PSEUDO-ROTATIONS AND STEENROD SQUARES

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Abstract. In this note we prove that if a closed monotone symplectic manifold $M$ of dimension $2n$, satisfying a homological condition that holds in particular when the minimal Chern number is $N > n$, admits a Hamiltonian pseudo-rotation, then the quantum Steenrod square of the point class must be deformed. This gives restrictions on the existence of pseudo-rotations. Our methods rest on previous work of the author, Zhao, and Wilkins, going back to the equivariant pair-of-pants product-isomorphism of Seidel.

1. Introduction

This paper deduces obstructions, in terms of pseudo-holomorphic curves, to the existence of Hamiltonian pseudo-rotations. The latter are special Hamiltonian diffeomorphisms of symplectic manifolds, characterized by having the minimal possible number of periodic points, of all integer periods.

The classical notion of pseudo-rotations of the two-sphere, or the two-disk, has appeared in the work of Anosov and Katok [1] (see also Fayad and Katok [10]) and was further investigated extensively in the field of conservative dynamics. Indeed, the simplest example of a pseudo-rotation is an irrational point in the Hamiltonian $S^1 = \mathbb{R}/\mathbb{Z}$ action on $S^2$ with Hamiltonian (a constant multiple of) the standard height function. However, [1, 10] construct, by means of the conjugation method, examples of pseudo-rotations of $S^2$ with different dynamical properties than those of such irrational rotations. For example, they admit precisely three ergodic measures: two fixed points, and the area measure.

Recent years saw a renewed interest in pseudo-rotations considered from the point of view of symplectic rigidity phenomena. For instance, Barney Bramham proved in [3, 4] that all Hamiltonian pseudo-rotations of the closed disk can be $C^0$ approximated by a sequence of conjugates of rational rotations, and that they are $C^0$ rigid in a suitable sense, provided that their rotation number is sufficiently Liouvillian. In the recent seminal paper [15] by Ginzburg and Gürel, the $C^0$-rigidity result of Bramham, as well as other results regarding the dynamics of Hamiltonian pseudo-rotations, were established for complex projective spaces of all dimensions.

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This paper, as well as [8], takes a different point of view, considering pseudo-rotations to be strong counter-examples to the Conley conjecture. From this perspective, a conjecture of Chance and McDuff, arising from [18], asserts that the existence of such counter-examples, and hence that of pseudo-rotations, must imply the existence of non-trivial algebraic counts of pseudo-holomorphic spheres in the manifold. Here we provide an instance of such an implication, ruling out in particular the existence of pseudo-rotations on a closed monotone symplectic manifold of dimension $2n$ with minimal Chern number $n+2$ or greater. Further extensions of our results have very recently appeared in [7, 27].

2. Setup and main results

In this paper, unless otherwise specified, we work with a closed monotone symplectic manifold $(M, \omega)$ of dimension $2n$, and rescale the symplectic form so that $[\omega] = \kappa \cdot c_1(TM)$ on the image of the Hurewicz map $\pi_2(M) \to H_2(M; \mathbb{Z})$ for $\kappa = 2$. Recall that the minimal Chern number of $(M, \omega)$ is the index

$$N = N_M = \left\{ \lambda : \text{image} \{ c_1(TM) : \pi_2(M) \to \mathbb{Z} \} \right\}. $$

For a Hamiltonian diffeomorphism $\phi \in \text{Ham}(M, \omega)$, we denote by $\text{Fix}(\phi)$ the set of contractible fixed points of $\phi$, and by $x^{(k)}$ for $x \in \text{Fix}(\phi)$ its image under the inclusion $\text{Fix}(\phi) \subset \text{Fix}(\phi^k)$. Contractible means that the homotopy class of the path $a(x, H) = (\phi^t_H(x))$ for a Hamiltonian $H \in C^\infty([0,1] \times M, \mathbb{R})$ generating $\phi$ as the time-one map $\phi^1_H = \phi$ of its Hamiltonian flow, is trivial. This class does not depend on the choice of Hamiltonian by a classical argument in Floer theory.

We say that a Hamiltonian diffeomorphism $\phi \in \text{Ham}(M, \omega)$ is a $\mathbb{F}_2$ Hamiltonian pseudo-rotation if

(i) It is perfect, that is for all iterations $k \geq 1$ of $\phi$, $\text{Fix}(\phi^k) = \text{Fix}(\phi)$ is finite. In other words, $\phi$ admits no simple periodic orbits of order $k > 1$.

(ii) For each $x \in \text{Fix}(\phi)$, the dimension of the local Floer homology of $\phi$ at $x$ satisfies $\dim_k HF^\text{loc}(\phi, x) \geq 1$, and furthermore, $\dim_k HF^\text{loc}(\phi^k, x^{(k)}) = \dim_k H_* (M)$ for all $k \geq 1$.

**Remark 1.** We observe that a perfect Hamiltonian diffeomorphism necessarily has no symplectically degenerate maxima (see [16]). Furthermore, if all the points in $\text{Fix}(\phi^k)$ are non-degenerate, for all $k \geq 1$, then condition (ii) is automatically satisfied, and all iterations are admissible, that is $\lambda^k \neq 1$ for all eigenvalues $\lambda \neq 1$ of $D(\phi)_x$. Furthermore, by the Smith inequality in local Floer homology [6, 28], conditions (i) and (iii) imply, for iterations of the form $k = 2^m$, the stronger statement that for all $x \in \text{Fix}(\phi)$, $\dim_k HF^\text{loc}(\phi^k, x^{(k)}) = \dim_k HF^\text{loc}(\phi, x)$. Moreover, [26, Theorem A] suggests that when a Hamiltonian diffeomorphism has a finite number of periodic points, then a condition like (iii) should be satisfied. Showing this would bridge the gap between the initial Chance–McDuff conjecture (see for example [16]) and the main result of this note, Theorem A. Finally, we include condition (ii) for compatibility with the literature: we do not use it below. We refer to [15] for further discussion of dynamics of Hamiltonian pseudo-rotations in higher dimensions.
We call a symplectic manifold strongly uniruled if there exists a non-trivial three point genus-zero Gromov–Witten invariant \( \langle [pt], a, b \rangle_\beta \), for \( \beta \in H_2(M, \mathbb{Z}) \sim [0]. \) By [18, Lemma 2.1], \( (M, \omega) \) is not strongly uniruled if and only if the \( \Lambda \)-linear subspace

\[
\mathcal{D}_- = H_{*-2n}(M) \otimes \Lambda \subset QH(M, \Lambda),
\]

where \( \Lambda \) is the minimal Novikov field of \( (M, \omega) \), with quantum variable \( q \) of degree \((-2N)\), is an ideal in the quantum homology ring \( QH(M, \Lambda) \). Recall that the quantum product in \( QH(M, \Lambda) \) is a deformation of the intersection product on homology given by three-point genus-zero Gromov–Witten invariants. Alternatively \( [pt] * r = 0 \) for all \( r \in \mathcal{D}_- \). Note that in this case, \( [pt] * [pt] = 0 \) in particular.

A generally different stronger notion than \( [pt] * [pt] = 0 \) is that the quantum Steenrod square \( \mathcal{D} (\langle pt \rangle) \), defined in [30], of the point class satisfy

\[
\mathcal{D} (\langle pt \rangle) = \hbar^{2n} [pt].
\]

This corresponds to there being no quantum corrections when passing from the classical Steenrod square of the point class \( M \) to its quantum version. In this case we say that \( M \) is not \( \mathbb{Z} / (2) \)-Steenrod uniruled.

**Remark 2.** Observe that when \( (M, \omega) \) is \( \mathbb{Z} / (2) \)-Steenrod uniruled, then by a Gromov compactness argument there exists a \( J \)-holomorphic curve through each point of \( M \). Furthermore, recall from [31] that setting the quantum variable to be of cohomological degree \( 2N \), \( \hbar \) to be of degree 1, and considering cohomological degree on the homology classes, \( \mathcal{D} (\langle pt \rangle) \) must be of degree \( 2 \deg([pt]) = 4n \). Hence if \( N > 2n = \dim(M) \), then \( (M, \omega) \) is automatically not \( \mathbb{Z} / (2) \)-Steenrod uniruled. By the same token, if \( N = 2n \), then being \( \mathbb{Z} / (2) \)-Steenrod uniruled is equivalent to \( [pt] * [pt] \neq 0 \). In fact, in this case \( [pt] * [pt] = q[M] \). We note that while the above choice of degrees was convenient for this remark, throughout the paper we work with homology and use different conventions for degrees.

**Remark 3.** The main result of [25] implies that if \( N > n \), and \( [\omega] \) lies in the lattice \( H^2(M, \mathbb{Z}) / \text{torsion} \subset H^2(M, \mathbb{R}) \), then \( (M, \omega) \) being \( \mathbb{Z} / (2) \)-Steenrod uniruled implies that \( (M, \omega) \) is strongly uniruled. In fact, in this case there exists a Gromov–Witten invariant

\[
\langle [pt], [pt], D \rangle_\beta
\]

that does not vanish modulo 2 for a suitable divisor class \( D \in H_{2n-2}(M; \mathbb{Z}) \). Note that when this holds, by a degree count, we obtain \( N = n + 1 \). Therefore Theorem A implies that there is no \( f_2 \) Hamiltonian pseudo-rotation for \( N > n + 1 \). Under the additional assumption that \( \langle [\omega]^n, [M] \rangle \) is odd, one may prove the above statement by a straightforward adaptation of the proof of [30, Lemma 6.1].

Finally, we say that \( (M, \omega) \) satisfies the Poincaré duality property, if for all \( \tilde{\phi} \in \text{Ham}(M, \omega) \), the following Poincaré duality identity of Hamiltonian spectral invariants (defined in Section 3.2 below) holds:

\[
c([M], \tilde{\phi}^{-1}) = -c([pt], \tilde{\phi}).
\]
It is well-known (see [9]) that \((M, \omega)\) with \(N > n\) satisfy this property. The main result of this note is the following.

**Theorem A.** Let \((M, \omega)\) be a closed monotone symplectic manifold satisfying the Poincaré duality property, and admitting an \(\mathbb{F}_2\) Hamiltonian pseudo-rotation \(\phi\). Then \((M, \omega)\) is \(\mathbb{Z}/(2)\)-Steenrod uniruled.

In view of Remarks 2 and 3 above we conclude the following result pertaining to the Chance–McDuff conjecture.

**Corollary 4.** Let \((M, \omega)\) be a closed monotone symplectic manifold with \(N > n\). If \(N > n + 1\) then \((M, \omega)\) does not admit \(\mathbb{F}_2\) Hamiltonian pseudo-rotations. If \(N = n + 1\), and \((M, \omega)\) admits an \(\mathbb{F}_2\) Hamiltonian pseudo-rotation, then \((M, \omega)\) satisfies \([pt] \ast [pt] \neq 0\), and in particular it is strongly uniruled.

**Remark 5.** The author was made aware that new relations between pseudo-rotations and holomorphic curves were also found in recent work of Çineli, Ginzburg, and Gürel [8].

**Remark 6.** The symplectic manifold \((\mathbb{C}P^n, \omega_{st})\) has \(N = n + 1\), and verifies the hypothesis and the conclusion of the theorem separately. Formally speaking, this result seems to be generally new even when \(\phi\) comes as the irrational rotation with respect to a Hamiltonian \(S^1\)-action with isolated fixed points (however it does not seem to give new examples in that case: see [17, 21, 5]). We remark that by a result of McDuff [18] all Hamiltonian \(S^1\)-manifolds are uniruled, the latter being defined with \(m\)-point genus 0 Gromov–Witten invariants with arbitrary \(m \geq 3\).

The strategy of the proof of the main result is the direct comparison between the following two results. First, the following Lusternik–Shnirelman type result was shown in [16]. Define for \(a \in \text{QH}(M)\), \(\tilde{\phi} \in \text{Ham}(M, \omega)\) the asymptotic spectral invariant by

\[
\overline{c}(a, \tilde{\phi}) = \lim_{k \to \infty} \frac{1}{k} c(a, \tilde{\phi}^k).
\]

Recall that a symplectic manifold is called *rational* if the period group of the symplectic form is a discrete subgroup of \(\mathbb{R}\).

**Theorem B.** Let \((M, \omega)\) be rational, and \(k\) be a ground field. Suppose \(\phi\) has isolated periodic points of each period, none of which is a symplectically degenerate maximum. Then for each \(k \geq 1\), and lift \(\tilde{\phi}\) of \(\phi\) to \(\text{Ham}(M, \omega)\),

\[
\frac{1}{k} c([M], \tilde{\phi}^k) > \overline{c}([M], \tilde{\phi}).
\]
We observe that in particular, when \( \phi \) is perfect, by [12, Theorem 1.18], none of the fixed points of \( \phi \) are symplectically degenerate maxima, and hence Theorem B applies. Hence, in the case when \( (M, \omega) \) satisfies the Poincaré duality property, we obtain by applying Theorem B to \( \tilde{\phi}^{-1} \), for \( \phi \) perfect, that

\[
\frac{1}{k} c([pt], \tilde{\phi}^k) < \overline{c}([pt], \tilde{\phi}).
\]

In particular there exists \( m \geq 1 \), such that for \( \tilde{\psi} = \tilde{\phi}^{2m} \),

\[
(1) \quad c([pt], \tilde{\psi}^2) > 2 \cdot c([pt], \tilde{\psi}).
\]

Second, we prove below the following statement:

**Theorem C.** Let \( \psi \) be an \( \mathbb{F}_2 \) Hamiltonian pseudo-rotation on \( (M, \omega) \) that is not \( \mathbb{Z}/(2) \)-Steenrod uniruled. Then

\[
(2) \quad c([pt], \tilde{\psi}^2) \leq 2 \cdot c([pt], \tilde{\psi})
\]

for each \( \tilde{\psi} \in \widetilde{\text{Ham}}(M, \omega) \) covering \( \psi \).

The proof of Theorem A is now the combination of (1) and (2).

### 3. Preliminaries

We present general preliminaries on Hamiltonian Floer homology and spectral invariants, to be applied in two settings below: usual and equivariant. To keep the paper short, we refer to [24, 30, 31, 28, 26] for preliminaries related to equivariant Floer homology, related natural operations and structures, and to [20] for further details on Hamiltonian Floer homology.

We denote by \( \mathcal{H} \subset C^\infty([0,1] \times M, \mathbb{R}) \) the space of time-dependent Hamiltonians on \( M \), that vanish near 0 and 1, where \( H_t(-) = H(t, -) \) is normalized to have zero mean with respect to \( \omega^n \). We shall consider \( H \in \mathcal{H} \) as a 1-periodic function on \( \mathbb{R} \times M \) in the \( \mathbb{R} \)-coordinate. The time-one maps of isotopies \( \{ \phi_t \}_{t \in [0,1]} \) generated by time-dependent vector fields \( X_t \), such that \( \iota_{X_t} \omega = -d(H_t) \), are called Hamiltonian diffeomorphisms and form the group \( \text{Ham}(M, \omega) \). For \( H \in \mathcal{H} \) we set \( \phi_k = \phi_{Ht}^k \). If the flow of \( H \) generates \( \tilde{\phi} \in \tilde{\text{Ham}}(M, \omega) \), then \( \phi_k \) generates \( \tilde{\phi}^k \).

Finally, let \( \mathcal{J}(M, \omega) \) be the space of \( \omega \)-compatible almost complex structures on \( M \).
3.1. **Hamiltonian Floer homology.** Consider $H \in \mathcal{H}$. Let $\mathcal{L}_{pt} M$ be the space of contractible loops in $M$. Let $c_M : \pi_1(\mathcal{L}_{pt} M) \to 2N_M \cdot \mathbb{Z}$ be the surjection given by $c_M(A) = 2 \langle c_1(M, \omega), A \rangle$, viewing $A$ as an element of $H_2(M; \mathbb{Z})$. Let $\tilde{\mathcal{L}}_{pt}^{\text{min}} M = \mathcal{L}_{pt} \times c_M (2N_M \cdot \mathbb{Z})$ be the cover of $\mathcal{L}_{pt} M$ associated to $c_M$. The elements of $\tilde{\mathcal{L}}_{pt}^{\text{min}} M$ can be considered to be equivalence classes of pairs $(x, \tilde{x})$ of $x \in \mathcal{L}_{pt} M$ and its capping $\tilde{x} : \mathbb{D} \to M$, $\tilde{x}|_{\partial \mathbb{D}} = x$. Of course $x$ is determined by $\tilde{x}$. The symplectic action functional

$$\mathcal{A}_H : \tilde{\mathcal{L}}_{pt}^{\text{min}} M \to \mathbb{R}$$

is given by

$$\mathcal{A}_H(\tilde{x}) = \int_0^1 H(t, x(t)) - \int_\tilde{x} \omega,$$

that is well-defined by monotonicity: $|\omega| = \kappa \cdot c_M$. Assuming that $H$ is non-degenerate, that is the graph graph$(\phi^1_t H) = \{(\phi^1_t H(x), x) \mid x \in M\}$ intersects the diagonal $\Delta_M \subset M \times M$ transversely, the generators over the base field $\mathbb{K}$ of the Floer complex $CF(H; J)$ are the lifts $\tilde{\mathcal{O}}(H)$ to $\tilde{\mathcal{L}}_{pt}^{\text{min}} M$ of 1-periodic orbits $\mathcal{O}(H)$ of the Hamiltonian flow $\{\phi^1_t H\}_{t \in [0, 1]}$. These are the critical points of $\mathcal{A}_H$, and we denote by $\text{Spec}(H) = \mathcal{A}_H(\tilde{\mathcal{O}}(H))$ the set of its critical values. Choosing a generic time-dependent $\omega$-compatible almost complex structure $|J_t \in \mathcal{J}(M, \omega)|_{t \in [0, 1]}$, and writing the asymptotic boundary value problem on maps $u : \mathbb{R} \times S^1 \to M$ defined by the negative formal gradient on $\mathcal{L}_{pt} M$ of $\mathcal{A}_H$, the count of isolated solutions with signs determined by a suitable orientation scheme, modulo $\mathbb{R}$-translation, gives a differential $d_{H,J}$ on the complex $CF(H; J)$, $d_{H,J}^2 = 0$. This complex is graded by the Conley–Zehnder index $CZ(H, \bar{x})$ [22, 23]. The Conley–Zehnder index has the property that the action of the generator $A = 2N_M$ of $2N_M \cdot \mathbb{Z}$ has the effect $CZ(H, \bar{x} \# A) = CZ(H, \bar{x}) - 2N_M$, and it is normalized to be equal to $n$ at a maximum of a small autonomous Morse Hamiltonian. Its homology $HF_* (H)$ does not depend on the generic choice of $J$. Moreover, considering generic families interpolating between different Hamiltonians $H, H'$ and writing the Floer continuation map, where the negative gradient depends on the $\mathbb{R}$-coordinate, we obtain that $HF_* (H)$ in fact does not depend on $H$ either. While $CF_*(H, J)$ is finite-dimensional in each degree, it is worthwhile to consider its completion in the direction of decreasing action. In this case it becomes a free graded module of finite rank over the Novikov field

$$\Lambda_{\mathbb{K}} = \Lambda_{M, \text{min}, \mathbb{K}} = \mathbb{K}[q^{-1}, q]$$

with $q$ being a variable of degree $(-2N_M)$. This field carries a non-Archimedean valuation $\nu : \Lambda_{\mathbb{K}} \to \mathbb{R} \cup \{+\infty\}$ given by $\nu(0) = \infty$, and

$$\nu(\sum a_j q^j) = j_0 \cdot A_M,$$

where $A_M = \kappa \cdot N_M$, and $j_0 = \min\{j \mid a_j \neq 0\}$. It satisfies the properties:

(i) $\nu(x) = +\infty$ if and only if $x = 0$,

(ii) $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in \Lambda_{\mathbb{K}}$,

(iii) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$, for all $x, y \in \Lambda_{\mathbb{K}}$. 


Moreover, we extend $\mathcal{A}_H$ to a non-Archimedean filtration on $CF(H, J)$ by
\begin{equation}
\mathcal{A}_H\{\sum \lambda_j \bar{x}_j\} = \max\{-\nu(\lambda_j) + \mathcal{A}_H(\bar{x}_j)\},
\end{equation}
for a $\Lambda_{\mathbb{K}}$-basis of $CF(H, J)$ consisting of capped orbits $\bar{x}_j \in \tilde{\mathcal{O}}(H)$.

Recall, following [29], that for a field $\Lambda$ with non-Archimedean valuation $\nu$, a function $l : C \to \mathbb{R} \cup \{-\infty\}$ on a finite-dimensional $\Lambda$-module $C$ is called a non-Archimedean filtration (function), if it satisfies the following properties:

(i) $l(x) = -\infty$ if and only if $x = 0$,
(ii) $l(\lambda x) = l(x) - \nu(\lambda)$ for all $\lambda \in \Lambda, x \in C$,
(iii) $l(x + y) \leq \max\{l(x), l(y)\}$, for all $x, y \in C$.

We call a complex $(C, d)$ with $C$ a finite-dimensional $\Lambda$-module with a non-Archimedean filtration $l$ filtered if $l(dy) \leq l(y)$ for all chains $y$ and strict if $l(dy) < l(y)$ for all chains $y \neq 0$. It is straightforward to see that $CF(H, J)$ with $\mathcal{A}_H$ is a strict filtered complex over $\Lambda_{\mathbb{K}}$.

Furthermore, for $a \in \mathbb{R} \sim \text{Spec}(H)$ the $\mathbb{K}$-subspace $CF(H, J)^{<a}$ spanned by all generators $\bar{x}$ with $\mathcal{A}_H(\bar{x}) < a$ forms a subcomplex with respect to $d_{H, J}$, and its homology $HF^{-a}(H)^{<a}$ does not depend on $J$. Arguing up to $c$, one can show that a suitable continuation map sends $HF^a(H)^{<a}$ to $HF^a(H)^{<a+\mathcal{E}_a(H-H^a)}$, for
\begin{equation}
\mathcal{E}_a(F) = \int_0^1 \max_{M}(F_t) \, dt.
\end{equation}

It shall also be useful to define $\mathcal{E}_a(F) = \mathcal{E}_a(-F)$, $\mathcal{E}_a(F) = \mathcal{E}_a(F) + \mathcal{E}_a(-F)$. Moreover, for an admissible action window, that is an interval $I = (a, b)$ with $a, b \in \mathbb{R} \sim \text{Spec}(H)$ and $a < b$, we define the Floer homology of $H$ in this window $HF^a(H)^I$ as the homology of the quotient complex
\begin{equation}
CF^a(H)^I = CF^a(H)^{<b}/CF^a(H)^{<a}.
\end{equation}

Finally, one can show that for each $a \in \mathbb{R}$, $HF(H)^{<a}$ as well as $HF(H)^I$ for an admissible action window, depends only on the class $\bar{\phi}_H$ of the path $\{\phi_H^t\}_{t \in [0,1]}$ in the universal cover $\tilde{\text{Ham}}(M, \omega)$ of the Hamiltonian group of $M$.

When we wish to emphasize the Novikov field that we use in the definition of Floer homology, we use the notation $HF(H, \Lambda_{\mathbb{K}})^{<a}$, $HF(H, \Lambda_{\mathbb{K}})^I$, even though for a and $I$ finite, these homology groups are not modules over $\Lambda_{\mathbb{K}}$. We shall mostly work with $\mathbb{K} = \mathbb{F}_2$ and $\mathbb{K} = \mathcal{K} = \mathbb{F}_2[h^{-1}, h]]$, for a formal variable $h$, and write $\Lambda = \Lambda_{\mathbb{K}}$ if the choice of $\mathbb{K}$ is clear.

In the case when $H$ is degenerate, we consider a perturbation $\mathcal{D} = (H^H, J^H)$, with $H^H \in \mathcal{H}$, such that $H^\mathcal{D} = H^\#K^H$ is non-degenerate, and $J^H$ is generic with respect to $H^\mathcal{D}$, and define the complex $CF(H; \mathcal{D}) = CF(H^\mathcal{D}; J^H)$ generated by $\mathcal{D}(H; \mathcal{D}) = \mathcal{D}(H^{\mathcal{D}})$, and filtered by the action functional $\mathcal{A}_{H; \mathcal{D}} = \mathcal{A}_{H^{\mathcal{D}}}$. An admissible action window $I = (a, b)$ for $H$ remains admissible for all $K^H$ sufficiently $C^2$-small, and the associated homology groups $HF(H; \mathcal{D})^{I}$ are canonically isomorphic for all $K^H$ sufficiently $C^2$-small. Hence $HF(H)^I$ is defined as the colimit of the associated indiscrete groupoid.

\footnote{Note that it is not a module over $\Lambda_{\mathbb{K}}$.}
3.2. Spectral invariants. Given a filtered complex \((C, \mathcal{A})\), to each homology class \(\alpha \in H(C)\), denoting by \(H(C)^{<a} = H(C^{-a})\), \(C^{<a} = \mathcal{A}^{-1}(-\infty, a)\), we define a spectral invariant by
\[
c(\alpha, (C, \mathcal{A})) = \inf \{ a \in \mathbb{R} \mid \alpha \in \text{im} \{ H(C)^{<a} \rightarrow H(C) \} \} \in \mathbb{R} \cup \{-\infty\}.
\]
For \((C, \mathcal{A}) = (CF(H; \mathcal{D}), \mathcal{A}_{H; \mathcal{D}})\) we denote \(c(\alpha, H; \mathcal{D}) = c(\alpha, (C, \mathcal{A}))\). Furthermore, one can obtain classes \(\alpha\) in the Hamiltonian Floer homology by the PSS isomorphism. This lets us define spectral invariants by
\[
c(\alpha_M, H; \mathcal{D}) = c(\text{PSS}(\alpha_M), (CF(H; \mathcal{D}), \mathcal{A}_{H; \mathcal{D}})),
\]
for \(\alpha_M \in QH(M)\). From the definition it is clear that the spectral invariants do not depend on the almost complex structure term in \(\mathcal{D}\). Moreover, if \(H\) is non-degenerate, we may choose the Hamiltonian term in \(\mathcal{D}\) to vanish identically, and denote the resulting invariants by \(c(-, H)\). Moreover, by [2, Section 5.4] spectral invariants remain the same under extension of coefficients, hence below we do not have to specify the Novikov field \(\Lambda\) that we work over. Spectral invariants enjoy numerous useful properties that hold for rational symplectic manifolds, the relevant ones of which we summarize below:

(i) **spectrality**: for each \(\alpha_M \in QH(M) \sim \{0\}\), and \(H \in \mathcal{H}\),
\[
c(\alpha_M, H) \in \text{Spec}(H).
\]
(ii) **non-Archimedean property**: \(c(-, H; \mathcal{D})\) is a non-Archimedean filtration function on \(QH(M)\), as a module over the Novikov field \(\Lambda\) with its natural valuation.
(iii) **continuity**: for each \(\alpha_M \in QH(M) \sim \{0\}\), and \(F, G \in \mathcal{H}\),
\[
|c(\alpha_M, F) - c(\alpha_M, G)| \leq \mathcal{E}(F - G),
\]
(iv) **triangle inequality**: for each \(\alpha_M, \alpha'_M \in QH(M)\), and \(F, G \in \mathcal{H}\),
\[
c(\alpha_M \ast \alpha'_M, F \# G) \leq c(\alpha_M, F) + c(\alpha'_M, G),
\]
(v) **invariance**: \(c(\alpha_M, H)\) depends only on the element \(\tilde{\phi} = [\{\phi_H^t\}_{t \in [0,1]}]\) in the universal cover \(\tilde{\text{Ham}}(M, \omega)\) generated by \(H\).

We remark that by the continuity property, the spectral invariants are indeed defined for all \(H \in \mathcal{H}\) and all the properties above apply in this generality. Finally, by the invariance property, we shall consider the spectral invariants as functions on \(\tilde{\text{Ham}}(M, \omega)\), and shall sometimes denote for brevity by \(\mathcal{A}_H\), the action functional \(\mathcal{A}_H\) for a certain Hamiltonian generating \(\tilde{\phi}\).

Below, we will sometimes be using the ground field \(\mathcal{K} = \mathbb{F}_2[[h^{-1}, h]]\), for a formal variable \(h\), for the Novikov field. This is the field of fractions of the ring \(\mathcal{L} = \mathbb{F}_2[[h]]\). In this case, we require the following observation.

**Lemma 7.** Let \(F \in \mathcal{H}\) be a non-degenerate Hamiltonian. Consider elements \(P \in CF(F; J; \Lambda_{\mathcal{F}_J}) \subset CF(F; J; \Lambda_{\mathcal{K}})\), and \(Q \in CF(F; J; \Lambda_{\mathcal{F}_J}) \subset CF(F; J; \Lambda_{\mathcal{K}})\). Then
\[
\mathcal{A}_F(P + hQ) = \max \{ \mathcal{A}_F(P), \mathcal{A}_F(Q) \}.
\]
The proof of this lemma is essentially immediate, since writing \( P = \sum \lambda_j \overline{x}_j \) with \( \lambda_j \in \Lambda_{\mathcal{F}_2} \) in the basis \( \{ \overline{x}_j \} \) and \( hQ = \sum h\mu_j \overline{x}_j \) with \( \mu_j \in \Lambda_{\mathcal{L}} \), we have
\[
P + hQ = \sum (\lambda_j + h\mu_j) \overline{x}_j.
\]
The lemma now follows from (3) and
\[
\nu(\lambda_j + h\mu_j) = \min \{ \nu(\lambda_j), \nu(\mu_j) \}
\]
for each \( j \). We prove the latter statement for a given fixed \( j \). Writing \( \lambda_j = \sum a_i q^{\sigma_i}, \mu_j = \sum b_i q^{\nu_i}, b_i \in \mathcal{L} \) with the sets \( \{ \sigma_i > c \mid a_i \neq 0 \} \), \( \{ \sigma_i > c \mid b_i \neq 0 \} \), finite for all \( c \in \mathbb{R} \), we get
\[
\lambda_j + h\mu_j = \sum (a_i + h b_i) q^{\sigma_i}.
\]
As \( a_i + h b_i = 0 \) if and only if \( a_i = 0 \) and \( b_i = 0 \), because \( a_i \in \mathbb{F}_2, h b_i \in h \mathcal{L} \), the conclusion now follows.

3.3. Homotopy-canonical filtered complexes and local Floer homology. The theme of this section is that the situation of filtered Hamiltonian Floer homology of \( H \in \mathcal{H} \) with \( \text{Fix}(\phi_H^i) \) finite is very similar to that of the non-degenerate case, once we allow a finite-dimensional graded vector space of generators to be supported at each \( x \in \text{Fix}(\phi_H^i) \). We first prove the following result, which is a chain-level enhancement of [15, Lemma 2.2]. Its proof below relies on homological perturbation techniques and constitutes a Novikov-field version of the canonical \( \Lambda^0 \)-complexes from [26]. While we formulate it for monotone symplectic manifolds to simplify notation, it extends almost verbatim to the case of rational symplectic manifolds.

**Theorem D.** Let \((M, \omega)\) be a closed monotone symplectic manifold. Consider the class \( \tilde{\phi} \in \tilde{\text{Ham}}(M, \omega) \) of the Hamiltonian flow \( \{ \phi_H^t \}_{t \in [0,1]} \) of \( H \in \mathcal{H} \), with \( \text{Fix}(\phi_H^1) \) finite. For a ground field \( \mathbb{K} \), there is a homotopy-canonical complex \( C(H) \) over the Novikov field \( \Lambda_{\mathbb{K}} \) on the action-completion of
\[
\oplus \mathcal{H}^\text{loc}_* \langle \tilde{\phi}, \overline{x} \rangle
\]
the sum running over all capped one-periodic orbits \( \overline{x} \in \tilde{\mathcal{O}}(H) \), that is free and graded over \( \Lambda_{\mathbb{K}} \), and is strict, that is \( \mathcal{A}_H(d_H y) < \mathcal{A}(y) \) for all \( y \in C(H) \), with respect to the non-Archimedean action-filtration \( \mathcal{A}_H \) on \( C(H) \) defined as follows:
\[
\mathcal{A}_H(\sum \lambda_j y_j) = \max \{ -\nu(\lambda_j), \mathcal{A}_H(y_j) \},
\]
\[
\mathcal{A}_H(y_j) = \mathcal{A}_H(\overline{x}_{i(j)})
\]
for a \( \Lambda \)-basis \( \{ y_j \} \) of \( C(H) \) determined by \( \{ y_j \} \) \( i(j) = i \) being a basis of \( \mathcal{H}^\text{loc}_* \langle \tilde{\phi}, \overline{x}_i \rangle \), where \( \text{Fix}(\phi) = \{ x_i \} \) and for each \( i \), \( \overline{x}_i \) is a choice of a lift of \( x_i \) to a capped orbit in \( \tilde{\mathcal{O}}(H) \). Furthermore the filtered homology \( HF(H)^{<a} \) is given by \( HF(C(H)^{<a}) \), \( C(H)^{<a} = (\mathcal{A}_H)^{-1}(-\infty, a) \), for all \( a \in \mathbb{R} \sim \text{Spec}(H) \). In particular,
\[
HF(H) = H(C(H), d_H) \equiv QH(M, \Lambda_{\mathbb{K}}).
\]

Moreover, all the Floer-theoretical operations that we consider extend naturally to this chain-level setting. We shall only need one instance of this naturality formulated in Proposition 9 below. We now outline a proof of Theorem D.
Proof of Theorem D. We let $H_1$ be a sufficiently $C^2$-small perturbation of a Hamiltonian $H$. It generates a Hamiltonian diffeomorphism $\phi_1$, whose contractible fixed points separate into clusters $\text{Fix}(\phi_1, x) \subset \text{Fix}(\phi_1)$ of fixed points of $\phi_1$ near $x \in \text{Fix}(\phi)$. Furthermore the corresponding capped periodic orbits $\tilde{\phi}(H_1)$ split into clusters $\tilde{\phi}(H_1, x)$ of orbits near $x \in \tilde{\phi}(H)$ in $\mathcal{L}^{\min}_{pt} M$, in such a way that the evaluation of a periodic orbit at $0$ is an isomorphism of sets $\tilde{\phi}(H_1, x) \to \text{Fix}(\phi_1, x)$ for each capping $x$ of $a(x, H)$, and for each $A \in (2N_M)\mathbb{Z}$, $\tilde{\phi}(H_1, x#A) = \tilde{\phi}(H_1, x)#A$. Following [23, 13] we observe that the elements of $\tilde{\phi}(H_1, x)$ and the Floer trajectories between them form a complex $CF^\text{loc}_*(H_1, x)$, with differential $d^\text{loc}$ whose homology $HF^\text{loc}(\phi, x)$ depends only on the class $\phi \in \text{Ham}(M, \omega)$ of the path $[\phi^t]_{t \in [0, 1]}$ (in particular it does not depend on the choice of $H_1$ given that it is sufficiently close to $H$). Furthermore, by [14, 16, 19], as well as [11, 28], there is a crossing energy $2\epsilon_0 > 0$ depending only on $H$ and $x \in \text{Fix}(\phi)$, such that each $H_1$ Floer trajectory $u(s, t)$ asymptotic to $a(x_1, H_1)$ as $s \to -\infty$ where $x_1 \in \text{Fix}(\phi_1, x)$ is either contained in a small isolating neighborhood $U_x$ of $a(x, H)$, in $S^1 \times M$, and hence connects two elements of $\tilde{\phi}(H_1, x)$ for each capping $x$ of $a(x, H)$, or has energy $E(u) = E_{H_1}(u) \geq 2\epsilon_0$. Finally, the actions $\mathcal{A}_{H_1}(x)$ and indices $CZ(H_1, x)$ for $x \in \tilde{\phi}(H_1, x)$ satisfy

\[
\left| \mathcal{A}_{H_1}(x) - \mathcal{A}(x) \right| < \delta < \epsilon_0, \\
\left| CZ(H_1, x) - \Delta(x) \right| \leq n.
\]

This implies that the Floer differential of the Floer complex $CF(H_1; \mathcal{D})$ splits as

(5) \[ d = d^\text{loc} + D \]

with $d^\text{loc} = \oplus_{x \in \tilde{\phi}(H)} d^\text{loc}(x)$ suitably completed, and $\mathcal{A}_{H_1}(Dy) \leq \mathcal{A}_{H_1}(y) - \epsilon_0$ for all chains $y \in CF(H_1; \mathcal{D})$. Observe that $d^\text{loc}$ is a differential of $CF(H_1; \mathcal{D})$ as a $\Lambda$-module.

Now choose subspaces $X^\text{loc}_*(\mathcal{D}) \subset \ker(d^\text{loc}) \subset CF^\text{loc}_*(H_1, x)$ such that

\[ X^\text{loc}_*(\mathcal{D})(2N_M)k \] for all $k \in \mathbb{Z}$, and the inclusion

\[ \iota_r : \left( X^\text{loc}_*(\mathcal{D}), 0 \right) \subset \left( CF^\text{loc}_*(H_1, x), d^\text{loc}(x) \right) \]

is a quasi-isomorphism. Choose projections

\[ \pi_r : \left( CF^\text{loc}_*(H_1, x), d^\text{loc}(x) \right) \to \left( X^\text{loc}_*(\mathcal{D}), 0 \right) \]

similarly compatible with the Novikov action, such that

\[ \pi_r \circ \iota_r = \text{id}_{X^\text{loc}_*(\mathcal{D})}, \]

\[ \iota_r \circ \pi_r = \text{id}_{CF^\text{loc}_*(H_1, x)} + d^\text{loc}(x) \Theta_{\mathcal{D}}^{\text{loc}}(x), \]

for homotopies

\[ \Theta_{\mathcal{D}} : CF^\text{loc}_*(H_1, x) \to CF^\text{loc}_{*+1}(H_1, x) \]

again compatible with the Novikov action, and satisfying $\Theta_{\mathcal{D}}^2 = 0$ and $\mathcal{A}(\Theta_{\mathcal{D}} y) < \mathcal{A}(y) + \delta$ for all chains $y$. We refer to [26, Section 6] for a detailed construction.
of such $Θ_τ, τ_π, π_τ$ in a similar setting, as the two local settings can be identified by fixing one capping $x_0$, making the choices for it, and then extending to all other cappings $x$ by compatibility with the Novikov action.

The homological perturbation formulae now yield a differential $d_H$ on the $Λ$-module $C_*(H)$ given by completing $ϕ X_*^{loc}$ with respect to the filtration $A_H$, as well as injection $i: C_*(H) → C_*(H; Δ)$, projection $π: C_*(H; Δ) → C_*(H)$, and homotopy $Θ: CF_*(H; Δ) → CF_{*+1}(H; Δ)$ satisfying

$$\overline{π} \circ i = id_{C_*(H)},$$

$$i \circ \overline{π} = id_{CF_*(H; Δ)} + dΘ + Θd,$$

and

$$A_H(Θy) < A_H(y) + δ$$

for all chains $y$. Furthermore

$$A_H(d_H(y)) ≤ A_H(y) - 2ε_0 + δ$$

for all chains $y \in C(H)$. In particular, for each $y$ and $z \in X_*^{loc}(x)$ the coefficient $⟨d_H(y), z⟩$ vanishes. Now define the filtration $A_H$ on $C(H)$ by setting

$$A_H(y) := A_H(x)$$

for all $y \in X_*^{loc}(x) \sim \{0\}$, and extending it naturally to the completion. This definition is easily seen to coincide with the description in Theorem D. Then

$$|A_H(y) - A_H(y)| < δ$$

for all chains $y \in C(H) \sim \{0\}$. In particular $(C(H), d_H)$ is strict with respect to $A_H$. Finally, it is easy to see by a filtration argument that the complexes $C(H)$ obtained from different sufficiently small perturbations $H_1$ of $H$ are all filtered-isomorphic: indeed the continuation maps between them yield chain maps that induce isomorphisms on the local homology groups, and split similarly to $d$ in (5). Hence the homological perturbation formulae produce maps on the complexes $C(H)$ that are of the form $γ^{loc} + Π$, where $A_H(Π(y)) ≤ A_H(y) - ε_0$ for all chains $y$, and $γ^{loc}$ is a linear isomorphism preserving the filtration. The conclusion follows.

Now we use monotonicity, or in fact rationality, to prove that the filtered complex $(C(H), A_H)$ calculates the filtered Floer homology of $H$. By rationality, and $ϕ$ having isolated contractible fixed points, we obtain that there is $ε_1 > 0$ such that

$$|A_H(x) - A_H(y)| ≥ ε_1$$

for each two capped orbits $x, y$ of $H$ with distinct actions. Now for $a \in R \sim Spec(H)$, by the independence of $C(H)$ up to filtered isomorphism on $H_1$, we choose $H_1$ such that $d(a, Spec(H)) > δ$ and hence $A_H(φ(H_1, x))$ is contained either in $(-∞, a)$ or in $(a, ∞)$ for all $x \in φ(H)$. Therefore $HF(H)^{<a} = HF(H_1)^{<a}$ by definition, and $HF(H_1)^{<a} = H(CF(H_1)^{<a}) = H(CF(H)^{<a})$ by construction.
Finally, it is not hard to see that the construction of $C(H)$ does not depend on the choices made in the proof up to filtration-preserving chain-homotopies. 

Moreover, the proof of Lemma 7 adapts tautologically to prove the following.

**Lemma 8.** Let $F \in \mathcal{X}$ be a Hamiltonian with $\text{Fix}(\phi_{\mathcal{X}}^1)$ finite. Let $P$ be in $C(F; \Lambda_{\mathcal{X}}) \subset CF(F; \Lambda_{\mathcal{X}})$. Let $Q$ be in $C(F; \Lambda_{\mathcal{X}}) \subset C(F; \Lambda_{\mathcal{X}})$. Then

$$\mathcal{A}_F(P + hQ) = \max\{\mathcal{A}_F(P), \mathcal{A}_F(Q)\}.$$

4. **Proof of Theorem A**

Consider the filtered Floer homology $HF(\tilde{\phi}, \Lambda)^{<c}$, $\Lambda = \Lambda_{\mathcal{X}}$, where $c \in \mathbb{R} \sim \text{Spec}(\tilde{\phi})$, and the filtered $\mathbb{Z}/(2)$-equivariant Tate Floer homology $HF(\tilde{\phi}^2)^{<c}$ for action level $c \in \mathbb{R} \sim \text{Spec}(\tilde{\phi}^2)$. The spectral invariant $c(a, \phi)$, for $a \in QH(M) \sim \{0\}$, is defined as

$$c(a, \tilde{\phi}) = \inf\{c \in \mathbb{R} \sim \text{Spec}(\tilde{\phi}) \mid PSS(a) \in \text{im}\{HF(\tilde{\phi})^{<c} \to HF(\tilde{\phi})\}\}.$$

For $a \in QH(M, \Lambda_{\mathcal{X}})$ let the $\mathbb{Z}/(2)$-equivariant Tate spectral invariant $\hat{c}(a, \tilde{\phi}^2)$ be

$$\hat{c}(a, \tilde{\phi}^2) = \inf\{c \in \mathbb{R} \sim \text{Spec}(\tilde{\phi}^2) \mid PSS_{\mathbb{Z}/(2)}(a) \in \text{im}\{HF(\tilde{\phi}^2)^{<c} \to HF(\tilde{\phi}^2)\}\},$$

where $PSS_{\mathbb{Z}/(2)}$ is the equivariant PSS isomorphism introduced in [31].

By the construction of Seidel [24] and Wilkins [31], combined with the action estimates as in [28] for example, the equivariant pair-of-pants product precomposed with Kaledin's quasi-Frobenius map yields an injective map, for $\mathcal{X} = \mathbb{F}_2[h^{-1}, h]$, and $c \in \mathbb{R}$,

$$\mathfrak{P}: HF(\tilde{\phi}, \Lambda_{\mathcal{X}})^{<c} \to HF_{\mathbb{Z}/(2)}(\tilde{\phi}^2)^{<2c}.$$

Furthermore, by [31], this map commutes with vertical maps

$$HF(\tilde{\phi}, \Lambda_{\mathcal{X}})^{<c} \to HF(\tilde{\phi}, \Lambda_{\mathcal{X}}),$$

$$HF_{\mathbb{Z}/(2)}(\tilde{\phi}^2)^{<2c} \to HF_{\mathbb{Z}/(2)}(\tilde{\phi}^2)^{<2c},$$

where $PSS$, $PSS_{\mathbb{Z}/(2)}$ may also be replaced by the inverse $PSS$ maps, and the horizontal map:

$$\mathcal{Z}: QH(M, \Lambda_{\mathcal{X}}) \to QH(M, \Lambda_{\mathcal{X}})$$

given by the quantum Steenrod square from [30]. From this commutativity, we immediately obtain the inequality

$$\hat{c}(\mathcal{Z}\mathcal{F}(y), \tilde{\phi}^2) \leq 2c(y, \tilde{\phi}).$$

We shall require one feature of the equivariant PSS map $PSS_{\mathbb{Z}/(2)}$: for each chain $z \in CM(f, \Lambda_{\mathcal{X}})$ in the Morse complex of a Morse function $f$ on $M$ we have the identity

$$PSS_{\mathbb{Z}/(2)}(z) = PSS(z) + hR(z),$$

where $hR(z) = \sum_{j \geq 1} h^j PSS_{\mathbb{Z}/(2), j}(z)$ is a collection of terms of higher order in $h$. Moreover, we require the following analogue of this statement in the case of isolated, but possibly degenerate, contractible fixed points. It is deduced from its non-degenerate version (7) by adapting the proof of Theorem D above to
the equivariant Tate complex, as in [26, Section 7] for the $\Lambda^0$ case, and inducing maps on the homotopy-canonical complexes from the PSS and equivariant PSS maps respectively. Indeed, the homological perturbation formula described therein, and whose convergence is a non-trivial technical point, gives a projection quasi-isomorphism

$$\overline{\pi}_{\text{Tate}} : CF_{\text{Tate}}(H^{(2)}_1) \to \tilde{C}(H^{(2)}) = C(H^{(2)}) \otimes_{\Lambda_\mathcal{K}} \Lambda_{\mathcal{K}}$$

from the Tate complex of $H^{(2)}_1$ to the canonical Tate complex of $H^{(2)}$, that satisfies

)$

(8) \overline{\pi}_{\text{Tate}} = \overline{\pi} + h\overline{\Pi}$

where $h\overline{\Pi}$ is a collection of terms of order at least 1 in $h$. Now, post-composing the equivariant PSS isomorphism for $H^{(2)}$ with $\overline{\pi}_{\text{Tate}}$, and the non-equivariant PSS isomorphism for $H^{(2)}_1$ with $\overline{\pi}$, we obtain the following result.

**Proposition 9.** Consider $H \in \mathcal{H}$ with $\text{Fix}(\phi_{H^{(2)}}^1)$ finite. Then the PSS isomorphism construction induces quasi-isomorphisms $\text{PSS} : C(f; \Lambda_\mathcal{K}) \to C(H^{(2)})$ and $\text{PSS}_{Z^{(2)}} : C(f; \Lambda_{\mathcal{K}}) \to \tilde{C}(H^{(2)}) = C(H^{(2)}) \otimes_{\Lambda_\mathcal{K}} \Lambda_{\mathcal{K}}$ for a Morse function $f$ on $M$. Furthermore, whenever (7) and (8) hold, we have this relation:

$$\text{PSS}_{Z^{(2)}} = \text{PSS} + hR$$

for $hR = \sum_{j=1}^{\infty} h^j \text{PSS}_{Z^{(2)}, j}$ is a collection of terms of higher order in $h$. In fact $\text{PSS}_{Z^{(2)}, j}$ is defined over $\Lambda_\mathcal{K}$.

Furthermore, we have the following key technical result specific to pseudo-rotations.

**Proposition 10.** For each class $z \in QH(M, \Lambda_{\mathcal{K}})$, and each $H \in \mathcal{H}$, with $\phi^1_H$ a pseudo-rotation, the chain representatives of $\text{PSS}(z)$ in $C(H)$ and $\text{PSS}_{Z^{(2)}(z)}$ in $\tilde{C}_{Z^{(2)}}(H^{(2)})$ are unique and their action levels coincide with their spectral invariants: $c(z, \phi)$ and $\tilde{c}(z, \phi^{(2)})$.

**Remark 11.** We shall use Proposition 10 to relate $c([pt], H^{(2)})$ and $\tilde{c}([pt], H^{(2)})$.

**Proof of Proposition 10.** First of all, by comparing the dimensions of $C(H)$ and $QH(M, \Lambda_{F_2})$ over $\Lambda_{F_2}$ condition (iii) of the pseudo-rotation $\phi^1_H$ implies that the differential $d_H$ on $C(H)$ vanishes. Hence each homology class $a \in C(H) \sim \{0\}$ has a unique representative. This is in particular true for $a = \text{PSS}(z)$ for $z \in QH(M, \Lambda_{F_2})$, and the compatibility of $C(H)$ with the action filtration shows that $\alpha_H(\text{PSS}(z)) = c(z, H)$. The same holds for coefficients in the field extension $\Lambda_{\mathcal{K}}$ of $\Lambda_{F_2}$.

A similar, yet slightly more complicated, argument applies to the equivariant Tate case. Indeed, by condition (i) of a pseudo-rotation, the homotopy-canonical equivariant Tate complex $\tilde{C}(H^{(2)})$ is given as the action-completion of

$$\Phi_{\overline{\xi} \in \mathcal{O}}(H^{(2)}) H^{\text{loc}}(H^{(2)}, \overline{\xi}) \otimes \mathcal{K}$$.
where each such capped $\bar{x}$ is a recapping of an iterated capped orbit $\bar{y}^{(2)}$ of $H$, and its differential, which is $\Lambda_{\bar{x}}$-linear, is given by

$$\hat{d}_{H^{(2)}} = \hat{a}_{H^{(2)}} + \hat{D}_{H^{(2)}},$$

where $\mathcal{A}_{H^{(2)}}(\hat{D}_{H^{(2)}(y)}) \leq \mathcal{A}_{H^{(2)}}(y) - \epsilon_0$ for all chains $y$. We claim that $\hat{d}_{H^{(2)}} = 0$. Indeed, by the equivariant PSS isomorphism, $(\hat{C}(H^{(2)}), \hat{d}_{H^{(2)}})$ is quasi-isomorphic as a $\Lambda_{\bar{x}}$-module to $QH(M, \Lambda_{\bar{x}})$. However, the dimension of $\hat{C}(H^{(2)})$ over $\Lambda_{\bar{x}}$ is given by $N(\phi^2, F_2) = \dim_{\bar{x}} QH(M, \Lambda_{\bar{x}})$ by condition (iii) of a pseudo-rotation. This finishes the proof.

**Remark 12.** In fact, for pseudo-rotations (6) becomes an equality, that is for $y \in QH(M, \Lambda_{\bar{x}})$, we have the identity of spectral invariants:

$$\bar{c}(\mathcal{D} \mathcal{F}(y), \tilde{\phi}^2) = 2c(y, \tilde{\phi}).$$

Indeed, consider $y \in QH(M)$, and let $c = c(y, \tilde{\phi})$ be its critical level. Since the equivariant pair of pants product is an isomorphism on local Floer homology (see [28, 26]), it gives an isomorphism of homologies in action windows

$$HF(\tilde{\phi}, \Lambda_{\bar{x}})^{(c - \epsilon, c + \epsilon)} \to \tilde{HF}_{Z^{(2)}}(\phi^2)^{(2c - 2\epsilon, 2c + 2\epsilon)} = HF(\tilde{\phi}, \Lambda_{\bar{x}})^{(2c - 2\epsilon, 2c + 2\epsilon)}$$

for all $\epsilon$ sufficiently small. This shows that $2c$ is the critical level of $\mathcal{D} \mathcal{F}(y)$ for $\tilde{\phi}^2$. Indeed, otherwise the non-zero image of the chain representative of $y$ in the leftmost homology would go to zero in the rightmost homology, for $\epsilon$ sufficiently small.

**Proof of Theorem C.** From estimate (6) we obtain the bound

$$\bar{c}(\mathcal{D} \mathcal{F}([pt]), \tilde{\phi}^2) \leq 2c([pt], \tilde{\phi}).$$

However,

$$\bar{c}(\mathcal{D} \mathcal{F}([pt]), \tilde{\phi}^2) = \bar{c}(h^{2n}[pt], \tilde{\phi}^2) = \bar{c}([pt], \tilde{\phi}^2).$$

Identifying between the class $[pt]$ and its chain level representative, by choosing a Morse function on $M$ with unique minimum, which represents the point class, by Proposition 10 the following identities hold:

$$\bar{c}([pt], \tilde{\phi}^2) = \mathcal{A}_{\tilde{\phi}^2}(PSS_{Z^{(2)}}([pt])),$$

$$c([pt], \tilde{\phi}^2) = \mathcal{A}_{\tilde{\phi}^2}(PSS([pt])).$$

Furthermore, by Proposition 9 combined with Lemma 8,

$$\bar{c}([pt], \tilde{\phi}^2) \geq c([pt], \tilde{\phi}^2).$$

This finishes the proof.

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