O(d+1,d+n+1)–invariant Formulation of Stationary Heterotic String Theory

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Abstract
We present a pair of symmetric formulations of the matter sector of the stationary effective action of heterotic string theory that arises after the toroidal compactification of \(d\) dimensions. The first formulation is written in terms of a pair of matrix potentials \(Z_1\) and \(Z_2\) which exhibits a clear symmetry between them and can be used to generate new families of solutions on the basis of either \(Z_1\) or \(Z_2\); the second one is an \(O(d+1,d+n+1)\)–invariant formulation which is written in terms of a matrix vector \(W\) endowed with an \(O(d+1,d+n+1)\)–invariant scalar product which linearizes the action of the \(O(d+1,d+n+1)\) symmetry group on the coset space \(O(d+1,d+n+1)/[O(d+1) \times O(d+n+1)]\); this fact opens as well a simple solution–generating technique which can be applied on the basis of known solutions. A special class of extremal solutions is indicated by assuming a simple ansatz for the matrix vector \(W\) that reduces the equation of motion to the Laplace equation for a real scalar function.
1 Introduction

At low energies the heterotic string theory leads to an effective field theory of massless fields which describes supergravity coupled to some matter fields. In [1]–[3] it was shown that when considering the toroidal compactification from $D$ to 3 dimensions using a Kaluza–Klein ansatz, the resulting theory turns out to be a nonlinear $\sigma$–model with values in the symmetric coset space $SO(d + 1, d + n + 1)/S[O(d + 1) \times O(d + n + 1)]$. It is well known that the theory of symmetric spaces provides a convenient framework for discussing and understanding the internal symmetries of the $\sigma$–models. Later on, an alternative representation of this effective field theory in terms of a couple of matrix Ernst potentials (MEP) was proposed in [4]. This formulation is, in fact, a matrix generalization of the nonlinear $\sigma$–model parametrization of the stationary Einstein–Maxwell (EM) theory and enables us to extrapolate the results obtained in the EM theory to the heterotic string realm; we review this formalism in Section 2. Further development in this direction was achieved in [5]–[6] by introducing a single rectangular matrix potential which transforms linearly under the action of the charging symmetry group. Several families of solutions have been constructed making use of the analogy between both theories (see, for instance, [7]–[10]), however, the full charge parametrized stationary axisymmetric black hole solution is still missing [11]. Most of the constructed solutions were obtained by making use of solution generating techniques based on the symmetries of the effective theory and their equations of motion.

In the framework of this motivation we present in Section 3 a particularly symmetric formulation of the matter sector of the effective field theory of heterotic string in terms of a pair of matrix potentials that enter the effective action in a completely symmetric way. This fact allows us to construct new solutions for the hole theory starting from a solution for the truncated theory in terms of one of the matrix potentials. We discuss as well the restrictions under which this procedure can consistently take place. Subsequently we give an explicit $O(d + 1, d + n + 1)$–invariant formulation of the effective theory in terms of a matrix vector endowed with an $O(d + 1, d + n + 1)$–invariant scalar product. This formulation linearizes the action of the $O(d + 1, d + n + 1)$ symmetry group on the coset space $O(d + 1, d + n + 1)/O[(d + 1) \times (d + n + 1)$ and open the possibility of applying a solution–generating technique in order to construct new families of solutions. We conclude this Section by choosing a simple ansatz for the matrix vector which reduces the equation of motion to the Laplace equation for a real scalar function and yields a family of extremal solutions. Finally, in Section 4 we present our conclusions and discuss on the further development and applications of this formalism.
The effective action and matrix Ernst potentials

Let us consider the effective action of the heterotic string at tree level

\[ S^{(D)} = \int d^{(D)}x \left| G^{(D)} \right|^2 e^{-\phi^{(D)}} \left( R^{(D)} + \phi_{(M)}^{(D)\cdot M} - \frac{1}{12} H^{(D)}_{MNP} H^{(D)MNP} - \frac{1}{4} F^{(D)I} F^{(D)IMN} \right), \]  

where

\[ F^{(D)}_{MN} = \partial_M A^{(D)I}_N - \partial_N A^{(D)I}_M, \quad H^{(D)}_{MNP} = \partial_M B^{(D)I}_{NP} - \frac{1}{2} A^{(D)I}_M F^{(D)I}_{NP} + \text{cycl. perms. of } M,N,P. \]

Here \( G^{(D)}_{MN} \) is the metric, \( B^{(D)}_{MN} \) is the anti-symmetric Kalb-Ramond field, \( \phi^{(D)} \) is the dilaton and \( A^{(D)I}_M \) is a set of Abelian \( U(1) \) vector fields \( (I = 1, 2, ..., n) \). In the consistent critical case \( D = 10 \) and \( n = 16 \), but we shall leave these parameters arbitrary for the sake of generality since when \( d = 1 \) and \( n = 6 \) the matter content of the considered effective field theory corresponds to that of \( D = N = 4 \) supergravity, and when \( d = n = 1 \), to that of Einstein–Maxwell Dilaton–Axion theory; moreover, several cases have been considered in the literature using different values of \( d \) and \( n \).

In \[2\]–\[3\] it was shown that after the compactification of this model on a \( D - 3 = d \)-torus, the resulting stationary theory possesses the \( SO(d+1, d+1+n) \) symmetry group and describes gravity coupled to the following set of three-dimensional fields:

a) scalar fields

\[ G \equiv G_{mn} = G^{(D)}_{m+3,n+3}, \quad B \equiv B_{mn} = B^{(D)}_{m+3,n+3}, \quad A \equiv A^I_m = A^{(D)I}_m, \quad \phi = \phi^{(D)} - \frac{1}{2} \ln |\det G|, \]  

b) tensor fields

\[ g_{\mu\nu} = e^{-2\phi} \left( G_{\mu\nu}^{(D)} - G^{(D)}_{m+3,m} G^{(D)}_{n+3,n} G^{mn} \right), \quad B_{\mu\nu} = B^{(D)}_{\mu\nu} - 4B_{mn} A^m_\mu A^n_\nu - 2 \left( A^{m+d}_\mu A^{n+d}_\nu - A^{m+d}_\nu A^{n+d}_\mu \right), \]  

c) vector fields \( A^{(a)}_\mu = (A_1^{m}, A_2^{m+d}, A_3^{2d+1}) \)

\[ (A_1)^m = \frac{1}{2} G^{mn} G_{n+3,m}, \quad (A_3)^I_{2d+2} = -\frac{1}{2} A_1^{(D)I} + A_1^{I+2d}_\mu, \quad (A_2)^{m+d}_\mu = \frac{1}{2} B^{(D)}_{m+3,m} - B_{mn} A^m_\mu + \frac{1}{2} A^{I+2d}_m A^{I+2d}_\mu \]  

where the subscripts \( m, n = 1, 2, ..., d \); and \( a = 1, ..., 2d+n \). Following \[3\], in this paper we set \( B_{\mu\nu} = 0 \) in order to remove the effective cosmological constant from our consideration.

In three dimensions all vector fields can be dualized on–shell as follows

\[ \nabla \times \vec{A}_1 = \frac{1}{2} e^{2\phi} G^{-1} \left( \nabla u + (B + \frac{1}{2} A A^T) \nabla v + A \nabla s \right), \]

\[ \nabla \times \vec{A}_3 = \frac{1}{2} e^{2\phi} \left( A \nabla v + A^T \nabla s \right) \]  

\[ \nabla \times \vec{A}_2 = \frac{1}{2} e^{2\phi} G \nabla v - (B + \frac{1}{2} A A^T) \nabla \times \vec{A}_1 + A \nabla \times \vec{A}_3. \]
Thus, the effective stationary theory describes gravity $g_{\mu\nu}$ coupled to the scalars $G, B, A, \phi$ and pseudoscalars $u, v, s$. All matter fields can be arranged in the following pair of MEP

$$\mathcal{X} = \begin{pmatrix} -e^{-2\phi} + v^T X v + v^T As + \frac{1}{2} s^T s & v^T X - u^T \\ X v + u + As & X \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} s^T + v^T A \\ A \end{pmatrix},$$

where $X = G + B + \frac{1}{2} AA^T$. These matrices have dimensions $(d + 1) \times (d + 1)$ and $(d + 1) \times n$, respectively. Some words about the physical meaning of the fields are in order. The relevant information of the gravitational field is encoded in $X$, whereas its rotational nature is hidden in $u$; $\phi$ is the dilaton field, $v$ is related to multidimensional components of the Kalb–Ramond field, $A$ and $s$ stand for electric and magnetic potentials, respectively.

In terms of MEP, the effective three–dimensional theory adopts the form

$$3 S = \int d^3 x | g |^{\frac{3}{2}} \{-R + \text{Tr} \left[ \frac{1}{4} (\nabla \mathcal{X} - \nabla AA^T) G^{-1} (\nabla \mathcal{X}^T - A \nabla A^T) G^{-1} + \frac{1}{2} \nabla A^T G^{-1} \nabla A \right] \},$$

where $X = G + B + \frac{1}{2} AA^T$, hence the matrices

$$G = \frac{1}{2} \left( \mathcal{X} + \mathcal{X}^T - AA^T \right) \quad \text{and} \quad B = \frac{1}{2} \left( \mathcal{X} - \mathcal{X}^T \right)$$

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$$G = \begin{pmatrix} -e^{-2\phi} + v^T G v & v^T G \\ G v & G \end{pmatrix}, \quad B = \begin{pmatrix} 0 & v^T B - u^T \\ B v + u & B \end{pmatrix}.$$  

In [4] it was shown that there exist a map between the stationary actions of the heterotic string and Einstein–Maxwell theories

$$\mathcal{X} \leftrightarrow E, \quad \mathcal{A} \leftrightarrow F,$$

$$\text{matrix transposition} \leftrightarrow \text{complex conjugation},$$

where $E$ and $F$ are the complex Ernst potentials of the Einstein–Maxwell theory [12]. Thus, the map (9) allows us to generalize the results obtained in the EM theory to the heterotic string one using the MEP formulism. It is worth noticing that in the right hand side we have complex functions, whereas in the left hand side we have real matrices (hence the transposition instead the complex conjugation) that obey the usual rules of matrix algebra.

### 3 Symmetric Formulations

We begin this Section by reformulating the matter sector of the three–dimensional effective field theory of the heterotic string (6). First we give a particularly symmetric representation of the
effective theory in terms of two potentials and then we rewrite it in an explicit $O(d+1, d+n+1)$-invariant form.

Thus, if we substitute $X = 2(Z_1 + \Sigma)^{-1} - \Sigma$ and $A = \sqrt{2}\Sigma - Z_2$, where $Z_1$ and $\Sigma$ are matrices of dimension $d+1$, $Z_2$ is a $(d+1) \times n$-matrix and $\Sigma = \text{diag}(-1, -1, 1, \ldots, 1)$, the matter sector of the action (7) takes the form

$$S_{\text{matt}} = \int d^3x \ |g|^{\frac{1}{2}} \text{Tr} \left[ \nabla Z_1 \left( \Sigma + \Sigma Z_1^T \Theta^{-1} Z_1 \Sigma \right) \nabla Z_1^T \Theta^{-1} + \nabla Z_1 \Sigma Z_1^T \Theta^{-1} Z_2 \nabla Z_2^T \Theta^{-1} + \nabla Z_2 \Theta^{-1} Z_1 \Sigma \nabla Z_2^T \Theta^{-1} + \nabla Z_2 \left( I_n + Z_2^T \Theta^{-1} Z_2 \right) \nabla Z_2^T \Theta^{-1} \right],$$

or, equivalently,

$$S_{\text{matt}} = \int d^3x \ |g|^{\frac{1}{2}} \text{Tr} \left( \Theta^{-1} \nabla Z_k Y_{kl} \nabla Z_l^T \right),$$

where the symmetric block–matrix reads

$$Y_{kl} = \begin{pmatrix} \Sigma + \Sigma Z_1^T \Theta^{-1} Z_1 \Sigma & \Sigma Z_1^T \Theta^{-1} Z_2 \\ Z_2^T \Theta^{-1} Z_1 \Sigma & I_n + Z_2^T \Theta^{-1} Z_2 \end{pmatrix},$$

$$\Theta = \Sigma - Z_1 \Sigma Z_1^T - Z_2 Z_2^T, \ I_n \ \text{is the unit matrix of dimension \( n \)} \ \text{and} \ \ k, l = 1, 2. \ \text{The corresponding equations of motion for the matrix potentials} \ Z_1 \ \text{and} \ Z_2 \ \text{are}

$$\nabla^2 Z_1 + 2 \left( \nabla Z_1 \Sigma Z_1^T + \nabla Z_2 Z_2^T \right) \Theta^{-1} \nabla Z_1 = 0
$$

$$\nabla^2 Z_2 + 2 \left( \nabla Z_1 \Sigma Z_1^T + \nabla Z_2 Z_2^T \right) \Theta^{-1} \nabla Z_2 = 0$$

This parametrization of the effective theory is a generalization of the Kähler $\sigma$–model representation of the stationary EM theory [13] in terms of a pair of real matrix potentials instead of complex functions. An important feature of this action is its evident invariance under the transformation

$$Z_2 \to Z_1 \tau,$$

if the rectangular matrix $\tau$ satisfies the following conditions

$$\tau \tau^T = \Sigma, \ \ \ \tau^T \Sigma \tau = I_n.$$ 

This symmetry mixes the gravitational and matter degrees of freedom of the theory. It recalls the Bonnor transformation of the Einstein–Maxwell theory [14]–[15], but in the heterotic string realm.
In the particular case when $\tau$ is a square matrix, only the first restriction is sufficient to ensure the map (13). This symmetry enables us to construct new classes of solutions for the whole theory on the basis of solutions for $Z_1$ or $Z_2$ making use of a simple solution generating procedure.

In principle, one can study the effective action under investigation formulated in terms of other dynamical variables in which the group $O(d + 1, d + n + 1)$ acts linearly on the coset space $O(d + 1, d + n + 1)/O[(d + 1) \times (d + n + 1)]$. In order to achieve this aim, let us introduce the $O(d + 1, d + n + 1)$–matrix vector $W = (W_1, W_2, W_3) \neq 0$ with components defined by the relations

$$Z_1 \equiv (W_2)^{-1}W_1, \quad Z_2 \equiv (W_2)^{-1}W_3,$$

where $W_1$ and $W_2$ are $(d + 1) \times (d + 1)$–matrices and the dimension of $W_3$ is $(d + 1) \times n$. Let us define as well the $O(d + 1, d + n + 1)$–invariant scalar product in the space of vectors $W$

$$\left( W, W^T \right) \equiv (W_1, W_2, W_3) \tilde{\mathcal{L}} (W_1, W_2, W_3)^T = -W_1 \Sigma W_1^T + W_2 \Sigma W_2^T - W_3 W_3^T,$$

where the matrix $\tilde{\mathcal{L}}$ determines the indefinite signature $\tilde{\mathcal{L}} = \text{diag}(\Sigma, \Sigma, -I_n)$ of the vector space.

In terms of the introduced vector our action adopts the form

$$S = -\int d^3x \ | g | \frac{1}{2} Tr \left\{ R + \left( W, W^T \right)^{-1} \left[ (\nabla W, \nabla W^T) - (\nabla W, W^T) \right] \right\},$$

the corresponding equations of motion are

$$R_{\mu\nu} = -Tr \left\{ \left( W, W^T \right)^{-1} \left[ (\nabla_\mu W, \nabla_\nu W^T) - (\nabla_\mu W, W^T) \left( W, W^T \right)^{-1} (W, \nabla_\nu W^T) \right] \right\},$$

$$\nabla^2 W - 2 \left( W, \nabla W^T \right) \left( W, W^T \right)^{-1} \nabla W = 0,$$

which is nothing else that a matrix vector generalization of the Ernst equation for $W$.

This new dynamical system is related to the original one in the following sense: any solution of the field equations (18) can be translated into a solution of the equations of motion for the original theory using the algebraic relations (15). This formulation of the theory and its equation of motion is explicitly $O(d + 1, d + n + 1)$–invariant and is a direct generalization of
the representation given in [13] and [16] in the framework of the stationary EM theory. The realization of the linear action of the $O(d+1,d+n+1)$ symmetry group on the coset space $O(d+1,d+n+1)/O[(d+1) \times (d+n+1)]$ is reached by means of the matrix transformation
\[ W' = U W \] (19)
where the matrix $U$ satisfies the following condition
\[ U \tilde{L} U^T = \tilde{L}, \] (20)
i.e., $U$ belongs to the $O(d+1,d+n+1)$ symmetry group.

In the simplest case when the matrix vector $W$ has the form
\[ W = \Psi^{-1} K, \] (21)
where the $\Psi$ is a real scalar function and $K$ is a constant matrix vector, the equation of motion (18) reduces to the Laplace equation for the function $\Psi$
\[ \nabla^2 \Psi = 0. \] (22)
The general solution of this equation is well known and in the simplest case we can consider the spherically symmetric solution
\[ \Psi = 1 - \frac{2m}{r}. \] (23)

4 Conclusion and discussion

We have presented a couple of symmetric formulations of the toroidally compactified stationary heterotic string theory. The first representation is written in terms of a pair of matrix potentials $Z_1$ and $Z_2$ that enter the effective action in a completely symmetric way, a fact that allows us to apply a simple solution–generating procedure on the basis of either $Z_1$ or $Z_2$. The second parametrization is expressed in terms of a matrix vector $W$ which linearizes the action of the $O(d+n+1)$ symmetry group on the coset space $O(d+1,d+n+1)/[O(d+1) \times O(d+n+1)]$ and can be exploited for generating new solutions on the basis of known ones. Under the assumption of a simple ansatz for the matrix vector $W$, its equation of motion reduces to the Laplace equation for a real scalar function giving rise to a family of extremal solutions.

As a further development of this formalism we would like to address the application of solution–generating techniques in the framework of both formulations ($Z_1, Z_2$ and $W$) in order to obtain and study more complicated field configurations. Another interesting issue is the investigation of the full symmetry group of the theory expressed in terms of the matrix vector $W$ since it introduces one more matrix dynamical variable in the formalism.
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References

[1] N. Marcus and J.H. Schwarz, Nucl. Phys. B228 (1983) 145.

[2] J. Maharana and J.H. Schwarz, Nucl. Phys. B390 (1993) 3.

[3] A. Sen, Nucl. Phys. B434 (1995) 179.

[4] A. Herrera and O. Kechkin, Int. J. Mod. Phys. A13 (1998) 393; A14 (1999) 1345.

[5] A. Herrera–Aguilar and O. Kechkin, Phys. Rev. D59 (1999) 124006.

[6] O. Kechkin, “Three–dimensional Heterotic String Theory: New Approach and Extremal Solutions”, to appear in Phys. Rev. D; hep–th/0110206.

[7] O. Kechkin and M. Yurova, Mod. Phys. Lett. A13 (1998) 219.

[8] A. Herrera–Aguilar and O. Kechkin, Class. Quant. Grav. 16 (1999) 1745.

[9] O.V. Kechkin, Phys. Lett. B522 (2001) 166.

[10] A. Herrera–Aguilar, “Charging Interacting Rotating Black Holes in Heterotic String Theory”, hep–th/0201126.

[11] D. Youm, Phys. Rept. 316 (1999) 1.

[12] F.J. Ernst, Phys. Rev. 167 (1968) 1175.

[13] P.O. Mazur, Act. Phys. Pol. B14 (1983) 219.

[14] W. Bonnor, Z. Phys. 190 (1966) 444.

[15] E. Fischer, J. Math. Phys. 20 (1979) 2547.

[16] W. Kinnersley, J. Math. Phys. 18 (1977) 529.