Transport Maps for $\beta$-Matrix Models in the Multi-Cut Regime

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Abstract

We use the transport methods developed in [3] to obtain universality results for local statistics of eigenvalues in the bulk and at the edge for $\beta$-matrix models in the multi-cut regime. We construct an approximate transport map between two probability measures from the fixed filling fraction model discussed in [6] and deduce from it universality in the initial model.

1 Introduction

The goal of this paper is to obtain universality results for local statistics of the eigenvalues for $\beta$-matrix models. The analysis of the local fluctuations of the eigenvalues was first done for the GUE and after the pioneer work of Gaudin, Dyson and Mehta the sine kernel law was exhibited (see [19]). Universality was then shown for classical values of $\beta$ ($\beta \in \{1, 2, 4\}$) and smooth potentials through the study of orthogonal polynomials (See the work of L. Pastur and M. Shcherbina [20] [21], and P. Deift et al. [12], [13]). For non classical values of $\beta$ and unless the potential is quadratic, there is however no known matrix representation behind the model and universality results cannot be obtained through orthogonal polynomial methods. For a quadratic potential, the log-gases can be viewed as the eigenvalues of tridiagonal matrices (see [14]) and the local behaviour of the eigenvalues in the bulk and at the edge have been made explicit thanks to the work of B. Virág, B. Valkó, J. Ramirez and B. Rider [22].

Recently, new techniques have been developed to study universality of the fluctuations. Thus, P. Bourgade, L. Erdős and H.T. Yau use dynamical methods and Dirichlet form estimates in [10], [7] to obtain the averaged energy universality of the correlations functions and fixed gap universality in the bulk (for $\beta > 0$), as well as universality at the edge ($\beta \geq 1$, see [9]) for smooth one-cut potentials. In the paper [24], M. Shcherbina uses change of variables to obtain the averaged energy universality of the correlation functions in both the one-cut case and multi-cut cases. The fluctuations of the linear statistics of the eigenvalues in the multi-cut regime were studied in [4] and [23], and rigidity in the multi-cut regime was recently obtained in [17]. In the paper [3], A. Figalli, A. Guionnet and the author construct approximate transport maps with an accurate dependence in the dimension. The dependence in $N$ allows to compare the local fluctuation of the eigenvalues under two different potentials. The potentials do not need to be analytic, but an important hypothesis made in this previous article was the connectedness of the support of the limit of the spectral measure. Here, we assume that the

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potentials are analytic but remove the one-cut assumption and use the same methods to construct approximate transport maps in the case where the filling fractions of each cut is fixed. As a result, we obtain universality of fixed eigenvalue gaps at the edge and in the bulk. The plan of this paper is as follows: In the first section we introduce some notations and state our main results. We reintroduce in section 2 a more general model discussed in [6] of $\beta$ log-gases with Coulomb interaction and construct an approximate transport map between two measures from this model when the number of particles in each cut is fixed. We will see how this approximate transport can lead to universality results in the fixed filling fractions case, and conclude for the initial model in Section 4. The main results are Theorems 1.3, 1.4 and 1.5.

We consider the general $\beta$-matrix model. For a subset $A$ of $\mathbb{R}$ union of disjoint (possibly semi-infinite or infinite) intervals and a potential $V : A \rightarrow \mathbb{R}$ and $\beta > 0$, we denote the measure on $A$

$$\mathbb{P}_{V,A}^N (d\lambda_1, \cdots, d\lambda_N) := \frac{1}{Z_{V,A}^N} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \exp \left( -N \sum_{1 \leq i \leq N} V(\lambda_i) \right) \prod d\lambda_i \; , \tag{1.1}$$

with

$$Z_{V,A}^N = \int_{A^N} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \exp \left( - \sum_{1 \leq i \leq N} V(\lambda_i) \right) \prod d\lambda_i .$$

It is well known (see [1], [2] and [11]) that under $\mathbb{P}_{V,A}^N$ the empirical measure of the eigenvalues converge towards an equilibrium measure:

**Proposition 1.1.** Assume that $V : A \rightarrow \mathbb{R}$ is continuous and if $\infty \in A$ assume that

$$\liminf_{x \to \infty} \frac{V(x)}{\beta \log |x|} > 1 .$$

then the energy defined by

$$E(\mu) = \int V(x) d\mu(x) - \frac{\beta}{2} \log |x_1 - x_2| d\mu(x_1) d\mu(x_2) \tag{1.2}$$

has a unique global minimum on the space $\mathcal{M}_1(A)$ of probability measures on $A$.

Moreover, under $\mathbb{P}_{V,A}^N$ the normalized empirical measure $L_N = N^{-1} \sum_{i=1}^N \delta_{\lambda_i}$ converges almost surely and in expectation towards the unique probability measure $\mu_V$ which minimizes the energy.

It has compact support $A$ and it is uniquely determined by the existence of a constant $C$ such that:

$$\beta \int_A \log |x - y| d\mu_V(y) - V(x) \leq C \; ,$$

with equality almost everywhere on the support. The support of $\mu_V$ is a union of intervals $A = \bigcup_{0 \leq h \leq g} [\alpha_h, -; \alpha_h, +]$ with $\alpha_h, - < \alpha_h, +$ and if $V$ is analytic on a neighbourhood of $A$,

$$\frac{d\mu_V}{dx} = S(x) \prod_{h=0}^g \sqrt{|x - \alpha_{h,-}| |x - \alpha_{h,+}|} .$$
with $S$ analytic on a neighbourhood of $A$.

We make the following assumptions:

**Hypothesis 1.2.**

- $V$ is continuous and goes to infinity faster than $\beta \log |x|$ if $A$ is semi-infinite.
- The support of $\mu_V$ is a union of $g+1$ intervals $A = \bigcup_{0 \leq h \leq g} A_h$ with $A_h = [\alpha_{h,-}; \alpha_{h,+}]$, $\alpha_{h,-} < \alpha_{h,+}$ and
  
  \[
  \frac{d\mu_V}{dx} = \rho_V(x) = S(x) \prod_{h=0}^{g} \sqrt{|x - \alpha_{h,-}| |x - \alpha_{h,+}|} \quad \text{with } S > 0 \text{ on } [\alpha_{h,-}; \alpha_{h,+}].
  \]

- $V$ extends to an holomorphic function on an open neighborhood $U$ of $A$, $U = \bigcup_{0 \leq h \leq g} U_h$ and $A_h \subset U_h$
- The function $V(\cdot) - \beta \int_A \log |\cdot - y| d\mu_V(y)$ achieves its minimum on the support only.

The last hypothesis is useful to ensure a control of large deviations. Before stating the main theorems, we will introduce some notations.

**Notations**

- For all $0 \leq h \leq g$, $\epsilon_{*,h} = \mu_V(A_h)$ and $\epsilon_* = (\epsilon_{*,0}, \cdots, \epsilon_{*,g})$.
- For all $0 \leq h \leq g$, $N_{*,h} = N\epsilon_{*,h}$, $N_* = N\epsilon_*$, and $|N_*| = (|N\epsilon_{*,0}|, \cdots, |N\epsilon_{*,g}|)$.
- For a configuration $\lambda \in \mathbb{R}^N$, $N(\lambda)$ denotes the vector such that for all $0 \leq h \leq g$, $(N(\lambda))_h$ is the number of eigenvalues in $U_h$.
- For an index $i$, we introduce the classical location $E_i^{V,N}$ of the $i-th$ eigenvalue by
  
  \[
  \int_{-\infty}^{E_i^{V,N}} \rho_V(x)dx = \frac{i}{N}.
  \]

In the case where the fraction $i/N$ exactly equals to the sum of the mass of the first cuts, we consider the smallest $E$ satisfying the equality.

- For a configuration $\lambda \in \mathbb{R}^N$, let $\lambda_{h,i}$ the $i$-th smallest eigenvalue in $U_h$.
- For a vector $x \in \mathbb{R}^{g+1}$ and $0 \leq h \leq g$, $[x]_h = x_0 + \cdots + x_h$ and $[x]_{-1} = 0$ .
- For a vector $x \in \mathbb{R}^{g+1}$, $0 \leq h \leq g$ and $i \in \mathbb{N}$ we write $i[h,x] = i - [x]_{h-1}$.
- For a signed measure $\nu$ and a function $f \in L^1(d|\nu|)$ we will write $\nu(f) = \int f d\nu$. 


The main goal of this paper is to prove universality results in the bulk and at the edge. Fixed eigenvalue gaps have been proved to be universal for regular one-cut potentials (see [3, 10]), and their convergence can be obtained using the translation invariance of the eigenvalue gaps as in [15] (see also [25] for the case of the GUE). More precisely, if \( V \) is the Gaussian potential \( G(\lambda) := \beta \lambda^2 \), we have for \( i \) away from the edge

\[
N_{\lambda V}(E_{\lambda V,N}(\lambda_{i+1} - \lambda_i) \xrightarrow{\mathcal{L}} G_\beta,
\]

where \( G_\beta \) is some distribution (corresponding to the Gaudin distribution for \( \beta = 2 \)).

Our first Theorem states that this result holds for any multi-cut potential satisfying Hypothesis 1.2.

**Theorem 1.3.** Let \( \beta > 0 \) and assume that \( V \) satisfies Hypothesis 1.2. Let \( i \leq N \) such that for some \( \varepsilon > 0 \) and \( h \in [0, g] \), \( \varepsilon N < i - |N_h|_{h-1} < N_{*,h} - \varepsilon N \). Then

\[
N_{\lambda V}(E_{\lambda V,N}(\lambda_{i+1} - \lambda_i) \xrightarrow{\mathcal{L}} G_\beta.
\]

We now state the results at the edge. Under a Gaussian potential and for general \( \beta \), the behaviour of the eigenvalues at the edge is described by the Stochastic Airy Operator (We refer to [22]). J. Ramírez, B. Rider and B. Virág have shown that under the Gaussian potential, the first rescaled eigenvalues \( (N^{2/3}(\lambda_1 + 2), \ldots, N^{2/3}(\lambda_k + 2)) \) converge in distribution to \( (\Lambda_1, \ldots, \Lambda_k) \) where \( \Lambda_i \) is the \( i \)-th smallest eigenvalue of the stochastic Airy operator SAO\( _\beta \).

In the following result, \( \Phi^h \) are smooth transport maps (defined later).

**Theorem 1.4.** Assume that \( V \) satisfies Hypothesis 1.2. Let \( \tilde{P}_{V,A}^N \) denote the distribution of the ordered eigenvalues under \( P_{V,A}^N \).

If for all \( 0 \leq h \leq g \), \( f_h : \mathbb{R}^m \rightarrow \mathbb{R} \) is Lipschitz and compactly supported we have:

\[
\lim_{N \rightarrow \infty} \int \prod_{0 \leq h \leq g} f_h(N^{2/3}(\lambda_{h,1} - \alpha_{h,-}), \ldots, N^{2/3}(\lambda_{h,m} - \alpha_{h,-})) d\tilde{P}_{V,A}^N = \prod_{0 \leq h \leq g} \mathbb{E}_{SAO_\beta} f_h(\Phi^h(-2)\Lambda_1, \ldots, \Phi^h(-2)\Lambda_m).
\]

It is also interesting to study the behaviour of the \( i \)-th eigenvalue where \( i = [\lfloor N_h \rfloor]_{h-1} + 1 \). This eigenvalue would be typically located at the right edge of the \( h \)-th cut or the left edge of the \( h+1 \)-st cut. The following theorem gives the limiting distribution of such eigenvalues. We will use the following fact proved by G.Borot and A.Guionnet in [3]: along the subsequences such that \( N_h \mod Z^{g+1} \rightarrow \kappa \) where \( \kappa \in [0; 1)^{g+1} \) and under \( P_{V,A}^N \), the vector \( N(\lambda) - [N_h] \) converges towards a random discrete Gaussian vector (not necessarily centered).

**Theorem 1.5.** Let \( 0 \leq h \leq g \), \( i = [\lfloor N_h \rfloor]_{h-1} + 1 \) and \( \Delta_h(\lambda) = [\lfloor N_h \rfloor]_{h-1} - [N(\lambda)]_{h-1} \).

Define
\[ \xi_h(\lambda) = 1_{\Delta_h(\lambda) \geq 0} \alpha_h^- + 1_{\Delta_h(\lambda) < 0} \alpha_h^+ , \]

where the expression above simplifies to \( \alpha_0^- \) for \( h = 0 \). Then along the subsequences \( N_* \mod \mathbb{Z}^g+1 \rightarrow \kappa \) and under \( \mathbb{P}_V^N \)

\[ \xi_h \xrightarrow{\mathcal{L}} 1_{\Delta_{h,n} \geq 0} \alpha_h^- + 1_{\Delta_{h,n} < 0} \alpha_h^+ , \]

\[ N^{2/3}(\lambda_i - \xi_h) \xrightarrow{\mathcal{L}} 1_{\Delta_{h,n} \geq 0} \Lambda_{\Delta_{h,n}+1} \Phi^h(-2) + 1_{\Delta_{h,n} < 0} \Lambda_{-\Delta_{h,n}} \Phi^{h-1}(2) , \]

where \((\Lambda_i)\) denote the eigenvalues of \( \text{SAO}_\beta \), \( \Phi^h \) is a transport map introduced later and \( \Delta_{h,n} \) is a discrete Gaussian random variable independent from \( \Lambda \) if \( 1 \leq h \leq g \), and equals to 0 if \( h = 0 \).

We could state a similar result about the joint distribution of \( k \) consecutive eigenvalues as well. We note also that using the transport methods of this paper, and adapting the methods presented in [16] (notably Lemma 4.1 and the proof of Corollary 2.8), we could prove universality of the correlation functions in the bulk. This would require rigidity estimate for the fixed filling fractions model introduced in the next section, which could be done as in [8], [17]. As this universality result has already been proved in [24], we do not continue in this direction.

In order to study the fluctuations of the eigenvalues we place ourselves in the setting of the fixed filling fraction model introduced in [4], in which the number of eigenvalues in each cut is fixed. The idea is to construct an approximate transport between our original measure, and a measure in which the interaction between different cuts has been removed. This measure can then be written as a product measure and we can use the results proved for the one cut regime in [3]. We will construct this map in the second section and show universality in the fixed filling fractions models in Section 3. We will deduce from it the proofs of Theorems 1.3, 1.4 and 1.5 in the fourth section.

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2 Fixed Filling Fractions

2.1 Introducing the model

We consider a slightly different model with a more general type of interaction between the particles and in which the number of particles in each cut is fixed. We will refer to [6] for the known results in this setting. For each \( 0 \leq h \leq g \), let \( B_h = [\beta_{h,-}; \beta_{h,+}] \) be a small enlargement of \( A_h = [\alpha_{h,-}; \alpha_{h,+}] \) included in \( U_h \) and \( B = \bigcup_{0 \leq h \leq g} B_h \). It is well known (see for instance [5]) that under our Hypothesis, the eigenvalues will leave \( B \) with an exponentially small probability and we can thus study the behaviour of the eigenvalues under \( \mathbb{P}_V^N \) instead of \( \mathbb{P}_V^N \) without loss of generality.
We fix \(N = (N_0, \cdots, N_g) \in \mathbb{N}^{g+1}\) such that \(\sum_{h=0}^g N_h = N\) and we want to consider a model in which the number of particles in each \(B_h\) is fixed equal to \(N_h\). Let \(\epsilon = N/N \in [0, 1]^{g+1}\) and for \(T : B \times B \to \mathbb{R}\) consider the probability measure on \(B = \prod_{h=0}^g (B_h)^{N_h} :\)

\[
\mathbb{P}_{T,B}^{N,\epsilon}(d\lambda) := \frac{1}{Z_{T,B}^{N,\epsilon}} \prod_{h=0}^g \prod_{1 \leq i < j \leq N_h} |\lambda_{h,i} - \lambda_{h,j}|^\beta \exp \left( -\frac{1}{2} \sum_{0 \leq h, h' \leq g} \sum_{1 \leq i \leq N_h} T(\lambda_{h,i}, \lambda_{h',j}) \right) \prod_{0 \leq h < h' \leq g} \prod_{1 \leq i \leq N_h} |\lambda_{h,i} - \lambda_{h',i}|^\beta \prod_{h=0}^g \prod_{i=1}^{N_h} 1_{B_h}(\lambda_{h,i}) d\lambda_{h,i}.
\]

(2.1)

Note that with \(T(\lambda_1, \lambda_2) = -(V(\lambda_1) + V(\lambda_2))\) and without the location constraints, we are in the same setting as in the previous section.

As in the original model, we can prove the following result (see [6]):

**Proposition 2.1.** Assume that \(T : B \times B \to \mathbb{R}\) is continuous.

Assume also that the energy defined by

\[
E(\mu) = -\frac{1}{2} \int T(x_1, x_2) + \beta \log |x_1 - x_2| d\mu(x_1) d\mu(x_2)
\]

(2.2)

has a unique global minimum on the space \(\mathcal{M}_1^T(B)\) of probability measures on \(B\) satisfying \(\mu(B_h) = \epsilon_h\).

Then under \(\mathbb{P}_{T,B}^{N,\epsilon}\) the normalized empirical measure \(L_N = N^{-1} \sum_{h=0}^g \sum_{i=1}^{N_h} \delta_{\lambda_{h,i}}\) converges almost surely and in expectation towards the unique probability measure \(\mu_T^\epsilon\) which minimizes the energy.

Moreover it has compact support \(A_T^\epsilon\) and it is uniquely determined by the existence of constants \(C_{\epsilon,h}\) such that:

\[
\beta \int_B \log |x - y| d\mu_T^\epsilon(y) + \int_B T(x, y) d\mu_T^\epsilon(y) \leq C_{\epsilon,h} \quad \text{on } B_h
\]

(2.3)

with equality almost everywhere on the support.

The support of \(\mu_T^\epsilon\) is a union of \(l + 1\) intervals \(A_T^\epsilon = \bigcup_{0 \leq h \leq l} [\alpha_{h,-}^{T,\epsilon}; \alpha_{h,+}^{T,\epsilon}]\) with \(\alpha_{h,-}^{T,\epsilon} < \alpha_{h,+}^{T,\epsilon}\), \(l \geq g\) and if \(T\) is analytic on a neighbourhood of \(A_T^\epsilon\),

\[
\frac{d\mu_T^\epsilon}{dx} = S_T^\epsilon(x) \prod_{h=0}^l \sqrt{|x - \alpha_{h,-}^{T,\epsilon}| |x - \alpha_{h,+}^{T,\epsilon}|}
\]

with \(S_T^\epsilon\) analytic on a neighbourhood of \(A_T^\epsilon\).

We point out the fact that the previous theorem is also valid in the unconstrained case. In that case, we denote by \(\mu_T\) the equilibrium measure. Let \(\epsilon_{*,T} = (\mu_T(B_h))_{0 \leq h \leq g}\). Then it is obvious that \(\mu_T^{\epsilon_{*,T}} = \mu_T\). It is shown in [6] that we have the following:
Lemma 2.2. If $T$ extends to an analytic function on a neighbourhood of $B$ and the energy defined in (2.2) has a unique minimizer over $M_1(B)$ then for $\varepsilon$ close enough from $\varepsilon_*$, the energy has a unique minimizer over $M_1^\varepsilon(B)$ and the number of cuts of the support of $\mu_\varepsilon T$ and $\mu_T$ are the same. Moreover, $\alpha_{h,-}^T, \alpha_{h,+}^T$ and $S^T_\varepsilon$ are smooth functions of $\varepsilon$ (for the $L^\infty$ norm on $B$).

They also prove a control of large deviations of the largest eigenvalue under $\mathbb{P}_{T,B}$. We define the effective potential as

$$
\tilde{T}^\varepsilon(x) = \beta \int_B \log |x - y|d\mu_\varepsilon^T(y) + \int_B T(x,y)d\mu_\varepsilon^T(y) - C_{\varepsilon,h} \text{ on } B_h. 
$$

(2.4)

Lemma 2.3. Let $T$ satisfy the conditions of the previous theorem. Then for any closed $F \subset B \setminus A_\varepsilon^T$ and open $O \subset B \setminus A_\varepsilon^T$ we have

$$
\begin{cases}
\limsup \frac{1}{N} \log \mathbb{P}_{T,B}^N(\exists i \lambda_i \in F) \leq \sup_{x \in F} \tilde{T}^\varepsilon(x), \\
\liminf \frac{1}{N} \log \mathbb{P}_{T,B}^N(\exists i \lambda_i \in O) \geq \sup_{x \in O} \tilde{T}^\varepsilon(x).
\end{cases}
$$

We consider a potential $V$ on $A$ satisfying Hypothesis 1.2 and the potentials $T_0(x,y) = -(V(x) + V(y))$ and $T_1(x,y) = -(\tilde{V}^\varepsilon(x) + \tilde{V}^\varepsilon(y) + W(x,y))$ where

$$
W(x,y) = \begin{cases}
\beta \log(x - y) \text{ if } x \in U_h, y \in U_{h'} h > h' \\
\beta \log(y - x) \text{ if } x \in U_h, y \in U_{h'} h < h' \\
0 \text{ if } x \in U_h, y \in U_h
\end{cases}
$$

and

$$
\tilde{V}^\varepsilon(x) = V(x) - \int W(x,y)d\mu_\varepsilon^V(y).
$$

The key point is that $d\mathbb{P}_{T_0,B}^N$ is a product measure as the interaction between cuts has been removed. Moreover, we can check by the characterization (2.3) that

$$
\mu_\varepsilon^V = \mu_\varepsilon^{T_0} = \mu_\varepsilon^{T_1}.
$$

We now consider

$$
T_t = (1-t)T_0 + t T_1, \ t \in [0;1].
$$

(2.5)

Still by (2.3) we can check that for all $t \in [0;1]$ we have:

$$
\mu_\varepsilon^{T_1} = (1-t)\mu_\varepsilon^{T_0} + t \mu_\varepsilon^{T_1} = \mu_\varepsilon^V.
$$
Remark 2.4. Note that, by Lemma 2.2, for $\epsilon$ in a small neighbourhood of $\epsilon_*$ (that we will denote $\tilde{E}$) the support $A^\epsilon$ of $\mu^\epsilon_{T_t}$ has $g + 1$ cut and we can write

$$d\mu^\epsilon_{T_t} = d\mu^\epsilon_{V} = S^\epsilon(x) \prod_{h=0}^{g} \sqrt{|x - \alpha^\epsilon_{h,-}| |x - \alpha^\epsilon_{h,+}|} dx,$$

(2.6)

with $S^\epsilon$ positive on $A^\epsilon$.

Remark 2.5. Note also that by the last point of Hypothesis 1.2 and by Lemma 2.2, if we fix a closed interval $F \subset B \setminus A$, then for $\epsilon$ close enough to $\epsilon_*$ and all $t \in [0;1]$ , $T_t^\epsilon < 0$ on $F$.

The goal is to build first an approximate transport map between the measures $d\bar{P}_{T_t,B}^N$ for a fixed $\epsilon$ in $\tilde{E}$ i.e find a map $X_1^{N,\epsilon}$ that satisfies for all $f : \mathbb{R}^N \to \mathbb{R}$ bounded measurable function

$$| \int f(X_1^{N,\epsilon}) d\bar{P}_{V,B}^N - \int f d\bar{P}_{T_t,B}^N | \leq C \|f\|_{\infty} \frac{(\log N)^3}{N}.
$$

(2.7)

We will see that we can build a transport map depending smoothly on $\epsilon$ and show universality in the fixed filling model. We will then use this result to prove universality in the original model.

Proposition 2.6. Assume that $V$ satisfies Hypothesis 1.2 and that $T_t$ is as defined previously. Let $N = (N_0, \ldots, N_g)$ such that $\epsilon = N/N$ is in $\tilde{E}$ and $\tilde{P}_{T_t,B}^N$ denote the distribution of the ordered eigenvalues under $P_{T_t,B}^N$. Then for a constant $C$ independent of $\epsilon$ and $N$, and if for all $0 \leq h \leq g$ $f_h : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz supported inside $[-M, M]^m$ we have:

1. Eigenvalue gaps in the Bulk

$$\left| \int \prod_{0 \leq h \leq g} f_h(N(\lambda_{h,i_h+1} - \lambda_{h,i_h}), \ldots, N(\lambda_{h,i_h+m} - \lambda_{h,i_h})) d\bar{P}_{V,B}^N - \int \prod_{0 \leq h \leq g} f_h(N(\lambda_{h,i_h+1} - \lambda_{h,i_h}), \ldots, N(\lambda_{h,i_h+m} - \lambda_{h,i_h})) d\bar{P}_{T_t,B}^N \right|$$

$$\leq C \frac{(\log N)^3}{N} \|f\|_{\infty} + C(\sqrt{m} \frac{(\log N)^2}{N^{1/2}} + M \frac{(\log N)}{N^{1/2}}) \|\nabla f\|_{\infty}$$

2. Eigenvalue gaps at the Edge

$$\left| \int \prod_{0 \leq h \leq g} f_h(N^{2/3}(\lambda_{h,1} - \alpha^\epsilon_{h,-}), \ldots, N^{2/3}(\lambda_{h,m} - \alpha^\epsilon_{h,-})) d\bar{P}_{V,B}^N - \int \prod_{0 \leq h \leq g} f_h(N^{2/3}(\lambda_{h,1} - \alpha^\epsilon_{h,-}), \ldots, N^{2/3}(\lambda_{h,m} - \alpha^\epsilon_{h,-})) d\bar{P}_{T_t,B}^N \right|$$

$$\leq C \frac{(\log N)^3}{N} \|f\|_{\infty} + C(\sqrt{m} \frac{(\log N)^2}{N^{3/6}} + \frac{\log N}{N^{1/2}}) \|\nabla f\|_{\infty}$$
where we defined \( f : \mathbb{R}^{m(g+1)} \rightarrow \mathbb{R} \) by \( f(x_0, \cdots, x_g) = \prod_{0 \leq h \leq g} f_h(x_h) \).

We deduce the following corollary from the results obtained in the one-cut regime in [3], and from the fact that \( \mathbb{P}_{T,\mathcal{E}}^{N,e} \) is a product measure.

**Corollary 2.7.** Assume the same hypothesis as in the precedent proposition. We write \( \mu_V = \sum_{0 \leq h \leq g} \mu_V^{e,h} \) where \( \mu_V^{e,h} \) has connected support. For some transport maps \( \Phi^{e,h} \) from \( \mu_V \) to \( \mu_V^{e,h} \),

1. **Eigenvalue gaps in the Bulk**

\[
\left| \int \prod_{0 \leq h \leq g} f_h(N(\lambda_{h,1} - \alpha_{h,1}^e), \cdots, N(\lambda_{h,g} - \alpha_{h,g}^e)) \mathbb{P}_{T,\mathcal{E}}^{N,e} \right| \leq C \frac{(\log N)^3}{N} \|f\|_\infty + C\left(\frac{\log N}{N^{1/2}} + \frac{M_1}{N^{1/2}}\right) \|\nabla f\|_\infty
\]

2. **Eigenvalue gaps at the Edge**

\[
\left| \int \prod_{0 \leq h \leq g} f_h(N^{2/3}(\lambda_{h,1} - \alpha_{h,1}^e), \cdots, N^{2/3}(\lambda_{h,g} - \alpha_{h,g}^e)) \mathbb{P}_{T,\mathcal{E}}^{N,e} \right| \leq C \frac{(\log N)^3}{N} \|f\|_\infty + C\left(\frac{\log N}{N^{5/6}} + \frac{M_1}{N^{1/3}}\right) \|\nabla f\|_\infty.
\]

The proof of the theorem will be similar to what has already been done in the one-cut case, one major difference being the inversion of the operator \( \Xi \) introduced in Lemma 3.2 of [3].

### 2.2 Approximate Monge Ampère Equation

The analysis done in the one-cut regime suggests to look at the transport as the flow of an approximate solution to the Monge Ampère equation \( Y^{N,e}_{k,t} = (Y^{N,e}_{0,t}, \cdots, Y^{N,e}_{g,t}) : \mathbb{R}^N \rightarrow \mathbb{R}^N \) where \( Y^{N,e}_{h,k,t} : \mathbb{R}^N \rightarrow \mathbb{R}^{N_h} \) solves the following equation:

\[
\text{div} \left( Y^{N,e}_{k,t} \right) = c^{N,e}_{k,t} - \beta \sum_{h=0}^{g} \sum_{1 \leq i \leq j \leq N_h} \frac{Y^{N,e}_{h,i,t} - Y^{N,e}_{h,j,t}}{\lambda_{h,i} - \lambda_{h,j}} - \beta \sum_{0 \leq k \leq g} \sum_{1 \leq i \leq j \leq N_h} \frac{Y^{N,e}_{h,i,t} - Y^{N,e}_{h,j,t}}{\lambda_{h,i} - \lambda_{h,j}} - \sum_{0 \leq h, k \leq g} \sum_{1 \leq i \leq j \leq N_h} \frac{\partial_i T(\lambda_{h,i}, \lambda_{h,j})}{N_h} Y^{N,e}_{h,k,t} - \frac{1}{2} W(\lambda_{h,i}, \lambda_{h,j}) - N \sum_{0 \leq k \leq g} \sum_{1 \leq i \leq N_h} \int W(\lambda_{h,i}, z) d\mu_V(z)
\]

(2.8)
Let

\[ c_t^{N,\varepsilon} = \int \left( N \sum_{0 \leq h < g} \sum_{1 \leq i \leq N_h} W(\lambda_{h,i}, z) d\mu_V(z) - \frac{1}{2} \sum_{0 \leq h, h' \leq g} \sum_{1 \leq j \leq N_{h'}} W(\lambda_{h,i}, \lambda_{h',j}) \right) d\mathbb{P}_T^{N,\varepsilon}(\lambda) \]

\[ = \partial_t \log(Z_{V_t,B}). \]

Let \( \mathcal{R}_t^{N,\varepsilon}(Y^{N,\varepsilon}) \) the error term defined as

\[ \mathcal{R}_t^{N,\varepsilon}(Y^{N,\varepsilon}) = \beta \sum_{h=0}^g \sum_{1 \leq i < j \leq N_h} \frac{Y_{h,i,t}^{N,\varepsilon} - Y_{h,j,t}^{N,\varepsilon}}{\lambda_{h,i} - \lambda_{h,j}} + \beta \sum_{0 \leq h < h' \leq g} \sum_{1 \leq j \leq N_{h'}} \frac{Y_{h,i,t}^{N,\varepsilon} - Y_{h',j,t}^{N,\varepsilon}}{\lambda_{h,i} - \lambda_{h',j}} \]

\[ + \sum_{0 \leq h, h' \leq g} \sum_{1 \leq i \leq N_h} \sum_{1 \leq j \leq N_{h'}} \int (\partial_t T(\lambda_{h,i}, \lambda_{h',j}) - \frac{1}{2} W(\lambda_{h,i}, \lambda_{h',j})) + N \sum_{0 \leq h \leq g} \sum_{1 \leq i \leq N_h} \int W(\lambda_{h,i}, z) d\mu_V(z) \]

\[ + \text{div}(Y_t^{N,\varepsilon}) - c_t^{N,\varepsilon}. \]

We have the following stability lemma

**Lemma 2.8.** Let \( Y_t^{N,\varepsilon} : \mathbb{R}^N \rightarrow \mathbb{R}^N \) be a smooth vector field and let \( X_t^{N,\varepsilon} \) be its flow:

\[ X_t^{N,\varepsilon} = Y_t^{N,\varepsilon}(X_0^{N,\varepsilon}) \quad X_0^{N,\varepsilon} = \text{Id.} \]

Assume that \( Y_t^{N,\varepsilon} \) vanishes on the boundary of \( B \).

Let \( f : \mathbb{R}^N \rightarrow \mathbb{R} \) be a bounded measurable function. Then

\[ \left| \int f(Y_t^{N,\varepsilon}) d\mathbb{P}_V^{N,\varepsilon} - \int f d\mathbb{P}_T^{N,\varepsilon} \right| \leq \| f \|_{\infty} \int_0^t \| \mathcal{R}_s^{N,\varepsilon}(Y^{N,\varepsilon}) \|_{L^1(\mathbb{P}_T^{N,\varepsilon})} ds. \]

**Proof.** Let

\[ \rho_t(\lambda) := \frac{1}{Z_{T,B}^{N,\varepsilon}} \prod_{h=0}^g \prod_{1 \leq i < j \leq N_h} |\lambda_{h,i} - \lambda_{h,j}|^\beta \exp \left( -\frac{1}{2} \sum_{0 \leq h, h' \leq g} \sum_{1 \leq i \leq N_{h'}} T_t(\lambda_{i,h}, \lambda_{j,h'}) \right) \]

\[ \prod_{0 \leq h < h' \leq g} \prod_{1 \leq i \leq N_h} \prod_{1 \leq j \leq N_{h'}} |\lambda_{h,i} - \lambda_{h',j}|^\beta \]

and \( JX_t^{N,\varepsilon} \) denote the Jacobian of \( X_t^{N,\varepsilon} \). As \( Y_t^{N,\varepsilon} \) vanishes on the boundary of \( B \), \( X_t^{N,\varepsilon}(B) = B \). By the change of variable formula we have

\[ \int f d\mathbb{P}_T^{N,\varepsilon} = \int_B f(\lambda) \rho_t(\lambda) d\lambda = \int_{X_t^{N,\varepsilon}(B)} f(\lambda) \rho_t(\lambda) d\lambda = \int B f(X_t^{N,\varepsilon}) \rho_t(X_t^{N,\varepsilon}) JX_t^{N,\varepsilon} d\lambda \]

Thus we have

\[ \left| \int f(Y_t^{N,\varepsilon}) d\mathbb{P}_V^{N,\varepsilon} - \int f d\mathbb{P}_T^{N,\varepsilon} \right| \leq \| f \|_{\infty} \int_B |\rho_0(\lambda) - \rho_t(X_t^{N,\varepsilon}) JX_t^{N,\varepsilon}| d\lambda. \]
Let
\[ \Delta t = \partial_t \int_B |\partial_0(\lambda) - \rho_t(X_t^{N, \epsilon}) J X_t^{N, \epsilon}| d\lambda. \]
Using \( \partial_t(J X_t^{N, \epsilon}) = \text{div} (Y_t^{N, \epsilon} J X_t^{N, \epsilon}) \) we have
\[
\Delta t \leq \int_B |\partial_t \left( J X_t^{N, \epsilon} \rho_t(X_t^{N, \epsilon}) \right)| d\lambda
\]
\[
= \int_B |\text{div} (Y_t^{N, \epsilon} J X_t^{N, \epsilon} \rho_t(X_t^{N, \epsilon})) + J X_t^{N, \epsilon} \nabla \rho_t(X_t^{N, \epsilon}) \dot{X}_t^{N, \epsilon}| d\lambda
\]
\[
= \int |\mathcal{R}_t^{N, \epsilon}(Y^{N, \epsilon})| dP_{T_t,B}^{N, \epsilon}
\]
and this gives the lemma.

### 2.3 Constructing an Approximate Solution

The construction of the approximate solution will be very similar to Section 3 of [3].

We fix \( t \in [0,1] \), \( N = (N_0, \cdots, N_g) \in \mathbb{N}^{g+1} \) such that \( \sum_{h=0}^{g} N_h = N \) and set \( \epsilon = N / N \in ]0,1[^{g+1} \).

Let
\[
L_N = \frac{1}{N} \sum_{h,i} \delta_{\lambda_{h,i}}, \quad M_N = \sum_{h,i} \delta_{\lambda_{h,i}} - N \mu_V.
\]
We look for a map \( Y_t^{N, \epsilon} = (Y_{0,1,t}^{N, \epsilon}, \cdots, Y_{g,N_t,t}^{N, \epsilon}) : \mathbb{R}^N \longrightarrow \mathbb{R}^N \) approximately solving (2.3). As in the one-cut regime, we make the following ansatz:

\[
Y_{h,i,t}^{N, \epsilon}(\lambda) = \frac{1}{N} y_{1,t}^{\epsilon}(\lambda_{h,i}) + \frac{1}{N} \xi_{1,t}^{\epsilon}(\lambda_{h,i}, M_N), \quad \xi_{1,t}^{\epsilon}(x, M_N) = \int z_{1,t}^{\epsilon}(x, y) dM_N(y)
\]  
(2.11)

for some functions \( y_{1,t}^{\epsilon} : \mathbb{R} \longrightarrow \mathbb{R} \) and \( z_{1,t}^{\epsilon} : \mathbb{R}^2 \longrightarrow \mathbb{R} \).

**Proposition 2.9.** Let \( V \) satisfy Hypothesis 1.2 and \( T_t \) is as in (2.7). Then there are \( y_{1,t}^{\epsilon} \) in \( C^\infty(\mathbb{R}) \) and \( z_{1,t}^{\epsilon} \) in \( C^\infty(\mathbb{R}^2) \) such that for a constant \( C \), for all \( t \in [0,1] \) and \( \epsilon \in \tilde{E} \):

\[
\| \mathcal{R}_t^{N, \epsilon}(Y^{N, \epsilon}) \|_{L^1(P_{T_t}^{N, \epsilon})} \leq C \frac{(\log N)^3}{N}.
\]

Using the substitution (2.11), we have to find equations for \( y_{1,t}^{\epsilon} \) and \( z_{1,t}^{\epsilon} \). To simplify the notations, we will write \( \mathcal{R} \) instead of \( \mathcal{R}_t^{N, \epsilon}(Y^{N, \epsilon}) \). We obtain:
where $c^N_i$ is a constant and for any measure $\nu$ we set

$$\eta(\nu) = \int \partial_2 z^\nu_t(x, y) d\nu(y).$$

We use equilibrium relations to recenter $L_N$ by $\mu^V_t$. Consider $f$ a bounded measurable function on $B$ and $\mu^V_{t, \delta} = \{x + \delta f(x)\} \# \mu^V_t$. Then as for $\delta$ small enough $\mu^V_{t, \delta}(B_h) = \varepsilon_h$ for all $0 \leq h \leq g$, we have $E(\mu^V_{t, \delta}) \geq E(\mu^V_t)$ where we defined the energy in (1.2). By differentiating at $\delta = 0$ we obtain

$$\frac{\beta}{2} \int \int \frac{f(x) - f(y)}{x - y} d\mu^V_t(x) d\mu^V_t(y) + \int \partial_1 T_t(x, y) f(x) d\mu^V_t(x) d\mu^V_t(y) = 0. \tag{2.12}$$

Thus, if we define the operator $\Xi$ acting on smooth functions $f : B \longrightarrow \mathbb{R}$ by

$$\Xi f(x) = \int \left[ \frac{\beta}{2} \frac{f(x) - f(y)}{x - y} + \partial_1 T_t(x, y) f(x) + \partial_2 T_t(x, y) f(y) \right] d\mu^V_t(y),$$

we obtain

$$\frac{\beta}{2} \int \int \frac{f(x) - f(y)}{x - y} dL_N(x) dL_N(y) + \int \partial_1 T_t(x, y) f(x) dL_N(x) dL_N(y)$$

$$= \frac{1}{N^2} \int \Xi f dM_N + \frac{1}{N^2} \int \left[ \frac{\beta}{2} \int \int \frac{f(x) - f(y)}{x - y} dM_N(x) dM_N(y) + \int \partial_1 T_t(x, y) f(x) dM_N(x) dM_N(y) \right].$$

Therefore we can write

$$\mathcal{R} = \int \left[ \Xi y^\nu_t + \left( 1 - \frac{\beta}{2} \right) \int \partial_1 z^\nu_t(z, \cdot) d\mu^V_t(z) \right] dM_N$$

$$+ \int \left[ \Xi z^\nu_t(\cdot, y) [x] - \frac{1}{2} W(x, y) \right] dM_N(x) dM_N(y) + C^{N, e}_N + E$$

with

$$\Xi z^\nu_t(\cdot, y) [x] = \int \left[ \beta z^\nu_t(x, y) - z^\nu_t(z, y) \frac{x - y}{x - z} + \partial_1 T_t(x, z) z^\nu_t(x, y) + \partial_2 T_t(x, z) z^\nu_t(z, y) \right] d\mu^V_t(z).$$
where $C_t^N, \epsilon$ is deterministic and $E$ is an error term:

$$E = \frac{1}{N} \int \partial_2 z_1^\epsilon(x, x)dM_N(x) + \frac{1}{N} \left(1 - \frac{\beta}{2}\right) \int y_{\epsilon, t}^x dM_N$$

$$+ \frac{1}{N} \left(1 - \frac{\beta}{2}\right) \int \int \partial_1 z_1^\epsilon(x, y)dM_N(x)dM_N(y)$$

$$+ \frac{1}{N} \int \int \left[\frac{\beta y_{\epsilon, t}^x(x) - y_{\epsilon, t}^x(y)}{2x - y} + \partial_1 T_t(x, y)y_{\epsilon, t}^x(x)\right] dM_N(x)dM_N(y)$$

$$+ \frac{1}{N} \int \int \left[\frac{\beta z_1^\epsilon(x, y) - z_1^\epsilon(z, y)}{x - z} + \partial_1 T_t(x, z)z_1^\epsilon(x, y)\right] dM_N(x)dM_N(y)dM_N(z)$$

(2.13)

To make $\mathcal{R}$ small we need

$$\begin{cases} \Xi z_1^\epsilon(\cdot, y)[x] = \frac{1}{2} W(x, y) + \kappa_1(x, y), \\ \Xi y_{\epsilon, t}^x = \left(\frac{\beta}{2} - 1\right) \int \partial_1 z_1^\epsilon(z, \cdot)d\mu^\epsilon_V(z) + \kappa_2, \end{cases}$$

where $\kappa_2$ and $\kappa_1(\cdot, y)$ are functions on $B$ constant on each $B_h$.

The following lemma shows how to invert $\Xi$ and will give us the desired functions. We will denote by $\mathcal{O}(U)$ the set of holomorphic functions on $U$.

**Lemma 2.10.** Let $V$ satisfy Hypothesis (2.7) and $T_t$ as in (2.9) and $\epsilon = N/N$ in $\tilde{E}$. The support of $\mu^\epsilon_V$ is a union of $g + 1$ intervals $A^\epsilon = \bigcup_{0 \leq h \leq g} [\alpha^\epsilon_{h, -}; \alpha^\epsilon_{h, +}]$ with $\alpha^\epsilon_{h, -} < \alpha^\epsilon_{h, +}$ and,

$$\frac{d\mu^\epsilon_V}{dx} = S(x) \prod \sqrt{|x - \alpha^\epsilon_{h, -}|x - \alpha^\epsilon_{h, +}|}$$

with $S$ positive on $A^\epsilon$.

Let $k \in \mathcal{O}(U)$ and set for $f \in \mathcal{O}(U)$

$$\Xi f(x) = \int \left[\frac{\beta f(x) - f(y)}{x - y} + \partial_1 T_t(x, y)f(x) + \partial_2 T_t(x, y)f(y)\right] d\mu^\epsilon_V(y) \quad \forall x \in U.$$

Then there exists a unique function $\kappa_k$ on $U$ constant on each $U_h$ such that the equation

$$\Xi f = k + \kappa_k$$

has a solution in $\mathcal{O}(U)$ . Moreover, for all $x \in U_h$

$$f(x) = -\frac{1}{2\beta \pi^2 \sigma(x) \sigma_h(x) S(x)} \left[\int \sigma_h(\xi) \left(k(\xi) + c_h^1\right) \frac{(\xi - x)}{\xi - x} d\xi + c_h^2\right],$$

(2.14)

where the contour surrounds $x$ and $A^\epsilon_h$ in $U_h$ and

$$\sigma^2(x) = \prod_{h'} (x - \alpha^\epsilon_{h', -}) (x - \alpha^\epsilon_{h', +})$$

$$\sigma(x) \sim x^{g + 1}$$

$$\sigma^2_h(x) = (x - \alpha^\epsilon_{h, -}) (x - \alpha^\epsilon_{h, +})$$

$$\sigma_h(x) \sim x$$

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and the constants $c^1_h$ and $c^2_h$ are chosen in a way such that the expression under the bracket vanishes at $x = \alpha_{h,-}^\epsilon$ and $x = \alpha_{h,+}^\epsilon$ for each $h$ (see the following Lemma).

Moreover $f$ satisfies for all $j$

$$\|f\|_{C^j(B)} \leq C_j \|k\|_{C^{j+2}(B)}$$

(2.15)

for some constants $C_j$. We will denote $f$ by $\Xi^{-1}k$.

Before proving this lemma we need another lemma

**Lemma 2.11.** Let $V \in \mathcal{O}(U)$ and $\mu^\epsilon_V$ as in the previous lemma.

Then for all $0 \leq h \leq g$ the linear operator

$$\Theta_h := \mathbb{C}^2 \rightarrow \mathbb{C}^2$$

$$(c^1, c^2) \rightarrow \left( c^1 \int \frac{\sigma_h(\xi)}{(\xi - \alpha_{h,-}^\epsilon)} d\xi + c^2, c^1 \int \frac{\sigma_h(\xi)}{(\xi - \alpha_{h,+}^\epsilon)} d\xi + c^2 \right)$$

is invertible and $\Theta_h^{-1}$ is analytic.

**Proof.** This comes easily from the fact that

$$\int_{\alpha_{h,-}^\epsilon}^{\alpha_{h,+}^\epsilon} \sqrt{(y - \alpha_{h,-}^\epsilon)(\alpha_{h,+}^\epsilon - y)} \frac{dy}{y - \alpha_{h,-}^\epsilon} = \pi \frac{\alpha_{h,+}^\epsilon - \alpha_{h,-}^\epsilon}{2}$$

and

$$\int_{\alpha_{h,-}^\epsilon}^{\alpha_{h,+}^\epsilon} \sqrt{(y - \alpha_{h,-}^\epsilon)(\alpha_{h,+}^\epsilon - y)} \frac{dy}{y - \alpha_{h,+}^\epsilon} = \pi \frac{\alpha_{h,-}^\epsilon - \alpha_{h,+}^\epsilon}{2}$$

$\Box$

**Proof of Lemma 2.10.** By the identity (2.12) with $f(x) = (z - x)^{-1}$ and $z$ outside the support, we obtain that the Stieltjes transform $G(z) = \int \frac{1}{z-y} d\mu^\epsilon_V(y)$ satisfies

$$\frac{\beta}{2} G(z)^2 + G(z) \int \partial_1 T_l(z, y) d\mu^\epsilon_V(y) + F(z) = 0$$

with $F(z) = \int \frac{\partial_1 T_l(\tilde{y}, z) - \partial_1 T_l(z, \tilde{y})}{\tilde{y} - z} d\mu^\epsilon_V(\tilde{y}) d\mu^\epsilon_V(y)$

and this gives

$$\beta G(z) + \int \partial_1 T_l(z, y) d\mu^\epsilon_V(y) = -\sqrt{\left( \int \partial_1 T_l(z, y) d\mu^\epsilon_V(y) \right)^2 - 2\beta F(z)}.$$ 

As $-\pi^{-1}IG(z)$ converges towards the density of $\mu^\epsilon_V$ as $z$ goes to the real axis (see for instance [1], Section 2.4 for the basic properties of the Stieltjes transform) and the quantity under the square root converges to a real number, this number has to be negative on the support (otherwise the density would vanish) and thus for $x \in A^\epsilon$

$$\frac{d\mu^\epsilon_V}{dx} = \frac{1}{\beta \pi} \sqrt{2\beta F(x) - \left( \int \partial_1 T_l(x, y) d\mu^\epsilon_V(y) \right)^2}.$$
Notice that \( \sigma \) becomes purely imaginary when \( z \) converges towards the support, we may write

\[
\beta G(z) + \int \partial_1 T_1(z,y) d\mu T_1(y) = \beta \pi \tilde{S}(z) \sigma(z)
\]

where \( \tilde{S} \) is an analytic extension of \( S \) in \( U \) (we can assume \( \tilde{S} \) non zero on \( U \) by possibly shrinking \( U \)). We will keep writing \( S \) for \( \tilde{S} \).

For \( f \) analytic in \( U \setminus A^c \) and \( z \in U \setminus A^c \) let

\[
\tilde{E} f(z) = \frac{i}{2} \int \left( \frac{\beta f(\xi)}{z - \xi} - \partial_2 T_1(z,\xi)f(\xi) \right) S(\xi) \sigma(\xi) d\xi
\]

where the contour surrounds \( z \) and each \( A^c_h \). Then \( \tilde{E} f \in \mathcal{O}(U \setminus A^c) \) and, noticing that

\[
-iS(x + i\delta)\sigma(x + i\delta) \rightarrow \frac{d\mu T}{dx},
\]

we have

\[
\Xi f(z) = -\int \left( \frac{\beta f(y)}{z - y} - \partial_2 T_1(z,y)f(y) \right) d\mu T_1(y) + f(z) \left( \int \partial_1 T_1(z,y)d\mu T_1(y) + \beta \int \frac{d\mu T_1(y)}{z - y} \right)
\]

\[
= \frac{i}{2} \int \left( \frac{\beta f(\xi)}{z - \xi} - \partial_2 T_1(z,\xi)f(\xi) \right) S(\xi) \sigma(\xi) d\xi + \beta \pi f(z) S(z) \sigma(z)
\]

\[
= \tilde{E} f(z)
\]

(2.17)

where the contour surrounds each \( A^c_h \) (but not \( z \)), and we used Cauchy’s formula and (2.16). If furthermore \( f \in \mathcal{O}(U) \), by continuity this formula extends to \( z \in U \).

Let \( k \in \mathcal{O}(U) \). We want to show that the function defined on each \( U_h \) by

\[
f(z) = -\frac{1}{2\pi^2 \sigma(z) \sigma_h(z) S(z)} \left[ \int \frac{i\sigma_h(\xi) (k(\xi) + c^1_h)}{(\xi - z)} d\xi + c^2_h \right]
\]

where the contour surrounds \( A^c_h \) and lays in \( U_h \), and \( c^1_h \) and \( c^2_h \) are defined as in the statement of the lemma, is a solution of \( \Xi f = k + \kappa_k \) in \( \mathcal{O}(U) \). The fact that \( f \in \mathcal{O}(U) \) is clear (the function is meromorphic and the poles are removable by construction of \( c^1 \) and \( c^2 \)). Thus, by previous remark, it suffices to prove that \( \tilde{E} f = k + \kappa_k \).

By (2.5) We have

\[
\tilde{E} f = (1 - t) \tilde{E}_0 f + t \tilde{E}_1 f + c_t
\]

(2.18)

where \( c_t \) is a function constant on each \( U_h \) depending on \( t \) and

\[
\left\{ \begin{array}{ll}
\tilde{E}_0 f(z) = \frac{\beta i}{2} \int \frac{f(\xi) \sigma(\xi) S(\xi)}{z - \xi} d\xi \\
\tilde{E}_1 f(z) = \frac{\beta i}{2} \int \frac{f(\xi) \sigma(\xi) S(\xi)}{z - \xi} d\xi
\end{array} \right.
\]

where the first contour surrounds \( z \) and each \( A^c_h \), whereas the second one surrounds \( z \) and \( A^c_h \) when \( z \in U_h \).
Let $f_0$ and $f_1$ be the functions analytic in $U \setminus A^*_x$ defined on each $U_h \setminus A^*_h$ by

$$
f_0(z) = -\frac{1}{2\beta^2 \sigma(z)\sigma_h(z)S(z)} \oint_C \frac{i\sigma_h(\xi)(k(\xi) + c_1^h)}{\xi - z} d\xi
$$

$$
f_1(z) = -\frac{c_2^h}{2\beta^2 \sigma(z)\sigma_h(z)S(z)}
$$

So that $f = f_0 + f_1$

$$
\tilde{\Xi}_0(f_0)(z) = -\frac{\beta i}{2} \sum_h \oint_{C_h} \frac{1}{z - \xi} \sigma_h(\xi)(k(\xi) + c_1^h) S(\xi) d\eta S(\xi) d\xi
$$

$$
= -\frac{\beta i}{2} \sum_h \oint_{C_h} \frac{1}{z - \xi} \sigma_h(\xi)(k(\xi) + c_1^h) S(\xi) d\eta S(\xi) d\xi
$$

where $C_h$ surrounds $z$ and $A^*_h$ (integral in $\xi$), and $C'_h$ surrounds $C_h$ (integral in $\eta$).

Cauchy formula gives

$$
\oint_{C'_h} \frac{\sigma_h(\eta)(k(\eta) + c_1^h)}{(\eta - \xi)} d\eta = 2i\pi(k(\xi) + c_1^h)\sigma_h(\xi) + \oint_{C''_h} \frac{\sigma_h(\eta)(k(\eta) + c_1^h)}{(\eta - \xi)} d\eta
$$

with $C_h$ surrounding $C''_h$. Thus:

$$
\tilde{\Xi}_0(f_0)(z) = \frac{1}{4\pi^2} \sum_h \oint_{C_h} \oint_{C''_h} \frac{\sigma_h(\eta)(k(\eta) + c_1^h)}{(z - \xi)(\eta - \xi)\sigma_h(\xi)} d\eta d\xi + \frac{1}{4\pi^2} \sum_h \oint_{C_h} \frac{2i\pi(k(\xi) + c_1^h)}{z - \xi} d\xi.
$$

Letting each $C_h$ go to infinity, we see that the first integral goes to zero and using Cauchy formula again we see that the second term equals $k(z) + c^1$.

We now prove $\tilde{\Xi}_0(f_1) = 0$.

$$
\tilde{\Xi}_0(f_1)(z) = -\frac{i}{4\pi^2} \sum_h \oint_{C_h} \frac{c_2^h S(\xi)\sigma(\xi)}{\sigma_h(\xi)(\sigma_h(\xi)S(\xi))(z - \xi)} d\xi = -\frac{i}{4\pi^2} \sum_h \oint_{C_h} \frac{c_2^h}{\sigma_h(\xi)(z - \xi)} d\xi = 0
$$

where we let the contours go to infinity.

By the exact same reasoning, we show that $\tilde{\Xi}_1(f_0) = k + c^1$ and $\tilde{\Xi}_1(f_1) = 0$.

By setting $\kappa_k = c_t + c_1^h$ on each $U_h$ we have the desired result. The unicity of $\kappa_k$ is implied by the previous lemma. Formula (2.15) can be easily deduced by (2.14).

\[\Box\]

**Remark 2.12.** By Lemma [2.11] and (2.14), if $k$ defined on $U \times U$ is analytic in each variable then $f$ defined on $U \times U$ and solution of

$$
\Xi f(\cdot, y) = k(x, y) + \kappa_k(x, y) \forall y \in U,
$$

with $\kappa(\cdot, y)$ constant on each $U_h$ is analytic in each variable.
We can now construct our approximate solution of the Monge-Ampère equation. As we want the domain \( B \) to be fixed by the flow of this approximate solution, we would like to choose \( y, \epsilon_{1,t} \) and \( z, \epsilon_{t} \) vanishing at the boundaries of \( B \) (and \( B \times B \)). Fix \( \delta > 0 \) small and denote \( B^\delta = \bigcup_{0 \leq h \leq g} [\beta h_-, \beta h_+ - \delta] \).

For a function \( f : B \rightarrow \mathbb{R} \) let \( \Upsilon(f) \) be the multiplication of \( f \) by a smooth plateau function equal to 1 on \( B^\delta \) and 0 outside \( B \). If we are given a function \( k \in O(U) \) and \( f \in O(U) \) satisfying \( \Xi(f) = k + \kappa k \), then:

- \( \Upsilon(f) = f \) on \( B^\delta \).
- \( \Upsilon(f) \) is \( C^\infty \) and has compact support in \( B \) (and can thus be extended by 0 to \( \mathbb{R} \)).
- \( \Xi(\Upsilon(f)) = k + \kappa k \) on \( B^\delta \) (By definition of \( \Xi \) and the fact that \( f \) and \( \Upsilon(f) \) coincide on \( B^\delta \)).
- \(|\Upsilon(f)|_{C^j(\mathbb{R})} \leq C_j |k|_{C^{j+2}(B)} \) for some constants \( C_j \).

Note that by Remark 2.5 possibly by shrinking \( \tilde{E} \) we can assume \( \tilde{T}_t < 0 \) outside \( B^\delta \). Thus for \( N \) large enough and a constant \( \eta > 0 \)

\[
\mathbb{P}^N_\mathbb{E} (\exists i \lambda_i \notin B^\delta) \leq \exp(-N\eta).
\]  

Moreover

\[
\int \left( \int |k - \Xi(\Upsilon(f))|dM_N \right) d\mathbb{P}^N_{T_i,B} \leq \int \left( \int |k - \Xi f|dM_N \right) d\mathbb{P}^N_{T_i,B} + \int \left( \int |\Xi f - \Xi(\Upsilon(f))|dM_N \right) d\mathbb{P}^N_{T_i,B}.
\]  

The first term on the right hand side is 0 as \( \kappa k \) is constant on each \( B_h \) and the second term is exponentially small by the large deviation estimate.

We first choose

\[
\begin{align*}
\tilde{z}^\epsilon(\cdot,y) &= \frac{1}{2} \Xi^{-1}(W(\cdot,y)) \quad \forall y \in B \\
\tilde{y}^\epsilon_{1,t} &= \left( \frac{\beta}{2} - 1 \right) \Xi^{-1} \left( \int \partial_1 \tilde{z}^\epsilon(z,\cdot) d\mu^\epsilon_{1,t}(z) \right)
\end{align*}
\]

and then

\[
\begin{align*}
z^\epsilon(\cdot,y) &= \Xi(\tilde{z}^\epsilon(\cdot,y)) \quad \forall y \in B \\
y^\epsilon_{1,t} &= \Xi(\tilde{y}^\epsilon_{1,t})
\end{align*}
\]

With this choice of function and by inequality (2.20) we have that

\[
\mathcal{R} = E + C_1^N + o \left( \frac{1}{N} \right).
\]  

We now have to control the error term \( E \). To do so we will use a direct consequence of the concentration result proved in Corollary 3.5 of [6] (adapted from a result from [18]):
Proposition 2.13. Let $V$ satisfy Hypothesis $[1,3]$ and $T_i$ is as in $(2.5)$. Then there exist constants $c$, $c'$ and $s_0$ such that for $N$ large enough, $s \geq s_0\sqrt{\frac{\log N}{N}}$, and for any $\epsilon = N/N \in \mathcal{E}$, $t \in [0; 1]$ we have

$$
\mathbb{E}_{T_i,B}^{N,\epsilon} \left( \sup_{\phi \in C^1(B)} \left| \int \phi(x)d(L_N - \mu_N^\epsilon)(x) \right| \geq s \right) \leq \exp(-cN^2s^2) + \exp(-c'N^2).
$$

(2.22)

In order to control the error term we will make use of the following three loop equations. We recall that $M_N = N(L_N - \mathbb{E}_{T_i,B}^{N,\epsilon}[L_N])$.

Lemma 2.14. Let $f \in C^1(B)$ such that for all $0 \leq h \leq g$, $f(\beta_{h,-}) = f(\beta_{h,+}) = 0$. Then

$$
\mathbb{E}_{T_i,B}^{N,\epsilon} \left( M_N(\Xi f) + \left(1 - \frac{\beta}{2}\right) L_N(f') + \frac{1}{N} \left[ \int \int \left( \frac{\beta f(x) - f(y)}{x - y} + \partial_1 T_i(x,y)f(x) \right) dM_N(x)dM_N(y) \right] \right) = 0.
$$

(2.23)

If $k_1$ is also in $C^1(B)$ then

$$
\mathbb{E}_{T_i,B}^{N,\epsilon} \left( L_N(fk_1') + M_N(\Xi f)M_N(k_1) + \left(1 - \frac{\beta}{2}\right) L_N(f')M_N(k_1) \right.
\left. + \frac{1}{N} \left[ \int \int \left( \frac{\beta f(x) - f(y)}{x - y} + \partial_1 T_i(x,y)f(x) \right) dM_N(x)dM_N(y) \right] \right) = 0.
$$

(2.24)

If $k_2$ and $k_3$ are also in $C^1(B)$ then

$$
\mathbb{E}_{T_i,B}^{N,\epsilon} \left( \sum_{\sigma} L_N(fk_{\sigma(1)}')M_N(k_{\sigma(2)})M_N(k_{\sigma(3)}) + M_N(\Xi f)M_N(k_1)M_N(k_2)M_N(k_3) \right.
\left. + \frac{1}{N} \left[ \int \int \left( \frac{\beta f(x) - f(y)}{x - y} + \partial_1 T_i(x,y)f(x) \right) dM_N(x)dM_N(y) \right] \right) = 0.
$$

(2.25)

where the sum ranges over the permutations of $\mathcal{S}_3$.

Proof. Using integration by parts we show

$$
\mathbb{E}_{T_i,B}^{N,\epsilon} \left( \int \int \left( \frac{\beta f(x) - f(y)}{x - y} + \partial_1 T_i(x,y)f(x) \right) dL_N(x)dL_N(y) + \frac{1}{N} \left(1 - \frac{\beta}{2}\right) L_N(f') \right) = 0
$$

(2.26)

we deduce the first loop equation by using the definition of $\Xi$.

The second loop equation is obtained by replacing in (2.26) $T_i(x,y)$ by $T_i(x,y) - \delta_1(k_1(x) + k_1(y))$ and differentiating at $\delta = 0$.

The third one is obtained by replacing in (2.26) $T_i(x,y)$ by $T_i(x,y) - \delta_1(k_1(x) + k_1(y)) - \delta_2(k_2(x) + k_2(y)) - \delta_3(k_3(x) + k_3(y))$ and differentiating at $\delta_1 = \delta_2 = \delta_3 = 0$. □
We will now put in use these loop equations and the concentration result of Proposition 2.13 to obtain some estimates.

**Lemma 2.15.** Let $k$ be an analytic function on $U$. Then for some constant $C$:

$$|\mathbb{E}^N_{T,B} (M_N(k))| \leq C \log N \|k\|_{C^6(B)}.$$  

$$\mathbb{E}^N_{T,B} (M_N(k)^2) \leq C (\log N)^2 \|k\|_{C^8(B)}^2.$$  

$$\mathbb{E}^N_{T,B} (M_N(k)^4) \leq C (\log N)^4 \|k\|_{C^6(B)}^4.$$  

**Proof.** We apply (2.23) to $f = \Upsilon(\Xi^{-1}k)$. Using (2.20) we obtain

$$\mathbb{E}^N_{T,B} (M_N(k) + \frac{1}{N} \left[ \frac{\beta}{2} \int \int \left( \frac{\Upsilon(\Xi^{-1}k)(x) - \Upsilon(\Xi^{-1}k)(y)}{x-y} + \frac{1}{2} L_N((\Upsilon(\Xi^{-1}k))') \right) dM_N(x)dM_N(y) \right]$$

$$+ \left( 1 - \frac{\beta}{2} \right) L_N((\Upsilon(\Xi^{-1}k))').$$

Let

$$A(k) = \frac{1}{N} \left[ \frac{\beta}{2} \int \int \left( \frac{\Upsilon(\Xi^{-1}k)(x) - \Upsilon(\Xi^{-1}k)(y)}{x-y} + \frac{1}{2} L_N((\Upsilon(\Xi^{-1}k))') \right) dM_N(x)dM_N(y) \right]$$

$$+ \left( 1 - \frac{\beta}{2} \right) L_N((\Upsilon(\Xi^{-1}k))').$$

Denoting by $\mathcal{F}$ the fourier transform operator (for functions of either one or several variables) we have

$$\int \int \frac{\Upsilon(\Xi^{-1}k)(x) - \Upsilon(\Xi^{-1}k)(y)}{x-y} dM_N(x)dM_N(y)$$

$$= i \int \left( \int_0^1 d\alpha \int e^{i\alpha \xi x} dM_N(x) \int e^{i(1-\alpha)\xi y} dM_N(y) \right) \mathcal{F}(\Upsilon(\Xi^{-1}k))(\xi) \xi d\xi$$

and

$$\int \int \partial_1 T_t(x,y) \Upsilon(\Xi^{-1}k)(x) dM_N(x)dM_N(y)$$

$$= \int \left( \int e^{i\xi x} dM_N(x) \int e^{i\xi y} dM_N(y) \right) \mathcal{F}(\partial_1 T_t \ Upsilon(\Xi^{-1}k))(\xi, \zeta) d\xi d\zeta.$$  

Now on the set $\Omega = \left\{ \sup_{\phi \in C_c^2(B)} |\int \phi(x) d(L_N - \mu_{\nu}) (x)| \leq s_0 \sqrt{\frac{\log N}{N}} \right\}$ we have

$$|\int e^{i\xi x} dM_N(x)| \leq |\int \Upsilon(e^{i\xi})(x) dM_N(x)| + 2N \epsilon^{-N\eta}$$

$$\leq C(1 + |\xi|) \sqrt{N \log N} + 2N \epsilon^{-N\eta}$$

consequently, on this set.
\[
\left| \int \frac{\Upsilon(\Xi^{-1}k)(x) - \Upsilon(\Xi^{-1}k)(y)}{x - y} dM_N(x) dM_N(y) \right| \leq C(N \log N) \int |\mathcal{F}(\Upsilon(\Xi^{-1}k))(\xi)|(1 + |\xi|)^3 d\xi + O(N e^{-N^\eta}).
\]

The integral is bounded by the norm \(\mathcal{H}^4(\mathbb{R})\) of \(\Upsilon(\Xi^{-1}k)\) and we have:
\[
\| \Upsilon(\Xi^{-1}k) \|_{\mathcal{H}^4(\mathbb{R})} \leq C \left( \| \Upsilon(\Xi^{-1}k) \|_{L^2(\mathbb{R})} + \| (\Upsilon(\Xi^{-1}k))^{(4)} \|_{L^2(\mathbb{R})} \right).
\]

As \(\Upsilon(\Xi^{-1}k)\) has its support in \(B\), the \(L^2(\mathbb{R})\) norm can be in turn controlled by the \(L^\infty(\mathbb{R})\) norm and we can use (2.15). Similarly on \(\Omega\) we have
\[
\left| \int \partial_t T_t(x, y) \Upsilon(\Xi^{-1}k)(x) dM_N(x) dM_N(y) \right| \leq C(N \log N) \|k\|_{C^6(B)} + O(N e^{-N^\eta})
\]

Note that here the constant depends on \(T_t\) but we can make it uniform in \(t\) and \(\epsilon \in \tilde{\mathcal{E}}\). On \(\Omega^c\) we can use the trivial bound
\[
\left| \int e^{i\xi x} dM_N(x) \right| \leq 2N
\]

to prove that \(|\Lambda(k)|\) is bounded everywhere by \(CN \|k\|_{C^6(B)}\). By using Proposition 2.13 we obtain
\[
|\mathbb{E}_{T_t, B}^N(\Lambda(k))| \leq C \left( (\log N) \|k\|_{C^6(B)} + N e^{-c_6 N \log N} \|k\|_{C^6(B)} \right)
\]

and we can conclude the proof of the first inequality.

To prove the second inequality, using (2.24) and (2.20) we have
\[
\mathbb{E}_{T_t, B}^N(M_N(k) \tilde{M}_N(k)) = -\mathbb{E}_{T_t, B}^N(\Lambda(k) \tilde{M}_N(k) + L_N(k' \Upsilon(\Xi^{-1}k))) + O(N^2 \|k\|_{L^\infty(B)}^2 e^{-N^\eta}).
\]

By splitting on \(\Omega\) and \(\Omega^c\) we see that
\[
\left| \mathbb{E}_{T_t, B}^N(M_N(k) \tilde{M}_N(k)) \right| \leq C \left( (\log N) \|k\|_{C^6(B)} \mathbb{E}_{T_t, B}^N(\tilde{M}_N(k)) \right) + \|k\|_{C^6(B)}^2 \left( 1 + N^2 e^{-c_6 N \log N} + N^2 e^{-N^\eta} \right)
\]

We notice that \(M_N(k) - \tilde{M}_N(k) = \mathbb{E}_{T_t, B}^N(M_N(k))\) is deterministic and that \(\mathbb{E}_{T_t, B}^N(\tilde{M}_N(k))\) vanishes. The term on the left is thus equal to \(\mathbb{E}_{T_t, B}^N(\tilde{M}_N(k)^2)\) and we obtain
\[
\mathbb{E}_{T_t, B}^N(\tilde{M}_N(k)^2) \leq C \left( (\log N) \|k\|_{C^6(B)} \sqrt{\mathbb{E}_{T_t, B}^N(\tilde{M}_N(k)^2)} + \|k\|_{C^6(B)}^2 \left( 1 + N^2 e^{-c_6 N \log N} + N^2 e^{-N^\eta} \right) \right).
\]

Elementary manipulations show that this implies that \(\mathbb{E}_{T_t, B}^N(\tilde{M}_N(k)^2) \leq C(\log N)^2 \|k\|_{C^6(B)}^2\) with a different constant.
Writing

\[ E_{T_i,B}^N (M_N(k)^2) \leq 2 \ E_{T_i,B}^N (\tilde{M}_N(k)^2) + 2 \ E_{T_i,B}^N ((\tilde{M}_N(k) - M_N(k))^2) \]

\[ = 2 \ E_{T_i,B}^N (\tilde{M}_N(k)^2) + 2 \ E_{T_i,B}^N (M_N(k))^2 \]

and using the first inequality yields to the second one.

Finally, to prove the last inequality, \([225]\) gives:

\[ E_{T_i,B}^N (\tilde{M}_N(k)^4) \leq C \left( \text{log } N \| k \|^4_{C^6(B)} + \| k \|^4_{C^6(B)} \right) \]

which shows \( E_{T_i,B}^N (\tilde{M}_N(k)^4) \leq C \left( \text{log } N \right)^4 \| k \|^4_{C^6(B)}. \) We conclude by using the identity

\[ E_{T_i,B}^N (M_N(k)^4) \leq 8 \ E_{T_i,B}^N (\tilde{M}_N(k)^4) + 8 \ E_{T_i,B}^N ((\tilde{M}_N(k) - M_N(k))^4). \]

\[ \square \]

We will need a last lemma to estimate the error \( E. \)

**Lemma 2.16.** There exists a constant \( C \) such that for \( \phi \in C^\infty(\mathbb{R}) \) (resp. \( \psi \in C^\infty(\mathbb{R}^2), \chi \in C^\infty(\mathbb{R}^3) \)) of compact support in \( B \) (resp. \( B^2, B^3 \)) we have

\[
E_{T_i,B}^N \left( \int \phi(x) dM_N(x) \right) \leq C \| \phi \|_{C^6(B)} \log N
\]

\[
E_{T_i,B}^N \left( \left\| \int \frac{\phi(x) - \phi(y)}{x-y} dM_N(x) dM_N(y) \right\| \right) \leq C \| \phi \|_{C^4(B)} \log N^2
\]

\[
E_{T_i,B}^N \left( \left\| \int \psi(x,y) dM_N(x) dM_N(y) \right\| \right) \leq C \| \psi \|_{C^4(B^2)} \log N^2
\]

\[
E_{T_i,B}^N \left( \left\| \int \int \chi(x,y,z) dM_N(x) dM_N(y) dM_N(z) \right\| \right) \leq C \| \chi \|_{C^4(B^3)} \log N^3
\]

\[
E_{T_i,B}^N \left( \left\| \int \int \frac{\psi(x,y) - \psi(z,y)}{x-z} dM_N(x) dM_N(y) dM_N(z) \right\| \right) \leq C \| \psi \|_{C^4(B^2)} \log N^3
\]

**Proof.** We will prove the last inequality as the other ones are simpler and can be proved the same way.

\[
\int \int \frac{\psi(x,y) - \psi(z,y)}{x-z} dM_N(x) dM_N(y) dM_N(z) = i \int \int \left( \int_0^1 dM_N(e^{i(1-\alpha)\xi}) M_N(e^{i\xi}) \right) \mathcal{F}(\partial_1 \psi)(\xi,\zeta) d\xi d\zeta
\]

and by using Hölder inequality we obtain

\[
E_{T_i,B}^N \left( \left\| \int \int \frac{\psi(x,y) - \psi(z,y)}{x-z} dM_N(x) dM_N(y) dM_N(z) \right\| \right)
\]

\[
\leq \int \int \left( \int_0^1 dM_N(e^{i(1-\alpha)\xi}) \left( \int_0^1 dM_N(e^{i\xi}) \right)^{\frac{1}{4}} \right)^4 \left( \int_0^1 dM_N(e^{i(1-\alpha)\xi}) \left( \int_0^1 dM_N(e^{i\xi}) \right)^{\frac{1}{4}} \right)^4 \left( \int_0^1 dM_N(e^{i\xi}) \right)^{\frac{1}{4}} |\xi| \mathcal{F}(\psi)(\xi,\zeta) d\xi d\zeta
\]

\[
\leq C(\text{log } N)^3 \int (1 + |\xi|^2)^2 (1 + |\zeta|^6) |\xi| \| \mathcal{F}(\psi)(\xi,\zeta) \| d\xi d\zeta
\]
where we used the last identity of Lemma 2.15. The last term is controled by the $H^{21}(\mathbb{R}^2)$ norm of $\psi$ and we have

$$
\|\psi\|_{H^{21}(\mathbb{R}^2)} \leq C\left(\|\psi\|_{L^2(\mathbb{R}^2)} + \sup_{|\beta| \leq 21} \left\|\partial^\beta \psi\right\|_{L^2(\mathbb{R}^2)}\right) \leq C \|\psi\|_{C^{21}(B^2)}.
$$

\[ \square \]

A direct application of this lemma shows that $\mathbb{E}_{T_1,B}^{N,\epsilon}(|E|) \leq C\left(\frac{\log N}{N}\right)^3$, and we could prove similarly using higher order loop equations that for all integer $k \geq 1$

$$
\left(\mathbb{E}_{T_1,B}^{N,\epsilon}(|E|^{2k})\right)^{1/2k} \leq C_k \left(\frac{\log N}{N}\right)^3.
$$

(2.27)

In order to prove Proposition 2.9 it remains to control the deterministic term $C_t^{N,\epsilon}$. Let

$$
\mathcal{L}(\mathbf{Y}) = \beta \sum_{h=0}^g \sum_{1 \leq i < j \leq N_h} \frac{\mathbf{Y}_{h,i} - \mathbf{Y}_{h,j}}{\lambda_{h,i} - \lambda_{h,j}} + \beta \sum_{0 \leq h < h' \leq g} \sum_{1 \leq j \leq N_{h'}} \frac{\mathbf{Y}_{h,i} - \mathbf{Y}_{h',j}}{\lambda_{h,i} - \lambda_{h',j}}
$$

$$
+ \sum_{0 \leq h, h' \leq g} \sum_{1 \leq i \leq N_h, 1 \leq j \leq N_{h'}} (\partial_i T(\lambda_{h,i}, \lambda_{h',j})\mathbf{Y}_{h,i,j}) + \text{div}(\mathbf{Y})
$$

Integration by parts show that any vector field $\mathbf{Y}$ that vanishes on the boundary of $B$ satisfies $\mathbb{E}_{T_1,B}^{N,\epsilon}(\mathcal{L}(\mathbf{Y})) = 0$. Thus

$$
\mathbb{E}_{T_1,B}^{N,\epsilon}(\mathcal{R}_t^{N,\epsilon}(\mathbf{Y}^{N,\epsilon})) = \mathbb{E}_{T_1,B}^{N,\epsilon}\left(\mathbf{R}_t^{N,\epsilon}(\mathbf{Y}^{N,\epsilon}) - c_t^{N,\epsilon}\right) = 0,
$$

and by (2.21)

$$
|c_t^{N,\epsilon}| = \mathbb{E}_{T_1,B}^{N,\epsilon}(\mathcal{R}_t^{N,\epsilon}(\mathbf{Y}^{N,\epsilon}) - E) + o\left(\frac{1}{N}\right) \leq C \left(\frac{\log N}{N}\right)^3.
$$

2.4 Obtaining the Transport map via the flow

In this section we will discuss the properties of the transport map given by the flow of the approximate solution $\mathbf{Y}^{N,\epsilon}$ of the Monge-Ampère equation. As the equilibrium measures of the initial potential and the target potential are the same, this map is equal to the identity at the first order. The smaller order are then given by the expansion (2.11) of $\mathbf{Y}^{N,\epsilon}$.

Lemma 2.17. Let $V$ satisfy Hypothesis 1.2, $T_t$ is as in (2.5) and $\epsilon = N/N \in \mathbb{E}$. Then the flow $X_t^{N,\epsilon}$ can be written

$$
X_t^{N,\epsilon} = \text{Id} + \frac{1}{N} X_t^{N,\epsilon,1} + \frac{1}{N^2} X_t^{N,\epsilon,2}
$$

(2.28)

where $X_t^{N,\epsilon,1}$ and $X_t^{N,\epsilon,2}$ are in $C^\infty(\mathbb{R}^N)$ supported in $B$, and for some constant $C > 0$
As in Lemma 2.16, we can prove that the last term is of order \( \log N \) and similarly, this proves (2.30). We now have to bound the norm of 

\[
\sup_{0 \leq h \leq g} \left\| X_{h,i,t}^{N,\epsilon,1} \right\|_{L^4(P_{V,B}^N)} \leq C \log N, \quad \left\| X_{h,i,t}^{N,\epsilon,2} \right\|_{L^2(P_{V,B}^N)} \leq C \sqrt{N} (\log N)^2 \tag{2.29}
\]

and with probability greater than \( 1 - N^{-\frac{1}{2}} \)

\[
\sup_{0 \leq h \leq g} \left| X_{h,i,t}^{N,\epsilon,1}(\lambda) - X_{h,j,t}^{N,\epsilon,1}(\lambda) \right| \leq C \sqrt{N} \log N |\lambda_{h,i} - \lambda_{h,j}| \tag{2.30}
\]

\[
\sup_{0 \leq h \leq g} \left| X_{h,i,t}^{N,\epsilon,2}(\lambda) - X_{h,j,t}^{N,\epsilon,2}(\lambda) \right| \leq C N \sqrt{N} \log N |\lambda_{h,i} - \lambda_{h,j}| \tag{2.31}
\]

\[
\sup_{0 \leq h \leq g} \left\| X_{h,i,t}^{N,\epsilon,1} \right\|_{\infty} \leq C \sqrt{N} \log N, \quad \sup_{0 \leq h \leq g} \left\| X_{h,i,t}^{N,\epsilon,2} \right\|_{\infty} \leq C N \sqrt{N} \log N \tag{2.32}
\]

**Proof.** The expansion (2.11) suggests to define \( X_{t}^{N,\epsilon,1} = (X_{0,i,t}^{N,\epsilon,1}, \ldots, X_{g,N,i,t}^{N,\epsilon,1}) \) as the solution of the linear ODE

\[
X_{h,i,t}^{N,\epsilon,1}(\lambda) = y_{1,t}(\lambda_{h,i}) + \int z_{t}^{\epsilon}(\lambda_{h,i}, y) dM_{N}(y) + \frac{1}{N} \sum_{0 \leq h' \leq g} \sum_{1 \leq i \leq N_{h'}} \partial_{2} x_{t}^{\epsilon}(\lambda_{h,i}, \lambda_{h',j}) X_{h',j,t}^{N,\epsilon,1}(\lambda) \tag{2.33}
\]

with initial condition \( X_{t}^{N,\epsilon,1}(\lambda) = 0 \). We then define \( X_{t}^{N,\epsilon,2} \) through the identity (2.28).

Using the fact that \( y_{1,t}^{\epsilon} \) and \( z_{t}^{\epsilon} \) have compact support and are thus bounded, along with equation (2.33), we obtain:

\[
\frac{d}{dt} \left( \sup_{0 \leq h \leq g} \left\| X_{h,i,t}^{N,\epsilon,1} \right\|_{L^4(P_{V,B}^N)} \right) \leq C \left( 1 + \sup_{0 \leq h \leq g} \left\| X_{h,i,t}^{N,\epsilon,1} \right\|_{L^4(P_{V,B}^N)} + \sup_{0 \leq h \leq g} \left\| \int z_{t}^{\epsilon}(\lambda_{h,i}, y) dM_{N}(y) \right\|_{L^4(P_{V,B}^N)} \right).
\]

As in Lemma 2.13, we can prove that the last term is of order \( \log N \). Using Grönwall’s Lemma, this proves

\[
\sup_{0 \leq h \leq g} \left\| X_{h,i,t}^{N,\epsilon,1} \right\|_{L^4(P_{V,B}^N)} \leq C \log N. \tag{2.34}
\]

Furthermore, Proposition 2.13 shows that for some constant \( C \), with probability greater than \( 1 - N^{-\frac{1}{2}} \) we have

\[
\left\| \int \partial_{2} x_{t}^{\epsilon}(\cdot, y) dM_{N}(y) \right\|_{\infty} \leq C \sqrt{N} \log N
\]

and similarly, this proves (2.30). We now have to bound the norm of \( X_{t}^{N,\epsilon,2} \). For \( s \in [0, 1] \) let

\[
X_{t}^{s,N,\epsilon} = Id + \frac{s}{N} X_{t}^{N,\epsilon,1} + \frac{s}{N^2} X_{t}^{N,\epsilon,2} = (1 - s) Id + s X_{t}^{N,\epsilon}
\]

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We then use the bounds

\[ \int f(y) dM_N^{X_{h,i,t}}(y) = \sum_{0 \leq h \leq g} \sum_{1 \leq i \leq N_h} f(X_{h,i,t}^{s,N}) - N \int f d\mu_t. \]

Then a Taylor expansion gives us an ODE for \( X_t^{N,e,2} \)

\[
\begin{align*}
X_{h,i,t}^{N,e,2}(\lambda) &= \int_0^1 (y_{h,i,t})' \left( X_{h,i,t}^{s,N,e}(\lambda) \right) ds \left( X_{h,i,t}^{s,N,e}(\lambda) + \frac{1}{N} X_{h,i,t}^{s,N}(\lambda) \right) \\
&\quad + \int_0^1 \left[ \int \partial_1 z_t^e \left( X_{h,i,t}^{s,N,e}(\lambda), y \right) dM_N^{X_{h,i,t}}(y) - \int \partial_1 z_t^e \left( \lambda_{h,i}, y \right) dM_N(y) \right] ds \left( X_{h,i,t}^{s,N,e}(\lambda) + \frac{1}{N} X_{h,i,t}^{s,N,e}(\lambda) \right) \\
&\quad + \int \partial_1 z_t^e \left( \lambda_{h,i}, y \right) dM_N(y) \left( X_{h,i,t}^{s,N,e}(\lambda) + \frac{1}{N} X_{h,i,t}^{s,N,e}(\lambda) \right) \\
&\quad + \sum_{0 \leq h' \leq g} \sum_{1 \leq j \leq N_{h'}} \int_0^1 \left[ \partial_2 z_t^e \left( X_{h,i,t}^{s,N,e}(\lambda), X_{h',j,t}^{s,N,e}(\lambda) \right) \right] ds \frac{X_{h',j,t}^{N,e,2}(\lambda)}{N}.
\end{align*}
\]

We then use the bounds

\[
\int_0^1 \left| \int \partial_1 z_t^e \left( X_{h,i,t}^{s,N,e}(\lambda), y \right) dM_N^{X_{h,i,t}}(y) \right| ds \\
\leq C |X_{h,i,t}^{s,N,e}| + \frac{C}{N} |X_{h,i,t}^{s,N,e,2}| + \frac{C}{N} \sum_{h',j} \left( |X_{h',j,t}^{s,N,e}| + \frac{1}{N} |X_{h',j,t}^{s,N,e,2}| \right),
\]

\[
\sum_{h',j} \int_0^1 \left| \partial_2 z_t^e \left( X_{h,i,t}^{s,N,e}(\lambda), X_{h',j,t}^{s,N,e}(\lambda) \right) \right| ds |X_{h',j,t}^{s,N,e}(\lambda)| \\
\leq \frac{C}{N} \sum_{h',j} \left( |X_{h',j,t}^{s,N,e}|^2 + \frac{1}{N} |X_{h',j,t}^{s,N,e,2}| |X_{h',j,t}^{s,N,e,1}| \right).
\]
to obtain

\[
\frac{d}{dt} \left\| X_{t}^{N,e,2} \right\|_{L^2(\mathbb{P}^N_{V,B})}^2 \leq C \mathbb{E}_{V,B} \left( \sum_{h,i} X_{h,i,t}^{N,e,1} \right)^2 + \frac{C}{N} \mathbb{E}_{V,B} \left( \sum_{h,i} X_{h,i,t}^{N,e,1} \right)^2 \\
+ C \mathbb{E}_{V,B} \left( \sum_{h,i} X_{h,i,t}^{N,e,2} \right)^2 + \frac{C}{N} \mathbb{E}_{V,B} \left( \sum_{h,i} X_{h,i,t}^{N,e,2} \right)^2 \\
+ \frac{C}{N^2} \mathbb{E}_{V,B} \left( \sum_{h,i} X_{h,i,t}^{N,e,3} \right)^2 \\
+ \frac{C}{N^2} \mathbb{E}_{V,B} \left( \sum_{h,i} X_{h,i,t}^{N,e,2} \right)^2 + \frac{C}{N^3} \mathbb{E}_{V,B} \left( \sum_{h,i} X_{h,i,t}^{N,e,2} \right)^2 \\
+ \frac{C}{N^2} \mathbb{E}_{V,B} \left( \sum_{h,i} X_{h,i,t}^{N,e,2} \right)^2 + \frac{C}{N^2} \mathbb{E}_{V,B} \left( \sum_{h,i} X_{h,i,t}^{N,e,2} \right)^2 \\
+ \frac{C}{N} \mathbb{E}_{V,B} \left( \sum_{h,i} X_{h,i,t}^{N,e,2} \right)^2 + \frac{C}{N} \mathbb{E}_{V,B} \left( \sum_{h,i} X_{h,i,t}^{N,e,2} \right)^2.
\]

(2.36)

Using the bounds \( \left\| \int \frac{d}{dt} \mathbb{P}^{N,e}(\lambda_{h,i}, y) dM_N(y) \right\|_{L^2(\mathbb{P}^N_{V,B})} \leq C \log N \) (see Lemma 2.16), \( |X_{h,i,t}^{N,e,1}| \leq C N \), \( |X_{h,i,t}^{N,e,2}| \leq C N^2 \) and inequalities such as

\[
\sum_{h,i} X_{h,i,t}^{N,e,1} \leq \frac{1}{2} \left( \sum_{h,i} X_{h,i,t}^{N,e,1} \right)^2 + \left( X_{h,i,t}^{N,e,2} \right)^2 \\
\sum_{h,i} X_{h,i,t}^{N,e,1} \leq \left( \sum_{h,i} X_{h,i,t}^{N,e,1} \right)^2 + \left( X_{h,i,t}^{N,e,2} \right)^2 + \left( X_{h,i,t}^{N,e,2} \right)^2.
\]

along with (2.31) and Hölder inequality, we get

\[
\frac{d}{dt} \left\| X_{t}^{N,e,2} \right\|_{L^2(\mathbb{P}^N_{V,B})}^2 \leq C \left( \left\| X_{t}^{N,e,2} \right\|_{L^2(\mathbb{P}^N_{V,B})}^2 \right) + N (\log N)^4.
\]

(2.37)

Using Grönwall’s Lemma, we can conclude the proof. The bounds (2.31) and (2.32) are proven the same way.
Remark 2.18. Using (2.32), (2.33) and (2.36) we see that we have in fact for all integer \( k \geq 1 \)

\[
\sup_{0 \leq h < g} \left\| X_{h,i,t}^{N,e,1} \right\|_{L^{2k}(P_{V,B}^{N,e})} \leq C_k \log N \quad \text{and} \quad \sup_{0 \leq h < g} \left\| X_{h,i,t}^{N,e,2} \right\|_{L^{2k}(P_{V,B}^{N,e})} \leq C_k \sqrt{N}(\log N)^2
\]

3 From Transport to Universality

In this section we will prove Proposition 2.6 and Corollary 2.7. We prove the results in the bulk as the proof is almost identical for the edge result.

Proof of Proposition 2.6. Note that by Lemma 2.8 and by our construction of \( Y_{i}^{N,e} \), \( X_{1}^{N,e} \) is an approximate transport map from \( P_{V,B}^{N,e} \) to \( P_{T_{i},B}^{N,e} \) in the sense that it satisfies (2.7). Now, keeping our notations from the previous section, set \( \tilde{X}^{N,e} = Id + \frac{1}{N} X_{1}^{N,e,1} \). Then for all \( f \in C^{1}(\mathbb{R}) \)

\[
\left| \int f(\tilde{X}^{N,e})dP_{V,B}^{N,e} - \int f(X_{1}^{N,e})dP_{V,B}^{N,e} \right| \leq \frac{\|\nabla f\|_{\infty}}{N^2} \int |X_{1}^{N,e,2}|dP_{V,B}^{N,e}
\]

\[
\leq \frac{\|\nabla f\|_{\infty}}{N^2} \left\| X_{1}^{N,e,2} \right\|_{L^{2}(P_{V,B}^{N,e})} \leq \|\nabla f\|_{\infty} \left( \frac{\log N}{N} \right)^2 \frac{(\log N)^2}{N^2}
\]

and thus

\[
\left| \int f(\tilde{X}^{N,e})dP_{V,B}^{N,e} - \int f dP_{T_{1},B}^{N,e} \right| \leq C\left( \frac{\log N}{N} \right)^{2} \|f\|_{\infty} + \|\nabla f\|_{\infty} \frac{(\log N)^2}{N^2}.
\]

Now for all \( 0 \leq h \leq g \) let \( R^{h} : B_{h}^{N,h} \rightarrow B_{h}^{N,h} \) the ordering map (i.e the map satisfying for all \( (\lambda_{1}, \ldots, \lambda_{N}) \in B_{h}^{N,h} \) \( R^{h,i}(\lambda_{1}, \ldots, \lambda_{N}) \leq R^{h,j}(\lambda_{1}, \ldots, \lambda_{N}) \) if \( i < j \) and \( \{\lambda_{1}, \ldots, \lambda_{N} \} = \{R^{h,1}(\lambda_{1}, \ldots, \lambda_{N}), \ldots, R^{h,N}(\lambda_{1}, \ldots, \lambda_{N}) \} \), so that if \( R(\lambda) = (R^{h,1}(\lambda_{1}, \ldots, \lambda_{N}), \ldots, R^{h,N}(\lambda_{1}, \ldots, \lambda_{N}, \lambda_{N})) \) we have \( R_{\sharp}dP_{B}^{N,e} = dP_{B}^{N,e} \).

Then if \( f_{h} \) is a function of \( m \) variables, we have \( \|\nabla(f_{h} \circ R^{h})\|_{\infty} \leq \sqrt{m} \|\nabla f_{h}\|_{\infty} \).

It is clear from (2.30) that \( \tilde{X}^{N,e} \) preserves the order of the eigenvalues with probability greater than \( 1 - \frac{N}{2} \). Thus, if we define \( f : \mathbb{R}^{m(g+1)} \rightarrow \mathbb{R} \) by \( f(x_{0}, \ldots, x_{g}) = \prod_{0 \leq h \leq g} f_{h}(x_{h}) \) where \( f_{h} : \mathbb{R}^{m} \rightarrow \mathbb{R} \) we obtain

\[
\left| \int \prod_{0 \leq h \leq g} f_{h}(N(\lambda_{h,i_{h}+1} - \lambda_{h,i_{h}}), \ldots, N(\lambda_{h,i_{h}+m} - \lambda_{h,i_{h}}))dP_{T_{1},B}^{N,e} \right.
\]

\[
\left. - \int \prod_{0 \leq h \leq g} f_{h}(N(\tilde{X}_{h,i_{h}+1}^{N,e}(\lambda), \ldots, N(\tilde{X}_{h,i_{h}+m}^{N,e}(\lambda) - \tilde{X}_{h,i_{h}}^{N,e}(\lambda)))dP_{V,B}^{N,e} \right| \leq C\left( \frac{\log N}{N} \right)^{3} \|f\|_{\infty} + \|\nabla f\|_{\infty} \sqrt{m} \left( \frac{(\log N)^2}{N^2} \right).
\]
Now, using (2.30) we notice that with probability greater than \(1 - N^{-2}\), for all \(1 \leq k \leq m\) and \(0 \leq h \leq g\)

\[
\hat{X}_{h,i+h+k}^N,\varepsilon (\lambda) - \hat{X}_{h,i}^N,\varepsilon (\lambda) = \lambda_{h,i+k} - \lambda_{h,i} + (\lambda_{h,i+h+k} - \lambda_{h,i+h}) O\left(\frac{\log N}{\sqrt{N}}\right).
\]

As \(f_h\) has compact support in \([-M, M]^m\), \((\lambda_{h,i+h+k} - \lambda_{h,i+h})\) remains bounded by \(\frac{2M}{N}\) and

\[
\hat{X}_{h,i+h+k}^N,\varepsilon (\lambda) - \hat{X}_{h,i}^N,\varepsilon (\lambda) = \lambda_{h,i+k} - \lambda_{h,i} + O\left(\frac{M \log N}{N\sqrt{N}}\right),
\]

we easily deduce the first part of Proposition 2.6.

\[\square\]

Before proving Corollary 2.7 we recall Theorem 1.5 of [3].

**Proposition 3.1.** Assume that \(W\) is a potential satisfying Hypothesis 1.2 with \(g = 0\). Then for a constant \(C\) and for all \(m \in \mathbb{N}^t\) and \(f : \mathbb{R}^m \rightarrow \mathbb{R}\) Lipschitz and compactly supported in \([-M, M]\) we have

1. **In the Bulk**

\[
\left| \int f(N(\lambda_{i+1} - \lambda_i), \ldots , N(\lambda_{i+m} - \lambda_i)) d\hat{\mathbb{P}}_W^N - \int f(N(\Phi)'(\lambda_{i+1} - \lambda_i), \ldots , N(\Phi)'(\lambda_{i+m} - \lambda_i)) d\hat{\mathbb{P}}_G^N \right| \\
\leq C \left( \frac{(\log N)^3}{N} \right) \|f\|_{\infty} + C \left( \sqrt{m} \frac{(\log N)^2}{N^{1/2}} + \frac{\log N}{N^{1/3}} + \frac{M^2}{N^{1/3}} \right) \|\nabla f\|_{\infty}
\]

2. **At the Edge**

\[
\left| \int f(N^{2/3}(\lambda_1 - \alpha_-), \ldots , N^{2/3}(\lambda_m - \alpha_-)) d\hat{\mathbb{P}}_W^N - \int f(N^{2/3}(\Phi)'(-2)(\lambda_{i+1} + 2), \ldots , N^{2/3}(\Phi)'(-2)(\lambda_{i+m} + 2)) d\hat{\mathbb{P}}_G^N \right| \\
\leq C \left( \frac{(\log N)^3}{N} \right) \|f\|_{\infty} + C \left( \sqrt{m} \frac{(\log N)^2}{N^{5/6}} + \frac{\log N}{N^{1/3}} + \frac{M^2}{N^{1/3}} \right) \|\nabla f\|_{\infty}
\]

where \(\Phi\) is a transport map from \(\mu_G\) to \(\mu_W\), and we recall that \(G\) denotes the Gaussian potential.

**Proof of Corollary 2.7** Noticing that \(d\hat{\mathbb{P}}_{\psi_{\varepsilon},\varphi}^N\) is a product measure we can write

\[
\int \prod_{0 \leq h \leq g} f_h(N(\lambda_{i+h+1} - \lambda_{i+h}), \ldots , N(\lambda_{i+h+m} - \lambda_{i+h})) d\hat{\mathbb{P}}_{\psi_{\varepsilon},\varphi}^N = \frac{1}{Z_{\psi_{\varepsilon},\varphi}^N} \int \prod_{0 \leq h \leq g} \prod_{1 \leq i \leq N_h} 1_{B_h}(\lambda_{h,i}) d\lambda_{h,i} \\
\left[ f_h(N(\lambda_{i+h+1} - \lambda_{i+h}), \ldots , N(\lambda_{i+h+m} - \lambda_{i+h})) \prod_{1 \leq i \leq N_h} |\lambda_{h,i} - \lambda_{h,j}|^\beta \exp \left( - N \sum_{1 \leq i \leq N_h} \hat{V}_{\varepsilon}(\lambda_{h,i}) \right) \right] \\
= \prod_{0 \leq h \leq g} \int f_h(N(\lambda_{i+h+1} - \lambda_{i+h}), \ldots , N(\lambda_{i+h+m} - \lambda_{i+h})) d\hat{\mathbb{P}}_{\psi_{\varepsilon},\varphi}^N_{\lambda_{h,i} B_h}
\]

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We notice using (2.3) that
\[ \mu_{\tilde{V}/\epsilon_h, B_h} = \mu_{\tilde{V}}^{e_h}. \]
We conclude using Proposition 3.1.

4 Universality in the initial model

To derive universality in the initial model, we expand the expectation of the quantity we want to compute in terms of the filling fractions, and we make use of Corollary 2.7.

First, we notice that for all \( 0 \leq h \leq g \) the map \( \Phi^{e,h} \) is smooth in \( \epsilon \in \tilde{E} \) and we have a bound
\[
(\Phi^{e,h})'(\lambda_{h,i}) = (\Phi^{e^*,h})'(\lambda_{h,i}) + O(|\epsilon - \epsilon|) \quad \text{uniformly in } \lambda_{h,i} \in B
\]
Indeed, it is shown in [3] (4.1) that our transport map \( \Phi^{e,h} \) is equal to \( X^\epsilon_1 \) where \( X^\epsilon_t \) solves the ordinary differential equation
\[
\dot{X}^\epsilon_t = y^\epsilon_t(X^\epsilon_t), \quad X^\epsilon_0 = \text{Id}
\]
and \( y^\epsilon_t \) is given by inverting \( \Xi \). By formula (2.14) and Lemma 2.2, we see that \( y^\epsilon_t \) is regular in \( \epsilon \), and from the standard theory of ordinary differential equations, so is \( \Phi^{e} \).

We will use the following result proved in section 8.2, equations (8.18) and (8.19) of [4].

Lemma 4.1. Along the subsequences such that \( N_\kappa \mod Z^{g+1} \rightarrow \kappa \) where \( \kappa \in [0; 1[^{g+1} \text{ and under } P^N_{V,B}, \text{ the vector } [N_\kappa] - N(\lambda) \text{ converges towards a random discrete Gaussian vector } \Delta_{h,\kappa}. \text{ In particular }
\]
\[
P^N_{V,B}(|N(\lambda) - |N_\kappa| \geq K) = O\left(\exp(-K^2)\right).
\]

Note that the limit is not necessarily centered, and although the result is proved for \( N_\kappa - N(\lambda) \), it obviously also holds for \( |N_\kappa| - N(\lambda) \) since we are only considering subsequences such that \( N_\kappa - |N_\kappa| \rightarrow \kappa. \) We will also need the following result, which can be proved using the previous result or Lemma 2.16

\[
\sum_{N=(N_0, \ldots, N_g)} \frac{N!}{\prod N_h!} \frac{Z^N_{V,B}}{Z^V_{V,B}} |\epsilon - \epsilon| \frac{E_{V,B}^N}{\mathbb{E}_{V,B}^N} \left( \sum_{0 \leq h \leq g} |L_N(B_h) - \mu_V(B_h)| \right) \leq C \frac{\log N}{N}. \quad (4.2)
\]

We now provide a proof of Theorem 1.3. Let \( f \) be a function of compact support and \( i \) such as in the hypothesis of the theorem. Using Corollary 2.7, we have
\[
\int f(N \rho_V(E^i_{1,N})(\lambda_{i+1} - \lambda_i)) d\tilde{P}^N_{V,B} \\
= \sum_{N=(N_0, \ldots, N_g)} \sum_{[N(\lambda) - [N_*]] \leq K} \frac{N!}{\prod N_h!} \frac{Z_{V,B}^{N,\epsilon}}{Z_N^{V,B}} \int f(N \rho_V(E^i_{1,N})(\lambda_{h,i+1}[h,N] - \lambda_{h,i}[h,N])) d\tilde{P}^N_{V,B} \\
= \sum_{N=(N_0, \ldots, N_g)} \sum_{[N(\lambda) - [N_*]] \leq K} \frac{N!}{\prod N_h!} \frac{Z_{V,B}^{N,\epsilon}}{Z_N^{V,B}} \int f(N \rho_V(E^i_{1,N})(\lambda_{h,i+1}[h,N] - \lambda_{h,i}[h,N])) d\tilde{P}^N_{V,B} \\
+ O \left( \|f\|_\infty \exp(-K^2) \right) \\
= \sum_{N=(N_0, \ldots, N_g)} \sum_{[N(\lambda) - [N_*]] \leq K} \frac{N!}{\prod N_h!} \frac{Z_{V,B}^{N,\epsilon}}{Z_N^{V,B}} \int f(N(\Phi_{\epsilon,h})'(\lambda_{i|h,N}) \rho_V(E^i_{1,N})(\lambda_{i+1}[h,N] - \lambda_{i}[h,N])) d\tilde{P}^N_{G} \\
+ O \left( (\exp(-K^2) + (\log N)^3 N) \right) \|f\|_\infty + \left( \sqrt{\frac{(\log N)^2}{N^{1/2}}} + \frac{(\log N)}{N^{1/2}} + M^2 \right) \|\nabla f\|_\infty \).
\]

If we manage to replace the term \( N(\Phi_{\epsilon,h})'(\lambda_{i|h,N}) \rho_V(E^i_{1,N}) \) by \( N_h \rho_G(E^i_{1,h,N}) \) then, using the convergence \((1.4)\) we can conclude.

By \((1.1)\) we can replace \( (\Phi_{\epsilon,h})'(\lambda_{i|h,N}) \) by \( (\Phi_{\epsilon,h})'(\lambda_{i|h,N}) \) in the last equation and obtain an error of order \( K/N \). Now, using that \( \Phi_{\epsilon,h} \) is a transport from \( \mu_G \) to \( \mu_{V,h}^{\epsilon,h} \) we see that

\[
(\Phi_{\epsilon,h})'(\lambda_{i|h,N}) = \frac{\rho_G(\lambda_{i|h,N})}{\rho_{V,h}^{\epsilon,h}(\Phi_{\epsilon,h}(\lambda_{i|h,N}))},
\]

\[
\int_{-\infty}^{\Phi_{\epsilon,h}(E^i_{1,h,N})} \rho_{V,h}^{\epsilon,h}(x) dx = \int_{-\infty}^{E^i_{1,N}} \rho_{V,h}^{\epsilon,h}(x) dx + O(K/N).
\]

Thus \( \Phi_{\epsilon,h}(E^i_{1,h,N}) = E^i_{1,N} + O(K/N) \) and using \( \rho_V = e_{s,h} \rho_{V,h}^{\epsilon,h} \) on \( A_h \) we see that

\[
N(\Phi_{\epsilon,h})'(\lambda_{i|h,N}) \rho_V(E^i_{1,N}) = N_{s,h} \rho_G(\lambda_{i|h,N}) \frac{\rho_{V,h}^{\epsilon,h}(E^i_{1,N})}{\rho_{V,h}^{\epsilon,h}(\Phi_{\epsilon,h}(\lambda_{i|h,N}))}.
\]

We can replace \( \lambda_{i|h,N} \) by \( E^i_{1,h,N} \) in the right hand side with an error term \( o(N) \) with high probability under \( \mathbb{P}^N_{\epsilon,h} \) using a very rough rigidity estimate that can be proved for instance using Proposition \((2.13)\). As \( (\Phi_{\epsilon,h})' \) is bounded by below and \( f \) is compact we notice that \( N(\lambda_{i+1}[h,N] - \lambda_{i}[h,N]) \) is of order 1 and we can conclude.

We can now proceed with the proof of Theorem \((1.3)\). To simplify the notations, we will do the proof when \( m = 1 \) but the proof for general \( m \) is identical.
\[
\int \prod_{h=0}^{g} f_h \left( N^{2/3} (\lambda_{h,1} - \alpha_{h,-}) \right) d\tilde{\mathbb{P}}_{V,B}^N \\
= \sum_{N=(N_0,\ldots,N_g)} N! \prod N_h! \frac{Z_{V,B}^{N,e} N_h}{Z_{V,B}^N} \prod_{h=0}^{g} f_h \left( N^{2/3} (\lambda_{h,1} - \alpha_{h,-}) \right) d\tilde{\mathbb{P}}_{V,B}^N \\
= \sum_{N=(N_0,\ldots,N_g)} N! \prod N_h! \frac{Z_{V,B}^{N,e} N_h}{Z_{V,B}^N} \prod_{h=0}^{g} f_h \left( N^{2/3} (\lambda_{h,1} - \alpha_{h,-}) \right) d\tilde{\mathbb{P}}_{V,B}^N \\
+ O \left( \sum_{N=(N_0,\ldots,N_g)} N! \prod N_h! \frac{Z_{V,B}^{N,e} N_h}{Z_{V,B}^N} \|f\|_\infty N^{2/3} \|\epsilon - \epsilon_*\| \right) \\
= \sum_{N=(N_0,\ldots,N_g)} N! \prod N_h! \frac{Z_{V,B}^{N,e} N_h}{Z_{V,B}^N} \prod_{h=0}^{g} f_h \left( N^{2/3} (\Phi^{e,h})'(-2)(\lambda_1 + 2) \right) d\tilde{\mathbb{P}}_{V,B}^N \\
+ O \left( \left( \exp(-K^2) + \frac{(\log N)^3}{N} \right) \|f\|_\infty + \left( \sqrt{m\frac{(\log N)^2}{N^{5/6}}} + \frac{\log N}{N^{1/3}} + \frac{M^2}{N^{4/3}} \right) \|\nabla f\|_\infty \right).
\]

Using the fact that \((\Phi^{e,h})'\) is bounded by below on \(B\) and that \(f_h\) is supported in \([-M; M]\) we obtain that \(|\lambda_1 + 2|\) remains bounded by \(\frac{CM}{N^{2/3}}\). Using (4.1) we get

\[
f_h \left( N^{2/3} (\Phi^{e,h})'(-2)(\lambda_1 + 2) \right) = f_h \left( N^{2/3} (\Phi^{e,0})'(-2)(\lambda_1 - \alpha_{G,-}) \right) + O(M \|\nabla f\|_\infty |\epsilon - \epsilon_*|).
\]

This equation, along with (4.2), shows that

\[
\int \prod_{h=0}^{g} f_h \left( N^{2/3} (\lambda_{h,1} - \alpha_{h,-}) \right) d\tilde{\mathbb{P}}_{V,B}^N \\
= \sum_{N=(N_0,\ldots,N_g)} N! \prod N_h! \frac{Z_{V,B}^{N,e} N_h}{Z_{V,B}^N} \prod_{h=0}^{g} f_h \left( N^{2/3} (\Phi^{e,h})'(-2)(\lambda_1 + 2) \right) d\tilde{\mathbb{P}}_{V,B}^N \\
+ O \left( \left( \exp(-K^2) + \frac{(\log N)^3}{N} \right) \|f\|_\infty + \left( \sqrt{m\frac{(\log N)^2}{N^{5/6}}} + \frac{\log N}{N^{1/3}} + \frac{M^2}{N^{4/3}} \right) \|\nabla f\|_\infty \right).
\]

As Theorem 1.1 of [22] ensures the convergence of the expectation, we can conclude.
We now come to the proof of Theorem 1.5. Let $0 \leq h \leq g$, $i = \lceil [N_*]_{h-1} + 1 \rceil$ and $\Delta_h(\lambda) = \lceil [N_*] \rceil_{h-1} - [N(\lambda)]_{h-1}$. As before we obtain

$$ \int f(N^{2/3}(\lambda_i - N^2)) d\tilde{\mathbb{P}}_{V,B} \sum_{N=(N_0,\ldots,N_g)} \prod_{\Delta_h(\lambda) \geq 0} N! \frac{Z_{V,B}^{N_*}}{Z_{V,B}^N} \int f(N^{2/3}(\lambda_i - \xi_h)) d\tilde{\mathbb{P}}_{V,B} \Delta_h(\lambda) \geq \alpha \geq 0$$

Noticing that $i[h,N] = \lceil [N_*] \rceil_{h-1} - [N]_{h-1} + 1$, we deduce the theorem from Lemma 4.1.

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