ON THE COMPUTATION OF TAMAGAWA NUMBERS AND NÉRON COMPONENT GROUPS OF JACOBIANS OF SEMISTABLE HYPERELLIPTIC CURVES

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Abstract. We describe an algorithm for calculating Tamagawa numbers of Jacobians of semistable hyperelliptic curves over local fields in terms of their reduction types. The computation is uniform across combinatorial families of reduction types, and thereby yields a finite algorithm to produce explicit characterisations of these Tamagawa numbers for all such curves of a fixed genus. As a corollary to the theory we develop, we derive new restrictions on the behaviour of these Tamagawa numbers as the base field is varied.

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1. Introduction

Fix a finite extension $K$ of $\mathbb{Q}_p$ with ring of integers $O_K$ and residue field $k$. The Tamagawa number $c_{X/K}$ of a (smooth, projective, geometrically integral) curve $X$ over $K$ is a numerical invariant that roughly controls the type of bad reduction of $X/K$, or more accurately of its Jacobian. Much is known about these Tamagawa numbers and their links to the reduction types of curves, particularly in the case that $X/K$ is semistable, and the central aim of this paper is to make this understanding as explicit as possible in the case that $X/K$ is semistable and hyperelliptic. More precisely, we would like to be able to address the following three motivating questions.

1. How do Tamagawa numbers of semistable hyperelliptic curves change as these curves vary in families?
2. How do Tamagawa numbers of semistable hyperelliptic curves change as we enlarge the base field?
3. How does one compute the Tamagawa number of a semistable hyperelliptic curve efficiently from an explicit equation $y^2 = f(x)$ (at least when $p \neq 2$)?

Let us say a few words about the significance of each of these questions. The third motivating question sits squarely in the realm of explicit arithmetic geometry, and is relevant for example in the context of computational verification of the Birch–Swinnerton-Dyer conjecture for Jacobians of curves [4, 10]. The second motivating question is also related to the Birch–Swinnerton-Dyer conjecture, but this time its relevance lies in understanding how the various terms of the BSD formula behave as one varies the base field; for instance, control of Tamagawa numbers of abelian varieties over certain field extensions, at least up to squares, plays a role in proofs of various known instances of the $p$-parity conjecture [3, 6, 7]. In this paper we will develop finer restrictions in the particular case of Jacobians of semistable hyperelliptic curves, which for example afford partial control of these Tamagawa numbers up to higher powers.

The original and main motivation for this paper, however, comes from the first motivating question, in particular in its relevance to the classification of reduction types of semistable hyperelliptic curves as developed in [8, 9]. Although there are infinitely many different possible reduction types for a given genus, these fall into finitely many denumerable combinatorial families, and the algorithms we will develop will be able to calculate the Tamagawa numbers of such curves (which are a function only of the reduction type) uniformly in such families. Hence the techniques developed in this paper reduce to a finite computation the problem of finding the Tamagawa numbers of all semistable hyperelliptic curves of a given genus as a function of their reduction type.

Before we give precise statements of the results we obtain, let us illustrate the kind of theory we will develop by recalling the corresponding picture for semistable elliptic curves. A semistable elliptic curve $X/K$ has either good reduction or multiplicative reduction of Kodaira type $I_n$ for some $n \in \mathbb{N}$, which is either split or non-split. Thus there are three families of reduction types for semistable elliptic curves:

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1 More properly, one should talk of the Tamagawa number of the Jacobian of such a curve – we will take the liberty of referring to this as the Tamagawa number of the curve itself in the interests of brevity.

2 For us, a hyperelliptic curve $X/K$ is a curve endowed with a degree 2 map $X \to \mathbb{P}^1_K$.
curves: good reduction; split type $I_n$ reduction for some $n \in \mathbb{N}$; and non-split type $I_n$ reduction for some $n \in \mathbb{N}$.

Moreover, for each of these three families of reduction types, the Tamagawa number of $X$ can be expressed as a function in the Kodaira parameter $n$. When $X/K$ has good reduction, its Tamagawa number is 1, while when $X/K$ has split (resp. non-split) type $I_n$ reduction, its Tamagawa number is $n$ (resp. $\text{hcf}(n,2)$) \cite{3, Example 1.2.1}. In this way, we obtain via the Kodaira classification a complete description of the Tamagawa numbers of all semistable elliptic curves. Moreover, Tate’s algorithm \cite[Chapter IV.9]{13} allows one to read off the Kodaira type of any given semistable elliptic curve from an explicit equation $y^2 = f(x)$ for $X/K$ in terms of the relative positions of the roots of $f$. Combined with the above description of Tamagawa numbers, this gives an efficient algorithm to compute Tamagawa numbers of semistable elliptic curves from explicit equations.

In extending this theory to hyperelliptic curves, we will replace the semistable part of the Kodaira classification with the classification scheme proposed in \cite{9}, which describes the reduction types of semistable hyperelliptic curves in terms of various equivalent forms of combinatorial data, called cluster pictures, BY trees or hyperelliptic graphs. The theory in \cite{8} (the relevant part of which we will recall in section 2.2) assigns to every semistable hyperelliptic curve $X/K$ a reduction type consisting of any of these equivalent forms of data, from which many important arithmetic invariants of $X/K$ can be recovered, including its local Galois representation \cite[Theorem 1.19]{8}, conductor \cite[Theorem 1.20]{8}, Tamagawa number \cite[Lemma 2.22]{8} and more. In this context, the analogue of the semistable part of Tate’s algorithm is found in \cite[Definition 1.1]{8} (at least when $p \neq 2$), in which it is explained how to read off the reduction type of $X/K$ (in the particular form of a cluster picture) from the relative position of the roots of $f$, where $y^2 = f(x)$ is a hyperelliptic equation for $X/K$.

Despite the innate calculability of cluster pictures of hyperelliptic curves, it will be most convenient for our purposes to work with reduction types in the equivalent form of BY trees. Since these BY trees will constitute a central object of study in this paper, let us now give a brief definition, to be elaborated on in section 2.2.

(Re)definition 1.0.1 (BY forests, cf. \cite[Section 3.2]{9}). A BY tree (resp. BY forest) is a triple $T = (T, S, \epsilon F)$ consisting of:

- a finite graph-theoretic tree (resp. forest) $T$ endowed with an integral metric, i.e. an edge-length function $l : E(T) \to \mathbb{N}$ assigning each edge of $T$ a positive integer length;
- a subgraph $S \subseteq T$; and
- a pair $\epsilon F = (F, \epsilon)$ consisting of:
  - an isometric isomorphism $F$ of $T$ fixing $S$ setwise; and

\cite{3}The definition we present here differs slightly from the definition in \cite[Definition 3.18]{9}, in that we do not record the genus attached to vertices, but on the other hand are interested in the action of a particular signed automorphism. Since we are simply working with a coarsened notion of what would be called a BY tree with automorphism in \cite{9}, we believe that this discrepancy should not cause any confusion.

\cite{4}The need to consider BY forests rather than just BY trees is simply an artefact of our algorithm, the first stage of which consists of disassembling a BY tree (or forest) into many simpler constituent parts.
– a sign function $\epsilon : \pi_0(T \setminus S) \to \{\pm 1\}$, where $T \setminus S$ denotes the complement of $S$ in the underlying topological space of $T$.

In our algorithms, we will frequently utilise two technical parity conditions which are automatically satisfied for BY trees associated to semistable hyperelliptic curves. These conditions will be called parity conditions (A) and (B); precise definitions can be found in proposition 2.2.3.

**Remark 1.0.2.** The unusual bracketing in definition 1.0.1 is there to suggest to the reader that they should think of the pair $\epsilon F$ as a single entity, namely as a signed automorphism of the pair $(T, S)$; this perspective will be developed in section 2.2.

Note that BY trees are naturally arranged in a discrete collection of denumerable families, where in each family the underlying homeomorphism type of the triple $(T, S, \epsilon F)$ is constant, and the BY trees within each family are parametrised by the lengths of the edges in $T$. The main aim, then, of this paper is to describe an algorithm to compute the Tamagawa number of a semistable hyperelliptic curve $X/K$ in terms of its associated BY tree which is sufficiently uniform to allow us to compute simultaneously all Tamagawa numbers of curves in any family of reduction types as a function in the edge-lengths in the BY tree. This will in particular answer motivating question 3: to find the Tamagawa number of a semistable hyperelliptic curve $X/K$ from an explicit hyperelliptic equation, first compute its cluster picture [8, Definition 1.1], translate this into a BY tree [9, Construction 4.13], and then finally apply our algorithm to extract the Tamagawa number of $X/K$.

1.1. **First algorithm: Tamagawa numbers.** Equipped with our definition of BY trees, we are now in a position to give a precise statement of our main algorithm. We will define in definition 2.3.1 a certain numerical invariant of a BY forest $T$, called its Tamagawa number and denoted $c_T$, which, when $T$ arises from a semistable hyperelliptic curve $X/K$ recovers its Tamagawa number $c_{X/K}$ in the usual sense. Our main algorithm then solves the (purely combinatorial) problem of computing the Tamagawa number $c_T$ of an arbitrary BY forest $T$ (possibly in the presence of parity conditions (A) and (B)) as a function in the edge-lengths of $T$.

1.1.1. **Reduction step.** The first step in our algorithm is to reduce the computation of the Tamagawa number of a potentially large BY forest to more straightforward calculations by breaking up the BY forest into smaller parts, using the lemma which follows.

**Lemma 1.1.1.** Let $T = (T, S, \epsilon F)$ be a BY forest.

- If $T_0$ is the BY forest formed by taking the disjoint union of the closures of the components of $T \setminus S$ (and giving this the induced signed automorphism and subgraph), then the Tamagawa numbers of $T$ and $T_0$ agree.
- If $T$ is a disjoint union of $F$-stable BY subforests $T_i$, then the Tamagawa number of $T$ is equal to the product of the Tamagawa numbers of the $T_i$.
- If $T$ consists of a single $F$-orbit of $q$ BY trees $T_0, FT_0, \ldots, F^{q-1}T_0$, then the Tamagawa number of $T$ is equal to that of $(T_0, S \cap T_0, (\epsilon F)^q)$ where $(\epsilon F)^q = \epsilon' F^q$ with $\epsilon'$ the product of the signs of the constituent trees of $F$.
- If $S = \emptyset$ or $S = T$, then the Tamagawa number of $T$ is 1.

5Strictly speaking, [9, Construction 4.13] produces an open BY tree [9, Definition 3.21]; to apply our methods, one should pass to the core of this open BY tree [9, Definition 3.25].
Moreover, in the first three points, if $T$ satisfies parity condition (A) (resp. (B)) from proposition 2.2.9, then so too do all the $T_i$.

1.1.2. Positive simple BY trees. After applying the previous lemma, we are reduced to calculating the Tamagawa numbers of BY trees which satisfy the following condition.

**Definition 1.1.2** (Simple BY trees). A BY tree $T = (T, S, eF)$ is said to be *simple* just when $S$ is a non-empty subgraph of $T$ with no edges and $T \setminus S$ consists of a single component. Put another way, $S$ is a non-empty set of degree 1 vertices of $T$. For a simple BY tree, the sign function $\epsilon : \pi_0(T \setminus S) \to \{\pm 1\}$ merely picks out a single choice of sign, either +1 or −1, and we will refer to $T$ as *positive* or *negative* accordingly.

In calculating the Tamagawa number of simple BY trees, we will treat the positive and negative cases separately. Of these, the case of a positive simple BY tree is more straightforward, being given by the following explicit formula.

**Theorem 1.1.3.** Let $T = (T, S, +F)$ be a positive simple BY tree satisfying parity condition (B) from proposition 2.2.9 and suppose that $S$ consists of $r + 1$ $F$-orbits, the product of whose sizes is $Q$. Let $T' = T/\langle F \rangle$ be the quotient tree, endowed with the metric whereby an edge $e'$ of $T'$ corresponding to an $F$-orbit of $q$ edges in $T$, each of length $l(e)$, is assigned a length of $l'(e') = l(e)/q$. Then the Tamagawa number of $T$ is

$$c_T = Q \sum_{e_1', \ldots, e_r'} \prod_{i=1}^r l'(e_i'),$$

where the sum is taken over all unordered $r$-tuples of edges of $T'$ whose removal disconnects the $r + 1$ points of $S/\langle F \rangle$ from one another.

**Remark 1.1.4.** In fact, the formula in theorem 1.1.3 is valid in greater generality: one needs only require that $S$ is a non-empty set of vertices of the tree $T$ and that the sign function $\epsilon$ is uniformly positive. BY trees of this form arise for us in the step of our algorithm dealing with negative simple BY trees, thereby allowing us to slightly shorten the calculations there.

1.1.3. Negative simple BY trees. The remaining case, that of a negative simple BY tree $T$, is slightly more complicated than the positive case. Rather than giving an explicit formula, we will instead express the Tamagawa number as an explicit correction factor (which is a power of 2) multiplied by the Tamagawa number of a new BY tree $T'$ (not necessarily simple), whose sign is uniformly positive. The Tamagawa number of $T'$ can then be calculated by using the reduction step (lemma 1.1.1) again to reduce to the case of a positive simple BY tree (theorem 1.1.3). Alternatively, one can just apply the formula in theorem 1.1.3 to $T'$ directly (see remark 1.1.4).

**Theorem 1.1.5.** Let $T = (T, S, -F)$ be a negative simple BY tree satisfying parity conditions (A) and (B) from proposition 2.2.9 and assume that $T$ is not a path (i.e. has a vertex of degree $\geq 3$). Write $S = S_0 \sqcup S_1$ where $S_0$ (resp. $S_1$) consists of those points in even-sized (resp. odd-sized) $F$-orbits. Write moreover the orbit-set $S_1/\langle F \rangle$ as $A_0 \sqcup A_1$, where $A_1$ consists exactly of the orbits of points in $S_1$ which are at the end of an odd-length twig (i.e. such that the distance to the nearest vertex of
degree $\geq 3$ is odd). Let $T'$ be the BY forest (satisfying parity condition (B)) formed by removing $S_1$ from $S$, contracting out all the edges of $T$ in odd-size $F$-orbits, adding into $S$ the unique $F$-fixed point of the resulting tree, and declaring the sign to be $+1$ everywhere. Then

$$c_T = \tilde{c}_1 c_{T'},$$

where

$$\tilde{c}_1 = \begin{cases} 
2 A_0^{-1} & \text{if } \#A_0 \geq 1, \\
2 & \text{if } \#A_0 = 0 \text{ and } \#S \text{ even}, \\
1 & \text{if } \#A_0 = 0 \text{ and } \#S \text{ odd}.
\end{cases}$$

For the remaining cases, we have that $c_T = 1$ if $\#S = 1$. If $T$ consists of a single path of length $l$ connecting two points of $S$, then

$$c_T = \begin{cases} 
2 & \text{if } l \text{ even and } T \text{ pointwise fixed by } F, \\
1 & \text{if } l \text{ odd and } T \text{ pointwise fixed by } F, \\
l & \text{if } T \text{ reversed by } F.
\end{cases}$$

1.1.4. Examples. Before we discuss other applications of the techniques developed in this paper, let us illustrate how this algorithm works in practice.

Example 1.1.6. As a basic example to illustrate the Tamagawa number algorithm in action, let us calculate the Tamagawa number of the following BY tree $T$ (which would arise from a curve of genus $\geq 3$). This BY tree satisfies parity conditions (A) and (B) from proposition 2.2.9.

**Figure 1.** A fairly straightforward example of a BY tree

In this diagram, the whole graph represents the tree $T$, while the blue/solid vertices represent the vertices of $S$ (which has no edges in this example) – by contrast, the vertices of $T$ not in $S$ are represented by yellow/open circles and the edges of $T$ not in $S$ are represented by yellow/squiggly lines. The lengths of the edges are indicated by the parameters $a$, $b$ and $c$, while the signed automorphism is indicated both with double-headed arrows for the underlying unsigned automorphism of $(T, S)$ (which here has order 2) and with $\pm$ signs next to each connected component of $T \setminus S$ (so here the sign is $+\$).

Since $T \setminus S$ consists of a single component, this is in fact a positive simple BY tree, so that we can apply theorem 1.1.3. In the notation of that theorem, the quotient tree $T'$ is

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[6] Here our diagrammatic conventions differ slightly from those in [9], where the sign labels are attached to each orbit of connected components of $T \setminus S$, and record the total sign of the automorphism over the entire orbit.
where again the blue/solid vertices indicate the subset $S' \subseteq T'$ and the labels indicate edge-lengths. The removal of any two of the three edges of this graph disconnects the three points of $S'$ from one another, and hence the formula in theorem 1.1.3 provides that the Tamagawa number of this positive simple BY tree is

$$c_T = 2 \cdot \left( \frac{a}{2} b + bc + \frac{a}{2} c \right) = a(b + c) + 2bc.$$  

**Example 1.1.7.** As an example to illustrate the application of our algorithm to Tamagawa numbers of hyperelliptic curves, consider the hyperelliptic curve $X/\mathbb{Q}_3$ given by the equation $y^2 = f(x)$ where

$$f(x) = x^8 - 2x^7 + 39x^6 - 19x^5 - 33x^4 - 5x^3 + 34x^2 - 27x - 27.$$  

To compute the Tamagawa number of $X/\mathbb{Q}_3$ (which we will soon see is semistable), the first step is to determine the relative position of the roots of $f$. Since $f$ is monic, all its roots are integral over $\mathbb{Z}_3$, and by reducing $f$ modulo 3 we see that $f$ has two roots in each of the residue discs about 0, 1 and $\pm i$. We wish to determine the distances between these roots, which boils down to determining the distances between the points in each of the four pairs.

To determine the distance between the two roots reducing to 0, note that one can see directly from the Newton polygon of $f$ that these two roots both have norm $3^{-3/2}$. Performing a suitable change of variables, we find that

$$\frac{1}{27} f(3^{3/2} w) \equiv w^2 - 1$$  

modulo $\sqrt{3}\mathbb{Z}_3[\sqrt{3}]$, and hence the two roots of $f$ in the residue disc about 0 are distinct modulo 9 (they are in fact congruent to $\pm 3^{3/2}$ modulo 9). Hence the distance between these two roots is $3^{-3/2}$.

Applying the same argument to the Newton polygons of the shifted polynomial $f(y + 1)$ (resp. $f(z \pm i)$) shows that the distance between the two roots of $f$ in the residue disc about 1 (resp. $\pm i$) is $3^{-1}$ (resp. $3^{-1/2}$). One can represent this information on the relative position of the roots of $f$ pictorially in the following cluster picture.

Here we are following the diagrammatic conventions of [8, 9]. The solid discs represent the roots of $f$, and the ovals around them represent those subsets which are cut out by discs. The subscripts on the ovals record the smallest radius of a disc cutting out exactly that subset (so a subscript $r$ denotes a radius of $3^{-r}$) and the horizontal line indicates that the final two pairs of roots are interchanged (setwise) by the action of Frobenius.

From this cluster picture, we already know that $X/\mathbb{Q}_3$ is semistable by the semistability criterion [8, Definition 1.7] (cf. [8, Theorem 7.1]). Indeed, the only
non-trivial part of this criterion to check is that the splitting field of \( f \) has ramification degree \( \leq 2 \) over \( \mathbb{Q}_3 \) – this follows since the action of inertia on \( \text{Root}(f) \) fixes setwise each of the four pairs of roots, and hence the square of any inertia element fixes the roots pointwise.

We now want to construct from this cluster picture the BY tree of \( X/\mathbb{Q}_3 \), for which we need to calculate extra sign data associated to \( f \) [8 Definition 1.12]. This sign data, \textit{a priori}, consists of a character \( \epsilon_s: G_{\mathbb{Q}_3} \to \{\pm 1\} \) for each cluster \( s \) of even size, but it follows immediately from the definition that in our case these five characters are all the same. Calculating the character associated to the top cluster using [8 Definition 1.12] immediately shows that the corresponding character is trivial (since the leading coefficient of \( f \) is square).

Using this sign data, one now reads off the BY tree of \( X/\mathbb{Q}_3 \) from the dictionary in [9 Construction 4.13] (and using [9 Definition 3.25]), obtaining the following BY tree.

![BY tree image]

But we already calculated the Tamagawa number of this BY tree in example 1.1.6, so the Tamagawa number of \( X/\mathbb{Q}_3 \) is 17.

**Example 1.1.8.** As a more comprehensive example to illustrate all the aspects of our algorithm, let us calculate the Tamagawa number of the following BY tree \( T \) (which would arise from a curve of genus \( \geq 11 \)). This BY tree satisfies parity conditions (A) and (B) provided that the edge-lengths \( c, w \) and \( x \) below are all even, which we will now assume.

![BY tree image]

Here, as before, the subgraph \( S \subseteq T \) is represented by blue/solid vertices and now also has a single edge, rendered blue/straight – the yellow/open circles and yellow/squiggly lines indicate the vertices and edges of \( T \) not in \( S \), respectively. The parameters \( a, b, c, w, x, y, z \) indicate edge-lengths. The signed automorphism...
is both indicated by the arrows (identifying the orbits of the underlying unsigned automorphism as it acts on edges – these orbits have sizes 2, 2, 2, 3 and 4 respectively), and the signs next to each connected component of $T \setminus S$ (so here the three signs are all $-$).

Since this is not a simple BY tree, we apply the first point of lemma 1.1.1., which shows that the Tamagawa number of $T$ is the same as the Tamagawa number of the following BY forest

![Diagram](image)

with three components, two of which are interchanged by the action of the signed automorphism. Thus by the second and third points of lemma 1.1.1. the Tamagawa number of $T$ is equal to the product of the Tamagawa numbers of the following two simple BY trees.

![Diagram](image)

We have already calculated the Tamagawa number of the right-hand BY tree in example 1.1.6, finding it to be $a(b+c)+2bc$, so it remains to calculate the Tamagawa number of the left-hand BY tree. This BY tree is simple and negative, so we may apply theorem 1.1.5. In the notation of that theorem, we have $\hat{c}_1 = \text{hcf}(z, 2)$ (since $x$ is even) and the tree $T'$ is the following BY tree.

![Diagram](image)

Finding the Tamagawa number of $T'$ (which is just $y$) is now straightforward, either with a second pass of lemma 1.1.1 to reduce to the formula in theorem 1.1.3 (in a trivial case), or simply by applying theorem 1.1.3 directly in the light of remark 1.1.4. Combining all this information, we find that the Tamagawa number of the original BY tree $T$ was

$$y \cdot (a(b+c)+2bc) \cdot \text{hcf}(z, 2).$$

Example 1.1.9. As a final example to illustrate the utility of our algorithm for classification questions, consider the classification of the reduction types of semistable genus 2 (hyperelliptic) curves from [9, Section 8.3]. This classification gives twenty-three families of reduction types, whose BY trees and Tamagawa numbers are given in table 1 below (adapted from [9, Table 9.3]).
Table 1. (From [9, Table 9.3].) The possible reduction types of semistable (hyperelliptic) curves of genus 2, along with their BY trees and Tamagawa numbers (the numbers on vertices represent extra data on the BY trees which is irrelevant for our purposes, but which we include for consistency with [9]). Here $\bar{x}$ is shorthand for $\text{hcf}(x, 2)$, $N = nm + mk + kn$ and $N_2 = \max \left\{ 1, \frac{\bar{n}mk}{2} \right\}$. Note that in types $I_n \sim_n$ and $I_n \times r I_n$ the signs of the two components of $T \setminus S$ are the same (the BY tree is independent of this choice of sign, up to a suitable notion of isomorphism).

| Type | BY tree $T$ | $c_T$ | Type | BY tree $T$ | $c_T$ |
|------|-------------|-------|------|-------------|-------|
| 2    | $\begin{array}{c} 1 \\ 2 \end{array}$ | 1     | $U^+_{n,m,k}$ | $\begin{array}{c} + k \\ m \\ n \end{array}$ | $N$ |
| $1 \times_n 1$ | $\begin{array}{c} 1 \\ r+s \\ n \end{array}$ | 1     | $U^-_{n,m,k}$ | $\begin{array}{c} - k \\ m \\ n \end{array}$ | $N_2$ |
| $1 \times_n 1$ | $\begin{array}{c} 2r \\ 1 \end{array}$ | 1     | $U^+_{n,n,k}$ | $\begin{array}{c} + k \\ n \\ n \end{array}$ | $n + 2k$ |
| $I_n^+$ | $\begin{array}{c} + n \\ n \end{array}$ | $n$   | $U^-_{n,n,k}$ | $\begin{array}{c} - n \\ n \end{array}$ | $n$ |
| $I_n^-$ | $\begin{array}{c} - n \\ n \end{array}$ | $\bar{n}$ | $U^+_{n,n,n}$ | $\begin{array}{c} + n \\ n \end{array}$ | $3$ |
| $1 \times_n I_n^+$ | $\begin{array}{c} n \\ r+s \\ n \end{array}$ | $n$   | $U^-_{n,n,n}$ | $\begin{array}{c} - n \\ n \end{array}$ | $1$ |
| $1 \times_n I_n^-$ | $\begin{array}{c} n \\ r+s \\ \bar{n} \end{array}$ | $\bar{n}$ |
| $I_{n,m}^+$ | $\begin{array}{c} + n \\ m \end{array}$ | $n \cdot m$ | $I_{n,m}^- \times I_m^+$ | $\begin{array}{c} + n \\ r+s \\ m \end{array}$ | $n \cdot \bar{m}$ |
| $I_{n,m}^-$ | $\begin{array}{c} - n \\ m \end{array}$ | $n \cdot \bar{m}$ | $I_{n,m}^- \times I_m^-$ | $\begin{array}{c} - n \\ r+s \\ \bar{m} \end{array}$ | $\bar{n} \cdot \bar{m}$ |
| $I_{n,m}^+$ | $\begin{array}{c} + n \\ m \end{array}$ | $\bar{n} \cdot \bar{m}$ |
| $I_{n,m}^-$ | $\begin{array}{c} - n \\ m \end{array}$ | $\bar{n} \cdot \bar{m}$ |
| $I_{n,n}^+$ | $\begin{array}{c} + n \\ n \end{array}$ | $n$ |
| $I_{n,n}^- \times I_n$ | $\begin{array}{c} + n \\ 2r \\ n \end{array}$ | $n$ |

Our Tamagawa number algorithm enables the Tamagawa numbers in such tables to be calculated rapidly, even in the absence of a computer. For example, to find all the Tamagawa numbers in table 1 takes approximately ten minutes with pen and paper.

1.2. Second algorithm: Néron component groups. The Tamagawa number of a curve $X/K$ is defined as the order of a certain finite abelian group, namely
the group of $k$-rational components of the special fibre of the Néron model of the Jacobian of $X/K$, which for brevity we will refer to as the Néron component group of $X/K$ and denote by $\Phi_{X/K}(k)$. As for Tamagawa numbers, there is a natural definition of the Néron component group $\Phi_T$ of an arbitrary BY forest $T$ (definition 2.3.1) which, when the BY tree arises from a semistable hyperelliptic curve $X/K$, recovers the usual Néron component group of $X/K$, and it is natural to wonder whether one can create an algorithm to calculate these Néron component groups much as we did for Tamagawa numbers.

This will be achieved by the algorithm to be presented in this section. However, the algorithm that we will present is of a very different character to the preceding Tamagawa number algorithm, and in particular doesn’t allow us to produce formulae calculating Néron component groups of BY trees varying in families. Nonetheless, coupled with the description of BY trees of semistable hyperelliptic curves in [8, Definitions 1.1 and D.6], this algorithm does yield an efficient way to explicitly compute the Néron component groups of individual semistable hyperelliptic curves, giving a refinement of our answer to motivating question 3. Furthermore, the techniques which we will develop to study this algorithm will have other, more abstract, consequences (which we will outline at the end of this introduction).

1.2.1. Reduction step. As in the case of our Tamagawa number algorithm, the first step in our calculation of the Néron component group of a BY forest is reduction to the particular case of a simple BY tree. This is afforded by the following lemma.

**Lemma 1.2.1.** Let $T = (T, S, \epsilon F)$ be a BY forest.

- If $T_0$ is the BY forest formed by taking the disjoint union of the closures of the components of $T \setminus S$ (and giving this the induced signed automorphism and subgraph), then the Néron component groups of $T$ and $T_0$ agree.
- If $T$ is a disjoint union of $F$-stable BY subforests, then the Néron component group of $T$ is isomorphic to the direct sum of the Néron component groups of the subforests.
- If $T$ consists of a single $F$-orbit of $q$ BY trees $T_0, FT_0, \ldots, F^{q-1}T_0$, then the Néron component group of $T$ is isomorphic to that of $(T_0, S \cap T_0, (\epsilon F)^q)$ where $(\epsilon F)^q = \epsilon' F^q$ with $\epsilon'$ the product of the signs of the constituent trees of $F$.
- If $S = \emptyset$ or $S = T$, then the Néron component group of $T$ is trivial.

Moreover, in the first three points, if $T$ satisfies parity condition (B) from proposition 2.2.9 then so too do all the $T_i$ (we do not need to consider parity condition (A) for this algorithm).

1.2.2. Positive simple BY trees. As in the Tamagawa number algorithm, lemma 1.2.1 reduces us to the problem of calculating the Néron component groups of simple BY trees, both positive and negative. Unlike the Tamagawa number algorithm, however, we are unable to give an explicit formula for these Néron component groups and our algorithm is again recursive, using the following slight variant on the notion of BY tree from earlier.

**Definition 1.2.2 (Marked BY tree).** A marked unsigned simple BY tree (henceforth just marked BY tree for brevity) is a quadruple $(T, S, F, \ast)$ consisting of an integrally metrised finite graph-theoretic tree $T$, a non-empty set $S$ of vertices of

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7That is, carrying an edge-length function $l : E(T) \to \mathbb{N}$ as in definition 1.0.1.
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of degree 1, an automorphism $F$ of the pair $(T, S)$, and an $F$-fixed vertex $* \in T$ not in $S$.

The significance of this definition is twofold. Firstly, it is obvious that every simple BY tree satisfying parity condition (B) from proposition 2.2.9 is of the form $(T, S, \pm F)$ or $(T, S \cup \{\ast \}, \pm F)$ for a marked BY tree $(T, S, F, \ast)$. Secondly, the class of marked BY trees admits several natural constructions, enabling several invariants of marked BY trees to be computed recursively from the simplest possible base case.

**Notation 1.2.3.** The class of all marked BY trees is generated from the base case of a single edge of length $l \in \mathbb{N}$, pointwise fixed by $F$ with one endpoint the sole element of $S$ and the other the marked vertex $\ast$, using the following four constructions.

- If $T = (T, S, F, \ast)$ is a marked BY tree, we denote by $T(l)$ for a positive integer $l$ the marked BY tree formed by grafting an edge of length $l$ onto $T$ at the marked point $\ast$, and shifting the marked point to the other end of this edge.
- Given marked BY trees $T_i$, we denote by $\bigvee_i T_i$ the marked BY tree formed by joining the trees $T_i$ at their marked points.
- Given a positive integer $q$ and a marked BY tree $T$ (whose automorphism we denote with $F^q$ instead of $F$), we denote by $\bigvee \text{Ind}_F^q T$ the marked BY tree formed by joining $q$ copies of $T$ together at their marked points, where we specify that $F$ maps the $i$th copy isomorphically onto the $(i+1)$th copy (mod $q$) so that the $q$-fold composite acts as $F^q$ on each factor.
- If $T_0 = (T_0, S_0, F, \ast)$ is a marked BY tree, we can produce trivial enlargements of $T_0$ by embedding it as a marked BY subtree of some $T = (T, S, F, \ast)$ where $S = S_0$ (i.e. no new vertices are added into $S_0$ in the enlargement). Such trivial enlargements will not affect any of the invariants we consider in this paper, so we will not fix any notation for them.

In order to compute the Néron component groups of positive simple BY trees, one might thus hope to find a recursive algorithm using the constructions in notation 1.2.3 to compute the Néron component groups of $(T, S, +F)$ and $(T, S \cup \{\ast \}, +F)$ for any marked BY tree $(T, S, F, \ast)$. However, it turns out that this does not work directly, and we must define a stronger invariant of marked BY trees which can be computed recursively, and which recovers both the Néron component group of $(T, S, +F)$ and of $(T, S \cup \{\ast \}, +F)$ naturally. This stronger invariant will be given a natural definition in section 5.2, but even without this definition, we can still precisely state the algorithm to compute it.

**Algorithm 1.2.4.** To every marked BY tree $(T, S, F, \ast)$ we assign a triple $(\Pi^F, y, \eta)$ consisting of a finitely generated abelian group $\Pi^F$ of torsion-free rank 1 and two non-torsion elements $y, \eta \in \Pi^F$. From this triple, one can read off the Néron component groups of $(T, S, +F)$ and $(T, S \cup \{\ast \}, +F)$ as $\Pi^F / \eta$ and $\Pi^F / y$ respectively. The triple $(\Pi^F, y, \eta)$ can be computed recursively, as follows:

- If $T$ consists of a single edge of length $l \in \mathbb{N}$, with one endpoint $\ast$ and the other in $S$, then $(\Pi^F, y, \eta) = (\mathbb{Z}, l, 1)$.

---

8Strictly speaking, this is only true if one works with BY trees up to (signed) metric equivalence, as discussed in remark 2.2.5. However, this makes no material difference to the theory.
9It is not true that every BY tree of such a form is simple. Thus the part of the algorithm in this section will also compute Néron component groups of certain non-simple BY trees.
• if $T = T_0(l)$ then $(\Pi^F, y, \eta) = (\Pi_0^F, y_0 + l\eta_0, \eta_0)$;
• if $T = T_0 \lor T_1$ then $(\Pi^F, y, \eta) = \left(\Pi_0^F, y_0 + \eta_0, (y_0, 0), (\eta_0, \eta_1)\right)$;
• if $T = \bigvee \text{Ind}_F^T, T_0$ then $(\Pi^F, y, \eta) = \left(\Pi_0^F, y_0, y_0, \eta_0\right)$, where $A(\frac{1}{q}a) = A_{\alpha}^{\otimes q}$ denotes the overgroup of $A$ formed by adjoining a formal one-q-th of $a \in A$;
• if $T = (T, S, F, \ast)$ contains a marked BY subtree $T_0 = (T_0, S_0, F, \ast)$ with $S_0 = S$, then $(\Pi^F, y, \eta) = (\Pi_0^F, y_0, \eta_0)$.

Here $(\Pi_0^F, y_0, \eta_0)$ always denotes the triple associated to the BY tree $T_i$.

1.2.3. Negative simple BY trees. The computation of the Néron component groups of negative simple BY trees proceeds in a similar manner, assigning to every marked BY tree $(T, S, F, \ast)$ a new invariant, which can be computed recursively via the constructions in notation 1.2.3, and from which the Néron component groups of $(T, S, \ast, F)$ and $(T, S \cup \{\ast\}, \ast, F)$ can be recovered.

Algorithm 1.2.5. To every marked BY tree $(T, S, F, \ast)$ we assign a triple $(\Pi^F, T, S, \ast)$ consisting of a finite abelian group $\Pi^F$, an element $\alpha \in \Pi^F/\Pi^F = \text{Ext}^1(\mathbb{Z}/2, \Pi^F)$, and a type $\tau \in \{0, 1, 2\}$. From this triple, one can read off the Néron component groups of $(T, S, \ast, F)$ and $(T, S \cup \{\ast\}, \ast, F)$ as

$$\Phi(T, S, \ast, F) \simeq \begin{cases} 
\Pi^F & \text{if } \tau \neq 0 \\
\Pi^F(T/2\alpha) & \text{if } \tau = 0
\end{cases}$$

The triple $(\Pi^F, T, S, \ast)$ can be computed recursively, as follows:

• if $T$ consists of a single edge of length $l \in \mathbb{N}$, with one endpoint $\ast$ and the other in $S$, then $(\Pi^F, T, S, \ast) = (0, 0, 2)$ or $(0, 0, 0)$ according as $l$ is odd or even;

• if $T = T_0(l)$ then $(\Pi^F, T, S, \ast) = (\Pi_0^F, T_0, S_0, \ast)$ and $T = \begin{cases} 
2 - \tau_0 & \text{if } l \text{ odd,} \\
\tau_0 & \text{if } l \text{ even,}
\end{cases}$

• if $T = T_0 \lor T_1$ then $\Pi^F = \begin{cases} 
\Pi_0^F \oplus \Pi_1^F & \text{if } \tau_0 = \tau_1 = 0, \\
\Pi_0^F \oplus \Pi_1^F & \text{otherwise,}
\end{cases}$

$$\tau = \tau_0 \tau_1 \mod 3,$$ and $\alpha = \begin{cases} 
(\alpha_0, 0) & \text{if } \tau_0 = 0, \\
(0, \alpha_1) & \text{if } \tau_1 = 0, \\
(\alpha_0, \alpha_1) & \text{else;}
\end{cases}$

• if $T = \bigvee \text{Ind}_F^T, T_0$ for odd $q$, then $(\Pi^F, T, S, \ast) = (\Pi_0^F, q, \alpha_0, 0)$;
• if $T = \bigvee \text{Ind}_F^T, T_0$ for even $q$, then $(\Pi^F, T, S, \ast) = (\Pi_0^F, q, \alpha_0, 1)$, where $(\Pi_0^F, q, \alpha_0, 1)$ is the triple computed recursively in algorithm 1.2.2 for $T_0$;
• if $T = (T, S, F, \ast)$ contains a marked BY subtree $T_0 = (T_0, S_0, F, \ast)$ with $S_0 = S$, then $(\Pi^F, T, S, \ast) = (\Pi_0^F, T_0, S_0, \ast, \ast)$.

Here $(\Pi_0^F, \alpha_0, \tau_0)$ always denotes the triple associated to the BY tree $T_i$.

1.2.4. Examples. Let us now illustrate how these algorithms play out in practice by computing Néron component groups of some specific BY trees.

Example 1.2.6. Let us calculate the Néron component group of the following positive simple BY tree $T$, from example 1.1.6.
In order to apply algorithm 1.2.4 we need to mark an $F$-fixed vertex on this BY tree. There are several options, but for the purposes of this example let us set $*$ equal to the central yellow/open vertex. If we let $T_l$ denote the marked BY tree with trivial $F$-action, then the marked BY tree $T_* := \bigvee \text{Ind}_F^F T_a \lor T_b \lor T_c$ looks like the following

so that $T$ is formed from this marked BY tree by forgetting the marking and setting the sign positive. Thus the Néron component group of $T$ is isomorphic to $\Pi^F/\eta$, where $(\Pi^F, y, \eta)$ is the triple associated to $T_*$ in algorithm 1.2.4, which we compute recursively.

Note that the triple associated to $T_l$ is $(\mathbb{Z}, l, 1)$ by the first two points of algorithm 1.2.4. It follows that the triple associated to $\bigvee \text{Ind}_F^F T_a$ is $(\mathbb{Z}(a/2), a/2, 1)$, and hence that $\Pi^F$ is the quotient of $\mathbb{Z}(a/2) \oplus \mathbb{Z} \oplus \mathbb{Z}$ by the identification $(a/2, 0, 0) \sim (0, b, 0) \sim (0, 0, c)$. The Néron component group of $T$ is then the quotient of $\Pi^F$ by the subgroup generated by $\eta = (1, 1, 1)$.

To make this concrete, we can explicitly realise $\mathbb{Z}(a/2)$ as the cokernel of the map $\mathbb{Z} \to \mathbb{Z}^\oplus 2$ represented by the matrix $(a -2)$, and hence $\Pi^F$ as the cokernel of the map $\mathbb{Z}^\oplus 3 \to \mathbb{Z}^\oplus 4$ represented by the matrix

$$
\begin{pmatrix}
 a & -2 & 0 & 0 \\
 0 & 1 & -b & 0 \\
 0 & 1 & 0 & -c
\end{pmatrix}.
$$

Under this identification, $\eta$ is the image of the vector $(1, 0, 1, 1)$, and hence the Néron component group of $T$ is the cokernel of the map represented by the matrix

$$
\begin{pmatrix}
 a & -2 & 0 & 0 \\
 0 & 1 & -b & 0 \\
 0 & 1 & 0 & -c \\
 1 & 0 & 1 & 1
\end{pmatrix}.
$$

A quick Smith normal form calculation then shows that the Néron component group is $C_{\text{hcf}(a,b,c)} \times C_{(a(b+c)+2bc)}/\text{hcf}(a,b,c)$.

**Example 1.2.7.** Let us calculate the Néron component group of the following negative simple BY tree $T$, which is the same as the previous example, but with the sign reversed.
Similarly to the previous example, we want to compute the triple \((\Pi^{F}, \alpha, \tau)\) associated to \(\sqrt[3]{\text{Ind}_{F}^{F} T_{a} \vee T_{b} \vee T_{c}}\) (in the notation of example 1.2.6), and then to read off the Néron component group of \(T\) as in algorithm 1.2.5. The steps of the recursive calculation of the triple associated to \(\sqrt[3]{\text{Ind}_{F}^{F} T_{a} \vee T_{b} \vee T_{c}}\) are detailed in the following table.

| Marked BY tree | Triple \((\Pi^{F}, \alpha, \tau)\) |
|----------------|----------------------------------|
| \(T_{b}\)     | \((0, 0, 0)\) \(b\) even        |
|                | \((0, 0, 2)\) \(b\) odd         |
| \(\sqrt[3]{\text{Ind}_{F}^{F} T_{a}}\) | \((\mathbb{Z}/a, 1, 1)\)          |
| \(\sqrt[3]{\text{Ind}_{F}^{F} T_{a} \vee T_{b}}\) | \((\mathbb{Z}/a, 0, 0)\) \(b\) even |
|                | \((\mathbb{Z}/a, 1, 2)\) \(b\) odd |
| \(\sqrt[3]{\text{Ind}_{F}^{F} T_{a} \vee T_{b} \vee T_{c}}\) | \(((\mathbb{Z}/a) \oplus (\mathbb{Z}/2), 0, 0)\) \(b\) and \(c\) even |
|                | \((\mathbb{Z}/a, 0, 0)\) one of \(b\) or \(c\) even |
|                | \((\mathbb{Z}/a, 1, 1)\) \(b\) and \(c\) odd |

\[
\left\{
\begin{array}{c}
\mathbb{Z}/a \oplus (\mathbb{Z}/2) \ b\text{ and } c\text{ even,} \\
\mathbb{Z}/a \ b\text{ or } c\text{ even,} \\
(\mathbb{Z}/a)(1/2) \ b\text{ and } c\text{ odd,}
\end{array}
\right\}
\begin{array}{c}
C_a \times C_2 \ b\text{ and } c\text{ even,} \\
C_a \ b\text{ or } c\text{ even,} \\
C_{2a} \ b\text{ and } c\text{ odd.}
\end{array}
\]

1.3. Application: growth of Tamagawa numbers in towers. The machinery which proves the validity of the Néron component group algorithm also yields abstract consequences for the growth of Tamagawa numbers of semistable hyperelliptic curves as we enlarge the base field, providing a partial answer to motivating question 2. On the level of BY trees, we will see that this amounts to analysing the Tamagawa numbers of the BY trees

\[T_{e,f} := (eT, eS, (eF)^f)\]

formed from a fixed BY tree \(T = (T, S, \epsilon F)\) by scaling all the edge-lengths in \(T\) by a factor of \(e \in \mathbb{N}\) and replacing the action of \(\epsilon F\) by \((\epsilon F)^f\) – if \(T\) arises as the BY tree associated to a semistable hyperelliptic curve \(X/K\), then \(T_{e,f}\) is the BY tree associated to \(X\) over any finite extension \(L/K\) of ramification degree \(e\) and residue class degree \(f\).

By combining techniques from both of the previous algorithms, we are able to prove the following strong restriction on the Tamagawa numbers of the BY trees \(T_{e,f}\) as a function of \(e\) and \(f\).

**Theorem 1.3.1.** Fix a BY forest \(T = (T, S, \epsilon F)\) satisfying parity conditions (A) and (B), and consider the family \(T_{e,f} = (eT, eS, (eF)^f)\) of BY forests indexed by \(e, f \in \mathbb{N}\). Then there are constants \((a_d, r_d, s_d) \in \mathbb{N} \times \mathbb{N}_0 \times \mathbb{Z}\) for each \(d \in \mathbb{N}\) (equal to \((1, 0, 0)\) for \(d\) not dividing the order of \(\epsilon F\)) such that the Tamagawa number of
each $T_{e,f}$ is given by
\[
\prod_{d \mid f} \left( a_d e^{e \cdot \text{hcf}(e, 2)^{s_d}} \phi(d) \right)
\]
where $\phi$ is Euler’s totient function.

The preceding theorem applies in particular to the case of Tamagawa numbers of a fixed semistable hyperelliptic curve $X/K$ over field extensions $L/K$ of ramification degree $e$ and residue class degree $f$. To pick just one very specific consequence, we obtain the following result about growth of Tamagawa numbers in unramified extensions of prime degree.

**Corollary 1.3.2.** Let $X$ be a semistable hyperelliptic curve over a $p$-adic number field $K$. Let $L/K$ be the unramified extension of prime degree $q$. Then the Tamagawa numbers $c_{X/L}$ and $c_{X/K}$ agree up to $(q - 1)$th powers.

If $X$ is defined and semistable over a $p$-adic subfield $K_0 \subseteq K$ of residue class degree $f$ under $K$, then $c_{X/L}$ and $c_{X/K}$ even agree up to $q^{\nu_q(f)}(q - 1)$th powers.

**Proof.** It suffices to prove the second statement. If we let $e = e(K/K_0) = e(L/K_0)$, then theorem 1.3.1 shows that
\[
\frac{c_{X/L}}{c_{X/K}} = \prod_{d \mid qf, d \mid f} \left( a_d e^{e \cdot \text{hcf}(e, 2)^{s_d}} \phi(d) \right).
\]
But for each such $d$ in the product $q^{\nu_q(f)}(q - 1) \mid \phi(d)$, so we are done. □

What makes this corollary particularly surprising is that the result becomes untrue if we remove the assumption that $X$ is hyperelliptic.

**Example 1.3.3.** Let $K$ be a $p$-adic number field with residue field $k$; write $k_5$ for the degree 5 extension of $k$ and $K_5/K$ the corresponding unramified extension of $K$. Let $X/K$ be a (smooth, projective, geometrically integral) curve with a regular semistable model $\mathcal{X}/\mathcal{O}_K$ such that the dual graph $\mathcal{G}$ of the geometric special fibre $\mathcal{X}_k$ is the following five-spoked wheel graph

![Wheel Graph](image)

where the edge-lengths are all 1, the vertices all have genus 0, and the induced action of Frobenius rotates the wheel by a one-fifth turn. In other words, the normalisation of $\mathcal{X}_k$ is $\mathbb{P}^1_k \sqcup \mathbb{P}^1_{k_5}$, where the copy of $\mathbb{P}^1_k$ meets the copy of $\mathbb{P}^1_{k_5}$ transversely at a point of degree 5 over $k$, and the copy of $\mathbb{P}^1_{k_5}$ meets itself transversely at a (different) point of degree 5 over $k$ such that, after base-changing from $k$ to $k_5$, each of the five components of $(\mathbb{P}^1_{k_5})_{k_5} \cong (\mathbb{P}^1_{k_5})^{15}$ meets its Frobenius conjugate.

(The existence of such a curve $\mathcal{X}/\mathcal{O}_K$ is provided by the classical deformation theory of stable curves and algebraisation of formal schemes. To sketch the argument,

\footnote{In other words, $\mathcal{X}/\mathcal{O}_K$ is proper, flat and regular, and its special fibre $\mathcal{X}_k$ is a reduced normal crossings divisor.}
one can write down regular formal deformations over \( \mathcal{O}_K \) of the completed local rings of the two nodes of the desired \( X_k \) (one \( k \)-rational and the other \( k_5 \)-rational), so that \[ \text{[5, Proposition 1.5]} \] then guarantees the existence of a global formal deformation \( X/\mathcal{O}_K \) of \( X_k \) inducing these deformations of the completed local rings. \( X \) is, \emph{a priori}, merely a formal \( \mathcal{O}_K \)-scheme, but since the tensor-cube of the relative canonical sheaf \( \omega_{X/\mathcal{O}_K} \) is very ample \[ \text{[5, Corollary to Theorem 1.2]} \], Grothendieck’s algebraisation theorem \[ \text{[11, Théorème 5.4.5]} \] shows that \( X \) is the formal completion of a projective \( \mathcal{O}_K \)-scheme, which we still denote \( X \). One can then easily check that \( X/\mathcal{O}_K \) is flat and regular, so we may take \( X/K \) to be the generic fibre of its model \( X/\mathcal{O}_K \).

We claim that the Tamagawa number of \( X \) over \( K \) is 1, but that over \( K_5 \) its Tamagawa number becomes \( 121 \), which is in particular not a fourth power (so that \( X/K \) does not satisfy the conclusion of corollary \[ \text{[1.3.2]} \]). These Tamagawa numbers can be computed directly from the dual graph \( G \) above (see section \[ \text{2.1} \]). Specifically, the homology lattice \( \Lambda = H_1(G, \mathbb{Z}) \) has a canonical embedding \( \Lambda \hookrightarrow \Lambda^\vee \) into its abstract dual induced by the intersection length pairing, and the Tamagawa number of \( X \) over \( K \) is then the order \( c_{X/K} = \#(\Lambda^\vee/\Lambda)^{\text{Frob}} \) of the Frobenius-invariants in the cokernel of this embedding.

The same description computes the Tamagawa number of \( X \) over \( K_5 \) once we replace \( \text{Frob} \) with its fifth power, which here acts trivially on \( G \). Hence over \( K_5 \), the Tamagawa number \( c_{X/K_5} = \#(\Lambda^\vee/\Lambda) \) is simply given by the discriminant of the intersection length pairing, which is easily computed to be \( 121 \).

To compute the Tamagawa number of \( X \) over \( K \) itself, we note that \( \Lambda \) is a free \( \mathbb{Z}[C_5] \)-module, and hence \( (\Lambda^\vee/\Lambda)^{\text{Frob}} = (\Lambda^\vee)^{\text{Frob}}/\Lambda^{\text{Frob}} \). Now on the one hand \( \Lambda^{\text{Frob}} \) is the group of \( \text{Frob} \)-invariant \( \mathbb{Z} \)-valued cycles on \( G \), which is \( \simeq \mathbb{Z} \), generated by the perimeter of the wheel. On the other hand, \( \Lambda^\vee \) can be identified (as an overlattice of \( \Lambda \)) with the group of \( \mathbb{Q} \)-valued cycles on \( G \) whose intersection pairing with any integer cycle is an integer, so the Frobenius-invariants \( (\Lambda^\vee)^{\text{Frob}} \) is again \( \simeq \mathbb{Z} \), generated by the perimeter of the wheel. Thus the Tamagawa number of \( X \) over \( K \) is \( c_{X/K} = 1 \) as desired.

1.4. Overview of sections. We will begin in section \[ \text{2} \] by reviewing the relationship between semistable hyperelliptic curves, their BY trees and their Tamagawa numbers, following closely the exposition in \[ \text{8} \]. This will be followed by the brief section \[ \text{3} \] which will establish lemmas \[ \text{1.1.1} \] and \[ \text{1.2.1} \], justifying the steps in our Tamagawa number and Néron component group algorithms allowing one to reduce to the case of simple BY trees. The justifications of these algorithms for simple BY trees forms the bulk of the work in this paper, and takes place in sections \[ \text{4} \] and \[ \text{5} \] respectively. Finally, we turn in section \[ \text{6} \] to the proof of theorem \[ \text{1.3.1} \] regarding the growth of Tamagawa numbers of semistable hyperelliptic curves as one enlarges the base field, drawing on the theory developed in the two previous sections. Appendix \[ \text{A} \] proves various technical results on fixpoint sets in integral representations.

2. BY forests and Tamagawa numbers

2.1. Semistable curves and dual graphs. The aim of this section is to define the BY tree associated to a semistable hyperelliptic curve \( X/K \), and to describe how to reconstruct from it the Tamagawa number of \( X/K \). To begin with, let us first recall the relationship between dual graphs of general semistable curves and
their Tamagawa numbers, which we will specialise in the hyperelliptic case to link BY trees and Tamagawa numbers of such curves. This general relationship certainly qualifies as well-known, but we will nonetheless state it carefully, in a form most closely suited to our applications.

**Theorem 2.1.1.** Let $X/K$ be a curve with a regular semistable model $X_\mathcal{O}_K$, and let $\mathcal{G}$ denote the dual graph of the geometric special fibre $X_{\mathcal{O}_K}$. Let $\Lambda = H_1(G, \mathbb{Z})$ be the homology lattice of $\mathcal{G}$ and embed $\Lambda$ in its abstract dual $\Lambda^\vee$ via the map $\Lambda \rightarrow \Lambda^\vee$ induced by the intersection pairing.

Then the cokernel $\Lambda^\vee/\Lambda$ of this embedding is canonically (and in particular Frobenius-equivariantly) identified with the group $\Phi_{X/K}(k)$ of $k$-points of the group-scheme of connected components of the special fibre of the Néron model of the Jacobian of $X/K$. In particular, the Néron component group $\Phi_{X/K}(k)$ of $X/K$ is canonically identified with the Frobenius invariants $(\Lambda^\vee/\Lambda)^{\text{Frob}}$, and the Tamagawa number $c_{X/K}$ of $X/K$ is computed by

$$c_{X/K} = \#(\Lambda^\vee/\Lambda)^{\text{Frob}}.$$ 

**Proof.** [12 Theorem 2.3] □

2.2. **BY trees of semistable hyperelliptic curves.** Having recalled the link between Tamagawa numbers and dual graphs of general semistable curves, let us now formally introduce the notion of a BY tree/forest and describe how to attach them to semistable hyperelliptic curves. At its most basic level, these BY trees are supposed to be simple combinatorial devices which completely encapsulate the dual graphs of semistable hyperelliptic curves, and fundamentally arise from the following useful observation.

**Theorem 2.2.1.** Let $X$ be a semistable hyperelliptic curve over a $p$-adic number field $K$ with minimal regular (semistable) model $X_\mathcal{O}_K$, and let $\mathcal{G}$ denote the dual graph of $X_k$. Then the quotient of (the underlying topological space of) $\mathcal{G}$ by the hyperelliptic involution $\iota$ is a (topological) tree.

**Proof.** [8 Theorem 5.18] proves this in the case $p \neq 2$. We will sketch a proof in general using the theory of analytic geometry in the sense of Berkovich. Let $X_{\mathcal{C}_K}^{\text{Berk}}$ denote the Berkovich analytification of $X$ over a completed algebraic closure $\mathcal{C}_K$ of $K$, and let $\iota$ denote the hyperelliptic involution on $X_{\mathcal{C}_K}^{\text{Berk}}$. Formula (⋆) in the proof of [2 Proposition 3.4.6] describes the fibres of the map $X_{\mathcal{C}_K}^{\text{Berk}} \rightarrow \mathbb{P}_{\mathcal{C}_K}^{1,\text{Berk}}$: if $x$ is a point of $\mathbb{P}_{\mathcal{C}_K}^{1,\text{Berk}}$ with residue field $\mathcal{H}(x)$, then its fibre is the Berkovich spectrum of a finite $\mathcal{H}(x)$-algebra of dimension 2 or 1, and the $\iota$-fixed subalgebra is exactly $\mathcal{H}(x)$.

It follows from this description that the map $|X_{\mathcal{C}_K}^{\text{Berk}}| \rightarrow |\mathbb{P}_{\mathcal{C}_K}^{1,\text{Berk}}|$ between underlying topological spaces is a set-theoretic quotient by the hyperelliptic involution $\iota$ – since the domain is compact and the codomain is Hausdorff, it is automatically a topological quotient.

Now it is well-known that the dual graph $\mathcal{G}$ of $X_{\mathcal{C}_K}$ embeds naturally inside $|X_{\mathcal{C}_K}^{\text{Berk}}|$ (see e.g. [1 Segment 4.9]). The image of this embedding is $\iota$-stable by naturality, and hence the quotient $\mathcal{G}/\iota$ is canonically identified with the image of $\mathcal{G}$ under $|X_{\mathcal{C}_K}^{\text{Berk}}| \rightarrow |\mathbb{P}_{\mathcal{C}_K}^{1,\text{Berk}}|$. This image is a topological tree, since on the one hand it is topologically a connected graph, and on the other, being a connected subspace of $|\mathbb{P}_{\mathcal{C}_K}^{1,\text{Berk}}|$, it is contractible by [2 Theorem 4.2.1]. □
Intuitively, the BY tree associated to $X/K$ will simply be this quotient tree $T = G/\iota$, endowed with enough extra data to reconstruct $G$ with its Frobenius action, at least up to non-canonical homeomorphism. This extra data arises as follows.

**Construction 2.2.2.** Let $G$ be a finite graph endowed with an involution $\iota$ and an automorphism $\text{Frob}$ commuting with $\iota$, and suppose that the topological quotient $G/\iota$ is a forest. We may then define a quadruple $(T, S, F, \epsilon)$ as follows:

- $T = G/\iota$ is the quotient forest, viewed as a graph by declaring the vertices of $T$ to be exactly those points which are images of vertices or midpoints of edges of $G$;
- $S \subseteq T$ is the subgraph of $T$ consisting of the ramification locus of the ramified double cover $G \to T$;
- $F$ is the automorphism of the graph-pair $(T, S)$ induced by the automorphism $\text{Frob}$ of $G$; and
- if we fix a section $\sigma$ of the trivial double cover $G\mid_{T S} \to T \setminus S$, then for every connected component $C$ of $T \setminus S$ we have an associated sign

$$\epsilon(C) := \begin{cases} +1 & \text{if } \sigma(F(C)) = \text{Frob}(\sigma(C)), \\ -1 & \text{if } \sigma(F(C)) = \epsilon\text{Frob}(\sigma(C)). \end{cases}$$

Taken together, this defines a sign function $\epsilon : \pi_0(T \setminus S) \to \{\pm 1\}$ (depending on $\sigma$).

We would like to show that the output $(T, S, F, \epsilon)$ of this construction is independent of the choice of section $\sigma$, at least up to some notion of isomorphism between quadruples $(T, S, F, \epsilon)$. It turns out that in trying to capture this notion, it is most natural to think of the pair $(F, \epsilon)$ as a kind of **signed isomorphism** $\epsilon F$ of the pair $(T, S)$. This then leads very naturally to the definition of a BY forest – a triple $(T, S, \epsilon F)$ such as is produced by construction 2.2.2, and moreover to the correct notion of isomorphism between such. When these definitions have been set up, it is essentially immediate that the output of construction 2.2.2 is well-defined (up to non-canonical isomorphism).

**Definition 2.2.3** (Signed isomorphisms). Suppose $(T, S)$ is a pair of an integrally metrised finite graph-theoretic forest $T$ and a subgraph $S \subseteq T$, and that $(T', S')$ is another such pair. A **signed isomorphism** $\epsilon F : (T, S) \cong (T', S')$ is a pair $\epsilon F = (F, \epsilon)$ consisting of an isometric isomorphism $F : (T, S) \cong (T', S')$ of graph-pairs, and a sign function $\epsilon : \pi_0(T \setminus S) \to \{\pm 1\}$, which we think of as encoding whether the signed automorphism acts positively or negatively on each component.

If $\epsilon F : (T, S) \cong (T', S')$ and $\epsilon' F' : (T', S') \cong (T'', S'')$ are two signed isomorphisms, then we define the **composite** $\epsilon' F' \circ \epsilon F : (T, S) \cong (T'', S'')$ to be the signed isomorphism whose underlying isomorphism is $F' F$ and whose sign function takes a component $C \in \pi_0(T \setminus S)$ to $\epsilon(C)\epsilon'(F(C)) \in \{\pm 1\}$.

**Definition 2.2.4** (BY forests). A **BY forest** (resp. **BY tree**) is a triple $T = (T, S, \epsilon F)$ consisting of an integrally metrised finite graph-theoretic forest (resp. tree) $T$, a subgraph $S$, and a signed automorphism $\epsilon F$ of the pair $(T, S)$. A **signed isomorphism** $\epsilon'' F'' : (T, S, \epsilon F) \cong (T', S', \epsilon' F')$ between two BY forests is a signed isomorphism $\epsilon'' F'' : (T, S) \cong (T', S')$ between the underlying pairs such that $(\epsilon'' F'') \circ (\epsilon F) = (\epsilon' F') \circ (\epsilon'' F'')$. 

If $T$ is a BY forest, we shall let $\text{div}(T)$ denote the BY forest obtained by replacing every edge $e$ of $T$ by a chain of $l(e)$ edges of length 1, and adjusting $S$ and $\epsilon_F$ accordingly. A signed metric equivalence from $T$ to $T'$ is then simply a signed isomorphism from $\text{div}(T)$ to $\text{div}(T')$, and we declare the composite of two signed metric equivalences $\tilde{T} \xrightarrow{\sim} \tilde{T}' \xrightarrow{\sim} \tilde{T}''$ to be the composite of the signed isomorphisms $\text{div}(T) \xrightarrow{\sim} \text{div}(T') \xrightarrow{\sim} \text{div}(T'')$. Every signed isomorphism canonically induces a signed metric equivalence, so we may and will view the category of BY forests with signed isomorphisms as a (non-full) subcategory of the category of BY forests with signed metric equivalences. Note that both these categories are groupoids, and that they have the same objects.

Remark 2.2.5. Although the category of BY forests and signed isomorphisms is sufficient for most of the arguments we will run in this paper, it does suffer from one technical inadequacy. Specifically, the BY tree associated to a semistable hyperelliptic curve $X/K$ will always have all edge-lengths equal to 1, and the effect of varying $X/K$ in a degenerating family is to increase the lengths of chains of length 1 edges in its BY tree. Thus in order to be able to apply our algorithms to such families, we need to be able to regard chains of $l$ edges of length 1 as equivalent to a single edge of length $l$; this is accomplished by working with signed metric equivalences in place of signed isomorphisms.

We will avoid heavy use of categorical language in this paper, so for the most part the only relevance of the above discussion is that we will throughout feel free to replace BY forests with signed metric equivalent ones. All the invariants of interest to us are functorial with respect to signed metric equivalences, so doing so will not cause us any problems.

Equipped with a rigorous definition of BY forests, we can now make good on our promise to show that the output of construction 2.2.2 is well-defined.

Proposition 2.2.6. Suppose that $T = (T, S, \epsilon_F)$ and $T' = (T, S, \epsilon'_F)$ are two BY forests (differing only in their sign functions) such that the products of the signs in $T$ and $T'$ over each $F$-orbit agree, i.e. such that for every $F$-orbit $C, FC, \ldots, F^{q-1}C$ of components of $T \setminus S$ we have

$$\prod_{i=0}^{q-1} \epsilon(C) = \prod_{i=0}^{q-1} \epsilon'(C).$$

Then $T$ and $T'$ are signed isomorphic. In particular:

- the BY forest arising from construction 2.2.2 is well-defined up to non-canonical signed isomorphism; and
- any BY forest is signed isomorphic to a BY forest where $\epsilon: \pi_0(T \setminus S) \to \{\pm 1\}$ takes the value $-1$ at most once per $F$-orbit.

Proof. It suffices to prove the result in the special case that

$$\epsilon'(C) = \begin{cases} -\epsilon(C) & \text{if } C = C_0 \text{ or } F^{-1}C_0, \\ \epsilon(C) & \text{otherwise,} \end{cases}$$

for some non-$F$-fixed component $C_0$ of $T \setminus S$. If we set

$$\epsilon''(C) = \begin{cases} -\epsilon(C) & \text{if } C = C_0, \\ \epsilon(C) & \text{otherwise,} \end{cases}$$

for some non-$F$-fixed component $C_0$ of $T \setminus S$, we have

$$\prod_{i=0}^{q-1} \epsilon''(C) = \prod_{i=0}^{q-1} \epsilon'(C).$$

Thus $T$ and $T'$ are signed isomorphic.
then it is easy to see that $\epsilon'' \text{id}_T : T \xrightarrow{\sim} T'$ is a signed isomorphism, as desired. For the first bullet point, simply note that changing the choice of $\sigma$ over the component $C_0$ (and keeping it the same elsewhere) amounts to changing the sign function $\epsilon$ to the $\epsilon'$ above, and hence doesn’t affect the signed isomorphism type of $(T, S, \epsilon F)$. □

Combining all this abstract combinatorial discussion, the definition of the BY tree associated to a semistable hyperelliptic curve is now particularly natural.

**Definition 2.2.7 (BY tree associated to a semistable hyperelliptic curve).** Let $X$ be a semistable hyperelliptic curve over a $p$-adic number field $K$, and let $X/\mathcal{O}_K$ be its minimal regular (semistable) model. Let $\mathcal{G}$ be the dual graph of $X$, $\iota \in \text{Aut}(\mathcal{G})$ the hyperelliptic involution and $\text{Frob} \in \text{Aut}(\mathcal{G})$ the induced action of Frobenius. The **BY tree associated to** $X/K$ is defined to be the BY tree produced from $(\mathcal{G}, \iota, \text{Frob})$ by construction 2.2.2. This is well-defined up to non-canonical signed isomorphism, and in particular up to non-canonical signed metric equivalence.

**Remark 2.2.8.** One important property of the BY tree of a semistable hyperelliptic curve $X/K$ is that it is explicitly calculable when the residue characteristic $p \neq 2$. Indeed, as explained in the introduction, given a hyperelliptic equation $y^2 = f(x)$ for $X$, [8, Definition 1.1] explains how to read off a certain combinatorial invariant called a **cluster picture** from the relative position of the roots of $f$, and then the dictionary in [9, Construction 4.13] explains how to translate this cluster picture into a BY tree. This BY tree is indeed the BY tree of $X/K$ by [8, Theorem 5.18].

### 2.2.1. Parity conditions

In studying the BY trees associated to semistable hyperelliptic curves, an important role will be played by two parity conditions, the significance of which is explained in the following proposition.

**Proposition 2.2.9.** A **BY forest** $T = (T, S, \epsilon F)$ arises from a graph, involution and automorphism $(\mathcal{G}, \iota, \text{Frob})$ as in construction 2.2.2 and only if, all its edge-lengths are equal to 1 and it satisfies the following two parity conditions\(^{11}\):

A) if two vertices of $T$ lie an odd distance apart, then at least one is either:
- a leaf (degree 1 vertex) and in $S$, with its incident edge not in $S$;
- degree 2 and not in $S$ (so its incident edges do not lie in $S$ also);
- degree 2 and in the interior of $S$ (so its incident edges all lie in $S$);

B) no iterate of $\text{Frob}$ inverts any edge of $T$ of odd length.

In particular, these parity conditions are automatically satisfied for the BY tree arising from a semistable hyperelliptic curve.

**Proof.** Firstly suppose that $(T, S, \epsilon F)$ is induced from $(\mathcal{G}, \iota, \text{Frob})$, so all the edges in $T$ have length 1. To check the first condition, consider any two vertices of $T$ that are an odd distance apart, so that when we lift these points to $\mathcal{G}$, exactly one of them must be a midpoint of an edge. There are then three cases, easily checked to correspond to the claimed trichotomy:
- the edge is inverted by the involution;
- the edge is sent to another by the involution;
- the edge is pointwise fixed by the involution.

\(^{11}\)These parity conditions are deliberately phrased so as to be invariant under signed metric equivalences, so are well-defined properties of signed metric equivalence classes of BY forests.
To check the second condition, suppose that $F^q$ fixes the midpoint of an edge $e$ of $T$. Lifting this to $G$, we find that Frobenius $Frob^q$ or $\iota Frob^q$ fixes a point one-quarter of the way along an edge of $G$. But then Frobenius $Frob^q$ or $\iota Frob^q$ must fix this edge pointwise, so $F^q$ fixes $e$ pointwise.

Finally, the proof of the converse direction proceeds in a similar fashion. The triple $(T, S, \epsilon F)$ determines a topological double cover $G = T \cup S T \to T$ with branch locus $S$, a topological involution $\iota$ of $G$ by interchanging the factors of $T$, and an automorphism Frobenius $Frob$ of $G$ covering $\epsilon F$. The parity conditions ensure that we can choose an $F$-stable bipartition of $T$ such that one of the vertex-classes does not contain any vertices satisfying any of the three properties. We then declare a point of $G$ to be a vertex iff it maps to a point in this vertex-class in $T$. It is straightforward to see that this gives $G$ the structure of a finite graph with involution, and Frobenius an automorphism thereof, as desired.

We will frequently use parity condition (B) in an alternative form.

**Proposition 2.2.10.** Let $T = (T, S, \epsilon F)$ be a BY forest. Then parity condition (B) is equivalent to the following condition:

(B') if some iterate of $F$ fixes a component of $T$ setwise, then it fixes a vertex of that component.

**Proof.** Any automorphism of a finite tree either fixes a vertex or inverts an edge, but not both. \square

### 2.3. Tamagawa numbers and BY trees

Combining the discussion in the previous two sections, let us now describe how to recover Tamagawa numbers (and Néron component groups) of semistable hyperelliptic curves from their BY trees. For the purposes of our algorithms, it will be most useful to give a purely combinatorial description of these invariants that makes sense for any BY forest, and then later to show that this recovers the corresponding invariants for semistable hyperelliptic curves.

**Definition 2.3.1 (Arithmetic invariants of BY forests).** Let $T = (T, S)$ be a pair of a finite graph-theoretic forest $T$ and a subgraph $S$. We let $\Lambda_T$ denote the relative homology lattice $\Lambda_T := H_1(T, S, \mathbb{Z})$, which carries a canonical $\mathbb{Z}$-valued symmetric intersection length pairing, denoted $\langle \cdot, \cdot \rangle$, where $\langle \gamma, \gamma' \rangle$ for two paths $\gamma$ and $\gamma'$ is defined to be the total length of the intersection $\gamma \cap \gamma'$ inside $T \setminus S$, interpreted in a suitably oriented manner. This pairing is automatically positive-definite, since it is the restriction of a positive-definite pairing on the free $\mathbb{Z}$-module on the oriented edges of $T \setminus S$.

This construction of a lattice with symmetric pairing is functorial with respect to signed isomorphisms, where the map induced by a signed isomorphism $\epsilon F: (T, S) \xrightarrow{\sim} (T', S')$ sends the class of a path $\gamma$ contained in the closure of a component $C$ of $T \setminus S$ to the class $\epsilon(C)[F\gamma] \in H_1(T', S', \mathbb{Z})$ – this map is easily checked to be isometric with respect to the intersection length pairings.

In particular, if $T = (T, S, \epsilon F)$ is a BY forest, then the relative homology lattice $\Lambda_T = H_1(T, S, \mathbb{Z})$ carries an action by $\epsilon F$ which is isometric for the intersection length pairing. We make the following definitions.

---

12 Strictly speaking, these conditions are equivalent only after replacing $T$ with a signed metric equivalent BY forest, for example div$(T)$ from definition 2.2.4.
• The geometric Néron component group

\[ \Phi_T := \Lambda_T^\vee / \Lambda_T \]

is defined to be the cokernel of the embedding \( \Lambda_T \hookrightarrow \Lambda_T^\vee \) of \( \Lambda_T \) into its abstract dual \( \Lambda_T^\vee = \text{Hom}(\Lambda_T, \mathbb{Z}) \) induced by the intersection length pairing. This is a finite \( \mathbb{Z} \)-module carrying an automorphism \( \epsilon F \).

• The Néron component group \( \Phi_T \) of \( T \) is defined to be the invariant submodule \( \Phi_T = \Phi_T \leq \Phi_{T_0} \).

• The Tamagawa number \( c_T \) of \( T \) is defined to be the order \( c_T := \# \Phi_T = \#(\Lambda_T^\vee / \Lambda_T)^{\epsilon F} \)
of the Néron component group of \( T \).

All the above constructions are functorial with respect to signed metric equivalences (in particular, with respect to signed isomorphisms) of BY forests, so that the Tamagawa numbers of signed-metric-equivalent BY forests agree.

**Proposition 2.3.2.** If \( T = (T, S, \epsilon F) \) is a BY forest and \( T_0 = (T_0, S, \epsilon F) \) is the convex hull of \( S \) (with the induced structure of a BY forest), then \( \Phi_T \cong \Phi_{T_0} \) equivariantly for the \( \epsilon F \)-action (and so \( \Phi_T \cong \Phi_{T_0} \) and \( c_T = c_{T_0} \)). Moreover, if \( T \) satisfies parity condition (A) (resp. (B)) of proposition 2.2.9 then so does \( T_0 \).

**Proof.** The map \( T_0 \to T \) induces an isomorphism on relative homology lattices, respecting the pairing and \( \epsilon F \) action. Preservation of the parity conditions is an easy check. \( \Box \)

It remains to show that definition 2.3.1 is sensible, i.e. that the invariants defined therein recover the expected invariants of semistable hyperelliptic curves. This requires little more than theorem 2.1.1 and an unwinding of definitions.

**Lemma 2.3.3.** Let \( G \) be a finite graph endowed with an involution \( \iota \) and an automorphism \( \text{Frob} \) commuting with \( \iota \), and suppose that the topological quotient \( G/\iota \) is a forest. Let \( T = (T, S, \epsilon F) \) be the BY forest constructed from \( (G, \iota, \text{Frob}) \) in construction 2.2.2. Then there is an isomorphism \( H_1(G, \mathbb{Z}) \cong H_1(T, S, \mathbb{Z}) \) between the homology lattice of \( G \) and the relative homology lattice of \( (T, S) \) which respects the intersection length pairings on either side and under which the induced \( \text{Frob} \) action on \( H_1(G, \mathbb{Z}) \) corresponds to the action of \( \epsilon F \) on \( H_1(T, S, \mathbb{Z}) \).

**Proof.** Choose a section \( \sigma \) of the ramified double cover \( G \to T \); we fix this choice of section (restricted to \( T \setminus S \)) for the definition of the sign function \( \epsilon \) in construction 2.2.2. Now \( G \) is covered by \( \sigma T \) and \( i\sigma T \), with intersection \( S \), so by excision \( \sigma \) induces an isomorphism

\[ H_1(T, S, \mathbb{Z}) \cong H_1(G, \iota\sigma T, \mathbb{Z}) \]
on relative homology. But since \( \iota\sigma T \) is a topological forest, the exact sequence on homology of a pair provides the the canonical map

\[ H_1(G, \mathbb{Z}) \cong H_1(G, \iota\sigma T, \mathbb{Z}) \]
is an isomorphism. We thus obtain an isomorphism \( H_1(T, S, \mathbb{Z}) \cong H_1(G, \mathbb{Z}) \) by combining the above isomorphisms. In detail, this sends the class of a cycle \( \gamma \) on \( T \) with \( \partial \gamma \subseteq S \) to the class of the cycle \( \sigma\gamma - i\sigma\gamma \) on \( G \). It is then easy to check that this isomorphism is isometric and takes the action of \( \epsilon F \) to the action of \( \text{Frob} \). \( \Box \)
Corollary 2.3.4. Let $X$ be a semistable hyperelliptic curve over a $p$-adic number field $K$ and let $T$ be its associated BY tree. Then:

- the geometric Néron component group $\Phi_T$ of $T$ is isomorphic to the group $\Phi_{X/K}(\overline{K})$ of $\overline{K}$-points of the group-scheme of connected components of the special fibre of the Néron model of the Jacobian of $X/K$ in such a way that the action of $\epsilon F$ corresponds to the action of Frobenius;

- the Néron component group $\Phi_T$ of $T$ is isomorphic to the group of $k$-points $\Phi_{X/K}(k)$ of the group-scheme of connected components of the special fibre of the Néron model of the Jacobian of $X/K$; and

- the Tamagawa number of $T$ is equal to the Tamagawa number of $X/K$.

Proof. Combine theorem 2.1.1 and lemma 2.3.3.

3. Reduction to simple BY trees

Having made explicit the basic definitions we will be using, we now turn our attention to the justification of the Tamagawa number and Néron component group algorithms. The bulk of the work will be in the consideration of simple BY trees, so let us first prove lemmas 1.1.1 and 1.2.1, which allow us to reduce to this case. In fact, we will prove a stronger result on the geometric Néron component group $\Phi_T = \Lambda_T^\lor / \Lambda_T$ equivariantly for the $\epsilon F$ action, from which both these lemmas are immediate consequences.

Lemma 3.0.1. Let $T = (T, S, \epsilon F)$ be a BY forest.

- If $T_0$ is the BY forest formed by taking the disjoint union of the closures of the components of $T \setminus S$ (and giving this the induced signed automorphism and subgraph), then $\Phi_T \cong \Phi_{T_0}$ as $\mathbb{Z}[\epsilon F]$-modules.

- If $T$ is a disjoint union of $F$-stable BY subforests $T_i$ then $\Phi_T \cong \bigoplus_i \Phi_{T_i}$ as $\mathbb{Z}[\epsilon F]$-modules.

- If $T$ consists of a single $F$-orbit of $q$ trees $T_0, FT_0, \ldots, F^{q-1}T_0$, then $\Phi_T \cong \text{Ind}_{\epsilon F^{q-1}}^{\epsilon F^q} \Phi_{T_0}$ as $\mathbb{Z}[\epsilon F]$-modules (here we consider $T_0$ as a BY tree with subgraph $S \cap T_0$ and signed automorphism $(\epsilon F)^q|_{T_0}$).

- If $S = \emptyset$ or $S = T$, then $\Phi_T = 0$.

Moreover, in the first three points, if $T$ satisfies parity condition (A) (resp. (B)), so do all the $T_i$.

Proof. It is straightforward to check that the parity conditions are preserved.

For the first point, one observes that the map $T_0 \to T$ induces an isomorphism $\Lambda_{T_0} \iso \Lambda_T$ on relative homology lattices (for instance, by contracting out the subgraphs $S$), which preserves the $\epsilon F$-action and intersection length pairing. Hence the induced map $\Phi_{T_0} = \Lambda_{T_0}^\lor / \Lambda_{T_0} \to \Lambda_T^\lor / \Lambda_T = \Phi_T$ is an isomorphism (of $\mathbb{Z}[\epsilon F]$-modules).

Similarly, for the second point, we see that we have an orthogonal decomposition $\Lambda_T = \bigoplus_i \Lambda_{T_i}$ of $\mathbb{Z}[\epsilon F]$-modules, so that $\Lambda_T^\lor \cong \bigoplus_i \Lambda_{T_i}^\lor$ and the map $\Lambda_T \to \Lambda_T^\lor$ is the direct sum of the maps $\Lambda_{T_i} \to \Lambda_{T_i}^\lor$. Hence we get the desired isomorphism

$$\Phi_T = \Lambda_T^\lor / \Lambda_T \cong \bigoplus_i (\Lambda_{T_i}^\lor / \Lambda_{T_i}) = \bigoplus_i \Phi_{T_i}.$$
The third point is only a little more complicated. In this case, in light of proposition 2.2.6 we may assume that the sign function $\epsilon$ is uniformly $+1$ on the components in the subtrees $T_0, FT_0, \ldots, F^{q-2}T_0$. Then just as in the above case, we have an $(\epsilon F)^q$-invariant orthogonal direct sum decomposition $\Lambda_T = \bigoplus_{j=0}^{q-1}(\epsilon F)^j\Lambda_{T_0} = \text{Ind}_{(\epsilon F)^q}\Lambda_{T_0}$. Moreover, this is even an $\epsilon F$-equivariant isomorphism, since the action of $\epsilon F$ on $\Lambda_T$ is given (with respect to the orthogonal decomposition) by a matrix

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & (\epsilon F)^q \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 
\end{pmatrix}.
\]

Then we also have $\Lambda_T^\vee \cong \text{Ind}_{(\epsilon F)^q}\Lambda_{T_0}^\vee$, and by orthogonality of the decomposition of $\Lambda_T$, the map $\Lambda_T \to \Lambda_T^\vee$ is just the induction of the map $\Lambda_{T_0} \to \Lambda_{T_0}^\vee$. Since $\text{Ind}_{(\epsilon F)^q}$ is an exact functor, we have $\Phi_T = \Lambda_T^\vee/\Lambda_T = \text{Ind}_{(\epsilon F)^q}(\Lambda_{T_0}^\vee/\Lambda_{T_0}) = \text{Ind}_{(\epsilon F)^q}\Phi_{T_0}$ as desired.

The final point is obvious, since the relative homology lattice $\Lambda_T$ is already zero.

□

4. THE TAMAGAWA NUMBER ALGORITHM

In this section we will complete the justification of our Tamagawa number algorithm from section 1.1 by proving theorems 1.1.3 and 1.1.5 computing the Tamagawa numbers of simple BY trees. Unsurprisingly, since these Tamagawa numbers are defined as a fixed-point count, group cohomology will play a large role in our proofs, for which we adopt the following notation.

Notation 4.0.1. If $M$ is a $\mathbb{Z}$-module on which an automorphism $\pm F$ acts with finite order, we will denote by $H^1(\pm F, M)$ the continuous Galois cohomology of the continuous action of the profinite cycle group $\hat{\mathbb{Z}}$ on the discrete group $M$ where a generator acts by $\pm F$.

4.1. POSITIVE SIMPLE BY TREES. We begin with the justification of theorem 1.1.3 computing the Tamagawa numbers of positive simple BY trees. As indicated in remark 1.1.4, these results are in fact valid in greater generality, namely whenever $S$ is a non-empty set of vertices of $T$ and the sign function $\epsilon$ is uniformly $+1$. In order to keep the statements as comprehensible as possible, we will only state the results below for positive simple BY trees (our intended application), but the reader will readily check that both the statements and proofs we give are valid also in this greater level of generality.

Proposition 4.1.1. Let $(T, S, +F)$ be a positive simple BY tree satisfying parity condition (B) with associated lattice $\Lambda = \Lambda_T = H_1(T, S)$, and let $m$ be the greatest common divisor of the sizes of the $F$-orbits in $S$. Then $H^1(F, \Lambda)$ is cyclic of order $m$, and the map

$$H^1(F, \Lambda) \to H^1(F, \Lambda^\vee)$$

induced by the intersection length pairing is the zero map.
Proof. First, note that the exact sequence on homology of a pair gives an exact sequence

\[ 0 \to \Lambda \to \mathbb{Z}[S] \to \mathbb{Z} \to 0, \]

where \( \mathbb{Z}[S] \) is the free \( \mathbb{Z} \)-module on \( S \) and the right-hand map is the sum-of-coordinates map. Taking \( F \)-fixed points, we obtain a sequence

\[ \mathbb{Z}[S]^F \to \mathbb{Z} \to H^1(F, \Lambda) \to H^1(F, \mathbb{Z}[S]) \]

and we identify the right-hand group as 0 by Shapiro’s lemma. \( \mathbb{Z}[S]^F \) is generated by the sums of elements in each \( F \)-orbit \( \omega \), which maps to \( \# \omega \) in \( \mathbb{Z} \), and hence we see that \( H^1(F, \Lambda) \) is cyclic of order \( m \), generated by the cocycle associated to \((1 - F)y, y \) for any choice of \( y \in S \).

It remains to show that this maps to zero in \( H^1(F, \Lambda^\vee) \). To do this, pick by proposition 2.2.10 any \( F \)-fixed vertex \( y_0 \) of \( T \), and let \( \alpha \in \Lambda^\vee \) be the map given by length of intersection with the path from \( y_0 \) to \( y \). Then the image of \((1 - F)y, y \) in \( \Lambda^\vee \) is given by intersection length with the path from \( Fy \) to \( y \), and hence is \((1 - F)\alpha \). In other words, the cocycle associated to \((1 - F)y, y \) maps to a coboundary in \( \Lambda^\vee \), as desired. \( \square \)

**Corollary 4.1.2.** Let \( T = (T, S, +F) \) be a positive simple BY tree satisfying parity condition (B), with associated lattice \( \Lambda = \Lambda_T = H_1(T, S) \), and let \( m \) be the greatest common divisor of the sizes of the \( F \)-orbits in \( S \). Then the cokernel of the inclusion \( \Lambda^F \to (\Lambda^\vee)^F \) has order \( c_T/m \).

**Proof.** This follows immediately from taking cohomology of the exact sequence

\[ 0 \to \Lambda \to \Lambda^\vee \to \Lambda^\vee/\Lambda \to 0. \]

\( \square \)

The content of corollary 4.1.2 is that the Tamagawa number is related to the restriction of the pairing to \( \Lambda^F \), and so is, up to a constant, the discriminant of this restricted pairing. Examining in more detail the constants involved, one is led to the following natural description of the Tamagawa number in terms of the discriminant of a pairing on the relative homology of a certain quotient tree.

**Lemma 4.1.3.** Let \( (T, S, +F) \) be a positive simple BY tree satisfying parity condition (B), with associated lattice \( \Lambda = \Lambda_T = H_1(T, S) \), and let \( Q \) be the product of the sizes of the \( F \)-orbits in \( S \). Let \( (T', S') \) be the quotient of \((T, S)\) by \( F \), and metric \( T' \) so that an edge \( e' \) corresponding to an \( F \)-orbit of \( q \) edges of length \( l(e) \) has length \( l(e') = l(e)/q \). Then

\[ c_T = Q \cdot \text{disc} \langle \cdot, \cdot \rangle' \]

where \( \langle \cdot, \cdot \rangle' \) is the intersection length pairing on \( \Lambda' = H_1(T', S') \).

**Proof.** Let \( \rho: \Lambda^F \to \Lambda' \) denote the map on relative homology induced by the quotient map \((T, S) \to (T', S')\), and let \( E: \Lambda' \to \Lambda^F \otimes \mathbb{R} \) be the map which takes (the class of) a path in \( T' \) to the average of the paths in \( T \) lying above it.

Now \( E \) and \( \rho \) are adjoint. To see this, observe that \( \rho \) and \( E \) naturally extend to all formal sums of oriented edges (not just those with zero boundary), so we need only check that \( \langle e, Ee' \rangle = \langle pe, e' \rangle \) for edges \( e, e' \) of \( T, T' \). But these are both easily seen to be \( l(e)/q = l(e') \) if \( e \) lies over \( e' \), and 0 otherwise, so that \( E \) and \( \rho \) are adjoint as claimed.

It follows from adjointness that we have a commuting square
Now all the vector spaces involved are equidimensional and have specified full-rank sublattices, which determine volume forms on each vector space (up to sign), and hence we may talk about the absolute determinant of each of these maps. The leftmost vertical map has determinant $\operatorname{disc} (\langle \cdot, \cdot \rangle') > 0$ by definition, and the rightmost has determinant $c_T m$ by corollary 4.1.2. It follows by taking determinants both ways around the square that

$$c_T = m \left| \frac{\det \rho^*}{\det E} \right| \operatorname{disc} (\langle \cdot, \cdot \rangle').$$

To complete the proof, it suffices to compute $|\det E|$ and $|\det \rho^*|$. To perform the computation of $|\det \rho^*|$, we note that since $H^1(F, \mathbb{Z}) = 0$, the exact sequence (dual to the sequence in the proof of proposition 4.1.1)

$$0 \to \mathbb{Z} \to \mathbb{Z}[S] \to \Lambda^F \to 0$$

remains exact when we take $F$-fixed points, and so identifies $(\Lambda^F)^F$ as the lattice of $F$-invariant $\mathbb{Z}$-valued functions on $S$, modulo constants. Yet by the same reasoning $(\Lambda')^\vee$ is the lattice of $\mathbb{Z}$-valued functions on $S' = S/\langle F \rangle$, modulo constants, so that $\rho^*: (\Lambda')^\vee \to (\Lambda^F)^F$ is an isomorphism of lattices. It follows that $|\det \rho^*| = 1$.

To perform the computation of $|\det E|$, we note that the proof of proposition 4.1.1 shows that the image of the sum-of-coordinates map $\mathbb{Z}[S]^F \to \mathbb{Z}$ had image $m\mathbb{Z}$. Thus we have a commuting diagram with exact rows

$$
\begin{array}{ccccccc}
0 & \to & \Lambda' \otimes \mathbb{R} & \xrightarrow{E} & \mathbb{Z}[S'] \otimes \mathbb{R} & \to & \mathbb{Z} \otimes \mathbb{R} & \to & 0 \\
& & \downarrow{E} & & \downarrow{E} & & \\
0 & \to & \Lambda^F \otimes \mathbb{R} & \xrightarrow{E} & \mathbb{Z}[S]^F \otimes \mathbb{R} & \to & m\mathbb{Z} \otimes \mathbb{R} & \to & 0.
\end{array}
$$

Again, each of these vertical maps goes between equidimensional vector spaces with specified full-rank sublattices, so they have well-defined absolute determinants, and the absolute determinant of the central map is the product of those of the outer two maps.

Yet if we let $y_i$ denote a collection of representatives for the $F$-orbits on $S$ and $q_i$ their sizes, it follows that $\rho y_i$ is a basis for $\mathbb{Z}[S']$ and $(1 + F + \cdots + F^{q_i-1}) y_i = q_i E \rho y_i$ is a basis for $\mathbb{Z}[S]^F$. Hence the central map has absolute determinant $\prod q_i^{-1} = Q^{-1}$.

Moreover the rightmost map clearly has absolute determinant $m^{-1}$, so that the absolute determinant of the leftmost map is $|\det E| = \frac{m}{Q}$. Combining this with the computed value of $|\det \rho^*| = 1$ and the above formula, this yields the desired result. \hfill $\square$

Lemma 4.1.3 in essence reduces the computation of Tamagawa numbers of positive simple $\mathbb{B}$-trees to a computation of a simple matrix determinant. This can now be done completely explicitly, which yields the claimed formula in theorem 1.1.3.
Lemma 4.1.4. Let $T$ be a tree (possibly with non-integral edge lengths) with a set $S$ of $r + 1$ marked points, and let $(\cdot, \cdot): \Lambda \otimes \Lambda \to \mathbb{R}$ be the intersection length pairing on $\Lambda = H_1(T, S)$ as usual. Then

$$\text{disc}((\cdot, \cdot)) = \sum_{e_1, \ldots, e_r} \prod_{i=1}^{r} l(e_i),$$

where the sum is taken over all unordered $r$-tuples of edges of $T$ whose removal disconnects the $r + 1$ points of $S$ from one another.

Proof. Note that, for any basis of $\Lambda$, the entries of the pairing matrix with respect to this basis are homogenous linear forms in the edge lengths of $T$, so that $\text{disc}((\cdot, \cdot))$ is a degree $r$ homogenous form in the edge lengths. We will find its coefficients by setting the edge lengths of $T$ to suitably chosen values.

Suppose first that $E$ is a set of edges of $T$ whose removal does not disconnect the points of $S$ from one another (this is certainly the case if $|E| < r$). Let us set the lengths of all edges not in $E$ to 0, and let those in $E$ be arbitrary. By assumption, there is a path between two points of $S$ not meeting $E$, and this path pairs to 0 with any other element of $H_1(T, S)$. Hence the pairing on $\Lambda$ is degenerate, so its discriminant is 0 independently of the lengths of the edges in $E$. It follows that $\text{disc}((\cdot, \cdot))$ does not contain any monomials only in edge lengths from $E$.

Thus we have shown that the only possible monomials that can appear in $\text{disc}((\cdot, \cdot))$ are products $l(e_1) \cdots l(e_r)$ where $e_1, \ldots, e_r$ are distinct edges whose removal disconnects the points of $S$. It remains to show that each of these monomials has coefficient 1.

To do this, set the lengths of $e_1, \ldots, e_r$ to 1 and all other edge lengths to 0. If we contract out all the edges of length 0, this does not make any of the points of $S$ collide (by assumption), and moreover does not affect the pairing on homology, so we may assume for this that $T$ is a tree with $r$ edges, all of length 1. But this means that $T$ only has $r + 1$ vertices in total, so that $S$ consists of all vertices of $T$. We can then choose a basis of $\Lambda = H_1(T, S)$ consisting of oriented edges, and with respect to this basis the intersection length pairing is represented by the identity matrix. It follows that the discriminant of the pairing is 1, which is what we wanted to show. \(\square\)

Corollary 4.1.5. Theorem 1.1.3 is true.

Proof. Just combine lemma 4.1.3 with lemma 4.1.4. \(\square\)

4.2. Negative simple BY trees. We now turn our attention to the corresponding computation for negative simple BY trees, where the group cohomology calculations are a little more complicated. In order to carry out these computations, we will use without comment the following calculation of the cohomology of permutation representations.

Proposition 4.2.1. Let $S$ be a finite set with an action by an automorphism $F$. Then $\mathbb{Z}[S]^{-F} \simeq \mathbb{Z}^{\#\{\text{even orbits in } S\}}$, generated by the classes of $s - Fs + F^2s - \cdots - F^{2m-1}s$ where $s$ is a representative of an orbit of size $2m$. Also $H^1(-F; \mathbb{Z}[S]) \simeq (\mathbb{Z}/2)^{\#\{\text{odd orbits in } S\}}$, generated by the classes of representatives for each odd orbit. The same applies to the dual representation $\mathbb{Z}^S$. 

The first step in our calculation is to separate out the contributions to the Tamagawa numbers arising from odd- and even-sized orbits in $S$.

**Lemma 4.2.2.** Let $(T, S, -F)$ be a negative simple BY tree, and partition $S = S_0 \sqcup S_1$ into the sets of points in even-sized, respectively odd-sized, $F$-orbits. Suppose that $S_0$ and $S_1$ are non-empty. Letting $T_i$ denote the restricted BY tree $(T, S_i, -F)$, we have an equality of Tamagawa numbers

$$c_T = \frac{1}{2} c_{T_0} c_{T_1}.$$

**Proof.** We will examine the terms of the cohomology exact sequence

$$\Lambda^{-F} \to ((\Lambda^\vee)^{-F} \to (\Lambda^\vee/\Lambda)^{-F} \to H^1(-F, \Lambda) \to H^1(-F, \Lambda^\vee)$$

where $\Lambda = \Lambda_T$ is the lattice associated to $T$, along with its analogues for the lattices $\Lambda_i = \Lambda_{T_i}$ associated to the trees $T_i$. In particular, we observe that we have equalities

$$c_{T_i} = \# coker(\Lambda_i^{-F} \to (\Lambda_i^\vee)^{-F}) \cdot \# \ker(H^1(-F, \Lambda_i) \to H^1(-F, \Lambda_i^\vee)).$$

Consider first the cohomology groups $H^j(-F, \Lambda)$ for $j = 0, 1$. From the short exact sequence

$$0 \to \Lambda \to \mathbb{Z}[S] \to \mathbb{Z} \to 0$$

we obtain a long exact sequence

$$0 \to \Lambda^{-F} \to \mathbb{Z}[S]^{-F} \to 0 \to H^1(-F, \Lambda) \to H^1(-F, \mathbb{Z}[S]) \to H^1(-F, \mathbb{Z}).$$

But $\mathbb{Z}[S] = \mathbb{Z}[S_0] \oplus \mathbb{Z}[S_1]$ and $\mathbb{Z}[S_1]^{-F} = 0$, so that $\Lambda_1^{-F} = 0$ and $\Lambda^{-F} = \Lambda_0^{-F}$. Also $H^1(-F, \mathbb{Z}[S_0]) = 0$, so that $H^1(-F, \Lambda_0) = 0$ and $H^1(-F, \Lambda) = H^1(-F, \Lambda_1)$.

Now let us perform the same analysis on the cohomology groups $H^j(-F, \Lambda^\vee)$, using the exact sequence

$$0 \to (\mathbb{Z}^S)^{-F} \to (\Lambda^\vee)^{-F} \to H^1(-F, \mathbb{Z}) \xrightarrow{\Delta} H^1(-F, \mathbb{Z}^S) \to H^1(-F, \Lambda^\vee) \to 0.$$ 

In this sequence we have $H^1(-F, \mathbb{Z}) \simeq \mathbb{Z}/2$, $H^1(-F, \mathbb{Z}^S) \simeq (\mathbb{Z}/2)^{\# \text{(odd orbits in } S)}$, and the map $\Delta$ is the diagonal. In particular, since $S_1 \neq \emptyset$, the map $\Delta$ is injective, so that $(\mathbb{Z}^S)^{-F} = (\Lambda^\vee)^{-F}$.

We can apply the same reasoning with $S$ replaced by $S_1$, in which case $(\mathbb{Z}^S_1)^{-F} = 0$ and $H^1(-F, \mathbb{Z}^S_1) = H^1(-F, \mathbb{Z}^S)$. Hence $\Lambda_1^{-F} = 0$ and $H^1(-F, \Lambda_1^\vee) = H^1(-F, \Lambda^\vee)$.

When we consider the same sequence with $S$ replaced by $S_0$, then $H^1(-F, \mathbb{Z}^S_0) = 0$ and $(\mathbb{Z}^S_0)^{-F} = (\mathbb{Z}^S)^{-F}$ (so in this case $\Delta$ is not injective). It follows that $H^1(-F, \Lambda_0^\vee) = 0$ and the natural inclusion $(\mathbb{Z}^S_0)^{-F} \to (\Lambda_0^\vee)^{-F}$ has cokernel $\mathbb{Z}/2$. In particular, the natural map $(\Lambda^\vee)^{-F} \to (\Lambda_0^\vee)^{-F}$ is injective with cokernel $\mathbb{Z}/2$.

Finally, we note that the pairings on $\Lambda_i$ are the restrictions of the pairing on $\Lambda$, so that we have commuting squares

$$
\begin{array}{ccc}
H^j(-F, \Lambda) & \longrightarrow & H^j(-F, \Lambda^\vee) \\
\downarrow & & \uparrow \\
H^j(-F, \Lambda_i) & \longrightarrow & H^j(-F, \Lambda_i^\vee)
\end{array}
$$
relating the pairing maps for Λ and Λ. Combining this with the above calculations, we see that
\[ \ker(H^1(-F, A_0) \to H^1(-F, \Lambda^\vee)) = 0 \]
\[ \text{coker}((\Lambda_1)^{-F} \to (\Lambda^\vee)^{-F}) = 0 \]
\[ \ker(H^1(-F, A_1) \to H^1(-F, \Lambda^\vee)) = \ker(H^1(-F, \Lambda) \to H^1(-F, \Lambda^\vee)) \]
\[ \text{coker}((\Lambda_0)^{-F} \to (\Lambda^\vee)^{-F}) \supset \text{coker}((\Lambda^{-F} \to (\Lambda^\vee)^{-F})) \]
where the final inclusion has cokernel \( \mathbb{Z}/2 \). Taking orders of these groups and multiplying together, we obtain the desired equality of Tamagawa numbers. \( \Box \)

**Corollary 4.2.3** (to the proof). Suppose that \( T = (T, S, -F) \) is a negative simple BY tree, and that all of the orbit sizes in \( S \) are even. Then the Tamagawa number \( c_T \) only depends on the lengths of the edges in even-sized \( F \)-orbits.

**Proof.** It follows from the proof of lemma 4.2.2 that \( c_T = \#\ker((\Lambda^{-F} \to (\Lambda^\vee)^{-F})) \). Now the two groups \( \Lambda^{-F} \) and \( (\Lambda^\vee)^{-F} \) do not depend on any edge lengths in \( T \), so it suffices to prove that the map \( \Lambda^{-F} \to (\Lambda^\vee)^{-F} \) is independent of the lengths of edges in odd-sized \( F \)-orbits. In other words, we want to show that the restriction of the pairing to \( \Lambda^{-F} \otimes \Lambda \) is independent of these lengths.

But if \( e \) is an (oriented) edge in an \( F \)-orbit of size \( 2m + 1 \) and \( \gamma \in \Lambda^{-F} \), then the number of times that \( e \) appears in \( \gamma \) is minus the number of times that \( Fe \) appears in \( -F\gamma = \gamma \). Iterating this \( 2m + 1 \) times, we see that the number of times \( e \) appears in \( \gamma \) is equal to its own negative, so \( e \) does not appear in \( \gamma \). It follows that if \( \gamma' \) is any other cycle, the edge \( e \) makes no contribution to \( \langle \gamma, \gamma' \rangle \), so that the restriction of the pairing to \( \Lambda^{-F} \otimes \Lambda \) does not depend on the length of \( e \). \( \Box \)

Now in order to complete the proof of theorem 1.1.5 we just have to deal separately with the cases that the \( F \)-orbits in \( S \) are all odd or all even. For the odd case, all the contributions to the Tamagawa numbers come from the cohomology group \( H^1(-F, \Lambda) \), so that \( c_T \) is a power of 2 and we can explicitly determine the exponent. The details of the calculation bear some similarities to the quotient tree calculation of lemma 4.1.3.

**Proposition 4.2.4.** Let \( T = (T, S, -F) \) be a negative simple BY tree satisfying parity condition (A), and assume that \( T \) is not a path (i.e. has a vertex of degree \( \geq 3 \)). Suppose moreover that all of the \( F \)-orbits in \( S \) have odd size. Write \( A_0 \subseteq S/\langle F \rangle \) for the set of orbits in \( S \) consisting of points which are at the end of an odd-length twig (i.e. such that the distance to the nearest vertex of degree \( \geq 3 \) is odd), and \( A_0 \) for the remaining orbits in \( S \). Then
\[
\begin{aligned}
c_T &= \begin{cases} 
2^{\#A_0-1} & \text{if } \#A_0 \geq 1 \\
2 & \text{if } \#A_0 = 0 \text{ and } \#A_1 \geq 2 \text{ even} . \\
1 & \text{else}
\end{cases}
\end{aligned}
\]

**Proof.** Let \( \omega_1, \ldots, \omega_{\#A_0 + \#A_1} \) be the orbits of \( F \) in \( S \), viewed as elements of \( \mathbb{Z}[S] \) in the natural way. Note that the proof of lemma 4.2.2 shows that we have \( c_T = \#\ker(H^1(-F, \Lambda) \to H^1(-F, \Lambda^\vee)) \), which is what we will compute.

Now from the proof of that lemma, we have an exact sequence
\[
0 \to H^1(-F, \Lambda) \to H^1(-F, \mathbb{Z}[S]) \to H^1(-F, \mathbb{Z}) \to 0,
\]
and can identify $H^1(-F,\mathbb{Z}[S])$ as the free $\mathbb{F}_2$-vector space on basis $\omega_1, \ldots, \omega_j \# A_0 + \# A_1$. It follows that an $\mathbb{F}_2$-basis of $H^1(-F,\Lambda)$ is given by $\gamma_j = (\# \omega_1 \cdot \omega_j - (\# \omega_j) \cdot \omega_1$ for $2 \leq j \leq \# A_0 + \# A_1$. Thinking of this element as a homology class, it is represented by the sum of all $(\# \omega_1)(\# \omega_j)$ paths from points in $\omega_1$ to points in $\omega_j$.

Moreover, we also have the dual sequence

$$0 \to H^1(-F,\mathbb{Z}) \to H^1(-F,\mathbb{Z}^\vee) \to H^1(-F,\Lambda^\vee) \to 0,$$

and can identify $H^1(-F,\mathbb{Z}^\vee)$ with the space of $\mathbb{F}_2$-valued $F$-invariant functions on $S$, so that $H^1(-F,\Lambda^\vee)$ is identified with this space modulo constant functions. It follows that $H^1(-F,\Lambda^\vee) \simeq \mathbb{F}_2^{\# A_0 + \# A_1 - 1}$, and a complete system of coordinate functions is given by the “evaluate at $\gamma_j$” functions.

Hence with respect to the given basis of $H^1(-F,\Lambda)$ and coordinates of $H^1(-F,\Lambda^\vee)$, the map $H^1(-F,\Lambda) \to H^1(-F,\Lambda^\vee)$ is given by the matrix $M_{jk} = \langle \gamma_j, \gamma_k \rangle$ reduced mod 2, where $\langle \cdot, \cdot \rangle$ is the intersection length pairing. In particular, $c_T = 2^{n(M)}$, where $n(M)$ denotes the nullity of $M$ mod 2.

Now suppose first of all that $\# A_0 \geq 1$, so that wlog we chose $\omega_1 \in A_0$, and let us compute $\langle \gamma_j, \gamma_k \rangle$ mod 2. Let us break up $T$ into paths connecting vertices of degree $\neq 2$. The paths of even length do not contribute to $\langle \gamma_j, \gamma_k \rangle$ mod 2, and so we need only consider the contributions from odd-length paths. However, parity condition (A) forces that the only odd length paths in $T$ are those with an endpoint in $S$, and these endpoints are precisely the vertices lying in orbits in $A_1$. In particular, such an edge appears in $\langle \gamma_j, \gamma_k \rangle$ mod 2 if and only if $j = k$, $\omega_j \in A_1$ and this edge is the one associated to $\omega_j$ (in which case the path occurs with odd multiplicity). Hence $\langle \gamma_j, \gamma_k \rangle = 1$ iff $j = k$ and $\omega_j \in A_1$, so that $M$ is diagonal with $\# A_0 - 1$ diagonal entries zero. Thus $n(M) = \# A_0 - 1$ as claimed.

In the remaining case, we can do the same analysis, except that now the odd-length paths starting from the points of $\omega_1$ contribute to every $\langle \gamma_j, \gamma_k \rangle$ in addition to the edges at $\omega_j$ and $\omega_k$. Hence in this case $M$ is the $(\# A_1 - 1)$-by-$(\# A_1 - 1)$ matrix with diagonal entries 0 and off-diagonal entries 1. A simple calculation verifies that $n(M) = 1$ if $\# A_1$ is even, and $n(M) = 0$ if $\# A_1$ is odd, which completes the proof.

Exactly the same proof works in the special case, provided we break up the path $T$ into $\geq 2$ edges at the arbitrarily selected point or points. \hfill \Box

Thus we have dealt with the case that the $F$-orbits in $S$ are all odd. It remains to deal with the even case, for which we wish to reduce to the case of positive simple BY trees dealt with previously. The bulk of the work in this reduction step is contained in corollary 4.2.3 which will allow us to reduce to the following case.

**Proposition 4.2.5.** Suppose that $T = (T, S, -F)$ is a negative simple BY tree such that every edge of $T$ lies in an even size $F$-orbit. Then $T$ has a unique $F$-fixed vertex (which cannot be a leaf, so $\ast \notin S$). Then $(T, S \cup \{\ast\}, +F)$ and $(T, S \cup \{\ast\}, -F)$ are BY trees with the same Tamagawa number. $(T, S, -F)$ satisfies parity condition (B), as does $(T, S \cup \{\ast\}, +F)$.

**Proof.** $T$ has an $F$-fixed point $\ast$, which a priori can be chosen to be either a vertex or a midpoint of an edge – it cannot be the latter since every edge of $T$ is contained in an even size $F$-orbit. The same condition also ensures that $\ast$ is unique, and that all of the $F$-orbits in $\pi_0(T \setminus (S \cup \{\ast\}))$ have even size. By proposition 2.2.6, the signed isomorphism class (and hence the Tamagawa number) of the BY tree $(T, S \cup \{\ast\}, -F)$ is unchanged if we change the sign function to (constant) +1.
All trees considered satisfy condition (B) since they have an $F$-fixed vertex (proposition 2.2.10).

**Corollary 4.2.6.** Theorem 1.1.5 is true.

**Proof.** Assume first that $T$ is not a path. Let us write

\[ \tilde{c}_T = \begin{cases} c_T & \text{if } S \neq \emptyset \\ 2 & \text{if } S = \emptyset \end{cases}, \]

so that lemma 4.2.2 gives that \( \tilde{c}_T = \frac{1}{2} \tilde{c}_T \tilde{c}_T \) irrespective of whether \( S_0 \) or \( S_1 \) is empty.

If we replace \( T_1 \) with the convex hull of \( S_1 \) in \( T \), then this does not change the Tamagawa number (the lattice \( \Lambda_{T_1} \) is unchanged in the notation of definition 2.3.1). The resulting tree satisfies parity condition (A), so that by propositions 2.3.2 and 4.2.4, we have \( \tilde{c}_{T_1} = \tilde{c}_1 \) in the notation of theorem 1.1.5.

Moreover, by corollary 4.2.3 the Tamagawa number of \( T_0 \) is unchanged if we set the lengths of the edges in odd-sized \( F \)-orbits to 0, and then contract them out (and nor does this operation cause parity condition (B) to be violated). The resulting tree differs from \( T' \) only in the addition of a single marked point and a change of sign (which does not affect the Tamagawa number or parity condition (B) by proposition 4.2.3), so by lemma 4.2.2 again, \( c_{T'} = \tilde{c}_{T'} = \frac{1}{2} \tilde{c}_{T_0} \tilde{c}_{T_1} = \tilde{c}_1 c_{T'} \) as desired.

Returning to the case when $T$ is a path, only the case of $T$ a path of length $l$ between two points of $S$ is non-trivial. But in this case \( \Psi_T \) is cyclic of order $l$ and \(-F\) acts by $-1$ or $+1$ according as $F$ fixes or reverses $T$. The result follows. \( \square \)

5. The Néron component group algorithm

In this section we will complete the justification of our Néron component group algorithm by proving the assertions in algorithms 1.2.4 and 1.2.5 computing the Néron component groups of simple BY trees.

5.1. Pipes. As mentioned in the introduction, we might hope to be able to construct a recursive algorithm to compute, for any marked BY tree \((T, S, F, \ast)\), the Néron component groups of the BY trees \((T, S, \pm F)\) and \((T, S \cup \{\ast\}, \pm F)\) in terms of the constructions in notation 1.2.3. However, this turns out not to be possible directly, and we instead proceed by defining two stronger invariants (the triples in algorithms 1.2.4 and 1.2.5) from which these invariants can be recovered, and which can itself be computed recursively. Both of these invariants will be derived from the following “master invariant”, which contains enough information to recover the entire geometric Néron component group of all of the above BY trees.

**Definition 5.1.1 (Pipes).** Suppose that \((T, S, \ast)\) is a triple consisting of an integrally metrised finite graph-theoretic forest \( T \), a non-empty set \( S \) of vertices of \( T \) and a marked vertex \( \ast \) of \( T \) not in \( S \). We let \( \Lambda = H_1(T, S \cup \{\ast\}, \mathbb{Z}) \) and \( \Sigma = H_1(T, S, \mathbb{Z}) \) denote the relative homology lattices, so that the inclusion of pairs \((T, S) \hookrightarrow (T, S \cup \{\ast\})\) induces an inclusion \( \Sigma \hookrightarrow \Lambda \) embedding \( \Sigma \) as a corank 1 sublattice and respecting the intersection length pairing. Hence we have a commuting square

\[
\begin{array}{ccc}
\Sigma & \hookrightarrow & \Sigma^\vee \\
\downarrow & & \uparrow \\
\Lambda & \hookrightarrow & \Lambda^\vee
\end{array}
\]
We will frequently identify $\Lambda$ with $\mathbb{Z}[S]$ by identifying a vertex $x \in S$ with the class of any path from $*$ to $x$. With this identification, the inclusion $\Sigma \hookrightarrow \Lambda$ becomes identified with the inclusion $\Sigma \hookrightarrow \mathbb{Z}[S]$ from the homology of the pair $(T, S)$, so that $\Sigma$ is the kernel of the sum-of-coordinates map $\mathbb{Z}[S] \to \mathbb{Z}$.

We now define the pipe of $(T, S, *)$ to be the triple $(\Pi, y, \eta)$ where $\Pi$ is the quotient $\Lambda^\vee/\Sigma$, $y \in \Pi$ is the image in $\Pi$ of any vertex $x \in S \subseteq \mathbb{Z}[S] \cong \Lambda$, and $\eta \in \Pi$ is the image in $\Pi$ of the sum-of-coordinates map $\eta \in \Lambda^\vee$. Note that $y$ does not depend on the chosen vertex.

The construction of $\Pi$ is clearly functorial with respect to isomorphisms between such triples $(T, S, *)$, so if $(T, S, F, *)$ is a marked BY tree, $\Pi$ is naturally a $\mathbb{Z}[F]$-module where $F$ acts with finite order.

**Proposition 5.1.2 (Basic properties of pipes).** Let $(T, S, *)$ be a triple as in definition 5.1.1, for instance a triple underlying a marked BY tree $(T, S, F, *)$, and let $(\Pi, y, \eta)$ be the associated pipe. Then $\Pi$ is a finitely generated $\mathbb{Z}$-module of torsion-free rank 1, and both $y$ and $\eta$ are non-torsion and of the same sign in $\Pi/\Pi_{\text{tors}} \cong \mathbb{Z}$. Moreover, both $y$ and $\eta$ are fixed by the action of any automorphism of $(T, S, *)$, so in particular, all automorphisms of $(T, S, *)$ act trivially on the quotient $\Pi/\Pi_{\text{tors}} \cong \mathbb{Z}$.

**Proof.** Since the inclusions $\Sigma \hookrightarrow \Lambda$ and $\Lambda \hookrightarrow \Lambda^\vee$ (in the notation of definition 5.1.1) have corank 1 and 0 respectively, it follows that $\Pi = \Lambda^\vee/\Sigma$ is finitely-generated of torsion-free rank 1.

Also, $\Lambda^\vee$ carries a positive definite dual pairing $\Lambda^\vee \otimes \Lambda^\vee \to \mathbb{Q}$ which, with respect to the inclusion $\Lambda \hookrightarrow \Lambda^\vee$, corresponds to the evaluation map $\Lambda^\vee \otimes \Lambda \to \mathbb{Z}$. Hence the map $\Lambda^\vee \to \mathbb{Q}$ given by $\langle \eta, - \rangle$ kills $\Sigma$ and so factors through $\Pi$. Now $\eta \in \Pi$ is taken to a positive element of $\mathbb{Q}$ under this map, so is non-torsion. Also, $y$ is taken to $\langle \eta, x \rangle = 1$, so it is also non-torsion, and of the same sign as $\eta$ (under both isomorphisms $\Pi/\Pi_{\text{tors}} \cong \mathbb{Z}$).

Moreover, for any automorphism $F$ of $(T, S)$, the sum-of-coordinates map in $\Lambda^\vee$ is clearly $F$-fixed, and $(1 - F)x \in \Sigma$ for all $x \in S$, so that $\eta$ and $y$ are $F$-fixed in $\Pi$, as desired. \hfill $\square$

The utility in the definition of the pipe is that from it one can recover the geometric Néron component groups of both $(T, S, \pm F)$ and $(T, S \cup \{*, \}, \pm F)$ in a simple way.

**Lemma 5.1.3 (Recovering Néron component groups).** Let $(T, S, *)$ be a triple as in definition 5.1.1, for instance a triple underlying a marked BY tree $(T, S, F, *)$. Then in the notation of definition 5.1.1 there are canonical isomorphisms

$$\Pi/y \sim \Lambda^\vee/\Lambda \text{ and } \Pi/\eta \sim \Sigma^\vee/\Sigma$$

compatible with the actions of automorphisms of $(T, S, *)$. In particular, for a marked BY tree $(T, S, F, *)$, the geometric Néron component groups of $(T, S, \pm F)$ and $(T, S \cup \{*, \}, \pm F)$ are given by

$$\overline{\Phi}(T, S, \pm F) \cong \Pi/\eta \text{ and } \overline{\Phi}(T, S \cup \{*, \}, \pm F) \cong \Pi/y,$$

where the action of $\pm F$ on $\Pi/\eta$ and $\Pi/y$ is $\pm 1$ times the induced action of $F$.

**Proof.** This is immediate from $\Pi = \Lambda^\vee/\Sigma$ and the facts that $\Lambda = \Sigma + \langle x \rangle$ (for any vertex $x \in S$) and $\Sigma^\vee = \Lambda^\vee/\eta$. \hfill $\square$
The quotient in the fourth point here is a quotient of \( \Pi F \), which we wanted to prove.

2.2.4, there is a natural notion of similarly to the definition of the category of BY forests in definition Remark 5.1.5.

is the quotient formed by identifying \( y \equiv y_0 \) for any vertex \( x \in S \), so that the induced maps \( \Lambda \to \Lambda^\vee \) agree on \( \Sigma \). Hence \( \Sigma = \Sigma_0 \) viewed as subgroups of \( \Lambda^\vee = \Lambda_0^\vee \) and so \( \Pi = \Pi_0^\vee \). Finally, the identity \( \langle x, \cdot \rangle = \langle x, -\rangle_0 + l_{y_0} \) tells us that \( y = y_0 + l_{y_0} \).

For the third, when \( T = T_0 \vee T_1 \), we have that \( \Lambda^\vee = \Lambda_0^\vee \oplus \Lambda_1^\vee \) and \( \Sigma = \Sigma_0 \oplus \Sigma_1 \oplus \langle (y_0, -y_1)\rangle \) inside \( \mathbb{Z}[S_0] \oplus \mathbb{Z}[S_1] \), so it follows that \( \Pi = \Pi_0 \oplus \Pi_1 \). Now if we pick a vertex \( x \) of \( T_0 \), the path from \(*\) to \( x \) is orthogonal to all the paths inside \( T_1 \) and hence \( x \) corresponds to the element \( (x, 0) \in \Lambda_0^\vee \oplus \Lambda_1^\vee \). Its image in \( \Pi \) is thus \( y = (y_0, 0) = (0, y_1) \). Finally, it is clear from the definition that \( \eta = (\eta_0, \eta_1) \) in \( \Lambda^\vee = \Lambda_0^\vee \oplus \Lambda_1^\vee \).

For the fourth point, when \( T = \text{Ind}_{T_0} F T_0 \), we have that \( T = T_0 \vee FT_0 \vee \ldots \vee F^{q-1} T_0 \) (where each \( F^i T_0 \) is a copy of \( T_0 \) with the \( F \)-action). Hence by repeated application of the previous step we see that we have an \( F^q \)-equivariant isomorphism \( \Pi \cong \Pi_0^{\oplus \infty} \oplus \Pi_1^{\oplus \infty} \oplus \Pi_0^{\oplus q-1} \oplus \Pi_1^{\oplus q-1} \) with \( y = y_0 = \ldots = F^{q-1} y_0 \) and \( \eta = (1 + F + \ldots + F^{q-1}) y_0 \). By inspection, the isomorphism respects the \( F \)-action, which is what we wanted to prove.

The final point follows immediately from proposition 2.3.2.

**Rem 5.1.5.** Similarly to the definition of the category of BY forests in definition 2.2.4, there is a natural notion of metric equivalence of marked BY trees. All the invariants of BY trees we consider only depend on the metric equivalence class of a marked BY tree (i.e., are functorial with respect to metric equivalences), and if one works with marked BY trees up to metric equivalence, then one need only consider the base case with \( l = 1 \) in algorithms 5.1.4 12.3 and 12.5. However, to avoid over-complicating our definitions, we have chosen not to emphasise this notion of equivalence, since the results we use are simple enough to state without it.

5.2. **Positive simple BY trees.** Let us now use algorithm 1.1.4 to derive algorithm 1.2.4 computing Néron component groups of \((T, S, +F)\) and \((T, S \cup \{\ast\}, +F)\).
for any marked BY tree \((T, S, F, \ast)\). This justification proceeds in three steps: giving a non-recursive definition of the triple \((\Pi^F, y, \eta)\) appearing in algorithm 1.2.4 checking that these triples recover the claimed Néron component groups; and checking that they satisfy the claimed recursive behaviour. Having developed the notion of pipes above, the definition of the triple \((\Pi^F, y, \eta)\) is essentially immediate – \(\Pi^F\) is just the \(F\)-fixed points of the pipe \(\Pi\), and \(y, \eta \in \Pi^F\) denote the same elements of \(\Pi\) as in definition 5.1.1 – and so we will now simply verify the latter two points.

**Proposition 5.2.1.** Let \((T, S, F, \ast)\) be a marked BY tree. Then the Néron component groups of \((T, S, +F)\) and \((T, S \cup \{\ast\}, +F)\) are, respectively, \(\Pi^F/\eta\) and \(\Pi^F/y\).  

**Proof.** Lemma 5.1.3 shows that the Néron component group of \((T, S, +F)\) is \((\Pi/\eta)^F\). To show this is equal to \(\Pi^F/\eta\), we consider the \(F\)-equivariant exact sequence 

\[
0 \longrightarrow \mathbb{Z}^0 \longrightarrow \Pi \longrightarrow \Lambda^\vee/\Lambda \longrightarrow 0.
\]

Since \(\mathbb{Z}\) is torsion-free, \(H^1(F, \mathbb{Z}) = 0\), so the sequence remains exact after taking \(F\)-invariants. Thus \(\Pi^F/\eta \cong (\Pi/\eta)^F\), and the case of \((T, S \cup \{\ast\}, F)\) follows exactly the same argument. \(\square\)

**Proposition 5.2.2.** The triples \((\Pi^F, y, \eta)\) associated to marked BY trees obey the recursive behaviour in algorithm 1.2.4.

**Proof.** The base case is trivial. The case \(T = T_0^{(l)}\) is immediate from algorithm 5.1.4.

When \(T = T_0 \vee T_1\), algorithm 5.1.4 provides us with an \(F\)-equivariant exact sequence

\[
0 \longrightarrow \mathbb{Z}^{(y_0 - y_1)} \Pi_0 \oplus \Pi_1 \longrightarrow \Pi \longrightarrow 0.
\]

Since \(H^1(F, \mathbb{Z}) = 0\), this remains exact after taking \(F\)-invariants, giving \(\Pi^F = \Pi_0^{F} \oplus \Pi_1^{F}\) as claimed. The claimed values of \(y\) and \(\eta\) are given by algorithm 5.1.4.

The third case regarding \(T = \bigvee \text{Ind}_{F_q}^F T_0\) is a little more complicated. If we let \(J_q \subseteq \mathbb{Z}[C_q]\) denote the augmentation ideal, viewed as a \(\mathbb{Z}[F]\)-module with \(F\) acting by multiplication by a generator, then algorithm 5.1.4 provides an \(F\)-equivariant exact sequence

\[
0 \longrightarrow J_q \longrightarrow \text{Ind}_{F_q}^F \Pi_0 \longrightarrow \Pi \longrightarrow 0
\]

where the first map is multiplication by \(y_0\). Taking \(F\)-fixed points gives us the exact sequence

\[
0 \longrightarrow (1 + F + \cdots + F^{q-1}) \Pi_0^{F} \longrightarrow \Pi^F \longrightarrow H^1(F, J_q)
\]

Since \(H^1(F, J_q) \cong \mathbb{Z}/q\), this tells us that \((1 + \cdots + F^{q-1})\Pi_0^{F} \cong \Pi_0^{F}\) is a submodule of \(\Pi^F\) of index at most \(q\).

On the other hand, \(y_0 \in \Pi_0 \leq \text{Ind}_{F_q}^F \Pi_0\) is (a representative of) an \(F\)-fixed element of \(\Pi\), since \((1 - F)y_0 \in J_qy_0\) is zero in \(\Pi\). Yet this has order \(q\) in \(\Pi_0^{F}\); \(a y_0 = 0\) in this group if and only if there is some \(z \in \Pi_0^{F}\) and integers \(a_0, \ldots, a_{q-1}\) with sum \(0\) such that \((a + \sum a_i F^i) y_0 = (1 + \cdots + F^{q-1}) z\) in \(\text{Ind}_{F_q}^F \Pi_0\). The only way this occurs is if \(a_0 = \cdots = a_{q-1}\) and \(z = a_1 y_0 = (a + (1 - q)a_1) y_0\). This is possible iff \(a = qa_1\) is divisible by \(q\), as desired.

Combining these two facts tells us that the index of \((1 + \cdots + F^{q-1})\Pi_0^{F}\) in \(\Pi^F\) must be exactly \(q\), and the quotient is generated by \(y_0\). Moreover, since \(q y_0 - (1 + \cdots + F^{q-1}) y_0 \in J_qy_0\), it follows that \(q y_0 = (1 + \cdots + F^{q-1}) y_0\) in \(\Pi^F\). Hence
(using the values of $y$ and $\eta$ from algorithm 5.1.4), $\Pi^F$ is the group generated by $(1 + \cdots + F^{q-1})\Pi^F_0$ and $y$ subject to $qy = (1 + \cdots + F^{q-1})\eta_0$, with $\eta = (1 + \cdots + F^{q-1})\eta_0$. Under the obvious isomorphism $(1 + \cdots + F^{q-1})\Pi^F_0 \cong \Pi^F_0$, this gives the triple claimed.

The final case is an immediate consequence of the corresponding point in algorithm 5.1.4.

\[ \square \]

5.3. Negative simple BY trees. To verify the validity of algorithm 1.2.5 we follow a similar strategy, with the important difference that here the definition of the triple $(\Pi^F, \alpha, \tau)$ is less obvious. By analogy with the positive case, we would expect that the Néron component groups of $(T, S, -F)$ and $(T, S \cup \{\ast\}, -F)$ for a marked BY tree $T = (T, S, F, \ast)$ should be recoverable in terms of extra data attached to the $-F$-invariants $\Pi^F$ in the pipe $\Pi$ of $T$, but the description is much less obvious than in the positive case. This difficulty is made explicit in the following basic proposition.

Proposition 5.3.1. Let $(T, S, F, \ast)$ be a marked BY tree, let $\gamma \in \Pi^F$ be non-torsion, and let $\gamma_* \in H^1(-F, \Pi)[2]$ denote the image of the non-trivial element of $H^1(-F, \mathbb{Z}/2) \simeq \mathbb{Z}/2$ under the map induced by $\mathbb{Z} \rightarrow \Pi$. Then the map $\Pi^F \rightarrow (\Pi/\gamma)^{-F}$ is either injective with cokernel $\mathbb{Z}/2$ or an isomorphism. More precisely, we have the following equivalent conditions:

- $(\Pi/\gamma)^{-F}$ is an extension of $\mathbb{Z}/2$ by $\Pi^F$ (resp. is equal to $\Pi^F$);
- $\gamma_* = 0$ (resp. $\gamma_* \neq 0$);
- there is $\beta \in \Pi$ with $(1 + F)\beta = \gamma$ (resp. no such $\beta$ exists).

Moreover, the $\beta$ in the third point is unique up to $\Pi^F$, and the element $\alpha = (1 - F)\beta \in \Pi^F / 2\Pi^F$ corresponds to the class $[(\Pi/\gamma)^{-F}] \in \text{Ext}^1(\mathbb{Z}/2, \Pi^{-F})$ under the canonical identification $\text{Ext}^1(\mathbb{Z}/2, \Pi^{-F}) \cong \Pi^F / 2\Pi^F$.

Proof. The equivalence of the first two points is immediate from the long exact sequence associated to

$$0 \rightarrow \mathbb{Z} \rightarrow \Pi \rightarrow \Pi/\gamma \rightarrow 0.$$ 

For the equivalence of the first and third point, let $\beta \in \Pi$ be any lift of an element of $(\Pi/\gamma)^{-F}$, so that $(1 + F)\beta = m\gamma$ for some $n$. Choosing the lift appropriately, we may assume that $m \in \{0, 1\}$. Hence $\beta$ represents a class in $(\Pi/\gamma)^{-F} \setminus \Pi^F$ iff $m = 1$, so that this set is non-empty iff there is a solution to $(1 + F)\beta = \gamma$, which is clearly unique up to $\Pi^F$.

In the case that $(\Pi/\gamma)^{-F}$ is an extension of $\mathbb{Z}/2$ by $\Pi^F$, the element of $\Pi^F / 2\Pi^F$ corresponding to the extension-class of $(\Pi/\gamma)^{-F}$ is given by doubling any element of $(\Pi/\gamma)^{-F} \setminus \Pi^F$. But such an element is represented by some $\beta \in \Pi$ such that $(1 + F)\beta = \gamma$, so that in $(\Pi/\gamma)^{-F}$ we have $2\beta = 2\beta - \gamma = (1 - F)\beta$, as claimed. \[ \square \]

Applying proposition 5.3.1 to the two Néron component groups $\Phi_{(T, S, -F)} \cong (\Sigma^0 / \Sigma)^{-F} \cong (\Pi/\eta)^{-F}$ and $\Phi_{(T, S \cup \{\ast\}, -F)} \cong (\Lambda^0 / \Lambda)^{-F} \cong (\Pi/\eta)^{-F}$, we see that these groups are either $\Pi^F$ or an extension of $\mathbb{Z}/2$ by $\Pi^F$. However, we need some extra input to determine which of these possibilities occur, which is given by the following surprising coincidence, whose proof we defer to the end of this section.

Proposition 5.3.2. Let $(T, S, F, \ast)$ be a marked BY tree, with pipe $(\Pi, y, \eta)$. Then, in the notation of proposition 5.3.1 exactly one of $y_*$, $\eta_*$, or $(y + \eta)_*$ is zero.
The trichotomy expressed in proposition 5.3.2 shows that exactly one of the groups \( \Pi/y \cdot F \), \( \Pi/\eta \cdot F \), and \( \Pi/y + \eta \cdot F \) is an extension of \( \mathbb{Z}/2 \) by \( \Pi \cdot F \), and the other two are \( \Pi \cdot F \). This very limited set of possibilities motivates the following definition of the triple \( (\Pi \cdot F, \alpha, \gamma) \) appearing in algorithm 1.2.5.

**Definition 5.3.3.** We associate to every marked BY tree a triple \( (\Pi \cdot F, \alpha, \gamma) \) as follows:

- \( \Pi \cdot F \) is the \( F \)-invariant subspace of the pipe \( \Pi \) of \( T \);
- the type \( \gamma \in \{0,1,2\} \) is defined to be
  \[
  \gamma = \begin{cases} 
  0 & \text{if } y_s = 0 \neq \eta_s, \\
  1 & \text{if } \eta_s = 0 \neq y_s, \\
  2 & \text{if } y_s = \eta_s \neq 0;
  \end{cases}
  \]
- the extension class \( \alpha \in \text{Ext}^1(\mathbb{Z}/2, \Pi \cdot F) = \Pi \cdot F/2\Pi \cdot F \) is the class of the extension \( (\Pi/\gamma) \cdot F \) of \( \mathbb{Z}/2 \) by \( \Pi \cdot F \) as in proposition 5.3.1 where \( \gamma = y, \eta, y + \eta \) according as \( \tau = 0,1,2 \). Explicitly, we have that \( \alpha \) is the class of \( (1 - F) \beta \in \Pi \cdot F \) where \( \beta \in \Pi \) is such that \( \gamma = (1 + F) \beta \) (so automatically \( \beta \in \Pi \cdot F^\ast \)).

It is obvious from this definition and the preceding two propositions that one may recover the Néron component groups of \((T, S, -F)\) and \((T, S \cup \{\ast\}, -F)\) from the triple \((\Pi \cdot F, \alpha, \gamma)\) as claimed in algorithm 1.2.5. It only remains to verify the recursive behaviour of these triples, which is a somewhat long and technical check.

**Proposition 5.3.4.** The triples \((\Pi \cdot F, \alpha, \gamma)\) associated to marked BY trees obey the recursive behaviour in algorithm 1.2.5.

**Proof.** The recursive behaviour of \( \gamma \) will follow immediately from the proof of proposition 5.3.2 so we now only consider the behaviour of \( \Pi \cdot F \) and \( \alpha \). Note that for the cases \( T = T_0^{(l)} \) and \( T = \text{Ind}_{F^*} T_0 \) \((q \text{ even})\) we need only prove the cases \( l = 1 \) and \( q = 2 \).

For the base case, we know that \( \Pi = \mathbb{Z} \) with the trivial \( F \)-action, so that \( \Pi \cdot F = 0 \). Hence \( \alpha \) must be \( 0 \) also.

For the case \( T = T_0^{(1)} \), we know that \( \Pi = \Pi_0 \) so that \( \Pi \cdot F = \Pi_0 \cdot F \). To check that \( \alpha \) remains the same, we use proposition 5.3.1. According to that proposition, there is \( \beta_0 \in \Pi_0 \) such that \( (1 + F) \beta_0 = y_0 \) or \( \eta_0 \) or \( y_0 + \eta_0 \) according as \( \tau_0 = 0 \) or \( 1 \) or 2. If we let \( \beta = \beta_0 + \eta_0 \) in the first case and \( \beta = \beta_0 \) otherwise, we see that \( (1 + F) \beta = y + \eta \) or \( \eta \) or \( y \), respectively. Hence we compute \( \alpha = (1 - F) \beta = (1 - F) \beta_0 = \alpha_0 \) in all cases (since \( (1 - F) \eta_0 = 0 \)), as desired.

For the case \( T = T_0 \cup T_1 \), we consider as usual the exact sequence
\[
0 \longrightarrow \mathbb{Z} \stackrel{(y_0 - y_1)}{\longrightarrow} \Pi_0 \oplus \Pi_1 \longrightarrow \Pi \longrightarrow 0
\]
and take \( F \)-fixed points to yield the exact sequence
\[
0 \longrightarrow \Pi_0 \cdot F \oplus \Pi_1 \cdot F \longrightarrow \Pi \cdot F \longrightarrow H^1(-F, \mathbb{Z}) \stackrel{(y_0 - y_1)}{\longrightarrow} H^1(-F, \Pi_0) \oplus H^1(-F, \Pi_1)
\]
Hence the inclusion \( \Pi_0 \cdot F \oplus \Pi_1 \cdot F \hookrightarrow \Pi \cdot F \) is an isomorphism unless both \( T_0 \) and \( T_1 \) have type 0, in which case it has cokernel \( \mathbb{Z}/2 \). We will shortly determine which extension it is in this case.
Now if one of $T_0$ or $T_1$ has type 0 (wlog $T_0$) then according to proposition 5.3.1 there is some $\beta_0 \in \Pi_0$ such that $(1 + F)\beta_0 = y_0$. Hence $\beta = (\beta_0, 0)$ satisfies $(1 + F)\beta = y$, and hence $\alpha = (1 - F)\beta = (\alpha_0, 0)$.

In the remaining cases we may take $\beta_i \in \Pi_i$ such that $(1 + F)\beta_i = \eta_i$ or $y_i + \eta_i$ according as $\tau_i = 1$ or 2. We thus set $\beta = (\beta_0, \beta_1 - y_1)$. Then by computation we see that $(1 + F)\beta = \eta$ or $y + \eta$ according as $\tau_0 = \tau_1 = 2$ in which case we set $\beta = (\beta_0, \beta_1 - y_1)$. Then by computation we see that $(1 + F)\beta = \eta$ or $y + \eta$ according as $\tau_0 = \tau_1$ or $\tau_0 \neq \tau_1$. Hence we see that $\alpha = (1 - F)\beta = (\alpha_0, \alpha_1)$ in all cases.

It remains to compute $\Pi^{-F}$ in the exceptional case $\tau_0 = \tau_1 = 0$, when it is an extension of $\mathbb{Z}/2$ by $\Pi_0^{-F} \oplus \Pi_1^{-F}$. In this case, we’ve seen that $(1 + F)(\beta_0, 0) = (1 + F)(0, \beta_1) = y$ and so $(\beta_0, -\beta_1) \in \Pi^{-F}$. Yet $(\beta_0, -\beta_1) \notin \Pi_0^{-F} \oplus \Pi_1^{-F}$, so this represents the non-trivial element of $\Pi^{-F}/(\Pi_0^{-F} \oplus \Pi_1^{-F})$. It doubles to $(\alpha_0, -\alpha_1)$, and hence $\Pi^{-F} = \Pi_0^{-F} \oplus \Pi_1^{-F} \langle 1/(2(\alpha_0, -\alpha_1)) \rangle$ as desired.

For the case $T = \text{Ind}_{F_2}^F T_0$, we consider the $F$-equivariant exact sequence

$$0 \rightarrow J_2 \rightarrow \text{Ind}_{F_2}^F \Pi_0 \rightarrow \Pi \rightarrow 0$$

where $J_2$ is the augmentation ideal of $\mathbb{Z}[C_2]$ endowed with the usual $F$-action (isomorphic to $\mathbb{Z}$ with the $F$-action by $-1$). Since $\Pi^1(-F, J_2) = 0$, when we take $-F$-fixed points we obtain the exact sequence

$$0 \rightarrow J_2 \rightarrow (1 - F)\Pi_0^{F^2} \rightarrow \Pi^{-F} \rightarrow 0$$

To determine $\alpha$ we note that $\eta = (1 + F)\eta_0$, so we may take $\beta = \eta_0$ in proposition 5.3.1. Thus we have $\alpha = (1 - F)\eta_0 \in (1 - F)\Pi_0^{F^2}$. Under the obvious isomorphism $(1 - F)\Pi_0^{F^2} \cong \Pi_0^{F^2}$ this maps to the value of $\alpha$ claimed.

For the case $T = \text{Ind}_{F_q}^F T_0$ with $q$ odd, we consider the $F$-equivariant exact sequence

$$0 \rightarrow J_q \rightarrow \text{Ind}_{F_q}^F \Pi_0 \rightarrow \Pi^{-F} \rightarrow 0$$

Since $\Pi^q(-F, J_q) = \Pi^1(-F, J_q) = 0$, the canonical map $(1 - F + \cdots + F^{q-1})\Pi_0^{-F^q} \rightarrow \Pi^{-F}$ is an isomorphism.

To check that $\alpha$ remains the same, pick $\beta_0 \in \Pi_0$ such that $(1 + F^q)\beta_0 = y_0$ or $\eta_0$ or $y_0 + \eta_0$ (according as $\tau_0 = 0$, 1 or 2). We thus set

$$\beta = \begin{cases} (1 - F + \cdots + F^{q-1})(1 + F + \cdots + F^{q-1})\beta_0 & \text{if } \tau_0 = 1 \\ (1 - F + \cdots + F^{q-1})(1 + F + \cdots + F^{q-1})\beta_0 - \frac{q-1}{2}y_0 & \text{else} \end{cases}$$

Hence by direct computation (using the fact that $y = y_0$ is $F$-fixed) we see that $(1 + F)\beta = y$ or $\eta$ or $y + \eta$. Thus

$$\alpha = (1 - F)\beta = (1 - F + \cdots + F^{q-1})(1 - F^q)\beta_0 = (1 - F + \cdots + F^{q-1})\alpha_0$$

in all cases. Under the obvious isomorphism $(1 - F + \cdots + F^{q-1})\Pi_0^{-F^q} \cong \Pi_0^{-F^q}$ this maps to the value of $\alpha$ claimed.

The final case is an immediate consequence of the corresponding point in algorithm 5.1.4.

Proof of proposition 5.3.2. We prove the proposition by structural induction over the class of all marked BY trees, using the constructors from notation 1.2.3. We adopt the obvious notational convention that $(\Pi_i, y_i, \eta_i)$ and $\tau_i$ will denote the pipe and type of a marked BY tree $T_i$ (and will only refer to the latter when it is inductively assumed to exist).
For the base case, we know that the pipe is \((Z, l, 1)\), so that if \(l\) is odd then \(y_s\) and \(\eta_s\) are both the non-trivial element of \(H^1(-F, \Pi) \cong Z/2\), and hence \(y_s + \eta_s \neq 0\). Hence this marked BY tree of type 2. The case when \(l\) is even follows similarly.

For the inductive step \(T = T_0^{(l)}\) with \(l\) odd, where we are inductively assuming that \(T_0\) satisfies proposition 5.3.2. Here we have \(\Pi = \Pi_0\), \(\eta = \eta_0\) and \(y = y_0 + \eta_0\), so we are done since \(y_s\) and \(\eta_s\) are 2-torsion. The case when \(l\) is even follows similarly.

For the inductive step \(T = \bigvee T_i\), it suffices to consider \(T = T_0 \vee T_1\) for two trees \(T_0\) and \(T_1\), which we assume to satisfy proposition 5.3.2. We have the exact sequence

\[
0 \rightarrow Z \xrightarrow{(y_0, -y_1)} \Pi_0 \oplus \Pi_1 \rightarrow \Pi \rightarrow 0.
\]

Hence we obtain the exact sequence

\[
H^1(-F, Z) \xrightarrow{(y_0, -y_1)} H^1(-F, \Pi_0) \oplus H^1(-F, \Pi_1) \rightarrow H^1(-F, \Pi)
\]

so that \(H^1(-F, \Pi)\) contains \(H^1(-F, \Pi_0) \oplus H^1(-F, \Pi_1)\) as a submodule. Note that this submodule contains both \(y_s = (y_0, 0)\) and \(\eta_s = (\eta_0, \eta_1)\).

Now \((y_0, -y_1) \in H^1(-F, \Pi_0) \oplus H^1(-F, \Pi_1)\) is a 2-torsion element, so that \(y_s = (y_0, 0)\) is 0 in \(H^1(-F, \Pi)\) iff either \((y_0, 0) = (0, 0)\) in \(H^1(-F, \Pi_0) \oplus H^1(-F, \Pi_1)\) – this occurs iff either \(\tau_0 = 0\) or \(\tau_1 = 0\).

Similarly we know that \(y_s = (y_0, 0)\) is 0 in \(H^1(-F, \Pi)\) iff either \((\eta_0, \eta_1)\) or \((\eta_0 + y_0, \eta_1)\) or \((\eta_0 + y_0, \eta_1)\) is 0 in \(H^1(-F, \Pi_0) \oplus H^1(-F, \Pi_1)\). Since \(\eta_0 + y_0 = \eta_0 + y_0\), this occurs iff \(\tau_0 = \tau_1 = 1\) or \(\tau_0 = \tau_1 = 2\).

Similarly again we know that \(\eta_s + y_s = 0\) in \(H^1(-F, \Pi)\) iff \((\eta_0 + y_0, \eta_1)\) or \((\eta_0, \eta_1 + y_1)\) or \((\eta_0 + y_0, \eta_1)\) is 0. This occurs iff \(\{\tau_0, \tau_1\} = \{1, 2\}\).

Between the three preceding points, we have determined which of \(y_s\), \(\eta_s\) and \(\eta_s + y_s\) vanish in \(H^1(-F, \Pi)\) for each possible pair \((\tau_0, \tau_1)\). In any case, we have shown that exactly one vanishes, for any \((\tau_0, \tau_1)\).

For the inductive step \(T = \text{Ind}^F_\ast T_0\) with \(q\) odd, we have an exact sequence

\[
0 \rightarrow J_q \xrightarrow{y_0} \text{Ind}^F_\ast \Pi_0 \rightarrow \Pi \rightarrow 0
\]

where \(J_q\) is the augmentation ideal of \(Z[C_q]\). Since \(H^1(-F, J_q) = 0\), we obtain an injection \(H^1(-F^q, \Pi_0) \cong H^1(-F, \text{Ind}^F_\ast \Pi_0) \hookrightarrow H^1(-F, \Pi)\) where the first map is the Shapiro isomorphism.

On the level of cocycles, the Shapiro isomorphism takes a cocycle \(a: qZ \rightarrow \Pi_0\) to the unique cocycle \(\tilde{a}: Z \rightarrow \text{Ind}^F_\ast \Pi_0\) such that \(\tilde{a}(1) = a(q)\). Hence the isomorphism takes \(y_0 \in H^1(-F^q, \Pi_0)\) to \(y_s \in H^1(-F, \Pi)\) and \(\eta_0\) to the class represented by the (unique) cocycle \(\tilde{a}: Z \rightarrow \Pi\) such that \(\tilde{a}(1) = \eta\). Yet \(F\) acts on \(H^1(-F, \Pi)\) by \(-1\), and so as elements of \(H^1\) we have \(\tilde{a} = (1 + F + \cdots + F^{q-1})\). Since \(y_0\) maps to \(y_s\) and \(\eta_0\) to \(\eta_s\), so that \(T\) satisfies proposition 5.3.2.

For the next inductive step, we need only consider \(T = \text{Ind}^F_\ast T_0\). Here we use that \(\eta_s\) and \(y_s\) are represented by cocycles

\[
(\eta_s)_i = \begin{cases} 
\eta & \text{if } i \text{ odd}, \\
0 & \text{if } i \text{ even}, 
\end{cases}
\]

and \((y_s)_i = \begin{cases} 
y & \text{if } i \text{ odd}, \\
0 & \text{if } i \text{ even}. 
\end{cases}\)

We will verify explicitly that the first of these is a coboundary, but the second isn’t.
The first case is straightforward: the element \( \eta_0 \in \text{Ind}^F_{\mathcal{E}_2} \Pi_0 \) is fixed by \( F^2 \) and hence has coboundary \((\delta \eta_0)_i = (1 - (-F)^i)\eta_0 = \begin{cases} (1 + F)\eta_0 & \text{if } i \text{ odd,} \\ 0 & \text{if } i \text{ even.} \end{cases}\) Since \((1 + F)\eta_0 = \eta \) in \( \Pi \), this shows that the cocycle \( \eta_* \) is a coboundary.

For the second case, suppose for contradiction that \( y_* \) were a coboundary of some \( w_0 + Fz_0 \in \text{Ind}^F_{\mathcal{E}_2} \Pi_0 \Pi_0 \) = \( \Pi \). Then in particular we would have some \( a \in \mathbb{Z} \) such that \((1 + F)(w_0 + Fz_0) = y_0 + a(1 - F)y_0 \) in \( \text{Ind}^F_{\mathcal{E}_2} \Pi_0 \). Equating coefficients and subtracting, we see that \((F^2 - 1)z_0 = (2a + 1)y_0 \). Yet this is impossible: \( F^2 \) acts trivially on the quotient \( \Pi_0/\Pi_0, \text{tors} \simeq \mathbb{Z} \) so \((F^2 - 1)z_0 \) is necessarily torsion while \((2a + 1)y_0 \) is not.

The final inductive step follows trivially from the final point in algorithm \ref{algorithm:5.1.4}.

\[
\Box
\]

\section{Growth of Tamagawa numbers in towers}

In this final part of this paper, we use the above techniques to examine how Tamagawa numbers of semistable hyperelliptic curves \( X/K \) vary as we enlarge the ground field \( K \). Recall that if \( X/K \) is any semistable curve and \( L/K \) is a finite extension of ramification degree \( e \) and residue class degree \( f \), then the dual graph \( \mathcal{G}_L \) of the geometric special fibre of the minimal regular (semistable) model of \( X \) over \( L \) is produced from the corresponding graph \( \mathcal{G}_K \) over \( K \) by scaling all the edge-lengths by a factor of \( e \), and restricting the Frobenius action from \( \text{Frob}_K \) to \( \text{Frob}_L = \text{Frob}_K^f \). From this description, it follows that BY trees of semistable hyperelliptic curves \( X/K \) also evolve in a predictable manner as one enlarges the base field \( K \): if \( T = (T, S, eF) \) denotes the BY tree of \( X/K \), then the BY tree of \( X \) over a finite extension \( L/K \) is the BY tree \( T_{e,f} = (eT, eS, (eF)^f) \) produced from \( T \) by scaling all the edge-lengths by a factor of \( e \) and replacing the signed automorphism \( eF \) by its \( f \)th power.

Thus the problem of controlling Tamagawa numbers of semistable hyperelliptic curves over all finite extensions of the base field reduces to the purely combinatorial problem of controlling the Tamagawa numbers of the BY trees \( T_{e,f} \) for a fixed BY tree \( T = T_{1,1} \), perhaps in the presence of parity conditions \((A)\) and \((B)\). To do this, we study the dependence on \( e \) and \( f \) separately, deriving restrictions on the Tamagawa numbers of \( T_{e,1} \) and \( T_{1,f} \) as functions of \( e \) and \( f \) respectively. Combining these results will yield the strong restrictions in theorem \ref{theorem:1.3.4} purely formally.

\subsection{Totally ramified extensions}

We begin with the easier problem of controlling the Tamagawa numbers of BY trees \( T_{e,1} = (eT, eS, eF) \) formed by scaling all the edge-lengths in \( T \) by a factor of \( e \). General tools to tackle this kind of problem particularly in terms of the relative homology lattice \( \Lambda_T = H_1(T, S, \mathbb{Z}) \) were developed in \[\text{[3]}\], but we will forgo these techniques, since our Tamagawa number algorithm already renders this particular problem essentially trivial.

\begin{proposition}
Fix a BY forest \( T = (T, S, eF) \) satisfying parity conditions \((A)\) and \((B)\), and consider the family \( T_{e,1} = (eT, eS, eF) \) of BY forests (also satisfying the parity conditions) indexed by \( e \in \mathbb{N} \), produced by scaling all the edge-lengths in \( T \) by a factor of \( e \). Then there are constants \( a \in \mathbb{N} \), \( r \in \mathbb{N}_0 \) and \( s \in \mathbb{Z} \) such that the Tamagawa number of \( T_{e,1} \) is given by \( c_{T_{e,1}} = ae^r(e, 2)^s \).
\end{proposition}
Proof. By lemma 3.0.1, it suffices to prove the result for a simple BY tree satisfying
parity conditions (A) and (B). In the case of a positive simple BY tree, theorem 1.1.3 shows that the Tamagawa number is a homogenous polynomial in the edge
lengths of $T$, so that $c_{T,e} = \alpha e^r$ for some $\alpha$ and $r$.

In the case of a negative simple BY tree, theorem 1.1.5 gives that the Tamagawa
number of a tree is $c_T = \tilde{c}_1 c_T'$, where $\tilde{c}_1$ is a certain power of 2 and $T'$ is a certain
BY tree obtained from $T$ whose sign is uniformly positive. It is easy to see that
scaling the edges of $T$ by $e$ scales the edges of $T'$ by $e$ also, so that its dependence
on $e$ is of the form $a'e^r$ for some constants $a'$ and $r$. As for $\tilde{c}_1$, it can change when
we scale the edge lengths by $e$, but it easy to see that it only depends on the parity
of $e$. Since it always a power of 2, its dependence on $e$ is of the form $2^{b/e^2}$
for some $b$ and $s$. Taking a product, we see that $c_{T,e} = \alpha e^r(2^b, 2)^s$, as claimed.

In fact, one can check from this description that $s \geq 0$, but we will not use this
fact. □

Remark 6.1.2. In the language of [3], the preceding proposition is equivalent to
the assertion that the group $B \Lambda$ is 2-torsion. In fact, this can be proved directly
(even in the absence of parity condition (A)), though the proof is more complicated
than the one given.

6.2. Unramified extensions. Having dealt with the dependence on $e$, it now
remains to control the dependence of the Tamagawa numbers of a family $T_{1,f} = (T, S, (\epsilon F)^f)$ of BY trees on the parameter $f$. Since

$$c_{T_{1,f}} = \# \Phi_{T_{1,f}} = \# \Phi_T(\epsilon F)^f$$

where $\Phi_T$ is the geometric Néron component group of the fixed BY tree $T$, this
amounts in essence to a problem of counting fixed points of powers of an automor-
phism of a finite abelian group. We will develop tools to deal with such problems in appendix A, but for now let us just record the definitions and basic properties
we will need.

Definition 6.2.1 (Fixpoint filtrations). Let $A$ be a $\mathbb{Z}$-module with an endomor-
phism $\sigma$. We define the $\sigma$-fixpoint filtration of $A$ to be the family of sub-
$\mathbb{Z}[\sigma]$-modules $A_{\sigma^d}$, which come with inclusions $A_{\sigma^d} \leq A_{\sigma^f}$ as $d$ divides $f$. We also
define the partial quotients of $A$ with respect to this filtration to be

$$\text{Gr}_{\sigma^d}(A) := \frac{A_{\sigma^d}}{\sum_{d|f, d \neq f} A_{\sigma^d}}.$$

Lemma 6.2.2. Let $A$ be a $\mathbb{Z}$-module with an endomorphism $\sigma$. Then, for any
$f \in \mathbb{N}$, $A_{\sigma^f}$ has an exhaustive, separated linearly ordered filtration with partial
quotients $\text{Gr}_{\sigma^d}(A)$ for $d | f$. In particular, if $A_{\sigma^f}$ is finite then

$$\# A_{\sigma^f} = \prod_{d | f} \# \text{Gr}_{\sigma^d}(A).$$

Moreover, the $\mathbb{Z}[\sigma]$-module structure on the partial quotient $\text{Gr}_{\sigma^d}(A)$ factors canonically
through the quotient $\mathbb{Z}[\sigma] \to \mathbb{Z}[\mu_f]$ sending $\sigma$ to a primitive $f$th root of unity.

Finally, if $B$ is another $\mathbb{Z}[\sigma]$-module and $C$ is a $\mathbb{Z}[\sigma^q]$-module for some $q \in \mathbb{N}$,
we have

$$\text{Gr}_{\sigma^d}(A \oplus B) \cong \text{Gr}_{\sigma^d}(A) \oplus \text{Gr}_{\sigma^d}(B)$$
and
\[ \text{Gr}^f_f (\text{Ind}^\sigma_{f/q} C) \cong \text{Gr}^\sigma_{f/q}(C) \otimes_{\mathbb{Z}_f} [\mathbb{Z}][\mu_q] \]
where \( f/q \) denotes the numerator of \( f/q \).

Proof. Deferred to appendix. \( \square \)

The concept of fixpoint filtrations and partial quotients of \( \mathbb{Z}[\sigma] \)-modules gives an integral analogue of the isotypic decomposition of \( \mathbb{Q}[\sigma] \)-modules (or at least those on which \( \sigma \) acts locally with finite order). However, while the isotypic pieces of \( \mathbb{Q}[\sigma] \)-modules are well-behaved, being free modules over the vector spaces \( \mathbb{Q}[\mu_n] \), much less can be expected in general for the partial quotients of \( \mathbb{Z}[\sigma] \)-modules. Thus, it will be useful for us to isolate a class of \( \mathbb{Z}[\sigma] \)-modules where the fixpoint filtration is well-behaved; the main theorem of this section will then assert that geometric Néron component groups are well-behaved in this sense.

**Definition 6.2.3.** We will say a \( \mathbb{Z}[\sigma] \)-module \( A \) is fixpoint-regular just when there is a collection \( (A_d)_{d \in \mathbb{N}} \) of \( \mathbb{Z} \)-modules such that we have \( \mathbb{Z}[\mu_d] \)-module isomorphisms \( \text{Gr}_f^f(A) \cong A_d \otimes_{\mathbb{Z}} [\mathbb{Z}][\mu_d] \) for all \( d \). (If \( \sigma \) acts on \( A \) with finite order, \( A_d \) are necessarily trivial for \( d \) not dividing the order.)

It is clear from lemma 6.2.2 that the class of fixpoint regular \( \mathbb{Z}[\sigma] \)-modules is closed under direct sums and induction from \( \mathbb{Z}[\sigma^q] \) to \( \mathbb{Z} \) for any \( q \in \mathbb{N} \).

**Theorem 6.2.4.** Let \( T = (T, S, \epsilon F) \) be a BY forest satisfying parity condition (B). Then the geometric Néron component group \( \Phi_T \) is fixpoint-regular (for the action of \( \epsilon F \)).

**Corollary 6.2.5.** Let \( T = (T, S, \epsilon F) \) be a BY forest satisfying parity condition (B), and consider the family \( T_{1, f} = (T, S, (\epsilon F)^f) \) of BY forests indexed by \( f \in \mathbb{N} \), produced by replacing the action of \( \epsilon F \) with the action of \( (\epsilon F)^f \). Then there are constants \( (a_d)_{d \in \mathbb{N}} \) (equal to 1 for \( d \) not dividing the order of \( \epsilon F \)) such that \( c_{T_{1, f}} = \prod_{d \mid f} a_d^\sigma(d) \) for all \( f \).

Proof. Since the geometric Néron component group \( \Phi_T \) is finite and fixpoint-regular, we may choose \( \mathbb{Z} \)-modules \( A_d \) as in definition 6.2.3. These are necessarily finite, and setting \( a_d = \#A_d \) yields the desired constants. \( \square \)

Let us now gradually build up to a proof of theorem 6.2.4. After a straightforward reduction to the case of simple BY trees, this will essentially amount to showing that for every marked BY tree \( (T, S, F, \ast) \), the geometric Néron component groups of \( (T, S, \pm F) \) and \( (T, S \cup \{\ast\}, \pm F) \) are all fixpoint regular. In proving this (by induction over marked BY trees), the crucial observation is that the geometric Néron component groups \( \Phi_{(T, S, \pm F)} \) and \( \Phi_{(T, S \cup \{\ast\}, \pm F)} \) are fixpoint-regular if and only if the pipe \( \Pi \) of \( (T, S, F, \ast) \) is fixpoint-regular (for the action of \( \pm F \)). This is a consequence of the following proposition.

**Proposition 6.2.6.** Let \( (T, S, F, \ast) \) be a marked BY tree, with pipe \( (\Pi, y, \eta) \). Then the quotient maps \( \Pi \rightarrow \Pi/y \) and \( \Pi \rightarrow \Pi/\eta \) induce isomorphisms on \( \text{Gr}_f^f \) for \( f > 1 \), and on \( \text{Gr}_f^F \) for \( f > 2 \).

Proof. We deal with the case of \( \Pi \rightarrow \Pi/y \), the other case being similar. For the action of \( \pm F \), proposition 6.2.1 applied to \( (T, S, F_d, \ast) \) tells us that the natural
map $\Pi^{Fd} \to (\Pi/y)^{Fd}$ is surjective with kernel $(y)$, so that the map $\Pi^{Fd}/\Pi^F \to (\Pi/y)^{Fd}/(\Pi/y)^F$ is an isomorphism for all $d$. But then for $f > 1$, we have

$$\text{Gr}_F^f(\Pi) := \frac{\Pi^{Ff}}{\sum_{d|f, d \neq f} \Pi^{Fd}} \cong \frac{\Pi^{Ff}/\Pi^F}{\sum_{d|f, d \neq f} \Pi^{Fd}/\Pi^F}$$

and similarly for $\text{Gr}_F^f(\Pi/y)$. So the induced map $\text{Gr}_F^f(\Pi) \to \text{Gr}_F^f(\Pi/y)$ is an isomorphism as desired.

For the action of $-F$, we have a similar natural isomorphism

$$\text{Gr}_{-F}^f(\Pi) \cong \frac{\Pi(\Pi^{(-F)^f})/\Pi(\Pi^{(-F)^{hd(f, 2)}})}{\sum_{d|f, d \neq f} \Pi(\Pi^{(-F)^d})/\Pi(\Pi^{(-F)^{hd(d, 2)}})}$$

for $f > 2$, and similarly for $\text{Gr}_{-F}^f(\Pi/y)$. It hence suffices to show that the induced maps $\Pi(\Pi^{(-F)^d})/\Pi(\Pi^{(-F)^{hd(d, 2)}}) \to (\Pi/y)(\Pi^{(-F)^d})/\Pi(\Pi^{(-F)^{hd(d, 2)}})$ are isomorphisms for all $d > 2$, i.e. that $\Pi^{(-F)^d}/\Pi^{(-F)} \to (\Pi/y)^{(-F)^d}/\Pi^F$ is an isomorphism for odd $d$, and $\Pi^{(-F)^d}/\Pi^{(-F)^2} \to (\Pi/y)^{(-F)^d}/\Pi^{(-F)^2}$ is an isomorphism for even $d$. The latter case we have already done, since $\Pi^{(-F)^2d} = \Pi^{(F^2)^d}$, and for the former we use the diagram with exact rows

$$\begin{array}{cccccc}
0 & \longrightarrow & \Pi^{-F} & \longrightarrow & (\Pi/y)^{-F} & \longrightarrow & H^1(-F, Z) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Pi^{-Fd} & \longrightarrow & (\Pi/y)^{-Fd} & \longrightarrow & H^1(-Fd, Z)
\end{array}$$

where the two rightmost vertical arrows are the restriction maps. Yet by the inflation-restriction sequence, the rightmost arrow has kernel $H^1(C_d, \Pi^{-Fd}[2]) = 0$ as $d$ odd, so is injective. In particular, the isomorphism $H^1(-F, Z) \cong H^1(-F^d, Z)$ induces an isomorphism between the kernels of the two maps labelled $y_*$. So by the snake lemma we find that the induced map $\Pi^{(-F)^d}/\Pi^{(-F)} \to (\Pi/y)^{(-F)^d}/\Pi^{(-F)}$ is an isomorphism, as desired. □

**Corollary 6.2.7.** Let $(T, S, F, *)$ be a marked BY tree, with pipe $(\Pi, y, \eta)$. Then the action of $\pm F$ on each of $\Pi$, $\Pi/y$ and $\Pi/\eta$ is fixpoint-regular.

**Proof.** Notice that by proposition 6.2.6, the action on $\pm F$ on $\Pi$, $\Pi/y$ and $\Pi/\eta$ is fixpoint-regular if it is fixpoint-regular on any one of them (since the fixpoint-regularity condition is automatic for $f \leq 2$). Armed with this observation, we will prove the result by structural induction over the class of marked BY trees, using algorithm 8.1.14.

For the trivial tree, the group $\Pi/y$ is trivial, so the action is trivially fixpoint-regular for both actions.

In the case $T = T_0(t)$, we have that $\Pi/\eta \cong \Pi_0/\eta_0$, which is fixpoint-regular for both actions by inductive assumption.

In the case that $T = T_0 \vee T_1$, we have that $\Pi/y \cong (\Pi_0/y_0) \oplus (\Pi_1/y_1)$, which is fixpoint-regular for both actions since both of its factors are by inductive assumption.

In the case that $T = \bigvee \text{Ind}_{\pm F}^F T_0$, we have that $\Pi/y \cong \text{Ind}_{\pm F}^{F}(\Pi_0/y_0)$. Since $\Pi_0/y_0$ is fixpoint-regular for both actions by inductive assumption and the class of
fixpoint-regular representations is closed under induction, we see that $\Pi/y$ is also fixpoint-regular for both actions by lemma 6.2.2.

In the case that $T = (T, S, F, *)$ contains a marked BY subtree $T_0 = (T_0, S_0, F, *)$ with $S_0 = S$, we have that $\Pi = \Pi_0$, which is fixpoint-regular for both actions by inductive assumption.

(Throughout this proof, we use the obvious convention that $(\Pi_i, y_i, \eta_i)$ denotes the pipe of marked BY tree $T_i$.)

Proof of theorem 6.2.4. It follows from lemma 6.2.2 that the class of fixpoint-regular representations is closed under direct sums and induction, so by lemma 3.0.1 it suffices to consider the case when $(T, S, \pm F)$ is a simple BY tree. But this is contained in the previous corollary.

6.3. General extensions. It remains to combine the separate considerations of unramified and totally ramified extensions to obtain a result on general extensions. This doesn’t require any extra input, and is a purely formal manipulation.

**Theorem 6.3.1** (= theorem 1.3.1). Fix a BY forest $T = (T, S, \epsilon F)$ satisfying parity conditions (A) and (B), and consider the family $T_{e,f} = (eT, eS, (\epsilon F)^f)$ a family of Tamagawa forests (also satisfying the parity conditions) indexed by $(e, f) \in \mathbb{N} \times \mathbb{N}$, produced by scaling all the edge-lengths in $T$ by a factor of $e$ and replacing the action of $\epsilon F$ with the action of $(\epsilon F)^f$. Then there are constants $(a_d, r_d, s_d) \in \mathbb{N} \times \mathbb{N}_0 \times \mathbb{Z}$ for each $d \in \mathbb{N}$ (equal to $(1, 0, 0)$ for $d$ not dividing the order of $\epsilon F$) such that

$$c_{T_{e,f}} = \prod_{d|f} (a_d e^{r_d(e, 2)s_d})^{\varphi(d)}$$

for all $e, f$, where $\varphi$ is Euler’s totient function.

Proof. By corollary 6.2.6 we know that, for each $e$ there are constants $a_d(e) \in \mathbb{N}$ (equal to 1 for $d$ not dividing the order of $\epsilon F$) such that

$$c_{T_{e,f}} = \prod_{d|f} a_d(e)^{\varphi(d)}$$

for all $f$. But by proposition 6.1.1 for fixed $f$ the Tamagawa number $c_{T_{e,f}}$ is of the form $ae^r(e, 2)^s$ for $a \in \mathbb{Q}^\times$ and $r, s \in \mathbb{Z}$. Applying the Möbius inversion formula to the product representation of $c_{T_{e,f}}$ above, we see that $a_d(e)^{\varphi(d)}$ must also be a function of $e$ of such a form. But $a_d(e)$ is a positive integer for all $e$, which forces it to be of the form $a_d(e) = a_d e^{r_d(e, 2)s_d}$ where $a_d \in \mathbb{N}$ and $r_d \in \mathbb{N}_0$ as desired.
Appendix A. Fixpoint filtrations

In this appendix, we set out the basic properties of fixpoint filtrations, aiming to justify the content of lemma 6.2.2. Recall that we are considering $\mathbb{Z}$-modules $A$ endowed with an endomorphism $\sigma$, and that we’re interested in the family of $\mathbb{Z}[\sigma]$-submodules $A^{\sigma^t}$, which we call the fixpoint filtration of $A$ (indexed by $\mathbb{N}$ with the divisibility ordering). We are also interested in the partial quotients of this filtration, by which we mean the $\mathbb{Z}[\sigma]$-modules

$$\text{Gr}_t^\sigma(A) := \frac{A^{\sigma^t}}{\sum_{d \mid f, d \neq f} A^{\sigma^d}}.$$

Our chief method of proof is careful calculations involving cyclotomic polynomials, for which we need a preparatory proposition.

**Proposition A.0.1.** Let $P_1, \ldots, P_m$ be integer polynomials, each of which is a product of some cyclotomic polynomials $P_i = \prod_j \Phi_{d_{i,j}}$. Then the $P_i$ generate a proper ideal of $\mathbb{Z}[t]$ if and only if, for each index $i$ we may choose an index $j_i$ such that each quotient $d_{i,j_i}/d_{i',j_i'}$ is a power of a fixed prime $\ell$ (the exponent of such a power is permitted to be any integer, including negative integers).

**Proof.** Let $R = \mathbb{Z}[t]/(P_1, \ldots, P_m)$, so that $(P_1, \ldots, P_m)$ is a proper ideal iff $R \neq 0$. Consider first the case $R \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[t]/(P_1, \ldots, P_m) \neq 0$ so that $R \neq 0$ and the $P_i$ have a common irreducible factor $\Phi_d$ in $\mathbb{Q}[t]$. It follows that we can select the indices $j_i$ such that each $d_{i,j_i} = d$, and hence the quotients $d_{i,j_i}/d_{i',j_i'} = 1$ are all powers of every prime. This deals with this case.

In the remaining case, we have $R \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and $R$ is finite over $\mathbb{Z}$, so that $R$ is a finite ring. Thus $R$ is non-zero iff it has any quotients which are finite fields, which occurs iff there is a ring homomorphism $R \to \overline{\mathbb{F}}_\ell$ for some prime $\ell$. Since ring homomorphisms $R \to \overline{\mathbb{F}}_\ell$ are in bijection with common roots in $\overline{\mathbb{F}}_\ell$ of the $P_i$, this occurs iff the $P_i$ have a common root in $\overline{\mathbb{F}}_\ell$.

Yet $\text{Root}_{\overline{\mathbb{F}}_\ell}(P_i) = \bigcup_j \text{Root}_{\overline{\mathbb{F}}_\ell}(\Phi_{d_{i,j}}) = \bigcup_j \mu_{d_{i,j}}(\overline{\mathbb{F}}_\ell)$ where $d_{i,j}$ is the largest $\ell$-free factor of $d_{i,j}$ and $\mu_d$ is the set of primitive $d$th roots of unity. Since the sets $\mu_{d_{i,j}}(\overline{\mathbb{F}}_\ell)$ for $\ell$-free $d$ are pairwise disjoint, the $P_i$ have a common root in $\overline{\mathbb{F}}_\ell$ iff we may choose indices $j_i$ for each index $i$ such that the $d_{i,j_i}$ are all equal. This is the same as requiring that each $d_{i,j_i}/d_{i',j_i'}$ be a power of $\ell$. \[ \square \]

For us, the main consequence of this proposition is that submodules of $\mathbb{Z}[\sigma]$-modules cut out by products of cyclotomic polynomials are particularly well-behaved.

**Lemma A.0.2.** For a finite subset $S \subseteq \mathbb{N}$ denote by 

$$\Phi_S = \prod_{d \in S} \Phi_d$$

If $S, S' \subseteq \mathbb{N}$ are finite subsets closed under divisors, then for every $\mathbb{Z}[\sigma]$-module $A$ we have 

$$A[\Phi_S \cap S'(\sigma)] = A[\Phi_S(\sigma)] \cap A[\Phi_{S'}(\sigma)]$$

and 

$$A[\Phi_{S \cup S'}(\sigma)] = A[\Phi_S(\sigma)] + A[\Phi_{S'}(\sigma)].$$

\[13\] The reader can feel free to replace “endomorphism” with “automorphism” in this section, as our definitions will only see the part of $A$ on which $\sigma$ acts as an automorphism (pointwise) of finite order.
Proof. Let $P = \Phi_{S|S'} = \Phi_{S}/\Phi_{S\cup S'} = \Phi_{S\cup S'}/\Phi_{S}$. and $P' = \Phi_{S'}) = \Phi_{S'/S} = \Phi_{S'/S}/\Phi_{S'}$. Since no element of $S\setminus S'$ divides any element of $S'\setminus S$ and vice versa, proposition A.0.1 ensures that $P$ and $P'$ generate the unit ideal of $\mathbb{Z}[t]$; there are integer polynomials $Q$ and $Q'$ such that $QP + Q'P' = 1$.

For the first equality, multiplying $QP + Q'P' = 1$ by $\Phi_{S\cup S'}$ we see that we have $Q\Phi_{S} + Q'\Phi_{S'} = \Phi_{S\cup S'}$ and hence $A[\Phi_{S}(\sigma)]\cap A[\Phi_{S'}(\sigma)] = A[\Phi_{S\cup S'}(\sigma)]$. As $\Phi_{S\cup S'}|\Phi_{S}, \Phi_{S'}$, the converse inclusion is clear.

For the second equality, consider some $a \in A[\Phi_{S\cup S'}(\sigma)]$. Now we have that $\Phi_{S}(\sigma)Q(\sigma)P(\sigma)a = Q(\sigma)\Phi_{S\cup S'}(\sigma)a = 0$ and $\Phi_{S'}(\sigma)Q'(\sigma)P'(\sigma)a = 0$ similarly, so $a = Q(\sigma)P(\sigma)a + Q'(\sigma)P'(\sigma)a = A[\Phi_{S}(\sigma)] + A[\Phi_{S'}(\sigma)]$. Hence we have the inclusion $A[\Phi_{S\cup S'}(\sigma)] \leq A[\Phi_{S}(\sigma)] + A[\Phi_{S'}(\sigma)]$, and the other inclusion is clear. □

Corollary A.0.3. Let $S$ be a finite subset of $\mathbb{N}$ closed under divisors, and $A$ a $\mathbb{Z}[\sigma]$-module. Then

$$A[\Phi_{S}(\sigma)] = \sum_{d \in S} A^{\sigma^{d}}$$

Proof. For each $d \in S$ let $S_{d}$ be the set of divisors of $d$, so that $S = \bigcup_{d \in S} S_{d}$. Since $A[\Phi_{S_{d}}(\sigma)] = A[\sigma^{d} - 1] = A^{\sigma^{d}}$, an iterated application of the preceding lemma provides the desired equality. □

Lemma A.0.2 has further important consequences. Firstly, we can use this lemma to turn the partially ordered fixpoint filtration $A^{\sigma^{f}}$ on a $\mathbb{Z}[\sigma]$-module $A$ into a totally order one, thereby justifying our use of the phrase “partial quotients” to describe the subquotients $Gr_{f}^{\sigma}(A)$, and secondly, we find that these partial quotients give something akin to an isotypic decomposition of the $\mathbb{Z}[\sigma]$-module $A$, in that the $\mathbb{Z}[\sigma]$-module structure on $Gr_{f}^{\sigma}(A)$ factors through $\mathbb{Z}[\mu_{f}]$.

Corollary A.0.4. Let $A$ be a $\mathbb{Z}[\sigma]$-module. Then for each $f \in \mathbb{N}$, $A^{\sigma^{f}}$ possesses an exhaustive separated $\mathbb{Z}$-indexed filtration whose partial quotients are $Gr_{f}^{\sigma}(A)$ for $d \mid f$ in some order.

Proof. Pick a sequence $\emptyset = S_{0} \subseteq S_{1} \subseteq \cdots \subseteq S_{m}$ of subsets of $\mathbb{N}$, each closed under divisors, so that each $S_{i+1} \setminus S_{i} = \{d_{i}\}$ has size 1, and $S_{m}$ is the set of divisors of $f$. We consider the linear filtration $0 = A_{0} \leq A_{1} \leq \cdots \leq A_{m}$ of $A^{\sigma^{f}}$ defined by

$$A_{i} = A[\Phi_{S_{i}}(\sigma)] = \sum_{d \in S_{i}} A^{\sigma^{d}}$$

so that $A_{0} = 0$ and $A_{m} = A^{\sigma^{f}}$.

Now lemma A.0.2 (applied to $S_{i}$ and the set of divisors of $d_{i}$) shows that $A_{i} \cap A^{\sigma^{d_{i}}} = A\left[\frac{\sigma^{d_{i}} - 1}{\Phi_{S_{i}}(\sigma)}\right] = \sum_{d \mid d_{i}, d \neq d_{i}} A^{\sigma^{d}}$ and $A_{i} + A^{\sigma^{d_{i}}} = A_{i+1}$. Hence the second isomorphism theorem, $A_{i+1}/A_{i} = A^{\sigma^{d_{i}}}/\sum_{d \mid d_{i}, d \neq d_{i}} A^{\sigma^{d}} = Gr_{d_{i}}^{\sigma}(A)$. Since $d_{i}$ runs through all the divisors of $f$, it follows that the partial quotients of the filtration are $Gr_{f}^{\sigma}(A)$ for each $d \mid f$, as desired. □

Corollary A.0.5. If $A$ is any $\mathbb{Z}[\sigma]$-module then $\Phi_{f}(\sigma)$ annihilates the partial quotient $Gr_{f}^{\sigma}(A)$ for all $f \in \mathbb{N}$.

Proof. $\Phi_{f}(\sigma)A^{\sigma^{f}} \leq A\left[\frac{\sigma^{f} - 1}{\Phi_{f}(\sigma)}\right] = \sum_{d \mid f, d \neq f} A^{\sigma^{d}}$. Hence the induced action of $\Phi_{f}(\sigma)$ on $Gr_{f}^{\sigma}(A)$ is zero. □
With these results, we have now justified all of lemma 6.2.2 save the behaviour of the partial quotients under direct sums and inductions. The case of direct sums is trivial, while the case of inductions involves some more technical manipulations.

**Lemma A.0.6.** Let \( A \) be an abelian group with an action by \( \sigma^q \), so that by corollary A.0.5 \( \text{Gr}^\sigma_f(A) \) can be viewed as a \( \mathbb{Z}[\mu_f] \)-module with the \( \sigma^q \)-action given by multiplication by \( \zeta_f \). Then

\[
\text{Gr}^\sigma_f(\text{Ind}^\sigma_f A) \simeq \text{Gr}^\sigma_{f/q}(A) \otimes_{\mathbb{Z}[\mu_{f/q}]} \mathbb{Z}[\mu_f]
\]

where \( f/q \) denotes the numerator of \( f/q \). In particular, when \( q \) is prime we have

\[
\text{Gr}^\sigma_f(\text{Ind}^\sigma_f A) \simeq \begin{cases} 
\text{Gr}^\sigma_{f/q}(A) \otimes_{\mathbb{Z}[\mu_{f/q}]} \mathbb{Z}[\mu_f] & \text{if } q \mid f \\
\text{Gr}^\sigma_f(A) & \text{else}
\end{cases}
\]

**Proof.** It suffices to prove the case when \( q \) is prime. Note that

\[
(\text{Ind}^\sigma_f A)^{\sigma^q} = \begin{cases} 
\text{Ind}^\sigma_f A^{\sigma^q} & \text{if } q \mid d \\
(1 + \sigma^d + \cdots + \sigma^{(q-1)d}) A^{\sigma^q} & \text{if } q \nmid d
\end{cases}
\]

Combining this classification with the definition

\[
\text{Gr}^\sigma_f(\text{Ind}^\sigma_f A) = \frac{(\text{Ind}^\sigma_f A)^{\sigma^q}}{\sum_{d \mid f, d \neq f} (\text{Ind}^\sigma_f A)^{\sigma^q}}
\]

we see immediately that when \( q \nmid f \) we have \( \text{Gr}^\sigma_f(\text{Ind}^\sigma_f A) = \frac{A^{\sigma^q}}{\sum_{d \mid f, d \neq f} A^{\sigma^q}} = \text{Gr}^\sigma_{f/q}(A) \), as desired.

When \( q^2 \mid f \) instead, then on the denominator we need only take those \( d \) such that \( q \mid d \), and hence by exactness of \( \text{Ind}^\sigma_f \) we have that \( \text{Gr}^\sigma_f(\text{Ind}^\sigma_f A) = \text{Ind}^\sigma_{f/q} \mathbb{Z}[\mu_{f/q}] \). Since \( \text{Gr}^\sigma_{f/q}(A) \) is a \( \mathbb{Z}[\mu_{f/q}] \)-module, this is the same as \( \text{Gr}^\sigma_{f/q}(A) \otimes_{\mathbb{Z}[\mu_{f/q}]} \mathbb{Z}[\mu_f] \), as desired.

The most difficult case is when \( q \) exactly divides \( d \). Here on the denominator we need only take those \( d \) such that \( q \mid d \), along with \( d = f/q \). In other words, we can identify \( \text{Gr}^\sigma_f(\text{Ind}^\sigma_f A) \) as the cokernel of the natural map

\[
\frac{(\text{Ind}^\sigma_f A)^{\sigma^q}}{\sum_{d \mid f, d \neq f/q} (\text{Ind}^\sigma_f A)^{\sigma^q}} \rightarrow \frac{(\text{Ind}^\sigma_f A)^{\sigma^q}}{\sum_{q \mid d, d \neq f} (\text{Ind}^\sigma_f A)^{\sigma^q}}.
\]

Exactly as we did above, we can identify the leftmost of these groups with \( \text{Gr}^\sigma_{f/q}(A) \), the rightmost with \( \text{Ind}^\sigma_{f/q} \text{Gr}^\sigma_f(A) \), and the map between them as multiplication by \( 1 + \sigma^f + \cdots + \sigma^{(q-1)f} \).

Yet we have an exact sequence of \( \mathbb{Z}[\mu_{f/q}] \)-modules

\[
0 \rightarrow \mathbb{Z}[\mu_{f/q}] \rightarrow \text{Ind}^\sigma_{f/q} \mathbb{Z}[\mu_{f/q}] \rightarrow \mathbb{Z}[\mu_f] \rightarrow 0
\]

where \( \mathbb{Z}[\mu_{f/q}] \) and \( \mathbb{Z}[\mu_f] \) are given \( \sigma^q \)- and \( \sigma \)-actions in the usual way. The first arrow is multiplication by \( 1 + \sigma^f + \cdots + \sigma^{(q-1)f} \), and the second (which is \( \sigma \)-equivariant) sends \( \sigma^q \rightarrow \zeta_q^{f/q} \). Since this sequence is an exact complex of flat \( \mathbb{Z}[\mu_{f/q}] \)-modules, it remains exact when we tensor with \( \text{Gr}^\sigma_{f/q}(A) \) and hence we obtain a
σ-equivariant exact sequence

\[ 0 \rightarrow \text{Gr}_{f/q}^\sigma(A) \rightarrow \text{Ind}_{\sigma}^\sigma \text{Gr}_{f/q}^\sigma(A) \rightarrow \text{Gr}_{f/q}^\sigma(A) \otimes_{\mathbb{Z}[\mu_{f/q}]} \mathbb{Z}[\mu_f] \rightarrow 0. \]

But we identified \( \text{Gr}_{f/q}^\sigma(\text{Ind}_{\sigma}^\sigma A) \) as the cokernel of the left-hand arrow, so that

\[ \text{Gr}_{f/q}^\sigma(\text{Ind}_{\sigma}^\sigma A) \cong \text{Gr}_{f/q}^\sigma(A) \otimes_{\mathbb{Z}[\mu_{f/q}]} \mathbb{Z}[\mu_f] \] as desired. □

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