Nonlinear Bessel potentials and generalizations of the Kato class

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Abstract

We study the scale of function spaces $M_p$ introduced by Zamboni. For these spaces, we get a characterization in terms of nonlinear Bessel potentials. This result is based on a known characterization of the Kato class $K_{n,s}$ of order $s$ in terms of Bessel potentials and the space of bounded uniformly continuous functions.

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1. Introduction

The Kato class $K_n$ was introduced and studied by Aizenman and Simon (see [7] and [2]). For $n \geq 3$, it consists of locally integrable functions $f$ on $\mathbb{R}^n$ such that

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-2}} dy = 0.$$ 

For $1 < p < n$, the following classes were defined by Zamboni (see [9]): the class $\tilde{M}_p$ of functions $f$ such that

$$\sup_{x \in \mathbb{R}^n} \left\{ \int_{B(x,r)} \frac{1}{|x-y|^{n-1}} \left( \int_{B(x,r)} \frac{|f(z)|}{|y-z|^{n-1}} dz \right)^{\frac{1}{p-1}} dy \right\}^{p-1} < \infty,$$

and the class $M_p$ of functions $f$ such that $f \in \tilde{M}_p$ and

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \left\{ \int_{B(x,r)} \frac{1}{|x-y|^{n-1}} \left( \int_{B(x,r)} \frac{|f(z)|}{|y-z|^{n-1}} dz \right)^{\frac{1}{p-1}} dy \right\}^{p-1} = 0.$$

In [3], Davies and Hinz introduced the scale $K_{n,s}$ of the Kato classes of order $s > 0$. It was shown by Gulisashvili (see [4] Theorem 1) that for a locally integrable function $f$ the following conditions are equivalent:

(a) $f \in K_{n,s}$ for $s > 0$;
(b) $J^{-s}|f| \in L_\infty$ and $\lim_{\alpha \to 0^+} \alpha^s \| J^{-s}(|f|)_\alpha \|_\infty = 0$;
(c) $J^{-s}|f| \in BUC$.

In (a) and (c), the symbol $J^{-s}$ stands for the Bessel potential of order $s$, $BUC$ denotes the space of bounded uniformly continuous functions on $\mathbb{R}^n$, and $|f|_\alpha(x) = |f(\alpha x)|$, $x \in \mathbb{R}^n$, $\alpha > 0$. Previously, this result was obtained for the Kato class $K_n$ and the Kato class of measures $\tilde{K}_n$, in [6] and [5], respectively.

In the present paper, we generalize the theorem formulated above for the classes $\tilde{M}_p$ and $M_p$, using the nonlinear Bessel potentials (see Theorems 1 and 2 below).

2. Definitions and Notation

In this section, we gather definitions and notation that will be used throughout the paper. We also include several simple lemmas. By $L_{loc}^1(\mathbb{R}^n)$ we will denote the space of functions which are locally integrable on $\mathbb{R}^n$, and by $L_{loc,u}^1$ the space of functions $f$ such that

$$\sup_{x \in \mathbb{R}^n} \int_{B(x,1)} |f(y)| dy < \infty.$$

**DEFINITION 1.** Let $f \in L_{loc}^1(\mathbb{R}^n)$. For any $1 < p < n$ and $r > 0$, we set
\[ \Phi(r) = \sup_{x \in \mathbb{R}^n} \left( \int_{B(x,r)} \frac{1}{|x-y|^{n-2}} \left( \int_{B(x,r)} \frac{|f(z)|dz}{|z-y|^{n-1}} \right)^{\frac{1}{p-1}} \, dy \right)^{p-1}, \]

where \( B(x,r) = \{ y : |x-y| < r \} \).

We say that \( f \) belongs to the space \( \tilde{M}_p(\mathbb{R}^n) \), if \( \Phi(r) < \infty \) for all \( r > 0 \).

**Definition 2.** We say that a function \( f \in M_p(\mathbb{R}^n) \) if

\[ \lim_{r \to 0} \Phi(r) = 0. \]

We are now ready to formulate some simple properties of the classes \( M_p \) and \( \tilde{M}_p \).

**Lemma 1.** (See [9], p. 151) For \( 1 < p < n \), we have

(i) \( M_p(\mathbb{R}^n) \subset \tilde{M}_p(\mathbb{R}^n) \), and

(ii) \( M_2(\mathbb{R}^n) = K_n \).

From Lemma 1 we conclude that both \( M_p(\mathbb{R}^n) \) and \( \tilde{M}_p(\mathbb{R}^n) \) are generalizations of \( K_n \).

**Remark 1.** The following example shows that \( K_n \) is properly contained in \( M_p(\mathbb{R}^n) \) for \( p > 2 \). It is known that the function \( f(x) = |x|^{-2} \) is not in the Kato class \( K_n \). However, \( f \in M_p \). Indeed,

\[
(2.1) \lim_{r \to 0} \sup_x \left\{ \int_{B(x,r)} \frac{1}{|x-y|^{n-2}} \left( \int_{B(x,r)} \frac{dz}{|z|^2 |z-y|^{n-1}} \right)^{\frac{1}{p-1}} \, dy \right\}^{p-1} = 0.
\]

This can be shown by splitting the domain of integration in the interior integral into the following three parts \( B(x,r) \cap \{|z| < \frac{T}{2} |y|\} \), \( B(x,r) \cap \{|z| \leq \frac{T}{2} |y|\} \) and \( B(x,r) \cap \{|z| \geq \frac{T}{2} |y|\} \). After routine calculations we see that

\[
\int_{B(x,r)} \frac{dz}{|z|^2 |z-y|^{n-1}} \]

is majorized by \( C |y|^{-1} \). Finally we have

\[
C \sup_x \left\{ \int_{B(x,r)} \frac{dy}{|y|^{p-1} |x-y|^{n-1}} \right\}^{p-1} \to 0 \quad \text{as} \quad r \to 0,
\]

this shows that (2.1) holds. Thus, \( f \in \bigcap_{p>2} M_p \).

**Remark 2.** (i) For \( 0 < r < 1 \), it is not hard to check that for \( 1 < p \leq 2 \) the expression
\[ \|f\|_{\tilde{M}_p(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \left( \int_{B(x,1)} \frac{1}{|x-y|^{n-1}} \left( \int_{B(x,1)} \frac{|f(z)|}{|z-y|^{n-1}} dz \right)^{\frac{1}{p-1}} dy \right)^{p-1} \]

defines a norm on \( \tilde{M}_p(\mathbb{R}^n) \).

(ii) For \( p > 2 \), the expression (2.2) satisfies the following inequality.

\[ \|f + g\|_{\tilde{M}_p(\mathbb{R}^n)} \leq 2^{p-2} \left( \|f\|_{\tilde{M}_p(\mathbb{R}^n)} + \|g\|_{\tilde{M}_p(\mathbb{R}^n)} \right), \]

for all \( f \) and \( g \) in \( \tilde{M}_p(\mathbb{R}^n) \). If \( U \) is a neighborhood of 0, from (2.3) we have \( 2^{p-1}U + 2^{p-1}U \subseteq U \), then \( \tilde{M}_p(\mathbb{R}^n) \) is a topological vector space.

**Lemma 2.** \( \tilde{M}_p(\mathbb{R}^n) \subset L^1_{\text{loc},u}(\mathbb{R}^n) \) for \( 1 < p < n \).

**Proof.** Let \( f \in \tilde{M}_p(\mathbb{R}^n) \), and fix \( r_0 > 0 \). Then there exists a positive constant \( C \) such that \( \Phi(r_0) \leq C \). It follows that

\[
\sup_{x \in \mathbb{R}^n} \left( \int_{B(x,r_0)} \frac{1}{|x-y|^{n-1}} \left( \int_{B(x,1)} \frac{|f(z)|}{|z-y|^{n-1}} dz \right)^{\frac{1}{p-1}} dy \right)^{p-1} \\
\geq \sup_{x \in \mathbb{R}^n} \left( \int_{B(x,r_0)} \frac{dy}{r_0^n} \left( \int_{B(x,1)} \frac{|f(z)|}{|z-y|^{n-1}} dz \right)^{\frac{1}{p-1}} \right)^{p-1} \\
\geq \sup_{x \in \mathbb{R}^n} \left( \frac{1}{2r_0^n} \left( \frac{n(B(x,r_0))}{r_0^{n-1}} \right)^{p-1} \int_{B(x,r_0)} f(z) \, dz \right) .
\]

Therefore

\[
\sup_{x \in \mathbb{R}^n} \int_{B(x,r_0)} f(z) \, dz < BC,
\]

where

\[
B = (2r_0)^{n-1}(r_0m(B(0,1)))^{p-1}.
\]

Finally, let \( B(x,1) \subseteq \bigcup_{k=1}^n B(x_k, r_0) \), then

\[
\sup_{x \in \mathbb{R}^n} \int_{B(x,1)} f(z) \, dz \leq \sum_{k=1}^n \sup_{x \in \mathbb{R}^n} \int_{B(x_k, r_0)} f(z) \, dz,
\]

so
\[
\sup_{x \in \mathbb{R}^n} \int_{B(x,1)} f(z) \, dz < \infty
\]

therefore
\[
\tilde{M}_p(\mathbb{R}^n) \subset L^1_{loc,u}(\mathbb{R}^n). \quad \square
\]

**Lemma 3.** For \(1 < p < n\), \(\tilde{M}_p(\mathbb{R}^n)\) is a complete space.

**Proof.** Let \(\{f_n\}_{n \in \mathbb{N}}\) be a Cauchy sequence in \(\mathcal{B}(0,r) = \{f \in \tilde{M}_p(\mathbb{R}^n) : f \tilde{M}_p(\mathbb{R}^n) \leq r\}\).

By Lemma 2, \(\{f_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence in \(L^1_{loc,u}(\mathbb{R}^n)\). Since this space is complete, there exists a function \(f \in L^1_{loc,u}(\mathbb{R}^n)\) such that \(f_n \to f\) in \(L^1_{loc,u}(\mathbb{R}^n)\).

By Fatou’s Lemma, we have \(f \tilde{M}_p(\mathbb{R}^n) \leq \liminf f_n \tilde{M}_p(\mathbb{R}^n) \leq r\).

Thus \(f \in \mathcal{B}(0,r)\), which means that \(\mathcal{B}(0,r)\) is complete with respect to the topology generated by \(L^1_{loc,u}(\mathbb{R}^n)\) - norm. By Corollary 2 of Proposition 9 in [4, Chapter III §3, no. 5] we obtain the assertion. \(\square\)

**Lemma 4.** If \(1 < p < n\), then \(M_p(\mathbb{R}^n)\) is closed in \(\tilde{M}_p(\mathbb{R}^n)\).

**Proof.** Let us define the map \(\varphi : \tilde{M}_p(\mathbb{R}^n) \to [0, \infty)\) by \(\varphi(f) = \lim_{r \to 0} \phi_f(r)\) (see definition 1). It is not hard to prove that the family \(\{\varphi_r\}_{r>0}\) where \(\varphi_r(f) = \phi_f(r)\) is equicontinuous and \(\varphi_r \to \varphi\) pointwise as \(r \to 0\). Since \(M_p(\mathbb{R}^n) = \varphi^{-1}(0)\). We obtain the result. \(\square\)

**Nonlinear Bessel Potentials**

In this section, we gather some well-known results concerning Riesz and Bessel potentials (see, e.g., [8]). Let
\[
G_\alpha(x) = \frac{1}{(2\pi)^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha-n}{2}} e^{-|x|^2 \frac{t}{4}} \frac{dt}{\sqrt{t}},
\]
denote the Bessel kernel of order \(\alpha > 0\). For more information on the Bessel kernel, we refer the reader to [8], Chapter 5.

**Definition 3.** For any \(f \in L^1_{loc}(\mathbb{R}^n)\), and \(\alpha > 0\), the function
\[
G_\alpha * (G_\alpha * f)^{\frac{1}{\alpha}}
\]
is called the nonlinear Bessel potential of \(f\), (see [1], p. 21).
The symbol $I_{\alpha}$ will stand for the Riesz potential kernel which is defined as follows:

$$I_{\alpha}(x) = \frac{\gamma_{\alpha}}{|x|^{n-\alpha}},$$

where $\gamma_{\alpha}$ is a certain constant (see [8], section V.1). It is known that

$$I_{\alpha}(x) = \frac{1}{(2\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty t^{\frac{n-\alpha}{2}} e^{-\frac{|x|^2}{4t}} dt,$$

where $0 < \alpha < n$. We have from (2.3) and (2.4) that

$$0 < G_{\alpha}(x) < I_{\alpha}(x) \text{ for } 0 < \alpha < n.$$

It is known that the local behavior of the Bessel potential kernel and the corresponding Riesz potential kernel is the same for $0 < \alpha \leq n$. It is also known that the Bessel potential kernels decay exponentially at infinity. More exactly, the following estimates hold: if $0 < \alpha < n$, then there exist $C_{\alpha} > 0$ and $\tilde{C}_{\alpha} > 0$ such that

$$\tilde{C}_{\alpha} |x|^{\alpha-n} \leq G_{\alpha}(x) \leq C_{\alpha} |x|^{\alpha-n},$$

for all $x$ with $0 < |x| < 1$. On the other hand, for every $\alpha > 0$ we have

$$G_{\alpha}(x) \leq C_{\alpha} e^{-c|x|},$$

for all $x \in \mathbb{R}^n$ with $|x| > 1$. We have from (2.7) and (2.8) that for all $x$ with $0 < |x| < \infty,

$$0 < G_{\alpha}(x) \leq C_{\alpha} \left( \frac{\chi_{B(0,1)}(x)}{|x|^{n-\alpha}} + e^{-c|x|} \right).$$

Main Results

In this section we will give a characterization of the classes $\tilde{M}_p(\mathbb{R}^n)$ and $M_p(\mathbb{R}^n)$ in terms of nonlinear Bessel potentials.

**REMARK 3.** It is not hard to prove that the following are equivalent

(a) $f \in M_p(\mathbb{R}^n),$

(b) $\lim_{r \to 0^+} \sup_{x \in \mathbb{R}^n} \int_{|y-x| \leq r} \frac{1}{|x-y|^{n-\tau}} \left( \int_{|y-z| \leq 1} |f(z)| |y-z|^{-\tau} dz \right)^{\frac{1}{p-\tau}} dy = 0,$

(c) $\lim_{r \to 0^+} \sup_{x \in \mathbb{R}^n} \int_{|y-x| \leq r} \frac{1}{|x-y|^{n-\tau}} \left( \int_{\mathbb{R}^n} G_1(y-z) |f(z)| dz \right)^{\frac{1}{p-\tau}} = 0.$
THEOREM 1. Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), and \( 1 < p < n \). Then \( f \in \tilde{M}_p(\mathbb{R}^n) \) if and only if \( \sup_{x \in \mathbb{R}^n} \{ G_1 \ast (G_1 \ast |f|)^{\frac{1}{p-1}} \} < \infty \).

**Proof.** Let \( f \in \tilde{M}_p(\mathbb{R}^n) \), \( G_{\text{in}} = \chi_{B(0,1)}G_1 \) and \( G_{\text{out}} = \chi_{\mathbb{R}^n \setminus B(0,1)}G_1 \). Since \( G_1 = G_{\text{in}} + G_{\text{out}} \) and using (2.9), we have \( \sup_{x \in \mathbb{R}^n} \{ G_1 \ast (G_1 \ast |f|)^{\frac{1}{p-1}} \} \)

\[
\leq \sup_{x \in \mathbb{R}^n} \{ G_{\text{in}} \ast (G_1 \ast |f|)^{\frac{1}{p-1}}(x) \}
\]

\[
+ \sup_{x \in \mathbb{R}^n} \{ G_{\text{out}} \ast (G_1 \ast |f|)^{\frac{1}{p-1}}(x) \}
\]

\[
= \sup_{x \in \mathbb{R}^n} \{ G_{\text{in}} \ast [(G_{\text{in}} + G_{\text{out}}) \ast |f|]^{\frac{1}{p-1}}(x) \}
\]

\[
+ \sup_{x \in \mathbb{R}^n} \{ G_{\text{out}} \ast (G_1 \ast |f|)^{\frac{1}{p-1}}(x) \}
\]

\[
= \sup_{x \in \mathbb{R}^n} \{ G_{\text{in}} \ast [G_{\text{in}} + G_{\text{out}}] \ast |f|)^{\frac{1}{p-1}}(x) \}
\]

\[
+ \sup_{x \in \mathbb{R}^n} \{ G_{\text{out}} \ast (G_1 \ast |f|)^{\frac{1}{p-1}}(x) \}
\]

by (2.9) we have

\[
\sup_{x \in \mathbb{R}^n} \{ G_1 \ast (G_1 \ast |f|)^{\frac{1}{p-1}}(x) \}
\]

\[
\leq \sup_{x \in \mathbb{R}^n} \int_{B(x,1)} \frac{1}{|y-x|^n} \left( \int_{B(y,1)} \frac{|f(z)|}{|y-z|^n} |dz| \right)^{\frac{1}{p-1}} dy
\]

\[
+ \sup_{x \in \mathbb{R}^n} \int_{B(x,1)} \frac{1}{|y-x|^n} \left( \int_{\mathbb{R}^n \setminus B(y,1)} e^{-|y-z|} |f(z)| |dz| \right)^{\frac{1}{p-1}} dy
\]

\[
+ \sup_{x \in \mathbb{R}^n} \int_{B(x,1)} e^{-|y-z|} \left( \int_{\mathbb{R}^n \setminus B(y,1)} G_1(y-z) |f(z)| |dz| \right)^{\frac{1}{p-1}} dy
\]

\[
\leq \sup_{x \in \mathbb{R}^n} \int_{B(x,1)} \frac{1}{|y-x|^n} \left( \int_{\mathbb{R}^n \setminus B(y,1)} |f(z)| |dz| \right)^{\frac{1}{p-1}} dy
\]
\[ + e^{-1} \sup_{x \in \mathbb{R}^n} \int_{B(x,1)} \left( \int_{B(y,1)} |f(z)|dz \right)^{\frac{1}{p-1}} dy \]

\[ + \sup_{x \in \mathbb{R}^n} \int_{B(x,1)} e^{-|x-y|} \left( \int_{\mathbb{R}^n} |f(z)|dz \right)^{\frac{1}{p-1}} \]

< \infty.

To prove the sufficiency in Theorem 1, let us assume that \( \sup_{x \in \mathbb{R}^n} (G_1 * |f|)^{\frac{1}{p-1}}(x) < \infty \) and \( 0 < r < 1 \). Then (2.7) gives

\[ C_p \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\chi_{B(0,r)}(x-y)}{|x-y|^{n-r}} \left( \int_{\mathbb{R}^n} \frac{\chi_{B(0,r)}(y-z)f(z)}{|y-z|^{n-r}} dz \right)^{\frac{1}{p-1}} dy \]

\[ \leq \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} G_1(x-y) \left( \int_{\mathbb{R}^n} G_1(y-z) f(z) dz \right)^{\frac{1}{p-1}} dy < \infty, \]

therefore, \( f \in M_p(\mathbb{R}^n) \).

This completes the proof of Theorem 1.

\[ \square \]

**THEOREM 2.** For \( 1 < p < n \), then \( f \in M_p(\mathbb{R}^n) \) if and only if \( G_1 * (G_1 * |f|)^{\frac{1}{p-1}} \in BUC \).

**Proof.** Let \( f \in M_p(\mathbb{R}^n) \), and \( \varphi \) be any function in \( C_c(\mathbb{R}^n) \) such that

\[ \varphi(x) = \begin{cases} 
1 & \text{if } |x| \leq 1/2, \\
0 & \text{if } |x| \geq 1
\end{cases} \]

with \( 0 \leq \varphi(x) \leq 1 \) for all \( x \in \mathbb{R}^n \) and \( \text{spt } \varphi \subseteq B(0,1) \).

Let us define \( G_{in,\alpha} = \varphi(\frac{1}{\alpha})G_1 \) and \( G_{out,\alpha} = (1 - \varphi(\frac{1}{\alpha}))G_1 \). Observe that \( G_{out,\alpha} \) is a continuous function. We claim that \( G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}} \in BUC \) for \( 1 < p < n \), to prove this let us consider

\[ \sup_{x \in \mathbb{R}^n} \left| G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}}(x+h) - G_{out,\alpha} * (G_1 * |f|)^{\frac{1}{p-1}}(x) \right| \]

\[ = \sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} [G_{out,\alpha}(x+h-y) - G_{out,\alpha}(x-y)] (G_1 * |f|)^{\frac{1}{p-1}}(y)dy \right| = I \]

by Lemma 2 we obtain

\[ I \leq \sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} (G_{out,\alpha}(x+h-y) - G_{out,\alpha}(x-y)) \right| \]

\[ \left( \sup_{x \in \mathbb{R}^n} \int_{B(x,1)} |f(z)| \right)^{\frac{1}{p-1}} \]

\[ \leq \left[ \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |G_1(x+h) - G_1(x)| + \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |G_{in,\alpha}(x+h) - G_{in,\alpha}(x)| \right] \rightarrow 0 \]
as \( h \to 0 \), and the claim is proved.

Next we want to show that \( G_1 \ast (G_1 \ast |f|)^{\frac{1}{p-1}} \) can be approximated by \( G_{out,\alpha} \ast (G_1 \ast |f|)^{\frac{1}{p-1}} \). Since we get \( G_1 = G_{in,\alpha} + G_{out,\alpha} \) we have

\[
\sup_{x \in \mathbb{R}^n} \left| G_1 \ast (G_1 \ast |f|)^{\frac{1}{p-1}}(x) - G_{out,\alpha} \ast (G_1 \ast |f|)^{\frac{1}{p-1}}(x) \right|
\]

\[
= \sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} G_{in,\alpha}(x - y) (\int_{\mathbb{R}^n} G_1(y - z)|f(x)|dz)^{\frac{1}{p-1}} dy \right|
\]

\[
= \sup_{x \in \mathbb{R}^n} \int_{|x - y| \leq \alpha/2} G_1(x - y) (\int_{\mathbb{R}^n} G_1(y - z)|f(z)|dz)^{\frac{1}{p-1}} dy
\]

by hypothesis and Remark 3 we have

\[
\leq \sup_x \int_{|x - y| \leq \alpha/2} \frac{1}{|x - y|^{n-\alpha}} (\int_{\mathbb{R}^n} G_1(y - z)|f(z)|dz)^{\frac{1}{p-1}} dy \to 0
\]
as \( \alpha \to 0 \).

Next, assume that \( G_1 \ast (G_1 \ast |f|)^{\frac{1}{p-1}} \in BUC \). Then by Theorem 1 in [5]

\[
\lim_{\alpha \to 0} \alpha \sup_{x \in \mathbb{R}^n} \left( \int_{\mathbb{R}^n} G_1(x - y) (\int_{\mathbb{R}^n} G_1(\alpha y - z)|f(z)|dz)^{\frac{1}{p-1}} dy \right) = 0,
\]

using (2.7), we see that

\[
\alpha \sup_{x \in \mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{\chi_{B(x,\alpha)}(u)}{\alpha^{n-\alpha}|u|^n} \left( \int_{\mathbb{R}^n} \frac{\chi_{B(x,\alpha)}(u - z)|f(z)|dz}{|u - z|^{n-\alpha}} \right)^{\frac{1}{p-1}} du \right)
\]

\[
= \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\alpha}} \left( \int_{\mathbb{R}^n} \frac{|f(z)|dz}{|u - z|^{n-\alpha}} \right)^{\frac{1}{p-1}} du \to 0,
\]
as \( \alpha \to 0 \). Applying Theorem 1 in [5] again, we get \( f \in M_p(\mathbb{R}^n) \).

This completes the proof of Theorem 2

\( \square \)

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