Finite-Time Stability of Adaptive Parameter Estimation and Control

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Abstract—This paper presents extensions of finite-time stability results to some prototypical adaptive control and estimation frameworks. First, we present a novel scheme of online parameter estimation that guarantees convergence of the estimation error in a fixed time under a relaxed persistence of excitation condition. Subsequently, we design a novel Model Reference Adaptive Control (MRAC) for a scalar system with finite-time convergence guarantees for both the state- and the parameter-error. Lastly, for a general class of strict-feedback systems with unknown parameters, we propose a finite-time stabilizing control based on adaptive backstepping techniques. We also present some numerical examples demonstrating the efficacy of our scheme.

I. INTRODUCTION

Adaptive control and estimation has been an area of ongoing research, and has had significant impact over the years in terms of practical applications. References such as [1]–[3] provide an overview of well known techniques that have been developed towards addressing a wide variety of problems encountered in control and estimation of dynamical systems. Classical adaptive control can be broadly classified into two categories, 1) Direct Adaptive Control, wherein the plant model is re-parameterized in terms of control parameters and the corresponding adaptive law is designed to adapt to these unknown parameters, and 2) Indirect Adaptive Control, wherein the unknown parameters of the plant are first estimated and then used to design the control. Adaptive control algorithms are typically designed to have asymptotic or exponential stability. However, it is often desirable to have stability and convergence guarantees in finite time. Finite-Time Stability (FTS) is a well-studied concept, motivated in part from a practical viewpoint due to properties such as achieving convergence in finite time, as well as exhibiting robustness with respect to (w.r.t) disturbances [4]. The seminal paper [5] presented the necessary and sufficient Lyapunov conditions for FTS. In [6], the authors provide geometric conditions for homogeneous systems to exhibit FTS. The authors in [7] extended the notion of finite-time stability to fixed-time stability, where the time of convergence is independent of initial condition.

The finite-time stability notion in adaptive control has gained much popularity in recent years [8], [9]. Following the work in [10], the authors in [11] studied systems in p-normal form, and designed a globally finite-time stabilizing controller in the presence of parametric uncertainty. More recently, in [12], a recursive algorithm was introduced for parameter estimation, which converges in finite time. The authors in [13] presented a method of parameter estimation under a relaxed persistence of excitation condition. Identification of time-varying parameter is studied in [14], in which the authors define the notion of Short-FTS and design an adaptation scheme in that framework. Authors in [15] design an Adaptive observer for LTI system with unknown parameters with fixed-time convergence guarantees. Finite-time MRAC is studied in [16], where the authors study finite-time convergence of the tracking error of a Single-Input Single-Output (SISO) system. In [17], the authors used an auxiliary-filter based sliding-mode technique so that the control and parameter errors converge in finite time. Semi-Global Practical FTS (SGPFS) has been utilized in [18], [19] to design a backstepping based controller, which guarantees that the error converges to a small neighborhood of the origin in finite time. In [20], the authors study systems in non-strict feedback form and design adaptive control which guarantees that the tracking error convergence to a small neighborhood of origin in finite time.

In this work, we extend the notion of finite-time stability to some prototypical cases of adaptive estimation and control algorithms. We present three results pertaining to FTS: 1) Online adaptive estimation of input-output model, where we relax the traditional assumptions on persistence of excitation, and show fixed-time convergence of parameter estimation error; 2) Scalar MRAC, an example of direct adaptive control, where we design a continuously differentiable control- and adaptation-law with finite-time convergence guarantees; 3) Adaptive Backstepping, an example of indirect adaptive control, wherein we consider a general class of systems in strict-feedback form with unknown parameters and design an adaptive control law to track a time-varying reference trajectory in finite time. Compared to aforementioned results, we assume very mild conditions on the system, design continuously differentiable control laws and guarantee the convergence of the state- and the parameter-errors in finite-time.

The organization of the rest of the paper is as follows. In Section II we study an example of online estimation of input-output model and design an adaptation scheme that guarantees fixed-time stability. In Section III we design a modified MRAC scheme with FTS guarantees for both the disturbance-free and additive disturbance case. In Section IV we study the systems in strict-feedback form with unknown
parameters and design adaptive controller with finite-time convergence guarantees for both the tracking- and parameter-estimation error. In Section VI, we illustrate the efficacy of our results with numerical simulations for each of these cases. We conclude the paper with suggestions for future work in Section VII.

II. FINITE-TIME ONLINE ESTIMATION

A. Notations

We denote \( \|x\| \) the Euclidean norm \( \|x\|_2 \) of vector \( x \), and \( |x| \) the absolute value of the scalar \( x \). Whenever clear from the context that a variable \( z(\cdot) \) is a function of \( t \), we drop the argument \( t \) for the sake of brevity. The set of non-negative integers as \( \mathbb{Z}_+ \). The smallest integer greater than or equal to \( x \) is denoted as \( \lceil x \rceil \). We denote \( [x]^c = \text{sign}(x)|x|^c \), where the function \( \text{sign} : \mathbb{R} \to \mathbb{R} \) is defined as:

\[
\text{sign}(x) = \begin{cases} 
-1, & x < 0; \\
0, & x = 0; \\
1, & x > 0.
\end{cases}
\] (1)

B. Fixed-time Adaptive Estimation

Consider the following model of the plant that illustrates online parameter estimation for an input-output model [1]:

\[
y(t) = \theta u(t),
\] (2)

where \( y \in \mathbb{R} \) is the system output, \( u \in \mathbb{R} \) is the system input and \( \theta \in \mathbb{R} \) is the constant, unknown input-output gain. In order to estimate the unknown parameter \( \theta \), we consider the plant model as:

\[
\hat{y}(t) = \hat{\theta}(t)u(t),
\] (3)

where \( \hat{\theta} \) is the estimate of the parameter \( \theta \). Define \( \hat{\theta}(t) = \theta - \theta(t) \) and \( \hat{y}(t) = y - \hat{y}(t) \) so that we have:

\[
\hat{y}(t) = \hat{\theta}(t)u(t).
\] (4)

The objective is to design an adaptation law for \( \hat{\theta} \) such that the error \( \hat{\theta}(t) \) converges to zero in a fixed time, independent of the initial condition. The commonly used assumption on persistency of excitation for \( u(t) \) in literature (e.g., [1], [12]) is that there exists constants \( \Delta, \alpha > 0 \) such that for all \( t \geq 0 \),

\[
\int_{t}^{t+\Delta} u(\tau)^2 d\tau \geq \alpha.
\] (5)

In this work, we relax this condition as:

Assumption 1: The input signal \( u(t) \) is continuous for all \( t \geq 0 \) and the following inequality holds for some \( \alpha, \Delta, K_1 > 0 \):

\[
\int_{k\Delta}^{(k+1)\Delta} u(\tau)^2 d\tau \geq \alpha,
\] (6)

for all \( k \in \mathbb{Z}_+, k \leq K_1 \), where \( K_1 \) is sufficiently large and positive integer.

Note that the difference between (5) and (6) is that in the latter case, we only need the persistence condition to hold in the disjoint intervals for a cumulative time \( 0 \leq t \leq K_1 \Delta \), unlike the former case where the inequality needs to hold for all \( t \geq 0 \). Before presenting the main result, we need the following Lemma:

Lemma 1: In each interval \( T_k = [k\Delta, (k+1)\Delta] \), \( k \in \mathbb{Z}_+ \), there exists a sub-interval \( \tau_k = [t_k, t_k + \delta_k] \) where \( \delta_k > 0 \) with \( t_k \geq k\Delta \) and \( t_k + \delta_k \leq (k+1)\Delta \), such that for all \( t \in \tau_k \),

\[
|u(t)| \geq \frac{u_{M_k}}{\sqrt{2}},
\] (7)

where \( u_{M_k} = \max_{t \in T_k} |u(t)| \).

Proof: Define \( z(t) = u(t)^2 \), so that continuity of \( u(t) \) implies that \( z(t) \) is also continuous. For any interval \( T_k = [k\Delta, (k+1)\Delta] \), \( k \geq 0 \), denote \( \hat{z}_{M_k} = \max_{t \in T_k} z(t) \) and \( z_{m_k} = \min_{t \in T_k} z(t) \). Note that (7) is equivalent to \( z(t) \geq \frac{1}{2} z_{M_k} \). Note that the fact that the interval \( T_k \) is closed and bounded implies that \( z(t) \) achieves the minimum and maximum values on the interval \( T_k \). If \( z_{m_k} \geq \frac{1}{2} z_{M_k} \), then we obtain the desired result with \( \delta_k = \Delta \). If \( z_{m_k} < \frac{1}{2} z_{M_k} \), denote the time instant \( t_1 \) when \( z(t_1) = z_{m_k} \). We know that there exists \( t_2 \in T_k \) satisfying \( t_2 \neq t_1 \), such that \( z(t_2) = z_{M_k} \). Let \( T_3 = \{t \in (\min(t_1, t_2), \max(t_1, t_2)) \mid z(t) = \frac{1}{2} z_{M_k} \} \), define \( \tilde{t} = \min_{t \in T_3} |t - t_1| \) and \( \delta_k \geq |t_2 - \tilde{t}| > 0 \) so that for all \( t \in \tau_k = [\min(t_2, \tilde{t}), \max(t_2, \tilde{t})] \), we have \( z(t) \geq \frac{1}{2} z_{M_k} \), which completes the proof.

Denote \( u_m = \min_{k \in \mathbb{Z}_+} u_{M_k} \), where \( \Sigma = \{1, 2, \cdots, K_1 \} \) and \( K_0 > 0 \) is a large positive integer. From Assumption 1, we know that \( u_{M_k} \geq \frac{1}{\sqrt{2}} \) for all \( k \geq 0 \) and hence, \( u_m > 0 \). Define \( c = \frac{u_m}{\sqrt{2}} \) and \( \delta = \min \delta_k > 0 \), so that from Lemma 1, we obtain that:

\[
u(t) \geq c,
\] (8)

for all \( t \in \Gamma = \bigcup_{k \in \Sigma} \tau_k \). Now we are ready to state the main result of this section.

Theorem 1: Let Assumption 7 hold for some \( K_1 > 0 \). Then, there exists \( T < \infty \) such that the parameter estimation error \( \hat{\theta}(t) = 0 \) for all \( t \geq T \), if the adaptation law for \( \hat{\theta}(t) \) is chosen as:

\[
\hat{\theta} = (k_1 \hat{y}^{\alpha_1} + k_2 \hat{y}^{\alpha_2} \hat{u}),
\] (9)

where \( k_1, k_2 > 0 \) and \( 0 < \alpha_1 < 1, \alpha_2 > 1 \).

Proof: Choose the candidate Lyapunov function

\[
V(\hat{\theta}) = \frac{1}{2} \hat{\theta}^2.
\]

The time derivative of this function reads:

\[
\dot{V} = \frac{1}{2} \hat{\theta} \dot{\hat{\theta}} = -\hat{\theta}[(k_1 \hat{y}^{\alpha_1} + k_2 \hat{y}^{\alpha_2} \hat{u})]
\]

\[
= -\hat{\theta}[(k_1 \hat{y}^{\alpha_1} |u|^{\alpha_1} + k_2 \hat{y}^{\alpha_2} |u|^{\alpha_2})]u
\]

\[
= -k_1 \hat{\theta}^{\alpha_1+1} |u|^{\alpha_1+1} - k_2 \hat{\theta}^{\alpha_2+1} |u|^{\alpha_2+1}
\]

From (8), we have that \( |u(t)|^{\alpha_1+1} \geq c^{\alpha_1+1} \triangleq c_1 \) and \( |u(t)|^{\alpha_2+1} \geq c^{\alpha_2+1} \triangleq c_2 \) for all \( t \in \Gamma \). Also, for \( t \notin \Gamma \), we have that \( |u(t)|^{\alpha_1+1} \geq 0 \) for \( i \in \{1, 2\} \). Define \( \alpha_i = k_i (2 - \alpha_i) \) and \( \beta_i = \frac{1}{2} \frac{|u|}{\sqrt{2}} \) so that we have:

\[
\dot{V} \leq \begin{cases} 
- a_1 \dot{V}_{\beta_1} - a_2 \dot{V}_{\beta_2}, & t \in \Gamma; \\
0, & \text{otherwise}
\end{cases}
\] (10)

where \( 0 < \beta_1 < 1 \) and \( \beta_2 > 1 \). We denote \( \dot{V}(t) = V(\hat{\theta}(t)) \).

Consider any interval \( T_k \). The length of the interval \( T_k \) satisfy
\[ V((k+1)\Delta)^{1-\beta_1} - V(k\Delta)^{1-\beta_1} \leq -a_1\delta, \]

for all \( k \geq 0 \). Define \( V_k = V(k(\Delta)) \) so that we have:

\[ V_{k+1}^{1-\beta_1} - V_k^{1-\beta_1} \leq -a_1\delta(1 - \beta_1), \]

\[ \implies \sum_k (V_{k+1}^{1-\beta_1} - V_k^{1-\beta_1}) \leq -K a_1 \delta(1 - \beta_1), \]

\[ \implies V_{K+1}^{1-\beta_1} - V_0^{1-\beta_1} \leq -K a_1 \delta(1 - \beta_1). \]

Hence, for \( K = \lceil \frac{V(\delta(0))^{1-\beta}}{a_1\delta(1-\beta)} \rceil \), we have \( V_{K+1} \leq 0 \). Since \( V \geq 0 \) it follows that \( V_{K+1} \leq 0 \). We have that for all \( t \geq T_1, \hat{\theta}(t) = 0 \), where \( T_1 \leq K\Delta = \lceil \frac{1}{a_1\delta(1-\beta)} \rceil \Delta \leq \frac{1}{a_1\delta(1-\beta)} \Delta \). Now, take the other case when \( V(0) \geq 1 \). We know that \( \dot{V} \leq -a_1V^{1-\beta_1} - a_2V^{1-\beta_2} \leq -a_2V^{1-\beta_2} \) for \( t \in \Gamma \). Let \( k = K_0 \) for which \( V_{k+1} \leq 1 \). Using this, we obtain for all \( k \leq K_0 \):

\[ V_{k+1}^{1-\beta_2} - V_k^{1-\beta_2} \leq -a_2\delta, \]

\[ \implies V_{k+1}^{1-\beta_2} - V_k^{1-\beta_2} \geq -a_2\delta(1 - \beta_2), \]

\[ \implies V_{K_0+1}^{1-\beta_2} - V_{K_0}^{1-\beta_2} \geq K_0 a_2 \delta(\beta_2 - 1), \]

\[ \implies 1 \geq V_{K_0+1}^{1-\beta_2} \geq K_0 a_2 \delta(\beta_2 - 1). \]

Hence, we have that after \( K_0 \) intervals, \( V \leq 1 \), where \( K_0 \leq \lceil \frac{\beta_2 - 1}{a_2\delta(\beta_2 - 1)} \rceil \). In other words, for \( t \geq T_2 \), \( V(t) \leq 1 \) where \( T_2 \leq \lceil \frac{1}{a_2\delta(\beta_2 - 1)} \rceil \Delta \). We know from the earlier analysis that \( V(t) = 0 \) for \( t \geq T_1, \) if \( V(0) \leq 1 \). Define \( T \triangleq T_1 + T_2 = \lceil \frac{1}{a_1\delta(1-\beta)} \rceil + \lceil \frac{1}{a_2\delta(\beta_2 - 1)} \rceil \Delta \) so that we have, for all \( t \geq T \), the error \( \hat{\theta}(t) = 0 \) for all \( \hat{\theta}(0) \). One can choose parameters \( k_1, k_2, a_1, a_2 \) so that the guarantee of convergence \( T \) satisfies \( \frac{T}{\Delta} \leq K_1 \), which guarantees that the error converges to zero for any \( K_1 > 0 \) in (11).

### III. FTS MRAC

#### A. Case 1: Normal system without Disturbance

In this section we further extend our investigations in finite-time stability to the case of model-reference adaptive control. We consider the case of scalar MRAC. The goal is to converge to a given reference trajectory and simultaneously adapt the parameters in the model. Consider the system:

\[ \dot{x}(t) = ax(t) + bu(t), \]

where the scalar parameters \( a, b \in \mathbb{R} \) are unknown with \( b \neq 0 \). We assume that the sign of \( b \) is known and without loss of generality, assume that \( b > 0 \). The reference model for (11) is given by:

\[ \dot{x}_m(t) = a_m x_m(t) + b_m r(t), \]

where the scalar parameters \( a_m, b_m \) are known and \( r(t) \) is a known, bounded signal. We assume that \( a_m < 0 \) and that the matching condition holds, i.e. there exists \( k^*_s \) and \( k^*_r \) such that:

\[ a + b k^*_s = a_m, \]  
\[ b k^*_r = b_m. \]

Furthermore, we assume that the persistence of excitation condition for \( r(t) \) is satisfied so that we can guarantee convergence of the error in parameters as well [2]. Define the state error as \( \bar{x} = x - x_m \). We design a controller as:

\[ u(t) = k_x(t)x(t) + k_r(t)r(t) - k[\bar{x}]^\alpha, \]

where \( k > 0 \) and \( 0 < \alpha < 1 \). Define \( \tilde{k}_x = k_x(t) - k^*_x \) and \( \tilde{k}_r = k_r(t) - k^*_r \), so that we obtain:

\[ \dot{\bar{x}} = a_m \bar{x} + b \tilde{k}_x x + b \tilde{k}_r r - b k[\bar{x}]^\alpha. \]

Define the adaptation laws for the parameters \( k_x \) and \( k_r \) as:

\[ \dot{k}_x = -\gamma_x [\bar{x}]^{2\alpha-1} x, \]

\[ \dot{k}_r = -\gamma_r [\bar{x}]^{2\alpha-1} r, \]

where \( \gamma_x, \gamma_r > 0 \) are constants. Since \( k^*_x \) and \( k^*_r \) are constants, we have that \( \tilde{k}_x(t) = \tilde{k}_x \) and \( \tilde{k}_r(t) = \tilde{k}_r \). Define the error vector \( z \triangleq [\bar{x} \ k_x \ k_r]^T \), so that the error dynamics reads:

\[ \dot{\bar{x}} = -bk[\bar{x}]^\alpha + \tilde{k}_x x + \tilde{k}_r r + \phi(\bar{x}), \]

\[ \dot{\tilde{k}}_x = -\gamma_x [\bar{x}]^{2\alpha-1} x(t), \]

\[ \dot{\tilde{k}}_r = -\gamma_r [\bar{x}]^{2\alpha-1} r(t). \]

We first analyze (17) assuming that the last term \( \phi(\bar{x}) \) is absent. Re-write (17) under this condition:

\[ \dot{\bar{x}} = -bk[\bar{x}]^\alpha + \tilde{k}_x x + \tilde{k}_r r, \]

\[ \dot{\tilde{k}}_x = -\gamma_x [\bar{x}]^{2\alpha-1} x(t), \]

\[ \dot{\tilde{k}}_r = -\gamma_r [\bar{x}]^{2\alpha-1} r(t). \]

We refer to (18) as the nominal system and (17) as the perturbation of the nominal system by the disturbance \( \phi(\bar{x}) \). First, we analyze (18) for FTS:

**Lemma 2:** The right-hand side of (18) is homogeneous with degree of homogeneity \( d = \alpha - 1 < 0 \).

*Proof:* Using the definition of homogeneity in [6], it is sufficient to show that there exist constants \( r_1, r_2, r_3 > 0 \) for the dilution \( \Delta_x = (e^{r_1\bar{x}}, e^{r_2\tilde{k}_x}, e^{r_3\tilde{k}_r}) \) and \( d \) such that:

\[ f_i(e^{r_1\bar{x}}, e^{r_2\tilde{k}_x}, e^{r_3\tilde{k}_r}) = e^{d+r_i} f_i(\bar{x}, \tilde{k}_x, \tilde{k}_r), \]

for all \( i \in \{1, 2, 3\}, \epsilon > 0 \). Choose \( r_1 = 1, r_2 = r_3 = \alpha \). With this choice of parameters, it is easy to verify that (19) holds for each \( i \) with \( d = \alpha - 1 \).■

**Theorem 2:** If \( \alpha > \frac{3}{2} \), then the origin is an asymptotically stable equilibrium for (18). Furthermore, all signals in the closed-loop system are bounded.

*Proof:* Choose the candidate Lyapunov function

\[ V(z) = \frac{1}{2\alpha} |\bar{x}|^{2\alpha} + \frac{b}{2\gamma_x} k_x^2 + \frac{b}{2\gamma_r} k_r^2. \]
Its time derivative along the trajectories of (18) reads:

\[ \dot{V} = [\ddot{x}]^{2\alpha-1}(-bk[\dot{x}]^{\alpha} + b\dot{k}_x x + b\dot{k}_r r) - b\dot{k}_x ([\dot{x}]^{2\alpha-1} x) - b\dot{k}_r ([\dot{x}]^{2\alpha-1} r) = -bk[\ddot{x}]^{3\alpha-1}. \]

Therefore, with \( V(z) > 0 \) and \( \dot{V}(z) \leq 0 \) for all \( t \geq 0 \), we have that \( V(z(t)) \leq V(z(0)) \), i.e., all the error terms remain bounded. This implies the estimates \( \dot{k}_x, \dot{k}_r \) remain bounded at all times. Now, since \( \alpha > \frac{2}{\gamma} \), we have that \( 3\alpha - 1 > 0 \) and hence, \( \ddot{x} = 0 \implies \dot{V} = 0 \). Now, taking the time derivative of \( \dot{V} \), we obtain \( \ddot{V} = k\dot{b}^2(3\alpha-1)[\ddot{x}]^{3\alpha-2}([\dot{x}]^{\alpha} - \text{sign}(\ddot{x})\dot{k}_x x - \text{sign}(\ddot{x})\dot{k}_r r) \). It is assumed that \( r(t) \) is bounded and the reference model \( [12] \) is stable, i.e. \( x_m \) is also bounded. Hence, we have that \( \ddot{x} \) is bounded. This completes the proof of the claim that all the closed-loop signals are bounded. For \( \alpha > \frac{2}{\gamma} \), we have that \( [\ddot{x}]^{3\alpha-2} \) is bounded for bounded \( \ddot{x} \). Hence, we have that \( \dot{V} \) is bounded. Using Barbalat’s lemma, we conclude that \( \dot{V} \to 0 \). Furthermore, from persistence of excitation condition, we have that the error terms \( \dot{k}_x, \dot{k}_r \) also go to zero as \( \ddot{x} \) goes to zero, which completes the proof.

We have so far shown that the system (18) is homogeneous with negative degree of homogeneity and that the origin is an asymptotically stable equilibrium. Hence, from [6, Theorem 7.1], we obtain that the origin of (18) is finite-time stable. We can now state the following result:

**Theorem 3:** Let \( \alpha > \frac{2}{\gamma} \). Then, the origin of the error dynamics (17) is a finite-time stable equilibrium.

**Proof:** Consider the error dynamics (17). As shown in Lemma 2 the right-hand side of (18) is homogeneous with respect to the dilation \( \Delta_\alpha = (\epsilon^{\alpha_1}, \epsilon^{\alpha_2}, \epsilon^{\alpha_3}) \). Using [6, Theorem 7.2], we have that there exists a positive definite, continuously differentiable function \( \tilde{V} : \mathbb{R}^3 \to \mathbb{R} \), such that its time derivative along (18) satisfies:

\[ \dot{\tilde{V}}(z) \leq -c\tilde{V}(z)^\beta, \quad (20) \]

for some \( 0 < \beta < \frac{1}{2} \) and \( c > 0 \). Note that \( \phi(\ddot{x}) = a_m \ddot{x} \) satisfies \( |\phi| \leq |a_m| |\ddot{x}| \leq |a_m| ||\ddot{x}|| \), i.e. the disturbances term is Lipschitz continuous with Lipschitz constant \( |a_m| \). Hence, using [5, Theorem 5.3], we have that the origin of (17) is a finite-time stable equilibrium.

**B. Case 2: System with Matched Disturbance**

Consider the system in the presence of matched disturbance \( f : \mathbb{R} \to \mathbb{R} \) given as:

\[ \dot{x}(t) = ax(t) + b(u(t) + f(x)), \quad (21) \]

where \( a, b \in \mathbb{R} \) are unknown. We make the same assumptions for the system and reference model as in Section III-A. Additionally, we make the following assumption for the disturbance \( f(x) \):

**Assumption 2:** The disturbance \( f(x) \) is of the form

\[ f(x) = \theta^T \psi(x), \quad (22) \]

where \( \theta \in \mathbb{R}^m \) is unknown constant and \( \psi : \mathbb{R} \to \mathbb{R}^m \) is a known, continuous function that satisfies the following property:

\[ |x| < \infty \implies ||\psi(x)|| < \infty, \quad (23) \]

that is, for bounded argument, the function \( \psi(\cdot) \) remains bounded.

In what follows, we simply use \( \psi \) in place of \( \psi(x) \). We define the control input as:

\[ u = k_x x + k_r r - k[\dot{x}]^\alpha - \hat{\theta}^T \psi, \quad (24) \]

where \( k > 0, 0 < \alpha < 1 \) and \( \hat{\theta} \) is the estimate of \( \theta \). The adaptation law for the parameters \( k_x, k_r, \hat{\theta} \) is chosen as:

\[ \dot{k}_x = -\gamma_x [\dot{x}]^{2\alpha-1} x, \quad \dot{k}_r = -\gamma_r [\dot{x}]^{2\alpha-1} r, \quad \dot{\hat{\theta}} = \gamma_\theta [\dot{x}]^{2\alpha-1} \psi, \quad (25) \]

where \( \gamma_x, \gamma_r, \gamma_\theta > 0 \) are constants. The error dynamics for the error vector \( \tilde{x} = [\dot{x} \dot{k}_x \dot{k}_r \theta]^T \) where \( \theta = \theta - \hat{\theta} \) is given as:

\[ \dot{\tilde{x}} = -bk[\ddot{x}]^{\alpha} + \ddot{k}_x x + b\dot{k}_r r + a_m \ddot{x} - b\tilde{\theta}^T \psi, \quad (26) \]

**Theorem 4:** If \( f(x) \) satisfies Assumption 2 and \( \alpha > \frac{2}{\gamma} \), the origin of the closed-loop system (26) is finite-time stable.

**Proof:** We follow the same logic as we used to prove that the origin of the error dynamics (17) is finite-time stable. First we prove that the nominal part of the error dynamics (26) is homogeneous and has the origin as an asymptotically stable equilibrium. Then, we show that the added disturbance is Lipschitz continuous, which renders the perturbed case finite-time stable. Denote the term \( \phi = a_m \ddot{x} \) as the disturbance in (26). Consider the nominal error dynamics in the absence of the disturbance term \( \phi \), given as:

\[ \dot{\tilde{x}} = -bk[\ddot{x}]^{\alpha} + \ddot{k}_x x + b\dot{k}_r r - b\tilde{\theta}^T \psi, \quad (27) \]

Similar to the analysis in Lemma 2 we can argue that the right-hand side of (27) is homogeneous with degree of homogeneity \( d = \alpha - 1 < 0 \) with respect to the dilation \( \Delta_\alpha = (\epsilon^{\alpha_1}, \epsilon^{\alpha_2}, \epsilon^{\alpha_3}) \) where \( r_1 = 1 \) and \( r_i = \alpha \) for \( i \in \{2, 3, 4\} \). Choose the candidate Lyapunov function as:

\[ V(z) = \frac{1}{2\alpha}[\ddot{x}]^{2\alpha} + \frac{b}{2\gamma_x} \ddot{k}_x^2 + \frac{b}{2\gamma_r} \ddot{k}_r^2 + \frac{b}{2\gamma_\theta} \tilde{\theta}^T \tilde{\theta}. \]

\(^1\)Note that with a slight abuse of notation, we use \( \epsilon^{\alpha_i} \) instead of \( \epsilon^{\alpha_{i1}}, \epsilon^{\alpha_{i2}}, \ldots, \epsilon^{\alpha_{im}} \), exploiting the symmetry of \( \theta \), which results into \( r_{i1} \equiv r_{i2} \equiv r_{i3} = \cdots = r_{i4m} \).
The time derivative of $V(z)$ along the trajectories of (27), after some calculations can be derived to be $\dot{V} = -bk|x|^{3\alpha-1}$. Hence, using the same arguments as in the proof of Theorem 2, we have that the origin of (27) is asymptotically stable for $\alpha > \frac{2}{3}$. Also, we know that there exists a positive constant $c$ and a positive-definite function $V(z)$ such that for $\beta \in (0, \frac{1}{2})$, its time derivative along the nominal dynamics (27) satisfies $\dot{V}(z) \leq -c\dot{V}(z)^{\beta}$. Now, the disturbance term $\phi$ can be bounded as $|\phi| = |a_{n}\bar{x}| \leq |a_n| \bar{x}$. This shows that the disturbance term $\phi$ is Lipschitz continuous. Hence, using [5, Theorem 5.3], we have that there exists $T < \infty$ such that for all $t \geq T$, $z(t) = 0$, i.e., the origin of (26) is FTS.

Remark 1: Unlike [16], our adaptation law and the result control input signals are continuously differentiable for all $t \geq 0$. We consider a general class of disturbance $f(x)$ in the system and show that finite-time stability can still be guaranteed.

IV. ADAPTIVE BACKSTEPPING WITH FINITE-TIME CONVERGENCE

In this section we consider the problem of trajectory tracking for a system with unknown parameters via the backstepping technique. We consider the system in the strict-feedback form (21, Chapter 2):

$$
\begin{align*}
\dot{x}_1 &= x_2 + \phi_1(x_1)^T\theta + \psi_1(x_1), \\
\dot{x}_2 &= x_3 + \phi_2(x_1, x_2)^T\theta + \psi_2(x_1, x_2), \\
\vdots &\quad\vdots \\
\dot{x}_{n-1} &= x_n + \phi_{n-1}(x_1, \ldots, x_{n-1})^T\theta + \psi_{n-1}(x_1, \ldots, x_{n-1}), \\
\dot{x}_n &= bu + \phi_n(x)^T\theta + \psi_n(x),
\end{align*}
$$

where $x = [x_1 \ x_2 \ \cdots \ x_n]^T \in \mathbb{R}^n$ is the state-vector, $\phi_i : \mathbb{R}^i \rightarrow \mathbb{R}$ and $\psi_i : \mathbb{R} \rightarrow \mathbb{R}$ are known functions for $i \in \{1, 3, \ldots, n\}$, $\theta \in \mathbb{R}^r$ is the constant vector of unknown parameters. The control gain $b \in \mathbb{R}$ is unknown, but its sign $\text{sign}(b)$ is assumed to be known. Without loss of generality, we assume that $b > 0$. The reference trajectory $x_r(t)$ is assumed to be bounded and has its $n$ derivatives continuous and bounded. The objective is to design the control input $u$ so that for any initial condition $x(0) \in D \subset \mathbb{R}^n$, there exists a finite time $T$ such that the closed-loop trajectories of (28) satisfy $x_1(t) - x_r(t) = 0$ for all $t \geq T$. Before proceeding with the control design, we make the following assumption on the functions $\phi_i$ and $\psi_i$:

Assumption 3: Each function $\phi_i$ and $\psi_i$ is at least $n - i$ times continuously differentiable, with all the $n-i$ derivatives as well as the functions $\phi_i(\cdot)$ and $\psi_i(\cdot)$ bounded for bounded input argument. Furthermore, the reference signal $x_r(t)$ is such that $\phi_1(x_r(t))$ is not constant for all times.

A. Backstepping Control Design

We adopt the technique of backstepping to achieve our objective. Consider the dynamics of state $x_1$ with $x_2$ as the control input. Define the error $e_1 = x_1 - x_r$. We seek the virtual controller $x_{2d}$ for the subsystem $x_1$ of the form:

$$
\dot{x}_{2d} = \dot{x}_r - \phi_1(x_1)^T\dot{\theta} - \psi_1(x_1) - k_1(x_1 - x_r),
$$

where $\dot{\theta} \in \mathbb{R}^r$ is the estimate of $\theta$. In what follows, we drop the arguments of the functions $\phi_1$ and $\psi_1$ for the sake of brevity. Define the error term $e_2 = x_2 - x_{2d}$ so that the dynamics of $e_1$ reads:

$$
\dot{e}_1 = \dot{x}_1 - \dot{x}_r = x_2 + \phi_1^T\dot{\theta} + \psi_1 - \dot{x}_r = e_2 + \phi_1^T\dot{\theta} - k_1 e_1,
$$

where $\tilde{\theta} = \theta - \hat{\theta}$ is the error in the estimate of $\theta$. Similarly, we can design the virtual controller, or the desired value of the $i$–th state for $i \in \{1, 2, \cdots, n-1\}$ as:

$$
x_{(i+1)d} = x_{id} - \phi_i^T\tilde{\theta} - \psi_1 - k_i e_i,
$$

with $x_{1d} = x_r$. Inspired from the control design in [6, Proposition 8.1], we design an estimator of $p = \frac{1}{b}$, denoted as $\tilde{p}$ and define the control input as:

$$
u = -\sum_{i=1}^{n} k_i [e_i]^{\alpha_i} + \tilde{p}(\dot{x}_{nd} - \phi_i^T\dot{\theta} - \psi_n),
$$

where $0 < \alpha_i < 1$ and $k_i > 0$ are such that the polynomial $p(s)$ given as:

$$p(s) = s^n + k_n s^{n-1} + \cdots + k_2 s + k_1,
$$

is Hurwitz. Define $p_c \triangleq (\dot{x}_{nd} - \phi_i^T\dot{\theta} - \psi_n)$, the error vector $e = [e_1 \ e_2 \ \cdots \ e_n]^T$ so that the closed-loop error dynamics reads:

$$
\begin{align*}
\dot{e}_1 &= -k_1 e_1 + e_2 + \phi_1^T\dot{\theta}, \\
\dot{e}_2 &= -k_2 e_2 + e_3 + \phi_2^T\dot{\theta}, \\
\vdots &\quad\vdots \\
\dot{e}_{n-1} &= -k_{n-1} e_{n-1} + e_n + \phi_{n-1}^T\dot{\theta}, \\
\dot{e}_n &= -b \sum_{i=1}^{n} k_i [e_i]^{\alpha_i} - \tilde{p}\dot{x}_{nd} - \psi_n
\end{align*}
$$

where $\tilde{p} = p - \hat{p}$. The difference $\tilde{\theta} = \theta - \hat{\theta}$ can be re-written as $\tilde{\theta} = \theta - \theta = \tilde{\theta} - \theta$. Using this and given that $pb = 1$, the last equation of (33) can be written as follows:

$$
\begin{align*}
\dot{e}_n &= -b \sum_{i=1}^{n} k_i [e_i]^{\alpha_i} - \tilde{p}\dot{x}_{nd} - \psi_n \\
&\quad\quad\quad + b\phi_n(x)^T(\dot{\theta}\tilde{p} - \theta p), \\
&= -b \sum_{i=1}^{n} k_i [e_i]^{\alpha_i} - \tilde{p}\dot{p} + \phi_n(x)^T\dot{\theta}
\end{align*}
$$

Consider the matrix $A = \begin{bmatrix} 0 & I_{n-1} \\ -k_1 & \cdots & -k_n \end{bmatrix}$ where $I_{n-1} \in \mathbb{R}^{(n-1) \times (n-1)}$ identity matrix and $0 \in \mathbb{R}^{n-1}$ is a vector with zero entries. Since $k_i$ are chosen as the coefficients of a Hurwitz polynomial, we have that $A$ is Hurwitz. Let $\bar{P}$ be the positive-definite solution of

$$PA + A^TP = -I_n,
$$

(34)
Let \( P_1, P_n \) denote the first and last column of the matrix \( P \). We define the adaptation law for \( \dot{p} \) and \( \dot{\theta} \) as:

\[
\dot{p} = -\gamma_p p_n \sum_{i=1}^{n} P_{ni}[e_i]^{\beta_i}, \quad (35a)
\]

\[
\dot{\theta} = \gamma_\theta \sum_{i=1}^{n} P_{1i}[e_i]^\gamma \phi_1, \quad (35b)
\]

where \( \gamma_p, \gamma_\theta > 0, P_{ni} \) and \( P_{1i} \) denote the \( i \)-th element of the vectors \( P_n \) and \( P_1 \), respectively. The exponents \( \beta_i, \gamma_i \) for \( i \in \{1, 2, \ldots, n\} \) are given as:

\[
\beta_i = \begin{cases} 2\alpha - 1/\alpha, & n \geq 2; \\ 2\alpha - 1, & n = 1 \end{cases}, \quad (36)
\]

\[
\gamma_i = \begin{cases} \alpha, & n \geq 2; \\ 2\alpha - 1, & n = 1 \end{cases}. \quad (37)
\]

Note that \( \dot{\theta} = -\dot{\theta} \) and \( \dot{p} = -\dot{p} \), using (35), we obtain:

\[
\dot{p} = \gamma_p p_n \sum_{i=1}^{n} P_{ni}[e_i]^{\beta_i}, \quad (38a)
\]

\[
\dot{\theta} = -\gamma_\theta \sum_{i=1}^{n} P_{1i}[e_i]^\gamma \phi_1, \quad (38b)
\]

### B. Convergence Analysis

**Theorem 5:** Let \( \alpha_i \) be chosen as:

\[
\alpha_{i-1} = \frac{\alpha_i \alpha_{i+1}}{2\alpha_{i+1} - \alpha_i}, \quad (39)
\]

for \( i \in \{2, 3, \ldots, n\} \) with \( \alpha_{n+1} = 1 \) and \( \alpha_n = \alpha \). Then, there exists \( \epsilon > 0 \) such that for each \( \alpha \in (1-\epsilon, 1) \), under the effect of the controller (31) with the adaptation law (35), holding follow:

i. The trajectories of \( (33), (38) \) satisfy \( [e(t)^T \quad \dot{\theta}(t)^T \quad \dot{p}(t)]^T = 0 \) for \( t \geq T \) where \( T < \infty \) for all \( [e(0)^T \dot{\theta}(0)^T \dot{p}(0)]^T \in \mathbb{V} \subset \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R} \), where \( \mathbb{V} \) is an open neighborhood of the origin.

ii. All the closed-loop signals, including the control input remain bounded at all times.

**Proof:** Theorem 5 equivalently states that the origin of the closed-loop error dynamics (33)-(38) is finite-time stable with input \( u(t) \) bounded for all \( t \geq 0 \). Note that for \( n = 1 \), we recover the error dynamics of the form (18), for which we have already shown that the origin is FTS. Hence, we continue the proof for \( n \geq 2 \). We follow the similar procedure of proving FTS of the error equations (33)-(38) as we followed in the proof of Theorem 4. We first show that the nominal error dynamics is homogeneous with negative degree of homogeneity. Define the disturbance vector \( \Phi : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}^n \) as:

\[
\Phi(e, \dot{\theta}, \dot{p}) = \begin{bmatrix} -k_1 e_1 + (1 - b)e_2 \\ -k_2 e_2 + \phi_2 \dot{\theta} + (1 - b)e_3 \\ \vdots \\ -k_{n-1} e_{n-1} + \phi_{n-1} \dot{\theta} + (1 - b)e_n \\ \phi_n \dot{\theta} \end{bmatrix} \quad (40)
\]

Consider the nominal error dynamics in the absence of \( \Phi \):

\[
\dot{e}_1 = be_2 + \phi_2 \dot{\theta}, \\
\dot{e}_2 = be_3, \\
\vdots \\
\dot{e}_{n-1} = be_{n}, \\
\dot{e}_n = -b \sum_{i=1}^{n} k_i [e_i]^{\alpha_i} - b p_p \quad (41)
\]

The nominal error dynamics (41) is homogeneous with degree of homogeneity \( d = \frac{\alpha}{\alpha_i} - 1 < 0 \). The claim can be verified using the definition of homogeneity for the dilation \( \Delta_i = ([e_1^\alpha, e_2^\alpha, \ldots, e_n^\alpha, e_p^\alpha, e_v^\alpha]) \), where \( r_i = \frac{1}{\alpha_i} \) for \( i \in \{1, 2, \ldots, n\} \), \( r_p = 1 \) and \( r_v = \frac{1}{\alpha} \).

Next, we show that the nominal error dynamics has the origin as an asymptotically stable equilibrium. Denote the right-hand side of (41) as \( f_h \), since the vector field depends upon the value of \( \alpha \). For \( \alpha = 1 \), we obtain that \( \alpha_i = 1 \) as well as \( \beta_i = \gamma_i = 1 \) for all \( i \in \{1, 2, \ldots, n\} \). Hence, the vector field \( f_1 \) is linear. Choose the candidate Lyapunov function as:

\[
V = e^T Pe + \frac{b}{\gamma_p} p^2 + \frac{1}{\gamma_\theta} \dot{\theta}^2 \dot{\theta}. \quad (42)
\]

The time derivative of (42) along the system trajectories of (41) for \( \alpha = 1 \), after some calculations, reads:

\[
\dot{V} = -be^T Qe + 2e^T P_{v_1} + 2e^T P_{v_2} \\
+ 2b p_p \sum_{i=1}^{n} P_{ni} e_i - 2 \dot{\theta} \sum_{i=1}^{n} P_{1i} e_i \phi_1, \\
\text{where} \quad v_1 = [\phi_1^T \dot{\theta} \quad 0 \quad \cdots \quad 0]^T \quad \text{and} \quad v_2 = [0 \quad 0 \quad \ldots \quad -b \phi_1]^T. \quad \text{Hence, we have} \quad e^T P v_1 = P_{1}^T e \phi_1^T \dot{\theta} \quad \text{and} \quad e^T P v_2 = P_{n}^T e (-b \phi_1). \quad \text{So,} \quad \dot{V} \text{ can be simplified as:}
\]

\[
\dot{V} = -be^T Qe + 2P_{1}^T e \phi_1^T \dot{\theta} - 2P_{p} e b p_p c, \\
+ 2b p_p P_{1}^T e - 2 \theta P_{1}^T e = -be^T Qe \leq 0.
\]

Hence, we have \( V \geq 0 \) and \( \dot{V} \leq 0 \), i.e., all the error signals are bounded. Taking the derivative of \( \dot{V} \), one can verify that \( \dot{V} \) is also bounded. Hence, using Barbalat’s Lemma, we obtain that \( \dot{V} \to 0 \) as \( t \to \infty \), i.e. the error vector \( e \) tends to zero. From (41) and Assumption 3 we obtain that for \( e_1(t) \) and \( e_n(t) \) to be identically zero, \( \dot{\theta} \) and \( \dot{p} \) also go to zero, respectively. Hence, we obtain that the origin of (41) is asymptotically stable for \( \alpha = 1 \). Using the same arguments as in [6, Proposition 8.1], we can argue that there exists \( \epsilon > 0 \) such that for all \( \alpha \in (1-\epsilon, 1) \), the origin of the error dynamics with right-hand side given by \( f_h \) is asymptotically stable. Since the (41) is also homogeneous with negative
degree of homogeneity, we have that the origin of (27) is FTS for \( \alpha \in (1 - \varepsilon, 1) \).

Now, consider the perturbed dynamics (33)-(38) in the presence of the disturbance \( \Phi \). From the above analysis, we know that the error terms \( \varepsilon_1 \) remain bounded at all times. From Assumption 3, we obtain that for bounded \( x \), the functions \( \phi_i \) and hence the vector \( \Phi \) remains bounded. Therefore, we have that the disturbance term linear in \( \theta, e \) with bounded coefficients \( \phi_i \) is Lipschitz continuous in \( z = [e^T \quad \tilde{p} \quad \tilde{\theta}^T]^T \). Using the same arguments as in the proof of Theorem 4, we can conclude that the origin is an FTS equilibrium for (33)-(38). Lastly, for all \( \alpha \in (1 - \varepsilon, 1) \), we have \( \alpha_i > 0 \) for all \( i \), which implies that the input \( u \) remains bounded at all times, which completes the proof.

**Remark 2:** Unlike previous work [10], [13], we guarantee that the parameter errors also converge to zero in finite time. Compared to [17] where a very specific class of systems is considered and very strong conditions on the reference signal are assumed, we consider a more general class of systems, and guarantee finite-time convergence with very mild conditions on the reference signal \( x_r(t) \).

**V. SIMULATIONS**

**A. Finite-time online estimation**

For the parameter estimation scheme presented in Section 4, we chose an arbitrary value of \( \theta = -1881 \) and simulate (9) for various initial conditions for the case when the input \( u(t) \) is given as per Figure 1. Figure 2 shows the trajectories of \( \tilde{\theta}(t) \) for various initial conditions \( \theta(0) \) and for \( \delta = 0.67, \Delta = 1, k_1 = k_2 = 10, \alpha_1 = 0.8, \alpha_2 = 1.2 \). It can be seen that error \( \tilde{\theta} \) converges to zero within \( T \leq 15 \).

**B. FTS MRAC**

We simulate the Case 2, i.e., the system with the disturbance \( f(x) \). We choose \( a_m = -1, b_m = 2, a = 100 \) and \( b = 50 \) so that \( k_2^* = -2.02 \) and \( k_2^* = 0.04 \). We choose the same reference signal that was used in [16], i.e. \( r(t) = 5 \cos(t) + 10 \cos(5t) \), while the disturbance term is chosen as \( f(x) = \theta^T \psi \), where \( \psi = [x \sin(5x) \quad x^2 \cos(x)]^T \) and \( \theta = [10 - 10]^T \). Figure 3 plots the system trajectory \( x(t) \) and the reference trajectory \( x_m(t) \). Figure 4 illustrates the evolution of error terms \( \tilde{x}, \tilde{k}_x, \tilde{k}_r, \tilde{\theta} \) for the choice of initial conditions given by \( x_m(0) = 20, x(0) = -100, k_x(0) = 200, k_r(0) = [-20 \ 20]^T \). The control parameters in (14) are chosen as \( k = 10 \) and \( \alpha = 0.9 \). It is evident from the figure that the error terms converge to zero as \( t \to 30 \) sec.

**C. FTS Adaptive backstepping**

We simulate the case of \( n = 2 \). The reference trajectory is chosen as \( x_r(t) = \sin(t) + 0.1t \) while the unknown parame-
ters are chosen as $\theta = \begin{bmatrix} -5 & 1 \end{bmatrix}^T$ and $b = 10$. The functions $\phi_1$ and $\psi_1$ are chosen as $\phi_1 = \begin{bmatrix} x_1 \sin(x_1) & \cos(x_1) \end{bmatrix}^T$, $\phi_2 = \begin{bmatrix} x_1 x_2 \sin(x_1 + x_2) & x_2 \cos(x_1 + x_2) \end{bmatrix}^T$, $\psi_1 = x_1 \cos(x_1)$ and $\psi_2 = x_2 \sin(x_1 x_2)$. The control gains are fixed as $k_1 = 10, k_2 = 20$ while the exponent $\alpha$ is chosen as $\alpha = 0.98$. The parameter estimation gains are chosen as $\gamma_p = 10, \gamma_0 = 10$. Figure 5 shows the system trajectory $x(t)$ and the reference trajectory $x_r(t)$ and it can be seen that the closed-loop trajectory tracks the unbounded reference in finite time. Figure 6 shows the error terms $e_1, \hat{\theta}_1, \hat{\theta}_2$ starting from initial condition $x_1(0) = 10, x_2(0) = 1$, $\hat{p}(0) = 1$, $\hat{\theta}_1(0) = 5, \hat{\theta}_2(0) = 10$. It is evident from the figure that all error terms converge to zero as $t \to 15$.

Fig. 5. FTS Adaptive Backstepping: The state $x_1(t)$ and reference trajectory $x_r(t)$ with time $t$.

Fig. 6. FTS Adaptive Backstepping: Error terms $e_1, \hat{\theta}_1, \hat{\theta}_2, \hat{p}$ with time $t$.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we presented a novel scheme of online parameter estimation under relaxed persistence of excitation condition with guarantees on convergence of the estimation error in finite time. We designed a novel MRAC with finite-time convergence guarantees for both state- and parameter-error in the presence of a class of matched parametric disturbance. We also considered a general class of strict-feedback systems with unknown parameters and designed an adaptive backstepping based finite-time stabilizing control which guarantees both tracking- and parameter-error convergence in finite time. In future, we would like to investigate methods of finite-time control design for adaptive systems with relaxed or no assumptions on the persistency of excitation. Future work also involves investigating the minimal set of system properties needed to be known in order to be able to design a finite-time stabilizing controller.

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