GESPAR: Efficient Phase Retrieval of Sparse Signals
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Abstract

We consider the problem of one dimensional (1D) phase retrieval, namely, recovery of a 1D signal from the magnitude of its Fourier transform. This problem is ill-posed since the Fourier phase information is lost. Therefore, prior information on the signal is needed in order to recover it. In this work we consider the case in which the prior information on the signal is that it is sparse, i.e., it consists of a small number of nonzero elements. We propose a fast local search method for recovering a sparse 1D signal from measurements of its Fourier transform magnitude. Our algorithm does not require matrix lifting, unlike previous approaches, and therefore is potentially suitable for large scale problems such as images. Simulation results indicate that the proposed algorithm is fast and more accurate than existing techniques.

I. INTRODUCTION

Recovery of a signal from the magnitude of its Fourier transform, also known as phase retrieval, is of great interest in applications such as optical imaging [1], crystallography [2], and more [3]. Due to the loss of Fourier phase information, the problem (in 1D) is generally ill-posed. A common approach to overcome this ill-posedness is to exploit prior information on the signal. A variety of methods have been developed that use such prior information, which may be the signal’s support (region in which the signal is nonzero), non-negativity, or the signal’s magnitude [4], [5]. A popular class of algorithms is based on the use of alternate projections between the different constraints. In order to increase the probability of correct recovery, these methods require the prior information to be very precise, for example, exact/or “almost” exact knowledge of the support set. Since the projections are generally not onto convex sets, convergence to a correct recovery is not guaranteed [6]. A more recent approach is to use matrix-lifting of the problem which allows to recast phase retrieval as a semi-definite programming (SDP) problem [7]. The algorithm developed in [7] does not require prior information about the signal but instead uses multiple signal measurements (e.g., using different illumination settings, in an optical setup).

In order to obtain more robust recovery without requiring multiple measurements, we develop a method that exploits signal sparsity. Existing approaches aimed at recovering sparse signals from their Fourier magnitude belong to two main categories: SDP-based techniques [8],[9],[10],[11] and algorithms that use alternate projections (Fienup-type methods) [12]. Phase retrieval of sparse signals can be viewed as a special case of the more general quadratic compressed sensing (QCS) problem considered in [8]. Specifically, QCS treats recovery of sparse vectors from quadratic measurements of the form \( y_i = x^T A_i x \), \( i = 1, \ldots, N \), where \( x \) is the unknown sparse vector to be recovered, \( y_i \) are the measurements, and \( A_i \) are known matrices. In (discrete) phase retrieval, \( A = F_i^T F_i \) where \( F_i \) is the \( i \)th row of the discrete Fourier transform (DFT) matrix. QCS is encountered, for example, when imaging a sparse object using partially spatially-incoherent illumination [8].

A general approach to QCS was developed in [8] based on matrix lifting. More specifically, the quadratic constraints where lifted to a higher dimension by defining a matrix variable \( X = xx^T \). The problem was then recast as an SDP involving minimization of the rank of the lifted matrix subject to the recovery constraints as

\[
\text{subject to } \| A_i X - y_i \|_F \leq \epsilon_i, \quad i = 1, \ldots, N.
\]

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well as row sparsity constraints on \( \mathbf{X} \). An iterative thresholding algorithm based on a sequence of SDPs was then proposed to recover a sparse solution. Similar SDP-based ideas were recently used in the context of phase retrieval \[9,10\]. However, due to the increase in dimension created by the matrix lifting procedure, the SDP approach is not suitable for large-scale problems.

Another approach for phase retrieval of sparse signals is adding a sparsity constraint to the well-known iterative error reduction algorithm of Fienup \[12\]. In general, Fienup-type approaches are known to suffer from convergence issues and often do not lead to correct recovery especially in 1D problems; simulation results show that even with the additional information that the input is sparse, convergence is still problematic and the algorithm often recovers erroneous solutions.

In this paper we propose an efficient method for phase retrieval which also leads to good recovery performance. Our algorithm is based on a fast 2-opt local search method (see \[13\] for an excellent introduction to such techniques) applied to a sparsity constrained non-linear optimization formulation of the problem. We refer to our algorithm as GESPAR: GrEedy Sparse PhAse Retrieval. Sparsity constrained nonlinear optimization problems have been considered recently in \[14\]; the method derived in this paper is motivated – although different in many aspects – by the local search-type techniques of \[14\]. In essence, GESPAR is a local-search method, where the support of the sought signal is updated iteratively, according to selection rules described in detail in Section \( \text{III} \). A local minimum of the objective function is then found given the current support using the damped Gauss Newton algorithm. We demonstrate through numerical simulations that the proposed algorithm is both efficient and more accurate than current techniques.

The remainder of the paper is organized as follows. We formulate the problem in Section \( \text{II} \). Section \( \text{III} \) describes our proposed algorithm in detail. Numerical performance is illustrated in Section \( \text{IV} \).

### II. Problem Formulation

We are given a vector of measurements \( \mathbf{y} \in \mathbb{R}^N \), that corresponds to the magnitude of an \( N \) point DFT of a vector \( \mathbf{x} \in \mathbb{R}^N \), i.e.:

\[
y_l = \left| \sum_{m=1}^{n} x_m e^{-\frac{2\pi i (m-1)(l-1)}{N}} \right|, \quad l = 1, \ldots, N, \tag{1}
\]

where \( \mathbf{x} \) is constructed by zero padding of a vector \( \bar{\mathbf{x}} \in \mathbb{R}^n \) with elements \( x_i, \ i = 1,2,\ldots,n \). In the simulations section we consider the setting \( N = 2n \) which corresponds to oversampling the DFT of \( \bar{\mathbf{x}} \) by a factor of 2. More generally, we assume that \( N \geq 2n - 1 \). This allows to determine the correlation sequence of \( \mathbf{x} \) from the given measurements, as we elaborate on further below. Denoting by \( \mathbf{F} \in \mathbb{C}^{N \times N} \) the DFT matrix with elements \( \exp \{ -\frac{2\pi i (m-1)(l-1)}{N} \} \), we can express \( \mathbf{y} \) as \( \mathbf{y} = |\mathbf{F}\mathbf{x}| \), where \( | \cdot | \) denotes the element-wise absolute value. The vector \( \mathbf{x} \) is known to be \( s \)-sparse, that is, it contains at most \( s \) nonzero elements, and in addition its last \( n \) elements are zero. Our goal is to recover \( \mathbf{x} \) given the measurements \( \mathbf{y} \) and the sparsity level \( s \).

The mathematical formulation of the problem that we consider consists of minimizing the sum of squared errors subject to the sparsity constraint:

\[
\begin{align*}
\min_{\mathbf{x}} & \quad \sum_{i=1}^{N} (|\mathbf{F}_i\mathbf{x}|^2 - y_i^2)^2 \\
\text{s.t.} & \quad \|\mathbf{x}\|_0 \leq s, \\
& \quad \text{supp}(\mathbf{x}) \subseteq \{1,2,\ldots,n\}, \\
& \quad \mathbf{x} \in \mathbb{R}^N,
\end{align*}
\tag{2}
\]

where \( \mathbf{F}_i \) is the \( i \)th row of the DFT matrix \( \mathbf{F} \), and \( \| \cdot \|_0 \) stands for the zero-“norm”, that is, the number of nonzero elements. Note that the unknown vector \( \mathbf{x} \) can only be found up to trivial degeneracies that are the result of the loss of Fourier phase information: circular shift, global phase, and signal “mirroring”.

To aid in solving the phase retrieval problem we will rely on the fact that the correlation sequence of \( \bar{\mathbf{x}} \) (the first \( n \) components of \( \mathbf{x} \)) can be determined from \( \mathbf{y} \). Specifically, let \( g_m = \sum_{i=1}^{n} x_i x_{i+m}, m = -(n-1), \ldots, n-1 \) denote the correlation sequence. Note that \( \{g_m\} \) is a sequence of length \( 2n - 1 \). Since the DFT length \( N \) satisfies \( N \geq 2n - 1 \), we can obtain \( \{g_m\} \) by the inverse DFT of the squared Fourier magnitude \( \mathbf{y} \). Throughout the paper, we assume that no support cancelations occur in \( \{g_m\} \), namely, if \( x_i \neq 0 \) and \( x_j \neq 0 \) for some \( i,j \), then \( g_{i-j} \neq 0 \). When the values of \( \mathbf{x} \) are random, this is true with probability 1. This fact is used in GESPAR in order to obtain
initial information on the support of $x$.

The information on the support is used to derive two sets, $J_1$ and $J_2$ from the correlation sequence $\{g_m\}$ in the following manner. We denote by $J_1$ the set of indices known in advance to be in the support, from the autocorrelation sequence. To derive the set $J_1$, note that due to the existing degree of freedom relating to shift-invariance of $x$, the index $1$ can be assumed to be in the support, thereby removing this degree of freedom; as a consequence, the index corresponding to the last nonzero element in the autocorrelation sequence is also in the support, i.e.

$$i_{\text{max}} = 1 + \arg\max_i \{i : g_i \neq 0\}.$$ 

Therefore, $J_1 = \{1, i_{\text{max}}\}$.

Next, we denote by $J_2$ the set of indices that are candidates for being in the support, meaning the indices that are not known in advance to be in the off-support (the complement of the support). Specifically, $J_2$ contains the set of all indices $k \in \{1, 2, \ldots, n\}$ such that $g_{k-1} \neq 0$. Obviously, since we assume that $x_k = 0$ for $k > n$, we have $J_2 \subseteq \{1, 2, \ldots, n\}$. As a concrete example, consider the signal $x = (2, 0, 0, -1, 0, 1.5)^T$. The corresponding 11 point autocorrelation function $g_m$ is then given by $g_m = (-3, 0, -2, 1.5, 0, 7.25, 0, 1.5, -2, 0, -3)^T$. The set $J_1$ is therefore $J_1 = \{1, 6\}$. Next, by examining the zeros of $g_m$, and using our assumption of no support-cancelations, we deduce that there are no two non-zero elements in $x$, $x_i \neq 0$ and $x_j \neq 0$ such that $|i-j| = 1, 4$. Therefore, forcing the first element in $x$ to be non-zero, which removes the shift-invariance degeneracy, immediately implies that $x_2 = x_5 = 0$. In this way $J_2$ is determined as $J_2 = \{1, 3, 4, 6\}$. Defining $A_i = R(F_i)^T R(F_i) + \beta \bar{A}_i 3(F_i)^T 3(F_i) \in \mathbb{R}^N \times \mathbb{R}^N$ and $c_i = y_i^2$ for $i = 1, 2, \ldots, N$, problem (2) along with the support information can be written as

$$\begin{align*}
\min_x & \quad f(x) \equiv \sum_{i=1}^N (x^T A_i x - c_i)^2 \\
\text{s.t.} & \quad \|x\|_0 \leq s, \\
& \quad J_1 \subseteq \text{supp}(x) \subseteq J_2, \\
& \quad x \in \mathbb{R}^N,
\end{align*}$$

which will be the formulation to be studied.

In the next section, we propose GESPAR - an iterative local-search based algorithm for solving (3). We note that although in the context of phase retrieval the parameters $A_i, J_1, J_2$ have special properties (e.g., $A_i$ is positive semidefinite of at most rank 2, $|J_1| = 2$), we will not use these properties in GESPAR. Therefore, our approach is capable of handling general instances of (3) with the sole assumption that $A_i$ is symmetric for any $i = 1, 2, \ldots, N$.

### III. Greedy Sparse Phase Retrieval (GESPAR)

#### A. The Damped Gauss-Newton Method

Before describing our algorithm, we begin by presenting the damped Gauss-Newton (DGN) method [15], [16] that is in fact the core step of our approach. The DGN method is invoked in order to solve the problem of minimizing the objective function $f$ over a given support $S \subseteq \{1, 2, \ldots, n\}$ ($|S| = s$): \n
$$\min \{ f(U_S z) : z \in \mathbb{R}^s \},$$

where $U_S \in \mathbb{R}^{n \times s}$ is the matrix consisting of the columns of the identity matrix $I_N$ corresponding to the index set $S$. With this notation, (4) can be explicitly written as

$$\min \left\{ g(z) \equiv \sum_{i=1}^N (z^T U_S^T A_i U_S z - c_i)^2 : z \in \mathbb{R}^s \right\}.$$ \n
Problem (5) is a nonlinear least-squares problem. A natural approach for tackling it is via the DGN iterations. This algorithm begins with an arbitrary vector $z_0$. In our simulations, we choose it to be an uncorrelated random Gaussian vector with zero mean and unit variance. At each iteration, all the terms inside the squares in $g(z)$ are linearized around the previous guess. Namely, we write $g(z)$ from (5) as:

$$g(z) = \sum_{i=1}^N h_i^2(z)$$
Algorithm 1 DGN for solving (8)

Input: \( (A_i, c_i, S, \varepsilon, L) \).
\( A_i \in \mathbb{R}^{N \times N}, i = 1, 2, \ldots, N \) - symmetric matrices.
\( c_i \in \mathbb{R}, i = 1, 2, \ldots, N \).
\( S \subseteq \{1, 2, \ldots, n\} - index\ set. \)
\( \varepsilon - stopping criteria parameter. \)
\( L - maximum\ allowed\ iterations. \)

Output: \( z - an\ optimal\ (or\ suboptimal)\ solution\ of\ (5). \)

Initialization: Set \( B_i = U_S^T A_i U_S, t_0 = 0.5, z_0 \) a random vector.

**General Step** \( k(k \geq 1): \) Given the iterate \( z_{k-1} \), the next iterate is determined as follows:
1. **Gauss-Newton Direction:** Let \( y_k \) be the solution of the linear least squares problem \( (8) \) given by:
   \[
y_k = (M^T M)^{-1} M^T b
   \]
   with the \( i \)th row of \( M \) being \( M_i = 2 (B_i z_{k-1})^T \), and \( b_i = c_i + z_{k-1}^T B_i z_{k-1} \) for \( i = 1, 2, \ldots, N \). The Gauss-Newton direction is \( d_k = y_k - z_{k-1} \).
2. **Stepsize Selection via Backtracking:** set \( u = \min\{2t_{k-1}, 1\} \). Choose a stepsize \( t_k \) as \( t_k = (\frac{1}{2})^m u \), where \( m \) is the minimal nonnegative integer for which \( g(z_{k-1} + (\frac{1}{2})^m u d_k) < g(z_{k-1}) \).
3. **Update:** set \( z_k = z_{k-1} + t_k d_k \).
4. **Stopping rule:** STOP if either \( \|z_k - z_{k-1}\| < \varepsilon \) or \( k > L \).

, with \( h_i(z) = z^T B_i z - c_i \), where \( B_i = U_S^T A_i U_S \). At each step we replace \( h_i \) by its linear approximation around \( z_{k-1} \):

\[
h_i \approx h_i(z_{k-1}) + \nabla h_i(z_{k-1})^T (y - z_{k-1}) = z_{k-1}^T B_i z_{k-1} - c_i + 2 (B_i z_{k-1})^T (y - z_{k-1})
\]

(7)

We then choose \( y_k \) to be the solution of the problem

\[
\min_y \sum_{i=1}^{N} (z_{k-1}^T B_i z_{k-1} - c_i + 2 (B_i z_{k-1})^T (y - z_{k-1}))^2.
\]

(8)

Problem \( (8) \) can be written as a linear least squares problem

\[
y_k = \arg\min \|M y - b\|^2_2
\]

(9)

with the \( i \)th row of \( M \) being \( M_i = 2 (B_i z_{k-1})^T \), and \( b_i = c_i + z_{k-1}^T B_i z_{k-1} \) for \( i = 1, 2, \ldots, N \). The solution \( y_k \) is given by \( M_i = (M^T M)^{-1} M^T b \). We then define a direction vector as \( d_k = y_k - z_{k-1} \). This direction is used to update the solution with an appropriate stepsize designed to guarantee the convergence of the method to a stationary point of \( g(z) \). The stepsize is chosen via a simple backtracking procedure. Note that the method requires \( M \) to be of a full column rank. In our numerical experiments we encountered cases in which this assumption was not valid, and in these cases, our implementation chose one of the optimal solutions of the corresponding least squares problem. We noticed that these cases had negligible effect on the results.

Algorithm 1 describes the DGN method in detail. In our implementation the stopping parameters were chosen as \( \varepsilon = 10^{-4} \) and \( L = 100 \).

B. The 2-opt Local Search Method

The GESPAR method consists of repeatedly invoking a local-search method on an initial random support set. In this section we describe the local search procedure. At the beginning, the support is chosen to be a set of \( s \)
with different measurements. Namely, the objective function used is actually chosen as $f$ is the incorporation of random weights added to the objective function, giving randomly different weights to the method is given in Algorithm 3. One element of our specific implementation that is not described in Algorithm 3.

The 2-opt method can have the tendency to get stuck at local optima points. Therefore, our final algorithm, which we call GESPAR, is a restarted version of 2-opt. The 2-opt method is repeatedly invoked with different initial random support sets until the resulting objective function value is smaller than a certain threshold (success) or the number of maximum allowed total number of swaps was passed (failure). A detailed description of the method is given in Algorithm 3. One element of our specific implementation that is not described in Algorithm 3 is the incorporation of random weights added to the objective function, giving randomly different weights to the different measurements. Namely, the objective function used is actually chosen as $f(x) = \sum_{i=1}^{N} w_i \cdot (x^T A_i x - c_i)^2$ with $w_i = 1$ or 2 with equal probability. The random generation of weights is done each time the DGN procedure

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**Algorithm 2 2-opt**

**Input:** $(A_i, c_i, M, P)$.
$A_i \in \mathbb{R}^{N \times N}, i = 1, 2, \ldots, N$ - symmetric matrices.
$c_i \in \mathbb{R}, i = 1, 2, \ldots, N$.
$M, P$ - positive integers.

**Output:** $x$ - a suggested solution for problem (3).
$T$ - total number of required swaps.

1) **Initialization:**
   a) Set $T = 0$.
   b) Generate a random index set $S_0(|S_0| = s)$ satisfying the support constraints ($J_1 \subseteq S_0 \subseteq J_2$).
   c) Invoke the DGN method with parameters $(A_i, c_i, S_0, 10^{-4}, 100)$ and obtain an output $z_0$. Set $x_0 = U_{S_0} z_0$.

2) **General Step** ($k = 1, 2, \ldots$):
   a) Let $D$ be the set of indices from $S_{k-1} \setminus J_1$ corresponding to the $M$ components of $x_{k-1}$ with the smallest absolute value. Let $E$ be the set of indices from $S_{k-1}^c \cap J_2$ corresponding to the $P$ components of $\nabla f(x_{k-1})$ with the highest absolute value.
   b) Set $\tilde{S} = S_{k-1}$. For each $i \in D$ and for each $j \in E$ make a swap between the indices $\tilde{S} = (S_{k-1} \setminus \{i\}) \cup \{j\}$.
      Invoke DGN with input $(A_i, c_i, \tilde{S}, 10^{-4}, 100)$ and obtain an output $\tilde{z}$. Set $\tilde{x} = U_{\tilde{S}} \tilde{z}$. Advance $T$: $T \leftarrow T + 1$.
      If $f(\tilde{x}) < f(x_{k-1})$, then set $S_k = \tilde{S}, x_k = \tilde{x}$, advance $k$ and goto 2.a.
   c) If none of the swaps resulted with a better objective function value, then STOP. The output is $x = x_{k-1}$ and $T$.

random indices chosen to satisfy the support constraints $J_1 \subseteq S \subseteq J_2$. Then, at each iteration a swap between a support and an off-support index is performed such that the resulting solution via the DGN method improves the objective function. Since at each iteration only two elements are changed (one in the support and one in the off-support), this is a so-called “2-opt” method (see [13]). The swaps are always chosen to be between the $M$ support indices corresponding to components in the current iterate $x_{k-1}$ with the smallest absolute value and the off-support indices corresponding to the $P$ components of $\nabla f(x_{k-1}) = 4 \sum_i (x_{k-1}^T A_i x_{k-1} - c_i) A_i x_{k-1}$ with the largest absolute value. This process continues as long as the objective function decreases and stops when no improvement can be made.

A detailed description of the method is given in Algorithm 2. Note that the maximum number of swaps at each iteration of the method is $MP$. In our implementation the parameters were chosen to be $M = 4, P = 8$, which proved empirically to work well. We also note that the order of the elements in the indices set $D$ and $E$ was chosen to be random.

**C. The GESPAR Algorithm**

The 2-opt method can have the tendency to get stuck at local optima points. Therefore, our final algorithm, which we call GESPAR, is a restarted version of 2-opt. The 2-opt method is repeatedly invoked with different initial random support sets until the resulting objective function value is smaller than a certain threshold (success) or the number of maximum allowed total number of swaps was passed (failure). A detailed description of the method is given in Algorithm 3. One element of our specific implementation that is not described in Algorithm 3 is the incorporation of random weights added to the objective function, giving randomly different weights to the different measurements. Namely, the objective function used is actually chosen as $f(x) = \sum_{i=1}^{N} w_i \cdot (x^T A_i x - c_i)^2$ with $w_i = 1$ or 2 with equal probability. The random generation of weights is done each time the DGN procedure
Algorithm 3 GESPAR

**Input:** \((A_i, c_i, \tau, \text{ITER})\).

- \(A_i \in \mathbb{R}^{N \times N}, i = 1, 2, \ldots, N\) - symmetric matrices.
- \(c_i \in \mathbb{R}, i = 1, 2, \ldots, N\).
- \(\tau\) - threshold parameter.
- \(\text{ITER}\) - Maximum allowed total number of swaps.

**Output:** \(x\) - an optimal (or suboptimal) solution of (3).

**Initialization.** Set \(C = 0, k = 0\).

- **Repeat**
  - Invoke the 2-opt method with input \((A_i, c_i, 4, 8)\) and obtain an output \(x\) and \(T\). Set \(x_k = x, C = C + T\) and advance \(k\): \(k \leftarrow k + 1\).
  - **Until** \(f(x) < \tau\) or \(C > \text{ITER}\).
- The output is \(x_\ell\) where \(\ell = \arg\min_{m=0,1\ldots k-1} f(x_m)\).

is invoked. We observed that this modification reduced the probability of the 2-opt procedure to get stuck in non-optimal points.

**IV. NUMERICAL SIMULATIONS**

In order to demonstrate the performance of GESPAR, we conduct several numerical simulations. The algorithm is compared to other existing methods, and is evaluated in terms of signal-recovery accuracy, computational efficiency, and robustness to noise.

**A. Signal-recovery Accuracy**

In this subsection we examine the accuracy of GESPAR as a function of sparsity - i.e. the recovery success rate as a function of the number of non-zero elements in the signal. A runtime comparison of the compared methods is also performed.

1) **Simulation details:** We choose \(\bar{x}\) as a random vector of length \(n\). The vector contains uniformly distributed values in the range \([-4, -3] \cup [3, 4]\) in \(s\) randomly chosen elements. The \(N\) point DFT of the signal is calculated, and its magnitude is taken as \(y\), the vector of measurements. The \(2n - 1\) point correlation is also calculated. In order to recover the unknown vector \(x\), the GESPAR algorithm is used with \(\tau = 10^{-4}\) and \(T = 20000\), as well as two other algorithms for comparison purposes: An SDP based algorithm (Algorithm 2, [9]), and an iterative Fienup algorithm with a sparsity constraint [12]. In our simulation \(n = 64\) and \(N = 128\). The Sparse-Fienup algorithm is run using 100 random initial points, out of which the chosen solution is the that best matches the measurements. Namely, \(\hat{x}\) is selected as the \(s\) sparse output of the Sparse-Fienup algorithm with the minimal cost \(f(x) = \sum_{i=1}^{N} (|F_i x|^2 - y_i^2)^2\) out of the 100 runs.

2) **Simulation Results:** Signal recovery results of the numerical simulation are shown in Fig. 1 where the probability for successful recovery is plotted for different sparsity levels. Successful recovery probability is defined as the ratio of correctly recovered signals \(x\) out of 100 signal-simulations. In each simulation both the support and the signal values are randomly selected. The three algorithms (GESPAR, SDP and Sparse-Fienup) are compared. The results clearly show that GESPAR outperforms the other algorithms in terms of probability of successful recovery - over 90% successful recovery up to \(s = 15\), vs. \(s = 8\) and \(s = 7\) in the other two algorithms.

Average runtime comparison of the three algorithms is shown in Table 1. The runtime is averaged over all successful recoveries. In all three algorithms the same signal sizes were used - i.e. \(n = 64\), \(N = 128\). The computer used to solve all problems has an intel i5 CPU and 4GB of RAM. As seen in the table, the SDP based algorithm is significantly slower than the other two, and the Fienup based algorithm, although fast, leads to a much lower success rate than GESPAR, which is fast and accurate.
Fig. 1. Recovery probability vs. sparsity (s)

TABLE I

|          | SDP          | Sparse-Fienup | GESPAR       |
|----------|--------------|---------------|--------------|
|          | recovery     | runtime       | recovery     | runtime       | recovery | runtime |
| s = 3    | 0.93         | 1.32 sec      | 0.98         | 0.09 sec      | 1        | 0.12 sec |
| s = 5    | 0.86         | 1.78 sec      | 0.97         | 0.12 sec      | 1        | 0.12 sec |
| s = 8    | 0.9          | 3.85 sec      | 0.82         | 0.50 sec      | 1        | 0.23 sec |

B. Sensitivity to exact sparsity knowledge

Since the exact value of the signal’s sparsity s is sometimes not known, the performance of GESPAR is examined when only an upper limit on s is known.

1) Simulation Details: A GESPAR simulation is run twice: Once with s known exactly at each realization, and once with only an upper limit on s being known. The upper limit is taken as 32. The other simulation settings are the same as in Section IV-A1, this time with \( T = 5000 \).

2) Simulation Results: Figure 2 shows the probability for successful recovery of the two simulations. The rather loose upper limit on s does not seem to affect the results significantly.

C. Robustness to noise

The performance of GESPAR as a function of SNR is examined, and compared with an existing method - sparse Fienup [12]. The SDP based method that is presented in [9] is not designed to deal with noise as presented, so it was left out of the current comparison. The SDP method presented in [10] is designed to deal with random measurements and does not produce comparable results from direct Fourier measurements, and therefore is not compared with in this work.

1) Simulation Details: As in Section IV-A1, we choose \( \tilde{x} \) as a vector of length n, with s randomly chosen elements containing uniformly distributed values, and the N point Fourier magnitude, y, is calculated. Now, white-gaussian noise \( v \) is added to the measurements, at different SNR values, defined as: \( SNR = 20 \log \frac{\|y\|}{\|v\|} \). In order to recover the unknown vector x, the GESPAR algorithm is used with \( \tau = 10^{-4} \) and \( T = 10000 \), as well as the
sparse-Fienup algorithm, for comparison purposes. In our simulation $n = 64$ and $N = 128$. The sparse-Fienup algorithm is run with a maximum of 1000 iterations, and with 100 random initial points.

Note that even with little noise, the information on the support obtained by the zeros of the autocorrelation is no longer available. This is due to the fact that in the presence of noise, there will be no true zeros in the measured (or calculated) autocorrelation. In this case, one might try to threshold the autocorrelation values, rendering small autocorrelation values as zeros. However, this might result in zeroing of small (yet non-zero) values of the true autocorrelation function. Therefore, in the noisy case, we do not use support information obtained by the autocorrelation function in GESPAR, namely $J_2 = \{1, 2, \ldots, n\}$. The noiseless case was simulated with the autocorrelation support information.

2) Simulation Results: Figure 3 shows the normalized mean squared reconstruction error (NMSE), defined as $NMSE = \frac{\|x - \hat{x}\|_2}{\|x\|_2}$ as a function of sparsity, for different SNR values. Each point represents an average over 300 different random realizations. The performance under different SNR values is plotted for GESPAR (full lines), and for sparse-Fienup (dashed-lines). The performance of GESPAR naturally improves as SNR increases, and it outperforms the sparse-Fienup in terms of noise-robustness.

D. Scalability

As the main advantage of GESPAR over SDP based methods is its ability to solve large problems efficiently, its performance for different vector sizes is examined.

1) Simulation Details: GESPAR is simulated for various values of $n$ (64, 128, 256, 512). In all simulated cases $N = 2n$. The other simulation parameters are as in Section [IV-B1].

2) Simulation Results: The recovery probability vs. sparsity $s$ for different vector lengths is shown in Fig. 4. The maximal sparsity allowing successful recovery is shown to increase with vector length $n$. The maximal value of $s$ that is recoverable with high probability seems to scale like $n^{1/3}$, which is consistent with the same scaling observation presented in [9]. The mean reconstruction time for a signal with $n = 512$, $s = 35$ from $N = 1024$ measurements, allowing $T = 5000$ replacements, is 25.92 seconds.
Fig. 3. Normalized MSE vs. sparsity level. The performance is plotted for several SNR values for GESPAR (full lines) and Sparse-Fienup (dashed lines).

Fig. 4. Scalability - recovery probability as a function of signal sparsity, for various vector lengths (64, 128, 256, 512)

V. CONCLUSION

We proposed and demonstrated GESPAR - a fast algorithm to recover a sparse vector from its Fourier magnitude. We showed via simulations that GESPAR outperforms alternative approaches suggested for this problem. The algorithm does not require matrix-lifting, and therefore is potentially suitable for large scale problems such as 2D images.
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