Coding Theorems for Repeat Multiple Accumulate Codes

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Abstract

In this paper the ensemble of codes formed by a serial concatenation of a repetition code with multiple accumulators connected through random interleavers is considered. Based on finite length weight enumerators for these codes, asymptotic expressions for the minimum distance and an arbitrary number of accumulators larger than one are derived using the uniform interleaver approach. In accordance with earlier results in the literature, it is first shown that the minimum distance of repeat-accumulate codes can grow, at best, sublinearly with block length. Then, for repeat-accumulate-accumulate codes and rates of 1/3 or less, it is proved that these codes exhibit asymptotically linear distance growth with block length, where the gap to the Gilbert-Varshamov bound can be made vanishingly small by increasing the number of accumulators beyond two. In order to address larger rates, random puncturing of a low-rate mother code is introduced. It is shown that in this case the resulting ensemble of repeat-accumulate-accumulate codes asymptotically achieves linear distance growth close to the Gilbert-Varshamov bound. This holds even for very high rate codes.

Index Terms

Multiple serial concatenation, repeat-accumulate codes, uniform interleaver, minimum distance growth rate coefficient, Gilbert-Varshamov bound

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I. INTRODUCTION

Since the invention of turbo codes, several new turbo-like coding schemes have been proposed. Among these are serially concatenated codes, a simple example of which is the repeat-accumulate (RA) code [2] consisting only of a repetition code, an interleaver, and an accumulator. Such a code has the advantage of very low decoding complexity compared to serially concatenated code constructions with more complex constituent codes. Another benefit of RA codes, compared to powerful code constructions such as LDPC codes, is their extremely low encoding complexity of $O(1)$, whereas LDPC codes have an encoding complexity of $O(g)$, where $g$, although much smaller than the block length [3], is greater than one. This makes RA codes well suited in power-limited environments; for example, for physical layer error correction in battery powered sensor network nodes or for spacecraft communications.

Design guidelines for achieving an interleaver gain with double serially concatenated code constructions have been given in [4] (and in [5] for more general parallel and serially concatenated code ensembles). In this paper we address multiple serially concatenated RA-type codes, where, in particular, we focus on the serial concatenation of an outer repetition code with two or more accumulators connected through random interleavers. The resulting code ensemble is then analyzed using the uniform interleaver approach [6] by averaging over all possible interleavers. Our work is mainly motivated by [7], [8], where a similar setup was considered. Whereas [7] considers the interleaver gain of repeat multiple accumulate (RMA) codes (and thus addresses a special case of the work in [5]), in [8] the authors show that, for an asymptotically large number of accumulators, there are codes in the ensemble whose minimum distance grows linearly with block length and which achieve the Gilbert-Varshamov bound (GVB). They also provide numerical calculations of the minimum distance for different numbers of accumulators and finite block lengths, but they do not give results for the asymptotic minimum distance growth rate coefficient in the practically more relevant case of a finite (small) number of accumulators. An extension of this work was presented in [9], where it is shown that, for a finite number of accumulators larger than one, linear minimum distance growth can be obtained, but an asymptotic growth rate coefficient for the minimum distance was only conjectured. This was also shown in [10] for the special case of two accumulators, but again an asymptotic growth rate coefficient was not given.

Lower bounds on the average minimum distance growth for finite block lengths are given in Fig. 1 for different code rates $R$ and two (RAA) and three accumulators (RAAA) along with the GVB. The bounds are computed such that half of the codes in each ensemble have a minimum distance of at least $d_{min}$. We observe that, for rate $R = 1/2$, the growth rate coefficient of the RAA code is much smaller
than the GVB; however, both the RAAA code and the RAA code punctured from a rate 1/3 mother code (R3-AAp) have a minimum distance growth rate coefficient very close to the GVB. Also, it can be seen that, for code rates of $R = 1/3$ and $R = 1/4$, the minimum distance growth rate coefficient is closer to the GVB than in the $R = 1/2$ case. For the rate $R = 1/3$ and $R = 1/4$ RAAA code ensembles the minimum distance growth rate coefficients coincide with the GVB and are therefore not shown in Fig. 1.

In contrast to the results given in Fig. 1, we are interested in the asymptotic behavior of the minimum distance for RMA codes with block lengths tending to infinity. In particular, an asymptotically ”good” code ensemble has the property that, for large block lengths, the minimum distance grows linearly with block length, which cannot be reliably determined from the finite length analysis shown in Fig. 1.

In the following, we extend previous results in the literature [8], [9], [10] and present an analysis that fully characterizes the asymptotic minimum distance behavior of RA and RMA code ensembles. The main result of the paper is that, for RAA codes and rates equal to 1/3 or smaller, these code ensembles exhibit linear distance growth with block length, where the gap to the GVB can be made arbitrarily small by increasing the number of accumulators beyond two. In addition, we consider random puncturing at the output of the last accumulator. We show that in this case the resulting ensemble of RAA codes achieves
linear distance growth close to the GVB for any code rate smaller than one.

The paper is organized as follows. In Section II we consider the ensemble average weight spectrum of RA codes and show that the minimum distance grows as a fractional power of the block length. Section III addresses the asymptotic minimum distance analysis of a double serially concatenated RAA code. This concept is extended to multiple serial concatenation with more than two accumulators in Section IV. Finally, random puncturing and its effect on minimum distance is discussed in Section V, and some concluding remarks are given in Section VI.

II. ENSEMBLE AVERAGE WEIGHT SPECTRUM FOR REPEAT-ACCUMULATE CODES

In this section we briefly address the minimum distance of RA codes and show that these codes cannot achieve linear distance growth with block length, i.e., they are not asymptotically good code ensembles. Related results have already been established in [9], [10], [11], and [12], where lower and upper bounds on minimum distance for more general serially concatenated codes have been derived. We restate these results for tutorial reasons and since we will use them in Section III where we analyze the RAA code ensemble.

The RA encoder is shown in Fig. 2. The binary input sequence \( u \) has length \( K \) and Hamming weight \( w \), and \( \mathcal{R} \) denotes the repetition code of rate \( R = 1/q \), which leads to a codeword of weight \( qw \) and length \( N = qK \). The subsequent interleaver \( \pi_1 \) permutes the symbols of the codeword. We consider the ensemble of all interleavers by using the uniform interleaver approach [6], where each possible interleaver realization has probability \( 1/N! \). The permuted output sequence is applied to the recursive convolutional code \( \mathcal{A}_1 \) with generator polynomial \( g(D) = 1/(1 + D) \) (accumulator), leading to an output sequence \( v \) of weight \( d \).

Fig. 2. Repeat-accumulate encoder.

We will characterize RA code ensembles by their input-output weight enumerating function (IOWEF) \( A_{d,w} \), which is the number of codewords having weight \( d \) that result from input sequences of weight \( w \). Let \( E(A_{d,w}) \) denote the expected value of the IOWEF using the uniform interleaver approach.
We also define the ensemble-average weight enumerating function (WEF) \( E(A_d) \), the expected number of codewords of weight \( d \), as

\[
E(A_d) \triangleq \sum_{w=1}^{K} E(A_{d,w}),
\]

and similarly the ensemble-average cumulative WEF \( E(A_{d \leq \delta}) \) specifying the expected number of codewords in the ensemble with weight not exceeding \( \delta \),

\[
E(A_{d \leq \delta}) \triangleq \sum_{d=1}^{\delta} E(A_d).
\]

For a given code, the minimum distance \( d_{\text{RA min}} \) is defined as the smallest value of \( \delta \) for which \( A_{d \leq \delta} \) is nonzero.

**Theorem 1.** In the ensemble of RA codes with block length \( N \to \infty \) and \( q \geq 3 \), almost all codes have minimum distance \( d_{\text{RA min}} \) lower bounded by the inequality

\[
d_{\text{RA min}} \geq N^{\frac{q-2}{q}} - \epsilon,
\]

where \( \epsilon \) is any fixed positive value.

**Proof:** The conditional probability that a weight \( d \) codeword is obtained at the output of the accumulator for a given input weight \( w \) can be expressed as [2, 6]

\[
\Pr(d|w) = \begin{cases} 
1, & \text{for } w = 0, \ d = 0, \\
0, & \text{for } w = 0, \ d \geq 1, \\
0, & \text{for } w \geq 1, \ d = 0, \\
\frac{d-1}{\left\lfloor \frac{qw}{2} \right\rfloor - 1} \left( \frac{qK-d}{\frac{qw}{2}} \right), & \text{for } w \geq 1, \ d \geq 1.
\end{cases}
\]

(3)

Since we are only interested in code words with nonzero weight, in the following we will only consider the case where \( w \geq 1 \). Note that from (3) we obtain the constraints

\[
\left\lfloor \frac{qw}{2} \right\rfloor \leq d \quad \text{and} \quad \frac{qw}{2} \leq qK - d. \tag{4}
\]

The total number of input sequences having weight \( w \) is \( \binom{K}{w} \). Then \( E(A_{d,w}) \) is given as

\[
E(A_{d,w}) = \binom{K}{w} \Pr(d|w).
\]

(5)
By using the fact that
\[ N^\ell > \prod_{\lambda=0}^{\ell-1} (N - \lambda) \geq \frac{N^\ell}{\varphi_N(\ell)} \quad \text{with} \quad \varphi_N(\ell) \triangleq \exp\left( \frac{\ell(\ell - 1)}{2\lambda} \right), \] (6)
we can show that
\[ \left( \frac{N}{\ell} \right)^\ell \frac{1}{\varphi_N(\ell)} \leq \left( \frac{N}{\ell} \right)^\ell \varphi_\ell(\ell). \] (7)
Thus we can upper bound (5) as
\[ E(A_{d,w}) \leq N^w d^{\lceil qw/2 \rceil - 1} N^{\lceil qw/2 \rceil} 2^{qw} \left[ \frac{qw}{2} \right] \varphi_N(w). \] (8)
The cumulative WEF is given by
\[ E(A_{d,w}) = \sum_{w=1}^{K} \sum_{d=1}^{\delta} E(A_{d,w}), \] (9)
and by using (8) the sum over \( d \) in (9) can be upper bounded as
\[ \sum_{d=1}^{\delta} E(A_{d,w}) \leq \sum_{d=1}^{\delta} \frac{N^w d^{\lceil qw/2 \rceil - 1} N^{\lceil qw/2 \rceil}}{q^w N^{qw}} 2^{qw} \left[ \frac{qw}{2} \right] \varphi_N(w) < \frac{\delta^{\lceil qw/2 \rceil}}{N^{qw} - w - \lceil qw/2 \rceil} \left( \frac{2q}{q} \right)^w \left[ \frac{qw}{2} \right] \varphi_N(w), \] (10)
where \( w \geq 1, \ d \geq 1 \). Now choosing \( \delta = N^{2-\epsilon} \), where \( \epsilon \) is any fixed positive value, we obtain
\[ \sum_{d=1}^{\delta} E(A_{d,w}) < N^{-qw/2} \left( \frac{2q}{q} \right)^w \left( e^{(w-1)/(2N)} \right)^w \left( (qw)^{1/w} \right)^w \]
\[ < N^{-qw/2} \left( \frac{2q}{q} \right)^w \left( e^{1/(2q)} \right)^w e^{qw/\epsilon} = \eta^w, \] (11)
where
\[ \eta = N^{-qc/2} \frac{2q}{q} e^{1/(2q)} e^{q/\epsilon}, \] (12)
and we have employed the inequalities
\[ \max_{1 \leq w \leq K} e^{(w-1)/(2N)} < e^{1/(2q)} \quad \text{and} \quad (qw)^{1/w} \leq e^{q/\epsilon}. \]
Next we choose an \( N_0 = N_0(\epsilon) \) such that
\[ \eta_0 = N_0^{-qc/2} \frac{2q}{q} e^{1/2q} e^{q/\epsilon} < \frac{1}{2}, \] (13)
and then, for \( N \geq N_0 \), by combining (9), (11), and (12) we obtain
\[ E(A_{d,w}) < \sum_{w=1}^{\infty} \eta^w = \frac{\eta}{1 - \eta} < 1. \] (14)
From (14) we conclude that there exists codes in the ensemble with block length \( N > N_0 \) whose minimum distance satisfies \( d_{\text{min}}^{RA} > N^{\frac{2-\epsilon}{q}} \). Since \( \eta/(1 - \eta) \to 0 \) as \( N \to \infty \), the fraction of codes in the ensemble with \( d_{\text{min}}^{RA} \leq N^{\frac{2-\epsilon}{q}} \) goes to zero, which proves the theorem. \( \blacksquare \)
In the case \( q = 2 \) the above bound does not hold. In particular, using (5) we can show that 
\[ E(A_{d=1,w=1}) = 1, \] i.e., on average each RA code has one codeword of weight one generated by a weight one input sequence.

III. ENSEMBLE AVERAGE WEIGHT SPECTRUM FOR REPEAT-ACCUMULATE-ACCUMULATE CODES

In this and the following section, we extend the encoder of Fig. 2 and consider a serial concatenation of \( M \) accumulators \( A_\ell \) with generator polynomials \( 1/(1 + D) \) separated by interleavers \( \pi_\ell, 1 \leq \ell \leq M \), as shown in Fig. 3. In particular, in this section we focus on \( M = 2 \) and, based on an average weight enumerator analysis, show that the resulting repeat double accumulate (RAA) code ensembles for \( q \geq 3 \) are asymptotically good, i.e., as the block length \( N \) tends to infinity, almost all codes in the ensemble exhibit linear distance growth with block length.

Analogous to (3), the conditional probability that a weight \( d_1 \) codeword is obtained at the output of the first accumulator and a weight \( d \) codeword at the output of the second accumulator for a given input weight \( w \) is given as

\[
\Pr(d, d_1 | w) = \Pr(d_1 | w) \Pr(d | d_1) 
\]

where \( w, d_1, \) and \( d \) must satisfy the constraints

\[
\left\lfloor \frac{qw}{2} \right\rfloor \leq d_1, \quad \left\lfloor \frac{qw}{2} \right\rfloor \leq qK - d_1, \quad \left\lceil \frac{d_1}{2} \right\rceil \leq d, \quad \text{and} \quad \left\lceil \frac{d_1}{2} \right\rceil \leq qK - d. \tag{16}
\]

After some straightforward manipulations and, recalling that \( N = qK \), (15) can be rewritten as

\[
\Pr(d, d_1 | w) = \frac{\binom{N - qw}{d_1 - \left\lfloor \frac{qw}{2} \right\rfloor} \binom{qw}{\left\lfloor \frac{d_1}{2} \right\rceil} \binom{d_1}{d - \left\lceil \frac{d_1}{2} \right\rceil} \binom{N - d_1}{\left\lfloor \frac{d_1}{2} \right\rceil} \binom{qw}{\left\lfloor \frac{d}{2} \right\rceil} \binom{qK - d}{\left\lfloor \frac{d_1}{2} \right\rceil}}{d_1 d}. \tag{17}
\]
We can now write the ensemble average conditional IOWEF as

$$E(A_{d,d_1,w}) = \frac{K}{w} \Pr(d,d_1|w).$$

(18)

An upper bound on the ensemble-average WEF is given by

$$E(A_d) = \sum_{w=1}^{K} \sum_{d_1=1}^{N} E(A_{d,d_1,w}) \leq K N \max_{1 \leq w \leq K} \max_{1 \leq d_1 \leq N} E(A_{d,d_1,w}),$$

(19)

and a lower bound is given by

$$E(A_d) \geq \max_{1 \leq w \leq K} \max_{1 \leq d_1 \leq N} E(A_{d,d_1,w}).$$

(20)

In a similar way, the ensemble-average cumulative WEF $E(A_{d \leq \delta})$ can be upper bounded by

$$E(A_{d \leq \delta}) \leq \delta K N \max_{1 \leq d \leq \delta} \max_{1 \leq w \leq K} \max_{1 \leq d_1 \leq N} E(A_{d,d_1,w}).$$

(21)

In the following, our goal is to show that the ensemble-average cumulative WEF tends to zero as $N \to \infty$ for all $\delta < \hat{\rho}_{\min} N$, where $\hat{\rho}_{\min}$ is a lower bound on the asymptotic minimum distance growth rate coefficient of the ensemble. We do this by showing that $N^3 E(A_{d,d_1,w}) \to 0$ for all values of $d$, $1 \leq d < \hat{\rho}_{\min} N$.

To this end, we write the weights $w$, $d_1$, and $d$ as

$$w = \alpha N^a, \quad d_1 = \beta N^b, \quad d = \rho N^c,$$

(22)

where $0 \leq a \leq b \leq c \leq 1$, and $\alpha$, $\beta$, $\rho$ are positive constants, and condition (16) must be satisfied.

We now divide the problem into the following two cases:

1) At least one of the weights $w$, $d_1$, and $d$ is of order $o(N)$, so at least one of the constants $a$, $b$, and $c$ is less than 1.

2) All the weights $w$, $d_1$, and $d$ can be expressed as fractions of the block length $N$, so $a = b = c = 1$.

**Lemma 1. In the ensemble of RAA codes with block length $N$ and $q \geq 3$, in the case where at least one of the weights $w$, $d_1$, and $d$ is of order $o(N)$, $N^3 E(A_{d,d_1,w}) \to 0$ as $N \to \infty$ for all values of $d$, $1 \leq d < N/2$.**

The proof of Lemma [1] can be found in Appendix [A]. From Lemma [1] we conclude that the contribution of the first case to the cumulative WEF tends to zero as the block length $N$ tend to infinity. Thus it is sufficient to only consider weights that are of the same order as the block length $N$. 
We now consider the second case when all the weights can be expressed as fractions of the block length $N$. Using Stirling’s approximation, we have that $\binom{N}{d} \approx e^{N \cdot H(d/N)}$, where $H(x) = -x \ln(x) - (1 - x) \ln(1 - x)$ is the binary entropy function. Applying this to (18), for $N \to \infty$ we obtain

$$E(A_{d,d,w}) = \exp \left( f(\alpha, \beta, \rho) N + o(N) \right),$$

where $\alpha = w/K = qw/N$, $\beta = d_1/N$, and $\rho = d/N$ are normalized weights, and the function

$$f(\alpha, \beta, \rho) = \frac{H(\alpha)}{q} - H(\beta) - H\left( \frac{\beta - \alpha/2}{1 - \alpha} \right) (1 - \alpha)$$

$$+ \alpha \ln 2 + H\left( \frac{\rho - \beta/2}{1 - \beta} \right) (1 - \beta) + \beta \ln 2. \quad (24)$$

Since the minimum distance of any linear code cannot be greater than $N/2$, we consider a fixed $\rho \leq 1/2$, and the function $f(\alpha, \beta, \rho)$ in (24) is defined on a region $G \subset \mathbb{R}^2$ with boundaries

- (a): $(\alpha = 0, 0 \leq \beta \leq 2\rho)$,
- (b): $(0 < \alpha \leq \min(1, 4\rho), \beta = \alpha/2)$,
- (c): $(0 < \alpha \leq \min(2 - 4\rho, 4\rho), \beta = 2\rho)$,
- (d): $(2 (1 - 2\rho) \leq \alpha \leq 1, \beta = 1 - \alpha/2)$ if $\rho > 0.25$.

where the boundaries follow from the constraints in (16). Note that boundary (a) should be excluded from region $G$ since we exclude the case $w = 0$. However, for the sake of clarity, we call it a boundary of $G$ since all points $(\alpha > 0, \beta > 0)$ arbitrarily close to boundary (a) in $G$ must be included. An example of $G$ is shown for $\rho < 0.25$ in Fig. 4. Now let $\tilde{f}_\rho(\alpha, \beta) \triangleq f(\alpha, \beta, \rho)$ be a function of $\alpha$ and $\beta$ for a
fixed $\rho < 1/2$. From (24) it can be seen that the value of the function $\tilde{f}_\rho(0,0)$ is zero for any given $\rho$.

A sufficient condition for linear distance growth can be stated as follows: if $\tilde{f}_\rho(\alpha, \beta)$ is strictly negative for all $0 < \rho < \hat{\rho}_\text{min}$ and all $\alpha$, $0 < \alpha \leq 2 \min(\beta, 1 - \beta)$, and $\beta$, $0 < \beta \leq 2 \min(\rho, 1 - \rho)$, then almost all codes in the ensemble have minimum distance $d_{\text{min}} = \rho_{\text{min}} N \geq \hat{\rho}_{\text{min}} N$. Thus, to show that $\hat{\rho}_{\text{min}}$ is a lower bound on the asymptotic minimum distance growth rate coefficient of the ensemble the maximum of $\tilde{f}_\rho(\alpha, \beta)$ for all $0 < \rho < \hat{\rho}_{\text{min}}$ must be negative in $\mathcal{G}$.

We now address the maximization of $f(\alpha, \beta, \rho)$ over $\alpha$ and $\beta$, where, in principle, a maximum of the function $\tilde{f}_\rho(\alpha, \beta)$ can occur inside the region $\mathcal{G}$ or on boundaries (b), (c), or (d). However, we will show below that the maximum only occurs inside the region $\mathcal{G}$.

**Lemma 2.** For any given $\rho \leq 1/2$, the stationary points of the function $\tilde{f}_\rho(\alpha, \beta)$ satisfy the following system of equations:

\[
\frac{\beta - \alpha/2}{1 - \alpha} \left( \frac{1 - \beta - \alpha/2}{1 - \alpha} \right) = \left( \frac{\alpha}{1 - \alpha} \right)^2, \quad (26)
\]

\[
4 \left( \frac{\rho - \beta/2}{1 - \beta} \right) \left( \frac{1 - \rho - \beta/2}{1 - \beta} \right) = \left( \frac{1 - \beta}{\beta} \cdot \frac{\beta - \alpha/2}{1 - \beta - \alpha/2} \right)^2. \quad (27)
\]

**Proof:** The partial derivatives of the function $\tilde{f}_\rho(\alpha, \beta)$ are given as

\[
\frac{\partial \tilde{f}}{\partial \alpha} = -\frac{1}{q} \ln \left( \frac{\alpha}{1 - \alpha} \right) + \frac{1}{2} \ln \left( \frac{\beta - \alpha/2}{1 - \alpha} \right) + \frac{1}{2} \ln \left( \frac{1 - \beta - \alpha/2}{1 - \alpha} \right) + \ln 2, \quad (28)
\]

\[
\frac{\partial \tilde{f}}{\partial \beta} = \ln \left( \frac{\beta}{1 - \beta} \right) - \ln \left( \frac{\beta - \alpha/2}{1 - \beta - \alpha/2} \right) + \frac{1}{2} \ln \left( \frac{\rho - \beta/2}{1 - \beta} \right) + \frac{1}{2} \ln \left( \frac{1 - \rho - \beta/2}{1 - \beta} \right) + \ln 2. \quad (29)
\]

At the stationary points of $\tilde{f}_\rho(\alpha, \beta)$, we have $\partial \tilde{f}/\partial \alpha = 0$ and $\partial \tilde{f}/\partial \beta = 0$. By setting (28) to zero and letting $x \triangleq \frac{\beta - \alpha/2}{1 - \alpha}$, we obtain

\[
4 x (1 - x) = \left( \frac{\alpha}{1 - \alpha} \right)^2,
\]

which is identical to (26). Likewise, by setting (29) to zero with $y \triangleq \frac{\rho - \beta/2}{1 - \beta}$ we obtain

\[
4 y^2 - 4 y + \left( \frac{1 - \beta}{\beta} \right)^2 \left( \frac{\beta - \alpha/2}{1 - \beta - \alpha/2} \right)^2 = 0,
\]

which leads to (27).

In order to find the stationary points of $\tilde{f}_\rho(\alpha, \beta)$, we must solve (26) and (27), where $\rho$ is expressed as a function of $\alpha$ and $\beta$. We first consider (27), which can be viewed as a quadratic equation in $\rho$ with variables $\alpha$ and $\beta$. It follows that, if $\beta > 1/2$, (27) has no real-valued zeros. On the other hand, if $\beta \leq 1/2$, then (27) contains two real-valued zeros: $\rho_1(\alpha, \beta) \leq 1/2$ and $\rho_2(\alpha, \beta) \geq 1/2$. Since we only
consider the case \( \rho \leq 1/2 \), the solution of the quadratic equation (27) in \( \rho \) then yields

\[
\rho = \rho_1(\alpha, \beta) = \frac{1}{2} - \frac{1 - \beta}{2} \sqrt{1 - \left( \frac{1 - \beta/2}{1 - \beta - \alpha/2} \right)^2},
\]  

(30)

which relates the quantities \( \alpha \), \( \beta \), and \( \rho \).

Now consider (26) as a quadratic equation in \( \beta \) with variable \( \alpha \). For \( \alpha > 1/2 \) there are no real-valued zeros, but for \( \alpha \leq 1/2 \) (26) has two real-valued zeros: \( \beta_1(\alpha) \leq 1/2 \) and \( \beta_2(\alpha) \geq 1/2 \). Since \( \rho_1(\alpha, \beta) \) is real-valued only if \( \beta \leq 1/2 \), the solution of the quadratic equation (26) in \( \beta \) is given by

\[
\beta = \beta_1(\alpha) = \frac{1}{2} - \frac{1 - \alpha}{2} \sqrt{1 - \left( \frac{\alpha}{1 - \alpha} \right)^2}.
\]  

(31)

A pair \((\alpha, \beta)\) maximizing \( \tilde{f}_\rho(\alpha, \beta) \) can be computed from (30) and (31) for a given \( \rho \). There is a lower limit \( \rho_0 \) for which the system of equations has a solution. If \( \rho < \rho_0 \), (30) and (31) have no solution and \( \tilde{f}_\rho(\alpha, \beta) \) has no stationary points within \( G \). For \( \rho > \rho_0 \), (30) and (31) yield two solutions, where one solution corresponds to a maximum and the other to a saddle point. Finally, there is only a single stationary point for \( \rho = \rho_0 \). Fig. 5 shows the maxima \( \tilde{f}_\rho^{\text{max}} \) inside the region \( G \) for different values of \( q \),

![Graph](image)

Fig. 5. Maxima \( \tilde{f}_\rho^{\text{max}} \) inside the region \( G \) for the RAA code ensemble with rates \( R = 1/3, 1/4, 1/5, 1/6 \).

where we observe a zero crossing at \( \rho = \hat{\rho}_{\text{min}}, 0 < \hat{\rho}_{\text{min}} \leq 0.5 \). To show that \( \hat{\rho}_{\text{min}} \) is indeed the asymptotic

\[1\]

The structure of (30) and (31) suggest a slightly different but equivalent approach, where \( \alpha \) is used as the free parameter and \( \beta \) and \( \rho \) are determined from (30) and (31).
minimum distance growth rate coefficient, we now address the behavior of the function $\tilde{f}_\rho(\alpha, \beta)$ at the boundaries of $\mathcal{G}$.

**Lemma 3.** The function $\tilde{f}_\rho(\alpha, \beta)$ cannot have a maximum on boundaries (a), (b), and (d) but can have a maximum on boundary (c).

**Proof:** We introduce the function $\varphi_{\rho,\beta}(\alpha) \triangleq f(\alpha, \beta, \rho)$ for fixed $\beta$ and $\rho$, which is defined on a line parallel to the $\alpha$-axis (see Fig. 6 for an illustration), and consider the following three cases:

1) For $0 < \beta \leq \min(0.5, 2\rho)$, we fix the normalized weight $\beta$ and allow $\alpha$ to vary between 0 and $2\beta$. In this case, $\varphi_{\rho,\beta}(\alpha)$ is defined on a line from boundary (a) to (b) in (25).

2) For $1/2 < \beta \leq 2\rho$, we fix $\beta$ and allow $\alpha$ to vary between 0 and $2(1 - \beta)$. In this case, the function $\varphi_{\rho,\beta}(\alpha)$ is defined on a line from boundary (a) to (d).

3) For $\beta = 2\rho$, the normalized input weight $\alpha$ varies between 0 and $\min(2 - 4\rho, 4\rho)$, i.e., $\varphi_{\rho,\beta}(\alpha)$ is defined on boundary (c).

The derivative $d\varphi_{\rho,\beta}(\alpha)/d\alpha$ is given by (28) in all three cases. Note that, at the stationary points, $\alpha$ and $\beta$ are related by (28), independent of $\rho$: if $\beta \leq 1/2$, then $\alpha$ can be obtained by solving (31), and if $\beta > 1/2$, we can find $\alpha$ by solving

$$\beta = \frac{1}{2} + \frac{1 - \alpha}{2} \sqrt{1 - \left(\frac{\alpha}{1 - \alpha}\right)^2}.$$  

The second derivative $d^2\varphi_{\rho,\beta}(\alpha)/d\alpha^2$ is given by

$$\frac{d^2\varphi_{\rho,\beta}(\alpha)}{d\alpha^2} = -\frac{1}{q} \frac{1}{\alpha (1 - \alpha)} + \frac{1}{2} \left(\frac{1 - 2x}{x(1-x)}\right) \frac{2\beta - 1}{(1-\alpha)^2},$$  

(32)

where $x \triangleq \frac{\beta - \alpha/2}{1 - \alpha}$ and $0 \leq x < 1/2$. At a stationary point with $\alpha = \alpha_0$, the corresponding $\beta$ can be obtained from (31). For such a pair $(\alpha_0, \beta)$, we then obtain from (32) that

$$\frac{d^2\varphi_{\rho,\beta}(\alpha)}{d\alpha^2} \bigg|_{\alpha=\alpha_0} = -\frac{1}{q(1 - \alpha_0)} \frac{(1 - 2\beta)^2}{(1 - \alpha_0)(\beta - \alpha_0/2)(1 - \beta - \alpha_0/2)} < 0.$$  

It follows that the stationary point on a constant $\beta$ line in the $(\alpha, \beta)$ plane corresponds to a maximum for each $(\alpha_0, \beta)$ pair satisfying (31). Consequently, for cases 1) and 2), the maximum of the function $\tilde{f}_\rho(\alpha, \beta)$ cannot be on the boundaries (a), (b), and (d). For the same reason, in case 3), the maximum of the line $\tilde{f}_\rho(\alpha, \beta)\big|_{\beta=2\rho}$ is located on the boundary (c).

**Lemma 4.** For any $\rho < 1/2$ and $q \geq 3$ the function $f(\alpha, \beta, \rho)$ is negative on the boundaries (a) and (b) of the region $\mathcal{G}$ except for the point $(\alpha, \beta) = (0, 0)$, where $f(0,0,\rho) = 0$ for any $\rho < 1/2$.  


The proof of Lemma 4 can be found in Appendix B.

A numerical analysis shows that the maximum value on boundary (c) is always less than the maximum inside the region $G$, if it exists, or strictly negative if there is no stationary point inside $G$. And since the function $f(\alpha, \beta, \rho)$ is always negative on the boundaries (a) and (b), except for the point $(\alpha, \beta) = (0, 0)$, we need not consider the values on the boundary of the region $G$ in (25), and we conclude that, for all $\rho < \rho_{\text{min}}$, the function $\tilde{f}_\rho(\alpha, \beta)$ is negative. Thus, we obtain from Lemmas 2, 3, and 4 that $\rho_{\text{min}}$ is a lower bound on the asymptotic minimum distance growth rate coefficient of the code ensemble.

We summarize our findings in the following theorem.

**Theorem 2.** In the ensemble of RAA codes of rate $R \leq 1/3$ and block length $N \to \infty$, almost all codes have minimum distance $d_{\text{min}}$ growing linearly with $N$. A lower bound on the asymptotic minimum distance growth rate coefficient $\rho_{\text{RAA}}$ can be obtained by solving the system of equations (30) and (31), i.e., by finding the maximum of the function $f(\alpha, \beta, \rho)$.

To illustrate the behavior of the function $\tilde{f}_\rho(\alpha, \beta)$, Fig. 6 shows two examples of contour plots of $\tilde{f}_\rho(\alpha, \beta)$ for the RAA code ensemble with $q = 3$ and different values of $\rho$. For $\rho = 0.1$, displayed in Fig. 6(a), there is no stationary point inside the region $G$. The function $\tilde{f}_\rho(\alpha, \beta)$ is decreasing monotonically from the origin towards boundary (c), located at the top of Fig. 6(a). By contrast, for $\rho = 0.2$, Fig. 6(b) clearly shows a maximum inside $G$. Note that the pairs $(\alpha_0, \beta)$ satisfying (31)
correspond to the dashed lines in Fig. 6, which indicate the possible locations of stationary points in the $(\alpha, \beta)$ plane.

The values of $\hat{\rho}_{\text{min}}$ are listed in Table I along with the corresponding values of the GVB, the estimated minimum distance growth rate coefficient $\tilde{\rho}_{\text{min}}$ based on a finite block length analysis analogous to the one presented in Fig. 1, and the values $\rho_0$. We see that, especially for lower code rates, the asymptotic minimum distance growth rate coefficient of the RAA ensemble is close to the GVB. Also, the results of the finite length analysis match the asymptotic results quite well.

| $q$ | $\hat{\rho}_{\text{min}}$ | $\tilde{\rho}_{\text{min}}$ | GVB | $\rho_0$ |
|-----|----------------|----------------|-----|--------|
| 3   | 0.1323         | 0.1339         | 0.1740 | 0.1225 |
| 4   | 0.1911         | 0.1935         | 0.2145 | 0.1742 |
| 5   | 0.2286         | 0.2312         | 0.2430 | 0.2075 |
| 6   | 0.2549         | 0.2570         | 0.2644 | 0.2309 |

For $q = 2$, there exists no such lower bound on the asymptotic minimum distance growth rate coefficient. In this case, for any $\bar{\rho} > 0$ the cumulative WEF of the RAA code ensemble can be lower bounded by

$$E(A_d \leq \bar{\rho}N) = \sum_{w=1}^{K} \sum_{d=1}^{N} \tilde{\rho}N \sum_{d_1=1}^{d_1} E(A_{d,d_1,w}) \geq \sum_{d=1}^{\tilde{\rho}N} E(A_{d,d_1=1,w=1}) = \sum_{d=1}^{\tilde{\rho}N} \frac{1}{N} = \bar{\rho}. \quad (33)$$

Even though we expect the majority of codes in the ensemble to have a minimum distance that grows linearly with block length [9], for any fixed $\bar{\rho} > 0$, there is a nonvanishing fraction of codes in the ensemble with minimum distance $d_{\text{min}} < \bar{\rho}N$. Thus, for the RAA ensemble with $q = 2$, we cannot give a lower bound on the asymptotic minimum distance growth rate coefficient.
IV. Ensemble Average Weight Spectrum for Repeat Multiple Accumulate Codes

We now generalize the results of the previous section to RMA codes with $M > 2$, i.e., with more than two accumulators (see Fig. 3). In this case, the conditional probability of the weight vector $d = [d_1, d_2, \ldots, d_M]$ for a given input weight can be written as

$$Pr(d|w) = Pr(d_1|w) \cdot \prod_{\ell=2}^{M} \left( \frac{d_\ell - 1}{qK - d_\ell} \right) \left( \frac{d_{\ell-1}}{d_{\ell-1}} \right),$$

(34)

where $Pr(d_1|w)$ is defined in (3). The ensemble average IOEWF is then given by

$$E(A_{d,w}) = \left( \frac{K}{w} \right) Pr(d|w) = \exp \left( f(\gamma) N + o(N) \right),$$

(35)

where Stirling’s approximation for large $N$ has again been employed. The vector $\gamma$ contains normalized weights and is given by

$$\gamma = [\beta_0, \beta_1, \beta_2, \ldots, \beta_M] = \left[ \alpha = \frac{w}{K}, \frac{d_1}{N}, \frac{d_2}{N}, \ldots, \rho = \frac{d_M}{N} \right],$$

where $\beta_0 \triangleq \alpha$ and $\beta_M \triangleq \rho$. The function $f(\gamma)$ in (35) can now be written as

$$f(\gamma) = \frac{H(\alpha)}{q} - \sum_{\ell=1}^{M-1} H(\beta_\ell) + \sum_{\ell=1}^{M} H \left( \frac{\beta_\ell - \beta_{\ell-1}/2}{1 - \beta_{\ell-1}} \right) \left( 1 - \beta_{\ell-1} \right) + \ln 2 \sum_{\ell=0}^{M-1} \beta_\ell,$$

(36)

which represents a generalization of the function defined in (24) to more than three normalized weight terms. Analogous to the derivation for the RAA case in Section III, $f(\gamma)$ must now be maximized over $\alpha, \beta_1, \ldots, \beta_{M-1}$. Here, the same arguments for the existence of stationary points on the boundary or inside an $M$-dimensional region $G_M$ can be made, analogous to the RAA case, where again only the maximum inside $G_M$ must be considered in the maximization problem. Thus, the $M+1$ tuple $(\alpha, \beta_1, \ldots, \beta_{M-1}, \rho)$ maximizing $f(\gamma)$ can be expressed by the following set of recursive equations:

$$\beta_1 = \frac{1}{2} - \frac{1 - \alpha}{2} \sqrt{1 - \left( \frac{\alpha}{1 - \alpha} \right)^2} \text{ and}$$

(37)

$$\beta_{\ell+1} = \frac{1}{2} - \frac{1 - \beta_\ell}{2} \sqrt{1 - \left( \frac{1 - \beta_\ell}{\beta_\ell} \frac{\beta_\ell - \beta_{\ell-1}/2}{1 - \beta_\ell - \beta_{\ell-1}/2} \right)^2},$$

(38)

$1 \leq \ell \leq M - 1$, for any $\alpha$ such that $0 < \alpha \leq 1/2$. The derivation of this set of equations follows from the derivation of (30) and (31) in a straightforward way: (37) is equivalent to (31) with $\beta$ replaced by $\beta_1$, and (38) is a generalization of (30) with $\rho$ replaced by $\beta_{\ell+1}$ and $\beta$ by $\beta_\ell$.

\footnote{In contrast to the $M = 2$ case, for $M > 2$ we are able to derive a lower bound on the asymptotic minimum distance growth rate coefficient for the rate $R = 1/2$ code ensemble.}
As an example, in Fig. 7 the values $\hat{\rho}_{\text{min}}$ for the asymptotic minimum distance growth rate coefficient are shown for $M = 2$ and $M = 3$ code ensembles for $q = 2, 3, 4, 5,$ and $6$ and compared to the GVB. The values of $\hat{\rho}_{\text{min}}$, along with the estimated values $\tilde{\rho}_{\text{min}}$ obtained from a finite-length analysis and the

![Graph showing $\hat{\rho}_{\text{min}}$ for different $q$ and $M$ values compared to GVB.]

Fig. 7. GVB and the corresponding asymptotic minimum distance growth rate coefficient lower bound $\hat{\rho}_{\text{min}}$ for RAA and RAAA code ensembles with rates $R = 1/2, 1/3, 1/4, 1/5,$ and $1/6$.

GVB, are listed in Table III. We observe that the resulting asymptotic minimum distance growth rate

| $q$ | $\hat{\rho}_{\text{min}}$ | $\tilde{\rho}_{\text{min}}$ | GVB |
|-----|----------------|----------------|-----|
| 2   | 0.1034         | 0.1109         | 0.1100 |
| 3   | 0.1731         | 0.1739         | 0.1740 |
| 4   | 0.2143         | 0.2150         | 0.2145 |
| 5   | 0.2428         | 0.2400         | 0.2430 |
| 6   | 0.2643         | 0.2627         | 0.2644 |
coefficients of the RAAA code ensemble for \( q \geq 3 \) essentially achieve the GVB, which is consistent with the results obtained in \cite{8} for finite block lengths and a number of accumulators tending to infinity.

Analogous to Theorem 2, we now state the following theorem.

**Theorem 3.** In the ensemble of RMA codes with \( M \) accumulators, \( M \geq 3 \), of rate \( R \leq 1/2 \) and block length \( N \to \infty \), almost all codes have minimum distance \( d_{\text{min}} \) growing linearly with \( N \). A lower bound on the asymptotic minimum distance growth rate coefficient \( \rho_{\text{min}}^{\text{RMA}} = d_{\text{min}}^{\text{RMA}} / N \) of the ensemble can be obtained by solving the system of equations \eqref{37} and \eqref{38}, i.e., by finding the maximum of the function \( f(\gamma) \).

V. Repeat multiple accumulate codes with random puncturing

The rate of RMA code ensembles is determined by the rate of the outer repetition code. Thus it is not possible to obtain rates higher than 1/2 without puncturing. This motivates us to employ random puncturing at the output of the inner accumulator in connection with a lower-rate RMA mother code for the purpose of achieving higher rate RMA code ensembles.

Let \( N' \) be the number of code symbols after puncturing, \( d' \) the corresponding codeword weight, and \( R' = R \cdot N / N' \) the code rate. We also define the ratios \( \eta = N'/N \), the normalized block length, and \( \rho' = d'/N' \), the normalized output weight, after puncturing. The conditional probability of a weight-\( d' \) sequence after puncturing is given by the hypergeometric distribution

\[
\Pr_{N'}(d'|d) = \binom{d}{d'} \binom{N - d}{N' - d'} / \binom{N}{N'},
\]

which for large \( N \) can be expressed as

\[
\Pr_{N'}(d'|d) = \exp \left\{ N \left[ H \left( \frac{\eta \rho'}{\rho} \right) \rho + H \left( \frac{\eta (1 - \rho')}{1 - \rho} \right) (1 - \rho) - H(\eta) \right] + o(N) \right\} . \tag{39}
\]

Considering the general case of repeat multiple accumulate codes, the ensemble average IOWEF can now be obtained from \eqref{35} as follows:

\[
E(A_{d,d';w}) = \binom{K}{w} \Pr(d|w) \Pr_{N'}(d'|d)
= \exp \left( F(\gamma, \rho', \eta) N + O(\ln N) \right),
\]

where \( \Pr(d|w) \) is defined in \eqref{34}, \( F(\gamma, \rho', \eta) \equiv f(\gamma) + \varphi(\rho', \rho, \eta) \), and

\[
\varphi(\rho', \rho, \eta) = H \left( \frac{\eta \rho'}{\rho} \right) \rho + H \left( \frac{\eta (1 - \rho')}{1 - \rho} \right) (1 - \rho) - H(\eta). \tag{40}
\]
Following the approach used for the RAA ensemble in Section III, the maximization of the function $F(\gamma, \rho', \eta)$ must now be carried out over all elements of the vector $\gamma$, including $\rho = \beta_M$. Again, for the same reasons as in the RAA case, we consider only stationary points of $F(\gamma, \rho', \eta)$ inside the $M + 1$-dimensional region spanned by the $M + 1$ tuple $\gamma$.

Note that $\varphi(\rho', \rho, \eta)$ in (40) does not depend on the variables $\alpha, \beta_1, \ldots, \beta_{M-1}$, since only the output of the inner encoder is punctured. Therefore we can still make use of (37) and (38) for $1 \leq \ell \leq M - 1$.

In addition, we need to compute the derivative $\partial F / \partial \rho$, which is given by

$$\frac{\partial F}{\partial \rho} = \ln \left( \frac{\rho^2 (1 - \rho - \beta_{M-1}/2)}{(1 - \rho)^2 (\rho - \beta_{M-1}/2)} \right) + \ln \left( \frac{1 - \rho - \eta + \rho'}{\rho - \rho'} \right). \quad (41)$$

We then solve $\partial F / \partial \rho = 0$ for $\rho'$, which yields

$$\rho' = \rho \left( \frac{c + 1 + \eta - 1}{1 + c} \right), \quad \text{where} \quad c = \frac{(1 - \rho)^2 (\rho - \beta_{M-1}/2)}{\rho^2 (1 - \rho - \beta_{M-1}/2)}. \quad (42)$$

We can now search for a maximum of $F(\gamma, \rho', \eta)$ by using (37) and (38), for $1 \leq \ell \leq M - 1$, and (42).

Fig. 8 considers the particularly interesting RAA case and shows the lower bound on the asymptotic minimum distance growth rate coefficient $\bar{\rho}_{\min}$ for different mother code rates $R$ and punctured code rates $R'$. We observe that, compared to the unpunctured RAA code ensemble considered in Section III, the asymptotic minimum distance growth rate coefficients are closer to the GVB for the punctured ensembles with rates $R' > R$. This behavior is due to the extra randomness added by puncturing the encoder output. We also see that the growth rate coefficients approach the GVB as the rate increases. We conjecture that this is due to the fact that a smaller value of $\eta$ leads to a larger random puncturing ensemble. In other words, a smaller $N'$ results in a more ”random-like” construction. Table III gives the lower bound on the asymptotic minimum distance growth rate coefficient $\bar{\rho}'_{\min}$, along with the estimated values from a finite-length analysis $\tilde{\rho}'_{\min}$ and the GVB, for a rate $R = 1/3$ mother code.

The following theorem is analogous to Theorems 2 and 3.

**Theorem 4.** Consider the ensemble of RMA codes with $M$ accumulators, $M \geq 2$, of rate $R'$ after random puncturing. If the block length $N' \to \infty$, almost all codes in the ensemble have minimum distance $d_{\min}$ growing linearly with $N'$. A lower bound on the asymptotic minimum distance growth rate coefficient $\bar{\rho}'_{\min}$ of the ensemble can be obtained by solving the system of equations (37), (38), and (42), i.e., by finding the maximum of the function $F(\gamma, \rho', \eta)$.

VI. CONCLUDING REMARKS

We have shown that RAA code ensembles for code rates equal to 1/3 or smaller are asymptotically good in the sense that their average minimum distance grows linearly with block length $N$ as $N \to \infty$. 
Fig. 8. GVB and the corresponding normalized asymptotic minimum distance lower bound $\hat{\rho}'_{\min}$ for the randomly punctured RAA code ensemble with mother code rates $R = 1/q = 1/3, 1/4, 1/5$.

TABLE III

| $R$  | $\hat{\rho}'_{\min}$ | $\tilde{\rho}'_{\min}$ | GVB   |
|------|----------------------|-------------------------|-------|
| 0.4  | 0.1242               | 0.1256                  | 0.1461|
| 0.5  | 0.1036               | 0.1039                  | 0.1100|
| 0.6  | 0.0771               | 0.0782                  | 0.0794|
| 0.7  | 0.0522               | 0.0530                  | 0.0532|
| 0.8  | 0.0306               | 0.0314                  | 0.0311|
| 0.9  | 0.0125               | 0.0133                  | 0.0130|

Moreover, we have shown that the asymptotic growth rate coefficients approach the GVB for small code rates. This extends the results of [8], where linear distance growth is only shown for an infinite number of accumulators. These new results also extend those in [9] and [10], where linear distance growth for the RAA ensemble is shown, but a growth rate coefficient is either not given or only conjectured. Similar results are also obtained for RMA code ensembles with $M \geq 3$ and code rates equal to $1/2$ or smaller. Further, by introducing random puncturing at the output of the inner accumulator, we demonstrate that
the resulting high rate RMA ensembles exhibit linear distance growth, where the asymptotic growth rate coefficient is close to the GVB if the mother code rate is sufficiently low.

Despite the fact that the RMA code ensembles considered in this paper are asymptotically good, the convergence behavior of these codes may not be sufficient to provide an iterative decoding threshold close to capacity, as can be seen from the simulation results presented in [8]. However, the results obtained may be useful in constructing similar code ensembles based on simple component codes with low encoding complexity, asymptotically linear distance growth, and good convergence behavior. In particular, for the interesting class of double serially concatenated codes, the RAA ensemble can be used as a starting point to design asymptotically good code constructions by replacing one or more of the constituent encoders with small memory convolutional codes, whose code polynomials can be chosen to improve iterative decoding convergence behavior. Some initial experimental results in this direction for different component encoders are presented in [13], [14]. Another approach is to consider hybrid concatenated coding schemes, where parts of the encoder are structurally equivalent to RAA encoders. Initial results for low code rates have shown that these schemes have improved threshold behavior compared to RAA codes, while still providing asymptotically linear distance growth, albeit with a smaller growth rate coefficient [15].

APPENDIX

A. Proof of Lemma 1

We consider the following five partial cases:

\begin{align*}
1 - \frac{2}{q} & \leq a = b \leq c < 1 \quad (43) \\
1 - \frac{2}{q} & \leq a = b < c \leq 1 \quad (44) \\
0 & \leq a < b \leq c < 1 \quad (45) \\
0 & \leq a < b < c = 1 \quad (46) \\
0 & \leq a < b = c = 1. \quad (47)
\end{align*}

From Theorem 1 it follows that \( d_1 \geq \beta N^{1-2/q} \) for almost all codes in the ensemble and thus we do not need to consider the case of \( b < 1 - \frac{2}{q} \).

In addition to \( 7 \), we will use the inequality \( 16 \)

\[
\sqrt{\frac{N}{8l(N-l)}} \exp \left( H \left( \frac{l}{N} \right) N \right) \leq \left( \frac{N}{l} \right) \leq \sqrt{\frac{N}{2\pi l(N-l)}} \exp \left( H \left( \frac{l}{N} \right) N \right), \quad (48)
\]
or the equivalent expression

\[
\binom{N}{l} = \exp \left( H \left( \frac{l}{N} \right) N + o(N) \right)
\]  

(49)

to bound binomial coefficients.

(a) Cases (43) and (44):

Using (7), (22), and (48), we can rewrite \( E(A_{d_1,w}) = \binom{K}{w} \Pr(d_1|w) \) as

\[
E(A_{d_1,w}) = \left( \frac{N}{q\alpha N^a} \right)^{\alpha N^a} \exp \left( H \left( \frac{q\alpha}{2\beta} \right) \right) \left( \frac{2N}{q\alpha N^a} \right)^{\frac{2}{q\alpha N^a}} \left( \frac{N}{q\alpha N^a} \right)^{-q\alpha N^a} c_N(w),
\]

(50)

where

\[
c_N(w) = \frac{\varphi_w(w) \left( \varphi_{qw} \left( \frac{qw}{2} \right) \right)^2}{\varphi_N(qw)} = \exp[o(N^a \ln N)]
\]

(51)
is a second order term. For simplicity we assume \( \left\lceil \frac{x}{2} \right\rceil = \left\lfloor \frac{x}{2} \right\rfloor = \frac{x}{2} \), which is valid for any even integer \( x \) and approximately valid for large odd \( x \). Then we have from (50) that

\[
E(A_{d_1,w}) = \exp \left[ \left( 1 - \frac{q}{2} \right) \alpha N^a \ln N + o(N^a \ln N) \right].
\]

(52)

Since \( E(A_{d_1,w}) \to 0 \) as \( N \to \infty \), it follows that \( \lim_{N \to \infty} N^3 E(A_{d,d_1,w}) = 0 \) independently of \( d \) for almost all codes in the ensemble.

(b) Case (45):

To make the expressions more compact, we will omit second order terms from now on. Again, using (7), (22), and (48), we can write

\[
E(A_{d_1,w}) \approx \left( \frac{N}{q\alpha N^a} \right)^{\alpha N^a} \left( \frac{2N}{N^b} \right)^{\frac{2}{N^b}} \left( \frac{2N}{q\alpha N^a} \right)^{\frac{2}{q\alpha N^a}} \left( \frac{N}{q\alpha N^a} \right)^{-q\alpha N^a}
\]

(53)

\[
= \exp \left[ \left( 1 - a \right) \left( 1 - \frac{q}{2} \right) \alpha + \left( b - a \right) \frac{q\alpha}{2} \right] N^a \ln N + o(N^a \ln N)
\]

and

\[
\Pr(d|d_1) \approx \left( \frac{2pN^c}{N^b} \right)^{\frac{2}{N^b}} \left( \frac{2N}{N^b} \right)^{\frac{2}{N^b}} \left( \frac{N}{N^b} \right)^{-\beta N^b}
\]

(54)

\[
= \exp \left[ -\left( 1 - b \right) \frac{\beta}{2} N^b \ln N + o(N^b \ln N) \right].
\]

Since \( b > a \), it follows from (53) and (54) that

\[
E(A_{d,d_1,w}) = \exp \left[ -(1 - b) \frac{\beta}{2} N^b \ln N + o(N^b \ln N) \right]
\]

(55)

and \( N^3 E(A_{d,d_1,w}) \to 0 \) as \( N \to \infty \).
(c) Case 46:

In this case $E(A_{d_1,w})$ is still given by (53), but

\[
\Pr(d|d_1) \approx \left( \frac{2\rho N}{\beta N^b} \right)^{\frac{\beta}{2} N^b} \left( \frac{2(1-\rho)N}{\beta N^b} \right)^{\frac{\beta}{2} N^b} \left( \frac{N}{\beta N^b} \right)^{-\beta N^b} = \exp \left( \frac{\beta}{2} N^b \ln(4\rho(1-\rho)) + o(N^b) \right).
\]

(56)

Since $b > a$, it follows from (53) and (56) that

\[
E(A_{d_1,w}) = \exp \left( \frac{\beta}{2} N^b \ln(4\rho(1-\rho)) + o(N^b) \right)
\]

(57)

and $N^3 E(A_{d_1,w}) \to 0$ as $N \to \infty$ for $\rho < 1/2$.

(d) Case 47:

In this case $E(A_{d_1,w})$ is still given by (53), but

\[
\Pr(d|d_1) = \left( \frac{d}{\lceil d_1/2 \rceil} \right) \left( \frac{N-d}{\lceil d_1/2 \rceil} \right) \left( \frac{d_1}{\lceil d_1/2 \rceil} \right) \left( \frac{N-d_1}{\lceil d_1/2 \rceil} \right) \left( \frac{d_1}{\lceil d_1/2 \rceil} \right) = \exp \left[ \left( \beta \ln 2 + H \left( \frac{\rho - \beta/2}{1-\beta} \right) \right) \left( 1 - \beta \right) - H(\rho) \right] N + o(N)
\]

(58)

The function

\[
F_\rho(\beta) = \beta \ln 2 + H \left( \frac{\rho - \beta/2}{1-\beta} \right) (1 - \beta) - H(\rho)
\]

(59)

is strictly negative for all $\rho < 1/2$ and $\beta \leq 2\rho$, which can be shown as follows. The derivative of $F_\rho(\beta)$,

\[
\frac{\partial F}{\partial \beta} = \frac{1}{2} \ln \left( \frac{\rho - \beta/2}{1-\beta} \right) + \frac{1}{2} \ln \left( \frac{1-\rho - \beta/2}{1-\beta} \right) + \ln 2
\]

(60)

is negative if $x = \frac{\rho - \beta/2}{1-\beta} \neq 1/2$ and equals zero if $x = 1/2$. $F_\rho(\beta)$ is thus negative for all $\rho \leq 1/2$ and all $0 < \beta \leq 2\rho$.

From (53) and (58) follows that

\[
E(A_{d_1,w}) = \exp \left[ \left( \beta \ln 2 + H \left( \frac{\rho - \beta/2}{1-\beta} \right) \right) \left( 1 - \beta \right) - H(\rho) \right] N + o(N)
\]

(61)

and $N^3 E(A_{d_1,w}) \to 0$ as $N \to \infty$ for all $\rho < 1/2$. 
B. Proof of Lemma 4

(a) On the boundary \((\alpha = 0, 0 \leq \beta \leq 2\rho)\), we have
\[
\frac{\partial f}{\partial \beta}_{|\alpha=0} = \frac{1}{2} \ln \left( \frac{\rho - \beta/2}{1 - \beta} \right) + \frac{1}{2} \ln \left( \frac{1 - \rho - \beta/2}{1 - \beta} \right) + \ln 2
\]
\[
= \frac{1}{2} \ln \left( 4 x_\beta (1 - x_\beta) \right) < 0
\] (62)
where \(x_\beta = \frac{\alpha - \beta/2}{1 - \beta}\). Since \(f(\alpha, \beta, \rho)_{|\alpha=0,\beta=0} = 0\), we obtain \(f(\alpha, \beta, \rho) < 0\) on the boundary (a) for all \(\beta > 0\) and \(\rho \leq 1/2\).

(b) On the boundary \((0 \leq \alpha \leq \min(1, 4\rho), \beta = \alpha/2)\) the total derivative with respect to \(\alpha\) is given as
\[
\frac{df}{d\alpha}_{|\beta=\alpha/2} = \frac{\partial f}{\partial \alpha}_{|\beta=\alpha/2} + \frac{\partial f}{\partial \beta}_{|\beta=\alpha/2} \frac{d\beta}{d\alpha}
\]
\[
= -\frac{1}{q} \ln \left( \frac{\alpha}{1 - \alpha} \right) + \frac{1}{2} \ln \left( \frac{\alpha/2}{1 - \alpha/2} \right)
\]
\[
+ \frac{1}{2} \left[ \frac{1}{2} \ln \left( \frac{\rho - \beta/2}{1 - \beta} \right) + \frac{1}{2} \ln \left( \frac{1 - \rho - \beta/2}{1 - \beta} \right) + \ln 2 \right].
\] (63)
Note that the inequality
\[
\frac{1}{q} \ln \left( \frac{\alpha}{1 - \alpha} \right) > \frac{1}{2} \ln \left( \frac{\alpha/2}{1 - \alpha/2} \right) + \ln 2
\] (64)
holds for \(q \geq 3\) and \(0 \leq \alpha < 1\). The last term on the right hand side of (63) is equivalent to (62). By inserting (62) and (64) into (63) we obtain \(df/d\alpha_{|\beta=\alpha/2} < 0\). Therefore, we conclude that \(f(\alpha, \beta, \rho)\) is negative along the boundary \((0 \leq \alpha \leq \min(1, 4\rho), \beta = \alpha/2)\) for all \(0 \leq \rho < 0.5\).

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