Isometric immersions and differential equations which describe pseudospherical surfaces

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Abstract

We provide families of second order non-linear partial differential equations describing pseudospherical surfaces (\textit{pss} equations) with the property of having local isometric immersions in $\mathbb{E}^3$ with principal curvatures depending on the finite order jets of solutions of the differential equations. These equations occupy a particularly special place amongst \textit{pss} equations since a series of recent papers on several classes of \textit{pss} equations seemed to suggest that only the sine-Gordon equation had the above property. Explicit examples are given, which include the short pulse equation and some generalizations.

Keywords: Nonlinear partial differential equations, Pseudospherical surfaces, Local isometric immersion, Sine-Gordon equation, Short pulse equation, Generalized short pulse equations

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1. Introduction

A differential equation $\mathcal{E}$, for $z = z(x,t)$, is said to be a \textit{pss equation} (or to describe pseudospherical surfaces) if it is equivalent to the structure equations of a surface with constant Gaussian curvature equal to $-1$, i.e.,

\[ d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_3, \quad d\omega_3 = \omega_1 \wedge \omega_2, \]

where $\omega_i = f_{ij} \, dx^j + f_{ik} \, dt$, $1 \leq i \leq 3$, are 1-forms with $f_{ij}$ smooth functions of $t$, $x$, $z$ and finite derivatives of $z$ with respect to $x$ and $t$, and such that $\omega_1 \wedge \omega_2 \neq 0$ for generic solutions. In view of (1), for any generic solution $z$ of $\mathcal{E}$ defined on an open subset $U \subset \mathbb{R}^2$, the 1-form $\omega_3 = \omega_1 dx + \omega_2 dt$ defines the Levi-Civita connection of the corresponding pseudospherical metric defined on $U$ by $I = (\omega_1)^2 + (\omega_2)^2$.

The best known example of a \textit{pss} equation is the sine-Gordon equation

\[ z_{tx} = \sin z, \]

with 1-forms

\[ \omega_1 = \frac{1}{\eta} \sin (z) \, dt, \quad \omega_2 = \eta \, dx \quad \omega_3 = \eta \, dz \]

where $\eta \in \mathbb{R} \setminus \{0\}$. In this case, one has the metric given by $I = \omega_1^2 + \omega_2^2 = \frac{1}{\eta^2} dt^2 + 2 \cos(z) \, dx \, dt + \eta^2 dx^2$.

Other well-known differential equations that describe pseudospherical surfaces include for example, the KdV equation, the MKdV, the Burgers, the nonlinear Schrödinger, the short pulse equation, etc.

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According to [8], pss equations have special properties that play an important role in the study of nonlinear evolution equations. In fact, pss equations may have Bäcklund transformations, infinite hierarchies of conservation laws and they may also be solved by applying the inverse scattering method (see for instance [8], [1] and references therein). Any pss equation is the integrability condition of a $2 \times 2$ and also a $3 \times 3$ linear problem determined by the 1-forms $\omega_i$, $i = 1, 2, 3$ satisfying [11]. In fact, the linear problem is given by
\[
\begin{align*}
\frac{\partial V}{\partial x} &= XV, & \frac{\partial V}{\partial t} &= TV,
\end{align*}
\]
where
\[
X = \frac{1}{2} \begin{pmatrix} f_{21} & f_{11} - f_{31} \\ f_{11} + f_{31} & -f_{21} \end{pmatrix}, \quad T = \frac{1}{2} \begin{pmatrix} f_{22} & f_{12} - f_{32} \\ f_{12} + f_{32} & -f_{22} \end{pmatrix}
\]
or
\[
X = \begin{pmatrix} 0 & f_{11} & f_{21} \\ f_{11} & 0 & f_{31} \\ f_{21} & -f_{31} & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & f_{12} & f_{22} \\ f_{12} & 0 & f_{32} \\ f_{22} & -f_{32} & 0 \end{pmatrix}.
\]
For recent developments on the study and classification of pss equations, the reader is referred to [2, 6, 10, 11, 12, 13].

Any pseudospherical surface described by a pss equation $\mathcal{E}$ admits a local isometric immersion into $\mathbb{E}^3$; this is due to the classical result that a pseudospherical surface always admits a local isometric immersion into $\mathbb{E}^3$. Hence, in view of Bonnet theorem, to any generic solution $z$ of $\mathcal{E}$ it is associated a pair $(I[z], II[z])$ of first and second fundamental forms, which satisfy the Gauss–Codazzi equations and describe a local isometric immersion into $\mathbb{E}^3$ of the associated pseudospherical surface. Recall that (see [12] and also [3, 11, 13, 2]) by introducing the 1-forms
\[
\omega_{13} = a\omega_1 + b\omega_2, \quad \omega_{23} = b\omega_1 + c\omega_2,
\]
the Gauss–Codazzi equations read
\[
ac - b^2 = -1, \quad d\omega_{13} = \omega_{12} \wedge \omega_{23}, \quad d\omega_{23} = \omega_{21} \wedge \omega_{13}
\]
and the second fundamental form writes as
\[
II = \omega_1 \cdot \omega_{13} + \omega_2 \cdot \omega_{23}.
\]
In particular, $H = (a + c)/2$ gives the mean curvature of the isometric immersion.

Thus for any generic solution $z$ of $\mathcal{E}$, defined on some open subset $U \subset \mathbb{R}^2$, there always exists a triple $(a, b, c)$ of differentiable functions that is locally defined on $U$ and satisfies the Gauss–Codazzi equations for the pseudospherical surface described by $z$. However, in general the dependence of $(a, b, c)$, and hence of $(I, II)$, on $z$ may be quite complicated, and it is not guaranteed that it depends on a jet of finite order of $z$, i.e., on $x, t$ and a finite number of derivatives of $z$ with respect to $x$ and $t$.

An example of a pss equation which admits local isometric immersions with $(a, b, c)$ (and hence principal curvatures) depending on a jet of finite order of $z$, is provided by the sine-Gordon equation: indeed by taking $\omega_1, \omega_2$ and $\omega_3$ as in [3], one has the first and second fundamental forms given by
\[
I = \frac{1}{\eta^2} dt^2 + 2 \cos(z) dx dt + \eta^2 dx^2 \quad II = \pm 2 \sin(z) dx dt,
\]
satisfying the Gauss–Codazzi equations. The corresponding 1-forms $\omega_{13}$ and $\omega_{23}$ are given by [5], where
\[
\begin{align*}
a &= \pm 2/\tan(z), \quad b = \mp 1, \quad c = 0.
\end{align*}
\]
In a series of recent papers [2, 7, 11, 12, 13], the existence of such local isometric immersions has been investigated for the families of pss equations previously studied in the papers [3, 6, 8, 10, 15]. Surprisingly, those papers showed that, except for the sine-Gordon equation, all local isometric immersions admitted by pss equations described in these papers have triples, $(a, b, c)$ which are “universal” in the sense that they
only depend on \(x\) and \(t\). Hence, contrary to the case of the sine-Gordon equation, for any pss equation \(E\) of the type described in \([3, 6, 8, 10, 13]\) the principal curvatures of the immersion are functions of \(x\), \(t\) and do not depend on the generic solutions. This fact shows that the sine-Gordon equation occupies a particularly special place amongst all pss equations, and clearly motivates the search for other examples of pss equations which admit local isometric immersions whose principal curvatures depend explicitly on their generic solutions. Families of equations of this type are provided in Section 2.

Motivated by our recent paper \([5]\), we start by introducing a family of pss differential equations depending on four arbitrary differentiable functions which satisfy generic conditions (see Proposition \([4]\)). Then, by specializing these functions, we obtain a family of pss equations that admit local isometric immersions with principal curvatures depending on the first order jet of solutions of the pss equations (see Corollary \([1]\) and Proposition \([2]\)). Next, we provide explicit examples, which include a generalization of the short pulse equation.

We point out that, since the nineteenth century, the sine-Gordon equation has been classically related to surfaces of constant negative curvature; moreover, it also appeared in a number of physical applications, including the relativistic field theory and the mechanical transmission lines. The short pulse equation, on the other hand, first appeared in the study of pss equations \([1, 14]\), and only recently it has found physical applications in the description of ultra-short light pulses in silica optical fibers \([17]\). It would be interesting to find out if the families of equations given in Section \([2]\) contain other equations, besides the short pulse equation, also describe physical phenomena.

2. Equations

Recently, in \([3]\) we obtained a classification for the class of quasilinear second order partial differential equations

\[
z_{tt} = A(z, z_x, z_t) z_{xx} + B(z, z_x, z_t) z_{xt} + C(z, z_x, z_t),
\]

which describes either pseudospherical surfaces or spherical surfaces, under suitable conditions for the functions \(f_{ij}\).

The following result is a consequence of Theorem 3.6 from \([5]\).

Proposition 1. A second order partial differential equation

\[
z_{xt} = \frac{1}{f_{12, z_t} - f_{11, z_x}} (f_{11, z_t} z_{tt} - f_{12, z_x} z_{xx} + f_{11, z_x} z_t - f_{12, z_t} z_x + f_{31} f_{22} - f_{21} f_{32}),
\]

where \(f_{ij} = f_{ij}(z, z_t, z_x), 1 \leq i \leq 3\) and \(1 \leq j \leq 2\), are differentiable functions such that \(f_{12, z_t} - f_{11, z_x} \neq 0\) and \(f_{21, z_t} = f_{31, z_x} = 0\), describes pss if, and only if,

\[
\begin{pmatrix}
    f_{11} \\
    f_{12}
\end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix}
    -\phi z_x - \psi_{21} & \varphi z_x + \psi_{31} \\
    -\phi z_t - \psi_{22} & \varphi z_t + \psi_{32}
\end{pmatrix} \begin{pmatrix}
    \psi_{21, z} & -\psi_{22, z} \\
    \psi_{31, z} & -\psi_{32, z}
\end{pmatrix} \begin{pmatrix}
    z_t \\
    z_x
\end{pmatrix}
\]

and

\[
\begin{align*}
    f_{21} &= \phi z_x + \psi_{21}, & f_{22} &= \phi z_t + \psi_{22}, \\
    f_{31} &= \varphi z_x + \psi_{31}, & f_{32} &= \varphi z_t + \psi_{32},
\end{align*}
\]

with \(f_{11, z_t} \neq 0\), \(\Delta = f_{32} f_{21} - f_{31} f_{22} \neq 0\) and \(f_{21} f_{12} - f_{11} f_{22} \neq 0\), where \(\phi, \varphi\) and \(\psi_{rs}\), \(r = 2, 3, s = 1, 2\) are differentiable functions of \(z\).

Remark 1. Since \(\Delta = f_{32} f_{21} - f_{31} f_{22}\), it is a straightforward computation to check that \([0]\) can be written in the more compact form

\[
D_x(f_{12}) - D_t(f_{11}) + \Delta = 0,
\]

where \(D_x\) and \(D_t\) are total derivatives.
The proposition above provides a huge family of pss differential equations. In fact, they are obtained by choosing six arbitrary differentiable functions of \( z \), satisfying generic conditions, namely \( \phi, \varphi \) and \( \psi_{rs} \), \( r = 2, 3, s = 1, 2 \), which determine \( f_{ij} \) by \((10)\) and \((11)\).

In what follows, we will specialize these functions to

\[
\phi(z) = 2\left( F_1(z) + \lambda^2 F_2(z) \right), \quad \varphi(z) = -2z F_1(z), \quad \psi_{21}(z) = \lambda, \quad \psi_{22}(z) = \lambda z^2 + \frac{1}{\lambda}, \quad \psi_{31}(z) = 0, \quad \psi_{32}(z) = -\epsilon z,
\]

where \( \epsilon = \pm 1; \ F_1(z), \ F_2(z) \) are differentiable functions and \( \lambda \in \mathbb{R} \setminus \{0\} \) is a free parameter. Then, as a consequence of Proposition\[11\] we get the following

**Corollary 1.** The differential equation

\[
D_x \left( \frac{(z^4 + 4F_2 z_t) z_x}{2G} \right) - D_t \left( \frac{(z^2 + 2F_2 z_x) z_x}{G} \right) + \frac{1}{z} G = 0,
\]

where

\[
G(z, z_x, z_t) = (F_1 z^2 - 2F_2) z_x - 2F_1 z_t - z^2,
\]

\( F_1(z) \) and \( F_2(z) \) are differentiable functions and \( G \neq 0 \), describes pss equations with

\[
\omega_1 = \frac{\lambda z_x}{2G} \left[ 2(z^2 + 2F_2 z_x) dx + (z^4 + 4F_2 z_t) dt \right],
\]

\[
\omega_2 = \left[ \frac{2(\lambda^2 F_2 + F_1)}{\lambda z^2} z_x + \lambda \right] dx + \left[ \frac{2(\lambda^2 F_2 + F_1)}{\lambda z^2} z_t + \frac{\lambda}{2} z^2 + \frac{1}{\lambda} \right] dt,
\]

\[
\omega_3 = -2\epsilon F_1 \frac{z_x}{z} dx + \left( -2\epsilon F_1 \frac{z_t}{z} - \epsilon z \right) dt,
\]

and \( \lambda \in \mathbb{R} \setminus \{0\} \).

The next result shows that \((9)\) is a pss equation that admits local isometric immersions, whose principal curvatures depend on a finite order jet of \( z \).

**Proposition 2.** For any solution \( z \) of \((9)\) which satisfies \( z_x \neq 0 \), the pseudospherical metric \( I = \omega_1^2 + \omega_2^2 \) has an isometric immersion in \( \mathbb{E}^3 \), with \( a, b \) and \( c \) depending on the first order jet of \( z \), which is given by

\[
\omega_{13} = \frac{-2\epsilon}{z^2 z_x} G \omega_1 + \omega_2, \quad \omega_{23} = \omega_1.
\]

**Proof.** Recalling that \( \omega_3 = \omega_{12} \) is the connection form of \( I = \omega_1^2 + \omega_2^2 \), a straightforward computation with \( \omega_1, \omega_2 \) and \( \omega_3 \) as in Corollary\[11\] entails that

\[
d\omega_{13} = \omega_{12} \wedge \omega_{23}, \quad d\omega_{23} = \omega_{21} \wedge \omega_{13} \quad \text{and} \quad ac - b^2 = -1.
\]

Before providing particular examples, we observe that for any choice of \( F_1(z) \) and \( F_2(z) \) as in Corollary\[11\] the corresponding differential equation \((9)\) is the compatibility condition of a linear problem (Lax pair or zero curvature condition) described by \((10)\), where the functions \( f_{ij} \) are explicitly given by \((11)\), in terms of \( z, z_x, z_t \) and a non zero parameter \( \lambda \), since \( \omega_i = f_{1i} dx + f_{2i} dt \).

Moreover, one may obtain infinite conservation laws for \((9)\) as a consequence of the following geometric properties. Assume that a differential equation \( \mathcal{E} \) for \( z(x, t) \) describes pseudospherical surfaces, with associated 1-forms \( \omega_1, \omega_2, \omega_3 \), then \( z \) is a solution of \( \mathcal{E} \) if and only if \( \omega_3 - d\rho \omega_1 + \sin \rho \omega_1 + \cos \rho \omega_2 = 0 \) is completely integrable for \( \rho \). For each solution \( z \) of \( \mathcal{E} \) and corresponding solution \( \rho \), the 1-form \( \cos \rho \omega_1 - \sin \rho \omega_2 \) is closed (see the proof of Proposition 4.2 in \[3\]). Whenever the 1-forms \( \omega_i, \ i = 1, 2, 3 \) are analytic on a parameter \( \lambda \), then \( \rho \) is also analytic on the parameter and hence the conservation laws may be obtained by the closed form written as a series in \( \lambda \).

We now exhibit a class of examples of \((9)\) which includes the short pulse equation.
Example 3. In view of (9), by considering $F_1(z) = \ell(z) z^2$ and $F_2(z) = 0$, where $\ell(z)$ is a differentiable function, one has

$$z_{xt} = \frac{1}{2[\ell(z^2 z_x + 2z_t) + 1]} \left\{ 4\ell z_x z_{xt} + z^2 (1 + 2\ell z_t) z_{xx} - \ell' (4z^2 z_x z_t - z^4 z^2 - 4z^2 z_x^2 ) z_x + + 2z^2 z_x^2 - 2z \left[ \ell(z^2 z_x - 2z_t) - 1 \right]^3 \right\}.$$  

This is a pss differential equation, whose generic solutions define pseudospherical metrics admitting local isometric immersions in $\mathbb{R}^3$, with first and second fundamental forms provided by

$$\omega_1 = \frac{\epsilon \lambda z_x}{\ell(z^2 z_x - 2z_t) - 1} \left( dx + \frac{z^2}{2} dt \right),$$

$$\omega_2 = \left( \frac{2\ell}{\lambda} z_x + \lambda \right) dx + \left( \frac{2 \ell}{\lambda} z_t + \frac{\lambda}{2} z^2 + \frac{1}{\lambda} \right) dt,$$

$$\omega_3 = -2\epsilon \ell z z_x dx - \epsilon z (2\ell z_t + 1) dt,$$

and

$$\omega_{13} = -\frac{2\ell}{z_x} [\ell(z^2 z_x - 2z_t) - 1] \omega_1 + \omega_2, \quad \omega_{23} = \omega_1.$$

Example 4. By choosing $\ell(z) = 0$ in the previous example (i.e. $F_1 = F_2 = 0$ in (9)), one obtains the short pulse equation

$$z_{xt} = z + \frac{1}{6} (z^3)_{xx},$$

also known as the Rabelo’s cubic equation [14], with first and second fundamental forms provided by

$$\omega_1 = -\epsilon \lambda z_x dx - \frac{\epsilon \lambda}{2} z^2 z_x dt, \quad \omega_2 = \lambda dx + \left( \frac{\lambda}{2} z^2 + \frac{1}{\lambda} \right) dt, \quad \omega_3 = -\epsilon z dt,$$

and

$$\omega_{13} = \frac{2\ell}{z_x} \omega_1 + \omega_2, \quad \omega_{23} = \omega_1.$$

Example 5. By choosing $\ell(z) = m \in \mathbb{R}$ in Example 3 one obtains the following equation

$$z_{xt} = \frac{1}{2[m(z^2 z_x + 2z_t) + 1]} \left\{ 4m z_x z_{xt} + z^2 (1 + 2m z_t) z_{xx} + 2z^2 z_{xx} - 2z \left[ m(z^2 z_x - 2z_t) - 1 \right]^3 \right\},$$

which is a pss equation with associated forms as in (11), where $\ell = m$.

Thus, the family of equations given by Example 3 (in particular by Example 5) provides pss equations that generalize the short pulse equation and admit local isometric immersions in $\mathbb{R}^3$ whose principal curvatures depend on the first order jet of its solutions. We note here that other generalizations of the short pulse equation have been discussed and investigated in [3, 4, 10].

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