ON THE PLURICANONICAL MAPS OF VARIETIES OF INTERMEDIATE KODAIRA DIMENSION

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Abstract. In this paper we will prove a uniformity result for the Iitaka fibration $f : X \to Y$, provided that the generic fiber has a good minimal model and the variation of $f$ is zero or that $\kappa(X) = \dim X - 1$.

1. Introduction

One of the main problems in complex projective algebraic geometry is to understand the structure of pluricanonical maps. Recently, Hacon and McKernan [HM06], Takayama [Tak06] and Tsuji [Tsu06] have proved a beautiful result stating that there is a universal constant $r_n$ such that if $X$ is a smooth projective variety of general type and dimension $n$, then the pluricanonical map

$$\phi_{rK_X} : X \dasharrow \mathbb{P}^r \left( H^0(X, \mathcal{O}_X(rK_X)) \right)$$

is birational for all $r \geq r_n$. In [HM06], Hacon and McKernan also proposed a related conjecture for the Iitaka fibration in the case $\dim X > \kappa(X) \geq 0$.

Conjecture 1.1 ([HM06 Conjecture 1.7]). Fix $n \in \mathbb{Z}_{\geq 0}$. There is a positive integer $r_n$ with the following property: Let $X$ be a smooth $n$-dimensional projective variety of non-negative Kodaira dimension. Then the rational map $\phi_{rK_X}$ is birationally equivalent to the Iitaka fibration for all sufficiently divisible integers $r \geq r_n$.

The purpose of this paper is to prove Conjecture 1.1 under the hypotheses that the Iitaka fibration is isotrivial or that $\kappa(X) = \dim X - 1$.

Theorem 1.2. For any positive integers $n, b, k$, there exists an integer $m(n, b, k) > 0$ such that if $f : X \to Y$ is the Iitaka fibration with $X$ and $Y$ smooth projective varieties, $\dim X = n$, with generic fiber $F$ of $f$ of Kodaira dimension zero, such that

1. the variation of $f$ is zero;
2. $F$ has a good minimal model;
3. $b$ is the smallest integer such that $h^0(F, \mathcal{O}_F(bK_F)) \neq 0$, and $\text{Betti}_{\dim(E')}(E') \leq k$, where $E'$ is a smooth model of the cover $E \to F$ of the generic fiber $F$ associated to $b$-th root of the unique element of $|bK_F|$;

then the pluricanonical map

$$\phi_{mK_X} : X \dasharrow \mathbb{P}^r \left( H^0(X, \mathcal{O}_X(mK_X)) \right)$$

is birationally equivalent to $f$, for any $m \in \mathbb{Z}_{>0}$ such that $m$ is divisible by $m(n, b, k)$.

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Theorem 1.3. Let $X$ be an $n$-dimensional smooth projective variety of Kodaira dimension $n-1$ with Iitaka fibration $f : X \to Y$. Then there exists a positive integer $m_n$ depending only on $n$ such that $\phi_{m_nK_X}$ is birationally equivalent to $f$ for all positive integers $m$ divisible by $m_n$.

Conjecture 1.3 has been extensively studied. In [FM00], Fujino and Mori prove that if $\kappa(X) = 1$, then (1.1) holds under the hypothesis (3) of Theorem 1.2. Viehweg and Zhang [VZ09] also obtain this uniformity result for $\kappa(X) = 2$ under the same hypothesis. A related result of [VZ09] for 3-folds has been obtained independently by Ringler [Rin07]. For arbitrary Kodaira dimension, Pacienza [Pac09] recently has given an affirmative answer to (1.1) assuming that $Y$ is not uniruled, the Iitaka fibration $f$ has maximal variation and the hypotheses (2) and (3) of Theorem 1.2.

We now sketch the proof of Theorem 1.2. The main idea is to follow the approach of [HM06], [Tak06] and [Tsu06]. By the Canonical Bundle Formula (cf. Section 3), there are two $\mathbb{Q}$-divisors $M_Y$ (the moduli part) and $B_Y$ (the boundary part) on $Y$, such that for all $i > 0$, $H^0(X, \mathcal{O}_X(ibNK_X)) \cong H^0(Y, \mathcal{O}_Y(\lfloor ibN(K_Y + M_Y + B_Y) \rfloor))$, where $N$ is a positive integer depending on the hypothesis (3) of Theorem 1.2. and $M_Y$ is $\mathbb{Q}$-linearly trivial by the hypotheses (1) and (2) (Theorem 3.6). In order to prove Theorem 1.2, it remains to bound a multiple $m$ of $bN$ for which $\phi_{m(K_Y + M_Y + B_Y)}$ is birational. We first show that there exists such $m$ of the form $\alpha(\text{vol}(Y, K_Y + M_Y + B_Y))^{-1/n'} + \beta$ that $\phi_{m(K_Y + M_Y + B_Y)}$ is birational, where $n' = \dim Y$ and $\alpha, \beta$ are constants depending only on $n, b$ and $k$. Then using techniques developed in [HMX10], we show that if $M_Y$ is $\mathbb{Q}$-linearly trivial, $\text{vol}(Y, K_Y + M_Y + B_Y)$ can be bounded from below. Hence $m$ admits a uniform bound.

The main difficulty is that for a very general point $y \in Y$, we need to construct an effective $\mathbb{Q}$-divisor $D_y$ which is $\mathbb{Q}$-linearly equivalent to $\lambda(K_Y + M_Y + B_Y)$, where $\lambda$ depends on $\text{vol}(Y, K_Y + M_Y + B_Y)$, such that $y$ is an isolated non-klt center of $(Y, D_y)$. There is a well established way for producing divisors with non-klt centers at $y$. The problem is that the smallest non-klt center $V$ containing $y$ may be of positive dimension. In order to produce an isolated non-klt center, we have to cut down the dimension of the non-klt centers. By [BCHM10], we can assume $Y$ is the log canonical model, so $K_Y + M_Y + B_Y$ is ample. Then by Subadjunction (see Section 5) we prove that $\text{vol}(V, (K_Y + M_Y + B_Y)|_V)$ is bounded by a number related to $\text{vol}(Y, K_Y + M_Y + B_Y)$. Using techniques developed in [KMWK02] (see Section 4), we can produce a new divisor with a smaller dimensional non-klt center at $y$. Repeating this procedure at most $n' - 1$ times, we get the desired divisor $D_y$, see Section 6.

In Theorem 1.3 the generic fiber of the Iitaka fibration $f$ is an elliptic curve, so the hypotheses (2) and (3) of Theorem 1.2 automatically hold and $12M_Y$ is linearly equivalent to a base point free divisor on $Y$. Then using techniques in [HMX10] and the above argument, we prove that (1.1) holds true for any $n$-dimensional variety $X$ of Kodaira dimension $n - 1$.

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2. Preliminaries

2.1. Notation and conventions. We work over the complex number field $\mathbb{C}$. Let $X$ be a normal variety. We say that two $\mathbb{Q}$-divisors $D_1, D_2$ on $X$ are $\mathbb{Q}$-linearly equivalent ($D_1 \sim_\mathbb{Q} D_2$) if there exists an integer $m > 0$ such that $mD_1$ are linearly equivalent. If $D = \sum d_i D_i$ is a $\mathbb{Q}$-divisor, then the round down of $D$ is $\lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$, where $\lfloor D \rfloor$ denotes the largest integer which is at most $d$, and the round up of $D$ is $\lceil D \rceil = \lfloor D \rfloor - [-D]$. A log pair $(X, \Delta)$ is a normal variety $X$ and an effective $\mathbb{Q}$-Weil divisor $\Delta$ on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that $(X, \Delta)$ is log smooth if $X$ is smooth and $\Delta$ is a $\mathbb{Q}$-divisor with simple normal crossings support. A projective morphism $\mu : Y \to X$ is a log resolution of the pair $(X, \Delta)$ if $Y$ is smooth and $\mu^{-1}(\Delta) \cup \{\text{exceptional set}\}$ is a divisor with simple normal crossings support. We write $K_Y = \mu^*(K_X + \Delta) + \Gamma$ and $\Gamma = \sum a_i \Gamma_i$ where $\Gamma_i$ are distinct reduced irreducible divisors. We call $a_i$ the discrepancy of the pair $(X, \Delta)$ at $\Gamma_i$. The pair $(X, \Delta)$ is kawamata log terminal, klt for short (resp. log canonical, lc for short), if there is a log resolution $\mu : Y \to X$ as above such that the discrepancies of $\Gamma$ are strictly greater than $-1$, i.e. $a_i > -1$ for all $i$ (resp. $a_i \geq -1$). A subvariety $V$ of $X$ is called a non-klt center of $(X, \Delta)$ if it is the image of a divisor of discrepancy at most $-1$. The non-klt locus $\text{Non-klt}(X, \Delta)$ of the pair $(X, \Delta)$ is the union of the non-klt centers. A non-klt center $V$ is called a pure log canonical center if $(X, \Delta)$ is log canonical at the generic point of $V$.

If $D$ is a Weil divisor on a normal projective variety $X$, then $\phi_D$ denotes the rational map $X \dasharrow \mathbb{P}H^0(X, \mathcal{O}_X(D))$ induced by global sections of $\mathcal{O}_X(D)$.

2.2. Volumes and bounded pairs.

Definition 2.1. Lex $X$ be an irreducible projective variety of dimension $n$ and $D$ be a $\mathbb{Q}$-divisor. The volume of $D$ is 

$$\text{vol}(X, D) = \limsup_{m \to \infty} \frac{n!h^0(X, \mathcal{O}_X(mD))}{m^n}.$$ 

We say that $D$ is big if $\text{vol}(X, D) > 0$.

We refer the reader to [Laz1] for further details.

Lemma 2.2 ([HM06, Lemma 2.2]). Let $X$ be a projective variety, $D$ a divisor such that $\phi_D$ is birational with image $Z$. Then the volume of $D$ is at least the degree of $Z$ and hence at least 1.

Lemma 2.3 ([HMX10, Lemma 2.3.4]). Let $X$ be a normal projective variety of dimension $n$ and let $D$ be a big $\mathbb{Q}$-Cartier divisor on $X$. If $\phi_D$ is birational, then $\phi_{K_X+(2n+1)(D+M)}$ is birational for any numerically trivial Cartier divisor $M$.

Definition 2.4 ([HMX10, Definition 2.4.2]). A set $\mathcal{D}$ of log pairs is log birationally bounded if there is a log pair $(Z, B)$ and a projective morphism $Z \to T$, where $T$ is of finite type, such that for every $(X, \Delta) \in \mathcal{D}$, there is a closed point $t \in T$ and a birational map $f : Z_t \dasharrow X$ such that the support of $B_t$ contains the support of the strict transform of $\Delta$ and any $f$-exceptional divisor.

Theorem 2.5 ([HMX10, Theorem 3.1]). Fix a positive integer $n$ and two constants $A$ and $\delta > 0$. Then the set of log pairs $(X, \Delta)$ satisfying

1. $X$ is projective of dimension $n$,
(2) $(X, \Delta)$ is log canonical,
(3) the coefficients of $\Delta$ are at least $\delta$,
(4) there is a positive integer $m$ such that $\text{vol}(X, m(K_X + \Delta)) \leq A$,
(5) $\phi_{K_X + m(K_X + \Delta)}$ is birational,

is log birationally bounded.

**Theorem 2.6** ([HMX10, Theorem 1.7]). Fix a set $I \subset [0, 1]$ which satisfies the DCC. Let $\mathcal{D}$ be a set of log smooth pairs $(X, \Delta)$, which is log birationally bounded, such that if $(X, \Delta) \in \mathcal{D}$, then the coefficients of $\Delta$ belong to $I$. Then the set \[
\{ \text{vol}(X, K_X + \Delta) | (X, \Delta) \in \mathcal{D} \},
\] satisfies the DCC.

2.3. **Multiplier ideals and singularities of pairs.** Let $X$ be a smooth variety. If $D$ is an effective $\mathbb{Q}$-divisor on $X$, then the multiplier ideal sheaf associated to $D$ is defined to be
\[
\mathcal{J}(X, D) = \mu_* \mathcal{O}_{X'}(K_{X'}/X - \lfloor \mu^* D \rfloor)
\]
where $\mu : X' \to X$ is a log resolution of $(X, D)$. It is known that a pair $(X, D)$ is klt (resp. non-klt) at a point $x$, if and only if
\[
\mathcal{J}(X, D)_x = \mathcal{O}_{X,x} \quad \text{(resp. } \mathcal{J}(X, D)_x \neq \mathcal{O}_{X,x})
\]
and a pair is klt if it is klt at each point $x \in X$. A pair $(X, D)$ is lc at a point $x$, if and only if
\[
\mathcal{J}(X, (1 - \varepsilon)D)_x = \mathcal{O}_{X,x}
\]
for all rational numbers $0 < \varepsilon < 1$ and a pair is lc if it is lc at each point $x \in X$.

Note that we have the following relation for non-klt locus
\[
\text{Non-klt}(X, D) = \text{Supp}(\mathcal{O}_X/\mathcal{J}(X, D))_{\text{red}}.
\]

The following is a useful way to produce non-klt pairs.

**Lemma 2.7** ([Laz2, Proposition 9.3.2]). Assume that $X$ is smooth of dimension $n$, and let $D$ be an effective $\mathbb{Q}$-divisor on $X$. If $\text{mult}_x D \geq n$ at some point $x \in X$, then $\mathcal{J}(X, D)$ is non-trivial at $x$, i.e. $\mathcal{J}(X, D) \subseteq \mathfrak{m}_x$, where $\mathfrak{m}_x$ is the maximal ideal of $x$.

We now recall Nakai’s vanishing theorem.

**Theorem 2.8** ([Laz2, Theorem 9.4.8]). Let $X$ be a smooth projective variety. Let $D$ be an effective $\mathbb{Q}$-divisor on $X$, and $L$ a divisor on $X$ such that $L - D$ is nef and big. Then, for all $i > 0$, we have
\[
H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(X, D)) = 0.
\]

2.4. **Iitaka fibration.** Here we recall some results regarding Iitaka fibrations.

Let $L$ be a line bundle on an irreducible projective variety $X$. The semigroup $\mathcal{N}(L)$ of $L$ is
\[
\mathcal{N}(L) = \{ m \in \mathbb{Z}_{>0} | H^0(X, mL) \neq 0 \}.
\]
Assuming $\mathcal{N}(L) \neq (0)$, all sufficiently large elements of $\mathcal{N}(L)$ are multiples of a largest single natural number $e = e(L) \geq 1$, which we call the **exponent** of $L$. If $\kappa(X, L) = \kappa \geq 0$, then $\dim(\phi_{mL}(X)) = \kappa$ for all sufficiently large $m \in \mathcal{N}(L)$.
Theorem 2.9 (Iitaka fibrations, see [Laz1, Theorem 2.1.33]). Let $X$ be a normal projective variety, and $L$ a line bundle on $X$ such that $\kappa(X, L) > 0$. Then for all sufficiently large $k \in \mathbb{N}$, there exists a commutative diagram of rational maps and morphisms

$$
\begin{array}{ccc}
X & \xrightarrow{u_\infty} & X_\infty \\
\phi_k & \downarrow & \downarrow \phi_\infty \\
Y & \xleftarrow{\nu_k} & Y_\infty
\end{array}
$$

where the horizontal maps are birational and $u_\infty$ is a morphism. One has $\dim Y_\infty = \kappa(X, L)$. Moreover, if we set $L_\infty = u_\infty^* L$, and $F$ is a very general fiber of $\phi_\infty$, we have $\kappa(F, L_\infty|F) = 0$.

In this paper, we only deal with the case $L = O_X(K_X)$ and simply write $\kappa(X) = \kappa(X, O_X(K_X))$. The following results are important for our induction in the proof of the main theorem.

Lemma 2.10. Let $X$ and $Y$ be smooth projective varieties and $T$ an algebraic variety. Assume that $f : X \to Y$ is the Iitaka fibration of $(X, K_X)$ and $\varphi : Y \to T$ is a surjective morphism. For a very general closed point $t \in T$, let $V = \varphi^{-1}(t)$ and $W = f^{-1}(\varphi^{-1}(t))$, then the restriction morphism $f_W : W \to V$ is the Iitaka fibration of $(W, K_W)$.

Proof. By assumption, we have the following diagram

$$
\begin{array}{ccc}
W & \xrightarrow{f_W} & X \\
\downarrow & & \downarrow f \\
V & \xrightarrow{\varphi} & Y \\
\downarrow t & & \downarrow \varphi \\
& \in & T
\end{array}
$$

Since $t$ is very general, we may assume $V$ and $W$ are smooth and the very general fiber of $f_W$ is just the very general fiber of $f$. Hence, in order to prove that $f_W$ is the Iitaka fibration, we only need to show $\dim V \leq \kappa(W)$.

Fix an ample divisor $H$ on $Y$, then there exists a positive integer $m$ such that $mK_X \geq f^*(H)$. Since $V$ is a smooth fiber, we have $K_X|_W = K_W$. It follows that $mK_W \geq f_W^*(H|_V)$, which implies

$$
h^0(W, O_W(mK_W)) \geq h^0(V, O_V(iH|_V)) \quad \forall \ i \in \mathbb{Z}_{>0}.
$$

Since $H|_V$ is ample on $V$, then $\kappa(W) \geq \dim V$. Therefore, $f_W$ is the Iitaka fibration. 

Theorem 2.11 ([Lai09, Theorem 4.4]). Let $X$ be a $\mathbb{Q}$-factorial normal projective variety with non-negative Kodaira dimension and at most terminal singularities. Suppose that the general fiber $F$ of the Iitaka fibration has a good minimal model, then $X$ has a good minimal model.
3. Canonical bundle formula

In this section, we collect some of the results regarding the direct image of the relative dualizing sheaf.

Let $X$ and $Y$ be smooth projective varieties and $f: X \rightarrow Y$ an algebraic fiber space with generic fiber $F$ of Kodaira dimension zero. Let $b$ be the smallest integer such that the $b$-th plurigenus $h^0(F, bK_F)$ of $F$ is non-zero. Then there exists a $Q$-divisor $L_{X/Y}$ on $Y$ such that

$$O_Y([iL_{X/Y}]) \cong (f_*O_X(iK_X))^\vee$$

and

$$H^0(Y, O_Y([ibK_Y + iL_{X/Y}])) \cong H^0(X, O_X(iK_X))$$

for all $i > 0$. We may write the divisor $L_{X/Y}$ as

$$L_{X/Y} = L_{X/Y}^{ss} + \Delta,$$

where $L_{X/Y}^{ss}$ is a $Q$-Cartier divisor, called the semistable part or the moduli part, and $\Delta$ is an effective $Q$-divisor, called the boundary part. Moreover, if $f$ satisfies the conditions as in [FM00, 4.4], then $L_{X/Y}^{ss}$ is nef and $\Delta$ has simple normal crossings support. Therefore, replacing $Y$ by a smooth birational model, we may always assume that $L_{X/Y}^{ss}$ is nef and $\Delta$ is a simple normal crossings divisor.

In applications, it is important to bound the denominator of $L_{X/Y}^{ss}$.

**Theorem 3.1** ([FM00, Theorem 3.1]). Under the above notations and assumptions, let $E \rightarrow F$ be the cover associated to the $b$-th root of the unique element of $|bK_F|$. Let $\overline{E}$ be a nonsingular projective model of $E$ and let $B_m$ be its $m$-th Betti number. Then there is a natural number $N = N(B_m)$ depending only on $B_m$ such that $NL_{X/Y}^{ss}$ is a divisor.

Let $\Delta = \sum_P s_P P$. We have the following result about the coefficients $s_P$.

**Proposition 3.2** ([FM00, Proposition 2.8]). Under the notations and the assumptions as above, let $N \in \mathbb{Z}_{>0}$ be such that $NL_{X/Y}^{ss}$ is a Weil divisor. Then we have

$$L_{X/Y} = L_{X/Y}^{ss} + \sum_P s_P P,$$

where $s_P \in \mathbb{Q}$ for every codimension one point $P$ of $Y$ is such that

1. For each $P$, there exists $u_P, v_P \in \mathbb{Z}_{>0}$, such that $0 < v_P \leq bN$ and $s_P = (bNu_P - v_P)/(Nu_P)$.
2. $s_P = 0$ if $f^*(P)$ has only canonical singularities or if $X \rightarrow Y$ has a semistable resolution in a neighbourhood of $P$.

Moreover, $s_P$ depends only on $f|_{f^{-1}(U)}$ where $U$ is an open set of $Y$ containing $P$.

For convenience, we write $M_Y = L_{X/Y}^{ss}/b$ and $B_Y = \Delta/b$, then all non-zero coefficients of $B_Y$ are contained in

$$A(b, N) := \left\{ \frac{bNu - v}{bNu} \mid u,v \in \mathbb{Z}_{>0}; 0 < v \leq bN \right\} \backslash \{0\}.$$

**Lemma 3.3** ([VZ09, Lemma 1.2]). Under the notations as above, the following hold true.
(1) The set $A(b, N)$ is a DCC set, and one has

$$\frac{1}{bN} \leq \inf A(b, N).$$

(2) $(Y, B_Y)$ is log smooth and has klt singularities.

(3) The $\mathbb{Q}$-divisor $K_Y + M_Y + B_Y$ is big.

(4) For every $s \in \mathbb{Z}_{>0}$, we have

$$H^0(Y, \mathcal{O}_Y(\lfloor sb(K_Y + M_Y + B_Y) \rfloor)) \cong H^0(X, \mathcal{O}_X(sbK_X));$$

further the map $\phi_{sbK_X}$ is birational to the Iitaka fibration $f$ if and only if $|sb(K_Y + M_Y + B_Y)|$ gives rise to a birational map.

(5) $bNM_Y$ is an integral nef Cartier divisor.

(6) If $m \in \mathbb{Z}_{>0}$ is divisible by $bN$, then $(mB_Y) \geq (m-1)B_Y$.

**Lemma 3.4.** Under the same notations and assumptions as in Lemma 3.3, $(Y, M_Y + B_Y)$ has a log terminal model and a log canonical model.

**Proof.** Since $K_Y + M_Y + B_Y$ is big, we may write $K_Y + M_Y + B_Y \sim_{\mathbb{Q}} A + E$, where $A$ is an ample $\mathbb{Q}$-divisor and $E$ is an effective $\mathbb{Q}$-divisor. By (2) of Lemma 3.3 $(Y, B_Y)$ is klt, so $(Y, B_Y + \epsilon E)$ is also klt for $0 < \epsilon \ll 1$. By (5) of Lemma 3.3, $M_Y$ is nef, so $M_Y + \epsilon A$ is ample. Thus there exist a sufficiently ample divisor $A'$ and a rational number $0 < \epsilon' \ll 1$ such that $M_Y + \epsilon A \sim_{\mathbb{Q}} \epsilon' A'$ and $(Y, B_Y + \epsilon E + \epsilon' A')$ is also klt. It follows that

$$(1 + \epsilon)(K_Y + M_Y + B_Y) \sim_{\mathbb{Q}} K_Y + M_Y + B_Y + \epsilon A + \epsilon E \sim_{\mathbb{Q}} K_Y + B_Y + \epsilon E + \epsilon' A'.$$

By [BCHM04], $(Y, B_Y + \epsilon E + \epsilon' A')$ has a log terminal model $Y^m$ and a log canonical model $Y^c$. It is easy to see that $Y^m$ (resp. $Y^c$) is also a log terminal model (resp. log canonical model) of $(Y, M_Y + B_Y)$. \hfill \Box

**Lemma 3.5.** Under the notations and assumptions as in Lemma 2.11, the boundary part $B_V$ of $f_W$ is the restriction of $B_Y$ to $V$ and the moduli part $M_V$ of $f_W$ is $\mathbb{Q}$-linearly equivalent to the restriction of $M_Y$.

**Proof.** Since $(Y, B_Y)$ is log smooth and $V$ is a very general fiber of $\varphi : Y \to T$, we may assume that $B_Y|_V$ has simple normal crossings support. Let $B_Y = \sum_P r_P P$ and $B_V = \sum_Q r'_Q Q$. Recall that $1 - r_P$ is the log canonical threshold of $f^* P$ with respect to $(X, -D_X/b)$ over the generic point of $P$ and $1 - r'_Q$ is the log canonical threshold of $f_W Q$ with respect to $(W, -D_W/b)$ over the generic point of $Q$, where $D_X = bK_X - f^* (bK_Y + L_X/Y)$ and $D_W = bK_W - f_W^* (bK_Y + L_W/V)$ (see [Fuj03 Definition 3.4]). Since $W$ is a very general fiber, we have $D_X|_W = D_W$. Hence $r'_Q = 0$ when $Q$ is not contained in the support of $B_Y|_V$ and $r'_Q = r_P$ when $Q$ is the restriction of some component $P$ of $B_Y$. Therefore $B_V = B_Y|_V$. On the other hand, we have $K_V + M_V + B_V \sim_{\mathbb{Q}} (K_Y + M_Y + B_Y)|_V$. Hence $M_Y \sim_{\mathbb{Q}} M_Y|_V$. \hfill \Box

**Variation.** Let $f : X \to Y$ be an algebraic fiber space. Let $K \supset \mathbb{C}$ be an algebraically closed field contained in $\mathbb{C}(Y)$ such that there is a finitely generated extension $L$ of $K$ such that $Q(L \otimes_K \mathbb{C}(Y)) \cong Q(\mathbb{C}(X) \otimes_{\mathbb{C}(Y)} \mathbb{C}(Y))$ over $\mathbb{C}(Y)$, where $Q$ denotes the fraction field. The minimum of $\text{tr.deg}_K K$ for all such $K$ is called the variation of $f$ and denoted by $\text{Var}(f)$. 
Theorem 3.6. Let \( f : X \to Y \) be the Iitaka fibration as in [FM00, 4.4]. If the generic fiber \( F \) of \( f \) has a good minimal model, then the following are equivalent:

1. \( M_Y \) is numerically trivial.
2. \( M_Y \sim_{\mathbb{Q}} 0 \).
3. \( \kappa(Y, M_Y) = 0 \).
4. \( \text{Var}(f) = 0 \).

Proof. (1) \( \iff \) (2) is followed by [Amb05, Theorem 3.5]. The implication (2) \( \implies \) (3) is trivial. Since \( F \) has a good minimal model, following [Kaw85, Theorem 1.1], we have (3) \( \iff \) (4) (cf. [Fuj03, Remark 3.9]). Finally, Fujino [Fuj03, Theorem 3.11] proves the implication (4) \( \implies \) (2).

\[ \square \]

4. Birational covering families of pure log canonical centers

In this section, we construct a birational covering family of pure log canonical centers.

Recall that a subset \( P \) of a variety \( Y \) is called countably dense if it is not contained in the union of countably many closed subsets of \( Y \).

Lemma 4.1. Let \((Y, \Delta)\) be a log pair, where \( Y \) is projective and let \( D \) be a big \( \mathbb{Q} \)-Cartier divisor on \( Y \). Suppose that for every point \( y \in P \), where \( P \) is a countably dense subset of \( Y \), we can find a pair \((\Delta_y, W_y)\) such that \( W_y \) is a pure log canonical center for \( K_Y + \Delta + \Delta_y \) at \( y \) and \( \Delta_y \sim_{\mathbb{Q}} D/w_y \) for some positive rational number \( w_y \). Then there exists a diagram

\[
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\downarrow \varphi & & \downarrow \\
T & & \\
\end{array}
\]

such that \( \varphi \) is a dominant morphism of normal projective varieties with connected fibers and for a general fiber \( V_t \) of \( \varphi \) there exists \( y \in \varphi(V_t) \) so that \( \varphi(V_t) \) is a pure log canonical center for \( K_Y + \Delta + \Delta_y \) at \( y \) and \( \Delta_y \sim_{\mathbb{Q}} D/w \) for some positive rational number \( w \). Also \( \pi \) is a generically finite and dominant morphism of normal varieties.

Proof. See [McK02, Lemma 3.2] or [Tod07, Lemma 3.2]. \( \square \)

Lemma 4.2 (McKernan). Let \((Y, \Delta)\) be a log pair, where \( Y \) is a normal projective variety of dimension \( n' \). Let \( D \) be a nef and big \( \mathbb{Q} \)-Cartier divisor. Let \((\Delta_t, V_t)\) be a covering family of weight less than \( w \) and dimension \( k \).

If \((\Delta_t, V_t)\) is not birational then we may find a covering family of \((\Gamma_s, W_s)\) of weight \( w/(n' - k) \) and dimension \( l \), where either

1. \( l > k \), or
2. \( l < k \) and \((\Gamma_s, W_s)\) is a birational family.

Remark 4.3. Lemma 4.2 still holds if we only assume that \( D \) is big instead of nef and big.

Proof. See [McK02, Lemma 4.2]. \( \square \)

Corollary 4.4. Let \((Y, \Delta)\) be a log pair, where \( Y \) is a normal projective variety of dimension \( n' \). Let \( D \) be a big \( \mathbb{Q} \)-Cartier divisor. Let \((\Delta_t, V_t)\) be a covering family of weight \( w \) and dimension \( k \). Then there exists a birational covering family of \((\Gamma_s, W_s)\) of weight \( w' \geq w/(n' - 1)! \).
Proof. This is immediate from Lemma 4.2.

By Lemma 3.3, $K_Y + M_Y + B_Y$ is a big $\mathbb{Q}$-divisor on $Y$, where $Y$ is a smooth projective variety of dimension $n'$, so for each point $y \in Y$, we can find a pair $(D_y, V_y)$ such that

1. $D_y \sim_{\mathbb{Q}} \lambda(K_Y + M_Y + B_Y)$, for some rational number $\lambda > 0$,
2. $V_y$ is a pure log canonical center of $(Y, D_y)$ at $y$.

Note that we can take the same $\lambda$ for every point in a countably dense subset of $Y$ with $\dim(V_y) = k$. Then by the previous corollary we obtain a diagram

\[
\begin{array}{c}
Y' \xrightarrow{\pi} Y \\
\downarrow \varphi \\
T
\end{array}
\]

such that

1. $\pi$ is birational and $\varphi$ is dominant.
2. Let $V_t = \pi(V'_t)$, where $V'_t$ is a general fiber of $\varphi$. Then there exists a $\mathbb{Q}$-divisor $D_t \sim_{\mathbb{Q}} \lambda'(K_Y + M_Y + B_Y)$ on $Y'$ such that $V_t$ is a pure log canonical center of $(Y, D_t)$ and $\lambda' \leq \lambda(n' - 1)$.

Proposition 4.5. Let $f : X \to Y$ be the Iitaka fibration satisfying the hypotheses of Theorem 1.2. Suppose that for any $y$ in a countably dense subset of $Y$, there is an effective $\mathbb{Q}$-divisor $D_y \sim_{\mathbb{Q}} \lambda(K_Y + M_Y + B_Y)$ such that $y \in \text{Non-klt}(Y, D_y)$. Then there exists a diagram

\[
\begin{array}{c}
X' \xrightarrow{f'} X \\
\downarrow f \\
Y' \xrightarrow{\pi} Y \\
\downarrow \varphi \\
T
\end{array}
\]

such that

1. $X'$ and $Y'$ are smooth projective varieties.
2. $\pi$ is birational, $\varphi$ is dominant with $\dim T \geq 0$ and $f'$ satisfies the hypotheses of Theorem 1.2.
3. For any very general fiber $V'_t$ of $\varphi$, there exists an effective $\mathbb{Q}$-divisor $D'_t \sim_{\mathbb{Q}} \lambda'(K_{Y'} + M_{Y'} + B_{Y'})$ on $Y'$ such that $V'_t$ is a pure log canonical center of $(Y', D'_t)$ and $\lambda' \leq \lambda(n' - 1)$, where $n' = \dim Y$.

Proof. By our discussions above, there exists a covering family $Y' \xrightarrow{\varphi} T$ such that $Y' \xrightarrow{\pi} Y$ is birational. Now replace $Y'$ by a smooth model and let $X'$ be the resolution of the main component of $X \times_Y Y'$. It is easy to see that $f'$ and $f$ have the same generic fiber. Hence, (1) and (2) are satisfied. We only need to show (3).

Let $V_t = \pi(V'_t)$. By our assumptions and previous discussions, there is an effective $\mathbb{Q}$-divisor $D_t \sim_{\mathbb{Q}} \lambda'(K_Y + M_Y + B_Y)$ on $Y$ such that $V_t$ is a pure log canonical
center of \((Y, D_t)\) and \(λ' ≤ λ(n' - 1)!\). Since \(π\) is birational, for all \(m \in \mathbb{Z}_{>0}\) sufficiently divisible, we have

\[
H^0(Y', \mathcal{O}_{Y'}(m(K_{Y'} + M_{Y'} + B_{Y'}))) \cong H^0(X', \mathcal{O}_{X'}(mK_X))
\]

\[
\cong H^0(X, \mathcal{O}_X(mK_X))
\]

\[
\cong H^0(Y, \mathcal{O}_Y(m(K_Y + M_Y + B_Y))).
\]

So there is an effective \(\mathbb{Q}\)-divisor \(D'_t \sim_\mathbb{Q} λ'(K_{Y'} + M_{Y'} + B_{Y'})\) on \(Y'\) such that \(π(D'_t) = D_t\). Since \(V'_t\) is a very general fiber of \(φ\), \((Y', D'_t, V'_t)\) and \((Y, D_t, V_t)\) are isomorphic at the generic point of \(V'_t\). Therefore, \(V'_t\) is a pure log canonical center of \((Y', D'_t)\).

\[\square\]

**Lemma 4.6 (M-K02 Lemma 5.3).** Let \((Y, Δ)\) be a log pair and let \(D\) be a \(\mathbb{Q}\)-divisor of the form \(A + E\) where \(A\) is ample and \(E\) is effective. Let \((Δ_t, V_t)\) be a covering family of weight greater than \(w\) and dimension \(k\). Let \(A_t\) be the restriction of \(A\) to \(V_t\). Suppose that for all very general points \(t ∈ U\) we may find a covering family \((Γ_{t,s}, W_{t,s})\) on \(V_t\) of weight, with respect to \(A_t\), greater than \(w'\).

Then we may find a covering family \((Γ_s, W_s)\) of \((Γ_{t,s}, W_{t,s})\) on \(V_t\) of weight less than \(k\) and weight

\[w'' \sim \frac{ww'}{w + w'}\]

Further if both \((Δ_t, V_t)\) and \((Γ_{t,s}, W_{t,s})\) are birational families then so is \((Γ_s, W_s)\).

5. **Subadjunction**

In his fundamental paper [Kaw98], Kawamata proves a remarkable subadjunction theorem. An immediate consequence of this theorem is that if \((X, D)\) is a log canonical pair, \(V\) is a non-klt center of \((X, D)\), then we have \((K_X + D)|_V \sim_\mathbb{Q} K_V + Δ_V\), where \(Δ_V\) is a pseudoeffective divisor on \(V\). Actually, one can prove a more precise result.

**Proposition 5.1** (Subadjunction). Let \(X\) be a normal variety and \(D\) an effective \(\mathbb{Q}\)-divisor on \(X\) such that \((X, D)\) is a log pair. If \(V\) is a pure log canonical center of \((X, D)\) and \(ν : V' → V\) is the normalization, then we have

\[(K_X + D)|_{V'} \sim_\mathbb{Q} K_{V'} + Δ_{V'},\]

where \(Δ_{V'}\) is an effective \(\mathbb{Q}\)-divisor.

**Remark 5.2.** Recently, Fujino and Gongyo [FG10] prove the much stronger result that if \((X, D)\) is an lc pair and \(V\) is a minimal non-klt center of \((X, D)\), then there exists an effective \(\mathbb{Q}\)-divisor \(Δ_V\) on \(V\) such that \((K_X + D)|_V \sim_\mathbb{Q} K_V + Δ_V\) and \((V, Δ_V)\) is klt.

This result depends on Ambro’s results on the moduli \((b-)\)divisor associated to an lc-trivial fibration.

**Theorem 5.3** (Ambro). Let \(f : (X, B) → Y\) be an lc-trivial fibration such that the generic geometric fiber \(X_{t_0} = X ×_Y \text{Spec}(k(Y))\) is a projective variety and \(B_{t_0}\) is effective. Then there exists a diagram
satisfying the following properties:

- $f' : (X', B') \to Y'$ is an lc-trivial fibration.
- $\tau$ is generically finite and surjective and $\varrho$ is surjective.
- There exists a nonempty open subset $U \subset \bar{Y}$ and an isomorphism

\[
(X, B) \times_Y \bar{Y}|_U \cong (X', B') \times_{Y'} \bar{Y}|_U
\]

- Let $M$ and $M'$ be the corresponding moduli $\mathbb{Q}$-b-divisors. Then $M'$ is $b$-nef and big and $\tau^* M = \varrho^*(M')$, which implies $M$ is $b$-nef and good. In particular, $M$ is $\mathbb{Q}$-linearly equivalent to an effective divisor.

Proof. See [Amb05, Theorem 3.3].

Before giving the proof of 5.1 we need the following useful lemmas.

**Lemma 5.4** (Hacon). Let $X$ be a normal quasi-projective variety and $B$ a boundary $\mathbb{R}$-divisor on $X$ such that $K_X + B$ is $\mathbb{R}$-Cartier. Then, there exists a projective birational morphism $f : Y \to X$ from a normal quasi-projective variety $Y$ with the following properties.

1. $Y$ is $\mathbb{Q}$-factorial.
2. $a(E, X, B) \leq -1$ for every $f$-exceptional divisor $E$ on $Y$.
3. We put

$$B_Y = f_*^{-1} B + \sum_{E \in \text{Ex}(f)} E.$$

Then $(Y, B_Y)$ is dlt and

$$K_Y + B_Y = f^*(K_X + B) + \sum_{a(E, X, B) < -1} (a(E, X, B) + 1)E.$$

In particular, if $(X, B)$ is lc, then $K_Y + B_Y = f^*(K_X + B)$. Moreover, if $(X, B)$ is dlt, then we can assume that $f$ is small, that is, $f$ is an isomorphism in codimension one.

Proof. See e.g. [Fuj09, Theorem 10.4].

**Remark 5.5.** Lemma [5.4] still holds if the coefficients of some components of $B$ are greater than 1. But we need to replace (3) by

3' Let

$$B_Y = f_*^{-1} B_{\leq 1} + \text{Supp} f_*^{-1} B_{> 1} + \sum_{E \in \text{Ex}(f)} E.$$
Then \((Y, B_Y)\) is dlt and
\[ K_Y + B_Y = f^*(K_X + B) + \sum_{a(F, X, B) < -1} (a(F, X, B) + 1)F. \]

**Lemma 5.6** (Adjunction for dlt pairs). Let \((X, D)\) be a dlt pair. We put \(S = |D|\) and let \(S = \sum_{i \in I} S_i\) be the irreducible decomposition of \(S\). Then, \(W\) is a non-klt center for the pair \((X, D)\) with \(\text{codim}_W W = k\) if and only if \(W\) is an irreducible component of \(S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}\) for some \(\{i_1, i_2, \ldots, i_k\} \subset I\). By adjunction, we obtain

\[ K_{S_{i_1}} + \text{Diff}(D - S_{i_1}) = (K_X + D)|_{S_{i_1}}, \]

and \((S_{i_1}, \text{Diff}(D - S_{i_1}))\) is dlt. Note that \(S_{i_1}\) is normal, \(W\) is a non-klt center for the pair \((S_{i_1}, \text{Diff}(D - S_{i_1}))\), \(S_{i_1}|_{S_{i_1}}\) is a reduced component of \(\text{Diff}(D - S_{i_1})\) for \(2 \leq j \leq k\), and \(W\) is an irreducible component of \(S_{i_2}|_{S_{i_1}} \cap (S_{i_3}|_{S_{i_1}}) \cap \cdots \cap (S_{i_k}|_{S_{i_1}})\).

By applying adjunction \(k\) times, we obtain a \(Q\)-divisor \(\Delta \geq 0\) on \(W\) such that
\[ (K_X + D)|_W = K_W + \Delta \]
and \((W, \Delta)\) is dlt.

**Proof.** See [Cor07, Proposition 3.9.2].

**Proof of Proposition 5.4.** Applying Lemma 5.4 and Remark 5.6 we may get a morphism \(f : Y \to X\) satisfying the properties of Lemma 5.4. Let \(D_Y = D|_{X_U} = f^*D + \sum_{E \in E(f)} \sum_{j} E_j\). Then we have
\[ f^*(K_X + D) = K_Y + D_Y - \sum_{a(F, X, D) < -1} (a(F, X, D) + 1)F, \]
and the pair \((Y, D_Y)\) is dlt. Since \(V\) is a pure log canonical center of \((X, D)\), \(F\) is vertical over \(V\) if \(a(F, X, D) < -1\).

Let \(W\) be a minimal non-klt center of \((Y, D_Y)\) over the generic point of \((X, D)\), \(\nu : V^\nu \to V\) the normalization of \(V\). We obtain the following diagram

\[
\begin{array}{ccc}
W^C & \xrightarrow{s} & Y \\
\downarrow g & & \downarrow f \\
U & \xrightarrow{t} & V^\nu \\
\downarrow s & & \downarrow \nu \\
U & \xrightarrow{t} & V^\nu \\
\end{array}
\]

where \(g : W \to V^\nu\) is the induced morphism and \(W \to U \to V^\nu\) is the Stein factorization of \(g\).

By Lemma 5.6 there exists a log pair \((W, \Delta_W)\), where \(\Delta_W \geq 0\), such that
\[ K_W + \Delta_W \sim_Q (K_Y + D_Y) - \sum_{a(F, X, D) < -1} (a(F, X, D) + 1)F|_W \sim_Q f^*(K_X + D)|_W, \]
and the non-klt centers of \((W, \Delta_W)\) are vertical over \(V^\nu\), so \((W, \Delta_W)\) has klt singularities over the generic point of \(V^\nu\). It follows that \((W, \Delta_W)\) is klt over the generic point of \(U\). Moreover,
\[ K_W + \Delta_W \sim_Q g^*((K_X + D)|_{V^\nu}) \sim_Q g^*s^*((K_X + D)|_U). \]
Therefore, \(s : (W, \Delta_W) \to U\) is an lc-trivial fibration as defined in [Amb04, Definition 2.1].
We may write \((K_X + D)|_U \sim_q K_U + M + B\), where \(M\) is the moduli part and \(B\) is the boundary part of this lc-trivial fibration. Since \(\Delta_W \geq 0\), \(B \geq 0\). By Theorem 5.3, we may assume that \(M\) is effective. Let \(\Delta_U = M + B\), then,

\[(K_X + D)|_U \sim_q K_U + \Delta_U\]

and \(\Delta_U \geq 0\). Since \(t : U \to V^\nu\) is finite and \(K_U + \Delta_U \sim_q t^*((K_X + D)|_{V^\nu})\), it is easy to see that there exists an effective \(\mathbb{Q}\)-divisor \(\Delta_U\) on \(V^\nu\) such that

\[(K_X + D)|_{V^\nu} \sim_q K_{V^\nu} + \Delta_{V^\nu}\].

\[\square\]

6. Creating isolated non-klt centers

**Proposition 6.1.** Assume that Theorem 1.2 holds for varieties of dimensions \(\leq n\). Let \(f : X \to Y\) be the Iitaka fibration satisfying the hypotheses of Theorem 1.2 with \(\dim X = n\) and \(\dim Y = n'\). Then there exist positive constants \(\alpha\) and \(\beta\) depending on \(n, b\) and \(k\), such that for any very general point \(y \in Y\) there is an effective \(\mathbb{Q}\)-divisor \(D_y\) such that

1. \(D_y \sim_q \lambda(K_Y + M_Y + B_Y)\), where \(\lambda < \frac{\alpha}{\text{vol}(Y, K_Y + M_Y + B_Y)^{1/n^\nu}} + \beta;\)

2. \(y\) is an isolated point of \(\text{Non-klt}(Y, D_y)\).

**Proof.** Take a very general point \(y \in Y\). Since \(K_Y + M_Y + B_Y\) is big, by the argument in the proof of [Pac09, Theorem 6.2], we can pick an effective \(\mathbb{Q}\)-divisor \(D_0 \sim_q \lambda_0(K_Y + M_Y + B_Y)\) which has multiplicity \(> n_0\) at \(y\), where \(n_0 = n'\) and \(\lambda_0 < n_0(\text{vol}(Y, K_Y + M_Y + B_Y))^{-1/n_0} + \varepsilon_0\) with \(1 \gg \varepsilon_0 > 0\). Hence there is a component \(V_0\) of \(\text{Non-klt}(Y, D_0)\) passing through \(y\). Multiplying \(D_0\) by a positive rational number \(\leq 1\), we can assume that \(V_0\) is a pure log canonical center of \((Y, D_0)\).

By Proposition 4.5, we may replace \(Y\) with a higher smooth birational model such that there exists a morphism \(\varphi : Y \to T\) satisfying the properties of 4.5. Therefore, the point \(y\) is contained in a very general fiber \(V_1\) of \(\varphi\) and there is an effective \(\mathbb{Q}\)-divisor \(D_1 \sim_q \lambda_1(K_Y + M_Y + B_Y)\) on \(Y\) with \(\lambda_1 \leq \lambda_0(n_0 - 1)! < n_0!(\text{vol}(Y, K_Y + M_Y + B_Y))^{-1/n_0} + \varepsilon_0(n_0 - 1)!\), such that \(V_1\) is a pure log canonical center of \((Y, D_1)\).

By Lemma 3.1, there is a log canonical model \(Y'\) of \((Y, M_Y + B_Y)\). Replacing \(Y\) with a higher smooth birational model, we may assume that there is a morphism \(\phi : Y \to Y'\). Let \(M_{Y'} = \phi_*M_Y\) and \(B_{Y'} = \phi_*B_Y\). Then \(K_{Y'} + M_{Y'} + B_{Y'}\) is \(\mathbb{Q}\)-Cartier and ample on \(Y'\).

By our assumption, the generic fiber of \(f\) has a good minimal model. Applying Theorem 2.11, there exists a good minimal model \(X'\) of \(X\). Replacing \(X\) with a higher smooth birational model, we may assume that there is a morphism \(\psi : X \to
X'. Hence, we obtain a diagram

\[
\begin{array}{c}
X \\
\downarrow f \\
Y \\
\downarrow \phi \\
T
\end{array} \quad \begin{array}{c}
\xrightarrow{\psi} \\
\xrightarrow{f'} \\
\xrightarrow{\phi'} \\
\xrightarrow{\phi'} \\
\xrightarrow{\phi'}
\end{array} \quad \begin{array}{c}
X' \\
\downarrow f' \\
Y' \\
\downarrow \phi' \\
\phi'
\end{array}
\]

where \(f'\) is the induced rational map.

Remark 6.2. The generic fiber of \(f\) may have changed after running the Minimal Model Program, so \(f\) may not satisfy the hypotheses of Theorem 1.2. But since our new \(X\) is a higher birational model of the original one, we do not change either \(M_Y\) or \(B_Y\) by the Canonical Bundle Formula.

Lemma 6.3. We have the following:

1. \(Y'\) is isomorphic to the weak canonical model \((X')^w\) of \(X'\) in the sense that

\[
(X')^w = \text{Proj} \bigoplus_{m \geq 0} H^0(X', \mathcal{O}_{X'}(mK_{X'})).
\]

2. \(f'\) is a morphism and \(K_{X'} \sim_\mathbb{Q} f'^* (K_{Y'} + M_{Y'} + B_{Y'})\).

Proof. \(X'\) is a good minimal model, so \(X'\) admits a morphism to its weak canonical model \((X')^w\). On the other hand, \(K_{Y'} + M_{Y'} + B_{Y'}\) is ample on \(Y'\), so

\[
Y' = \text{Proj} \bigoplus_{m \geq 0} H^0(Y', \mathcal{O}_{Y'}(\lfloor m(K_{Y'} + M_{Y'} + B_{Y'}) \rfloor)).
\]

If \(m \in \mathbb{Z}_{>0}\) is sufficiently divisible, by the Canonical Bundle Formula we have

\[
H^0(X', \mathcal{O}_{X'}(mK_{X'})) \cong H^0(X, \mathcal{O}_X(mK_X))
\cong H^0(Y, \mathcal{O}_Y(\lfloor m(K_Y + M_Y + B_Y) \rfloor))
\cong H^0(Y', \mathcal{O}_{Y'}(\lfloor m(K_{Y'} + M_{Y'} + B_{Y'}) \rfloor)).
\]

Hence \(Y'\) is the weak canonical model of \(X'\) and (2) follows from (1). \(\square\)

Now let \(y' = \phi(y)\), \(V'_i = \phi(V_i)\), and \(D'_i = \phi_*(D_i)\) and let \(n_1 = \dim V_1 = \dim V'_1\). Since \(V_1\) is a pure log canonical center of \((Y, D_1)\) and \(y'\) is very general, it follows that \(V'_1\) is a pure log canonical center of \((Y', M_{Y'} + B_{Y'} + D'_1)\) at \(y'\). Let \(W_1 = f^{-1}(V_1)\), \(W'_1 = f'^{-1}(V'_1)\), \(V'_1\) the normalization of \(V_1\), \(W'_1\) the normalization of \(W_1\) and
\[ \gamma : W_1 \to V_1 \] the induced morphism. We have the following diagram

\[
\begin{array}{c}
W_1 \xleftarrow{f_{W_1}} W_1'' \xrightarrow{\gamma} W_1' \\
X \xleftarrow{t_1} V_1 \xrightarrow{\nu} V_1'' \xrightarrow{\nu'} Y' \\
Y \xrightarrow{\phi} T
\end{array}
\]

By Lemma 6.4 and Lemma 6.5, the morphism \( f_{W_1} : W_1 \to V_1 \) is the Iitaka fibration of \( (W_1, K_{W_1}) \) and the moduli part \( M_{V_1} \) of \( f_{W_1} \) is \( \mathbb{Q} \)-linearly equivalent to the restriction of \( M_{V'} \) to \( V_1 \). Thus we can assume that \( f_{W_1} \) satisfies the hypotheses of Theorem 17.2.

**Remark 6.4.** As in Remark 6.2, the generic fiber of \( f_{W_1} \) may be different from the original one. However this does not affect the computation of \( M_{V_1} \) and \( B_{V_1} \).

**Lemma 6.5.** There exists a constant \( \delta > 0 \) depending on \( n-1, b \) and \( k \), such that \( \text{vol}(V_1, K_{V_1} + M_{V_1} + B_{V_1}) \geq \delta \).

**Proof.** Since \( \dim W_1 \leq n \), by our assumptions in Proposition 6.4, there exists a positive integer \( m_1 \) depending on \( n-1, b \) and \( k \), such that \( \phi_{m_1(K_{V_1} + M_{V_1} + B_{V_1})} \) gives a birational map. Then \( \text{vol}(V_1, m_1(K_{V_1} + M_{V_1} + B_{V_1})) \geq 1 \) by Lemma 2.12.

Therefore,

\[
\text{vol}(V_1, K_{V_1} + M_{V_1} + B_{V_1}) = \frac{1}{m_1} \text{vol}(V_1, m_1(K_{V_1} + M_{V_1} + B_{V_1})) \geq \frac{1}{m_1} \geq \frac{1}{m_1^{n-1}}.
\]

Now let \( \delta = 1/m_1^{n-1} \). \( \square \)

We have the following fact.

**Lemma 6.6.** \( \text{vol}(V_1', (K_{V'} + M_{V'} + B_{V'} + D'_1)|_{V_1'}) \geq \delta \).

**Proof.** By Lemma 6.3, we have \( K_{V'} \sim_{\mathbb{Q}} f'^*(K_{Y'} + M_{Y'} + B_{Y'}) \). \( V_1' \) is a pure log canonical center of \( (Y', M_{Y'} + B_{Y'} + D'_1) \) and \( y' \) is a very general point of \( Y' \), so \( W_1' \) is a pure log canonical center of \( (X', f'^*D'_1) \).

By Proposition 5.1, there exists an effective \( \mathbb{Q} \)-divisor \( \Delta_{W_1'} \) on \( W_1' \), such that

\[
(K_{X'} + f'^*D'_1)|_{W_1'} \sim_{\mathbb{Q}} K_{W_1'} + \Delta_{W_1'}.
\]

On the other hand,

\[
(K_{X'} + f'^*D'_1)|_{W_1'} \sim_{\mathbb{Q}} \gamma'^*((K_{Y'} + M_{Y'} + B_{Y'} + D'_1)|_{V_1'}).
\]

For all \( m \in \mathbb{Z}_{>0} \) sufficiently divisible, by the Projection Formula we have

\[
h^0(W_1', \mathcal{O}_{W_1'}(m(K_{W_1'} + \Delta_{W_1'}))) = h^0(V_1', \mathcal{O}_{V_1'}(m(K_{Y'} + M_{Y'} + B_{Y'} + D'_1)|_{V_1'})). \]

(*
By the Canonical Bundle Formula, 

$$h^0(W_1, \mathcal{O}_{W_1}(mK_{W_1})) = h^0(V_1, \mathcal{O}_{V_1}(m(K_{V_1} + M_{V_1} + B_{V_1}))).$$

(***)

Since $W_1$ is smooth and $\Delta_{W_1} \geq 0$, it follows that 

$$h^0(W_1', \mathcal{O}_{W_1'}(m(K_{W_1'} + \Delta_{W_1'}))) \geq h^0(W_1, \mathcal{O}_{W_1}(mK_{W_1})).$$

Therefore, by equations (***) and (**), 

$$h^0(V_1', \mathcal{O}_{V_1'}(m(K_{V_1'} + M_{V_1'} + B_{V_1'} + D'_1)|_{V_1'})) \geq h^0(V_1, \mathcal{O}_{V_1}(m(K_{V_1} + M_{V_1} + B_{V_1}))).$$

which implies 

$$\text{vol}(V_1', (K_{V_1'} + M_{V_1'} + B_{V_1'} + D'_1)|_{V_1'}) \geq \text{vol}(V_1, K_{V_1} + M_{V_1} + B_{V_1}).$$

Note that the normalization $\nu : V_1' \to V_1'$ is birational. Thus we have 

$$\text{vol}(V_1', (K_{V_1'} + M_{V_1'} + B_{V_1'} + D'_1)|_{V_1'}) \geq \text{vol}(V_1, K_{V_1} + M_{V_1} + B_{V_1}) \geq \delta.$$

\[\square\]

Let $\phi_{V_1} : V_1 \to V_1'$ be the restriction of $\phi$ to $V_1$. We have 

$$\phi^*(K_{V_1'} + M_{V_1'} + B_{V_1'})|_{V_1} \sim_{\mathbb{Q}} \phi_{V_1}'((K_{V_1'} + M_{V_1'} + B_{V_1'})|_{V_1'}).$$

Recall that $D'_1 \sim_{\mathbb{Q}} \lambda_1(K_{V_1'} + M_{V_1'} + B_{V_1'})$, so by Lemma 6.3 it follows that 

$$\text{vol}(V_1, \phi^*(K_{V_1'} + M_{V_1'} + B_{V_1'})|_{V_1'}) = \frac{\text{vol}(V_1', (K_{V_1'} + M_{V_1'} + B_{V_1'})|_{V_1'})}{(1 + \lambda_1)^{n_1}} \geq \frac{\delta}{(1 + \lambda_1)^{n_1}}.$$

Hence for any very general fiber $V_t$ of $\phi$, we always have 

$$\text{vol}(V_t, \phi^*(K_{V_1'} + M_{V_1'} + B_{V_1'})|_{V_t}) \geq \delta(1 + \lambda_1)^{-n_1}.$$

Then for any point $p \in V_t$, there exists an effective $\mathbb{Q}$-divisor $E_{t,p} \sim_{\mathbb{Q}} \lambda_{t,p}(\phi^*(K_{V_1'} + M_{V_1'} + B_{V_1'})|_{V_t})$ on $V_t$ such that $\text{mult}_p E_{t,p} > n_1$ and 

$$\lambda_{t,p} < \frac{n_1}{\text{vol}(V_t, \phi^*(K_{V_1'} + M_{V_1'} + B_{V_1'})|_{V_t})^{1/n_1}} + \varepsilon_1$$

$$< \frac{n_1}{\delta^{1/n_1} \text{vol}(Y, K_Y + M_Y + B_Y)^{1/n_0}} + (1 + \varepsilon_0(n_0 - 1)!) \frac{n_1}{\delta^{1/n_1}} + \varepsilon_1$$

where $0 < \varepsilon_1 \ll 1$. This implies that there is a component of Non-klt($V_t, E_{t,p}$) passing through $p$. Multiplying $E_{t,p}$ by a positive rational number $\leq 1$, we can assume that $p$ is contained in a pure log canonical center of $(V_t, E_{t,p})$.

Applying Lemma 4.4 and Corollary 4.4 there exists a birational covering family of $(\Gamma_{t,s}, W_{t,s})$ on $V_t$ of weight $w'$ with respect to $\phi^*(K_{V_1'} + M_{V_1'} + B_{V_1'})|_{V_t}$ such that $\Gamma_{t,s} \sim_{\mathbb{Q}} \left(1/w'^n\right)\phi^*(K_{V_1'} + M_{V_1'} + B_{V_1'})|_{V_t}$ and the image of $W_{t,s}$ on $V_t$ is a pure log canonical center of $(V_t, \Gamma_{t,s})$, where 

$$\frac{1}{w'} < \frac{n_0!n_1!}{\delta^{1/n_1} \text{vol}(Y, K_Y + M_Y + B_Y)^{1/n_0}} + (1 + \varepsilon_0(n_0 - 1)!) \frac{n_1!}{\delta^{1/n_1}} + \varepsilon_1(n_1 - 1)!.$$
By Lemma 6.6, we can find a new birational covering family of $(D', V'')$ on $Y'$ of dimension less than $n_1$ and weight $w''$ such that

\[
\frac{1}{w''} = \lambda_1 + \frac{1}{w'} < \frac{n_0!n_1!\delta^{-1/n_1} + n_0!}{\text{vol}(Y, K_Y + M_Y + B_Y)_{1/n_0}} + (1 + \varepsilon_0(n_0 - 1)!\frac{n_1!}{\delta^{1/n_1}} + \varepsilon_1(n_1 - 1)! + \varepsilon_0(n_0 - 1)!).
\]

Therefore, we obtain the following diagram

\[
\begin{array}{c}
Y'' \\
\phi'' \downarrow \downarrow \\
Y' \\
\phi' \downarrow \\
S
\end{array}
\]

where $\phi''$ is birational and $\phi''$ is surjective. For the very general point $y' \in Y'$, there are an effective $\mathbb{Q}$-divisor $D'_y \sim_{\mathbb{Q}} \lambda_2(K_{Y'} + M_{Y'} + B_{Y'})$ on $Y'$ with $\lambda_2 = 1/w''$ and a very general fiber $V''_y$ of $\phi''$ such that $V''_y = \phi''(V'')$ is a pure log canonical center of $(Y', M_{Y'} + B_{Y'} + D'_y)$ at $y'$ with $\dim V''_y < \dim V'_y = n_1$. Replacing $Y''$ with the common higher smooth model of $Y, Y'$ and $Y''$, we can assume that $Y''$ is smooth and the dimension of any very general fiber of $\phi'' : Y'' \rightarrow S$ is strictly less than that of $\phi : Y \rightarrow T$. The moduli part $M_{Y''}$ on $Y''$ is still $\mathbb{Q}$-linearly trivial, since it is the pullback of $M_Y$.

Repeating above procedure at most $n' - 1$ times, there exists an effective $\mathbb{Q}$-divisor $D' \sim_{\mathbb{Q}} \lambda(K_{Y'} + M_{Y'} + B_{Y'})$ on $Y'$ with $\lambda < \alpha(\text{vol}(Y, K_Y + M_Y + B_Y))^{-1/n'} + \beta$, where $\alpha$ and $\beta$ depend only on $n, k$ and $b$, such that $y'$ is a pure log canonical center of $(Y', M_{Y'} + B_{Y'} + D')$. By the standard tie-breaking technique, we can assume that $y'$ is the unique non-klt center of $(Y', M_{Y'} + B_{Y'} + D')$ on a neighborhood of $y'$, i.e. $y'$ is an isolated point of Non-klt$(Y', M_{Y'} + B_{Y'} + D')$. Since $Y'$ and $Y$ are birational, there is a unique effective $\mathbb{Q}$-divisor $D_y \sim_{\mathbb{Q}} \lambda(K_Y + M_Y + B_Y)$ on $Y$ such that $\phi_*(D_y) = D'$. Then $D_y$ satisfies the requirements in Proposition 6.1. This completes the proof.

Remark 6.7. If we assume Theorem 1.2 without the hypothesis (1) holds for varieties of dimension $< n$ (i.e. we do not assume that $M_Y \sim_{\mathbb{Q}} 0$), then for any Iitaka fibration $f : X \rightarrow Y$ satisfying the hypotheses (2) and (3) of Theorem 1.2 with $\dim X = n$ and $\dim Y = n'$, the conclusion of Proposition 6.1 still holds. Therefore, if Theorem 1.3 holds for varieties of dimension $< n$, then for any $n$-dimensional variety $X$ of Kodaira dimension $n - 1$ with Iitaka fibration $f : X \rightarrow Y$, there exist positive constants $\alpha$ and $\beta$ depending only on $n$ such that for any very general point $y \in Y$ there is an effective $\mathbb{Q}$-divisor $D_y$ satisfying (1) and (2) of Proposition 6.1.

7. PROOF OF 1.2 AND 1.3

Lemma 7.1. Let $f : X \rightarrow Y$ be the Iitaka fibration satisfying the hypotheses of Theorem 1.2. Let $m_0$ be a positive integer and assume that for any very general point $y \in Y$, there exists an effective $\mathbb{Q}$-divisor $D_y \sim_{\mathbb{Q}} \lambda(K_Y + M_Y + B_Y)$ where $\lambda \leq m_0 - 1$, such that $y$ is an isolated point in Non-klt$(Y, D_y)$. Then for all $m \geq m_0$ such that $mM_Y$ is an integral divisor, i.e. $m$ is divisible by $bN$, we have $h^0(X, O_X(mK_X)) > 0$ and moreover, if $m \geq 2m_0$, then $h^0(X, O_X(mK_X)) \geq 2$.
Proof. Since $K_Y + M_Y + B_Y$ is big, there exist an ample $\mathbb{Q}$-divisor $H$ and an effective $\mathbb{Q}$-divisor $G$ on $Y$ such that $K_Y + M_Y + B_Y \sim_\mathbb{Q} H + G$. Pick a very general point $y \in Y$ not contained in the support of $G + B_Y$. By Lemma 3.3, the divisor $|(mB_Y)-(m-1)B_Y|$ is effective. Let $D'_y = D_y + (m-1-\lambda)G + |mB_Y|-(m-1)B_Y$. Then

$$[m(K_Y + M_Y + B_Y)] - K_Y - D'_y \sim_\mathbb{Q} (m-1-\lambda)H + M_Y$$

is ample so that $H^1(Y, \mathcal{O}_Y([m(K_Y + M_Y + B_Y)]) \otimes \mathcal{J}(Y, D'_y)) = 0$.

Consider the short exact sequence of coherent sheaves on $Y$

$$0 \to \mathcal{O}_Y([m(K_Y + M_Y + B_Y)]) \otimes \mathcal{J}(Y, D'_y) \to \mathcal{O}_Y([m(K_Y + M_Y + B_Y)]) \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ denotes the corresponding quotient. By the discussion above, the map

$$H^0(Y, \mathcal{O}_Y([m(K_Y + M_Y + B_Y)])) \to H^0(Y, \mathcal{Q})$$

is surjective. Since $y$ is an isolated point in $\text{Non-klt}(Y, D'_y)$, $\mathcal{C}_y$ is a direct summand of $H^0(Y, \mathcal{Q})$. Thus, we have

$$h^0(X, \mathcal{O}_X(mK_X)) = h^0(Y, \mathcal{O}_Y([m(K_Y + M_Y + B_Y)])) > 0.$$

Pick a very general point $y_1 \in Y$. Then there is an effective $\mathbb{Q}$-divisor $D_{y_1} \sim_\mathbb{Q} \lambda(K_Y + M_Y + B_Y)$ such that $y_1$ is an isolated point in $\text{Non-klt}(Y, D_{y_1})$. Now we may pick a very general point $y_2 \in Y$ not contained in the support of $D_{y_1}$, and pick a very general divisor $D_{y_2} \sim_\mathbb{Q} \lambda(K_Y + M_Y + B_Y)$ such that $y_2$ is an isolated point in $\text{Non-klt}(Y, D_{y_2})$ and $y_1$ is not contained in the support of $D_{y_2}$. Hence $y_1$ and $y_2$ are isolated points in $\text{Non-klt}(Y, D_{y_1} + D_{y_2})$. Then $h^0(X, \mathcal{O}_X(mK_X)) \geq 2$ by an argument similar to the discussion above.

Lemma 7.2. Let $f : X \to Y$ be the Iitaka fibration satisfying the hypotheses of Theorem 1.3. Let $m'_0$ be a positive integer divisible by $bN$. Assume that $h^0(X, mK_X) \geq 2$ for all $m \geq m'_0$ such that $m$ is divisible by $bN$. Let $X' \to Y' \to \mathbb{P}^1$ be any morphism induced by sections of $\mathcal{O}_X(m'_0K_X)$ on an appropriate birational model $f' : X' \to Y'$ of $f : X \to Y$. Let $p \in \mathbb{P}^1$ be a very general point. $f'_W : W \to V$ denotes the restriction of $f'$ to the fiber over $p$. If there is a positive integer $s$ divisible by $bN$ such that $|sK_W|$ induces the Iitaka fibration for any very general point $p$, then $|tK_X|$ induces the Iitaka fibration for all $t \geq m'_0(2s + 2) + s$ such that $t$ is divisible by $bN$.

Proof. Following [Kol86, Theorem 4.6] and its proof, $|(m'_0(2s + 1) + s)K_X|$ gives the Iitaka fibration. Since $mK_X$ is effective for all $m \geq m'_0$ such that $m$ is divisible by $bN$, the assertion follows.

Proof of Theorem 1.3. Since the moduli part is $\mathbb{Q}$-linearly trivial by Theorem 3.6, we always have $	ext{vol}(Y, K_Y + M_Y + B_Y) = \text{vol}(Y, K_Y + B_Y)$. The proof is by induction on the dimension of $X$. It is well known that the theorem holds for $n = 1$. Assume that the theorem holds when $\dim X \leq n - 1$. Let $f : X \to Y$ be the Iitaka fibration.
satisfying the hypotheses of Theorem 1.2 with \( \dim X = n \) and \( \dim Y = n' \). By Proposition 6.1, for any very general point \( y \in Y \), there exists an effective \( \mathbb{Q} \)-divisor \( D_y \sim_\mathbb{Q} \lambda(K_Y + M_Y + B_Y) \) with \( \lambda < \alpha(\text{vol}(Y, K_Y + M_Y + B_Y))^{-1/n'} + \beta \), where \( \alpha \) and \( \beta \) are two positive constants depending only on \( n, b \) and \( k \), such that \( y \) is an isolated point in \( \text{Non-klt}(Y, D_y) \).

By Proposition 6.1, Lemma 7.1, and Lemma 7.2 imply that there exists a positive integer \( m \) only depending on \( n, b \) and \( k \) such that \( mK_X \) gives the Iitaka fibration if \( m \geq m_n \) and divisible by \( bn \).

By induction, there exists a positive integer \( s \) such that \([sK_W] \) gives the Iitaka fibration for all \( W \) with \( \dim W \leq n - 1 \) satisfying the hypotheses of Theorem 1.2. By Proposition 6.1, Lemma 7.1, and Lemma 7.2, \( |mK_X| \) induces the Iitaka fibration, for

\[
m = 8bN s \left[ \frac{\alpha}{\text{vol}(Y, K_Y + M_Y + B_Y)^{1/n'} + \beta + 1} \right],
\]

so \( \phi_{m(K_Y + M_Y + B_Y)} \) gives a birational map. As \( mK_Y \) is a \( \mathbb{Q} \)-linearly trivial Cartier divisor, \( \phi_{K_Y + (2n' + 1)m(K_Y + B_Y)} \) is also birational by Lemma 2.2. We have

\[
\text{vol}(Y, (2n' + 1)m(K_Y + B_Y)) = (2n' + 1)^n m^n \text{vol}(Y, K_Y + B_Y) \\
\leq (2n' + 1)^n (8bNs)^n(\alpha + \beta + 2)^n \\
\leq (2n + 1)^n (8bNs)^n(\alpha + \beta + 2)^n.
\]

It follows that there is a constant \( A \) such that \( \text{vol}(Y, (2n' + 1)m(K_Y + B_Y)) \leq A \). Then Lemma 3.3 and Theorem 2.5 imply that the set of such log pairs \((Y, B_Y)\) is log birationally bounded.

By Theorem 2.6, there exists a constant \( \delta_n > 0 \) such that

\[
\text{vol}(Y, K_Y + B_Y) \geq \delta_n.
\]

So we are done by applying Proposition 6.1, Lemma 7.1, and Lemma 7.2 again. \( \square \)

**Proof of Theorem 1.3** By Remark 6.7, Lemma 7.1, and the argument in the proof of Theorem 1.2, we only need to show that \( \text{vol}(Y, K_Y + M_Y + B_Y) \) is bounded from below.

Since Kodaira dimension of \( X \) is \( n - 1 \), the general fiber of \( f \) is an elliptic curve. The \( j \)-invariant defines a rational map \( J : Y \dashrightarrow \mathbb{P}^1 \). Replacing \( Y \) by a higher model, we may assume that \( J \) is a morphism. Then by [PS09, 7.16], we have

\[
M_Y \sim_\mathbb{Q} \frac{1}{12} J^*(\mathcal{O}_{\mathbb{P}^1}(1)).
\]

\( J^*(\mathcal{O}_{\mathbb{P}^1}(1)) \) is base point free on \( Y \). Picking a general member \( D_Y \in |J^*(\mathcal{O}_{\mathbb{P}^1}(1))| \), we can assume that \( \frac{1}{12} D_Y + B_Y \) is simple normal crossings and

\[
K_Y + M_Y + B_Y \sim_\mathbb{Q} K_Y + \frac{1}{12} D_Y + B_Y.
\]

If \( \text{vol}(Y, K_Y + \frac{1}{12} D_Y + B_Y) = \text{vol}(Y, K_Y + M_Y + B_Y) \geq 1 \), we are done. So we can assume that \( \text{vol}(Y, K_Y + \frac{1}{12} D_Y + B_Y) = \text{vol}(Y, K_Y + M_Y + B_Y) < 1 \).

By the same argument in the proof of Theorem 1.2, it is easy to see that the set \( \{\text{vol}(Y, K_Y + \frac{1}{12} D_Y + B_Y)\} \) satisfies the DCC. Therefore, \( \text{vol}(Y, K_Y + M_Y + B_Y) \) is bounded from below. \( \square \)
Remark 7.3. In [PS09, Conjecture 7.13], Prokhorov and Shokurov list a series of conjectures concerning the effective Iitaka fibration problem. If one can prove the effective adjunction conjecture ([PS09, Conjecture 7.13(3)]), i.e. there exists a positive integer $I$ depending only on the dimension of $X$ such that $IM_Y$ is linearly equivalent to a base point free divisor on $Y$, then by Remark 6.7 and the argument in the proof of 1.3, one can prove Theorem 1.2 without the hypothesis (1).

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