The classical and quantum formalism for a $p$-adic and adelic harmonic oscillator with time-dependent frequency is developed, and general formulae for main theoretical quantities are obtained. In particular, the $p$-adic propagator is calculated, and the existence of a simple vacuum state as well as adelic quantum dynamics is shown. Space discreteness and $p$-adic quantum-mechanical phase are noted.

1. Introduction

In quantum-mechanical experiments, as well as in all measurements, numerical results belong to the field of rational numbers $\mathbb{Q}$. In principle, the corresponding theoretical models could be made using only $\mathbb{Q}$, but it would missed usual effectiveness and beauty of mathematical analysis. So, instead of $\mathbb{Q}$ one traditionally applies the field of real numbers $\mathbb{R}$ in classical mechanics and the field of complex numbers $\mathbb{C}$ in quantum mechanics. $\mathbb{R}$ is completion of $\mathbb{Q}$ with respect to the metric induced by the absolute value and $\mathbb{C}$ is an algebraic extension of $\mathbb{R}$. In addition to $\mathbb{R}$ there exist the fields of $p$-adic numbers $\mathbb{Q}_p$ as completions of $\mathbb{Q}$ with respect to $p$-adic norms ($p$ = a prime number) [1]. According to the Ostrowski theorem, $\mathbb{R}$ and $\mathbb{Q}_p$ (for every $p$) exhaust all possible completions of $\mathbb{Q}$. Thus $\mathbb{Q}$ is dense not only in $\mathbb{R}$ but also in each $\mathbb{Q}_p$. Therefore, in the last decade there has been a lot and successful interest in construction of theoretical models with $p$-adic numbers (for a review, see, Refs. 2-5).

There is a common belief that none separated prime number $p$ plays a special role in physics and that $p$-adic models have to be taken together for all primes. It is clear that $p$-adic models, having some physical meaning, must be somehow connected with the ordinary (real) ones. The space of adeles [6] $\mathbb{A}$ is a mathematical instrument which enables us to consider real and $p$-adic numbers simultaneously and as a whole. Thus it is natural to expect that adelic approach provides a more complete description of a physical system than the ordinary one.
\[ p \text{-Adic numbers exhibit ultrametric (non-archimedean) properties, which may be realized in quantum systems at very short distances. Possibility that space-time at the Planck scale exhibits } p \text{-adic and adelic structure is one of the main physical motivations to investigate the corresponding models.} \]

In order to start with a systematic approach to \( p \)-adic models of quantum systems, \( p \)-adic quantum mechanics \([7,8]\) was formulated. Quantization is done along the Weyl procedure. The corresponding Hilbert space \( L_2(\mathbb{Q}_p) \) contains complex-valued square integrable functions on \( \mathbb{Q}_p \). Instead of the Schrödinger equation, the dynamical evolution and the spectral problem of a system are related to the unitary representation of the evolution operator \( U_p(t) \) on \( L_2(\mathbb{Q}_p) \). As a generalization and unification of \( p \)-adic and ordinary quantum mechanics, recently was formulated adelic quantum mechanics \([9]\).

So far a rather small number of physical systems has been treated in \( p \)-adic and adelic quantum mechanics: a non-relativistic free particle \([7]\), a harmonic oscillator \([7,9]\), a particle in a constant field \([8]\), the de Sitter minisuperspace model of the universe \([10]\) and a relativistic free particle \([11]\). It is doubtless that evaluation of some other physical systems, which exhibit \( p \)-adic and adelic properties, will give new insights into this subject and new directions for future investigations at the Planck scale.

In this paper we show existence and some properties of \( p \)-adic and adelic harmonic oscillator with time-dependent frequency (HOTDF). Model of the HOTDF has vast applications from quantum optics \([12]\) to quantum cosmology \([13]\). Nevertheless, many properties of classical and quantum motion can be found without specifying the time dependence of \( \omega(t) \).

### 2. \( p \)-Adic numbers and adeles

To make this paper more self contained, we give here a very short review of some basic facts on \( p \)-adic numbers and adeles.

Any rational number \( x \neq 0 \) can be presented as \( x = p^{\nu} \frac{m}{n} \), where \( \nu, m, n \in \mathbb{Z} \) and \( p \) is a given prime number which divides neither \( m \) nor \( n \). By definition, \( p \)-adic norm of \( x \) is

\[ |x|_p = p^{-\nu}, \quad |0|_p = 0, \quad (2.1) \]

and holds the strong triangle inequality:

\[ |x + y|_p \leq \max(|x|_p, |y|_p). \quad (2.2) \]

A norm (valuation) with the property (2.2) is called non-archimedean or ultrametric norm. Every \( p \)-adic number \( x \) can be uniquely presented by the canonical expansion

\[ x = p^\nu \sum_{i=0}^{+\infty} x_ip^i, \quad x_i \in \{0, 1, ..., p - 1\}, \quad x_0 \neq 0, \quad \nu \in \mathbb{Z} \quad (2.3) \]
The expansion (2.3) is convergent with respect to the metric induced by \( p \)-adic norm, i.e. 
\[ d_p(x, y) = |x - y|_p. \]

There are mainly two kinds of analysis on \( \mathbb{Q}_p \) based on two different maps: \( \mathbb{Q}_p \to \mathbb{Q}_p \) and \( \mathbb{Q}_p \to \mathbb{C} \). We use both of these analyses.

Elementary \( p \)-adic functions, like \( \exp x \), \( \sin x \) and \( \cos x \) are given by series of the same form as in the real case. However, the region of convergence is rather restricted and it is \( |x|_p < |2|_p \) for the above functions. Derivatives of \( p \)-adic valued functions are defined as in the real case, but using \( p \)-adic norm instead of the absolute value.

For complex-valued functions of \( p \)-adic argument there is well-defined integration with the Haar measure. In particular, we use the Gauss integral [3]

\[
\int_{|x|_p \leq p^{-\nu}} \chi_p(\alpha x^2 + \beta x)dx = \begin{cases} 
p^{-\nu} \Omega(p^{-\nu}|\beta|_p), & |\alpha|_p \leq p^{-2\nu}, \\
\lambda_p(\alpha)|2\alpha|^{-1/2}_p \chi_p\left(-\frac{\beta^2}{4\alpha}\right) \Omega(p^{-\nu}|\beta|_p), & |4\alpha|_p > p^{-2\nu}.
\end{cases} (2.4)
\]

\( \chi_p(u) = \exp(2\pi i \{u\}_p) \) is a \( p \)-adic additive character, where \( \{u\}_p \) denotes the fractional part of \( u \in \mathbb{Q}_p \). \( \lambda_p(\alpha) \) is an arithmetic complex-valued function with the following basic properties [3]:

\[
\lambda_p(0) = 1, \quad \lambda_p(a^2\alpha) = \lambda_p(\alpha), \quad \lambda_p(\alpha)\lambda_p(\beta) = \lambda_p(\alpha + \beta)\lambda_p(\alpha^{-1} + \beta^{-1}), \quad |\lambda_p(\alpha)|_\infty = 1. \quad (2.5)
\]

\( \Omega(|u|_p) \) is the characteristic function on \( \mathbb{Z}_p \), i.e.

\[
\Omega(|u|_p) = \begin{cases} 
1, & |u|_p \leq 1, \\
0, & |u|_p > 1,
\end{cases} (2.6)
\]

where \( \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} \) is the ring of \( p \)-adic integers.

An adele [6] \( a \in \mathbb{A} \) is an infinite sequence

\[
a = (a_\infty, a_2, \ldots, a_p, \ldots), \quad (2.7)
\]

where \( a_\infty \in \mathbb{R} \) and \( a_p \in \mathbb{Q}_p \) with the restriction that \( a_p \in \mathbb{Z}_p \) for all but a finite set \( S \) of primes \( p \). The set of all adeles \( \mathbb{A} \) can be written in the form

\[
\mathbb{A} = \bigcup_S \mathcal{A}(S), \quad \mathcal{A}(S) = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p. \quad (2.8)
\]

\( \mathbb{A} \) is a topological space. It is a ring with respect to componentwise addition and multiplication. There is a natural generalization of analysis on \( \mathbb{R} \) and \( \mathbb{Q}_p \) to analysis on \( \mathbb{A} \).

3. Classical oscillator: real, \( p \)-adic and adelic case
Classical HOTDF is given by the Lagrangian

\[ L(x, \dot{x}, t) = \frac{m}{2} \dot{x}^2 - \frac{m\omega^2(t)}{2} x^2, \]  

(3.1)

where \( m \in \mathbb{Q} \). Time-dependent frequency \( \omega(t) = \sum_{n \geq 0} \omega_n t^n \), where \( \omega_n \in \mathbb{Q} \), is assumed to be an analytic function on \( \mathbb{D}_\infty \subset \mathbb{R} \) and on \( \mathbb{D}_p \subset \mathbb{Z}_p \) for all \( p \). In other words, when \( t \in \mathcal{A}(S) \) then \( \omega(t) \in \mathcal{A}(S') \), where \( S \) and \( S' \) are some finite sets of primes \( p \). In the real case \( m, x, \dot{x}, t, \omega(t) \in \mathbb{R} \equiv \mathbb{Q}_\infty \) (in the sequel index \( \infty \) denotes quantities defined on \( \mathbb{R} \) or \( \mathbb{C} \)) and the analogous situation is for the \( p \)-adic counterparts. Because of formal similarity of analyses, evaluation of (3.1) is the same in real and \( p \)-adic dynamics. Thus, in real and \( p \)-adic cases, the equation of motion is

\[ \ddot{x}(t) + \omega^2(t) x(t) = 0 \]  

(3.2)

with general solution [14]

\[ x(t) = G(t)[C_1 \cos \gamma(t) + C_2 \sin \gamma(t)]. \]  

(3.3)

The amplitude \( G(t) \) and phase \( \gamma(t) \) satisfy equations

\[ G^3(t) \dot{G}(t) + \omega^2(t) G(t) = C^2, \quad \dot{\gamma}(t) G(t) = C, \]  

(3.4)

where \( C \) is a constant (\( 0 < C \in \mathbb{R}, \quad C \in 1 + p\mathbb{Z}_p \)) and can be taken \( C = 1 \). We are interested in analytic solution of (3.3), where \( G(t) \) and \( \gamma(t) \) are power series in \( t \) with rational coefficients. Differential equation for \( G(t) \) is non-linear. However, it does not lead to non-linear algebraic equations for unknown coefficients \( G_n \) in expansion \( G(t) = \sum_{n \geq 0} G_n t^n \) and any \( G_n \) can be presented as a rational number, which is the same in the real and all \( p \)-adic cases. Note that usual power series with rational coefficients which are convergent on \( \mathbb{D}_\infty \subset \mathbb{R} \) in the real case are also \( p \)-adically convergent in some region \( \mathbb{D}_p \subset \mathbb{Z}_p \).

As an illustration of analytic solutions of the equations (3.4) (with \( C = 1 \)) we present two simple examples.

**Example 1.** Let \( \omega(t) = \omega_0/(1 + at)^2 \), where \( \omega_0 = b^{-2} \) and \( a, b \in \mathbb{N} \). Then

\[ G(t) = b(1 + at), \quad \gamma(t) = \frac{1}{b^2} \frac{t}{1 + at}. \]

Since \( \gamma(t) \) is argument of trigonometric functions in (3.3) one obtains that common region of convergence for all analytic expansions is \( |t|_p < 2b^2 |p| \) for each \( p \).

**Example 2.** Let \( \omega(t) = \omega_0/(1 + at) \), where \( \omega_0 = b^{-2}(1 + a^2 b^4/4)^{1/2} \) and \( a, b \in 2\mathbb{N} \). Then in an analogous way to the Example 1 we get:

\[ G(t) = b(1 + at)^{1/2}, \quad \gamma(t) = \frac{1}{ab^2} \ln(1 + at), \quad \) |t|_p < 2b^2 |p| \) .
Thus, there exist non-trivial adelic solutions for \( G(t) \), and \( \gamma(t) \), and consequently for \( x(t) \) in the form (3.3).

To determine constants \( C_1 \) and \( C_2 \) we use two kinds of conditions on the classical trajectory.

**3.1 Solution with the end point conditions**

The classical trajectory that links two space-time points \( (x', t') \) and \( (x'', t'') \) is

\[
x(t) = \frac{G(t)}{\sin(\gamma'' - \gamma')} \left[ \frac{x'}{G'} \sin(\gamma'' - \gamma(t)) + \frac{x''}{G''} \sin(\gamma(t) - \gamma') \right],
\]

where \( x' = x(t') \), \( x'' = x(t'') \), \( G' = G(t') \), \( G'' = G(t'') \), \( \gamma' = \gamma(t') \) and \( \gamma'' = \gamma(t'') \). Note that condition \( \gamma'' - \gamma' \neq m\pi, \ m \in \mathbb{Z} \), must be satisfied in the real case. Recall also that \( |\gamma'' - \gamma'|_p < |2|_p \). As we shall see later it is useful to write the corresponding momentum in the form

\[
k(t) = m\dot{x}(t) = m\frac{\dot{G}(t)}{G(t)}x(t) + \frac{mG(t)\dot{\gamma}(t)}{\sin(\gamma'' - \gamma')} \left[ \frac{x''}{G''} \cos(\gamma(t) - \gamma') - \frac{x'}{G'} \cos(\gamma'' - \gamma(t)) \right].
\]

**3.2 Solution with the initial conditions**

Imposing the initial conditions \( x^0 = x(t^0), \ k^0 = m\dot{x}(t^0) \) we find evolution of the classical state as follows:

\[
x(t) = \left[ \frac{G(t)}{G^0} \cos(\gamma(t) - \gamma^0) - \frac{G(t)\dot{\gamma}^0}{C} \sin(\gamma(t) - \gamma^0) \right] x^0 + \frac{G(t)G^0}{mC} \sin(\gamma(t) - \gamma^0)k^0,
\]

\[
k(t) = \left[ m \left( \frac{\dot{G}(t)}{G^0} - \frac{G(t)\dot{\gamma}(t)\dot{G}^0}{C} \right) \cos(\gamma(t) - \gamma^0) - m \left( \frac{\dot{G}(t)G^0}{C} + \frac{G(t)\dot{\gamma}(t)}{G^0} \right) \sin(\gamma(t) - \gamma^0) \right] x^0
\]

\[
\quad + \frac{G^0}{C} \left[ G(t)\dot{\gamma}(t) \cos(\gamma(t) - \gamma^0) + \dot{G}(t) \sin(\gamma(t) - \gamma^0) \right] k^0,
\]

where \( G^0 = G(t^0) \) and \( \gamma^0 = \gamma(t^0) \). Putting first \( t = t' \) and then \( t = t'' \) in the first equation of (3.7) one can find \( x^0 \) and \( k^0 \) as functions of \( x' \) and \( x'' \). Inserting these \( x^0 = x^0(x', x'') \) and \( k^0 = k^0(x', x'') \) into the second equation of (3.7) one gets the same formula (3.6) for \( k(t) \).

A suitable way to calculate the corresponding classical action

\[
\bar{S}(x'', t''; x', t') = \frac{m}{2} \int_{t'}^{t''} [\dot{\omega}(t)^2 - \omega^2(t)x^2(t)]dt \quad (3.8)
\]
is integrating by parts and using the equation of motion (3.2). It leads to
\[ \bar{S}(x'', t''; x', t') = \frac{m}{2} (x'' \dot{x}'' - x' \dot{x'}) . \] (3.9)

In virtue of (3.6), that gives \( \dot{x} \) as function of \( x \), we find action in the form quadratic in \( x'' \) and \( x' \), i.e.
\[ \bar{S}(x'', t''; x', t') = \frac{m}{2} \left[ \left( \frac{\dot{x}''}{\tan(\gamma'' - \gamma')} + \frac{\dot{G}''}{G''} \right) x''^2 
- \frac{2\sqrt{\gamma'' \gamma'}}{\sin(\gamma'' - \gamma')} x'' x' + \left( \frac{\dot{x}'}{\tan(\gamma'' - \gamma')} - \frac{\dot{G}'}{G'} \right) x'^2 \right] , \] (3.10)
where we used equality
\[ \frac{G'' \dot{\gamma}''}{G'} + \frac{G' \dot{\gamma}'}{G''} = 2\sqrt{\gamma'' \gamma'} , \]
which is derived by means of (3.4).

Note that the above \( p \)-adic formalism has the same form as its real counterpart, or in other words, the classical HOTDF is invariant under change of the number field \( \mathbb{R} \) and \( \mathbb{Q}_p \), for every \( p \). This may be regarded as a necessary condition for existence of an adelic classical HOTDF, which we construct in the following way. Let the position \( x \), momentum \( k \) and time \( t \) be adelic quantities, like (2.7). The corresponding adelic Lagrangian is given by
\[ L(x, \dot{x}, t) = (L(x_{\infty}, \dot{x}_{\infty}, t_{\infty}), L(x_2, \dot{x}_2, t_2), ..., L(x_p, \dot{x}_p, t_p), ...) , \]
where \( L(x_v, \dot{x}_v, t_v) = m[\dot{x}_v^2 - \omega^2(t_v)x_v^2]/2 \) with \( v = \infty, 2, ..., p, ..., \) and \( |L(x_p, \dot{x}_p, t_p)|_p \leq 1 \) for all but a finite number of primes \( p \). Also, all the other above introduced quantities, regarded as real and \( p \)-adic, can be generalized to the adelic ones. For instance, adelic classical action is
\[ \bar{S}(x'', t''; x', t') = (\bar{S}(x''_{\infty}, t''_{\infty}; x'_{\infty}, t'_{\infty}), \bar{S}(x''_2, t''_2; x'_2, t'_2), ..., \bar{S}(x''_p, t''_p; x'_p, t'_p), ...) , \] (3.11)
where real and \( p \)-adic ingredients have the form (3.10).

4. Quantum oscillator: real, \( p \)-adic and adelic case

The unique formalism of ordinary quantum mechanics which enables \( p \)-adic and adelic generalization with complex-valued wave functions is a triple [7,9]
\[ (L_2(\mathbb{R}), W(z_\infty), U(t_\infty)) \] , (4.1)
where \( L_2(\mathbb{R}) \) is the Hilbert space, \( z_\infty \) is a point of real classical phase space, \( W(z_\infty) \) is a unitary representation of the Heisenberg-Weyl group on \( L_2(\mathbb{R}) \), and \( U(t_\infty) \) is a unitary
representation of the evolution operator on $L_2(\mathbb{R})$. Hence, under $p$-adic and adelic quantum mechanics we understand $p$-adic and adelic analogues of (4.1), i.e.

$$
(L_2(\mathbb{Q}_p), W(z_p), U(t_p)) ,
$$

$$
(L_2(\mathbb{A}), W(z), U(t)) ,
$$

(4.2) (4.3)

respectively. Thus, to find adelic eigenstate and its evolution of a HOTDF given by $U(t'', t')$, one has to solve the equation

$$
U(t'', t')\Psi_S^{(\alpha)}(x', t') = \chi[\alpha(\gamma'' - \gamma')]|\Psi_S^{(\alpha)}(x', t') ,
$$

(4.4)

where $\alpha = (\alpha_\infty, \alpha_2, ..., \alpha_p, ...)$ is an adelic analogue of energy, $\chi(u) = \prod_v \chi_v(u_v) = \exp(-2\pi i u_\infty) \prod_p \exp(2\pi i \{u_p\}_p)$ and

$$
\Psi_S^{(\alpha)}(x, t) = \Psi_S^{(\alpha_\infty)}(x_\infty, t_\infty) \prod_{p \in S} \Psi_p^{(\alpha_p)}(x_p, t_p) \prod_{p \not\in S} \Omega(|x_p|_p) .
$$

(4.5)

The evolution operator $U(t'', t') = \prod_v U_v(t''_v, t'_v)$ acts componentwise as follows:

$$
[U_v \Psi_v](x''_v, t''_v) = \int_{Q_v} K_v(x''_v, t''_v; x'_v, t'_v) \Psi_v(x'_v, t'_v) dx'_v .
$$

(4.6)

The kernel $K_v(x''_v, t''_v; x'_v, t'_v)$ is defined by the Feynman path integral

$$
K_v(x''_v, t''_v; x'_v, t'_v) = \int \chi_v \left(- \frac{1}{\hbar} S[x]\right) Dx = \int \chi_v \left(- \frac{1}{\hbar} \int_{t'_v}^{t''_v} L(x_v, \dot{x}_v, t_v) dt_v\right) \prod_{t_v} dx(t_v) ,
$$

(4.7)

where $\hbar$ is the Planck constant. The kernel $K_v$ also called the quantum-mechanical propagator, is of central importance not only in ordinary but also in $p$-adic and adelic quantum mechanics.

The $p$-adic Feynman path integral for classical actions quadratic in $x''$ and $x'$ is calculated in [15] and has the same form as its real counterpart. Namely, if $\bar{S}(x''_v, t''_v; x'_v, t'_v)$ is quadratic in $x''_v$ and $x'_v$ then

$$
K_v(x''_v, t''_v; x'_v, t'_v) = \lambda_v \left(- \frac{1}{2\hbar} \frac{\partial^2 \bar{S}}{\partial x''_v \partial x'_v}\right) \left( \frac{1}{\hbar} \frac{\partial^2 \bar{S}}{\partial x''_v \partial x'_v} \right)^{1/2} \chi_v \left(- \frac{1}{\hbar} \bar{S}(x''_v, t''_v; x'_v, t'_v)\right) ,
$$

(4.8)

where $\lambda_\infty(\alpha) = (1 - i \text{ sign } \alpha)/\sqrt{2}$, $\lambda_\infty(0) = 1$ and satisfies properties (2.5).

Applying formula (4.8) to the HOTDF and using (3.10), we get

$$
K_v(x''_v, t''_v; x'_v, t'_v) = \lambda_v \left( \frac{m}{2\hbar} \frac{\sqrt{\gamma'' - \gamma'}}{\sin(\gamma'' - \gamma')} \right) \left( \frac{m}{\hbar} \frac{\sqrt{\gamma'' - \gamma'}}{\sin(\gamma'' - \gamma')} \right)^{1/2}
$$
\[
\chi_v \left\{ -\frac{m}{2\hbar} \left[ \left( \frac{\hat{\gamma}''}{\tan(\gamma'' - \gamma')} + \frac{\hat{G}''}{G''} \right) x''^2 - \frac{2\sqrt{\hat{\gamma}''\hat{\gamma}'}\tan(\gamma'' - \gamma')} \sin(\gamma'' - \gamma') x'' x' + \left( \frac{\hat{\gamma}'}{\tan(\gamma'' - \gamma')} - \frac{\hat{G}'}{G''} \right) x'^2 \right] \right\},
\]

that contains the earlier obtained result in the real case (see, e.g. Ref. 14). One can explicitly show that the propagator (4.9) satisfies all usual properties of the probability amplitude for a quantum particle to go from a space-time point \((x'_v, t'_v)\) to a space-time point \((x''_v, t''_v)\).

The corresponding adelic propagator is

\[
\mathcal{K}(x'', t''; x', t') = \prod_v \mathcal{K}_v(x''_v, t''_v; x'_v, t'_v),
\]

where \(\mathcal{K}_v(x''_v, t''_v; x'_v, t'_v)\) is given by (4.9). Product in (4.10) is divergent, but it may be regarded as an adelic functional on the space of test functions which are the adelic Schwartz-Bruhat functions (see, also [16]).

In \(p\)-adic quantum mechanics a significant role plays the eigenstate \(\Omega(|x_p|_p)\) (2.6), which is invariant under \(U_p(t''_p, t'_p)\) transformation and may be regarded as \(p\)-adic vacuum state since it has \(\{\alpha_p(\gamma''_p - \gamma'_p)\}_p = 0\). Due to (4.5), the existence of \(\Omega(|x_p|_p)\) for all but a finite number of \(p\) is a necessary condition for unification of ordinary and \(p\)-adic quantum mechanics in the form of adelic one. This \(\Omega\)-state exists iff

\[
\int_{|x'_p|_p \leq 1} \mathcal{K}_p(x''_p, t''_p; x'_p, t'_p) dx'_p = \Omega(|x''_p|_p)
\]

is satisfied.

Inserting (4.9) with \(v = p\) into (4.11), and using the integral (2.4) for \(\nu = 0\), we can derive some conditions on \(G(t_p)\), \(\gamma(t_p)\) and \(m\), which provide \(\Omega\)-eigenstate. For example, if \(\gamma(t_p) = \gamma_0 + \gamma_1 t_p + \gamma_2 t_p^2 + \ldots + \gamma_n t_p^n + \ldots\) and

\[
\left| \frac{\hat{G}'}{G'} \right|_p < \left| \frac{\hat{\gamma}'}{\tan(\gamma'' - \gamma')} \right|_p > \left| \frac{\hbar}{2m} \right|_p
\]

then \(\Omega(|x_p|_p)\) exists for all \(p \neq 2\). It is worth noting that not every HOTDF has \(\Omega\)-state and may be adelically generalized.

As the simplest illustration of the above expressions one can take frequency \(\omega(t) = \omega_0\) and recover earlier obtained result [9].

5. Concluding remarks

According to (4.5) adelic wave function \(\Psi(x, t)\) offers more information on a physical system than only its standard part \(\Psi_\infty(x_\infty, t_\infty)\). Let us note here space discreteness and \(p\)-adic phase, which are generic and mainly follow from adelic quantum formalism.
For example, adelic state

\[
\Psi(x, t) = \Psi_\infty(x_\infty, t_\infty) \prod_p \Omega(|x_p|_p)
\]

exhibits discrete structure of the space at the length \( l_0 = (\hbar m^{-1} - 1(\omega)^{-1})^{1/2} \). Namely, according to the usual interpretation of the wave function we have to consider \( |\Psi(x, t)|_\infty^2 \) at rational points \( x \) and \( t \). In the above adelic case we get

\[
|\Psi(x, t)|_\infty^2 = |\Psi_\infty(x, t)|_\infty^2 \prod_p \Omega(|x|_p) = \begin{cases} |\Psi_\infty(x, t)|_\infty^2, & x \in \mathbb{Z}, \\
0, & x \in \mathbb{Q} \setminus \mathbb{Z} . \end{cases}
\]

Here we used the following properties of \( \Omega \)-function: \( \Omega^2(|x|_p) = \Omega(|x|_p) \), \( \prod_p \Omega(|x|_p) = 1 \) if \( x \in \mathbb{Z} \), and \( \prod_p \Omega(|x|_p) = 0 \) if \( x \in \mathbb{Q} \setminus \mathbb{Z} \). Thus, it means that position \( x \) may have only discrete values: \( x/l_0 = 0, \pm 1, \pm 2, \ldots \). To verify this space discreteness experimentally one has to examine physical system in its vacuum state and at distances characterized by the length \( l_0 \). When system is in a rather mixed state

\[
\Psi(x, t) = \sum_{S, \alpha} C(S, \alpha) \Psi^{(\alpha)}_S(x, t)
\]

the sharpness of the discrete structure disappears and space demonstrates usual continuous properties. So, this space discreteness is a quantum effect and depends on adelic quantum state.

Adelic wave function gives also a framework to investigate a new kind of phase, which may be called \( p \)-adic phase. In fact, (4.5) contains

\[
\Psi_p(x_p, t_p) = \chi_p[\alpha_p(\gamma(t_p) - \gamma^0)] \Psi_p(x_p, 0) ,
\]

where \( \chi_p[\alpha_p(\gamma(t_p) - \gamma^0)] \) presents a \( p \)-adic dynamical phase. This may be observed investigating the fine structure of interference phenomena.

At real distances which are very large in comparison with \( l_0 \), \( p \)-adic effects become hidden and adelic quantum mechanics reduces to the ordinary one. In such case we have to integrate \( |\Psi(x, t)|^2 \) over \( p \)-adic components of adelic space. Since \( \int_{|x_p|_p \leq 1} dx = 1 \) and \( \int_{\mathbb{Q}_p} |\Psi_p(x_p, t_p)|_\infty^2 dx_p = 1 \) we have

\[
\int_{\mathcal{A}(S) \setminus \mathbb{R}} |\Psi_S(x, t)|_\infty^2 dx = |\Psi_\infty(x_\infty, t_\infty)|_\infty^2 dx_\infty \prod_p \int_{\mathbb{Q}_p} |\Psi_p(x_p, t_p)|_\infty^2 dx_p \\
\times \prod_{p \notin S} \int_{\mathbb{Z}_p} \Omega(|x_p|_p) dx_p = |\Psi_\infty(x_\infty, t_\infty)|_\infty^2 dx_\infty .
\]
Hence, ordinary quantum theory may be regarded as an effective approximation of the more profound adelic one.

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