Leggett-type $N$-partite scenarios for testing nonlocal realistic theory

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Abstract

The nonlocal realistic theory might be the last cornerstone of classical physics confronting to the quantum theory, which was found mostly untenable in the bipartite system [Nature 446, 871 (2007)]. We extend the Leggett-type nonlocal realistic model to arbitrary $N$-partite systems with polarizer settings, and obtain some stronger inequalities to distinguish quantum mechanics from nonlocal realistic theories. For illustration, with certain measurement settings the quantum violations of Leggett-type inequalities are found for Greenberger-Horne-Zeilinger (GHZ) state. Our results, say the nonlocal realism in multipartite systems, are testable in experiment.

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Quantum physics only tells us what is the probability $P(x|X)$ that the measurement of $X$ performed in the state $\rho$ will lead to the outcomes $x$:

$$P(x|X) = \text{Tr}[\rho_X],$$

where $X_x$ is the projector on the subspace associated to the measurement result $x$. For the case of a single system, it is possible to simulate the same statistics probability $P(x|X)$ as in the quantum experiment based only on the description of the state $\rho$ and of the measurement $X$ to be performed on it. Of the bipartite or multipartite system, correlations are the no-signaling (NS) probability distributions characterizing the outcomes of measurements performed on quantum states [1]. For a bipartite system, when we measure observables $X^{(1)}$ and $X^{(2)}$ respectively, we can calculate similar quantum joint probability

$$P(x_1, x_2|X^{(1)}, X^{(2)}) = \text{Tr}[\rho_{x_1 x_2}X^{(1)} \otimes X^{(2)}].$$

In attempt to reproduce this joint probability, one have ever employed various of hidden variable models in which the correlation information is preestablished by the common hidden variable $\lambda$. Based on the local hidden variable model, the joint probability can be decomposed as

$$P(x_1, x_2|X^{(1)}, X^{(2)}) = \int d\lambda \rho(\lambda) P_{\lambda}(x_1|X^{(1)}) P_{\lambda}(x_2|X^{(2)}),$$

which is the well-known Bell’s model. However, Bell model fails to reproduce the correlation containing in Eq. (2), which has been testified by the violation of Bell inequality in various different physical systems [2–9]. Note, very recently Bell-CH inequality has been generalized to test local realism in high energy physics [10]. While ample experiments confirm quantum predictions, and therefore exclude the local realistic theories to a great extent, the non-local realism is still tenable.
In 2003, Leggett introduced a class of nonlocal models, which relax the requirement of locality while still keep the realism [11], and formulated an incompatible theorem between the nonlocal realism and quantum physics in terms of the Leggett inequality. Soon after, experiments with photon were performed and shown in conflict with the Leggett model and agree with the quantum predictions [12–16]. Lately, the falsifications of Leggett model using neutron matter wave [17] and with solid state spins [18] were carried out. Very recently, the violation of Leggett inequality has been discovered in high energy physics [19, 20]. Again, which supports the statement, in the interactions which obey only quantum theory, any future extension of quantum theory that is in agreement with experiments must abandon certain features of realistic description [12]. In the literature, there are also some alternative nonlocal models constructed [21], which had been as well falsified in experiments [22, 23].

Whereas, going from two parties to three makes the situation much complicated as discussion about whether the nonlocality is genuine or weak comes into the mix. In the multipartite case, nonlocality displays much richer or more complicated structure compared to the case of bipartite. This makes the study and the characterization of multipartite nonlocal correlations an interesting, but challenging problem. Although much work has been done for multipartite Bell nonlocality [24–27] but multipartite Leggett-type nonlocality stays silent. In this work, we study $N$-partite Leggett model and build the generalized Leggett-type inequalities for $N$-qubit system which is different from bipartite [15] and tripartite case in Ref. [28]. We obtain $N$-pair inequalities which originate from constraints of realism on $N$ particles. We check for the violations for GHZ states as a specific example. Due to symmetry of GHZ state, we only consider one pair of inequalities out of all equivalent pairs. In this case, we demonstrate quantum physics violations of Leggett-type inequalities in the specific measurement settings. The letter is structured as
in the following section we give an introduction for $N$-partite Leggett model. We present
Leggett-type inequalities for $N$-partite in section 3 and then we discuss violations of
tripartite GHZ state in specific measurement settings in section 4. The letter is concluded
in the last section.

2 $N$-partite Leggett Model

Here, we consider the case of qubits which can be encoded in the polarization of pho-
tons. For this case, the observable $\mathcal{X}$ can be parameterized by unit vectors $\vec{n}$ on the
Poincaré sphere, that is, $\mathcal{X} = \vec{\sigma} \cdot \vec{n}$ where $\sigma_\mu$ are Pauli matrices. In order to formulate
$N$-partite Leggett model, it is convenient to use the notations: $x := x_1, x_2, \ldots, x_N$, $\lambda := 
\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N$, $\mathcal{X} := \mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \ldots, \mathcal{X}^{(N)}$. Here, $x_i = \{\pm 1\}$ is binary outcomes corre-
sponding to observable $\mathcal{X}^{(i)}$ of i’th particle. $\vec{u}_i$ is unit vector on the Poincaré sphere and
denotes polarization directions of i’th particle.

Leggett model [11, 12] upholds realism, while allowing for nonlocal influences. Specif-
ically, this model supposes that the source emits $N$ photons which can be described by
polarization product states $\otimes_{i=1}^{N} |\vec{u}_i\rangle$, that is, each single photon would always be in a
definite polarization state, which requires that the local expectation of the single photon
satisfies Malus’ law

$$
\langle x_i \rangle_\lambda = \langle \vec{u}_i | \mathcal{X}^{(i)} | \vec{u}_i \rangle = \langle \vec{u}_i | \vec{\sigma} \cdot \vec{n}_i | \vec{u}_i \rangle = \vec{u}_i \cdot \vec{n}_i^{(i)} .
$$

(4)
For Leggett model, the $N$-partite joint probability can be written as

$$\mathcal{P}(x|\mathcal{X}) = \int d\lambda \rho(\lambda)\mathcal{P}_\lambda(x|\mathcal{X}) ,$$  \tag{5}$$

and

$$\mathcal{P}_\lambda(x|\mathcal{X}) = \frac{1}{2^N} \left[ 1 + \sum_{i=1}^{N} x_i \mathcal{C}_\lambda^{(i)}(\mathcal{X}) + \sum_{i_1 < i_2} x_{i_1} x_{i_2} \mathcal{C}_\lambda^{(i_1,i_2)}(\mathcal{X}) + \sum_{i_1 < i_2 < i_3} x_{i_1} x_{i_2} x_{i_3} \mathcal{C}_\lambda^{(i_1,i_2,i_3)}(\mathcal{X}) + \cdots + \sum_{i_1 < i_2 \cdots < i_{N-1}} x_{i_1} x_{i_2} \cdots x_{i_{N-1}} \mathcal{C}_\lambda^{(i_1,i_2,\cdots,i_{N-1})}(\mathcal{X}) + x_1 x_2 \cdots x_N \mathcal{C}_\lambda^{(1,2,\cdots,N)}(\mathcal{X}) \right] . \tag{6}$$

Here, $\rho(\lambda)$ denotes a normalized polarization distribution function. The marginals $\mathcal{C}_\lambda^{(i)}(\mathcal{X})$, $\mathcal{C}_\lambda^{(i_1,\cdots,i_j)}(\mathcal{X})$ and correlation coefficient $\mathcal{C}_\lambda^{(1,\cdots,N)}(\mathcal{X})$ are

$$\mathcal{C}_\lambda^{(i)}(\mathcal{X}) = \sum_{x_1,\cdots,x_N} x_i \mathcal{P}_\lambda(x|\mathcal{X}) , \tag{7}$$

$$\mathcal{C}_\lambda^{(1,\cdots,N)}(\mathcal{X}) = \sum_{x_1,\cdots,x_N} x_1 \cdots x_N \mathcal{P}_\lambda(x|\mathcal{X}) , \tag{8}$$

$$\mathcal{C}_\lambda^{(i_1,\cdots,i_j)}(\mathcal{X}) = \sum_{x_1,\cdots,x_N} x_{i_1} \cdots x_{i_j} \mathcal{P}_\lambda(x|\mathcal{X}) , \tag{9}$$

where $i, j \in \{1, 2, \cdots, N\}$ and $j = 1, 2, \cdots, N - 1$. The marginals should satisfy NS condition i.e., $\mathcal{C}_\lambda^{(i)}(\mathcal{X}) = \mathcal{C}_\lambda^{(i)}(\mathcal{X}^{(i)})$ and $\mathcal{C}_\lambda^{(i_1,\cdots,i_j)}(\mathcal{X}) = \mathcal{C}_\lambda^{(i_1,\cdots,i_j)}(\mathcal{X}^{(i_1,\cdots,i_j)})$. Different from Bell’s model, we only ask that the single side means $\mathcal{C}_\lambda^{(i)}(\mathcal{X}^{(i)})$ should satisfy

$$\mathcal{C}_\lambda^{(i)}(\mathcal{X}^{(i)}) = \vec{u}_i \cdot \vec{n}^{(i)} , \tag{10}$$

which is Malus’ law and $i = 1, 2, \cdots, N$. And we have no else constraints for the correlation coefficients $\mathcal{C}_\lambda^{(i_1,\cdots,i_j)}(\mathcal{X})$ and $\mathcal{C}_\lambda^{(1,\cdots,N)}(\mathcal{X})$, which keeps the possible nonlocal correlations.
3 \textbf{$N$-partite Leggett-type inequalities}

Employing the nonnegativity of the probability $P_\lambda(x|\mathcal{X})$, we obtain the following $N$-partite Leggett-type inequalities

\begin{align}
3 \sum_{k=1}^{N} \min (|\langle \alpha_+ \rangle_k - \beta_k|, |\langle \alpha_+ \rangle_k + \beta_k|) & \leq 6 - 2|\cos(\theta/2)|, \quad (11a) \\
3 \sum_{k=1}^{N} \min (|\langle \alpha_- \rangle_k - \beta_k|, |\langle \alpha_- \rangle_k + \beta_k|) & \leq 6 - 2|\sin(\theta/2)|. \quad (11b)
\end{align}

Here,

\begin{align}
(\alpha_{\pm})_k &= \mathcal{C}(\mathcal{X}^{(i_1)}_k, \cdots, \mathcal{X}^{(i_N-1)}_k) \pm \mathcal{C}(\mathcal{X}^{(i_1)'}_k, \cdots, \mathcal{X}^{(i_N-1)'}_k), \\
(\beta_k) &= \mathcal{C}(\mathcal{X}^{(1)}_k, \cdots, \mathcal{X}^{(i)}_k, \cdots, \mathcal{X}^{(N)}_k) + \mathcal{C}(\mathcal{X}^{(1)'}_k, \cdots, \mathcal{X}^{(i)'}_k, \cdots, \mathcal{X}^{(N)'}_k),
\end{align}

where $i_1, i_2, \cdots, i_{N-1} \neq i$, $i_1 < i_2 < \cdots < i_{N-1}$, $i, i_1, i_2, \cdots, i_{N-1} \in \{1, 2, \cdots, N\}$. The detailed derivations can be found in Sect. A of Supplementary Material (SM). For each particle, we obtain one pair of inequalities like Eqs. (11a) and (11b). Therefore, we obtain $N$-pair Leggett-type inequalities for $N$-qubit system. It is notable that there only is one pair of inequalities for the totally symmetrical and totally antisymmetrical states, which is because the correlation functions $\mathcal{C}(\mathcal{X}_k)$ and $\mathcal{C}(\mathcal{X}^{(i_1)}_k, \cdots, \mathcal{X}^{(i_{N-1})}_k)$ are invariant under the interchange of any pair particles. Then, we discuss Leggett-type inequalities Eqs. (11a) and (11b) in some specific cases.

\textbf{GHZ state.} For GHZ state, if the number of particles is odd, $\mathcal{C}(\mathcal{X}^{(i_1)}_k, \cdots, \mathcal{X}^{(i_{N-1})}_k)$ is only related to the third component of $\vec{n}^{(i)}$, i.e. $\prod_{i=1}^{N-1} n^{(i)}_3$, otherwise zero. The detailed calculations can be found in Sect. B of SM. Thus, if the number of particles is even, we
have the following Leggett-type inequalities for GHZ state

\[ \sum_{k=1}^{3} |\beta_k| \leq 6 - 2|\cos(\theta/2)|, \quad (14a) \]
\[ \sum_{k=1}^{3} |\beta_k| \leq 6 - 2|\sin(\theta/2)|. \quad (14b) \]

**Bipartite case.** It is worth noting that when \( N = 2 \), we get the following inequalities for the Bell states

\[ \frac{1}{3} \sum_{k=1}^{3} |C(\vec{n}_k^{(1)}, \vec{n}_k^{(2)}) + C(\vec{n}_k^{(1)'}, \vec{n}_k^{(2)'})| \leq 2 - \frac{2}{3}|\sin(\theta/2)|, \quad (15) \]
\[ \frac{1}{3} \sum_{k=1}^{3} |C(\vec{n}_k^{(1)}, \vec{n}_k^{(2)}) + C(\vec{n}_k^{(1)'}, \vec{n}_k^{(2)'})| \leq 2 - \frac{2}{3}|\cos(\theta/2)|. \quad (16) \]

However, they are different from Leggett-type inequality in Ref. [15]. In fact, for bipartite case we assume that one photon is in a definite polarization state which requires Malus’ law and another photon has no similar constraint and its local expectation is calculated by the reduced state of the single state, while Leggett-type inequality in Ref. [15] only involve one photon satisfying Malus’ law.

\((\alpha_{\pm})_k = 0\) case. Derivation of Leggett-type inequalities depends on the choice of measurement settings. If the measurement settings satisfy \((\alpha_{\pm})_k = 0\), we obtain the following Leggett-type inequalities

\[ \mathcal{L}_+ \equiv \sum_{k=1}^{3} |\beta_k| + 2|\cos(\theta/2)| \leq 6, \quad (17a) \]
\[ \mathcal{L}_- \equiv \sum_{k=1}^{3} |\beta_k| + 2|\sin(\theta/2)| \leq 6, \quad (17b) \]

which are tighter than Eqs. (11a) and (11b) since \(\min((|\alpha_{\pm})_k - \beta_k|, |(\alpha_{\pm})_k + \beta_k|) \leq |\beta_k|\). In addition, when we take \(X^{(i_1)}_k = X^{(i_1)'}_k, \ldots, X^{(i_{N-1})}_k = X^{(i_{N-1})'}_k\), we have \((\alpha_-)_k = 0\). Thus,
we obtain the following Leggett-type inequality from Eq. (11b)

\[ \sum_{k=1}^{3} |C(X_k^{(1)}, X_k^{(2)}, \ldots, X_k^{(N)}) + C(X_k^{(1)'}, X_k^{(2)'}, \ldots, X_k^{(N)')})| + 2|\sin(\theta/2)| \leq 6 , \]  

(18)

which is just the result of Ref. [28].

4 Tripartite Leggett-type inequalities and their violations for GHZ state

For the sake of formulating tripartite Leggett-type inequalities, we consider three 6-tuples of settings \((\vec{a}_k, \vec{a}_k', \vec{b}_k, \vec{b}_k', \vec{c}_k, \vec{c}_k')\) with the same angle \(\theta \in [0, \pi]\) between all pairs \((\vec{a}_k, \vec{a}_k')\), and such that \(\vec{a}_k + \vec{a}_k' = 2\cos(\theta/2)\vec{e}_k\) and \(\vec{a}_k - \vec{a}_k' = 2\sin(\theta/2)\vec{e}_k'\), where \(\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}\) and \(\{\vec{e}_1', \vec{e}_2', \vec{e}_3'\}\) form two orthogonal bases; see Fig. 1(a) for \(\vec{a}_k, \vec{a}_k'\) and see Fig. 1(b) for \(\vec{b}_k, \vec{b}_k', \vec{c}_k, \vec{c}_k'\). In light of notations of the above section, we have \(\beta_k = C(\vec{a}_k, \vec{b}_k, \vec{c}_k) + C(\vec{a}_k', \vec{b}_k', \vec{c}_k')\) for tripartite case. Then we obtain the following Leggett-type inequalities for tripartite GHZ state

\[ \mathcal{L}_{\pm}^{\text{GHZ}} \leq 6 , \]  

(19)

where,

\[ \mathcal{L}_{+}^{\text{GHZ}} \equiv \sum_{k=1}^{3} |C(\vec{a}_k, \vec{b}_k, \vec{c}_k) + C(\vec{a}_k', \vec{b}_k', \vec{c}_k')| + 2\cos(\theta/2) , \]  

(20a)

\[ \mathcal{L}_{-}^{\text{GHZ}} \equiv \sum_{k=1}^{3} |C(\vec{a}_k, \vec{b}_k, \vec{c}_k) + C(\vec{a}_k', \vec{b}_k', \vec{c}_k')| + 2\sin(\theta/2) . \]  

(20b)

In light of the measurement settings in Fig. 1, for tripartite GHZ state quantum physics predictions \(\mathcal{L}_{\pm}^{\text{GHZ}}\) should be

\[ \mathcal{L}_{+}^{\text{GHZ}} = 2\sqrt{5}\sin(\theta_0 + \theta/2) + 2|\cos(\theta + \phi + \varphi)/2| , \]  

(21a)

\[ \mathcal{L}_{-}^{\text{GHZ}} = 2\sqrt{5}\cos(\theta_0 - \theta/2) + 2|\cos(\theta + \phi + \varphi)/2| . \]  

(21b)
FIG. 1. Measurement settings at A(lice), B(ob) and C(harlie)'s site for the test of nonlocal realism of tripartite system. All observables have been parameterized by unit vectors on the Poincaré sphere. Here, we take three 6-tuples of settings \((\vec{a}_k, \vec{a}'_k, \vec{b}_k, \vec{b}'_k, \vec{c}_k, \vec{c}'_k)\) with the same angle \(\theta \in [0, \pi]\) between all pairs \((\vec{a}_k, \vec{a}'_k)\), and such that \(\vec{a}_k + \vec{a}'_k = 2\cos(\theta/2)\vec{e}_k\) and \(\vec{a}_k - \vec{a}'_k = 2\sin(\theta/2)\vec{e}'_k\), where \(\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}\) and \(\{\vec{e}'_1, \vec{e}'_2, \vec{e}'_3\}\) form two orthogonal bases. \(\vec{b}_k, \vec{b}'_k, \vec{c}_k, \vec{c}'_k\) are arranged in the \(x-y\) plane which ensures \((\alpha_{\pm})_k = 0\).

Here, \(\theta_0 = \arctan 2\), \(\theta \in [0, \pi]\) and \(\phi, \varphi \in [0, 2\pi]\). The detailed calculations can be found in Sect. C of SM. Next, we discuss about the violations of the inequalities Eqs. (20a) and (20b). When \(\theta = 0, \pi\), we have \(\mathcal{L}^{\text{GHZ}}_{\pm} \leq 6\), and there is no violation found. And when \(\theta \neq 0, \pi\), the whole ranges of the parameters \(\phi \in [0, 2\pi]\) can violate Eq. (20a) or Eq. (20b) or both as depicted in Fig. 2. \(\mathcal{L}^{\text{GHZ}}_+\) and \(\mathcal{L}^{\text{GHZ}}_-\) can reach the maximal quantum violation \(2(\sqrt{5} + 1)\) which is more violation than result in Ref. [28], when \(\theta = \pi - 2\arctan 2\) and \(\theta = 2\arctan 2\) respectively.
When the measurement settings are arranged in accordance with Fig. 1, quantum physics violates Eqs. (21a) and (21b) obviously. Especially, violations occur for arbitrary angle $\phi \in [0, 2\pi]$ when $\theta \neq 0, \pi$. The $L_{GHZ}^+$ and $L_{GHZ}^-$ can reach the maximal quantum violation $2(\sqrt{5} + 1)$ when $\theta = \pi - 2 \arctan 2$ and $\theta = 2 \arctan 2$ respectively.

### 5 Conclusions

In pursuit of understanding nonlocality in multipartite case specifically Leggett-type nonlocality which is almost unknown, in this work, we developed Leggett-type inequalities for $N$-qubit system. It is found that the constraint originating from realism on each
particle will result in one pair of Leggett-type inequalities. As such, we obtain $N$-pair Leggett-type inequalities for $N$-qubit system. It should be noted that we only need to consider one pair of inequalities out of all equivalent pairs for totally symmetrical and totally antisymmetrical states. Moreover, in case of bipartite, we formulate one pair of inequalities which are attached with the additional constraint. As a specific example, we study multipartite Leggett model for GHZ state and formulate one pair of Leggett-type inequalities in the specific measurement settings, which can be compared with quantum physics experimentally. Quantum physics violates these inequalities obviously, which demonstrates that Leggett nonlocal model fails to reproduce quantum correlations of GHZ state. It is remarkable that the correlation function in GHZ state behaves differently for odd and the even number of particles, which may lead to different violations for Leggett-type inequalities.

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A Derivation of \(N\)-partite Leggett-type inequalities

Here, we formulate the Leggett’s nonlocal model with the following specific photon source. Suppose the source emits \(N\) photons with definite polarizations \(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N\) to \(A_1, A_2, \ldots, A_N\). According to Leggett model, the joint probability distribution for \(N\)-partite can be written as

\[
P(x|\mathcal{X}) = \int d\lambda \rho(\lambda) P_\lambda(x|\mathcal{X}),
\]

\[
P_\lambda(x|\mathcal{X}) = \frac{1}{2^N} \left[ 1 + \sum_{i=1}^{N} x_i C^{(i)}(\mathcal{X}) + \sum_{i_1 < i_2} x_{i_1} x_{i_2} C^{(i_1,i_2)}(\mathcal{X}) + \sum_{i_1 < i_2 < i_3} x_{i_1} x_{i_2} x_{i_3} C^{(i_1,i_2,i_3)}(\mathcal{X}) + \cdots + \sum_{i_1 < i_2 < \cdots < i_{N-1}} x_{i_1} x_{i_2} \cdots x_{i_{N-1}} C^{(i_1,i_2,\cdots,i_{N-1})}(\mathcal{X}) + x_1 x_2 \cdots x_N C^{(1,2,\cdots,N)}(\mathcal{X}) \right].
\]

Here, we have used the notation \(x := x_1, x_2, \ldots, x_N\), \(\mathcal{X} := \mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \ldots, \mathcal{X}^{(N)}\) and \(\lambda := \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N\). \(\rho(\lambda)\) is a normalized polarization distribution function. \(\mathcal{X}^{(i)} = \vec{\sigma} \cdot \vec{n}^{(i)}\) is measurement along \(\vec{n}^{(i)}\) for \(i\)’th photon and \(x_i = \{\pm 1\}\) are measurement results.
accordingly. And $C^{(i)}_\lambda(\mathcal{X}), C^{(i_1,i_2,\cdots,i_j)}_\lambda(\mathcal{X}), C^{(1,2,\cdots,N)}_\lambda(\mathcal{X})$ are

$$C^{(i)}_\lambda(\mathcal{X}) = \sum_{x_1,x_2,\cdots,x_N} x_i P_\lambda(x|\mathcal{X}),$$

(3)

$$C^{(1,2,\cdots,N)}_\lambda(\mathcal{X}) = \sum_{x_1,x_2,\cdots,x_N} x_1 x_2 \cdots x_N P_\lambda(x|\mathcal{X}),$$

(4)

$$C^{(i_1,i_2,\cdots,i_j)}_\lambda(\mathcal{X}) = \sum_{x_1,x_2,\cdots,x_N} x_{i_1} x_{i_2} \cdots x_{i_j} P_\lambda(x|\mathcal{X}),$$

(5)

where $i, i_j \in \{1,2,\cdots,N\}$ and $j = 1,2,\cdots,N-1$. NS condition is $C^{(i)}_\lambda(\mathcal{X}) = C^{(i)}_\lambda(\mathcal{X}^{(i)})$ and $C^{(i_1,i_2,\cdots,i_j)}_\lambda(\mathcal{X}) = C^{(i_1,i_2,\cdots,i_j)}_\lambda(\mathcal{X}^{(i_1)},\mathcal{X}^{(i_2)},\cdots,\mathcal{X}^{(i_j)})$. Leggett model asks that each single photon would always be in a definite polarization state, which requires that the local expectation of the single photon satisfies Malus' law

$$C^{(i)}_\lambda(\mathcal{X}^{(i)}) = \vec{n}^{(i)} \cdot \vec{u}_i.$$  

(6)

Together with nonnegativity of probability $P_\lambda(x|\mathcal{X})$, we can derive the following constraints for correlation coefficients $C^{(i_1,i_2,\cdots,i_{N-1})}_\lambda(\mathcal{X}), C^{(1,2,\cdots,N)}_\lambda(\mathcal{X})$:

$$\begin{cases}
|C^{(i)}_\lambda(\mathcal{X}^{(i)})| \pm C^{(i_1,i_2,\cdots,i_{N-1})}_\lambda(\mathcal{X}) \leq 1 \pm C^{(1,2,\cdots,N)}_\lambda(\mathcal{X}), \\
i_1, i_2,\cdots,i_{N-1} \neq i, \ i_1 < i_2 < \cdots < i_{N-1}, \\
i, i_1, i_2,\cdots,i_{N-1} \in \{1,2,\cdots,N\}.
\end{cases}$$  

(7)

As a matter of fact, Eq. (7) can also be obtained by the following simple algebraic method. If $x, y$ only can take values $\{\pm 1\}$, the following equality always holds

$$|x \pm y| \mp xy = 1.$$  

(8)

We take $x = x_1$ and $y = x_2 x_3 \cdots x_N$, then

$$|x_1 \pm x_2 x_3 \cdots x_N| \mp x_1 x_2 \cdots x_N = 1.$$  

(9)
After multiplying the two sides by probability weight $\mathcal{P}_\lambda(x|\mathcal{X})$ and taking summation, we have

$$
\sum_{x_1, x_2, \cdots, x_N} |x_1 \pm x_2 x_3 \cdots x_N| \mathcal{P}_\lambda(x|\mathcal{X}) + C^{(1,2,\cdots,N)}_\lambda(\mathcal{X}) = 1,
$$

(10)

$$
|C^{(1)}_\lambda(\mathcal{X}) \pm C^{(2,3,\cdots,N)}_\lambda(\mathcal{X}')| \leq 1 \pm C^{(1,2,\cdots,N)}_\lambda(\mathcal{X}).
$$

(11)

Similarly, we can prove other cases and then we arrive at Eq. (7). If we take one pair of measurements $\mathcal{X} (\mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \cdots, \mathcal{X}^{(N)})$ and $\mathcal{X}' (\mathcal{X}'^{(1)}, \mathcal{X}'^{(2)}, \cdots, \mathcal{X}'^{(N)})$ for each photon, we obtain the following four inequalities

$$
\begin{align*}
|C^{(i)}_\lambda(\mathcal{X}^{(i)}) \pm C^{(i_2,i_3,\cdots,i_{N-1})}_\lambda(\mathcal{X})| &\leq 1 \pm C^{(1,2,\cdots,N)}_\lambda(\mathcal{X}), \\
|C^{(i)}_\lambda(\mathcal{X}^{(i)}) \pm C^{(i_2,i_3,\cdots,i_{N-1})}_\lambda(\mathcal{X}')| &\leq 1 \pm C^{(1,2,\cdots,N)}_\lambda(\mathcal{X}').
\end{align*}
$$

(12)

For convenience, we give the following notations:

$$
\begin{align*}
\gamma^{(i)}_\pm(\lambda) &= C^{(i)}_\lambda(\mathcal{X}^{(i)}) \pm C^{(i)}_\lambda(\mathcal{X}^{(i)}'), \\
\beta(\lambda) &= C^{(1,2,\cdots,N)}_\lambda(\mathcal{X}) + C^{(1,2,\cdots,N)}_\lambda(\mathcal{X}'), \\
\alpha_\pm(\lambda) &= C^{(i_1,i_2,\cdots,i_{N-1})}_\lambda(\mathcal{X}) \pm C^{(i_1,i_2,\cdots,i_{N-1})}_\lambda(\mathcal{X}').
\end{align*}
$$

(13)

(14)

(15)

After simple but tedious calculations, we can obtain the following inequalities from Eq. (12)

$$
\begin{align*}
\gamma^{(i)}_\pm(\lambda) &\leq 2 \mp [\alpha_\pm(\lambda) - \beta(\lambda)], \\
-\gamma^{(i)}_\pm(\lambda) &\leq 2 \mp [\alpha_\pm(\lambda) + \beta(\lambda)].
\end{align*}
$$

(16)

(17)

Equivalently, we can reformulate these inequalities as the compact form

$$
\begin{align*}
\gamma^{(i)}_\pm(\lambda) &\leq 2 - |\alpha_\pm(\lambda) - \beta(\lambda)|, \\
-\gamma^{(i)}_\pm(\lambda) &\leq 2 - |\alpha_\pm(\lambda) + \beta(\lambda)|.
\end{align*}
$$

(18)

(19)
These inequalities are combined as

\[ |\gamma^{(i)}_{\pm}(\lambda)| \leq 2 - \min (|\alpha_{\pm}(\lambda) - \beta(\lambda)|, |\alpha_{\pm}(\lambda) + \beta(\lambda)|) \, . \] (20)

Integrating Eq. (20) over \( \rho(\lambda) \) and using the fact that \( \int d\lambda \rho(\lambda) = 1 \), we obtain the following inequalities

\[ \min (|\alpha_{\pm} - \beta|, |\alpha_{\pm} + \beta|) \leq 2 - \gamma^{(i)}_{\pm} \, . \] (21)

Here,

\[ \beta \equiv \int d\lambda \rho(\lambda) \beta(\lambda) = C(\mathcal{X}) + C(\mathcal{X}') \, , \] (22)

\[ \gamma^{(i)}_{\pm} \equiv \int d\lambda \rho(\lambda) |\gamma^{(i)}_{\pm}(\lambda)| = \int \left| \vec{u}_i \cdot \left( \vec{n}^{(i)}_{\pm} \pm \vec{n}'^{(i)}_{\pm} \right) \right| \rho(\lambda) d\lambda \, , \] (23)

\[ \alpha_{\pm} \equiv \int d\lambda \rho(\lambda) \alpha_{\pm}(\lambda) = C \left( \mathcal{X}^{(i_1)}, \ldots, \mathcal{X}^{(i_{N-1})} \right) \pm C \left( \mathcal{X}'^{(i_1)}, \ldots, \mathcal{X}'^{(i_{N-1})} \right) \, , \] (24)

and

\[ C \left( \mathcal{X}^{(i_1)}, \ldots, \mathcal{X}^{(i_{N-1})} \right) \equiv \int d\lambda \rho(\lambda) C^{(i_1,i_2,\ldots,i_{N-1})}_{\lambda}(\mathcal{X}) \, , \] (25)

\[ C(\mathcal{X}) = C \left( \mathcal{X}^{(1)}, \ldots, \mathcal{X}^{(N)} \right) \equiv \int d\lambda \rho(\lambda) C^{(1,2,\ldots,N)}_{\lambda}(\mathcal{X}) \, . \] (26)

Let us consider three 2N-tuple of settings \( \left( \mathcal{X}^{(1)}_k, \mathcal{X}'^{(1)}_k, \ldots, \mathcal{X}^{(N)}_k, \mathcal{X}'^{(N)}_k \right) \) with the same angle \( \theta \) between all pairs \( \mathcal{X}^{(i)}_k, \mathcal{X}'^{(i)}_k \) for a specific \( i \), and such that \( \vec{n}^{(i)}_k + \vec{n}'^{(i)}_k = 2 \cos(\theta/2)\vec{e}_k \) and \( \vec{n}^{(i)}_k - \vec{n}'^{(i)}_k = 2 \sin(\theta/2)\vec{e}'_k \), where \( \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \) and \( \{\vec{e}'_1, \vec{e}'_2, \vec{e}'_3\} \) form an orthogonal basis respectively. Via the fact that \( \sum_{k=1}^{3} |\vec{u}_i \cdot \vec{e}_k| \geq 1 \), we obtain the following N-partite Leggett-type inequalities

\[ \sum_{k=1}^{3} \min ((|\alpha_+)_k - \beta_k|, (|\alpha_+)_k + \beta_k|) \leq 6 - 2|\cos(\theta/2)| \, , \] (27a)

\[ \sum_{k=1}^{3} \min ((|\alpha_-)_k - \beta_k|, (|\alpha_-)_k + \beta_k|) \leq 6 - 2|\sin(\theta/2)| \, . \] (27b)
Here,
\[(\alpha_{\pm})_k = C\left(X^{(i_1)}_k, \ldots, X^{(i_{N-1})}_k\right) \pm C\left(X^{(i_1)'}_k, \ldots, X^{(i_{N-1})'}_k\right), \quad (28)\]
\[\beta_k = C(X^{(1)}_k, \ldots, X^{(i)})_k + C(X^{(1)'}_k, \ldots, X^{(i)'})_k), \quad (29)\]
where \(i_1, i_2, \ldots, i_{N-1} \neq i, \quad i_1 < i_2 < \cdots < i_{N-1}, \quad i, i_1, i_2, \ldots, i_{N-1} \in \{1, 2, \ldots, N\}.\) Therefore we get \(N\)-pair Leggett-type inequalities for \(i = 1, 2, \ldots, N.\)

In order to uncover the possible violation of Leggett-type inequalities, what matters most is to choose the optimal measurement settings. In fact, Leggett-type inequalities above imply how to choose. Due to \(\min(|(\alpha_{\pm})_k - \beta_k|, |(\alpha_{\pm})_k + \beta_k|) \leq |\beta_k|,\) when measurement settings satisfy \((\alpha_{\pm})_k = 0,\) we obtain the tighter Leggett-type inequalities
\[\mathcal{L}_+ \equiv \sum_{k=1}^{3} |\beta_k| + 2|\cos(\theta/2)| \leq 6, \quad (30a)\]
\[\mathcal{L}_- \equiv \sum_{k=1}^{3} |\beta_k| + 2|\sin(\theta/2)| \leq 6. \quad (30b)\]

### B Correlation functions \(C(\mathcal{X})\) and \(C\left(X^{(i_1)}_k, \ldots, X^{(i_{N-1})}_k\right)\)

for \(N\)-partite GHZ state

The density matrix for \(N\)-partite GHZ state \(|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|0\rangle^{\otimes N} + |1\rangle^{\otimes N}),\)
\[
\rho = \frac{1}{2} |\text{GHZ}\rangle \langle \text{GHZ}| = \frac{1}{2} \begin{pmatrix}
1 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 1
\end{pmatrix}_{2^n \otimes 2^n}. \quad (31)
\]

Correlation function \(C(\mathcal{X}).\)
\[
C(\mathcal{X}) = C\left(\vec{\sigma} \cdot \vec{n}^{(1)}, \ldots, \vec{\sigma} \cdot \vec{n}^{(N)}\right), \quad (32)
\]
\[= \text{Tr} \left[ \rho \left( \bigotimes_{j=1}^{N} \vec{\sigma} \cdot \vec{n}^{(j)} \right) \right]. \quad (33)
\]
Since $\rho$ only has 4 nonzero entries, it is readily to calculate $\mathcal{C}(\mathcal{X})$, i.e.,

$$\mathcal{C}(\mathcal{X}) = \frac{1}{2} \left( \left[ \bigotimes_{j=1}^{N} \vec{\sigma} \cdot \vec{n}^{(j)} \right]_{1,1} + \left[ \bigotimes_{j=1}^{N} \vec{\sigma} \cdot \vec{n}^{(j)} \right]_{1,2} \right) + \left[ \bigotimes_{j=1}^{N} \vec{\sigma} \cdot \vec{n}^{(j)} \right]_{2,1} + \left[ \bigotimes_{j=1}^{N} \vec{\sigma} \cdot \vec{n}^{(j)} \right]_{2,2}$$

$$= \frac{1}{2} \left( \prod_{j=1}^{N} n_{3}^{(j)} + (-1)^{N} \prod_{j=1}^{N} n_{3}^{(j)} \right) + \text{Re} \left[ \prod_{j=1}^{N} (n_{1}^{(j)} + i n_{2}^{(j)}) \right]$$

$$= \begin{cases} 
\text{Re} \left[ \prod_{j=1}^{N} (n_{1}^{(j)} + i n_{2}^{(j)}) \right], & N \text{ is odd.} \\
\prod_{j=1}^{N} n_{3}^{(j)} + \text{Re} \left[ \prod_{j=1}^{N} (n_{1}^{(j)} + i n_{2}^{(j)}) \right], & N \text{ is even.} 
\end{cases}$$

(34)

If we take the spherical coordinate frame $\vec{n}^{(j)} = (\sin \vartheta_{j} \cos \varphi_{j}, \sin \vartheta_{j} \sin \varphi_{j}, \cos \vartheta_{j})^T$, we have

$$\mathcal{C}(\mathcal{X}) = \begin{cases} 
\cos \varphi \prod_{j=1}^{N} \sin \vartheta_{j}, & N \text{ is odd.} \\
\prod_{j=1}^{N} \cos \vartheta_{j} + \cos \varphi \prod_{j=1}^{N} \sin \vartheta_{j}, & N \text{ is even.} 
\end{cases}$$

(35)

Here, $\varphi = \sum_{j=1}^{N} \varphi_{j}$. It is interesting that the correlation function $\mathcal{C}(\mathcal{X})$ of GHZ state occurs different cases for the even and odd number of particles. For the odd number of particles, we only need to measure in $x$-$y$ plane. In particular, when measurements only involve $x$-$y$ plane, we have

$$\mathcal{C}(\mathcal{X}) = \cos \left( \sum_{j=1}^{N} \varphi_{j} \right).$$

(36)

Under the spherical coordinate frame, the unit vector $\vec{n}$ can be parameterized by polar angle $\vartheta$ and azimuthal angle $\varphi$. Hereafter, we denote the unit vector by

$$\vec{n} = \{ \vartheta, \varphi \},$$

(37)
which may notably simplify calculations of the correlation function $C(X)$. Next, we calculate the correlation function $C(X)$, in which case the measurement settings are arranged as follows,

$$\vec{n}_1^{(1)} = \{\frac{\pi}{2}, \frac{\theta}{2}\}, \quad \vec{n}_2^{(1)} = \{\frac{\pi}{2}, \frac{\theta}{2}\}, \quad \vec{n}_3^{(1)} = \{\frac{\theta}{2}, 0\},$$

$$\vec{n}_1^{(1)\prime} = \{\frac{\pi}{2}, -\frac{\theta}{2}\}, \quad \vec{n}_2^{(1)\prime} = \{\frac{\pi}{2}, \frac{\theta}{2}\}, \quad \vec{n}_3^{(1)\prime} = \{\frac{\theta}{2}, \pi\},$$

which ensures that $\{\vec{n}_k^{(1)} + \vec{n}_k^{(1)\prime}\}_k^3 = 1$ and $\{\vec{n}_k^{(1)} - \vec{n}_k^{(1)\prime}\}_k^3 = 1$ form orthogonal base, respectively.

Thus, these measurements should satisfy Leggett-type inequalities derived above. If we arrange $\vec{n}_k^{(j)} = \{\theta_{kj}, \phi_{kj}\}, \vec{n}_k^{(j)\prime} = \{\theta_{kj}', \phi_{kj}'\}$ and $j = 2, \ldots, N$, it is straightforward to calculate $C(X)$ in light of Eq. (35). Particularly, if $\vec{n}_k^{(j)}$ and $\vec{n}_k^{(j)\prime}$ are in $x$-$y$ plane, i.e., $\{\theta_{kj} = \theta_{kj}' = \pi\}_j^N$, for the odd number of particles, we have

$$C(X_1) = \cos(\phi_1)/2, \quad C(X_1') = \cos(\phi_1')/2,$$

$$C(X_2) = -\cos(\theta/2) \sin(\phi_2/2), \quad C(X_2') = -\cos(\theta/2) \sin(\phi_2'/2),$$

$$C(X_3) = \sin(\theta/2) \cos(\phi_3/2), \quad C(X_3') = -\sin(\theta/2) \cos(\phi_3'/2).$$

Here, $\phi_k = \sum_{j=2}^N \phi_{kj}, \phi_k' = \sum_{j=2}^N \phi_{kj}'$ and $k = 1, 2, 3$.

**Correlation function** $C(\mathcal{X}^{(i_1)}_k, \ldots, \mathcal{X}^{(i_{N-1})}_k)$. The reduced states after tracing out $i$'th particle

$$\rho_{i_1, \ldots, i_{N-1}} = \text{Tr}_{(i)}[\rho] = \frac{1}{2} \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}_{2^{N-1} \otimes 2^{N-1}}.$$  

We take $\mathcal{X}^{(i_1)}_k = \vec{\sigma} \cdot \vec{n}^{(1)}, \ldots, \mathcal{X}^{(i_{N-1})}_k = \vec{\sigma} \cdot \vec{n}^{(N-1)}$ as a specific example and similarly for
C. Quantum physics predictions of $\mathcal{L}_{\pm}^{\text{GHZ}}$ for tripartite GHZ state

The measurement settings are arranged by Fig. 1 of the main text, that is,

\begin{align*}
\vec{a}_1 &= \left\{ \pi/2, \theta/2 \right\}, \quad \vec{a}_2 = \left\{ \pi/2, -\theta/2 \right\}, \quad \vec{a}_3 = \left\{ \pi/2, 0 \right\}, \\
\vec{a}'_1 &= \left\{ \pi/2, -\theta/2 \right\}, \quad \vec{a}'_2 = \left\{ \pi/2, \theta/2 \right\}, \quad \vec{a}'_3 = \left\{ \pi/2, \pi \right\}, \\
\vec{b}_1 &= \left\{ \pi/2, \phi/2 \right\}, \quad \vec{b}_2 = \left\{ \pi/2, 0 \right\}, \quad \vec{b}_3 = \left\{ \pi/2, 0 \right\}, \\
\vec{b}'_1 &= \left\{ \pi/2, -\phi/2 \right\}, \quad \vec{b}'_2 = \left\{ \pi/2, 0 \right\}, \quad \vec{b}'_3 = \left\{ \pi/2, 0 \right\}, \\
\vec{c}_1 &= \left\{ \pi/2, \varphi/2 \right\}, \quad \vec{c}_2 = \left\{ \pi/2, \pi/2 \right\}, \quad \vec{c}_3 = \left\{ \pi/2, 0 \right\}, \\
\vec{c}'_1 &= \left\{ \pi/2, -\varphi/2 \right\}, \quad \vec{c}'_2 = \left\{ \pi/2, \pi/2 \right\}, \quad \vec{c}'_3 = \left\{ \pi/2, \pi \right\}.
\end{align*}

Via the results of Eqs. (40) to (42), it is readily to calculate quantum physics predictions of $\mathcal{L}_{\pm}^{\text{GHZ}}$

\begin{align*}
\mathcal{L}_+^{\text{GHZ}} &= 4 \cos(\theta/2) + 2 \sin(\theta/2) + 2 |\cos(\theta + \phi + \varphi)/2|, \quad (54a) \\
\mathcal{L}_-^{\text{GHZ}} &= 2 \cos(\theta/2) + 4 \sin(\theta/2) + 2 |\cos(\theta + \phi + \varphi)/2|. \quad (54b)
\end{align*}
Making use of trigonometric formulas, we can obtain

\[ \mathcal{L}^{\text{GHZ}}_+ = 2\sqrt{5}\sin(\theta_0 + \theta/2) + 2|\cos(\theta + \phi + \varphi)/2|, \quad (55a) \]

\[ \mathcal{L}^{\text{GHZ}}_- = 2\sqrt{5}\cos(\theta_0 - \theta/2) + 2|\cos(\theta + \phi + \varphi)/2| . \quad (55b) \]

Here, \( \theta_0 = \arctan 2, \theta \in [0, \pi] \) and \( \phi, \varphi \in [0, 2\pi] \). When \( \theta = 0 \), we have

\[ \mathcal{L}^{\text{GHZ}}_+ = 4 + 2|\cos(\phi + \varphi)/2| \leq 6 , \quad (56a) \]

\[ \mathcal{L}^{\text{GHZ}}_- = 2 + 2|\cos(\phi + \varphi)/2| \leq 6 . \quad (56b) \]

And when \( \theta = \pi \), we have

\[ \mathcal{L}^{\text{GHZ}}_+ = 2 + 2|\sin(\phi + \varphi)/2| < 6 , \quad (57a) \]

\[ \mathcal{L}^{\text{GHZ}}_- = 4 + 2|\sin(\phi + \varphi)/2| \leq 6 . \quad (57b) \]

Therefore, there is no violation for \( \theta = 0, \pi \). When \( \theta \neq 0, \pi \), violations can be found for proper \( \varphi, \phi \), see Fig. 1. And when \( \theta = \pi - 2\theta_0 \) for \( \mathcal{L}^{\text{GHZ}}_+ \) and \( \theta = 2\theta_0 \) for \( \mathcal{L}^{\text{GHZ}}_- \), we have

\[ \mathcal{L}^{\text{GHZ}}_+ = 2\sqrt{5} + 2|\sin(2\theta_0 - \phi - \varphi)/2| \quad (58a) \]

\[ \mathcal{L}^{\text{GHZ}}_- = 2\sqrt{5} + 2|\cos(2\theta_0 + \phi + \varphi)/2| . \quad (58b) \]

In this case, the maximal violation \( 2(\sqrt{5} + 1) \) can always occur for arbitrary angle \( \phi \in [0, 2\pi] \) and \( \mathcal{L}^{\text{GHZ}}_+ \) reaches the maximal violation, when \( \varphi \) satisfies

\[ \begin{cases} 
\varphi = \pi + 2\theta_0 - \phi , & \phi \in [0, \pi + 2\theta_0] , \\
\varphi = 3\pi + 2\theta_0 - \phi , & \phi \in [\pi + 2\theta_0, 2\pi] .
\end{cases} \quad (59) \]

\( \mathcal{L}^{\text{GHZ}}_- \) reaches the maximal violation, when \( \varphi \) satisfies

\[ \begin{cases} 
\varphi = 2\pi - 2\theta_0 - \phi , & \phi \in [0, 2\pi - 2\theta_0] , \\
\varphi = 4\pi - 2\theta_0 - \phi , & \phi \in [2\pi - 2\theta_0, 2\pi] .
\end{cases} \quad (60) \]
FIG. 1. Quantum violations of Leggett-type inequalities for tripartite GHZ state. When the measurement settings are arranged in accordance with Fig.1 of the main text, quantum physics violates Eqs. (55a) and (55b) obviously. Especially, violations occur for arbitrary angle $\phi \in [0, 2\pi]$ when $\theta \neq 0, \pi$. The $L_{\text{GHZ}}^+$ and $L_{\text{GHZ}}^-$ can reach the maximal quantum violation $2(\sqrt{5} + 1)$ when $\theta = \pi - 2 \arctan 2$ and $\theta = 2 \arctan 2$ respectively.