Jaqueline da Costa Ferreira and Marcone Corrêa Pereira

A nonlocal Dirichlet problem with impulsive action: estimates of the growth for the solutions

Volume 358, issue 11-12 (2020), p. 1119-1128.

<https://doi.org/10.5802/crmath.109>

© Académie des sciences, Paris and the authors, 2020.

Some rights reserved.

This article is licensed under the Creative Commons Attribution 4.0 International License.
http://creativecommons.org/licenses/by/4.0/
A nonlocal Dirichlet problem with impulsive action: estimates of the growth for the solutions

Jaqueline da Costa Ferreira\textsuperscript{a} and Marcone Corrêa Pereira\textsuperscript{a, b}

\textsuperscript{a} Depto. Matemática, CCE, Universidade Federal do Espírito Santo, Av. Fernando Ferrari 514, Vitória - ES, Brazil
\textsuperscript{b} Depto. Matemática Aplicada, IME, Universidade de São Paulo, Rua do Matão 1010, São Paulo - SP, Brazil
E-mails: jaqueline.c.ferreira@ufes.br (J. da Costa Ferreira), marcone@ime.usp.br (M. Corrêa Pereira)

Abstract. Through this paper we deal with the asymptotic behaviour as $t \to +\infty$ of the solutions for a nonlocal diffusion problem with impulsive actions and Dirichlet condition. We establish a decay rate for the solutions assuming appropriate hypotheses on the impulsive functions and the nonlinear reaction.

Funding. Jaqueline da Costa Ferreira is partially supported by PRPPG N\textsuperscript{o} 8373/2017 and FAPES N\textsuperscript{o} 21/2018 - UNIVERSAL, Brazil. Marcone Corrêa Pereira is partially supported by CNPq 303253/2017-7, Brazil.

Manuscript received 4th July 2020, revised 23rd August 2020, accepted 27th August 2020.

1. Introduction

Our main purpose in this paper is to study the asymptotic behaviour of the solutions of the following nonlinear problem with impulsive actions

\begin{equation}
\begin{aligned}
    & u_t(x,t) = \int_\Omega f(x-y)u(y,t)\,dy - u(x,t) + f(x,u(x,t)), \quad (x,t) \in \Gamma, \\
    & u(x,0) = u_0(x), \quad x \in \Omega, \\
    & u(x,t_k) = g_k(u(x,t_{k-1})), \quad x \in \Omega, \quad k = 1,2, \ldots
\end{aligned}
\end{equation}

exploring the properties of the integral operator

$K_f(u)(x,t) := \int_\Omega f(x-y)u(y,t)\,dy.$

* Corresponding author.
Along the whole paper we assume

(H1) $\Omega \subset \mathbb{R}^n$ is an open bounded domain;

(H2) $0 = t_0 < t_1 < t_2 \ldots < t_k < \ldots$ are fixed instants of times with $\lim_{k \to +\infty} t_k = +\infty$;

(H3) $\Gamma = \bigcup_{k=0}^{\infty} \Gamma_k$ where $\Gamma_k = \{ (x,t) \in \mathbb{R}^{n+1} : t \in (t_k, t_{k+1}), x \in \Omega \}$ for $k = 0, 1, \ldots$;

(H4) The impulsive functions $g_k : \mathbb{R} \to \mathbb{R}$ are continuous satisfying $|g_k(x)| \leq M_k|x|$ for positive constants $M_k$ for $k = 1, 2, \ldots$. We still set $M_0 = 1$ for convenience;

(H5) The local reaction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in the second variable $s \in \mathbb{R}$ uniformly in $\Omega$. Also, we assume there exist constants $C > 0$ and $D \geq 0$ such that

\[ f(\cdot, s)s \leq Cs^2 + D. \]

(H6) The kernel $J$ of the integral operator $K_J$ is supposed to be non-singular. It satisfies

\[ J \in C(\mathbb{R}^n, \mathbb{R}) \]

is a non-negative function with $J(0) > 0$,

\[ J(-x) = J(x) \]

for every $x \in \mathbb{R}^n$ and $\int_{\mathbb{R}^n} J(x) \, dx = 1$.

It is worth noticing that according to the pioneering works [2, 6, 7, 10] the nonlocal problem considered here can be seen as a nonlocal analogous to a reaction-diffusion problem given by the Laplacian with homogeneous Dirichlet boundary condition and impulsive actions

\[
\begin{align*}
    u_r(x, t) &= \Delta u(x, t) + f(x, u(x, t)), \quad (x, t) \in \Gamma, \\
    u(x, t) &= 0, \quad \text{on } \partial \Omega, \\
    u(x, 0) &= u_0(x), \quad x \in \Omega, \\
    u(x, t_k) &= g_k(u(x, t_k^-)), \quad x \in \Omega, \quad k = 1, 2, \ldots
\end{align*}
\]

Notice that, if $u(x, t)$ sets the population density of a species at the position $x$ and time $t$, and $J(x-y)$ is treated of as the probability distribution of jumping from spot $y$ to spot $x$, then $\int_{\mathbb{R}^n} J(x-y)u(y, t) \, dy = (J * u)(x, t)$ is the rate at which individuals are arriving at position $x$ from all other positions $y \in \mathbb{R}^n$ and $-u(x, t) = -\int_{\mathbb{R}^n} J(x-y)u(x, t) \, dy$ is the rate at which they are leaving location $x$ to move to all other positions. As consequence, the density $u$ satisfies the integral equation $u_r = J * u - u$, in the absence of external or internal sources.

Additionally, one may notice situations in nature where abrupt changes such as droughts, tropical storms or many other environmental accidents may appear. Thus, the need to consider models with some kind of impulse becomes essential. Our proposal mix both situations. We combine a nonlocal reaction-diffusion equation with impulsive actions in fixed times.

The literature [5, 11–13, 16, 17, 19, 20] is recommended for more knowledge on nonlocal diffusion equations and applications. In [18], examples of integral equations whose solutions blow-up in finite time are considered. On the other side, the classical literature [14, 21] deals with the impulsive differential equations in fixed times. For recent works, we refer to [1, 4, 9] and references therein.

To the best of our knowledge, the corresponding theory for partial differential equations with impulses for the operator $K_J$ has not been investigated yet. In this paper, we intend to make some contributions in this class of problems. Our goal is to secure the existence of global solutions taking estimates for the impulsive solutions as $t \to +\infty$.

In our analysis, the first eigenvalue $\lambda_1$ of the operator $\mathcal{D}_\Omega : L^2(\Omega) \to L^2(\Omega)$ given by

\[ \mathcal{D}_\Omega(u)(x) = u(x) - K_J(u)(x), \quad x \in \Omega, \]  \hspace{0.5cm} (2)

plays an important role. Its existence is guaranteed by [8, Theorem 2.1] and hypotheses (H1) and (H6). It can be taken by the following expression

\[ \lambda_1 = \inf \left\{ \frac{\int_{\Omega} \int_{\Omega} f(x-y)(u(y) - u(x))^2 \, dy \, dx}{\int_{\Omega} u^2(x) \, dx} : u \neq 0 \text{ with } u \in L^2(\Omega) \right\}. \]  \hspace{0.5cm} (3)
We also mention [3] where recent results concerning to the eigenvalues of the operator $\Omega$ has been considered. The main result of the paper is the following.

**Theorem 1.** Let $u$ be the global solution of the impulsive nonlocal equation (1) with initial condition $u_0 \in L^2(\Omega)$. Set $\alpha = 2(\lambda_1 - C)$ and $\beta = 2D|\Omega|$ where $C$ and $D$ are the non-negative constants given by assumption (H5), $\lambda_1$ is the first eigenvalue introduced by (3), and $|\Omega|$ is the Lesbegue measure of the open bounded set $\Omega$. Suppose there exists $L_k > 0$ such that

\[
\max \left\{ \prod_{j=1}^{k} M_j^2, \prod_{j=2}^{k} M_j^2, \ldots, M_k^2, 1 \right\} \leq L_k
\]

where $M_k$ are the constants given by condition (H4).

Then, we have the following estimate

\[
\|u(\cdot, t)\|^2_{L^2(\Omega)} \leq e^{-\alpha t} \left[ \left( \prod_{j=0}^{k} M_j^2 \right) \|u_0(\cdot)\|^2_{L^2(\Omega)} + L_k \frac{\beta}{|\alpha|} \sum_{j=0}^{k} e^{\alpha t_j} \right] \quad \forall \ t_k \leq t < t_{k+1} \text{ if } \alpha \neq 0,
\]

and

\[
\|u(\cdot, t)\|^2_{L^2(\Omega)} \leq \left( \prod_{j=0}^{k} M_j^2 \right) \|u_0(\cdot)\|^2_{L^2(\Omega)} + L_k \beta t \quad \forall \ t_k \leq t < t_{k+1} \text{ if } \alpha = 0.
\]

Furthermore, assume that there exist positive constants $L$ and $\gamma$ such that $Le^t \geq L_k$ for all $t \in [0, t_{k+1})$ and $k \in \mathbb{N}$, and there exists $\delta > 0$ such that $|t_i - t_j| > \delta$ for all $i, j$ where $t_i$'s are the impulsive instants. Then, if $\alpha < 0$, we have

\[
\|u(\cdot, t)\|^2_{L^2(\Omega)} \leq Le^{(\gamma - \alpha)t} \left[ \|u_0(\cdot)\|^2_{L^2(\Omega)} + \frac{\beta}{\delta |\alpha|} \left( \frac{1}{1 - e^{\alpha \delta}} - 1 \right) \right] \quad \forall \ t > 0,
\]

and, when $\alpha > 0$, we get

\[
\|u(\cdot, t)\|^2_{L^2(\Omega)} \leq Le^{(\gamma - \alpha)t} \left[ \|u_0(\cdot)\|^2_{L^2(\Omega)} + \frac{\beta}{\delta |\alpha|} \left( e^{\alpha \delta} - 1 \right) \right] \quad \forall \ t \in [0, t_{k+1}).
\]

For instance, let us assume $D = 0$ at condition (H5). Then, $\beta = 0$ at Theorem 1 and $f(x, 0) = 0$ in $\Omega$ which implies that $u(x, t) = 0$ is an equilibrium solution to (1). Now, let us suppose there exist positive constants $K$ and $\zeta$ such that $\prod_{j=0}^{k} M_j^2 \leq Ke^{\zeta t}$ for all $t \in [0, t_{k+1})$ and $k \in \mathbb{N}$. Hence, it follows from Theorem 1 that

\[
\int_{\Omega} u^2(x, t) dx \leq Ke^{(\zeta - \alpha)t} \int_{\Omega} u_0^2(x) dx \quad \forall \ 0 \leq t < +\infty.
\]

Thus, if $\zeta - \alpha < 0$, we can conclude that $u(x, t) = 0$ is an asymptotically stable equilibrium solution for (1), and then, the dynamics defined by equation (1) is trivial. In this way, conditions on the impulsive functions can set the null equilibrium as asymptotically stable.

Next, let us mention our last result concerning to the estimates for the solutions with initial conditions in $L^\infty(\Omega)$. Here, we take $D = 0$ at assumption (H5) in order to obtain a workable estimate.

**Theorem 2.** Let $u$ be the global solution of the impulsive nonlocal equation (1) with initial condition $u_0 \in L^\infty(\Omega)$. Also, let us assume $D = 0$ at assumption (H5).

Then, for all $\epsilon > 0$, there exists $C_0 > 0$ such that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \left( \prod_{j=0}^{k} C_0 M_j \right) \|u_0(\cdot)\|_{L^\infty(\Omega)} e^{(C_0 + \epsilon - \lambda_1) t} \quad \text{for all } t_k \leq t < t_{k+1}
\]

where $C$ is the constant given by condition (H5) and $\lambda_1 > 0$ is the first eigenvalue of the operator $\Omega$ which is set by (3).
As we will see at the proof of Theorem 2, the positive constant $C_0$ comes from the estimate of the linear semigroup of the bounded linear operator $\mathcal{D}_\Omega$. It depends on the spectral set of $\mathcal{D}_\Omega$, and is needed since we are using the variation of constant formula to estimate the solutions of (1). Moreover, if there exist $K > 0$ and $\zeta \in \mathbb{R}$ such that \(\prod_{j=0}^{k} C_0 M_j \leq K e^{\zeta t}\) for all $t \in (0, t_{k+1})$ and $k \in \mathbb{N}$, then, we can proceed as in Theorem 1 obtaining from Theorem 2 that

$$
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K \|u_0(\cdot)\|_{L^\infty(\Omega)} e^{(\zeta + C_0 C + e - \lambda_1) t} \quad \text{for all } t \geq 0.
$$

Hence, the null equilibrium is asymptotically stable if one can get $\zeta + C_0 C + e - \lambda_1 < 0$.

Finally, it is worth mentioning that one may combine Theorem 1 and Theorem 2 in order to obtain estimates in $L^p(\Omega)$ for any $p \in (2, +\infty)$ by interpolation. Indeed, since we are working in bounded domains, it is a direct consequence of the Riesz–Thorin Theorem (it will be left to the interested reader).

The organization of this article is as follows. In Section 2, we briefly discuss the existence of global solutions for the nonlocal equation without impulses getting estimates which guarantee the existence of solutions for problem (1) with impulsive actions. In Section 3 we prove the main results of the paper establishing estimates for the impulsive solutions with initial conditions in $L^2(\Omega)$ and $L^\infty(\Omega)$.

### 2. Existence of solutions

In this section, we give some conditions in order to guarantee the existence of solutions for the nonlocal problem with impulsive actions (1). First let us notice that the impulsive linear case

\[
\begin{align*}
\begin{cases}
  u_t(x, t) &= (K_f - I)u(x, t), & (x, t) \in \Gamma, \\
  u(x, 0) &= u_0(x), & x \in \Omega, \\
  u(x, t_k) &= g_k(u(x, t_{k-1}^+)), & x \in \Omega, \quad k = 1, 2, \ldots,
\end{cases}
\end{align*}
\]

where $u(x, t_{k-1}^+) = \lim_{t \to t_{k-1}^-} u(x, t)$ is well defined since the solutions of the non impulsive equation

\[
\begin{align*}
\begin{cases}
  u_t(x, t) &= (K_f - I)u(x, t), & x \in \Omega, \quad t > 0, \\
  u(x, t) &= 0, & x \notin \Omega, \quad t > 0, \\
  u(x, 0) &= u_0(x), & x \in \Omega,
\end{cases}
\end{align*}
\]

are globally defined for any $u_0 \in L^p(\Omega)$ with $p \in [1, \infty]$ (see for instance [20]).

Thus, for each $k$, we have

$$
\lim_{t \to t_k^-} u(x, t) = u(x, t_k),
$$

and then, the solution for the impulsive equation (6) is a piecewise continuous function $u$ such that in each interval $[t_k, t_{k+1})$ satisfies

\[
\begin{align*}
\begin{cases}
  u_t(x, t) &= (K_f - I)u(x, t), & (x, t) \in \Gamma_k, \\
  u(x, t_k) &= g_k(u(x, t_{k-1}^+)), & x \in \Omega.
\end{cases}
\end{align*}
\]

Consequently, a enough condition to define the solution to the impulsive equation (1) is to ensure that the associated problem

\[
\begin{align*}
\begin{cases}
  u_t(x, t) &= (K_f - I)u(x, t) + f(x, u(x, t)), & x \in \Omega, \quad t > 0, \\
  u(x, 0) &= u_0(x), & x \in \Omega,
\end{cases}
\end{align*}
\]

possesses solution globally defined for appropriated initial conditions.

The local existence and uniqueness is proved using a fixed point argument with the variation of constants formula on the right side of

$$
u(\cdot, t) = e^{-t}u_0(\cdot) + \int_0^t e^{-(t-s)} \left[(K_f u)(\cdot, s) + f(\cdot, u(\cdot, s))\right] ds.
$$

C. R. Mathématique, 2020, 358, no 11-12, 1119-1128
Considering the formula on the right side of (8) as an operator defined from $L^1([0,T],L^2(\Omega))$ into $\mathcal{C}([0,T],L^2(\Omega))$, one can get a $T > 0$ in such way that a contraction is established since function $f$ is locally Lipschitz in the second variable. Hence, local existence and uniqueness to (7) is obtained applying abstract theorems from [15, p. 109]. Indeed, a strong solution is gotten in $\mathcal{C}^1([0,T],L^2(\Omega))$.

Analogously, one can show local existence and uniqueness to (8) in $\mathcal{C}^1([0,T],L^\infty(\Omega))$ for initial conditions in $L^\infty(\Omega)$. The next estimates ensure that the solutions are upper bounded for a function defined for all $t \geq 0$ which also guarantees that they are globally defined in $[0, + \infty)$.

**Proposition 3.** Let $u : [0, T) \rightarrow L^2(\Omega)$ be the solution of (7) with initial condition $u_0 \in L^2(\Omega)$ and nonlinearity $f$ satisfying hypothesis (H5). Then

$$\|u(\cdot)\|^2_{L^2(\Omega)} \leq \gamma(t) \quad \forall \ t \in [0, T)$$

where

$$\gamma(t) = (\beta/\alpha) \left(1 - e^{-\alpha t}\right) + e^{-\alpha t} \|u_0(\cdot)\|^2_{L^2(\Omega)}$$

with $\beta = 2D|\Omega|$ and

$$\alpha = 2(\lambda_1 - C) \neq 0 \quad \text{or} \quad \gamma(t) = \|u_0(\cdot)\|^2_{L^2(\Omega)} + \beta t \quad \text{if} \quad \alpha = 0.$$

Consequently, the solution $u$ with initial condition $u_0$ can be extended to a global solution of (7) in $[0, + \infty)$.

**Proof.** First, let us assume $\lambda_1 \neq C$. Then,

$$\frac{\partial}{\partial t} \left[ \frac{1}{2} \int_{\Omega} u^2(x,t) \, dx \right] = \frac{1}{2} \int_{\Omega} 2u(x,t) \frac{\partial}{\partial t} u(x,t) \, dx = \int_{\Omega} u(x,t) \left[(K_I - I)u(x,t) + f(x,u(x,t))\right] \, dx$$

$$= -\int_{\Omega} u(x,t)(I-K_I)u(x,t) \, dx + \int_{\Omega} u(x,t)f(x,u(x,t)) \, dx$$

$$\leq -\lambda_1 \int_{\Omega} u^2(x,t) \, dx + \int_{\Omega} (Cu^2(x,t) + D) \, dx$$

by condition (H5). Hence,

$$\frac{\partial}{\partial t} \left[ \int_{\Omega} u^2(x,t) \, dx \right] \leq \left(2\lambda_1 - 2C\right) \int_{\Omega} u^2(x,t) \, dx \leq 2 \int_{\Omega} D \, dx \leq 2D|\Omega|.$$ 

Thus,

$$\int_{0}^{t} \frac{\partial}{\partial s} \left[ e^{\alpha s} \int_{\Omega} u^2(x,s) \, dx \right] \, ds \leq \int_{0}^{t} \beta e^{\alpha s} \, ds$$

$$\Rightarrow e^{\alpha t} \int_{\Omega} u^2(x,t) \, dx - \int_{\Omega} u^2_0(x) \, dx \leq \frac{\beta}{\alpha} e^{\alpha t} - \frac{\beta}{\alpha}$$

$$\Rightarrow \int_{\Omega} u^2(x,t) \, dx \leq \frac{\beta}{\alpha} \left(1 - e^{-\alpha t}\right) + e^{-\alpha t} \int_{\Omega} u^2_0(x) \, dx = \gamma(t).$$

In particular, we have that

$$\int_{\Omega} u^2(x,t) \, dx \leq e^{-\alpha t} \int_{\Omega} u^2_0(x) \, dx + \frac{\beta}{\alpha} \quad \text{if} \quad \alpha > 0 \quad (9)$$

and

$$\int_{\Omega} u^2(x,t) \, dx \leq e^{-\alpha t} \left[ \int_{\Omega} u^2_0(x) \, dx - \frac{\beta}{\alpha} \right] \quad \text{if} \quad \alpha < 0. \quad (10)$$
Finally, let us suppose \( \lambda_1 = C \). Arguing as before, one has
\[
\frac{\partial}{\partial t} \left[ \int_\Omega u^2(x,t) \, dx \right] + (2\lambda_1 - 2C) \int_\Omega u^2(x,t) \, dx \leq 2 \int_\Omega D \, dx = 2D |\Omega|
\]
\[
\Rightarrow \left[ \frac{\partial}{\partial t} \int_\Omega u^2(x,t) \, dx \right] \leq \beta \Rightarrow \int_0^t u^2(x,t) \, dx - \int_0^t u_0^2(x) \, dx \leq \int_0^t \beta \, ds = \beta t
\]
which completes the proof of Proposition 3.

It follows from hypothesis (H5) that \( f(\cdot, s) \leq Cs + D/s \) whenever \( s > 0 \) and \( f(\cdot, s) \geq Cs + D/s \) if \( s < 0 \) for all \( x \in \Omega \). Also, we have that \( f \) is locally Lipschitz in the second variable uniformly in \( \Omega \). Thus, there exist \( D_1 > 0 \) such that
\[
|f(\cdot, s)| \leq C|s| + D_1 \quad \text{for all} \quad s \in \mathbb{R}.
\]
Hence, we can estimate the solutions with initial conditions in \( L^\infty(\Omega) \) in the following way.

**Proposition 4.** Let \( u : [0, T) \to L^\infty(\Omega) \) be the solution of (7) with initial condition \( u_0 \in L^\infty(\Omega) \) and assume nonlinearity \( f \) satisfies hypothesis (H5).

Then, for all \( \epsilon > 0 \), there exists \( C_0 > 0 \) such that
\[
\| u(t) \|_{L^\infty(\Omega)} \leq C_0 \left[ \| u_0(\cdot) \|_{L^\infty(\Omega)} + \frac{D_1}{(\lambda_1 - \epsilon)} \left( e^{(\lambda_1 - \epsilon)t \Omega} - 1 \right) \right] e^{(C_0 + \epsilon - \lambda_1)t}
\]
where \( C \) and \( D_1 \) are the constants given by (11) and \( \lambda_1 > 0 \) is the first eigenvalue of the linear operator \( \mathcal{D}_D \) given by (3).

**Proof.** Let us consider the linear semigroup \( e^{-\mathcal{D}_D t} := \sum_{k \geq 0} \frac{t^k (-\mathcal{D}_D)^k}{k!} \) of the problem (7). Since \( D_0 \) set in (2) is a bounded linear operator in \( L^\infty(\Omega) \), it follows from [15] that \( e^{-\mathcal{D}_D t} \) is well defined and we can rewrite \( u \) as
\[
u(x, t) = e^{-\mathcal{D}_D t} u_0(x) + \int_0^t e^{-\mathcal{D}_D (t-s)} f(x, u(x,s)) \, ds
\]
for all \( t \in [0, T) \) and \( x \in \Omega \).

Notice that \( \text{Re}(\sigma(-\mathcal{D}_D)) \leq -\lambda_1 \) where \( \lambda_1 \) is the first eigenvalue of \( \mathcal{D}_D \) set in (3). Hence, by [20, Section 4.3] we get that for all \( \epsilon > 0 \) there exists \( C_0 > 0 \) such that
\[
\left\| e^{-\mathcal{D}_D t} u_0(\cdot) \right\|_{L^\infty(\Omega)} \leq C_0 e^{-(\lambda_1 - \epsilon)t} \| u_0(\cdot) \|_{L^\infty(\Omega)} \quad \forall \ t \geq 0.
\]

Then, due to condition (H5) and (11), we have
\[
\| u(\cdot, t) \|_{L^\infty(\Omega)} \leq C_0 e^{-(\lambda_1 - \epsilon)t} \| u_0(\cdot) \|_{L^\infty(\Omega)} + \int_0^t C_0 e^{-(\lambda_1 - \epsilon)(t-s)} \| f(\cdot, u(\cdot, s)) \|_{L^\infty(\Omega)} \, ds
\]
\[
\leq C_0 e^{-(\lambda_1 - \epsilon)t} \| u_0(\cdot) \|_{L^\infty(\Omega)} + \int_0^t C_0 e^{-(\lambda_1 - \epsilon)(t-s)} \left[ C \| u(\cdot, s) \|_{L^\infty(\Omega)} + D_1 \right] \, ds.
\]
Consequently,
\[
e^{(\lambda_1 - \epsilon)t} \| u(\cdot, t) \|_{L^\infty(\Omega)} \leq C_0 \left[ \| u_0(\cdot) \|_{L^\infty(\Omega)} + \frac{D_1}{(\lambda_1 - \epsilon)} \left( e^{(\lambda_1 - \epsilon)t \Omega} - 1 \right) \right] e^{(C_0 + \epsilon - \lambda_1)t}
\]
and then, by Grönwall’s Lemma, we obtain that
\[
e^{(\lambda_1 - \epsilon)t} \| u(\cdot, t) \|_{L^\infty(\Omega)} \leq C_0 \left[ \| u_0(\cdot) \|_{L^\infty(\Omega)} + \frac{D_1}{(\lambda_1 - \epsilon)} \left( e^{(\lambda_1 - \epsilon)t \Omega} - 1 \right) \right] e^{C_0 t}
\]
which proves the result.

\[\square\]
It follows from Propositions 3 and 4 that the solutions of our nonlocal problem with impulsive actions are unique for initial conditions in $L^2(\Omega)$ and $L^\infty(\Omega)$. Indeed, the solutions are piecewise continuous functions $u$ satisfying

$$
\begin{cases}
u_t(x, t) = (K_j - I)u(x, t) + f(x, u(x, t)), & t \in (t_k, t_{k+1}) \\
u(x, 0) = u_0(x) \in L^2(\Omega) \cup L^\infty(\Omega), & x \in \Omega. \\
u(x, t_k) = g_k\{u(x, t^-_k)\}, & k = 1, 2, \ldots
\end{cases}
$$

3. Proof of Theorems 1 and 2.

Next, we show Theorems 1 and 2.

**Proof of Theorem 1.**

**Case $\alpha < 0$:** From (10) in $0 \leq t < t_1$, we get

$$
\int_\Omega u^2(x, t^-_1) \, dx \leq e^{-\alpha t_1} \left[ \int_\Omega u_0^2(x) \, dx - \frac{\beta}{\alpha} \right].
$$

Using (10) again, we have by (H4) that

$$
\int_\Omega u^2(x, t) \, dx \leq e^{-\alpha(t-t_1)} \left[ \int_\Omega g_1^2\{u(x, t^-_1)\} \, dx - \frac{\beta}{\alpha} \right] \quad \text{for} \quad t_1 \leq t < t_2
$$

which then gives

$$
\int_\Omega u^2(x, t^-_2) \, dx \leq e^{-\alpha t_2} \left[ M_1^2 \left( \int_\Omega u_0^2(x) \, dx - \frac{\beta}{\alpha} e^{\alpha t_1} \right) - \frac{\beta}{\alpha} e^{\alpha t_1} \right].
$$

Working as before, we obtain

$$
\int_\Omega u^2(x, t) \, dx \leq e^{-\alpha(t-t_2)} \left[ \int_\Omega g_2^2\{u(x, t^-_2)\} \, dx - \frac{\beta}{\alpha} \right] \quad \text{for} \quad t_2 \leq t < t_3
$$

Arguing analogously, for $t_k \leq t < t_{k+1}$, we conclude that

$$
\int_\Omega u^2(x, t) \, dx \leq e^{-\alpha t} \left[ \prod_{j=1}^{k} \left( \int_\Omega u_0^2(x) \, dx - \frac{\beta}{\alpha} \right) - \beta e^{\alpha t_1} - \beta e^{\alpha t_2} - \ldots - \frac{\beta}{\alpha} e^{\alpha t_k} \right].
$$

Thus,

$$
\int_\Omega u^2(x, t) \, dx \leq e^{-\alpha t} \left[ \prod_{j=0}^{k} M_j^2 \left( \int_\Omega u_0^2(x) \, dx - \beta e^{\alpha t} \right) - \beta e^{\alpha t_1} - \beta e^{\alpha t_2} - \ldots - \frac{\beta}{\alpha} e^{\alpha t_k} \right] \quad \text{for} \quad t_k \leq t < t_{k+1}. \quad (12)
$$

**Case $\alpha > 0$:** From (9), we get

$$
\int_\Omega u^2(x, t^-_1) \, dx \leq e^{-\alpha t_1} \int_\Omega u_0^2(x) \, dx + \frac{\beta}{\alpha}
$$
which implies
\[ \int_{\Omega} u^2(x, t) \, dx \leq e^{-\alpha (t - t_1)} M_1^2 \left[ e^{-\alpha t_1} \int_{\Omega} u_0^2(x) \, dx + \frac{\beta}{\alpha} \right] + \frac{\beta}{\alpha} \left[ M_1^2 e^{-\alpha (t - t_1)} + 1 \right] \quad \text{for } t_1 \leq t < t_2. \]

Analogously, one can get
\[ \int_{\Omega} u^2(x, t) \, dx \leq e^{-\alpha (t - t_1)} M_2^2 \left[ e^{-\alpha t_2} M_1^2 \left( \int_{\Omega} u_0^2(x) \, dx + \frac{\beta}{2} \left( M_1^2 e^{-\alpha (t - t_1)} + 1 \right) \right) + \frac{\beta}{\alpha} \right. \]
\[ \left. \leq e^{-\alpha t} M_2^2 M_1^2 \int_{\Omega} u_0^2(x) \, dx + \frac{\beta}{\alpha} \left[ M_1^2 M_2^2 e^{-\alpha (t - t_2)} + M_2^2 e^{-\alpha (t - t_2)} + 1 \right] \right] \]

whenever \( t_2 \leq t < t_3. \) Hence, for any \( t_k \leq t < t_{k+1} \) we have
\[ \int_{\Omega} u^2(x, t) \, dx \leq e^{-\alpha t} \prod_{j=1}^{k} M_j^2 \left[ \prod_{j=1}^{k} \int_{\Omega} u_0^2(x) \, dx \right. \]
\[ + \frac{\beta}{\alpha} \left( \prod_{j=1}^{k} M_j^2 e^{-a(t-t_1)} + \prod_{j=2}^{k} M_j^2 e^{-a(t-t_2)} + \ldots + M_k^2 e^{-a(t-t_k)} + 1 \right) \]
\[ \leq e^{-\alpha t} \left( \prod_{j=0}^{k} M_j^2 \right) \int_{\Omega} u_0^2(x) \, dx + \frac{\beta}{\alpha} \sum_{j=0}^{k} e^{-a(t-t_j)}. \]

**Case \( \alpha = 0 \):** Using Proposition (3) again, we have that
\[ \int_{\Omega} u^2(x, t) \, dx \leq \int_{\Omega} u_0^2(x) \, dx + \beta \, t \quad \text{for } 0 \leq t < t_1 \quad \text{and} \quad \int_{\Omega} u^2(x, t_1) \, dx \leq \int_{\Omega} u_0^2(x) \, dx + \beta \, t_1. \]

Hence, for \( t_1 \leq t < t_2 \), we get
\[ \int_{\Omega} u^2(x, t) \, dx \leq \int_{\Omega} g_1^2 \left( u(x, t_1) \right) \, dx + \beta \, (t - t_1) \]
\[ \leq M_1^2 \int_{\Omega} u_0^2(x) \, dx + M_1^2 \beta \, t_1 + \beta \, (t - t_1) \]

which also implies for \( t_2 \leq t < t_3 \) that
\[ \int_{\Omega} u^2(x, t) \, dx \leq \int_{\Omega} g_2^2 \left( u(x, t_2) \right) \, dx + \beta \, (t - t_2) \]
\[ \leq M_2^2 \left[ M_1^2 \int_{\Omega} u_0^2(x) \, dx + M_1^2 \beta \, t_1 + \beta \, (t_2 - t_1) \right] + \beta \, (t - t_2) \]
\[ \leq M_2^2 M_1^2 \int_{\Omega} u_0^2(x) \, dx + M_2^2 M_1^2 \beta \, (t_1 - t_0) + M_2^2 \beta \, (t_2 - t_1) + \beta \, (t - t_2). \]

Thus, for all \( t_k \leq t < t_{k+1} \) one can get
\[ \int_{\Omega} u^2(x, t) \, dx \leq \prod_{j=1}^{k} M_j^2 \int_{\Omega} u_0^2(x) \, dx \]
\[ + \prod_{j=1}^{k} M_j^2 \beta (t_1 - t_0) + \prod_{j=2}^{k} M_j^2 \beta (t_2 - t_1) + \ldots + M_k^2 \beta (t_k - t_{k-1}) + \beta \, (t - t_k) \]
\[ \leq \prod_{j=0}^{k} M_j^2 \int_{\Omega} u_0^2(x) \, dx + L_k \beta \, t. \]
Now let us see the last two estimate. Assume \( \alpha < 0 \) and set \([t_j] = \max\{n \in \mathbb{Z} : n \leq t_j\}\) the integer part of the impulsive instant \( t_j \). Define \( n_i \) for the cardinality of \( \{t_j : i \leq t_j < i+1\} \) setting \( N_k = \max_{i \in [0, [t_k]]} n_i \). Then,

\[
\begin{align*}
\sum_{j=0}^{k} e^{\alpha t_j} & \leq \sum_{j=0}^{k} e^{\alpha [t_j]} \leq \sum_{j=0}^{[t_k]} n_j e^{\alpha j} \leq N_k \sum_{j=0}^{[t_k]} e^{\alpha j} < \frac{N_k}{1 - e^\alpha}.
\end{align*}
\]

Thus, it follows from (12) that

\[
\int \Omega u^2(x, t) dx \leq e^{-\alpha t} \left[ \prod_{j=1}^{k} M_j \int \Omega u^2_0(x) dx - L_k \frac{1}{\alpha} \left( \frac{N_k}{1 - e^\alpha} \right) \right].
\]

As \(|t_i - t_j| > \delta\) for some \( \delta > 0 \) and all \( i, j \), we ensure \( N_k < \delta^{-1} \) for all \( k \). Hence

\[
\int \Omega u^2(x, t) dx \leq L e^{(\gamma - \alpha t)} \left[ \int \Omega u^2_0(x) dx - \frac{\beta}{\alpha} \delta^{-1} \left( \frac{1}{1 - e^\alpha} \right) \right] \quad \text{for} \quad t \geq 0
\]

since \( L_k \leq L e^\gamma \) for all \( t \in [0, t_{k+1}] \) and for all \( k \in \mathbb{N} \) proving (4).

On the other side, if \( \alpha > 0 \), we have

\[
\sum_{j=0}^{k} e^{\alpha t_j} \leq \delta^{-1} \int_0^{t_{k+1}} e^{\alpha s} ds = \frac{1}{\delta} (e^{\alpha t_{k+1}} - 1).
\]

Hence, we can argue as in (16) to obtain (5) completing the proof of Theorem 1. \( \square \)

Finally, let us show Theorem 2.

**Proof of Theorem 2.** From Proposition (4) taking \( D_1 = 0 \) we get

\[
\| u(\cdot, t^1) \|_{L^\infty(\Omega)} \leq C_0 \| u_0(\cdot) \|_{L^\infty(\Omega)} e^{(C_0 + \epsilon - \lambda_1)t_1}.
\]

Hence, from (H4) and Proposition (4) again, we obtain

\[
\| u(\cdot, t) \|_{L^\infty(\Omega)} \leq C_0 \| g_1 \left( u(\cdot, t^1) \right) \|_{L^\infty(\Omega)} e^{(C_0 + \epsilon - \lambda_1)(t-t_1)} \quad \text{for} \quad t_1 \leq t \leq t_2
\]

which implies

\[
\| u(\cdot, t) \|_{L^\infty(\Omega)} \leq C_0 M_1 \| u(\cdot, t^1) \|_{L^\infty(\Omega)} e^{(C_0 + \epsilon - \lambda_1)(t-t_1)} \leq C_0 M_1 \| u_0(\cdot) \|_{L^\infty(\Omega)} e^{(C_0 + \epsilon - \lambda_1)t}
\]

for all \( t_1 \leq t \leq t_2 \). Consequently,

\[
\| u(\cdot, t^2) \|_{L^\infty(\Omega)} \leq C_0 M_1 \| u_0(\cdot) \|_{L^\infty(\Omega)} e^{(C_0 + \epsilon - \lambda_1)t_2}.
\]

Arguing in a similar way for \( t_k \leq t < t_{k+1} \), one can have

\[
\| u(\cdot, t) \|_{L^\infty(\Omega)} \leq (C_0) k^1 M_1 M_2 \ldots M_k \| u_0(\cdot) \|_{L^\infty(\Omega)} e^{(C_0 + \epsilon - \lambda_1)t}
\]

which completes the proof of Theorem 2. \( \square \)

**Acknowledgements**

We would like to thank the anonymous referee whose comments have considerably improved the writing of the paper.
References

[1] D. D. Bainov, E. Minchev, K. Nakagawa, “Asymptotic behaviour of solutions of impulsive semilinear parabolic equation”, *Nonlinear Anal., Theory Methods Appl.* 30 (1997), no. 5, p. 2725-2734.
[2] P. W. Bates, P. C. Fife, X. Ren, X. Wang, “Travelling waves in a convolution model for phase transitions”, *Arch. Ration. Mech. Anal.* 138 (1997), no. 2, p. 105-136.
[3] R. D. Benguria, M. C. Pereira, “Remarks on the spectrum of a nonlocal Dirichlet problem”, https://arxiv.org/abs/1911.05803v3, 2019.
[4] A. Boucherif, A. S. Al-Qahtani, B. Chanane, “Existence of solutions for impulsive parabolic partial differential equations”, *Numer. Funct. Anal. Optim.* 36 (2015), no. 6, p. 730-747.
[5] A. Boucherif, A. S. Al-Qahtani, B. Chanane, “Existence of solutions for impulsive parabolic partial differential equations”, *Numer. Funct. Anal. Optim.* 36 (2015), no. 6, p. 730-747.
[6] X. Chen, “Existence, uniqueness and asymptotic stability of travelling waves in nonlocal evolution equations”, *Adv. Differ. Equ.* 2 (1997), no. 1, p. 125-160.
[7] C. Cortazar, M. Elgueta, J. D. Rossi, “Nonlocal diffusion problems that approximate the heat equation with Dirichlet boundary conditions”, *Isr. J. Math.* 170 (2009), p. 53-60.
[8] P. C. Fife, “Some nonclassical trends in parabolic and parabolic-like evolutions”, in *Trends in Nonlinear Analysis. On the occasion of the 60th birthday of Willi Jäger*, Springer, 2003, p. 153-191.
[9] J. García-Melián, J. D. Rossi, “On the principal eigenvalue of some nonlocal diffusion problems”, *J. Differ. Equations* 246 (2009), no. 1, p. 21-38.
[10] E. M. Hernández, S. M. T. Aki, H. Henríquez, “Global solutions for impulsive abstract partial differential equations”, *Comput. Math. Appl.* 56 (2008), no. 5, p. 1206-1215.
[11] L. I. Ignat, D. Pinasco, J. D. Rossi, A. San Antolín, “Decay estimates for nonlinear nonlocal diffusion problems in the whole space”, *J. Anal. Math.* 122 (2014), p. 375-401.
[12] L. I. Ignat, J. D. Rossi, “Asymptotic behaviour for a nonlocal diffusion equation on a lattice”, *Z. Angev. Math. Phys.* 59 (2008), no. 5, p. 918-925.
[13] S. Kindermann, S. Osher, P. W. Jones, “Deblurring and denoising of images by nonlocal functionals”, *Multiscale Model. Simul.* 4 (2005), no. 4, p. 109-1115.
[14] V. Lakshmikantham, D. D. Bainov, e. P. S. Simeonov, *Theory of impulsive differential equations*, Series in Modern Applied Mathematics, vol. 6, World Scientific, 1989.
[15] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer, 1983.
[16] M. C. Pereira, J. D. Rossi, “Nonlocal problems in thin domains”, *J. Differ. Equations* 263 (2017), no. 3, p. 1725-1754.
[17] ———, “Nonlocal problems in perforated domains”, *Proc. R. Soc. Edinb., Sect. A. Math.* 150 (2020), no. 1, p. 305-335.
[18] M. Pérez-Llanos, J. D. Rossi, “Blow-up for a non-local diffusion problem with Neumann boundary conditions and a reaction term”, *Nonlinear Anal., Theory Methods Appl.* 70 (2009), no. 4, p. 1629-1640.
[19] A. Rodríguez-Bernal, S. Sastre-Gómez, “Nonlinear Nonlocal Reaction-Diffusion Equations”, in *Advances in Differential Equations and Applications*, SEMA SIMAI Springer Series, vol. 4, Springer, 2014, p. 53-61.
[20] ———, “Linear nonlocal diffusion problems in metric measure spaces”, *Proc. R. Soc. Edinb., Sect. A. Math.* 146 (2016), no. 4, p. 833-863.
[21] A. M. Samoilenko, N. A. Perestyuk, *Impulse Differential Equations*, World Scientific Series on Nonlinear Science. Series A, vol. 14, World Scientific, 1995.