Weak measurement of arrival time

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The arrival time probability distribution is defined by analogy with the classical mechanics. The difficulty of requirement to have the values of non-commuting operators is circumvented using the concept of weak measurements. The proposed procedure is suitable to the free particles and to the particles subjected to an external potential, as well. It is shown that such an approach imposes an inherent limitation to the accuracy of the arrival time determination.

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I. INTRODUCTION

Time plays a special role in quantum mechanics. Unlike other observables, time remains a classical variable. It is well-known, that the self-adjoint operator of time does not exist for bounded Hamiltonians. The problems with the time rise from the fact that in quantum mechanics many quantities cannot have definite values simultaneously. However, the detection of the particles in time-of-flight and coincidence experiments are common, and quantum mechanics should give the method for the calculation of the arrival time. The arrival time distribution may be useful solving the tunneling time problem, as well. Therefore, the quantum description of the arrival time has attracted much attention [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Aharonov and Bohm introduced the arrival time operator [1]

$$\hat{T}_{AB} = \frac{m}{2} \left( (X - \hat{x}) \frac{1}{p} + \frac{1}{p}(X - \hat{\dot{x}}) \right).$$  (1)

By imposing several conditions (normalization, positivity, minimum variance, and a symmetry with respect to the arrival point $X$) a quantum arrival time distribution for the free particle has been obtained by Kijowski [2]. The Kijowski’s distribution may be associated with the positive operator valued measure generated by the eigenstates of $\hat{T}_{AB}$. However, Kijowski’s set of conditions cannot be applied in a general case [2]. Nevertheless, the arrival time operators can be built even if the particle is not free [15, 16].

In this paper we take another approach. Since the mean arrival time even in the classical mechanics can be infinite or the particle may not arrive at all, it is convenient to deal not with the mean arrival time and corresponding operator $\hat{T}$, but with the probability distribution of the arrival time. The probability distribution of the arrival time can be obtained from the suitable classical definition. The non-commutativity of the operators in quantum mechanics is circumvented by using the concept of weak measurements. Such an approach has several advantages. It gives, in principle, the procedure of measuring the obtained quantity. Since in the classical mechanics all quantities can have definite values simultaneously, weak measurements give the correct classical limit. The concept of weak measurements has been already applied to the time problem in quantum mechanics [17].

We proceed as follows. In Sec. II we discuss various definitions of the arrival time in the classical mechanics. The weak measurement of the quantum arrival time distribution is presented in Sec. III. The properties of the obtained quantity are analyzed in Sec. IV. Section V summarizes our findings.

II. ARRIVAL TIME IN CLASSICAL MECHANICS

In classical mechanics the particle moves along the trajectory $H(x, p) = \text{const}$ as $t$ increases. This allows us to work out the time of arrival at the point $x(t) = X$, by identifying the point $(x_0, p_0)$ of the phase space where the particle is at $t = 0$, and then following the trajectory that passes by this point, up to the arrival at the point $X$. If multiple
crossings are possible, one may define a distribution of arrival times with contributions from all crossings, when no distinction is made between first, second and $n$th arrivals. In this article we will consider such a distribution.

We can ask, whether there is a definition of the arrival time, which can be valid both in classical and quantum mechanics. In our opinion, the words “the particle arrives from the left at the point $X$ at the time $t$” mean that: (i) at time $t$ the particle has been in the region $x < X$ and (ii) at time $t + \Delta t$ ($\Delta t \to 0$) the particle has been found in the region $x > X$. Now we apply this definition, given by (i) and (ii), to the time of arrival in the classical case.

Since quantum mechanics deals with probabilities, it is convenient to use probabilistic description of the classical mechanics, as well. Therefore, we will consider an ensemble of non interacting classical particles. The probability density in the phase space is $\rho(x, p; t)$.

Let us denote the region $x < X$ as $\Gamma_1$ and the region $x > X$ as $\Gamma_2$. The probability that the particle arrives from the region $\Gamma_1$ to the region $\Gamma_2$ at the time between $t$ and $t + \Delta t$ is proportional to the probability that the particle is in the region $\Gamma_1$ at time $t$ and it is in the region $\Gamma_2$ at time $t + \Delta t$. This probability is

$$\Pi_+(t) \Delta t = \frac{1}{N_+} \int_{\Omega} dp dx \rho(x, p; t),$$

where $N_+$ is the constant of normalization and the region of the phase space $\Omega$ has the following properties: (i) the coordinates of the points in $\Omega$ are in the space region $\Gamma_1$ and (ii) if the phase trajectory goes through the point of the region $\Omega$ at time $t$ then the particle at time $t + \Delta t$ is in the space region $\Gamma_2$. Since $\Delta t$ is infinitesimal, the change of the coordinate during the time interval $\Delta t$ is equal to $\frac{\Delta t}{m}\Delta t$. Therefore, the particle arrives from the region $\Gamma_1$ to the region $\Gamma_2$ only if the momentum of the particle in the point $X$ is positive. The phase space region $\Omega$ consists from the points with positive momentum $p$ and with coordinates between $X - p/m\Delta t$ and $X$. Then from Eq. (2) we have the probability of arrival time

$$\Pi_+(t) \Delta t = \frac{1}{N_+} \int_{0}^{\infty} dp \int_{X - \frac{p}{m}\Delta t}^{X} dx \rho(x, p; t).$$

Since $\Delta t$ is infinitesimal and the momentum of every particle is finite, we can replace $x$ in Eq. (3) by $X$ and obtain the equality

$$\Pi_+(t, X) = \frac{1}{N_+} \int_{0}^{\infty} dp \frac{p}{m} \rho(X, p; t).$$

The obtained arrival time distribution $\Pi_+(t, X)$ is well-known and has appeared quite often in the literature (see, e.g., review [13] and references therein).

The probability current in classical mechanics is

$$J(x; t) = \int_{-\infty}^{+\infty} \frac{p}{m}\rho(x, p; t) dp.$$

From Eqs. (4) and (5) it is clear that the time of arrival is related to the probability current. This relation, however, is not straightforward. We can introduce the “positive probability current”

$$J_+(x; t) = \int_{0}^{\infty} \frac{p}{m}\rho(x, p; t) dp$$

and rewrite Eq. (4) as

$$\Pi_+(t, X) = \frac{1}{N_+} J_+(X; t).$$

The proposed [14, 18, 19] various quantum versions of $J_+$, even in the case of the free particle can be negative (the so-called backflow effect). Therefore, classical expression (7) for the time of arrival becomes problematic in quantum mechanics.

Similarly, for the arrival from the right we obtain the probability density

$$\Pi_-(t, X) = \frac{1}{N_-} J_-(X; t),$$

where the negative probability current is

$$J_-(x; t) = \int_{-\infty}^{0} \frac{|p|}{m}\rho(x, p; t) dp.$$
We see that our definition, given at the beginning of this section, leads to the proper result in the classical mechanics. The conditions (i) and (ii) does not involve the concept of the trajectories. We can try to use this definition also in quantum mechanics.

III. WEAK MEASUREMENT

The proposed definition of the arrival time probability distribution can be used in the quantum mechanics only if the determination of the region, in which the particle is, does not disturb the motion of the particle. This can be achieved using the weak measurements of Aharonov, Albert and Vaidman [20, 21, 22, 23, 24, 25]. We have the detector in the initial state $|\Phi\rangle$. The detector interacts with the particle only in the region $\Gamma_1$. In order the weak measurements can provide the meaningful information, the measurements must be performed on an ensemble of identical systems. Each system with its own detector is prepared in the same initial state. The readings of the detectors are collected and averaged.

We take the operator describing the interaction between the particle and the detector of the form[17, 26, 27]

$$\hat{H}_I = \lambda \hat{q} \hat{p}_1,$$

(10)

where $\hat{P}_1$ is the projection operator, projecting into the region $\Gamma_1$ and $\lambda$ characterizes the strength of the interaction with the detector. The interaction operator (10) only slightly differs from the one used by Aharonov, Albert and Vaidman [21]. The similar interaction operator has been considered by von Neumann [28] and has been widely used in the strong measurement models (e.g., [24, 30, 31, 32, 33, 34] and many others).

The measurement time is $\tau$. We assume that the interaction strength $\lambda$ and the time $\tau$ are small. A very small parameter $\lambda$ ensures the undisturbance of the particle’s evolution. The operator $\hat{q}$ acts in the Hilbert space of the detector. We require a continuous spectrum of the operator $\hat{q}$. For simplicity, we can consider this operator as the coordinate of the detector. The momentum, conjugate to $q$, is $p_q$.

Since interaction strength $\lambda$ and the duration of the measurement $\tau$ are small, the probability $\langle \hat{P}_1 \rangle$ to find the particle in the region $\Gamma_1$ does not change significantly during the measurement. The action of the Hamiltonian (10) results in the small change of the mean detector’s momentum $\langle \hat{p}_q \rangle - \langle \hat{p}_q \rangle_0 = -\lambda \tau \langle \hat{P}_1 \rangle$, where $\langle \hat{p}_q \rangle_0 = \langle \Phi(0) | \hat{p}_q | \Phi(0) \rangle$ is the mean momentum of the detector at the beginning of the measurement and $\langle \hat{p}_q \rangle = \langle \Phi(\tau) | \hat{p}_q | \Phi(\tau) \rangle$ is the mean momentum of the detector after the measurement. Therefore, in analogy to Ref. [21], we define the “weak value” of the probability to find the particle in the region $\Gamma_1$,

$$W(1) \equiv \langle \hat{P}_1 \rangle = \frac{\langle \hat{p}_q \rangle_0 - \langle \hat{p}_q \rangle}{\lambda \tau}.$$  

(11)

In order to obtain the arrival time probability using the definition from Sec. III we measure the momenta $p_q$ of each detector after the interaction with the particle. After the time $\Delta t$ we perform the final, postselection measurement on the particles of our ensemble and measure if the particle is found in the region $\Gamma_2$. Then we collect the outcomes $p_q$ only for the particles found in the region $\Gamma_2$.

The projection operator projecting into the region $\Gamma_2$ is $\hat{P}_2$. In the Heisenberg representation this operator is

$$\hat{P}_2(t) = \hat{U}(t)^\dagger \hat{P}_2 \hat{U}(t),$$

(12)

where $\hat{U}$ is the evolution operator of the free particle. After the measurement the state of the particle and the detector is $\hat{U}_M(\tau) |\Phi\rangle |\Psi\rangle$, where $|\Psi\rangle$ is the initial state of the particle and $\hat{U}_M$ is the evolution operator of the particle interacting with the detector. The joint probability that the detector has the momentum $p_q$ and the particle after the time $\Delta t$ is found in the region $\Gamma_2$ is

$$W(p_q, 2) = \langle \Psi | \langle \Phi | \hat{U}_M(\tau)^\dagger |p_q\rangle \langle p_q \hat{P}_2(\Delta t) \hat{U}_M(\tau) | \Phi \rangle |\Psi\rangle,$$

(13)

where $|p_q\rangle$ is the eigenfunction of the momentum operator $\hat{p}_q$. $\hat{P}_2$ is the projection operator, projecting into the region $\Gamma_2$. In quantum mechanics the probability that two quantities simultaneously have definite values does not always exist. If the joint probability does not exist then the concept of the conditional probability is meaningless. However, in our case operators $\hat{P}_2$ and $|p_q\rangle \langle p_q |$ act in different spaces and commute, therefore, the joint probability $W(p_q, 2)$ exists.

Let us define the conditional probability, i.e., the probability that the momentum of the detector is $p_q$, provided that the particle after the time $\Delta t$ is found in the region $\Gamma_2$. This probability is given according to the Bayes’s theorem

$$W(p_q|2) = \frac{W(p_q, 2)}{W(2)},$$

(14)
where $W(2)$ is the probability that the particle after the time $\Delta t$ is found in the region $\Gamma_2$. The average momentum of the detector on condition that the particle after the time $\Delta t$ is found in the region $\Gamma_2$ equals to
\begin{equation}
\langle \hat{p}_q \rangle_2 = \int p_q W(p_q | 2) dp_q = \frac{1}{W(2)} \langle \Psi | \hat{U}_M(\tau) \hat{P}_2(\Delta t) \hat{U}_M(\tau) | \Phi \rangle | \Psi \rangle,
\end{equation}
where $\hat{P}_2(\Delta t) = \hat{U}(\Delta t) \hat{P}_2 \hat{U}(\Delta t)$ is the operator $\hat{P}_2$ in the Heisenberg picture. According to Eq. (11)
\begin{equation}
W(1|2) = \frac{\langle \hat{p}_q \rangle_0 - \langle \hat{p}_q \rangle_2}{\lambda \tau} = \frac{\langle \hat{p}_q \rangle_0 W(2) - \langle \hat{p}_q \rangle_2 W(2)}{\lambda \tau W(2)}
\end{equation}
is the weak value of probability to find the particle in the region $\Gamma_1$ on condition that the particle after the time $\Delta t$ is in the region $\Gamma_2$. The probability that the particle is in the region $\Gamma_1$ and after the time $\Delta t$ is in the region $\Gamma_2$ equals to
\begin{equation}
W(1, 2) = W(2)W(1|2).
\end{equation}
When the measurement time $\tau$ is sufficiently small, the influence of the Hamiltonian of the particle can be neglected and the evolution operator of the particle and the detector can be expressed as
\begin{equation}
\hat{U}_M(\tau) = \exp \left(- \frac{i}{\hbar} \lambda \tau \hat{q} \hat{P}_1 \right).
\end{equation}
We expand the operator $\hat{U}_M(\tau)$ into the series of the parameter $\lambda$, assuming that $\lambda$ is small. In the first-order approximation, using Eqs. (10) and (14), we obtain
\begin{equation}
W(1, 2) \approx \frac{1}{2} \langle \hat{P}_2(\Delta t) \hat{P}_1 + \hat{P}_1 \hat{P}_2(\Delta t) \rangle + \frac{i}{\hbar} \langle \hat{p}_q \rangle | \psi \rangle - \Re \langle \hat{q} \hat{p}_q \rangle \langle \hat{P}_1, \hat{P}_2(\Delta t) \rangle.
\end{equation}
The probability $W(1, 2)$ is constructed using the conditions (i) and (ii) from Sec. II: the weak measurement is performed to determine if the particle is in the region $\Gamma_1$ and after the time $\Delta t$ the strong measurement determines if the particle is in the region $\Gamma_2$. Therefore, according to Sec. III the quantity $W(1, 2)$ after normalization can be considered as the weak value of the arrival time probability distribution.

Eq. (18) consists of two terms and we can introduce two quantities
\begin{equation}
\Pi^{(1)} = \frac{1}{2\Delta t} \langle \hat{P}_1 \hat{P}_2(\Delta t) + \hat{P}_2(\Delta t) \hat{P}_1 \rangle
\end{equation}
and
\begin{equation}
\Pi^{(2)} = \frac{1}{2i\Delta t} \langle \hat{P}_1, \hat{P}_2(\Delta t) \rangle.
\end{equation}
Then
\begin{equation}
W(1, 2) = \Pi^{(1)} \Delta t - \frac{2\Delta t}{\hbar} \langle \hat{p}_q \rangle | \langle \hat{q} \rangle - \Re \langle \hat{q} \hat{p}_q \rangle \rangle \Pi^{(2)}.
\end{equation}
If the commutator $[\hat{P}_1, \hat{P}_2(\Delta t)]$ in Eqs. (19)–(21) is not zero then, even in the limit of the very weak measurement, the measured value depends on the particular detector. This fact means that in such a case we cannot obtain the definite value for the arrival time probability. Moreover, the coefficient $\langle \hat{p}_q \rangle | \langle \hat{q} \rangle - \Re \langle \hat{q} \hat{p}_q \rangle \rangle$ may be zero for the specific initial state of the detector, e.g., for the Gaussian distribution of the coordinate $q$ and momentum $p_q$.

The quantities $W(1, 2)$, $\Pi^{(1)}$ and $\Pi^{(2)}$ are real. However, it is convenient to consider the complex quantity
\begin{equation}
\Pi_C = \Pi^{(1)} + i\Pi^{(2)} = \frac{1}{\Delta t} \langle \hat{P}_1 \hat{P}_2(\Delta t) \rangle.
\end{equation}
We name it “complex arrival probability”. We can introduce the corresponding operator
\begin{equation}
\hat{\Pi}_+ = \frac{1}{\Delta t} \hat{P}_1 \hat{P}_2(\Delta t).
\end{equation}
By analogy, the operator
\[ \hat{\Pi}_- = \frac{1}{\Delta t} \hat{P}_2 \hat{P}_1 (\Delta t). \] (24)
corresponds to the arrival from the right.

The introduced operator \( \hat{\Pi}_+ \) has some properties of the classical positive probability current. From the conditions \( \hat{P}_1 + \hat{P}_2 = 1 \) and \( \hat{P}_1 (\Delta t) + \hat{P}_2 (\Delta t) = 1 \) we have
\[ \hat{\Pi}_+ - \hat{\Pi}_- = \frac{1}{\Delta t} (\hat{P}_2 (\Delta t) - \hat{P}_2). \]
In the limit \( \Delta t \to 0 \) we obtain the probability current, \( \hat{J} = \lim_{\Delta t \to 0} (\hat{\Pi}_+ - \hat{\Pi}_-) \), as it is in the classical mechanics. However, the quantity \( \langle \hat{\Pi}_+ \rangle \) is complex and the real part can be negative, in contrary to the classical quantity \( J_+ \).

The reason for this is the non-commutativity of the operators \( \hat{\Pi}_1 \) and \( \hat{\Pi}_2 (\Delta t) \). When the imaginary part is small, the quantity \( \langle \hat{\Pi}_+ \rangle \) after normalization can be considered as the approximate probability distribution of the arrival time.

IV. ARRIVAL TIME PROBABILITY

The operator \( \hat{\Pi}_+ \) is obtained without specification of the Hamiltonian of the particle and is suitable to the free particles and to the particles subjected to an external potential as well. In this section we consider the arrival time of the free particle.

The calculation of the “weak arrival time distribution” \( W(1,2) \) involves the average \( \langle \hat{\Pi}_+ \rangle \). Therefore, it is useful to have the matrix elements of the operator \( \hat{\Pi}_+ \). It should be noted, that matrix elements of the operator \( \hat{\Pi}_+ \) as well as the operator itself are only auxiliary and does not have independent meaning.

In the basis of momentum eigenstates \( |p\rangle \), normalized according the condition \( \langle p_1 | p_2 \rangle = 2\pi \hbar \delta(p_1 - p_2) \), the matrix elements of the operator \( \hat{\Pi}_+ \) are
\[ \langle p_1 | \hat{\Pi}_+ | p_2 \rangle = \frac{1}{\Delta t} \langle p_1 | \hat{P}_1 \hat{U}(\Delta t)\hat{P}_2 \hat{U}(\Delta t) | p_2 \rangle = \frac{1}{\Delta t} \int_{-\infty}^{X} dx_1 \int_{X}^{\infty} dx_2 e^{-\frac{i}{\hbar} p_1 x_1} (x_1 \hat{U}(\Delta t)\hat{P}_2 | x_2 \rangle e^{i p_2 x_2 - \frac{\hbar^2}{2m} \Delta t}. \] (25)
After performing the integration we obtain
\[ \langle p_1 | \hat{\Pi}_+ | p_2 \rangle = \frac{i \hbar}{2\Delta t (p_2 - p_1)} \exp \left( \frac{i}{\hbar} (p_2 - p_1) X \right) \times \left( e^{\frac{\hbar}{\Delta t} (p_1^2 - p_2^2)} \text{erfc} \left( -p_1 \sqrt{\frac{\Delta t}{2 \hbar m}} \right) - \text{erfc} \left( -p_2 \sqrt{\frac{\Delta t}{2 \hbar m}} \right) \right), \] (26)
where \( \sqrt{t} = \exp(i\pi/4) \). When
\[ \frac{1}{\hbar} \frac{\Delta t}{2m} (p_1^2 - p_2^2) \ll 1, \quad p_1 \sqrt{\frac{\Delta t}{2 \hbar m}} > 1, \quad p_2 \sqrt{\frac{\Delta t}{2 \hbar m}} > 1, \]
the matrix elements of the operator \( \hat{\Pi}_+ \) are
\[ \langle p_1 | \hat{\Pi}_+ | p_2 \rangle \approx \frac{p_1 + p_2}{2m} \frac{\hbar}{2 \Delta t} \exp \left( \frac{i}{\hbar} (p_2 - p_1) X \right). \] (27)
This equation coincides with the expression for the matrix elements of the probability current operator.

From Eq. (26) we obtain the diagonal matrix elements of the operator \( \hat{\Pi}_+ \)
\[ \langle p | \hat{\Pi}_+ | p \rangle = \frac{p}{2m} \text{erfc} \left( -p \sqrt{\frac{i \Delta t}{2 \hbar m}} \right) + \frac{\hbar}{\sqrt{2 \pi m \Delta t}} e^{-\frac{\hbar^2}{2m} \Delta t}. \] (28)
The real part of the quantity \( \langle p | \hat{\Pi}_+ | p \rangle \) is shown in Fig. 1 and the imaginary part in Fig. 2.
Using the asymptotic expressions of the function erfc we obtain from Eq. (28) that

$$\lim_{p \to +\infty} \langle p | \hat{\Pi}_+ | p \rangle \to \frac{p}{m}$$

and

$$\langle p | \hat{\Pi}_+ | p \rangle \to 0, \quad \text{when} \quad p \to -\infty,$$

i.e., the imaginary part tends to zero and the real part approaches the corresponding classical value as the modulus of the momentum $|p|$ increases. Such a behaviour is evident from Figs. 1 and 2, too.

The asymptotic expressions of the function erfc are valid when the argument of the erfc is large, i.e., $|p| \sqrt{\frac{\Delta t}{2\hbar m}} > 1$ or

$$\Delta t > \frac{\hbar}{E_k}. \tag{29}$$

Here $E_k$ is the kinetic energy of the particle. The dependence of the quantity $\text{Re} \langle p | \hat{\Pi}_+ | p \rangle$ from $\Delta t$ is shown in Fig. 3. For small $\Delta t$ the quantity $\langle p | \hat{\Pi}_+ | p \rangle$ is proportional to $1/\sqrt{\Delta t}$. Therefore, unlike the classical mechanics, in quantum mechanics $\Delta t$ cannot be zero. Eq. (29) imposes the lower bound to the resolution time $\Delta t$. It follows that our model does not permit determination of the arrival time with the resolution greater than $\hbar / E_k$. The relation, similar to Eq. (29), based on measurement models has been obtained by Aharonov et al. \cite{29}. The time-energy uncertainty relations, associated with time of arrival distribution, are also discussed in Refs. \cite{8, 35}.

V. CONCLUSION

The definition of density of one sided arrivals is proposed. This definition is extended into quantum mechanics, using the concept of weak measurements by Aharonov et al. The proposed procedure is suitable to the free particles and to the particles subjected to an external potential, as well. It gives not only the mathematical expression for the arrival time probability distribution but also the way to measure the obtained quantity. However, this procedure gives no unique expression for the arrival time probability distribution.

In analogy with complex tunneling time the complex arrival time “probability distribution” is introduced (Eq. (22)). It is shown that the proposed approach imposes an inherent limitation, Eq. (29), to the resolution time of the arrival
FIG. 2: The imaginary part of the quantity $\langle p|\hat{\Pi}_+|p\rangle$. Used parameters are the same as in Fig. 11.

time determination.

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FIG. 3: The dependence of the quantity $\text{Re}(p|\Pi_+|p)$ according to Eq. (28) from the resolution time $\Delta t$. The corresponding classical positive probability current is shown with the dashed line. The used parameters are $\hbar = 1$, $m = 1$, and $p = 1$. In these units, the time $\Delta t$ is dimensionless.

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