Abstract. In this paper, we investigate the many-valued version of coalgebraic modal logic through predicate lifting approach. Using techniques from abstract algebraic logic, we prove soundness and completeness theorem by developing one-step logic. In the final section, we also propose some questions deserved for future works.

1. Introduction

Coalgebraic modal logic, first proposed by L. Moss [13] in 1999, provides a uniform framework to various semantics of modal logics using the theory of coalgebra [10]. This framework includes different class of models and many reasoning principles. Basically, there are two approaches toward coalgebraic modal logic—relation lifting [13] and predicate lifting [14]. The logic, often called $\nabla$-logic, given by relation lifting approach encodes the modality in any set functor $T$ that preserves weak pullbacks. In this logic system, there are only one modal similarity type, namely the $\nabla$, in which the semantic is provided by the set functor $T$. However, this logic system has unusual syntax makes it not easy to work with. For example, in [17], the authors proposed the first axiom system for $\nabla$-logic and use complicated techniques from coalgebra theory to prove the soundness and completeness of this system. The second approach, predicate lifting, provides coalgebraic logics with a more standard modal syntax. But this requires a second parameter—the modal similarity type which is not fixed in the logic system given by predicate lifting. Since the syntax is simpler than $\nabla$-logic, the proof of soundness and completeness of the logical system in [14] is less difficult and it can be developed using one-step logic. Also, coalgebraic modal logic allow applications in computer science and philosophy.

Institute of Information Science, Academia Sinica, No 128, Academia Road, Section 2 Nankang, Taipei 115201, Taiwan.

E-mail address: maxcylin@iis.sinica.edu.tw, liaucj@iis.sinica.edu.tw.

Key words and phrases. Many-valued modal logic, coalgebraic logic, many-valued logic, modal logic.
For the exclusive survey of coalgebraic modal logic, we referred to the article written by two leading experts [12].

On the other hands, consider the reasoning in modal notion with vague concepts, like belief, uncertainty, knowledge, etc., the fuzzy logic over residuated lattices (like in [9]) appear as a suitable framework for developing logical systems. Thus, many-valued modal logic has been developed in response to the investigation of connection between modality and vagueness. The first systematic study of many-valued modal logic using Heyting algebra was by M. Fitting [6] [7]. Then, through abstract algebraic logic, F. Bou et.al. [3] developed minimum many-valued modal logic over a finite residuated lattice. This key paper affects the following development of many-valued modal logics.

It is then natural to combine the many-valued modal logic and coalgebraic modal logic to develop coalgebraic many-valued modal logic. This direction of research was first studied by M. Bílková and M. Dostál both in relation lifting approach [1] and predicate lifting approach [2]. They showed that one can define many-valued semantics in both approach and prove Hennessy-Milner property with some further assumptions. In both papers, however, the authors did not propose any sound and complete axiom systems for coalgebraic many-valued modal logic.

Therefore, in this paper, we adopt the semantic given in [2] and using one-step logic to show that there exists a sound and complete axiom system for coalgebraic many-valued modal logic. The syntax in this paper follows from [4] which is widely used in many-valued logic and fuzzy logic.

2. Syntax

2.1. Language. We first introduce the language we used in this paper. Let $P$ be a set of propositional symbols and $\Lambda$ be a set of predicate liftings. Our $n$-ary predicate lifting is defined as a natural tranformation

$$\lambda : Hom(-, \mathbb{A}^n) \Rightarrow Hom(T(-), \mathbb{A})$$

with functor $T : \text{Set} \rightarrow \text{Set}$ and commutative Full-Lambek integral algebra $\mathbb{A}$.

**Definition 1.** We say $\mathbb{A} = (A, \lor, \land, \rightarrow, \odot, 0, 1)$ is a commutative integral Full-Lambek algebra (FL-algebra) if

- $\langle A, \lor, \land, 0, 1 \rangle$ is a bounded lattice,
- $\langle A, \odot, 1 \rangle$ is a commutative monoid,
• We can define ordering \( \leq \) as \( a \leq b \) iff \( a \land b = b \) iff \( a \lor b = a \),
• \( \odot \) is residuated with \( \to \), i.e. for all \( a, b, c \in A \), \( a \odot b \leq c \) iff \( b \leq a \to c \),
• \( a \leq 1 \) for all \( a \in A \).

In many-valued modal logic, we usually generalized the classical modal logic with Boolean algebra \( \mathbb{2} \) to \( A \), which provide semantics for wide class of substructural logics.

In this paper, we use \( \lor, \land, \& \), \( \to \), \( \top \), \( \bot \), and the modal similarity types \( \odot \lambda \) as our logical symbols and \( \bar{c} \) as our constant symbols. The language \( \mathcal{ML} \) is defined inductively as follows:

\[ \phi ::= p \in P | \varhexagon | \bot | \top | \varhexagon \phi_0 \lor \phi_1 | \varhexagon \phi_0 \land \phi_1 | \varhexagon \phi_0 \& \phi_1 | \varhexagon \phi_0 \to \varhexagon \phi_1 | \varhexagon \lambda(\phi_0, \ldots, \phi_{n-1}) \]

where \( \lambda \) is a \( n \)-ary predicate lifting. Then \( \varhexagon \phi_0 \leftrightarrow \phi_1 \) can be defined as \( (\varhexagon \phi_0 \to \varhexagon \phi_1) \land (\varhexagon \phi_1 \to \varhexagon \phi_0) \).

We define the rank-0 language \( \mathcal{L}_0 \) and rank-1 languages \( \mathcal{L}_1 \) as follows. Let \( \Phi \) be a nonempty set. Define \( T_\Lambda(\Phi) \) to be the following set

\[ \{ \varhexagon \lambda(a_1, \ldots, a_n) : \lambda \in \Lambda, a_1, \ldots, a_n \in \Phi \} \]

with \( \lambda \) being \( n \)-ary predicate lifting. Then the rank-0 language \( \mathcal{L}_0 \) is defined inductively as

\[ \pi ::= p \in P | \varhexagon | \bot | \top | \varhexagon \pi_0 \lor \pi_1 | \varhexagon \pi_0 \land \pi_1 | \varhexagon \pi_0 \& \pi_1 | \varhexagon \pi_0 \to \pi_1 \]

The rank-1 language \( \mathcal{L}_1 \) is defined inductively as

\[ \alpha ::= \alpha \in T_\Lambda(\mathcal{L}_0) | \top | \bot | \top | \alpha_0 \lor \alpha_1 | \alpha_0 \land \alpha_1 | \alpha_0 \& \alpha_1 | \alpha_0 \to \alpha_1 \]

Therefore, one can easily show that

\[ \mathcal{ML} = \bigcup_{i=0}^{\infty} \mathcal{L}_i \]

where \( \mathcal{L}_i = T_\Lambda(\mathcal{L}_{i-1}) \).

In order to prove the existence lemma, we will then expand our language \( \mathcal{ML} \) by introducing a new symbol \( \Delta \). This expansion is widely used in \([9]\).

**Definition 2.** The language \( \mathcal{ML}_\Delta \) is the language \( \mathcal{ML} \) expanded by the unary connective as follows:

If \( \phi \in \mathcal{ML} \) then \( \Delta \phi \in \mathcal{ML}_\Delta \)

Note that in similar way, we can define enriched rank-0 and rank-1 language \( \mathcal{L}_{0,\Delta} \) and \( \mathcal{L}_{1,\Delta} \).
For the simplicity of notation, we will denote $\mathcal{ML}_\Delta$ as $\mathcal{ML}$ in the remaining part of this paper.

Similarly, for any FL-algebra $A$, we can also enrich the algebraic structure by $\delta$ as unary operation on $A$ in following way.

**Definition 3.** A $\Delta$-algebra $A_\Delta$ is a FL-algebra $A$ expanded by an unary operation such that $\Delta(1) = 1$ and $\Delta(a) = 0$ for any $a \in \text{dom}(A)$ and satisfies the following axioms: for any $a, b \in A$

- $\Delta a \lor (\Delta a \rightarrow \bot) = 1$
- $\Delta(a \lor b) \leq \Delta a \lor \Delta b$
- $\Delta a \leq a$
- $\Delta a \leq \Delta \Delta a$
- $\Delta a \odot (\Delta(a \rightarrow b)) \leq \Delta b$

We will denote $A_\Delta$ as $A$ in the same manner.

**Remark 1.** In some literature, our language $\mathcal{ML}$ is actually $Fm_{\mathcal{ML}}$, the set of formulas generated by logical and nonlogical symbols in $\mathcal{ML}$ under some rules. To simplify our notation, we do not make such a difference.

In fact, we can view the construction of $\mathcal{ML}$ as a free generating operation. Therefore, $\mathcal{ML}$ is a absolutely free-algebra generated by $P \cup \{\bar{c}\}$ under the formal operation $\lor, \land, \&$, $\rightarrow$, $\otimes$, $\lambda$, and $\Delta$.

### 2.2. Logical System

In this subsection, we will develop some logic at the level of rank-0 formulas, rank-1 formulas and $\mathcal{ML}$-formulas.

**Definition 4.** (Logical rule)

A *logical rule* in language $\mathcal{ML}$ is a pair $\langle \Gamma, \phi \rangle$, where $\Gamma \cup \{\phi\} \subseteq \mathcal{ML}$. Clearly, each subset $\mathbb{D}$ of the set of all logical rules can be realized as a relation between set of formulas and formulas.

Since we will use one-step logic in this paper, the definition of logical rule can be refined as one-step version.

**Definition 5.** (One-step logical rule) A *one-step logical rule* $\langle \Gamma, \phi \rangle$ is a logical rule where $\Gamma \subseteq \mathcal{L}_0$ and $\phi \in \mathcal{L}_1$.

Examples of one-step logical rules are the congruence rule $C_\lambda$:

\[
\begin{array}{c}
\pi_0 \leftrightarrow \pi_0' \cdots \pi_{n-1} \leftrightarrow \pi_{n-1}' \\
\odot_\lambda(\pi_0, \ldots, \pi_{n-1}) \leftrightarrow \odot_\lambda(\pi_0', \ldots, \pi_{n-1}')
\end{array}
\]
and the monotonicity rule $M_{\lambda}$

\[
\frac{\pi_0 \to \pi_0' \ldots \pi_{n-1} \to \pi_{n-1}'}{\bigcup_{\lambda}(\pi_0, \ldots, \pi_{n-1}) \to \bigcup_{\lambda}(\pi_0', \ldots, \pi_{n-1}')}
\]

that we will associate with an $n$-ary modality $\bigcup_{\lambda}$.

**Notation 2.** We denote $\langle \Gamma, \phi \rangle \in \mathcal{D}$ as $\Gamma \vdash \mathcal{D} \phi$.

If a set $\mathcal{D}$ of logical rules satisfying some special properties, we call this set a logic. To elaborate what a logic is, we can use substitution to define instances of logical rules.

**Definition 6.** A *substitution* is a map $\sigma : P \to \mathcal{ML}$. We will use the notation $\phi/v$ for the substitution that maps the variable $v$ to the formula $\phi$ which remain identical in other variables.

**Definition 7.** A set of logical rules in the language $\mathcal{ML}$ is called a *logic* $\mathcal{L}$ when it satisfies the following properties for each $\Gamma \cup \Phi \cup \{\phi\} \subseteq \mathcal{ML}$:

- If $\phi \in \Gamma$, then $\Gamma \vdash_{\mathcal{L}} \phi$.
- If $\Phi \vdash_{\mathcal{L}} \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash_{\mathcal{L}} \phi$, then $\Phi \vdash_{\mathcal{L}} \phi$.
- If $\Gamma \vdash_{\mathcal{L}} \phi$, then $\sigma[\Gamma] \vdash_{\mathcal{L}} \sigma(\phi)$ for each substitution $\sigma$.

If the logical rules are all one-step, then we call $\mathcal{L}$ a *one-step logic*.

Now we introduce the notion of axiomatic system similar to logic.

**Definition 8.** An *axiomatic system* $\mathcal{A}$ in the language $\mathcal{ML}$ is a set of logical rules closed under arbitrary substitutions. Then elements of $\mathcal{A}$ of the form $\langle \Gamma, \phi \rangle$ are called *axioms* if $\Gamma = \emptyset$ and we call $\phi$ an axiom of $\mathcal{A}$.

Therefore, in the same manner, we call $\mathcal{A}$ a one-step axiomatic system if the logical rules in consideration are all one-step.

**Definition 9.** (Proof) A *proof* of a formula $\phi$ from a set of formula $\Gamma$ in an axiomatic system $\mathcal{A}$ is a well-founded tree (with no infinite branch) labeled by the formulas such that

- its root is labeled by $\phi$ and leaves by axioms of $\mathcal{A}$ or elements of $\Gamma$ and
- if a node is labeled by $\psi$ and $\Phi \neq \emptyset$ is the set of labels of its preceding nodes, then $\langle \Phi, \psi \rangle \in \mathcal{A}$.

We write $\Gamma \vDash_{\mathcal{A}} \phi$ if there is a proof of $\phi$ from $\Gamma$ in $\mathcal{A}$. If $\Gamma = \emptyset$, then we say that $\phi$ is $\mathcal{A}$-derivable. It is customary to identify $\vDash_{\mathcal{A}}$ as a special set of logical rules. This motivates the following definition.
Definition 10. (Logical system)
Let $L$ be a logic in $\mathcal{ML}$ and $\mathcal{A}$ be an axiomatic system in $\mathcal{ML}$. We say that the logic $L$ is a logical system if $L = \Vdash_\mathcal{A}$. Similarly, if $L$ is a one-step logic and $\mathcal{A}$ is a one-step axiomatic system then we call $L$ a one-step logical system.

If a formula $\phi$ is $\mathcal{A}$-derivable in the logical system $L$, we say $\phi$ is $L$-derivable. From the generalization of Theorem 2.1.25 in [5], we know that the logical system can also defined as the definition of logic in [3] through non-modal homomorphisms.

Definition 11. We say that a mapping $h: \mathcal{ML} \rightarrow \mathcal{A}$ is a non-modal homomorphism if the following conditions hold:

- $h(\bar{c}) = c, h(\top) = 1, h(\bot) = 0$;
- $h(\phi_1 \ast \phi_2) = h(\phi_1) \ast^h h(\phi_2)$ where $\ast \in \{\lor, \land, \&\}$ and $\ast^h \in \{\lor, \land, \circ, \rightarrow\}$;
- $h(\Delta \phi) = \Delta h(\phi)$.

We denote the set of non-modal homomorphisms as $Hom(\mathcal{ML}, \mathcal{A})$.

Then given a logical system $L$, we say that for all sets of formulas $\Gamma \cup \{\phi\} \subseteq \mathcal{ML}$,

$\Gamma \vdash_L \phi \iff \forall h \in Hom(\mathcal{ML}, \mathcal{A})$, if $h[\Gamma] \subseteq \{1\}$ then $h(\phi) = 1$.

In this article, we will use logical systems $L$ consist of (1) a set of logical rules $R = \Gamma_R / \gamma_R$ where $\Gamma_R \subseteq \mathcal{ML}$ and $\gamma \in \mathcal{ML}$ (2) all axioms and rules from $\Lambda(\mathcal{A})$, the non-modal logic of residuated lattice $\mathcal{A}$, (3) the congruence rule $(C_\lambda)$ for $\lambda \in \Lambda$, (4) the following axioms for unary connective $\Delta$: for any $\phi, \psi \in \mathcal{ML}$,

- $\Delta \phi \lor (\Delta \phi \rightarrow \bot)$,
- $\Delta(\phi \lor \psi) \rightarrow (\Delta \phi \lor \Delta \psi)$,
- $\Delta \phi \rightarrow \phi$,
- $\Delta \phi \rightarrow \Delta \Delta \phi$,
- $\Delta(\phi \rightarrow \psi) \rightarrow (\Delta \phi \rightarrow \Delta \psi)$.

The detailed axioms and logical rules for $\Lambda(\mathcal{A})$ can be found in the appendix of [3]. In the literature on abstract algebraic logic, our definition of logical system is assertional logic of the class of FL-algebra [8]. Both the definition through morphisms or trees will be used in the proof of soundness and completeness theorem in this article.
3. COALGEBRAIC SEMANTICS

3.1. COALGEBRAIC MANY-VALUED LOGIC. In this section, we introduce the model we used in this paper. There are three different models which correspond to the layer structure of $\mathcal{ML}$. First, we write $(S, \sigma)$ as $T$-coalgebra where $\sigma$ is a function from $S$ to $TS$. Different from the definition in first section, we denote the set of functions from $S$ to $\Delta$-algebra $A$ as $\text{Hom}(S, A)$. For the full language $\mathcal{ML}$, the semantic is defined as follows.

Definition 12. Let $T$ be a set functor. A $T$-model $S = (S, \sigma, V, A)$ consists of a nonempty set $S$, a $T$-coalgebra map $\sigma: S \to TS$, a $\Delta$-algebra $A$, and a valuation of propositional symbols $V: P \to \text{Hom}(S, A)$. We define the semantics $\|\phi\|_{\sigma}: S \to A$ of $\phi \in \mathcal{ML}$ inductively: for all $s \in S$

- $\|p\|_{\sigma} := V(p)$ for all $p \in P$, $\|c\|_{\sigma}(s) := c$ with $c \in A$,
- $\|T\|_{\sigma}(s) := 1$, $\|\bot\|_{\sigma}(s) := 0$,
- $\|\phi_0 \ast \phi_1\|_{\sigma}(s) := \|\phi_0\|_{\sigma}(s) \ast \|\phi_1\|_{\sigma}(s)$ for $\ast \in \{\lor, \land, \&\}$ and $\ast \in \{\lor, \land, \circ, \to\}$, the corresponding algebraic operation on the FL-algebra $A$,
- $\|\square_{\lambda}(\phi_0, \ldots, \phi_{n-1})\|_{\sigma}(s) := \lambda_S(\|\phi_0\|_{\sigma}, \ldots, \|\phi_{n-1}\|_{\sigma})(\sigma(s))$, where $\lambda$ is a $n$-ary predicate lifting,
- $\|\Delta\phi\|_{\sigma}(s) := \Delta(\|\phi\|_{\sigma})$.

We then define the satisfaction relation $s \models_{\sigma} \phi$ between $S$ and $\mathcal{ML}$ as

$s \models_{\sigma} \phi = \|\phi\|_{\sigma}(s)$ for any $s \in S$.

Definition 13. We say a formula $\phi$ of $\mathcal{ML}$ is valid in a $T$-model $S$ if $s \models_{\sigma} \phi = 1$ for all $s \in S$. Then $\phi$ is called valid if it is valid in all $T$-models.

3.2. ONE STEP LOGIC. In this subsection, we localize the semantic define in the previous subsection to rank-0 language $L_0$ and rank-1 language $L_1$. We say $m: S \to \text{Hom}(P, A)$ is a $P$-marking if $m$ is a mapping from $S$ to the set $\text{Hom}(P, A)$.

Definition 14. Let $T$ be a set endofunctor. A one-step $T$-frame is a pair $\langle S, \delta \rangle$ with $\delta \in TS$. A one-step $T$-model over a set $P$ of propositional symbols is a triple $\langle S, \delta, m, A \rangle$ such that $\langle S, \delta \rangle$ is a one-step T-frame, $\Delta$ a and $m: S \to \text{Hom}(P, A)$ a P-marking on $S$. 

Following the definition of coloring in classical one-step logic \[15\], we define the coloring of \( m \) as 
\[ m^\flat : \text{P} \rightarrow \text{Hom}(S, A) \].

**Definition 15.** Given a marking \( m : S \rightarrow \text{Hom}(P, A) \), we define the 0-step interpretation \([\pi]^0_m : S \rightarrow A\) of \( \pi \in \mathcal{L}_0 \) by induction: for all \( s \in S \)

- \([p]^0_m(s) := m^\flat(p)\),
- \([\top]^0_m(s) := 1\), \([\bot]^0_m(s) := 0\),
- \([\pi_0 \ast \pi_1]^0_m(s) := [\pi_0]^0_m(s) \ast^A [\pi_1]^0_m(s)\) for \( \ast \in \{\lor, \land, \& , \rightarrow\} \) and \( \ast^A \in \{\lor, \land, \circ, \rightarrow\} \), the corresponding algebraic operation on the FL-algebra \( A \),
- \([\Delta \pi]^0_m(s) := \Delta [\pi]^0_m(s)\).

We write \( S, m, s \models^0 \pi \) to be \([\pi]^0_m(s)\) with \( s \in S \) and the marking \( m \). Similarly, we can define the 1-step interpretation \([\alpha]^1_m\) of \( \alpha \in \mathcal{L}_1 : TS \rightarrow A \) in the following way.

**Definition 16.** Let \( m \) be a marking and \( \lambda \) is a n-ary predicate lifting. The 1-step interpretation of \( \alpha \in \mathcal{L}_1 \) is defined as

\[ [\circ_\lambda(\pi_0, \ldots, \pi_{n-1})]^1_m(\delta) := \lambda_S([\pi_0]^0_m, \ldots, [\pi_{n-1}]^0_m)(\delta) \]

and standard clauses applying for \( \lor, \land, \& , \rightarrow\) and \( \Delta \) in the same manner as in \( \mathcal{L}_0 \).

Given an one-step T-model \( \langle S, \delta, m \rangle \), we write \( S, \delta, m \models^1 \alpha \) for the function \([\alpha]^1_m(\delta)\).

4. **One-step Soundness and Completeness**

In this section, we fix a one-step logical system \( \mathcal{L}_1 \) which consists of (1) a set of logical rules \( R = \langle \Gamma_R, \gamma_R \rangle \) where \( \Gamma_R \subseteq \mathcal{L}_0 \) and \( \gamma \in \mathcal{L}_1 \), (2) all axioms and logical rules from \( \Lambda(A) \), the non-modal logic of a residuated lattice \( A \), (3) the congruence rule \( (C_\lambda) \) for \( \lambda \in \Lambda \) and (4) the following axioms for unary connective \( \Delta \): For any \( \phi, \psi \in \mathcal{L}_0 \),

- \( \Delta \phi \lor (\Delta \phi \rightarrow \bot) \),
- \( \Delta(\phi \lor \psi) \rightarrow (\Delta \phi \lor \Delta \psi) \),
- \( \Delta \phi \rightarrow \phi \),
- \( \Delta \phi \rightarrow \Delta \Delta \phi \),
- \( \Delta(\phi \rightarrow \psi) \rightarrow (\Delta \phi \rightarrow \Delta \psi) \).

**Definition 17.** A one-step logical rule \( R = \langle \Gamma, \gamma \rangle \) where \( \Gamma \subseteq \mathcal{L}_0 \) and \( \gamma \in \mathcal{L}_1 \) is called one-step sound if \([\pi]^0_m = [\top]^0_m\) for every \( \pi \in \Gamma \) implies
||\gamma||_m^1 = ||\top||_m^1. The one-step logical system \(L_1\) is one-step sound if all one-step logical rules are one-step sound.

**Definition 18.** We say \(\pi \in L_0\) is a true propositional fact of a marking \(m : S \to \text{Hom}(P, A)\) if \(||\pi||_m^0 = ||\top||_m^0\). We let \(TPF(m)\) to be the following set
\[\{\pi \in L_0 : ||\pi||_m^0 = ||\top||_m^0\}\]

**Definition 19.** A one-step derivation system \(L_1\) is one-step complete if for every marking \(m : S \to \text{Hom}(P, A)\) and every \(\alpha \in L_1\), we have
\[||\alpha||_m^1 = ||\top||_m^1\] implies \(TPF(m) \vdash_{L_1} \alpha\)

5. **Soundness and Completeness**

In this section, we are going to prove soundness and completeness theorem of coalgebraic many-valued modal logic by one-step approach. we then first give the definition of soundness and completeness.

**Definition 20.** Let \(L\) be the logical system satisfied the requirement in section 2. We say that \(L\) is sound if all \(L\)-derivable formulas are valid, and complete if all valid formulas are \(L\)-derivable.

As [3] indicates, \(\Lambda(A)\) is sound and strongly complete with respect to \(A\). Thus, it is also sound and complete.

**Definition 21.** A set of \(ML\)-formulas \(\Psi\) is called the closure of a set of \(ML\)-formulas \(\Phi\) iff
\[
\begin{align*}
& (1) \ \Phi \subseteq \Psi \\
& (2) \ \Psi \text{ is closed under subformulas} \\
& (3) \ \Psi \text{ contains } \top \text{ and } \bot
\end{align*}
\]

\(\Phi\) is said closed if the closure \(\Psi\) of \(\Phi\) equals to \(\Phi\). To prove the theorem, first let \(P_\Phi\) be the set
\[\{a_\phi : \phi \in \Phi\}\]
where \(\Phi\) is finite closed set of \(ML\)-formulas and \(a_\phi\) are new propositional symbols for every \(\phi \in \Phi\). Then we use \(L_0(P_\Phi)\) and \(L_1(P_\Phi)\) to denote the rank-0 and rank-1 language which substitutes the set \(P\) of propositional symbols with \(P_\Phi\).

Define \(Thm\) to be the set of formulas \(\phi \in ML\) such that \(\phi\) is \(L\)-derivable. Now, let \(s : ML \to A\) be a non-modal homomorphism such that \(s[Thm] = 1\). Consider the set \(\tilde{S}\) of all such homomorphisms \(s\). We define the marking \(m : \tilde{S} \to \text{Hom}(P_\Phi, A)\) which maps \(s\) to \(h_s : P_\Phi \to A\) with \(h_s(a_\phi) = s(\phi)\). Then we have the following proposition.
Proposition 1. With the definition above, we can formulate the following equality
\[\|a_\phi\|_m^0(s) = s(\phi)\] for any \(s \in \bar{S}\).

Proof. This can be shown by \(\|a_\phi\|_m^0(s) = m^\delta(a_\phi)(s) = m(s)(a_\phi) = h_s(a_\phi) = s(\phi).\) \(\square\)

Let \((\phi/a_\phi : \phi \in \Phi)\) denote the natural substitution replacing all the variable \(a_\phi\) with the original formula \(\phi\). Therefore, for any formula \(\pi \in L_0(P_\Phi)\) and \(\alpha \in L_1(P_\Phi)\), we use \(\hat{\pi}, \hat{\alpha} \in \mathcal{ML}\) respectively, to denote \(\hat{\pi} := \pi(\phi/a_\phi : \phi \in \Phi)\), and \(\hat{\alpha} := \alpha(\phi/a_\phi : \phi \in \Phi)\).

We then have the following lemma.

Lemma 1. (Stratification Lemma) Let \(L_1\) be a one-step logical system which is one-step sound and complete. Then

1. For any formula \(\pi \in L_0(P_\Phi)\), \(\vdash_{L_1} \hat{\pi} \iff \langle \bar{S}, m \rangle \vdash^0 \pi = 1\)
2. For any formula \(\alpha \in L_1(P_\Phi)\), \(\vdash_{L_1} \hat{\alpha} \iff \langle \bar{S}, m \rangle \vdash^1 \alpha = 1\)

Proof. To prove part one of this lemma, we first claim that \(\|\pi\|_m^0(s) = s(\bar{\pi})\) for any \(\pi \in L_0(P_\Phi)\) and \(s \in \bar{S}\). This can be done by induction on the complexity of formulas in \(L_0(P_\Phi)\). For the base step, \(\pi\) is of the form \(a_\phi \in P_\Phi\) and \(\bar{\pi}\) is \(\phi\) by definition. Then \(\|a_\phi\|_m^0(s) = s(\phi) = s(\bar{\pi})\) using Proposition 1. For the inductive step, let \(* \in \{\lor, \land, \&\}\) and \(\pi_1, \pi_2 \in L_0(P_\Phi)\). By the inductive hypothesis, we then have
\[\|\pi_1*\pi_2\|_m^0(s) = \|\pi_1\|_m^0(s)^*\delta \|\pi_2\|_m^0(s) = s(\bar{\pi_1})^*\delta s(\bar{\pi_2}) = s(\bar{\pi_1*\pi_2}) = s(\hat{\pi_1*\pi_2}).\]

For \(\pi \in L_0(P_\Phi)\),
\[\|\Delta\pi\|_m^0(s) = \Delta\|\pi\|_m^0(s) = \Delta s(\bar{\pi}) = s(\hat{\Delta\pi})\]
using inductive hypothesis. This proves the claim.

Now, suppose that \(\vdash_{L_1} \hat{\pi}\). From the definition of one-step provability, every non-modal homomorphism \(s : \mathcal{ML} \to A\) satisfies \(s(\bar{\pi}) = 1\). From the claim above,
\[\|\pi\|_m^0(s) = s(\bar{\pi}) = 1\]
which proves the result we want. On the other side, assume that \(\not\vdash \hat{\pi}\). Then \(\hat{\pi} \notin \text{Thm}\) which means there exists a non-modal homomorphism \(h : \mathcal{ML} \to A\) such that \(h(\bar{\pi}) \neq 1\) but \(h[\text{Thm}] \subseteq \{1\}\). By the claim again, we have \(\|\pi\|_m^0(h) = h(\bar{\pi}) \neq 1\). This implies \(\hat{\pi}\) is not valid which is absurd.
Now, suppose that $\langle \bar{S}, m \rangle \models^{\perp_1} \alpha = 1$. Then by one-step completeness, $TPF(m) \vdash_{L_1} \alpha$. We prove by induction on the complexity of proof in $L_1$, that this implies $\vdash_{L_1} \hat{\alpha}$.

For the base case, we are considering $\pi \in L_0(\Phi)$ which is a true propositional fact about $\bar{S}$. The first part of Stratification lemma implies that $\vdash_{L_1} \hat{\pi}$. □

**Definition 22.** Let $\bar{S}$ be defined as above. We define a syntactical evaluation $|\phi| : \bar{S} \to A$ for every $\phi \in ML$ as $s \mapsto s(\phi)$.

This is clearly a well-defined function. Thus, we will have

$$\|a_\phi\|^0_m = |\phi|$$

since $\|a_\phi\|^0_m(s) = s(\phi) = |\phi|(s)$ for every $s \in \bar{S}$ by definition. Now, we prove the key lemma that will be used in the proof of completeness theorem.

**Lemma 2. (Existence Lemma)**

Let $\Phi$ be a finite closed set of $ML$-formulas. There is a map $\sigma : \bar{S} \to T\bar{S}$ such that for all $s \in \bar{S}$ and all formulas of the following form $\Diamond_{\lambda}(\phi_0, \ldots, \phi_{n-1}) \in \Phi$, we have

$$s(\Diamond_{\lambda}(\phi_0, \ldots, \phi_{n-1})) = \lambda_{\bar{S}}(|\phi_0|, \ldots, |\phi_{n-1}|)(\sigma(s))$$

**Proof.** We prove by assuming for a contradiction that for some $s \in \bar{S}$ there is no $\sigma(s)$ satisfying the equality. List all formulas of the form $\Diamond_{\lambda}(\phi_0, \ldots, \phi_{n-1})$ and denote them as $\Diamond_{\lambda}\psi_0, \ldots, \Diamond_{\lambda}\psi_k-1$ if there are $k$-numbers of such formula. Each $\psi_i$ is in fact a n-vector $(\phi_0, \ldots, \phi_{n-1})_i$ if $\lambda$ is a n-ary predicate lifting. Suppose that $s(\Diamond_{\lambda}(\psi_i)) = c_i$ for all $c_i \in A$ and $i = 0, \ldots, k - 1$. Define the following formula $\alpha \in L_1(\Phi)$

$$\alpha := \Delta(\bigwedge_{i=0}^{k-1}(\Diamond_{\lambda}a_{\psi_i} \leftrightarrow c_i)) \rightarrow \bot.$$

Note that the new propositional symbols $a_{\psi_i}$ is actually n-vectors $(a_{\phi_0}, \ldots, a_{\phi_{n-1}})_i$ whenever $\lambda$ is n-ary predicate lifting. We denote it as
Consider $\lambda : \bar{\mathcal{S}} \to T\bar{\mathcal{S}}$ to simplify our notation. Then for any $\delta \in T\bar{\mathcal{S}}$

$$\|\alpha\|_m^{1}(\delta) = \|\Delta(\bigwedge_{i=0}^{k-1}(\Box a_{\psi_i} \leftrightarrow \bar{c}_i)) \to \bot\|_m(\delta)$$

$$= \|\Delta(\bigwedge_{i=0}^{k-1}(\Box a_{\psi_i} \leftrightarrow \bar{c}_i))\|_m^{1}(\delta) \to \|\bot\|_m(\delta)$$

$$= \Delta(\|\bigwedge_{i=0}^{k-1}(\Box a_{\psi_i} \leftrightarrow \bar{c}_i)\|_m^{1}(\delta)) \to \|\bot\|_m(\delta)$$

$$= \Delta(\bigwedge_{i=0}^{k-1}(\lambda \bar{S}(\|a_{\psi_i}\|_m(\delta)) \leftrightarrow c_i)) \to 0$$

$$= \Delta(\bigwedge_{i=0}^{k-1}(\lambda \bar{S}(\|\psi_i\|))(\delta) \leftrightarrow c_i) \to 0$$

Since $\lambda \bar{S}(\|\psi_i\|)(\delta) \neq c_i$ by our assumption, we have

$$\bigwedge_{i=0}^{k-1}(\lambda \bar{S}(\|\psi_i\|))(\delta) \leftrightarrow c_i \neq 1.$$

That is, $\|\alpha\|_m(\delta) = 0 \to 0 = 1$. From the Stratification Lemma, we get $\vdash_{L_1} \hat{\alpha}$ which means $s(\hat{\alpha}) = 1$. However, compute $s(\hat{\alpha})$ gives us

$$s(\hat{\alpha}) = s(\Delta(\bigwedge_{i=0}^{k-1}(\Box a_{\psi_i} \leftrightarrow \bar{c}_i)) \to \bot)$$

$$= \Delta(\bigwedge_{i=0}^{k-1}s(\Box a_{\psi_i} \leftrightarrow \bar{c}_i)) \to 0$$

$$= \Delta(1) \to 0$$

$$= 0$$

This contradicts to the value of $\hat{\alpha}$ we have above. \qed

**Lemma 3.** *(Truth Lemma)*

Let $\Phi$ be a finite, closed set of $\mathcal{ML}$-formulas and $\sigma : \bar{\mathcal{S}} \to T\bar{\mathcal{S}}$ be the map satisfying the Existence Lemma. Then we have

$$s(\phi) = s \models_{\sigma} \phi$$

for all $s \in \bar{\mathcal{S}}$ and $\phi \in \Phi$.
Proof. We prove this by a straightforward formula induction. Let $S$ be a T-model given by $\sigma$ as above and a valuation $V : P \to \text{Hom}(S, A)$ defined as $V(p)(s) = s(p)$ for $s \in S$ and $p \in P$. Then $s(p) = V(p)(s) = \|p_{\sigma}\|_\sigma(s)$, $s(\top) = 1 = \|\top\|_\sigma(s)$, $s(\bot) = 0 = \|\bot\|_\sigma(s)$, and $s(\bar{c}) = c = \|\bar{c}\|_\sigma$. For the logical connectives,

$$s(\phi \ast \psi) = s(\phi) \ast_\Lambda s(\psi) = \|\phi\|_\sigma(s) \ast_\Lambda \|\psi\|_\sigma(s) = \|\phi \ast \psi\|_\sigma(s)$$

for $\ast \in \{\lor, \land, \&\}$ and $\ast_\Lambda \in \{\lor, \land, \circ, \to\}$. Given $\phi \in \mathcal{ML}$, $s(\Delta \phi) = \Delta(s(\phi)) = \Delta(\|\phi\|_\sigma(s)) = \|\Delta \phi\|_\sigma(s)$. Last, for $\lambda \in \Lambda$ which is n-ary predicate lifting,

$$s(\circ_\lambda(\phi_0, \ldots, \phi_{n-1})) = \lambda_{\bar{s}}(\|\phi_0\|, \ldots, \|\phi_{n-1}\|)(\sigma(s))$$

$$= \lambda_{\bar{s}}(\|\phi_0\|_\sigma, \ldots, \|\phi_{n-1}\|_\sigma)(\sigma(s))$$

$$= \|\circ_\lambda(\phi_0, \ldots, \phi_{n-1})\|_\sigma(s)$$

Note that the first equality is given by the Existence Lemma. Also, the second equality is by the inductive hypothesis $s(\phi) = \|\phi\|_\sigma(s)$ and the definition $\|\phi\|_\sigma(s) = s(\phi)$. \qed

Now we are ready for proving the soundness and completeness theorem.

**Theorem 3.** Let $\Lambda$ be a modal signature for the set functor $T$, and let $L_1$ be a one-step logical system for $\Lambda$ which is one-step sound and complete for $T$. Then $L_1$ is also a sound and complete logical system for the set of $\mathcal{ML}$-validities in $T$.

Proof. For the proof of soundness, it suffices to show that all four part of axioms and logical rules of $L_1$ are valid. By one-step soundness, we only need to check one-step valid implies validity. Given any T-model $\langle S, \sigma, V, A \rangle$ with semantic $\|\cdot\|_\sigma$. We then let the coloring $m^\delta$ to be $V$ and the marking is defined as $m : S \to \text{Hom}(P, A)$ with $m(s)(p) := V(p)(s)$ for any $s \in S$ and $p \in P$. Also, define $\delta$ to be $\sigma(s)$. The routine induction shows that $\|\pi\|_{m^\delta}(s) = \|\pi\|_\sigma(s)$ for $\pi \in \mathcal{L}_0$ and $\|\alpha\|_{m^\delta}(\sigma(s)) = \|\alpha\|_\sigma(s)$.

For the proof of completeness, suppose that there is a $\mathcal{ML}$-formula $\psi$ which is valid but not $L_1$-derivable. Then $\text{Thm} \nvdash_{L_1} \psi$. Then there exists a non-modal homomorphism $h : \mathcal{ML} \to A$ such that $h[\text{Thm}] \subseteq \{1\}$ but $h(\psi) \neq 1$. Take $\Psi$ to be the closure of $\{\psi\}$. From Existence Lemma, there is a map $\sigma : S \to TS$ satisfying the equality. Let $\langle S, \sigma, V \rangle$ be the T-model as in the proof of truth lemma, then we have $h(\psi) = \|\psi\|_\sigma(h) \neq 1$ which implies that $\psi$ is not valid. \qed
6. Concluding remarks

In this article, we prove that there could be a sound and complete logical system for many-valued modal logic under the assumption of one-step sound and complete. This gives another way to show the soundness and completeness of many-valued modal logic. However, the main difficulties is to show the logic you aim for is one-step sound and complete. In the classical case, this can be done using the equivalent concept of one-step consistency and satisfiable \[15\]. However, there is no corresponding definition of such equivalence in many-valued logic. Thus, showing one-step soundness and completeness become more difficult in many-valued case. We left find a one-step sound and complete many-valued logic as an open problem. Also, we use the $\Delta$ operator in the proof of Existence lemma. Removing the $\Delta$ both from FL-algebras and our language $\mathcal{ML}$ would make the theory more general. The authors would like to know whether there is a proof of Existence lemma without the $\delta$ operator. Finally, from \[11\], there exists a sound and weak complete logical system for coalgebraic propositional dynamic logic. The proof was using predicate lifting and one-step approach. Besides, the soundness and weak completeness of finitely-valued propositional dynamic logic has also been proved using logical system in abstract algebraic logic \[16\]. Therefore, one can try to generalize the methods in this article to prove the soundness and completeness of coalgebraic many-valued propositional dynamic logic. We left this direction of research as future study.

References

[1] Marta Bílková and Matěj Dostál. Many-valued relation lifting and moss’ coalgebraic logic. In *International Conference on Algebra and Coalgebra in Computer Science*, pages 66–79. Springer, 2013.

[2] Marta Bílková and Matěj Dostál. Expressivity of many-valued modal logics, coalgebraically. In *International Workshop on Logic, Language, Information, and Computation*, pages 109–124. Springer, 2016.

[3] Félix Bou, Francesc Esteva, Lluís Godo, and Ricardo Oscar Rodríguez. On the minimum many-valued modal logic over a finite residuated lattice. *Journal of Logic and computation*, 21(5):739–790, 2011.

[4] Petr Cintula and Carles Noguera. Chapter ii: A general framework for mathematical fuzzy logic. *Handbook of Mathematical Fuzzy Logic-Volume 1 Studies in Logic, Mathematical Logic and Foundations*, 37, 01 2011.

[5] Petr Cintula and Carles Noguera. Two-layer modal logics: from fuzzy logics to a general framework. In *TACL*, pages 43–47. Citeseer, 2013.

[6] Melvin Fitting. Many-valued modal logics. *Fundam. Inform.*, 15(3-4):235–254, 1991.
[7] Melvin Fitting. Many-valued model logics ii. *Fundam. Inform.*, 17(1-2):55–73, 1992.

[8] J.M. Font. *Abstract Algebraic Logic: An Introductory Textbook*. Studies in logic and the foundations of mathematics. College Publications, 2016.

[9] Petr Hájek. *Metamathematics of fuzzy logic*, volume 4. Springer Science & Business Media, 2013.

[10] Bart Jacobs. *Introduction to Coalgebra*, volume 59. Cambridge University Press, 2017.

[11] C Kupke and HH Hansen. Weak completeness of coalgebraic dynamic logics. In *FICS 2015: Proceedings of the 10th International Workshop on Fixed Points in Computer Science, Berlin, Germany, 11-12 September 2015*. Cornell university Library, 2015.

[12] Clemens Kupke and Dirk Pattinson. Coalgebraic semantics of modal logics: an overview. *Theoretical Computer Science*, 412(38):5070–5094, 2011.

[13] Lawrence S Moss. Coalgebraic logic. *Annals of Pure and Applied Logic*, 96(1-3):277–317, 1999.

[14] Dirk Pattinson. Coalgebraic modal logic: Soundness, completeness and decidability of local consequence. *Theoretical Computer Science*, 309(1-3):177–193, 2003.

[15] Lutz Schröder and Dirk Pattinson. Rank-1 modal logics are coalgebraic. *Journal of Logic and Computation*, 20(5):1113–1147, 2010.

[16] Igor Sedlár. Finitely-valued propositional dynamic logics. In *Advances in Modal Logic, Volume 13*, pages 561–579. College Publications, 2020.

[17] Yde Venema, Alexander Kurz, and Clemens Kupke. Completeness for the coalgebraic cover modality. *Logical Methods in Computer Science*, 8, 2012.