Supersymmetry in Dimensions Beyond Eleven

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Spacetime superalgebras with 64 or less number of real supercharges, containing the type IIB Poincaré superalgebra in $(9, 1)$ dimensions and the $N = 1$ Poincaré superalgebra in $(10, 1)$ are considered. The restriction $D \leq 14$, and two distinct possibilities arise: The $N = (1, 0)$ superalgebra in $(11, 3)$ dimensions, and the $N = (2, 0)$ superalgebra in $(10, 2)$ dimensions. Emphasizing the former, we describe superparticle and super Yang-Mills systems in $(11, 3)$ dimensions. We also propose an $N = (2, 1)$ superstring theory in $(n, n)$ dimensions as a possible origin of super Yang-Mills in $(8 + n, n)$ dimensions.

1. Introduction

It is by now well known that the type IIA string in ten dimensions is related to M-theory on $S^1$, and the $E_8 \times E_8$ heterotic string is related to M-theory on $S^3/Z_3$ (See [1] for a review). However, the connection between M-theory and type IIB theory, $SO(32)$ heterotic string and the type I string theory is less direct. One needs to consider at least a dimensional reduction to nine dimensions to see a connection.

One may envisage a unification of the type IIA and type IIB strings in the framework of a higher than eleven dimensional theory. The simplest test for such an idea is to show that the Poincaré superalgebras of the IIA/B theories are both contained in a spacetime superalgebra in $D \geq 10$ dimensions. The downside of this reasoning is an old result due to Nahm [2], who showed that, with certain assumptions made, supergravity theories are impossible in more than $(10, 1)$ dimensions (and supersymmetric Yang-Mills theories in more than $(9, 1)$ dimensions). He assumed Lorentzian signature, and required that no spin higher than two occurs. Much later, an analysis of super p-brane scan allowing spacetimes with non-Lorentzian signature, the possibility of a $(2, 2)$ brane in $(10, 2)$ dimensions was suggested.

More recently, various studies in M-theory have also indicated the possibility of higher than eleven dimensions [1, 11]. (See also, [11, 23]). In most of these approaches, however, one needs to introduce constant null vectors into the superalgebra which break the higher dimensional Poincaré symmetry. Accordingly, one does not expect the usual kind of supergravity theory in higher than eleven dimensions.

An approach which maintains higher dimensional Poincaré symmetry has been proposed [12]. However, much remains to be done to determine the physical consequences of this approach [13, 24], since it requires a nonlinear version of the finite dimensional super-Poincaré algebra, in which the anticommutator of two supercharges is proportional to a product of two or more translation generators (see also [14, 18]). Simplest realizations of these types of algebras involves multi-particles, as was shown first for bosonic systems in [15, 20], and later for superparticles in [16, 21, 23, 25]. Putting all particles but one on-shell yields an action for a superparticle in which the constant momenta of the other particles appear as null vectors.

In what follows, we shall focus our attention on the superalgebraic structures in $D > 11$ that may
suggest a IIA/B unification and their field theoretic realizations which involves null vectors, or certain tensorial structures, explicitly. We shall then summarize and extend the results of [16,13] for the three-superparticle system and its coupling to (11, 3) dimensional super Yang-Mills. Finally, we shall outline the structure of an $N = (2, 1)$ superstring theory in $(n, n)$ dimensions as a possible origin of super Yang-Mills in $(8 + n, n)$ dimensions.

2. Unification of IIA/B Algebras in $D > 10$

To begin with, let us recall the properties of spinors and Dirac $\gamma$-matrices in $(s, t)$ dimensions where $s(t)$ are the number of space(time) coordinates. The possible reality conditions are listed in Table 1, where $M, PM, SM, PSM$ stand for Majorana, pseudo Majorana, symplectic Majorana and pseudo symplectic Majorana, respectively [31]. An additional chirality condition can be imposed for $s - t = 0 \text{ mod } 4$.

The symmetry properties of the charge conjugation matrix $C$ and the $\gamma$-matrix $(\gamma^\mu C)_{\alpha\beta}$ are listed in Table 2. The parameters $\epsilon_0$ and $\epsilon_1$ arise in the relation

$$C^T = \epsilon_0 C, \quad (\gamma^\mu C)^T = \epsilon_1 (\gamma^\mu C). \quad (1)$$

This information is sufficient to deduce the symmetry of $(\gamma^{\mu_1 \cdots \mu_r} C)_{\alpha\beta}$ for any $r$, since the symmetry property alternates for $r \text{ mod } 2$.

Using Tables 1 and 2, it is straightforward to deduce the structure of the type IIA/B superalgebras in $(9, 1)$ dimensions. The $N = (1, 1)$ super Poincaré algebra (i.e. type IIA) contains a single 32 component Majorana-Weyl spinor generator $Q^\alpha$, with $\alpha = 1, \ldots, 32$ and a set of 528 bosonic generators, including the translation generator $P^\mu$, that span a symmetric $32 \times 32$ dimensional symmetric matrix. The non-trivial part of the algebra reads

$$D = (9, 1), \text{ IIA:} \quad (2)$$

$$\{Q^\alpha, Q^\beta\} = \gamma^\mu_{\alpha\beta} P_\mu + (\gamma_{11})_{\alpha\beta} Z + (\gamma_{11} \gamma^\mu)_{\alpha\beta} Z_\mu + \gamma^{\mu\nu}_{\alpha\beta} Z_{\mu\nu} + (\gamma_{11} \gamma^{\mu_1 \cdots \mu_5})_{\alpha\beta} Z_{\mu_1 \cdots \mu_5} + \epsilon_0 \gamma^{\mu}_{\alpha\beta} Z_{\mu} + \epsilon_1 \gamma^{\mu\nu}_{\alpha\beta} Z_{\mu\nu},$$

where $\epsilon_0$ and $\epsilon_1$ are the $2 \times 2$ matrices $\tau_0 = (\sigma_3, \sigma_1, 1)$. We can make the identification $Z^a_\mu \equiv P^\mu$.

| $\epsilon$ | $\eta$ | $(s - t) \text{ mod } 8$ | SpinorType |
|------------|--------|--------------------------|------------|
| +          | +      | 0, 1, 2                  | M          |
| +          | -      | 0, 6, 7                  | PM         |
| -          | +      | 4, 5, 6                  | SM         |
| -          | -      | 2, 3, 4                  | PSM        |

Table 1

| $t \text{ mod } 4$ | $\epsilon_0$ | $\epsilon_1$ |
|---------------------|---------------|---------------|
| 0                   | $+\epsilon$   | $+\epsilon\eta$ |
| 1                   | $-\epsilon\eta$ | $+\epsilon$ |
| 2                   | $-\epsilon$   | $-\epsilon\eta$ |
| 3                   | $+\epsilon\eta$ | $-\epsilon$ |

Table 2

Symmetries of $C$ and $\gamma^\mu C$, see [3]. All the generators labelled by $Z$ in this algebra, and all the algebras below, commute with each other. The generators of the Lorentz group can be added to all these algebras, and the $Z$-generators transform as tensors under Lorentz group, as indicated by their indices. It is clear that the algebra (2) can be written in a $(10, 1)$ dimensional covariant form

$$D = (10, 1), N = 1: \quad (3)$$

$$\{Q^\alpha, Q^\beta\} = \gamma^\mu_{\alpha\beta} P_\mu + \gamma^{\mu\nu}_{\alpha\beta} Z_{\mu\nu} + \gamma^{\mu_1 \cdots \mu_5}_{\alpha\beta} Z_{\mu_1 \cdots \mu_5}.$$
Various aspects of the algebras discussed above have been treated in [30] in the context of brane charges, and in [18] in the context of higher dimensional unification of IIA/B superalgebras.

In order to unify the superalgebras (3) and (4), we consider superalgebras in

\[ D = (10 + m, 1 + n), \quad m, n = 0, 1, \ldots \quad (5) \]
dimensions. To simplify matters, we shall restrict the number real supercharges to be

\[ \dim Q \leq 64. \quad (6) \]

Using Table 1 and Table 2, we learn that this restriction requires dimensions (3) with

\[ m + n \leq 3. \quad (7) \]

Examining all dimensions (3) for which \((m, n)\) obey this condition, we find that there are two distinct possibilities:

- \( N = (1, 0) \) algebra in \( D = (11, 3) \)
- \( N = (2, 0) \) algebra in \( D = (10, 2) \)

Both of these algebras contain 64 real supercharges, and the second one is not contained in the first.

The \( N = (2, 0) \) algebra in \( (10, 2) \) dimensions has two Majorana-Weyl spinor generators \( Q^\alpha_a (\alpha = 1, \ldots, 32, \ i = 1, 2) \) obeying the anticommutator

\[ D = (10, 2), \ N = (2, 0) : \]

\[ \{Q^i_\alpha, Q^j_\beta\} = \tau^{ij}_{\alpha\beta} \left( \gamma^\mu_{\alpha\beta} Z^\mu_{ij} + \gamma^{\mu_1 \ldots \mu_7}_{\alpha\beta} Z^+_{\mu_1 \ldots \mu_7} \right) \]

\[ + \epsilon^{ij}_{\alpha\beta} \left( C_{\alpha\beta} Z + \gamma^{\mu_1 \ldots \mu_4}_{\alpha\beta} Z^-_{\mu_1 \ldots \mu_4} \right). \]

The \( N = (1, 0) \) algebra in \( D = (11, 3) \) dimensions, on the other hand, takes the form

\[ D = (11, 3), \ N = (1, 0) : \]

\[ \{Q_\alpha, Q_\beta\} = (\gamma^\mu_{\alpha\beta}) Z^\mu_{ij} + (\gamma_{\mu_1 \ldots \mu_7}) Z^+_{\mu_1 \ldots \mu_7}. \]

Here and in (8), the \( \gamma \)-matrices are chirally projected. Factors of \( C \) used to raise or lower indices of \( \gamma \)-matrices are suppressed for notational simplicity. Note also that both (3) and (4) have 2080 generators on their right hand sides, spanning 64 \times 64 dimensional symmetric matrices.

Various dimensional reductions of the algebra (8) yield:

- \( N = 1 \) algebras in \( D = (11, 2), (10, 3) \)
- \( N = 1 \) algebra in \( D = (11, 1) \)
- \( N = (1, 1) \) algebra in \( D = (10, 2) \)
- \( N = 2 \) algebra in \( D = (10, 1) \)

all of which have 64 real supercharges, and contain the \( (9, 1) \) dimensional IIB and \( (10, 1) \) dimensional \( N = 1 \) algebras. The algebra (8) reduces to the last one in the above list.

The master algebras (3) and (4) also give the \( N = 1 \) algebra in \( D = (9, 3) \), the \( N = 2 \) algebra in \( D = (9, 2) \), the \( N = (2, 2) \) and \( N = (2, 1) \) algebras in \( D = (9, 1) \), all of which contain the IIA/B algebras of \( D = (9, 1) \), but not the \( N = 1 \) algebra of \( D = (10, 1) \).

We conclude this section by showing the embedding of the IIA/B and heterotic algebras of \( D = (9, 1) \) in the master algebras (3) and (4).

In the case of (3), the spinor of \( SO(10, 2) \) decomposes under \( SO(10, 2) \rightarrow SO(9, 1) \times SO(2) \) into two left-handed spinors \( Q^i_\alpha \) and two right-handed spinors \( \tilde{Q}^{\dot{\alpha}}_\dot{\beta} \). We are using chiral notation in which lower and upper spinor indices refer to opposite chiralities and there can be no raising or lowering of these indices. We keep only \( Z^a_{\dot{\mu}\dot{\nu}} \) (\( \dot{\mu} = 0, 1, \ldots, 11 \)) for simplicity, and make the ansatz \( Z^a_{\dot{\mu}\dot{\nu}} \rightarrow \tilde{Z}^a_{\dot{\mu}\dot{\nu}} = P_{\dot{\mu}_1\dot{\nu}_1} v^{ij}_\alpha \). In the reduction to \( (9, 1) \) dimensions we set \( P_{\dot{\mu}_1\dot{\nu}_1} = (P_{\dot{\mu}}; 0, 0) \) and \( v^{ij}_\alpha \rightarrow (\tilde{v}_1, v^0) \), where \( r = 10, 11 \). Now the nonvanishing part of the algebra (3) reads:

\[ \{Q^i_\alpha, Q^j_\beta\} = \gamma^\mu_{\alpha\beta} P_{\dot{\mu}} v^{ij}_\alpha, \]

\[ \{Q^{\dot{\alpha}}_\dot{\beta}, Q^{\dot{\alpha}'}_\dot{\beta}'\} = \gamma^{\mu\dot{\alpha}\dot{\beta}} P_{\dot{\mu}} v^{\dot{ij}}_\alpha, \]

where \( v_\pm = \frac{1}{2} (v_{10} \pm v_{11}) \). The desired embeddings are then obtained by setting

IIA: \( v^{ij}_\alpha = \delta^{ij}_\alpha, \quad v_\pm = 0 \).

IIA: \( v^{ij}_\alpha = v_\pm = \frac{1}{2} (1 + \sigma^3) \delta^{ij}_\alpha, \quad v_+ = 0 \).

In the case of (4), the spinor of \( SO(11, 3) \) decomposes under \( SO(11, 3) \rightarrow SO(9, 1) \times SO(2, 2) \) into two left-handed spinors \( Q^i_\alpha \) and two right-handed spinors \( \tilde{Q}^{\dot{\alpha}}_\dot{\beta} \) where \( A, A' = 1, 2 \) label
left- and right-handed spinors of $SO(4)$. We keep only $Z^{\mu\dot{\nu}}$ ($\mu = 0, 1, \ldots, 13$) for simplicity, and make the ansatz $Z_{\mu\dot{\nu}} = P_{[\mu} F_{\dot{\nu}\rho]}$. We now set $P_{\mu} = (P_{\mu}^0, 0, 0, 0)$, $F_{\mu\nu} = 0 = F_{\nu\rho}$ and $F_{rs} = (\sigma_{rs})^{AB} v_{AB} + (\sigma_{rs})^{AB} F_{AB}$, where $r, s = 10, 11, 12, 13$ and $\sigma$-matrices are the van der Waerden symbols of $SO(4)$. Now the non-vanishing part of the algebra (\ref{1}) reads:

\[
\{Q_{\alpha A}, \bar{Q}_{\beta B}\} = \gamma_{\alpha\beta}^{\mu} P_{\mu} v_{AB} , \quad (12)
\]

\[
\{Q_{\alpha A}, \bar{Q}_{\beta B}\} = \gamma_{\alpha\beta}^{\mu} P_{\mu} v_{AB} . \quad (13)
\]

The desired embeddings are then obtained by setting

\[
\text{IIB: } \det(v_{AB}) \neq 0 , \quad v_{AB} = 0 ,
\]

\[
\text{IIA: } v_{AB} = u_{(A} u_{B)} , \quad v_{AB} = u_{(A} u_{B)} ,
\]

\[
\text{Het: } v_{AB} = u_{(A} u_{B)} , \quad v_{AB} = 0 ,
\]

where $u_{A}$, $u_{\bar{A}}$ are constant spinors. Note that in the case of IIB, the symmetric matrix $v_{AB}$ has rank two and it can be chosen to be $\delta_{AB}$ in a suitable basis.

For the heterotic case, which we will focus on in the rest of this paper, the embedding can be equally well realized by choosing (dropping the hats)

\[
Z_{\mu\nu\rho} = P_{[\mu} v_{\nu\rho]} ,
\]

where

\[
v_{\mu\nu} \equiv n_{[\mu} m_{\nu]} ,
\]

and $n, m$ are constant and mutually orthogonal null vectors:

\[
m^{\mu} n_{\mu} = 0 , \quad n^{\mu} n_{\mu} = 0 , \quad m^{\mu} n_{\mu} = 0 .
\]

With these choices, it is clear that the matrix \{\(Q_\alpha, Q_\beta\) in (4)\} has rank 16, as appropriate for the heterotic algebra.

3. Superparticle Action and Super Yang-Mills Equations in (11,3) Dimensions

The simplest brane action in which the symmetry algebra (4) may be realized is that of a 0-brane, i.e. a superparticle \[13,21\]. In \[19\], an action for superparticle in the background of a second and third superparticle was obtained essentially by putting the background superparticles on-shell. The null vectors $m_{\mu}$ and $n_{\mu}$ satisfying (3) are the constant momenta of the second and third superparticles. The following action for a superparticle in (11,3) dimensions was derived \[19\]:

\[
I = \int d\tau \left[ -\frac{1}{2} e P_{\mu} P_{\mu} + P_{\mu} (\Pi^\mu - A n^\mu - B m^\mu) \right] , \quad (17)
\]

where $A$ and $B$ are Lagrange multiplier fields similar to the einbein $e$, and

\[
\Pi^\mu = \partial_{\tau} X^\mu - \frac{1}{12} \bar{e} \gamma_{\mu\nu\rho} \partial_{\tau} v_{\nu\rho} , \quad \delta e = \epsilon , \quad \delta A = \epsilon , \quad \delta B = 0 .
\]

The action is invariant under the bosonic $\xi, \Lambda$ and $\Sigma$-transformations

\[
\delta e = \partial_{\tau} \xi , \quad \delta A = \partial_{\tau} \Lambda , \quad \delta B = \partial_{\tau} \Sigma , \quad \delta X^\mu = \xi P^\mu + A n^\mu + B m^\mu , \quad (19)
\]

and the global supersymmetry transformations

\[
\delta_{\epsilon} X^\mu = \frac{1}{12} \bar{e} \gamma_{\mu\nu\rho} v_{\nu\rho} , \quad \delta_{\epsilon} e = \epsilon , \quad \delta_{\epsilon} A = 0 , \quad \delta_{\epsilon} B = 0 . \quad (20)
\]

The action is also invariant under the local fermionic $\kappa, \eta$ and $\omega$ transformations

\[
\delta \theta = \gamma^\mu P_{\mu} \kappa + \gamma^\mu n_{\mu} \eta + \gamma^\mu m_{\mu} \omega , \quad \delta X^\mu = \frac{1}{4} \bar{\theta} \gamma_{\mu\nu\rho} \eta v_{\nu\rho} (\delta_{\eta} \theta + \bar{\delta}_{\eta} \theta) , \quad \delta P_{\mu} = 0 , \quad (17)
\]

\[
\delta e = -\frac{1}{3} (\bar{e} \gamma_{\mu\nu\rho} \partial_{\tau} \eta) v_{\mu\nu} , \quad \delta A = -\frac{1}{3} (\bar{e} \gamma_{\mu\nu\rho} \partial_{\tau} \theta) m_{\mu} n_{\nu} - \frac{1}{4} (\bar{\delta}_{\kappa} \gamma_{\mu\nu\rho} \partial_{\tau} \theta) v_{\mu\nu} , \quad \delta B = -\frac{1}{3} (\bar{\delta}_{\kappa} \gamma_{\mu\nu\rho} \partial_{\tau} \theta) m_{\mu} n_{\nu} - \frac{1}{4} (\bar{\delta}_{\kappa} \gamma_{\mu\nu\rho} \partial_{\tau} \theta) v_{\mu\nu} .
\]

Superparticle actions have also been constructed in \[21\]. Our results essentially agree with each other. Some apparent differences in fermionic symmetry transformations are presumably due to field redefinitions and symmetry transformations proportional to the equations of motion \[25\].

\[3\]See \[18\] which achieves the embeddings of the full IIA/B algebras (4) and (14), by using multi $F$-tensors and taking into account $Z^{(1,1)}$.
We next describe the coupling of Yang-Mills background. To this end, it is convenient to work in the second-order formalism. Elimination of $P^\mu$ in (17) gives

$$I_0 = \frac{1}{2} \int d\tau \, e^{-1} \Pi^\mu \left( \Pi_{\mu} - A n_{\mu} - B m_{\mu} \right) . \tag{22}$$

The bosonic and fermionic symmetries of this action can be read off from (19) and (21) by making the substitution $P^\mu \rightarrow e^{-1}(\Pi^\mu - A n_{\mu} - B m_{\mu})$. To couple super Yang-Mills background to this system, we introduce the fermionic variables $\psi^r$, $r = 1, \ldots, 32$, assuming that the gauge group is $SO(32)$. The Yang-Mills coupling can then be introduced as

$$I_1 = \int d\tau \, \psi^r \partial_\tau \psi^s \partial_\tau Z^M A^r_M . \tag{23}$$

where $Z^M$ are the coordinates of the $(11,3|32)$ superspace, and $A^r_M$ is a vector superfield in that superspace.

The torsion super two-form $T^A = dE^A$ can be read from the superalgebra (3) (with (14) understood):

$$T^a = e^a \wedge (\gamma^c de_b)_{ab} \psi^c , \quad T^a = 0 , \tag{24}$$

where the basis one-forms defined as $e^a = dZ^M E^a = dz^M A^r_M$ satisfy

$$d e^a = e^a \wedge (\gamma^c de_b)_{ab} \psi^c , \quad d e^a = 0 , \tag{25}$$

and $a, b, c, \ldots$ are the $(11,3)$ dimensional tangent space indices.

Using these equations, a fairly standard calculation shows that the total action $I = I_0 + I_1$ is invariant under the fermionic gauge transformations provided that the Yang-Mills super two-form is given by

$$F = e^a \wedge (\gamma^c = \lambda_r \gamma_{ab} (\gamma^c \lambda_r - 2(\gamma_b \lambda_r) + \frac{1}{2} e^a \wedge e^b F_{ab} . \tag{26}$$

and that the transformation rules for $e, A$ and $B$ pick up the extra contributions. These contributions are determined by the requirement of the cancellation of the terms proportional to $\Pi^2$, $\Pi \cdot n$, and $\Pi \cdot m$, respectively, in the fermionic variation of the total action. They are easy determine, but as their form is not particularly illuminating we shall not give them here (see [3]). For the case of superparticle in (10,2) dimensions, the fermionic field $\psi^r$ must be assigned the fermionic transformation rule

$$\delta \psi^r = -\delta \theta^s A^r_s \psi^s . \tag{27}$$

In (26), we have introduced the chiral spinor superfield $\chi_\alpha$ and the anti-chiral spinor superfield $\lambda$. These fields and $F$ must satisfy certain constraints so that the Bianchi identity $DF = 0$ is satisfied. These constraints are [22]

$$n^a F_{ab} = 0 , \quad m^a F_{ab} = 0 , \tag{28}$$

$$n^a \gamma_\alpha \lambda = 0 , \quad m^a \gamma_\alpha \lambda = 0 , \tag{29}$$

$$D_\alpha \chi_\lambda = \left( \gamma^a \right)_\alpha \beta F_{ab} \psi^c , \tag{30}$$

$$D_\alpha \lambda = \left( \gamma^a \psi^c \right)_\alpha \beta F_{ab} \psi^c . \tag{31}$$

The above constraints are sufficient to solve the super Bianchi identity $DF = 0$, which can be shown [22] to yield the the super Yang-Mills system in $(11,3)$ dimensions [16]. Special $\gamma$-matrix identities similar to those required for the existence of the usual super $p$-branes are not needed here. In showing the vanishing of the term proportional to $e^a \wedge e^b \wedge e^c$, for example, it is sufficient to do a Fierz rearrangement, and use the constraints (29). One also find that the spinor superfield $\chi$ is unphysical, as it drops out the equations of motion.

The component form of the super Yang-Mills equations are [16]

$$\gamma^a D_{\mu} \lambda = 0 , \tag{34}$$

$$D^\sigma F_{[\mu} v_{\nu]} + \frac{1}{12} \lambda \gamma_{\mu
u} \lambda = 0 . \tag{35}$$

In addition to the manifest Yang-Mills gauge symmetry, these equations are invariant under the supersymmetry transformations

$$\delta'_\epsilon A_\mu = \bar{\epsilon} \gamma_\mu \lambda , \tag{36}$$

The extra bosonic local gauge transformation

$$\delta_{\Omega} A_\mu = -v_{\mu\nu} \Omega^\nu , \quad \delta_{\Omega \lambda} = 0 , \tag{38}$$

provided that the constraints (28)-(30) hold, and that

$$v_{\mu\nu} D_\mu \Omega^\nu = 0 , \quad v_\mu \epsilon_\nu \lambda D_\mu \Omega^\nu = 0 . \tag{39}$$
The commutator of two supersymmetry transformations closes on shell, and yields a generalized translation, the usual Yang-Mills gauge transformation and an extra gauge transformation with parameters $\xi^\mu$, $\Lambda$, $\Omega^\mu$, respectively, as follows:

$$[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_\xi + \delta_\Lambda + \delta_\Omega,$$  \hspace{1cm} (40)

where the composite parameters are given by

$$\xi^\mu = \bar{\epsilon}_2 \gamma^{\mu\rho} \epsilon_1 v_{\rho},$$ \hspace{1cm} (41)

$$\Lambda = -\xi^\mu A_\mu,$$ \hspace{1cm} (42)

$$\Omega^\mu = \frac{1}{2} \bar{\epsilon}_2 \gamma^{\mu\rho} \epsilon_1 F_{\rho}. $$ \hspace{1cm} (43)

The global part of the algebra (40) indeed agrees with (10), (11), (12). Note the symmetry between the parameters $\xi^\mu$ and $\Omega^\mu$. The former involves a contraction with $v_{\mu\nu}$, and the latter one with $F_{\mu\nu}$.

In (12), an obstacle was encountered in extending the above construction of super Yang-Mills system to higher than 14 dimensions. For example, in (12, 4) dimensions, while everthing goes through in much the same fashion as in (11, 3) dimensions, the supersymmetrical variation of the Dirac equation gave rise to a term proportional to $\lambda\lambda$, which appeared to be nonvanishing, and hence problematic in obtaining the correct Yang-Mills equation. However, as has been observed in (21), this term actually vanishes due to the constraints (29). In fact, super Yang-Mills systems in (8 + n, n) dimensions, for any $n \geq 1$ have been constructed in (22). More recently, an action for (10,2) dimensional super Yang-Mills, which can presumably be generalized to higher dimensions, has also been found in (24).

4. $N = (2,1)$ Superstring in $(n, n)$ Dimensions

In the previous section we have described super Yang-Mills theory in higher than (10,2) dimensions. The (10,2) dimensional super Yang-Mills theory can be derived from a critical heterotic string theory based on the $N = (2,1)$ superconformal algebra (22). In this section we shall describe the underlying critical string theories of the super Yang-Mills theories in higher than (10,2) dimensions using a generalization of the heterotic $N = (2,1)$ string of (22). The model is based on an $N = 1$ superconformal algebra for left-movers in $(8 + n, n)$ dimensions and an $N = 2$ superconformal algebra for right-movers in $(n, n)$ dimensions. Both these algebras are extended with null currents.\footnote{For a construction which uses null-extended $N = (1, 1)$ superconformal algebras realized in (10,2) dimensions, see (23).} The null-extended $N = 1$ algebra is realized in terms of free scalars $X^{\hat{\mu}}$ and fermions $\psi^{\hat{\mu}}$ and makes use of $n - 1$ mutually orthogonal null vectors $v^{i}_{\mu} \psi^{i}_{\mu j} = 0$, $i, j = 1, \ldots, n-1$, $\hat{\mu} = 1, \ldots, 8 + 2n$. (44)

The left-moving $N = 1$ algebra is realized as

$$T = -\frac{1}{2} \eta_{\hat{\mu}\hat{\nu}} \partial X^{\hat{\mu}} \partial X^{\hat{\nu}} + \frac{1}{2} \eta_{\hat{\mu}\hat{\nu}} \partial \psi^{\hat{\mu}} \partial \psi^{\hat{\nu}}, \hspace{1cm} (45)$$

$$G = \sqrt{2} \psi^{\hat{\mu}} \partial X^{\hat{\nu}} \eta_{\hat{\mu}\hat{\nu}}, \hspace{1cm} (46)$$

$$J^i = v^{i}_{\mu} \partial X^{\mu}, \hspace{1cm} (47)$$

$$\Gamma^i = v^{i}_{\mu} \psi^{\mu}. \hspace{1cm} (48)$$

The basic OPE’s are $X^{\hat{\mu}}(z) X^{\hat{\nu}}(0) = -\eta_{\hat{\mu}\hat{\nu}} \log z$ and $\psi^{\hat{\mu}}(z) \psi^{\hat{\nu}}(0) = -\eta_{\hat{\mu}\hat{\nu}} z^{-1}$. The OPE’s of the energy momentum tensor $T$ has central charge $c = 12 + 3n$, and its OPE’s with the currents $G, J^i, \Gamma^i$ imply that they have conformal spin $\frac{d}{2}, 1, \frac{d}{2}$, respectively. Thus, the ghost anomaly is $c_g = -26 + 11 - (n - 1) \times (2 + 1) = -(12 + 3n)$.

The null-extended $N = 2$ algebra is realized in terms of scalars $X^{\mu}$, and fermions $\psi^\mu$ and makes use of a real structure $I_{\mu \nu}$ in $(n, n)$ dimensions obeying

$$I_{\mu \nu} = - I_{\nu \mu}, \hspace{1cm} I_{\mu \rho} I_{\rho \nu} = \delta^\nu_\nu, \hspace{1cm} \mu = 1, \ldots, 2n,$$ \hspace{1cm} (49)

where $I_{\mu \nu} = I_{\mu \rho} \eta_{\rho \nu}$. The real structure has $n$ eigenvectors of eigenvalue $+1$, and $n$ eigenvectors of eigenvalue $-1$. The crucial property of these eigenvectors that allows us to write down a critical algebra in $(n, n)$ dimensions is that the inner products of two eigenvectors of the same eigenvalue vanish. Hence, in particular all the eigenvectors are null. Pick $(n - 2)$ of these null vectors, $\tilde{v}^r_\mu$, say of eigenvalue $+1$. They satisfy

$$I_{\mu \nu} \tilde{v}^r_\mu = \tilde{v}^r_\nu, \hspace{1cm} (50)$$

$$\tilde{v}^r_\mu \tilde{v}^s_\mu = 0, \hspace{1cm} r, s = 1, \ldots, n - 2. \hspace{1cm} (51)$$
The right-moving \( N = 2 \) algebra is then realized as

\[
\bar{T} = -\frac{1}{2} \eta_{\mu\nu} \bar{\partial} X^\mu \bar{\partial} X^\nu + \frac{1}{2} \eta_{\mu\nu} \psi^\mu \bar{\partial} \psi^\nu \quad (52)
\]

\[
\bar{G}_\pm = \frac{1}{\sqrt{2}} (\eta_{\mu\nu} \pm I_{\mu\nu}) \psi^\mu \bar{\partial} \psi^\nu \quad (53)
\]

\[
J = -\frac{1}{2} I_{\mu\nu} \psi^\mu \psi^\nu \quad (54)
\]

\[
J^r = \bar{v}^r_\mu \bar{\partial} X^\mu \quad (55)
\]

\[
\Gamma^r = \bar{v}^r_\mu \psi^\mu \quad (56)
\]

The energy momentum tensor \( \bar{T} \) has central charge \( \bar{c} = 3n \), and its OPE’s with the currents \( \bar{G}_\pm, J, J^r, \Gamma \) imply that they have conformal spin \( \frac{3}{2}, 1, 1, \frac{3}{2} \) respectively. Note that the closure of the algebra requires the eigen property of the vectors \( \bar{v}^r \) as well as their nullness. Hence, in this case the ghost anomaly is assumes the critical value \( \bar{c}_g = -26 + 2 \times 11 - 2 - (n - 2)(2 + 1) = -3n \).

Following the usual BRST quantization scheme one constructs the right-moving supercharges \( [33] \)

\[
Q_{\hat{\alpha}} = \int dz \, \Sigma_{gh} \, S_{\hat{\alpha}} , \quad (57)
\]

where \( S_{\hat{\alpha}} \) are the right-moving spin fields of \( \psi^\hat{\mu} \), and \( \Sigma_{gh} = \exp(-\bar{\phi}/2 - \phi_1/2 - \cdots - \phi_{n-1}/2) \) is the spin field of the commuting ghosts. The index \( \hat{\alpha} \) labels the Majorana spinor of \( O(8 + n, n) \). The single-valuedness of OPE algebra of fermionic vertex operators in the Ramond sector require that \( S_{\hat{\alpha}} \), and therefore \( Q_{\hat{\alpha}} \), are Majorana-Weyl.

BRST invariance requires the null-conditions

\[
\bar{v}_i Q = 0 \quad , \quad i = 1, \ldots, n - 1 . \quad (58)
\]

The standard form of the target space super-algebra is obtained by considering the anti-commutator of \( (57) \) with its picture-change \( Q' \):

\[
Q' = ZZ_1 \cdots Z_{n-1} Q \quad , \quad (59)
\]

where \( Z = \{ Q_{BRST}, \xi X \} \) is the picture changing operation built from the BRST charge and the zero-modes \( \xi, \xi_1, \ldots, \xi_{n-1} \) of the \( (\xi, \eta) \) systems used for bosonizing the commuting ghosts. The supercharges \( (57) \) and \( (58) \) obey the algebra \( \{ Q_{\hat{\alpha}}, Q'_{\hat{\beta}} \} = (\bar{\phi}_1 \cdots \bar{\phi}_n \phi_1 \cdots \phi_{n-1}) \delta_{\hat{\alpha}}^{\hat{\beta}} \), which reduces to

\[
\{ Q_{\hat{\alpha}}, Q'_{\hat{\beta}} \} = (\bar{v}^{\hat{\mu}_1} \cdots \bar{v}^{\hat{\mu}_{n-1}} v_1^{\mu_1} \cdots v_{n-1}^{\mu_{n-1}}) P_{\mu_n} \quad (60)
\]

in the BRST-invariant sector. The case of \( n = 2 \) has also been discussed in [34].

The spectrum of states depend on the choices for the null vectors \( v_i \) and \( \bar{v}^r \). The choices for \( \bar{v}^r \) break \( SO(n,n) \) down to \( SO(2,2) \), and the choices for \( v_i \) break \( SO(2,2) \) down to \( SO(2,1) \) or less. Generically, one obtains massless states which assemble into a super Yang-Mills multiplet in \( (8 + n, n) \) dimensions, which effectively has the \( 8 + 8 \) degrees of freedom of the usual \( (9,1) \) dimensional super Yang-Mills, after all the physical states conditions are imposed (see [35] for the \( n = 2 \) case). There is a subtlety in the present case, however, having to do with the spectral flows induced by the null-currents in the left-moving sector. They shift the (nonchiral) \( (n,n) \) dimensional momentum with multiples of \( \bar{v}_r \). The physical state conditions then force \( \bar{v}_r \) to be orthogonal to \( v_i \).

We do not yet know the exact feature of the target space field theory. We expect, however, that it will be of the kind studied in [30], where the Yang-Mills field strength satisfies a generalized self-duality condition.

5. Comments

We started out by considering an algebraic unification of the \( (9,1) \) dimensional IIA/B superalgebras in higher dimensions, with emphasis on \( (11,3) \) dimensional \( N = (1,0) \) algebra [3]. Having made the choices [34] and [30], however, we have restricted ourselves to the embedding of a supersymmetric theory with only 16 supercharges. While this is useful in understanding how the null vectors arise in a field theoretic realization, ideally one should seek a master field theoretic realization in which both IIA and IIB (and hence heterotic) symmetries are realized according to the suitable choices to be made for the three-form charge occuring in [1].

We have focused our attention on zero-brane and the super Yang-Mills system it couples to, but the considerations reviewed here apply to higher branes as well [37,2].

We have also restricted our attention to the IIA/B unification in a maximal dimensional spacetime (with 64 real supercharges), namely...
$D = (11, 3)$. Null reduction of our results for multi-superparticles and super Yang-Mills, yield corresponding results for $N = (1, 0)$ supersymmetric models in $(10, 2)$ dimensions. However, $N = (2, 0)$ supersymmetric results in $(10, 2)$ dimensions cannot be obtained in this way. In fact, a IIA/B unification in the framework of the $N = (2, 0)$ algebra in $(10, 2)$ dimensions does not seem to have attracted attention previously, and it may be interesting to investigate this case further.

One of the dividends of a higher than eleven dimensional unification of IIA/B systems should be a more manifest realization of various duality symmetries among the ten dimensional strings/branes. As it has been stressed in [9,12], these symmetries are to be interpreted as the similarity transformations of the $64 \times 64$ symmetric matrix $\{Q_\alpha, Q_\beta\}$ which leaves the BPS condition $\det \{Q_\alpha, Q_\beta\} = 0$ invariant. An explicit realization of these symmetries at the level of brane actions would be desirable.

The introduction of structures, e.g. null vectors which break the higher dimensional Poincaré symmetry may give the impression that not much is gained by a higher dimensional formulation, and that it may amount to a rewriting of the original theory. This is not quite so, even if one considers the embedding of a single type of algebra in higher dimensions, when one considers the null vectors as the averages of certain quantities, e.g. momenta, attributed to other branes co-existing with the brane under consideration, as has been illustrated in [4,13]. Furthermore, as mentioned briefly in the introduction, there exists now a simple realization of $D > 11$ superalgebras which involve the momenta multi-superparticles [23]. These do not involve constant null vectors and maintain manifest covariance in $D > 11$. The multi-brane extension of these results and the nature of target space field theories they imply, are interesting open problems.

Another aspect of the theories considered here is that their reductions to lower than ten dimensions give rise to new kinds of super Yang-Mills theories which, together with supergravity sector which can be included, are candidates to be the low energy limits of certain $N = (2, 1)$ strings. The utility of such strings lies in the fact that they provide a unified picture of various branes, e.g. string and membranes [22], resulting from different choices for the null vectors. Here we have generalized the construction of [23] to higher than $(2, 2)$ dimensional targets, indeed to $(n, n)$ dimensional ones, with $n \geq 2$. The description of their effective target space models is an interesting problem. We expect that the field equations in these models are related to the generalized self-dual Yang-Mills systems studied in [30].

In all the algebras considered here the $Z$-type generators commute with each other. However, there exist interesting extensions of the Poincaré superalgebra in $(10, 1)$ dimensions that includes super two-form [34,35], and super five-form [36] generators. The most general such algebra with $N = 1$ supersymmetry in $(10, 1)$ dimensions has been called the $M$-algebra. In this algebra, there are non-vanishing (anti) commutators of super two-form generators. The role of the charges, some of which are bosonic and some fermionic, has not been understood yet in the context of $M$-theory. However, we expect them to play an important role. It would be interesting, therefore, to determine if the $M$-algebra generalizes to the $(10, 2)$ and $(11, 3)$ dimensional algebras reviewed here, and if such an algebraic structure can help in arriving at a more unifying picture of a wealth of $M$-theory phenomena.

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