PERIODIC POINTS OF QUASIANALYTIC HAMILTONIAN BILLIARDS

M. NOVITSKII AND YU. SAFAROV

January 1999

Abstract. We study absolutely periodic points and trajectories of Hamiltonian systems. Our main result is a necessary and sufficient for a Hamiltonian system to have the following property: if there exists one absolutely periodic trajectory then all trajectories are periodic with the same period.

Introduction

Periodic and absolutely periodic points of Hamiltonian systems play an important role in the study of spectral and semi-classical asymptotics for partial differential operators (see, for example, [DG, PP2, SV2]). For instance, if the set of periodic points of the geodesic flow on a compact Riemannian manifold is of measure zero then the classical Weyl two-term asymptotic formula holds, which means, roughly speaking, that the eigenvalues of $\sqrt{-\Delta}$ ($\Delta$ being the Laplace-Beltrami operator) are distributed uniformly on the real line. If this set has a positive measure then the spectrum of $\sqrt{-\Delta}$ may contain clusters — contracting groups of eigenvalues of unusually high total multiplicity [DG, SV2]. Similar results have been also obtained for

(i) general self-adjoint elliptic partial differential operators on manifolds without boundary, where one has to consider the Hamiltonian flow generated by the principal symbol [DG, SV2];
(ii) boundary value problems, where instead of the Hamiltonian flows one has to deal with so-called Hamiltonian billiards [I, SV2];
(iii) semi-classical problems, where the principal symbol is defined in a different way and is not necessarily a homogeneous function [PP2].

General two-terms asymptotic formulae for the counting function of a partial differential operator contain an oscillating term, given by an integral over the set of periodic points and involving the period function [PP2, SV2]. These formulae often imply that the spectrum does contain clusters whenever the period function is constant [SV2]. Therefore it is essential to know

(i) whether the set of periodic points is of measure zero and, if not,
(ii) whether the period function is constant.

The first author would like to thank the Department of Mathematics of King’s College, London, where this work was performed, for its hospitality. The first author was supported by the Royal Society and the second author by the British Engineering and Physical Sciences Research Council, Grant B/93/AF/1559. Both authors are very grateful to D. Vassiliev for valuable discussions.
Unfortunately, there are very few results on these problems. Even in the simplest case of the standard Euclidean billiard in a convex smooth domain it is still unknown whether the set of periodic points is of measure zero.

The main aim of this paper is to show how one can obtain results on the above mentioned problems with the use of theory of quasianalytic functions. We shall take advantage of the notion of absolutely periodic point introduced in [DG]. An absolutely periodic point can be roughly described as a periodic point at which all the derivatives of the Hamiltonian flow coincide with that of the identical map (see Definition 4.1).

The set of periodic points may have a very complicated structure. In the generic case, the set of absolutely periodic points is much poorer. However, it is known that these two sets are of the same measure (see Lemma 4.4). Therefore it is tempting to find conditions under which there are no absolutely periodic points and, consequently, the set of periodic points is of measure zero.

Sometimes such conditions can be formulated in terms of smoothness. In [SV2] the authors considered, in particular, the Hamiltonian systems generated by analytic homogeneous Hamiltonians. It was proved that in the analytic case the existence of one absolutely periodic point implies that all sufficiently close points are absolutely periodic. This allows one to find sufficient conditions for the set of periodic points to be empty.

In this paper we use the scale of Carleman spaces, which contains the class of analytic functions as a particular case. This scale can be divided into two parts: quasianalytic and nonquasianalytic classes. Our main observation is that the technique suggested in [SV2] is also applicable in the quasianalytic case. We extend results obtained in [SV2] to the quasianalytic Carleman classes and construct counterexamples which imply that these results do not hold for nonquasianalytic Hamiltonians. Thus, we obtain necessary and sufficient conditions in terms of the Carleman spaces.

The paper is organized as follows. In Section 1 we define the Carleman classes and recall some results, including a criterion of quasianalyticity. Section 2 and 3 are devoted to definitions and basic results concerning Hamiltonian flows and billiards. In Section 4 we define periodic and absolutely periodic points and trajectories and formulate our main results (note that Lemma 4.2 is new even in the analytic case). These results are proved in Section 6. In Section 5 we give an explicit description of geodesics on a surface of revolution, which we use later in Section 5 in order to construct the counterexamples.

As we have mentioned before, the results on periodic and absolutely points have applications in spectral theory. Without getting into details, we formulate a simple corollary of our Corollary 4.6 and [SV2, Theorem 1.6.1].

**Corollary.** Let $M$ be a compact quasianalytic Riemannian manifold with strictly convex boundary and $N(\lambda)$ be the counting function of $\sqrt{-\Delta}$, where $\Delta$ is the Laplace-Beltrami operator subject to Dirichlet or Neumann boundary condition. Then

$$N(\lambda) = c_0 \lambda^d + c_1 \lambda^{d-1} + o(\lambda^{d-1}), \quad \lambda \to +\infty,$$

where $c_0$ and $c_1$ are the classical Weyl coefficients. In particular, (0.1) is valid for the usual Laplacian on a convex domain with quasianalytic strictly convex boundary.
1. Carleman spaces $C(m_n)$

Let $\{m_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of positive numbers satisfying the following conditions:

1. $m_n$ grow faster than any power of $n$;
2. $m_{n+1}/m_n \leq n C^n$ for some positive constant $C$;
3. the sequence $\{\ln(m_n/n!)/n\}_{n=1}^{\infty}$ is convex with respect to $n$.

**Definition 1.1.** Let $D \subset \mathbb{R}^d$ be a bounded domain. The Carleman space $C(m_n, D)$ is defined as the set of all functions $u \in C^\infty(D)$ such that

$$\sup_{x \in D} |\partial^\alpha u| \leq C^{|\alpha|} m_{|\alpha|}, \quad \forall \alpha,$$

where $\partial^\alpha u$ are the partial derivatives of $u$ and $C$ is a positive constant which may depend on $u$ but is independent of $\alpha$. We say that $u \in C(m_n, D)$ if $u \in C(m_n, D)$ for any bounded open subset $D$ of its domain of definition.

**Example 1.2.** If $m_n = n!$ then $C(m_n, D)$ coincides with the class of real analytic functions.

**Example 1.3.** If $s \geq 1$ then the sequence $m_n = n^s n^s$ satisfy the conditions (1)–(3). The space $C(m_n)$ defined by such a sequence is called the Gevrey class.

In a similar way one defines classes $C(m_n)$ of vector-valued and matrix-valued functions. We shall take advantage of the following results (see [D, Section 5]).

**Lemma 1.4.** Composition of $C(m_n)$-functions is a $C(m_n)$-function.

**Lemma 1.5.** Let $F(t, z)$ be a real $C(m_n)$-function of variables $t \in \mathbb{R}, z \in \mathbb{R}^N$. If $\partial_t F(t_0, z_0) \neq 0$ then the local $t$-solution $t^*(z)$ of the equation $F(t, z) = 0$, defined in a neighbourhood of $z_0$, belongs to $C(m_n)$.

**Lemma 1.6.** If the Cauchy problem for the a system of ordinary differential equations

$$\frac{\partial z}{\partial t} = f(t, z(t)), \quad z(0) = z_0 \in \mathbb{R}^N,$$

has a continuously differentiable solution $z(t; z_0)$ and $f \in C(m_n)$ then $z(t; z_0)$ also belongs to $C(m_n)$ as a function of $(t, z_0)$.

In view of Lemma 1.4 one can define $C(m_n)$-manifolds and $C(m_n)$-functions on $C(m_n)$-manifolds. The scale of spaces $C(m_n)$ on real analytic manifolds was used for the study of the wave front sets of solution of linear differential equations with $C(m_n)$-coefficients [H, Chapter 8] and some problems of function theory [D].

**Remark 1.7.** One can define the classes $C(m_n)$ assuming only that $m_n$ grow faster than any power of $n$. The sequences $\{m_n\}$ satisfying the conditions (2) and (3) are said to be regular. Lemmas 1.4–1.6 were proved in [D] only for the regular sequences. It is quite possible that these lemmas hold true under less restrictive conditions (for instance, one can try to apply the regularization theory [M]), and then all our further results remain valid as well.

**Definition 1.8.** The class $C(m_n)$ is said to be quasianalytic if every function $u \in C(m_n)$ which has an infinite order zero is identically equal to zero.
Example 1.9. The analytic functions are quasianalytic.

The following theorem gives a criterion of quasianalyticity for the classes $C(m_n)$.

Theorem 1.10 [M]. $C(m_n)$ is quasianalytic if and only if the series $\sum_{n=1}^{\infty} m_n^{-1/n}$ is divergent.

2. Hamiltonian flows

1. Hamiltonian flows in $\mathbb{R}^d \times \mathbb{R}^d$. Let $h(x, \xi)$ be a real infinitely differentiable function on $\mathbb{R}^d \times \mathbb{R}^d$, where $x = (x_1, \ldots, x_d)$ and $\xi = (\xi_1, \ldots, \xi_d)$ are $d$-dimensional variables. The systems of ordinary differential equations

$$
\dot{x}^*(t; y, \eta) = h_\xi(x^*(t; y, \eta), \xi^*(t; y, \eta)), \quad x^*(0; y, \eta) = y,
$$
$$
\dot{\xi}^*(t; y, \eta) = -h_x(x^*(t; y, \eta), \xi^*(t; y, \eta)), \quad \xi^*(0; y, \eta) = \eta,
$$

is said to be a Hamiltonian system, and the function $h$ is called the Hamiltonian. If the first and second derivatives of $h$ are uniformly bounded then the Hamiltonian system has a unique global solution $(x^*, \xi^*)$ for every set of initial data $(y, \eta) \in \mathbb{R}^d \times \mathbb{R}^d$ (see, for example, [Ha]).

A solution

$$(x^*(t), \xi^*(t)) := (x^*(t; y, \eta), \xi^*(t; y, \eta)), \quad t \in \mathbb{R}, \quad (2.1)$$

of the Hamiltonian system is usually interpreted as a trajectory in the phase space $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$ originating from the initial point $(y, \eta)$. The group of shifts along the Hamiltonian trajectories is said to be the Hamiltonian flow. It is easy to see that the Hamiltonian $h$ is constant along the Hamiltonian trajectories. Moreover, the Hamiltonian flow preserves the canonical 2-form $dx \wedge d\xi$ and, consequently, the canonical measure $dx \wedge d\xi$ on $\mathbb{R}^{2d}$ [A,CFS].

2. Hamiltonian flows on manifolds without boundary. If we identify the phase space $\mathbb{R}^{2d}$ with the cotangent bundle $T^*\mathbb{R}^d$ (which simply means that the choice of coordinates $x$ determines, in a standard way, the choice of coordinates $\xi$) then the solution $\xi^*(t; y, \eta)$ behaves under change of coordinates as a covector over the point $\xi^*(t; y, \eta)$. Therefore the above construction can be generalized to the case where the Hamiltonian is a function on the cotangent bundle $T^*M$ over a smooth $d$-dimensional manifold $M$ without boundary.

If $M$ is a manifold and $h$ is a smooth function on $T^*M$ then the Hamiltonian equations are understood in local coordinates, $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ being local coordinates on (or points of) $M$, $\xi = (\xi_1, \ldots, \xi_d)$ and $\eta = (\eta_1, \ldots, \eta_d)$ being the dual coordinates on (or points of) the fibres $T^*_xM$ and $T^*_yM$ respectively. In this case the solution (2.1) is a smooth trajectory in $T^*M$. As before, the Hamiltonian flow preserves the Hamiltonian, the canonical 2-form and canonical measure on $T^*M$ (which are defined as $dx \wedge d\xi$ and $dx \wedge d\eta$ in every local coordinate system).

3. Homogeneous Hamiltonian flows. Let $h$ be a function on $T^*M$ satisfying the following conditions:

1. $h(x, \xi) > 0$ whenever $\xi \neq 0$;
2. $h$ is infinitely smooth outside $\{\xi = 0\}$;
3. $h$ is positively homogeneous of degree 1 in $\xi$, that is, $h(x, \lambda \xi) = \lambda h(x, \xi)$ for all positive $\lambda$ and all $(x, \xi) \in T^*M$. 
Clearly, such a function is not smooth at \( \{ \xi = 0 \} \). However, we can consider the Hamiltonian trajectories and Hamiltonian flow generated by \( h \) on the invariant sets

\[
T' M := \{ (x, \xi) \in T^* M : h(x, \xi) \neq 0 \} = \{ (x, \xi) \in T^* M : \xi \neq 0 \}
\]

and

\[
S^* M := \{ (x, \xi) \in T' M : h(x, \xi) = 1 \}.
\]

If conditions (1)–(3) are fulfilled then the Hamiltonian flow has the following additional properties (see, for example, [SV2]):

(i) the solutions \( x^* \) and \( \xi^* \) of the Hamiltonian system are positively homogeneous in \( \eta \) of degrees 0 and 1 respectively;

(ii) \( \dot{x}^* \) does not vanish, that is, the trajectory \( x^*(t) \subset M \) cannot stop;

(iii) the Hamiltonian flow preserves the canonical 1-form \( \xi \cdot dx \).

In the theory of elliptic (pseudo) differential operators one usually deals with the homogeneous Hamiltonian flow generated by the principal symbol to the power \( 1/m \), \( m \) being the order of the operator under consideration (see, for example [MS, SV2]).

**Example 2.1.** Let \( M \) be a Riemannian manifold and \( h(x, \xi) = |\xi|_x \), where \( |\xi|_x \) is the length of the covector \( \xi \in T^*_x M \). Then \( x^*(t) \) are geodesics and

\[
\xi^*_k(t) = \sum_{j=1}^{d} g_{jk}(x^*(t)) \dot{x}^*_j(t),
\]

where \( \{g_{jk}\} \) is the metric tensor and \( \dot{x}^* = \{\dot{x}^*_1, \ldots, \dot{x}^*_d\} \) is the tangent vector at the point \( x^* \). The homogeneous Hamiltonian flow generated by this Hamiltonian is called a geodesic flow. Note that it is more common to define geodesics with the use of the Hamiltonian

\[
|\xi|^2_x = \sum_{j,k=1}^{d} g^{jk}(x) \xi_j \xi_k,
\]

where \( \{g^{jk}\} := \{g_{jk}\}^{-1} \). Since the Hamiltonian is constant along the trajectories, this simply means that the parameter \( t \) is chosen in a different way; if \( h(x, \xi) = |\xi|_x \) then the geodesics are parameterized by their length.

### 3. Homogeneous Hamiltonian billiards

Throughout this section we assume that the Hamiltonian satisfies the conditions (1)–(3) of subsection 2.3. The definitions and results quoted below can be found in [SV1, SV2].

**1. Billiard trajectories.** Let \( M \) be a smooth \( d \)-dimensional manifold with boundary and \( h \) be a homogeneous Hamiltonian on \( T^* M \). Near \( \partial M \) we shall use special coordinates \( x = (x', x_d) \) such that \( x' \in \mathbb{R}^{d-1} \), \( \partial M = \{x_d = 0\} \) and \( x_d > 0 \) for points in the interior of \( M \). Then \( \xi = (\xi', \xi_d) \), where \( \xi_d \) is the so-called conormal component of \( \xi \).

Let \( (y, \eta) \in T'(M \setminus \partial M) \), that is, \( y \not\in \partial M \). Assume that, at some time \( t = \tau \), the Hamiltonian trajectory (2.1) originating from \( (y, \eta) \) hits the boundary at some point

\[
(x^*(\tau, 0), \xi^*(\tau, 0)) \in T' M.
\]
(in other words, \(x^*(\tau - 0) \in \partial M\)). Then \((x^*(\tau - 0), \xi^*(\tau - 0))\) is said to be a point of incidence. At a point of incidence we always have

\[
h_{\xi_d}(x^*(\tau - 0), \xi^*(\tau - 0)) \leq 0.
\]

The incoming trajectory is said to be transversal if

\[
h_{\xi_d}(x^*(\tau - 0), \xi^*(\tau - 0)) < 0.
\]

**Definition 3.1.** We say that \((x^*(\tau + 0), \xi^*(\tau + 0)) \in T'_\partial M M\) is a point of reflection if

(i) \(h(x^*(\tau - 0), \xi^*(\tau - 0)) = h(x^*(\tau + 0), \xi^*(\tau + 0)),\)

(ii) \(x^*(\tau - 0) = x^*(\tau + 0),\)

(iii) the covectors \(\xi^*(\tau - 0)\) and \(\xi^*(\tau + 0)\) differ only in their conormal component,

(iv) \(h_{\xi_d}(x^*(\tau + 0), \xi^*(\tau + 0)) > 0.\)

Since \(h_{\xi_d}(x^*(\tau + 0), \xi^*(\tau + 0)) > 0\), the Hamiltonian trajectory originating from the point \((x^*(\tau + 0), \xi^*(\tau + 0))\) is well defined. It is called a reflected trajectory.

**Remark 3.2.** Note that, generally speaking, there may be several points of reflection and reflected trajectories corresponding to one point of incidence.

**Definition 3.3.** The trajectory obtained by consecutive transversal reflection is called a billiard trajectory.

A billiard trajectory originating from an interior point \((y, \eta)\) may not be defined for all \(t \in \mathbb{R}\) for one of the following reasons:

(i) at some moment the incoming part of the trajectory is not transversal,

(ii) the trajectory experiences an infinite number of reflections in a finite time.

In the first case the trajectory is said to be grazing and in the second case it is called a dead-end.

2. Billiard flows. If in our special coordinates

\[
(x^*(\tau - 0), \xi^*(\tau - 0)) = ((x^*)_t, 0, (\xi^*)_t, \xi_{d}\),

\( (x^*(\tau + 0), \xi^*(\tau + 0)) = ((x^*)_t, 0, (\xi^*)_t, \xi_{d}^{+}) \),

then the reflection law can be written as

\[
h((x^*)_t, 0, (\xi^*)_t, \xi_{d}^{-}) = h((x^*)_t, 0, (\xi^*)_t, \xi_{d}^{+}),

h_{\xi_d}(x^*_t, 0, (\xi^*)_t, \xi_{d}^{+}) > 0.\]

**Definition 3.4.** We say that the simple reflection condition is fulfilled if, for every \((x', \xi') \in T'\partial M\), the function \(h(x', 0, \xi', \cdot)\) has the only local (and hence global) minimum \(\xi_{d}^{st}\). We say that the strong simple reflection condition is fulfilled if, in addition, \(h_{\xi_d}(x', 0, \xi', \xi_{d}^{st}) \neq 0\).

If the simple reflection condition is fulfilled then

\[
\text{sign} h_{\xi_d}(x', 0, \xi', \xi_{d}^{st}) = \text{sign} (\xi_{d}^{st}), \quad \forall \xi_{d} \in \mathbb{R}.
\]
Therefore for every transversal incoming trajectory there exists the only reflected trajectory and we have
\[ \xi_d^{-} < \xi_d^{st} < \xi_d^{+}. \]
Under the simple reflection condition one can define the group of shifts along the billiard trajectories, which is called the billiard flow. The billiard flow is defined for all \( t \in \mathbb{R} \) on a set of full measure [CFS] (that is, the set of starting points of grazing and dead-end trajectories is of measure zero) and has the same properties as the Hamiltonian flow (see subsections 2.2 and 2.3).

**Example 3.5.** The Hamiltonian \( h(x, \xi) = |\xi|_x \) on a Riemannian manifold with boundary generates so-called geodesic billiard. In this case \( x^*(t) \) are geodesics (see Example 2.1) and the reflection law takes the usual form: the angle of incidence is equal to the angle of reflection. Clearly, the geodesic billiard satisfies the strong simple reflection condition.

### 3. Hamiltonian billiards with nonnegative Hamiltonian curvature.

If the simple reflection condition is fulfilled then the function
\[ k(x', \xi') = \{h_{\xi_d}, h\}_{x_d=0, \xi_d=\xi_d^{st}(x', \xi')} \]
is said to be the Hamiltonian curvature of \( \partial M \). Here \( (x', \xi') \in T'\partial M \) and \( \{\cdot, \cdot\} \) are the Poisson brackets.

**Definition 3.6.** We say that the Hamiltonian billiard (or billiard flow) is convex if the strong simple reflection condition is fulfilled and
\[ k(x', \xi') \geq 0, \quad \forall (x', \xi') \in T'\partial M. \tag{3.1} \]

**Example 3.7.** If \( M \) is a domain in the Euclidean space and \( h(x, \xi) = |\xi| \) then (3.1) is equivalent to the usual definition of convexity.

In Section 6 we shall use the following result.

**Lemma 3.8** [SV2, Lemma 1.3.17]. In a convex billiard there are no grazing and dead-end trajectories.

### 4. Periodic and absolutely periodic trajectories

**1. Homogeneous Hamiltonian and billiard flows.** Throughout this subsection we assume that

(i) the Hamiltonian is defined on \( T'\partial M \) and satisfies the conditions (1)–(3) of subsection 2.3;

(ii) either \( \partial M = \emptyset \) or the corresponding homogeneous Hamiltonian billiard satisfies the simple reflection condition.

For the sake of convenience we shall regard homogeneous Hamiltonian flows on manifolds without boundary as a particular case of homogeneous billiard flows and assume that, by definition, the Hamiltonian flows on manifolds without boundary are convex.
Definition 4.1. Let $T > 0$. A trajectory $\left( x^*(t; y_0, \eta_0), \xi^*(t; y_0, \eta_0) \right)$ and its starting point $(y_0, \eta_0) \in T'(M \setminus \partial M)$ are said to be

1. $T$-periodic if $\left( x^*(T; y_0, \eta_0), \xi^*(T; y_0, \eta_0) \right) = (y_0, \eta_0)$;
2. absolutely $T$-periodic if the function
   \[ |x^*(T; y, \eta) - y|^2 + |\xi^*(T; y, \eta) - \eta|^2 \] of the variables $(y, \eta) \in T'(M \setminus \partial M)$ has an infinite order zero at $(y_0, \eta_0)$;
3. absolutely $(T, l)$-periodic if they are absolutely $T$-periodic and the trajectory hits the boundary $l$ times as $t \in (0, T)$;
4. periodic if they are $T$-periodic for some $T > 0$;
5. absolutely periodic if they are absolutely $T$-periodic for some $T > 0$.

We shall denote the sets of periodic, absolutely periodic, absolutely $T$-periodic points and absolutely $(T,l)$-periodic points lying in $S^* (M \setminus \partial M)$ by $\Pi$, $\Pi^a$, $\Pi_T^a$ and $\Pi_{T,l}^a$ respectively.

The following two lemmas suggest that the points lying in a path-connected component of $\Pi$ have a common period. However, we do not know whether it is true in the general case.

Lemma 4.2. Let the manifold $M$ and Hamiltonian $h$ belong to a quasianalytic class $C(m_n)$, and let $\Omega$ be a path-connected subset of $\Pi^a$. Then either $\Omega \cap \Pi_{T,l}^a = \emptyset$ or $\Omega \subset \Pi_{T,l}^a$.

Lemma 4.3. If $\gamma$ is a smooth path in $\Pi$ and the period $T(y, \eta)$ is a continuous function of $(y, \eta) \in \gamma$ then $T(y, \eta)$ is constant on $\gamma$.

Note that in Lemma 4.2 the set $\Pi^a$ may be disconnected even if $M$ is connected. Indeed, if there exist grazing or dead-end trajectories then, generally speaking, the billiard flow is well defined for $t \in [0, T]$ only on a subset of $T'M$, which may well be disconnected.

The following lemma implies, in particular, that the set $\Pi \setminus \Pi^a$ is of measure zero.

Lemma 4.4 [SV1, SV2]. The set of points which are $T$-periodic but not absolutely $T$-periodic for some $T > 0$ is of measure zero.

Theorem 4.5. Let $M$ be a compact connected manifold and let the billiard flow generated by a Hamiltonian $h$ be convex. If $M$ and $h$ belong to a quasianalytic class $C(m_n)$ then the existence of one absolutely $(T, l)$-periodic billiard trajectory implies that all trajectories are $(T, l)$-periodic.

Theorem 4.5 was proved in [SV2] in the analytic case. If $\partial M = \emptyset$ then it takes the following form.

Theorem 4.5'. Let $M$ be a compact connected manifold without boundary. If $M$ and $h$ belong to a quasianalytic class $C(m_n)$ then the existence of one absolutely $T$-periodic Hamiltonian trajectory implies that all trajectories are $T$-periodic.

Theorem 4.5 implies the following important corollary.
Corollary 4.6. Let $M$ be a compact connected manifold, $\partial M \neq \emptyset$, and let the billiard flow generated by a Hamiltonian $h$ be convex. If $M$ and $h$ belong to a quasianalytic class $C(m_n)$ and $k \neq 0$ then the set of periodic points is of measure zero.

The next two theorems show that the quasianalyticity condition in Theorems 4.5 and 4.5′ cannot be removed.

Theorem 4.7. If the class $C(m_n)$ is not quasianalytic then there exists a Riemannian $C(m_n)$-manifold $M$ without boundary such that the geodesic flow on $M$ satisfies the following conditions:

1. for some $T$ the set $\Pi_T$ has a positive measure,
2. but there are nonperiodic trajectories; moreover, $S^*M \setminus \Pi$ is a set of positive measure.

Theorem 4.8. If the class $C(m_n)$ is not quasianalytic then there exists a Riemannian $C(m_n)$-manifold $M$ with boundary such that

1. the geodesic billiard on $M$ is convex with $k \equiv 0$,
2. for some $T$ and $l$ the set $\Pi_{T,l}$ has a positive measure,
3. but there are nonperiodic billiard trajectories; moreover, $S^*M \setminus \Pi$ is a set of positive measure.

One can generalize the definition of absolutely periodic points in the following way.

Definition 4.9. Let $T^*(y, \eta)$ be a smooth function defined in a neighbourhood of $(y_0, \eta_0)$. We say that the point $(y_0, \eta_0)$ is absolutely $T^*$-periodic if the function

$$|x^*(T^*(y, \eta); y, \eta) - y|^2 + |\xi^*(T^*(y, \eta); y, \eta) - \eta|^2$$

of variables $(y, \eta)$ has an infinite order zero at $(y_0, \eta_0)$.

However, in the quasianalytic case this is equivalent to Definition 4.1. Indeed, if the manifold and functions $h$ and $T^*$ are quasianalytic then the function (4.2) is also quasianalytic. If it has an infinite order zero at $(y_0, \eta_0)$ then it is identically equal to zero in a neighbourhood of $(y_0, \eta_0)$, which means that all points $(y, \eta)$ of this neighbourhood are periodic with period $T^*(y, \eta)$. Now Lemma 4.3 implies that $T^* \equiv T^*(y_0, \eta_0)$, and therefore the point $(y_0, \eta_0)$ is absolutely $T^*(y_0, \eta_0)$-periodic.

2. Branching Hamiltonian billiards. If the simple reflection condition is not fulfilled then the corresponding billiard is said to be branching. In this case there may exist infinitely many billiard trajectories originating from a fixed point $(y, \eta) \in T'(M \setminus \partial M)$, moreover, the set of these trajectories may well be uncountable. Therefore Theorem 4.5, as it is stated above, is unlikely to be true even for the simplest branching billiards.

For homogeneous branching billiards it is possible to prove that the set of starting points of grazing trajectories is of measure zero [SV2]. But, even in the analytic case, the measure of the set of starting points of dead-end trajectories may be positive [SV1].

One can classify trajectories by the type of their reflections, introduce the notion of periodic and absolutely periodic trajectories and prove statements similar to Lemmas 4.2.4.4 [SV2]. However, it is of a little interest in applications unless...
we have effective sufficient conditions for the set of starting points of dead-end trajectories to be of measure zero. To the best of our knowledge, the only result in this direction (without requiring simple reflection) was obtained in [V1].

3. Nonhomogeneous Hamiltonians. From the geometric point of view, the nonhomogeneous Hamiltonian flows are more difficult to study because they do not preserve the canonical one form. If the Hamiltonian \( h \) is not homogeneous then one has to consider the restriction of the Hamiltonian flow to

\[
\Sigma_\lambda := \{(x, \xi) \in T^*M : h(x, \xi) = \lambda\}
\]

for each fixed \( \lambda \) separately. If \( \lambda \) is not a critical value of \( h \) then \( \Sigma_\lambda \) is a smooth \((2d-1)\)-dimensional submanifold, and one can define the sets \( \Pi \subset \Sigma_\lambda \) and \( \Pi^a \subset \Sigma_\lambda \) in the same way as for homogeneous flows (with (4.1) being considered as a function on \( \Sigma_\lambda \)).

In the nonhomogeneous case the problems discussed in Introduction become much more difficult. Not only may the answer depend on \( \lambda \), but also the structure of the sets \( \Pi \) and \( \Pi^a \) on a fixed energy surface \( \Sigma_\lambda \) may be more complicated. The following simple observation shows that Lemma 4.3 does not necessarily hold for the nonhomogeneous flows.

Example 4.10. Assume that zero is not a critical value of the Hamiltonian \( h \) and let \( h_g := gh \), where \( g \) is a smooth strictly positive function. Then \( h_g \) vanishes on \( \Sigma_0 \) and the Hamiltonian trajectories \( (x_g^*, \xi_g^*) \) of \( h_g \) lying on \( \Sigma_0 \) are defined by the equations

\[
\begin{align*}
\dot{x}_g^*(t; y, \eta) &= g(x_g^*, \xi_g^*) h_\xi(x_g^*, \xi_g^*), & x_g^*(0; y, \eta) &= y, \\
\dot{\xi}_g^*(t; y, \eta) &= -g(x_g^*, \xi_g^*) h_\xi(x_g^*, \xi_g^*), & \xi_g^*(0; y, \eta) &= \eta.
\end{align*}
\]

This implies that \( x_g^*(t; y, \eta) = x^*(f(t; y, \eta); y, \eta) \) and \( \xi_g^*(t; y, \eta) = \xi^*(f(t; y, \eta); y, \eta) \), where

\[
f(t; y, \eta) = \int_0^t g(x_g^*(s; y, \eta), \xi_g^*(s; y, \eta)) \, ds.
\]

Therefore every point \( (y, \eta) \in \Pi \subset \Sigma_0 \) is periodic with respect to the Hamiltonian flow generated by \( h_g \) and its periods \( T \) and \( T_g \) are related as follows

\[
T(y, \eta) = f(T_g(y, \eta); y, \eta).
\]

Clearly, the period \( T_g \) may vary from one point to another even if \( T \) is constant.

In [PP1, PP2] the authors proved an analogue of Lemma 4.4 and, under certain additional restrictions, the analytic version of Theorem 4.5’ for a class of nonhomogeneous Hamiltonians, including Hamiltonians of the form \( h(x, \xi) = |\xi|^2 + V(x) \). The latter result is likely to remain valid in the quasianalytic case.

5. Geodesic flows on surfaces of revolution

In this section we consider the geodesic flow on a 2-dimensional surface of revolution provided with the standard metric. In the first subsection we write down the differential equations for geodesics in an explicit form. We use the arguments suggested by D. Vassiliev in [V2], where 2-dimensional analytic manifolds whose geodesics are closed with the same length were described (see also [B, Chapter 4]). In the second subsection we study the set of absolutely periodic points.
1. Differential equations for geodesics. Let $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $0 \leq \varphi \leq 2\pi$.

Consider a surface of revolution $M \subset \mathbb{R}^3$ defined by

$$q_1 = \cos \theta \sin \varphi, \quad q_2 = \cos \theta \cos \varphi, \quad q_3 = \int_0^\theta \sqrt{[1 + f(\psi)]^2 - \sin^2 \psi} \, d\psi, \quad (5.1)$$

where $q_k$ are coordinates in $\mathbb{R}^3$ and $f(\theta)$ is a smooth function such that

1. $f(0) = 0$,
2. $1 + f(\psi) > |\sin \psi|$ for all $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$,
3. $f$ can be extended to a smooth function on $\mathbb{R}$ which is even with respect to the points $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ in some neighbourhoods of these points,
4. the Taylor expansions of the function $[1 + f(\psi)]^2 - \sin^2 \psi$ at the points $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ start with terms of degree $2q_-$ and $2q_+$ respectively, where $q_-$ and $q_+$ are arbitrary odd positive integers.

In view of (2), the surface $M$ is of the same smoothness as the function $f$ outside the poles $\{\theta = \pm \frac{\pi}{2}\}$. The conditions (3) and (4) imply that the same is true in a neighbourhood of the poles. Indeed, one can easily prove that, under conditions (3) and (4), $M$ can be defined in a neighbourhood of a pole by the equation $z = F(r)$, where $r = \sqrt{x^2 + y^2}$ and $F$ is an even function which belongs to the same class $C(m_n)$ as $f$.

Example 5.1. If $f \in C^\infty_0(-\frac{\pi}{2}, \frac{\pi}{2})$ and $f$ satisfies the conditions (1) and (2) then $f$ also satisfies (3) and (4).

The geodesic flow on a 2-dimensional surface in $\mathbb{R}^3$ can be interpreted as the motion of a particle, with velocity and mass equal to one, in the field of inertial forces. It is described by the Euler–Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = 0, \quad i = 1, 2,$$

where $x_i = x_i(t)$, $\dot{x}_i = \dot{x}_i(t)$ is the tangent vector and

$$L = L(x_1, x_2, \dot{x}_1, \dot{x}_2) := \sum_{k=1}^3 \left( \frac{d}{dt} q_k(x_1, x_2) \right)^2$$

is the Lagrangian, $x_i$ being coordinates on the surface (see, for example [DFN]). The Euler–Lagrange equations imply that the function

$$I_1(x_1, x_2, \dot{x}_1, \dot{x}_2) := \dot{x}_1 \left( \frac{\partial L}{\partial \dot{x}_1} \right) + \dot{x}_2 \left( \frac{\partial L}{\partial \dot{x}_2} \right) - L(x_1, x_2, \dot{x}_1, \dot{x}_2)$$

is constant along the trajectories $(x(t), \dot{x}(t))$. Note that this function turns into the corresponding Hamiltonian if we replace $\dot{x}_i$ with $\sum_k g^{ik} \xi_k$, where $\{g^{ik}\}$ is the metric tensor.

If $M$ is defined as above and $x_1 := \theta$, $x_2 := \varphi$ then

$$L(\theta, \dot{\theta}, \varphi) = I_1(\theta, \dot{\theta}, \varphi) = \varphi^2 \cos^2 \theta + \dot{\theta}^2 [1 + f(\theta)]^2,$$

and, by the second Euler–Lagrange equation, the function

$$L(\theta, \dot{\theta}) = \dot{\theta} \cos^2 \theta,$$
is constant along the trajectories.

Having found two motion integrals $I_1$ and $I_2$ for the Euler–Lagrange equations, we can write down equations for the geodesics in an explicit form. Assume that

(i) the starting point of a geodesic $(\varphi^*(t), \theta^*(t))$ lies on the equator, that is, $\theta^*(0) = 0$ and so

$$I_1 = \dot{\varphi}^*(t) \cos^2 \theta^*(t) = \dot{\varphi}^*(0); \quad (5.2)$$

(ii) the geodesic $(\varphi^*(t), \theta^*(t))$ is parametrized by its length, that is,

$$I_2 = (\dot{\varphi}^*(t))^2 \cos^2 \theta^*(t) + (\dot{\theta}^*(t))^2 \left[1 + f(\theta^*(t))\right]^2 = 1. \quad (5.3)$$

Clearly, $\dot{\theta}^*(0) = \cos \alpha$, where $\alpha \in [0, \pi/2]$ is the angle between the geodesic and equator at the starting point. Therefore we can rewrite (5.2) as

$$\dot{\varphi}^*(t) = \frac{\cos \alpha}{\cos^2 \theta^*(t)}. \quad (5.4)$$

Now (5.3) and (5.4) imply

$$|\dot{\theta}^*(t)| = \sqrt{\cos^2 \theta^*(t) - \cos^2 \alpha} \frac{\cos \theta^*(t)}{\left[1 + f(\theta^*(t))\right] \cos \theta^*(t)}. \quad (5.5)$$

Note that $|\theta^*(t)| \leq \alpha$ and therefore $\cos^2 \theta^*(t) \geq \cos^2 \alpha$. Indeed, if $\dot{\theta}^*(t) = 0$ then, in view of (5.3) and (5.4), $|\theta^*(t)| = \alpha$. In particular, if $\theta^*(t)$ has a local maximum at $t_+$ and a local minimum at $t_-$ then

$$\theta^*(t_+) = -\theta^*(t_-) = \alpha. \quad (5.6)$$

The last observation together with the fact that $(\dot{\theta}^*(t))^2 = 1 - \cos^2 \alpha$ whenever $\theta^*(t) = 0$ imply that the function $\theta^*(t)$ is periodic. Moreover, if

$$\dot{\theta}^*(t_1) = \dot{\theta}^*(t_2) = \dot{\theta}^*(t_3) = 0, \quad t_1 < t_2 < t_3,$$

and $\dot{\theta}^*(t) \neq 0$ for $t \in (t_1, t_2) \cup (t_2, t_3)$ then its period coincides with $t_3 - t_1$. In view of (5.5) and (5.6) we have

$$t_2 - t_1 = t_3 - t_2 = \int_{-\alpha}^{\alpha} \frac{\left[1 + f(\theta)\right] \cos \theta}{\sqrt{\cos^2 \theta - \cos^2 \alpha}} \, d\theta = \pi + \int_{-\alpha}^{\alpha} \frac{\tilde{f}(\theta) \cos \theta}{\sqrt{\cos^2 \theta - \cos^2 \alpha}} \, d\theta,$$

where

$$\tilde{f}(\theta) := \frac{1}{2} (f(\theta) + f(-\theta)).$$

Therefore

$$\theta^*(t + T_*(\alpha)) = \theta^*(t) \quad (5.7)$$

for all $t \in \mathbb{R}$, where

$$T_*(\alpha) := 2 \pi + 2 \int_{-\alpha}^{\alpha} \frac{\tilde{f}(\theta) \cos \theta}{\sqrt{\cos^2 \theta - \cos^2 \alpha}} \, d\theta.$$
is the period of \( \theta^*(t) \).

Similarly, in view of (5.4)–(5.6),

\[
\varphi^*(t_2) - \varphi^*(t_1) = \varphi^*(t_3) - \varphi^*(t_2) = \cos \alpha \int_{-\alpha}^{\alpha} \frac{1 + f(\theta)}{\cos \theta \sqrt{\cos^2 \theta - \cos^2 \alpha}} d\theta.
\]

Using

\[
\frac{1}{\cos \theta \sqrt{\cos^2 \theta - \cos^2 \alpha}} = \frac{d}{d\theta} \left( \arctan \left( \frac{\cos \alpha \sin \theta}{\sqrt{\cos^2 \theta - \cos^2 \alpha}} \right) \right)
\]
we obtain

\[
\varphi^*(t_3) - \varphi^*(t_1) = 2\pi + R(\alpha),
\]
where

\[
R(\alpha) := 2\cos \alpha \int_{-\alpha}^{\alpha} \frac{f(\theta)}{\cos \theta \sqrt{\cos^2 \theta - \cos^2 \alpha}} d\theta.
\]

Since the function \( \theta^*(t) \) is \( T_*(\alpha) \)-periodic, (5.4) and (5.8) imply that for all \( t \in \mathbb{R} \)

\[
\varphi^*(t + T_*(\alpha)) = \varphi^*(t) + R(\alpha) \pmod{2\pi}.
\]

2. Absolutely periodic points. In the previous subsection we have proved that any geodesic starting at the equator can be written as \( (\varphi^*(t), \theta^*(t)) \), where the functions \( \varphi^*(t) \) and \( \theta^*(t) \) satisfy the differential equations (5.4), (5.5). If we identify vectors and covectors as in Example 2.1 then the corresponding Hamiltonian trajectory has the form \( (\varphi^*(t), \theta^*(t), \varphi^*(t), \theta^*(t)) \). In particular, the equator is a \( 2\pi \)-periodic geodesic (with \( \varphi^*(0) = 1 \)), and the corresponding Hamiltonian trajectory is \( \Gamma(t) = (t + \varphi^*(0), 0, 1, 0) \), \( t \in \mathbb{R} \). The following lemma gives a necessary and sufficient condition for this trajectory to be absolutely \( 2\pi \)-periodic.

**Lemma 5.2.** The Hamiltonian trajectory \( \Gamma(t) \) is absolutely \( 2\pi \)-periodic if and only if all even derivatives of the function \( f(\theta) \) vanish at \( \theta = 0 \), that is,

\[
f^{(2k)}(0) = 0, \quad k = 0, 1, 2, \ldots
\]

**Proof.** Let \( (\theta^*(t; \varphi, \dot{\varphi}, \theta, \dot{\theta}), \varphi^*(t; \varphi, \dot{\varphi}, \theta, \dot{\theta})) \) be a geodesic starting at the point \( (\varphi, \theta, \dot{\varphi}, \dot{\theta}) \) and \( \alpha^*(\theta, \phi) \) be the angle of intersection of this geodesic with equator (here we consider \( \varphi, \theta, \dot{\varphi}, \dot{\theta} \) as independent variables). Assume, for the sake of definiteness, that \( \varphi = 0 \). We have to prove that the functions

\[
\varphi^*(2\pi; \varphi, \theta, \dot{\varphi}, \dot{\theta}) - \varphi, \theta^*(2\pi; \varphi, \theta, \dot{\varphi}, \dot{\theta}) - \theta, \varphi^*(2\pi; \varphi, \theta, \dot{\varphi}, \dot{\theta}) - \varphi, \dot{\theta}^*(2\pi; \varphi, \theta, \dot{\varphi}, \dot{\theta}) - \dot{\theta}
\]

have infinite order zeros at the point \( \{0, 0, 1, 0\} \) if and only if (5.10) holds true.

According to (5.4)

\[
\cos \alpha^* = \varphi \cos^2 \theta
\]
and, by (5.7),

\[
\theta = \theta^*(T_*(\alpha^*); \varphi, \theta, \dot{\varphi}, \dot{\theta}).
\]

If the function \( \theta^*(2\pi; \varphi, \theta, \dot{\varphi}, \dot{\theta}) - \theta \) has an infinite order zero at \( \{0, 0, 1, 0\} \) then, in view of (5.5) and (5.12), the function \( T_*(\alpha^*) - 2\pi \) has an infinite order zero at (\( \varphi, \theta, \dot{\varphi}, \dot{\theta} \)).
\{ \theta = 0, \dot{\phi} = 1 \}$. This fact and (5.11) imply that $T_*(\alpha) - 2\pi$ has an infinite order zero at $\alpha = 0$, which is equivalent to (5.10).

Assume now that all even derivatives of $f$ vanish at $\theta = 0$. Then the functions $R(\alpha)$ and $T_*(\alpha^*) - 2\pi$ have infinite order zeros at \{ $\theta = 0, \dot{\phi} = 1$ \}, so (5.12) and (5.9) respectively imply that $\theta^*(2\pi; \varphi, \theta, \dot{\varphi}, \dot{\theta}) - \theta$ and $\varphi^*(2\pi; \varphi, \theta, \dot{\varphi}, \dot{\theta}) - \varphi$ have infinite order zeros at \{ $0, 0, 1, 0$ \}. By (5.4) and (5.5) the same is true for the functions $\dot{\varphi}(2\pi; \varphi, \theta, \dot{\varphi}, \dot{\theta}) - \dot{\varphi}$ and $\dot{\theta}(2\pi; \varphi, \theta, \dot{\varphi}, \dot{\theta}) - \dot{\theta}$. □

**Lemma 5.3.** If $f$ is odd on an interval $[-\alpha_0, \alpha_0]$, $0 < \alpha_0 \leq \frac{\pi}{2}$ then all the geodesics intersecting the equator at an angle $\alpha \leq \alpha_0$ are $2\pi$-periodic. In particular, if $f$ is odd on $[-\pi/2, \pi/2]$ then all geodesics are $2\pi$-periodic.

**Proof.** Under conditions of the lemma

$$T_*(\alpha) = 2\pi, \quad R(\alpha) = 0, \quad \forall \alpha \in [-\alpha_0, \alpha_0].$$

Therefore the required result immediately follows from (5.7) and (5.9). □

### 6. Proofs

1. **The period function.** Let $(y_0, \eta_0)$ be a $T$-periodic point of the homogeneous Hamiltonian or billiard flow. We are going to define in a neighbourhood of this point a positive function $t^*(y, \eta)$ with the following properties:

   (1) $t^*(y_0, \eta_0) = T$ and $t^*$ is positively homogeneous in $\eta$ of degree 0;
   (2) $t^*$ belongs to the same Carleman class $C(m_n)$ as the pair $(h, M)$;
   (3) $\nabla_{y, \eta} t^*(y, \eta) = 0$ if and only if the point $(y, \eta)$ is periodic with period $t^*(y, \eta)$,
   (4) $\nabla_{y, \eta} t^*$ has an infinite order zero at $(y, \eta)$ if and only if the point $(y, \eta)$ is absolutely $t^*(y, \eta)$-periodic.

Note that the function $t^*$ is not uniquely defined by (1)–(4). Indeed, outside the set $\Pi$ we only assume (1) and (2). Moreover, if $(y, \eta) \in \Pi$ or $(y, \eta) \in \Pi^a$ but the corresponding period does not coincide with $t^*(y, \eta)$ then we do not impose any restrictions on the derivatives of $t^*$ at $(y, \eta)$.

**Lemma 6.1.** Let the manifold $M$ and Hamiltonian $h$ belong to a class $C(m_n)$ and let $(y_0, \eta_0)$ be a $T$-periodic point. Then there exists a positive function $t^*$ satisfying the conditions (1)–(4) in a neighbourhood of $(y_0, \eta_0)$.

The proof is based on the following technical lemma (see [SV2, Lemma 2.3.2] or [T]).

**Lemma 6.2.** Under conditions of Lemma 6.1 there exists a coordinate system in a neighbourhood of $y_0$ such that $\det \xi_\eta(T; y_0, \eta_0) \neq 0$.

Note that in [SV2, T] the authors only proved the existence of $C^\infty$-coordinates for which the matrix $\det \xi_\eta(T; y_0, \eta_0)$ is nondegenerate. However, it is clear from the proof that in the case of $C(m_n)$-manifold one can find $C(m_n)$-coordinates with the same property.

**Proof of Lemma 6.1.** Let us choose coordinates as in Lemma 6.2 and consider the function

$$\Phi(t; x, y, \eta) = (x - \xi^*(t; x, y, \eta)), \quad \xi^*(t; x, y, \eta).$$
defined in a neighbourhood of the point \((T; y_0, \eta_0)\). Using the fact that the Hamiltonian and billiard flows preserve the Hamiltonian and canonical 1-form on \(T'M\), one can prove that in a neighbourhood of \((T; y_0, \eta_0)\)
\[
\Phi_\eta(t; x, y, \eta) = 0 \iff x = x^*(t; y, \eta),
\]
\[
\Phi_x(t; x, y, \eta)|_{x=x^*(t; y, \eta)} = \xi^*((t; y, \eta),
\]
\[
\Phi_y(t; x, y, \eta)|_{x=x^*(t; y, \eta)} = -\eta,
\]
\[
\Phi_t(t; x, y, \eta)|_{x=x^*(t; y, \eta)} = -h(y, \eta)
\]
(see [SV2, Sections 2.3, 2.4] for details).

Define
\[
\Psi(t; y, \eta) := \Phi(t; y, y, \eta).
\]
In view of (6.4), in a neighbourhood of \((y, \eta)\) the equation
\[
\Psi(t; y, \eta) = 0
\]
has the only \(t\)-solution \(t^*(y, \eta)\). Since the point \((y_0, \eta_0)\) is \(T\)-periodic we have \(\Psi(t, y, \eta) = 0\), and therefore \(t^*(y_0, \eta_0) = T\). Clearly, the function \(t^*\) is positively homogeneous in \(\eta\) and, by Lemma 1.5, it belongs to the class \(C(m_\eta)\).

From (6.1)–(6.3) it easily follows that \(\nabla_{y, \eta} \Psi(t, y, \eta) = 0\) if and only if the point \((y, \eta)\) is \(t\)-periodic, and \(\nabla_{y, \eta} \Psi(t, y, \eta)\) has an infinite order zero at \((y_0, \eta_0)\) (as a function of \((y, \eta)\)) if and only if the point \((y_0, \eta_0)\) is absolutely \(t\)-periodic (see [SV2, Section 4.1] for details). Differentiating the identity
\[
\Psi(t^*(y, \eta); y, \eta) = 0
\]
in \(y\) and \(\eta\) and taking into account (6.4), we see that \(\nabla_{y, \eta} \Psi(t, y, \eta)|_{t=t^*(y, \eta)} = 0\) if and only if \(\nabla t^*(y, \eta) = 0\), and \(\partial_{y} \partial_{\eta}^2 \Psi(t, y, \eta)|_{t=t^*(y, \eta_0)} = 0\) for all nonzero \(\alpha, \beta\) if and only if \(\nabla t^*(y, \eta)\) has an infinite order zero at \((y_0, \eta_0)\). This implies (3) and (4).

2. Proof of Lemma 4.2. Let \((y_0, \eta_0) \in \Pi_{T, l}^o\). Since we consider only transversal reflections, the billiard trajectories \((x^*(t; y, \eta), \xi^*(t; y, \eta))\) are well defined for all \(t \in [0, T]\) and all \((y, \eta)\) lying in a neighbourhood of \((y_0, \eta_0)\). Moreover, every trajectory starting in this neighbourhood hits the boundary \(l\) times as \(t \in [0, T]\).

Let \(t^*\) be the period function defined on a smaller neighbourhood of \((y_0, \eta_0)\). In view of (4) the gradient of this function has an infinite order zero at \((y_0, \eta_0)\). Under conditions of the lemma \(t^*\) is quasianalytic, and therefore it is identically equal to \(T\) in this neighbourhood. Thus, every point \((y, \eta) \in \Pi_{T, l}^o\) has a neighbourhood \(U(y, \eta) \subset \Pi_{T, l}^o\).

Let \(\gamma\) be a path in \(\Omega\) and, for \((y, \eta) \in \gamma\), let \(T(y, \eta) := \min\{T : (y, \eta) \in \Pi_{T}^o\}\). If \((y^{(n)}, \eta^{(n)}) \to (y, \eta)\) and \(T(y^{(n)}, \eta^{(n)}) \to T\) as \(n \to \infty\) then, obviously, \((y, \eta) \in \Pi_{T}^o\). This implies that the function \(T(y, \eta)\) takes its minimal value on \(\gamma\) at some point \((y_0, \eta_0) \in \gamma\). Consider the set \(\gamma \cap \Pi_{T_0, l_0}^o\). By the above, this set is open in \(\gamma\). On the other hand, since the restriction of the billiard flow
\[
(y, \eta) \to (x^*(T_0; y, \eta), \xi^*(T_0; y, \eta))
\]
to a neighbourhood of \(\gamma\) is a smooth map, the set \(\gamma \cap \Pi_{T_0, l_0}^o\) is closed in \(\gamma\). Therefore \(\gamma \subset \Pi_{T_0, l_0}^o\). Since any two points of \(\Omega\) can be joined by a path, this implies that \(\Omega \subset \Pi_{T_0, l_0}^o\).
3. Proof of Lemma 4.3. Under conditions of the lemma, in a neighbourhood of every point \((y_0, \eta_0)\) the function \(T(y, \eta)\) coincides with the restriction of the period function \(t^*\) to \(\gamma\). Now (3) implies that \(T(y, \eta)\) is locally and hence globally constant on \(\gamma\).

4. Proof of Lemma 4.4. The lemma immediately follows from the fact that the set of zeros of \(\nabla y, \eta\) is of the same measure as the set of infinite order zeros.

5. Proof of Theorem 4.5. By Lemma 3.8, under conditions of the theorem the trajectories \((x^*(t; y, \eta), \xi^*(t; y, \eta))\) are well defined for all \(t \in \mathbb{R}\) and all \((y, \eta) \in T'(M \setminus \partial M)\). Let \((y_0, \eta_0) \in \Pi^s_{T,l}\) and \(\gamma\) be a path in \(T'(M \setminus \partial M)\) from \((y_0, \eta_0)\) to another point \((y, \eta)\). In the same way as in the proof of Lemma 4.2 one can show that \(\gamma \subset \Pi^s_{T,l}\). Since the set \(T'(M \setminus \partial M)\) is path-connected, this implies that \(\Pi^s_{T,l} = T'(M \setminus \partial M)\).

6. Proof of Corollary 4.6. In view of Theorem 4.5 it is sufficient to show that, given \(l\) and \(T\) we can find at least one trajectory which is not \((T, l)\)-periodic. If \(k(x', \xi') \neq 0\) then, by choosing the starting point close to \((x', 0, \xi', 0)\), we can construct a trajectory which experiences arbitrary many reflections in any given time (see proof of Lemma 1.3.34 in [SV2]). This proves the corollary.

7. Proof of Theorems 4.7 and 4.8. Every nonquasianalytic class \(C(m_0)\) contains nonnegative \(C^\infty\)-functions [M]. Let \(f \in C(m_0) \cap C^\infty_0(\mathbb{R})\), \(f \geq 0\) and \(f = 0\) outside the interval \((\frac{1}{2}, 1)\). Consider the surface of revolution \(M\) defined by (5.1). Then, according to Lemma 5.3, all the geodesics intersecting the equator at an angle \(\alpha \leq \frac{1}{2}\) are \(2\pi\)-periodic, which implies that the set \(\Pi^s_{2\pi}\) is of positive measure.

On the other hand, in view of (5.7) and (5.9), a geodesics intersecting the equator at an angle \(\alpha\) is periodic only if

\[
k R(\alpha) = 2k \cos \alpha \int_{-\alpha}^{\alpha} \frac{\tilde{f}(\theta)}{\cos \theta \sqrt{\cos^2 \theta - \cos^2 \alpha}} \, d\theta = 0 \pmod{2\pi} \quad (6.6)
\]

for some integer \(k\). Since \(f\) has an infinite order zero at \(\alpha = \frac{1}{2}\) and \(\cos \alpha\) is a decreasing function on \((0, \frac{\pi}{2})\), the function \(R(\alpha)\) is strictly decreasing on an interval \([\frac{1}{2}, \alpha']\), \(\alpha' \in (\frac{1}{2}, \frac{\pi}{2})\). Therefore, for each \(k\), (6.6) can only be true for a finite number of points \(\alpha \in [\frac{1}{2}, \alpha']\). This implies that the set of periodic points corresponding to the trajectories with \(\alpha \in [\frac{1}{2}, \alpha']\) is of measure zero. The measure of the set of starting points of all Hamiltonian trajectories with \(\alpha \in [\frac{1}{2}, \alpha']\) is not zero, so \(S^* M \setminus \Pi\) is a set of positive measure.

In order to prove Theorem 4.8 it is sufficient to notice that our surface is symmetric with respect to the plane \(\{x = 0\}\). Therefore the billiard trajectories on the manifold with boundary \(M_+ = \{(x, y, z) \in M : x \geq 0\}\) behave in the same way as the geodesics on \(M\); namely, the reflected trajectories of the billiard flow are obtained by the reflection with respect to the plane \(\{x = 0\}\) of the parts of geodesics lying in the second half of \(M\). This implies that, for the function \(f\) described above, \(\Pi^s_{2\pi, \alpha}\) and \(S^* M \setminus \Pi\) are sets of positive measure.

The fact that \(k = 0\) is easily checked by a direct calculation.
References

[A] V. Arnol’d, Mathematical methods of classical mechanics, “Nauka”, Moscow, 1974 (Russian); English transl., Springer, New York, 1989.

[B] A. Besse, Manifolds all of whose geodesics are closed, Springer, New York, 1978.

[CFS] I. Cornfeld, S. Fomin and Ya. Sinai, Ergodic theory, “Nauka”, Moscow, 1980 (Russian); English transl., Springer, New York, 1982.

[D] E. Dyn’kin, Pseudoanalytic extensions of smooth functions. The uniform scale, AMS translations (2) 115 (1980), 33–58.

[DFN] B. Dubrovin, A Fomenko and S. Novikov, Modern geometry: methods and applications, “Nauka”, Moscow, 1979 (Russian); English transl., Springer, New York, 1984.

[DG] J. Duistermaat and V. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, Invent. Math. 25 (1975), 39–79.

[H] L. Hörmander, The analysis of linear partial differential operators I, Springer-Verlag, Berlin Heidelberg New York, 1983.

[Ha] P. Hartman, Ordinary differential Equations, John Wiley & Sons, New York London Sydney, 1964.

[I] V. Ivrii, On the second term of the spectral asymptotics for the Laplace-Beltrami operator on manifolds with boundary, Funktsional. Anal. i Prilozhen. 14 (1980), 25–34; English transl. in Functional Anal. Appl. 14 (1980), 98–106.

[M] S. Mandelbrojt, Series Adherrentes, regularisation des suites, applications, Gauthier-Villars, Paris, 1952.

[MS] R. Melrose and J. Sjostrand, Singularities of boundary value problem I, Comm. Pure Appl. Math. 31(5) (1978), 593–617.

[PP1] V. Petkov and G. Popov, On the Lebesgue measure of the periodic points of a contact manifold, Math. Z. 218 (1995), 91–102.

[PP2] , Semi-classical trace formula and clustering of eigenvalues for Schrödinger operators, Ann. Inst. Henri Poincaré 68(1) (1998), 17–83.

[PR] V. Petkov and D. Robert, Asymptotiques semi-clasiques du spectre d’hamiltoniens quantiques et trajectoires classiques périodiques, Comm. Part. Diff. Equations 10 (1985), 365–390.

[SV1] Yu. Safarov and D. Vassiliev, Branching Hamiltonian billiards, Dokl. AN SSSR 301 (1988), 271–275 (Russian); English transl. in Soviet Math. Dokl. 38 (1989), 64–68.

[SV2] , The asymptotic distribution of eigenvalues of partial differential operators, AMS, 1996.

[T] F. Trèves, Introduction to pseudodifferential and Fourier integral operators, vols. 1, 2, Plenum Press, N.Y., 1982.

[V1] D. Vassiliev, Asymptotics of the spectrum of a boundary value problem, Trudy Moskov. Mat. Obshch. 49 (1986), 167–237 (Russian); English transl. in Trans. Moscow Math. Soc. (1987), 173–245.

[V2] , On the closed geodesics on two-dimensional the surface of revolution, Preprint (unpublished), 1977.

Institute for Low Temperature Physics, Kharkov

E-mail address: novitskii@ilt.kharkov.ua

Department of Mathematics, King’s College, Strand, London WC2R 2LS, UK

E-mail address: ysafarov@mth.kcl.ac.uk