Modified entanglement purification scheme with doubly entangled photon state

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Recently Xiao et al. proposed a scheme for entanglement purification based on doubly entangled photon states (Phys. Rev. A 77(2008) 042315). We modify their scheme for improving the efficiency of entanglement purification. This modified scheme contains two steps, i.e., the bit-flip error correction and the entanglement purification of phase-flip errors. All the photon pairs in the first step can be kept as all the bit-flip errors are corrected. For purifying the phase-flip errors, a wavelength conversion process is needed. This scheme has the advantage of high efficiency and it requires the original fidelity of the entangled state wanted fay lower than other schemes, which makes it more feasible in a practical application.

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I. INTRODUCTION

During the past decades, quantum entanglement presented many useful properties in quantum information processing and transmission, such as quantum key distribution\textsuperscript{1,2,3,4}, quantum dense coding\textsuperscript{5}, quantum teleportation\textsuperscript{6}, and so on. In quantum communication, entangled states are used to built quantum channel for information transmission. However, in a practical condition the noise of the channel will inevitably affect the entangled quantum states and even make them be mixed states. The reduced entanglement of the quantum systems will decrease the success probability of quantum teleportation of an unknown state and even make a quantum key distribution insecure. For accomplishing the task of secret quantum communication, people should obtain some maximally entangled states from a less-entanglement ensemble, called entanglement purification. In 1996, Bennett et al.\textsuperscript{2} proposed the first protocol for entanglement purification of Werner states with controlled-not (CNOT) gates and bilateral rotations. In 2001, Pan et al.\textsuperscript{3} proposed an entanglement purification protocol with polarization beam splitters (PBSs) and sophisticated single photon detectors. In 2003, they experimentally demonstrated entanglement purification of bit-flip error by using PBSs and four-path coincidence photon counters.\textsuperscript{4}. Also, entanglement purification based on the parametric down conversion (PDC) source was presented by Simon and Pan \textsuperscript{5}. Recently, Sheng, Deng and Zhou\textsuperscript{6} introduced a perfect protocol for entanglement purification not only for PDC source but also for ideal source with nonlinear optics. So far, entanglement purification have been widely studied by many groups\textsuperscript{7,8,9}.

Considering the novel idea of entanglement with multiple degrees of freedom, many quantum communication protocols can be improved. For instance, Aolita and Walborn\textsuperscript{10} proposed a quantum communication protocol based on polarization and mode entangled state, which has a high capacity. Barreiro et al.\textsuperscript{11} had demonstrated a superdense coding experiment based on two degrees of freedom of photons, which beats the channel capacity in quantum communication. In 2005, Ravaro et al.\textsuperscript{12} produced the doubly entangled state with two degrees of freedom, i.e., the frequency and the polarization of photons. They generated two-photon states using semiconductor waveguides pumped by lasers. The spontaneous parametric down conversion process generates a pair of entangled photons with discrete frequencies. Recently, Xiao et al.\textsuperscript{13} studied the properties of doubly entangled photon state (DEPS) and proposed a entangled purification protocol for DEPSs. This protocol can be realized with two steps, i.e., the entanglement purification for bit-flip errors and that for phase-flip errors. Two wavelength-division multiplexing (WDM) devices are used in the first step to exclude the states with bit-flip errors from the mixed ensemble. After the entanglement purification for bit-flip error, the two parties select two states from the remaining photon states and perform the PBS operations to distinguish the states from those with phase-flip errors. However, there is a problem in the second step as it requires that the two photon pairs should be in the same state, which means both of them should be in the state \(\Phi^+\) or in the state \(\Phi^-\). In this way, the second step cannot purify the ensemble in a mixed state with the unit fidelity.

In this paper, we modify Xiao’s entanglement purification protocol for improving the success probability for entanglement purification of bit-flip errors and decreasing phase-flip errors. In the first step for the purification of bit-flip errors, we add two half wave plates (HWPs)
in the setups in Xiao’s protocol. Just this modification will make all the entangled states with or without bit-flip errors kept as the bit-flip errors will be canceled by the spatial modes and the HWPs. We complete the purification of phase-flip errors in the second step following some ideas from the protocol proposed by Pan et al. This protocol has the advantage of high success probability and works more efficiently than Xiao’s protocol.

II. MODIFIED ENTANGLEMENT PURIFICATION PROTOCOL BASED ON DEPS

A. bit-flip error correction for a DEPS

The DEPS generated in Ravaro’s experiment can be described as \( \rho \):

\[
|\Phi^{\pm}_{ab}\rangle = \frac{1}{\sqrt{2}}(|H, \omega_s\rangle|H, \omega_i\rangle \pm |V, \omega_s\rangle|V, \omega_i\rangle). \tag{1}
\]

Here \( H \) and \( V \) represent the horizontal and the vertical polarizations of photons, respectively, and \( \omega_s(\omega_i) \) and \( \omega_i(\omega_s) \) correspond to the frequencies of entangled photons. Considering the noisy channel transmission, we can not avoid the state to be disturbed. The bit-flip errors or phase-flip errors will take place on one of the particle or on both of the two particles. Then the original state will be changed to

\[
|\Phi^{\pm}_{ab}\rangle = \frac{1}{\sqrt{2}}(|H, \omega_s\rangle|H, \omega_i\rangle \pm |V, \omega_s\rangle|V, \omega_i\rangle); \tag{2}
\]

\[
|\Psi^{\pm}_{ab}\rangle = \frac{1}{\sqrt{2}}(|H, \omega_s\rangle|V, \omega_i\rangle \pm |V, \omega_s\rangle|H, \omega_i\rangle); \tag{3}
\]

\[
|\Gamma^{\pm}_{ab}\rangle = \frac{1}{\sqrt{2}}(|V, \omega_s\rangle|H, \omega_i\rangle \pm |H, \omega_s\rangle|V, \omega_i\rangle); \tag{4}
\]

\[
|\Upsilon^{\pm}_{ab}\rangle = \frac{1}{\sqrt{2}}(|V, \omega_s\rangle|V, \omega_i\rangle \pm |H, \omega_s\rangle|H, \omega_i\rangle). \tag{5}
\]

In this time, a pure state system transmitted may become a mixed state ensemble described by the Werner state

\[
\rho = F|\Phi^{\pm}_{ab}\rangle \langle \Phi^{\pm}_{ab}| + \frac{1-F}{7}|\Phi^{\pm}_{ab}\rangle \langle \Psi^{\pm}_{ab}| + \frac{1-F}{7}|\Psi^{\pm}_{ab}\rangle \langle \Phi^{\pm}_{ab}| + \frac{1-F}{7}|\Psi^{\pm}_{ab}\rangle \langle \Psi^{\pm}_{ab}| + \frac{1-F}{7}|\Gamma^{\pm}_{ab}\rangle \langle \Gamma^{\pm}_{ab}| + \frac{1-F}{7}|\Upsilon^{\pm}_{ab}\rangle \langle \Upsilon^{\pm}_{ab}|. \tag{6}
\]

The coefficient \( F = \langle \Phi^{\pm}_{ab}| \rho |\Phi^{\pm}_{ab}\rangle \) is the fidelity of initial state \( |\Phi^{\pm}_{ab}\rangle\).

Since a DEPS exhibits two degrees of freedom, we can utilize the frequencies in our purification procedures to purify the polarizations of photons. In our protocol, we exploit the entanglement of frequencies to correct the bit-flip errors. Its principle is shown in Fig.1. When a DEPS enters the device, one photon goes to the left wavelength-division multiplexing device (WDM) and the other to the right one. Photons with different frequencies can be distinguished by recording their port information from WDMs. Since the two photons with different frequencies leave their respective WDM in either \( \omega_s/\omega_i \) port or \( \omega_i/\omega_s \) port, each photon passes through a polarization beam splitter and then enters the port 1, 2, 3 or 4. Table I illustrates the correspondence between the ports that each photon leaves the device and the states.

From Table I one can see that the state has no bit-flip error if the two photons come out of the port 1 and the port 2, respectively. If a photon comes out of the lower spatial mode (the port 3 or the port 4), a bit-flip error takes place on it. We can exploit the two HWPs on the port 3 and the port 4 to correct the bit-flip errors in the photon pair. For example, for the bit-flip error state \( |\Gamma^{\pm}_{ab}\rangle \), the two photons a and b come out of the port 2 and the port 3, respectively. That is, a bit-flip error takes place on the photon a coming out of the port 3. The HWPs on the port 3 will accomplish the transformation \( |H\rangle \to |V\rangle \), which means that the state of the photon pair becomes \( |\Phi^{\pm}_{ab}\rangle \) after the setup shown in Fig.1.

After the first step in our entanglement purification scheme, all the states in the mixed ensemble are preserved and the bit-flip errors are corrected. The DEPS system remains only the original state \( |\Phi^{\pm}_{ab}\rangle \) and the phase-flip error state \( |\Phi^{-}_{ab}\rangle \). That is, the initial Werner state becomes to

\[
\rho' = \frac{4F + 3}{7}|\Phi^{\pm}_{ab}\rangle \langle \Phi^{\pm}_{ab}| + \frac{4(1-F)}{7}|\Phi^{-}_{ab}\rangle \langle \Phi^{-}_{ab}|. \tag{7}
\]

Compared with the Xiao’s protocol, we only add two HWPs on the two ports 3 and 4, but just this modification improves the efficiency of the entanglement purification of bit-flip errors largely. In an ideal condition, its efficiency for bit-flip error correction is 100%.

**TABLE I:** The correspondence between the ports and the states

| Triggered Port | Corresponding states |
|----------------|----------------------|
| 1,2            | \( |\Phi^{\pm}_{ab}\rangle \) |
| 1,4            | \( |\Psi^{\pm}_{ab}\rangle \) |
| 3,2            | \( |\Gamma^{\pm}_{ab}\rangle \) |
| 3,4            | \( |\Upsilon^{\pm}_{ab}\rangle \) |

**FIG. 1:** The setup for bit-flip error correction in the first step. HW and PBS represent half wave plate and polarizing beam splitter, respectively.
B. entanglement purification for phase-flip error

In the second step, we purify the phase-flip error in the state $\rho'$. The key element before the second step is the wavelength conversion process [19, 20] which transforms the DEPSs to traditional Bell states by a up-conversion process:

$$|\Phi_{ab}^{\pm}\rangle \rightarrow |\Phi_{Bell}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|H\rangle_a|H\rangle_b + |V\rangle_a|V\rangle_b).$$  

This conversion process is performed in the first step after the two WDMs. The wavelength conversion process consists of a WDM coupler, a periodically poled lithium niobate (PPLN) waveguide and a filter. One photon with the frequency $\omega_a$ pass through the WDM as a signal light and pumped by a high power laser with a certain wavelength. They are sent to PPLN waveguide and the sum-frequency process generates the photon with the frequency $\omega$ needed. The filter is used to filter the pump light. The frequencies of both the $a$ and $b$ photons are performed by this wavelength conversion to the same frequency, and then the DEPSs are transformed to Bell states.

In the nonlocal entanglement purification procedures, the DEPS with phase-flip errors can be further distill with the ways in other schemes existing, such as that with PBSs and sophisticated single photon detectors [8], shown in Fig. 2. In detail, the two parties, Alice and Bob, select two pairs of Bell states (marked with $a_1, a_2$ and $b_1, b_2$) randomly and perform a Hadamard operation on each of the photons. After these operations, the phase-flip error state $|\Phi_{Bell}\rangle$ will be transformed into $|\Psi_{Bell}\rangle = \frac{1}{\sqrt{2}}(|H\rangle_a|V\rangle_b + |V\rangle_a|H\rangle_b)$ while the initial state $|\Phi_{Bell}\rangle$ remains unchanged. That is, the state $\rho'$ becomes

$$\rho'' = \frac{4F + 3}{7}|\Phi_{ab}^{+}\rangle\langle\Phi_{ab}^{+}| + \frac{4(1 - F)}{7}|\Psi_{ab}^{+}\rangle\langle\Psi_{ab}^{+}|.$$

Alice and Bob choose the four-mode instances for their entanglement purification, same as that in Ref. [3]. Alice and Bob performs $\sigma_z = \{ |+\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle), |\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)\}$ measurement on the photons coming out of the port 3 and the port 4, shown in Fig. 2. If their outcomes are antiparallel, Alice and Bob need to take a phase-flip operation on the photon pair coming from the up modes. After this purification process, the state becomes

$$\rho'' = F'|\Phi_{ab}^{+}\rangle\langle\Phi_{ab}^{+}| + (1 - F')|\Psi_{ab}^{+}\rangle\langle\Psi_{ab}^{+}|,$$

where

$$F' = \frac{(4F + 3)^2}{32F^2 - 8F + 25}.$$

In summary, we have improved the entangled purification scheme based on doubly entangled photon states. High fidelity DEPSs can be achieved by two-step purification operations after the transmission over a noisy channel. We have modified the purification setups using WDMs, polarization beam splitters and high efficiency nonlinear processes. In the first step operation, all the bit-flip error states can be distinguished and corrected. The second step operation can efficiently purify the phase-flip errors without the wavelength conversion process.

In summary, we have improved the entangled purification scheme based on doubly entangled photon states. High fidelity DEPSs can be achieved by two-step purification operations after the transmission over a noisy channel. We have modified the purification setups using WDMs, polarization beam splitters and high efficiency nonlinear processes. In the first step operation, all the bit-flip error states can be distinguished and corrected. The second step operation can efficiently purify the phase-flip errors. Moreover, the original fidelity of the entangled state wanted in our scheme is much lower than others [6, 8, 9, 10, 11, 12, 13]. In experiment, the probability of the wavelength conversion process is not unit, but it approaches unit [19]. If the two parties possess some cross-Kerr nonlinear media, it is possible for them to purify the phase-flip errors without the wavelength conversion process.

III. DISCUSSION AND SUMMARY

Compared to the traditional purification protocols, our scheme is more efficient as the first step can correct all the bit-flip errors. This good feature makes the original fidelity of the entangled state wanted in our scheme much lower than others [6, 8, 9, 10, 11, 12, 13]. In the second step for the purification of phase-flip errors, we can also use the ways in other protocols to improve the fidelity, such as that with nonlinear optics [11]. In experiment, the probability of the wavelength conversion process is not unit, but it approaches unit [19]. If the two parties possess some cross-Kerr nonlinear media, it is possible for them to purify the phase-flip errors without the wavelength conversion process.

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