Interaction of F- and D-strings in the Matrix Model

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Abstract

We study a configuration of a parallel F- (fundamental) and D-string in IIB string theory by considering its T-dual configuration in the matrix model description of M-theory. We show that certain non-perturbative features of string theory such as $O(e^{-\frac{1}{f^4}})$ effects due to soliton loops, the existence of bound state (1,1) strings and manifest S-duality, can be seen in matrix models. We discuss certain subtleties that arise in the large-N limit when membranes are wrapped around compact dimensions.
1 Introduction

Recently matrix models have been proposed as a non-perturbative description of M-theory [1] and of Type IIB string theory [2, 3]. Conceptually the idea is very interesting since one can explicitly write down matrices that represent interesting configurations such as D-p-branes [1, 2, 6, 4, 5, 7, 10]. Quite a bit of work has been done in extending these ideas as well as in checking the validity of these models by comparing with results from perturbative string theory and supergravity ([12] - [34]).

In an earlier paper, one of us [35] had investigated certain aspects of the IIB model [2]. A very important property of the IIB string theory is its SL(2,Z) duality invariance. How this symmetry manifests itself in the IIB matrix model is an open question although some suggestions were made in [35]. Being a non-perturbative symmetry, which exchanges F-(fundamental) strings with D-strings, this is impossible to see directly in the usual perturbative formulation of string theory. However the matrix model purports to be a non-perturbative description and this should be manifest. In fact in the IIA matrix model [1, 4], we can make this manifest if we recall the origin of this symmetry from M-theory [36, 37]. It corresponds to interchanging two cycles of a torus on which M-theory can be compactified to give IIA theory compactified on $S^1$. Since the IIA and IIB theories are related by T-duality one can then see the SL(2,Z) symmetry of IIB theory.

Another interesting question is whether one can see the effects of soliton loops which are expected to give terms of $O(e^{-1/g_s})$ [39]. Thus in the interaction between two D-branes one should expect loops of an open D-string stretched between the two objects. In particular, if one studies an F-string and a D-string, the object that stretches between them, cannot be an F-string (see Fig. 1). Flux conservation says that it has to be a (1,-1) string [37], which can be thought of as a bound state of an F-string and a D-string. Thus, if one can see such effects, one would also be seeing a bound state [40, 41, 42, 43, 44, 45].

In this paper we will try to do the above calculation. We consider an F-string wound around some direction (say, $X^7$), and a D-string, also wound around the same direction a certain distance away. We calculate the interaction between the two (one loop approximation) and look for the above effects. This calculation is done in the T-dual IIA string theory or equivalently in M-theory, using the matrix model of ref. [1]. In the M-theory matrix model
one can construct such a configuration in the form of a membrane (D-2-brane of IIA) wound around $X^7$ and $X^8$ and another membrane wound around $X^9$ and $X^7$. The latter membrane gives an F-string wound around $X^7$, if $X^9$ is taken to be the “eleventh” dimension of M-theory, whose radius fixes the IIA coupling constant and string tension. Interchanging the eighth and ninth directions then corresponds to S-duality of the T-dual IIB string theory. Note that the IIB string theory is the one obtained on T-dualizing $X^8$. We find that, indeed, non-perturbative effects described above can be seen. In the process of doing this calculation, we uncover some subtleties that arise at one loop, when a membrane is wound around a compact direction. This situation has been considered at tree level in [47, 54].

This paper is organized as follows. In section II we give a short review of matrix model formalism and discuss in general terms a membrane wound around a compact direction. In Section III we focus on a specific configuration of interest, calculate the one loop effective action and study the non-perturbative issues mentioned above. Section IV has some concluding comments.
2 Generalities

2.1 Review of M-theory matrix model

The matrix model for M-theory \[1\] is essentially a D-0-brane action, reinterpreted as a light cone gauge action where the “eleventh” dimension, of radius \(R_{11}\), is part of the light cone coordinates. The \(P^+\) component of momentum in the infinite momentum frame is thus approximately \(\frac{N}{R_{11}}\) (where \(N\) is the number of D-0branes), and is like a ten dimensional mass. When \(P^+\) is large (as it is in an infinite momentum frame) the action is that of a non relativistic particle with mass \(\frac{N}{R_{11}}\). Thus the evolution operator \(P^- = \frac{P^2 + m^2}{2P^+}\) is the Hamiltonian obtained from the D-0-brane action. The bosonic part of the action for \(N\) D-0-branes

\[
S = \frac{1}{2g_s} \int \frac{dt}{l_s} Tr\{(\partial_t X^i)^2 + \frac{1}{4\pi^2 l_s^4}[X^i, X^j]^2\} \quad (2.1.1)
\]

We have explicitly written factors of \(l_s\), to make the action dimensionless. Our conventions are as follows: \(g_s\) is the (IIA) string coupling constant and we have defined the inverse string tension to be \(2\pi\alpha'\) with \(\alpha' = \frac{l_s^2}{\pi}\). The parameters of M-theory are the radius \(R_{11}\) and the eleven dimensional Planck length \(l_{11}^p\). \(l_{11}^p\) is defined by the membrane tension which we have taken to be \(\frac{1}{(2\pi)^2(l_{11}^p)^3}\). This fixes \(g_s^2 = \left(\frac{R_{11}}{l_{11}^p}\right)^3\) and also \(\alpha' = \frac{(l_{11}^p)^3}{R_{11}}\). These relations also imply that \(g_s l_s = R_{11}\). Thus the kinetic term is essentially \(\frac{1}{2}m v^2\) with \(m = \frac{1}{R_{11}}\). The coefficient of the potential term can be seen to be \(\frac{R_{11}^2}{8\pi^2(l_{11}^p)^6}\).

The normalization of the potential term is chosen so that the classical mass of a D-2 brane comes out right. This will become clear as we proceed (see equation 2.3.5).

2.2 Matrix Formalism

A general \(N \times N\) matrix can be expanded in a basis,

\[
A = \sum_{m,n} A_{mn} e^{impe^{lnq}} \quad (2.2.1)
\]

\[\text{1 This supersymmetric action has also been studied in a different context by [48]}\]
where \( q, p \) are \( N \times N \) hermitian matrices that satisfy

\[
[q, p] = \frac{2\pi i}{N}
\]  (2.2.2)

There is the usual caveat that \( N \) has to be infinity for this to work.\(^2\) To make things concrete, consider a particle in a one dimensional box of length \( L \), with periodic boundary conditions. Also assume an ultraviolet cutoff i.e. a lattice spacing “\( a \)”. The number of sites is \( L/a \equiv N \). We introduce canonical momentum and position operators \( P \) and \( Q \). The momentum eigenfunctions are of the form

\[
\phi_n = \frac{1}{\sqrt{L}} e^{ip_n x}
\]  (2.2.3)

where \( p_n = \frac{2\pi n}{L} = \frac{2\pi n}{Na} \) is the \( n^{th} \) eigenvalue of the momentum operator \( P \). Let us choose \( a = \sqrt{\frac{2\pi}{N}} \) and \( L = \sqrt{N2\pi} \). Then

\[
p_n = \frac{2\pi n}{\sqrt{N2\pi}} \quad 0 \leq n \leq N - 1
\]  (2.2.4)

The operator \( p = P \sqrt{\frac{2\pi}{N}} \), has eigenvalues \( \frac{2\pi n}{N} \). The matrix \( e^{iP\sqrt{\frac{2\pi}{N}}} \) then is

\[
e^{ip} = e^{iP\sqrt{\frac{2\pi}{N}}} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^{N-1}
\end{pmatrix}
\]  (2.2.5)

where \( \omega \) are the \( N^{th} \) roots of unity. \( Q \) has possible values \( 0, a, 2a, ..., na, ...(N-1)a \) which is the same as

\[
(0, \frac{1}{\sqrt{N}}, ..., \frac{N-1}{\sqrt{N}})\sqrt{2\pi}
\]  (2.2.6)

the same as \( P \). In this normalization \( [Q, P] = i \). If I choose \( q = \sqrt{\frac{2\pi}{N}}Q \) so that

\[
[q, p] = \frac{2\pi i}{N}
\]  (2.2.7)

\(^2\) For a careful treatment of the finite \( N \) analog of this construction see [49].
it has eigenvalues $\frac{2\pi}{N} i$, the same as $p$.

$$e^{iq} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \ddots & 1 \\ 1 & 0 & \ddots & \ddots \end{pmatrix}$$ (2.2.8)

Note again that the matrix $q$ satisfying (2.2.7) and (2.2.8) only exists in the limit $N \to \infty$ which is also the limit $L \to \infty$ or $a \to 0$. In this limit $q$ and $p$ have continuous eigenvalues ranging from 0 to $2\pi$.

In the one loop calculation we will need to define adjoint action of matrices. If we define $\bar{O} = \text{Adjoint}(O)$ as

$$[O, X] = \bar{O}X$$ (2.2.9)

Then, clearly

$$\bar{p} = -\frac{2\pi i}{N} \frac{d}{dq}; \bar{q} = \frac{2\pi i}{N} \frac{d}{dp}$$ (2.2.10)

Note that

$$[\bar{p}, \bar{q}] = 0$$ (2.2.11)

Another useful operator is $\sigma_3$ whose adjoint $\bar{\sigma}_3$ has eigenvalues $\pm 2, 0, 0$.

The following can also be easily checked:

$$\overline{A \otimes B} = \overline{A} \otimes B + A \otimes \overline{B} - \overline{A} \otimes \overline{B}$$ (2.2.12)

### 2.3 Membrane wrapped around $S^1$

When one space dimension is compactified, a collection of D-0-branes is physically equivalent to a D-1-brane wrapped around the compact dimension $[46]$. An explicit construction has been given in [47]. Matrix model compactifications on torii have been discussed in the literature [50, 51, 52, 53, 54, 55, 56, 57, 58]. In this section we concentrate on the $S^1$ case and investigate what happens at one loop. This is in preparation for Section 3.

The action is that of 9+1 dimensional Supersymmetric $U(N)$ Yang-Mills theory reduced to 1+1 dimension.

$$S = \frac{1}{2g_s} \int \frac{dt}{l_s} \int_0^{2\pi L_5} \frac{dx}{2\pi L_5^5} Tr \{ (\partial_t X^i)^2 - (D_x X^i)^2 +$$
\[ D_x = \partial_x + i A_9 \]

\[ D_x = \partial_x + i A_9 \]

is the covariant derivative in a direction \( X^9 \), which is T-dual to \( X^9 \), and is of radius \( L_9^{\ast} \). \( x \) is thus a coordinate along a D-1-brane wound around \( X^9 \). If \( X_{mn}^i \) is a matrix describing the lattice of D-0-branes, \( (m, n = 0...M) \) then following [47], we have

\[
A^9 = \frac{1}{2\pi\alpha'} \sum_{n=0}^{M} e^{i m x L_9^{\ast}} X_{0n}^9 \]  

(2.3.2)

Note that \( X_{00} \) is the original D-0-brane matrix of the uncompactified theory.

Let us now consider a configuration where a membrane of M-theory, or equivalently a D-2-brane of IIA string theory is wrapped around this compact dimension. This can be described as a non-trivial background for the gauge field in the 1+1 dimensional theory. In the D-0-brane language we can describe the membrane by using, as membrane coordinates, the \( N \times N \) matrices \( p, q \) and the wrapping by [1, 4]

\[
X^9 = L_9 p \]  

(2.3.3)

The eigenvalues of \( p \) range from \( 0 - 2\pi \), thus (2.3.3) describes a membrane with one side wrapped around a circle of radius \( L_9 \). If we set \( X_{00} \) of eqn.(2.3.2) to be that given in (2.3.3) we get the background gauge field configuration corresponding to a wrapped membrane.

We can also wrap the other side of the membrane around another compact dimension (say \( X^7 \)), if we want, by setting

\[
X^7 = L_7 q \]  

(2.3.4)

One can check that the value of the classical Lagrangian given in (2.3.1) for this membrane configuration is

\[
\frac{(L_7 L_9)^2}{2 R_{11}} \]  

(2.3.5)

This is equal to

\[
\frac{(m_{\text{membrane}})^2}{2 p^+} \]  

(2.3.6)
as expected.

Let us consider the spectrum of small fluctuations around this background. We will need to do this in calculating the one loop effective action in a similar situation in the next section.\footnote{Of course this particular configuration is a BPS state, hence the one loop action vanishes.}

The relevant operator is

\[ D_x = \partial_x \otimes I + \text{Adj}(I \otimes iA^0) \tag{2.3.7} \]

Since the \( x \)-derivative acts on a different space than the matrix indices we have used a direct product notation for clarity. Also since \( A^0 \) acts by commutation we have used the adjoint notation introduced in the last sub-section.

The space of eigenfunctions that it is acting on is of the type

\[ e^{i r \frac{2\pi}{N}} e^{i m p} e^{i n q} \tag{2.3.8} \]

Here \( r, n, m \) are integers. Using the fact that \( \text{adj}(I \otimes p) = I \otimes -\frac{2\pi i}{N} \partial_q \) we find that the eigenvalues are \( \frac{rN + n}{N^2} \).

\[ \text{Figure 2: } \text{The translation operator } \partial_x \otimes I + I \otimes \frac{1}{N} \partial_q \text{ moves along the spiral shown here.} \]

\[ \text{Diagram with a spiral trajectory showing } x, q, \text{ and } \frac{2\pi}{N}. \]
Note that the covariant derivative $D_x$ has become

$$\partial_x \otimes I + I \otimes \frac{1}{N} \partial_q$$  \hspace{1cm} (2.3.9)$$

While $x$ is a coordinate along the compact direction, in the usual interpretation where a wrapped membrane gives a string, $q$ is a coordinate along the string (which is also along the unwrapped direction of the membrane). Thus the operator $D_x$ generates a translation along a spiral, as shown in Fig 2. Thus a rotation by $2\pi$ along the $x$ direction is accompanied by a shift $\frac{2\pi}{N}$ in the transverse direction. Only when it winds $N$ times around does it come back to the starting point.

Thus the effective (dual) radius in the presence of a wrapped membrane is $NL_9^*$, rather than $L_9^*$. This is reminiscent of the “long string” phenomenon that has cropped up in different places [59, 60, 8, 9]. We will see this same effect in the next section where we discuss an F-string and a D-string.

3 F-D interaction

3.1 The Configuration

We consider M-theory with three compact directions with radii $R_{11}, L_9, L_8$. $R_{11}$ is the radius of what we have thus far referred to as the eleventh dimension and defines a IIA string theory with an inverse tension $\alpha' = \frac{l_{11}}{R_{11}}$. But one can equally well consider a IIA string theory defined by an inverse tension $\beta' = \frac{l_{11}}{L_9}$. Let us refer to the first description as the $\alpha'$ description and the second one as the $\beta'$ description. Our strategy will be to define a matrix model using the $\alpha'$ description and perform calculations involving two membranes wrapped around $L_9$ and $L_8$ respectively. In this description these are two wrapped D-2-branes. On the other hand, in the $\beta'$ description one of these (the one wrapped around $X^9$) becomes the fundamental or F-string and the other is a wrapped D-2-brane. We can now consider the T-dual of the $\beta'$-description with the duality being performed on $X^8$ (i.e. the dual radius $\tilde{L}_8 = \frac{\beta'}{L_9}$). Now our configuration is a D-1-brane (or a D-string) and an F-string of IIB theory. Thus in the T-dual $\beta'$ description we have the configuration that we wanted viz, an F-string and a D-string. For convenience we can wind these two around a large compact coordinate $X^7$ of radius $L_7$. 


We will assume $L_7$ is large enough that the winding sectors are unimportant for the dynamics. We will place these two strings at a distance $b$ from each other in the $X^6$ direction. All the calculations will however be performed using the matrix model describing D-0-branes of the $\alpha'$ description. Thus, to summarize, in the $\alpha'$ description we have two D-2-branes wrapped around $L_9, L_7$ and $L_8, L_7$ respectively. We calculate the one loop effective potential using the matrix model. In the end we will interpret the result in terms of the T-dual of the $\beta'$ description.

The string coupling constant in the $\alpha'$ description is $g_{s\alpha'}^2 = (\frac{R_{11}}{l_{11}^p})^3$ while in the $\beta'$ description it is $g_{s\beta'}^2 = (\frac{L_9}{l_{11}^p})^3$. When we T-dualise, it becomes $\tilde{g}_{s\beta'} = g_{s\beta'} \frac{L_8}{l_{11}^p} = \frac{L_9}{L_8}$. This is the usual relation between M-theory on $T^2$ and IIB string theory [36, 37, 38].

As explained in the introduction, this is a configuration where one should be able to see S-duality. S-duality merely interchanges the F- and D-strings and inverts the coupling constant. Furthermore the open string connecting an F-string to a D-string cannot be an F-string. In fact the simplest possibility is a (1,-1) string which is expected to exist and can also be considered to be a bound state of a (1,0) and a (0,-1) string. Thus in effect we have soliton loops and one expects terms of $O(e^{-\frac{1}{g_s}})$ in the effective action. These are the two aspects that we would like to see explicitly. So let us proceed with the calculation.

### 3.2 One loop effective action

Following section 2 we can describe the above configuration by the following $2N \times 2N$ matrices:

$$
X^9_{0,0} = \begin{pmatrix} L_9p & 0 \\ 0 & 0 \end{pmatrix} \\
X^8_{0,0} = \begin{pmatrix} 0 & 0 \\ 0 & L_8p \end{pmatrix} \\
X^7_{0,0} = \begin{pmatrix} L_7q & 0 \\ 0 & L_7q \end{pmatrix} \\
X^6_{0,0} = \begin{pmatrix} \frac{b}{2} & 0 \\ 0 & -\frac{b}{2} \end{pmatrix}
$$

(3.2.1)
The subscripts on the matrices represent the additional indices introduced by the compactification of the 8th and 9th directions as described in [47]. The action is now a 2+1 SYM theory on radii $L_8^*, L_9^*$. Although $X^7$ is compact, since we have assumed that $L_7$ is large enough that the winding sector does not contribute to the dynamics, we do not write a 3+1 SYM theory. The string length is defined by $\alpha'$ as before. (Remember that all calculations are in the $\alpha'$ description.) It is $(i, j : 1-7)$

$$\frac{1}{g_s} \int \frac{dt}{t_s} \int_0^{2\pi L_0^*} dy \int_0^{2\pi L_0^*} dx \frac{dx}{2\pi L_8^*} \text{Tr} \{ (\partial_t X^i)^2 - (D_x X^i)^2 - (D_y X^i)^2 + (F_{09})^2 + (F_{08})^2 + (F_{89})^2 + \frac{1}{4\pi^2 l_s^4} ([X^i, X^j])^2 \} \quad (3.2.2)$$

Note that $g_s L_8^* L_9^* = g_s' l_s^2$, where $g_s'$ is the coupling constant of the IIA theory after two T-dualities in the 8th and 9th directions [4]. The covariant derivatives are as in section 2: $D_x = \partial_x + iA_8$ and $D_y = \partial_y + iA_9$. Equation (3.2.1) implies that $A_{8,9}$ have expectation values which are

$$< A_8 > = \frac{1}{2\pi \alpha'} X_{0,0}^8$$

$$< A_9 > = \frac{1}{2\pi \alpha'} X_{0,0}^9 \quad (3.2.3)$$

In addition $X^6, X^7$ have expectation values.

In calculating the one loop effective action we have to identify a part of the action that is quadratic in small fluctuations about a background. This can be done as in [3]. The result is that the fluctuation operator for the bosonic fluctuations is

$$\mathcal{H}_{\text{bos}} = P_\lambda^2 \delta_{\mu\nu} - 2i F_{\mu\nu} \quad (3.2.4)$$

Here $P_\lambda$ are (adjoint) operators, and are defined below. We will assume that they all have dimensions of momentum. And,

$$F_{\mu\nu} = i[P_\mu, P_\nu] \quad (3.2.5)$$

is also an adjoint operator.

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4The reader should not confuse this with $\tilde{g}_s$ which was defined as the IIB coupling obtained after a T-duality along $X^8$ in the $\beta'$ description.
Let

\[ p_6 = \frac{1}{2\pi\alpha'} X^6_{0,0} \]
\[ p_7 = \frac{1}{2\pi\alpha'} X^7_{0,0} \]
\[ p_8 = \langle A_8 \rangle \]
\[ p_9 = \langle A_9 \rangle \]  

(3.2.6)

Then,

\[ P_6 = I \otimes \mathrm{Adj}(p_6) \]
\[ P_7 = I \otimes \mathrm{Adj}(p_7) \]
\[ P_8 = -i\partial_x \otimes I + I \otimes \mathrm{Adj}(p_8) \]
\[ P_9 = -i\partial_y \otimes I + I \otimes \mathrm{Adj}(p_9) \]
\[ P_0 = -i\partial_t \otimes I \]

(3.2.7)

We have split the space into direct product of functions of \(x, y, t\) on which the derivatives act and the space of matrices on which the adjoint of the matrices act.

We will now make a rotation of the axes in the (8,9) plane into \((8', 9')\) such that \(f_{79'}\) is proportional to the identity. This will make \(F_{79'} = 0\). The required transformation is

\[ p'_9 = \frac{L_0 p_8 + L_8 p_9}{\sqrt{L_8^2 + L_9^2}} \]
\[ p'_8 = \frac{L_9 p_9 - L_8 p_8}{\sqrt{L_8^2 + L_9^2}} \]

(3.2.8)

This gives (we have set \(\alpha' = 1\)) using (3.2.1) and in an obvious notation where \(2N \times 2N\) matrices are expressed as a direct product of an \(N \times N\) matrix and a \(2 \times 2\) matrix,

\[ p'_9 = \frac{1}{2\pi\sqrt{L_8^2 + L_9^2}} [L_8 L_0 p \otimes I] \]
\[ p'_8 = \frac{1}{2\pi\sqrt{L_8^2 + L_9^2}} [\frac{L_9^2 - L_8^2}{2} p \otimes I + \frac{L_8^2 + L_9^2}{2} p \otimes \sigma_3] \]

(3.2.9)
Using (2.2.12) we can calculate the adjoints of these and substitute into (3.2.7). We get (in units where $\alpha' = 1$)

\[
P'_8 = -\frac{i}{\sqrt{L_8^2 + L_9^2}}(L_9 \partial_y - L_8 \partial_x) \otimes I \otimes I + I \otimes P'_{8\text{I}} + I \otimes P'_{8\text{II}}
\]

\[
P'_{8\text{I}} = \frac{\sqrt{L_8^2 + L_9^2}}{4\pi}i(-\frac{2\pi i}{N} \partial \otimes \sigma_3 + p \otimes \bar{\sigma}_3 + \frac{2\pi i}{N} \partial \otimes \bar{\sigma}_3)
\]

\[
P'_{8\text{II}} = \frac{i}{4\pi\sqrt{L_8^2 + L_9^2}}(\frac{2\pi i}{N} \partial \otimes I)
\]

\[
P_9' = \frac{-i}{\sqrt{L_8^2 + L_9^2}}(L_9 \partial_x + L_8 \partial_y) \otimes I + \frac{L_8 L_9}{\sqrt{L_8^2 + L_9^2}}(\frac{-i}{N} \partial \otimes I)
\]

\[
P_7 = I \otimes L_7(\frac{i}{N} \partial \otimes I)
\]

\[
P_6 = I \otimes \frac{b}{4\pi} I \otimes \bar{\sigma}_3
\]

One can also check the following commutators:

\[
[P'_8, P'_9] = [P_7, P'_9] = 0
\]

\[
i[P_7, P'_8] = F_{78} = -\frac{\pi L_7 \sqrt{L_8^2 + L_9^2}}{4\pi^2 N} \bar{\sigma}_3
\]

$P_6$ commutes with all of them. The non-zero eigenvalues of $F_{78}$ are thus

\[
\pm \omega = \pm \frac{L_7 \sqrt{L_8^2 + L_9^2}}{2\pi N}.
\]

Thus on this subspace $P_7$ and $P_{8\text{I}}$ behave like harmonic oscillator variables and one concludes that the eigenvalues of $P_{7\text{I}}^2 + P_{8\text{I}}^2$ are given by

\[
E_n = \frac{4\pi L_7}{N} \sqrt{L_8^2 + L_9^2}(\bar{n} + \frac{1}{2}) = 2\omega(\bar{n} + \frac{1}{2})
\]

These operators act on a space spanned by the matrix functions:

\[
\Phi_{mnrs} = e^{im\phi} e^{in\psi} e^{ir \frac{\tau_+}{2}} e^{is \frac{\tau_-}{2}}
\]

These are to be multiplied by a $2 \times 2$ matrix on which $\bar{\sigma}_3$ gives 2.
The harmonic oscillator eigenfunctions are to be made by taking linear combinations of \( e^{imp} \). If we denote these by \( H_n(p) \), then the eigenfunctions \( H_n(p) e^{in \frac{r}{L^2} e^{is \frac{n}{L^2}}} \) for any value of \( r, s, n \) have the same eigenvalue \( \bar{E}_n \).

\( P_0 \) has eigenfunctions \( \Phi_{mnrs} \) for any values of \( m, n, r, s \). Its eigenvalues are

\[
\bar{p}_0 = \frac{L_8 L_9}{\sqrt{L_8^2 + L_9^2}} ([r + s + n] \frac{N}{R})
\]

If we define

\[
R = \frac{\sqrt{L_8^2 + L_9^2}}{L_8 L_9} = \sqrt{L_8^2 + L_9^2}
\] (3.2.17)

we see that the eigenvalues are \( \frac{1}{NR} [N(r + s) + n] \). The range of \( n \) is 1 - \( N \) and so we see that the combination \( N(r + s) + n \) runs over the entire range 1 to \( NM \) without repetition. (We have assumed that the combination \( (r + s) \) and \( r - s \) have ranges from 1 to \( M \) when \( M \) is some cutoff). Thus the only combination of integers that does not appear in the expression for the eigenvalues is \( (r - s) \), thus leading to a degeneracy of \( O(M) \). To summarize

\[
E = P_0^2 + \left( \frac{b}{2\pi\alpha'} \right)^2 + (2\bar{n} + 1)\omega + \left( \frac{N(r + s) + n}{NR} \right)^2
\] (3.2.18)

are the eigenvalues of \( P_0^2 \) (in the subspace where \( F_{78} \) is non-zero) and the degeneracy of each eigenvalue is \( M \). Note that \( NR \) is thus the effective radius of the dual theory and has acquired an extra factor \( N \) just as we saw in section 2. Finally the \( P_0 \) eigenfunctions are of the form \( e^{ip_0 t} \). We can assume a spacing of \( \frac{1}{T} \) for the eigenvalues \( p_0 \), where \( T \) is an infrared cutoff in the time direction. This brings a factor of \( T \) when converting the trace over eigenvalues of \( P_0 \) to an integral over \( p_0 \). Finally, the fermionic determinant also involves the same operators \( \bar{H} \).

The calculation of the one loop effective action is straightforward once we have all the above information. The final answer for the one-loop effective action is

\[
T2M \int \frac{dt}{t} \int dp_0 \sum_v e^{\exp\{-\left(\frac{p_0^2 + \left(\frac{v}{NR}\right)^2 + \left(\frac{b}{2\pi\alpha'}\right)^2}{\omega}\right) t\}2\tanh\left(\frac{t}{4}\right)\sinh^2\left(\frac{t}{4}\right)}
\] (3.2.19)

\[6\]Terms involving \( (r - s) \) have the effect of translating “p” so that \( H_n(p) \) becomes \( H_n(p - \text{const}) \). This does not affect \( \bar{E}_n \).
The coefficient of $T$ can be interpreted as the potential energy of the configuration. The various factors of $M, N$ in (3.2.19) imply that they are parameters of the theory that have some physical significance. If we are to take the limit $M, N \to \infty$ one has to worry about finiteness of the final answer. Furthermore some terms that describe physical effects may go to zero. Let us therefore first investigate the physics of the formula (3.2.19) for finite $N, M$. Later we will describe a renormalization of the other parameters that eliminates all $M, N$ dependence.

We can consider the limit where $b$ is much larger than all other lengths $L_7, L_8, L_9, l_s$. Thus we can consider $t$ to be small and expand the hyperbolic function, which starts off as $t^3$. We also use a Poisson resummation to extract systematically the effects of a finite $R$. We find the following:

$$V = \frac{M \omega^3 3\pi}{64} \left[ \frac{4NR}{3b^4} + \sum_{m=1}^{\infty} \frac{8m^2 \pi^2 (NR)^3}{3b^2} K_2(2bm\pi NR) \right]$$

(3.2.20)

$$= \frac{M \omega^3 3\pi}{64} \left[ \frac{4NR(\alpha')^4}{3b^4} + \sum_{m=1}^{\infty} \frac{8m^2 \pi^2 (NR)^3(\alpha')^2}{3b^2} \sqrt{\frac{\alpha'}{4mbNR}} e^{-\frac{mbNR}{\alpha'}} \{1+O(1/b)\} \right]$$

(3.2.21)

where we have inserted appropriate powers of $\alpha'$. The $b^{-4}$ is as expected when there are three compact dimensions. The power of $b^{-4}$ for the Yukawa terms is also as expected and corresponds to a simple pole in momentum space. As explained earlier, this configuration can be viewed, either as two D-2-branes wound around $X^8, X^7$ and $X^9, X^7$ or in the T-dual picture as an F-string and a D-string, both wound around $X^7$. In the latter case the string coupling $\tilde{g}_{s\beta'}$ of the IIB theory is defined as $\frac{L_8}{\tilde{L}_8}$, and the string tension by $\frac{1}{\tilde{g}_{s\beta'}} = \frac{L_8}{(l_{11})^3}$. Thus $\frac{2\sqrt{g_s}}{\alpha'}$, which can be written as $N \sqrt{\tilde{L}_8^{1/2} + \tilde{L}_8^{1/2}/\alpha'}$ can now be reexpressed as $\frac{N\tilde{L}_8}{\sqrt{g_{s\beta'}}} \sqrt{1 + \frac{1}{g_{s\beta'}^{1/2}}} \tilde{L}_8$, where $\tilde{L}_8$ is the dual of $L_8$ with respect to $\beta'$ (i.e. $\tilde{L}_8 = \frac{\tilde{L}_8}{L_8}$). This is the mass of a $(1,1)$ or $(1,-1)$ string wound around a radius of $N\tilde{L}_8!$. Thus the Yukawa terms in the potential can be understood as the contribution (in the closed string channel) of closed $(1,-1)$ strings of the IIB theory wound a radius $N\tilde{L}_8$. The fact that there is a simple pole (in momentum space) proves the existence of a bound state. These terms are also the $O(e^{-1/\tilde{g}_{s\beta'}})$ terms that one expects in non-perturbative string theory.
The value of $\tilde{L}_8$ in terms of the original M-theory parameters is $\frac{(l_{11}^9)^3}{L_0 l_8}$. This is the usual connection between M-theory on a 2-torus and IIB on a circle \cite{36, 38}. The factor $N$ comes from the presence of the wrapped membrane. It is reminiscent of a “long string”. Somehow a wrapped membrane has additional low energy excitations that make it look as if the radius of compactification (in this case the dual radius) is $N$ times larger. In the limit $N \to \infty$ these terms drop out. However as will be shown later, it is possible to renormalize the parameters such that this $N$ dependence disappears.

We thus see that the Yukawa terms represent the non-perturbative $e^{-\frac{1}{\tilde{g}^2}}$ effects that are expected in string theory due to soliton loops. Here the soliton is the (1,-1) open string connecting a D-string with an F-string. The fact that a (1,-1) string connects an F-string with a D-string is also expected when you consider flux conservation \cite{37}. In fact, as mentioned above, one can view the above calculation as a demonstration of the existence of a (1,-1) string.

Another point worth mentioning is that S-duality (which consists of interchanging $L_8$ and $L_9$) is manifest in this formalism.

Finally, there are a couple of simple checks to see that the answer (3.2.21) is correct. The long distance force between a D-string and an anti-D-string should be exactly twice that between an F-string and a D-string (when they are degenerate in mass, i.e. $\tilde{g}_s g_s' = 1$). This is because the $B^{ij}$ contributes to the force in one case but not in the other. If one repeats the calculation with two membranes wound around $X^9$ after setting $L_0 = L_8$ one finds that the net effect is to replace $\omega$ by $\sqrt{2}\omega$ and $R$ by $\frac{R}{\sqrt{2}}$. Making these changes in the leading part of (3.2.21) we see that we get a factor of 2.

The other check is to compare the answer with what one expects for the gravitational potential between two objects in Newtonian gravity. We consider the case where $L_8 = L_9$. The mass of a D-2-brane is $m = \frac{L_8 L_7}{g_s l_s^2}$. The gravitational coupling constant is $G \approx \frac{(l_s^8 g_s^2)}{L_7 L_8 L_9}$. The potential is thus

$$G \frac{m^2}{b^4} \approx \frac{l_s^2 L_7}{b^4} \quad (3.2.22)$$

We can also do this in the T-dual $\beta'$ description, pretending that we are in a IIB theory and, of course, get the same answer. Note that the answer is manifestly T-dual symmetric when we perform T-duality in $X^8$ and $X^9$. 

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The configuration we started off with was manifestly symmetric under this T-duality, hence the final expression should also be so. To compare this with (3.2.21) we have to map this potential to an M-theory calculation as was done for instance in [15]. In M-theory matrix model we calculate\[\approx m_{int}^2 \int P_{11} \int ,\]
where \(m_{int}\) is the contribution of the interaction to the rest mass and \(P_{11}^{int}\) is the momentum due to this mass. This is \(\approx \frac{m_{int}}{\gamma}\), where \(\gamma\) is the boost factor. As shown in [15] there is an extra factor of \(\gamma\) due to the fact that in the boosted frame the distance between clusters of D-0-brane shrinks, and so we get a larger answer. So we get two powers of \(\gamma\): \(\frac{m_{int}}{\gamma^2}\). We can use
\[
\gamma = \frac{P^{11}}{L_8 L_7} \tag{3.2.23}
\]
\[
= \frac{N(l_p^{11})^3}{R_{11} L_8 L_7} = \frac{N l_p^2}{L_8 L_7} \tag{3.2.24}
\]
Here we have used \(P^{11} = \frac{N}{R_{11}}\). Putting all this together we get
\[
\frac{L_8^3 L_9^3 R_{11}}{N^2 b_1 l_p^{11} L_3^3} \tag{3.2.25}
\]
One can check that [3.2.21] gives this answer (except for the factor of \(M\) which can be absorbed in \(T\)). Note that the limit \(N \to \infty\) does not affect the result for the potential energy. It just makes the boost factor \(\gamma\) infinite. However it does affect the Yukawa terms.

One can also renormalize the parameters \(L_7, L_8, L_9, b, R_{11}, l_p^{11}\) such that physical quantities such as a) rest mass of the membrane , and b) the potential energy calculated in the rest frame, have no \(N, M\) dependences. The rescaling is done by requiring that all the dependence on \(N, M\) are in the boost factors \(\gamma\) for the non interacting piece (2.3.5) and \(\gamma^2\) for the interacting piece (3.2.21). Thus the physics becomes \(N, M\) independent. In fact we find that even the renormalized \(\gamma\) can be made \(N, M\) independent, so nothing depends on \(N, M\). This is intriguing. Perhaps when additional processes are considered the situation will be clarified. The rescaling is as follows:(Barred quantities are held fixed as \(N, M \to \infty\))
\[L_{8,9} = N M \bar{L}_{8,9}\]
\[ L_7 = M \bar{L}_7 \]
\[ R_{11} = N \bar{R}_{11} \]
\[ b = M \bar{b} \]
\[ (l_p^{11})^3 = NM^2 (\bar{l}_p^{11})^3 \quad (3.2.26) \]

The above rescalings also imply that
\[ \alpha' = M^2 \bar{\alpha'} \]

One can check that if we plug in these rescalings, the final answer has no \( N, M \) dependence. The masses of the membranes (which were finite and \( N, M \) independent to begin with) are unchanged by this rescaling. The exponent in the Yukawa potential can be seen to be finite: \( \frac{\bar{b} \bar{b}}{\alpha'} \). Other processes need to be investigated before we fully understand the significance of this rescaling.

4 Conclusions

In this paper we have calculated the one loop effective action for a configuration of an F-string and a parallel D-string some distance away. The motivation was to see, within the context of the matrix model, whether non-perturbative effects mentioned in the introduction viz. S-duality and \( O(e^{-\frac{1}{gs}}) \) effects can be seen. The calculation demonstrates that it is indeed possible. In fact the soliton that produces this effect are the (1,-1) strings and thus this calculation also is an independent proof of the existence of these bound states.

The intriguing factor of \( N \) in (3.2.19) needs an explanation. Somehow wrapping a membrane has introduced light excitations that make the dual radius much larger. This does not affect the physics as far as the leading term is concerned. But in higher order terms the exponential fall-off with distance is affected by \( N \). Presumably analogous phenomena exist on higher dimensional torii.

The significance of the renormalization is also a little unclear. If we are given that freedom, the significance of the parameter \( N \) seems to disappear. One has to check whether this renormalization can be consistently done for all processes.
Finally it would be interesting to repeat this calculation in the IKKT matrix model for the IIB superstring.

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