Coefficient Estimates for Certain Analytic Functions

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Abstract

Let $\lambda \in [0, 1]$ and $p \in \mathbb{N}$, we introduce

$$S^*_{\lambda,p}(\varphi) = \left\{ f \in \mathcal{A}_p : \frac{zf'(z)}{(1 - \lambda)f(z) + \lambda z^p} \prec \varphi(z) \right\},$$

where $\varphi$ is a Ma-Minda function. We obtain sharp bound for a particular case of the generalized Zalcman conjecture and for some combinations involving the initial coefficients of $f$ in $S^*_{\lambda,p}(\varphi)$. Further sharp estimate for the fifth coefficient of functions belonging to $S^*_{0,1}(\varphi) =: S^*(\varphi)$ is deduced. Applications appear for several well-known classes for the particular choices of $\lambda$, $p$ and $\varphi(z)$ along with a dominance result for the class $S^*(\varphi)$.

Keywords- Analytic functions; Starlike functions; p-Valent functions; Fifth coefficient; Zalcman functional; Ma-Minda class.

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1 Introduction

Let $\mathcal{A}_p$ be the class of analytic functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \in \mathbb{N},$$

in the unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ with $\mathcal{A}_1 =: \mathcal{A}$. Assume $\overline{\mathbb{D}} = \{ z \in \mathbb{C} : |z| \leq 1 \}$, $\mathcal{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ containing univalent functions. Recall that the analytic function $f$ is subordinate to another analytic function $F$, if there exists a Schwarz function $\omega$ such that $f(z) = F(\omega(z))$ for all $z \in \mathbb{D}$ and simply write $f \prec F$. By $\mathcal{B}_0$, we represent the class of Schwarz functions having the form

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n.$$

Let $\mathcal{P}$ be the Carathéodory class, contains analytic functions $p$ defined in $\mathbb{D}$ which satisfy $\text{Re} p(z) > 0$ with

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in \mathbb{D}.\quad (3)$$

Clearly, the function

$$L(z) = \frac{1 + z}{1 - z}, \quad z \in \mathbb{D}$$

satisfies

$$\frac{zf'(z)}{(1 - \lambda)f(z) + \lambda z^p} \prec L(z),$$

where $\lambda$ is a real number in $[0, 1]$. Let $\varphi$ be a Ma-Minda function and $\mathcal{S}^* = \{ \varphi \}$. The family $\mathcal{S}^*_{\lambda,p}(\varphi)$ is defined by

$$\mathcal{S}^*_{\lambda,p}(\varphi) = \left\{ f \in \mathcal{S}^* : \frac{zf'(z)}{(1 - \lambda)f(z) + \lambda z^p} \prec \varphi(z) \right\}.$$
is a member of $\mathcal{P}$ as it maps the unit disk on to the right half-plane. We say function $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is dominated by function $G(z) = \sum_{n=0}^{\infty} A_n z^n$ if $|a_n| \leq A_n$, for each integer $n \geq 0$ and write $g(z) \ll G(z)$, where series for function $g$ and $G$ are convergent in $\mathbb{D}$.

Ma and Minda \[21\] introduced the class

$$S^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \varphi(z) \right\},$$

(5)

where $\varphi$ is an analytic univalent function having positive real part with $\varphi'(0) > 0$ and mapping $\mathbb{D}$ onto a domain starlike with respect to 1, which is symmetric about the real axis. Suppose, for $z \in \mathbb{D}$, $\varphi$ has the following series expansion

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad B_1 > 0.$$

(6)

Since $\varphi(\mathbb{D})$ is symmetric with respect to the real axis, we have $\varphi(\mathbb{D}) = \varphi(z)$, which gives all $B_i$'s, $i \in \mathbb{N}$ are real. For the family $S^*(\varphi)$ sharp growth, distortion theorems and estimates for the coefficient functional $|a_2 - ra_2^2|$ are known, where $r \in \mathbb{R}$. In fact, the class $S^*(\varphi)$ unifies several subfamilies of starlike functions. For instance, if we take $\varphi(z) = (1 + z)/(1 - z)$ and $(1 + (1 - 2\alpha)z)/(1 - z)$, $0 \leq \alpha < 1$, $S^*(\varphi)$ reduces to the known classes $S^*$ of starlike functions and $S^*(\alpha)$ of starlike functions of order $\alpha$ respectively. For $\varphi(z) = 1 + (2/\pi^2)(\log((1 + \sqrt{z})/(1 - \sqrt{z})))^2$, we obtain the class of parabolic starlike functions $S_p$, which is introduced and studied by Renning \[30\]. In the past, the class $S^*(\varphi)$ is studied for various alternatives to $\varphi(z)$, some of them are listed below in Table 1.

Table 1: Subclasses of starlike functions obtained for different Choices of $\varphi(z)$

| Class       | $\varphi(z)$                              | Reference          |
|-------------|-------------------------------------------|--------------------|
| $S_C^*$     | $1 + 4z/3 + 2z^2/3$                        | K. Sharma, N. K. Jain and V. Ravichandran |
| $S_{SG}^*$  | $2/(1 + e^{-z})$                           | P. Goel and S. Kumar |
| $S_x^*$     | $1 + ez^2$                                | S. S. Kumar and K. Gangania |
| $S_L^*$     | $\sqrt{1 + z^2}$                          | J. Sokol            |
| $S_{NE}^*$  | $1 + z - z^3/3$                           | L. A. Wani and A. Swaminathan |
| $S_{gb}^*$  | $\sqrt{1 + b_1 z}$, $b \in (0,1)$        | J. Sokol            |
| $\Delta^*$  | $z + \sqrt{1 + z^2}$                      | R. K. Raina and J Sokol |
| $S_p^*$     | $e^z$                                     | R. Mendirattra, S. Nagpal, and V. Ravichandran |
| $S_{hpl}(t)$| $1/(1 - z)^{t'}, t \in (0,1]$             | S. Kanas, V. S. Masih and A. Ebadian |
| $S_{p'}^*$  | $1 + \frac{k + 1}{k + 2}$, $k = 1 + \sqrt{2}$ | S. Kumar and V. Ravichandran |
| $S_{hpl}^*$ | $\sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1 + z}{1 + 2(\sqrt{2} - 1)z}}$ | R. Mendirattra, S. Nagpal and V. Ravichandran |

Recently, R. M. Ali et al. \[2\] studied some subclasses of $\mathcal{A}_p$, which generalize many classes including $S^*(\varphi)$ and obtained sharp bound of $|a_{p+2} - ra_{p+2}^2|$ and $|a_{p+3}|$, where $r \in \mathbb{R}$. Also, Raina and Sokol \[27\] introduced and studied the coefficient and radius results for the following class when $\lambda \in [0,1]$:

$$F_{\lambda} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{(1 - \lambda)f(z)} < z + \sqrt{1 + z^2} \right\}.$$

Motivated by these works, here below, we define a new subclass of $\mathcal{A}_p$.

**Definition 1.** Let $\varphi(z)$ be given by (5) and $\lambda \in [0,1]$, $p \in \mathbb{N}$. Then the class $S_{p,\lambda}^*(\varphi)$ is the collection of $f \in \mathcal{A}_p$ satisfying

$$\frac{1}{p} \left( \frac{zf'(z)}{(1 - \lambda)f(z) + \lambda z^p} \right) < \varphi(z).$$

Note that as a special case, this class includes several well known classes. For instance, when $p = 1$ and $\varphi(z) = z + \sqrt{1 + z^2}$, we have $S_{1,1}^*(\varphi) = F_{\lambda}$. For $\lambda = 0$ and $p = 1$, $S_{p,\lambda}^*$ reduces to the class $S^*(\varphi)$ and for $\lambda = p = 1$, we obtain the new defined class

$$S_{1,1}^*(\varphi) = \mathcal{R}(\varphi) = \left\{ f \in \mathcal{A} : f'(z) < \varphi(z) \right\}.$$

For $\varphi(z) = (1 + z)/(1 - z)$, $\mathcal{R}(\varphi)$ reduces to the class $\mathcal{R}$. By taking $p = 1$, we have $S_{1,1}^*(\varphi) = S_{1}^*(\varphi)$, which is more general form of the class $F_{\lambda}$. Finding the upper bound of coefficients comes under the
coefficient problem. Since the growth, covering and distortion theorems for functions \( f \in \mathcal{S} \) can be proved using the fact \( |a_2| \leq 2 \), so the coefficient bound plays a major role in identifying the geometric nature of the function and this in fact, is the foundation for our study of the classes introduced here. Note that sharp bound for the initial coefficients \( |a_2|, |a_3| \text{ and } |a_4| \) of functions in \( S^*(\varphi) \) are already known, see [2, 3, 37], however, for \( |a_5| \), it is still open. Sokö [33] conjectured that \( |a_n| \leq 1/(2(n-1)) \) for \( f \in S^*_L \) and Mendiratta et al. [23] conjectured that \( |a_n| \leq (5 - 3\sqrt{2})/(2(n-1)) \) for \( f \in S^*_L \). V. Ravichandran and S. Verma [28] settled these conjectures for \( n = 5 \). For different choices of \( \varphi(z) \) the bound for \( |a_5| \) is known [10, 13, 17], but not for the general case. Recently, O. S. Kwon et al. [19] established fifth coefficient bound for functions belonging to Bazilevic class. Motivated by this paper, we obtain sharp bound for \( |a_5| \) of functions in \( S^*(\varphi) \), under special circumstances, covering most of the cases.

In 1999, Ma [20] proposed a conjecture for \( f \in \mathcal{S} \) that

\[
|J_{j,k}| := |a_ja_k - a_{j+k-1}| \leq (j - 1)(k - 1), \quad (j, k \in \mathbb{N} \setminus \{1\}).
\]  

(7)

It is also called generalized Zalcman conjecture because it is a generalized form of Zalcman conjecture \( |a_n^2 - 2a_{n-1}| \leq (n-1)^2 \). Ma proved the inequality (7) for univalent functions with real coefficients and starlike functions. The Importance of Zalcman’s conjecture can be seen as it also implies Bieberbach conjecture. For the interesting history and relevant work, see [20, 29]. Many authors have computed upper bounds for the functional \( |J_{2,3}| = |a_2a_3 - a_4| \), a specific case of the generalized Zalcman coefficient functional \( J_{j,k} \), for various classes, see [4, 9, 10]. In Section 3 we obtain sharp bound of \( |J_{2,3}(f)| \) for the above specified classes.

We require the following lemmas to prove our results:

**Lemma 1.** [24] Lemma I] If the functions \( 1 + \sum_{n=1}^{\infty} b_n z^n \) and \( 1 + \sum_{n=1}^{\infty} c_n z^n \) are in \( \mathcal{P} \), then the same holds for the function

\[
1 + \frac{1}{2} \sum_{n=1}^{\infty} c_n z^n.
\]

**Lemma 2.** [24] Lemma II] Let \( h(z) = 1 + u_1 z + u_2 z^2 + \cdots \) and \( 1 + G(z) = 1 + d_1 z + d_2 z^2 + \cdots \) be functions in \( \mathcal{P} \), and set

\[
\gamma_n = \frac{1}{2^n} \left[ 1 + \frac{1}{2} \sum_{k=1}^{n} \binom{n}{k} u_k \right], \quad \gamma_0 = 1.
\]

If \( A_n \) is defined by

\[
\sum_{n=1}^{\infty} (-1)^{n+1} \gamma_{n-1} G_n^o(z) = \sum_{n=1}^{\infty} A_n z^n,
\]

then \( |A_n| \leq 2 \).

It is worth recalling the Möbius function \( \Psi_\zeta \) which maps the unit disk onto the unit disk and given by

\[
\Psi_\zeta(z) = \frac{z - \zeta}{1 - \zeta z}, \quad \zeta \in \mathbb{D}.
\]  

(8)

**Lemma 3.** [8] Lemma 2.4] If \( p \in \mathcal{P} \), then for some \( \zeta_i \in \mathbb{D}, i \in \{1, 2, 3\} \),

\[
p_1 = 2\zeta_1,
\]

\[
p_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2,
\]

\[
p_3 = 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^4)\zeta_1 \overline{\zeta_2} + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3.
\]

(10)

For \( \zeta_1, \zeta_2 \in \mathbb{D} \) and \( \zeta_3 \in \mathbb{T} \), there is a unique function \( p = L \circ \omega \in \mathcal{P} \) with \( p_1, p_2 \) and \( p_3 \) as in (9) - (11), where

\[
\omega(z) = z \Psi_{-\zeta_1}(z \Psi_{-\zeta_2}(z \Psi_{-\zeta_3}(z))), \quad z \in \mathbb{D},
\]

(12)

that is

\[
p(z) = \frac{1 + \zeta_2 \zeta_3 + \zeta_2 \zeta_1 + \zeta_1 z + \zeta_1 \zeta_2 z^2 + \zeta_3 z^3}{1 + \zeta_2 \zeta_3 + \zeta_1 z}, \quad z \in \mathbb{D}.
\]
Conversely, if \( \zeta_1, \zeta_2 \in \mathbb{D} \) and \( \zeta_3 \in \overline{\mathbb{D}} \) are given, then we can construct a (unique) function \( p \in \mathcal{P} \) of the form (3) so that \( p_i, i \in \{1, 2, 3\} \), satisfy the identities in (9)-(11). For this, we define

\[
\omega(z) = \omega_{\zeta_1, \zeta_2, \zeta_3}(z) = z \Psi_{\omega_{\zeta_1, \zeta_2, \zeta_3}}(z), \quad z \in \mathbb{D},
\]

where \( \Psi_{\omega_{\zeta_1, \zeta_2, \zeta_3}} \) is the function given as in (8). Then \( \omega \in \mathcal{B}_0 \). Moreover, if we define \( p(z) = (1 + \omega(z))/(1 - \omega(z)), z \in \mathbb{D} \), then \( p \) is represented by \( \mathcal{K} \), where \( p_1, p_2 \) and \( p_3 \) satisfy the identities in (9)-(11) (see the proof of \( \mathcal{K} \) Lemma 2.4).  

Lemma 4. [25] Lemma 2] If \( \omega \in \mathcal{B}_0 \), then for any real number \( q_1 \) and \( q_2 \), the following sharp estimate holds:

\[
|c_3 + q_1c_2 + q_2c_1^3| \leq L(q_1, q_2),
\]

where \( L(q_1, q_2) \) is as given in the original Lemma cited above.

## 2 Estimation of the Fifth Factor

We begin with the following lemma:

**Lemma 5.** If \( -1 < \sigma < 1 \), then \( F(z) = (1 + 2\sigma z + z^2)/(1 - z^2) \) belongs to \( \mathcal{P} \).

**Proof.** Let us consider

\[
\omega(z) = \frac{F(z) - 1}{F(z) + 1} = \frac{z + \sigma}{1 + \sigma z}.
\]

From (8)

\[
\omega(z) = z \Psi_{\sigma}(z), \quad z \in \mathbb{D}.
\]

Since \( \Psi_{\sigma}(z) \) is a conformal automorphism of \( \mathbb{D} \), \( |w(z)| < 1 \) and \( w(0) = 0 \). Which means that \( w \) is a Schwarz function and \( F \in \mathcal{P} \).

To prove our next result, we require the following assumption, which in general is satisfied by many functions:

**Assumption.** Let \( \varphi(z) \) be as defined in (9), whose coefficients, satisfy the following conditions:

\[
\begin{align*}
\text{C1} & : 0 < B_1 (2 - B_1) - 2B_2 < 4B_1, \\
\text{C2} & : 0 < 3B_1^4 + 2B_1^3 + 2B_1^2 (10B_2 - 9) + 18B_2^2 - 9B_1B_3 < 3 (B_1^2 + 2B_1 + 2B_2) (2B_1 - 3B_1 + 3B_2), \\
\text{C3} & : 0 < -(B_1^2 + 2B_1 + 2B_2) \left( 9B_1^6 - 120B_1^5 + B_1^4 (48B_2 + 163) - B_1^3 (530B_2 - 216) - 18B_1(35B_2^2 - 2B_2 (9 + 10B_3 - 18B_4) - 162 (B_3^2 - 2B_2^2 - 2B_3^2 + 2B_2 (2B_3 + B_4)) + B_1^2 (468B_2 + 79B_2^2 - 18 (5B_3 + 9B_4 + 18)) \right) < 16 \left( 9B_1^7 + 9B_1^6 + 63B_2^2 - 52 + B_1^4 (32B_2 - 27) + B_1^3 (16B_2^2 - 18B_2^2 - 9B_2^2) + 16B_2^2 B_3 \right), \\
\text{C4} & : 0 < 4B_1^2 + 6(B_2 - B_1) < 3B_1^2 + 6(B_2 - B_1).
\end{align*}
\]

**Theorem 6.** Let \( \varphi(z) \) be as defined in (9), whose coefficients satisfy the above conditions C1 to C4. If \( f \in \mathcal{S}^*(\varphi) \), then

\[
|a_5| \leq \frac{|B_1|}{4}.
\]

The inequality is sharp.

**Proof.** Suppose \( f \in \mathcal{S}^*(\varphi) \), then

\[
\frac{zf'(z)}{f(z)} = \varphi(\omega(z)),
\]

where \( \omega \in \mathcal{B}_0 \). If we choose \( \omega(z) = (p(z) - 1)/(p(z) + 1) \), where \( p \in \mathcal{P} \), given by (10), then by comparing the coefficients obtained by series expansion of \( f(z) \) together with \( p(z) \) and \( \varphi(z) \), yields:

\[
a_5 = \frac{B_1}{8}I,
\]

(14)
where
\[ I = p_4 + I_1p_1^4 + I_2p_1^2p_2 + I_3p_1p_3 + I_4p_2^2, \] (15)
with
\[ I_1 = \frac{B_1^4 - 6B_1^2 + 11B_1^2 + 6B_1^2B_2 - 6B_1 + 3B_2^2 - 22B_1B_2 + 18B_2 - 18B_3 + 8B_1B_3 + 6B_4}{48B_1}, \]
\[ I_2 = \frac{3B_1^2 - 11B_1^2 + 9B_1 - 18B_2 + 11B_1B_2 + 9B_3}{12B_1}, \quad I_3 = \frac{2B_1^2 - 3B_1 + 3B_2}{3B_1}, \]
and
\[ I_4 = \frac{B_1^2 - 2B_1 + 2B_2}{4B_1}. \]

Let \( q(z) = 1 + b_1z + b_2z^2 + \cdots \) be in \( \mathcal{P} \). Then Lemma 1 yields
\[ 1 + \frac{1}{2}(p(z) - 1) * (q(z) - 1) = 1 + \frac{1}{2} \sum_{n=1}^{\infty} b_n p_n z^n \in \mathcal{P}. \]

If we assume \( h(z) = 1 + \sum_{n=1}^{\infty} u_n z^n \in \mathcal{P} \) and take \( 1 + G(z) := 1 + \frac{1}{2} \sum_{n=1}^{\infty} b_n p_n z^n \), then Lemma 2 yields
\[ |A_4| \leq 2, \]
where
\[ A_4 = \frac{1}{2} \gamma_0 b_4 p_4 - \frac{1}{4} \gamma_1 b_2 p_2^2 - \frac{1}{2} \gamma_1 b_1 b_3 p_1 p_3 + \frac{3}{8} \gamma_2^2 b_2^2 p_1^2 p_2 - \frac{1}{16} \gamma_3^4 b_1^4 p_4^4 \] (16)
and for \( i \in \{0, 1, 2, 3, 4\} \), \( \gamma_i \)'s are given by \( \gamma_0 = 1, \gamma_1 = \frac{1}{2} \left( 1 + \frac{u_1}{2} \right), \gamma_2 = \frac{1}{4} \left( 1 + u_1 + \frac{u_2}{2} \right), \gamma_3 = \frac{1}{8} \left( 1 + \frac{3}{2} u_1 + \frac{3}{2} u_2 + \frac{1}{2} u_3 \right) \). \( \) (17)

So from (15) and (16), we can observe that if there exist \( q, h \in \mathcal{P} \) such that
\[ b_4 = 2, \quad I_1 = -\frac{1}{128} \left( 1 + \frac{3}{2} u_1 + \frac{3}{2} u_2 + \frac{1}{2} u_3 \right) b_1^4, \quad I_2 = \frac{3}{32} \left( 1 + u_1 + \frac{u_2}{2} \right) b_1^2 b_2, \]
\[ I_3 = -\frac{1}{4} \left( 1 + \frac{u_1}{2} \right) b_1 b_3, \quad I_4 = -\frac{1}{8} \left( 1 + \frac{u_1}{2} \right) b_2^2, \]
then we have
\[ I = A_4. \] (18)

The bound for \( |A_4| \) can be obtained from Lemma 2 consequently, we can estimate the bound for \( |I| \) and thus we arrive at the desired bound by using (14). To prove the theorem, we construct the functions \( q \) and \( h \) in such a way that we obtain (18).

From Lemma 3 suppose that the functions \( q \) and \( h \) are constructed by taking \( \zeta_1, \zeta_2 \in \mathbb{D}, \zeta_3 \in \mathbb{D} \) and \( \xi_1, \xi_2 \in \mathbb{D}, \xi_3 \in \mathbb{D} \) respectively, as follows:
\[ q = L \circ \omega_1 \quad \text{and} \quad h = L \circ \omega_2, \] (19)
where
\[ \omega_1(z) = z \Psi_{-\zeta_1}(z \Psi_{-\zeta_2}(\zeta_3 z)), \quad \omega_2(z) = z \Psi_{-\xi_1}(z \Psi_{-\xi_2}(\xi_3 z)) \] (20)
and \( L \) is the function given by (14). So again from Lemma 3 the \( b_i \)'s and \( u_i \)'s, \( i \in \{1, 2, 3\} \) are given by
\[ b_1 = 2\zeta_1, \quad b_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_1, \]
\[ b_3 = 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1 \zeta_2 - 2(1 - |\zeta_1|^2)\overline{\zeta_1} \zeta_2^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\xi_3, \]
and
\[ u_1 = 2\zeta_1, \quad u_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2, \]
\[ u_3 = 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1 \zeta_2 - 2(1 - |\zeta_1|^2)\overline{\zeta_1} \zeta_2^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\xi_3. \]
There may be many solutions for the above set of equations. For our purpose, we impose some restrictions on the parameters. We take all $\xi_i \in \mathbb{R}$, then
\[
\begin{align*}
    u_1 &= 2\xi_1, \quad u_2 = 2\xi_1^2 + 2(1 - \xi_1^2)\xi_2, \\
    u_3 &= 2\xi_3^3 + 4(1 - \xi_1^2)\xi_1\xi_2 - 2(1 - \xi_1^2)\xi_1\xi_2^2 + 2(1 - \xi_1^2)(1 - \xi_2^2)\xi_3.
\end{align*}
\]  

Further, if we define
\[
\begin{align*}
    \xi_1 &= -\frac{B_1^2 + 2B_2}{2B_1}, \quad \xi_2 = \frac{B_1^2 - B_1B_2 + 18B_2^2 - 18B_3}{3(B_1^2 + 2B_1 + 2B_2)(2B_1 - 3B_1 + 3B_2)}, \\
    \xi_3 &= \left(-9B_1^3 + 30B_1^2 - B_1^2 (66B_2 - 5) + 2B_1^2 (85B_2 - 63) + 4B_1^2 (5B_2 (11B_2 - 18B_3 - 9))
    + 27B_3^2 + 4B_1^2 (B_2^3 - 36B_2^2 - 81B_2^2 + 45B_2B_3 + 162 (B_2 - 1) B_4) - 144B_1 (5B_2 - 9) B_3B_3
    + 324B_2^2 (B_1^3 + B_2 ((B_2 - 2) B_2 + 2B_4)) + 18B_1^3 (9B_4 + 5B_3 + 6) - 5B_1^3 B_2 (35B_2
    - 2))\right)\left(8(3B_1^4 + 2B_1^2 + 18B_2^2 + B_1^2 (10B_2 - 9) - 9B_1 B_3) (B_1 (3B_1^2 + B_3 + 11B_2 - 9) + 9B_3)\right),
\end{align*}
\]  

then the conditions C1, C2 and C3 on the coefficients $B_1, B_2, B_3$ and $B_4$ yield
\[
    |\xi_1| < 1, \quad |\xi_2| < 1, \quad |\xi_3| < 1.
\]

Using these $\xi_i$ in (21), we can obtain $u_i$, which in turn by using (17) gives
\[
\begin{align*}
    \gamma_1 &= \frac{1}{4} \left(2 - B_1 - \frac{2B_2}{B_1}\right), \\
    \gamma_2 &= \frac{(B_1^2 + 2B_2 - 2B_1) (3B_1^2 - 11B_1^2 + B_1 (11B_1 + 9) + 9 (B_3 - 2B_2))}{2B_1 (2B_1^2 + 3B_2 - 3B_1)}, \\
    \gamma_3 &= -\frac{1}{64B_1 (2B_1^2 + 3B_2 - 3B_1)^2} \left(3 (B_1^2 + 2B_2 - 2B_1) (B_1^4 - 6B_1^3 + B_1^2 (6B_2 + 11)
    + B_1 (8B_3 - 22B_2 - 6) + 3 (B_2^2 + 6B_2 - 6B_3 + 2B_4))\right).
\end{align*}
\]  

Let us consider
\[
    q(z) = \frac{1 + 2\sigma z + z^2}{1 - z^2},
\]
with $\sigma = \sqrt{(4B_1^2 + 6B_2 - B_1)/((3B_2^2 + 6B_2 - B_1))}$, then
\[
    b_1 = b_3 = 2\sigma \quad \text{and} \quad b_2 = b_4 = 2. 
\]  

If we choose $B_1$ and $B_2$ such that $0 < \sigma < 1$, which is equivalent to condition C4. Then by Lemma 5, we have $q \in \mathcal{P}$. On putting the values of $b_i$’s and $\gamma_i$’s obtained from (23) and (24) respectively in (16), we get (18), which together with (14) gives the desired bound for $|a_5|$. Let the function $H : \mathbb{D} \rightarrow \mathbb{C}$ be given by
\[
    H(z) = z \exp \int_0^z \frac{\varphi(t^4) - 1}{t} dt = z + \frac{B_1}{4} z^5 + \frac{1}{32} (B_1^2 + 4B_2) z^9 + \cdots,
\]
where coefficients of $\varphi(z)$ satisfies the conditions C1 to C4. Then $H(0) = 0$, $H'(0) = 1$ and $zH'(z)/H(z) = \varphi(z^4)$ and hence the function $H \in S^* (\varphi)$, proving the result to be sharp for the function $H$. \hspace{1cm} \blacksquare

Choose $\varphi(z) = 1 + \sin z$, whose coefficients satisfy C1 to C4, to obtain the following result:

**Corollary 6.1.** If $f(z) \in S^* (\varphi)$, where $\varphi(z) = 1 + \sin z$, then
\[
    |a_5| \leq \frac{1}{4}.
\]

The bound is sharp.
For all below mentioned choices of ϕ conditions C1 to C4 are valid. Therefore, the bounds of |a₅| for some of the known classes are obtained from our result as a special case.

Remark. 1. [13] Theorem 4.1] If ϕ(z) = 2/(1 + e⁻ᶻ), then f ∈ S₅ unlawful |a₅| ≤ 1/8.

2. [25] Theorem 3.1] If ϕ(z) = √₁ + ᶩ, then f ∈ S₅ unlawful |a₅| ≤ 1/8.

3. [14] Theorem 3.1] If ϕ(z) = √₁ + bẑ, then f ∈ S₅ unlawful, |a₅| ≤ b/8, where b ∈ (0, 1].

4. [25] Theorem 3.1] If f ∈ S₅ unlawful, then |a₅| ≤ (5 − 3√₂)/8.

5. [4] If ϕ(z) = ((1 + ᶩ)/(1 − ᶩ) ᶪ, then the conditions of Theorem 3 are satisfied only for 0 < δ ≤ δ₀ ≈ 0.350162. Therefore, |a₅| ≤ δ/2 for f ∈ S₅(ϕ) where 0 < δ ≤ δ₀.

3 Zalcman’s Functional

The following result provides the bound for the combination of coefficients of functions in the class S₅ₗₚ(ϕ). Throughout this section, we use the notations L(q₁, q₂), Dₖ’s, i = 1 to 12, ω₋₁(ẑ), ω₋₂(ẑ) and ω₀(ẑ), carry their meaning as given in [25] Lemma 2.

Theorem 7. If f ∈ S₅ₗₚ(ϕ), then for real numbers r, s and u, the following holds

\[ |r a_{p+3} + sa_{p+2}a_{p+1} + ua_{p+1}^3| ≤ \frac{pB_r}{(\lambda p + 3)} L(q₁, q₂), \quad B₁ > 0, r ≠ 0, \]

with

\[ q₁ = \frac{pB₁ \left( s (3 + pλ) + r (1 − λ) (3 + 2pλ) \right) + 2rB₂ \left( p²λ² + 3pλ + 2 \right)}{B₁ r (1 + pλ) (2 + pλ)}, \quad (25) \]

and

\[ q₂ = \left( \frac{B₁² p² \left( r (λ − 1)² (pλ + 1)² − (pλ + 3) s (λ − 1) (pλ + 1) − u (pλ + 2) \right)}{B₁ B₂ pB₁ pλ + 1)² (−s (pλ + 3) + r (λ − 1) (2pλ + 3)) + B₃ r (pλ + 2) (p³λ³ + 3p²λ² + 3pλ + 1)} \right) \left( B₁ r (pλ + 1)³ (λp + 2) \right). \]

The result is sharp.

Proof. Since f ∈ S₅ₗₚ(ϕ), there is a function ω ∈ B₀ of the form (2) satisfying

\[ \frac{1}{p} \frac{zf'(z)}{(1 − λ)f(z) + λzf(z)} = ϕ(ẑ), \]

where and is given by (2). By comparing the coefficients of ẑ, z² and z³, we obtain

\[ a_{p+1} = \frac{pB₁}{(λp + 1) c₁}, \quad a_{p+2} = \frac{p \left( (pB₁² (1 − λ) + B₂ (λp + 1)) c₁² + B₁ (λp + 1) c₂ \right)}{(λp + 1) (λp + 2)}, \quad (27) \]

and

\[ a_{p+3} = p \left( B₁² p² c₁² (λ − 1)² − p c₁ c₂ B₁² (λ − 1) (2pλ + 3) + B₃ c₁² (p²λ² + 3pλ + 2) + 2B₂ c₁² (p²λ² + 3pλ + 2) + 3pλ + 2) + B₁ \left( B₂ p c₁² (1 − λ) (2pλ + 3) + c₂ (p²λ² + 3pλ + 2) \right) \right) / (p³λ³) \]

+ 11pλ + 6p²λ² + 6).

Using the above values of aₚ₊₁, aₚ₊₂ and aₚ₊₃ together with the real numbers r, s and u, a simple calculation yields

\[ |r a_{p+3} + sa_{p+2}a_{p+1} + ua_{p+1}^3| = \frac{pB_r}{(λp + 3)} |c₁ + q₁ c₁ c₂ + q₂ c₁²|, \quad (28) \]
where $q_1$ and $q_2$ are given by (23) and (29). The desired result follows at once from (28) and Lemma 4 and the extremal functions depend on the values of $q_1$ and $q_2$. Now we give the extremal functions $H_i: \mathbb{D} \to \mathbb{C}$, satisfying:

$$
\frac{1}{p} \left( \frac{z H_i'(z)}{(1 - \lambda) H_i(z) + \lambda z^p} \right) = \begin{cases} 
\varphi(z^3) & \text{for } i = 1, \ (q_1, q_2) \in D_1 \cup D_2 \cup \{(2, 1)\}, \\
\varphi(z) & \text{for } i = 2, \ (q_1, q_2) \in \cup_{k=3}^7 D_k \cup \{(2, 1)\}, \\
\varphi(\omega_{-1}(z)) & \text{for } i = 3, \ (q_1, q_2) \in D_8 \cup D_9, \\
\varphi(\omega_0(z)) & \text{for } i = 4, \ (q_1, q_2) \in D_{10} \cup D_{11} \setminus \{(2, 1)\}, \\
\varphi(\omega_{-2}(z)) & \text{for } i = 5, \ (q_1, q_2) \in D_{12}.
\end{cases}
$$

Which completes the proof. □

If we put $s = u = 0$ and $r = 1$ in Theorem 7 we obtain the following result:

**Corollary 7.1.** If $f \in S_{x,p}^+(\varphi)$, then

$$
|a_{p+3}| \leq \frac{pB_1}{\lambda p + 3} L(q_1, q_2),
$$

where

$$
q_1 = \frac{B_1^2 p (1 - \lambda) \left(2p \lambda + 3\right) + 2B_2 \left(p^2 \lambda^2 + 3p \lambda + 2\right)}{B_1 (p \lambda + 1) (p \lambda + 2)},
$$

$$
q_2 = \frac{B_3 \left(p^2 \lambda^2 + 3p \lambda + 2\right) + B_1 p (1 - \lambda) \left(B_1^2 p (1 - \lambda) + B_2 (2p \lambda + 3)\right)}{B_1 (p \lambda + 1) (p \lambda + 2)}.
$$

For $s = 1 = u = 0$, then $|a_{p+3}| \leq \frac{pB_1}{\lambda p + 3} L(q_1, q_2),

where

$$
q_1 = \frac{B_1^2 p (p - 2p \lambda - 3) + 2B_2 \left(p^2 \lambda^2 + 3p \lambda + 2\right)}{B_1 (p \lambda + 1) (p \lambda + 2)},
$$

$$
q_2 = \frac{B_3 \left(p \lambda + 1\right)^2 (p \lambda + 2) + B_1 p \left(B_1^2 p (\lambda - 1) (p \lambda^2 + \lambda + 2) + B_2 \left(p^2 (\lambda - 2\lambda^2) - 5p \lambda + p - 3\right)\right)}{B_1 (p \lambda + 1)^2 (p \lambda + 2)}.
$$

Similarly, by choosing different values for $\lambda, s, p, r, u$ and $\varphi(z)$ in Theorem 7, we obtain several results pertaining to many known classes. Now we proceed to applications pertaining to the classes $S^x_{x} (\varphi), F_{x}, S^x (\varphi)$ and $R(\varphi)$.

### 3.1 Applications to the classes $S^x_{x} (\varphi)$ and $F_{x}$

For $p = s = 1 = r = 0$, then $|a_{2a_3} - a_4|$, which is $|J_{2,3}|$, for the class $S^x_{x} (\varphi)$.

**Corollary 7.3.** If $f \in S^x_{x} (\varphi)$, then

$$
|J_{2,3}(f)| \leq \frac{B_1}{(\lambda + 3)} L(q_1, q_2),
$$

where

$$
q_1 = \frac{-2B_1^2 \lambda + 2B_2 (\lambda + 2)}{B_1 (\lambda + 2)}, \quad q_2 = \frac{B_1 \left(\lambda^3 + \lambda - 2\right) - 2B_1 B_2 \lambda (\lambda + 1)^2 + B_3 (\lambda + 1)^2 (\lambda + 2)}{B_1 (\lambda + 1)^2 (\lambda + 2)}.
$$

If we put $\varphi(z) = z + \sqrt{1 + z^2}$ in corollary 7.3, then we obtain the bound of $|J_{2,3}|$ for $f \in F_{x}$. 

8
Corollary 7.4. If \( f \in \mathcal{F}_\lambda \), then

\[
|J_{2,3}(f)| \leq \begin{cases} 
\frac{1}{\lambda + 3} & \lambda \in [\lambda_0, 1], \\
\frac{2(1 + \lambda^2)}{(\lambda + 1)^2(\lambda + 2)(\lambda + 3)} & \lambda \in [0, \lambda_0),
\end{cases}
\]

where \( \lambda_0 \approx 0.0639089 \) is the root of the equation \( 27\lambda^4 + 216\lambda^3 + 439\lambda^2 + 284\lambda - 20 = 0 \). Extremal functions \( f_1 \) and \( f_2 \) for \( \lambda \in [\lambda_0, 1] \) and \( \lambda \in [0, \lambda_0) \), respectively are given by

\[
\frac{zf_1'(z)}{(1 - \lambda)f_1(z) + \lambda z} = z^3 + \sqrt{1 + z^6}, \quad \frac{zf_2'(z)}{(1 - \lambda)f_2(z) + \lambda z} = z + \sqrt{1 + z^2}.
\]

3.2 Applications to the Class \( S^*(\varphi) \)

By taking \( \lambda = 0 \), \( p = s = 1 \), \( r = -1 \) and \( u = 0 \) in Theorem 7, we obtain the bound of \( |J_{2,3}(f)| \) for \( f \in S^*(\varphi) \).

Corollary 7.5. If \( f \in S^*(\varphi) \), then

\[
|J_{2,3}(f)| \leq \frac{B_3}{3} L(q_1, q_2),
\]

where

\[
q_1 = \frac{2B_2}{B_1}, \quad q_2 = -\frac{B_1}{B_1} \frac{B_3}{B_1}.
\]

Note that Corollary 7.5 yields bound of \( |J_{2,3}| \) for different classes by choosing the appropriate choices of \( \varphi(z) \). In Table 2, we provide bounds for \( |J_{2,3}(f)| \) with extremal functions for some classes.

| Class   | Bound of \( |J_{2,3}| \) | Extremal functions |
|---------|--------------------------|--------------------|
| \( S_C^* \) | 64/81 | \( z \exp ((4z + z^2)/3) \) |
| \( S_{Ne}^* \) | 4/9 | \( z \exp ((9t - t^4)/9) \) |
| \( S_{L}^* \) | 1/6 | \( z \exp \int_0^1 (\sqrt{1 + t^6} - 1)/t \, dt \) |
| \( S_{SG}^* \) | 1/6 | \( z \exp \int_0^1 (e^{t^3} - 1)/(t(e^{t^3} + 1)) \, dt \) |
| \( S_{\varphi_0}^* \) | 1/(3k), \( k = 1 + \sqrt{2} \) | \( z \exp \int_0^1 (t^2(k + t^3))/(k(k - t^3)) \, dt \) |
| \( S_P^* \) | 8/(3π^2) | \( z \exp \int_0^1 (2/\pi^2) \left( (1 + \sqrt{t^2})/(1 - \sqrt{t^2}) \right)^2 \, dt \) |

Corollary 7.6. 1. If \( f \in S_{\varphi_0}^* \), then

\[
|J_{2,3}(f)| \leq \frac{2}{3} \sqrt{\frac{2}{5}},
\]

and the equality holds for the function \( f(z) \) given by \( z f'(z)/f(z) = \varphi(\omega_{-1}(z)) \), where \( \omega_{-1}(z) = c_1^{-1}z + c_2^{-1}z^2 + c_3^{-1}z^3 + \cdots \) with \( c_1^{-1} = \sqrt{2}/5 \), \( c_2^{-1} = -3/5 \) and \( c_3^{-1} = -(3/5)\sqrt{(2/5)} \).

2. If \( f \in S_C^* \), then

\[
|J_{2,3}(f)| \leq \frac{8}{9\sqrt{7}},
\]

and the equality holds for the function \( f(z) \) given by \( z f'(z)/f(z) = \varphi(\omega_{-1}(z)) \), where \( \omega_{-1}(z) = c_1^{-1}z + c_2^{-1}z^2 + c_3^{-1}z^3 + \cdots \) with \( c_1^{-1} = 2/\sqrt{7} \), \( c_2^{-1} = -3/7 \) and \( c_3^{-1} = -6/(7\sqrt{7}) \).

Remark. For different choices of \( \varphi(z) \) in Corollary 7.5, we obtain several known results, which are listed below:
1. If \( f \in \Delta^* \), then \( |J_{2,3}(f)| \leq 4\sqrt{6}/27 \) [10 Theorem 2.1].
2. If \( f \in S_{\rho}^* \), \( b \in (0,1) \), then \( |J_{2,3}(f)| \leq b/6 \) [14 Theorem 3.1].
3. For \( f \in S^* \), then \( |J_{2,3}(f)| \leq 2 \) [4 Theorem 3.3].
4. For \( \varphi(z) = (1 + (1 - 2a)z)/(1 - z) \), where \( a \in (0,1) \), Corollary 7.5 reduces to a result due to Cho et al. [2] Theorem 7.5.

For \( s = u = \lambda = 0 \) and \( p = r = 1 \), Theorem 4 gives the following bound of \( |a_4| \) for the class \( S^*(\varphi) \), which we can also obtain from [2] Theorem 1.1.

**Corollary 7.7.** If \( f \in S^*(\varphi) \), then \( |a_4| \leq (B_1/3) L(q_1, q_2) \), where \( q_1 = (3B_1^2 + 4B_2) / (2B_1) \), and \( q_2 = (B_1^2 + 3B_1B_2 + 2B_3) / (2B_1). \)

For different choices of \( \varphi(z) \) in Corollary 7.7 we obtain the bound of fourth coefficient for different classes.

**Corollary 7.8.**
1. If \( f \in S_{\rho}^* \), then \( |a_4| \leq 68/81 \). The result is sharp for the function \( f(z) \) such that \( z(f''(z)/f(z)) = 1 + (4/3)z + (2/3)z^2. \)
2. If \( f \in S_{\rho}^* \), then \( |a_4| \leq 1/2. \) The result is sharp for the function \( f(z) \) such that \( z(f''(z)/f(z)) = \varphi(z^3) \), where \( \varphi(z) = 1 + z - z^3/3. \)
3. If \( f \in S^* \), then \( |a_4| \leq 1/3. \) The result is sharp for the function \( f(z) \) such that \( z(f''(z)/f(z)) = \varphi(z^3) \), where \( \varphi(z) = 1 + z^2. \)
4. If \( f \in S^* \), then \( |a_4| \leq 4. \) The result is sharp for the function \( f(z) \) such that \( z(f''(z)/f(z)) = (1 + z)/(1 - z). \)

Some of the already known results, which we obtain for various choices of \( \varphi(z) \) in Corollary 7.7 are listed below.

**Remark.**
1. If \( f \in S_{\rho}^* \), then \( |a_4| \leq 5/6 \) [17 Theorem 4.2].
2. If \( f \in S_{\rho}^* \), then \( |a_4| \leq 17/36 \) [23 Theorem 2.3].
3. If \( f \in S_{SG}^* \), then \( |a_4| \leq 1/6 \) [13 Theorem 4.1].
4. If \( f \in S^*(q) \), then \( |a_4| \leq 5/12 \) [10 Theorem 2.1].
5. If \( f \in S_{RL}^* \), then \( |a_4| \leq 1/(3) \) [23 Theorem 2.2].
6. If \( f \in S_{L} \), then \( |a_4| \leq 1/6 \) [23 Theorem 2].
7. If \( f \in S_{\rho}^* \) and \( |a_4| \leq b/6 \), where \( b \in (0,1) \) [17 Theorem 3.1].
8. If \( f \in S_{\rho}^*(t), t \in (0,1) \). Then we have

\[
|a_4| \leq \begin{cases} 
{t/3} & \text{if } t \in (0,1), \\
(17t^2 + 15t^2 + 4t)/36 & \text{if } t \in (\sqrt{769} - 15)/34,1].
\end{cases}
\]

The first inequality is sharp for the function \( f(z) \) given by \( z(f''(z)/f(z)) = \varphi(z^3) \) and for the second inequality extremal function \( f(z) \) is given by \( z(f''(z)/f(z)) = \varphi(z) \), where \( \varphi(z) = 1/(1 - z)^3, t \in [0,1] \) [17 Theorem 11].

### 3.3 Applications to the Class \( \mathcal{R}(\varphi) \)

For \( \lambda = p = s = 1, r = -1 \) and \( u = 0 \), Theorem 4 gives the following bound of \( |J_{2,3}| \) for \( f \in \mathcal{R}(\varphi) \).

**Corollary 7.9.** If \( f \in \mathcal{R}(\varphi) \), then \( |J_{2,3}| \leq (B_1/4) L(q_1, q_2) \), where \( q_1 = (-2B_1^2 + 6B_2) / (3B_1) \) and \( q_2 = (-2B_1B_2 + 3B_3) / (3B_1). \)

**Example.** On taking \( \varphi(z) = \sqrt{1 + z} \) and \( \varphi(z) = z + \sqrt{1 + z^2} \) in Corollary 7.9, the following examples follow immediately.
1. Let \( \varphi(z) = \sqrt{1 + z} \) and \( f \in \mathcal{R}(\varphi) \), then \( |J_{2,3}(f)| \leq 1/8 \). Extremal function is given by \( f(z) \) such that \( f'(z) = \varphi(z^3) \).

2. Let \( \varphi(z) = z + \sqrt{1 + z^2} \) and \( f \in \mathcal{R}(\varphi) \), then \( |J_{2,3}(f)| \leq 1/4 \). Extremal function is given by \( f(z) \) such that \( f'(z) = \varphi(z^3) \).

**Remark.** For \( \varphi(z) = (1 + z)/(1 - z) \) in Corollary 7.3, we obtain the result of K. Babalola [4] Theorem 3.1.

For \( s = u = 0 \) and \( \lambda = p = r = 1 \), Theorem 8 gives the following sharp bound of \( |a_4| \) for the class \( \mathcal{R}(\varphi) \).

**Corollary 7.10.** If \( f \in \mathcal{R}(\varphi) \), then \( |a_4| \leq \left( B_1/4 \right) L(q_1, q_2) \) where \( q_1 = 2B_2/B_1 \) and \( q_2 = B_3/B_1 \).

For different choices of \( \varphi(z) \) in Corollary 7.10, we obtain the following example.

**Example.** 1. Let \( \varphi(z) = \sqrt{1 + z} \) and \( f \in \mathcal{R}(\varphi) \), then \( |a_4| \leq 1/8 \). Extremal function is given by \( f(z) \) such that \( f'(z) = \varphi(z^3) \).

2. Let \( \varphi(z) = z + \sqrt{1 + z^2} \) and \( f \in \mathcal{R}(\varphi) \), then \( |a_4| \leq 1/4 \). Extremal function is given by \( f(z) \) such that \( f'(z) = \varphi(z^3) \).

3. Let \( \varphi(z) = (1 + z)/(1 - z) \) and \( f \in \mathcal{R}(\varphi) \), then \( |J_{2,3}(f)| \leq 1/2 \). Extremal function is given by \( f(z) \) such that \( f'(z) = \varphi(z^3) \).

In the following theorem, we obtain the bounds of \( |a_{p+1}| \) and \( |a_{p+2}| \) when \( f \in \mathcal{S}_{2,3}^*(\varphi) \).

**Theorem 8.** If \( f \in \mathcal{S}_{2,3}^*(\varphi) \), then

\[
|a_{p+1}| \leq \frac{pB_1}{\lambda p + 1}, \quad |a_{p+2}| \leq \frac{p}{(\lambda p + 1)(\lambda p + 2)} \left[ B_2(\lambda p + 1) + B_2^2(1 - \lambda)p \right].
\]

These bounds are sharp.

**Proof.** From [27], we have

\[
|a_{p+1}| = \frac{pB_1}{1 + \lambda p} |c_1|.
\]

Using the bound \( |c_1| \leq 1 \) (see [14] Vol I, p. 84-85)), we get the required bound of \( |a_{p+1}| \). Again from [27], we obtain

\[
|a_{p+2}| \leq \frac{p}{(\lambda p + 1)(\lambda p + 2)} \left[ (B_2(\lambda p + 1) + B_2^2(1 - \lambda)p) |c_1|^2 + |B_1(1 + \lambda p)| |c_2| \right].
\]

Using the bound \( |c_2| \leq 1 - |c_1|^2 \) (see [14] Vol II, p. 78)) in the above inequality, required bound of \( |a_{p+3}| \) follows. Equality sign holds for \( f \in \mathcal{A}_p \), which satisfy

\[
\frac{1}{p} \left( \frac{z f'(z)}{(1 - \lambda)f(z) + \lambda z^p} \right) = \varphi(z).
\]

This completes the proof. \( \square \)

Finally, we prove a result about dominance for the class \( \mathcal{S}^*(\varphi) \). Brannan et al. [6] proved some results about dominance for the class \( \mathcal{C}(\alpha) \) of convex functions of order \( \alpha \). Similar result is proved by Keopf [16] for the class \( \mathcal{C}(\alpha) \), with \( f \in \mathcal{A}_p \). Similarly, we prove the following result for the class \( \mathcal{S}^*(\varphi) \).

**Theorem 9.** Let \( f \in \mathcal{S}^*(\varphi) \), then

\[
f'(z) \ll k'(z), \quad z \in \mathbb{D},
\]

where \( k(z) = z/(1 - z)^2 \) is the Koebe function.
Proof. Suppose $f \in S^*(\varphi)$, so there exists $\omega \in B_0$ such that

$$f'(z) = \left( \frac{f(z)}{z} \right) \varphi(\omega(z)).$$

Since $S^*(\varphi) \subset S^*$, so $f$ is a starlike function. From [7], Theorem 3, we can write $f$ in the following form

$$f(z) = \int_{|z|=1} \frac{z}{(1-xz)^2} d\mu,$$

where $\mu$ is a Borel probability measure on the unit circle. Thus we have

$$f'(z) = \left( \int_{|z|=1} \frac{1}{(1-xz)^2} d\mu \right) \varphi(\omega(z)).$$

We can rewrite the above equation

$$f'(z) = \left( \int_{|z|=1} \frac{1}{1-x^2z^2} \left( \frac{1+xz}{1-xz} \right) d\mu \right) \varphi(\omega(z)).$$

For fixed $x \in \partial \mathbb{D}$, $p_x := \left( \frac{1+xz}{1-xz} \right) \varphi(\omega(z))^{1/2} \in \mathcal{P}$. By applying lemma [5] (see e.g. [31], Theorem 2.21), we have

$$p^2_x(z) \ll \left( \frac{1+z}{1-z} \right)^2.$$

Thus we get

$$f'(z) = \int_{|z|=1} \frac{d\mu}{1-x^2z^2} \cdot p^2_x(z)$$

$$= \sum_{k=0}^{\infty} z^{2k} \left( \int_{|z|=1} x^{2k} p^2_x(z) d\mu \right)$$

$$\ll \sum_{k=0}^{\infty} z^{2k} \left( \frac{1+z}{1-z} \right)^2 = \frac{(1+z)^3}{(1-z)^3},$$

since $\mu$ has total mass one.

Remark. For $\varphi(z) = (1+z)/(1-z)$, Theorem 9 gives the well known bound $|a_n| \leq n$ for $f \in S^*$.

Conflict of interest
The authors declare that they have no conflict of interest.

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