Abstract

We give a condition ensuring that the operators in a nilpotent Lie algebra of linear operators on a finite dimensional vector space have a common eigenvector.

INTRODUCTION

Throughout this paper $V$ is a vector space of positive dimension over a field $f$ and $g$ is a nilpotent Lie algebra over $f$ of linear operators on $V$. An element $u \in V$ is an eigenvector for $S \subset g$ if $u$ is an eigenvector for every operator in $S$. If $V$ has a basis $(e_1, \ldots, e_n)$ representing each element of $g$ by an upper triangular matrix, then $e_1$ is an eigenvector for $g$. Such a basis exists when $f$ is algebraically closed and $g$ is solvable (Lie’s Theorem), and also when every element of $g$ is a nilpotent operator (Engel’s Theorem). Our results are further conditions guaranteeing existence eigenvectors.

The minimal and characteristic polynomials of a linear operator $A$ on $V$ are denoted respectively by $\pi_A, \mu_A \in f[t] =$ the ring of polynomials over $f$, written $\#S$.

Let $k$ be a Galois extension field of $f$ of degree $d := [k : f]$, and define $M \subset N$ to be the additive monoid generated by zero and the prime divisors $d$.

Consider the conditions:

(C1) $\mu_A$ splits in $k$ for every $A \in g$

(C2) $\dim V \not\in M$

Our main result is:

Theorem 1 If (C1) and (C2) hold then $g$ has an eigenvector.
The proof is preceded by some applications.

When (C1) holds, Theorem I shows that there is an eigenvector in every invariant subspace whose dimension is not in \( M \). This is exploited to yield the following two results:

**Corollary 2** If a nilpotent Lie algebra of linear operators on \( \mathbb{R}^n \) does not have an eigenvector, every nontrivial invariant subspace has odd dimension.

**Proof** When \( \mathfrak{f} \) is the real field \( \mathbb{R} \) and \( \mathfrak{k} \) is the complex field \( \mathbb{C} \), \( M \) consists of the positive even integers. ■

**Corollary 3** Let (C1) hold. Assume \( \mathfrak{g} \) preserves a direct sum decomposition \( V = \oplus_i W_i \), and let \( D \subset \mathbb{N} \) denote the set of dimensions of the subspaces \( W_i \).

(i) If \( \mathfrak{g} \) does not have an eigenvector then \( D \subset M \).

(ii) If \( V' \subset V \) is a maximal subspace spanned by eigenvectors of \( \mathfrak{g} \) then \( \dim(V') \geq \# \{ D \ \setminus \ M \} \).

**Proof** Assertion (i) follows from Theorem I. To prove (ii) order the \( W_i \) so that \( W_1, \ldots, W_m \) are the only summands whose dimensions are not in \( M \). For each \( j \in \{1, \ldots, m\} \) we choose an eigenvector \( e_j \in W_j \) by Theorem I. The \( e_j \) are linearly independent and belong to \( V' \) by maximality of \( V' \), whence (ii). ■

**Example 4**
Assume \( n \not\in M \) and let \( \alpha \in \mathfrak{f}[t] \) be a monic polynomial that splits in \( \mathfrak{k}[t] \). Denote by \( \mathcal{A}(\alpha) \) the set of \( n \times n \) matrices \( \mathcal{T} \) over \( \mathfrak{f} \) such that \( \alpha(\mathcal{T}) = 0 \). Then every pairwise commuting family \( \mathcal{T} \subset \mathcal{A}(\alpha) \) has an eigenvector in \( \mathfrak{f}^n \). This follows from Theorem I applied to the Lie algebra \( \mathfrak{g} \) of linear operators on \( \mathfrak{f}^n \) generated by \( \mathcal{T} \). Being abelian, \( \mathfrak{g} \) can be triangularized over \( \mathfrak{k} \), hence (C1) holds.

**Example 5**
The assumption that \( n \in M \) is essential to Theorem I. For instance, take \( \mathfrak{f} = \mathbb{R} \), \( \mathfrak{k} = \mathbb{C} \), \( V = \mathbb{R}^2 \). The abelian Lie algebra of \( 2 \times 2 \) of real skew symmetric matrices does not have an eigenvector in \( \mathbb{R}^2 \).

**Example 6**
The hypothesis of Theorem I cannot be weakened to \( \mathfrak{g} \) being merely solvable. For a counterexample with \( \mathfrak{f} = \mathbb{R}, \mathfrak{k} = \mathbb{C} \), take \( \mathfrak{g} \) to be the solvable 3-dimensional real Lie algebra with basis \((X, U, V)\) such that \([X, U] = -V, [X, V] = U, [U, V] = 0\).

A Lie algebra \( \mathfrak{b} \) over \( \mathfrak{f} \) is **supersolvable** if the spectrum of the linear map \( \text{ad} A : \mathfrak{b} \to \mathfrak{b} \) lies in \( \mathfrak{f} \) for all \( A \in \mathfrak{b} \). If \( \mathfrak{b} \) is not supersolvable it need not have an eigenvector, as is shown by Example 6. We don’t know if Theorem I extends to supersolvable Lie algebras, except for the following special case:
Theorem 7 A supersolvable Lie algebra $\mathfrak{b}$ of linear transformations of $\mathbb{R}^3$ has an eigenvector.

Proof Lacking an algebraic proof, we use a dynamical argument. Let $G \subset GL(3, \mathbb{R})$ be the connected Lie subgroup having Lie algebra $\mathfrak{b}$. The natural action of $G$ on the projective plane $\mathbb{P}^2$ of lines in $\mathbb{R}^3$ through the origin fixes some $L \in \mathbb{P}^2$. This follows from supersolvability because $\dim(\mathbb{P}^2) = 2$, the action on $\mathbb{P}^2$ is effective and analytic, and the Euler characteristic of $\mathbb{P}^2$ is nonzero (Hirsch & Weinstein [1]). The nonzero points of $L$ are eigenvectors for $\mathfrak{b}$.

Proof of Theorem 1 We rely on Jacobson’s Primary Decomposition Theorem [2, II.4, Theorem 5]. This states that $V$ has a $\mathfrak{g}$-invariant direct sum decomposition $\bigoplus V_i$ where each primary component $V_i$ has the following property: For each $A \in \mathfrak{g}$ the minimal polynomial of $A|V_i$ is a prime power in $f[t]$.

Condition (C2) implies the dimension of some primary component is $\notin M$. To prove Theorem 1 it therefore suffices to apply the following result to such a primary component:

Theorem 8 Assume (C1) and (C2). If $\pi_A$ is a prime power in $f[t]$ for each $A \in \mathfrak{g}$ then the following hold:

(a) $\pi_A(t) = (t - r_A)^n$, $r_A \in f$

(b) there is a basis putting $\mathfrak{g}$ in triangular form

Assertion (a) is equivalent to $\pi_A$ having a root $r_A \in f$. Therefore (a) follows from:

Lemma 9 Let $\alpha \in f[t]$ be a polynomial of degree $n$ that splits in $k[t]$. If $n \notin M$ then $\alpha$ has a root in $f$, and the sum of the multiplicities of such roots is $\notin M$.

Proof Let $R \subset k$ denote the set of roots of $\pi$, and $R_j \subset R$ the set of roots of multiplicity $j$. The Galois group $\Gamma$ has order $[k : f]$ and acts on $R$ by permutations. The cardinality of each orbit divides $[k : f]$, and $R \cap f$ is the set of fixed points of this action. Each $R_j$ is a union of orbits, as is $R_j \setminus f$. It follows that $\#(R_j \setminus f) \in M$.

Let $k \leq n$ denote the sum of the multiplicities of the roots that are not in $f$. Then

$$k = \sum_{j=2}^{n} j \cdot \#(R_j \setminus f)$$

Therefore $k \in M$ because $M$ is closed under addition. By hypothesis $n \notin M$, hence $n - k \notin M$ and $n - k > 0$. As $n - k$ is the sum of the multiplicities of the roots in $f$, the conclusion follows.

Now that (a) of Theorem 8 is proved, assertion (b) is a consequence of the following result:
Lemma 10 Let $\mathfrak{h}$ be a nilpotent Lie algebra of linear operators on $V$. Assume that for all $A \in \mathfrak{h}$ there exists $r_A \in \mathfrak{f}$ such that $\pi_A(t) = (t - r_A)^n$. Then $V$ has a basis putting $\mathfrak{h}$ in triangular form.

Proof Every $A \in \mathfrak{h}$ can be written uniquely as $r_A I + N_A$ with $N_A$ nilpotent and $I$ the identity map of $V$. It is easy to see that the set comprising the $N_A$ is closed under commutator brackets. Therefore $V$ has a basis triangularizing all the $N_A$ (Jacobson [2 II.2, Theorem 1']), and such a basis triangularizes $\mathfrak{h}$. This completes the proof of Theorem 1.

References

[1] M. Hirsch & A. Weinstein, Fixed points of analytic actions of supersoluble Lie groups on compact surfaces, Ergod. Th. Dyn. Sys. 21 (2001), no. 6, 1783–1787. See also http://front.math.ucdavis.edu/math.DS/0002013

[2] N. Jacobson, “Lie Algebras”, Interscience Tracts in Pure Mathematics No. 10. John Wiley, New York (1962)