Form factors approach to current correlations in one dimensional systems with impurities.

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We show how to compute analytically time and space dependent correlations in one dimensional quantum integrable systems with an impurity. Our approach is based on a description of these systems in terms of massless scattering of quasiparticles. Correlators follow then from matrix elements of local operators between multiparticle states, the “massless form factors”. Although an infinite sum of these form factors has to be considered in principle, we find that for current, spin, and energy operators, only a few (typically two or three) are necessary to obtain an accuracy of more than 1%, for arbitrary coupling strength, that is all the way from short to large distances. As examples we compute, at zero temperature, the frequency dependent conductance in a Luttinger liquid with impurity, the spectral function in the double well problem of dissipative quantum mechanics and part of the space dependent susceptibility in the Kondo model.
1. Introduction.

One dimensional quantum impurity problems arise in diverse areas of solid-state physics. Of recent interest are the tunneling through a point contact in the fractional quantum Hall effect [1], the Kondo problem [2] describing electrons interacting with dilute impurities in a metal, and the dynamics of double or multiwell systems coupled to an ohmic bath in the context of dissipative quantum mechanics [3],[4]. A common feature to all these models is that they can be reduced to a model described by massless excitations in the bulk interacting with an impurity at the boundary. The absence of a mass gap leads to a power law behaviour for the current correlators in both the ultra-violet and the infra-red regime. The cross-over between these two regimes is non-trivial because of the renormalisation group flow induced by the impurity. A standard approach to these systems would be to use perturbation theory but it fails to capture all the physics, and new methods are needed.

Beside (largely numerical) renormalization group approaches, another possibility for progress is provided by the integrability of some of these systems. Albeit not all of them are integrable, surprisingly, many are, and exact results can be obtained. The oldest such results concern the Toulouse limit [5] of the Kondo model, which is a special point in the parameter space of the anisotropic model that is equivalent to free fermions. Other examples include the thermodynamic properties of the Kondo problem [6] which were computed using the Bethe ansatz, and more recently the tunneling through a point contact in the $\nu = 1/3$ quantum Hall effect [7], where thermodynamic and as well some static transport properties were computed by a combination of Bethe ansatz and Boltzmann equations. Apart from the latter solution, transport properties as well as space and time-dependent properties are difficult to obtain because they require knowledge of correlation functions.

Impurity problems of physical interest have a very simple bulk hamiltonian, typically a free boson. All the difficulty lies in the impurity interaction. In the basis where the bulk problem is simple this interaction is difficult to handle: in the classical limit, plane waves are scattered into very complicated wave packets by the impurity. If the theory is integrable, there is however another basis onto which the impurity has a simple effect: in the classical limit, there are some special wave packets that scatter simply at the impurity [8]. The price to pay of course is that in this new basis the bulk hamiltonian looks more complicated. Typically, this will lead us to describe a simple free boson as a gas of massless
quasiparticles (solitons, antisolitons and breathers) with non trivial scattering! The net result however is that both bulk and impurity are now manageable, and physical properties can be computed.

Let us now recall that integrable massive theories, eg the bulk sine-Gordon model, can be described as a gas of quasiparticles with factorized scattering. A convenient formalism is to introduce creation and annihilation operators for these quasiparticles, like one would normally do for say a theory of free Fermions. If we denote by $Z_\epsilon^*(\theta) (Z_\epsilon(\theta))$ the creation (annihilation) operator of a quasiparticle of type $\epsilon$, the bulk interaction is encoded in the following Fadeev-Zamolodchikov relations:

$$Z_\epsilon^1(\theta_1)Z_\epsilon^2(\theta_2) = S_{\epsilon_1^1\epsilon_1^2}(\theta_1 - \theta_2)Z_{\epsilon_1^2}^*(\theta_2)Z_{\epsilon_1^1}^*(\theta_1)$$

$$Z_\epsilon^*(\theta_1)Z_\epsilon^*(\theta_2) = S_{\epsilon_1^1\epsilon_1^2}(\theta_1 - \theta_2)Z_{\epsilon_1^*}^*(\theta_2)Z_{\epsilon_1^*}(\theta_1)$$

$$Z_\epsilon^1(\theta_1)Z_{\epsilon_2}^*(\theta_2) = S_{\epsilon_2^1\epsilon_2^1}(\theta_1 - \theta_2)Z_{\epsilon_2^1}^*(\theta_2)Z_{\epsilon_2^1}^*(\theta_1) + 2\pi\delta_{\epsilon_1^1}\delta(\theta_1 - \theta_2).$$

(1.1)

In these expressions, the $\theta_i$’s are rapidity variables. In the discussion so far the theory is massive, and there is a meaningful mass scale $m$ related to the coupling constant of some bulk perturbation. The energy and the momentum of quasiparticles can be then parametrized in the form $E_\epsilon = m_\epsilon \cosh(\theta)$ and $P_\epsilon = m_\epsilon \sinh(\theta)$, with $m_\epsilon \propto m$. For the relations (1.1) to make sense, constraints have to be satisfied by the $S$ matrix, in particular it has to be a solution of the Yang-Baxter equation.

It is then convenient to think of a massless theory as the limit of a massive theory when the bulk mass scale $m$ goes to zero; for instance one can think of a free boson theory as the limit of the bulk sine-Gordon model when the amplitude of the bulk cosine perturbation vanishes\footnote{Such a description requires in particular that the space of fields of the massive field theory considered as a perturbed conformal field theory and its massless ultraviolet limit are identical, which is the case in the “superrenormalizable” theories we consider here\footnote{.}}. Consider thus a massive theory, and take the following (massless) limit: $\theta = \theta_0 \pm \beta$ and $m \to 0$ such that $M = m e^{\theta_0}/2$ remains finite. Excitations split into left and right movers, and dispersion relations become:

$$E_\epsilon = P_\epsilon = M_\epsilon e^{\beta} \quad \text{rightmovers}$$

$$E_\epsilon = -P_\epsilon = M_\epsilon e^{-\beta} \quad \text{leftmovers.}$$

(1.2)

Here $M$ is an arbitrary energy scale without physical meaning. The previous relations (1.1) are then decorated by a supplemental L or R subscript. The $S$ matrices for the interaction
between movers of the same chirality are unchanged, since in the initial massive theory they depended only on the rapidity difference $\theta$, but the right-left scattering becomes a constant phase (we refer the reader to [8] for a more detailed discussion). This trivial left-right scattering will lead in the form factors approach to a left-right factorization, and will simplify matters drastically.

This massless basis was not explicitly used in the original exact approaches to the Kondo problem. In the corresponding works, bare hamiltonians were used instead, and explicitly diagonalized via the Bethe ansatz. The point of view we take here is quite different: we work directly in the renormalized theory which is assumed to be factorizable [10] (this following of course from the integrability of the bare hamiltonian). This makes a tremendous difference when dealing with matrix elements of fields. While their computation from the bare theory is extremely arduous, they can be obtained rather easily in the context of the factorized renormalized theory by using an axiomatic approach. The massless basis has proven very convenient in recent works [11], in particular for what concerns static transport properties, like the DC conductance and the DC noise [7], [12] in the problem of tunneling between Hall edge states. Here, following the idea described in the pioneering work of [13], we show how this basis also permits the computation of time (or space) dependent properties.

The natural approach to get correlations is to use matrix elements in the basis (1.1), the so-called form-factors. In the massive theory, these matrix elements are computed using a set of “axioms” [14], [15], [16], generalizing Watson’s theorem [17]. Form-factors $\mathcal{F}$ of an operator $\mathcal{O}$ in a bulk theory are defined as:

$$ f(\theta_1, \ldots, \theta_n)_{\epsilon_1, \ldots, \epsilon_n} = \langle 0| \mathcal{O}(0, 0) Z_{\epsilon_1}^*(\theta_1) \ldots Z_{\epsilon_n}^*(\theta_n)|0 \rangle $$

where $|0\rangle$ is the ground state, and their determination results form the following axioms:

$$ f(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n)_{\epsilon_1, \ldots, \epsilon_i, \epsilon_{i+1}, \ldots, \epsilon_n} \mathcal{S}_{\epsilon_i, \epsilon_{i+1}}(\theta_i - \theta_{i+1}) = f(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n)_{\epsilon_1, \ldots, \epsilon_{i+1}, \epsilon_i, \ldots, \epsilon_n}. $$

$^2$ The $S$ matrix depends on the relativistic invariant $(E_1 + E_2)^2 - (P_1 + P_2)^2$ in the massive case, $E_1/E_2$ in the massless case, both functions of the rapidity difference using our parametrization

$^3$ These are actually called “generalized form-factors” since no order of the rapidities is prescribed
which is a consequence of (1.1), and:
\[
f(\theta_1, \ldots, \theta_n + 2\pi i)\epsilon_1, \ldots, \epsilon_n = f(\theta_n, \theta_1, \ldots, \theta_n + 2\pi i)\epsilon_n, \epsilon_1, \ldots, \epsilon_{n-1},
\]
which is a generalisation of the two particle form factor monodromy equations [15]. The S matrix in the first relation has annihilation poles and this will induce similar poles in the form factors. Thus when \(\theta_n = \theta_{n-1} + i\pi\) (which correspond to the mass thresholds in the two particles case for example) we have:
\[
i \text{res} f(\theta_1, \ldots, \theta_n)\epsilon_1, \ldots, \epsilon_n = f(\theta_1, \ldots, \theta_{n-2})\epsilon_1, \ldots, \epsilon_{n-2} C_{\epsilon_n, \epsilon_{n-1}} \times \\
\left(1 - S_{\tau_1 \epsilon_1}^{\epsilon_n-1 \epsilon_1'} (\theta_{n-1} - \theta_1) \cdots S_{\tau_{n-3} \epsilon_{n-3}}^{\epsilon_n-3 \epsilon_{n-3}} (\theta_{n-1} - \theta_{n-3}) S_{\epsilon_n, \epsilon_{n-2}}^{\epsilon_n-2, \epsilon_{n-2}} (\theta_{n-1} - \theta_{n-2})\right).
\]
Also, if the theory contains bound states, there are further relations involving the corresponding residues in the S matrix. For example, if there is a bound-state at \(\theta_n = \theta_{n-1} + iu\), corresponding to a diagram depicted in figure 1.

![Fig. 1: Bound state diagram.](image)

Then, the corresponding relation for the form factor is:
\[
i \text{res} f(\theta_1, \ldots, \theta_n)\epsilon_1, \ldots, \epsilon_n = a_\gamma (-1)^{(2\epsilon_n+1)/2} C_{\epsilon_n-1, \epsilon_n} f(\theta_1, \ldots, \theta_{n-1} - iu)\epsilon_1, \ldots, \epsilon_{n-2}, \gamma,
\]
and here \(a_\gamma\) correspond to the square root of the corresponding residue in the S matrix for this process.

Solving this set of equations is the most convenient method to obtain the matrix elements (although other methods based on lattice regularizations might also be applicable [18]). This is still a difficult task, but results are available for the sine-Gordon [15] and the sinh-Gordon [16] models, which is all what we need in this paper. Form-factors for the massless theory will be defined simply by taking the foregoing limit of massive form-factors (trying to formulate axioms for a massless theory per se might lead to some ambiguities). Our strategy will then be to compute correlators simply by decomposing on intermediate states and using the exact matrix elements. This might sound a priori hopeless since there
are actually an infinite sum of relevant terms for operators of physical interest; however, we shall see that in many cases, the problem simplifies drastically.

The paper is organized as follows. In section 2 we introduce the models and the quantities of physical interest which we want to compute. In section 3, the technique of form-factors is introduced with the example of the sinh-Gordon model. Although it is not of immediate physical relevance, this model is very simple since it has a single quasi-particle excitation, and appears useful pedagogically. In section 4, we introduce similarly the form-factors for the sine-Gordon model and compute the frequency dependent conductance in the $\nu = 1/3$ quantum Hall effect. In section 5, further applications are discussed; in particular we compute the spectral function in dissipative quantum mechanics and the uniform part of the space dependent susceptibility in the anisotropic Kondo model related to the screening-cloud problem.

2. The models.

The impurity problems we shall discuss here can all be mapped onto a hamiltonian of the form:\footnote{In all what follows we set $e = h = 1$.}

$$H = \frac{1}{2} \int_{-\infty}^{0} dx \left[8\pi g \Pi^2 + \frac{1}{8\pi g} (\partial_x \phi)^2 \right] + B.$$  \hfill (2.1)

where $B$ is a problem dependent boundary interaction and the fields are defined on the negative half line. The geometry of the problem is shown in figure 2.

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{geometry.png}
\caption{Geometry of the problem.}
\end{figure}
\end{center}
The method we present is quite general and in this section we proceed to an enumeration of the models we study and the corresponding quantities we want to compute:

2.1. Sinh-Gordon model.

In this model, the boundary interaction is of the form:

$$B = \lambda \cosh \frac{1}{2} \phi(x = 0, t).$$

(2.2)

Although we have no physical application for this model, it is much simpler than the models we will explore subsequently. The spectrum of massless excitations consist of only one particle and the form factors are much simpler than those used later in the sine-Gordon model. The quantities we will compute are the current-current correlation and the equivalent of the conductance in the Hall model.

Our motivation for its study is to develop, in a simple fashion, the techniques used later in the real physical problems. We will show in a more lengthy manner how to obtain the current correlation functions and how the boundary affects these correlations. All features found in the other models are found in this one as well so it is a very good exercise.

2.2. $\nu = 1/3$ Hall effect.

To start, let us recall the relation between the physical sine-Gordon model with an impurity and the problem in the half space. Let us start with the Hamiltonian:

$$H = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[ 8\pi \nu \Pi^2 + \frac{1}{8\pi \nu} (\partial_x \phi)^2 \right] + \lambda \delta(x) \cos(\varphi_L - \varphi_R),$$

(2.3)

where the L and R components depend on $x, t$ as $\varphi_L(x+t), \varphi_R(x-t)$. The bulk part of this Hamiltonian physically describes the low energy edge degrees of freedom in the quantum Hall effect at filling fraction $\nu = \frac{1}{2s+1}$. When this Hall sample is subjected to a point contact constriction, transfer of fractionally charged excitations is possible. For generic filling fraction, many types of quasi-particles contributes as relevant charge transfer. The $\nu = 1/3$ is peculiar in that only the $Q = e/3$ charged Laughlin quasi-particle is relevant [1]. This is described by the impurity term in (2.3). As discussed in [4] in order to map this to a boundary problem, it is convenient to proceed in two steps. First introduce:

$$\phi^c(x + t) = \frac{1}{\sqrt{2}} [\varphi_L(x, t) + \varphi_R(-x, t)]$$

$$\phi^o(x + t) = \frac{1}{\sqrt{2}} [\varphi_L(x, t) - \varphi_R(-x, t)]$$

(2.4)
which are both left moving. It is clear that the interaction term does not affect the even field, which therefore remains free. As for the odd term, it can be mapped onto a boundary problem as follows. Define:

$$
\phi^o_L(x,t) = \sqrt{2} \phi^o(x+t), \ x < 0,
$$

$$
\phi^o_R(x,t) = \sqrt{2} \phi^o(-x+t), \ x < 0.
$$

The odd hamiltonian then reads:

$$
H = \frac{1}{2} \int_{-\infty}^{0} \left[ 8\pi g (\Pi^o)^2 + \frac{1}{8\pi g} (\partial_x \phi^o)^2 \right] + \lambda \delta(x) \cos \frac{1}{2} \phi^o, \quad (2.5)
$$

and in the following we will write $\phi \equiv \phi^o$ and $g$ instead of $\nu$. Thus, for this problem, $B = \lambda \cos \frac{1}{2} \phi(x = 0, t)$.

The quantity of interest in this case is the AC conductance at vanishing temperature. A standard way of representing it is through the Kubo formula:

$$
G(\omega_M) = -\frac{1}{8\pi \omega_M L^2} \int_{-L}^{L} dx \int_{-\infty}^{\infty} dy \ e^{i\omega_M y} \left< j(x,y)j(x',0) \right>, \quad (2.7)
$$

where $\omega_M$ is a Matsubara frequency, $y$ is imaginary time, $y = it$. One gets back to real physical frequencies by letting $\omega_M = -i\omega$. In (2.7), $j$ is the physical current in the unfolded system, $j = \partial_t (\varphi_L - \varphi_R)$. Without impurity, the AC conductance of the Luttinger liquid is frequency independent, $G = g$. When adding the impurity, it becomes $G = \frac{g}{2} + \Delta G$.

After some simple manipulations using the folding, one finds:

$$
\Delta G(\omega_M) = \frac{1}{8\pi \omega_M L^2} \int_{-L}^{0} dx dx' \int_{-\infty}^{\infty} dy \ e^{i\omega_M y} \left[ < \partial_z \phi(x,y) \partial_{z'} \phi(x',0) > + < \partial_z \phi(x,y) \partial_{z'} \phi(x',0) > \right], \quad (2.8)
$$

where $z = x + iy$.

2.3. Dissipative quantum mechanics and Kondo model.

The second model we will be interested in is the anisotropic Kondo model:

$$
H = \frac{1}{2} \int_{-\infty}^{0} dx \left[ 8\pi g \Pi^2 + \frac{1}{8\pi g} (\partial_x \phi)^2 \right] + \lambda \left( S_+ e^{i\phi(0)/2} + S_- e^{-i\phi(0)/2} \right). \quad (2.9)
$$
It can be related to dissipative quantum mechanics [3] where it describes the dynamics of a double well system subjected to Ohmic dissipation. The Hamiltonian for such a system is given by:

\[
H = -\frac{\hbar \Delta}{2} \sigma_x + \frac{\hbar \epsilon}{2} \sigma_z + \sum_\alpha \left[ \frac{p_\alpha^2}{2m_\alpha} + \frac{1}{2}m_\alpha \omega_\alpha^2 \left( x_\alpha - \frac{C_\alpha}{m_\alpha \omega_\alpha^2} \sigma_z \right)^2 \right].
\]

(2.10)

In this expression, the Pauli matrices \( \sigma_x, \sigma_z \) act on the two dimensional space of states. \( \Delta \) is a tunnelling matrix element and \( \epsilon \) denotes a bias between the two states. Dissipation comes from the coupling of this two states system to an environment of oscillators described by the last part of the Hamiltonian. The effect of these oscillators (with mass, frequencies and coupling \( m_\alpha, \omega_\alpha, C_\alpha \)) can be entirely encoded in the environment spectral function:

\[
J(\omega) = \frac{\pi}{2} \sum_\alpha \left( \frac{C_\alpha^2}{m_\alpha \omega_\alpha} \right) \delta(\omega - \omega_\alpha).
\]

(2.11)

It was shown [3] that this system can be mapped, in the so called ohmic dissipation case, \( J(\omega) = 2\pi g\omega \), to an anisotropic Kondo model. The conduction electrons in the latter play the role of dissipation and the \( z \) value of the spin is associated with the two states. In this paper we will work at zero bias (the bias would be equivalent to a magnetic field in (2.3)). The hopping \( \Delta \) is related the the strength of the impurity \( \lambda \) the precise relation being given in [3]. This system has numerous physical applications [19].

The quantity of physical interest in the two state system is the effect of the bath on the tunneling between the two (degenerate minima). It is conveniently encoded into the correlator:

\[
C(t) = \frac{1}{2} < [S_z(t), S_z(0)] > .
\]

(2.12)

It describes the probability to be in a state \( S_z(t) \) given that the system was in state \( S_z(0) \) at \( t = 0 \). We will show how this is related to a current correlation and can therefore be addressed by the form factor approach.

Another problem of interest is the screening cloud problem in the Kondo model. The three dimensional Kondo model can be reduced to one dimension by restricting to the s wave:

\[
\psi(r) = \frac{1}{2\sqrt{2\pi r}} [e^{-ik_Fr}\psi_L(r) - e^{ik_Fr}\psi_R(r)] + \cdots
\]

(2.13)

with the \( \cdots \) denoting higher harmonics. In this language the Hamiltonian is given by:

\[
H = v_F \int_0^\infty dr \: \psi_L^\dagger(r) \frac{id}{dr} \psi_L(r) - \psi_R^\dagger(r) \frac{id}{dr} \psi_R(r)
\]

(2.14)
and the boundary interaction is:

\[ \mathcal{B} = \lambda \psi^\dagger_L(0) \frac{\sigma_z}{2} \psi_L(0) \cdot \mathbf{S}_{\text{imp}}. \] (2.15)

The screening cloud problem can be addressed by computing the local susceptibility which in 3d is given by:

\[ \chi(r) = \langle \psi^\dagger(r) \frac{\sigma_z}{2} \psi(r) \int dt \, S^z_{\text{tot}} \rangle, \] (2.16)

with:

\[ S^z_{\text{tot}} = S^z_{\text{imp}} + \int d^3r \, \psi^\dagger(r) \frac{\sigma_z}{2} \psi(r). \] (2.17)

Here \( S_{\text{imp}} \) correspond to the impurity spin and the second part is the electron spin (the spin indices of the fermions are contracted with the Pauli matrix). When reducing these quantities to one dimension, we find following [20] a uniform and \( 2k_F \) part written:

\[ \chi(r) = \frac{1}{8\pi^2r^2} [\chi_{\text{un}} + 2\chi_{2k_F} \cos(2k_Fr)] \] (2.18)

with:

\[ \chi_{\text{un}} = \langle \left[ \psi^\dagger_L(r) \frac{\sigma_z}{2} \psi_L(r) + \psi^\dagger_R(r) \frac{\sigma_z}{2} \psi_R(r) \right] \int dt S^z_{\text{tot}} \rangle \] (2.19)

and the total spin now given by:

\[ S_{\text{tot}} = S^z_{\text{imp}} + \frac{1}{2\pi} \int_0^\infty dr \left[ \psi^\dagger_L(r) \frac{\sigma_z}{2} \psi_L(r) + \psi^\dagger_R(r) \frac{\sigma_z}{2} \psi_R(r) \right]. \] (2.20)

After having established the quantities we want to compute, we can bosonise [21]. Two bosonic fields are necessary: one associated with charge and one with spin. The charge field decouples completely and only the spin charge has interaction at the boundary. The action for the spin field is of the form (2.9). The \( S_z \) term in the action has been eliminated by a unitary rotation of the Hamiltonian (which is unity at \( g = 1 \), the isotropic model). In terms of this bosonic field, the uniform part of the susceptibility at \( g = 1 \) takes the simple form:

\[ \chi_{\text{un}}(r) = \frac{1}{2} \left\langle \partial_r \phi(r) \int dt \, S^z_{\text{tot}} \right \rangle \] (2.21)
with now:

\[ S_{\text{tot}}^z = S_{\text{imp}}^z + \frac{1}{4\pi} \int_{-\infty}^{0} dx \partial_x \phi(x). \] (2.22)

We will show how to compute this uniform part at zero temperature.

The problems are posed and our task is then to compute these correlation functions. This is a long story, and to explain the method we start by considering as a toy model the boundary sinh-gordon theory.

3. Formalism: The sinh-Gordon model.

3.1. The bulk current-current correlators.

In most of this paper, we shall work in Euclidian space with \( x, y \) coordinates. Imaginary time is at first considered as running along \( x \). We consider the sinh-Gordon model with action:

\[ S = \frac{1}{16\pi g} \int_{-\infty}^{\infty} dx dy \left[ (\partial_x \phi)^2 + (\partial_y \phi)^2 + \Lambda \cosh \phi \right]. \] (3.1)

This is a massive theory which is integrable; the conformal weights of the perturbing operator are \( h = \bar{h} = -g \). The spectrum is very simple and consists of a single particle of masse \( m \), and S matrix:

\[ S(\theta) = \frac{\tanh \frac{1}{2} (\theta - i\pi B)}{\tanh \frac{1}{2} (\theta + i\pi B)}, \] (3.2)

where:

\[ B = \frac{2g}{1 + g} \]

and \( \theta \) is the usual rapidity, \( P = m \sinh \theta \), \( E = m \cosh \theta \). Recall the duality of the S matrix in \( B \to 2 - B \), i.e. \( g \to 1/g \).

Consider the massless limit where \( \Lambda \to 0 \). In that limit, the current correlators are trivial and are given by:

\[ < \partial_z \phi(z, \bar{z}) \partial_{z'} \phi(z', \bar{z}') > = -\frac{2g}{(z - z')^2} \]

\[ < \partial_{\bar{z}} \phi(z, \bar{z}) \partial_{\bar{z}'} \phi(z', \bar{z}') > = -\frac{2g}{(\bar{z} - \bar{z}')^2}. \] (3.3)

5 This expression has to be considered with some caution but will be enough for the screening cloud computations of section 5.
On the other hand, we can formally describe this limit still using a scattering theory. As described in the introduction, this is done by writing \( \theta = \pm (\theta_0 + \beta) \) and taking simultaneously \( \theta_0 \to \infty \) and \( m \to 0 \) with \( me^{\theta_0}/2 \to M \), \( M \) finite. In that case, the spectrum separates into Right and Left movers with respectively \( E = P = Me^\beta \) and \( E = -P = Me^\beta \). The scattering of R and L movers is still given by (3.2) where \( \theta \to \beta \). The RL and LR scattering becomes a simple phase, \( e^{\mp i\pi B/2} \). This phase will turn out to cancel out at the end of all computations, but is confusing to keep along. We just set it equal to unity in the following, that is we consider all L and R quantities as commuting.

In this new description of the massless theory, we will need form factors in order to compute (3.3). The form factors of the massive sinh-Gordon theory are well known \[22\]. We will only use the form factors of the fundamental field \( \phi \) in what follows. By taking the massless limit of the formulas in \[22\], it is easy to check that \( \phi \) can alter only the right or left content of states; in other words, matrix elements of \( \phi \) between states which have different content both in the left and right sectors vanish.

Our conventions are conveniently summarized by giving the one particle form factor of the sinh-Gordon field:

\[
<0|\phi(x,y)|\beta>_R = \mu \exp [Me^\beta (x+iy)],
<0|\phi(x,y)|\beta>_L = \mu \exp [Me^\beta (x-iy)].
\]

(3.4)

For the field \( \phi \), form factors with an even number of particles vanish. This is because \( \phi \) as well as the creation operators of the sinh-gordon particle are odd under the \( Z_2 \) symmetry \( \phi \to -\phi \). In the following we will use the notation:

\[
f(\beta_1, \ldots, \beta_{2n+1})=<0|\phi|\beta_1, \ldots, \beta_{2n+1}>_{R, \ldots, R},
\]

(3.5)

with the normalization of asymptotic states \( R < \beta|\beta'>_R = 2\pi \delta(\beta - \beta') \). Here, \( f \) depends on \( g \), but we do not indicate it explicitly for simplicity. We have the following properties:

\[
<0|\phi|\beta_1, \ldots, \beta_{2n+1}>_{R, \ldots, R} = ( <0|\phi|\beta_1, \ldots, \beta_{2n+1}>_{L, \ldots, L} )^*,
<0|\phi|\beta_1, \ldots, \beta_{2n+1}>_{R, \ldots, R} = <0|\phi|\beta_{2n+1}, \ldots, \beta_1>_{L, \ldots, L},
\]

(3.6)

These form factors are expressed as follows \[22\]. Introduce:

\[
f_{\min}(\beta) = N \exp \left\{ 8 \int_{0}^{\infty} \frac{dx}{x} \frac{\sinh(xB/2) \sinh[x(2-\beta)/2] \sinh(x/2)}{\sinh^2(x)} \sin^2 \left[ \frac{x(i\pi - \beta)}{2\pi} \right] \right\},
\]

(3.7)
\[ N = \exp \left[ -4 \int_0^\infty dx \frac{\sinh(xB) \sinh(x(2-B)) \sinh(x)}{x \sinh^2(x)} \right] \]

Then,

\[
f(\beta_1, \ldots, \beta_{2n+1}) = \mu \left( \frac{4 \sin \frac{\pi B}{2}}{F_{\text{min}}(i\pi, B)} \right)^n \sigma_{2n+1}^{(2n+1)} P_{2n+1}(x_1, \ldots, x_{2n+1}) \prod_{i<j} \frac{f_{\text{min}}(\beta_i - \beta_j)}{x_i + x_j},
\]

where we introduced \( x \equiv e^\beta \) and the \( \sigma \)'s are the basic symmetric polynomials:

\[ \sigma_p = \sum_{i_1 < i_2 < \cdots < i_p} x_{i_1} x_{i_2} \cdots x_{i_p}, \]

with the convention \( \sigma_0 = 1 \) and \( \sigma_p = 0 \) if \( p \) is greater than the number of variables. The \( P_{2n+1} \)'s are symmetric polynomials, which can be obtained by solving LSZ recursion relations. The first ones read:

\[
P_3(x_1, \ldots, x_3) = 1 \\
P_5(x_1, \ldots, x_5) = \sigma_2 \sigma_3 - c_1^2 \sigma_5 \\
P_7(x_1, \ldots, x_7) = \sigma_2 \sigma_3 \sigma_4 \sigma_5 - c_1^2 (\sigma_4 \sigma_5^2 + \sigma_1 \sigma_2 \sigma_5 \sigma_6 + \sigma_2^2 \sigma_3 - c_1^2 \sigma_2 \sigma_5) \\
- c_2 (\sigma_1 \sigma_6 \sigma_7 + \sigma_1 \sigma_2 \sigma_4 \sigma_7 + \sigma_3 \sigma_5 \sigma_6) + c_1 c_2^2 \sigma_7^2.
\]

with \( c_1 = 2 \cos \frac{\pi B}{2}, c_2 = 1 - c_1^2 \). Observe that except for the overall normalization \( \mu(g) \), these expressions are invariant in the duality transformation \( g \to \frac{1}{g} \).

In (3.4), \( \mu \) is an overall normalization for the form-factors. It is usually chosen by reference to the IR limit. However, we will require that the result (3.3) be recovered, and this involves a more complex computation. Using form factors, this two point function expands, assuming \( \text{Re} \ z < \text{Re} \ z' \), as:

\[
< 0|\partial_z \phi(z, \bar{z}) \partial_{z'} \phi(z', \bar{z}')|0 >= - \sum_{n=0}^\infty \frac{d\beta_1 \ldots d\beta_{2n+1}}{(2\pi)^{2n+1}(2n+1)!} M^2 \left( e^{\beta_1} + \ldots + e^{\beta_{2n+1}} \right)^2 \\
\exp \left[ M(z - z') \left( e^{\beta_1} + \ldots + e^{\beta_{2n+1}} \right) \right] |f(\beta_1, \ldots, \beta_{2n+1})|^2.
\]

Now, by relativistic invariance, all the form factors depend only on differences of rapidities. Setting \( M(z - z') \equiv e^{\beta_0} \), (where \( \beta_0 \) will in general be complex), one can shift all the \( \beta \)'s
by $\beta_0$ to factor out, for any $2n + 1$ particle contributions, a factor $\frac{1}{(z-z')^2}$. Hence, the form factor expansion gives the result (3.3) provided $N$ is chosen such that

$$\sum_{n=0}^{\infty} I_{2n+1} = 2g,$$

(3.11)

where

$$I_{2n+1} = \int \frac{d\beta_1 \ldots d\beta_{2n+1}}{(2\pi)^{2n+1}(2n+1)!} (e^{\beta_1} + \ldots + e^{\beta_{2n+1}})^2 e^{-(e^{\beta_1} + \ldots + e^{\beta_{2n+1}})} |f(\beta_1, \ldots, \beta_{2n+1})|^2.$$

(3.12)

In practice, this sum cannot be computed analytically, but it can be easily evaluated numerically. The convergence is extremely fast with $n$, and for most practical purposes, the consideration of up to five particles is enough to get correct results up to $10^{-4}$. Similar convergence properties were observed in [23].

It must be emphasized that this result is very peculiar to the current operator. For most other chiral operators, the correct $(z-z')$ dependence involves a non trivial anomalous dimension, instead of the naive engineering dimension. Hence, this dependence is not obtained term by term, as observed here, but rather once the whole series is summed up. Truncating the series to any finite $n$ does not, in such cases, give reliable results all the way from short to large distances. The current is therefore an extremely favorable case, as would be the stress tensor, and we are fortunate it has a lot of physical meaning.

3.2. Current current correlators with a boundary

Having fixed the form-factors normalization, let us now consider the theory with a boundary. The geometry of the problem was illustrated in figure 2, where the boundary stands at $x = 0$ and runs parallel to the $y = it$ axis. The action is now:

$$S = \frac{1}{16\pi g} \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \left[ (\partial_x \phi)^2 + (\partial_y \phi)^2 + \Lambda \cosh \phi \right] + \lambda \int_{-\infty}^{\infty} dy \cosh \frac{1}{2} \phi(x = 0, y).$$

(3.13)

This model is also integrable for any choice of $\Lambda, \lambda$. The boundary dimension of the perturbing operator is $x = -g$. We can in particular take the limit where $\Lambda \to 0$ while $\lambda$ remains finite. It then describes a theory which is conformal invariant in the bulk but has a boundary interaction that breaks this invariance and induces a flow from Neumann boundary conditions at small $\lambda$ to Dirichlet boundary conditions at large $\lambda$. As discussed in [8], the boundary interaction is characterized by an energy scale, which one can represent
as $T_B = M e^{\beta_B}$. $T_B$ is related with the bare coupling in the action (3.13) by $\lambda \propto T_B^{1+g}$. In the following, since obviously changes of $M$ (which is not a physical scale) can be absorbed in rapidity shifts, we set $M = 1$. The effect of the boundary is then expressed by the reflection matrix:

$$R(\beta) = \tanh \left( \frac{\beta}{2} - i \frac{\pi}{4} \right).$$  

(3.14)

In the picture where imaginary time is along $x$, the effect of the boundary is represented by a boundary state. Following [24] we can represent it in terms of the boundary scattering matrix:

$$|B> = \exp \left[ \int_{-\infty}^{\infty} \frac{d\beta}{2\pi} K(\beta_B - \beta) Z^*_L(\beta) Z^*_R(\beta) \right] |0>.$$  

(3.15)

In this formula, $Z^*$ denote the Zamolodchikov Fateev creation operators, $K$ is related with the reflection matrix by:

$$K(\beta) = R \left( \frac{i\pi}{2} - \beta \right) = - \tanh \frac{\beta}{2}.$$  

(3.16)

One can expand the boundary state into the convenient form:

$$|B> = \sum_{n=0}^{\infty} \int_{0<\beta_1<...<\beta_n} K(\beta_B - \beta_1) \ldots K(\beta_B - \beta_n) Z^*_L(\beta_1) \ldots Z^*_L(\beta_n) Z^*_R(\beta_1) \ldots Z^*(\beta_n).$$  

(3.17)

Observe now that by analyticity, the matrix elements of $\partial_z \phi$ between the ground state and any state with at least one L moving particle are identically zero. More generally, the only non vanishing matrix elements of $\partial_z \phi$ are those where bra and ket have the same L moving part. The same results apply by exchanging $\partial_z$ with $\partial_{\bar{z}}$ and L with R moving particles. As a result one gets immediately two of the four current correlators:

$$< 0| \partial_z \phi(z, \bar{z}) \partial_{z'} \phi(z', \bar{z}') |0> = -\frac{2g}{(z - z')^2}.$$  

$$< 0| \partial_{\bar{z}} \phi(z, \bar{z}) \partial_{z'} \phi(z', \bar{z}') |0> = -\frac{2g}{(\bar{z} - \bar{z}')^2},$$  

(3.18)

which are identical with the ones without a boundary.

The two other correlators are more difficult to get. Let us consider for instance:

$$< 0| \partial_{\bar{z}} \phi(z, \bar{z}) \partial_{z'} \phi(z', \bar{z}') |B>.$$  

(3.19)
The first non trivial contribution comes from the two particle term in the expansion of the boundary state:

\[
\int_{-\infty}^{\infty} \frac{d\beta}{2\pi} K(\beta_B - \beta) < 0|\partial_z \phi(z, \bar{z}) \partial_z \phi(z', \bar{z}') Z_L^\dagger(\beta) Z_R^\dagger(\beta)|0 > = \\
\mu^2 \int_{-\infty}^{\infty} \frac{d\beta}{2\pi} K(\beta_B - \beta) e^{2\beta} \exp \left[ e^\beta (\bar{z} + z') \right].
\]

(3.20)

More generally, because \(|B>| is a superposition of states with equal numbers of left and right moving particles, and \(\partial_z \phi\), respectively \(\partial_{\bar{z}} \phi\) act only on \(R\), respectively \(L\), particles, the expansion of (3.19) takes a very simple form:

\[
\sum_{n=0}^{\infty} \int \frac{d\beta_1 \ldots d\beta_{2n+1}}{(2\pi)^{2n+1}(2n+1)!} K(\beta_B - \beta_1) \ldots K(\beta_B - \beta_{2n+1}) \left( e^{\beta_1} + \ldots + e^{\beta_{2n+1}} \right)^2 \\
\exp \left[ (\bar{z} + z') \left( e^{\beta_1} + \ldots + e^{\beta_{2n+1}} \right) \right] |f(\beta_1, \ldots, \beta_{2n+1})|^2.
\]

(3.21)

This correlation function depends on the product \(e^{\beta_B}(\bar{z} + z')\). It is scale invariant in the UV and IR fixed point. These correspond respectively to sending \(\beta_B\) to \(\mp \infty\), that is the coupling \(\lambda\) in the action to 0 or \(\infty\), in other words Neumann or Dirichlet boundary conditions. In the first case, \(K = 1\), in the second, \(K = -1\). Comparing with (3.10) and (3.11) we find, as expected, that:

\[
< 0|\partial_z \phi(z, \bar{z}) \partial_z \phi(z', \bar{z}')|B > = \pm \frac{2g}{(\bar{z} + z')^2},
\]

(3.22)

for Neumann, respectively Dirichlet boundary conditions. Although trivial, this result shows that the form factor expansion is well behaved, and allows us to study the correlator all the way from the UV to the IR fixed point when there is a boundary perturbation. In figures 3 and 4 we show the one particle (which is independent of \(B\)) and three particles contributions. We observe that indeed the convergence, by looking at the respective contributions, is very rapid.
Fig. 4: Three particles contribution for $B = 1, 0.1$.

The only drawback of this expansion is that it is not suited for studying the correlation of two operators right at the boundary. Indeed in that case, $\text{Re}(\bar{z} + z') = 0$, and the integrals in (3.21) do not converge. To solve this problem, we can introduce a modular transformed picture. We now consider the imaginary time as running along the $y$ axis. Now the boundary is not represented as a state; rather, the whole space of states is different, since now we have only a half space to deal with. The asymptotic states are not pure L or R moving, but are mixtures. For instance, one particle states are:

$$||\beta\rangle = |\beta\rangle_R + R(\beta)|\beta\rangle_L.$$ (3.23)

More generally, asymptotic states are obtained by adding to $|\beta_1, \ldots, \beta_n\rangle_R, \ldots, R$ all combinations with different choices of $R$ particles transformed into $L$ particles, via action of the boundary. Only the following two terms contribute:

$$||\beta_1, \ldots, \beta_n\rangle = |\beta_1, \ldots, \beta_n\rangle_R, \ldots, R + \ldots + R(\beta_1) \ldots R(\beta_n)|\beta_n, \ldots, \beta_1\rangle_L, \ldots, L + \ldots.$$ (3.24)

Although we used the same notation as previously, different things are meant by L,R. To make it clear, we now use the conventions:

$$<0|\phi(x, y)|\beta\rangle_R = \mu \exp[e^\beta(-y + ix)]$$

$$<0|\phi(x, y)|\beta\rangle_L = \mu \exp[e^\beta(-y - ix)].$$ (3.25)

To keep the notations as uniform as possible, we introduce the new coordinates:

$$w(z) \equiv iz = -y + ix.$$ (3.26)
so here R movers depend on \( w \), L movers on \( \bar{w} \). The normalization \( N \) is of course the same as before, and as before the LL and RR correlators do not depend on the boundary interaction. One finds:

\[
\begin{align*}
\langle 0 | \partial_w \phi(w, \bar{w}) \partial_{w'} \phi(w', \bar{w}') | 0 \rangle &= -\frac{2g}{(w - w')^2}, \\
\langle 0 | \partial_{\bar{w}} \phi(w, \bar{w}) \partial_{\bar{w}'} \phi(w', \bar{w}') | 0 \rangle &= -\frac{2g}{(\bar{w} - \bar{w}')^2},
\end{align*}
\]

where we used the fact that \(|R(\beta)|^2 = 1\). When compared with (3.18), these correlators have an overall minus sign due to the dimension \( h = 1, \bar{h} = 0 \) (resp. \( h = 0, \bar{h} = 1 \)) of the operators.

Let us now consider:

\[
\begin{align*}
\langle 0 | \partial_{\bar{w}} \phi(w, \bar{w}) \partial_w \phi(w', \bar{w}') | 0 \rangle.
\end{align*}
\]

To compute it, we insert a complete set of states which are of the form (3.23). In the massless case however, since \( \partial_w \phi \) is a R operator, \( \partial_{\bar{w}} \phi \) a L operator, the only terms that contribute are in fact the ones with either all L or all R moving particles, as written in (3.24). Thus, (3.28) expands simply as:

\[
-\sum_{n=0}^{\infty} \int \frac{d\beta_1 \ldots d\beta_{2n+1}}{(2\pi)^{2n+1}} R(\beta_1 - \beta_B) \ldots R(\beta_{2n+1} - \beta_B) (e^{\beta_1} + \ldots + e^{\beta_{2n+1}})^2 \exp \left[ (\bar{w} - w')(e^{\beta_1} + \ldots + e^{\beta_{2n+1}}) \right] |f(\beta_1, \ldots, \beta_{2n+1})|^2.
\]

Observe the crucial minus sign when compared to (3.21). It occurs because in one geometry the correlator depends on \( \bar{z} + z' \), while in the other on \( \bar{w} - w' \). This now converges provided \( y > y' \), even if \( x = x' = 0 \) i.e the operators are sitting right on the boundary. Now, using the fact that from factors depend only on differences of rapidities, this expression can be mapped with (3.21) if we formally set \( \beta = \beta' + i\frac{\pi}{2} \), provided one has:

\[
K(\beta) = R \left( i\frac{\pi}{2} - \beta \right),
\]

as claimed above.

To summarize, we can write the left right current current correlator in two possible ways. By using the boundary state one finds:

\[
\langle \partial_x \phi(x, y) \partial_{x'} \phi(x', y') \rangle = \int_0^\infty dE \mathcal{G}(E) \exp [E(x + x') - iE(y - y')],
\]

where \( \mathcal{G}(E) \) is the spectral function.
(recall that $x, x' < 0$). One obtains $G(E)$ simply by fixing the energy to a particular value in (3.21). When this is done, the remaining integrations occur on a finite domain for each of the individual particle energies since $\sum_{i=1}^{2n+1} e^{\beta_i} = E$, and there is no problem of convergence anymore. One then gets:

$$G(E) = \sum_{n=0}^{\infty} \int_{-\infty}^{\ln E} \frac{d\beta_1 \ldots d\beta_{2n}}{(2\pi)^{2n+1}(2n+1)!} \frac{E^2}{E - e^{\beta_1} - \ldots - e^{\beta_{2n}}}$$

$$K(\beta_B - \beta_1) \ldots K(\beta_B - \beta_{2n}) K \left[ \beta_B - \ln \left( E - e^{\beta_1} - \ldots - e^{\beta_{2n}} \right) \right]$$

$$|f \left[ \beta_1 \ldots \beta_{2n}, \ln \left( E - e^{\beta_1} - \ldots - e^{\beta_{2n}} \right) \right]|^2,$$

with the constraint $\sum_{i=1}^{2n} e^{\beta_i} \leq E$. The denominator might suggest some possible divergences; it is important however to realize that it vanishes if and only if the particle with rapidity $\beta_{2n+1}$ has vanishing energy, in which case the form factor vanishes too. We can now shift the integrands to write equivalently:

$$G(E) = E \sum_{n=0}^{\infty} \int_{-\infty}^{\ln E} \frac{d\beta_1 \ldots d\beta_{2n}}{(2\pi)^{2n+1}(2n+1)!} \frac{1}{1 - e^{\beta_1} - \ldots - e^{\beta_{2n}}}$$

$$K(\ln(T_B/E) - \beta_1) \ldots K(\ln(T_B/E) - \beta_{2n}) K \left[ \ln(T_B/E) - \ln (1 - e^{\beta_1} - \ldots - e^{\beta_{2n}}) \right]$$

$$|f \left[ \beta_1 \ldots \beta_{2n}, \ln (1 - e^{\beta_1} - \ldots - e^{\beta_{2n}}) \right]|^2,$$

where the constraint $\sum_{i=1}^{2n} e^{\beta_i} \leq 1$ is implied, we used the fact that form-factors depend only on rapidity differences, and $T_B \equiv e^{\beta_B}$. By using the dual picture, one finds:

$$< \partial_x \phi(x,y) \partial_y \phi(x',y') > = \int_0^{\infty} dE F(E) \exp \left[ -iE(x + x') - E(y - y') \right],$$

(3.34)

where:

$$F(E) = -E \sum_{n=0}^{\infty} \int_{-\infty}^{\ln E} \frac{d\beta_1 \ldots d\beta_{2n}}{(2\pi)^{2n+1}(2n+1)!} \frac{1}{1 - e^{\beta_1} - \ldots - e^{\beta_{2n}}}$$

$$R(\beta_1 - \ln(T_B/E)) \ldots R(\beta_{2n} - \ln(T_B/E)) R \left[ \ln (1 - e^{\beta_1} - \ldots - e^{\beta_{2n}}) - \ln(T_B/E) \right]$$

$$|f \left[ \beta_1 \ldots \beta_{2n}, \ln (1 - e^{\beta_1} - \ldots - e^{\beta_{2n}}) \right]|^2,$$

(3.35)

where in (3.33) and (3.35) the constraint $\sum_{i=1}^{2n} e^{\beta_i} \leq 1$ is implied. The two expressions are in correspondence by the simple analytic continuation:

$$G(E) = iF(iE).$$

(3.36)
3.3. The analog of the conductance

Although such a quantity does not have much physical meaning, we can formally define a conductance in the sinh-Gordon case using the current current correlators. It is instructive to carry out this computation now.

To use the Kubo formula of the first section, we adopt the first point of view where the boundary is taken into account through the introduction of the boundary state $|B\rangle$. Write again:

$$<\partial_{\bar{z}}\phi(x,y)\partial_{z'}\phi(x',0)> = \int_0^\infty dE \mathcal{G}(E) \exp [E(x+x')-iEy]. \quad (3.37)$$

This is the only correlation contributing to $\Delta G$ for positive Matsubara frequencies, and

$$\Delta G(\omega_M) = \frac{G(\omega_M)}{4\omega_M}. \quad (3.38)$$

Here we have used the fact that $\omega_M L \ll 1$, i.e., the system, although large, is much smaller than the wavelength associated with the (modulus of the) AC frequency.

To go to real frequencies, we can simply substitute $\omega_M \rightarrow -i\omega$ in the $K$ matrices in the integrals (3.33):

$$\Delta G(\omega) = \frac{1}{4\omega} \text{Im} G(-i\omega) = \frac{1}{4\omega} \text{Re} F(\omega). \quad (3.39)$$

Recall that $K(\beta) = -\tanh(\beta/2)$. So we can expand the product of $K$ matrices as a series, using

$$K[\ln(T_B/i\omega)-\beta] = \left(\frac{i\omega}{T_B} e^{\beta} - 1\right) \sum_{n=0}^\infty \left(\frac{-i\omega}{T_B}\right)^n e^{n\beta}. \quad (3.40)$$

Computing term by term gives $\Delta G$ as a power series in $(\omega/T_B)^2$. This is an IR expansion, valid for strong barriers $T_B > \omega$.

To get an UV expansion, holding for weak barriers $T_B < \omega$, we have to split each integration into two pieces, $\int_{-\infty}^{-\ln(\omega/T_B)}$ and $\int_{-\ln(\omega/T_B)}^0$. In the first integral, we expand $K$ as in (3.40) but in the second case we expand it as

$$K[\ln(T_B/i\omega)-\beta] = \left(1 - \frac{T_B}{i\omega} e^{-\beta}\right) \sum_{n=0}^\infty \left(\frac{-T_B}{i\omega}\right)^n e^{-n\beta}. \quad (3.41)$$

This gives $\Delta G$ as a power series in $(T_B/\omega)^2$.

A nice feature of the sinh-Gordon case is that the problem is well defined both for coupling $g$ and for its dual $\frac{1}{g}$. This is because the dimension of the perturbing operator
being negative, it is always relevant, and none of these two couplings is plagued by problems of irrelevant perturbation theory, like what happens in the sine-Gordon case. We can therefore consider exactly the same problem with a coupling $1/g$. On the other hand, these two cases are related by the sinh-Gordon duality, under which form factors are identical, up to an overall scale (due to the choice of $\mu$ as related with the two point function of the field $\phi$). We deduce therefore the identity

$$G\left( g, \frac{\omega}{T_B} \right) = g^2 G\left( \frac{1}{g}, \frac{\omega}{T_B} \right),$$

(3.42)

where the right hand term is computed using exactly the same formula as (3.21) and (3.38) but with the formal replacement $g \to \frac{1}{g}$. $T_B = e^{\beta_B}$ is the same in both cases. Of course, since the correspondence between $T_B$ and the coupling $\lambda$ in (3.13) depends on $g$, (3.42) maps the conductance for $g$ and $1/g$ with different boundary couplings. In other words, if we introduce the function $T_B(g, \lambda)$ we have that

$$G(g, \omega, \lambda) = g^2 G\left( \frac{1}{g}, \omega, \lambda' \right),$$

(3.43)

where $\lambda'$ follows from

$$T_B(g, \lambda) = T_B \left( \frac{1}{g}, \lambda' \right).$$

(3.44)

Related duality properties are expected in the sine-Gordon model, but unfortunately are much more difficult to establish.

4. $\nu = 1/3$ Hall effect.

In this section we follow the same line of thought for the sine-Gordon model. This is the massive deformation of the free boson which preserves integrability with either boundary interactions $B$ used in the Hall problem and the anisotropic Kondo model. Thus the form factors of the sine-Gordon model in the massless limit will be the quantities we need. The solitons/anti-solitons and breathers quasi-excitations make the problem more complicated but the results presented before hold with the addition of a few indices (and much more complicated form factors).
4.1. Expressions for the sine-Gordon form factors.

We now consider the same problem in the sine-Gordon case, with action:

\[ S = \frac{1}{16\pi g} \int_{-\infty}^{\infty} dx dy \left[ (\partial_x \phi)^2 + (\partial_y \phi)^2 + \Lambda \cos \phi \right]. \]  

(4.1)

To compare with standard normalizations, one has \( \beta^2 = 8\pi g \). The form factors approach is formally the same, albeit more complicated because the particle content is much richer, and depends on \( g \). For \( 1/2 < g < 1 \), only solitons/anti-solitons appear in the spectrum of the theory. This is the so called repulsive case, with \( g = 1/2 \) the Toulouse limit. When \( 0 < g < 1/2 \), the particle content is enriched by \( [1/g - 2] \) bound states, called breathers. In the following we will denote by the indices \( \epsilon = \pm \) the solitons and anti-solitons, and \( \epsilon = 1, 2, \ldots, [1/g - 2] \) the breathers. The solitons form factors in the massive case were written by Smirnov \footnote{17} and we obtain the massless form factors by taking the appropriate limit of the massive ones. Only right and left moving form factors survive in this limit, as in the sinh-Gordon case. Moreover, the symmetry of the action dictates that only form factors with total topological charge zero are non-zero for the current operator. As an example, the soliton/anti-soliton form factor is given by:

\[ <0|\partial_\phi(z, \bar{z})|\beta_1, \beta_2>_{RR}^{\epsilon\epsilon'} = \\
\epsilon' \mu M(2\pi d)^2 e^{(\beta_1+\beta_2)/2} \frac{\zeta(\beta_1 - \beta_2)}{\cosh \left( \frac{1-g}{2g} (\beta_1 - \beta_2 + i\pi) \right)} \exp \left[ M(e^{\beta_1} + e^{\beta_2})z \right], \]  

(4.2)

with \( \epsilon + \epsilon' = 0 \) and \( \epsilon = \pm \) stands for soliton (resp. antisoliton). From \footnote{17} one has:

\[ \zeta(\beta) = c \sinh \frac{\beta}{2} \exp \left( \int_0^{\infty} \frac{\sin^2 \frac{x(\beta+i\pi)}{2}}{x \sinh \frac{\pi g x}{2(1-g)} \sinh \pi x \cosh \frac{\pi x}{2}} dx \right), \]  

(4.3)

with the constant \( c \) given by:

\[ c = \left( \frac{4(1-g)}{g} \right)^{1/4} \exp \left( \frac{1}{4} \int_0^{\infty} \frac{\sin \frac{\pi x}{2} \sinh \frac{\pi(1-2g)x}{2(1-g)}}{x \sinh \frac{\pi g x}{2(1-g)} \cosh \frac{\pi x}{2}} dx \right), \]  

(4.4)

and \( d \) by:

\[ d = \frac{1}{2\pi c} \frac{(1-g)}{g}, \]  

(4.5)

and \( \mu \) is a normalization constant to be determined as before. Observe that in the free case \( g = 1/2 \), the form factors \footnote{12} reduce to trivial kinetic terms since \( \zeta(\beta) \propto \sinh(\beta/2) \).
The other soliton-antisoliton form factors follow from the analysis of [15]. Their expression simplifies in the case $g = \frac{1}{t}$, $t$ an integer. This is the physically relevant case for the $\nu = \frac{1}{t}$ fractional quantum Hall effect. One then finds:

$$< 0 | \partial_z \phi(z, \bar{z}) | \beta_1, \ldots, \beta_{2n} >_{R, \ldots, R} = \mu M (8\pi^2 d)^n e^{(\beta_1 + \ldots + \beta_{2n})/2} \prod_{i<j} \zeta(\beta_i - \beta_j)$$

$$\sinh \left[ \frac{t-1}{2} \sum_{p=1}^{n} (\beta_{p+n} - \beta_p - i\pi) \right] \prod_{p=1}^{n} \prod_{q=n+1}^{2n} \sinh^{-1} (t-1)(\beta_q - \beta_p) \ det H.$$ (4.6)

The matrix $H$ is obtained as follows. First introduce the function:

$$\psi(\alpha) = 2^t \prod_{j=1}^{t-2} \sinh \left[ \frac{1}{2} \left( \alpha - i \frac{\pi j}{t-1} + i \frac{\pi}{4} \right) \right].$$ (4.7)

One then defines the matrix elements as:

$$H_{ij} = \frac{1}{2i\pi} \int_{-2i\pi}^{0} d\alpha \prod_{k=1}^{k=2n} \psi(\alpha - \beta_k) \exp \left[ \left( n - 2j - 1 \right) \alpha + \left( n - 2i \right) \left( t - 1 \right) \alpha \right],$$ (4.8)

where $i, j$ run over $1, \ldots, n - 1$. It is not difficult to convince oneself that this produce a symmetric polynomial of the right degree. Although cumbersome, it is an easy task to extract these determinants, as examples we find for $g = 1/3$ :

$$\det H = \exp \left( - \frac{1}{2} \sum_{i=1}^{2n} \beta_i \right) \sigma_1(e^{\beta_1}), \ n = 2,$$

$$\det H = \exp(- \sum_{i=1}^{2n} \beta_i) \sigma_1(e^{\beta_1}) \sigma_3(e^{\beta_1}), \ n = 3,$$ (4.9)

up to irrelevant phases and with the $\sigma_q$’s defined previously. Having these expression we can get all form factors using the axioms they were constructed upon [15]. For example, the solitons form factors with different positions of the indices $\epsilon_i$ we use the symmetry property:

$$f(\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots, \beta_n)_{\epsilon_1, \ldots, \epsilon_i, \epsilon_{i+1}, \ldots, \epsilon_n} s_{\epsilon_i, \epsilon_{i+1}}^{\epsilon_i, \epsilon_{i+1}}(\beta_i - \beta_{i+1})$$

$$= f(\beta_1, \ldots, \beta_{i+1}, \beta_i, \ldots, \beta_n)_{\epsilon_1, \ldots, \epsilon_{i+1}, \epsilon_i, \ldots, \epsilon_n}.$$ (4.10)

Here again, we omit the distinction between left and right moving form factor, they are simply related by complex conjugation. At the points $g = 1/t$ the soliton $S$ matrix used in the last expression is reflectionless and basically just permutes the rapidities up to a phase. When there are breathers, the soliton $S$ matrix has poles corresponding to the
bound states at the points \( \beta = i\pi - \frac{i\pi g}{(1 - g)} m \) for the \( m \)'th breather. In view of the last relation, this induces poles in the form factors. We obtain the breather form factors from these poles:

\[
\text{res} f(\beta_1, \ldots, \beta_{n-1}, \beta_n)_{\epsilon_1, \ldots, \epsilon_{n-1}, \epsilon_n} = a_m(-1)^{\frac{\epsilon_n + 1}{2}} C_{\epsilon_{n-1}, \epsilon_n} f(\beta_1, \ldots, \beta_{n-1} + i\frac{\pi}{2} - i\frac{\pi g}{2(1 - g)})_{\epsilon_1, \ldots, \epsilon_{n-2}, \epsilon_n},
\]

and \( a_m \) is given by the residue at \( \beta = i\pi - \frac{i\pi g}{(1 - g)} m \):

\[
a_m = \left( \text{res} S_{\epsilon_{n-1}, \epsilon_n}(\beta) \right)^{1/2}.
\]

Having these relations, we possess all ingredients to compute all form factors for \( g = 1/t \).

Then, using them for the computation of the current correlations is merely an extension of the previous results for sinh-Gordon with indices. The normalisation of the form factors, \( \mu \), is chosen such that (3.18) is reproduced. This is fixed by introducing a complete basis of states:

\[
1 = \sum_{n=0}^{\infty} \sum_{\epsilon_i} \int \frac{d\beta_1 \ldots d\beta_n}{(2\pi)^n n!} |\beta_1, \ldots, \beta_n >_{\epsilon_1, \ldots, \epsilon_n} | < \beta_n, \ldots, \beta_1|
\]

and computing the correlations exactly like in the sinh-Gordon case. It is interesting to observe however, that in the sine-Gordon case, there is another - a priori independent - way to fix the normalisation \( \mu \). Indeed, the operator \( \partial \phi \) being related with the \( U(1) \) charge, we need that

\[
\frac{1}{R} < \beta_1| \int_{-\infty}^{\infty} \partial_x \phi| \beta_2 >_{+} = 2\pi \delta(\beta_1 - \beta_2),
\]

using the fact that a soliton for the bulk theory (4.1) obeys \( \phi(\infty) - \phi(-\infty) = 2\pi \). Using (4.6) we get the requirement that

\[
\mu = \frac{1}{2\pi d\zeta(-i\pi)} = \frac{2\pi g}{(1 - g)}.
\]

Remarkably, this involves only the two particle form factor while the requirement that (3.18) is obeyed involves a sum over an infinity of form-factors. However, the two should be identical if the description is consistent, which we checked is the case.

Knowing the normalisation before performing the sum (3.11) gives us an a priori estimate of how many form-factors will be necessary to compute the full correlator. Indeed for \( g = 1/3 \), the one breather and 2 solitons form factors normalised to 3.14 which is very
close to the exact $\pi$. Similarly for $g = 1/4$ we found from the contributions up to two solitons that $\mu = 2.05$ to compare with $2.094 = 2\pi/3$.

Moreover the considerations concerning the correlations involving the boundary state follow in this case with now the boundary state given by:

$$|B> = \sum_{n=0}^{\infty} \int_{0<\beta_1<...<\beta_n} K^{a_1b_1}(\beta_B - \beta_1) ... K^{a_nb_n}(\beta_B - \beta_n) Z^{a_1}_{L}^{*}(\beta_1) Z^{b_1}_{R}(\beta_1) ... Z^{a_nb_n}_{R}(\beta_n),$$

(4.16)

with an implicit sum on the indices implied in this expression. The matrix $K^{a b}$ is related to the boundary $R$ matrix in the following way:

$$K^{a b}(\beta) = R^{a \bar{b}}_{\bar{b} b}(i\pi/2 - \beta).$$

(4.17)

The $\bar{b}$ means that we take the conjugate of the indices ie. $\pm \rightarrow \mp$ and $m \rightarrow m$.

From the previous expressions, we can compute the current-current correlation function in the presence of a boundary for $g = 1/t$. The results we will get depends on the boundary interaction, in the next subsection we present some general results for all values of $g$ when the boundary is of the form $(2.6)$. This is of relevance to the conductance in the quantum Hall effect.

4.2. General remarks and analytic form of the conductance.

To proceed we need the reflection matrix of the boundary sine-Gordon theory. For generic values of the coupling $g$, the amplitude for the processes $+ \rightarrow +$ and $- \rightarrow -$ is $R_{\pm}^{\pm}(\beta - \beta_B)$, and for the processes $+ \rightarrow -$ and $- \rightarrow +$ it is $R_{\mp}^{\pm}(\beta - \beta_B)$ with:

$$R_{\mp}^{\pm}(\beta) = e^{i(1-g)\beta/2g} R(\beta),$$

$$R_{\pm}^{\pm}(\beta) = i e^{(g-1)\beta/2g} R(\beta)$$

(4.18)

where the function $R$ reads:

$$R(\beta) = \frac{e^{i\gamma}}{2 \cosh \left[ \frac{(1-g)\beta}{2g} - \frac{i\pi}{4} \right]} \prod_{l=0}^{\infty} \frac{Y_l(\beta)}{Y_l(-\beta)}$$

$$Y_l(\beta) = \frac{\Gamma \left( \frac{3}{4} + l \frac{(1-g)}{2g} - \frac{i(1-g)\beta}{2\pi g} \right) \Gamma \left( \frac{1}{4} + (l + 1) \frac{(1-g)}{2g} - \frac{i(1-g)\beta}{2\pi g} \right)}{\Gamma \left( \frac{1}{4} + (l + \frac{1}{2}) \frac{(1-g)}{2g} - \frac{i(1-g)\beta}{2\pi g} \right) \Gamma \left( \frac{3}{4} + (l + \frac{1}{2}) \frac{(1-g)}{2g} - \frac{i(1-g)\beta}{2\pi g} \right)}.$$

In (4.18), our conventions are such that in the UV limit ($\beta_B \rightarrow -\infty$) the scattering is totally off-diagonal so a soliton bounces back as an anti-soliton, in agreement with classical limit
results for Neumann boundary conditions. A useful integral representation of $R$ is given by:

$$R(\beta) = \frac{e^{i\gamma}}{2 \cosh \left[ \frac{(1-g)\beta}{2g} - i \frac{\pi}{4} \right]} \exp \left( i \int_{-\infty}^{\infty} \frac{dy}{2g} \sin \frac{2(1-g)\beta y}{g\pi} \frac{\sinh \left( \frac{1-2g}{g} y \right)}{\sinh 2y \cosh \left( \frac{1-2g}{g} y \right)} \right). \quad (4.19)$$

Recall that the spectrum is made of one breather and the pair soliton antisoliton in the whole domain $1/3 \leq g < 1/2$. More breathers appear for $g < 1/3$, moreover the reflection matrix of the 1-breather is the same as in the sinh-Gordon case. There are no breathers for $g > 1/2$.

In all these regimes, the form factors are known. They are quite complicated for generic $g$, and expressions for the correlators are more involved because the S matrix is non diagonal. We can however extract some features of the UV and IR expansions easily, following the same logic as in the sinh-Gordon case. To do so, consider the soliton antisolitons reflection matrix. Evaluating the integral in (4.19) by the residues method leads to a double expansion of the elements $R_{\epsilon \epsilon'}$ in powers of $\exp(\beta)$ and $\exp(\frac{1}{g} - 1)\beta$. This leads for the conductance to a double power series in $(\omega/T_B)^{-2+2/g}$ and $(\omega/T_B)^2$ in the IR, $(T_B/\omega)^2-2g$ and $(T_B/\omega)^2$ in the UV. Breathers do not change this result. For instance for the 1-breather, since the reflection matrix is the same as in the sinh-Gordon case, and therefore $g$ independent, the contributions expand as a series in $(\omega/T_B)^2$ in the IR, $(T_B/\omega)^2$ in the UV. This holds for any coupling $g$. Therefore, as first argued by Guinea et al. [25], at low frequency, the conductance goes as $\omega^2$ for $g < 1/2$, $\omega^{-2+2/g}$ for $g > 1/2$. The $\omega^2$ power would seem to indicate that there should be a $T^2$ term in the DC conductance, but this is not correct because only the modulus square of $R_{\epsilon \epsilon'}$ contribute to the DC conductance, and these expand only as powers of $\exp(\frac{1}{g} - 1)\beta$.

The presence of analytic terms in the IR is a straightforward consequence of the fact that IR perturbation theory involves an infinity of counter-terms, in particular polynomials in derivatives of $\phi$ [25]. More surprising maybe is the fact that we find analytical terms in the UV. This requires some discussion. The UV terms follow from the short distance behaviour of the correlation function of the current. For any operator $O$ we could write formally,

$$< O(x', y')O(x, y) > = \sum_{n=0}^{\infty} (\lambda)^{2n} \int d1 \ldots dn < \tilde{O}(x', y')\tilde{O}(x, y) \cos \frac{1}{2} \phi(1) \ldots \cos \frac{1}{2} \phi(n) >,$$

(4.20)
where $\tilde{O}$ is the $\lambda \to 0$ limit of the field $O$. From (4.20), one would naively expect that the two point function of the current expands as a power series in $\lambda^2$, which would lead to a power series in $(\omega/T_B)^{2g-2}$. This is incorrect however because, even if integrals are convergent at short distance for $g < 1/2$, they are always divergent at large distances. It is known that these IR divergences give precisely rise to non analyticity in the coupling constant $\lambda$. One usually writes:
\[
\langle O(x', y')O(x, y) \rangle = \sum_i C^i_{OO}(x' - x, y' - y)O_i(x, y), \tag{4.21}
\]
where $O_i$ are a complete set of local operators in the theory and the $C_i$'s are structure functions. These, being local quantities, have analytic behaviour in $\lambda$. However, $\langle O_i(x, y) \rangle$ being non local is in general non analytic - actually, on dimensional grounds,
\[
\langle O_i(x, y) \rangle = \lambda^{\Delta/(1-g)} \propto T_B^\Delta, \tag{4.22}
\]
where $\Delta = h + \bar{h}$ is the (bulk) dimension of the field $O_i$. If we computed the conductance perturbatively using Matsubara formula, we would use (4.20) with $O$ the electrical current operator. The case $O_i$ the identity operator gives rise to an analytical expression in $\lambda$, but eg the case $O_i = \partial_z \partial_{\bar{z}} \phi$ gives $\lambda^{2/(1-g)}$ times an analytical expression in $\lambda$ (its mean value can be non zero because there is a boundary). More generally, since the only operators $O_i$ appearing in the case of the current are polynomials in derivatives of $\phi$, all with integer dimensions, we expect that the two point function of the current will expand as a double series of the form $\lambda^{2n}\lambda^{2m/(1-g)}$, ie going back to $T_B$ variable, that the conductance will expand as a double series of the form $(T_B/\omega)^{2n(1-g)}(T_B/\omega)^{2m}$, in agreement with the form factors result.

4.3. The free case.

In the case $g = 1/2$ one has simply:
\[
R_{\pm \pm}(\beta) = P(\beta) = \frac{e^\beta}{e^\beta + i},
\]
\[
R_{\pm \pm}(\beta) = Q(\beta) = \frac{i}{e^\beta + i}. \tag{4.23}
\]
In that case, only the soliton-antisoliton form factor is non zero, $f(\beta_1, \beta_2) = \mu e^{\beta_1/2} e^{\beta_2/2}$, where the normalization is easily evaluated $\mu = 2\pi$ and we have set $M = 1$. Hence, $\mathcal{F}(\omega)$ is readily evaluated
\[
\mathcal{F}(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\beta_1 d\beta_2 \delta(e^{\beta_1} + e^{\beta_2} - \omega) [Q(\beta_1)Q(\beta_2) - P(\beta_1)P(\beta_2)] e^{\beta_1} e^{\beta_2}, \tag{4.24}
\]
so

\[ F(\omega) = \omega \int_0^1 dx \frac{x(1-x) + 1}{(x + i\frac{T_B}{\omega})(\omega - x + i\frac{T_B}{\omega})}, \quad (4.25) \]

from which it follows that

\[ \Delta G(\omega) = \frac{1}{4} - \frac{T_B}{2\omega} \tan^{-1}(\omega/T_B). \quad (4.26) \]

Thus we find

\[ G(\omega) = \frac{1}{2} \left( 1 - \frac{T_B}{\omega} \tan^{-1}(\omega/T_B) \right). \quad (4.27) \]

This is in agreement with the solution of [20].

4.4. \(G(\omega)\) at \(g = 1/3\).

The conductance for \(g = 1/3\) has a direct application to the quantum Hall effect. Comparing with the free case, previously treated, we now have a breather in the spectrum and non zero form factors for all number of rapidities. Still the convergence is such that evaluating the first few form factors give results to a very good accuracy, independently of the regime, UV or IR, in which we make the computation.

In this case, the first few non zero form factors are \(f_1, f_{\pm, \mp}, f_{\pm, \mp, 1}, f_{1, 1, 1}, \text{etc...}\) Here the subscript “1” denotes the breather. The first step is to compute the normalisation in order to satisfy (3.11). When computing this normalisation, we find that the first two form factors account for the whole result to more than one percent accuracy. Then including the 1 breather-2 solitons form factor is sufficient to get the result to a very good accuracy \((F_{1,1,1} \text{ is negligible})\). Actually one observes that the speed of convergence of the form factor expansion varies geometrically with the number of solitons (counting the breathers as 2 solitons).

In order to get the conductance we need the reflection matrices, they were given in previous expressions and reduce to a simpler form for this value of \(g\) :

\[ R(\beta) = \frac{1}{2 \cosh(\beta - i\frac{\pi}{4})} \frac{\Gamma(3/8 - \frac{i\beta}{2\pi})\Gamma(5/8 + \frac{i\beta}{2\pi})}{\Gamma(5/8 - \frac{i\beta}{2\pi})\Gamma(3/8 + \frac{i\beta}{2\pi})} \quad (4.28) \]

and the breather reflection matrix is :

\[ R_1(\beta) = \tanh\left(\frac{\beta}{2} - \frac{i\pi}{4}\right). \quad (4.29) \]
From the pole of the 2 solitons form factor, the one breather form factor is found using (4.11) and its contributions to the conductance is:

$$\Delta G^{(1)}(\omega) = -\mu^2 \frac{2 \pi d^2}{8} \Re e \ \tanh\left(\frac{\log(\sqrt{2T_B})}{2} - i \pi/4\right),$$  \hspace{1cm} (4.30)

here $\mu = 3.14$ is fixed by (3.18) and $d = 0.1414...$. The contribution from the two solitons form factors is computed similarly, we find:

$$\Delta G^{(2)}(\omega) = -\frac{\mu^2 d^2}{2} \Re e \int_{-\infty}^{0} d\beta \frac{R(\beta + \log(\omega/T_B))R[(1 - e^\beta)(\omega/T_B)]}{\cosh^2(\beta - \log(1 - e^\beta))} |\zeta[\beta - \log(1 - e^\beta)]|^2$$

$$e^\beta \left[ e^\beta (1 - e^\beta)(\omega/T_B)^2 + \frac{1}{e^\beta (1 - e^\beta)(\omega/T_B)^2} \right],$$  \hspace{1cm} (4.31)

where $\zeta(\beta)$ is the function defined in (4.2). We can similarly write the following contribution, and we find that these last two expressions are sufficient for any reasonable purpose, they give the frequency dependent conductance to more than one percent accuracy. We give the full function $G(\omega)$ in figure 5.

![Fig. 5: Frequency dependent conductance at T=0.](image)

Observe that in the UV and in the IR we obtain the $\omega$ dependance discussed previously, even with the truncation to a few form-factors. The form-factors expansion is indeed very different from the perturbative expansion in powers of the coupling constant in the UV, or
in powers of the inverse coupling constant in the IR. Each form-factor contribution has by itself the same analytical structure as the whole sum; contributions with higher number of particles simply determine coefficients to a greater accuracy.

5. Anisotropic Kondo model and dissipative quantum mechanics.

In section 2 we explained how the anisotropic model was related to dissipative quantum mechanics. The bosonised form of the Hamiltonian is:

\[
H = \frac{1}{2} \int_{-\infty}^{0} dx \left[ 8\pi g \Pi^2 + \frac{1}{8\pi g} (\partial_x \phi)^2 \right] + \frac{\lambda}{2} \left( S_y e^{i\phi(0)/2} + S_x e^{-i\phi(0)/2} \right),
\]

(5.1)

where \( S \) are Pauli matrices. As for the quantum Hall problem, we keep using as a basis the massless excitations of the sine-Gordon model; however the boundary interaction is different. This will result in different reflection matrices. Another difference with the previous section will be the quantities we compute, the first step will be to relate them to current correlation functions and then using our by now well known techniques to get results.

5.1. Dissipative quantum mechanics.

We first work in imaginary time. We consider therefore the anisotropic Kondo problem at temperature \( T \). Let us consider the quantity \( X(y) \equiv \langle [S^z(y) - S^z(0)]^2 \rangle \). On the one hand, using that \( S^z = \pm 1 \), it reads \( 2[1 - C(y)] \), where \( C(y) \) is the usual spin correlation

\[
C(y) = \frac{1}{2} \left[ \langle S^z(y) S^z(0) \rangle + \langle S^z(0) S^z(y) \rangle \right].
\]

(5.2)

On the other hand, we can write a perturbative expansion for \( X(y) \) by expanding evolution operators in powers of the coupling constant \( \lambda \). At every order, we get ordered monomials which are a product of a monomial in \( S^\pm \) and vertex operators of charge \( \pm 1/2 \). We must then evaluate \( S^z(y) - S^z(0) \) for each such term, trace over the two possible spin states, and average over the quantum field. Since we deal with spin 1/2, terms \( S^+ \) and \( S^- \) must alternate, and there must be an overall equal number of \( S^+ \) and \( S^- \), and an equal number of 1/2 and -1/2 electric charges.

Now, since each \( S^+(y) \) comes with a \( e^{-i\phi(y)/2} \) and each \( S^-(y) \) comes with a \( e^{i\phi(y)/2} \), \( S^z(y) = S^z(0) \) if there is a vanishing electric charge inserted between 0 and \( y \), and \( S^z(y) = -S^z(0) \) if the charge inserted between 0 and \( y \) is non zero (and then it has to be \( \pm 1/2 \)).
Therefore, we can write the perturbation expansion of $X(y)$ in such a way that the spin contributions all disappear:

$$X(y) = \frac{1}{Z} \sum_{n=0}^{\infty} \lambda^{2n} \sum_{\text{alternating} \epsilon_i = \pm} \sum_{p=0}^{2n} \int_0^y dy_1 \int_0^{y_1} dy_2 \ldots \int_0^{y_{p-1}} dy_p \int_{y_p}^{1/T} dy_{p+1} \ldots \int_{y_{2n-1}}^{1/T} dy_{2n}$$

$$4(\epsilon_1 + \ldots + \epsilon_p)^2 \left\langle e^{-i\epsilon_1 \phi(y_1)/2} \ldots e^{-i\epsilon_{2n} \phi(y_{2n})/2} \right\rangle_N,$$

(5.3)

where $Z$ is the partition function, the factor 4 occurs because of the normalization $S^z = \pm 1$, for every configuration of $\epsilon$'s, only one value of $S^z(0)$ gives a non vanishing contribution. Here, the label $N$ indicates correlation functions for the free boson evaluated with Neumann boundary conditions (the conditions as $\lambda \to 0$).

On the other hand, let us consider the correlator

$$<\partial_x \phi(x, y) \phi(0, y')>_N = -8g \frac{x}{x^2 + (y - y')^2},$$

(5.4)

which goes to $-8g\pi \delta(y - y')$ as $x \to 0$. We have then, by Wick’s theorem,

$$<e^{-i\epsilon_1 \phi(y_1)/2} \ldots e^{-i\epsilon_{2n} \phi(y_{2n})/2} \partial_x \phi(x, y)>_N =$$

$$8ig \left( \sum_{i=1}^{2n} \epsilon_i \frac{x}{x^2 + (y - y_i)^2} \right) \left\langle e^{i\epsilon_1 \phi(y_1)/2} \ldots e^{i\epsilon_{2n} \phi(y_{2n})/2} \right\rangle_N,$$

(5.5)

and therefore

$$\left\langle e^{-i\epsilon_1 \phi(y_1)/2} \ldots e^{-i\epsilon_{2n} \phi(y_{2n})/2} : \partial_x \phi(x, y) \partial_x \phi(x, y') : \right\rangle_N =$$

$$- (8g)^2 \left( \sum_{i=1}^{2n} \epsilon_i \frac{x}{x^2 + (y - y_i)^2} \right) \left( \sum_{i=1}^{2n} \epsilon_i \frac{x}{x^2 + (y' - y_i)^2} \right) \left\langle e^{-i\epsilon_1 \phi(y_1)/2} \ldots e^{-i\epsilon_{2n} \phi(y_{2n})/2} \right\rangle_N,$$

(5.6)

where contractions between the dots are discarded. In (5.4), contractions between the dots would lead to a term factored out as the product of the two point function of $\partial_x \phi$ and the 2n point function of vertex operators, both evaluated with $N$ boundary conditions. Now, we are going to be interested in the $x \to 0$ limit where, with $N$ boundary conditions, $\partial_x \phi$ vanishes. As a result we can actually forget the subtraction in (5.6), and write simply obtain

$$X(y, \lambda) = - \frac{1}{(4g\pi)^2} \lim_{x \to 0} \int_0^y \int_0^y dy' dy'' < \partial_x \phi(x, y') \partial_x \phi(x, y'') >_\lambda,$$

(5.7)

where the label $\lambda$ designates the correlator evaluated at coupling $\lambda$, $N$ corresponding to $\lambda = 0$. Hence, we can get $C(y)$ from the current current correlator. The latter can then
be obtained using form factors along the above lines. The only difference is the boundary matrix. If we restrict to the repulsive regime where the bulk spectrum contains only a soliton and an antisoliton, one has:

$$R^\pm = \tanh \left( \frac{\beta}{2} - \frac{i\pi}{4} \right), \quad R^\pm = 0. \quad (5.8)$$

Here again our conventions are such that a soliton bounces back as an antisoliton, in agreement with the UV and the IR limit that have Neumann boundary conditions. In the attractive regime we need to add the breathers with:

$$R_m^m = \frac{\tanh \left( \frac{\beta}{2} - \frac{i\pi m}{4(1/g - 1)} \right)}{\tanh \left( \frac{\beta}{2} + \frac{i\pi m}{4(1/g - 1)} \right)} . \quad (5.9)$$

Writing:

$$<\partial_z \phi(x, y') \partial_z \phi(x, y'')>_\lambda = \int_0^\infty G(E, \beta_B) \exp \left[ 2Ex - iE(y' - y'') \right] , \quad (5.10)$$

we have that:

$$\lim_{x \to 0} <\partial_x \phi(x, y') \partial_x \phi(x, y'')>_\lambda = \int_0^\infty dE \left[ G(E, \beta_B) - G(E, -\infty) \right] \exp \left[ -iE(y' - y'') \right] + c.c., \quad (5.11)$$

where the $<\partial_z \phi \partial_z \phi>$ part and its complex conjugate (which are $\lambda$ independent) have been evaluated by requiring that the correlator vanishes as $\lambda \to 0$ due to $N$ boundary conditions. Hence, using the fact that $G$ is real,

$$X(y) = \frac{1}{2(g\pi)^2} \int_0^\infty \frac{dE}{E^2} \left[ G(E, \beta_B) - G(E, -\infty) \right] \sin^2(Ey/2). \quad (5.12)$$

Therefore, if we write:

$$C(y) - 1 = \int_0^\infty A(\omega_M) \cos(\omega_M y) \, d\omega_M, \quad (5.13)$$

where $\omega_M$ is a Matsubara frequency, we have:

$$A(\omega_M) = \frac{1}{(2g\pi)^2} \frac{1}{\omega_M^2} \left[ G(\omega_M, \beta_B) - G(\omega_M, -\infty) \right]. \quad (5.14)$$

An observation is now in order. From the foregoing results we see that

$$<S^z(0)S^z(y)> - 1 = \frac{1}{(2g\pi)^2} \int_0^\infty \frac{dE}{E^2} \left[ G(E, \beta_B) - G(E, -\infty) \right] \cos Ey. \quad (5.15)$$
On the other hand, consider the expression
\[< \int_{-\infty}^{0} dx' \int_{-\infty}^{0} dx'' \left[ \partial_x \phi(x', y) \partial_x \phi(x'', 0) > \chi - \partial_x \phi(x', y) \partial_x \phi(x'', 0) > N \right]. \tag{5.16} \]

By using the same representation \((5.10)\), this is
\[\int_{-\infty}^{0} dx' \int_{-\infty}^{0} dx'' \int_{0}^{\infty} dE \left[ G(E, \beta_B) - G(E, -\infty) \right] \exp \left[ E(x' + x'') - iEy \right] + cc, \]
which coincides with \((5.13)\) after performing the integrations. We conclude that
\[< S^z(0)S^z(y) > -1 = < J_x(0)J_x(y) > - < J_x(0)J_x(y) > N, \tag{5.17} \]
where we defined
\[J_x = \frac{1}{2g\pi} \int_{-\infty}^{0} \partial_x \phi(x, y) dx. \tag{5.18} \]

We find also by the same manipulations that
\[< S^z(0)S^z(y) > -1 = < J_y(0)J_y(y) > - < J_y(0)J_y(y) > N, \tag{5.19} \]
where
\[J_y = \frac{1}{2g\pi} \int_{-\infty}^{0} \partial_y \phi(x, y) dx. \tag{5.20} \]

We now continue to real frequencies to find the response function:
\[\chi''(\omega) \equiv \frac{1}{2} \int dt e^{i\omega t} \langle [S^z(t), S^z(0)] \rangle, \tag{5.21} \]
to find:
\[\chi''(\omega) = \frac{1}{(2g\pi)^2} \frac{1}{\omega^2} \text{Re} \left[ G(-i\omega, \beta_B) - G(-i\omega, -\infty) \right]. \tag{5.22} \]

As a first example, let us consider the so called Toulouse limit or free fermion case. Then the only contribution comes from the soliton antisoliton form factors, which as discussed above is \(f(\beta_1, \beta_2) = \mu e^{\beta_1/2} e^{\beta_2/2} \). Hence,
\[\chi''(\omega) = \frac{1}{\pi^2} \text{Re} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\beta_1 d\beta_2 \frac{e^{\beta_1} e^{\beta_2}}{(e^{\beta_1} + i\beta_B)(e^{\beta_2} + i\beta_B)} \frac{1}{e^{\beta_1} + e^{\beta_2}} \delta(e^{\beta_1} + e^{\beta_2} - \omega), \tag{5.23} \]
that is
\[\chi''(\omega) = \frac{2}{\pi^2} \frac{T_B}{\omega} \text{Im} \left( \int_{0}^{\omega} dx \frac{1}{(x + iT_B)(\omega - x + iT_B)} \right), \tag{5.24} \]
32
and
\[
\chi''(\omega) = \frac{4}{\pi^2} \frac{T_B}{\omega} \text{Im} \frac{1}{\omega + 2iT_B} \ln \left( \frac{\omega + iT_B}{iT_B} \right)
\]
\[= \frac{1}{\pi^2} \frac{4T_B^2}{\omega^2 + 4T_B^2} \left[ \frac{1}{\omega} \ln \left( \frac{T_B^2 + \omega^2}{T_B^2} \right) + \frac{1}{T_B} \tan^{-1} \frac{\omega}{T_B} \right]. \tag{5.25}\]

In general, observe that, since the reflection matrix for solitons and antisolitons expands as a series in \( e^\beta \), \( \chi''(\omega) \), will, for any coupling, expand as a series of the form \( (\omega/T_B)^{2n} \) in the IR. In particular, this leads to a behaviour \( C(t) \propto \frac{1}{t^2}, t >> 1 \) for any \( g \). In the UV, one has to split integrals in two pieces as explained above in (3.41). Since the soliton-soliton form factors expansion involves powers of \( \exp(\frac{1}{g} - 1) \beta \), \( \chi''(\omega) \) expands as a double series in \( (T_B/\omega)^{2-2g} \) and \( (T_B/\omega)^2 \) in the UV. Hence at short times, \( C(t) - 1 \propto t^{2-2g} \). This is in agreement with the qualitative analysis of [27].

Results for \( g \neq 1/2 \) are more involved because there are non zero form factors at all levels. Still, when working out the first few form factors we observe a very rapid convergence with the number of rapidities and again we can give precise results for different values of \( g \). As an example, let us show the results for \( g = 1/3 \).

The computation for \( g = 1/3 \) is very similar to the previous conductance computations. The boundary matrices are much more simpler though. In this case we have :
\[
R_{\mp}^+ = \tanh(\frac{\beta}{2} - \frac{i\pi}{4}), \quad R_{\pm}^+ = 0, \tag{5.26}
\]
and :
\[
R_1^+ = \frac{\tanh(\frac{\beta}{2} - \frac{i\pi}{8})}{\tanh(\frac{\beta}{2} + \frac{i\pi}{8})}. \tag{5.27}
\]
Then, as was found for the conductance, we find that the first two contributions are sufficient for most purposes, they are given by :
\[
\delta \chi''(\omega)^{(1)} = -\frac{9\mu^2 d^2}{8\pi \omega} \text{Re} \left[ \frac{\log(\frac{\sqrt{2}T_B}{\omega} - i\frac{\pi}{8})}{\log(\frac{\sqrt{2}T_B}{\omega} + i\frac{\pi}{8})} - 1 \right]. \tag{5.28}
\]
and :
\[
\delta \chi''(\omega)^{(2)} = -\left( \frac{3\mu d}{2\pi} \right)^2 \frac{1}{\omega} \text{Re} \int_{-\infty}^{0} d\beta \frac{\zeta(\beta - \log(1 - e^\beta))}{\cosh^2(\beta - \log(1 - e^\beta))} e^\beta \left[ R_+^+(\beta + \log(\omega/T_B))R_-^+(\log((1 - e^\beta)\omega/T_B)) - 1 \right]. \tag{5.29}
\]
Fig. 6: Spectral function for $T_B = 0.1$.

Again these two expressions are sufficient to get a very precise result. Similar computations give rise to the results in figure 6 where we plotted $\chi''(\omega)/\omega$ for the values, $g = 3/5, 1/2, 1/3, 1/4$.

When making these calculations we have to be careful about which terms are needed for a good convergence. Our observation is that keeping the form factors up to two rapidities give very good results precise to 1%. It is possible to go further and get a better precision if needed. The last statements are true for $g \in [0.6, 0.2]$ and we believe even further (we can get rough bounds on the higher contributions and have an idea of the precision). Still, for the moment the isotropic Kondo point is difficult to treat.

A surprising result is that the emergence of quasi-particle peaks in $S(\omega)$ is found at $g = 1/3$ and not at $g = 1/2$ as was expected from other means of calculations. Physically this means that the behaviour of the two state system goes from coherent to incoherent behaviour at that value of $g$. This is supported by a recent RG numerical study [28].

5.2. Shiba’s Relation.

Up until now, we showed results for certain values of $g$ more or less limited by our ability (or tenacity) to write the form factors corresponding to that value of the anisotropy, and make them converge. It is not impossible to find general relations though; for example the behaviours in the UV and the IR in different models were inferred in all generality.
Here we present a generalisation of Shiba’s relation [29] which was proven for the Anderson model and generalised to Luttinger liquids by Sassetti and Weiss [30]. The relation states that:

\[
\lim_{\omega \to 0} \frac{\chi''(\omega)}{\omega} = 2\pi g\chi_0^2, \tag{5.30}
\]

with \(\chi_0\) the static susceptibility. If we look at the quantity:

\[
G(E) = E \sum_{n=0}^{\infty} \int_{-\infty}^{0} \frac{d\beta_1 \ldots d\beta_{2n}}{(2\pi)^{2n+1}(2n+1)!} \frac{1}{1 - e^{\beta_1} - \ldots - e^{\beta_{2n}}} \\
K^{a_1b_1}(\ln(T_B/E) - \beta_1) \ldots K^{a_{n-1}b_{n-1}}(\ln(T_B/E) - \beta_{n-1}) \\
K^{a_nb_n} \left[ \ln(T_B/E) - \ln \left(1 - e^{\beta_1} - \ldots - e^{\beta_{2n}} \right) \right] \left[ f \left[ \beta_1 \ldots \beta_{2n}, \ln \left(1 - e^{\beta_1} - \ldots - e^{\beta_{2n}} \right) \right] \right]^2, \tag{5.31}
\]

insert it in the expression for \(\chi''(\omega)\) and expand it around \(E \simeq 0\) we find that the contributions from the \(K\) matrices all cancel and only a constant is left (we have to take into account the fact that the soliton/anti-solitons \(K\) matrices always appear in pair). Then comparing this with the UV normalisation we find that:

\[
\lim_{\omega \to 0} \frac{\chi''(\omega)}{\omega} = \frac{1}{\pi^2 gT_B^2}. \tag{5.32}
\]

The total susceptibility is \(\chi = \chi' + i\chi''\) and the static susceptibility \(\chi_0\) which is the zero frequency limit of \(\chi'\) can also be inferred from the previous expressions for the spin-spin correlation. We just need to take the real part when continuing (5.14) to real frequencies, which leads to:

\[
\chi_0 = \frac{1}{\pi^2 gT_B}. \tag{5.33}
\]

Then in order to make contact with the previous expression we need to renormalise the spins to 1/2 and put the correct normalisation. This amounts to multiplying the each of the previous expressions by \(\pi/2\) which leads to the correct result:

\[
\lim_{\omega \to 0} \frac{\chi''(\omega)}{\omega} = 2\pi g\chi_0^2. \tag{5.34}
\]

5.3. Screening Cloud problem.

Another problem that can be addressed using the form factor techniques is the screening cloud problem for the anisotropic Kondo model. This has been a long standing problem.\(^6\)

\(^6\) We compute \(\frac{1}{2}[S^z, S^z]\) but the susceptibility has \(\frac{1}{2\pi}\) in front instead of 1/2 thus at \(h = 1\) we have to renormalise by 2\(\pi\).
and many theoretical studies were devoted to it\cite{31}. Our one dimensional formulation here follows from\cite{20}.

Our aim here is to compute the uniform part of the susceptibility, defined in the introduction. We restrict to the case $g = 1$ in what follows. The electron spin density is given by:

$$\frac{1}{2} \int_{-\infty}^{0} \partial_x \phi. \quad (5.35)$$

We can proceed to the evaluation of $\chi_{un}$, which is given by:

$$\Delta \chi_{un}(r) = \frac{1}{8\pi} \int dt \ <\partial_r \phi(r) S_{tot}(t)>. \quad (5.36)$$

In the following, we will always subtract the free part ($T_B = 0, \beta_B = -\infty$) and denote the corresponding quantity by the symbol $\Delta$. To find correct results for the zero temperature susceptibilities, we always need to take the length of the system to infinity first before taking $T \to 0$. In this case, this means that we always do the time integrals last.

Let us compute separately the static electron-electron and electron-impurity, $\Delta \chi_{ee}$, $\Delta \chi_{ei}$ susceptibilities. The electron-electron contribution is the appropriate integral of the correlator $<\partial_x \phi \partial_x \phi>-<\partial_x \phi \partial_x \phi>_N$:

$$\Delta \chi_{ee}(r, T = 0) = \frac{1}{8\pi} \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \int_{0}^{E} dE \left[ G(E, T_B) - G(E, 0) \right] e^{E(x+r)-iy} + c.c.,$$

$$= \frac{1}{8\pi} \lim_{E \to 0} \left[ G(E, T_B) - G(E, 0) \right] e^{E r},$$

$$= 0, \quad r \neq 0. \quad (5.37)$$

with $y$ the imaginary time. For $r = 0$, the integral over $E$ is divergent and this computation does not make sense. In fact, $\Delta \chi_{ee}(r, T = 0)$ has a delta like contribution at $r = 0$ as we now show. Let us integrate instead $\chi_{ee}(r, T = 0)$ over $r$ first, and then over $y$ since we are at $T = 0$. The result is the static electron-electron contribution to the susceptibility:

$$\Delta \chi_{ee}(T = 0) = \frac{1}{16\pi^2} \lim_{E \to 0} \left[ G(E, T_B) - G(E, 0) \right] \frac{1}{E^2}, \quad (5.38)$$

where now the integration over $y$ made sense since the $E$ integral was convergent. Hence we conclude

$$\Delta \chi_{ee}(r, T = 0) = \frac{1}{4\pi^2 T_B} \delta(r). \quad (5.39)$$
Let us now come to the electron-impurity contribution and show that it is equal and opposite to the previous contribution. It needs a little trick to be converted to a current correlation: Look at the quantity:

$$\frac{\partial_r \phi(r, y)}{2} [S_{imp}^z(y') - S_{imp}^z(-\infty)].$$

(5.40)

For finite $y$, the term $< \partial_r \phi(r, y) S_{imp}^z(-\infty) >$ will be zero because of the infinite time separation between the operators, and the average of (5.40) will reproduce what we want. On the other hand, the difference of spins can be computed following the transformations in section 5.1. We then find that:

$$\frac{1}{2} < \partial_r \phi(r, y) S_{imp}^z(y') > = \frac{i}{8\pi g} \lim_{X \to 0} \int_{-\infty}^{y'} d\xi <: \partial_r \phi(r, y) \partial_X \phi(X, \xi) :>.$$  

(5.41)

In order for the integral over $\xi$ to converge, we need to take now the modular transformed representation of the current correlation. We finally obtain the expression:

$$\frac{i}{8\pi g} \int_0^{\infty} \frac{dE}{E} [\mathcal{F}(E, T_B) - \mathcal{F}(E, 0)] e^{-iEr + Ey'} + c.c.$$  

(5.42)

Then using analytical continuation, we find the contribution to be the same as the previous one up to a sign $\Delta\chi_{ei}(r) = -\Delta\chi_{ee}(r)$. This is consistent with Lowenstein [32]. This is also consistent with results recently found in perturbation theory at finite $T$ showing that these susceptibilities are zero everywhere outside $r = 0$. This is inconsistent with the results of [33] in that in their case, the extrapolation of the perturbative results towards zero temperature show that both $\Delta\chi_{ie}$ and $\Delta\chi_{ee}$ go to zero (not the sum). This last behaviour is also expected from a physical argument. We don’t have a clear understanding of the discrepancy yet, since making contact between the two calculations is not easy. Our guess is that there is a subtlety when the $T \to 0$ limit is taken at the same time as the coupling to the impurity and $g \to 1$, and that working directly at zero temperature is not the correct way to proceed. We hope to be able to verify this explicitly by making a finite temperature calculation in the same formalism soon and check whether the discrepancy comes from the limit $g \to 1$ or $T \to 0$. In that sense, the last results, should be taken with caution.

All these results are related to the so called uniform part of the susceptibility. The $2k_f$ part, defined in section 2, involves a different set of operators once bosonised: $e^{i\phi/2}$. These operators have different anomalous dimension in the IR and UV and their treatment using massless form factors encounters serious complications which are discussed in the next section.
6. Form-factors failures

For operators with naive engineering dimension such as the current, the massless scattering approach is thus seen to work quite nicely. Another candidate for which things work is the stress energy tensor, that is basically $(\partial_+ \phi(x))^2$, describing the density of energy at some distance from the boundary (the impurity). Unfortunately, things are very different for operators with a dimension that is not constrained by any symmetry. An example is $\cos \frac{1}{2} \phi(x)$. This operator is of crucial physical interest: for instance knowing its correlators would lead to an exact determination of Friedel’s oscillations [34]. Unfortunately key difficulties appear here with the form-factors approach. Form-factors of the operator $\cos \frac{1}{2} \phi$ in the bulk massive sine-Gordon model have been determined [15]. A first feature is that they depend only on rapidity differences, and do not exhibit any overall factor depending on the rapidity scale and fixing the (naive) dimension, as for the current. We can then take massless limit as described above. What one finds is that, at least for $g$ = integer, the only form-factors whose limit is not cancelled exponentially by powers of $e^{-\beta_0}$ are those which are left and right neutral - for instance form-factors between the ground state and any state made only of breathers. A first problem then arises when one tries to reproduce properties of the operator $\cos \frac{1}{2} \phi$ in the bulk massless theory. For instance one finds that the one breather form-factor [35] is a pure, rapidity independent, number,

$$<0| \cos \frac{1}{2} \phi |\theta> = c$$

In the massless limit, its contribution to the two point function will therefore be of the form

$$|c|^2 \int_{-\infty}^\infty e^{M z e^g} d\beta$$

an integral that is IR divergent - recall that for the current it was the term giving the naive engineering dimension that made the integrals converge in the IR. One might think that this problem has a simple solution: put by hand the anomalous dimension, ie multiply in the massless limit each of the form-factors by a factor $m^{g/2}$, where $m$ is the bulk mass. The above contribution reads then

$$m^g |c|^2 \int_{-\infty}^\infty \exp m [\cosh \beta x + i \sinh \beta y] d\beta$$

Unfortunately, in the limit $m \to 0$, it does not behave any better, and does not reproduce the expected $|x|^{-g}$ behaviour. Recall how, for the current, the dimension followed trivially
from the scaling properties of each individual form-factor, and was reproduced by every term. Here, the 1-breather example suggests that the correct behaviour can be reproduced only when the whole series is summed up at finite $m$ and then only $m$ is sent to zero. There is thus no hope to reproduce even approximately the bulk-behaviour by massless form-factors. A related situation has been encountered in massless flows in [13].

We have tried to study the effect of the boundary by putting in by hand the correct massless bulk behaviour and concentrating on the corrections induced by the boundary - for instance by studying ratios of correlators with different boundary couplings. Similar problems unfortunately occur. A simple way of seeing this is to consider the expected form of the correlations of this operator in the UV and IR limit. In the IR the field $\phi$ obeys Dirichlet boundary conditions, and one has

$$<\phi(z, \bar{z})\phi(z', \bar{z}')> = -4g \ln \left| \frac{z - z'}{\bar{z} + \bar{z}'} \right|^g,$$

so, restricting to arguments on the $x$ axis,

$$<\cos\frac{\phi}{2}(x)\cos\frac{\phi}{2}(x')> = \left| \frac{x + x'}{x - x'} \right|^g.$$

In the UV on the other hand it obeys Neumann boundary conditions and thus

$$<\cos\frac{\phi}{2}(x)\cos\frac{\phi}{2}(x')> = \frac{1}{|x - x'|^{2g}}.$$

We see thus that the anomalous dimension of the field $\cos\frac{\phi}{2}$ is changing, being equal to $g/2$ in the IR and to $g$ in the UV. This is of course very different from the case of derivatives of $\phi$. Thus, at the present time, the form-factors approach fails for that operator.

7. Conclusions.

In this paper we presented a method to obtain current-current correlations in massless theories with interaction at the boundary. The technique, as presented, is quite general and has been applied to different problems successfully. Several generalizations along the same lines appear possible. The most interesting would be to study similar quantities but in the presence of a bias (or voltage) and temperature. In that case the ground state is no longer empty but filled with quasi-particles in a way determined by the thermodynamic Bethe ansatz. A combination of the TBA and the technique presented here can hopefully allow the determination of the current-current correlation in that case.
Other generalizations appear more challenging. As explained in the previous section, the approach, when applied to operators with a non-trivial dimension, naively fails, preventing us eg to study Friedel oscillations for the moment. The approach also fails when one gets too close to the isotropic point $g = 1$. Although this should not be too catastrophic in practice because results can be extrapolated from the $g < 1$ regime, this remains a big challenge.

Acknowledgements.

We thank I. Affleck and G. Mussardo for many interesting discussions. This work was supported by the Packard Foundation, the National Young Investigator Program and the DOE. F. Lesage was also partly supported by a Canadian NSERC postdoctoral Fellowship.
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