Large deviations for grey Gaussian processes

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Abstract
The main results in this paper concern large deviations for families of non-Gaussian processes obtained as suitable perturbations of continuous centered multivariate Gaussian processes which satisfy a large deviation principle. We apply these results to a class of processes which contains the generalized grey Brownian motion as a particular case, and to some other related processes with independent components. We also obtain logarithmic estimates for suitable exit probabilities.

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1 Introduction
Gaussian processes are widely studied and, in particular, an important part of the literature concerns their asymptotic behavior. In this paper we are interested in the theory of large deviations which gives an asymptotic computation of small probabilities on exponential scale (see e.g. [11] as a reference of this topic); an important feature of large deviations for continuous Gaussian processes is that the rate functions are expressed in terms of norms of suitable reproducing kernel Hilbert spaces associated to some covariance functions.

The main results in this paper (Theorem 3.2 and Corollary 3.3) concern a general setting and allow to generalize some results in [25] (see Remark 3.1 for more details). These main results concern families of non-Gaussian processes (on some finite time interval $[0,T]$), obtained as suitable perturbations of continuous centered $p$-variate Gaussian processes $(X^n)_{n \in \mathbb{N}}$ which satisfies a large deviation principle as $n \to \infty$. In particular, in order to have results with more explicit rate function expressions, we assume that the processes $(X^n)_{n \in \mathbb{N}}$ have independent components.

The approach of the main results is quite abstract, and it is motivated by the applications presented in this paper. More precisely we mean the applications to some generalizations of processes in the literature, with some suitable scalings. In Section 4.1 we consider a generalization of the $p$-variate generalized grey Brownian motion, and we shall use the term multivariate grey Gaussian process. In Section 4.2 we consider $p$-variate processes with independent components, and each component is an univariate grey Gaussian processes (as the one of the processes defined in Section 4.1 with $p = 1$).

Now we recall some references on the generalized grey Brownian motion. We start with [24] and [24]; actually both references concern the univariate case, while here we consider a multivariate case. More precisely we refer to the presentation in [4], where the authors investigate the representation

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of the generalized grey Brownian motion in terms of a weighted integral of a stochastic process which is a solution of a certain stochastic differential equation. In [4] the authors also highlight some connections between the univariate case and the multivariate case (see eqs. (8) and (9) in that reference; see also [19] cited therein).

Here for completeness we recall some other references: [27] and [28] for an introduction to the grey noise (where the reader can find stochastic models for slow-anomalous diffusions described by the time fractional diffusion equations); [13] (see also the previous related paper [12]) for some connections with the fractional heat equation; [30] for some connections with some other random diffusivity models.

In the final part of the paper we study the asymptotic behavior of suitable exit probabilities of the processes studied in Sections 4.1 and 4.2. We mean exit probabilities from a halfspace (see Section 5.1) and form a quadrant (see Section 5.2); in both cases we get the limit in (14), and we compute the value $w$ by using the Lagrange multipliers method (this is a standard method used to prove similar results for Gaussian processes in the literature; see for instance the results for univariate processes in [2]). Moreover, again in both cases, the exit probabilities can be interpreted as level crossing probabilities over a finite time horizon; we recall that level crossing probabilities, or equivalently the distributions of first passage times, for univariate generalized grey Brownian motions are studied in the literature (see e.g. [29]; moreover see [20] for a more general discussion concerning also other topics and models). We also recall that, when we deal with the exit probabilities from a halfspace, we have linear combinations of the components of the $p$-variate processes, and therefore in some sense we deal with univariate processes. Moreover, when we deal with the exit probabilities from a quadrant, we mean that the exit occurs when there exists a single component of the multivariate process that crosses a level.

The exit from a quadrant has a natural interpretation when one considers risk models with different lines of business. In such a case the “ruin” event can be defined in different ways; moreover, motivated by diffusion approximation approach, some papers study multidimensional Brownian motions with drift, and with some possible correlation between components. Logarithmic asymptotic estimates (similar to the limit in (14) in this paper) can be found in [9] and in [10]; however we recall that in those references the levels go to infinity while, in this paper, we have fixed levels and some scaling factors that tend to zero (as $n \to \infty$). Among the recent references with different kind of results here we recall [8] with exact asymptotic estimates, and [7] with approximations and bounds.

We conclude with the outline of the paper. We start with some preliminaries in Section 2. We present some general results in Section 3 and their applications to multivariate grey Gaussian processes (and for the processes with independent components which are univariate grey Gaussian processes) in Section 4. Finally we conclude with the asymptotic estimates for some exit probabilities in Section 5, together with some comparisons between asymptotic rates in Section 6.

2 Preliminaries

In this section we recall some preliminaries on reproducing kernel Hilbert spaces and on large deviations. A final part is devoted to recall a useful result which provides large deviation principles on product spaces.

2.1 Reproducing kernel Hilbert spaces and large deviations

In this section we briefly recall some main facts related to reproducing kernel Hilbert space (RKHS) and large deviations for Gaussian measures on the Banach space of the continuous functions. For a detailed development of this very wide theory we can refer, for example, to the following classical references: for large deviations, see Section 3.4 in [15] and Chapter 4 (in particular Sections 4.1,
4.2 and 4.5 in [11]; for reproducing kernel Hilbert spaces, see Chapter 4 (in particular Section 4.3) in [16], Chapter 2 (in particular Sections 2.2 and 2.3) in [5], Section 2 in [21], Section 3 in [1] and Section 4.3 in [17].

We introduce the setting that we are going to consider throughout the paper. From now on, given \( T > 0 \) and \( p \geq 1 \), we will denote with \( C([0,T], \mathbb{R}^p) \) the space of \( \mathbb{R}^p \)-valued continuous functions on \([0,T] \) endowed with the topology induced by the sup-norm \( \| \cdot \|_\infty \), i.e. for \( f = (f_1, \ldots, f_p) \) then

\[
\|f\|_\infty = \sup_{0 \leq t \leq T} \|f(t)\|,
\]

where \( \| \cdot \| \) is the euclidean norm in \( \mathbb{R}^p \). The dual set of \( C([0,T], \mathbb{R}^p) \) is the set of vector of measures \( \lambda = (\lambda_1, \ldots, \lambda_p) \) on \([0,T] \) and we will denote it with \( \mathcal{M}^p[0,T] \). The action of \( \mathcal{M}^p[0,T] \) on \( C([0,T], \mathbb{R}^p) \) is denoted by

\[
\langle \lambda, h \rangle = \sum_{i=1}^{p} \int_0^T h_i(t) \, d\lambda_i(t)
\]

for every \( \lambda \in \mathcal{M}^p[0,T] \) and \( h = (h_1, \ldots, h_p) \in C([0,T], \mathbb{R}^p) \). In what follows, we will always suppose our processes to be continuous.

A Gaussian process \((U(t))_{t \in [0,T]}\) is characterized by the mean function and the covariance function, i.e.

\[
m : [0,T] \to \mathbb{R}^p, \quad m_i(t) = \mathbb{E}[U_i(t)] \quad i = 1, \ldots, p,
\]

and

\[
k : [0,T] \times [0,T] \to \mathbb{R}^{p \times p}, \quad k_{ij}(t,s) = \text{Cov}(U_i(t),U_j(s)) \quad i,j = 1, \ldots, p.
\]

Recall that the covariance function \( k \) of any Gaussian process is a symmetric, positive definite function. Since there is a one-to-one correspondence between centered Gaussian processes and its covariance, we can talk of RKHS relative to a positive definite kernel \( k \). Then let \((U(t))_{t \in [0,T]}\) be a continuous centered Gaussian process, with covariance function \( k \) as above, defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Moreover, define the set

\[
\mathcal{D} = \left\{ x \in C([0,T], \mathbb{R}^p) : x(t) = \int_0^T k(t,s) \, d\lambda(s), \ \lambda \in \mathcal{M}^p[0,T] \right\},
\]

where \( x(t) = \int_0^T k(t,s) \, d\lambda(s) \) means

\[
x_i(t) = \sum_{j=1}^{p} \int_0^T k_{ij}(t,s) \, d\lambda_j(s) \quad (\text{for all } i = 1, \ldots, p).
\]

As we shall see (see Remark 2.1 just after Definition 2.1) the RKHS \( \mathcal{H} \) relative to the kernel \( k \) can be constructed as the completion of the set \( \mathcal{D} \) with respect to a suitable norm. Consider the set of (real) Gaussian random variables

\[
\Gamma = \{ Y : Y = \langle \lambda, U \rangle, \ \lambda \in \mathcal{M}^p[0,T] \} \subset L^2(\Omega, \mathcal{F}, \mathbb{P}).
\]

We have that, for \( Y_1, Y_2 \in \Gamma \), say \( Y_i = \langle \lambda^i, U \rangle, \ i = 1, 2, \)

\[
\langle Y_1, Y_2 \rangle_{L^2(\Omega, \mathcal{F}, \mathbb{P})} = \text{Cov}(Y_1, Y_2)
\]

\[
= \text{Cov} \left( \int_0^T \sum_{j=1}^{p} U_j(t) \, d\lambda^1_j(t), \int_0^T \sum_{\ell=1}^{p} U_\ell(t) \, d\lambda^2_\ell(t) \right)
\]

\[
= \int_0^T \int_0^T \sum_{j,\ell=1}^{p} k_{j\ell}(t,s) \, d\lambda^1_j(t) d\lambda^2_\ell(s).
\]
Moreover, define the set
\[ H = \mathfrak{F}^{\| \cdot \|_{L^2(\Omega, \mathcal{F}, \mathbb{P})}}. \]
Then, since \( L^2 \)-limits of Gaussian random variables are still Gaussian, we have that \( H \) is a closed subspace of \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) consisting of real Gaussian random variables. Moreover, it becomes a Hilbert space when endowed with the inner product
\[ \langle Y_1, Y_2 \rangle_H := \langle Y_1, Y_2 \rangle_{L^2(\Omega, \mathcal{F}, \mathbb{P})} \quad Y_1, Y_2 \in H. \]

Consider now the following map,
\[ \mathcal{S} : H \to C([0, T], \mathbb{R}^p) \]
\[ Y \mapsto (\mathcal{S}Y)(\cdot) = \mathbb{E}(U(\cdot)Y). \]

The map \( \mathcal{S} \) is the Loève isometry (see, Theorem 35 in [3]).

**Definition 2.1.** Let \( U = (U(t))_{t \in [0, T]} \) be a continuous centered Gaussian process. We define the reproducing kernel Hilbert space relative to the Gaussian process \( U \) as
\[ \mathcal{H} := \mathcal{S}(H) = \{ h \in C([0, T], \mathbb{R}^p) : h(t) = (\mathcal{S}Y)(t), Y \in H \} \]
with an inner product defined as
\[ \langle h_1, h_2 \rangle_{\mathcal{H}} := \langle \mathcal{S}^{-1}h_1, \mathcal{S}^{-1}h_2 \rangle_H = \langle \mathcal{S}^{-1}h_1, \mathcal{S}^{-1}h_2 \rangle_{L^2(\Omega, \mathcal{F}, \mathbb{P})}, \quad h_1, h_2 \in \mathcal{H}. \]

Now we present some remarks and an example.

**Remark 2.1.** For any \( \lambda \in \mathcal{M}^p[0, T] \), and \( t \in [0, T] \)
\[ (\mathcal{S}\langle \lambda, U \rangle)(t) = \mathbb{E} \left( U(t) \int_0^T \sum_{j=1}^p U_j(s)d\lambda_j(s) \right) = \int_0^T k(t, s)d\lambda(s) \]
and thus \( \mathcal{S}(\Gamma) = \mathcal{D} \). Then, since \( \Gamma \) is dense in \( H \), and \( \mathcal{S} \) is an isometry, we have that
\[ \mathcal{H} = \mathcal{S}(H) = \mathcal{S}(\Gamma)^{\| \cdot \|_{\mathcal{H}}} = \mathcal{S}(\Gamma)^{\| \cdot \|_{\mathcal{H}}}. \]

**Remark 2.2.** Let \( x \in \mathcal{H} \). Then
\[ \| x \|_\infty \leq C_k \| x \|_{\mathcal{H}}, \quad \text{where} \quad C_k = \sup_{t \in [0, T]} \sum_{j=1}^p k_{jj}(t, t)^{1/2}. \]
In fact if, \( x \in \mathcal{H}, x(t) = \mathbb{E}[YU(t)] \) (there exists \( Y \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \)), and therefore
\[ \| x \|_\infty = \sup_{t \in [0, T]} \| x(t) \| \leq (\mathbb{E}[Y^2])^{1/2} \sup_{t \in [0, T]} \sum_{j=1}^p k_{jj}(t, t)^{1/2} = C_k \| x \|_{\mathcal{H}}. \]

**Remark 2.3.** If the components of the process \( U \) are independent, then the set \( \mathcal{D} \) is
\[ \mathcal{D} = \left\{ x \in C([0, T], \mathbb{R}^p) : x_i(t) = \int_0^T k_{ii}(t, s)d\lambda_i(s), \ i = 1, \ldots, p, \ \lambda \in \mathcal{M}^p[0, T] \right\}. \tag{1} \]
and for \( x \in \mathcal{D} \), we have
\[ \| x \|_{\mathcal{H}}^2 = \int_0^T \int_0^T \sum_{j=1}^p k_{jj}(t, s)d\lambda_j(t)d\lambda_j(s). \]
Example 2.1. Let \((B(t))_{t \in [0,T]}\) be a Brownian motion. It is well known that the RKHS is the Cameron-Martin space
\[
H^1_B[0,T] = \{ \varphi \in C([0,T], \mathbb{R}) : \varphi(0) = 0, \varphi \in L^2[0,T] \},
\]
with the inner product defined as
\[
\langle \varphi, \psi \rangle_{H^1_B[0,T]} = \int_0^T \dot{\varphi}(t)\dot{\psi}(t)\,dt \quad \varphi, \psi \in H^1_B[0,T].
\]
More generally let \((\hat{B}(t))_{t \in [0,T]}\) be a Volterra type Gaussian process with a square integrable kernel \(K\), i.e., for \(t \in [0,T]\)
\[
\hat{B}(t) = \int_0^t K(t,s)dB(s).
\]
Then the reproducing kernel Hilbert space \(\mathcal{H}_B\) is
\[
\mathcal{H}_B = \left\{ \varphi \in C_0([0,T], \mathbb{R}) : \varphi(t) = \int_0^t K(t,u)\hat{f}(u)\,du, t \in [0,T], f \in H^1_B[0,T] \right\}
\]
with the inner product defined as
\[
\langle \varphi, \psi \rangle_{\mathcal{H}_B} = \int_0^T \hat{f}(t)\hat{g}(t)\,dt,
\]
where \(\varphi(t) = \int_0^t K(t,u)\hat{f}(u)\,du\) and \(\psi(t) = \int_0^t K(t,u)\hat{g}(u)\,du\), for \(f, g \in H^1_B[0,T]\).

Now we present some basic definitions on large deviations.

Definition 2.2. Let \(E\) be a topological space, \(\mathcal{B}(E)\) the Borel \(\sigma\)-algebra and \((\mu_n)_{n \in \mathbb{N}}\) a family of probability measures on \(\mathcal{B}(E)\); let \(\gamma : \mathbb{N} \rightarrow \mathbb{R}^+\) be a function, such that \(\gamma_n \rightarrow +\infty\) as \(n \rightarrow +\infty\). We say that the family of probability measures \((\mu_n)_{n \in \mathbb{N}}\) satisfies a large deviation principle (LDP) on \(E\) with the rate function \(I\) and the speed \(\gamma_n\) if, for any open set \(\Theta\),
\[
- \inf_{x \in \Theta} I(x) \leq \liminf_{n \rightarrow +\infty} \frac{1}{\gamma_n} \log \mu_n(\Theta)
\]
and, for any closed set \(\Gamma\),
\[
\limsup_{n \rightarrow +\infty} \frac{1}{\gamma_n} \log \mu_n(\Gamma) \leq - \inf_{x \in \Gamma} I(x). \tag{2}
\]
A rate function is a lower semicontinuous mapping \(I : E \rightarrow [0, +\infty]\). A rate function \(I\) is said good if the sets \(\{I \leq a\}\) are compact for every \(a \geq 0\).

Definition 2.3. Let \(E\) be a topological space, \(\mathcal{B}(E)\) the Borel \(\sigma\)-algebra and \((\mu_n)_{n \in \mathbb{N}}\) a family of probability measures on \(\mathcal{B}(E)\); let \(\gamma : \mathbb{N} \rightarrow \mathbb{R}^+\) be a function, such that \(\gamma_n \rightarrow +\infty\) as \(n \rightarrow +\infty\). We say that the family of probability measures \((\mu_n)_{n \in \mathbb{N}}\) satisfies a weak large deviation principle (WLDP) on \(E\) with the rate function \(I\) and the speed \(\gamma_n\) if the upper bound (2) holds for compact sets.

Remark 2.4. We say that a family of continuous processes \(((U^n(t))_{t \in [0,T]}\)_{n \in \mathbb{N}}, satisfies a LDP if the family of their laws satisfies a LDP on \(C([0,T], \mathbb{R}^p)\).

Definition 2.4. A family of continuous processes \(((U^n(t))_{t \in [0,T]}\)_{n \in \mathbb{N}} is exponentially tight at the speed \(\gamma_n\) if, for every \(R > 0\) there exists a compact set \(K_R\) such that
\[
\limsup_{n \rightarrow +\infty} \gamma_n^{-1} \log \mathbb{P}(U^n \notin K_R) \leq -R.
\]
If the mean and the covariance functions of an exponentially tight family of Gaussian processes have a good limit behavior, then the family satisfies a LDP, as stated in the following theorem which is a consequence of the classic abstract Gärtner-Ellis Theorem (see e.g. Baldi Theorem 4.5.20 and Corollary 4.6.14 in [11]).

**Theorem 2.5.** Let \( (U^n(t))_{t \in [0,T]} \) be an exponentially tight family of continuous Gaussian processes with respect to the speed function \( \gamma_n \). Suppose that, for any \( \lambda \in \mathcal{M}[0,T] \),

\[
\lim_{n \to +\infty} \mathbb{E}[\langle \lambda, U^n \rangle] = \lim_{n \to +\infty} \sum_{i=1}^p \mathbb{E}[\langle \lambda_i, U^n_i \rangle] = 0
\]

and the limit

\[
\lim_{n \to +\infty} \gamma_n \text{Var} (\langle \lambda, U^n \rangle) = \sum_{i,j=1}^p \int_0^T \int_0^T k_{ij}(t, s) d\lambda_i(t) d\lambda_j(s)
\]

exists, for some continuous, symmetric, positive definite function \( k = (k_{ij})_{i,j=1,...,p} \), that is the covariance function of a continuous Gaussian process. Then \( (U^n(t))_{t \in [0,T]} \) satisfies a LDP on \( C([0,T], \mathbb{R}^p) \), with the speed \( \gamma_n \) and the good rate function

\[
I(h) = \begin{cases} 
\frac{1}{2} \| h \|_{\mathcal{H}}^2 & \text{if } h \in \mathcal{H} \\
+\infty & \text{otherwise}
\end{cases}
\]

where \( \mathcal{H} \) and \( \| \cdot \|_{\mathcal{H}} \) denote, respectively, the reproducing kernel Hilbert space and the related norm associated to the covariance function \( k \).

Next theorem that is a classical result for Gaussian measures immediately follows (for details see, for example, [15]).

**Theorem 2.6.** Define \( U^n = n^{-1/2}U \), where \( U \) is a continuous, centered Gaussian process. Then \( \{U^n\}_{n \in \mathbb{N}} \) satisfies a large deviation principle on \( C([0,T], \mathbb{R}^p) \) with the speed \( n \) and the (good) rate function

\[
I(h) = \begin{cases} 
\frac{1}{2} \| h \|_{\mathcal{H}}^2 & \text{if } h \in \mathcal{H} \\
+\infty & \text{otherwise}
\end{cases}
\]

where \( \mathcal{H} \) is the RKHS associated to the process \( U \).

We can simply write \( I(h) = \frac{1}{2} \| h \|_{\mathcal{H}}^2 \) for every continuous function \( h \) if we take (as a slight abuse of notation) \( \| h \|_{\mathcal{H}}^2 = +\infty \) when \( h \notin \mathcal{H} \).

**Remark 2.5.** Let \( (U^n)_{n \in \mathbb{N}} \) be a family of \( \mathbb{R}^p \)-valued processes with independent components. Suppose that the family of the \( i \)-th component \( (U^n_i)_{n \in \mathbb{N}} \) satisfies a LDP on \( C([0,T], \mathbb{R}) \) with the speed \( \gamma_n \) and the good rate function \( I_i \). Then the family of processes \( (U^n)_{n \in \mathbb{N}} \) satisfies a LDP on \( C([0,T], \mathbb{R}^p) \) with the speed \( \gamma_n \) and the good rate function

\[
I(h) = \sum_{i=1}^p I_i(h_i).
\]

In particular, if we refer to Theorem 2.5 the last equality holds and \( I_i \) is defined by

\[
I_i(h_i) = \begin{cases} 
\frac{1}{2} \| h_i \|_{\mathcal{H}_i}^2 & \text{if } h_i \in \mathcal{H}_i \\
+\infty & \text{if } h_i \notin \mathcal{H}_i, \quad \text{(for } h_i \in C([0,T], \mathbb{R})\text{)},
\end{cases}
\]

where \( \mathcal{H}_i \subset C([0,T], \mathbb{R}) \) is the RKHS associated to the covariance function \( k_{ii} \).
2.2 Large deviations for joint and marginal distributions

In this section we recall the Chaganty Theorem (see e.g. Theorem 2.3 in [6]) which provides the LDP for sequences of probability measures (joint distributions) on a product space $E_1 \times E_2$ when one has the LDP for the sequence of marginal distributions on $E_1$, and some suitable hypotheses for the conditional distributions on $E_2$ precised below (see Definition 2.7). In particular, as a consequence, it is also possible to derive the LDP for the sequence of marginal distributions on $E_1$.

We start by introducing some notation. Let $Y$ and $Z$ be two random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values, respectively, in the measurable spaces $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$. Let us denote by $\eta_1$ the (marginal) laws of $Y$, by $\eta_2$ the marginal of $Z$ and by $\eta$ the joint distribution of $(Y, Z)$ on $(E, \mathcal{E}) = (E_1 \times E_2, \mathcal{E}_1 \times \mathcal{E}_2)$. A family of probabilities $(\eta_2(\cdot|y))_{y \in E_1}$ on $(E_2, \mathcal{E}_2)$ is a regular version of the conditional law of $Z$ given $Y$ if

1. For every $B \in \mathcal{E}_2$, the map $((E_1, \mathcal{E}_1) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})), y \mapsto \eta_2(B|y)$ is $\mathcal{E}_1$-measurable.

2. For every $B \in \mathcal{E}_2$ and $A \in \mathcal{E}_1$, $\mathbb{P}(Y \in A, Z \in B) = \int_A \eta_2(B|y)\eta_1(dy)$.

In this case we have

$$\eta(dy, dz) = \eta_2(dz|y)\eta_1(dy).$$

From now on $(E_1, \mathcal{B}_1)$ and $(E_2, \mathcal{B}_2)$ are two Polish spaces. Moreover, in what follows, given a sequence of probability measures $(\eta_n)_{n \in \mathbb{N}}$ on the product space $(E_1 \times E_2, \mathcal{B}_1 \times \mathcal{B}_2)$ (the sequence of the joint distributions), for $i = 1, 2$ we denote by $(\eta_{in})_{n \in \mathbb{N}}$ the sequence of the marginal distributions on $(E_i, \mathcal{B}_i)$, and we denote by $(\eta_{2n}(\cdot|x_1), x_1 \in E_1)_{n \in \mathbb{N}}$ the sequence of the conditional distributions on $(E_2, \mathcal{B}_2)$, i.e.

$$\eta_n(B_1 \times B_2) = \int_{B_1} \eta_{2n}(B_2|x_1)\eta_{1n}(dx_1)$$

for every $B_1 \times B_2$, with $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$.

Now we are ready to recall a definition and some results in [6].

**Definition 2.7.** Let $(E_1, \mathcal{B}_1), (E_2, \mathcal{B}_2)$ be two Polish spaces and $x_1 \in E_1$. We say that the sequence of conditional laws $(\eta_{2n}(\cdot|x_1))_{n \in \mathbb{N}}$ on $(E_2, \mathcal{B}_2)$ satisfies the LDP continuously in $x_1$ with the rate function $J(\cdot|x_1)$ and the speed $\gamma_n$, or simply, the LDP continuity condition holds, if

(a) For each $x_1 \in E_1$, $J(\cdot|x_1)$ is a good rate function on $E_2$.

(b) For any sequence $(x_{1n})_{n \in \mathbb{N}}$ in $E_1$ such that $x_{1n} \to x_1$, the sequence $(\eta_{2n}(\cdot|x_{1n}))_{n \in \mathbb{N}}$ satisfies a LDP on $E_2$ with the rate function $J(\cdot|x_1)$ and the speed $\gamma_n$.

(c) $J(\cdot|\cdot)$ is lower semicontinuous as a function of $(x_1, x_2) \in E_1 \times E_2$.

**Theorem 2.8.** [Theorem 2.3 in [6]] Let $(E_1, \mathcal{B}_1), (E_2, \mathcal{B}_2)$ be two Polish spaces. Let $(\eta_{1n})_{n \in \mathbb{N}}$ be a sequence of probability measures on $(E_1, \mathcal{B}_1)$. For $x_1 \in E_1$ let $(\eta_{2n}(\cdot|x_1))_{n \in \mathbb{N}}$ be the sequence of the conditional laws on $(E_2, \mathcal{B}_2)$. Suppose that the following two conditions are satisfied:

(i) $(\eta_{1n})_{n \in \mathbb{N}}$ satisfies a LDP on $E_1$ with the good rate function $I_1(\cdot)$ and the speed $\gamma_n$.

(ii) for every $x_1 \in E_1$, the sequence $(\eta_{2n}(\cdot|x_1))_{n \in \mathbb{N}}$ obeys the LDP continuity condition with the rate function $J(\cdot|x_1)$ and the speed $\gamma_n$.

Then the sequence of joint distributions $(\eta_n)_{n \in \mathbb{N}}$, given by (3), satisfies a WLDP on $E = E_1 \times E_2$ with the speed $\gamma_n$ and the rate function

$$I(x_1, x_2) = I_1(x_1) + J(x_2|x_1),$$
for \(x_1 \in E_1\) and \(x_2 \in E_2\). Furthermore the sequence of the marginal distributions \((\eta_{2n})_{n \in \mathbb{N}}\) defined on \((E_2, \mathcal{B}_2)\), satisfies a LDP with the speed \(\gamma_n\), and the rate function
\[
I_2(x_2) = \inf_{x_1 \in E_1} I(x_1, x_2).
\]

Moreover, \((\eta_n)_{n \in \mathbb{N}}\) satisfies a LDP if \(I(\cdot, \cdot)\) is a good rate function, and in this case, also \(I_2(\cdot)\) is a good rate function.

Finally we conclude with a useful sufficient condition (presented in \([6]\)) on the rate functions \(I_1(\cdot)\) and \(J(\cdot | \cdot)\) which guarantees that \(I(\cdot, \cdot)\) is a good rate function.

**Lemma 2.1.** [Lemma 2.6 in \([6]\)] In the same hypotheses of Theorem 2.8, if the set
\[
\bigcup_{x_1 \in K_1} \{x_2 : J(x_2 | x_1) \leq L\}
\]
is a compact subset of \(E_2\) for any \(L \geq 0\) and for any compact set \(K_1 \subset E_1\), then \(I(\cdot, \cdot)\) is a good rate function (and therefore also \(I_2(\cdot)\) is a good rate function).

### 3 General results

For \(u = (u_1, \ldots, u_p)\) and \(v = (v_1, \ldots, v_p)\) (from now on we will use this notation for every vector (also random) in \(\mathbb{R}^p\)) denote \(u \circ v\) the Hadamard product, i.e.
\[
u \circ v = (u_1v_1, \ldots, u_pv_p).
\]

Let \(A^n = (a^n_1, \ldots, a^n_p)\) be a random vector in \([0, +\infty)^p\). In this section we always assume that the following condition holds.

**Condition (C).** \((X^n(t))_{t \in [0,T]}\) is a family of continuous centered Gaussian processes, starting from zero, with independent components, which satisfy the hypothesis of Theorem 2.8.

Consider the family \(((A^n, B^n), Z^n)_{n \in \mathbb{N}}\), where \(((A^n, B^n))_{n \in \mathbb{N}}\) is a family of processes with paths in \([0, +\infty)^p \times C([0, T], \mathbb{R}^p)\) independent of \(((X^n(t))_{t \in [0,T]}\) and, for \(n \in \mathbb{N}\),
\[
Z^n(t) = A^n \circ X^n(t) + B^n(t) \quad t \in [0, T].
\]

For simplicity, sometimes, we will write \(Z^n = A^n \circ X^n + B^n\) (not making explicit the dependence on time). Suppose that \(((A^n, B^n))_{n \in \mathbb{N}}\) satisfies a LDP with the good rate function \(I_{(A,B)}\) and the speed \(\gamma_n\). Our aim is to prove a LDP for \(((A^n, B^n), Z^n)_{n \in \mathbb{N}}\). We start with some lemmas.

**Lemma 3.1.** Let \(((X^n(t))_{t \in [0,T]}\) be as in Condition (C). Moreover, set \(y = (a,b) \in [0, +\infty)^p \times C([0, T], \mathbb{R}^p)\). Then \(((aX^n(t) + b(t))_{t \in [0,T]}\) satisfies a LDP with the good rate function
\[
\mathcal{J}(z|(a,b)) = \sum_{i=1}^p J_i(z_i|(a_i, b_i)),
\]
where
\[
J_i(z_i|(a_i, b_i)) = \begin{cases} \frac{1}{2a_i^2}z_i^2 + \frac{1}{2}z_i^2 & a_i > 0 \\ 0 & z_i = b_i \mbox{ and } a_i = 0, \\ \infty & \mbox{otherwise,} \end{cases}
\]
and \(\mathcal{H}_i\) is the RKHS associated to the covariance function \(k_{ii}\).
Proof. Note that the family \(((aX^n(t) + b(t))_{t \in [0,T]}\)_{n \in \mathbb{N}} is still a family of continuous Gaussian processes and
\[
\lim_{n \to +\infty} \gamma_n \text{Var}((\lambda, aX^n)) = \sum_{i=1}^{p} \int_{0}^{T} \int_{0}^{T} a_i^2 k_{ii}(t,s) \, d\lambda_i(t) d\lambda_i(s).
\]
Then we obtain the LDP in the statement of lemma as a consequence of Theorem 4.2.13 in [11], it is enough to show that \(\big((\lambda, aX^n(t))\big)_{t \in [0,T]}\)_{n \in \mathbb{N}} satisfies the same LDP as the family \(((b(t) + aX^n(t)))\)_{n \in \mathbb{N}}.

Lemma 3.2. Let \(((X^n(t))_{t \in [0,T]}\)_{n \in \mathbb{N}} be as in Condition (C). Moreover, set \(y = (a,b) \in [0, +\infty)^p \times C([0, T], \mathbb{R}^p)\). We also assume that \(((a^n, b^n))_{n \in \mathbb{N}} \subset [0, +\infty)^p \times C([0, T], \mathbb{R}^p)\) such that \((a^n, b^n) \to (a,b)\) in \([0, +\infty)^p \times C([0, T], \mathbb{R}^p)\). Then the family of processes \(((b^n(t) + a^nX^n(t)))_{t \in [0,T]}\)_{n \in \mathbb{N}} satisfies the same LDP as the family \((b(t) + aX^n(t)))\)_{n \in \mathbb{N}}.

Proof. Set \(Z^n(t) = b(t) + aX^n(t)\) and \(\tilde{Z}^n(t) = b^n(t) + a^nX^n(t)\) for \(t \in [0, T], n \in \mathbb{N}\). Then, thanks to Theorem 4.2.13 in [11], it is enough to show that \((Z^n)_{n \in \mathbb{N}}\) and \((\tilde{Z}^n)_{n \in \mathbb{N}}\) are exponentially equivalent (see e.g. Definition 4.2.10 in [11]). For any \(\delta > 0\),
\[
\mathbb{P}(||Z^n - \tilde{Z}^n||_\infty > \delta) \leq \mathbb{P}(\max_{1 \leq i \leq p} |a^n_i - a_i| > \frac{\delta}{2}) + \mathbb{P}(||b^n - b||_\infty > \frac{\delta}{2}).
\]
For \(n\) large enough \(||b^n - b||_\infty \leq \frac{\delta}{4}\); furthermore the family \(((X^n(t))_{t \in [0,T]}\)_{n \in \mathbb{N}} is exponentially tight (at the speed \(\gamma_n\)) and then, thanks to Definition 2.4. recalling that a compact set in \(C([0, T], \mathbb{R}^p)\) is bounded, we have
\[
\limsup_{n \to +\infty} \frac{1}{\gamma_n} \log \mathbb{P}(\max_{1 \leq i \leq p} |a^n_i - a_i| > \frac{\delta}{2}) = -\infty;
\]
and therefore
\[
\limsup_{n \to +\infty} \frac{1}{\gamma_n} \log \mathbb{P}(||Z^n - \tilde{Z}^n||_\infty > \delta) = -\infty.
\]

Now we want to prove the lower semicontinuity of the function \((z, (a,b)) \mapsto J(\cdot | (a,b))\).

Lemma 3.3. If \((z^n, (a^n, b^n)) \to (z, (a,b))\) in \(C([0, T], \mathbb{R}^p) \times ([0, +\infty)^p \times C([0, T], \mathbb{R}^p))\), then
\[
\liminf_{n \to +\infty} J(z^n| (a^n, b^n)) \geq J(z| (a,b)).
\]

Proof. Thanks to equation (3) it is enough to show that, for \(i = 1, \ldots, p\), the function \((z_i, (a_i, b_i)) \mapsto J_i(z_i | (a_i, b_i))\) is lower semicontinuous.

For \(a_i > 0\), since \(h_i^n = \frac{z_i - b_i}{a_i^n} \xrightarrow{a^n_{\mathcal{H}}} h_i = \frac{z_i - b_i}{a_i}\), thanks to the lower semicontinuity of \(||\cdot||_{\mathcal{H}}^2\), we have
\[
\liminf_{h_i^n \to h_i} \frac{1}{2} ||h_i^n||_{\mathcal{H}}^2 \geq \frac{1}{2} ||h_i||_{\mathcal{H}}^2 \geq \frac{1}{2} ||z_i - b_i||_{\mathcal{H}}^2 = J_i(z_i | (a_i, b_i)).
\]

If \(a_i = 0\) and \(z_i = b_i\) then it is trivial since \(J_i(z_i | (a_i, b_i)) = 0\). If \(a_i = 0\) and \(z_i \neq b_i\) then it is enough to observe that \(\liminf_{(a_i^n, b_i^n) \to ((a_i, b_i), z_i)} J_i(z_i^n | (a_i^n, b_i^n)) = +\infty\). Therefore equation (5) immediately follows.

Now we are ready to establish the WLDP for \(((A^n, B^n), (Z^n))_{n \in \mathbb{N}}\).
**Proposition 3.1.** Let \( ((X^n(t))_{t \in [0,T]}))_{n \in \mathbb{N}} \) be as in Condition (C). Moreover, we consider the family \( ((A^n, B^n), Z^n)_{n \in \mathbb{N}} \), where \( ((A^n, B^n))_{n \in \mathbb{N}} \) is a family of processes with paths in \([0, +\infty)^p \times C([0,T], \mathbb{R}^p)\) independent of \((X^n)_{n \in \mathbb{N}}\) and, for \( n \in \mathbb{N} \),
\[
Z^n = A^n \circ X^n + B^n.
\]
We also assume that \( ((A^n, B^n))_{n \in \mathbb{N}} \) satisfies a LDP with the good rate function \( I_{(A,B)} \) and the speed \( \gamma_n \). Then \( ((A^n, B^n), Z^n)_{n \in \mathbb{N}} \) satisfies the WLDP with the speed \( \gamma_n \) and the rate function
\[
I_{(A,B), Z}((a, b), z) = I_{(A,B)}(a, b) + J(z|(a, b)),
\]
where \( J(z|(a, b)) \) is defined by \([3]\) and \([4]\).

**Proof.** The family of processes \( ((A^n, B^n), Z^n)_{n \in \mathbb{N}} \) satisfies the hypotheses of Chaganty Theorem (Theorem 2.8) by Lemmas 3.1, 3.2 and 3.3. So the thesis holds.

We have proved that the family of processes \( ((A^n, B^n), Z^n)_{n \in \mathbb{N}} \) satisfies a WLDP. In the next lemma we will prove that \( I_{(A,B), Z} \) is a good rate function and therefore (by Theorem 2.8 i.e. the Chaganty main result) the family satisfies a (full) LDP.

**Lemma 3.4.** The rate function \( I_{(A,B), Z} \) is a good rate function.

**Proof.** It is enough to show (see Lemma 2.1) that
\[
\bigcup_{(a, b) \in K} \{ z \in C([0,T], \mathbb{R}^p) : J(z|(a, b)) \leq L \}
\]
is a compact subset of \( C([0,T], \mathbb{R}^p) \), for any \( L \geq 0 \) and for any compact set \( K \subset [0, +\infty)^p \times C([0,T], \mathbb{R}^p) \). Let \( K \) be a compact subset of \([0, +\infty)^p \times C([0,T], \mathbb{R}^p)\); for \((a, b) \in K\) define
\[
A^L_{(a,b)} = \{ z \in C([0,T], \mathbb{R}^p) : J(z|(a, b)) \leq L \} = \left\{ z \in C([0,T], \mathbb{R}^p) : \sum_{i=1}^p J_i(z_i|(a_i, b_i)) \leq L \right\}.
\]

\( A^L_{(a,b)} \) is a compact subset of \( C([0,T], \mathbb{R}^p) \) since \( J(\cdot|(a, b)) \) is a good rate function. We want to show that every sequence in \( \bigcup_{(a, b) \in K} A^L_{(a,b)} \) has a convergent subsequence. Let \((z_n)_{n \in \mathbb{N}} \subset \bigcup_{(a, b) \in K} A^L_{(a,b)}\); then, for every \( n \in \mathbb{N} \), there exists \((a^n, b^n) \in K\) such that \( z_n \in A^L_{(a^n,b^n)} \) (i.e. \( J(z^n|(a^n,b^n)) \leq L \) and therefore \( J_i(z^n_i|(a^n_i, b^n_i)) \leq L \) for \( i = 1, \ldots, p \)). Since \( ((a^n, b^n))_{n \in \mathbb{N}} \subset K \), up to a subsequence, we can suppose that \( (a^n, b^n) \rightarrow (a, b) \) in \( [0, +\infty)^p \times C([0,T], \mathbb{R}^p) \), as \( n \rightarrow +\infty \), with \((a, b) \in K\). For every \( i = 1, \ldots, p \), we have two cases: \( a_i > 0 \) or \( a_i = 0 \). If \( a_i > 0 \) then there exists a constant \( N > 0 \) such that, for every \( n \in \mathbb{N} \),
\[
J_i(z^n_i|(a_i, b^n_i)) = \left( \frac{a^n_i}{a_i} \right)^2 J_i(z^n_i|(a^n_i, b^n_i)) \leq N.
\]
Therefore the sequence \( (z^n_i - b^n_i)_{n \in \mathbb{N}} \) is contained in a level set of \( \| \cdot \|_{H^i} \) which is a compact set. Up to a subsequence we can suppose that \( (z^n_i - b^n_i)_{n \in \mathbb{N}} \) converges and therefore \( (z^n_i)_{n \in \mathbb{N}} \) converges to some \( z_i \) (remember that \( b^n \rightarrow b \)). If \( a_i = 0 \) then \( (z^n_i - b^n_i)_{n \in \mathbb{N}} \) converges to zero in \( \| \cdot \|_{H^i} \) and, thanks to Remark 2.2, a fortiori in \( C([0,T], \mathbb{R}^p) \) (for details see, for example, Section 2.1 and Remark 3.4 in [26]), and therefore again \( (z^n_i)_{n \in \mathbb{N}} \) converges to \( b_i \). Then we have showed that \( (z^n)_{n \in \mathbb{N}} \) converges to some \( z \) and, since the function \( J(\cdot|(a, b)) \) is semicontinuous (see Lemma 3.3), we have
\[
J(z_i|(a_i, b_i)) \leq \liminf_{n \rightarrow +\infty} J(z^n_i|(a^n_i, b^n_i)) \leq L,
\]
that implies \( z \in A^L_{(a,b)} \). 

\[\Box\]
We can summarize the results obtained in the following theorem.

**Theorem 3.2.** Let $((X^n(t))_{t \in [0,T]})_{n \in \mathbb{N}}$ be as in Condition (C). Moreover, we consider the family $((A^n, B^n), Z^n)_{n \in \mathbb{N}}$, where $((A^n, B^n))_{n \in \mathbb{N}}$ is a family of processes with paths in $[0, +\infty)^p \times C([0, T], \mathbb{R}^p)$ independent of $(X^n)_{n \in \mathbb{N}}$ and, for $n \in \mathbb{N}$,

$$Z^n = A^n \circ X^n + B^n.$$  

We also assume that $((A^n, B^n))_{n \in \mathbb{N}}$ satisfies a LDP with the good rate function $I_{(A, B)}$ and the speed $\gamma_n$. Then $((A^n, B^n), Z^n)_{n \in \mathbb{N}}$ satisfies the LDP with the speed $\gamma_n$ and the rate function $I_{(A, B), Z}$ defined in equation (6) (see also (3) and (4)). Moreover $(Z^n)_{n \in \mathbb{N}}$ satisfies the LDP with the speed $\gamma_n$ and the good rate function

$$I_Z(z) = \inf_{(a, b) \in [0, +\infty)^p \times C([0, T], \mathbb{R}^p)} \{I_{(A, B)}(a, b) + J(z | (a, b))\}.$$

In view of what follows we present a consequence of Theorem 3.2 for the case in which the components of the vectors of the sequence $(A^n)_{n \in \mathbb{N}}$ are all coincident. So in the next corollary we think to have vectors $((A^n, \ldots, A^n))_{n \in \mathbb{N}}$ where $A^n$ is common one-dimensional component of the vector, and we can consider an usual product instead of the Hadamard product.

**Corollary 3.3.** Let $((X^n(t))_{t \in [0,T]})_{n \in \mathbb{N}}$ be as in Condition (C). Moreover, we consider the family $((A^n, B^n), Z^n)_{n \in \mathbb{N}}$, where $((A^n, B^n))_{n \in \mathbb{N}}$ is a family of processes with paths in $[0, +\infty) \times C([0, T], \mathbb{R}^p)$ independent of $(X^n)_{n \in \mathbb{N}}$ and, for $n \in \mathbb{N}$,

$$Z^n = A^nX^n + B^n.$$  

We also assume that $((A^n, B^n))_{n \in \mathbb{N}}$ satisfies a LDP with the good rate function $I_{(A, B)}$ and the speed $\gamma_n$. Then $((A^n, B^n), Z^n)_{n \in \mathbb{N}}$ satisfies the LDP with the speed $\gamma_n$ and the rate function

$$I_{(A, B), Z}((a, b), z) = I_{(A, B)}(a, b) + \sum_{i=1}^{p} J_i(z_i | (a, b_i)),$$

where

$$J_i(z_i | (a, b_i)) = \begin{cases} \frac{1}{2a^2} \|z_i - b_i\|^2_{\mathcal{F}_i} & a > 0 \\ 0 & z_i - b_i = 0 \text{ and } a = 0. \\ \infty & \text{otherwise.} \end{cases} \quad (7)$$

Moreover $(Z^n)_{n \in \mathbb{N}}$ satisfies the LDP with the speed $\gamma_n$ and the good rate function

$$I_Z(z) = \inf_{(a, b) \in [0, +\infty) \times C([0, T], \mathbb{R}^p)} \left\{I_{(A, B)}(a, b) + \sum_{i=1}^{p} J_i(z_i | (a, b_i)) \right\}.$$

**Remark 3.1.** The results in this section have some relationship with the results in [25]. In that reference it is proved a LDP for a family $((A^n, B^n), Z^n)_{n \in \mathbb{N}}$, where $Z_n = A_nX_n + B_n$ (as we see in this case we can avoid to refer to the Hadamard product), and the following hypotheses hold:

- $((X^n(t))_{t \in [0,T]})_{n \in \mathbb{N}}$ is a family of continuous univariate processes as in Condition (C) (and therefore we can neglect the hypothesis of independent components);
- $(A^n, B^n)_{n \in \mathbb{N}}$ is a family of continuous processes with paths in $C_\alpha([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R})$ (where $C_\alpha([0, 1], \mathbb{R}) = \{y \in C([0, 1], \mathbb{R}) : y(t) \geq \alpha, t \in [0, 1]\}$, equipped with the uniform norm on compact sets) which satisfies the same hypotheses as in Theorem 3.2.

So, in order to explain the differences with the model studied in [25], we can say that in this paper $(A^n)_{n \in \mathbb{N}}$ could be equal to zero, or arbitrarily close to zero, but it is a sequence of random variables (and not a family of stochastic processes); moreover in this paper $(A^n)_{n \in \mathbb{N}}$ and $((X^n(t))_{t \in [0,T]})_{n \in \mathbb{N}}$ can be multivariate (and not univariate).
4 Applications to grey Gaussian processes

In this section we present two sequences of processes with a suitable small scaling factors. In both cases we refer to a random variable $L^\beta$ as in the next definition.

**Definition 4.1.** For $\beta \in (0, 1)$, let $L^\beta$ be a random variable with density

$$M_\beta(\tau) = \sum_{k=0}^{\infty} \frac{(-\tau)^k}{k! \Gamma(-\beta k + 1 - \beta)}$$

(see Proposition 3 and eq. (A.1) in [24]).

Some large deviation results for a random variable $L^\beta$ as in Definition 4.1 are given in Lemma 4.1 presented below. In view of this we recall (see e.g. equation (A.4) in [24]) that the moment generating function of $L^\beta$ is

$$\mathbb{E}\left[\exp(\eta L^\beta)\right] = E_\beta(\eta) \quad \text{for all } \eta \in \mathbb{R}, \quad (8)$$

where $E_\beta(z) := \sum_{h=0}^{\infty} \frac{z^h}{\Gamma(zh + 1)}$ is the Mittag-Leffler function. Note that equation (A.4) in [24] and some other references provides this formula only for $\eta \leq 0$; however this restriction is not needed because we can refer to the analytic continuation of the Laplace transform with complex argument.

The first sequence will be studied in Section 4.1. In such a case we refer to the framework of Theorem 3.2. Now we recall some preliminary results collected in the next lemma. In view of this we recall (see e.g. equation (A.4) in [24]) that the moment generating function of $L^\beta$ is

$$\mathbb{E}\left[\exp(\eta L^\beta)\right] = E_\beta(\eta) \quad \text{for all } \eta \in \mathbb{R}, \quad (8)$$

where $E_\beta(z) := \sum_{h=0}^{\infty} \frac{z^h}{\Gamma(zh + 1)}$ is the Mittag-Leffler function. Note that equation (A.4) in [24] and some other references provides this formula only for $\eta \leq 0$; however this restriction is not needed because we can refer to the analytic continuation of the Laplace transform with complex argument.

**Theorem 3.2.**

1. The sequence $(\sqrt{L^\beta} X(t))_{t \in [0,T]}$ is a continuous centered Gaussian process with independent components, and independent of the random variable $L^\beta$. We also suppose that the process $(X(t))_{t \in [0,T]}$ is non degenerate and $X(0) = 0$. In this case we say that $(\sqrt{L^\beta} X(t))_{t \in [0,T]}$ is a multivariate grey Gaussian process; indeed we recover the case of the generalized grey Brownian motion (ggBM) when $(X(t))_{t \in [0,T]}$ is a fractional Brownian motion with Hurst parameter $\frac{\alpha}{2}$, with $\alpha \in (0, 2)$ (see e.g. Proposition 3 in [24]), and therefore

$$k_{ij}(t, s) = \left\{ \begin{array}{ll}
\frac{1}{2}(t^\alpha + s^\alpha - |t - s|^\alpha) & \text{if } i = j \\
0 & \text{otherwise.}
\end{array} \right.$$ We also remark that the grey Brownian motion (gBM) concerns the case $\alpha = \beta$; moreover, when we have $\alpha = \beta = 1$, we can refer to some formulas in Example 2.4.

The second sequence will be studied in Section 4.2. In such a case we have a multivariate case with independent univariate grey Gaussian processes, namely

$$((\sqrt{L^\beta_1} X_1(t), \ldots, \sqrt{L^\beta_p} X_p(t)))_{t \in [0,T]},$$

where the components are independent, and each one is an univariate grey Gaussian process according the presentation in Section 4.1 (with $p = 1$), with some possible different values $\beta_1, \ldots, \beta_p$ in place of $\beta$. We have again some scalings (i.e. $n(-1+\beta_1)/2, \ldots, n(-1+\beta_p)/2$ for each component) and additive some perturbations; in this case it is useful to refer to the framework of Theorem 3.2.

Now we recall some preliminary results collected in the next lemma.

**Lemma 4.1.** Let $L^\beta$ be a random variable as in Definition 4.1 for some $\beta \in (0,1)$. Then we have the following results.

(i) The sequence $(n^{-1+\beta} L^\beta)_{n \in \mathbb{N}}$ satisfies the LDP, on $[0, +\infty)$, with good rate function $I_{L^\beta}$ defined by

$$I_{L^\beta}(b) := K_\beta b^{1/(1-\beta)},$$

where $K_\beta := \beta^{\beta/(1-\beta)} - \beta^{1/(1-\beta)}$.

(ii) The sequence $(n^{-1+\beta}/2 \sqrt{L^\beta})_{n \in \mathbb{N}}$ satisfies the LDP, on $[0, +\infty)$ with good rate function $J_{L^\beta}$ defined by

$$J_{L^\beta}(a) := K_\beta a^{2/(1-\beta)}.$$
Proof. The proof of the LDP in (i) is a straightforward application of the Gärtner Ellis Theorem (see e.g. Theorem 2.3.6 in [11]) with the speed \( n \to +\infty \). Then, by taking into account the moment generating function in (3.4.14)-(3.4.15) in Proposition 3.6 in [14], we have

\[
\Lambda(\eta) = \lim_{n \to +\infty} \frac{1}{n} \log \mathbb{E} \left[ \exp((n n^{-1+\beta} \eta L^\beta)) \right] = \lim_{n \to +\infty} \frac{1}{n} \log E_\beta(n^\beta \eta) = \begin{cases} 
\eta^{1/\beta} & \text{if } \eta \geq 0 \\
0 & \text{otherwise.}
\end{cases}
\]

Then, since the function \( \Lambda \) is differentiable, the desired LDP holds with good rate function \( I_{L^\beta} \) defined by

\[
I_{L^\beta}(b) := \sup_{\eta \in \mathbb{R}} \{ \eta b - \Lambda(\eta) \},
\]

which coincides with the rate function \( I_{L^\beta} \) in the statement of the proposition as one can readily check (note that, actually, we should consider \( b \in \mathbb{R} \); however here we already know that we can neglect the case \( b < 0 \) because we deal with non-negative random variables).

The proof of the LDP in statement (ii) is a straightforward consequence of the LDP in the statement (i), and an application of the contraction principle (see e.g. Theorem 4.2.1 in [11]). In fact, since \( n(1+\beta/2)\sqrt{L^\beta} = (n^{-1+\beta} L^\beta)^{1/2} \) and the function \( y \mapsto y^{1/2} \) is continuous, the desired LDP holds with good rate function \( J_{L^\beta} \) defined by

\[
J_{L^\beta}(a) := \inf \{ I_{L^\beta}(b) : \sqrt{b} = a \} = I_{L^\beta}(a^2),
\]

and indeed it coincides with the rate function \( J_{L^\beta} \) in the statement of the proposition. \( \square \)

4.1 Multivariate grey Gaussian processes

In this section we refer to Corollary 3.3 with

\[
(A^n)_{n \in \mathbb{N}} = (n^{-(1+\beta)/2}\sqrt{L^\beta})_{n \in \mathbb{N}}, \tag{9}
\]

that is the sequence in Lemma 4.1(ii).

Proposition 4.2. Let \((X(t))_{t \in [0,T]}\) be a continuous centered Gaussian process with independent components, and independent of the random variable \( L^\beta \) as in Definition 4.1. Moreover, let \((B^n)_{n \in \mathbb{N}}\) be a family of processes, independent of \((A^n)_{n \in \mathbb{N}}\) as in (4) and of \((X(t))_{t \in [0,T]}\), and assume that \((B^n)_{n \in \mathbb{N}}\) satisfies a LDP with the speed \( n \) and the good rate function \( I_B \). Then the sequence of processes \(((A^n, B^n), Z^n)_{n \in \mathbb{N}}\) defined by

\[
((A^n, B^n), Z^n)_{n \in \mathbb{N}} = \left((n^{-(1+\beta)/2}\sqrt{L^\beta}, B^n), n^{-1+\beta/2}\sqrt{L^\beta} X + B^n \right)_{n \in \mathbb{N}}
\]

satisfies a LDP with the speed \( n \) and the good rate function

\[
I_{(A,B), Z}((a,b), z) := I_B(b) + K_\beta a^{2/(1-\beta)} + \sum_{i=1}^p J_i(z_i|\{a, b_i\}),
\]

where the function \( J_i \) is defined by (7). So we have

\[
I_{(A,B), Z}((a,b), z) = \begin{cases} 
I_B(b) & \text{if } a = 0 \text{ and } z_i - b_i = 0, i = 1, \ldots, p. \\
I_B(b) + K_\beta a^{2/(1-\beta)} + \frac{1}{2\beta^2} \sum_{i=1}^p \|z_i - b_i\|_{\mathbb{R}^d}^2 & \text{if } a > 0 \\
\infty & \text{otherwise.}
\end{cases}
\]

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Moreover the sequence \((Z^n)_{n \in \mathbb{N}}\) satisfies a LDP with the speed \(n\) and the good rate function

\[
I_Z(z) := \inf_{(a,b) \in [0,+\infty) \times C([0,T],\mathbb{R}^p)} \left\{ I_B(b) + K_\beta a^{2/(1-\beta)} + \sum_{i=1}^p J_i(z_i|(a,b_i)) \right\}
\]

\[
= \inf_{b \in C([0,T],\mathbb{R}^p)} \left\{ I_B(b) + C_\beta \left( \sum_{i=1}^p \frac{1}{2} \| z_i - b_i \|^2 \right)^{1/(2-\beta)} \right\},
\]

where \(C_\beta := \left( (1-\beta) \frac{1}{2-\beta} + (1-\beta) \frac{\beta-1}{2-\beta} \right) K_\beta^{(1-\beta)/(2-\beta)}\) and \(K_\beta\) is as in Lemma 4.1.

Proof. Thanks to Theorem 2.6 if we denote the covariance function of the continuous process \((X(t))_{t \in [0,T]}\) by \(k\), the sequence \(((n^{-1/2}X(t))_{t \in [0,T]}{n \in \mathbb{N}}\) satisfies the LDP with limit covariance \(k\) and the speed \(\gamma_n = n\). By Lemma 4.1(ii) the sequence \((A^n)_{n \in \mathbb{N}}\) satisfies the LDP with rate function \(I_A = J_L\) and the speed \(n\). Then the statements to prove follow from an application of Corollary 3.3: indeed \(((n^{-1/2}X(t))_{t \in [0,T]}{n \in \mathbb{N}}\) satisfies Condition (C) because \((X(t))_{t \in [0,T]}\) has independent components. In particular the last equality for \(I_Z\) in the statement holds noting that

\[
\inf_{a \in [0,+\infty)} \left\{ K_\beta a^{2/(1-\beta)} + \sum_{i=1}^p J_i(z_i|(a,b_i)) \right\} = C_\beta \left( \sum_{i=1}^p \frac{1}{2} \| z_i - b_i \|^2 \right)^{1/(2-\beta)}
\]

(this can be checked with some easy computations).

In view of the results presented in Section 3 it is useful to refer to the sequence \((Z^n)_{n \in \mathbb{N}}\) when \((B^n)_{n \in \mathbb{N}}\) is a constant deterministic sequence equal to some \(\hat{b} = (\hat{b}_1, \ldots, \hat{b}_p) \in C([0,T],\mathbb{R}^p)\); in such a case we have \(I_B(b) = 0\) if \(b = \hat{b}\), and infinity otherwise, and therefore we can say that \(I_Z(z) = I_Z^{(gGp)}(z)\), where

\[
I_Z^{(gGp)}(z) := C_\beta \left( \sum_{i=1}^p \frac{1}{2} \| z_i - \hat{b}_i \|^2 \right)^{1/(2-\beta)}.
\]

(10)

4.2 Multivariate case with independent univariate grey Gaussian processes

In this section we refer to Theorem 3.2 where \((A^n)_{n \in \mathbb{N}}\) is a sequence of \(p\)-variate random variables defined by

\[
A^n_i = n^{-(-1+\beta_i)/2} \sqrt{L^\beta_i}, \quad i = 1, \ldots, p,
\]

where, for some \(\beta_1, \ldots, \beta_p \in (0,1)\), \(L^\beta_1, \ldots, L^\beta_p\) are independent random variables as the random variable \(L^\beta\) in Definition 4.1.

**Proposition 4.3.** Let \((X(t))_{t \in [0,T]}\) be a continuous centered Gaussian process with independent components, and independent of the random variables \(L^\beta_1, \ldots, L^\beta_p\) cited above. Let \((B^n)_{n \in \mathbb{N}}\) be a family of processes, independent of \((A^n)_{n \in \mathbb{N}} = ((A^n_i)_{i=1,\ldots,p}{n \in \mathbb{N}}\) as in (11) and of \((X(t))_{t \in [0,T]}\), and assume that \((B^n)_{n \in \mathbb{N}}\) satisfies a LDP with the speed \(n\) and the good rate function \(I_B\). Then the sequence of processes \(((A^n, B^n), Z^n)_{n \in \mathbb{N}}\) defined by

\[
((A^n, B^n), Z^n)_{n \in \mathbb{N}} = ((A^n, B^n), n^{-1/2}A^n \circ X + B^n)_{n \in \mathbb{N}}
\]

satisfies a LDP with the speed \(n\) and the good rate function

\[
I_{(A,B),Z}((a,b), z) := I_B(b) + I_A(a) + \mathcal{J}(z|(a,b)),
\]

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where $I_A(a) := \sum_{i=1}^{p} K_{\beta_i} a_i^{2/(1-\beta_i)}$ (and $K_{\beta}$ is as in Lemma 4.1), and $\mathcal{J}(z|(a,b))$ is defined by (3) and (4). So we have

$$I_{(A,B),Z}((a,b),z) = I_B(b) + \sum_{i=1}^{p} \left\{ K_{\beta_i} a_i^{2/(1-\beta_i)} + J_i(z_i|(a_i, b_i)) \right\}$$

and, if we set

$$\mathcal{I}^+(a) = \{1 \leq i \leq p : a_i > 0\} \text{ and } \mathcal{I}^0(a) = \{1 \leq i \leq p : a_i = 0\},$$

we have

$$I_{(A,B),Z}((a,b),z) = \begin{cases} I_B(b) + \sum_{i \in \mathcal{I}^+(a)} \left\{ K_{\beta_i} a_i^{2/(1-\beta_i)} + \frac{1}{2a_i^2} \|z_i - b_i\|^2_{\mathcal{H}_i} \right\} & z_i - b_i = 0 \text{ for all } i \in \mathcal{I}^0(a), \\ \infty & \text{otherwise.} \end{cases}$$

Moreover the sequence $(Z^n)_{n \in \mathbb{N}}$ satisfies a LDP with the speed $n$ and the good rate function

$$I_Z(z) := \inf_{(a,b) \in [0,\infty)^p \times C([0,T],\mathbb{R}^p)} \left\{ I_B(b) + I_A(a) + \mathcal{J}(z|(a,b)) \right\}$$

$$= \inf_{(a,b) \in [0,\infty)^p \times C([0,T],\mathbb{R}^p)} \left\{ I_B(b) + \sum_{i=1}^{p} \left( K_{\beta_i} a_i^{2/(1-\beta_i)} + J_i(z_i|(a_i, b_i)) \right) \right\}$$

$$= \inf_{b \in C([0,T],\mathbb{R}^p)} \left\{ I_B(b) + \sum_{i=1}^{p} C_{\beta_i} \left( \frac{1}{2} \|z_i - b_i\|^2_{\mathcal{H}_i} \right)^{1/(2-\beta_i)} \right\},$$

where $C_{\beta_i} = \left( (1 - \beta_i)^{-\frac{1}{2(1-\beta_i)}} + (1 - \beta_i)^{-\frac{1}{2(\beta_i - 1)}} \right) K_{\beta_i}^{(1-\beta_i)/(2-\beta_i)}$ as in Proposition 4.2 (with $\beta_i$ in place of $\beta$).

**Proof.** We follow the same lines of the proof of Proposition 4.2. We can repeat what we said in that proof for the LDP $((n^{-1/2} X(t))_{t \in [0,T]} )_{n \in \mathbb{N}}$. By a standard consequence of Lemma 4.1(iii) the sequence $(A^n)_{n \in \mathbb{N}}$ satisfies the LDP with rate function $I_A$ defined by $I_A(a) := \sum_{i=1}^{p} J_{L_{\beta_i}}(a_i)$ and the speed $n$. Then the statements to prove follow from an application of Theorem 3.2 indeed $((n^{-1/2} X(t))_{t \in [0,T]} )_{n \in \mathbb{N}}$ satisfies Condition (C) because $(X(t))_{t \in [0,T]}$ has independent components. In particular the last equality for $I_Z$ in the statement holds noting that

$$\inf_{a \in [0,\infty)^p} \sum_{i=1}^{p} \left\{ K_{\beta_i} a_i^{2/(1-\beta_i)} + J_i(z_i|(a_i, b_i)) \right\} = \sum_{i=1}^{p} C_{\beta_i} \left( \frac{1}{2} \|z_i - b_i\|^2_{\mathcal{H}_i} \right)^{1/(2-\beta_i)}$$

(this can be checked with some easy computations).

\[\square\]

In view of the results presented in Section 5 it is useful to refer to the sequence $(Z^n)_{n \in \mathbb{N}}$ when $(B^n)_{n \in \mathbb{N}}$ is a constant deterministic sequence equal to some $\hat{b} = (\hat{b}_1, \ldots, \hat{b}_p) \in C([0, T], \mathbb{R}^p)$; in such a case we have $I_B(b) = 0$ if $b = \hat{b}$, and infinity otherwise, and therefore we can say that $I_Z(z) = I_Z^{(\text{ind})}(z)$, where

$$I_Z^{(\text{ind})}(z) := \sum_{i=1}^{p} C_{\beta_i} \left( \frac{1}{2} \|z_i - \hat{b}_i\|^2_{\mathcal{H}_i} \right)^{1/(2-\beta_i)} . \quad (12)$$
5 Asymptotic results for some exit probabilities

In this section we obtain some asymptotic estimates for two exit probabilities concerning the processes \((Z^n)_{n \in \mathbb{N}}\) presented in Propositions 4.2 and 4.3; more precisely, in both cases, we restrict our attention to the case in which \((B^n)_{n \in \mathbb{N}}\) is a constant deterministic sequence (that is \(B^n = \hat{b}\) for every \(n \in \mathbb{N}\), for some \(\hat{b} = (\hat{b}_1, \ldots, \hat{b}_p) \in C([0,T], \mathbb{R}^p)\)). So the rate function \(I_{Z}^{\text{(exp)}}\) in Proposition 4.2 coincides with \(I_{Z}^{\text{(exp)}}(g_{G_{p}})\) in (10), and the rate function \(I_{Z}^{\text{(ind)}}\) in Proposition 4.3 coincides with \(I_{Z}^{\text{(ind)}}\) in (12).

We consider exit probabilities from a halfspace in Section 5.1, and from a quadrant in Section 5.2; moreover, in both cases, the set of paths which lead to exit will be denoted by \(A\) and it is a closed set. We deal with two sequences of exit probabilities \((p_n)_{n \in \mathbb{N}}\) (see (17) for the exit of a halfspace, and (20) for the exit from a quadrant); then, by also taking into account that we have \(\gamma(n) = n\) for the LDPs stated in Section 4, we have

\[
-\inf_{z \in A^o} I_{Z}(z) \leq \liminf_{n \to +\infty} \frac{1}{n} \log(p_n) \leq \limsup_{n \to +\infty} \frac{1}{n} \log(p_n) \leq -\inf_{z \in A} I_{Z}(z),
\]

where \(A^o\) is the interior of the set \(A\), and \(I_{Z}\) is the rate function in (10) for the processes in Section 4.1 or the rate function in (12) for the processes in Section 4.2. Moreover, in all the cases studied below, we check that

\[
w := \inf_{z \in A^o} I_{Z}(z) = \inf_{z \in A} I_{Z}(z),
\]

which yields

\[
\lim_{n \to +\infty} \frac{1}{n} \log(p_n) = -w.
\]

In particular we also have \(w = I_{Z}(z^*)\) for some \(z^* \in A\) and, in the fashion of large deviations, \(z^*\) is said to be “a most likely path leading to exit”. As far as the equality in (13) is concerned, in general we trivially have

\[
\inf_{z \in A^o} I_{Z}(z) \geq \inf_{z \in A} I_{Z}(z),
\]

and therefore only the inverse inequality has to be checked.

We recall that the processes \((Z^n)_{n \in \mathbb{N}}\) presented in Propositions 4.2 and 4.3 are defined in terms of a continuous centered Gaussian process \((X(t))_{t \in [0,T]}\) with independent components; so we can refer to the dense set \(\mathcal{D}\) as in (1) in Remark 2.3. Moreover, in what follows we also take into account that

\[
\inf_{z \in A} I_{Z}(z) = \inf_{z \in A \cap \mathcal{D}_b} I_{Z}(z),
\]

where

\[
\mathcal{D}_b := \mathcal{D} + \hat{b} = \{z = y + \hat{b} : y \in \mathcal{D}\}.
\]

More precisely we can say that \(z \in \mathcal{D}_b\) if and only if, for some \(\lambda_1, \ldots, \lambda_p \in \mathcal{M}^1[0,T]\), we have

\[
z_i(u) = \int_0^T k_{ii}(u,v) \, d\lambda_i(v) + \hat{b}_i(u), \quad u \in [0,T], \ i = 1, \ldots, p; \tag{15}
\]

then, if \(z\) is as in (15), we have

\[
\|z_i - \hat{b}_i\|_{\mathcal{H}_i}^2 = \int_0^T \int_0^T k_{ii}(u,v) \, d\lambda_i(u) \, d\lambda_i(v). \tag{16}
\]
5.1 Exit probabilities from a halfspace

In this section we consider
\[ p_n := \mathbb{P} \left( \sup_{t \in [0,T]} \langle Z^n(t), \xi \rangle \geq x \right), \tag{17} \]
for \( 0 \neq \xi \in \mathbb{R}^p, \xi_i \geq 0 \) for \( i = 1, \ldots, n \), and \( x > 0 \) such that \( x - \langle \hat{b}(t), \xi \rangle > 0 \) for every \( t \in [0,T] \).

Note that, in some sense, we deal with a sequence of univariate processes \((Z^n(t), \xi)_{n \in \mathbb{N}}\). Moreover the set \( \mathcal{A} \) is defined by
\[ \mathcal{A} := \left\{ z \in C([0,T], \mathbb{R}^p) : \sup_{t \in [0,T]} \langle z(t), \xi \rangle \geq x \right\}. \]

Then we have
\[ \mathcal{A} = \bigcup_{t \in [0,T]} \mathcal{A}_t, \text{ where } \mathcal{A}_t := \{ z \in C([0,T], \mathbb{R}^p) : \langle z(t), \xi \rangle = x \}. \tag{18} \]

**Remark 5.1.** Here we check (13), and therefore we only have to check the inequality
\[ \inf_{z \in \mathcal{A}^o} I_Z(z) \leq \inf_{z \in \mathcal{A}} I_Z(z). \]

There exists \( z^* \in \mathcal{A} \) such that \( I_Z(z^*) = \inf_{z \in \mathcal{A}} I_Z(z) \). If \( z^* \in \mathcal{A}^o \) this is trivial, and therefore here we assume that \( z^* \notin \mathcal{A}^o \). In this case we have
\[ \sup_{0 \leq t \leq T} \langle z^*(t), \xi \rangle = x; \]
so, for some \( t^* \in [0,T] \), we have \( \langle z^*(t^*), \xi \rangle = x \) and
\[ \langle z^*(t^*) - \hat{b}(t^*), \xi \rangle = x - \langle \hat{b}(t^*), \xi \rangle > 0. \]

Then, for every \( \varepsilon > 0 \), let \( z^{*\varepsilon} \) be defined by
\[ z^{*\varepsilon}(t) := (1 + \varepsilon)(z^*(t) - \hat{b}(t)) + \hat{b}(t); \]
so \( z^{*\varepsilon} \in \mathcal{A}^o \) because
\[ \sup_{0 \leq t \leq T} \langle z^{*\varepsilon}(t), \xi \rangle \geq \langle z^{*\varepsilon}(t^*), \xi \rangle = (1 + \varepsilon)(z^*(t^*) - \hat{b}(t^*), \xi) + \langle \hat{b}(t^*), \xi \rangle = (1 + \varepsilon)(x - \langle \hat{b}(t^*), \xi \rangle) + \langle \hat{b}(t^*), \xi \rangle = x + \varepsilon(x - \langle \hat{b}(t^*), \xi \rangle) > x. \]

Finally we get
\[ \inf_{z \in \mathcal{A}^o} I_Z(z) \leq I_Z(z^{*\varepsilon}) \rightarrow I_Z(z^*) \text{ as } \varepsilon \rightarrow 0, \]
where the limit can be checked by taking \( I_Z \) as \( I_Z^{(gGp)} \) in (10) or \( I_Z^{(ind)} \) in (12).

In the proof of the following Propositions 5.1 and 5.2 we compute \( w \) noting that
\[ w = \inf_{z \in \mathcal{A}} I_Z(z) = \inf_{t \in [0,T]} \inf_{z \in \mathcal{A}_t} I_Z(z); \tag{19} \]
moreover, for every \( t \in [0,T] \), we compute \( \inf_{z \in \mathcal{A}_t} I_Z(z) \) by taking \( z \) in the dense set \( \mathcal{D}_b \) defined above (see eq. (15)) by applying the Lagrange multipliers methods. Now we are ready to prove the results when \((Z^n)_{n \in \mathbb{N}}\) are the sequence of processes in Section 4. Here, for the processes in Section 4.2 the parameters \( \beta_1, \ldots, \beta_p \) should be coincident.
Proposition 5.1. Let $p_n$ be as in (17) with $(Z^n)_{n \in \mathbb{N}}$ be as in Section 4.1. Then (14) holds with $w = w_H^{(gGp)}$, where

$$w_H^{(gGp)} := \inf_{t \in [0,T]} C_\beta \left[ \frac{1}{2} \left( \langle \dot{b}(t), \xi \rangle + x \right)^2 \right]^{1/(2-\beta)}.$$

Proof. We have to compute $w$ in (19) where $\mathcal{M}_1$ is as in (18) and $I_Z$ is the rate function $I_Z^{(gGp)}$ in (10). Thus, by referring to the elements in the dense set of paths $\mathcal{D}_b$ (see (15) and (16)), we have to minimize

$$C_\beta \left( \sum_{i=1}^p \frac{1}{2} \int_0^T \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \, d\lambda_i(v) \right)^{1/(2-\beta)}$$

with respect to the vector of measures $\lambda$, subjected to the following constraint

$$\langle z(t), \xi \rangle = \sum_{i=1}^p \xi_i \int_0^T k_{ii}(t, v) \, d\lambda_i(v) + \langle \dot{b}(t), \xi \rangle = x.$$

So we use the method of Lagrange multipliers and, for $\gamma \in \mathbb{R}$, we have to find the stationary points of

$$\mathcal{L}(\lambda, \gamma) := C_\beta \left( \sum_{i=1}^p \frac{1}{2} \int_0^T \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \, d\lambda_i(v) \right)^{1/(2-\beta)} - \gamma \left( \sum_{i=1}^p \xi_i \int_0^T k_{ii}(t, v) \, d\lambda_i(v) + \langle \dot{b}(t), \xi \rangle - x \right).$$

Then, for every $\eta \in \mathcal{M}^1[0,1]$, we have

$$\frac{C_\beta}{2-\beta} \left( \frac{1}{2} \sum_{i=1}^p \int_0^T \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \, d\lambda_i(v) \right)^{\frac{\beta-1}{\beta}} \int_0^T \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \, d\eta(v) +$$

$$-\gamma \xi_i \int_0^T k_{ii}(t, v) \, d\eta(v) = 0, \quad i = 1, \ldots, p;$$

therefore we have to find a vector of measures $\lambda$ such that, for every $v \in [0,T]$,

$$\frac{C_\beta}{2-\beta} \left( \frac{1}{2} \sum_{i=1}^p \int_0^T \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \, d\lambda_i(v) \right)^{\frac{\beta-1}{\beta}} \int_0^T k_{ii}(u, v) \, d\lambda_i(u)$$

$$-\gamma \xi_i k_{ii}(t, v) = 0, \quad i = 1, \ldots, p.$$

A solution is $\lambda_i = c_i \delta_{\{t\}}$ for some $c_i \in \mathbb{R}$ (for every $i = 1, \ldots, p$). So we have to consider the following system

$$\begin{cases}
C_\beta \left( \frac{1}{2} \sum_{i=1}^p c_i^2 k_{ii}(t, t) \right)^{\frac{\beta-1}{\beta}} c_i - \gamma \xi_i = 0 & i = 1, \ldots, p \\
\sum_{i=1}^p \xi_i c_i k_{ii}(t, t) = -\langle \dot{b}(t), \xi \rangle + x,
\end{cases}$$

which can be explicitly solved; in fact we can check that

$$\gamma = \frac{C_\beta}{2-\beta} \left( \frac{1}{2} \right)^{\frac{\beta-1}{\beta}} \frac{(-\langle \dot{b}(t), \xi \rangle + x)^{\beta/(2-\beta)}}{\left( \sum_{i=1}^p c_i^2 k_{ii}(t, t) \right)^{1/(2-\beta)}}.$$
and
\[ c_i = \frac{\xi_i(\langle \dot{b}(t), \xi \rangle + x)}{\sum_{j=1}^{p} \xi_j^2 k_{jj}(t,t)}, \quad i = 1, \ldots, p. \]

So this is a solution of the Lagrange multipliers problem, and it is therefore a critique point for the functional we want to minimize. Moreover, since this is a strictly convex functional restricted on a linear subspace of \( \mathcal{H}^p[0,1] \), it is still strictly convex, and therefore the critique point is actually its unique point of minimum. In conclusion we easily get the desired expression for \( w \) by \([19]\); indeed we have \( \inf_{z \in Z} I_Z(z) = C_\beta \left[ \sum_{i=1}^{p} \xi_i^2 k_{ii}(t,t) \right]^{1/(2-\beta)} \) with \( c_1, \ldots, c_p \) computed above.

**Proposition 5.2.** Let \( p_n \) be as in \([17]\) with \( (Z^n)_{n \in \mathbb{N}} \) be as in Section 4.2 with \( \beta_1 = \cdots = \beta_p = \beta \). Then \([14]\) holds with \( w = w_H^{(\text{ind})} \), where
\[
w_H^{(\text{ind})} := \inf_{t \in [0,T]} C_\beta \left[ \frac{1}{2} \left( \frac{(-\langle \dot{b}(t), \xi \rangle + x)^2}{\sum_{j=1}^{p} \xi_j^2 k_{jj}(t,t) \xi_j^{1/(2-\beta)}} \right)^{1/(2-\beta)} \right].
\]

**Proof.** We remark that we take into account the hypothesis \( \beta_1 = \cdots = \beta_p = \beta \) only in the final part of the proof in order to explain where it is actually useful. We have to compute \( w \) in \([19]\) where \( \mathcal{A}_t \) is as in \([18]\) and \( I_Z \) is the rate function \( I^{(\text{ind})}_Z \) in \([12]\). We follow the same lines of the proof of Proposition 5.1 and we omit some details. Then we use again the method of Lagrange multipliers and, for \( \gamma \in \mathbb{R}, \) we have to find the stationary points of
\[
\mathcal{L}(\lambda, \gamma) := \sum_{i=1}^{p} C_\beta_i \left( \frac{1}{2} \int_0^T \int_0^T k_{ii}(u,v) \ d\lambda_i(u) \ d\lambda_i(v) \right)^{1/(2-\beta_i)}
- \gamma \left( \sum_{i=1}^{p} \xi_i \int_0^T k_{ii}(t,v) \ d\lambda_i(v) + \langle \dot{b}(t), \xi \rangle - x \right).
\]
Then, for every \( \eta \in \mathcal{M}^1[0,1], \) we have
\[
\frac{C_\beta_i}{2-\beta_i} \left( \frac{1}{2} \int_0^T \int_0^T k_{ii}(u,v) \ d\lambda_i(u) \ d\lambda_i(v) \right)^{\frac{\beta_i-1}{2-\beta_i}} \int_0^T \int_0^T k_{ii}(u,v) \ d\lambda_i(u) \ d\eta(v) +
- \gamma \xi_i \int_0^T k_{ii}(t,v) d\eta(v) = 0, \quad i = 1, \ldots, p;
\]
therefore we have to find a vector of measures \( \lambda \) such that, for every \( v \in [0,T], \)
\[
\frac{C_\beta_i}{2-\beta_i} \left( \frac{1}{2} \int_0^T \int_0^T k_{ii}(u,v) \ d\lambda_i(u) \ d\lambda_i(v) \right)^{\frac{\beta_i-1}{2-\beta_i}} \int_0^T \int_0^T k_{ii}(u,v) \ d\lambda_i(u)
- \gamma \xi_i k_{ii}(t,v) = 0, \quad i = 1, \ldots, p.
\]
Then, again, a solution is \( \lambda_i = c_i \delta(t) \) for some \( c_i \in \mathbb{R} \) (for every \( i = 1, \ldots, p \)). So we have to consider the following system
\[
\begin{align*}
\frac{C_\beta_i}{2-\beta_i} \left( \frac{1}{2} k_{ii}(t,t) \right)^{\frac{\beta_i-1}{2-\beta_i}} c_i \beta_i/(2-\beta_i) - \gamma \xi_i &= 0 \quad i = 1, \ldots, p, \\
\sum_{i=1}^{p} \xi_i c_i k_{ii}(t,t) &= -\langle \dot{b}(t), \xi \rangle + x.
\end{align*}
\]
From now on we take $\beta_1 = \cdots = \beta_p = \beta$ as in the statement of the proposition, and the system can be explicitly solved; in fact we can check that

$$\gamma = \left( \frac{-\langle \hat{b}(t), \xi \rangle + x}{\sum_{i=1}^p \xi_i^2 \left( \frac{2-\beta}{C\beta} \right)^\beta \left( \frac{1}{2} \right)^{\frac{1}{\beta}} k_{ii}(t,t)^\frac{1}{\beta} } \right)^{\frac{\beta}{2-\beta}}$$

and

$$c_i = \frac{2-\beta}{2-\beta} \frac{2-\beta}{C\beta} \left( \frac{2-\beta}{C\beta} \right)^{\beta} \left( \frac{1}{2} \right)^{\frac{1}{\beta}} k_{ii}(t,t)^{\frac{1}{\beta}} = (-\langle \hat{b}(t), \xi \rangle + x) \frac{\xi_i^{\frac{1}{\beta}-1}}{\sum_{j=1}^p \xi_j^2 k_{jj}(t,t)^{\frac{1}{\beta}}} k_{ii}(t,t)^{\frac{1}{\beta}-1}, \quad i = 1, \ldots, p.$$

So we can conclude following the same lines of the final part of the proof of Proposition 5.1. In particular we easily get the desired expression for $w$ by (19); indeed we have $\inf_{z \in \mathcal{A}} I_Z(z) = C\beta \sum_{i=1}^p \left[ \frac{1}{2} \xi_i^2 k_{ii}(t,t) \right]^{1/(2-\beta)}$ with $c_1, \ldots, c_p$ computed above (actually here we need some more computations with respect to the case of Proposition 5.1). $\square$

**On the most likely paths leading to exit.** A closer of the proof of Proposition 5.1 reveals that the following function $z^*$ is a most likely path leading to exit:

$$z^*_i(u) := \frac{\xi_i (-\langle \hat{b}(t^*), \xi \rangle + x)}{\sum_{j=1}^p \xi_j^2 k_{jj}(t^*, t^*)} k_{ii}(u, t^*), \quad u \in [0, T], \quad i = 1, \ldots, p,$$

where

$$t^* = \arg\min_{t \in [0, T]} \left[ \frac{1}{2} \left( -\langle \hat{b}(t), \xi \rangle + x \right)^2 \right]^{1/(2-\beta)}.$$

Similarly, for Proposition 5.2 we can define $z^*$ as follows:

$$z^*_i(u) := (-\langle \hat{b}(t^*), \xi \rangle + x) \frac{\xi_i^{\frac{1}{\beta}-1}}{\sum_{j=1}^p \xi_j^2 k_{jj}(t^*, t^*)^{\frac{1}{\beta}}} k_{ii}(u, t^*), \quad u \in [0, T], \quad i = 1, \ldots, p,$$

where

$$t^* = \arg\min_{t \in [0, T]} \left[ \frac{1}{2} \left( -\langle \hat{b}(t), \xi \rangle + x \right)^2 \right]^{1/(2-\beta)}.$$

We also remark that $k_{ii}(0, 0) = 0$ for all $i = 1, \ldots, p$; so, in both cases, $t^* \in (0, T]$.

### 5.2 Exit probabilities from a quadrant

In this section we consider

$$p_n := \mathbb{P} \left( \bigcap_{i=1}^p \left\{ \sup_{t \in [0, T]} Z_i^n(t) \geq x_i \right\} \right), \quad (20)$$

for $\hat{b}_1, \ldots, \hat{b}_p$ such that $x_1 - \hat{b}_1(t), \ldots, x_p - \hat{b}_p(t) > 0$ for $t \in [0, T]$, and $x_1, \ldots, x_p > 0$. Moreover the set $\mathcal{A}$ is defined by

$$\mathcal{A} := \left\{ z \in C([0, T], \mathbb{R}^p) : \sup_{t \in [0, T]} z_i(t) \geq x_i, \quad i = 1, \ldots, p \right\}.$$
Then we have

$$\mathcal{A} = \bigcup_{t_1, \ldots, t_p \in [0, T]} \mathcal{A}_{t_1 \ldots t_p}, \text{ where } \mathcal{A}_{t_1 \ldots t_p} := \{ z \in C([0, T], \mathbb{R}^p) : z_i(t_i) = x_i, \ i = 1, \ldots, p \}. \quad (21)$$

**Remark 5.2.** Here we check (13), and we follow the same lines of Remark 5.1 (where there is a different set $\mathcal{A}$). Again we have to check an inequality. Moreover we can say again that there exists $z^* \in \mathcal{A}$ such that $I_Z(z^*) = \inf_{z \in \mathcal{A}} I_Z(z)$ and, to avoid trivialities, we assume that $z^* \notin \mathcal{A}^c$.

In this case we have sup$_{0 \leq t \leq T} z^*_i(t) \geq x_i$ and, moreover, sup$_{0 \leq t \leq T} z^*_i(t) = x_i$ if and only if $i$ belongs to a suitable nonempty set of indices $\mathcal{I}$. Therefore, for some $t^*_i, \ldots, t^*_p \in [0, T]$, we have $z^*_i(t^*_i) = x_i - \hat{b}_i(t^*_i)$ if $i \in \mathcal{I}$, and $z^*_i(t^*_i) > x_i - \hat{b}_i(t^*_i)$ otherwise. Then, for every $\varepsilon > 0$, let $z^{*, \varepsilon} = (z^*_1, \ldots, z^*_p)$ be defined by

$$z^{*, \varepsilon}_i(t_i) := \begin{cases} (1 + \varepsilon)(z^*_i(t_i) - \hat{b}_i(t_i)) + \hat{b}_i(t_i) & \text{if } i \in \mathcal{I} \\ z^*_i(t_i) & \text{otherwise}. \end{cases}$$

So $z^{*, \varepsilon} \in \mathcal{A}^c$. Indeed, if $i \notin \mathcal{I}$, we have

$$\sup_{0 \leq t \leq T} z^{*, \varepsilon}_i(t_i) \geq z^*_i(t_i) = (1 + \varepsilon)(z^*_i(t_i) - \hat{b}_i(t_i)) + \hat{b}_i(t_i)$$

$$= (1 + \varepsilon)(x_i - \hat{b}_i(t^*_i)) + \hat{b}_i(t^*_i) > x_i;$$

on the other hand, if $i \notin \mathcal{I}$, we have sup$_{0 \leq t \leq T} z^{*, \varepsilon}_i(t_i) \geq z^*_i(t^*_i) = z^*_i(t^*_i) > x_i$. Finally we conclude as we did in Remark 5.1.

In the following Propositions 5.3 and 5.4 we compute $w$ noting that

$$w = \inf_{z \in \mathcal{A}} I_Z(z) = \inf_{t_1, \ldots, t_p \in [0, T]} \inf_{z \in \mathcal{A}_{t_1 \ldots t_p}} I_Z(z); \quad (22)$$

moreover, for every $t_1, \ldots, t_p \in [0, T]$, we compute $\inf_{z \in \mathcal{A}_{t_1 \ldots t_p}} I_Z(z)$ by taking $z$ in the dense set $\mathcal{D}$ defined above (see eq. (15)) by applying the Lagrange multiplied methods.

Now we are ready to prove the results when $(Z^n)_{n \in \mathbb{N}}$ are the sequence of processes in Section 4. Here, for the processes in Section 4.2 there are no restrictions on the parameters $\beta_1, \ldots, \beta_p$.

**Proposition 5.3.** Let $p_n$ be as in (20) with $(Z^n)_{n \in \mathbb{N}}$ be as in Section 4.4. Then (14) holds with $w = w_Q^{(gGp)}$, where

$$w_Q^{(gGp)} := \inf_{t_1, \ldots, t_p \in [0, T]} C_\beta \left[ \frac{1}{2} \sum_{i=1}^{p} \frac{(-\hat{b}_i(t_i) + x_i)^2}{k_{ii}(t_i, t_i)} \right]^{1/(2-\beta)}.$$

**Proof.** We have to compute $w$ in (22) where $\mathcal{A}_{t_1 \ldots t_p}$ is as in (21) and $I_Z$ is the rate function $I_Z^{(gGp)}$ in (10). Thus, by referring to the elements in the dense set of paths $\mathcal{D}_b$ (see (15) and (16)), we have to minimize

$$C_\beta \left[ \sum_{i=1}^{p} \frac{1}{2} \int_{0}^{T} \int_{0}^{T} k_{ii}(u, v) \ d\lambda_i(u) \ d\lambda_i(v) \right]^{1/(2-\beta)}$$

with respect to the vector of measures $\lambda$, subjected to the following constraint

$$\int_{0}^{T} k_{ii}(t, v) \ d\lambda_i(v) = x_i - \hat{b}_i(t_i), \ i = 1, \ldots, p.$$
In particular we immediately get

\[ L(\lambda, \gamma) := C_\beta \left( \sum_{i=1}^{p} \frac{1}{2} \int_0^T \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \, d\lambda_i(v) \right)^\frac{1}{1/(2-\beta)} - \sum_{i=1}^{p} \gamma_i \left( \int_0^T k_{ii}(t_i, v) \, d\lambda_i(v) - x_i + \hat{b}_i(t_i) \right). \]

Then, for every \( \eta \in \mathcal{M}^1[0,1] \), we have

\[
\frac{C_\beta}{2 - \beta} \left( \frac{1}{2} \sum_{i=1}^{p} \int_0^T \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \, d\lambda_i(v) \right)^\frac{\beta}{2-\beta} \int_0^T \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \, d\eta(v) + \\
- \gamma_i \int_0^T k_{ii}(t_i, v) \, d\eta(v) = 0, \quad i = 1, \ldots, p;
\]

therefore we have to find a vector of measures \( \lambda \) such that, for every \( v \in [0,T] \),

\[
\frac{C_\beta}{2 - \beta} \left( \frac{1}{2} \sum_{i=1}^{p} \int_0^T \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \, d\lambda_i(v) \right)^\frac{\beta}{2-\beta} \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \\
- \gamma_i k_{ii}(t, v) = 0, \quad i = 1, \ldots, p.
\]

A solution is \( \lambda_i = c_i \delta_{\{t_i\}} \) for some \( c_i \in \mathbb{R} \) (for every \( i = 1, \ldots, p \)). So we have to consider the following system

\[
\left\{ \begin{array}{l}
\frac{C_\beta}{2 - \beta} \left( \frac{1}{2} \sum_{i=1}^{p} c_i^2 k_{ii}(t_i, t_i) \right)^\frac{\beta}{2-\beta} c_i - \gamma_i = 0 \quad i = 1, \ldots, p \\
c_i k_{ii}(t_i, t_i) = -\hat{b}_i(t_i) + x_i. \quad i = 1, \ldots, p.
\end{array} \right.
\]

In particular we immediately get

\[
c_i = \frac{-\hat{b}_i(t_i) + x_i}{k_{ii}(t_i, t_i)}, \quad i = 1, \ldots, p.
\]

We have a solution of the Lagrange multipliers problem, which is a critique point for the functional we want to minimize. Moreover, since this is a strictly convex functional restricted on a linear subspace of \( \mathcal{M}^p[0,1] \), it is still strictly convex, and therefore the critique point is actually its unique point of minimum. In conclusion we immediately get the desired expression for \( w \) by \((22)\); indeed we have \( \inf_{z \in \mathcal{Z}_1, \ldots, \mathcal{Z}_p} I_Z(z) = C_\beta \left[ \sum_{i=1}^{p} \frac{1}{2} c_i^2 k_{ii}(t_i, t_i) \right]^{1/(2-\beta)} \) with \( c_1, \ldots, c_p \) computed above.

**Proposition 5.4.** Let \( p_n \) be as in \((26)\) with \( (Z^n)_{n \in \mathbb{N}} \) be as in Section 4.2. Then \((14)\) holds with \( w = w_Q^{(\text{ind})} \), where

\[
w_Q^{(\text{ind})} := \inf_{t_1, \ldots, t_p \in [0,T]} \left[ \sum_{i=1}^{p} C_\beta_i \left[ \frac{1}{2} \left( \frac{-\hat{b}_i(t_i) + x_i}{k_{ii}(t_i, t_i)} \right)^2 \right]^{1/(2-\beta_i)} \right].
\]

**Proof.** We have to compute \( w \) in \((22)\) where \( \mathcal{Z}_1, \ldots, \mathcal{Z}_p \) is as in \((21)\) and \( I_Z \) is the rate function \( I_Z^{(\text{ind})} \) in \((12)\). We follow the same lines of the proof of Proposition \( 5.3 \) and we omit some details. Then we
Then, for every \( \eta \) therefore we have to find a vector of measures \( \lambda \) such that, for every \( v \in [0, T] \),

\[
\frac{C_{\beta_i}}{2 - \beta_i} \left( \frac{1}{2} \int_0^T \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \, d\lambda_i(v) \right)^{\frac{\beta_i - 1}{2 - \beta_i}} \int_0^T \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \, d\eta(v) + \gamma_i \int_0^T k_{ii}(t_i, v) \, d\xi(v) = 0, \quad i = 1, \ldots, p;
\]

therefore we have to find a vector of measures \( \lambda \) such that, for every \( v \in [0, T] \),

\[
\lambda_i = c_i \delta_{\{t_i\}} \quad \text{for some} \quad c_i \in \mathbb{R} \quad \text{(for every} \quad i = 1, \ldots, p). \quad \text{So we have to consider the following system}
\]

\[
\begin{align*}
&\frac{C_{\beta_i}}{2 - \beta_i} \left( \frac{1}{2} \int_0^T \int_0^T k_{ii}(u, v) \, d\lambda_i(u) \, d\lambda_i(v) \right)^{\frac{\beta_i - 1}{2 - \beta_i}} c_i^{\beta_i/(2 - \beta_i)} - \gamma_i = 0, \quad i = 1, \ldots, p \\
c_i k_{ii}(t_i, t_i) &= -\hat{b}_i(t_i) + x_i, \quad i = 1, \ldots, p.
\end{align*}
\]

In particular we immediately get

\[
c_i = -\frac{\hat{b}_i(t_i) + x_i}{k_{ii}(t_i, t_i)}, \quad i = 1, \ldots, p.
\]

So we can conclude following the same lines of the final part of the proof of Proposition 5.3. In particular we immediately get the desired expression for \( \lambda \) by (22); indeed we have

\[
\inf_{z \in \mathcal{z}_{1, \ldots, p}} I_Z(z) = \sum_{i=1}^p C_{\beta_i} \left[ \frac{1}{2} c_i^2 k_{ii}(t_i, t_i) \right]^{1/(2 - \beta_i)} \text{ with } c_1, \ldots, c_p \text{ computed above.} \quad \square
\]

On the most likely paths leading to exit. This is the analogue of the discussion presented in Section 5.1. A closer of the proofs of Propositions 5.3 and 5.4 reveal that, in both cases, we can define a most likely path \( z^* \) as follows

\[
z_i^*(u) := \frac{-\hat{b}_i(t_i^*) + x_i}{k_{ii}(t_i^*, t_i^*)}, \quad u \in [0, T], \quad i = 1, \ldots, p,
\]

with two different definitions of \((t_1^*, \ldots, t_p^*)\). In the case of Proposition 5.3 we have

\[
(t_1^*, \ldots, t_p^*) = \arg\min_{t_1, \ldots, t_p \in [0, T]} \left[ \frac{1}{2} \sum_{i=1}^p \frac{(-\hat{b}_i(t_i) + x_i)^2}{k_{ii}(t_i, t_i)} \right]^{1/(2 - \beta)},
\]
and in the case of Proposition 5.4 we have
\[
(t_1^*, \ldots, t_p^*) = \arg\min_{t_1, \ldots, t_p \in [0, T]} \sum_{i=1}^p C_{\beta_i} \left[ \frac{1}{2} \left(-\hat{b}_i(t_i) + x_i\right)^2 \right]^{1/(2-\beta_i)},
\]
that is \( t_i^* = \arg\min_{t_i \in [0, T]} \left[ \frac{1}{2} \left(-\hat{b}_i(t_i) + x_i\right)^2 \right]^{1/(2-\beta_i)} \) for every \( i = 1, \ldots, p \). We also remark that \( k_{ii}(0, 0) = 0 \) for all \( i = 1, \ldots, p \); so, in both cases, \( t_i^* \in (0, T] \) for all \( i = 1, \ldots, p \).

6 Comparisons between asymptotic rates

Inequalities between rate functions allow to compare the convergence of two sequences of stochastic processes. Moreover, by taking into account the limit 14, inequalities between rate functions allow to compare the exponential decay rates of exit probabilities (indeed, as we pointed out, \( w \) is equal to the infimum of the rate function over a suitable set of paths). In order to present our results we need the following lemma.

Lemma 6.1. Let \( x_1, \ldots, x_n \) be non negative number. Then: for \( r \in (0, 1) \) \((r \in (1, +\infty), \text{ resp.})\)
\[
\sum_{i=1}^p x_i^r \geq \left( \sum_{i=1}^p x_i \right)^r \quad \left( \sum_{i=1}^p x_i^r \leq \left( \sum_{i=1}^p x_i \right)^r \text{ resp.} \right),
\]
and the equality holds if and only if the set \( \{ i \in \{1, \ldots, p \} : x_i \neq 0 \} \) has at most one element.

Proof. The case \( x_1 = \ldots = x_n = 0 \) is trivial. We remark that \( y^r \geq y \) for \( y \in [0, 1] \) if \( r \in (0, 1) \), and \( y^r \leq y \) for \( y \in (0, 1] \) if \( r \in (1, \infty) \); moreover we have \( y^r = y \) if and only if \( y = 0 \) or \( y = 1 \). Now we assume that \( x_i \neq 0 \) for some \( i \in \{1, \ldots, p\} \), and we have \( S(x) = \sum_{i=1}^p x_i > 0 \); then, if \( r \in (0, 1) \), we have
\[
\sum_{i=1}^p x_i^r = S(x)^r \sum_{i=1}^p \left( \frac{x_i}{S(x)} \right)^r \geq S(x)^r \sum_{i=1}^p \frac{x_i}{S(x)} = S(x)^r,
\]
and the inverse inequality for \( r \in (1, +\infty) \). Finally it is easy to see that the inequalities turn into equalities if and only if the set \( \{ i \in \{1, \ldots, p\} : x_i \neq 0 \} \) has at most one element.

We are ready to present the following proposition.

Proposition 6.1. The following three statements hold.
(i) Let \( I_Z^{(gGp)} \) be the rate function in 14, and let \( I_Z^{(ind)} \) be the rate function in 12 \((\beta_1 = \cdots = \beta_p = \beta)\). Then \( I_Z^{(gGp)}(z) \leq I_Z^{(ind)}(z) \) for all \( z \in C([0, T], \mathbb{R}^p) \). Moreover the equality holds if and only if the set \( \{ i \in \{1, \ldots, p\} : z_i \neq 0 \} \) has at most one element.
(ii) Let \( w_H^{(gGp)} \) and \( w_H^{(ind)} \) be the exponential decay rates in Propositions 5.1 and 5.2. Then \( w_H^{(gGp)} \leq w_H^{(ind)} \). Moreover the equality holds if and only if the set \( \{ i \in \{1, \ldots, p\} : \xi_i \neq 0 \} \) has at most one element.
(iii) Let \( w_Q^{(gGp)} \) and \( w_Q^{(ind)} \) be the exponential decay rates in Propositions 5.3 and 5.4, and take the second one with \( \beta_1 = \cdots = \beta_p = \beta \). Then \( w_Q^{(gGp)} \leq w_Q^{(ind)} \). Moreover the equality holds if and only if \( p = 1 \).

Proof. We prove the statements separately.
(i) We trivially have \( I_Z^{(gGp)}(z) = I_Z^{(ind)}(z) = 0 \) if \( z = 0 \), i.e. if all \( z_1, \ldots, z_p \) are null functions. If
not, we can apply Lemma 6.1 with \( x_i = \frac{1}{2}\|z_i\|_{\beta}^2 \) for \( i \in \{1, \ldots, p\} \) and \( r = \frac{1}{2}\beta \in (0, 1) \).

(ii) By Lemma 6.1 with \( x_i = \xi_i^2 k_{ii}(t, t) \) for \( i \in \{1, \ldots, p\} \) and \( r = \frac{1}{\beta} \in (1, +\infty) \) we get

\[
\left( \sum_{i=1}^{p} \xi_i^2 k_{ii}(t, t) \right)^{1/\beta} \geq \sum_{i=1}^{p} \xi_i^2 k_{ii}(t, t)^{1/\beta};
\]

then we obtain

\[
\left[ \frac{1}{2} \sum_{j=1}^{p} \xi_j^2 k_{jj}(t, t) \right]^{1/(2 - \beta)} \leq \left[ \frac{1}{2} \left( \sum_{i=1}^{p} \xi_i^2 k_{ii}(t, t)^{1/\beta} \right)^{\beta} \right]^{1/(2 - \beta)},
\]

and the equality holds if and only if the set \( \{ i \in \{1, \ldots, p\} : \xi_i \neq 0 \} \) has one element. If this is not the situation, then we get the desired strict inequalities between the infima with respect to \( t \in [0, T] \) (in both left and right hand sides the infima are attained).

(iii) By Lemma 6.1 with \( x_i = \frac{1}{2} (-\hat{b}_i(t, t_i) + x_i)^2 \) for \( i \in \{1, \ldots, p\} \) (note that they are all positive) and \( r = \frac{1}{2\beta} \in (0, 1) \) we get

\[
C_\beta \sum_{i=1}^{p} \left[ \frac{1}{2} (-\hat{b}_i(t_i) + x_i)^2 \right]^{1/(2 - \beta)} \geq C_\beta \left[ \sum_{i=1}^{p} \left( \frac{1}{2} k_{ii}(t_i, t_i) \right)^{1/(2 - \beta)} \right],
\]

and the equality holds if and only if \( p = 1 \). If this is not the situation, then we get the desired strict inequalities between the infima with respect to \( t_1, \ldots, t_p \in [0, T] \) (in both left and right hand sides the infima are attained).

The inequalities between exponential decay rates of exit probabilities (see statements (ii) and (iii) in Proposition 6.1) have some analogies with some inequalities in the literature. Here we recall the inequalities in Proposition 4.1 in [18] concerning a multivariate risk process with delayed claims in insurance; in such a case the exponential decay rate is called Lundberg parameter. In particular one has a smaller exponential decay rate when the joint distribution of the claims is larger with respect to the supermodular order. We also recall that, as discussed in [18] (see Section 4.2), comonotonicity is the strongest case of dependence with respect to the supermodular order (see [22], Theorem 3.9.5 (c), p. 114; see also [3], Theorem 2.1) when one compares two joint distributions with the same marginal distributions; indeed this can be related to the hypothesis \( \beta_1 = \cdots = \beta_p = \beta \) in Proposition 6.1 which allows to have the same marginal one-dimensional distributions of the case of the multivariate grey Gaussian processes. We also recall that, when we deal with the grey Gaussian processes, the random vectors \((A^n)_{n \in N}\) in Corollary 3.3 are comonotonic because the components are all coincident.

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