HOPF ALGEBRA STRUCTURE ON GRAPH

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Abstract. In this paper, we construct a bialgebraic and further a Hopf algebraic structure on top of subgraphs of a given graph. Further, we give the dual structure of this Hopf algebraic structure. We study the algebra morphisms induced by graph homomorphisms, and obtain a functor from a graph category to an algebra category.

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1. Introduction

The Hopf algebraic theory has a long history and broad applications in mathematics and physics [3, 9, 15, 23, 25, 28, 29, 31, 33]. The Hopf algebra of rooted forests originated from the work of A. Dür [6] and had a very rich theory. For example, A. Connes and D. Kreimer introduced the Connes-Kreimer Hopf algebra of rooted forests in [3, 24] to study the renormalization of perturbative quantum field theory. In [3, 4, 5], they introduced the commutative Hopf algebras of graphs to study the combinatorial structure of renormalization. It is also related to the Loday-Ronco Hopf algebra [27] and Grossman-Larson Hopf algebra [14] of rooted trees. L. Foissy has done a lot of work on the Hopf algebra of rooted forests [9, 10, 11, 12]. As related results, various infinitesimal bialgebras on decorated rooted forests have been established in [17, 18, 32], via different Hochschild 1-cocycle conditions. However, the Hopf algebra on a given graph does not have similar results with the Hopf algebra of rooted trees.

A totally assigned graph is a graph with a total order on its edges. In [7], G. E. Duchamp et al. built a Hopf algebra of totally assigned graphs. L. Foissy [13] constructed a Hopf algebra on

Date: December 24, 2018.
2010 Mathematics Subject Classification. 16W99, 16T05, 16T10, 16T30.
Key words and phrases. Hopf algebra, graph category, algebra category, covariant functor.
all graphs to insert chromatic polynomial into the theory of combinatorial Hopf algebra, and gave the new proof of some classical results.

The concept of graph algebras was introduced by Schmitt [30]. Namely, a graph algebra is a commutative, cocommutative, graded, connected Hopf algebra, whose basis elements correspond to finite graphs. Humpert and Martin [21] obtained a new nonrecursive formula for the antipode of a graph algebra. In [26], the author introduced the 4-bialgebra of graphs that satisfies some relations.

The study of Hopf algebras of graphs is extensive, but the research on Hopf algebras of subgraphs has no result yet. Subgraph plays an important role in the study of graph theory. For example, we can study the prefect graph by the independence number and clique number of the induced subgraph. We can determine if a graph is an interval map by the induced subgraph and many more.

In the present paper, we construct a Hopf algebraic structure on subgraphs of a given graph. In particular, a combinatorial description of the coproduct is given. Narrowing oneself to the algebraic part of the aforementioned Hopf algebra on subgraphs of a given graph, we obtain an algebra morphism from a graph homomorphism. Using the language of categories, we obtain a covariant functor from the category of graphs to the category of algebras.

Here is the structure of the paper. In Section 2, we first review the definitions of free monoid and free module. Then we proceed to give an algebraic structure on subgraphs of a graph (Lemma 2.7). Next, we construct a coalgebraic structure on subgraphs of a graph (Lemma 2.17), thereby we construct a bialgebra on the subgraph (Theorem 2.21). Finally, to make the coproduct $\Delta_G$ more explicit, we describe it in a combinatorial method (Equation (5)).

In Section 3, continuing the line in Section 2, a Hopf algebra structure on subgraphs of graph is given (Theorem 3.5). Further, we study the dual Hopf algebra structure. We conclude with a description of the dual Hopf algebra(Lemmas 3.7, 3.9). This section is also devoted to an algebra morphism induced by a graph homomorphism (Theorem 3.12). As an application, we obtain a functor from graph category to algebra category (Theorem 3.15).

Notation. Graphs considered in this paper are connected and undirected graphs without multiple edges and loops. We will be working over a unitary commutative base ring $k$. By an algebra we mean a unitary associative $k$-algebra and by a coalgebra we mean a counitary coassociative $k$-coalgebra, unless otherwise stated. Linear maps and tensor products are taken over $k$. Denote by $[n]$ the set $\{1, 2, \cdots, n\}$. For an algebra $A$, we view $A \otimes A$ as an $A$-bimodule via

(1) \[ a \cdot (b \otimes c) := ab \otimes c \quad \text{and} \quad (b \otimes c) \cdot a := b \otimes ca. \]

2. Bialgebra on subgraphs of a given graph.

2.1. An algebraic structure on subgraphs of a graph. Let us first review some notations on graphs and algebras, which are used throughout the remainder of the paper. For a graph $G$, denote by $V(G)$ and $E(G)$ the vertex set and edge set of $G$, respectively.

Definition 2.1. [8] Let $G$ be a graph.

(a) If there is a path between any two vertices of $G$, then $G$ is connected.
(b) A subgraph of a graph $G$ is simply a graph, all of whose vertices belong to $V(G)$ and all of whose edges belong to $E(G)$.
(c) For $S \subseteq V(G)$, the induced subgraph $G[S]$ in $G$ by $S$ is the graph whose edge set consists of all of the edges in $G$ that have both endpoints in $S$. 

In this section, we build a bialgebraic structure on top of subgraphs of a given graph.
The concepts of free monoid and free module are needed later.

**Definition 2.2.** [19]

(a) A **semigroup** is a nonempty set $S$ together with a binary operation $\cdot : S \times S \to S$ which is associative:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \text{ for all } x, y, z \in S.$$  

A semigroup $S$ is called a **monoid** if it contains an element $1$ with the property that

$$x \cdot 1 = 1 \cdot x = x \text{ for all } x \in S.$$  

(b) Let $(S, \cdot_S, 1_S)$ and $(T, \cdot_T, 1_T)$ be monoids. A map $\phi : S \to T$ is called a **monoids morphism** if

$$\phi(x \cdot_S y) = \phi(x) \cdot_T \phi(y) \text{ and } \phi(1_S) = 1_T \text{ for all } x, y \in S.$$  

(c) The **free semigroup** on a set $X$ is a semigroup $S$ together with a map $i : X \to S$ with the property that, for any semigroup $T$ and map $\phi : X \to T$, there exists a unique morphism $\psi : S \to T$ such that $\psi \circ i = \phi$. Adjoining an identity $1$ to $S$, the **free monoid** on $X$ is obtained.

**Definition 2.3.** [22]

(a) Let $k$ be a ring. A (left) **$k$-module** is an additive abelian group $M$ together with a function $k \times M \to M$, $(k, x) \mapsto kx$ such that for all $k, l \in k$ and $x, y \in M$:

(i) $k(x + y) = kx + ky$.

(ii) $(k + l)x = kx + lx$.

(iii) $k(lx) = (kl)x$.

(b) Let $M$ and $N$ be modules over ring $k$. A map $f : M \to N$ is a **$k$-module morphism** if

$$f(x + y) = f(x) + f(y) \text{ and } f(kx) = kf(x) \text{ for all } x, y \in M \text{ and } k \in k.$$  

(c) The **free $k$-module** on a set $X$ is a $k$-module $F$ together with a map $i : X \to F$ with the property that, for any $k$-module $M$ and map $f : X \to M$, there exists a unique $k$-module morphism $\tilde{f} : F \to M$ such that $\tilde{f} \circ i = f$.

A $k$-module $M$ is free if and only if it has a $k$-linear basis [22].

In the rest of this paper, let $G$ be a graph and $\mathcal{G}$ the set of nonempty connected subgraphs of $G$. Let $M(\mathcal{G}) = \langle \mathcal{G} \rangle$ be the free monoid on $\mathcal{G}$ in which the multiplication is the concatenation, denoted by $m_\mathcal{G}$ and usually suppressed. The unit in $M(\mathcal{G})$ is the empty graph, denote by $\emptyset$. An element $F$ in $M(\mathcal{G})$ is a noncommutative product of connected subgraph in $\mathcal{G}$:

$$F = \Gamma_1 \cdots \Gamma_n, \text{ where } n \geq 0 \text{ and } \Gamma_1, \cdots, \Gamma_n \in \mathcal{G}.$$  

Here we employ the convention that $F = \emptyset$ whenever $n = 0$. We define $\text{bre}(F) := n$ to be the **breath** of $F$. For example,

$$\text{bre}(\emptyset) = 0, \text{bre}(\cdot_1) = 1, \text{bre}(1_1^2 1_2^2 \cdot_1 1_2) = 4.$$  

Denote by

$$H(\mathcal{G}) := kM(\mathcal{G})$$  

the free $k$-module spanned by $M(\mathcal{G})$. Then $H(\mathcal{G})$ is closed under the multiplication $m_\mathcal{G}$.

**Example 2.4.** Here are some examples of $H(\mathcal{G})$.

(a) If $G = \bullet_1$, then $\mathcal{G} = \{\bullet_1\}$, $M(\mathcal{G}) = \langle \bullet_1 \rangle$ and $H(\mathcal{G}) = k\langle \bullet_1 \rangle$.

(b) If $G = 1_1^2$, then

$$\mathcal{G} = \{\cdot_1, \cdot_2, 1_1^2\}, M(\mathcal{G}) = \langle \cdot_1, \cdot_2, 1_1^2 \rangle \text{ and } H(\mathcal{G}) = k\langle \cdot_1, \cdot_2, 1_1^2 \rangle.$$
(c) If \( G = \overline{\square}_3 \), then
\[
\mathcal{G} = \{*, 1, 2, 3, 4, 1_1, 1_2, 1_{-1}, 1_{-2}, 1_{-3}, 1_{-4}, 2, 3, 4, \overline{1}_1, \overline{1}_2, \overline{1}_{-1}, \overline{1}_{-2}, \overline{1}_{-3}, \overline{1}_{-4}, 1_3, 1_4, \overline{1}_3, \overline{1}_4, 1_{-3}, 1_{-4}, 1_3, 1_4, \overline{1}_3, \overline{1}_4, 1_{-3}, 1_{-4}, 1_3, 1_4, \overline{1}_3, \overline{1}_4, 1_{-3}, 1_{-4}\},
\]
\[
M(\mathcal{G}) = \langle *, 1, 2, 3, 4, 1_1, 1_2, 1_{-1}, 1_{-2}, 1_{-3}, 1_{-4}, 2, 3, 4, \overline{1}_1, \overline{1}_2, \overline{1}_{-1}, \overline{1}_{-2}, \overline{1}_{-3}, \overline{1}_{-4}, 1_3, 1_4, \overline{1}_3, \overline{1}_4, 1_{-3}, 1_{-4}, 1_3, 1_4, \overline{1}_3, \overline{1}_4, 1_{-3}, 1_{-4}, 1_3, 1_4, \overline{1}_3, \overline{1}_4, 1_{-3}, 1_{-4}\rangle,
\]
\[
H(\mathcal{G}) = \kappa \langle *, 1, 2, 3, 4, 1_1, 1_2, 1_{-1}, 1_{-2}, 1_{-3}, 1_{-4}, 2, 3, 4, \overline{1}_1, \overline{1}_2, \overline{1}_{-1}, \overline{1}_{-2}, \overline{1}_{-3}, \overline{1}_{-4}, 1_3, 1_4, \overline{1}_3, \overline{1}_4, 1_{-3}, 1_{-4}, 1_3, 1_4, \overline{1}_3, \overline{1}_4, 1_{-3}, 1_{-4}\rangle.
\]

The following is the concepts of tensor products and algebras.

**Definition 2.5.** [28, p. 24]

(a) Let \( M, N, P \) be three \( k \)-modules. A mapping \( f : M \times N \rightarrow P \) is said to be **k-bilinear** if for each \( x \in M \) the mapping \( y \mapsto f(x, y) \) of \( N \) into \( P \) is \( k \)-linear, and for each \( y \in N \) the mapping \( x \mapsto f(x, y) \) of \( M \) into \( P \) is \( k \)-linear.

(b) Let \( M, N \) be \( k \)-modules. Then there exists a pair \( (T, g) \) consisting of an \( k \)-module \( T \) and a \( k \)-bilinear mapping \( g : M \times N \rightarrow T \), with the following property: for any \( k \)-module \( P \) and any \( k \)-bilinear mapping \( f : M \times N \rightarrow P \), there exists a unique \( k \)-linear mapping \( f' : T \rightarrow P \) such that \( f = f' \circ g \). The module \( T \) is called the **tensor product** of \( M \) and \( N \), and is denoted by \( M \otimes N \).

**Definition 2.6.** [28, Definition 1.1] An **algebra** \( (A, m, u) \) over \( k \) is a \( k \)-module \( A \) together with morphisms of \( k \)-modules \( m : A \otimes A \rightarrow A \), called the multiplication, and \( u : k \rightarrow A \), called the unit, such that the diagrams

\[
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{m \otimes id_A} & A \otimes A \\
id_A \otimes m & & m \\
A \otimes A & \xrightarrow{m} & A
\end{array}
\]

and

\[
\begin{array}{ccc}
k \otimes A & \xrightarrow{u \otimes id_A} & A \otimes A \\
\alpha_l & & \alpha_r \\
A & \xrightarrow{m} & A \otimes k
\end{array}
\]

are commutative. Here \( \alpha_l \) and \( \alpha_r \) are the isomorphisms defined by

\[
\alpha_l : k \otimes A \rightarrow A, \ k \otimes x \mapsto kx,
\]
\[
\alpha_r : A \otimes k \rightarrow A, \ x \otimes k \mapsto kx, \ k \in k, x \in A.
\]

The empty graph \( \mathbb{1} \) can also be viewed as a linear map \( 1 : k \rightarrow H(\mathcal{G}) \) given by \( 1_k \rightarrow 1 \).

**Lemma 2.7.** The triple \( (H(\mathcal{G}), m_G, 1) \) is an algebra.

**Proof.** It follow from

\[
F_1(F_2F_3) = (F_1F_2)F_3 \text{ for } F_1, F_2, F_3 \in H(\mathcal{G})
\]

and

\[
\mathbb{1}F = F = F\mathbb{1} \text{ for } F \in H(\mathcal{G}).
\]

This completes the proof. \( \square \)
2.2. A coproduct on subgraphs of a graph. In this subsection, we construct a coproduct on subgraphs of a graph. Let us first recall

**Definition 2.8.** [28, Definition 2.1] A coalgebra $(C, \Delta, \varepsilon)$ over $k$ is a $k$-module $C$ together with morphisms of $k$-modules $\Delta := C \to C \otimes C$, called the coproduct, and $\varepsilon := C \to k$, called the counit, such that the diagrams

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\downarrow & & \downarrow \\
C \otimes C & \xrightarrow{\Delta \otimes \text{id}_C} & C \otimes C \otimes C
\end{array}
\]

and

\[
\begin{array}{ccc}
k \otimes C & \xrightarrow{\varepsilon \otimes \text{id}_C} & C \otimes C \\
\downarrow & & \downarrow \\
C & \xrightarrow{\text{id}_C \otimes \varepsilon} & C \otimes k
\end{array}
\]

are commutative. Here $\beta_1$ and $\beta_r$ are the isomorphisms defined by

\[
\beta_1 : C \to k \otimes C, \quad x \mapsto 1_k \otimes x,
\]

\[
\beta_r : C \to C \otimes k, \quad x \mapsto x \otimes 1_k, \quad x \in C.
\]

**Definition 2.9.** [1, p. 54] Let $(C, \Delta, \varepsilon)$ be a coalgebra. A $k$-linear subspace $D$ of a coalgebra $C$ is called a $k$-subcoalgebra of $C$. If $D$ is a coalgebra with the restriction of $\Delta$ and $\varepsilon$.

Now we define a coproduct $\Delta_G$ on $H(G)$. By linearity, we only need to define $\Delta_G(F)$ for basis elements $F \in M(G)$. If $\bre(F) = 0$, then $F = 1$ and define

\[
\Delta_G(F) := \Delta_G(1) := 1 \otimes 1.
\]

If $\bre(F) = 1$, then $F \in G$ and we define

\[
\Delta_G(F) := \sum_{V_1 \cup V_2 = \bre(F)} F[V_1] \otimes F[V_2],
\]

where $F[V_i]$ are induced subgraphs of $F$ by vertex sets $V_i$ with $i = 1, 2$. Note that $F[V_1], F[V_2] \in M(G)$ and $\Delta_G(F) \in H(G) \otimes H(G)$.

If $\bre(F) > 1$, then $F = \Gamma_1 \cdots \Gamma_n$ for some $n \geq 2$ and $\Gamma_1, \ldots, \Gamma_n \in G$, and define

\[
\Delta_G(F) := \Delta_G(\Gamma_1) \cdots \Delta_G(\Gamma_n).
\]

Again note that

\[
\Delta_G(F) \in H(G) \otimes H(G) \text{ by } \Delta_G(\Gamma_i) \in H(G) \otimes H(G) \text{ for each } i = 1, \ldots, n.
\]

Let us give some examples for better insight into $\Delta_G$.

**Example 2.10.** Let $G = 1^2_1$. Then $1^2_1$ and $1^2_1 1^2_1$ are in $M(G)$, and

\[
\Delta_G(1^2_1) = 1 \otimes 1^2_1 + 1^1_1 \otimes 1^1_2 + 1^1_2 \otimes 1^1_1 + 1^1_2 \otimes 1,
\]

\[
\Delta_G(1^2_1 1^2_1) = \Delta_G(1^2_1) \Delta_G(1^2_1)
\]

\[
= 1 \otimes 1^2_1 1^2_1 + 1^1_1 \otimes 1^2_1 1^1_2 + 1^1_2 \otimes 1^1_1 1^2_1 + 1^1_2 1^2_1
\]

\[
+ 1^1_1 \otimes 1^2_1 1^2_1 + 1^1_2 1^2_1 1^1_2 + 1^1_2 1^2_1 1^1_1 + 1^1_1 1^2_1 1^2_1
\]

\[
+ 1^1_2 \otimes 1^2_1 1^2_1 + 1^1_2 1^2_1 1^1_2 + 1^1_2 1^2_1 1^1_1 + 1^1_1 1^2_1 1^2_1.
\]
and so

Example 2.11. Let $\Gamma = 1_{1 \cdot 2}$. If $F = 1_{1 \cdot 2}$ is a subgraph of $\Gamma$, then

$$V(F) = \{1, 2\} \cup \emptyset = \{1\} \cup \{2\} = \{1\} \cup \{2\} = \emptyset \cup \{1, 2\},$$

and so

$$\Delta_G(1_{1 \cdot 2}) = 1_{1 \cdot 2} \otimes 1 + 1 \otimes 1_{1 \cdot 2} + 1_{1 \cdot 2} \otimes 1 + 1_{1 \cdot 2} \otimes 1_{1 \cdot 2}.$$

Further, the coproduct $\Delta_G$ can be extended to $H(G) = kM(G)$ by linearity. Let us compute some examples for better understanding of Eq. (5).

Example 2.12. Consider $\Gamma = 1_{1 \cdot 3}$. Then

$$\Delta_G(1_{1 \cdot 3}) = 1_{1 \cdot 3} \otimes 1 + 1 \otimes 1_{1 \cdot 3} + 1_{1 \cdot 3} \otimes 1 + 1_{1 \cdot 3} \otimes 1_{1 \cdot 3},$$

and

$$\Delta_G(1_{1 \cdot 3}^5) = 1_{1 \cdot 3}^5 \otimes 1 + 1 \otimes 1_{1 \cdot 3}^5 + 1_{1 \cdot 3} \otimes 5 + 1_{1 \cdot 3} \otimes 1 + 1 \otimes 1_{1 \cdot 3}^5 + 5 \otimes 1_{1 \cdot 3}^5 + 2 \otimes 1_{1 \cdot 3}^5 + 1 \otimes 1_{1 \cdot 3}^5 + 1_{1 \cdot 3}^5 \otimes 1 + 1_{1 \cdot 3}^5 \otimes 1_{1 \cdot 3}^5 + 1_{1 \cdot 3}^5 \otimes 1_{1 \cdot 3}^5 + 1_{1 \cdot 3}^5 \otimes 1_{1 \cdot 3}^5 + 1_{1 \cdot 3}^5 \otimes 1_{1 \cdot 3}^5.$$
We have in Subsection 2.2. For this, it suffices to show that \( \Delta_G \) in Eq. (5) satisfies Eqs. (2)—(4). For the cases of Eqs. (2) and (3), it follows directly from the definition of \( \Delta_G \) in Eq. (5). For the case of Eq. (4), we have

\[
\Delta_G(\Gamma_1 \cdots \Gamma_n) = \Delta_G(\Gamma_1) \cdots \Delta_G(\Gamma_n).
\]

**Proof.** We have

\[
\Delta_G(\Gamma_1 \cdots \Gamma_n) = \sum_{U \subseteq V = V(\Gamma_1, \cdots, \Gamma_n)} (\Gamma_1 \cdots \Gamma_n)[U] \otimes (\Gamma_1 \cdots \Gamma_n)[V] \quad (\text{by Eq. } (5))
\]

\[
= \sum_{U_i \subseteq V_i = V(\Gamma_i)} (\Gamma_1 \cdots \Gamma_n)[U_1 \cup \cdots \cup U_n] \otimes (\Gamma_1 \cdots \Gamma_n)[V_1 \cup \cdots \cup V_n]
\]

\[
= \sum_{U_i \subseteq V_i = V(\Gamma_i)} \Gamma_1[U_1] \cdots \Gamma_n[U_n] \otimes \Gamma_1[V_1] \cdots \Gamma_n[V_n]
\]

\[
= \left( \sum_{U_i \subseteq V_i = V(\Gamma_i)} \Gamma_1[U_1] \otimes \Gamma_1[V_1] \right) \cdots \left( \sum_{U_n \subseteq V_n = V(\Gamma_n)} \Gamma_n[U_n] \otimes \Gamma_n[V_n] \right)
\]

\[
= \Delta_G(\Gamma_1) \cdots \Delta_G(\Gamma_n) \quad (\text{by Eq. } (5)).
\]

This completes the proof. \( \square \)

So we conclude

**Proposition 2.14.** The \( \Delta_G \) given in Eq. (5) coincides with the \( \Delta_G \) given in Subsection 2.2.

**Remark 2.15.** The coproduct \( \Delta_G \) is cocommutative by the Definition 5.

### 2.4. A coalgebraic structure on subgraphs of a given graph.

In this subsection, we obtain a coalgebraic structure on subgraphs of a given graph \( G \). The following result shows that \( H(G) \) is closed under the coproduct \( \Delta_G \).

**Lemma 2.16.** Let \( G \) be a graph and \( F \in H(G) \). Then \( \Delta_G(F) \in H(G) \otimes H(G) \).

**Proof.** By linearity, it is suffices to consider basis elements \( F \in M(G) \). Then by Eq. (5), we have

\[
\Delta_G(F) = \sum_{U \subseteq V = V(F)} F[U] \otimes F[V].
\]

Then \( F[U], F[V] \in H(G) \) by definition and so \( \Delta_G(F) \in H(G) \otimes H(G) \). \( \square \)
Define a linear map $\varepsilon_G : H(\mathcal{G}) \to k$ by taking

$$
\varepsilon_G(F) := \begin{cases} 
1_k, & \text{if } F = 1, \\
0, & \text{if } F \neq 1.
\end{cases}
$$

(6)

**Lemma 2.17.** Let $G$ be a graph. Then the triple $(H(\mathcal{G}), \Delta_G, \varepsilon_G)$ is a coalgebra.

**Proof.** We first show the coassociativity

$$(\text{id} \otimes \Delta_G) \Delta_G(F) = (\Delta_G \otimes \text{id}) \Delta_G(F) \text{ for } F \in H(\mathcal{G}).$$

By linearity, it suffices to consider basis elements $F \in M(\mathcal{G})$. We have

$$(\text{id} \otimes \Delta_G) \Delta_G(F) = (\text{id} \otimes \Delta_G) \left( \sum_{U \oplus V = V(F)} F[U] \otimes F[V] \right) \text{ (by Eq. (5))}$$

$$= \sum_{U \oplus V = V(F)} F[U] \otimes \left( \sum_{V' \oplus V'' = V(F[V])} (F[V])[V'] \otimes (F[V])[V''] \right) \text{ (by Eq. (5))}$$

$$= \sum_{U \oplus V = V(F)} F[U] \otimes \sum_{V' \oplus V'' = V(F[V])} F[V'] \otimes F[V''] \text{ (by } V(F[V]) = V)$$

$$= \sum_{U \oplus V = V(F)} (F[U \oplus V'])[U] \otimes (F[U \oplus V']')[V'] \otimes F[V'']$$

$$= \sum_{W \oplus V' = V'(F)} \left( \sum_{U \oplus V = W} F[W][U] \otimes F[W][V'] \right) \otimes F[V''] \text{ (by } V(F[W]) = W)$$

$$= \sum_{W \oplus V' = V'(F)} \Delta_G(F[W]) \otimes F[V''] \text{ (by } U \oplus V' = W)$$

$$= (\Delta_G \otimes \text{id}) \left( \sum_{W \oplus V' = V'(F)} F[W] \otimes F[V''] \right)$$

Thus $\Delta_G$ is coassociative. Next we prove the counicity of $\varepsilon_G$. For $F \in M(\mathcal{G})$,

$$(\varepsilon_G \otimes \text{id}) \Delta_G(F) = (\varepsilon_G \otimes \text{id}) \left( \sum_{U \oplus V = V(F)} F[U] \otimes F[V] \right) \text{ (by Eq. (5))}$$

$$= (\varepsilon_G \otimes \text{id}) \left( 1 \otimes F + F \otimes 1 + \sum_{U \oplus V = V(F)} F[U] \otimes F[V] \right)$$
\[ \varepsilon_G(\mathbb{1}) \otimes F + \varepsilon_G(F) \otimes \mathbb{1} + \sum_{U \cup V = V(F) \setminus \{F\}} \varepsilon_G(F[U]) \otimes F[V] \]

\[ = 1_k \otimes F = \beta_l(F) \quad (\text{by Eq. (6)}). \]

With the same argument, we can show \((\text{id} \otimes \varepsilon_G)\Delta_G = \beta_r\). Here \(\beta_l\) and \(\beta_r\) are the isomorphisms defined in Definition 2.8. This completes the proof. \(\square\)

**Corollary 2.18.** Let \(G\) be a graph and \(G'\) be a subgraph of graph \(G\). Then the triple \((H(G'), \Delta_G, \varepsilon_G)\) is a subcoalgebra of coalgebra \((H(G), \Delta_G, \varepsilon_G)\).

**Proof.** It follows from the Lemma 2.17. \(\square\)

**Corollary 2.19.** Let \(G\) be a graph. Then \((H(\bullet), \Delta_G, \varepsilon_G)\) is a simple subcoalgebra, for any \(\bullet \in V(G)\).

**Proof.** The triple \((H(\bullet), \Delta_G, \varepsilon_G)\) is a subcoalgebra by Corollary 2.18. Let \((H, \Delta_G, \varepsilon_G)\) be a subgraph of \((H(\bullet), \Delta_G, \varepsilon_G)\) and \(H \neq 0, H \neq H(\bullet)\). Then for any \(F \in H\), we have

\[ \Delta_G(F) = \sum_{U \cup V = V(F) \setminus \{F\}} F[U] \otimes F[V] = \bullet \otimes F[V(F) \setminus \bullet] + \sum_{U \cup V = V(F) \setminus \{F\}} F[U] \otimes F[V] \notin H \otimes H. \]

Our proof is complete. \(\square\)

**2.5. A bialgebra structure on subgraphs of a given graph.** Combing the algebraic structure obtained in Subsection 2.1 and the coalgebraic structure obtained in Subsection 2.4, we build a bialgebraic structure on top of subgraphs of a given graph. Let us review the concept of bialgebras.

**Definition 2.20.** (a) [16, p. 49] Let \((H, m_H, u_H)\) and \((L, m_L, u_L)\) be two algebras. A map \(f: H \rightarrow L\) is called an algebra morphism if

\[ m_L \circ (f \otimes f) = f \circ m_H \quad \text{and} \quad u_L = f \circ u_H. \]

(b) [16, p. 51] A bialgebra is a quintuple \((H, m, u, \Delta, \varepsilon)\), where \((H, m, u)\) is an algebra and \((H, \Delta, \varepsilon)\) is a coalgebra such that \(\Delta: H \rightarrow H \otimes H\) and \(\varepsilon: H \rightarrow k\) are morphisms of algebras.

Now, we arrive at our main result in this subsection.

**Theorem 2.21.** Let \(G\) be a graph. Then the quintuple \((H(G), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)\) is a bialgebra.

**Proof.** The triple \((H(G), m_G, \mathbb{1})\) is an algebra by Lemma 2.7 and the triple \((H(G), \Delta_G, \varepsilon_G)\) is a coalgebra by Lemma 2.17. Further, the coproduct \(\Delta_G\) is an algebra morphism by Eq. (4). We are left to prove that \(\varepsilon_G\) is an algebra morphism. For this, let \(F_1, F_2 \in M(G)\). Then by Eq. (6)

\[ \varepsilon_G(F_1 F_2) = \begin{cases} 1_k, & \text{if } F_1 \text{ and } F_2 \text{ are empty graphs,} \\ 0, & \text{others,} \end{cases} \]

and

\[ \varepsilon_G(F_1) \varepsilon_G(F_2) = \begin{cases} 1_k, & \text{if } F_1 \text{ and } F_2 \text{ are empty graphs,} \\ 0, & \text{others.} \end{cases} \]

Thus, \(\varepsilon_G(F_1 F_2) = \varepsilon_G(F_1) \varepsilon_G(F_2)\) and \(\varepsilon_G\) is an algebra morphism. This completes the proof. \(\square\)
3. Hopf algebra on subgraphs of a graph

In this section, we construct a Hopf algebra structure on subgraphs of a given graph $G$.

3.1. Hopf algebraic structure on subgraphs of a graph. Let us review some concepts needed later.

**Definition 3.1.** [1, p. 61] Let $(C, \Delta, \varepsilon)$ be a coalgebra and $(A, m, u)$ an algebra. For $f, g \in \text{Hom}(C, A)$,

$$f \ast g := m \circ (f \otimes g) \circ \Delta$$

is said to be the convolution of $f$ and $g$.

**Definition 3.2.** [1, p. 61] Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra. Then $\text{Hom}(H, H)$ is a $k$-algebra with structure maps

$$m_{\text{Hom}(H, H)}(f \otimes g) = f \ast g \quad \text{and} \quad u_{\text{Hom}(H, H)}(k) = ku \circ \varepsilon.$$  

When the identity map $\text{id}_H$ of $H$ is a regular element of $\text{Hom}(H, H)$ with respect to multiplication on $\text{Hom}(H, H)$, the inverse $S$ of $\text{id}_H$ is called the antipode of $H$. The antipode $S$ satisfies the following condition:

$$S \ast \text{id}_H = \text{id}_H \ast S = u \circ \varepsilon.$$  

A bialgebra with an antipode is called a **Hopf algebra**.

**Definition 3.3.** [16, Definition 2.3.1] A bialgebra $(H, m, u, \Delta, \varepsilon)$ is called a graded bialgebra if there are $k$-submodules $H^{(n)}$, $n \geq 0$, of $H$ such that

\begin{enumerate}
  \item $H = \bigoplus_{n=0}^{\infty} H^{(n)}$;
  \item $H^{(p)}H^{(q)} \subseteq H^{(p+q)}$, $p, q \geq 0$;
  \item $\Delta(H^{(n)}) \subseteq \bigoplus_{p+q=n} H^{(p)} \otimes H^{(q)}$, $n \geq 0$.
\end{enumerate}

Elements of $H^{(n)}$ are said to have degree $n$. $H$ is called **connected** if $H^{(0)} = k$ and $\ker \varepsilon = \bigoplus_{n \geq 1} H^{(n)}$.

**Lemma 3.4.** [3, 12, 16] Any connected graded bialgebra $H$ is a Hopf algebra. The antipode $S$ is given by:

$$S(x) = \sum_{k \geq 0} (e - \text{id}_H)^k(x).$$

It is also defined by $S(1_H) = 1_H$ and recursively by any of the two formulas:

$$S(x) = -x - \sum_{(x)} S(x')x''$$  

$$S(x) = -x - \sum_{(x)} x'S(x''), \quad x \in \ker \varepsilon.$$  

Here $x'$, $x''$ are from

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{(x)} x' \otimes x''.$$
We proceed to prove that $H(\mathcal{G})$ is a connected graded bialgebra. For this, denoted by

$$H(\mathcal{G})^{(n)} = k\left\{ F \in M(\mathcal{G}) \left| |V(F)| = n \right. \right\} \quad \text{for } n \geq 0,$$

where $V(F)$ is the multiset of vertices of $F$. Now we arrive at one of our main results in this section.

**Theorem 3.5.** Let $G$ be a graph. Then

(a) The quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a connected graded bialgebra.

(b) The quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a Hopf algebra with antipode $S = \sum_{k \geq 0} (e - \text{id}_H)^\varepsilon_k$.

**Proof. (a).** By the definition of $H(\mathcal{G})$, we obtain

$$H(\mathcal{G}) = \bigoplus_{n=0}^{\infty} H(\mathcal{G})^{(n)}.$$ 

Let $F_1 \in H(\mathcal{G})^{(p)}$ and $F_2 \in H(\mathcal{G})^{(q)}$ with $p, q \geq 0$. Then

$$|V(F_1)| = p, \quad |V(F_2)| = q \quad \text{and} \quad |V(F_1F_2)| = pq,$$

whence

$$H(\mathcal{G})^{(p)}H(\mathcal{G})^{(q)} \subseteq H(\mathcal{G})^{(p+q)}.$$ 

By Eq. (5),

$$\Delta_G(H(\mathcal{G})^{(n)}) \subseteq \bigoplus_{p+q=n} H(\mathcal{G})^{(p)} \otimes H(\mathcal{G})^{(q)}.$$ 

Thus, the quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a graded bialgebra. Further,

$$H(\mathcal{G})^{(0)} = k\{\mathbb{1}\} = k \quad \text{and} \quad \ker \varepsilon = \bigoplus_{n\geq1} H(\mathcal{G})^{(n)} \quad \text{by Eq. (6)}.$$ 

Therefore, the quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a connected graded bialgebra.

**(b).** The quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a bialgebra by Theorem 2.21 Further, it is a Hopf algebra with antipode $S = \sum_{k \geq 0} (e - \text{id}_H)^\varepsilon_k$ by Lemma 3.4 and Item (a). \hfill \Box

### 3.2. The dual Hopf algebra.

In this subsection, we consider the dual Hopf algebra of $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$. The following elementary result is fundamental [12].

**Lemma 3.6.** Let $H = \bigoplus_{n\in\mathbb{N}} H^{(n)}$ be a graded space. Then

(a) The graded dual $H^*$ is $\bigoplus_{n\in\mathbb{N}} H^{(n)^*}$. Note that $H^*$ is also a graded space, and $H^{**} \approx H$.

(b) $H \otimes H$ is also a graded space, with $(H \otimes H)(n) = \sum_{i=1}^{n} H(i) \otimes H(n-i)$ for all $n \in \mathbb{N}$. Moreover,

$$H \otimes H \simeq H^* \otimes H^*.$$ 

Where the symbol “$\approx$” is isomomorphism.

The quintuple $(H(\mathcal{G}), m_G, \mathbb{1}, \Delta_G, \varepsilon_G)$ is a graded Hopf algebra by Theorem 3.5. Then its graded dual inherits also a graded Hopf algebra, writed $(H(\mathcal{G})^*, \Delta_G^*, \varepsilon_G^*, m_G^*, \mathbb{1}^*)$, by Lemma 3.6.

For any $F \in M(\mathcal{G})$, we define the following element of the graded dual $H(\mathcal{G})^*$,

$$(7) \quad Z_F = \left\{ \begin{array}{ll} H(\mathcal{G}) & \rightarrow k, \\
G \in M(\mathcal{G}) & \rightarrow \delta_{F,G}, \end{array} \right.$$ 

where $\delta_{F,G}$ is Kronecker function. Then $(Z_F)_{F\in M(\mathcal{G})}$ is a basis of $H(\mathcal{G})^*$. 
Lemma 3.7. Let $G$ be a graph. The product of $Z_F$ and $Z_G$ is given by

$$(8) \quad Z_F Z_G = \sum_H Z_{H,*} \quad \text{for } F, G, H \in M(\mathcal{G}),$$

where $H$ is the induced subgraph in $M(\mathcal{G})$ by $V(F)$ and $V(G)$.

Proof. Let $G$ be a graph. We consider the basis elements of $H(\mathcal{G})^*$. For any $F, G, H \in M(\mathcal{G})$, we have

$$Z_F Z_G(H) = (Z_F \otimes Z_G)(\Delta_G(H))$$

$$= (Z_F \otimes Z_G)\left( \sum_{U \subseteq V(H)} H[U] \otimes H[V] \right)$$

$$= \sum_{U \subseteq V(H)} Z_F(H(U)) \otimes Z_G(H(V))$$

$$= \sum_{U \subseteq V(H)} \delta_{F,H(U)} \otimes \delta_{G,H(V)},$$

this equation not equal to 0 if and only if $H$ is the induced subgraph in $M(\mathcal{G})$ by $V(F)$ and $V(G)$. \hfill \Box

Example 3.8. Let $G$ be a graph in Item (3) of Example 2.4. We consider the product of $H(\mathcal{G})^*$.

(a) Let $F = \cdot_1$ and $G = 1^1$. Then

$$Z_{,1} Z_{1^1} = Z_{,1} 1^1 + Z_{1^1,1}.$$

(b) Let $F = \cdot_3$ and $G = 1^1$. We have

$$Z_{,3} Z_{1^1} = Z_{,3} 1^1 + Z_{1^1,3} + Z_{3,3}.$$

The coproduct $m^*_G$ of $(H(\mathcal{G})^*)^*$, $\Delta_G^*$, $e_G^*$, $m^*_G$, $1^*$ is given by

Lemma 3.9. Let $G$ be a graph. The coproduct of $Z_{\Gamma_1 \cdots \Gamma_n}$ is given by

$$(9) \quad m^*_G(Z_{\Gamma_1 \cdots \Gamma_n}) = \sum_{i \in [1, \ldots, n]} Z_{\Gamma_{i,1} \cdots \Gamma_{i,n}} \otimes Z_{\Gamma_{i+1} \cdots \Gamma_n}, \quad \text{for } \Gamma_1, \cdot \cdot \cdot , \Gamma_n \in \mathcal{G},$$

where $\{1, \ldots, n\} = \{i, \cdot \cdot \cdot, j\} \cup \{k, \cdot \cdot \cdot, l\}$.

Proof. We consider the basis elements of $H(\mathcal{G})^*$. For any $\Gamma_1, \cdot \cdot \cdot , \Gamma_n, F, G \in M(\mathcal{G})$, we have

$$m^*_G(Z_{\Gamma_1 \cdots \Gamma_n})(F \otimes G) = Z_{\Gamma_1 \cdots \Gamma_n}(FG) = \delta_{\Gamma_1 \cdots \Gamma_n, FG},$$

this equation not equal to 0 if and only if $FG = \Gamma_1 \cdots \Gamma_n$ by commutative, as desired. \hfill \Box

Example 3.10. Let $G$ be a graph in Item (3) of Example 2.4. We consider the coproduct of $H(\mathcal{G})^*$. Let $\Gamma_1 = \cdot_1$ and $\Gamma_2 = 1^1$. Then

$$Z_{,1} 1^1 = Z_{,1} 1^1 \otimes Z_{,1} 1^1 \otimes Z_{,1} 1^1 + Z_{,1} 1^1 \otimes Z_{,1} 1^1 + Z_{,1} 1^1 \otimes Z_{,1} 1^1.$$
3.3. **Algebra morphisms induced from graph homomorphisms.** In this subsection, we are going to consider algebra morphisms induced by graph homomorphisms. For this, let us recall the concept of a graph homomorphism.

**Definition 3.11.** [20] p. 4] Let \( G_1 \) and \( G_2 \) be two graphs. A homomorphism of \( G_1 \) to \( G_2 \), written as \( f : G_1 \to G_2 \), is a mapping \( f : V(G_1) \to V(G_2) \) such that \( (f(u), f(v)) \in E(G_2) \) whenever \( (u, v) \in E(G_1) \).

Next, we give the main result in this subsection. Let \( f : G_1 \to G_2 \) be a graph homomorphism. Then \( f \) can be restricted to any subgraphs of \( G_1 \), still denoted by \( f \). We define the linear map \( H(f) : H(G_1) \to H(G_2) \) given by

\[
H(f)(1_{G_1}) := 1_{G_2}
\]
and

(10) \( H(f)(F) := H(f)(\Gamma_1 \cdots \Gamma_n) := f(\Gamma_1) \cdots f(\Gamma_n) \) for \( F = \Gamma_1 \cdots \Gamma_n \in M(G_1) \) with \( n \geq 1 \).

**Theorem 3.12.** The \( H(f) : H(G_1) \to H(G_2) \) is an algebra morphism.

**Proof.** Let \( F_1 = \Gamma_1 \cdots \Gamma_m \) and \( F_2 = \Gamma'_1 \cdots \Gamma'_n \in M(G_1) \).

Then by Eq. (10),

\[
H(f)(F_1 F_2) = H(f)(\Gamma_1 \cdots \Gamma_m \Gamma'_1 \cdots \Gamma'_n) = f(\Gamma_1) \cdots f(\Gamma_m)f(\Gamma'_1) \cdots f(\Gamma'_n)
\]
\[
= H(f)(\Gamma_1 \cdots \Gamma_m)H(f)(\Gamma'_1 \cdots \Gamma'_n)
\]
\[
= H(f)(F_1)H(f)(F_2).
\]

Thus \( H(f) \) is an algebra morphism. This completes the proof. \( \square \)

As an application of Theorem 3.12, we obtain a functor from the graph category to the algebra category. Let us recall

**Definition 3.13.** [22] Definition 7.1] A **category** is a class \( \mathcal{C} \) of objects (denoted \( A, B, C, \cdots \)) together with

(a) a class of disjoint sets, denoted \( \text{hom}(A, B) \), one for each pair of objects in \( \mathcal{C} \); (an element \( f \) of \( \text{hom}(A, B) \) is called a morphism from \( A \) to \( B \) and is denoted \( f : A \to B \))
(b) for each triple \( (A, B, C) \) of objects of \( \mathcal{C} \) a function:

\[
\text{hom}(B, C) \times \text{hom}(A, B) \to \text{hom}(A, C);
\]

(for morphisms \( f : A \to B, g : B \to C \), this function is written \( (g, f) \mapsto g \circ f \) and \( g \circ f : A \to C \) is called the composite of \( f \) and \( g \)); all subject to the two axioms:

(i) **Associativity.** If \( f : A \to B, g : B \to C, h : C \to D \) are morphisms of \( \mathcal{C} \), then
\[
h \circ (g \circ f) = (h \circ g) \circ f.
\]

(ii) **Identity.** For each object \( B \) of \( \mathcal{C} \) there exists a morphism \( 1_B : B \to B \) such that for any \( f : A \to B, g : B \to C \),

\[
1_B \circ f = f \quad \text{and} \quad g \circ 1_B = g.
\]

**Definition 3.14.** [22] Definition 1.1.] Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories. A **covariant functor** \( T \) from \( \mathcal{C} \) to \( \mathcal{D} \) (denoted \( T : \mathcal{C} \to \mathcal{D} \)) is a pair of functions (both denoted by \( T \)), an object function that assigns to each object \( C \) of \( \mathcal{C} \) an object \( T(C) \) of \( \mathcal{D} \) and a morphism function which assigns to each morphism \( f : C \to D \) of \( \mathcal{C} \) a morphism

\[
T(f) : T(C) \to T(D)
\]
of $\mathcal{C}$, such that

(a) $T(1_C) = 1_{T(C)}$ for every identity morphism $1_C$ of $\mathcal{C}$;
(b) $T(g \circ f) = T(g) \circ T(f)$ for any two morphisms $f, g$ of $\mathcal{C}$ whose composite $g \circ f$ is defined.

Let $\mathcal{G}$ be the class of all graphs, for $G_1, G_2 \in \mathcal{G}$, $\text{hom}(G_1, G_2)$ is the set of all graph homomorphisms $f : G_1 \rightarrow G_2$. Then $\mathcal{G}$ is a category, called the graph category. Let $\mathcal{A}$ be the $k$-algebra category whose objects are all $k$-algebra; the $\text{hom}(A_1, A_2)$ is the set of all $k$-algebra morphisms $\varphi : A_1 \rightarrow A_2$.

Now we arrive at our main result of this section.

**Theorem 3.15.** Let $\mathcal{G}$ be the graph category and $\mathcal{A}$ the $k$-algebra category. Define

$$H(-) : \mathcal{G} \rightarrow \mathcal{A}, \quad G \mapsto H(G), \quad f \mapsto H(f),$$

where $G$ is a graph in $\mathcal{G}$, $f : G_1 \rightarrow G_2$ is a graph homomorphism and $H(f) : H(G_1) \rightarrow H(G_2)$ is the algebra morphism obtained in Theorem 3.12. Then $H(-)$ is a covariant functor.

**Proof.** By Lemma 2.7 and Theorem 3.12 it suffices to prove Items (a) and (b) in Definition 3.14. For Item (a), let $F = \Gamma_1 \cdots \Gamma_n \in M(\mathcal{G})$. Then by Eq. (10),

$$H(1_G)(F) = H(1_G)(\Gamma_1 \cdots \Gamma_n) = 1_G(\Gamma_1) \cdots 1_G(\Gamma_n) = \Gamma_1 \cdots \Gamma_n,$$

where the third equation employs the fact that the image of a connected subgraph of a graph homomorphism is a connected subgraph. Thus $H(g) \circ H(f)(F) = H(g \circ f)(F)$. This completes the proof.

**Acknowledgments:** This work was supported by the National Natural Science Foundation of China (Grant No. 11771191, 11501267 and 11861051), Fundamental Research Funds for the Central Universities (Grant No. lzujbky-2017-162), the Natural Science Foundation of Gansu Province (Grant No. 17JR5RA175).

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