Exponential quasi-ergodicity
for processes with discontinuous trajectories

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Abstract
This paper tackles the issue of establishing a lower-bound on the asymptotic ratio of survival probabilities between two different initial conditions, asymptotically in time for a given Markov process with extinction. Such a comparison is a crucial step in recent techniques for proving exponential convergence to a quasi-stationary distribution. We introduce a weak form of the Harnack inequality as the essential ingredient for such a comparison. This property is actually a consequence of the convergence property we intend to prove. Its complexity appears as the price to pay for the level of flexibility required by our applications. We show in our illustrations how simply and efficiently it can be used nonetheless. As illustrations, we consider two continuous-time processes on $\mathbb{R}^d$ that do not satisfy the classical Harnack inequalities, even in a local version. The first one is a piecewise deterministic process while the second is a pure jump process with restrictions on the directions of its jumps.

Keywords: continuous-time and continuous-space Markov process; jumps; quasi-stationary distribution; survival capacity; Q-process; Harris recurrence

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1 Introduction

1.1 General presentation
This work is concerned with the long time behavior of quite general strong Markov processes, conditionally upon the fact that this process has not been absorbed in some "cemetery state" (i.e. that it is not "extinct"). The eventual interest is on the analogous of stationary distributions when such a conditioning is taken into account, namely quasi-stationary distributions (QSD).

In the aftermath of recent works by Champagnat and Villemonais, notably in [17], we are interested in highlighting key properties that ensure convergence results at exponential
rate towards QSD. While the approach was initiated in the framework of a convergence in total variation that is uniform over the initial condition, how to deal with heterogeneity in the initial condition has already been the concern of further studies [22, 57, 2].

These works are inspired by the Harris recurrence techniques that is exploited for the proof of convergence towards a stationary distribution for conservative semi-groups (cf Section 2 in [45] or [48], notably chapter 15, for more details). The core of these techniques is a Doeblin minorization condition where the density of the marginal law is lower-bounded uniformly over some initial conditions. A Lyapunov criterion is then exploited to deal with the heterogeneity in the initial conditions. While refinements of these two properties can be identified in [22, 57, 2], another key property is introduced for the generalization to non-conservative semi-groups that compare survival between different initial conditions.

This kind of property has a different nature as the others in that it is expressed asymptotically in large time. In this paper, as a representative of those properties, we focus on the following property stated in Equation (1.1). It is more simply expressed in that it is stated uniformly over the initial condition and the denominator is taken as a single reference measure $\zeta \in M_1(\mathcal{X})$.

$$\limsup_{t \to \infty} \sup_{x \in \mathcal{X}} \frac{\mathbb{P}_x(t < \tau_\partial)}{\mathbb{P}_\zeta(t < \tau_\partial)} < \infty,$$

In the expression, $\tau_\partial$ denotes the extinction time and the initial condition of the process is written in subscript. Although $\zeta$ is meant to relate to small sets, no property on $\zeta$ should be needed for the proof of (1.1). (1.1) is the archetype of the comparison properties of asymptotic survival probability, that are introduced in the abstract.

The derivation of a property like (1.1) is well understood in the cases of discrete-space processes (cf the Proof of Theorem 4.1 p14 in [17]), and processes that satisfy either two-sided estimates (cf. Section 2 in [15]), gradient estimates (cf. Section 3 in [15]) or the Harnack inequality (step 4 of the proof given in Section 4 of [21], or more generally stated in Subsection 4.2.3 in [57]). In the case of many processes involving jumps, and notably in multi-dimensional settings, none of these three estimates seem to be applicable. It seems difficult for the other techniques that we know of (notably Lemma 5 in [36]) to determine to what extent they could be generalised.

This is why we worked at a weakened form of the Harnack inequality with more flexibility, (that we call "almost perfect harvest", see Subsection 1.3.3), that would still suffice to obtain (1.1), when we exploit the other key properties requested for exponential convergence towards QSD. The aim is to capture a much larger range of processes. In practice, the statement of our main result (see Theorem 2.1) relies on the other key properties proposed in [57].

This choice is not simply due to the fact that we were more involved in the proofs of [57], but also because the fact that the approach is trajectorial gives more insight into the estimates and because the proofs in [57] were designed with the current article already in mind. Potential generalizations are nevertheless discussed in Subsection 1.4.2.

As a direct consequence, we present in Subsection 2.3 the new set of key conditions that is proved sufficient for the general results of [57] to be deduced. It implies not only the existence and uniqueness of the QSD, but also several results of exponential convergence.
The paper is organized as follows. In the next Subsection 1.2 we present two illustrative models that contribute to motivate the new conditions introduced in this article. With these two practical issues in mind, a concise presentation of the results can more readily be addressed in Subsection 1.3. There, we first specify the convergence objective, then how the contribution of [57] leads to proving such convergence results and finally how the present contribution eases the verification of one of the main conditions of [57, Section 2]. In Subsection 1.4 we then place our method in the context of existing results on quasi-stationarity.

In complement to this concise overview of our results, a more detailed description is provided in Section 2. The focus is then on the main contribution of the article, namely the proposed key properties implying (1.1). The relation to the set of criteria given in [57] and the applicability of our combined results in practice is thus postponed to Subsection 2.3. Yet, it is also convenient to introduce or recall concise and precise notations at use, which we do first in Subsection 2.1. Subsection 2.2 is then devoted to the main contribution of this article, namely the derivation of Property (1.1) thanks to our new criterion. Subsection 2.3 is thus more generally concerned on the implications for the proof of exponential convergence to a unique QSD. The above-mentioned complements to this result are given in Subsection 2.3.3 that are reciprocal statements and the uniformity in a localization procedure.

We detail our proofs in Section 3. Sections 4 and 5 are finally dedicated to each of our applications, which are to be introduced in their simpler form in the next paragraph.

1.2 Applications

The two illustrations of the current paper are meant to help the reader get insight on the adaptability of our new criterion of “almost perfect harvest”. They fall into the class of piecewise deterministic Markov processes, the second one being actually a pure jump process. For the broader view of the applications we have in mind, we refer to the Discussion Section, and more precisely Subsection 6.1.

We first consider the following process on $\mathbb{R}^d$ for $d \geq 1$:

$$X_t = x - v t e_1 + \sum_{i \leq N_t} W_i,$$

where $x \in \mathbb{R}^d$ is the deterministic initial condition, $v > 0$, $e_1$ the first unit vector, $N_t$ a Poisson process with intensity 1 and the family $(W_i)_{i \in \mathbb{Z}^+}$ is made up of i.i.d. normal variables with mean 0 and covariance matrix $\sigma^2 I_d$. This process dies at a state-dependant rate. The extinction event, whose time is denoted $\tau_\partial$, is occurring at rate $t \mapsto \rho(X_t)$, where $\rho(x) := ||x||^2$ (with the euclidian norm).

Then, as stated in Theorem 1.1 there exists a unique quasi-stationary distribution associated to this dynamics and it is the only attractor of the conditioned marginals as $t$ goes to infinity.

While the one-dimensional case is treated in [23], with a stronger result than in [30], the convergence result is new for the multidimensional setting. For more details on the interpretation of this model and some generalization of its parameters, we refer to Section 4.
The second illustration, presented in Section 5, concerns a pure jump process on \( \mathbb{R}^d \), for \( d \geq 2 \):

\[
X_t := x + \sum_{\{i \leq N_t\}} \sigma W_i \cdot e_{D_i}.
\]

In this formula, \( x \) is the initial condition, \( N_t \) a standard Poisson process on \( \mathbb{Z}_+ \), while, for any \( i \geq 1 \), \( W_i \) is a standard 1d. normal random variable and \( D_i \) is uniform over \([1,d]\). Moreover, all these random variables are independent from each others. Similarly as the previous example, this process dies at a state-dependent rate given by \( \rho_e : x \in \mathbb{R}^d \mapsto \|x\|_\infty^2 \), where \( \|x\|_\infty := \sup_{i \leq d} |x_i| \).

Then, as stated in Theorem 5.1, \( \sigma \leq 1/8 \) is a sufficient condition for the existence of a unique quasi-stationary distribution associated to this dynamics.

Note that jumps are restricted to happen along the vectors of an orthonormal basis \((e_1, \ldots, e_d)\). One jump is thus unsufficient to erase the singularities with respect to the Lebesgue measure on \( \mathbb{R}^d \). To our knowledge, there is no other result of quasi-stationarity for processes with such restrictions on the jumps. The result presented in [31] for the pure-jump case appears the closest to ours. In this multidimensional setting however, the jump effect is assumed to have a density with respect to the Lebesgue measure on \( \mathbb{R}^d \).

More details on the interpretation of this model and some generalization of its parameters are presented in Section 5.

1.3 Specification of the results and techniques derived from [57] and the current contribution

1.3.1 Current statement of the convergence we aim at

For any Markov process \( X = (X_t)_{t \in \mathbb{R}_+} \) with extinction time \( \tau_0 \), the property of quasi-stationary convergence that we aim to prove for these processes is stated in the current article as follows. The statement depends on a bounded function \( h \) on the state space \( \mathcal{X} \) and on a probability measure \( \alpha \) on \( \mathcal{X} \) that are uniquely defined by imposing that \( \int_{\mathcal{X}} h(x) \alpha(dx) = 1 \) in addition to the property.

There exists \( C, \gamma > 0 \) such that the following inequality holds for any \( t \geq 0 \) and any probability measure \( \mu \) on \( \mathcal{X} \) such that \( \int_{\mathcal{X}} h(x) \mu(dx) > 0 \):

\[
\| P_\mu (X_t \in dx; t < \tau_0) - \alpha(dx) \int_{\mathcal{X}} h(y) \mu(dy) \|_{TV} \leq C \| \mu - \alpha \|_{TV} e^{-\gamma t}.
\] (1.2)

The reader is referred to Definition 1 in Section 2.3.2 for a more precise statement and to Subsection 1.4.1 for the comparison with the result given in [57].

1.3.2 Main properties leading to the result following [57]

Up to minor adjustment justified in Subsection 3.2, it is the purpose of [57] that the proof of this property (1.2) can be deduced by the proofs of the following 4 properties. They depend on a probability measure \( \zeta \) on \( \mathcal{X} \) and on a subset \( E \) of \( \mathcal{X} \) that have to be the same. The names are altered to ease the comparaison with the assumptions proposed in the current article.
(A0) “Exhaustion of \(X\)”: There exists a sequence \((D_\ell)_{\ell \geq 1}\) of closed subsets of \(X\) such that for any \(\ell \geq 1\), \(D_\ell\) is included in the interior of \(D_{\ell+1}\) and such that their union makes up the whole state space \(X\).

(A1) “Mixing property”: For any \(\ell \geq 1\), there exists \(L > \ell\) and \(c, t > 0\) such that:

\[
\forall x \in D_\ell, \quad P_x [X_t \in dx; \ t < \tau_\partial \wedge T_{D_L}] \geq c \zeta(dx),
\]

where \(T_{D_L}\) is the first exit time of \(D_L\).

(A2) “Escape from the Transitory domain”: There exists \(\rho > \rho_S\) such that the following boundedness property holds, where \(\tau_E\) is the exit time of \(E\) and \(E\) is required to be included in \(D_\ell\) for \(\ell\) sufficiently large:

\[
\sup_{\{x \in X\}} E_x \left( \exp \left[ \rho (\tau_\partial \wedge \tau_E) \right] \right) < \infty,
\]

while the value \(\rho_S\) is defined as a survival estimate with the following definition:

\[
\rho_S := \sup \left\{ \gamma \in \mathbb{R} \mid \sup_{L \geq 1} \inf_{t > 0} e^{\gamma t} P_\zeta(t < \tau_\partial \wedge T_{D_L}) = 0 \right\}.
\]

(A3) “Asymptotic Comparison of Survival”:

\[
\limsup_{t \to \infty} \sup_{x \in E} \frac{P_x(t < \tau_\partial)}{P_\zeta(t < \tau_\partial)} < \infty.
\]

1.3.3 Current proposal of an alternative the last property

Proving the three first properties is quite straightforward for the two presented applications, as one can check in Subsections 4.3 and 5.4. The proof of Property (A3) is on the other hand much less direct as it involves an asymptotic in large time for the proposed ratio. It is only slightly easier than Equation (1.1) in that the inequality can be stated on a convenient subset of \(X\) provided the exponential moment given in (A2).

As mentioned just after Equation (1.1) in the introduction, neither the two-sided estimates nor the gradient estimate or the Harnack inequality are manifestly applicable. This is due to the fact that any singularity of the initial condition is maintained with just a decay in time (and possibly a translation in space), due to the jump event taking time.

This is why we make the emphasis on the following property that is quite easily proved in our examples. It still depends on \(E\) and \(\zeta\) but also on a value \(\rho\) such that (A2) holds true and on a value \(\epsilon > 0\) that is deduced from (A1) and (A2) as an intricate quantity. In practice, we thus expect the property to be deduced whatever this value of \(\epsilon\), by adjusting the other estimates accordingly. Notably, the property involves the design of a specific stopping time \(U_H\), that we require to be infinite after a given threshold \(t_F\) in time. The design of the other stopping time \(V\) is to be adjusted according to \(U_H\), without specific restrictions.

(A3F) “Almost Perfect Harvest”: There exist \(t_F, c > 0\) such that for any \(x \in E\) there exists two stopping time \(U_H\) and \(V\) with the following properties:

\[
P_x(X(U_H) \in dx'; \ U_H < \tau_\partial) \leq c P_\zeta(X(V) \in dx'; \ V < \tau_\partial),
\]
including the next conditions on $U_H$:
\[ \mathbb{P}_x(U_H = \infty, t_F < \tau_0) \leq \epsilon \exp(-\rho t_F), \quad \text{where } \{\tau_0 \wedge t_F \leq U_H \} = \{U_H = \infty\}. \]

Additional regularity condition of $U_H$ are also required with respect to the Markov property, that we provide in detail in Subsection 2.2.2. Although technical, these conditions are expected to hold for any reasonable choice of $U_H$, generally defined through the meeting of specific conditions at some stopping time.

Thanks to Theorem 2.1 in Subsection 2.2.3, assuming (A0), we deduce from (A1), (A2) and (A3$F$) that (A3) holds also.

### 1.3.4 Implications of the convergence results

Thanks to Theorem 2.2 which mostly relies on Theorems 2.1 and 2.2 of [57] together with previous Theorem 2.1, the property of quasi-stationary convergence is then deduced from (A1), (A2) and (A3$F$) (as well as from (A1), (A2) and (A3)). We then infer in Corollary 2.3.4 the following convergence result for $\alpha$, that justifies its identification as a quasi-stationary distribution:

**“Convergence to $\alpha$”:** For any $t \geq 0$ and $\mu \in \mathcal{M}_1(\mathcal{X})$ such that $\langle \mu \mid h \rangle > 0$:
\[ \| \mathbb{P}_\mu [X_t \in dx \mid t < \tau_0] - \alpha(dx) \|_{TV} \leq C \frac{\|\mu - \alpha\|_{TV}}{\langle \mu \mid h \rangle} e^{-\gamma t}. \quad (1.3) \]

It is actually justified in Theorem 2.1 and 2.2 that the family of sets $(\mathcal{D}_\ell)$ may not cover the whole set $\mathcal{X}$ for the new formulation given in Equation (1.2) to hold. Unexpectedly at first, this flexibility has been a great help for the proofs exploited in [60] and [46], even if $h$ is still strictly positive in the latter case. In addition to the convergence depending on $\langle \mu \mid h \rangle$, it raises the question of identifying lower-bounds of $h$, which we tackle in Proposition 2.3.5 by proving the following property:

**“Lower-bounds of $h$”:** $h$ is uniformly bounded away from zero on any set $H \subset \mathcal{X}$ for which there exists $t > 0$ and $\ell \geq 1$ such that $\inf_{x \in H} \mathbb{P}_x(\tau_{\mathcal{D}_\ell} < t \wedge \tau_0) > 0$.

This justifies the identification of the domain of $h$ as follows:
\[ \mathcal{H} := \{x \in \mathcal{X}; h(x) > 0\} = \{x \in \mathcal{X}; \exists \ell \geq 1, \mathbb{P}_x(\tau_{\mathcal{D}_\ell} < \tau_0) > 0\}. \]

As in [57] Theorem 2.3, we additionally deduce the existence of the $Q$-process, that is of a Markov process living on this space $\mathcal{H}$ whose generator $(Q_x)_{x \in \mathcal{H}}$ satisfies the following asymptotic property:
\[ \lim_{t \to \infty} \mathbb{P}_x(\Lambda_s \mid t < \tau_0) = Q_x(\Lambda_s), \]
for any $x \in \mathcal{H}$, $s > 0$ and $\Lambda_s$ any $\mathcal{F}_s$-measurable set.

As an extension of property (1.2), we also infer in Corollary 2.3.7 the following convergence property towards a unique stationary distribution $\beta$ of this $Q$-process.

**“Convergence to $\beta$”:** for any probability measure $\mu$ on $\mathcal{H}$ satisfying $\int_{\mathcal{H}} \frac{\mu(dx)}{h(x)} < \infty$ and $t \geq 0$:
\[ \|Q_\mu [X_t \in dx] - \beta(dx)\|_{TV} \leq C e^{-\gamma t} \cdot \left\|\mu(dx) - \left(\int_{\mathcal{H}} \frac{\mu(dy)}{h(y)}\right) \cdot \beta(dx)\right\|_{1/h}, \]

\[ 6 \]
where
\[ Q_\mu(dw) := \int H \mu(dx) Q_x(dw), \quad \|\mu\|_{1/h} := \left\| \frac{\mu(dx)}{h(x)} \right\|_{TV}. \]

### 1.3.5 Additional robustness properties of these results

**Approximation over restricted state space.** The constants involved in the convergences are explicitly related to the parameters involved in the presented assumptions. Although the specific relation is very intricate, it implies that one can fairly approximate the quasi-stationary regime by restricting the state space \(\mathcal{X}\). Indeed, for any \(L \geq 1\) (thought to be large), let us consider the following approximation \(\tau^L_0\) of the extinction event that restricts the process \(X\) to remain in \(D_L\) in the time-interval \([0, \tau^L_0]\).

Then, as stated in Theorem 2.3 by proving any of the two sets of assumptions (for the extinction time \(\tau_0\)) we deduce that all the above results hold uniformly in \(L\) with the extinction time replaced by \(\tau^L_0\). We mean that we can choose the constants \(C\) independent of \(L\) for the convergences to \(\alpha_L, h_L, (Q^L_t)\) and \(\beta_L\). Moreover, as \(L\) goes to infinity, \(\lambda_L\) converges to \(\lambda\) and \(\alpha_L, h_L\) converge to \(\alpha, h\) in total variation and pointwise respectively. Also, we deduce \(\rho_S = \lambda\).

**Reciprocal results.** It is always satisfying to check that the sufficient conditions that one is attempting to verify are not too restrictive on the process. This is why we have also been concerned with proving of properties as analogous as possible to our key assumptions, starting from the core result (1.2). In particular, given this property of quasi-stationary convergence, Proposition 2.3.9 namely ensures that Property (A2) holds for a certain parameter \(\rho > \lambda\), and a certain \(\ell \geq 1\) regardless of the choice of the sequence \(D_L\) satisfying Assumption (A0). Proposition 2.3.10 then asserts, under the same conditions and for the same value of \(\rho\), that Property (A3_F) is effectively met, where \(\zeta\) can be chosen as any probability measure on \(X\).

### 1.4 Comparison with the literature

#### 1.4.1 Similar recent contributions

The interest of our current result is mostly on generalizations of the Harris recurrence principle, among which [17, 22, 57, 2, 23] present the most general statements for homogeneous-in-time processes. The upper-bounds that are derived can be compared with the following inequality, that takes a more general form than Equation 1.3

\[ \| P_\mu [ X_t \in dx \mid t < \tau_0 ] - \alpha(dx) \|_{TV} \leq C(\mu) e^{-\gamma t}, \quad (1.4) \]

where the main differences in the conclusions come from different expressions and interpretations of the constant \(C(\mu)\). [17] first highlighted two necessary and sufficient conditions for this convergence result to be uniform, that is with \(C(\mu)\) taken as a constant. The argument of a contraction in total variation norm is then simpler, yet inspired the other extensions considering a non-trivial dependency \(C(\mu)\) related to the forms of the key properties required.
When considering extinction, the property of linearity over the initial condition is lost, so that linear expressions of $C(\mu)$ in terms of $\mu$ is not as natural as in the conservative case. This explains also why we came back to linear convergence statements in Equation 1.2.

The alternative techniques proposed in [2], [22] or [23] involve contraction estimates of the operator in specific norms that are weighted by specific functions. These functions satisfy some properties with respects to the semi-group $(P_t)$ that justify their association with the principles of Lyapunov contraction in Harris recurrence techniques. With the help of such Lyapunov function $W$ from $X$ to $\mathbb{R}_+$, we may generalize exponential moments as our property (A2) and interpret the sequence $D_\ell$ as level sets of $W$ (i.e. $D_\ell = \{x \in X; W(x) \leq \ell\}$). With intricate boundary conditions, providing an efficient definition of such functions in practice remains however challenging.

1.4.2 Generalization of our approach

Given the close interplay between our assumptions (A2) and (A3F), adapting the reasoning around (A3F) is not obvious. Notably, the contraction estimates exploited in [2] and [23] do not appear to relate as easily to a crucial property for our argument (namely [22] about the decay estimate of survival probability).

Doob's "$h$-transform" is a typical technique to deduce asymptotic results of a generally non-conservative semi-group $(P_t)_t$ from the study of a related sub-Markovian semi-group. Provided that there exists a positive measurable function $\psi$ on $X$ and $\rho_\psi > 0$ such that for any $t > 0$, $P_t \psi \leq e^{\rho_\psi t} \psi$, the following definition indeed provides a sub-Markovian semi-group:

$$P_t^\psi(x, dy) := \frac{\psi(y)}{\psi(x)} e^{\rho_\psi t} P_t(x, dy).$$

Implicitly, it means that $\psi$ is exploited to weight the state space $X$, as in the norms weighted by Lyapunov functions in [2, 22, 23]. Note that the uniqueness property of the QSD given as $\psi(y)\alpha(dy)$ for $P^\psi$ corresponds exactly to the fact that $\alpha$ is the unique QSD of $P$ in the space $\mathcal{M}(\psi) := \{\mu \in \mathcal{M}(X); \langle \mu | \psi \rangle < \infty\}$. This allows for other probability distributions $\nu$ to be QSD, in which case $\langle \nu | \psi \rangle = \infty$ holds necessarily.

On the other hand, our proofs would be easy to adapt to processes in discrete-time. Our techniques should generalize naturally to time-inhomogeneous processes, given the recent adaptations presented in [20, 2, 23, 22]. It can probably be extended to semi-Markov processes, i.e. pure jump processes for which the waiting time between jumps is not necessarily exponential.

1.4.3 Other frameworks

General surveys like [24], [38] or more specifically for population dynamics [47] give an overview on the huge literature dedicated to QSD, for which Pollett has collected quite an impressive bibliography, cf. [50].

When jumps in continuous space are involved, the reversibility property is generally expected not to hold true. They are even more exceptional when the state space is multi-dimensional: cf Appendix A of [10].
Comparison of survival is also an essential part of perturbation techniques as in Chapter 12 of [34] or in [35], yet it is mostly exploited for finite time and compared to an intrinsic convergence rate. In [30], results on the non-conservative semi-group are deduced from the study of the Q-process as in the R-theory (cf [1] for pure jump processes, [24] or [55] for the original discrete-time setting). Our approach may provide guidance in dealing with estimates of the poorly known survival capacity.

The other methods appear to bring less quantitative insight in terms of uniqueness (except possibly [1]) or rate of convergence. Besides the classical use of the Krein-Rutman theorem (we recall [30, 10]), extensions from fixed point argument [25] and the above-mentioned R-theory, we also refer to “renewal theory” [39].

The compactness of the semi-group is actually not required for our approach. We recall that many classical approaches rely on this property to deduce the existence of a QSD (cf e.g. the reviews [24], [47]), often thanks to the Ascoli-Arzelà theorem. Since the process is allowed in the illustrations given in Section 1.2 either not to jump or to have a large number of jumps in any time-interval [0, t], we could not rely on this technique.

2 Detailed description of our results

Before we present our results in more details, it is convenient to use efficient notations, that we introduce or recall in the next subsection. We can then focus in Subsection 2.2 on the main contribution of the current article before clarifying in Subsection 2.3 the implications of this result combined with the ones of [57, Theorems 2.1-3] on the quasi-stationary convergence.

2.1 Notations

In Subsection 2.1.1 we describe our general notations, in Subsection 2.1.2 our specific setup of a càdlàg strong Markov process with extinction and clarify in Subsecton 2.1.3 some notations made to express various event restrictions.

2.1.1 Elementary notations

The most classical sets of integers are denoted as follows: \( \mathbb{Z}_+ := \{0, 1, 2, \ldots\} \), \( \mathbb{N} := \{1, 2, 3, \ldots\} \), \([m, n] := \{m, m+1, \ldots, n-1, n\} \) (for \( m \leq n \)). By the notation :=, we simply makes explicit that the equality is meant to explicit some notation. For maxima and minima, we use the following abbreviations: \( s \lor t := \max\{s, t\} \), \( s \land t := \min\{s, t\} \). In the paper, we may write \( k \geq 1 \) instead of \( k \in \mathbb{N} \) and \( t \geq 0 \) (resp. \( c > 0 \)) instead \( t \in \mathbb{R}_+ := [0, \infty) \) (resp. \( c \in \mathbb{R}_+ := (0, \infty) \)), when there is no real ambiguity.

The state space is denoted \( \mathcal{X} \cup \partial \), where the cemetery \( \partial \) is assumed to be isolated from the topology \( \mathcal{B} \) on the Polish space \( \mathcal{X} \). In the study of the process, we will need to apply the Markov property at first entry times of either closed or open subsets. This is why we assume that the time homogeneous process \( X \) is strong Markov for the filtration \( (\mathcal{F}_t)_{t \geq 0} \) that is right-continuous and complete and that it has càdlàg paths (left limited and right continuous). The hitting time (resp. the exit time out) of \( D \), for some domain \( D \subset \mathcal{X} \), will...
generally be denoted by \( \tau_D \) (resp. by \( T_D \)). These are stopping times for any \( D \) that is either closed or open, cf. Theorem 52 in [37], or more recently Theorem 2.4 in [4].

Exploiting the same notations as in [51], Definition III.1.1, \( P_t \) would then be the semi-group of the process and the latter shall satisfy the usual measurability assumptions and the Chapman-Kolmogorov equation. The law of the process starting from initial condition \( x \in \mathcal{X} \cup \partial \) will be given by the probability \( \mathbb{P}_x \).

2.1.2 The stochastic process with absorption

Here, we consider a strong Markov processes absorbed at \( \partial \): the cemetery. More precisely, we assume that \( X_s = \partial \) implies \( X_t = \partial \) for all \( t \geq s \). This implies that the extinction time: \( \tau_\partial := \inf \{ t \geq 0; \ X_t = \partial \} \) is a stopping time. Thus, we rather consider the family \(( P_t )_{t \geq 0} \) as a non-conservative semigroup of operators on the set \( \mathcal{B}_+(\mathcal{X}) \) (resp. \( \mathcal{B}_b(\mathcal{X}) \)) of positive (resp. bounded) \((\mathcal{X},\mathcal{B})\)-measurable real-valued functions.

For any probability measure \( \mu \) on \( \mathcal{X} \), and \( f \in \mathcal{B}_+(\mathcal{X}) \) (or \( f \in \mathcal{B}_b(\mathcal{X}) \)), we use the notations:

\[
\mathbb{P}_\mu(\cdot) := \int_{\mathcal{X}} P_x(\cdot) \mu(dx), \quad \langle \mu | f \rangle := \int_{\mathcal{X}} f(x) \mu(dx).
\]

We also denote by \( \mathbb{E}_x \) (resp. \( \mathbb{E}_\mu \)) the expectation according to \( \mathbb{P}_x \) (resp. \( \mathbb{P}_\mu \)).

The set of respectively probability measures on \( \mathcal{X} \), of positive and of signed measures are denoted respectively \( \mathcal{M}_1(\mathcal{X}) \), \( \mathcal{M}_+(\mathcal{X}) \) and \( \mathcal{M}(\mathcal{X}) \). For any \( B \in \mathcal{B} \) and \( \mu \in \mathcal{M}_1(\mathcal{X}) \), \( \mu P_t(B) \) is clearly defined as \( \mathbb{P}_\mu( X_t \in B ) \). The fact that \( t < \tau_\partial \) immediately follows from \( B \) being a subset of \( \mathcal{X} \) and \( X \) being absorbed in \( \partial \). Yet, we wish to avoid confusion for instance in the examples given in Subsection 1.2 where extinction is prescribed through some state-dependent rate defined a posteriori from an internal dynamics of \( X \) on \( \mathcal{X} \). This is why we usually make the restriction on the event \( t < \tau_\partial \) explicit even when it is not required, notably in the definition of the action of \( P_t \) on \( \mu \in \mathcal{M}_1(\mathcal{X}) \) as follows:

\[
\mu P_t(dy) := \mathbb{P}_\mu(X_t \in dy \ ; \ t < \tau_\partial), \quad \text{or} \quad \langle \mu P_t | f \rangle = \langle \mu | P_t f \rangle = \mathbb{E}_\mu[f(X_t) \ ; \ t < \tau_\partial],
\]

where \( f \in \mathcal{B}_+(\mathcal{X}) \) or \( f \in \mathcal{B}_b(\mathcal{X}) \). Let us then define the family of conditioned operators \(( A_t )_{t \geq 0} \) acting as follows on any probability measure \( \mu \in \mathcal{M}_1(\mathcal{X}) \):

\[
\mu A_t(dy) := \mathbb{P}_\mu(X_t \in dy \ ; \ t < \tau_\partial), \quad \langle \mu A_t | f \rangle = \mathbb{E}_\mu[f(X_t) \ ; \ t < \tau_\partial] = \frac{\mathbb{E}_\mu[f(X_t) \ ; \ t < \tau_\partial]}{\mathbb{P}_\mu[t < \tau_\partial]}.
\]

\( \mu A_t \) is what we call the MCNE at time \( t \), with initial distribution \( \mu \), as it is "the Marginal distribution (at time \( t \)) Conditioned upon the fact that No Extinction has yet occurred" (also at time \( t \)).

In this setting, the family \(( P_t )_{t \geq 0} \) (resp. \(( A_t )_{t \geq 0} \)) defines a linear but non-conservative semigroup (resp. a conservative but non-linear semigroup) of operators on \( \mathcal{M}_1(\mathcal{X}) \) endowed with the total variation norm. We consider the following definition of this norm, generally for any signed measure \( \mu \in \mathcal{M}(\mathcal{X}) \):

\[
\| \mu \|_{TV} := \sup \{ |\mu(A)| \ ; \ A \in \mathcal{B} \}.
\]

While the semigroup \( P \) is directly generalized by linearity for any signed measure, note that it is not as clear for the semi-group \( A \) because \( \mathbb{P}_\mu[t < \tau_\partial] \) could then be equal to zero for some \( t > 0 \).
A probability measure $\alpha \in \mathcal{M}_1(X)$ is said to be the \textit{quasi-limiting distribution} of an initial condition $\mu \in \mathcal{M}_1(X)$ if: $\forall B \in \mathcal{B}$, $\lim_{t \to \infty} \mu_{A_t}(B) = \lim_{t \to \infty} \mu_{A_t}(B) = \lim_{t \to \infty} \mu_{A_t}(B) = \alpha(B)$.

It is now classical (cf e.g. Proposition 1 in [47]) that $\alpha$ is then a \textit{quasi-stationary distribution} or QSD, in the sense that: $\forall t \geq 0$, $\alpha_{A_t}(dy) = \alpha(dy)$.

Also, for any such QSD, there exists a unique extinction rate $\lambda > 0$ such that: $\forall t \geq 0$, $P_{\alpha}(t < \tau^\partial) = \exp[-\lambda t]$.

### 2.1.3 Conditions, stopping times and random events

While dealing with the Markov property between different stopping times, we wish to clearly indicate with our notation that we introduce a copy of $X$ (ie a process with the same semigroup $P_t$) independent of $X$ given its initial condition. This copy (and the associated stopping times) is then denoted with a tilde ($\tilde{X}$, $\tilde{\tau}^\partial$, $\tilde{T}^D$ etc.).

For instance in the notation $P_X(\tau^E < \tilde{\tau}^\partial)$, $\tau^E$ and $X(\tau^E)$ refer to the initial process $X$ while $\tilde{\tau}^\partial$ refers to the copy $\tilde{X}$.

Besides, some notations of semi-colons and commas have meanings that are specific to our probabilistic notations and currently very efficient given that we often consider restrictions on various events. For expectations, the terms after the semi-colon indicate the sets of conditions under which the former term is counted (it is replaced by 0 if the conditions are not met). For sets, and notably random ones, the terms after the semi-colon indicate the sets of conditions under which the former term is included. Different conditions may be separated by commas, notably when combinations of those are introduced through indices. For instance, given any random variables $F$, $(X_i)_{i \leq d}$, for $d \geq 1$, $T, T'$ and $t > 0$, the following notation can be translated as follows:

$$E(F; \forall i \leq d, X_i \geq 0, t < T) = E(F \cdot \prod_{i \leq d} 1_{\{X_i \geq 0\}} \cdot 1_{\{t < T\}}).$$

To give another example, the following notation:

$$\{s \geq T'; \forall i \leq d, X_i \geq 0, s < T\}$$

is to be understood as the random set $E$ (that is thus dependent on $\omega \in \Omega$) defined as follows. If there exists $i$ such that $X_i(\omega)$ is negative or if $T'(\omega) \geq T(\omega)$, then $E = \emptyset$. Otherwise, $E$ consists of all values $s \in \mathbb{R}_+$ such that $s \in [T'(\omega), T(\omega))$. The infimum of an empty set is generally to be taken as $\infty$. Without such semi-colon, the set has to be considered as the part of $\Omega$ for which the conditions are satisfied, for instance:

$$\{\forall i \leq d, X_i \geq 0, t < T\} = \cap_{i \leq d} \{X_i \geq 0\} \cap \{t < T\}$$

consists of all $\omega \in \Omega$ such that for any $i \leq d$, $X_i(\omega) \geq 0$ and $t < T(\omega)$.

### 2.2 Coupling approach including failures

The main contribution of this article is presented in this Subsection 2.2, namely the derivation of Property 1.1 thanks to our new criterion. We start in Subsection 2.2.1 by explaining the key properties derived from [57, Section 2] and the role of the associated parameters involved in our new criterion. After presenting this new criterion of ”Almost Perfect Harvest” in Subsection 2.2.2, we state our main Theorem 2.1 in the next Subsection 2.2.3.
2.2.1 The associated assumptions

Our core property of "almost perfect harvest" exploits several parameters ($\rho > 0$, $E$ and $\zeta$) whose choices are strongly tied to several key properties highlighted in [57]. We shall thus start by explaining the key properties of [57] on which we rely and thus the role of these parameters.

The approach is trajectory-based and designed to handle specific dependencies on the initial condition, so it appears efficient to consider a customizable covering by an increasing sequence of sets, as specified in the next property:

\[(A_0)\]: “Specification of space” There exists a sequence \((D_\ell)_{\ell \geq 1}\) of closed subsets of \(\mathcal{X}\) such that for any \(\ell \geq 1\), \(D_\ell \subset \text{int}(D_{\ell+1})\) (with \(\text{int}(D)\) the interior of \(D\)).

This sequence serves as a reference for the other key properties, notably through the following notation, for subsets that are "regular" with respect to this specification:

\[D := \{D; \ D \text{ is closed and } \exists \ell \geq 1, \ D \subset D_\ell\}. \quad (2.1)\]

As it is stated in [57, Section 2] (and by extension in the introduction), this property \((A_0)\) is often strengthened as follows. This additional condition of complete covering enables to obtain uniqueness of the QSD as such and not only among the QSD with minimal extinction rate.

\[\text{(A0)} : \text{the sequence } (D_\ell)_{\ell \geq 1} \text{ satisfying (A0) is such that } \bigcup_{\ell \geq 1} D_\ell = \mathcal{X}.\]

The next assumption consists in a minoration of the density of the process after a given time. This minoration extends the classical Doeblin inequality where the marginal law is lower-bounded uniformly over the initial conditions. This is the step that produces the mixing of the past dependencies. As in Harris recurrence technique, this lower-bound needs only to be uniform locally in the initial condition, at the expense of an associated contraction estimate. In our case, we further restrict the probability to trajectories that remain locally confined and exploit for practical convenience the minoration with the restriction on the event of survival \(\{t < \tau_\partial\}\) rather than with the conditioning on the event. We recall the following definitions for the exit and first entry times of any set \(D\):

\[T_D := \inf \{t \geq 0; \ X_t \notin D\} , \quad \tau_D := \inf \{t \geq 0; \ X_t \in D\} .\]

\[\text{(A1)[\zeta]}: \text{ “Mixing property” The probability measure } \zeta \in \mathcal{M}_1(\mathcal{X}) \text{ is such that, for any } \ell \geq 1, \text{ there exists } L > \ell \text{ and } c, t > 0 \text{ such that:}
\]
\[\forall x \in D_\ell, \quad \mathbb{P}_x [X_t \in dx; \ t < \tau_\partial \wedge T_{D_L}] \geq c \zeta(dx).\]

The last assumption corresponds to the contraction estimate in Harris recurrence techniques, in the form of an exponential moment of return to some reference subset \(E\) of \(\mathcal{X}\). To combine this property with the mixing stated in (A1), we require this set \(E\) to be included in one of the \(D_\ell\). For practical convenience, we consider the moment estimate on
the stopping time \( \tau_0 \wedge \tau_E \) for the unconditioned expectation. For the contraction to be of relevance for the conditioned process, an exponential moment estimate with a parameter larger than the extinction rate of the process appears crucial the more so as the extinction rate is not directly accessible and has to be evaluated.

\[ (A2)[\rho, E] : \text{“Escape from the Transitory domain”} \quad \text{The value } \rho > 0 \text{ and the subset } E \in D \text{ are such that:} \]

\[
\sup_{x \in X} \mathbb{E}_x (\exp [\rho (\tau_0 \wedge \tau_E)]) < \infty.
\]

\( \rho \) as stated in (A2) and (A3_F) is required in the following theorems to be strictly larger than the following “survival estimate”:

\[
\rho_S := \sup \{ \gamma \in \mathbb{R} \left| \sup_{t \geq 1} \inf_{t>0} e^{\gamma t} \mathbb{P}_\zeta (t < \tau_0 \wedge T_{D_1}) = 0 \right. \}.
\]

Moreover, the set \( E \) shall be common for (A2) and (A3_F).

From the proof of Corollary 5.2.1 in [57], we deduce the following property of survival as a direct consequence the above properties.

**Proposition 2.2.1.** Assume that (A0), (A1)[\zeta] and (A2)[\rho, E] hold for some \( \zeta \in \mathcal{M}_1 (X) \), \( \rho > \rho_S \) and \( E \in D \). Then, there exists \( t_S, c_S > 0 \) and \( \hat{\rho}_S \in (\rho_S, \rho) \) such that:

\[
\forall u \geq 0, \forall t \geq u + t_S, \quad P_\zeta (t - u < \tau_0) \leq c_S e^{\hat{\rho}_S u} \mathbb{P}_\zeta (t < \tau_0). \tag{2.2}
\]

It means that for sufficiently large times, the exponential decay in time with parameter \( \hat{\rho}_S \) provides a relevant estimation of the decay of the probability of survival starting from the initial condition \( \zeta \).

### 2.2.2 The new key property

We are now in position to introduce the following weak form of Harnack inequality, whose purpose is to imply the property (2.1):

\[ (A3_F)[\rho, E, \zeta] : \text{“Almost perfect harvest”} \quad \text{The value } \rho > 0, \text{ the subset } E \in D \text{ and the probability measure } \zeta \in \mathcal{M}_1 (X) \text{ are such that the following condition holds:} \]

* For any \( \epsilon \in (0, 1) \), there exist \( t_F, c > 0 \) such that for any \( x \in E \) there exists two stopping time \( U_H \) and \( V \) with the following properties:

\[
\mathbb{P}_x (X(U_H) \in dx'; U_H < \tau_0) \leq c \mathbb{P}_x (X(V) \in dx'; V < \tau_0), \tag{2.3}
\]

including the next conditions on \( U_H \):

\[
\{ \tau_0 \wedge t_F \leq U_H \} = \{ U_H = \infty \} \quad \text{and} \quad \mathbb{P}_x (U_H = \infty, t_F < \tau_0) \leq \epsilon \exp(-\rho t_F). \tag{2.4}
\]

As a regularity condition of \( U_H \) with respect to the Markov property, we additionally require that \( U_H \) can be expanded in the following sense by at least some stopping time \( U_H^\infty \) such that:

* \( U_H^\infty := U_H \) on the event \( \{ \tau_0 \wedge U_H < \tau_1 \} \), where \( \tau_1 := \inf \{ s \geq t_F \mid X_s \in E \} \).
* On the event \( \{ \tau_1 < \tau_0 \wedge U_H \} \) and conditionally on \( \mathcal{F}_{\tau_1} \), the law of \( U_H^\infty - \tau_1 \) coincides with the one of \( \tilde{U}_H^\infty \) for a realization \( \tilde{X} \) of the Markov process \( (X_t, t \geq 0) \) with initial condition \( \tilde{X}_0 := X(\tau_1) \) and independent of \( X \) conditionally on \( X(\tau_1) \).
Interpretation of the core assumption

This property is to be compared to the more classical Harnack inequality, which should take the following form (as can be seen in [57]):

The subset $E \in D$ and the probability measure $\zeta \in M_1(X)$ are such that there exist $t_F, v, c > 0$ for which the following inequality holds for any $x \in E$:

$$P_x(X_{t_F} \in dx'; t_F < \tau_\theta) \leq c P_{\zeta}(X_v \in dx'; v < \tau_\theta).$$

(2.5)

Thanks to (A3F), we will also be able to couple the trajectories of the processes starting with the initial conditions resp. $x \in E$ and $\zeta$ so that they coincide with a time lag for sufficiently large times. In other words, we want to embed the “tail trajectories” of the process starting from $x$ into the trajectories starting from $\zeta$, a time lag being allowed. When (2.5) holds true, the trajectories with initial condition $x$ can namely be coupled after time $t_F$, with a time-lag of $t_F - v$.

This embedding procedure is what we describe as the “harvest” of the tail trajectories. With (A3F), this embedding is here to occur after time $U_H^\infty$, the “harvesting time”, the time-lag being more flexible and notably allowed to be random.

The Markov property being granted, inequality (2.3) is what makes this embedding possible. This embedding is directly obtained on the event $\{U_H < t_F \land \tau_\theta\}$, in which case $U_H$ and $U_H^\infty$ coincide. The Markov property being exploited for the coupling, we request both $U_H$ and $V$ to be stopping time.

It is however rarely expected that $U_H$ can be obtained upper-bounded by a uniform constant $t_F$ such that inequality (2.3) still holds. Possibly several attempts of coupling may thus be required under some specific conditions, that we describe as failures. A “failure” to do the coupling is then characterized by the event $\{\tau_\theta \land t_F \leq U_H\}$ (which we require to coincide with $\{U_H = \infty\}$). The objective of establishing the embedding of tail trajectories except in case of rare events in probability is what motivates our denomination of “almost perfect harvest”.

In case of failures, $U_H^\infty$ must be larger than $t_F$, and a new attempt can directly be implemented only after $\tau_1^E$. We need to ensure that the events of recurrent failures play a negligible role in the probability of long-term survival. This is why, in complement to the factor $\epsilon$ being sufficiently small, we require the penalisation by $\exp(-\rho t_F)$ to compensate for the extinction rate (during the time-interval $[0, t_F]$ for the process with initial condition $\zeta$). For such a compensation to be exploitable, the value of $\rho$ will be required to be greater than the known lower bound $\rho_S$ for the decay of the survival probability.

The condition of failure where $\{U_H = \infty\}$ has to be adjusted for convenience to the model. As the allowed error $\epsilon$ goes to 0, we expect to see the failure conditions stated through certain thresholds being less and less stringent. This certainly leads to larger and larger constants $c$ in (2.3), and often to larger and larger constants $t_F$.

Because the attempts of coupling are meant to be iterated after each failure, the condition related to the Markov property on $U_H^\infty$ is requested. It takes into account the waiting time before the process comes back to $E$ and a new attempt can be initiated. Provided that the definition of $U_H$ do not depend in a singular way on the initial condition $x \in E$, this condition on $U_H^\infty$ should be easily satisfied.
2.2.3 The central result of the paper

**Theorem 2.1.** Assume that there exist $\zeta \in \mathcal{M}_1(\mathcal{X})$, $t_S, c_S > 0$, $\rho \geq \hat{\rho}_S > \rho_S > 0$, and $E \in \mathcal{D}$ such that inequality (2.2), assumptions $(A_0)$, $(A1)[\zeta]$, $(A2)[\rho, E]$ and $(A3_F)[\rho, E, \zeta]$ hold. Then:

$$\limsup_{t \to \infty} \sup_{x \in E} \frac{\mathbb{P}_x(t < \tau_\theta)}{\mathbb{P}_\zeta(t < \tau_\theta)} < \infty.$$  \hspace{1cm} (2.6)

For the implication that (1.1) is a consequence ($(A0)$, $(A1)$ and $(A2)$ being granted) of this more local property (which is exactly $(A3)$), we refer to [57, Theorem 5.2]. A careful check of the proof ensures that the additional requirement $\bigcup_{\ell \geq 1} D_\ell = \mathcal{X}$ is actually not exploited therein.

**Remark 2.2.2.** In $(A1)$, a confinement in some subspace $D_L$ is required. This confinement part of the assumption is actually not involved in the proof of Theorem 2.1, but is in fact exploited to deduce Theorem 2.2. We kept it to avoid too frequent variations of our assumptions.

2.3 Exponential convergence to a unique QSD

Thanks to Theorem 2.1, we see that the general conclusions of [57, Theorems 2.1-3] can be derived from a new set of assumptions, as we present it in this Subsection.

2.3.1 Two general sets of assumptions

We say that Assumption $(A_F)$ holds, whenever:

"There exists $\zeta \in \mathcal{M}_1(\mathcal{X})$ such that $(A1)[\zeta]$ holds for a specific sequence $(D_\ell)$ satisfying $(A0)$. Moreover, there exists $\rho > \rho_S$ and $E \in \mathcal{D}$ such that assumptions $(A2)[\rho, E]$ and $(A3_F)[\rho, E, \zeta]$ hold."

Slightly adapting [57] (regarding $(A0)$), we say that Assumption $(A)$ holds, whenever:

"There exists $\zeta$ such that $(A1)[\zeta]$ holds for a specific sequence $(D_\ell)$ satisfying $(A0)$. Moreover, there exists $\rho > \rho_S$ and $E \in \mathcal{D}$ such that assumptions $(A2)[\rho, E]$ and $(A3)[E, \zeta]$ hold."

**Remark 2.3.1.** $\star$ The purpose of Theorem 2.1 is to prove that Assumption $(A_F)$ actually implies Assumption $(A)$. For this, we refer to Subsection 2.2.

$\star$ Almost sure extinction is not at all needed for our proof (which would in fact include the case where there is no extinction, or only in some “transitory domain”).

$\star$ In our set of assumptions, the case where $D_\ell := \mathcal{X}$ for any $\ell$ is actually allowed and fits in the setting of uniform convergence covered in [17].

2.3.2 Main consequences of our main result

The main property that we deduce in Theorem 2.2 and in our subsequent applications can be stated as in the following definition:
Definition 1. For any linear positive and bounded semi-group \((P_t)_{t \geq 0}\) acting on a state space \(\mathcal{X}\), we say that \(P\) displays a uniform exponential quasi-stationary convergence with characteristics \((\alpha, h, \lambda) \in \mathcal{M}_1(\mathcal{X}) \times B(\mathcal{X}) \times \mathbb{R}\) if \(\langle \alpha | h \rangle = 1\) and there exists \(C, \gamma > 0\) such that for any \(t > 0\) and for any measure \(\mu \in \mathcal{M}(\mathcal{X})\) with \(\|\mu\|_{TV} \leq 1\):

\[
\left\| e^{\lambda t} \mu P_t - \langle \mu | h \rangle \alpha \right\|_{TV} \leq C e^{-\gamma t}.
\] (2.7)

This definition slightly extends the one given in introduction, cf [1.2], by considering a general class of semi-groups and allows the value \(\lambda\) to be negative. Since \(\langle \alpha | h \rangle = 1\) and since \(P_t\) is linear, elementary computations show that the above property implies the apparently stronger one given in (1.2) with \(\|\mu - \alpha\|_{TV}\) as additional factor.

The property given in Definition 1 implies that \(\alpha\) is a QSD with extinction rate \(\lambda\), as stated in the following fact, whose proof is deferred to the appendix.

Fact 2.3.2. If a semi-group \(P\) displays a uniform exponential quasi-stationary convergence with characteristics \((\alpha, h, \lambda)\), then for any \(t > 0\), \(\alpha P_t(ds) = e^{-\lambda t} \alpha(ds)\).

Remark 2.3.3. \(*\) It is elementary that \(h_t : x \mapsto e^{\lambda t} \langle \delta_x P_t | 1 \rangle\) converges in the uniform norm to \(h\), implying that \(h\) is non-negative. Since \(h_{t+t'} = e^{\lambda t} P_t h_{t'}\), one can also easily deduce that \(e^{\lambda t} P_t h = h\).

\(*\) By “characteristics”, we express that they are uniquely defined.

Theorem 2.2. Assume that either \((A_F)\) or \((A)\) holds. Then, \(P\) displays a uniform exponential quasi-stationary convergence with some characteristics \((\alpha, h, \lambda)\) in \(\mathcal{M}_1(\mathcal{X}) \times B(\mathcal{X}) \times \mathbb{R}_+\). It also holds true that \(h\) is bounded away from zero on any \(D_\ell\), that \(h\) is upper-bounded on \(\mathcal{X}\) and that \(\alpha(D_\ell) > 0\) for \(\ell\) sufficiently large.

The convergence to \(\alpha\) is made more precise by the following corollary:

Corollary 2.3.4. Assume (2.8). Then for any \(t \geq 0\) and \(\mu \in \mathcal{M}_1(\mathcal{X})\) such that \(\langle \mu | h \rangle > 0\):

\[
\| \mathbb{P}_\mu [X_t \in dx \mid t < \tau_0] - \alpha(dx) \|_{TV} \leq C \frac{\|\mu - \alpha\|_{TV}}{\langle \mu | h \rangle} e^{-\gamma t}.
\] (2.8)

Thanks to Theorem 2.2 it is elementary that \(h\) is positive under assumption (A0). It might be useful otherwise to exploit the following proposition to identify a posteriori the domain on which \(h\) is positive.

Proposition 2.3.5. Assume that \((A_F)\) or \((A)\) holds. Then, the survival capacity \(h\) is uniformly bounded away from zero on any set \(H \subset \mathcal{X}\) that satisfies the following condition:

\((H_0)\) : there exists \(t > 0\), \(\ell \geq 1\) such that

\[
\inf_{\{x \in H\}} \mathbb{P}_x(\tau_{D_\ell} < t \land \tau_0) > 0.
\]

It implies the following identification:

\[
\mathcal{H} := \{x \in \mathcal{X}; h(x) > 0\} = \{x \in \mathcal{X}; \exists \ell \geq 1, \mathbb{P}_x(\tau_{D_\ell} < \tau_0) > 0\}.
\]

Remark 2.3.6. Corollary 2.3.4 implies that if \(\nu\) were a QSD different from \(\alpha\), then \(\nu A_\ell \neq \alpha\) thus \(\langle \nu | h \rangle = 0\). This implies \(\nu(\mathcal{H}) = 0\). So by contraposition, the previous proposition provides a practical way to ensure the uniqueness of the QSD a posteriori.
Once the set $\mathcal{H}$ is clarified, we can study Doob’s $h$-transform of the semi-group $P$ with weight given by the survival capacity. This is the so-called $Q$-process that is actually a conservative Markov process as stated in the next corollary:

**Corollary 2.3.7.** Under again $(A_F)$ or $(A)$, with $(\alpha, h, \lambda)$ the characteristics of exponential convergence of $P$, the following properties hold:

(i) **Existence of the $Q$-process:**

There exists a family $(Q_x)_{x \in \mathcal{H}}$ of probability measures on $\Omega$ defined by:

$$\lim_{t \to \infty} \mathbb{P}_x (A_s | t < \tau_0) = Q_x (A_s),$$

(2.9)

for any $x \in \mathcal{H}$, $s > 0$ and $A_s$ any $\mathcal{F}_s$-measurable set. The process $(\Omega; (\mathcal{F}_t)_{t \geq 0}; (X_t)_{t \geq 0}; (Q_x)_{x \in \mathcal{H}})$ is an $\mathcal{H}$-valued homogeneous strong Markov process.

(ii) **Weighted exponential ergodicity of the $Q$-process:**

The measure $\beta (dx) := h(x) \alpha (dx)$ is the unique invariant probability measure under $Q$. Moreover, for any $\mu \in M_1(\mathcal{H})$ satisfying $\langle \mu | 1/h \rangle < \infty$ and $t \geq 0$:

$$\| Q_\mu [X_t \in dx] - \beta (dx) \|_{TV} \leq C \| \mu - \langle \mu | 1/h \rangle \beta \|_{1/h} e^{-\gamma t},$$

(2.10)

where $Q_\mu (dw) := \int_{\mathcal{H}} \mu (dx) Q_x (dw)$, $\| \mu \|_{1/h} := \left\| \frac{\mu (dx)}{h(x)} \right\|_{TV}$.

The constant $\langle \mu | 1/h \rangle$ before $\beta$ (2.10) is optimal up to a factor 2, due to the following fact, that is proved in the appendix. This is also the case for the implicit constant 1 before $\alpha$ in (2.8).

**Fact 2.3.8.** For any $u > 0$:

$$\| \mu - u \alpha \|_{1/h} \geq (1/2) \cdot \| \mu - \langle \mu | 1/h \rangle \beta \|_{1/h}, \quad \| \mu - u \alpha \|_{TV} \geq (1/2) \cdot \| \mu - \alpha \|_{TV}.$$

2.3.3 Additional robustness properties of the results

**Reciprocal results** The following propositions are stated to justify that our assumptions are not particularly restrictive. We shall see in the following Theorem 2.3 that $\rho_S$ actually equals $\lambda$ if $(A_F)$ is satisfied. We thus aim at stating reciprocal statements in terms of a parameter $\rho$ satisfying $\rho > \lambda$. The first proposition concerns $(A2)$, that is derived under the assumption of exponential convergence together with property $(A0)$.

**Proposition 2.3.9.** Assume that $\mathcal{X} = \bigcup_{\ell \geq 1} D_\ell$ where $(D_\ell)_{\ell \geq 1}$ is an increasing subsequence. Assume that (2.7) is satisfied. Then, for $\rho := \lambda + \gamma/2$, there exists $\ell \geq 1$ such that:

$$\sup_{x \in \mathcal{X}} \mathbb{E}_x \left( \exp \left( \rho (\tau_0 \wedge \tau_{D_\ell}) \right) \right) < \infty.$$

Since $(A3_F)$ is the most intricate property, one may more likely suspect that it leads to restrictions. The following proposition shows to what extend we may generally reply that this is not the case, in that the convergence result actually implies a general form of property $(A3_F)$.

Since $(A3_F)$ is the most intricate property, one may more likely suspect that it leads to restrictions. The following proposition shows to what extend we may generally reply that this is not the case, in that the convergence result actually implies a general form of property $(A3_F)$. Note that $\mu$ in the proposition is meant to represent the reference measure $\zeta$ in the original statement of $(A3_F)$. 17
Proposition 2.3.10. Assume that (2.7) is satisfied. Then, for \( \rho := \lambda + \gamma / 2 \), there exist \( c_F > 0 \) such that for any \( \mu \in \mathcal{M}_1(\mathcal{X}) \), and any \( t_F > 0 \) sufficiently large, there exists for any \( x \in \mathcal{X} \) a stopping time \( U_H \) satisfying the three following properties:

\[
\mathbb{P}_x(t_F < \tau_\partial \wedge U_H) \leq \frac{c_F}{\langle \mu \mid h \rangle} e^{-\rho t_F}, \quad \text{where } \{ \tau_\partial \wedge t_F \leq U_H \} = \{ U_H = \infty \};
\]

and

\[
\mathbb{E}_x(X_{U_H} \in dy; U_H < \tau_\partial) \leq \frac{\| h \|_\infty}{\langle \mu \mid h \rangle} \mathbb{E}_\mu(X_{t_F} \in dy; t_F < \tau_\partial).
\]

Furthermore, the definition of \( U_H \) can be extended to derive a stopping time \( U_H^\infty \) as stated in (A3_F).

To deduce that (A3_F) holds for any measure \( \mu \) satisfying \( \langle \mu \mid h \rangle > 0 \), one has simply to remark that the following inequality holds with \( \hat{\rho} := \lambda + \gamma / 4 \) for any \( \epsilon > 0 \) and any \( t_F \) sufficiently large:

\[
(c_F / \langle \mu \mid h \rangle) \cdot e^{-\rho t_F} \leq \epsilon \cdot e^{-\hat{\rho} t_F},
\]

then to take \( V := t_F \).

The proof of this Proposition follows in Subsection 3.4 the one of Proposition 2.3.9.

This proof provides also a relevant intuition on the role of the parameters involved in the expression of (A3_F).

Remark 2.3.11. The derivation of the mixing property (A1) from a convergence property in total variation such as (2.7) appears much more difficult to obtain. It indeed requires a lower-bound with respect to some measure \( \zeta \) that is uniform over the initial condition in some \( D_t \), while the total variation informs about the discrepancy between two measures. Given our additional restriction involving \( T_{DL} \), a property of uniform convergence would also at least be required, as stated in the next Theorem 2.3.

Uniformity in the localization procedure The constants involved in the convergences are explicitly related to the ones in Assumptions (A_F) or (A). Although the specific relation is very intricate, it implies the following corollary of approximation:

Theorem 2.3. Assume that (A_F) or (A) holds. Then all the preceding results hold true with the same constants involved in the convergences when \( \tau_\partial \) is replaced by \( \tau_\partial^L := \tau_\partial \wedge T_{DL} \) for any \( L \geq 1 \) sufficiently large.

Let \( \alpha^L, \lambda^L, h^L \) be the corresponding QSD, extinction rates and survival capacities. Then as \( L \) goes to infinity, \( \lambda_L \) converges to \( \lambda \) and \( \alpha^L, h^L \) converge to \( \alpha, h \) in total variation and pointwise respectively. Also, we deduce \( \rho_S = \lambda \).

3 Proofs of the general results

3.1 Proof of Theorem 2.1

Recall the expression of \( c_S \) in Equation (2.2). Thanks to Assumption (A2) that controls the escape time from the transitory domain, we may define the constant \( e_T \) as follows:

\[
e_T := \sup_{x \in \mathcal{X}} \mathbb{E}_x \left( \exp \left[ \rho (\tau_\partial \wedge \tau_E) \right] \right).
\]

(3.1)
In the following, every time we will apply Assumption (A3\(_F\)), we will exploit the following parameter \( \epsilon := (2c_S e_T)\). For \( t_F \) the associated deterministic upper-bound on \( U_H \), we define \( t_A := t_F + t_S \).

The first step consists in the following lemma:

**Lemma 3.1.1.** Assume that inequality (2.2) and assumption (A3\(_F\)) hold with the above parameters. Then, there exists \( C_0 > 0 \) such that the following upper-bound holds for any \( x \in E \) and any \( t \geq t_A \):

\[
\mathbb{P}_x(U_H < t < \tau_\theta) \leq C_0 \mathbb{P}_\zeta(t < \tau_\theta).
\]

For \( \epsilon \) sufficiently small, \( \{U_H < t < \tau_\theta\} \) is meant to be the leading part of \( \mathbb{P}_x(t < \tau_\theta) \). We will prove indeed that the extension of survival during the failed coupling procedure, for a time-length \( t_F \), then outside \( E \) (before \( \tau_E^1 \)) is not sufficient to compensate the cost of the complementary event of failure, i.e. \( \{\tau_\theta \land t_F \leq U_H\} = \{U_H = \infty\} \).

Our idea is to distinguish the events according to the number of failures, and treat them inductively. So if a first failure is observed (implying that \( \tau_\theta < \tau_H^1 \land t_F \)), we start a new turn by replacing \( x \) by \( X(\tau_E^1) \) and \( t \) by \( t - \tau_E^1 \). To combine it efficiently with Lemma 3.1.1, the induction is in fact not stated with a dependency on the value of \( J \), but rather on the value of the following random variable:

\[
J(t) := \sup \left\{ j \geq 0 ; \, \tau_E^j < (t - t_A) \land \tau_\theta \land U_H^\infty \right\}, \tag{3.2}
\]

where we recall from Equation (6.1) the inductive definition the sequence \( (\tau_E^j) \):

\[
\tau_E^{j+1} := \inf \{ s \geq \tau_E^j + t_F ; \, X_s \in E \} \land \tau_\theta, \text{ and } \tau_E^0 = 0.
\]

Thanks to the following lemma, whose proof marks the second step, we will finish the initialization with an upper-bound on the probability of the event \( \{J(t) = 0\} \):

**Lemma 3.1.2.** Assume that assumption (A2), inequalities (2.2) and assumption (A3\(_F\)) hold with \( \rho > \hat{\rho}_S \). Then, there exists \( C_F > 0 \) such that the following upper-bound holds for any \( x \in E \) and \( t \geq t_A \):

\[
\mathbb{P}_x(U_H = \infty, \, t \leq \tau_E^1, \, t < \tau_\theta) \leq C_F \mathbb{P}_\zeta(t < \tau_\theta).
\]

The induction property will then be propagated thanks to the following lemma, whose proof marks the third step.

**Lemma 3.1.3.** Assume that assumption (A2), inequalities (2.2) and assumption (A3\(_F\)) hold with \( \rho > \hat{\rho}_S \). If there exists \( j \geq 0 \) and \( C_j > 0 \) such that the following upper-bound holds for any \( x \in E \) and any \( t \geq t_A \):

\[
\mathbb{P}_x(t < \tau_\theta, \, J(t) = j) \leq C_j \mathbb{P}_\zeta(t < \tau_\theta), \tag{3.3}
\]

then the inequality holds also in the next step \( j + 1 \) as follows for any \( x \in E \) and any \( t \geq t_A \):

\[
\mathbb{P}_x(t < \tau_\theta, \, J(t) = j + 1) \leq \frac{C_j}{t_F} \mathbb{P}_\zeta(t < \tau_\theta) \tag{3.4}
\]

With these three lemmas, we will then finish the proof of Theorem 2.1.
3.1.1 Step 1: Proof of Lemma 3.1.1

The Markov property is exploited as follows in relation to (A3F):

\[ P_x(U_H < t < \tau_0) = E_x \left[ P_{X|U_H}(t - U_H < \tau_0); U_H < \tau_0 \right] \leq C E_{\zeta} \left[ P_{X|V}(t - t_F < \tau_0); V < \tau_0 \right] \leq C P_{\zeta} [t - t_F < \tau_0], \]

where we exploited (3.3) and the fact that

\[ P_x(U_H < t < \tau_0) \leq C \cdot c_S \cdot e^{\rho t_F} P_{\zeta}[t < \tau_0]. \]

Lemma 3.1.1 is thus satisfied with: \[ C_0 := C \cdot c_S \cdot e^{\rho t_F}. \]

\[ \square \]

3.1.2 Step 2: Proof of Lemma 3.1.2

Thanks to Equation (3.1), the following upper-bound holds a.s. on \( \{U_H = \infty, t_F < \tau_0\} \):

\[ E_{X_{t_F}} \left[ \exp (\rho [\tau_E \wedge \tau_0]) \right] \leq e^T. \]

Thanks to the Markov inequality, this implies the next upper-bound a.s. on the same event:

\[ P_{X_{t_F}} (t - t_A - t_F \leq \tau_E; t - t_F < \tau_0) \leq e^T \cdot e^{-\rho [t - t_A - t_F]}. \]

In combination with the Markov property, we deduce the next upper-bound:

\[ P_x(U_H = \infty, t - \tau_E \leq \tau_E^1, t < \tau_0) \leq e^T \cdot e^{-\rho [t - t_A - t_F]} P_x [t_F < \tau_0, U_H = \infty]. \]

Thanks to Inequality (2.2), we can relate the decay \( e^{-\rho t} \) to \( P_{\zeta}(t < \tau_0) \). Thanks to (A3F) and recalling the definition of \( \epsilon \) as \( (2c_S e^T)^{-1} \), we then conclude the proof of Lemma 3.1.2 where:

\[ C_F := \frac{e^T \cdot c_S \cdot \epsilon}{e^{\rho t_S} P_{\zeta}(t_S < \tau_0)} = (2e^{\rho t_S} P_{\zeta}(t_S < \tau_0))^{-1} > 0. \]

\[ \square \]

3.1.3 Step 3: Proof of Lemma 3.1.3

Thanks to the Markov property assumed on \( U_H^\infty \) (see assumption (A3F)) and to definition (3.2):

\[ P_x(t < \tau_0, J(t) = j + 1) = E_x \left[ P_{X[\tau_k]}(t - \tau_1^E < \tau_0, \tau(t - \tau_1^E) = j); J(t) \geq 1 \right] \leq C_j E_x \left[ P_{\zeta}(t - \tau_1^E < \tau_0); J(t) \geq 1 \right], \]

where we exploited (3.3) and the fact that \( t - \tau_1^E \geq t_A \) on the event \{\( J(t) \geq 1 \)\}. Then, thanks to Inequality (2.2):

\[ P_x(t < \tau_0, J(t) = j + 1) \leq C_j \cdot c_S \cdot P_{\zeta}(t < \tau_0) \cdot E_x \left[ \exp (\rho \tau_1^E); J(t) \geq 1 \right]. \quad (3.5) \]
We then decompose $\tau_1^E$ into the sum of $t_F$ and $\tilde{\tau}_E$ to arrive at the next upper-bound on the expectation in the right-hand side:

\[
\mathbb{E}_x \left[ \exp(\rho \tau_1^E); \ J(t) \geq 1 \right] \\
\leq \mathbb{E}_x \left[ \mathbb{E}_{X_{t_F}} \left( \exp(\rho \tilde{\tau}_E); \ \tilde{\tau}_E < (t - t_F - t_A) \wedge \tilde{\tau}_0 \right) \cdot \exp(\rho t_F); \ U_H = \infty, \ t_F < \tau_0 \right]. \quad (3.6)
\]

Now, thanks to assumption (A2), the following upper-bound holds a.s. on \( \{U_H = \infty, \ t_F < \tau_0\} \):

\[
\mathbb{E}_{X_{t_F}} \left( \exp(\rho \tilde{\tau}_E); \ \tilde{\tau}_E < (t - t_F - t_A) \wedge \tilde{\tau}_0 \right) \leq e_T. \quad (3.7)
\]

The next upper-bound is derived for any \( x \in E \) thanks to assumption (A3):

\[
\mathbb{E}_x \left[ \exp(\rho t_F); \ U_H = \infty \right] \leq \epsilon = \frac{1}{2c_s e_T}. \quad (3.8)
\]

Combining the four inequalities (3.5), (3.6), (3.7) and (3.8) yields the following upper-bound for any \( x \in E \) and any \( t \geq t_A \):

\[
\mathbb{P}_x(t < \tau_0, \ J(t) = j + 1) \leq C \mathbb{P}_\zeta(t < \tau_0).
\]

which concludes the proof of Lemma 3.1.3. \( \square \)

3.1.4 The end of the proof of Theorem 2.1

With \( C := 2(C_0 + C_F) \), thanks to Lemmas 3.1.1-3, we deduce that the following upper-bound holds for any \( x \in E \), any \( t \geq t_A \) and any \( j \in \mathbb{Z}_+ \):

\[
\mathbb{P}_x(t < \tau_0, \ J(t) = j) \leq 2^{-j-1} C \cdot \mathbb{P}_\zeta(t < \tau_0).
\]

With this decomposition, we simply conclude the proof of Theorem 2.1 as follows:

\[
\mathbb{P}_x(t < \tau_0) = \sum_{j \geq 0} \mathbb{P}_x(t < \tau_0, \ J(t) = j) \leq C \cdot \mathbb{P}_\zeta(t < \tau_0). \quad \square
\]

3.2 A more refined convergence result

By Corollary 5.2.1 in [57] and Theorem 2.1, Assumption (A) of [57] is nearly implied by our Assumption (A_F), except that \( \bigcup_{\ell \geq 1} D_\ell = \mathcal{X} \) is no longer assumed. We let the reader check that the proofs given in Section 5 of [57] apply mutatis mutandis, except for the following facts:

\begin{itemize}
  \item As presented in several examples in [60], \( \alpha \) might not be the unique QSD for \( \mathcal{X} \).
  
  By adapting the argument given in the following proof of Proposition 2.3.5 we can nonetheless deduce that any other QSD \( \nu \) must satisfy that: \( \nu(\bigcup_{\ell} D_\ell) = 0 \).
  
  \item The lower-bound on \( h(x) \) are only obtained for \( x \in D_\ell \), so that \( h(x) \) might be equal to 0 for \( x \in \mathcal{X} \setminus \bigcup_{\ell} D_\ell \).
  
  \item The reasoning on the Q-process can only be applied for initial conditions on \( \mathcal{H} := \{x \in \mathcal{X}; h(x) > 0\} \supset \bigcup_{\ell} D_\ell \).
\end{itemize}

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The linear decomposition of $\mu$ that the four following upper-bounds hold on particular, this implies the following identification:

\[ \forall t > 0, \|e^{\lambda \mathbb{P}(t < \tau_0)} - h\|_\infty \leq C_h e^{-\gamma t}. \quad (3.9) \]

Moreover, there exists also $\alpha \in M_1(\mathcal{X})$ and a family of constants $C_\alpha(\ell, \xi) > 0$, defined for any $\ell \geq 1$ and $\xi \in (0, 1]$, such that the following property holds true for any $\mu \in M_1(\mathcal{X})$ such that $\mu(\mathcal{D}_\ell) \geq \xi$ and any $t > 0$:

\[ \|\mu A_t(dx) - \alpha(dx)\|_{TV} \leq C_\alpha(\ell, \xi) e^{-\gamma t}. \quad (3.10) \]

Moreover, $\langle \alpha | h \rangle = 1$. The first condition on the $Q$-process is also directly deduced. It only remains to prove the convergence results as they are stated.

### 3.2.1 Proof of Theorem 2.2

We recall that $\mu$ is chosen in Definition 1 such that $\|\mu\|_{TV} \leq 1$. Denote by $\mu_+$ (resp. $\mu_-$) the positive (resp. negative) component of $\mu$ so that: $\mu = \mu_+ - \mu_-$ and $\|\mu\|_{TV} = 1 = \mu_+(\mathcal{X}) + \mu_-(\mathcal{X})$. Let $y \in \mathcal{D}_1$ and define the positive measures $\hat{\mu}_+$ and $\hat{\mu}_-$ as follows:

\[ \hat{\mu}_+(dx) := \frac{1}{1 + \mu_+(\mathcal{X})} [\delta_y + \mu_+(dx)] \geq 0, \quad \hat{\mu}_-(dx) := \frac{1}{1 + \mu_-(\mathcal{X})} [\delta_y + \mu_-(dx)] \geq 0. \]

Note that $\hat{\mu}_+(\mathcal{X}) = \hat{\mu}_-(\mathcal{X}) = 1$, so that both $\hat{\mu}_+$ and $\hat{\mu}_-$ are probability measures. In particular, this implies the following identification:

\[ \mu_+.(e^{-t \lambda} P_t)(dx) = \langle \hat{\mu}_+ | h_t \rangle \cdot \hat{\mu}_+. A_t(dx) \quad (3.11) \]

and similarly for $\mu_-$. These measures are also constructed so as to satisfy the two following properties:

\[ \mu = [1 + \mu_+(\mathcal{X})] \cdot \hat{\mu}_+ - [1 + \mu_-(\mathcal{X})] \cdot \hat{\mu}_-, \]

\[ \hat{\mu}_+(\mathcal{D}_1) \wedge \hat{\mu}_-(\mathcal{D}_1) \geq \frac{1}{1 + \|\mu\|_{TV}} \geq 1/2. \]

Thus, first thanks to (3.10), then to (3.9), there exists $\gamma, C > 0$ independent from $\mu$ such that the four following upper-bounds hold on $\mu_+$ and $\mu_-$ for any $t \geq 0$:

\[ \|\hat{\mu}_+. A_t(dx) - \alpha(dx)\|_{TV} \leq C e^{-\gamma t}, \quad \|\hat{\mu}_-. A_t(dx) - \alpha(dx)\|_{TV} \leq C e^{-\gamma t}, \]

\[ \|\langle \hat{\mu}_+ | h_t - h \rangle\| \leq C e^{-\gamma t}, \quad \|\langle \hat{\mu}_- | h_t - h \rangle\| \leq C e^{-\gamma t}. \]

The linear decomposition of $\mu$ between $\mu_+$ and $\mu_-$ thus implies the next estimations, thanks also to Equation (3.11) and to the fact that $h$ and the family $h_t$ are uniformly bounded:

\[ \mu.(e^{-t \lambda} P_t)(dx) = [1 + \mu_+(\mathcal{X})] \cdot \langle \hat{\mu}_+ | h_t \rangle \cdot \hat{\mu}_+. A_t(dx) - [1 + \mu_-(\mathcal{X})] \cdot \langle \hat{\mu}_- | h_t \rangle \cdot \hat{\mu}_-. A_t(dx) \]

\[ = \langle \mu | h \rangle \cdot \alpha(dx) + O_{TV}(e^{-t \gamma}). \]
Thus, we conclude the proof of Theorem 2.2 in that there exists some $C' > 0$ such that the following upper-bound holds for any signed measure such that $\|\mu\|_{TV} \leq 1$, and any $t \geq 0$:

$$\|\mu, (e^{-t\lambda} P_t) - \langle \mu | h \rangle \alpha \|_{TV} \leq C' \exp[-t\gamma].$$

$$\square$$

3.2.2 Proof of Corollary 2.3.4

Let $\bar{\mu} = \mu - \alpha$. We recall that by definition, $\langle \alpha | h_t \rangle = \langle \alpha | h \rangle = 1$. Then we can express the difference between $\mu A_t$ and $\alpha$ as follows in terms of $\bar{\mu}$ (recall also the analogous of Equation (3.11) for $\mu$)

$$\mu A_t - \alpha = \frac{\exp[t\lambda] \mu P_t - \langle \mu | h \rangle \alpha}{\langle \mu | h \rangle} + \frac{\langle \mu | h - h_t \rangle}{\langle \mu | h \rangle} \exp[t\lambda] \mu P_t$$

$$= \frac{\exp[t\lambda] \bar{\mu} P_t - \langle \bar{\mu} | h \rangle \alpha}{\langle \mu | h \rangle} + \frac{\langle \bar{\mu} | h - h_t \rangle}{\langle \mu | h \rangle} \mu A_t.$$  

Thanks to Theorem 2.2, this immediately implies the estimate on the convergence to $\alpha$. $\square$

3.2.3 Proof of Proposition 2.3.5

From ($H_0$), we consider $H \subset \mathcal{X}$, $t, c > 0$ and $\ell \geq 1$ such that the following lower-bound holds for any $x \in H$:

$$\mathbb{P}_x(\tau_{D_\ell} < t \wedge \tau_\theta) \geq c.$$

Recalling that the proof in [57] ensures that $h$ is bounded away from zero by a positive constant on any $D_\ell$, let $h_\ell$ be such lower-bound. The property of $h$ being an eigenfunction of the semi-group $(P_t)$ can be rephrased by saying that $(h(X_t)e^{\rho_0 t}1_{\{t < \tau_\theta\}})$ is a martingale. We also recall that $h$ is non-negative. The following upper-bound for any $x \in H$ is derived thanks to the martingale property, then the definition of $h_\ell$ and finally Inequality (3.12):

$$h(x) = \mathbb{E}_x[h(X(\tau_{D_\ell} \wedge t)) \exp[\rho_0(\tau_{D_\ell} \wedge t)] ; \tau_{D_\ell} \wedge t < \tau_\theta]$$

$$\geq h_\ell \mathbb{P}_x(\tau_{D_\ell} < t \wedge \tau_\theta) \geq c \cdot h_\ell > 0.$$  

This proves the uniform lower-bound of $h$ on $H$.

For the second point, recalling that the sequence of stopping times $\tau_{D_\ell}$ is necessarily decreasing, we deduce the following inclusion, that leads to the first desired inclusion:

$$\{x \in \mathcal{X} ; \exists \ell \geq 1, \mathbb{P}_x(\tau_{D_\ell} < \tau_\theta) > 0\} = \bigcup_{\ell \geq 1} \{x \in \mathcal{X} ; \mathbb{P}_x(\tau_{D_\ell} < \ell \wedge \tau_\theta) \geq 1/\ell\}$$

$$\subset \{x \in \mathcal{X}; h(x) > 0\}.$$  

For the reciprocal inclusion, let any $x$ be such that $h(x) > 0$. Thanks to Corollary 2.3.4, $\delta_x A_t$ converges to $\alpha$. Choosing $\ell \geq 1$ by Theorem 2.2 such that $\alpha(D_\ell) > 0$, it implies that $\delta_x A_t(D_\ell) > 0$ for $t$ sufficiently large, thus $\mathbb{P}_x(\tau_{D_\ell} < (t + 1) \wedge \tau_\theta) > 0$. This ends the proof of Proposition 2.3.5. $\square$
3.2.4 Proof of (ii) in Corollary 2.3.7

Assume that $\mu \in M_1(\mathcal{X})$ satisfies $\langle \mu | 1/h \rangle < \infty$. We may define $\nu(dx) := \frac{\mu(dx)}{h(x) \langle \mu | 1/h \rangle}$, which trivially satisfies that $\nu \in M_1(\mathcal{X})$ and that $\nu \cdot B[h] = \mu$. Let $\bar{\nu} = \nu - \alpha$. The difference between $\nu \cdot B[h] \cdot Q_t$ and $\beta$ is expressed as follows in terms of $\bar{\nu}$ and of a real-valued bounded measurable function $f$:

$$
\langle \nu \cdot B[h] \cdot Q_t | f \rangle - \langle \beta | f \rangle = \frac{\langle \nu | e^{\lambda t} P_t | h \cdot f \rangle - \langle \nu | h \rangle \cdot \langle \alpha | h \cdot f \rangle}{\langle \nu | h \rangle} 
\leq \frac{\langle \bar{\nu} | e^{\lambda t} P_t | h \cdot f \rangle - \langle \bar{\nu} | h \rangle \cdot \langle \alpha | h \cdot f \rangle}{\langle \nu | h \rangle} 
\leq \frac{\| \bar{\nu} \|_{TV} \cdot \| h \|_{\infty} \cdot \| f \|_{\infty} \cdot e^{-\gamma t}}{\langle \nu | h \rangle},
$$

thanks to Theorem 2.2. Moreover, by the definition of $\beta$ and $\nu$:

$$
\frac{\| \nu - \alpha \|_{TV}}{\langle \nu | h \rangle} = \left\| \frac{\mu}{\| \mu | 1/h \|} - \beta \right\|_{1/h} = \frac{\langle \mu | 1/h \rangle}{\langle \mu | 1 \rangle} = \| \mu - \langle \mu | 1/h \rangle \beta \|_{1/h}.
$$

Injecting this equality into the preceding upper-bound yields (ii) and conclude the proof of Corollary 2.3.7. □

3.3 Proof of Propositions 2.3.9

For $x \in \mathcal{X}$ and $t \geq 0$, define $\nu_{x,t}(dy) := (e^{\lambda t} \delta_x P_t - h(x)\alpha)_+(dy) \geq 0$. Thanks to (2.7), we know that there exists $C, \gamma > 0$ such that $\nu_{x,t}(\mathcal{X}) \leq Ce^{-\gamma t}$. Let $\rho := \lambda + \gamma/2$ and $t_E := 2 \cdot [\gamma \cdot \log(4C)]^{-1}$. We thus ensure that $C \cdot e^{-(\lambda+\gamma)t_E} \leq (1/4) \cdot e^{-\rho t_E}$.

Since $\mathcal{X} = \bigcup_{t \geq 1} \mathcal{D}_t$, we choose $\ell$ sufficiently large to ensure $\alpha(\mathcal{D}_\ell^c) \leq \frac{e^{-\gamma t_E/2}}{4\| h \|_{\infty}}$. Recalling the definition of $\nu_{x,t_E}$, and denoting as $\mathcal{D}_\ell$ the complementary of $\mathcal{D}_t$, the above results imply the following upper-bound for any $x \in \mathcal{X}$:

$$
\delta_x P_{t_E} \mathcal{D}_\ell^c \leq e^{-\lambda t_E} h(x)\alpha(\mathcal{D}_\ell^c) + \nu_{x,t_E}(\mathcal{X}) 
\leq e^{-\rho t_E} / 2.
$$

We split $\mathbb{R}_+$ into time-intervals of length $t_E$, so as to decompose the following expectation for any $K \geq 2$:

$$
\mathbb{E}_x \left( \exp \left[ \rho (\tau_0 \wedge \tau_{\mathcal{D}_t} \wedge (Kt_E)) \right] \right) 
\leq e^{\rho t_E} + \sum_{k=1}^{K-1} \exp((k+1)t_E) \mathbb{P}_x(\forall k' \in [1, k], X_{k't_E} \notin \mathcal{D}_\ell \wedge kt_E < \tau_0). \quad (3.14)
$$

Thanks to the Markov property and to Inequality (3.13), an elementary induction proves the following upper-bound for any $k \geq 1$:

$$
\mathbb{P}_x(\forall k' \in [1, k], X_{k't_E} \notin \mathcal{D}_\ell \wedge kt_E < \tau_0) \leq \frac{e^{-\rho k t_E}}{2^k}. \quad (3.15)
$$
Thanks to Inequalities (3.15) and (3.14), we obtain the following upper-bound:
\[ E_x \left( \exp \left[ \rho (\tau_0 \wedge \tau_{D_1} \wedge (Kt_E)) \right] \right) \leq e^{\rho t_E} (1 + \sum_{k \geq 1} 2^{-k}) \leq 2e^{\rho t_E} < \infty. \]
Letting \( K \) tend to infinity concludes the proof of Propositions 2.3.9. \( \square \)

3.4 Proof of Propositions 2.3.10

Let \( x \in \mathcal{X}, \mu \in M_1(\mathcal{X}) \) and \( t_F > 0 \).

**Step 1: Definition of \( U_H \)**

We define:
\[ \nu_{x,\mu}^{t_F} := \left( \delta_x P_{t_F} - \frac{\|h\|_\infty}{\langle \mu \mid h \rangle} \mu P_{t_F} \right)_. \]
We impose that \( U_H \) takes values \( t_F \) or \( \infty \) in such a way that \( \nu_{x,\mu}^{t_F} \) exactly correspond to the harvested measure:
\[ E_x (X_{t_F} \in dy; U_H = t_F < \tau_0) = \delta_x P_{t_F}(dy) - \nu_{x,\mu}^{t_F}(dy) \geq 0. \]  
(3.16)

A natural choice of \( U_H \) is defined through \( U \) being a uniform random variable on \((0,1)\), independent of the process \( X \). The choice of \( \nu_{x,\mu}^{t_F} \) was made to ensure, by the Radon-Nikodym Theorem, that \( \frac{d\nu_{x,\mu}^{t_F}}{d\delta_x P_{t_F}} : \mathcal{X} \to [0,1] \) can be defined as a density for the measure \( \delta_x P_{t_F} \). So we simply impose \( U_H = \infty \) if \( U \geq \frac{d\nu_{x,\mu}^{t_F}}{d\delta_x P_{t_F}}(X_{t_F}) \), and \( U_H = t_F \) otherwise. With this choice, (3.16) is satisfied and \( U_H \) can clearly be extended into \( U_H^\infty \) by exploiting a new uniform random variable, independent of the previous construction, for the coupling after each failure.

**Step 2: Control of the densities**

Thanks to the definitions of \( U_H \) and \( \nu_{x,\mu}^{t_F} \), the following inequality is straightforward:
\[ E_x (X_{U_H} \in dy; U_H < \tau_0) \leq \frac{\|h\|_\infty}{\langle \mu \mid h \rangle} E_{\mu}(X_{t_F} \in dy; t_F < \tau_0). \]

**Step 3: Control of failures**

With this definition of \( U_H \), recalling (3.16):
\[ P_x(U_H = \infty) = 1 - (\delta_x P_{t_F}(\mathcal{X}) - \nu_{x,\mu}^{t_F}(\mathcal{X})) = \nu_{x,\mu}^{t_F}(\mathcal{X}), \]  
(3.17)
which is thus the quantity for which we want an upper-bound.

Consider the following positive measure:
\[ \nu_{x,\mu}^{t_F} := e^{-\lambda t_F} \cdot \left( (e^{\lambda t_F} \delta_x P_{t_F} - h(x)\alpha)_+ + \frac{\|h\|_\infty}{\langle \mu \mid h \rangle} \left( \langle \mu \mid h \rangle \alpha - e^{\lambda t_F} \mu P_{t_F} \right)_+ \right). \]
Thanks to Inequality (2.7) with $\rho := \lambda + \gamma/2$, the mass of this measure can be efficiently upper-bounded:

$$\hat{\nu}_{x,\mu}^t(F) \leq C e^{-(\lambda + \gamma)t_F} \left( 1 + \frac{\|h\|_\infty}{\langle \mu \mid h \rangle} \right) \leq \frac{C_F}{\langle \mu \mid h \rangle} e^{-\rho t_F},$$

(3.18)

where $c_F := 2C \cdot \|h\|_\infty$ is independent of $x$, $\mu$ and $t_F$.

On the other hand, $\hat{\nu}_{x,\mu}^t$ is such that the following property holds:

$$\delta_x P_{t_F}(dy) \leq \frac{\|h\|_\infty}{\langle \mu \mid h \rangle} \mu P_{t_F}(dy) + \hat{\nu}_{x,\mu}^t(dy),$$

which implies that $\nu_{x,\mu}^t \leq \hat{\nu}_{x,\mu}^t$. Combining it with (3.17) and (3.18), the intended inequality is obtained:

$$P_x(U_H = \infty) \leq \frac{c_F}{\langle \mu \mid h \rangle} e^{-\rho t_F}.$$

Since $t_F$ is indeed allowed to take any sufficiently large values, this concludes the proof of Proposition 2.3.10.

\[\square\]

### 3.5 Proof of Theorem 2.3

Recall that we wish to describe the approximations of the previous dynamics when extinction happens at $\tau^L_{\partial} := \tau_0 \wedge T_{D_L}$ instead of $\tau_\partial$.

There is an explicit relation between all the constants introduced in the proofs of Theorems 2.2-4 (requiring also the proofs in [57]). Moreover, the proof actually relies on a single value of $\rho > \rho_S$ and a specific set $E$. Note that for any $L$ such that $E \subset D_L$, we have:

$$\sup_{x \in X} E_x \left( \exp \left[ \rho (\tau^L_{\partial} \wedge \tau_E^L) \right] \right) \leq \sup_{x \in X} E_x \left( \exp \left[ \rho (\tau_\partial \wedge \tau_E) \right] \right) := e_T.$$

Likewise, Assumption (A3$_F$) extends naturally for $\tau^F_\partial$. (A1) is stated with extinction already occurring at the exit of some set $D_L$ prescribed by the value of $\ell$. Considering the proof of Theorem 2.2 in Subsection 3.2.1 we see that (3.3) and (3.4) are only required for $\ell = 1$ and $\xi = 1/2$. Once these two values are fixed, the proof given in [57] Section 5) treats uniformly initial conditions $\mu$ such that $\mu(D_1) \geq 1/2$ by exploiting (A1) a finite number of times. One can thus identify an upper-bound $L_{\xi} \geq 1$ of the values $L(\ell)$ involved in the successive applications of (A1). So it suffices to take $L$ such that $D_{L_{\xi}} \subset D_L$ to ensure that all the results extend for the extinction time $\tau^L_\partial$ instead of $\tau_\partial$. Under this condition, our proof ensures that uniform exponential quasi-stationary convergence also holds for the process with extinction at time $\tau^L_\partial$ and that the constants involved in the convergences can be taken uniformly over these values $L$.

To compare $\lambda$ to $\lambda_L$, we can observe that for any $t > 0$:

$$\frac{-1}{t} \log P_\zeta(t < \tau_\partial) \leq \frac{-1}{t} \log P_\zeta(t < \tau^L_\partial)$$

so that $\lambda \leq \lambda_L$ is deduced by taking the limit $t \to \infty$ and exploiting the convergence to the survival capacities. The same argument ensures that $\lambda_L$ is a decreasing sequence in $L$.
Assume then by contradiction that there exists $\eta > 0$ such that $\lim_{t} \lambda_{L} \geq \lambda + \eta$. Recall that we have an explicit upper-bound $\|h_{*}\|_{\infty}$ that is valid uniformly for the functions $h^{L}$ and $h$. Thanks to Theorem 2.2 and the analogous result with $\tau_{0}$, for $t$ sufficiently large, the next two estimates on the survival probability holds:

$$e^{\lambda t} P_{\zeta}(t < \tau_{0}) \geq \frac{1}{2} \langle \zeta \mid h \rangle, \quad e^{\lambda L t} P_{\zeta}(t < \tau_{0}^{L}) \leq 2\|h_{*}\|_{\infty}.$$ 

By exploiting the property of $\eta$, we deduce:

$$0 < \langle \zeta \mid h \rangle \leq 2e^{(\lambda - \lambda_{L})t} \cdot e^{\lambda L t} \cdot \mathbb{P}_{\zeta}(t < \tau_{0}^{L}) + 2e^{\lambda t} |\mathbb{P}_{\zeta}(t < \tau_{0}) - \mathbb{P}_{\zeta}(t < \tau_{0}^{L})|$$

$$\leq 4\|h_{*}\|_{\infty} e^{-\eta t} + 2e^{\lambda t} |\mathbb{P}_{\zeta}(t < \tau_{0}) - \mathbb{P}_{\zeta}(t < \tau_{0}^{L})|.$$  (3.19)

The first term in the upper-bound becomes negligible uniformly over $L$ by taking $t$ sufficiently large. In order to obtain a contradiction, we merely have to prove that $\mathbb{P}_{\zeta}(t < \tau_{0}^{L})$ converges to $\mathbb{P}_{\zeta}(t < \tau_{0})$ at any fixed (large) time $t$. The difference is $\mathbb{P}_{\zeta}(T_{D_{L}} < t < \tau_{0})$ thus upper-bounded by $\mathbb{P}_{\zeta}(T_{D_{L}} < t)$. So our first aim is to prove that a.s. $\lim_{L} T_{D_{L}} = \infty$.

Assume by contradiction that the limit $T_{\infty}$ of this increasing sequence is at a finite value. Yet, thanks to (4.10) and to the fact that $X$ is càd-làg, $X_{T_{\infty} -} \in \mathcal{X}$ belongs to $D_{M}$ for some $M$. Thus, there exists a vicinity to the left of $T_{\infty}$ on which $T_{D_{L}}$ for $L > M$ cannot happen. But this precisely contradicts the definition of $T_{\infty}$. Consequently, $\lim_{L} T_{D_{L}} = \infty$ holds a.s. and $\mathbb{P}_{\zeta}(t < \tau_{0}^{L})$ converges to $\mathbb{P}_{\zeta}(t < \tau_{0})$ as $L \to \infty$. The contradiction with (3.19) makes us conclude that $\lambda_{L}$ tends to $\lambda$ as $L \to \infty$.

The next step is to look at the survival capacities, thanks again to Theorem 2.2 (with the measures evaluated on $\mathcal{X}$). The following upper-bound holds for any $x \in \mathcal{X}$:

$$|h(x) - h_{L}(x)| \leq e^{\lambda t}|\mathbb{P}_{x}(t < \tau_{0}) - \mathbb{P}_{x}(t < \tau_{0}^{L})| + e^{\lambda t} - e^{\lambda L t} + C e^{-\gamma t}.$$ 

Again, we can choose $t$ sufficiently large to make $C e^{-\gamma t}$ negligible. We already know that $\lambda_{L}$ tends to $\lambda$ and as previously, we prove that $\mathbb{P}_{x}(t < \tau_{0}^{L})$ tends to $\mathbb{P}_{x}(t < \tau_{0})$, as $L \to \infty$. This concludes the punctual convergence of $h_{L}$ to $h$. The conclusion would be the same if one replaces $x$ by any probability measure $\mu$, for instance $\alpha$.

Concerning the QSD:

$$\|\alpha - \alpha_{L}\|_{TV} \leq \left\|e^{\lambda L t} \delta_{\alpha} P_{t}^{L} - (\alpha \mid h_{L})\alpha_{L}\right\|_{TV} + |e^{\lambda L t} - e^{\lambda t}|$$

$$+ |(\alpha \mid h_{L} - h)| + e^{\lambda t} \left\|\delta_{\alpha} P_{t}^{L} - \delta_{\alpha} P_{t}\right\|_{TV},$$

where as $L \to \infty$, for $t$ fixed, we have just shown that the following quantity tends to 0 by proving that $\lim_{L} T_{D_{L}} = \infty$ holds a.s.:

$$\left\|\delta_{\alpha} P_{t}^{L} - \delta_{\alpha} P_{t}\right\|_{TV} = \mathbb{P}_{\alpha}(T_{D_{L}} < t < \tau_{0}) \to 0.$$ 

Thanks again to Theorem 2.2 and the previous convergence results to $\alpha_{L}$, $h$ and $\lambda$, the right-hand side can be made negligible by taking $t$ then $L$ sufficiently large, concluding the convergence of $\alpha_{L}$ to $\alpha$ in total variation. This concludes the proof of Theorem 2.3. \hfill $\square$
Applications

4 Mutations compensating a drift leading to extinction

4.1 A first simple process

We recall that we wish to prove uniform exponential quasi-stationary convergence for the following process:

\[ X_t = x - v t e_1 + \sum_{i \leq N_t} W_i, \]

with a state-dependent extinction rate given by \( \rho_e : x \mapsto \|x\|^2 \). The number \( N_t \) of mutations at time \( t \) is given as a classical Poisson process on \( \mathbb{R}_+ \). Each mutation effect \( W_i \) is distributed as a normal variable with covariance matrix \( \sigma^2 I_d \), and drawn independently of each others and of \( N_t \). Between jumps, the process is translated at constant speed \( v > 0 \) along the first coordinate (i.e. along \( e_1 \)).

**Theorem 4.1.** Consider \( P \) the semi-group associated to the process \( X \) as above (including the extinction). Then, for any \( v, \sigma > 0 \), \( P \) displays a uniform exponential quasi-stationary convergence with some characteristics \((\alpha, h, \lambda) \in M_1(\mathbb{R}^d) \times B(\mathbb{R}^d) \times \mathbb{R}_+ \) (cf Definition 7). Moreover, \( h \) is positive and bounded.

4.2 The main required properties

This application is related to non-local reaction-diffusion equations with a drift term. The one dimensional case has been studied recently by [30], with existence results obtained with compactness argument, and in Section 2 of [23], with the use of Lyapunov functions.

To highlight the generality of our approach, we specify next the main properties of \( X \) that we exploit. We consider generally a càdlàg process \( X \) on \( \mathbb{R}^d \), confronted to an extinction at a state-dependent rate given by \( \rho_e : \mathbb{R}^d \mapsto \mathbb{R}_+ \), and of the following form:

\[ X_t = x - v t e_1 + \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}_+} w 1_{\{u \leq g(X_{s-},w)\}} M(ds,dw,du). \]  

where \( M \) is a Poisson Random Measure (PRaMe) over \( \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \), with intensity \( \pi(ds,dw,du) = ds dw du \), while \( g(x,w) \) describes the jump rate from \( x \) to \( x + w \). In our focal example,

\[ g(x,w) := \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^d \exp\left( - \frac{\|w\|^2}{2\sigma^2} \right). \]

The infinitesimal generator \( \mathcal{L} \) of such generic process is defined on all \( C^1 \) and bounded function \( f \) on \( \mathbb{R}^d \) as follows:

\[ \mathcal{L} f(x) := -v \partial_{x_1} f(x) + \int_{\mathbb{R}^d} (f(x+w) - f(x)) \cdot g(x,w) dw - \rho_e(x) f(x). \]

The dynamics prescribed by the dual \( \mathcal{L}^* \) of \( \mathcal{L} \) by the equation \( \partial_t u = \mathcal{L}^* u \), which is the starting point of [30], then corresponds to the dynamics in time of the density of the measure-valued process \( \mu P_t \) (see notably Section 2 in [23]).
The next properties are stated for these measurable functions \( g \) and \( \rho_e \), with \( B(x,r) \) the open ball around \( x \) of radius \( r \) for the Euclidean norm. The fact that an upper-bound holds locally in \( x \) means that for any compact subset \( K \) of \( \mathcal{X} \), the upper-bound holds uniformly for \( x \in K \).

**Assumption (P)** (for Piecewise-Deterministic):

(P1) \( \rho_e \) is locally upper-bounded and \( \lim_{\|x\| \to \infty} \rho_e(x) = +\infty \).

Also, explosion implies extinction: \( \tau_\theta \leq \sup_{\{t \geq 1\}} T_{D_\ell} \).

(P2) The jump-rate \( \rho_J(x) := \int_{\mathbb{R}^d} g(x,w) \, dw \) is locally upper-bounded.

(P3) Locally in \( x \), there exists \( 0 < \eta < a \) such that the restriction of \( g \) to \( \mathcal{X} \times B(a \cdot e_1, \eta) \) is lower-bounded.

(P4) The jump size is tight locally in \( x \).

(P5) The density for each jump vector \( w \) is upper-bounded locally in \( x \).

**Theorem 4.2.** Provided the above conditions (P) are satisfied, \( P \) displays a uniform exponential quasi-stationary convergence with some characteristics \((\alpha, h, \lambda) \in \mathcal{M}_1(\mathbb{R}^d) \times B(\mathbb{R}^d) \times \mathbb{R}_+\). Moreover, \( h \) is positive and bounded.

Besides, the \( Q \)-process exists and is exponentially ergodic with weight \( 1/h \) as stated in Corollary 2.3.4 while the uniformity in the localization procedure holds as stated in Theorem 2.3 for \( D_{\ell} := B(0, \ell a) \).

The proof of Theorem 4.2 is given in Subsection 4.3. It entails the proof of Theorem 4.1 since it is elementary that the process described in (4.1) satisfies (P). Let us nonetheless first clarify the meaning of these different properties.

To fix ideas, let us consider any compact \( K \) subset of \( \mathcal{X} \). Thanks to Property (P1) there exists \( \rho_v \) such that \( \rho_v(x) \leq \rho_v \) for any \( x \in K \). This makes it possible to have simple lower-bounds on the survival of a given trajectory of \( (X_t)_{t \geq 0} \) provided it remains confined in the compact \( K \). On the other hand, the fact that \( \rho_v(x) \) tends to infinity as \( x \) tends to infinity makes it possible to justify the complementary of \( B(0, \ell) \) as transitory for \( \ell \) sufficiently large (according to property (A2)).

Thanks to Property (P2), there exists \( \rho_J^\gamma > 0 \) such that for any \( x \in K, \rho_J(x) \leq \rho_J^\gamma \). We can thus consider events of positive probabilities such that the corresponding trajectories of \( X \) have no other jumps than the one we carefully describe.

According to Property (P3), there exists also \( g_v > 0 \) and \( 0 < \eta < a \) such that the following inequality holds for any \( x \in K \) and any \( w \in B(a e_1, \eta) \), \( g(x,w) \geq g_v \). This will make it possible to consider trajectories in which each jump compensate the drift of the previous time-interval, with a small variation.

In addition, according to Property (P1), for any \( \epsilon > 0 \), there exists \( w_v \) such that the following upper-bound holds uniformly in \( x \in K \): \( \int_{\mathbb{R}^d} g(x,w) \, dw \leq \epsilon \cdot \rho_J(x) \). This makes it possible to restrict the size of the jumps with a probability close to 1.
Finally, according to Property \((P5)\), there exists \(g\) such that the following upper-bound holds uniformly in \(x \in K\) and \(w \in \mathbb{R}^d\): \(g(x, w) \leq g \cdot \rho_J(x)\). Thanks to this property, we will deduce upper-bounds of the marginal density of \(X_t\) according to the Lebesgue measure after some jumps.

4.3 Proof of Theorem 4.2

We aim at proving Assumption \((A_\Phi)\) for the sequence \(D_\ell := B(0, \ell \cdot a)\).

\((A0)\) is clearly satisfied. The proof of \((A1)\) is deduced from the following proposition, whose proof is deferred to the end of this subsection:

**Proposition 4.3.1.** Under \((P1, 2, 3)\), for any \(\ell \geq 1\), with \(L := \ell + 2\), there exists \(c, t > 0\) such that:

\[
\forall x \in D_\ell, \quad \mathbb{P}_x [X_t \in dx; \ t < \tau_\partial \wedge T_{D_L}] \geq c \cdot 1_{D_\ell}(dx).
\]

In particular, it implies that Assumption \((A1)\) holds with \(\zeta\) uniform over \(D_1\). From Lemma 3.0.2 in [57], noting that \(\zeta(D_1) > 0\) in particular, we can (explicitly) deduce a strict upper-bound \(\rho\) of \(\rho_S\). Since the extinction rate outside of \(D_\ell\) tends to infinity while \(\ell \to \infty\), for any \(\rho > 0\), we can find some \(L \geq 1\) such that assumption \((A2)\) holds true for \(E := D_\ell\) (cf. Subsection 4.1.2 in [57]). The proof of Assumption \((A3_\Phi)\) for these choices is a clear consequence of the next proposition, whose proof is given in the next Subsection 4.3.1:

**Proposition 4.3.2.** Suppose that Assumption \((P)\) holds true. Consider any \(\rho_E > \hat{\rho}_S\) and \(\ell_E \geq 1\) such that the set \(E = \bar{B}(0, \ell_E)\) satisfies \(\forall y \notin E, \rho_e(y) \geq \rho_E\). Set also any \(\epsilon > 0\). Then, there exists \(t_F, t_V, c > 0\), such that for any \(x \in E\), there exists a stopping time \(U_H\) with the following properties:

\[
\mathbb{P}_x(U_H = \infty, \ t_F < \tau_\partial) \leq \epsilon \cdot \exp[\hat{\rho}_S \cdot t_F], \quad \text{where } \{\tau_\partial \wedge t_F \leq U_H\} = \{U_H = \infty\}, \quad (4.3) \\
\mathbb{P}_x(X(U_H) \in dx'; \ U_H < \tau_\partial) \leq c \cdot \mathbb{P}_x(X(t_V) \in dx'; \ t_V < \tau_\partial), 
\]

including the fact that its definition can be extended into the one of some \(U_H^\infty\) as specified in Assumption \((A3_\Phi)\).

Given these two propositions, we can conclude that Assumption \((A_\Phi)\) holds true. By Theorem 2.2 Corollary 2.3.7 and Theorem 2.3 it directly implies Theorem 4.2.

4.3.1 Proof of Proposition 4.3.2

With the notations of the proposition, we first define \(t_F\) by the relation:

\[
\exp[\hat{\rho}_S \cdot (2 \ell_E/v) - (\rho_E - \hat{\rho}_S) \cdot (t_F - 2 \ell_E/v)] = \epsilon/2.
\]

The left-hand side is decreasing and converges to 0 when \(t_F \to \infty\), so that \(t_F\) is well-defined. Let \(T_{J_1}\) be the first jump time of \(X\). On the event \(\{t_F < T_{J_1}\}\), we set \(U_H = \infty\). The choice of \(t_F\) is done to ensure that the probability associated to the failure is indeed exceptional enough (with threshold \(\epsilon/2\) and time-penalty \(\hat{\rho}_S\)). Any jump occurring before \(t_F\) occurs
from a position \( X(T_j-) \in \tilde{B}(0, \ell_E + v t_F) := K \). Thanks to Assumption (P4), we can then define \( w_\varphi \) such that:

\[
\forall x \in K, \quad \int_{\mathbb{R}^d} g(x, w) 1_{\{\|w\| \geq w_\varphi\}} \, dw \leq \epsilon/2 \cdot \exp[\hat{\rho}_S t_F].
\]

A jump size larger than \( w_\varphi \) is then the other criterion of failure.

On the event \( \{T_j \leq t_F\} \cap \{T_j < \tau_\partial\} \cap \{\|W\| \leq w_\varphi\} \), where \( W \) is the size of the first jump (at time \( T_j \)), we thus set \( U_H := T_j \leq t_F \). Otherwise \( U_H := \infty \).

The proof that \( U_H \) extends to some \( U_H^\infty \) as stated in (A3_\( F \)) is elementary and the reader will be spared these details. Let us just say that

\[
\text{Proposition 4.3.2.}\quad \square
\]

where \( \tau \) is the first jump time of \( X \) after \( \tau_\partial \) for some \( i \geq 0 \) (cf. (6.1)); \( T_j' \leq \tau_\partial + t_F \) for this value of \( i \); and \( \|\Delta X_{T_j'}\| \leq w_\varphi \).

In particular, \( \{\tau_\partial \wedge t_F \leq U_H\} = \{U_H = \infty\} \) is clearly satisfied.

We prove next that the failures are indeed exceptional enough:

\[
\mathbb{P}_x(U_H = \infty, t_F < \tau_\partial) \leq \mathbb{P}_x(t_F \leq T_j \wedge \tau_\partial) + \mathbb{P}_x(T_j < \tau_\partial \wedge t_F, \|W\| > w_\varphi).
\]

By the definition of \( w_\varphi \), we deal with the second term:

\[
\mathbb{P}_x(T_j < \tau_\partial \wedge t_F, \|W\| > w_\varphi) \leq \mathbb{P}_x(\|W\| > w_\varphi | T_j < \tau_\partial \wedge t_F) \leq \epsilon/2 \cdot \exp[\hat{\rho}_S t_F].
\]

On the event \( \{t_F \leq T_j \wedge \tau_\partial\} \) it holds a.s. for any \( t \leq t_F \) that \( X_i = x - v t e_1 \). Thus, \( X \) is outside of \( E \) in the time-interval \((2\ell_E/v, t_F)\), with an extinction rate at least \( \rho_E \). By the definition of \( t_F \), it implies:

\[
\mathbb{P}_x(t_F \leq T_j \wedge \tau_\partial) \leq \exp[-\rho_E (t_F - 2 \ell_E/v)] \leq \epsilon/2 \cdot \exp[\hat{\rho}_S t_F].
\]

This concludes (13).

On the other hand, recall that a.s. on the event \( \{U_H < \infty\} \), \( X_{U_H} = W + X(T_j-) \) where \( X(T_j-) \in K \). Thus, thanks to Assumption (P5), there exists \( g_\varphi > 0 \) such that the following upper-bound on the density of \( X(U_H) \) holds uniformly in \( x \in E \):

\[
\mathbb{P}_x(X(U_H) \in dx') ; T_j < t_F \wedge \tau_\partial, \|W\| \leq w_\varphi \leq g_\varphi 1_{\{x' \in B(0, \ell_E + v t_F + w_\varphi)\}} dx'.
\]

We know also from Proposition (13.1) that there exists \( t_M, c_M > 0 \) such that:

\[
\mathbb{P}_\zeta(X(t_M) \in dx ; t_M < \tau_\partial) \geq c_M 1_{\{x' \in B(0, \ell_E + v t_F + w_\varphi)\}} dx'.
\]

With \( t := t_M, c := g_\varphi/c_M \) and thanks to Inequality (13), this concludes the proof of Proposition (13.2) \( \square \)
4.3.2 Proof of Proposition 4.3.1

We consider a characteristic length of dispersion given by \( r := \eta / 4 \). Given some \( \ell \geq 1 \), \( x_I \in D_\ell \), \( L := \ell + 2 \) and \( c > 0 \), we propose the following definition of the range of triplets time/density/position that can be reasonably reached by the process, starting from the vicinity of \( x_I \in D_\ell \) and restricted on specific domains (of the form \( D_L \)):

\[
\mathcal{R}^{(L)} := \left\{ (t,c,x_F) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d ; \forall x_0 \in B(x_I,r), \right. \\
\left. \mathbb{P}_{x_0}(X_t \in dx ; t < T_{D_L} \land \tau_0) \geq c \cdot 1_{B(x_F,r)}(x) dx \right\}.
\]

The proof then relies on the following three elementary lemmas. As a first step formalized in the next Lemma 4.3.3, we show that \( \mathcal{R}^{(L)} \) contains the product of a non-empty time-interval and of a vicinity of \( x_I \), for a sufficiently small density factor \( c_0 \).

**Lemma 4.3.3.** Given any \( L \geq 3 \), there exists \( c_0, t_0, \delta > 0 \), such that the following inclusion holds for any \( x_I \in D_\ell \) with \( \ell = L - 2 \):

\[
[t_0, t_0 + \delta] \times [0, c_0] \times B(x_I, r) \subset \mathcal{R}^{(L)}.
\]

The lemma is proved in Subsection 4.3.3 by compensating the drift component with exactly one jump and adjusting the time to the allowed variations in jump vector. Thanks to the lemma, we may start to consider trajectories from any initial position in \( D_\ell \). In addition, a time-interval is considered in order to adjust as a last step the durations of the trajectories that link the various reference points (corresponding to both \( x_I \) and \( x_F \)). As the second step formalized in the next Lemma 4.3.3, we show that we may expand the range in the feature dimension \( \mathcal{X} \), at the expense of a specific increase of time and reduction of the reference density:

**Lemma 4.3.4.** For any \( L \geq 3 \), with \( \ell = L - 2 \), there exists \( t_a, c_a > 0 \) such that the following implication holds uniformly for any \( x_I \in D_\ell \) and any \( t, c > 0 \):

\[
(t,c,x) \in \mathcal{R}^{(L)} \quad \Rightarrow \quad \{t + t_a\} \times \{c \cdot c_a\} \times B(x,r) \subset \mathcal{R}^{(L)}.
\]

The lemma is proved in Subsection 4.3.3 once more with the compensation of the drift by one jump with enough flexibility. This lemma is exploited inductively in the final step, cf Subsection 4.3.3, until the whole \( D_\ell \) belongs to the feature component of the range (for a large enough time and a small enough reference density). We also need to justify that we can make the associated durations coincide, up to a reduction of the density factor, which exploits the next Lemma in combination with Lemma 4.3.3.

**Lemma 4.3.5.** There exists \( c_P > 0 \) (only depending on \( r \) and \( d \)) such that the following implication holds for any \( L \geq 3 \), for any \( x_I \in D_\ell \) with \( \ell = L - 2 \), any \( t, c > 0 \) such that \((t,c,x_I) \in \mathcal{R}^{(L)}\), any \( t', c' > 0 \) and any \( x' \in D_L \):

\[
(t',c',x') \in \mathcal{R}^{(L)} \Rightarrow (t + t', c \cdot c_P \cdot c', x') \in \mathcal{R}^{(L)}.
\]

The lemma is proved as the third step in Subsection 4.3.3. The forth step concludes the proof of Proposition 4.3.1 will be achieved.
4.3.3 Step 1 : Initialisation - proof of Lemma 4.3.3

Thanks to Assumptions (P2, 3) (recalling that \( r = \eta/4 \) as stated in (P3)), there exists \( g_\wedge > 0 \) such that the following lower-bound hold for any \( x \in D_L \) and \( w \in B(a \cdot e_1, 4r) \):

\[
\rho_f(x)^{-1} \cdot g(x, w) \geq g_\wedge. \tag{4.4}
\]

Let \( T_1^I, T_2^I \) be respectively the first and second time of jump of \( X \). Thanks to Assumptions (P1, 2), exploiting that \( D_L \) is convex, there exists \( p_\wedge > 0 \) such that the following lower-bound holds for any \( x \in D_L \) and \( t \geq 0 \) such that \( x - v \cdot t \cdot e_1 \in D_L \):

\[
\mathbb{P}_x(t < T_1^I \wedge \tau_0) = \exp\left[ - \int_0^t \rho_f(x - v \cdot s \cdot e_1) ds \right] \geq p_\wedge. \tag{4.5}
\]

On the other hand, thanks to Assumption (P1, 3), there exists also \( q_\wedge > 0 \) such that the following lower-bound holds for any \( x \in D_L \) and \( t \geq a/v \) such that \( x - v \cdot t \cdot e_1 \in D_L \):

\[
\mathbb{P}_x(T_1^I < t \wedge \tau_0) \geq q_\wedge. \tag{4.6}
\]

Let \( t_0 := a/v, \delta := r/v, x_0 \in B(x_I, r) \) and \( t_* \in [t_0, t_0 + \delta] \). Concerning the constraint \( t_* < T_{DL} \), note that the following set is part of \( B(x_I, 6r) \):

\[
A := \{x_0 - v \cdot s \cdot e_1; s \leq t_0 + \delta \} \cup \{x_0 - v \cdot s \cdot e_1 + w; s \leq t_0 + \delta, w \in B(a \cdot e_1, 4r)\}.
\]

Since in addition \( x_I \in D_\ell = B(0, \ell \cdot a), 6r = 3\eta/2 \leq 2a \) and \( L = \ell + 2 \), this set is itself a subset of \( D_L \). Thus, by imposing at most one such jump, with a jump effect \( w \in B(a \cdot e_1, 4r) \), we keep the process inside of \( D_L \). Let us denote by \( W = \Delta X_{T_1^I} \) the size of the first jump. We therefore restricts our analysis to the following event:

\[
\mathcal{J} := \{T_1^I < t_* < T_2^I \wedge \tau_0\} \cap \{W \in B(a \cdot e_1, 4r)\}. \tag{4.7}
\]

Both \( t_* < T_{DL} \) and \( X(t_*) = x_0 - v \cdot t_* \cdot e_1 + W \) hold a.s. on \( \mathcal{J} \). It implies the following lower-bound for any real-valued positive test function \( f \) on \( X \):

\[
\mathbb{E}_{x_0}(f[X(t_*)]; t_* < T_{DL} \wedge \tau_0) \geq \mathbb{E}_{x_0}(f[x_0 - v \cdot t_* \cdot e_1 + W]; \mathcal{J}). \tag{4.8}
\]

Let \( Y := X(T_1^I -) \) be the position of the process just before the first jump. Given the definition of \( p_\wedge \) in (4.5) and since \( A \subset D_L \), the following lower-bound holds a.s. on the event \( \{T_1^I < t_* \wedge \tau_0\} \cap \{W \in B(a \cdot e_1, 4r)\} \):

\[
\mathbb{P}_Y(t_* - T_1^I < \tilde{T}_1^I \wedge \tilde{\tau}_0) \geq p_\wedge. \tag{4.9}
\]

In combination with (4.8) and the Markov inequality, this implies the next lower-bound:

\[
\mathbb{E}_{x_0}(f[X(t_*)]; t_* < T_{DL} \wedge \tau_0) \geq p_\wedge \mathbb{E}_{x_0}(f[x_0 - v \cdot t_* \cdot e_1 + W]; T_1^I < t_* \wedge \tau_0, W \in B(a \cdot e_1, 4r)). \tag{4.10}
\]

Conditionally on \( Y \) (or equivalently on the time \( T_1^I \)), the law of \( W \) is given by the measure \( \rho_f(Y)^{-1} \cdot g(Y, w) \cdot dw \). Given the definition of \( g_\wedge \) in (4.4), the law of \( W \) is lower-bounded by
the measure \( g_\Lambda \cdot 1_{B(a, e_1, 4r)}(w) dw \). Given also the definition of \( g_\Lambda \) in (4.10), we can derive from (4.10) the next lower-bound:

\[
\mathbb{E}_{x_0} \left( f[X(t_\ast)]; t_\ast < T_{D_L} \land \tau_\delta \right) \geq p_\Lambda \cdot g_\Lambda \cdot g_\Lambda \cdot \int_{B(a, e_1, 4r)} f[x_0 - v \cdot t_\ast \cdot e_1 + w] dw. \tag{4.11}
\]

Recalling that \( x_0 \in B(x_I, r) \) and \( t_\ast \in [a/v, (a + r)/v] \), we note that the following inclusions is valid for any \( x_F \in B(x_I, r) \):

\[
B(x_F, r) \subset B(x_I, 2r) \subset x_0 - v \cdot t_\ast \cdot e_1 + B(a \cdot e_1, 4r).
\]

With the fact that (4.11) holds for any positive test-function \( f \), this concludes the proof of Lemma 4.3.3 where \( c_0 := p_\Lambda \cdot g_\Lambda \cdot g_\Lambda > 0 \). \(\square\)

4.3.4 Step 3: Expansion - proof of Lemma 4.3.4

By definition, the fact that \( (t, c, \hat{x}) \in \mathcal{R}(L) \) means that the following lower-bound holds uniformly in \( x_0 \in B(x_I, r) \):

\[
\mathbb{P}_{x_0}(X(t) \in dx_1; t < T_{D_L} \land \tau_\delta) \geq c_0 \cdot 1_{B(\hat{x}, r)}(x_1) dx_1. \tag{4.12}
\]

We see in the above proof of Lemma 4.3.3 that the definitions of \( t_0, \delta, c_0 \) can be stated in terms of \( L \) uniformly in \( x_I \) within \( D_\ell \). In particular, this implies the following lower-bound for any \( \hat{x} \in D_\ell \) and any \( x_1, x_F \in B(\hat{x}, r) \):

\[
\mathbb{P}_{x_1}(X(t_0) \in dx; t_0 < T_{D_L} \land \tau_\delta) \geq c_0 \cdot 1_{B(x_F, r)}(x) dx. \tag{4.13}
\]

In combination with the Markov property, (4.12) and (4.13) imply the following lower-bound uniformly in \( x_0 \in B(x_I, r) \):

\[
\mathbb{P}_{x_0}(X(t + t_0) \in dx; t + t_0 < T_{D_L} \land \tau_\delta) \geq c_0 \cdot c_a \cdot \text{Leb}(B(\hat{x}, r)) \cdot 1_{B(x_F, r)}(x) dx.
\]

Note that \( c_a = c_0 \cdot \text{Leb}(B(\hat{x}, r)) = c_0 \cdot r^d \cdot \text{Leb}(B(0, 1)) \) is a positive constant independent of \( \hat{x} \). With \( t_a = t_0 \), this implies that \( \{t + t_a\} \times \{c \cdot c_a\} \times B(\hat{x}, r) \subset \mathcal{R}(L) \), concluding the proof of Lemma 4.3.4 \(\square\)

4.3.5 Step 3: Time-adjustment- proof of Lemma 4.3.5

Exploiting the Markov property as in the previous step, the properties of \( (t, c, x_I) \) and \( (t', c', x'_{r}) \) imply the following lower-bound for any \( L \geq 3, x_I \in D_\ell \) and \( x_0 \in B(x_I, r) \):

\[
\mathbb{P}_{x_0}(X(t + t') \in dx; t + t' < T_{D_L} \land \tau_\delta) \geq c \cdot c' \cdot \text{Leb}(B(x_I, r)) \cdot 1_{B(x', r)}(x) dx.
\]

With \( c_p := r^d \cdot \text{Leb}(B(0, 1)) > 0 \), this concludes the proof of Lemma 4.3.5 \(\square\)
4.3.6 Step 4: Conclude the proof of Proposition 4.3.1

Let us first consider $x_I$ and $x_F$ any points of $D_t$. Let $K > \|x_F - x_I\|/r$. For $0 \leq k \leq K$, let $x_k := x_I + k(x_F - x_I)/K$, which belongs to $D_t$ (by convexity of the set). Exploiting Lemma 4.3.3 we choose $(t_0, c_0)$ such that $(t_0, c_0, x_I) \in \mathcal{R}(L)$. Since $x_{k+1} \in B(x_k, r)$ for each $k$, by induction for any $k \leq K$ thanks to Lemma 4.3.4 there exists $t_k, c_k > 0$ such that: $(t_k, c_k, x_k) \in \mathcal{R}(L)$. In particular with $k = K$, there exists $t_f, c_f > 0$ such that $(t_f, c_f, x_F) \in \mathcal{R}(L)$.

A priori, these constants $t_f, c_f > 0$ still depend on the choices of $x_I$ and $x_F$. We thus look at adjusting the different values $t_f, c_f$ with a finite number of such focal trajectories. By compactness, there indeed exists $(x^j)_{j \leq J}$ such that: $D_t \subset \bigcup_{j \leq J} B(x^j, r)$. Let $t \forall$ be the larger time $t_f$ needed to reach the vicinity of any $x_F \in \{x^j\}$ from any $x_I \in \{x^j\}$.

To adjust the arrival time, we make the process stay some time around $x_I$. Thanks to Lemmas 4.3.3 and 4.3.5, we deduce iteratively in $k \geq 1$ the following inclusion $[k t_0, k t_0 + k \delta] \times \{x_I\} \subset \mathcal{R}(L)(e^k)$. The time-intervals $[k t_0, k t_0 + k \delta]$ for $k \geq 1$ cover $[t_A, \infty)$ where $t_A := t_0 \cdot [1 + t_0/\delta]$. Thus, for any $t \geq t_A$, we can find some $c > 0$ for which $(t, c, x_I) \in \mathcal{R}(L)$. Thanks to Lemma 4.3.5 it ensures, with $t_M := t \vee t_A$, that there exists $c_M > 0$ such that the following lower-bound on the density at time $t_M$ holds for any $j, j' \leq J$ and any $x_0 \in B(x^j, r)$:

$$
\mathbb{P}_{x_0} [X(t_M) \in dx \mid t_M < T_{D_L} \wedge \tau_0] \geq c_M 1_{B(x^j, r)}(x) dx.
$$

Since $D_t \subset \bigcup_{j \leq J} B(x^j, r)$, this completes the proof of Proposition 4.3.1.

Now that Propositions 4.3.2 and 4.3.1 are proved, as mentioned just after their statements, the proof of Theorem 4.2 is achieved. As noted after the statement of Theorem 4.2 this also ends the proof of Theorem 4.1.

5 The case of jumps occurring as in a Gibbs sampler

5.1 The core typical example

$X$ is a pure jump process on $\mathbb{R}^d$, for $d \geq 2$, again confronted to a death rate at state $x$ given by $\rho(x) := \|x\|_\infty^2$, where $\|x\|_\infty := \sup_{i \leq d} |x_i|$. Jumps are restricted to happen along the vectors of an orthonormal basis $(e_1, ..., e_d)$. Independently of these directions and of previous jumps, each jump occurs at rate 1 and its size follows an exponential distribution with mean $\sigma$. This entails the following representation:

$$
X_t := x + \sum_{i \leq N_t} \sigma W_i e_{D_i}.
$$

In this formula, $x$ is the initial condition, $N_t$ a standard Poisson process on $\mathbb{Z}_+$, while, for any $i \geq 1$, $W_i$ is a standard normal random variable on $\mathbb{R}$, $D_i$ is uniform over $[1, d]$. Moreover, all these random variables are independent from each others.

Theorem 5.1. Consider $P$ the semi-group associated to the process $X$ given by equation (5.1) and weighted by the extinction event at rate $\rho$. Assume that $\sigma \leq 1/8$. Then, $P$ displays a uniform exponential quasi-stationary convergence with some characteristics $(\alpha, h, \lambda) \in \mathcal{M}_1(\mathbb{R}^d) \times B(\mathbb{R}^d) \times \mathbb{R}_+$ (cf Definition 7). Moreover, $h$ is positive on bounded.
5.2 The main required properties

The process under consideration in Section 5 is a specific instance of pure jump processes. We refer to [31] for a detailed presentation of existence results of a QSD for a pure jump process. Contrary to the former approaches given in [10, 26, 27, 29, 52, 53] and relying on adaptations of the Krein-Rutman theorem, the one in [31] exploits some maximum principle, which makes it possible to obtain uniqueness. It appears that no quantitative results of convergence are known.

To our knowledge, the restriction of having jumps only along specific directions seems not to have been analyzed until the current article. As a motivation, the process \( X \) could for instance characterize an ecosystem where each coordinate corresponds to a single species.

Remark 5.2.1. In order to prevent concentration effects, several assumptions are additionally provided by the authors, see also [11]. Their connection to our assumption of "almost perfect harvest" is a topic of interest for a future work.

To highlight the generality of our approach, we specify also in this case the main properties that we exploit. Let \((X_t)_{t \geq 0}\) be the pure jump process on \( \mathcal{X} := \mathbb{R}^d \) defined by:

\[
X_t := x + \sum_{i \leq d} \int_{[0,t] \times \mathbb{R} \times \mathbb{R}^+} w \mathbf{e}_i \mathbf{1}_{\{u \leq g_i(X_{s^-},w)\}} M_i(ds, dw, du),
\]

where \( M_i \) are mutually independent PRaMes on \( \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^+ \) with intensities \( ds dw du \), and the \((g_i)_{i \in [1,d]}\) are real-valued measurable function on \( \mathbb{R}^d \times \mathbb{R} \). The process is also associated to a state-dependent extinction rate given by \( \rho_e : \mathbb{R}^d \to \mathbb{R}^+ \).

In our focal example, \( \rho_e(x) := \|x\|_{\infty}^2 \) and \( g_i \) is defined as follows for any \( i \leq d \):

\[
g_i(x, w) := \frac{1}{\sqrt{2\pi \sigma}} \exp\left(-\frac{|w|^2}{2\sigma^2}\right).
\]

Remark that the infinitesimal generator \( \mathcal{M} \) of such generic process is defined on all \( C^1 \) and bounded function \( f \) on \( \mathbb{R}^d \) as follows:

\[
\mathcal{M}f(x) := \sum_{i \leq d} \int_{\mathbb{R}^d} (f(x + we_i) - f(x)) \cdot g_i(x, w) dw - \rho_e(x)f(x).
\]

\((X_t)_{t \geq 0}\) is a Markov Process with piecewise constant trajectories. Conditionally upon \( X_t = x \), the waiting-time and size of the next jump are independent, the law of the waiting-time is exponential of rate \( \rho J(x) := \sum_{i \leq d} \rho_i J(x) \), where the jump rate can be decomposed along each direction \( i \in [1,d] \) according to \( \rho_i J(x) := \int_{\mathbb{R}^d} g_i(x, w) dw < \infty \).

The jump occurs on the \( i \)-th coordinate with probability \( \rho_i J(x)/\rho J(x) \), then with size given by \( g_i(x, w)/\rho_i J(x) dw \).

- **Assumption (J) (for Jumps)**
  
  (J1) The global jump rate \( \rho J \) is upper-bounded locally in \( x \).

  (J2) Locally in \( x \), there exists \( \eta > 0 \) such that the restriction of \( g \) to \( \mathcal{X} \times B(0, \eta) \) is lower-bounded.
The jump size has a tight law locally in $x$.

The density for each jump vector is upper-bounded locally in $x$.

The probability that each direction gets involved in the jump is uniformly lower-bounded.

$\rho_e$ is bounded away from zero by $\rho > \rho_S$ outside some compact set. Moreover, $\rho_e$ is locally bounded and explosion implies extinction: $\tau_\partial \leq \sup\{\ell \geq 1\} T_{D_\ell}$.

No stable subset: $\rho_S < \rho_F$ where $\rho_F := \inf_{x \in \mathbb{R}^d, i \leq d} \{\rho_i^J(x) + \rho_e(x)\}$.

**Theorem 5.2.** Provided the above conditions (J) are satisfied, $P$ displays a uniform exponential quasi-stationary convergence with some characteristics $(\alpha, h, \lambda) \in M_1(\mathbb{R}^d) \times B(\mathbb{R}^d) \times \mathbb{R}_+$. Moreover, $h$ is positive and bounded.

Besides, the $Q$-process exists and is exponentially ergodic with weight $1/h$ as stated in Corollary 2.3.7 while the uniformity in the localization procedure holds as stated in Theorem 2.3 for $D_\ell := B(0, \ell)$.

We also refer to Subsections 6.2.2 for the connection with reaction-diffusion equations, which holds in the same way, this time for a non-local dispersion operator of the form:

$$\mathcal{M}^* u(x) := \sum_{i \leq d} \left[ \int_{\mathbb{R}} g_i(x - w_i, e_i) u(x - w_i, e_i) dw_i - \left( \int_{\mathbb{R}} g_i(x, w_i) dw_i \right) u(x) \right]$$

Theorem 5.1 is deduced from Theorem 5.2 once we prove that the process indeed satisfies (J), for which only (J7) is not elementary. Let us first clarify the meaning of these assumptions.

The main difference with Assumption (P) is in the last three assumptions. Remark that we could not avoid the comparison with the quantity $\rho_S$ in the last two. Note however that, as we can see in Subsection 5.3 Lemma 3.0.2 of [57] provide an efficient way to get upper-bounds of $\rho_S$. It is hopefully enough to ensure such properties as (J6, 7). While (J6) plays a similar role as (P1), (J5, 7) are really specific to the fact that some jump directions are restricted. In this illustrative example, (J5) is required to make sure that the different directions can efficiently be explored by the process. Thanks to it, there exists $p_\wedge$ such that the following lower-bound holds uniformly for $x \in \mathbb{R}^d$ and $i \leq d$: $\rho_i^J(x) \geq p_\wedge \cdot \rho_J(x)$.

(J7) is required to prevent some directions in $\mathbb{R}^d$ from being avoided by the process, meaning that the probability to do so for a long time becomes negligible even compared to extinction.

To be clear with the other properties, let us consider a compact $K$ subset of $X$. Thanks to (J1), there exists $\rho_i^J > 0$ such that the following upper-bound holds uniformly for $x \in K$: $\rho_j^J(x) \leq \rho_i^J$. This is the same as (P2).

Thanks to (J2), there exist $r, g_\wedge > 0$ such that the following upper-bound holds uniformly for $x \in K$, $i \in [1, d]$ and $w \in B(0, r)$: $g_i(x, w) \geq g_\wedge$. This is analogous to (P3) except that there is no drift to compensate here.
Thanks to (J3), for any $\varepsilon > 0$, there exists $w_\varepsilon$ such that the following upper-bound holds uniformly for $x \in K$, and $t \in [1, d]$: $\int_{\mathbb{R}} g_t(x, w) 1_{\{\|w\|_\infty \geq w_\varepsilon\}} dw \leq \varepsilon \cdot \rho_f(x)$. This is analogous to (P4).

Finally, thanks to (J4) there exists $g_\varepsilon$ such that the following upper-bound holds uniformly for $x \in K$, $i \in [1, d]$ and $w \in \mathbb{R}$: $g_t(x, w) \leq g_\varepsilon \cdot \rho_f(x)$. This is the same as (P5).

### 5.3 Proof of (J7) for our typical example

In this example, $\rho^j \equiv 1$ while the minimal value of $\rho_e$ is simply 0. Thus, we need to prove that provided $\sigma \leq 1/8$, $\rho_S < \rho_F = 1$ holds. We rely on the criteria proposed in Lemma 3.0.2 of [57] and aim at finding some set $D_S$, $L \geq 1$ and $t > 0$ such that:

$$\inf_{\{x \in D_S\}} \mathbb{P}_x(X_t \in D_S, t < \tau_\theta \land T_{D_L}) > e^{-t}.$$  

We justify next our choice of $t := (4/3) \cdot d \cdot \log 4$, $D_S$ and $D_L$ being of the form respectively $B(0, a)$ and $B(0, 2a)$ for $a := 1/4$. Since the jumps of $X$ occur at a uniform rate $1/d$ along each direction and with a distribution independent of the position, the increase process $(X^i_t - x^i)_{t \geq 0, i \leq d}$ on each coordinate can be expressed as an i.i.d. family of processes whose law is given by:

$$Y_t := \sigma \sum_{j \leq N^j_t} W_j,$$

where $(N^j_t)_{t \geq 0}$ is a standard Poisson process on $\mathbb{Z}_+$ with intensity $(1/d)$ while for any $j \geq 1$ $W_j$ is a normal random variable. $N'$ and the family $(W_j)_j$ are independent. We remark that $Y$ is a martingale with predictable quadratic variation $\langle Y \rangle_t := \sigma^2 t/d$, with the same law as $-Y$ as a symmetry.

Exploiting also the fact that $\rho_e$ is upper-bounded by $4a^2 = 1/4$ on $D_L$ and thanks to the symmetries of the process, we deduce:

$$\inf_{\{x \in D_S\}} \mathbb{P}_x(X_t \in D_S, t < \tau_\theta \land T_{D_L}) \geq e^{-t/4} \inf_{\{x \in D_S\}} \mathbb{P}_x(\sup_{s \leq t} \|X_s - x\|_\infty \leq a, \forall i \leq d, x_i \cdot (X^i_t - x^i) \leq 0) \geq e^{-t/4} \left[(1/2) \cdot \mathbb{P}(\sup_{s \leq t} |Y_s| \leq a)\right]^d.$$  

Thanks to the Doob inequality, and recalling our expressions for $a$ and $t$:

$$\mathbb{P}(\sup_{s \leq t} |Y_s| \geq a) \leq \frac{\mathbb{E}(\langle Y \rangle_t)}{a^2} = \frac{16 \sigma^2 t}{d} = \frac{64 \sigma^2 \log 4}{3} < 1/2,$$

provided $\sigma \leq 1/8 < \sqrt{3/[128 \cdot \log(4)]}$. Since the definition of $t$ is made such that $e^{-t/4} / 4^d \geq e^{-t}$, this concludes the following uniform lower-bound:

$$\inf_{\{x \in D_S\}} \mathbb{P}_x(X_t \in D_S, t < \tau_\theta \land T_{D_L}) > e^{-t}.$$  

Thanks to Lemma 3.0.2 of [57], $\rho_S$ is thus necessarily smaller than 1, which concludes the proof of (J7) for our example.  

□
Remark 5.3.1. i) The condition $\sigma \leq 1/8$ comes only from the way we prove (J7) and is likely not to be optimal. For too large values of $\sigma$ however, singular concentration effects around 0 may play a substantial role, as in [11]. The event consisting of forbidding any jump when starting at a Dirac Mass around 0 might lead to a lower rate of decay in probability than the one consisting of accumulating jumps, because these jumps mostly send the process to deadly regions.

ii) The purpose of assumption (J7) is to bound the time $T_c$ at which either extinction occurs or all of the coordinates have changed. Assumption (J7) indeed ensures an exponential moment with parameter $\hat{\rho}_S$ (cf. (5.3) below).

5.4 Proof of Theorem 5.2

For this example, we consider the family $(D_\ell)_{\ell \geq 1}$ as the open balls $D_\ell := \bar{B}(0, \ell)$, now for the supremum norm $\| \cdot \|_\infty$ for commodity.

Remark 5.4.1. Because this norm is equivalent to the Euclidian norm, it is not difficult to see that the statements of Assumption (A_F) are actually equivalent for these two choices.

Assumption (A0) is clearly satisfied. The proof of (A1) as stated in the following proposition is very similar to the one of Proposition 4.3.1. By these means, we deal with each coordinate one by one so as to get a uniform lower-bound of the density on a subspace of inductively increasing dimension. The reader will be spared further details.

Proposition 5.4.2. Assumptions (J1, 2, 5) imply Assumption (A1), with $\zeta$ the uniform distribution over $D_1$. More generally, for any $\ell \geq 1$, there exist $L > \ell$ and $t, c > 0$ such that the following inequality holds for any $\forall x \in D_\ell$,

$$\mathbb{P}_x [X(t) \in dy; t < \tau_\partial \wedge T_{D_L}] \geq c \mathbf{1}_{y \in D_1} dy$$

Thanks to this proposition, we know in particular that Assumption (A1) holds true for the uniform distribution over $D_1$, i.e.: $\zeta(dy) := \mathbf{1}_{y \in D_1} / \text{Leb}(D_1) dy$. From Lemma 3.0.2 in [57], we can (explicitly) deduce a strict upper-bound $\rho$ of $\rho_S$. Assumption (A2) with this value of $\rho$ is clearly implied by Assumption (J6) for $E := D_L$ where $L \geq 1$ is chosen sufficiently large. $L$ is simply chosen so that the extinction rate outside of $E$ is larger than $\rho$. The proof of Assumption (A3_F) for these choices is a clear consequence of the next proposition, whose proof is given in the next subsection:

Proposition 5.4.3. Assumption (J) implies that for any $E \in D$ and $\rho > 0$, Assumption (A3_F) holds.

With this result, we can conclude that Assumption (A_F) holds true. By Theorem 2.2 Corollary 2.3.7 and Theorem 2.3 it directly implies Theorem 5.2.

5.4.1 Proof of Proposition 5.4.3

We consider here three types of “failed attempts”. Either the process has not done all of its required jumps despite a very long time of observation, or there are too many of these
Proof: The property is proved by induction over \( N \), where one needs to adjust at each step both \( \epsilon \) and \( w_\vee \). The initialization is directly implied by assumption (J3). For some \( N \) and \( w_\vee^N > 0 \), consider the event \( \mathbb{W}_N(w_\vee^N) \) according to which the \( N \) first jumps have a size that is upper-bounded by \( w_\vee^N > 0 \), that is \( \mathbb{W}_N(w_\vee^N) := \{ \sup_{i \leq N} \| W_i \|_\infty \leq w_\vee^N \} \).
Assume by the induction hypothesis that \( w_N^\vee \) is chosen such that the following lower-bound is ensured uniformly for \( x \in \bar{B}(0, L) \):
\[
\mathbb{P}_x(\|W_N(w_N^\vee)\|_\infty \leq L) \geq 1 - \epsilon / 2.
\]
On the event \( \mathbb{W}_N(w_N^\vee) \), with \( \|x\|_\infty \leq L \), we deduce that \( \|X(T_N)\|_\infty \leq L + N \cdot w_N^\vee \). Recall that \( \mathcal{F}_{T_N^J} \) describe the information of the process up to its \( N \)-th jump time. Thanks to Assumption (J3), there exists \( w_{N+1}^\vee \geq w_N^\vee \) such that the event \( \{\|W_{N+1}\|_\infty \leq w_{N+1}^\vee\} \) occurs with probability greater than \( 1 - \epsilon / 2 \) conditionally on \( \mathcal{F}_{T_N^J} \) restricted to the event \( \mathbb{W}_N(w_N^\vee) \) and uniformly on \( x \in \bar{B}(0, L) \). Note also the following inclusion:
\[
\mathbb{W}_N(w_N^\vee) \cap \{\|W_{N+1}\|_\infty \leq w_{N+1}^\vee\} \subset \mathbb{W}_{N+1}(w_{N+1}^\vee).
\]
Thanks to the Markov property, the following upper-bound is then derived for any \( x \in \bar{B}(0, L) \):
\[
\mathbb{P}_x(\|W_{N+1}\|_\infty \leq w_{N+1}^\vee | \mathcal{F}_{T_N^J}) \geq 1 - \epsilon / 2 \geq 1 - \epsilon.
\]
The induction over \( N \) then concludes the proof of the lemma.

Thanks to Lemma 5.4.4, we can choose a value \( w_\vee > 0 \) such that
\[
\epsilon \cdot \exp\left(-\hat{\rho}_S t_F/3\right) \leq \mathbb{P}_x(X(T_{N+1}) \in dx', \ U_H = \infty).
\]
On the event that at time \( T_d^J \leq t_F \), none of the three following conditions have been violated:
\( (i) \) \( T_d^J \) still has not occurred at time \( t_F \), or \( (ii) \) the \( n_\vee^J \)-th jump has occurred, or \( (iii) \) a jump of size larger than \( w_\vee \) has occurred (before time \( T_d^J \)),
we set \( U_H := T_d^J \). Otherwise \( U_H := \infty \).

Given our construction (see (5.4), (5.5) and the above definition of \( w_\vee \)), it is clear that:
\[
\{t_\theta \leq U_H\} = \{U_H = \infty\} \quad \text{and} \quad \mathbb{P}_x(U_H = \infty, t_F < t_\theta) \leq \epsilon \exp(-\rho t_F).
\]
The proof of Proposition 5.4.3 is then completed with Lemma 5.4.2 and the following complementary lemma, whose proof constitutes the last step:

**Lemma 5.4.5.** Assume that (J3\(-7\)) hold, with the preceding notations. Then, there exists \( c > 0 \) such that:
\[
\mathbb{P}_x(X(U_H) \in dx', \ U_H < t_\theta) \leq c \mathbf{1}_{\{x' \in D_L\}} dx'.
\]
\[ \square \]
5.4.2 Proof of Lemma 5.4.5

The proof is based on an induction on the coordinates affected by jumps in the time-interval $[0, t_F]$. We recall that, thanks to our criterion of exceptionality, we can restrict ourselves to trajectories where any coordinate is affected by at least one jump in the time-interval $[0, t_F]$, while at most $n_J^*$ jumps have occurred in this time-interval. We consider the sequence of directions that the process follows at each successive jumps. There is clearly a finite number of such sequences. In order to deduce the upper-bound on the density of $X(U_H)$ presented in Lemma 5.4.5, we merely need to prove the restricted versions for any such possible sequence of directions.

Let $(i(k))$ for $k \leq n_J \leq n_J^*$ be a given sequence of directions in $[1, d]$ such that, at $k = n_J$, all the $d$ directions have been listed. Let also $I(k) \in [1, d]$ for $k \leq n_J^*$, be the sequence of random directions followed by the $n_J^*$ first successive jumps of $X$. Let $U_k^J$ be the time of the $k$-th jump of $X$.

Since in our model, all directions are defined in a similar way, we can simplify a bit our notations without loss of generality by relabeling some of the directions. Since we will go backwards to progressively forget about the conditioning, we order the coordinates by the time they appear for the last time in the sequence $(i(k))_{k \leq n_J}$.

It means that $i(n_J) = d$ and that, up to the relabeling, we exploit the unique non-decreasing function $j : [1, n_J] \to [1, d]$ such that for any $K \in [1, n_J]$, $i(k); K \leq k \leq n_J = \lfloor j(K), d \rfloor$. Let then $K[j]$ be the largest integer $k \leq n_J$ such that $j(k) \leq j$. With this definition, it holds for any $j \in [1, d]$ that $i(K[j]) = j$ and that for any $k \in [K[j] + 1, n_J]$, $i(k) \in [j + 1, d]$.

Remark 5.4.6. In our case, $n_J$ is naturally chosen as the first integer for which all the directions have been listed. Yet, our induction argument is more clearly stated if we do not assume this condition on $n_J$.

We define the sequence $(A(k))_{k \leq n_J}$ of events that encode the fact that $U_k^J$ has not reached $\tau_0 \wedge t_F$ and that, up to the $k$-th jump, the random sequence of directions coincide with the sequence $i$ and the size of the jumps remain uniformly bounded by $w_\vee$. Namely, for $K \in [1, n_J]$:

$$A(K) := \{ U_k^J < \tau_0 \wedge t_F \} \cap \{ \forall k \leq K - 1, I(k) = i(k), \| \Delta X(U_k^J) \|_\infty \leq w_\vee \}.$$ 

Then, we look for a lower-bound that is uniform in $x \in E$ on the following expectation that involves any given non-negative and measurable functions $(f_j)_{j \leq d}$:

$$E^d := \mathbb{E}_x \left[ \prod_{j \leq d} f_j \left[ X^J(U_{n_J}^J) \right]; A(n_J) \right].$$

Define the information up to time $U_{n_J}^J$ deprived from the last jump size as follows:

$$\mathcal{F}_{U_{n_J}^J}^{*} := \sigma \left( \mathcal{F}_{U_{n_J}^J - 1}^{*}, \{ I(n_J) = d \} \cap \{ U_{n_J}^J < \tau_0 \wedge t_F \} \right).$$

To compute $E^d$, we then need to compute the expectation of the following quantity:

$$\prod_{j \leq d - 1} f_j \left[ X^J(U_{n_J}^J - 1) \right] \cdot \mathbb{E}_x \left[ f_d \left[ X^d(U_{n_J}^J) \right]; \| \Delta X^d(U_{n_J}^J) \| \leq w_\vee \right| \mathcal{F}_{U_{n_J}^J}^{*},$$

(5.6)
restricted to the following event:

\[ \mathcal{A}(n_j - 1) \cap \{ U_{n_j}^J < \tau_0 \land t_F \} \cap \{ I(n_j) = d \}. \]

Note that \( X(U_{n_j}^J) = X(U_{n_j}^J - 1) \) is \( \mathcal{F}_{U_{n_j}^J} \)-measurable, since we consider a pure jump process. Thanks to the Markov property, the law of the next jump only depends on \( x' := X(U_{n_j}^J - 1) \) through the functions \( (w \mapsto g_j(x', w))_{j \leq d} \). With the \( \sigma \)-algebra \( \mathcal{F}_{U_{n_j}^J}^* \), we include the knowledge of the direction of the jump at time \( U_{n_j}^J \), so that only the size of this jump (possibly negative) remains random. With \( L := \ell_E + n_j^\gamma \cdot w_\nu \), which is clearly independent of \( n_j \) and of the particular choice of the sequence \( (i(k)) \), we note the following containment property:

\[ \| X^d(U_{n_j}^J - 1) \|_\infty \lor \| X^d(U_{n_j}^J) \|_\infty \leq L. \]

This implies thanks to assumption \((J4)\) that the following inequality holds a.s. on the event \( \mathcal{A}(n_j - 1) \cap \{ U_{n_j}^J < \tau_0 \land t_F \} \cap \{ I(n_j) = d \} \):

\[ \mathbb{E}_x \left[ f_d(X^d(U_{n_j}^J)) \mid |X^d(U_{n_j}^J)| \leq w_\nu \mid \mathcal{F}_{U_{n_j}^J}^* \right] \leq g_\nu \int_{[-L,L]} f_d(x^d) \, dx^d. \tag{5.7} \]

In what follows, the probability of the event \( \{ U_{n_j}^J < \tau_0 \land t_F \} \cap \{ I(n_j) = d \} \) is simply upper-bounded by 1. Combining inequalities \((5.6), \,(5.7)\), and our ordering with the definition of \( K[j] \), we deduce:

\[ E^d \leq g_\nu \int_{[-L,L]} f_d(x^d) \, dx^d \cdot \mathbb{E}_x \left[ \prod_{j \leq d} f_j[X^d(U_{n_j}^J - 1)] \mid \mathcal{A}(n_j - 1) \right] \]

\[ \leq g_\nu \int_{[-L,L]} f_d(x^d) \, dx^d \cdot \mathbb{E}_x \left[ \prod_{j \leq d} f_j[X^d(U_{K[d-1]}^J)] \mid \mathcal{A}(K[d-1]) \right]. \]

Recall in particular that \( i(K[d - 1]) = d - 1 \) and that \( K[d - 1] \leq n_j^\gamma \). The procedure can be iterated as follows:

\[ E^{(d-1)} = \mathbb{E}_x \left[ \prod_{j \leq d-1} f_j[X^d(U_{K[d-1]}^J)] \mid \mathcal{A}(K[d-1]) \right] \]

\[ \leq g_\nu \int_{[-L,L]} f_{d-1}(x^{d-1}) \, dx^{d-1} \cdot \mathbb{E}_x \left[ \prod_{j \leq d-2} f_j[X^d(U_{K[d-2]}^J)] \mid \mathcal{A}(K[d-2]) \right], \]

and so on until finally:

\[ E^d \leq (g_\nu)^d \cdot \prod_{i \leq d} \left( \int_{[-L,L]} f_i(x) \, dx \right). \]

We then sum over all sequences \( (i(k)) \) possibly observed up to time \( T_d^i \). With the definition of the range of sequence \( i \) up to step \( n \geq 1 \) as \( R_n^i := \{ i(k) : k \leq n \} \), the set of these sequences can be rigorously defined as follows:

\[ \{(i(k))_{k \leq n_j} \in [1, d]^{n_j} : n_j \leq n_j^\gamma, \ n_j = \min\{n \geq 1; R_n^i = [1, d]\}\}. \]
There are clearly less than $d^{n \gamma}$ possibilities (there is a surjection from the set of all sequences of length $n \gamma$). Since for any positive and measurable functions $(f_j)_{j \leq d}$, the following upper-bound is deduced uniformly for any $x \in E$:

$$
\mathbb{E}_x \left[ \prod_{j=1}^{d} f_j[X(U_H)]; U_H < \tau_0 \right] \leq d^{n \gamma} \cdot (g_\gamma)^d \int_{B(0,L)} \prod_{j=1}^{d} f_j(x_j) dx_1...dx_d,
$$

it is classical that it implies the following lower-bound on the marginal density:

$$
\forall x \in E, \quad \mathbb{P}_x [X(U_H) \in dx; U_H < \tau_0] \leq d^{n \gamma} \cdot (g_\gamma)^d 1_{\{x \in \bar{B}(0,L)\}} dx.
$$

It concludes the proof of Lemma 5.4.5.

Recall with the statement just before Lemma 5.4.5 that the proof of Proposition 5.4.3 is now completed. Note also with the statement just after Proposition 5.4.3 that the proof of Theorem 5.2 is then completed. With the statement just after Theorem 5.2, this also concludes the proof of Theorem 5.1.

6 Discussion

6.0.1 Assumption $(A3_F)$ of ”Almost Perfect Harvest”

How the different parameters have to be adjusted? In fact, we will exploit this assumption only for a given single value of $\epsilon > 0$, which is explicitly related to the other parameters (cf Subsection 3.1). But in generic proofs, this explicit value is not expected to be really tractable.

The random variable $U_H$ and $V$ are thus expected to depend both on $x \in E$ and on $\epsilon$, and to be related to $t_F$ and $c$, while these two constants must be uniform in $x$.

Is it really important to consider failures? The purpose of introducing failure is to handle singularities, i.e. events which are rare in probability but for which comparison estimates are poor or simply impossible.

Notably in pure jump models, waiting for a jump is a priori needed to loosen the dependency on the initial condition (especially when the latter is a Dirac mass). Yet, this implies that the event of a very late jump (being one condition of failure in the harvest) has to be considered carefully, to prove that its probability is negligible compared to the whole survival probability. In multidimensional model, we may also need to wait for a jump on a specific coordinate to happen, while there is often a positive probability for very singular behavior to happen meanwhile on the other coordinates. It is generally needed to adjust the singularity level (and implicitly the efficiency of the coupling), for the associated probability to be sufficiently small and for such events to be treated as a failure in the harvest.

If this issue is made easily manageable in the applications of Section 4 and 5, this is mainly because we allow for both random stopping times and failure events.

Considering failures can also be of interest in order to exploit a Girsanov transform to simplify the dynamics of the process. As can be seen in [59] and [46], this transform is very efficient to relate the original dynamics to one that is more easily described, notably
by decoupling different components of the dynamics. Namely, the original and simplified models are related through a change of the densities by a multiplicative factor that is upper-bounded except for rare events in probability. The statement of our property \((A3_F)\) is very adapted to deal with such imprecision: exceptional behavior is treated as a failure, so that a uniform bounds on the multiplicative factor can be ensured. From these bounds, we can then deduce the appropriate constant \(c\) in \((2.7)\).

Is it difficult to check the Markov properties for the harvesting time \(U_H^\infty\)? As in our applications, the definition of \(U_H^\infty\) should be naturally derived from the way \(U_H\) is defined and the proofs that both are stopping times should be similar.

Although the law of \(U_H^\infty\) is defined uniquely (which is what we need), it is a priori unclear how to define it generally. If \(U_H\) is defined directly in terms of the trajectories of \(X\), where \(X\) has independent increments like Brownian Motions, Poisson Random Measures or say Levy processes, \(U_H^\infty\) can be expressed through these increments in the time-intervals \([\tau^i_E, \tau^{i+1}_E]\), for \(i \in [0, \infty]\), where recursively:
\[
\tau^{i+1}_E := \inf\{s \geq \tau^i_E + t_F; X_s \in E\} \wedge \tau_\partial, \text{ and } \tau^0_E = 0. \tag{6.1}
\]

The Markov property on the incremental process shall then imply the condition on \(U_H^\infty - \tau^1_E\).

Could we improve the assumption with less restrictions on the parameters? Would it be worth it? The first condition in \((2.4)\) means indeed that \(U_H\) is required to be less than \(t_F\) for a first success in the harvest to be achieved. This implies the following equalities: \(\{U_H < \tau_\partial\} = \{U_H < t_F\} = \{U_H = \infty\}\). The requirement that \(U_H\) must be less than \(t_F\) is however not as stringent as it might seem and it makes the statement of \((A3_F)\) much more tractable.

We believe it is generally compatible with the upper-bound on the failure event that one restricts any candidate \(\tilde{U}_H\) to be less than \(t_F\), provided \(t_F\) is large enough, possibly by reducing the considered value of \(\rho\) towards a value closer to \(\rho_S\) and by considering the event \(\{\tilde{U}_H \geq t_F\}\) as an additional criterion of failure.

Nonetheless, a refinement of the assumption with a looser upper-bound on \(U_H\) may still provide a better estimate of the constants involved. Note simply that it requires to specify the times at which failures are stated, since there is no more reason for each step to end before time \(t_F\). Since the statement would be much more technical, it is not included in the current article. Still, one may find a version of the proofs adapted for this context in the second ArXiv version \([58]\) of the current article.

6.1 Brief overview of the intended applications

Although greatly simplified, the two applications of the current article relate to eco-evolutionary models. The growth rate or the persistence of a population is related to the individual characteristics of its members, in other words their “features” or “traits”. This effect shall be represented by the state-dependent extinction rate. The dynamics of these traits may depend on mutations, a changing environment or the ageing of the individuals, for which
our applications provide archetypal models. These effects are expected to be represented in a “discontinuous” fashion, i.e. with brutal transitions, for which our approach is adapted.

Eco-evolutionary models thus form a large class of applications. Our assumption of a constant drift term in our first application is merely taken for simplicity given our multidimensional state space. Our approach could simply be adjusted for a drift term depending on the position as in [30] or [24], as long as it brings the process to infinity. The proof is then much more specific to the biological motivation.

Such a drift term could as well be interpreted as the ageing of individuals in age-structured population models (as in [56]), or as the growth rate of the units in growth-fragmentation models (as in [49] [13] [32]). Our hope is to see our technique fruitful for these applications extended to a multidimensional setting (where age or size is coupled to other individual features). The applications are not restricted to ecology, and may for instance come from chemistry (notably for polymer growth as in [43]), neuroscience (see the elapsed-time models e.g. in [56]) or epidemiology (notably when the infection rate depends on the elapsed time after the infection as presented in Subsection 1.1.2 of [12]).

More detailed ecological models have also been studied thanks to the theorems of the current paper. Notably in [59], we couple a diffusive process specifying the population size to a piecewise deterministic process specifying the adaptation of the population. The proof is more involved than in the current paper, notably because we use the Girsanov transform to decouple the diffusive and the piecewise deterministic components of the system. In [46], we study another related application, in which accumulation of deleterious mutations is slowed down by natural selection: the conditions of the present article are exploited to obtain the convergence to a unique QSD of a diffusion in an infinite dimensional state space.

More generally, our techniques provide conditions ensuring the existence and uniqueness of the positive eigenvector of general linear non-local reaction diffusion equations (see notably Subsection 6.2.2 for some partial results and Subsection 4.1 for the related conditions). The long-time behavior of structured branching processes can typically be captured by such results, thanks to the many-to-one formula (see [42]).

6.2 Practical implications of these results

6.2.1 Biological motivations

The processes presented in our applications can be seen as models for the adaptation of a population to its environment. In the first application, forcing by a regularly changing environment is considered, whereas in the second application, dependent but distinct subpopulations contribute to global adaptation to an otherwise fixed environment.

The environmental change in the first application is represented by a translation of the fitness landscape at constant speed v. We can consider X as a summary of the individual characters of the population. Then, the jumps come from the fixation of new mutations in the population, whose rate depends on the adaptation of the mutant subpopulation (with trait $X_{I+} + w$) as compared to the resident individuals (with trait $X_{I-}$). A much more detailed description is proposed in [59]. There, we extend the proof to a coupled process involving additionally continuous fluctuations of the population size.

Considering distinct directions of jumps in the second application is motivated by the
interpretation that each of these directions corresponds to the variation of a single species, where the various species contribute to the survival of the community. Many communities are then subjects to death and reproduction events and we can describe the state of the meta-community in this formalism as in [60].

6.2.2 Connection with reaction-diffusion equations

The quasi-stationary regime of the process generally prescribed in (4.2) is expected to be related to the behavior of the solution \((u(t, x))_{t \geq 0, x \in \mathbb{R}^d}\) at low densities \((u \approx 0)\) to reaction-diffusion evolution equations of the form:

\[
\partial_t u(t, x) := v \partial_x u(t, x) + \int_{\mathbb{R}^d} g(y, x - y) u(t, y) \, dy - \left( \int_{\mathbb{R}^d} g(x, w) \, dw \right) u(t, x) + r(x, u(t, x)) u(t, x).
\] (6.2)

At low density, the approximation of the growth rate \(r(x, u)\) by \(r_0(x) := r(x, 0)\) is usually valid, so as to linearize (6.2). Looking at the linear problem provides a criterion for the possibility and rate of invasion, cf for instance [7] and [28].

Also, if we consider \(r_0(x)\) as an upper-bound of \(r(x, u)\) for any density \(u\), the solution \(\bar{u}\) to the linear problem with \(r_0\) shall provide an upper-bound of \(u\) by maximum principle approaches. If the eigenvalue \(\lambda^*\) of the linear problem is negative, the solution \(u\) is expected to asymptotically decline at least quicker than at rate \(-\lambda^*\). Thus, results such as ours have implications as a criteria for non-persistence, as in [5], [6], [7], [8], [9], [14].

In view of these interpretations, several authors are looking at characterizing such eigenvalue problems when there is possibly no regular eigenvector (see e.g. [26], [31], [41], [44], [30]). For now, we simply conjecture that, provided Theorem 4.2 applies (with a translation of the growth rate by a constant to deduce an extinction rate), they all coincide to the value prescribed with \(\lambda\).

6.2.3 Ecological relevance of the results

In practice, the dynamics for which results of quasi-stationarity can be derived usually come as an approximation. The relevance of the approximation is then of course at stake, yet the quasi-stationary regime may provide insight on the conditions of relevance.

Considering for instance our first application in Section 4.2, the population is certainly doomed to extinction for too strong environmental drift. When population size strongly declines, the estimation of individual features through the marginal of \(X\) is not relevant. On the other hand, our result of convergence for the process \(X\) is not directly affected: it holds for any value of \(v\). As \(v\) tends to \(\infty\), we shall simply have the asymptotic extinction rate \(\lambda\) going to \(-\infty\). Considering the asymptotic extinction rate, notably in comparison to the convergence rate, can nonetheless inform about the validity of the marginal of \(X\) to capture the individual features.
Appendix: Elementary facts in the absorbed setting

Proof of Fact 2.3.2 Let us demonstrate that the property given in Definition 2.3 implies that $\alpha$ is a QSD with extinction rate $\lambda$. For $u > 0$, let us define $\mu_u(ds) := e^\lambda u \alpha P_u(ds) - \alpha(ds)$. Then, for any $t > 0$:

$$
\left\| e^{\lambda t} \alpha P_t - \alpha \right\|_{TV} \leq \left\| e^{\lambda t} \alpha P_t - e^{\lambda(t+u)} \alpha P_{t+u} \right\|_{TV} + \left\| e^{\lambda(t+u)} \alpha P_{t+u} - \alpha \right\|_{TV}
$$

$$
= \left\| e^{\lambda u} \mu_u P_t \right\|_{TV} + \left\| \mu_{u+t} \right\|_{TV}
$$

$$
\leq Ce^{-\gamma u}(e^{\lambda t} \| P_t \| + e^{-\gamma t}).
$$

Letting $u$ tends to $\infty$ concludes the equality $\alpha P_t(ds) = e^{-\lambda t} \alpha(ds)$. □

Proof of Fact 2.3.8 For any $u \geq 0$, $\| \mu - \alpha \|_{TV} \leq \| \mu - u \alpha \|_{TV} + |1 - u|$ because $\mu - u \alpha = \mu - \alpha + (1 - u) \alpha$. On the other hand $\| \mu - u \alpha \|_{TV} \geq |\mu(X) - u \alpha(X)| = |1 - u|$. By combining these two estimates, we conclude that $\| \mu - u \alpha \|_{TV} \geq \| \mu - \alpha \|_{TV}/2$.

Let $\mu$ be such that $\langle \mu | 1/h \rangle < \infty$ and define the biased probability distribution:

$$
\nu(dx) := \frac{1/h(x)}{\langle \mu | 1/h \rangle} \mu(dx).
$$

Exploiting the previous inequality, we deduce that for any $u > 0$:

$$
\| \mu - u \beta \|_{1/h} = \langle \mu | 1/h \rangle \cdot \| \nu - (u/\langle \mu | 1/h \rangle) \alpha \|_{TV}
$$

$$
\geq \frac{\langle \mu | 1/h \rangle}{2} \cdot \| \nu - \alpha \|_{TV} = \frac{\| \mu - \langle \mu | 1/h \rangle \beta \|_{1/h}}{2}.
$$

□

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