}|_S = 0 \).
4.2.2. The (proper) KID equations. Again, we would like to replace the non-intrinsic conditions (b), (c) and $\mathcal{Y}$ by conditions which do not involve transverse derivatives of $X$ and $Y$. For the latter two this can be done as in the light-cone case. We just note that the ODEs for $\Upsilon_N$, $a = 1, 2$, corresponding to (4.34), need to be supplemented by the boundary condition $\Upsilon_N|_S = \partial_0 \mathcal{Y}$. To replace (b) one needs to take into account that, due to (4.39), we have

$$\kappa_1 = 0 \iff A_1 = 0 = S_{211}|_N = S_{112}|_N.$$  

Furthermore, (b) and (c) imply

$$S_{12}|_S = 2V(A_{12})_ - \nabla_1 A_1 = 2V(A_{21}),$$

i.e. (d) can be replaced by the condition

$$0 = S_{12}|_S = 2V_1 X_2 - 2R_1 A^i X_i - 4V(A Y_2) + 2V_2 Y_1$$

As a direct consequence of theorem (4.12) we end up with the result:

**Theorem 4.13.** Assume we have been given a 3 + 1-dimensional space-time $(M, g, \Theta)$, with $(g, \Theta)$ a smooth solution of the conformal field equations. Let $\bar{X}$ be a vector field and $\bar{Y}$ a function defined on two transversally intersecting null hypersurfaces $N_a \subset M$, $a = 1, 2$, with transverse intersection along a smooth two-dimensional submanifold $S$. Then there exists a smooth vector field $X$ with $\bar{X} = \bar{X}$ and $\nabla_s \bar{X}^s = \bar{Y}$ satisfying the unphysical Killing equations (3.1) on $D^s (N_1 \cup N_2)$ (i.e. representing a Killing field of the physical space-time) if and only if the KID equations are fulfilled (we suppress the dependence of $D_i$ on $N_a$):

(i) $\bar{X}^\mu \nabla_\mu \Theta = \bar{Y} = 0$,

(ii) $\bar{X}^\mu \nabla_\mu \bar{X}^\nu + \bar{Y} \bar{X}^\nu D_\nu \bar{Y} - \bar{Y} \bar{X}^\nu D_\nu \bar{X}^\nu - (\Upsilon_N) \partial_0 \bar{X} = 0$,

(iii) $D_i \bar{X}^i = 0$,

(iv) $D_1 X_1 = D_2 X_2 = 0$,

(v) $2 \bar{X}^\nu \nabla_\nu X_{2i} = 2 \bar{Y} (2 D_i \bar{X}^i + \bar{X}^i D_i \bar{Y}) = 0$, $i = 2, A$,

(vi) $2 \bar{X}^\nu \nabla_\nu X_{1i} = 2 \bar{Y} (2 D_i \bar{X}^i + \bar{X}^i D_i \bar{Y}) = 0$, $i = 1, A$,

(vii) $2 \bar{Y} \bar{X}^\nu \nabla_\nu X_{3i} = 2 \bar{Y} (2 D_i \bar{X}^i + \bar{X}^i D_i \bar{Y}) + \Upsilon_N \bar{X}^\nu \nabla_\nu X_{3i} = 0$, $a = 1, 2$,

(viii) $2 \bar{X}^\nu \nabla_\nu X_{3i} = 2 \bar{Y} \bar{X}^\nu \nabla_\nu X_{3i} = 0$,

where $\Upsilon_N$ is given by $\Upsilon_N|_S = D_1 \bar{Y}$ and

$$\left( \frac{\partial_2 + \frac{T_N}{2}}{2} - \Gamma_1^1 \right) \bar{X} - \Gamma_1^1 \bar{X}^i D_i \bar{X} + \frac{1}{2} \bar{Y} \left( \bar{X}^2 D_2 \bar{X} + 2 \bar{X}^2 D_2 \bar{X} + \bar{X}^2 D_2 \bar{Y} + \frac{1}{6} \bar{X}^2 \nabla_\mu \bar{X} + \frac{1}{3} \bar{Y} \right) = 0,$$

similarly on $N_2$.

The condition (iv) is not needed on the closure of those sets on which the expansion $\tau$ is non-zero.
Proof. Once (i)–(viii) have been solved one uses the solutions $\dot{X}$ and $\dot{Y}$ as initial data for the wave equations (3.3) and (3.4). A solution exists due to [12], and the rest follows from the considerations above. □

Remark 4.14. As in [3] one could replace the condition $\mathcal{G}^{AB}D_A\dot{X}_B - 2\dot{Y} = 0$ of (iii) by certain conditions on $S$ if one makes sure that (iv) holds regardless of the (non-)vanishing of $\tau$.

Remark 4.15. Theorem 4.13 can e.g. be applied to two null hypersurfaces intersecting transversally with one of them being part of $\mathcal{F}^-$. 

5. KID equations on the light-cone $C_{i^-}$

Let us analyse now in detail the case where the initial surface is the light-cone $C_{i^-}$ with vertex at past timelike infinity $i^-$ in $3 + 1$-space-time dimensions (note that this requires a vanishing cosmological constant $\lambda$). In particular that means

$$\Theta = 0. \quad (5.1)$$

That the corresponding initial value problem is well-posed for suitably prescribed data has been shown in [4]. Our aim is to apply theorem 4.4 and analyse the KID equations in this special case.

5.1. Gauge freedom and constraint equations

To make computations as easy as possible it is useful to impose a convenient gauge condition. We will adopt the gauge scheme explained in [11, section 2.2 & 4.1], where the reader is referred to for further details. Let us start with a brief overview over the relevant gauge degrees of freedom.

The freedom to choose the conformal factor $\Theta$, regarded as an unknown in the conformal field equations (2.1)–(2.6), is comprised in the freedom to prescribe the Ricci scalar $R$ and the function $\overline{s}_{\alpha\beta}$, where the latter one needs to be the restriction to $C_{i^-}$ of a smooth function, non-vanishing at $i^-$ (which ensures $\Theta|_{\mathcal{F}^-} \neq 0$).

As above, we will choose adapted null coordinates $(\nu^0 = u, \nu^1 = r, \nu^A), A = 2, 3,$ on $C_{i^-}$. The freedom to choose coordinates off the cone is reflected in the freedom to prescribe an arbitrary vector field $\sigma^\nu$ for the $\hat{g}$-generalized wave-map gauge condition

$$H^\nu := g^{\sigma\rho}(\Gamma^\sigma_{\alpha\beta} - \hat{\Gamma}^\sigma_{\alpha\beta}) - W^\nu = 0,$$

where $\hat{g}$ denotes some target metric. The choice $W^\nu = 0$ is called wave-map gauge.

This still leaves the freedom to parameterize the null geodesics generating $C_{i^-}$, due to which it is possible to additionally prescribe the function

$$\kappa := v^0\partial_0v_0 - \frac{1}{2}\sigma - \frac{1}{2}v_0(\mathcal{G}^{\mu\nu}\Gamma^0_{\mu\nu} + W^0).$$

The choice $\kappa = 0$ corresponds to an affine parameterization. Moreover, when $H^\nu = 0$ it holds that

$$\kappa = \frac{\Theta}{\nu}|_{\nu^0}.$$ 

Henceforth we choose as in [4, 11]

$$R = 0, \quad \overline{s} = -2, \quad W^\nu = 0, \quad \kappa = 0, \quad \hat{g} = \eta. \quad (5.2)$$

where

$$\eta := -(du)^2 + 2du dr + r^2s_{AB}d\nu^A d\nu^B$$
denotes the Minkowski metric in adapted null coordinates, with \( s = s_{AB} \, dx^A \, dx^B \) being the standard metric on \( S^2 \).

Let us assume we have been given a smooth solution \((g, \Theta)\) of the conformal field equations (2.1)–(2.6) in the \((R = 0, \bar{s} = -2, \kappa = 0, \bar{g} = \eta)\)-wave-map gauge.\(^3\) It is shown in [11, section 4] that then the following equations are valid on \( C^-\),
\[
\bar{g}_{\mu\nu} = \eta_{\mu\nu}, \quad \bar{T}_{1\mu} = 0, \quad \bar{T}_{AB} = 0, \quad \bar{T}_{0A} = \frac{1}{2} \nabla_B \lambda_A^B,
\]
(5.3)
\[
\frac{\partial_1 \Theta}{\delta_{\delta_{\partial_1}}} = -2r, \quad \frac{\partial_1 \bar{g}_{1\mu}}{\delta_{\delta_{\partial_1}}} = 0,
\]
(5.4)
\[
\tau = \frac{2}{r}, \quad \xi_A := -2\bar{T}_{1A} = 0, \quad \zeta := 2\bar{g}^{AB} \bar{T}_{1A}^B + \tau = -2/r,
\]
(5.5)
\[
(\partial_1 - r^{-1}) \lambda_{AB} = -2\omega_{AB}, \quad \lambda_A^A = \omega_A^A = 0,
\]
(5.6)
where \( \lambda_{AB} := \frac{\partial_1 \bar{g}_{AB}}{\delta_{\delta_{\partial_1}}} = O(r^3) \). The operator \( \tilde{\nabla} \) denotes the Levi-Civita connection of \( \bar{g} := \bar{g}_{AB} dx^A dx^B \). The \( s \)-trace-free tensor \( \omega_{AB} = O(r^3) \) may be regarded as representing the free initial data in the corresponding characteristic initial value problem [4, 11].

For convenience we give a list of the Christoffel symbols in adapted null coordinates on \( C^-\), which are easily obtained from (5.3)–(5.6) and the formulae in [2, appendix A],
\[
\begin{align*}
\tilde{\Gamma}_{00}^0 &= \tilde{\Gamma}_{ij}^j = \tilde{\Gamma}_{11}^j = \bar{\Gamma}_{00j} = \tilde{\Gamma}^j_{00} = \bar{\Gamma}^j_{10} = 0, \\
\tilde{\Gamma}^i_{00} &= \frac{1}{2} \partial_0 \delta_{00}, \\
\tilde{\Gamma}^c_{00} &= \bar{g}^{CD} \partial_0 \delta_{00}, \\
\tilde{\Gamma}^0_{AB} &= -\frac{1}{r} \bar{g}_{AB}, \\
\tilde{\Gamma}^c_{1A} &= \frac{1}{r} \delta_A^C,
\end{align*}
\]
(5.5)
\[
\tilde{\Gamma}^c_{0A} = \frac{1}{2} \lambda_A^C, \\
\tilde{\Gamma}^1 = -\frac{1}{r} \bar{g}_{AB} - \frac{1}{2} \lambda_{AB}, \\
\tilde{\Gamma}^c_{AB} = \tilde{\Gamma}^c_{AB} = S^c_{AB}.
\]

5.2. Analysis of the KID equations

5.2.1. The conditions \( \tilde{\phi} = 0, \tilde{\phi}_{\text{aux}} = 0, \tilde{\phi}_j = 0 \) and \( \tilde{\phi}_{10} = 0 \). With \( \tilde{\phi} = 0 \) and \( \partial_0 \tilde{\phi} = -2r \) it immediately follows that (recall that \( \phi \) has been defined in theorem 4.1)
\[
\tilde{\phi} = 0 \iff \tilde{X}^0 = 0,
\]
(5.7)
i.e. any vector field satisfying the unphysical Killing equations necessarily needs to be tangent to \( C^- \).

Taking further into account that \( \bar{s} = -2 \) and \( v_0 = 1 \) we obtain (recall that \( \tilde{\phi}_{\text{aux}} \) has been defined in theorem 4.4)
\[
\tilde{\phi}_{\text{aux}} = 0 \iff (\partial_1 - r^{-1}) \tilde{\phi} = 0 \iff \tilde{\phi} = c(x^4) r,
\]
(5.8)
for some angle-dependent function \( c \). The condition \( \tilde{A}_{11} = 0 \) is then automatically fulfilled. Furthermore, one readily checks that (we denote by \( \tilde{\nabla} \) the Levi-Civita connection associated to the standard metric on \( S^2 \))
\[
\begin{align*}
\tilde{\Gamma}^0_{1A} &= 0 \iff \partial_1 \tilde{X}^A = 0 \iff \tilde{X}^A = d^A(\tilde{x}^B),
\end{align*}
\]
(5.9)
\[
\begin{align*}
\bar{g}^{AB} \tilde{A}_{1B} &= 0 \iff \tilde{X}^1 = -\frac{1}{2} r \partial_A d^A + cr^2,
\end{align*}
\]
(5.10)
\[
\tilde{A}_{1B} = 0 \iff d^A \text{ is a conformal Killing field on } (S^2, s_{AB}).
\]
(5.11)
Here and in what follows \( \tilde{\cdot} \) denotes the \( s_{AB} \)- (equivalently the \( \bar{g}_{AB} \)-) trace-free part of the corresponding rank-2 tensor field.

Since \( \tau = 2/r > 0 \) the condition \( \tilde{S}_{10} = 0 \) holds automatically for all \( r > 0 \).

\(^3\) In fact it is not necessary here to require the rescaled Weyl tensor to be regular at \( r^- \).
5.2.2. The conditions $\mathcal{B}_{ii} = 0$ and $\mathcal{B}_{ij}^{\text{intr}} = 0$. First we solve (4.40) for $Y$, which in our gauge becomes

$$(\partial_t + r^{-1}) Y + \frac{1}{2} r^{-2} \Delta_t \hat{Y} + r^{-1} \partial_t \hat{Y} = 0,$$  
(5.12)

where we have set $\Delta_t := s^{AB} \mathcal{P}_A \mathcal{P}_B$. With $\hat{Y} = c r$ and $Y = O(1)$ we obtain as the unique solution of (5.12)

$$Y = -\frac{1}{2} (\Delta_t + 2 c).$$  
(5.13)

For $\mathcal{B}_{ii}$ we find

$$\mathcal{B}_{11} = \hat{X}^\alpha \nabla_\alpha \hat{L}_{11} + 2 \hat{L}_{(1)} \hat{X}^\alpha + D_1 D_1 \hat{Y} = 0,$$
$$\mathcal{B}_{1A} = \hat{X}^\alpha \nabla_\alpha \hat{L}_{1A} + 2 \hat{L}_{(A)} \hat{X}^\alpha + D_1 D_A \hat{Y} = 0,$$
$$\mathcal{B}_{0A} = \hat{X}^\alpha \nabla_A \hat{L}_{0A} + \omega_{AB} \partial_1 \hat{X}^A + \partial_A (\partial_t - r^{-1}) \hat{Y} = 0,$$

without any further restrictions on $\hat{X}, \hat{Y}$ or the initial data $\omega_{AB}$. It remains to determine $\mathcal{B}_{AB}^{\text{intr}},$

$$\mathcal{B}_{AB}^{\text{intr}} = \hat{X}^\alpha \nabla_\alpha \hat{L}_{AB} + 2 \hat{L}_{(A)} \hat{D}_B \hat{X}^\alpha + 2 \hat{L}_{(C)} \hat{D}_B \hat{X}^C + D_A D_B \hat{Y} + r^{-1} \hat{L}_{AB} Y = 0,$$

We first compute its trace,

$$\mathcal{B}_{AB}^{\text{intr}} = 2 \omega_{AB} (\hat{V}_A \hat{X}_B) + \Delta_t \hat{Y} + 2 r^{-1} \partial_t \hat{Y} + 2 r^{-1} Y = 0,$$

again without any further restrictions. For its traceless part we find

$$\mathcal{B}_{AB}^{\text{intr}} = \hat{X}^\alpha \partial_1 \omega_{A\beta} + \hat{X}^\alpha \nabla_\alpha \hat{V}_{C\beta} \omega_{AB} + 2 \omega_{C(A} \hat{V}_{B)} \hat{X}^C - \hat{L}_{AB} \omega^{CD} (\nabla_C \hat{X}_D Y) + (\nabla_A \hat{V}_B) \hat{Y} + \frac{1}{2} \lambda_{AB} \partial_t \hat{Y}$$

$$= \mathcal{L}_d \omega_{AB} - \frac{1}{2} r \partial_t \omega_{AB} \mathcal{P}_C d^C + cr^2 \partial_1 \omega_{AB} + \frac{1}{2} c \lambda_{AB} + r (\mathcal{P}_A \mathcal{P}_B c).$$

Recall that regularity of the metric requires $\omega_{AB} = O(r^2)$ and $\lambda_{AB} = O(r^3)$, in particular $\mathcal{L}_d \omega_{AB} = O(r^2)$. Hence $\mathcal{B}_{AB}^{\text{intr}} = 0$ if and only if

$$\hat{V}_A$$

is a conformal Killing field on $(S^2, s_{AB} \, dx^A \, dx^B)$.

$$\mathcal{L}_d \omega_{AB} - \frac{1}{2} r \partial_1 \omega_{AB} \mathcal{P}_C d^C + cr^2 \partial_1 \omega_{AB} + \frac{1}{2} c \lambda_{AB} = 0.$$  
(5.15)

5.2.3. Summary. By way of summary the conditions (a)–(f) in theorem 4.4 hold if and only if

$$\dot{X}^0 = 0,$$  
(5.16)

$$\dot{X}^A = d^A,$$  
(5.17)

$$\dot{X}^1 = -\frac{1}{2} r \mathcal{P}_A d^A + cr^2,$$  
(5.18)

$$\dot{Y} = cr,$$  
(5.19)

such that

$$\mathcal{P}_A c$$

and $d_A$ are conformal Killing fields on $(S^2, s_{AB} \, dx^A \, dx^B)$.

$$\mathcal{L}_d \omega_{AB} - \frac{1}{2} r \mathcal{P}_A d^A \partial_1 \omega_{AB} + cr^2 \partial_1 \omega_{AB} + \frac{1}{2} c \lambda_{AB} = 0.$$  
(5.20)
In section 4.1.4 we have shown that solutions of the KID equations are restrictions to the light-cone of smooth space-time fields. On $C_\Gamma$ this turns out to be a trivial issue anyway: the candidate fields satisfying (5.16)–(5.20) are explicitly known\(^4\) and coincide independently of the choice of initial data $\omega_{AB}$, with the restriction to $C_\Gamma$ of the Minkowskian Killing fields.

While in the Minkowski case $\omega_{AB} = 0$ every candidate field does extend to a Killing vector field, equation (5.21) provides an obstruction equation for non-flat data. We call (5.21) the reduced KID equations.

As a corollary of theorem 4.4 we obtain:

**Theorem 5.1.** Assume that we have been given a $3+1$-dimensional ‘unphysical’ space-time $(\mathcal{M}, g, \Theta)$ which contains a regular $C_\Gamma$-cone (the cosmological constant $\lambda$ thus needs to vanish) and where $(g, \Theta)$ is a smooth solution of the conformal field equations in the $(R = 0, \mathcal{L} = -2, \kappa = 0, \tilde{g} = \eta)$-wave-map gauge. Then there exists a smooth vector field $X$ satisfying the unphysical Killing equations (3.1) on $\mathcal{M} - (C_\Gamma)$ (i.e. representing a Killing field of the physical space-time) if and only if there exist a function $c$ and a vector field $dA$ on $S^2$ with $\mathcal{D}_{AB}$ and $dA$ conformal Killing fields on $(S^2, s_{AB} \, dx^A \, dx^B)$ such that the reduced KID equations

$$\mathcal{L}_{dA} \lambda_{AB} - \frac{1}{2} \mathcal{L}_{\omega_{AB}} \mathcal{D}_{cd} c - cr^2 \partial_t \lambda_{AB} + \frac{1}{4} \mathcal{D}_{cd} c - 2cr \lambda_{AB} = 0$$

(5.22)

are satisfied on $C_\Gamma$ (recall that $\lambda_{AB}$ is the unique solution of $(\partial_1 - r^{-1}) \lambda_{AB} = -2\omega_{AB}$ with $\lambda_{AB} = O(r^3)$).

The Killing field satisfies

$$\overline{X}^0 = 0, \quad \overline{X}^A = d^A, \quad \overline{X}^1 = -\frac{1}{2} r \mathcal{D}_A d^A + cr^2, \quad \overline{\nabla}_\mu \overline{X}^\nu = 4cr.$$  

(5.23)

**Remark 5.2.** The reduced KID equations (5.22) can be replaced by one of their equivalents (i)–(iii) in lemma 5.3.

5.3. Analysis of the reduced KID equations

5.3.1. Equivalent representations of the reduced KID equations. We provide some alternative formulations of the reduced KID equations.

**Lemma 5.3.** The reduced KID equations (5.22) are equivalent to each of the following equations:

(i) $\mathcal{L}_d \lambda_{AB} - (\frac{1}{4} \mathcal{D}_{cd} c - cr^2) \partial_t \lambda_{AB} + (\frac{1}{4} \mathcal{D}_{cd} c - 2cr) \lambda_{AB} = 0$,

(ii) $(\partial_1 - r^{-1}) \mathcal{L}_d \omega_{AB} - \frac{1}{2} \mathcal{L}_{\omega_{AB}} \mathcal{D}_{cd} c + cr^2 \partial_t (\partial_1 + r^{-1}) \omega_{AB} = 0$,

(iii) $2 \mathcal{L}_d \mathcal{L}_{\omega_{AB}} + (1 - r^2) \mathcal{L}_{\omega_{AB}} \mathcal{D}_{cd} c + r\omega_{AB} \mathcal{D}_{cd} c - 2cr^2 \partial_t \mathcal{L}_{\omega_{AB}} - (2\omega_{AB} + r^{-1} \lambda_{AB}) \mathcal{D}_{cd} c = 0$

(recall that $\mathcal{L}_{\omega_{AB}} = \frac{1}{2} \mathcal{D}_{\mu} \omega_{AB}^\mu$).

**Proof.** (i) Applying $(\partial_1 - r^{-1})$ to (i) yields (5.22), equivalence follows from regularity.

(ii) Applying $(\partial_1 - r^{-1})$ to (5.22) yields (ii), equivalence follows from regularity.

(iii) We use the fact that on $(S^2, s_{AB} \, dx^A \, dx^B)$ the equations $\omega_{AB} = 0$ and $\mathcal{D}_{cd} c = 0$ with $w_{AB}$ trace-free are equivalent. Taking the divergence of (i) and invoking the conformal Killing field $d^A$ complete the equivalence proof.

Both $\omega_{AB}$ or $\lambda_{AB}$ may be regarded as the freely prescribable initial data. So (i) and (ii) in lemma 5.3 provide formulations of the reduced KID equations which involve exclusively explicitly known quantities for all admissible initial data. In the case of an ordinary cone, treated in [3], this is not possible: For generic KIDs there, neither the candidate fields nor all the relevant metric components can be computed analytically.

\(^4\) The function $c$ satisfies the equation $\mathcal{D}_{\lambda} (\Delta_\lambda + 2)c = 0$ and can thus be written as linear combination of $\ell = 0, 1$ spherical harmonics. Conformal Killing fields on the round 2-sphere are discussed in appendix B.
5.3.2. Some special cases. We continue with a brief discussion of some special cases. There exists a vector field $X$ satisfying the unphysical Killing equations (3.1) on $D^+$ ($C^-$) with

1. $\nabla_\mu \hat{X}^\mu = 0 \iff \exists$ a conformal Killing field $d^A$ on $(S^2, s_{AB} dx^A dx^B)$ with $\mathcal{L}_{d^A} \omega_{AB} = \frac{1}{2} r^2 c_{\mu AB} \partial c d^\mu$,

2. $\hat{X}^i = 0 \iff \exists$ a Killing field $d^A$ on $(S^2, s_{AB} dx^A dx^B)$ with $\mathcal{L}_{d^A} \omega_{AB} = 0$,

3. $\hat{X}^i = 0 \iff \partial_i (\partial_1 + r^{-1}) \omega_{AB} = 0 \iff \omega_{AB} = O(r^2)$ is a necessary condition for the Schouten tensor to be regular at $i^\pm$.

The third case shows that the property $\hat{X}^A = 0$ is compatible only with the Minkowski case (supposing that $i^\pm$ is a regular point). In the non-flat case any non-trivial vector field satisfying the unphysical Killing equations has a non-trivial component $\hat{X}^A = d^A \neq 0$. Since

$$\hat{g}_{\mu \nu} \hat{X}^\mu \hat{X}^\nu = \hat{g}_{AB} \hat{X}^A \hat{X}^B = r^2 s_{AB} d^A d^B,$$

we see that there are no non-trivial vector fields satisfying the unphysical Killing equations which are null on $C^-$. To put it differently, possibly apart from certain directions determined by the zeros of $d^A$, any isometry of a non-flat, asymptotically flat vacuum space-time is necessarily spacelike sufficiently close to $\mathcal{I}^{}$. This leads to the following version of a classical result of Lichnerowicz [9]:

**Theorem 5.4.** Minkowski space-time is the only stationary vacuum space-time which admits a regular $C^-$-cone.

5.3.3. Structure of the solution space. Let $X$ and $\hat{X}$ be two distinct non-trivial solutions of the unphysical Killing equations (3.1). Since solutions of these equations form a Lie algebra, $\hat{X} := [X, \hat{X}]$ is another, possibly trivial solution. We have

$$\hat{X}^0 = [X, \hat{X}]^0 = 0,$$

$$\hat{X}^A = [X, \hat{X}]^A = [d, \hat{d}]^A,$$

$$\hat{X}^i = [X, \hat{X}]^i = -\frac{1}{r} \mathcal{D}_B [d, \hat{d}]^B + r^2 (d^B \mathcal{D}_B \hat{c} - \hat{d}^B \mathcal{D}_B + \frac{1}{2} c \mathcal{D}_B \hat{d}^B - \frac{1}{2} \hat{c} \mathcal{D}_B \hat{d}^B).$$

Hence, by their derivation, the reduced KID equations are fulfilled with

$$\hat{d}^A = [d, \hat{d}]^A,$$

$$\hat{c} = d^B \mathcal{D}_B \hat{c} - \hat{d}^B \mathcal{D}_B + \frac{1}{2} c \mathcal{D}_B \hat{d}^B - \frac{1}{2} \hat{c} \mathcal{D}_B \hat{d}^B,$$

and $\hat{d}^A$ and $\mathcal{D}_A \hat{c}$ are conformal Killing fields on the standard 2-sphere. Indeed, via the relation $\mathcal{L}_{[d, \hat{d}]} \lambda_{AB} = [\mathcal{L}_d, \mathcal{L}_{\hat{d}}] \lambda_{AB}$, this can be straightforwardly checked. We refer the reader to appendix B where the conformal Killing fields on the standard 2-sphere are explicitly given.

Let us consider for the moment flat initial data $\lambda_{AB} = 0$ which generate Minkowski space-time. Then one has ten independent isometries.

- The four translations are generated by the tuples $(c, d^A = 0)$ with $c$ being a spherical harmonic function of degree $\ell = 0$ or 1.
- The three rotations are generated by the tuples $(c = 0, d^A)$ with $d^A$ being a Killing field on $(S^2, s_{AB} dx^A dx^B)$.
- The three boosts are generated by the tuples $(c = 0, d^A = \mathcal{D}_A f)$ with $f$ being a spherical harmonic function of degree $\ell = 1$.

We have already seen above that translations (in the above sense) cannot exist in the non-flat case $\lambda_{AB} \neq 0$ if the Schouten tensor is assumed to be regular at $i^\pm$. 

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**Proposition 5.5.** Minkowski space-time is the only vacuum space-time with a regular $C^\infty$-cone which admits translational Killing vector fields.

This is linked with another observation. Since, in the non-flat case, any non-trivial Killing field of the physical space-time (i.e. a vector field satisfying the unphysical Killing equations) has a non-trivial $d^A$, for a given $d^A$ there can be at most one $c$ such that $(c, d^A)$ solves the reduced KID equations. Now the standard 2-sphere admits six independent conformal Killing vector fields $d^A$. We thus have:

**Proposition 5.6.** Any non-flat vacuum space-time with a regular $C^\infty$-cone admits at most six independent Killing vector fields.

Now let us assume that there are two distinct rotations, i.e. two Killing fields $d^{(1)}$ and $d^{(2)}$ on $(S^2, s_{AB} dx^a dx^b)$ such that $(c = 0, d = d^{(i)})$, $i = 1, 2$, solves the reduced KID equations. Then $(c = 0, d = d^{(3)})$ with $d^{(3)} = [d^{(1)}, d^{(2)}]$ provides another independent, non-trivial solution of the reduced KID equations. Altogether we have

\[ \mathcal{L}_{d^A} \lambda_{AB} = 0, \quad i = 1, 2, 3 \implies \lambda_{AB} \propto s_{AB} \implies \lambda_{AB} = 0, \]

since $\lambda_{AB}$ is trace-free. This recovers the well-known fact that two rotational symmetries imply Minkowski space-time.

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**Appendix A. Fuchsian ODEs**

As we have not been able to find an adequate reference, we state and prove here a key result about Fuchsian ODEs which is used in our work.

**Lemma A.1.** Let $a > 0$, and for $r \in (0, a)$ consider a first-order ODE-system of the form

\[ \partial_r \phi = r^{-1} A \phi + M(r) \phi, \quad (A.1) \]

for a set of fields $\phi = (\phi^I)$, $I = 1, \ldots, N$, where $A$ is an $N \times N$-matrix, and where $M(r)$ is a continuous map on $[0, a)$ with values in $N \times N$-matrices which satisfies $r \|M(r)\|_{op} = o(1)$. Let $\lambda$ denote the smallest number so that

\[ \langle \phi, A \phi \rangle \leq \lambda \| \phi \|^2. \]

Suppose that there exists $\epsilon > 0$ such that

\[ \phi = O(r^{\lambda + \epsilon}). \]

Then

\[ \phi \equiv 0. \]

**Proof.** The proof is done by a simple energy estimate. Set

\[ \langle \phi, \psi \rangle := \sum_I \phi^I \psi^I, \quad \| \phi \|^2 := \langle \phi, \phi \rangle, \]

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then for any $k \in \mathbb{R}$
\[
\varphi(r-2k\|\phi\|^2) = 2r^{-2k}\phi \partial_r \phi - 2kr^{-2k-1}\|\phi\|^2 \\
= 2r^{-2k-1}(\phi, A\phi) + r \langle \phi, M(r)\phi \rangle - k\|\phi\|^2 \\
\leq 2r^{-2k-1}(\lambda - k + r\|M(r)\|_{op})\|\phi\|^2.
\]
Applying $\int_{r_0}$ yields (assume $r_0 < r$)
\[
\varphi(r-2k\|\phi\|^2) \leq \int_{r_0}^{r} (\lambda - k + \tilde{r}\|M(\tilde{r})\|_{op})\tilde{r}^{-2k-1}\|\phi\|^2 d\tilde{r} \\
\leq \int_{r_0}^{r} (\lambda - k + \sup_{0 < \tilde{r} < r} (\tilde{r}\|M(\tilde{r})\|_{op}))\tilde{r}^{-2k-1}\|\phi\|^2 d\tilde{r}.
\]
Due to our assumption $\phi = O(r^{1+\epsilon})$ any $\lambda < k_0 < \lambda + \epsilon$ satisfies $r^{-2k_0}\|\phi\|^2 = O(r^{2\epsilon})$, where $\delta := \lambda - k_0 + \epsilon > 0$. We then take the limit $r_0 \to 0$,
\[
\varphi(r-2k_0\|\phi\|^2) \leq \int_{0}^{r} (\lambda - k_0 + \sup_{0 < \tilde{r} < r} (\tilde{r}\|M(\tilde{r})\|_{op}))\tilde{r}^{-2k_0-1}\|\phi\|^2 d\tilde{r} \\
\leq 0 \quad \text{for sufficiently small } r.
\]
Thus $\phi$ vanishes for small $r$, but then it needs to vanish for all $r$. \hfill \Box

**Appendix B. Conformal Killing fields on the round 2-sphere**

We consider the 2-sphere equipped with the standard metric
\[
s = s_{AB} dx^A dx^B = d\theta^2 + \sin^2 \theta \, d\phi^2.
\]
It admits the maximal number of independent conformal Killing vector fields, which is 6. There are three independent Killing vector fields,
\[
K_{(1)} = \partial_\theta, \\
K_{(2)} = \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi, \\
K_{(3)} = \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi,
\]
and three independent conformal Killing fields which are not Killing fields,
\[
C_{(1)} = \sin \theta \partial_\theta, \\
C_{(2)} = \cos \theta \cos \phi \partial_\theta - \sin^{-1} \theta \sin \phi \partial_\phi, \\
C_{(3)} = \cos \theta \sin \phi \partial_\theta + \sin^{-1} \theta \cos \phi \partial_\phi.
\]
All the $C_{(i)}$’s turn out to be gradients of $\ell = 1$-spherical harmonics,
\[
C_{(1)}^A = \mathcal{D}^A c_{(1)}, \quad \text{where} \quad c_{(1)} = \cos \theta, \\
C_{(2)}^A = \mathcal{D}^A c_{(2)}, \quad \text{where} \quad c_{(2)} = \sin \theta \cos \phi, \\
C_{(3)}^A = \mathcal{D}^A c_{(3)}, \quad \text{where} \quad c_{(3)} = \sin \theta \sin \phi.
\]
Moreover,
\[
\mathcal{D}_A C_{(i)}^A = \Delta_A c_{(i)} = -2c_{(i)}, \quad i = 1, 2, 3.
\]
The conformal Killing fields satisfy the commutation relations
\[
[K_{(i)}, K_{(j)}] = \sum_k \varepsilon_{ijk} K_{(k)}, \\
[C_{(i)}, C_{(j)}] = -\sum_k \varepsilon_{ijk} K_{(k)}, \\
[K_{(i)}, C_{(j)}] = \sum_k \varepsilon_{ijk} C_{(k)},
\]

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i.e. they form a Lie algebra isomorphic to the Lie algebra $so(3, 1)$ of the Lorenz group in four dimensions. The Killing fields form a Lie subalgebra.

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