STOCHASTIC INVARIANCE FOR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION WITH NON-LIPSCHITZ COEFFICIENTS

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Abstract. In this paper, by the use of martingale property and spectral decomposition theory, we investigate the stochastic invariance for neutral stochastic functional differential equations (NSFDEs) and provide necessary and sufficient conditions for the invariance of closed sets of $\mathbb{R}^d$ with non-Lipschitz coefficients. A pathwise asymptotic estimate example is given to illustrate the feasibility and effectiveness of obtained result.

1. Introduction. In this paper, we are mainly concerned with the general neutral stochastic functional differential equation (NSFDE) of the form:

$$d[X(t) - G(X_t)] = f(X_t)dt + g(X_t)dw(t), \quad t \geq 0,$$

with the initial value $X_0 = \xi \in L^F_2([-\tau, 0]; \mathbb{R}^d); \ X_t(\theta) = X(t + \theta)$ and $\theta \in [-\tau, 0];$ where $\tau$ is a positive constant and

$$\begin{cases}
  f : C([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}^d, \\
  g : C([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}^{d \times d}, \\
  G : C([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}^d.
\end{cases}$$

Assume that $f$ and $g$ satisfy the following linear growth condition (H1) and $G$ satisfies the contractive condition (H2):

(H1) $|f(X_t)|^2 + |g(X_t)|^2 \leq K(1 + \|X_t\|^2),$ where the constant $K > 0$;

(H2) $|G(X_t) - G(X_s)|^p \leq k^p\|X_t - X_s\|^p,$ $|G(X_t)| \leq k\|X_t\|,$ $k \in (0, 1),$ $G(0) = 0,$ and $G(X_t)$ is differentiable;

(H3) $C$ can be extended to a $C^{1,1}_{loc}(\mathbb{R}^d, \mathbb{S}^d)$ function that coincides with $gg^T$ on $\mathcal{D}$.

NSFDEs in Hilbert space were studied by some authors and many valuable results on the existence, uniqueness and stability of the solution were established, for example [1, 15, 20]. The papers (mentioned above) were focused on the stability in Lyapunov sense of NSFDEs, which required the existence and uniqueness of equilibrium points. However, in many real physical systems, especially in nonlinear...
and non-autonomous dynamical systems, the equilibrium point sometimes does not exist. Therefore, an interesting subject is to discuss the stability in Lagrange sense.

The main difference between Lagrange stability and Lyapunov stability is that Lagrange stability refers to the stability of total systems, not the stability of equilibrium points. When studying the stability in Lagrange sense, we do not make any assumption on the numbers of the equilibrium points. Especially as to the asymptotal stability in Lagrange sense, only if all the solutions of a system ultimately enter into an attractive compact set, we call the system to be asymptotically stable in Lagrange sense or ultimately bounded. Lagrange stability can be used in analyzing both monostable systems and multistable systems, see [11]. And in the special case where the attractive compact set is a single point, the system is globally stable in Lyapunov sense and the attractive compact set is the unique equilibrium point.

Stochastic invariance is one of the most important methods to study Lagrange stability of dynamical systems. After the first result about stochastic invariance given by [12], the stochastic invariance of dynamical systems have been extensively studied over the past few decades and various results were reported. For deterministic differential systems with or without delays, see [5, 22]. For partial differential systems, see [24]. For stochastic or random systems, see [8, 13, 27]. Basically, the invariant set is given by the means of defining the tangent cone, the viscous solution or the distance function, then the main goal is to find the conditions which guarantee the solution of the equation in the stochastic invariant set almost surely. To the best of our knowledge, there exist no articles which deal with the stochastic invariance of closed sets for NSFDEs. Although, there have been three articles in the study of the attracting and invariant sets of neutral differential equations which are another way to study Lagrange stability, see [16, 17, 26]. However, they have at least one thing in common: they are limited on Lipschitz coefficients condition, which made all of these existing results difficult to apply in practice. We shall make the first attempt to study stochastic invariance of closed sets for NSFDEs on non-Lipschitz coefficients condition. Our results are inspired by [13] where the stochastic invariance theory for equation 1 with $\tau = 0$ and $G(X_t) = 0$ was studied.

The paper is organized as follows: In Section 2, we present some preliminary definitions. In Section 3, we give several lemmas which lay good foundation for our main result in the following, and then we give the proof of necessity and sufficiency. In Section 4, we give an example to illustrate Theorem 3.7 below.

2. Preliminary results. All identities involving random variables have to be considered in the a.s. sense, the probability space and the probability measure have been given by the context. Elements of $\mathbb{R}^d$ are viewed as column vectors. We use the standard notion $I_d$ to denote the $d \times d$ identity matrix and $\mathbb{M}^d$ to denote the collection of $d \times d$ matrices. We say that $A \in \mathbb{S}$ if it is a symmetric elements of $\mathbb{M}$. Given $x = (x^1, \cdots, x^d) \in \mathbb{R}^d$, $\text{diag}[x] \in \mathbb{R}$ denotes the diagonal matrix whose $i$-th diagonal component is $x^i$. If $A$ is a symmetric positive semi-definite matrix, then $A^{\frac{1}{2}}$ denotes its symmetric square-root. Let the Banach space $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^d)$ given the supremum norm $\| \cdot \|_C$. Let $0 < \alpha < \frac{1}{2}$ and $\mathcal{C}^\alpha = \mathcal{C}^\alpha([-\tau, 0]; \mathbb{R}^d)$, the Banach space of all $\alpha$-Hölder continuous path $\eta : [-\tau, 0] \to \mathbb{R}^d$ with the $\alpha$-Hölder norm

$$
\|\eta\|_{C^\alpha} = \|\eta\|_C + \sup \left\{ \frac{|\eta(s_1) - \eta(s_2)|}{|s_1 - s_2|^{\alpha}} : s_1, s_2 \in [-\tau, 0], s_1 \neq s_2 \right\}.
$$
Now, for convenience, we give some useful definitions which will be used throughout the whole paper.

**Definition 2.1.** A closed subset $\mathcal{D} \subset \mathbb{R}^d$ is said to be stochastically invariant with respect to NSFDE 1 if, for all $x \in \mathcal{D}$, there exists a weak solution $(X, W)$ to equation 1 starting at $X(0) = x$ such that $X(t) \in \mathcal{D}$ almost surely for all $t \geq 0$.

Let $C_0^\infty(\mathbb{R}^d)$ denote the space of the real-valued, infinitesimally differentiable functions on $\mathbb{R}^d$ with compact support. For any $\phi \in C_0^\infty(\mathbb{R}^d)$, we define an operator $\mathcal{L}
\mathcal{L}\phi(\varphi) = (\dot{G}(\varphi)+f(\varphi))D\phi(\varphi(0))+\frac{1}{2}Tr(D^2\phi(\varphi(0))gg^T(\varphi)), \; \varphi \in C([−\tau,0];\mathbb{R}^d), \quad (2)
$ where $\dot{G}(\varphi)$ dotes the derivative of function $G(\varphi)$.

Furthermore, fix a $\mathbb{R}^d$-valued continuous function $\xi$ on $[−\tau,0]$, we shall give a definition of the local martingale problem (see [18]) for the operator in 2.

**Definition 2.2.** A probability measure $Q_\xi$ on $(\Omega, M)$ solves the local martingale problem associated with $f$ and $C = gg^T$ with initial data $\xi$ if

1. $Q_\xi(X_0 = \xi) = 1$,
2. $\phi(X^\circ(t)) - \int_0^t \mathcal{L}\phi(X_s^\circ)ds, \; t \geq 0$

is a local $(M(t),Q_\xi)$-martingale for all $\phi \in C_0^\infty(\mathbb{R}^d)$.

Now we show the relation between local martingale problem of Definition 2.2 and weak solutions of NSFDE. Firstly, assume that there exists a weak solution of 1 with initial data $\xi$:

$X_0 = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\} \in L^2([−\tau,0];\mathbb{R}^d),$

then there exists a sextuple $(\Omega, \mathcal{F}, \mathcal{F}_t, P, B, X)$ such that

$X_i(t) = \xi_i(0) - G_i(\xi) + G_i(X_t) + \int_0^t f_i(X_s)ds + \sum_{j=1}^d \int_0^t g_{ij}(X_s)dB_j(s),$

holds a.s. or equivalently

$dX_i(t) = dG_i(X_t) + f_i(X_t)dt + \sum_{j=1}^d g_{ij}(X_t)dB_j(s), \quad (3)$

where $t \geq 0$, $i = 1, \cdots, d$. Define the probability measure

$Q_\xi(A) = P(x \in A), \; A \in M.$

Then, $Q_\xi$ solves the local martingale problem of Definition 2.2 for the coefficients $f$ and $C = gg^T$, where $g^T$ denotes the transpose of $g$. For $\phi \in C_0^\infty(\mathbb{R}^d)$, by Itô formula, we have

$\phi(X(t)) = \phi(X(0)) + \sum_{i=1}^d \int_0^t \frac{\partial \phi}{\partial x_i}(X(s))dX_i(s)$

$+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 \phi}{\partial x_i \partial x_j}(X(s))d<X_i,X_j>(s), \; t \geq 0.$

Then, using equality 3 and $d < X_i, X_j > (s) = gg^T_{ij}(X_s)ds$, we obtain that
\[ M(t) = \phi(X(t)) - \phi(X(0)) - \sum_{i=1}^{d} \int_{0}^{t} \frac{\partial \phi}{\partial x_i}(X(s))dG_i(X_s) \]
\[ - \sum_{i=1}^{d} \int_{0}^{t} f_i(X_s) \frac{\partial \phi}{\partial x_i}(X(s))ds - \frac{1}{2} \sum_{i,j=1}^{d} \int_{0}^{t} (gg^T)_{ij}(X_s) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(X(s))ds \]
\[ = \phi(X(t)) - \phi(X(0)) - \int_{0}^{t} \mathcal{L}\phi(X_s)ds \]

is a local \((\mathcal{F}_t, P)\)-martingale. Then by transformation of measures it holds that
\[ \phi(X^\circ(t)) - \int_{0}^{t} \mathcal{L}\phi(X^\circ_s)ds, \ t \geq 0 \]
is a local \((M_t, Q_\xi)\)-martingale on the canonical space \(\Omega = \mathcal{C}([-\tau, 0]; \mathbb{R}^d)\). This shows that \(Q_\xi\) is the distribution of the solution process and solves the local martingale problem.

**Definition 2.3.** (see [14], Definition 7.20) An adapted process \(M(t)\) is called a local martingale if there exists a sequence of stopping times \(\tau_n\) such that for all \(n\) the stopped processes \(M(t \wedge \tau_n)\) is a uniformly integrable martingale in \(t\).

**Definition 2.4.** Let \(\mathcal{N}_D^1\), \(\mathcal{N}_D^2\) and \(\mathcal{N}_D^{1,\text{prox}}(x)\) be respectively the first order normal cone, the second order normal cone and proximal cone at the point \(x\).
\[ \mathcal{N}_D^1(x) = \{ u \in \mathbb{R}^d : \langle u, y - x \rangle \leq o(\|y - x\|), \ \forall \ y \in D \}; \]
\[ \mathcal{N}_D^2(x) = \{ (u, v) \in \mathbb{R}^d \times S^d : \langle u, y - x \rangle + \frac{1}{2} \langle v(y - x), y - x \rangle \leq o(\|y - x\|^2), \ \forall \ y \in D \}; \]
\[ \mathcal{N}_D^{1,\text{prox}}(x) = \{ u \in \mathbb{R}^d, \| u \| = d_D(x + u) \}, \]
in which \(d_D\) is the distance function to \(D\).

3. **Main result.** In this section, we shall give the main results of this paper. First of all, we need to prepare several lemmas for the latter stochastic invariance analysis.

**Lemma 3.1.** (see [14], Corollary 7.22) Let \(M(t), 0 \leq t < \infty\), be a local martingale such that for all \(t\), \(\mathbb{E}(\sup_{s \leq t} | M(s) |) < \infty\). Then \(M(t)\) is a martingale, and as such it is uniformly integrable on any finite interval \([0, T]\).

**Lemma 3.2.** (see [20], Theorem 6.4.5) Let \(p \geq 2\) and \(\mathbb{E}(\|\xi\|^p) < \infty\). Let \((H1)-(H2)\) hold. Then
\[ \mathbb{E}( \sup_{-\tau \leq s \leq t} |X(s)|^p) \leq (1 + C\mathbb{E}(\|\xi\|^p))e^{Ct}, \]
where \(C\) and \(\bar{C}\) are positive constants.

**Lemma 3.3.** \((H1)-(H3)\) imply that, for any positive integer \(p \geq 2\), there exist a pair of positive constants \(C^*\) and \(\bar{C}^*\) such that
\[ \mathbb{E}(\|X(t) - X(s)\|^p) \leq C^*|t - s|^\frac{p}{2}, \]
\[ \mathbb{E}(\|X_t - X_s\|^p) \leq \sup_{-\tau \leq \theta \leq 0} \mathbb{E}(\|X(t + \theta) - X(s + \theta)\|^p) \leq \bar{C}^*|t - s|^\frac{p}{2}, \]
for any \(s, t > 0\) and \(0 \leq t - s < \tau\).
Proof. Since all the sample paths of $\xi(\cdot)$ are continuous on $[-\tau, 0]$, the dominated convergence theorem implies that $\xi(\cdot)$ is Lipschitz continuous, hence uniformly Lipschitz continuous on $[-\tau, 0]$. Therefore, for any $-\tau \leq s < t \leq 0$, there exists a $\beta_{t-s} > 0$ such that

$$E|\xi(t) - \xi(s)|^p \leq \beta_{t-s}(t-s)^{\frac{p}{2}}.$$ 

Let $p \geq 2$, $E||\xi||^p < \infty$. Applying (H1)−(H2), we have

$$E|X(t) - X(s)|^p 
\leq (1 + \varepsilon \cdot \frac{1}{r^\tau})^{p-1} \left( E|X(t) - X(s) - G(X_t) + G(X_s)|^p + \frac{1}{\varepsilon} E|G(X_t) - G(X_s)|^p \right) 
\leq (1 + \varepsilon \cdot \frac{1}{r^\tau})^{p-1} \left( E \left| \int_s^t f(X_r)dr + \int_s^t g(X_r)dw(r) \right|^p + \frac{k^p}{\varepsilon} E||X_t - X_s||^p \right) 
\leq (1 + \varepsilon \cdot \frac{1}{r^\tau})^{p-1} \left( 2^{p-1} (t-s)^{p-1} E \int_s^t |f(X_r)|^p dr 
+ \frac{1}{2} \cdot 2[p(p-1)]^\frac{p}{2} (t-s)^{\frac{p}{2}} E \int_s^t |g(X_r)|^p dr \right) + (1 + \varepsilon \cdot \frac{1}{r^\tau})^{p-1} \frac{k^p}{\varepsilon} E||X_t - X_s||^p 
\leq C_1(t-s)^{\frac{p-1}{2}} E \int_s^t (1 + ||X_r||^2) dr + (1 + \varepsilon \cdot \frac{1}{r^\tau})^{p-1} \frac{k^p}{\varepsilon} E||X_t - X_s||^p 
\leq C_2(t-s)^{\frac{p}{2}} + (1 + \varepsilon \cdot \frac{1}{r^\tau})^{p-1} \frac{k^p}{\varepsilon} E||X_t - X_s||^p,$$

where $C_1$ and $C_2$ are positive constants. Letting $\varepsilon = \left[ \frac{k}{1-k} \right]^{p-1}$, we have

$$E|X(t) - X(s)|^p \leq k E||X_t - X_s||^p + C_2(t-s)^{\frac{p}{2}}.$$ 

For any $0 < t < T$, we have

$$\sup_{0 < s \leq t} E|X(t) - X(s)|^p \leq k \sup_{-\tau < \theta < 0} E|X(t + \theta) - X(s + \theta)|^p + C_2(t-s)^{\frac{p}{2}} 
\leq k \sup_{-\tau < \theta < 0} E|X(t) - X(s)|^p + C_2(t-s)^{\frac{p}{2}} 
\leq k \sup_{-\tau < s \leq t} E|\xi(t) - \xi(s)|^p + k \sup_{0 < s \leq t} E|X(t) - X(s)|^p 
+ C_2(t-s)^{\frac{p}{2}}.$$ 

Therefore,

$$E|X(t) - X(s)|^p \leq \frac{C_2}{1-k} (t-s)^{\frac{p}{2}} + \beta_{t-s} \frac{k}{1-k} (t-s)^{\frac{p}{2}} \leq C^*(t-s)^{\frac{p}{2}}.$$

Hence, The Kolmogorov’s continuity criterion ensures that the sample paths of $X$ are (locally) $\alpha$-Hölder continuous for any $\alpha \in (0, 1/2)$.

**Lemma 3.4.** Assume that $C \in C_{loc}^{1,1}(\mathbb{R}^d, S^d)$ is defined in Definition 6.1 and 6.8. Let $X_0 = \xi(\theta) \in \mathcal{D}$, $\theta \in [-\tau, 0]$. The normalized matrix of $C(X_0)$ is given by

$$C(X_0) = Q(X_0) \text{diag} [\lambda_1(X_0), \ldots, \lambda_r(X_0), 0, \ldots, 0] Q^T(X_0),$$

where

$$\lambda_1(X_0) > \lambda_2(X_0) > \cdots > \lambda_r(X_0) > 0$$

and

$$Q(X_0)Q^T(X_0) = I_d, \ r \leq d.$$
Then there exist an open (bounded) neighborhood $N(X_0)$ of $X_0$ and two measurable $M^d$-valued functions on $\mathbb{R}^d$

$$Y \mapsto Q(Y) = [q_1(Y), \cdots, q_d(Y)]$$

and

$$Y \mapsto \Lambda(Y) = \text{diag}([\lambda_1(Y), \cdots, \lambda_d(Y)]]$$

such that

1. $C(Y) = Q(Y)\Lambda(Y)Q^T(Y)$ and $Q(Y)Q^T(Y) = I_d$, for all $Y \in \mathbb{R}^d$;
2. $\lambda_1(Y) > \lambda_2(Y) > \cdots > \lambda_d(Y) > \max\{\lambda_i(Y), \ r+1 \leq i \leq d\} \vee 0$, for all $Y \in N(X_0)$;
3. $g : Y \mapsto Q(Y)\Lambda(Y)^{1/2}$ is $C^{1,1}(N(X_0), M^d)$, in which

$$\bar{Q} = [q_1, \cdots, q_r, 0, \cdots, 0]$$

and

$$\bar{\Lambda} = \text{diag}([\lambda_1, \cdots, \lambda_r, 0, \cdots, 0]).$$

Moreover, we have

$$\langle u, \sum_{j=1}^d D\bar{g}^j(X_0)\bar{g}^j(X_0)^T \rangle = \langle u, \sum_{j=1}^d DC^j(X_0)(CC^+)j(X_0) \rangle,$$

for all $u \in \ker(C(X_0))$.

Proof. Note the fact that $(g_i)$ can be chosen measurable for the form:

$$g^j(X_0) = k_j(1 + \xi^2_{j}(\theta))^{1/2},$$

which is guaranteed when $(C, \Lambda)$ is measurable by the each eigenvector solving a quadratic minimization problem (see [4]). Since $u \in \ker(C(X_0))$, we have

$$u^T\bar{Q}(X_0) = u^T\bar{g}(X_0) = 0.$$

Since $C \in C^{1,1}_{\text{loc}}$, we can get $\bar{C} = \bar{g}\bar{g}^T$ is differentiable at $X_0$. Combing with Definition 6.4 and 6.6, $\bar{g}\bar{g}^T = C\bar{Q}\bar{Q}^T$ and $\bar{Q}(X_0)\bar{Q}^T(X_0) = C(X_0)C(X_0)^+$, we have

$$\langle u, \sum_{j=1}^d D\bar{g}^j(X_0)\bar{g}^j(X_0)^T \rangle = \sum_{j=1}^d u^T(D\bar{g}^j(X_0)e_j\bar{g}^j(X_0)^T)e_j$$

$$= \sum_{j=1}^d u^T(e_j^T\otimes I_d)D\bar{g}^j(X_0)\bar{g}^j(X_0)^Te_j$$

$$= Tr[(I_d \otimes u^T)D\bar{g}(X_0)\bar{g}(X_0)^T]$$

$$= \langle u, \sum_{j=1}^d DC^j(X_0)(CC^+)j(X_0) \rangle.$$

\[\square\]

**Lemma 3.5.** Given $g \in C^{1,1}_{\text{loc}}(\mathbb{R}^d, S^d)$ (i.e. $g$ is differentiable with a bounded and a globally Lipchitz derivative). Then

$$C := gg^T \in C^{1,1}_{\text{loc}}(\mathbb{R}^d, S^d_+),$$

$$\langle u, \sum_{j=1}^d Dg^j(X_0)g^j(X_0)^T \rangle = \langle u, \sum_{j=1}^d DC^j(X_0)(CC^+)j(X_0) \rangle,$$

for all $X(0) = x \in D$ and $u \in \ker g(X_0)$. 
Lemma 3.6. Let \( \{u(t)\}_{t \geq 0} \) be a standard \( d \)-dimensional Brownian motion on a filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \). Let \( \alpha \in \mathbb{R}^d \), \( \{\beta_t\}_{t \geq 0} \in \mathbb{R}^d \), \( \{\gamma_t\}_{t \geq 0} \in \mathbb{M}^d \) and \( \{\theta_t\}_{t \geq 0} \in \mathbb{R} \) satisfy

1. \( \beta \) is bounded;
2. \( \int_0^t \|\gamma_s\|^2 ds < \infty \), for all \( t \geq 0 \);
3. there exists a random variable \( \eta > 0 \), such that a.s
   \[ \int_0^t \|\gamma_s - \gamma_0\|^2 ds = o(t^{1+\eta}), \quad t \to 0; \]
4. \( \theta \) is a.s. continuous at 0.

Suppose that for all \( t \geq 0 \),

\[ \int_0^t \theta_s ds + \int_0^t \left( \alpha + \int_0^s \beta_r dr + \int_0^s \gamma_r dw(r) \right)^T dw(s) \leq 0. \]

Then
Proof. Observe that the condition of Lemma 3.6 is different from the one of ([5], Lemma 2.1), but it’s result is the exactly same as one of ([5], Lemma 2.1). So our main aim is to reduce the case where ([5], Lemma 2.1) holds (e.g. \(R_t = \circ(t)\)).

Since

\[
\Theta_0 - \frac{1}{2} \text{Tr} (\gamma_0) \leq 0,
\]

and inequality 5, we have

\[
\Theta_0 - \frac{1}{2} \text{Tr} (\gamma_0) + \sum_{i=1}^{d} \alpha_i \Theta_i + \sum_{i=1}^{d} \frac{\gamma_{ii}^0}{2} (\Theta_i)^2 + \sum_{1 \leq i \neq j \leq d} \gamma_{ij}^0 \int_0^t \Theta_i d\Theta_j + R_t \leq 0,
\]

where

\[
R_t = \int_0^t (\Theta_s - \Theta_0) ds + \int_0^t \left( \int_0^s \beta_r dr \right) T dw(s) + \int_0^t \left( \int_0^s (\gamma_r - \gamma_0) dw(r) \right) T dw(s)
\]

\[
= R_1^t + R_2^t + R_3^t.
\]

Since \(\Theta\) is continuous at 0, we get

\[
R_1^t = \circ(t) \ a.s.
\]

Moreover, in view of ([7], Proposition 3.9), we can have

\[
R_2^t = \circ(t) \ a.s.,
\]

as \(\beta\) is bounded. Define

\[
M_{ij}^t = \gamma_{ij}^t - \gamma_{ij}^0
\]

and

\[
M_i = \int_0^t \sum_{j=1}^{d} M_{ij}^t dw(r)^j \text{ for } i, j = 1, 2, \ldots, d,
\]

we can deduce that

\[
\langle M_i \rangle = o(s^{1+\eta}) \ a.s.
\]

By using the Dambis-Dubins-Schwarz theorem, we know that \(M_i^t\) is a time changed Brownian motion. By the law of iterated logarithm for Brownian motion

\[
\langle M_i^t \rangle^2 = o(s^{1+\eta} \gamma) \ a.s.,
\]

we have

\[
\langle R_2^t \rangle = o(t^{2+\eta} \gamma) \ a.s.
\]

By applying the Dambis-Dubins-Schwarz theorem and law of iterated logarithm for Brownian motion again, we get that

\[
R_3^t = \circ(t) \ a.s.
\]

\(\square\)

\textbf{Theorem 3.7.} (Invariance characterization) Let \(D\) be closed, assume that \(f, g\) and \(G\) satisfy assumptions (H1)-(H3). Then, the set \(D\) is stochastically invariant with respect to the equation 1 if and only if
\[ C(X_0)u = 0, \tag{6} \]

\[ \left\langle u, f(X_0) + \dot{G}(X_0) - \frac{1}{2} D C^j(X_0)(C C^+)^j(X_0) \right\rangle \leq 0, \tag{7} \]

for every initial data \( X(0) = x \in D \) and all \( u \in N_D(x). \)

3.1. **Necessary condition.** In this subsection, we prove that the conditions of Theorem 3.7 are necessary for \( D. \) Our general strategy is similar to \([13]\), the main idea consists of using the spectral decomposition of \( C \) in the form \( QAQ^T \) in which \( Q \) is an orthogonal matrix and \( \Lambda \) is diagonal positive semi-definite. Then, divide the rest proof into 3 cases (I, II, III).

I. The case of distinct and non-zero eigenvalues

Since \( C \in C_{\text{loc}}^1(\mathbb{R}^d, S^d), \) \( C(X_0) \) has distinct and non-zero eigenvalues, we can reduce to the case where \( Q \) and \( \Lambda \) are smooth enough and \( \Lambda \) has only (strictly) positive entries.

Consider a smooth function \( \phi : \mathbb{R}^d \to \mathbb{R} \) such that \( \max_D \phi = \phi(x) \) for \( x = X(0), \) we can deduce that \( \phi, \| D \phi \| \) and \( \| D^2 \phi \| \) are bounded because \( \phi \) has compact support. Apply the stochastic Taylor expansion formula (see \([5]\) or \([3]\)), for all \( t \geq 0, \) p-a.s.,

\[
\phi(X(t)) = \phi(X(0)) + \sum_{i=1}^{d} g^i \phi(X(0)) w^i(t) + \sum_{i=1}^{d} (g^i)^2 \phi(X(0)) \frac{(w^i(t))^2}{2} + \sum_{i\neq j} g^i g^j(X_0) \int_0^t w(s)^i dw(s)^j + \tilde{f}(X(0)) t + R_t,
\]

where \( R_t \) satisfies \( R_t/t \to 0 \) in probability as \( t \to 0, \) we apply operator notation

(1) \( g^i \phi(X(0)) = \langle g(X_0), D \phi(X(0)) \rangle, \)
(2) \( \tilde{f}(X(0)) = \langle f(X_0), D \phi(X(0)) \rangle, \)
(3) \( \tilde{f}(X_0) = f(X_0) + \dot{G}(X_0) - \frac{1}{2} \sum_{i=1}^{d} \langle D g^i(X_0), g^i(X_0) \rangle. \)

Since \( D \) is stochastically invariant and \( \max_D \phi = \phi(X(0)) \), then we have

\[ \phi(X(t)) \leq \phi(X(0)). \]

Thus, p-a.s., for fixed \( t \geq 0, \)

\[
\sum_{i=1}^{d} g^i \phi(X(0)) w^i(t) + \sum_{i=1}^{d} (g^i)^2 \phi(X(0)) \frac{(w^i(t))^2}{2} + \sum_{i\neq j} g^i g^j(X_0) \int_0^t w(s)^i dw(s)^j + \tilde{f}(X(0)) t + R_t \leq 0.
\]

Then, we can apply \((5), \) Lemma 1.5) and obtain

\[ \langle g(X_0), D \phi(X(0)) \rangle = D \phi(X(0)) gg^T(X_0) = 0 \]

and

\[ \tilde{f}(X(0)) = \langle f(X_0) + \dot{G}(X_0) - \frac{1}{2} \sum_{i=1}^{d} D g^i(X_0) g^i(X_0)^T, D \phi(X(0)) \rangle \leq 0. \]

Under appropriate regularity conditions, we can choose a suitable test function \( \phi \) i.e. such that \( D \phi(X(0)) = u^T. \) Further, by using Lemma 3.5, we get \( 6 \) and \( 7. \)

II. The case of distinct eigenvalues
Assume that \( \mathcal{D} \) is stochastically invariant with respect to the equation 1. Let \( X(0) = x \in \mathcal{D} \) and \( C \) has distinct eigenvalues. Then 6 and 7 hold at point \( x \), for all \( u \in \mathcal{N}_D^1(x) \).

**Proof.** Let \((X, W)\) denote a weak solution of equation 1 with the initial value \( X(0) = x \) such that \( X(t) \in \mathcal{D} \), for all \( t \geq 0 \). If \( x \) is in the interior of \( \mathcal{D} \), then \( \mathcal{N}_D^1(x) = \{0\} \) and 6 and 7 clearly hold. Therefore, from now on, we assume that \( x \in \partial \mathcal{D} \), \( u \in \mathcal{N}_D^1(x) \), divide the rest proof into 4 steps.

**Step 1.** There exists a function \( \phi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}) \) with compact support in \( N(x) \) such that max\( \mathcal{D} \phi = \phi(x) = 0 \) and \( D\phi(x) = u^T \), indeed, it follows from ([23], Chapter 6.E) that one can find \( \phi \) such that \( \langle u, y - x \rangle \leq \frac{\lambda}{2} \|y-x\|^2 \) for all \( y \in \mathcal{D} \).

**Step 2.** Since \( \mathcal{D} \) is invariant under the point \( x \), \( \phi(X(t)) \leq \phi(x) \), for all \( t \geq 0 \). From now on, we use the notations of Lemma 3.4. Applying Itô formula to \( \phi(X(t)) \), we have

\[
\begin{align*}
\int_0^t \mathcal{L}\phi(X_s)ds + \int_0^t D\phi(X(s))g(X_s)dw(s) \\
= \int_0^t \mathcal{L}\phi(X_s)ds + \int_0^t D\phi(X(s))Q\Lambda^{1/2}Q(X_s)dw(s) \\
& \leq 0.
\end{align*}
\]

Define a Brownian motion \( B_t = \int_0^t Q(X_s)^T dw(s) \), we first verify that \( B_t \) satisfies the condition of Lemma 3.1. Since \( B_t \) is a local martingale, combining with BDG's inequality, we get

\[
\begin{align*}
\mathbb{E} \left[ \sup_{s \in [0,t]} |B_s| \right] &= \mathbb{E} \left[ \sup_{s \in [0,t]} \left| \int_0^t Q(X_s)^T dw(s) \right| \right] \\
& \leq 16\sqrt{2} \mathbb{E} \left[ \int_0^t |Q(X_s)|^2 ds \right]^{1/2} \\
& \leq 16\sqrt{2} \int_0^t |I_d| ds \\
& < \infty.
\end{align*}
\]

Hence, we can deal with real martingales instead of local martingales. Recall that \( Q \) is orthogonal together with

\[
\begin{align*}
\mathcal{B} &= \mathcal{N}(X_t)^{\perp} B = (B^1, \cdots, B^r, 0, \cdots, 0)^T, \\
\mathcal{B}^{\perp} &= (I_d - \mathcal{N}(X_t)^{\perp}) B = (0, \cdots, 0, B^{r+1}, \cdots, B^d).
\end{align*}
\]

Since \( Q\mathcal{N}^{\perp} = \mathcal{N}^{\perp} \), the left-hand of inequality 8 can be written in the form

\[
\int_0^t \mathcal{L}\phi(X_s)ds + \int_0^t D\phi(X(s))\mathcal{B} Q(X_s)ds + \int_0^t D\phi(X(s))Q\Lambda^{1/2}Q^{\perp} d\mathcal{B}^{\perp}_s \leq 0.
\]

Let \((F_{s})_{s \geq 0}\) be a complete filtration generated by \( \mathcal{B} \), combining with ([25], Lemma 14.2) and the fact that the martingale \( \mathcal{B}^{\perp} \) is independent of \( \mathcal{B} \). Then, we have

\[
\int_0^t \mathbb{E}_{F_s} \mathcal{L}\phi(X_s)ds + \int_0^t \mathbb{E}_{F_s} D\phi(X(s))\mathcal{B} Q(X_s)ds \leq 0.
\]
Applying Itô formula to $D\phi(X(s))\bar{g}(X_s)$, we get
\[
\int_0^t E_F^T L\phi(X_s)ds + \int_0^t \left\{ E_F^T(D\phi(X(0))\bar{g}(X_0)) + \int_0^s E_F^T D\phi(X(r))\bar{g}(X_r)dr \right. \\
+ \left. \int_0^s E_F^T[D\phi(X(r))\bar{g}(X_r)]\bar{g}(X_r)dw(r) \right\} dB_s \\
\leq 0.
\]

**Step 3.** Check that we can apply Lemma 3.6. First note that all the above processes are bounded, because Lemma 3.4, (H1) and the fact that $\phi$ has compact support. In addition, given $T > 0$, the independence of the increments of $B$ implies that
\[
\theta_s = E_F^T[L\phi(X_s)]
\]
for all $s \leq T$, due to that $\theta$ is a.s continuous at 0. Similarly,
\[
r_s = E_F^T[D\phi(X(s))\bar{g}(X_s)]\bar{g}(X_s) \quad \text{on} \quad [0,T].
\]
Moreover, assume that
\[
F = D(D\phi(X(s))\bar{g}(X_s))\bar{g}(X_s)
\]
\[
= (\bar{g}(X_s)^T \otimes I_d)D^2\phi(X(s))\bar{g}(X_s) + (I_d \otimes D\phi(X(s)))D\bar{g}(X_s).
\]
Since $D^2\phi(X(s))$, $D\phi(X(s))$ and $\phi(X(s))$ are bounded, by using Jensen’s inequality and Lemma 3.3, we can derive
\[
E[\|r_s - r_r\|^4] \leq E[\|F(X_s) - F(X_r)\|^4] \leq L'|s - r|^2,
\]
for all $s, r \in [0,T]$, where $L'$ is a positive constant. By Kolmogorov’s continuity criterion, up to considering a suitable modification, $r$ has $\alpha$-Hölder sample paths for $0 < \alpha < 1/2$. In particular
\[
\int_0^t \|r_s - r_0\|^2ds = o(t^{3+\alpha}),
\]
for $0 < \alpha < 1/2$.

**Step 4.** In view of Step 3, by using Lemma 3.6, we get
\[
D\phi(X(0))\bar{g}(X_0) = 0
\]
(9)
and
\[
L\phi(X_0) - \frac{1}{2}Tr(D(D\phi(X(0))\bar{g}(X_0))\bar{g}(X_0)) \leq 0.
\]
(10)
Applying 9 and $D\phi(X(0)) = u^T$, we have
\[
D\phi(X(0))\bar{g}(X_0) = u^T Q\bar{A}^{1/2}A^{1/2} Q^T(X_0) = u^T C(X_0) = 0,
\]
or equality 9 implies that
\[
C(X_0)u = 0
\]
on owing to the symmetry of $C(X_0)$. In terms of inequality 10, $D\phi(X(0)) = u^T$ and Definition 6.7, we have
\[ \mathcal{L} \phi(X_0) - \frac{1}{2} \text{Tr} \left[ \mathcal{G}(X_0) D^2 \phi(X(0)) \mathcal{G}(X_0) + (I_d \otimes u^T) D \mathcal{G}(X_0) \right] \]
\[ = u^T (f(X_0) + \dot{G}(X_0)) - \frac{1}{2} \text{Tr} \left[ (I_d \otimes u^T) D \mathcal{G}(X_0) \right] \]
\[ = \langle u, f(X_0) + \dot{G}(X_0) - \frac{1}{2} \text{Tr} D \mathcal{G}(X_0) \rangle \]
\[ \leq 0, \]
which is equivalent to 7 due to Lemma 3.5 and equality 4.

III. The case of the same eigenvalues.

**Proposition 1.** Let the condition (H1)–(H3) hold. Assume that \( C \) has the same eigenvalues and \( \mathcal{D} \) is stochastically invariant with respect to the equation 1. Then conditions 6 and 7 hold for all \( X(0) = x \in \mathcal{D} \) and \( u \in N_{D}^1(x) \).

**Proof.** Since \( C \) has the same eigenvalues, we will make a change of variable to derive the conditions of the case I or II. Let \( \lambda_1(X_0) \geq \cdots \geq \lambda_d(X_0) \). First, we assume that
\[ A^\varepsilon = Q(X_0) \text{diag} \left[ \sqrt{1-\varepsilon}, \sqrt{(1-\varepsilon)^2}, \cdots, \sqrt{(1-\varepsilon)^d} \right] Q(X_0)^T, \]
for \( 0 < \varepsilon < 1 \). Since \( \mathcal{D} \) is invariant with respect to \( X \), hence, \( \mathcal{D}^\varepsilon = A^\varepsilon \mathcal{D} \) is invariant with respect to \( X^\varepsilon := A^\varepsilon X \).

Note that
\[ dX^\varepsilon = \dot{G}_\varepsilon(X^\varepsilon_t) + f_\varepsilon(X^\varepsilon_t)dt + C^2_\varepsilon(X^\varepsilon_t)dw(t), \]
where \( f_\varepsilon = A^\varepsilon f((A^\varepsilon)^{-1}) \), \( C_\varepsilon = A^\varepsilon C((A^\varepsilon)^{-1}) (A^\varepsilon)^T \) and \( \dot{G}_\varepsilon = A^\varepsilon \dot{G}((A^\varepsilon)^{-1}) \) have the same regularity and growth as \( f, C \) and \( \dot{G} \). On the one hand, \( C_\varepsilon \) has non-zero-eigenvalue, because the positive eigenvalues of \( C_\varepsilon \) are all distinct at \( X_0^\varepsilon = A^\varepsilon X_0 \). So we can apply the case I to \((X^\varepsilon, \mathcal{D}^\varepsilon)\), then we get
\[ \begin{cases} 
C_\varepsilon(X_0^\varepsilon)u_\varepsilon = 0, \\
\langle u_\varepsilon, f_\varepsilon(X_0^\varepsilon) + \dot{G}_\varepsilon(X_0^\varepsilon) - \frac{1}{2} \sum_{j=1}^{d} DC^j_\varepsilon(X_0^\varepsilon)(C_\varepsilon C^+_\varepsilon)^j(X_0^\varepsilon) \rangle \leq 0.
\end{cases} \tag{11} \]

On the other hand, \( C_\varepsilon \) has zero-eigenvalue, because the positive eigenvalues of \( C_\varepsilon \) are all distinct at \( X_0^\varepsilon = A^\varepsilon X_0 \), as
\[ C_\varepsilon(X_0^\varepsilon) = Q(X_0) \text{diag} \left[ (1-\varepsilon)\lambda_1(X_0), \cdots, (1-\varepsilon)^d \lambda_d(X_0) \right] Q(X_0)^T. \]
Therefore, we can apply the case II to \((X^\varepsilon, \mathcal{D}^\varepsilon)\), then we get 11. We can verify that \( N_{A^\varepsilon D}(x^\varepsilon) = (A^\varepsilon)^{-1} N_D^1(x) \). By the Definition 2.4 and continuity of \( \varepsilon \), set \( \varepsilon \to 0 \) in 11, we have
\[ \begin{cases} 
C(X_0)u = 0, \\
\langle u, f(X_0) + \dot{G}(X_0) - \frac{1}{2} \sum_{j=1}^{d} DC^j(X_0)(CC^+_0)^j(X_0) \rangle \leq 0,
\end{cases} \]
for all \( u \in N_{D}^1(x) \). \( \square \)

**3.2. Sufficient condition.** In this section, we prove that the necessary conditions of Theorem 3.7 are also sufficient.

We will show that 6 and 7 imply that the generator \( \mathcal{L} \) of \( X \) satisfies the positive maximum principle (see [9], p165): If \( \phi \in C^2(\mathbb{R}^d, \mathbb{R}), x \in \mathcal{D} \) and \( \max_{\mathcal{D}} \phi = \phi(x) \geq 0, \)
we have \( \mathcal{L}\phi(x) \leq 0. \)

**Proposition 2.** Assume that 6 and 7 hold for all \( x = X(0) \in \mathcal{D} \) and \( u \in N_{D}^1(x) \). Then the generator \( \mathcal{L} \) satisfies the positive maximum principle.
Proof. It is similar to the proof of ([13], Proposition 4.1)
\[ \text{Tr}(D^2 \phi(x))C(X_0)) \leq -\langle D\phi(x)^T, \sum_{j=1}^d DC^j(X_0)(CC^+)^j(X_0) \rangle, \quad (12) \]
for any smooth function \( \phi \) such that \( \max D \phi = \phi(x) \geq 0 \).

Utilizing inequality 7 and 12, we have
\[
\mathcal{L} \phi(x) = D\phi(x)(f(X_0) + \dot{G}(X_0)) + \frac{1}{2} \text{Tr}(D^2 \phi(x)gg^T(X_0))
\leq D\phi(x)(f(X_0) + \dot{G}(X_0)) - \frac{1}{2} \langle D\phi(x)^T, \sum_{j=1}^d DC^j(X_0)(CC^+)^j(X_0) \rangle
= \langle D\phi(x)^T, f(X_0) + \dot{G}(X_0) - \frac{1}{2} \sum_{j=1}^d DC^j(X_0)(CC^+)^j(X_0) \rangle
= \langle u, f(X_0) + \dot{G}(X_0) - \frac{1}{2} \sum_{j=1}^d DC^j(X_0)(CC^+)^j(X_0) \rangle
\leq 0.
\]

\[ \square \]

**Remark 1.** Linear operator \( \mathcal{L} \) satisfying the positive maximum principle is dissipative, therefore the proof of Theorem 3.7 can also apply the theory of dissipative structure.

**Remark 2.** We assume that the NSFDE 1 is conservative in the following sense (see [10]): there exists a function \( H(x) \) defined on \( \mathbb{R}^d \) such that
\[
\begin{align*}
\langle \nabla H(x), g(x) \rangle &= 0, \\
\langle \nabla H(x), f(x) + \dot{G}(x) \rangle &= 0.
\end{align*}
\]
for any \( x \in \mathbb{R}^d \). We know that equation 1 is invariant on the conditions 6 and 7. Choosing
\[
H(x) = \phi(x),
\]
we can derive that\[
\langle \nabla H(x), g(x) \rangle = 0,
\]
but we can not get \[
\langle \nabla H(x), f(x) + \dot{G}(x) \rangle = 0.
\]
Therefore, equation 1 is not conservative unless
\[
\frac{1}{2} D^2 \phi(x)gg^T(x) = 0
\]
holds.

**Proposition 3.** Under the assumptions of Theorem 3.7, assume that condition 6 and 7 hold for all \( X(0) = x \in \mathcal{D} \) and \( u \in N^1_{\mathcal{D}}(x) \). Then \( \mathcal{D} \) is stochastically invariant with respect to the equation 1.

**Proof.** Lemma 3.1 implies that local martingale may be converted to real martingale. We shall verify the following integrability condition
\[
\mathbb{E} \sup_{s \in [0,t]} |M(s)| < \infty, \quad t \geq 0.
\]
Note that

\[ M(t) = \phi(X(t)) - \phi(X(0)) - \int_0^t \mathcal{L}\phi(X_s)ds \]

\[ = \sum_{i=1}^d \int_0^t \frac{\partial \phi}{\partial x_i} X(s) \sum_{j=1}^d g_{i,j}(X_s)dB_j(s), \]

where \( \frac{\partial \phi}{\partial x_i} \) is bounded, which follows from \( \phi \in C^\infty_b(\mathbb{R}^d, \mathbb{R}) \) with compact support in \( N(x) \). We assume that \( \mathcal{D}\phi(X_t) < C_1 \), for \( C_1 > 0 \), by using BDG’s inequality, Jensen’s inequality, condition (H1) and Lemma 3.1, we have

\[ \mathbb{E} \sup_{s \in [0,t]} |M(s)| = \mathbb{E} \left[ \sup_{s \in [0,t]} |\phi(X(t)) - \phi(X(0)) - \int_0^t \mathcal{L}\phi(X_s)ds| \right] \]

\[ = \mathbb{E} \left[ \sup_{s \in [0,t]} \left| \sum_{i=1}^d \int_0^t \frac{\partial \phi}{\partial x_i} X(s) \sum_{j=1}^d g_{i,j}(X_s)dB_j(s) \right| \right] \]

\[ \leq 16\sqrt{2} \max_{s \in [0,t]} \left( \frac{\partial \phi}{\partial x_i}(X(s)) \right) \mathbb{E} \left[ \left| \int_0^t \sum_{i=1}^d \sum_{j=1}^d g_{i,j}(X_s)ds \right|^2 \right]^{\frac{1}{2}} \]

\[ \leq 16\sqrt{2}C_1 \mathbb{E} \left[ \int_0^t \|g(X_s)\|^2 ds \right]^{\frac{1}{2}} \]

\[ \leq K \mathbb{E} \left[ \int_0^t (1 + \|X_s\|^2) ds \right]^{\frac{1}{2}} \]

< \infty,

where \( K \) is a positive constant. Hence, we can deal with real martingale problem instead of local martingale problem. We know that \( \mathcal{L} \) satisfies the positive maximum principle, then there exists a compact subset \( \mathcal{D}_{E}[\tau, T] \) of \( \mathcal{D}_{E}[0, \infty) \) (see [21], Theorem 4.6), where \( (E, r) \) denotes a metric space (see [9], p122). Then, (9), Theorem 4.5.4) yields the existence of a solution to the martingale problem associated to \( \mathcal{L} \) with sample paths in the space of \( \text{cadlag} \) functions with values in \( \mathcal{D}^\Delta = \mathcal{D} \cup \Delta \) which is the one-point compactification of \( \mathcal{D} \). Recall Lemma 3.3 and (9), Proposition 5.3.5), then we get that the solution has a modification with continuous sample paths in \( \mathcal{D} \). Finally, (9), Theorem 5.3.3) implies the existence of weak solution \( (X_., W) \) such that \( X(t) \in \mathcal{D} \) almost surely for all \( t > 0 \). \( \square \)

**Remark 3.** When \( \tau = 0 \) and \( G(X_t) = 0 \), Theorem 3.7 is equivalent to ([13], Theorem 2.3).

**4. Application.** In this section, we will show how Theorem 3.7 can be applied in the practical equation. Then, we give the result of the pathwise asymptotic estimates of solution for this equation, which supports the stochastic invariance of the solution of following equation. The follow-up study is restricted to an one-dimensional setting for ease of computation and notation.

**Example 4.1.** Consider the Tanaka equation

\[ d(X(t) - 0.1X_t) = 0.1\text{sign}(X_t)dt + \text{sign}(X_t)dw(t), \] (13)
where
\[
\text{sign}(x) = \begin{cases} 
+1 & \text{if } x \geq 0, \\
-1 & \text{if } x < 0.
\end{cases}
\]

Then, with the previous notations, \( \phi : z \to 1 - z^2 \). Therefore, the first order normal cone given by Definition 2.4 reads
\[
\mathcal{N}_D^1(x) = \{ -2z \in \mathbb{R} : \langle -2z, y - x \rangle \leq o(\|y - x\|), \forall y \in D \}.
\]

Then, we get \( D \subset \mathbb{R} \) and the set \( D \) is stochastically invariant with respect to the equation 13 if and only if
\[
\begin{cases}
-2X(0)\text{sign}(X_0) = 0, \\
\langle -2X(0), 0.1\text{sign}(X_0) + 0.1 \rangle \leq 0. \quad (14)
\end{cases}
\]

4.1. **Sufficient condition.**

**Proof.** Since
\[
\begin{cases}
|0.1\text{sign}(X_t)|^2 \lor |\text{sign}(X_t)|^2 \leq 1 + \|X_t\|^2, \\
0.1X_t \leq 0.1\|X_t\|, \\
E[0.1X_t - 0.1X_s]^p \leq 0.1^p \sup_{-\tau \leq \theta \leq 0} E[X(t + \theta) - X(s + \theta)]^p.
\end{cases} \quad (15)
\]

Hence, \((H1)-(H2)\) are satisfied. We can derive from 14
\[
X(0) = 0, \quad X_0 \leq 0.
\]

Choose
\[
\max_D \phi = 1 - X(0)^2 \geq 0,
\]

we have
\[
\mathcal{L}\phi(X(0)) = [f(X_0) + \dot{G}(X_0)]D\phi(X(0)) + \frac{1}{2} Tr(D^2\phi(X(0))gg^T(X_0))
\]
\[
= -2X(0)(0.1\text{sign}(X_0) + 0.1) - \text{sign}^2(X_0)
\]
\[
= -1
\]
\[
< 0.
\]

Hence, the generation \( \mathcal{L} \) satisfies the positive maximum principle, by using of ([9], Theorem 5.3.3), we get that there exists a weak solution \((X, W)\) such that \( X(t) \in D \) almost surely for all \( t > 0 \).

4.2. **Necessary condition.** Assume that \( X(0) = x \in D \) and \( \phi(x) = 1 - x^2 \). Since the set \( D \) is stochastically invariant with respect to the equation 13, there exist
\[
\max_D \phi = 1 - x^2,
\]

and
\[
\phi(X(t)) \leq \phi(x), \quad t > 0. \quad (16)
\]

Applying Itô formula twice to inequality 16, we obtain
\[
\int_0^t [-2X(s)(0.1\text{sign}(X_s) + 0.1) - \text{sign}^2(X_s)] \, ds
\]
\[ + \int_0^t \left\{ -2 \text{sign}(X_0)X(0) + \int_0^s -0.2 \text{sign}^2(X_r) - 0.2 \text{sign}(X_r)dr \\
+ \int_s^t [-2 \text{sign}^2(X_r)]dw(r) \right\} dw(s) \leq 0. \]

Combing with Lemma 3.6, we can derive
\[
\begin{cases}
-2X(0)\text{sign}^2(X_0) = 0, \\
(-2X(0), 0.1\text{sign}(X_0) + 0.1) \leq 0,
\end{cases}
\]
which are equivalent to condition 14. \(\square\)

4.3. Verification. Since (H1) and (H2) are satisfied. By using pathwise asymptotic estimate theory ([20], Corollary 6.4.8), for any given initial data, there exists a solution \(X(t, \xi)\) of equation 13 and this solution has the property
\[
\limsup_{t \to \infty} \frac{1}{t} \ln(|X(t)|) \leq \left( \frac{1}{1 - 0.1} \right)^2 [2(1 + 0.1) + 65],
\]
that is
\[
P\{w \in \Omega : \limsup_{t \to \infty} \frac{1}{t} \ln(|X(t)|) \leq 83\} = 1.
\]
Then, we have
\[
\frac{1}{t} \ln |X(t)| \leq \limsup_{t \to \infty} \frac{1}{t} \ln(|X(t)|) \leq 83 \text{ a.s.}
\]
Therefore,
\[
|X(t)| \leq e^{83t} \text{ a.s.} \tag{17}
\]
Let \(F = [-e^{83t}, e^{83t}] \subseteq \mathbb{R}\). Then \(F \subseteq D \subseteq \mathbb{R}\). Hence, 17 implies that the solution of 13 is stochastic invariance in \(D\).

Remark 4. In pathwise asymptotic estimate theory one can get the approximate trajectory of the solution for the equation; in stochastic invariance theory one can get the coverage area of all weak solutions for the equation. This is a balance. In addition, one can still obtain specific initial value in stochastic invariance theory, but one can not get it in asymptotic estimate theory.

5. Conclusion. In this work, stochastic invariance theory for NSFDEs has been studied. Some necessary and sufficient conditions for the Theorem 3.7 have been established. These obtained results extend and improve the results in [13]. Moreover, an example is given to illustrate our results.

6. Appendix. For the readers’ convenience, we collect some definitions and properties of matrix tools intensively used in the proofs throughout the article in this Appendix.

Definition 6.1. Let \(a_{ij}(x)(i, j = 1, 2, \cdots, d)\) is a real-valued function defined on \([-\tau, 0]\). Then \(C(x) = (a_{ij}(x))_{d \times d}\) is a function matrix defined on \([-\tau, 0]\).

In particular, if
\[
g(X_0) = (g_1(X_0), \cdots, g_d(X_0))^T \text{ and } X_0 = \xi(\theta),
\]
Then

\[ C(\xi(\theta)) = \begin{pmatrix}
\sum_{i=1}^{d} g_{1}^{2}(\xi_{i}(\theta)) & \sum_{i=1}^{d} g_{1}g_{2}(\xi_{i}(\theta)) & \cdots & \sum_{i=1}^{d} g_{1}g_{d}(\xi_{i}(\theta)) \\
\sum_{i=1}^{d} g_{2}g_{1}(\xi_{i}(\theta)) & \sum_{i=1}^{d} g_{2}^{2}(\xi_{i}(\theta)) & \cdots & \sum_{i=1}^{d} g_{2}g_{d}(\xi_{i}(\theta)) \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i=1}^{d} g_{d}g_{1}(\xi_{i}(\theta)) & \sum_{i=1}^{d} g_{d}g_{2}(\xi_{i}(\theta)) & \cdots & \sum_{i=1}^{d} g_{d}^{2}(\xi_{i}(\theta))
\end{pmatrix}
\]

since \( g(X_{0}) \) satisfies linear growth condition.

Let

\[ g_{j}(\xi_{i}(\theta)) = k_{j}(1 + \xi_{i}^{2}(\theta))^{\frac{1}{2}}, \quad \text{where} \quad k_{j} > 0. \]

Then,

\[ C(\xi(\theta)) = \begin{pmatrix}
k_{1}^{2} & k_{1}k_{2} & \cdots & k_{1}k_{d} \\
k_{2}k_{1} & k_{2}^{2} & \cdots & k_{2}k_{d} \\
\vdots & \vdots & \ddots & \vdots \\
k_{d}k_{1} & k_{d}k_{2} & \cdots & k_{d}^{2}
\end{pmatrix}
\]

\( (d + \sum_{i=1}^{d} \xi_{i}^{2}(\theta)) = C(d + |\xi(\theta)|^{2}). \)

Hence, \( C \) is a real-valued symmetric matrix (or a real Hermitian matrix), there exists an orthogonal matrix \( Q \) which satisfies \( QQ^{T} = I_{d} \), such that

\[ C = Q\text{diag}(\lambda_{i})Q^{T}, \quad i = 1, 2, \cdots, r, \]

where \( r \leq d \) and \( \lambda_{1}, \lambda_{2}, \cdots, \lambda_{r} \in \mathbb{R} \) are distinct eigenvalues of \( C \) (see [19], Theorem 3.2.4). Then,

\[ C(\xi(\theta)) = Q\text{diag}(\lambda_{i}(d + |\xi(\theta)|^{2}))Q^{T}. \]

Since \( 0 < d + |\xi(\theta)|^{2} < \infty \), therefore, the eigenvalue of \( C(\xi(\theta)) \) is only dependent on matrix \( C \) and has nothing to do with \( \theta \).

**Definition 6.2.** Fix \( A \in M^{m \times n} \). The Moore-Penrose pseudoinverse of \( A \) is the unique \( n \times m \) matrix \( A^{+} \) satisfying: \( AA^{+}A = A \), \( A^{+}AA^{+} = A^{+} \), \( AA^{+} \) and \( A^{+}A \) are Hermitian.

For any given matrix \( A \in M^{m \times n} \), there exists the unique Moore Penrose pseudoinverse \( A^{+} \in M^{n \times m} \) satisfying: \( (\alpha A)^{+} = \alpha^{-1}A^{+}, \alpha \in R, \alpha \neq 0 \) (see [2], Theorem 7.6.5).

**Definition 6.3.** (the decomposition theorem of Hermitian Matrix) Let \( A \in M^{d} \) is a real Hermitian Matrix, if and only if there exists a real valued orthogonal matrix \( Q \in M^{d} \) and a real valued diagonal matrix \( \Lambda = \text{diag}([\lambda_{i}]_{i \leq d}) \in M^{d} \) such that \( A = QAQ^{T} \).

**Proposition 4.** If \( A \in M^{d} \) has the spectral decomposition \( QAQ^{T} \) for some orthogonal matrix \( Q \in M^{d} \) and a diagonal matrix

\[ \Lambda = \text{diag}([\lambda_{i}]_{i \leq d}) \in M^{d}. \]

Then,

\[ A^{+} = QA^{+}Q^{T} \]

in which

\[ \Lambda^{+} = \text{diag}([\lambda_{i}^{-1}1_{\lambda_{i} \neq 0}]_{i \leq d}) \]

and

\[ AA^{+} = Q\text{diag}([1_{\lambda_{i} \neq 0}]_{i \leq d}]Q^{T}. \]
Moreover, if $A$ is positive semi-definite and $B \triangleright$, then

$$B^+ = Q(A^+) \frac{1}{2} Q^T.$$ 

**Definition 6.4.** Let $A = (a_{ij}) \in M^{m_1 \times n_1}$ and $B \in M^{m_2 \times n_2}$. The Kronecker product $(A \otimes B)$ is defined as the $m_1 m_1 \times n_1 n_2$ matrix

$$A \otimes B = \begin{pmatrix}
  a_{11}B & \cdots & a_{1n_1}B \\
  \vdots & \ddots & \vdots \\
  a_{m_11}B & \cdots & a_{m_1n_1}B
\end{pmatrix}.$$ 

**Definition 6.5.** Let $A$ and $B$ be as in Definition 6.3, $C \in M^{n_1 \times n_3}$ and $D \in M^{n_2 \times n_4}$. Then

$$(A \otimes B)(C \otimes D) = (AC \otimes BD),$$

$$(A \otimes B) = A(I_{n_1} \otimes B), \quad \text{if} \quad m_2 = 1,$$

$$(A \otimes B) = B(A \otimes I_{n_2}), \quad \text{if} \quad m_1 = 1.$$ 

**Definition 6.6.** Let $F$ be a differential map: $M^{n,q} \to M^{m,p}$. The Jacobian matrix $DF(X)$ of $F$ at $X$ is defined as the following $mp \times nq$ matrix (see [19]):

$$DF(X) = \frac{\partial \text{vec}(F(X))}{\partial \text{vec}(X)}.$$ 

**Definition 6.7.** Let $F$ be a differentiable map from $M^{n,q}$ to $M^{m,p}$ and $H$ be a differentiable map from $M^{n,q}$ to $M^{p,l}$. Then

$$D(GH) = (H^T \otimes I_m)DG + (I_l \otimes G)DH.$$ 

**Definition 6.8.** (see [6]) Let $V$ and $W$ be Banach spaces, and $U \subset V$ be an open subset of $V$. A function $f : U \to W$ is called Frchet differentiable at $x \in U$ if there exists a bounded linear operator $A : V \to W$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ah\|_W}{\|h\|_V} = 0,$$

the limit here is meant in the usual sense of a limit of a function defined on a metric space. Equivalently, the first-order expansion holds, in Landau notation

$$f(x+h) = f(x) + Ah + o(h).$$

If there exists such an operator $A$, it is unique, so we write $Df(x) = A$ and call it the Frchet derivative of $f$ at $x$.

The Frchet derivative in finite-dimensional spaces is the usual derivative. In particular, it is represented in coordinates by the Jacobian matrix. Suppose that $f$ is a map, $f : \mathbb{R}^n \to \mathbb{R}^m$ with $U$ an open set. If $f$ is Frchet differentiable at a point $x \in U$, then its derivative is

$$Df(x)(v) = J_f(x)v,$$

where $J_f(x)$ denotes the Jacobian matrix of $f$ at $x$. 

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