ON THE FORWARD DYNAMICAL BEHAVIOR OF NONAUTONOMOUS SYSTEMS

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Dedicated to Professor Peter E. Kloeden on the occasion of his 70th birthday

Abstract. This paper is concerned with the forward dynamical behavior of nonautonomous systems. Under some general conditions, it is shown that in an arbitrary small neighborhood of a pullback attractor of a nonautonomous system, there exists a family of sets \( \{ A_\varepsilon(p) \}_{p \in P} \) of phase space \( X \), which is forward invariant such that \( \{ A_\varepsilon(p) \}_{p \in P} \) uniformly forward attracts each bounded subset of \( X \). Furthermore, we can also prove that \( \{ A_\varepsilon(p) \}_{p \in P} \) forward attracts each bounded set at an exponential rate.

1. Introduction. It is well known that the theory of attractors plays a fundamental role in the study of dynamical systems. If a system has an attractor, then all its long-term dynamics near the attractor will be captured by the attractor. For autonomous dynamical systems, the theory of attractors has been fully developed in the past decades, in both finite and infinite dimensional cases. However, the research of the dynamics of nonautonomous systems seems to be far more difficult, and even the notions of attractors are still undergoing investigations.

For a nonautonomous dynamical system (NDS in short), usually one can define two different types of attractors: pullback attractor and forward attractor, corresponding to pullback attraction and forward attraction, respectively. These two attractors can be regarded as a natural generalization of global attractors of autonomous systems. Concerning the existence of a pullback attractor, there are quite general existence results on pullback attractors under some hypotheses similar to those of attractors for autonomous systems; see e.g. [4, 11, 9, 10, 23, 24]. This has received great attention in a systematic study on pullback attractors in

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recent years. On the other hand, in most cases we are naturally more concerned with the evolution of a system in the future, see e.g. [3, 20]. However, a pullback attractor may not be forward attracting generally and provide little information on this aspect; see e.g. [19, Example 1.1].

Theoretically, the notion of a forward attractor sounds to be quite suitable to describe the forward dynamical behavior of a nonautonomous system. Unfortunately the general existence criteria for such type of attractors remain open problems except in some particular cases such as the asymptotically autonomous and the periodic ones, and are still under investigations; see e.g. Cheban et al. [8], Wang et al. [25], Kloeden [21], Carvalho et al. [5, pp. 595] and Ju et al. [19]. We all know that for a skew-product flow \((\theta, \phi)\) on the space \(P \times X\), where \(P\) is a compact base space and \(X\) is a complete metric space, if the pullback attraction of a pullback attractor \(A = \{A(p)\}_{p \in P}\) is uniform with respect to \(p \in P\), then it is also a forward attractor. Based on this result the authors [8] proved that if the section \(A(p)\) of \(A\) is lower semicontinuous in \(p\), then \(A\) is a forward attractor (see also [25]). However, in general it is not easy to verify the lower semicontinuity of a family of subsets of a metric space.

Inspired by these works mentioned above, in this paper we are concerned with the forward dynamical behavior of a nonautonomous system. We first construct a family of sets in an arbitrary small neighborhood of pullback attractors, which may be used to describe the forward behavior of a general NDS. Specifically, let \((\theta, \phi)\) be a skew-product flow consisting of a base flow \(\{\theta_t\}_{t \in \mathbb{R}}\) acting on a base space \(P\) and a cocycle mapping \(\phi\) on a phase space \(X\). Under some general conditions, we will show that for any small \(\varepsilon\)-neighborhood of the pullback attractor \(A(p)\) for \(p \in P\), there exists a family of sets \(\{A_{\varepsilon}(p)\}_{p \in P}\), denoted by \(A_{\varepsilon}(\cdot)\) which is forward invariant under the acting of the skew-product flow \((\theta, \phi)\), such that \(A_{\varepsilon}(\cdot)\) uniformly forward attracts each bounded subset of \(X\).

Our second purpose of this work is to consider the forward attraction rate of the set \(A_{\varepsilon}(\cdot)\). As we all know, an exponential attractor of autonomous systems, which was first made by Eden et al. [14], attracts the trajectories more fast than global attractors and is more robust under perturbations. This exponential attractor is a compact set, which is positively invariant and has a finite fractal dimensionality. As for nonautonomous systems, pullback exponential attractors are extensively studied, see e.g. [6, 7, 15, 17]. For instance, in [17] the authors constructed pullback exponential attractors of nonautonomous systems in the framework of a process on Banach spaces. The interested readers are referred to [1, 2, 12, 13, 16, 26] for some concrete systems on the existence and construction of exponential attractors. In this paper we further consider the forward exponentially attracting property of the set \(A_{\varepsilon}(\cdot)\). Precisely, we will show that the set \(A_{\varepsilon}(\cdot)\) uniformly forward exponentially attracts each bounded subsets of the phase space under some suitable conditions.

Finally, as an application of our main results, we consider the following evolution equation:

\[
u_t - \Delta u = \mu u + g(x, t), \quad x \in \Omega
\]

associated with the Dirichlet boundary condition, where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) and \(g\) is a bounded continuous function. We show that under some conditions, for any \(\varepsilon > 0\) there exists a family of sets \(\{A_{\varepsilon}(p)\}_{p \in P}\) for some base space \(P\), such that it is forward invariant and uniformly forward exponentially attracts bounded subsets of the phase space.
The paper is organized as follows. In section 2, we make some preliminaries. Section 3 is devoted to our main results. Finally, in section 4 we give an example to illustrate our results.

2. Preliminaries. For the reader’s convenience, we first collect some basic notions on dynamical systems, one can also see [4, 12] etc., for details.

Let $X$ be a complete metric space and $P$ a metric space with the metrics $d$ and $\rho$, respectively. Given two subsets $A,B$ of $X$. Define the Hausdorff-semidistance and Hausdorff distance of $A$ and $B$, respectively, as

$$d_H(A, B) = \sup_{x \in A} d(x, B), \quad \delta_H(A, B) = \max\{d_H(A, B), d_H(B, A)\}.$$ 

The $\varepsilon$-neighborhood of $A$ is defined by the set

$$N_\varepsilon(A) = \{x \in X : d(x, A) < \varepsilon\}.$$ 

A skew-product flow $(\theta, \phi)$ consists of a base flow $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ on a metric space $P$, and a flow $\phi$ on the phase space $X$ which is driven by $\theta$ on $P$. More precisely, the base flow $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ is a dynamical system on $P$, i.e. a group of homeomorphisms from $P$ to itself such that

(i) $\theta_0 = \text{id}_P$;
(ii) $\theta_{t+s} = \theta_t \theta_s$, for all $t, s \in \mathbb{R}$;
(iii) the mapping $(t, p) \to \theta_t p$ is continuous.

In addition, the dynamics on the phase space $X$ is given by a family of mappings

$$\mathbb{R}_+ \times P \ni (t, p) \mapsto \phi(t, p)$$

that enjoys the following properties:

(i) $\phi(0, p) = \text{id}_X$ for all $p \in P$;
(ii) $\phi(t+s, p) = \phi(t, \theta_s p)\phi(s, p)$ for all $t, s \in \mathbb{R}_+$ and $p \in P$;
(iii) the mapping $(t, p) \to \phi(t, p)x$ is continuous for $x \in X$.

A family of sets $A = \{A(p)\}_{p \in P}$ is said to be forward invariant under the acting of a skew-product flow $(\theta, \phi)$ if

$$\phi(t, p)A(p) \subset A(\theta_t p) \quad \text{for all } p \in P \text{ and } t \in \mathbb{R}_+.$$ 

$A$ is called invariant under $(\theta, \phi)$ if

$$\phi(t, p)A(p) = A(\theta_t p) \quad \text{for all } p \in P \text{ and } t \in \mathbb{R}_+.$$ 

By $A(\cdot)$ we denote a family of subsets $\{A(p)\}_{p \in P}$ of $X$, for convenience in statement, we call the set $A(\cdot)$ a nonautonomous set of $X$.

**Definition 2.1.** A family of compact sets $\{A(p)\}_{p \in P}$ is called a (global) pullback attractor for $(\theta, \phi)$ if it is invariant and pullback attracts any bounded subset $B$ of $X$ in the sense that

$$\lim_{t \to \infty} d_H(\phi(\theta_{-t} p) B, A(p)) = 0, \quad \forall p \in P.$$ 

**Definition 2.2.** A nonautonomous set $B(\cdot)$ of $X$ is said to be uniformly absorbing for $(\theta, \phi)$ if for any bounded set $B \subset X$, there exists a $T_B > 0$ independent of $p$ such that

$$\phi(t, p)B \subset B(\theta_t p) \quad \text{for all } t \geq T_B \text{ and all } p \in P.$$
**Definition 2.3.** A skew-product flow \((\theta, \phi)\) is called uniformly asymptotically compact if there exists a compact set \(K \subset X\) such that for any bounded subset \(B\) of \(X\),

\[
\lim_{t \to \infty} \sup_{p \in P} d_H(\phi(t, p)B, K) = 0.
\]

Finally, let us introduce a definition of pullback exponential attractors for a skew-product flow.

Let \((\theta, \phi)\) be a skew-product flow on the space \(X \times P\). Let \(p_0 \in P\) be fixed. Define a mapping on \(X\) by

\[
U(t, s)x = \phi(t - s, \theta_s p_0)x, \quad t \geq s, \quad x \in X.
\]

Then the family \(\{U(t, s)\}_{t \geq s}\) is a process on \(X\).

**Remark 1.** In general, the base space \(P\) may be defined as follows:

\[
P := P[h] = \{h(\tau + \cdot) : \tau \in \mathbb{R}\}^{C_b(\mathbb{R}, X)}
\]

for some \(h \in C_b(\mathbb{R}, X)\), where \(C_b(\mathbb{R}, X)\) denotes the set of bounded continuous functions from \(\mathbb{R}\) to \(X\) with the uniform convergence topology.

Based on this relationship (1) and the notion of pullback exponential attractors (see [17, 27]), we give a definition of pullback exponential attractors for the skew-product flow \((\theta, \phi)\).

**Definition 2.4.** A nonautonomous set \(M(\cdot)\) is said to be a pullback exponential attractor for \((\theta, \phi)\) on \(P \times X\), if it satisfies the following properties:

1. \(M(p)\) is a compact subset of \(X\) for any \(p \in P\), and its fractal dimension is finite uniformly with respect to \(p \in P\), i.e. \(\sup_{p \in P} \dim_f M(p) < \infty\).
2. It is forward invariant under \((\theta, \phi)\):

\[
\phi(t, p)M(p) \subset M(\theta t p) \quad \forall t \geq 0, \quad \forall p \in P.
\]
3. There exists \(\alpha > 0\) such that for each \(p \in P\) and any bounded subset \(B\) of \(X\), there exists \(T_{p, B} > 0\) with

\[
d_H(\phi(t, \theta_{-t} p)B, M(p)) \leq C_B e^{-\alpha t}, \quad \forall t \geq T_{p, B},
\]

where \(C_B\) is a positive constant depending on \(B\).

### 3. Main results

In this section we consider the forward dynamical behavior of a skew-product flow \((\theta, \phi)\) on \(P \times X\), where \(P\) is a compact metric space and \(X\) is a Banach space.

Let \(\{A(p)\}_{p \in P}\) be a (global) pullback attractor for \((\theta, \phi)\). We first construct a nonautonomous set \(A_\varepsilon(\cdot)\) in an arbitrary small \(\varepsilon\)-neighborhood of the pullback attractor \(\{A(p)\}_{p \in P}\), which is forward invariant under \((\theta, \phi)\) such that \(A_\varepsilon(\cdot)\) uniformly forward attracts bounded subsets of \(X\).

For the sake of convenience, we define an autonomous semigroup \(\Phi(t)\) on \(P \times X\) corresponding to the skew-product flow \((\theta, \phi)\) by

\[
\Phi(t)(p, x) = (\theta_t p, \phi(t, p)x), \quad (p, x) \in P \times X.
\]

**Theorem 3.1.** Assume that the skew-product flow \((\theta, \phi)\) is uniformly asymptotically compact. Then for any \(\varepsilon > 0\), there exists a nonautonomous set \(A_\varepsilon(\cdot)\) with

\[
A(p) \subset A_\varepsilon(p) \subset N_\varepsilon(A(p)) \quad \text{for all } p \in P,
\]

where \(N_\varepsilon(x)\) is the \(\varepsilon\)-neighborhood of \(x\).
such that it is forward invariant under $(\theta, \phi)$, where $N_{\epsilon}(A(p))$ is an $\epsilon$-neighborhood of $A(p)$. Moreover, $A_c(\cdot)$ uniformly forward attracts each bounded subset $B$ of $X$:

$$\lim_{t \to \infty} \sup_{p \in P} d_H(\phi(t, p) B, A_c(\theta t(p))) = 0. \quad (2)$$

**Proof.** Since the system $(\theta, \phi)$ is uniformly asymptotically compact and the base space $P$ is compact, it follows from [3, 9] that the associated autonomous semigroup $\Phi(t)$ has a global attractor $\mathcal{A}$ on $P \times X$ given by

$$\mathcal{A} = \bigcup_{p \in P} (\{p\} \times A(p)), \quad (3)$$

where $\{A(p)\}_{p \in P}$ is the pullback attractor of $(\theta, \phi)$, moreover, the compact set $D = \bigcup_{p \in P} A(p)$ is a uniform attractor of $(\theta, \phi)$.

For any fixed $\epsilon > 0$, write $D = P \times D$. Since $\mathcal{A}$ attracts $D$ under $\Phi$, one can pick a $T := T_\epsilon > 0$ independent of $p \in P$ such that

$$\Phi(t) D \subset N_{\epsilon}(A), \quad t \geq T.$$

Now set $\mathcal{A}_\epsilon = \bigcup_{s \geq T} \Phi(s) D$. It is easy to check that the set $\mathcal{A}_\epsilon$ is $\Phi$-positively invariant and the $p$-section $\mathcal{A}_\epsilon(p)$ of $\mathcal{A}_\epsilon$ satisfies

$$\mathcal{A}_\epsilon(p) = \bigcup_{s \geq T} \phi(s, \theta s p) D.$$

Consequently, we have

$$\Phi(t) \bigcup_{p \in P} (\{p\} \times \mathcal{A}_\epsilon(p)) \subset \bigcup_{p \in P} (\{p\} \times \mathcal{A}_\epsilon(p)) \quad \text{for all } t \geq 0.$$ 

That is

$$\bigcup_{p \in P} (\{\theta t p\} \times \phi(t, p) \mathcal{A}_\epsilon(p)) \subset \bigcup_{p \in P} (\{\theta t p\} \times \mathcal{A}_\epsilon(\theta t p)) \quad t \geq 0,$$

from which it can be deduced that the family $\{A_\epsilon(p)\}_{p \in P}$ is forward invariant under $(\theta, \phi)$. Clearly,

$$A(p) \subset \mathcal{A}_\epsilon(p) \subset N_{\epsilon}(A(p)).$$

Let $A^T_\epsilon(p) = \phi(T, \theta - T p) D$. Then the set-valued mapping $p \to A^T_\epsilon(p)$ is continuous at $p \in P$. Indeed, for the fixed $T > 0$ and the compact set $D \subset X$, it follows from the definition of $\phi$ that the map $\phi(T, \cdot) D$ is continuous from $P$ to $X$ in the sense of Hausdorff distance. Hence we deduce from the construction of $A^T_\epsilon(p)$ that the mapping $p \to A^T_\epsilon(p)$ is continuous at $p \in P$.

In what follows we show that the family of sets $\{A_c(p)\}_{p \in P}$ uniformly forward attracts each bounded set $B \subset X$. To this end, we first prove that $A^T_\epsilon(p)$ enjoys the property (2). We argue by contraction and suppose the result fails to be true. Then there would exist an $\epsilon > 0$, sequences $x_n \in B, p_n \in P$ and $t_n$ with $t_n \to \infty$ as $n \to \infty$ such that for each $n$,

$$d(\phi(t_n, p_n) x_n, A^T_\epsilon(\theta t_n p_n)) \geq \epsilon.$$ 

On the other hand, note that $P$ is compact. Then there are a subsequence of $\{\theta t_n p_n\}$, still denoted by $\{\theta t_n p_n\}$ and a $p_0 \in P$ so that

$$\theta t_n p_n \to p_0 \quad \text{as } n \to \infty.$$ 

Recalling that

$$d((\theta t_n p_n, \phi(t_n, p_n) x_n), A) = 0, \quad \text{as } n \to \infty,$$
we conclude that
\[ \phi(t_n, p_n)x_n \to x_0 \quad \text{as} \quad n \to \infty \]
for some \( x_0 \) with \( x_0 \in A(p_0) \subset A^T_x(p_0) \). By the continuity of \( A^T_x(p) \) at \( p \in P \), one can deduce that
\[ d_H(A^T_x(p_0), A^T_x(\theta_n p_n)) < \epsilon/2 \]
for \( n \) sufficiently large. Thus for sufficiently large \( n \), we have
\[ d(\phi(t_n, p_n)x_n, A^T_x(\theta_n p_n)) \leq d(\phi(t_n, p_n)x_n, A^T_x(p_0)) + d_H(A^T_x(p_0), A^T_x(\theta_n p_n)) < \epsilon, \]
which leads to a contradiction.

Now, let \( B \subset X \) be a bounded set. Then there exists \( T_B > 0 \) (independent of \( p \in P \)) such that
\[ \sup_{p \in P} d_H(\phi(t, p)B, A^T_x(\theta_t p)) < \epsilon, \quad \text{for all} \quad t \geq T_B. \]
Because for each \( p \in P \),
\[ d_H(\phi(t, p)B, A_x(\theta_t p)) \leq d_H(\phi(t, p)B, A^T_x(\theta_t p)), \]
we deduce that if \( t \geq T_B \),
\[ d_H(\phi(t, p)B, A_x(\theta_t p)) \leq \sup_{p \in P} d_H(\phi(t, p)B, A^T_x(\theta_t p)) < \epsilon \quad \text{for all} \quad p \in P, \]
from which one can immediately conclude that \( \{A_x(p)\}_{p \in P} \) uniformly forward attracts any bounded set \( B \subset X \).

**Remark 2.** In the above theorem, we see that for any \( \epsilon > 0 \),
\[ A(p) \subset A_x(p) \subset N_\epsilon(A(p)) \]
for each \( p \in P \). Then
\[ \delta_H(A_x(p), A(p)) \to 0 \quad \text{as} \quad \epsilon \to 0. \]
Moreover, the nonautonomous set \( A_x(\cdot) \) is forward attracting. Thus the set \( A_x(\cdot) \) we constructed may be used to describe the forward dynamical behavior of nonautonomous systems.

**Remark 3.** The continuity of the family \( \{A^T_x(p)\}_{p \in P} \) at \( p \in P \) implies some synchronous properties of \( A^T_x(p) \) with \( p \). For instance, if \( p \in P \) is (almost) periodic for \( t \in \mathbb{R} \), then the mapping \( t \to A^T_x(\theta_t p) \) is (almost) periodic.

Next, we continue to discuss the forward attraction rate of the nonautonomous set \( A_x(\cdot) \). For this purpose, we assume that the following conditions hold:

1. **(H)*** There exists an \( \epsilon_0 \)-neighborhood \( \{N_{\epsilon_0}(A(p))\}_{p \in P} \) of the pullback attractor \( \{A(p)\}_{p \in P} \) such that
   (i) There exists a \( \tau^* > 0 \) such that for each \( p \in P \), the operator \( \phi(\tau^*, p) \) is a compact perturbation of the contraction on \( N_{\epsilon_0}(A(p)) \):
   \[ \|\phi(\tau^*, p)u - \phi(\tau^*, p)v\|_X \leq \delta\|u - v\|_X + \|K(p)u - K(p)v\|_X \]
   for all \( u, v \in N_{\epsilon_0}(A(p)) \), where \( 0 < \delta < 1/2 \) and \( K(p) \) is an operator from \( N_{\epsilon_0}(A(p)) \) to \( Y \), which is a Banach space compactly embedded into \( X \) and satisfies
   \[ \|K(p)u - K(p)v\|_Y \leq L_1\|u - v\|_X, \quad u, v \in N_{\epsilon_0}(A(p)) \]
   with \( L_1 > 0 \) independent of \( p \).
Assume that the skew-product flow

\[ \theta, \phi \]

established in [17] to show that the skew-product flow \((\theta, \phi)\) has a pullback exponential attractor.

Theorem 3.2. Assume that the skew-product flow \((\theta, \phi)\) is uniformly asymptotically compact and let the assumption (H) hold true. Then the system \((\theta, \phi)\) has a pullback exponential attractor \(M(\cdot) = \{M(p)\}_{p \in P}\) with

\[ M(p) \subset N_{\epsilon_0}(A(p)) \quad \text{for all } p \in P, \]

where \(N_{\epsilon_0}(A(p))\) is given in assumption (H).

Proof. Since the system \((\theta, \phi)\) is uniformly asymptotically compact and the base space \(P\) is compact, we see that \(A\) defined by (3) is a global attractor of the autonomous semigroup \(\Phi(t)\).

For any fixed \(\epsilon > 0\), similar to the construction of \(A_\epsilon(\cdot) = \{A_\epsilon(p)\}_{p \in P}\) in the proof of Theorem 3.1, one can construct a family of sets \(B_\epsilon(\cdot) = \{B_\epsilon(p)\}_{p \in P}\) such that it is forward invariant and satisfies

\[ B_\epsilon(p) \subset N_{\epsilon_0}(A(p)) \quad \text{for all } p \in P. \]

Observing that

\[ A \subset \bigcup_{p \in P} \{\{p\} \times N_{\epsilon_0}(A(p))\}, \]

we see that for each bounded set \(B \subset X\), there is \(T_B > 0\) independent of \(p \in P\) such that

\[ \Phi(t)B \subset \bigcup_{p \in P} \{\{p\} \times N_{\epsilon_0}(A(p))\} \quad t \geq T_B, \]

where \(B = P \times X\). Then by the definition of \(\Phi(t)\), we have if \(t \geq T_B,\)

\[ \bigcup_{p \in P} \{\{\theta_p\} \times \phi(t, p) B\} \subset \bigcup_{p \in P} \{\{\theta_p\} \times N_{\epsilon_0}(A(\theta_p))\}, \]

which implies that the family \(\{N_{\epsilon_0}(A(p))\}_{p \in P}\) is uniformly absorbing for the skew-product flow \((\theta, \phi)\). Thus we may suppose that the \(T\) (independent of \(p \in P\)) in the construction of \(\{B_\epsilon(p)\}_{p \in P}\) is sufficiently large so that

\[ B_\epsilon(p) \subset N_{\epsilon_0}(A(p)) \quad \text{for all } p \in P. \]

Similarly, one can easily verify that the family \(\{B_\epsilon(p)\}_{p \in P}\) is uniformly absorbing as well. Note that \(\{B_\epsilon(p)\}_{p \in P}\) satisfies conditions (i) and (ii) in assumption (H) and is uniformly bounded with respect to \(p \in P\).

Now, based on the relationship between the notations of skew-product flows and processes (see (1)), we prove the existence of a pullback exponential attractor \(\{M(p)\}_{p \in P}\) for \((\theta, \phi)\) with \(M(p) \subset B_\epsilon(p)\) for all \(p \in P\). Observe that for each \(s \in \mathbb{R}, \theta_s : P \to P\) is a homeomorphism. Then for each \(p \in P\), there exists \(s \in \mathbb{R}\) such that \(p = \theta_s p_0\). Write

\[ B_\epsilon(s) := B_\epsilon(\theta_s p_0). \]

Clearly, \(\{B_\epsilon(t)\}_{t \in \mathbb{R}}\) is positively invariant for the process \(\{U(t, s)\}_{t \geq s}\) defined by (1).
We first show that the family of sets \( \{ B_c(s) \}_{c \in \mathbb{R}} \) satisfies the conditions (1)-(5) in [17, pp. 656], and hence the process \( \{ U(t,s) \}_{t \geq s} \) has a pullback exponential attractor \( \{ \tilde{M}(t) \}_{t \in \mathbb{R}} \) with \( \tilde{M}(t) \subset B_c(t) \) for all \( t \in \mathbb{R} \). Indeed, since \( \{ B_c(p) \}_{p \in P} \) satisfies conditions (i) and (ii) in assumption (H), we deduce that for any \( p \in P \) with \( p = \theta_s p_0 \) for some \( s \in \mathbb{R} \), and \( \tau^* > 0 \), we have

\[
\| \phi(\tau^*, \theta_s p_0) u - \phi(\tau^*, \theta_s p_0) v \|_X \leq \delta \| u - v \|_X + \| K(\theta_s p_0) u - K(\theta_s p_0) v \|_X
\]

for all \( u, v \in B_c(\theta_s p_0) \) and

\[
\| K(\theta_s p_0) u - K(\theta_s p_0) v \|_Y \leq L_1 \| u - v \|_X, \quad u, v \in B_c(\theta_s p_0).
\]

Moreover, for all \( \tau \in [0, \tau^*] \),

\[
\| \phi(\tau, \theta_s p_0) u - \phi(\tau, \theta_s p_0) v \|_X \leq L_2 \| u - v \|_X, \quad u, v \in B_c(\theta_s p_0).
\]

It is straightforward to get that

\[
\| U(\tau^* + s, s) u - U(\tau^* + s, s) v \|_X \leq \delta \| u - v \|_X + \| K(s) u - K(s) v \|_X
\]

for all \( u, v \in B_c(s) \),

\[
\| K(s) u - K(s) v \|_Y \leq L_1 \| u - v \|_X, \quad u, v \in B_c(s)
\]

and for all \( \tau \in [0, \tau^*] \),

\[
\| U(\tau + s, s) u - U(\tau + s, s) v \|_X \leq L_2 \| u - v \|_X, \quad u, v \in B_c(s),
\]

from which it can be seen that the family of sets \( \{ B_c(s) \}_{s \in \mathbb{R}} \) satisfies the conditions (4)-(5) in [17, pp. 656] for the process \( \{ U(t,s) \}_{t \geq s} \). Similarly, one can also check that \( \{ B(s) \}_{s \in \mathbb{R}} \) satisfies conditions (1)-(3) in [17, pp. 656]. Thus we conclude from the result on existence of pullback exponential attractors in [17, Theorem 2.1] that the process \( \{ U(t,s) \}_{t \geq s} \) defined by (1) has a pullback exponential attractor \( \{ \tilde{M}(t) \}_{t \in \mathbb{R}} \) with \( \tilde{M}(t) \subset B_c(t) \) for any \( t \in \mathbb{R} \).

For each \( t \in \mathbb{R} \), we define

\[
M(p) = \tilde{M}(t), \quad p = \theta_t p_0.
\]

It is clear that

\[
M(p) \subset B_c(p), \quad p \in P.
\]

Finally, we verify that the family \( \{ M(p) \}_{p \in P} \) is a pullback exponential attractor of the skew-product flow \((\theta, \phi)\). First, for each \( p \in P \), there exists \( s \in \mathbb{R} \) such that \( p = \theta_s p_0 \) (Note that for each \( s \in \mathbb{R} \), \( \theta_s : P \to P \) is a homeomorphism). Then it follows from the compactness of the section \( \tilde{M}(s) \) that \( M(p) \) is compact. Let us estimate the fractal dimension of \( M(p) \). Since the fractal dimension of \( \tilde{M}(t) \) is finite and uniformly bounded with respect to \( t \in \mathbb{R} \), we deduce that the fractal dimension of \( M(p) \) is finite and uniformly bounded for \( p \in P \) as well. Now we show that the family of sets \( \{ M(p) \}_{p \in P} \) is forward invariant under the action of \((\theta, \phi)\). By (1), (4) and the positive invariance of \( \{ \tilde{M}(t) \}_{t \in \mathbb{R}} \) for \( \{ U(t,s) \}_{t \geq s} \), we have that for \( \tau \geq 0 \) and \( p = \theta_s p_0 \),

\[
\phi(\tau, p) M(p) = \phi(\tau, \theta_s p_0) M(\theta_s p_0) = U(\tau + s, s) \tilde{M}(s) \subset \tilde{M}(\tau + s) = M(\theta_{\tau + s} p_0) = M(\theta_s p).
\]

To complete the argument, it remains to check the exponential attractivity of \( \{ M(p) \}_{p \in P} \). Let \( B \subset X \) be a bounded set. Then it follows from the definition
of the pullback exponential attractor $\{\tilde{M}(t)\}_{t \in \mathbb{R}}$ for $\{U(t,s)\}_{t \geq s}$ that for $\tau \geq 0$ and $p = \theta_{\tau} p_0$, there exist constants $\alpha > 0$, $C_B$ and $T_{p,B}$ depending on $T_{s,B}$ such that

$$d_H(\phi(\tau, \theta_{-\tau} p) B, M(p)) = d_H(\phi(\tau, \theta_{-\tau + \tau} p_0) B, M(\theta_{s} p_0))$$

$$= d_H(U(s, s - \tau) B, \tilde{M}(s)) \leq C_B e^{-\alpha \tau}, \quad \tau \geq T_{p,B}.$$ 

In a word, the family of sets $\{M(p)\}_{p \in P}$ is a pullback exponential attractor for the skew-product flow $(\theta, \phi)$ satisfying

$$M(p) \subset B_{t}(p) \subset N_{\epsilon_0}(A(p))$$

for all $p \in P$. The proof of the theorem is complete.

Now we further show that the family of subsets $\{A_{\epsilon}(p)\}_{p \in P}$ given in Theorem 3.1 uniformly forward exponentially attracts each bounded subset of $X$.

**Theorem 3.3.** Assume the skew-product flow $(\theta, \phi)$ is uniformly asymptotically compact and let the condition (H) hold true.

Then for any $\epsilon > 0$, the nonautonomous set $A_{\epsilon}(\cdot)$ constructed in Theorem 3.1 uniformly forward exponentially attracts each bounded subset of $X$.

**Proof.** Since the skew-product flow $(\theta, \phi)$ is uniformly asymptotically compact, we infer from [4] that the system $(\theta, \phi)$ has a (global) pullback attractor $\{A(p)\}_{p \in P}$, moreover, the set $A$ defined by (3) is the global attractor for $(\theta, \phi)$.

For any $\epsilon > 0$, note that the nonautonomous set $A_{\epsilon}(\cdot)$ is forward invariant under $(\theta, \phi)$. Furthermore, similar to the argument of the nonautonomous set $B_{\epsilon}(\cdot)$ in the proof of Theorem 3.2, it can be shown that $A_{\epsilon}(\cdot)$ is uniformly absorbing and $A_{\epsilon}(p) \subset N_{\epsilon_0}(A(p))$, for all $p \in P$.

Since the set $A_{\epsilon}(\cdot)$ satisfies the same conditions as those of $B_{\epsilon}(\cdot)$ in the proof of Theorem 3.2, there exists a pullback exponential attractor $\{M_{\epsilon}(p)\}_{p \in P}$ of $(\theta, \phi)$ such that

$$M_{\epsilon}(p) \subset A_{\epsilon}(p)$$

for all $p \in P$. As the pullback attractor $\{A(p)\}_{p \in P}$ is invariant under $(\theta, \phi)$, it holds that $A(p) \subset M_{\epsilon}(p)$ for all $p \in P$. Therefore, for each $p \in P$, we have

$$A(p) \subset M_{\epsilon}(p) \subset A_{\epsilon}(p) \subset N_{\epsilon}(A(p)).$$

(5)

Now for each bounded set $B \subset X$, it follows from the construction of a pullback exponential attractor (see [17, pp. 661]) and its definition that

$$d_H(\phi(t, \theta_{-t} p) A_{\epsilon}(\theta_{-t} p), M_{\epsilon}(p)) \leq C e^{-\alpha t}, \quad \text{for all } p \in P \text{ and } t \geq 0,$$ 

(6)

where $C, \alpha$ are some positive constants independent of $p \in P$, and $C$ depends on the boundedness of the family $\{A_{\epsilon}(p)\}_{p \in P}$ uniformly for $p \in P$. Recalling that $\{A_{\epsilon}(p)\}_{p \in P}$ is uniformly absorbing, we infer from (6) that there exists $T_B > 0$ (independent of $p$) such that

$$d_H(\phi(t, \theta_{-t} p) B, M_{\epsilon}(p)) \leq C_1 e^{-\alpha t}, \quad \text{for all } t \geq T_B,$$

where $C_1$ is a positive constant depending on $C$ and $T_B$. As $M_{\epsilon}(p) \subset A_{\epsilon}(p)$ for all $p \in P$, one concludes that

$$d_H(\phi(t, \theta_{-t} p) B, A_{\epsilon}(p)) \leq d_H(\phi(t, \theta_{-t} p) B, M_{\epsilon}(p)).$$

Thus if $t \geq T_B$, it holds

$$d_H(\phi(t, \theta_{-t} p) B, A_{\epsilon}(p)) \leq C_1 e^{-\alpha t}.$$
for all \( p \in P \). Thus
\[
\sup_{p \in P} d_H (\phi(t, \theta_{-p} p) B, A_\varepsilon(p)) \leq C_1 e^{-\alpha t}, \quad \text{for all } t \geq T_B,
\]
which shows that
\[
\sup_{p \in P} d_H (\phi(t, p) B, A_\varepsilon(\theta_p)) \leq C_1 e^{-\alpha t}, \quad \text{for all } t \geq T_B.
\]
This completes the proof of the theorem. \( \square \)

**Remark 4.** By (5), one can deduce that
\[
\delta_H (M_\varepsilon(p), A(p)) \to 0, \quad \delta_H (A_\varepsilon(p), M_\varepsilon(p)) \to 0 \quad \text{as } \varepsilon \to 0.
\]

In the proof of the above theorem, one can see that the existence of pullback exponential attractors plays an important role in showing the exponential attraction of the set \( A_\varepsilon(\cdot) \). Thus combining Theorem 3.3 with the results on the existence of pullback exponential attractors, see [27, Theorem 2], we can obtain the following results on a Hilbert space \( H \), which seem to be more convenient in applications.

**Theorem 3.4.** Let \( (\theta, \phi) \) be a skew-product flow on \( P \times H \). Assume that there exists a compact uniformly absorbing set \( B_0 \) for \( (\theta, \phi) \), so there is a \( T_0 := T(B_0) > 0 \) (independent of \( p \)) with
\[
\phi(t, p) B_0 \subset B_0, \quad \text{for all } t \geq T_0 \text{ and all } p \in P.
\]

In addition, suppose that

\( H_1 \) there exist \( T^* > T_0 \) and \( L = L_{T^*} > 0 \) such that for all \( p \in P \),
\[
\| \phi(t, p) u - \phi(t, p) v \|_H \leq L \| u - v \|_H, \quad u, v \in B_0, \quad t \in [T_0, T^*];
\]

\( H_2 \) there exist a number \( 0 < \lambda < 1/2 \) and a finite dimensional subspace \( H_m \) along with a projection \( P_m : H \to H_m \) such that for all \( p \in P \),
\[
\|(I - P_m)(\phi(T^*, p) u - \phi(T^*, p) v)\|_H \leq \lambda \| u - v \|_H, \quad u, v \in B_0,
\]
where \( \gamma \) and \( m \in \mathbb{N} \) (depending on \( T^* \)) are independent of \( p \).

Then for any \( \varepsilon > 0 \), there exists a family of sets \( \{ A_\varepsilon(p) \}_{p \in P} \), which is forward invariant under the acting of system \( (\theta, \phi) \), such that \( \{ A_\varepsilon(p) \}_{p \in P} \) uniformly forward exponentially attracts any bounded subset of \( H \).

**Proof.** First, since the set \( B_0 \) is compact uniformly absorbing, it follows that the system \( (\theta, \phi) \) is uniformly asymptotically compact. Define
\[
B(p) = \bigcup_{s \geq T_0} \phi(s, \theta_{-s} p) B_0.
\]

Then it is trivial to see that the family \( \{ B(p) \}_{p \in P} \) is forward invariant under \( (\theta, \phi) \). Because \( B_0 \) is uniformly absorbing, one can easily verify that the family of sets \( \{ B(p) \}_{p \in P} \) is uniformly absorbing as well.

For any fixed \( \varepsilon > 0 \), let \( A_\varepsilon(\cdot) \) be the nonautonomous set constructed in Theorem 3.1. Then we may assume that the \( T \) (independent of \( p \) in \( P \)) in the construction of \( A_\varepsilon(\cdot) \) is sufficiently large so that \( A_\varepsilon(p) \subset B(p) \) for each \( p \in P \). By \( H_2 \), we conclude that for any \( p \in P \) and \( u, v \in A_\varepsilon(p) \),
\[
\| \phi(T^*, p) u - \phi(T^*, p) v \|_H
\leq \lambda \| u - v \|_H + \| P_m (\phi(T^*, p) u - \phi(T^*, p) v) \|_H.
\]
Thus by the construction of pullback exponential attractors in the proof of [27, Theorem 1] (see also [27, Theorem 2]) and the proof of Theorem 3.4, one can conclude that the system \((\theta, \phi)\) has a pullback exponential attractor \(\{M_\epsilon(p)\}_{p \in P}\) with \(M_\epsilon(p) \subset A_\epsilon(p) \subset \mathcal{B}(p)\) for all \(p \in P\). Repeating the same argument below (5), it can be shown that the family of sets \(\{A_\epsilon(p)\}_{p \in P}\) fulfill all the requirements of the theorem. □

4. Applications. As an application of our main results, we consider the following initial boundary value problem:

\[
\begin{cases}
    u_t - \Delta u = \mu u + g(x, t), & x \in \Omega; \\
    u(x, t) = 0, & x \in \partial \Omega,
\end{cases}
\] (7)

where \(\Omega \subset \mathbb{R}^n\) is a bounded domain, \(\mu > 0\), and \(g\) is a bounded continuous function.

Let \(X = L^2(\Omega)\) and \(Y = H_0^1(\Omega)\). Denote \((\cdot, \cdot)\) and \(\| \cdot \|\) the usual inner product and norm on \(X\), respectively. The norm \(\| \cdot \|_Y\) on \(Y\) is defined by

\[\|u\|_Y = \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2}, \quad u \in Y.\]

By \(A\) we denote the operator \(-\Delta\) associated with the homogenous Dirichlet boundary condition. Then \(A : H^2(\Omega) \cap Y \to X\) is a sectorial operator. Note that for each \(\lambda \in \rho(A)\), the resolvent set of \(A\), \((\lambda I - A)^{-1}\) is an operator from \(X\) to \(H^2(\Omega) \cap Y\). We deduce that \((\lambda I - A)^{-1}\) is a compact operator as for the embedding \(H^2(\Omega) \cap Y \hookrightarrow X\) is compact. Thus \(A\) has a compact resolvent. Let \(\{e_i\}_{i \in \mathbb{N}}\) be a complete orthogonal basis of \(X\) with

\[Ae_i = \lambda_i e_i, \quad (e_i, e_j) = \delta_{i,j}, \quad i, j = 1, 2, \ldots.\]

It is easy to see that

\[0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_i \leq \cdots \quad \text{and} \quad \lambda_i \to \infty \quad \text{as} \quad i \to \infty.\]

Set

\[X_m = \text{span}\{e_1, e_2, \ldots, e_m\}\]

and \(\widetilde{X}_m = (X_m)\perp\). Define the projector operator \(P_m : X \to X_m\). Then for each \(u \in X\), it holds that

\[u = P_m u + (I - P_m) u := u_1 + u_2, \quad u_1 \in X_m, \quad u_2 \in \widetilde{X}_m.\]

Denote by \(C_b(\mathbb{R}, X)\) the set that consists of bounded continuous functions from \(\mathbb{R}\) to \(X\). We equip \(C_b(\mathbb{R}, X)\) with the compact-open topology generated by the metric

\[g(f_1, f_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\max_{t \in [-n,n]} \|f_1(t) - f_2(t)\|}{1 + \max_{t \in [-n,n]} \|f_1(t) - f_2(t)\|}.\]

Then \(C_b(\mathbb{R}, X)\) is a complete metric space.

Let \(g(x, t)\) be a function in (7). Define \(\hat{g}(t) = g(\cdot, t)\). Let \(\hat{g}(t) \in C_b(\mathbb{R}, X)\) and define the hull \(\mathcal{H}(\hat{g})\) of \(\hat{g}\) as

\[P := \mathcal{H}(\hat{g}) = \left\{ \hat{g}(\tau + \cdot) : \tau \in \mathbb{R} \right\}^{C_b(\mathbb{R}, X)}.\]

The translation operator \(\theta = \{\theta_t\}_{t \in \mathbb{R}}\) on \(P\) is defined by

\[\theta_t p = p(t + \cdot), \quad p \in P, \quad t \in \mathbb{R}.\]

Henceforth we assume that \(\hat{g}(t)\) and the constant \(\mu\) satisfy the following conditions:
Remark 5. The space $X^\alpha$ is a continuous function. Thanks to the basic theory on evolution equations in Banach spaces (see Henry [18] or [22]), the Cauchy problem of (8) is well-posed. Specifically, for each $p \in P$ and $t_0 \in \mathbb{R}$, $u_0 \in X^\alpha(0 \leq \alpha < 1)$, the equation (8) has a unique solution $u(t) = u(t, t_0; u_0, p)$ with $u(t_0) = u_0$ such that
\[
\begin{align*}
\frac{du}{dt} + A u &= \mu u + p(t), \quad p(t) \in P, \\
\end{align*}
\]
which is a continuous function.

Remark 5. The space $X^\alpha(\alpha \geq 0)$ with the norm $\| \cdot \|_\alpha$ defined by
\[
\| u \|_\alpha = \| A^\alpha u \|, \quad u \in X^\alpha
\]
is called the fractional power of $X$, see [18, Chapter I] for details. In particular, $X^0 = X$.

Set
\[
\phi(t, p)u_0 = u(t, 0; u_0, p) := u(t), \quad u_0 \in X^\alpha, \quad p \in P.
\]
Then $\phi$ is a cocycle semiflow on $X^\alpha$, driven by the translation group $\theta$ on $P$.

Denote $\sigma(A)$ the spectral of $A$ and write $\Re \sigma(A) = \{ \Re z : z \in \sigma(A) \}$.

Lemma 4.1. Let the assumption (G) hold true. Then the NDS $(\theta, \phi)$ has a compact uniformly absorbing set $B_0 \subset X^\alpha$. 

Proof. Let $R > 0$ so that $\sup_{t \in \mathbb{R}} \| \tilde{g}(t) \| \leq R$. Let $\beta \in (\alpha, 1)$. For any $p \in P$, by (9), we have
\[
\| u(t) \|_\beta \leq \| e^{-(A-\mu I)(t-t_0)}u_0 \|_\beta + \int_{t_0}^{t} \| e^{-(A-\mu I)(t-s)}p(s) \|_\beta ds.
\]
By some simple fundamental computations, one has
\[
\| u(t) \|_\beta \leq C_2(t-t_0)^{-(\beta-\alpha)}e^{-\delta(t-t_0)}\| u_0 \|_{\alpha} + C_2R \int_{t_0}^{t}(t-s)^{-\beta}e^{-\delta(t-s)}ds,
\]
where $C_2$ and $\delta$ are positive constants with $\Re \sigma(A - \mu I) \geq \delta$. Set
\[
R_\infty := 2C_2R \int_{0}^{\infty} u^{-\beta}e^{-\delta u}du
\]
and define $B_0 := \{ u \in X : \| u \|_\beta \leq R_\infty \}$. Note that if $\alpha < \beta < 1$, then the inclusion $X^\beta \subset X^\alpha$ is compact. We conclude from (10) that the set $B_0$ is the desired compact uniformly absorbing subset of $X^\alpha$.

By Lemma 4.1, we deduce that there exists $T_0 > 0$ (independent of $p \in P$) such that
\[
\phi(t, p)B_0 \subset B_0 \quad \text{for all } t \geq T_0 \text{ and all } p \in P.
\]
Lemma 4.2. For any $T > T_0$, there exists a $L_T > 0$ (independent of $p \in P$) such that for all $p \in P$ and $t \in [T_0, T]$,
\[
\| \phi(t, p) u_0 - \phi(t, p) v_0 \| \leq L_T \| u_0 - v_0 \|, \quad \forall u_0, v_0 \in B_0.
\]

Proof. Let
\[
w(t) = \phi(t, p) u_0, \quad v(t) = \phi(t, p) v_0, \quad w(t) = u(t) - v(t).
\]
Then
\[
\frac{d}{dt} w - \Delta w = \mu w.
\]
Taking the inner product of (11) with $w$, we have
\[
\frac{1}{2} \frac{d}{dt} \| w \|^2 + \| \nabla w \|^2 = \mu \| w \|^2.
\]
It follows that
\[
\frac{d}{dt} \| w \|^2 \leq 2 \mu \| w \|^2.
\]
Applying Gronwall inequality on $[T_0, T]$ yields
\[
\| w(t) \|^2 \leq e^{C_T} \| w_0 \|^2, \quad t \in [T_0, T],
\]
where $C_T = 2 \mu$. The proof is complete. \qed

Lemma 4.3. There exist positive constants $T^*, \gamma \in [0, 1)$ and an $M$-dimensional orthogonal projection $P_M : X \to X_M$ such that for each $p \in P$,
\[
\| (I - P_M) (\phi(T^*, p) u_0 - \phi(T^*, p) v_0) \| \leq \gamma \| u_0 - v_0 \|, \quad u_0, v_0 \in B_0.
\]

Proof. Let $u_0, v_0 \in B_0$ and
\[
w(t) = \phi(t, p) u_0 - \phi(t, p) v_0, \quad w_0 = u_0 - v_0, \quad p \in P.
\]
Set
\[
w = P_m w + (I - P_m) w := w_1 + w_2, \quad w_1 \in X_m, \quad w_2 \in \bar{X}_m.
\]
Taking the inner product of (11) with $w_2$ in $X$, we have
\[
\frac{1}{2} \frac{d}{dt} \| w_2 \|^2 + \| \nabla w_2 \|^2 = \mu \| w_2 \|^2.
\]
Note that $\| \nabla w_2 \|^2 \geq \lambda_{m+1} \| w_2 \|^2$. Then one can pick an $M$ such that $\lambda_{M+1} > \mu$. Thus it follows from (12) that
\[
\frac{d}{dt} \| w_2 \|^2 + 2 (\lambda_{M+1} - \mu) \| w_2 \|^2 \leq 0.
\]
By Gronwall inequality on $[0, t]$, we have
\[
\| w_2(t) \|^2 \leq e^{-2(\lambda_{M+1} - \mu) t} \| w_2(0) \|^2.
\]
Now pick a $T^* > T_0$ sufficiently large so that $e^{-2(\lambda_{M+1} - \mu) T^*} < 1/4$. Therefore we have
\[
\| w_2(T^*) \| \leq \gamma \| w(0) \|,
\]
where $\gamma = e^{-(\lambda_{M+1} - \mu) T^*}$, which completes the proof of the Lemma. \qed

By virtue of Lemmas 4.1-4.3 and Theorem 3.4, we can obtain the following results for (8).

Theorem 4.4. Let the assumption (G) hold true. Then for any $\varepsilon > 0$, there exists a family of sets $\{ A_\varepsilon(p) \}_{p \in P}$ of the system (7), which is forward invariant such that $\{ A_\varepsilon(p) \}_{p \in P}$ uniformly forward exponentially attracts each bounded subsets of $X$. 

Remark 6. In the above example, we only consider the linear equation (7) to illustrate our results. In fact, we may study the nonlinear equation with the term \( \mu u \) in (7) replaced by some nonlinear term \( f(u) \). Under some suitable conditions on \( f \), one can also show the vitality of Theorem 4.4. Here we will not pursue the details.

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