Self-Similarity Among Energy Eigenstates

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In a quantum system, different energy eigenstates have different properties or features, allowing us to define a classifier to divide them into different groups. We find that the ratio of each type of energy eigenstate in an energy shell \([E_c - \Delta E/2, E_c + \Delta E/2]\) is invariant with changing width \(\Delta E\) or Planck constant \(\hbar\) as long as the number of eigenstates in the shell is statistically large enough. We give an argument that such self-similarity in energy eigenstates is a general feature for all quantum systems, which is further illustrated numerically with various quantum systems, including circular billiard, double top model, kicked rotor, and Heisenberg XXZ model.

I. INTRODUCTION

Energy eigenvalues and eigenstates are constitutive to the properties of quantum systems. They have already been thoroughly studied in various aspects. For eigenvalues, the well-known results include the Weyl law\cite{1} and its generalization\cite{2–10}, which describes the asymptotic behavior of the number of energy eigenvalues below an increasing energy. The distribution of nearest energy level spacings\cite{11–17} is now widely used to characterize quantum systems: Wigner-Dyson distribution for chaotic systems and Poisson distribution for integrable systems. The degeneracy in eigen-energies and their differences has been shown to be related to ergodicity and mixing in quantum systems\cite{18–20}.

For energy eigenstates, there are also many interesting results. These earliest studies have focused on the correlation and amplitude distribution of a single energy eigenstate\cite{11, 21–27}. This line of studies ultimately leads to a well known hypothesis by Berry: each energy eigenstate has a Wigner function concentrated on the region explored by a typical orbit over infinite times in the semiclassical limit; or, equivalently, each energy eigenstate becomes a minimal invariant ensemble distribution in classical phase space in the semiclassical limit\cite{28}. Recently, there have been studies on the single energy eigenstate in spin systems, which have no well defined semiclassical limit\cite{29–35}.

In this work we focus on a sequence of energy eigenstates in an energy shell \(S(E_c, \Delta E) = [E_c - \Delta E/2, E_c + \Delta E/2]\). As these eigenstates have different physical properties or features, we define a classifier for a given physical property or feature and divide these eigenstates into different groups. We find self-similarity among energy eigenstates for all quantum systems in the following sense: if the ratio of the energy eigenstates having property \(A\) is \(f\) in the energy shell \(S(E_c, \Delta E)\), the ratio is still \(f\) in the sub-shell \(S(E'_c, \Delta E') \subset S(E_c, \Delta E)\) as long as the number of eigenstates in the sub-shell is statistically large enough. The self-similarity is particularly pronounced in the semiclassical limit \(\hbar \to 0\), where the number of eigenstates in a very narrow energy shell is very large.

We first illustrate such self-similarity with a simple model, circular billiard with analytical results and extensive numerical computation. We then give an analysis, arguing that such self-similarity is generic feature for any quantum system that has a well-defined semiclassical limit. We finally illustrate the self-similarity with more examples, which include coupled tops, kicked rotor, and Heisenberg XXZ model. The result for the XXZ model is of particular interest as it shows that the self-similarity exists even in quantum systems that have no well-defined semiclassical limits. In the end, we argue that such a self-similarity offers a good explanation why the microcanonical ensemble in quantum statistical mechanics, which is established on the equal probability hypothesis, works for all quantum systems regardless of their integrability.

II. SELF-SIMILARITY IN ENERGY EIGENSTATES

Before general discussion, we study a simple but illustrative example, a quantum circular billiard\cite{36}, where the self-similarity in its energy eigenstates can be demonstrated convincingly through analysis and extensive numerical calculation.

A. Circular Billiard

For a quantum particle of mass \(M\) moving in a circular billiard of radius \(R_0\), its energy eigenstates can be expressed analytically in terms of the Bessel function \(J_m\),

\[
\psi_{nm}(r, \theta) = \frac{1}{\sqrt{\pi R_0} J_m(k_{nm} R_0)} J_m(k_{nm} r) e^{im\theta},
\]

where \(n\) is the radial quantum number and \(m\) is the angular quantum number with \(k_{nm}\) being determined by
we introduce a classifier, used to characterize the eigenstate $|\psi_{nm}\rangle$ of a classical particle bouncing elastically inside the circular billiard, as one can see clearly from Fig. 1(b), for a classical particle bouncing elastically inside the circular billiard, if its initial angular momentum is $MvR_b$, it never moves inside the circle of radius $R_b$. Since the motion of a classical particle in a billiard is independent of the size of its moment $Mv$, the radius $R_b$ of such a blank region can be regarded as a kind of normalized angular momentum. With this understanding in mind, for an energy eigenstate $|\psi_{nm}\rangle$, we define a normalized angular momentum $\Re_{nm}$ as

$$\Re_{nm} = \frac{\langle \psi_{nm} | L_z | \psi_{nm} \rangle}{\sqrt{2E_{nm}}} = \frac{m}{k_{nm}}. \quad (3)$$

As indicated in Fig.1(c), $\Re_{nm}$ defined in such a way can be regarded as the radius of the “blank region” of $|\psi_{nm}\rangle$.

Our discussion above shows that the radius $\Re_{nm}$ can be used to characterize the eigenstate $|\psi_{nm}\rangle$. To be precise, we introduce a classifier,

$$C(R_b; |\psi_{nm}\rangle) = \begin{cases} 1 & \text{if } 0 < \Re_{nm} < R_b, \\ 0 & \text{if } R_b \leq \Re_{nm} < R_0. \end{cases} \quad (4)$$

It says that if the radius $\Re_{nm}$ of an eigenstate $|\psi_{nm}\rangle$ is smaller than $R_0$ then $C(R_b; |\psi_{nm}\rangle) = 1$; otherwise it is zero. We consider an energy shell $S(E_c, \Delta E) = [E_c - \Delta E/2, E_c + \Delta E/2]$, which is centered at $E_c$ and with a width of $\Delta E$. We are interested in how many eigenstates in the shell $S(E_c, \Delta E)$ have their blank region radii $\Re_{nm} < R_b$. For this purpose, we define a ratio

$$f(R_b; E_c, \Delta E) = \frac{\sum_{\Re_{nm} \in S} C(R_b; |\psi_{nm}\rangle)}{\text{number of eigenstates in } S}. \quad (5)$$

![FIG. 1. Circular billiard. (a) A typical quantum energy eigenfunction $|\psi_{nm} + \psi_{n,-m}\rangle/\sqrt{2}$; (b) a typical classical trajectory; (c) the square of a Bessel function $J_m(k_{nm}r)$ with $R_0$ indicated. In this figure, $n = 7$ and $m = 12$.](image)

![FIG. 2. (a) Ratio $f(R_0/2; E_c, \Delta E)$ as a function of $\Delta E$. (b) Ratio $f(R_0/2; E_c, \Delta E)$ as a function of $R_0/R_0$. In this numerical results, we take $\hbar = 1$ and fix $k_c = 515$ (corresponding $E_c = k_c^2/2$, while changing $\Delta k$ from 0.1 to 5 (corresponding $\Delta E = 2k_c \Delta k$). The narrowest energy window (i.e. $\Delta k = 0.2$) has merely 28 energy eigenstates, thus $\Delta k$ cannot be further narrowed. Here we plot the ratio of energy eigenstates with “blank region radius” less then $R_b$ vs. $R_0$. Except for $\Delta k = 0.1$ and 0.25 (i.e. energy window is too narrow), other curve (with moderate or large energy window) embody a perfect self-similarity. The black line represents the theoretical result Eq. 8.

We have numerically computed the ratio $f(R_b; E_c, \Delta E)$. One set of the results are plotted in Fig. 2(a), which shows how the ratio $f(R_b; E_c, \Delta E)$ changes with $\Delta E$ with $R_0$ fixed at $R_0/2$. It is clear from this figure that the ratio $f$ stays almost constant once the shell width $\Delta E$ is large enough. For this specific example, the figure shows that in a not-too-narrow energy shell, there are always about 61% of the eigenfunctions $|\psi_{nm}\rangle$ whose blank region radius $R_b$ is smaller.
than $R_0/2$. This is self-similarity. Fig. 2(b) shows that this kind of self-similarity exists for all values of $R_0$ not just for $R_b = R_0/2$. In particular, as seen from this figure, the curves $f(R_b; E_c, \Delta E)$ for different widths $\Delta E$ approach a limiting curve when $\Delta E$ increases. Note that the above results are not sensitive to the center $E_c$ of the energy shell as long as it is not too close to the ground state.

For any quantum system, when $\hbar$ becomes smaller, more energy eigenstates enter a given energy shell. In the semi-classical limit $\hbar \to 0$, the energy spectrum becomes continuous.

For any quantum system, when $\hbar$ becomes smaller, more energy eigenstates enter an energy shell $S(E_c, \Delta E)$ with fixed center $E_c$ and width $\Delta E$ (see Fig. 3). For this billiard system, as its eigen-energy $E_{nm} = \hbar^2 k_{nm}/(2M)$, decreasing $\hbar$ reduces the gap between nearest energy levels and is roughly equivalent to enlarge the width of energy shell. (See appendix A for a detail discussion of the relation between decreasing $\hbar$ and increasing $\Delta E$.) Therefore, the self-similarity demonstrated in Fig. 2 implies that at the smaller Planck constant, the fraction of $f(R_b; \Delta E)$ would get little changed. Since the system becomes classical in the limit of $\hbar \to 0$, the limit of the ratio $f(R_b; \Delta E)$ is likely to have a classical interpretation. This is indeed the case as we shall see.

Let us consider the classical circular billiard. For a classical system, the energy shell $S(E_c, \Delta E)$ specifies a volume in its phase space. For a billiard system, as its dynamics is the same for all different energies, we can focus on an isoenergetic surface in the volume. We define another ratio

$$
g(R_b; E_c) = \frac{\int_{r<R_b} dx dy dp_x dp_y}{\pi \int_0^{\pi} dx dy dp_x dp_y}.
$$

6

Here the nominator is the phase-space volume of a constant energy surface with energy $E = p^2/2$. For the trajectories in a classical billiard are the same for different momentum, we take $p = 1$ for simplicity. The denominator is the volume occupied by all the trajectories with blank region radius smaller than $R_b$. For a trajectory starting at point $r = (x, y)$ in the circular billiard, it is completely determined by the direction of its momentum $p$. For this trajectory to enter the circle of radius $R_b$, the direction of its motion must be limited in the angle $\alpha(x, y; R_b) = 2 \arcsin(R_b/r)$, where $r = \sqrt{x^2 + y^2}$, as shown in Fig. 4. As a result, we have

$$
g(R_b) = \frac{\int_{r<R_b} \alpha(x, y; R_b) dx dy}{\pi \int_0^{\pi} dx dy}.
$$

7

Evaluating this integral in the polar coordinate, we obtain

$$
g(R_b) = \frac{2}{\pi} (u \sqrt{1 - u^2} + \arcsin u),
$$

8

where $u = R_b/R_0$. This theoretical result is plotted as a black line in Fig. 2(b), where we see that it agrees very well with our numerical results for $f(R_b; E_c, \Delta E)$.

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region radius or normalized angular momentum $R_b$ is bigger than $r$, the motion is integrable; otherwise, it is chaotic. We find that the number of chaotic energy eigenstates, e.g., the one in the upper-right corner of Fig. 5, in an energy shell is proportional to the volume of chaotic trajectories in the classical phase space. For this system, $R_b$ is an indicator of chaotic motion.

B. General discussion

With the circular billiard, we have found a self-similarity in energy eigenstates that is intimately related to the classical dynamics. This is in fact a general feature that exists in all quantum systems as our analysis below shows.

![Fig. 6](image)

**FIG. 6.** The schematic of a quantum system approaching its semiclassical limit $\hbar \rightarrow 0$ in phase space. The two black curves represent two constant energy surfaces with difference $\Delta E$; the area enclosed by them is the volume of a given energy shell $S(E_c, \Delta E)$. The dark shaded area is a region where the system has a certain physical property $A$. As $\hbar$ decreases towards zero, the Planck cells (squares in the figure) become smaller. As a result, there are more energy eigenstates in the energy shell $S(E_c, \Delta E)$ and more eigenstates have property $A$. However, the proportional of energy eigenstates having property $A$ in the energy shell $S(E_c, \Delta E)$ quickly approach a limit, which is the overlap of the energy shell and the dark area.

We consider a general quantum system, which has a well-defined classical counterpart. Its quantum phase space can be obtained by dividing the classical phase space into Planck cells[38] as shown in Fig. 6. An energy shell $S(E_c, \Delta E)$ in phase space is a volume enclosed by two constant energy surfaces around $E_c$, which are plotted as two black curves in Fig. 6. The dark shaded area in the figure is a collection of all the states (quantum or classical) that have physical property $A$. For the above example of the circular billiard, the property $A$ is the blank region radius smaller than $R_b$. According to quantum mechanics, the number of Planck cells in a phase space volume is the same as the number of energy eigenstates[39]. As a result, the number of energy eigenstates with property $A$ in the energy shell $S(E_c, \Delta E)$ is equal to the number of Planck cells in the overlap region, the dark area between two black curves in Fig. 6. When $\hbar$ decreases, the Planck cells become smaller so that the number of energy eigenstates in different categories, e.g., in an energy shell or having property $A$, increases. However, the ratio between numbers in these different categories will quickly saturate and reach a limit that is set by the ratio between volumes of corresponding categories in the classical phase space. For the circular billiard, the former is $f(R_b; E_c, \Delta E)$ and the latter is $g(R_b; E_c)$.

The above analysis motivates us to generalize the classifier $C$ and related ratio $f$ in Eqs.(4,5). For a given quantum system, a classifier $C$ maps its energy eigenstates into binary value $C(|E\rangle) \in \{0, 1\}$, that is, divides the eigenstates into two groups. Formally, we can write

$$C(\xi; |E\rangle) = \begin{cases} 0 & \mathcal{A}(|E\rangle) < \xi \\ 1 & \mathcal{A}(|E\rangle) \geq \xi \end{cases}, \quad (9)$$

where $\mathcal{A}$ is a function that characterizes physical property $A$ of an eigenfunction $|E\rangle$ and $\xi$ is a certain value. The choice of property $A$ and related function $\mathcal{A}$ depends on the quantum system that is being considered. For the circular billiard, we have chosen the blank region radius $R_b$ and related function $U_{nm}$. For a quantum chaotic system, one possible choice is the effective occupation number $\mathcal{A}(|E\rangle) = (\sum_x |\langle x|E\rangle|^4)^{-1}[40]$, which characterizes how widely a wave function spreads in space. More examples will be given later. Note that one can certainly define a classifier that maps the energy eigenstates into three or more groups. We for simplicity focus on the above definition.

For an energy shell $S(E_c, \Delta E, \hbar) = \{|E\rangle : E \in [E_c - \Delta E/2, E_c + \Delta E/2]\}$, with the above classifier, we define the following ratio

$$f(\varphi, E_c, \Delta E) = \frac{\sum_{|E\rangle \in S} C(|E\rangle)}{\text{number of eigenstates in } S}. \quad (10)$$

Here $\varphi$ is a parameter that controls the number of energy eigenstates in the energy shell $S(E_c, \Delta E)$. When the quantum system has a well-defined classical limit, $\varphi$ is just the Planck constant $\hbar$ or the effective Planck constant. The self-similarity in energy eigenstates means that the ratio $f(\varphi, E_c, \Delta E)$ is independent of the control parameter $\varphi$ and the width $\Delta E$ as long as the number of eigenstates in the shell $S$ is statistically large enough. In particular, one can divide the shell $S$ into many small sub-shells and tune the control parameter $\varphi$ so that each sub-shell contains enough energy eigenstates. In this case, within statistical fluctuations, the ratio $f$ is the same for every sub-shell.

For a quantum system with a well-defined classical counterpart, such self-similarity is rooted in the correspondence between the energy eigenstates and invariant distributions in classical phase space[28, 41]. Without loss of generality, we choose $\varphi = \hbar$ and focus on time-independent systems in the following discussion. By a well-defined classical counterpart, we mean that the quantum dynamics starting with a wave function that is well-localized in phase space follows the classical trajectory in the semiclassical limit $\hbar \rightarrow 0$. In the Planck cell notation[38], such a correspondence can be written as

$$\lim_{\hbar \rightarrow 0} \langle x'| \hat{U}(t) |x \rangle ^2 = \delta_{x', gix}, \quad (11)$$
in which \( \hat{U}(t) = e^{-i\hat{H}t/\hbar} \) is the propagator of time evolution during \( t \) while \( g_t \) is the corresponding classical time evolution by canonical equations\[42\]. The basis \( |x\rangle = |Q, P\rangle \) is the Planck cell basis at a discretized phase space\[38\]. As a result, for an energy eigenstate \( |E\rangle \), we have

\[
\lim_{\hbar \to 0} |\langle x|E \rangle|^2 = \lim_{\hbar \to 0} |\langle x|\hat{U}(t)|E \rangle|^2 = \lim_{\hbar \to 0} |\langle g_t^{-1}x|E \rangle|^2.
\]  

(12)

This shows that at the semiclassical limit, each energy eigenstates becomes a distribution in phase space which is invariant under classical dynamics\[28, 41\].

For a classical system, its isoenergetic surface is usually filled with different invariant distributions, which do not overlap with each other\[43\]. The Poincaré sections in Figs.7\&9 offers some glimpses of such a structure: the invariant distributions represented by the chaotic seas do not overlap with the distributions represented by smooth lines in integral islands. Consider an energy shell \( S \) with a very small width \( \Delta E \) so that each isoenergetic surface within the shell is filled with similar non-overlapping invariant distributions. In this way, with \( \Delta E \) one can legitimately say that an invariant distribution occupies a volume in phase space. Due to the quantum-classical correspondence discussed above, these different invariant distributions are the limits of different energy eigenstates when \( h \) goes to zero. As the energy is continuous in classical mechanics, the energy shell width \( \Delta E \) can be arbitrary small. And for a given width \( \Delta E \), no matter how small it is, we can always choose a small enough \( h \) so that there are large number of eigenstates in the shell \( S \), in which the number of each type of eigenstates is proportional to the volume of the corresponding invariant distribution. This gives rise to the self-similarity that we have found in eigenstates.

Our above analysis has been done with quantum systems that have well-defined classical limits. However, such self-similarity appears very general and exists in all quantum systems. This is indicated by our numerical computation in the next section, where a model of Heisenberg spin chain is studied. This system has no well-defined classical limit, and we still find self-similarity in its eigenstates. It is not clear why self-similarity exists in such quantum systems.

Before we present more examples, we use an analogy to summarize our finding. For a quantum system, if we regard each of its energy eigenstates as a small ball, then all the eigenstates lie on a one-dimensional line in the order of their corresponding eigen-energies. Such a line has at least one end, which is the ground state. Suppose that a fraction \( f \) of these balls are red, representing that the corresponding eigenstates have property \( A \). And for this red balls that are mixed with other balls. As a result, for any segment of line that contains large number of balls, the fraction of red balls on this segment is exactly \( f \) if we ignore the statistical fluctuations.

In quantum chaotic systems with no apparent symmetries, degeneracy rarely happens. People often refer to it as energy level repulsion. Our finding can also be regarded as a repulsion phenomenon: energy eigenstates with similarity properties tend to “repel” each other and scatter rather evenly among other eigenstates. And this kind of repulsion exists for all quantum systems, not limited to chaotic systems.

### III. EXAMPLES EXHIBITING SELF-SIMILARITY

Below are three examples. The first example, quantum coupled top, is a time-independent system; the second example, quantum kicked rotor, is a periodically-driven system; the third example, a Heisenberg chain, is a quantum system that has no well defined classical limit. Self-similarity in energy eigenstates is evident in all of them. The third examples suggests that the self-similarity exists also in quantum systems that have no well defined classical limits.

#### A. Quantum Coupled Top

The quantum coupled top is a famous model that were used to study quantum chaos\[44, 45\]. It describes the interaction between two identical angular momentum \( \hat{L}_1 \) and \( \hat{L}_2 \), which is governed by the following Hamiltonian\[46, 47\]

\[
\hat{H} = \hat{L}_{1z} + \hat{L}_{2z} + \frac{\mu}{J} \hat{L}_{1x} \hat{L}_{2x},
\]  

(13)

where \( \mu \) is the magnitude of the angular momentum, and \( \mu \) denotes the coupling constant.

This system has a well defined classical counterpart, whose Hamiltonian is obtained by simply replacing the operators \( \hat{L}_1 \) and \( \hat{L}_2 \) with two variables of angular momentum \( L_1 \) and \( L_2 \). For the classical model, it is convenient to introduce a different set of canonical variables

\[
L_i = J(\sqrt{1 - P_i^2} \cos Q_i, \sqrt{1 - P_i^2} \sin Q_i, P_i),
\]  

(14)

where \( i \) is either 1 or 2. The classical Hamiltonian becomes

\[
H = J(\sqrt{1 - P_i^2} \cos Q_i + \sqrt{1 - P_j^2} \cos Q_j + \mu P_i P_j).
\]  

(15)

(rigorous derivation of quantum-classical correspondence is illustrated in appendix B) The classical dynamics is described by the following canonical equations:

\[
\dot{P}_i = -\sqrt{1 - P_i^2} \sin Q_i, \quad \dot{Q}_i = \frac{P_i}{\sqrt{1 - P_i^2}} \cos Q_i + \mu P_j,
\]  

(16)

where \( i, j \in \{1, 2\} \) and \( i \neq j \). The classical dynamics is a mixture of regular and chaotic motions as indicated by the classical Poincaré section at \( Q_1 = \pi \) with \( \sin Q_2 > 0 \) in Fig. 7(a). The parameters for the figure are \( J = 2, \mu = 0.5 \) and the energy \( E_i = -0.9 \).
and the uncertainties of lar momentum of state also have excellent classical meaning: the average angular momentum quantum number \(j\) will neglect dependent on angular momentum quantum number \(j\). The subscript \(j\) in the four dimensional phase space, we plot a section, \(J\) fixed to \(\frac{1}{2}\). Limited by computational resources, we take \(4\) Plank cells, i.e. \(mL^2\). Here \(L = 2m + 1\) and \(L^2 = 2j + 1\). Limited by computational resources, we take \(L = 10\). For each Plank cell, we assign a localized quantum state:\[|Q_1, P_1; Q_2, P_2\rangle = |Q_1, P_1\rangle \otimes |Q_2, P_2\rangle,\] where \(|Q, P\rangle\) is defined as \[|Q, P\rangle = \frac{1}{\sqrt{L}} \sum_{n=-m}^{m} e^{-iQ(n + P_j)n + P_j}.\]

The subscript \(j\) means that Planck cells \(|Q, P\rangle\) are dependent on angular momentum quantum number \(j\). We will neglect \(j\) afterwards for simplicity. These quantum states not only form a complete basis for the system but also have excellent classical meaning: the average angular momentum of state \(|Q, P\rangle\) is just the same as Eq. 14 and the uncertainties of \(J_x, J_y, J_z\) are all proportional to \(h\sqrt{j}\), which decreases as \(1/\sqrt{\hbar}\) with decreasing \(\hbar\) and fixed \(J\); see appendix C.

As it is impossible to plot a complete energy eigenstate in the four dimensional phase space, we plot a section, which we call quantum Poincaré section. In our calculation, for an energy eigenstate \(|E\rangle\), we first set the section at \(Q_1 = \pi\) and then compute the projection amplitude, \(\rho_{Q,sec}(P_1, P_2; E) = |\langle \pi, P_1; Q_2, P_2|E\rangle|^2\), where

\[Q_2 = \arccos \left(\frac{\langle H - J\rangle - \mu P_1 P_2 - \sqrt{1 - P_1^2} \cos Q_1}{\sqrt{1 - P_2^2}}\right).\]

The results \(\rho_{Q,sec}\) for five different eigenstates are shown in Figs. 7(b-f). The energy eigenvalues are chosen around \(E = -0.9\), the energy for the classical Poincaré section in Fig. 7(a). It is clear that each quantum Poincaré section resembles a part of the classical Poincaré section. For example, Fig. 7(d) resembles integrable islands located at the upper right and lower left corners of Fig. 7(a); Fig. 7(c) corresponds to the whole chaotic sea in Fig. 7(a); Fig. 7(e) resembles two integrable island located at the upper right and lower left corners of Fig. 7(a). The most interesting is that the five quantum Poincaré sections combined just fill up the classical Poincaré section. This feature is general in our numerical results: for any classical Poincaré section at a given energy \(E_c\), we can always find energy eigenstates around \(E_c\) whose quantum Poincaré sections just fill up the classical Poincaré section. This feature is a signature of self-similarity in eigenstates.

We next put the above observation on a quantitative ground by focusing on how wide the eigenstates spread in phase space. For an energy eigenstate \(|E\rangle\) and its distribution \(\rho_{Q,sec}\), we define the following variance

\[\text{Var}(P^2) = \frac{\sum P_1 \sum P_2 \rho_{Q,sec}(P)(P - \bar{P})^2}{\sum P_1 \sum P_2 \rho_{Q,sec}(P)},\] (20)

where we abbreviate \((P_1, P_2)\) as a vector \(P\) and denote \(\bar{P}\) as the mean of \(P\) for the left half of Poincaré section. If \(|E\rangle\) is chaotic, its \(\text{Var}(P^2)\) is large, while \(\text{Var}(P^2)\) will be small if it is regular. The classifier is defined as

\[C(\delta; |E\rangle) = \begin{cases} 0 & \text{if } 0 < \text{Var}(P^2) < \delta \\ 1 & \text{if } \text{Var}(P^2) \geq \delta \end{cases},\] (21)

where \(\delta\) is a chosen threshold. For this classifier, the control parameter is \(\Delta E\), the width of the energy shell. Our numerical results are summarized in Fig.8. The first three panels (a,b,c) are the histograms of the variance for different widths \(\Delta E\). They are very similar to each other. In the last panel (d), the ratio function \(f(E_c, \Delta E)\) in Eq. 10 is plotted, and it saturates quickly with \(\Delta E\). They all demonstrate the self-similarity in eigenstates.

For this model, the last numerical check is done by counting the numbers of integrable and chaotic eigenstates. We compute all the eigenstates in the energy shell \([-0.95, -0.85]\), then examine each of them to see whether it is an integrable or chaotic eigenstate. We find that there are \(N_I = 122\) integrable eigenstates and \(N_C = 59\) chaotic ones. The volume of integrable islands and the total volume of the constant energy surface at
$E_c = -0.9$ in classical phase space are evaluated numerically. Our result is $V_f / V = 0.60 \pm 0.02$, close to $N_f / (N_f + N_c) = 0.68$, confirming the self-similarity. For details, see appendix D.

### B. Quantum Kicked Rotor

The Hamiltonian of kicked rotor is [48]

$$H = \frac{p^2}{2} + K \cos q \sum_{n=\infty}^{\infty} \delta(t - n),$$

where $K$ is the kicking strength. Its classical dynamics can be reduced to the Chirikov map on the toric phase space $(q, p) \in [0, 2\pi)^{\otimes 2}$ by focusing the state before each kick, i.e., $(q_n, p_n) = (q, p)(t = n - 0)$. The dynamics reads [48]

$$\begin{cases} 
q_{n+1} = q_n + p_{n+1} \mod 2\pi, \\
p_{n+1} = p_n + K \sin q_n \mod 2\pi.
\end{cases}$$

Parameter $K$ controls the dynamics of kicked rotor. For $K \geq K_c \approx 0.972$ [48], the phase space is roughly separated into chaotic sea and integrable islands. See Fig. 9 for the case of $K = 1.1$.

As there is a periodic kicking, the quantum dynamics of the kicked rotor is given by the following unitary Floquet evolution

$$\hat{U} = e^{-i\hat{p}^2/2\hbar} e^{-iK \cos \hat{q}/\hbar}.$$  

The analogous to energy eigenstates and eigenvalues here are the eigenstates of $\hat{U}$ (Floquet states) and their pseudo-energies [49–51]. For the continuity of notation, we still denote them as $|E\rangle$ and $E$, i.e., $\hat{U} |E\rangle = e^{-iE} |E\rangle$.

We point out that there still holds the correspondence between Floquet states and the classical invariant distributions as we argued in Sec. II. This is illustrated in Fig. 10 with the same setup as in Ref. [38, 40, 52]. The phase space is discretized into $m \times m$ Planck cells and the effective Planck constant is $\hbar_{\text{eff}} \approx 0.0039$, i.e., there are $40 \times 40$ Planck cells in total.

![Diagram](image_url)

FIG. 8. Histograms the variance $\text{Var}(P^2)$ at three different widths (a) $\Delta E = 0.2$, (b) $\Delta E = 0.1$, (c) $\Delta E = 0.04$. (d) The ratio function $f(E_c, \Delta E)$ with the threshold $\delta = 0.16$. The energy shells are all centered at $E_c = -0.9$.

![Diagram](image_url)

FIG. 9. The Poincaré section of the classical kicked rotor. The initial points are uniformly sampled along $p$-axis with fixed $q$ near $\pi$.

![Diagram](image_url)

FIG. 10. The typical chaotic(a) and integrable(c) trajectories of classical kicked rotor and their corresponding quantum Floquet states (b), (d). The parameter $K = 1.1$ and the effective Planck constant is $\hbar_{\text{eff}} \approx 0.0039$, i.e., there are $40 \times 40$ Planck cells in total.
$h_{\text{eff}}$ is shown in Fig. 11(a) with the classifier

$$C_{\text{width}}(|E|) = \begin{cases} 1 & W(|E|) > 0.8\pi \\ 0 & \text{otherwise} \end{cases}.$$  \hfill (25)

Here $W(|E|)$ is the width of Floquet state in phase space defined with Planck cell basis $|Q, P\rangle$ \cite{38}, i.e.,

$$W(|E|)^2 = \sum_{Q,P} \left[ (Q - \pi)^2 + (P - \pi)^2 \right] |\langle Q, P|E\rangle|^2.$$  \hfill (26)

Geometrically, $W(|E|)$ quantifies the “radius” of Floquet states to the center point $(q,p) = (\pi,\pi)$. With the same classifier, the self-similarity with pseudo-energy shell width $\Delta E$ as the control parameter is shown in Fig. 11(b), where the ratio $f(E_c, \Delta E)$ quickly saturates to a similar value as in Fig. 11(a). This demonstrates that decreasing $h_{\text{eff}}$ fixing $\Delta E$ is equivalent to increasing $\Delta E$ with fixed $h_{\text{eff}}$ as discussed in Appendix A.

Although the agreement between the two saturation values in Fig. 11(a) and Fig. 11(b) vindicates our argument in Appendix A, we note that our analysis in the Appendix only works at the vicinity of $\hbar \sim 0$ and for small $\Delta E$, and the agreement at such large $\Delta E$ is still surprising. This is related to that the pseudo-energies are confined into $[0, 2\pi]$ due to the periodic kicking. This means that the different eigen-energies without kicking are folded into this finite interval. Thus the kicked rotor has the same dynamical feature at pseudo-energy shells centered at different pseudo-energies, similar to the billiard systems where the dynamics is also not sensitive to the energy. This is supported by the results in Fig. 11(c), where the number of eigenstates in the energy shell grows linearly with the shell width. This can be further demonstrated by that the entropy of the renormalized distributions of Floquet states in given pseudo-energy shells is roughly independent of the center of the shell. Formally, we compute

$$p_{Q,P}(E_c, \Delta E) = \frac{\sum_{E' \in S(E_c, \Delta E)} |\langle Q, P|E'\rangle|^2}{|S(E_c, \Delta E)|},$$  \hfill (27)

which is actually the probability distribution $p_{Q,P} = \text{Tr}[|Q, P\rangle \langle Q, P|\rho(E_c, \Delta E)]$ of maximally mixed state in the energy shell $\rho(E_c, \Delta E) \propto \sum_{E' \in S(E_c, \Delta E)} |E'\rangle \langle E'|$. We then compute its generalized Wigner-von Neumann entropy (GWvNE)\cite{53} on Planck cell basis (with normalization)

$$\Gamma(E_c, \Delta E) = -\frac{1}{\log m^2} \sum_{Q,P} p_{Q,P}(E_c, \Delta E) \log p_{Q,P}(E_c, \Delta E).$$  \hfill (28)

The function log is the natural logarithm. By normalization, we make sure $\Gamma \in [0, 1]$. In Fig. 11(d), that the entropy is almost constant as the center energy $E_c$ changes demonstrates the independence of dynamics to $E_c$ in the kicked rotor.

**FIG. 11.** The self-similarity with quantum kicked rotor. (a) The ratio $f(E_c, \Delta E)$ as a function of the inverse of the effective Planck constant with $\Delta E = 2$ and $E_c = 0$. (b) The ratio as the function of the width of energy shell $\Delta E$ with $h_{\text{eff}} \approx 0.0039$ and $E_c = 0$. (c) The number of Floquet states as a function of the width of energy shell $\Delta E$. (d) The dependence of the generalized Wigner-von Neumann entropy $\Gamma(E_c, \Delta E)$ on the center pseudo-energy $E_c$. The pseudo-energy shell width $\Delta E = 0.4$ here.

**C. Heisenberg XXZ Model**

Though our explanation of self-similarity relies on the quantum-classical correspondence, we find that it exists for quantum systems that do not have well defined classical limits. Here is an example, one-dimensional Heisenberg XXZ model\cite{54}, whose Hamiltonian with open boundary condition is

$$H = \sum_{i=1}^{N-1} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \rho \sigma_i^z \sigma_{i+1}^z),$$  \hfill (29)

where $\sigma_i^{x,y,z}$ are Pauli operators on the $i$-th site. We use five different functions $F(|\psi_n\rangle)$ to characterize the energy eigenstates $|\psi_n\rangle$. The first four are expectations of different operators $O$, $F(O, |\psi_n\rangle) = \langle \psi_n |O|\psi_n\rangle$. These four different operators are: (i) single spin, here we choose $\sigma_i^{[N/2]}$; (ii) total magnetization $M_z = \sum_{i=1}^{N} \sigma_i^z$; (iii) two-point correlation $C_{34} = \sigma_3^x \sigma_4^x$; (iv) global-X-correlation product $N_x = \prod_{i=1}^{N} \sigma_i^x$. The fifth function the second-Rényi entropy\cite{55} of the first spin, i.e.,

$$F(|\psi_n\rangle) = -\log[H^2] = -\log[\text{Tr}(\rho_1)],$$  \hfill (30)

where $\rho_1 = \text{Tr}_{2,...,N} |\psi_n\rangle \langle \psi_n|$ is the reduced density matrix for the first spin. For each of these five different functions, the classifier is defined as

$$C(|\psi_n\rangle; \xi) = \begin{cases} 1 & F(|\psi_n\rangle) \geq \xi, \\ 0 & \text{otherwise}, \end{cases}$$  \hfill (31)

where $\xi$ is a chosen threshold value.
All calculations are done by exact diagonalization and the results are shown in Fig. 12. For all the local operators \[56\], the total magnetization, single spin and two-point-correlation operator, the ratio functions \( f(\Delta E) \) quickly saturate as \( \Delta E \), indicating the self-similarity. The result for the the second-Rényi entropy is similar. However, for the global-X-correlation, the ratio does not saturate at a constant value even if there are hundreds of energy eigenstates in energy shell \([E_c - \Delta E/2, E_c + \Delta E/2]\). We take the length of spin chain to be \( N = 12 \), parameter \( \rho = 0.4 \) and central energy \( E_c = 4 \).

The three quantum systems that we studied in Sec. III have a common feature: their Hilbert spaces have finite dimensions. When we enlarge their Hilbert spaces, for example, by increasing \( j \) (or \( N \)) or adding more spins, each of these quantum systems will have more eigenstates. The self-similarity suggests that despite that the Hilbert space is getting bigger, the composition of eigenstates will remain roughly the same. It is analogue to shuffled cards: when you insert one deck of cards randomly into, one hundred decks of thoroughly shuffled cards, nothing changes significantly besides there are more cards. One may use this to define the classical limit of a quantum system, in particular, the quantum system that has no well defined classical counterpart: when the composition of different types of eigenstates no longer changes with the dimension of the Hilbert space, the quantum system reaches its classical limit.

**IV. CONCLUSION AND DISCUSSION**

We have discovered self-similarity among energy eigenstates of a quantum system. If all the eigenstates are ordered according to their corresponding eigenvalues and are divided into different groups according to their properties or features, then the members of the same group tend to “repel” each other and scatter rather evenly among all the eigenstates. In other words, inside any energy shell \( S(E_c, \Delta E) \), the composition of eigenstates from different groups is very similar. Just like a shoreline, any piece looks similar to the other piece as long as the piece is not too short. We have offered an explanation why this self-similarity should exist in quantum systems that have well-defined classical limits. Our numerical results show that this self-similarity also exists in quantum systems that have no well-defined classical limits.

Modern statistical mechanics has a very basic hypothesis, the postulate of equal a priori probability\[57\]. It is regarded as a working hypothesis, that is, its justification comes not from that it is derived from the fundamental microscopic theory but from the fact that the conclusions derived from it agree with experiments\[57\]. The self-similarity among energy eigenstates offers a reasonable justification for this hypothesis from a microscopic perspective. According to this postulate, for a quantum system, the micro-canonical ensemble average of any physical observable \( \hat{O}_{\text{phy}} \) should be \( \langle \hat{O}_{\text{phy}} \rangle_{\Delta E} = \text{Tr}[\hat{O}_{\text{phy}} \rho_{\Delta E}] \).
with

$$\hat{\rho}_{\Delta E} = \frac{1}{\Omega(E_c, \Delta E)} \left[ \sum_{E \in S} |E\rangle \langle E| \right], \quad (32)$$

where $\Omega(E_c, \Delta E)$ is the total number of energy eigenstates in the energy shell $S(E_c, \Delta E)$. Let us project every eigenstate $|E\rangle$ into the phase space, then the probability of the system at the Planck cell $|Q, P\rangle$ is

$$\text{Prob}(Q, P) = \frac{1}{\Omega(E_c, \Delta E)} \left[ \sum_{E \in S} |(Q, P|E\rangle|^2 \right], \quad (33)$$

where $Q, P$ are multi-dimensional variables. The self-similarity implies that this probability is zero outside of the shell $S(E_c, \Delta E)$ and the same for every Planck cell in the shell. This holds for any quantum system, integrable or chaotic. For a fully chaotic system, every eigenstate $|E\rangle$ inside the shell looks about the same; consequently, the density matrix (32) is still valid even as the width $\Delta E$ is so small that there is only one eigenstate in the shell. This is just the well known eigenstate thermalization hypothesis[58, 59]. For systems that are not fully chaotic, the width $\Delta E$ has to be large enough so that different types of eigenstates are properly represented statistically. Overall, this explains why there is no need to consider the integrability of the system when the micro-canonical density matrix (32) is used.

V. ACKNOWLEDGEMENT

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Appendix A: Relation Between Decreasing Planck Constant and Enlarging Energy Window

When discussing the circular billiard, we have used the fact that the decreasing of \( h \) is equivalent to simply increasing the shell width \( \Delta E \) with fixed \( E_c \). This is clearly right for the case of billiards; for a general case, it needs some more clarification.

Let us discuss the general case and the formal definition of the eigenstates in the energy shell here. Consider a general quantum system near the limit \( h \to 0 \), the energy levels in the interval of \( S(E_c, \Delta E) \) could be expressed by a quantum number relative to \( E_c \), i.e., by letting \( E_c = E_n(h) \), then \( E_n(h) = E_{n_c} + (n - n_c)\hbar \omega \approx E_c + (n - n_c)^2 \hbar^2 \omega \approx E_c + \hbar^2 G_{n-n_c}(E_c) \). Here \( \alpha > 0 \) and \( G_n \) is some function independent of \( h \). We note that the nearest energy spacing in a fixed energy shell \( S(E_c, \Delta E) \) should vanish as \( h \to 0 \), thus we always have the factor of \( \hbar^2 \). Here are two examples. For a one-dimensional billiard, we have \( E_n \propto \hbar^2 n^2 \Rightarrow E_n \approx E_c + 2E_c\hbar^2(n - n_c) \) with \( \alpha = 2 \). And for systems similar to hydrogen atoms, we have \( E_n \propto 1/\hbar^2 n^2 \), and thus \( E_n \approx E_c - 2(n - n_c)\hbar^2/E_c^{3/2} \), which means \( \alpha = 1 \).

With the above consideration, the collection of eigen-
states within a given energy shell can then be defined as
\[
\Upsilon(E_c, \Delta E, \hbar) = \{ \psi_{n,h} : E_n(h) \in S(E_c, \Delta E) \} = \{ \psi_{n,h} : \hbar^\alpha G_n \in S(0, \Delta E) \} \tag{A1}
\]
With \( \hbar \) decreasing by a factor of \( w > 1 \), we have \( \Upsilon(E_c, \Delta E, \hbar/w) = \{ \psi_{n,h/w}, \hbar^\alpha G_n/w^\alpha \in S(0, \Delta E) \} = \{ \psi_{n,h/w}, \hbar^\alpha G_n \in S(0, \hbar^\alpha \Delta E) \} \). If the physics of eigenstates of fixed level is not sensitive to \( \hbar \) (this is true for quantum systems who have clear classical counterparts, since eigenstates become classical invariant distributions in phase space as \( \hbar \to 0 \)), we have the asymptotic behavior for the collection of eigenstates,
\[
\Upsilon(E_c, \Delta E, \hbar/w) = \Upsilon(E_c, \hbar^\alpha \Delta E, \hbar). \tag{A2}
\]
This means that changing \( \hbar \) and changing \( \Delta E \) at the opposite directions are the same on the population of eigenstates, so does it on any statistical properties of eigenstates, e.g., the self-similarity.

This equivalence of increasing \( \Delta E \) and decreasing \( \hbar \) relies on two assumptions. (1) The expansion of \( E_n + (n - n_c) \approx E_c + \hbar^\alpha G_n \) near \( E_c \) has a high precision; (2) the eigenstates are not sensitive to \( \hbar \). For billiards, both of these two assumptions are satisfied exactly. The energy levels has a natural cut-off at \( \alpha = 2 \) while the eigenfunction is the actual solution of \( (\nabla^2 + k_{nm}^2)\psi_{nm} = 0 \) which is independent of \( \hbar \). For other systems discussed in the main text, these two assumptions are not exactly satisfied. For the double-top and kicked rotor, the equivalence still works well as their classical counterparts are well-defined. For the XXZ model, we expect that there is difference between enlarging \( \Delta E \) and reducing \( \hbar \).

We also note that the center of energy shell is the same on both sides of Eq. (A2). This holds when \( E_c = E_{n_c} \) has a solution. In general, we can only choose \( n_c \) as the closest quantum energy level near \( E_c \) and this would make the right hand side of Eq. (A2) has a different \( E_c \). However, we expect when \( \hbar \) goes to zero, this shifting would vanish since the energy levels become sufficient dense and \( E_c = E_{n_c} \) would have a solution of arbitrarily high precision.

**Appendix B: Classical Correspondence of Quantum Coupled Top**

We start with Hamiltonian (13), and use path integral method to rigorously derive the system’s classical counterpart. We first review some basics of coherent states of spin-\( J \), which are defined as
\[
|\mu\rangle = \frac{1}{(1 + |\mu|^2)^J} \sum_{n=0}^{2J} \sqrt{\frac{(2J)!}{(2J-n)!n!}} \mu^n |L_z = J - n\rangle, \tag{B1}
\]
where \( \mu = e^{i\beta} \tan(\theta/2) \). We have following properties for spin coherent states,
\[
\langle \mu' | \mu \rangle = \frac{(1 + \mu'^* \mu)^2J}{(1 + |\mu'|^2)^J(1 + |\mu|^2)^J}, \tag{B2}
\]
\[
\langle \mu' | \hat{L}_+ | \mu \rangle = \frac{2J\mu}{1 + \mu'^* \mu} \langle \mu' | \mu \rangle, \tag{B3}
\]
\[
\langle \mu' | \hat{L}_z | \mu \rangle = \frac{1 - \mu'^* \mu}{1 + \mu'^* \mu} \langle \mu' | \mu \rangle, \tag{B4}
\]
\[
1 = \int \frac{2J + 1}{4\pi} d\Omega(\mu) \langle \mu | \mu \rangle. \tag{B5}
\]
We want to evaluate the propagator \( \langle \mu(t')|e^{-iHT}|\mu(t)\rangle \) with \( T = t' - t \). Here \( \mu \equiv (\mu_1, \mu_2) \), \( |\mu\rangle \equiv |\mu_1\rangle \otimes |\mu_2\rangle \), which corresponds a double-spin coherent state. We divide the time interval \( T \) into \( N + 1 \) equal pieces of duration \( \delta t = T/(N + 1) \), and denote \( \mu_i = |\mu(t)\rangle + i\delta t \). After inserting \( N \) coherent states’ closure-relations (B5), we get
\[
\langle \mu(t)|e^{-iHT}|\mu(t)\rangle
= \prod_{i=1}^{N} \left( \frac{2J + 1}{4\pi} d\Omega_{1,i} d\Omega_{2,i} \langle \mu_{i+1} | e^{-i\delta t} | \mu_i \rangle \right)
= D[\mu] \prod_{i=1}^{N} \langle \mu_{i+1} | \mu_i \rangle \left[ 1 - i\delta t \left( -\frac{2J Re \mu_{1,i}}{1 + \mu_{1,i+1} \mu_{1,i}} \right) \right]
\tag{B6}
\]
Here \( D[\mu] \equiv \prod_{i=1}^{N} \left( \frac{2J + 1}{4\pi} \right)^2 d\Omega_{1,i} d\Omega_{2,i} \) is the functional integration measure, and \( d\Omega_{1,i} (d\Omega_{2,i}) \) is the solid angle measure of the first (second) top at lattice \( \mu_i \). The discrete form of action immediately reads
\[
S[\mu, \mu'] = \sum_{i=1}^{N} -i \log \prod_{\alpha=1,2} \frac{1 + \mu_{\alpha,i+1}^* \mu_{\alpha,i}}{\sqrt{(1 + |\mu_{\alpha,i}^*|^2)(1 + |\mu_{\alpha,i}|^2)}} + \delta t \left( \frac{Re \mu_{1,i}}{1 + \mu_{1,i+1} \mu_{1,i}} + \frac{Re \mu_{2,i}}{1 + \mu_{2,i+1} \mu_{2,i}} + \frac{\mu_1 - \mu_{1,i+1} \mu_{1,i}}{2} \frac{\mu_{2,i+1} \mu_{2,i}}{1 + \mu_{2,i+1} \mu_{2,i}} \right). \tag{B7}
\]
In the continuous limit, substitute \( \mu_{i+1} = \mu_i + \delta t \mu, \) we get
\[
\frac{1 + \mu^* \mu}{\sqrt{(1 + |\mu|^2)(1 + |\mu|^2)}} = \frac{1}{2} \frac{1 + \mu \mu^* - \mu^* d\mu}{1 + |\mu|^2}. \tag{B8}
\]
FIG. 13. The coupled-top’s classical Poincaré sections of various $Q_1$ with $E = -0.9$ fixed. The partition we take is \{0.1, 0.7, 1.0, 1.3, 1.6, 1.9, 2.2, 2.5, 2.8, 3.1, 3.4, 3.7, 4.0, 4.3, 4.6, 4.9, 5.2, 5.8\} (from left to right, up to down).

Then, the action can be written as the functional
\[
S[\mu, \mu'] = \int dt \left[ -i \frac{\mu_1 \dot{\mu}_1^* - \mu_1^* \dot{\mu}_1}{1 + |\mu_1|^2} - i \frac{\mu_2 \dot{\mu}_2^* - \mu_2^* \dot{\mu}_2}{1 + |\mu_2|^2} 
+ \cos \phi_1 \sin \theta_1 + \cos \phi_2 \sin \theta_2 + \mu \cos \theta_1 \cos \theta_2 \right].
\] (B9)

The (normalized) Lagrangian reads
\[
L = -(1 - \cos \theta_1) \dot{\phi}_1 - (1 - \cos \theta_2) \dot{\phi}_2 
+ \cos \phi_1 \sin \theta_1 + \cos \phi_2 \sin \theta_2 + \mu \cos \theta_1 \cos \theta_2.
\] (B10)

In the classical limit, i.e., $J \to +\infty$ (or equivalently, $\hbar \to 0$), the propagator is permutation matrix with the jumping governed by the classical canonical equation of motion 16.

**Appendix C: Angular Momentum Planck Cells**

In this section of appendix, we give the proof of properties regarded to Planck cells 18. Recall the definition of Planck cells[38]:
\[
|Q, P \rangle = \frac{1}{\sqrt{\Delta_q \Delta_p}} \int_{Q}^{P+\Delta_q} dp |p \rangle e^{-iQp/\hbar},
\] (C1)
where $\Delta_q$ and $\Delta_p$ are size of cells along $q, p$ axis with $\Delta_q \Delta_p = 2\pi \hbar$. Express it in a discrete formalism, we obtain the Planck cells for a single angular momentum:
\[
|Q, P \rangle = \frac{1}{\sqrt{L}} \sum_{n=-m+P}^{m+P} e^{-iQn} |L_z = n \rangle,
\] (C2)

which is 18. It is unambiguous that the $\hat{L}_z$ has mean value approximate to $P_j$ while variation is in a same magnitude with $m$, i.e. proportional to $1/\sqrt{\hbar}$. In the classical limit, this form of Planck cell is equivalent to[38]
\[
|Q, P \rangle^{(position)} = \frac{1}{\sqrt{\Delta_q}} \int_{Q}^{Q+\Delta_q} dq |q \rangle e^{iPq/\hbar},
\] (C3)

which means the $\hat{Q}$ also has mean value approximate to its classical correspondence with variation proportional to $1/\sqrt{\hbar}$. The above analysis can also be done by brute force calculation using algebraic relations of quantum angular momentum.

FIG. 14. Plot of integral $A(\Omega; Q_1)$. As for black line, $\Omega$ represents ‘total isoenergetic surface’; while for dashed line $\Omega$ denotes integrable islands.
Appendix D: Coupled-Top’s Phase Space Volume

Coupled-top’s phase space is a 4-dimensional manifold. Therefor the integrable islands and chaotic seas on a specific isoenergetic surface are 3-dimensional hypersurface. However, only sections of isoenergetic surface are visible to us. In this part of appendix, we show how to calculate the area of integrable islands and chaotic seas via plotting Poincaré sections. The overall integration measure for evaluating area of 3-dimensional hypersurface \( \Omega \) is

\[
V(\Omega) = \int_0^{2\pi} dQ_1 A(\Omega; Q_1), \tag{D1}
\]

where

\[
A(\Omega; Q_1) = \iint_{P \in \Omega} dP_1 dP_2 C_1(E; Q_1, P_1, P_2) C_2(E; Q_1, P_1, P_2). \tag{D2}
\]

with two projection factors \( C_1 \) and \( C_2 \) defined as

\[
C_1(E; Q_1, P_1, P_2) = \sqrt{1 + \left( \frac{\partial Q_2}{\partial P_1} \right)^2 + \left( \frac{\partial Q_2}{\partial P_2} \right)^2}, \tag{D3}
\]

\[
C_2(E; Q_1, P_1, P_2) = \frac{1}{\sqrt{1 - \left( \frac{\partial H/\partial Q_1}{|\nabla H|} \right)^2}}. \tag{D4}
\]

Here \( Q_2 \) is interpreted as a function of \((E; Q_1, P_1, P_2)\), see Eq. (19); and \( H \) is interpreted as a function of \((Q_1, Q_2, P_1, P_2)\) with \( Q_2 \) evaluated via Eq.(19) after partial derivative. The 3-dimensional area of region \( \Omega \) can be approximated as a summation

\[
V(\Omega) \approx \sum_{i=1}^{N-1} \frac{A(\Omega, Q_{1,i}) + A(\Omega, Q_{1,i+1})}{2} \Delta Q_{1,i}, \tag{D5}
\]

where \( \{Q_{1,1}, \ldots, Q_{1,N}\} \) is a partition of \([0, 2\pi]\). The two-dimensional integral \( A(\Omega; Q_1) \) is done in each Poincaré section numerically, and the results are plotted in Fig.14.