Number Parity effect in the normal state of $\text{SrTiO}_3$

Xing Yang
School of Physics and Electronics, Hunan University, Hunan, China, 410082
Physics Department, University of Notre Dame, Notre Dame, Indiana, USA, 46565

Quanhui Liu
School of Physics and Electronics, Hunan University, Hunan, China, 410082

Boldizsár Jankó
Physics Department, University of Notre Dame, Notre Dame, Indiana, USA, 46565

(Dated: March 18, 2020)

We study the recently discovered even-odd effect in the normal state of single-electron devices manufactured at strontium titanium oxide/lanthanum aluminum oxide interfaces (STO/LAO). Within the framework of the number parity-projected formalism and a phenomenological fermion-boson model we find that, in sharp contrast to conventional superconductors, the crossover temperature $T^*$ for the onset of number parity effect is considerably larger than the superconducting transition temperature $T_c$ due to the existence of the superconducting gap above $T_c$. Furthermore, the finite lifetime of the preformed pairs reduces by several orders of magnitude the effective number of states $N_{\text{eff}}$ available for the unpaired quasiparticle in the odd parity state of the Coulomb blockaded STO/LAO island. Our findings are in qualitative agreement with the experimental results reported by Levy and coworkers for STO/LAO based single electron devices.

PACS numbers: 73.23.Hk, 73.40.-c, 73.63.-b, 74.25.-q, 74.78.-w

I. INTRODUCTION

Number parity effect in superconductors were expected as soon as the Bardeen-Cooper-Schrieffer (BCS) microscopic model was developed [1]. Indeed, the BCS ground state corresponds to a coherent superposition of pair states in which the number of particles has even parity and the total number $N$ is not fixed. Under these circumstances, the charge displacement operator $\exp(i\phi)$ (canonically conjugate with the number operator $N$) has a fixed expectation value, which leads to the common notion that a macroscopic BCS superconductor has a complex order parameter $\Delta$ with a rigid phase $\phi$. As soon as the BCS state is projected onto fixed $N$ [2], it becomes clear that one has to differentiate between two cases: (a) if the total number $N = 2n$ is even, all particles can participate in pair states and the ground state resembles the usual grand canonical BCS ground state; (b) if $N = 2n + 1$ is odd, the ground state will inevitably contain not only pairs, but also an unpaired electron (more precisely: the ground state will contain a Bogoliubov quasiparticle).

Intuitively, one would expect that the $N$ vs $N + 1$ (even/odd) difference in a superconductor or any kind of paired fermionic state must be experimentally observable only if $N$ is relatively small. Indeed, inspired by the success of the BCS theory, Bohr, Mottelson and Pines [3] were the first of many who studied pairing and even-odd effect in nuclear matter, with particle numbers around $N \sim 10^2$. It was therefore, even more surprising, when Mooij et al. [4], Tinkham and coworkers [5], as well as Devoret and his colleagues [6, 7] showed experimentally measurable difference between Coulomb blockaded microscopic superconducting islands that contain a billion, and a billion plus one electrons. As it turns out, the magnitude of $N$ was less important. Instead, the quality of the Coulomb blockade turned out to be crucial: the superconducting islands had to be isolated from their environment with ultrasmall tunnel junctions and highly resistive electromagnetic environment, in order to ensure that $N$ is a fixed, good quantum number.

The pioneering experiments on number parity effect in conventional superconductors were performed on single-electron (SET) devices consisting of lithographically patterned aluminum islands [8]. Even-odd effect emerged below a crossover temperature $T^*$ that was always much lower than the superconducting transition temperature: $T^* < T_c$. Rather than being directly correlated with $T_c$, $T^*$ is set by the experimentally measurable even-odd free energy difference $\delta F_{c/o} \sim \Delta_0 - k_B T \log N_{\text{eff}}$. Here $\Delta_0$ is the low temperature energy gap, and $N_{\text{eff}}$ is the effective number of states [5–7, 9] available for the unpaired electron to explore in the odd number parity state of the superconducting island. Within this parity projected framework [8, 9] $T^*$ corresponds to the temperature at which $\delta F_{c/o}$ becomes negligibly small: $T^* \sim \Delta_0 / (k_B \log N_{\text{eff}})$. For typical device parameters in these early experiments, the crossover temperature was measured to be around $T^* \sim 10^2 \text{mK}$ for aluminum island with $T_c \sim 1 \text{K}$. Consequently, the effective number of states was typically around $N_{\text{eff}} \sim 10^3$.

The experiments by Levy and his coworkers [10] on SET devices constructed on STO/LAO provided experimental evidence for a spectacular departure from the conventional number parity effect described above. Levy and his colleagues detected $T^* \sim 900 \text{mK}$, much higher than the superconducting transition temperature $T_c \sim 300 \text{mK}$.
measured for these devices. Even-odd effect remained detectable well into the "normal" phase of the superconductor, and persisted in magnetic fields $B^* \sim 1 \sim 4T$, much higher than the upper critical field of the device. Furthermore, the extracted $N_{\text{eff}}$ is also drastically different: $N_{\text{eff}} \sim 2 - 3$.

A possible and relatively straightforward interpretation of novel experimental developments suggest that preformed pairs \cite{11,12} persist into the normal state of STO/LAO well above the superconducting transition temperature. Consequently, fundamental changes must be made to the theoretical description of the number parity effect in this novel preformed pair phase. This paper is devoted to the presentation a phenomenological theoretical framework aimed at providing a description of number parity effect in the normal phase of STO/LAO devices. Given the fact that the details of the microscopic mechanism behind the superconducting and preformed pair state of STO/LAO are not yet established, we use a phenomenological fermion-boson model \cite{13} that allows us to describe a normal phase where both pairs and unpaired particles are present. Furthermore, the model allows pairs to decay into unpaired particles, and particles to form pairs. This theoretical picture provides in a natural way a finite pair lifetime \cite{10} in the preformed pair state. We find, after performing the number parity projection developed earlier by Ambegaokar, Smith and one of us \cite{9}, that the finite pair lifetime has drastic effect on the magnitude of $N_{\text{eff}}$. In fact, as we will show in detail below, the theoretical framework we develop in this paper can reproduce not only $T^* \gg T_c$, but also $N_{\text{eff}} \sim O(1)$.

Generally speaking, the materials for making the single-electron devices can be separated into three categories (see Fig. 1): gapless materials, BCS and unconventional superconductors. Their density of states are shown respectively in Panel (a), (b) and (c), while the single electron transistors with ultra small islands and discrete energy spectrum are also investigated extensively \cite{14,15}. According to our calculations, the density of states at Fermi level should be vanishingly small in order to obtain a finite even/odd free energy difference. As a result, the single electron transistors made from BCS superconductors and unconventional superconductors (as shown in Panel (b) and (c)) are expected to show even-odd effect, and a superconducting gap above $T_c$ is necessary to cause $T^* \gg T_c$. The effective excitation number for the unpaired electrons in the odd parity states is highly dependent on the density of states at $E = \Delta$, since the smallest excitation energy is assumed to be $\Delta$. \cite{5–7}. In BCS superconductors (see Panel (b)), the density of states at $E = \Delta$ is known as the van Hove singularity, and this results in a large $N_{\text{eff}} \sim 10^4$. In the unconventional superconductors (see Panel (c)), the van Hove singularities is broadened by the decays and formations of the electron pairs, which results in a small $N_{\text{eff}} \sim O(1)$.

Recently many possible microscopic superconducting mechanisms of the electron system at the STO/LAO interface are proposed by different groups. Kedem, et al. \cite{16,17} related the mechanism to the ferroelectric mode and Arce-Gamboa and Guzmán-Verri \cite{18} discover the influence of strain force to the ferroelectric mode and obtain the phase diagram of superconducting transition temperature and cation substitutions. Ruhman and Lee \cite{19} suggest a plasmon-induced superconducting mechanism. On the contrary, Wölfl and Balastsky \cite{20} propose the transverse optical phonons may be the glue for electron pairing. While the theories can explain the origin of superconducting gap self-consistently, some facts in the

![Diagram](image-url)
experiments are neglected. Firstly, the superconducting gap should persist above \( T_c \). This is very important for even-odd effect to happen above \( T_c \) in the single electron transistors, and also, the experiment has proven the fact \[21\]. Next, the van Hove singularity in the density of states should be smoothened. This can lead to a small \( N_{\text{eff}} \) as shown in Levy’s experiments \[10\]. The detailed discussions are presented in the below sections.

This paper is organized in four sections. In Section II, two important physical quantities, the even/odd free energy difference and the effective excitation number for the unpaired electron in odd parity state, are related to the density of states with Dynes’ model, and the results are also plotted. The phenomenological boson fermion model is introduced, and the analytic form of its Green’s function is provided in Section III. The density of states and the physical quantities of the even-odd effect predicted by the model is calculated and plotted as shown in Levy’s experiments \[10\]. The detailed discussions are presented in the below sections. This paper is organized in four sections. In Section II, two important physical quantities, the even/odd free energy difference and the effective excitation number for the unpaired electron in odd parity state, are related to the density of states with Dynes’ model, and the results are also plotted. The phenomenological boson fermion model is introduced, and the analytic form of its Green’s function is provided in Section III. The density of states and the physical quantities of the even-odd effect predicted by the model is calculated and plotted as shown in Levy’s experiments \[10\]. The detailed discussions are presented in the below sections.

II. EVEN/ODD FREE ENERGY DIFFERENCE AND EFFECTIVE EXCITATION NUMBER FOR THE UNPAIRED ELECTRON

A. even/odd free energy difference

The electron system of the quantum dot at the STO/LAO interface is described by a general Hamiltonian \( \hat{H} = \hat{H}_0 + \hat{H}_1 \) where \( \hat{H}_0 \) is the kinetic energy of electrons and \( \hat{H}_1 \) is a small perturbation of Hamiltonian \( \hat{H}_0 \). Within the number parity projection formalism, the canonical partition function with even/odd number parity is :

\[
Z_{e/o} = \text{Tr} \left\{ \frac{1}{2} \left[ 1 + (-1)^N \right] e^{-\beta \hat{H}} \right\}
\]

where the symbol \( e/o \) corresponds to the even/odd parity, \( N \) is the electrons’ number operator, while \( \beta = \frac{1}{k_B T} \) where \( k_B \) is Boltzmann constant and \( T \) is the temperature.

From Eq. 1 the difference between the free energy of a system with an odd and even number of particles is

\[
F_o - F_e = \frac{1}{\beta} \ln \left[ 1 + \frac{\langle (-1)^N \rangle}{1 - \langle (-1)^N \rangle} \right],
\]

where \( \langle ... \rangle \equiv \text{tr}(e^{\beta \hat{H}} ...) \). The expectation value \( \langle (-1)^N \rangle \) is the order parameter that signals the presence or absence of even-odd effect. When \( \langle (-1)^N \rangle = 0 \), even-odd effect will not be observable. Let’s assume that the Hamiltonian can be expressed in a more compact form \( \hat{H} = \sum_{k, \sigma} c_k^\dagger \sigma \hat{c}_{k, \sigma} \). From the above, \( \langle (-1)^N \rangle \) is

\[
\langle (-1)^N \rangle = \prod_k \frac{\left( 1 - e^{\beta \epsilon_k} \right)^2}{\left( 1 + e^{\beta \epsilon_k} \right)^2} = \prod_k \tanh^2 \left( \frac{\beta \epsilon_k}{2} \right). \tag{3}
\]

If it is defined that \( A \) as \( e^A \equiv \langle (-1)^N \rangle \), then

\[
A = 2 \sum_k \ln \left| \tanh \frac{\beta \epsilon_k}{2} \right| = 2 \int_{-\infty}^{+\infty} D(E) \ln \left| \tanh \frac{\beta E}{2} \right| dE
\]

where \( D(E) \) is the density of states. Notice that the factor \( \ln \left| \tanh \frac{\beta E}{2} \right| \) in the integrand is divergent when \( E = 0 \). This suggests that an energy gap is necessary for a system to show even-odd effect. At the interface of STO/LAO, the even-odd effect appears above the superconducting transition temperature \( T_c \). This implies the existence of an energy gap above \( T_c \), and scanning tunneling spectroscopy experiments also proves that an energy gap persists above \( T_c \) at the interface. However, the experiments \[10\] at the interface of STO/LAO discover a very small \( N_{\text{eff}} \sim 1 \). This suggests the van Hove singularity needs to be smoothened in order to reduce \( N_{\text{eff}} \).

B. effective excitation number for the unpaired electron

With the assumption that the smallest excitation energy for electrons is \( \Delta \), we can calculate the effective excitation number for the unpaired electron with the formula \[5\[7\]

\[
N_{\text{eff}} = \int_{\Delta_0}^{\infty} D(E) \exp(-\beta(E - \Delta)) dE. \tag{5}
\]

In Eq. 5 the density of states at \( E = \Delta \) contribute most to the effective excitation number for the unpaired electron in the odd parity states. If it is assumed that the singularity exists, it can easily produce a large \( N_{\text{eff}} \sim 10^4 \) or more in BCS superconductors. However, the experiments \[10\] at the interface of STO/LAO discover a very small \( N_{\text{eff}} \sim 1 \). This suggests the van Hove singularity needs to be smoothened in order to reduce \( N_{\text{eff}} \).

C. even-odd effect with Dynes’ model

As we can see in the above discussion that the even-odd effect are related to the density of states of the small island in the single electron transistors. In order to reproduce the experimental results, at least, the van Hove singularity should be broadened. This just can be provided by the lifetime effects of electron pairs. As a result, the Dynes’ model is adopted in the calculations of the even/odd free energy difference \( \delta F_{e/o} \) and the effective excitation number for the unpaired electron \( N_{\text{eff}} \) and its density of states is
The superconducting gap of the 2D electron system at the interface of STO/LAO, Δ vanishes at \(T_s \sim 300\text{mK}\), and it turns into superconducting state at \(T_c \sim 190\text{mK}\) [21]. This suggests between \(T_c\) and \(T_s\), the superconducting phase is destroyed, but the superconducting gap are preserved, which defines the preformed pair state. The present model is devoted to studying the preformed pair state and superconducting state. Notice that the coherence length of pairs is \(\sim 70 - 100\text{nm}\) in (001)-STO/LAO and \(40 - 75\text{nm}\) for (011)-STO/LAO [22], which is very small compared with that in BCS theory. This proves self-consistently that the electron pairs can be approximated as a bosonic field. A bosonic field \(b_q\) is introduced with elementary charge unit \(-2e\). For a small momentum \(q\), the dispersion of the pairs is approximated as \(\xi_q = \xi_0 + \hbar v|q| - \mu_b\) [2], and the Hamiltonian for the bare bosonic field is:

\[
\hat{H}_{0p} = \sum_q \xi_q \hat{b}_q^\dagger \hat{b}_q
\]  

where \(\hat{b}_q^\dagger\) and \(\hat{b}_q\) are defined to commute with \(\hat{c}_q^{\dagger}_{k,\sigma}\) and \(\hat{c}_k^{\dagger}_{\sigma}\). The interaction Hamiltonian between the fermions and bosons is assumed to be:

\[
\hat{H}_1 = \sum_{k,q} \frac{V_1(q)}{\sqrt{n_0}} \hat{b}_q^\dagger \hat{c}^{\dagger}_{k+\frac{q}{2},\downarrow} \hat{c}_{k+\frac{q}{2},\uparrow} + H.c.
\]  

III. THE BOSON FERMION MODEL

In order to reproduce Dynes-like density of states, the phenomenological boson fermion model is introduced. For a single band model, electrons have an approximated Bogoliubov quasi-particle dispersion \(E_k = \sqrt{\xi_k^2 + \Delta^2}\), where \(\xi_k = \frac{\hbar^2 k^2}{2m} - \mu_f\). The Hamiltonian of the electrons can be written as:

\[
\hat{H}_{0e} = \sum_{k,\sigma} E_k \hat{c}_k^{\dagger}_{\sigma} \hat{c}_{k,\sigma}.
\]

The superconducting gap of the 2D electron system at the interface of STO/LAO, \(\Delta\) vanishes at \(T_s \sim 300\text{mK}\), and it turns into superconducting state at \(T_c \sim 190\text{mK}\) [21]. This suggests between \(T_c\) and \(T_s\), the superconducting phase is destroyed, but the superconducting gap are preserved, which defines the preformed pair state. The present model is devoted to studying the preformed pair state and superconducting state. Notice that the coherence length of pairs is \(\sim 70 - 100\text{nm}\) in (001)-STO/LAO and \(40 - 75\text{nm}\) for (011)-STO/LAO [22], which is very small compared with that in BCS theory. This proves self-consistently that the electron pairs can be approximated as a bosonic field. A bosonic field \(b_q\) is introduced with elementary charge unit \(-2e\). For a small momentum \(q\), the dispersion of the pairs is approximated as \(\xi_q = \xi_0 + \hbar v|q| - \mu_b\) [2], and the Hamiltonian for the bare bosonic field is:

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where \(\hat{b}_q^\dagger\) and \(\hat{b}_q\) are defined to commute with \(\hat{c}_q^{\dagger}_{k,\sigma}\) and \(\hat{c}_k^{\dagger}_{\sigma}\). The interaction Hamiltonian between the fermions and bosons is assumed to be:

\[
\hat{H}_1 = \sum_{k,q} \frac{V_1(q)}{\sqrt{n_0}} \hat{b}_q^\dagger \hat{c}^{\dagger}_{k+\frac{q}{2},\downarrow} \hat{c}_{k+\frac{q}{2},\uparrow} + H.c.
\]

with

\[
V_1 = V_c \sqrt{\frac{(\xi_q - E_{q-k})^2}{(\xi_q - E_{q-k})^2 + \Delta^2}}
\]

where \(V_c\) is the strength of the coupling, and \(n_0\) is the total number of quasiparticles in the quantum dot. \(n_0\) is around 500 [10]. The momentum dependent interaction kernel is introduced as an example to reproduce a Dynes-like density of states. Notice that the factor \(\xi_q - E_{q-k}\) in fact is equivalent to the frequency of the electron in Green’s function \(\omega\). From the above, the total Hamiltonian is

\[
\hat{H} = \hat{H}_0 + \hat{H}_1
\]

where

\[
H_\theta = H_\theta + \hat{H}_0
\]

The total particle number is defined as \(\hat{N} = \sum_{k,\sigma} \hat{c}_k^{\dagger}_{k,\sigma} \hat{c}_{k,\sigma} + 2 \sum_q \hat{b}_q^\dagger \hat{b}_q\), and it can be proved that \([\hat{N}, \hat{H}] = 0\). The first order approximation of the self-energy is

\[
\Sigma(k, \omega) = \frac{1}{\hbar^2} \int \frac{L|V_i(q)|^2 dq}{2\pi n_0} \frac{1}{\omega - \frac{(\xi_q - E_{q-k})}{\hbar} + i\eta} \left( \frac{1}{e^{\beta \xi_q} - 1} + \frac{1}{e^{\beta E_{q-k}} + 1} \right)
\]
where $L$ is the length of the quantum dot in the middle of the single electron transistor. Notice that the superconductivity of the STO/LAO system is considered to be one-dimensional \[10\], which is an important factor that makes the proposed theory to be one-dimensional either. The calculations of the self-energy are presented in Appendix A.

IV. EVEN-ODD EFFECT WITH BOSON-FERMION MODEL

With equation (A2, A3), the one-particle Green’s function can be written as:

$$G(k, \omega) = \frac{1}{\omega - \epsilon_k - \Sigma' - i\Gamma} \quad (14)$$

where $\epsilon_k = \sqrt{\Delta^2 + \frac{\hbar^2 (k^2 - k_f^2)}{2m^*}} \approx \sqrt{\Delta^2 + \frac{\hbar v_f (k - k_f)}{2m^*}}$, $\Sigma' = \Sigma'_k(\omega)|_{k=k_f}$, $\Gamma = \Gamma_k(\omega)|_{k=k_f}$ and numerical calculations show that $\Sigma' \ll \Delta$ at low temperature and is negligible.

The density of states is

$$D(\omega) = \int_{-\infty}^{\infty} LA(k, \omega) dk \quad (15)$$

where $A(k, \omega) = \frac{1}{\pi} \text{Im}(G(k, \omega))$ and $L \approx 500$nm is the length of the small island in the middle of the single electron transistor. Notice that only $\epsilon_k$ is dependent on the momentum $k$ in the spectral weight function $A(k, \omega)$, and it results in the mathematical expression of $D(\omega)$ with the residue theorem (see Appendix B).

As shown in Fig. 3, the decay and formation of the electron pairs can produces many gap states and broaden the van Hove singularity in the density of states. The effects can reduce the even/odd free energy difference $\delta F_{e/o}$, and the effective excitation number for the unpaired electron in the odd parity state $N_{\text{eff}}$, and this can be measured in experiments. In addition, zero superconducting gap makes a finite spectral function at Fermi level, and in that case, the density of states, $D(\omega)$, is finite at the Fermi level. This can be destructive to the even-odd effect. As shown in Fig. 3, the results on the even-odd effect calculated by the boson fermion model is very similar to that from Dynes’ model, since both of the models generate the similar densities of states.

V. CONCLUSION

In the present paper, we proposed three rules for the density of states summed from the fit to the even-odd effect experiments. One is the density of states at Fermi level should be zero below and above $T_c$. A superconducting gap is required for the electron system to demonstrate even-odd effect above $T_c$. Next, the gap states are necessary to reduce the even/odd free energy difference and weaken the even-odd effect. Finally, the van Hove singularity needs to be smoothened in order to obtain a small $N_{\text{eff}}$. These rules for the density of states are neglected by the previous theories. On the other hand, the theories are meaningful in explaining the microscopic origin of the superconducting gap $\Delta$ in the dispersion of Bogoliubov quasiparticles.

Moreover, the broadening of the van Hove singularity in the density of states is a fingerprint for the lifetime effects of electron pairs. The signals in the single electron transistor experiments are very sensitive for detecting the lifetime effects, for example, the existence of the superconducting gap, the gap states and the broadening of the van Hove singularity. Compared to the experimental condition of scanning tunneling spectroscopy, that of
the single electron devices can be relatively more easily satisfied in some electron systems. Furthermore, the decay and formation of electron pairs may widely exist in many kinds of superconductors, including BCS superconductors. The applications of the single electron transistor in the area are very promising.

Finally, the microscopic origin of phenomenological interaction potential is still unknown, while phonon-electron interactions, electron-electron interactions, etc. may participate in the mechanism. Further theoretical and experimental investigations are needed.

Acknowledgments

One of us (X. Y.) gratefully acknowledges the support from China Scholarship Council under Grant No. 201506130054. We thank Peter B. Littlewood, Anthony Ruth, Alexander Edelman and Xiaoyu Ma for helpful discussions. This work is financially supported by National Natural Science Foundation of China under Grant No. 11675051.

Appendix A: Calculations of the self-energy

The decay rate of quasiparticles is defined to be \( \Gamma = 3\text{Im}(\Sigma(k, \omega)) \). In order to facilitate the calculations, it is assumed that \( \Sigma(k, \omega) = \Sigma(k, \omega)|_{|k|=k_f} \). From Eq. (A3) and the above approximations, the decay rate of electron pairs is

\[
\Gamma(\omega) = \frac{1}{R^2} \int \frac{d^2q}{2n_0} \delta(\omega - \sqrt{\xi_0^2 - E_q^2}) \\
\times \frac{1}{e^{\beta E_q} - 1 + \frac{1}{e^{\beta E_q} + 1}}.
\]

Appendix B: Calculations of the Density of States

The denominator of the spectral weight function is \( \omega - \sqrt{\Delta^2 + (\hbar v_f(k - k_f))^2 - \Sigma')^2 + \Gamma^2} \). If the denominator equals to zero, there are four solutions of the momentum \( k \) that two solutions are in upper half-plane of the complex plane of \( k \) and two solutions are in the lower half. Furthermore, there are four different cases, if we set \( a = (\omega - \Sigma')^2 - \Delta^2, b = 2\Gamma(\omega - \Sigma') \)

\[
\begin{cases}
  a > 0, b > 0 & \text{...................i} \\
  a > 0, b < 0 & \text{...................ii} \\
  a < 0, b > 0 & \text{...................iii} \\
  a < 0, b < 0 & \text{...................iv.}
\end{cases}
\]

For case i, four solutions of the momentum \( k \) in the upper half-plane are

\[
\begin{align}
  k_1 &= k_f - \frac{1}{v_f} \sqrt{Re \frac{a}{\theta}} \\
  k_2 &= k_f + \frac{1}{v_f} \sqrt{Re \frac{a}{\theta}} \\
  k_3 &= k_f - \frac{1}{v_f} \sqrt{Re \frac{a}{\theta}} \\
  k_4 &= k_f + \frac{1}{v_f} \sqrt{Re \frac{a}{\theta}}.
\end{align}
\]

For case ii,

\[
\begin{align}
  k_1 &= k_f + \frac{1}{v_f} \sqrt{Re \frac{a}{\theta}} \\
  k_2 &= k_f - \frac{1}{v_f} \sqrt{Re \frac{a}{\theta}}.
\end{align}
\]

For case iii,

\[
\begin{align}
  k_1 &= k_f - \frac{1}{v_f} \sqrt{Re \frac{a}{\theta}} \\
  k_2 &= k_f + \frac{1}{v_f} \sqrt{Re \frac{a}{\theta}}.
\end{align}
\]

For case iv,

\[
\begin{align}
  k_1 &= k_f - \frac{1}{v_f} \sqrt{Re \frac{a}{\theta}} \\
  k_2 &= k_f + \frac{1}{v_f} \sqrt{Re \frac{a}{\theta}}.
\end{align}
\]

After applying Jordan’s lemma and the residue theorem, we obtain the density of states, for case i,

\[
D(\omega) = \frac{2L}{v_f} \frac{\omega - \Sigma'}{\sqrt{R}} \cos(\theta \frac{\omega}{2}) + \frac{\Gamma}{\sqrt{R}} \sin(\theta \frac{\omega}{2}).
\]

For case ii,

\[
D(\omega) = -\frac{2L}{v_f} \frac{\omega - \Sigma'}{\sqrt{R}} \cos(\theta \frac{\omega}{2}) + \frac{\Gamma}{\sqrt{R}} \sin(\theta \frac{\omega}{2}).
\]
For case iii,
\[ D(\omega) = \frac{2L}{v_f} \left( \frac{\omega - \Sigma'}{\sqrt{R}} \cos\left(\frac{\theta + \pi}{2}\right) + \frac{\Gamma}{\sqrt{R}} \sin\left(\frac{\theta + \pi}{2}\right) \right). \] (B11)

For case iv,
\[ D(\omega) = -\frac{2L}{v_f} \left( \frac{\omega - \Sigma'}{\sqrt{R}} \cos\left(\frac{\theta - \pi}{2}\right) + \frac{\Gamma}{\sqrt{R}} \sin\left(\frac{\theta - \pi}{2}\right) \right). \] (B12)

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