Excited discrete spectrum states
wave functions of quantum integrable
N-particle systems in an external field

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Abstract

A problem of constructing excited state wave functions of the discrete spectrum of completely integrable quantum systems is considered. Recurrence relations defining wave functions up to the normalizing constant are obtained.
1. Introduction

Explicit solutions of classical and quantum completely integrable systems still attract much attention[1-5].

By now a lot of completely integrable systems with \(N\) degrees of freedom have been discovered[3-5]. However for few of such systems explicit solutions have been found. Some methods use additional constants of motion to reduce and simplify corresponding eigenvalue problems.

In [6] we proposed a method of finding discrete spectrum energies and wave functions. This method doesn’t use the fact of existence of additional constants of motion. In [7] we applied this method to the quantum systems with the following Hamilton function

\[
H = \sum_{i=1}^{N} \left( \frac{p_{i}^2}{2} + W(x_i) \right) + \alpha(\alpha + 1) \sum_{i>j}^{N} \left[ V(x_i - x_j) + \epsilon V(x_i + x_j) \right]
\]  (1)

with pair interaction potentials

\[
V(x) = \frac{1}{sh^{2}x}, \quad \epsilon = 0, 1
\]

External field for \(\epsilon = 1\) has the form

\[
W(x) = \frac{\mu(\mu - 1)}{2} \frac{1}{sh^{2}x} - \frac{\lambda(\lambda + 1)}{2} \frac{1}{ch^{2}x}, \quad \lambda > 0
\]  (2a)

whereas for \(\epsilon = 0\) it takes the form

\[
W(x) = 2A^2(e^{4x} - 2e^{2x})
\]  (2b)

The first step of the method proposed in [6] is the transformation of variables \(t_j = (chx_j)^{-2}, \quad j = 1, \ldots, N\), which enables us to reduce the order of
singularities in Schrodinger equation $H \psi = E \psi$. Then we perform further transformation to variables symmetrical in $t_j, \ j = 1, ..., N$

$$a_1 = \prod_{j=1}^{N} t_j, \quad a_l = \frac{\hat{D}^{l-1}}{(l-1)!} a_1, \quad \hat{D} = \sum_{j=1}^{N} \frac{\partial}{\partial t_j}, \ l = 1, ..., N$$

which enables us to convert Schrodinger equation into the form convenient for solving it explicitly

$$[H_1(\beta) + H_2(\beta)]\kappa = [E - \epsilon(\beta)]\kappa$$

$$H_1(\beta) = 2 \sum_{l,m=1}^{N} \sum_{\tau=1}^{l-1} \sum_{\nu=0}^{1} ((l+m-2\tau) - \nu)a_{\tau}a_{l+m-\tau-\nu} - (N-m+2)a_{m-1}a_l$$

$$-(N-l+2)a_{m}a_{l-1} - (N+m+1)a_{l}a_{m}] \frac{\partial^2}{\partial a_l \partial a_m}$$

$$-\sum_{l=1}^{N} [a_{l-1}(N-l+2)(4\beta+2\mu+3+2\alpha(N-l-3)) + 2a_l(n-l+1)(2\beta+1+\alpha(N+l-2))] \frac{\partial}{\partial a_l},$$

$$H_2(\beta) = a_N(2 \sum_{l,m=1}^{N} a_la_m \frac{\partial^2}{\partial a_l \partial a_m}$$

$$+ \sum_{l=1}^{N} a_l \frac{\partial}{\partial a_l} (4\beta+2\mu+3+4\alpha(N-1) + p(\beta))$$

Here parameter $\beta$ and $\epsilon(\beta), p(\beta)$ are to be determined from normalizing condition. Wave functions are polynomials in symmetrical variables $a_j$:

$$\psi(x_1, ..., x_N) = \kappa(a_1, ..., a_N) \prod_{j=1}^{N} z_j^{\mu} (1 - z_j^2)_{\beta} \prod_{j>k} (z_j^{1+\epsilon} - z_k^{1+\epsilon})$$

$z_j = thx_j$ and

$$\kappa(a_1, ..., a_N) = \sum_{p=0}^{n} \sum_{0 \leq j_1, ..., j_N \leq N} c_{j_1, ..., j_N} a_1^{j_1} ... a_N^{j_N}$$
Energy levels are determined by a set of $N$ integers $j_1, \ldots, j_N$, $0 \leq j_1, \ldots, j_N \leq n$, $j_1 + \ldots + j_N = p \leq n$, where $n$ is the highest power of the polynomial $\kappa$. One can conclude from the normalizing condition that the upper boundary for $n$ is

$$n < \frac{\lambda - \mu}{2} - \alpha(N - 1), \quad \epsilon = 1$$

$$n < A - \alpha(N - 1) - \frac{1}{2}, \quad \epsilon = 0$$

Energy levels are the following

$$-E_{j_1, \ldots, j_N} = -\epsilon(\beta) - 4 \sum_{l > m}^{N} lj_l j_m + 2(N + 1)p(\beta)(p(\beta) + \lambda - \mu - 2n - \alpha N)$$

$$+ 2 \sum_{l=1}^{N} lj_l [\alpha(2N + 1) + 2n + \mu - \lambda - j_l - \alpha l]$$ (5)

In [7] we only considered the simplest case $n = 1$. Here we find wave functions of excited levels with $n = 2$.

**2. Poschl-Teller potential**

Expression (2) is a generalized Poschl-Teller potential. Having demanded wave functions to have the following form

$$\kappa(a_1, \ldots, a_N) = \sum_{l=0}^{N} c_l a_l + c_{N+1} + \sum_{l \leq k} b_{lk} a_l a_k$$ (6)

and taking into account the normalizing condition we come to

$$\beta = \frac{\lambda - \mu}{2} - \alpha(N - 1) - 2$$

$$p(\beta) = 1 - 2\lambda$$ (7)
\[ \epsilon(\beta) = -\frac{N}{2} \left[ \frac{\alpha^2}{3} (N^2 - 1) + (\lambda - \mu - \alpha(N - 1) - 4)^2 \right] \]

Substituting (5), (6) and (7) into (3), we come to a system of \( \frac{N(N+1)}{2} \) linear algebraic equations with zero determinant. Unlike the case \( n = 1 \), there is a contribution from the first term in \( H_1(\beta) \) with second order derivatives in \( a_j \). This leads to coefficients \( c_l, b_{lk} \) being dependent on all non-zero coefficients \( c_m, m > l \), \( b_{mn}, m > l, n > k \). Performing induction one can obtain recurrence relations for coefficients in (6). As a result we come to 3 different sets of recurrent relations for different meanings of quantum numbers \( j_1, ..., j_N \).

1. In this case \( j_l = 2\delta_{ls}, \ s = 1, ..., N, \ p = 2; \ s' = 2s - N - 1, \ n' = 2s - j, \ K = (N - s + 1)(\lambda - \mu - 2 - \alpha(N - s)); \ b_{ss} = d_s \), where \( d_s \) is an arbitrary constant to be determined from the normilizing condition, and

\[
\begin{align*}
b_{jl} &= 0, \ j + l > 2s; \quad c_j = 0, \ j > s' \\
b_{jn'} &= \frac{4(n' - j)}{F_{j00}} \sum_{\sigma=j+1}^{s} b_{\sigma n' - \sigma + j} 
\end{align*}
\]

where \( F_{jkl} \) are defined below.

2. In this case \( j_l = \delta_{ls} + \delta_{lN}, \ s = 1, ..., N - 1, \ p = 2; \ s' = s - 1, \ n' = N, \ K = \mu - \lambda + 1; \ b_{sN} = d_s \), where \( d_s \) is an arbitrary constant to be determined from the normilizing condition, and

\[
\begin{align*}
b_{jl} &= 0, \ j + l > s + N; \quad c_j = 0, \ j > s' \\
\end{align*}
\]

Other relations have the form

\[ b_{sN} = \frac{1 - 2\lambda}{Q_s} d_s \]
\[ b_{s-mN} = \frac{1}{Q_{s-m}} \{-2[(N-s+m+1)(2\lambda - 3 - 2\alpha(N-s+m) + 1)b_{s-m+1N} + 4(N-s+m)\sum_{k=1}^{m} \sum_{\tau=0}^{1} b_{s-m+kN-k+\tau} + (1-2\lambda)c_{s-m}\}, \quad m = 1, \ldots, s-1 \]

In this relation summation is performed under the condition \(2k \leq N-s+m+\tau\).

It must be stressed here that Hamiltonian (1) is self-conjugated if and only if \((2\lambda - 1) > 10 + 4\alpha(N-1)\), i.e. the coefficients \(c_j\) and \(b_{jk}\) are not independent.

3. In this case \(j_l = \delta_{ls} + \delta_{lN-m}\), \(s = 1, \ldots, N-2\), \(m = 1, \ldots, N-s+1\), \(b_{sN-m} = d_{sN-m}\), \(s' = s-m-1\), \(n' = N+s-m-j\), where \(d_s\) is an arbitrary constant, and

\[ b_{jl} = 0, \quad j + l > s'; \quad c_j = 0, \quad j + l > N+s-m \]

Coefficients \(b_{jn'}\) are determined by (8). Suppose

\[ R_{jkl} = 2^l \prod_{\tau=1}^{l} [(s-j+k-\tau)(\lambda - \mu - 3 - \alpha(2N-s-j+k-\tau)) - K], \]

\[ F_{jkl} = 2^{k-l} \prod_{\tau=0}^{k-l} [(s-j)(\lambda - \mu - 3 - \alpha(2N-s-j+1)) \]

\[ + (N-n'+k-\tau+1)(\lambda - \mu - 1 - \alpha(N-n'+k-\tau)) - K], \]

\[ S_j = R_{j11} + N-j+1, \]

\[ Q_j = R_{j11} + 2(N-s+1)(\lambda - \mu - 3 - \alpha(N-3)) + 2(2\lambda - 2\mu - 7). \]

Then the common part of the three recurrence relations sets is the following:

\[ c_{s'-k+1} = \frac{(N-s'+k)!}{\Gamma((2\lambda-5)/(2\alpha) - N+s'-k)}[(-2\alpha)^k} \]
\[
\Gamma \left( \frac{(2\lambda - 5)/(2\alpha) - N + s'}{N - s'} \right) \frac{c_{s'+1}}{(N - s')!R_{s'k}} \\
+ 4 \sum_{l=1}^{k} (-2\alpha)^{l-1} \frac{\Gamma \left( \frac{(2\lambda - 5)/(2\alpha) - N + s' - k + l - 1}{N - s' + k - l} \right)}{(N - s' + k - l)!R_{s'kl}} \\
\sum_{\sigma=1}^{s-s'+k-\tau-l} \sum_{\tau=0}^{1} b_{s' - k + l + \sigma + \tau} N - \sigma + 1 (1 - \delta_{\tau1}\delta_{l,k}], \quad k = 1, \ldots, s'
\]

In this relation summation is performed under the condition \(2\sigma \leq N + 1 + k - s' - l - \tau\).

\[
b_{jn' - k} = (-4\alpha)^{k} \frac{(N - n' + k + 1)!}{\Gamma \left( \frac{(\lambda - 2)/\alpha - N + n' - k}{N - n' + 1} \right)} \frac{\Gamma \left( \frac{(\lambda - 2)/\alpha - N + n' - l}{N - n' + l + 1} \right)}{F_{jkl}} b_{jn'}
\]

\[
-2 \sum_{l=1}^{k} (n' - l - j) \sum_{\sigma=j+1}^{s} \sum_{\tau=0}^{1} (-4\alpha)^{k} \frac{\Gamma \left( \frac{(\lambda - 2)/\alpha - N + n' - l}{N - n' + l + 1} \right)}{F_{jkl}} [\\
(2(N - j + 1)(\lambda - 2 - \alpha(N - 1)) + n' - l + j)b_{j+1n'-l} + \sum_{\sigma=j+1}^{s} \sum_{\tau=0}^{1} b_{\sigma n' - l - \sigma + \tau + j} (1 - \delta_{\tau1}\delta_{\sigma j+1})], \quad k = 1, \ldots, n' - j + 1
\]

In this relation summation is performed under the condition \(2\sigma \leq n' - l + 1 + \tau + j\).

\[
b_{jj} = \frac{2(N - j + 1)(2\lambda - 3 - 2\alpha(N - j))}{S_{j}} b_{jj+1}.
\]

### 3. Morse potential

One can come to a case of molecular Morse potential by a limiting procedure \(x_j = q, \ q \to \infty\). As a result potential (2) turns into (2b), and

\[
\lambda - \mu \to 2A + 1
\]

(9)
However it’s easier to find wavefunctions directly from (3) and perform the
calculations similar to those in section 2. Substituting (9) into (5) and
(7), one can derive energy levels in Morse potential. Taking into account
\( p(\beta) = -8A \) recurrence relations for the coefficients in (6) turn out to be
the following

\[
c_{s'-k+1} = (N - s' + k)! \frac{(-2A)^k}{(N - s')! R_{s'kk}} c_{s'+1}
\]

\[
+ 4 \sum_{l=1}^{k} \frac{(-2A)^{l-1}}{(N - s' + k - l)! R_{s'kl}} \sum_{\sigma=1}^{s - s' + k - \tau - l} b_{s'-k+l+\sigma+\tau N-\sigma+1}, \quad k = 1, ..., s'
\]

In this relation summation is performed under the condition \( 2\sigma \leq N + 1 + k - s' - l \).

\[
b_{jn'-k} = (-4A)^k (N - n' + k + 1)! \frac{b_{jn'}}{(N - n' + 1)! F_{jk1}}
\]

\[
- 2 \sum_{l=1}^{k} \frac{(-4A)^{l-1}}{(N - n' + l + 1)! F_{jkl}} [(2A(N - j + 1)b_{j+1n'-l - 2(n' - l - j)} \sum_{\sigma=j+1}^{s} b_{\sigma n' - l - \sigma + j}]}, \quad k = 1, ..., n' - j + 1
\]

In this relation summation is performed under the condition \( 2\sigma \leq n' - l + 1 + j \).

\[
b_{s-mN} = \frac{1}{Q_{s-m}} \{[(N - s + m + 1)(1 - 4A) - 1]b_{s-m+1N}
\]

\[
+ 4(N - s + m) \sum_{k=1}^{m} b_{s-m+kN-1} - 4Ac_{s-m}\}, \quad m = 1, ..., s - 1
\]
In this relation summation is performed under the condition $2k \leq N-s+m$.

$$b_{jj} = \frac{4A(N - j + 1)}{S_j} b_{jj+1}.$$  

It is sufficiently to perform $\lambda - \mu \to 2A + 1$ in the other relations.

Therefore we have found recurrence relations which determine coefficients $c_j$ and $b_{jk}$ for arbitrary $N$ for Poschl-Teller and Morse potentials. Wave functions are determined by means of (6).

Let us stress here that at $0 < \lambda - \mu - 2\alpha(N - 1) < 6$ and $1 < 2A - 2\alpha(N - 1) < 7$ wave functions found together with wave functions for $n = 1$, which were found in [7], give the whole discrete spectrum of the systems in question.

As an example we have found four particle wave function of one of the excited states in Morse potential (see Appendix).

The results obtained could be used for testing of various approximate methods of solving many particle quantum system as well as in supersymmetrical generalizations of such systems. They could also be useful for investigations of systems close to integrable.

Let us note in conclusion that wave functions of higher levels with such values of parameters $A$ and $\alpha$ that $n \geq 3$ can be found in the same way.

**Appendix.**

Here we write down explicit form of the coefficients $c_j$ and $b_{jk}$ for the 14th exited state for four-particle system in Morse potential. Parameters $A$ and $\alpha$ satisfy inequality $1 < 2A - 6\alpha < 7$. Discrete spectrum consists of 15 levels: the ground state, 4 levels with $n = 1$ ($p = 1$) and 10 levels with $n = 2$ ($p = 2$). The energy of the level in question corresponds to
\[ j_l = \delta_{l3} + \delta_{l4} \] and is equal to

\[ E = -2(4A^2 - 12\alpha A - 16A + 14\alpha^2 + 30\alpha + 19) \]

Suppose

\[ F(A, \alpha) = (1 - 2A)(12A - 5\alpha - 23)(4A - 3\alpha - 6)(4A - 3\alpha - 5) \]

\[ T(A, \alpha) = 2(24A^3 - 104A^2 + 42\alpha A + 116A - 3)(4A - 3\alpha - 5)/F(A, \alpha) \]

\[ X(A, \alpha) = 8A(4608A^4 - 5376\alpha A^3 - 15720A^3 + 1035\alpha^2 A^2 \]
\[ + 9504\alpha A^2 + 13144A^2 - 42\alpha A - 116A + 3)/F(A, \alpha) \]

\[ Y(A, \alpha) = 3(73216A^5 - 248565A^4 - 85632\alpha A^4 + 150328\alpha A^3 \]
\[ + 149893\alpha^2 A^3 + 204520A^3 + 1888\alpha A^2 + 2285A^2 \]
\[ - 624A - 126\alpha^2 A - 606\alpha A + 9\alpha + 15)/[(8A - 6\alpha - 9)F(A, \alpha)] \]

\[ Z(A, \alpha) = 64A^2(32A^2 - 24\alpha A - 36\alpha - 3)(14A - 10\alpha - 9)/(8A - 6\alpha - 9) \]

\[ N(A, \alpha) = \frac{296A - 366\alpha - 421}{8A - 6\alpha - 9} \]

\[ M(A, \alpha) = \frac{2A^2[3Z(A, \alpha) - N(A, \alpha)Y(A, \alpha)]}{(5A - 6\alpha - 7)(3A - 4(\alpha + 1))} \]

Then the coefficients of (6) depend on the potential depth \( A \) and pair interaction constant \( \alpha \) in the following way

\[ c_1 = 2 \left( \frac{2}{3A - 4\alpha - 5}(T(A, \alpha) + \frac{8A}{2A - 1}) - \frac{3A(A - 3\alpha - 6)}{(2A - 2\alpha - 3)(4A - 3\alpha - 6)^2} \right) \]

\[ c_2 = \frac{6}{4A - 3\alpha - 6} \]
$$b_{11} = M(A, \alpha) \frac{4A}{3A - 4(\alpha + 1)} d$$

$$b_{12} = M(A, \alpha) d$$

$$b_{13} = \frac{-4A}{5A - 6\alpha - 7} [Y(A, \alpha) + Z(A, \alpha)] d ; \quad b_{14} = Y(A, \alpha) d$$

$$b_{22} = \frac{12A}{8A - 6\alpha - 9} X(A, \alpha) d ; \quad b_{23} = X(A, \alpha) d$$

$$b_{24} = T(A, \alpha) d ; \quad b_{33} = \frac{8A}{2A - 1} d ; \quad b_{34} = d$$

Here $d$ is an arbitrary constant to be fixed by normalizing condition.

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