HANKEL DETERMINANTS OF DIRICHLET SERIES

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Abstract. We derive a general expression for the Hankel determinants of a Dirichlet series \( F(s) \) and derive the asymptotic behavior for the special case that \( F(s) \) is the Riemann zeta function. In this case the Hankel determinant is a discrete analogue of the Selberg integral and can be viewed as a matrix integral with discrete measure. We briefly comment on its relation to Plancherel measures.

1. Introduction

In this paper we will consider the Hankel determinants

\[
H^{(0)}_1[\zeta] = \zeta(2), \quad H^{(0)}_2[\zeta] = \begin{vmatrix} \zeta(2) & \zeta(3) & \zeta(4) \\ \zeta(3) & \zeta(4) & \zeta(5) \end{vmatrix}, \quad H^{(0)}_3[\zeta] = \begin{vmatrix} \zeta(2) & \zeta(3) & \zeta(4) \\ \zeta(3) & \zeta(4) & \zeta(5) \\ \zeta(4) & \zeta(5) & \zeta(6) \end{vmatrix} \ldots
\]

and

\[
H^{(1)}_1[\zeta] = \zeta(3), \quad H^{(1)}_2[\zeta] = \begin{vmatrix} \zeta(3) & \zeta(4) \\ \zeta(4) & \zeta(5) \end{vmatrix}, \quad H^{(1)}_3[\zeta] = \begin{vmatrix} \zeta(3) & \zeta(4) & \zeta(5) \\ \zeta(4) & \zeta(5) & \zeta(6) \\ \zeta(5) & \zeta(6) & \zeta(7) \end{vmatrix} \ldots
\]

and various generalizations of them. These determinants go very rapidly to zero as the dimension of the matrix becomes large e.g.

\[
H^{(0)}_{100}[\zeta] \approx 4.9 \times 10^{-16684}
\]

\[
H^{(1)}_{100}[\zeta] \approx 4.3 \times 10^{-16871}
\]

The ratios have (experimentally) a surprisingly simple asymptotic expansion:

\[
-\frac{H^{(0)}_{n+1}[\zeta]H^{(1)}_n[\zeta]}{H^{(0)}_n[\zeta]H^{(1)}_{n+1}[\zeta]} = \frac{1}{2n+1} + \frac{2}{(2n+1)^2} - \frac{7}{3(2n+1)^3} + \frac{16}{5(2n+1)^4} - \frac{41}{9(2n+1)^5} + \ldots
\]

\[
-\frac{H^{(0)}_{n+1}[\zeta]H^{(1)}_n[\zeta]}{H^{(0)}_n[\zeta]H^{(1)}_{n+1}[\zeta]} = \frac{1}{2n} - \frac{1}{(2n)^2} + \frac{2}{3(2n)^3} - \frac{6}{5(2n)^4} + \frac{56}{45(2n)^5} + \ldots
\]

This recursion can be solved to yield explicit expressions for \( H^{(r)}_n[\zeta] \). In fact more detailed numerical experiments by D. Zagier confirmed our findings with the result

\[
H^{(0)}_n[\zeta] = A^{(0)}\left( \frac{2n+1}{e\sqrt{e}} \right)^{-\left(n+\frac{1}{2}\right)^2} \left( 1 + \frac{1}{24(2n+1)^2} - \frac{12319}{259200} \frac{1}{(2n+1)^4} + \frac{504407873}{217728} \frac{1}{(2n+1)^6} \ldots \right)
\]

and

\[
H^{(1)}_n[\zeta] = A^{(1)}\left( \frac{2n}{e\sqrt{e}} \right)^{-\left(n+\frac{1}{2}\right)^2} \left( 1 - \frac{17}{240(2n)^2} - \frac{199873}{7257600} \frac{1}{(2n)^4} - \frac{90789413}{174182400} \frac{1}{(2n)^6} \ldots \right)
\]
with the interesting observation that

\[ A^{(0)} \approx 0.351466738331 \]
\[ A^{(1)} = \frac{e^{3/8}}{\sqrt{6}} A^{(0)}. \]

We do not know how to prove the full asymptotic expansions but will describe a method that lets one at least understand the weaker asymptotic form

\[ \log H_n^{(0)}[\zeta] \sim \log H_n^{(1)}[\zeta] \sim -n^2 (\log(2n) - \frac{3}{2}). \]

We will also discuss interesting relations with a continuous version (Selberg integral) and with the Plancherel measure of the symmetric group. We start with the presentation of some results on the Hankel determinants of Dirichlet series.

2. Hankel determinants of Dirichlet series

**Definition.** Let \( F(s) \) be a Dirichlet series with coefficients \( f(n) \) i.e.

\[ F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}. \]

For \( n \) integer, \( n > 0 \), \( r \) integer, \( r \geq 0 \) we define the Hankel determinant \( H_n^{(r)}[F] \):

\[ H_n^{(r)}[F] = \det (F(i + j + r))_{1 \leq i, j \leq n} \]

We will also use the notation \( H_n[F] = H_n^{(0)}[F] \). Our first result is:

**Theorem 2.1.** \( H_n^{(r)}[F] \) is given by

\[ H_n^{(r)}[F] = \frac{1}{n!} \sum_{m_1, m_2, \ldots, m_n \geq 1} \prod_{i=1}^{n} \frac{f(m_i)}{m_i^{2n+r}} \prod_{i<j} (m_i - m_j)^2. \]

**Proof.** We prove the case \( r = 0 \) first. Using the definition of the determinant

\[
H_n^{(0)}[F] = \sum_{\pi \in S_n} (-1)^\pi F(1 + \pi(1))F(2 + \pi(2)) \cdots F(n + \pi(n)) \\
= \sum_{m_1, m_2, \ldots, m_n \geq 1} \sum_{\pi \in S_n} (-1)^\pi \frac{f(m_1)f(m_2) \cdots f(m_n)}{m_1^{1+\pi(1)}m_2^{2+\pi(2)} \cdots m_n^{n+\pi(n)}} \\
= \sum_{m_1, m_2, \ldots, m_n \geq 1} \frac{f(m_1)f(m_2) \cdots f(m_n)}{m_1^2m_2^3 \cdots m_n^{n+1}} \sum_{\pi \in S_n} (-1)^\pi \frac{\pi(1)m_1^{\pi(2)} \cdots m_1^{\pi(n)}}{m_2^{\pi(2)} \cdots m_n^{\pi(n)}} \\
= \sum_{m_1, m_2, \ldots, m_n \geq 1} \frac{f(m_1)f(m_2) \cdots f(m_n)}{m_1^2m_2^3 \cdots m_n^{n+1}} \prod_{i<j} \left( \frac{1}{m_i} - \frac{1}{m_j} \right)
\]

where \( S_n \) is the symmetric group and \( (-1)^\pi \) is the sign of the permutation \( \pi \). In the last line we have used the Vandermonde determinant

\[
\sum_{\pi \in S_n} (-1)^\pi \frac{1}{m_1^{\pi(1)}m_2^{\pi(2)} \cdots m_n^{\pi(n)}} = \frac{1}{m_1m_2 \cdots m_n} \prod_{i<j} \left( \frac{1}{m_i} - \frac{1}{m_j} \right).
\]
Interchanging two summation variables say $m_i$ and $m_j$ with $i \neq j$ the Vandermonde determinant changes sign. Summing over all permutations of \{1, 2, \ldots, n\} and dividing by the number of permutations we obtain:

$$H_n^{(0)}[F] = \frac{1}{n!} \sum_{m_1 \ldots m_n \geq 1} \frac{f(m_1) f(m_2) \cdots f(m_n)}{m_1 m_2 \cdots m_n} \sum_{\pi \in \mathfrak{S}_n} \frac{(-1)^{\pi}}{m_{\pi(1)}^1 m_{\pi(2)}^2 \cdots m_{\pi(n)}^n} \prod_{i<j} \left( \frac{1}{m_i} - \frac{1}{m_j} \right)$$

$$= \frac{1}{n!} \sum_{m_1 \ldots m_n \geq 1} \frac{f(m_1) f(m_2) \cdots f(m_n)}{m_1^2 m_2^2 \cdots m_n^2} \prod_{i<j} \left( \frac{1}{m_i} - \frac{1}{m_j} \right)^2$$

$$= \frac{1}{n!} \sum_{m_1 \ldots m_n \geq 1} \frac{f(m_1) f(m_2) \cdots f(m_n)}{(m_1 m_2 \cdots m_n)^{2n}} \prod_{i<j} (m_i - m_j)^2$$

For $r > 0$ we replace $f(n) \to f(n)n^{-r}$ which gives

$$H_n^{(r)}[F] = \frac{1}{n!} \sum_{m_1 \ldots m_n \geq 1} \frac{f(m_1) f(m_2) \cdots f(m_n)}{(m_1 m_2 \cdots m_n)^{2n+r}} \prod_{i<j} (m_i - m_j)^2$$

which completes the proof. \(\square\)

**Corollary 2.2.** If $f(n)$ is multiplicative it follows trivially from Theorem 2.1

$$H_n^{(r)}[\zeta] = \frac{1}{n!} \sum_{m_1 m_2 \cdots m_n = 1} f(m_1 m_2 \cdots m_n) \prod_{i<j} (m_i - m_j)^2.$$

$$= \sum_{m=1}^{\infty} \frac{f(m)}{m^{2n+r}} \left( \frac{1}{n!} \sum_{m_1 m_2 \cdots m_n = m} \prod_{i<j} (m_i - m_j)^2 \right)$$

which is again a Dirichlet series. Note that

$$\sum_{m_1 m_2 \cdots m_n = m} \prod_{i<j} (m_i - m_j)^2 \geq 0$$

and vanishes if the number of prime factors of $m$ is less than $n-1$ and is divisible by $\left( \prod_{i=1}^{n} (i-1)! \right)^2$ since each summand $\prod_{i<j} (m_i - m_j)^2$ is divisible by $\left( \prod_{i=1}^{n} (i-1)! \right)^2$. For a proof see [2].

**Lemma 2.3.** If $f(n) > 0$ for all integers $n > 0$ then

$$H_n^{(r)}[F] > 0.$$

**Proof.** The smallest $m$ contributing in the sum is $m = 1 \cdot 2 \cdots n = n!$. For $m_i = \{1, 2, \ldots, n\}$ we have

$$\sum_{m_1 m_2 \cdots m_n = m} \prod_{i<j} (m_i - m_j)^2 = \prod_{i=1}^{n} n \prod_{j=i+1}^{n} (j-i)^2 = \left( \prod_{i=1}^{n} (i-1)! \right)^2 > 0$$

so there is at least one term greater than zero in the sum in Corollary 2.2. The other terms in the sum are all greater or equal to zero so the sum is positive which proves the lemma. \(\square\)
Definition 2.4. For each integer $n > 0$ and integer $m > 0$ we define the function $h_n(m)$ by

$$h_n(m) = \frac{1}{m!} \sum_{m_1 \cdots m_n = m} \prod_{i < j} (m_i - m_j)^2.$$ 

For $n = 2$ this function can be expressed in terms of arithmetic functions:

$$h_2(m) = \sigma_2(m) - md(m)$$

where $\sigma_2(m)$ is the divisor function $\sigma_2(m) = \sum_{d|m} d^2$ and $d(m) = \sum_{d|m}$ gives the number of divisors of $m$. We are now in the position to state our second theorem.

Theorem 2.5. For a Dirichlet series $F(s)$ with coefficients multiplicative coefficients $f$ the Hankel determinant $H_n[F]$ is given by a Dirichlet series with coefficients $h_n(m)f(m)$:

$$H_n^{(r)}[F] = \sum_{m=1}^{\infty} \frac{h_n(m)f(m)}{m^{2n+r}}.$$ 

Proof. Inserting the definition of $h_n(m)$ in Corollary 2.2 gives the right hand expression. □

Example 2.6. Our first example, already discussed in the introductions, is $f(n) = 1$, so that $F(s) = \zeta(s)$ where $\zeta(s)$ is the Riemann zeta function. Using Theorem 2.1 we obtain:

$$H_n^{(r)}[\zeta] = \frac{1}{n!} \sum_{m_1 \cdots m_n \geq 1} \frac{1}{(m_1 m_2 \cdots m_n)^{2n+r}} \prod_{i < j} (m_i - m_j)^2.$$ 

In particular $H_n^{(r)}[\zeta] > 0$ for any $n$. Specializing to e.g. $n = 2$ and $r = 0$ we obtain from Theorem 2.1:

$$H_2[\zeta] = \frac{1}{2} \sum_{m_1, m_2 \geq 1} \frac{1}{m_1 m_2} (m_1 - m_2)^2$$

$$= \frac{1}{2} \sum_{m_1, m_2 \geq 1} \left( \frac{1}{m_1^2} - \frac{2}{m_1 m_2} + \frac{1}{m_2^2} \right)$$

$$= \zeta(2)\zeta(4) - \zeta(3)^2$$

$$= \left| \begin{array}{cc} \zeta(2) & \zeta(3) \\ \zeta(3) & \zeta(4) \end{array} \right|$$

The Hankel determinant can also be expressed as a linear combination of multiple zeta values

$$H_n[\zeta] = \sum_{\pi \in S_n} (-1)^\pi \sum_{m_n > \cdots > m_1 \geq 1} \frac{1}{m_1^{1+\pi(1)} m_2^{2+\pi(2)} \cdots m_n^{n+\pi(n)}}$$

$$= \sum_{\pi \in S_n} (-1)^\pi \zeta(1+\pi(1), 2+\pi(2), \ldots, n+\pi(n)).$$

Example 2.7. The second nontrivial example is

$$\det \left( \frac{1}{\zeta(i+j)} \right)_{1 \leq i, j \leq n} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{2n}} \left( \frac{1}{n!} \sum_{m_1 \cdots m_n = m} \prod_{i < j} (m_i - m_j)^2 \right)$$

where $\mu(n)$ is the Möbius function.
Remark. Multiple Dirichlet series have been investigated for some time for a recent work see e.g. [3] and [4].

3. A Probabilistic Model for the Asymptotic Behavior of \( H_n[\zeta] \)

In this section we determine the behavior of \( H_n[\zeta] \) as \( n \to \infty \). The basic idea is to find the dominant contribution to the sum. We note that all contributions are positive and that the Vandermonde determinant is only nonzero if the \( m_i \) are pairwise different. We can reorder them so that \( m_n > \ldots > m_2 > m_1 \). There are precisely \( n! \) contributions with \( m_1, m_2 \ldots m_n \) in the unordered sum. We write

\[
H_n[\zeta] = \sum_{m_n > \ldots > m_2 > m_1 \geq 1} \exp(\Phi(m_1, m_2 \ldots m_n))
\]

with

\[
\Phi(m_1, m_2 \ldots m_n) = -2n \sum_{i=1}^{n} \log(m_i) + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \log|m_j - m_i|.
\]

Finding the largest \( \Phi(m_1, m_2 \ldots m_n) \) is a discrete combinatorial optimization problem. We can also view \( \Phi \) as an energy functional of a one-dimensional Coulomb gas problem on a lattice where a large \((-2n)\) attractive charge is placed at zero and the charges placed at positions \( m_i \) integer repel each other with a logarithmic potential. Unlike in the standard one-dimensional Coulomb gas problem, the charges cannot have a distance smaller than one. Let us consider the case \( n \gg 1 \).

Building up the configuration by adding a charge at \( m_i \) one by one for the first few \( m_1, m_2 \ldots \) the first term is dominant so that \( \Phi \) is optimized by placing the first charge at \( m_1 = 1 \), the second at \( m_2 = 2 \) and so on. Adding more charges slowly builds up the second term which produces a repulsive potential which makes it more favorable to place charges at \( m_i > i \). From this analogy we expect the density of the \( m_i \) to be one from \( i = 1 \) up to some \( i_{\text{max}} < n \) and then to decay to zero as \( i \to n \).

We define the distribution function for each configuration \( \{m_1, m_2 \ldots m_n\} \)

\[
\rho(x) = \sum_{i=1}^{n} \delta(nx - m_i)
\]

with \( \delta(x) \) being the delta distribution. The integral over \( \rho(x) \) is

\[
\int_{0}^{\infty} \rho(x) dx = 1.
\]

by definition. We call a distribution which obeys Eq. (3.4) normalized. The functional \( \Phi(m_1, m_2 \ldots m_n) \) can be expressed as a functional of the distribution function \( \rho(x) \):

\[
\Phi[\rho] = -n^2 \left( 2 \int_{0}^{\infty} \rho(x) \log(nx) dx - \int_{0}^{\infty} \log |n(x - y)| \rho(x) \rho(y) dx dy \right)
= -n^2 \left( 2 \log(n) + 2 \int_{0}^{\infty} \rho(x) \log(x) dx - \log(n) + \int_{0}^{\infty} \rho(x) \rho(y) \log |x - y| dx dy \right)
= -n^2 \log(n) - n^2 \left( 2 \int_{0}^{\infty} \rho(x) \log(x) dx - \int_{0}^{\infty} \rho(x) \rho(y) \log |x - y| dx dy \right).
\]

(3.5)
We seek a continuous test function $\rho(x)$ which maximizes the functional Eq. 3.5 subject to the constraint that $0 < \rho(x) < 1$ for all $x$ and $\rho(x) = 0$ for $x < 0$ and Eq. 3.4. We assume that such a continuous functions exists. The arguments for its existence can probably be made much more rigorous by using Poissonization, see e.g. [7, 8]. The constraint Eq. 3.4 can be taken care of by introducing a Lagrange multiplier $\lambda \in \mathbb{R}$ and finding a extremum of $\Phi[\rho] - \lambda \int_0^\infty \rho(x)dx$. The problem of finding the extremum of $\Phi[\rho]$ is thus reduced to the integral equation

$$2\log(x) - 2\int_0^\infty \rho(y) \log |x-y| = \lambda. \quad (3.6)$$

This integral equation only applies when $\rho(x)$ can actually be varied, i.e. for $\rho(x) \neq 1$ and $\rho(x) \neq 0$. Differentiating with respect to $x$ eliminates the Lagrange multiplier and finally leads to

$$\frac{1}{x} = \int_0^\infty \frac{\rho(y)}{x-y}dy \quad (3.7)$$

where $\int$ denotes the principal value integral. Assume $\rho(x) < 1$ for all $x \in \mathbb{R}$ then the integral equation has to be fulfilled for all $x \in [0, \infty)$. The integral equation has as only solution $\rho(x) = \delta(x)$ where $\delta(x)$ is the Dirac distribution which diverges for $x \to 0$ and does not fulfill the constraint. Therefore the constraint $\rho(x) \leq 1$ has to be sharp for some positive real number $a$. We will use an Ansatz for $\rho(x)$ and verify that it obeys the integral equation and the constraint.

**Theorem 3.1.** Define the function $\rho(x)$ for $x \in [0, \infty)$:

$$\rho(x) = \begin{cases} 
1 & (0 \leq x \leq \frac{1}{2}) \\
\frac{1}{\pi} \left( \arctan \left( \frac{1}{\sqrt{2x-1}} \right) - \sqrt{2x-1} \right) & (x > \frac{1}{2})
\end{cases} \quad (3.8)$$

Then $\rho(x)$ is the continuous solution of the stationary condition Eq. 3.7 for $x > 1/2$, is normalized and fulfills the condition $1 \geq \rho(x) \geq 0$ for all positive $x \in \mathbb{R}$.

**Proof.** The derivative of $\rho$ is:

$$\rho'(x) = \begin{cases} 
0 & (x \leq 1/2) \\
-\frac{1}{\pi} \sqrt{\frac{1}{2x-1}} & (x > 1/2)
\end{cases} \quad (3.9)$$

Let $\epsilon \in \mathbb{R}$ with $\epsilon > 0$. Define $A_\epsilon(x)$ as

$$A_\epsilon(x) = \frac{1}{2} \int_0^\infty \log |(x-y)^2 + \epsilon^2| \rho'(y)dy \quad (3.9)$$

where $\log(z)$ denotes the principal branch of the logarithm ($0 \leq \arg(z) \leq \pi$). Using 3.8 and substituting $y = (s^2 + 1)/2$ we have

$$A_\epsilon(x) = -\frac{4}{\pi} \int_0^\infty \frac{1}{2} \log \left( \frac{(s^2 + (1 - 2x)) + \epsilon^2}{(s^2 + 1)^2} \right) ds.$$

$$= -\frac{1}{\pi} \int_{-\infty}^{+\infty} \log \left( \frac{(s^2 + (1 - 2x))^2 + \epsilon^2}{(s^2 + 1)^2} \right) ds + \log(2)$$
First consider the case $x > 1/2$. Using contour integration above the real axis closing the contour in the upper half plane gives

$$A_{\epsilon}(x) = 2\pi i \frac{d}{ds} \left( -\frac{1}{\pi} \log \left( \frac{(s^2 + 1 - 2x)^2 + \epsilon^2}{(1+s)^2} \right) \right)_{s=i} + \log(2)$$

$$= -2i \left[ -\frac{4s(s^2 + 1 - 2x)}{(1+s^2 - 2x)^2 + \epsilon^2 (s+i)^2} \frac{1}{(s+i)^3} - 2\log\left( \frac{(1+s^2 - 2x)^2 + \epsilon^2}{(s+i)^3} \right) \right]_{s=i} + \log(2)$$

$$= \frac{x}{x^2 + \epsilon^2} - \log \left( \sqrt{x + \epsilon^2} \right).$$

Taking the limit $\epsilon \to 0$ we have

$$\lim_{\epsilon \to 0^+} A_{\epsilon}(x) = \frac{1}{x} - \log |x|.$$

Using integration by parts on Eq. 3.9 we have for all $\epsilon > 0$

$$A_{\epsilon}(x) = \frac{1}{2} \log \left( (x-y)^2 + \epsilon^2 \right) \rho(x) \bigg|_{x=\infty}^{x=\frac{1}{2}} - \int_{1/2}^{\infty} \frac{1}{(x-y)^2 + \epsilon^2} \rho(y) dy$$

In the limit $\epsilon \to 0$ we have

$$\lim_{\epsilon \to 0^+} A_{\epsilon}(x) = -\log(x) - \int_{0}^{\infty} \frac{\rho(y)}{x-y} dy$$

where $\int$ denotes the principal value integral. Combining the two expressions for $A_{\epsilon}(x)$ we obtain

$$\frac{1}{x} = \int_{0}^{\infty} \frac{\rho(x)}{x-y}$$

for $x > 1/2$ which is the first statement of the theorem.

The normalization integral is 1 since

$$\int_{-\infty}^{+\infty} \rho(x) dx = \frac{1}{2} + 2\pi \int_{1/2}^{\infty} \left( \arctan \left( \frac{1}{\sqrt{2x-1}} \right) - \frac{\sqrt{2x-1}}{2x} \right) dx$$

$$= \frac{1}{2} + \frac{4}{\pi} \int_{1/2}^{\infty} \frac{1}{y^3} \left( \arctan(y) - \frac{y}{1+y^2} \right) dy$$

$$= \frac{1}{2} + \frac{1}{\pi} \left. \arctan(y) \right|_{y=\infty}^{y=1/2} - 2 \int_{1/2}^{\infty} \frac{1}{y(1+y^2)} dy$$

$$= 1.$$ 

The function $\rho(x)$ is continuous at $1/2$ since for $\epsilon \in \mathbb{R}$ with $\frac{1}{2} > \epsilon > 0$

$$\rho \left( \frac{1}{2} + \epsilon \right) = 1 - O \left( \sqrt{\epsilon} \right).$$

and $\rho \left( \frac{1}{2} - \epsilon \right) = 1$. The function $\rho(x)$ is continuous and monotonically decreasing with increasing $x$ since $\rho'(x) < 0$ for $x \in (1/2, \infty)$. The maximum of $\rho(x)$, $x \in [0, \infty)$ is one and the inmum is 0 since for $x \in [0, \infty)$ with $x \gg 1$

$$\rho(x) = \frac{\sqrt{2}}{3\pi} x^{-3/2} + O \left( x^{-5/2} \right) > 0.$$

which completes the proof. □
To determine $\Phi[\rho]$ we need to evaluate the integral $\int_0^\infty \rho(y) \log |x - y|$. For $x > 1/2$ we can use the stationary condition Eq. 3.6 however we have to determine $\lambda$ first.

**Proposition 3.2.** The function $\rho(x) x \in (0, \infty)$ of Theorem 3.1 has the property

$$\int_0^\infty \log |x - y| \rho(y)dy = \begin{cases} \log(2) - \frac{1}{2} & (x = 0) \\ -\sqrt{1 - 2x} - \log(2) + 2(x - 1) \log (1 + \sqrt{1 - 2x}) + x \log(2x) & (0 < x < 1/2) \\ -\sqrt{1 - 2x} - 2 & (1/2 \leq x) \end{cases}$$

**Proof.** Consider the case $x \geq 1/2$ first. From theorem 3.1 we have

$$\int_0^\infty \rho(y) \log |x - y| dy = \log(x) + \frac{1}{2} \lambda.$$ 

To prove $\lambda = 0$ it is sufficient to prove it for one $x \geq 1/2$. We choose $x = 1/2$. Using the definition of $\rho(x)$ we have

$$\int_0^\infty \rho(y) \log \left| \frac{1}{2} - y \right| dy = \int_0^{1/2} \log \left| \frac{1}{2} - y \right| dy + \int_{1/2}^\infty (\log (2y - 1) - \log(2)) \rho(y)dy$$

$$= \log \left( \frac{1}{2} \right) - \frac{1}{2} + \int_{1/2}^\infty \log(2y - 1) \rho(y)dy$$

$$= \log \left( \frac{1}{2} \right) - \frac{1}{2} - \int_{1/2}^\infty ((2y - 1) \log (2y - 1) - (2y - 1)) \rho'(y)dy$$

$$= \log \left( \frac{1}{2} \right) - \frac{1}{2} + \frac{2}{\pi} \int_0^\infty \frac{2s^2 \log(s) - s^2}{(s^2 + 1)^2} ds$$

$$= \log \left( \frac{1}{2} \right).$$

where we have substituted $s = 1/\sqrt{2x - 1}$. The integral over $s$ can be done by contour integration. Finally we have

$$\int_0^\infty \rho(y) \log \left| \frac{1}{2} - y \right| = \log \left( \frac{1}{2} \right) = \log \left( \frac{1}{2} \right) + \frac{1}{2} \lambda.$$ 

This implies $\lambda = 0$ which proves the case for $x > 0$.

Next consider the case $0 < x < 1/2$. We split the integral in two parts and integrate by parts

$$\int_0^\infty \rho(y) \log |x - y| = \int_0^{1/2} \log |x - y| dy + \int_{1/2}^\infty \rho(y) \log (y - x) dy$$

$$= -x + x \log(x) + \int_{1/2}^\infty \rho'(y) ((y - x) \log (y - x) - (y - x)) dy$$
The second integral can be done by substituting \( s = \sqrt{2y - 1} \). The second term in the last line is
\[
\int_{1/2}^{\infty} \rho'(y)(y - x)(\log(y - x) - 1) \, dy = \frac{2}{\pi} \int_0^\infty \frac{\log(s^2 + 1 - 2x) - \log(2) - 1}{(s^2 + 1)^2} (s^2 + 1 - 2x) \, ds
\]
\[
= (\log(2) + 1) (x - 1) + \int_0^\infty \frac{\log(s^2 + 1 - 2x)}{(s^2 + 1)^2} (s^2 + 1 - 2x) \, ds
\]
\[
= (\log(2) + 1) (x - 1) + 2 \log\left(1 + \sqrt{1 - 2x}\right) + \frac{x}{1 + \sqrt{1 - 2x}}
\]
where the last integral was evaluated using contour integration (see e.g. [5], 4.295, integral 7). Collecting all terms we obtain the \( x < 1/2 \) case of the proposition.

Finally consider the limit \( x \to 0 \) with \( x > 0 \). We have to leading order in \( x \)
\[\sqrt{1 - 2x} - \log(2) + 2\left(x - 1\right) \log\left(1 + \sqrt{1 - 2x}\right) + \frac{x}{1 + \sqrt{1 - 2x}}\]
In the limit \( x \to 0, x > 0 \) we find
\[
\lim_{x \to 0^+} \int_0^\infty \rho(y) \log|x - y| \, dy = -1 + \log(2)
\]
which completes the proof. \( \square \)

We are now in the position to evaluate \( \phi(\rho) \). Using Proposition 3.2 all integrals of Eq. 3.1 can be reduced to elementary integrals with the result
\[
\Phi[\rho] = -n^2 \log(n) - n^2 \left(2 \int_0^\infty \rho(x) \log(x) \, dx - \int_0^\infty \rho(x) \rho(y) \log|x - y| \, dx \, dy\right)
\]
\[
= -n^2 \log(n) - n^2 \left(2 \left(\log(2) - 1\right) - \left(\log(2) - \frac{1}{2}\right)\right)
\]
\[
= -n^2 \left(\log(2) - \frac{3}{2}\right).
\]
From this we conclude that the dominant contribution to \( H_n(\zeta) \) is
\[
\log(H_n(\zeta)) \approx -n^2 \left(\log(2) - \frac{3}{2}\right)
\]
which agrees with numerical findings.

4. RELATION TO THE SELBERG INTEGRAL

We next discuss the relation of \( H_n^{(r)}(\zeta) \) to the Selberg integral (Selberg’s extension of the beta integral [11], for a detailed explanation see [1] chapter 8) which plays a central role in random matrix theory (see [10], chapter 17) and is given by
\[
S_n(\alpha, \beta, \gamma) = \int_0^1 \cdots \int_0^1 t_1^{\alpha-1} \cdots t_n^{\alpha-1} (1 - t_1)^{\beta-1} \cdots (1 - t_n)^{\beta-1} \prod_{i<j} (t_i - t_j)^{2\gamma} \, dt_1 \cdots dt_n.
\]
Substituting \( t_i \to 1/m_i \) we write
\[
S_n(\alpha, \beta, \gamma) = \int_1^{\infty} \cdots \int_1^{\infty} \prod_{i=1}^n \frac{1}{m_i^{\alpha-1}} (1 - \frac{1}{m_i})^{\beta-1} \prod_{i<j} \left(\frac{1}{m_i} - \frac{1}{m_j}\right)^{2\gamma} \, dm_1 \cdots dm_n.
\]
For $\alpha = 1 + r$, $\beta = 1$ and $\gamma = 1$ we find

$$\mathcal{S}_n(r+1,1) = \int_1^\infty \cdots \int_1^\infty \frac{1}{(m_1 \cdots m_n)^{2n+r}} \prod_{i<j} (m_i - m_j)^2 \, dm_1 \cdots dm_n$$

The similarity between Eq. (4.1) and Eq. (2.1) is striking and can be generalized easily. It can be seen immediately from the definition that $S_n(1,1,1) > 0$ for each $n > 0$.

However as we will prove now for $n \gg 1$ the Selberg integral $S_n(1,1,1)$ is much larger than $H_n(\zeta)$ excluding a naive application of Euler-MacLaurin summation formula to $H_n(\zeta)$. It is instructive to repeat the saddle point analysis of the previous chapter for $S_n(1,1,1)$ in the limit $n \to \infty$. First note that

$$S_n(1,1,1) = \int_1^\infty \cdots \int_1^\infty \exp(\Phi(\rho_S)) \, dm_1 \cdots dm_n,$$

with the density

$$\rho_S(x) = \frac{1}{n} \sum_{i=1}^n \delta(x - m_i).$$

Note that we did not rescale $x$. The density is normalized to one

$$\int_{-\infty}^{+\infty} \rho_S(x) \, dx = 1.$$

**Proposition 4.1.** The asymptotic density $\rho_S$ maximizing $\Phi[\rho]$ is

$$\rho_S(x) = \begin{cases} 
0 & x \leq 1 \\
\frac{1}{\pi x \sqrt{x-1}} & x > 1
\end{cases}.$$

**Proof.** The condition that $\Phi[\rho]$ is stationary is

$$\frac{1}{x} = \int_1^\infty \frac{\rho_S(y)}{x-y} \, dy$$

the only difference to Eq. (3.6) is that here no constraint $\rho_S(x) \leq 1$ has to be imposed since there is no restriction for the difference $|m_i - m_j|$ for two integration variables. The integral equation can again be solved by standard methods [6] and yields the result stated above. Evaluating $\Phi[\rho_S]$ we find:

$$\Phi[\rho_S] = -n^2 \left( 2 \int \rho_S(x) \log(x) \, dx - \int \int \rho_S(x) \rho_S(y) \log|x-y| \, dx \, dy \right)$$

$$= -n^2 \left( 4 \log(2) - 2 \log(2) \right)$$

$$= -2 \log(2) n^2$$

giving $S_n(1,1,1) \approx \exp(-2 \log(2) n^2 + \ldots)$ so that $S_n(1,1,1) \gg H_n(\zeta) > 0$. This is in fact the correct behavior as we will prove now. \[ \Box \]

**Proposition 4.2.** The asymptotic behavior of $S_n(1,1,1)$ as $n \to \infty$ is

$$\log(S_n(1,1,1)) = -2 \log(2) n^2 + \log(2\pi n) - n + O(1).$$

**Proof.** The asymptotic behavior can be derived from the exact expression (Theorem (8.1.1) in [1])

$$\log(S_n(1,1,1)) = \sum_{j=1}^n (2 \log(\Gamma(j)) + \log(\Gamma(j+1)) - \log(\Gamma(n+j)))$$

$$= 3 \log(G(n+1)) + \log(G(n+2)) - \log(G(2n+1)).$$
where $\Gamma(x)$ is the Euler Gamma function and $G(n) = \prod_{i=0}^{n-2} i!$ for $n \in \mathbb{N}$, $n \geq 0$ is the Barnes function. Inserting the asymptotic expansion of the Barnes function

$$\log(G(1+z)) = z^2 \left( \frac{1}{2} \log(z) - \frac{3}{4} \right) + \frac{1}{2} \log(2\pi) z - \frac{1}{12} \log(z) + O(1)$$

in the previous expression we find

$$\log(S_n(1, 1, 1)) = -2 \log(2)n^2 + (\log(2\pi n) - 1)n + O(1),$$

which completes the proof. \qed

5. Relation to the Plancherel measure

We finally remark on the relation of our results to the asymptotic behavior of Plancherel measures. The Plancherel measure is defined as

$$P(\lambda) = \left( \frac{\dim \lambda}{|\lambda|!} \right)^2 = \frac{\prod_{1 \leq i < j \leq n} (m_i - m_j)^2}{\prod_{i=1}^{n} (m_i!)^2}$$

where $\dim \lambda$ is the dimension of the representation of the symmetric group $\Sigma(|\lambda|)$ indexed by the partition $\lambda$. The sum over all partitions is given by

$$Z_n = \sum_{0 \leq m_1 < \ldots < m_n} \frac{1}{(m_1! \cdots m_n!)^2 \prod_{i<j} (m_i - m_j)^2}.$$

An analysis similar to the one above yields the following integral equation for the density

$$\log(x) = \int_0^\infty \frac{\rho_P(y)}{x-y} dy.$$

In this case the solution has finite support and is given by

$$\rho_P(x) = \begin{cases} \frac{1}{\pi} \arccos(1-x) & x \in [0, 2] \\ 0 & \text{otherwise} \end{cases}.$$

In this case the constraint $\rho_P(0) \leq 1$ does not restrict the solution because $\rho_P(0) = 1$ and $\rho_P(x)$ is monotonically decreasing for $x > 0$. Below we show for comparison the density $\rho(x)$ (solid line) and $\rho_P(x)$ (dashed line).

Integrating $\rho_P(x)$ over $x$ we recover the famous asymptotic behavior of Plancherel measure [12, 9].
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