1 Abstract

In this talk I will first give a short discussion of some lattice results for QCD at finite temperature. I will then describe in some detail the technique of dimensional reduction, which in principle is a powerful technique to obtain results on the long distance properties of the quark-gluon plasma. Finally I will describe some new results, which test the technique in a simpler model, namely three dimensional gauge theory.

2 Introduction

The physical challenges for lattice computations of high temperature QCD are to obtain information, which can be of use for the planning and analysis of recent experiments, as well as to give theoretical insight into the non perturbative properties of QCD. Some of the more important goals are the determination of the value of the transition temperature in physical units, the equation of state, the order or even the existence of a phase transition to a high temperature quark-gluon plasma, and the long distance properties of the plasma phase.

A simple picture of the finite temperature properties of QCD, as first discussed by Cabibbo and Parisi, predicts a low temperature confined phase, and a high temperature phase which is essentially an ideal quark-gluon gas. The transition temperature is expected to be of the order of the Hagedorn temperature, i.e. around 140 MeV. Up to now only lattice calculations can give more quantitative information. Reviews of the lattice results can be found in the proceedings of the yearly lattice conferences, the most recent published review is reference. Here I will only emphasize some of the results on the quantities mentioned above.

In the pure gluon theory it has been shown that the thermodynamics can be essentially completely solved, by continuum extrapolation of the lattice re-
sults. In this way, one can estimate the ratio between the critical temperature and the square root of the string tension to be 0.630(5). A similar extrapolation using a different lattice action gives 0.650(5). The difference seems to come from the measurement of the string tension, which is difficult to estimate precisely, and where systematic errors come from the uncertainties in the non asymptotic terms in the spatial distance. Still, the estimates agree within 3%.

Using a value $\sqrt{\sigma} = 425$ MeV, one obtains a rather high critical temperature of around 270 MeV.

Unfortunately, as far as the order of the transition and the critical temperature goes, the pure gluon theory is not a good approximation to full QCD with dynamical quarks. Direct lattice calculations with systematic continuum extrapolation in this case are far too time consuming for present computers, and would, with the methods currently available, need computing power in the range of 10 to 100 Teraflops. From the present results one expects the critical temperature e.g. in ratio to the string tension to be considerably lower. In physical units one estimates the critical temperature to be in the range of 150 - 190 MeV, where the latest results are near 170 MeV. It is difficult to estimate the systematic errors. The order of the transition may be very sensitive to the strange quark. A crossover as a shadow of a nearby second or first order transition is favoured by the data, and also by general universality arguments.

In the pure gluon theory, the pressure and energy density as a function of temperature can also be extrapolated to the continuum limit, and also here results from different calculations agree within about 3%. The main message from these calculations is that the free gluon gas limit is approached rather slowly, and that even at temperatures of two times the critical temperature, which corresponds to energy densities at least 16 times the energy density at the transition, there are 15 - 20% corrections to the ideal gas limit. Qualitatively, one observes a similar behaviour in full QCD. In this case a continuum extrapolation has not been attempted, and one is restricted to results on rather coarse lattices. One should remark that the experiments planned at Brookhaven and even LHC will operate in a range of temperatures, where the deviations from the ideal gas behaviour should be substantial. A better understanding of the non perturbative effects is certainly needed. Resummation techniques of the perturbative series, essentially introducing a gluon mass, do agree with the lattice results for sufficiently high temperature (from about $2T_c$). The systematic errors in this procedure are also not very well known, and further investigation of the long distance properties of the plasma is certainly needed. Dimensional reduction is a technique, which is well suited for studying these long range properties. In the following I will give a short
review of the general technique, and report on some recent results, which test
the approach in detail.

3 Dimensional Reduction

Dimensional reduction is in principle a powerful technique that was introduced
in the early eighties as a mean to treat high temperature field theories. In
the Euclidean formulation such theories are defined in a volume with one
compact dimension of extent $1/T$. As the temperature $T$ becomes large, one
may expect that the non-static modes in the temperature direction can be
neglected, and that one is left with a theory in one dimension less $10,11$. This
naive reduction is, however, only exact in the classical limit. In general, one
has to integrate over the nonstatic modes to obtain the effective action $12,13,14$.

As was shown in Refs 12, the effective action for the long distance phenomena
will, however, only contain a limited number of local terms at sufficiently high
temperature, because the integral over the nonstatic modes does not contain
infrared divergencies. Furthermore, the coefficients can be determined from
a perturbative expansion of the integral over the non-static modes. This
effective action is then expected to describe correctly the infrared behaviour
of the full theory. It has been successfully applied to the electroweak phase transition $15$ in QCD the action is of the form

$$S = \frac{1}{g^2} \int_0^{1/T} d\tau \int d^4x \left[ F_{\mu\nu}(\tau, \vec{x}) + \bar{\psi} D\psi \right]$$

where possible ghost and gauge fixing terms have been suppressed. It is
convenient to analyse the theory in the static Landau gauge, defined by

$$A_0(\tau, \vec{x}) = A_0(\vec{x})$$
$$\int_0^{1/T} d\tau \partial_i A_i = 0.$$  

The bosonic and fermionic fields are periodic and antiperiodic respectively in
the temperature direction, and the corresponding Matsubara frequencies are

$$\omega_n = 2\pi n T \quad n = 0, \pm 1, \pm 2, ... \quad \text{Bosons}$$
$$\omega_n = (2n + 1)\pi T \quad \text{Fermions}$$

Note that all modes become infinitely heavy in the high temperature limit,
apart from the static modes in the bosonic case. If one neglects the non static
modes, or at least integrate them out perturbatively, a much simpler theory
describes the long distance properties of the plasma. At the tree level, one is left with the effective action in one less dimension

\[ S \rightarrow \frac{1}{g^2 T} \int d^d x \left[ F_{ij}^{st} \theta_2 (\vec{x}) + (D_i A_0^s (\vec{x}))^2 \right]. \tag{5} \]

In the quantum theory, one should integrate over the non static modes. This gives in principle a very complicated non local effective action. If one observes that the integration over these modes is infrared finite, one may, however, develop the action in a series of local terms, which describe the leading behaviour of correlation functions at large T and small momenta.

If we introduce a field \( \phi \) instead of \( A^s_0 \)

\[ A_0^s (\vec{x}) = \sqrt{T} \phi (\vec{x}) \tag{6} \]

we can see essentially from power counting and symmetry arguments that the dominating terms in perturbation theory at high temperature and long distances are the first orders in a polynomial in \( \phi \):

\[
\begin{align*}
3 + 1 & \rightarrow 3 & 2 + 1 & \rightarrow 2 \\
g^2 T^2 \phi^2 & & g_3^2 T \phi^2 \\
g^4 T \phi^4 & & g_3^4 \phi^4 \\
g^6 \phi^6 & & g_3^6 / T \phi^6
\end{align*}
\]

The numerical value of the coefficients can be calculated in perturbation theory.

Although this is a very nice result, which means essentially that e.g. full QCD in 3+1 dimensions is replaced by a purely bosonic theory in three dimensions, the range of validity of the approximation is not yet known, although several results indicate that it may be valid down to a temperature of about two times the critical temperature.

In a recent work we have investigated how dimensional reduction works, both for correlations between Polyakov loops, which are related to chromoelectric screening and for spatial Wilson loops, which are related to the chromomagnetic sector. We have performed this investigation on a simpler model, three dimensional \( SU(3) \) gauge theory. Thereby we can obtain a high statistical accuracy in the comparison. The reduced model is a two dimensional
adjoint Higgs model. This model is interesting in its own right, unfortunately it has not been analytically solved.

Three dimensional gauge theories have several similarities with the full four dimensional case, as e.g. confinement and asymptotic freedom. They are, however, superrenormalizable, and have infrared singularities, which are stronger than in four dimensions. Even the leading perturbative definition of the Debye screening mass is not well defined, because of a infrared logarithmic singularity.

I will not go into the technical details of the reduction, but instead I refer to our publication [9]. We have simulated the three dimensional theory in a wide range of temperature, for lattices with four steps in the temperature direction, using the Wilson action [21]. We also calculated in the one loop approximation, the effective action in the lattice regularization. Again the action can be written in terms of lattice variables. In the scaling limit we get the continuum form

$$L_{\text{eff}} = \frac{1}{4} \sum_{c=1}^{8} F_{ij}^c F_{ij}^c + \text{tr} [D_i \phi]^2 + \frac{g_2^2}{32\pi} \left( \frac{g_2}{T} \right)^2 \text{tr} \phi^4 + L_{\text{CT}},$$

with its canonical dimension one in energy, sets the scale. The non kinetic quadratic term is the counterterm $L_{\text{CT}}$, suited to a lattice UV regularization with spacing $a$. As is explained in our paper [9], it is very important to include properly the regularized divergence in the $\phi^2$ term.

On the lattice, the three dimensional model is defined on a $L_0 \times L_2^2$ lattice and the two dimensional model on a $L_0^2$ lattice. In the reduced model, the spatial vector fields $A_i$ are replaced by the group elements $U_i$, as usual, and the space part of the kinetic term is replaced by the Wilson action. The field $\phi$ is kept as a scalar field defined on the sites of the lattice, and the appropriate covariant definition of the covariant derivative is used. Thereby the explicit $Z_3$ symmetry of the original three dimensional theory is lost. It can be restored formally, e.g. by substituting $\phi$ by a group element. This substitution is, however, not uniquely determined by perturbation theory. One proposal has been made recently by Pisarski [22]. We have investigated some similar $Z_3$ symmetric models. The hope would be to get a description,
Figure 1. Physical screening masses $M_S$ in units of $g_3 \sqrt{T}$ versus $g_3^2 / T$, in $(2 + 1)D$ (black points) and 2D (squares). Also shown are the masses obtained with the numerical value of the two point coupling (triangles) instead of its scaling form, and with the tree level reduced action (diamonds).

which is valid all the way down to the phase transition.

To the physical variables $g_3$ and $T$ correspond the lattice parameters $L_0 = 1 / a T$ and $\beta_3 = 6 / a g_3^2$. It follows that as $a \rightarrow 0$, scaling (constant physics) corresponds to

$$L_0 \rightarrow \infty, \quad \beta_3 \rightarrow \infty, \quad \tau \equiv \frac{\beta_3}{6 L_0} = \frac{T}{g_3^2} = \text{constant}. \quad (9)$$

The dimensionless quantity $\tau$ thus measures temperature in units of the scale $g_3^2$. High temperature means large $\tau$ values. From numerical simulations we obtain that $T_c / g_3^2 \approx 0.61 [4]$. Also, on the lattice, $L_s$ must be kept much larger than the largest spatial correlation length in lattice units.

The two quantities, which we have studied with precision are the corre-
lation between Polykov loops and the spatial string tension. In Fig. 1 is shown the screening mass, as defined from the inverse correlation length of the Polyakov loops. The details of the fit are discussed in the article. It is important to mention that the best fit is obtained by a simple pole, although in resummed perturbation theory one might expect a cut, because two gluons have to be exchanged, as the Polyakov loops are colour singlets. As can be seen from the figure the agreement between the reduced and the full model is very good down to about \(1.5T_c\). Going to the critical temperature, the mass in the full model goes to zero, while in the reduced model, which does not have the corresponding phase transition, it stays finite. The data for the full model comes from lattice calculations with \(L_0 = 4\). We have, however, investigated scale breaking effects in the reduced model, and found them to be small.

Another important message is that the terms coming from the quantum effects in the reduction are essential for the agreement between the reduced and the full model. The naively reduced model gives masses, which are more than 50\% larger.

The spatial Wilson loop in three dimensions is not a static operator, and thus not predicted to assume the same value at high temperature and in the reduced two dimensional model. However, there is a finite string tension in both models, and one may expect that the non static corrections to this observable are small. We have also compared the string tension in the full model to the string tension in two dimensional \(SU(3)\) theory, which can be easily calculated analytically. In fact one expects

\[
\frac{\sigma_0^2}{g_5^2T} = \frac{2}{3} + \frac{7}{36} \frac{g_5^2}{T} (aT)^2 + \mathcal{O}\left[\frac{(g_5^2)^2}{T} (aT)^4\right],
\]

showing that \(\sigma_0^2\) scales as \(2g_5^2T/3\), up to scaling violations at finite \(T\) of order \((aT)^2 = 1/L_0^2\), where \(\sigma_0^2\) is the string tension in two dimensional \(SU(3)\). In Fig. 2 is shown the comparison between the spatial string tension in the full and the reduced model, as well as the analytical result for two dimensional \(SU(3)\). One should note that this is done only for a fixed value of \(L_0 = 4\). In this case we have not estimated scale breaking effects.

4 Conclusions

Although the thermodynamics of pure gluon theory is essentially solved numerically, the same is not true for full QCD. Furthermore, the long distance properties are even more difficult to extract. Dimensional reduction gives potentially a simpler and systematic alternative to direct QCD calculations at
high temperature and large distances. We have found that in 2 + 1 dimensional $SU(3)$ it is a very good approximation down to about $1.5 T_c$, but the dynamics of the phase transition is not captured by the present approach. It is, however, important to notice that we can describe both chromoelectric and chromomagnetic quantities consistently, without extra parameters, which has not been shown in the 3 + 1 dimensional case. Systematic extensions of the method would be very important, in particular since the potential gain in computer time is enormous.

Acknowledgments

I want to thank my collaborators P. Bialas, C. Legeland, A. Morel, K. Petrov and T. Reisz for a very stimulating collaboration, which led to the results
described here. I also thank Xiang-Qian Luo and Eric Gregory for a very interesting conference.

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