What is the number of decompositions of torus into given number of regions by unions of geodesics?

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Abstract

We prove some preliminary results concerning two questions of O. Karpenkov:

1. What is the number of decompositions (up to $SL(2,\mathbb{Z})$) of two-dimensional torus into given number $f$ of regions by unions of $n$ geodesics?

2. On the plane there are $n$ circles not in general position, every pair of circles has at least one common point. What is the set of all possible numbers of regions?

Introduction

Let us consider a flat two-dimensional torus $T$ (quotient space of real euclidean plane for an action of lattice — abelian group with two generators). In a fixed homology basis on torus $T$ a closed oriented geodesic is defined (up to parallel shifting) by a pair of coprime integers. The matrix of changing from a homology basis to any other is an integer $2 \times 2$ matrix with determinant $\pm 1$. We shall consider arrangements of nonoriented geodesics up to changing homology basis. Let $f$ be the number of connected components of the complement in two-dimensional torus $T$ to the union of $n$ geodesics. The set $F(n)$ of all possible numbers $f$ for given $n > 1$ is the following (see [1])

$$F(n) = \{n - 1, n\} \cup \{m \in \mathbb{N} \mid m \geq 2n - 4\}.$$

Let $t_i$ be the number of intersection points, which are incident to $i$ geodesics of the arrangement. If not all geodesics of the arrangement are parallel, then $f = \sum (i - 1) t_i$. For example, geodesics of types $(1, 0), (0, 1), (k, 1)$ form $k$ or $k + 1$ regions if they intersect in one point or not. If in arrangement of geodesics $\gamma_1, \ldots, \gamma_n$ are at least two non-parallel, then

$$f \leq \sum_{i<j} |\gamma_i \cap \gamma_j|$$

where $|\gamma_i \cap \gamma_j|$ is the number of intersection points of non-parallel geodesics. For arrangements of general position the inequality turns to equality.

Question for torus

In connection with the theory of high-dimensional chain fractions O. Karpenkov asked:

What is the number of decompositions of two-dimensional torus into given number $f$ of regions by unions of $n$ geodesics?

Lemma 1. If two geodesics intersect in $k$ points in the two-dimensional flat torus, then we may change bases so that geodesics will be of type $(1, 0)$ and $(x, k)$, where integer $1 \leq x \leq k - 1$ is such that $\gcd(x, k) = 1$ and is defined uniquely up to change $x \leftrightarrow k - x$.

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Lemma 2. For $f = n$ and $f = n - 1$ there is a unique arrangement of $n$ geodesics in the two-dimensional torus which divides torus into $f$ regions. The number $f = 2n - 4$ is realised as the number of regions by $n - 1$ arrangements for $n \geq 7$ (and at least by 8 arrangements for $n = 6$).

Proof. Let us take $n - 2$ geodesics of type $(1,0)$ a geodesic of type $(a,1)$ and a geodesic of type $(0,1)$, where $0 \leq a \leq n - 2$, and all intersection points of the last two geodesics are incident to some of the first $n - 2$ geodesics. Let $m$ be the maximal number of parallel (homologically equal) geodesics in an arrangement.

Lemma 3. If $f \leq cn^{6/5}$, then $m \geq n - f + O(1)$, for suitable positive constant $c$.

Corollary 1. For $f \leq cn^{6/5}$ in arrangement of geodesics almost all geodesics are homologically equal and so the number of arrangements which realize $f$ as the number of regions may be counted explicitly.

Question for circles in the plane

O.Karpenkov asked the following “On the plane there are $n$ circles not in general position, every pair of circles has at least one common point. What is the set of all possible numbers of regions for given $n$?”

For complete solution of this problem one need to determine the possible number of tangent points of a circle and special arrangements of $n - a$ lines for $n > ca^2$.

Let us denote by $C_n$ the set of numbers $f$ which are formed by $n$ circles in the plane not in general position such that every two circles have at least one common point. Let us denote by $L_n$ the set of numbers of regions in the plane, formed by $n$ distinct lines not in general position (without any requirements on intersection points). Let $m$ be the maximal number of circles, incident to one point.

Lemma 4. We have $C_n \supseteq L_n$.

Proof. Let us take any arrangement of lines in the plane with $f \in L_n$, make an inversion and get the suitable arrangement of $n$ circles with $f$ regions.

If $m = n$, i.e. all circles have one common point, then the number of regions $f \in L_n$.

Lemma 5. If $m = n - 1$, then $f$ may be any number of the sets $L_{n-1} + 2n - 2$, $L_{n-1} + 2n - 3$, $L_{n-1} + 2n - 4$, and this list of possibilities is uncomplete (here we sum a number to every element of $L_{n-1}$).

The numbers $3n - 4, 4n - 4 \in C_n$ and $3n - 4, 4n - 4 \notin L_n$. We have

$$L_n = \{n + 1, 2n, 3n - 3, 3n - 2, 4n - 8, 4n - 7, 4n - 6, 4n - 5, 5n - 15, \ldots \}$$

Conjecture 1. The set $C_n$ contains all integers between $\frac{n(n-1)}{2} + 1$ and $n(n - 1) + 2$, which are the maximal elements of $L_n$ and $C_n$ correspondingly.

References

[1] I.N. Shmurnikov, On the number of regions formed by arrangements of closed geodesics on flat surfaces, *Math. Notes* 90, N 3 – 4 (2011), 619 – 622.