Polynomially Isometric Matrices in Low Dimensions

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Abstract. Given two $d \times d$ matrices, say $A$ and $B$, when do $p(A)$ and $p(B)$ have the same “size” for every polynomial $p$? In this article, we provide definitive results in the cases $d = 2$ and $d = 3$ when the notion of size used is the spectral norm.

1. INTRODUCTION. Given a square matrix $A$, there is no ambiguity in what “squaring a matrix” should mean; $A^2$ is the product of $A$ with itself. This simple notion can be extended in a natural way to nonnegative integer powers of matrices: $A^n$ is the product of $n$ copies of $A$ when $n > 0$, while $A^0$ is the identity matrix $I$. Likewise, one may construct other matrices associated with $A$, namely, polynomial functions $p(A)$ of $A$. That is, given a polynomial with complex coefficients $p(z) = c_0 + c_1 z + \cdots + c_m z^m$, $p(A)$ denotes the square matrix $c_0 I + c_1 A + \cdots + c_m A^m$. This definition also works just as well if $A$ is an operator (i.e., a linear transformation) on a complex vector space; the only difference is that the operation of composition is used instead of matrix product. The assignment $p \mapsto p(A)$ induced by $A$ is often referred to as its polynomial functional calculus.

Where do polynomial functions of a matrix or operator come about, and why are they important? Although answers abound, let us mention only a couple of places where they are encountered, perhaps in disguise, in the undergraduate curriculum. For other types of functional calculus and their applications, we refer the interested reader to [15, Chapter 10] for the Riesz–Dunford holomorphic functional calculus (for the graduate student) and to [11] for a discussion of the Dyn'kin nonholomorphic functional calculus (for the advanced scholar).

First, a fundamental result in linear algebra states that every operator $A$ on a finite-dimensional complex vector space (e.g., a square matrix) has an eigenvalue. (As usual, $\lambda$ is an eigenvalue of $A$ if there is a nonzero vector $\mathbf{v}$ so that $A\mathbf{v} = \lambda \mathbf{v}$.) A quick proof of that result amounts to observing that given a nonzero vector $\mathbf{w}$, there is a polynomial $p$ so that $p(A)\mathbf{w} = \mathbf{0}$ (see [2, p. 145] for details). Second, a linear inhomogeneous differential equation (DE) with constant coefficients can be seen as an operator equation of the differentiation operator $D$. For instance, the DE

$$y''(t) + c_1 y'(t) + c_0 y(t) = f(t)$$

(1)

can be written as $p(D)y = f$, where $p(z) = z^2 + c_1 z + c_0$ is a quadratic polynomial and $y$ is the unknown function. In this case, if $p_j(z) = z - \lambda_j$ for $j = 1, 2$ are such that $p_1(z)p_2(z) = p(z)$, then the functional calculus for $D$ gives a way to solve (1): solve consecutively the first-order DEs $y_1' - \lambda_1 y_1 = p_1(D)y_1 = f$ for $y_1$ and $y' - \lambda_2 y = p_2(D)y = y_1$ for $y$.

In applications, not only is a function of a matrix important, so is its size. For instance, when $A$ is a square matrix, $\|A^k\|$ and $\|(zI - A)^{-1}\|$ arise naturally in the models of discrete-time evolution processes and responses of forced systems, respectively [18, Chapter 47]. Such quantities are used to describe behavior, and so it is

\begin{align*}
&\text{doi.org/10.1080/00029890.2021.1898872} \\
&\text{MSC: Primary 47A10, Secondary 15A60; 15A18; 47A56}
\end{align*}
natural to question what conditions a matrix $B$ might satisfy to ensure that its behavior is the same as that of $A$.

More precisely and following [9], we say that $A$ and $B$ are polynomially isometric\(^1\) (under the spectral norm) if

\[
\|p(A)\| = \|p(B)\| \quad \text{for all polynomials } p. \tag{2}
\]

Thus, in this article, we consider the following question:

Given a pair of square matrices $A$ and $B$, what set of invariants (e.g., spectra, Frobenius norms, etc.) are necessary and sufficient to ensure that $A$ and $B$ are polynomially isometric?

One might think that unitary similarity characterizes (2), but the condition turns out to be too strong. Unitary similarity is certainly sufficient to ensure that two matrices are polynomially isometric; after all, if there is a unitary matrix $U$ (i.e., $U^*U = UU^* = I$) so that $B = UAU^*$, then $\|p(B)\| = \|p(A)\|$ holds for every polynomial $p$ since $p(B) = Up(A)U^*$. Furthermore, unitary similarity is also necessary for (2) to hold if the matrices $A$ and $B$ are $2 \times 2$ (see Theorem 2). However, if

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

then $\|p(A)\| = \max\{|p(1)|, |p(0)|\} = \|p(B)\|$ holds for all polynomials $p$, i.e., $A$ and $B$ are polynomially isometric, but $A$ and $B$ cannot be unitarily similar because they have different ranks.

One might then turn to equality of spectra. After all, equality of spectra is a necessary condition for (2) and furthermore, by the spectral theorem, $\|p(N)\| = \max\{|p(\lambda)| : \lambda \in \sigma(N)\}$ holds whenever $N$ is a normal matrix (i.e., $N^*N = NN^*$). However, the condition is not sufficient (see (4)) and this is not an isolated case; numerical analysts have long known that knowledge of the spectrum of a matrix alone is not enough to describe the behavior of nonnormal matrices. On the other hand, pseudospectral analysis has proven to be a useful tool to better understand the behavior of matrices that arise in scientific applications; i.e., matrices that are nonnormal and of large dimension. For instance, the reader can find a wealth of examples in the book [18] (see also [17]) that illustrate how pseudospectra\(^2\) may capture the “spirit” of a (nonnormal) matrix more effectively. Unfortunately, it is observed in [7] that for (2) to hold, it is necessary, but not sufficient, that $A$ and $B$ have identical pseudospectra (see (5)).

But hope is not lost! The condition of identical pseudospectra does not suffice in general, but for square matrices of small dimension, we show below that it does the trick. We address the question in the context of $2 \times 2$ and $3 \times 3$ matrices, and prove the necessity and sufficiency of identical pseudospectra for (2) to hold. Not only does this article serve as “food for thought” for the linear-algebra enthusiast, it is aimed to provide clarity for newcomers to matrix analysis concerning the precise connections between some related notions encountered in the field, namely, polynomially isometric matrices, identical pseudospectra, super-identical pseudospectra, and unitary similarity.

\(^1\)The terminology used in this article was introduced in [9]. However, the same notion has appeared previously as “$A$ and $B$ have the same norm behavior,” e.g., see [7] and [18, Chapter 47].

\(^2\)Roughly, a pseudospectral plot for a matrix $A$ consists of contour plots of the norm $\| (zI - A)^{-1} \|$ of its resolvent $(zI - A)^{-1}$.

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2. TERMINOLOGY AND THE MAIN RESULT. Let $\mathbb{C}^d$ denote complex Euclidean $d$-dimensional space, and let $\mathbb{M}_d$ be the algebra of complex $d \times d$ matrices. For $T \in \mathbb{M}_d$, $\text{tr} \, T$ denotes the trace of $T$. We denote the Frobenius (or Hilbert–Schmidt) norm of $T$ by $\|T\|_F$, and the spectral norm of $T$ by $\|T\|$. That is, $\|T\|_F = \sqrt{\text{tr} \, T^* T}$, where $T^*$ is the conjugate transpose of $T$, and $\|T\| = \sup \{ \|Tv\|_{\mathbb{C}^d} : \|v\|_{\mathbb{C}^d} = 1 \}$ is the operator norm induced by the Euclidean norm on $\mathbb{C}^d$. The minimal and characteristic polynomials of $T$ are denoted by $m_T$ and $\chi_T$, respectively. That is, $m_T$ is the monic polynomial $p$ of minimal degree such that $p(T) = 0$ while

$$\chi_T(z) = \det(zI - T),$$

where the determinant $\det(A)$ is the product of the eigenvalues of $A \in \mathbb{M}_d$ (taking into account multiplicities). As usual, the spectrum $\sigma(T)$ of $T$ is the set of eigenvalues of $T$, i.e.,

$$\sigma(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible} \}.$$

Finally, the singular values $s_1(T), \ldots, s_d(T)$ of $T$ are the nonnegative square roots of the eigenvalues of $T^* T$ listed in nonincreasing order. Thus, $s_1(T) = \|T\|$, $s_1^2(T) = s_2^2(T) + \cdots + s_d^2(T) \geq \|T\|^2$, and $s_d(T) = \|T^{-1}\|^{-1}$ whenever $T$ is invertible. We refer the reader to [2] and [8] for further explanations and results concerning these concepts.

At this point, one may be wondering what can (and cannot) be expected of polynomially isometric matrices $A$ and $B$. Surely, they need not have the same characteristic polynomials. This is demonstrated by the pair of matrices in (3). Must they have the same spectra? Absolutely. In fact, (2) implies (for matrices $A$ and $B$ of arbitrary size) that the minimal polynomials $m_A$ and $m_B$ must be equal.

On the other hand, equality of minimal polynomials is not enough to guarantee the converse: the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

have minimal polynomial $m_A(z) = m_B(z) = z^2$ but

$$\|A\| \leq \|A\|_F = 1 \quad \text{while} \quad \|B\| \geq \|(0, 1, 2)\|_{\mathbb{C}^3} = \sqrt{5}.$$ 

Thus, (2) fails with $p(z) = z$.

As mentioned in Section 1, it is known that having identical pseudospectra is also a necessary condition for matrices to be polynomially isometric [7]. To be precise, let us agree that two square matrices $A$ and $B$ (not necessarily of the same size) have **identical pseudospectra** if

$$\|(zI - A)^{-1}\| = \|(zI - B)^{-1}\| \quad \text{for all} \quad z \in \mathbb{C}.$$  \hspace{1cm} (5)

Now since (5) implies $m_A = m_B$ (see Theorem 12), one may ask whether a pair of matrices are polynomially isometric precisely when they have identical pseudospectra. If at least one of $A$ or $B$ is normal, an affirmative answer is known [4]. However,
an example from [7] shows (after padding a matrix with zeros) that there are $5 \times 5$ (nonnormal) matrices having identical pseudospectra for which the condition in (2) fails with $p(z) = z$. (This example also appears in [18, Chapter 47].) Furthermore, [5] and [14] contain examples of $4 \times 4$ matrices having identical pseudospectra but whose squares have distinct norms, i.e., (2) fails with $p(z) = z^2$. Nevertheless, the following holds.

**Theorem 1.** The following statements are equivalent for $A, B \in \mathbb{M}_d$ when $d = 2, 3$.

1. $A$ and $B$ have identical pseudospectra.
2. $A$ and $B$ are polynomially isometric.

In the case $d = 2$, the equivalence in Theorem 1 was mentioned (without proof) in [7] and so in Section 3, we establish a slight improvement that either of the two statements listed in Theorem 1 is equivalent to saying that $A$ and $B$ are unitarily similar. In Section 3, we also introduce the stronger notion of super-identical pseudospectra. The main result of that section clarifies which $3 \times 3$ matrices with identical pseudospectra also have super-identical pseudospectra. This, coupled with the fact that matrices with identical pseudospectra have the same minimal polynomials, leads to the reduction of the proof of Theorem 1 to the case of matrices with quadratic minimal polynomials in Section 4. Furthermore, in that context, we establish an easy-to-check necessary and sufficient condition in terms of the Frobenius norm that determines when a pair of matrices have identical pseudospectra. Finally, in Appendix A, we include proofs of two technical results concerning $d \times d$ matrices having identical pseudospectra that are used in Sections 3 and 4.

3. **SUPER-IDENTICAL PSEUDOSPECTRA.** Recall that polynomially isometric matrices must have identical pseudospectra (regardless of their size), but the converse need not hold. Even more surprisingly, it is known that there are pairs of matrices $A$ and $B$ with identical pseudospectra for which the corresponding norms $\|A^k\|$ and $\|B^k\|$ for $k \geq 2$ are completely unrelated (see [13, Theorem 2.3] for details). This can be attributed to the fact that, roughly speaking, parts of a matrix may not actively play a role when computing its spectral norm, e.g., see the proof of Theorem 11. So, in an attempt to prevent such parts from being “hidden” and drawing inspiration from the way that pseudospectra are computed, Fortier Bourque and Ransford introduced in [5] the notion of super-identical pseudospectra of matrices belonging to the same class $\mathbb{M}_d$.

Matrices $A$ and $B$ in $\mathbb{M}_d$ are said to have **super-identical pseudospectra** if

$$s_k(zI - A) = s_k(zI - B) \quad \text{for all } z \in \mathbb{C}, k = 1, \ldots, d.$$  

(6)

Thus, since the condition in (5) is equivalent to

$$s_d(zI - A) = s_d(zI - B) \quad \text{for all } z \in \mathbb{C},$$

the requirement in (6) is stronger than (5).

It can be shown (see [13, Theorem 3.6]) that if $A$ and $B$ have super-identical pseudospectra, then the norms of $p(A)$ and $p(B)$ are at least comparable; more specifically,

$$\frac{1}{\sqrt{d}} \|p(B)\| \leq \|p(A)\| \leq \sqrt{d} \|p(B)\|$$

holds for all polynomials $p$. This result suggests that pairs of matrices having super-identical pseudospectra may be polynomially isometric, but this need not be the case;
in fact, the examples of $4 \times 4$ matrices from [5] and [14] mentioned in Section 2 have super-identical pseudospectra but are not polynomially isometric.

On the other hand, for matrices $A, B \in \mathbb{M}_d$ in low dimensions $d = 2$ or $d = 3$, it was shown in [5] that a sufficient condition for $A$ and $B$ to be polynomially isometric is that $A$ and $B$ have super-identical pseudospectra. However, the failure of necessity can already be seen by the pair of $3 \times 3$ matrices in (3). For $2 \times 2$ matrices, it turns out that the notions of identical pseudospectra, polynomial isometry, and super-identical pseudospectra are all equivalent.

**Theorem 2.** The following statements are equivalent for $A, B \in \mathbb{M}_2$.

1. $A$ and $B$ have identical pseudospectra.
2. $A$ and $B$ are polynomially isometric.
3. $A$ and $B$ have super-identical pseudospectra.
4. $A$ and $B$ are unitarily similar.

As previously mentioned, the equivalence “1 $\iff$ 2” was stated in [7]. The equivalence “3 $\iff$ 4” was established in [5]. Our proof of Theorem 2 is based on Lemma 3 whose proof is left to the reader. Before stating that lemma, we need a definition.

Given a polynomial $p$, define $D_p : \mathbb{C}^2 \to \mathbb{C}$ by

$$D_p(\alpha, \beta) := \begin{cases} p'(\alpha) & \text{if } \alpha = \beta \\ \frac{p(\alpha) - p(\beta)}{\alpha - \beta} & \text{if } \alpha \neq \beta \end{cases}.$$  

**Lemma 3.** The family of $2 \times 2$ matrices $t(\alpha, \beta, \delta) = \begin{bmatrix} \alpha & \delta \\ 0 & \beta \end{bmatrix}$ has the following properties for $\alpha, \beta, \delta \in \mathbb{C}$.

1. $s_1(t(\alpha, \beta, \delta)) = s_2(t(\alpha, \beta, \delta))$ if and only if $|\alpha| = |\beta|$ and $\delta = 0$.
2. $s_1(t(\alpha, \beta, \delta))$ and $s_2(t(\alpha, \beta, \delta))$ are, respectively, strictly increasing and strictly decreasing in $|\delta|$.
3. $s_1(t(\alpha, \beta, 0)) = \max\{|\alpha|, |\beta|\}$ and $s_2(t(\alpha, \beta, 0)) = \min\{|\alpha|, |\beta|\}$.
4. $s_j(t(\alpha, \beta, \delta)) = s_j(t(\beta, \alpha, \delta))$ for $j = 1, 2$.
5. $p(t(\alpha, \beta, \delta)) = t(p(\alpha), p(\beta), \delta D_p(\alpha, \beta))$ for any polynomial $p$.

**Proof of Theorem 2.** By our preliminary remarks, it suffices to show that matrices with identical pseudospectra must be unitarily similar. Suppose $A$ and $B$ have identical pseudospectra. Then $A$ and $B$ have the same eigenvalues, say $\alpha$ and $\beta$. Consequently, $A$ and $B$ are unitarily similar to upper triangular matrices of the form

$$\begin{bmatrix} \alpha & \delta_A \\ 0 & \beta \end{bmatrix} \text{ and } \begin{bmatrix} \alpha & \delta_B \\ 0 & \beta \end{bmatrix},$$

for some $\delta_A, \delta_B \geq 0$ (e.g., see [8, Chapter 25]). Since the singular values of a matrix are invariant under multiplication by unitary matrices on the right and left,

$$s_2(t(z - \alpha, z - \beta, -\delta_A)) = \|(zI - A)^{-1}\|^{-1}$$

and

$$s_2(t(z - \alpha, z - \beta, -\delta_B)) = \|(zI - B)^{-1}\|^{-1}$$

are equal. Hence, by Lemma 3, $\delta_A = \delta_B$ and so $A$ and $B$ are unitarily similar.
Remark. In light of Murnaghan’s criterion [10] for unitary similarity of matrices in \( M_2 \), an easy-to-check necessary and sufficient condition for any (or all) of the statements in Theorem 2 is that \( \text{tr} A^*A = \text{tr} B^*B \) and \( \text{tr} A^k = \text{tr} B^k \) for \( k = 1, 2 \). Thus, these three traces form a complete set of invariants to determine when matrices have identical pseudospectra.

As an amusing consequence, we state the following corollary and leave its proof to the reader.

Corollary 4. There are similar matrices \( A, B \in M_2 \) that do not have identical pseudospectra.

It is worth mentioning that if two \( d \times d \) matrices have super-identical pseudospectra, then they must be similar [1]. However, by Corollary 4, the converse need not hold.

What about the case of \( 3 \times 3 \) matrices? In this context, matrices having super-identical pseudospectra need not be unitarily similar. After all, a matrix \( A \) and its transpose \( A^t \) always have super-identical pseudospectra, but there are known examples of \( 3 \times 3 \) matrices \( A \) that are not unitarily similar to \( A^t \) (see after Theorem 10). Even better, it is proved in [5] that \( A, B \in M_3 \) have super-identical pseudospectra if and only if \( A \) is unitarily similar to \( B \) or to \( B^t \); consequently, \( A \) and \( B \) must be polynomially isometric. Furthermore, in analogy to Pearcy’s or Sibirskii’s criteria for unitary similarity of \( 3 \times 3 \) matrices (see [12] and [16]), the following six trace conditions are necessary and sufficient for \( 3 \times 3 \) matrices \( A \) and \( B \) to have super-identical pseudospectra [13]: \( \text{tr} (A^*A) = \text{tr} (B^*B), \text{tr} (A^*A^2) = \text{tr} (B^*B^2), \text{tr} (A^{*2}A^2) = \text{tr} (B^{*2}B^2), \text{tr} A^k = \text{tr} B^k \) for \( k = 1, 2, 3 \). Although these results provide characterizations for matrices having super-identical pseudospectra, they do not appear to answer these simple questions: If a pair of matrices have identical pseudospectra, what condition may ensure that they have super-identical pseudospectra? Is that condition necessary and sufficient? We now close this gap.

Theorem 5. The following statements are equivalent for \( A, B \in M_3 \).

1. \( A \) and \( B \) have identical pseudospectra and \( \chi_A = \chi_B \).
2. \( A \) and \( B \) have super-identical pseudospectra.

To prove Theorem 5, we employ two lemmas. We postpone the proof of Lemma 6 to Appendix A and leave that of Lemma 7 to the reader. As usual, for \( T \in M_d \), \( \text{Re} T = (T + T^*)/2 \).

Lemma 6. Let \( A \) and \( B \) be square matrices (not necessarily of the same size). If \( A \) and \( B \) have identical pseudospectra, then the largest eigenvalues of the matrices \( \text{Re} A \) and \( \text{Re} B \) coincide, as do the smallest eigenvalues.

Lemma 7. If \( x_1, y_1, x_2, y_2 \in \mathbb{R} \) satisfy \( x_1 + y_1 = x_2 + y_2 \) and \( x_1 \cdot y_1 = x_2 \cdot y_2 \), then either \( (x_1, y_1) = (x_2, y_2) \) or \( (x_1, y_1) = (y_2, x_2) \).

Proof of Theorem 5. If \( A, B \in M_3 \) have super-identical pseudospectra (see (6)), then \( A \) and \( B \) necessarily have identical pseudospectra and the same characteristic polynomials, as

\[
|\det(zI - A)| = \prod_{k=1}^3 s_k(zI - A) = \prod_{k=1}^3 s_k(zI - B) = |\det(zI - B)|. \tag{7}
\]
Suppose now that $A$ and $B$ have identical pseudospectra and equal characteristic polynomials. Since

$$\sum_{k=1}^{3} s_k^2(zI - A) = \text{tr}[(zI - A)^*(zI - A)] = 3|z|^2 - \bar{z} \text{tr} A - z \text{tr} A^* + \text{tr} A^* A,$$

we see that

$$\sum_{k=1}^{3} s_k^2(zI - A) = \sum_{k=1}^{3} s_k^2(zI - B) \tag{8}$$

holds provided $\text{tr} A = \text{tr} B$ and $\text{tr} A^* A = \text{tr} B^* B$. Clearly, $\chi_A = \chi_B$ implies $\text{tr} A^k = \text{tr} B^k$ for $k = 1, 2, 3$. In particular, $\text{tr} A = \text{tr} B$ and so $\text{tr} \Re A = \text{tr} \Re B$; therefore, by Lemma 6, $\Re A$ and $\Re B$ have the same eigenvalues (counting multiplicities) and so $\text{tr}(\Re A)^2 = \text{tr}(\Re B)^2$, or equivalently,

$$\text{tr} A^2 + 2 \text{tr} A^* A + \text{tr} A^{*2} = \text{tr} B^2 + 2 \text{tr} B^* B + \text{tr} B^{*2}.$$

Thus, $\text{tr} A^* A = \text{tr} B^* B$ and (8) is established.

Recalling that $A$ and $B$ have identical pseudospectra, (7) and (8) simplify to

$$\prod_{k=1}^{2} s_k(zI - A) = \prod_{k=1}^{2} s_k(zI - B) \quad \text{and} \quad \sum_{k=1}^{2} s_k^2(zI - A) = \sum_{k=1}^{2} s_k^2(zI - B).$$

Hence, the fact that $A$ and $B$ have super-identical pseudospectra follows now from Lemma 7.

Although matrices with identical pseudospectra are known to have the same minimal polynomials (see Theorem 12), they need not have the same characteristic polynomials; for an example, consider again the diagonal matrices $A$ and $B$ in (3). This example demonstrates that the assumption $\chi_A = \chi_B$ in Theorem 5 is not superfluous. More strikingly, these $A$ and $B$ have identical pseudospectra and yet none of Ransford’s six trace criteria (which characterize super-identical pseudospectra as found in [13] and stated just before Theorem 5) hold. Hence, no five of those six traces alone suffice to characterize when a pair of (generic) matrices have identical pseudospectra!

Nevertheless, by Theorem 5, Ransford’s six trace criteria may be used to confirm whether (or not) a pair of matrices have identical pseudospectra and the same minimal polynomial of degree 3. Instead, in the case of matrices with common minimal polynomial of degree 2, we find and present in the next section another easy-to-check criterion to confirm that they have identical pseudospectra.

4. MATRICES WITH QUADRATIC MINIMAL POLYNOMIALS. In this section, we complete the proof of Theorem 1 via the proof of Theorem 9. To do so, we state an analog of Lemma 3 for $3 \times 3$ matrices whose proof is left to the interested reader.

**Lemma 8.** The family of matrices $T(\gamma, \alpha, \beta, \delta) = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \alpha & \delta \\ 0 & 0 & \beta \end{bmatrix}$ has the following properties for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. 

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1. Following the notation of Lemma 3, the singular values of $T(\gamma, \alpha, \beta, \delta)$ consist of those of $t(\alpha, \beta, \delta)$ and $|\gamma|$. In particular,

$$s_1(T(\gamma, \alpha, \beta, \delta)) = s_1(t(\alpha, \beta, \delta)) \text{ when } |\gamma| \leq \max\{|\alpha|, |\beta|\},$$

$$s_2(T(\gamma, \alpha, \beta, \delta)) = s_2(t(\alpha, \beta, \delta)) \text{ when } |\gamma| \geq \min\{|\alpha|, |\beta|\}.$$

2. $p(T(\gamma, \alpha, \beta, \delta)) = T(p(\gamma), p(\alpha), p(\beta), \delta D_p(\alpha, \beta))$ for any polynomial $p$.

**Theorem 9.** The following statements are equivalent for $A, B \in \mathbb{M}_3$.

1. $A$ and $B$ have identical pseudospectra.
2. $A$ and $B$ are polynomially isometric.

Moreover, if $A$ and $B$ also have the same minimal polynomial of degree 2, then the above statements are equivalent to

3. $\|A - \gamma_A I\|_F = \|B - \gamma_B I\|_F$, where $\gamma_A$ and $\gamma_B$ are the eigenvalues corresponding to $A$ and $B$, respectively, of largest multiplicity.

**Proof.** It is shown in [7] that $A$ and $B$ have identical pseudospectra whenever they are polynomially isometric. To prove the converse, assume $A$ and $B$ have identical pseudospectra. By Theorem 12 in Appendix A, $A$ and $B$ have the same minimal polynomials. If their common minimal polynomial has degree one, then $A = \alpha I$ and $B = \beta I$ for some $\alpha, \beta \in \mathbb{C}$; it follows readily from this that $\alpha = \beta$ and so $A$ and $B$ are polynomially isometric. Thus, in light of Theorem 5, we assume that $A$ and $B$ have common quadratic minimal polynomial $p(z) = (z - \alpha)(z - \beta)$, where $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq \beta$, and such that $\chi_A \neq \chi_B$; i.e., $A$ and $B$ have the same eigenvalues but with distinct multiplicities. Since the three statements listed in the theorem are invariant under unitary similarity, we assume further without loss of generality that $A$ and $B$ are upper triangular matrices of the form

$$A = \begin{bmatrix} 
\alpha & 0 & 0 \\
0 & \alpha & \delta_A \\
0 & 0 & \beta 
\end{bmatrix} \text{ and } B = \begin{bmatrix} 
\beta & 0 & 0 \\
0 & \alpha & \delta_B \\
0 & 0 & \beta 
\end{bmatrix},$$

where $\delta_A, \delta_B > 0$ ([8, Chapter 25]).

By Lemmas 3 and 8, $A = T(\alpha, \alpha, \beta, \delta_A)$ and $B = T(\beta, \alpha, \beta, \delta_B)$ satisfy

$$\|(zI - A)^{-1}\|^{-1} = s_2(t(z - \alpha, z - \beta, -\delta_A)),$$

$$\|(zI - B)^{-1}\|^{-1} = s_2(t(z - \alpha, z - \beta, -\delta_B)),$$

$$\|p(A)\| = s_1(t(p(\alpha), p(\beta), \delta_A D_p(\alpha, \beta))), \text{ and}$$

$$\|p(B)\| = s_1(t(p(\alpha), p(\beta), \delta_B D_p(\alpha, \beta))).$$

We now see that $A$ and $B$ have identical pseudospectra if and only if $\delta_A = \delta_B$. Likewise, $A$ and $B$ are polynomially isometric if and only if $|\delta_A D_p(\alpha, \beta)| = |\delta_B D_p(\alpha, \beta)|$ for all polynomials $p$, or equivalently, $\delta_A = \delta_B$. On the other hand, $\delta_A = \delta_B$ is equivalent to $\|A - \alpha I\|_F = \|B - \beta I\|_F$. $lacksquare$

Another look at the proof of Theorem 9 reveals the validity of the following result which complements Theorem 5. We omit the details.
Theorem 10. The following statements are equivalent for \( A, B \in \mathbb{M}_3 \) having equal quadratic minimal polynomials.

1. \( A \) and \( B \) have identical pseudospectra and \( \chi_A = \chi_B \).
2. \( A \) and \( B \) are unitarily similar.

Note, however, that the equivalence in Theorem 10 need not hold if \( A \) and \( B \) have equal cubic minimal polynomials. For an example, let

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]

and \( B = A' \), and note that \( A \) and \( B \) cannot be unitarily equivalent because

\[
\text{tr}(AA^*A^2A^{*2}) \neq \text{tr}(BB^*B^2B^{*2})
\]

(see the third example in [16]). It also worth noting that unitary similarity of \( d \times d \) matrices having equal quadratic minimal polynomials has been characterized in [6] as those matrices having the same eigenvalues and the same singular values.

Although the third condition in Theorem 9 is easy to check, it does not lend itself to generalization. For instance, by Lemma 3, the \( 4 \times 4 \) matrices

\[
A = \begin{bmatrix}
1 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\text{ and } B = \begin{bmatrix}
1 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

have identical pseudospectra, equal quadratic minimal polynomials, and yet the Frobenius norms \( \|A - \gamma_A I\|_F \) and \( \|B - \gamma_B I\|_F \) are not equal whether one interprets \( \gamma_A \) and \( \gamma_B \) as 0 or 1. Nevertheless, the following theorem holds.

Theorem 11. The following statements are equivalent for square matrices \( A \) and \( B \) (not necessarily of the same size) with quadratic minimal polynomials.

1. \( A \) and \( B \) have identical pseudospectra.
2. \( A \) and \( B \) are polynomially isometric.
3. \( \|A\| = \|B\| \).

Sketch of the proof. In view of the assumptions on \( A \) and \( B \), we may assume that \( A \) and \( B \) have the form ([8, Chapter 25])

\[
A = \alpha I_r \oplus \beta I_s \oplus \begin{bmatrix} \alpha & \delta_1 \\ 0 & \beta \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \alpha & \delta_t \\ 0 & \beta \end{bmatrix}
\text{ and } B = \alpha I_u \oplus \beta I_v \oplus \begin{bmatrix} \alpha & \gamma_1 \\ 0 & \beta \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \alpha & \gamma_w \\ 0 & \beta \end{bmatrix}
\]

where \( \alpha \) and \( \beta \) are the zeros of \( m_A \) and \( m_B \), \( I_n \) denotes the \( n \times n \) identity matrix, and

\[
\delta_1 \geq \cdots \geq \delta_t \geq 0 \quad \text{and} \quad \gamma_1 \geq \cdots \geq \gamma_w \geq 0.
\]

In this case, the conclusion follows in an analogous manner as in that of Theorem 9 after observing that \( \|A\| = \|B\| \) precisely when \( \delta_1 = \gamma_1 \) by Lemma 3. 

June–July 2021} POLYNOMIALLY ISOMETRIC MATRICES
A. APPENDIX: TWO RESULTS ON IDENTICAL PSEUDOSPECTRA. In this section, we prove two technical results used in this article on matrices (of arbitrary size \( d \)) having identical pseudospectra. The first, stated as Lemma 6 above, concerns equality of the smallest and largest eigenvalues of their “real parts.” The second concerns equality of minimal polynomials.

To begin, recall that the numerical range (or field of values), \( W(T) \), of \( T \in \mathbb{M}_d \) is defined by \( W(T) = \{ \alpha^* T \alpha : \| \alpha \|_C = 1 \} \). It is well known that \( W(T) \) is a compact convex subset of \( \mathbb{C} \) that contains \( \sigma(T) \). As such, \( W(T) \) is the intersection of all closed half-planes \( H \) containing it. With these notions available, we are ready to prove Lemma 6.

Proof of Lemma 6. By a result from [3], a closed half-plane \( H \) satisfies \( W(T) \subseteq H \) if and only if \( \sigma(T) \subseteq H \) and \( \| (z I - T)^{-1} \| \leq 1/\text{dist}(z, H) \) for all \( z \notin H \). Therefore, \( W(A) = W(B) \) when \( A \) and \( B \) have identical pseudospectra, and the desired conclusion is now at hand; after all, the smallest and largest eigenvalues of the self-adjoint matrix \( \text{Re} \ T \) are given by, respectively,

\[
\min_{\| x \|_C = 1} \langle (\text{Re} \ T)x, x \rangle = \min_{z \in W(T)} \text{Re} \ z \quad \text{and} \quad \max_{\| x \|_C = 1} \langle (\text{Re} \ T)x, x \rangle = \max_{z \in W(T)} \text{Re} \ z. \]

Next, we turn to the second result. In [8, Chapter 23], it is stated that matrices having identical pseudospectra must have the same minimal polynomials. Since we were unable to find a direct proof of this fact, we include one for completeness.

Theorem 12. Let \( A \) and \( B \) be square matrices (not necessarily of the same size). If \( A \) and \( B \) have identical pseudospectra, then \( A \) and \( B \) have the same minimal polynomial.

Proof. Recall that the minimal polynomial \( m_T \) of \( T \in \mathbb{M}_d \) is given by

\[
m_T(z) = \prod_{\lambda \in \sigma(T)} (z - \lambda)^{n_T(\lambda)}, \tag{9}\]

where \( n_T(\lambda) \) denotes the index of \( \lambda \in \sigma(T) \), the smallest nonnegative integer such that \( \ker(T - \lambda I)^{n_T(\lambda)} = \ker(T - \lambda I)^d \).

Moreover, the resolvent \( (z I - T)^{-1} \) of \( T \) at \( z \) is given by

\[
(z I - T)^{-1} = \sum_{\lambda \in \sigma(T)} \sum_{k=0}^{n_T(\lambda)-1} (z - \lambda)^{-(k+1)} (T - \lambda I)^k E_{\lambda}, \quad z \notin \sigma(T). \tag{10}\]

Here, \( E_{\lambda_1}, \ldots, E_{\lambda_s} \) are the uniquely defined orthogonal projections on \( \mathbb{C}^d \) such that \( I = E_{\lambda_1} + \cdots + E_{\lambda_s} \) and so that the range \( \text{ran}(E_{\lambda_j}) = \ker(T - \lambda I)^{n_T(\lambda_j)} \) for \( j = 1, \ldots, s \). (The formula in (10) may be verified by multiplying its right-hand side by \( (z I - T) = (z - \lambda)I - (T - \lambda I) \).) In light of (10), the product \( |z - \lambda|^\ell \cdot \|(z I - T)^{-1}\| \) is bounded near \( \lambda \in \sigma(T) \) if and only if \( \ell \geq n_T(\lambda) \).

Now suppose \( A \) and \( B \) are square matrices such that \( \|(z I - A)^{-1}\| \leq \|(z I - B)^{-1}\| \) for all \( z \in \mathbb{C} \). Then \( \sigma(A) \subseteq \sigma(B) \), and \( n_B(\alpha) \geq n_A(\alpha) \) holds for all \( \alpha \in \sigma(A) \) because \( |z - \alpha|^n_B(\alpha) \cdot \|(z I - A)^{-1}\| \) is bounded near \( \alpha \). Hence, \( m_A \) divides \( m_B \) by (9). Reversing the roles of \( A \) and \( B \) then yields the result.

Remark. One can also use the Jordan canonical form to prove Theorem 12. For a proof following that approach, it suffices to note that \( n_T(\lambda) \) equals the size of the largest Jordan block corresponding to \( \lambda \) and then proceed to compute the resolvent.
of the Jordan matrix. From this, one can argue that (as in the proof above) $|z - \lambda|^\ell \cdot \|(zI - T)^{-1}\|$ is bounded near $\lambda \in \sigma(T)$ if and only if $\ell \geq \nu_T(\lambda)$.

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Grades Are Not Money

Your grade is not a bounty paid
for capturing the answer and bringing it dead or alive,
dumping it headfirst into my bucket.
I don’t want my car designed by an engineer
who didn’t learn to engineer
but, by chance, stumbled on the answer.

Your grade is not given in trade
for good behavior or extra credit, without learning,
like nice clothes on a corpse.
I don’t want my child taught by a teacher
who didn’t learn to teach
but did some easy project and came to class on time.

Your grade is not snatched in a raid
from gullible village guards who failed to protect it,
then roaring, raising plunder overhead.
I don’t want my surgery done by a doctor
who didn’t learn to doctor
but got the answers from a website or a friend.

—Submitted by Vadim Ponomarenko, San Diego State University

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