CRAMER-CASTILLON ON A TRIANGLE’S INCIRCLE AND EXCIRCLES

DOMINIQUE LAURAIN, PETER MOSES, AND DAN REZNIK

Abstract. The Cramer-Castillon problem (CCP) consists in finding one or more polygons inscribed in a circle such that their sides pass cyclically through a list of $N$ points. We study this problem where the points are the vertices of a triangle and the circle is either the incircle or one of the excircles. We find that (i) in each case there is always a pair of solutions (total of 8 new triangles and 24 vertices); (ii) each pair shares all Brocard geometry objects, (iii) barycentric coordinates are laden with the golden ratio; and (iv) simple operations on the barycentrics of a single vertex out of the 24 yield all other 23.

Keywords Golden ratio, triangle, Brocard, symmedian.

MSC 51M04 and 51N20 and 51N35

1. Introduction

The Cramer-Castillon problem (CCP) consists in finding one or more $N$-gons inscribed in a circle $C$ such that their sides pass cyclically through a set of points $P_i, i = 1 \cdots N$. In Figure 1 this is illustrated for the $N = 3$ case. The solutions to CCP are given\(^1\) by the roots of a quadratic equation (see [7, Section 6.9] [10]), i.e., there can be 0, 1, or 2 real solutions. Geometric conditions for solution existence, though not germane to this article, are described in the aforementioned references.

Referring to Figure 2, we investigate the CCP for the case where the $P_i$ are the vertices $A, B, C$ of a reference triangle and $C$ is the incircle or one of the excircles. Our findings include:

- For any triangle, the CCP on either the incircle or an excircle has exactly two solutions (total of 8 new triangles and 24 new vertices).
- We derive barycentric coordinates for the 4 pairs of solutions and notice they are laden with the golden ratio $\phi = (\sqrt{5}+1)/2$.
- Each pair shares circumcenter, symmedian point, and all “Brocard geometry” objects [4], e.g., the Brocard points, Brocard circle and inellipse, Lemoine and Brocard axis, isodynamic points, etc.
- The four distinct Brocard axes shared by each pair concur on the de Longchamps point [11] of the reference triangle.
- Given barycentrics for a single vertex out of the 24 newly generated, all other 23 can be obtained with simple cyclic substitutions.
- Solving the CCP for a triangle’s arbitrary inconic is equivalent to solving it (via a projectivity) for the incircle case.

\(^1\)In the hyperbolic plane, corresponding sides of the two solutions are polar-orthogonal with respect to the ideal circle [1].
Figure 1. The Cramer-Castillon problem (CCP) in the $N = 3$ case. In the left (resp. right) picture, two points are exterior and one is interior (resp. all exterior) to the target circle. In each case, two solutions are shown (magenta and orange).

**Article organization.** The Cramer-Castillon Problem (CCP) on the incircle is covered in Section 2. Its shared Brocard objects are examined in Section 3. The CCP on the excircles are analyzed in Section 4. In Appendix A we provide a list of correspondences between triangle centers in the incircle CCP solutions and the reference triangle.

**Notation.** We shall use barycentric coordinates [11] and refer to triangle centers using Kimberling’s notation $X_k$ [5].

2. Cramer-Castillon on the Incircle

Referring to Figure 2, consider a triangle $T = ABC$ with the sidelengths $a, b, c$ and $s = (a + b + c)/2$ its semiperimeter. Below we use $\phi$ to denote the golden ratio, $\phi = (1 + \sqrt{5})/2$.

**Proposition 1.** The CCP on a triangle $T$ and its incircle $C$ admits exactly two solutions $T_1$ and $T_2$, whose barycentric vertex matrices with respect to $T$ are given by:

$$T_1 = \begin{bmatrix}
(1 - \phi)^2vw & uw & (2 - \phi)^2uv \\
(2 - \phi)^2vw & (1 - \phi)^2uw & uw \\
vw & (2 - \phi)^2uw & (1 - \phi)^2uv
\end{bmatrix}$$

$$T_2 = \begin{bmatrix}
(1 - \phi)^2vw & (2 - \phi)^2uw & uw \\
vw & (1 - \phi)^2uw & (2 - \phi)^2uv \\
(2 - \phi)^2vw & uw & (1 - \phi)^2uv
\end{bmatrix}$$

where $u = (s - a)$, $v = (s - b)$, $w = (s - c)$.

**Proof.** While barycentric entries in $T_1, T_2$ can be obtained as roots of a quadratic equation [7, Section 6.9], we also provide a synthetic construction for the vertices.

Referring to ??, define three paths $(A_1, A_2, A_3, A_4), (B_1, B_2, B_3, B_4)$ and $(C_1, C_2, C_3, C_4)$. Cramer-Castillon requires that $A_iA_{i+1}$ (or $B_iB_{i+1}$ or $C_iC_{i+1}$) be a circle chord.
The points $B_1$, $C_1$ are tangent points of $C$ with $AC$ and $AB$ while $A_1$ is the reflection of a tangent point with $BC$ with respect to $X_1$.

The three paths are non-closing: $A_4$, $B_4$ and $C_4$ are respectively different from $A_1$, $B_1$ and $C_1$.

Intersect the two pairs of segments $(A_1B_4,A_4B_1)$ and $(A_1C_4,A_4C_1)$ to get points $H_1$ and $H_2$ on their perspectrix.

Intersecting the homography line with $C$ gives $M_1$ and $M_4$, two vertices of the solution triangles $M_1M_2M_3$ and $M_4M_5M_6$. □

By definition, both solutions share their circumcenter $X_3$, located at the incenter $X_1$ of the reference. Interestingly:

**Proposition 2.** The two solutions $T_1$ and $T_2$ have a common symmedian point $X_6$ which coincides with the Gergonne point $X_7$ of $T$.

**Proof.** Using a CAS, we obtain $[1/(s-a) : 1/(s-b) : 1/(s-c)]$ as barycentrics for the symmedian point of $T_1,T_2$ with respect to $T$, which are precisely those of the Gergonne $X_7$ of $T$ [5]. □

3. Shared Brocard objects

Recall (i) the Brocard circle of a triangle has segment $X_3X_6$ as diameter, and (ii) the Brocard axis is the line that passes through said diameter [11].

Since by definition $T_1,T_2$ share their circumcenter $X_3$, and per Proposition 2 their symmedian as well:

**Corollary 1.** $T_1$ and $T_2$ share their Brocard axis and Brocard circles.
Figure 3. **top**: The construction for the two points $H_1$ and $H_2$ which define the homography axis used in Proposition 1. The sequence $C_1, \ldots, C_4$ is not shown, but is as $A_1 \ldots A_4$ and $B_1 \ldots B_4$.
**bottom**: a slight zoom-in on the region where $H_1$ and $H_2$ are.

Let $\delta = |X_6 - X_3|$. The Brocard angle $\omega$ to a triangle is given by [2, Prop. 3, p. 209]:

$$\tan \omega = \frac{\sqrt{3}}{3} \sqrt{1 - \left(\frac{\delta}{R}\right)^2}$$

**Corollary 2.** $T_1$ and $T_2$ have the same Brocard angle $\omega$.

Referring to Figure 4, recall the two Brocard points of a triangle lie on the Brocard circle and the line joining them is perpendicular to the Brocard axis $X_3 X_6$. 
In [9] the following formula is given for the distance between the two Brocard points \( \Omega_1, \Omega_2 \) which only depends on the circumradius \( R \) and the Brocard angle \( \omega \):
\[
|\Omega_1 - \Omega_2|^2 = 4c^2 = 4R^2 \sin^2 \omega (1 - 4 \sin^2 \omega)
\]

**Corollary 3.** \( T_1 \) and \( T_2 \) share their Brocard points \( \Omega_1 \) and \( \Omega_2 \).

**Proposition 3.** The shared Brocard points \( \Omega_1 \) and \( \Omega_2 \) of \( T_1 \) and \( T_2 \) are triangle centers of \( T \) given by the following barycentric coordinates:
\[
\Omega_1 = [\alpha/u : \beta/v : \gamma/w], \quad \Omega_2 = [\gamma/u : \alpha/v : \beta/w]
\]
where:
\[
\begin{align*}
\alpha &= (a-b)^2 - (a+b)c \\
\beta &= (b-c)^2 - (b+c)a \\
\gamma &= (c-a)^2 - (c+a)b
\end{align*}
\]
and \( u, v, w \) are as in Proposition 1.

Let \( a, b \) be the semi-axes of the Brocard inellipse of a triangle (whose foci are the Brocard points). The following relation was derived in [8, Lemma 2]:
\[
[a, b] = R [\sin \omega, 2 \sin^2 \omega]
\]

**Corollary 4.** \( T_1, T_2 \) share their Brocard inellipses.

Recall the two isodynamic points \( X_{15} \) and \( X_{16} \) are the limiting points of the Schoute pencil \([3]\), defined by the circumcircle and the Brocard circle (and orthogonal to the Apollonius circles), whose radical axis is the Lemoine axis.

**Corollary 5.** \( T_1 \) and \( T_2 \) share their isodynamic points \( X_{15} \) and \( X_{16} \) and Lemoine axis.

The intersection of the Lemoine axis with the Brocard axis is \( X_{187} \) [5].

**Corollary 6.** \( T_1 \) and \( T_2 \) share their \( X_{187} \).

**Generalizing the CCP to any inconic.** Referring to Figure 5(top), let \( \mathcal{I} \) be some inconic of a triangle \( T = ABC \).

**Proposition 4.** The CCP of \( A, B, C \) on \( \mathcal{I} \) has two solutions which circumscribe a single inconic \( \mathcal{B} \).

**Proof.** The CCP is projectively-invariant since only incidences are involved. Let \( T' \) be the image of \( T \) under a projectivity \( \Pi \) that sends \( \mathcal{I} \) to a circle \( \mathcal{C'} \), see Figure 5(bottom). Clearly, \( \mathcal{C'} \) is the incircle or an excircle of \( T' \). Per Proposition 1 and Corollary 4, the CCP on \( (T', \mathcal{C'}) \) has two solutions with a common Brocard inellipse \( \mathcal{B}' \). Thus \( \mathcal{B} \) is the latter’s pre-image under \( \Pi \), with all tangencies preserved. \( \square \)

**Proposition 5.** Consider a general inconic \( \mathcal{I} \) with perspector \([p, q, r]\). The two solutions \( T_1 \) and \( T_2 \) of the CCP on \( \mathcal{I} \) have A-vertices given by:
\[
T_1(A-vertex) = [(2-\phi)p, q, (1+\phi)r] \\
T_2(A-vertex) = [(2-\phi)p, (1+\phi)r, q]
\]
Figure 4. The two solutions (orange, magenta) of CCP on a triangle’s incircle (black) share all Brocard geometry objects, to be sure: their Brocard points $\Omega_1, \Omega_2$, Brocard circle and axis (dashed purple), Brocard inellipse (light blue) whose foci are the Brocard points, the two isodynamic points $X_{15}$, and $X_{16}$ (not shown) and the Lemoine axis (solid purple).

Figure 5. Since the CCP is projectively-invariant, its solution on a $ABC$ (top) with respect to a generic inconic (black) can be regarded as the pre-image of a perspectivity $\Pi$ which sends said inconic to a circle (black, bottom). Clearly, this circle is an incircle of the new triangle $A'B'C'$. This implies that the two solutions in the original case will envelop a conic (light blue, top) which is the pre-image of the Brocard inellipse (light blue, bottom) under $\Pi$. 
Corollary 7. If $\mathcal{I}$ is the Steiner inellipse (perspector $X_2$), the two $A$-vertices are given by:

$$T_1(A-vertex) = [2 - \phi, 1, 1 + \phi]$$
$$T_2(A-vertex) = [2 - \phi, 1 + \phi, 1]$$

Notice that if $\mathcal{I}$ is the incircle, whose perspector is the Gergonne point $X_7 : [1/(s - a) : 1/(s - b) : 1/(s - c)]$, we recover Proposition 1.

4. Cramer-Castillon on the Excircles

In this section we extend the CCP to the excircles of a given triangle $T$. Referring to Figure 6:

**Proposition 6.** The first and second solutions $T_{1,exc}$ and $T_{2,exc}$ of the CCP on the $A$-excircle are given by the following barycentric vertex matrices:

$$T_{1,exc} = \begin{bmatrix}
-vw(\phi - 1)^2 & sv & sw(\phi - 2)^2 \\
-vw(\phi - 2)^2 & sv(\phi - 1)^2 & sw \\
-vw & sv(\phi - 2)^2 & sw(\phi - 1)^2
\end{bmatrix}$$
$\begin{bmatrix}
-vw(\phi - 1)^2 & sv(\phi - 2)^2 & sw \\
-vw & sv(\phi - 1)^2 & sw(\phi - 2)^2 \\
vw(\phi - 2)^2 & sv & sw(\phi - 1)^2
\end{bmatrix}$

where $u, v, w$ are as in Proposition 1.

**Proposition 7.** The symmedian $X_6$ of the A-excircle is shared by the two solution and is the A-exversion of the reference’s Gergonne $X_7$, i.e., its barycentrics are given by $[-vw, sv, sw]$, coinciding with the Gergonne point $X_7$ of the “outer” contact triangle inscribed in the A-excircle.

**Corollary 8.** Each pair of solutions of the CCP on an excircle shares all of their Brocard objects.

**Proposition 8.** The Brocard axis of the incircle-CCP as well as the 3 Brocard axes of the excircle-CCP solutions concur on the de Longchamps point $X_{20}$ of the reference.

**Proof.** The shared Brocard axis of the incircle-CCP solutions contains, by definition, $X_3$ and $X_6$ of either solution triangle. We saw above these correspond to $X_1$ and $X_7$ of the reference, i.e., it is the Soddy line of the reference [6], which is known to pass through $X_{20}$. The Brocard axis shared by the A-excircle solutions are the A-exversion ($a \rightarrow -a$) of the incircle Brocard axis, and similarly for the B- and C-excircles. It can be shown these 4 lines meet at $X_{20}$. □

**Twenty-three from one.** There are a total of 4 pairs of triangles which are solutions to the CCP on both incircle and excircle, i.e., there are eight triangles and a total of 24 vertices.

**Proposition 9.** Twenty-three of said vertices can be directly derived from a single vertex of a solution triangle of the CCP in the incircle.

**Proof.** As seen in Proposition 1, the A-vertex of the first solution $T_1$ on the incircle is given by:

$$[(1 - \phi)^2(s - b)(s - c), (s - a)(s - c), (2 - \phi)^2(s - a)(s - b)]$$

Perform a bicentric substitution, i.e., $b \rightarrow c$ and $c \rightarrow b$, and swap positions 2 and 3 to arrive at:

$$[(\phi - 1)^2(s - b)(s - c), (\phi - 2)^2(s - a)(s - c), (s - a)(s - b)]$$

i.e., the A-vertex of $T_2$. The other vertices can be computed cyclically. Now, derive the A-excircle solution from the incircle one by performing an “A-exversion”, i.e., changing every $a \rightarrow -a$, obtaining the $T_1$ A-vertex of the A-excircle:

$$[(\phi - 1)^2(b - s)(s - c), (s - b)s, (\phi - 2)^2(s - c)s]$$

For the B-vertex of the A-excircle, we first perform a cyclic substitution, then the exversion $a \rightarrow -a$, obtaining:

$$[(\phi - 1)^2(s - b)(s - c), (s - a)(s - c), (\phi - 2)^2(s - a)(s - b)]$$

which leads to:

$$[(\phi - 2)^2(s - b)(s - c), (\phi - 1)^2(s - a)(s - c), (s - a)(s - b)]$$
and then:

\[ (\phi - 2)^2(s - b)(s - c), (\phi - 1)^2s(b - s), s(c - s) \]

The remaining vertices can be obtained similarly.

\[ \square \]

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APPENDIX A. CENTER CORRESPONDENCES

Let \([i, k]\) indicate that \(X_i\) of either solution of the incircle CCP coincides with \(X_j\) of the reference triangle. The following is a list of corresponding pairs: [3, 1], [6, 7], [15, 3638], [16, 3639], [32, 10481], [182, 5542], [187, 1323], [371, 482], [372, 481], [511, 516], [512, 514], [575, 43180], [576, 30424], [1151, 176], [1152, 175], [1350, 390], [1351, 4312], [1384, 21314], [2076, 14189], [3053, 279], [3098, 30331], [3311, 1373], [3312, 1374], [3592, 21169], [5017, 42309], [5023, 3160], [5085, 11038], [5585, 31721], [5611, 10652], [5615, 10651], [6200, 31538], [6221, 1371], [6396, 31539], [6398, 1372], [6409, 17805], [6410, 17802], [6419, 21171], [6425, 31601], [6426, 31602], [6431, 21170], [6437, 17804], [6438, 17801], [6449, 17806], [6450, 17803], [11824, 31567], [11825, 31568], [12305, 30333], [12306, 30334], [14810, 43179], [15815, 5543], [21309, 20121], [31884, 8236], [43118, 30342], [43119, 30341], [43120, 31570], [43121, 31569].

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D. LAURAIN, ENSEENH, TOULOUSE, FRANCE, dominique.laurain31@orange.fr
P. MOSES, MOPARMATIC INC., WORCESTERSHIRE, ENGLAND, moparmatic@gmail.com
D. REZNIK, DATA SCIENCE CONSULTING LTD., RIO DE JANEIRO, BRAZIL, dreznik@gmail.com