A nonperturbative method for the scalar field theory

Renata Jora

\textit{National Institute of Physics and Nuclear Engineering PO Box MG-6, Bucharest-Magurele, Romania}

(Dated: April 8, 2014)

We compute an all order correction to the scalar mass in the $\Phi^4$ theory using a new method of functional integration adjusted also to the large couplings regime.

PACS numbers: 11.10.Ef, 11.15.Tk

I. INTRODUCTION

Currently very much is known about the perturbative behavior of many theories with or without gauge fields. Beta functions for the $\Phi^4$ theory and QED is known up to the fifth order whereas for QCD is known up to the fourth order [1]-[7]. However there is limited knowledge regarding the non-perturbative behavior of the same theories. Recently [8] have been made for determining the existence in some renormalization scheme of all order beta functions for gauge theories with various representations of fermions. It is rather useful to search for alternative methods which may reveal either the higher orders of perturbation theories or even the non-perturbative regime.

Here we shall consider the massive $\Phi^4$ theory as a laboratory for implementing a method that can be further applied to more comprehensive models. There is an ongoing debate with regard to the behavior of the renormalized coupling $\lambda$ at small momenta referred to as "the triviality problem" [9]-[11]. With the hope that our approach might shed light even on this problem we introduce a new variable in the path integral formalism which allows for a more tractable functional integration and series expansion. Then we compute in this new method the corrections to the mass of the scalar in all order of perturbation theory. This approach should be regarded as an alternate renormalization procedure. Since the corresponding mass anomalous dimension $\gamma(m^2) = \frac{d\ln m^2}{d\mu^2}$ has the first order (one loop) coefficient universal we verify that the first order correction is correct. However we expect that the next orders are different.

II. THE SET-UP

We shall illustrate our approach for a simple scalar theory, given by the Lagrangian:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

$$\mathcal{L}_0 = \frac{1}{2} (\partial_{\mu} \Phi)(\partial^{\mu} \Phi) - \frac{1}{2} m^2 \Phi^2$$

$$\mathcal{L}_1 = -\frac{\lambda}{4!} \Phi^4,$$

We will work both in the Minkowski and Euclidian space upon convenience.

The generating functional in the euclidian space has the expression:

$$W[J] = \int d\Phi \exp[-\int d^4 x \left( \frac{1}{2} (\frac{\partial \Phi}{\partial \tau})^2 + \frac{1}{2} (\Delta \Phi)^2 + \frac{1}{2} m^2 \Phi^2 + \frac{\lambda}{4!} \Phi^4 + J \Phi \right)]$$

and can be written as:

$$W[J] = \exp[\int d^4 x \mathcal{L}_1(\delta J)]W_0[J]$$

where,

$$W_0[J] = \int d\Phi \exp[\int d^4 x (\mathcal{L}_0 + J \Phi)]$$

† Email: rjora@theory.nipne.ro
From Eq. (3) is clear how the perturbative approach can work. If \( \lambda \) is a small parameter one can expand the exponential in terms of \( \lambda \) and solve successive contributions accordingly. However we are interested in the regime where \( \lambda \) is large and one cannot use the above expansion.

We will illustrate our approach simply on a simple function. Assume we have the following one-dimensional integral which cannot be solved analytically:

\[
I = \int dx \exp[-af(x)], \tag{5}
\]

where \( f \) is polynomial of \( x \). For a small the expansion in \( a \) makes sense. For \( a \to \infty \) the Taylor expansion uses:

\[
\lim_{a \to \infty} \frac{d^n}{da^n} \exp[-af(x)] = 0 \tag{6}
\]

which does not lead to a correct answer.

We shall use however a simple trick. We replace in the polynomial \( f \) some of the variables \( x \) with a new variable \( y \) (for example \( x^4 \to x^2y^2 \)). Then we write:

\[
I = \int dx dy \delta(x - y) \exp[-af(x, y)] = \int dx dy dz \exp[-i(x - y)z] \exp[-af(x, y)] =
\]

\[
\int dx dy dz \exp[-i(x - y)z - af(x, y)] \tag{7}
\]

This does not help too much in the present form. However if \( f(x, y) = x^2y^2 \) or any other function that contains \( x^2 \) we can form the perfect square:

\[
-ixz - ax^2y^2 = -(\sqrt{axy} + \frac{iz}{2\sqrt{ay}})^2 - \frac{z^2}{4ay^2}. \tag{8}
\]

Introduced in Eq. (7) this leads:

\[
I = \text{const} \int \frac{1}{\sqrt{ay}} dz \exp[-\frac{z^2}{4ay^2}] \exp[iyz] \tag{9}
\]

Then expansion in \( \frac{1}{a} \) makes sense and one can write:

\[
I = \text{const} \int dx dz \frac{1}{\sqrt{ay}} [1 - \frac{z^2}{4ay^2} + ...] \exp[iyz] \tag{10}
\]

This expansion may seem ill defined and highly divergent. For example if one integrates over \( z \) already encounters infinities. However in the functional method one is dealing with functions instead of simple variables and one encounters divergences also in the usual expansion in small parameters. Such that we will consider the above approach as our starting point and solve the problem of divergences as they appear.

We will start with the simple partition function for a \( \Phi^4 \) theory without a source:

\[
W[0] = \int d\Phi \exp[i \int d^4x [\mathcal{L}_0 + \mathcal{L}_1]] \tag{11}
\]

We consider the extended functional \( \delta \) defined in the Minkowski space as (see the Appendix):

\[
\delta(\Phi) = \text{const} \int dK \exp[i \int d^4x M K \Phi] \tag{12}
\]

which in the euclidian space becomes:

\[
\delta(\Phi) = \text{const} \int dK \exp[- \int d^4x K \Phi] \tag{13}
\]

We then rewrite Eq. (11) in Minkowski space as:

\[
W[0] = \int d\Phi d\Psi \delta(\Phi - \Psi) \exp[i \int d^4x [\mathcal{L}_0 - \frac{\lambda}{8} \Phi^2 \Psi^2] = \]

\[
\text{const} \int d\Phi d\Psi dK \exp[i \int d^4x K (\Phi - \Psi)] \exp[i \int d^4x [\mathcal{L}_0 - \frac{\lambda}{8} \Phi^2 \Psi^2] = \]

\[
\text{const} \int \frac{1}{\sqrt{\lambda}} d\Phi dK \exp[i \int d^4x \frac{2}{\lambda} K^2] \exp[i \int d^4x K \Phi^2] \exp[i \int d^4x [\mathcal{L}_0] \tag{14}
\]
In order to obtain this result we made the following change of variable in the second line of Eq. (14): $K \rightarrow K\Phi$, $\Psi \rightarrow \frac{\Psi}{\Phi \sqrt{\lambda}}$. Note that the $\lambda$ term gets rescaled by 3 such that to take into account the various contribution of the Fourier modes.

We will estimate the first order of the integral in Eq. (14) given by:

$$\text{const} \int \frac{1}{\sqrt{\lambda}} d\Phi dK \frac{1}{\exp} [i \int d^4x K \Phi^2] \exp [i \int d^4x L_0]$$

(15)

In order to solve the integral we write:

$$\int d^4x [K \Phi^2 + L_0] = \int d^4x [\frac{1}{V^2} \sum_{k_n} (\text{Re}K_m + i\text{Im}K_m)(\text{Re}\Phi_n + i\text{Im}\Phi_n)(\text{Re}\Phi_p + i\text{Im}\Phi_p) - \frac{1}{2V} \sum_{k_n} (m^2 - p_n^2)(\text{Re}\Phi_n)^2 + (\text{Im}\Phi_n)^2)]$$

(16)

We denote the bilinear form in the exponential in the Eq. (16) by:

$$\Phi^T K \frac{1}{V^2} \frac{1}{2K_0} - \frac{1}{V} \sum_{n=0}^{\infty} \left( m^2 - \frac{p_n^2}{2} \delta_{2n+1,2n+1} + \delta_{2n+2,2n+2} \right) \Phi$$

(17)

where the counting starts from $n = 0$ and we arranged for example the $\text{Re}\Phi_n$ and $\text{Im}\Phi_n$ components in the $2n + 1$, respectively $2n + 2$ columns of an infinitely dimensional vector.

Then the integral in Eq. (15) can be solved easily as a gaussian integral:

$$\text{const} \int \frac{1}{\sqrt{\lambda}} d\Phi dK \exp [i \int d^4x K \Phi^2] \exp [i \int d^4x L_0] =$$

$$= \int dK \frac{1}{\det \left[ \frac{K}{V^2} + \frac{1}{2V} \sum_{n=0}^{\infty} \left( m^2 - \frac{p_n^2}{2} \delta_{2n+1,2n+1} + \delta_{2n+2,2n+2} \right) \right]^{1/2}}$$

(18)

Note that one can write also a result for the full partition function in Eq. (14):

$$\text{const} \int \frac{1}{\sqrt{\lambda}} d\Phi dK \exp [i \int d^4x K \Phi^2] \exp [i \int d^4x L_0] =$$

$$= \int dK \exp [i \int d^4x \frac{2}{\lambda} K^2] \frac{1}{\det \left[ \frac{K}{V^2} + \frac{1}{2V} \sum_{n=0}^{\infty} \left( m^2 - \frac{p_n^2}{2} \delta_{2n+1,2n+1} + \delta_{2n+2,2n+2} \right) \right]^{1/2}}$$

(19)

The next step is to determine through this procedure the propagator.

III. THE PROPAGATOR

The propagator is given by:

$$\langle \Omega | T\Phi(x_1)\Phi(x_2) | \Omega \rangle = \lim_{T \rightarrow \infty} \frac{\int d\Phi (x_1) \Phi(x_2) \exp [i \int^T_{-T} d^4x L]}{\int d\Phi \exp [i \int^T_{-T} d^4x L]}$$

(20)

For our partition function the Eq. (20) is rewritten as:

$$\langle \Omega | T\Phi(x_1)\Phi(x_2) | \Omega \rangle = \frac{\int \frac{1}{\sqrt{\lambda}} d\Phi dK \Phi(x_1)\Phi(x_2) \exp [i \int d^4x \frac{2}{\lambda} K^2] \exp [i \int d^4x K \Phi^2] \exp [i \int d^4x L_0]}{\int d\Phi dK \exp [i \int d^4x K \Phi^2] \exp [i \int d^4x L_0]} =$$

$$\frac{1}{V^2} \sum_m \exp [-ip_m(x_1 - x_2)] V^{\delta(m \rightarrow \lambda)} \frac{\int d\Phi dK \exp [i \int d^4x \frac{2}{\lambda} K^2] \exp [i \int d^4x K \Phi^2] \exp [i \int d^4x L_0]}{\int d\Phi dK \exp [i \int d^4x K \Phi^2] \exp [i \int d^4x L_0]}$$

(21)

Note that as opposed to the standard functional approach one has contributions only from the quadratic terms. In the Feynman diagram language in this approach one cannot form propagators with the other terms in the exponential, fact that simplifies significantly the calculations.
Since the quantity \( m^2 - k_m^2 \) appears only in the determinant in Eq. (19) we can compute:

\[
\frac{\delta}{\delta(m^2 - p_n^2)} \left[ \frac{K}{V^2} + \frac{2K_0}{V} - (m^2 - p_n^2)\delta_{2n+1,2n+1} + \delta_{2n+2,2n+2} \right]^{-1/2} =
-\frac{1}{2} \frac{\langle \det \left[ \frac{K}{V^2} + \frac{2K_0}{V} - (m^2 - p_n^2)\delta_{2n+1,2n+1} + \delta_{2n+2,2n+2} \right] \rangle}{V} \times
\]

\[
\frac{1}{2} \frac{\langle \det \left[ \frac{K}{V^2} + \frac{2K_0}{V} - (m^2 - p_n^2)\delta_{2n+1,2n+1} + \delta_{2n+2,2n+2} \right] \rangle}{V} =
\]

\[
\frac{1}{2} \frac{\langle \det \left[ \frac{K}{V^2} + \frac{2K_0}{V} - (m^2 - p_n^2)\delta_{2n+1,2n+1} + \delta_{2n+2,2n+2} \right] \rangle}{V} =
\]

Then Eq. (21) becomes:

\[
\begin{aligned}
\langle \Omega | T \Phi(x_1) \Phi(x_2) | \Omega \rangle &= \frac{1}{V^2} \sum_m \exp[-ip_m(x_1 - x_2)]iV \times \\
\int dK \exp[i \int d^4x \frac{K^2}{2}] \frac{1}{\det \left[ \frac{K}{V^2} + \frac{2K_0}{V} - (m^2 - p_n^2)\delta_{2n+1,2n+1} + \delta_{2n+2,2n+2} \right]^{-1/2}} \\
\int dK \exp[i \int d^4x \frac{K^2}{2}] \frac{1}{\det \left[ \frac{K}{V^2} + \frac{2K_0}{V} - (m^2 - p_n^2)\delta_{2n+1,2n+1} + \delta_{2n+2,2n+2} \right]^{-1/2}} =
\end{aligned}
\]

We denote:

\[
\begin{aligned}
\frac{1}{\lambda V} = b_0 \\
m^2 - p_n^2 = c^2 \\
\frac{2}{V} = a_0 \\
det \left( \frac{K}{V^2} + \frac{2K_0}{V} - (m^2 - p_n^2)\delta_{2n+1,2n+1} + \delta_{2n+2,2n+2} \right) = \det[a_0K_0 + B]
\end{aligned}
\]

We need to evaluate:

\[
\begin{aligned}
\int dK dK_0 \exp[2i\pi b_0K_0^2] \frac{1}{\det[a_0K_0 + B]^{1/2}} \\
\int dK dK_0 \exp[i \int d^4x \frac{K_0^2}{2}] \frac{1}{\det[a_0K_0 + B]^{1/2}} =
\end{aligned}
\]

We extracted a factor of \( \frac{1}{V} \) from the determinant and dropped the corresponding constant factor everywhere. In order to determine the ratio in Eq. (25) we evaluate each term in the expansion in the denominator:

\[
I_n = \int dK_0 dK \left( \frac{a_0K_0}{c_0^{2n}} \right)^n \exp[2i\pi b_0K_0^2] \frac{1}{\det[a_0K_0 + B]^{-1/2}} =
\]

\[
\int dK_0 dK \left( \frac{a_0K_0}{c_0^{2n}} \right)^n \exp[2i\pi b_0K_0^2] \frac{1}{\det[a_0K_0 + B]^{-1/2}} =
\]

\[
-\int dK_0 dK \left( \frac{a_0K_0}{c_0^{2n}} \right)^n \exp[2i\pi b_0K_0^2] \frac{1}{\det[a_0K_0 + B]^{-1/2}} =
\]

\[
-\int dK_0 dK \left( \frac{a_0K_0}{c_0^{2n}} \right)^n \exp[2i\pi b_0K_0^2] \frac{1}{\det[a_0K_0 + B]^{-1/2}} =
\]

\[
-\frac{4ib_0}{a_0(n+1)} I_{n+2} = \int dK dK_0 \left( \frac{a_0K_0}{c_0^{2n}} \right)^n \sum_m \frac{a_0K_0 - c_m}{a_0K_0 - c_m} \exp[2i\pi b_0K_0^2] \frac{1}{\det[a_0K_0 + B]^{-1/2}}.
\]
Here we used the formula of differentiation of a determinant. 

From Eqs. (26) and (33) we obtain the following recurrence formula:

\[(n + 1)I_n + I_{n+2}c^4 \left( \frac{4ib}{a_0} + \sum_m \frac{1}{c_{m}^4} \right) + I_{n+1}c^2 \sum_m \frac{1}{c_{m}^2} + \ldots + I_{n+k}c^{2k} \sum_m \frac{1}{c_{m}^{2k}} + \ldots = 0 \]  

(27)

First we multiply the whole Eq. (27) by \( \frac{1}{V} \) and then introduce \( I_n c^{2n} = J_n \) to get the new recurrence formula:

\[\frac{1}{V}(n + 1)J_n + J_{n+1} \frac{1}{V} \sum_m \frac{1}{c_{m}^{2}} + J_{n+2} \frac{2ib}{V} \sum_m \frac{1}{c_{m}^{4}} + \ldots = 0 \] 

(28)

Finally since we denoted the partition function by \( I_0 \) from Eqs. (37) and (28) one can derive:

\[\text{Propagator} = -\frac{i}{c^2} \sum_n \frac{I_n}{I_0} = -\frac{1}{c^2} \sum_n \frac{1}{c^{2n}} J_n / I_0, \]

(29)

where \( J_0 = I_0 \) is the full partition function. Before going further we need to determine the coefficients in Eq. (28). For that we first state,

\[\frac{1}{V} \sum_m \frac{1}{c_{m}^{2k}} = \frac{1}{V} \sum_m \frac{1}{(m^2 - p_{m}^2)^k} = (-1)^k \int d^4 p \frac{1}{(p^2 - m^2)^k} = q_k. \]

(30)

Note that only the integral with \( k = 1, 2 \) are divergent whereas the other ones are finite. We shall use a simple cut-off the regularize them upon the case. Then we get:

\[q_1 = \frac{1}{16 \pi^2} \left[ \Lambda^2 - m^2 \ln \left( \frac{\Lambda^2}{m^2} \right) \right] \]
\[q_2 = \frac{1}{16 \pi^2} \left[ -1 + \ln \left( \frac{\Lambda^2}{m^2} \right) \right] \]
\[q_{n,n>2} = \frac{1}{16 \pi^2} \left( \frac{(m^2)^{2-n}}{(n-1)(n-2)} \right). \]

(31)

IV. DISCUSSION AND CONCLUSIONS

The terms \( J_n \) in the two point function in Eq. (29) correspond to various loop corrections and one can cut the series to obtain results in various orders of perturbation theory. However we shall not attempt to do this here. We will rather aim to obtain if possible an all order result for the correction to the mass of the scalar. We do this with the hope that the approach initiated here can be extended easily to theories with spontaneous symmetry breaking and even to the standard model. It is clear that an approach that could estimate the correction to the Higgs boson mass could prove of great interest. One can write quite generally an exact expression for the propagator of a scalar:

\[\frac{i}{p^2 - m^2 - M^2(p^2)} \]

(32)

where \( m \) is the physical mass and \( M^2(p^2) \) is the one particle irreducible self energy. In our approach the propagator is given by:

\[\frac{i}{p^2 - m_0^2} \sum_n (-1)^n \frac{J_n}{I_0} \frac{1}{(p^2 - m_0^2)^n} \]

(33)

Now if we identify Eq. (29) with Eq. (33) and expand the first equation in series in \( \frac{1}{(p^2 - m_0^2)^n} \) we obtain:

\[(-1)^n \frac{J_n}{I_0} = [m^2 - m_0^2 - M^2(p^2)]^n \]
\[\frac{J_n}{I_0} = [m_0^2 - m^2 - M^2(p^2)]^n. \]

(34)
We denote,

\[ X = [m_0^2 - m^2 - M^2(p^2)], \]

and sum in the recurrence formula in Eq. (28) all terms with the indices \( n + k, k \geq 3 \).

\[
\sum_{k \geq 3} \frac{J_{n+k}}{J_0} q_k = X^n \sum_k \frac{i}{16\pi^2} m_0^4 \frac{X}{m_0^2} (n-1)(n-2) = \\
\frac{i}{16\pi^2} X^{n+1}[X + (m_0^2 - X) \ln\left(\frac{m_0^2 - X}{m_0^2}\right)]
\]

Then the recurrence formula becomes:

\[
(n + 1)a_0 X^n + q_1 X^{n+1} + \left(\frac{2i}{\lambda} + q_2\right) X^{n+2} + \frac{i}{16\pi^2} X^{n+1}[X + (m_0^2 - X) \ln\left(\frac{m_0^2 - X}{m_0^2}\right)] = 0
\]

\[
(n + 1)a_0 \frac{1}{X} + q_1 + \left(\frac{2i}{\lambda} + q_2\right) X + \frac{i}{16\pi^2} \left[X + (m_0^2 - X) \ln\left(\frac{m_0^2 - X}{m_0^2}\right)\right] = 0
\]

\[
q_1 + \left(\frac{2i}{\lambda} + q_2\right) X + \frac{i}{16\pi^2} \left[X + (m_0^2 - X) \ln\left(\frac{m_0^2 - X}{m_0^2}\right)\right] = 0.
\]

Here in the last line we took the limit \( a_0 = \frac{1}{x} \to 0 \).

Now we shall consider the standard renormalization conditions which state:

\[
M^2(p^2)_{p^2=m^2} = 0 \]
\[
\frac{dM^2(p^2)}{dp^2} \bigg|_{p^2=m^2} = 0
\]

With these X will become \( X_0 = m_0^2 - m^2 \). Note that although we used the conditions in Eq. (38) we should not consider our approach equivalent with any of the standard renormalization procedures.

Then Eq. (39) will become:

\[
q_1 + (m_0^2 - m^2)\left[\frac{2i}{\lambda} + q_2\right] + \frac{i}{16\pi^2} \left[(m_0^2 - m^2) + m^2 \ln\left(\frac{m_0^2 - m^2}{m_0^2}\right)\right] = 0
\]

The equation above determines the physical mass in terms of the bare mass and of the cut-off scale. Instead we observe that for a large cut-off scale one can divide the Eq. (39) by \( q_1 \) and retain the first and second term. Then,

\[
m^2 \approx m_0^2 + \frac{q_1}{\frac{2i}{\lambda} + q_2} \approx m_0^2 + \frac{\Lambda^2 - m_0^2 \ln\left(\frac{\Lambda^2}{m_0^2}\right)}{1 + \frac{\lambda}{32\pi^2}[-1 + \ln\left(\frac{\Lambda^2}{m_0^2}\right)]} \lambda
\]

Note that this result leads to the same first order coefficient of the mass anomalous dimension as in the standard renormalization procedures.

**Acknowledgments**

The work of R. J. was supported by a grant of the Ministry of National Education, CNCS-UEFISCDI, project number PN-II-ID-PCE-2012-4-0078.

**Appendix A**

In the following we will show that the relation in Eq. (13) make sense perfect sense in the functional approach. We start with:

\[
\int dK \exp[-\int d^4 x K \Phi] = \prod_{k^2_0 > 0} \int dReK_n dImK_n \exp[-\frac{1}{V} (Re\Phi_n + iIm\Phi_n)(ReK_n + iImK_n)] = \\
= \prod_{k^2_0 > 0} \int dReK_n dImK_n \exp[-\frac{1}{V} Re\Phi_n ReK_n - i\frac{1}{V} Im\Phi_n ReK_n] \exp[i\frac{1}{V} Im\Phi_n ImK_n - i\frac{1}{V} Re\Phi_n ImK_n].
\] (A1)
Next let us consider a regular integral of the type:

\[ \int dx \exp[-ipx - ap] = \int dx[1 - (ap) + \frac{1}{2}(ap)^2 + ... \exp[-ipx] = \int dx[1 - (-i)a \frac{\delta}{\delta x} + \frac{1}{2}(-i)^2 \frac{\delta^2}{\delta x^2} + ... \exp[-ipx] = [1 - (-i)a \frac{\delta}{\delta x} + \frac{1}{2}(-i)^2 \frac{\delta^2}{\delta x^2} + ...] \delta(x) = g(x) \] (A2)

Let us apply this result to Eq. (A1) with the variable a replaced depending on the case by ReΦ or ImΦ:

\[ \int dK d\Phi f(\Phi) \exp[- \int d^4xK \Phi] = \text{const} \prod_{k_0 > 0} d\text{Re}\Phi_n d\text{Im}\Phi_n f(\text{Re}\Phi_k, \text{Im}\Phi_k) \times (1 - \text{Re}\Phi_n(-i) \frac{\delta}{\delta \text{Im}\Phi_n} + ...) \delta(\text{Im}\Phi_n) \times (1 - \text{Im}\Phi_n(-i) \frac{\delta}{\delta \text{Re}\Phi_n} + ...) \delta(\text{Re}\Phi_n) = \text{const} f(0, 0) \] (A3)

We will prove that by considering a few terms in the above expansion. The zeroth order term contains two delta functions and clearly leads to \( f(0, 0) \). Another possible term is:

\[ \int \prod_{k_0 > 0} d\text{Re}\Phi_n d\text{Im}\Phi_n f(\text{Re}\Phi_k, \text{Im}\Phi_k) \text{Re}\Phi_n \frac{\delta}{\delta \text{Im}\Phi_n} \delta(\text{Im}\Phi_n) \delta(\text{Re}\Phi_n) = 0 \] (A4)

by virtue of the \( \delta(\text{Re}\Phi_n) \) function. Another possible term is,

\[ \int \prod_{k_0 > 0} d\text{Re}\Phi_n d\text{Im}\Phi_n f(\text{Re}\Phi_k, \text{Im}\Phi_k) \text{Re}\Phi_n \frac{\delta}{\delta \text{Im}\Phi_n} \delta(\text{Im}\Phi_n) \text{Im}\Phi_n \frac{\delta}{\delta \text{Re}\Phi_n} \delta(\text{Re}\Phi_n) = -\prod_{k_0 > 0} \text{Re}\Phi_n \text{Im}\Phi_n \frac{\delta}{\delta \text{Re}\Phi_n} f(0, \text{Im}\Phi_n) = f(0, 0) \] (A5)

It can be shown that all other terms are either zero or proportional to \( f(0, 0) \) which concludes our proof that the integral in Eq. (A3) gives a well defined delta function.