ON A RANKIN-SELBERG INTEGRAL OF THE \( L \)-FUNCTION FOR \( \widetilde{SL}_2 \times GL_2 \)

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Abstract. We present a Rankin-Selberg integral on the exceptional group \( G_2 \) which represents the \( L \)-function for generic cuspidal representations of \( SL_2 \times GL_2 \). As an application, we show that certain Fourier-Jacobi type periods on \( G_2 \) are non-vanishing.

1. Introduction

Let \( F \) be a global field with the ring of adeles \( \mathbb{A} \). We assume that the characteristics of \( F \) is not 2. We present in this paper a Shimura type integral on the exceptional group \( G_2 \) which represents the \( L \)-function

\[
L(s, \pi \times (\chi \otimes \tau))L(s, \pi \otimes (\chi \otimes \omega_\tau)),
\]

where \( \pi \) is an irreducible genuine cuspidal representation of \( \widetilde{SL}_2(\mathbb{A}) \), \( \tau \) is an irreducible generic cuspidal representation of \( GL_2(\mathbb{A}) \) and \( \chi \) is the quadratic character of \( F^\times \backslash \mathbb{A}^\times \) defined by \( \chi(a) = \prod_v (a_v, -1)_{F_v} \), where \( a = (a_v)_v \in \mathbb{A}^\times \) and \( (,)_F \) is the Hilbert symbol on \( F_v \).

To give more details about the integral, we introduce some notations. The group \( G_2 \) has two simple roots and we label the short root by \( \alpha \) and the long root by \( \beta \). Let \( P = MV \) (resp. \( P' = M'V' \)) be the maximal parabolic subgroup of \( G_2 \) such that the root space of \( \beta \) is in the Levi \( M \) (resp. the root space of \( \alpha \) is in the Levi \( M' \)). The Levi subgroups \( M \) and \( M' \) are isomorphic to \( GL_2 \).

Let \( J \) be the subgroup of \( P \) which is isomorphic to \( SL_2 \times V \). Let \( \widetilde{SL}_2(\mathbb{A}) \) be the metaplectic double cover of \( SL_2(\mathbb{A}) \). There is a Weil representation \( \omega_\psi \) of \( \widetilde{SL}_2(\mathbb{A}) \) for a nontrivial additive character \( \psi \) of \( F \backslash \mathbb{A} \). Let \( \tilde{\theta}_\phi \) be a corresponding theta series associated with a function \( \phi \in S(\mathbb{A}) \). Let \( \tau \) be an irreducible cuspidal automorphic representations of \( GL_2(\mathbb{A}) \). For \( f_s \in \text{Ind}_{P_0(\mathbb{A})}^{SL_2(\mathbb{A})} (\tau \otimes \delta_{P_0}) \), we can form an Eisenstein series \( E(g, f_s) \) on \( G_2(\mathbb{A}) \). Let \( \pi \) be an irreducible genuine cuspidal automorphic forms of \( \widetilde{SL}_2(\mathbb{A}) \). For a cusp form \( \tilde{\varphi} \in \pi \), we consider the integral

\[
I(\tilde{\varphi}, \phi, f_s) = \int_{SL_2(F) \backslash SL_2(\mathbb{A})} \int_{V(F) \backslash V(\mathbb{A})} \bar{\varphi}(g) \tilde{\theta}_\phi(vg) E(vg, f_s) dv dg.
\]

Our main result is the following

Theorem 1.1. The above integral is absolutely convergent for \( \text{Re}(s) \gg 0 \) and can be meromorphically continued to all \( s \in \mathbb{C} \). When \( \text{Re}(s) \gg 0 \), the integral \( I(\tilde{\varphi}, \phi, f_s) \) is Eulerian. Moreover, at an unramified place \( v \), the local integral represents the \( L \)-function

\[
\frac{L(3s - 1, \pi_v \times (\chi_v \otimes \tau_v))L(6s - 5/2, \pi_v \otimes (\chi_v \otimes \omega_{\tau_v}))}{L(3s - 1/2, \tau_v)L(6s - 2, \omega_{\tau_v})L(9s - 7/2, \tau_v \otimes \omega_{\tau_v})}.
\]

Here \( \chi_v \) is the unramified nontrivial quadratic character of \( F_v^\times \).

This is Theorem 3.1 and Proposition 4.6. We remark that Ginzburg-Rallis-Soudry gave integral representations for \( L \)-functions of generic cuspidal representations of \( \text{Sp}_{2n} \times GL_m \) in \([GRS98]\) using symplectic groups. It is still interesting to have different integral representations. As an application of Theorem 1.1, we show that if \( Wd_{\psi}(\pi) = \chi \otimes \tau \), then a Shimura type period with respect to \( \pi \) and the residue of Eisenstein series on \( G_2 \) is non-vanishing, where \( Wd_{\psi} \) is the Shimura-Waldspurger lift. It is an interesting theme in number theory to investigate the relations between poles of \( L \)-functions and non-vanishing of automorphic periods. There are many examples of this kind relations. See
[JS, Gi93, GRS97] for some examples. The non-vanishing results of automorphic periods have many interesting applications in automorphic forms. We expect the non-vanishing period in our case would be useful on problems related to the residue spectrum of $G_2$.

There are several known Rankin-Selberg integrals on $G_2$ which represents different $L$-functions and have many applications, see [Gi91, Gi93, Gi95] for example. The integral $I(\tilde{\varphi}, \phi, f_s)$ can be viewed as a dual integral of the standard $G_2$ $L$-function integral in [Gi93] in the following sense. The integral $I(\tilde{\varphi}, \phi, f_s)$ is an integral of a triple product of a cusp form on $SL_2(\A)$, a theta series and an Eisenstein series on $G_2(\A)$, while the integral in [Gi93] is an integral of a triple product of a cusp form on $G_2(\A)$, a theta series and an Eisenstein series on $SL_2(\A)$. The integral in [Gi95] is also in a similar pattern, which is an integral of a triple product of a cusp form on $SL_2(\A)$, a theta series and an Eisenstein series on a cover of $G_2(\A)$. The results presented here were known for D. Ginzburg. But we still think that it might be useful to write up the details.

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2. The group $G_2$

2.1. Roots and Weyl group for $G_2$. Let $G_2$ be the split algebraic reductive group of type $G_2$ (defined over $\mathbb{Z}$). The group $G_2$ has two simple roots, the short root $\alpha$ and the long root $\beta$. The set of the positive roots is $\Sigma^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$. Let $\langle , \rangle$ be the inner product in the root system and $\langle , \rangle$ be the pair defined by $\langle \gamma_1, \gamma_2 \rangle = \frac{2 \langle \gamma_1, \gamma_2 \rangle}{\langle \gamma_2, \gamma_2 \rangle}$. For the root space $G_2$, we have the relations:

$$\langle \alpha, \beta \rangle = -1, \langle \beta, \alpha \rangle = -3.$$ 

For a root $\gamma$, let $s_\gamma$ be the reflection defined by $\gamma$, i.e., $s_\gamma(\gamma') = \gamma' - \langle \gamma', \gamma \rangle \gamma$. We have the relation

$$s_\alpha(\beta) = 3\alpha + \beta, s_\beta(\alpha) = \alpha + \beta.$$ 

The Weyl group $W = W(G_2)$ of $G_2$ has 12 elements, which is explicitly given by

$$W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\beta s_\alpha s_\beta, s_\alpha s_\beta s_\alpha, (s_\alpha s_\beta)^2, (s_\beta s_\alpha)^2, s_\beta(s_\alpha s_\beta)^2, s_\alpha(s_\beta s_\alpha)^2, (s_\alpha s_\beta)^3\}.$$ 

For a root $\gamma$, let $U_\gamma \subset G$ be the root space of $\gamma$, and let $x_\gamma : F \to U_\gamma$ be a fixed isomorphism which satisfies various Chevalley relations, see Chapter 3 of [St]. Among other things, $x_\gamma$ satisfies the following commutator relations:

\begin{align*}
[x_\alpha(x), x_\beta(y)] &= x_{\alpha + \beta}(-xy)x_{2\alpha + \beta}(-x^2y)x_{3\alpha + \beta}(x^3y)x_{3\alpha + 2\beta}(-2x^3y^2) \\
[x_\alpha(x), x_{\alpha + \beta}(y)] &= x_{2\alpha + \beta}(-2xy)x_{3\alpha + \beta}(3x^2y)x_{3\alpha + 2\beta}(3xy^2) \\
[x_\beta(x), x_{3\alpha + \beta}(y)] &= x_{3\alpha + 2\beta}(xy) \\
[x_{\alpha + \beta}(x), x_{2\alpha + \beta}(y)] &= x_{3\alpha + 2\beta}(3xy) \\
[& (2.1)] \\
[x_{\alpha + \beta}(x), x_{3\alpha + \beta}(y)] &= x_{3\alpha + 2\beta}(3xy).
\end{align*}

For all the other pairs of positive roots $\gamma_1, \gamma_2$, we have $[x_{\gamma_1}(x), x_{\gamma_2}(y)] = 1$. Here $[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2$ for $g_1, g_2 \in G_2$. For these commutator relationships, see [Re].

Following [St], we denote $w_\gamma(t) = x_{\gamma}(t)x_{-\gamma}(-t^{-1})x_{\gamma}(t)$ and $w_\gamma = w_\gamma(1)$. Note that $w_\gamma$ is a representative of $s_\gamma$. Let $h_\gamma(t) = w_\gamma(t)w_{-1}$. Let $T$ be the subgroup of $G$ which consists of elements of the form $h_\alpha(t_1)\tilde{h}_\beta(t_2), t_1, t_2 \in T$ and $U$ be the subgroup of $G_2$ generated by $U_\gamma$ for all $\gamma \in \Sigma^+$. Let $B = TU$, which is a Borel subgroup of $G_2$. 
For \( t_1, t_2 \in \mathbb{C}_m \), denote \( h(t_1, t_2) = h_\alpha(t_1 t_2)h_\beta(t_1^2 t_2). \) From the Chevalley relation \( h_{\gamma_1}(t) \tau_{\gamma_1}(r)h_{\gamma_1}(t)^{-1} = \tau_{\gamma_2}(t^{(\gamma_2 \gamma_1)} r) \) (see [St, Lemma 20, (c)]), we can check the following relations

\[
\begin{align*}
    h^{-1}(t_1, t_2)\tau_\alpha(r)h(t_1, t_2) &= \tau_\alpha(t_2^{-1} r), \\
    h^{-1}(t_1, t_2)\tau_\beta(r)h(t_1, t_2) &= \tau_\beta(t_1^{-1} t_2 r), \\
    h^{-1}(t_1, t_2)\tau_{\alpha + \beta}(r)h(t_1, t_2) &= \tau_{\alpha + \beta}(t_1^{-1} r), \\
    h^{-1}(t_1, t_2)\tau_{2\alpha + \beta}(r)h(t_1, t_2) &= \tau_{2\alpha + \beta}(t_1^{-1} t_2^{-1} r), \\
    h^{-1}(t_1, t_2)\tau_{3\alpha + 2\beta}(r)h(t_1, t_2) &= \tau_{3\alpha + 2\beta}(t_1^{-1} t_2^{-1} r).
\end{align*}
\]  

(2.2)

Thus the notation \( h(a, b) \) agrees with that of [Gi93].

One can also check that \( w_\alpha h(t_1, t_2)w_\alpha^{-1} = h(t_1 t_2, t_2^{-1}), \quad w_\beta h(t_1, t_2)w_\beta^{-1} = h(t_2, t_1). \)

2.2. Subgroups. Let \( F \) be a field and denote \( G = G_2(F) \). The group \( G \) has two proper parabolic subgroups. Let \( P = M \times V \) be the parabolic subgroup of \( G \) such that \( U_\beta \subset M \cong \text{GL}_2 \). Thus the unipotent subgroup \( V \) consists of root spaces of \( \alpha, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta \), and a typical element of \( V \) is of the form

\[
\tau_\alpha(r_1)\tau_{2\alpha + \beta}(r_2)\tau_{3\alpha + 2\beta}(r_3)\tau_{3\alpha + 3\beta}(r_4)\tau_{3\alpha + 4\beta}(r_5), \quad r_i \in F.
\]

To ease the notation, we will write the above element as \([r_1, r_2, r_3, r_4, r_5]\). Denote by \( J \) the following subgroup of \( P \)

\[
J = \text{SL}_2(F) \ltimes V.
\]

Let \( V_1 \) (resp. \( Z \)) be the subgroup of \( V \) which consists root spaces of \( 3\alpha + \beta \) and \( 3\alpha + 2\beta \) (resp. \( 2\alpha + \beta, 3\alpha + \beta \) and \( 3\alpha + 2\beta \)). Note that \( P \) and hence \( J \) normalizes \( V_1 \) and \( Z \). We will always view \( \text{SL}_2(F) \) as a subgroup of \( G \) via the inclusion \( \text{SL}_2(F) \subset M \). Denote by \( A_{\text{SL}_2}, \ N_{\text{SL}_2} \) and \( B_{\text{SL}_2} \) the standard torus, the upper triangular unipotent subgroup and the upper triangular Borel subgroup of \( \text{SL}_2(F) \). Note that the torus element \( h(a, b) \) can be identified with

\[
\begin{pmatrix} a \\ b \end{pmatrix} \in \text{GL}_2(F) \cong M,
\]

and thus \( A_{\text{SL}_2} = \{ h(a, a^{-1}) | a \in F^\times \} \) and \( B_{\text{SL}_2} = A_{\text{SL}_2} \times U_\beta \).

Let \( P' = M' \ltimes V' \) be the other maximal parabolic subgroups of \( G \) with \( U_\alpha \) in the Levi subgroup \( M' \). The Levi \( M' \) is isomorphic to \( \text{GL}_2(F) \), and from relations in \( (2.2) \), one can check that one isomorphism \( M' \cong \text{GL}_2(F) \) can be determined by

\[
\begin{align*}
    \tau_\alpha(r) &\mapsto \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \\
    h(a, b) &\mapsto \begin{pmatrix} ab & 1 \\ 0 & a \end{pmatrix}.
\end{align*}
\]

In particular, we see that \( h(a, 1) \in T \subset M' \) can be identified with \( \text{diag}(a, a) \). Let \( \delta_{P'} \) be the modulus character of \( P' \). One can check that \( \delta_{P'}(m') = |\det(m')|^3 \) for \( m' \in M' \), where \( \det(m') \) can be computed using the above isomorphism \( M' \cong \text{GL}_2(F) \).

2.3. Weil representation of \( \tilde{\text{SL}}_2(\mathbb{A}) \ltimes V(\mathbb{A}) \). In this subsection, we assume that \( F \) is a global field and \( \mathbb{A} \) is its ring of adeles. In \( \text{SL}_2(F) \), we denote \( t(a) = \text{diag}(a, a^{-1}), a \in F^\times \) and

\[
n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \in F.
\]

Denote \( w^1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \), which represents the unique nontrivial Weyl element of \( \text{SL}_2(F) \). Under the embedding \( \text{SL}_2(F) \subset M \subset G \), the element \( w^1 \) can be identified with \( w_\beta \).
Let $\widetilde{SL}_2(\mathbb{A})$ be the metaplectic double cover of $SL_2(\mathbb{A})$. Then we have an exact sequence
\[0 \to \mu_2 \to \widetilde{SL}_2(\mathbb{A}) \to SL_2(\mathbb{A}) \to 0,\]
where $\mu_2 = \{\pm 1\}$.

We will identify $SL_2(\mathbb{A})$ with the symplectic group of $\mathbb{A}^2$ with symplectic structure defined by
\[
\langle (x_1, y_1), (x_2, y_2) \rangle = -2x_1y_2 + 2x_2y_1.
\]

Let $\mathcal{H}(\mathbb{A})$ be the Heisenberg group of the symplectic space $(\mathbb{A}^2, \langle , \rangle)$, i.e., $\mathcal{H}(\mathbb{A}) = \mathbb{A}^3$ with group law
\[
(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 - x_1y_2 + y_1x_2).
\]

Let $SL_2(\mathbb{A})$ act on $\mathcal{H}(\mathbb{A})$ from the right side by
\[
(x_1, y_1, z_1)g = ((x_1, y_1)g, z_1), g \in SL_2(\mathbb{A}),
\]
where $(x_1, y_1)g$ is the usual matrix multiplication.

We then can form the semi-direct product $SL_2(\mathbb{A}) \ltimes \mathcal{H}(\mathbb{A})$, where the product is defined by
\[
(g_1, h_1)(g_2, h_2) = (g_1g_2, (h_1g_2)h_2), g_1, g_2 \in SL_2(\mathbb{A}), h_1, h_2 \in \mathcal{H}(\mathbb{A}), i = 1, 2.
\]

Let $\psi$ be a nontrivial additive character of $F^\times \mathbb{A}$. Then there is a Weil representation $\omega_\psi$ of $\widetilde{SL}_2(\mathbb{A}) \ltimes \mathcal{H}(\mathbb{A})$. The space of $\omega_\psi$ is $S(\mathbb{A})$, the Bruhat-Schwartz functions on $\mathbb{A}$.

For $\phi \in S(\mathbb{A})$, we have the well-know formulas:
\[
\omega_\psi(n(b))\phi(x) = \psi(bx^2)\phi(x), b \in \mathbb{A}
\]
\[
(\omega_\psi(r_1, r_2, r_3)\phi)(x) = \psi(r_3 - 2xr_2 - r_1r_2)\phi(x + r_1), (r_1, r_2, r_3) \in \mathcal{H}(\mathbb{A}).
\]

The above formulas could be found in [Ku].

Recall that for $r_1, r_2, r_3, r_4, r_5 \in \mathbb{A}$, the notation $[r_1, r_2, r_3, r_4, r_5]$ is an abbreviation of
\[
x_{\alpha}(r_1)x_{\alpha + \beta}(r_2)x_{\alpha + 2\beta}(r_3)x_{3\alpha + \beta}(r_4)x_{3\alpha + 2\beta}(r_5) \in V(\mathbb{A}).
\]

Define a map $\text{pr} : V(\mathbb{A}) \to \mathcal{H}(\mathbb{A})$
\[
\text{pr}([r_1, r_2, r_3, r_4, r_5]) = (r_1, r_2, r_3 - r_1r_2).
\]

From the commutator relation (2.1), we can check that $\text{pr}$ is a group homomorphism and defines an exact sequence
\[0 \to V_1(\mathbb{A}) \to V(\mathbb{A}) \to \mathcal{H}(\mathbb{A}) \to 0.
\]

Recall that $V_1$ is the subgroup of $V$ which is generated by the root space of $3\alpha + \beta, 3\alpha + 2\beta$. Note that there is a typo in the formula of the projection map $\text{pr}$ in [Gi93, p.316].

For $g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(F) \subset M$, we can check that
\[
g^{-1}[r_1, r_2, r_3, 0, 0]g = [r_1', r_2', r_3', r_4', r_5'],
\]
where $r_1' = ar_1 - cr_2, r_2' = -br_1 + dr_2, r_3' = r_3 - r_1r_2$.

Consider the map $\text{pr} : J(\mathbb{A}) = SL_2(\mathbb{A}) \times V(\mathbb{A}) \to SL_2(\mathbb{A}) \ltimes \mathcal{H}(\mathbb{A})$
\[
(g, v) \mapsto (g^*, \text{pr}(v)), g \in SL_2(\mathbb{A}), v \in V(\mathbb{A}).
\]

where $g^* = \left( \begin{array}{cc} a & -b \\ -c & d \end{array} \right) = d_1gd_1^{-1}$, where $d_1 = \text{diag}(1, -1) \in \text{GL}_2(F)$. From the above discussion, the map $\text{pr}$ is a group homomorphism and its kernel is also $V_1(\mathbb{A})$. We will also view $\text{pr}$ as a homomorphism $\widetilde{SL}_2(\mathbb{A}) \ltimes V(\mathbb{A}) \to \widetilde{SL}_2(\mathbb{A}) \ltimes \mathcal{H}(\mathbb{A})$.

In the following, we will also view $\omega_\psi$ as a representation of $\widetilde{SL}_2(\mathbb{A}) \ltimes V(\mathbb{A})$ via the projection map $\text{pr}$. For $\phi \in S(\mathbb{A})$, we form the theta series
\[
\tilde{\theta}_{\phi}(vg) = \sum_{\xi \in F} \omega_\psi(vh)\phi(\xi), v \in V(\mathbb{A}), g \in \widetilde{SL}_2(\mathbb{A}).
\]

Note that given a genuine cusp form $\tilde{\varphi}$ on $\widetilde{SL}_2(\mathbb{A})$, the product
\[
\tilde{\varphi}(g)\tilde{\theta}_{\phi}(vg), v \in V(\mathbb{A}), g \in \widetilde{SL}_2(\mathbb{A})
\]
can be viewed as a function on \( J(\mathbb{A}) = SL_2(\mathbb{A}) \ltimes V(\mathbb{A}) \).

2.4. **An Eisenstein series on** \( G_2 \). Let \( \tau \) be a cuspidal automorphic representation on \( GL_2(\mathbb{A}) \).

We will view \( \tau \) as a representation of \( M'(\mathbb{A}) \) via the identification \( M' \cong GL_2 \). We then consider the induced representation \( I(s, \tau) = Ind_{M'(\mathbb{A})}^{G_2(\mathbb{A})} (\tau \otimes \delta_P) \). A section \( f_s \in I(s, \tau) \) is a smooth function satisfying

\[
f_s(v'm'g) = \delta_P(m')^{s} f_s(g), \forall v' \in V'(\mathbb{A}), m' \in M'(\mathbb{A}), g \in G_2(\mathbb{A}).
\]

For \( f_s \in I(s, \tau) \), we consider the Eisenstein series

\[
E(g, f_s) = \sum_{\delta \in P'(F) \backslash G_2(F)} f_s(\delta g), g \in G_2(\mathbb{A}).
\]

3. **A global integral**

Let \( \tilde{\pi} \) be a genuine cuspidal automorphic representation on \( \tilde{SL}_2(\mathbb{A}) \), and \( \tau \) be a cuspidal automorphic representation of \( GL_2(\mathbb{A}) \). For \( \tilde{\varphi} \in \tilde{V}_\tau, \phi \in S(\mathbb{A}) \) and \( f_s \in I(s, \tau) \), we consider the integral

\[
I(\tilde{\varphi}, \phi, f_s) = \int_{SL_2(\mathbb{A}) \backslash SL_2(\mathbb{A})} \int_{U_{\alpha, \beta}(\mathbb{A}) \backslash V(\mathbb{A})} \tilde{\varphi}(g) \tilde{\theta}_\phi(vg) E(vg, f_s) dv dg.
\]

Let \( \gamma = w_\beta w_\alpha w_\beta w_\alpha \in G_2(\mathbb{F}) \).

**Theorem 3.1.** The integral \( I(\tilde{\varphi}, \phi, f_s) \) is absolutely convergent when \( \Re(s) \gg 0 \) and can be meromorphically continued to all \( s \in \mathbb{C} \). Moreover, when \( \Re(s) > 0 \), we have

\[
I(\tilde{\varphi}, \phi, f_s) = \sum_{\delta \in P'(F) \backslash G_2(F)} \int_{\tilde{SL}_2(\mathbb{F}) \backslash SL_2(\mathbb{A})} \int_{V^\delta(\mathbb{F}) \backslash V(\mathbb{A})} \tilde{\varphi}(g) \tilde{\theta}_\phi(vg) f_s(\delta vg) dv dg,
\]

where \( W_{\tilde{\varphi}}(g) = \int_{F \backslash \mathbb{A}} \tilde{\varphi}(x_{\beta}(r)g) \psi(r) dr \), and

\[
W_{f_s}(\gamma vg) = \int_{F \backslash \mathbb{A}} f_s(x_{\alpha}(r)\gamma vg) \psi(-2r) dr.
\]

**Proof.** The first assertion is standard. We only show that the above integral is Eulerian when \( \Re(s) > 0 \). Unfolding the Eisenstein series, we can get

\[\begin{align*}
I(\tilde{\varphi}, \phi, f_s) &= \sum_{\delta \in P'(F) \backslash G_2(F) \cap P(F)} \int_{\tilde{SL}_2(\mathbb{F}) \backslash SL_2(\mathbb{A})} \int_{V^\delta(\mathbb{F}) \backslash V(\mathbb{A})} \tilde{\varphi}(g) \tilde{\theta}_\phi(vg) f_s(\delta vg) dv dg,
\end{align*}\]

where \( X^\delta = \delta^{-1} P \delta \cap X \) for \( X \subset G_2(\mathbb{F}) \). We can check that a set of representatives of the double coset \( P'(\mathbb{F}) \backslash G_2(\mathbb{F}) / P(\mathbb{F}) \) can be taken as \( \{1, w_\beta w_\alpha, \gamma = w_\beta w_\alpha w_\beta w_\alpha\} \). For \( \delta = 1, w_\beta w_\alpha \), or \( \gamma = w_\beta w_\alpha w_\beta w_\alpha \), denote

\[\begin{align*}
I_\delta &= \int_{\tilde{SL}_2(\mathbb{F}) \backslash SL_2(\mathbb{A})} \int_{V^\delta(\mathbb{F}) \backslash V(\mathbb{A})} \tilde{\varphi}(g) \tilde{\theta}_\phi(vg) f_s(\delta vg) dv dg.
\end{align*}\]

If \( \delta = 1 \), the above integral \( I_1 \) has an inner integral

\[
\int_{U_{2\alpha + \beta}(\mathbb{F}) \backslash U_{2\alpha + \beta}(\mathbb{A})} \tilde{\theta}_\phi(x_{2\alpha + \beta}(r)vg) f_s(x_{2\alpha + \beta}(r)vg) dr,
\]

which is zero because \( f_s(x_{2\alpha + \beta}(r)vg) = f_s(vg), \tilde{\theta}_\phi(x_{2\alpha + \beta}(r)vg) = \psi(r) \tilde{\theta}_\phi(vg) \) and \( \int_{F \backslash \mathbb{A}} \psi(r) dr = 0 \). The last equation follows from the fact that \( \psi \) is non-trivial.

We next consider the term when \( \delta = w_\beta w_\alpha \). We write

\[
\tilde{\theta}_\phi(vg) = \omega_\psi(vg) \phi(0) + \sum_{\xi \in F^*} \omega_\psi(vg) \phi(\xi).
\]

The contribution of the first term to the integral \( I_\delta \) is

\[
\int_{\tilde{SL}_2(\mathbb{F}) \backslash SL_2(\mathbb{A})} \int_{V^\delta(\mathbb{F}) \backslash V(\mathbb{A})} \tilde{\varphi}(g) \omega_\psi(vg) \phi(0) f_s(\delta vg) dv dg.
\]
Note that $\delta x_\beta(r)\delta^{-1} \subset U_{2\alpha+\beta} \subset V'$, we have $f_s(\delta x_\beta(r)g) = f_s(\delta x_\beta(-r)v x_\beta(r)g)$. On the other hand, we have $\omega_\psi(x_\beta(r)vg)\phi(0) = \omega_\psi(vg)\phi(0)$. After a changing variable on $v$, we can see that the above integral contains an inner integral
\[ \int_{F \setminus A} \bar{\varphi}(x_\beta(r)vg) dr, \]
which is zero since $\bar{\varphi}$ is cuspidal. Thus the contribution of the term $\omega_\psi(vg)\phi(0)$ is zero when $\delta = w_\beta w_a$. The contribution of $\sum_{\xi \in F^\times} \omega_\psi(vg)\phi(\xi)$ is
\[ \int_{SL_2(F) \setminus SL_2(A)} \int_{V^\times(F) \setminus V(A)} \bar{\varphi}(g) \sum_{\xi \in F^\times} \omega_\psi(vg)\phi(\xi) f_s(\delta vg) dvg. \]
We consider the inner integral on $U_{\alpha+\beta}(F) \setminus U_{\alpha+\beta}(A)$. Note that $U_{\alpha+\beta} \subset V$ and $\delta U_{\alpha+\beta} \delta^{-1} = U_{2\alpha+\beta} \subset V'$, we get $f_s(\delta x_{\alpha+\beta}(r)vg) = f_s(\delta vg)$. On the other hand, we have $\omega_\psi(x_{\alpha+\beta}(r)vg)\phi(\xi) = \psi(-2r\xi)\omega_\psi(vg)\phi(\xi)$. Thus the above integral has an inner integral
\[ \int_{F \setminus A} \int_{\xi \in F^\times} \psi(-2r\xi)\omega_\psi(vg)\phi(\xi) dr \]
Thus when $\delta = w_\beta w_a$, the corresponding term is zero. Thus we get
\[ I(\bar{\varphi}, \phi, f_s) = \int_{SL_2(F) \setminus SL_2(A)} \int_{V^\times(F) \setminus V(A)} \bar{\varphi}(g) \tilde{\theta}_\phi(vg) f_s(\gamma vg) dvg. \]
We have $SL_2^* = B_{SL_2}$ and $V^\gamma = U_{\alpha+\beta}$. We decompose $\tilde{\theta}_\phi$ as
\[ \tilde{\theta}_\phi(vg) = \omega_\psi(vg)\phi(0) + \sum_{\xi \in F^\times} \omega_\psi(vg)\phi(\xi) = \omega_\psi(vg)\phi(0) + \sum_{\alpha \in F^\times} \omega_\psi(t(a)vg)\phi(1). \]
Recall that $t(a) = \text{diag}(a, a^{-1}).$ Since $\gamma U_{\beta}\gamma^{-1} \subset U_{3\alpha+\beta} \subset V'$, we have
\[ f_s(\gamma v x_\beta(r)g) = f_s(\gamma x_\beta(-r) v x_\beta(r)g). \]
On the other hand we have $\omega_\psi(v x_\beta(r)g)\phi(0) = \omega_\psi(x_\beta(-r) v x_\beta(r)g)\phi(0)$. Thus after a changing variable on $v$, we can get that the contribution of $\omega_\psi(vg)\phi(0)$ to $I(\bar{\varphi}, \phi, f_s)$ has an inner integral
\[ \int_{F \setminus A} \bar{\varphi}(x_\beta(r)g) dr, \]
which is zero by the cuspidality of $\bar{\varphi}$. Thus we get
\[ I(\bar{\varphi}, \phi, f_s) = \int_{B_{SL_2}(F) \setminus SL_2(A)} \int_{U_{\alpha+\beta}(F) \setminus V(A)} \bar{\varphi}(g) \sum_{\alpha \in F^\times} \omega_\psi(t(a)vg)\phi(1) f_s(\gamma vg) dvg. \]
Collapsing the summation with the integration, we then get
\[ I(\bar{\varphi}, \phi, f_s) \]

\[ = \int_{N_{SL_2}(F) \setminus SL_2(A)} \int_{U_{\alpha+\beta}(F) \setminus V(A)} \bar{\varphi}(g) \omega_\psi(vg)\phi(1) f_s(\gamma vg) dvg \]
\[ = \int_{N_{SL_2}(A) \setminus SL_2(A)} \int_{U_{\alpha+\beta}(A) \setminus V(A)} \int_{F \setminus A} \bar{\varphi}(x_\beta(r)g) \omega_\psi(v x_\beta(r)g)\phi(1) f_s(\gamma v x_\beta(r)g) drdvg. \]
Note that we have $\omega_\psi(v x_\beta(r)g)\phi(1) = \omega_\psi(x_\beta(r) v x_\beta(-r) v x_\beta(r)g)\phi(1) = \psi(r)\omega_\psi(x_\beta(-r) v x_\beta(r)g)\phi(1).$ On the other hand, we have $\gamma x_\beta(r)\gamma^{-1} \subset U_{3\alpha+\beta} \subset V'$. Thus $f_s(\gamma v x_\beta(r)g) = f_s(\gamma x_\beta(-r) v x_\beta(r)g).$ After a changing of variable on $v$, we get
\[ I(\bar{\varphi}, \phi, f_s) = \int_{N_{SL_2}(A) \setminus SL_2(A)} \int_{U_{\alpha+\beta}(F) \setminus V(A)} W_{\bar{\varphi}}(g) \omega_\psi(vg)\phi(1) f_s(\gamma vg) dvg, \]
where
\[ W_{\bar{\varphi}}(g) = \int_{F \setminus A} \bar{\varphi}(x_\beta(r)g) \psi(r) dr. \]
We can further decompose the above integral as

\[ I(\bar{f}, \phi, f_s) = \int_{N_{\tilde{S}L_2}(\mathfrak{A})/SL_2(\mathfrak{A})} \int_{U_{\alpha + \beta}(\mathfrak{A})/V(\mathfrak{A})} W_{\bar{f}}(g) \omega_\psi(x_{\alpha + \beta}(r)vg) \phi(1)f_s(\gamma x_{\alpha + \beta}(r)vg) \, dr \, dv. \]

Note that \( \omega_\psi(x_{\alpha + \beta}(r)vg) \phi(1) = \psi(-2r)\omega_\psi(vg) \phi(1) \) and \( f_s(\gamma x_{\alpha + \beta}(r)vg) = f_s(x_{\alpha}(r)\gamma vg) \) since \( \gamma x_{\alpha + \beta}(r) \gamma^{-1} = x_{\alpha}(r). \) We then get

\[ I(\bar{f}, \phi, f_s) = \int_{N_{\tilde{S}L_2}(\mathfrak{A})/SL_2(\mathfrak{A})} W_{\bar{f}}(g) \omega_\psi(vg) \phi(1)W_{f_s}(\gamma vg) \, dv \, gdg. \]

where

\[ W_{f_s}(\gamma vg) = \int_{F\setminus A} f_s(x_{\alpha}(r)\gamma vg) \psi(-2r) \, dr. \]

This concludes the proof. \( \square \)

### 4. Unramified calculation

In this section, let \( F \) be a \( p \)-adic field with \( p \neq 2 \). Let \( \mathfrak{o} \) be the ring of integers of \( F \), and let \( p \) be a uniformizer of \( \mathfrak{o} \) by abuse of notation. Let \( q \) be the cardinality of the residue field \( \mathfrak{o}/(p) \).

#### 4.1. Local Weil representations

Let \( \psi \) be an additive character of \( F \) and let \( \gamma(\psi) \) be the Weil index and let \( \mu_\psi(a) = \frac{\gamma(\psi)}{\gamma(\psi_2)}. \) Let \( \omega_\psi \) be the Weil representation of \( \tilde{S}L_2(F) \ltimes V \) on \( S(F) \) via the projection \( \tilde{S}L_2(F) \ltimes V \to \tilde{S}L_2(F) \ltimes \mathcal{H}. \) For \( \phi \in S(F) \), we have the well-know formulas:

\[
\begin{align*}
(\omega_\psi(w^1)\phi)(x) &= \gamma(\psi)\hat{\phi}(x), \\
(\omega_\psi(n(b))\phi)(x) &= \psi(bx^2)\phi(x), \quad b \in F \\
(\omega_\psi(t(a))\phi)(x) &= |a|^{1/2}\mu_\psi(a)\phi(ax), \quad a \in F^\times \\
(\omega_\psi((v_1, v_2))\phi)(x) &= \psi(r_3 - 2xv_2 - r_1v_3)\phi(x + r_1), \quad (r_1, r_2, r_3) \in \mathcal{H}(F).
\end{align*}
\]

where \( \hat{\phi}(x) = \int_{F} \phi(y)\psi(2xy) \, dy \) is the Fourier transform of \( \phi \) with respect to \( \psi \). Note that under the embedding \( S\tilde{L}_2(F) \to G_2(F) \), we have \( w^1 = w_\beta, n(b) = x_\beta(b) \) and \( t(a) = h(a, a^{-1}) \).

#### 4.2. Unramified calculation

In this subsection, we compute the local integral in last section. The strategy is similar as the unramified calculation in [Gi95].

Let \( \pi \) be an unramified genuine representation of \( S\tilde{L}_2(F) \) with Satake parameter \( a \), and let \( \tau \) be an unramified irreducible representation of \( GL_2(F) \) with Satake parameters \( b_1, b_2 \). Let \( \bar{W} \in \mathcal{W}(\pi, \psi) \) with \( \bar{W}(1) = 1 \). Let \( v_0 \in V_\tau \) be an unramified vector and \( \lambda \in \text{Hom}_\mathbb{C}(V_\tau, \psi) \) such that \( \lambda(v_0) = 1 \). Let \( f_s : G_2 \to V_\tau \) be the unramified section in \( I(s, \tau) \) with \( f_s(e) = v_0 \). Let

\[ W_{f_s} : G_2 \times GL_2(F) \to \mathbb{C} \]

be the function \( W_{f_s}(g, a) = \lambda(\tau(a)f_s(g)) \). We will write \( W_{f_s}(g) \) for \( W_{f_s}(g, 1) \) in the following. By assumption and Shintani formula, we have

\[
W_{f_s}(h(p^k, p^l)) = q^{-3s(2k+l)}\lambda(\tau(\text{diag}(p^{k+l}, p^k)v_0))
= q^{-3s(2k+l)}W_{v_0}(\text{diag}(p^{k+l}, p^k))
= \begin{cases}
q^{-3s(2k+l)}\frac{(b_1b_2)^{k+l}}{b_1^{k+l}b_2^{k+l}}(b_1^{k+l} - b_2^{k+l}), & \text{if } l \geq 0, \\
0, & \text{if } l < 0.
\end{cases}
\]

Let \( \phi \in S(F) \) be the characteristic function of \( \mathfrak{o} \). We need to compute the integral

\[ I(\bar{W}, W_{f_s}, \phi) = \int_{N_{\tilde{S}L_2}(F)\setminus \tilde{S}L_2(F)} \int_{U_{\alpha + \beta} \setminus V} \bar{W}(g)\omega_\psi(vg)\phi(1)W_{f_s}(\gamma vg) \, dv \, gdg. \]

In the following, we fix the Haar measure such that \( \text{vol}(dr, \mathfrak{o}) = 1 \). Thus \( \text{vol}(d^*r, \mathfrak{o}^*) = 1 - q^{-1} \).
Using the Iwasawa decomposition $\text{SL}_2(F) = N_2(F)A_2(F)\text{SL}_2(\mathfrak{o})$, we have

$$
I(\tilde{W}, W_f, \phi) = \int_{F^\times} \int_{F^3} \tilde{W}(t(a))\omega_\psi([r_1, 0, r_3]t(a))\phi(1)W_f(\gamma(r_1, 0, r_3, r_4, r_5)t(a))|a|^{-2}dr_3 dr_4 dr_5 d^\times a
$$

$$
= \int_{F^\times} \int_{F^3} \tilde{W}(t(a))\omega_\psi(t(a)[r_1, 0, r_3])\phi(1)W_f(\gamma(t(a)(r_1, 0, r_3, r_4, r_5))|a|^{-3}dr_3 dr_4 dr_5 d^\times a
$$

If $\tilde{W}(t(a)) \neq 0$, then $|a| \leq 1$. On the other hand, we have

$$
\omega_\psi(t(a)[r_1, 0, r_3])\phi(1) = \mu_\psi(a)|a|^{1/2}\psi(r_3)\phi(a + r_1).
$$

If $\phi(a + r_1) \neq 0$ and $a \in \mathfrak{o}$, then $r_1 \in \mathfrak{o}$. Thus the domain for $a$ and $r_1$ in the above integral is $\{a \in F^\times \cap \mathfrak{o}, r_1 \in \mathfrak{o}\}$. Note that $\gamma(t(a) = h(1, a)\gamma = h(1, a)w_\beta w_\alpha w_\alpha$. Thus, if we conjugate $w_\alpha x_\alpha(r_1)$ to the right side, we can get

$$
h(1, a)\gamma[r_1, 0, r_3, r_4, r_5] = h(1, a)w_\beta w_\alpha w_\beta x_\alpha + (r_3)x_\beta(-r_4 - 3r_1 r_3)x_3 + 2\beta(r_5)w_\alpha x_\alpha(r_1).
$$

Since $w_\alpha x_\alpha(r_1) \in K$ for $r_1 \in \mathfrak{o}$, by changing of variables, we get

$$
I(\tilde{W}, W_f, \phi) = \int_{|a|\leq 1} \tilde{W}(t(a))|a|^{-5/2}\mu_\psi(a)
$$

$$
\cdot \int_{F^3} W_f(h(1, a)w_\beta w_\alpha w_\beta x_\alpha + (r_3)x_\beta(r_4)x_3 + 2\beta(r_5))\psi(-r_3)dr_3 dr_4 dr_5 d^\times a
$$

$$
= \sum_{n \geq 0} \tilde{W}(t(p^n))q^{5n/2}\mu_\psi(p^n)J(n),
$$

where

$$
J(n) = \int_{F^3} W_f(h(1, p^n)w_\beta w_\alpha w_\beta x_\alpha + (r_3)x_\beta(r_4)x_3 + 2\beta(r_5))\psi(-r_3)dr_3 dr_4 dr_5.
$$

By dividing the domain of $r_3$ into two parts, we can write $J(n) = J_1(n) + J_2(n)$, where

$$
J_1(n) = \int_{|r_3|\leq 1} \int_{F^2} W_f(h(1, p^n)w_\beta w_\alpha w_\beta x_\alpha + (r_3)x_\beta(r_4)x_3 + 2\beta(r_5))\psi(-r_3)dr_3 dr_4 dr_5
$$

$$
= \int_{F^2} W_f(h(1, p^n)w_\beta w_\alpha w_\beta x_\beta(r_4)x_3 + 2\beta(r_5))dr_4 dr_5,
$$

and

$$
J_2(n) = \int_{|r_3|> 1} \int_{F^2} W_f(h(1, p^n)w_\beta w_\alpha w_\beta x_\alpha + (r_3)x_\beta(r_4)x_3 + 2\beta(r_5))\psi(-r_3)dr_3 dr_4 dr_5.
$$

**Lemma 4.1.** Set

$$
I(n) = \int_F W_f(h(1, p^n)w_\beta x_\beta(r))dr.
$$

Then

$$
I(n) = \frac{q^{-3(n+1/2)}}{b_1 - b_2}\left\{[b_1^{n+1} - b_2^{n+1}] +(1 - q^{-1})\frac{b_1 b_2 X}{(1 - b_1 X)(1 - b_2 X)}(b_1^n - b_2^n - b_1^{n+1}X + b_2^{n+1}X + b_1 X(b_1b_2 X)^n - b_2 X(b_1b_2 X)^n)\right\},
$$

where $X = q^{-(3n-3/2)}$. 
Proof. We have
\[ I(n) = \int_F W_{f_\tau}(h(1,p^n)w_{\beta}x_\beta(r))dr \]
\[ = \int_{|r|\leq 1} W_{f_\tau}(h(1,p^n)w_{\beta}x_\beta(r))dr \]
\[ + \int_{|r|> 1} W_{f_\tau}(h(1,p^n)w_{\beta}x_\beta(r))dr \]
\[ = W_{f_\tau}(h(1,p^n)) + \int_{|r|> 1} W_{f_\tau}(h(1,p^n)w_{\beta}x_\beta(r))dr. \]

To deal with the integral when $|r| > 1$, we consider the following Iwasawa decomposition of $w_{\beta}x_\beta(r)$:

\[ w_{\beta}x_\beta(r) = x_\beta(-r^{-1})h(-r^{-1}, -r)x_{-\beta}(r^{-1}). \]

Since $x_{-\beta}(r^{-1})$ is in the maximal compact subgroup for $|r| > 1$, we have

\[ W_{f_\tau}(h(1,p^n)w_{\beta}x_\beta(r)) = W_{f_\tau}(h(1,p^n)x_{-\beta}(r^{-1})h(-r^{-1}, -r)) = W_{f_\tau}(h(1,p^n)h(r^{-1}, r)), \]

where we used $U_\beta \subset V'$. For $|r| > 1$, we can write $r = p^{-m}u$ for some $m \geq 1$ and $u \in \mathfrak{o}^\times$. We then have $dr = q^m du$. Note that $\text{vol}(\mathfrak{o}^\times) = 1 - q^{-1}$. Thus we have

\[ I(n) = W_{f_\tau}(h(1,p^n)) + \sum_{m \geq 1} (1 - q^{-1})q^m W_{f_\tau}(h(p^m,p^{n-m})). \]

Note that $h(p^m,1) \mapsto \text{diag}(p^m,p^m)$ under the isomorphism $M' \cong \text{GL}_2$. Thus we have

\[ W_{f_\tau}(h(p^m,1)h(1,p^{n-m})) = q^{-6sm} \omega_r(p)^m W_{f_\tau}(h(1,p^{n-m})). \]

Thus we get

\[ I(n) = W_{f_\tau}(h(1,p^n)) + \sum_{m \geq 1} (1 - q^{-1})q^{(-6s+1)m} \omega_r(p)^m W_{f_\tau}(h(1,p^{n-m})). \]

By (4.1), we have

\[ W_{f_\tau}(h(1,p^{n-m})) = \begin{cases} q^{-(3s+1)/2} b_{1-b_2}^n (b_1^{n-m+1} - b_2^{n-m+1}), & \text{if } n \geq m, \\ 0, & \text{if } n < m. \end{cases} \]

Thus for $n \geq 1$, we have

\[ I(n) = \frac{q^{-(3s+1)/2} n b_2}{b_1-b_2} \left( b_1^{n+1} - b_2^{n+1} \right) + \sum_{m=1}^{n} (1 - q^{-1})q^{-(3s+3/2)m} (b_1^{n+1} b_2^m - b_2^{n+1} b_1^m). \]

Thus result can be computed using the geometric summation formula. One can check that the given formula also satisfies $I(0) = 1$. \qed

Lemma 4.2. We have

\[ J_1(n) = \frac{1 - q^{-6s+1}b_1b_2}{1 - q^{-6s+2}b_1b_2} I(n). \]

Proof. To compute $J_1(n)$, we break up the domain of integration in $r_4$ and get

\[ J_1(n) = \int_F \int_{|r_4| \leq 1} W_{f_\tau}(h(1,p^n)w_{\alpha}w_{\beta}x_\beta(r_4)x_{3\alpha+2\beta}(r_5))dr_4dr_5 \]
\[ + \int_F \int_{|r_4| > 1} W_{f_\tau}(h(1,p^n)w_{\alpha}w_{\beta}x_\beta(r_4)x_{3\alpha+2\beta}(r_5))dr_4dr_5 \]
\[ := J_{11}(n) + J_{12}(n), \]
where

\[ J_{11}(n) = \int_F \int_{|r_4| \leq 1} W_{f_4}(h(1, p^n)w_\beta w_\alpha w_\beta x_\beta(r_4) x_{3\alpha+2\beta}(r_5)) dr_4 dr_5 \]
\[ = \int_F \int_{|r_4| \leq 1} W_{f_4}(h(1, p^n)w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(r_5) w_\beta^{-1} w_\alpha^{-1} w_\alpha w_\beta x_\beta(r_4)) dr_4 dr_5 \]
\[ = \int_F W_{f_4}(h(1, p^n)w_\beta x_\beta(r_5)) dr_5 \]
\[ = I(n), \]
and

\[ J_{12}(n) = \int_F \int_{|r_4| > 1} W_{f_4}(h(1, p^n)w_\beta w_\alpha w_\beta x_\beta(r_4) x_{3\alpha+2\beta}(r_5)) dr_4 dr_5 \]
\[ = \int_F \int_{|r_4| > 1} W_{f_4}(h(1, p^n)w_\beta w_\alpha w_\beta x_{3\alpha+2\beta}(r_5) w_\beta^{-1} w_\alpha^{-1} w_\alpha w_\beta x_\beta(r_4)) dr_4 dr_5 \]
\[ = \int_F \int_{|r_4| > 1} W_{f_4}(h(1, p^n)w_\beta x_\beta(r_5) w_\alpha w_\beta x_\beta(r_4)) dr_4 dr_5. \]

We have the Iwasawa decomposition of \( w_\beta x_\beta(r_4) \):

\[ w_\beta x_\beta(r_4) = x_\beta(-r_4^{-1})h(-r_4^{-1}, -r_4)x_{-\beta}(r_4^{-1}). \]

Since \( x_{-\beta}(r_4^{-1}) \) is in the maximal compact subgroup for \(|r_4| > 1\), we then get

\[ J_{12}(n) = \int_F \int_{|r_4| > 1} W_{f_4}(h(1, p^n)w_\beta x_\beta(r_5) w_\alpha x_\beta(-r_4^{-1})h(r_4^{-1}, r_4)) dr_4 dr_5 \]
\[ = \int_F \int_{|r_4| > 1} W_{f_4}(h(1, p^n)h(r_4^{-1}, 1)w_\beta x_\beta(r_4^{-1} r_5)) dr_4 dr_5 \]
\[ = \int_F \int_{|r_4| > 1} |r_4| W_{f_4}(h(1, p^n)h(r_4^{-1}, 1)w_\beta x_\beta(r_5)) dr_4 dr_5 \]
\[ = \sum_{m \geq 1} (1-q^{-1})q^{2m} \int_F W_{f_4}(h(p^m, 1)h(1, p^n)w_\beta x_\beta(r_5)) dr_5, \]

where in the second equality, we conjugated \( x_{-\beta}(r_4^{-1})h(r_4^{-1}, r_4) \) to the left, and in the third equality, we wrote \( r_4 = p^{-m}u \) for \( m \geq 1, u \in \mathfrak{o}^\times \) and used \( dr_4 = q^{m} du, \text{vol}(\mathfrak{o}^\times) = 1-q^{-1} \). Note that \( h(p^m, 1) \) is in the center of \( M' \), and thus

\[ W_{f_4}(h(p^m, 1)g) = q^{-6sm} \omega_\tau(p)^m W_{f_4}(g), \]

we get

\[ J_{12}(n) = (1-q^{-1}) \sum_{m \geq 1} q^{-6sm+2m} \omega_\tau(p)^m \int_F W_{f_4}(h(1, p^n)w_\beta x_\beta(r_5)) dr_5. \]

Thus we get

\[ J_1(n) = I(n) + \sum_{m \geq 1} (1-q^{-1})q^{(-6s+2)m}(b_1 b_2)^m I(n). \]

A simple calculation gives the formula of \( J_1(n) \). \( \square \)

We next consider the term

\[ J_2(n) = \int_{|r_3| > 1} \int_{F^2} W_{f_4}(h(1, p^n)w_\beta w_\alpha w_\beta x_{\alpha+\beta}(r_3)x_\beta(r_4) x_{3\alpha+2\beta}(r_5)) \psi(-r_3) dr_3 dr_4 dr_5. \]

For \(|r_3| > 1\), we can write \( r_3 \in p^{-m}u \) with \( m \geq 1, u \in \mathfrak{o}^\times \). We then have,

\[ J_2(n) = \int_{F^2} \sum_{m \geq 1} q^m W_{f_4}(h(1, p^n)w_\beta w_\alpha w_\beta (p^{-m}u)x_{\alpha+\beta}(p^{-m}u)x_\beta(r_4) x_{3\alpha+2\beta}(r_5)) \psi(-p^{-m}u) du dr_4 dr_5. \]
Write $x_{\alpha + \beta}(p^{-m}u) = h(u, u^{-1})x_{\alpha + \beta}(p^{-m})h(u^{-1}, u)$, and by conjugation and changing of variables, we get

$$J_2(n) = \int_{F^2} \sum_{m \geq 1} q^m W_{f_s}(h(u^{-1}, p^n)w_{\beta}w_{\alpha}x_{\alpha + \beta}(p^{-m})x_\beta(r_4)x_{3\alpha + 2\beta}(r_5)) \psi(-p^{-m}u) du dr_4 dr_5,$$

where we used $h(u, u^{-1})$ is in the maximal compact subgroup of $G_2(F)$. Since $h(u^{-1}, 1)$ maps to the center of $M'$ and $|\omega_\tau(u)| = 1$, we have

$$W_{f_s}(h(u^{-1}, p^n)w_{\beta}w_{\alpha}x_{\alpha + \beta}(p^{-m})x_\beta(r_4)x_{3\alpha + 2\beta}(r_5)) = W_{f_s}(1, p^n)w_{\beta}w_{\alpha}x_{\alpha + \beta}(p^{-m})x_\beta(r_4)x_{3\alpha + 2\beta}(r_5)).$$

Thus we get

$$J_2(n) = \int_{F^2} \sum_{m \geq 1} q^m W_{f_s}(h(1, p^n)w_{\beta}w_{\alpha}x_{\alpha + \beta}(p^{-m})x_\beta(r_4)x_{3\alpha + 2\beta}(r_5)) \psi(-p^{-m}u) du dr_4 dr_5.$$

Since

$$\int_{\mathbf{A}^*} \psi(p^k u) du = \begin{cases} 1 - q^{-1}, & \text{if } k \geq 0, \\ -q^{-1}, & \text{if } k = -1, \\ 0, & \text{if } k \leq -2, \end{cases}$$

we get $J_2(n) = -R(n)$, where

$$R(n) = \int_{F^2} W_{f_s}(h(1, p^n)w_{\beta}w_{\alpha}x_{\alpha + \beta}(p^{-1})x_\beta(r_4)x_{3\alpha + 2\beta}(r_5)) du dr_4 dr_5.$$

To evaluate $R(n)$, we split the domain of $r_4$, and write $R(n) = R_1(n) + R_2(n)$, where

$$R_1(n) = \int_{|r_4| \leq 1} \int_{F} W_{f_s}(h(1, p^n)w_{\beta}w_{\alpha}x_{\alpha + \beta}(p^{-1})x_\beta(r_4)x_{3\alpha + 2\beta}(r_5)) dr_4 dr_5,$$

$$= \int_{F} W_{f_s}(h(1, p^n)w_{\beta}w_{\alpha}x_{\alpha + \beta}(p^{-1})x_{3\alpha + 2\beta}(r_5)) dr_5,$$

and

$$R_2(n) = \int_{|r_4| > 1} \int_{F} W_{f_s}(h(1, p^n)w_{\beta}w_{\alpha}x_{\alpha + \beta}(p^{-1})x_\beta(r_4)x_{3\alpha + 2\beta}(r_5)) dr_4 dr_5.$$

We now compute $R_1(n)$. We conjugate $w_{\alpha}w_{\beta}x_{\alpha + \beta}(p^{-1})$ to the right and then get

$$R_1(n) = \int_{F} W_{f_s}(h(1, p^n)w_{\beta}x_\beta(r_5)w_{\alpha}w_{\beta}x_{\alpha + \beta}(p^{-1})) dr_5$$

$$= \int_{F} W_{f_s}(h(1, p^n)w_{\beta}x_\beta(r_5)w_{\alpha}(-p^{-1})) dr_5$$

Next, we use the Iwasawa decomposition of $w_{\alpha}x_\alpha(p^{-1})$:

$$w_{\alpha}x_\alpha(-p^{-1}) = x_\alpha(p)h(p^{-1}, p^2)x_{-\alpha}(-p)$$

to get

$$R_1(n) = \int_{F} W_{f_s}(h(1, p^n)w_{\beta}x_\beta(r_5)x_\alpha(p)h(p^{-1}, p^2)) dr_5.$$

Next, we use the commutator relation

$$x_\beta(r_5)x_\alpha(p) = x_{\alpha + \beta}(pr_5)ux_\alpha(p)x_\beta(r_5),$$

where $u$ is in the root space of $2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta$. Then we get

$$R_1(n) = \int_{F} W_{f_s}(h(1, p^n)w_{\beta}x_{\alpha + \beta}(pr_5)ux_\alpha(p)x_\beta(r_5)h(p^{-1}, p^2)) dr_5.$$
Note that \(w_\beta x_\alpha(r)w_\beta(1) \in V\), and \(h(1, p^n)w_\beta x_{\alpha+\beta}(pr_5)(h(1, p^n)w_\beta)^{-1} = x_\alpha(-p^{n+1}r_5)\), and 
\(W_{f_\alpha}(x_\alpha(r)g) = \psi(2r)W_{f_\alpha}(g)\), we get
\[
R_1(n) = \int_F W_{f_\alpha}(h(1, p^n)w_\beta x_\beta(r_5)h(p^{-1}, p^2))\psi(-2p^{n+1}r_5)dr_5 = \int_F W_{f_\alpha}(h(p^2, 1)h(1, p^{n-1})w_\beta x_\beta(p r_5))\psi(-2p^{n+1}r_5)dr_5
\]
\[
= q^{-12s+3}\omega_\tau(p^2) \int_F W_{f_\alpha}(h(1, p^{n-1})w_\beta x_\beta(r_5))\psi(-2p^{n-2}r_5)dr_5,
\]
where the last equality comes from a changing of variable on \(r_5\) and the fact that \(h(p^2, 1) \mapsto \text{diag}(p^2, p^2)\) under the isomorphism \(M' \cong \text{GL}_2\). We next break up the integral on \(r_5\) and get
\[
R_1(n) = q^{-12s+3}\omega_\tau(p^2)W_{f_\alpha}(h(1, p^{n-1})) \int_{|r_5| \leq 1} \psi(-2p^{n-2}r_5)dr_5 + q^{-12s+3}\omega_\tau(p^2) \int_{|r_5| > 1} W_{f_\alpha}(h(1, p^{n-1})w_\beta x_\beta(r_5))\psi(-2p^{n-2}r_5)dr_5.
\]
Using the Iwasawa decomposition of \(w_\beta x_\beta(r_5)\), we have
\[
R_1(n) = q^{-12s+3}\omega_\tau(p^2) \left( W_{f_\alpha}(h(1, p^{n-1})) \int_{|r_5| \leq 1} \psi(-2p^{n-2}r_5)dr_5 + \sum_{m=1}^{\infty} W_{f_\alpha}(h(p^m, p^{n-m-1}))q^m \int_{q^m} \psi(-2p^{n-m-2}u)du \right).
\]

**Lemma 4.3.** We have \(R_1(n) = 0\) if \(n \leq 1\), and
\[
R_1(n) = q^{-12s+3}\omega_\tau(p^2)I(n-1) - q^{-6s(n+1)+n+2}\omega_\tau(p)I(n^2),
\]
for \(n \geq 2\).

**Proof.** Note that \(\int_{|r| \leq 1} \psi(p^kr)dr = 0\) if \(k < 0\) and \(\int_{|r| \leq 1} \psi(p^kr)dr = 1\) if \(k \geq 0\). Moreover, we have
\[
\int_{q^m} \psi(p^k u)du = \begin{cases} 1 - q^{-1}, & \text{if } k \geq 0, \\ -q^{-1}, & \text{if } k = -1, \\ 0, & \text{if } k \leq -2. \end{cases}
\]
Thus we get \(R_1(n) = 0\) for \(n \leq 1\). For \(n \geq 2\), we have
\[
R_1(n) = q^{-12s+3}\omega_\tau(p^2)
\]
\[
\cdot \left( W_{f_\alpha}(h(1, p^{n-1})) + \sum_{m=1}^{n-2} (1 - q^{-1})q^m W_{f_\alpha}(h(p^m, p^{n-m-1})) - q^{-1}q^{n-1}W_{f_\alpha}(h(p^{n-1}, 1)) \right).
\]
\[
= q^{-12s+3}\omega_\tau(p^2)
\]
\[
\cdot \left( W_{f_\alpha}(h(1, p^{n-1})) + \sum_{m=1}^{n-1} (1 - q^{-1})q^m W_{f_\alpha}(h(p^m, p^{n-m-1})) - q^{-1}q^{n-1}W_{f_\alpha}(h(p^{n-1}, 1)) \right)
\]
\[
= q^{-12s+3}\omega_\tau(p^2)I(n-1) - q^{-12s+3+n-1}\omega_\tau(p^2)W_{f_\alpha}(h(p^{n-1}, 1)),
\]
where in the last equation, we used the formula in the computation of \(I(n)\). Since \(h(p^{n-1}, 1)\) is in the center of \(M'\), we have \(W_{f_\alpha}(h(p^{n-1}, 1)) = q^{-6s(n+1)\omega_\tau(p)}I^{n-1}\). The result follows.

We next consider
\[
R_2(n) = \int_{|r_4| > 1} \int_F W_{f_\alpha}(h(1, p^{n})w_\beta x_\alpha w_\beta x_{\alpha+\beta}(p^{-1})x_\beta(r_4)x_{3\alpha+2\beta}(r_5))dr_4 dr_5.
\]
Conjugating \(w_\beta\) to the right side and using the Iwasawa decomposition of \(w_\beta x_\beta(r_4)\), we can get
\[
R_2(n) = \int_F \int_{|r_4| > 1} W_{f_\alpha}(h(1, p^{n})w_\alpha x_\alpha(p^{-1})x_{3\alpha+\beta}(r_4)h(r_4^{-1}, r_4))dr_4 dr_5.
\]
From the commutator relation, we have
\[
x_\alpha(p^{-1})x_\beta(r_4^{-1}) = x_\beta(r_4^{-1})x_\alpha(p^{-1})x_{2\alpha + \beta}(p^{-2}r_4^{-1})u,
\]
for some u in the group generated by roots subgroups of \(\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\). Like in the computation of \(R_1(n)\), we have
\[
R_2(n) = \int_{F} \int_{|r_4| > 1} W_{f_4}(h(1, p^n) w_\beta w_\alpha x_\alpha(p^{-1})x_{3\alpha + \beta}(r_5) h(r_4^{-1}, r_4)) \psi(-2p^{-2}r_4^{-1}) dr_4 dr_5
\]
\[
= \int_{F} \int_{|r_4| > 1} W_{f_4}(h(1, p^n) h(r_4^{-1}, 1) w_\beta x_\beta(r_5 r_4^{-1}) w_\alpha x_\alpha(p^{-1} r_4^{-1})) \psi(-2p^{-2}r_4^{-1}) dr_4 dr_5
\]
\[
= \int_{F} \int_{|r_4| > 1} |r_4| W_{f_4}(h(1, p^n) h(r_4^{-1}, 1) w_\beta x_\beta(r)) \psi(-2p^{-2}r_4^{-1}) dr_4 dr
\]
\[
= I(n) \int_{|r_4| > 1} |r_4|^{-6s+1} \omega_r(r_4^{-1}) \psi(-2p^{-2}r_4^{-1}) dr_4
\]
\[
= I(n) \sum_{m=1}^{\infty} q^{(-6s+2)m} \omega_r(p)^m \int_{E^*} \psi(-2p^{m+n-2}u) du.
\]

**Lemma 4.4.** We have
\[
R_2(n) = \begin{cases} 
I(0)q^{-6s+2} \omega_r(p) \left(-q^{-1} + (1 - q^{-1}) \frac{q^{-6s+2} \omega_r(p)}{1 - q^{-6s+2} \omega_r(p)} \right), & n = 0, \\
I(n)(1 - q^{-1}) \frac{q^{-6s+2} \omega_r(p)}{1 - q^{-6s+2} \omega_r(p)}, & n \geq 1.
\end{cases}
\]

**Proof.** If \(n \geq 1\), then \(\int_{E^*} \psi(p^{m+n-2}u) du = (1 - q^{-1})\) for \(m \geq 1\). Thus, we have
\[
R_2(n) = I(n) \sum_{m=1}^{\infty} q^{(-6s+2)m} \omega_r(p)^m (1 - q^{-1})
\]
\[
= I(n)(1 - q^{-1}) \frac{q^{-6s+2} \omega_r(p)}{1 - q^{-6s+2} \omega_r(p)}.
\]
If \(n = 0\), then \(\int_{E^*} \psi(p^{m+n-2}u) du = (1 - q^{-1})\) for \(m \geq 2\), and \(\int_{E^*} \psi(p^{m+n-2}u) du = -q^{-1}\) for \(m = 1\). Thus, we have
\[
R_2(0) = I(0)(-q^{-1} q^{-6s+2} \omega_r(p) + (1 - q^{-1}) \sum_{m=2}^{\infty} q^{(-6s+2)m} \omega_r(p)^m)
\]
\[
= I(0)q^{-6s+2} \omega_r(p) \left(-q^{-1} + (1 - q^{-1}) \frac{q^{-6s+2} \omega_r(p)}{1 - q^{-6s+2} \omega_r(p)} \right).
\]
The completes the proof of the lemma. \(\square\)

Combining the above results, we get the following

**Lemma 4.5.** We have
\[
R(n) = \begin{cases} 
-I(0)q^{-6s+1} \omega_r(p) \frac{1 - q^{-6s+3} \omega_r(p)}{1 - q^{-6s+2} \omega_r(p)}, & n = 0, \\
I(1)(1 - q^{-1}) \frac{q^{-6s+2} \omega_r(p)}{1 - q^{-6s+2} \omega_r(p)}, & n \geq 1,
\end{cases}
\]
and
\[
J(n) = J_1(n) - R(n)
\]
\[
= \begin{cases} 
1 + Y, & n = 0 \\
I(1), & n = 1,
I(n) - q^{-1} Y^2 I(n - 1) + q^{-n} Y^{n+1}, & n \geq 2.
\end{cases}
\]
where \(Y = q^{-6s+2} \omega_r(p)\)
By the main result of [BFH], we have
\[
\tilde{W}(t(p^n)) = \frac{\mu_\psi(p^n)q^{-n}}{a - a^{-1}} \left(1 - \chi(p)q^{-1/2}a^{-1}\right)a^{n+1} - (1 - \chi(p)q^{-1/2}a)a^{-(n+1)},
\]
where \(\chi(p) = (p,p)_F = (p,-1)_F\). Note that the notation \(\gamma(a)\) in [BFH] is our \(\mu_\psi(a)^{-1}\). Note that \(\mu_\psi(p^n)\mu_\psi(p^n) = (p^n,p^n)_F = \chi(p^n)\). Thus
\[
I(\tilde{W}, W_{f,s}, \phi) = \sum_{n \geq 0} \frac{g^{3n/2}\chi(p)^n}{a - a^{-1}} \left(1 - \chi(p)q^{-1/2}a^{-1}\right)a^{n+1} - (1 - \chi(p)q^{-1/2}a)a^{-(n+1)} \right)J(n).
\]
Plugging the formula \(J(n)\) into the above equation, we can get that
\[
I(\tilde{W}, W_{f,s}, \phi) = \frac{(1 - b_1q^{-1}X)(1 - b_2q^{-1}X)(1 - b_1b_2q^{-1}X^2)(1 - b_1^2b_2q^{-1}X^3)(1 - b_1b_2^2q^{-1}X^3)}{(1 - \chi(p)a^{-1}b_1b_2q^{-1/2}X^2)(1 - \chi(p)ab_1b_2q^{-1/2}X^2)}
\]
\[
\prod_{i=1}^2(1 - \chi(p)a^{-1}b_iq^{-1/2}X) \prod_{i=1}^2(1 - \chi(p)ab_iq^{-1/2}X)
\]
\[
L(3s - 1, \bar{\pi} \times (\chi \otimes \tau))L(6s - 5/2, \bar{\pi} \otimes (\chi \otimes \omega_\tau)).
\]
Here
\[
L(s, \bar{\pi} \otimes (\chi \otimes \tau)) = \frac{1}{(1 - a\chi(p)b_1b_2q^{-s})((1 - a^{-1}\chi(p)b_1b_2q^{-s})}
\]
is the \(L\) function of \(\bar{\pi}\) twisted by the character \(\chi \otimes \omega_\tau\), and
\[
L(s, \bar{\pi} \times (\chi \otimes \tau)) = \frac{1}{\prod_{i=1}^2(1 - \chi(p)a^{-1}b_iq^{-s}) \prod_{i=1}^2(1 - \chi(p)ab_iq^{-s})}
\]
is the Rankin-Selberg \(L\)-function of \(\bar{\pi}\) twisted by \(\chi \otimes \tau\).

We record the above calculation in the following

**Proposition 4.6.** Let \(\tilde{W} \in W(\bar{\pi}, \psi)\) be the normalized unramified Whittaker function, \(f_s\) be the normalized unramified section in \(I(s, \tau)\) and \(\phi \in S(F)\) is the characteristic function of \(\varnothing\), we have
\[
I(\tilde{W}, W_{f,s}, \phi) = \frac{L(3s - 1, \bar{\pi} \times (\chi \otimes \tau))L(6s - 5/2, \bar{\pi} \otimes (\chi \otimes \omega_\tau))}{L(3s - 1/2, \tau)L(6s - 2, \omega_\tau)L(9s - 7/2, \tau \otimes \omega_\tau)}
\]

5. Some local theory

In this section, let \(F\) be a local field, which can be archimedean or non-archimedean. If \(F\) is non-archimedean, let \(\varnothing\) be the ring of integers of \(F\), \(p\) be a uniformizer of \(\varnothing\) and \(q = \varnothing/(p)\). Let \(\bar{\pi}\) be an irreducible genuine generic representation of \(SL_2(F)\), \(\tau\) be an irreducible generic representation of \(GL_2(F)\). Let \(\psi\) be a nontrivial additive character of \(F\).

**Lemma 5.1.** Let \(\tilde{W} \in W(\bar{\pi}, \psi), f_s \in I(s, \tau), \phi \in S(F)\), then the integral \(I(\tilde{W}, W_{f,s}, \phi)\) converges absolutely for \(\text{Re}(s)\) large and has a meromorphic continuation to the whole \(s\)-plane. Moreover, if \(F\) is a \(p\)-adic field, then \(I(\tilde{W}, W_{f,s}, \phi)\) is a rational function in \(q^{-s}\).

The proof is similar to [Gi93, Lemma 4.2-4.7] and [Gi95, Lemma 3.10, Lemma 3.3]. We omit the details.

**Lemma 5.2.** Let \(s_0 \in \mathbb{C}\). Then there exists \(\tilde{W} \in W(\bar{\pi}, \psi), f_{s_0} \in I(s_0, \tau), \phi \in S(F)\) such that \(I(\tilde{W}, W_{f_{s_0}, \phi}) \neq 0\).

**Proof.** The proof is similar to the proof of [Gi93, Lemma 4.4.4.7], [Gi95, Proposition 3.4]. We omit the details.
6. Nonvanishing of certain periods on $G_2$

6.1. Poles of Eisenstein series on $G_2$. Let $\tau$ be a cuspidal unitary representation of $GL_2(\mathbb{A}) \cong M'(\mathbb{A})$. Let $K$ be a maximal compact subgroup of $G_2(\mathbb{A})$. Given a $K \cap GL_2(\mathbb{A})$-finite cusp form $f$ in $\tau$, we can extend $f$ to a function $\tilde{f} : G_2(\mathbb{A}) \rightarrow \mathbb{C}$ as in [Sh, §2]. We then define

$$\Phi_{\tilde{f},s}(g) = \tilde{f}(g)\delta_P(m')s^{3+1/2},$$

for $g = v'm'k$ with $v' \in V'(\mathbb{A}), m' \in M'(\mathbb{A}), k \in K$. Then $\Phi_{\tilde{f},s}$ is well-defined and $\Phi_{\tilde{f},s} \in I(\hat{d}, \tau)$. Then we can consider the Eisenstein series

$$E(s, \tilde{f}, g) = \sum_{P'(F) \backslash G_2(F)} \Phi_{\tilde{f},s}(\gamma g).$$

**Proposition 6.1.** The Eisenstein series $E(s, \tilde{f}, g)$ has a pole on the half plane $\text{Re}(s) > 0$ if and only if $s = \frac{3}{4}, \omega_\tau = 1$ and $L(\frac{3}{4}, \tau) \neq 0$.

For a proof of the above proposition, see [Za, §1] or [Kim, §5]. If $\omega_\tau = 1$ and $L(\frac{3}{4}, \tau) \neq 0$, denote by $\mathcal{R}(\frac{3}{4}, \tau)$ the space generated by the residues of Eisenstein series $E(s, \tilde{f}, g)$ defined as above. Note that an element $R \in \mathcal{R}(\frac{3}{4}, \tau)$ is an automorphic form on $G_2(\mathbb{A})$.

6.2. On the Shimura-Waldspurger lift. Let $\pi$ be a genuine cuspidal automorphic representation of $\tilde{SL}_2(\mathbb{A})$. Let $Wd_\psi(\pi)$ be the Shimura-Waldspurger lift of $\pi$. Then $Wd_\psi(\pi)$ is a cuspidal representation of $\text{PGL}(\mathbb{A})$. A cuspidal automorphic representation $\tau$ is in the image of $Wd_\psi$ if and only if $L(\frac{1}{2}, \tau) \neq 0$. Moreover, the correspondence $\pi \mapsto Wd_\psi(\pi)$ respects the Rankin-Selberg $L$-functions. For these assertions, see [Wald] or [G].

6.3. A period on $G_2$.

**Theorem 6.2.** Let $\pi$ be a genuine cuspidal automorphic representation of $\tilde{SL}_2(\mathbb{A})$ and $\tau$ be a unitary cuspidal automorphic representation of $GL_2(\mathbb{A})$. Assume that $\omega_\tau = 1$ and $L(\frac{1}{2}, \tau) \neq 0$. In particular, $\tau$ can be viewed as a cuspidal automorphic representation of $\text{PGL}(\mathbb{A})$. If $Wd_\psi(\pi) = \chi \otimes \tau$, then there exists $\bar{\varphi} \in \mathcal{V}_{\pi}, \phi \in \mathcal{S}(\mathbb{A}), R \in \mathcal{S}(\frac{1}{2}, \tau)$ such that the period

$$\mathcal{P}(\bar{\varphi}, \bar{\theta}_\phi, R) = \int_{\tilde{SL}_2(F) \backslash \tilde{SL}_2(\mathbb{A})} \int_{V(F) \backslash V(\mathbb{A})} \bar{\varphi}(g)\bar{\theta}_\phi(vg)R(vg)dvgd\psi$$

is non-vanishing.

**Proof.** For $\bar{\varphi} \in \mathcal{V}_{\pi}, \phi \in \mathcal{S}(\mathbb{A})$ and a good section $\Phi_{\tilde{f},s}$ as in §6.1, by Theorem 3.1 and Proposition 4.6, we have

$$I(\bar{\varphi}, \phi, \tilde{f}, s) = \int_{\tilde{SL}_2(F) \backslash \tilde{SL}_2(\mathbb{A})} \int_{V(F) \backslash V(\mathbb{A})} \bar{\varphi}(g)\bar{\theta}_\phi(vg)E(vg, \Phi_{\tilde{f},s})dvgd\psi$$

$$= \int_{\tilde{SL}_2(\mathbb{A})} \int_{U(\mathbb{A})} W_\varphi(g)\omega_\psi(\phi) \phi(1) W_{\Phi_{\tilde{f},s}}(\gamma vg)dvgd\psi$$

$$= I_S \cdot L^S(s + \frac{1}{2}, \chi \otimes \tau)L^S(2s + \frac{1}{2}, \chi \otimes \omega_\tau).$$

Here $S$ is a finite set of places of $F$ such that for $v \notin S$, $\pi_v, \tau_v$ are unramified, and $I_S$ is the product of the local zeta integrals over all places $v \in S$ and $L^S$ denotes the partial $L$-function which is the product of all local $L$-function as the place $v$ runs over $v \notin S$. Note that $\tau \cong \tau^\vee$ since $\omega_\tau = 1$. Suppose that $Wd_\psi(\pi) = \chi \otimes \tau = \chi \otimes \tau^\vee$, then $L^S(s + 1/2, \chi \otimes \tau)$ has a pole at $s = 1/2$. Note that at $s = 3/4$, $L^S(2s + 1/2, \chi \otimes (\chi \otimes \omega_\tau))$ is holomorphic and nonzero, while $L^S(s + 1, \tau)L^S(2s + 1, \omega_\tau)L^S(3s + 1, \tau \otimes \omega_\tau)$ is holomorphic. Moreover, $I_S$ can be chosen to be nonzero. Thus we get that $I(\bar{\varphi}, \phi, \tilde{f}, s)$ has a pole at $s = 1/2$, which means that there exists a residue $R(g, \tilde{f})$ of $E(s, \tilde{f}, g)$ such that

$$\mathcal{P}(\bar{\varphi}, \bar{\theta}_\phi, R) = \int_{\tilde{SL}_2(F) \backslash \tilde{SL}_2(\mathbb{A})} \int_{V(F) \backslash V(\mathbb{A})} \bar{\varphi}(g)\bar{\theta}_\phi(vg)R(vg, \tilde{f})dvgd\psi \neq 0.$$
This completes the proof. □

Remark 6.3. For an $L^2$-automorphic form $\eta \in L^2(G_2(F)\backslash G_2(\mathbb{A}))$, one can form the period

$$\eta_{\tilde{\phi}, \tilde{\theta}}(g) = \int_{SL_2(F)\backslash SL_2(\mathbb{A})} \int_{V(F)\backslash V(\mathbb{A})} \tilde{\varphi}(h)\tilde{\theta}(v)\eta(vhg)dvdh,$$

for a genuine cusp form $\tilde{\phi}$ of $\tilde{\SL_2}(\mathbb{A})$ and $\phi \in S(\mathbb{A})$. Theorem 6.2 says that if $\eta$ varies in $S(\frac{1}{2}, \tau)$, then under the condition $Wd_\psi(\tilde{\pi}) = \chi \otimes \tau$, the period $\eta_{\tilde{\varphi}, \tilde{\theta}}$ is non-vanishing for certain $\tilde{\varphi} \in V_\mathbb{A}$ and $\phi \in S(\mathbb{A})$. For general $\eta$, one can ask under what conditions the period $\eta_{\tilde{\varphi}, \tilde{\theta}}$ is not identically zero as $\tilde{\phi}$ varies in $V_\mathbb{A}$ and $\phi \in S(\mathbb{A})$. In the classical group case, this is the global Gan-Gross-Prasad conjecture for Fourier-Jacobi case, see [GGP]. It is natural to ask if it is possible to extend the GGP-conjecture to the $G_2$-case.

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