Determination of $\Delta \Gamma_s$ from Analysis of Untagged Decays $B_s^0 \to J/\psi \phi$

by Using the Method of Angular Moments

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Abstract

The performance of the method of angular moments on the $\Delta \Gamma_s$ determination from analysis of untagged decays $B_s^0(t), \overline{B}_s^0(t) \to J/\psi(\to l^+l^-) \phi(\to K^+K^-)$ is examined. The results of Monte Carlo studies with evaluation of measurement errors are presented. The method of angular moments gives stable results for the estimate of $\Delta \Gamma_s$ and is found to be an efficient and flexible tool for the quantitative investigation of the $B_s^0 \to J/\psi \phi$ decay. The statistical error of the ratio $\Delta \Gamma_s/\Gamma_s$ for values of this ratio in the interval [0.03, 0.3] was found to be independent on this value, being 0.015 for $10^5$ events.

1 Introduction

The study of decays $B_s^0(t), \overline{B}_s^0(t) \to J/\psi(\to l^+l^-) \phi(\to K^+K^-)$, which is one of the gold plated channels for $B$-physics studies at the LHC, looks very interesting from the physics point of view. It presents several advantages related to the dynamics of these decays, characterized by proper-time-dependent angular distributions, which can be described in terms of bilinear combinations of transversity amplitudes. Their time evolution involves, besides the values of two transversity amplitudes at the proper time $t = 0$ and their relative strong phases, the following fundamental parameters: the difference and average value of decay rates of heavy and light mass eigenstates of $B_s^0$ meson, $\Delta \Gamma_s$ and...

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The angular analysis of the decays $B_s^0(t), \overline{B_s^0}(t) \rightarrow J/\psi(\rightarrow l^+l^-) \phi(\rightarrow K^+K^-)$ provides complete determination of the transversity amplitudes and, in principle, gives the access to all these parameters.

In the present paper we examine the performance of the angular-moments method \[1\] applied to the angular analysis of untagged decays $B_s^0(t), \overline{B_s^0}(t) \rightarrow J/\psi(\rightarrow l^+l^-) \phi(\rightarrow K^+K^-)$ for the determination of $\Delta \Gamma_s$. After giving the physics motivation in Section 2, we describe in the next section the method of angular moments based on weighting functions introduced in Ref. \[1\]. For the case of $\Delta \Gamma_s$ determination this method is properly modified in Section 4. The SIMUB-package \[2\] for physics simulation of $B$-meson production and decays has been used for Monte Carlo studies with two sets of weighting functions. In Section 5 we present the results of these studies and concentrate on the evaluation of measurement errors and their dependence on statistics.

2 Phenomenological description of the decays

$$B_s^0(t), \overline{B_s^0}(t) \rightarrow J/\psi(\rightarrow l^+l^-) \phi(\rightarrow K^+K^-)$$

The angular distributions for decays $B_s^0(t), \overline{B_s^0}(t) \rightarrow J/\psi(\rightarrow l^+l^-) \phi(\rightarrow K^+K^-)$ are governed by spin-angular correlations (see \[3\]-\[5\]) and involve three physically determined angles. In case of the so-called helicity frame \[5\], which is used in the present paper, these angles are defined as follows (see Fig. 1):

- The $z$-axis is defined to be the direction of $\phi$-particle in the rest frame of the $B_s^0$. The $x$-axis is defined as any arbitrary fixed direction in the plane normal to the $z$-axis. The $y$-axis is then fixed uniquely via $y = z \times x$ (right-handed coordinate system).
- The angles $(\Theta_{t^+}, \chi_{t^+})$ specify the direction of the $l^+$ in the $J/\psi$ rest frame while $(\Theta_{K^+}, \chi_{K^+})$ give the direction of $K^+$ in the $\phi$ rest frame. Since the orientation of the $x$-axis is a matter of convention, only the difference $\chi = \chi_{t^+} - \chi_{K^+}$ of the two azimuthal angles is physically meaningful.

In the most general form the angular distribution for the decay $B_s^0(t) \rightarrow J/\psi(\rightarrow l^+l^-) \phi(\rightarrow K^+K^-)$ in case of a tagged $B_s^0$ sample can be expressed as

$$\frac{d^4 N_{tag}(B_s^0)}{d \cos \Theta_{t^+} d \cos \Theta_{K^+} d \chi dt} = \frac{9}{32 \pi} \sum_{i=1}^{6} \mathcal{O}_i(t) g_i(\Theta_{t^+}, \Theta_{K^+}, \chi).$$

(1)

Here $\mathcal{O}_i \ (i = 1, ..., 6)$ are time-dependent bilinear combinations of the transversity amplitudes $A_0(t), A_{||}(t)$ and $A_{\perp}(t)$ for the weak transition $B_s^0(t) \rightarrow J/\psi \phi$ \[7\] (we treat these combinations as observables):

$$\mathcal{O}_1 = |A_0(t)|^2, \quad \mathcal{O}_2 = |A_{||}(t)|^2, \quad \mathcal{O}_3 = |A_{\perp}(t)|^2,$$

$$\mathcal{O}_4 = \text{Im}(A_{\perp}(t)A_{||}(t)), \quad \mathcal{O}_5 = \text{Re}(A_0(t)A_{||}(t)), \quad \mathcal{O}_6 = \text{Im}(A_0^*(t)A_{\perp}(t)).$$

(2)
Figure 1: Definition of physical angles for description of decays $B^0_s(t), \overline{B}^0_s(t) \rightarrow J/\psi(\rightarrow l^+l^-) \phi(\rightarrow K^+K^-)$ in the helicity frame

and the $g_i$ are functions of the angles $\Theta_{l^+}, \Theta_{K^+}, \chi$ only [5]:

$$
g_1 = 2\cos^2 \Theta_{K^+} \sin^2 \Theta_{l^+},
$$

$$
g_2 = \sin^2 \Theta_{K^+} (1 - \sin^2 \Theta_{l^+} \cos^2 \chi),
$$

$$
g_3 = \sin^2 \Theta_{K^+} (1 - \sin^2 \Theta_{l^+} \sin^2 \chi),
$$

$$
g_4 = -\sin \Theta_{K^+} \sin^2 \Theta_{l^+} \sin 2\chi,
$$

$$
g_5 = \frac{1}{\sqrt{2}} \sin 2\Theta_{l^+} \sin 2\Theta_{K^+} \cos \chi,
$$

$$
g_6 = \frac{1}{\sqrt{2}} \sin 2\Theta_{l^+} \sin 2\Theta_{K^+} \sin \chi.
$$

For the decay $\overline{B}^0_s(t) \rightarrow J/\psi(\rightarrow l^+l^-) \phi(\rightarrow K^+K^-)$ in case of a tagged $\overline{B}_s^0$ sample the angular distribution is given by

$$
\frac{d^4 N^{\text{tag}}(\overline{B}^0_s)}{d\cos \Theta_{l^+} d\cos \Theta_{K^+} d\chi dt} = \frac{9}{32\pi} \sum_{i=1}^{6} \mathcal{O}_i(t) g_i(\Theta_{l^+}, \Theta_{K^+}, \chi)
$$

with the same angular functions $g_i$ and

$$
\mathcal{O}_1 = |\tilde{A}(t)|^2, \quad \mathcal{O}_2 = |\tilde{A}_{\|}(t)|^2, \quad \mathcal{O}_3 = |\tilde{A}_\perp(t)|^2,
$$

$$
\mathcal{O}_4 = \text{Im}(\tilde{A}_{\|}^*(t)\tilde{A}_\perp(t)), \quad \mathcal{O}_5 = \text{Re}(\tilde{A}_0^*(t)\tilde{A}_{\|}(t)), \quad \mathcal{O}_6 = \text{Im}(\tilde{A}_0^*(t)\tilde{A}_\perp(t)),
$$

where $\tilde{A}_0(t), \tilde{A}_{\|}(t)$ and $\tilde{A}_\perp(t)$ are the transversity amplitudes for the transition $\overline{B}_s^0(t) \rightarrow J/\psi \phi$. 

3
The time dependence of the transversity amplitudes for the transitions \( B_s^0(t), \overline{B}_s^0(t) \rightarrow J/\psi \phi \) is not of purely exponential form due to the presence of \( B_s^0 - \overline{B}_s^0 \) mixing. This mixing arises due to either a mass difference or a decay-width difference between the mass eigenstates of the \( (B_s^0 - \overline{B}_s^0) \) system. The time evolution of the state \( |B_s^0(t)\rangle \) of an initially, i.e. at time \( t = 0 \), present \( B_s^0 \) meson can be described in general form as follows:

\[
|B_s^0(t)\rangle = g_+(t)|B_s^0\rangle + g_-(t)|\overline{B}_s^0\rangle, \quad g_+(t = 0) = 1, \quad g_-(t = 0) = 0,
\]
i.e., the state \( |B_s^0(t)\rangle \) at time \( t \) is a mixture of the flavor states \( |B_s^0\rangle \) and \( |\overline{B}_s^0\rangle \) with probabilities defined by the functions \( g_+(t) \) and \( g_-(t) \). In analogous way, the time evolution of the state \( |\overline{B}_s^0(t)\rangle \) of an initially present \( \overline{B}_s^0 \) meson is described by the relation

\[
|\overline{B}_s^0(t)\rangle = \tilde{g}_+(t)|B_s^0\rangle + \tilde{g}_-(t)|\overline{B}_s^0\rangle, \quad \tilde{g}_+(t = 0) = 0, \quad \tilde{g}_-(t = 0) = 1.
\]

Diagonalization of the full Hamiltonian (see [8] for more details) gives

\[
g_+(t) = \frac{1}{2} (e^{-i\mu_L t} + e^{-i\mu_H t}), \quad g_-(t) = \frac{\alpha}{2} (e^{-i\mu_L t} - e^{-i\mu_H t}),
\]

\[
\tilde{g}_+(t) = g_-(t)/\alpha^2, \quad \tilde{g}_-(t) = g_+(t).
\]

Here \( \mu_{L/H} \equiv M_{L/H} - (i/2)\Gamma_{L/H} \) are eigenvalues of the full Hamiltonian corresponding to the masses and total widths of “light” and “heavy” eigenstates \( |B_{L/H}\rangle \), and \( \alpha \) is a phase factor defining the \( CP \) transformation of flavor eigenstates of the neutral \( B_s \)-meson system: \( CP|B_s^0\rangle = \alpha|\overline{B}_s^0\rangle \). In the case \( |\alpha| \neq 1 \) the probability for \( B_s^0 \) to oscillate to a \( \overline{B}_s^0 \) is not equal to the probability of a \( \overline{B}_s^0 \) to oscillate to a \( B_s^0 \). Such an asymmetry in mixing is often referred to as indirect \( CP \) violation, which is negligibly small in case of the neutral \( B \)-meson system.

The time evolution of the transversity amplitudes \( A_f(t) \) \( (f = 0, ||, \perp) \) is given by the equations

\[
A_f(t) = A_f(0)[g_+(t) + g_-(t)] \frac{1}{\eta_{CP}\alpha} \xi_f^{(s)}], \quad \tilde{A}_f(t) = A_f(0)[\tilde{g}_+(t) + \tilde{g}_-(t)] \frac{1}{\eta_{CP}\alpha} \xi_f^{(s)}].
\]

Here \( \eta_{CP}^f \) are eigenvalues of \( CP \)-operator acting on the transversity components of the final state which are eigenstates of \( CP \)-operator:

\[
CP|J/\psi \phi\rangle_f = \eta_{CP}^f |J/\psi \phi\rangle_f, \quad (f = 0, ||, \perp),
\]

\[
\eta_{CP}^0 = 1, \quad \eta_{CP}^|| = 1, \quad \eta_{CP}^\perp = -1,
\]

and \( \xi_f^{(s)} \) is the \( CP \)-violating weak phase [9]:

\[
\xi_f^{(s)} = e^{-i\phi_c^{(s)}}, \quad \phi_c^{(s)} = 2[\arg(V_{ts}^* V_{tb}) - \arg(V_{cq}^* V_{cb})] = -2\delta\gamma,
\]

where \( \delta \) is the complex phase in the standard parameterization of the CKM matrix elements \( V_{ij} \) \( (i \in \{u,c,t\}, j \in \{d,s,b\}) \), and \( \gamma \) is the third angle of the unitarity triangle.
The phase $\phi_c^{(s)}$ is very small and vanishes at leading order in the Wolfenstein expansion. Taking into account higher-order terms in the Wolfenstein parameter $\lambda = \sin \theta_C = 0.22$ gives a non-vanishing result \cite{10}:

$$\phi_c^{(s)} = -2\lambda^2 \eta = -2\lambda^2 R_b \sin \gamma.$$  

Here

$$R_b \equiv \frac{|V_{ub}|}{\lambda |V_{cb}|}$$

is constrained by present experimental data as $R_b = 0.36 \pm 0.08$ \cite{11}. Using the experimental estimate $\gamma = (59 \pm 13)^o$ \cite{12}, the following constrain can be obtained for the phase $\phi_c^{(s)}$:

$$\phi_c^{(s)} = -0.03 \pm 0.01.$$

According to Eq. (7) at time $t = 0$, the transversity amplitudes of $B^0_s \rightarrow J/\psi \phi$ decays depend on the same observables $|A_0(0)|$, $|A_{||}(0)|$, $|A_{\perp}(0)|$ and on the two CP-conserving strong phases, $\delta_1 \equiv \arg[A^*_0(0)A_{\perp}(0)]$ and $\delta_2 \equiv \arg[A^*_0(0)A_{||}(0)]$. Time-reversal invariance of strong interactions forces the form factors parameterizing quark currents to be all relatively real and, consequently, naive factorization leads to the following common properties of the observables:

$$\text{Im}[A^*_0(0)A_{||}(0)] = 0, \quad \text{Im}[A^*_0(0)A_{\perp}(0)] = 0, \quad \text{Re}[A^*_0(0)A_{||}(0)] = \pm |A_0(0)A_{||}(0)|.$$  

Moreover, in the absence of strong final-state interactions, $\delta_1 = \pi$ and $\delta_2 = 0$.

In the framework of the effective Hamiltonian approach the two body decays, both $B^0_s \rightarrow J/\psi \phi$ and $B^0_d \rightarrow J/\psi K^*$, correspond to the transitions $\bar{b} \rightarrow \bar{s}c\bar{c}$ with topologies of color-suppressed spectator diagrams shown in Fig. 2. Factorizing the hadronic matrix elements of the four-quark operators of the effective Hamiltonian into hadronic matrix elements of quark currents, the transversity amplitudes $|A_0(0)|$, $|A_{||}(0)|$, $|A_{\perp}(0)|$ of decays $B^0_q \rightarrow J/\psi V((q, V) \in \{(s, \phi), (d, K^*)\})$ can be expressed in terms of effective Wilson coefficient functions, constants of $J/\psi$ decay, and form factors of transitions $B_q \rightarrow V$ induced by quark currents \cite{1}. In Table II we collect the predictions of Ref. \cite{4} for the transversity amplitudes of $B^0_s \rightarrow J/\psi \phi$ ($B^0_d \rightarrow J/\psi K^*$) calculated with $B \rightarrow K^*$ form factors given by different models \cite{13, 14, 15}. The $B \rightarrow K^*$ form factors can be related to the $B \rightarrow \phi$ case by using SU(3) flavor symmetry. The most precise polarization measurements performed recently in decays $B \rightarrow J/\psi K^*$:

$$|A_0(0)|^2 = 0.60 \pm 0.04, \quad |A_{\perp}(0)|^2 = 0.16 \pm 0.03 \quad \text{(BaBar \cite{16})},$$

$$|A_0(0)|^2 = 0.62 \pm 0.04, \quad |A_{\perp}(0)|^2 = 0.19 \pm 0.04 \quad \text{(Belle \cite{17})},$$

confirm the predictions based on the model \cite{15}. 


Figure 2: Color suppressed diagrams for decays $B^0_q \rightarrow J/\psi V ((q, V) \in \{(s, \phi), (d, K^*)\})$

Table 1: Predictions for $B^0_s \rightarrow J/\psi \phi$ (in brackets – for $B^0_d \rightarrow J/\psi K^*$) observables obtained in Ref. [1] for various model estimates of the $B \rightarrow K^*$ form factors [13, 14, 15] (the normalization condition $|A_0(0)|^2 + |A_{||}(0)|^2 + |A_{\perp}(0)|^2 = 1$ is implied)

| Observable | BSW [13] | Soares [14] | Cheng [15] |
|------------|----------|------------|------------|
| $|A_0(0)|^2$ | 0.55 (0.57) | 0.41 (0.42) | 0.54 (0.56) |
| $|A_{\perp}(0)|^2$ | 0.09 (0.09) | 0.32 (0.33) | 0.16 (0.16) |

3 Angular-moments method

The angular distributions for decays $B^0_s(t), \overline{B}^0_s(t) \rightarrow J/\psi(t^+ l^-) \phi(\rightarrow K^+ K^-)$ in case of tagged $B^0_s$ and $\overline{B}^0_s(t)$ samples (see Eqs. (1) and (4), respectively) as well as in case of the untagged sample can be expressed in the most general form in terms of observables $b_i(t)$:

$$f(\Theta_{l^+}, \Theta_{K^+}, \chi; t) = \frac{9}{32\pi} \sum_{i=1}^{6} b_i(t) g_i(\Theta_{l^+}, \Theta_{K^+}, \chi).$$ \hspace{1cm} (9)

The explicit time dependence of observables is given by the following relations:

$$b_1(t) = |A_0(0)|^2 G_L(t),$$
$$b_2(t) = |A_{||}(0)|^2 G_L(t),$$
$$b_3(t) = |A_{\perp}(0)|^2 G_H(t),$$
$$b_4(t) = |A_{||}(0)||A_{\perp}(0)|Z_1(t),$$
$$b_5(t) = |A_0(0)||A_{||}(0)|G_L(t)\cos(\delta_2 - \delta_1),$$
$$b_6(t) = |A_0(0)||A_{\perp}(0)|Z_2(t),$$ \hspace{1cm} (10)

where we have used the general compact notations:

$$G_{L/H}(t) = \frac{1}{2} \left[ (1 \pm \cos \phi_c(s)) e^{-\Gamma_L t} + (1 \mp \cos \phi_c(s)) e^{-\Gamma_H t} \right],$$
$$Z_{1,2}(t) = \frac{1}{2} (e^{-\Gamma_H t} - e^{-\Gamma_L t}) \cos \delta_{1,2} \sin \phi_c(s)$$
for observables $b_i \equiv (\mathcal{O}_i + \overline{\mathcal{O}}_i)/2$ in case of the untagged sample with equal initial numbers of $B^0_s$ and $\overline{B}^0_s$, while
\[
G_{L/H}(t) = G_{L/H}(t) \pm e^{-\Gamma_t} \sin(\Delta Mt) \sin\phi_c(s),
\]
\[
Z_{1,2}(t) = Z_{1,2}(t) \pm e^{-\Gamma_t} \left[ \sin\delta_{1,2}\cos(\Delta Mt) - \cos\delta_{1,2}\sin(\Delta Mt)\sin\phi_c(s) \right]
\]
- for observables $b_i^{(B^0_s)} \equiv \mathcal{O}_i$ and $b_i^{(B^0_s)} \equiv \overline{\mathcal{O}}_i$ in case of tagged $B^0_s$ and $\overline{B}^0_s(t)$ samples, respectively, with $\Gamma_s \equiv (\Gamma_L + \Gamma_H)/2$. It is easy to see that both in the tagged and untagged case we have
\[
G_{L/H}(t)|_{\phi_c(s) = 0} = e^{-\Gamma_L/Ht}.
\]
According to Ref. [1], the observables $b_i(t)$ can be extracted from distribution function (9) by means of weighting functions $w_i(\Theta_{t+}, \Theta_{K+}, \chi)$ for each $i$ such that
\[
\frac{9}{32\pi} \int d\cos\Theta_{t+}d\cos\Theta_{K+}d\chi \ w_i(\Theta_{t+}, \Theta_{K+}, \chi) \ g_j(\Theta_{t+}, \Theta_{K+}, \chi) = \delta_{ij},
\]
projecting out the desired observable alone:
\[
b_i(t) = \int d\cos\Theta_{t+}d\cos\Theta_{K+}d\chi \ f(\Theta_{t+}, \Theta_{K+}, \chi; t) \ w_i(\Theta_{t+}, \Theta_{K+}, \chi).
\]
The angular-distribution function (11) obeys the condition
\[
L(t) \equiv \int d\cos\Theta_{t+}d\cos\Theta_{K+}d\chi \ f(\Theta_{t+}, \Theta_{K+}, \chi; t) = b_1(t) + b_2(t) + b_3(t).
\]
For decays $B \rightarrow J/\psi(\rightarrow l^+l^-) \phi(\rightarrow K^+K^-)$, the explicit expressions of weighting functions, given in Table 5 of Ref. [1] for physically meaningful angles in the transversity frame, get the following form (Set A) after transformation into the helicity frame:
\[
\begin{align*}
w_1^{(A)} &= 2 - 5 \cos^2 \Theta_{t+}, \\
w_2^{(A)} &= 2 - 5 \sin^2 \Theta_{t+} \cos^2 \chi, \\
w_3^{(A)} &= 2 - 5 \sin^2 \Theta_{t+} \sin^2 \chi, \\
w_4^{(A)} &= - \frac{5}{2} \sin^2 \Theta_{K+} \sin 2\chi, \\
w_5^{(A)} &= \frac{25}{4\sqrt{2}} \sin 2\Theta_{K+} \sin 2\Theta_{t+} \cos \chi, \\
w_6^{(A)} &= \frac{25}{4\sqrt{2}} \sin 2\Theta_{K+} \sin 2\Theta_{t+} \sin \chi.
\end{align*}
\]
The expressions of Eq. (14) are not unique and there are many legitimate choices of weighting functions. A particular set can be derived by linear combination of angular functions $g_i$ (see [1] for more discussions):
\[
w_i(\Theta_{t+}, \Theta_{K+}, \chi) = \sum_{j=1}^{6} \lambda_{ij} g_j(\Theta_{t+}, \Theta_{K+}, \chi),
\]
(15)
where the 36 unknown coefficients $\lambda_{ij}$ are solutions of 36 equations

$$\frac{9}{32\pi} \sum_{j=1}^{6} \lambda_{ij} \int d\cos \Theta_{t+} \, d\cos \Theta_{K+} \, d\chi \, g_j(\Theta_{t+}, \Theta_{K+}, \chi) \, g_k(\Theta_{t+}, \Theta_{K+}, \chi) = \delta_{ik}. \quad (16)$$

The weighting functions (set B) corresponding to the linear combination of the angular functions (3) are given by

$$w_{1(B)} = \frac{1}{12} \left[ 28 \cos^2 \Theta_{K+} \sin^2 \Theta_{t+} - 3 \sin^2 \Theta_{K+} (1 + \cos^2 \Theta_{t+}) \right],$$

$$w_{2(B)} = -\frac{1}{8} \left[ 4 \cos^2 \Theta_{K+} \sin^2 \Theta_{t+} - 29 \sin^2 \Theta_{K+} (1 - \sin^2 \Theta_{t+} \cos^2 \chi) \right. \right.$$ \nonumber

$$\left. + 21 \sin^2 \Theta_{K+} (1 - \sin^2 \Theta_{t+} \sin^2 \chi) \right],$$

$$w_{3(B)} = -\frac{1}{8} \left[ 4 \cos^2 \Theta_{K+} \sin^2 \Theta_{t+} + 21 \sin^2 \Theta_{K+} (1 - \sin^2 \Theta_{t+} \cos^2 \chi) \right. \right.$$ \nonumber

$$\left. - 29 \sin^2 \Theta_{K+} (1 - \sin^2 \Theta_{t+} \sin^2 \chi) \right],$$

$$w_{4(B)} = -\frac{25}{8} \sin^2 \Theta_{K+} \sin^2 \Theta_{t+} \sin 2\chi,$$

$$w_{5(B)} = w_{5(A)},$$

$$w_{6(B)} = w_{6(A)}. \quad (17)$$

For a limited number of experimental events $N$ in the time bin around the fixed value of the proper time $t$, distributed according to the angular function (9), it is convenient to introduce the normalized observables

$$\bar{b}_i(t) \equiv b_i(t)/L(t) \quad (18)$$

with normalization factor $L(t)$ given by Eq. (13). Then, as it follows from the Eq. (12), the observables $\bar{b}_i(t)$ (18) are measured experimentally by

$$\bar{b}_i^{(exp)} = \frac{1}{N} \sum_{j=1}^{N} w_i^j \quad (19)$$

with summation over events in a time bin around $t$. Here $w_i^j \equiv w_i(\Theta_{t+}^j, \Theta_{K+}^j, \chi^j)$, where $\Theta_{t+}^j, \Theta_{K+}^j$ and $\chi^j$ are angles measured in the $j$-th event. The statistical measurement error of the observable (19) can be estimated as

$$\delta \bar{b}_i^{(exp)} = \frac{1}{N} \sqrt{\sum_{j=1}^{N} (\bar{b}_i^{(exp)} - w_i^j)^2},$$

with summation over all events in the same time bin.

4 Time-integrated observables

For data analysis it is rather convenient to use the time-integrated observables defined as

$$\bar{b}_i(T_0) = \frac{1}{L(T)} \int_0^{T_0} dt \int d\cos \Theta_{t+} \, d\cos \Theta_{K+} \, d\chi \, w_i(\Theta_{t+}, \Theta_{K+}, \chi) \, f(\Theta_{t+}, \Theta_{K+}, \chi; t) \quad (20)$$

8
with argument $T_0 \leq T$, where $T$ is the maximal value of the $B$-meson proper time measured for the sample of events being used, and $\tilde{L}(T)$ is a new normalization factor, which has the form:

$$
\tilde{L}(T) \equiv \int_0^T L(t) = \int_0^T dt \int d\cos \Theta_{t^+} \int d\cos \Theta_{K^+} \int d\chi \ f(\cos \Theta_{t^+}, \cos \Theta_{K^+}, \chi; t) = \left( |A_0(0)|^2 + |A_{||}(0)|^2 \right) \tilde{G}_L(T) + |A_{\perp}(0)|^2 \tilde{G}_H(T), \tag{21}
$$

where, in the compact notations used in Eq. (10),

$$
\tilde{G}_{L/H}(T) \equiv \int_0^T dt \ G_{L/H}(t).
$$

The following normalization condition is valid for the observables (20): $\tilde{b}_1 + \tilde{b}_2 + \tilde{b}_3 = 1$. For a limited number of experimental events $N(T)$, measured in the proper time region $t \in [0, T]$, Eq. (20) reduces to

$$
\tilde{b}_i^{(\text{exp})}(T_0) = \frac{1}{N(T)} \sum_{j=1}^{N(T_0)} w_j^i \tag{22}
$$

with summation over all events $N(T_0)$ in the time interval $t \in [0, T_0]$ for $T_0 \leq T$. In case of the untagged sample we have

$$
\tilde{G}_{L/H}(T) = -\frac{1}{2} \left[ (1 \pm \cos \phi_c^{(s)}) \frac{e^{-\Gamma_L T}}{\Gamma_L} - 1 + (1 \mp \cos \phi_c^{(s)}) \frac{e^{-\Gamma_H T}}{\Gamma_H} - 1 \right] \tag{23}
$$

and

$$
\tilde{Z}(T) \equiv \frac{1}{\cos \delta_1, \cos \phi_c^{(s)}} \int_0^T dt \ Z_{1,2}(T) = -\frac{1}{2} \left[ \frac{(e^{-\Gamma_H T} - 1)/\Gamma_H - (e^{-\Gamma_L T} - 1)/\Gamma_L}{(e^{-\Gamma_H T} - 1)/\Gamma_H} \right].
$$

For the untagged sample the explicit form of time-integrated normalized observables (20) in terms of the functions $\tilde{G}_{L/H}(T)$ and $\tilde{Z}(T)$ is given by

$$
\begin{align*}
\tilde{b}_1(T_0) &= |A_0(0)|^2 \tilde{G}_L(T_0)/\tilde{L}(T), \\
\tilde{b}_2(T_0) &= |A_{||}(0)|^2 \tilde{G}_L(T_0)/\tilde{L}(T), \\
\tilde{b}_3(T_0) &= |A_{\perp}(0)|^2 \tilde{G}_H(T_0)/\tilde{L}(T), \\
\tilde{b}_4(T_0) &= |A_{||}(0)| |A_{\perp}(0)| \tilde{Z}(T_0) \cos \delta_1 \sin \phi_c^{(s)}/\tilde{L}(T), \\
\tilde{b}_5(T_0) &= |A_0(0)| |A_{||}(0)| \tilde{G}_L(T_0) \cos (\delta_2 - \delta_1)/\tilde{L}(T), \\
\tilde{b}_6(T_0) &= |A_0(0)| |A_{\perp}(0)| \tilde{Z}(T_0) \cos \delta_2 \sin \phi_c^{(s)}/\tilde{L}(T).
\end{align*} \tag{24}
$$

In the Standard Model (SM) $\sin \phi_c^{(s)} \approx 0$ and the observables $\tilde{b}_{4,5}(T_0)$ are vanishing. In case of a new physics signal the values of $\sin \phi_c^{(s)}$ and $\tilde{b}_{4,5}(T_0)$ can be sizable, however.
The following relations are valid for the observables \([24]\):

\[
\begin{align*}
\tilde{b}_4(T_0) &= \cos\delta_1 \sin\phi^{(s)} \tilde{Z}(T_0) \sqrt{\frac{\tilde{b}_2(T_0) \tilde{b}_3(T_0)}{G_L(T_0) G_H(T_0)}}, \\
\tilde{b}_5(T_0) &= \cos(\delta_2 - \delta_1) \sqrt{\tilde{b}_1(T_0) \tilde{b}_2(T_0)}, \\
\tilde{b}_6(T_0) &= \cos\delta_2 \sin\phi^{(s)} \tilde{Z}(T_0) \sqrt{\frac{\tilde{b}_1(T_0) \tilde{b}_3(T_0)}{G_L(T_0) G_H(T_0)}}.
\end{align*}
\]

If we introduce the function

\[
\tilde{\gamma}(T) \equiv \tilde{G}_H(T)/\tilde{G}_L(T),
\]

then, the values of initial transversity amplitudes at \(t = 0\) and the strong-phase difference \((\delta_2 - \delta_1)\) are determined from the observables \(\tilde{b}_i(T) \equiv \tilde{b}_i(T = T_0)\) by

\[
\begin{align*}
|A_0(0)|^2 &= \frac{\tilde{b}_1(T)}{\tilde{b}_1(T) + \tilde{b}_2(T) + \tilde{b}_3(T)/\tilde{\gamma}(T)}, \\
|A_{||}(0)|^2 &= \frac{\tilde{b}_2(T)}{\tilde{b}_1(T) + \tilde{b}_2(T) + \tilde{b}_3(T)/\tilde{\gamma}(T)}, \\
|A_{\perp}(0)|^2 &= \frac{\tilde{b}_3(T)/\tilde{\gamma}(T)}{\tilde{b}_1(T) + \tilde{b}_2(T) + \tilde{b}_3(T)/\tilde{\gamma}(T)}, \\
\cos(\delta_2 - \delta_1) &= \frac{\tilde{b}_5(T)}{\sqrt{\tilde{b}_1(T) \tilde{b}_2(T)}},
\end{align*}
\]

where we consider the initial amplitudes normalized as \(|A_0(0)|^2 + |A_{||}(0)|^2 + |A_{\perp}(0)|^2 = 1\). We have also:

\[
\sin\phi^{(s)} \cos\delta_{1,2} = \frac{\tilde{b}_{4,6}(T)}{\sqrt{\tilde{b}_{2,1}(T) \tilde{b}_3(T)}} \sqrt{\tilde{G}_L(T) \tilde{G}_H(T)} \tilde{Z}(T).
\]

For extraction of the \(B^0_s\)-width difference \(\Delta\Gamma_s \equiv \Gamma_H - \Gamma_L\) from experimental data it is convenient to use a special set of the time-integrated normalized observables:

\[
\hat{b}_i(T_0) = \frac{1}{L(T)} \int_0^{T_0} dt \int d\cos\theta_t d\cos\theta_\perp d\chi w_i(\theta_t, \theta_\perp, \chi) e^{\Gamma'_t} f(\theta_t, \theta_\perp, \chi; t),
\]

where \(\Gamma'\) is some arbitrary initial approximation of the \(B^0_s\)-meson total decay width. These observables can be extracted from the experimental events \(N(T)\), measured in the proper time region \(t \in [0, T]\), by using the formula

\[
\hat{b}_i^{(exp)}(T_0) = \frac{1}{N(T)} \sum_{j=1}^{N(T_0)} W_i^j,
\]

where \(W_i^j \equiv e^{\Gamma'_j} w_i^j\), and summation is performed over all events \(N(T_0)\) in the time interval \(t^j \in [0, T_0]\).
For the untagged sample, the explicit expressions for the time-integrated observables \( \tilde{b}_i \), \( \tilde{G}_{L/H} \) and \( \tilde{Z} \) can be easily obtained by replacing \( \tilde{b}_i \), \( \tilde{G}_{L/H} \) and \( \tilde{Z} \) in the expressions of Eq. (24) by \( \hat{b}_i \), \( \hat{G}_{L/H} \) and \( \hat{Z} \), respectively, (with the same normalization factor (21)) after introducing the following notations

\[ \hat{G}_{L/H}(T) \equiv \int_0^T dt \ e^{\Gamma t} G_{L/H}(t) \]

\[ = (1 \pm \cos\phi_c^{(s)}) \frac{e^{\Delta \Gamma_L T/2} - 1}{\Delta \Gamma_L} - (1 \mp \cos\phi_c^{(s)}) \frac{e^{-\Delta \Gamma_H T/2} - 1}{\Delta \Gamma_H}, \quad (30) \]

\[ \hat{Z}(T) \equiv \frac{1}{\cos\delta_{1,2}\sin\phi_c^{(s)}} \int_0^T dt \ e^{\Gamma t} Z_{1,2}(T) \]

\[ = \frac{1 - e^{\Delta \Gamma_L T/2}}{\Delta \Gamma_L} + \frac{1 - e^{-\Delta \Gamma_H T/2}}{\Delta \Gamma_H}. \]

where \( \Delta \Gamma_{L/H} \) are auxiliary parameters given by

\[ \Delta \Gamma_L = 2(\Gamma' - \Gamma_L), \quad \Delta \Gamma_H = -2(\Gamma' - \Gamma_H). \quad (31) \]

Eq. (26) is also valid after such a replacement.

5 Monte Carlo studies

For Monte Carlo studies of the estimation of physical parameters by applying the angular-moments method, untagged samples of events of \( B_s^0(t) \rightarrow J/\psi \phi \) decays have been generated by using the package SIMUB [2] with various sets of the input values of initial amplitudes \( |A_0(0)| \) and \( |A_{\perp}(0)| \) and \( \Delta \Gamma_s \). Other parameters are fixed as follows:

\[ \delta_1 = \pi, \quad \delta_2 = 0, \quad \Gamma_s = 1/\tau_s = 2.278 \ {\text{[mm/c]}^{-1}}, \quad \phi_c^{(s)} = 0.04. \]

The value of \( \Gamma_s \) used corresponds to the lifetime \( \tau_s = 1.464 \ {\text{ps}} \) [12], while the CP-violating weak phase \( \phi_c^{(s)} \) was fixed as the upper limit of the constrain [8]. The value of \( \Delta \Gamma_s \) is expected to be negative in the SM. The combined experimental result for \( |\Delta \Gamma_s|/\Gamma_s \) is not precise: \(|\Delta \Gamma_s|/\Gamma_s < 0.52 \) at 95% CL [12]. In the approximation of the equal \( B_s^0 \) and \( B_d^0 \) lifetimes, the \( |\Delta \Gamma_s| \) extraction can be improved [12]: \(|\Delta \Gamma_s|/\Gamma_s < 0.31 \) at 95% CL. A set of the untagged-event samples has been generated with \( \Delta \Gamma_s/\Gamma_s \in [-0.3, -0.01] \) to study the influence of \( \Delta \Gamma_s \) value on the estimation of \( B_s^0(t) \rightarrow J/\psi \phi \) decay parameters from data analysis.

The values of the time integrated observables \( \tilde{b}_i^{(exp)}(T_0) \), defined by Eq. (20), can be extracted from data according to Eq. (22) by summation of weighting functions for each event. The statistical error of \( \tilde{b}_i(T_0) \) is defined by

\[ (\Delta \tilde{b}_i)^{\text{stat}} = \frac{1}{N(T)} \sqrt{\sum_{j=1}^{N(T_0)} (\tilde{b}_i^{(exp)} - w_i^j)^2}, \quad (32) \]
while a systematic error due to limited precision of angular measurements can be estimated as

\[
(\delta \hat{b}_i)^{\text{(sys)}} = \sqrt{\frac{\sum_{j=1}^{N(T_0)} \Delta^2_j}{N(T)}}.
\]

Here

\[
\Delta^2_j = \left[ \frac{\partial w^j_i}{\partial \cos \Theta_{t^i}} \Delta(\cos \Theta_{t^i}) \right]^2 + \left[ \frac{\partial w^j_i}{\partial \cos \Theta_{K^+}} \Delta(\cos \Theta_{K^+}) \right]^2 + \left[ \frac{\partial w^j_i}{\partial \chi} \Delta(\chi) \right]^2.
\]

In a similar way, the values of the observables \( \hat{b}_i^{\text{(exp)}}(T_0) \), defined by Eq. (28), can be extracted from the data according to Eq. (29). The formulae for statistical and systematic errors for \( \hat{b}_i^{\text{(exp)}}(T_0) \) can be obtained by replacement of \( w^j_i \) to \( W^j_i \) in Eq. (32) and the following redefinition of \( \Delta^2_j \) in Eq. (33):

\[
\Delta^2_j = \left[ \frac{\partial W^j_i}{\partial \cos \Theta_{t^i}} \Delta(\cos \Theta_{t^i}) \right]^2 + \left[ \frac{\partial W^j_i}{\partial \cos \Theta_{K^+}} \Delta(\cos \Theta_{K^+}) \right]^2 + \left[ \frac{\partial W^j_i}{\partial \chi} \Delta(\chi) \right]^2 + \left[ \frac{\partial W^j_i}{\partial t} \Delta(t) \right]^2.
\]

Eq. (33) can be applied to estimate the systematic errors related both to the measurement precision of the detector and to the limited resolution of the Monte Carlo generator. In the SIMUB generator, for each variable \( V \in \{\cos \Theta_{t^i}, \cos \Theta_{K^+}, \chi, t\} \) randomly generated for decays \( B^0_s(t) \to J/\psi(\to l^+l^-) \phi(\to K^+K^-) \), the number of bins in the region \([V_{\text{min}}, V_{\text{max}}]\) was set as \( N^{\text{gen}} = 50,000 \). The generation precision for the variable \( V \) is defined as \( \Delta(V) = (V_{\text{max}} - V_{\text{min}})/N^{\text{gen}} \) and systematic errors are proportional to \((N^{\text{gen}})^{-1/2}\). The \( B \)-meson proper time was generated within the interval \( t \in [0, T = 2 \text{ mm/c}] \) which includes 99.3% of all \( B \)-decays. We have used samples with a maximum of 100,000 events of the decay \( B^0_s \to J/\psi \phi \) because a statistics of about 83,800 events is expected to be obtained per year at the CMS detector at the LHC low luminosity under realistic triggering conditions [19].

Table 2 shows the values of the observables \( \hat{b}_i^{\text{(exp)}}(T) \equiv \hat{b}_i^{\text{(exp)}}(T_0 = T) \) extracted from the Monte Carlo data by applying the sets A and B of weighting functions, given by Eqs. (14) and (17), respectively. Various theoretical models for estimation of the transversity amplitudes \(|A_0(0)|\) and \(|A_\perp(0)|\) (see Table 1) have been considered to fix these parameters in the SIMUB generator. It can be seen from Table 2 that the choice of \( N^{\text{gen}} = 50,000 \) provides negligibly small systematic errors for the observables as compared with the statistical ones. Moreover, both errors slightly depend on the values of the observables. For observables obtained by using the set-B weighting functions, the statistical errors are significantly smaller than in case of the set-A weighting functions. We should also note that even with the statistics of 100,000 events, the values of observables \( \hat{b}_i^{\text{(exp)}}(T) \) and − as consequence of Eq. (27) − the combination \( \cos \delta_1 \sin \phi \) cannot be extracted from the data if the CP-violating weak phase \( \phi^{(s)} \) is small according to the SM expectation [39]. In this case, these parameters can be estimated only by using a statistics which is not less than \( 3 \times 10^9 B^0_s(t) \to J/\psi \phi \) decays.

Analysis of the same Monte Carlo data leads to similar conclusions concerning the behavior of statistical and systematic errors for the observables \( \hat{b}_i^{\text{(exp)}}(T) \equiv \hat{b}_i^{\text{(exp)}}(T_0 = T) \).
Table 2: Comparison of the observables $\tilde{b}_i^{(exp)}(T)$, extracted from Monte Carlo data, with their values $\tilde{b}_i^{(th)}(T)$ corresponding to various theoretical models for $|A_0(0)|$ and $|A_+(0)|$. A sample of 100,000 decay events generated with $\Delta \Gamma_s/\Gamma_s = -0.15$ was used. The first errors are statistical while the second errors correspond to the systematic uncertainties.

a) BSW model [13]:

| $i$ | $\tilde{b}_i^{(th)}(T)$ | $\tilde{b}_i^{(exp)}(T)$ (set A) | $\tilde{b}_i^{(exp)}(T)$ (set B) |
|-----|--------------------------|-----------------------------------|----------------------------------|
| 1   | 0.5425                   | 0.5409 ± 0.0044 ± 0.0003          | 0.5432 ± 0.0024 ± 0.0002         |
| 2   | 0.3551                   | 0.3619 ± 0.0047 ± 0.0004          | 0.3579 ± 0.0036 ± 0.0004         |
| 3   | 0.1024                   | 0.0972 ± 0.0049 ± 0.0004          | 0.0991 ± 0.0034 ± 0.0004         |
| 4   | -0.00055                 | -0.0017 ± 0.0037 ± 0.0004         | -0.0021 ± 0.0033 ± 0.0003        |
| 5   | -0.4389                  | -0.4344 ± 0.0050 ± 0.0003         | -0.4344 ± 0.0050 ± 0.0003        |
| 6   | 0.00067                  | 0.0037 ± 0.0055 ± 0.0003          | 0.0037 ± 0.0055 ± 0.0003         |

b) Model by Soares [14]:

| $i$ | $\tilde{b}_i^{(th)}(T)$ | $\tilde{b}_i^{(exp)}(T)$ (set A) | $\tilde{b}_i^{(exp)}(T)$ (set B) |
|-----|--------------------------|-----------------------------------|----------------------------------|
| 1   | 0.3908                   | 0.3900 ± 0.0046 ± 0.0003          | 0.3955 ± 0.0023 ± 0.0002         |
| 2   | 0.2574                   | 0.2617 ± 0.0049 ± 0.0004          | 0.2551 ± 0.0037 ± 0.0004         |
| 3   | 0.3518                   | 0.3483 ± 0.0047 ± 0.0004          | 0.3509 ± 0.0037 ± 0.0004         |
| 4   | -0.00086                 | -0.0083 ± 0.0040 ± 0.0004         | -0.0017 ± 0.0035 ± 0.0003        |
| 5   | -0.3171                  | -0.3156 ± 0.0052 ± 0.0003         | -0.3156 ± 0.0052 ± 0.0003        |
| 6   | 0.0011                   | 0.0008 ± 0.0052 ± 0.0003          | 0.0008 ± 0.0052 ± 0.0003         |

c) Model by Cheng [15]:

| $i$ | $\tilde{b}_i^{(th)}(T)$ | $\tilde{b}_i^{(exp)}(T)$ (set A) | $\tilde{b}_i^{(exp)}(T)$ (set B) |
|-----|--------------------------|-----------------------------------|----------------------------------|
| 1   | 0.5271                   | 0.5228 ± 0.0045 ± 0.0003          | 0.5267 ± 0.0024 ± 0.0001         |
| 2   | 0.2928                   | 0.2980 ± 0.0048 ± 0.0004          | 0.2950 ± 0.0036 ± 0.0004         |
| 3   | 0.1801                   | 0.1791 ± 0.0048 ± 0.0004          | 0.1778 ± 0.0035 ± 0.0004         |
| 4   | -0.00066                 | -0.0030 ± 0.0037 ± 0.0003         | -0.0034 ± 0.0034 ± 0.0003        |
| 5   | -0.3928                  | -0.3927 ± 0.0051 ± 0.0003         | -0.3927 ± 0.0051 ± 0.0003        |
| 6   | 0.00088                  | -0.0019 ± 0.0054 ± 0.0003         | -0.0019 ± 0.0054 ± 0.0003        |
Table 3: Comparison of the values of observables $\hat{b}_i^{(exp)}(T)$, with $\Gamma' = \Gamma_s$, extracted from the Monte Carlo data, with their values $\hat{b}_i^{(th)}(T)$ corresponding to the model of Cheng [15] for initial transversity amplitudes

| $i$ | $\hat{b}_i^{(th)}(T)$ | $\hat{b}_i^{(exp)}(T)$ (set A) | $\hat{b}_i^{(exp)}(T)$ (set B) |
|-----|----------------------|-------------------------------|-------------------------------|
| 1   | 2.2036               | 2.176 ± 0.044 ± 0.003         | 2.206 ± 0.026 ± 0.001         |
| 2   | 1.2242               | 1.282 ± 0.045 ± 0.004         | 1.245 ± 0.034 ± 0.004         |
| 3   | 0.9187               | 0.917 ± 0.045 ± 0.004         | 0.930 ± 0.034 ± 0.004         |
| 4   | -0.0073              | -0.099 ± 0.036 ± 0.003        | -0.094 ± 0.032 ± 0.003        |
| 5   | -1.6425              | -1.618 ± 0.048 ± 0.003        | -1.618 ± 0.048 ± 0.003        |
| 6   | 0.0098               | 0.067 ± 0.050 ± 0.003         | 0.067 ± 0.050 ± 0.003         |

To illustrate the performance of our method in this case, only the results obtained for transversity amplitudes, corresponding to the Cheng’s model [15], are shown in Table 3.

Fig. 3 shows the dependence of the observables $\tilde{b}_i(T)$ on the value of the ratio $\Delta \Gamma_s / \Gamma_s$. For $\Delta \Gamma_s = 0$ we have

$$\tilde{b}_{1,2,3}(T)|_{\Delta \Gamma_s=0} = \tilde{b}_{1,2,3}(T)|_{\Delta \Gamma_s=0} = |A_{0,||,\perp}(0)|^2.$$

The observables $\tilde{b}_i(T)$ slightly depend on $\Delta \Gamma_s$. The rather strong dependence of the observables $\tilde{b}_i(T)$ on the decay width difference $\Delta \Gamma_s$, shown in Fig. 3, can be used for extraction of this parameter from the data analysis as it will be discussed below.

Under the assumption $\phi_c^{(s)} = 0$, we have from Eq. (30):

$$\hat{G}_{L/H}^{(0)}(T) = \frac{1 - e^{-\Gamma_s T}}{\Gamma_s T} \cdot \hat{b}_i(T) \quad (i = 1, 2, 3)$$

Therefore, the values of the auxiliary parameters $\Delta \Gamma_{L/H}$, defined by Eq. (31), can be determined separately by using the ratios of observables $\hat{b}_i^{(exp)}(T)/\hat{b}_i^{(exp)}(T_0)$, extracted from the data analysis, and solving numerically the equations which arise from one of the following relations:

$$\hat{b}_i(T) / \hat{b}_i(T_0) = \hat{G}_L(T) / \hat{G}_L(T_0) \quad (i = 1, 2, 5)$$

– to determine $\Delta \Gamma_L$, and the relation

$$\hat{b}_3(T) / \hat{b}_3(T_0) = \hat{G}_H(T) / \hat{G}_H(T_0)$$

– to determine $\Delta \Gamma_H$. Then, the decay-width parameters $\Gamma_s$, $\Delta \Gamma_s$ and $\Gamma_{L/H}$ can be determined via $\Gamma'$ and $\Delta \Gamma_{L/H}$ as

$$\Gamma_s = \Gamma' - \frac{\Delta \Gamma_L - \Delta \Gamma_H}{4}, \quad \Delta \Gamma_s = \frac{\Delta \Gamma_L + \Delta \Gamma_H}{2}, \quad \Gamma_{L/H} = \Gamma' + \frac{\Delta \Gamma_L}{2}. \quad (36)$$
Figure 3: Dependence of the observables $\tilde{b}_i(T)$ and $\tilde{b}'_i(T) \equiv \tilde{b}_i(T)[1 - \exp(-\Gamma T)]/(\Gamma T)$ ($i = 1, 2, 3$) on the value of $\Delta \Gamma_s/\Gamma_s$. The observables have been calculated for the case of Cheng’s model for transversity amplitudes.

So, by using some reasonable approximation for $\Gamma'$ as a starting point for the data analysis, the experimental value of $\Gamma_s$ can be essentially improved simultaneously with determination of $\Delta \Gamma_s$. The statistical error of $\Gamma_s$ determination is expected to be twice smaller than for $\Delta \Gamma_s$ determination.

The direct numerical calculations have shown that the difference between the values of observables $\tilde{b}_i(T)$ ($i = 1, 2, 3, 5$), calculated with $\phi^{(s)}_c = 0$ and $\phi^{(s)}_c = 0.04$, does not exceed 0.01%. Even in case of statistics of 100 000 events this difference is negligibly small as compared with statistical errors for these observables (see Table 3). Therefore, the assumption $\phi^{(s)}_c = 0$ is a good approximation for $\Gamma_s$, $\Delta \Gamma_s$ and $\Gamma_{L/H}$ determination by the method described above.

Table 4 shows the results of determination of the decay-width parameters after applying the described procedure to the Monte Carlo data. The sample of 100 000 events generated in case of Cheng’s model with $\Delta \Gamma_s/\Gamma_s = -0.15$ has been used. Both sets A and B of weighting functions have been applied to extract the observables $\tilde{b}_i^{(exp)}$. The value of $\Gamma'$, which is treated as some arbitrary initial approximation for the total decay width of $B_s^0$-meson, was fixed as $\Gamma' = 1.05 \Gamma_s$, i.e. it was shifted by 5% relative to the “true” value of $\Gamma_s$ fixed in the Monte Carlo generator SIMUB. The value of $T_0 = 0.1 T$ was chosen as it provides the minimal statistical errors to determine the ratios $\tilde{b}^{(exp)}_i(T)/\tilde{b}^{(exp)}_i(T_0)$. In Table 4 we present the result for $\Delta \Gamma_L$ obtained from the ratio $\tilde{b}^{(exp)}_1(T)/\tilde{b}^{(exp)}_1(T_0)$ only, which gives the best precision. Table 4 shows that the set-B weighting functions give more precise and stable results than the set-A functions.

To improve the precision of $\Delta \Gamma_s$ determination, the same procedure should be repeated with $\Gamma'$ fixed to be equal to the value of $\Gamma_s$ determined at the first step. Because of $\Delta \Gamma_s = \Delta \Gamma_L = \Delta \Gamma_H$ in case of $\Gamma' = \Gamma_s$, the value of $\Delta \Gamma_s$ is defined at the second
Table 4: Results of determination of the decay-width parameters (in units (mm/c)^{-1}) based on extraction of the observables \( \tilde{b}_1^{(exp)} \) from analysis of 100,000 Monte Carlo events. The input value of \( \Delta \Gamma_s \) corresponds to \( \Delta \Gamma_s/\Gamma_s = -0.15 \)

| Parameter | Input value | Measurement (set A) | Measurement (set B) |
|-----------|-------------|---------------------|---------------------|
| \( \Delta \Gamma_L \) | -0.1139 | -0.103 ± 0.058 ± 0.003 | -0.110 ± 0.034 ± 0.002 |
| \( \Delta \Gamma_H \) | -0.5969 | -0.478 ± 0.137 ± 0.012 | -0.554 ± 0.101 ± 0.012 |
| \( \Gamma_L \) | 2.4493 | 2.444 ± 0.029 ± 0.002 | 2.447 ± 0.017 ± 0.001 |
| \( \Gamma_H \) | 2.1076 | 2.154 ± 0.068 ± 0.006 | 2.115 ± 0.050 ± 0.006 |
| \( \Gamma_s \) | 2.2784 | 2.299 ± 0.037 ± 0.003 | 2.281 ± 0.027 ± 0.003 |
| \( \Delta \Gamma_s \) | -0.3418 | -0.290 ± 0.074 ± 0.006 | -0.332 ± 0.053 ± 0.006 |

step to be equal to the value of \( \Delta \Gamma_L \) determined from the ratio \( \tilde{b}_1^{(exp)}(T)/\tilde{b}_1^{(exp)}(T_0) \) using Eq. (34). Using the values of \( \Delta \Gamma_s \) from Table 4 as an input value of \( \Gamma' \) at the second step, we have obtained finally the following results (to be compared with the input value \( \Delta \Gamma_s = -0.3418 \) set in the SIMUB generator):

\[
\Delta \Gamma_s^{exp} = -0.330 \pm 0.057 \pm 0.004 \quad \text{(set A)},
\]
\[
\Delta \Gamma_s^{exp} = -0.338 \pm 0.034 \pm 0.002 \quad \text{(set B)}.
\]

This way one can reduce not only the statistical error but also essentially improve the stability of the \( \Delta \Gamma_s \) result even in case of using the set-A weighting functions.

Table 5 shows the statistical errors of \( \Delta \Gamma_s/\Gamma_s \) determination by the described approach applied to different statistics of Monte Carlo events generated with various ”true” values of \( \Delta \Gamma_s \). The lack of numbers in the table corresponds to cases when the approach is not able to give a certain result for \( \Delta \Gamma_s \). The use of set-B weighting functions gives more stable results even in case of too small statistics and values of \( \Delta \Gamma_s \), for which the same approach does not work with set-A functions. The sensitivity of the method is measured by the statistical error of \( \Delta \Gamma_s/\Gamma_s \), which only slightly depends on the value of this ratio and is proportional to \( 1/\sqrt{N} \), where \( N \) is the number of events. In particular, for a statistics 100,000 events, the statistical error is about 0.015, while for 1000 events – about 0.15.

In principle, the value of \( \Delta \Gamma_s \) can be determined similarly by using the ratios \( \tilde{b}_i^{(exp)}(T)/\tilde{b}_i^{(exp)}(T_0) \) or \( \hat{b}_i^{(exp)}(T)/\hat{b}_i^{(exp)}(T) \), extracted from the data analysis with \( \Gamma' = \Gamma_s \), and solving the equations arising from the relations

\[
\tilde{b}_i(T)/\tilde{b}_i(T_0) = \tilde{G}_L(T)/\tilde{G}_L(T_0) \quad (i = 1, 2, 5),
\]
\[
\tilde{b}_3(T)/\tilde{b}_3(T_0) = \tilde{G}_H(T)/\tilde{G}_H(T_0)
\]
or

\[
\hat{b}_i(T)/\hat{b}_i(T) = \hat{G}_L(T)/\hat{G}_L(T) \quad (i = 1, 2, 5),
\]
\[
\hat{b}_3(T)/\hat{b}_3(T) = \hat{G}_H(T)/\hat{G}_H(T).
\]
Table 5: Statistical errors of $\Delta \Gamma_s$ extraction (in units [mm/c]−1) obtained by applying the angular-moments method with set-B (set-A) weighting functions to the Monte Carlo data samples with different numbers of events

| $\Delta \Gamma_s/\Gamma_s$ | 200 events | 500 events | $10^3$ events | $10^4$ events | $10^5$ events |
|---------------------------|------------|------------|---------------|---------------|---------------|
| -0.03                     | -          | -          | -             | 0.035 (-)     | 0.014 (0.023) |
| -0.05                     | -          | -          | -             | 0.046 (-)     | 0.014 (0.022) |
| -0.1                      | -          | -          | 0.11 (-)      | 0.046 (0.079) | 0.014 (0.024) |
| -0.15                     | -          | -          | 0.13 (0.19)   | 0.045 (0.078) | 0.014 (0.024) |
| -0.2                      | -          | 0.23 (-)   | 0.12 (0.18)   | 0.048 (0.072) | 0.015 (0.026) |
| -0.3                      | 0.21 (-)   | 0.23 (-)   | 0.18 (0.20)   | 0.050 (0.083) | 0.016 (0.028) |

But in both these cases the precision of $\Delta \Gamma_s$ determination turns out to be worse than in the approach based on the ratios $\hat{b}_i^{(\exp)}(T)/\hat{b}_i^{(\exp)}(T_0)$ because of the weak $\Delta \Gamma_s$-dependence of the $\tilde{b}_i(T)$ observables.

The initial transversity amplitudes and strong-phase difference can be recalculated from the values of observables $\tilde{b}_i^{(\exp)}(T)$ according to Eq. (26). The results of such determination of the parameters $|A_f(0)|^2$ ($f = 0, ||, \perp$) and $\cos(\delta_2 - \delta_1)$ are shown in Table 6 for different statistics. We have used the Monte Carlo sample generated with the theoretical values of the amplitudes $|A_0(0)|$ and $|A_\perp(0)|$ corresponding to Cheng’s model [15]. To extract the observables $\tilde{b}_i^{(\exp)}(T)$, the set B of the weighting function has been applied to Monte Carlo data. To estimate the statistical errors for parameters $|A_f(0)|^2$ ($f = 0, ||, \perp$) and $\cos(\delta_2 - \delta_1)$, the standard error-propagation method has been applied to the statistical errors of the observables $\tilde{b}_i^{(\exp)}(T)$, taking into account the correlation between pairs of different observables. The systematic errors of the observables related to the limited generator resolution are neglected. The total errors for parameters $|A_f(0)|^2$ ($f = 0, ||, \perp$) should also include the additional uncertainty related to the error of calculation of $\tilde{\gamma}(T)$ caused by the error of $\Delta \Gamma_s$ (see definition of $\tilde{\gamma}(T)$ in Eq. (25) and Eq. (28)). In Table 6 we also show these errors calculated by assuming $\Delta \Gamma_s = -0.15 \Gamma_s$ (see Table 5)

$$\frac{\delta(\Delta \Gamma_s)}{|\Delta \Gamma_s|} = \begin{cases} 30\% & \text{for 10000 events,} \\ 9.3\% & \text{for 100000 events.} \end{cases}$$

(37)

6 Conclusion

For the decay $B_s^0 \rightarrow J/\psi \phi$ in the framework of the method of angular moments a non-fit scheme for separate estimation of the parameters $\Delta \Gamma_s$, $\Gamma_s$ and $|A_f(0)|^2$ ($f = 0, ||, \perp$) has been proposed, based on analysis of an untagged sample, and studied by Monte Carlo method. A strong dependence of statistical measurement errors on the choice of the weighting functions has been demonstrated. The statistical error of the ratio $\Delta \Gamma_s/\Gamma_s$ for
Table 6: Determination of initial transversity amplitudes and strong-phase difference by using the values of observables $\tilde{b}_i^{(\text{exp})}(T)$ extracted from Monte Carlo data. The events sample has been generated for the case of Cheng’s model \[\text{[15]}\] for transversity amplitudes and with $\Delta \Gamma_s = 0.15 \Gamma_s$. The first errors are statistical while the second errors are caused by uncertainties of $\Delta \Gamma_s$ determination.

| Parameter | Input value | 10 000 events | 100 000 events |
|-----------|-------------|---------------|----------------|
| $|A_0(0)|^2$ | 0.54 | $0.527 \pm 0.007 \pm 0.012$ | $0.5398 \pm 0.0023 \pm 0.0011$ |
| $|A_{||}|^2$ | 0.30 | $0.337 \pm 0.011 \pm 0.008$ | $0.3023 \pm 0.0036 \pm 0.0006$ |
| $|A_{\perp}|^2$ | 0.16 | $0.136 \pm 0.010 \pm 0.020$ | $0.1579 \pm 0.0032 \pm 0.0018$ |
| $\cos(\delta_2 - \delta_1)$ | -1 | $-1.021 \pm 0.044$ | $-0.9962 \pm 0.015$ |

values of this ratio in the interval $[0.03, 0.3]$ was found to be independent on this value and about 0.015 for $10^5$ events. The method of angular moments gives stable results for the estimate of $\Delta \Gamma_s$ and is found to be an efficient and flexible tool for the quantitative investigation of the $B_s^0 \rightarrow J/\psi \phi$ decay.

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