Global Strong Solutions to the Compressible Magnetohydrodynamic Equations with Slip Boundary Conditions in 3D Bounded Domains

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Abstract
We deal with the barotropic compressible magnetohydrodynamic equations in three-dimensional (3D) bounded domain with slip boundary condition and vacuum. By a series of a priori estimates, especially the boundary estimates, we prove the global well-posedness of classical solution and the exponential decay rate to the initial-boundary-value problem of this system for the regular initial data with small energy but possibly large oscillations. The initial density of such a classical solution is allowed to contain vacuum states. Moreover, it is also shown that the oscillation of the density will grow unboundedly with an exponential rate when the initial state contains vacuum.

Keywords: compressible magnetohydrodynamic equations; global classical existence; priori estimates; slip boundary condition; vacuum.

AMS subject classifications: 35Q55, 35K65, 76N10, 76W05

1 Introduction

In this paper, we consider the viscous compressible magnetohydrodynamic (MHD) equations for barotropic flows in a domain $\Omega \subset \mathbb{R}^3$, which can be written as

$$\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \mu \Delta u + (\mu + \lambda)\nabla \text{div} u + (\nabla \times H) \times H, \\
H_t - \nabla \times (u \times H) = -\nu \nabla \times (\nabla \times H), \\
\text{div} H = 0,
\end{cases}\tag{1.1}$$

where $(x, t) \in \Omega \times (0, T]$, $t \geq 0$ is time, and $x = (x_1, x_2, x_3)$ is the spatial coordinate. The unknown functions $\rho, u = (u^1, u^2, u^3), P = P(\rho)$, and $H = (H^1, H^2, H^3)$ denote the fluid density, velocity, pressure and magnetic field, respectively. Here we consider the barotropic flows with $\gamma$-law pressure $P(\rho) = a \rho^\gamma$, where $a > 0$ and $\gamma > 1$ are some physical parameters. The constants $\mu$ and $\lambda$ are the shear viscosity and bulk coefficients respectively satisfying the following physical restrictions $\mu > 0$ and $2\mu + 3\lambda \geq 0$. The constant $\nu > 0$ is the resistivity coefficient which is inversely proportional to the electrical conductivity constant and acts as the magnetic diffusivity of magnetic fields. In addition, the system is solved subject to the given initial data

$$\begin{align*}
\rho(x, 0) &= \rho_0(x), & \rho u(x, 0) &= \rho_0 u_0(x), & H(x, 0) &= H_0(x), & x \in \Omega, \tag{1.2}
\end{align*}$$
and slip boundary conditions

\begin{align}
  u \cdot n &= 0, \quad \text{curl}\, u \times n = 0, & \text{on } \partial \Omega, \\
  H \cdot n &= 0, \quad \text{curl}\, H \times n = 0, & \text{on } \partial \Omega,
\end{align}

where \( n \) is the unit outward normal vector to \( \partial \Omega \).

The choice of boundary conditions is very important for hydrodynamics. For the velocity field, one of the well-accepted choices is the no-slip boundary condition (i.e. Dirichlet boundary condition), which has been successfully applied to many hydrodynamical problems on the macroscopic scale. However, experimental studies reveal that for flows on the micro- and nanoscale, the empirical non-slip boundary condition may break down, depending on the interfacial roughness and interfacial interactions between solids and fluids. Another choice is the slip boundary condition, which has different behaviors between the macroscopic scale and microscale. The earliest slip boundary condition is proposed by Navier, which indicates that there is a stagnant layer of fluid close to the wall allowing a fluid to slip and the slip velocity is proportional to the shear stress, that is

\begin{equation}
  u \cdot n = 0, \quad (D(u)n)_\tau + \vartheta u_\tau = 0 \quad \text{on } \partial \Omega,
\end{equation}

where \( D(u) = (\nabla u + (\nabla u)^*)/2 \) is the shear stress, \( \vartheta \) is a scalar friction function, subscript \( \tau \) denotes the tangential component on \( \partial \Omega \). With the development of micro/nano test technologies and molecular-dynamic simulation technology, the Navier-type slip boundary has been more concerned and well studied in numerical studies and analysis for various fluid mechanical problems, see, for instance [4, 6, 9, 16, 25, 41–43] and the references therein. Additional, as shown in [6, 41], the Navier-slip condition \((1.5)\) is written to the following generalized one

\begin{equation}
  u \cdot n = 0, \quad \text{curl}\, u \times n = -Au \quad \text{on } \partial \Omega,
\end{equation}

where \( A \) is a smooth symmetric matrix defined on \( \partial \Omega \), especially when \( A = 0 \), it is strongly related to \((1.5)\). Hence, the boundary condition \((1.3)\) presented in this paper can be regarded as a Navier-type slip boundary condition. For the magnetic field, the boundary condition \((1.4)\) describes that the boundary \( \partial \Omega \) is a perfect conductor (see [14, 42]), that means the magnetic field is confined inside and separated from the exterior. We also observed that \((1.4)\) is adaptable to the system since it ensured the boundary balance of the quantities on the boundary. Therefore, it is appropriate to consider the compressible MHD equations with the boundary conditions \((1.3)-(1.4)\).

The compressible MHD system \((1.1)\) has been attracted a lot of attention of physicists and mathematicians due to its physical importance and mathematical challenges, including the strong coupling and interplay interaction between fluid motion and magnetic field, and significant progress has been made in the analysis of the well-posedness and dynamic behavior to the solutions of the system, see, for example, [7, 8, 10–14, 17–21, 26, 27, 30–32, 36, 37, 39, 40, 44–46] and their references. Among them, we briefly review the results related to well-posedness of solutions for the multi-dimensional compressible MHD equation. The local existence of strong solutions to the compressible MHD equations was obtained by Vol’pert and Khudiaev [37] for the Cauchy problem with large initial data and the initial density being strictly positive. Fan and Yu [14] extended the result to the case that the initial density may contain vacuum for the whole space or a bounded domain with non-slip boundary condition. Lv and Huang [31] obtained the
local existence of strong and classical solutions in $\mathbb{R}^2$ with vacuum as far field density. The global existence of solutions to the compressible MHD equations has been studied in many works. Kawashima [26] obtained the global existence of smooth solutions to the general electro-magneto-fluid equations in two dimensions when the initial data are small perturbations of a given constant state. Hu and Wang [18, 21] and Fan and Yu [13] proved the global existence of renormalized solutions to the compressible MHD equations for general large initial data. Recently, Li, Xu and Zhang [27] established the global existence and uniqueness of classical solutions with constant state as far field in $\mathbb{R}^3$ with large oscillations and vacuum. Hong, Hou, Peng and Zhu [17] generalized the result for large initial data when $\gamma - 1$ and $\nu^{-1}$ are suitably small. Lv, Shi and Xu [32] got the global existence of unique classical solutions in two-dimensional space and obtained some better a priori decay with rates.

However, all of the above results only concern with the whole space or with non-slip boundary conditions. It is rather complicated to investigate the well-posedness and dynamical behaviors of the compressible MHD system with slip boundary condition due to the compatibility issues of the nonlinear terms with the slip boundary conditions. Tang and Gao [36] consider the local strong solutions to the compressible MHD equations with initial vacuum, in which the velocity field satisfies the Navier-slip condition. Considering the full compressible MHD system, Xi and Hao [40] proved the local existence of the classical solutions to the initial-boundary value problem with slip boundary condition for the full compressible MHD system without thermal conductivity, where the initial data contains vacuum and satisfies some initial layer compatibility condition. However, to our best knowledge, whether the strong (classical) solution for general bounded smooth domains $\Omega$ with density containing vacuum initially to the MHD system exists globally in time is still open. Recently, for the barotropic compressible Navier-Stokes equations in a bounded domain $\Omega$ with slip boundary condition, Cai and Li [6] proved that the classical solution of the initial-boundary-value problem exists globally with vacuum and small energy but possibly large oscillations and adopt some new techniques to obtain necessary a priori estimates, especially the boundary estimates, compared with the work in [24, 29] for the Cauchy problem of the compressible Navier-Stokes equations. The main purpose of this paper is to establish the global well-posedness of classical solutions of the compressible MHD system (1.1)-(1.4) in a bounded domain $\Omega \subset \mathbb{R}^3$. We would like to obtain the time-independent upper bound of the density and the time-dependent higher-norm estimates of $(\rho, u, H)$ and extend the classical solution globally in time, which is motivated by the works of Cai and Li [6] and Li, Xu and Zhang [27].

Before formulating our main result, we first explain the notation and conventions used throughout the paper. For integer $k \geq 1$ and $1 \leq q < +\infty$, we denote the standard Sobolev space by $W^{k,q}(\Omega)$ and $H^k(\Omega) \triangleq W^{k,2}(\Omega)$. For some $s \in (0, 1)$, the fractional Sobolev space $H^s(\Omega)$ is defined by

$$H^s(\Omega) \triangleq \left\{ u \in L^2(\Omega) : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dxdy < +\infty \right\},$$

with the norm:

$$\|u\|_{H^s(\Omega)} \triangleq \|u\|_{L^2(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dxdy \right)^{\frac{1}{2}}.$$  

For simplicity, we denote $L^q(\Omega)$, $W^{k,q}(\Omega)$, $H^k(\Omega)$ and $H^s(\Omega)$ by $L^q$, $W^{k,q}$, $H^k$ and $H^s$. 


provided the initial total energy $C_0 \leq \varepsilon$. Moreover, for any $r \in [1, \infty)$ and $p \in [1, 6]$, there exist positive constants $C$ and $\eta_0$ depending only on $\mu$, $\lambda$, $\gamma$, $a$, $\bar{\rho}$, $\hat{\rho}$, $\Omega$, $M_1$, $M_2$, $r$ and $p$ such that for $t > 0$,

$$
\|\rho(\cdot, t) - \bar{\rho}\|_{L^r} + \|u(\cdot, t)\|_{W^{1,p}} + \|\sqrt{\rho}u(\cdot, t)\|_{L^2}^2 + \|H(\cdot, t)\|_{H^2} \leq Ce^{-\eta_0 t}.
$$

(1.15)
Then, thanks to the exponential decay rate \((1.15)\), taking the similar procedure as in \([6,28]\), we can directly deduce the following large-time behavior of the gradient of the density when the initial density contains vacuum state.

**Theorem 1.2** Under the conditions of Theorem 1.1, assume further that there exists some point \(x_0 \in \Omega\) such that \(\rho_0(x_0) = 0\). Then the unique global classical solution \((\rho, u, H)\) to the problem \((1.1)-(1.4)\) obtained in Theorem 1.1 satisfies that for any \(r_1 > 3\), there exist positive constants \(\tilde{C}_1\) and \(\tilde{C}_2\) depending only on \(\mu, \lambda, \gamma, a, s, \bar{\rho}, \rho, \Omega, M_1, M_2\) and \(r_1\) such that for any \(t > 0\),

\[
\|\nabla \rho(\cdot, t)\|_{L^{r_1}} \geq \tilde{C}_1 e^{\tilde{C}_2 t}.
\]  

\((1.16)\)

**Remark 1.1** From Sobolev’s inequality and \((1.14)_1\) with \(q > 3\), it follows that

\[
\rho, \nabla \rho \in C(\bar{\Omega} \times [0,T]).
\]  

\((1.17)\)

Moreover, it also follows from \((1.14)_{2-5}\) that

\[
\begin{align*}
    u, H, \nabla u, \nabla H, \nabla^2 u, \nabla^2 H, u_t, H_t & \in C(\bar{\Omega} \times [\tau,T]),
\end{align*}
\]  

\((1.18)\)

due to the following simple fact that

\[
L^2(\tau,T;H^1) \cap H^1(\tau,T;H^{-1}) \hookrightarrow C([\tau,T];L^2).
\]

Finally, by \((1.1)_1\), we have

\[
\rho_t = -u \cdot \nabla \rho - \rho \text{div} u \in C(\bar{\Omega} \times [\tau,T]).
\]  

\((1.19)\)

Hence the solution obtained in Theorem 1.1 becomes a classical one away from the initial time.

**Remark 1.2** When we consider the general slip boundary \((1.6)\) for the velocity field, and assume that the matrix \(A\) is smooth and positive semi-definite, and even if the restriction on \(A\) is relaxed to \(A \in H^3\) and the negative eigenvalues of \(A\) (if exist) are small enough, Theorem 1.1 will still hold. This can be achieved by a similar way as in [6]. When \(H = 0\), i.e., there is no electromagnetic field effect, the compressible MHD system \((1.1)\) turns to be the compressible Navier-Stokes equations, and Theorem 1.1 is the same as the result of Cai and Li [6]. Roughly speaking, we generalize the results of [6] to the compressible MHD equations.

**Remark 1.3** When the initial state contains vacuum, Theorem 1.2 implies that the oscillation of the density will grow unboundedly with an exponential rate, which is somewhat surprisingly compared with the Cauchy problem \([27]\) where there is no results concerning the growth rate of the gradient of the density.

**Remark 1.4** In our case, \(\Omega\) is a bounded domain in \(\mathbb{R}^3\), for the small initial energy, we need the boundedness assumptions on the \(H^s\)-norm, for \(s \in (1/2,1]\), of the initial velocity and magnetic field, which is analogous to the \(H^3\)-norm in the whole space case \([27,29]\). Thus, compared with the results in \([26]\), the conditions on the initial velocity may be optimal under the smallness conditions on the initial energy.
We now sketch the main idea used in the proof of Theorem 1.1. Similar to the argument in [6,27], the key issue in our proof is to derive the time-independent upper bound of the density in Proposition 3.1 and the time-dependent higher-norm estimates of \((\rho, u, H)\). It is worth pointing out that the effective viscous flux \(F\) and the vorticity \(\omega\) (see (2.5) for the definition) play an important role in the proof. However, unlike [27], it can not get the \(L^p\)-norm \((2 \leq p \leq 6)\) of \(\nabla u\) by the standard elliptic estimate, due to the bounded domain with the slip boundary condition (1.3). To deal with this difficulty, we consider the Petrovsky type Lamé’s system (A.5) (see [1]) and obtain the estimates of \(W^{k,a}\)-norm to the slip boundary condition (see Lemma A.3). Thanks to [2,38], Lemma A.4 shows that the inequality \(\|\nabla u\|_{L^q} \leq C(\|\text{div} u\|_{L^q} + \|\text{curl} u\|_{L^q})\) for any \(q \geq 1\) holds for \(u \in W^{1,q}\) with \(u \cdot n = 0\) on \(\partial \Omega\). These fact allows us to control \(\nabla u\) by means of \(\text{div} u\) and \(\text{curl} u\). For the magnetic field, with the help of the magnetic diffusivity structure, we obtain the estimates of \(\text{curl} H\) and \(\text{curl}^2 H\) to control \(\nabla H\) and \(\nabla^2 H\), which can be used to deal with the strong coupling and interplay interaction between the fluid motion and the magnetic field, such as the magnetic force \((\nabla \times H) \times H\) and the convection term \(\nabla \times (u \times H)\). In addition, the slip boundary also makes the time-independent estimates of \(A_1(T)\) and \(A_2(T)\) more difficult. Our observation is that due to the boundary condition \(u \cdot n = 0\), which yields \(u \cdot \nabla u \cdot n = -u \cdot \nabla n \cdot u\). This equality is the key to estimate the integrals on the boundary \(\partial \Omega\) and we obtain the estimate of \(\dot{u}\) and \(\nabla \dot{u}\) (see (2.34) and (2.35)).

The rest of the paper is organized as follows. In Section 2, we derive the elementary energy estimates for the system (1.1)-(1.4) and some key a priori estimates. Section 3 and Section 4 are devoted to deriving the necessary time-independent lower-order estimates and time-dependent higher-order estimates, which can guarantee the local classical solution to be a global classical one. In Section 5, the proof of Theorem 1.1 will be completed. In Appendix A, we list some elementary inequalities and important lemmas that we use intensively in the paper.

2 Preliminaries

In this section, we derive the elementary energy estimates for the system (1.1)-(1.4) and some key a priori estimates. Let \(T > 0\) be a fixed time and \((\rho, u, H)\) be a smooth solution to (1.1)-(1.4) on \(\Omega \times (0, T)\). For \(H\) and \(u\) sufficiently smooth, there are some formulas based on \(\text{div} H = 0\):

\[
\begin{align*}
(\nabla \times H) \times H &= \text{div}(H \otimes H - \frac{1}{2}|H|^2 I_3) = H \cdot \nabla H - \frac{1}{2} \nabla |H|^2, \\
\nabla \times (u \times H) &= (H \cdot \nabla)u - (u \cdot \nabla)H - H \text{div} u.
\end{align*}
\]

(2.1)

Then we rewrite (1.1) in the following form:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho u_t + \rho u \cdot \nabla u - (\lambda + 2\mu)\nabla \text{div} u + \mu \nabla \times \omega + \nabla(P - \bar{P}) &= H \cdot \nabla H + \frac{|H|^2}{2}, \\
H_t + u \cdot \nabla H - H \cdot \nabla u + H \text{div} u &= -\nu \nabla \times \text{curl} H, \\
\text{div} H &= 0,
\end{align*}
\]

(2.2)

where we used the fact \(-\Delta u = -\nabla \text{div} u + \nabla \times \omega\) and \(\omega \triangleq \nabla \times u\), \(\text{curl} H \triangleq \nabla \times H\). Multiplying (2.2) by \(G'(\rho)\), (2.2) by \(u\) and (2.2) by \(H\) respectively, integrating by
parts over $\Omega$, summing them up, by (1.3) and (1.4), we have
\[
\left( \int \left( G(\rho) + \frac{1}{2} \rho |u|^2 + \frac{1}{2} |H|^2 \right) \right)_t + (\lambda + 2\mu) \int (\text{div} u)^2 \, dx \\
+ \mu \int |\omega|^2 \, dx + \nu \int |\text{curl} H|^2 \, dx = 0,
\]
which, integrated over $(0,T)$, leads to the following elementary energy estimates.

**Lemma 2.1** Let $(\rho, u, H)$ be a smooth solution of (1.1)-(1.4) on $\Omega \times (0,T)$. Then
\[
\sup_{0 \leq t \leq T} \left( \frac{1}{2} \|\rho \frac{\partial u}{\partial t}\|_{L^2}^2 + \|G(\rho)\|_{L^1} + \frac{1}{2} \|H\|_{L^2}^2 \right) \\
+ \int_0^T (\lambda + 2\mu) \|\text{div} u\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \nu \|\text{curl} H\|_{L^2}^2) \, dt \leq C_0.
\]

Next, similarly to the compressible Navier-Stokes equations, let us set
\[
F \triangleq (\lambda + 2\mu) \text{div} u - (P - \bar{P}) - \frac{1}{2} |H|^2,
\]
where $F$ denotes the effective viscous flux, which plays an important role in our following analysis. For $F$, $\omega$, $\nabla u$ and $\nabla H$, we give the following conclusion, which is a key to a priori estimates.

**Lemma 2.2** Let $(\rho, u, H)$ be a smooth solution of (1.1)-(1.4) on $\Omega \times (0,T)$. Then for any $p \in [2,6]$, $1 < q < +\infty$, there exists a positive constant $C$ depending only on $p$, $q$, $\mu$, $\lambda$, $A$ and $\Omega$ such that
\[
\|\nabla u\|_{L^p} \leq C(\|\text{div} u\|_{L^q} + \|\omega\|_{L^q}),
\]
\[
\|\nabla H\|_{L^q} \leq C\|\text{curl} H\|_{L^q},
\]
\[
\|\nabla F\|_{L^p} \leq C(\|\rho \frac{\partial u}{\partial t}\|_{L^2} + \|H \cdot \nabla H\|_{L^2}),
\]
\[
\|\nabla \omega\|_{L^p} \leq C(\|\rho \frac{\partial u}{\partial t}\|_{L^2} + \|H \cdot \nabla H\|_{L^2} + \|\nabla u\|_{L^2}),
\]
\[
\|F\|_{L^p} \leq C(\|\rho \frac{\partial u}{\partial t}\|_{L^2} + \|H \cdot \nabla H\|_{L^2})^{(3p-6)/(2p)}(\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} + \|H\|_{L^4}^2)^{(6-p)/(2p)} \\
+ C(\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} + \|H\|_{L^4}^2),
\]
\[
\|\omega\|_{L^p} \leq C(\|\rho \frac{\partial u}{\partial t}\|_{L^2} + \|H \cdot \nabla H\|_{L^2})^{(3p-6)/(2p)}\|\nabla u\|_{L^2}^{(6-p)/(2p)} + C\|\nabla u\|_{L^2},
\]
Moreover,
\[
\|\nabla u\|_{L^p} \leq C(\|\rho \frac{\partial u}{\partial t}\|_{L^2} + \|H \cdot \nabla H\|_{L^2})^{(3p-6)/(2p)}(\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} + \|H\|_{L^4}^2)^{(6-p)/(2p)} \\
+ C(\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} + \|H\|_{L^4}^2).
\]

**Proof.** The inequality (2.6)-(2.7) is a direct result of Lemma A.4, since $u \cdot n = 0, H \cdot n = 0$ on $\partial \Omega$. Moreover, noticing that (1.1)$_3$ and $H \cdot n = 0, \text{curl} H \times n = 0$ on $\partial \Omega$, by Lemma A.4-A.5, for any integer $k \geq 1$, we obtain
\[
\|H\|_{W^{k+1,q}} \leq C\|\text{curl} H\|_{W^{k,q}} \leq C(\|\text{curl}^2 H\|_{W^{k-1,q}} + \|\text{curl} H\|_{L^p}),
\]
where $\text{curl}^2 H \triangleq \text{curl}\text{curl} H$ and we have used the fact $\text{div} \text{curl} H = 0$. 

By \((1.1)_2\), \((2.1)\) and the slip boundary condition \((1.3)\), one can find that the viscous flux \(F\) satisfies
\[
\begin{cases}
\Delta F = \text{div}(\rho \dot{u} - H \cdot \nabla H) \quad \text{in } \Omega, \\
\frac{\partial F}{\partial n} = (\rho \dot{u} - H \cdot \nabla H) \cdot n \quad \text{on } \partial \Omega.
\end{cases}
\]

It follows from Lemma 4.27 in [35] that
\[
\|\nabla F\|_{L^q} \leq C(\|\rho \dot{u}\|_{L^q} + \|H \cdot \nabla H\|_{L^q}),
\]
which gives \((2.8)\). Moreover, for any integer \(k \geq 0\),
\[
\|\nabla F\|_{W^{k+1,q}} \leq C(\|\rho \dot{u}\|_{L^q} + \|H \cdot \nabla H\|_{L^q} + \|\nabla(\rho \dot{u})\|_{W^{k,q}} + \|\nabla(H \cdot \nabla H)\|_{W^{k,q}}).
\]

On the other hand, one can rewrite \((1.1)_2\) as
\[
\mu \nabla \times \omega = \nabla F - \rho \dot{u} + H \cdot \nabla H.
\]
Noticing that \(\omega \times n = 0\) on \(\partial \Omega\) and \(\text{div} \, \omega = 0\), by Lemma A.5, we get
\[
\|\nabla \omega\|_{L^q} \leq C(\|\nabla \times \omega\|_{L^q} + \|\omega\|_{L^q}) \leq C(\|\rho \dot{u}\|_{L^q} + \|H \cdot \nabla H\|_{L^q} + \|\omega\|_{L^q}),
\]
and for any integer \(k \geq 0\),
\[
\begin{align*}
\|\nabla \omega\|_{W^{k+1,q}} & \leq C(\|\nabla \times \omega\|_{W^{k+1,q}} + \|\omega\|_{L^q}) \\
& \leq C(\|\rho \dot{u}\|_{L^q} + \|H \cdot \nabla H\|_{L^q} + \|\nabla(\rho \dot{u})\|_{W^{k,q}} + \|\nabla(H \cdot \nabla H)\|_{W^{k,q}} + \|\omega\|_{L^q}),
\end{align*}
\]
where we have taken advantage of \((2.15)\) and \((2.16)\). By Sobolev’s inequality and \((2.18)\), for \(p \in [2, 6]\),
\[
\|\nabla \omega\|_{L^p} \leq C(\|\rho \dot{u}\|_{L^p} + \|H \cdot \nabla H\|_{L^p} + \|\omega\|_{L^p}) \\
\leq C(\|\rho \dot{u}\|_{L^p} + \|H \cdot \nabla H\|_{L^p} + \|\rho \dot{u}\|_{L^2} + \|\nabla \omega\|_{L^p} + \|\omega\|_{L^2}) \\
\leq C(\|\rho \dot{u}\|_{L^p} + \|H \cdot \nabla H\|_{L^p} + \|\nabla u\|_{L^2}),
\]
which implies \((2.9)\).

Furthermore, one can deduce from \((A.1)\) and \((2.8)\) that for \(p \in [2, 6]\),
\[
\|F\|_{L^p} \leq C(\|F\|_{L^2}^{(6-p)/(2p)} \|\nabla F\|_{L^2}^{(3p-6)/(2p)} + C\|F\|_{L^2} \\
\leq C(\|\rho \dot{u}\|_{L^2} + \|H \cdot \nabla H\|_{L^2})^{(3p-6)/(2p)} \|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} + \|H\|_{L^4}^2)^{(6-p)/(2p)} \\
+ C(\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} + \|H\|_{L^4}^2),
\]
similarly, by \((A.1)\) and \((2.18)\),
\[
\|\omega\|_{L^p} \leq C(\|\omega\|_{L^2}^{(6-p)/(2p)} \|\nabla \omega\|_{L^2}^{(3p-6)/(2p)} + C\|\omega\|_{L^2} \\
\leq C(\|\rho \dot{u}\|_{L^2} + \|H \cdot \nabla H\|_{L^2} + \|\nabla u\|_{L^2})^{(3p-6)/(2p)} \|\nabla \omega\|_{L^2}^{(6-p)/(2p)} + C\|\nabla u\|_{L^2},
\]
and so \((2.10)-(2.11)\) are established. By virtue of \((2.6), (2.10), (2.11), (2.8)\) and \((2.18)\),
it implies that \((2.12)\) holds. This completes the proof. □
Remark 2.1 It is easy to check that there exists a positive constant \(C\) depending only on \(a, \gamma, \hat{\rho}, \bar{\rho}\) such that
\[
C^{-1}(\rho - \bar{\rho})^2 \leq G(\rho) \leq C(\rho - \bar{\rho})^2,
\]
(2.23)
which together with (2.4), (2.6) and (2.7) gives
\[
\sup_{0 \leq t \leq T} \|\rho - \bar{\rho}\|_{L^2}^2 + \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) dt \leq CC_0.
\]
(2.24)

Remark 2.2 From (2.13), we can get the estimate of \(\|\nabla^2 H\|_{L^p}\) and \(\|\nabla^3 H\|_{L^p}\) for \(p \in [2, 6]\),
\[
\|\nabla^2 H\|_{L^p} \leq C \|\text{curl} H\|_{W^{1,p}} \leq C (\|\text{curl}^2 H\|_{L^p} + \|\text{curl} H\|_{L^p}),
\]
(2.25)
and
\[
\|\nabla^3 H\|_{L^p} \leq C \|\text{curl} H\|_{W^{2,p}} \leq C (\|\text{curl}^2 H\|_{W^{1,p}} + \|\text{curl} H\|_{L^p}).
\]
(2.26)
On the other hand, we can get the estimates of \(\|\nabla^2 u\|_{L^p}\) and \(\|\nabla^3 u\|_{L^p}\) for \(p \in [2, 6]\) by Lemma A.4, which will be devoted to giving higher order estimates in Section 4. In fact, by Lemma A.4, (2.15) and (2.16), for \(p \in [2, 6]\),
\[
\|\nabla^2 u\|_{L^p} \leq C (\|\text{div} u\|_{W^{1,p}} + \|\omega\|_{W^{1,p}})
\leq C (\|\rho \hat{\nabla}\|_{L^p} + \|H \cdot \nabla H\|_{L^p} + \|\nabla P\|_{L^p} + \|P - \bar{P}\|_{L^p}
\leq \|H\|_{L^p} + \|\nabla H\cdot H\|_{L^p} + \|\nabla u\|_{L^2}),
\]
(2.27)
and
\[
\|\nabla^3 u\|_{L^p} \leq C (\|\text{div} u\|_{W^{2,p}} + \|\omega\|_{W^{2,p}})
\leq C (\|\nabla (\rho \hat{\nabla})\|_{L^p} + \|\nabla (H \cdot \nabla H)\|_{L^p} + \|\nabla^2 P\|_{L^p} + \|\nabla (\nabla H\cdot H)\|_{L^p} + \|H \cdot \nabla H\|_{L^p}
\leq \|\rho \hat{\nabla}\|_{L^p} + \|\nabla P\|_{L^p} + \|P - \bar{P}\|_{L^p} + \|H\|_{L^p} + \|\nabla H\cdot H\|_{L^p} + \|\nabla u\|_{L^2}).
\]
(2.28)
The lemma below gives an a priori estimate on the \(L^2(\Omega \times (0,T))\)-norm of \(\rho - \bar{\rho}\).

Lemma 2.3 Let \((\rho, u, H)\) be a smooth solution of (1.1)-(1.4) on \(\Omega \times (0, T)\). Then there exists a positive constant \(C\) depending only on \(p, q, \mu, \lambda, A\) and \(\Omega\) such that
\[
\int_0^T \int (\rho - \bar{\rho})^2 dx dt \leq CC_0.
\]
(2.29)
Proof. From Lemma A.7, multiplying (1.1) by \(\mathcal{B}[\rho - \bar{\rho}]\) and integrating over \(\Omega\), one has
\[
\int (P - \bar{P})(\rho - \bar{\rho}) dx
= \left(\int \rho u \cdot \mathcal{B}[\rho - \bar{\rho}] dx\right)_t - \int \rho u \cdot \nabla \mathcal{B}[\rho - \bar{\rho}] \cdot u dx - \int \rho u \cdot \mathcal{B}[\rho_1] dx
+ \mu \int \nabla u \cdot \nabla \mathcal{B}[\rho - \bar{\rho}] dx + (\lambda + \mu) \int (\rho - \bar{\rho}) \text{div} u dx
\]
- \int H \cdot \nabla B[\rho - \bar{\rho}] \cdot H \, dx - \int (\rho - \bar{\rho}) |H|^2 / 2 \, dx
\leq \left( \int \rho u \cdot B[\rho - \bar{\rho}] \, dx \right)_t + C \|\rho \|_{L^2}^2 \|\rho - \bar{\rho}\|_{L^2}^2 + C \|\rho u\|_{L^2}^2
+ C \|\rho - \bar{\rho}\|_{L^2} \|\nabla u\|_{L^2} + C \|\rho - \bar{\rho}\|_{L^2} \|H\|_{L^4}^2
\leq \left( \int \rho u \cdot B[\rho - \bar{\rho}] \, dx \right)_t + \delta \|\rho - \bar{\rho}\|_{L^2}^2 + C(\delta)(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2),
(2.30)
integrating it over \((0, T]\) together with (2.24) gives
\[\int_0^T \int (\rho - \bar{\rho})^2 \, dx \, dt \leq C \left( \sup_{0 \leq t \leq T} (\|\rho \|_{L^2}^2 + \|\rho - \bar{\rho}\|_{L^2}^2) + \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) \, dt \right) \leq CC_0, \quad (2.31)\]
and we finish the proof. \qed

Next we need the estimates on the material derivative of \(u\). Since \(u \cdot n = 0\) on \(\partial \Omega\), it follows that
\[u \cdot \nabla u \cdot n = -u \cdot \nabla n \cdot u, \quad (2.32)\]
which implies
\[(\dot{u} + (u \cdot \nabla n) \times u^\perp) \cdot n = 0 \text{ on } \partial \Omega, \quad (2.33)\]
where \(u^\perp \triangleq -u \times n\) on \(\partial \Omega\). We review the following Poincare-type inequality of \(\dot{u}\), which depends on this observation (see [6], Lemma 3.2).

**Lemma 2.4** If \((\rho, u, H)\) is a smooth solution of (1.1) with slip condition (1.3)-(1.4), then there exists a positive constant \(C\) depending only on \(\Omega\) such that
\begin{align*}
\|\dot{u}\|_{L^6} & \leq C(\|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^2}^2), \\
\|\nabla \dot{u}\|_{L^2} & \leq C(\|\text{div} \dot{u}\|_{L^2} + \|\text{curl} \dot{u}\|_{L^2} + \|\nabla u\|_{L^4}^2).
\end{align*}
(2.34)\hspace{1cm}(2.35)

To this end, we recall the following local existence theorem of classical solution of (1.1)-(1.4), which can be proved in a similar manner as that in [36, 40], base on the standard contraction mapping principle.

**Lemma 2.5** Assume that the initial date \((\rho_0, u_0, H_0)\) satisfy the conditions (1.9), (1.10) and (1.12). Then there exist a positive time \(T_0 > 0\) and a unique classical solution \((\rho, u, H)\) of the system (1.1)-(1.4) in \(\mathbb{R}^3 \times (0, T_0]\), satisfying that \(\rho \geq 0\), and that for \(\tau \in (0, T_0)\),
\begin{align*}
(\rho - \bar{\rho}, P - \bar{P}) & \in C([0, T_0); H^2 \cap W^{2,q}), \\
\nabla u & \in C([0, T_0); H^1) \cap L^\infty(\tau, T_0; H^2 \cap W^{2,q}), \\
u_t & \in L^\infty(\tau, T_0; D^1 \cap H^2) \cap H^1(\tau, T_0; H^1), \\
H & \in C([0, T_0); H^2) \cap L^\infty(\tau, T_0; H^4), \\
H_t & \in C([0, T_0); L^2) \cap H^1(\tau, T_0; H^1) \cap L^\infty(\tau, T_0; H^2).
\end{align*}
(2.36)
A priori estimates(I): lower order estimates

In this section, we will establish the time-independent a priori bounds of the solutions of the problem (1.1)-(1.4). Let \( T > 0 \) be a fixed time and \((\rho, u, H)\) be a smooth solution to (1.1)-(1.4) on \( \Omega \times (0, T] \) with smooth initial data \((\rho_0, u_0, H_0)\) satisfying \( u_0 \in H^s, H_0 \in H^s \) for some \( s \in (\frac{1}{2}, 1] \) and \( 0 \leq \rho_0 \leq 2\hat{\rho} \).

Set \( \sigma = \sigma(t) \triangleq \min\{1, t\} \), we define

\[
A_1(T) \triangleq \sup_{0 \leq t \leq T} \sigma \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) \\
+ \int_0^T \sigma \left( (\rho|\dot{u}|^2 + |\text{curl}^2 H|^2 + |H_t|^2) \right) dt,
\]

(3.1)

\[
A_2(T) \triangleq \sup_{0 \leq t \leq T} \sigma^2 \int_\Omega \left( (\rho|\dot{u}|^2 + |\text{curl}^2 H|^2 + |H_t|^2) \right) dx
\]

\[
+ \int_0^T \sigma^2 \left( \|\nabla \dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 \right) dt,
\]

(3.2)

\[
A_3(T) \triangleq \sup_{0 \leq t \leq T} \int_\Omega |H|^3 dx,
\]

(3.3)

\[
A_4(T) \triangleq \sup_{0 \leq t \leq T} \sigma^{\frac{3+2s}{2}} \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right)
\]

\[
+ \int_0^T \sigma^{\frac{3+2s}{2}} \left( (\rho|\dot{u}|^2 + |\text{curl}^2 H|^2 + |H_t|^2) \right) dt,
\]

(3.4)

\[
A_5(T) \triangleq \sup_{0 \leq t \leq T} \int_\Omega |\rho|^3 dx,
\]

(3.5)

where \( s \in (\frac{1}{2}, 1] \) and \( \dot{v} = v_t + u \cdot \nabla v \) is the material derivative.

Now we will give the following key a priori estimates in this section, which guarantees the existence of a global classical solution of (1.1)-(1.4).

**Proposition 3.1** Under the conditions of Theorem 1.1, for \( \delta_0 \triangleq \frac{2s+1}{4s} \in (0, \frac{1}{2}] \), there exists a positive constant \( \varepsilon \) depending on \( \mu, \lambda, \nu, a, \gamma, \tilde{\rho}, \tilde{\rho}, s, \Omega, M_1 \) and \( M_2 \) such that if \((\rho, u, H)\) is a smooth solution of (1.1)-(1.4) on \( \Omega \times (0, T] \) satisfying

\[
\begin{aligned}
\sup_{\Omega \times [0, T]} \rho &\leq 2\hat{\rho}, \quad A_1(T) + A_2(T) \leq 2C_0^{1/2},
\end{aligned}
\]

\[
\begin{aligned}
A_3(T) &\leq 2C_0^{\delta_0}, \quad A_4(\sigma(T)) + A_5(\sigma(T)) \leq 2C_0^{\delta_0},
\end{aligned}
\]

(3.6)

then the following estimates hold

\[
\begin{aligned}
\sup_{\Omega \times [0, T]} \rho &\leq 7\tilde{\rho}/4, \quad A_1(T) + A_2(T) \leq C_0^{1/2},
\end{aligned}
\]

\[
\begin{aligned}
A_3(T) &\leq C_0^{\delta_0}, \quad A_4(\sigma(T)) + A_5(\sigma(T)) \leq C_0^{\delta_0},
\end{aligned}
\]

(3.7)

provided \( C_0 \leq \varepsilon \).

**Proof.** Proposition 3.1 is a consequence of the following Lemmas 3.3, 3.7-3.9 below. \(\square\)

In the following, we will use the convention that \( C \) denotes a generic positive constant depending on \( \mu, \lambda, \nu, \gamma, a, \tilde{\rho}, \tilde{\rho}, s, \Omega, M_1 \) and \( M_2 \) and use \( C(\alpha) \) to emphasize that \( C \) depends on \( \alpha \). We begin with the following standard energy estimate for \((\nabla u, \nabla H)\).
Lemma 3.2  Let \((\rho, u, H)\) be a smooth solution of (1.1)-(1.4) satisfying (3.6). Then there is a positive constant \(C\) such that
\[
\int_0^T (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4)dt \leq C C_0^{2\delta_0},
\]
provided \(C_0 \leq 1\).

Proof. By (3.6), together with (2.4), (2.24), we have
\[
\int_0^T (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4)dt 
\leq C \int_0^{\sigma(T)} (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4)dt + \int_{\sigma(T)}^T (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4)dt 
\leq C \sup_{0 \leq t \leq \sigma(T)} \left( \sigma^{\frac{3}{2}-\frac{3}{4}} (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) \right)^2 \int_0^{\sigma(T)} \sigma^{\frac{2n-3}{2}} dt 
+ C \sup_{\sigma(T) \leq t \leq T} \sigma (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) \int_{\sigma(T)}^T (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2)dt 
\leq C C_0^{2\delta_0} + CC_0^{3/2} \leq C C_0^{2\delta_0},
\]
(3.9)
since \(s \in (1/2, 1]\) and \(\delta_0 \in (0, 1/9]\). The proof of Lemma 2.1 is completed. \(\square\)

Now, we give the estimate of \(A_3(T)\).

Lemma 3.3  Let \((\rho, u, H)\) be a smooth solution of (1.1)-(1.4) satisfying (3.6). Then there is a positive constant \(\varepsilon_1 > 0\), depending on \(\mu, \lambda, \nu, \alpha, \gamma, \bar{\rho}, \bar{\rho}\) and \(\Omega\) such that
\[
A_3(T) \leq C_0^{\delta_0},
\]
(3.10)
provided \(C_0 \leq \varepsilon_1\).

Proof. Multiplying (2.2) by \(3|H|H\) and integrating by parts over \(\Omega\), we have
\[
\frac{d}{dt} \|H\|_{L^3}^3 = -3\nu \int \text{curl} H \cdot \text{curl}(|H| H) dx + 3 \int |H| H \cdot \nabla u \cdot H dx - 2 \int |H|^3 \text{div} u dx 
\leq C \|H\|_{L^n} \|\nabla H\|_{L^2}^2 + C \|\nabla u\|_{L^2} \|H\|_{L^6}^3 
\leq C \|\nabla H\|_{L^2}^{5/2} \|\text{curl} H\|_{L^2}^{1/2} + C \|\nabla H\|_{L^2}^{2} + C \|\nabla H\|_{L^2}^{4/2} + C \|\nabla u\|_{L^2}^4,
\]
(3.11)
which together with (3.6) and (3.8) indicates that
\[
\sup_{0 \leq t \leq T} \|H\|_{L^3}^3 
\leq \|H_0\|_{L^3}^3 + C \int_0^T \|\nabla H\|_{L^2}^{5/2} \|\text{curl} H\|_{L^2}^{1/2} dt + CC_0 + CC_0^{2\delta_0} 
\leq \|H_0\|_{L^3}^3 + C \int_0^{\sigma(T)} \left( \sigma^{\frac{3}{2}-\frac{3}{4}} \|\nabla H\|_{L^2}^2 \right)^{5/4} \left( \sigma^{\frac{3}{2}-\frac{3}{4}} \|\text{curl} H\|_{L^2}^2 \right)^{1/4} \left( \sigma^{\frac{4(2s-1)}{s}} \right) dt 
+ C \|\nabla H\|_{L^2} \int_{\sigma(T)}^T (\|\nabla H\|_{L^2}^2)^{3/4} (\sigma \|\text{curl} H\|_{L^2}^2)^{1/4} dt + CC_0 + CC_0^{2\delta_0} 
\leq C_1 C_0^{3\delta_0/2},
\]
(3.12)
where in the last inequality we have used the simple fact
\[
\|H_0\|_{L^2}^3 \leq C \|H_0\|_{L^2}^{3(2s-1)} \|\frac{\partial}{\partial t} H_0\|_{L^2}^2 \leq C(M_2) C_0^{-2/9}.
\] (3.13)

Thus it follows from (3.13) that (3.10) holds provided that \(C_0 \leq \varepsilon_1 \triangleq \min\{1, C_1^{-2/9}\}\). The proof of 3.3 is completed.

The following lemma shows the preliminary \(L^2\) bounds for \(\nabla H\).

**Lemma 3.4** Let \((\rho, u, H)\) be a smooth solution of (1.1)–(1.4) satisfying (3.6). Then there is a positive constant \(C\) such that
\[
\sup_{0 \leq t \leq T} (\sigma \|\nabla H\|_{L^2}^2) + \int_0^T \sigma (\|\text{curl}^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2) \, dt \leq C C_0.
\] (3.14)

Moreover, for any \(\theta \in [0, 1]\), one has
\[
\sup_{0 \leq t \leq T} (\sigma^{-1-\theta} \|\nabla H\|_{L^2}^2) + \int_0^T \sigma^{-1-\theta} (\|\text{curl}^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2) \, dt \leq C \|H_0\|_{H^0}^2.
\] (3.15)

**Proof.** Multiplying (2.2) by \(H\) and integrating by parts over \(\Omega\), by (1.4), (A.1) and (2.7), we have
\[
\left(\frac{1}{2} \|H\|_{L^2}^2\right)_t + \nu \|\text{curl} H\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 \|H\|_{L^4}^2 \leq \frac{\nu}{2} \|\text{curl} H\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|H\|_{L^2}^2,
\] (3.16)

which together with (2.7), (3.8) and Gronwall inequality gives
\[
\sup_{0 \leq t \leq T} \|H\|_{L^2}^2 + \int_0^T \|\nabla H\|_{L^2}^2 \, dt \leq C \|H_0\|_{L^2}^2.
\] (3.17)

By Lemma 2.2, one easily deduces from (2.2) and (1.4) that
\[
\left(\frac{\nu}{2} \|\text{curl} H\|_{L^2}^2\right)_t + \nu^2 \|\text{curl}^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2 \leq \int |H \cdot \nabla u - u \cdot \nabla H - H \text{div} u|^2 \, dx \leq C \|\nabla u\|_{L^2}^2 \|H\|_{L^4}^2 + C \|u\|_{L^6}^2 \|\nabla H\|_{L^2}^2 \leq C \|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 + C \|H\|_{L^2}^2 \|\nabla H\|_{L^2}^2 \leq \frac{\nu^2}{2} \|\text{curl}^2 H\|_{L^2}^2 + C (\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4) \|\nabla H\|_{L^2}^2,
\] (3.18)

using (2.7), (3.8) and Gronwall inequality, we get
\[
\sup_{0 \leq t \leq T} \|\nabla H\|_{L^2}^2 + \int_0^T (\|\text{curl}^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2) \, dt \leq C \|\nabla H_0\|_{L^2}^2.
\] (3.19)

On the other hand, multiplying (3.18) by \(\sigma\) and integrating it over \((0, T)\), by (3.8) and (3.17), we obtain
\[
\sup_{0 \leq t \leq T} (\sigma \|\nabla H\|_{L^2}^2) + \int_0^T \sigma (\|\text{curl}^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2) \, dt \leq C \|H_0\|_{L^2}^2 \leq C C_0.
\] (3.20)
This finishes the proof of (3.14). Note that for fixed \( u \) (smooth), the solution operator \( H_0 \mapsto H(\cdot, t) \) is linear, by the standard Stein-Weiss interpolation argument [5], one can deduce from (3.19) and (3.20) that (3.15) holds for any \( \theta \in [0, 1] \). The proof of Lemma 3.4 is completed.

Then, we give the estimate of \( A_1(T) \) and \( A_2(T) \).

**Lemma 3.5** Let \((\rho, u, H)\) be a smooth solution of (1.1)-(1.4) satisfying (3.6). Then there is a positive constant \( \varepsilon_2 \) depending only on \( \mu, \lambda, \nu, a, \gamma, \tilde{\rho}, \Omega \) and \( A \) such that

\[
A_1(T) \leq C C_0 + C \int_0^T \int \sigma |\nabla u|^3 dx dt, \tag{3.21}
\]

\[
A_2(T) \leq C C_0^{1/2 + 2b_0/3} + C A_1(T) + C \int_0^T \int \sigma^2 |\nabla u|^4 dt, \tag{3.22}
\]

provided \( C_0 \leq \varepsilon_2 \).

**Proof.** Let \( m \geq 0 \) be a real number which will be determined later. Multiplying (1.1) by \( \sigma^m \dot{u} \) and then integrating the resulting equality over \( \Omega \) lead to

\[
\int \sigma^m \rho |\dot{u}|^2 dx = - \int \sigma^m \dot{u} \cdot \nabla P dx + (\lambda + 2\mu) \int \sigma^m \nabla \text{div} u \cdot \dot{u} dx
- \mu \int \sigma^m \nabla \times \omega \cdot \dot{u} dx + \int \sigma^m (H \nabla H - \nabla |H|^2/2) \cdot \dot{u} dx
\triangleq I_1 + I_2 + I_3 + I_4. \tag{3.23}
\]

We neglect the calculation details similar to that in [6] and only show the part related to the magnetic field and the boundary terms. Firstly, By (1.1) and Lemma 2.2, a direct calculation gives

\[
I_1 = - \int \sigma^m u_t \cdot \nabla (P - \tilde{P}) dx - \int \sigma^m u \cdot \nabla u \cdot \nabla P dx
= \left( \int \sigma^m (P - \tilde{P}) \text{div} u dx \right)_t - m \sigma^{m-1} \sigma' \int (P - \tilde{P}) \text{div} u dx
+ \int \sigma^m P \nabla u : \nabla u dx + (\gamma - 1) \int \sigma^m P (\text{div} u)^2 dx - \int_{\partial \Omega} \sigma^m P u \cdot \nabla u \cdot n ds
\leq \left( \int \sigma^m (P - \tilde{P}) \text{div} u dx \right)_t + C \|\nabla u\|_{L^2}^2 + C \|\rho - \tilde{\rho}\|_{L^2}^2. \tag{3.24}
\]

Similarly, by (2.32), it indicates that

\[
I_2 = (\lambda + 2\mu) \int_{\partial \Omega} \sigma^m \text{div} u (\dot{u} \cdot n) ds - (\lambda + 2\mu) \int \sigma^m \text{div} u \text{div} \dot{u} dx
= (\lambda + 2\mu) \int_{\partial \Omega} \sigma^m \text{div} u (u \cdot \nabla u \cdot n) ds - \frac{\lambda + 2\mu}{2} \left( \int \sigma^m (\text{div} u)^2 dx \right)_t
+ \frac{\lambda + 2\mu}{2} \int \sigma^m (\text{div} u)^3 dx - (\lambda + 2\mu) \int \sigma^m \text{div} u \cdot \nabla u dx
+ \frac{m(\lambda + 2\mu)}{2} \sigma^{m-1} \sigma' \int (\text{div} u)^2 dx. \tag{3.25}
\]
For the first term on the righthand side of (3.25), we obtain

\[ (\lambda + 2\mu) \int_{\Omega} \sigma^m \text{div} (u \cdot \nabla u \cdot n) \, ds \]

\[ = - \int_{\Omega} \sigma^m F u \cdot \nabla n \cdot u \, ds - \int_{\Omega} \sigma^m (P - \bar{P}) u \cdot \nabla n \cdot u \, ds - \int_{\Omega} \sigma^m \frac{|H|^2}{2} u \cdot \nabla n \cdot u \, ds \]

\[ \leq C \left( \int_{\Omega} \sigma^m |F||u|^2 \, ds + \int_{\Omega} \sigma^m |u|^2 \, ds + \int_{\Omega} \sigma^m |H|^2 |u|^2 \, ds \right) \]

\[ \leq C \sigma^m (||\nabla F||_{L^2} ||u||_{L^2}^2 + ||F||_{L^6} ||u||_{L^6} ||\nabla u||_{L^2} + ||F||_{L^2} ||u||_{L^4}^2 ) + C \sigma^m ||\nabla u||_{L^2}^2 \]

\[ + C \sigma^m (||\nabla H||_{L^2} ||H||_{L^6} ||u||_{L^6}^2 + ||H||_{L^6} ||u||_{L^6} ||\nabla u||_{L^2}^2 + ||H||_{L^4} ||u||_{L^4}^2 ) \]

\[ \leq \frac{1}{2} \sigma^m ||\rho u||_{L^2}^2 + C \sigma^m ||\text{curl}^2 H||_{L^2}^2 + C \sigma^m (||\nabla u||_{L^2}^2 + ||\nabla H||_{L^2}^2) (||\nabla u||_{L^2}^2 + 1), \quad (3.26) \]

where we have used

\[ ||H \cdot \nabla H||_{L^2} \leq C ||H||_{L^4} ||\nabla H||_{L^6} \leq C C_0^{\delta_0/3} (||\text{curl}^2 H||_{L^2} + ||\nabla H||_{L^2}). \quad (3.27) \]

Therefore,

\[ I_2 \leq - \frac{\lambda + 2\mu}{2} \left( \int \sigma^m (\text{div} u)^2 \, dx \right)_t + C \sigma^m ||\nabla u||_{L^2}^2 + \frac{1}{2} \sigma^m ||\rho u||_{L^2}^2 + C \sigma^m ||\text{curl}^2 H||_{L^2}^2 \]

\[ + C \sigma^m (||\nabla u||_{L^2}^2 + ||\nabla H||_{L^2}^2) ||\nabla u||_{L^2}^2 + C (||\nabla u||_{L^2}^2 + ||\nabla H||_{L^2}^2). \quad (3.28) \]

Next, by (1.3), a straightforward computation shows that

\[ I_3 = - \frac{\mu}{2} \left( \int \sigma^m |\omega|^2 \, dx \right)_t + \frac{\mu m}{2} \sigma^m - 1 \sigma^t \int |\omega|^2 \, dx \]

\[ - \mu \int \sigma^m (\nabla u^i \times \nabla u^j) \cdot \omega \, dx + \frac{\mu}{2} \int \sigma^m |\omega|^2 \, \text{div} u \, dx \]

\[ \leq - \frac{\mu}{2} \left( \int \sigma^m |\omega|^2 \, dx \right)_t + C m \sigma^m - 1 \sigma^t ||\nabla u||_{L^2}^2 + C \sigma^m ||\nabla u||_{L^2}^3. \quad (3.29) \]

Finally, by (1.3), a direct calculation yields

\[ I_4 = \left( \int \sigma^m (H \cdot \nabla H - \nabla |H|^2/2) \cdot u \, dx \right)_t - m \sigma^m - 1 \sigma^t \int (H \cdot \nabla H - \nabla |H|^2/2) \cdot u \, dx \]

\[ + \int \sigma^m ((H \otimes H) \cdot \nabla u - (\nabla |H|^2/2) \text{div} u) \, dx + \int \sigma^m (H \cdot \nabla H - \nabla |H|^2/2) \cdot u \cdot \nabla u \, dx \]

\[ \leq \left( \int \sigma^m (H \cdot \nabla H - \nabla |H|^2/2) \cdot u \, dx \right)_t + C (||\nabla H||_{L^2}^2 + ||\nabla u||_{L^2}^2) \]

\[ + C \sigma^m (||H||_{L^2}^2 + ||\text{curl}^2 H||_{L^2}^2) + C \sigma^m ||\nabla u||_{L^2}^3 + C \sigma^m ||\nabla H||_{L^2}^2 ||\nabla u||_{L^2}^2 \]

\[ + C \sigma^m ||\nabla H||_{L^2}^2 (||\nabla H||_{L^2} + ||\nabla u||_{L^2}^4), \quad (3.30) \]

Making use of the results (3.24), (3.28), (3.29) and (3.30), it follows from (3.23) that

\[ \left( \int \sigma^m (P - \bar{P}) \, \text{div} u \, dx \right)_t + \left( \int \sigma^m (H \cdot \nabla H - \nabla |H|^2/2) \cdot u \, dx \right)_t \]

\[ \leq \left( \int \sigma^m (P - \bar{P}) \, \text{div} u \, dx \right)_t + \left( \int \sigma^m (H \cdot \nabla H - \nabla |H|^2/2) \cdot u \, dx \right)_t \]
Let us estimate where in the third equality we have used \( j = 1 \) together with (3.33), we get

\[
\sigma^m \|
abla u\|^2_{L^2} + \rho \|\dot{u}\|^2_{L^2} dt
\]

\[
\leq C_0 + C \int_0^T \sigma^m (\|H_t\|_{L^2}^2 + \|\nabla u\|^2_{L^2}) dt
\]

\[
+ C \int_0^T \sigma^m (\|\nabla H\|^2_{L^2} + \|\nabla u\|^2_{L^2})(\|\nabla H\|^2_{L^2} + \|\nabla u\|^4_{L^2}) dt
\]

\[
+ C \int_0^T \sigma^m \|
abla u\|^2_{L^2} + \|\nabla H\|^2_{L^2} + \|\nabla u\|^2_{L^2} + 1, \quad (3.31)
\]

integrating over \((0, T)\), by (2.6), (2.29), Lemma 2.1 and Young’s inequality, we conclude that for any \( m > 0 \),

\[
\sigma^m \|
abla u\|^2_{L^2} + \rho \|\dot{u}\|^2_{L^2} dt
\]

\[
\leq C_0 + C \int_0^T \sigma^m (\|H_t\|_{L^2}^2 + \|\nabla u\|^2_{L^2}) dt
\]

\[
+ C \int_0^T \sigma^m \|
abla u\|^2_{L^2} + \|\nabla H\|^2_{L^2} + \|\nabla u\|^2_{L^2} + 1, \quad (3.32)
\]

Choose \( m = 1 \), together with (3.6), (3.8) and (3.14), we obtain (3.21).

Now we will claim (3.22). Operating \( \sigma^m \dot{u}^j [\partial_j \partial_t + \text{div}(u \cdot \cdot)] \) to (2)\(^j\), summing with respect to \( j \), and integrating over \( \Omega \), together with (1.1)\(_1\), we get

\[
\left( \frac{\sigma^m}{2} \int \rho \|\dot{u}\|^2 dx \right)_t - \frac{m}{2} \sigma^m \dot{\sigma} \int \rho \|\dot{u}\|^2 dx
\]

\[
= \int \sigma^m (\dot{u} \cdot \nabla F_t) + \dot{u} ^j \text{div}(u \partial_j F)) dx
\]

\[
+ \mu \int \sigma^m (-\dot{u} \cdot \nabla \times \omega t - \dot{u} ^j \text{div}((\nabla \times \omega)^j u)) dx
\]

\[
+ \int \sigma^m (\dot{u} \cdot (\text{div}(H \otimes H))_t + \dot{u} ^j \text{div}((\text{div}(H \otimes H^j) u)) dx
\]

\[
\triangleq J_1 + J_2 + J_3, \quad (3.33)
\]

Let us estimate \( J_1, J_2 \) and \( J_3 \). By (1.3) and (2.2)\(_1\), a direct computation yields

\[
J_1 = \int_{\partial \Omega} \sigma^m F_t \dot{u} \cdot n ds - \int \sigma^m F_t \text{div} \dot{u} dx + \int \sigma^m u \cdot \nabla \dot{u} \partial_j F dx
\]

\[
= \int_{\partial \Omega} \sigma^m F_t \dot{u} \cdot n ds - (\lambda + 2\mu) \int \sigma^m (\text{div} \dot{u})^2 dx + (\lambda + 2\mu) \int \sigma^m \text{div} \dot{u} \nabla u : \nabla u dx
\]

\[
- \gamma \int \sigma^m P \text{div} \dot{u} dx + \int \sigma^m \text{div} \dot{u} u \cdot \nabla F dx - \int \sigma^m u \cdot \nabla \dot{u} \partial_j F dx
\]

\[
+ \int \sigma^m \text{div} \dot{u} H \cdot H_t dx + \int \sigma^m \text{div} \dot{u} u \cdot \nabla H \cdot H dx
\]

\[
\leq \int_{\partial \Omega} \sigma^m F_t \dot{u} \cdot n ds - (\lambda + 2\mu) \int \sigma^m (\text{div} \dot{u})^2 dx + \frac{\delta}{12} \sigma^m \|\dot{u}\|^2_{L^2} + C \sigma^m \|\nabla u\|^4_{L^4}
\]

\[
+ C \sigma^m (\|\nabla u\|^2_{L^2} \|\nabla F\|^2_{L^2} + C_{\delta} \|\nabla H_t\|^2_{L^2} + \|\nabla u\|^2_{L^2} \|\nabla H\|^2_{L^2} \|\nabla H\|^2_{L^2}) \quad (3.34)
\]

where in the third equality we have used

\[
F_t = (2\mu + \lambda) \text{div} \dot{u} - (2\mu + \lambda) \nabla u : \nabla u - u \cdot \nabla F + \gamma P \text{div} u - H \cdot H_t.
\]
For the first term on the righthand side of (3.34), we have

\[
\int_{\partial \Omega} \sigma^m F_t \hat{u} \cdot nds = - \int_{\partial \Omega} \sigma^m F_t (u \cdot \nabla n \cdot u) ds
\]

\[
= - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t + m \sigma^{m-1} \sigma' \int_{\partial \Omega} (u \cdot \nabla n \cdot u) F ds
\]

\[
+ \int_{\partial \Omega} \sigma^m (F \cdot \nabla n \cdot u + Fu \cdot \nabla n \cdot \hat{u}) ds
\]

\[
- \int_{\partial \Omega} \sigma^m (F(\nabla u \cdot \nabla n \cdot u + Fu \cdot \nabla n \cdot (u \cdot \nabla)u) ds
\]

\[
\leq - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t + Cm \sigma^{m-1} \sigma' \|\nabla u\|_{L^2}^2 \|F\|_{H^1}
\]

\[
+ \frac{\delta}{12} \sigma^m \|\nabla \hat{u}\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^2}^4 + C \sigma^m \|\nabla u\|_{L^2}^2 \|F\|_{H^1}
\]

\[
+ C \|\nabla F\|_{L^6} \|\nabla u\|_{L^2}^2 + C \|F\|_{H^1} \|\nabla u\|_{L^2} \left( \|\nabla u\|_{L^4}^2 + \|\nabla u\|_{L^2}^2 \right).
\]

Together with Lemma 2.2, (2.9), (2.34), (2.35), (3.4), (3.34) and (3.35), we have

\[
J_1 \leq -(\lambda + 2\mu) \int \sigma^m (\text{div}\hat{u})^2 dx - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds \right)_t
\]

\[
+ \frac{\delta}{3} \sigma^m \|\nabla \hat{u}\|_{L^2}^2 + C \sigma^m C_0^{2k_0/3} \|\nabla H_t\|_{L^2}^2 + C \sigma^m \|\nabla H\|_{L^2}^4 + \|\nabla u\|_{L^2}^4 \|\text{curl}^2 H\|_{L^2}^2
\]

\[
+ C \sigma^m \|\nabla u\|_{L^2}^4 + C \sigma^m (\|\frac{\partial}{\partial t} \hat{u}\|_{L^2}^2 + \|\text{curl}^2 H\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2)
\]

\[
+ Cm \sigma^{m-1} \sigma' (\|\frac{\partial}{\partial t} \hat{u}\|_{L^2}^2 + \|\text{curl}^2 H\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2)
\]

\[
+ Cm \sigma^{m-1} \sigma' (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) \|\nabla u\|_{L^2}^2.
\]

Next, by \(\omega_t = \text{curl} \hat{u} - u \cdot \nabla \omega - \nabla u \times \partial_i u\) and a straightforward calculation leads to

\[
J_2 = -\mu \int \sigma^m |\text{curl}\hat{u}|^2 dx + \mu \int \sigma^m \text{curl}\hat{u} \cdot (\nabla u^i \times \partial_i u) dx
\]

\[
+ \mu \int \sigma^m u \cdot \nabla \omega \cdot \text{curl}\hat{u} dx + \mu \int \sigma^m u \cdot \nabla \cdot (\omega \times n) dx
\]

\[
\leq -\mu \int \sigma^m |\text{curl}\hat{u}|^2 dx + \frac{\delta}{3} \sigma^m \|\nabla \hat{u}\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^4}^4
\]

Finally, a directly computation shows that

\[
J_3 = - \int \sigma^m \nabla \hat{u} : (H \otimes H) dx - \mu \int \sigma^m H \cdot \nabla H^2 u \cdot \nabla \hat{u} dx
\]

\[
\leq C \sigma^m (\|\nabla \hat{u}\|_{L^2}^2 \|H\|_{L^4} \|H_t\|_{L^5} + \|\nabla \hat{u}\|_{L^2}^2 \|H\|_{L^6} \|\nabla H\|_{L^5} \|u\|_{L^6})
\]

\[
\leq \frac{\delta}{3} \sigma^m \|\nabla \hat{u}\|_{L^2}^2 + C \sigma^m (\|\nabla H\|_{L^2}^4 + \|\nabla u\|_{L^2}^4 \|\text{curl}^2 H\|_{L^2}^2)
\]

\[
+ C \sigma^m C_0^{2k_0/3} \|\nabla H_t\|_{L^2}^2 + C \sigma^m \|\nabla H\|_{L^2}^4 \|\nabla u\|_{L^2}^2.
\]

Combining (3.36), (3.37) with (3.38), we deduce from (3.33) that

\[
\left( \frac{\sigma^m}{2} \|\frac{\partial}{\partial t} \hat{u}\|_{L^2}^2 \right)_t + (\lambda + 2\mu) \sigma^m \|\text{div}\hat{u}\|_{L^2}^2 + \mu \sigma^m \|\text{curl}\hat{u}\|_{L^2}^2
\]
\begin{align*}
&\leq -\left(\int_{\partial\Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds\right) t + \delta \sigma^m \|\nabla \dot{u}\|^2_{L^2} + C \sigma^m \|H_t\|^2_{L^2} \\
&+ C \sigma^m \|\nabla u\|^4_{L^2} + C \sigma^m \|\nabla H\|^2_{L^2} + C \sigma^m \|\nabla u\|^2_{L^2} \|\nabla u\|^2_{L^2} \\
&+ C \sigma^m \|\nabla u\|^2_{L^2} \left(1 + \|\nabla u\|^2_{L^2} + \|\nabla H\|^2_{L^2} + \|\nabla \sigma\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) \\
&+ C m \sigma^m \|\nabla u\|^2_{L^2} + ||\nabla H_{\sigma}||_{L^2}^2 \|\nabla u\|^2_{L^2}. \quad (3.39)
\end{align*}

By (2.35) and Lemma 2.1, choosing \(\delta\) small enough, and integrating (3.39) over \((0, T)\), for \(m > 0\), we get

\begin{align*}
&\sigma^m \|ho \frac{1}{\tau} \dot{u}\|^2_{L^2} + \int_0^T \sigma^m \|\nabla \dot{u}\|^2_{L^2} dt \\
&\leq -\int_{\partial\Omega} \sigma^m (u \cdot \nabla n \cdot u) F ds + C C_0^{2\delta_0/3} \int_0^T \sigma^m \|\nabla H_t\|^2_{L^2} dt \\
&+ C \int_0^T \sigma^m \|\nabla u\|^4_{L^2} dt + C C_0^{2\delta_0} \sup_{0 \leq t \leq T} \sigma^m \|\nabla H\|^2_{L^2} + \|\rho \frac{1}{\tau} \dot{u}\|^2_{L^2} \\
&+ C C_0^{2\delta_0} \sup_{0 \leq t \leq T} \sigma^m \|\nabla u\|^2_{L^2} + \|\nabla H\|^2_{L^2} + \|\nabla u\|^2_{L^2} + C_0 \sup_{0 \leq t \leq T} \sigma^m \|\nabla u\|^2_{L^2} \\
&+ C \int_{\sigma(T)} \sup_{0 \leq t \leq T} \sigma^m \|\nabla u\|^2_{L^2} + \|\nabla H\|^2_{L^2} dt + C C_0. \quad (3.40)
\end{align*}

For the boundary term in the right-hand side of (3.40), from Lemma 2.2, we have

\begin{align*}
&\int_{\partial\Omega} (u \cdot \nabla n \cdot u) F ds \leq C \|\nabla u\|^2_{L^2} \|F\|_{H^1} \\
&\leq \frac{1}{2} \|ho \frac{1}{\tau} \dot{u}\|^2_{L^2} + C C_0^{2\delta_0/3} \|\nabla H\|^2_{L^2} + C (\|\nabla u\|^2_{L^2} + \|\nabla H\|^2_{L^2} + \|\nabla u\|^2_{L^2}). \quad (3.41)
\end{align*}

Therefore,

\begin{align*}
&\sigma^m \|ho \frac{1}{\tau} \dot{u}\|^2_{L^2} + \int_0^T \sigma^m \|\nabla \dot{u}\|^2_{L^2} dt - C C_0^{2\delta_0/3} \int_0^T \sigma^m \|\nabla H_t\|^2_{L^2} dt \\
&\leq C \int_0^T \sigma^m \|\nabla u\|^4_{L^2} dt + C C_0^{2\delta_0} \sup_{0 \leq t \leq T} \sigma^m \|\nabla H\|^2_{L^2} + \|\rho \frac{1}{\tau} \dot{u}\|^2_{L^2} \\
&+ C C_0^{2\delta_0} \sup_{0 \leq t \leq T} \sigma^m \|\nabla u\|^2_{L^2} + \|\nabla H\|^2_{L^2} + \|\nabla u\|^2_{L^2} + C_0 \sup_{0 \leq t \leq T} \sigma^m \|\nabla u\|^2_{L^2} \\
&+ C \int_{\sigma(T)} \sup_{0 \leq t \leq T} \sigma^m \|\nabla u\|^2_{L^2} + \|\nabla H\|^2_{L^2} dt + C C_0 \\
&+ C C_0^{2\delta_0/3} \sigma^m \|\nabla H\|^2_{L^2} + C \sigma^m \|\nabla u\|^2_{L^2} + \|\nabla H\|^2_{L^2} + \|\nabla u\|^2_{L^2}. \quad (3.42)
\end{align*}

Next, we need to estimate the term \(\|\nabla H_t\|_{L^2}\). Noticing that

\begin{align*}
\begin{cases}
H_{tt} - \nu \nabla \times (\nabla H) = (H \cdot \nabla u - u \cdot \nabla H - H \text{div} u), & \text{in } \Omega, \\
H_t \cdot n = 0, & \text{curl} H \times n = 0, \quad \text{on } \partial\Omega,
\end{cases} \quad (3.43)
\end{align*}

and after directly computations we obtain

\begin{align*}
&\left(\frac{\sigma^m}{2} \|H_t\|^2_{L^2}\right) t + \sigma^m \|\nabla H_t\|^2_{L^2} - \frac{m}{2} \sigma^m \|H_t\|^2_{L^2} \\
&+ \sigma^m \|\nabla H_t\|^2_{L^2} - \frac{m}{2} \sigma^m \|H_t\|^2_{L^2}.
\end{align*}
By Sobolev trace theorem and Lemma A.1, we have

\[ K_1 \leq C_\sigma \|H_t\|_{L^3} \|H_t\|_{L^6} \|\nabla u\|_{L^2} + \|u\|_{L^6} \|H_t\|_{L^3} \|\nabla H_t\|_{L^2} \]

\[ \leq \frac{\delta}{4} \sigma^m \|\nabla H_t\|_{L^2}^2 + C_\sigma \|\nabla u\|_{L^2}^4 \|H_t\|_{L^2}^2. \]  

(3.45)

Similarly,

\[ K_2 \leq C_\sigma \|H\|_{L^3} \|H_t\|_{L^6} \|\nabla \dot{u}\|_{L^2} - \int_{\partial \Omega} \sigma^m (\dot{u} \cdot n)(H \cdot H_t) ds 
+ \int \sigma^m \text{div} \dot{u} H \cdot H_t dx + \int \sigma^m \dot{u} \cdot \nabla H_t \cdot H dx \]

\[ \leq \int_{\partial \Omega} \sigma^m(u \cdot \nabla n \cdot u)(H \cdot H_t) ds + CC^{\sigma_0/3} \sigma^m (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla H_t\|_{L^2}^2), \]  

(3.46)

where in the last inequality we have used the fact that \( \dot{u} \cdot n = -u \cdot \nabla n \cdot u \) on \( \partial \Omega \). By Sobolev trace theorem and Lemma 2.2, it indicates that

\[ \int_{\partial \Omega} \sigma^m(u \cdot \nabla n \cdot u)(H \cdot H_t) ds \]

\[ \leq C_\sigma \sigma^m(\|u\|_{L^6} \|\nabla u\|_{L^2} \|H\|_{L^6} \|H_t\|_{L^6} + \|u\|_{L^6}^2 \|\nabla H_t\|_{L^2} \|H_t\|_{L^6} 
+ \|u\|_{L^6}^2 \|\nabla u\|_{L^2} \|H\|_{L^6} + \|u\|_{L^4}^2 \|H\|_{L^3} \|H_t\|_{L^6} \]

\[ \leq \frac{\delta}{4} \sigma^m \|\nabla H_t\|_{L^2}^2 + C_\sigma \sigma^m(\|\nabla u\|_{L^2}^4 \|H\|_{L^2}^2 
+ CC^{\sigma_0/3} \sigma^m (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla H_t\|_{L^2}^2). \]  

(3.47)

Combining (3.46) and (3.47), we have

\[ K_2 \leq \frac{\delta}{4} \sigma^m \|\nabla H_t\|_{L^2}^2 + C_\sigma \sigma^m(\|\nabla u\|_{L^2}^4 \|\nabla H\|_{L^2}^2 
+ CC^{\sigma_0/3} \sigma^m (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla H_t\|_{L^2}^2). \]  

(3.48)

By Sobolev trace theorem and Lemma 2.2 again, a direct computation yields

\[ K_3 = \int \sigma^m H \cdot \nabla H_t \cdot (u \cdot \nabla u) dx - \int \sigma^m(u \cdot \nabla u) \cdot \nabla H \cdot H_t dx 
- \int_{\partial \Omega} \sigma^m H \cdot H_t (u \cdot \nabla u \cdot n) ds + \int \sigma^m u \cdot \nabla H \cdot H_t dx 
+ \int \sigma^m u \cdot \nabla H_t \cdot H dx \]

\[ \leq \frac{\delta}{2} \sigma^m \|\nabla H_t\|_{L^2}^2 + C_\sigma \sigma^m(\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^2(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + 1) 
+ CC^{\sigma_0/3} \sigma^m (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^4 + \|\nabla H_t\|_{L^2}^2). \]  

(3.49)
Putting (3.45), (3.48) and (3.49) into (3.44), choosing $\delta$ small enough, we have
\[
(s^m\|H_t\|_{L^2}^2 + \sigma^m\|
abla H_t\|_{L^2}^2 - CC_{\delta_0}^{1/3}\sigma^m\|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) \\
\leq C\sigma^m(\|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2)(\|\rho^{1/2}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) \\
+ C\sigma^m\|\nabla u\|_{L^2}^2\|\nabla H_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 + 1) + Cm\sigma^{m-1}\|H_t\|_{L^2}^2. \tag{3.50}
\]
Integrating over $(0, T]$, then by Lemma 2.1, for $m > 0$, we get
\[
\sigma^m\|H_t\|_{L^2}^2 + \int_0^T \sigma^m\|\nabla H_t\|_{L^2}^2 dt - C_2C_{\delta_0}^{1/3}\int_0^T \sigma^m(\|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) dt \\
\leq CC_0^{2\delta_0}\sup_{0 \leq t \leq T} \sigma^m(\|\rho^{1/2}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) + CC_0\sup_{0 \leq t \leq T} \sigma^m\|\nabla u\|_{L^2}^2 \\
+ CC_0^{2\delta_0}\sup_{0 \leq t \leq T} \sigma^m(\|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) + C\int_0^T m\sigma^{m-1}\|H_t\|_{L^2}^2 dt. \tag{3.51}
\]
Now take $m = 2$ in (3.42) and (3.51), we deduce after adding them together that
\[
\sigma^2(\|\rho^{1/2}u\|_{L^2}^2 + \|H_t\|_{L^2}^2) + \int_0^T \sigma^2(\|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) dt \\
- C_2C_0^{2\delta_0/3}\int_0^T \sigma^2(\|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) dt \\
\leq C\int_0^T \sigma^2(\|\rho^{1/2}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) dt \\
+ CC_0^{2\delta_0}\sup_{0 \leq t \leq T} \sigma^2(\|\rho^{1/2}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) \\
+ C\int_0^T \sigma^2(\|\rho^{1/2}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) dt \\
+ CC_0^{2\delta_0/3}\sigma^2(\|\nabla H_t\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) \\
\leq C\int_0^T \sigma^2(\|\rho^{1/2}u\|_{L^2}^2 + \|H_t\|_{L^2}^2) dt + CC_0^{2\delta_0+1/2} + CA_1(T) + CC_0 + CC_0^{2\delta_0/3+1/2}. \tag{3.52}
\]
Thus we have
\[
\sup_{0 \leq t \leq T} \sigma^2(\|\rho^{1/2}u\|_{L^2}^2 + \|H_t\|_{L^2}^2) + \int_0^T \sigma^2(\|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) dt \\
\leq C\int_0^T \sigma^2(\|\rho^{1/2}u\|_{L^2}^2 + \|H_t\|_{L^2}^2) dt + CA_1(T) + CC_0^{2\delta_0/3+1/2}. \tag{3.53}
\]
provided that $C_0$ is chosen to satisfy
\[
C_0 \leq \varepsilon_2 \triangleq \min\{\varepsilon_1, (4C_2)^{-3/2\delta_0}, (4C_3)^{-3/\delta_0}\}.
\]
Finally, by Lemma A.1 and (1.1)3, it holds
\[
\|\nabla H_t\|_{L^2} \leq C(\|H_t\|_{L^2} + \|\nabla u\|_{L^2}^{1/2}\|\nabla H_t\|_{L^2}^{1/2} + \|\nabla H_t\|_{L^2} + \|\nabla u\|_{L^2}) \\
\leq \frac{1}{2}\|\nabla H_t\|_{L^2} + C(\|H_t\|_{L^2} + \|\nabla H_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla H_t\|_{L^2}). \tag{3.54}
\]
Thus, by (3.6) and (3.54), we have
\[
\sup_{0 \leq t \leq T} \sigma^2(\|\nabla H_t\|_{L^2}^{2} \\
\leq CA_1(T) + CC_0^{2\delta_0/3+1/2}.
\]
\[ \begin{align*}
&= C \sup_{0 \leq t \leq T} \sigma^2(\|H_t\|^2 + \|\nabla H\|^2_{L^2} \|\nabla u\|^2_{L^2} + \|\nabla H\|^2_{L^2} \|\nabla u\|^2_{L^2}) \\
&\leq C \int_0^T \sigma^2 \|\nabla u\|^2_{L^2} dt + CA_1(T) + CC_0^{2\delta_0/3+1/2}. \tag{3.55}
\end{align*} \]

Combining (3.53) and (3.55), we give (3.22) and complete the proof of Lemma 3.5. \( \square \)

**Lemma 3.6** Assume that \((\rho, u, H)\) is a smooth solution of (1.1)-(1.4) satisfying (3.6) and the initial data condition (1.10), then there exist positive constants \(C\) and \(\varepsilon_3\) depending only on \(\mu, \lambda, \nu, \gamma, a, \bar{\rho}, \bar{\rho}, s, \Omega, M_1\) and \(M_2\) such that

\[ \begin{align*}
&\sup_{0 \leq t \leq \sigma(T)} t^{1-s} \|\nabla u\|^2_{L^2} + \int_0^{\sigma(T)} t^{1-s} \rho |\dot{u}|^2 dx dt \leq C(\dot{\rho}, M_1, M_2), \tag{3.56} \\
&\sup_{0 \leq t \leq \sigma(T)} t^{2-s} \int (|\dot{u}|^2 + |\text{curl} H|^2 + |H_t|^2) dx \\
&+ \int_0^{\sigma(T)} t^{2-s} (|\nabla \dot{u}|^2 + |\nabla H_t|^2) dx dt \leq C(\dot{\rho}, M_1, M_2), \tag{3.57}
\end{align*} \]

provide that \(C_0 < \varepsilon_3\).

**Proof.** Suppose \(w_1(x, t), w_2(x, t)\) and \(w_3(x, t)\) solve problems

\[ \begin{align*}
&Lw_1 = 0 \quad \text{in } \Omega, \\
&w_1(x, 0) = w_{10}(x) \quad \text{in } \Omega, \\
&w_1 \cdot n = 0 \text{ and curl} w_1 \times n = 0 \text{ on } \partial \Omega,
\end{align*} \tag{3.58} \]

and

\[ \begin{align*}
&Lw_2 = -\nabla(P - \bar{P}) \quad \text{in } \Omega, \\
&w_2(x, 0) = 0 \quad \text{in } \Omega, \\
&w_2 \cdot n = 0 \text{ and curl} w_2 \times n = 0 \text{ on } \partial \Omega,
\end{align*} \tag{3.59} \]

and

\[ \begin{align*}
&Lw_3 = (\nabla \times B) \times H \quad \text{in } \Omega, \\
&w_3(x, 0) = 0 \quad \text{in } \Omega, \\
&w_3 \cdot n = 0 \text{ and curl} w_3 \times n = 0 \text{ on } \partial \Omega,
\end{align*} \tag{3.60} \]

where \(Lf \triangleq \rho \dot{f} - \mu \Delta f - (\lambda + \mu) \nabla \text{div} f\), \(B\) is the solution of \((1.1)_3\) with fixed smooth \(u\) and initial data \(B_0(x)\). Note that \(B\) satisfies (3.14) and (3.15) of Lemma 3.4.

Just as we have done in the proof of Lemma 2.2, by Lemma A.3 and Sobolev’s inequality, for any \(p \in [2, 6]\), we have

\[ \begin{align*}
\|\nabla^2 w_1\|_{L^2} &\leq C(\|\rho \dot{w}_1\|_{L^2} + \|w_1\|_{L^2}) \leq C(\|\rho \dot{w}_1\|_{L^2} + \|\nabla w_1\|_{L^2}), \tag{3.61} \\
\|\nabla w_1\|_{L^p} &\leq C \|w_1\|_{W^{2, 2}} \leq C(\|\rho \dot{w}_1\|_{L^2} + \|\nabla w_1\|_{L^2}), \tag{3.62} \\
\|\nabla F_{w_3}\|_{L^p} &\leq C(\|\nabla F_{w_3}\|_{L^2} + \|F_{w_3}\|_{L^2}) \leq C(\|\rho \dot{w}_2\|_{L^2} + \|\nabla w_2\|_{L^2}), \tag{3.63} \\
\|\nabla w_3\|_{L^p} &\leq C(\|\rho \dot{w}_3^2\|_{L^2} \|\nabla w_3\|_{L^2} + \|P - \bar{P}\|_{L^2})^{6-p} + C(\|\nabla w_3\|_{L^2} + \|P - \bar{P}\|_{L^2}), \tag{3.64}
\end{align*} \]
where \( F_{w_2} = (\lambda + 2\mu) \text{div} w_2 - (P - \tilde{P}) \).

A similar way as for the proof of (2.4) shows that

\[
\sup_{0 \leq t \leq \sigma(T)} \int_0^{\sigma(T)} \int \rho |w_1|^2 dx + \int_0^{\sigma(T)} \int \nabla w_1 \cdot \nabla w_1 dx dt \leq C \int |w_{10}|^2 dx, \tag{3.66}
\]

\[
\sup_{0 \leq t \leq \sigma(T)} \int_0^{\sigma(T)} \int |\nabla w_3|^2 dx dt \leq C \int_0^{\sigma(T)} \int \nabla B \cdot \nabla B dx dt \leq C \|B_0\|_{L^2}^2. \tag{3.67}
\]

and

\[
\sup_{0 \leq t \leq \sigma(T)} \int_0^{\sigma(T)} \int \rho |w_3|^2 dx dt + \int_0^{\sigma(T)} \int |\nabla w_3|^2 dx dt \leq C \int_0^{\sigma(T)} \int |\nabla w_3|^2 dx dt \leq C \|B_0\|_{L^2}^2. \tag{3.68}
\]

Multiplying (3.58) by \( w_{1t} \) and integrating over \( \Omega \), by (3.62), (3.6), Sobolev’s and Young’s inequalities, we obtain

\[
\int_0^{\sigma(T)} \int (\text{div} w_1)^2 dx + \frac{\mu}{2} \int |\text{curl} w_1|^2 dx \leq \int \rho |w_1|^2 dx
\]

\[
= \int \rho w_1 \cdot (u \cdot \nabla w_1) dx \leq C_4 \|\rho^{\frac{1}{2}}w_1\|^2_{L^2} + \|\nabla w_1\|^2_{L^2}. \tag{3.69}
\]

Together with (3.66), and by Gronwall’s inequality and Lemma A.4, it yields

\[
\sup_{0 \leq t \leq \sigma(T)} \|\nabla w_1\|^2_{L^2} + \int_0^{\sigma(T)} \int \rho |w_1|^2 dx dt \leq C \|\nabla w_{10}\|^2_{L^2}, \tag{3.70}
\]

and

\[
\sup_{0 \leq t \leq \sigma(T)} t \|\nabla w_1\|^2_{L^2} + \int_0^{\sigma(T)} t \int \rho |w_1|^2 dx dt \leq C \|w_{10}\|^2_{L^2}, \tag{3.71}
\]

provided \( C_0 < \tilde{\varepsilon}_1 \triangleq (2C_4)^{-\frac{3}{2}} \).

Since the solution operator \( w_{10} \mapsto w_1(\cdot, t) \) is linear, by the standard Stein-Weiss interpolation argument \([5]\), one can deduce from (3.70) and (3.71) that for any \( \theta \in [s, 1] \),

\[
\sup_{0 \leq t \leq \sigma(T)} t^{1-\theta} \|\nabla w_1\|^2_{L^2} + \int_0^{\sigma(T)} t^{1-\theta} \int \rho |w_1|^2 dx dt \leq C \|w_{10}\|^2_{H^{\theta}}, \tag{3.72}
\]

with a uniform constant \( C \) independent of \( \theta \).

Multiplying (3.59) by \( w_{2t} \) and integrating over \( \Omega \) give that

\[
\int \frac{(\lambda + 2\mu)}{2} \int (\text{div} w_2)^2 dx + \frac{\mu}{2} \int |\text{curl} w_2|^2 dx - \int (P - \tilde{P}) \text{div} w_2 dx \right)_t + \int \rho |w_2|^2 dx
\]

\[
= \int \rho w_2 \cdot (u \cdot \nabla w_2) dx - \frac{1}{\lambda + 2\mu} \int (P - \tilde{P})(F_{w_2} \text{div} u + \nabla F_{w_2} \cdot u) dx
\]

\[
- \frac{1}{2(\lambda + 2\mu)} \int (P - \tilde{P})^2 \text{div} u dx + \gamma \int P \text{div} w_2 dx
\]

\[
\leq C(\|\rho^{\frac{1}{2}}w_2\|^2_{L^2} \|\rho^{\frac{1}{2}}u\|^2_{L^2} \|\nabla w_2\|^2_{L^2} + \|\nabla u\|^2_{L^2} \|F_{w_2}\|^2_{L^2} + \|\nabla F_{w_2}\|^2_{L^2} \|u\|^2_{L^2})
\]
\[
\begin{align*}
&+ C(\|P - \bar{P}\|_{L^2}\|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla w_2\|_{L^2}) \\
&\leq C_5 C_0^{\delta_0} \|\rho^{\frac{1}{2}} \dot{w}_2\|_{L^2}^2 + \frac{1}{4} \|\rho^{\frac{1}{2}} \dot{w}_2\|_{L^2}^2 + C(\|\nabla w_2\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|P - \bar{P}\|_{L^2}^{2/3}),
\end{align*}
\]

where we have utilized (3.6), (3.63)-(3.65), Hölder’s, Poincaré’s and Young’s inequalities. As a result,
\[
\begin{align*}
&\left((\lambda + 2\mu)\|\text{div} w_2\|_{L^2}^2 + \mu\|\text{curl} w_2\|_{L^2}^2 - 2 \int (P - \bar{P})\text{div} w_2 dx\right) + \int \rho |\dot{w}_2|^2 dx \\
&\leq C \left(\|\nabla w_2\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|P - \bar{P}\|_{L^2}^{\frac{2}{3}}\right),
\end{align*}
\]

provide that \(C_0 < \bar{\varepsilon}_2 \triangleq (4C_5)^{-\frac{a_0}{\lambda_0}}\).

By Gronwall’s inequality, (3.67) and Lemmas A.4, 2.1, one has
\[
\sup_{0 \leq t \leq \sigma(T)} \|\nabla w_2\|_{L^2}^2 + \int_0^{\sigma(T)} \int \rho |\dot{w}_2|^2 dx dt \leq CC_0^{1/3}.
\]

Similarly, multiplying (3.60) by \(w_3t\) and integrating over \(\Omega\) give that
\[
\begin{align*}
&\frac{d}{dt} L_0 + L_1 + L_2.
\end{align*}
\]

By (A.1) and (3.10), we have
\[
L_0 \leq C \|H\|_{L^3}\|\nabla B\|_{L^2}\|w_3\|_{L^6} \leq \frac{\mu}{4} \|\nabla w_3\|_{L^2}^2 + CC_0^{2\delta_0/3}\|\nabla B\|_{L^2}^2.
\]

Using (3.14), a directly computation yields
\[
\begin{align*}
L_1 &\leq C(\|H_t\|_{L^2}\|\nabla B\|_{L^2}\|\nabla w_3\|_{L^2} + \|B_t\|_{L^2}\|H\|_{L^3}\|\nabla w_3\|_{L^6} + \|B_t\|_{L^2}\|\nabla w_3\|_{L^2}\|\nabla H\|_{L^3}) \\
&\leq CC_0^{\delta_0/3}(\|\rho^{1/2} \dot{w}_3\|_{L^2}^2 + \|\nabla w_3\|_{L^2}^2) + C(\|B_t\|_{L^2}^2 + \|\text{curl}^2 B\|_{L^2}^2 + \|\nabla B\|_{L^2}^2) \\
&+ \|\text{curl}^2 B\|_{L^2}^{1/2}\|\nabla B\|_{L^2}^{1/2}\|H_t\|_{L^2}\|\nabla \nabla w_3\|_{L^2} + \|\text{curl}^2 B\|_{L^2}^{1/2}\|\nabla B\|_{L^2}^{1/2}\|H_t\|_{L^2}\|\nabla w_3\|_{L^2}
\end{align*}
\]

where we have used the fact (3.65). Similarly, using Lemma A.1 yields
\[
L_2 \leq CC_0^{\delta_0/3}(\|\rho^{1/2} \dot{w}_3\|_{L^2}^2 + \|\text{curl}^2 B\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 + \|\nabla w_3\|_{L^2}^2).
\]

Putting (3.77), (3.78) and (3.79) into (3.76), we obtain
\[
\begin{align*}
&\|\nabla w_3\|_{L^2}^2 + \int \rho |\dot{w}_3|^2 dx - (CC_0^{2\delta_0/3}\|\nabla B\|_{L^2}^2) t \\
&\leq C_6 C_0^{\delta_0/3}(\|\rho^{1/2} \dot{w}_3\|_{L^2}^2 + \|\nabla w_3\|_{L^2}^2) + C(\|B_t\|_{L^2}^2 + \|\text{curl}^2 B\|_{L^2}^2 + \|\nabla B\|_{L^2}^2)
\end{align*}
\]
Thus, if $C_0$ is chosen to be such that $C_0 \leq \hat{\epsilon}_3 \triangleq (2C_0)^{-3/\delta_0}$, we obtain
\[
\sup_{0 \leq t \leq \sigma(T)} \| \nabla w_3 \|_{L^2}^2 + \int_0^{\sigma(T)} \int \rho |\dot{w}_3|^2 \, dx \, dt \\
\leq C \| B_0 \|_{L^2}^2 + C C_0^{\delta_0} \sup_{0 \leq t \leq \sigma(T)} \| \nabla w_3 \|_{L^2}^2 + C C_0^{\delta_0/2} \sup_{0 \leq t \leq \sigma(T)} (\| \nabla B \|_{L^2}^2 + \| \nabla w_3 \|_{L^2}^2) \\
+ C C_0^{\delta_0/2} \left( \int_0^{\sigma(T)} \| \text{curl}^2 B \|_{L^2}^2 \, dt \right)^{1/4} \sup_{0 \leq t \leq \sigma(T)} \| \nabla B \|_{L^2}^{1/2} \| \nabla w_3 \|_{L^2} \\
\leq C \| B_0 \|_{L^2}^2 + C \gamma C_0^{\delta_0/2} \sup_{0 \leq t \leq \sigma(T)} \| \nabla w_3 \|_{L^2}^2,
\] (3.81)
where we have used (3.14) and (3.8) (with $H, H_0$ being replaced by $B, B_0$, respectively). Thus, it follows from (3.81) that
\[
\sup_{0 \leq t \leq \sigma(T)} \| \nabla w_3 \|_{L^2}^2 + \int_0^{\sigma(T)} \int \rho |\dot{w}_3|^2 \, dx \, dt \leq C \| B_0 \|_{L^2}^2,
\] (3.82)
provided $C_0 \leq \hat{\epsilon}_4 \triangleq \min\{\hat{\epsilon}_3, (2C_7)^{-2/\delta_0}\}$. Similarly, multiplying (3.80) by $t$, integrating it over $(0, \delta(T))$, for $C_0 \leq \hat{\epsilon}_4$, we obtain that
\[
\sup_{0 \leq t \leq \sigma(T)} (t \| \nabla w_3 \|_{L^2}^2) + \int_0^{\sigma(T)} t \int \rho |\dot{w}_3|^2 \, dx \, dt \leq C \| B_0 \|_{L^2}^2,
\] (3.83)
which we have used (3.6), (3.8) and (3.14). Since the solution operators $B_0 \mapsto B$ and $B \mapsto w_3$ are linear, by the standard Stein-Weiss interpolation argument [5], one can deduce from (3.81) and (3.82) that for any $\theta \in [s, 1]$,
\[
\sup_{0 \leq t \leq \sigma(T)} t^{1-\theta} \| \nabla w_3 \|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-\theta} \int \rho |\dot{w}_3|^2 \, dx \, dt \leq C \| B_0 \|_{H^\theta}^2,
\] (3.84)
Now let $w_{10} = u_0$ and $B_0 = H_0$, so that $w_1 + w_2 + w_3 = u$ and $B = H$, we derive (3.56) from (3.72), (3.75) and (3.84) directly under certain condition $C_0 < \hat{\epsilon_3} \triangleq \min\{\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_4\}$.

In order to prove (3.57), taking $m = 2 - s$ in (3.39), (3.51), and integrating over $(0, \sigma(T))$ instead of $(0, T)$, in a similar way as we have gotten (3.52), we obtain
\[
\sup_{0 \leq t \leq \sigma(T)} \int_0^{\sigma(T)} \sigma^{2-s} (\| \partial_t u \|_{L^2}^2 + \| H_t \|_{L^2}^2) + \int_0^{\sigma(T)} \sigma^{2-s} (\| \nabla u \|_{L^2}^2 + \| \nabla H_t \|_{L^2}^2) \, dt \\
\leq C \int_0^{\sigma(T)} \sigma^{2-s} \| \nabla u \|_{L^4}^2 \, dt + C C_0^{2\delta_0/3} \sigma^{2-s} \| \text{curl}^2 H \|_{L^2}^2 + C (\rho, M_1, M_2),
\] (3.85)
where we have taken advantage of (3.56). Next, from (3.54), we have
\[
\sup_{0 \leq t \leq \sigma(T)} t^{2-s} \| \text{curl}^2 H \|_{L^2}^2
\]
Then combining (3.85) and (3.86), we obtain

\[
\sup_{0 \leq t \leq \sigma(T)} \sigma^{2-s} \left( \| \frac{1}{2} \dot{u} \|^2_{L^2} + \| H_t \|^2_{L^2} \right) + \int_0^{\sigma(T)} \sigma^{2-s} \left( \| \nabla \dot{u} \|^2_{L^2} + \| \nabla H_t \|^2_{L^2} \right) dt \\
\leq C \int_0^{\sigma(T)} \sigma^{2-s} \| \nabla u \|^4_{L^4} dt + C(\rho, M_1, M_2),
\]

(3.87)

provided \( C_0 \leq \varepsilon_2 \). By (2.12) and (3.56), we have

\[
\int_0^{\sigma(T)} t^{2-s} \| \nabla u \|^4_{L^4} dt \\
\leq C \int_0^{\sigma(T)} t^{2-s} \left( \| \rho \frac{1}{2} \dot{u} \|^3_{L^2} + \| \text{curl}^2 H \|^3_{L^2} \right) \| \nabla u \|_{L^2} + \| P - \hat{P} \|_{L^2} + \| \nabla H \|_{L^2} dt \\
+ C \int_0^{\sigma(T)} t^{2-s} \left( \| \nabla u \|_{L^2} + \| P - \hat{P} \|_{L^2} + \| P - \hat{P} \|_{L^4} \right) \| \nabla H \|_{L^4} + \| H \|_{L^4} dt \\
\leq CC_0^{\delta_0} \sup_{0 \leq t \leq \sigma(T)} \left( t^{2-s} \left( \| \rho \frac{1}{2} \dot{u} \|^2_{L^2} + \| \text{curl}^2 H \|^2_{L^2} \right) \right) + C,
\]

(3.88)

since it follows from (3.6) that for \( s \in (1/2, 1] \),

\[
\int_0^{\sigma(T)} t^{2-s} \| \rho \frac{1}{2} \dot{u} \|^3_{L^2} \| \nabla u \|_{L^2} + \| \nabla H \|_{L^2} dt \\
\leq C \int_0^{\sigma(T)} t^{\frac{3-2s}{4}} \left( \| \nabla u \|^2_{L^2} + \| \nabla H \|^2_{L^2} \right) \left( t^{2-s} \| \rho \frac{1}{2} \dot{u} \|^2_{L^2} \right) \left( t^{2-s} \| \rho \frac{1}{2} \dot{u} \|^2_{L^2} \right) dt \\
\leq CC_0^{\delta_0} \sup_{0 \leq t \leq \sigma(T)} \left( t^{2-s} \| \rho \frac{1}{2} \dot{u} \|^2_{L^2} \right),
\]

(3.89)

and

\[
\int_0^{\sigma(T)} t^{2-s} \| H \|_{L^4}^4 dt \leq C \int_0^{\sigma(T)} t^{2-s} \| H \|_{L^\infty}^4 \| H \|_{L^4} dt \\
\leq C \int_0^{\sigma(T)} t^{2-s} \left( \| \nabla H \|_{L^2}^4 \| \text{curl}^2 H \|^2_{L^2} + \| \nabla H \|_{L^2}^6 \right) dt \\
\leq CC_0^{\delta_0} \sup_{0 \leq t \leq \sigma(T)} \left( t^{2-s} \| \text{curl}^2 H \|^2_{L^2} \right) + C.
\]

(3.90)

Then combining (3.87) and (3.88), we have

\[
\sup_{0 \leq t \leq \sigma(T)} \sigma^{2-s} \left( \| \rho \frac{1}{2} \dot{u} \|^2_{L^2} + \| H_t \|^2_{L^2} \right) + \int_0^{\sigma(T)} \sigma^{2-s} \left( \| \nabla \dot{u} \|^2_{L^2} + \| \nabla H_t \|^2_{L^2} \right) dt \\
\leq C_8 C_0^{\delta_0} \sup_{0 \leq t \leq \sigma(T)} \sigma^{2-s} \left( \| \rho \frac{1}{2} \dot{u} \|^2_{L^2} + \| H_t \|^2_{L^2} \right) + C(\rho, M_1, M_2),
\]

(3.91)

Therefore, if we choose \( C_0 \) to be such that \( C_0 \leq \varepsilon_3 \triangleq \min\{ \varepsilon_3, (2C_8)^{-1/\delta_0} \} \), (3.91) and (3.86) implies (3.57). The proof of Lemma 3.6 is completed. \( \square \)
Lemma 3.7 If \((\rho, u, H)\) is a smooth solution of (1.1)-(1.4) satisfying (3.6) and the initial data condition (1.10), then there exists a positive constant \(\varepsilon_4\) depending only on \(\mu, \lambda, \nu, \gamma, a, \bar{\rho}, \bar{\rho}, s, \Omega, M_1\) and \(M_2\) such that
\[
A_4(\sigma(T)) + A_5(\sigma(T)) \leq C_0^{\delta_0},
\]
provided \(C_0 < \varepsilon_4\).

**Proof.** Multiplying (1.1) by \(3|u||u|\), and integrating the resulting equation over \(\Omega\) lead to
\[
\left( \int \rho|u|^3 dx \right)_t \leq C \int |u||\nabla u|^2 dx + C \int |P - \bar{P}||u||\nabla u| dx + C \int |H||\nabla H||u|^2 dx
\]
\[
\leq C \|\nabla u\|_2^2 (\|\rho u\|_2^2 + \|\text{curl}^2 H\|_2^2) + C \|\nabla u\|_2^2 \|\nabla H\|_2 + C C_0^{\frac{1}{8}} \|\nabla u\|_2^2
\]
\[
+ C \|\nabla u\|_2^2 + C C_0^{\frac{3}{8}} \|\nabla u\|_2^2 + C (\|\nabla H\|_2^2 + \|\nabla u\|_2^2).
\]
Hence, integrating (3.93) over \((0, \sigma(T))\) and using (3.6), (3.8), we get
\[
\sup_{0 \leq t \leq \sigma(T)} \int \rho|u|^3 dx
\]
\[
\leq C \int_0^{\sigma(T)} \left( t \frac{3-2s}{2} \|\nabla u\|_2^2 \right)^\frac{5}{2} \left( t \frac{3-2s}{2} (\|\rho u\|_2^2 + \|\text{curl}^2 H\|_2^3) \right) \frac{1}{4} \, dt
\]
\[
+ C \int_0^{\sigma(T)} \left( t \frac{3-2s}{2} \|\nabla u\|_2^2 \right)^\frac{5}{2} \left( t \frac{3-2s}{2} \|\nabla H\|_2^2 \right) \frac{1}{4} \, dt
\]
\[
+ C C_0^{\frac{3}{8}} \int_0^{\sigma(T)} \left( t \frac{3-2s}{2} \|\nabla u\|_2^2 \right)^\frac{5}{2} \left( t \frac{5(2s-3)}{16} \right) \, dt + C C_0^{\frac{3}{8}} + \int \rho_0|u_0|^3 dx
\]
\[
\leq C C_0^{\frac{3}{8}} + \int \rho_0|u_0|^3 dx \leq C_0 C_0^{\frac{3}{8} / 2},
\]
where we have used the fact \(\delta_0 \in (0, 1/9), s \in (1/2, 1)\) and
\[
\int \rho_0|u_0|^3 dx \leq C \|\rho_0 u_0\|_L^2 \|H\|_H^s \leq C C_0^{\frac{3}{8} / 2},
\]
Finally, set \(\varepsilon_4 \triangleq \min\{\varepsilon_3, (C_9)^{-\frac{3}{8}}\}\), we get \(A_5(\sigma(T)) \leq C_0^{\delta_0}\). Next, it remains to estimate \(A_4(\sigma(T))\). Using (3.56), we have
\[
A_4(\sigma(T)) \leq \sup_{0 \leq t \leq \sigma(T)} \left( t^{1-s} \|\nabla u\|_2^2 \right)^\frac{2s-1}{16} \sup_{0 \leq t \leq \sigma(T)} \left( t \|\nabla u\|_2^2 \right)^\frac{2s-1}{16}
\]
\[
+ \sup_{0 \leq t \leq \sigma(T)} \left( t^{1-s} \|\nabla H\|_2^2 \right)^\frac{2s-1}{4s} \sup_{0 \leq t \leq \sigma(T)} \left( t \|\nabla H\|_2^2 \right)^\frac{2s-1}{4s}
\]
\[
\leq C A_1(T)^\frac{2s-1}{4s} \leq C C_0^{\frac{2s-1}{8s}} \leq C_0^{\delta_0},
\]
provided \(C_0 \leq \varepsilon_4\) and \(\delta_0 \triangleq \frac{2s-1}{8s} < \frac{2s-1}{8s}\). The proof of Lemma 3.7 is completed. □
Lemma 3.8 Let \((\rho, u, H)\) be a smooth solution of (1.1)-(1.4) on \(\Omega \times (0, T]\) satisfying (3.6). Then there exists a positive constant \(\varepsilon_5\) depending only on \(\mu, \lambda, \nu, \gamma, a, s, \bar{\rho}, \bar{\rho}, \Omega, M_1\) and \(M_2\) such that

\[
A_1(T) + A_2(T) \leq C_0^{1/2},
\]

provided \(C_0 \leq \varepsilon_5\).

Proof. By (A.1) and (2.24), one can check that

\[
\int_0^T \sigma \|\nabla u\|^3_{L^3} dt \leq C \int_0^T \sigma (\|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^4}^2 + \|\nabla u\|^3_{L^2}) dt
\]

\[
\leq CC_0 + C \int_0^T \sigma^2 \|\nabla u\|^4_{L^4} dt,
\]

which, along with (3.21) and (3.22) gives

\[
A_1(T) + A_2(T) \leq C(C_0^{1/2+\delta_0/2} + \int_0^T \sigma^2 \|\nabla u\|^4_{L^4} dt).
\]

So it reduces to estimate \(\int_0^T \sigma^2 \|\nabla u\|^4_{L^4} dt\). On one hand, by (2.12), (3.6), (2.29) and Lemma 2.1 again, it indicates that

\[
\int_0^T t^2 \|\nabla u\|^4_{L^4} dt
\]

\[
\leq C \int_0^T t^2 (\|\rho \frac{\partial}{\partial t} u\|^3_{L^2} + \|\text{curl}^2 H\|^3_{L^2}) (\|\nabla u\|_{L^2} + \|P - \bar{P}\|_{L^2} + \|\nabla H\|_{L^2}) dt
\]

\[
+ C \int_0^T t^2 (\|\nabla u\|^4_{L^4} + \|P - \bar{P}\|^4_{L^2} + \|P - \bar{P}\|^4_{L^2} + \|\nabla H\|^4_{L^2} + \|H\|^2_{L^4}) dt
\]

\[
\leq CC_0^{1/2+\delta_0} + CC_0 + CC_0^{1/2+2\delta_0} \leq CC_0^{1/2+\delta_0},
\]

since it follows from (3.6) that for \(s \in (1/2, 1]\),

\[
\int_0^T t^2 \|\rho \frac{\partial}{\partial t} u\|^3_{L^2} (\|\nabla u\|_{L^2} + \|\nabla H\|_{L^2}) dt \leq C \int_0^T t^2 (t^{3-2s} (t^{2s-3} (\|\nabla u\|^2_{L^2} + \|\nabla H\|^2_{L^2}))^{1/2} (t^{2s-3} \|\rho \frac{\partial}{\partial t} u\|^2_{L^2})^{1/2} (t^2 \|\rho \frac{\partial}{\partial t} u\|^2_{L^2}) dt
\]

\[
\leq CC_0^{1/2+\delta_0},
\]

and

\[
\int_0^T t^2 \|H\|^2_{L^4} dt \leq C \int_0^T t^2 \|H\|^2_{L^6} \|\nabla H\|^4_{L^4} dt
\]

\[
\leq C \int_0^T t^2 (\|\nabla H\|^4_{L^2} + \|\text{curl}^2 H\|^2_{L^2} + \|\nabla H\|^6_{L^2}) dt \leq CC_0^{1/2+2\delta_0}.
\]

On the other hand, by (3.6), (2.24) and Lemma 3.6, we have

\[
\int_0^T t^2 \|\nabla u\|^4_{L^4} dt
\]
\[ \begin{align*}
&\leq C \int_{\sigma(T)}^{T} \sigma^2(\frac{1}{2} |\rho|^2 |\dot{u}|^2 L^2 + \|\text{cur} |H|^2| L^2) (\|\nabla u\| L^2 + \|P - \bar{P}\| L^2 + \|\nabla H\| L^2) dt \\
&+ C \int_{\sigma(T)}^{T} \sigma^2(\|\nabla u\|^4 L^2 + \|P - \bar{P}\|^4 L^2 + \|P - \bar{P}\|^4 L^2 + \|\nabla H\|^4 L^2 + \|H|^2 L^2) dt \\
&\leq CC_0^1 + CC_0^{1/2 + 2\delta_0} \leq CC_0^{1/2 + 2\delta_0}.
\end{align*} \] (3.103)

Combining (3.100) and (3.103), it follows from (3.99) that

\[ A_1(T) + A_2(T) \leq C_10 C_0^{1/2 + 2\delta_0}. \] (3.104)

Set \( \varepsilon_5 \triangleq \min \{\varepsilon_4, (C_10^{-2/\delta_0}) \} \), from (3.104), (3.97) holds when \( C_0 < \varepsilon_5 \). The proof of Lemma 3.8 is completed.

We now proceed to prove the uniform (in time) upper bound for the density.

**Lemma 3.9** Let \((\rho, u, H)\) be a smooth solution of (1.1)-(1.4) on \( \Omega \times (0, T) \) satisfying (3.6). Then there exists a positive constant \( \varepsilon_6 \) depending only on \( \mu, \nu, \gamma, \alpha, s, \bar{\rho}, \rho, \Omega, M_1 \) and \( M_2 \) such that

\[ \sup_{0 \leq t \leq T} \|\rho(t)\|_{L^\infty} \leq \frac{\varepsilon_6}{4}. \] (3.105)

provided \( C_0 \leq \varepsilon_6 \).

**Proof.** First, the equation of mass conservation (1.1) can be equivalently rewritten in the form

\[ D_t \rho = g(\rho) + b'(t), \] (3.106)

where

\[ D_t \rho \triangleq \rho_t + u \cdot \nabla \rho, \quad b(t) \triangleq -\frac{\rho(P - \bar{P})}{2\mu + \lambda}, \quad b(t) \triangleq -\frac{1}{2\mu + \lambda} \int_0^t \rho(F + \frac{|H|^2}{2}) dt. \] (3.107)

Naturally, we shall prove our conclusion by Lemma A.2. It is sufficient to check that the function \( b(t) \) must verify (A.3) with some suitable constants \( N_0, N_1 \).

For \( t \in [0, \sigma(T)] \), one deduces from (A.1), (A.2), (2.8), (2.9), (2.34), (3.6) and Lemmas 2.1, 3.6 that for \( \delta_0 \) as in Proposition 3.1 and for all \( 0 \leq t_1 \leq t_2 \leq \sigma(T) \),

\[ \begin{align*}
|b(t_2) - b(t_1)| &\leq \frac{1}{\lambda + 2\mu} \left| \int_{t_1}^{t_2} \rho(F + \frac{|H|^2}{2}) dt \right| \\
&\leq C \int_0^{\sigma(T)} \|F\|_{L^6} \|\nabla F\|_{L^3} dt \\
&+ C \int_0^{\sigma(T)} \|F\|_{L^{3/2}} dt \\
&\leq C \int_0^{\sigma(T)} (\|\rho \dot{u}\|_{L^2} + \|\text{cur} |H|^2|_{L^2}) \|\nabla \dot{u}\|_{L^2} dt \\
&+ C \int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2} + \|\text{cur} |H|^2|_{L^2}) \|\nabla H\|_{L^2} dt \\
&\leq C \int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2} + \|\text{cur} |H|^2|_{L^2}) \|\nabla u\|_{L^2} + \|\nabla H\|_{L^2} dt
\end{align*} \]
+ C \int_0^{\sigma(T)} \left( \| \nabla u \|_{L^2}^2 + \| \nabla H \|_{L^2}^2 + \| P - \bar{P} \|_{L^2}^2 \right)\| \nabla u \|_{L^2}^2 \frac{t^{2-\gamma}}{t^{6a-11}} dt \\
+ C \int_0^{\sigma(T)} \left( \| \nabla u \|_{L^2}^2 + \| \nabla H \|_{L^2}^2 + \| P - \bar{P} \|_{L^2}^2 \right)\| \nabla u \|_{L^2}^2 \| \nabla \sigma \|_{L^2}^2 \frac{t^{2-\gamma}}{t^{6a-11}} dt \\
+ C \int_0^{\sigma(T)} \left( \| \nabla u \|_{L^2}^2 + \| \nabla H \|_{L^2}^2 + \| P - \bar{P} \|_{L^2}^2 \right)(\| \nabla u \|_{L^2}^2 + \| \nabla H \|_{L^2}) dt \\
+ C \int_0^{\sigma(T)} \| \nabla H \|_{L^2}^2 (\| \nabla H \|_{L^2}^2 + \| \text{curl}^2 H \|_{L^2}^2) dt \triangleq \sum_{i=1}^{8} B_i. \quad (3.108)

We have to estimate $B_i$, $i = 1, 2, \cdots, 8$ one by one. A directly computation gives

\begin{align*}
B_1 & \leq C \int_0^{\sigma(T)} \left( t^{3-2\alpha} \left( \| \rho^\frac{1}{2} \nabla u \|_{L^2}^2 + \| \text{curl}^2 H \|_{L^2}^2 \right) \right)^{\frac{1}{4}} \left( t^{2-\gamma} \| \nabla u \|_{L^2}^2 \right)^{\frac{1}{4}} \left( t^{6a-11} \right)^{\frac{1}{4}} dt \\
& \leq C \left( \int_0^{\sigma(T)} t^{\frac{3-2\alpha}{4}} \left( \| \rho^\frac{1}{2} \nabla u \|_{L^2}^2 + \| \text{curl}^2 H \|_{L^2}^2 \right) \right)^{\frac{1}{4}} \left( \int_0^{\sigma(T)} t^{\frac{6a-11}{4}} dt \right)^{\frac{1}{4}} \leq CC_0^{\beta_0/4}, \quad (3.109)
\end{align*}

similarly,

\begin{align*}
B_2 & \leq C \int_0^{\sigma(T)} \left( t^{2-\gamma} (\| \rho^\frac{1}{2} \nabla u \|_{L^2}^2 + \| \text{curl}^2 H \|_{L^2}^2) \right)^{\frac{1}{4}} \left( t^{\frac{3-2\alpha}{4}} \| \nabla H \|_{L^2}^2 \right)^{\frac{1}{4}} \left( t^{6a-11} \right)^{\frac{1}{4}} dt \\
& \leq C C_0^{\beta_0/2}, \quad (3.110)
\end{align*}

\begin{align*}
B_3 & \leq C \int_0^{\sigma(T)} \left( \| \rho^\frac{1}{2} \nabla u \|_{L^2}^2 + \| \text{curl}^2 H \|_{L^2}^2 \right)^{\frac{1}{4}} \left( t^{\frac{3-2\alpha}{4}} \left( \| \nabla u \|_{L^2}^2 + \| \nabla H \|_{L^2}^2 + \| P - \bar{P} \|_{L^2}^2 \right) \right)^{\frac{1}{4}} \left( t^{3(24a-3)} \right)^{\frac{1}{4}} dt \\
& \leq CC_0^{\beta_0/4} \quad (3.111)
\end{align*}

\begin{align*}
B_4 & \leq C \int_0^{\sigma(T)} \left( \| \nabla u \|_{L^2}^2 + \| \nabla H \|_{L^2}^2 + \| P - \bar{P} \|_{L^2}^2 \right)^{\frac{1}{4}} \left( t^{3-2\gamma} \| \nabla u \|_{L^2}^2 \right)^{\frac{1}{4}} \left( t^{2-\gamma} \| \nabla H \|_{L^2}^2 \right)^{\frac{1}{4}} \left( t^{6a-11} \right)^{\frac{1}{4}} dt \\
& \leq CC_0^{1/4}, \quad (3.112)
\end{align*}

\begin{align*}
B_5 & \leq C \int_0^{\sigma(T)} \left( t^{3-2\alpha} \left( \| \nabla u \|_{L^2}^2 + \| \nabla H \|_{L^2}^2 \right) \right)^{\frac{1}{4}} \left( t^{3-2\alpha} \| \nabla H \|_{L^2}^2 \right)^{\frac{1}{4}} \left( t^{6a-11} \right)^{\frac{1}{4}} dt \\
& + CC_0^{\frac{1}{4}} \int_0^{\sigma(T)} \left( t^{3-2\alpha} \| \nabla H \|_{L^2}^2 \right)^{\frac{1}{4}} \left( t^{3-2\alpha} \| \text{curl}^2 H \|_{L^2}^2 \right)^{\frac{1}{4}} \left( t^{6a-11} \right)^{\frac{1}{4}} dt \\
& \leq CC_0^{\beta_0/4} + CC_0^{1/4+\beta_0/2} \leq CC_0^{\beta_0/4}, \quad (3.113)
\end{align*}

\begin{align*}
B_6 & \leq C \int_0^{\sigma(T)} \left( t^{3-2\alpha} \left( \| \nabla u \|_{L^2}^2 + \| \nabla H \|_{L^2}^2 \right) \right)^{\frac{1}{4}} \left( t^{3-2\alpha} \| \text{curl}^2 H \|_{L^2}^2 \right)^{\frac{1}{4}} \left( t^{6a-11} \right)^{\frac{1}{4}} dt \\
& + CC_0^{\frac{1}{4}} \int_0^{\sigma(T)} \left( t^{3-2\alpha} \left( \| \nabla u \|_{L^2}^2 + \| \nabla H \|_{L^2}^2 \right) \right)^{\frac{1}{4}} \left( t^{3-2\alpha} \| \text{curl}^2 H \|_{L^2}^2 \right)^{\frac{1}{4}} \left( t^{6a-11} \right)^{\frac{1}{4}} dt \\
& \leq CC_0^{\beta_0/4} + CC_0^{1/4+\beta_0/2} \leq CC_0^{\beta_0/4}, \quad (3.114)
\end{align*}
\[
B_7 = \int_0^{\sigma(T)} (\|\nabla u\|_{L^2} + \|\nabla H\|_{L^2} + \|P - \bar{P}\|_{L^2}) dt \leq CC_0^{1/2},
\]
\[
B_8 \leq \int_0^{\sigma(T)} \left( t^{\frac{3-2s}{4}} \|\nabla H\|^2_{L^2} \right)^\frac{1}{2} \left( t^{\frac{3-2s}{4}} \|\text{curl}H\|^2_{L^2} \right)^\frac{3}{4} + CC_0 dt
\leq CC_0^{1/2+\delta_0/2}.
\]

Putting (3.109)-(3.116) into (3.108), we have
\[
|b(t_2) - b(t_1)| \leq C_1 C_0^{\delta_0/4}.
\]
Combining (3.117) with (3.106) and choosing \(N_1 = 0, N_0 = C_11C_0^{\delta_0/4}, \tilde{\zeta} = \hat{\rho} \) in Lemma A.2 give
\[
\sup_{t \in [0,\sigma(T)]} \|\rho\|_{L^\infty} \leq \hat{\rho} + C_11C_0^{\delta_0/4} \leq \frac{3\hat{\rho}}{2},
\]
provided \(C_0 \leq \hat{\varepsilon}_6 \triangleq \min\{\hat{\varepsilon}_5, \left(\frac{\hat{\rho}}{2C_11}\right)^{\frac{\delta_0}{20}}\}\).

On the other hand, for \(t \in [\sigma(T), T]\), \(\sigma(T) \leq t_1 \leq t_2 \leq T\), it follows from (2.8), (2.34), (3.6), (2.29) and Lemma 2.1 that
\[
|b(t_2) - b(t_1)| \leq C \int_{t_1}^{t_2} (\|F\|_{L^\infty} + \|H\|^2_{L^\infty}) dt
\leq \frac{a}{\lambda + 2\mu} (t_2 - t_1) + C \int_{t_1}^{t_2} \|F\|^4_{L^\infty} dt + C \int_{t_1}^{t_2} \|H\|^2_{L^\infty} dt
\leq \frac{a}{\lambda + 2\mu} (t_2 - t_1) + CC_0^{\frac{1}{2}} \int_{\sigma(T)}^T (\|\nabla \hat{u}\|^2_{L^2} + \|\nabla H\|_{L^2}^2 + \|\text{curl}H\|^2_{L^2} + \|\nabla H\|^4_{L^2}) dt
\leq CC_0 + C \int_{t_1}^{t_2} (\|\nabla H\|_{L^2}^2 + \|\text{curl}H\|_{L^2} + \|\nabla H\|_{L^2}^2) dt
\leq \frac{a}{\lambda + 2\mu} (t_2 - t_1) + C_12C_0^{2/3}.
\]

Now we choose \(N_0 = C_12C_0^{2/3}, N_1 = \frac{a}{\lambda + 2\mu}\) in (A.3) and set \(\tilde{\zeta} = \frac{3\hat{\rho}}{2}\) in (A.4). Since for all \(\zeta \geq \tilde{\zeta} = \frac{3\hat{\rho}}{2} > \hat{\rho} + 1\),
\[
g(\zeta) = -\frac{a\zeta}{2\mu + \lambda} (\zeta^2 - \hat{\rho}^2) \leq -\frac{a}{\lambda + 2\mu} = -N_1.
\]
Together with (3.106) and (3.119), by Lemma A.2, we have
\[
\sup_{t \in [\sigma(T), T]} \|\rho\|_{L^\infty} \leq \frac{3\hat{\rho}}{2} + C_12C_0^{2/3} \leq \frac{7\hat{\rho}}{4},
\]
provided \(C_0 \leq \hat{\varepsilon}_6 \triangleq \min\{\hat{\varepsilon}_6, \left(\frac{\hat{\rho}}{2C_12}\right)^{3/2}\}\). The combination of (3.118) with (3.120) completes the proof of Lemma 3.9.

4 A priori estimates (II): higher order estimates

In this section, we derive the time-dependent higher order estimates, which are necessary for the global existence of classical solutions. Here we adopt the method of the
article [6, 27, 29], and follow their work with a few modifications. We sketch it here for completeness. Let \((\rho, u, H)\) be a smooth solution of (1.1)-(1.4) satisfying Proposition 3.1 and the initial energy \(C_0 \leq \varepsilon_0\), and the positive constant \(C\) may depend on \(T, \mu, \lambda, \nu, a, \gamma, \rho, \bar{\rho}, s, \Omega, M_1, M_2, \|\nabla u_0\|_{H^1}, \|\nabla H_0\|_{H^1}, \|\rho_0 - \bar{\rho}\|_{W^{2, \sigma}}, \|g\|_{L^2}, \|P(\rho_0) - \bar{P}\|_{W^{2, \sigma}}\) for \(q \in (3, 6)\) where \(g \in L^2(\Omega)\) is given by compatibility condition (1.12).

**Lemma 4.1** There exists a positive constant \(C\), such that

\[
\sup_{0 \leq t \leq T} \left( \|\partial_t u\|^2_{L^2} + \|\nabla H\|^2_{L^2} \right) + \int_0^T \left( \|\rho \partial_t u\|^2_{L^2} + \|H_t\|^2_{L^2} + \|\nabla^2 H\|^2_{L^2} \right) dt \leq C, \tag{4.1}
\]

\[
\sup_{0 \leq t \leq T} \left( \|\rho \partial_t^2 u\|^2_{L^2} + \|H_t\|^2_{L^2} + \|\nabla^2 H\|^2_{L^2} \right) + \int_0^T \left( \|\partial_t u\|^2_{L^2} + \|\nabla H_t\|^2_{L^2} \right) dt \leq C, \tag{4.2}
\]

\[
\sup_{0 \leq t \leq T} \left( \|\nabla \rho\|_{L^6} + \|u\|_{L^2} \right) + \int_0^T \left( \|\nabla \partial_t u\|^2_{L^2} + \|\nabla^2 u\|^2_{L^2} \right) dt \leq C. \tag{4.3}
\]

**Proof.** First, taking \(\theta = 1\) in (3.15) and taking \(s = 1\) in (3.36) along with (2.25) gives (4.1). Then choosing \(m = 0\) in (3.39) and (3.50), integrating them over \((0, T)\), by (3.41), (4.1) and the compatibility condition (1.12), we have

\[
\sup_{0 \leq t \leq T} \left( \|\rho \partial_t^2 u\|^2_{L^2} + \|H_t\|^2_{L^2} + \|\nabla^2 H\|^2_{L^2} \right) + \int_0^T \left( \|\partial_t u\|^2_{L^2} + \|\nabla H_t\|^2_{L^2} \right) dt \leq C + \frac{1}{2} \sup_{0 \leq t \leq T} \left( \|\rho \partial_t^2 u\|^2_{L^2} + \|\nabla^2 H\|^2_{L^2} \right), \tag{4.4}
\]

where we have also used Lemma A.1, Lemma 2.2, (3.27) and (3.54), then we deduce (4.2) from (4.4). Based on the Beale-Kato-Majda type inequality (see Lemma A.6), we can derive (4.3), the estimates on the gradient of density and velocity, in arguments similar to [6]. This finishes the proof. \(\square\)

**Lemma 4.2** There exists a positive constant \(C\) such that

\[
\sup_{0 \leq t \leq T} \|\rho \partial_t^2 u_t\|^2_{L^2} + \int_0^T \int |\nabla u_t|^2 dx dt \leq C, \tag{4.5}
\]

\[
\sup_{0 \leq t \leq T} \left( \|\partial \rho\|_{H^2} + \|P - \bar{P}\|_{H^2} + \|\rho_t\|_{H^1} + \|P_t\|_{H^1} \right) + \int_0^T \left( \|\rho_{tt}\|^2_{L^2} + \|P_{tt}\|^2_{L^2} \right) dt \leq C, \tag{4.6}
\]

\[
\sup_{0 \leq t \leq T} \|\nabla u_t\|^2_{L^2} + \|\nabla H_t\|^2_{L^2} + \int_0^T \|\rho \partial_t^2 u_t\|^2_{L^2} + \|H_{tt}\|^2_{L^2} \right) dt \leq C. \tag{4.7}
\]

**Proof.** Based on Lemma 4.1, (4.5)-(4.6) can be obtained by the same method as that in [6]. It remains to prove (4.7). Introducing the function

\[
K(t) = (\lambda + 2\mu) \int (\text{div} u_t)^2 dx + \mu \int |\omega_t|^2 dx + \nu \int |\text{curl} H_t|^2 dx.
\]

Since \(u_t \cdot n = 0, H_t \cdot n = 0\) on \(\partial \Omega\), by Lemma A.4, we have

\[
\|\nabla u_t\|^2_{L^2} + \|\nabla H_t\|^2_{L^2} \leq C(\Omega)K(t). \tag{4.8}
\]
Differentiating $(1.1)_{2,3}$ with respect to $t,$

\[
\rho_{tt} - (\lambda + 2\mu) \nabla \text{div} u_t + \mu \nabla \times \omega_t = -\nabla P_t - \rho_t u_t - (\rho u \cdot \nabla) u_t + (H \cdot \nabla H - \nabla |H|^2/2)_t, \tag{4.9}
\]

and

\[
H_{tt} - \nu \nabla \times \text{curl} H_t = (H \cdot \nabla u - u \cdot \nabla H - H \text{div} u)_t, \tag{4.10}
\]

then multiplying $(4.9)$ by $2u_{tt}$, multiplying $(4.10)$ $2H_{tt}$ respectively, we obtain

\[
\frac{d}{dt} K(t) + 2 \int (\rho |u_{tt}|^2 + |H_{tt}|^2) dx = \frac{d}{dt} \left( \int (\rho |u_{tt}|^2 - |H_{tt}|^2) dx \right) + \int \rho u_{tt} \cdot \nabla u_t \cdot u_t dx - 2 \int \rho_t u \cdot \nabla u \cdot u_t dx + 2 \int \rho_t u_t^2 dx + 2 \int_\Omega (H \cdot \nabla H - u \cdot \nabla H - H \text{div} u)_t \cdot H_{tt} dx - 2 \int_\Omega P_t \text{div} u_t dx \tag{4.11}
\]

Let us estimate $K_i, i = 0, 1, \cdots, 6.$ We conclude from $(1.1)_1, (2.34), (4.2), (4.3), (4.5), (4.6), (4.8)$ and Sobolev’s, Poincaré’s inequalities that

\[
K_0 \leq \left| \int \text{div}(\rho u) |u_{tt}|^2 dx \right| + C \|\rho_t\|_{L^1} \|u\|_{L^\infty} \|\nabla u\|_{L^2} \|u_t\|_{L^6} + C \|P_t\|_{L^2} \|\nabla u_t\|_{L^2} + \frac{1}{2} K(t) + C, \tag{4.12}
\]

\[
K_1 \leq \left| \int \rho_t |u_t|^2 dx \right| = \left| \int \text{div}(\rho u) |u_t|^2 dx \right| = 2 \left| \int (\rho u + \rho u) \cdot \nabla u_t \cdot u_t dx \right| \\ \leq C \|\nabla u_t\|_{L^2}^2 K(t) + C \|\nabla u_t\|_{L^2}^2 + C, \tag{4.13}
\]

\[
K_2 + K_3 + K_4 \leq C \|\rho_{tt}\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2 + \|\rho^{1/2} u_{tt}\|_{L^2}^2 + C \|P_t\|_{L^2}^2 + C, \tag{4.14}
\]

\[
K_5 \leq \frac{1}{2} \|H_{tt}\|_{L^2}^2 + C \|H_t\|_{L^2}^2 K(t) + C \left( \|\nabla H_t\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2 \right), \tag{4.15}
\]

\[
K_6 \leq \frac{1}{2} \|H_{tt}\|_{L^2}^2 + C \left( \|\nabla H_t\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2 \right). \tag{4.16}
\]

Consequently, multiplying $(4.11)$ by $\sigma$, together with $(4.13)-(4.16)$, we get

\[
\frac{d}{dt} (\sigma K(t) - \sigma K_0) + \sigma \int (\rho |u_{tt}|^2 + |H_{tt}|^2) dx \leq C (1 + \|\nabla u_t\|_{L^2}^2) \sigma K(t) + C (1 + \|\nabla u_{tt}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 + \|\rho_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2), \tag{4.17}
\]
By Gronwall’s inequality, (4.2), (4.5), (4.6) and (4.12), we derive that
\[
\sup_{0 \leq t \leq T} (\sigma K(t)) + \int_0^T \sigma (\|\rho^{\frac{1}{2}} u_{tt}\|^2_{L^2} + \|H_{tt}\|^2_{L^2}) dt \leq C. \tag{4.18}
\]

As a result, by (4.8), we get (4.7). This finishes the proof. \(\square\)

**Lemma 4.3** There exists a positive constant \(C\) so that for any \(q \in (3, 6)\),
\[
\sup_{t \in [0,T]} (\|\rho - \bar{\rho}\|_{W^{2,q}} + \|P - \bar{P}\|_{W^{2,q}}) \leq C, \tag{4.19}
\]
\[
\sup_{t \in [0,T]} \sigma (\|\nabla u\|_{H^2}^2 + \|\nabla H\|_{H^2}^2)
\]
\[
+ \int_0^T (\|\nabla u\|_{H^2}^2 + \|\nabla H\|_{H^2}^2 + \|\nabla^2 u\|_{W^{1,q}}^2 + \sigma \|\nabla u_t\|_{H^1}^2) dt \leq C, \tag{4.20}
\]
where \(p_0 = \frac{b_0 - 6}{12b_0} \in (1, \frac{7}{6})\).

**Proof.** Let’s start with (4.20). By Lemma 4.1 and Poincaré’s, Sobolev’s inequalities, one can check that
\[
\|\nabla (\rho u_t)\|_{L^2} \leq \|\nabla \rho\|_{L^2} \|u_t\|_{L^2} + \|\rho \nabla u_t\|_{L^2} + \|\nabla \rho\|_{L^2} \|u\|_{L^2} + \|\rho \nabla u\|_{L^2} + \|\rho u\|_{L^2} \|\nabla^2 u\|_{L^2} 
\]
\[
\leq C + C \|\nabla u_t\|_{L^2}. \tag{4.21}
\]
Consequently, together with (4.6) and Lemma 4.1, it yields
\[
\|\nabla^2 u\|_{H^2} \leq C (\|\rho u_t\|_{H^1} + \|H \cdot \nabla H\|_{H^1} + \|P - \bar{P}\|_{H^2} + \|H\|_{H^2}^2 + \|u\|_{L^2})
\]
\[
\leq C + C \|\nabla u_t\|_{L^2}. \tag{4.22}
\]
It then follows from (4.22), (4.3), (4.5) and (4.7) that
\[
\sup_{0 \leq t \leq T} \sigma \|\nabla u\|_{H^2}^2 + \int_0^T \|\nabla u\|_{H^2}^2 dt \leq C. \tag{4.23}
\]
Next, from (2.2)_3, (2.26), it follows
\[
\|\nabla^2 H\|_{H^1} \leq C (\|H_t\|_{H^1} + \|u \cdot \nabla H\|_{H^1} + \|H \cdot \nabla u\|_{H^1} + \|H \cdot \text{div} u\|_{H^1} + \|\nabla H\|_{L^2})
\]
\[
\leq C + C \|\nabla H_t\|_{L^2}. \tag{4.24}
\]
Similarly, from (4.21), (4.1) and (4.3), we obtain
\[
\sup_{0 \leq t \leq T} \sigma \|\nabla H\|_{H^2}^2 + \int_0^T \|\nabla H\|_{H^2}^2 dt \leq C. \tag{4.25}
\]
Next, we deduce from Lemma 4.1 and (4.6) that
\[
\|\nabla^2 u_{tt}\|_{L^2} \leq C (\|\rho u_{tt}\|_{L^2} + \|\nabla P_t\|_{L^2} + \|((\nabla \times H) \times H)_t\|_{L^2} + \|u_t\|_{L^2})
\]
\[
\leq C \|\rho^{\frac{1}{2}} u_{tt}\|_{L^2} + C \|\nabla u_t\|_{L^2} + C \|\nabla H_t\|_{L^2} + C, \tag{4.26}
\]
where in the first inequality, we have utilized the \(L^p\)-estimate for the following elliptic system
\[
\begin{cases}
\mu \Delta u_t + (\lambda + \mu) \nabla \text{div} u_t = (\rho u_t)_t + \nabla P_t + ((\nabla \times H) \times H)_t & \text{in } \Omega, \\
u_t \cdot n = 0 \text{ and } \omega_t \times n = 0 & \text{on } \partial \Omega.
\end{cases} \tag{4.27}
\]
Together with (4.26) and (4.7) yields
\[
\int_0^T \sigma \| \nabla u_t \|^2_{H^1} dt \leq C. \tag{4.28}
\]

By Sobolev’s inequality, (2.34), (4.3), (4.6) and (4.7), we get for any \(q \in (3,6)\),
\[
\| \nabla (\rho \dot{u}) \|_{L^q} \leq C \| \nabla \rho \|_{L^q} (\| \nabla \dot{u} \|_{L^2} + \| \nabla u \|^2_{L^2}) + C \| \nabla \dot{u} \|_{L^q} \leq C \sigma^{-\frac{1}{2}} + C \| \nabla u \|_{H^2} + C \sigma^{-\frac{1}{2}} \left( \frac{\| \nabla u_t \|^2_{H^1}}{\| \nabla u \|_{H^2}} \right) + C. \tag{4.29}
\]

Integrating this inequality over \([0, T]\), by (4.2) and (4.28), we have
\[
\int_0^T \| \nabla (\rho \dot{u}) \|_{L^q}^{p_0} dt \leq C. \tag{4.30}
\]

On the other hand, (4.6) gives
\[
(\| \nabla^2 P \|_{L^q})_t \leq C \| \nabla u \|_{L^\infty} (\| \nabla^2 P \|_{L^q} + \| \nabla^2 u \|_{W^{1,q}} + \| \nabla (\rho \dot{u}) \|_{L^q} + \| \nabla \dot{u} \|_{L^q} + \| \nabla u_t \|_{L^2} + \| \nabla \dot{u} \|_{L^q}), \tag{4.31}
\]
where in the last inequality we have used the following simple fact that
\[
\| \nabla^2 u \|_{W^{1,q}} \leq C (1 + \| \nabla u_l \|_{L^2} + \| \nabla (\rho \dot{u}) \|_{L^q} + \| \nabla \dot{u} \|_{L^q}), \tag{4.32}
\]
due to (2.27), (2.28), (4.2) and (4.6).

Hence, applying Gronwall’s inequality in (4.31), we deduce from (4.3), (4.5) and (4.30) that
\[
\sup_{t \in [0,T]} \| \nabla^2 P \|_{L^q} \leq C, \tag{4.33}
\]
which along with (4.5), (4.6), (4.32) and (4.30) also gives
\[
\sup_{t \in [0,T]} \| P - \bar{P} \|_{W^{2,q}} + \int_0^T \| \nabla^2 u \|_{W^{1,q}}^{p_0} dt \leq C. \tag{4.34}
\]

Similarly, one has
\[
\sup_{0 \leq t \leq T} \| \rho - \bar{\rho} \|_{W^{2,q}} \leq C, \tag{4.35}
\]
which together with (4.34) gives (4.19). The proof of Lemma 4.3 is finished. \( \square \)

**Lemma 4.4** There exists a positive constant \(C\) such that
\[
\sup_{0 \leq t \leq T} \sigma \left( \| \rho \|^1_2 u_{tt} \|_{L^2} + \| H_{tt} \|_{L^2} + \| \nabla u_t \|_{H^1} + \| \nabla H_t \|_{H^1} + \| \nabla^2 H \|_{H^2} + \| \nabla u \|_{W^{2,q}} \right)
\]
\[
+ \int_0^T \sigma (\| \nabla u_t \|_{L^2}^2 + \| \nabla H_t \|_{L^2}^2) dt \leq C, \tag{4.36}
\]
for any \(q \in (3,6)\).
Proof. Differentiating (1.1)$_{2,3}$ with respect to $t$ twice, multiplying them by $2u_{tt}$ and $2H_{tt}$ respectively, and integrating over $\Omega$ lead to

$$
\frac{d}{dt} \int (\rho |u_{tt}|^2 + |H_{tt}|^2) dx \\
+ 2(\lambda + 2\mu) \int (\text{div}u)^2 dx + 2\mu \int |\omega_{tt}|^2 dx + 2\nu \int |\text{curl}H_{tt}|^2 dx \\
= -8 \int (\rho u_t \cdot \nabla u_t) dx - 2 \int (\rho u_t \cdot [\nabla (u_t \cdot u_t) + 2\nabla u_t \cdot u_t]) dx \\
- 2 \int (\rho u_t + 2\rho u_t) \cdot \nabla u \cdot u_{tt} dx - 2 \int (\rho u_{tt} \cdot \nabla u \cdot u_{tt} - P_{tt} \text{div}u_{tt}) dx \\
- 2 \int (H \cdot \nabla H - \nabla |H|^2/2)_{tt} u_{tt} dx + 2 \int (H \cdot \nabla u - u \cdot \nabla H - H \text{div}u_{tt}) H_{tt} dx
$$

\( \Delta \sum_{i=1}^{6} R_i. \) \hspace{1cm} (4.37)

Let us estimate $R_i$ for $i = 1, \cdots, 6$. Hölder’s inequality and (4.3) give

$$R_1 \leq C\|\sqrt{\rho} u_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2}\|u\|_{L^\infty} \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C(\delta) \|\sqrt{\rho} u_{tt}\|_{L^2}^2. \hspace{1cm} (4.38)$$

By (4.2), (4.5), (4.6) and (4.7), we conclude that

$$R_2 \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C(\delta) \|\nabla u_t\|_{L^2}^3 + C(\delta) \|\nabla u_t\|_{L^2}^2, \hspace{1cm} (4.39)$$

$$R_3 \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C(\delta) \|\nabla u_t\|_{L^2}^2 + C(\delta) \|\nabla u_t\|_{L^2}^2, \hspace{1cm} (4.40)$$

$$R_4 \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C(\delta) \|\sqrt{\rho} u_{tt}\|_{L^2}^2 + C(\delta) \|P_{tt}\|_{L^2}^2, \hspace{1cm} (4.41)$$

$$R_5 \leq \delta \|\nabla u_{tt}\|_{L^2}^2 + C(\delta) \|H_{tt}\|_{L^2}^2 + C(\delta) \|\nabla H_t\|_{L^2}^3, \hspace{1cm} (4.42)$$

$$R_6 \leq \delta (\|\nabla u_{tt}\|_{L^2}^2 + \|\nabla H_{tt}\|_{L^2}^2) + C(\delta) \|H_{tt}\|_{L^2}^3 \hspace{1cm} (4.43)$$

Substituting these estimates of $R_i (i = 1, \cdots, 6)$ into (4.37), utilizing the fact that

$$\|\nabla u_{tt}\|_{L^2} \leq C(\|\text{div}u_{tt}\|_{L^2} + \|\omega_{tt}\|_{L^2}), \hspace{1cm} \|\nabla H_{tt}\|_{L^2} \leq C\|\text{curl}H_{tt}\|_{L^2}, \hspace{1cm} (4.44)$$

due to Lemma A.4 since $u_{tt} \cdot n = 0, H_{tt} \cdot n = 0$, on $\partial \Omega$, and then choosing $\delta$ small enough, we can get

$$\frac{d}{dt} (\|\sqrt{\rho} u_{tt}\|_{L^2}^2 + \|H_{tt}\|_{L^2}^2) + \|\nabla u_{tt}\|_{L^2}^2 + \|\nabla H_{tt}\|_{L^2}^2 \leq C(\|\sqrt{\rho} u_{tt}\|_{L^2}^2 + \|H_{tt}\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2 + \|\nabla H_{tt}\|_{L^2}^2) \hspace{1cm} (4.45)$$

which together with (4.6), (4.7), and by Gronwall’s inequality yields that

$$\sup_{0 \leq t \leq T} \sigma^2 (\|\sqrt{\rho} u_{tt}\|_{L^2}^2 + \|H_{tt}\|_{L^2}^2) + \int_0^T \sigma^2 (\|\nabla u_{tt}\|_{L^2}^2 + \|\nabla H_{tt}\|_{L^2}^2) dt \leq C \int_0^T \sigma (\|\sqrt{\rho} u_{tt}\|_{L^2}^2 + \|H_{tt}\|_{L^2}^2) dt + \int_0^T \sigma^2 (\|\rho_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2) dt \hspace{1cm} (4.46)$$

$$+ \int_0^T \sigma^2 (\|\nabla u_t\|_{L^2}^3 + \|\nabla H_t\|_{L^2}^3 + \|\nabla u_{tt}\|_{L^2}^2 \|\nabla H_t\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2 \|\nabla H_t\|_{L^2}^2) dt \hspace{1cm} (4.47)$$

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Furthermore, it follows from (2.26), (2.28), (4.26) and (4.7) that
\[
\sup_{0 \leq t \leq T} (\sigma \|\nabla^2 u_t\|_{L^2} + \sigma \|\nabla^2 H_t\|_{L^2}) 
\leq C\sigma (1 + \|\rho^{1/2} u_t\|_{L^2} + \|H_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\nabla H_t\|_{L^2}) \leq C. \tag{4.46}
\]
Finally, we deduce from (4.7), (4.19), (4.20), (4.24), (4.29), (4.32), (4.46) and (4.47) that
\[
\sigma \|\nabla^2 u\|_{W^{1,q}} \leq C\sigma (1 + \|\nabla u_t\|_{L^2} + \|\nabla H_t\|_{L^2} + \|\nabla (\rho u)\|_{L^q} + \|\nabla^2 P\|_{L^2}) 
\leq C(1 + \sigma \|\nabla u\|_{H^2} + \sigma^{5/2} (\sigma \|\nabla u_t\|_{H^1}^2)^{\frac{2(q-2)}{4q}}) \leq C + C\sigma^{5/2} (\sigma^{-1})^{\frac{2(q-2)}{4q}} \leq C, \tag{4.48}
\]
and
\[
\sigma \|\nabla^2 H\|_{H^2} \leq C\sigma (1 + \|\nabla H_t\|_{H^1} + \|\nabla^2 u\|_{H^2} \|\nabla H\|_{H^2}) \leq C, \tag{4.49}
\]
Together with (4.46) and (4.47) yields (4.36) and this completes the proof of Lemma 4.4. \hfill \Box

5 Proof of Theorem 1.1

With all the a priori estimates in Section 3 and Section 4 at hand, we are going to prove the main result of the paper in this section.

Proof of Theorem 1.1. By Lemma 2.5, there exists a $T_0 > 0$ such that the system (1.1)-(1.4) has a unique classical solution $(\rho, u, H)$ on $\Omega \times (0, T_0]$. One may use the a priori estimates, Proposition 3.1 and Lemmas 4.2-4.4 to extend the classical solution $(\rho, u, H)$ globally in time.

First, by the definition of (3.1)-(3.5), the assumption of the initial data (1.10) and (3.95), one immediately checks that
\[
0 \leq \rho_0 \leq \bar{\rho}, \ A_0(0) + A_2(0) = 0, \ A_3(0) \leq C_0^{\delta_0}, \ A_4(0) + A_5(0) \leq C_0^{\delta_0}. \tag{5.1}
\]
Therefore, there exists a $T_1 \in (0, T_\ast]$ such that
\[
\begin{cases}
0 \leq \rho_0 \leq 2\bar{\rho}, \ A_1(T) + A_2(T) \leq 2C_0^{1/4}, \\
A_3(T) \leq 2C_0^{\delta_0}, \ A_4(\sigma(T)) + A_5(\sigma(T)) \leq 2C_0^{\delta_0},
\end{cases} \tag{5.2}
\]
hold for $T = T_1$. Next, we set
\[
T^\ast = \sup\{T \mid (5.2) \text{ holds}\}. \tag{5.3}
\]
Then $T^\ast \geq T_1 > 0$. Hence, for any $0 < \tau < T \leq T^\ast$ with $T$ finite, it follows from Lemmas 4.1-4.4 that
\[
\begin{cases}
\rho - \bar{\rho} \in C([0, T]; H^2 \cap W^{2,q}), \\
u, H) \in C([\tau, T]; H^2), \quad (\nabla u_t, \nabla H_t) \in C([\tau, T]; L^q);
\end{cases} \tag{5.4}
\]
where one has taken advantage of the standard embedding
\[
L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C(\sigma, T; L^q), \quad \text{for any } q \in [2, 6].
\]
Due to (4.5), (4.7), (4.36) and (1.1), we obtain

\[
\int_\tau^T \left( \left( \int \rho |u_t|^2 \, dx \right)_t \right) \, dt \leq \int_\tau^T \left( \|\rho_t u_t\|_{L^3}^2 + 2\|\rho u_t \cdot u_{tt}\|_{L^1} \right) \, dt
\]

\[
\leq C \int_\tau^T \left( \|\rho^{\frac{1}{2}} u_t\|_{L^2}^2 \|\nabla u\|_{L^\infty} + \|u\|_{L^6} \|\nabla \rho\|_{L^2} \|u_t\|_{L^6} + \|\rho^{\frac{1}{2}} u_{tt}\|_{L^2} \right) \, dt \leq C,
\]

which together with (5.4) yields

\[
\rho^{1/2} u_t, \quad \rho^{1/2} \dot{u} \in C([\tau, T]; L^2).
\]

Finally, we claim that

\[
T^* = \infty.
\]

Otherwise, \( T^* < \infty \). Then by Proposition 3.1, it holds that

\[
\begin{cases}
0 \leq \rho \leq \frac{2}{T^*} \hat{\rho}, \quad A_1(T^*) + A_2(T^*) \leq C_0^0, \\
A_3(T^*) \leq C_1^0, \quad A_4(\sigma(T^*)) + A_5(\sigma(T^*)) \leq C_0^0,
\end{cases}
\]

(5.7)

It follows from Lemmas 4.3, 4.4 and (5.5) that \((\rho(x, T^*), u(x, T^*), H(x, T^*))\) satisfies the initial data condition (1.9)-(1.10), (1.12), where \(g(x) \triangleq \sqrt{\rho} \hat{u}(x, T^*), \ x \in \Omega\). Thus, Lemma 2.5 implies that there exists some \( T^{**} > T^* \) such that (5.2) holds for \( T = T^{**} \), which contradicts the definition of \( T^* \). As a result, (5.6) holds. By Lemmas 2.5 and 4.1-4.4, it indicates that \((\rho, u, H)\) is in fact the unique classical solution defined on \( \Omega \times (0, T] \) for any \( 0 < T < T^* = \infty \).

Finally, with (2.3), (2.6), (3.17), (2.23), (2.30), (3.18) and (3.31) at hand, (1.15) can be obtained in similar arguments as used in [6], and we omit the details. The proof of Theorem 1.1 is finished.

A Some basic theories and lemmas

In this appendix, we review some elementary inequalities and important lemmas that are used extensively in this paper.

First, we recall the well-known Gagliardo-Nirenberg inequality (see [34]).

Lemma A.1 (Gagliardo-Nirenberg) Assume that \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^3 \). For \( p \in [2, 6], q \in (1, \infty), \) and \( r \in (3, \infty) \), there exist two generic constants \( C_1, C_2 > 0 \) which may depend on \( p, q \) and \( r \) such that for any \( f \in H^1(\Omega) \) and \( g \in L^q(\Omega) \cap D^{1,r}(\Omega), \)

\[
\|f\|_{L^p(\Omega)} \leq C_1 \|f\|_{L^2}^{\frac{6-p}{2}} \|\nabla f\|_{L^2}^{\frac{3p-6}{2}} + C_2 \|f\|_{L^2},
\]

(A.1)

\[
\|g\|_{C(\overline{\Omega})} \leq C_1 \|g\|_{L^q(\Omega)}^{q(3r-3)/(3r+q(r-3))} \|\nabla g\|_{L^r}^{3r/(3r+q(r-3))} + C_2 \|g\|_{L^2}.
\]

(A.2)

Moreover, if \( f \cdot n|_{\partial \Omega} = 0 \) and \( g \cdot n|_{\partial \Omega} = 0 \), then the constant \( C_2 = 0 \).

In order to get the uniform (in time) upper bound of the density \( \rho \), we need the following Zlotnik inequality in [48].
Lemma A.2 Suppose the function $y$ satisfy
\[ y'(t) = g(y) + b'(t) \text{ on } [0, T], \quad y(0) = y^0, \]
with $g \in C(R)$ and $y, b \in W^{1,1}(0, T)$. If $g(\infty) = -\infty$ and
\[ b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1) \quad (A.3) \]
for all $0 \leq t_1 < t_2 \leq T$ with some $N_0 \geq 0$ and $N_1 \geq 0$, then
\[ y(t) \leq \max \{y^0, \zeta\} + N_0 < \infty \text{ on } [0, T], \]
where $\zeta$ is a constant such that
\[ g(\zeta) \leq -N_1 \text{ for } \zeta \geq \zeta. \quad (A.4) \]

Consider the Lamé's system
\[
\begin{aligned}
-\mu \Delta u - (\lambda + \mu) \nabla \text{div} u &= f \text{ in } \Omega, \\
u \cdot n &= 0 \text{ and } \text{curl} u \times n &= 0 \text{ on } \partial \Omega,
\end{aligned}
\]
(A.5)

Then, the following estimate is standard (see [1]).

Lemma A.3 For the Lamé's equation (A.5), one has

(1) If $f \in W^{k,q}$ for some $q \in (1, \infty)$, $k \geq 0$, then there exists a unique solution $u \in W^{k+2,q}$, such that
\[ \|u\|_{W^{k+2,q}} \leq C(\|f\|_{W^{k,q}} + \|u\|_{L^q}); \]

(2) If $f = \nabla g$ and $g \in W^{k,q}$ for some $q \geq 1$, $k \geq 0$, then there exists a unique weak solution $u \in W^{k+1,q}$, such that
\[ \|u\|_{W^{k+1,q}} \leq C(\|g\|_{W^{k,q}} + \|u\|_{L^q}). \]

The following two lemmas are given in Theorem 3.2 in [38] and Propositions 2.6-2.9 in [2].

Lemma A.4 Let $k \geq 0$ be an integer, $1 < q < +\infty$, and assume that $\Omega$ is a simply connected bounded domain in $\mathbb{R}^3$ with $C^{k+1,1}$ boundary $\partial \Omega$. Then for $v \in W^{k+1,q}$ with $v \cdot n = 0$ on $\partial \Omega$, it holds that
\[ \|v\|_{W^{k+1,q}} \leq C(\|\text{div} v\|_{W^{k,q}} + \|\text{curl} v\|_{W^{k,q}}). \]
In particular, for $k = 0$, we have
\[ \|\nabla v\|_{L^q} \leq C(\|\text{div} v\|_{L^q} + \|\text{curl} v\|_{L^q}). \]

Lemma A.5 Let $k \geq 0$ be a integer, $1 < q < +\infty$. Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^3$ and its $C^{k+1,1}$ boundary $\partial \Omega$ only has a finite number of 2-dimensional connected components. Then for $v \in W^{k+1,q}$ with $v \times n = 0$ on $\partial \Omega$, we have
\[ \|v\|_{W^{k+1,q}} \leq C(\|\text{div} v\|_{W^{k,q}} + \|\text{curl} v\|_{W^{k,q}} + \|v\|_{L^q}). \]
In particular, if $\Omega$ has no holes, then
\[ \|v\|_{W^{k+1,q}} \leq C(\|\text{div} v\|_{W^{k,q}} + \|\text{curl} v\|_{W^{k,q}}). \]
Next, similar to [3, 22, 23], we need a Beale-Kato-Majda type inequality with respect to the slip boundary condition (1.3) which is given in [6].

**Lemma A.6** For $3 < q < \infty$, assume that $u \cdot n = 0$ and $\text{curl}u \times n = 0$ on $\partial \Omega$, $u \in W^{2,q}$, then there is a constant $C = C(q, \Omega)$ such that the following estimate holds

$$\|\nabla u\|_{L^\infty} \leq C (\|\text{div}u\|_{L^\infty} + \|\text{curl}u\|_{L^\infty}) \ln(e + \|\nabla^2 u\|_{L^q}) + C \|\nabla u\|_{L^2} + C.$$

Finally, we consider the problem

$$\begin{cases}
\text{div}v = f \text{ in } \Omega, \\
v = 0 \text{ on } \partial \Omega.
\end{cases} \quad (A.6)$$

One has the following conclusion (see [15], Theorem III.3.1).

**Lemma A.7** There exists a linear operator operator $B = [B_1, B_2, B_3]$ enjoying the properties:

1) $B : \{ f \in L^p(\Omega) \vert \int_\Omega f \, dx = 0 \} \mapsto (W^{1,p}_0(\Omega))^3$ is a bounded linear operator, that is,

$$\|B[f]\|_{W^{1,p}_0(\Omega)} \leq C(p) \|f\|_{L^p(\Omega)}, \text{ for any } p \in (1, \infty), \quad (A.7)$$

2) The function $v = B[f]$ solve the problem (A.6).

3) if $f$ can be written in the form $f = \text{div}g$ for a certain $g \in L^r(\Omega), g \cdot n|_{\partial \Omega} = 0$, then

$$\|B[f]\|_{L^r(\Omega)} \leq C(r) \|g\|_{L^r(\Omega)}, \text{ for any } r \in (1, \infty). \quad (A.8)$$

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