A NOTE ON GABRIEL DIMENSION FOR IDIOMS

ANGEL ZALDÍVAR CORICHI

ABSTRACT. The aim of this note is to illustrate that the definition and construction of the Gabriel dimension for modular lattices in the sense of [1] is the same as the module case in the document [2].

1. INTRODUCTION

The following definition is taken, literally, from [1], top of the page 135:

"Let be a modular upper-continuous lattice with 0 and 1. We define the Gabriel dimension of , denoted by Gdim(A), using transfinite recursion. We put Gdim(A) = 0 if and only if A = {0}. Let α be a nonlimit ordinal and assume that the Gabriel dimension Gdim(A') = β has already been defined for lattices with β < α. We say that A is α-simple if for each in we have Gdim[0, a] ≠ α and Gdim[a, 1] < α. We then say that Gdim(A) = α if Gdim(A) < α but for every in there exist a b > 0 such that [a, b] is β-simple for some β ≤ α."

In the second paragraph of the same page, it is stated the following:

"Consider a ∈ A. If Gdim(0, a) = α then we say that α is the Gabriel dimension of a and we write Gdim(a) = α. If [0, a] is α-simple then a is said to be an α-simple element of A."

We will rewrite this definition in the idiom context, mimicking the construction of Gabriel dimension, in the module category, given in [2]. Basically, the proofs are the same as in [2]. In fact, the two constructions are related via the slicing technique, for more details about the slicing and relation with dimension in module categories and lattices we refer to [2] and [6].

2. GABRIEL DIMENSION FOR IDIOMS

To begin with, fix an idiom A (that is a complete, modular, upper-continuous lattice ), let [a, b] = {x ∈ A | a ≤ x ≤ b} the interval of a ≤ b. Denote by 3(A) the set of all intervals of A and by the set of all trivial intervals, that is, for an element a ∈ A the trivial interval of it is [a, a] = {a}. Next we recall the definition of the Gabriel dimension for an idiom.

An interval [a, b] is simple if [a, b] = {a, b} observe now that this is equivalent to say:

An interval [a, b] is simple if for every a ≤ x ≤ b one has a = x or b = x

and immediately this is also equivalent to :

An interval [a, b] is simple if for every a ≤ x ≤ b one has [a, x] ∈ or [x, b] ∈ with this in mind the relative version of the relation simple is direct, that is, given a set of intervals B ⊆ 3(A) an interval [a, b] is B-simple if for every a ≤ x ≤ b one has [a, x] ∈ B or [x, b] ∈ B. Observe now that this produce an operation in the set all sets of intervals on
A more over this operation is defined in a particular kind of sets of intervals. As in the case
with module classes closed under certain kind of operations one introduce the following,
mimicking the module idea:

Given two intervals $I = [a, b]$ and $J = [a', b']$, we say that $I$ is a subinterval of $J$,
denoted by $I \rightarrow J$, if $I = [a, b]$ and $J = [a', b']$ with $a' \leq a \leq b \leq b'$ in $A$. We say that $J$
and $I$ are similar, denoted by $J \sim I$, if there are $l, r \in A$ with associated intervals
$$L = [l, l \lor r] \quad [l \land r, r] = R$$
where $J = L$ and $I = R$ or $J = R$ and $I = L$. Clearly, this a reflexive and symmetric
relation. Moreover, if $A$ is modular, this relation is just the canonical lattice isomorphism
between $L$ and $R$.

A set of intervals $A \subseteq J(A)$ is abstract if is not empty and it is closed under $\sim$, that is,
$$J \sim I \in A \Rightarrow J \in A.$$  

An abstract set $B$ is a basic set of intervals if it is closed by subintervals, that is,
$$J \rightarrow I \in B \Rightarrow J \in B$$
for all intervals $I, J$. A set of intervals $C$ is a congruence set if it is basic and closed under
abutting intervals, that is,
$$[a, b], [b, c] \in C \Rightarrow [a, c] \in C$$
for elements $a, b, c \in A$. A basic set of intervals $B$ is a pre-division set if
$$\forall x \in X \quad [a, x] \in B \Rightarrow [a, \bigvee X] \in B$$
for each $a \in A$ and $X \subseteq [a, 1]$. A set of intervals $D$ is a division set if it is a congruence
set and a pre-division set. Put $D(A) \subseteq C(A) \subseteq B(A) \subseteq A(A)$ the set of all division,
congruence, basic and abstract set of intervals in $A$. This gadgets can be understood like
certain classes of modules in a module category $R$-Mod, that is, classes closed under
isomorphism, subobjects, extensions and coproducts. From this point of view $C(A)$ and
$D(A)$ are the idioms analogues of the Serre classes and the torsion (localizations) classes
in module categories.

Is straightforward to see that $B(A)$ and $A(A)$ are frames also $D(A)$ and $C(A)$ are frames
too this is not directly, the details of these are in [5]. Let be $\text{Smp}(B)$ the set of all $B$-simples
intervals, this set is basic provided $B$ is a basic set. To define the gabriel dimension of an
idiom, specifically the Gabriel dimension of an interval we need to produce a filtration. This
filtration is related with the simples and with critical intervals that is, let be $B \in B(A)$ and
denote by $\text{Crit}(B)$ the set of intervals such that for all $a \leq x \leq b$ we have $a = x$ or
$[x, b] \in B$; this is the set of all $B$-critical intervals. Note that $\text{Smp}(\emptyset) = \text{Crit}(\emptyset)$ and
$\text{Crit}(B) \leq \text{Smp}(B)$.

As we mention before the set $D(A)$ is a frame in particular is a complete lattice therefore
for any basic set $B$ there exists the least division set that contains it $D_{\text{vs}}(B)$, this description
set up an operation in the frame of basic sets of intervals, that is, a function $\text{Kpr} : B(A) \rightarrow
B(A)$ such that $B \leq \text{Kpr}(B)$, $\text{Kpr}(B) \leq \text{Kpr}(A)$ whenever $B \leq A$ and $\text{Kpr}(\bigcap
A) = \text{Kpr}(B) \cap \text{Kpr}(A)$, this kind of functions are called pre-nucleus a nucleus is an
idempotent pre-nucleus. The $D_{\text{vs}}$ construction is a nucleus on $B(A)$ with this we can
set up $\text{Gab} := D_{\text{vs}} \circ \text{Crit}$ this is the Gabriel pre-nucleus of $A$ and one can prove that
$D_{\text{vs}} \circ \text{Crit} = \text{Gab} = D_{\text{vs}} \circ \text{Smp}$. We can iterate $\text{Gab}$ over all ordinals to obtain a chain
of division sets $\text{O}(A) \leq \text{Gab}(0) \leq \ldots \leq \text{Gab}^\alpha(0) \leq \ldots$ where $\text{Gab}^\alpha(0)$ is defined by
$\text{Gab}^{\alpha + 1}(0) := \text{Gab}(\text{Gab}^{\beta}(0))$ and $\text{Gab}^\alpha(0) = D_{\text{vs}}(\bigcup\{\text{Gab}^\beta(0) \mid \beta < \alpha\})$ for non-limit
and limit ordinals. Now with this filtration we can define the Gabriel dimension of an
interval \([a, b]\) to be the extended ordinal \(G(a, b) \leq \alpha\) if and only if \([a, b] \in \mathcal{G}ab^\alpha(0)\). The central objective of this short note is to illustrate that the construction of \([1]\) produce this filtration but with another point of view. For a more detail treatment of this construction and related topics with dimension and inflator theory the reader is refer to \([6, 7]\) and \([8]\).

**Definition 2.1.** We define the *Gabriel dimension*, \(\text{Gdim}\) of \([a, b]\) as follows:

1. \(\text{Gdim}(a, b) = 0 \iff a = b\).
2. \(\text{Gdim}(a, b) = \alpha' \iff \text{Gdim}(a, b) \not\in \alpha\) and
   \[(\forall a \leq x < b) \ [\exists x < y \leq b] \ [\exists \beta \leq \alpha'] \ [[x, y] \text{ is } \beta\text{-simple}] \text{,} \]
   for ordinals \(\alpha\) and \(\alpha'\) its successor.
3. \(\text{Gdim}(a, b) = \lambda \iff (\forall a \leq x < b) \ (\exists x < y \leq b) \ [\exists \beta < \lambda] \ [[x, y] \text{ is } \beta\text{-simple}] \text{,} \) for limit ordinals \(\lambda\).

Here, \(\beta\text{-simple}\) means that for the successor ordinal \(\beta\), the interval \([a, b]\) is \(\beta\text{-simple}\) if:

\[
(\forall a < x \leq b) [\text{Gdim}(a, x) \not\in \beta \text{ and } \text{Gdim}(x, b) < \beta]
\]

Following \([2]\), we say that the only \(0\text{-simple}\) and \(\lambda\text{-simple}\) intervals, for all limit ordinals \(\lambda\), are the trivial ones, that is, \(O(A)\). Then, condition (3) of Definition \(2.1\) is reinterpreted as:

\[
\text{Gdim}(a, b) = \lambda \iff (\forall a \leq x < b) \ (\exists x < y \leq b) \ [\exists \beta < \lambda] \ [[x, y] \text{ is } \beta\text{-simple}] \text{.}
\]

Next we make these definitions accumulative. Following \([2]\), define the set \(S[\alpha]\) of \(\alpha\text{-simple}\) intervals, with \(\alpha\) an ordinal, as

\[
[a, b] \in S[\alpha] \iff (\forall a < x \leq b) [\text{Gdim}(a, x) \not\in \alpha \text{ and } \text{Gdim}(x, b) \leq \alpha]\text{,}
\]

and then proceed step by step as follows:

1. \(\mathcal{D}(0) = O(A)\).
2. \(\mathcal{D}(\alpha') = \mathcal{D}(\alpha) \cup S[\alpha]\)
3. \(\mathcal{D}(\lambda) = \bigcup \{ \mathcal{D}(\alpha) \mid \alpha < \lambda \}\text{,}

for each ordinal \(\alpha\) and limit ordinal \(\lambda\).

In Definition \(2.1\) there is a (strange) quantification \((\exists \beta)\) in items (2) and (3). To deal with this quantification and make everything more clear, we introduce the following definitions:

**Definition 2.2.** For each \(C \subseteq I(A)\), set:

\[
[a, b] \in (\forall \exists)(C) \iff (\forall a \leq x < b) \ (\exists x < y \leq b) \ [[x, y] \in C]\text{.}
\]

Immediately one observes that, if \(C\) is basic then \((\forall \exists)(C) = \mathcal{D}_{vs}(C)\). Note also that the operator \((\forall \exists)(\cdot)\) is monotone. (For the details about the \(\mathcal{D}_{vs}\)-construction see \([5\text{, Theorem 5.6}]\) With this we redefine:

**Definition 2.3** (\(\mathcal{L}\)-construction). For each interval \([a, b]\) and for each ordinal \(\alpha\) and limit ordinal \(\lambda\), we set:

1. \([a, b] \in \mathcal{L}[0] \iff a = b\).
2. \([a, b] \in \mathcal{L}[\alpha'] \iff [a, b] \in (\forall \exists)(\mathcal{D}(\alpha')) \text{ and } [a, b] \notin \mathcal{L}(\alpha)\).
3. \([a, b] \in \mathcal{L}[\lambda] \iff [a, b] \in (\forall \exists)(\mathcal{D}(\lambda))\text{,}

where:

1. \(\mathcal{L}(0) = O(A)\)
2. \(\mathcal{L}(\alpha') = \mathcal{L}(\alpha) \cup \mathcal{L}[\alpha']\)
3. \(\mathcal{L}(\lambda) = \bigcup \{ \mathcal{L}(\alpha) \mid \alpha < \lambda \} \cup \mathcal{L}[\lambda]\text{,}
Theorem 2.6. Then, we must first show:

Proof.

Definition 2.7. \( \bigvee \) and \( \bigwedge \) and the aim of this note is to show that \( \mathbb{G}(a, b) = \alpha \) and \( \mathcal{L}(\alpha) = \bigcup \{ \mathcal{L}(\beta) \mid \beta \leq \alpha \} \) the set of intervals with Gabriel dimension \( \mathbb{G}(a, b) \leq \alpha \).

Lemma 2.4. For each ordinal \( \alpha \) we have

\[ \mathcal{L}(\alpha') = \mathcal{L}(\alpha) \cup (\forall \exists)(\mathcal{D}(\alpha')). \]

Proof. For each interval \([a, b]\) we have:

\[ [a, b] \in \mathcal{L}(\alpha') \iff [a, b] \in \mathcal{L}(\alpha) \text{ or } \mathcal{L}[\alpha'] \]

\[ \iff [a, b] \in \mathcal{L}(\alpha) \text{ or } ([a, b] \in (\forall \exists)(\mathcal{D}(\alpha')) \text{ and } [a, b] \notin \mathcal{L}(\alpha)) \]

\[ \iff [a, b] \in \mathcal{L}(\alpha) \text{ or } [a, b] \in (\forall \exists)(\mathcal{D}(\alpha')). \]

Definition 2.5 (Accumulative \( \mathcal{L} \)-construction). For each ordinal \( \alpha \) and limit ordinal \( \lambda \), introduce:

\[ \mathcal{L}(0) = \mathcal{O}(A) \quad \mathcal{D}(0) = \mathcal{O}(A) \]

\[ \mathcal{L}(\alpha') = \mathcal{L}(\alpha) \cup (\forall \exists)(\mathcal{D}(\alpha')) \quad \mathcal{D}(\alpha') = \mathcal{D}(\alpha) \cup \mathcal{S}[\alpha] \]

\[ \mathcal{L}(\lambda) = \bigcup \{ \mathcal{L}(\alpha) \mid \alpha < \lambda \} \cup (\forall \exists)(\mathcal{D}(\lambda)) \quad \mathcal{D}(\lambda) = \bigcup \{ \mathcal{D}(\alpha) \mid \alpha < \lambda \} \]

Where, again, in the step:

\[ [a, b] \in \mathcal{S}[\alpha] \iff (\forall \alpha < x \leq b) [[a, x] \notin \mathcal{L}(\alpha) \text{ and } [x, b] \in \mathcal{L}(\alpha)] \]

for each interval.

As Simmons says, this is getting easier to read, and the construction gives two ascending chains of sets of intervals

\[ \mathcal{L}(0) \subseteq \cdots \subseteq \mathcal{L}(\alpha) \subseteq \cdots \quad \text{and} \quad \mathcal{D}(0) \subseteq \cdots \subseteq \mathcal{D}(\alpha) \subseteq \cdots, \]

and the aim of this note is to show that \( \mathcal{L}(-) \) produces the Gabriel filtration in \( A \) for \( \mathcal{O}(A) \), that is,

\[ \mathcal{L}(\alpha) = \mathbb{G}(a, b^\alpha(\mathcal{O}(A))). \]

Then, we must first show:

Theorem 2.6. For each ordinal \( \alpha \), the collection \( \mathcal{L}(\alpha) \) is a division set in \( A \).

Proof. Clearly the set \( \mathcal{L}(\alpha) \) is an abstract set. Now, for the proofs of the basic congruences and \( \bigvee \)-closed properties, we invoke Proposition 3.4.1, Corollary 3.4.2, and 3.4.3 of \([1]\). \( \square \)

Definition 2.7. For each ordinal \( \alpha \) let be

\[ \mathcal{E}(\alpha) = \mathcal{L}(\alpha) \cup \mathcal{S}[\alpha] \]

where \( \mathcal{S}[\alpha] \) is the set of all \( \alpha' \)-simple intervals.

Lemma 2.8. For each ordinal \( \alpha \),

\[ \mathcal{E}(\alpha) = \mathcal{E}(\mathcal{L}(\alpha)). \]
Theorem 2.10. Theorem 2.10. Clear from the fact that $L(\alpha)$ is in particular basic. If $[a, b] \in S[\alpha]$, consider $a \leq x \leq b$; thus by definition of this set we have that $[x, b] \in L(\alpha)$.

Reciprocally, if $[a, b] \in S[\alpha]$ there is nothing to prove. Thus, suppose $[a, b] \notin S[\alpha]$, then there is a $a < x \leq b$ such that $[a, x] \in L(\alpha)$ or $[x, b] \notin L(\alpha)$. But the condition says that $[x, b] \in L(\alpha)$ and $L(\alpha)$ is a congruence set, thus $[a, b] \in L(\alpha)$.

Proposition 2.9. We have: $D(\alpha) \subseteq L(\alpha)$ for each ordinal $\alpha$.

Proof. By induction, the case $\alpha = 0$ being obvious because, $D(0) = O(A) = L(0)$ by definition of these sets. For the step $\alpha \mapsto \alpha'$, suppose that $[a, b] \in D(\alpha')$. The definition of this set gives two possibilities: First, if $[a, b] \in D(\alpha)$ then from the induction hypothesis $[a, b] \in L(\alpha) L(\alpha')$. Now, if $[a, b] \notin L(\alpha)$ then $[a, b] \in S[\alpha]$ and in this case we will show that $[a, b] \in (\forall \exists)(D(\alpha'))$ and using $(\forall \exists)(D(\alpha')) \subseteq L(\alpha')$, we will be done. To prove our claim, consider $a \leq x < b$. We will produce a $x < y \leq b$ with $[x, y] \in D(\alpha')$ and show that $y = b$ is the required element. If $a = x$, there is nothing to prove. If $a \neq x$ then $[a, b] \in S[\alpha]$ gives $[a, x] \notin L(\alpha)$ and $[x, b] \in L(\alpha)$. If $[a, x] \notin L(\alpha)$, the induction hypothesis gives $[x, b] \in L(\alpha) \subseteq D(\alpha) \subseteq D(\alpha')$, and we are done.

Now, for the limit case $\lambda$ we have $D(\lambda) = \bigcup \{D(\alpha) \mid \alpha < \lambda\} \subseteq \bigcup \{L(\lambda) \mid \alpha < \lambda\} \subseteq L(\lambda)$, where the inclusion $\bigcup \{D(\alpha) \mid \alpha < \lambda\} \subseteq \bigcup \{L(\lambda) \mid \alpha < \lambda\}$ is by the induction hypothesis.

From Proposition 2.9 Lemma 2.8 and Definition 2.7 it follows that $L(\alpha) \cup S[\alpha] = C(\alpha) = Crt(L(\alpha))$.

From the fact that $C(\alpha)$ is basic upon applying $Gab$ we have $G(L(\alpha)) = Dvs(C(\alpha)) = (\forall \exists)(C(\alpha))$ since the two operators $Dvs$ and $(\forall \exists)$ agree on basic sets. All this is summarized in the following

Theorem 2.10. With the above notation we have $Gabol(\alpha) = L(\alpha')$ for each ordinal $\alpha$.

Proof. From Proposition 2.9 and the definition of $D(\alpha')$ we have $C(\alpha) = L(\alpha) \cup S[\alpha] \subseteq L(\alpha) \cup D(\alpha') \subseteq L(\alpha')$. It follows that $Gabol(\alpha) = Dvs(C(\alpha)) \subseteq L(\alpha')$ by the remark before this theorem and the fact that $L(\alpha')$ is a division set. For other inclusion we have $D(\alpha') = D(\alpha) \cup S[\alpha] \subseteq L(\alpha) \cup S[\alpha] = C(\alpha)$ again by Proposition 2.9. From the monotonicity of $(\forall \exists)(\_\_)$ it follows that $(\forall \exists)(D(\alpha')) \subseteq (\forall \exists)(C(\alpha)) = Gabol(\alpha)$, and then $L(\alpha') = L(\alpha) \cup (\forall \exists)(D(\alpha')) \subseteq Gabol(\alpha)$ since $Gabol$ is an inflator.

We can now prove the main result of this note:
Theorem 2.11. With the same notation we have
\[ \mathcal{L}(\alpha) = \mathcal{G}ab^\alpha(\emptyset) \]
for each ordinal \( \alpha \). Here \( \emptyset = O(A) \).

Proof. By induction on \( \alpha \), the base case \( \alpha = 0 \), being clear. The induction step is just Theorem 2.10. For the limit case \( \lambda \) let \( \mathcal{L} = \bigcup \{ \mathcal{L}(\alpha) \mid \alpha < \lambda \} \). Since \( \mathcal{L}(\alpha) \) is basic for each ordinal, then \( \mathcal{L} \) is also basic. Thus, the induction hypothesis gives
\[ \mathcal{G}ab^\lambda(\emptyset) = \mathcal{D}vs(\mathcal{L}), \]
and by the accumulative \( \mathcal{L} \)-construction
\[ \mathcal{L}(\lambda) = \mathcal{L} \cup (\forall \exists)(\mathcal{D}(\lambda)) \]
and
\[ (\forall \exists)(\mathcal{D}(\lambda)) = (\forall \exists)(\bigcup \{ \mathcal{D}(\alpha) \mid \alpha < \lambda \}) \]
\[ = (\forall \exists)(\bigcup \{ \mathcal{D}(\alpha') \mid \alpha < \lambda \}) \subseteq (\forall \exists)(\bigcup \{ \mathcal{L}(\alpha') \mid \alpha < \lambda \}) = (\forall \exists)(\mathcal{L}) \]
\[ = \mathcal{D}vs(\mathcal{L}) \]
where the first equality is the definition of \( \mathcal{D}(\lambda) \) in the limit case, the second equality is because the construction \( \mathcal{D}(\cdot) \) is an ascending chain. The inclusion in the second row is from Theorem 2.10 and the monotonicity of \( (\forall \exists)(\cdot) \). The last equality is because the operators \( \mathcal{D}vs \) and \( (\forall \exists) \) agree on basic sets. Finally, with this and the description of \( \mathcal{L}(\cdot) \) in the limit case we conclude that
\[ \mathcal{L}(\lambda) = \mathcal{L} \cup \mathcal{D}vs(\mathcal{L}) = \mathcal{D}vs(\mathcal{L}) = \mathcal{G}ab^\lambda(\emptyset) \]
\[ \square \]

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E-mail address: zaldivar@matem.unam.mx

INSTITUTO DE MATEMÁTICAS, CIUDAD UNIVERSITARIA, UNAM, MÉXICO, D. F., 04510, MÉXICO.