ON POTENTIAL FUNCTION OF GRADIENT STEADY RICCI SOLITONS

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Abstract

In this paper we study potential function of gradient steady Ricci solitons. We prove that infimum of potential function decays linearly; in particular, potential function of rectifiable gradient steady Ricci solitons decays linearly. As a consequence, we show that a gradient steady Ricci soliton with bounded potential function must be trivial, and no gradient steady Ricci soliton admits uniformly positive scalar curvature.

1 Introduction

A complete Riemannian metric $g$ on a smooth manifold $M^n$ is called a gradient Ricci soliton if there is a smooth function $f : M^n \to \mathbb{R}$ such that

$$Ric + Hess f = \lambda g,$$

(1)

for some constant $\lambda$. The function $f$ is called a potential function for $g$. When $\lambda > 0$ the Ricci soliton is shrinking, when $\lambda = 0$ it is steady, and when $\lambda < 0$ it is expanding. When $f$ is constant the gradient Ricci soliton is simply an Einstein manifold. Thus Ricci solitons are natural extensions of Einstein metrics, an Einstein manifold with constant potential function is called a trivial gradient Ricci soliton. Gradient Ricci solitons play an important role in Hamilton’s Ricci flow as they correspond to self-similar solutions, and often arise as singularity models. They are also related to smooth metric measure spaces, since equation (1) is equivalent to $\infty$-Bakry-Emery Ricci tensor $Ric_f = 0$. In physics, a smooth metric space $(M^n, g, e^{-f}dvol)$ with $Ric_f = \lambda g$ is called quasi-Einstein manifold. Therefore it is important to study geometry and topology of gradient Ricci solitons and their classifications. Recently there have been a lot of work on gradient solitons, see [3, 8, 10, 11, 14, 16, 17, 18, 20, 21, 22, 23] for example; and [1, 2] for excellent surveys.

The growth of the potential function of gradient Ricci solitons has been an interesting problem. For gradient shrinking solitons, H.-D. Cao and D. Zhou [5] proved that the potential function grows quadratically, and based on this estimate they proved that gradient shrinking Ricci solitons have at most Euclidean volume growth.

In [12] R. Hamilton proved the following identity for gradient Ricci solitons

$$R + |\nabla f|^2 - 2\lambda f = \Lambda.$$

(2)

where $R$ is the scalar curvature of $(M^n, g)$, and $\Lambda$ is a constant. For gradient steady Ricci solitons, equation (2) becomes

$$R + |\nabla f|^2 = \Lambda.$$

where $\Lambda$ is a constant. B.-L. Chen [6] proved that the scalar curvature $R \geq 0$, hence $|\nabla f| \leq \sqrt{\Lambda}$, and the potential function has at most linear growth. It is natural to ask that whether potential function of a gradient steady Ricci soliton grows linearly at infinity.
Recently H.-D. Cao and Q. Chen [4] partially confirmed this under additional assumption of positive Ricci curvature, they proved that if a gradient steady Ricci soliton has positive Ricci curvature and scalar curvature attains its maximum at some point, then the potential function decays linearly.

In general one cannot expect potential function to grow or decay linearly along all directions at infinity, because of the product property: the product of any two gradient steady Ricci solitons is also a gradient steady Ricci soliton. Consider for example \((\mathbb{R}^2, g_0, f)\), where \(g_0\) is the standard Euclidean metric, \(f(x_1, x_2) = x_1\). \(f\) is constant along \(x_2\) direction, so without additional conditions, \(f\) may not have linear growth at infinity. We prove that though the potential function may not be linear, its infimum must decay linearly,

**Theorem 1.1** Let \((M^n, g, f)\) be a gradient steady Ricci soliton with \(R + |\nabla f|^2 = \Lambda\). For any \(x \in M^n\), there exists \(r_0 > 0\), such that for any \(r \geq r_0\),

\[
-\sqrt{\Lambda}r \leq \inf_{y \in \partial B_r(x)} f(y) - f(x) \leq -\sqrt{\Lambda}r + \sqrt{2n\sqrt{\Lambda}r + 1}.
\]

Therefore, a gradient steady Ricci soliton with bounded potential function must be trivial.

**Remark 1.2** The author was told that Ovidiu Munteanu and Natasa Sesum [15] also got the same result using a different method.

As a consequence, we have

**Corollary 1.3** Let \((M^n, g, f)\) be a complete noncompact gradient steady Ricci soliton. Then for any \(x \in M^n\),

\[
\limsup_{y \in B_r(x), r \to \infty} |\nabla f|(y) = \sqrt{\Lambda}, \quad \liminf_{y \in B_r(x), r \to \infty} R(y) = 0.
\]

In another word, no gradient steady Ricci soliton admits uniformly positive scalar curvature.

In [20] P. Petersen and W. Wylie introduced the concept of rectifiable Ricci soliton and studied their rigidity. A gradient Ricci soliton is called rectifiable if its potential function \(f = f(r)\), where \(r\) is the distance function. In this case \(\inf_{y \in \partial B_r(x)} f = f(r)\), by the same argument for proof of Theorem 1.1 we show that its potential function decays linearly,

**Theorem 1.4** Let \((M^n, g, f)\) be a complete noncompact rectifiable gradient steady Ricci soliton. Then there exists \(r_0 > 0\), such that when \(r = d(x, \cdot) \geq r_0\),

\[
-\sqrt{\Lambda}r \leq f(r) - f(0) \leq -\sqrt{\Lambda}r + \sqrt{n\sqrt{\Lambda}(r + 1)}.
\]

**Remark 1.5** We in fact obtained an estimate of potential function for almost all known gradient steady Ricci soliton examples, since almost all of them are rectifiable, see for instance solitons constructed in [9] and [13].

Our tool is \(f\)-volume comparison theorem for smooth metric measure spaces that were developed by G. Wei and W. Wylie [24]. In Section 2 we improve the \(f\)-volume comparison theorem for smooth metric measure spaces with nonnegative Bakry-Emery Ricci tensor, and apply to gradient steady Ricci solitons to obtain an upper bound of the \(f\)-volume. In Section 3 we deduce a lower bound for the \(f\)-volume and prove the main theorem.

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2 Volume Comparison for Smooth Metric Measure Spaces

In this section, we improve \( f \)-volume comparison theorem for smooth metric spaces under the condition \( f \) is at most linear. Recall that a smooth metric measure space is triple \((M^n, g, e^{-f}dvol_g)\), where \((M^n, g)\) is a Riemannian manifold, \( f \) is a smooth real valued function on \( M \). Smooth metric measure spaces play an essential role in Perelman’s work on the Ricci flow, and they arise as smooth collapsed measured Gromov-Hausdorff limits.

The Ricci curvature of smooth metric measure space, which is called Bakry-Emery Ricci curvature, is defined as \( \text{Ric}_f = \text{Ric}_g + \text{Hess} f \). The self-adjoint Laplacian with respect to the weighted measure \( e^{-f}dvol_g \) is \( \Delta_f = \Delta - \nabla f \nabla f \), the weighted (or \( f \)-)mean curvature is defined as \( m_f = m - \langle \nabla f, \nabla r \rangle = \Delta_f (r) \), where \( r \) is the distance function, and the weighted (or \( f \)-)volume is defined as \( \text{Vol}_f (B_r (x)) = \int_{B_r (x)} e^{-f}dvol_g \). Fix \( x \in M^n \), under exponential polar coordinates around \( x \) we write the volume element \( dvol_g = A(r, \theta)dr \wedge d\theta_{n-1} \), where \( d\theta_{n-1} \) is the standard volume element on the unit sphere \( S^{n-1} \). Let \( A_f (r, \theta) = e^{-f}A(r, \theta) \), it is easy to check that \( (\ln [A_f (r, \theta)]) = m_f (r) \), and \( \text{Vol}_f (B_r (x)) = \int_0^r \int_{S^{n-1}} A_f (t, \theta)dt d\theta \).

Denote \( \text{Vol}^n_R (B_r) \) be the volume of the radius \( r \)-ball in the model space \( M^n_R \). G. Wei and W. Wylie [24] proved the following \( f \)-volume comparison theorem for smooth metric measure spaces.

**Theorem 2.1** (\( f \)-Volume comparison) (Theorem 1.2 in [24]).
Suppose \((M^n, g, e^{-f}dvol)\) is a smooth metric measure space with \( \text{Ric}_f \geq (n-1)H \). Fix \( x \in M \). If \( |f| \leq \Lambda \). Then for \( R \geq r > 0 \) (assume \( R \leq \pi/4\sqrt{H} \) if \( H > 0 \))
\[
\frac{V_f (B_R (x))}{V_f (B_r (x))} \leq \frac{V^{n+4\Lambda}_H (B_R)}{V^{n+4\Lambda}_H (B_r)}.
\]

**Remark 2.2** If \( |\nabla f| \leq a \), without loss of generality, assume \( f (x) = 0 \), then \( f (y) \leq aR \) for \( y \in B_R (x) \), and Theorem 2.1 becomes
\[
\frac{V_f (B_R (x))}{V_f (B_r (x))} \leq \frac{V^{n+4Ra}_H (B_R)}{V^{n+4Ra}_H (B_r)}.
\]

For our purpose, we concentrate on the case \( H = 0 \). Denote \( \text{Vol}^n_R (B_r) \) be the volume of the radius \( r \)-ball in \( \mathbb{R}^n \). When \( f \) has at most linear growth, we obtain the following:

**Theorem 2.3** Suppose \((M^n, g, e^{-f}dvol)\) is a smooth metric measure space with \( \text{Ric}_f \geq 0 \), \( |\nabla f| \leq a \); and in addition \( \frac{f(y) - f(x)}{d(y, x)} \geq -a + \epsilon \) for \( y \in B_R (x) \setminus B_s (x) \) for \( R \geq s > 0 \), then for \( s < S < R, s < r < R \), we have
\[
\frac{V_f (B_R (x) \setminus B_r (x))}{V_f (B_S (x) \setminus B_s (x))} \leq \frac{V^{n+Ra}_R (B_R \setminus B_r)}{V^{n+Ra}_R (B_S \setminus B_s)}.
\]

where \( \bar{a} = a - e^2/2a \). In particular, if \( \frac{f(y) - f(x)}{d(y, x)} \geq -a + \epsilon \) for \( y \in B_R (x) \), then for \( r < R \), we have
\[
\frac{V_f (B_R (x))}{V_f (B_r (x))} \leq \frac{V^{n+Ra}_R (B_R)}{V^{n+Ra}_R (B_r)}.
\]

**Remark 2.4** In Theorem 2.3 we improved the dimension of the model space from \( n + 4ar \) to \( n + \bar{a}r \), which is crucial in our proof of the main theorem.
To prove the $f$-volume comparison theorem, we first prove the following $f$-mean curvature comparison theorem, and Theorem 2.3 follows directly from the argument in G. Wei and W. Wylie [24] and Lemma 3.2 in [25].

**Theorem 2.5 (f-Mean Curvature Comparison).** Suppose $(M^n, g, e^{-f}dvol)$ is a smooth metric measure space with $\text{Ric}_f \geq 0$. If $|\nabla f| \leq a$ along a minimal geodesic segment from $x$, and $\frac{f(y)-f(x)}{d(y,x)} \geq -a + \epsilon$ for $y \in \partial B_r(x)$, then

$$m_f(r) \leq \tilde{a} + \frac{n-1}{r} = m_{\mathbb{R}}^{n+\tilde{a}}(r).$$

along that minimal geodesic segment from $x$, where $\tilde{a} = a - \epsilon^2/2a$.

**Proof.** From inequality (2.21) in [24]

$$m_f(r) \leq \frac{n-1}{r} - 2 \frac{r}{r^2} f(r) + 2 \frac{2}{r^2} \int_0^r f(t)dt.$$

Suppose $f(y) - f(x) = (-a + \epsilon)r$ for some $y \in \partial B_r(x)$, we will maximize

$$-\frac{2}{r^2} f(y) + 2 \frac{2}{r^2} \int_0^r f(t)dt = \frac{2}{r^2} \int_0^r (f(t) - f(y))dt$$

along a minimal geodesic segment from $x$ to $y$.

Since $|\nabla f| \leq a$, along a minimal geodesic segment from $x$ to $y$, $f(t) - f(x)$ is bounded from above by

$$F(t) = \begin{cases} at, & 0 \leq t \leq \frac{\epsilon}{2a}r, \\ -at + \epsilon r, & \frac{\epsilon}{2a}r \leq t \leq r, \end{cases}$$

Thus

$$\int_0^r (f(t) - f(y))dt = \int_0^r [(f(t) - f(x)) - (f(y) - f(x))]dt$$

$$\leq \int_0^r (F(t) - (-a + \epsilon)t)dt$$

$$= \frac{r^2}{2} (a - \frac{\epsilon^2}{2a}).$$

Hence if $\frac{f(y)-f(x)}{d(y,x)} \geq -a + \epsilon$ on $\partial B_r(x)$, we obtain the following $f$-mean curvature comparison

$$m_f(r) \leq \tilde{a} + \frac{n-1}{r} = m_{\mathbb{R}}^{n+\tilde{a}}(r).$$

### 3 Proof of the Main Theorems

**Proof of Theorem 1.1.** To prove that infimum of the potential function decays linearly, we will derive an inequality on the left hand side of the volume comparison in Theorem 2.3.

Taking trace of the soliton equation (1), we get $R + \Delta f = 0$. Add to equation (2) we obtain

$$-\Delta_f f = -\Delta f + |\nabla f|^2 = \Lambda.$$ (3)
Hence by maximal principle, \( f \) has no local minimum.

Choose \( x \in M, \delta > 0 \). Suppose \( \inf \frac{\frac{d}{dy}(f(y)-f(x))}{d(y,x)} = -\sqrt{\Lambda} + \epsilon \) in \( B_{\sqrt{\Lambda}+\delta}(x) \setminus B_{\sqrt{\Lambda}}(x) \), so \( \Lambda = \sqrt{\Lambda} - \epsilon^2/2\sqrt{\Lambda} \). Since \( f \) has no local minimum, we have \( \epsilon \leq \sqrt{\Lambda} \), and \( \Lambda \geq \sqrt{\Lambda}/2 \). Choose a smooth cut-off function \( \phi \), such that

\[
\phi(y) = \begin{cases} 
1, & y \in B_{\sqrt{\Lambda}+\delta}(x) \\
0, & y \in M \setminus B_{\sqrt{\Lambda}+\delta}(x),
\end{cases}
\]

and \( |\nabla \phi| \leq \frac{1+\delta}{\delta} \).

Integrate equation (3) in \( B_{\sqrt{\Lambda}+\delta}(x) \) and apply stokes formula,

\[
\Lambda \int_{B_{\sqrt{\Lambda}+\delta}(x)} e^{-f} \phi d\text{vol} = -\int_{B_{\sqrt{\Lambda}+\delta}(x)} \Delta f e^{-f} \phi d\text{vol}
\]

\[
= \int_{B_{\sqrt{\Lambda}+\delta}(x) \setminus B_{\sqrt{\Lambda}}(x)} \langle \nabla f, \nabla \phi \rangle e^{-f} d\text{vol}
\]

\[
\leq \int_{B_{\sqrt{\Lambda}+\delta}(x) \setminus B_{\sqrt{\Lambda}}(x)} |\nabla f| |\nabla \phi| e^{-f} d\text{vol}
\]

\[
\leq \frac{(1+\delta)\sqrt{\Lambda}}{\delta} \int_{B_{\sqrt{\Lambda}+\delta}(x) \setminus B_{\sqrt{\Lambda}}(x)} e^{-f} d\text{vol}.
\]

Therefore

\[
\Lambda \int_{B_{\sqrt{\Lambda}+\delta}(x) \setminus B_{\sqrt{\Lambda}}(x)} e^{-f} d\text{vol} \leq \Lambda \int_{B_{\sqrt{\Lambda}+\delta}(x)} e^{-f} \phi d\text{vol}
\]

\[
\leq \frac{(1+\delta)\sqrt{\Lambda}}{\delta} \int_{B_{\sqrt{\Lambda}+\delta}(x) \setminus B_{\sqrt{\Lambda}}(x)} e^{-f} d\text{vol}.
\]

So we get

\[
\frac{\delta\sqrt{\Lambda}}{1+\delta} \leq \frac{V_f(B_{\sqrt{\Lambda}+\delta}(x) \setminus B_{\sqrt{\Lambda}}(x))}{V_f(B_{\sqrt{\Lambda}+\delta}(x) \setminus B_{\sqrt{\Lambda}}(x))}.
\]

On the other hand, by Theorem 2.3, we have

\[
\frac{V_f(B_{\sqrt{\Lambda}+\delta}(x) \setminus B_{\sqrt{\Lambda}}(x))}{V_f(B_{\sqrt{\Lambda}}(x) \setminus B_{\sqrt{\Lambda}}(x))} \leq \frac{V_{\mathbb{R}}^{n+(r+\sqrt{\Lambda})\Lambda}(B_{\sqrt{\Lambda}+\delta}) - V_{\mathbb{R}}^{n+(r+\sqrt{\Lambda})\Lambda}(B_{\sqrt{\Lambda}})}{V_{\mathbb{R}}^{n+(r+\sqrt{\Lambda})\Lambda}(B_{\sqrt{\Lambda}}) - V_{\mathbb{R}}^{n+(r+\sqrt{\Lambda})\Lambda}(B_r)}
\]

\[
= \frac{(r+\sqrt{\Lambda})^{n+(r+\sqrt{\Lambda})\Lambda} - (r+\sqrt{\Lambda})^{n+(r+\sqrt{\Lambda})\Lambda}}{(r+\sqrt{\Lambda})^{n+(r+\sqrt{\Lambda})\Lambda} - r^{n+(r+\sqrt{\Lambda})\Lambda}}.
\]

Therefore,

\[
\frac{\delta\sqrt{\Lambda}}{1+\delta} \leq \frac{(r+\sqrt{\Lambda})^{n+(r+\sqrt{\Lambda})\Lambda} - (r+\sqrt{\Lambda})^{n+(r+\sqrt{\Lambda})\Lambda}}{(r+\sqrt{\Lambda})^{n+(r+\sqrt{\Lambda})\Lambda} - r^{n+(r+\sqrt{\Lambda})\Lambda}}.
\]

Divide both sides by \( \delta \) and let \( \delta \to 0 \), we get

\[
\sqrt{\Lambda} \leq \frac{\frac{n}{r+\sqrt{\Lambda}} + \tilde{\Lambda}}{1 - (\frac{1}{\sqrt{\Lambda}})^{-n-(r+\sqrt{\Lambda})\Lambda}.}
\]
Therefore,
\[ \epsilon \leq \sqrt{\frac{2n\sqrt{\Lambda}}{r + \sqrt{r}} + 2\Lambda(1 + \frac{1}{\sqrt{r}})^{-n\sqrt{\Lambda}(r + \sqrt{r})/2}}. \]

Thus when \( r \) is sufficiently large, \( \epsilon \leq \sqrt{\frac{2n\sqrt{\Lambda}}{r}} \), hence
\[ \inf_{y \in B_{r_0}(x) \setminus B_r(x)} \frac{f(y) - f(x)}{d(x, y)} \leq -\sqrt{\Lambda} + \sqrt{\frac{2n\sqrt{\Lambda}}{r}}. \]

Now suppose \( \inf \frac{f(y) - f(x)}{d(y, x)} \) is attained by \( y_0 \in B_{r_0}(x) \) for some \( r \leq r_0 \leq r + \sqrt{r} \). Let \( z = \partial B_r(x) \cap \gamma(t) \), where \( \gamma(t) \) is the minimal geodesic from \( x \) to \( y_0 \), then
\[ f(y_0) - f(x) \leq -\sqrt{\Lambda}r_0 + \sqrt{\frac{2n\sqrt{\Lambda}}{r}}r_0, \]
\[ f(z) - f(y_0) \leq \sqrt{\Lambda}(r_0 - r). \]

Therefore,
\[ \inf_{y \in \partial B_r(x)} f(y) - f(x) \leq f(z) - f(x) \leq -\sqrt{\Lambda}r + \sqrt{2n\sqrt{\Lambda}(\sqrt{r} + 1)}. \]

Proof of Theorem 1.4. For rectifiable gradient steady Ricci solitons, since \( f(r) \) has no local minimum, it attains its maximum at \( r = 0 \), and is monotonically nonincreasing, hence the function \( F(t) \) in the proof of Theorem 2.5 can be replaced by
\[ F(t) = \begin{cases} 
0, & 0 \leq t \leq \frac{\epsilon}{a}r, \\
-at + \epsilon r, & \frac{\epsilon}{a}r \leq t \leq r,
\end{cases} \]

Therefore the dimension of the model space in Theorem 2.5 becomes \( \tilde{a} = a - \epsilon^2/a \). Hence by the above argument we obtain
\[ -\sqrt{\Lambda}r \leq f(r) - f(0) \leq -\sqrt{\Lambda}r + \sqrt{2n\sqrt{\Lambda}(\sqrt{r} + 1)}. \]

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