QUASIMODULAR FORMS AS SOLUTIONS OF MODULAR DIFFERENTIAL EQUATIONS

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Abstract. We study quasimodular forms of depth \( \leq 4 \) and determine under which conditions they occur as solutions of modular differential equations. Furthermore, we study which modular differential equations have quasimodular solutions. We use these results to investigate extremal quasimodular forms as introduced by M. Kaneko and M. Koike \([16]\) further. Especially, we prove a conjecture stated by these authors concerning the divisors of the denominators occurring in their Fourier expansion.

1. Introduction

The notion of “quasimodular form” was coined by M. Kaneko and D. Zagier in \([18]\). Since then quasimodular forms have gained increasing attention as they have intrinsic connections to very different fields of mathematics and beyond. For two excellent introductions to the subject we refer to \([29, 36]\).

There has been an extensive study of linear differential equations, whose solution set is invariant under modular transformations (see \([1, 9, 14, 15, 17, 17, 19, 26]\)). Such differential equations can be used to study families of modular forms and quasimodular forms. The question, under which conditions such equations have modular or quasimodular solutions, is very prominent in many of these papers. The present paper will shed some new light on that and gives a unified view on the subject.

Quasimodular forms gained new interest since they occurred prominently in the construction of certain Fourier eigenfunctions with prescribed zeros. These were used in the proof that in dimensions 8 and 24 the \( E_8 \) and the Leech lattice achieve the best packing (see \([4, 33]\)), as well as in the proof of universal optimality of these lattices (see \([5]\)). For a survey on the construction of such Fourier eigenfunctions we refer to \([7]\). In this paper modular differential equations are used to

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encode the asymptotic behaviour of quasimodular forms in a concise and tractable way. The differential equations are then used to derive linear recurrence relations for the quasimodular forms of interest.

In a series of papers M. Knopp and many coauthors have introduced and studied vector valued modular forms [8, 9, 20–23, 25–28]. Modular differential equations also play a role in this context, as they are a method to capture properties of the components of a vector valued form in a concise way (see [9, 26]). These give rise to representations of the modular group. In this context the action of the map $T : z \mapsto z + 1$ is always diagonalisable. We will study a similar concept for quasimodular forms, where the action of $T$ will turn out not to be diagonalisable.

The paper is organised as follows. In Section 2 we recall some basic facts and definitions about modular forms and quasimodular forms. We shortly recall the Frobenius ansatz method for finding holomorphic solutions of differential equations.

In Section 3 we introduce and study quasimodular vectors as an analogue to vector valued modular forms. We derive several properties that have been known for the modular case and will be used later in this paper.

In Section 4 we introduce the notion of balanced quasimodular forms. These are forms $f$, which exhibit certain patterns for the vanishing orders at $i\infty$ of the quasimodular forms occurring in the transformation behaviour of $f$ under the modular group. These forms turn out to be solutions of modular differential equations of the simplest form.

In Section 5 we study modular differential equations which have (balanced) quasimodular solutions. It turns out that this can only be the case, if the depth of the form is $\leq 4$, which also means that the order of the differential equation is $\leq 5$. There is only one degenerate exception to this rule, namely powers of $\Delta$, which occur as solutions of modular differential equations of any order.

In Section 6 we use the results of Section 5 to derive differential recurrences for extremal quasimodular forms. Such forms are defined by the property that they have maximal possible order of vanishing at $i\infty$ (see [16]). The recursions obtained are then used to prove a conjecture stated in [16] concerning the divisors of the denominators of the Fourier coefficients of such forms.

In an Appendix we collect some huge expressions for polynomials and modular forms that occur in Section 6 for the case of depth 4.
2. Basics

In this section we collect some basic facts about modular and quasimodular forms and give a short exposition of the Frobenius ansatz method to solve linear differential equations.

2.1. Modular forms. The modular group $\Gamma$ is the group of $2 \times 2$-matrices with integer entries and determinant 1

$$\Gamma = \text{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$  

The group $\Gamma$ is generated by

$$Sz = -\frac{1}{z}, \quad Tz = z + 1,$$

which satisfy the relations $S^2 = \text{id}$ and $(ST)^3 = \text{id}$. It acts on the upper half plane $\mathbb{H} = \{ z \in \mathbb{C} \mid \Im z > 0 \}$ by Möbius transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a weakly holomorphic modular form of weight $w$, if it satisfies

$$\frac{(cz+d)^{-w}}{(cz+d)^{-w}} f \left( \frac{az+b}{cz+d} \right) = f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

The space of weakly holomorphic modular forms is denoted by $\mathcal{M}_w^!(\Gamma)$. This space is non-trivial only for even values of $w$. A form $f$ is called holomorphic, if

$$f(i\infty) := \lim_{z \to i\infty} f(z)$$

exists. The subspace $\mathcal{M}_w(\Gamma)$ of holomorphic modular forms is non-trivial only for even $w \geq 4$. Its dimension equals

$$\dim \mathcal{M}_w(\Gamma) = \begin{cases} \left\lfloor \frac{w}{12} \right\rfloor & \text{for } w \equiv 2 \pmod{12} \\ \left\lfloor \frac{w}{12} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$
The most prominent examples of modular forms are the Eisenstein series

\begin{align}
E_2 &= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \\
E_4 &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \\
E_6 &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,
\end{align}

where \( \sigma_k(n) = \sum_{d \mid n} d^k \) denotes the divisor sum of order \( k \). Every modular form can be expressed as a polynomial in \( E_4 \) and \( E_6 \). Furthermore, by the invariance under \( T \), every holomorphic modular form \( f \) has a Fourier expansion

\[ f(z) = \sum_{n=0}^{\infty} a_f(n) e^{2\pi i n z} = \sum_{n=0}^{\infty} a_f(n) q^n, \]

where we have used the convention to denote \( q = e^{2\pi i z} \), which will follow in the sequel by freely switching between dependence on \( z \) and \( q \).

We will make frequent use of the transformation behaviour of \( E_2 \)

\[ z^{-2} E_2(Sz) = E_2(z) + \frac{6}{\pi i z}. \]

A holomorphic form \( f \) is called a \textit{cusp form}, if \( f(i\infty) = 0 \). The prototypical example of a cusp form is

\[ \Delta = \frac{1}{1728} (E_4^3 - E_6^2). \]

The space of cusp forms is denoted by \( \mathcal{S}_w(\Gamma) \). Since we only deal with modular forms for the full modular group \( \Gamma \), we will omit reference to the group in the sequel.

For a detailed introduction to the theory of modular forms we refer to [2, 3, 6, 13, 21, 30, 31].

2.2. \textbf{Quasimodular forms.} The space of quasimodular forms of weight \( w \) and depth \( \leq r \) is given by

\[ \mathcal{Q}
\mathcal{M}^r_w = \bigoplus_{\ell=0}^{r} E_{2\ell} M_{w-2\ell}. \]

Quasimodular forms occur naturally as derivatives of modular forms (see [20, 36]).
For later reference we notice that the dimension of the space $\mathcal{QM}_w^r$ equals

(7) \quad \dim \mathcal{QM}_w^r = \frac{w(r + 1)}{12} + 1

if $r \leq 4$ and $w(r + 1) \equiv 0 \pmod{12}$. More generally, the dimension equals

$$\dim \mathcal{QM}_w^r = \frac{w}{2} \left\lfloor \frac{r}{6} \right\rfloor + \dim \mathcal{QM}_w^{(\pmod{6})} - \left\lfloor \frac{r}{6} \right\rfloor \left( r - 3 \left\lfloor \frac{r}{6} \right\rfloor - 2 \right)$$

This formula can be obtained from

$$\dim \mathcal{QM}_w^r = \sum_{k=0}^{r} \dim \mathcal{M}_{w-2k}$$

by splitting the summation over $k$ in intervals of length 6. The values of the dimension for $r = 0, \ldots, 5$ are given by

- $\dim \mathcal{QM}_w^0 = \left\lfloor \frac{w}{12} \right\rfloor + 1 - [w \equiv 2 \pmod{12}]$
- $\dim \mathcal{QM}_w^1 = \left\lfloor \frac{w}{6} \right\rfloor + 1$
- $\dim \mathcal{QM}_w^2 = \left\lfloor \frac{w}{4} \right\rfloor + 1$
- $\dim \mathcal{QM}_w^3 = \left\lfloor \frac{w}{3} \right\rfloor + 1$
- $\dim \mathcal{QM}_w^4 = \left\lfloor \frac{5w}{12} \right\rfloor + 1 - [w \equiv 10 \pmod{12}]$
- $\dim \mathcal{QM}_w^5 = \frac{w}{2}$

The notation $[P]$ means 1, if the condition $P$ is satisfied and 0 otherwise.

From this the formula

$$\dim \mathcal{QM}_w^r = \left\lfloor \frac{w(r + 1)}{12} \right\rfloor - \left\lfloor \frac{r + 1}{6} \right\rfloor \left( r - 3 \left\lfloor \frac{r + 1}{6} \right\rfloor - 1 \right) - \left\lfloor \frac{r}{6} \right\rfloor + 1 - [w(r + 1) \equiv 2 \pmod{12}]$$

can be obtained by a case distinction $r \pmod{6}$ and $w \pmod{12}$.

We will follow the convention to denote the derivative by

$$f' = \frac{1}{2\pi i} \frac{df}{dz} = q \frac{df}{dq}.$$
With this notation Ramanujan’s identities read
\begin{align}
E'_2 &= \frac{1}{12} (E_2^2 - E_4) \\
E'_4 &= \frac{1}{3} (E_2 E_4 - E_6) \\
E'_6 &= \frac{1}{2} (E_2 E_6 - E_4^2).
\end{align}
(8)
These give rise to the definition of the Serre derivative
\[ \partial_w f = f' - \frac{w}{12} E_2 f, \]
where \( w \) is (related to) the weight of \( f \). We will use the product rule
\[ \partial_{w_1 + w_2} (fg) = (\partial_{w_1} f) g + f (\partial_{w_2} g) \]
and also make frequent use of the following immediate consequence of (8)
\begin{align}
\partial_1 E_2 &= -\frac{1}{12} E_4 \\
\partial_4 E_4 &= -\frac{1}{3} E_6 \\
\partial_6 E_6 &= -\frac{1}{2} E_4^2.
\end{align}
(9)
From the second and third equation together with the fact that every holomorphic form is a polynomial in \( E_4 \) and \( E_6 \), it follows immediately that for a form \( f \in M_w \) we have \( \partial_w f \in M_{w+2} \), and for \( f \in S_w \) we have \( \partial_w f \in S_{w+2} \).

**Lemma 1.** The Serre derivative \( \partial_{w-r} \) maps quasimodular forms of weight \( w \) and depth \( r \) to quasimodular forms of weight \( w+2 \) and depth \( \leq r \).

**Proof.** A quasimodular form of weight \( w \) and depth \( r \) can be written as
\[ f_w = \sum_{k=0}^r A_{w-2k} E_2^k \]
with \( A_{w-2k} \in M_{w-2k} \). Then
\[ \partial_{w-r} f_w = \sum_{k=0}^r \left( (\partial_{w-2k} A_{w-2k}) E_2^k + A_{w-2k} (\partial_{2k-r} E_2^k) \right). \]
Inserting
\[ \partial_{2k-r} E_2^k = -\frac{k}{12} E_4 E_2^{k-1} + \frac{r-k}{12} E_2^{k+1} \]
yields
\[
\partial_{w-r}f_w = \partial_w A_w - \frac{1}{12} E_4 A_{w-2} + \sum_{k=1}^{r-1} \left( \partial A_{w-2k} - \frac{k+1}{12} E_4 A_{w-2k-2} + \frac{r-k+1}{12} A_{w-2k+2} \right) E_2^k
\]
\[+ \left( \partial_{w-2r} A_{w-2r} + \frac{1}{12} A_{w-2r+2} \right) E_2^r, \]
which is a quasimodular form of weight \( w + 2 \) and depth \( \leq r \). \( \square \)

**Lemma 2.** Let \( f : \mathbb{H} \to \mathbb{C} \) be holomorphic. Then
\[
\partial_w \left( z^{-w} f(Sz) \right) = z^{-w-2} \left( \partial_w f \right)(Sz).
\]

**Proof.** We compute
\[
\partial_w \left( z^{-w} f(Sz) \right) \\
= -\frac{w}{2\pi i} z^{-w-1} f(Sz) + z^{-w-2} f'(Sz) - \frac{w}{12} E_2(z) z^{-w} f(Sz) \\
= -\frac{w}{2\pi i} z^{-w-1} f(Sz) + z^{-w-2} f'(Sz) \\
- \frac{w}{12} \left( z^{-2} E_2(Sz) - \frac{12}{2\pi i} z^{-w} f(Sz) \right) \\
= z^{-w-2} \left( f'(Sz) - \frac{w}{12} E_2(Sz) f(Sz) \right) = z^{-w-2} \left( \partial_w f \right)(Sz).
\]
\( \square \)

We will use the following convention for iterated Serre derivatives throughout the paper:
\[
\partial_w^0 f = f, \quad \partial_w^{k+1} = \partial_w^{2k} \left( \partial_w^k f \right).
\]

We will consider differential equations of the form
\[
K_B f = B_m \partial_{w-r}^{r+1} f + B_{m+2} \partial_{w-r}^{r+2} f + \cdots + B_{m+2r+2} f = 0,
\]
where \( B = (B_m, \ldots, B_{m+2r+2}) \) are modular forms of respective weights \( m, m+2, \ldots, m+2r+2 \) with \( B_m(i\infty) = 1 \).

**Lemma 3.** For every holomorphic solution \( f : \mathbb{H} \to \mathbb{C} \) of the differential equation \( K_B f = 0 \), \( f(Tz) \) and \( z^{-w} f(Sz) \) are also solutions. Thus, for any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), \( (cz+d)^{-w} f(\gamma z) \) is also a solution.

**Proof.** Let \( f : \mathbb{H} \to \mathbb{C} \) be a holomorphic solution of \( K_B f = 0 \). Then by Lemma 2 we have
\[
K_B(z^{-w} f(Sz)) = z^{-r-w-m-2} K_B(f)(Sz) = 0.
\]
Similarly, since all coefficient functions and all Serre derivatives are invariant under $T$, we have
\[ K_B(f(Tz)) = K_B(f(Tz)) = 0. \]
The last assertion follows from the fact that $S$ and $T$ generate $\Gamma$. □

We will be mostly interested in the case that $m = 0$ and thus $B_m = 1$, in which we call the corresponding equation normalised.

2.3. The Frobenius ansatz method. F. G. Frobenius [10] devised a method to find holomorphic (=power series) solutions of differential equations of the form
\[ f(n)(z) + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f(z) = 0, \]
where $a_0, \ldots, a_{n-1}$ are meromorphic functions on some $U \subset \mathbb{C}$. Under the condition that $a_k$ has a pole of order at most $n - k$ at $z_0 \in U$ (this is called a regular singularity in this context), there exists a solution of (12) of the form
\[ f(z) = (z - z_0)^\lambda \sum_{n=0}^{\infty} f_n(z - z_0)^n, \]
where $\lambda$ is a solution of the so called indicial equation, a polynomial equation arising from inserting this ansatz into (12) and requiring $f_0 \neq 0$. Originally, the method was developed for equations of degree $n = 2$.

In our case the situation is slightly different, since we are interested in power series in $q = e^{2\pi iz}$, the derivatives still being with respect to $z$. Furthermore, we have expressed all derivatives in terms of Serre derivatives. We are looking for solutions of (11) of the form
\[ q^\lambda \sum_{n=0}^{\infty} a(n)q^n. \]
For such a solution to exist $\lambda$ has to be a root of the indicial equation
\[ p_B(x) = \sum_{\ell=0}^{r} B_{m+2\ell}(i\infty) q_{r-\ell}(x, w) = 0, \]
where
\[ q_\ell(x, w) = \left( \frac{x - w - r}{12} \right) \left( \frac{x - w - r + 2}{12} \right) \cdots \left( \frac{x - w - r + 2\ell - 2}{12} \right). \]
Then $p_B(x)$ is a polynomial of degree $r + 1$ with roots $\lambda_0, \ldots, \lambda_r \in \mathbb{C}$ called the Frobenius exponents of (11). As long as these exponents are pairwise different and none of the pairwise differences $\lambda_k - \lambda_\ell$ ($k \neq \ell$)
is an integer, the ansatz method immediately gives $r + 1$ linearly independent solutions of (11) by successively solving for $a(1), a(2), \ldots$. In our case we are especially interested in the opposite situation, namely that all exponents are positive integers and thus all the pairwise differences are integers. In the classical situation this is the case, where the monodromy representation associated to a fundamental system of solutions is not diagonalisable in general.

We order the exponents in decreasing order $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0$. Then we start with the solution

$$f_0(z) = q^{\lambda_0} \sum_{n=0}^{\infty} a_0(n) q^n.$$

A second solution can then be found using the ansatz

$$f_1(z) = C z f_0(z) + q^{\lambda_1} \sum_{n=0}^{\infty} a_1(n) q^n,$$

where $C$ has to be chosen so that the computation of the coefficient $a_1(\lambda_0 - \lambda_1)$ is possible. Further solutions can be found by making an ansatz

$$f_\ell(z) = C_\ell^{(\ell)} z^\ell f_0(z) + C_{\ell-1}^{(\ell)} z^{\ell-1} q^{\lambda_1} \sum_{n=0}^{\infty} a_1(n) q^n + \cdots + q^{\lambda_\ell} \sum_{n=0}^{\infty} a_\ell(n) q^n,$$

where the constants $C_\ell^{(\ell)}, \ldots, C_1^{(\ell)}$ have to be chosen so that the computation of the coefficients $a_\ell(\lambda_{\ell-1} - \lambda_\ell), \ldots, a_\ell(\lambda_0 - \lambda_\ell)$ is possible. For more details on the method we refer to [11, 12, 32].

### 3. Quasimodular vectors

In this section we will use the transformation behaviour of quasimodular forms to define vector valued functions that encode this transformation behaviour. This has some analogy to vector valued modular forms as studied in [8, 9, 20, 22, 23, 25, 26, 28], but also exhibits some differences.

Let $f$ be a holomorphic quasimodular form of weight $w$ and depth $s$. Then $f$ can be written as

$$f(z) = \sum_{\ell=0}^{s} E_2(z)^\ell h_\ell(z),$$

where $h_\ell$ ($\ell = 0, \ldots, s$) are modular forms of weight $w - 2\ell$. Define quasimodular forms $g_\ell$ ($\ell = 0, \ldots, s$) of weight $w - 2\ell$ and depth $s - \ell$
by

\[
\binom{r}{\ell} g_\ell(z) = \left( \frac{6}{\pi i} \right)^{\ell} \sum_{m=0}^{s-\ell} \binom{\ell + m}{m} E_2(z)^m h_{\ell+m}(z);
\]

notice that \( f = g_0 \). Then we have

\[
f(Tz) = f(z) \quad \text{and} \quad z^{-w} f(Sz) = \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{1}{z^\ell} g_\ell(z),
\]

which follows from (14).

**Definition 1.** Let \( f \) be a quasimodular form of weight \( w \) and depth \( s \leq r \) given by (14). Let then the forms \( g_\ell (\ell = 0, \ldots, s) \) be given by (15). Use these to define

\[
f_k(z) = \sum_{\ell=0}^{\min(k, s)} \binom{k}{\ell} z^{k-\ell} g_\ell(z)
\]

for \( k = 0, \ldots, r \). A holomorphic vector valued function \( \vec{F} : \mathbb{H} \to \mathbb{C}^{r+1} \) is called quasimodular, if there is a quasimodular form \( f \) such that \( \vec{F} \) is given by

\[
\vec{F}(z) = (f_0(z), f_1(z), \ldots, f_r(z))^T.
\]

If \( s < r \), we call \( \vec{F} \) degenerate.

**Proposition 1.** Let \( \vec{F} = (f_0, \ldots, f_r)^T \) be a holomorphic vector function on \( \mathbb{H} \). Then \( \vec{F} \) is quasimodular, if and only if it has the following behaviour under the generators \( S \) and \( T \) of \( \Gamma \):

\[
z^{r-w} \vec{F}(Sz) = \rho(S) \vec{F}(z)
\]

\[
\vec{F}(Tz) = \rho(T) \vec{F}(z)
\]

with

\[
\rho(S) = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & -1 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 \\
(-1)^r & 0 & \cdots & 0 & 0
\end{pmatrix}
\]
and

\[
\rho(T) = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 1 & 0 & \ldots & 0 & 0 \\
1 & 2 & 1 & \ldots & 0 & 0 \\
1 & 3 & 3 & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
1 & \binom{r}{1} & \binom{r}{2} & \ldots & \binom{r}{r-1} & 1
\end{pmatrix}.
\]

Proof. Let

\[ f(z) = \sum_{\ell=0}^{r} E_2^\ell h_\ell \]

be a quasimodular form of weight \( w \) and depth \( \leq r \); the case \( s < r \) is included by setting \( h_\ell = 0 \) for \( \ell > s \). Then \( h_\ell \) (\( \ell = 0, \ldots, r \)) is a modular form of weight \( w - 2\ell \). The transformation behaviour of \( f \) under \( S \) and \( T \) is given by (16).

The forms \( g_\ell \) given by (15) transform under \( S \) by

\[ z^{2\ell-w} g_\ell(Sz) = \sum_{m=0}^{r-\ell} \binom{r-\ell}{m} \frac{1}{z^m} g_{m+\ell}(z). \]

Using this we obtain

\[
z^{r-w} f_k(Sz) = \sum_{\ell=0}^{k} \left(-\frac{1}{z}\right)^{\ell} z^{r-w} g_{k-\ell}(Sz)
\]

\[ = (-1)^k \sum_{m=0}^{r} z^{r-p-m} g_m(z) \sum_{\ell=0}^{m} (-1)^{\ell} \binom{k}{\ell} \binom{r-\ell}{r-m}. \]

The inner sum equals \( \binom{r-k}{m} \), which gives

\[ z^{r-w} f_k(Sz) = (-1)^k \sum_{m=0}^{r-k} \binom{r-k}{m} z^{r-k-m} g_m(z) = (-1)^k f_{r-k}(z). \]

Similarly, we have for the transformation behaviour under \( T \)

\[ f_k(Tz) = \sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \binom{\ell}{m} \binom{k}{\ell} z^m g_{k-\ell}(z) \]

\[ = \sum_{p=0}^{k} \binom{k}{p} \sum_{m=0}^{p} \binom{p}{m} z^m g_{p-m}(z) = \sum_{p=0}^{k} \binom{k}{p} f_p(z). \]

Assume now that \( \vec{F} = (f_0, \ldots, f_r)^T \) is an \( (r+1) \)-dimensional vector valued function satisfying (17) and (18). Then \( f_0 \) is \( T \)-invariant, thus
admits a power series representation in $q$, which we denote by $g_0$ for
simplifying the notation in the following argument. By (16) the second
coordinate $f_1$ satisfies
\[ f_1(Tz) = f_1(z) + f_0(z); \]
we consider the function
\[ g_1(z) = f_1(z) - zf_0(z), \]
which is $T$-invariant. Thus we can write
\[ f_1(z) = zg_0(z) + g_1(z), \]
where $g_1$ is a power series in $q$. Assume now by induction that we have
already shown that
\[ f_m(z) = m \sum_{\ell=0}^\infty \binom{m}{\ell} z^\ell g_{m-\ell}(z) \]
for $0 \leq m < k$ where each of the functions $g_0, \ldots, g_{k-1}$ is a power series
in $q$. We define the function
\[ g_k(z) = f_k(z) - \sum_{\ell=1}^k \binom{k}{\ell} z^\ell g_{k-\ell}(z). \]
Then we have
\[ g_k(Tz) = \sum_{\ell=0}^k \binom{k}{\ell} f_\ell(z) - \sum_{\ell=1}^k \binom{k}{\ell} \sum_{m=0}^\ell z^m g_{k-\ell}(z) \]
\[ = f_k(z) + \sum_{\ell=0}^{k-1} \binom{k}{\ell} \sum_{m=0}^\ell \binom{\ell}{m} z^m g_{\ell-m}(z) - \sum_{\ell=1}^k \binom{k}{\ell} \sum_{m=0}^\ell z^m g_{k-\ell}(z). \]
A similar rearrangement as before then gives $g_k(Tz) = g_k(z)$, which
shows that $g_k$ can be expressed as a power series in $q$. Summing up, it
follows from (18) that there are power series in $q$, $g_0, \ldots, g_r$, such that
each of the functions $f_k$, $k = 0, \ldots, r$ can be expressed as
\[ f_k(z) = \sum_{\ell=0}^k \binom{k}{\ell} z^\ell g_{k-\ell}(z). \]
Assume now that in addition (17) holds. Then we have
\[ z^{r-w} f_0(Sz) = f_r(z) = \sum_{\ell=0}^r \binom{r}{\ell} z^\ell g_{r-\ell}(z). \]
Now we have (recall that $f_0 = g_0$)
\[
z^{r-w} f_1(Sz) = z^{r-w} \left( \left( \frac{1}{z} \right) g_0(Sz) + g_1(Sz) \right)
= - \frac{1}{z} \sum_{\ell=0}^{r} \binom{r}{\ell} z^{\ell} g_{r-\ell}(z) + z^{r-w} g_1(Sz)
= - f_{r-1}(z) = - \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} z^{\ell} g_{r-1-\ell}(z),
\]
from which we derive
\[
z^{r-w} g_1(Sz) = \sum_{\ell=0}^{r} \binom{r}{\ell} z^{\ell-1} g_{r-\ell}(z) - \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} z^{\ell} g_{r-1-\ell}(z)
= \frac{1}{z} g_r(z) + \sum_{\ell=0}^{r-1} \left[ \binom{r}{\ell+1} - \binom{r-1}{\ell} \right] z^{\ell} g_{r-1-\ell}(z)
= \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} z^{\ell-1} g_{r-1-\ell}(z),
\]
which gives
\[
z^{2-w} g_1(Sz) = \sum_{\ell=0}^{r-1} \binom{r-1}{\ell} \frac{1}{z^\ell} g_{r+\ell}(z).
\]
Assume now by induction that we have already shown
\[
z^{2m-w} g_m(Sz) = \sum_{\ell=0}^{r-m} \binom{r-m}{\ell} \frac{1}{z^\ell} g_{m+\ell}(z)
\]
for $0 \leq m < k$.

Applying $S$ to $f_k$ gives
\[
z^{r-w} f_k(Sz) = z^{r-w} \sum_{\ell=0}^{k} \binom{k}{\ell} \left( \frac{1}{z} \right)^\ell g_{k-\ell}(Sz)
= z^{r-w} g_k(Sz) + \sum_{\ell=1}^{k} (-1)^\ell \binom{k}{\ell} z^{r+\ell-2k} \sum_{m=0}^{r+k-\ell} \binom{r+\ell-k}{m} \frac{1}{z^m} g_{k-\ell+m}(z)
= z^{r-w} g_k(Sz) + \sum_{m=0}^{r} z^{m-k} g_{r-m}(z) \sum_{\ell=1}^{k} (-1)^\ell \binom{k}{\ell} \binom{r+\ell-k}{m}.\]
The inner sum evaluates to
\[
\sum_{\ell=1}^{k} (-1)^{\ell} \binom{k}{\ell} \binom{r + \ell - k}{m} = (-1)^k \binom{r - k}{m - k} - \binom{r - k}{m},
\]
which gives
\[
(21) \quad z^{r-w} f_k(Sz) = z^{r-w} g_k(Sz) + (-1)^k \sum_{m=k}^{r} \binom{r - k}{m - k} z^{m-k} g_{r-m}(z)
\]
\[- \binom{r - k}{m} z^{m-k} g_{r-m}(z).
\]

On the other hand we have
\[
z^{r-w} f_k(Sz) = (-1)^k f_{r-k}(z) = (-1)^k \sum_{\ell=0}^{r-k} \binom{r - k}{\ell} z^\ell g_{r-k-\ell}(z),
\]
which equals the first sum after shifting the index of summation. Inserting this into (21) gives
\[
z^{r-w} g_k(Sz) = \sum_{m=0}^{r-k} \binom{r - k}{m} z^{m-k} g_{r-m}(z),
\]
which proves the assertion. \(\square\)

In order to get a better understanding of quasimodular vectors, we study the modular Wronskian of a quasimodular vector \(\vec{F}\):
\[
W(z) = W_{\vec{F}}(z) = \det \left( \vec{F}, \partial_{w-r} \vec{F}, \ldots, \partial_{w-r}^{r-1} \vec{F} \right).
\]
For vector valued modular forms this has been studied in [26]. The most important property of \(W\) is its modularity.

**Proposition 2.** Let \(\vec{F}\) be a quasimodular vector of weight \(w\) and dimension \(r + 1\). Then the corresponding modular Wronskian \(W_{\vec{F}}\) is a modular form of weight \(w(r + 1)\).

**Proof.** Let \(\vec{F}\) be a quasimodular vector. Then \(\vec{F}(Tz) = \rho(T) \vec{F}(z)\) and therefore \(W(Tz) = W(z)\).

For the transformation behaviour under \(S\), we notice that by Lemma 2
\[
\rho(S) \partial_{w-r}^{\ell} \vec{F}(z) = \partial_{w-r}^{\ell} z^{r-w} \vec{F}(Sz) = z^{r-w-2\ell} \left( \partial_{w-r}^{\ell} \vec{F} \right) (Sz),
\]
from which we derive
\[
z^{-w(r+1)} W(Sz) = \det(\rho(S)) W(z).
\]
Since \(\det(\rho(S)) = 1\), this gives the assertion. \(\square\)
For the fundamental system of a normalised modular differential equation we have a far more precise statement. This is the analogue to [26, Theorem 4.3].

**Proposition 3.** Let \( f_0, f_1, \ldots, f_r \) be a fundamental system of solutions of the normalised modular differential equation

\[
\partial_{w-r}^{r+1} f + B_4 \partial_{w-r}^{r-1} f + \cdots + B_{2r+2} f = 0,
\]

where \( B_4, B_6, \ldots, B_{2r+2} \) are modular form of respective weights \( 4, 6, \ldots, 2r+2 \). Assume further that the solutions of the indicial equation \( \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0 \) are all integers. Then the modular Wronskian of \( f_0, \ldots, f_r \) equals \( c \Delta^{w(r+1)/12} \) for some constant \( c \neq 0 \).

**Proof.** Without loss of generality, we take \( f_0, f_1, \ldots, f_r \) as the solutions obtained by the Frobenius ansatz in this order. Notice that \( \lambda_0 + \lambda_1 + \cdots + \lambda_r = w^{(r+1)/12} \). Let \( \vec{F} = (f_0, \ldots, f_r)^T \). Then there exist matrices \( \rho(S) \) and \( \rho(T) \), such that

\[
\vec{F}(Tz) = \rho(T) \vec{F}(z) \quad \text{and} \quad z^{r-w} \vec{F}(Sz) = \rho(S) \vec{F}(z).
\]

By the construction of \( \vec{F} \) from the Frobenius ansatz it follows that \( \rho(T) \) is lower triangular with 1s on the diagonal, which gives \( \det(\rho(T)) = 1 \). On the other hand we have

\[
\rho(S)^2 = (-1)^r \text{id},
\]

from which we derive \( \det(\rho(S))^2 = 1 \). Furthermore, we have

\[
(\rho(S)\rho(T))^3 = (-1)^r \text{id},
\]

which implies \( \det(\rho(S))^3 = 1 \). Thus we finally have \( \det(\rho(S)) = 1 \).

Applying Lemma 2 we obtain

\[
\partial_{w-r}^k \left( z^{r-w} \vec{F}(Sz) \right) = z^{r-w-2k} \left( \partial_{w-r}^k \vec{F} \right)(Sz).
\]

The Wronskian \( W = \det(\vec{F}, \partial_{w-r} \vec{F}, \ldots, \partial_{w-r}^r \vec{F}) \) then satisfies

\[
z^{-w(r+1)} W(Sz)
= \det \left( z^{r-w} \vec{F}(Sz), z^{r-w-2} \left( \partial_{r-w} \vec{F} \right)(Sz), \ldots, z^{r-w} \left( \partial_{r-w}^r \vec{F} \right)(Sz) \right)
= \det \left( \rho(S) \vec{F}(z), \rho(S) \partial_{w-r} \vec{F}(z), \ldots, \rho(S) \partial_{w-r}^r \vec{F}(z) \right)
= \det(\rho(S)) W(z) = W(z).
\]

Similarly, we have \( W(Tz) = W(z) \). Thus \( W \) is a modular form with weight \( w(r+1) \). From the vanishing orders of \( f_i \) we obtain that \( W \) vanishes to order (at least) \( \frac{w(r+1)}{12} \) at \( i \infty \). Since \( W(z) \neq 0 \) for \( z \in \mathbb{H} \),
this implies that $W$ has to be a non-zero multiple of $\Delta^{\frac{w(r+1)}{12}}$, thus proving the assertion. \qed

4. Balanced quasimodular forms

We start with a proposition that clarifies the possible orders of vanishing of quasimodular vectors and the underlying quasimodular forms. This answers a question posed in [16]. The assertion of the proposition is the analogue to [26, Theorem 3.7 and Corollary 3.8].

Proposition 4. Let $f$ be a quasimodular form of weight $w$ and depth $r$ given by

$$f(z) = \sum_{\ell=0}^{r} E_{2}^{\ell} h_{\ell},$$

where $h_{\ell}$ ($\ell = 0, \ldots, r$) are modular forms of weight $w - 2\ell$. Let the functions $g_{\ell}$ given by (15) have vanishing orders $\lambda_{\ell} < \dim QM_{w-2\ell}$ ($\ell = 0, \ldots, r$) with $\lambda_{0} \geq \lambda_{1} \geq \cdots \geq \lambda_{r} \geq 0$, i.e.

$$g_{\ell}(z) = q^{\lambda_{\ell}} \sum_{n=0}^{\infty} a_{\ell}(n)q^{n}, \quad \text{with } a_{\ell}(0) \neq 0.$$  

Then

$$\lambda_{0} + \ldots + \lambda_{r} \leq \frac{w(r+1)}{12}. \quad (23)$$

Proof. Under the assumptions of the proposition the term in the definition of $f_{k}$, which does not carry a positive power of $z$, has vanishing order $\lambda_{k}$ and by the ordering of the exponents $\lambda_{\ell}$, this is the order of vanishing of $f_{k}$ at $i\infty$ (except possibly for terms multiplied by $z$). The same holds for the derivatives $\partial_{w-r}^{\ell}f_{k}$. Thus the Wronskian $W$ vanishes to order $\lambda_{0} + \cdots + \lambda_{r}$ (the terms carrying a $z$ are eliminated by the determinant by Proposition 2). Thus it can be written as

$$W = \Delta^{\lambda_{0} + \cdots + \lambda_{r}} H$$

for a holomorphic form $H$. Comparing the weights gives (23). \qed

Definition 2. Let $f$ be a quasimodular form of weight $w$ and depth $r$; thus there are quasimodular forms $g_{\ell}$ ($\ell = 0, \ldots, r$) of weights $w - 2\ell$ and depth $r - \ell$ such that

$$z^{-w}f(Sz) = f(z) + \sum_{\ell=1}^{r} \binom{r}{\ell} \frac{1}{z^{\ell}} g_{\ell}(z).$$

The form $f$ is called balanced, if there are non-negative integers

$$\lambda_{\ell} < \dim QM_{w-2\ell}^{r-\ell} \quad \text{for } \ell = 0, \ldots, r \quad (24)$$
and

\begin{equation}
\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_r,
\end{equation}

such that

\[ g_\ell(z) = q^{\lambda_\ell} \sum_{n=0}^\infty a_\ell(n)q^n \]

with \( a_\ell(0) \neq 0 \) for \( \ell = 0, \ldots, r \) and furthermore

\[ \lambda_0 + \cdots + \lambda_r = \dim \mathcal{QM}_w^r - 1 \]

holds.

Thus a balanced form \( f \) has the maximum possible order of vanishing distributed amongst the functions \( f_0, g_1, \ldots, g_r \) under the restriction (25). Such forms do exist for any \( r \geq 0 \) and \( w \) large enough by dimension considerations using (24). Extremal quasimodular forms as studied in [16] are special cases, namely \( \lambda_1 = \cdots = \lambda_r = 0 \), which gives the maximum possible order of vanishing of a quasimodular form of weight \( w \).

**Remark 1.** Notice that for \( r \leq 4 \) and \( (r+1)w \equiv 0 \) (mod 12)

\[ \dim \mathcal{QM}_w^r - 1 = \frac{w(r+1)}{12}. \]

Thus there is equality in (23) for balanced forms of depth \( \leq 4 \).

**Remark 2.** Notice that as opposed to the situation studied in [26] the assumption (25) on the ordering of the vanishing orders is a restriction in our case. Nevertheless, this restrictive condition will be satisfied in our later applications.

**Theorem 1.** Every balanced quasimodular form of depth \( r \leq 4 \) and weight \( w \) with \( (r+1)w \equiv 0 \) (mod 12) is a solution of a modular differential equation of the form

\begin{equation}
\partial_{w-r}^{r+1}f + a_4 E_4 \partial_{w-r}^{r-1}f + \cdots + a_{2r+2} E_{2r+2} f = 0
\end{equation}

with \( a_4, a_6, \ldots, a_{2r+2} \in \mathbb{Q} \).

**Proof.** Let \( f \) be a form satisfying the assumptions of the theorem. Then choose \( a_4, \ldots, a_{2r+2} \) so that the indicial equation of (26)

\begin{equation}
\left( \lambda - \frac{w+r}{12} \right) \left( \lambda - \frac{w+r-2}{12} \right) \cdots \left( \lambda - \frac{w-r}{12} \right)
+ a_4 \left( \lambda - \frac{w+r-4}{12} \right) \cdots \left( \lambda - \frac{w-r}{12} \right) + \cdots + a_{2r+2} = 0
\end{equation}
has solutions $\lambda_0, \ldots, \lambda_r$ (counted with multiplicity). The $\lambda^r$-term comes from the first summand and has coefficient $-\frac{w(r+1)}{12}$, which is an integer by assumption. Thus we have

$$\lambda_0 + \cdots + \lambda_r = \frac{w(r+1)}{12} = \dim \mathcal{M}_w^r - 1$$

by (7). The coefficients $a_4, \ldots, a_{2r+2}$ are then all rational.

With this choice of coefficients define the differential operator

$$K = \partial_w^{r+1} + a_4 E_4 \partial_w^{r-1} + \cdots + a_{2r+2} E_{2r+2}.$$

By Lemma 1, $\phi = K f$ is then a quasimodular form of weight $w + 2r + 2$ and depth $\leq r$. Following Definition 1 we define the functions $f_\ell$ ($\ell = 0, \ldots, r$) from $f$ as in Definition 1 and set

$$\phi_\ell = K f_\ell.$$

Then we have

$$z^{-r-w-2} \phi_\ell(S z) = K (z^{-r-w} f_\ell(S z)) = (-1)^\ell K f_{r-\ell}(z) = (-1)^\ell \phi_{r-\ell}(z)$$

and

$$\phi_\ell(T z) = K (f_\ell(T z)) = \sum_{k=0}^\ell \binom{\ell}{k} K f_k(z) = \sum_{k=0}^\ell \binom{\ell}{k} \phi_k(z),$$

which shows that $(\phi_0, \ldots, \phi_r)^T$ satisfies (17) and (18) and is thus a quasimodular vector of weight $w + 2r + 2$.

From the fact that the indicial equation of $K$ has the roots $\lambda_0, \ldots, \lambda_r$ it follows that

$$\phi_\ell(z) = K f_\ell(z) = \mathcal{O}(z^{\mu-1} q^{\lambda_\ell+1}),$$

where $\mu$ is the multiplicity of $\lambda_\ell$. Thus $\phi$ is a quasimodular form of weight $w + 2r + 2$ and depth $\leq r$, such that the corresponding orders of vanishing sum up to

$$(\lambda_0 + 1) + \cdots + (\lambda_r + 1) = \frac{(w + 12)(r+1)}{12} > \frac{(w + 2r + 2)(r+1)}{12},$$

which shows that $\phi$ has to vanish identically and $f$ is a solution of $K f = 0$. \hfill \Box

**Remark 3.** Since for every $r$ the sum of the solutions of the indicial equation equals $\frac{w(r+1)}{12}$, which is larger than the dimension of the space $\dim \mathcal{M}_w^r$ for $r \geq 5$, the assertion of the theorem is false for $r \geq 5$: no quasimodular form of depth $\geq 5$ is the solution of a normalised differential equation of the form (26).
Remark 4. The functions \( f, f_1, \ldots, f_r \) are the functions that would be obtained by solving (26) using the Frobenius ansatz (see [11,12,32]) in this order.

Theorem 1 has an obvious converse.

**Theorem 2.** Let \( r \leq 4 \) be a natural number, \( w \) such that \( w(r + 1) \equiv 0 \pmod{12} \), and \( a_4, a_6, \ldots, a_{2r+2} \) be rational numbers such that the equation (27) has only non-negative integer solutions \( \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_r \). Then the solution of the differential equation

\[
\partial_{w-r}^{r+1} f + a_4 E_4 \partial_{w-r}^{r-1} f + \cdots + a_{2r+2} E_{2r+2} f = 0
\]

with \( q \)-expansion

\[
f(z) = q^{\lambda_0} \sum_{n=0}^{\infty} a_0(n) q^n, \quad a_0(0) = 1
\]

is a balanced quasimodular form of weight \( w \) and depth \( r \), if at least one of the inequalities in (25) is strict. If \( \lambda_0 = \cdots = \lambda_r \) (which implies that \( w \equiv 0 \pmod{12} \)), then the functions \( z^\ell \Delta^{w} \) \( (\ell = 0, \ldots, r) \) form a fundamental system of (28).

**Proof.** It only remains to show that \( \Delta^{w} \) is the solution of a normalised quasimodular differential equation for any \( r \geq 0 \) with coefficients chosen so that \( \lambda_0 = \frac{w}{12} \) is an \( (r + 1) \)-fold zero of the indicial equation. For this purpose we observe that

\[
\partial_{w-r}^k \Delta^{w} = \Delta^{w} Q^{(r)}_k,
\]

where \( Q^{(r)}_k \) is a quasimodular form of weight \( 2k \) and depth \( \leq k \). These forms satisfy the recurrence relation

\[
Q^{(r)}_k = \frac{r + 1 - k}{12} E_2 Q^{(r)}_{k-1} + \partial_{k-1} Q^{(r)}_{k-1}
\]

with initial condition \( Q^{(k)}_0 = 1 \). This recursion shows that the depth increases with \( k \), except for \( k = r+1 \), where the first term vanishes. Thus \( Q^{(r)}_{r+1} \) has depth \( \leq r \); indeed it has depth \( r - 1 \), since there is no quasimodular form of weight \( 2r+2 \) and depth \( r \). By successively subtracting the highest power of \( E_2 \) we obtain modular forms \( B_4, B_6, \ldots, B_{2r} \) such that

\[
Q^{(r)}_{r+1} + B_4 Q^{(r)}_{r-1} + \cdots + B_{2r-2} Q^{(r)}_1 + B_{2r} Q^{(r)}_0 = 0.
\]

Now take \( B = (1, 0, B_4, \ldots, B_{2r}) \). The corresponding modular differential operator \( K_B \) then annihilates \( \Delta^{w} \). A fundamental system of solutions of \( K_B f = 0 \) is given by

\[
\Delta^{w}, z \Delta^{w}, \ldots, z^r \Delta^{w},
\]
which can be derived from the fact that \( z^{r-w} \Delta^{w}(S z) = z^{r} \Delta^{w} \) is also a solution by the invariance properties of \( K_{B} \). The other elements of the fundamental system can be found by applying \( T \) and taking differences. \( \square \)

**Remark 5.** The proof shows that for every \( r \geq 0 \) there is a linear differential equation (28) such that the functions \( z^{\ell} \Delta^{w} \), \( \ell = 0, \ldots, r \) form a fundamental system of solutions.

5. **Modular differential equations for quasimodular forms of depth \( \leq 4 \)**

In this section we discuss the consequences of Theorems 1 and 2 for finding balanced quasimodular forms of depths \( r \leq 4 \) also for weights \( w \), which do not satisfy \( w(r+1) \equiv 0 \) (mod 12). We include the case \( r = 0 \) for completeness, even if the results are rather trivial (see Table 1).

In this section we will use the notation

(29) \( (\lambda_{0}, \lambda_{1}, \ldots, \lambda_{r}) \ominus 1 \)

for to denote a new set of exponents satisfying the order condition (25), but with one exponent diminished by 1.

| \( w \) (mod 12) | \( f \) | differential equation |
|-----------------|-----------------|----------------------|
| 0 (mod 12)      | \( \Delta^{w} \) | \( \partial_w f = 0 \) |
| 2 (mod 12)      | \( E_{4}E_{6}\Delta^{w/12} \) | \( E_{4}E_{6}\partial_w f + \frac{1}{3}(3E_{4}^{3} + 4E_{6}^{2})f = 0 \) |
| 4 (mod 12)      | \( E_{4}\Delta^{w/12} \) | \( E_{4}\partial_w f + \frac{1}{3}E_{6}f = 0 \) |
| 6 (mod 12)      | \( E_{6}\Delta^{w/12} \) | \( E_{6}\partial_w f + \frac{1}{2}E_{4}^{2}f = 0 \) |
| 8 (mod 12)      | \( E_{4}^{2}\Delta^{w/12} \) | \( E_{4}\partial_w f + \frac{2}{3}E_{6}f = 0 \) |
| 10 (mod 12)     | \( E_{4}E_{6}\Delta^{w/12} \) | \( E_{4}E_{6}\partial_w f + \frac{1}{6}(3E_{4}^{3} + 2E_{6}^{2})f = 0 \) |

**Table 1.** The forms and differential equations for \( r = 0 \).

5.1. **Depth 1.** In this case the dimension of the space of quasimodular forms is given by

(30) \( \dim QM_{w}^{1} = \left\lfloor \frac{w}{6} \right\rfloor + 1 \).
Theorem 3. Let $w \equiv a \pmod{6}$ ($a = 0, 2, 4$) and let $\frac{w-a}{12} \leq \lambda \leq \frac{w-a}{6}$.

Let

$$f(z) = q^\lambda \sum_{n=0}^{\infty} a(n)q^n.$$ 

\(a = 0\): If \(f\) is a solution of

$$\left(\frac{12\lambda - w - 1)(12\lambda - w + 1) + 1}{144} \right) E_4f_w = 0,$$

then \(f\) is a balanced quasimodular form of weight \(w\) and depth 1, if $\lambda > \frac{w}{12}$. If $\lambda = \frac{w}{12}$, the functions $\Delta \frac{w}{12}$ and $z\Delta \frac{w}{12}$ form a fundamental system of (31).

\(a = 2\): If \(f\) is a solution of

$$E_4\partial_{w-1}^2f_w + \frac{1}{3} E_6\partial_{w-1}f_w - \left(\frac{12\lambda - w + 1)(12\lambda - w + 3)}{144} \right) E_4^2f_w = 0,$$

then \(f\) is a balanced quasimodular form of weight \(w\) and depth 1, if $\lambda > \frac{w-2}{12}$. If $\lambda = \frac{w-2}{12}$, the functions $E_2\Delta \frac{w-2}{12}$ and $zE_2\Delta \frac{w-2}{12} + \frac{6}{12}\Delta \frac{w-2}{12}$ form a fundamental system of (32).

\(a = 4\): If \(f\) is a solution of

$$E_4^2\partial_{w-1}^2f_w + \frac{2}{3} E_4E_6\partial_{w-1}f_w
- \left(\frac{(12\lambda - w + 3)(12\lambda - w + 5)}{144} + \frac{1}{6} \right) E_4^3 - \frac{2}{9} E_6^2 \right) f_w = 0,$$

then \(f\) is a balanced quasimodular form of weight \(w\) and depth 1, if $\lambda > \frac{w-4}{12}$. If $\lambda = \frac{w-4}{12}$, the functions $E_4\Delta \frac{w-4}{12}$ and $zE_4\Delta \frac{w-4}{12}$ form a fundamental system of (33).

Proof. For $a = 0$ this is the statement of Theorem 2.

For $a = 2$, we take $g$ to be solution of (31) for $w-2$. Then $f = \partial_{w-3}g$ is a solution of (32). Similarly, for $a = 4$ we take $g$ to be a solution of (31) for $w-4$. Then $E_4g$ is a solution of (33). In both cases, neither application of the Serre derivative, nor multiplication by $E_4$ change the vanishing orders. Thus the resulting forms are still balanced. 

5.2. Depth 2. In this case the dimension of the space $\mathcal{QM}_w^2$ is given by

$$\dim \mathcal{QM}_w^2 = \left\lfloor \frac{w}{4} \right\rfloor + 1.$$ 

Theorem 4. Let $w \equiv a \pmod{4}$ ($a = 0, 2$) and let

$$\lambda_0 \geq \lambda_1 \geq \lambda_2$$
be positive integers with \( \lambda_0 + \lambda_1 + \lambda_2 = \frac{w-a}{4} \) and set

\[
A = \frac{1}{144} \left(4 - 3(w - a)^2 + \lambda_0 \lambda_1 + \lambda_0 \lambda_2 + \lambda_1 \lambda_2 \right)
\]

\[
B = -\left(\lambda_0 - \frac{w - a - 2}{12}\right) \left(\lambda_1 - \frac{w - a - 2}{12}\right) \left(\lambda_2 - \frac{w - a - 2}{12}\right)
\]

Let

\[f(z) = q^{\lambda_0} \sum_{n=0}^{\infty} a(n)q^n.\]

\(a=0\): If \(f\) is a solution of

\[
\partial_{w-2}^3 f + AE_4 \partial_{w-2} f + BE_6 f = 0
\]

then \(f\) is a balanced quasimodular form of weight \(w\) and depth 2, if there is at least one strict inequality in (35). If \(\lambda_0 = \lambda_1 = \lambda_2 = \frac{w}{12}\) the functions \(\Delta^{\frac{w}{2}}, z\Delta^{\frac{w}{2}},\) and \(z^2 \Delta^{\frac{w}{2}}\) form a fundamental system of (36).

\(a=2\): If \(f\) is a solution of

\[
E_6 \partial_{w-2}^3 f + \frac{1}{2} E_4^2 \partial_{w-2}^2 f + AE_4 E_6 \partial_{w-2} f + \left(\frac{1}{2} AE_4^3 + \frac{1}{3} (3B - A) E_6^2 \right) f = 0
\]

then \(f\) is a balanced quasimodular form of weight \(w\) and depth 2, if there is at least one strict inequality in (35). If \(\lambda_0 = \lambda_1 = \lambda_2 = \frac{w}{12}\), the functions \(E_2 \Delta^{\frac{w}{12}}, zE_2 \Delta^{\frac{w}{12}} + \frac{2}{\pi i} \Delta^{\frac{w}{12}},\) and \(z^2 E_2 \Delta^{\frac{w}{12}} + \frac{6z}{\pi i} \Delta^{\frac{w}{12}}\) form a fundamental system of (37).

**Proof.** The case \(a = 0\) is covered by Theorem 2. For \(a = 2\) we observe that if \(g\) is a solution of (36) for \(w - 2\), then \(\partial_{w-4} g\) is a solution of (37). \(\square\)

### 5.3. Depth 3.

In this case the dimension of the space \(\mathcal{Q}M_w^3\) is given by

\[
\dim \mathcal{Q}M_w^3 = \left\lfloor \frac{w}{3} \right\rfloor + 1.
\]

We denote

\[
\sigma_2 = \lambda_0 \lambda_1 + \cdots + \lambda_2 \lambda_3
\]

\[
\sigma_3 = \lambda_0 \lambda_1 \lambda_2 + \cdots + \lambda_1 \lambda_2 \lambda_3
\]
the elementary symmetric functions in \( \lambda_0, \ldots, \lambda_3 \). Using this notation we set

\[
A_0 = -\frac{w^2}{24} + \sigma_2 + \frac{5}{72},
\]

\[
B_0 = -\frac{w^3}{216} + \frac{w^2}{72} + \frac{w - 2}{6} \sigma_2 - \sigma_3 - \frac{5}{216},
\]

\[
C_{0,2,4} = \left( \lambda_0 - \frac{w - 3}{12} \right) \left( \lambda_1 - \frac{w - 3}{12} \right) \left( \lambda_2 - \frac{w - 3}{12} \right) \left( \lambda_3 - \frac{w - 3}{12} \right),
\]

and

\[
A_2 = -\frac{(w - 2)^2}{24} + \sigma_2 + \frac{5}{72},
\]

\[
B_2 = -\frac{(w - 2)^3}{216} + \frac{w - 2}{6} \sigma_2 - \sigma_3,
\]

\[
D_2 = \frac{16}{3} (w - 2)^3 - 16(w - 2)^2 + \frac{80}{3} - 192(w - 4)\sigma_2 + 1152\sigma_3,
\]

and

\[
A_4 = -\frac{(w - 1)^2}{24} + \frac{1}{36} + \sigma_2
\]

\[
B_4 = -\frac{2w^3 - 9w^2 + 12w - 3}{432} - \sigma_3 + \frac{w - 2}{6} \sigma_2
\]

\[
D_4 = \frac{4}{3} \left( 2w^3 - 21w^2 + 72w - 79 \right) - 96(w - 2)\sigma_2 + 576\sigma_3.
\]

**Theorem 5.** Let \( w \equiv 0, 2, 4 \pmod{6} \) and let

\[
\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3
\]

be positive integers with \( \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = \left\lfloor \frac{w}{3} \right\rfloor \) and not all equal. Let \( A_0, B_0, C_0, A_2, \ldots, D_2, \) and \( A_4, \ldots, D_4 \) be given as above and let

\[
f(z) = q^{a_0} \sum_{n=0}^{\infty} a(n)q^n.
\]

**w \equiv 0 \pmod{6}:** If \( f \) is a solution of

\[
\partial_{w-3}^4 f + A_0 E_4 \partial_{w-3}^2 f + B_0 E_6 \partial_{w-3} f + C_0 E_4^2 f = 0,
\]

then \( f \) is a balanced quasimodular form of weight \( w \) and depth \( 3 \).

**w \equiv 2 \pmod{6}:** If \( f \) is a solution of

\[
E_4 \partial_{w-3}^4 f + \frac{2}{3} E_6 \partial_{w-3}^3 f + A_2 E_4^2 \partial_{w-3} f + B_2 E_4 E_6 \partial_{w-3} f + (C_2 E_4^2 + D_2 \Delta) f = 0,
\]
then $f$ is a balanced quasimodular form of weight $w$ and depth $3$.

$w \equiv 4 \pmod{6}$: If $f$ is a solution of

$$
E_4 \partial_{w-3}^4 f + \frac{1}{3} E_6 \partial_{w-3}^3 f + A_4 E_4^2 \partial_{w-3}^2 f + B_4 E_4 \partial_{w-3} f + (C_4 E_4^3 + D_4 \Delta) f = 0,
$$

(43)

then $f$ is a balanced quasimodular form of weight $w$ and depth $3$.

Proof. For $w \equiv 0 \pmod{6}$ this is the assertion of Theorem 2. For $w \equiv 2 \pmod{6}$ let $g$ be the solution of the form (40) of (41) for $w - 2$. Then $\partial_{w-5} g$ is a balanced quasimodular form of weight $w$ and depth 3 satisfying (42).

For $w \equiv 4 \pmod{6}$ we have to take into account that $\lfloor \frac{w}{3} \rfloor = \lfloor \frac{w-4}{3} \rfloor + 1$, which means that the total order of vanishing for a balanced quasimodular form of weight $w$ is one higher than that of a form of weight $w - 4$. We take $g$ as the solution of (41) for the exponents $(\lambda_0, \ldots, \lambda_3) \ominus 1$, where we call $\lambda$ the exponent that has been decreased. We then set

$$
f = \partial_{w-7}^2 g - \left( \lambda - \frac{w + 5}{12} \right) \left( \lambda - \frac{w + 7}{12} \right) E_4 g.
$$

(44)

Then $f$ is a balanced quasimodular form of weight $w$. Notice that $g$ has order $\lambda - 1$ for the corresponding vanishing order, whereas $f$ has again $\lambda$ by the choice of the operator applied to $g$. Then $f$ is a solution of (43).

5.4. Depth 4. In this case the dimension of the space $\mathcal{QM}_w^4$ is given by

$$
\dim \mathcal{QM}_w^4 = \left\lfloor \frac{5w}{12} \right\rfloor + \begin{cases} 
1 & \text{if } w \equiv 0, 2, 4, 6, 8 \pmod{12} \\
0 & \text{if } w \equiv 10 \pmod{12}.
\end{cases}
$$

(45)

In principle, a similar theorem to Theorems 2–5 could be given. For depth 4 it would give six modular differential equations according to the residue classes $0, 2, 4, 6, 8, 10 \pmod{12}$. The equation for $w \equiv (\mod{12})$ can just be determined from $\lambda_0, \ldots, \lambda_4$ by requiring that these are the solutions of the indicial equation. For $w \equiv 2 \pmod{12}$ the balanced quasimodular form of weight $w$ can be obtained from the form of weight $w - 2$ by applying $\partial_{w-6}$. Similarly, for $w \equiv 4 \pmod{12}$ the balanced form of weight $w$ can be obtained from the form of weight $w - 4$ by a similar operator as (44). For $w \equiv 6, 8 \pmod{12}$ operators of orders 3 and 4 have to be used, which are determined by two respectively three roots of their indicial equation. For $w \equiv 10 \pmod{12}$ the
form can again be obtained by applying $\frac{\partial}{\partial w} - 6$ to the form obtained for $w \equiv 8 \pmod{12}$.

6. Recursions for extremal quasimodular forms

In this section we adopt the notation used in [14] that an equation number with subscript $w + a$ means that the parameter $w$ in this equation is replaced by $w + a$. Also, throughout this section we take $f_w$ to denote a normalised form of weight $w$; recall that we call a form normalised, if its leading coefficient equals 1.

6.1. Depth 1. As stated in [14, 16, 34] extremal quasimodular forms of depth 1 satisfy the differential equation

\begin{equation}
\frac{\partial^2}{\partial w^2} f_w - \frac{w^2 - 1}{144} E_4 f_w = 0 \tag{46}
\end{equation}

for $w \equiv 0 \pmod{6}$, which we assume throughout this subsection. We observe that

\begin{equation}
K_{w}^{\text{up}} f_w = E_4 \frac{\partial}{\partial w} f_w - \frac{w + 1}{12} E_6 f_w \tag{47}
\end{equation}

is a solution of (46) if $f_w$ is a solution of (46). This can be seen from the fact that (47) is a quasimodular form of weight $w + 6$ of depth 1 with one order of vanishing higher than $f_w$, thus an extremal quasimodular form of weight $w + 6$.

On the other hand,

\begin{equation}
K_{w}^{\text{down}} f_w = \frac{1}{\Delta} \left( E_4 \frac{\partial}{\partial w} f_w + \frac{w - 1}{12} E_6 f_w \right) \tag{48}
\end{equation}

is a solution of (46) if $f_w$ is a quasimodular form of weight $w - 6$ vanishing to one order less than $f_w$. The holomorphy follows from the fact that the differential operator in parenthesis applied to constants gives $O(q)$. The leading coefficient of $K_{w}^{\text{down}} f_w$ equals $\frac{w}{6}$, thus

\[ K_{w}^{\text{down}} f_w = \frac{w}{6} f_{w-6}. \]

Furthermore, we have

\begin{equation}
K_{w+6}^{\text{down}} K_{w}^{\text{up}} f_w = 12 (w + 1)(w + 5) f_w \tag{49}
\end{equation}

using (46). Defining $c_{w+6}$ by

\[ K_{w}^{\text{up}} f_w = c_{w+6} f_{w+6}, \]

we have

\[ c_{w+6} K_{w+6}^{\text{down}} f_{w+6} = c_{w+6} \frac{w + 6}{6} f_w. \]
which allows to compute $c_{w+6}$ from \[49\.\] Summing up, we have proved the following proposition.

**Proposition 5.** Let $(f_w)_{w \in 2\mathbb{N}}$ ($w \geq 6$) denote the sequence of normalised extremal quasimodular forms of depth 1. Then for $w \equiv 0 \pmod{6}$

\[
f_6 = \frac{1}{720} (E_2 E_4 - E_6)
\]

and

\[
f_{w+6} = \frac{w + 6}{72(w + 1)(w + 5)} \left(E_4 \partial_{w-1} f_w - \frac{w + 1}{12} E_6 f_w\right).
\]

Furthermore, we have

\[
f_{w+2} = \frac{12}{w + 1} \partial_{w-1} f_w
\]

\[
f_{w+4} = E_4 f_w
\]

### 6.2. Depth 2.

The extremal quasimodular forms of weight $w$ and depth 2 satisfy the differential equation

\[
\partial_{w-2}^3 f = \frac{3w^2 - 4}{144} E_4 \partial_{w-2} f - \frac{(w + 1)(w - 2)^2}{864} E_6 f = 0
\]

for $w \equiv 0 \pmod{4}$, which we assume for this subsection. If $f_w$ satisfies \[53\], then

\[
K^\text{up}_w f_w = \frac{w(w + 1)}{36} E_4 f_w - \partial_{w-2}^2 f_w
\]

satisfies \[53\] and

\[
K^\text{down}_w f_w = \frac{1}{\Delta} \left( E_4 \partial_{w-2}^2 f_w + \frac{w - 1}{6} E_6 \partial_{w-2} f_w + \frac{(w - 2)^2}{144} E_4^2 f_w \right)
\]

satisfies \[53\] and \[53\]. As before this can be seen from the fact that $K^\text{up}_w f_w$ and $K^\text{down}_w f_w$ are quasimodular forms of respective weights $w + 4$ and $w - 4$, which vanish to respective orders $\frac{w}{4} + 1$ and $\frac{w}{4} - 1$, thus being extremal. Notice that the indicial equation of $\Delta K^\text{down}_w$ has a double root at 0, thus it maps linear functions in $z$ to $q \times (\text{linear functions in } z)$. In \[16\] essentially the same operator expressed in terms of the Rankin-Cohen bracket (see \[33\]) has been used.

As before, define $c_{w+4}$ by

\[
K^\text{up}_w f_w = c_{w+4} f_{w+4}
\]

and observe that

\[
K^\text{down}_w f_w = \left(\frac{w}{4}\right)^2 f_{w-4}.
\]
Furthermore, we have
\[ K_{w+4}^{\down} K_{w}^{\up} f_w = \frac{1}{3} (w + 1)(w + 2)^2(w + 3). \]

Putting these together gives
\[ c_{w+4} = \frac{16(w + 1)(w + 3)(w + 2)^2}{3w^2}. \]

Thus we have proved

**Proposition 6.** Let \((f_w)_{w \in \mathbb{N}} \ (w \geq 4)\) denote the sequence of normalised extremal quasimodular forms of depth 2. Then for \(w \equiv 0 \pmod{4}\)

\[ f_4 = \frac{1}{288} (E_4 - E_2^2) \]  

and

\[ f_{w+4} = \frac{3w^2}{16(w + 1)(w + 2)^2(w + 3)} \left( \frac{(w - 1)w}{36} E_4 f_w - \partial_{w-2}^2 f_w \right). \]

Furthermore, we have

\[ f_{w+2} = \frac{6}{w + 1} \partial_{w-2} f_w. \]

**6.3. Depth 3.** For depth 3 the extremal quasimodular forms of weight \(w\) satisfy the differential equation

\[ \partial_{w-3}^4 f - \frac{3w^2 - 5}{72} E_4 \partial_{w-3}^2 f - \frac{w^3 - 3w^2 + 5}{216} E_6 \partial_{w-3} f \]

\[ - \frac{(w + 1)(w - 3)^3}{6912} E_4 f = 0 \]

for \(w \equiv 0 \pmod{6}\), which we assume for this subsection.

We define

\[ K_{w}^{\up} f = 48(7w^2 + 42w + 60) \partial_{w-3}^3 f \]

\[ - (15w^4 + 96w^3 + 151w^2 - 30w - 116) E_4 \partial_{w-3} f \]

\[ - \frac{1}{6} (w + 1)(9w^4 + 45w^3 + 40w^2 + 24w + 144) E_6 f \]

Then \(K_{w}^{\up} f_w\) is a solution of \((56)\) if \(f_w\) is a solution of \((59)\). by a similar reasoning as before.

For the corresponding operator \(K_{w}^{\down}\) we make an ansatz

\[ L_w = A_{12} \partial_{w-3}^3 + A_{14} \partial_{w-3}^2 + A_{16} \partial_{w-3} + A_{18} \]

with unknown modular forms \(A_{12}, \ldots, A_{18}\) of respective weights 12, \ldots, 18.

We apply \(L_w\) to the solution \(f_w\) of \((59)\) with \(f_w = q^w (1 + O(q)) \) (a quasimodular form of weight \(w\) and depth 3). Then \(L_w f_w\) is a quasimodular
form of weight $w + 18$. From this form we define the quasimodular forms $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3$ as in Section 3. We define the map

$$\Phi : M_{12} \oplus \cdots \oplus M_{18} \to \mathbb{C}^6$$

$$(A_{12}, \ldots, A_{18}) \mapsto \text{coefficients of } 1 \text{ and } q \text{ of } \tilde{g}_1, \tilde{g}_2, \tilde{g}_3.$$  

Since the dimension of $M_{12} \oplus \cdots \oplus M_{18}$ is 7, this map has a non-trivial kernel. We take $(A_{12}, \ldots, A_{18})$ to be in this kernel. Then the functions $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3$ vanish to order 2, $L_w f_w$ vanishes to order $\frac{w}{3}$, thus $\frac{1}{\Delta^2} L_w f_w$ is a holomorphic form of weight $w - 6$ vanishing to order $\frac{w}{3} - 2$ at $i\infty$, thus an extremal quasimodular form of this weight.

This gives

$$K_{\text{down}} f_w = \frac{1}{2\pi^4} \left( \left( 9w^2 - 54w + 84 \right) E_4^2 + \left( 7w^2 - 42w + 60 \right) E_6^2 \right) \partial_{w-3}^2 f$$
$$+ 4(w-3)^2(w-1)E_4^2 E_6 \partial_{w-3}^3 f$$
$$+ \frac{1}{2\pi^4} \left( 39w^4 - 336w^3 + 1099w^2 - 1626w + 924 \right) E_4 E_6^2$$
$$+ 3(3w^4 - 48w^3 + 231w^2 - 450w + 316) E_4^2 \partial_{w-3} f$$
$$+ \frac{1}{2\pi^4} (w-3)^2 E_6 \left( 3(3w^3 - 24w^2 + 64w - 56) E_4^3 - (w^3 - 24w + 48) E_6^3 \right) f. \tag{61}$$

Then $K_{\text{down}} f_w$ is a holomorphic form of weight $w - 6$ with vanishing order $\frac{w}{3} - 2$, which solves (59) for a solution $f_w$ of (59). This gives

$$K_{\text{down}} K_{\text{up}} f_w = 5184(w + 1)(w + 2)^3(w + 3)^2(w + 4)^3(w + 5)f_w. \tag{62}$$

From the fact that

$$K_{\text{down}} f_w = 768(w - 3)^2 \left( \frac{w}{3} \right)^3 q^{w - 2} + \cdots = 768(w - 3)^2 \left( \frac{w}{3} \right)^3 f_{w-6},$$

we obtain with $K_{\text{up}} f_w = c_{w+6} f_{w+6}$ that

$$K_{\text{down}} K_{\text{up}} f_w = 768(w + 3)^2 \left( \frac{w + 6}{3} \right)^3 c_{w+6} f_w$$
$$= 5184(w + 1)(w + 2)^3(w + 3)^2(w + 4)^3(w + 5)f_w,$$

which gives $c_{w+6}$.

Furthermore, the Frobenius ansatz for $f_w$ gives

$$f_w = q^w \left( 1 + \frac{w(w^2 + 15w - 18)}{(w + 3)^2} q + O(q^2) \right).$$
Inserting this into $\partial_{w-3} w - 3$ and $(w+1)(3w+1) \frac{48}{48} E_4 f_w - \partial_{w-3}^2 f_w$ gives

$$\partial_{w-3} f_w = q^\frac{w}{3} \left( \frac{w+1}{4} + O(q) \right)$$

$$(w+1)(3w+1) \frac{48}{48} E_4 f_w - \partial_{w-3}^2 f_w = q^\frac{w}{3} \left( \frac{27(w+1)(w+2)^3}{2(w+3)^2} q + O(q^2) \right).$$

These two forms then have respective weights $w + 2$ and $w + 4$ and vanishing orders $\left\lfloor \frac{w+2}{3} \right\rfloor$ and $\left\lfloor \frac{w+4}{3} \right\rfloor$ and are thus extremal. The leading coefficients can be read off.

Thus we have proved

**Proposition 7.** Let $(f_w)_{w \in \mathbb{N}}$ $(w \geq 6)$ denote the sequence of normalised extremal quasimodular forms of depth 2. Then

$$(63) \quad f_6 = \frac{5E_2^3 - 3E_2 E_4 - 2E_6}{51840}$$

and

$$(64) \quad f_{w+6} = \frac{4(w+6)^3}{(w+1)(w+2)^3(w+4)^3(w+5)} K_{w}^\text{up} f_w$$

with $K_{w}^\text{up}$ given by (60). Furthermore, we have

$$(65) \quad f_{w+2} = \frac{4}{w+1} \partial_{w-3} f_w$$

$$(66) \quad f_{w+4} = \frac{2(w+3)^2}{27(w+1)(w+2)^3} \left( \frac{(w+1)(3w+1)}{48} E_4 f_w - \partial_{w-3}^2 f_w \right).$$

**6.4. Depth 4.** For depth 4 the extremal quasimodular forms of weight $w$ satisfy the differential equation

$$(66) \quad \partial_{w-4}^5 f = \frac{5}{72} (w^2 - 2) E_4 \partial_{w-4}^3 f - \frac{5}{432} (w^3 - 3w^2 + 6) E_6 \partial_{w-4}^2 f$$

$$- \frac{15w^4 - 120w^3 + 280w^2 - 496}{20736} E_2^2 \partial_{w-4} f - \frac{(w-4)^4(w+1)}{62208} E_4 E_6 f = 0$$

for $w \equiv 0 \pmod{12}$, which we assume for this subsection.

We define

$$K_{w}^\text{up} f = -p_0(w) E_4 \partial_{w-4}^5 + \frac{(w+4)^4}{12} p_1(w) E_6 \partial_{w-4}^3 f + \frac{1}{720} p_2(w) E_6^2 \partial_{w-4}^2 f$$

$$+ \frac{1}{8640} p_3(w) E_4 E_6 \partial_{w-4} f + \frac{w+1}{25920} p_4(w) E_2^3 f + \frac{(w+1)(w+4)^4}{15} p_5(w) \Delta f.$$
Similarly, we make an ansatz for an operator
\[(68)\]
\[K_{w\down} f = \frac{1}{\Delta^5} \left( C_{40} \partial_{w-4}^4 f + C_{42} \partial_{w-4}^3 f + C_{44} \partial_{w-4}^2 f + C_{46} \partial_{w-4} f + C_{48} f \right), \]
where \(C_{40}, \ldots, C_{48}\) are modular forms of weights 40, \ldots, 48. We call the operator \(\Delta^5 K_{w\down}\) (the term in parenthesis) \(L_w\) for the following arguments. First we notice that for any such operator \(L_w\), \(L_w f\) is a quasimodular form of weight \(w + 48\) and depth 4. Furthermore, if \(f\) vanishes to some order at \(i\infty\), then \(L_w f\) vanishes to at least this order. Now take \(f_w\) to be a solution of (66) with \(f_w = \mathcal{O}(q^{5w})\). Then, as in Section 3 we form the quasimodular forms \(\tilde{g}_1, \ldots, \tilde{g}_4\) from the function \(L_w f_w\) and consider the linear map
\[
\Phi : M_{40} \oplus \cdots \oplus M_{48} \to \mathbb{C}^{20}
\]
\[(C_{40}, \ldots, C_{48}) \mapsto \text{coefficients of } 1, q, \ldots, q^4 \text{ of } \tilde{g}_1, \ldots, \tilde{g}_4.\]

The space \(M_{40} \oplus \cdots \oplus M_{48}\) has dimension 21, thus the map \(\Phi\) has a non-trivial kernel. We choose \((C_{40}, \ldots, C_{48})\) in the kernel to form the operator \(L_w\) (and then \(K_{w\down}\)). Then the functions \(L_w f_w, \tilde{g}_1, \ldots, \tilde{g}_4\) all vanish to order at least 5 at \(i\infty\). If we divide these functions by \(\Delta^5\), we still obtain holomorphic functions. Thus \(K_{w\down} f_w\) is a holomorphic quasimodular form of weight \(w - 12\) and depth 4. Furthermore, it vanishes to order \((at least) \frac{5(w-12)}{12}\) at \(i\infty\) and is thus extremal and solves (66) \(w - 12\). The choice of the forms is given in (79)–(83) in the Appendix.

With these two operators we obtain
\[
K_{w+12} K_{w\up} f_w = \frac{729}{40} (w+1)(w+2)^5(w+3)^5(w+4)(w+5)(w+6)^4 \\
\times (w+7)(w+8)^5(w+9)^5(w+10)^5(w+11) f_w.
\]

On the other hand by the construction of \(K_{w\down}\) we have
\[
K_{w\down} f_w = \frac{5^4}{2^{31} 3^{17}} w^4(5w - 12)^2(5w - 24)^2(5w - 36)^2(5w - 48)^4 f_{w-12}.
\]

Furthermore, the Frobenius ansatz gives
\[
f_w = q^{\frac{5w}{12}} \left( 1 + \frac{2w(211w^4 + 4440w^3 + 12960w^2 - 20736)}{(5w + 12)^4} q + \cdots \right)
\]
with explicit coefficients for $q^2, \ldots, q^6$. From this we obtain

\begin{align*}
\text{(69)} \quad & \partial_{w-4} f_w = \frac{w+1}{18} q^{\frac{5w}{2} + 3 + \ldots} \\
& \frac{1}{18} (w+1)(2w+1) E_w f_w - \partial_{w-4} f_w = \frac{1552(w+1)(w+3)^3}{(5w+12)^4} q^{\frac{5w}{2} + 3 + \ldots} \\
& \left(17w^2 + 78w + 90\right) \partial_{w-4} f_w - \frac{1}{18\pi} \left(192w^4 + 1008w^3 + 1504w^2 + 192w - 576\right) E_w \partial_{w-4} f_w \\
& = \frac{27308928(w+1)(w+2)^5}{(5w+12)^4} \left(12w^4 + 60w^3 + 50w^2 + 25w + 5\right) q^{\frac{5w}{2} + 3 + \ldots} \\
& - \left(1313w^6 + 28678w^5 + 255122w^4 + 11383008w^3 + 3016512w^2 + 4012416w + 2177280\right) \partial_{w-4} f_w \\
& + \frac{1}{18\pi} \left(13423w^8 + 295800w^7 + 2645368w^6 + 12166080w^5 + 29311504w^4 + 29202416w^3 - 1565376w^2 \right. \\
& \left. - 56692224w^2\right) + 33094656 \left(31w^6 + 139994w^5 + 1139536w^4 + 4759344w^3 + 10294016w^2 + 11541472w^1 + 14671104w^0 \right) \\
& + 41398272w^2 + 63016704w + 31974912 \left(31w^6 + 139994w^5 + 1139536w^4 + 4759344w^3 + 10294016w^2 + 11541472w^1 + 14671104w^0 \right) \\
& - 41419562w^2 - 20570112w + 30855168 \left(31w^6 + 139994w^5 + 1139536w^4 + 4759344w^3 + 10294016w^2 + 11541472w^1 + 14671104w^0 \right) \\
& = \frac{2674591344788(w+1)(w+2)^5}{(5w+12)^4} \left(5w^4 + 20w^3 + 35w^2 + 35w + 10\right) \left(5w^4 + 20w^3 + 35w^2 + 35w + 10\right) q^{\frac{5w}{2} + 3 + \ldots} \\
& \left(293w^4 + 4332w^3 + 22968w^2 + 51192w + 40824\right) \left(3w^3 + 15w^2 + 270w + 2605\right) q^{\frac{5w}{2} + 3 + \ldots} \\
& - \frac{1}{4} \left(3311w^6 + 512334w^5 + 291550w^4 + 731040w^3 + 717696w^2 - 2592w - 256608\right) \left(31w^6 + 139994w^5 + 1139536w^4 + 4759344w^3 + 10294016w^2 + 11541472w^1 + 14671104w^0\right) \\
& - \frac{1}{4\pi} \left(1313w^6 + 19430w^5 + 104354w^4 + 251616w^3 + 310464w^2 + 300672w + 248832\right) \left(31w^6 + 139994w^5 + 1139536w^4 + 4759344w^3 + 10294016w^2 + 11541472w^1 + 14671104w^0\right) \\
& = \frac{128290546548864(w+1)(w+2)^5}{(5w+12)^4} \left(3w^4 + 15w^3 + 35w^2 + 35w + 10\right) \left(3w^4 + 15w^3 + 35w^2 + 35w + 10\right) q^{\frac{5w}{2} + 3 + \ldots} \\
& \frac{128290546548864(w+1)(w+2)^5}{(5w+12)^4} \left(3w^4 + 15w^3 + 35w^2 + 35w + 10\right) \left(3w^4 + 15w^3 + 35w^2 + 35w + 10\right) q^{\frac{5w}{2} + 3 + \ldots}. \\
\end{align*}

These are extremal quasimodular forms of respective weights $w+2, \ldots, w+10$, whose leading coefficient can be read off.

**Proposition 8.** Let $(f_w)_{w \in \mathbb{N}}$ ($w \geq 12$) denote the sequence of normalised extremal quasimodular forms of depth 4. Then for $w \equiv 0 \pmod{12}$

\begin{align*}
\text{(70)} \quad & f_{12} = \frac{13025E_2^2 - 12796E_2^3 + 3852E_2E_1E_3 - 2706E_2E_1^2 + 27500E_2^2E_3 - 28875E_2^3}{744232832323} \\
\text{and} \quad & f_{w+12} = \frac{5^3(w+12)^4(5w+12)^4(5w+24)^4(5w+36)^4(5w+48)^4}{244233(w+1)(w+2)(w+3)(w+4)(w+5)(w+6)^4(w+7)(w+8)^3(w+9)^3(w+10)^3(w+11)} \cdot K_w f_w.
\end{align*}
Furthermore, the functions

\[
\begin{align*}
    f_{w+2} &= \frac{3}{w+1} \partial_{w-4} f_w, \\
    f_{w+4} &= \frac{(5w+12)^4}{15552(w+1)(w+2)^2} \left( \frac{(w+1)(2w+1)}{18} E_4 f_w - \partial_{w-4}^2 f_w \right), \\
    f_{w+6} &= \frac{(5w+12)^4 (5w+24)^4}{573309828(w+1)(w+2)^5 (w+3)^3 (w+4)(w+5)} \\
    &\quad \times \left( 17w^2 + 78w + 96 \right) \partial_{w-4}^3 f_w - \cdots, \\
    f_{w+8} &= \frac{(5w+12)^4 (5w+24)^4}{26748301444768(w+1)(w+2)^8 (w+3)^5 (w+4)(w+5)(w+6)^4 (w+7)} \\
    &\quad \times \left( -1313w^6 + \cdots + 2177280 \right) \partial_{w-4}^4 f_w - \cdots, \\
    f_{w+10} &= \frac{(5w+12)^4 (5w+24)^4 (5w+36)^4}{1283918464548864(w+1)(w+2)^{10} (w+3)^5 (w+4)(w+5)(w+6)^4 (w+7)(w+8)} \\
    &\quad \times \left( 293w^4 + 4332w^3 + 22968w^2 + 51192w + 40824 \right) E_4 \partial_{w-4}^5 f_w - \cdots
\end{align*}
\]

are normalised extremal quasimodular forms of weights \(w+2, \ldots, w+10\) (the omitted terms indicated by “…” are given in (69)).

As an immediate consequence of Propositions 5–8, we obtain the following fact conjectured in [16].

**Theorem 6.** The denominators of the coefficients of the normalised extremal quasimodular forms of depth \(\leq 4\), are divisible only by primes \(< w\).

**Appendix A. Coefficients of the operators for depth 4**

In this appendix we collect the more elaborate formulas used especially in Section 6.4. The polynomials \(p_0, \ldots, p_5\) in equations (73)–(78) and the modular forms \(C_{40}, \ldots, C_{48}\) in equations (79)–(83) were computed with the help of Mathematica.
\[ p_0(w) = 53567w^{14} + 4499628w^{13} + 173318340w^{12} + 4055616864w^{11} + 64374205218w^{10} + 732790207224w^9 + 616510065840w^8 + 38914973459904w^7 \\
+ 185044363180416w^6 + 659055640624128w^5 + 172905893794176w^4 + 3237068849283072w^3 + 4084118362128384w^2 + 3105388005949440w \\
+ 107271835180800 \] (73)

\[ p_1(w) = 21257w^{11} + 1465884w^{10} + 45186990w^9 + 9759703548w^7 + 79588527156w^6 + 453687847200w^5 + 1804779218520w^4 \\
+ 4900200364800w^3 + 8628400143360w^2 + 8845395333120w + 3990767616000 \] (74)

\[ p_2(w) = 2662740w^{16} + 224120550w^{15} + 6848003840w^{14} + 202621853220w^{13} + 36586266504480w^{11} + 30665823496380w^{10} \\
+ 191356528986240w^9 + 8970889439482816w^8 + 3086647757195600w^7 + 75319919247624192w^6 + 11866493675630592w^5 \\
+ 8329602154783136w^4 - 82769055163929352w^3 - 25879055163929352w^2 - 245119018746249216w - 86822757140004864 \] (75)

\[ p_3(w) = 4272785w^{17} + 351970350w^{16} + 13234823080w^{15} + 300533087760w^{14} + 459260872932w^{13} + 49787752253076w^{12} + 39286825495684w^{11} \\
+ 227486661846720w^{10} + 959711895486912w^9 + 2878990167644544w^8 + 58741997991303168w^7 + 790176901335181056w^6 \\
+ 10007199924086912w^5 + 278562611915585984w^4 + 779359222970449920w^3 + 1260737947219525632w^2 + 105465307357366816w + 3557360617012592 \] (76)

\[ p_4(w) = 517135w^{17} + 40772970w^{16} + 1455719580w^{15} + 31076826800w^{14} + 44103424168w^{13} + 4375275488634w^{12} + 3108479608256w^{11} \\
+ 1600907863104w^{10} + 608772267089664w^9 + 183142879379392w^8 + 5229385586024448w^7 + 1577597750304768w^6 + 40287913631023104w^5 \\
+ 57115900062203904w^4 - 19258645489385742w^3 - 22428530886564864w^2 - 34361693472345928w - 182090547421249536 \] (77)

\[ p_5(w) = 531441w^{13} + 36690686w^{12} + 1135566168w^{11} + 20680195920w^{10} + 247548700336w^9 + 2043291298652w^8 + 11897624359104w^7 + 4918566645388w^6 \\
+ 143692776009216w^5 + 293687697411072w^4 + 418695721574400w^3 + 42653249928064w^2 + 316421756411904w + 135523565862912. \] (78)
\[ C_{40} = 2880 \left( w^2 - 12w + 27 \right) \left( 26359w^{14} - 2214156w^{13} + 85087955w^{12} - 1981432728w^{11} + 31214109018w^{10} - 351608948568w^9 + 2918019038293w^8 - 18107342458608w^7 + 84336226011558w^6 - 293055231675096w^5 + 74693804608792w^4 - 1352381369540544w^3 + 1642191216192000w^2 - 1195772294131200w + 393589456128000 \right) E_4 E_6^6 + \frac{45}{2} \left( 79223933w^{16} - 760547568w^{15} + 33767250615w^{14} - 9198636642300w^{13} + 171985015024288w^{12} - 233876897600320w^{11} + 2391364697149624w^{10} - 18742121239029760w^9 + 1137138517815826176w^8 - 535537663756038784w^7 + 194995500452841088w^6 - 542824755426850752w^5 + 113193191971235304w^4 - 17084233956890080512w^3 + 175944268854662529024w^2 + 31829998802618548224 \right) E_7 E_8^2 + \frac{9}{2} \left( 5190001w^{10} - 4982401824w^9 + 22122681745w^8 - 602740068900w^7 + 112718884720404w^6 - 1533303103011000w^5 + 1568391042924776w^4 - 122977017767207520w^3 + 74651198878954304w^2 - 3517545090663659520w^1 + 12814220784607361280w^0 - 35687146233130066944w^1 + 7443730389344393184w^0 \right) E_4 E_8^4 + \frac{9}{2} \left( 5782232065w^{16} - 555094278240w^{15} + 24644076043200w^{14} - 671264352864000w^{13} + 12548411370416640w^{12} - 170603540496691200w^{11} + 17439135674707202880w^{10} - 13663290082790784000w^9 + 8286895211175704320w^8 - 390124920526971863040w^7 + 1419971443258483998720w^6 - 3951669805653747370240w^5 + 8238643867091026575360w^4 - 124344665529068469800960w^3 + 12809293284994355036160w^2 - 8045822842460386099200w + 2321594527761517510656 \right) E_6^4 \]
\[
C_{42} = \frac{1}{512} \left( 28531797265w^{17} - 2767249697185w^{16} + 12430670493720w^{15} - 343209882778080w^{14} + 65176240693977120w^{13} \\
- 902586178757415360w^{12} + 94292036530667080w^{11} - 7582028600125952000w^{10} + 474523918515578357760w^9 \\
- 2321707796212305162240w^8 + 886784591197522295680w^7 - 26250234048546423275520w^6 + 5937956548182225828256w^5 \\
- 10028750958268215851401480w^4 + 121837232013953539276800w^3 - 100036267763074313748480w^2 \\
+ 49373556262052078103680w - 1096977079319657803776 \right) \cdot E_4^{2} E_6 + \frac{15}{12} \left( 1163484751w^{17} - 112856838504w^{16} + 5070769249855w^{15} - 140055576225720w^{14} + 266115783570438w^{13} \\
- 368792321705000w^{12} + 38564356618660224w^{11} - 310476547784411520w^{10} + 19461087664110921728w^9 \\
- 95397247219725824w^8 + 3652097184623397440w^7 - 1084063809755019441276w^6 + 246027390426188440064w^5 \\
- 4171277972304558931968w^4 + 5090325905990328383536w^3 - 4200806019645381279744w^2 + 2085099726008498061312w \\
- 465981816686361182208 \right) \cdot E_6^{2} E_3 + 3 \left( 59652285w^{17} - 5787076015w^{16} + 260099656175w^{15} - 7187596573325w^{14} + 13666636404880w^{13} - 1895868881231940w^{12} \\
+ 19850576141606160w^{11} - 160077510633435000w^{10} + 1005448818786370020w^9 - 4941095849873520120w^8 + 18973862172105675840w^7 \\
- 565280805160938382400w^6 + 128841634960315258752w^5 - 2195443573790026006080w^4 + 269458849731356651520w^3 \\
- 23803434829495398400w^2 + 111855694290511822848w - 25177794683564851200 \right) \cdot E_7^{2} E_4 + 120 \left( 21257w^{17} - 2062778w^{16} + 92763360w^{15} - 2565735040w^{14} + 48848958061w^{13} - 678845453898w^{12} + 7124298569246w^{11} - 5762105111720w^{10} \\
+ 363254019564612w^9 - 1793230468030980w^8 + 6923704540090248w^7 - 20761290287314920w^6 + 4768120453380720w^5 - 8195631029191844w^4 \\
+ 101570826817332296w^3 - 85256494285271040w^2 + 4308478067865000w - 9805316032512000 \right) \cdot E_6^7
\]
\[
C_{44} = \frac{1}{242} \left( 56490942435w^{18} - 574145660265w^{17} + 27210408942670w^{16} - 797473001419560w^{15} + 16188499652022320w^{14} - 241595348932913040w^{13} + 274575557618982320w^{12} - 24286261064854871040w^{11} + 169405063961957458432w^{10} - 938429115674081925120w^9 + 4136645160744791040000w^8 - 1447098700702782578680w^7 + 39867414308202588041260w^6 - 8531692540371471630336w^5 + 13871213617398221271040w^4 - 1635669205696631924551680w^3 + 136184996014666316513280w^2 - 6917932937614268192640w + 1632713379795811989504 \right) E_8^4 E_6^4 + \\
\frac{1}{54} \left( 208766802w^{18} - 20738706663w^{17} + 95761061114w^{16} - 27294236030727w^{15} + 53782163388004w^{14} - 777587655849948w^{13} + 8544305359966780w^{12} - 72930404535384352w^{11} + 489965034544430160w^{10} - 260951375555730257504w^9 + 110418717465901382976w^8 - 37031590893799143936w^7 + 977199822897307158528w^6 - 2002217065285338178560w^5 + 3117189914871002634240w^4 - 35612078020870173120w^3 + 28143293915032053540w^2 - 1374584400000255337984w + 312699758864962682880 \right) E_8^4 E_6^4 + \\
+ \left( 12532755w^{18} - 1228914345w^{17} + 55930244285w^{16} - 1568787184755w^{15} + 30362687859910w^{14} - 430260670247520w^{13} + 4622655124809540w^{12} - 38470537186614840w^{11} + 25121107502837014w^{10} - 129586432043809260w^9 + 5290237710212019760w^8 - 17044686985346563680w^7 + 4301394131829395456w^6 - 83886321769065465408w^5 + 1237187422294774200w^4 - 13328666284184299008w^3 + 98925565236150067200w^2 - 45228008659406118912w + 960919638953863680 \right) E_8^4 E_6^4 + \\
+ \left( 382542395w^{18} - 104423413845w^{17} + 10600504613695w^{16} - 519813084715w^{15} + 5621227460265w^{14} - 17869147986785w^{13} + 1079152909161055w^{12} - 632588806695450w^{11} + 46674079571890978w^{10} - 27277230862661490w^9 + 1264955711494116660w^8 - 461445797912537120w^7 + 13369905344637454464w^6 - 29817133546427702208w^5 + 5034845969983539840w^4 - 62124101951090918400w^3 + 52769732557888954368w^2 - 27556002547299164160w + 6664708975492582088 \right) E_8^4 E_6^4 \right)
\]
\[ C_{16} = \frac{(w-3)E_4 E_6}{24576} \left( 16\left(5935658055w^{18} - 60413068965w^{17} + 28644817180790w^{16} - 840407590446360w^{15} + 17090822555729336w^{14} - 255749220384903192w^{13} + 2917535343452941424w^{12} - 25935973597843047168w^{11} + 182093301447056161536w^{10} - 101703389895633205248w^9 + 45288288587682666256w^8 - 16038621164741078679552w^7 + 4483503478019289409344w^6 - 97590317268828859711488w^5 + 1617702554323054828032w^4 - 1970741037785235301324w^3 + 16617636364625804587008w^2 - 8656763297344309395456w + 2007297908173738524672 \right) E_4^2 E_6^2 + 512 \left( 35364885w^{18} - 4591508505w^{17} + 225948191030w^{16} - 6896125710270w^{15} + 146241363847352w^{14} - 2287548585907344w^{13} + 27345401226776308w^{12} - 25532979535427776w^{11} + 188773674575447016w^{10} - 1112745637158347328w^9 + 5240976036830260176w^8 - 196716615900284178432w^7 + 583891074756704448000w^6 - 135158686033677955584w^5 + 23856689003430076535808w^4 - 309751241580910210816w^3 + 2785135409919820431360w^2 - 1547183341402221772800w + 39552833743361771520 \right) E_4^4 E_6^4 \] + \left( 35224547665w^{18} - 3533943045270w^{17} + 16195191875370w^{16} - 4757220455686080w^{15} + 94946288308939008w^{14} - 1391879999449512576w^{13} + 15524090529690134912w^{12} - 1364232412052963823104w^{11} + 919695494473819799552w^{10} - 49840155975802493376w^9 + 21464444623798611689472w^8 - 732469409666966519660544w^7 + 196469687001100291670016w^6 - 408349070730564361519104w^5 + 642734721946789924700160w^4 - 73866389322361661868320w^3 + 583143789019164227665920w^2 - 281900069458357354758144w + 62720415667791014658048 \right) E_4^8 \] + 4096 \left( 21385w^{18} - 3193005w^{17} + 208163655w^{16} - 806148895w^{15} + 210389101452w^{14} - 3957360451044w^{13} + 5584317936048w^{12} - 606173995705176w^{11} + 5140474585608580w^{10} - 3435902597735320w^9 + 181571882448971520w^8 - 757327716121328640w^7 + 2475717370348610880w^6 - 6259999331043132480w^5 + 11971955047567243392w^4 - 16719922187261236224w^3 + 16057775935936217088w^2 - 9464734983391911936w + 2576906536236318720 \right) E_6^8 \)
\[ C_{48} = \left( \frac{w-4}{w+4} \right) \left( 16 \cdot 2314158061 w^{16} - 22259070574 w^{15} + 1015610960565 w^{14} - 281276718421650 w^{13} + 5354355751604718 w^{12} - 74250742207449696 w^{11} + 775561858607479616 w^{10} - 6221336083117357440 w^9 + 38716279294460726016 w^8 - 18745490457102346444 w^7 + 703499359272507740160 w^6 - 2024100636419312971776 w^5 + 437539017386854600704 w^4 - 6867228800076590481408 w^3 + 7378429565839006236672 w^2 - 4847796658391713579008 + 1467072833849567903232 \right) E_6^3 E_6^2 \]

\[ + 256 \left( 372768821 w^{16} - 35518352589 w^{15} + 1564351553185 w^{14} - 42250630382915 w^{13} + 782728613878590 w^{12} - 10539973948217796 w^{11} + 10664213440104444 w^{10} - 826431392449118340 w^9 + 4953943715462672760 w^8 - 23029565126880689280 w^7 + 82687822898011670016 w^6 - 22672406051812308224 w^5 + 465078193737971627520 w^4 - 689454957866260193280 w^3 + 696182683156551475200 w^2 - 247583296582966296576 w + 120287920179051823104 \right) E_6^3 E_6^2 \]

\[ + 8192 \left( 2457393 w^{16} - 236482912 w^{15} + 10518088690 w^{14} - 286781981590 w^{13} + 5360469353751 w^{12} - 72765839079438 w^{11} + 741255904301260 w^{10} - 577351886309360 w^9 + 347022058059736 w^8 - 16125875214334112 w^7 + 576389068472688264 w^6 - 156508822375950920 w^5 + 3157725662801879472 w^4 - 45643558505896435648 w^3 + 4443610057900592640 w^2 - 2593014575868933120 w + 67965900571883520 \right) E_6^3 E_6^2 \]

\[ + 3 \left( 177147 w^{16} - 494579648 w^{15} + 374950146040 w^{14} - 15829704501920 w^{13} + 408049598642272 w^{12} - 7176728984768912 w^{11} + 911594198565376 w^{10} - 863556865787120640 w^9 + 620750445150781952 w^8 - 34121559189028110336 w^7 + 143531014974408622080 w^6 - 4563192720061953603072 w^5 + 108040295478832895152 w^4 - 184110895214567306032 w^3 + 21312302194096553502848 w^2 - 149821029075330022912 w + 42082510332022719744 \right) E_6^{12} \]

\[ + 65536 \left( 16 w^{16} - 22569 w^{15} + 1981200 w^{14} - 81695870 w^{13} + 206609124 w^{12} - 35382892521 w^{11} + 436987914760 w^{10} - 400175045380 w^9 + 27614352153240 w^8 - 14446510096140 w^7 + 57162815749656 w^6 - 169223576264444 w^5 + 366913984800560 w^4 - 5618633951049120 w^3 + 5708545872899184 w^2 - 34178058682113056 w + 899813453132160 \right) E_6^8 \]
References

1. Y. Arike, M. Kaneko, K. Nagatomo, and Y. Sakai, Affine vertex operator algebras and modular linear differential equations, Lett. Math. Phys. 106 (2016), no. 5, 693–718.
2. B. C. Berndt and M. I. Knopp, Hecke’s theory of modular forms and Dirichlet series, Monographs in Number Theory, vol. 5, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.
3. J. H. Bruinier, G. van der Geer, G. Harder, and D. Zagier, The 1-2-3 of modular forms, Universitext, Springer-Verlag, Berlin, 2008, Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad.
4. H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. Viazovska, The sphere packing problem in dimension 24, Ann. of Math. (2) 185 (2017), no. 3, 1017–1033.
5. A. S. Feigenbaum, P. J. Grabner, and D. P. Hardin, Eigenfunctions of the Fourier Transform with specified zeros, https://arxiv.org/abs/1907.08558, July 2019.
6. C. Franc and G. Mason, Fourier coefficients of vector-valued modular forms of dimension 2, Canad. Math. Bull. 57 (2014), no. 3, 485–494.
7. F. G. Frobenius, Über die Integration der linearen Differentialgleichungen durch Reihen, J. reine angew. Math. 76 (1873), 214–235.
8. P. Henrici, Applied and computational complex analysis. Vol. 2, Wiley Interscience [John Wiley & Sons], New York-London-Sydney, 1977, Special functions—integral transforms—asymptotics—continued fractions.
9. E. Hille, Ordinary differential equations in the complex domain, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1976, Pure and Applied Mathematics.
10. H. Iwaniec, Topics in classical automorphic forms, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997.
11. M. Kaneko and M. Koike, On modular forms arising from a differential equation of hypergeometric type, Ramanujan J. 7 (2003), no. 1-3, 145–164, Rankin memorial issues.
12. M. Kaneko and D. Zagier, A generalized Jacobi theta function and quasimodular forms, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 165–172.
13. F. G. Frobenius, Über die Integration der linearen Differentialgleichungen durch Reihen, J. reine angew. Math. 76 (1873), 214–235.
14. P. Henrici, Applied and computational complex analysis. Vol. 2, Wiley Interscience [John Wiley & Sons], New York-London-Sydney, 1977, Special functions—integral transforms—asymptotics—continued fractions.
15. E. Hille, Ordinary differential equations in the complex domain, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1976, Pure and Applied Mathematics.
16. H. Iwaniec, Topics in classical automorphic forms, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997.
17. M. Kaneko and M. Koike, On modular forms arising from a differential equation of hypergeometric type, Ramanujan J. 7 (2003), no. 1-3, 145–164, Rankin memorial issues.
18. M. Kaneko and D. Zagier, A generalized Jacobi theta function and quasimodular forms, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 165–172.
19. A. S. Feigenbaum, P. J. Grabner, and D. P. Hardin, Eigenfunctions of the Fourier Transform with specified zeros, https://arxiv.org/abs/1907.08558, July 2019.
20. C. Franc and G. Mason, Fourier coefficients of vector-valued modular forms of dimension 2, Canad. Math. Bull. 57 (2014), no. 3, 485–494.
21. F. G. Frobenius, Über die Integration der linearen Differentialgleichungen durch Reihen, J. reine angew. Math. 76 (1873), 214–235.
22. P. Henrici, Applied and computational complex analysis. Vol. 2, Wiley Interscience [John Wiley & Sons], New York-London-Sydney, 1977, Special functions—integral transforms—asymptotics—continued fractions.
23. E. Hille, Ordinary differential equations in the complex domain, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1976, Pure and Applied Mathematics.
24. H. Iwaniec, Topics in classical automorphic forms, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997.
25. M. Kaneko and M. Koike, On modular forms arising from a differential equation of hypergeometric type, Ramanujan J. 7 (2003), no. 1-3, 145–164, Rankin memorial issues.
26. M. Kaneko and D. Zagier, A generalized Jacobi theta function and quasimodular forms, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 165–172.
27. A. S. Feigenbaum, P. J. Grabner, and D. P. Hardin, Eigenfunctions of the Fourier Transform with specified zeros, https://arxiv.org/abs/1907.08558, July 2019.
19. K. Kawasetsu and Y. Sakai, *Modular linear differential equations of fourth order and minimal W-algebras*, J. Algebra 506 (2018), 445–488.

20. M. Knopp and G. Mason, *On vector-valued modular forms and their Fourier coefficients*, Acta Arith. 110 (2003), no. 2, 117–124.

21. , *Vector-valued modular forms and Poincaré series*, Illinois J. Math. 48 (2004), no. 4, 1345–1366.

22. , *Logarithmic vector-valued modular forms*, Acta Arith. 147 (2011), no. 3, 261–262.

23. , *Logarithmic vector-valued modular forms and polynomial-growth estimates of their Fourier coefficients*, Ramanujan J. 29 (2012), no. 1-3, 213–223.

24. S. Lang, *Introduction to modular forms*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 222, Springer-Verlag, Berlin, 1995, With appendixes by D. Zagier and Walter Feit, Corrected reprint of the 1976 original.

25. C. Marks and G. Mason, *Structure of the module of vector-valued modular forms*, J. Lond. Math. Soc. (2) 82 (2010), no. 1, 32–48.

26. G. Mason, *Vector-valued modular forms and linear differential operators*, Int. J. Number Theory 3 (2007), no. 3, 377–390.

27. , *2-dimensional vector-valued modular forms*, Ramanujan J. 17 (2008), no. 3, 405–427.

28. , *On the Fourier coefficients of 2-dimensional vector-valued modular forms*, Proc. Amer. Math. Soc. 140 (2012), no. 6, 1921–1930.

29. E. Royer, *Quasimodular forms: an introduction*, Ann. Math. Blaise Pascal 19 (2012), no. 2, 297–306.

30. G. Shimura, *Modular forms: basics and beyond*, Springer Monographs in Mathematics, Springer, New York, 2012.

31. W. Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, vol. 79, American Mathematical Society, Providence, RI, 2007, With an appendix by Paul E. Gunnells.

32. G. Teschl, *Ordinary differential equations and dynamical systems*, Graduate Studies in Mathematics, vol. 140, American Mathematical Society, Providence, RI, 2012.

33. M. S. Viazovska, *The sphere packing problem in dimension 8*, Ann. of Math. (2) 185 (2017), no. 3, 991–1015.

34. T. Yamashita, *On a construction of extremal quasimodular forms of depth two*, Master’s thesis, Tsukuba University, 2010, Japanese.

35. D. Zagier, *Modular forms and differential operators*, Proc. Indian Acad. Sci. Math. Sci. 104 (1994), no. 1, 57–75, K. G. Ramanathan memorial issue.

36. , *Elliptic modular forms and their applications*, in *The 1-2-3 of modular forms* [29], pp. 1–103.

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