THE COARSE BAUM-CONNES CONJECTURE FOR RELATIVELY HYPERBOLIC GROUPS

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Abstract. We study a group which is hyperbolic relative to a finite family of infinite subgroups. We show that the group satisfies the coarse Baum-Connes conjecture if each subgroup belonging to the family satisfies the coarse Baum-Connes conjecture and admits a finite universal space for proper actions. Especially, the group satisfies the analytic Novikov conjecture.

1. Introduction

Let $X$ be a proper metric space. We say that $X$ satisfies the coarse Baum-Connes conjecture if the following coarse assembly map $\mu_X$ of $X$ is an isomorphism:

$$\mu_X : KX_*(X) \to K_*(C^*(X)).$$

If a countable group $G$ equipped with a proper invariant metric satisfies the coarse Baum-Connes conjecture, and if $G$ admits a finite $G$-simplicial complex which is a universal space for proper actions, then, by a descent principle, $G$ satisfies the analytic Novikov conjecture. For details, see [17, Theorem 8.4] and also [7, Theorem 12.6.3].

There are several studies on the coarse Baum-Connes conjecture for relatively hyperbolic groups. Let $G$ be a group which is hyperbolic relative to a finite family of infinite subgroups $\mathbb{P} = \{P_1, \ldots, P_k\}$. Osin [14] showed that $G$ has finite asymptotic dimension if each subgroup $P_i$ has finite asymptotic dimension. Ozawa [15] showed that $G$ is exact if each subgroup $P_i$ is exact. Dadarlat and Guentner [2] showed that $G$ is uniformly embeddable in a Hilbert space if each subgroup $P_i$ is uniformly embeddable in a Hilbert space. Due to Yu’s works [19][20], those results imply the coarse Baum-Connes conjecture for such groups.

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In the present paper, we show the following:

**Theorem 1.1.** Let $G$ be a finitely generated group and $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a finite family of infinite subgroups. Suppose that $(G, \mathcal{P})$ is a relatively hyperbolic group. If each subgroup $P_i$ satisfies the coarse Baum-Connes conjecture, and admits a finite $P_i$-simplicial complex which is a universal space for proper actions, then $G$ satisfies the coarse Baum-Connes conjecture.

We note that $G$ admits a finite $G$-simplicial complex which is a universal space for proper actions (see Appendix B).

Here we summarize the proof of Theorem 1.1. Let $X(G, \mathcal{P}, S)$ be the augmented space obtained by attaching horoballs to the Cayley graph $\Gamma(G, S)$ along the left cosets of subgroups $P \in \mathcal{P}$ where $S$ is a finite generating set (Definition 2.1 and Definition 2.2). Since $X(G, \mathcal{P}, S)$ is $\delta$-hyperbolic, $X(G, \mathcal{P}, S)$ satisfies the coarse Baum-Connes conjecture. We fix an order on horoballs. Let $X_n$ be a subspace obtained by removing the first $n - 1$ horoballs from $X(G, \mathcal{P}, S)$ (Notation 5.1). By Mayer-Vietoris arguments, we show inductively that $X_n$ satisfies the coarse Baum-Connes conjecture (Section 5.1). To study the coarse assembly map for $X_\infty = \bigcap X_n$, which is coarsely equivalent to $G$, we need to analyze the coarse $K$-homology of the projective limit. We might expect a so-called Milnor exact sequence

$$0 \rightarrow \lim\limits_{\leftarrow} \ker KX_{p+1}(X_n) \rightarrow KX_p(X_\infty) \rightarrow \lim\limits_{\rightarrow} KX_p(X_n) \rightarrow 0.$$

Unfortunately, (1) is not necessarily exact, in general. A simple counterexample is given by $Y_n = \mathbb{R} \setminus [-n, n]$. Thus we introduce a contractible space $EX(G, \mathcal{P})$. The following isomorphism (Proposition 3.1) is crucial to the proof of Theorem 1.1

$$KX_*(X(G, \mathcal{P}, S)) \cong K_*(EX(G, \mathcal{P})).$$

Sections 2 and 3 are devoted to a proof of this isomorphism. For the projective limit of locally compact Hausdorff spaces, there is a Milnor exact sequence in $K$-homology (Section 5.2). Combining this with an exact sequence in $K$-theory of $C^*$-algebras (Proposition 5.3), we complete the proof.

2. **Coarse K-homology of the augmented space**

Let $G$ be a finitely generated group with a finite family of infinite subgroups $\mathcal{P} = \{P_1, \ldots, P_k\}$. Groves and Manning [4] introduced a space obtained by attaching “combinatorial horoballs” to $G$ along the left cosets of subgroups $P \in \mathcal{P}$. Their construction is
suitable for Mayer-Vietoris arguments to compute the coarse K-homology of \( G \) in terms of that of \( P \in \mathbb{P} \). We review the construction and study the coarse K-homology of the resulting space.

2.1. The augmented space.

**Definition 2.1.** Let \((P, d)\) be a proper metric space. The *combinatorial horoball* based on \( P \), denoted by \( \mathcal{H}(P) \), is the graph defined as follows:

1. \( \mathcal{H}(P)^{(0)} = P \times (\mathbb{N} \cup \{0\}) \).
2. \( \mathcal{H}(P)^{(1)} \) contains the following two type of edges:
   a. For each \( l \in \mathbb{N} \cup \{0\} \) and \( p, q \in P \), if \( 0 < d(p, q) \leq 2^l \) then there is a *horizontal edge* connecting \((p, l)\) and \((q, l)\).
   b. For each \( l \in \mathbb{N} \cup \{0\} \) and \( p \in P \), there is a *vertical edge* connecting \((p, l)\) and \((p, l + 1)\).

Here \( \mathbb{N} \) denotes the set of positive integers. We endow \( \mathcal{H}(P) \) with the graph metric.

For a closed subset \( I \subset \mathbb{R} \), let \( \mathcal{H}(P; I) \) denote the full subgraph of \( \mathcal{H}(P) \) spanned by \( P \times (I \cap (\mathbb{N} \cup \{0\})) \).

Let \( G \) be a finitely generated group with a finite family of infinite subgroups \( \mathbb{P} = \{P_1, \ldots, P_k\} \). We take a finite generating set \( S \) for \( G \). We assume that \( S \) is symmetrized, so that \( S = S^{-1} \). We endow \( G \) with the left-invariant word metric \( d_S \) with respect to \( S \). We choose a sequence \( g_1, g_2, \ldots \) in \( G \) such that for each \( r \in \{1, \ldots, k\} \), the map \( \mathbb{N} \to G/P_r : a \mapsto g_{ak+r}P_r \) is bijective. For \( i = ak + r \in \mathbb{N} \), let \( P_{(i)} \) denote a subgroup \( P_r \). Thus the set of all cosets \( \bigsqcup_{r=1}^{k} G/P_r \) is indexed by the map \( \mathbb{N} \ni i \mapsto g_{i}P_{(i)} \). Each coset \( g_{i}P_{(i)} \) has a proper metric \( d_{i} \) which is the restriction of \( d_S \). Let \( \Gamma \) be the Cayley graph of \((G, S)\). There exists a natural embedding \( \psi_i : \mathcal{H}(g_{i}P_{(i)}; \{0\}) \hookrightarrow \Gamma \) such that \( \psi_i(x, 0) = x \) for all \( x \in g_{i}P_{(i)} \).

**Definition 2.2.** The *augmented space* \( X(G, \mathbb{P}, S) \) is obtained by pasting \( \mathcal{H}(g_{i}P_{(i)}) \) to \( \Gamma \) by \( \psi_i \) for all \( i \in \mathbb{N} \). Thus we can write it as follows:

\[
X(G, \mathbb{P}, S) = \Gamma \cup \bigsqcup_{i \in \mathbb{N}} \mathcal{H}(g_{i}P_{(i)}).
\]
We endow $X(G, \mathbb{P}, S)$ with the graph metric. For positive integer $N$, set

$$X(N) = \Gamma \cup \bigcup_{i \in \mathbb{N}} \mathcal{H}(g_i P_{(i)}; [0, N]);$$
$$Y(N) = \bigsqcup_{i \in \mathbb{N}} \mathcal{H}(g_i P_{(i)}; [N, \infty]);$$
$$Z(N) = \bigsqcup_{i \in \mathbb{N}} \mathcal{H}(g_i P_{(i)}; \{N\}).$$

**Remark 2.3.** The vertex set of $X(G, \mathbb{P}, S)$, denoted by $X(G, \mathbb{P}, S)^{(0)}$, can naturally be identified with the set of 2-tuple $(x, t)$, where $x \in \bigcup_{i \in \mathbb{N}} g_i P_{(i)}$ and $t \in \mathbb{N}$, or $x \in G$ and $t = 0$. We endow $X(G, \mathbb{P}, S)^{(0)}$ with the metric from the graph structure.

**Definition 2.4.** The pair $(G, \mathbb{P})$ is a relatively hyperbolic group if the augmented space $X(G, \mathbb{P}, S)$ is $\delta$-hyperbolic for some $\delta \geq 0$.

**Remark 2.5.** Groves and Manning [4, Theorem 3.25] show that the above definition is equivalent to other various definitions. See also [9].

### 2.2. An anti-Čech system.

We form an anti-Čech system $\{U(j)\}_j$ of $X(G, \mathbb{P}, S)^{(0)}$ as follows: For $i \geq 1, (x, t) \in g_i P_{(i)} \times \mathbb{N}$ and $j \geq 1$, a column centered at $(x, t)$ with the size $j$ is

$$B((x, t), j) = \{(y, l) \in g_i P_{(i)} \times \mathbb{N} : d_S(x, y) \leq 2^{t+j}, t \leq l \leq t + j\}.$$  

For $x \in G$ and $j \geq 1$, a column centered at $(x, 0)$ with the size $j$ is

$$B((x, 0), j) = \{(y, l) \in X(G, \mathbb{P}, S)^{(0)} : d_S(x, y) \leq 2^j, 0 \leq l \leq j\}.$$  

The locally finite cover $U(j)$ is made up of all those columns with size $j$, that is,

$$U(j) = \{B((x, t), j) : (x, t) \in X(G, \mathbb{P}, S)^{(0)}\}.$$  

When $j \leq j'$, the map $U(j) \rightarrow U(j')$ is defined by sending $B((x, t), j)$ to $B((x, t), j')$. 

2.3. *Mayer-Vietoris sequences.* Set \( j_n = 3^n, N_n = 3^n + 1 \) for \( n \geq 0 \). We introduce a decomposition of \( \mathcal{U}(j_n) \) as follows:

\[
\mathcal{U}_n = \mathcal{U}(j_n);
\]

\[
\mathcal{X}_n = \{ B \in \mathcal{U}(j_n) : B \cap X(N_n) \neq \emptyset \};
\]

\[
\mathcal{Y}_n = \{ B \in \mathcal{U}(j_n) : B \cap Y(N_n) \neq \emptyset \};
\]

\[
\mathcal{Z}_n = \{ B \in \mathcal{U}(j_n) : B \cap Z(N_n) \neq \emptyset \};
\]

\[
\mathcal{Z}_i^n = \{ B \in \mathcal{Z}_n : B \cap H(g_i P(i)) \neq \emptyset \}.
\]

We remark that \( \mathcal{U}_n = \mathcal{X}_n \cup \mathcal{Y}_n, \mathcal{X}_n \cap \mathcal{Y}_n = \mathcal{Z}_n \) and \( \mathcal{Z}_n = \bigsqcup_i \mathcal{Z}_i^n \). Then the pair \( (\mathcal{X}_n, \mathcal{Y}_n) \) forms an excision pair of \( \mathcal{U}_n \) and the map \( \mathcal{U}_n \to \mathcal{U}_{n+1} \) preserves the pairs. Thus we have the following exact sequence:

\[
\cdots \to \lim_{\leftarrow} K_p(|\mathcal{X}_n|) \to \lim_{\leftarrow} K_p(|\mathcal{X}_n|) \oplus \lim_{\leftarrow} K_p(|\mathcal{Y}_n|) \to \lim_{\leftarrow} K_p(|\mathcal{U}_n|) \to \lim_{\leftarrow} K_{p-1}(|\mathcal{Z}_n|) \to \cdots.
\]

Since \( \{\mathcal{U}_n\}_n \) forms an anti-Čech system of \( X(G, \mathbb{P}, S)^{(0)} \), we have \( \lim_{\leftarrow} K_*(|\mathcal{U}_n|) = KX_*(X(G, \mathbb{P}, S)) \).

In this section, we compute \( \lim_{\leftarrow} K_*(|\mathcal{X}_n|) \) and \( \lim_{\leftarrow} K_*(|\mathcal{Y}_n|) \).

**Lemma 2.6.** The inductive limit of \( K_*(|\mathcal{X}_n|) \) is isomorphic to \( KX_*(X(1)) \).

**Proof.** For \( N \geq j + 1 \geq 0 \), we define that the subset \( \mathcal{U}(N, j) \) of \( \mathcal{U}(j) \) is made up of all columns \( B((x, t), j) \in \mathcal{U}(j) \) which intersect with \( X(N) \). We remark that \( \mathcal{X}_n = \mathcal{U}(N_n, j_n) \).

We define simplicial maps \( \alpha_n, \beta_n, \gamma_n \) by

\[
\alpha_n : \mathcal{U}(N_n, j_n) \to \mathcal{U}(N_n, j_n)
\]

: \( B((x, t), j_n) \mapsto B((x, t), j_n) \),

\[
\beta_n : \mathcal{U}(N_n, j_n) \to \mathcal{U}(N_n, j_n+1)
\]

: \( B((x, t), j_n) \mapsto \begin{cases} B((x, 1), j_{n+1}) & (t \geq 1) \\ B((x, 0), j_{n+1}) & (t = 0), \end{cases} \)

\[
\gamma_n : \mathcal{U}(N_n, j_n) \to \mathcal{U}(N_n+1, j_{n+1})
\]

: \( B((x, t), j_n) \mapsto B((x, t), j_{n+1}) \).

Clearly \( \alpha_{n+1} \circ \beta_n \) and \( \gamma_n \) belong to the same contiguity class. Since two simplicial maps belonging to the same contiguity class define continuous maps which are homotopic [18, Lemma 5.5.2.], we have the following commutative diagram:

\[
\begin{array}{c}
K_*(|\mathcal{U}(1, j_n)|) \xrightarrow{\alpha_n^*} K_*(|\mathcal{U}(N_n, j_n)|) \\
\downarrow \quad \downarrow \quad \downarrow \\
K_*(|\mathcal{U}(1, j_{n+1})|) \xrightarrow{\alpha_{n+1}^*} K_*(|\mathcal{U}(N_{n+1}, j_{n+1})|).
\end{array}
\]
It follows that \( \lim_{\to} K_*(|\mathcal{U}(1, j_n)|) \cong \lim_{\to} K_*(|\mathcal{U}(N_n, j_n)|) \).

Let \( \mathcal{U}(1, j_n) \cap X(1) \) denote the cover of \( X(1) \) which consists of all \( B \cap X(1) \) for \( B \in \mathcal{U}(1, j_n) \). Then \( \mathcal{U}(1, j_n) \cap X(1) \) forms an anti-Čech system of \( X(1) \). Since \( |\mathcal{U}(1, j_n)| \) \( X(1) \), we have \( K_* X_*(X(1)) = \lim_{\to} K_* (|X_n|) \). \( \square \)

**Lemma 2.7.** The inductive limit of \( K_*(|X_1|) \) is trivial.

**Proof.** For an integer \( s \geq 0 \), we define a simplicial map \( q_{n,s} : Y_n \to Y_{n+1} \) by

\[
q_{n,s}(B((x, t), j_n)) = \begin{cases} 
B((x, t), j_{n+1}) & \text{if } t \geq s, \\
B((x, s), j_{n+1}) & \text{if } t < s.
\end{cases}
\]

Clearly \( q_{n,s} \) and \( q_{n,s+1} \) are contiguous. Let \( h_{n,s} : [s, s+1] \times |Y_n| \to |Y_{n+1}| \) be a proper homotopy between geometric realizations of \( q_{n,s} \) and \( q_{n,s+1} \). We define a proper map \( q_n : \mathbb{R}_{\geq 0} \times |Y_n| \to |Y_{n+1}| \) by \( q_n(\theta, x) = h_{n,[\theta]}(\theta, x) \), where \( \theta \in \mathbb{R}_{\geq 0} \), \( x \in |Y_n| \), and \( [\theta] \) denotes the largest integer not greater than \( \theta \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
|Y_n| & \xrightarrow{q_{n,s}} & |Y_{n+1}| \\
\downarrow & & \downarrow \\
\mathbb{R}_{\geq 0} \times |Y_n| & \xrightarrow{q_n} & \mathbb{R}_{\geq 0} \times |Y_{n+1}|
\end{array}
\]

Here the horizontal arrow is the canonical map and the map \( |Y_n| \to \mathbb{R}_{\geq 0} \times |Y_n| \) is given by the inclusion onto \( \{0\} \times |Y_n| \). Since \( \mathbb{R}_{\geq 0} \times |Y_n| \) is contractible (see [7, Remark 7.1.4]), the homomorphism \( K_*(|Y_n|) \to K_*(|Y_{n+1}|) \) factors through zero. Therefore, \( \lim_{\to} K_*(|Y_n|) = 0 \). \( \square \)

By the cluster axiom of \( K \)-homology (see [7, Definition 7.3.1]), we have \( K_* (|X_1|) \cong \prod_{i \geq 1} K_* (|Z_{1i}|) \). Therefore we have the following exact sequence:

\[
\cdots \to \lim_{\to} \prod_{i \geq 1} K_* (|Z_{1i}|) \to K_* (X(1)) \to K_* (X(G, \mathbb{P}, S)) \to \cdots
\]

(3)

We remark that \( K_* (X(1)) \cong K_* (G) \) since \( X(1) \) and \( G \) are coarsely equivalent. In the next section, we will show \( \lim_{\to} \prod_{i \geq 1} K_* (|Z_{1i}|) \cong \prod_{i \geq 1} K_* (g_i P_i) \) with the aid of finite universal spaces \( E P_1, \ldots, E P_k \).
3. Contractible models

In this section, we take \((G, \mathbb{P})\) in Theorem 1.1. Let \(\mathcal{E}G\) be a finite \(G\)-simplicial complex which is a universal space for proper actions. For \(r \in \{1, \ldots, k\}\), let \(\mathcal{E}P_r\) be a finite \(P_r\)-simplicial complex which is a universal space for proper actions. In the rest of this paper, we assume that all \(\mathcal{E}P_r\) are embedded in \(\mathcal{E}G\). We also assume that \(G\) is naturally embedded in the set of vertices of \(\mathcal{E}G\) and \(g_i P_{(i)}\) is embedded in \(g_i \mathcal{E}P_{(i)}\). If \((G, \mathbb{P})\) satisfies conditions in Theorem 1.1, then we can take \(\mathcal{E}G\) satisfying these conditions (see Appendix A). We take a finite subcomplex \(\Delta \subset \mathcal{E}G\) containing a fundamental domain of \(\mathcal{E}G\). We may assume that \(\Delta \cap \mathcal{E}P_r\) contains a fundamental domain of \(\mathcal{E}P_r\) for \(r = 1, \ldots, k\) without loss of generality.

Now, we introduce a contractible model of \(X(G, \mathbb{P}, S)\). We define an embedding \(\varphi_i: g_i \mathcal{E}P_{(i)} \times \{0\} \hookrightarrow \mathcal{E}G\) by \(\varphi_i(x, 0) = x\).

A contractible model for \(X(G, \mathbb{P}, S)\) is obtained by pasting \(g_i \mathcal{E}P_{(i)} \times [0, \infty)\) to \(\mathcal{E}G\) by \(\varphi_i\) for all \(i \in \mathbb{N}\). Thus we can write it as follows:

\[
EX(G, \mathbb{P}) = \mathcal{E}G \cup \bigcup_{i \in \mathbb{N}} (g_i \mathcal{E}P_{(i)} \times [0, \infty)).
\]

Contractible models for \(X(1), Y(1)\) and \(\mathcal{H}(g_i P_{(i)}; \{1\})\) are also defined as follows:

\[
EX(1) = \mathcal{E}G \cup \bigcup_{i \in \mathbb{N}} (g_i \mathcal{E}P_{(i)} \times [0, 1]);
\]

\[
EY(1) = \bigsqcup_{i \in \mathbb{N}} (g_i \mathcal{E}P_{(i)} \times [1, \infty));
\]

\[
EZ^i = g_i \mathcal{E}P_{(i)} \times \{1\}.
\]

We remark that \(EX(G, \mathbb{P})\) admits a proper metric such that \(EX(G, \mathbb{P})\) is coarsely equivalent to \(X(G, \mathbb{P}, S)\), but it is neither of bounded geometry nor uniformly contractible, if \(\mathbb{P}\) is not empty. Thus \(EX(G, \mathbb{P})\) is not coarsening of \(X(G, \mathbb{P}, S)\) in the sense of [17, Definition 2.4]. However \(EX(G, \mathbb{P})\) is a “weakly coarsening” of \(X(G, \mathbb{P}, S)\) in the following sense:

**Proposition 3.1.** The coarse K-homology of \(X(G, \mathbb{P}, S)\) can be computed by the contractible model, that is, \(K_* (X(G, \mathbb{P}, S)) \cong K_* (EX(G, \mathbb{P})).\)

Proposition 3.1 is no direct consequence of [6, Proposition 3.8]. Our strategy is cutting off horoballs by Mayer-Vietoris arguments.
3.1. **Proof of Proposition 3.1.** We construct a locally finite cover \( \mathcal{E}_n \) of \( EX(G, \mathbb{P}) \) as follows: for \( x \in g_iP(i) \) and \( j \geq 1 \), the ball in \( g_iEP(i) \) centered at \( x \) with the size \( j \) is

\[
(4) \quad EB(x, j) = \bigcup y(\Delta \cap EP(i))
\]

where the union is taken over all \( y \in g_iP(i) \) such that \( d_S(x, y) \leq 2^j \). A contractible column centered at \( (x, t) \in g_iP(i) \times \mathbb{N} \) with the size \( j \) is

\[
EB((x, t), j) = EB(x, t + j) \times [t, t + j].
\]

For \( x \in G \), a contractible column centered at \( (x, 0) \in G \times \{0\} \) with the size \( j \) is

\[
EB((x, 0), j) = \bigcup \left(y\Delta \cup \bigcup_{i \in \mathbb{N}} ((y\Delta \cap g_iEP(i)) \times [0, j])\right)
\]

where the first union is taken over all \( y \in G \) such that \( d_S(x, y) \leq 2^j \). We define that the cover \( \mathcal{E}_n \) of \( EX(G, \mathbb{P}) \) consists of all those columns \( EB((x, t), j_n) \) for \( (x, t) \in X(G, \mathbb{P}, \mathcal{S})^{(0)} \). Taking subsequence if necessary, we define a simplicial map \( \mathcal{E}_n \to \mathcal{U}_{n+1} \) by \( EB((x, t), j_n) \mapsto B(((x, t), j_{n+1}) \).

A partition of the unity gives a continuous map \( h_n : EX(G, \mathbb{P}) \to |\mathcal{E}_n| \). The composite of \( h_2 \) and \( |\mathcal{E}_2| \to |\mathcal{U}_3| \) induces a homomorphism \( K_*EX(G, \mathbb{P}) \to KX_*(X(G, \mathbb{P}, \mathcal{S})) \).

Next, for each \( i \in \mathbb{N} \), we construct an anti-Čech system \( \{EZ^n_i\}_n \) of \( EZ^i \) as follows: the cover \( EZ^n_i \) of \( EZ^i \) consists of all balls \( EB(x, j_n) \times \{1\} \) for \( x \in g_iP(i) \). Then \( \{EZ^n_i\}_n \) forms an anti-Čech system.

We define a simplicial map \( EZ^n_i \to EZ^{i+1}_n \) by \( B((x, s), j_n) \mapsto EB((x, j_{n+1}) \times \{1\} \). We also define a simplicial map \( EZ^n_i \to EZ^{i+1}_n \) by \( EB((x, j_n) \times \{1\} \mapsto B(((x, 1), j_{n+1}) \). Then we have a commutative diagram

\[
\begin{array}{ccc}
\prod_{i \in \mathbb{N}} K_*(|Z^n_i|) & \longrightarrow & \prod_{i \in \mathbb{N}} K_*(|EZ^n_{i+1}|) \\
\downarrow & & \downarrow \\
\prod_{i \in \mathbb{N}} K_*(|Z^n_{i+2}|) & \longrightarrow & \prod_{i \in \mathbb{N}} K_*(|EZ^n_{i+3}|).
\end{array}
\]

Hence \( \varprojlim \prod_{i \in \mathbb{N}} K_*(|Z^n_i|) \cong \varprojlim \prod_{i \in \mathbb{N}} K_*(|EZ^n_i|) \). The partition of the unity gives a continuous map \( h^n_i : EZ^i \to |EZ^n_i| \) for \( i \) and \( n \geq 1 \). By the proof of [6, Proposition 3.8], taking a subsequence if necessary (not depending on \( i \)), the induced map \( (h^n_i)_* : K_*(EZ^i) \to K_*(|EZ^n_i|) \) is an isomorphism onto the image of the map \( K_*(|EZ^n_{i-1}|) \to K_*(|EZ^n_i|) \). See
also [5, Lemma 7.11]. It follows that

\[
\prod_{i \in \mathbb{N}} K_* (EZ^i) \cong \lim_{\rightarrow} \prod_{i \in \mathbb{N}} K_* (|EZ^i_n|) \cong \lim_{\rightarrow} \prod_{i \in \mathbb{N}} K_* (|Z^i_n|).
\]

By arguments similar to that in the case of $EZ^i$, we can show the following isomorphism:

\[
K_* (EX(1)) \cong \lim_{\rightarrow} K_* (|U(1,j_n)|) = KX_* (X(1)).
\]

By the Mayer-Vietoris sequence for $EX(G, \mathbb{P}) = EX(1) \cup EY(1)$, the exact sequence (3) and the fact that $K_* (EY(1)) = 0$, we have the following commutative diagram with two horizontal exact sequences:

\[
\begin{array}{cccccc}
\prod_{i \in \mathbb{N}} K_* (EZ^i(1)) & \rightarrow & K_* (EX(1)) & \rightarrow & K_* (EX(G, \mathbb{P})) & \rightarrow \\
\lim_{\rightarrow} \prod_{i \in \mathbb{N}} K_* (|Z^i_n|) & \rightarrow & KX_* (X(1)) & \rightarrow & KX_* (X(G, \mathbb{P}, S)) & \rightarrow
\end{array}
\]

By (5), (6) and the five lemma, all vertical maps are isomorphisms. This completes the proof of Proposition 3.1.

4. Coarse Mayer-Vietoris sequences

Higson, Roe and Yu [8] introduced a coarse Mayer-Vietoris sequence in the K-theory of the Roe algebras. It is used to prove a Lipschitz homotopy invariance of the K-theory of the Roe algebras [17, Theorem 9.8].

We first recall a notion of “excision pair” in coarse category. For a metric space $M$, a subspace $A$, and a positive number $R$, we denote by $\text{Pen}(A; R)$ the $R$-neighbourhood of $A$ in $M$, that is, $\text{Pen}(A; R) = \{ p \in M : d(p,A) \leq R \}$.

**Definition 4.1.** Let $M$ be a proper metric space, and let $A$ and $B$ be closed subspaces with $M = A \cup B$. We say that $M = A \cup B$ is an $\omega$-excisive decomposition, if for each $R > 0$ there exists some $S > 0$ such that

\[
\text{Pen}(A; R) \cap \text{Pen}(B; R) \subset \text{Pen}(A \cap B; S).
\]

We summarize results in [8] (see also [12] and [13]) on coarse assembly maps and Mayer-Vietoris sequences as follows:
Theorem 4.2. Suppose that \( M = A \cup B \) is an \( \omega \)-excisive decomposition. Then the following diagram is commutative and horizontal sequences are exact:

Here vertical arrows are coarse assembly maps.

5. Proof of theorem 1.1

In this section, we give a proof of Theorem 1.1 which is divided into two parts. In the first part, we show inductively the coarse Baum-Connes conjecture for the space obtained by removing the first \( n-1 \) horoballs from \( X(G, \mathcal{P}, \mathcal{S}) \). In the second part, we compute the coarse K-homology and the K-theory of the Roe algebra of \( G \) which is the intersection of a decreasing sequence of subspaces of \( X(G, \mathcal{P}, \mathcal{S}) \).

5.1. The first part.

Notation 5.1. We introduce the following notations:

\[
X_n = \Gamma \cup \bigcup_{i \geq n} \mathcal{H}(g_i P(i)); \quad X_\infty = \bigcap_{n \geq 1} X_n; \quad EX_n = E G \cup \bigcup_{i \geq n} (g_i E P(i) \times [0, \infty)) \quad EX_\infty = \bigcap_{n \geq 1} EX_n.
\]

We remark that \( X_1 = X(G, \mathcal{P}, \mathcal{S}), X_\infty = \Gamma, EX_1 = EX(G, \mathcal{P}) \) and \( EX_\infty = E G \).

Since \( X_1 \) is \( \delta \)-hyperbolic for some \( \delta \geq 0 \), by the result of Higson-Roe [6 Corollary 8.2], the coarse assembly map \( \mu: K_* (X_1) \rightarrow K_* (C^* (X_1)) \) is an isomorphism. See Appendix B. In fact, by Proposition 3.1, the coarse assembly map

\[
\mu: K_* (EX_1) \rightarrow K_* (C^* (X_1))
\]

is an isomorphism. By assumption and [6 Proposition 3.8], \( \mu: K_* (g_n E P(n)) \rightarrow K_* (C^* (g_n P(n))) \) is an isomorphism for all \( n \geq 1 \).

Lemma 5.2. For any \( n \geq 0 \), the coarse assembly map \( \mu_n: K_* (EX_n) \rightarrow K_* (C^* (X_n)) \) is an isomorphism.
Proof. We assume that $\mu_n$ is an isomorphism. Since $X_n = X_{n+1} \cup \mathcal{H}(g_nP_n)$ is an $\omega$-excisive decomposition, it follows from (coarse) Mayer-Vietoris sequences and the five lemma that $\mu_{n+1}$ is an isomorphism. □

5.2. The second part. Let $(EX_n)^+$ denote the one-point compactification of $EX_n$. It is clear that $(EX_\infty)^+ = \bigcap_{n \in \mathbb{N}} (EX_n)^+$. By the Milnor exact sequence [7, Proposition 7.3.4], we have

$$0 \to \lim_{\leftarrow} K_{p+1}((EX_n)^+) \to K_p((EX_\infty)^+) \to \lim_{\leftarrow} K_p((EX_n)^+) \to 0. \quad (9)$$

Since the $K$-homology of $EX_n$ is just the reduced $K$-homology of $(EX_n)^+$, we have $K_*(((EX_n)^+)) \cong K_*(EX_n) \oplus K_*(\{+\})$ where $\{+\}$ denotes a one-point space. This is also a direct consequence of an exact sequence [7, Definition 7.1.1(b)]. Thus we can replace $K_*(((EX_n)^+))$ in (9) by $K_*(EX_n)$.

Next, we consider the $K$-theory of the Roe algebras. Let $H$ be a Hilbert space and $\rho: C_0(X_1) \to \mathfrak{B}(H)$ is an ample representation where $\mathfrak{B}(H)$ is the set of all bounded operators on $H$. The Roe algebra $C^*(X_1, H)$ is the norm closure of the algebra of locally compact, controlled operators on $H$ (see [7, Definition 6.3.8]). The restriction $\rho: C_0(X_n) \to \mathfrak{B}(C_0(X_n)H)$ gives an ample representation of $C_0(X_n)$. The Roe algebra $C^*(X_n, C_0(X_n)H)$ can be naturally identified with a sub-$C^*$-algebra of $C^*(X_1, H)$, in fact, we have

$$C^*(X_n, C_0(X_n)H) = \{ T \in C^*(X_1, H) : \text{supp} T \subset X_n \times X_n \}.$$

We abbreviate $C^*(X_n, C_0(X_n)H)$ to $C^*(X_n)$. Now it is easy to see that $C^*(X_\infty) = \bigcap_{n \geq 1} C^*(X_n)$.

Phillips [16] studied the $K$-theory of the projective limit of $C^*$-algebras.

Proposition 5.3 ([16, Theorem 5.8(5)]). The following sequence is exact.

$$0 \to \lim_{\leftarrow} K_{p+1}(C^*(X_n)) \to K_p(C^*(X_\infty)) \to \lim_{\leftarrow} K_p(C^*(X_n)) \to 0.$$

By Proposition 5.3 and (9), we have the following commutative diagram such that upper and lower horizontal sequences are exact:

$$\begin{array}{ccccccc}
0 & \to & \lim_{\leftarrow} K_{p+1}(EX_n) & \to & K_p(EX_\infty) & \to & \lim_{\leftarrow} K_p(EX_n) & \to & 0. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \lim_{\leftarrow} K_{p+1}(C^*(X_n)) & \to & K_p(C^*(X_\infty)) & \to & \lim_{\leftarrow} K_p(C^*(X_n)) & \to & 0.
\end{array}$$
By Lemma 5.2 and the five lemma, every vertical map is an isomorphism. This completes the proof of Theorem 1.1.

**Remark 5.4.** In the proof of Theorem 1.1, we use $\delta$-hyperbolicity of the augmented space only for the first step of the induction in section 5.1 and the existence of a universal space $EG$ mentioned in the beginning of Section 3.

**Appendix A. A finite universal space for proper actions of a relatively hyperbolic group**

In this appendix we prove the following (refer to [3, Theorem 0.1] on the case of torsion free groups):

**Theorem A.1.** Let a countable group $G$ be hyperbolic relative to a finite family of infinite subgroups $\mathbb{P}$. Suppose that every $P \in \mathbb{P}$ admits a finite $P$-simplicial complex which is a universal space for proper actions. Then $G$ admits a finite $G$-simplicial complex which is a universal space for proper actions. In fact, $G$ has a finite $G$-simplicial complex $EG$ with an embedding $i \colon G \hookrightarrow EG$ and each $P \in \mathbb{P}$ has a finite $P$-simplicial complex $EP$ which is a subcomplex of $EG$ such that $i(P) \subset EP$.

See [10] for universal spaces for proper actions.

Let a countable group $G$ be finitely generated relative to a finite family of infinite subgroups $\mathbb{P}$. We denote the family of all left cosets by $a := \bigsqcup_{P \in \mathbb{P}} G/P$. We take a left invariant, proper metric $d_G$ on $G$ such that $G$ is generated by $\{g \in G \mid d_G(e, g) \leq 1\} \cup \bigcup_{P \in \mathbb{P}} P$. We remark that $\{g \in G \mid d_G(e, g) \leq 1\}$ is a finite set.

Now we recall the definition of the augmented space $X(G, \mathbb{P}, d_G)$ (see [4, Section 3] and also [9]). Its vertex set $V(G, \mathbb{P}, d_G)$ is $G \sqcup \bigsqcup_{A \in a} (A \times \mathbb{N})$ where $\mathbb{N}$ is the set of positive integers. We often denote the subset $G \subset V(G, \mathbb{P})$ by $G \times \{0\}$. Also we often regard $A \in a$ as a subset $A \times \{0\}$ of $G \times \{0\}$. Its edge is either a vertical edge or a horizontal edge: a vertical edge is a pair $\{(a, t_1), (a, t_2)\} \subset A \times (\{0\} \sqcup \mathbb{N})$ such that $|t_1 - t_2| = 1$ for $A \in a$; a horizontal edge is a pair $\{(a_1, t), (a_2, t)\} \subset A \times \mathbb{N}$ such that $0 < d_G(a_1, a_2) \leq 2t$ for $A \in a$ or a pair of $\{g_1, g_2\} \subset G$ such that $d_G(g_1, g_2) = 1$.

Since $G$ is generated by $\{g \in G \mid d_G(e, g) \leq 1\} \cup \bigcup_{P \in \mathbb{P}} P$, the augmented space $X(G, \mathbb{P}, d_G)$ is connected. This graph structure induces a metric on $V(G, \mathbb{P}, d_G)$. When we consider for $P \in \mathbb{P}$, a left invariant proper metric $d_P := d_G|_{P \times P}$ on $P$, then $X(P, \{P\}, d_P)$ is nothing but the full subgraph of $P \sqcup (P \times \mathbb{N})$ in $X(G, \mathbb{P}, d_G)$. Moreover we can confirm that $X(P, \{P\}, d_P)$ is an isometrically embedded subgraph of $X(G, \mathbb{P}, d_G)$.
We consider the Rips complex \( R_D(V(G, \mathbb{P}, d_G)) \) for a positive integer \( D \). We denote the full subcomplexes of

\[
V(G, \mathbb{P}, d_G)_r = \bigsqcup_{A \in a} (A \times \{r, \ldots\});
\]

\[
V(G, \mathbb{P}, d_G)^R = G \sqcup \bigsqcup_{A \in a} (A \times \{1, \ldots, R\});
\]

\[
V(G, \mathbb{P}, d_G)_{r}^R = \bigsqcup_{A \in a} (A \times \{r, \ldots, R\}) = V(G, \mathbb{P}, d_G)_r \cap V(G, \mathbb{P}, d_G)^R,
\]

in \( R_D(V(G, \mathbb{P}, d_G)) \) by \( R_D(V(G, \mathbb{P}, d_G))_r \), \( R_D(V(G, \mathbb{P}, d_G))^{R} \) and \( R_D(V(G, \mathbb{P}, d_G))_{r}^{R} \), respectively, where \( r, R \in \mathbb{N} \) such that \( r \leq R \).

**Remark A.2.** If \( r + D \leq R \), then we have \( R_D(V(G, \mathbb{P}, d_G)) = R_D(V(G, \mathbb{P}, d_G))_r \cup R_D(V(G, \mathbb{P}, d_G))^{R} \) and \( R_D(V(G, \mathbb{P}, d_G))_{r}^{R} \).

\( G \) is hyperbolic relative to \( \mathbb{P} \) if and only if \( V(G, \mathbb{P}, d_G) \) is \( \delta \)-hyperbolic for some \( \delta \geq 0 \) (see [11, Theorem 3.25]). Since \( V(G, \mathbb{P}, d_G) \) is \( \delta \)-hyperbolic, there exists some positive number \( D_\delta \) such that for any \( D \in \mathbb{N} \) such that \( D \geq D_\delta \), the Rips complex \( R_D(V(G, \mathbb{P}, d_G)) \) is contractible. Moreover we have the following:

**Proposition A.3.** Let a countable group \( G \) be hyperbolic relative to a finite family of infinite subgroups \( \mathbb{P} \). Suppose that \( V(G, \mathbb{P}, d_G) \) is \( \delta \)-hyperbolic, where \( \delta \) is a non-negative number. Then there exists some positive number \( D_\delta' \) such that for any integer \( D \) such that \( D \geq D_\delta' \), the first barycentric subdivision of the Rips complex \( R_D(V(G, \mathbb{P}, d_G)) \) is a \( G \)-simplicial complex which is a universal space for proper actions.

If \( \mathbb{P} \) is empty on the above, then \( G \) is a hyperbolic group. The above for this case is known ([11]). Since arguments in the proof of [11, Theorem 1] can be applied to the above, we omit its proof.

**Proof of Theorem A.1.** We take a left invariant proper metric \( d_G \) on \( G \) such that \( G \) is generated by \( \{g \in G \mid d_G(e, g) \leq 1\} \cup \bigcup_{P \in \mathbb{P}} P \). We denote by \( d_P \) a left invariant proper metric \( d_G|_{P \times P} \) on \( P \in \mathbb{P} \).

Suppose that \( V(G, \mathbb{P}, d_G) \) is \( \delta \)-hyperbolic. Then for every \( P \in \mathbb{P} \), the vertex set \( V(P, \{P\}, d_P) \) is \( \delta \)-hyperbolic because \( X(P, \{P\}, d_P) \) is an isometrically embedded subgraph of \( X(G, \mathbb{P}, d_G) \). We fix \( D \in \mathbb{N} \) such that \( D \geq D_\delta' \), where \( D_\delta' \) is a constant in Proposition A.3. We take \( P \in \mathbb{P} \) and \( r, R \in \mathbb{N} \) such that \( r + D \leq R \). Also we take for every \( P \in \mathbb{P} \), a finite \( P \)-simplicial complex \( \overline{E}P \) which is a universal space for proper actions.
Since the first barycentric subdivision of $R_D(V(P, \{P\}, d_P))_r$ is a $P$-simplicial complex which is a universal space for proper actions by Proposition A.3, we have a $P$-homotopy equivalent map $h_P : R_D(V(P, \{P\}, d_P))_r \to EP$. It follows from an equivariant version of simplicial approximation theorem (see [1] Exercise 6 for Chapter 1) that there exist a natural number $n$ and a $P$-simplicial map $f_P : R_D^{(n)}(V(P, \{P\}, d_P))_r \to EP$ which is $P$-homotopy equivalent to $h_P$ where $R_D^{(n)}(V(P, \{P\}, d_P))_r$ is the $n$-th barycentric subdivision of $R_D(V(P, \{P\}, d_P))_r$. We can take $n$ independently of $P$ because $\mathbb{P}$ is a finite family. We consider mapping cylinders
\[(R_D^{(n)}(V(P, \{P\}, d_P))_r \times [0, 1]) \cup_{j_P} R_D^{(n)}(V(P, \{P\}, d_P))_r;\]
\[(R_D^{(n)}(V(P, \{P\}, d_P))_r \times [0, 1]) \cup_{q_P} EP,\]
whose pasting maps are
\[j_P : R_D^{(n)}(V(P, \{P\}, d_P))_r \times \{1\} \ni (x, 1) \mapsto x \in R_D^{(n)}(V(P, \{P\}, d_P))_r;\]
\[q_P : R_D^{(n)}(V(P, \{P\}, d_P))_r \times \{1\} \ni (x, 1) \mapsto f_P(x) \in EP,\]
respectively. Then the maps $id_{R_D^{(n)}(V(P, \{P\}, d_P))_r}$ and $f_P$ induce a map
\[\tilde{f}_P : (R_D^{(n)}(V(P, \{P\}, d_P))_r \times [0, 1]) \cup_{j_P} R_D^{(n)}(V(P, \{P\}, d_P))_r \to\]
\[(R_D^{(n)}(V(P, \{P\}, d_P))_r \times [0, 1]) \cup_{q_P} EP,\]
which is a $P$-homotopy equivalent map. In fact we can confirm that $\tilde{f}_P$ is a $P$-homotopy equivalent map relative to $R_D^{(n)}(V(P, \{P\}, d_P))_r \times \{0\}$. Now we construct two $G$-simplicial complex $R_D^{(n)}(V(G, \mathbb{P}, d_G))_1$ and $R_D^{(n)}(V(G, \mathbb{P}, d_G))_2$ as follows: First, $R_D^{(n)}(V(G, \mathbb{P}, d_G))_1$ is obtained by, for every $P \in \mathbb{P}$, pasting $G$-equivariantly, $(R_D^{(n)}(V(P, \{P\}, d_P))_r \times [0, 1]) \cup_{j_P} R_D^{(n)}(V(P, \{P\}, d_P))_r$, to $R_D^{(n)}(V(G, \mathbb{P}, d_G))_R$ by the pasting map
\[R_D^{(n)}(V(P, \{P\}, d_P))_r \times [0, 1] \to R_D^{(n)}(V(P, \{P\}, d_P))_r.\]
Second, $R_D^{(n)}(V(G, \mathbb{P}, d_G))_2$ is obtained by, for every $P \in \mathbb{P}$, pasting $G$-equivariantly, $R_D^{(n)}(V(P, \{P\}, d_P))_r \times [0, 1] \cup_{q_P} EP$ to $R_D^{(n)}(V(G, \mathbb{P}, d_G))_R$ by the same pasting map. Then they are $G$-homotopy equivalent by the induced map by $id_{R_D^{(n)}(V(G, \mathbb{P}, d_G))_R}$ and $\tilde{f}_P$ for any $P \in \mathbb{P}$. Since $R_D^{(n)}(V(G, \mathbb{P}, d_G))_r$ is clearly $G$-homeotopic to $R_D^{(n)}(V(G, \mathbb{P}, d_G))_1$ by Remark A.2, we have $R_D^{(n)}(V(G, \mathbb{P}, d_G))_2$ is $G$-homeotopic to $R_D^{(n)}(V(G, \mathbb{P}, d_G))_2$. It follows from Proposition A.3 that $R_D^{(n)}(V(G, \mathbb{P}, d_G))_2$ is a $G$-simplicial complex which is a universal space for proper actions. It is also clear that $R_D^{(n)}(V(G, \mathbb{P}, d_G))_2$ is a finite $G$-simplicial complex by the construction. $G$ is naturally embedded in $R_D^{(n)}(V(G, \mathbb{P}, d_G))_2$. 
$R_D^{(n)}(V(P, \{P\}, d_P))_2$ is a subcomplex of $R_D^{(n)}(V(G, \mathbb{P}, d_G))_2$ and is a finite universal $P$-simplicial complex with the natural embedding of $P$. □

APPENDIX B. THE COARSE BAUM-CONNES CONJECTURE FOR HYPERBOLIC METRIC SPACES

Higson and Roe [6, Corollary 8.2] proved the coarse Baum-Connes conjecture for hyperbolic metric spaces. The following Proposition B.1 plays an important role in their proof.

**Proposition B.1.** Let $Y$ be a compact metric space and let $O_Y$ denote an open cone of $Y$. Then the coarsening map

$$\mu: K_*(O_Y) \to KX_*(O_Y)$$

is an isomorphism.

Higson and Roe [6, Proposition 4.3] proved this proposition assuming that the dimension of $Y$ is finite. Here we prove it without assuming that.

**Proof.** Any compact metric space can be embedded in the separable Hilbert space $l_2$. In fact, the stereographic projection gives an embedding in the unit ball of $l_2$. So we assume $Y \subset \{x \in l_2 : ||x|| = 1\}$. Then the open cone of $Y$ is given by $O_Y = \{tx \in l_2 : x \in Y, t \in [0, \infty)\}$. For $I \subset (0, \infty)$, set

$$Y \times I = \{tx \in l_2 : x \in Y, t \in I\}.$$

Since $Y$ is compact, for each $n \in N$, there exist $p_1^n, \ldots, p_{a_n}^n \in Y \times \{n\}$ such that

$$\bigcup_{m=1}^{a_n} B(p_m^n, 1) \supset Y \times \{n\}. \quad (10)$$

Here $B(x, r)$ denotes a ball of radius $r$ centered at $x$. Then we have

$$\bigcup_{m=1}^{a_n} B(p_m^n, 2) \supset Y \times [n - 1, n + 1].$$

For each $i \in \mathbb{N}$, we form a cover $\mathcal{U}_i$ of $O_Y$ as follows:

$$U_m^n(i) = B(p^n_m, 3^i) \cap O_Y, \quad m = 1, \ldots, a_n,$$

$$\mathcal{U}_i = \bigcup_{n \geq 1} \{U_1^n(i), \ldots, U_{a_n}^n(i)\}.$$
It is clear that $\mathcal{U}_i$ is a locally finite cover and thus we obtain an anti-Čech system $\{\mathcal{U}_i\}_{i \geq 1}$.

By the definition, it follows that
\[
\bigcup_{n \geq 1} \bigcup_{m=1}^{a_n} B(p^n_m, 3^i) \subset \text{Pen}(\mathcal{O}Y, 3^i).
\]

Then the method used in the proof of [6, Proposition 4.3] can be applied to $\{\mathcal{U}_i\}_{i \geq 1}$. This completes the proof of Proposition B.1. □

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