STABILITY OF SOME FUNCTIONAL EQUATIONS ON BOUNDED DOMAINS

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Abstract. In this paper, we investigate the Hyers-Ulam stability of the functional equations

\[ f(x+y) + f(x-y) = 2f(x), \]
\[ f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \]
\[ f(px + (1-p)y) + f((1-p)x + py) = f(x) + f(y) \]

for \( p = \frac{1}{3} \) and \( p = \frac{1}{4} \), where \( f \) is a mapping from a bounded subset of \( \mathbb{R}^{N>1} \) into a Banach space \( E \).

1. Introduction

It is well-known that the Hyers-Ulam stability problems of functional equations originated from a question of Ulam [12] in 1940, concerning the stability of group homomorphisms. In other words, the concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Hyers [1] gave a first affirmative partial answer to the question of Ulam for Banach spaces. It is interesting to consider a functional equation satisfying on a bounded domain or satisfying under a restricted condition. Skof [9] was the first author to solve Ulam problem for additive mapping on a bounded domain. Indeed, Skof proved that if a function \( f \) from \( [0,c) \) into a Banach space \( E \) satisfies the functional inequality \( \| f(x+y) - f(x) - f(y) \| \leq \delta \) for all \( x, y \in [0,c) \) with \( x+y \in [0,c) \), then there exists an additive function \( A : \mathbb{R} \to E \) such that \( \| f(x) - A(x) \| \leq 3\delta \) for all \( x \in [0,c) \). Z. Kominek [5] extended this result on a bounded domain \( [0,c)^N \) of \( \mathbb{R}^N \) for any positive integer \( N \). He also proved a more generalized theorem concerning the stability of the additive Cauchy equation and Jensen equation on a bounded domain of \( \mathbb{R}^N \). Skof [331] also proved the Hyers–Ulam stability of the additive Cauchy equation on an unbounded and restricted domain. She applied this result and obtained an interesting asymptotic behavior of additive functions: The function \( f : \mathbb{R} \to \mathbb{R} \) is additive if and only if \( f(x+y) - f(x) - f(y) \to 0 \) as \( |x| + |y| \to +\infty \). F. Skof and S. Terracini [11] investigated the problem of stability of the quadratic functional equations for functions defined on bounded real domains with values in a Banach space. For more general information on this subject, we refer the reader to [3, 6, 8].

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2. Stability of \( f(x+y) + f(x-y) = 2f(x) \) on bounded subsets of \( \mathbb{R} \)

In this section \( r > 0 \) and \( \delta \geq 0 \) are real numbers and we assume that \( E \) is a Banach space.

**Theorem 1.** Let \( f : [0, r) \to E \) be a function with \( f(0) = 0 \) and satisfy

\[
\|f(x+y) + f(x-y) - 2f(x)\| \leq \delta, \tag{1}
\]

for some \( \delta > 0 \) and all \( (x,y) \in T(r) \), where

\[
T(r) = \{(x,y) \in [0,r) \times [0,r) : 0 \leq x \pm y < r\}.
\]

Then there exists an additive function \( A : \mathbb{R} \to E \) such that

\[
\|f(x) - A(x)\| \leq 11\delta, \quad x \in [0, r). \tag{2}
\]

**Proof.** Let \( u, v \in [0, r) \). We can choose \( x, y \in [0, r) \) such that \( x \pm y \in [0, r) \), \( x + y = u \) and \( x - y = v \). Then it follows from (1) that

\[
\|f(u) + f(v) - 2f\left(\frac{u+v}{2}\right)\| \leq \delta. \tag{3}
\]

Letting \( v = 0 \) in (3), we get

\[
\|f(u) - 2f\left(\frac{u}{2}\right)\| \leq \delta, \quad u \in [0, r). \tag{4}
\]

We extend the function \( f \) to \([0, +\infty)\). For this we represent an arbitrary \( x \geq 0 \) by \( x = n(r/2) + \alpha \), where \( n \) is an integer and \( 0 \leq \alpha < r/2 \). Then we define a function \( \phi : [0, +\infty) \to E \) by \( \phi(x) = nf(r/2) + f(\alpha) \). It is clear that \( \phi(x) = f(x) \) for all \( x \in [0, r/2) \). If \( x \in [r/2, r) \), then \( \phi(x) = f(r/2) + f(x - r/2) \), and we get from (3) and (4) that

\[
\|\phi(x) - f(x)\| = \|f\left(\frac{r}{2}\right) + f\left(x - \frac{r}{2}\right) - f(x)\|
\leq \|f\left(\frac{r}{2}\right) + f\left(x - \frac{r}{2}\right) - 2f\left(\frac{x}{2}\right)\| + \|2f\left(\frac{x}{2}\right) - f(x)\|
\leq 2\delta.
\]

So

\[
\|\phi(x) - f(x)\| \leq 2\delta, \quad x \in [0, r). \tag{5}
\]

We now show that \( \phi \) satisfies

\[
\|\phi(x) + \phi(y) - 2\phi\left(\frac{x+y}{2}\right)\| \leq 3\delta, \quad x, y \in [0, +\infty). \tag{6}
\]
For given \( x, y \geq 0 \), let \( x = n(r/2) + \alpha \) and \( y = m(r/2) + \beta \), where \( m \) and \( n \) are integers and \( 0 \leq \alpha, \beta < r/2 \). Then

\[
\frac{x+y}{2} = \frac{m+n}{2} \left( \frac{r}{2} \right) + \frac{\alpha + \beta}{2}, \quad m+n \text{ is even};
\]

\[
\frac{x+y}{2} = \frac{m+n+1}{2} \left( \frac{r}{2} \right) + \frac{\alpha + \beta}{2} - \frac{r}{4}, \quad m+n \text{ is odd and } \alpha + \beta \geq \frac{r}{2};
\]

\[
\frac{x+y}{2} = \frac{m+n-1}{2} \left( \frac{r}{2} \right) + \frac{\alpha + \beta}{2} + \frac{r}{4}, \quad m+n \text{ is odd and } \alpha + \beta < \frac{r}{2}.
\]

Therefore we have

\[
\varphi \left( \frac{x+y}{2} \right) = \frac{m+n}{2} f \left( \frac{r}{2} \right) + f \left( \frac{\alpha + \beta}{2} \right), \quad m+n \text{ is even};
\]

\[
\varphi \left( \frac{x+y}{2} \right) = \frac{m+n+1}{2} f \left( \frac{r}{2} \right) + f \left( \frac{\alpha + \beta}{2} - \frac{r}{4} \right), \quad m+n \text{ is odd and } \alpha + \beta \geq \frac{r}{2};
\]

\[
\varphi \left( \frac{x+y}{2} \right) = \frac{m+n-1}{2} f \left( \frac{r}{2} \right) + f \left( \frac{\alpha + \beta}{2} + \frac{r}{4} \right), \quad m+n \text{ is odd and } \alpha + \beta < \frac{r}{2}.
\]

To prove (6) we have the following cases.

(i) If \( m+n \) is even, then

\[
\left\| \varphi(x) + \varphi(y) - 2 \varphi \left( \frac{x+y}{2} \right) \right\| = \left\| f(\alpha) + f(\beta) - 2 f \left( \frac{\alpha + \beta}{2} \right) \right\| \leq \delta.
\]

(ii) If \( m+n \) is odd and \( \alpha + \beta \geq \frac{r}{2} \), then

\[
\left\| \varphi(x) + \varphi(y) - 2 \varphi \left( \frac{x+y}{2} \right) \right\| = \left\| f(\alpha) + f(\beta) - f \left( \frac{r}{2} \right) - 2 f \left( \frac{\alpha + \beta}{2} - \frac{r}{4} \right) \right\|
\leq \left\| f(\alpha) + f(\beta) - 2 f \left( \frac{\alpha + \beta}{2} \right) \right\|
+ \left\| f \left( \alpha + \beta - \frac{r}{2} \right) - 2 f \left( \frac{\alpha + \beta}{2} - \frac{r}{4} \right) \right\|
+ \left\| 2 f \left( \frac{\alpha + \beta}{2} \right) - f \left( \frac{r}{2} \right) - f \left( \alpha + \beta - \frac{r}{2} \right) \right\|
\leq 3\delta.
\]

(iii) If \( m+n \) is odd and \( \alpha + \beta < \frac{r}{2} \), then

\[
\left\| \varphi(x) + \varphi(y) - 2 \varphi \left( \frac{x+y}{2} \right) \right\| = \left\| f(\alpha) + f(\beta) + f \left( \frac{r}{2} \right) - 2 f \left( \frac{\alpha + \beta}{2} + \frac{r}{4} \right) \right\|
\leq \left\| f(\alpha) + f(\beta) - 2 f \left( \frac{\alpha + \beta}{2} \right) \right\|
+ \left\| 2 f \left( \frac{\alpha + \beta}{2} \right) - f(\alpha + \beta) \right\|
+ \left\| f(\alpha + \beta) + f \left( \frac{r}{2} \right) - 2 f \left( \frac{\alpha + \beta}{2} + \frac{r}{4} \right) \right\|
\leq 3\delta.
\]
We show that if $\epsilon > 0$, then there exists an additive function $\epsilon$ such that $h^{(e)} \geq \|h(\epsilon) \epsilon - (x)\|$. Hence

**Corollary 1.** Let be a function with $(\epsilon) > 0$. If $h^{(e)} \geq \|h(\epsilon) \epsilon - (x)\| + \|h(\epsilon) \epsilon - (x)\| \geq \|h(\epsilon) \epsilon - (x)\|$

Therefore satisfies and then according to (5), there exist an additive function $\epsilon$, such that $\epsilon \geq (0, \epsilon) _x$. Since $\epsilon \geq (0, \epsilon) _x$ for all $x \geq (0, \epsilon) _x$. Therefore satisfies and then according to (7), there exist an additive function $\epsilon$, such that $\epsilon \geq (0, \epsilon) _x$
Corollary 2. Let \( f : (-r, r) \to E \) be a function with \( f(0) = 0 \) and satisfy
\[
\|f(x+y) + f(x-y) - 2f(x)\| \leq \delta,
\]
(8)
for some \( \delta > 0 \) and all \((x, y) \in T(r)\). Then there exists an additive function \( A : \mathbb{R} \to E \) such that
\[
\|f(x) - A(x)\| \leq 12\delta, \quad x \in (-r, r).
\]

Proof. Letting \( x = 0 \) in (8), we get \( \|f(y) + f(-y)\| \leq \delta \) for all \( y \in (-r, r) \). By Theorem 1, there exists an additive function \( A : \mathbb{R} \to E \) such that \( \|f(x) - A(x)\| \leq 11\delta \) for all \( x \in [0, r) \). If \( x \in (-r, 0) \), then
\[
\|f(x) - A(x)\| \leq \|f(x) + f(-x)\| + \|A(-x) - f(-x)\| \leq 12\delta.
\]
This completes the proof.

Theorem 2. Let \( f : (-r\sqrt{2}, r\sqrt{2}) \to E \) be a function with \( f(0) = 0 \) and satisfy
\[
\|f(x+y) + f(x-y) - 2f(x)\| \leq \delta,
\]
(9)
for some \( \delta > 0 \) and all \((x, y) \in \mathbb{R}^2\), where \( x^2 + y^2 \leq r^2 \). Then there exists an additive function \( A : \mathbb{R} \to E \) such that
\[
\|f(x) - A(x)\| \leq 19\delta, \quad x \in (-r\sqrt{2}, r\sqrt{2}).
\]

Proof. It is clear that if \( |x\pm y| \leq r \), then \( x^2 + y^2 \leq r^2 \). Therefore \( f \) satisfies (1) for all \((x, y) \in T(r)\). By Theorem 1, there exist an additive function \( A : \mathbb{R} \to E \) satisfying (2) for all \( x \in [0, r) \). Let \( \varphi \) and \( g \) be given as in the proof of Theorem 1. Then
\[
\varphi(x) = g(x), \quad \|\varphi(x) - f(x)\| \leq 2\delta, \quad x \in [0, r).
\]
(11)
If \( r \leq x < r\sqrt{2} \), then \((x/2)^2 + (x/2)^2 < r^2 \), and we infer from (9) that
\[
\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \delta, \quad x \in \left[r, r\sqrt{2}\right).
\]
Since \( \varphi(x) = g(x) \) for all \( x \geq 0 \), we get from (6) that
\[
\left\| g(x) - 2g\left(\frac{x}{2}\right) \right\| \leq 3\delta, \quad x \in [0, +\infty).
\]
Therefore from the above inequalities, we have
\[
\|f(x) - g(x)\| \leq \left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| + \left\| 2g\left(\frac{x}{2}\right) - g(x) \right\| + 2\left\| f\left(\frac{x}{2}\right) - g\left(\frac{x}{2}\right) \right\|
\leq 8\delta, \quad x \in \left[r, r\sqrt{2}\right).
\]
For the case \(-r\sqrt{2} < x < 0\), from the definition of \(g\), (9) and (11), we have
\[
\|f(x) - g(x)\| = \|f(x) + \varphi(-x)\|
\leq \left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| + 2\left\|f\left(\frac{x}{2}\right) + f\left(-\frac{x}{2}\right)\right\|
+ 2\left\|\varphi\left(-\frac{x}{2}\right) - f\left(-\frac{x}{2}\right)\right\| + \left\|\varphi(-x) - 2\varphi\left(-\frac{x}{2}\right)\right\|
\leq 10\delta.
\]

Hence we get
\[
\|f(x) - g(x)\| \leq 10\delta, \quad x \in \left(-r\sqrt{2}, r\sqrt{2}\right).
\]
Since \(\|g(x) - A(x)\| \leq 9\delta\) for all \(x \in \mathbb{R}\) (see the proof of Theorem 1), it follows from the last inequality that
\[
\|f(x) - A(x)\| \leq \|f(x) - g(x)\| + \|g(x) - A(x)\| \leq 19\delta, \quad x \in \left(-r\sqrt{2}, r\sqrt{2}\right),
\]
which ends the proof.

**THEOREM 3.** Let \(f : (-r, r) \rightarrow E\) be a function with \(f(0) = 0\) and satisfy
\[
\|f(x + y) + f(x - y) - 2f(x)\| \leq \delta,
\]
for some \(\delta > 0\) and all \((x, y) \in D(r)\), where
\[
D(r) = \{(x, y) \in (-r, r) \times (-r, r) : |x \pm y| < r\}.
\]
Then there exists an additive function \(A : \mathbb{R} \rightarrow E\) such that
\[
\|f(x) - A(x)\| \leq 5\delta, \quad x \in (-r, r).
\]

**Proof.** Letting \(y = x\) and \(x = 0\) in (12), respectively, we get
\[
\|f(2x) - 2f(x)\| \leq \delta, \quad \|f(y) + f(-y)\| \leq \delta, \quad |2x|, |y| < r.
\]
For an arbitrary \(x \in \mathbb{R}\), we set \(x = n(r/2) + \mu\), where \(n\) is an integer and \(0 \leq \mu < r/2\). Hence we can define a function \(g : \mathbb{R} \rightarrow E\) by \(g(x) = nf(r/2) + f(\mu)\). We show that \(\|g(x) - f(x)\| \leq 2\delta\) for all \(x \in (-r, r)\). For this we have the following cases:

1. For \(0 \leq x < r/2\), we have \(g(x) = f(x)\).
2. For \(r/2 \leq x < r\), we have \(x = r/2 + \mu\). Then it follows from (12) and (14) that
\[
\|g(x) - f(x)\| = \left\|f\left(\frac{r}{2}\right) + f(\mu) - f(x)\right\|
\leq \left\|f\left(\frac{r}{2}\right) + f(\mu) - 2f\left(\frac{x}{2}\right)\right\| + \left\|2f\left(\frac{x}{2}\right) - f(x)\right\|
\leq \delta + \delta = 2\delta.
\]
3. For \(-r/2 \leq x < 0\), we have \(x = -(r/2) + \mu\). Then
\[
\|g(x) - f(x)\| = \left\| -f\left(\frac{r}{2}\right) + f(\mu) - f(x) \right\| \\
\leq \left\| f(x) + f\left(\frac{r}{2}\right) - 2f\left(\frac{\mu}{2}\right) \right\| + \left\| 2f\left(\frac{\mu}{2}\right) - f(\mu) \right\| \\
\leq \delta + \delta = 2\delta.
\]

4. For \(-r < x < -(r/2)\), we have \(x = -2(r/2) + \mu\). Then
\[
\|g(x) - f(x)\| = \left\| -2f\left(\frac{r}{2}\right) + f(\mu) - f(x) \right\| \\
\leq \left\| f(\mu) + f(-x) - 2f\left(\frac{r}{2}\right) \right\| + \|f(-x) + f(x)\| \\
\leq \delta + \delta = 2\delta.
\]

We now show that \(g\) satisfies
\[
\|g(x + y) + g(x - y) - 2g(x)\| \leq 3\delta, \quad x, y \in \mathbb{R}.
\] (15)

For given \(x, y \in \mathbb{R}\), let \(x = n(r/2) + \alpha\) and \(y = m(r/2) + \beta\), where \(n\) and \(m\) are integers and \(\alpha, \beta \in [0, r/2)\). Therefore
\[
x + y = (n + m)\frac{r}{2} + (\alpha + \beta), \quad 0 \leq \alpha + \beta < r,
\]
\[
x - y = (n - m)\frac{r}{2} + (\alpha - \beta), \quad -\frac{r}{2} \leq \alpha - \beta < \frac{r}{2}.
\]

We consider following cases:

1. If \(0 \leq \alpha \pm \beta < r/2\), then
\[
\|g(x + y) + g(x - y) - 2g(x)\| = \|f(\alpha + \beta) + f(\alpha - \beta) - 2f(\alpha)\| \leq \delta.
\]

2. If \(0 \leq \alpha + \beta < r/2\) and \(-r/2 \leq \alpha - \beta < 0\), then
\[
\|g(x + y) + g(x - y) - 2g(x)\| = \left\| f(\alpha + \beta) + f\left(\alpha - \beta + \frac{r}{2}\right) - f\left(\frac{r}{2}\right) - 2f(\alpha) \right\| \\
\leq \left\| f(\alpha + \beta) + f(\alpha - \beta) - 2f(\alpha) \right\| \\
+ \left\| f(\alpha - \beta) + f\left(\frac{r}{2}\right) - f\left(\alpha - \beta + \frac{r}{2}\right) \right\| \\
= \|f(\alpha + \beta) + f(\alpha - \beta) - 2f(\alpha)\| \\
+ \|f(\alpha - \beta) - g(\alpha - \beta)\| \\
\leq \delta + 2\delta = 3\delta.
\]
3. If \( r/2 \leq \alpha + \beta < r \) and \( 0 \leq \alpha - \beta < r/2 \), then

\[
\|g(x+y) + g(x-y) - 2g(x)\| = \left\| f\left(\frac{r}{2}\right) + f\left(\alpha + \beta - \frac{r}{2}\right) + f(\alpha - \beta) - 2f(\alpha) \right\|
\leq \|f(\alpha + \beta) + f(\alpha - \beta) - 2f(\alpha)\|
+ \left\| f\left(\frac{r}{2}\right) + f\left(\alpha + \beta - \frac{r}{2}\right) - f(\alpha + \beta) \right\|
= \|f(\alpha + \beta) + f(\alpha - \beta) - 2f(\alpha)\|
+ \|g(\alpha + \beta) - f(\alpha + \beta)\|
\leq \delta + 2\delta = 3\delta.
\]

4. If \( r/2 \leq \alpha + \beta < r \) and \(-r/2 \leq \alpha - \beta < 0\), then

\[
\|g(x+y) + g(x-y) - 2g(x)\| = \left\| f\left(\frac{r}{2}\right) + f\left(\alpha - \beta + \frac{r}{2}\right) - 2f(\alpha) \right\| \leq \delta.
\]

Therefore \( g \) satisfies (15). It is easy to show that

\[
\left\| \frac{g(2^n x)}{2^n} - \frac{g(2^m x)}{2^m} \right\| \leq \sum_{i=m+1}^{n} \frac{3\delta}{2_i}, \quad n > m, \ x \in \mathbb{R}.
\]

Hence \( \{2^{-n}g(2^n x)\} \) is a Cauchy sequence for every \( x \in \mathbb{R} \). Since \( E \) is a Banach space, we can define a function \( A : \mathbb{R} \to E \) by

\[
A(x) = \lim_{n \to \infty} \frac{g(2^n x)}{2^n}.
\]

Letting \( m = 0 \) and taking the limit as \( n \to \infty \) in (16), we obtain

\[
\|A(x) - g(x)\| \leq 3\delta, \quad x \in \mathbb{R}.
\]

Since \( \|g(x) - f(x)\| \leq 2\delta \) on \( (-r, r) \), we get

\[
\|f(x) - A(x)\| = \|f(x) - g(x)\| + \|g(x) - A(x)\| \leq 5\delta, \quad x \in (-r, r).
\]

It follows from (15) that

\[
\|g(2^n x + 2^n y) + g(2^n x - 2^n y) - 2g(2^n x)\| \leq 3\delta, \quad x, y \in \mathbb{R}, \ n \geq 1.
\]

Dividing by \( 2^n \) and letting \( n \to \infty \) in this inequality, we infer that \( A \) is an additive function.

3. Stability of Drygas functional equation on bounded subsets of \( \mathbb{R} \)

We now prove the stability of Drygas functional equation on a restricted domain. First, we introduce a theorem of Skof and Terracini [11].
Theorem 4. [11] Let $E$ be a Banach space and let a function $f : (-r, r) \to E$ satisfy the inequality
\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \delta,
\]
for some $\delta > 0$ and all $x, y \in \mathbb{R}$ with $|x \pm y| < r$. Then there exists a quadratic function $Q : \mathbb{R} \to E$ such that
\[
\|f(x) - Q(x)\| \leq \frac{81}{2} \delta, \quad x \in (-r, r).
\]

Using ideas from [5], we can state the following proposition which is a generalization of Theorem 4.

Proposition 1. Let $E$ be a Banach space and let $D$ be a bounded subset of $\mathbb{R}$. Assume, moreover, that there exist a non-negative integer $n$ and a positive number $c > 0$ such that
\begin{enumerate}[(i)]
  \item $D \subseteq 2D$,
  \item $(-c, c) \subseteq D$,
  \item $D \subseteq (-2^n c, 2^n c)$.
\end{enumerate}
If a function $f : D \to E$ satisfies the functional inequality (17) for some $\delta \geq 0$ and for all $x, y \in D$ with $x \pm y \in D$, then there exists a quadratic function $Q : \mathbb{R} \to E$ such that
\[
\|f(x) - Q(x)\| \leq \frac{82.4^n - 1}{2} \delta, \quad x \in D.
\]

Proof. By Theorem 4, there exists a quadratic function $Q : \mathbb{R} \to E$ such that
\[
\|f(x) - Q(x)\| \leq \frac{81}{2} \delta, \quad x \in (-c, c).
\]

For $x \in D$, the conditions (i) and (iii) imply that $2^{-k}x \in D$ for $k = 1, 2, \ldots, n$ and $2^{-n}x \in (-c, c)$. It follows from (17) that for each $x \in D$
\[
\left\|4^{k-1}f\left(\frac{x}{2^{k-1}}\right) - 4^kf\left(\frac{x}{2^k}\right) + 4^{k-1}f(0)\right\| \leq 4^{k-1} \delta, \quad k = 1, 2, \ldots, n.
\]

Therefore
\[
\left\|f(x) - 4^n f\left(\frac{x}{2^n}\right) + \frac{4^n - 1}{3} f(0)\right\| \leq \frac{4^n - 1}{3} \delta.
\]

Using the above inequalities and $2\|f(0)\| \leq \delta$, we get
\[
\|f(x) - Q(x)\| \leq \left\|f(x) - 4^n f\left(\frac{x}{2^n}\right) + \frac{4^n - 1}{3} f(0)\right\| + \left\|4^n f\left(\frac{x}{2^n}\right) - Q(x)\right\| + \frac{4^n - 1}{3} \|f(0)\|
\leq \frac{82.4^n - 1}{2} \delta, \quad x \in D.
\]

This completes the proof.
THEOREM 5. Let $f : (-r, r) \to E$ be a function with $f(0) = 0$ and satisfy
\[
\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leq \delta, \quad (18)
\]
for some $\delta > 0$ and all $(x, y) \in D(r)$, where
\[
D(r) = \{(x, y) \in (-r, r) \times (-r, r) : |x \pm y| < r\}.
\]
Then there exist a quadratic function $Q : \mathbb{R} \to E$ and an additive function $A : \mathbb{R} \to E$ such that
\[
\|f(x) - A(x) - Q(x)\| \leq \frac{91}{2} \delta, \quad x \in (-r, r). \quad (19)
\]

Proof. We denote by $g$ and $h$ the even and odd part of $f$, respectively, i.e.,
\[
g, h : (-r, r) \to E, \quad g(x) = \frac{f(x) + f(-x)}{2}, \quad h(x) = \frac{f(x) - f(-x)}{2}.
\]
It is clear that $g$ and $h$ satisfy in (18) for all $(x, y) \in D(r)$. Since $g$ is even and $h$ is odd, we have
\[
\|g(x+y) + g(x-y) - 2g(x) - 2g(y)\| \leq \delta, \quad x, y \in D(r), \quad (20)
\]
\[
\|h(x+y) + h(x-y) - 2h(x)\| \leq \delta, \quad x, y \in D(r). \quad (21)
\]
By Theorems 3 and 4, there exist an additive function $A : \mathbb{R} \to E$ and a quadratic function $Q : \mathbb{R} \to E$ such that
\[
\|g(x) - Q(x)\| \leq \frac{81}{2} \delta, \quad \|h(x) - A(x)\| \leq 5\delta, \quad x \in (-r, r).
\]
Since $f = g + h$, we get (19).

PROPOSITION 2. Let $E$ be a Banach space and let $D$ be a symmetric bounded subset of $\mathbb{R}$. Assume, moreover, that there exist a non-negative integer $n$ and a positive number $c > 0$ such that
\[
(i) \quad D \subseteq 2D,
\]
\[
(ii) \quad (-c, c) \subseteq D,
\]
\[
(iii) \quad D \subseteq (-2^n c, 2^n c).
\]
If a function $f : D \to E$ satisfies the functional inequality (18) for some $\delta > 0$ and for all $x, y \in D$ with $x \pm y \in D$, then there exist a quadratic function $Q : \mathbb{R} \to E$ and an additive function $A : \mathbb{R} \to E$ such that
\[
\|f(x) - A(x) - Q(x)\| \leq \left[6.2^n + 41.4^n - \frac{3}{2}\right] \delta, \quad x \in D.
\]
Proof. Let \( g \) and \( h \) be the even and odd part of \( f \), respectively. Since \( D \) is symmetric, \( g \) satisfies (20) and \( h \) satisfies (21) for all \( x, y \in D \) with \( x \pm y \in D \). By Proposition 1, there exists a quadratic function \( Q : \mathbb{R} \to E \) such that

\[
\|g(x) - Q(x)\| \leq \frac{8.4^n - 1}{2} \delta, \quad x \in D. \tag{22}
\]

Similarly, as in the proof of Proposition 1, it follows from (21) that for each \( x \in D \)

\[
\left\| 2^{k-1}h\left(\frac{x}{2^{k-1}}\right) - 2^k h\left(\frac{x}{2^k}\right) \right\| \leq 2^{k-1} \delta, \quad k = 1, 2, \ldots, n.
\]

Therefore

\[
\left\| h(x) - 2^{n}h\left(\frac{x}{2^n}\right) \right\| \leq (2^n - 1) \delta, \quad x \in D. \tag{23}
\]

On the other hand, by Theorem 3, there exists an additive function \( A : \mathbb{R} \to E \) such that \( \| h(x) - A(x) \| \leq 5 \delta \) for all \( x \in (-c, c) \). Using the above inequalities, we get

\[
\| h(x) - A(x) \| \leq \left( h(x) - 2^n h\left(\frac{x}{2^n}\right) \right) + \left( 2^n h\left(\frac{x}{2^n}\right) - A(x) \right) \leq (6.2^n - 1) \delta, \quad x \in D.
\]

Since \( f = g + h \), the result follows from (22) and (23). Theorem 4 was generalized by Jung and Kim [4]. They proved the following result:

**Theorem 6.** Let \( E \) be a Banach space and let \( r, \delta > 0 \) be given constants. If a function \( f : [-r, r]^n \to E \) satisfies the inequality

\[
\| f(x + y) + f(x - y) - 2f(x) - 2f(y) \| \leq \delta
\]

for all \( x, y \in [-r, r]^n \) with \( x \pm y \in [-r, r]^n \), then there exists a quadratic function \( Q : \mathbb{R}^n \to E \) such that

\[
\| f(x) - Q(x) \| \leq (2912n^2 + 1872n + 334) \delta,
\]

for any \( x \in [-r, r]^n \).

4. **Stability of** \( f(px + (1 - p)y) + f((1 - p)x + py) = f(x) + f(y) \) **on bounded subsets of** \( \mathbb{R}^{N \geq 1} \) **for** \( p = \frac{1}{3} \) **and** \( p = \frac{1}{4} \)

In this section \( r > 0 \) and \( \delta \geq 0 \) are real numbers and we assume that \( E \) is a normed space. We will now start this section with the following lemma presented by Kominek [5] (see also [3]).

**Lemma 1.** Let \( E \) be a Banach space and let \( N \) be a positive integer. Suppose \( D \) is a bounded subset of \( \mathbb{R}^N \) containing zero in its interior. Assume, moreover, that there exist a nonnegative integer \( n \) and a positive number \( c > 0 \) such that

(i) \( D \subseteq 2D \),
(ii) \((-c, c)^N \subseteq D\),

(iii) \(D \subseteq (-2^n c, 2^n c)^N\).

If a function \(f : D \to E\) satisfies the functional inequality

\[
\|f(x + y) - f(x) - f(y)\| \leq \delta
\]

for some \(\delta \geq 0\) and for all \(x, y \in D\) with \(x + y \in D\), then there exists an additive function \(A : \mathbb{R}^N \to E\) such that

\[
\|f(x) - A(x)\| \leq (2^n 5N - 1)\delta, \quad x \in D.
\]

**Theorem 7.** Let \(f : (-r, r) \to E\) be a function with \(f(0) = 0\) and satisfy

\[
\left\|f\left(\frac{1}{3}x + \frac{2}{3}y\right) + f\left(\frac{2}{3}x + \frac{1}{3}y\right) - f(x) - f(y)\right\| \leq \delta, \quad x, y \in (-r, r).
\]

Then

\[
\left\|f(x + y) - f(x) - f(y)\right\| \leq 9\delta, \quad x, y \in \left(-\frac{2r}{9}, \frac{2r}{9}\right).
\]

**Proof.** Replacing \(x\) by \(3x\) and \(y\) by \(3y\) in (24), we have

\[
\|f(x + 2y) + f(2x + y) - f(3x) - f(3y)\| \leq \delta, \quad x, y \in \left(-\frac{r}{3}, \frac{r}{3}\right).
\]

By replacing \(x\) by \(\frac{2y - x}{3}\) and \(y\) by \(\frac{2x - y}{3}\) in (25), we get

\[
\|f(x) + f(y) - f(2x - y) - f(2y - x)\| \leq \delta, \quad x, y \in \left(-\frac{r}{3}, \frac{r}{3}\right).
\]

Replacing \(y\) by \(-y\) in (26), we have

\[
\|f(2x + y) + f(-2y - x) - f(x) - f(-y)\| \leq \delta, \quad x, y \in \left(-\frac{r}{3}, \frac{r}{3}\right).
\]

Replacing \(y = 0\) in (25), we infer

\[
\|f(x) + f(2x) - f(3x)\| \leq \delta, \quad x \in \left(-\frac{r}{3}, \frac{r}{3}\right),
\]

and replacing \(x\) by \(-x\) in (28), we have

\[
\|f(-x) + f(-2x) - f(-3x)\| \leq \delta, \quad x \in \left(-\frac{r}{3}, \frac{r}{3}\right).
\]

Letting \(y = -x\) in (25), we have

\[
\|f(-x) + f(x) - f(3x) - f(-3x)\| \leq \delta, \quad x \in \left(-\frac{r}{3}, \frac{r}{3}\right).
\]
Using (28), (29) and (30), we have \( \|f(2x) + f(-2x)\| \leq 3\delta \), for all \( x \in \left(-\frac{r}{3}, \frac{r}{3}\right) \). Therefore
\[
\|f(x) + f(-x)\| \leq 3\delta, \quad x \in \left(-\frac{2r}{3}, \frac{2r}{3}\right).
\] (31)
Putting \( y = -2x \) in (25), we get
\[
\|f(-3x) - f(3x) - f(-6x)\| \leq \delta, \quad x \in \left(-\frac{r}{6}, \frac{r}{6}\right). \] (32)
Using the triangle inequality, it follows from (31) and (32) that
\[
\|2f(-3x) - f(-6x)\| \leq 4\delta, \quad x \in \left(-\frac{r}{6}, \frac{r}{6}\right).
\]
Then
\[
\|2f(x) - f(2x)\| \leq 4\delta, \quad x \in \left(-\frac{r}{2}, \frac{r}{2}\right). \] (33)
It follows from (31) that \( \|f(-2y - x) + f(2y + x)\| \leq 3\delta \) for all \( x, y \in \left(-\frac{2r}{9}, \frac{2r}{9}\right) \). Hence (25), (27) and (28) imply
\[
\|2f(2x + y) - f(2x) - 2f(x) - f(2y) - f(y) - f(-y)\| \leq 7\delta, \quad x, y \in \left(-\frac{2r}{9}, \frac{2r}{9}\right). \]
Using this inequality and applying (31) and (33), we obtain
\[
\|f(2x + y) - f(2x) - f(y)\| \leq 9\delta, \quad x, y \in \left(-\frac{2r}{9}, \frac{2r}{9}\right). \] (34)
Then we have
\[
\|f(x + y) - f(x) - f(y)\| \leq 9\delta, \quad x, y \in \left(-\frac{2r}{9}, \frac{2r}{9}\right). \]
A similar argument as in the proof of Theorem 7 yields the following results in the case of functions defined on certain subsets of \( \mathbb{R}^N \) (\( N \) is a positive integer) with values in a normed space.

**Theorem 8.** Suppose that \( D \) is a symmetric and bounded subset of \( \mathbb{R}^N \) containing zero. Let \( f : D \to E \) be a function with \( f(0) = 0 \) and satisfy
\[
\|f\left(\frac{1}{3}x + \frac{2}{3}y\right) + f\left(\frac{2}{3}x + \frac{1}{3}y\right) - f(x) - f(y)\| \leq \delta, \] (35)
for some \( \delta \geq 0 \) and for all \( x, y \in D \) with \( 2x + y \in 3D \). Then
\[
\|f(x + y) - f(x) - f(y)\| \leq 9\delta, \quad x, y \in (2/9)D.
\]
COROLLARY 3. Let \( f : (-r, r)^N \rightarrow E \) be a function with \( f(0) = 0 \) and satisfy
\[
\|f\left(\frac{1}{3}x + \frac{2}{3}y\right) + f\left(\frac{2}{3}x + \frac{1}{3}y\right) - f(x) - f(y)\| \leq \delta, \quad x, y \in (-r, r)^N.
\]
Then
\[
\|f(x + y) - f(x) - f(y)\| \leq 9\delta, \quad x, y \in \left(-\frac{2r}{9}, \frac{2r}{9}\right)^N.
\]

Using Lemma 1 and Theorem 8 we prove the stability of the functional equation
\[
f\left(\frac{1}{3}x + \frac{2}{3}y\right) + f\left(\frac{2}{3}x + \frac{1}{3}y\right) = f(x) + f(y)
\]
on a restricted domain.

THEOREM 9. Let \( E \) be a Banach space and let \( f : (-r, r)^N \rightarrow E \) be a function with \( f(0) = 0 \) and satisfy (35) for all \( x, y \in (-r, r)^N \). Then there exists an additive function \( A : \mathbb{R}^N \rightarrow E \) such that
\[
\|f(x) - A(x)\| \leq 9(5N - 1)\delta, \quad x \in \left(-\frac{2r}{9}, \frac{2r}{9}\right)^N.
\]

THEOREM 10. Let \( E \) be a Banach space and let \( N \) be a positive integer. Suppose \( D \) is a symmetric and bounded subset of \( \mathbb{R}^N \) containing zero in its interior. Assume, moreover, that there exist a nonnegative integer \( n \) and a positive number \( c > 0 \) such that
\[
\begin{align*}
(i) & \quad D \subseteq 2D, \\
(ii) & \quad (-c, c)^N \subseteq D, \\
(iii) & \quad D \subseteq (-2^nc, 2^nc)^N.
\end{align*}
\]
If a function \( f : D \rightarrow E \) satisfies \( f(0) = 0 \) and the functional inequality
\[
\|f\left(\frac{1}{3}x + \frac{2}{3}y\right) + f\left(\frac{2}{3}x + \frac{1}{3}y\right) - f(x) - f(y)\| \leq \delta,
\]
for some \( \delta \geq 0 \) and for all \( x, y \in D \) with \( 2x + y \in 3D \), then there exists an additive function \( A : \mathbb{R}^N \rightarrow E \) such that
\[
\|f(x) - A(x)\| \leq 9(2^nN - 1)\delta, \quad x \in (2/9)D.
\]

Proof. Let \( G = (2/9)D \) and \( r = (2/9)c \). Then \( G \subseteq 2G, \ (-r, r)^N \subseteq G \) and \( D \subseteq (-2^nr, 2^nr)^N \). By Theorem 8, \( f \) satisfies
\[
\|f(x + y) - f(x) - f(y)\| \leq 9\delta, \quad x, y \in G.
\]
Therefore on account of Lemma 1, we get the result.
THEOREM 11. Let \( f : (-r, r) \to E \) be a function with \( f(0) = 0 \) and satisfy

\[
\left\| f\left(\frac{1}{4}x + \frac{3}{4}y\right) + f\left(\frac{3}{4}x + \frac{1}{4}y\right) - f(x) - f(y)\right\| \leq \delta, \quad x, y \in (-r, r).
\]

Then

\[
\left\| f(x+y) - f(x) - f(y)\right\| \leq 9\delta, \quad x, y \in \left(\frac{-3r}{16}, \frac{3r}{16}\right).
\]

Proof. Replacing \( x \) by 4\( x \) and \( y \) by 4\( y \) in (36), we have

\[
\left\| f(x+3y) + f(3x+y) - f(4x) - f(4y)\right\| \leq \delta, \quad x, y \in \left(-\frac{r}{4}, \frac{r}{4}\right).
\]

By replacing \( x \) by \( \frac{3y-x}{4} \) and \( y \) by \( \frac{3x-y}{4} \) in (37), we have

\[
\left\| f(2x) + f(2y) - f(3x-y) - f(3y-x)\right\| \leq \delta, \quad x, y \in \left(-\frac{r}{4}, \frac{r}{4}\right).
\]

If we replace \( y \) by \(-y \) in the last inequality, we obtain

\[
\left\| f(3x+y) + f(-3y-x) - f(2x) - f(-2y)\right\| \leq \delta, \quad x, y \in \left(-\frac{r}{4}, \frac{r}{4}\right).
\]

Putting \( x = 0 \) in (38), we get

\[
\left\| f(y) + f(-3y) - f(-2y)\right\| \leq \delta, \quad y \in \left(-\frac{r}{4}, \frac{r}{4}\right).
\]

Putting \( y = 0 \) in (37), we have

\[
\left\| f(x) + f(3x) - f(4x)\right\| \leq \delta, \quad x \in \left(-\frac{r}{4}, \frac{r}{4}\right).
\]

If we put \( y = -x \) in (37), we obtain

\[
\left\| f(-2x) + f(2x) - f(-4x) - f(4x)\right\| \leq \delta, \quad x \in \left(-\frac{r}{4}, \frac{r}{4}\right),
\]

and then

\[
\left\| f(-x) + f(x) - f(-2x) - f(2x)\right\| \leq \delta, \quad x \in \left(-\frac{r}{2}, \frac{r}{2}\right).
\]

It follows from (40) that

\[
\left\| f(-x) + f(x) + f(-3x) + f(3x) - f(-4x) - f(4x)\right\| \leq 2\delta, \quad x \in \left(-\frac{r}{4}, \frac{r}{4}\right).
\]

Hence we get from (42) and (43) that

\[
\left\| f(-2x) + f(2x) + f(-3x) + f(3x) - f(-4x) - f(4x)\right\| \leq 3\delta, \quad x \in \left(-\frac{r}{4}, \frac{r}{4}\right).
\]
Using the triangle inequality for (41) and (44), we obtain
\[ \|f(-3x) + f(3x)\| \leq 4\delta, \quad x \in \left(-\frac{r}{4}, \frac{r}{4}\right). \]  
\[(45)\]

Therefore
\[ \|f(-x) + f(x)\| \leq 4\delta, \quad x \in \left(-\frac{3r}{4}, \frac{3r}{4}\right), \]
\[ \|f(-3y-x) + f(3y+x)\| \leq 4\delta, \quad x, y \in \left(-\frac{3r}{16}, \frac{3r}{16}\right). \]  
\[(46)\]

Using the last inequality (46) and inequalities (37) and (38), we get
\[ \|2f(3x+y) - f(4x) - f(4y) - f(2x) - f(2y)\| \leq 6\delta, \quad x, y \in \left(-\frac{3r}{16}, \frac{3r}{16}\right). \]  
\[(47)\]

If we consider (40) with \(x\) and \(y\), then it follows by (47) that
\[ \|2f(3x+y) - f(3x) - f(3y) - f(x) - f(2x) - f(-2y)\| \leq 8\delta, \]
for all \(x, y \in \left(-\frac{3r}{16}, \frac{3r}{16}\right)\). Consider the inequality (39) for \(y\) and \(-x\), and using the above inequality, we obtain
\[ \|2f(3x+y) - 2f(3x) - f(3y) - f(-3y) - f(x) - f(-x) - 2f(y)\| \leq 10\delta, \]
for all \(x, y \in \left(-\frac{3r}{16}, \frac{3r}{16}\right)\). Hence this inequality with the inequalities (45) and (46) imply
\[ \|2f(3x+y) - 2f(3x) - 2f(y)\| \leq 18\delta, \quad x, y \in \left(-\frac{3r}{16}, \frac{3r}{16}\right). \]

Therefore
\[ \|f(x+y) - f(x) - f(y)\| \leq 9\delta, \quad x, y \in \left(-\frac{3r}{16}, \frac{3r}{16}\right). \]

By a similar way as in the proof of Theorem 11 we obtain the following results on restricted domains of \(\mathbb{R}^N\).

**Theorem 12.** Suppose that \(D\) is a symmetric and bounded subset of \(\mathbb{R}^N\) containing zero. Let \(f : D \to E\) be a function with \(f(0) = 0\) and satisfy
\[ \left\|f\left(\frac{1}{4}x + \frac{3}{4}y\right) + f\left(\frac{3}{4}x + \frac{1}{4}y\right) - f(x) - f(y)\right\| \leq \delta, \]

for some \(\delta \geq 0\) and for all \(x, y \in D\) with \(3x+y \in 4D\). Then
\[ \|f(x+y) - f(x) - f(y)\| \leq 9\delta, \quad x, y \in (3/16)D. \]
Let $f : (-r,r)^N \to E$ be a function with $f(0) = 0$ and satisfy
\[
\left\| f\left(\frac{1}{4}x + \frac{3}{4}y\right) + f\left(\frac{3}{4}x + \frac{1}{4}y\right) - f(x) - f(y) \right\| \leq \delta, \quad x, y \in (-r,r)^N. \tag{48}
\]
Then
\[
\left\| f(x+y) - f(x) - f(y) \right\| \leq 9\delta, \quad x, y \in \left(\frac{-3r}{16}, \frac{3r}{16}\right)^N.
\]

Using Lemma 1 and Theorem 13 we prove the stability of the functional equation
\[
f\left(\frac{1}{3}x + \frac{2}{3}y\right) + f\left(\frac{2}{3}x + \frac{1}{3}y\right) = f(x) + f(y)
\]
on a restricted domain.

**Theorem 14.** Let $E$ be a Banach space and let $f : (-r,r)^N \to E$ be a function with $f(0) = 0$ and satisfy (48) for all $x, y \in (-r,r)^N$. Then there exists an additive function $A : \mathbb{R}^N \to E$ such that
\[
\left\| f(x) - A(x) \right\| \leq 9(5N-1)\delta, \quad x, y \in \left(\frac{-3r}{16}, \frac{3r}{16}\right)^N.
\]

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