Towards Clarifying the Theory of the Deconfounder

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Abstract

Wang and Blei (2019a) studies multiple causal inference and proposes the deconfounder algorithm. The paper discusses theoretical requirements and presents empirical studies. Several refinements have been suggested around the theory of the deconfounder. Among these, Imai and Jiang clarified the assumption of “no unobserved single-cause confounders.” Using their assumption, this paper clarifies the theory. Furthermore, Ogburn et al. (2020) proposes counterexamples to the theory. But the proposed counterexamples do not satisfy the required assumptions.

Wang and Blei (2019a) studies multiple causal inference and proposes the deconfounder algorithm. It discusses theoretical requirements and presents empirical studies. Wang and Blei (2019a) was discussed at JSM 2019 and later in print by D’Amour (2019), Athey et al. (2019), Imai and Jiang (2019), and Ogburn et al. (2019); Wang and Blei (2019b) responds to the comments.

Several refinements have been suggested around the theory of the deconfounder. Among these, Imai and Jiang clarified the assumption of “no unobserved single-cause confounders,” which is discussed in Wang and Blei (2019b). Using the refined assumption, this paper discusses the theory. Further, in a continuation of their commentary, Ogburn et al. (2020) proposes counterexamples to Wang and Blei (2019a). This paper also shows how the proposed counterexamples do not satisfy the required assumptions.

The theoretical results below are the same as those in Wang and Blei (2019a). The original paper constructs the substitute confounder with a probabilistic factor model and then states the assumptions under which that construction satisfies weak
unconfoundedness. This paper first states the assumptions under which a substitute confounder exists, shows that it satisfies weak unconfoundedness, and then shows how the deconfounder algorithm constructs it.

1 Clarifying the theory of the deconfounder

Begin by defining a multi-cause separator, a type of random variable.

**Definition 1: Multi-cause separator.** Consider all the causes \( \mathbf{A} = \{A_1, \ldots, A_m\} \). A multi-cause separator \( U \) is a smallest \( \sigma \)-algebra that renders all the causes conditionally independent,

\[
P(A_1, \ldots, A_m \mid U) = \prod_{j=1}^{m} P(A_j \mid U),
\]

and where none of the conditionals \( P(A_j \mid U) \) is a point mass.

The concept of the smallest \( \sigma \)-algebra defines the sense in which the variable \( U \) is “multi-cause.” If \( U \) contains information about a single cause then it is not the smallest separating \( \sigma \)-algebra. (Appendix A shows why.)

The following assumption was suggested by Imai and Jiang at JSM 2019. Here \( X \) is a set of observed covariates and \( Y(\mathbf{a}) \) are potential outcomes.

**Assumption 1: No unobserved single-cause confounders.** There exists a random variable \( U \) that satisfies the following two requirements:

1. It is a multi-cause separator.

2. Together with the observed covariates \( X \), it satisfies weak unconfoundedness,

\[
A_1, \ldots, A_m \perp Y(\mathbf{a}) \mid U, X \quad \forall \mathbf{a} \in \mathcal{A}.
\]

The first part ensures that \( U \) only involves multiple causes. The second part ensures that the variable \( U \) contains all multi-cause confounders. (It can contain other multi-cause variables as well.)
Why is Assumption 1 called “no unobserved single-cause confounders”? The variable $U$ is a multi-cause separator: it cannot capture single-cause variables; it must capture ancestors of multiple causes; and it cannot capture descendants of multiple causes. For $U$ and the observed covariates $X$ to satisfy weak unconfoundedness, the observed covariates must include all single-cause confounders.

The next assumption is that every multi-cause separator is pinpointed by a single function of the observed causes. It is called a substitute confounder, though only after the theorem below will it deserve this name.

**Assumption 2: The substitute confounder.** All multi-cause separators $Z$ are pinpointed by a single deterministic function of the causes,

$$P(Z \mid A_1, \ldots, A_m) = \delta_{f(A_1, \ldots, A_m)}, \tag{3}$$

where $\delta_{f(\cdot)}$ denotes a point mass at $f(\cdot)$.

**Theorem 1: Weak unconfoundedness for the substitute confounder.** Suppose Assumptions 1 and 2 hold. Consider a multi-cause separator $Z$. It satisfies weak unconfoundedness,

$$A_1, \ldots, A_m \perp \perp Y(a) \mid Z, X \forall a \in A. \tag{4}$$

Appendix B proves the theorem. Assumption 1 posits the existence of a multi-cause separator that also satisfies unconfoundedness. Assumption 2 implies there is only one multi-cause separator $Z$, it is unique. (The next paragraph discusses why.) These two assumptions together imply that the multi-cause separator $Z$ also satisfies weak unconfoundedness.

Assumption 2 pinpoints the separator as a function of the causes. Why does this assumption imply the uniqueness of the multi-cause separator, particularly across the probability space expanded with all potential outcomes? Eq. 3 implies

$$P(Z \mid A_1, \ldots, A_m, \{Y(a)\}_{a \in \mathcal{A}}, X) = \delta_{f(A_1, \ldots, A_m)}. \tag{5}$$

If two variables $Z_1$ and $Z_2$ both satisfy Eq. 5 then they must be equal. The reason is that $Z_1 = Z_2 = f(A)$ in the full probability space $\mathcal{A} \times X \times \{Y(a)\}_{a \in \mathcal{A}}$. Thus Assumption 2 implies the uniqueness of the multi-cause separator.

The identification theorems of Wang and Blei (2019a) rely on the weak unconfoundedness of the substitute confounder $Z$ and consider adjustments as if $Z$ were
explicitly observed. The theorems rest on further technical assumptions because a pinpointed $Z$ violates overlap of the whole set of causes $A$. (These technicalities are distinct from the fact that $Z$ satisfies weak unconfoundedness.) Note that if $Z$ is not pinpointed then there is uncertainty about the separator, even with infinite data. But with further assumptions, point identification is still possible. See Wang and Blei (2019a) (Appendix C) and Imai and Jiang (2019).1

2 From the theory to the algorithm

The deconfounder algorithm of Wang and Blei (2019a) operationalizes this theory. If the investigator finds a deterministic function of the causes that renders them conditionally independent then the output of that function can be used as a substitute confounder in a downstream causal inference. Assumption 1 is that the observed covariates $X$ and multi-cause separator $U$ provide weak unconfoundedness: there are no unobserved single-cause confounders. Assumption 2 is that there is a single $f(A)$ that provides the substitute confounder.

The algorithm uses a probabilistic factor model and posterior predictive check to find $f(A)$. Suppose a factor model describes well the distribution of the causes. Then its local latent variable renders the causes conditionally independent. When the number of causes is large and the local latent variable is low-dimensional, this inference approaches a deterministic function, satisfying Eq. 3. The deconfounder infers the local latent variables and calls them substitute confounders.

Why are the inferred confounders multi-cause separators? Why do they form the smallest $\sigma$-algebra that renders the causes conditionally independent? The reason has two parts. (1) The factor model implies that its latent variable renders the causes conditionally independent. (2) The $\sigma$-algebra of a pinpointed separator cannot pick up single-cause variables; Wang and Blei (2019b) provides a proof.

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1The theory in this paper is the same as in Wang and Blei (2019a,b) (WB), though it is clarified here. For readers interested in the mapping: WB Definition 4 says there exists a smallest-$\sigma$-algebra variable that renders the causes independent (WB Eq 40), and it satisfies weak unconfoundedness. For weak unconfoundedness, fix WB Eq 39 by adding $A_{i,-j}$ to its conditioning or, equivalently, use Imai and Jiang’s assumption articulated in Wang and Blei (2019b). WB Definition 5 says that there the substitute confounder is pinpointable through a probabilistic factor model; because it’s probabilistic, none of the factors is a point mass. Finally, the proof of Theorem 1 in this paper appears at the end of the proof of WB Lemma 2.
Assumption 2 requires a single deterministic function that provides the separator; the algorithm uses a factor model to find it. Kruskal (1989) and Allman et al. (2009) provide conditions that guarantee the uniqueness of the factor model that captures the distribution of the causes. Bai and Li (2012) and Chen et al. (2019) study conditions under which the latent variables of factor models are identifiable. With many causes and a low-dimensional factor model, inference of its local variable approaches a deterministic function.

Finally, the deconfounder uses posterior predictive checks (Guttman, 1967; Rubin, 1984; Gelman et al., 1996) to assess the fidelity of the distribution of causes that is provided by the factor model. Specifically, the check evaluates the predictive distribution on sets of held-out causes. This strategy uses ideas from Bayarri and Berger (2000); Robins et al. (2000); Ranganath and Blei (2019) to provide a better-calibrated check.

3 The proposed counterexamples violate the assumptions

Ogburn et al. (2020) propose counterexamples to the theory of Wang and Blei (2019a). Using the theory, as outlined above, the proposed counterexamples do not satisfy the required assumptions.

Example 1. There are two independent causes \(A_1\) and \(A_2\) and a substitute confounder \(Z \sim \text{Bernoulli}(0.5)\) that is independent of all other variables \((A_1, A_2, Y(a))\). Here the substitute confounder \(Z\) does not satisfy Assumption 2; its conditional distribution \(P(Z | A_1, A_2) = \text{Bernoulli}(0.5)\) is not a point mass.

Example 2. There are two causes \(A_1\) and \(A_2\), and another variable \(U\). Assume that \(A_j \perp Y(a) | U, j = 1, 2\) and \(U\) is the smallest \(\sigma\)-algebra that renders \(A_1\) and \(A_2\) conditionally independent. Set the substitute confounder \(Z = U\). Again \(P(Z | A_1, A_2)\) is not a point mass, violating Assumption 2. Further, \(U\) does not satisfy weak unconfoundedness because \((A_1, A_2) \perp Y(a) | U\), violating Assumption 1.\(^2\)

Example 3. A variant of the second example involves a third cause \(A_3\) and sets the substitute confounder \(Z = A_3\). This example violates Assumption 2. A pinpointed

\(^2\)This example does satisfy the “no unobserved single-cause confounder” assumption as stated in Definition 4 of Wang and Blei (2019a). But it violates the refined assumption in Assumption 1, due to Imai and Jiang at JSM and also published in in Wang and Blei (2019b).
substitute cannot be a function of only one cause (Wang and Blei, 2019b).

The importance of conditional independence. Ogburn et al. (2020) claim that the conditional independence requirement of factor models does not “drive the success” of deconfounder-like methods. But the conditional independence requirement, along with pinpointability, plays an important role in confirming the assumptions required by the algorithm.

1. Requiring conditional independence outlines the class of confounders that the deconfounder targets; they must be multi-cause confounders. This requirement is why, with the assumption of no unobserved single-cause confounder, the deconfounder handles all confounders.

2. Requiring conditional independence prevents the substitute confounder from capturing multi-cause colliders or mediators; capturing such variables violates the conditional independence requirement.

3. As for single-cause post-treatment variables, Wang and Blei (2019b) shows that substitute confounders that satisfy Assumption 2 cannot capture any single-cause variables. Thus, along with point #2 above, the substitute confounder does not capture any post-treatment variables.

4 Discussion

To reiterate Wang and Blei (2019b), the deconfounder is not a turnkey solution to causal inference. It does not relieve the researcher from trying to measure confounders. As for all causal inference with observational data, it comes with uncheckable assumptions. In particular, Assumption 1 is that there are no unobserved single-cause confounders.

This paper clarifies the theoretical foundations of Wang and Blei (2019a). The refinements of Imai and Jiang make clearer the assumptions required for identification, and they help simplify the proof that the substitute confounder satisfies weak unconfoundedness.

Both Ogburn et al. (2019) and Ogburn et al. (2020) question the correctness of the theory behind the deconfounder. The objections are incorrect, discussed above and in Wang and Blei (2019b). But more broadly, Ogburn et al.’s resistance to
the idea stems from an important misunderstanding. Both commentaries reiterate
the fact that no information about unobserved confounders can be inferred from
observational data. Wang and Blei (2019a) does not challenge this fact. Rather,
the theory finds confounders that are *effectively observed*, even if not explicitly so,
and embedded in the multiplicity of the causes. The deconfounder extracts this
information for causal inference.

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**A Smallest \(\sigma\)-algebra**

The restriction of the smallest \(\sigma\)-algebra ensures that \(U\) can only pick up multiple-
cause variables. The proof is by contradiction: if \(U\) involves both multi-cause and
single-cause variables then \(U\) cannot be the smallest \(\sigma\)-algebra.

Formally, suppose the variable \(U\) contains a single cause component and a multi-
cause component, \(U = (U^s, U^*)\). Without loss of generality, suppose the single-
cause component only depends on the first cause \(A_1\). Then Eq. 1 implies

\[
P(A_1, \ldots, A_m \mid U) = \prod_{j=1}^{m} P(A_j \mid U) \tag{6}
\]

\[
= P(A_1 \mid U^s, U^*) \cdot \prod_{j=2}^{m} P(A_j \mid U^*). \tag{7}
\]
This implies that
\[
P(A_1, \ldots, A_m \mid U^*) = \int P(A_1, \ldots, A_m \mid U) \cdot P(U^* \mid U^*) \, dU^* \tag{8}
\]
\[
= \prod_{j=2}^m P(A_j \mid U^*) \cdot \int P(A_1 \mid U^*, U^s) \cdot P(U^s \mid U^*) \, dU^s \tag{9}
\]
\[
= P(A_1 \mid U^*) \cdot \prod_{j=2}^m P(A_j \mid U^*), \tag{10}
\]
which means that \( U = (U^s, U^*) \) is not the smallest \( \sigma \)-algebra that renders the causes independent.

\section*{B Proof of Theorem 1}

Eq. 3 implies the following conditional distribution of the substitute confounder given all of the other variables (including the potential outcomes),
\[
P(Z \mid A_1, \ldots, A_m, \{Y(a)\}_{a \in \mathcal{A}}, X) = \delta_{f(A_1, \ldots, A_m)}. \tag{11}
\]
Assumption 2 only concerns the probability space of the observed causes \( \mathcal{A} \). But Eq. 11 holds because \( P(Z \mid A_1, \ldots, A_m) \) is a point mass \( \delta_{f(A_1, \ldots, A_m)} \), which satisfies \( \delta_{f(A_1, \ldots, A_m)} \perp \perp \{Y(a)\}_{a \in \mathcal{A}}, X \).

Eq. 11 implies that the substitute confounder \( Z \) is unique. The reason is that if two variables \( Z_1 \) and \( Z_2 \) satisfy Eq. 11 then they must be equal in the whole probability space, \( Z_1 = Z_2 = f(A_1, \ldots, A_m) \) on \( \mathcal{A} \times X \times \{Y(a)\}_{a \in \mathcal{A}} \).

The uniqueness of \( Z \), together with Assumption 1, implies that the substitute confounder satisfies weak unconfoundedness,
\[
A_1, \ldots, A_m \perp Y(a) \mid Z, X \quad \forall a \in \mathcal{A}. \tag{12}
\]
Why? Assumption 1 asserts the \textit{existence} of a random variable that is (1) the smallest \( \sigma \)-algebra that renders the causes conditionally independent and (2) satisfies weak unconfoundedness. The argument above establishes the \textit{uniqueness} of the random variable that satisfies (1). Thus the variable satisfying (1) must also satisfy (2). This unique random variable is the substitute confounder \( Z \).

Note that this proof simplifies the proof of Lemma 2 in \textit{Wang and Blei} (2019a).
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