QUANTUM AFFINE REFLECTION ALGEBRAS
OF TYPE $d_n^{(1)}$ AND REFLECTION MATRICES

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Abstract. Quantum affine reflection algebras are coideal subalgebras of quantum affine algebras that lead to trigonometric reflection matrices (solutions of the boundary Yang-Baxter equation). In this paper we use the quantum affine reflection algebras of type $d_n^{(1)}$ to determine new $n$-parameter families of non-diagonal reflection matrices. These matrices describe the reflection of vector solitons off the boundary in $d_n^{(1)}$ affine Toda field theory. They can also be used to construct new integrable vertex models and quantum spin chains with open boundary conditions.

1. Introduction

Solutions to the reflection equation (also known as the boundary Yang-Baxter equation) \[ \mathfrak{R} \], called reflection matrices, are required for the construction of quantum integrable models with boundaries \[ \mathfrak{R} \]. The reflection equation at the boundary is similar to the Yang-Baxter equation in the bulk, the study of which led to the axiomatization of quantum groups \[ \mathfrak{R} \]. Intertwiners of these quantum groups are solutions of the Yang-Baxter equation (called R-matrices) \[ \mathfrak{R} \].

An analogous method to find reflection matrices using intertwiners of reflection equation algebras \[ \mathfrak{R} \] embedded as coideal subalgebras in the corresponding Yangian or quantum affine algebra has recently been developed \[ \mathfrak{R} \]. While there is a general construction of these subalgebras in terms of the universal R-matrix \[ \mathfrak{R} \], it is of practical importance to have a simple set of generators for them. Such simple generators have been found for coideal subalgebras of Yangians \[ \mathfrak{R} \] and for coideal subalgebras of quantum affine algebras \[ \mathfrak{R} \], which we refer to as “quantum affine reflection algebras”. With these expressions it is easy to solve the intertwining condition, which has already led to several new reflection matrices \[ \mathfrak{R} \].

The purpose of this letter is to demonstrate the practical utility of these quantum affine reflection algebras by using them to derive hitherto unknown reflection matrices corresponding to the vector representation of $so(2n)$. These reflection matrices are needed, for example, to describe the reflection of solitons in $d_n^{(1)}$ affine Toda theory off an integrable boundary \[ \mathfrak{R} \].

2. Quantum affine algebras

In this section we summarize some essential facts about quantum affine algebras. More details can be found in, for example, \[ \mathfrak{R} \].

Let \( \hat{\mathfrak{g}} \) be an affine Lie algebra with generalized Cartan matrix \( (a_{ij})_{i,j=0,...,n} \). The quantum affine algebra \( U_h(\hat{\mathfrak{g}}) \) is the unital associative algebra over \( \mathbb{C}[\hbar] \) with
generators $x_i^+, x_i^-, h_i$, $i = 0, \ldots, n$ and relations
\begin{equation}
[h_i, h_j] = 0, \quad [h_i, x_j^\pm] = \pm a_{ij} x_j^\pm,
\end{equation}
\begin{equation}
[x_i^+, x_j^-] = \delta_{ij} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}},
\end{equation}
\begin{equation}
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k!} \right] (x_i^\pm)^k x_j^\pm (x_i^\pm)^{1-a_{ij}-k} = 0 \quad i \neq j.
\end{equation}

Here $\left[ \begin{array}{c} a \\ b \end{array} \right]_q$ are the $q$-binomial coefficients. We have defined $q_i = e^{d_i h}$ where the $d_i$ are such that $d_i a_{ij}$ is a symmetric matrix.

The Hopf algebra structure of $U_h(\hat{g})$ is given by the comultiplication $\Delta : U_h(\hat{g}) \to U_h(\hat{g}) \otimes U_h(\hat{g})$ defined by
\begin{equation}
\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i,
\end{equation}
\begin{equation}
\Delta(x_i^\pm) = x_i^\pm \otimes q_i^{-h_i/2} + q_i^{h_i/2} \otimes x_i^\pm,
\end{equation}
and the antipode $S$ and counit $\epsilon$ defined by
\begin{equation}
S(h_i) = -h_i, \quad S(x_i^\pm) = -q_i^{\mp 1} x_i^\pm, \quad \epsilon(h_i) = \epsilon(x_i^\pm) = 0.
\end{equation}

For any $(n+1)$-tuple $\{\sigma_0, \ldots, \sigma_n\}$ of invertible constants in $\mathbb{C}[[h]]$ there is a Hopf-algebra automorphism $\sigma$ of $U_h(\hat{g})$ defined by
\begin{equation}
\sigma : x_i^\pm \mapsto \sigma_i^{\mp 1} x_i^\pm, \quad h_i \mapsto h_i.
\end{equation}

Given some finite-dimensional $U_h(\hat{g})$-module $V$ with representation map $\pi : U_h(\hat{g}) \to \text{End}(V)$ one can define a spectral parameter dependent representation $\pi_x : U_h(\hat{g}) \to \text{End}(V_x)$ by
\begin{equation}
\pi_x = \pi \circ \sigma_x
\end{equation}
where $\sigma_x$ is an automorphism of the form (4) with the choice $\sigma_0 = x$, $\sigma_i = 1$, $i = 1, \ldots, n$. The tensor product modules $V_{x_1} \otimes V_{x_2} \otimes \cdots \otimes V_{x_n}$ are irreducible for generic values of the spectral parameters $x_i$.

Let $\hat{R}(x)$ be the unique module homomorphism $\hat{R}(x) : V_x \otimes V_1 \mapsto V_1 \otimes V_x$, i.e., let $\hat{R}(x)$ be a solution of the intertwining equation
\begin{equation}
\hat{R}(x) (\pi_x \otimes \pi_1)(\Delta(Q)) = (\pi_1 \otimes \pi_x)(\Delta(Q)) \hat{R}(x), \quad \text{for all} \quad Q \in U_h(\hat{g}).
\end{equation}
Then $\hat{R}(x)$ automatically satisfies (3) the Yang-Baxter equation
\begin{equation}
(\hat{R}(y) \otimes 1)(1 \otimes \hat{R}(xy))(\hat{R}(x) \otimes 1) = (1 \otimes \hat{R}(x))(\hat{R}(xy) \otimes 1)(1 \otimes \hat{R}(y)).
\end{equation}

3. QUANTUM AFFINE REFLECTION ALGEBRAS

In this paper we are concerned with subalgebras $B_h(\hat{g}) \subset U_h(\hat{g})$ generated by
\begin{equation}
\hat{Q}_j = q_j^{h_j/2} (x_j^+ + x_j^-) + \hat{\epsilon}_j \left( q_j^{h_j/2} - 1 \right), \quad j = 0, \ldots, n,
\end{equation}
for certain choices of the parameters $\hat{\epsilon}_j \in \mathbb{C}[[h]]$. We refer to these algebras as quantum affine reflection algebras. It was shown in [4] how they are related to Sklyanin’s reflection equation algebras [5]. The algebras $B_h(\hat{g})$ are left coideal subalgebras of $U_h(\hat{g})$ because
\begin{equation}
\Delta(\hat{Q}_j) = \hat{Q}_j \otimes 1 + q_j^{h_j} \otimes \hat{Q}_j \in U_h(\hat{g}) \otimes B_h(\hat{g}).
\end{equation}
Generic, the irreducible evaluation modules $V_x$ of $U_h(\hat{g})$ and their tensor products are also irreducible as modules of $B_h(\hat{g})$.

A rescaling automorphism $\sigma$ of the form (4) maps the subalgebra $B_h(\hat{g})$ to a different subalgebra of $U_h(\hat{g})$ which is isomorphic to $B_h(\hat{g})$ as an algebra and as a $U_h(\hat{g})$-coideal. In particular the sign of $\epsilon_j$ in (8) can be changed by such an automorphism with $\sigma_j = -1$.

Let us assume that for some constant $\eta$ there exists an intertwiner $K(x) : V_{\eta x} \to V_{\eta x}$, i.e., a matrix $K(x)$ satisfying

\[(10) \quad K(x) \pi_{\eta x}(\hat{Q}) = \pi_{\eta x}(\hat{Q}) K(x), \quad \forall \hat{Q} \in B_h(\hat{g}).\]

Then $K(x)$ is automatically a solution of the reflection equation

\[(11) \quad (1 \otimes K(y)) \hat{R}(x,y)(1 \otimes K(x)) \hat{R}(x,y) = \hat{R}(x,y)(1 \otimes K(x)) \hat{R}(x,y)(1 \otimes K(y))\]

This is so because both sides of the equation are intertwiners between the irreducible modules $V_{\eta x} \otimes V_y$ and $V_{\eta x} \otimes V_{\eta y}$. By Schur’s lemma the two sides are proportional. That they are equal then follows from the fact that they have the same determinant and are equal at $h = 0$.

4. Vector representation of $d_n^{(1)}$

We now specialize to the case of the $2n$-dimensional representation of $\hat{g} = d_n^{(1)} = so(2n)$. We choose $d_i = 1$ in (3). The $U_h(\hat{g})$ generators are represented by

\[(12) \quad \pi(x_j^+) = E_{j,j+1} - E_{j+1,j-1}, \quad \pi(x_j^-) = E_{j+1,j} - E_{j,j-1},\]

\[(13) \quad \pi(h_j) = E_{j,j} - E_{j+1,j+1} - E_{j,j+1} - E_{j+1,j} + 1,\]

\[(14) \quad \pi(h_0) = E_{1,1} - E_{2,2} + E_{1,2} - E_{2,1} + E_{1,\bar{1}} - E_{\bar{1},1} - E_{2,\bar{1}} - E_{\bar{1},2} + 1,\]

where $E_{i,j}$ is the matrix with a 1 in the $i$-th row and the $j$-th column and all other entries 0, and $\bar{1} = 2n + 1 - j$.

The intertwining equation (3) was solved by Jimbo (3) to give the trigonometric $R$-matrix

\[\hat{R}(x) = (x - q^{-2})(x - \xi)\sum_{a \neq \overline{\pi}} E_{a,a} \otimes E_{a,a} + (x - 1)(x - \xi)/q \sum_{a \neq \overline{b}} E_{a,b} \otimes E_{b,a}\]

\[+ (1 - q^2)(x - \xi) \left( \sum_{a < b, a \neq \overline{b}} + x \sum_{a > \overline{b}, a \neq \overline{b}} \right) E_{a,a} \otimes E_{b,b}\]

\[+ \sum_{a \neq \overline{b}} b_{ab}(x) E_{\overline{a},a} \otimes E_{\overline{b},b},\]

where $\xi = q^{2n-2n}$ and

\[(17) \quad b_{ab}(x) =\]

\[
\begin{cases}
\frac{(q^{-2}x - \xi)(x - 1)}{(q^{-2} - 1)} & \text{for } a = b \\
\frac{(q^{-2} - 1)(\xi q^{b-a'})(x - 1) - \delta_{a,b}(x - \xi)}{q^{-2} - 1} & \text{for } a < b \\
\frac{(q^{-2} - 1)(\xi q^{a-a'})(x - 1) - \delta_{a,b}(x - \xi)}{q^{-2} - 1} & \text{for } a > b
\end{cases}
\]

\]
with
\begin{equation}
    a' = \begin{cases} 
    a + 1/2 & \text{if } a \leq n \\
    a - 1/2 & \text{if } a > n 
    \end{cases}
\end{equation}

A Hopf-algebra automorphism \( \sigma \) of the form (14) corresponds to a change of basis and a change of spectral parameter, i.e.,
\begin{equation}
    \pi_x(\sigma(Q)) = \Sigma^{-1} \pi_x(Q) \Sigma \quad \forall Q \in U_h(\hat{g}),
\end{equation}

where
\begin{equation}
    \zeta = \sigma_0 \sigma_1 (\sigma_2 \ldots \sigma_{n-2})^2 \sigma_{n-1} \sigma_n,
\end{equation}

\begin{equation}
    \Sigma = \text{diag}(\Sigma_1, \ldots, \Sigma_{2n})
\end{equation}

\begin{align*}
    \Sigma_j &= (\sigma_j \ldots \sigma_{n-2})^{-1}, \quad \Sigma_j = \sigma_j \ldots \sigma_n, \quad j = 1, \ldots, n-2 \\
    \Sigma_{n-1} &= 1, \quad \Sigma_n = \sigma_{n-1}, \quad \Sigma_{n-1} = \sigma_{n-1} \sigma_n.
\end{align*}

The R-matrix satisfies
\begin{equation}
    (\Sigma^{-1} \otimes \Sigma^{-1}) \tilde{R}(x)(\Sigma \otimes \Sigma) = \tilde{R}(x).
\end{equation}

5. Reflection matrix for \( d_n^{(1)} \)

It is now straightforward to solve the intertwining equation (14) for the vector representation of \( d_n^{(1)} \) to obtain the solutions of the reflection equation (11). One finds that a solution exists only if
\begin{equation}
    \eta = \pm (-1)^n q^{1-n}, \quad \hat{\epsilon}_j = \pm i (q^{1/2} - q^{-1/2})^{-1}, \quad j = 0, \ldots, n.
\end{equation}

The signs of the \( \hat{\epsilon}_j \)'s can be changed by a rescaling automorphism, as discussed above, and the sign of \( \eta \) can be changed by \( x \rightarrow -x \). We thus now choose the upper signs in (22). We write the solution as
\begin{equation}
    K(x) = \sum K_{ab}(x) E_{a,b}.
\end{equation}

We only need to give the entries on and above the antidiagonal because the others are determined by the symmetry
\begin{equation}
    K_{a'b'}(x) = (-1)^n \tilde{K}_{ab}(1/x),
\end{equation}

where the tilde indicates the change \( q \rightarrow 1/q \). The entries are
\begin{equation}
    K_{ab}(x) = \begin{cases} 
    i^{-a} q^{b-n} k(x) & \text{for } a + b \leq 2n \text{ and } a \neq b, b' \\
    i^{-a} q^{b-n} (k(x) + (-1)^n \tilde{k}(1/x))(q+1)^{-1} & \text{for } b = \pi \text{ and } a \leq n \\
    i^{-a} q^{b-n} k(x) - (-1)^n q^{1/2} k(x) + ((-1)^n q^{(n-1)/2} + q^{(1-n)/2}) & \text{for } b = a
    \end{cases}
\end{equation}

where
\begin{equation}
    a = \begin{cases} 
    a & \text{for } a \leq n \\
    \pi & \text{for } a > n,
    \end{cases}
\end{equation}

the notation \( a' \) was defined in (18), and
\begin{equation}
    k(x) = q^{n/2-1} (1-x).
\end{equation}
Using the invariance (21) of the R-matrix with respect to the automorphisms (4) we can write an $n$-parameter family of solutions $K^+_\sigma$ of the reflection equation (11)

$$K^+_\sigma(x) = \Sigma^{-1} K(x) \Sigma.$$  

A second family $K^-_\sigma(x)$ of solutions is obtained by choosing the opposite sign for $\eta$ in (22), leading to $K^-_\sigma(x) = K^+_\sigma(-x)$. Furthermore, multiplying any of these solutions by an arbitrary function of $x$ gives another solution because the reflection equation is linear.

6. Discussion

The main new result of this letter is the hitherto unknown trigonometric reflection matrices (25) for the vector representation of $\mathfrak{so}(2n)$. It proves the practical utility of the quantum affine reflection algebras with their simple generators (8) and makes clear the possibility of generalization to other representations and other algebras, which is the topic of continuing research [13, 14].

In order to use the reflection matrices (25) to describe soliton reflection in $\mathfrak{a}_n^{(1)}$ affine Toda field theory they need to be transformed to the principal gradation. This is achieved by an automorphism $\sigma$ of the form (4) with $\sigma_0 = 1, \sigma_i = x, i = 1, \ldots, n$. Furthermore the normalisation of the reflection matrix has to be fixed by the requirements of unitarity and crossing symmetry, as was done for $a_2^{(1)}$ in [15].

An equation very similar to the classical limit of our intertwining condition (10) appeared in [16] in the context of constructing classically integrable boundary conditions in affine Toda field theory.

Coideal subalgebras of non-affine quantized enveloping algebras $U_h(\hat{g})$ have been studied [17, 18] in the context of quantum symmetric spaces and $q$-orthogonal polynomials. Expressions similar to (8) appear there.

The properties of quantum affine reflection algebras and their representation theory deserve further study. For example a knowledge of how irreducible representations of $U_h(\hat{g})$ branch into irreducible representations of $B_h(\hat{g})$ will have physical applications to the spectrum of boundary states in integrable models.

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