Entangled quantum tunneling of two-component Bose-Einstein condensates

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Quantum tunneling of macroscopically coherent systems is an intriguing phenomena well known in the context of Josephson junction effects in superconducting electronic systems. For superfluids consisting of neutral particles, detailed investigations of tunneling is aided by the recent realization Bose-Einstein condensation of atomic vapor in a well controllable environment. Indeed, recent experiments have successfully demonstrated quantum tunneling for condensates confined in an array of optical potentials. One prominent feature of tunneling in Bose condensates is the nonlinear dynamics arising from the interaction between atoms. Quite remarkably, for single-component condensates trapped in double-well configurations, previous studies have indicated that a self-trapping mechanism can suppress the tunneling rate significantly by increasing the atom-atom interaction strength.

An interesting extension of the tunneling problem involves Bose condensates of two interacting species (Fig. 1). The main issue is how the interspecies interaction affects the tunneling process, and particularly the quantum coherence as the two condensates mix together. Previous studies of the general properties of two-component Bose condensates have emphasized the important role of the interspecies interaction, which leads to novel features, such as the components separation, cancellation of quantum phase diffusion. However, the investigation of the influence of interspecies interaction on tunneling dynamics has only just begun.

In this paper we present a novel tunneling mechanism for a two-component condensate trapped in a double-well (see Fig. 1). The atoms of the component $A$ (initially prepared in the left potential well) and $B$ (initially prepared in the right potential well) are initially prepared in the left (right) potential well. We discover the condition under which the interspecies interaction can eliminate the self-trapping effect and thus enhances the tunneling significantly. Such an enhanced tunneling originates from the correlated quantized motion of the two condensates. We also show that atoms of different species tunnel through the barrier as correlated pairs in opposition directions, i.e., a form of quantum entangled tunneling. Therefore tunneling serves as a mechanism to build up a strong correlation among atoms of different species, and this leads to the generation of quantum entanglement between two multi-particle systems.

The configuration of our double-well system is sketched in Fig. 1. Our focus in this paper is the quantum dynamics beyond the mean field description. An exact many-body description is difficult even for single-component condensate problems. The usual method to capture the essential physics is based on the two-mode approximation in which the evolution is confined by the left and right localized mode functions associated with the respective potential wells. Such an approximation is valid when each potential well is sufficiently deep so that higher modes of the wells essentially do not participate in the dynamics.

In the two-mode approximation, the system is modeled by the Hamiltonian ($\hbar = 1$),

$$H = \frac{\Omega}{2}(a^\dagger_L a_R + a_R^\dagger a_L + b^\dagger_R b_L + b_R^\dagger b_L) + \frac{\kappa_L}{2} (a^\dagger_L a_L)^2 + (a_{R}^\dagger a_{R})^2$$
$$- \frac{\kappa_R}{2} (b^\dagger_R b_L)^2 + (b^\dagger_R b_R)^2$$
$$+ \kappa (a^\dagger_L a_R b^\dagger_R b_L + a_R^\dagger a_{R} b_L^\dagger b_R).$$

Here the subscripts $L$ and $R$ respectively denote the localized modes in the left and right potential wells. Since there are two modes available for each component, the...
model in fact consists of four mode operators. We use \( a_j^\dagger \) and \( b_j^\dagger \) \( (j = L, R) \) to denote the creation operators of the component \( A \) and \( B \) respectively. The parameters \( \Omega \), \( \kappa_A(\kappa_B) \) and \( \kappa \) describe the tunneling rate, self-interaction strength of the component \( A(B) \) and the interspecies interaction strength respectively.

To gain insight of the quantum correlation developing in the tunneling process, we first consider the exactly solvable case with only one \( A \) atom in the left well and one \( B \) atom in the right well. In this case the system is spanned by four basis vectors: \(| 1, 0 \rangle_A | 1, 0 \rangle_B \), \(| 1, 0 \rangle_A | 0, 1 \rangle_B \), \(| 0, 1 \rangle_A | 1, 0 \rangle_B \) and \(| 0, 1 \rangle_A | 0, 1 \rangle_B \), where \(| p, q \rangle \) denotes the state with \( p \) atoms of species \( S \) \( (S = A, B) \) in the left well and \( q \) atoms of species \( S \) in the right well. The eigenvalues and eigenvectors of \( H \) can be found straightforwardly. In the regime where the interspecies interaction is sufficiently strong such that \( \kappa \gg \Omega \), the state vector evolves as

\[
|\Psi(t)\rangle = e^{-i(\kappa_A t + \kappa_B t)/2 - \omega_0 t}[\cos \omega_0 t |1, 0\rangle_A |1, 0\rangle_B \\
+ i \sin \omega_0 t |0, 1\rangle_A |0, 1\rangle_B ] + O(\Omega/\kappa). \tag{2}
\]

In writing Eq. (2) we have defined \( \omega_0 = \Omega^2/2\kappa \) as an effective tunneling frequency. Because of the strong interaction between the atoms, the probability of finding both particles in the same well at any time \( t \) is negligible (of order \( \Omega^2/\kappa^2 \)). The tunneling motion of the two atoms are anti-correlated in the sense that the atom \( A \) and the atom \( B \) always move in opposite directions. Such an anti-correlated tunneling motion gives rise to quantum entanglement between the two atoms. At time \( t = (n + 1/4)\pi/\omega_0 \), \( n \) \( (\text{integer}) \), the state is a form of Bell’s state that is maximally entangled in the two-particle two-mode subspace.

Now we examine the multiple atoms case. In order to facilitate the discussion, we assume the number of particles are the same for the two components, i.e., \( N_A = N_B = N \), and the condensates shares the same interaction strength i.e., \( \kappa_A = \kappa_B = \kappa \). The latter condition is a good approximation to \(^{87}\text{Rb} \) condensate of atoms in hyperfine spins states \(| F = 2, m_f = 1 \rangle \) and \(| F = 1, m_f = -1 \rangle \), which share similar scattering lengths [8]. However, we emphasize that these assumptions are not crucial, we shall relax these conditions later in the paper.

We shall limit our study to the \( 4\kappa \gg N\Omega \) regime where the nonlinear interaction is dominant. As before we consider the initial condition in which all atoms in the component \( A(B) \) are localized in the left (right) potential well. The general form of the state vector at time \( t \) is given by: \( |\Psi(t)\rangle = e^{-iN^2t} \sum_{n=0}^{N} \sum_{m=0}^{N} c_{n,m}(t) |n, N - n\rangle_A |m, N - m\rangle_B \). The amplitudes \( c_{n,m}(t) \) are governed by the Schrödinger equation according to the Hamiltonian [8]:

\[
i\dot{c}_{n,m} = \frac{\Omega}{2} \left[ \sqrt{(n+1)(N-n)}c_{n+1,m} \\
+ \sqrt{n(N-n+1)}c_{n-1,m} \right]
\]

with the initial condition \( c_{n,m}(0) = \delta_{n,N} \delta_{m,0} \).

Let us first present the typical results obtained from the numerical solutions of Eq. (3). In Fig. 2, we show the particle number difference \( W \equiv (a_A^\dagger a_L - a_R^\dagger a_R) \) of species \( A \) between the two wells as a function of time. The occurrence of tunneling is revealed by the decrease of \( W \). At longer times \( W \) approaches zero, therefore the numbers of \( A \) atoms in the potential wells are roughly equalized. We emphasize that the nonzero interspecies interaction is responsible for the tunneling to occur. If the two species do not interact with each other (i.e., \( \kappa = 0 \)), then a sufficiently strong self-interaction \( \kappa_j = N\Omega \) \( (j = a, b) \) can suppress the tunneling almost completely by the self-trapping effect [8].

![FIG. 2. Particle number difference of the component \( A \) between the two potential wells as a function of dimensionless time \( (\Omega^2/\kappa)t \) with \( N = 5 \) and \( \kappa = \kappa_A = \kappa_B = 10\Omega \). The dashed line is the solution based on the effective Hamiltonian in Eq. (6). The inset shows the overlap probability \( P \) of finding the system belonging to the degenerate set of states \( |\phi_q\rangle \).](image-url)

To understand the mechanism of the two-component tunneling, we note that the state vector (3) is a superposition of \( (N+1)^2 \) states of the form \(|n, N - n\rangle_A |m, N - m\rangle_B \). However, only a few number of them are actively involved. This is intuitively understood because \(|n, N - n\rangle_A |m, N - m\rangle_B \) have different energies for different values of \( n \) and \( m \), and only those with energies closer to that of the initial state are more accessible.

The difference in energies is significant in the regime \( 4\kappa \gg N\Omega \) considered here. We find that the states \(|\phi_q\rangle = |q, N - q\rangle_A |N - q, q\rangle_B \) \( (q = 0, 1, 2, ..., N) \) (4) are approximately degenerate energy eigenvectors of \( H \).
amplitudes, the Schrödinger equation of

\[ E_n \left( \phi_n \right) \] is \( \left| \phi_q \right\rangle \) by an amount of order of \( \kappa \) or higher. Since the initial state \( \left| \Psi(0) \right\rangle = \left| \phi_q \right\rangle \), \( \left| \Psi(t) \right\rangle \) is mainly a superposition of \( \left| \phi_q \right\rangle \) at any time \( t \) according to the energy argument above. We find that this is indeed the case. To provide a numerical evidence of our finding, we show in the inset of Fig. 2 the overlap probability defined by \( P(t) = \sum_{q=0}^{N} | \left< \phi_q | \Psi(t) \right> |^2 \). For the parameters used in this figure, the set of \( \left| \phi_q \right\rangle \) contributes more than 90% of \( \left| \Psi(t) \right\rangle \). Our further numerical tests suggest that \( P \rightarrow 1 \) in the limit \( N \Omega/\kappa \rightarrow 0 \).

We remark that all \( \left| \phi_q \right\rangle \) have the same number of particles (component \( A \) plus component \( B \)) in the left potential well, and the same also holds for the right well. Therefore \( P(t) \approx 1 \) implies small fluctuations of total particle number in the each potential well. In the case of Fig. 2, we find that the fluctuation of total particle number in the left well \( \langle \Delta N_L \rangle \) is much smaller than \( \langle N_L \rangle^{1/2} \), i.e., a sub-Poissonian distribution.

The time-dependent problem now is significantly simplified because the evolution of the condensate states mainly involves the set of degenerate states \( \left| \phi_q \right\rangle \). For those states that do not have the same energy as \( \left| \phi_q \right\rangle \), they act as intermediate states that are rarely populated. We may eliminate such intermediate states with the adiabatic following method which is well known in quantum optics. The last line in Eq. (3) indicates that the states with \( n + m = N + r \) (\( r \) = integer) has the diagonal term \( r^2 \kappa \) that can be interpreted as the energy of the states if the tunneling interaction \( \Omega \) were switched off. Since \( \kappa \) is a large parameter here, the states with \( n + m = N + 1 \) have a much different energy than that of the states \( \left| \phi_q \right\rangle \) with \( n + m = N \). Any transition (due to \( \Omega \)) from the manifold \( n + m = N \) to the manifolds \( n + m = N + 1 \) must quickly return to \( n + m = N \). In other words, the states with \( n + m = N + 1 \) are intermediate states that are hardly occupied at any time. The amplitudes associated with the \( n + m = N + 1 \) manifold are: \( c_{n+1,N-n} \), \( c_{n-1,N-n} \), \( c_{n,N-n+1} \) and \( c_{n,N-n-1} \), and they can be found approximately by adiabatic following under the condition \( 4\kappa \gg N \Omega \). With these approximate amplitudes, the Schrödinger equation of \( c_{n,N-n} \) is given by:

\[
\begin{align*}
    i & \dot{c}_{n,N-n} \approx \Omega^2 \left( \frac{1}{2\kappa} (n+1)(N-n)c_{n+1,N-n-1} - \frac{1}{2\kappa} n(N-n+1)c_{n-1,N-n+1} - \frac{\Omega^2}{\kappa} [n(N-n)+N]c_{n,N-n-n} \right)
\end{align*}
\]

This corresponds to the Schrödinger equation governed by the effective Hamiltonian:

\[
H_{eff} = -\frac{\Omega^2}{2\kappa} \left( a_R^\dagger b_L^\dagger b_R^\dagger b_L + a_L a_R^\dagger b_L^\dagger b_R + a_L^\dagger a_R b_L b_R^\dagger \right),
\]

apart from a constant term proportional to the total number of particles. The effective Hamiltonian \( H_{eff} \) is an approximation that captures the essential tunneling mechanism in the \( 4\kappa \gg N \Omega \) limit. We have tested the validity of the effective Hamiltonian numerically. For example, in Fig. 2 the dashed line, which is obtained from the evolution based on \( H_{eff} \), agrees well with the exact numerical solution.

The physical picture becomes transparent according to \( H_{eff} \). The interaction term \( a_R^\dagger b_L a_L b_R^\dagger \) suggests that every time an atom \( A \) moves from the left well to the right well there must be an atom \( B \) moves from the right well to the left well. This explains why the small fluctuations in the total particle number in each potential well. The reverse process is described by \( a_L^\dagger b_R^\dagger a_L b_R \) in \( H_{eff} \). In other words, the atoms \( A \) and \( B \) must be moving in pair in opposite directions during the tunneling process.

It is worth noting that \( H_{eff} \) can be cast in the form

\[
H_{eff} = -\frac{\Omega^2}{4\kappa} (K_+ K_- + K_- K_+)
\]

where \( K_+ = a_L^\dagger a_R + b_L^\dagger b_R \) and \( K_- = a_L a_R^\dagger + b_L b_R^\dagger \) satisfy the angular momentum commutation relation: \( [K_+ K_-] = 2K_\zeta, [K_\zeta, K_\pm] = \pm K_\zeta \), where \( K_\zeta = (a_L^\dagger a_R + b_L^\dagger b_R - a_R^\dagger a_L - b_R^\dagger b_L)/2 \). Therefore \( K_\pm \) and \( K_\zeta \) are analogous to collective spin operators, and the eigenvectors and eigenvalues of \( H_{eff} \) can be solved analytically using angular momentum algebra. This shares a common feature in spinor condensates where the spin mixing problem can be solved in a similar fashion. The nonlinear interaction between collective spins is the key to generate nonclassical correlations between different spin components, and particularly, the notion of multi-particle entanglement has been discussed in the context of squeezed spin states. We show here that quantum entanglement between two multi-particle subsystems (\( A \) and \( B \)) can be achieved in the double-well tunneling process.

To this end we quantify the degree of entanglement between the two species based on entanglement entropy: \( E = -\text{tr}(\rho_A \ln \rho_A) = -\text{tr}(\rho_B \ln \rho_B) \), where \( \rho_A \) and \( \rho_B \) are reduced density matrices of the respective subsystems, i.e., \( \rho_A = \text{tr}_B \rho_{AB} \) and \( \rho_B = \text{tr}_A \rho_{AB} \) with \( \rho_{AB} = \left| \Psi(t) \right\rangle \left\langle \Psi(t) \right| \) being the density matrix of the whole system. A disentangled state (for example the initial state above) has zero entanglement entropy. The more entangled the systems, the larger the value of \( E \) is. As an illustration, we show in Fig. 3 how the entanglement is established in time for various particle numbers. As the time increases the value of \( E \) increases until a saturated value is reached. Since there are \( N + 1 \) degenerate states \( \left| \phi_q \right\rangle \) mainly involved in the evolution, \( E \approx \ln (N + 1) \) is a maximum if all \( \left| \phi_q \right\rangle \) have equal contributions to the state of the system. In the case of Fig. 3, the value of \( E \) can reach as high as 90% of the value \( \ln (N + 1) \).

Finally we would like to address the conditions for the entangled tunneling to occur. Our discussion above has been restricted to the simplest symmetric situation:
$N_a = N_b = N$ and $\kappa_a = \kappa_b = \kappa$, in order to illustrate the essential mechanism under the condition $4\kappa \gg N\Omega$. The same analysis can be performed to examine the general situations. We have examined the system with unequal particle numbers $(N_b - N_a) \equiv D > 1$ and unequal coupling strengths $\kappa_a = \kappa + \delta, \kappa_b = \kappa - \delta$ with $|\delta| \ll \kappa$. We find that if the tunneling strength $\Omega$ is sufficiently weak or the self interaction strength $\kappa$ is sufficiently strong such that

$$4|\kappa(D - 1) - |N_a(D - 2)|\delta||/2 \gg \Omega N_b.$$  \hspace{1cm} (8)

then the system mainly evolves among the states $|n, N_a - n\rangle_A |N_a - n, N_b - N_a + n\rangle_B$, where $n = 0, 1, 2, \ldots N_a$. In other words, the tunneling under condition (8) is characterized by entanglement generation term $a_L^\dagger b_R a_L b_R$ as before. However, we point out that the tunneling are generally less efficient for the cases with non-zero $\delta$ and $D$. This is because the states $|n, N_a - n\rangle_A |N_a - n, N_b - N_a + n\rangle_B$ are not as degenerate as that in the symmetric case with $\delta$ and $D$ are both zero. We also remark that the condition (8) does not apply to the special case $N_b - N_a = \pm 1$ in which some of the states in the $n + m = N$, and $n + m = N \pm 1$ manifolds are accidentally degenerated.

To conclude, we have presented a novel mechanism of double-well tunneling involving Bose condensates of two interacting components, based on the two-mode approximation model in the strong coupling regime. We find that the interplay of intraspecies and interspecies interactions permits a set of energy degenerate states, a small tunneling coupling can push the system to ‘explore’ through these degenerate states and thus result in a substantial tunneling not limited by the self trapping effect. The most interesting feature is the strongly correlated tunneling motion. We have derived an exactly solvable effective Hamiltonian to capture the mixing dynamics. In addition, we have shown that high degree of quantum entanglement between two macroscopically coherent systems can be achieved.

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![Graph](image_url)  \hspace{1cm} FIG. 3. The time dependence of the entanglement entropy $E$ for the different particle numbers: $N = 2$, $N = 10$ and $N = 50$ with $\kappa = \kappa_a = \kappa_b = 10\kappa$.  

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