Forecasting the distribution of long-horizon returns with time-varying volatility

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1 Background
The study of long-horizon returns has received a great deal of attention in recent years (see, for example, Boudoukh, Richardson, and Whitelaw (2008), Neuberger (2012) and Lee (2013), Fama and French (2018)). While most of the discussions are concerned with some practical issues in investment, few have touched the important aspect on risk management. The approach adopted in this article is to predict the future distribution of the returns of a fixed long-horizon by which the risk measures of interest that come in the form of a distributional functional such as the value at risk (VaR) and the conditional tail expectation (CTE) can be easily derived. The characteristic feature of our approach which requires no specification of the volatility dynamics nor parametric assumptions of the shock distribution extends the work by Ho et al. (2016) and Ho (2017) to a more general volatility dynamics that includes both the widely-used SV model and the GARCH model (Bollerslev, 1986) as special cases.

2 Model
We consider a general time-varying volatility (GV) model for the return $r_t$ which is in the form of

$$r_t = \mu + h_t u_t, \quad h_t = f(\eta_{t-1}, \eta_{t-2}, \ldots),$$

and satisfies the following two conditions.

(i) $\{u_t\}$ is the sequence of iid symmetric shocks with zero mean and unit variance.

(ii) $\{\eta_i, -\infty < i < +\infty\}$ is a strictly stationary sequence independent of $\{u_t\}$, and $f$ is a positive measurable function such that $\sigma^2 = Eh_t^2 = E(r_t - \mu)^2 < \infty$.

The GV model defined in (1) generalizes a host of popular models proposed in the literature for financial econometrics, in particular, the stochastic volatility model (SV) and the ARCH-type model.
(i) If \( \{\eta_t\} \) is an iid sequence independent of the normal shocks \( \{u_t\} \), and \( f \) is
\[
f(x_1, x_2, \ldots) = \delta_0 \exp \left\{ \sum_{i=0}^{\infty} \phi^i x_i / 2 \right\}, \quad 0 < \phi < 1,
\]
then (1) represents the standard SV model (Taylor, 1994).

(ii) Suppose the volatility component \( h_t \) follows the square GARCH(p,q) dynamic equation (Bollerslev, 1986)
\[
h_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i h_{t-i}^2 + \sum_{j=1}^{q} \beta_j (r_{t-j} - \mu)^2.
\]
By using the shift-operator method, it is not difficult to see that \( h_t^2 \) can be expressed as an infinite moving average of \( (r_t - \mu)^2 \)’s, i.e.,
\[
h_t^2 = \sum_{i=1}^{\infty} g_i (r_{t-i} - \mu)^2
\]
for some summable sequence \( \{g_i\} \). Equation (1) hence covers the case of GARCH (p,q) sequence with \( f(x_1, x_2, \ldots) = (\sum_{i=0}^{\infty} g_i x_i)^{1/2} \) and \( \eta_{t-1} = (r_{t-1} - \mu)^2 \).

3 Forecasting distribution

Let \( \{r_t\} \) be the underlying return sequence modeled by (1). For a fixed \( T \), let \( R_{t,T} = \sum_{s=t+1}^{t+T} r_s \) denote the integrated returns of horizon \( T \) from \( t+1 \) to \( t+T \), and let \( F^R_{t,T}(\cdot|\mathcal{F}) \) be the distribution of \( R_{N,T} \) conditional on the information set \( \mathcal{F} \) generated by a set of past returns. We replace \( F^R_{t,T}(\cdot|\mathcal{F}) \) by \( F^R_{t,T}(\cdot) \) when \( \mathcal{F} \) is generated by \( \{r_s, 1 \leq s \leq N\} \).

Model (1) entails an interesting observation. Let \( \Psi_t \) denote the information set generated by \( \{|r_i - \mu|, 1 \leq i \leq t\} \), and define the sign function \( \pi_i \) of the shock \( u_i \), i.e., \( \pi_i = u_i / |u_i| \). Let \( (a_1, \ldots, a_t) \in \{-1, 1\}^t \) with \( a_i = -1 \) or \( 1 \) be an outcome of the random vector \( (\pi_1, \ldots, \pi_t) \). Set \( I_i = (0, +\infty) \) if \( a_i = 1 \), or \((-\infty, 0) \) if \( a_i = -1 \). Then
\[
P ( (\pi_1, \ldots, \pi_t) = (a_1, \ldots, a_t) | \Psi_t )
= P ( (\pi_1, \ldots, \pi_t) = (a_1, \ldots, a_t) | |u_1|, \ldots, |u_t| )
= P (u_1 \in I_1, \ldots, u_t \in I_t | |u_1|, \ldots, |u_t| )
= P (u_i \in (0, +\infty), 1 \leq i \leq t | |u_1|, \ldots, |u_t| )
= \prod_{i=1}^{t} P (u_i \in (0, +\infty))
= 2^{-t} = \prod_{i=1}^{t} P (\pi_i = a_i),
\]
where the third equality holds due to the assumption of symmetric \( u_t \). We formalize (2) in the following statement.

**Property I.** Conditional on \( \Psi_t \), \( \{\pi_i, 1 \leq i \leq t\} \) forms an iid Rademacher sequence with \( P(\pi_i = 1) = P(\pi_i = -1) = 1/2 \).

Suppose for the time being \( \mu \) of \( r_t \) in (1) is known. Let \( r_t - \mu = m_t \delta_t \), where \( m_t = |r_t - \mu| \) and \( \delta_t = 1 \) or \( -1 \) is the sign of \( r_t - \mu \). Given \( \{m_t, 1 \leq t \leq N\} \), denote by \( \hat{m}_{n,N,h} \) with \( h \geq 1 \) the best linear predictor of \( m_{N+h} \) based on the past \( n \) returns of \( \{m_t, N-n+1 \leq t \leq N\} \), and let \( \Delta_h(N,n) = m_{N+h} - \hat{m}_{n,N,h} \) be the prediction error. Write

\[
R_{N,T} - T\mu = \sum_{h=1}^{T} (r_{N+h} - \mu) = \sum_{h=1}^{T} m_{N+h} \delta_{N+h} = \sum_{h=1}^{T} (\hat{m}_{n,N,h} + \Delta_h(N,n)) \delta_{N+h}.
\]

From **Property I** we know that \( \{\delta_t\} \) is an iid Rademacher sequence. What left to be specified in order to approximate \( m_{N+h} \delta_{N+h} \) is the \( \Delta_h(N,n) \) on the righthand side of the last equation in (3). Because \( \Delta_h(N,n) \) is the prediction errors of future observations (with \( N \) being the current time) that are perpendicular to \( \{m_t, 1 \leq t \leq N\} \), it would not be feasible to predict their values. Yet the distribution of \( \Delta_h(N,n) \) for different \( h \)'s can be retrieved empirically from the past prediction errors. We therefore propose using \( \hat{m}_{n,N,h} + W_{n,h} \) as the proxy for the forecast of \( m_{N+h} \) where \( W_{n,h} \) is an independent copy of the \( h \)-step prediction error \( \Delta_h(t,n) = m_{t+h} - \hat{m}_{n,t,h} \) for fixed \( n \) and \( h \). We choose \( W_{n,h} \) not only because it is identically distributed with \( \Delta_h(N,n) \) but also that the two random vectors \( (\hat{m}_{n,N,1} + W_{n,1}, \ldots, \hat{m}_{n,N,T} + W_{n,T}) \) and \( (\hat{m}_{n,N,1} + \Delta_1(N,n), \ldots, \hat{m}_{n,N,T} + \Delta_T(N,n)) \) have the same covariance matrix, since

\[
\text{cov}(\hat{m}_{n,N,h} + W_{n,h}, \hat{m}_{n,N,k} + W_{n,k}) = \text{cov}(\hat{m}_{n,N,h} + \Delta_h(N,n), \hat{m}_{n,N,k} + \Delta_k(N,n)) = \text{cov}(\hat{m}_{n,N,h}, \hat{m}_{n,N,k})
\]

for \( h,k = 1, \ldots, T \). To construct the best \( h \)-step linear predictor \( \hat{m}_{n,N,h} \) for \( m_{N+h} \) based on \( \{r_t, N-n+1 \leq t \leq N\} \), we adopt the widely used innovation algorithm which only requires the covariance function of \( \{m_t\} \). More specifically, let \( m^*_t = m_t - \mu_m \) where \( Em_t = \mu_m \), then the one-step predictor \( \hat{m}_{n,t,1}^* \) of \( m_{t+1}^* \) based on \( \{m^*_s, t-n+1 \leq s \leq t\} \) can be expressed as

\[
\hat{m}_{n,t,1}^* = \sum_{j=1}^{n} \theta_{n,j} (m_{t+1,j}^* - \hat{m}_{n,j,t,j,1}^*),
\]

(4)
where the coefficients $\theta_{n,j}$ are determined by a set of recursive equations built on the covariances of $\{m_t^*\}$ (Proposition 5.2.2, Brockwell and Davis, 1991). In (4) we set $\hat{m}^*_{t,y,z}$ to be zero if $x = 0$. For $h$-step prediction, it follows from equation (4) that

$$\hat{m}^*_{n,t,h} = \sum_{j=h}^{n-h+1} \theta_{n+h-1,j} \left( m_{i+1-j}^* - \hat{m}^*_{n-j,t-j,1} \right),$$  \hspace{1cm} (5)

using the fact that the prediction error $m^*_{t+h-j} - \hat{m}^*_{n-j,t+h-j,1}$ for $h - j \geq 1$ is orthogonal to the linear span of $\{m^*_s, t - n + 1 \leq s \leq t\}$ (Section 5.2, Brockwell and Davis, 1991). Note that in (4) and (5) the underlying time series $\{m_t^*\}$ for which the prediction is carried out is of mean zero. Thus, it is necessary to add the mean back to $\hat{m}^*_{n,t,h}$ to obtain the predictor $\hat{m}_{n,t,h} = \hat{m}^*_{n,t,h} + \mu_m$ for the observation $m_{t+h}$. When the mean $\mu_m$ is unknown, we modify the predictor to be $\hat{m}_{n,t,h} = \max\{\hat{m}^*_{n,t,h} + \hat{\mu}_m, 0\}$ to conform to the non-negativity of $m_t$.

For $h = 1, \ldots, T$, let $E_h$ be the set of all the observed prediction errors $\Delta_h(t, n)$ created by the past observations, that is,

$$E_h(N, n) = \{ \Delta_h(t, n) = m_{t+h} - \hat{m}_{n,t,h}, t = n, n+1, \ldots, N \}. \hspace{1cm} (6)$$

Define

$$\hat{R}_{n,N,T} = T\hat{\mu} + \sum_{h=1}^{T} \left( \hat{m}_{n,N,h} + \hat{W}_{n,h} \right) \delta^*_h \hspace{1cm} (7)$$

where $\hat{W}_{n,h}$ is the random variable designed to approximate $W_{n,h}$ and having the same distribution as the empirical distribution constructed by the elements of $E_h(N, n)$, and $\{\delta^*_h, h = 1, \ldots, T\}$ is an iid Rademacher sequence independent from $\{m_t\}$. For $h = 1, \ldots, T$ we draw independent samples $\{\hat{W}^{(i)}_{n,h}, h = 1, \ldots, T\}$ and $\{\delta^{(i)}_h, h = 1, \ldots, T\}$ with the former from $E_h(N, n)$ to form

$$\hat{R}^{(i)}_{n,N,T} = T\hat{\mu} + \sum_{h=1}^{T} \left( \hat{m}_{n,N,h} + \hat{W}^{(i)}_{n,h} \right) \delta^{(i)}_h \hspace{1cm} (8)$$

Then the empirical distribution $\hat{F}^R_{R,T}(\cdot)$ built on $\{\hat{R}^{(i)}_{N,T}, i = 1, \ldots, B\}$ for a large $B$ is the forecast we propose for the distribution $F_{R,T}(\cdot)$ of $R_{N,T}$ conditional on $\{r_t, 1 \leq t \leq N\}$.

### 4 Presence of stochastic trends

The GV model previously described gives a white noise sequence which is a common stylized facts exhibited by the returns of market indexes (Asset Price Dynamics, Volatility, and Prediction by SJ Taylor, 2005). For individual stocks, however, the return sequence tends to show serial correlations. In this section
we extend the forecasting method discussed in Section 3 to the case where the returns are serially correlated. For ease of presentation, we from now on use $x_t$ to denote the return and $S_{N,T} = \sum_{t=N+1}^{T} x_t$ for its integrated returns. Let $\hat{F}_{N,T}^x(\cdot)$ be the distribution of $S_{N,T}$ conditional on $\{x_t, 1 \leq t \leq N\}$. Assume that $x_t$ is invertible with respect to $\{r_t\}$, that is, for some smooth function $f(x) = \sum_{i=0}^{\infty} a_i x^i$ with $a_0 = 1$,

$$r_t = f(B)(x_t - \mu) = \sum_{i=0}^{\infty} a_i (x_{t-i} - \mu), \quad (9)$$

where $B$ is the shift operator and $\{r_t\}$ is a zero-mean innovation sequence following the GV model. Model (9) includes the familiar ARMA process with ARCH-type or SV innovations. Given a sample $\{x_t, 1 \leq t \leq N\}$ of size $N$, similar to the notation used in Section 3 set $x_t^* = x_t - \mu$ and let $\hat{x}_{n,N,h}^*$ denote the best linear $h$-step predictor of $x_{N+h}^*$ base on $\{x_t^*, N-n+1 \leq t \leq N\}$. Using the innovation algorithm, we express each future $x_{N+h}^*$ as a weighted sum of prediction errors,

$$x_{N+h}^* = \left(x_{N+h}^* - \hat{x}_{n,N+h-1,1}^*\right) + \hat{x}_{n,N+h-1,1}^* = \sum_{j=0}^{n} \theta_{N+h-1,j} \left(x_{N+h-j}^* - \hat{x}_{n,N+h-1-j,1}^*\right)$$

$$= \sum_{j=0}^{h-1} \theta_{N+h-1,j} \left(x_{N+h-j}^* - \hat{x}_{n,N+h-1-j,1}^*\right) + \sum_{j=0}^{h-1} \theta_{N+h-1,j} \left(x_{N+h-j}^* - \hat{x}_{n,N+h-1-j,1}^*\right), \quad (10)$$

where $\theta_{0,0} = 1$. Then we write the integrated return $S_{N,T}$ of $x_t^*$ as

$$S_{N,T} - T\mu = \sum_{h=1}^{T} x_{N+h}^*$$

$$= \sum_{h=1}^{T} \sum_{j=0}^{n} \theta_{N+h-1,j} \left(x_{N+h-j}^* - \hat{x}_{n,N+h-1-j,1}^*\right)$$

$$= \sum_{h=1}^{T} \sum_{j=0}^{h-1} \theta_{N+h-1,j} \left(x_{N+h-j}^* - \hat{x}_{n,N+h-1-j,1}^*\right) = \mathcal{T}_{n,N,T} + \mathcal{I}_{n,N,T} \quad (11)$$

By grouping the coefficients $\theta_{i,j}$ that associated with the same estimated innovation term $x_{N+h-j}^* - \hat{x}_{n,N+h-1,j,1}^*$, we can write $\mathcal{T}_{n,N,T}$ and $\mathcal{I}_{n,N,T}$ as

$$\mathcal{T}_{n,N,T} = \sum_{j=1}^{n} \left(\sum_{h=1}^{T} \theta_{N+h-1,j}\right) \left(x_{N+1-j}^* - \hat{x}_{n,N-j,1}^*\right),$$
\[ \mathcal{I}_{n,N,T} = \sum_{j=N+1}^{N+T} \left( \sum_{h=0}^{n+T-j} \theta_{j+h-1,h} \right) (x_j^* - \hat{x}_{n,j-1,1}^*). \]

In the decomposition (11) of \( S_{N,T} \), \( \mathcal{T}_{n,N,T} \) represents the linear forecast of \( S_{N,T} \) derived from the given sample \( \{x_i^*, N-n+1 \leq t \leq N\} \), and \( \mathcal{I}_{n,N,T} \) consists of the one-step prediction errors not directly observed. Thus an acceptable candidate for forecasting the distribution of \( S_{N,T} \) would be in the form of \( \mathcal{T}_{n,N,T} + \hat{F}_{N,T}^I(x) \) where \( \hat{F}_{N,T}^I(x) \) is a good estimate of the conditional distribution \( F_{N,T}^I(x) \) of \( \mathcal{I}_{n,N,T} \) given \( \{x_i^*, 1 \leq t \leq N\} \). Before we proceed to find \( \hat{F}_{N,T}^I(x) \), we first note that due to assumption (10) the one-step prediction error \( x_i^* - \hat{x}_{n,t-1,1}^* \) is close to \( r_t \) for each \( t \) if \( n \) is sufficiently large. Therefore we may regard \( \mathcal{I}_{n,h} \) as a weighted sum of uncorrelated \( r_t \)’s. Define \( z_j(N,n) = x_j^* - \hat{x}_{n,j-1,1}^* \), \( \tilde{m}_j(N,n) = |z_j(N,n)| \), and \( \delta_j(N,n) = z_j(N,n)/\tilde{m}_j(N,n) \). For \( j = N+1,\ldots,N + T \), let \( \tilde{m}_j(N,n) \) denote the best linear forecast of \( m_j(N,n) \) based on the previous prediction errors \( \{\tilde{m}_i(N,n) = |x_i^* - \hat{x}_{n,t-1,1}^*|, n+1 \leq t \leq N\} \).

Similar to finding an estimate for the conditional distribution \( F_{N,T}^R(\cdot) \) as presented in Section (3) (cf. (6), (7) and (8)), we introduce the random variable \( \hat{W}_{n,h} \) that is identically distributed with \( \tilde{\Delta}_h(t,n) = \hat{m}_{t+h} - \hat{m}_{n,t,h} \), and the set \( \mathcal{E}_h^I(n,N) = \{\tilde{\Delta}_h(t,n), t = n,\ldots,N\} \) consisting of \( h \)-step prediction errors for \( h = 1,\ldots,T \). By the similar technique used in (7), we approximate \( \mathcal{I}_{n,N,T} \) by
\[
\hat{\mathcal{I}}_{n,N,T} = \sum_{j=N+1}^{N+T} \left( \sum_{h=0}^{n+T-j} \theta_{j+h-1,h} \right) (\hat{m}_{n,N,j-N} + \hat{W}_{n,j-N}) \delta_{j-N}^i
\]
where \( \hat{W}_{n,h} \) is the random variable having the same distribution as the empirical distribution constructed by the elements of \( \mathcal{E}_h^I \); and \( \{\delta_{h}^i, h = 1,\ldots,T\} \) is an independent Rademacher sequence. Let \( \{\hat{W}_{n,h}, h = 1,\ldots,T\} \) and \( \{\delta_{h}^i, h = 1,\ldots,T\} \) be an independent copy of \( \{W_{n,h}, h = 1,\ldots,T\} \) and \( \{\delta_{h}, h = 1,\ldots,T\} \), respectively. We propose using the empirical distribution formed by the independent samples
\[
\hat{I}_{N,T}^{(i)} = \sum_{j=N+1}^{N+T} \left( \sum_{h=0}^{n+T-j} \theta_{j+h-1,h} \right) \left( \hat{m}_{n,N,j-N} + \hat{W}_{n,j-N} \right) \delta_{j-N}^i, \quad i = 1,\ldots,B,
\]
as the estimate \( \hat{F}_{N,T}^I(x) \) of the conditional distribution \( F_{N,T}^I(x) \) of \( \mathcal{I}_{n,N,T} \). Combining (11) and the preceding derivation of \( \hat{F}_{N,T}^I(x) \) yields the desired forecast \( T\mu + \mathcal{T}_{n,N,T} + \hat{F}_{N,T}^I(x) \) of the conditional distribution of \( S_{N,T} \). Replace \( \mu \) by the sample mean \( \hat{\mu} \) if \( \mu \) is unknown.
5 Non-symmetric shocks

While the symmetry assumption on the shocks \( \{u_t\} \) in model (1) is quite common in studies concerning the conditional heteroscedastic model (Christian Francq and Jean-Michel Zakoian 2010), many works also point out that using the non-symmetric shocks such as the skewed normal or skewed-t can bring some performance improvements (see, e.g., Dongming Zhu and John W. Galbriath, 2010, 2011, and references there in). In this section we discuss how to extend our prediction procedure to the case where \( \{u_t\} \) is not symmetric. We first focus on the GV model with \( r_t = \mu + h_t u_t \) where the iid zero-mean-unit-variance shock sequence \( \{u_t\} \) need not be symmetric. Define \( m_t = |r_t^*| \). For a small \( \lambda > 0 \), we discretize \( r_t^* \) as \( r_{t,\lambda}^* = m_t \delta_t \) where

\[
m_{t,\lambda} = j \lambda \quad \text{if} \quad m_t \in I_{j,\lambda} = [j \lambda, (j + 1) \lambda), \quad j = 0, 1, \ldots.
\]

Let \( X_\lambda \) be a random variable having the same distribution as \( r_{t,\lambda}^* \). Conditional on \( \{m_{t,\lambda}, t = 1, \ldots, T\} \), define

\[
\delta_{t,\lambda}^*(m_{t,\lambda} = j \lambda) = \begin{cases} 
1 & \text{with probability} \quad P(X_\lambda \in I_{j,\lambda})/P(|X_\lambda| \in I_{j,\lambda}) \\
-1 & \text{with probability} \quad P(X_\lambda \in -I_{j,\lambda})/P(|X_\lambda| \in I_{j,\lambda})
\end{cases}
\]

Note that if \( u_t \) is symmetric, then the conditional probability that \( \delta_{t,\lambda}^*(m_t) = 1 \) or \(-1\) is always 1/2. Define

\[
r_{t,\lambda}^* = m_{t,\lambda} \delta_{t,\lambda}^*(m_{t,\lambda}). \tag{12}
\]

Then it is not difficult to see that \( r_{t,\lambda}^* \) and \( r_{t,\lambda}^* \) have the same distribution. Since as \( \lambda \to 0 \), \( r_{t,\lambda}^* \) converges to \( r_t^* \) for each \( t \), so dose their distribution of \( r_{t,\lambda}^* \) to that of \( r_t \).

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