Construction of supercharacter theories of finite groups

Anders O.F. Hendrickson

Department of Mathematics and Computer Science, Concordia College, Moorhead, MN 56562, USA

Abstract

Much can be learned about a finite group from its character table, but sometimes that table can be difficult to compute. Supercharacter theories are generalizations of character theory, defined by P. Diaconis and I.M. Isaacs in [8], in which certain (possibly reducible) characters called supercharacters take the place of the irreducible characters, and a certain coarser partition of the group takes the place of the conjugacy classes.

In particular, if \( K \) is a partition of a finite group \( G \), there may exist a compatible partition \( X' \) of the irreducible characters of \( G \), along with a character \( \chi_X \) for every \( X \in X' \) with the elements of \( X \) as its irreducible constituents, so that each \( \chi_X \) is constant on each \( K \in K \) and \( |X'| = |K| \). If every irreducible character is a constituent of some \( \chi_X \), then the ordered pair \((X', K)\) is called a supercharacter theory.

We present five new ways to construct new supercharacter theories out of supercharacter theories already known to exist, including a direct product, a lattice-theoretic join, two products over normal subgroups, and a duality for supercharacter theories of abelian groups.

Key words: Finite groups, Characters, Supercharacter theories

1. Introduction

Let \( G \) be a finite group with \( n \) conjugacy classes, and let \( \text{Irr}(G) \) be the set of complex irreducible characters of \( G \). Now the \( n \) conjugacy classes partition the group \( G \), and each of the \( n \) irreducible characters is constant on every conjugacy class of \( G \). The relationship between the irreducible characters and the conjugacy classes is of great use in studying a finite group, but for some groups \( \text{Irr}(G) \) is quite difficult to compute.

For example, let \( U_n(F_q) \) be the group of upper triangular matrices over the field of size \( q \) with all diagonal entries one. The irreducible characters of \( U_n(F_q) \) are quite difficult to compute, but Carlos André [1, 2, 3] and Ning Yan [12] developed a theory of “basic characters” or “transition characters” as an approximation to the full character table of \( U_n(F_q) \). In this theory certain reducible characters take the place of irreducible characters and the role of conjugacy classes is played by certain unions of conjugacy classes. Although these theories are simple enough to be computed explicitly, they are
rich enough to handle some problems that traditionally required knowing the full character theory [7]. Persi Diaconis and I. Martin Isaacs [8] have generalized the work of André and Yan to the concept of a supercharacter theory of a finite group, which is defined as follows. All groups mentioned will be finite.

**Definition 1.1.** Let $G$ be a finite group, let $\mathcal{K}$ be a partition of $G$, and let $\mathcal{X}$ be a partition of the set $\text{Irr}(G)$. Suppose that for every part $X \in \mathcal{X}$ there exists a character $\chi_X$ whose irreducible constituents lie in $X$, and suppose the following three conditions hold.

1. Each of the characters $\chi_X$ is constant on every part $K \in \mathcal{K}$.
2. $|\mathcal{X}| = |\mathcal{K}|$.
3. Every irreducible character is a constituent of some $\chi_X$.

Then we call the characters $\chi_X$ supercharacters, we call the members of $\mathcal{K}$ superclasses, and we say that the ordered pair $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory. If $C = (\mathcal{X}, \mathcal{K})$ is a supercharacter theory, then we define $|C|$ to be the integer equal to both $|\mathcal{X}|$ and $|\mathcal{K}|$. We write $\text{Sup}(G)$ for the set of all supercharacter theories of $G$.

Assume that $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of a group $G$, and for every subset $X$ of $\text{Irr}(G)$ let $\sigma_X$ be the character $\sum_{\psi \in X} \psi(1)\psi$. Diaconis and Isaacs prove in [8, Lemma 2.1] that $\{1\} \in \mathcal{K}$, that $\{1_G\} \in \mathcal{X}$, and that for each $X \in \mathcal{X}$, the supercharacter $\chi_X$ must be a constant multiple of $\sigma_X$. It is therefore no loss to assume that $\chi_X = \sigma_X$, and we shall make that assumption throughout this paper. It is also shown in [8, Theorem 2.2(c)] that if $C = (\mathcal{X}, \mathcal{K})$ is a supercharacter theory, then each of the partitions $\mathcal{X}$ and $\mathcal{K}$ uniquely determines the other.

This paper investigates several ways to discover new supercharacter theories from supercharacter theories already known to exist, as an aid to future work classifying $\text{Sup}(G)$ for given groups $G$. Let us begin by reviewing the five constructions of supercharacter theories given in [8].

**Definition 1.2.** Let $G$ be a group. Then the minimal supercharacter theory $m(G)$ is given by the partitions of $\text{Irr}(G)$ into singleton sets and of $G$ into its conjugacy classes. The maximal supercharacter theory $M(G)$, on the other hand, is given by the partition $\{\{1_G\}, \text{Irr}(G) - \{1_G\}\}$ of $\text{Irr}(G)$ and the partition $\{\{1\}, G - \{1\}\}$ of $G$.

Because the superclasses of the minimal theory are the conjugacy classes and its supercharacters are scalar multiples of the irreducible characters of $G$, the minimal theory is just the ordinary character theory of $G$. The supercharacters of the maximal theory, by contrast, are the principal character $1_G$ and $\rho_G - 1_G$, where $\rho_G$ is the regular character of $G$. It is routine to verify that these two theories satisfy the conditions of Definition 1.1.

Next, let $A$ be a group acting on $G$ by automorphisms. Then $A$ permutes both $\text{Irr}(G)$ and the set of conjugacy classes of $G$. The partition of the conjugacy classes into $A$-orbits yields a partition $\mathcal{K}$ of the group $G$, and taking $\mathcal{X}$ to be the partition of $\text{Irr}(G)$ into $A$-orbits, a straightforward calculation shows that for each $A$-orbit $X \in \mathcal{X}$, the character $\sigma_X$ is constant on each member of $\mathcal{K}$. A lemma of Richard Brauer guarantees that $|\mathcal{X}| = |\mathcal{K}|$, and therefore $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of $G$. 

2
For the fourth construction given in [8], let $G$ be a group and let $H$ be a group of field automorphisms of $C$; then a supercharacter theory can be obtained by taking $X$ to be the orbit decomposition of $\text{Irr}(G)$ under the action of $H$. Finally, the bulk of [8], like the subsequent papers by Diaconis, Otto, Thiem, Venkateswaran, and Marberg [9, 12, 13, 14], generalizes the theory of André and Yan to a particularly well-behaved supercharacter theory of a certain family of groups called algebra groups. In other papers, André and Neto have recently begun to describe a supercharacter theory for Sylow subgroups of symplectic and orthogonal groups [4, 5, 6].

In this article we present several new ways to obtain supercharacter theories of a finite group $G$. Section 2 defines direct products of supercharacter theories, and Sections 3 and 4 derive a lattice-theoretic join operation on supercharacter theories. Sections 5 through 9 introduce new supercharacter theories of a group $G$ as products of supercharacter theories of quotient groups and normal subgroups of $G$, and investigate properties of these products. Restricting our attention to abelian groups, in the closing sections we describe a duality relation between the supercharacter theories of an abelian group $G$ and those of the group $\text{Irr}(G)$ of its irreducible characters.

2. Direct products

New supercharacter theories can often be found by combining supercharacter theories that are already known. Our first approach will be to form a direct product of supercharacter theories. Let $M$ and $N$ be groups, and let $G$ be the direct product $M \times N$; then we know that $\text{Irr}(G) = \text{Irr}(M) \times \text{Irr}(N)$. Given two supercharacter theories $(X, K) \in \text{Sup}(M)$ and $(Y, L) \in \text{Sup}(N)$, we can form a natural “product” theory $(X \times Y, K \times L)$ by forming a new supercharacter for every element of the cartesian product $X \times Y$ and a new superclass for every element of $K \times L$.

**Lemma 2.1.** Using the above notation, $(X \times Y, K \times L) \in \text{Sup}(G)$.

**Proof.** Certainly $|Z| = |X||Y| = |K||L| = |M|$. So it suffices to show for all sets $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, $K \in \mathcal{K}$, and $L \in \mathcal{L}$ that the character $\sigma_{X \times Y}$ is constant on the set $K \times L$. For all $m \in M$ and $n \in N$, we have

$$\sigma_{X \times Y}((m, n)) = \sum_{\varphi \in X} \sum_{\theta \in Y} (\varphi \times \theta)((1, 1)) \cdot (\varphi \times \theta)((m, n))$$

$$= \sum_{\varphi \in X} \sum_{\theta \in Y} \varphi(1)\theta(1)\varphi(m)\theta(n)$$

$$= \sum_{\varphi \in X} \varphi(1)\varphi(m) \sum_{\theta \in Y} \theta(1)\theta(n)$$

$$= \sigma_X(m)\sigma_Y(n).$$
Thus for all $m, m' \in K$ and all $n, n' \in L$, 
\[
\sigma_{X \times Y}((m, n)) = \sigma_X(m)\sigma_Y(n) = \sigma_X(m')\sigma_Y(n') = \sigma_{X \times Y}((m', n')).
\]

Thus $\sigma_{X \times Y}$ is in fact constant on $K \times L$. We conclude that $(Z, \mathcal{K})$ is indeed a supercharacter theory of $G$. \hfill \square

Of course, not every supercharacter theory of a direct product arises in this manner; in particular, the maximal supercharacter theory $M(M \times N)$ does not lie in $\text{Sup}(M) \times \text{Sup}(N)$ unless either $M$ or $N$ is trivial.

3. Supercharacter theories and subalgebras

To prepare for our next construction, we investigate the connection between supercharacter theories of a finite group $G$ and certain subalgebras of the center of the group algebra $\mathbb{C}[G]$. For every subset $K \subseteq G$, let $\hat{K} = \sum_{x \in K} x$; recall that for every subset $X \subseteq \text{Irr}(G)$, $\sigma_X$ denotes $\sum_{\psi \in X} \psi(1)\psi$. Also recall that every character $\chi \in \text{Irr}(G)$ has a corresponding central idempotent $e_\chi = \frac{1}{|G|} \chi(1) \sum_{g \in G} \overline{\chi(g)} g$, and that the set $\{e_\chi : \chi \in \text{Irr}(G)\}$ is a basis for $\mathbb{Z}[\mathbb{C}[G]]$. For every subset $X \subseteq \text{Irr}(G)$, let $f_X = \sum_{\psi \in X} e_\psi$. Then $f_X$ and $\sigma_X$ are closely related. Because $e_\chi = \frac{1}{|G|} \sum_{g \in G} \sigma(\chi)(g) g$, by the linearity of the $\sigma$ operator we have
\[
f_X = \sum_{\chi \in X} e_\chi = \frac{1}{|G|} \sum_{g \in G} \sigma_X(g) g. \tag{1}
\]

Diaconis and Isaacs show in [8, Theorem 2.2(b)] that if $G$ is a group and if $\mathcal{C} = (X, \mathcal{K}) \in \text{Sup}(G)$, then the set of superclass sums $\{\hat{K} : K \in \mathcal{K}\}$ and the set of sums of idempotents $\{f_X : X \in \mathcal{X}\}$ are two bases for the same subalgebra of $\mathbb{Z}[\mathbb{C}[G]]$, which we shall denote $\mathcal{A}(\mathcal{C})$.

For each partition $\mathcal{X}$ of $\text{Irr}(G)$, let $A_{\mathcal{X}}$ denote the subspace span $\{f_X : X \in \mathcal{X}\}$. Because the $f_X$ are central idempotents and $f_Xf_Y = 0$ if $X \neq Y$, the subspace $A_{\mathcal{X}}$ is actually a subalgebra of $\mathbb{Z}[\mathbb{C}[G]]$. For example, if $\mathcal{C} = (\mathcal{X}, \mathcal{K})$ is a supercharacter theory of $G$, then $A_{\mathcal{X}} = \mathcal{A}(\mathcal{C})$. It turns out that every subalgebra of $\mathbb{Z}[\mathbb{C}[G]]$ arises from a partition of $\text{Irr}(G)$ in this way.

**Lemma 3.1.** Let $G$ be a group and let $A$ be a subalgebra of $\mathbb{Z}[\mathbb{C}[G]]$. Then there exists a unique partition $\mathcal{Z}$ of $\text{Irr}(G)$ such that $\{f_Z : Z \in \mathcal{Z}\}$ is a basis for $A$.

**Proof.** Because $\mathbb{Z}[\mathbb{C}[G]]$ is isomorphic to a direct sum of copies of $\mathbb{C}$, it contains no nilpotent elements, so neither does its subalgebra $A$; hence the Jacobson radical $\mathfrak{J}(A) = 0$. Then by Wedderburn’s theorem $A$ is a direct sum of full matrix rings; but since $A$ is commutative, those are rings of $1 \times 1$ matrices, so $A$ too is a direct sum of copies of $\mathbb{C}$. Hence $A$ is the linear span of some idempotents $f_1, \ldots, f_r$ whose sum is 1 and whose pairwise products are 0. But every idempotent in $\mathbb{Z}[\mathbb{C}[G]]$ is a sum of some distinct $e_\chi$, and because $\sum_{i=1}^r f_i = 1 = \sum_{\chi \in \text{Irr}(G)} e_\chi$ but the product $f_if_j = 0$ for $i \neq j$, every $e_\chi$ must appear in exactly one $f_i$. Then there exists a partition $\mathcal{Z}$ such that $\{f_Z : Z \in \mathcal{Z}\} = \{f_1, \ldots, f_r\}$, and this is the desired basis for $A$. 

To show uniqueness, suppose \( \mathcal{Y} \) is also a partition of \( \text{Irr}(G) \) such that
\[
\text{span}\left\{ f_{Y} : Y \in \mathcal{Y} \right\} = A = \text{span}\left\{ f_{Z} : Z \in \mathcal{Z} \right\}.
\]
Let \( \chi \in \text{Irr}(G) \), and let \( Y_{0} \in \mathcal{Y} \) and \( Z_{0} \in \mathcal{Z} \) be the parts containing \( \chi \). Then because \( f_{Y_{0}} \in A = \text{span}\left\{ f_{Z} : Z \in \mathcal{Z} \right\} \), the set \( Y_{0} \) must be a union of parts of \( \mathcal{Z} \); in particular \( Z_{0} \subseteq Y_{0} \). But by symmetry \( Y_{0} \subseteq Z_{0} \), so \( Y_{0} = Z_{0} \). Since the parts of \( \mathcal{Y} \) and \( \mathcal{Z} \) containing \( \chi \) are identical for all \( \chi \in \text{Irr}(G) \), it follows that \( \mathcal{Y} = \mathcal{Z} \). \( \square \)

This Lemma enables us to determine whether an arbitrary partition \( \mathcal{K} \) of \( G \) corresponds to a supercharacter theory, using only computations in \( \mathbb{C}[G] \) and making no mention of characters.

**Proposition 3.2.** Let \( \mathcal{K} \) be a partition of \( G \). Then there exists a partition \( \mathcal{X} \) of \( \text{Irr}(G) \) such that \( (\mathcal{X}, \mathcal{K}) \in \text{Sup}(G) \) if and only if \( \text{span}\left\{ \hat{K} : K \in \mathcal{K} \right\} \) is a subalgebra of \( \mathbb{Z}[\mathbb{C}[G]] \).

**Proof.** As noted above, if \( (\mathcal{X}, \mathcal{K}) \) is a supercharacter theory, then Diaconis and Isaacs have proved \(^{3}\) Theorem 2.2(b)\) that \( \text{span}\left\{ \hat{K} : K \in \mathcal{K} \right\} \) is a subalgebra of \( \mathbb{Z}[\mathbb{C}[G]] \). So now suppose \( \text{span}\left\{ \hat{K} : K \in \mathcal{K} \right\} \) is a subalgebra \( A \) of \( \mathbb{Z}[\mathbb{C}[G]] \). Because \( A \subseteq \mathbb{Z}[\mathbb{C}[G]] \), each part \( K \in \mathcal{K} \) is a union of conjugacy classes of \( G \). Because the parts of \( \mathcal{K} \) are disjoint, we know that the set \( \{ \hat{K} : K \in \mathcal{K} \} \) is linearly independent, and hence a basis for \( A \). By Lemma 3.1 there exists some partition \( \mathcal{X} \) of \( \text{Irr}(G) \) so that \( \{ f_{X} : X \in \mathcal{X} \} \) is also a basis for \( A \). So \( |\mathcal{X}| = \dim A = |\mathcal{K}| \), and it only remains to show that \( \sigma_{X} \) is constant on \( K \) for all \( X \in \mathcal{X} \) and all \( K \in \mathcal{K} \). Now an element \( y \in \mathbb{C}[G] \) lies in \( \text{span}\left\{ \hat{K} : K \in \mathcal{K} \right\} \) if and only if the function from \( G \) to \( \mathbb{C} \) which maps each group element to its coefficient in \( y \) is constant on all \( K \in \mathcal{K} \). But
\[
f_{X} = \sum_{g \in G} \left( \frac{1}{|G|} \overline{\sigma_{X}(g)} \right) g
\]
as we saw in Eq. \(^{11}\); then because \( f_{X} \in A \), the function \( g \mapsto \frac{1}{|G|} \overline{\sigma_{X}(g)} \) is constant on all \( K \in \mathcal{K} \), so \( \sigma_{X} \) must be constant on all \( K \in \mathcal{K} \) as well. Therefore \( (\mathcal{X}, \mathcal{K}) \) is a supercharacter theory of \( G \). \( \square \)

### 4. Joins of supercharacter theories

Our next method of obtaining new supercharacter theories is to perform a lattice-theoretic “join” on two supercharacter theories already known. Let us first recall some well-known facts about partitions of a set.

For each finite set \( S \), the set \( \text{Part}(S) \) of partitions of \( S \) into disjoint subsets forms a partially ordered set under the relation “\( \preceq \)” where \( \mathcal{X} \preceq \mathcal{Y} \) if and only if every part of \( \mathcal{X} \) is contained in some part of \( \mathcal{Y} \). This poset in fact forms a lattice, which is called the **partition lattice** of \( S \). (See \(^{10}\) pp. 192ff for more details.) Thus for any two partitions \( \mathcal{X} \) and \( \mathcal{Y} \) of \( S \), their join \( \mathcal{X} \vee \mathcal{Y} \) and meet \( \mathcal{X} \wedge \mathcal{Y} \) are defined. The statement \( \mathcal{X} \preceq \mathcal{Y} \) is equivalent to the statement \( \mathcal{X} \vee \mathcal{Y} = \mathcal{Y} \) and to the statement \( \mathcal{X} \wedge \mathcal{Y} = \mathcal{X} \). We also note without proof the following easy facts.

**Lemma 4.1.** Let \( \mathcal{X}, \mathcal{Y} \in \text{Part}(S) \). Then the following hold:

1. \( \mathcal{X} \vee \mathcal{Y} = \mathcal{Y} \) if and only if \( \mathcal{X} \wedge \mathcal{Y} = \mathcal{X} \) if and only if for each part \( K \in \mathcal{X} \), there exists a part \( Y \in \mathcal{Y} \) containing \( K \).
2. \( \mathcal{X} \wedge \mathcal{Y} \) is the finest partition \( \mathcal{A} \) in \( \text{Part}(S) \) which refines both \( \mathcal{X} \) and \( \mathcal{Y} \).
3. \( \mathcal{X} \vee \mathcal{Y} \) is the coarsest partition \( \mathcal{B} \) in \( \text{Part}(S) \) which contains both \( \mathcal{X} \) and \( \mathcal{Y} \).

---

\(^{1}\) We have made use of results from our previous paper \(^{1}\) in our work here.
(a) Suppose \( \mathcal{X} \preceq \mathcal{Y} \). Then each part of \( \mathcal{Y} \) is the union of some parts of \( \mathcal{X} \).
(b) Let \( T \) be a set and let \( f : S \to T \) be a function. Suppose \( f \) is constant on each member of \( \mathcal{X} \) and each member of \( \mathcal{Y} \). Then \( f \) is constant on each member of \( \mathcal{X} \cup \mathcal{Y} \).

We saw in Section 3 that to every partition \( \mathcal{X} \) of \( \text{Irr}(G) \) corresponds a central subalgebra \( A_{\mathcal{X}} = \text{span}\{f_X : X \in \mathcal{X}\} \), and that every central subalgebra arises in this way. This bijection between \( \text{Part}(\text{Irr}(G)) \) and the subalgebras of \( Z(\mathbb{C}[G]) \) is also order-reversing with respect to partial orders.

**Lemma 4.2.** The map \( \mathcal{X} \mapsto A_{\mathcal{X}} \) is a bijection from \( \text{Part}(\text{Irr}(G)) \) to the set of subalgebras of \( Z(\mathbb{C}[G]) \). Moreover, if \( \mathcal{X}, \mathcal{Y} \in \text{Part}(\text{Irr}(G)) \), then
(a) \( \mathcal{X} \preceq \mathcal{Y} \) if and only if \( A_\mathcal{Y} \subseteq A_\mathcal{X} \).
(b) \( A_{\mathcal{X} \cup \mathcal{Y}} = A_{\mathcal{X}} \cap A_{\mathcal{Y}} \).
(c) \( A_{\mathcal{X} \cap \mathcal{Y}} = \langle A_\mathcal{X}, A_\mathcal{Y} \rangle \), the subalgebra generated by \( A_\mathcal{X} \) and \( A_\mathcal{Y} \).

**Proof.** [Lemma 3.3] shows that the map \( \mathcal{X} \mapsto A_{\mathcal{X}} \) is invertible. For part (a), suppose \( \mathcal{X} \preceq \mathcal{Y} \) and let \( \mathcal{Y} \) be a part of \( \mathcal{Y} \). Then \( \mathcal{Y} \) is the union of some parts of \( \mathcal{X} \), so \( f_\mathcal{Y} \) is the sum of the corresponding idempotents \( f_X \). Then \( f_\mathcal{Y} \in A_\mathcal{X} \); hence \( A_\mathcal{Y} \subseteq A_\mathcal{X} \). Conversely, suppose \( A_\mathcal{Y} \subseteq A_\mathcal{X} \). Let \( \mathcal{Y} \) be a part of \( \mathcal{Y} \). Then \( f_\mathcal{Y} \) is an idempotent in \( A_\mathcal{X} \), so it is a sum of some of the spanning idempotents \( \{f_X : X \in \mathcal{X}\} \) of \( A_\mathcal{X} \). It follows that \( \mathcal{Y} \) must be a union of parts of \( \mathcal{X} \), as desired. Thus the map \( \mathcal{X} \mapsto A_{\mathcal{X}} \) is an order-reversing bijection between \( \text{Part}(\text{Irr}(G)) \) and the partially ordered set of the subalgebras of \( Z(\mathbb{C}[G]) \) under inclusion.

Then for part (b), because \( \mathcal{X} \cup \mathcal{Y} \) is the least upper bound for \( \mathcal{X} \) and \( \mathcal{Y} \) in \( \text{Part}(\text{Irr}(G)) \), the subalgebra \( A_{\mathcal{X} \cup \mathcal{Y}} \) must be the largest subalgebra contained in both \( A_\mathcal{X} \) and \( A_\mathcal{Y} \); namely \( A_\mathcal{X} \cap A_\mathcal{Y} \). Similarly for part (c), we observe that \( \mathcal{X} \cap \mathcal{Y} \) is the greatest lower bound for \( \mathcal{X} \) and \( \mathcal{Y} \), so \( A_{\mathcal{X} \cap \mathcal{Y}} \) is the smallest subalgebra containing both \( A_\mathcal{X} \) and \( A_\mathcal{Y} \), which by definition is \( \langle A_\mathcal{X}, A_\mathcal{Y} \rangle \).

The map from the partitions \( \mathcal{K} \) of \( G \) to the subspaces \( \text{span}\{\hat{K} : K \in \mathcal{K}\} \) forms no such bijection, not even when we require \( \mathcal{K} \) to be coarser than the conjugacy classes and those subspaces actually to be subalgebras. The proof of [Lemma 3.1] which established the bijection used to prove [Lemma 4.2] relied heavily on the orthogonality of the \( \chi_g \)'s, but the \( \hat{K} \)'s exhibit no such orthogonality. An analogue of [Lemma 4.2(b)] does hold, however.

**Lemma 4.3.** Let \( \mathcal{K}, \mathcal{L} \in \text{Part}(G) \), and write \( \mathcal{M} = \mathcal{K} \cup \mathcal{L} \). Then
\[
\text{span}\left\{\hat{M} : M \in \mathcal{M}\right\} = \text{span}\left\{\hat{K} : K \in \mathcal{K}\right\} \cap \text{span}\left\{\hat{L} : L \in \mathcal{L}\right\}.
\] (2)

**Proof.** Since each part \( M \in \mathcal{M} = \mathcal{K} \cup \mathcal{L} \) is a union of some parts of \( \mathcal{K} \), the sum \( \hat{M} \) lies in \( \text{span}\{\hat{K} : K \in \mathcal{K}\} \). Likewise \( \hat{M} \in \text{span}\{\hat{L} : L \in \mathcal{L}\} \), so the left side of (2) is contained in the right hand side.

On the other hand, each element \( d \) on the right side of (2) may be written as \( d = \sum_{K \in \mathcal{K}} a_K \hat{K} = \sum_{L \in \mathcal{L}} b_L \hat{L} \) for some coefficients \( a_K, b_L \in \mathbb{C} \). Recall that each element \( g \in \hat{G} \) occurs in exactly one \( K \) and in exactly one \( L \), and that \( G \) is a basis for \( \mathbb{C}[G] \).

Now the function mapping \( g \) to the coefficient of \( g \) in \( d \) is constant on each \( K \in \mathcal{K} \), and also constant on each \( L \in \mathcal{L} \). Hence it is constant on each member of \( \mathcal{K} \cup \mathcal{L} = \mathcal{M} \) by [Lemma 4.1(b)] and so \( d \) lies in the span of \( \{\hat{M} : M \in \mathcal{M}\} \). □
These two lemmas allow us to define a binary “join” operation on supercharacter theories of a group $G$:

**Proposition 4.4.** Let $G$ be a group. Let $\mathcal{X}$ and $\mathcal{Y}$ be partitions of $\text{Irr}(G)$ and $K$ and $L$ be partitions of $G$ such that $(\mathcal{X}, K)$ and $(\mathcal{Y}, L)$ are each supercharacter theories of $G$. Then $(\mathcal{X} \lor \mathcal{Y}, K \lor L)$ is also a supercharacter theory of $G$, which is denoted $(\mathcal{X}, K) \lor (\mathcal{Y}, L)$.

**Proof.** Let $\mathcal{Z} = \mathcal{X} \lor \mathcal{Y}$ and $\mathcal{M} = K \lor L$. To show that the functions $\{\sigma_z : z \in \mathcal{Z}\}$ are constant on the sets $M \in \mathcal{M}$, let $Z \in \mathcal{Z}$, let $M \in \mathcal{M}$, and let $g, h \in M$. Now $Z = \bigcup_{X \in \mathcal{X}} X$ for some subset $I \subseteq \mathcal{X}$, so $\sigma_Z = \sum_{X \in I} \sigma_X$ must be constant on each $K \in K$ because $(\mathcal{X}, K)$ is a supercharacter theory. On the other hand, by symmetry $\sigma_Z$ is also constant on each $L \in L$. So by Lemma 4.1(b) it follows that $\sigma_Z$ is constant on each $M \in \mathcal{M}$. It only remains to show that $|\mathcal{Z}| = |\mathcal{M}|$.

Recall that Diaconis and Isaacs showed [8, Theorem 2.2(b)] that $\{f_X : X \in \mathcal{X}\}$ and $\{\hat{K} : K \in K\}$ are two different bases for the algebra $A((\mathcal{X}, K))$. Hence

\[
\text{span} \{f_z : z \in \mathcal{Z}\} = \text{span} \{f_X : X \in \mathcal{X}\} \cap \text{span} \{f_Y : Y \in \mathcal{Y}\} \quad \text{(by Lemma 4.2(b))}
\]

\[
= \text{span} \{\hat{K} : K \in K\} \cap \text{span} \{\hat{L} : L \in L\} \quad \text{by [8, Theorem 2.2(b)]}
\]

\[
= \text{span} \{\hat{M} : M \in \mathcal{M}\} \quad \text{(by Lemma 4.3)}
\]

But since both $\{f_z : z \in \mathcal{Z}\}$ and $\{\hat{M} : M \in \mathcal{M}\}$ are linearly independent sets over $\mathbb{C}$, both $|\mathcal{Z}|$ and $|\mathcal{M}|$ must equal the dimension of the algebra in question, so $|\mathcal{Z}| = |\mathcal{M}|$ as desired. We conclude that $(\mathcal{X} \lor \mathcal{Y}, K \lor L)$ is a supercharacter theory of $G$. □

If both $(\mathcal{X}, K)$ and $(\mathcal{Y}, L)$ are supercharacter theories of $G$, it is in general not true that $(\mathcal{X} \land \mathcal{Y}, K \land L)$ is again a supercharacter theory. For example, let $G$ be the symmetric group on six letters, whose eleven conjugacy classes may be denoted by their cycle structures. Then the two partitions $K = \{1, 2^1 \cup 2^1 \cup 2^1, 3^1 \cup 2^1, 2^2, 3^1 \cup 3^2, 4^1 2^1, 5^1\}$ and $L = \{1, 2^1 \cup 2^1 \cup 4^1 \cup 6^1 \cup 3^1 2^1, 2^2, 3^1, 3^2, 4^1 2^1, 5^1\}$ both correspond to supercharacter theories, but their meet does not.

We conclude this discussion of joins by making $\text{Sup}(G)$ into a partially ordered set. Because every supercharacter theory in $\text{Sup}(G)$ corresponds to a partition $X \in \text{Part}(\text{Irr}(G))$, one possibility would be to declare that $(\mathcal{X}, K) \preceq (\mathcal{Y}, L)$ if $X \preceq Y$. On the other hand, we could just as well partially order $\text{Sup}(G)$ with respect to the partition of $G$ given by each theory’s superclasses; according to this method, we would write $(\mathcal{X}, K) \preceq (\mathcal{Y}, L)$ if $K \preceq L$. Thanks to Proposition 4.4, we can show that each of these two alternatives produces the same partial ordering of $\text{Sup}(G)$.

**Corollary 4.5.** Let $(\mathcal{X}, K)$ and $(\mathcal{Y}, L)$ be supercharacter theories of $G$. Then $\mathcal{X} \preceq \mathcal{Y}$ if and only if $K \preceq L$.

**Proof.** ($\Leftarrow$) Suppose $K \preceq L$. Since $(\mathcal{X}, K)$ and $(\mathcal{Y}, L)$ are supercharacter theories for $G$, so is $(\mathcal{X} \lor \mathcal{Y}, K \lor L)$, which is equal to $(\mathcal{X} \lor \mathcal{Y}, L)$. But because superclasses and supercharacters determine one another, there is only one supercharacter theory with superclasses $L$, namely $(\mathcal{Y}, L)$. So $\mathcal{X} \lor \mathcal{Y} = \mathcal{Y}$ and $\mathcal{X} \preceq \mathcal{Y}$ as desired.

($\Rightarrow$) Suppose $\mathcal{X} \preceq \mathcal{Y}$. Then $(\mathcal{X} \lor \mathcal{Y}, K \lor L) = (\mathcal{Y}, K \lor L)$ is a supercharacter theory for $G$. Because $(\mathcal{Y}, L)$ is the unique supercharacter theory with supercharacters from $\mathcal{Y}$, we must have $K \lor L = L$ and $K \preceq L$. □
We are therefore not breaking symmetry between superclasses and supercharacters when we define a partial ordering of \( \text{Sup}(G) \) as follows.

**Definition 4.6.** Let \((\mathcal{X}, \mathcal{K})\) and \((\mathcal{Y}, \mathcal{L})\) be supercharacter theories of a group \( G \). Then we write \((\mathcal{X}, \mathcal{K}) \preceq (\mathcal{Y}, \mathcal{L})\) if \( \mathcal{X} \preceq \mathcal{Y} \).

5. The \(*\)-product

In this section, we show that if \( N \) is a normal subgroup of \( G \), then some supercharacter theories of \( N \) can be combined with supercharacter theories of \( G/N \) to form supercharacter theories of the full group \( G \). We can thus construct supercharacter theories of large groups by combining those of smaller groups.

Let a group \( G \) act on another group \( H \); then \( G/C_G(H) \) embeds naturally in \( \text{Aut}(H) \), so there exists a supercharacter theory \( m_G(H) = (\mathcal{X}, \mathcal{K}) \in \text{Sup}(H) \) such that \( \mathcal{X} \) is the partition of \( \text{Irr}(H) \) into \( G \)-orbits and \( \mathcal{K} \) is the finest partition of \( H \) into unions of conjugacy classes such that each part is \( G \)-invariant. Then for another supercharacter theory \((\mathcal{Y}, \mathcal{L}) \in \text{Sup}(H)\), every part \( L \in \mathcal{L} \) is \( G \)-invariant if and only if \( \mathcal{K} \preceq \mathcal{L} \), which is true if and only if \( \mathcal{X} \preceq \mathcal{Y} \) (by Corollary 4.5), which is true if and only if every part \( Y \in \mathcal{Y} \) is \( G \)-invariant. We may thus unambiguously speak of a \( G \)-invariant supercharacter theory.

**Definition 5.1.** Let \( G \) and \( H \) be groups and let \( G \) act on \( H \) by automorphisms. We say that \((\mathcal{X}, \mathcal{K}) \in \text{Sup}(H)\) is \( G \)-invariant if the action of \( G \) fixes each part \( K \in \mathcal{K} \) setwise. We denote by \( \text{Sup}_G(H) \) the set of all \( G \)-invariant supercharacter theories of \( H \).

For example, if \( N \trianglelefteq G \), then \( C \in \text{Sup}(N) \) is \( G \)-invariant if and only if its superclasses are unions of \( G \)-conjugacy classes. Also, every supercharacter theory of \( G \) is \( G \)-invariant, so \( \text{Sup}_G(G) = \text{Sup}(G) \). Note that if \( M, N \trianglelefteq G \) with \( N < M \), then a supercharacter theory of \( M/N \) is \( G/N \)-invariant if and only if it is \( G \)-invariant. As our notation indicates, the supercharacter theory \( m_G(H) \) is the minimal \( G \)-invariant member of \( \text{Sup}(H) \).

We would like to define a product

\[
* : \text{Sup}_G(N) \times \text{Sup}(G/N) \longrightarrow \text{Sup}(G).
\]

So suppose \( C = (\mathcal{X}, \mathcal{K}) \in \text{Sup}_G(N) \) and \( D = (\mathcal{Y}, \mathcal{L}) \in \text{Sup}(G/N) \). Let us first consider the superclasses: \( \mathcal{K} \) is a \( G \)-invariant partition of \( N \) and \( \mathcal{L} \) a \( G \)-invariant partition of \( G/N \), one part of which is the coset \( N \). We can “inflate” \( \mathcal{L} \) to be a partition of \( G \), one part of which will be the set \( N \), which we can then replace with the partition \( \mathcal{K} \) of \( N \). To express this formally, for each subset \( L \subseteq G/N \) let \( \tilde{L} \) denote the inflation \( \bigcup_{N \in L} Ng \).

Extend this notation to a set \( \mathcal{L} \) of subsets of \( G/N \) by \( \tilde{\mathcal{L}} = \{ \tilde{L} : L \in \mathcal{L} \} \), and let \( \tilde{\mathcal{L}}^\circ \) denote \( \mathcal{L} - \{ \{N\} \} \). Then for \( \mathcal{K} \in \text{Part}(N) \) and \( \mathcal{L} \in \text{Part}(G/N) \), we have a partition

\[
\mathcal{K} \cup \tilde{\mathcal{L}}^\circ \in \text{Part}(G).
\]

For the supercharacters, recall that if \( N \) is a normal subgroup of \( G \) and \( \psi \in \text{Irr}(N) \), then \( \text{Irr}(G/\psi) \) denotes the set of irreducible characters \( \chi \) of \( G \) such that \([\chi_N, \psi] > 0\). If \( Z \subseteq \text{Irr}(N) \) is a union of \( G \)-orbits, then define the subset \( Z^G \) of \( \text{Irr}(G) \) to
be $\bigcup_{\psi \in Z} \text{Irr}(G|\psi)$. Extend this notation to a set $Z$ of subsets of $\text{Irr}(N)$ by letting $Z^G = \{ Z^G : Z \in Z \}$, and let $Z^\circ = Z - \{ \{1_N\} \}$.

Now consider $(\mathcal{X}, \mathcal{K}) \in \text{Sup}_G(N)$ and $(\mathcal{Y}, \mathcal{L}) \in \text{Sup}(G/N)$ as before. Since $\mathcal{X}$ is a partition of $\text{Irr}(N)$ into unions of $G$-orbits, it follows that $\mathcal{X}^G$ is a partition of $\text{Irr}(G)$. Since $\{1_N\} \in \mathcal{X}$, one part of $\mathcal{X}^G$ is $\{1_N\}^G = \{ \chi \in \text{Irr}(G) : N \subseteq \ker \chi \}$, which we identify with $\text{Irr}(G/N)$ in the usual way. Thus we can replace that part of $\mathcal{X}^G$ with the partition $\mathcal{Y}$ of $\text{Irr}(G/N)$, obtaining a partition of $G$

$$\mathcal{Y} \cup (\mathcal{X}^\circ)^G \in \text{Part}(\text{Irr}(G))$$

(4)

incorporating information from both $\mathcal{X}$ and $\mathcal{Y}$.

We shall show that the partitions of (3) and (4) do form a supercharacter theory of $G$, by way of a brief lemma demonstrating the suitability of the notation “$X^G$.”

Lemma 5.2. Let $N \triangleleft G$ and let $X \subseteq \text{Irr}(N)$ be a union of $G$-orbits. Then

$$\sigma(X^G) = (\sigma_X)^G.$$  

Proof. It suffices to prove the statement when $X$ is a single $G$-orbit. Let $Z$ be the partition of $\text{Irr}(N)$ into $G$-orbits, so that $Z^G$ is a partition of $\text{Irr}(G)$. Then the regular character $\rho_N = \sum_{Z \in Z} \sigma_Z$ and so

$$\rho_G = (\rho_N)^G = \sum_{Z \in Z} (\sigma_Z)^G.$$  

Now the characters $(\sigma_Z)^G$ have no irreducible constituents in common with one another, so $(\sigma_X)^G = \sigma_Y$ where $Y$ is the set of irreducible constituents of $(\sigma_X)^G$. But $Y = X^G$, so we conclude that $\sigma_X^G = (\sigma_X)^G$. $\square$

Theorem 5.3. Let $G$ be a group and let $N \triangleleft G$. Suppose $(\mathcal{X}, \mathcal{K}) \in \text{Sup}_G(N)$ and suppose $(\mathcal{Y}, \mathcal{L}) \in \text{Sup}(G/N)$. Then

$$\left( \mathcal{Y} \cup (\mathcal{X}^\circ)^G, \mathcal{K} \cup \mathcal{L}^\circ \right)$$

is a supercharacter theory of $G$.

Proof. Let $Z = \mathcal{Y} \cup (\mathcal{X}^\circ)^G \in \text{Part}(\text{Irr}(G))$ and let $\mathcal{M} = \mathcal{K} \cup \mathcal{L}^\circ \in \text{Part}(G)$. Now $|Z| = |\mathcal{Y}| + |\mathcal{X}| - 1 = |\mathcal{L}| + |\mathcal{K}| - 1 = |\mathcal{M}|$, so it remains to show that for each part $Z \in Z$, the character $\sigma_Z$ is constant on each subset $M \in \mathcal{M}$.

One possibility is that $Z$ lies in $\mathcal{Y}$. In this case, because $Z \subseteq \text{Irr}(G/N)$, the character $\sigma_Z$ has $N$ in its kernel, so it is certainly constant on every part $K \in \mathcal{K}$. Moreover, because $(\mathcal{Y}, \mathcal{L})$ is a supercharacter theory of $G/N$, we know that $\sigma_Z$ (viewed as a character of $G/N$) is constant on each superclass $L \in \mathcal{L}$; viewed as a character of $G$, it is therefore constant on each $\tilde{L} \in \mathcal{L}^\circ$. So $\sigma_Z$ is constant on each set $M \in \mathcal{M}$ in the case that $Z \in \mathcal{Y}$.

The other possibility is that $Z = X^G$ for some part $X \in \mathcal{X}$. Now because $(\mathcal{X}, \mathcal{K})$ is $G$-invariant, we know that $X$ is a union of $G$-orbits of $\text{Irr}(N)$, so we can calculate by Lemma 5.2 that $\sigma_Z = \sigma(X^G) = (\sigma_X)^G$. Because $N \triangleleft G$, we know that $(\sigma_X)^G$ vanishes outside of $N$; therefore $\sigma_Z$ is constant on every set $\tilde{L} \in \mathcal{L}^\circ$. Moreover, because the set $X$ is $G$-invariant, the character $\sigma_X$ of $N$ is invariant under the action of $G$, so the restriction
\((\sigma_Z)_N = ((\sigma_X)^G)_N = |G : N| \sigma_X\). Because \(\sigma_X\) is constant on every part \(K \in \mathcal{K}\), so too is \(\sigma_Z\).

Thus \(\sigma_Z\) is constant on \(M\) for every \(Z \in \mathcal{Z}\) and every \(M \in \mathcal{M}\), so we conclude that 
\((\mathcal{Z}, \mathcal{M})\) is a supercharacter theory of \(G\). \(\square\)

We may therefore define a product of supercharacter theories as follows.

**Definition 5.4.** Let \(G\) be a group and let \(N \triangleleft G\). Let \(C = (\mathcal{X}, \mathcal{K}) \in \text{Sup}_G(N)\) and \(D = (\mathcal{Y}, \mathcal{L}) \in \text{Sup}(G/N)\). Then their \(*\)-product, written \(C \ast_N D\), is the supercharacter theory of \(G\)

\[
\left(\mathcal{Y} \cup (\mathcal{X}^N)^G, \mathcal{K} \cup \mathcal{L}^G\right).
\]

We say that \(C \ast_N D\) is a \(*\)-product over \(N\) or that \(C \ast_N D\) factors over \(N\). When \(N\) is clear from context, we omit the subscript and write simply \(C \ast D\).

6. Factoring

Let us investigate some properties of the \(*\)-product. First, we would like to recognize which supercharacter theories of a group \(G\) arise as \(*\)-products. One characteristic of a \(*\)-product over \(N\) is that \(N\) is a union of some of the superclasses.

**Definition 6.1.** Let \(G\) be a group and let \(C \in \text{Sup}(G)\). Then a subgroup \(N\) of \(G\) which is a union of some superclasses of \(C\) is called \(C\)-normal.

Recall that \(m(H)\) denotes the minimal supercharacter theory of a group \(H\), namely its ordinary character theory; likewise \(M(H)\) denotes the supercharacter theory with exactly two superclasses, 1 and \(H - 1\). If \(N \triangleleft H\), then \(M(N) \ast M(G/N)\) is a supercharacter theory of \(G\) with the three superclasses 1, \(N - 1\), and \(G - N\); the corresponding partition of \(\text{Irr}(G)\) is \(\{\{1_G\}, \text{Irr}(G/N) - \{1_G\}, \text{Irr}(G) - \text{Irr}(G/N)\}\). We shall denote this supercharacter theory by \(\text{MM}_N(G)\). Let \(C = (\mathcal{X}, \mathcal{K}) \in \text{Sup}(G)\); then \(N\) is \(C\)-normal if and only if \(C \leq \text{MM}_N(G)\), which is true if and only if \(\text{Irr}(G/N)\) is the union of some members of \(\mathcal{X}\).

A "minimal" counterpart to \(\text{MM}_N(G)\) is \(m_G(N) \ast m(G/N)\), whose superclasses are those conjugacy classes of \(G\) which lie in \(N\), together with the nontrivial conjugacy classes of \(G/N\) pulled back to \(G\). Note that within the partition lattice of \(G\), this is the finest partition of \(G\) into unions of conjugacy classes such that each nontrivial \(N\)-coset lies entirely within some superclass. We shall denote this supercharacter theory by the symbol \(\text{mm}_N(G)\). In its partition of \(\text{Irr}(G)\), the characters in \(\text{Irr}(G/N)\) are in singleton parts, while every part outside \(\text{Irr}(G/N)\) is of the form \(\text{Irr}(G|\psi)\) for some \(\psi \in \text{Irr}(N)\).

We noted above that if \(C\) is a \(*\)-product over a normal subgroup \(N\) of \(G\), then \(N\) is \(C\)-normal. The converse is not true: a supercharacter theory \(C\) may not factor over \(N\) even if \(N\) is \(C\)-normal. We nevertheless can construct related supercharacter theories of \(N\) and of \(G/N\) from such a supercharacter theory. Since \(N\) is \(C\)-normal, those superclasses of \(C\) which lie in \(N\) partition \(N\), and we shall show that this partition belongs to a supercharacter theory \(C_N\) of \(N\). Likewise we shall prove that those supercharacters of \(C\) which have \(N\) in their kernels are the supercharacters of a supercharacter theory \(C^{G/N}\) of \(G/N\).

To prove these statements, we need a little notation. If \(Z \subseteq \text{Irr}(G)\) is a union of sets of the form \(\text{Irr}(G|\psi)\) for various \(\psi \in \text{Irr}(N)\), let \(f(Z)\) denote the set of all irreducible
Definition 6.2. Let $G$ be a group, let $C \in \text{Sup}(G)$, and let $N$ be a $C$-normal subgroup of $G$. Writing $C \lor \text{mm}_N(G) = (Z, M)$ and defining $\varphi$ and $f$ as above, let

$$C_N = \left\{ \{f(Z) : Z \in Z, \ Z \not\subseteq \text{Irr}(G/N)\} \cup \{1_N\}, \{M \in M : M \subseteq N\} \right\}$$

and

$$C^{G/N} = \left\{ \{Z \in Z : Z \subseteq \text{Irr}(G/N)\}, \{\varphi(M) : M \in M, \ M \not\subseteq N\} \cup \{N\} \right\}.$$

Note that replacing $C$ with $C \lor \text{mm}_N(G)$ does not change the superclasses which lie within $N$, nor that portion of the partition of $\text{Irr}(G)$ which lies within $\text{Irr}(G/N)$. Therefore the supercharacters of $C^{G/N}$ are indeed those supercharacters of $C$ with $N$ in their kernels, and the superclasses of $C_N$ are those superclasses of $C$ which lie in $N$. In the proof of the following lemma, let us say “$X$ is constant on $K$” to mean that for each $X \in X$, the character $\sigma_X$ is constant on every part $K \in K$.

Lemma 6.3. Let $N$ be a subgroup of a group $G$, let $C \in \text{Sup}(G)$, and suppose $N$ is $C$-normal. Then $C_N$ is a $G$-invariant supercharacter theory of $N$ and $C^{G/N}$ is a supercharacter theory of $G/N$. Moreover,

$$C \lor \text{mm}_N(G) = C_N * C^{G/N}.$$ 

Proof. Let $B = C \lor \text{mm}_N(G)$ and write $B = (Z, M)$. Now since both $C \preceq \text{MM}_N(G)$ and $\text{mm}_N(G) \preceq \text{MM}_N(G)$, it follows that $B = C \lor \text{mm}_N(G) \preceq \text{MM}_N(G)$, so $N$ is $B$-normal. Let

$$\mathcal{K} = \{M \in M : M \subseteq N\},$$

$$\mathcal{L} = \{\varphi(M) : M \in M, \ M \not\subseteq N\} \cup \{N\}$$

$$\mathcal{X} = \{f(Z) : Z \in Z, \ Z \not\subseteq \text{Irr}(G/N)\} \cup \{1_N\},$$

and

$$\mathcal{Y} = \{Z \in Z : Z \subseteq \text{Irr}(G/N)\},$$

so that by definition $C_N = (\mathcal{X}, \mathcal{K})$ and $C^{G/N} = (\mathcal{Y}, \mathcal{L})$. Let us now verify that the sets $\mathcal{X}$, $\mathcal{K}$, $\mathcal{Y}$, and $\mathcal{L}$ are partitions of the appropriate sets.

Since $N$ is $B$-normal, it follows that $\mathcal{K} = \{M \in M : M \subseteq N\}$ is a partition of $N$, and the set $\{M \in M : M \not\subseteq N\}$ is a partition of $G - N$. Since $\text{mm}_N(G) \preceq B$, each element of this latter set is a union of $N$-cosets, and thus $\mathcal{L}^o = \{\varphi(M) : M \in M, \ M \not\subseteq N\}$ is a partition of $G/N - \{N\}$, so $\mathcal{L}$ is a partition of $G/N$. Because $\varphi(M) = M$, we may also note that $\mathcal{M} = \mathcal{K} \cup \mathcal{L}^o$.

As for the characters, $\mathcal{Y} = \{Z \in Z : Z \subseteq \text{Irr}(G/N)\}$ is a partition of $\text{Irr}(G/N)$ because $N$ is $B$-normal. Likewise the set $\{Z \in Z : Z \not\subseteq \text{Irr}(G/N)\}$ is a partition of $\text{Irr}(G) - \text{Irr}(G/N)$, and since $\text{mm}_N(G) \preceq B$, each $Z$ in this set is a union of sets of the form $\text{Irr}(G|\psi)$. Thus $\mathcal{X}^o = \{f(Z) : Z \in Z, \ Z \not\subseteq \text{Irr}(G/N)\}$ is a well-defined partition of $\text{Irr}(N) - \{1_N\}$, so $\mathcal{X}$ is a partition of $\text{Irr}(N)$. Moreover, since $(f(Z))^G = Z$, we also see that $Z = \mathcal{Y} \cup (\mathcal{X}^o)^G$. 

11
To show that $C_N$ and $C^{G/N}$ are indeed supercharacter theories, it remains to show that the purported supercharacters are actually constant on the superclasses, that $|\mathcal{X}| = |\mathcal{K}|$, and that $|\mathcal{Y}| = |\mathcal{L}|$.

Consider first $C_N$. Let $X \in \mathcal{X}$. If $X = \{1_N\}$, then $\sigma_X = 1_N$ is trivially constant on all $K \in \mathcal{K}$; otherwise, $X = f(Z)$ for some $Z \in \mathcal{Z}$, so $Z = X^G$. Then because

$$ (\langle \sigma \rangle)_N = (\langle \sigma \circ \iota \rangle)_N = (\langle \sigma_X \rangle)_N = (\langle \sigma_X \rangle)_N = |G : N| \sigma_X $$

is constant on each $K \in \mathcal{K}$, so too is $\sigma_X$. We conclude that $\mathcal{X}$ is constant on $\mathcal{K}$.

Now consider $(\mathcal{Y}, \mathcal{L})$, and let $Y \in \mathcal{Y} \subseteq Z$. Then $\sigma_Y$ is constant on every superclass $M \in \mathcal{M}$, and in particular, on those superclasses outside $N$. But since $\sigma_Y$ has $N$ in its kernel, when viewed as a character of $G/N$ it is constant on the images of those superclasses, namely the members of $\mathcal{L}^G$. Moreover, $\sigma_Y$ is certainly constant on the singleton set $\{N\} \subseteq G/N$. We conclude that $\mathcal{Y}$ is constant on $\mathcal{L}$.

Now because $\mathcal{X}$ is constant on $\mathcal{K}$ and $\mathcal{Y}$ is constant on $\mathcal{L}$, by [8, Theorem 2.2] we know that $|\mathcal{X}| \leq |\mathcal{K}|$ and $|\mathcal{Y}| \leq |\mathcal{L}|$. But then

$$ |K| + |Y| - 1 \leq |K| + |L| - 1 = |M| = |Z| = |Y| + |X| - 1 \leq |Y| + |K| - 1, $$

so equality must hold throughout; hence $|\mathcal{X}| = |\mathcal{K}|$ and $|\mathcal{Y}| = |\mathcal{L}|$. We conclude that $C_N = (\mathcal{X}, \mathcal{K})$ is a supercharacter theory of $N$ and $C^{G/N} = (\mathcal{Y}, \mathcal{L})$ a supercharacter theory of $G/N$; the former is $G$-invariant because its superclasses are also superclasses of $C$. Finally, by definition

$$ C_N \ast C^{G/N} = (\mathcal{X}, \mathcal{K}) \ast (\mathcal{Y}, \mathcal{L}) = (\mathcal{Y} \cup (\mathcal{X}^\circ)^G, \mathcal{K} \cup \mathcal{L}^G) = (\mathcal{Z}, \mathcal{M}) = \mathcal{C} \cup \mathsf{mm}_N(G) $$

as desired. \qed

With the help of the preceding lemma, we can determine whether a supercharacter theory $E$ of $G$ factors over a normal subgroup $N$.

**Corollary 6.4.** Let $G$ be a group, let $N \triangleleft G$, and let $E$ be a supercharacter theory of $G$. Then $E$ factors over $N$ if and only if $N$ is $E$-normal and every superclass outside $N$ is a union of $N$-cosets. Moreover, if $E = C \ast_N D$, then $C = E_N$ and $D = E^{G/N}$.

**Proof.** Suppose $E = C \ast_N D$; write $C = (\mathcal{X}, \mathcal{K})$ and $D = (\mathcal{Y}, \mathcal{L})$, so that by definition $E = (\mathcal{Y} \cup (\mathcal{X}^\circ)^G, \mathcal{K} \cup \mathcal{L}^G)$. Then $N = \bigcup_{K \in \mathcal{K}} K$ is $E$-normal, and each superclass of $E$ outside $N$ is the preimage of some $L \in \mathcal{L}$, and hence a union of $N$-cosets. Moreover, the superclasses of $E_N$ are those superclasses of $E$ that lie in $N$, namely the superclasses of $C$; thus $E_N = C$. Likewise the supercharacters of $E^{G/N}$ are the supercharacters of $E$ with $N$ in their kernels, namely the supercharacters of $D$, so $E^{G/N} = D$.

Now suppose for the converse that $N$ is $E$-normal and that every superclass of $E$ outside $N$ is a union of $N$-cosets. Then $\mathsf{mm}_N(G) \subseteq E$, so

$$ E = E \cup \mathsf{mm}_N(G) = E_N \ast E^{G/N} $$

by Lemma 6.3. \qed

Since $E \leq \mathsf{MM}_N(G)$ if and only if $N$ is $E$-normal and $\mathsf{mm}_N(G) \leq E$ if and only if every superclass outside $N$ is a union of $N$-cosets, the preceding corollary can be rephrased in terms of the partial order on $\text{Sup}(G)$: $E$ factors over $N$ if and only if $\mathsf{mm}_N(G) \leq E \leq \mathsf{MM}_N(G)$. 12
7. Associativity

Up to this point, we have been working with a fixed normal subgroup $N$ of $G$. Suppose now that we have two normal subgroups $N$ and $M$, with $1 \leq N \leq M \leq G$, and three supercharacter theories $C \in \text{Sup}(N)$, $D \in \text{Sup}(M/N)$, and $E \in \text{Sup}(G/M)$. We may summarize this situation in a diagram:

\[
\begin{array}{ccc}
1 & \leq & N \\
\leq & C & \leq M \\
\leq & D & \leq G \\
\leq & E & \leq \end{array}
\]

Suppose $C$ and $D$ are $G$-invariant. Then in particular $C$ is $M$-invariant, so we can form the product $C *_N D \in \text{Sup}(M)$, which is also $G$-invariant. Thus we can form $(C *_N D) *_{M/N} E \in \text{Sup}(G)$. On the other hand, we can also form $D *_{M/N} E \in \text{Sup}(G/N)$ and then form $C *_N (D *_{M/N} E) \in \text{Sup}(G)$ as well. Fortunately, the two products $(C * D) * E$ and $C * (D * E)$ are the same; in other words, the $*$-product is associative.

**Lemma 7.1.** Let $G$ be a group and let $N$ and $M$ be normal subgroups with $N \leq M$. Suppose $C \in \text{Sup}_G(N)$, $D \in \text{Sup}_G(M/N)$, and $E \in \text{Sup}(G/M)$. Then

\[(C *_N D) *_{M/N} E = C *_N (D *_{M/N} E)\]

**Proof.** Let $C = (\mathcal{W}, \mathcal{J})$, let $D = (\mathcal{X}, \mathcal{K})$, and let $E = (\mathcal{Y}, \mathcal{L})$. Consider the partitions of characters. For $C * D$, the partition of characters is $\mathcal{X} \cup (\mathcal{W}^o)^M$; thus for $(C * D) * E$, the partition of characters is

\[\mathcal{Y} \cup ((\mathcal{X} \cup (\mathcal{W}^o)^M)^G) = \mathcal{Y} \cup (\mathcal{X}^o \cup (\mathcal{W}^o)^M)^G = \mathcal{Y} \cup (\mathcal{X}^o)^G \cup (\mathcal{W}^o)^G.\]

On the other hand, the partition of characters for $D * E$ is $\mathcal{Y} \cup (\mathcal{X}^o)^G$, so the partition of characters for $C * (D * E)$ is also $\mathcal{Y} \cup (\mathcal{X}^o)^G \cup (\mathcal{W}^o)^G$. Thus $(C *_N D) *_{M/N} E$ and $C *_N (D *_{M/N} E)$ are two supercharacter theories of $G$ with identical supercharacters, and hence are equal. \(\square\)

8. Unique factorization

Although [Corollary 6.4](#) showed that a supercharacter theory $E$ of $G$ can be written as a $*$-product over $N$ either in a unique way or not at all, [Lemma 7.1](#) shows that a supercharacter theory might factor over two different normal subgroups $N$ and $M$. However, this can only happen if $N$ contains $M$ or vice versa.

**Lemma 8.1.** Let $G$ be a group with normal subgroups $N$ and $M$. Let $C \in \text{Sup}(G)$, and suppose $C$ is a $*$-product both over $N$ and over $M$. Then either $N \leq M$ or $M \leq N$.

**Proof.** Suppose $N \nleq M$, and choose an element $n \in N - M$. Then because $C$ factors over $M$ and $n \notin M$, the entire coset $Mn$ must lie in the same superclass of $C$ as $n$. But then because $N$ is $C$-normal, we must have $Mn \subseteq N$ and thus $M \leq N$. \(\square\)
Let us investigate further what happens when $C \ast_N D = E \ast_M F$. Without loss of generality, we have normal subgroups $N$ and $M$ of $G$ with $1 \leq N \leq M \leq G$, where $C \in \text{Sup}_G(N)$, $D \in \text{Sup}(G/N)$, $E \in \text{Sup}_G(M)$, and $F \in \text{Sup}(G/M)$. We may portray this setup in a diagram:

\[
\begin{array}{c}
\text{C} \\
\text{1} \leq N \leq M \leq G \quad \text{E} \\
\text{D} \\
\text{F}
\end{array}
\]

In this situation, we shall show that there exists a $G$-invariant supercharacter theory $G \in \text{Sup}_G(M/N)$ such that our supercharacter theory of $G$ is simply $C \ast_N G \ast_M F$, with $E = C \ast_N G$ and $D = G \ast_{M/N} F$, as in this diagram:

\[
\begin{array}{c}
\text{E} \\
\text{1} \leq N \leq M \leq G \quad \text{F} \\
\text{D} \\
\text{C} \quad \text{G}
\end{array}
\]

**Lemma 8.2.** Let $G$ be a group, let $M, N \triangleleft G$, and suppose $C \ast_N D = E \ast_M F \in \text{Sup}(G)$. Without loss of generality, suppose $N \leq M$. Then there exists some supercharacter theory $G \in \text{Sup}_G(M/N)$ such that $E = C \ast_N G$ and $D = G \ast_{M/N} F$.

**Proof.** First we shall show that $E$ factors over $N$. Let $B = C \ast_N D = E \ast_M F$. Now the superclasses of $E$ are precisely those superclasses of $B$ which lie in $M$. Because $B$ factors over $N$, we know that $N$ is $B$-normal; hence $N$ is $E$-normal. Also because $B$ factors over $N$, every superclass of $B$ outside $N$ is a union of $N$-cosets. In particular, every superclass of $E$ outside $N$ is a union of $N$-cosets. Then by [Corollary 6.4](#), we know $E$ must be a $\ast$-product over $N$. Because the superclasses of $E$ lying in $N$ are exactly the superclasses of $B$ lying in $N$, namely the superclasses of $C$, we have $E = C \ast_N G$ for some supercharacter theory $G \in \text{Sup}(M/N)$. Since $C$ is $G$-invariant, so too is $G$.

Then $B = E \ast_M F = (C \ast_N G) \ast_M F = C \ast_N (G \ast_{M/N} F)$ by [Lemma 7.1](#), but we also know $B = C \ast_N D$. By [Corollary 6.4](#) we conclude that $D = G \ast_{M/N} F$, as desired. \(\square\)

The important implication of these lemmas is that every supercharacter theory of a group $G$ can be factored uniquely into a $\ast$-product of one or more supercharacter theories that cannot themselves be written as $\ast$-products in a nontrivial way.

**Definition 8.3.** Let $G$ be a group and let $C \in \text{Sup}(G)$. We say $C$ is *decomposable* if $C$ is a $\ast$-product over a proper nontrivial normal subgroup of $G$. We say $C$ is *indecomposable* if it is not decomposable and $|G| > 1$.

**Theorem 8.4.** Let $G$ be a nontrivial group and let $C \in \text{Sup}(G)$. Then there exists a unique chain of normal subgroups $1 = N_0 < N_1 < \cdots < N_r = G$ and unique indecomposable supercharacter theories $D_i \in \text{Sup}_G(N_i/N_{i-1})$ for $i = 1, \ldots, r$ such that

$$C = D_1 \ast D_2 \ast \cdots \ast D_r.$$ 

**Proof.** Among all chains of normal subgroups $1 = N_0 < N_1 < \cdots < N_r = G$ for which there exist (possibly decomposable) supercharacter theories $D_i \in \text{Sup}_G(N_i/N_{i-1})$ such
that $C = D_1 \ast \cdots \ast D_r$, choose one of maximal length $r$. This can be done because $G$ is finite.

Now if $D_j$ were decomposable for some $j \in \{1, \ldots, r\}$, then there would exist a normal subgroup $M$ of $N_j$ such that $N_{j-1} < M < N_j$, with the property that $D_j = E \ast M_{j}/N_{j-1} F$ for some $E \in \text{Sup}(N_j/M)$ and $F \in \text{Sup}(N_j/M)$. Since $D_j$ is $G$-invariant, in fact $M$ would be normal in $G$ and both $E$ and $F$ would be $G$-invariant. Then

$$1 = N_0 < \cdots < N_{j-1} < M < N_j < \cdots < N_r = G$$

would be a chain of length $r + 1$ with $C = D_1 \ast \cdots \ast D_{j-1} \ast E \ast F \ast D_{j+1} \ast \cdots \ast D_r$, contradicting the maximality of our choice. Hence each $D_i$ is indecomposable, and we have proven the existence of a factorization into indecomposable supercharacter theories.

Now to show uniqueness, induct on $|G|$. Suppose there are two chains of normal subgroups $1 = N_0 < \cdots < N_r = G$ and $1 = M_0 < \cdots < M_s = G$ and two sets of indecomposable supercharacter theories $D_i \in \text{Sup}(N_i/N_{i-1})$ for $i = 1, \ldots, r$ and $E_i \in \text{Sup}(M_i/M_{i-1})$ for $i = 1, \ldots, s$ such that

$$C = D_1 \ast \cdots \ast D_r = E_1 \ast \cdots \ast E_s.$$

Suppose $N_1 \neq M_1$; then without loss of generality, by Lemma 8.1 we may assume that $N_1 < M_1$. Then we have

$$\begin{array}{c}
D_1 \\
\hline
1 < N_1 < M_1 \leq G \\
\hline
E_1 \\
\end{array}$$

and by Lemma 8.2 we know $E_1$ must factor over $N_1$, contradicting the indecomposability of $E_1$. Hence $N_1 = M_1$, so

$$C = D_1 \ast_{N_1} (D_2 \ast \cdots \ast D_r) \text{ and } C = E_1 \ast_{N_1} (E_2 \ast \cdots \ast E_s)$$

are identical products over $N_1$. Then Corollary 6.4 implies that $D_1 = E_1$ and that $D_2 \ast \cdots \ast D_r = E_2 \ast \cdots \ast E_s$. By applying the inductive hypothesis to $G/N$, we see both that $r - 1 = s - 1$ and that $N_i/N_1 = M_i/N_1$ and $D_i = E_i$ for all $i \in \{2, \ldots, r\}$. Therefore $r = s$ and $N_i = M_i$ and $D_i = E_i$ for all $i \in \{1, \ldots, r\}$, proving the uniqueness of the factorization. 

\[\square\]

9. The $\triangle$-product

We have investigated the $\ast$-product constructed from a supercharacter theory $C$ of a normal subgroup $M \triangleleft G$ and a supercharacter theory $D$ of the quotient group $G/M$. A similar construction can still be done in the more general situation when $D$ is a supercharacter theory of a quotient of $G$ by a smaller normal subgroup $N$, provided that $C$ and $D$ satisfy certain conditions. We are considering the situation of the following diagram:

$$\begin{array}{c}
\begin{array}{c}
C \\
\hline
1 \leq N \leq M \leq G \\
\hline
D
\end{array}
\end{array}$$
In order to put $C$ and $D$ together to form a supercharacter theory for $G$, we will of course want $N$ to be $C$-normal and $M/N$ to be $D$-normal, but we will also want the “overlap” of the two theories on $M/N$ to be the same; more explicitly, we will require $C^{M/N} = D_{M/N}$.

**Theorem 9.1.** Let $G$ be a group with subgroups $N \leq M \leq G$. Suppose $C \in \text{Sup}_G(M)$ and $D \in \text{Sup}(G/N)$ such that

(a) $N$ is $C$-normal,
(b) $M/N$ is $D$-normal, and
(c) $C^{M/N} = D_{M/N}$.

Then there exists a unique supercharacter theory $E \in \text{Sup}(G)$ such that $E_M = C$ and $E^{G/N} = D$ and every superclass outside $M$ is a union of $N$-cosets.

Using our earlier notation, if $C = (\mathcal{X}, \mathcal{K})$ and $D = (\mathcal{Y}, \mathcal{L})$, then

$$E = \left( \mathcal{Y} \cup \{ X^G : X \in \mathcal{X}, \, X \not\subseteq \text{Irr}(M/N) \}, \mathcal{K} \cup \{ \tilde{L} : L \in \mathcal{L}, \, L \not\subseteq M/N \} \right).$$

**Proof.** For every superclass $L$ of $D$ lying outside $M/N$, take its preimage $\tilde{L}$ in $G$; because $M/N$ is $D$-normal, this gives a partition of $G - M$. To this set add all the superclasses of $C$; since these partition $M$, the resulting set $\mathcal{K} \cup \{ L : L \in \mathcal{L}, \, L \not\subseteq M/N \}$ is a partition of $G$ which we shall call $\mathcal{J}$. Recalling that $|C|$ denotes the number of superclasses of $C$, note that $|\mathcal{J}| = |C| + (|D| - |D_{M/N}|)$.

Now because $N$ is $C$-normal, the subset $\text{Irr}(M/N)$ is a union of parts of $\mathcal{X}$, as discussed in Section 6. Hence $\{ X \in \mathcal{X} : X \not\subseteq \text{Irr}(M/N) \}$ partitions $\text{Irr}(M) - \text{Irr}(M/N)$, so the set $\{ X^G : X \in \mathcal{X}, \, X \not\subseteq \text{Irr}(M/N) \}$ partitions $\text{Irr}(G) - \text{Irr}(G/N)$ since $C$ is $G$-invariant.

Since $\mathcal{Y}$ is a partition of $\text{Irr}(G/N)$, the union $\mathcal{Y} \cup \{ X^G : X \in \mathcal{X}, \, X \not\subseteq \text{Irr}(M/N) \}$ is a partition of $\text{Irr}(G)$; call it $\mathcal{W}$. Note that

$$|\mathcal{W}| = |D| + (|C| - |C^{M/N}|) = |C| + (|D| - |D_{M/N}|) = |\mathcal{J}|.$$

Then to prove that $(\mathcal{W}, \mathcal{J})$ is a supercharacter theory of $G$, it remains only to show that $\sigma_W$ is constant on $J$ for each $W \in \mathcal{W}$ and each $J \in \mathcal{J}$.

Let $W \in \mathcal{W}$. It may be that $W \in \mathcal{Y}$, so that $\sigma_W$ is a supercharacter of $D$. In this case, there are three sorts of sets $J \in \mathcal{J}$ to consider: those that lie within $G - M$, those within $M - N$, and those within $N$. First, the supercharacter $\sigma_W$ of $D$ is constant on each superclass $L$ of $D$ lying outside $M/N$, so $\sigma_W$ (viewed as a character of $G$) is constant on each preimage $\tilde{L}$ in $G - M$. Thus $\sigma_W$ is constant on each set $J \in \mathcal{J}$ that lies in $G - M$. Next note that $\sigma_W$ is constant on the nontrivial superclasses of $D_{M/N} = C^{M/N}$.

Therefore $\sigma_W$ is constant on the preimages of these superclasses, which are exactly the superclasses of $C_N \ast C^{M/N}$ which lie outside $N$. Now $C \preceq C \ast \text{mm}_{N}(G) = C_N \ast C^{M/N}$, so every superclass $K$ of $C$ lying outside of $N$ is contained within a superclass of $C_N \ast C^{M/N}$ outside $N$; thus $\sigma_W$ is constant on that superclass $K$. Hence $\sigma_W$ is constant on every set $J \in \mathcal{J}$ that lies in $M - N$. Finally, $\sigma_W$ has $N$ in its kernel, so it is constant on those sets $J \in \mathcal{J}$ that lie within $N$. Therefore $\sigma_W$ is constant on every member of $\mathcal{J}$, under the supposition that $W \in \mathcal{W}$.

The other possibility is that $W = X^G$ for some part $X \in \mathcal{X}$ not lying in $\text{Irr}(M/N)$. Since $C$ is $G$-invariant, the set $X$ must be a union of $G$-orbits, so $\sigma_W = \sigma_{X^G} = (\sigma_X)^G$ by Lemma 5.2. Then because $M \triangleleft G$, we know that $\sigma_W$ vanishes outside $M$, and hence...
is constant on all parts \( J \in \mathcal{J} \) that lie outside \( M \). On the other hand, each part \( J \in \mathcal{J} \) that lies in \( M \) is a superclass of \( C \), and when \( \sigma_W \) is restricted to \( M \), the character \((\sigma_W)_M = (\sigma_X)^G)_M = [G:M] \sigma_X\) is constant on \( J \) because \( \sigma_X \) is.

Hence \( \sigma_W \) is constant on each part \( J \in \mathcal{J} \) for all parts \( W \in \mathcal{W} \), and we conclude that \((\mathcal{W}, \mathcal{J})\) is a supercharacter theory of \( G \). Let \( \mathcal{E} = (\mathcal{W}, \mathcal{J}) \); we need to show that \( \mathcal{E} \) satisfies the conclusions of the \( \text{Theorem} \). By construction, the superclasses of \( \mathcal{E} \) that lie in \( M \) are the superclasses of \( C \), so \( \mathcal{E} = \mathcal{E}_M \). Likewise the supercharacters of \( \mathcal{E}^{G/N} \) are those supercharacters of \( \mathcal{E} \) that have \( N \) in their kernels, namely the supercharacters of \( D \); hence \( \mathcal{D} = \mathcal{E}^{G/N} \). Third, by construction the superclasses of \( \mathcal{E} \) outside \( M \) are preimages of certain superclasses of \( D \), so they are unions of \( N \)-cosets.

Finally, to show uniqueness, suppose \( \mathcal{F} \in \text{Sup}(G) \) satisfies the conditions that \( \mathcal{F}_M = C \), that \( \mathcal{F}^{G/N} = D \), and that every superclass of \( \mathcal{F} \) outside \( M \) is a union of \( N \)-cosets. Then \( \mathcal{F}_M = C = \mathcal{E}_M \), so \( \mathcal{E} \) has the same superclasses within \( M \) as does \( \mathcal{F} \). Moreover, because the superclasses of \( \mathcal{F} \) outside of \( M \) are unions of \( N \)-cosets, the set

\[
\{\text{superclasses of } \mathcal{F} \text{ outside } M\} = \{\text{superclasses of } \mathcal{F} \text{ outside } M\} = \{\text{superclasses of } \mathcal{F} \text{ outside } M\} = \{\text{superclasses of } \mathcal{E} \text{ outside } M\}.
\]

Therefore \( \mathcal{F} \) has the same superclasses as \( \mathcal{E} \), so \( \mathcal{F} = \mathcal{E} \) as desired. \( \square \)

We may therefore define a \( \triangle \)-product as follows.

**Definition 9.2.** Let \( G \) be a group with subgroups \( N \) and \( M \) such that \( N \leq M \), and suppose \( \mathcal{C} = (\mathcal{X}, \mathcal{K}) \in \text{Sup}_G(M) \) and \( \mathcal{D} = (\mathcal{Y}, \mathcal{L}) \in \text{Sup}(G/N) \). If \( N \) is \( \mathcal{C} \)-normal, if \( M/N \) is \( \mathcal{D} \)-normal, and if \( \mathcal{C}^{M/N} = \mathcal{D}_{M/N} \), then we define their \( \triangle \)-product, written \( \mathcal{C} \triangle \mathcal{D} \), to be the supercharacter theory of \( G \)

\[
(\mathcal{Y} \cup \{X^G : X \in \mathcal{X}, X \not\in \text{Irr}(M/N)\}, \mathcal{K} \cup \{L : L \in \mathcal{L}, L \not\in M/N\}).
\]

We say that \( \mathcal{C} \triangle \mathcal{D} \) is a \( \triangle \)-product over \( N \) and \( M \). If \( 1 < N \) and \( M < G \), then we say that \( \mathcal{C} \triangle \mathcal{D} \) is a nontrivial \( \triangle \)-product.

If \( N = M \), then the condition that \( \mathcal{C}^{M/N} = \mathcal{D}_{M/N} \) holds trivially and \( \mathcal{C} \triangle \mathcal{D} = \mathcal{C} \ast \mathcal{D} \). Thus the \( \triangle \)-product is a generalization of the \( \ast \)-product. An analogue of **Corollary 6.4** holds for \( \triangle \)-products, giving necessary and sufficient conditions for a supercharacter theory to be a \( \triangle \)-product.

**Proposition 9.3.** Let \( G \) be a group, let \( \mathcal{E} \in \text{Sup}(G) \), and let \( N \) and \( M \) be \( \mathcal{E} \)-normal subgroups of \( G \) with \( N \leq M \). Then \( \mathcal{E} \) is a \( \triangle \)-product over \( N \) and \( M \) if and only if every superclass outside \( M \) is a union of \( N \)-cosets. In this case, \( \mathcal{E} = \mathcal{E}_M \triangle \mathcal{E}^{G/N} \).

**Proof.** Suppose \( \mathcal{E} = \mathcal{C} \triangle \mathcal{D} \) over \( N \) and \( M \). Then by definition the superclasses of \( \mathcal{E} \) outside \( M \) are unions of \( N \)-cosets.

So suppose for the converse that every superclass of \( \mathcal{E} \) outside \( M \) is a union of \( N \)-cosets. Let \( \mathcal{C} = \mathcal{E}_M \) and let \( \mathcal{D} = \mathcal{E}^{G/N} \); we want to show that \( \mathcal{E} = \mathcal{C} \triangle \mathcal{D} \), but
first we need to show that this $\triangle$-product is defined. Because $N$ and $M$ are $E$-normal, it follows that $N$ is $C$-normal and $M/N$ is $D$-normal. Let $\varphi : G \rightarrow G/N$ be the quotient homomorphism. Then the set
\[
\{\text{nontrivial superclasses of } D_{M/N}\} = \{\text{nontrivial superclasses of } D \text{ in } M/N\}
\]
\[
= \varphi (\{\text{superclasses of } E_N \ast D \text{ in } M - N\})
\]
\[
= \varphi (\{\text{superclasses of } E_N \ast E^{G/N} \text{ in } M - N\})
\]
\[
= \varphi (\{\text{superclasses of } E \lor \text{mm}_N(G) \text{ in } M - N\})
\]
\[
= \varphi (\{\text{superclasses of } C \lor \text{mm}_N(M) \text{ in } M - N\})
\]
\[
= \varphi (\{\text{superclasses of } C_N \ast C^{M/N} \text{ in } M - N\})
\]
\[
= \{\text{nontrivial superclasses of } C^{M/N}\}.
\]
Therefore $C^{M/N} = D_{M/N}$, so we can form the product $C \triangle D$.

But then $E$ is a supercharacter theory of $G$ with $E_M = C$ and $E^{G/N} = D$ and every superclass outside $M$ a union of $N$-cosets, and [Theorem 9.1] guarantees that there is only one such supercharacter theory, namely $C \triangle D$. So $E = C \triangle D$ as desired. \hfill $\square$

10. Dual supercharacter theories

We conclude this article by restricting our attention to abelian groups and investigating a bijection between $\text{Sup}(G)$ and $\text{Sup}(\text{Irr}(G))$, constructed using the natural isomorphism $\sim : G \rightarrow \text{Irr}(\text{Irr}(G))$. Recall that $\tilde{g}(\chi)$ is defined to be $\chi(g)$ for all $\chi \in \text{Irr}(G)$ and all $g \in G$. If $K$ is a subset of $G$, then we define $\tilde{K}$ to be $\{g : g \in K\}$; likewise if $\mathcal{K}$ is a partition of $G$, we define $\tilde{\mathcal{K}}$ to be $\{\tilde{K} : K \in \mathcal{K}\}$, which is a partition of $\text{Irr}(\text{Irr}(G))$. We shall show that if $(\mathcal{X}, \mathcal{K}) \in \text{Sup}(G)$, then $(\tilde{\mathcal{K}}, \mathcal{X}) \in \text{Sup}(\text{Irr}(G))$. The proof requires us to define a matrix corresponding to a partition of a set.

**Definition 10.1.** Let $S$ be a set of size $n$ and let $\mathcal{R}$ be a partition of $S$ into $k$ parts. Fix an ordering $S = \{s_1, \ldots, s_n\}$ and an ordering $\mathcal{R} = \{R_1, \ldots, R_k\}$. Then the partition matrix of $\mathcal{R}$ is the $k \times n$ matrix $\mathbf{R}$ given by

\[
R_{ij} = \begin{cases} 
1, & \text{if } s_j \in R_i \\
0, & \text{if } s_j \not\in R_i.
\end{cases}
\]

Let $G$ be an abelian group of order $n$ and fix orderings $G = \{g_1, \ldots, g_n\}$ and $\text{Irr}(G) = \{\chi_1, \ldots, \chi_n\}$. Now $\mathbb{C}[G]$ has two different bases, an element basis $\{g_1, \ldots, g_n\}$ and an idempotent basis $\{e_{\chi_1}, \ldots, e_{\chi_n}\}$. Therefore there exists a nonsingular $n \times n$ change-of-basis matrix $\mathbf{T}$ such that if $x$ is a row vector giving the idempotent coordinates of some element $x$ of $\mathbb{C}[G]$, then the row vector $x\mathbf{T}$ gives the element coordinates for $x$.

**Lemma 10.2.** Let $G$ be an abelian group, let $\mathcal{K} \in \text{Part}(G)$, and let $\mathcal{X} \in \text{Part}(\text{Irr}(G))$. Fix orderings of $G$, $\text{Irr}(G)$, $\mathcal{K}$, and $\mathcal{X}$. Let $\mathbf{K}$ and $\mathbf{X}$ be the partition matrices of $\mathcal{K}$ and $\mathcal{X}$, respectively. Let $\mathbf{T}$ be the change of basis matrix from the idempotent coordinates to the element coordinates. Then $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of $G$ if and only if

\[
\text{rowspace}(\mathbf{K}) = \text{rowspace}(\mathbf{X}\mathbf{T})).
\]
Proof. If \((\mathcal{X}, \mathcal{K}) \in \text{Sup}(G)\), then \(\text{span}\{\bar{K} : K \in \mathcal{K}\} = \text{span}\{f_X : X \in \mathcal{X}\}\). On the other hand, if \(\text{span}\{\bar{K} : K \in \mathcal{K}\} = \text{span}\{f_X : X \in \mathcal{X}\}\), then in particular the subspace \(\text{span}\{\bar{K} : K \in \mathcal{K}\}\) is a subalgebra of \(\mathbb{Z}[C[G]]\), so by Proposition 3.2 there is some partition \(\mathcal{Y}\) of \(\text{Irr}(G)\) such that \((\mathcal{Y}, \mathcal{K}) \in \text{Sup}(G)\). But then
\[
\text{span}\{f_Y : Y \in \mathcal{Y}\} = \text{span}\{\bar{K} : K \in \mathcal{K}\} = \text{span}\{f_X : X \in \mathcal{X}\},
\]
so \(\mathcal{X} = \mathcal{Y}\) by Lemma 3.1. Thus \((\mathcal{X}, \mathcal{K})\) is a supercharacter theory of \(G\) if and only if \(\text{span}\{\bar{K} : K \in \mathcal{K}\} = \text{span}\{f_X : X \in \mathcal{X}\}\).

Now the rows of \(X\) are the idempotent coordinates of the members of \(\{f_X : X \in \mathcal{X}\}\), so the rows of \(XT\) are the element coordinates of those same idempotent sums. Likewise the rows of \(K\) are the element coordinates of the members of \(\{\bar{K} : K \in \mathcal{K}\}\). So \(\{f_X : X \in \mathcal{X}\}\) and \(\{\bar{K} : K \in \mathcal{K}\}\) have the same linear span if and only if the matrices \(XT\) and \(K\) have the same row space.

With the aid of this lemma we now can prove that each supercharacter theory of an abelian group \(G\) corresponds to a supercharacter theory of \(\text{Irr}(G)\).

**Theorem 10.3.** Let \(G\) be an abelian group, and let \((\mathcal{X}, \mathcal{K})\) be a supercharacter theory of \(G\). Then \((\mathcal{K}, \mathcal{X}')\) is a supercharacter theory of \(\text{Irr}(G)\).

Proof. Let \(n = |G|\) and \(k = |\mathcal{K}|\). Fix orderings of \(G\), \(\text{Irr}(G)\), \(\mathcal{X}\), and \(\mathcal{K}\), and let \(X\) and \(K\) be the partition matrices corresponding to \(\mathcal{X}\) and \(\mathcal{K}\), respectively. Order \(\bar{G} = \text{Irr}(\text{Irr}(G))\) and \(\bar{K}\) in the natural way by letting \(\bar{g}_i = (g_i)\) and \(\bar{K}_i = (K_i)\); then the partition matrix corresponding to \(\bar{K}\) is also \(K\).

Now for the algebra \(\mathbb{C}[G]\), the matrix \(T\) which changes the basis from idempotent coordinates to element coordinates is the \(n \times n\) matrix whose \(i\)th row consists of the element coordinates of \(e_{\chi_i} = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) g\); hence
\[
T = \left(\frac{1}{|G|} \chi_i(g_j)\right)_{ij} = \frac{1}{|G|} \left(\chi_i(g_j)\right)_{ij} = \frac{1}{|G|} C,
\]
where \(C\) is the character table of \(G\) viewed as a matrix.

On the other hand, the character table of the group \(\text{Irr}(G)\) viewed as a matrix is just \(C^T\). Hence the argument above, when applied to the group algebra \(\mathbb{C}[\text{Irr}(G)]\), shows that the matrix \(S\) which changes coordinates from the idempotent basis \(\{e_{\bar{g}_1}, \ldots, e_{\bar{g}_n}\}\) to the element basis \(\{\chi_1, \ldots, \chi_n\}\) is \(\frac{1}{|G|} C^T\). Since \(C^T = C^{-1}\) by the orthogonality of the irreducible characters, we have \(S = \frac{1}{|G|} C^{-1}\).

Now because \((\mathcal{X}, \mathcal{K}) \in \text{Sup}(G)\), we have \(\text{rowspace}(K) = \text{rowspace}(XT)\) by Lemma 10.2, so \(\text{rowspace}(K) = \text{rowspace}(XCC^{-1})\). Now both \(K\) and \(X\) are real matrices, so taking complex conjugates of both sides gives \(\text{rowspace}(K) = \text{rowspace}(XC)\). If two matrices have the same row space, then right multiplication by any matrix yields two matrices which again have the same row space. Therefore
\[
\text{rowspace}(KC^{-1}) = \text{rowspace}(XCC^{-1}) = \text{rowspace}(X).
\]
Finally, because \(S\) is a scalar multiple of \(C^{-1}\), we have \(\text{rowspace}(KS) = \text{rowspace}(X)\). Then because \(K\) is the partition matrix of \(\bar{K}\), we conclude by Lemma 10.2 that \((\bar{K}, \mathcal{X}')\) is a supercharacter theory of \(\text{Irr}(G)\).
Definition 10.4. Let $G$ be an abelian group and let $C = (\mathcal{X}, \mathcal{K}) \in \text{Sup}(G)$. Then the dual supercharacter theory $\tilde{C}$ is defined to be $(\tilde{\mathcal{K}}, \tilde{\mathcal{X}}) \in \text{Sup}(\text{Irr}(G))$.

The word “dual” is appropriate because $\tilde{\tilde{C}} = (\tilde{\mathcal{X}}, \tilde{\mathcal{K}})$, where $\tilde{\mathcal{X}}$ denotes the image of $\mathcal{X}$ under the natural isomorphism from $\text{Irr}(G)$ to $\text{Irr(Irr(Irr(G))))}$. Thus $\tilde{\tilde{C}}$ is exactly the image of $C$ under the natural isomorphism.

Corollary 10.5. Let $G$ be an abelian group. Then the map $C \mapsto \tilde{C}$ defines a bijection from $\text{Sup}(G)$ to $\text{Sup}(\text{Irr}(G))$.

Proof. Suppose $C, D \in \text{Sup}(G)$ such that $\tilde{\tilde{C}} = \tilde{\tilde{D}}$; then the superclasses of $\tilde{C}$ are the same as those of $\tilde{D}$. Then $C$ and $D$ correspond to the same partition of $\text{Irr}(G)$, so $C = D$. Therefore the map $C \mapsto \tilde{C}$ is injective. But $|\text{Sup}(G)| = |\text{Sup}(\text{Irr}(G))|$ because $G$ and $\text{Irr}(G)$ are isomorphic, so the map must be a bijection. \[\square\]

Let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory of an abelian group $G$. If $X \in \mathcal{X}$ and $K \in \mathcal{K}$, then we know that $\sigma_X(g) = \sum_{\chi \in X} \chi(g)$ is constant for all $g \in K$. Thus in the submatrix of the character table whose rows lie in $X$ and whose columns lie in $K$, the column sums are identical. Theorem 10.3 shows that $\sigma_K(\chi) = \sum_{g \in K} \tilde{g}(\chi) = \sum_{g \in K} \chi(g)$ is constant for all $\chi \in X$, so the row sums of that submatrix are also all identical. Letting $z$ be the sum of all entries in the submatrix, however, we see that each row sum is $z/|X|$ and each column sum is $z/|K|$; thus the row sums will not equal the column sums unless $|K| = |X|$.  

11. Duality of $C$-normal subgroups and $\triangle$-products

We next investigate the behavior of $C$-normal subgroups and $\triangle$-products under the dual map. Let $G$ be an abelian group, and recall that every subgroup of $\text{Irr}(G)$ is of the form $\text{Irr}(G/N)$ for some subgroup $N \leq G$. If $C \in \text{Sup}(G)$, a similar connection holds for $C$-normal subgroups.

Lemma 11.1. Let $G$ be an abelian group, let $N$ be a subgroup of $G$, and let $C \in \text{Sup}(G)$. Then $N$ is $C$-normal if and only if $\text{Irr}(G/N)$ is a $\tilde{C}$-normal subgroup of $\text{Irr}(G)$.

Proof. Write $C = (\mathcal{X}, \mathcal{K})$. Recall from Section that $N$ is $C$-normal if and only if $\text{Irr}(G/N)$ is a union of members of $\mathcal{X}$, which by definition means that $\text{Irr}(G/N)$ is $\tilde{C}$-normal. \[\square\]

Moreover, there is a strong connection between the supercharacter theories of $G$ that arise as $\triangle$-products and the $\triangle$-product supercharacter theories of $\text{Irr}(G)$. We shall show that $C \in \text{Sup}(G)$ is a $\triangle$-product if and only if $\tilde{C}$ is a $\tilde{\triangle}$-product. To compute the factors of $\tilde{C}$ explicitly, we define a new map. Let $G$ be abelian, let $M \leq G$, and let $\theta \in \text{Irr}(M)$. Since $G$ is abelian, the set $\text{Irr}(G/\theta)$ of irreducible characters of $G$ that lie over $\theta$ consists
of all extensions of \( \theta \) to \( G \). By Gallagher’s Theorem \[11\], Corollary 6.17, this set is a coset of \( \text{Irr}(G/M) \) in \( \text{Irr}(G) \). Moreover, the map

\[
\kappa : \text{Irr}(M) \longrightarrow \text{Irr}(G) / \text{Irr}(G/M)
\]

\[
\theta \longrightarrow \text{Irr}(G/\theta)
\]

is an isomorphism of groups. If \( X \) is a subset of \( \text{Irr}(M) \), let \( X^\bullet = \{ \theta^\bullet : \theta \in X \} \); note that the set \( X^G \) is the full preimage of \( X^\bullet \) with respect to the canonical map from \( \text{Irr}(G) \rightarrow \text{Irr}(G)/\text{Irr}(G/M) \). If \( \mathcal{X} \) is a subset of \( \text{Irr}(M) \), let \( \mathcal{X}^\bullet = \{ X^\bullet : X \in \mathcal{X} \} \).

Now if \( \hat{\mathcal{C}} \) is a supercharacter theory of \( \text{Irr}(M) \), then the superclasses of \( \hat{\mathcal{C}} \) partition \( \text{Irr}(M) \), so their images under \( \kappa \) partition \( \text{Irr}(G)/\text{Irr}(G/M) \). Because \( \kappa \) is a group isomorphism, these images are the superclasses of a supercharacter theory of \( \text{Irr}(G)/\text{Irr}(G/M) \).

To find the partition of \( \text{Irr}(\text{Irr}(G)/\text{Irr}(G/M)) \) corresponding to this supercharacter theory, note that \( \text{Irr}(\text{Irr}(G)/\text{Irr}(G/M)) \) is the set of all irreducible characters of \( \text{Irr}(G) \) that have \( \text{Irr}(G/M) \) in the kernel. But \( \bar{g}(\psi) = \psi(g) \) equals 1 for all \( \psi \in \text{Irr}(G/M) \) if and only if \( g \in \bigcap_{\psi \in \text{Irr}(G/M)} \ker \psi = M \). Thus \( \text{Irr}(\text{Irr}(G)/\text{Irr}(G/M)) \) is the image of \( M \) under the map \( \sim : G \rightarrow \text{Irr}(\text{Irr}(G)) \). So if \( \mathcal{K} \) is a partition of \( M \), it follows that \( \mathcal{K} \) is a partition of \( \text{Irr}(\text{Irr}(G)/\text{Irr}(G/M)) \).

We are now ready for the following lemma.

**Lemma 11.2.** Let \( G \) be an abelian group, let \( M \) be a subgroup of \( G \), and let \( \hat{\mathcal{X}} \) be the isomorphism from \( \text{Irr}(M) \) to \( \text{Irr}(G)/\text{Irr}(G/M) \) defined in \[4\]. Let \( \mathcal{C} = (\mathcal{X}, \mathcal{K}) \) be a supercharacter theory of \( M \). Then \( (\hat{\mathcal{K}}, \mathcal{X}^\bullet) \) is a supercharacter theory of \( \text{Irr}(G)/\text{Irr}(G/M) \).

**Proof.** We have seen that \( \mathcal{X}^\bullet \) is a partition of \( \text{Irr}(G)/\text{Irr}(G/M) \) because \( \mathcal{X} \) is a partition of \( \text{Irr}(M) \). The discussion above has also established that \( \hat{\mathcal{K}} \) is a partition of \( \text{Irr}(\text{Irr}(G)/\text{Irr}(G/M)) \) because \( \mathcal{K} \in \text{Part}(M) \). Certainly \( |\hat{\mathcal{K}}| = |\mathcal{K}| = |\mathcal{X}| = |\mathcal{X}^\bullet| \), so it suffices to show that \( \sigma_{\hat{\mathcal{K}}} \) is constant on \( \mathcal{X}^\bullet \) for each part \( K \in \mathcal{K} \) and each part \( X \in \mathcal{X} \).

Let \( K \in \mathcal{K} \) and let \( X \in \mathcal{X} \). For the duration of this proof, let \( \hat{\mathcal{X}} \) denote the natural isomorphism from \( M \) to \( \text{Irr}(\text{Irr}(M)) \); by **Theorem 10.3** \( (\hat{\mathcal{K}}, \hat{\mathcal{X}}) \) is a supercharacter theory of \( \text{Irr}(M) \), so there exists a complex number \( c \) such that \( \sigma_{\hat{\mathcal{K}}} (\theta) = c \) for all characters \( \theta \in X \). Now every member of \( \mathcal{X}^\bullet \) is of the form \( \chi \text{ Irr}(G/M) \) where \( \chi_M \in X \); hence

\[
\sigma_{\hat{\mathcal{K}}} (\chi \text{ Irr}(G/M)) = \sigma_{\hat{\mathcal{K}}} (\chi) = \sum_{m \in \mathcal{K}} \bar{m} (\chi) = \sum_{m \in \mathcal{K}} \chi (m) = \sum_{m \in \mathcal{K}} \chi_M (m) = \sum_{m \in \mathcal{K}} \bar{m} (\chi_M) = \sigma_{\hat{\mathcal{K}}} (\chi_M) = c
\]

is the same for all members \( \chi \text{ Irr}(G/M) \) of \( \mathcal{X}^\bullet \). Thus \( \hat{\mathcal{K}} \) is constant on \( \mathcal{X}^\bullet \), and we conclude that \( (\hat{\mathcal{K}}, \mathcal{X}^\bullet) \) is a supercharacter theory of \( \text{Irr}(G)/\text{Irr}(G/M) \). \( \Box \)

Let us give a name to the supercharacter theory of **Lemma 11.2**.

**Definition 11.3.** Let \( G \) be an abelian group, let \( M \) be a subgroup of \( G \), and let \( \mathcal{C} = (\mathcal{X}, \mathcal{K}) \) be a supercharacter theory of \( M \), so that \( \hat{\mathcal{C}} = (\hat{\mathcal{K}}, \mathcal{X}^\bullet) \in \text{Sup} (\text{Irr}(M)) \). Then \( \hat{\mathcal{C}}^\bullet \) denotes the supercharacter theory \( (\hat{\mathcal{K}}, \mathcal{X}^\bullet) \) of \( \text{Irr}(G)/\text{Irr}(G/M) \).
Using this definition, if a supercharacter theory \( E \) of an abelian group is a \( \triangle \)-product, at last we can write \( \hat{E} \) as a \( \triangle \)-product and give the factors explicitly.

**Proposition 11.4.** Let \( G \) be an abelian group with subgroups \( N \leq M \leq G \), and let \( \hat{\star} \) be the isomorphism from \( \text{Irr}(M) \to \text{Irr}(G)/\text{Irr}(G/M) \) defined in (5). Let \( C \triangle D \) be a \( \triangle \)-product over \( N \) and \( M \). Then

\[
\hat{C} \triangle \hat{D} = \hat{D} \triangle \hat{C}^\star,
\]

where the second \( \triangle \)-product is over \( \text{Irr}(G/M) \) and \( \text{Irr}(G/N) \).

**Proof.** Let \( C = (X,K) \) and \( D = (Y,L) \). The relevant subgroups of \( \text{Irr}(G) \) are

\[
1 \leq \text{Irr}(G/M) \leq \text{Irr}(G/N) \leq \text{Irr}(G),
\]

and the superclasses of \( \hat{C} \triangle \hat{D} \) are the members of the set

\[
\{ X^G : X \in X, X \not\subseteq \text{Irr}(M/N) \}. \tag{6}
\]

Now every superclass outside \( \text{Irr}(G/N) \), being of the form \( X^G \) for some part \( X \in X \), is a union of sets of the form \( \text{Irr}(G|\psi) \) where \( \psi \in \text{Irr}(M) \), and therefore is a union of \( \text{Irr}(G/M) \)-cosets. Then by Proposition 9.3 we know that \( \hat{C} \triangle \hat{D} \) factors over \( \text{Irr}(G/M) \) and \( \text{Irr}(G/N) \) as

\[
\hat{C} \triangle \hat{D} = (\hat{C} \triangle \hat{D})_{\text{Irr}(G/N)} \triangle (\hat{C} \triangle \hat{D})_{\text{Irr}(G)/\text{Irr}(G/M)}. \tag{7}
\]

But as we saw in (6), the superclasses of \( \hat{C} \triangle \hat{D} \) lying in \( \text{Irr}(G/N) \) are the members of \( \mathcal{Y} \), which are the superclasses of \( \hat{D} \). Hence \( (\hat{C} \triangle \hat{D})_{\text{Irr}(G/N)} = \hat{D} \).

On the other hand, let \( \mathcal{Z} \) be the partition of \( \text{Irr}(\text{Irr}(G)) \) corresponding to \( \hat{C} \triangle \hat{D} \); then the members of \( \mathcal{Z} \) are the images of the superclasses of \( C \triangle D \) under the natural isomorphism \( \sim : G \to \text{Irr}(\text{Irr}(G)) \). We saw in the discussion before Lemma 11.2 that \( \text{Irr}(\text{Irr}(G)/\text{Irr}(G/M)) \) is exactly \( \tilde{M} \), so the members of \( \mathcal{Z} \) lying in \( \text{Irr}(\text{Irr}(G)/\text{Irr}(G/M)) \) are the images of those superclasses of \( C \triangle D \) that lie in \( M \), namely the superclasses of \( C \).

Hence the partition of \( \text{Irr}(\text{Irr}(G)/\text{Irr}(G/M)) \) corresponding to \( (\hat{C} \triangle \hat{D})_{\text{Irr}(G)/\text{Irr}(G/M)} \) is exactly \( \tilde{K} \); but by Lemma 11.2 this is also the partition of \( \text{Irr}(\text{Irr}(G)/\text{Irr}(G/M)) \) corresponding to \( \hat{C}^\star \). Therefore \( (\hat{C} \triangle \hat{D})_{\text{Irr}(G)/\text{Irr}(G/M)} = \hat{C}^\star \). We conclude from Eq. (7) that \( \hat{C} \triangle \hat{D} = \hat{D} \triangle \hat{C}^\star \), as desired. \( \square \)

**Corollary 11.5.** Let \( G \) be an abelian group with subgroups \( N \leq M \leq G \). Let \( E \) be a supercharacter theory of \( G \). Then \( E \) is a \( \triangle \)-product over \( N \) and \( M \) if and only if \( \hat{E} \) is a \( \triangle \)-product over \( \text{Irr}(G/M) \) and \( \text{Irr}(G/N) \).
Proof. If \( E \) is a \( \triangle \)-product over \( N \) and \( M \), then Proposition 11.4 implies that \( \tilde{E} \) is a \( \triangle \)-product over \( \text{Irr}(G/M) \) and \( \text{Irr}(G/N) \). So now suppose for the converse that \( \tilde{E} \) is a \( \triangle \)-product over \( \text{Irr}(G/M) \) and \( \text{Irr}(G/N) \). Write \( E = (X, K) \), so that \( \tilde{E} = (\tilde{K}, \tilde{X}) \). Then applying Proposition 11.4 with \( \tilde{E} \) in the place of \( E \), we may conclude that the supercharacter theory \( \tilde{\text{E}} = (\tilde{\tilde{X}}, \tilde{\tilde{K}}) \) of \( \text{Irr}(\text{Irr}(G/M)) \) is a \( \triangle \)-product over \( \text{Irr}(\text{Irr}(G/M))/\text{Irr}(G/M)) = \tilde{N} \) and \( \text{Irr}(\text{Irr}(G/M))/\text{Irr}(G/M)) = \tilde{M} \).

But then the members of \( \tilde{K} \) are the superclasses of a \( \triangle \)-product over \( \tilde{N} \) and \( \tilde{M} \). Since the map \( \sim: G \rightarrow \text{Irr}(\text{Irr}(G)) \) is a group isomorphism, we conclude that \( K \) is the set of superclasses of a \( \triangle \)-product over \( N \) and \( M \). Hence \( E \) is a \( \triangle \)-product over \( N \) and \( M \), completing the proof. \( \blacksquare \)

The most important application of Proposition 11.4 and Corollary 11.5 occurs when \( N = M \), in which case we have the following immediate corollary:

**Corollary 11.6.** Let \( G \) be an abelian group with a subgroup \( M \leq G \), and let \( \star \) be the isomorphism from \( \text{Irr}(M) \) to \( \text{Irr}(G)/\text{Irr}(G/M) \) defined in (3). Let \( E \in \text{Sup}(G) \); then \( E \) is a \( \star \)-product over \( M \) if and only if \( \tilde{E} \) is a \( \star \)-product over \( \text{Irr}(G/M) \). Moreover, if \( C \in \text{Sup}(M) \) and \( D \in \text{Sup}(G/M) \), then

\[
\tilde{C} \star_{M} \tilde{D} = \tilde{D} \star_{\text{Irr}(G/M)} \tilde{C}.
\]

**12. Conclusion**

We have presented five operations for constructing new supercharacter theories out of existing ones: the direct product \( (\times) \), the join operation \( (\lor) \), the \( \star \)-product and its generalization the \( \triangle \)-product, and the dual operation \( \tilde{\cdot} \). In forthcoming papers we shall show that these operators, together with the original supercharacter theory constructions given by Diaconis and Isaacs in [8], suffice to produce all the supercharacter theories of certain infinite families of finite groups, including cyclic \( p \)-groups of odd order and cyclic groups of order \( pq \) and \( pqr \).

**References**

[1] C. A. M. André, Basic characters of the unitriangular group, J. Algebra 175 (1995) 287–319.
[2] C. A. M. André, Irreducible characters of finite algebra groups, in: Matrices and Group Representations (Coimbra, 1998), vol. 19 of Textos Mat. Sér B, 1999, pp. 65–80.
[3] C. A. M. André, Basic characters of the unitriangular group (for arbitrary primes), Proc. Amer. Math. Soc. 130 (2002) 1943–1954.
[4] C. A. M. André, A. M. Neto, Super-characters of finite unipotent groups of types \( B_n, C_n \) and \( D_n \), J. Algebra 305 (2002) 394–429.
[5] C. A. M. André, A. M. Neto, A supercharacter theory for the Sylow \( p \)-subgroups of the finite symplectic and orthogonal groups, preprint (2008). arXiv:0810.3761v2[math.GR]
[6] C. A. M. André, A. M. Neto, Supercharacters of the Sylow \( p \)-subgroups of the finite symplectic and orthogonal groups, preprint (2008). arXiv:0804.4285v1[math.GR]
[7] E. Arias-Castro, P. Diaconis, R. Stanley, A super-class walk on upper-triangular matrices, J. Algebra 278 (2004) 739–765.
[8] P. Diaconis, I. M. Isaacs, Supercharacters and superclasses for algebra groups, Trans. Amer. Math. Soc. 360 (2008) 2359–2392.
[9] P. Diaconis, N. Thiem, Supercharacter formulas for pattern groups, Trans. Amer. Math. Soc., to appear. [arXiv:0810161v1[math.RT]]

[10] G. A. Grätzer, General Lattice Theory, Academic Press, New York, 1978.

[11] I. M. Isaacs, Character Theory of Finite Groups, AMS Chelsea Publishing, Providence, Rhode Island, 2006.

[12] B. A. Otto, Supercharacters of algebra groups, Ph.D. thesis, University of Wisconsin-Madison (2009).

[13] N. Thiem, E. Marberg, Superinduction for pattern groups, J. Algebra, to appear. [arXiv:0712.1228v1[math.RT]]

[14] N. Thiem, V. Venkateswaran, Restricting supercharacters of the finite group of unipotent uppertriangular matrices, Electronic Journal of Combinatorics 16, research paper 23.

[15] N. Yan, Representation theory of the finite unipotent linear groups, Ph.D. thesis, University of Pennsylvania (2001).