ON RELATIONS BETWEEN VERTEX OPERATORS, QUASICLASSICAL OPERATORS, AND PHASE SPACE COORDINATES

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Abstract

For certain situations we give a geometrical background for quasiclassical KP calculations based on an explicit connection to quantum mechanics and the collapse of coherent states to coadjoint orbits for classical operators.

1. INTRODUCTION.

Despite the fact that soliton equations such as KdV, KP, NLS, or DS often arise physically from water wave problems for example, the background mathematics involves tau functions, wave functions, Lax operators, Grassmannians, etc. and has a combinatorial nature at times with many quantum mechanical features. There are also many direct connections of soliton mathematics to quantum mechanics, conformal field theory, quantum gravity, strings, etc. and we mention here only [1-4;8;13;18;20;25;42;46;49] which have some direct relation to the matters we discuss. If one looks at this background mathematics for the KP hierarchy for example one sees the variables $x = t_1$, and $t_n\ (n \geq 2)$ arising via vertex operators in the bosonization process in representation theory a la [20], or directly as in [9] (cf. also [8;34;46]). In terms of the mathematics this should then be regarded as the basic meaning; thus for example $\partial \sim a$ and $x \sim a^\dagger$ with $[a, a^\dagger] = 1$. We do not dwell here upon the origins of KP and KdV equations via algebraic curves, which should also be regarded as another fundamental point of departure (cf. [8;30;31;38]). It seems especially important to regard the $t_n\ (n \geq 1)$, in this way since they play different roles in different theories. For example their role in theories of (p,q) minimal matter coupled to 2-D gravity seems no less (and perhaps more) fundamental than their role in water waves. In such theories $x$ plays the role of a cosmological constant ($\partial \sim$ a puncture operator), the $t_n$ are coupling constants ($\sim$ deformation parameters), and the tau function corresponds to a partition function. Thus the $x, t_n\ (n \geq 2)$ do have a coordinate
aspect but it is not clear why techniques of fast and slow variable scaling and averaging (which arise naturally in water wave problems), are appropriate here. Also, there is a sense in which one can think of the coupling constants as emerging out of the phase space of \((x, \partial)\) (cf. [49]) so their coordinate nature is somewhat secondary. Another feature here is that in passing to quasiclassical limits in physics or to dispersionless limits in water waves via a scaling \(\varepsilon t_n = T_n (\varepsilon x = X)\) the background mathematics passes from quantum mechanical to classical (or quasiclassical) and the averaging or scaling procedure has nothing intrinsically quantum mechanical about it, nor anything geometrical. Hence we were led to try to provide an underlying phase space or geometrical context in which to view the scaling mechanism.

In this paper we show how, in certain situations at least, the scaling can be related to quantum mechanical procedures in [17;47] where coherent states collapse onto a coadjoint orbit (cf. also [48]). This seems, to me at least, much more satisfactory than thinking of fast and slow variables, or averaging, which appear too mysterious as a general directive. The coherent state idea can be carried further in the soliton context as indicated in [34;35] but we do not pursue this here (see also comments and suggestions in section 4). We also give some further results on the semiclassical action principle of [3;4], relating the integrand to a limit of the Sato equation of KP theory. Finally some heuristic comments are made concerning relations of the Maslov canonical operator to dressing ideas and its possible use in quasiclassical or dispersionless soliton mathematics.

In terms of providing a theme in our presentation we suggest starting with the boson operators \(\partial \sim a, x \sim a^\dagger\) acting on a vacuum \(|0\rangle = 1\). One creates then states via \(|n\rangle = (a^\dagger)^n|0\rangle / \sqrt{n!}\) for example or coherent states via \(|z\rangle = D(z)|0\rangle\) as in Appendix A. One is working in a Fock space \(\hat{\mathcal{B}}\) here with no a priori reference to physics. Then one asks about classical and/or quasiclassical objects related to \(\hat{\mathcal{B}}\) and the state vectors above. For this one needs some sort of physical object having a classical counterpart so we create such an object via \(\hat{q} = (a + a^\dagger)/\sqrt{2}\) and \(i\hat{p} = (a - a^\dagger)/\sqrt{2}\) with \(\hat{Q} \sim \sqrt{\hbar}\hat{q}\) and \(\hat{P} \sim \sqrt{\hbar}\hat{p}\). This leads to the development in the text. Thus given \(\tilde{g}_h\) the Lie algebra generated by \(e_1 = i\hat{P}/\hbar, e_2 = i\hat{Q}/\hbar,\) and \(e_3 = i/\hbar\) with corresponding Weyl-Heisenberg group \(G_h = \{ U_h = exp[\frac{i}{\hbar}(u + \frac{1}{2}\pi \xi)]U(\pi, \xi)\}\) where \(U(\pi, \xi) \sim exp[\frac{i}{\hbar}(\pi\hat{Q} - \xi\hat{P})] \sim D(z), \) \(z = \frac{1}{\sqrt{2\pi}}(\xi + i\pi),\) one generates coherent states \(|u\rangle = U_h|0\rangle_{>h}\) \((|0\rangle_{>h}\) being a peaked vacuum for \(\hat{Q}\)). This leads to coadjoint orbits \(\Gamma = \{ \zeta = Ad_u^*\zeta_0, u \in G_h, \zeta_0 \in \tilde{g}_h^*\\}\) and for \(\lim < 0|\frac{1}{\hbar}\hat{A}|0 \rangle = < \zeta_0, \lambda > (\lambda \sim (\pi, \xi, z)\) in \(\tilde{g}_h; \hat{A} \sim (\pi e_1 - \xi e_2, ue_3)\) one has \(< u|\frac{1}{\hbar}\hat{A}|u \rangle_{>h} = < \zeta_0, Ad_u^{-1}\lambda > = AD_{\nu}^*\zeta_0, \lambda >\). The variables \((\pi, \xi)\) provide coordinates on \(\Gamma\) and there is a symplectic structure, etc. Such operators \(\hat{A}\) generate classical operators via \(\hat{A} = \int d\lambda f(\lambda) exp[\hbar\hat{A}]\) and covariant
symbols \( A_h(u) = \langle u | \hat{A} | u \rangle_h \) lead to functions \( a(\zeta) \) on \( \Gamma \). Expectation values of \( \frac{\hbar}{i} \hat{A} \) distinguish classically inequivalent states and coherent states thus collapse to \( \Gamma \) via an equivalence relation. Therefore \( \Gamma \) provides a classical geometric underpinning for \( \hat{B} \) via the ad hoc prescription of \( (\hat{p}, \hat{q}) \) based on \( (a, a^\dagger) \). This is all in the mathematics and the procedure works for any \( (a, a^\dagger) \). The only physical objects are the contrived \( (\hat{p}, \hat{q}) \). Now once this is set up one can refer to the results of Hepp involving the natural convergence \( \hat{Q} \sim \sqrt{\hbar} \hat{q} \to \xi, \hat{P} \sim \sqrt{\hbar} \hat{p} \to \pi \), etc. to discuss quasiclassical operators defined by (4.38) via \( \hat{D}_z(z) = \exp\left[\frac{i}{\hbar}(\hat{p}\hat{Q} - \hat{Q}\hat{P})\right] \) \( (\hat{\xi} = \sqrt{2}\xi, \hat{\pi} = \sqrt{2}\pi) \) where \( \epsilon^2 \sim \hbar \). These correspond in fact to certain vertex operator situations from dKP theory and allow one to give a geometrical underpinning to dispersionless or quasiclassical limits via the phase space \( \Gamma \). The original \( \partial \) and \( x \), determining \( a, a^\dagger \) in this particular situation, can then be recovered in a scaled form \( x \to X/\epsilon, \partial \to \epsilon \partial \) via (4.39) in terms of \( (\xi, \pi) \). This makes the scaling procedure part of a general mathematical framework related to an underlying phase space. Let us mention also that the idea of classicalization is of course rather more complicated than simply letting \( \hbar \) tend to 0 and an extensive theory of the quantum-classical correspondence is developed in [50]. Generally there are a number of classical systems corresponding to a given quantum system and in [50] for example one works with a classicalization in terms of the removal of quantum fluctuation effects in the physical observables, which is not necessarily the usual limit \( \hbar \to 0 \). This really has no effect on the basic mathematical theory involved here in which quasiclassical operators are defined and related to the dispersionless KP theory for example. On the other hand in terms of suppression or accommodation of quantum fluctuations (\( \sim \) rapid oscillations) the ideas of [50] may well have meaning and applicability in discussing weak and generalized solutions of KdV type water wave problems (cf. [27]).

2. SOME QUANTUM MECHANICS AND COHERENT STATES.

We begin with a sketch of some ideas in [17]. Consider a classical system

\[
(2.1) \quad \mathcal{H} = \frac{\pi^2}{2m} + V(\xi); \quad m\dot{\xi} = \pi; \quad \dot{\pi} = -V'(\xi);
\]

\[
\xi(\alpha, 0) = \xi; \quad \pi(\alpha, 0) = \pi; \quad \alpha = \frac{\xi + i\pi}{\sqrt{2}}
\]

The associated quantum mechanical system has the form

\[
(2.2) \quad i\hbar \psi_t = -\frac{\hbar^2}{2m} \psi_{qq} + V \psi; \quad H = \frac{p^2}{2m} + V_h; \quad ip_h = \hbar \partial_q;
\]
\[ \psi_t = U_h \psi_0; V_h \sim V(h^{\frac{3}{2}} \xi); U_h = \exp(-it \frac{H_h}{\hbar}) \]

(here \( H_h \) is some selfadjoint extension of the symmetric operator indicated and we use \( h \) for Dirac's slash \( h \)). In this context \( q \sim x, \hat{q} \sim \text{operator} \), and

\[ [\hat{q}_h, \hat{p}_h] = \hbar i; a = \frac{(\hat{q} + i\hat{p})}{\sqrt{2}}; a^\dagger = \frac{(\hat{q} - i\hat{p})}{\sqrt{2}}; \hat{q} = \frac{(a + a^\dagger)}{\sqrt{2}}; \hat{p} = \frac{(a - a^\dagger)}{\sqrt{2}} \]

We note that

\[ a = a_h = \frac{(\hat{q}_h + i\hat{p}_h)}{\sqrt{2h}}; a^\dagger = a_h^\dagger = \frac{(\hat{q}_h - i\hat{p}_h)}{\sqrt{2h}} \]

and this is related to scaling as follows. Let \( q \to \epsilon q = Q, \partial_q \to \epsilon \partial_Q, q = \frac{Q}{\epsilon} \)

(the situation for quasiclassical limit calculations in soliton mathematics).

Then

\[ \frac{\hat{Q}}{\epsilon} = \frac{[a + a^\dagger]}{\sqrt{2}}; [a, a^\dagger] = [a_h, a_h^\dagger] = 1; \frac{i\hat{p}}{\epsilon} = \epsilon \partial_Q = \frac{(a(a) - a^\dagger(a))}{\sqrt{2}} \]

Consider now \( z = h^{-\frac{1}{2}} \alpha \) and \( U(p, q) \sim D(z) \) (cf. (A.5)-(A.6)) with \( U^* = U^\dagger \sim D^\dagger(z) = D(-z) \sim U(-p, -q) \). Look at a typical expression \( |h^{-\frac{1}{2}} \alpha > = U(h^{-\frac{1}{2}} \alpha)|0 > \)

\[ \Xi = < h^{-\frac{1}{2}} \alpha |(\hat{q} - \frac{\xi}{\sqrt{h}})(\hat{p} - \frac{\pi}{\sqrt{h}})|h^{-\frac{1}{2}} \alpha > = \]

\[ < 0|U^*(h^{-\frac{1}{2}} \alpha)(\hat{q} - \frac{\xi}{\sqrt{h}})UU^*(\hat{p} - \frac{\pi}{\sqrt{h}})U|0 > \]

Using (A.8) one has \( a = a_h \)

\[ U^*(z)[(\hat{q} + i\hat{p})/\sqrt{2}]U(z) = \frac{(\hat{q} + i\hat{p})}{\sqrt{2}} + \frac{(\xi + i\pi)}{\sqrt{2h}} \]

which implies

\[ U^*(z)\hat{q}U(z) = \hat{q} + \frac{\xi}{\sqrt{h}}; U^*(z)\hat{p}U(z) = \hat{p} + \frac{\pi}{\sqrt{h}} \]
Consequently from (2.7)

\[ (2.10) \quad \Xi = \langle 0 | \hat{q} \hat{p} | 0 \rangle \]

This implies

\[ (2.11) \quad < h^{-\frac{1}{2}} \alpha | (\hat{q}_h - \xi)(\hat{p}_h - \pi) | h^{-\frac{1}{2}} \alpha > \to 0; < h^{-\frac{1}{2}} \alpha | \hat{Q} \hat{P} | h^{-\frac{1}{2}} \alpha > \to \xi \pi \]

Thus the scaled variables \( Q \sim \hat{q}_h \to \xi \), and \( \hat{P} \sim \hat{p}_h \to \pi \), where \( \xi, \pi \) correspond to classical values of \( \hat{q}, \hat{p} \).

Now we want to relate this to [47] where it is shown how coherent states collapse onto coadjoint orbits as \( h \to 0 \). Thus, using the notation of [47]

\[ (2.12) \quad U_h(p, q, u) = e^{\frac{iu}{\hbar} Q} e^{\frac{-iup}{\hbar}} \]

with \( e_1 = \frac{i}{\hbar} p, e_2 = \frac{iQ}{\hbar} \), and \( e_3 = \frac{i}{\hbar} \) generating a Lie algebra \( \hat{g}_h \) (\( [e_1, e_2] = e_3 \sim [\hat{Q}, \hat{P}] = i\hbar \)). Evidently, via the Baker-Campbell-Hausdorff (BCH) formula for the present situation

\[ (2.13) \quad e^A e^B = e^{\frac{1}{2}[A,B]} e^{A+B} = e^{[A,B]} e^A \]

(valid when \( [[A,B],A] = [[A,B],B] = 0 \)) one has (cf. (A.6))

\[ (2.14) \quad U_h(p, q, u) = e^{\frac{i}{\hbar}(u + \frac{1}{2}pq)} U(p, q) \]

In [17] one writes \( \lambda = \sum \lambda_i e_i \in \hat{g}_h \) and \( \xi = \sum \xi_i e'_i \in \hat{g}^* \) (for Lie groups, coadjoint orbits, etc. see e.g. [5;7;8;21;33]). The Weyl-Heisenberg (WH) group \( G_h \) generated by the \( U_h \) has Lie algebra \( \hat{g}_h \) and one has coadjoint orbits \( \Gamma = Ad^*_u \zeta_0 \) for \( \zeta_0 \in \hat{g}_h^* \) fixed. Thus a coherent state \( |u> = U_h |0> \) gives rise to a point \( \zeta = Ad^*_u \zeta_0 \in \Gamma(|0> \) will be a peaked vacuum as in (A.5)) and the set \( [u] \) of equivalence classes of coherent states corresponds to \( \Gamma \). The tangent space \( T_{\zeta} \Gamma \) at \( \zeta \) is now generated by \( ad^*_u(\lambda), \lambda \in \hat{g}_h \langle ad^*_u(\zeta), \eta >= - < \zeta, [\lambda, \eta] > \rangle \) and \( T_{\zeta} \Gamma \) can be identified with equivalence classes \( [\lambda], \lambda \in \hat{g}_h \), via \( [\lambda] = \{ \lambda' \in \hat{g}_h; ad^*_u \lambda' = ad^*_u \lambda \} \). Let \( H_\zeta \) = isotropy group of \( \zeta = \{ u \in G; Ad^*_u \zeta = \zeta \} \) so \( \Gamma \sim \frac{G}{H_\zeta} \). Explicitly, for the WH group coadjoint orbits correspond to planes \( \zeta_3 = \text{constant} \neq 0 \) since

\[ (2.15) \quad Ad^*_u(\sum \zeta_i e'_i) = \sum \hat{\zeta}_i e'_i; \hat{\zeta}_3 = \zeta_3; \hat{\zeta}_1 = \zeta_1 + p \zeta_3; \hat{\zeta}_2 = \zeta_2 + q \zeta_3 \]
Relabeling \( \zeta_1 \sim p, \zeta_2 \sim q \) one has natural coordinates \((p,q)\) on \(\Gamma\) and a symplectic structure with Poisson brackets, etc. can be written down. In particular \(\{f,g\} = \zeta_3 [f_p g_q - f_q g_p]\).

Now associated to \(\lambda = (p,q,u)\) one has the operator \(\hat{\Lambda} = (i \hat{Q}, \frac{i q}{\hbar}, \frac{i u}{\hbar})\) and \(\hbar \hat{\Lambda}\) is a ”classical operator”, i.e., \(< \zeta_0, \lambda> = \lim(\frac{1}{\hbar}) <\zeta_0|\hat{\Lambda}|0>\) exists. Classical operators are then defined via a symbolic representation

\[
(2.16) \quad \hat{A} = \int d\lambda f(\lambda) e^{\hbar \hat{\Lambda}}
\]

\((\exp(\hbar \Lambda) \sim U_h(hp, hq, hu) \text{ via BCH})\). The covariant symbol (following Berezin - cf. [39]) \(A_h(u)\) is defined as the set of coherent state expectation values

\[
(2.17) \quad A_h(u) = <u|\hat{A}|u>_{h}
\]

where \(|u>_{h} \sim U_h|0>_{h}, |0>_{h}\) being the peaked state vacuum which we simply write as \(|0>\) when no confusion can arise. Given e.g.

\[
(2.18) \quad \lim(\frac{1}{\hbar}) <0|\hat{U}^{-1}\hat{\Lambda}\hat{U}|0> = <\zeta_0, A_{\lambda^{-1}}(\lambda) = <\zeta_0, \hat{A}_{\lambda}(\lambda)\>
\]

we distinguish classically equivalent states as those \(|u>_{h}\) mapped onto a given point \(\zeta \in \Gamma\) (i.e. expectation values of \(\hbar \hat{\Lambda}\) distinguish classically inequivalent states). Then for any classical operators \(\hat{A}, \hat{B}\) as in (2.16) one has

\[
(2.19) \quad A_h(u) \rightarrow a(\zeta); (AB)_h(u) \rightarrow a(\zeta)b(\zeta); \frac{i}{\hbar}[A, B]_h(u) \rightarrow \{a(\zeta), b(\zeta)\}
\]

A more or less precise development of all this is given in [47] (cf. also [16;48]).

3. SOME SOLITON VERSIONS.

In dealing with vertex operators in soliton mathematics one will encounter terms \(\exp(x\lambda - \lambda^{-1}\partial)\) which can be viewed in several ways. As indicated in Appendix A, essentially we will gratuitously introduce a coordinate representation and coherent states and treat \((\partial, x)\) as boson operators \((a, a^\dagger)\). First scale \(x \rightarrow \epsilon x = Q, \partial_x \rightarrow \epsilon \partial_Q\) to obtain (cf. (2.5))

\[
\lambda \frac{\hat{Q}}{\epsilon} - \lambda^{-1} \epsilon \partial_Q = \frac{(\lambda - \lambda^{-1})}{\sqrt{2}} a(\epsilon) + \frac{(\lambda + \lambda^{-1})}{\sqrt{2}} a^\dagger(\epsilon); a(\epsilon) = \frac{1}{\sqrt{2}}(\hat{Q} + \epsilon^2 \partial_Q);
\]

\[
a^\dagger(\epsilon) = \frac{1}{\epsilon \sqrt{2}}(\hat{Q} - \epsilon^2 \partial_Q); \quad \frac{\hat{Q}}{\epsilon} = \frac{1}{\sqrt{2}}(a(\epsilon) + a^\dagger(\epsilon)); \quad \epsilon \partial_Q = \frac{1}{\sqrt{2}}(a(\epsilon) - a^\dagger(\epsilon))
\]
One can generate peaked states exactly as before, using $a(\epsilon)$ and $a^\dagger(\epsilon) (\epsilon^2 \sim h)$. Writing $\hat{P} = \epsilon^2 \partial_Q$ (no i) we can set

$$\frac{(\lambda \hat{Q} - \lambda^{-1} \hat{P})}{\epsilon} = \alpha a^\dagger(\epsilon) + \beta a(\epsilon); \alpha = \frac{(\lambda + \lambda^{-1})}{\sqrt{2}}; \beta = \frac{(\lambda - \lambda^{-1})}{\sqrt{2}}$$

In particular if $\lambda \in S^1$ so $\lambda^{-1} = \bar{\lambda}$ one obtains

$$\frac{(\lambda \hat{Q} - \lambda^{-1} \hat{P})}{\epsilon} = \sqrt{2}(\text{Re}(\lambda)a^\dagger(\epsilon) + i \text{Im}(\lambda)a(\epsilon))$$

which however we do not exploit.

Alternatively one can think of $x \sim a^\dagger$ and $\partial \sim a$ with (cf. (2.5))

$$a = \frac{(\hat{q} + \partial_q)}{\sqrt{2}}; a^\dagger = \frac{(\hat{q} - \partial_q)}{\sqrt{2}}; a \sim a(\epsilon) = \frac{(\hat{q} + \epsilon^2 \partial_q)}{\epsilon \sqrt{2}};$$

$$a^\dagger \sim a^\dagger(\epsilon) = \frac{(\hat{Q} - \epsilon^2 \partial_Q)}{\epsilon \sqrt{2}}; \lambda a^\dagger - \lambda^{-1} a = \frac{[(\lambda - \lambda^{-1}) \hat{Q} - (\lambda + \lambda^{-1}) \epsilon^2 \partial_Q]}{\epsilon \sqrt{2}}$$

For $\lambda \in S^1$ this becomes $(a, a^\dagger$ are based on $(\partial, x)$)

$$\lambda a^\dagger - \lambda^{-1} a = \lambda a^\dagger(\epsilon) - \bar{\lambda} a(\epsilon) = \frac{i\sqrt{2}[\text{Im}(\lambda) \hat{Q} + i \text{Re}(\lambda) \epsilon^2 \partial_Q]}{\epsilon}$$

Looking at (A.7) one has $za^\dagger_h - \bar{z}a_h = (\frac{i}{\lambda})(p\hat{Q} - q\hat{P})$ with $z = \frac{(q + ip)}{\epsilon \sqrt{2}}$ and we think of $\epsilon^2 \sim h$ so $\epsilon^2 \partial_Q \sim i\hat{P}$. Then compare with (3.5) rewritten as

$$\frac{(\lambda a^\dagger(\epsilon) - \lambda a(\epsilon))}{\epsilon \sqrt{2}} = \frac{i\sqrt{2}[\text{Im}(\lambda) \hat{Q} + i \text{Re}(\lambda) \epsilon^2 \partial_Q]}{\epsilon^2}$$

**THEOREM 3.1.** For $\lambda \in S^1$ (3.5) is quantum mechanical in nature with $p = \text{Im}(\lambda), q = \text{Re}(\lambda)$ ($p^2 + q^2 = 1$), and $z = \frac{(q + ip)}{\epsilon \sqrt{2}} = \frac{\lambda}{\epsilon \sqrt{2}}$. The $\hat{Q}$ operator however arises as in (3.4) and is not directly a scaling of $\hat{x}$ (cf. (4.21) for connections).

**REMARK 3.2.** Let now $\lambda$ be general and look at (3.2) and (3.4), with $\epsilon^2 \partial_Q = \hat{P}$. For both cases one will have the right scaling to fit in the framework of (3.6) ($\frac{(\alpha, \beta)}{\epsilon}$ in (3.2) or $\frac{\lambda}{\epsilon}$ in (3.4)) but there seems to be no way to phrase this in terms of unitary operators if $\lambda \notin S^1$ (one would want say $\lambda$ real in (3.2) and $(\alpha, \beta)$ real in (3.4)). Thus we will develop the situation of Theorem 3.1. In this case we note that the measure $d\mu$ in (A.7) is inappropriate and one thinks rather of $\oint \frac{dz}{z}$ suitably normalized. For example consider from (A.5)
(3.7) \( u_n(z) = \langle n|z\rangle = e^{-\frac{1}{2}z^n}\sqrt{n!} \)

One can then generate an orthonormal set \( w_n = c_n u_n = e^{\frac{1}{2}z^n}\sqrt{n!}u_n = z^n \) so that \( z \in S^1 \)

(3.8) \( \langle w_n|w_m\rangle = \bar{c}_n c_m e^{\frac{1}{2}i\pi} \oint \frac{z^n z^m dz}{z\sqrt{n!m!}} = \delta_{mn} \)

4. CONNECTIONS TO DISPERSIONLESS LIMITS.

We refer here to [2-4;6;8;23;24;42;43] for background. For classical KP one has a Lax operator \( L = \partial + \sum_1^\infty u_{n+1}\partial^{-n} \) and a gauge operator \( W = 1 + \sum_1^\infty w_{n}\partial^{-n} \) satisfying \( L = W\partial W^{-1}(\partial = \partial_x, x \sim t_1 - \text{some authors use } x + t_1 \text{ in the } t_1 \text{ position}). \) An operator \( M \) (or \( G \)) is defined via

(4.1) \( M = W(\sum_1^\infty kt_k\partial^{k-1})W^{-1} = G + \sum_2^\infty kt_k L^{k-1}; \)

\[ G = WxW^{-1}; [L,M] = [L,G] = 1 \]

The operator \( M \) is connected to the wave function \( We^\xi = (1+\sum_1^\infty w_{n}\lambda^{-n})e^\xi = w, \xi = \sum_1^\infty t_{n}\lambda^{n}, \text{ via } \partial_{\lambda}w = MW, \) and \( M \) arises in the study of nonisospectral symmetries for example (see [3;4;7;11;12;32]). Further

(4.2) \( M = \sum_1^\infty jt_j L^{j-1} + \sum_1^\infty V_{j+1} L^{-j-1}, V_{j+1} = -js_j(-\bar{\partial})log(\tau); \)

\[ G = x + \sum_1^\infty V_{j+1}L^{-j-1} \]

where \( \tau \) is the tau function (cf. [3;4;6-8;11;42;43] for discussion). In particular one should recall that \( (\partial_n = \frac{\partial}{\partial t_n}) \)

(4.3) \( X(\lambda)\tau = w\tau; X^*(\lambda)\tau = \bar{w}\tau; X(\lambda) = e^{\sum t_n\lambda^n}e^{-\sum \frac{\partial}{\partial t_n}}; \)

\[ X^*(\lambda) = e^{-\sum t_n\lambda^n}e^{\sum \frac{\partial}{\partial t_n}}; w^* = W^{*-1}e^{-\xi} = (1+\sum_1^\infty w_{i}^*\lambda^{-1})e^{-\xi} \]

Now to go to dispersionless KP (i.e. quasiclassical limits) one writes \( t_n \rightarrow \epsilon t_n = T_n, x = x_1 \rightarrow \epsilon x = X, \partial_n \rightarrow \epsilon \frac{\partial}{\partial t_n} = \epsilon \partial_n, \) with

(4.4) \( L_\epsilon = \epsilon \partial + \sum_1^\infty u_{n+1}(\epsilon,T)(\epsilon \partial)^{-n}; w \sim e^{\frac{1}{2}S(\lambda,T)+O(1)}; \tau \sim e^{\frac{1}{2}F(T)+O(1)} \)
The latter expression for \( \tau \) is really an "ansatz" (cf. \[3;4;51;52\]) which is verified in examples (see e.g. \[27\]). One assumes that \( u_{n+1}(\epsilon, T) \to \tilde{u}_{n+1}(T) \) as \( \epsilon \to 0 \) and we call this again \( u_{n+1}(T) \); we omit a discussion of the philosophy of slow variables, averaging, etc. (cf. \[3;4;13;18;25;42;43\]). We recall that the standard embellished (with \( M \)) KP hierarchy equations \( \partial_n L = [B_n, L], \partial_n M = [B_n, M], B_n = L_+^n, Lw = \lambda w, \partial_n w = B_n w, \) become

\[
(L, M) = \epsilon; \epsilon \partial_n w = B_n w; w = \frac{\tau(\epsilon, T, n)}{\tau(\epsilon, T)} \sum_{i=1}^{\infty} \lambda_i^{w(\frac{\epsilon}{\tau})}
\]

etc. Write now \( P = \partial S = \frac{\partial S}{\partial T}, (T_1 = X) \) and \( \epsilon i \partial^i w \to P^i w \) as \( \epsilon \to 0 \). Then \( Lw = \lambda w \) becomes

\[
\lambda = P + \sum_{n=1}^{\infty} u_{n+1} P^{-n}; P = \lambda - \sum_{n=1}^{\infty} P_i \lambda^{-i}
\]

(the latter equation being simply the inversion of the first). From \( B_n w = \sum_{n=1}^{\infty} b_{n,m}(\epsilon \partial)^m w \) one gets (with some abuse of notation)

\[
B_n \to B_n = \sum_{n=1}^{\infty} b_{n,m} P^n = \lambda^n; \partial_n S = B_n(P); \partial_n P = \hat{\partial}B_n(P) = \partial_X B_n + \partial P B_n \frac{\partial P}{\partial X}; M \to \mathcal{M} = \sum_{n=1}^{\infty} n T_n \lambda^n - 1 + \sum_{n=1}^{\infty} V_{n+1} \lambda^{-n-1}
\]

and writing \( \{A, B\} = \partial_p A \partial_p B - \partial_p A \partial_p B \) there results

\[
\partial_n \lambda = \{B_n, \lambda\}; \partial_n \mathcal{M} = \{B_n, \mathcal{M}\}; \{\lambda, \mathcal{M}\} = 1; \partial_n \mathcal{S} = B_n; \partial_\lambda \mathcal{S} = \mathcal{M};
\]

\[
S = \sum_{n=1}^{\infty} T_n \lambda^n + \sum_{n=1}^{\infty} S_{n+1} \lambda^{-n}; \partial S_{n+1} = -P_n; V_{n+1} = -n S_{n+1} = \partial_n \log(\tau) = \partial_n F
\]

Then one defines \( \tau^{dKP} = e^F, F = F(X, \dot{T}). \)

In \[3;4\] we showed how this framework is related to the Hamilton- Jacobi theory of \[32\] for dKP. In particular it is important to rescale the \( T_n \) to \( T_n = n T_n, \partial_n \to \frac{n}{\partial T_n} \) (this rescaling is also involved in the Landau-Ginsburg equation and connections to gravity (cf. \[2;4;13;25;42;43\;49\]). Then one gets

\[
\partial_n S = \frac{\lambda^n}{n}; \partial_n \lambda = \{Q_n, \lambda\}; Q_n = B_n \frac{\lambda^n}{n}; \partial_n P = \hat{\partial}Q_n = \partial Q_n + \partial P Q_n \partial P
\]

Further with \( (P, X, T_n), n \geq 2 \), as Hamiltonian variables, \( P = P(X, T_n), -Q_n = -Q_n(P, X, T_n) = \) Hamiltonian, there results

\[
\hat{P}_n = \frac{\partial P}{\partial T_n} = \partial Q_n; \hat{X}_n = -\partial P Q_n
\]
where

\[ \hat{\xi} = S_\lambda - \sum_2^n T_n' \lambda^{n-1} = \text{lim} W x W^{-1} = \text{lim} G = M - \sum_2^n T_n' \lambda^{n-1} \]

Let us also remark that, posing \( w^* = \exp(\sum \frac{D_n}{\epsilon} + O(1)) \), one obtains as above

\[ (w^*) = e^{-\xi(\frac{x}{\epsilon})} \]

\[ \log(w^*) \sim \frac{S^*}{\epsilon} + O(1) \sim -\frac{1}{\epsilon} \sum_1^\infty T_n \lambda^n + \frac{1}{\epsilon} \sum_1^\infty \frac{\delta a F}{n \lambda} \]

Consequently \( S^* = -S \).

We recall next that tau functions are generated via Bäcklund type actions from the Clifford group (cf. [8;9;19]). In the bosonic picture this translates into generation by vertex operators (cf. (4.3))

\[ X(\lambda, \zeta) = \frac{(\lambda-\zeta)}{\epsilon} X^*(\zeta) X(\lambda) = \frac{(\lambda-\zeta)}{\epsilon} X(\lambda) X^*(\zeta) \]

\[ = e^{\xi(x,\lambda)-\xi(x,\zeta)} e^{-\xi(\bar{\lambda},\lambda^{-1})+\xi(\bar{\lambda},\zeta^{-1})} \]

\( (\bar{\theta} = (\partial_1, \frac{1}{2} \partial_2, ...)) \). Write now e.g. \( \xi'(x, \lambda) = \sum_2^\infty x_n \lambda^n \) with \( \lambda \in S^1 \) so via (2.13)

\[ X(\lambda) = e^{x \lambda + \xi'(x, \lambda)} e^{-\bar{\lambda} \theta - \xi'(\bar{\lambda}, \lambda)} = e^{x \lambda} e^{\xi'(x, \lambda)} e^{-\bar{\lambda} \theta} e^{-\xi'(\bar{\lambda}, \lambda)} \]

\[ = e^{\xi'(x, \lambda)} e^{-\xi'(\bar{\lambda}, \lambda)} e^{-\frac{1}{2} [x, \theta]} e^{x \lambda - \bar{\lambda} \theta} = e^{\frac{1}{2} \epsilon} e^{\xi'(x, \lambda)} e^{-\xi'(\bar{\lambda}, \lambda) \hat{D}(\lambda)} \]

where \( \hat{D}(\lambda) = e^{\lambda a_1 - \lambda a} \) (cf. (A.6) where \( D(z) = \exp(z a_1^1 - z a_1) \) and recall \( \frac{1}{\epsilon \sqrt{2}} \sim z \) from (3.6)). Similarly for \( \zeta \in S^1 \)

\[ X^*(\zeta) = e^{-\xi'(x, \zeta)} e^{\xi'(\bar{\lambda}, \zeta)} e^{\frac{1}{2} \hat{D}(-\zeta)} \]

This leads to

\[ X(\lambda, \zeta) = \frac{(\lambda-\zeta)}{\epsilon} \hat{X}(\lambda, \zeta) \hat{D}(\lambda) \hat{D}(-\zeta); \]

\[ \hat{X}(\lambda, \zeta) = e^{\xi'(x, \lambda)} e^{-\xi'(\bar{\lambda}, \lambda)} e^{-\xi'(x, \zeta)} e^{\xi'(\bar{\lambda}, \zeta)} \]

One could continue and pass any finite number of Bose operators \( a_n \sim \)}
\[ \partial_n, a_n^\dagger \sim x_n \] to the right (but we do not know at this time how to construct the limiting geometrical object \( \sim \) dispersionless Grassmannian). Here we simply want to distinguish the \((x, \partial)\) variables as determining a phase space in the spirit of dispersionless limits (cf. (4.4)-(4.11)) and hence we will treat \((x, \partial)\) as special and think of the other \(x_n \sim t_n\) as time parameters.

One can also think of multisoliton tau functions of the form \( X^2(\lambda, \zeta) = 0 \)

\[ (4.17) \quad \tau_N = \prod_1^N (1 + a_j X(\lambda_j, \zeta_j)) \cdot 1 \]
in the bosonic picture, and more generally one considers a limiting procedure with \( N \to \infty \) (cf. [9;19]). It is folkloric that such multisoliton constructions will be dense in some sense but we will not try to clarify that here. Such \( \tau_N \) arise from a construction \( \tau(x, g) = \langle 0 \mid e^{\int H(x)} g \mid 0 \rangle \) for \( H(x) = \sum_1^\infty x_k J_k, J_k \to \partial_k, g = e^{\int \sum_1^N (\lambda_j - \bar{\lambda}_j)} \psi(\lambda_j) \psi^*(\lambda_j) \) where the \( \psi, \psi^* \) operators are built up from free fermion operators and need not concern us here (cf. [8;9;19]).

For the quasiclassical or dispersionless situation one must insert \( \epsilon \) at appropriate places to arrive e.g. at (cf. [42;43])

\[ (4.18) \quad \tau_\epsilon \sim \prod_1^N (1 + \frac{2\epsilon}{\epsilon} X(\frac{T_\epsilon}{\epsilon}, \lambda_j, \zeta_j)) \cdot 1 \]

and we will assume \( \lambda_j, \zeta_j \in S^1 \) in what follows. The vacuum vector \( 1 \in \mathcal{B} = \) polynomial Fock space corresponds to the boson representation and we can think of \( \tau_\epsilon \in \hat{\mathcal{B}} = \) Fock space based on \((x, \partial)\) with the \( x_n \sim t_n \) or \( T_n \) variables as parameters. Then the vacuum \( 1 = |0\rangle \) can also be represented in terms of peaked states by our linking procedure and we can look at a phase space \( \sim \) coadjoint orbit based on coordinates \( p, q \) as in section 2. The question then is to relate this to the phase space based on \( X, P \) or \( \lambda, \hat{\xi} \) obtained in (4.4)-(4.11).

**REMARK 4.1** Let us note that for \( \lambda \in S^1, \lambda a^\dagger - \bar{\lambda} a \) in (3.5) (cf. Theorem 3.1) has the form \((i\sqrt{2}/\epsilon)(p\hat{Q} - q\hat{P})\) with

\[ (4.19) \quad \frac{1}{\sqrt{2}}(\lambda a^\dagger - \bar{\lambda} a) = \zeta a^\dagger(\epsilon) - \bar{\zeta} a(\epsilon) = \frac{i}{\epsilon} (p\hat{Q} - q\hat{P}) \]

so \( \lambda a^\dagger - \bar{\lambda} a \) lies inbetween a classical operator \( i(\bar{p}\hat{Q} - \bar{q}\hat{P}) \) and a coherent state generator \( \frac{i}{\epsilon} (\bar{p}\hat{Q} - \bar{q}\hat{P}) \), where \( \bar{p} = \sqrt{2}Im(\lambda) \) and \( \bar{q} = \sqrt{2}Re(\lambda)((\bar{p}, \bar{q}) = \sqrt{2}(p, q)) \) Now via section 2 one has for \( z = h^{-\frac{1}{2}} \alpha \sim \frac{1}{\sqrt{2}}(\xi + i\pi) \)

\[ (4.20) \quad < z|((\hat{Q} - \xi)(\hat{P} - \pi))|z \to 0 \]
so $\hat{Q} \to \xi$ and $\hat{P} \to \pi$ (note $(\xi, \pi) \neq (q,p)$). Observe that if we write in (3.4) $a \sim \epsilon \partial_X$ and $a^\dagger \sim \frac{X}{\epsilon}$ then

$$\frac{X}{\epsilon} = \frac{(\hat{Q} - i\hat{P})}{\epsilon \sqrt{2}}, \quad e^2 \partial_X = \frac{1}{\epsilon \sqrt{2}} (\hat{Q} + i\hat{P})$$

(4.21)

Formally it follows that $X \to \frac{(\xi - i\pi)}{\sqrt{2}}$ and $e^2 \partial_X \to \frac{(\xi + i\pi)}{\sqrt{2}}$ (cf. (4.33) for further validation). Thus phase space variables $\xi, \pi$ coming from $z$ can be compared to the scaled $X$ variable, which arises in the dispersionless KP situation. We emphasize here that $(\pi, \xi) \neq (p, q)$, where we stipulate that $(p, q) \sim \zeta$ now.

**REMARK 4.2** Assume we have built up a tau function as in (4.18) and think of it in $\hat{B}$. One can carry all $\partial_n(n \geq 2)$ to the right to work on the 1 vacuum of $\mathcal{B}$ for example so that there remains a sum of terms = functions of $(T_n, \epsilon), n \geq 2$, times operators $\hat{D}_\epsilon(\lambda_i) \hat{D}_\epsilon(-\xi_i) \frac{\lambda_{i} - \xi_{i}}{\lambda_{n}}$ as in (4.16) (see below for $\hat{D}_\epsilon$). Note how $\epsilon$ arises in (4.18) so analogously to (4.16) we would have terms

$$\frac{a}{\epsilon} \hat{X} (\hat{\xi} \epsilon, \lambda_j, \xi_j) \frac{(\lambda_j - \xi_j)}{\lambda_j} \hat{D}_\epsilon(\lambda_j) \hat{D}_\epsilon(-\xi_j);$$

$$\hat{D}_\epsilon(\lambda_j) = e^{\frac{1}{\epsilon \lambda_j} X - \epsilon \partial_X \lambda_j} = e^{\frac{i}{\epsilon} (\hat{p}_j \hat{Q} - \hat{q}_j \hat{P})}$$

(4.22)

One can use $a(\epsilon), a^\dagger(\epsilon)$, based on $(\frac{X}{\epsilon}, \epsilon \partial_X) \equiv (x, \partial)$ interchangeably with $(x, \partial)$, consistent with (4.21). The $\hat{D}_\epsilon(\lambda)$ in (4.22) lie inbetween a classical operator and a coherent state generator and we will refer to them as semiclassical or quasiclassical operators.

**THEOREM 4.3.** Let $\hat{D}_\epsilon(\lambda)$ be semiclassical. Then one can write

$$\sqrt{2} |\hat{D}_\epsilon(\lambda)| z > \sim e^{-\frac{1}{2} \epsilon |(\frac{\hat{Q}}{\sqrt{2}}) - (\hat{\xi} - \hat{\eta})|}$$

(4.23)

where $z \sim h^{-\frac{1}{2}} \alpha = h^{-\frac{1}{2}} \frac{(\xi + i\pi)}{\sqrt{2}}, h = \epsilon^2, \hat{p} = \sqrt{2}p, \hat{q} = \sqrt{2}p, p = \text{Im}(\lambda), q = \text{Re}(\lambda), \lambda a^\dagger(\epsilon) - \bar{\lambda} a(\epsilon)) = \frac{i}{\epsilon} (\hat{p} \hat{Q} - \hat{q} \hat{P}), \hat{Q} = \hat{q}_h = \epsilon \hat{q}; \hat{P} = \hat{p}_h = \epsilon \hat{p}$, and $\hat{D}_\epsilon(\lambda) = \text{exp}(\lambda a^\dagger(\epsilon) - \bar{\lambda} a(\epsilon))$.

**Proof:** First consider (2.11) in the form

$$<z|\hat{q} - \frac{\xi}{\sqrt{h}}|z> = <z|\frac{(\hat{Q} - \hat{\xi})}{\sqrt{h}}|z> = 0$$

(4.24)

$$\Rightarrow <z|\hat{Q} - \xi|z> = \sqrt{h} <0|\hat{q}|0> \to 0$$

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Similarly \(< z|(^{\hat{P}} - \pi)|0 >\) = \(h^{\frac{1}{2}} < 0|\hat{p}|0 >\) \(\rightarrow 0\). Here \(z = h^{-\frac{1}{2}} \alpha = h^{-\frac{1}{2}}(\xi + i\pi)\sqrt{2}\).

Consider \(< z|(\lambda a^\dagger - \lambda \alpha)|z >\) for general \(z\). Write \(\frac{\lambda}{\epsilon\sqrt{2}} = \zeta\) so

\[\text{(4.25)} \quad \zeta a^\dagger(\epsilon) - \zeta a(\epsilon) = \frac{i}{\epsilon}(p\hat{Q} - q\hat{P}); \lambda a^\dagger(\epsilon) - \lambda a(\epsilon) = \frac{i}{\epsilon}\sqrt{2}(p\hat{Q} - q\hat{P})\]

Let \(z \sim h^{-\frac{1}{2}} \alpha\) as above, based on \((\xi, \pi)((\xi, \pi) \neq (q, p))\). We know

\[\text{(4.26)} \quad < z|(\hat{Q} - \xi)|z > = \epsilon < 0|\hat{q}|0 > \rightarrow \frac{i}{\epsilon}p\sqrt{2} < z|(^{\hat{Q}} - \xi)|z > = -i\epsilon q\sqrt{2} < 0|\hat{p}|0 >\]

Consequently

\[\text{(4.27)} \quad < z|(\lambda a^\dagger - \lambda \alpha(\epsilon) - (\frac{i\sqrt{2}}{\epsilon})(p\xi - q\pi))|z > = \frac{i\sqrt{2}}{\epsilon} < z|(p\hat{Q} - q\hat{P}) - (p\xi - q\pi)|z > = i\sqrt{2}(p < 0|\hat{q}|0 > - q < 0|\hat{p}|0 >)\]

We can assume \(0|\hat{q}|0 > = 0|\hat{p}|0 > = 0\) (see below) so

\[\text{(4.28)} \quad < z|(\lambda a^\dagger(\epsilon) - \lambda \alpha(\epsilon))|z > = \frac{i\sqrt{2}}{\epsilon}(p\xi - q\pi)\]

where \(p, q\) come from \(\lambda\) and \(\pi, \xi\) from \(z\). Note for \(z = 0, \xi = 0\) we have

\(< 0|\hat{Q} - 0|0 > = 0 \Rightarrow < 0|\hat{q}|0 >= 0\) etc. This suggests that functions like \({\hat{D}}_\epsilon(\lambda) = \exp(\lambda a^\dagger(\epsilon) - \lambda \alpha(\epsilon))\) should also have limit expressions. In particular

\[\text{(4.29)} \quad < z|{\hat{D}}_\epsilon(\lambda)|z > \sim ce^{(\frac{i\sqrt{2}}{\epsilon})(p\xi - q\pi)}\]

should be valid for a suitable \(c\) (see below). For confirmation we recall first from [47] that for classical operators \({\hat{A}}_h(\lambda) = < z|\hat{A}|z >_h\)

\[\text{(4.30)} \quad (AB)_h = \int d\mu(|\langle u|u'\rangle|^2)[\frac{<u'|\hat{A}|u'>}{<u'|u'>} <u'|\hat{B}|u'>] + o(1) = A_h(u)B_h(u) + o(1)\]

One could surely base a proof of (4.31) and a determination of \(c\) based upon (4.30) but there is a simpler approach using a theorem from [17]. First note in (4.21) with \(X \rightarrow \frac{(\xi - i\pi)}{\sqrt{2}}, \epsilon^2 \partial_X \rightarrow \frac{(\xi + i\pi)}{\sqrt{2}}\), based on section 2 (as in (4.28)), that \((\rightarrow \text{in the sense of (4.28)})\)
and one can use this this as a validation of the identification in (4.21). Moreover the arrow $\rightarrow$ is unnecessary since in fact $<z|\hat{Q}|z> = \xi <z|z> = \xi$ etc.

Now to confirm (4.29) we refer to [17] and recall from section 2, $U(h^{-\frac{1}{2}}\alpha) \sim D(z), z = h^{-\frac{1}{2}}\alpha$. For suitable problems (2.1)-(2.2) one specializes Theorem 2.1 in [17] (cf. also [45]) to $t = 0$ to obtain (limit in strong operator topology)

\begin{equation}
U(h^{-\frac{1}{2}}\alpha)^\dagger e^{i[r(h^{-\frac{1}{2}}\xi) + s(h^{-\frac{1}{2}}\pi)]} U(h^{-\frac{1}{2}}\alpha) \rightarrow e^{i(r\hat{q} + s\hat{p})}
\end{equation}

Since we are at liberty to choose any problem (2.1)-(2.2) giving peaked states (based on the identification (3.4)) there is no problem in using Theorem 2.1 of [17]. Then since $\hat{D}_r(\lambda) = \exp\left(\frac{i}{\epsilon}(\hat{p}\hat{Q} - \hat{q}\hat{P})\right), h \sim \epsilon^2, \hat{Q} \sim \hat{q}_h$, etc. we take $r = \hat{p}$ and $s = -\hat{q}$ in (4.32) to obtain

\begin{equation}
<z|e^{i\frac{1}{\epsilon}[\hat{p}(\hat{Q} - \xi) - \hat{q}(\hat{P} - \pi)]}|z> \rightarrow<0|e^{i(\hat{p}\hat{q} - \hat{q}\hat{p})}|0> = c(\hat{p}, \hat{q})
\end{equation}

But $i(\hat{p}\hat{q} - \hat{q}\hat{p}) \sim \lambda a^\dagger - \lambda a$ with $a|0> = 0$ so by (2.13) $c(\hat{p}, \hat{q}) = e^{-\frac{1}{\epsilon}}$ and

\begin{equation}
<z|e^{i\frac{1}{\epsilon}(\hat{p}\hat{Q} - \hat{q}\hat{P})}|z> \sim e^{-\frac{1}{\epsilon}} e^{i\frac{1}{\epsilon}(\hat{p}\xi - \hat{q}\pi)}
\end{equation}

which corresponds to (4.29). QED

From the construction (2.16) for classical operators one sees an immediate generalization for quasiclassical operators. Thus eliminate the $u$ term in (2.16) and write (cf. (2.14))

\begin{equation}
\hat{A} = \int \int dp dq \tilde{f}(p, q) U(hp, hq)
\end{equation}

(e.g. $u = -\frac{1}{\epsilon^2}pq$ removes $u$ and factors of $h$ in $\tilde{f}$ automatically vanish - or simply integrate out the $u$ term). Now $U(hp, hq) \sim \exp[i(p\hat{Q} - q\hat{P})]$ so we suggest that general quasiclassical operators $\hat{A}_{QC}$ can be obtained via

\begin{equation}
\hat{A}_{QC} = \int \int d\tilde{p} dq \tilde{f}(\tilde{p}, q) \hat{D}_r(\lambda); \hat{D}_r(\lambda) = e^{i\frac{1}{\epsilon}[\hat{p}\hat{Q} - \hat{q}\hat{P}]}
\end{equation}

In view of (4.23), for this to make sense one would specify that

\begin{equation}
\frac{1}{\epsilon}\lambda X - \epsilon\lambda \partial \rightarrow \frac{1}{\sqrt{2} \epsilon}(q + ip)(\xi - i\pi) - (q - ip)(\xi + i\pi)
\end{equation}

= \frac{i\sqrt{2}}{\epsilon}(p\xi - q\pi)
be valid. Thus the integral should be well defined and the error term suitably small. Heuristically we state here (cf. Appendix B for more structure)

**COROLLARY 4.4.** One can heuristically define a class of quasiclassical operators via (4.36).

**REMARK 4.5.** We note that the idea of quasiclassical objects in [42;43] uses a KP based $\hbar$ instead of $\epsilon$, which overlooks the type of explicit connection to quantum mechanical ideas indicated in the present paper. Physically one can perhaps think of the $\epsilon \sim \sqrt{\hbar}$ smoothing of quantum fluctuations as related to an interaction between dispersionless limits and weak solutions in fluid dynamics.

**REMARK 4.6.** Theorem 4.3 shows that phase space calculations based on $\hat{D}_\epsilon(\lambda)$ have an asymptotic character as in (4.23) and this agrees (up to a constant $c(\tilde{p}, \tilde{q})$ with a direct calculation based on $\frac{1}{\epsilon}X$ and $\epsilon^2 \partial$ as in (4.21) and (4.31) (cf. (4.22)). The soliton calculation, not using $\hat{B}$, would pass $\partial_X$ to the right where it would act on 1, thus eliminating it’s contribution, and this would change the exponential factor corresponding to (4.23). However (4.23) can be recast via (4.31) in terms of $\frac{1}{\epsilon}X$ and $\epsilon^2 \partial \sim i \mathcal{P}$, (cf. Remark 4.7), and eliminating the $i \mathcal{P}$ contribution one obtains the equivalent soliton calculation. Note that the soliton calculation approach does not a priori inject $S$ via $P \sim \partial S$ into the equations; $S$, and thence $P$, appears as a result of the calculations. In this connection we note also that a finite product of terms $\hat{D}_\epsilon(\lambda_j) \hat{D}_\epsilon(-\zeta_j)$ arising out of (4.18) for example leads to (cf. (4.22), (2.13))

$$
\prod \left( \frac{\epsilon^2}{\epsilon} \right)^{(\lambda_j - \zeta_j)} \hat{X} \left( \frac{\epsilon^2}{\epsilon}, \lambda_j, \zeta_j \right) \prod \hat{D}_\epsilon(\lambda_j) \hat{D}_\epsilon(-\zeta_j) = \tilde{\phi}(\lambda_j, \zeta_j, \hat{T}, \epsilon) \hat{D}_\epsilon(\sum \lambda_j - \sum \zeta_j)
$$

where $
\tilde{\phi} = \phi \exp(-iM \sum \lambda_j \tilde{\zeta}_j) \cdot \Xi, \quad \Xi = \exp[(\lambda_1 - \zeta_1) \sum_2^N ((\tilde{\lambda}_j - \tilde{\zeta}_j) + (\lambda_2 - \zeta_2) \sum_3^N (\tilde{\lambda}_j - \tilde{\zeta}_j) + ... + (\lambda_{N-1} - \zeta_{N-1}) (\tilde{\lambda}_N - \tilde{\zeta}_N)].
$$

Thus all terms (4.38) are expressed as operators in $\hat{B}$ via $\hat{D}_\epsilon(\Lambda - Z), \Lambda = \sum \lambda_j, Z = \sum \zeta_j,$ and the estimates obtained via (4.23) apply. It follows that the asymptotic estimates $\tau \sim \exp(\frac{1}{\epsilon}F)$ for KP based on quasiclassical soliton calculations are unchanged and our approach gives a geometrical background for the quasiclassical soliton procedure. Conceptually one can avoid the original scaling step in $x$ by arguing via $a \sim \partial$ and $a^\dagger \sim x$. Thus one has given various $\hat{D}(\lambda) = \exp(\lambda a^\dagger - \lambda a)$ and the insertion of $\epsilon$ can be thought of as a way of introducing peaked states and thence coadjoint orbit variables.
REM 4.7. Let us write $\epsilon^2 \partial_X \sim i\mathcal{P}$ and rephrase (4.31) as

$$
(4.39) \quad X \sim \frac{(\xi - i\pi)}{\sqrt{2}}; i\mathcal{P} \sim \frac{(\xi + i\pi)}{\sqrt{2}}; \xi \sim \frac{(X + i\mathcal{P})}{\sqrt{2}}; i\pi \sim \frac{(i\mathcal{P} - X)}{\sqrt{2}}
$$

Here $(\xi, \pi) \sim z, (q, p) \sim \lambda$, and $(X, \mathcal{P}) \sim a(\epsilon)$. From section 2 the z coordinates appear in Poisson brackets $\{ f, g \} = \zeta_3(f_\pi g_\xi - f_\xi g_\pi)$ while from [3;4] Poisson brackets for $X$ and $P = \partial S$ are eventually defined via $\{ A, B \} = \partial_P A \partial_X B - \partial_X A \partial_P B$ so that $\{ P, X \} = 1$. From (4.39)

$$
(4.40) \quad \partial_\pi = \frac{i}{\sqrt{2}}(\partial_P - i\partial_X); \partial_\xi = \frac{i}{\sqrt{2}}(\partial_X - i\partial_P); f_\pi g_\xi - f_\xi g_\pi = f_\pi g_X - f_X g_\pi
$$

so one has a natural identification of $P$ and $\mathcal{P}$ for $\zeta_3 = 1$. Note also that (4.10) becomes then $\dot{\xi} = -\partial_\pi \mathcal{Q}_n$ and $\dot{\pi} = \partial_\xi \mathcal{Q}_n$.

REM 4.8. There is actually no restriction of the form $(\lambda_j, \zeta_j) \in S^1$ as long as we keep our coherent states $|z>$ based as before on unitary operators. Thus e.g. in (4.38) replace $\tilde{D}_\pi(\lambda_j)$ and $\tilde{D}_\lambda(-\zeta_j)$ arising out of (4.18) by $exp(\frac{i}{\xi}(\lambda_j x - \epsilon \partial \lambda_j^{-1})) = exp(\lambda_j a^\dagger(\epsilon) - \lambda_j^{-1} a(\epsilon)) = \tilde{D}(\lambda_j, -\lambda_j^{-1})$. Then $\tilde{D}(\lambda_j, -\lambda_j^{-1}) \tilde{D}(-\zeta_j, -\zeta_j^{-1}) = exp(\frac{i}{2}((\zeta_j \lambda_j^{-1} - \lambda_j \zeta_j^{-1}))) \tilde{D}(\lambda_j - \zeta_j, \zeta_j^{-1} - \lambda_j^{-1})$ so finite products as in (4.38) become

$$
(4.41) \quad \phi(\lambda_j, \zeta_j, \tilde{T}, \epsilon) \prod e^{\frac{i}{2}(\zeta_j \lambda_j^{-1} - \lambda_j \zeta_j^{-1})} \tilde{D}(\lambda_j - \zeta_j, \zeta_j^{-1} - \lambda_j^{-1}) = \phi(\lambda_j, \zeta_j, \tilde{T}, \epsilon) \tilde{D}(\sum(\lambda_j - \zeta_j), \sum(\zeta_j^{-1} - \lambda_j^{-1})) = \hat{\phi}\tilde{D} (\mu, \nu)
$$

For specific choices of $(\lambda_j, \zeta_j)$ one could use (4.23), etc.

REM 4.9. One can deal with coherent state manifolds based on $gl(\infty)$ in a general context suggested in [34;35]. Given e.g. imaginary time coordinates $t_n$, Lagrange equations can be developed via path integrals, etc. (cf. [26;34;35;37;41]) but a geometrical transition to quasiclassical or dispersionless limits via collapse of coherent states to coadjoint orbits seems unclear at this time. We expect that some variation on the geometric ideas indicated here via the connection to quantum mechanics should also apply in the general soliton situation. That is, the coherent state manifold should collapse onto coadjoint orbits, and the corresponding Kähler structures should be related, etc. We have not yet made this all explicit however.

REM 4.10. Let us be explicit about what has been accomplished here. We start basically with $\tilde{D}(\lambda) \cdot 1 = exp(\lambda x - \tilde{\lambda} \partial) \cdot 1 = exp(\lambda X)$ which becomes $\tilde{D}_\pi(\lambda) \cdot 1$ or $exp(\frac{i}{\xi} \lambda x)$ by scaling. On the other hand given $e^{\lambda a^\dagger - \bar{\lambda} a}$ ($a^\dagger \sim x, a \sim \partial$), we insert $\epsilon$ to generate peaked states via $a = a(\epsilon) = \ldots$
\[ \frac{1}{\epsilon \sqrt{2}}(\hat{Q} + \epsilon^2 \partial_Q), a^\dagger = a^\dagger(\epsilon) = \frac{1}{\epsilon \sqrt{2}}(\hat{Q} - \epsilon^2 \partial_Q) \] as in (3.4)-(3.5). Then Theorem 4.3 applies and one obtains \[ \langle z | \hat{D}_\epsilon(\lambda) | z \rangle \sim e^{-\frac{\epsilon}{2} \exp \left[ i \hat{Q} - \hat{P} \hat{\xi} - \hat{Q} \pi \right]} = \hat{d}(\epsilon, \lambda, \xi, \pi) \] as the quasiclassical object associated with the operator \( \hat{D}_\epsilon(\lambda) \). Then via (4.21), (4.31), (4.39) one creates an \( X \) variable (incidentally the same as \( X \) obtained by scaling) and rewrites \( \hat{d}(\epsilon, \lambda, \xi, \pi) \) as the quasiclassical object associated with the operator \( \hat{D}_\epsilon(\lambda) \).

Then via (4.21), (4.31), (4.39) one creates an \( X \) variable (incidentally the same as \( X \) obtained by scaling) and rewrites \( \hat{d}(\epsilon, \lambda, \xi, \pi) \) in terms of \( X \) and \( P \). In the situation where \( \hat{D}(\lambda) \) acts on 1 we need only concern ourselves with \( \check{d}(\lambda) = e^{\lambda x} \) in which case the corresponding quasiclassical object \( \check{d}(\epsilon, \lambda, \xi, \pi) \) becomes \( \check{d}(\epsilon, \lambda, X) = e^{\frac{\lambda}{2} X} \) (cf. also (4.30)). This may seem like a lot of work to go from \( e^{\lambda x} \) to \( e^{\frac{\lambda}{2} X} \) but conceptually we have eliminated the ideas of averaging or scaling the \( x \) variable (via fast and slow variables, etc.); such notions have been replaced by a more geometrical construction. Evidently one may apply such techniques to any finite number of \( t_n, \partial_n \) as well, and this could provide a background structure for some kind of limiting geometrical object (Grassmannian) alluded to in Remark 4.9. In the event that the \((\xi, \pi)\) phase space is based on a constrained region (e.g. \( S^1 \)) one has recourse to peaked state constructions on such regions (cf. [10]).

5. PHASE SPACES, ACTION, AND WKB.

Let us put some material from [3;4] (cf. (4.1)-(4.12)) in a broader perspective and indicate at the same time some calculations in the dispersionless theory. We recall from [3;4] and section 4

\[ \tilde{S} = S - \sum_2^\infty T_n \lambda^n = X \lambda - \sum_1^\infty \frac{\partial_n F}{n} \lambda^{-n} = - \int_X [P(X', \lambda, \hat{T}_n) - \lambda] dX' + \lambda X; \frac{\partial_n F}{n} = - \int_X P_n dX'; P - \lambda = - \sum_1^\infty P_j \lambda^{-j} \]

\[ \hat{\xi} = \hat{S}_\lambda; \hat{S}_X = P = S_X; \hat{\partial}_n \hat{S} = \hat{Q}_n - R_n; \hat{Q}_n = \frac{1}{n} B_n = \hat{\partial}_n S; \]

\[ T'_n = n T_n; R_n = \frac{1}{n} \lambda^n; S_\lambda = \mathcal{M}; K_n = - R_n \]

Now for \( H_n = -Q_n \) we can write

\[ PdX - H_n dT'_n = -\hat{\xi} d\lambda - K_n dT'_n + d\tilde{S} \]

which reveals \( \tilde{S} \) as a generating function of type \( \tilde{S}(X, \lambda, \hat{T}_n) = F_1(X, \lambda, \hat{T}_n) \) for a canonical transformation \((X, P) \rightarrow (\lambda, -\hat{\xi})\), and \((\lambda, -\hat{\xi})\) are action-angle variables with
\begin{equation}
\frac{d\lambda}{dt} = 0; \frac{d\xi}{dt} = \partial_n K_n
\end{equation}

(and \(\hat{P}_n = \partial Q_n\) with \(\hat{X}_n = -\partial P Q_n\)). Recall that there are two main types of generating functions \(F_1(q, Q, t)\) and \(F_2(q, P, t)\) for canonical transformations \((q, p) \rightarrow (Q, P)\) satisfying \(p\dot{q} - H = \hat{P} \dot{Q} - K + dF\) where \(H\) and \(K\) are Hamiltonians. One has in particular \(K = H + \partial_t F, \hat{P} = -H, \hat{q} = H, \hat{P} = -K_Q,\) and \(\hat{Q} = K_P\) in both cases and

\begin{equation}
p = \partial_q F_1; P = -\partial_q F_1; p = \partial_q F_2; Q = \partial_p F_2
\end{equation}

(\(\text{thus } -\dot{\xi} = -\partial_n \tilde{S}\) and \(-\dot{\hat{\xi}} = -K_{\lambda}\) as required)

Now in our semiclassical action principle of [3;4], based on the Jevicki-Yoneya action principle of [18;49], the quantity \(S-\xi\) was introduced via heuristic considerations and served very well (see below). Its origin was the use of \(\log(W) \sim -H\) as an action ingredient in [18;49], corresponding to \(W \sim \exp(-1/2 H)\) in the semiclassical version. Thus a basic action was posited to be \(Sp(H) \sim \int \frac{dH}{2\pi i} \) in [18;49] and we observed that \(W \exp(\xi) \sim w \sim \exp(1/\epsilon S)\) corresponds to \(-H = S - \xi\). This identification has an interesting interpretation in terms of the quasiclassical limit of the Sato equation

\begin{equation}
(\partial_n W)W^{-1} = -L^n
\end{equation}

Thus from (4.7)-(4.9), writing e.g. \(W(T, \epsilon) = 1 + \sum_1^\infty w_n(\epsilon, T)(\epsilon \partial)^{-n}\), one obtains

\begin{equation}
\epsilon(\partial_n W)W^{-1} \rightarrow -\lambda^n
\end{equation}

Note formally for \(W \sim \exp(-1/2 H), \epsilon \partial_n \log(W) = \epsilon(\partial_n W)W^{-1} = -\partial_n H \sim \partial_n(S - \xi)\) while directly from (4.7) or (5.1)-(5.3)

\begin{equation}
\partial_n(S - \xi) = B_n - \lambda^n = \lambda^n_+ - \lambda^n_0 = -\lambda^n_-
\end{equation}

Consequently one has (cf. also [42;43] for some related ideas)

**Theorem 5.1.** The term \(S - \xi\) arising in the semiclassical action principle of [3;4] can be connected to \(W\) via a semiclassical limit of the Sato equation and has the form \(\partial_n(S - \xi) = -\lambda^n_-\).

It would also be of interest to investigate the sense in which \(S - \xi\) relates to action (via residue calculation).
REMARK 5.2. It is clear that the Maslov canonical operator (cf. [28]) is connected with semiclassical soliton theory and we wrote this out in an earlier version of this paper. There does seem to be good motivation for pursuing this (cf. also [43]).

REMARK 5.3. We mention a few connections of our work to the development in [43], which has just come to our attention. We hope to return to this in more detail at another time. We note that in [43] one writes
\[ \lambda = \exp(ad(\phi))P, \quad (P \sim k, \lambda \sim L) \]
via a dressing function \( \phi \), with
\[ \exp(ad(\phi))X = X + \sum_{i=1}^{\infty} V_{i+1} \lambda^{-i-1}. \]
This last expression corresponds to our relation \( WxW^{-1} = G \to \xi = X + \sum_{i=1}^{\infty} V_{i+1} \lambda^{-i-1} \), derived in [3;4] (cf. also (4.11)). Thus \( \lambda = \exp(ad(\phi))P \) and \( \hat{\xi} = \exp(ad(\phi))X \) where (\( \lambda, -\hat{\xi} \)) are the action-angle variables of [3;4;24], and \( ad(\phi) = \{ \phi, \psi \} \). There is also a connection to [49] (cf. also [3;4]) where
\[ \lambda = \lim_{\epsilon \to 0} \exp(-\frac{1}{\epsilon}H \circ P) \]
is used with \( (f \circ g)(x, k) = f(x, k) \exp(\epsilon \partial_k \partial_x g(x, k)) \) and \( \{ f, g \} = \frac{1}{\epsilon} (f \circ g - g \circ f) (W \sim \exp(-\frac{1}{\epsilon}H)) \). Finally let us remark that in [43] one provides an important quasiclassical limit of certain Hirota equations via a quasiclassical differential Fay identity which has as a consequence the formula (a minus sign typo is corrected)

\[ \sum_{m,n=1}^{\infty} \mu^{-m} \lambda^{-n} \frac{\partial_m \partial_n F}{mn} = \log[1 + \sum_{i=1}^{\infty} \frac{\lambda^{n-\mu} - \mu^{n-\lambda}}{\mu - \lambda} \frac{\partial_n F}{n}] \]

where \( \log(\tau)^{dKP} \sim F \) and \( \tau(\epsilon, T) \sim \exp(\frac{1}{\epsilon}F) \) (cf. (4.12)). This formula has an interesting version in terms of the quantity \( P(\mu) - P(\lambda) \) which plays an important role in the Hamilton-Jacobi theory of [24] (cf. also [3;4]). Thus in (4.6) one writes \( P(\lambda) = \lambda - \sum_{n=1}^{\infty} P_n \lambda^{-n} \) where (cf. (4.8)) \( -P_n = \partial S_{n+1} \) and \( -n S_{n+1} = \partial_n F \) which implies \( \partial \partial_n F = -n \partial S_{n+1} = n P_n \) (note also \( -P_n = q_{n+1} \) in the notation of [42;43]). Hence in (5.9) \( \sum_{i=1}^{\infty} \lambda^{n-\mu} \frac{\partial_n F}{n} = \sum_{i=1}^{\infty} \lambda^{-n} P_n = \lambda - P(\lambda) \) which implies

\[ \sum_{m,n=1}^{\infty} \mu^{-m} \lambda^{-n} \frac{\partial_m \partial_n F}{mn} = \log[\frac{P(\mu) - P(\lambda)}{\mu - \lambda}] \]

We anticipate that this formula might prove interesting in terms of phase space geometry, topological field theory, etc. (cf. [2-4;13;24;25;42;43;49]). In particular, in [6] we show how to extract the dispersionless Hirota type equations from (5.16), using (5.17). These are nonlinear partial differential equations involving \( \partial_m \partial_n F \), and \( \partial \partial_n F \) which should characterize \( F \).

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APPENDIX A. Coherent states.

We collect here a few remarks and formulas concerning coherent states and various representations in quantum mechanics (cf. [25;41;49] for background material). We think of $Q \sim \hat{q}_h, P \sim \hat{p}_h$ as in (2.6) with $a \sim a_h, a^\dagger \sim a^\dagger_h$. There are various representations of vectors in terms of coordinates, momenta, number operators, coherent states, etc. and we describe this briefly. Then, given boson operators $a, a^\dagger$ with $[a, a^\dagger] = 1$ one can choose a vacuum vector $|0 >$ with $a|0 >= 0$ and normalized vectors ($<0|0>=1$)

(A.1) $|n> = (a^\dagger)^n|0> / \sqrt{n!}; <m|n> = \delta_{m,n}$

As an example, for $z \in S^1, a \sim \partial_z, a^\dagger \sim z, |n> = \frac{z^n}{\sqrt{n!}}, <n|m> = \frac{1}{2\pi} \int z^n \bar{z}^m \frac{dz}{z}$. The number operator is $a^\dagger a$ with $a^\dagger a|n> = n|n>$, $a|n> = \sqrt{n}|n-1>$, and

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\[ a^\dagger |n> = \sqrt{n+1} |n+1> \]. In a more quantum mechanical spirit (cf. [44]) one has position and momentum representations via (recall \( \hat{Q} \) means operator \( \hat{Q} |Q'> = Q'|Q'>, <Q'|Q'> = \delta(Q''-Q'), 1 = \int dQ'(|Q'> < Q'|), (f = \int_{-\infty}^{\infty}) \), and for states \(|\alpha>||\beta>||\alpha> = \int dQ'(|Q'> < Q'|\alpha>) \), with

(A.2) \[ <Q'|\alpha> = \psi_{\alpha}(Q'); <\beta|\alpha> = \int dQ' <\beta|Q'> <Q'|\alpha> = \int dQ' \tilde{\psi}_{\beta}(Q')\psi_{\alpha}(Q') \]

Also in general \(<\beta|\hat{A}|\alpha> = \int \int dQ' dQ'' \tilde{\psi}_{\beta} <Q'|\hat{A}|Q'' > \psi_{\alpha}(Q'')\). For the momentum representation \( P \sim \hat{p}_h \) with \( \hat{P} = h\partial_Q \), \( \hat{P}|P'> = P'|P'> \),

\(<P'|P''> = \delta(P' - P''), 1 = \int dP'(|P'> < P'|), |\alpha> = \int dP'(P'> < P'|\alpha>) \), and

(A.3) \[ <P'|\alpha> = \phi_{\alpha}(P'); <Q'|\hat{P}|\alpha> = -i\hbar \partial_Q <Q'|\alpha> ; \]

\(<Q'|\tilde{\hat{P}}|Q''> = -i\hbar \partial_Q \delta(Q' - Q''); <Q'|P'> = \frac{1}{\sqrt{2\hbar \pi}} e^{\frac{2\pi}{\hbar} P'Q'} \]

In this context vacuum vectors \(|0>\) such that \( a|0> = 0 (\sim \hat{a}_h|0> = 0)\) can be represented in a peaked state or Schrödinger form via the coordinate \( Q\). Thus (cf. [36]) from \(<Q'|a|0> = 0\) we have \((Q' + \hbar \partial_Q) <Q'|0> = 0\) or

(A.4) \[ <Q'|0> = (\hbar \pi)^{-\frac{1}{4}} e^{-\frac{Q^2}{2\hbar}} ; <Q'|n> = \frac{(Q' - \hbar \partial_Q)^n}{(\hbar \pi)^{\frac{1}{4}} n!(2\hbar)^n} e^{-\frac{Q^2}{2\hbar}} H_n\left(\frac{Q'}{\sqrt{\hbar}}\right) / (\hbar \pi)^{\frac{1}{4}} \sqrt{n!2^n} \]

These will be referred to as oscillator eigenfunctions or the Schrödinger representation \((H_n \sim \text{Hermite polynomial})\).

Next, coherent states are defined via

(A.5) \[ |z> = D(z)|0> = e^{-\frac{1}{2}z^2} \sum_0^\infty \frac{z^n}{\sqrt{n!}} |n> ; D(z) = e^{za_h - \bar{z}a_h} ; z = \frac{1}{\sqrt{2\hbar}} (q + ip) \]

We write also \(|z> \sim |p,q>\) and set

(A.6) \[ U(p,q) = e^{\frac{\hbar}{i}(p\hat{q}_h - q\hat{p}_h)} = e^{\frac{\hbar}{i}(p\hat{Q} - q\hat{P})} \]

Note \( a_h = \frac{1}{\sqrt{2\hbar}} (\hat{q}_h + i\hat{p}_h), a_h^\dagger = \frac{1}{\sqrt{2\hbar}} (\hat{q}_h - i\hat{p}_h)\), so \( za_h - \bar{z}a_h = (\frac{1}{\hbar})(p\hat{q}_h - q\hat{p}_h)\). Thus \(|z> \sim U(p,q)|0>\) and one records
\( (A.7) \quad a_h|z| > = |z| z > ; D^\dagger(z) = D^{-1}(z) = D(-z); D^\dagger(z) a_h D(z) = a_h + z; \)
\[
D(z) = e^{-\frac{\pi}{2}|z|^2} e^{z a_h} e^{-\bar{z} a_h} = e^{\frac{\pi}{2}|z|^2} e^{-\bar{z} a_h} e^{z a_h} ; < z|z' > = e^{-\frac{\pi}{2}|z|^2 + i \bar{z} z' - \frac{1}{2}|z'|^2} ;
\]
\[
< z'|a_h|z> = \bar{z} < z'| z> ; < z'|a_h^\dagger|z> = \bar{z} < z'| z> = \bar{z} < z'| z> ;
\]
\[
1 = \int (|z > < z|) d\mu = \frac{1}{\pi} d\bar{z}_1 d\bar{z}_2 ; D(z)|\zeta> = e^{i \text{Im}(z \zeta)} |z + \zeta>
\]
The measure \( d\mu \) will change for \( z \in S^1 \) or for other constraints. (cf. (3.11)-(3.14)). One is also interested in the Bargman representation through \( \tilde{\psi}(z) \) where
\[
(A.8) \quad \psi(z) = < z|\psi > = e^{-\frac{1}{2}|z|^2} \sum_0^\infty \frac{\bar{z}^n}{\sqrt{n!}} < n|\psi > = e^{-\frac{1}{2}|z|^2} \tilde{\psi}(\bar{z})
\]
Here \( \tilde{\psi}(\bar{z}) \) is an analytic function of \( \bar{z} \). Finally let us record a formula from [40] (for \([a, a^\dagger] = 1\))
\[
(A.9) \quad [a, g(a^\dagger)] = (\frac{\partial}{\partial a^c}) g(a^\dagger) ; [f(a), a^\dagger] = (\frac{\partial}{\partial a^c}) f(a)
\]

ATTENDNENT B. Operator structure.

We elaborate here on (4.35)-(4.37) and Corollary 4.4. First observe from [47] that given \( \hat{A} \sim \zeta_0 \) (cf. remarks before (2.16)) and \( \zeta = Ad_u^* \zeta_0 \) defined as in (2.18), the coordinates of \( \zeta \) can be taken as \( \zeta_1 = \zeta_1^0 + p \zeta_3, \zeta_2 = \zeta_2^0 + q \zeta_3, \) and \( \zeta_3 = \zeta_3^0 \) (with \( \zeta_3^0 = 1 \) say). Hence \((p, q)\) (note the order) serve as coordinates on a given coadjoint orbit \( \Gamma = \{ Ad_u^* \zeta_0 \} \) and if we think of \( U \sim U(\pi, \xi) \sim D(z) \) with \( z = \frac{1}{\sqrt{2h_0}}(\xi + i \tau) \) as in (4.23), then \((\pi, \xi)\) determines coordinates on \( \Gamma \subset \tilde{g}^* \) (cf. (2.15) - \( \tilde{g} \sim g_1 \) or a generic \( \tilde{g}_h \)). Thus \( U \in G_h \) gives rise to \( \zeta \in \Gamma \subset \tilde{g}^* \) and \( \Gamma \sim \frac{\partial}{\partial \zeta^0} (H_0 = H_{g_0}) \) as indicated before (2.15). The map \( J : \{ U \} \rightarrow \zeta : \frac{\partial}{\partial \zeta^0} = M \rightarrow \Gamma \) is in fact a sort of momentum map (cf. [8]).
Recall one defines \( J \) via \( J : M \rightarrow \tilde{g}^*; J_* : TM \rightarrow T\tilde{g}^* \simeq \tilde{g}^*; J^* : T^* \tilde{g}^* \simeq \tilde{g} \rightarrow T^* M; J^* \xi = d\tilde{J}(\xi) \) for \( \xi \in \tilde{g} \) where \( \tilde{J} : \tilde{g} \rightarrow C^\infty(M) \) and \( \xi_M(m) = X_{J^*(\xi)}(m) \) for \( \xi_M(m) = D_{\phi}(exp(t\xi))m|_{t=0} \) and \( X_f(g) = \{ g, f \} \). In order to erect some heuristic structure we will not belabor details here. Now consider operators \( \hat{A}, \hat{A}_{QC} \) as in (4.35)-(4.36) so that
\[
(B.1) \quad < z|\hat{A}|z> = \int \int d\bar{p} d\bar{q} \tilde{f}(\bar{p}, \bar{q}) < z|e^{i(\bar{p}\hat{Q} - \bar{q}\hat{P})} |z> ;
\]
\[
< z|\hat{A}_{QC}|z> = \int \int d\bar{p} d\bar{q} \tilde{f}(\bar{p}, \bar{q}) < z|e^{i(\bar{p}\hat{Q} - \bar{q}\hat{P})} |z>
\]
From Theorem 4.3 we know e.g. that \( < z|exp(\frac{i}{\hbar}(\bar{p}\hat{Q} - \bar{q}\hat{P})|z> \sim e^{-\frac{\bar{q}}{\hbar}} exp(\frac{i}{\hbar}(\bar{p}\xi-\bar{q}\hat{P})|z>
\]

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\( (z \sim (\xi, \pi)) \) and we assume the error term is suitably small etc. when writing (B.1) (for \( \hat{A} \) with \( < z | \exp[i(\hat{p}\hat{Q} - \hat{q}\hat{P})] | z > \) this is surely OK for reasonable \( f \)). Now from remarks before (2.16) we can write (here think of \( \hat{\Lambda} = \frac{A}{\hbar} \sim \frac{\Lambda}{\hbar} \in \tilde{g}_h, \lambda \sim (-\tilde{q}, \tilde{p}) \) and the \( e_3 \) terms can be neglected as in sections 2 and 4)

\[(B.2) \quad \lim_{(\frac{\epsilon}{\hbar})} < z | \hat{A} | z > = Ad_{U(\pi, \xi)}^{\epsilon} \zeta_0, \lambda >
\]
\[= < \zeta, \lambda > \sim < (\pi, \xi), (-\tilde{q}, \tilde{p}) > = \tilde{p} \xi - \tilde{q} \pi
\]
\((\zeta \sim (\pi, \xi) \) comes from \( z, \zeta_0 \sim (0, 0), \) and \( \lambda \sim (-\tilde{q}, \tilde{p}) \)). Consequently our quasiclassical operator \( \hat{A}_{QC} \) has the approximate symbol

\[(B.3) \quad < z | \hat{A}_{QC} | z > \sim \int d\tilde{p} d\tilde{q} \tilde{f}(\tilde{p}, \tilde{q}) e^{i<\zeta, \lambda>} = a(\zeta) \sim \int_{\tilde{g}} d\lambda \tilde{F}(\lambda) e^{i<\zeta, \lambda>} = a(\zeta)
\]

where \( \tilde{g} \sim \tilde{g}_1 \) say, \( \zeta \in \Gamma, \) and \( e^{-\frac{\epsilon}{\hbar}} \) terms are ignored for convenience. One can in fact take equivalence classes \([\lambda] \in \tilde{g}, \) as indicated before (2.15) so that the integrals in (B.3) are over \( T_\xi \Gamma \) but \( \tilde{g} \) should be retained for inversion. Thus one enters the realm of coherent state transforms, oscillatory integrals, Weyl-Wigner-Moyal theory, etc. with many possibilities for further development (cf. [14;28;29;39]). In particular one can recover \( \tilde{F}(\lambda) \) via Fourier inversion which amounts to determining the asymptotic forms for \( \hat{A} \) or \( \hat{A}_{QC} \) from their diagonal symbols.