Anomaly-induced charges in baryons

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We show that quantum chiral anomaly of QCD in magnetic backgrounds induces a novel structure of electric charge inside baryons. To illustrate the anomaly effect, we employ the Skyrme model for baryons, with the anomaly-induced gauged Wess-Zumino term \( \sim (\pi_0 + \text{multi-pion}) F \cdot \vec{B} \). Due to this term, the Skyrmions giving a local pion condensation \( \langle (\pi_0 + \text{multi-pion}) \rangle \neq 0 \) necessarily become a local charge source, in the background magnetic field \( \vec{B} \neq 0 \). We present detailed evaluation of the anomaly effects, and calculate the total induced charge, for various baryons in the magnetic field.

I. INTRODUCTION

The chiral anomaly is one of the central concepts in QCD, and it manifests nature of quantum field theories in an explicit way in our hadronic world. As the chiral anomaly is essentially coupled to electromagnetic sector since the electromagnetism is a part of the chiral symmetry, the introduction of nontrivial electromagnetic backgrounds should add a good flavor of physics onto the chiral anomaly. In this paper, we report an interesting new effect induced by the chiral anomaly, for baryons in a background magnetic field.

Our finding is that baryons in a constant magnetic background acquire additional electric charge distribution due to the chiral anomaly. The result that this would generate even a total net charge is quite surprising, but the mechanism is quite simple. It is well-known that Wess-Zumino-Witten (WZW) term \[ \frac{1}{2} \epsilon_{\mu
u\rho} F^{\mu\nu} A^{\rho} \] actually captures the chiral anomaly in terms of the hadronic degrees of freedom. In particular, this term serves as a manifestation of the famous \( \sigma \rightarrow 2 \gamma \) decay. Now, any baryon carries a cloud of pions around it, and so it is a source of the pions. Once we replace one of the two \( \gamma \)'s in the Wess-Zumino-Witten term by the background magnetic field, we immediately see that the baryon can have an additional charge structure due to the chiral anomaly and the pion cloud. The schematic picture of this mechanism is illustrated in Fig. 1.

In this paper, we explicitly demonstrate this mechanism in detail, with a help of a concrete model of the pion-cloud picture of the baryons, the Skyrme model \[ \Box \]. In the Skyrme model, baryons are given as a solitonic object made of a local pion condensate \( \langle \pi (x) \rangle \neq 0 \). Plugging the Skyrme solution to the Wess-Zumino-Witten term, it can be shown that the magnetic field background can induce a novel charge structure inside the baryon (Skyrmion).

In particular, we give an argument that the total charge can also be generated, and resultantly the Gell-Mann–Nishijima formula for baryon charges can be corrected under the magnetic field due to the anomaly,\[
Q_e = e \left( I_3 + \frac{N_B}{2} \right) + \frac{Q_{anm}}{2}.
\] (1)

Here in the modified formula, \( Q_e \) is the electric charge of the baryon, \( I_3 \) is the third component of isospin, \( N_B \) is the baryon number, and the new term \( Q_{anm} \) is the charge generated by the anomaly and the background magnetic field.

One may be suspicious on this generation of electric charges. However, for example in the renowned Witten effect \[ \Box \], monopoles are accompanied with electric charges, in the presence of the \( \theta \) term. We may regard our WZW term as an analogue of the \( \theta \) term for the Witten effect. In addition, the chiral magnetic effect \[ \Box \] in heavy ion collisions shares the same property too. So, it is fare to say that the generation of the electric charge is an interesting effect due to the chiral anomaly in magnetic backgrounds.
charge is not a unique feature of our investigation, but is a common feature among parity-violating effects.

Quantum anomaly is literally quantum-mechanical, and thus is a tiny effect. However, when the coupled magnetic field is strong, this effect may be enhanced. So our physical motivation for this work is primarily oriented to the situation in which strong magnetic field is present with a finite density of baryonic matter. For this, one can come up with two important physical cases: one is a neutron star, at which neutrons are very dense and with a strong magnetic field, and the other is a heavy ion collision at which nuclei are smashed and a strong electromagnetic field is expected to be created instantly. In this paper, we do not go into these concrete cases. We concentrate on providing a basis for that, and in particular evaluate in detail the anomaly WZW term with the quantized Skyrmions, under a constant magnetic field.

The organization of this paper is as follows. In section II, we provide a review of the Skyrme model and the WZW term, with a brief introduction to the Skyrmion solution. In section III, we shall see explicitly that the background magnetic field generates an additional charge structure in the Skyrmions (baryons). We evaluate the Skyrmion and evaluate the anomaly-induced electric current for an arbitrary baryon state. In section IV, we evaluate the multipole moments of the anomaly-induced electric current and found a quadrupole, with a pion-mass dependence. In section V, we discuss possible other effect due to the background magnetic field on the baryon. In section VI, we discuss classical the anomaly-induced charge for higher-charge (=multiple) Skyrmions. The final section is for our conclusion and discussions. Appendix A is a study of the generated charge in a nonconstant magnetic field. In Appendix B, we show that the induced charge is due to a multi-pion effect (i.e. a pion cloud), and we compare our result with a point particle description of baryons. The letter version of this paper is 10.

II. THE SKYRMIONS AND THE ANOMALY-INDUCED CHARGES

As briefly described in the introduction, it is indeed almost straightforward to calculate the effect of the anomaly term for baryons in the presence of the magnetic field background, once we adopt a concrete model of the pion cloud. Here, we first review the Skyrme model which realizes baryons as a condensation of the pions, and also review the gauged WZW term which manifests the chiral anomaly in QCD.

A. The model

1. The Skyrme model

The chiral symmetry $SU(N)_L \times SU(N)_R$ acts on left-handed and right-handed quarks as

$$q_L \rightarrow U_L q_L, \quad q_R \rightarrow U_R q_R,$$

with $U_{L,R} \in SU(N)_{L,R}$. (2)

When the chiral condensate $\bar{q}q$ develops a non-zero vacuum expectation value by some non-perturbative effects

$$(\bar{q}q) = -v^3 1_N, \quad (v = \mathcal{O}(\Lambda_{QCD})),$$

the axial-part of the chiral symmetry is spontaneously broken as

$$SU(N)_L \times SU(N)_R \rightarrow SU(N)_{L+R}.$$ (4)

This gives rise to Nambu-Goldstone (NG) bosons, namely the pions, which takes value in the coset space $SU(N)_L \times SU(N)_R / SU(N)_{L+R}$,

$$U(x) = \exp \left( \frac{4i\pi^a(x)}{F_\pi} T^a \right), \quad (a = 1, 2, \cdots, N^2 - 1).$$ (5)

Here $F_\pi = 108[\text{MeV}]$ is the pion decay constant and $T^a$ is a generator of $SU(N)$ and we use the following standard normalization

$$\text{Tr}[T^a T^b] = \frac{1}{2} \delta^{ab}.$$ (6)

The chiral symmetry acts on the NG modes as

$$U \rightarrow U_L U_R^\dagger.$$ (7)

For later convenience, let us define left- and right-invariant Maurer-Cartan one-forms by

$$L_\mu \equiv U^\dagger \partial_\mu U, \quad R_\mu \equiv \partial_\mu U U^\dagger.$$ (8)

These take their values in the algebra of $SU(N)_R$ and $SU(N)_L$, respectively. The chiral symmetry acts on them as

$$L_\mu \rightarrow U_R L_\mu U_R^\dagger, \quad R_\mu \rightarrow U_L R_\mu U_L^\dagger.$$ (9)

We can think of $U$ as an effective low-energy field. Its effective Lagrangian of the leading order to $\mathcal{O}(\partial^2)$ can be uniquely determined as

$$\mathcal{L}^{(2)} = \left. \frac{F_\pi^2}{16} \text{Tr} \left[ \partial_\mu U \partial^\mu U^\dagger + M_\pi^2 (U + U^\dagger - 2) \right] \right|_\mathcal{O}(\partial^2)$$

$$= \left. \frac{F_\pi^2}{16} \text{Tr} \left[ -R_\mu R_\mu^\dagger + M_\pi^2 (U + U^\dagger - 2) \right] \right|_\mathcal{O}(\partial^2).$$ (10)

Here $M_\pi$ stands for the pion mass $M_\pi = 137[\text{MeV}]$ and our metric is $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. By expanding $L_\mu$ and $R_\mu$ with respect to $1/F_\pi$, one gets

$$L_\mu = \left. 4i \frac{\partial_\mu \pi^a}{F_\pi} T^a + 8i \epsilon^{abc} \pi^a \frac{\partial_\mu \pi^b}{F_\pi} T^c + \cdots \right|_\mathcal{O}(\partial^2),$$ (11)

$$R_\mu = \left. 4i \frac{\partial_\mu \pi^a}{F_\pi} T^a - 8i \epsilon^{abc} \pi^a \frac{\partial_\mu \pi^b}{F_\pi} T^c + \cdots \right|_\mathcal{O}(\partial^2).$$ (12)
Plugging this into $\mathcal{L}^{(2)}$, one obtain a standard kinetic term of the pions and corrections,

$$\mathcal{L}^{(2)} = \frac{1}{2} \partial_{\mu} \pi^a \partial_{\mu} \pi^a - \frac{M_\pi^2}{2} \pi^a \pi^a - \frac{2}{3F_\pi^2} \left( \pi^a \pi^a \partial_\mu \pi^b \partial_\mu \pi^b - \pi^a \pi^b \partial_\mu \pi^a \partial_\mu \pi^b \right) + \frac{2M_\pi^2}{3F_\pi^2} (\pi^a \pi^a)^2 + \cdots . \quad (13)$$

We are interested in a topological soliton made by the pions in this work. The topological winding number is given by

$$\pi_3 (SU(N)) = N_B \in \mathbb{Z} . \quad (14)$$

As will be shown, $N_B$ is identified with the baryon number via the WZW term. However, it is easy from a simple scaling argument that no topological solitons can survive from collapsing in the theory with $\mathcal{L}^{(2)}$. So one needs higher derivative corrections to $\mathcal{L}^{(2)}$. Therefore, we take a term of order $O(\partial^4)$ which is so-called the Skyrme term

$$\mathcal{L}^{(4)} = \frac{1}{32e_s^2} \text{Tr} \left( [R_\mu, R_\nu][R^\mu, R^\nu] \right), \quad (15)$$

with $e_s$ being a dimensionless coupling constant. We will choose the parameter $e_s = 4.84$ by following Ref. [12].

Now we are ready to write down the Skyrme model with the right-invariant one-form as

$$\mathcal{L} = \frac{F_\pi^2}{16} \text{Tr} \left[ -R_\mu R^\mu + M_\pi^2 \left( U + U^\dagger - 2 \right) \right] + \frac{1}{32e_s^2} \text{Tr} \left( [R_\mu, R_\nu][R^\mu, R^\nu] \right). \quad (16)$$

A Noether current of $SU(N)_L$ can be obtained by performing a local and infinitesimal $SU(N)_L$ rotation

$$\delta R_\mu = i \partial_\mu \phi_L. \quad (17)$$

Variation of the Skyrme Lagrangian is given by

$$\delta \mathcal{L} = \text{Tr} \left\{ \frac{i}{8} \left( -F_\pi^2 R_\mu + \frac{1}{e_s^2} [R_\mu, [R^\mu, R^\nu]] \right) \partial_\mu \phi_L \right\}. \quad (18)$$

Then the conserved current is given by

$$j_\mu^L = \frac{i}{8} \left( F_\pi^2 R_\mu - \frac{1}{e_s^2} [R^\mu, [R_\mu, R_\nu]] \right). \quad (19)$$

Similarly, the $SU(N)_R$ current takes the form

$$j_\mu^R = \frac{i}{8} \left( -F_\pi^2 L_\mu + \frac{1}{e_s^2} [L^\mu, [L_\mu, L_\nu]] \right). \quad (20)$$

These currents are related by

$$j_\mu^L = -U j_\mu^R U^\dagger \quad (21)$$

where we have used

$$U L_\mu U^\dagger = U (U^\dagger \partial_\mu U) U^\dagger = R_\mu. \quad (22)$$

The equation of motion of the Skyrme model is identical to the current conservation law if the pion mass is zero

$$\partial_\mu j_\mu^L = 0, \quad \partial_\mu j_\mu^R = 0. \quad (23)$$

When the pion mass is non-zero, the equation of motion becomes

$$\partial_\mu j_\mu^L = -\frac{iF_\pi^2 M_\pi^2}{16} \text{Tr} \left[ U - U^\dagger \right]. \quad (24)$$

The vector and axial conserved currents are defined by

$$j_\mu^V = \frac{j_\mu^L + j_\mu^R}{2}, \quad j_\mu^A = \frac{j_\mu^L - j_\mu^R}{2}. \quad (25)$$

The vector $SU(N)_{L+R}$ is nothing but the isospin, so we write its conserved charge as

$$I^a = \int d^3x \ j_\nu^a \ = \int d^3x \ \text{Tr}[j_\nu^0 \ U^\dagger T^a]. \quad (26)$$

2. Electromagnetic interaction

Let us next take the electromagnetic interaction into account. For simplicity, hereafter, we concentrate on the minimal case with two flavors $N = 2$. Since the electric charges of $u$ and $d$ quarks are $2/3$ and $-1/3$ respectively, the NG modes are rotated under the electromagnetic $U(1)$ as

$$U \rightarrow e^{-ieQ} U e^{ieQ} = e^{-ieT^3} U e^{ieT^3}, \quad Q = \frac{1}{2} \ 1 + T^3, \quad (27)$$

where $T^3 = \tau^3/2$ with the Pauli matrix $\tau^a$. Thus the electromagnetic $U(1)_{em}$ is a subgroup of $SU(2)_{L+R}$.

Interactions of the NG modes and the electromagnetic fields are introduced by gauging the $U(1)_{em} \subset SU(2)_{L+R}$ and replacing the partial derivative $\partial_\mu$ by a covariant derivative

$$\mathcal{D}_\mu U = \partial_\mu U + ieA_\mu [T^3, U]. \quad (28)$$

The left- and right-invariant one-forms are then replaced as

$$R_\mu \rightarrow \tilde{R}_\mu \equiv \mathcal{D}_\mu U U^\dagger, \quad L_\mu \rightarrow \tilde{L}_\mu \equiv U^\dagger \mathcal{D}_\mu U. \quad (29)$$

Then the total Lagrangian can be read as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{F_\pi^2}{16} \text{Tr} \left[ -\tilde{R}_\mu \tilde{R}^\mu + (U + U^\dagger - 2) \right] + \frac{1}{32e_s^2} \text{Tr} \left( \tilde{R}_\mu, R_\nu \right) \tilde{R}^\mu, \quad (30)$$
The classical equation of motion is derived by variational method as before. One can easily obtain

\[ D_\mu j^\mu_L = -i F^2 M^2 \frac{1}{16} \text{Tr} \left[ U - U^\dagger \right], \]  

\[ j^\mu_L = \frac{i}{8} \left( F^2 \tilde{R}^\mu - \frac{1}{e_s} \text{Tr} \left[ \tilde{R}_\nu, \dot{R}^\nu \right] \right). \]  

The first term in the current \( j_B^\mu \) gives a topological number associated with \( \pi_3(SU(2)_L - R) \). Indeed, the integration of it over the space gives the topological winding number

\[ N_B = \int d^3 x \frac{1}{24 \pi^2} e^{ijk} \text{Tr} [R_l R_j R_k]. \]  

One can also express the E.O.M. of the right-invariant one-form \( \tilde{L}_\mu \) by just replacing \( R_\mu \) with \( \tilde{L}_\mu \). Note that since the electromagnetic charge \( Q \) breaks the chiral anomaly. Actually, plugging Eqs. (11) and (12) into Eq. (35), one finds the famous \( \pi^0 \to 2 \gamma \) term by the anomaly

\[ - \frac{N_c e^2}{48 \pi^2 F_\pi} e^{\mu \nu \rho \sigma} A_\mu F_{\nu \rho} \partial_\sigma \pi^0 + O(F^{-3}) \]

\[ = - \frac{N_c e^2}{12 \pi^2 F_\pi} \pi^0 \tilde{E} \cdot \tilde{B} + O(F^{-3}), \]

where the equality holds up to a total derivative. Let us next obtain the electric charge coupled to a photon fluctuation \( a_\mu \) under a background electromagnetic field \( A_\mu \). To this end, we expand the gauge field as

\[ A_\mu = \bar{A}_\mu + a_\mu. \]

Then the WZW action linear in \( a_\mu \) gives

\[ S_{\text{WZW}}[A_\mu] = \int d^4 x \frac{e}{2} j^\mu_B A_\mu, \]

with the baryonic current

\[ j^\mu_B = \frac{1}{24 \pi^2} e^{\mu \nu \rho \sigma} \text{Tr} [R_\nu R_\rho R_\sigma] - \frac{e}{8 \pi^2} e^{\mu \nu \rho \sigma} \partial_\nu (A_\rho P_\sigma). \]  

Here we use \( e^{0123} = -1 \) and define

\[ P_\sigma \equiv \frac{i}{2} \text{Tr} [\tau^3 (L_\sigma + R_\sigma)]. \]  

This baryon current is clearly conserved due to the antisymmetric tensor \( e^{\mu \nu \rho \sigma} \). On the other hand, \( j_B^\mu \) appears to depend on gauge choices, at a glance. But this is not the case. One can rewrite the baryonic current as

\[ j^\mu_B = \frac{1}{24 \pi^2} e^{\mu \nu \rho \sigma} \text{Tr} [\tilde{R}_\nu \tilde{R}_\rho \tilde{R}_\sigma]
\]

\[ - \frac{i e}{32 \pi^2} e^{\mu \nu \rho \sigma} F_{\nu \rho} \text{Tr} [\tau^3 (\tilde{L}_\sigma + \tilde{R}_\sigma)]. \]  

This is manifestly gauge invariant.
invariant and conserved electric charge which includes an extra term to the well-known Gell-Mann–Nishijima formula. The last term is a contribution from the chiral anomaly.

Note that, as we will see below shortly, the surface term in Eq. (43) does not contribute to the net electric charge if the pion mass is non-zero. So, we focus on the new last term of Eq. (46) coming from Eq. (44), in the following of this paper.

B. The Skyrmion: A nucleon as a topological soliton

Let us find a solution of the classical equations of motion derived previously,

\[ \mathcal{D}_{\mu} j_{L}^{\mu} = 0, \quad \partial_{\mu} F^{\nu \mu} = e j_{Y}^{\nu}. \]  

We solve these by dealing with the electromagnetic interaction as a perturbation. Then we expand the chiral field with respect to the electromagnetic coupling constant \( e \) as

\[ U = \exp \left( i \vec{\tau} \cdot (\vec{f}_0 + \vec{f}_1 + \cdots) \right), \]

\[ A_\mu = A_\mu^{(0)} + A_\mu^{(1)} + \cdots. \]  

At the zeroth order the chiral fields and electromagnetic fields are decoupled, so that the equations of motion are those of the Skyrme model without the electromagnetic interaction and the Maxwell equation without a source

\[ \partial_{\mu} j_{L}^{(0)\mu} = -\frac{i F_{\pi}^{2} M_{\pi}^{2}}{16} \text{Tr} \left[ U - U^\dagger \right], \quad \partial_{\mu} F^{(0)\nu \mu} = 0, \]  

with

\[ j_{L}^{(0)\mu} = i \left( F_{\pi}^{2} R^{(0)\mu} - \frac{1}{e_{\pi}^{2}} [R^{(0)\mu}, [R^{(0)\mu}, R^{(0)\nu}]] \right). \]  

The second equation in Eq. (44) is solved by considering a constant background magnetic field, say along the \( x_i \)-axis

\[ \frac{1}{2} \epsilon^{ijk} F_{jk} = B^i \]  

with \( B^i \) being a constant.

In order to solve the first equation in Eq. (44), it is useful to introduce a dimensionless coordinate

\[ x_\mu \to \frac{1}{e_{\pi} F_{\pi}} x_\mu, \quad \partial_\mu \to e_{\pi} F_{\pi} \partial_\mu. \]  

In terms of this new coordinate, the Skyrme equation is written as

\[ \partial_\mu \left( R^{(0)\mu} - [R^{(0)\mu}, [R^{(0)\mu}, R^{(0)\nu}]] \right) = \frac{m_{\pi}^{2}}{2} \text{Tr} \left[ U^\dagger - U \right] \]  

with a dimensionless mass in unit of \( 1/(e_{\pi} F_{\pi}) \)

\[ m_{\pi} = \frac{M_{\pi}}{e_{\pi} F_{\pi}}. \]  

Here and after, we will use this notation.

Let us make a standard hedgehog (radial) ansatz, for a static and topologically non-trivial solution with \( N_B = 1 \),

\[ U_0(\vec{x}) = \exp \left( i \vec{f}_0 \cdot \vec{r} \right) = \exp \left( i f(r) \vec{r} \cdot \hat{x}_i \right), \]

\[ \hat{x}_i = \frac{x_i}{r}. \]  

One can express \( U \) in a different fashion as

\[ U_0 = 1_2 \cos |\vec{f}_0| + i \frac{\vec{f}_0}{|\vec{f}_0|} \cdot \vec{r} \sin |\vec{f}_0| \]

\[ = 1_2 \cos f + i \vec{r} \cdot \vec{r} \sin f. \]  

Then we obtain for a static configuration

\[ R_{i}^{(0)} = \left( \frac{1}{4} + \frac{2 \sin^{2} f}{r^{2}} \right) f'' + \frac{1}{2 r} f' \]

\[ + \frac{\sin 2 f}{r^{2}} \left( f'' - \frac{1}{4} - \frac{\sin^{2} f}{r^{2}} \right) = \frac{m_{\pi}^{2}}{4} \sin f. \]  

The solution with a unit winding number corresponds to

\[ \lim_{r \to \infty} f(r) = \pi, \quad \lim_{r \to 0} f(r) = 0. \]  

Numerical solutions with different \( m_{\pi} \)’s are given in Fig. 2. We adopt the physical pion mass \( m_{\pi}^{\text{phys}} = 0.263 \) which was determined from the mass splitting between nucleon and \( \Delta \) [12]. We also show the profile functions for \( m_{\pi} = 2.63 \) and 0.0263 in order to demonstrate a characteristic property of the profile function. We see that the larger (smaller) \( m_{\pi} \) gives the thinner (fatter) Skyrmion. The profile function with \( m_{\pi} = 0.0263 \) almost coincides to that with \( m_{\pi} = 0 \), see Fig. 2.

Asymptotic behavior of \( f \) can be found by solving the linearized equations of motion for large \( r \)

\[ f'' + \frac{2}{r} f' - \frac{2 f}{r^{2}} - \frac{m_{\pi}^{2}}{4} f = 0. \]  

This is solved by

\[ f \simeq \left( \frac{C}{r^{2}} + A \right) e^{-m_{\pi} r}, \]  

where \( C \) and \( A (= C m_{\pi} ) \) are constant. For massless pion, we find

\[ f \simeq \frac{C}{r^{2}}, \]  

which was determined from the mass splitting between nucleon and \( \Delta \).
with \( C \approx 8.638 \). For massive pion, the asymptotic form decays exponentially,
\[
f \simeq \frac{A}{r} e^{-m_r r},
\]
where \( A \) is a constant which depends on \( m_r \). For example, we find \( A \approx 2 \) for \( m_r = 0.263 \).

### III. ANOMALY-INDUCED CHARGE

We substitute the Skyrme solution to the electromagnetic current calculated from the WZW term, to evaluate the anomaly-induced electric charge of the baryons. We here use classical Skyrmions for an illustration first, then we move onto quantized Skyrmions, to obtain a formula for the anomaly-induced charge for baryon quantum states. The anomaly-induced charge is the last term of Eq. (40). Since we are considering a constant magnetic field background, it is enough to see \( P_i \) defined in Eq. (37).

#### A. Classical evaluation

To evaluate \( P_i \), let us first write down the left- and right-invariant one-forms as
\[
R_i = i \left( \vec{\tau} \cdot \vec{x} \right) f' \hat{x}_i + \frac{i}{2} \left( \vec{\tau} \cdot \partial_i \vec{x} \right) \sin 2f + \left[ \left( \partial_i \vec{x} \cdot \vec{x} \right) 1 + i \left( \partial_i \vec{x} \times \vec{x} \right) \cdot \vec{\tau} \right] \sin^2 f,
\]
\[
L_i = i \left( \vec{\tau} \cdot \vec{x} \right) f' \hat{x}_i + \frac{i}{2} \left( \vec{\tau} \cdot \partial_i \vec{x} \right) \sin 2f + \left[ \left( \partial_i \vec{x} \cdot \vec{x} \right) 1 - i \left( \partial_i \vec{x} \times \vec{x} \right) \cdot \vec{\tau} \right] \sin^2 f.
\]
Plugging these into Eq. (37), we have
\[
P_i = -f' \hat{x}_i \hat{x}_3 - \frac{1}{2} \left( \partial_i \hat{x}_3 \right) \sin 2f.
\]

The topological charge density, \( P_1 \) and \( P_3 \) are shown in Fig. 3 at \( m_r = 0 \). The induced electric charge densities with nonzero \( m_r \) (see Fig. 4) are quite similar to those for the massless case in Fig. 3. A tiny difference comes form the similar but a little different behaviors in profile functions \( f(r) \) as shown in Fig. 2.

The spatial integrations of \( P_1 \) and \( P_3 \) become
\[
\int d^3x P_1 = 0, \quad \int d^3x P_3 = -\frac{4\pi}{3(e_r F_r)^2} c_0,
\]
with
\[
c_0 \equiv \int dr \left( r^2 f' + r \sin 2f \right).
\]
Note that the integration variable \( r \to r/(e_r F_r) \) is the dimensionless coordinate, so that \( c_0 \) is a dimensionless number. This means that the net induced charge is zero for \( \vec{B} \propto (1, 0, 0) \), and \((0, 1, 0)\) whereas it is non-zero for \( \vec{B} \propto (0, 0, 1) \) where non-zero correction appears to the Gell-Mann–Nishijima formula. The numerical coefficient \( c_0 \) can be rewritten as
\[
c_0 = \left[ r^2 f \right]_0^\infty + \int dr \left( -2rf + r \sin 2f \right).
\]
The first term is surface contribution which becomes non-zero only at \( m_r = 0 \) due to distinct behavior at large \( r \) shown in Eq. (64), and the second term expresses a pion-cloud effect discussed in Appendix B. Computational results of \( c_0 \) become
\[
c_0 = -14.1 + C \quad (m_r = 0),
\]
\[
c_0 = -10.2 \quad (m_r = 0.263),
\]
where \( C \) is asymptotic factor shown in Eq. (63). Computational results for other values of the pion masses are also summarized in Table I.

Finally, we evaluate contributions from the surface term in Eq. (38). To this end, we need to compute
\[
W = -\int d^3x \epsilon^{ijk} \partial_i (A_j P_k).
\]
Interestingly, plots of the integrand (density) are quite similar to those in Fig. 3 if we choose a gauge \( A_i =
To evaluate the anomalous current and charge for each baryon state, the Skyrmion is quantized as a slowly rotating soliton. Quantizing the collective coordinates of soliton’s moduli space $G$ is achieved by the canonical quantization of a particle on a manifold $G$. In the case of two flavors, $G = SU(2) \simeq S^3$. We construct the angular momentum operators acting on baryon states and the harmonic functions on $G$ corresponding to baryon wave function. Using these, we evaluate the expectation values of the anomalous currents.

We evaluate the expectation values $(j_{\text{anm}}^i)_{B}$ and the anomalous charge $Q_{\text{anm}}^B$ for each baryon state $B$ in the presence of a background magnetic field. We show that the spatial components of the current vanish: $(j_{\text{anm}}^i)_{B} = 0$. We also obtain the anomalous charge $Q_{\text{anm}}$, which is given by integrating $(j_{\text{anm}}^0)_{B}$.

1. Angular momentum operators and spherical harmonics on $S^3$

Let $g \in G$ be a group element of a group manifold $G$. Then there are operators $\mathcal{L}_a$ and $\mathcal{R}_a$ acting on $g$ from left and right, respectively, and satisfying the commutation relations

$$[\mathcal{L}_a, \mathcal{L}_b] = i f_{abc} \mathcal{L}_c, \quad [\mathcal{R}_a, \mathcal{R}_b] = i f_{abc} \mathcal{R}_c. \quad (75)$$

Here the roman indices correspond to those of the tangent space of $G$, and $f_{abc}$ is the structure constant of the Lie algebra of $G$: $[T_a, T_b] = i f_{abc} T_c$ and $\text{Tr}(T_a T_b) = \delta_{ab}/2$. The actions of $\mathcal{L}_a$ and $\mathcal{R}_a$ on $g$ are

$$\mathcal{L}_a g = -T_a g, \quad \mathcal{L}_a g^{-1} = g^{-1} T_a, \quad \mathcal{R}_a g = g T_a, \quad \mathcal{R}_a g^{-1} = -T_a g^{-1}. \quad (76)$$

We restrict ourselves to the case $G = SU(2)$, and thus $f_{abc} = \epsilon_{abc}$. This is the case of the two-flavor Skyrmion.

The operators $\mathcal{L}_a$ and $\mathcal{R}_a$ are precisely the angular momentum operators with respect to the isometry $SU(2)_L \times SU(2)_R$ of $S^3$. We introduce the scalar spherical harmonics on $S^3$, $Y_{jm\hat{m}}$, where $J$ is the same magnitude spins of both $SU(2)_L$ and $SU(2)_R$, and $m$ and $\hat{m}$ are the eigenvalues of their third-components, respectively. The actions of the operators on the spherical harmonics are

$$\mathcal{L}^2 Y_{jm\hat{m}} = \mathcal{R}^2 Y_{jm\hat{m}} = \sqrt{J(J + 1)} Y_{jm\hat{m}},$$
$$\mathcal{L}^\pm Y_{jm\hat{m}} = \sqrt{(J \pm m)(J \pm m + 1)} Y_{j(m\pm1)\hat{m}},$$
$$\mathcal{R}^\pm Y_{jm\hat{m}} = \sqrt{(J \pm \hat{m})(J \pm \hat{m} + 1)} Y_{j(m\pm1)\hat{m}},$$
$$\mathcal{L}_3 Y_{jm\hat{m}} = m Y_{jm\hat{m}}, \quad \mathcal{R}_3 Y_{jm\hat{m}} = \hat{m} Y_{jm\hat{m}}, \quad (77)$$

where $\mathcal{L}_\pm = \mathcal{L}_1 \pm i \mathcal{L}_2$ and $\mathcal{R}_\pm = \mathcal{R}_1 \pm i \mathcal{R}_2$.

It is convenient to introduce a D-function $D_{ab}(g)$ so as to see the relation of Eq. (76) and Eq. (77). It is defined

$$B_1(0, -z/2, y/2) \text{ or } \tilde{A}_1 = B_3(-y/2, x/2, 0).$$
The surface term can be evaluated as

$$W = \int d\Omega_2 \left[ \ddot{x}_A \varepsilon^{ijk} \tilde{A}_j \left( f' \ddot{x}_k \ddot{x}_3 + \frac{(\partial_k \ddot{x}_3) \sin 2f}{2} \right) \right]_{r \to \infty}. \quad (74)$$

As shown in Eq. (74), the profile function for $m_\pi \neq 0$ is exponentially small at large $r$, so that this integration vanishes.

Note that in the massless case the asymptotic behavior given in Eq. (72) leads to $W \neq 0$ as

$$W = \frac{32\pi i C}{3(e_s F_\pi)^2} B_3. \quad (74)$$

We see that this surface term with the constant $C$ given in Eq. (72) cannot cancel the last term of Eq. (10). Anyway, since the physical pion mass is not zero, we consider the new last term in Eq. (10) as the anomaly-induced electric charge.\[22\]
by the adjoint action of $g$,
\[ gT_\alpha g^\dagger = T_bD_{ab}(g). \] (78)

The action of $\mathcal{L}_a$ and $\mathcal{R}_a$ on $D_{ab}(g)$ becomes
\[
\mathcal{L}_c D_{ab}(g) = i\epsilon_{cde} D_{de}(g),
\mathcal{R}_c D_{ab}(g) = -i\epsilon_{cde} D_{de}(g).
\] (79)

With a little more algebra, it can be shown that appropriate linear combinations of $D_{ab}(g)$ precisely give the harmonic functions $Y_{Jm\bar{m}}$.

Note that $\mathcal{L}_a$ and $\mathcal{R}_a$ are respectively the isospin and the spin operators for baryon states. See, for instance, Ref. [15] for a discussion in three-flavor case. The adjoint action of $g$ to the hedgehog Skyrmion gives $U(x) = gU_0(x)g^\dagger = U_0(x^{\text{rot}})$. That is, $g\hat{x}\cdot\hat{\tau}g^\dagger = \hat{x}^{\text{rot}}\cdot\hat{\tau}$. The transformation of the unit vector $\hat{x}$ under the spatial rotation caused by $D(g)$ is
\[ \hat{x}_a \rightarrow \hat{x}^{\text{rot}}_b = D_{ab}(g)\hat{x}_b. \] (80)

It is natural to identify $SU(2)g$ as the baryon spin, where $g \rightarrow gk_R$ with $k_R \in SU(2)_R$.

### 2. Absence of the spatial anomalous current

To evaluate $\langle j_{\text{ann}}^i \rangle_B$ in the presence of a background magnetic field, we need to focus only on $P_0$ owing to the index structure of the WZW term. We first write down $P_0$ and then quantize it. Substituting a slowly rotating Skyrmion $U(x) = g(t)U_0(x)g^\dagger(t)$ for $P_0$ [37], we obtain
\[ P_0 = 2\sin(2f)\epsilon_{abc}D_{ab}(g)\hat{x}_bTr[\tau_c\hat{g}^\dagger]. \] (81)

In the procedure of the canonical quantization, the time-derivative part is replaced with the angular momentum as follows [16]:
\[ \mathcal{L}_a = i\Lambda Tr[\tau_a\hat{g}^\dagger], \]
\[ \Lambda = \frac{8\pi}{3} \int d^2r \sin^2 f \left[ 1 + 4 \left( f'^2 + \frac{\sin^2 f}{r^2} \right) \right]. \] (82)

Hence $P_0$ can be written in terms of raising- and lowering-operators,
\[ P_0 = -\frac{1}{\sqrt{3}\Lambda} \sin(2f) [\hat{x}_+ (Y_{1-} - L_+ + Y_{1+} - L_-) - \hat{x}_- (Y_{1-} + L_+ + Y_{1+} + L_-) + \hat{x}_3 (Y_{10} - L_+ + Y_{10} + L_-)]_{\text{Weyl}}, \] (83)

where the indices $\pm$ in $Y_{Jm\bar{m}}$ mean $\pm 1$, and $\hat{x}_3 = \hat{x}_1 \pm i\hat{x}_2$. The Weyl ordering for the operators is understood.

Integrating a product of three spherical harmonics over $S^3$ gives
\[
\int \frac{d\Omega_3}{2\pi^2} (Y_{J_1m_1\bar{m}_1})^* Y_{J_2m_2\bar{m}_2} Y_{J_3m_3\bar{m}_3} = \sqrt{(2J_2 + 1)(2J_3 + 1)C_{J_1m_1J_2m_2J_3m_3}^* C_{J_2m_2J_3m_3J_1m_1}^{J_1m_1}}.
\] (84)

where $C_{J_{m_1}m_{2}J_{m_{2}}m_{3}J_{m_{3}}}^{J_{m_{1}}}$ is a Clebsch-Gordan coefficient of $SU(2)$. The wave function of each baryon state is given by $Y_{Jm\bar{m}}$. Our primary interest is in nucleons ($I = J = 1/2$) and $\Delta$ baryons ($I = J = 3/2$), but here we can keep $J$ arbitrary. We use Eq. (84) to evaluate $P_0$ projected onto each baryon state.

In Eq. (83), we need to focus only on the last line:
\[ \langle O_W \rangle_B = \int \frac{d\Omega_3}{2\pi^2} (Y_{J_1-} - S_3)^* O_W Y_{J_1S_3} = \sqrt{3}C_{J_1S_3}^{J_2S_2} \left( \sqrt{(J - I)(J + I + 1)}C_{J_4-1}^{J_4+1} \right) + \sqrt{(J + I)(J - I + 1)}C_{J_4+1}^{J_4-1} \right)
\] (86)

where the minus sign appearing in front of $S_3$ is due to Eq. (79). This exactly vanishes once values of the Clebsch-Gordan coefficients are substituted [17]:
\[ C_{J_1J_2}^{J_3} = \frac{\gamma}{\sqrt{J(J + 1)}}, \]
\[ C_{111}^{J_3} = \frac{(J \pm \gamma)(J \mp \gamma + 1)}{2J(J + 1)}. \] (87)

Thus we see $\langle j_{\text{ann}}^i \rangle_B = 0$.

### 3. Anomalous electric charge in baryons

In $j_{\text{ann}}^0$, rotation of the Skyrmion is encoded in $\hat{x}_3^{\text{rot}}$. Hence, it is sufficient to evaluate $\langle \hat{x}_3^{\text{rot}} \rangle_B$, which directly leads us to $\langle j_{\text{ann}}^0 \rangle_B$. Since this part does not contain derivatives in time, we simply integrate Eq. (80) by using Eq. (84). The result is
\[ \langle \hat{x}_3^{\text{rot}} \rangle_B = -\frac{I_3S_3}{J(J + 1)} \hat{x}_3. \] (88)

Below we will mainly focus on the case $J = 1/2$ for simplicity. However, thanks to Eq. (88), it is straightforward to consider higher-spin cases. For instance, this gives $\langle \hat{x}_3^{\text{rot}} \rangle^N = -4I_3S_3\hat{x}_3/N$ for nucleons, and $\langle \hat{x}_3^{\text{rot}} \rangle^G = -4I_3S_3\hat{x}_3/15$ for $\Delta$ baryons.

Let us calculate the total electric charge from the anomalous effect. The matrix elements of $P_\mu$ are eva-
uated by applying Eq. (88),
\[\langle P_0\rangle_{I_3, S_3}^N = 0, \quad (89)\]
\[\langle P_3\rangle_{I_3, S_3}^N = -\frac{16i}{3} I_3 S_3 \left( f' - \frac{\sin(2f)}{2r} \right) x_3, \quad (90)\]
\[\langle P_3\rangle_{I_3, S_3}^N = -\frac{16i}{3} I_3 S_3 \left[ \left( f' - \frac{\sin(2f)}{2r} \right) x_3^2 + \sin(2f) \right]. \quad (91)\]

The anomalous charge density under a constant magnetic field \( B \) is indeed induced in nucleons:
\[\langle j_{\text{anm}}^0\rangle_{I_3, S_3}^N = \frac{i e^2 N_c}{48\pi^2} B_i \langle P_i\rangle_{I_3, S_3}^N. \quad (92)\]

The integration of \( \langle P_i\rangle_{I_3, S_3}^N \) over the whole space yields
\[\int d^3x \langle P_i\rangle_{I_3, S_3}^N = \left\{ \begin{array}{ll}
0 & (i = 1, 2), \\
-\frac{16\pi i}{9} (4I_3 S_3) c_0 & (i = 3),
\end{array} \right. \]
where \( c_0 = \int dr (r^2 f' + r \sin(2f)) \). Numerical value of \( c_0 \) will be shown in Table I for several pion masses. From Eq. (92), we obtain the anomalous charge for nucleons
\[Q_{\text{anm}}^N = \frac{4e^2 N_c}{27\pi} I_3 S_3 \frac{c_0 B_3}{(e_s F_\pi)^2}. \quad (93)\]

In this final expression we restored the rescaling factor \( e_s F_\pi \) by a dimensional counting. Equation (93) shows that an electric charge is actually induced by the anomalous effect even for a neutron.

As seen from Eq. (88), dividing the result in Eq. (88) by a factor of 5 gives the anomalous charge of \( \Delta \) baryons.

The plot of the charge density of \( \langle j_{\text{anm}}^0\rangle \) for the quantized Skyrmion shows exactly the same as Fig. 3. For a magnetic field along \( x^3 \) direction, the charge density plot is symmetric, thus the total charge vanishes. However, obviously multipoles, in particular a quadrupole, may show up. In the next section, we calculate multipoles in \( \langle j_{\text{anm}}^0\rangle \).

IV. MULTIPOLe MOMENTS OF ANOMALOUS CHARGES AND PION-MASS DEPENDENCE

When one regards charged baryons as point-like particles, multipole moments are suitable physical quantities to describe an original charge distribution. In this section, we extend the calculations to the higher multipole moments due to the anomalous-charge distributions, and estimate pion-mass dependence of the anomalous charges and the multipole moments.

First, we can easily find that the dipole moment due to the anomalous charge vanishes:
\[D_i = \int d^3x x_i \langle j_{\text{anm}}^0\rangle^N = 0 \quad (i = 1, 2, 3). \quad (94)\]
On the other hand, the quadrupole moment:

\[ Q_{ij} \equiv \int d^3x \left( 3x_i x_j - r^2 \delta_{ij} \right) \langle j^0_{\text{ann}} \rangle^N, \tag{95} \]

is calculated for the nucleon as

\[ Q_{ij} = c_2 \left[ 2N_c I_3 S_3 \hat{Q}_{ijk} eB_k (e_e F_e)^4 c_2, \tag{96} \right. \]

\[ \hat{Q}_{ijk} = \left( \begin{array}{ccc} -2 \delta_{k1} & 0 & 3 \delta_{k1} \\ 0 & -2 \delta_{k2} & 3 \delta_{k2} \\ 3 \delta_{k1} & 3 \delta_{k2} & 4 \delta_{k3} \end{array} \right) \right]_{ij}, \tag{97} \]

where the numerical coefficient,

\[ c_2 = \int dr \left[ 2r^4 f' - r^3 \sin(2f) \right], \tag{98} \]

is shown in Tab. I for several pion masses. This means that the leading multipole due to the anomalous contribution is the quadrupole moment. We note that the quadrupole is induced in response to all directions of the external magnetic fields, although the anomalous charge is induced only by \( B_3 \) (see Eq. (93)).

In order to extract the pion-mass dependence of the anomalous charge and the quadrupole moment, we calculate the Skyrme profile function \( f(r) \) for wide pion-mass range \((0.1 \leq m_\pi/m_\pi^{\text{phys}} \leq 100)\). Behavior of \( f(r) \) for several pion masses is shown in Fig. 5, where the solid line is \( f(r) \) at \( m_\pi = m_\pi^{\text{phys}} \). We find that the wave function shrinks with the pion mass increasing.

Since the pion mass dependence of the anomalous charge and the quadrupole moment appears in the numerical coefficients, \( c_0 \) and \( c_2 \), via \( r \) integration with \( f(r; m_\pi) \), we focus on these coefficients. Figure 6 shows results of \( -c_0 \) (top) and \( -c_2 \) (bottom) as a function of \( m_\pi/m_\pi^{\text{phys}} \) in log-log scale. Numerical values of \( c_0 \) and \( c_2 \) are also summarized in Tab. I. We can see that \( c_0 \) becomes almost plateau at small pion-mass \((m_\pi/m_\pi^{\text{phys}} < 1)\), whereas that decreases linearly at large pion-mass \((m_\pi/m_\pi^{\text{phys}} > 10)\). We fit the results by a function, \( A/m_\pi^n \), with \( A \) and \( n \) being free parameters, and obtain \( c_0 \sim -6.0(7)/m_\pi^{0.97(5)} \) in a range of \( 30 \leq m_\pi/m_\pi^{\text{phys}} \leq 100 \), shown by the dashed line on Fig. 5 (top). This implies that the pion-mass dependence of the anomalous charge is \( Q_{\text{ann}} \propto 1/m_\pi \) at large pion-mass.

On the other hand, \( c_2 \) behaves linearly for all pion-mass region. We also fit the results by \( A/m_\pi^n \), and obtain \( c_2 \sim -127(15)/m_\pi^{2.00(7)} \) in a range of \( 20 \leq m_\pi/m_\pi^{\text{phys}} \leq 100 \), shown by the dashed line on Fig. 6 (bottom). Although the fit is performed at large pion-mass region, almost all results of \( c_2 \) is located around the dashed line. This implies that the quadrupole moment due to the anomaly behaves as \( Q_{ij} \propto 1/m_\pi^2 \).

Note that we have evaluated \( j^0_{\text{ann}} \), which is only a part of the total electromagnetic current.\[23\]

| \( m_\pi/m_\pi^{\text{phys}} \) | \( c_0 \)  | \( c_2 \) |
|----------------|--------|--------|
| 0             | \(-1.41\times10^4\) | \(-\infty\) |
| 0.1           | \(-1.37\times10^4\) | \(-4.90\times10^5\) |
| 0.125         | \(-1.37\times10^3\) | \(-2.99\times10^5\) |
| 0.25          | \(-1.34\times10^4\) | \(-6.68\times10^4\) |
| 0.5           | \(-1.23\times10^4\) | \(-1.50\times10^4\) |
| 1             | \(-1.02\times10^3\) | \(-3.27\times10^3\) |
| 2             | \(-7.32\) | \(-6.91\times10^2\) |
| 3             | \(-5.67\) | \(-2.76\times10^2\) |
| 4             | \(-4.62\) | \(-1.44\times10^2\) |
| 5             | \(-3.88\) | \(-8.63\times10^1\) |
| 10            | \(-2.20\) | \(-1.89\times10^1\) |
| 20            | \(-1.15\) | \(-4.14\) |
| 30            | \(-7.97\times10^{-1}\) | \(-1.74\) |
| 50            | \(-4.99\times10^{-1}\) | \(-6.90\times10^{-1}\) |
| 100           | \(-2.38\times10^{-1}\) | \(-1.35\times10^{-1}\) |

TABLE I: Numerical results of the coefficients \( c_0 \) and \( c_2 \). We neglect the surface contribution at \( m_\pi = 0 \) shown in Eq. (94) due to distinct behavior of the Skyrme profile function \( f(r) \) at \( r \rightarrow \infty \), i.e. \( f(r) \sim 1/r^2 \), discussed in Sec. [13].

V. NO CONTRIBUTION FROM OTHER CORRECTIONS

In this section, we study other effects of the background magnetic field to the the electric charge of the nucleon. Our aim is to show that the anomaly-induced electric charge is not cancelled by the other electromagnetic effects.

The total electric charge is written as a modified Gell-Mann–Nishijima formula,\[Q_e = e I_3 + \frac{e N_B}{2} + \frac{Q_{\text{ann}}}{2} \tag{99}\]

The first term stems from the electromagnetic current in the original Skyrme model. The second term is due to the baryon number coupling to the electromagnetic potential. The last one is the anomaly-induced electric charge which is nonzero only when we have the background electromagnetic field.

We are working with the perturbative expansion with respect to the background magnetic field \( eB \). What we found for the anomaly-induced charge is\[Q_{\text{ann}} = O(e^2 B). \tag{100}\]

In the total charge formula (99), the second term is due to the baryon charge which is a topological charge for the Skyrmion model, thus not corrected by the background magnetic field. On the other hand, the first term can be corrected in the presence of \( B \). If a correction of the order \( O(eB) \) appears from the \( I_3 \) term in the charge formula, then it may possibly cancel our anomaly-induced charge. In the following, we shall present an argument showing that there is no such correction of \( O(eB) \) to the \( I_3 \) term.
First, let us examine if there is a correction to the electromagnetic current itself in the Skyrme model. One may naively think that since the Skyrme solution itself is corrected by the background electromagnetic field, the current may also be corrected. However, this is not the case for the Skyrmion. The reason is that for the Skyrmion, the electromagnetic $U(1)$ is identical to a part of the isospin, and the action itself has the isospin structure from the first place. In fact, the relevant $I_3$ is indeed expressed by a part of the flavor $SU(2)$ rotation, and thus the current $I_3$ is universally expressed as

$$ I_3 = \frac{i}{2} \left[ a_0 \frac{\partial}{\partial a_3} - a_3 \frac{\partial}{\partial a_0} - a_1 \frac{\partial}{\partial a_2} + a_2 \frac{\partial}{\partial a_1} \right] $$  \hspace{1cm} (101)

Here, there is no room for $eB$ to show up, thus we can safely use this expression for $I_3$ in the electric charge formula \( [19] \).

Then, the issue is whether the expectation value of $I_3$ in the background magnetic field is corrected or not. The background magnetic field modifies the wave function of the Skyrmion, so in principle this corrected wave function may give a correction to $\langle I_3 \rangle$, which is of importance for us. We shall show in the following that there is no such correction at $O(eB)$.

To proceed, we need to know how the Skyrmie moduli wave function is corrected. Due to the background magnetic field, there appears a potential in the moduli space, then two of the moduli parameters are lifted to become pseudo-moduli parameters. The corrected quantum mechanics of the moduli and the pseudo-moduli is written as

$$ S = 2\lambda \sum_{i=0}^{3} (\hat{a}_i)^2 - eB_3 \hat{V}(\hat{a}) $$  \hspace{1cm} (102)

where the potential of the quantum mechanics is of the form $\hat{V}(\hat{a}) = ((a_1)^2 + (a_2)^2) \hat{V}(\hat{a})$. The function $\hat{V}(\hat{a})$ is a polynomial of $a_\mu$ (with a finite order), with just numerical coefficients. The potential $\hat{V}(\hat{a})$ breaks the $SU(2)$ symmetry of the system down to the diagonal $U(1)$.

We briefly explain how to derive the form \( [112] \) of the induced potential. In the Skyrmie model, the electromagnetic interaction enters as

$$ \hat{R}_\mu = D_\mu U U^\dagger, \quad D_\mu U \equiv \partial_\mu + ieA_\mu[q, U]. $$  \hspace{1cm} (103)

For a background magnetic field $B_3$, we consider $A_1 = -B_3 x^3$, thus among $\hat{R}_\mu$, the electromagnetic contribution appears only in $\hat{R}_1$ as

$$ \hat{R}_1 = \partial_1 U U^\dagger + \delta \hat{R}_1, $$  \hspace{1cm} (104)

$$ \delta \hat{R}_1 \equiv -ieB_3 x^2 [q, GU_0 G^\dagger]GU_0 G^\dagger. $$  \hspace{1cm} (105)

Here we have already substituted the Skyrmie solution. Assuming that $G$ is dependent on $t$ and plugging this $\hat{R}$ into the action, we obtain a correction to the Skyrmie Lagrangian at $O(eB)$ as

$$ \delta L = \frac{F^2}{8} \text{Tr}[\hat{R}_1 \delta \hat{R}_1] + \frac{1}{8e^2} \text{Tr}[\hat{R}_\mu, \hat{R}_1][R^n, \delta \hat{R}_1]. $$  \hspace{1cm} (106)

Since we know that the Skyrme solution has the particular dimension dependence $x \rightarrow x/(e, F^2)$, we obtain

$$ \delta S = \int d^4x \, \delta L = \int dt \frac{1}{e^2 F^4} eB_3 \hat{V}(\hat{a}), $$  \hspace{1cm} (107)

where $\hat{V}(\hat{a})$ is a polynomial in $a_\mu$, with only dimensionless numerical coefficients. This is nothing but the potential in Eq. \( [112] \). As the potential $\hat{V}(\hat{a})$ should vanish when $G$ corresponds to the electromagnetic direction, i.e. the $r_3$ direction, $\hat{V}(\hat{a})$ is proportional to $(a_1)^2 + (a_2)^2$.

With the potential, the Skyrmie wave function $\psi(\hat{a})$ is modified. We may apply a well-known perturbation technique for quantum mechanics, and obtain the corrected nucleon wave function as

$$ |l = 1/2\rangle = |l = 1/2\rangle_0 + eB_3 \sum_{n=1}^{\infty} \frac{V_{l=1/2, l=n+1/2} I_{l=1/2} - E_{l=n+1/2}}{E_{l=1/2} - E_{l=n+1/2}} l = n + 1/2\rangle_0 + O((eB)^2). $$  \hspace{1cm} (108)

Here $V_{l=1/2, l=n+1/2}$ is the matrix element of the operator $V$ appearing in the quantum mechanics \( [112] \), and the state with subscript 0 is the one without the perturbation. In the current case the states have degenerate energy, but the expression above is universal.

Now, using this corrected wave function, we evaluate the expectation value of $I_3$. Since we have

$$ \langle l = n+1/2 | I_3 | l = 1/2\rangle_0 = 0 $$  \hspace{1cm} (109)

for $n \geq 1$, we obtain

$$ \langle I_3 \rangle = \langle I_3 \rangle_0 + O((eB)^2), $$  \hspace{1cm} (110)

where $\langle I_3 \rangle_0$ is the third component of the isospin of the leading (uncorrected) order wave function. Therefore, the electromagnetic correction to the charge formula starts at $O((eB)^2)$, which is at higher order compared to the anomaly-induced charge $Q_{\text{ann}}$. This means that our anomaly-induced charge $Q_{\text{ann}}$ is the leading-order correction of $O(eB)$, and cannot be cancelled by the other effect of the background magnetic field.

Here we have presented an argument that the total induced charge due to the anomaly is not cancelled by other corrections due to the magnetic field. This argument may be reinforced and supplemented by an explicit computation of a back-reaction to the Skyrmie configuration itself, from the magnetic field. The calculation of the back-reaction is quite complicated, so we leave it to our future work.

VI. HIGHER-CHARGE SKYRMIONS

In this section, we study classical higher-charge Skyrmions and the anomaly-induced charges. To this end, we will utilize the so-called rational map ansatz \( [19] \).
which is a reasonable method giving a good approximation. In this section we use another notation based on a standard textbook [20]. A main difference from the previous sections is the dimensionless coordinate

\[ x_\mu \rightarrow \frac{2x_\mu}{e_\Sigma F_\pi}, \quad \partial_\mu \rightarrow \frac{e_\Sigma F_\pi \partial_\mu}{2}. \quad (111) \]

Let us first give a brief review on the rational map ansatz. A solution of the Skyrmion \( U(x) \) with \( U(x) \rightarrow 1 \) as \( |x| \rightarrow \infty \) gives a map from \( \mathbb{R}^3 + \{ \infty \} \simeq S^3 \) to \( SU(2) \simeq S^3 \). The map is characterized by the homotopy group \( \pi_3(SU(2)) = \mathbb{Z} \). More explicitly, it can be expressed as

\[ U(x) = \exp(i f_B(r) \bar{r} \cdot \vec{n}(\theta, \phi)), \quad (112) \]

where \( \{ f_B, \vec{n} \} \) is a coordinate of \( SU(2) \) under a constraint \( f_B \in [0, \pi] \) and \( |\vec{n}| = 1 \). Namely, we decompose \( SU(2) \) into \([0, \pi] \times S^2\). The parameters \((r, \theta, \phi)\) are standard spherical coordinates on the space \( \mathbb{R}^3 \simeq \mathbb{R}_{\geq 0} \times S^2 \). In order to get the map of degree \( N_B \in \mathbb{Z} \), we assume that \( f_B \) is a one-to-one map from \( \mathbb{R}_{\geq 0} \rightarrow [0, \pi] \). Then \( \vec{n}(\theta, \phi) \) should give a map \( S^2 \rightarrow S^2 \) of degree \( N_B \). Let us introduce the stereographic projection which is useful to find the generic map of degree \( N_B \),

\[ z(\theta, \phi) = e^{i \phi} \tan \frac{\theta}{2}. \quad (113) \]

For example, the \( N_B = 1 \) hedgehog ansatz (the one-to-one map from \( S^2 \) to \( S^2 \)) can be expressed by

\[ \vec{n}(z, \bar{z}) = \left( \frac{z + \bar{z}}{1 + |z|^2}, -i \frac{z - \bar{z}}{1 + |z|^2}, 1 - |z|^2 \right). \quad (114) \]

This can be easily extended to a \( N_B \)-to-one map from \( S^2 \) to \( S^2 \) by replacing \( z \) with any rational maps \( w(z) : \mathbb{C} \rightarrow \mathbb{C} \),

\[ w(z) = \frac{P(z)}{Q(z)}. \quad (115) \]

Here \( P(z) \) and \( Q(z) \) are holomorphic functions in \( z \) and we set \( N_B = \text{max} \{ \deg P, \deg Q \} \). Thus, we obtain the map from \( S^3 \) to \( S^2 \) of degree \( N_B \),

\[ \vec{n} = \left( \frac{w + \bar{w}}{1 + |w|^2}, -i \frac{w - \bar{w}}{1 + |w|^2}, 1 - |w|^2 \right). \quad (116) \]

Plugging this into Eq. (112), we reach the map from \( S^3 \) to \( SU(2) \) of degree \( N_B \). This is called the rational map ansatz.

The baryon number can be expressed by

\[ N_B = -\int \frac{f_B'}{2\pi^2} \left( \frac{\sin f_B}{r} \frac{1 + |z|^2}{1 + |w|^2} \right)^2 |dw/dz|^2 r^2 d\Omega_z, \quad (117) \]

where \( d\Omega_z \) is the usual area element on a 2-sphere

\[ d\Omega_z = \frac{2i dz d\bar{z}}{(1 + |z|^2)^2} = \sin \theta d\theta dr. \quad (118) \]

By making use of the following pull-back

\[ \left( \frac{1 + |z|^2}{1 + |w|^2} \right)^2 \left( \frac{dw}{dz} \right)^2 d\Omega_z = d\Omega_w, \quad (119) \]

it is easy to change the integral area over \( S^2 \) to the target space of the rational map \( S^2 \) as

\[ \text{r.h.s of Eq. (117)} = -\int \frac{f_B'}{2\pi^2} \sin^2 f_B drd\Omega_w \]

\[ = \frac{2N_B}{\pi} \int_0^\infty \sin^2 f_B df_B = N_B, \quad (120) \]

where we have used \( \int d\Omega_w = 4\pi N_B \). The Skyrmion energy in the \( F_\pi/(4e_\pi) \) energy unit and \( 2/(e_\pi F_\pi) \) length unit with the rational map ansatz is given by

\[ E = 4\pi \int_0^\infty dr \left[ \frac{e^2 f_B^2}{r^2} + 2N_B (f_B^2 + 1) \sin^2 f_B \right. \quad \left. + 8m_\pi^2 \right] \left. (1 - \cos f_B) \right], \quad (121) \]

where we have introduced

\[ I \equiv \frac{1}{4\pi} \int \left( \frac{1 + |z|^2}{1 + |w|^2} \right)^4 \left| \frac{dw}{dz} \right|^4 d\Omega_z. \quad (122) \]

In order to find the best approximation, we need to seek an appropriate rational map \( w(z) \). We should choose \( w \) in such a way that \( I \) is minimized. Though this is not easy task, by using a numerical method, the rational maps \( w(z) \) for several \( N_B \) were found in Ref. [19]. For instance, the following rational maps for \( N_B = 1, 2, \ldots, 8 \) are known as

\[ w_1 = z, \quad \frac{z^2}{(1 + |z|^2)^2}, \quad (123) \]

\[ w_2 = z^2, \quad \frac{z^4}{(1 + |z|^2)^3}, \quad (124) \]

\[ w_3 = \frac{z^3 - 3iz}{\sqrt{3}z^2 - 1}, \quad (125) \]

\[ w_4 = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}, \quad (126) \]

\[ w_5 = \frac{z(z^4 + 2az^2 + a)}{az^4 - bz^2 + 1}, \quad (127) \]

\[ w_6 = \frac{z^4 + 4c}{z^2 (idz^4 + 1)}, \quad (128) \]

\[ w_7 = \frac{z^7 - 3z^5 - 7z^2 - 1}{z^7 + 7z^5 - 7z^2 + 1}, \quad (129) \]

\[ w_8 = \frac{z^6 - d}{z^2 (dz^6 + 1)}, \quad (130) \]

with \( a = 3.07, b = 3.94, c = 0.16 \) and \( d = 0.14 \).

The last task is to determine the profile function \( f_B \). Because no analytic solutions have been known, we need
to solve equations of motion numerically,
\[
\left(1 + \frac{2N_B}{r^2} \sin^2 f_B \right) f_B'' + \frac{2}{r} f_B' + \frac{N_B \sin 2f_B}{r^2} \left( f_B^2 - 1 - \frac{1}{N_B} \sin^2 f_B \right) - 4m_π^2 \sin f_B = 0, \tag{131}
\]
with the boundary condition \( f_B(0) = \pi \) and \( f_B(\infty) = 0 \). We show several numerical solutions for \( N_B = 2 \) with different pion masses in Fig. 7.

Now we are ready to evaluate the anomaly induced electric charge from Eqs. (37) and (46). As before, what we need is only \( P_i \) which can be obtained from
\[
R_i = i(\vec{\tau} \cdot \vec{n}) f_B' \hat{x}_i + \frac{i}{2} (\vec{\tau} \cdot \partial_i \vec{n}) \sin 2f_B \\
+ \{(\partial_i \vec{n} \cdot \vec{n}) + i(\partial_i \vec{n} \times \vec{n}) \cdot \vec{\tau}\} \sin 2f_B, \tag{132}
\]
\[
L_i = i(\vec{\tau} \cdot \vec{n}) f_B' \hat{x}_i + \frac{i}{2} (\vec{\tau} \cdot \partial_i \vec{n}) \sin 2f_B \\
+ \{(\partial_i \vec{n} \cdot \vec{n}) - i(\partial_i \vec{n} \times \vec{n}) \cdot \vec{\tau}\} \sin 2f_B. \tag{133}
\]
By plugging these into Eq. (37), we get
\[
P_i = -f_B' n_3 \hat{x}_i - \frac{1}{2} (\partial_i n_3) \sin 2f_B. \tag{134}
\]
Note that, as expected, replacement \( n_3 \) with \( \hat{x}_3 \) gives us \( N_B = 1 \) hedgehog solution.

The induced charge densities for the \( N_B = 2 \) solution are shown in Fig. 8. As one can see, the \( N_B = 1 \) and \( N_B = 2 \) charge distributions are quite similar, even though the baryon charge distributions are totally different. However, one can find differences if paying attention to the detail structures. As can be seen in Figs. 4 and 9, the \( N_B = 2 \) densities are fatter than those of \( N_B = 1 \). Also the \( N_B = 2 \) configuration has an internal structure.

The anomaly induced charges of the classical Skyrmions with \( N_B = 1, 2, \cdots, 8 \) under a constant back-ground magnetic field \( \vec{B} = (0, 0, B_3) \)
\[
Q_{\text{anom}}^{\text{classical}} = \frac{e^2}{16\pi^2(e_s F_\pi)^2} B_3 \tilde{c}_0, \tag{135}
\]
\[
\tilde{c}_0 = 4 \int d^3x \, P_3, \tag{136}
\]
are summarized in the Table II. The pre-factor 4 is
TABLE II: The anomaly induced charge of $N_B = 1, 2, \cdots, 8$ Skyrmions under a constant magnetic field background. The dimensionless pion mass is chosen to be $m_\pi = 0.263$.

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$N_B$ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
$c_0$ & -43.2 & -105 & -60.3 & 0.00 & -13.3 & 28.7 & 0.00 & -11.6 \\
\hline
\end{tabular}
\end{center}

needed because of the dimensionless coordinate $x_\mu \to 2x_\mu/e_sF_\pi$. Note that $c_0$ and $\tilde{c}_0$ for $N_B = 1$ are related by $\tilde{c}_0 = 4\pi c_0/3$. We find that the classical anomaly-induced charge is not proportional to the baryon charge $N_B$. It is intriguing that $N_B = 4$ and 7 Skyrmions have zero induced electric charge. From the values given in the Table II we observe that higher-charge Skyrmions tend to cancel the total induced charge. A natural reason for this cancellation is as follows. Each Skyrmion has a classical orientation in spin and isospin space, and to form a bound state of the Skyrmions the orientations should be arranged to cancel each other. Our formula of the anomaly-induced charge depends on the signs of the quantum spin and isospin, so, accordingly the total anomaly-induced charge would tend to cancel each other. Although we have not performed quantization of the higher-charge Skyrmions, we expect that this cancellation should occur even at the quantized level.

Let us finally display the baryon number densities and anomaly induced electric charges of the Skyrmions with $N_B = 3, 4, \cdots, 8$ and $N_B = 17$ [10], see Fig. 10. The anomaly-induced charge densities exhibit amusing shapes. Possible interpretation of the shape is an open question.

VII. CONCLUSION AND DISCUSSION

We have evaluated the gauged WZW term for quantized Skyrmions under a background magnetic field in the expansion of the electromagnetic coupling constant. We have found that there is an anomaly-induced charge structure due to the gauged WZW term. The detailed analysis of the total induced charge suggests that the pion cloud of the baryons can induce a net charge. The magnitude of the induced charge structure is $O(e^2B)$, so it is quite small except for the case with strong magnetic field background.

We have calculated the anomaly-induced electric charge for any baryons which appear as quantum excitations of Skyrmions (Section III). The induced charge is non-vanishing when the magnetic field is present along the axis of the quantization of the spin and the isospin of the baryon. In Section IV we argued that this induced charge may not be cancelled by other possible electromagnetic corrections to the Skyrmion, although a complete verification may need an explicit calculation of the back-reaction of the Skyrmion solution in the magnetic field. We further examined explicitly the anomaly-induced quadrupole moment (Section IV) and also the cases with multi-baryons (Section VI).

It is nontrivial that an additional electric charge of baryons is generated in magnetic fields. It may have an observable effect on physics related neutron stars and heavy ion collisions [10].
Finally we discuss possible origin of the anomaly-induced charge. One may wonder if the constant magnetic field may be too artificial and it might be a reason for the anomaly-induced charge. In Appendix A we considered a magnetic field generated by a circular electric current, and we found that the calculated induced charge is again nonzero. It suggests that the induced charge is not an artifact of the everywhere-constant magnetic field.

Then what is the origin of the additional charge? A good indication comes from the peculiar property of baryons. As shown in Appendix B, we found that the total induced charge is due to the multi-pion effect in the nonlinear sigma model. As the Skyrmion profile extends to the spatial infinity, the charge distribution also has a tail which elongates to the spatial infinity. This would be the origin of the generation of the additional electric charge. Obviously, if quarks are completely confined, the total charge of any baryon should be quantized to be a half-integer. However, in reality, any baryon is surrounded by a pion cloud, which means that quark-antiquark pair can percolate out of the mean volume of the baryon. We can interpret our anomaly-induced charge as an effective charge carried by the pion cloud surrounding the baryon. To make sure our interpretation, it is important to calculate the complete effect of the magnetic field, i.e. the back-reaction to the Skyrmion profile due to the magnetic field.

The anomaly-induced charge may appear to violate the charge conservation. In general, any electromagnetic current should be conserved at on-shell when the total system is gauge-invariant, and this applies surely to our case. However, we considered in our paper only a static situation, so we have not considered the situation where one turns on the magnetic field gradually from zero to a nonzero value, in a time-dependent manner. To understand the origin of the additional charge concretely, one needs to calculate the back-reaction and also the time-dependent magnetic fields. We leave it to our future work.

Acknowledgment. — The authors would like to thank Aleksey Chernia, Kenji Fukushima, Deog-Ki Hong, Nicholas Manton, Makoto Oka, Masashi Wakamatsu, Nodoka Yamanaka, and Koichi Yazaki for useful comments and discussions. The work of M. E. is supported in part by Grant-in Aid for Scientific Research (No. 23740226). K. H. is supported in part by the Japan Ministry of Education, Culture, Sports, Science and Technology. H.I. is supported by Grant-in-Aid for Scientific Research on Innovative Areas (No. 23105713). T. I. was supported by JSPS Research Fellowships for Young Scientists.

Appendix A: Anomaly induced charge in circular electric-current

In the above argument of the anomaly induced charge, we have assumed a uniform external magnetic-field. However the magnetic field should be always closed unless the magnetic monopole appears. In this appendix, we consider the anomaly induced charge in the external magnetic field generated by a circular electric-current, which is instructive for us because the magnetic field is closed with finite circular radius, whereas that becomes uniform when a radius of the circular electric-current becomes infinity. Here we suppose that an electric field is not induced by the electric current. We will show that the anomalous charge is induced in the circular electric-current even with the finite radius.

Let us suppose the circular electric-current density with a radius $a$ on xy-plane as,

$$j(r) = \frac{j_0 a}{2\pi} \delta(x) \delta(y) \left(-\sin \zeta, \cos \zeta, 0\right),$$

where we assume that magnitude of the electric-current is proportional to the radius $a$. The magnetic field generated by the electric-current density can be given by,

$$B(r) = \frac{\mu}{4\pi} \text{rot} \int \frac{d\tilde{r} \cdot \tilde{j}(\tilde{r})}{|\tilde{r} - r|}.$$  \hspace{1cm} (A2)

with $\mu$ being a magnetic permeability. For simplicity, we omit the factor $\mu/4\pi$ in the following. One can easily see that, in the large radius limit ($a \rightarrow \infty$), the magnetic field becomes

$$B_1 = B_2 = 0, \quad B_3 = j_0.$$  \hspace{1cm} (A3)

This is the same situation with the uniform external magnetic-field to z-direction.

When the nucleon is located at the center of the circular electric-current, the anomalous charge in the external magnetic field is given by an integration of $\langle j_{anm} \rangle_{I_3, S_3}$, shown in Eq. (A2), over the whole space,

$$Q_{anm} = \frac{ieN_c}{48\pi^2} \int d^3x B_i(P_{\hat{I}_3, S_3}).$$

Notice that the magnetic field is also functions of the coordinate variables. Performing the integration over the whole angular-space, we can separate three components of the anomalous charge:

$$\rho_{xy}(r) = \int d\Omega_2 \tilde{x}_1 \tilde{x}_3 \tilde{B}_1 = \int d\Omega_2 \tilde{x}_2 \tilde{x}_3 \tilde{B}_2,$$

$$\rho_{z1}(r) = \int d\Omega_2 \tilde{B}_3, \quad \rho_{z2}(r) = \int d\Omega_2 \tilde{x}_3 \tilde{B}_3,$$

where $\tilde{B}_i = B_i/j_0$. Then the anomalous charge can be rewritten as

$$Q_{anm} = \frac{4eN_c}{27\pi} (I_3 S_3) \left(\frac{e_j}{(e_s F_3)^2}\right)(c_{xy} + c_{z1} + c_{z2}),$$

where $c_{xy}$, $c_{z1}$, and $c_{z2}$ are as in Eq. (A4).
the anomalous charge is induced by not only $B$-term, but also $z$-term, which implies that $c_{z,1} + c_{z,2} \sim c_0$. Furthermore, since the anomalous charge is also induced by $B_x$ and $B_y$ at finite radius discussed above, there is finite contribution, $|c_{xy}| > 0$. This extra contribution gives larger induced charge than that induced by the uniform magnetic field.

We also calculate the multipole moment due to the anomaly, and find that the results are similar to the case of the uniform magnetic field: the dipole moment vanishes, whereas the quadrupole moment $Q_{ij}$ shows finite values only for diagonal parts ($i = j$).

Appendix B: Multi-pion effect and comparison with point-particle picture

Here we argue that the anomaly-induced electric charge is due to the pion cloud which exists around any baryon. The pion “cloud,” which is the multi-pion effect, in the anomaly term is simply the terms with higher powers in the $\pi$ field. The anomaly term in the gauged WZW term $S_{WZW}$ can be expanded as

\begin{equation}
S_{WZW} \sim \int d^4 x \ A_0 B_3 P_3 \\
\sim \int d^4 x \ \text{Tr}[\gamma_5 U^\dagger \partial U] A_0 B_3 \\
\sim \int d^4 x \ [\partial_0 + \pi \pi \partial_\pi + \cdots] A_0 B_3. \quad (B1)
\end{equation}

The first term is responsible for the famous $\pi_0 \to 2\gamma$ interaction, while the remaining terms are the pion cloud.

In the following, we shall see that, only with the first term, the anomaly-induced total charge $Q_{\text{ann}}$ vanishes. So, our anomaly-induced total charge is due to the pion cloud.

For the Skyrme solution, we have $\pi_0 \sim f(r) \hat{x}_3$, so the total electric charge induced by the first term in Eq. (B1)
is proportional to
\[
\int d^3x \, \partial_3 \pi_0 = \int d^3x \partial_3 (f(r) \xi_3) = 2\pi \int r^2 \sin \theta d\theta \, \sin \theta \, \left[ (f' - f) \cos^2 \theta + f \right]
\]
\[
= \frac{4\pi}{3} \int_0^\infty dr \, \left( r^2 f' + 2rf \right)
\]
\[
= \frac{4\pi}{3} \left[ r^2 f \right]_{r=0}^{r=\infty}.
\]
(B2)

The last expression vanishes for nonzero pion mass, because \( f(r) \) decays exponentially at large \( r \), and \( f(0) \) is finite. So, the anomaly-induced total charge vanishes if one use only the single-pion term in the anomaly term [18].

It was discussed in [18] that the anomaly-induced total charge of nucleon vanishes, by using a generic argument without using the specific Skyrme model. The argument [18] uses only the single-pion term, so our result is consistent with it.

Before going to the multi-pion term, we note that, in the chiral limit where the pion mass vanishes, the last expression is nonzero, since \( f \sim r^{-2} \) at large \( r \) (see [11]). So, in the chiral limit, contribution which comes from the single-pion term is nonzero. This is again consistent with the discussion in [18] where the pion momentum is neglected compared to the pion mass to show the vanishing total charge. Note that this discussion on the chiral limit is suggestive but not so firm since various observables in the Skyrme model diverges in the chiral limit.

Now, let us evaluate the multi-pion term in Eq. (B1). The representative 3-pion term is evaluated as
\[
\int d^3x \, \pi \pi \partial \pi \sim \int_0^\infty r f(r)^3 dr,
\]
which is nonzero for any pion mass. Therefore, we conclude that our anomaly-induced total charge is due to the multi-pion effect.

The point-particle picture of [18] shows that the quadrupole moment is induced as a leading moment. So let us compare conclusion of the Skyrmion with that of the point-particle picture.

The anomaly-induced quadrupole moment has been written in the point-particle picture as [18]:
\[
Q_{ij}^{pp} = \frac{N_c \alpha}{6\pi} \frac{g_A}{(f_\pi m_\pi)^2} N^i \sigma^j r^3 NB^j,
\]
where \( \alpha = e^2/4\pi \), and \( g_A \) and \( N \) are the axial coupling constant and the nucleon wave function, respectively. In the Skyrmion, the quadrupole moment due to the anomaly is given in Eq. (99), where the pion-mass dependence of the coefficient becomes,
\[
c_2 \simeq \frac{A}{(m_\pi/c_r F_r)^2}.
\]
(B5)

Using the formula of \( g_A = -\pi D/3e_\pi^2 \) in the Skyrme model, where \( D \) is the numerical coefficients of \( r \) integration including the pion profile function [11], we can rewrite the quadrupole moment with familiar physical observables as,
\[
Q_{ij} = -\frac{8N_c \alpha A}{45\pi} \frac{g_A}{F_r m_\pi^2} \frac{g_A}{(F_r m_\pi)^2} \hat{Q}_{ijk} B_k.
\]
(B6)

We have checked that the numerical coefficient \( D \) does not show singular dependence on \( m_\pi \). This implies that the pion-mass dependence of the quadrupole moment is qualitatively consistent between the point-particle picture and the Skyrme picture.

As a consequence of the comparison, we find no contradiction between the point-particle picture and the Skyrme picture. For further understanding of the anomaly-induced charge, calculations of the multi-pion effect in the point-particle picture are required.
[20] N. S. Manton and P. M. Sutcliffe, Cambridge, UK: Univ. Pr., 493 (2004).

[21] Note that, when we make the spatial integration, the last term in (43) drops off as it is a total derivative term. For massive pions, the pion profile of the Skyrmion always decay exponentially asymptotically, so the surface integral derived from the integration of this total derivative term always vanish.

[22] As long as we think of the massless pion as the limit $m_\pi \to 0$, the surface term contribution can be always ignored.

[23] It is possible that the last term in (43) gives additional multipoles although it is negligible for the total charge. However, the term itself is not gauge-invariant and once combined with the baryon number term (the second term in (43)) it becomes gauge-invariant. In this paper we evaluate only the gauge-invariant $j_{\mu n m}$ for the quadrupole moment, and the other terms (which can be evaluated if a back-reaction to the Skyrmion profile can be computed) are left for our future work.