On the predictions of dip-effect in $Q^2$ dependence of electromagnetic and electroweak formfactors of $\pi$-meson decays and their experimental verification.

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Abstract

In present note the arguments in favour of high statistics measurements of the form-factors that describe $\pi$-meson decays into a lepton-antilepton pair plus a photon are given. It is shown that these formfactors may contain an important information on the dynamics of quark motion inside a hadron.

Key-words: meson decays, decay formfactors, quarks, bound state

1 Introduction.

The aim of the present note is to discuss the arguments in favour of future high statistic measurements of $\pi$-mesons decay formfactors that contain the information on quark structure of mesons. The decays of light $\pi$-meson such as $\pi^0 \rightarrow \gamma + e^+ e^-$ and $\pi^\pm \rightarrow \gamma + e^\pm \nu$ (in what follows they would be denoted as $\pi^0 \rightarrow \gamma + ee$ and $\pi \rightarrow \gamma + e\nu$ processes) are of a big interest from the viewpoint of studying the dynamics of quark motion inside a hadron.

The pioneering works on kinematics of these processes and the relation between the vector $F_V$ and the axial-vector $F_A$ formfactors as well as on the relation with the width of $\pi^0 \rightarrow \gamma \gamma$ process and also on the account of Inner Bresstrahlung amplitude contribution one can find in [1].

There is a plenty of different models based on current algebra, vector dominance model, chiral field theories and etc. (see, for instance, the reviews [2]) that were proposed to describe the formfactors dependence on a square of the invariant mass of a final state lepton pair. This variable can also be interpreted as a square of a 4-momentum $Q^2$ transfered from the hadronic block of the corresponding Feynman diagram to a leptonic pair. Only one particular prediction on a shape of a $Q^2$-dependence of the decay formfactor, namely, on the appearance of a dip in a region of small values of $x = Q^2/M_{\pi}^2$ variable [3], would be discussed here. It should be noted that the high statistic experiment done at Saclay 6 years later had presented the data [4] (for more details see the reviews in [3] and [4]) that may be interpreted as an experimental confirmation of the prediction done in [3]. Nevertheless, the systematic errors quoted in [4] are rather high, so more precise measurements, especially in a region of small values of $x$, would be obviously very important. In this note
it will be shown also that the analogous dip-effect may reveal itself in the formfactors of
the electroweak decay $\pi^\pm \rightarrow \gamma + e^\pm \nu$. So, it would be very interesting to perform with
good systematic errors new high statistic measurements of the behavior of $F_{\pi^0 \rightarrow \gamma e^+e^-}(Q^2)$
formfactor in a region of small values of $Q^2$ as well as to collect the analogous data on
$F_{\pi^\pm \rightarrow \gamma e^\pm \nu}(Q^2)$ formfactor.

The prediction done in [3] was obtained in the framework of the relativistic constituent
quark model (see references in [4–11]) which make use of a covariant equation for two-body wave function [12] that was derived on the basis of 3-dimensional quasipotential
approach to two-particle relativistic equations in quantum field theory (QFT) [14].

The important point of this model is that its mathematical apparatus incorporates,
in difference with ordinary QFT amplitudes, the bound state wave functions of quarks.
Really, pion is a bound state of light quark and antiquark tighten together by forces
caused by gluon exchange. Therefore, an application of Feynman diagrams (originally
proposed in QFT for calculation of scattering amplitudes of particles that are free in an
initial state) for describing the processes, that include the bound states in initial state,
may serve, definitely, only as some perturbative model approximation applied in a region
where the nonperturbative effects play an essential role. To this reason it is natural
to expect that the consistent amplitudes to be used to describe the decay processes of
mesons have to include the wave functions that take into account the bound state nature
of mesons.

Finally the prediction of the appearance of a dip is a sequence of three main features
of the considered model:
1. relativistic motion of quarks bounded in spin 0 $\pi$-meson state;
2. Standard Model (perturbative QCD and SM Feynman diagram technique) form
of a quark propagator that enter the quarks interaction amplitude describing the photon
and lepton-antilepton ($e^+e^-$ or $e^\pm \nu$) pair production in a final state;
3. large value of a binding energy in a pion considered as $q\bar{q}$ bound state.

2 Main formulas.

To explain the above statements the main points of the analysis performed in [3] for
$\pi^0 \rightarrow \gamma + ee$ case would be sketched below. The schematic view (not a Feynman
diagram! of $\pi$-meson decay processes that illustrates the corresponding invariant amplitudes
$M_{\pi \rightarrow f}(P|q_1, q_2)$ at quark level ($k_1$ and $k_2$ are quark and antiquark moments, while $p_l$ and
$p_{\bar{l}}$ are the moments of the lepton and the antilepton respectively) is shown in Fig. 1.

In Fig.1 $f$ denotes a final state, i.e. $f = \gamma + l\bar{l}$ for $\pi^0$-decay and $f = \gamma + e\bar{e}$ for
$\pi^\pm$-decay. These amplitudes may be parameterized through the decay formfactors in the
following way ($q_1 = p_+ + p_- = p_l + p_{\bar{l}}$):

$$M_{\pi^0 \rightarrow \gamma e^+e^-}(q_1, q_2|P) = \frac{F_{\pi^0 \rightarrow \gamma e^+e^-}(q_1^2)}{q_1^2} V_{\mu\nu}(q_2|P)(e_j^\mu)(p_+, p_-)e_\nu(q_2),$$  (1)
$$M_{\pi\rightarrow\gamma e} = \left[ F_V(q_1^2) \cdot V_{\mu\nu}(q_2|P) + F_A(q_1^2) \cdot A_{\mu\nu}(q_2|P) \right] \times$$

$$\times \left( (e) \frac{G_F V_{ud}}{\sqrt{2}} \right) \cdot j_{\bar{V}-A}(p_2, p_{\nu}) \cdot e^\nu(q_2) .$$

Here $e^\nu(q_2)$ is a polarization 4-vector of a real photon with the 4-momentum $q_2$, $(e)$ is an electric charge of a lepton and $G_F$ is Fermi coupling constant of weak interaction and $V_{ud} \sim \cos(\theta_c)$ is the element of Cabibbo-Kobayashi-Maskawa matrix. Also the following notations are used for two (“orthogonal” to each other) structure tensors $V_{\mu\nu}$ and $A_{\mu\nu}$:

$$V_{\mu\nu}(q_1|P) = \epsilon_{\mu\nu\alpha\beta} P^\alpha q_1^\beta$$

$$A_{\mu\nu}(q_1|P) = g_{\mu\nu}(P \cdot q_1) - P_\mu q_1^\nu,$$

that define, respectively, the vector and axial formfactors. Both electromagnetic (EM) and electroweak (EW) final state currents can be defined by one and the same formula:

$$j_{\bar{V}-A}(p_1, p_2) = \bar{u}(p_1) \gamma^\mu(V - A) u(p_2),$$

with the factor $(V - A)$ defined as

$$V - A = \begin{cases} V = 1; A = 0; & \text{for } e^+e^-; \\ V = 1; A = \gamma^5; & \text{for } e^\pm\nu . \end{cases}$$

To make clear, why the same arguments that were used in [3] may be valid also for the $F_V(q_1^2)$ and $F_A(q_1^2)$ formfactors of $\pi^\pm \rightarrow \gamma + e\nu$ decay, let us start with the definition of the amplitude of the process.

An amplitude $M_{\pi \rightarrow f}(q_1, q_2|P)$ of a $\pi$-meson (considered as a $q\bar{q}$ bound state) decay process may be presented in a framework of the relativistic quark model as a convolution (with the relativistic invariant differential volume element of the momentum space $\frac{d^3k}{(2\pi)^3}$) of a covariant 2-body bound state (B) wave function $\Psi_{\sigma_1\sigma_2 BP}(k_1)$ and the final state interaction amplitude $T_{q\bar{q} \rightarrow f}$, taken in a form of Feynman matrix element (see Fig.1).

Let us consider a general case of $\pi \rightarrow \gamma^* V^*$ decay, where by $V^*$ the virtual photon $\gamma^*$ or virtual $W^*$ boson (see Fig.1) is denoted. The transition amplitude is defined, following the guide line of [3] and [2], like
\begin{equation}
M_{\pi^0 \to \gamma^* V^*}(q_1, q_2| P) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k_1}{2k_1^0} \Psi_{\sigma_1 \sigma_2 BP}(k_1) T_{q q \to \gamma^* V^*}(q_1, q_2; k_1| P) .
\end{equation}

In (7) the summation over quark polarizations \( \sigma_1, \sigma_2 \) is supposed. Now one may define the formfactor of \( \pi^0 \to \gamma^* \gamma^* \) transition as follows

\begin{equation}
M_{\pi^0 \to \gamma^* \gamma^*}(q_1, q_2| P) = F_{\pi^0 \to \gamma^* \gamma^*}(q_1^2, q_2^2)e_1^\mu e_2^\nu \epsilon_{\mu\nu\rho\sigma} q_1^\rho P^\sigma =

= F_{\pi^0 \to \gamma^* \gamma^*}(q_1^2, q_2^2)e_1^\mu e_2^\nu V_{\mu\nu}(q_1| P) ,
\end{equation}

where \( e_1^\mu \) and \( e_2^\nu \) are the the polarization 4-vectors of two virtual photons with the 4-moments (off mass shell) \( q_1 \) and \( q_2 \) respectively.

After these definitions one can determine the formfactor of \( \pi^0 \to \gamma e^+ e^- \) decay as (see [11], [3])

\begin{equation}
F_{\pi^0 \to \gamma e^+ e^-}(q_1^2) = F_{\pi^0 \to \gamma^* \gamma^*}(q_1^2, 0) .
\end{equation}

The wave function in (7) is a solution of a covariant two-body equation [1], [9], [12] that has a three dimensional form due to the use of a covariant single-time method of describing of a relative motion of quarks in a system where \( \pi \)-meson has the 4-momentum \( P \). It should be mentioned that the relativistic wave function is connected with the vertex function \( \Gamma(P| k_1, k_2) \) according to formula

\begin{equation}
\Psi_{BP}(k_1) = \frac{\Gamma(P| k_1, k_2)}{(k_1^2 - m^2)(k_2^2 - m^2)}
\end{equation}

(we take the masses of quark and antiquark to be equal to \( m_q \)). The vertex function is used in diagram technique to include the interactions between particles, in our case quarks, which lines in Feynman diagrams do enter this vertex. Formula (10) allows to establish a more close analogy with the Feynman diagram approach and with a quark triangle diagram often used to describe the decay process.

Let us note that in perturbative leading order the vertex function, or the triangle Feynman matrix element (corresponding to decay amplitude), is set to be \( \Gamma(P| k_1, k_2) = 1 \). It should be mentioned also that the three-dimensional nature of the integration in (7) over the 3-vector \( k_1 \) moment components (as well as a three-dimensional form of the wave function in (8)) is caused by a passing to a single-time formalism [13], [14] and by the fact that in difference with the ordinary Feynman diagram technique, where the virtual particle moments are “off the mass shell”, in 3-dimensional approach the momenta of particles are on a mass shell, i.e. \( p^2 = m^2 \), but the equations for the bound state wave function are written “off the energy shell” like in “old-fashioned” perturbation theory.
The interaction amplitude $T_{q_1^2}^{a_1a_2} (q_1; k_1 | P)$ has the standard QFT form:

$$T_{q_1^2}^{a_1a_2} (q_1; k_1 | P) = \frac{4\pi \sqrt{\alpha g_s q} \bar{u}_q^{a_2} (k_2) e_1 (k_1 - q_1 + m_q) \bar{e}_2 (V - A) u_q^{a_1} (k_1)}{(k_1 - q_1)^2 - m_q^2} + (q_1 \leftrightarrow q_2).$$

(11)

Here $e_2 \equiv \gamma^\nu e_2 (q_2)$ with $e_2 (q_2)$, being the polarization 4-vector of a photon (that would be finally treated as a real one) with the 4-momentum $q_2$, while $e_1 \equiv \gamma^\alpha e_1 (q_1)$ with $e_1 (q_1)$ being a polarization 4-vector of a virtual boson (photon or $W^{\pm}$) with the 4-momentum $q_1$. The value $s_q = \sqrt{n_c} \sum e_q^2$ includes a number of colors $n_c$ and the summation is done over the squared charges of quarks appearing in a fermion loop of a diagram shown in Fig.1. Factor $\alpha$ is the electromagnetic coupling constant while $g$ is equal to $\sqrt{\alpha}$ in a case of $\pi^0 \to \gamma + e^+ e^-$ decay and $g = (eV_{ud})/2\sqrt{2}\sin (\theta_w)$ in a case of the process $\pi \to \gamma + e\nu$.

Factor $(V - A)$ is defined by (6) and it takes into account the structure of the vertex (see Fig.1) corresponding to an intermediate $V^* = (\gamma^*/W^{\pm})$ boson coupling to quarks.

The spin structure of $\pi$-meson wave function is taken according to [3], [11] as follows:

$$\Psi_{\sigma_1\sigma_2 BP} (k_1) = \bar{u}^{a_2} (k_2) \gamma^5 u^{a_1} (k_1) \frac{\delta_{BP} (k_1)}{2P \cdot k_1/M}.$$  

(12)

where $\sigma_1$ and $\sigma_2$ are quark polarizations and $\delta_{BP} (k_1)$ is taken to be a scalar function because in what follows we shall consider $q\bar{q}$ $s$-state (i.e. with zero orbital angular momentum $l = 0$). It should be mentioned that if one shall put (12) (with setting $\frac{\delta_{BP} (k_1)}{2P \cdot k_1/M} = 1$) into (7) and then substitute the wave function by the vertex function according to formula (10) and take there $\Gamma (P | k_1, k_2) = 1$, then an exact expression of QFT Feynman matrix element would appear under the sign of the integral. The nature of a 3-dimensial form of the integration can be easily understood on the basis of widely used rather straightforward way of passing to a 3-dimensial formalism. In this approach one starts with the expression of a decay amplitude taken as an integral convolution (with the 4-dimensional integration volume element) of a two-time 4-dimensional Bethe-Salpeter wave function with a 4-dimensional Feynman amplitude. Then by performing the subsequent equating of fermion and antifermion individual times in Bethe-Salpeter wave function by introdution of $\delta$-function, having the difference of these 2 times as its argument (see [13]– [16]) one gets a 3-dimensial equation in the momentum space.

3 Dip-effect in $\pi^0 \to \gamma + e^+ e^-$ decay formfactor.

In a case of $\pi^0 \to \gamma + ee$ only the $V=1$ term in (11) is taken in the amplitude, so it has a pure quantum electrodynamical QED form.

After substituting of such a final state interaction amplitude into (7) one has to perform the summation over spin polarizations, what leads to the appearance of the corresponding
trace of $\gamma$ matrices, including those that were summed up with two polarization 4-vectors $e_{1\mu}(q_1)$ and $e_{2\nu}(q_2)$ in (11). The Lorentz indexes of these $\gamma^\mu$ and $\gamma^\nu$ matrices, associated with two fermion-boson interaction vertices shown in Fig.1, define the Lorentz index structure of the expresion for the calculated trace which is equal to $V_{\mu\nu}(q_1P)$ in (3).

Two polarization 4-vectors of photons $e_{1\mu}(q_1)$ and $e_{2\nu}(q_2)$ are not included, according to the definition (8), into the expression for the decay formfactor. The last one, thus, is a Lorentz scaler and is defined only through the quark block of the diagram shown in Fig.1, that includes only two quark wave function and quark variables which enter the amplitude (11).

After calculation of the trace and the separation of its convolution with photon polarization vectors from the amplitude one may pass to performing the integration over the angular variables. Those are left, in a case of $s$-state, only in the denominator of quark propagator in (11)

$$\frac{1}{(k_1 - q_1)^2 - m^2_q} = \frac{1}{2q \cdot k_1} \cdot \frac{1}{A + z},$$

where

$$A = \frac{q_1^2 - 2q_1^0 k_1^0}{2q \cdot k_1}.$$ (13)

Here $z = \cos(\theta) = (\vec{q}_1 \cdot \vec{k}_1)/qk_1$ and the notations $k_1 = |\vec{k}_1|$ ; $q = |\vec{q}_1|$ are used. The following relations are valid also: for a real photon momentum we have $q_2^2 = q_{20}^2 - (\vec{q}_2)^2 = 0$, i.e. $q_2^0 = |\vec{q}_2|$; for quark 4-momentum $k_1^2 = m^2$, because, as it was mentioned before, within the approach used in [3] the 4-moments of particles are “on the mass shell” but out of the covariantly defined ”energy shell” [12].

We denote the square of the 4-momentum $q_1=(q_1^0, \vec{q}_1)$ of a virtual vector boson $V$ that produce a final state lepton-antilepton pair as $q_1^2 = Q^2$, keeping for the modulus $|\vec{q}_1|$ a notation $q = |\vec{q}_1|$. Thus, $A = (Q^2 - 2q_1^0 k_1^0)/(2q \cdot k_1)$ and we get an expression

$$F_{\pi^0\rightarrow\gamma\gamma}(q_1^2) = \frac{8m_q s_q \alpha}{\sqrt{2\pi} M_\pi} \left\{ 2\pi \int_0^\infty \frac{dk_1 \cdot k_1^2}{2k_1^0} \cdot \frac{\tilde{\phi}_{BP}(k_1)}{2q \cdot k_1} \cdot \int_{-1}^{+1} \frac{dz}{q_1^2 - 2q_1^0 k_1^0 + z} \right\}.$$ (15)

The appearance of the factor $q = |\vec{q}_1|$ in the denominator of (13) and finally in (15) has an important sequence that, possibly, may be experimentally observed. Really, due to the relation $q_2^2 = (P - q_1)^2$, following from the 4-momentum conservation law $P = q_1 + q_2$ we get a relation $M_\pi^2 - 2Pq_1 + q_1^2 = 0$. This invariant formula can be rewritten as

$$2Pq_1 = M_\pi^2(1 + x), \quad x = Q^2/(M_\pi)^2,$$

where from one can get in the pion rest frame ($\vec{P} = 0$) the relations for the components of the 4-vector $q_1 = (q_1^0, \vec{q}_1)$ (keeping in mind our notation $q = |\vec{q}_1|$ and the definition
\[ q^2 \equiv |\tilde{q}_1|^2 = q_{10}^2 - (\tilde{q}_1)^2: \]
\[ q_1^0 \equiv \frac{M_\pi^2 + Q^2}{2M_\pi} = M_\pi(1 + x)/2, \]
\[ q \equiv |\tilde{q}_1| = \frac{M_\pi^2 - Q^2}{2M_\pi} = M_\pi(1 - x)/2. \]

Thus we see that after integration of the propagator in (15) over \( z = \cos \theta \) one gets the prediction [17], [3]

\[ F(q_1^2) = F(Q^2) = F(x) \sim (1 - x)^{-1}. \]

So, a possible growth of \( F(x) \) in the region of \( x \sim 1 \), if it can be observed in a data, may serve as a confirmation that the choice of the propagator in (11), as well as of the quark amplitude as a whole in a form of (11), i.e. in a standard for perturbative QFT form, may be quite a reasonable one and a consistent with data.

A specific theoretical prediction of [3], that is also connected with the form of the propagator (13) from (11) (but is not completely defined by it only) is about the formfactor behavior at small values of \( x \).

Firstly let us mention that the investigation performed in [17], where the so called “static” approximation for the wave function was used (what is in fact equivalent to ignoring of the effects of quarks motion inside the meson) have shown that in this highly nonrelativistic approximation the slope of the formfactor has a positive value.

In [3] another limiting case of ultrarelativistic quark motion inside pion (i.e. when to the integral (15) large values of \( k_1 \) contribute mainly) was considered. It was shown that in this limit the derivative of the formfactor has a negative value at sufficiently small \( x \).

Combining this observation, based on analytical calculations only, with the discussed above \( \sim (1 - x)^{-1} \) behaviour of the formfactor at \( x \sim 1 \), i.e. where it has a positive slope, one can suppose that for the relativistic bound state systems (and a light \( \pi \)-meson is a good candidate in such a case) described with the relativistic wave functions, the formfactor \( F(x) \) may have a minimum and, therefore, a changing sign of its slope. (It is worth mentioning that the present data for this slope, see, for instance, [4], include the positive as well as negative values.)

The origin of such possible prediction can be understood from the analysis of the structure of formulae (15) without applying to any concrete form of the wave function.

Really, the integration over \( z \)-variable in (15) leads to an appearance of a logarithmic function [17], [3] (which may have different sign in different regions of its argument) under the integral sign

\[ F_{\pi^0 \rightarrow \gamma^+ \gamma}(q_1^2) = \frac{8m_\pi s_\pi \alpha}{\sqrt{2\pi} M_\pi} \left\{ 2\pi \int_0^\infty dk_1 \cdot \frac{k_1^2}{2k_1^0} \cdot \frac{\bar{\phi}_{BP}(k_1)}{2q_1 \cdot k_1} \cdot \ln \left| \frac{q_1^2 - 2k_1^0 q_1^0 + 2q \cdot k_1}{q_1^2 - 2k_1^0 q_1^0 - 2q \cdot k_1} \right| \right\} \]
According to \cite{3} the expression for the decay formfactor $F_{\pi^0 \to \gamma^* \gamma}(q_1^2)$ being normalized to the constant of $\pi^0 \to 2\gamma$ decay

$$F_{\pi^0 \to \gamma^* \gamma}(q_1^2) = f_{\pi^0 \to \gamma \gamma}(0) \tilde{F}_{\pi^0 \to \gamma^* \gamma}(q_1^2).$$  \hspace{1cm} (20)

may be transformed to a form:

$$\tilde{F}_{\pi^0 \to \gamma^* \gamma}(x) = \frac{1}{1-x} \left\{ 1 + \frac{1}{4} \int_0^\infty d\chi_k \phi(\chi_k) \ln |X(x, \chi_k)| \right\},$$  \hspace{1cm} (21)

$$X(x, \chi_k) = \frac{1 - xe^{-\chi_k}(M_p/m_q - e^{-\chi_k})}{1 - xe^{\chi_k}(M_p/m_q - e^{\chi_k})}. \hspace{1cm} (22)$$

where $4\pi\phi(\chi_k) = k_1 \tilde{\phi}_{BP}(k_1)$ with $k_1 = |\vec{k}_1|$. The quark rapidity $\chi_k = \ln[(k_1^0 + k_1)/m]$ corresponds to the following parametrization of 4-vector components:

$$k_1^0 = m_q c h \chi_k; \hspace{1cm} k_1 = |\vec{k}_1| = m_q s h \chi_k. \hspace{1cm} (23)$$

The integral in the denominator of (21) defines the $\pi^0 \to \gamma\gamma$ decay constant \cite{8,11,17,3}.

$$f_{\pi^0 \to \gamma \gamma} = \frac{32(2\pi)^{3/2} m_q \alpha_s}{M_p^2} \int_0^\infty d\chi_k \phi(\chi_k) \chi_k. \hspace{1cm} (24)$$

It is clear from (22) that at $x = 0$ one gets: $\tilde{F}(0) = 1$ and thus

$$F_{\pi^0 \to \gamma^* \gamma}(0) = f_{\pi^0 \to \gamma \gamma}. \hspace{1cm} (25)$$

Formula (21) presents the formfactor normalized to unity as a product of two factors. It may be also treated as a sum of two terms. One of them is $(1 - x)^{-1}$. It defines the monotonic growth of formfactor near $x \sim 1$ without a changing of the sign of a curve slope.

The second one contains (besides the integral over the wave function multiplied by log of (22)) also the factor $(1 - x)$ in the denominator and serves as a small correction to the main term $(1 - x)^{-1}$. In \cite{3} it was shown that the sign of this correction depends on the sign of the logarithmic function $\ln|X|$ under the integral sign and on what region of the integration of this $\ln|X|$ over the rapidity dominates. The last circumstance depends on a shape of the wave function.

The wave function $\phi(\chi_k)$ takes into account the bound state effect and, thus, has a nonperturbative nature. It serves in integral as a weight factor for $\ln|X|$ and defines what
region of quark rapidity may give the most contribution to the integral. Thus, the sign of the logarithmic function \( \ln |X| \) in this region would define the sign of a small additional integral term (appearing here as a correction to the leading term \( (1 - x)^{-1} \) in (21)) and thus it defines finally the sign of the formfactor \( F(x) \) slope in a region of small values of \( x \).

In a case of \( M_p/m_q \leq 1 \) the numerator and the denominator in (22) are both positive, so the modulus sign in \( \ln |X| \) can be omitted. Then it is easy to check that for the values of quark rapidity \( \chi_k \), satisfying the relation \( M_\pi \leq 2m_qch(\chi_k) \), the numerator in (22) would be less than the denominator and thus the \( \ln |X| \) function would have a negative sign. So, we see that in a case when the binding energy of pion (as of the \( q\bar{q} \) system) is negative (i.e. it may be parametrized as follows: \( M_\pi = 2m_q\cos\beta \)) the condition \( M_\pi \leq 2m_qch(\chi_k) \) would be satisfied and thus the additional term to \( (1-x)^{-1} \) would be negative. It would lead, in principle, to a negative sign of the formfactor slope at small values of \( x \). The value of a slope parameter depends on a shape of a particular wave function.

From what was said above it is clear that the logarithmic function includes the information about perturbative amplitude (11). In this sense the log function in (21) fulfils a job of a perturbative probe by help of which one can get under the sign of the integral and test the shape of the \( q\bar{q} \) bound state wave function in momentum space by means of variation the of external kinematic parameter \( x = Q^2/M_\pi^2 \) value.

Now after these general considerations it is a time to mention the results of [3] where for numerical calculation of the formfactor behaviour two types of relativistic wave functions obtained in [3] as the solutions of 2-body relativistic three-dimensional equations with QCD inspired model potentials were used as well as one wave function that is an exact solution of relativistic oscillator model. All of these model wavefunctions have lead to a negative sign of the formfactor slope at small values of \( x \) (see below Fig.2 where the result obtained within the relativistic oscillator model is presented). The results of other two QCD models give curves with the position of their minimums being from 20 to 40 percents higher than that one of a curve shown in Fig.2.

In this connection it is worth mentioning the result of paper [18] where, for a sake of testing the method used in paper [3], a decay of purely QED system of muon-antimuon bound state into a real photon and \( e^+e^- \) pair was considered. The invariant mass of two bound muons is much more higher than the value of two electron masses. So there is a more wide interval of \( Q^2 \) values may be attainable as comparing with a case of \( \pi \)-meson decay.

At the same time the well studied apparatus of QED is known to be experimentally checked quite well and the tools for calculation of relativistic wave functions in a case of QED interactions in purely electromagnetic systems are also existing. So a muon-antimuon bound state is a good place to test the idea of the approach discussed above.

The output of this kind of work done in [18] was finding out that the same dip in the formfactor of \( (\mu\bar{\mu}) \to \gamma + e^+e^- \) decay exists but its depth is of about one order smaller than that one found in \( F_{\pi^0 \to \gamma e^+e^-} \) formfactor. This difference in a value of dip was obtained with the same expression for the amplitude (7) (with \( A = 0 \)), but with the Coulomb-like relativistic wave function. Thus, from here one may make a conclusion that the depth of
the formfactor dip is defined by a value of binding energy of a system, which was found to be higher in a case of $\pi$-meson within the theoretical models considered in [3].

So, from comparison of the results of these two works, one may conclude that the value of the depth in dip, being measured at experiment, may give the information about the value of the binding energy in $\pi$-meson as a $q\bar{q}$ bound state system.

4 Dip-effect in $\pi^\pm \to \gamma + e^\pm\nu$ decay formfactor.

The detailed discussion of structure of the used expressions, performed in a previouse Section, allows now to pass easily to a case of $\pi \to \gamma + e\nu$ processes.

First let us note that the factors 1) $\frac{1}{q_1^2}$; 2) $e$; 3) $j_\mu^\nu(p_+, p_-)$ that follow each other in the expression (1) do represent, respectively:

1. the virtual photon propogator,

$$g^{\alpha\mu}/q_1^2,$$

(26)

It contains the metric tensor $g^{\alpha\mu}$ which convolutes the Lorentz index of the $\gamma^\alpha$- matrix (that was included into $e_1 \equiv \gamma^\alpha e_{1a}(q_1)$ in amplitude (11)) with the Lorentz index $\mu$ of lepton current $j_\mu^\nu(p_+, p_-)$. This $\gamma^\alpha$- matrix was splited from the polarization 4-vector of a photon $e_{1a}(q_1)$ while extraction (according to formula (8)) of the formfactor $F_{\pi^0 \to \gamma e^+ e^-}(q_1^2)$ from the amplitude (7):
2. current \( j_\mu^V(p_+, p_-) \), that describes the lepton-antilepton pair production and is characterized by

3. external to current \( j_\mu^V(p_+, p_-) \) factor \( e \), i.e. an electric charge of electron - the QED coupling constant, not included into the expression for the current (5) for a sake of convention and for keeping the universal structure of current definition.

All these three factors do correspond to the lines on the diagram of Fig.1 that are external to the formfactor, which one, as it was discussed previously, contains only \( q\bar{q} \) wave function and quark components of amplitude (11).

Let us rewrite the formula (2) in an analogous way. For this aim we shall write the factors that have to appear according to the Feynman rules of Standard Model if we shall take them for the diagram shown at Fig.1:

1.1 the virtual W-boson propagator

\[
\frac{[g^{\alpha\mu} - (q_1^\alpha q_1^\mu)/M_W^2]}{(q_1^2 - M_W^2)},
\]

(27)

2.1 current \( j_\mu^V(p_\pm, p_\nu) \) defined by (5) and (6),

3.1 external to current \( j_\mu^V(p_\pm, p_\nu) \) factor of SM coupling constants, defined for a case of \( W \) exchange and the production of a pair of electron and neutrino in a final state \( f = \gamma + e\nu \) for \( \pi^\pm \)-decay as

\[
e/(2\sqrt{2}\sin(\theta_w)),
\]

(28)

while for the vertex of diagram in Fig.1, where \( W \) couples to quarks as

\[
(eV_{ud})/(2\sqrt{2}\sin(\theta_w)).
\]

(29)

The combination of all of this factors allows to write down the expression (2) in the following form

\[
M_{\pi^\pm \rightarrow \gamma e^\pm \nu}(q_1, q_2|P) = \left[ F_V(q_1^2) \cdot V_\alpha\nu(q_2|P) + F_A(q_1^2) \cdot A_\alpha\nu(q_2|P) \right] \times
\]

\[
\times \left( e^2 \frac{V_{ud}}{8sin^2(\theta_w)} \right) \cdot \frac{[g^{\alpha\mu} - (q_1^\alpha q_1^\mu)/M_W^2]}{(q_1^2 - M_W^2)} \cdot j_\mu^V(p_\pm, p_\nu)e^\nu(q_2)
\]

(30)

which has the structure analogous to formula (1) for QED process of \( \pi^0 \rightarrow \gamma + e^+e^- \) decay.

If we shall consider in the last formula the limit \( q_1^2 \ll M_W^2 \) and take into account the relation

\[
e/(2\sqrt{2}\sin(\theta_w))^2 = M_W^2(G_F/\sqrt{2})
\]

(31)

then we come to formula (2) that parametrizes the amplitude of \( \pi^\pm \rightarrow \gamma + e^\pm \nu \) process through the formfactors \( F_V \) and \( F_A \).
Thus it is shown that the formula (30) has the same QFT structure as the formula (1), discussed in the previous Section. Now if we shall write down the analog of formula (8) for a case of \( \pi^\pm \to \gamma + e^\pm \nu \) decay and present the amplitude (7) as a sum of two terms according to two \( V \) and \( A \) parts of \((V-A)\) factor in the amplitude (11), then defining by help of polarization vectors \( e_{\mu}^1 \) and \( e_{\nu}^2 \) of virtual photon and the virtual \( W \)-boson for each part the corresponding formfactor, we shall see that both of them would be defined by one and the same expression (21) as they are given by one and the same quark bloc like that one considered in a previous Section.

From here it is clear that the behaviour of the formfactors in a case of \( \pi^\pm \to \gamma + e^\pm \nu \) decay must have the same dip as shown in Fig.2 for \( \pi^0 \to \gamma + e^+e^- \) process.

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