MULTIDIMENSIONAL INTEGRABLE VACUUM COSMOLOGY WITH TWO CURVATURES

V. R. Gavrilov, V. D. Ivashchuk and V. N. Melnikov

Center for Gravitation and Fundamental Metrology, VNIIMS,
31 M. Ulyanovoy St., Moscow, 117313, Russia
e-mail: mel@cvsi.rc.ac.ru

ABSTRACT
The vacuum cosmological model on the manifold $R \times M_1 \times \ldots \times M_n$ describing the evolution of $n$ Einstein spaces of non-zero curvatures is considered. For $n = 2$ the Einstein equations are reduced to the Abel (ordinary differential) equation and solved, when $(N_1 = \dim M_1, N_2 = \dim M_2) = (6, 3), (5, 5), (8, 2)$. The Kasner-like behaviour of the solutions near the singularity $t_s \to +0$ is considered ($t_s$ is synchronous time). The exceptional ("Milne-type") solutions are obtained for arbitrary $n$. For $n = 2$ these solutions are attractors for other ones, when $t_s \to +\infty$. For $\dim M = 10, 11$ and $3 \leq n \leq 5$ certain two-parametric families of solutions are obtained from $n = 2$ ones using "curvature-splitting" trick. In the case $n = 2$, $(N_1, N_2) = (6, 3)$ a family of non-singular solutions with the topology $R^7 \times M_2$ is found.

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1 Introduction

Usually within multidimensional cosmology (see, for instance, [1]-[19] and refs. therein) the space-time is considered as a manifold

\[ M = R \times M_1 \times \ldots \times M_n, \]

where \( R \) is the time axis, one of the manifolds \( M_1, \ldots, M_n \) is interpreted as 3-dimensional external space and the others as so-called internal spaces. In most models \( M_1, \ldots, M_n \) are considered as spaces of constant Riemann curvature, or Einstein spaces. From the very beginning it is supposed that the usual three spatial dimensions and extra spatial dimensions are on the same footing and this assumption is followed from believing (of some physicists) that the early Universe is multidimensional. The separation of extra dimensions from the usual one is attained in multidimensional cosmology by the so-called dynamical compactification of internal spaces to unobservable sizes when their scale factors decrease to the lengths of the Planck order \((10^{-33} \text{cm})\) during the Universe evolution.

The study of dynamical properties of the multidimensional Universe such as, for instance, compactification of internal spaces or expanding of the external space in detail demands exact solutions. However, in all known integrable multidimensional cosmological models the chain of the constant curvature spaces \( M_1, \ldots, M_n \) may contain at most one space of non-zero curvature (see, for example, [10]). As far as we know, there are no integrability conditions or explicit integration methods for the models describing evolution of two or more Einstein spaces with non-zero Ricci tensors. The aim of this paper is to integrate the vacuum model on the manifold \( R \times M_1 \times M_2 \), where the Einstein spaces \( M_1 \) and \( M_2 \) have non-zero Ricci tensors.

This paper is organized as follows. In Sect. 2 we describe the model and get the equations of motion. We show, that their integrability by quadratures is reduced to the integrability of some ordinary differential equation. The latter appeared to be Abel’s equation. In Sect. 3, we show, that for the models with \((\dim M_1, \dim M_2) = (6, 3), (5, 5), (8, 2)\) the Abel equation belongs to the integrable class described by Zaitsev and Polyanin [20, 21] within discrete-group analysis methods. We integrate the (vacuum) Einstein equations for these three models (see formula (3.38)) for non-exceptional case and present the trajectories of motion on the scale factors configuration plane. In Sect. 4 the exceptional ”Milne-type” solutions (see (4.1) and generalization for arbitrary \( n \) in (4.2)) are presented. The Kasner-like behaviour near the singularity (for \( t_s \to +0 \)) is investigated in Sect. 5. In Sect. 6 certain two-parametric families of solutions with \( n \) curvatures for \( 3 \leq n \leq 5 \) are presented. Some non-singular solutions with topology \( R^7 \times M_2 \) are considered in Sect. 7.

2 The model

At first, following our previous papers [7]-[15] we consider here the general formalism for description of multidimensional vacuum cosmological models. It is supposed, that \( D \)-dimensional space-time manifold \( M \) is defined by relation (1.1). The manifold \( M \) is equipped with the
metric
\[ g = - \exp[2\gamma(t)] dt \otimes dt + \sum_{i=1}^{n} \exp[2x^i(t)] g^{(i)}, \] (2.1)

where \( \gamma(t) \) is an arbitrary function determining the time \( t \). It is supposed that the manifold \( M_i \) (see (1.1)) for \( i = 1, \ldots, n \) is the Einstein space of dimension \( N_i \) with the metric \( g^{(i)} \), i.e.
\[ R_{m,n}[g^{(i)}] = \lambda_i g^{(i)}_{m,n}; \quad m_i, n_i = 1, \ldots, N_i, \] (2.2)

where \( \lambda_i \) is constant. (For the manifold \( M_i \) of constant Riemann curvature \( K_i \) the constant \( \lambda_i \) reads: \( \lambda_i = K_i(N_i - 1) \)).

The non-zero components of the Ricci tensor for the metric (2.1) are the following \[ \lambda \]
\[ R^i_{k,l} = \delta^i_k \left( \lambda_i \exp[-2x^i] + \left[ x^i + \dot{x}^i \left( \sum_{j=1}^{n} N_j \dot{x}^j - \dot{\gamma} \right) \right] \exp[-2\gamma] \right), \] (2.4)

where indices \( k_i \) and \( l_i \) for \( i = 1, \ldots, n \) run over from \( 1 + \sum_{j=1}^{i-1} N_j \) to \( \sum_{j=1}^{i} N_j \).

After the gauge fixing \( \gamma = F(x^i) \) vacuum Einstein equations \( R^0_A = 0 \) for \( A, B = 1, \ldots, D - 1 \) and equation \( R^0_A - \delta^0_A R/2 = 0 \) may be presented as the Lagrange-Euler equations and zero-energy constraint correspondingly obtained from some Lagrangian \( L(x^i, \dot{x}^i) = L_F(x^i, \dot{x}^i) \). For the so-called harmonic time gauge
\[ \gamma = \gamma_0 \equiv \sum_{i=1}^{n} N_i x^i \] (2.5)

this Lagrangian has the following form \[ \lambda \]
\[ L(x^i, \dot{x}^i) = \frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} \sum_{i=1}^{n} \lambda_i N_i \exp[-2x^i + 2\gamma_0], \] (2.6)

where
\[ G_{ij} = N_i \delta_{ij} - N_i N_j \] (2.7)

are components of the minisuperspace metric. The minisuperspace metric has the pseudo-Euclidean signature \((-+, \ldots, +)\). (This is a well-known property of the Hilbert-Einstein action \[ \lambda \].)

If all spaces \( M_1, \ldots, M_n \) are Ricci-flat, i.e. \( \lambda_i = 0 \) for \( i = 1, \ldots, n \), or only one of them has non-zero Ricci tensor, then equations of motion following from the Lagrangian (2.6) are integrable and exact solutions are obtained in \[ \lambda \]. But nearly nothing is known about integrability of the models for two or more Einstein spaces with non-zero Ricci tensor. Obtaining exact solutions for such models is connected with a general problem of integrability of the so-called pseudo-Euclidean Toda-like systems \[ \lambda \]. These systems are described by the Lagrangian of the following form
\[ L = \frac{1}{2} \sum_{a,b=0}^{n-1} \eta_{ab} \dot{z}^a \dot{z}^b - \sum_{s=1}^{m} A^{(s)} \exp[\sum_{a=0}^{n-1} \eta_{a} \dot{z}^a], \] (2.8)
where all \( u_s^a, \) \( A^{(s)} \) are real constants and \( (\eta_{ab}) = \text{diag}(-1, +1, \ldots, +1) \). The Lagrangian (2.6) may be reduced to the Lagrangian (2.8) by some linear transformation of the configuration variables \( x^1, \ldots, x^n \). It should be noted, that the Lagrangian (2.8) describes both vacuum multidimensional cosmological models or models with multicomponent perfect fluid source \([12, 14]\). The well-known 4-dimensional Bianchi-IX model (the so-called ”mixmaster model”) is also described by the Lagrangian (2.8) with \( n = 3, m = 6 \) and certain \( u_s^a, A^{(s)} \) \([23]\).

There are extensive literature devoted to the integrability of the Euclidean Toda-like systems \((\eta_{ab}) = \text{diag}(+1, \ldots, +1)\) in (2.8) and methods of their explicit integration (see, for instance, \([24]\)), but, as far as we know, the integrability of the pseudo-Euclidean systems is not well studied yet. In our previous paper \([14]\) we singled out two integrable by quadratures classes of these systems and developed their integration. The first class contains the pseudo-Euclidean systems trivially reducible to the Euclidean systems. It is easy to see that under the conditions: \( u_s^0 = 0 \) for \( s = 1, \ldots, m \) it follows from (2.8) that \( \dot{z}^0 = \text{const.} \) Then the Lagrangian (2.8) effectively ”loses” negative term in the kinetic energy, so for \( z^1, \ldots, z^{n-1} \) the Euclidean system arises. The second class appears provided vectors \( u^1, \ldots, u^n \in \mathbb{R}^n \), where \( u_s = (u_s^0, \ldots, u_s^{n-1}) \) are collinear or form the orthogonal set with respect to the symmetrical bilinear form on \( \mathbb{R}^n \) with matrix \((\eta_{ab})\) (see for details \([14]\)). But the models describing evolution of two or more Einstein spaces with non-zero Ricci tensors are out of these classes and until now we had neither any integrability conditions nor explicit integration methods for them. In this paper we show that at least the models with \( n = 2, \) i.e. multidimensional vacuum cosmological models with metric (2.1) on the manifold (1.1), where \( M_1 \) and \( M_2 \) are Einstein spaces of certain dimensions with non-zero Ricci tensor, are integrable by quadratures, and develop their integration procedure.

So, let us consider the Lagrangian (2.6) for \( n = 2 \) and \( \lambda_1, \lambda_2 \neq 0 \). After the following coordinate transformation diagonalizing the minisuperspace metric:

\[
x = x^1 - x^2 + \frac{1}{2} \ln \left| \frac{\lambda_2 N_2}{\lambda_1 N_1} \right|, \\
y = \alpha \beta \left( N_1 x^1 + N_2 x^2 \right),
\]

where \( \alpha \) and \( \beta \) are defined by the relations

\[
\alpha = \sqrt{\frac{N_1 + N_2 - 1}{N_1 N_2}}, \quad \beta = \sqrt{\frac{N_1 + N_2}{N_1 N_2}},
\]

we obtain the following expression for the Lagrangian (2.6) in the new variables \( x \) and \( y \)

\[
L = \frac{1}{2 \beta^2} (x^2 - y^2) - V(x, y),
\]

where the potential \( V(x, y) \) has the form

\[
V(x, y) = V_0 \exp(2\alpha \beta^{-1} y) [\text{sgn}(\lambda_1) \exp(2\beta_1 x) + \text{sgn}(\lambda_2) \exp(2\beta_2 x)].
\]

Here we denoted

\[
V_0 = -\frac{1}{2} [\lambda_1 N_1]^{\beta_2} [\lambda_2 N_2]^{-\beta_1}, \quad \beta_1 = -\frac{N_2}{N_1 + N_2}, \quad \beta_2 = \frac{N_1}{N_1 + N_2}.
\]
We note that $\beta_2 - \beta_1 = 1$. The corresponding zero-energy constraint reads

$$E \equiv \frac{1}{2\beta^2} (\dot{x}^2 - \dot{y}^2) + V(x, y) = 0.$$  \hspace{1cm} (2.15)

The Lagrangian (2.12) leads to the following equations of motion

$$\ddot{x} = -2\beta^2 V_0 \exp(2\alpha \beta^{-1} y) [\beta_1 \text{sgn}(\lambda_1) \exp(2\beta_1 x) + \beta_2 \text{sgn}(\lambda_2) \exp(2\beta_2 x)], \hspace{1cm} (2.16)$$

$$\ddot{y} = 2\alpha \beta V(x, y). \hspace{1cm} (2.17)$$

It is easy to see, that the set of equations (2.16), (2.17) does not admit solutions with $\dot{x} = \dot{y} = 0 \ \forall t$. Indeed, such solutions are possible only if $\lambda_1 \lambda_2 < 0$ and $\beta_1 = \beta_2$. But the latter condition is impossible in the model under consideration (see (2.14)). So, our model has no static solutions for any dimensions of $M_1$ and $M_2$.

To integrate the equations of motion (2.16), (2.17) under the zero-energy constraint (2.15) we consider at first the following procedure proposed in [6]. Using the relation

$$y'' \equiv \frac{d^2 y}{dx^2} = \frac{\ddot{y} - \dot{x} y'}{\dot{x}^2}, \hspace{1cm} (2.18)$$

equations of motion (2.16),(2.17) and the zero-energy constraint (2.15) in the form

$$\dot{x}^2 = 2\beta^2 \frac{V(x, y)}{(y/x)^2 - 1} = 2\beta^2 \frac{V(x, y)}{(y')^2 - 1}, \hspace{1cm} (2.19)$$

we obtain the following ordinary differential equation of the second order

$$y'' = \left[(y')^2 - 1\right] \left\{\frac{1}{2}(\beta_1 + \beta_2 + f(x))y' + \frac{\alpha}{\beta}\right\}, \hspace{1cm} (2.20)$$

where

$$f(x) = \begin{cases} \tanh(x), & \text{if } \lambda_1 \lambda_2 > 0, \\ \coth(x), & \text{if } \lambda_1 \lambda_2 < 0. \end{cases} \hspace{1cm} (2.21)$$

$$\frac{\alpha}{\beta} = \frac{R}{N_1 + N_2}, \hspace{1cm} (2.22)$$

After this procedure is done the following solutions may be lost: $\dot{x} = \pm \dot{y} \ \forall t$ and $\dot{x} = 0 \ \forall t$. It is easy to see, that under the zero-energy constraint the equations of motion (2.16), (2.17) have no solutions with $\dot{x} = \pm \dot{y} \ \forall t$. The solutions with $\dot{x} = 0 \ \forall t$ appear only in the case $\lambda_1 < 0$ and $\lambda_2 < 0$. So, by integration of the equation (2.20) we obtain all possible trajectories on the configuration plane $(x, y)$ except the trajectory $x = \frac{1}{2} \ln[-\beta_1/\beta_2]$ for the model with $\lambda_1 < 0$ and $\lambda_2 < 0$. If the trajectories of motion are known we may get the law of motion for each trajectory by integration of the equation (2.19). Thus, the problem of the integrability by quadratures of the equations (2.16), (2.17) under the constraint (2.15) is reduced to the problem of the integrability of the equation (2.20). In the paper [6] the equation (2.20) was studied by qualitative methods, but here we consider its explicit integration.

For the considered model we have from (2.11) and (2.14)

$$\beta_1 + \beta_2 = \frac{N_1 - N_2}{N_1 + N_2}, \hspace{1cm} \frac{\alpha}{\beta} = \frac{R}{N_1 + N_2}, \hspace{1cm} (2.23)$$
where
\[ R = R(N_1, N_2) \equiv \sqrt{N_1N_2(N_1 + N_2 - 1)}. \] (2.24)

We also rewrite (2.21), (2.22) as follows
\[ f(x) = \frac{\exp(2x) - \varepsilon}{\exp(2x) + \varepsilon}, \] (2.25)

where here and below
\[ \varepsilon = \varepsilon_1\varepsilon_2, \quad \varepsilon_i = \text{sgn}(\lambda_i), \] (2.26)
i = 1, 2.

It should be noted, that the right side of the equation (2.20) does not contain the unknown function \( y \) due to the factorization of the potential \( V(x, y) \). So, for the function \( z \equiv y' \) we have the first order equation of the form
\[ z' = f_0(x) + f_1(x)z + f_2(x)z^2 + f_3(x)z^3. \] (2.27)

An equation of the form (2.27) is well-known as Abel’s equation (see, for instance [20, 21]). There are no methods to integrate Abel’s equation with arbitrary functions \( f_0(x), f_1(x), f_2(x) \) and \( f_3(x) \).

### 3 Exact solutions

If any solution \( z_0(x) \) to the Abel equation (2.27) is known, then by the transformation [21]
\[ u = \frac{E(x)}{z - z_0}, \quad E(x) = \exp \left[ \int \left( 3f_3(x)z_0^2 + 2f_2(x)z_0 + f_1(x) \right) dx \right] \] (3.1)
it may be written as
\[ uu' + F_1(x) + F_2(x)u = 0, \] (3.2)
where
\[ F_1(x) = f_3(x)(E(x))^2, \quad F_2(x) = (3f_3(x)z_0 + f_2(x))E(x). \] (3.3)

For the new variable
\[ q = -\int F_2(x)dx \] (3.4)
the Abel equation has the so-called canonical form [21]
\[ u\frac{du}{dq} - u = F(q), \] (3.5)
where we denoted
\[ F(q) = \frac{F_1(x)}{F_2(x)}. \] (3.6)

The Abel equation in the form (3.5) was recently studied by a discrete-group analysis methods and a number of new integrable equations were described [20, 21]. In principle, the Abel
equation (2.20) may be written in the canonical form as its solutions \( y' = \pm 1 \) are known, but some difficulty to obtain the function \( F(q) \) in an explicit form for arbitrary parameters \( \alpha, \beta \), and \( \beta_i \) arises. It is not hard to verify, that if one of the two following conditions holds

\[-\alpha = \beta(2\beta_1 - \beta_2), \quad \alpha = \beta(2\beta_2 - \beta_1),\]

(3.7)

then the function \( F(q) \) for the equation (2.20) may be obtained in an explicit form by solving some algebraic second order equation. Using (2.11) and (2.14) we get corresponding relations for the dimensions of \( M_1 \) and \( M_2 \).

\[ N_1 = \frac{4N_2}{N_2 - 1}, \quad N_2 = \frac{4N_1}{N_1 - 1}. \]

(3.8)

It can be easily seen that the first condition holds in the following three cases: i) \( N_1 = 6 \) and \( N_2 = 3 \); ii) \( N_1 = 8 \) and \( N_2 = 2 \), iii) \( N_1 = 5 \) and \( N_2 = 5 \). The second condition corresponds to the inverse numbering of the spaces \( M_1 \) and \( M_2 \).

Under one of the conditions (3.8) the Abel equation (2.20) in the canonical form may be reduced to the Emden-Fowler equation of the form: \( d^2Y/dX^2 = \text{const} Y^a \), which may be easily integrated.

### 3.1 \((1+6+3)\)-model

First, let us consider only the model with \( N_1 = 6 \) and \( N_2 = 3 \). This \((1+6+3)\)-model is of most interest, because the 3-dimensional space \( M_2 \) may be interpreted as our (external) space. Omitting technical details we give at once the final Emden-Fowler equation for this model. One may check that the Abel equation (2.20) for \( N_1 = 6 \) and \( N_2 = 3 \) by the transformation [21]

\[ 1 + \varepsilon \exp[2x] = \frac{3X}{2Y} \frac{dY}{dX}, \]

(3.9)

\[ y' = 1 + \frac{3X \frac{dY}{dX} - 2}{1 + \frac{a}{Y} \left[ \frac{dY}{dX} \right]^2 - 1}, \]

(3.10)

where \( a \neq 0 \) is arbitrary constant, may be reduced to the following Emden-Fowler equation

\[ \frac{d^2Y}{dX^2} = aY^{-2}, \]

(3.11)

which is well-known in classical mechanics as describing the one-dimensional motion of a charged particle in the field of the (attractive or repulsive) Coulomb center. We remind that \( \varepsilon = \text{sgn}(\lambda_1 \lambda_2) \). So, by integrating (3.11) and using the transformation (3.9), (3.10) we obtain the general solution to the equation (2.20) for \( N_1 = 6 \) and \( N_2 = 3 \). In a parametrical form the result looks as follows

\[ \varepsilon \exp[2x] = \Phi(\tau, -\delta, C_1) = \frac{3}{2} \delta \left[ \tau^2 - \delta \right] \left[ 1 + \tau \left( g(\tau, -\delta) + C_1 \right) \right] + \frac{1}{2}, \]

(3.12)

\[ y' = \frac{(1 + 3\delta \tau^2) \exp[2x] + \varepsilon}{(1 - 3\delta \tau^2) \exp[2x] + \varepsilon} , \]

(3.13)

\[ y = x - \frac{3}{2} \ln \left| \tau^2 - \delta \right| + C_2, \]

(3.14)
where
\[ g(\tau, -\delta) = \begin{cases} 
\frac{1}{2} \ln \left| \frac{\tau - 1}{\tau + 1} \right|, & \delta = +1, \\
\arctan(\tau), & \delta = -1.
\end{cases} \] (3.15)

By \( \tau \) we denote the parameter (time), \( C_1 \) and \( C_2 \) are arbitrary constants. Formulas (3.12), (3.13) together with \( y' = \pm 1 \) present the general solution to the Abel equation (2.20) for \( N_1 = 6 \) and \( N_2 = 3 \). This may be also verified by a straightforward calculation using the following relation
\[ \dot{\Phi} = -\frac{\delta}{\tau(\tau^2 - \delta)} \left( (1 - 3\delta\tau^2) \Phi + 1 \right). \] (3.17)

The function (3.12) is depicted in Figs. 1, 2 for different values of \( \delta \) and \( C_1 \). The limits \( \Phi(\tau) \to \infty \) and \( \Phi(\tau) \to 0 \) (for non-exceptional \( C_1 \)) correspond to Kasner-like behaviour near the singularity (see Sec. 5 below).

**Figs. 1, 2**
Using relations (3.12), (3.17), zero-energy constraint (2.15) and the expression for the potential (2.13) we obtain the following relation between harmonic time and \( \tau \)-variable
\[ \left( \frac{d\tau}{dt} \right)^2 = A(-\delta\varepsilon_2)|\tau^2 - \delta|^{-2} \exp \left( 4x + \frac{8}{3}C_2 \right), \] (3.18)
where
\[ A = \frac{1}{6}|6\lambda_1|^{2/3}|3\lambda_2|^{1/3}. \] (3.19)

Thus, from (3.18) we obtain
\[ \delta = -\varepsilon_2 = -\text{sgn}(\lambda_2). \] (3.20)

The transformation inverse to (2.9), (2.10) for \( N_1 = 6, N_2 = 3 \) looks as follows
\[ x^1 = \frac{1}{6} \left( 2x + y - \ln \left| \frac{\lambda_2}{2\lambda_1} \right| \right), \] (3.21)
\[ x^2 = \frac{1}{6} \left( -4x + y + 2 \ln \left| \frac{\lambda_2}{2\lambda_1} \right| \right). \] (3.22)

Using (2.5), (3.12), (3.14), (3.18)-(3.22) we get the following relation for the metric (2.1) \( (n = 2) \)
\[ g = c (f_1 f_2)^{-\frac{1}{2}} \{-2f_1^{-2}d\tau \otimes d\tau + f_2|\lambda_1|g^{(1)} + |\lambda_2|g^{(2)} \}. \] (3.23)

Here \( c \neq 0 \) is an arbitrary constant and
\[ f_1 = f_1(\tau, \varepsilon_2) = |\tau^2 + \varepsilon_2|, \] (3.24)
\[ f_2 = f_2(\tau, \varepsilon_1, \varepsilon_2, C_1) = 2\varepsilon_1\varepsilon_2 \Phi(\tau, \varepsilon_2, C_1) \]
\[ = -3\varepsilon_1 \left( \tau^2 + \varepsilon_2 \right) [1 + \tau \{ g(\tau, \varepsilon_2) + C_1 \}] + \varepsilon_1\varepsilon_2 > 0. \] (3.25)

We recall that \( \varepsilon_i = \text{sgn}(\lambda_i), i = 1, 2 \). It should be noted that although originally \( c > 0 \) since \( c = 2^{-1/2}A^{-1} \exp(C_2/3) \), the negative \( c \) also gives us the solution to vacuum Einstein
equations. The trajectories on the plane of scale factors \( a_i > 0, i = 1, 2 \), corresponding to the solution (3.23), where

\[
a_i^2 = c (f_1 f_2)^{-\frac{1}{2}} f_2 |\lambda_1|, \quad a_2^2 = c (f_1 f_2)^{-\frac{1}{2}} |\lambda_2|,
\]

are depicted in Figs. 3-5 (we put \( c = |\lambda_1| = |\lambda_2| = 1 \)). The points \( \tau_i = \tau_i(C_1), i = 1, 2, 3 \), are zeros of \( f_2 \).

Figs. 3 - 5

### 3.2 The (1+8+2)- and (1+5+5)-models

Some modification of the Anzatz (3.12)-(3.14) may be used for obtaining the solutions to the Abel equation for other two cases \((N_1, N_2) = (8, 2), (5, 5)\). The solutions read

\[
\varepsilon \exp[2x] = \Phi_k(\tau, -\delta, C_1) \tag{3.26}
\]

\[
y' = \frac{1 + k\delta \tau^2}{1 - k\delta \tau^2} \exp[2x] + \varepsilon \tag{3.27}
\]

\[
y = x - \frac{k}{2} \ln |\tau^2 - \delta| + C_2, \tag{3.28}
\]

where

\[
k = (N_1 + N_2)/N_2 = 3, 2, 5 \quad \text{for } (N_1, N_2) = (6, 3), (5, 5), (8, 2), \tag{3.29}
\]

respectively and functions \( \Phi_k = \Phi_k(\tau, -\delta, C_1) \) satisfy to the equations

\[
\dot{\Phi}_k = -\frac{\delta}{\tau(\tau^2 - \delta)} \left[ (1 - k\delta \tau^2) \Phi_k + 1 \right]. \tag{3.30}
\]

The solutions to (3.30) are the following: \( \Phi_3 = \Phi \) (see (3.12)) and

\[
\Phi_2(\tau, -\delta, C_1) = 1 + 2\delta \left( \tau^2 - \delta \right) + C_1 \delta \tau \left| \tau^2 - \delta \right|^{\frac{1}{2}}, \tag{3.31}
\]

\[
\Phi_5(\tau, -\delta, C_1) = \frac{1}{4} - \frac{5}{8} \left( \tau^2 - \delta \right) \left[ 3\tau^2 - 2\delta + 3\tau \left( \tau^2 - \delta \right) (g(\tau, -\delta) + C_1) \right]. \tag{3.32}
\]

Using relations (3.26), (3.30) zero-energy constraint (2.15) and the expression for potential (2.13) we obtain the following relation between harmonic time and \( \tau \)-variable

\[
\left( \frac{d\tau}{dt} \right)^2 = A(-\delta \varepsilon_2)|\tau^2 - \delta|^{1-k} \exp \left[ 4x + 2 \left( 1 + \frac{1}{k} \right) C_2 \right] \tag{3.33}
\]

where here

\[
A = \frac{1}{N_1} |N_1 \lambda_1|^{1-\frac{k}{2}} |N_2 \lambda_2|^{\frac{k}{2}}. \tag{3.34}
\]

From (3.33) we obtain the relation (3.20).
The transformation inverse to (2.9), (2.10) is the following

\[ x^1 = \frac{N_2}{N_1 + N_2} \left( x - \frac{1}{2} \ln \left| \frac{\lambda_2 N_2}{\lambda_1 N_1} \right| \right) + \frac{qy}{N_1 + N_2}, \]

\[ x^2 = -\frac{N_1}{N_1 + N_2} \left( x - \frac{1}{2} \ln \left| \frac{\lambda_2 N_2}{\lambda_1 N_1} \right| \right) + \frac{qy}{N_1 + N_2}, \]

where

\[ q = (\alpha \beta)^{-1} = \sqrt{\frac{N_1 N_2}{N_1 + N_2 - 1}}. \]

Using (2.5), (3.20), (3.26), (3.28), (3.33)-(3.37) we get the relation for the metric (2.1) \((n = 2)\)

\[ g = c (f_1 f_2)^{-r} \left\{ -\frac{N_1 N_2}{f_1^2} f_1^{-2} d\tau \otimes d\tau + f_2 |\lambda_1| g^{(1)} + |\lambda_2| g^{(2)} \right\}. \]

Here \(c \neq 0\),

\[ r = q/N_2 = 2 - q = \frac{1}{2}, \frac{1}{3}, \frac{2}{3} \]
for \((N_1, N_2) = (6, 3), (5, 5), (8, 2)\) respectively, \(f_1 = f_1(\tau, \varepsilon_2)\) is defined in (3.24) and

\[ f_2 = f_2(\tau, \varepsilon_1, \varepsilon_2, C_1, N_1, N_2) = \frac{N_1}{N_2} \varepsilon_1 \varepsilon_2 \Phi_k(\tau, \varepsilon_2, C_1), \]

where functions \(\Phi_k\) are defined in (3.12), (3.31) and (3.32).

### 4 Exceptional solution for negative curvatures

Now, we consider the exceptional solution with \(\dot{x} = 0\). This solution takes place, when \(\lambda_1, \lambda_2 < 0\), and has the following form

\[ g = -dt_s \otimes dt_s + \sum_{i=1}^{2} \frac{\lambda_i}{(2 - D)} t_s^2 g^{(i)}, \]

where \(D = 1 + N_1 + N_2\). Here the synchronous-time parametrization is used. Formula (4.1) may be obtained by solving eqs. (2.15)-(2.17) and performing the coordinate transformation (2.9), (2.10) and appropriate time reparametrization.

Moreover, it is not difficult to verify that the metric

\[ g = -dt_s \otimes dt_s + \sum_{i=1}^{n} \frac{\lambda_i}{(2 - D)} t_s^2 g^{(i)}, \]

defined on the manifold (1.1), where \((M_i, g^{(i)})\) are Einstein spaces satisfying (2.2) \((\lambda_i \neq 0)\) and \(\text{dim} M = D\), is a solution to the vacuum Einstein equations (or, equivalently, is Ricci-flat). (See (2.3), (2.4)).
Remark 1. We note that the metric
\[ g^* = \sum_{i=1}^{n} \frac{\lambda_i}{(2-D)} g^{(i)} \]  
(4.3)
defined on the manifold \( M_1 \times \ldots \times M_n \) satisfies the same relation \( R_{mn}[g^*] = -(D-2)g_{mn}^* \) as the metric \( g(H^{D-1}) \) of \((D-1)\)-dimensional Lobachevsky space. We also note that the metric
\[ \eta = -dt_s \otimes dt_s + t_s^2 g(H^{D-1}) \]  
(4.4)
is flat. For \( D = 3 \) (4.4) coincides with the well-known Milne solution [25]. So, the metric (4.2) may be called a "quasi-flat" or "Milne-like" metric.

Remark 2. It may be proved that, when \((M_i, g^{(i)})\) are spaces of constant (non-zero) curvature, \( i = 1, \ldots, n \), and \( n \geq 2 \), the metric (4.2) has a divergent Riemann-tensor squared as \( t_s \to +0 \)
\[ I[g] = R_{MNPQ}[g]R^{MNPQ}[g] \to +\infty. \]  
(4.5)
In this case the solution (4.2) is singular. The relation (4.5) follows from the formula
\[ I[g] = \mathcal{A}t_s \]  
(4.6)
where
\[ \mathcal{A} = (D-2)^2 \sum_{i=1}^{n} \frac{I[g^{(i)}]}{\lambda_i^2} - 2(D-1)(D-2). \]  
(4.7)
(see [13, 16]). For spaces of constant negative curvature
\[ \mathcal{A} = 2(D-2)^2 \left( \sum_{i=1}^{n} \frac{N_i}{N_i-1} - \frac{D-1}{D-2} \right) > 0, \]  
(4.8)
if \( n \geq 2 \).

**Isotropization.** Now, we show that the solution (4.1) is an attractor for solutions (3.38) with \( \varepsilon_1 < 0, \varepsilon_2 < 0 \) and \( c > 0 \), as \( \tau \to 1 \) (or \( t_s \to +\infty \)). Using the relation \( f_2(\tau) \to 1 \) as \( \tau \to 1 \), we get the asymptotical formula for the synchronous-time variable \( t_s \) \((c > 0)\)
\[ c^{1/2} t_s \sim \sqrt{\frac{N_1}{N_2}} \frac{1-r}{r^{2r/2}} |\tau - 1|^{-r/2} \text{ as } \tau \to 1. \]  
(4.9)
Hence
\[ (f_1 f_2)^{-r} \sim \frac{r^2 N_2}{N_1} t_s^2 = \frac{t_s^2}{D-2} \text{ as } \tau \to 1. \]  
(4.10)
Thus, we obtain the relation for the metric
\[ g = -dt_s \otimes dt_s + b_1(t_s)|\lambda_1|g^{(1)} + b_2(t_s)|\lambda_2|g^{(2)}, \]  
(4.11)
where
\[ b_i(t_s) \sim \frac{t_s^2}{D-2} \text{ as } t_s \to +\infty. \]  
(4.12)
5 Kasner-like behaviour for $t_s \to +0$

Let us consider the solutions (3.38) written in the synchronous-time parametrization

$$g = -dt_s \otimes dt_s + a_1^2(t_s)g^{(1)} + a_2^2(t_s)g^{(2)},$$

(5.1)

where

$$t_s = \pm \int_{\tau_0}^{\tau} d\tau' (f_1 f_2)^{-r/2} f_1^{-1} \left( c N_1 N_2 \right)^{1/2} + t_{0s},$$

(5.2)

($f_i = f_i(\tau)$). As follows from [13] almost all solutions for the considered model should have a Kasner-like behaviour for small $t_s$, i.e.

$$a_i(t_s) \sim A_i t_s^{\alpha_i} \text{ as } t_s \to +0,$$

(5.3)

where $A_i > 0$ are constants and the Kasner parameters $\alpha_i$ satisfy the relations

$$\sum_{i=1}^{2} N_i \alpha_i = \sum_{i=1}^{2} N_i \alpha_i^2 = 1.$$

(5.4)

Here a suitable choice of the constant $t_{0s} = t_{0s}(\tau_0)$ and the sign in (5.2), such that $v = a_1^{N_1} a_2^{N_2} \to +0$ corresponds to $t_s \to +0$, is assumed. We note, that although in [13] models with minisuperspace dimensions $n \geq 3$ were considered, the results are also applicable for the case $n = 2$.

Solving the equations (5.4) we get [11]

$$\alpha_1 = \frac{N_1 \pm R}{(N_1 + N_2) N_1}, \quad \alpha_2 = \frac{N_2 \mp R}{(N_1 + N_2) N_2},$$

(5.5)

where $R = R(N_1, N_2)$ is defined in (2.24). For $(N_1, N_2) = (6, 3), (5, 5), (8, 2)$ $R$ is integer

$$R = N_1 + 2N_2 = 12, \ 15, \ 12$$

(5.6)

correspondingly. We obtain from (5.5)

$$\left( \alpha_1, \alpha_2 \right)_+ = \left( \frac{1}{3}, -\frac{1}{3} \right), \ \left( \frac{2}{5}, -\frac{1}{5} \right), \ \left( \frac{1}{4}, -\frac{1}{2} \right),$$

(5.7)

$$\left( \alpha_1, \alpha_2 \right)_- = \left( -\frac{1}{9}, \frac{5}{9} \right), \ \left( -\frac{1}{5}, \frac{2}{5} \right), \ \left( -\frac{1}{20}, \frac{7}{10} \right),$$

(5.8)

for $(N_1, N_2) = (6, 3), (5, 5), (8, 2)$ respectively. Here $(\alpha_1, \alpha_2)_{\pm}$ corresponds to "±" in (5.5).

Now, we return to the metric (3.38) written in the synchronous-time form (5.1). The Kasner-like behaviour (5.3) with $\alpha_i$ from (5.7) takes place if $\tau \to \tau_*$, where $f_2(\tau_*) = 0$ and $f_2'(\tau_*) \neq 0$. Indeed, from (5.2) we get

$$t_s \sim \text{const}|\tau - \tau_*|^{1-\frac{2}{r}} \text{ as } \tau \to \tau_*$$

(5.9)

($r$ is defined in (3.39)) and using (3.38) we obtain the Kasner-like behaviour (5.3) with the parameters

$$\alpha_1 = \frac{1-r}{2-r}, \quad \alpha_2 = \frac{-r}{2-r},$$

(5.10)
coinciding with those from eq. (5.7)

The Kasner-like asymptotes (5.3) with parameters from (5.8) take place, when \( \tau \to \pm \infty \), for non-exceptional values of the constant \( C_1 \):

\[
C_1 \neq 0, \text{ if } \varepsilon_2 = -1 \text{ and } C_1 \neq \mp \frac{\pi}{2}, \text{ if } \varepsilon_2 = +1, \text{ for } (N_1, N_2) = (6, 3), (8, 2) \tag{5.11}
\]

\[
C_1 \neq \pm 2 \text{ for } (N_1, N_2) = (5, 5). \tag{5.12}
\]

In this case

\[
f_2(\tau) \sim B_{\pm} |\tau|^k \text{ as } \tau \to \pm \infty, \tag{5.13}
\]

where \( B_{\pm} \) are constants (integer \( k \) is defined in (3.29)) and

\[
t_s \sim \text{const} |\tau|^{-1 - \frac{s}{2}} \text{ as } \tau \to \pm \infty. \tag{5.14}
\]

Here \( s = (k + 2)r \). From (3.38), (5.13) and (5.14) we get the Kasner-like behaviour (5.3) with parameters

\[
\alpha_1 = \frac{s - k}{2 + s}, \quad \alpha_2 = \frac{s}{2 + s}. \tag{5.15}
\]

coinciding with (5.8).

According to the results of \[16\] the Riemann tensor squared for the solutions with asymptotically Kasner-like behaviour for \( t_s \to +0 \) is divergent

\[
I[g] = R_{MNPQ}[g] R^{MNPQ}[g] \to +\infty \tag{5.16}
\]

as \( t_s \to +0 \) for fixed \((x_1, x_2) \in M_1 \times M_2 \). If \( I[g^{(i)}] \geq c_i \) for some \( c_i, i = 1, 2 \), then the relation (5.16) takes place uniformly on \((x_1, x_2) \in M_1 \times M_2 \) (see Theorem in \[16\]).

6 Special solutions with \( n \leq 5 \) curvatures

The obtained above solutions may be used for generating some special (two-parametric) classes of the solutions to vacuum Einstein equations. This may be done using the “curvature-splitting” method described below.

Let us consider a set of \( k \) Einstein manifolds \((\mathcal{M}_i, h^{(i)})\) with non-zero curvature, i.e.

\[
\text{Ric}(h^{(i)}) = \mu_i h^{(i)}, \tag{6.1}
\]

where \( \mu_i \neq 0 \) is a real constant, \( i = 1, \ldots, k \). Here and below we denote by \( \text{Ric}(h) \) the Ricci-tensor corresponding to the metric \( h \). Let \( \mu \neq 0 \) be a real number. Then

\[
h = \sum_{i=1}^{k} \frac{\mu_i}{\mu} h^{(i)} \tag{6.2}
\]

is an Einstein metric, (correctly) defined on the manifold

\[
\mathcal{M} = \mathcal{M}_1 \times \ldots \times \mathcal{M}_k \tag{6.3}
\]
and satisfying
\[ \text{Ric}(h) = \mu h. \] (6.4)

Indeed,
\[ \text{Ric}(h) = \sum_{i=1}^{k} \text{Ric}(\frac{\mu_i}{\mu} h^{(i)}) = \sum_{i=1}^{k} \text{Ric}(h^{(i)}) = \sum_{i=1}^{k} \mu_i h^{(i)} = \mu h. \] (6.5)

Remark 3. In (6.2) (like in (2.1)) we identify the metric \( h^{(i)} \) on \( \mathcal{M}_i \) with its canonical extension to the manifold \( \mathcal{M} \) (6.3). It is more correct to write (instead of (6.2))
\[ h = \sum_{i=1}^{k} \frac{\mu_i}{\mu} p_i^* h^{(i)}, \] (6.6)

where \( p_i : \mathcal{M} \to \mathcal{M}_i \) is the canonical projection. Analogously, \( \text{Ric}(h^{(i)}) \) in (6.5) should be understood as \( p_i^* \text{Ric}(h^{(i)}) \) etc.

Now, using the suggested trick, we may consider the following Einstein spaces \( (M_i, g^{(i)}) \) \( (i = 1, 2) \) in (3.38):
\[ g^{(i)} = \sum_{j=1}^{n_i} \frac{\lambda_{ij}}{\lambda_i} g^{(ij)}, \] (6.7)
\[ M_i = M_{i1} \times \ldots \times M_{in_i}, \] (6.8)

where \( (M_{ij}, g^{(ij)}) \) are Einstein spaces of non-zero curvature
\[ \text{Ric}(g^{(ij)}) = \lambda_{ij} g^{(ij)}, \quad \lambda_{ij} \neq 0, \] (6.9)

\( j = 1, \ldots , n_i; i = 1, 2 \). Clearly that
\[ N_i = \sum_{j=1}^{n_i} N_{ij} \] (6.10)

where
\[ N_{ij} = \dim M_{ij} > 1, \] (6.11)

\( j = 1, \ldots , n_i; i = 1, 2 \). It follows from (6.10), (6.11) that
\[ 1 \leq n_i \leq \left\lfloor \frac{N_i}{2} \right\rfloor, \] (6.12)

\( i = 1, 2 \); where \( \lfloor x \rfloor \) denotes the integer part of \( x \).

Substituting (6.7) into (3.38) we get the following solutions to (vacuum) Einstein equations:
\[ g = c (f_1 f_2)^{-r} \left\{-\frac{N_1}{N_2} f_1^{-2} d\tau \otimes d\tau + \sum_{j=1}^{n_1} f_2 \varepsilon_1 \lambda_{1j} g^{(1j)} + \sum_{l=1}^{n_2} \varepsilon_2 \lambda_{2l} g^{(2l)} \right\}, \] (6.13)

defined on the manifold
\[ M = R \times M_{11} \times \ldots \times M_{1n_1} \times M_{21} \times \ldots \times M_{2n_2}. \] (6.14)
The solution (6.13), (6.14) describes the evolution of \( n = n_1 + n_2 \) Einstein spaces \((M_{ij}, g^{(ij)})\), satisfying (6.9)-(6.12). Here \((N_1, N_2) = (6, 3), (5, 5), (8, 2); c \neq 0, \varepsilon_i = \pm 1; r = r(N_1, N_2)\) and \(f_i = f_i(\tau)\) are defined in subsections 3.1 and 3.2, \(i = 1, 2\).

The relations (6.13), (6.14) give us

\[
(1 + 4 + 2 + 3), \ (1 + 3 + 3 + 3), \ (1 + 2 + 2 + 2 + 3)
\]

solutions for \((N_1, N_2) = (6, 3)\);

\[
(1 + 5 + 3 + 2), \ (1 + 3 + 2 + 3 + 2)
\]

solutions for \((N_1, N_2) = (5, 5)\) and

\[
(1 + 6 + 2 + 2), \ (1 + 5 + 3 + 2), \ (1 + 4 + 4 + 2), \ (1 + 4 + 2 + 2 + 2 + 2 + 2)
\]

solutions for \((N_1, N_2) = (8, 2)\).

In the last case \((8, 2)\) we have two families (of solutions) of the same type as in (6.16). (The corresponding solutions from (6.16) and (6.17) seem to be different.) Thus, here we obtained some special two-parametrical families of solutions to the problem of cosmological evolution of \(n\) curvatures for \(n = 3, 4, 5\).

### 7 Non-singular solutions

Here we show that there exist non-singular solutions among the considered ones. Let us restrict ourselves to the \((1 + 6 + 3)\)-case, with two negative curvatures \(\varepsilon_1 = \varepsilon_2 = -1\) and \(C_1 = 0\). In this case the functions \(g(\tau)\),

\[
f_2(\tau) = 3(\tau^2 - 1) \left[ 1 + \frac{1}{2} \tau \ln \left| \frac{\tau - 1}{\tau + 1} \right| \right] + 1
\]

and \(f_1(\tau)f_2(\tau)\) are holomorphic in \(\bar{\Omega} = \Omega \cup \{+\infty\}\), where \(\Omega = C \setminus [-1, 1] \) (\(\bar{\Omega}\) is the complex Riemann sphere with a cut),

\[
g(\tau) = - \sum_{k=0}^{\infty} \frac{1}{2k + 1} \frac{\tau^{-2k-1}}{f_2(\tau)}
\]

\[
f_2(\tau) = \frac{2}{5\tau^2} + \sum_{k=2}^{\infty} f_{2,k} \tau^{-2k}
\]

\[
f_1(\tau)f_2(\tau) = \frac{2}{5} + \sum_{k=1}^{\infty} f_{k} \tau^{-2k}
\]

\(|\tau| > 1\).

Now we introduce a new time variable defined by the relation

\[
\ln \frac{\rho}{\rho_0} = - \int_{\tau_0}^{\tau} d\tau' \sqrt{\frac{2}{5f_{1,2}^2f_2}}.
\]
where \( \tau_0 > 1, \rho_0 > 0 \). We may rewrite (7.5) as
\[
\rho = \frac{\rho_0 \tau_0}{\tau} \exp[-I(\tau_0, \tau)],
\] (7.6)
where
\[
I(\tau_0, \tau) = \int_{\tau_0}^{\tau} d\tau' \left[ \sqrt{\frac{2}{5f_1^2 f_2}} - \frac{1}{\tau} \right].
\] (7.7)
The function (7.7) may be analytically continued to the neighbourhood of \( \infty \), i.e. in \( \{ |\tau| > T \} \cup \{ \infty \} \) for some \( T \), and, clearly that
\[
I(\tau_0, \tau) = I(\tau_0, +\infty) + \sum_{k=1}^{\infty} I_k \left( \frac{1}{\tau} \right)^{2k}.
\] (7.8)
We put
\[
\rho_0 \tau_0 \exp[-I(\tau_0, +\infty)] = 1.
\] (7.9)
Then \( \rho \sim \frac{1}{\tau} \) as \( \tau \to \infty \). The function \( \rho = \rho(\tau) \) (7.6) is smooth and monotonically decreasing on \((1,+\infty)\) and may be analytically continued to \( \{ |\tau| > T \} \cup \{ \infty \} \), where
\[
\rho(\tau) = \frac{1}{\tau} \left[ 1 + \sum_{k=1}^{\infty} \rho_k \left( \frac{1}{\tau} \right)^{2k} \right].
\] (7.10)
For \( \tau \to 1 \) from (7.5) we get
\[
\rho(\tau) \sim \text{const}(\tau - 1)^{-1/\sqrt{10}}.
\] (7.11)
Now, we put also
\[
M_1 = S^6, \quad g^{(1)} = -g(S^6) = -d\Omega_6^2,
\] (7.12)
where \( d\Omega_6^2 \) is the standard metric on 6-dimensional sphere, normalized by the condition
\[
\text{Ric} (d\Omega_6^2) = 5d\Omega_6^2.
\] (7.13)
For this special case we get from (3.23), (7.5), (7.12), (7.13)
\[
g = cF_1(\rho) \left\{ -F_2(\rho) \left[ d\rho \otimes d\rho + \rho^2 d\Omega_6^2 \right] + |\lambda_2| g^{(2)} \right\},
\] (7.14)
where
\[
F_1(\rho) = \left[ f_1(\tau(\rho)) f_2(\tau(\rho)) \right]^{-1/2},
\] (7.15)
\[
F_2(\rho) = \frac{5 f_2(\tau(\rho))}{\rho^2}.
\] (7.16)
Here the inverse function \( \tau = \tau(\rho) \) is smooth for \( \rho \in (0, +\infty) \) and may be analytically continued to the domain \( \{ |\rho| < \delta \} \) for some \( \delta > 0 \). The same is valid for the functions
\begin{align}
F_1(\rho) &= \left(\frac{2}{5}\right)^{-1/2} \left[1 + \sum_{k=1}^{\infty} F_{1,k} \rho^{2k}\right], \\
F_2(\rho) &= 2 + \sum_{k=1}^{\infty} F_{2,k} \rho^{2k}
\end{align}
(7.17), (7.18)

There exist functions \(\Phi_i(w)\), \(i = 1, 2\), defined on \(\{|w| < \delta^2\} \cup (0, +\infty)\), satisfying
\[\Phi_i(\rho^2) = F_i(\rho),\] (7.19)
holomorphic in \(\{|w| < \delta^2\}\) and smooth on \((0, +\infty)\). In terms of these functions the solution (7.14) may written as
\[g = c\Phi_1(|\vec{x}|^2) \left\{-\Phi_2(|\vec{x}|^2) \sum_{i=1}^{7} dx^i \otimes dx^i + |\lambda_2|g^{(2)}\right\},\] (7.20)
where \(|\vec{x}|^2 = \sum_{i=1}^{7} (x_i)^2\). Clearly, (7.20) is a smooth metric on the manifold
\[R^7 \times M_2.\] (7.21)

The solution (7.20), (7.21) is an extension of the special solution defined for \(\tau \in (0, +\infty)\) to the semi-interval \(\tau \in (0, +\infty]\). It may be interpreted in two ways. First, we may say that the metric (7.20) describes an extension of the 4-dimensional \(R \times M_2\) cosmological solution to the case of 7-dimensional Euclidean time manifold. On the other hand it may be also considered as a spherically-symmetric (\(O(7)\)-symmetric) solution with a curved time manifold \(M_2\) (of negative curvature).

It seems likely that the procedure considered here may be also applied to the special solution with \(C_1 = \frac{\pi}{2}, \varepsilon_1 = -1, \varepsilon_2 = +1\) for \(N_1 = 6, N_2 = 3\) and for other special solutions in \((1 + 8 + 2)\)- and \((1 + 5 + 5)\)-models. We may also obtain the "Milne-like" solution, when \((M_1, g^{(1)})\) is the 6-dimensional Lobachevsky space \((H^6, g(H^6))\).

\section{Discussion}

In the considered paper we have integrated the vacuum Einstein equations for the model describing the evolution of two Einstein spaces \(M_1\) and \(M_2\) with dimensions \((N_1, N_2) = (6, 3), (5, 5), (8, 2)\). To our knowledge these are the first non-trivial cosmological solutions describing the evolution of more than one Einstein spaces of non-zero curvatures.

The Kasner-like behaviour near the singularity (for \(t_s \to +0\)) is investigated. The Kasner parameters (5.7), (5.8) are rational for all considered three cases. We may consider the following hypothesis: the Abel equation (2.20) may be integrated by methods described in [20, 21] (or by its extension) for the dimensions satisfying
\[R(N_1, N_2) \equiv \sqrt{N_1 N_2 (N_1 + N_2 - 1)} = m \in \mathbb{Z}.\] (8.1)
(m is integer). In this (and only in this case) the Kasner parameters (5.7), (5.8) are rational. 
(The relation (8.1) is satisfied for \((N_1, N_2) = (13, 13), (25, 3), (25, 25), (41, 41)\) etc).

Here we also received some special solutions to Einstein equations, i.e. "Milne-type" solutions for arbitrary \(n\) (see (4.2)), and two-parametric families for \(3 \leq n \leq 5\) (see (6.13)). We also obtained non-singular solutions with topology \(R^7 \times M_2\) (sec. 7). (It may be shown that the non-singular solutions with topology \(R^{N_1+1} \times M_2^{N_2}\) exist also for other dimensions \((N_1, N_2) = (5, 5), (8, 2)\).)

The considered in this paper solutions to the vacuum Einstein equations are defined on the manifold \(M\) of dimension \(D = \text{dim}M = 10, 11\). These solutions satisfy the equations of motion for \(D = 10\) supergravity of superstring origin and for \(D = 11, N = 1\) supergravity respectively. For certain manifolds \(M_1, M_2\) it is possible to use the obtained here solutions for generation of other classical solutions with non-zero matter fields (dilatonic, Kalb-Ramond etc) in \(D = 10, 11\) supergravities. (This may be done, for example, by using the duality transformations.)

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Fig. 1. The graphical representation for the function $2\Phi(\tau, -\delta, C_1)$ (3.12) for $\delta = +1$ and a) $C_1 > 0$, b) $C_1 = 0$, c) $C_1 < 0$, respectively.

Fig. 2. The graphical representation for the function $2\Phi(\tau, -\delta, C_1)$ (3.12) for $\delta = -1$ and a) $0 \leq C_1 < \frac{\pi}{2}$, b) $C_1 \geq \frac{\pi}{2}$.

Fig. 3. The trajectories on the plane of scale factors $a_i = a_i(\tau), i = 1, 2$, (corresponding to the solution (3.23)) with two negative curvatures ($\varepsilon_1 = \varepsilon_2 = -1$). Here a) $\tau \in (1, +\infty)$, for $C_1 \geq 0$; $\tau \in (1, \tau_3(C_1))$ for $C_1 < 0$; b) $\tau \in (\tau_2(C_1), 1)$.

Fig. 4. The trajectories on the plane of scale factors $a_i = a_i(\tau), i = 1, 2$, for the case $\varepsilon_1 = +1, \varepsilon_2 = -1$, (6-dimensional space has positive curvature and 3-dimensional space has negative curvature). Here a) $\tau \in (\tau_3(C_1), +\infty)$ for $C_1 < 0$; b) $\tau \in (\tau_1(C_1), \tau_2(C_1))$ for $C_1 \geq 0$.

Fig. 5. The trajectories on the plane of scale factors $a_i = a_i(\tau), i = 1, 2$, for the case $\varepsilon_2 = +1$ (3-dimensional space has positive curvature) and a) $\varepsilon_1 = +1$ (6-dimensional space has positive curvature), $\tau \in (-\infty, \tau_1(C_1))$ for $C_1 > \frac{\pi}{2}$; b) $\varepsilon_1 = -1$ (6-dimensional space has negative curvature), $\tau \in (-\infty, +\infty)$ for $C_1 \leq \frac{\pi}{2}$; $\tau \in (\tau_1(C_1), +\infty)$ for $C_1 > \frac{\pi}{2}$. 
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