Independent Set Size Approximation in Graph Streams *

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Abstract

We study the problem of estimating the size of independent sets in a graph G defined by a stream of edges. Our approach relies on the Caro-Wei bound, which expresses the desired quantity in terms of a sum over nodes of the reciprocal of their degrees, denoted by \( \beta(G) \). Our results show that \( \beta(G) \) can be approximated accurately, based on a provided lower bound on \( \beta \). Stronger results are possible when the edges are promised to arrive grouped by an incident node. In this setting, we obtain a value that is at most a logarithmic factor below the true value of \( \beta \) and no more than the true independent set size. To justify the form of this bound, we also show an \( \Omega(n/\beta) \) lower bound on any algorithm that approximates \( \beta \) up to a constant factor.

1 Introduction

For very large graphs, the model of streaming graph analysis, where edges are observed one by one, is a useful lens. Here, we assume that the graph of interest is too large to store in full, but some representative summary is maintained incrementally. We seek to understand how well different problems can be solved in this model, in terms of the size of the summary, time taken to process each edge and answer a query, and the accuracy of any approximation obtained. Variants arise in the model depending on whether edges can also be removed as well as added, or if edges arrived grouped in some order, and so on.

We study questions pertaining to independent sets within graphs. Independent sets play a fundamental role in graph theory, and have many applications in optimization and scheduling problems. Given a graph, an independent set is a set of nodes such that there is no edge between any pair. There are conceptual links with matchings in graphs, since the dual problem (find a set of edges such that no pair shares a node in common) encodes the matching problem. However, while matching permits efficient algorithms in the offline setting, for independent set, the maximization problem is NP-hard, and remains hard to approximate within \( n^{1-\epsilon} \) for any \( \epsilon \).

The matching problem has received significant interest in the streaming setting, and a large number of approximation algorithms are known, with variations based on number of passes over the input data, and whether the edges are weighted or unweighted. Yet Independent Set is much less well understood. In this paper, we provide algorithms and lower bounds that characterize how well we can approximate the independent set problem in the data stream model. We focus on the cardinality version of the problem: the objective is to output an estimate of the independent set size. The size of the independent set can be linear in the number of nodes, while we show that in some cases its cardinality can be estimated in polylogarithmic space.

Our results rely on a combinatorial characterization of the independent set size in terms of the degrees of nodes. This reduces the focus to approximating a simple to describe function (denoted as \( \beta \)), yet this is still challenging in the streaming model. Indeed, this \( \beta \) function is hard to approximate when applied to an arbitrary input sequence. However we show that we can obtain good approximations to \( \beta \) when

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the input sequence corresponds to a graph. We further distinguish the cases based on ordering in the input. For arbitrary arrival orders, we obtain an approximation algorithm for $\beta$, where the space reduces based on an assumed lower bound on $\beta$ (which can be obtained based on average degree, for example). We then show that in the vertex arrival order, where each node arrives along with all edges to nodes already in the graph, we can do much better: the space cost drops to polylogarithmic, albeit for a slightly different notion of approximation. Our lower bounds characterize the minimum space cost necessary for any algorithm which follows this approach, and help to explain why some stronger approximation results are not possible in either model.

We proceed as follows. First, we provide necessary definitions and notation; this allows us to state our results more formally (Section 2). After surveying related work, we present our algorithms for the two arrival models – arbitrary edge arrivals (Section 3), and vertex-grouped edges arrival (vertex arrival for short, Section 5). Last, we show space lower bounds for this problem based on a reduction to a hard problem in communication complexity (Section 6).

2 Definitions and Statement of Results

The Independent Set problem is most naturally modeled as a problem over graphs $G = (V, E)$. A set $U \subseteq V$ is an independent set if for all pairs $u, w \in U$ we have $\{u, w\} \notin E$, i.e. there is no edge between $u$ and $w$.

We consider graphs defined by streams of edges. That is, we observe a sequence of unordered pairs $\{u, w\}$ which collectively define the (current) edge set $E$. We do not require $V$ to be given explicitly, but take it to be defined implicitly as the union of all nodes observed in the stream. In the (arbitrary, possibly adversarial) edge arrival model, no further constraints are placed on the order in which the edges arrive. In the vertex arrival model, there is a total ordering on the vertices $\prec$ which is revealed incrementally. Given the final graph $G$, node $v$ “arrives” so that all edges $\{u, v\} \in E$ such that $u \prec v$ are presented sequentially before the next vertex arrives. We do not assume that there is any further ordering among this group of edges.

Let $\alpha(G)$ be the independence number of graph $G$, i.e., the size of a maximum independent set in $G$. Let $\beta(G) = \sum_{v \in V(H)} \frac{1}{\deg_G(v) + 1}$ denote the Caro-Wei bound. It is well-known that $\alpha(G) \geq \beta(G)$, for every graph $G$. Our results focus on the problem of approximating $\beta(G)$ for graphs presented as streams of edges. We show the following three main results:

1. In the Vertex Arrival Order model, we exhibit a one-pass randomized streaming algorithm that w.h.p. computes a value $\beta'$ such that $\beta' = \Omega(\beta(G)/\log n)$ and $\beta' \leq \alpha(G)$ using space $O(\log^2 n)$ bits.

2. A lower bound of $\Omega\left(\frac{n}{\beta(G)^{\gamma}}\right)$ for computing a $\gamma$-approximation to $\beta(G)$. The lower bound holds for the vertex arrival order and hence also in the (weaker) edge arrival order.

3. In the (adversarial) Edge Arrival Order, we present a one-pass randomized streaming algorithm that with high probability computes a $\phi$-approximation to $\beta(G)$ using space $O(\frac{n \polylog n}{\gamma \polylog(\gamma)})$, where $\gamma$ is an arbitrary lower bound on $\beta(G)$. We also show a version of this algorithm which gives a $(1 + \epsilon)$-approximation using $O\left(\frac{n \polylog n}{\epsilon \polylog(\epsilon)}\right)$ space. Quantity $\beta(G)$ is bounded from below by $\frac{n}{d+1}$, where $d$ is the average degree of the input graph. Using $\gamma = \frac{n}{d+1}$, the space of our algorithm becomes poly-logarithmic in $n$ for graphs of constant average degree, such as planar graphs or bounded arboricity graphs.

3 Related Work

There has been substantial interest in the topic of streaming algorithms for graphs in the last two decades. Indeed, the introduction of the streaming model focused on problems over graphs [16]. McGregor provides a survey that outlines key results on well-studied problems such as finding sparsifiers, identifying connectivity structure, and building spanning trees and matchings [19]. We expand on results related to matchings, due to the similarity in problem statement. For the unweighted case, the trivial greedy
algorithm achieves a maximal matching which is a 2-approximation to the size of the maximum matching \[9\]. In the weighted case, a sequence of results have improved the constant of approximation for this problem. Most recently, a \((2 + \epsilon)\) factor approximation was presented \[20\]. In tandem with this line of work there has been a line of work that seeks to approximate the cardinality of the maximum (unweighted) matching in the stream. This requires a distinct set of techniques. Here, results have been recently shown by Assadi et al. \[2\]. For the dynamic version of the problem (edges can be removed and inserted), they give an algorithm with a cost proportional to \(n^2/\alpha^4\), where \(n\) denotes the number of vertices and \(\alpha\) the quality of the approximation, and for the insert-only case, for arbitrary graphs, they give an algorithm with space cost proportional to \(n/\alpha^2\). When the graph is sparse, characterized by having arboricity at most \(c\), a sampling-based algorithm can achieve an exponential improvement in the space cost of \(O(c \log^2 n)\) in order to provide an approximation factor proportional to \(c\) \[6\]. Our aim in this paper is to provide similar guarantees for estimating the cardinality of independent sets.

Computing a maximum independent set is NP-hard on general graphs \[15\] and is even hard to approximate within factor \(n^{1-\epsilon}\), for any \(\epsilon > 0\) \[15, 23\]. For this reason, often either special graph classes are considered that admit reasonable approximations, or a different quality measure on the size of independent sets is used. The Turán bound \[21\] implies that every graph has an independent set of size \(n/(d+1)\), where \(d\) is the average degree of the input graph. Caro \[3\] and Wei \[22\] improved this bound independently to \(\beta(G)\). The quantity \(\beta(G)\) is an attractive bound on the size of a maximum independent set since it is given by the degree sequence alone of a graph. It is known that a simple greedy algorithm for maximum independent sets computes an independent set of size at least \(\beta(G)\) \[22, 11\]. The algorithm iteratively picks a node of minimum degree, and removes all neighbors from consideration — note that this cannot be simulated in the streaming model with small space. There are parallels to other graph problems: for example, the minimum vertex coloring problem is also NP-hard and hard to approximate within a factor of \(n^{1-\epsilon}\). There is however a huge interest in computing \(\Delta+1\)-colorings, which is a quality bound also given by the degrees of the input graph.

It is known that the Caro-Wei bound gives polylogarithmic approximation guarantees on graphs which are of \textit{polynomially bounded-independence} \[13\], which means (informally) that the size of the independent set in \(r\)-neighborhood around a node is bounded in size by a polynomial in \(r\). This graph class includes unit interval and unit disc graphs. The problem of finding independent sets themselves in the streaming model has received some recent attention. Halldórsson et al. showed that an independent set of expected size \(\beta(G)\) can be computed in the edge arrival model using \(O(n \log n)\) space \[12\]. The streaming independent set problem has been studied on interval graphs: In this model, the intervals arrive one-by-one. The goal is to compute an independent set of intervals. There is an algorithm that computes a \(2\)-approximation on general interval graphs and a \(1.5\)-approximation on unit interval graphs which uses space linear in the size of the computed independent set \[3\]. Cabello and Pérez-Lantero gave polylogarithmic space streaming algorithms, which approximate the size of independent sets of intervals \[4\].

Our work is concerned with streaming approximations of the size of the maximum independent set. In \[13\], very strong space lower bounds on approximating the size of a maximum independent set are given: Every \(c\)-approximation algorithm requires \(\Omega(n^2/c)\) space (which can also be achieved by sampling an induced subgraph and computing a maximum independent set in it using exponential time). This strong lower bound provides a strong motivation for considering related measures such as \(\beta(G)\) instead. Approximating \(\beta(G)\) is essentially the same as approximating the \(-1\) negative frequency moment or harmonic mean of a frequency vector derived from the graph stream. This approach has been addressed via sampling approaches in the property testing literature \[10, 7, 1\], but has received less attention from the perspective of streaming algorithms. Braverman and Chestnut studied the problem of approximating the negative frequency moments \[3\] for general frequency vectors. They consider only \((1 + \epsilon)\)-approximations and relate the space complexity to the stream length, i.e., the total weight of the input stream. Our results evade their lower bounds, since the additional constraint of being the degree distribution of a graph limits the shape of the derived frequency vector, and precludes the pathological cases.

4 Algorithm in the Edge-arrival Model

Suppose that we are given a bound \(\gamma\) such that \(\gamma < \beta(G)\). We first give an algorithm with space \(O\left(\frac{n \log^2 n}{\epsilon^4 \gamma}\right)\) which approximates \(\beta(G)\) within a factor of \(1 + \epsilon\) with high probability. We then show how
Algorithm 1 takes a uniform random sample \( S \) which implies
\[
\sum |i| \leq \frac{1}{\delta^2}.
\]
Then, via Inequality 3, this gives an accurate estimate of \( \beta \) of these vertices while processing the stream. In a post-processing step, vertices of \( V \) and by plugging Inequality 1 into Inequality 2, we conclude
\[
\sum_{i} \beta_i(G) \leq \beta(G)(1 - \frac{\gamma}{g}).
\]
Let \( g > 1 \) be a parameter we set subsequently to control the approximation factor. Let \( I_1 \) be the set of class indices \( i \) such that \( \beta_i(G) \geq \frac{\beta(G)}{|\log n|g} \), and let \( I_0 \) be all other indices. We call a degree class \( i \) (or \( V_i \)) heavy, if \( i \in I_1 \), otherwise it is light. We will argue that in order to obtain a good approximation to \( \beta(G) \), it is enough to approximate \( \beta_i(G) \) for every heavy degree class \( i \). We have
\[
\sum_{i \in I_0} \beta_i(G) \leq \sum_{i \in I_0} \frac{\beta(G)}{|\log n|g} \leq \frac{\beta(G)}{g}.
\]
which implies \( \sum_{i \in I_1} \beta_i(G) \geq \beta(G)(1 - \frac{1}{g}) \). Furthermore, we obtain
\[
\sum_{i \in I_1} \beta_i(G) \leq \beta(G) \leq \frac{g}{g - 1} \sum_{i \in I_1} \beta_i(G),
\]
and by plugging Inequality 1 into Inequality 2 we conclude
\[
\sum_{i \in I_1} \beta_i(G) \leq \beta(G) \leq \frac{ge}{g - 1} \sum_{i \in I_1} \beta_i(G).
\]
Last observe that for \( i \in I_1 \), we have
\[
\frac{\beta(G)}{|\log n|g} \leq \beta_i(G) = \sum_{v \in V_i} \frac{1}{\deg_G(v) + 1} \leq \frac{|V_i|}{c^i + 1},
\]
which implies \( |V_i| \geq \frac{\beta(G)}{|\log n|g} \), i.e., we establish a lower bound on the size of every heavy degree class.

### 4.2 A \( (1 + \epsilon) \)-approximation Algorithm

Algorithm 1 takes a uniform random sample \( S \) of the vertices of the input graph and maintains the degrees of these vertices while processing the stream. In a post-processing step, vertices of \( S \) are partitioned into degree classes \( (S_i) \), in the same way \( V \) was partitioned in the previous subsection. Large enough sets \( S_i \) then contribute to our estimate for \( \beta(G) \). By adjusting the parameters correctly, we ensure that heavy degree classes \( V_i \) give large samples \( S_i \) with high probability, and we can accurately estimate \( |V_i| \) via \( |S_i| \). Then, via Inequality 3, this gives an accurate estimate of \( \beta(G) \).

The main analysis of Algorithm 1 is conducted in Lemma 1, which gives approximation and space bounds depending on parameters \( \delta, g \) and \( c \). In Theorem 1, we optimize these parameters so that space is minimized for obtaining a \( (1 + \epsilon) \)-approximation. Last, in Theorem 2 we show how this algorithm can be used to obtain a \( \phi \)-approximation, for an arbitrary value of \( \phi \).

**Lemma 1.** Let \( \delta > 0 \). If a value \( \gamma \leq \beta(G) \) is given to the algorithm, then Algorithm 1 is a randomized one-pass streaming algorithm with space \( O(\frac{n \alpha^2 |(n)|g}{\gamma^2 \beta^2 \log c}) \) in the edge arrival model. With high probability it outputs a value \( \beta' \) such that
\[
\frac{1}{1 + \delta} \sum_{i \in I_1} \beta_i(G) \leq \beta' \leq (1 + \delta) \beta(G).
\]
If \( \gamma > \beta(G) \), then the upper bound \( \beta' \leq (1 + \delta) \beta(G) \) still holds w.h.p.
Algorithm 1 Sampling based algorithm

Require: real value $\delta > 0$, real value $\epsilon > 1$, real value $g > 1$, $\gamma \leq \beta(G)$

1: $C \leftarrow \frac{\epsilon}{\delta^2}$, $v_0 \leftarrow \frac{\gamma}{\log n/g}$, $p \leftarrow \frac{C \log n}{v_0}$
2: $S \leftarrow$ subset of vertices obtained by sampling every vertex u.a.r. with probability $p$
3: while Processing the stream do
4:   For every $v \in S$: Compute degree $\deg_G(v)$
5: end while
6: Post-processing:
7: $S_i \leftarrow$ subset of vertices $v$ with $c^i \leq \deg_G(v) < c^{i+1}$
8: $\beta' \leftarrow 0$
9: for $i = 0 \ldots \left\lfloor \frac{\log n}{\log g} \right\rfloor - 1$ do
10:   if $|S_i| \geq v_0p/(1 + \delta)$ then
11:      $\beta' \leftarrow \beta' + \frac{|S_i|}{(c^{i+1} + 1)p}$
12: end if
13: end for
14: return $\beta'$

Proof. Suppose that $\gamma \leq \beta(G)$. First, we prove that for every $i$ with $|V_i| \geq v_0 = \frac{\gamma}{\log n/g}$, the probability that the size of set $S_i$ deviates from its expectation by more than a factor of $1 + \delta$ is small. To this end, suppose indeed that $|V_i| \geq v_0$. Then, $\mu = \mathbb{E}[S_i] \geq v_0p$, and

$$P\left(|S_i - \mu| \geq \delta \mu\right) \leq 2 \exp\left(-\frac{C \log n \delta^2}{2}\right) \leq n^{-\frac{\delta^2 C \log n}{2}} \leq n^{-2},$$

for $C \geq \frac{16}{\delta^2}$, applying a standard Chernoff bound. This proves that with high probability the condition in Line 10 is fulfilled for every heavy degree class defined by the threshold $v_0$. Next, suppose that $|V_i| \leq v_0/(1 + \delta)^2$. Then, $\mathbb{E}[S_i] \leq v_0p/(1 + \delta)^2$, and by a similar Chernoff bound argument,

$$P\left(|S_i| \geq \frac{v_0p}{(1 + \delta)^2}\right) \leq P\left(|S_i| > \frac{v_0p}{(1 + \delta)^2} \cdot (1 + \delta)\right) \leq \exp\left(-\frac{\delta^2 C \log n}{2(1 + \delta)^2}\right) \leq n^{-\frac{\delta^2 C}{2(1 + \delta)^2}} \leq n^{-2},$$

for $C \geq \frac{24}{\delta^2}$. Thus, degree classes with fewer than $v_0/(1 + \delta)^2$ vertices are not considered in Line 10 with high probability.

Since w.h.p. degree classes with fewer than $v_0/(1 + \delta)^2$ nodes do not contribute to the output value $\beta'$ (i.e., the condition in Line 10 evaluates to false), and for all degree classes $i$ with $|V_i| \geq v_0$, the size $|S_i|$ is concentrated around its mean within a factor of $1 + \delta$ w.h.p., the following lower bound on the output $\beta'$ holds w.h.p.:

$$\beta' \geq \sum_{i \in I_1} \frac{|S_i|}{(c^{i+1} + 1)p} \geq \sum_{i \in I_1} \frac{p|V_i|}{(c^{i+1} + 1)p} = \sum_{i \in I_1} \frac{|V_i|}{(c^{i+1} + 1)(1 + \delta)} = \frac{1}{1 + \delta} \sum_{i \in I_1} \beta'_i.$$

Furthermore, using the same argument as above, i.e., the fact that the sizes of all sets $S_i$ that contribute to $\beta'$ are concentrated around their means within a factor of $1 + \delta$, we obtain the following upper bound on the output $\beta'$:

$$\beta' \leq \sum_{i : |V_i| \geq v_0/(1 + \delta)^2} \frac{|S_i|}{(c^{i+1} + 1)p} \leq \sum_{i : |V_i| \geq v_0/(1 + \delta)^2} \frac{p|V_i|(1 + \delta)}{(c^{i+1} + 1)p} \leq (1 + \delta)\beta(G).$$

This concludes the first part of the proof.

For the second part, to see that the upper bound $\beta' \leq (1 + \delta)\beta(G)$ still holds if $\gamma > \beta(G)$, recall that the sizes of all sets $S_i$ that contribute to $\beta'$ are concentrated around its expected size within a factor of
1 + δ. Since the sampling probability becomes smaller as γ increases, fewer degree classes contribute to β' and the upper bound thus equally holds.

Last, concerning space requirements of our algorithm, in expectation we sample \( n \cdot p = O\left(\frac{n \log^2(n) g}{\gamma \delta \log c}\right) \) nodes and compute the degree for each node. Hence, space \( O\left(\frac{n \log^2(n) g}{\gamma \delta \log c}\right) \) bits are sufficient. Using a Chernoff bound, it can be seen that this also holds with high probability. 

We now use the previous lemma to establish our main theorem.

**Theorem 1.** Let \( \gamma \leq \beta(G) \). Then, there is a randomized one-pass approximation streaming algorithm in the edge arrival model with space \( O\left(\frac{n \log^2(n) g}{\gamma \delta \log c}\right) \) that approximates \( \beta(G) \) within a factor of \( 1 + \epsilon \), with high probability. If \( \gamma > \beta(G) \), then the algorithm uses the same space and with high probability outputs a value \( \beta' \) with \( \beta' \leq (1 + \delta) \beta(G) \).

**Proof.** Suppose first that \( \gamma \leq \beta(G) \). We run Algorithm 1 using values for \( \delta, c \) and \( g \), which we determine later. By Lemma 1 the algorithm returns a value \( \beta' \) such that \( \frac{1}{1 + \delta} \sum_{i \in I'} \beta_i'(G) \leq \beta' \leq (1 + \delta) \beta(G) \). Using Inequality 3 this gives

\[
\frac{\beta'}{1 + \delta} \leq \beta(G) \leq \frac{gc}{g - 1}(1 + \delta)\beta'.
\]

Thus, we obtain a \((1 + \epsilon)\)-approximation, if \( \frac{gc}{g - 1}(1 + \delta) \leq 1 + \epsilon \). It can be verified that this is fulfilled if we set \( g = \frac{16}{3} \), \( c = 1 + \frac{1}{37} \), and \( \delta = \frac{1}{37} \). The space requirements thus are \( O\left(\frac{n \log^2(n) g}{\gamma \delta \log c}\right) \) = \( O\left(\frac{n \log^2(n) g}{\gamma \delta \log c}\right) \), using the fact that \( \log(1 + \epsilon) < \epsilon \), for any \( \epsilon < 1 \).

Last, if \( \gamma > \beta(G) \), then \( \beta'(G) \leq \beta' \leq (1 + \delta) \beta(G) \) equally applies, by Lemma 1 and the upper bound equally holds.

Last, we turn the algorithm of the previous theorem into an approximation factor \( \phi \).

**Theorem 2.** Let \( \phi > 2 \) and suppose that \( \gamma' \leq \beta(G) \) is a given lower bound on \( \beta(G) \). There is a randomized one-pass approximation streaming algorithm in the edge arrival model with space \( O\left(\frac{n \log^2(n) g}{\gamma' \phi^2}\right) \) that approximates \( \beta(G) \) within a factor of \( \phi \), with high probability.

**Proof.** We run the algorithm as stated in Theorem 1 with values \( \gamma = \gamma' \cdot \phi^2 \) and \( \epsilon = 1/4 \). Let \( \beta' \) be the output of the algorithm of Theorem 1. Then our algorithm returns the value \( \beta' \) if \( \beta' \geq \gamma/(1 + \epsilon) \), and \( \gamma' \phi \) otherwise.

First, suppose that \( \beta(G) \geq \gamma \). By Theorem 1, with high probability, it holds \( \beta(G)/(1 + \epsilon) \leq \beta' \leq \beta(G)(1 + \epsilon) \), and thus the output of our algorithm is \( \beta' \), which constitutes a \((1 + \epsilon)\)-approximation.

Next, suppose that \( \beta(G) \leq \gamma/2 \). By Theorem 1, with high probability, it holds \( \beta' \leq \beta(G)(1 + \epsilon) \) and thus the output of our algorithm is \( \gamma' \phi \). Since \( \beta(G) \leq \gamma/2 \) (and larger than \( \gamma' \)), this constitutes a \( \phi \)-approximation.

Last, if \( \gamma/2 \leq \beta(G) \leq \gamma \), then both outputs \( \beta(G) \) and \( \gamma' \phi \) give \( \phi \)-approximations.

## 5 Algorithm in the Vertex-arrival Model

Let \( v_1, \ldots, v_n \) be the order in which the vertices appear in the stream. Let \( G_i = G_i[v_1, \ldots, v_i] \) be the subgraph induced by the first \( i \) vertices.

Let \( n_{d,i} := |\{v \in V(G_i) : \deg_{G_i}(v) \leq d\}| \) be the number of vertices of degree at most \( d \) in \( G_i \), and let \( n_d = \max n_{d,i} \). We first give an algorithm, DecTest\((d, \epsilon)\), which with high probability returns a \((1 + \epsilon)\)-approximation of \( n_d \) using \( O\left(\frac{1}{\epsilon^2} \log^2 n\right) \) bits of space.

In the description of the algorithm, we suppose that we have a random function \( \text{Coin} : [0, 1] \rightarrow \{\text{false}, \text{true}\} \) such that \( \text{Coin}(p) = \text{true} \) with probability \( p \) and \( \text{Coin}(p) = \text{false} \) with probability \( 1 - p \). Furthermore, the outputs of repeated invocations of \( \text{Coin} \) are independent.

Algorithm DecTest\((d, \epsilon)\) maintains a sample \( S \) of at most \( c \log n \) vertices. It ensures that all vertices \( v \in S \) have degree at most \( d \) in the current graph \( G_i \) (notice that \( \deg_{G_j}(v) \leq \deg_{G_i}(v) \), for every \( j \geq i \)). Initially, \( p = 1 \), and all vertices of degree at most \( d \) are stored in \( S \). Whenever \( S \) reaches the limiting size of \( c \log n \), we downsample \( S \) by removing every element of \( S \) with probability \( \frac{1}{1 + \epsilon} \) and update...
Algorithm 2 Algorithm DegTest$(d, \epsilon)$

**Require:** Degree bound $d$, $\epsilon$ for a $1 + \epsilon$ approximation

1. $p \leftarrow 1$, $S \leftarrow \emptyset$, $m \leftarrow 0$, $c' \leftarrow \epsilon/2$, $c \leftarrow \frac{28}{c'}$
2. **while** stream not empty **do** {The current subgraph is $G_i$}
3. \hspace{1em} $v \leftarrow$ next vertex in stream
4. \hspace{2em} **if** COIN$(p)$ **then** $S \leftarrow S \cup \{v\}$ **end if** {Sample vertex with probability $p$}
5. Update degrees of vertices in $S$, i.e., ensure that for every $u \in S$ $\deg_{G_i}(u)$ is known
6. Remove every vertex $u \in S$ if $\deg_{G_i}(u) > d$
7. **if** $p = 1$ **then** $m \leftarrow \max\{m, |S|\}$ **end if**
8. **if** $|S| = c \log(n)$ **then**
9. \hspace{1em} $m \leftarrow c \log(n)/p$
10. \hspace{2em} Remove each element from $S$ with probability $\frac{1}{1 + c'}$
11. \hspace{2em} $p \leftarrow p/(1 + c')$
12. **end if**
13. **end while**
14. **return** $m$

$p \leftarrow p/(1 + c')$. This guarantees that throughout the algorithm $S$ constitutes a uniform random sample (with sampling probability $p$) of all vertices of degree at most $d$ in $G_i$.

The algorithm outputs $m \leftarrow c \log(n)/p$ as the estimate for $n_d$, where $p$ is the largest value of $p$ that occurs during the course of the algorithm. It is updated whenever $S$ reaches the size $c \log n$, since $S$ is large enough at this moment to be used as an accurate predictor for $n_{d,i}$, and hence also for $n_d$.

**Lemma 2.** Let $0 < \epsilon \leq 1$. DegTest$(d, \epsilon)$ (Algorithm 2) approximates $n_d$ within a factor $1 + \epsilon$ with high probability, i.e.,

$$\frac{n_d}{1 + \epsilon} \leq \text{DegTest}(d, \epsilon) \leq (1 + \epsilon)n_d,$$

and uses $O(\frac{1}{\epsilon} \log^2 n)$ bits of space.

**Proof.** First, suppose that $n_d < c \log n$. Then the algorithm never downsamples the set $S$ and computes $n_d$ exactly (and makes no error).

Assume now that $n_d \geq c \log n$. For $i \geq 0$, let $j_i$ be the smallest index $j$ such that $n_{d,j} \geq c \log n(1 + \epsilon')^j(1 + \epsilon'/2)$. We say that the algorithm is in phase $i$, if $p = 1/(1 + \epsilon')^i$.

First, for any $i$, we argue that in iteration $k \leq j_i$, the algorithm is in a phase at most $i + 1$ w.h.p. Let $E_{k,i}$ be the event that the transition from phase $i + 1$ to $i + 2$ occurs in iteration $k \leq j_i$, and let $E$ be the event that at least one of the events $E_{k,i}$, for every $k$ and $i$, occurs. For $E_{k,i}$ to happen, it is necessary that the algorithm is in phase $i + 1$ in iteration $k$. Assume that this is the case. Then, since $n_{d,k} \leq n_{d,j_i}$, the expected size of $S$ in iteration $k$ is

$$\mathbb{E}[S] = \frac{n_{d,k}}{p} \leq \frac{c \log(n)(1 + \epsilon')^i(1 + \epsilon'/2)}{(1 + \epsilon')^{i+1}} = \frac{c \log(n)(1 + \epsilon'/2)}{1 + \epsilon'},$$

and thus, by a Chernoff bound,

$$\mathbb{P}[|S| \geq c \log n] \leq \exp \left( -\frac{(1 + \epsilon'/2)^2}{2 + (1 + \epsilon'/2)^2} \cdot \frac{c \log(n)(1 + \epsilon'/2)}{1 + \epsilon'} \right) = \exp \left( -\frac{(1 + \epsilon')^2 c \log(n)}{2} \cdot \frac{1}{1 + (1 + \epsilon')^2} \right) \leq \exp \left( -\frac{c \log(n)}{3 + 2\epsilon'} \right) \leq n^{-3},$$

for $c \geq 21$. Thus, by the union bound, the probability that $E$ occurs is at most $n^{-2}$.

We assume from now on that $E$ does not occur. Let $F_i$ be the event that at the end of iteration $j_i$, the algorithm is in phase $i + 1$. We prove now by induction that all $F_i$ occur with high probability. Consider first $F_0$. Conditioned on $\neg E$, the algorithm is in phase 0 or 1 after iteration $j_0$. We argue that with high probability, the algorithm is in phase 1 after iteration $j_0$. Suppose that the algorithm is
Algorithm in the Vertex-arrival Order

Let \( V \) be the output of Algorithm 3. Then, the following holds with high probability:

\[
\Pr \{ |S| \leq c \log n \} \leq \exp \left( -c \log n \frac{1 + \epsilon^2 / 2}{4 + 2\epsilon^2} \right) = \exp \left( -c \log n \frac{\epsilon^2}{4 + 2\epsilon^2} \right) \leq n^{-2},
\]

for \( c \geq \frac{\sqrt{2}}{2}, \) and hence, if the algorithm was in phase 0 at the beginning of iteration \( j_0 \), then, with high probability, the transition to phase 1 would occur.

Assume now that both \( \neg E \) and \( F_1 \) hold. Then, the algorithm is in phase \( i + 1 \) or \( i + 2 \) at the end of iteration \( j_{i+1} \). Suppose we are in phase \( i + 1 \) at the beginning of iteration \( j_{i+1} \). Then, \( \mathbb{E}[S] = \frac{n_d v_d}{p} = n_{d,e} = c \log n(1 + \epsilon^2 / 2) \). Thus, by a Chernoff bound, 

\[
\Pr \{ |S| \leq c \log n \} \leq \exp \left( -c \log n \frac{1 + \epsilon^2 / 2}{4 + 2\epsilon^2} \right) = \exp \left( -c \log n \frac{\epsilon^2}{4 + 2\epsilon^2} \right) \leq n^{-2},
\]

Concerning the space requirements of the algorithm, at most \( c \log n \) vertex degrees are stored, which requires \( O(\frac{c}{\log^2 n}) \) bits of space.

Next, we run multiple copies of \( \text{DegTest} \) in order to obtain our main algorithm, Algorithm 3.

**Algorithm 3 Algorithm in the Vertex-arrival Order**

```plaintext
for every \( i \in \{0, 1, \ldots, \lfloor \log n \rfloor \} \); run in parallel:
\( n_{2^i} = \text{DegTest}(2^i, 1/2) \)
end for
return \( \max \left\{ \frac{\bar{n}_{2^i}}{2(2^i + 1)} : i \in \{0, 1, \ldots, \lfloor \log n \rfloor \} \right\} \)
```

**Theorem 3.** Let \( \gamma \) be the output of Algorithm 3. Then, the following holds with high probability:

1. \( \gamma = \Omega(\frac{\beta(G)}{\log n}) \), and
2. \( \gamma \leq \alpha(G) \).

Furthermore, the algorithm uses space \( O(\log^3 n) \) bits.

**Proof.** For \( 0 \leq i < \lfloor \log n \rfloor \), let \( V_i \subseteq V \) be the subset of vertices with \( \deg_G(v) \in \{ 2^i, 2^{i+1} - 1 \} \). Then,

\[
\beta(G) = \sum_{v \in V} \frac{1}{\deg_G(v) + 1} = \sum_i \sum_{v \in V_i} \frac{1}{\deg_G(v) + 1} \leq \sum_i \frac{|V_i|}{2^i + 1}.
\]

Let \( i_{\text{max}} := \arg \max \frac{|V_i|}{2^{i_{\text{max}} + 1}} \). Then, we further simplify the previous inequality as follows:

\[
\beta(G) \leq \cdots \leq \sum_i \frac{|V_i|}{2^i + 1} \leq \lfloor \log(n) \rfloor \cdot \frac{|V_{i_{\text{max}}}|}{2^{i_{\text{max}} + 1}} \leq \lfloor \log(n) \rfloor \cdot \frac{|V_{\leq i_{\text{max}}}|}{2^{i_{\text{max}} + 1}} \leq \lfloor \log(n) \rfloor \cdot \frac{|V_{\leq i_{\text{max}}}|}{2^{i_{\text{max}} + 1}}
\]

(4)

where \( V_{\leq i} = \cup_{j \leq i} V_j \). Let \( d_{\text{max}} = 2^{i_{\text{max}}} \). Since \( |V_{i_{\text{max}}}| \leq n d_{\text{max}} \) and \( \bar{n}_{d_{\text{max}}} = \text{DegTest}(d_{\text{max}}, 1/2) \) is a 1.5-approximation to \( n d_{\text{max}} \), we obtain \( \gamma = \Omega(\frac{\beta(G)}{\log n}) \), which proves Item 1.

Concerning Item 2, notice that for every \( i \) and \( d \), it holds

\[
\alpha(G) \geq \alpha(G_i) \geq \beta(G_i) = \sum_{v \in V(G_i)} \frac{1}{\deg_G(v) + 1} \geq \sum_{v \in V(G_i), \deg_G(v) \leq d} \frac{1}{\deg_G(v) + 1} \geq \frac{n_{d,e}}{d + 1},
\]

and, in particular, the inequality holds for \( n d_{\text{max}} = n_{i_{\text{max}}, d_{\text{max}}} \). Since the algorithm returns a value bounded by \( \frac{n_{d_{\text{max}}, e}}{d_{\text{max}} + 1} \), and \( \bar{n}_{d_{\text{max}}} \) constitutes a 1.5-approximation of \( n d_{\text{max}} \), Item 2 follows.

Concerning the space requirements, the algorithm runs \( O(\log n) \) copies of Algorithm 2, which itself requires \( O(\log^2 n) \) bits of space.
6 Space Lower Bound

Our lower bound follows from a reduction using a well-known hard problem from communication complexity. Let DISJ$_n$ refer to the two-party set disjointness problem for inputs of size $n$. In this problem we have two parties, Alice and Bob. Alice knows $X \subseteq [n]$, while Bob knows $Y \subseteq [n]$. Alice and Bob must exchange messages until they both know whether $X \cap Y = \emptyset$ or $X \cap Y \neq \emptyset$.

Using $R(DISJ_n)$ to refer to the randomised (bounded error probability) communication complexity of DISJ$_n$, the following theorem is known.

**Theorem 4** (Kalyanasundaram and Schintger [17]).

\[ R(DISJ_n) \in \Omega(n) \]

To get our lower bound, we will show a reduction from randomised set disjointness to randomised $c$-approximation of $\beta(G)$.

**Theorem 5.** Every randomized constant error one-pass streaming algorithm that approximates $\beta(G)$ within a factor of $c$ uses space $\Omega(\frac{n}{c \log c})$, even if the input stream is in vertex arrival order.

**Proof.** Let ALG$_{c,n}$ be any streaming algorithm which takes as input a vertex arrival stream of an $n$-vertex graph $G$ and returns a $c$-approximation of $\beta(G)$ with probability $\frac{2}{3}$.

Suppose we are given an instance of DISJ$_k$. We will construct a graph $G$ from $X$ and $Y$ which we can use to tell whether $X \cap Y = \emptyset$ by checking a $c$-approximation of $\beta(G)$.

Let $z \geq 2$ be an arbitrary integer. Set $q = 2zc^2$ and $a = kq$. Let $G = (V, E)$, where $V$ is partitioned into disjoint subsets $A$, $B$, $C$, and $U_i$ for $i \in [k]$. These are of size $|A| = |B| = a$, $|C| = z$, and $|U_i| = q$. So $n := |V| = kq + 2a + z = 3kq + z = z(6kc^2 + 1)$. Thus, $k \in \Theta(\frac{n}{z})$ holds.

First consider the set of edges $E_0$ consisting of all $\{u, v\}$ with $u, v \in A \cup B, u \neq v$. Setting $E = E_0$ makes $A \cup B$ a clique, while all other vertices remain isolated.

Figure [14] shows this initial configuration. For clarity, we represent the structure using super-nodes and super-vertices. A super-node is a subset of $V$ (in this case we use $A$, $B$, $C$, and each $U_i$). Between the super-nodes, we have super-edges representing the existence of all possible edges between constituent vertices. So a super-edge between super-nodes $Z_1$ and $Z_2$ represents that $\{z_1, z_2\} \in E$ for every $z_1 \in Z_1$ and $z_2 \in Z_2$. The lack of a super-edge between $Z_1$ and $Z_2$ indicates that none of these $\{z_1, z_2\}$ are in $E$.

Now we add dependence on $X$ and $Y$. Let

\[ E_X = \bigcup_{i \in [n] \setminus X} \left( \bigcup_{u \in U_i, v \in A} \{u, v\} \right) \quad \text{and} \quad E_Y = \bigcup_{i \in [n] \setminus Y} \left( \bigcup_{u \in U_i, v \in B} \{u, v\} \right). \]

So $E_X$ contains all edges from vertices in $U_i$ to vertices in $A$ exactly when index $i$ is not in the set $X$. Similarly for $E_Y$ with $B$, and $Y$.

Now let $E = E_0 \cup E_X \cup E_Y$. Adding these edge sets corresponds to adding a super-edge to Figure [14] between $U_i$ and $A$ (or $B$) whenever $i$ is not in $X$ (or $Y$). Figures [14] and [15] illustrate this. In Figure [15] the intersection is non-empty, which creates a set of isolated nodes that push up the value of $\beta(G)$. Meanwhile, there is no intersection in Figure [15] so the only isolated nodes are those in $C$.

Now consider $\beta(G)$. In the case where $X \cap Y = \emptyset$, we will have a super-edge connecting each $U_i$ to at least one of $A$ and $B$, so the degree of each vertex in each $U_i$ is either $a$ or $2a$. Similarly, $A \cup B$ is a clique, so each vertex has degree at least $2a - 1$. There are $2a$ such vertices, so they contribute at most $\frac{2a}{2a-1} = 1$ to $\beta$. Vertices in $C$ are isolated and contribute exactly $z$ to $\beta$. Therefore, $z \leq \beta(G) \leq \frac{kq}{a} + 1 + z = z + 2$.

Now consider the case where $X \cap Y \neq \emptyset$. This means that there exists some $i \in X \cap Y$, and so $U_i$ will have no super-edges. So each vertex in $U_i$ is isolated, and contributes exactly $1$ to $\beta$. There are $q$ such vertices, and also accounting for the contribution of vertices in $C$, we obtain $\beta(G) \geq q + z = 2(2c^2 + 1)$. Since the minimum possible ratio of the $\beta$-values between graphs in the two cases is at least $\frac{2(2c^2 + 1)}{c^2} > c^2$ (using $z \geq 2$), a $c$-approximation algorithm for $\beta(G)$ would allow us to distinguish between the two cases.

Now, return to our instance of DISJ$_k$. We can have Alice initialise an instance of ALG$_{c,n}$ and have all vertices in $A, C$, and each $U_i$ arrive in any order. This only requires knowledge of $X$ because only
edges in $E_0$ and $E_X$ are between these vertices and these are the only edges that will be added so far in the vertex arrival model. Alice then communicates the state of $\text{ALG}_{c,n}$ to Bob. Bob can now have all vertices in $B$ arrive in any order. This only requires knowledge of $Y$ because only edges in $E_0$ and $E_Y$ are still to be added. Bob can then compute a $c$-approximation of $\beta(G)$ with probability at least $\frac{2}{3}$, determining which case we are in and solving $\text{DISJ}_k$.

From Theorem 4, we know that Alice and Bob must have communicated at least $\Omega(k)$ bits. However, all they communicated was the state of $\text{ALG}_{c,n}$. Therefore, $\Omega(k) = \Omega(\frac{\alpha}{z^c})$ bits was being used by $\text{ALG}_{c,n}$ at the time.

Consider again the graph $G$. The above argument shows that in order to compute a $c$-approximation to $\beta(G)$, space $\Omega(\frac{\alpha}{z^c})$ is needed. Since $\beta(G) \geq z$ in both cases, we obtain the space bound $\Omega(\frac{\beta(G)}{z^c})$.

Last, recall that $z$ and thus $\beta(G)$ can be chosen arbitrarily. The theorem hence holds for any value of $\beta(G)$. 

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