The simplicity of Kac modules for the quantum superalgebra $U_q(gl(m, n))$

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1 Introduction

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the general linear Lie superalgebra over the complex number field $\mathbb{C}$. The quantum superalgebra $U_q(\mathfrak{g})$ in the present paper was defined by R. Zhang [12]. The Kac module $K(M)$ is the $U_q(\mathfrak{g})$-module induced from a simple $U_q(\mathfrak{g}_0)$-module $M$. Assume $M$ is a weighted $U_q(\mathfrak{g}_0)$-module which is generated by a primitive vector of weight $\lambda$. Then $\lambda$ is called typical if $K(M)$ is simple. The typical weights in both generic case and the case where $q$ is a primitive root of unity were first studied in [12]. Also in [5], a sufficient condition for the typicality is given in generic case.

One of the main goals of the present paper is to determine the typical weights. We prove that in the case where $K(M)$ is weighted, the typical weights are determined by a polynomial. Then we determine the polynomial using the method provided by [11]. Let us note that our polynomial coincides with one given in [12], despite the fact that the order of the product for the elements $F_{ij}$ with $(i,j) \in \mathcal{I}_1$ used in [12] to define the polynomial is completely different from ours.

The paper is organized as follows. Sec. 3 is the preliminaries. In Sec. 4, we give some identities in $U_q(\mathfrak{g})$. In Sec. 5 we discuss the simplicity of the Kac modules, which is determined by a polynomial. The polynomial is determined in Sec. 6. In Sec. 7, we study the simple modules in the case where $q$ is a $l$th root of unity. We prove that, under certain conditions, the algebras $u_{\eta,\lambda}(\mathfrak{g}_0)$ and $u_{\eta,\lambda}$ are Morita equivalent.
2 Notations

Throughout the paper we use the following notation.

\[ [1, m + n] = \{1, 2, \ldots, m + n\}. \]

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\[ A^{m+n} \] the set of all \( m + n \)-tuples \( z = (z_1 \ldots z_{m+n}) \) with \( z_i \in A \) for all \( i = 1, \ldots, m + n \).

\[ I_0 = \{ (i, j) | 1 \leq i < j \leq m \text{ or } m + 1 \leq i < j \leq m + n \} \]

\[ I_1 = \{ (i, j) | 1 \leq i \leq m < j \leq m + n \} \]

\[ I = I_0 \cup I_1 \]

\[ A^B \] the set of all tuples \( \psi = (\psi_{ij})_{(i,j) \in B} \) with \( \psi_{ij} \in A \), where \( B = I_0 \) or \( B = I_1 \)

\[ A \] the set of all \( \bar{z} = (z_1 \ldots z_{m+n}) \) with \( z_i \in A \) for all \( i = 1, \ldots, m + n \).

\[ h(V) \] the set of all homogeneous elements in a \( \mathbb{Z}_2 \)-graded vector space \( V = V_0 \oplus V_1 \)

\[ x \] the parity of the homogeneous element \( x \in V = V_0 \oplus V_1 \).

\[ U(L) \] the universal enveloping superalgebra for the Lie superalgebra \( L \).

3 The quantum deformation of \( gl(m, n) \)

The general linear Lie superalgebra \( g = g_0 \oplus g_1 \) has the standard basis ([7]) \( e_{ij} \), \( 1 \leq i, j \leq m+n \). We denote \( e_{ij} \) with \( i < j \) also by \( f_{ij} \). Then we get \( g = g_{-1} \oplus g_0 \oplus g_1 \), where

\[ g_1 = \langle e_{ij} | (i, j) \in I_1 \rangle \quad g_{-1} = \langle f_{ij} | (i, j) \in I_1 \rangle. \]

The parity of the basis elements is given by

\[ \bar{e}_{ij} = \bar{f}_{ij} = \begin{cases} 0, & \text{if } (i, j) \in I_0 \text{ or } i = j \\ 1, & \text{if } (i, j) \in I_1. \end{cases} \]

Let \( H = \langle e_{ii} | 1 \leq i \leq m + n \rangle \). Then the set of positive roots of \( g \) relative to \( H \) is \( \Phi^+ = \Phi_0^+ \cup \Phi_1^+ \), where

\[ \Phi_0^+ = \{ e_i - e_j | (i, j) \in I_0 \}, \Phi_1^+ = \{ e_i - e_j | (i, j) \in I_1 \}. \]

Let \( \Lambda = \mathbb{Z} \epsilon_1 + \cdots + \mathbb{Z} \epsilon_{m+n} \subseteq H^* \). There is a symmetric bilinear form defined on \( \Lambda \) as follows ([10]):

\[ (\epsilon_i, \epsilon_j) = \begin{cases} \delta_{ij}, & \text{if } i < m \\ -\delta_{ij}, & \text{if } i > m. \end{cases} \]

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Let $q$ be an indeterminate over $\mathbb{C}$. Then the quantum supergroup $U_q(\mathfrak{g})$ (see [12, p.1237]) is defined as the $\mathbb{C}(q)$-superalgebra with the generators $K_j, K_j^{-1}, E_{i,j+1}, F_{i,j+1}, i \in [1, m + n]$, and relations

\begin{align*}
(R1) & \quad K_i K_j = K_j K_i, K_i K_i^{-1} = 1, \\
(R2) & \quad K_i E_{j,j+1} K_i^{-1} = q_i^{(\delta_{ij} - \delta_{i,j+1})} E_{j,j+1}, \quad K_i F_{j,j+1} K_i^{-1} = q_i^{-(\delta_{ij} - \delta_{i,j+1})} F_{j,j+1}, \\
(R3) & \quad [E_{i,i+1}, F_{j,j+1}] = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_{i+1}^{-1} K_i}{q_i - q_i^{-1}}, \\
(R4) & \quad E_{m,m+1} = F_{m,m+1}^2 = 0, \\
(R5) & \quad E_{i,i+1} E_{j,j+1} = E_{j,j+1} E_{i,i+1}, \quad F_{i,i+1} F_{j,j+1} = F_{j,j+1} F_{i,i+1}, |i - j| > 1, \\
(R6) & \quad E_{i,i+1}^2 E_{j,j+1} - (q + q^{-1}) E_{i,i+1} E_{j,j+1} E_{i,i+1}^2 + E_{j,j+1} = 0 \quad (|i-j| = 1, i \neq m), \\
(R7) & \quad F_{i,i+1}^2 F_{j,j+1} - (q + q^{-1}) F_{i,i+1} F_{j,j+1} F_{i,i+1}^2 + F_{j,j+1} = 0 \quad (|i-j| = 1, i \neq m), \\
(R8) & \quad [E_{m-1,m+2}, E_{m,m+1}] = [F_{m-1,m+2}, F_{m,m+1}] = 0,
\end{align*}

where

$$q_i = \begin{cases} q, & \text{if } i \leq m \\ q^{-1}, & \text{if } i > m. \end{cases}$$

Most often, we shall use $E_{\alpha_i}$ (resp. $F_{\alpha_i}$; $K_{\alpha_i}$) to denote $E_{i,i+1}$ (resp. $F_{i,i+1}$; $K_{i+1}^{-1}$) for $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

Remark: (1) For each pair of indices $(i, j) \in I$, the notation $E_{ij}, F_{ij}$ are defined by

$$E_{ij} = E_{ic} E_{cj} - q_c^{-1} E_{c j} E_{ic}, \quad F_{ij} = -q_c E_{ic} E_{cj} + F_{c j} F_{ic}, \quad i < c < j.$$ 

The relation (R2) then implies that, for $s \in [1, m + n], (i, j) \in I$,

$$K_s E_{ij} K_s^{-1} = q_s^{\delta_{is} - \delta_{sj}} E_{ij}, \quad K_s F_{ij} K_s^{-1} = q_s^{-(\delta_{is} - \delta_{sj})} F_{ij}.$$ 

(2) The parity of the elements $E_{ij}, F_{ij}, K_{s}^{\pm 1}$ is defined by $\bar{E}_{ij} = \bar{F}_{ij} = \bar{e}_{ij} \in \mathbb{Z}_2$, $K_{s}^{\pm 1} = 0$.

(3) The bracket product in $U_q(\mathfrak{g})$ is defined by

$$[x, y] = xy - (-1)^{xy}yx, x, y \in h(U_q(\mathfrak{g})).$$

A bijective (even)$\mathbb{F}$-linear map $f$ from an $\mathbb{F}$-superalgebra $\mathfrak{A}$ into itself is called an anti-automorphism (resp. $\mathbb{Z}_2$-graded anti-automorphism) if

$$f(xy) = f(y)f(x) \text{ (resp. } f(xy) = (-1)^{xy} f(y)f(x) \text{)}$$

for any $x, y \in h(\mathfrak{A})$.

It is easy to show that
Lemma 3.1. [10, 12] There is an anti-automorphism $\Omega$ and a $\mathbb{Z}_2$-graded anti-automorphism $\Psi$ of $U_q(\mathfrak{g})$ such that

$$\Omega(E_{\alpha_i}) = F_{\alpha_i}, \Omega(F_{\alpha_i}) = E_{\alpha_i}, \Omega(K_j) = K_j^{-1}, \Omega(q) = q^{-1}$$

$$\Psi(E_{\alpha_i}) = E_{\alpha_i}, \Psi(F_{\alpha_i}) = F_{\alpha_i}, \Psi(K_j) = K_j, \Psi(q) = q^{-1},$$

for all $i \in [1, m+n], j \in [1, m+n]$.

From the lemma it is easily seen that

$$\Omega(E_{ij}) = F_{ij}, \Psi(E_{ij}) = q^z E_{ij}, \Psi(F_{ij}) = q^z F_{ij}, z \in \mathbb{Z}$$

for any $(i, j) \in I$.

We abbreviate $U_q(\mathfrak{g})$ to $U_q$ in the following.

4 Some formulas in $U_q$

In this section we present some formulas in $U_q$, most of which are given in [12]. To keep the paper self-contained, each formula will be proved unless an explicit proof can be found elsewhere.

For $i \in [1, m+n] \setminus m$, the automorphism $T_{\alpha_i}$ of $U_q$ is defined by (see [12, Appendix A] and also [8, 1.3])

$$T_{\alpha_i}(E_{\alpha_j}) = \begin{cases} -F_{\alpha_i}K_{\alpha_i}, & \text{if } i = j \\ E_{\alpha_j}, & \text{if } |i - j| > 1 \\ -E_{\alpha_i}E_{\alpha_j} + q^{-1}E_{\alpha_i}E_{\alpha_j}, & \text{if } |i - j| = 1. \end{cases}$$

$$T_{\alpha_i}F_{\alpha_j} = \begin{cases} -K_{\alpha_i}^{-1}E_{\alpha_i}, & \text{if } i = j \\ F_{\alpha_j}, & \text{if } |i - j| > 1 \\ -F_{\alpha_j}F_{\alpha_i} + qF_{\alpha_i}F_{\alpha_j}, & \text{if } |i - j| = 1. \end{cases}$$

$$T_{\alpha_i}K_j = \begin{cases} K_{i+1}, & \text{if } j = i \\ K_i, & \text{if } j = i + 1 \\ K_j, & \text{if } j \neq i, i + 1. \end{cases}$$

$T_{\alpha_i}$ is an even automorphism for $U_q$, that is,

$$T_{\alpha_i}(uv) = T_{\alpha_i}(u)T_{\alpha_i}(v), \quad \text{for all } u, v \in h(U_q).$$

By a straightforward computation ([12, A3]), one obtains for each $i \in [1, m+n] \setminus m$ the inverse map $T_{\alpha_i}^{-1}$:

$$T_{\alpha_i}^{-1}E_{\alpha_j} = \begin{cases} -K_{\alpha_i}^{-1}F_{\alpha_i}, & \text{if } i = j \\ E_{\alpha_j}, & \text{if } |i - j| > 1 \\ -E_{\alpha_j}E_{\alpha_i} + q^{-1}E_{\alpha_i}E_{\alpha_j}, & \text{if } |i - j| = 1. \end{cases}$$

$$T_{\alpha_i}^{-1}F_{\alpha_j} = \begin{cases} -K_{\alpha_i}^{-1}F_{\alpha_i}, & \text{if } i = j \\ E_{\alpha_j}, & \text{if } |i - j| > 1 \\ -E_{\alpha_j}E_{\alpha_i} + q^{-1}E_{\alpha_i}E_{\alpha_j}, & \text{if } |i - j| = 1. \end{cases}$$

$$T_{\alpha_i}^{-1}K_j = \begin{cases} K_{i+1}, & \text{if } j = i \\ K_i, & \text{if } j = i + 1 \\ K_j, & \text{if } j \neq i, i + 1. \end{cases}$$
$$T_{\alpha_i}^{-1}F_{\alpha_j} = \begin{cases} -E_{\alpha_i}K_{\alpha_i}, & \text{if } i = j \\ F_{\alpha_j}, & \text{if } |i - j| > 1 \\ -F_{\alpha_i}F_{\alpha_j} + q_iF_{\alpha_j}F_{\alpha_i}, & \text{if } |i - j| = 1. \end{cases}$$

$$T_{\alpha_i}^{-1}K_j = \begin{cases} K_{i+1}, & \text{if } j = i \\ K_i, & \text{if } j = i + 1 \\ K_j, & \text{if } j \neq i, i + 1. \end{cases}$$

It follows from the definition that

$$(b1) \quad E_{ij} = (-1)^{j-i-1}T_{\alpha_i}T_{\alpha_{i+1}} \cdots T_{\alpha_{j-1}}E_{j-1,j} = (-1)^{j-i-1}T_{\alpha_{j-1}}^{-1} \cdots T_{\alpha_{i+1}}^{-1}E_{i,i+1},$$

$$(b2) \quad F_{i,j} = (-1)^{j-i-1}T_{\alpha_i}T_{\alpha_{i+1}} \cdots T_{\alpha_{j-1}}F_{j-1,j} = (-1)^{j-i-1}T_{\alpha_{j-1}}^{-1} \cdots T_{\alpha_{i+1}}^{-1}F_{i,i+1}.$$\]

By the defining relation (3), (4) and the formulas above we get

$$(1) \quad E_{ij}^2 = F_{ij}^2 = 0, \quad (i, j) \in \mathcal{I},$$

$$(2)([12]) \quad [E_{ij}, F_{ij}] = \frac{K_iK_j^{-1} - K_j^{-1}K_i}{q_i - q_i^{-1}}, (i, j) \in \mathcal{I}.$$\]

Let $V = V_0 \oplus V_1$ be a vector superspace over a field $\mathbb{F}$. A $\mathbb{F}$-linear mapping $f : V \rightarrow V$ is called $\mathbb{Z}_2$-graded with parity $\bar{f} = \bar{i} \in \mathbb{Z}_2$ if $f(V_k) \subseteq V_{k+\bar{i}}$ for any $\bar{k} \in \mathbb{Z}_2$. Let $A = A_0 \oplus A_1$ be an associative $\mathbb{F}$-superalgebra. A $\mathbb{Z}_2$-graded $\mathbb{F}$-linear mapping $\delta$ from $A$ into itself is called a derivation if

$$\delta(xy) = \delta(x)y + (-1)^{\bar{x}\bar{y}}x\delta(y) \quad \text{for any} \quad x, y \in h(A).$$

Denote by $\text{Der}_\mathbb{F}A$ the set of all derivations on $A$. For any $x, y \in h(A)$, we define $[x, y] = xy - (-1)^{\bar{x}\bar{y}}yx$. Clearly we have

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x].$$

For each $x \in h(A)$, it is easy to see that $[x, -], [-, x] \in \text{Der}_\mathbb{F}A$.

**Lemma 4.1.** ([12]) The following identities hold in $U_q$.

$$(1) \quad F_{s,j}F_{s,i} = (-1)^{\bar{F}_{s,i}}q_{s}F_{s,i}F_{s,j}, s < i < j,$$

$$(2) \quad F_{i,s}F_{j,s} = (-1)^{\bar{F}_{i,s}}q_{s}^{-1}F_{j,s}F_{i,s}, i < j < s.$$\]

For $c < i < j$,

$$(3) \quad [F_{c,j}, E_{c,i}] = F_{i,j}K_cK_i^{-1}q_i, \quad (4) \quad [F_{c,i}, E_{c,j}] = E_{i,j}K_c^{-1}K_i,$$

$$(5) \quad [E_{i,j}, F_{c,j}] = F_{c,j}K_i^{-1}K_j, \quad (6) \quad [E_{c,j}, F_{i,j}] = E_{c,j}K_iK_j^{-1}q_i^{-1},$$

$$(7) \quad [F_{s,t}, F_{i,j}] = -(q_j - q_j^{-1})F_{s,j}F_{s,t}, \quad i < s < j < t.$$
Proof. (1) and (2) follow from a short computation using the formulas provided by Remark (1) in Sec. 3.1.

(3) By Remark (1) in Sec. 3.1, we have

$$[F_{cj}, E_{ci}] = [F_{ij} F_{ci} - q_i F_{ci} F_{ij}, E_{ci}].$$

Since $$[-, E_{ci}]$$ is a derivation on $$U_q$$ and $$[F_{ij}, E_{ci}] = 0$$, we have

$$[F_{cj}, E_{ci}] = F_{ij}[F_{ci}, E_{ci}] - q_i(-1)E_{ci}[F_{ci}, E_{ci}] F_{ij}.$$

Let us note that at least one of the $$E_{ci}, F_{ij}$$ is 0 ∈ $$\mathbb{Z}_2$$. Then Using the formula (2) we have that

$$[F_{cj}, E_{ci}] = -(-1)^{E_{ci} F_{ci}} F_{ij} K_{c} K_{j}^{-1} q_i - q_i^{-1} F_{ij} K_{c} K_{j}^{-1} q_i^{-1} F_{ij}.$$

It is easy to see that $$\Omega([x, y]) = [\Omega(y), \Omega(x)]$$ for any $$x, y \in h(U_q)$$, applying which to (3) we obtain (4).

(5), (6) can be proved similarly.

(7) follows from an application of $$\Omega$$ to [10, Lemma 4.2(6)].

Lemma 4.2. [10]

1. $$[F_{ij}, F_{st}] = 0, \quad i < s < t < j,$$
2. $$[E_{ij}, F_{st}] = 0, \quad i < s < t < j,$$
3. $$[F_{ij}, E_{st}] = 0, \quad i < s < t < j.$$

Lemma 4.3. For $$i < s < j < t$$, we have

$$(a) \quad [E_{ij}, F_{st}] = (q_j^{-1} - q_j)(K_s K_j^{-1}) F_{jt} E_{is},$$
$$(b) \quad [E_{st}, F_{ij}] = (q_j - q_j^{-1}) F_{is} E_{jt} K_s K_j^{-1}. $$

Proof. It suffices to prove (a), (b) follows from the application of $$\Omega$$ to (a). Since $$[E_{ij}, -]$$ is a derivation of $$U_q$$, we have

$$[E_{ij}, F_{st}]$$

$$= [E_{ij}, F_{jt} F_{sj} - q_j F_{sj} F_{jt}]$$

$$= F_{jt} [E_{ij}, F_{sj}] - q_j [E_{ij}, F_{sj}] F_{jt}$$

(Using Lemma 4.1(6))

$$= F_{jt} E_{is} K_s K_j^{-1} q_s^{-1} - q_j E_{is} K_s K_j^{-1} q_s^{-1} F_{jt}$$

$$= (q_j^{-1} - q_j)(K_s K_j^{-1}) F_{jt} E_{is}.$$

\(\square\)
5 The simplicity of Kac modules

There is an order $<$ defined on the set of elements $E_{ij}, (i,j) \in \mathcal{I}([10])$:

$$E_{ij} < E_{st} \text{ if } (i,j) \in \mathcal{I}_0 \text{ and } (s,t) \in \mathcal{I}_1$$

or

$$(i,j), (s,t) \in \mathcal{I}_0, \theta = 0, 1, i < s \text{ or } i = s \text{ and } j < t,$$

$$F_{ij} < F_{st} \text{ if and only if } E_{ij} > E_{st}.$$  

For each $\delta \in \{0,1\}^\mathcal{I}_1$, let $E_1^\delta$ denote the product $\Pi_{(i,j)\in \mathcal{I}_1} E_{ij}^\delta$, in the order given above. Let $F_1^\delta = \Omega(E_1^\delta)$.

Set

$$N_1 = \langle E_1^\delta | \delta \in \{0,1\}^\mathcal{I}_1 \rangle, N_{-1} = \langle F_1^\delta | \delta \in \{0,1\}^\mathcal{I}_1 \rangle,$$

$$N_1^+ = \langle F_1^\delta | \sum \delta_{ij} > 0 \rangle, N_1^- = \langle E_1^\delta | \sum \delta_{ij} > 0 \rangle.$$

By [10], these are subalgebras of $U_q$, and

$$U_q = N_{-1} U_q(\mathfrak{g}_0) N_1 \cong N_{-1} \otimes U_q(\mathfrak{g}_0) \otimes N_1.$$  

By applying the $\mathbb{Z}_2$-graded anti-automorphism $\Psi$, we get

$$U_q = N_1 U_q(\mathfrak{g}_0) N_{-1} \cong N_1 \otimes U_q(\mathfrak{g}_0) \otimes N_{-1}.$$  

The subalgebra $U_q(\mathfrak{g}_0) N_1$ (resp. $N_{-1} U_q(\mathfrak{g}_0)$) has a nilpotent ideal $U_q(\mathfrak{g}_0) N_{1}^+$ (resp. $N_{-1}^+ U_q(\mathfrak{g}_0)$), by which each simple $U_q(\mathfrak{g}_0)$-module is annihilated. Therefore, each simple $U_q(\mathfrak{g}_0) N_1$-module can be identified with a simple $U_q(\mathfrak{g}_0)$-module(cf. [10]).

Let $U^0$ be the subalgebra of $U_q$ generated by the elements $K_i^{\pm 1}, i \in [1, m+n]$. Then by the PBW theorem([10]), $U^0$ is a polynomial algebra in variables $K_i^{\pm 1}, i \in [1, m+n]$. Let $K^\mu = \Pi_{i=1}^{m+n} K_i^{\mu_i}$ for $\mu = \sum_{i=1}^{m+n} \mu_i \epsilon_i \in \Lambda$. Denote

$$X(U^0) =: \text{Hom}_{\mathbb{C}(q)^*}(U^0, \mathbb{C}(q)).$$

Each $\lambda \in X(U^0)$ is completely determined by $\lambda(K_i) \in \mathbb{C}(q)^*, i \in [1, m+n]$. Then $X(U^0)$ is an additive group with the addition defined by

$$(\lambda_1 + \lambda_2)(K^\mu) = \lambda_1(K^\mu) \lambda_2(K^\mu), \mu \in \Lambda.$$  

Each $\lambda \in X(U^0)$ is called a weight for $U_q$. Note that $\Lambda$ can be canonically imbedded in $X(U^0)$ by letting

$$\mu(K_i) = q_i^\mu_i, i \in [1, m+n], \mu = \sum_{i=1}^{m+n} \mu_i \epsilon_i.$$  

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A weight in $\Lambda$ is called integral. Clearly we have $\lambda(K^\mu) = q^{(\lambda,\mu)}$ for $\lambda, \mu \in \Lambda$.

Let $M$ be a $U_q(\mathfrak{g}_0)$-module and let $\lambda \in X(U^0)$, let

$$M_\lambda = \{x \in M | ux = \lambda(u)x, u \in U^0\}.$$

A nonzero vector $v \in M_\lambda$ is called a maximal vector of weight $\lambda$ if $E_{ij}v^+ = 0$ for all $(i, j) \in I_0$. If $M$ is finite dimensional, then $M = \sum M_\lambda([6, \text{Prop. 5.1}])$. If $M$ is a finite dimensional simple $U_q(\mathfrak{g}_0)$-module, then there is a maximal vector, unique up to scalar multiple, which generates $M$. In this case we denote $M$ by $M(\lambda)$. Regard $M(\lambda)$ as a $U_q(\mathfrak{g}_0)\mathcal{N}_1$-module annihilated by $U_q(\mathfrak{g}_0)\mathcal{N}_1^+$. Define the Kac module

$$K(\lambda) = U_q \otimes_{U_q(\mathfrak{g}_0)\mathcal{N}_1} M(\lambda).$$

Then we have $K(\lambda) = \mathcal{N}_{-1} \otimes_{\mathbb{F}} M(\lambda)$ as $\mathcal{N}_{-1}$-modules.

To study the simplicity of $K(\lambda)$, we define a new order on $I_1$ by

$$(i, j) \prec (s, t) \text{ if } j > t \text{ or } j = t \text{ but } i < s.$$  

We denote $(i, j) \preceq (s, t)$ if $(i, j) \prec (s, t)$ or $(i, j) = (s, t)$. We define $F_{ij} \prec F_{st}$ if $(i, j) \prec (s, t)$.

For each subset $I \subseteq I_1$, denote by $F_I$ the product $\prod_{(i, j) \in I} F_{ij}$ in the new order. In particular, we let $F_\emptyset = 1$. For each $I \subseteq I_1$, set $E_I = \Omega(F_I)$.

For each $(i, j) \in I_1$, denote by $> (i, j)$ (resp. $\geq (i, j)$; $< (i, j)$; $\leq (i, j)$) the subset

$$\{(s, t) \in I_1 | (s, t) > (i, j)\}$$

(resp. $\{(s, t) \in I_1 | (s, t) \geq (i, j)\}$;

$$\{(s, t) \in I_1 | (s, t) < (i, j)\};$$

$$\{(s, t) \in I_1 | (s, t) \leq (i, j)\}$$

For $(i, j), (s, t) \in I_1$ with $(i, j) \prec (s, t)$, set

$$((i, j), (s, t)) = \{(i', j') \in I_1 | (i, j) \prec (i', j') \prec (s, t)\}.$$  

Then we have

$$F_{> (m, m+1)} = F_{< (1, m+n)} = 1 \text{ and } F_{I_1} = F_{< (i, j)} F_{> (i, j)} = F_{\leq (i, j)} F_{> (i, j)}$$

for any $(i, j) \in I_1$.

**Lemma 5.1.**  (a) $\mathcal{N}_{-1}$ (resp. $\mathcal{N}_{-1}^+$) has a basis $F_I$, $I \subseteq I_1$ (resp. $\phi \neq I \subseteq I_1$).

(b) $\mathcal{N}_1$ (resp. $\mathcal{N}_1^+$) has a basis $E_I$, $I \subseteq I_1$ (resp. $\phi \neq I \subseteq I_1$).

**Proof.** Since $\mathcal{N}_1 = \Omega(\mathcal{N}_{-1})$, (b) follows from the application of $\Omega$ to (a).

(a). Clearly the number of the above elements is equal to $\dim \mathcal{N}_{-1}$. We only need to show that the elements $F_I$ span $\mathcal{N}_{-1}$.
First we claim that any product $F_{ij} F_{st}$, $(i, j), (s, t) \in \mathcal{I}$ can be written as an $\mathbb{Z}[q, q^{-1}]$-linear combination of products in the new order. The case where $j = t$ and $i > s$ follows from Lemma 4.1(2). The cases where $j < t$ and $i \geq s$ follow from Lemma 4.1(1) and Lemma 4.2(1). The only case left is $i < s \leq m < j < t$, in which we have by Lemma 4.1(7) that

$$F_{ij} F_{st} = -F_{st} F_{ij} - (q_j - q_j^{-1}) F_{sj} F_{it}$$

(Under Lemma 4.2(1)) $$= -F_{st} F_{ij} + (q_j - q_j^{-1}) F_{it} F_{sj}.$$

Thus, the claim follows.

Since $\mathcal{I}$ is a finite set, by induction on the cardinality $|I|$ of $I$ we obtain that each product $\Pi_{(i,j) \in I \subseteq \mathcal{I}} F_{ij}$ in any order can be written as a $\mathbb{Z}[q, q^{-1}]$-linear combination of elements $F_{I'}$, $I' \subseteq \mathcal{I}$.

Lemma 5.2. Let $(i, k) \in \mathcal{I}$. Then $F_{st} F_{\geq (i, k)} = 0$ for any $F_{st} \geq F_{ik}$ or, equivalently, $F_{st} F_{> (i, k)} = 0$ for any $F_{st} \succ F_{ik}$.

Proof. Denote the set $\geq (i, k)$ by $I$. We proceed with induction on $|I|$. The case $|I| = 1$ is trivial. Assume the lemma for $|I| = d > 1$.

Note that $F_I = F_{i,k} F_{> (i, k)}$. By Lemma 4.1(2) and the formula (1) in the preceding section, we have $F_{st} F_I = 0$ for any $(s, t) \in \mathcal{I}$ with $t = k$.

Suppose $t < k$. If $s \geq i$, by Lemma 4.1(2) and the formulas (1) in Sec. 4 we have

$$F_{st} F_I = \pm q^z F_{ik} F_{st} F_{> (i, k)} = 0, z \in \mathbb{Z},$$

where the last equality is given by the induction hypothesis.

If $s < i$, then we must have $s < i \leq m < t < k$. Note that $F_{it} \succ F_{ik}$ and $F_{st} \succ F_{ik}$. Then using Lemma 4.1(7) and the induction hypothesis we obtain

$$F_{st} F_I = -F_{ik} F_{st} F_{> (i, k)} + (q_t - q_t^{-1}) F_{sk} F_{it} F_{> (i, k)} = 0.$$

By a similar proof we can show that

Lemma 5.3. Let $(i, k) \in \mathcal{I}$. Then $F_{\leq (i, k)} F_{st} = 0$ for any $F_{st} \leq F_{ik}$ or, equivalently, $F_{< (i, k)} F_{st} = 0$ for any $F_{st} \prec F_{ik}$.

Proposition 5.4. For every $(i, j) \in \mathcal{I}$, there are $z_1, z_2 \in \mathbb{Z}$ such that

(1) $F_{ij} F_{< (i, j)} = \pm q^{z_1} F_{\leq (i, j)}$

(2) $F_{> (i, j)} F_{ij} = \pm q^{z_2} F_{\geq (i, j)}$. 
Proof. (1) Let \( (k,s) \in I_1 \). By Lemma 4.1, 4.2 we have,

\[
F_{i,j}F_{k,s} = \begin{cases} 
-F_{ks}F_{i,j} & \text{if } k < i \text{ and } s > j \\
q_kF_{ks}F_{ij} & \text{if } i = k \text{ and } s > j \\
-F_{ks}F_{ij} + (q_j - q_j^{-1})F_{i,s}F_{kj} & \text{if } i < k < j < s.
\end{cases}
\]

Let \((k_1, s_1), \ldots, (k_p, s_p)\) be all the pairs in the set \(\leq (i, j)\) such that \(i < k_t < j < s_t, t = 1, \ldots, p\), so that \(F_{i,s} < F_{k_t,s_t}\). Then there are integers \(z_1', \ldots, z_p'\) such that

\[
F_{ij}F_{\leq (i,j)} = F_{ij}F_{\leq (k_1,s_1)}F_{k_1,s_1}F_{((k_1,s_1),(i,j))} = \pm q^{z_1'}F_{\leq (k_1,s_1)}(F_{ij}F_{k_1,s_1})F_{((k_1,s_1),(i,j))} = \pm q^{z_1'}F_{\leq (k_1,s_1)}(-F_{k_1,s_1}F_{ij} + (q_j - q_j^{-1})F_{i,s}F_{kj})F_{((k_1,s_1),(i,j))}
\]

(Using Lemma 5.3) = \(\pm q^{z_1'}F_{\leq (k_1,s_1)}F_{ij}F_{((k_1,s_1),(i,j))} = \ldots = \pm q^{z'}F_{\leq (i,j)}F_{ij} = \pm q^{z'}F_{\leq (i,j)}\).

(2) can be verified similarly. \(\square\)

As an immediate consequence, we have

**Corollary 5.5.** Let \((i,j), (s, t) \in I_1\) with \((s, t) \preceq (i, j)\). Then \(F_{st}F_{\leq (i,j)} = 0\).

**Lemma 5.6.** Each nonzero submodule of \(K(\lambda)\) contains \(F_{I_1} \otimes v\) for some \(0 \neq v \in M(\lambda)\).

**Proof.** Let \(I, I'\) be two nonempty subsets of \(I_1\). We define \(I < I'\) if, with respect to the order in \(I_1\), the first pair \((s, t) \notin I \cap I'\) is in \(I'\). Then we have by Prop. 5.4 that \(F_{st}F_I = \pm q^zF_{I \cap (s, t)}\) for some \(z \in \mathbb{Z}\) and \(F_{st}F_{I'} = 0\).

Let \(N = N_0 \oplus N_1\) be a nonzero submodule of \(K(\lambda)\). Take a nonzero element \(x = \sum_{I \subseteq I_1} F_I \otimes v_I \in N, v_I \neq 0\) for all \(I\). Let \(\bar{I}\) be the minimal subset appeared in the expression.

We proceed with induction on the order of \(\bar{I}\). If \(\bar{I} = I_1\), that is, \(x = F_{I_1} \otimes v\), the lemma follows. Suppose \(\bar{I} \neq I_1\). Let \((s, t) \in I_1\) be the first pair such that \((s, t) \notin \bar{I}\). Then by definition we have \((i, j) \in I\) for all \((i, j) < (s, t)\) and all \(I\) appeared above. Applying \(F_{st}\) to \(x\) and using Prop. 5.4, we have \(F_{st}x \neq 0\), and the minimal \(I\) appeared in \(F_{st}x\), denoted \(I'\), satisfies \(I' > \bar{I}\). Then the induction hypothesis yields the lemma. \(\square\)

**Lemma 5.7.** For any \((i, j) \in I_0\), there is \(z \in \mathbb{Z}\) such that \(F_{ij}F_{I_1} = q^zF_{I_1}F_{ij}\).

**Proof.** Recall that \(F_{ij} = -q_cF_{ic}F_{cj} + F_{cj}F_{ic}, \ i < c < j\). Then it suffices to consider the case \(j = i + 1\).
By Lemma 4.1(1), (2) and Lemma 4.2(1) we have

\[ F_{i,i+1}F_{sk} = \begin{cases} 
q_k F_{sk}F_{k,k+1} + F_{s,k+1}, & \text{if } i = k \\
q_s^{-1}(F_{i+1,k}F_{i,i+1} - F_{ik}), & \text{if } s = i + 1 \\
q^2 F_{sk}F_{i,i+1}, & \text{otherwise,}
\end{cases} \]

for some \( z \in \mathbb{Z} \). Since \((i, i + 1) \in \mathcal{I}_0 \), we have that \( F_{i,i+1} \) commutes, up to multiple of \( q^z, z \in \mathbb{Z} \), with all \( F_{sk} \), \((s, k) \in \mathcal{I}_1 \), but the case \( s = i + 1 \) if \( i < m \) and the case \( i = k \) if \( i > m \).

Assume \( i < m \). Then we have

\[
F_{i,i+1}F_{iz} = F_{i,i+1}F_{<(i+1,m+n)}F_{i+1,m+n}F_{>(i+1,m+n)} = q^{z_1}F_{<(i+1,m+n)}(F_{i,i+1}F_{i+1,m+n})F_{>(i+1,m+n)}
\]

(Using \( F_{<(i+1,m+n)}F_{i,m+n} = 0 \))

\[
= q^{z_1-1}F_{<(i+1,m+n)}(F_{i,i+1}F_{i,i+1})F_{>(i+1,m+n)}
= q^{z_2}F_{<(i+1,m+n-1)}(F_{i,i+1}F_{i,i+1,m+n-1})F_{>(i+1,m+n-1)}
= \ldots
= q^z F_{iz} F_{i,i+1}.
\]

Similarly one verifies that \( F_{i,i+1}F_{iz} = q^z F_{iz} F_{i,i+1} \) for some \( z \in \mathbb{Z} \), if \( i > m \). This completes the proof.

**Lemma 5.8.** For any \((i, j) \in \mathcal{I}_0 \), we have \( E_{ij}F_{iz} = F_{iz}E_{ij} \).

**Proof.** By the formula \( E_{ij} = E_{ic}E_{cj} - q_s^{-1}E_{cj}E_{ic} \), it suffices to assume \( j = i + 1 \). Recall the (even) derivation \([E_{i,i+1}, -] \) of \( U_q \).

Using the definition of \( U_q \) and Lemma 4.1(1), (2) we have, for any \((s, k) \in \mathcal{I}_1 \),

\[
[E_{i,i+1}, F_{sk}] = \begin{cases} 
-F_{i+1,k} K_{i,i+1}^{-1} q_i, & \text{if } i = s \\
F_{si} K_{i,i+1}^{-1} q_i, & \text{if } i + 1 = k \\
0, & \text{otherwise.}
\end{cases}
\]

Then we have

\[
[E_{i,i+1}, F_{iz}] = \sum_{(s, k) \in \mathcal{I}_1} F_{<(s,k)}[E_{i,i+1}, F_{sk}]F_{>(s,k)}
= \begin{cases} 
\sum_{s=1} F_{<(s,k)}(-F_{i+1,k} K_{i,i+1}^{-1} q_i)F_{>(s,k)}, & \text{if } i < m \\
\sum_{k=i+1} F_{<(s,k)}(F_{si} K_{i+1}^{-1} q_i)F_{>(s,k)}, & \text{if } i > m 
\end{cases}
= 0.
\]

where the last equality is given by the fact that \( F_{i+1,k} \succ F_{sk} \) if \( s = i \) and \( F_{si} \succ F_{sk} \) if \( k = i + 1 \). Then the lemma follows. \( \square \)
Let $E_{I_1} = \Omega(F_{I_1})$. Using the triangular decomposition $U_q = U_q^- \otimes U^0 \otimes U_q^+$ we have

$$E_{I_1}F_{I_1} = f(K) + \sum u_i^-u_i^0u_i^+ , u_i^\pm \in U_q^\pm, f(K), u_i^0 \in U^0.$$ 

Note that $U_q$ is a $U^0$-module under the conjugation:

$$K_i \cdot u = K_iuK_i^{-1}, 1 \leq i \leq m + n.$$ 

Since the $U^0$-weight of $E_{I_1}F_{I_1}$ is zero, we get $u_i^+ = 0$ if and only if $u_i^- = 0$.

Let $v_\lambda$ be a maximal vector in $M(\lambda) \subseteq K(\lambda)$. Then we get

$$E_{I_1}F_{I_1}v_\lambda = f(K)v_\lambda = f(K)(\lambda)v_\lambda, f(K)(\lambda) \in C(q).$$

As $\lambda \in X(U^0)$ varies, one obtains a function $f(K)(\lambda)$. We denote it by $f_{m,n}(\lambda)$.

**Proposition 5.9.** The $U_q$-module $K(\lambda)$ is simple if and only if $f_{m,n}(\lambda) \neq 0$.

**Proof.** Assume $f_{m,n}(\lambda) \neq 0$. Let $N = N_0 \oplus N_1$ be a nonzero submodule of $K(\lambda)$. By Lemma 5.6, we have $F_{I_1} \otimes v \in N$ for some $0 \neq v \in M(\lambda)$. Since $K_iF_{I_1} = q^aF_{I_1}K_i$ for some $a_i \in \mathbb{Z}$, we may assume $v$ is a weight vector. Since $M(\lambda)$ contains a unique (up to scalar multiple) maximal vector $v_\lambda$, there is a sequence of elements $E_{\alpha_{i_1}} \cdots E_{\alpha_{i_s}} \in U_q(\mathfrak{g}_0)$ such that

$$E_{\alpha_{i_1}} \cdots E_{\alpha_{i_s}} v = v_\lambda.$$ 

Then Lemma 5.8 implies that $F_{I_1} \otimes v_\lambda \in N$, and hence $E_{I_1}F_{I_1} \otimes v_\lambda = f_{m,n}(\lambda) \otimes v_\lambda \in N$. It follows that $v_\lambda \in N$ and hence $N = K(\lambda)$, so that $K(\lambda)$ is simple.

Suppose $K(\lambda)$ is simple. By Lemma 5.7, 5.8, the subspace $F_{I_1} \otimes M(\lambda) \subseteq K(\lambda)$ is a $U_q(\mathfrak{g}_0)$-submodule, and hence simple. Note that Corollary 5.5 says that $N^{-1}_1F_{I_1} \otimes M(\lambda) = 0$, so that $F_{I_1} \otimes M(\lambda)$ is a simple $N^{-1}_1U_q(\mathfrak{g}_0)$-module annihilated by $N^{-1}_1U_q(\mathfrak{g}_0)$. Since $K(\lambda)$ is simple, we have

$$K(\lambda) = N_1U_q(\mathfrak{g}_0)N^{-1}_1F_{I_1} \otimes M(\lambda) = N_1F_{I_1} \otimes M_0(\lambda).$$

Since $\dim N^{-1}_1 = \dim N_1$, we have that $K(\lambda)$ has a basis

$$E_{I_1}F_{I_1} \otimes v_i, I \subseteq I_1, i = 1, \ldots, s,$$

with $v_1, \ldots, v_s$ a basis of $M(\lambda)$. We can choose $v_1 = v_\lambda$. Then we get

$$0 \neq F_{I_1}F_{I_1}v_\lambda = f_{m,n}(\lambda)v_\lambda,$$

so that $f_{m,n}(\lambda) \neq 0.$

}$\square$
6 The polynomial $f_{m,n}(\lambda)$

This section is devoted to the determination of the polynomial $f_{m,n}(\lambda)$, for $\lambda \in X(U^0)$. Let us note that R. Zhang defined in [12] a polynomial using a different order of the product $\prod_{(i,j) \in \mathcal{I}_t} F_{ij}$.

**Lemma 6.1.** For $1 \leq i \leq m$, we have $E_{i,m+n}F_{> (i,m+n)}v_{\lambda} = 0$.

*Proof.* Using the formulas from Lemma 4.1, 4.3, we have, for any $(s,t) \succ (i, m+n)$,

$$[E_{i,m+n}, F_{st}] = \begin{cases} 
E_{t,m+n}K_s^{-1}K_t^{-1}, & \text{if } s = i, t < m+n \\
E_{is}K_sK_{m+n}q_s^{-1}, & \text{if } s > i, t = m+n \\
(q_t - q_t^{-1})F_{si}E_{t,m+n}K_i^{-1}K_t^{-1}, & \text{if } s < i < t < m+n \\
0, & \text{otherwise}.
\end{cases}$$

Then we have

$$E_{i,m+n}F_{> (i,m+n)}v_{\lambda} = [E_{(i,m+n)}, F_{> (i,m+n)}]v_{\lambda}$$

$$= \sum_{F_{st} > F_{i,m+n}} (-1)^{\alpha_{st}}F_{((i,m+n),(s,t))}[E_{i,m+n}, F_{st}]F_{> (s,t)}v_{\lambda}$$

$$= \sum_{s > i, t = m+n} (-1)^{\alpha_{st}}F_{((i,m+n),(s,t))}(E_{is}K_sK_{m+n}q_s^{-1})F_{> (s,m+n)}v_{\lambda}$$

$$+ \sum_{s = i, t < m+n} (-1)^{\alpha_{st}}F_{((i,m+n),(s,t))}(E_{t,m+n}K_i^{-1}K_t)F_{> (s,t)}v_{\lambda}$$

$$+ \sum_{s < i < t < m+n} (-1)^{\alpha_{st}}F_{((i,m+n),(s,t))}((q_t - q_t^{-1})F_{si}E_{t,m+n}K_i^{-1}K_t)F_{> (s,t)}v_{\lambda},$$

where $\alpha_{st} \in \mathbb{Z}_2$. Note that the second and the third summation are equal to zero, since $E_{t,m+n}$ commutes with all $F_{ij}((i, j) \in \mathcal{I}_1)$ with $F_{ij} \succ F_{st}$.

We claim that the first summation is also equal to zero. In fact, we have, in the case where $s > i, t = m+n$,

$$E_{is}F_{> (s,m+n)}v_{\lambda} = [E_{is}, F_{> (s,m+n)}]v_{\lambda}$$

$$= \sum_{j = m+n-1}^{m+1} \sum_{k=i}^{s-1} F_{((s,m+n),(k,j))}[E_{is}, F_{kj}]F_{> (k,j)}v_{\lambda}.$$ 

For $k = i, m+1 \leq j \leq m+n - 1$, we have by Lemma 4.1(3) that

$$[E_{is}, F_{kj}]F_{> (k,j)}v_{\lambda} = q_s F_{sj}(K_iK_s^{-1})F_{> (k,j)}v_{\lambda} = 0,$$

where the last equality is given by the fact that $(s,j) \succ (k,j)$.

For $i < k \leq s - 1$, we have by using Lemma 4.3(a) that

$$[E_{is}, F_{kj}]F_{> (k,j)}v_{\lambda} = (q_s^{-1} - q_s)(K_iK_s^{-1})E_{ik}F_{sj}F_{> (k,j)}v_{\lambda} = 0,$$
where the last equality follows from the fact that \((s, j) \succ (k, j)\). Thus, the claim follows.

\[ \square \]

For \((i, j) \in \mathcal{I}\), let \(K_{ij} = K_iK_j^{-1}\). Let us denote

\[
[(\lambda + \rho)(K_{ij})] = \frac{(\lambda + \rho)(K_{ij}) - (\lambda + \rho)(K_{ij}^{-1})}{q - q^{-1}}.
\]

Then we see that \([(\lambda + \rho)(K_{ij})] = [(\lambda + \rho, \epsilon_i - \epsilon_j)]\) if \(\lambda\) is integral.

**Theorem 6.2.** Let \(\lambda \in X(U^0)\). Then \(f_{m,n}(\lambda) = \Pi_{(i,j)\in \mathcal{I}_1}[(\lambda + \rho)(K_{ij})]\). In particular, \(f_{m,n}(\lambda) = \Pi_{(i,j)\in \mathcal{I}_1}[(\lambda + \rho, \epsilon_i - \epsilon_j)]\) if \(\lambda\) is integral.

**Proof.** Using the formula (2) in Sec. 4, we have

\[
E_{\mathcal{I}_i}F_{\mathcal{I}_i}v_\lambda = E_{(1,m+n)}(E_{1,m+n}F_{1,m+n})F_{(1,m+n)}v_\lambda
\]

\[
= E_{(1,m+n)}(\frac{K_{1,m+n} - K_{1,m+n}^{-1}}{q - q^{-1}})F_{(1,m+n)}v_\lambda
\]

\[
- E_{(1,m+n)}F_{1,m+n}E_{1,m+n}F_{(1,m+n)}v_\lambda
\]

(Using Lemma 6.1) = \[
E_{(1,m+n)}\frac{K_{1,m+n} - K_{1,m+n}^{-1}}{q - q^{-1}}F_{(1,m+n)}v_\lambda
\]

\[
= \[(\lambda + \alpha_1)(K_{1,m+n})]E_{(1,m+n)}F_{(1,m+n)}v_\lambda,
\]

where \(\lambda + \alpha_1\) is the weight of \(F_{(1,m+n)}v_\lambda\).

Next we compute \(E_{(i,m+n)}F_{(i,m+n)}v_\lambda\) in a similar way. Continue the process, we get

\[
E_{\mathcal{I}_i}F_{\mathcal{I}_i}v_\lambda = \[(\lambda + \alpha_1)(K_{1,m+n})]E_{(1,m+n)}F_{(1,m+n)}v_\lambda
\]

\[
= \[(\lambda + \alpha_1)(K_{1,m+n})][\lambda + \alpha_2)(K_{2,m+n})]E_{(2,m+n)}F_{(2,m+n)}v_\lambda
\]

= \[\cdots\]

\[
= \Pi_{i=1}^m[(\lambda + \alpha_i)(K_{i,m+n})]E_{(i,m+n)}F_{(i,m+n)}v_\lambda,
\]

where \(\lambda + \alpha_i\) is the weight of \(F_{(i,m+n)}v_\lambda\), \(1 \leq i \leq m\). It is easily seen that

\[
\lambda + \alpha_i = \lambda - 2\rho_1 + \sum_{k=1}^i(\epsilon_k - \epsilon_{m+n}).
\]

By the proof of [11, Th.4], we have

\[
(\alpha_i, \epsilon_i - \epsilon_{m+n}) = (\rho, \epsilon_i - \epsilon_{m+n}),
\]

so that

\[
(\lambda + \alpha_i)(K_{i,m+n}) = \lambda(K_{i,m+n})q^{(\rho, \epsilon_i - \epsilon_{m+n})} = \(\lambda + \rho)(K_{i,m+n})
\]

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for any \( i \leq m \), which gives

\[
f_{m,n}(\lambda) = \prod_{k=1}^{m}[(\lambda + \rho)(K_{k,m+n})]E_{\geq(1,m+n-1)}F_{\geq(1,m+n-1)}\psi_\lambda.
\]

We now prove the proposition by induction on \( n \). The case \( n = 1 \) follows immediately from the equation above. Assume the proposition for \( n-1 \). To proceed, let us denote by \( \rho_{m,n-1} \) the \( \rho \) for Lie superalgebra \( gl(m, n - 1) \). By the proof of [11, Th.4], we have

\[
(\rho_{m,n-1}, \epsilon_i - \epsilon_j) = (\rho, \epsilon_i - \epsilon_j)
\]

for \( i < m < j \leq m + n - 1 \). Applying the induction hypothesis, we have

\[
f_{m,n}(\lambda)
= \prod_{k=1}^{m}[(\lambda + \rho)(K_{k,m+n})]f_{m,n-1}(\lambda)
= \prod_{k=1}^{m}[(\lambda + \rho)(K_{k,m+n})]\prod_{i<m<j\leq m+n-1}(\lambda(\rho_{m,n-1})(K_{ij}) - q - q^{-1})
= \prod_{k=1}^{m}[(\lambda + \rho)(K_{k,m+n})]\prod_{i<m<j\leq m+n-1}\frac{\lambda(K_{ij})q^{(\rho_{m,n-1}, \epsilon_i - \epsilon_j)} - \lambda(K_{ij})^{-1}q^{-1}}{q - q^{-1}}
= \prod_{(i,j)\in I_1}[(\lambda + \rho)(K_{ij})].
\]

\[\square\]

7 Representations of \( U_q \) at roots of unity

7.1 Simple \( U_q \)-modules

Let \( l \) be an odd number \( \geq 3 \) and let \( \eta \) be a primitive \( l \)th root of unity. For \( 1 \leq i \leq m \), let \( \eta_i = \begin{cases} \eta, & \text{if } i \leq m \\ \eta^{-1}, & \text{if } i > m \end{cases} \)

Set

\( \mathcal{A}' = \{ f(q)/g(q) |, f(q), g(q) \in \mathcal{A}, g(\eta) \neq 0 \} \).

Let \( U_{\mathcal{A}'} \) be the \( \mathcal{A}' \)-subalgebra of \( U_q \) generated by the elements

\[
E_{i,i+1}, F_{i,i+1}, K_{j}^{\pm1}, i \in [1, m+n], j \in [1, m+n].
\]

For \( \psi = (\psi_{ij}) \in \mathbb{N}^{I_1} \), let \( E_0^\psi \) denote the product \( \prod_{(i,j)\in I_1} E_{ij}^{\psi_{ij}} \) in the order given in Sec.5 and let \( F_0^\psi = \Omega(E_0^\psi) \). Recall the notion \( E_I, F_I, I \subseteq I_1 \). Then by Lemma 5.1 and the PBW theorem of \( U_q \)(see [10]) we have

**Corollary 7.1.** The \( \mathcal{A}' \)-superalgebra \( U_{\mathcal{A}'} \) has an \( \mathcal{A}' \)-basis

\[
F_I F_0^\psi K^\mu E_0^\psi E_{I'} \psi, I, I' \subseteq I_1, \psi, \psi' \in \mathbb{N}^{I_1}, \mu \in \Lambda.
\]
Let $U_{A'}(g_0)$ (resp. $N_{-1,A'}$; $N_{-1,A}$) be the $A'$-subalgebra of $U_A$ generated by elements $E_{a_i}, F_{a_i}, K_{a_i}^\pm, i \in [1, m+n] \setminus m, j \in [1, m+n]$ (resp. $E_{ij}, (i, j) \in I_1; F_{ij}, (i, j) \in I_1$). Then we have by Sec. 5 that

$$U_{A'} = N_{-1,A'} U_{A'}(g_0) N_{1,A'}.$$ 

Moreover, we have from the above corollary that there is an $A'$-module isomorphism;

$$U_{A'} \cong N_{-1,A'} \otimes U_{A'}(g_0) \otimes N_{1,A'}.$$ 

Lemma 5.1 says that $N_{-1,A'}$ (resp. $N_{1,A'}$) has an $A'$-basis $F_I$ (resp. $E_I$), $I \subseteq I_1$.

Let $N_{1,A'}^+$ (resp. $N_{-1,A'}^+$) be the $A'$-submodule of $N_1$ (resp. $N_{-1}$) generated by elements $E_I$ (resp. $F_I$), $I \neq \emptyset$. Then by [10] $N_{1,A'}^+$ (resp. $N_{-1,A'}^+$) is an $A'$-subalgebra of $U_A$. Moreover, using the formulas from Sec. 4 it is easy to see that $U_{A'}(g_0) N_{1,A'}$ and $N_{-1,A'} U_{A'}(g_0)$ are $A'$-subalgebras of $U_A$ having $U_{A'}(g_0) N_{1,A'}$ and $N_{-1,A'} U_{A'}(g_0)$ as nilpotent ideals respectively.

Let $B_{A'}$ (resp. $B^{-}_{A'}; U^{0}_{A'}$) be the $A'$-subalgebra of $U_{A'}(g_0)$ generated by elements $E_{a_i}, i \neq m$ (resp. $F_{a_i}, i \neq m; K_{a_i}^\pm, 1 \leq i \leq m+n$). By [6, Th. 4.21], we have

$$U_{A'}(g_0) \cong B_{A'} \otimes U_{A'}^0 \otimes B^{-}_{A'}.$$ 

Moreover, the $A'$-algebra $B_{A'}$ (resp. $B^{-}_{A'}$) is the algebra generated by the elements $E_{a_i}$ (resp. $F_{a_i}$), $i \neq m$ with relations (R5), (R6)(resp. (R5), (R7)). Set

$$U_\eta = U_{A'} \otimes A' \mathbb{C}, \quad U_\eta(g_0) = U_{A'}(g_0) \otimes A' \mathbb{C}$$

$$N_{-1,\eta} = N_{-1,A'} \otimes \mathbb{C}, \quad N_{1,\eta} = N_{1,A'} \otimes \mathbb{C}$$

$$N_{1,\eta}^+ = N_{1,A'}^+ \otimes \mathbb{C}, \quad N_{-1,\eta}^+ = N_{-1,A'}^+ \otimes \mathbb{C}$$

$$B_\eta = B_{A'} \otimes \mathbb{C}, \quad B^{-}_\eta = B^{-}_{A'} \otimes \mathbb{C},$$

$$U^0_\eta = U_{A'} \otimes \mathbb{C}.$$ 

where $\mathbb{C}$ is viewed as an $A'$-algebra with $q$ acting as multiplication by $\eta$. Then $U_\eta(g_0), N_{\pm 1,\eta}, N_{1,\eta}^+$ can be viewed as $\mathbb{C}$-subalgebras of $U_\eta$. We also have $\mathbb{C}$-algebra isomorphisms:

$$U_\eta \cong N_{-1,\eta} \otimes U_\eta(g_0) \otimes N_{1,\eta}$$

$$U_\eta(g_0) \cong B^{-}_\eta \otimes U^0_\eta \otimes B_\eta.$$ 

For $x \in U_A$, we denote $x \otimes 1 \in U_\eta$ also by $x$. Then $B_\eta$ (resp. $B^{-}_\eta$) is the algebra generated by the elements $E_{a_i}$ (resp. $F_{a_i}$), $i \neq m$ with relations (R5), (R6)(resp. (R5), (R7)) in which $q$ is replaced by $\eta$.

**Corollary 7.2.** (PBW theorem) The $\mathbb{C}$-superalgebra $U_\eta$ has a basis

$$F_I F_0^\psi K^{\mu} F_0^{\psi'} E_I', I, I' \subseteq I_1, \psi, \psi' \in N^{I_0}, \mu \in \Lambda.$$ 

The center of the $\mathbb{C}$-superalgebra $U_\eta$ is defined by

$$Z(U_\eta) = \{ x \in (U_\eta)_0 | xu = ux \quad \text{for all} \quad u \in U_\eta \}.$$
Let \((i,j) \in \mathcal{I}_0, s \in [1, m + n]\). Then it is easy to see that
\[
x_{i j} =: F_{i j}^{l}, y_{i j} =: F_{i j}^{l}, z_{s}^{\pm 1} =: K_{s}^{\pm l}
\]
are all contained in \(Z(U_{\eta})\). By the PBW theorem for \(U_{\eta}\), the \(\mathbb{C}\)-subalgebra \(Z_{0}\) generated by these elements is a polynomial algebra in variables \(x_{i j}, y_{i j}, z_{s}^{\pm 1}\). Set
\[
\Lambda_{l} =: \{k_{i} \epsilon_{i} + \cdots + k_{m+n} \epsilon_{m+n} \in \Lambda | 0 \leq k_{i} < l, i = 1, \cdots, m+n]\).

Clearly we have

**Lemma 7.3.** \(U_{\eta}\) is a free \(Z_{0}\)-module having a basis
\[
F_{i} F_{i}^{\psi} K_{1}^{\mu} E_{1}^{\psi} E_{1}'^{\mu}, I, I' \subseteq \mathcal{I}, \psi, \psi' \in [0, l], \mu \in \Lambda_{l}.
\]

Let \(M = M_{0} \oplus M_{1}\) be a simple \(U_{\eta}\)-module. For any \(z \in Z_{0}\), we define a linear mapping
\[
\phi_{z} : M \rightarrow M, \phi_{z}(x) = zx, x \in M.
\]
Clearly \(\phi_{z}\) is an even \(U_{\eta}\)-module homomorphism. Since \(\ker \phi_{z}\) is a \(\mathbb{Z}_{2}\)-graded submodule of \(M\), either \(\ker \phi_{z} = M\) or \(\ker \phi_{z} = 0\). In the former case, we have \(\phi_{z} = 0\); in the latter case, the simplicity of \(M\) says that \(\phi_{z}(M) = M\), so that \(\phi_{z}\) is an (even) isomorphism.

**Lemma 7.4.** ([9, Lemma 2.1, Ch.5]) Let \(R\) be a commutative ring with unity and suppose that \(I \subseteq R\) is an ideal of \(R\). Let \(V\) be a finitely generated unitary \(R\)-module with annihilator \(\text{ann}_{R}(V) = \{r \in R | rv = 0 \text{ for all } v \in V\}\). If \(IV = V\), then \(I + \text{ann}_{R}(V) = R\).

**Proposition 7.5.** Let \(M = M_{0} \oplus M_{1}\) be a simple \(U_{\eta}\)-module. Then \(M\) is finite dimensional.

*Proof.* Let \(V = V_{0} \oplus V_{1}\) be a simple \(U_{\eta}\)-module. Since \(U_{\eta}\) is a finitely generated \(Z_{0}\)-module by Lemma 7.3, \(V\) is a finitely generated \(Z_{0}\)-module. Given any ideal \(I \subseteq Z_{0}\), \(IV\) is a \(U_{\eta}\)-submodule of \(V\). Then either \(IV = V\) or \(IV = 0\). Since \(I \in Z_{0}\), \(\text{ann}_{Z_{0}}(V) \neq Z_{0}\). Let \(I \neq Z_{0}\) be any ideal containing \(\text{ann}_{Z_{0}}(V)\). If \(IV = V\), then by the above lemma we get \(Z_{0} = \text{ann}_{Z_{0}}(V) + I = I\), a contradiction. Therefore, we have \(IV = 0\); that is \(I = \text{ann}_{Z_{0}}(V)\), which implies that \(\text{ann}_{Z_{0}}(V)\) is a maximal ideal of \(Z_{0}\). By Hilbert’s nullstellensatz, \(Z_{0}/\text{ann}_{Z_{0}}(V)\) is finite dimensional over \(\mathbb{C}\). Since \(V\) is finite dimensional over \(Z_{0}/\text{ann}_{Z_{0}}(V)\), \(V\) is finite dimensional over \(\mathbb{C}\).

**Lemma 7.6.** For each simple \(U_{\eta}\)-module \(V = V_{0} \oplus V_{1}\), there is a \(\mathbb{C}\)-algebra homomorphism \(\chi : Z_{0} \rightarrow \mathbb{C}\) such that \((z - \chi(z))M = 0\) for any \(z \in Z_{0}\).

*Proof.* Let \(z \in Z_{0}\). Since \(\mathbb{C}\) is algebraically closed and \(V\) is finite dimensional, there is \(\chi(z) \in \mathbb{C}\) and nonzero \(v \in V\) such that \(zv = \chi(z)v\). Then
\[
V_{\chi} =: \{v \in V | zv = \chi(z)v\} \neq 0.
\]
Since \(z \in (U_{\eta})_{0}, V_{\chi}\) is \(\mathbb{Z}_{2}\)-graded. Clearly \(V_{\chi}\) is a \(U_{\eta}\)-submodule of \(V\). Thus, we have \(V = V_{\chi}\); that is, \(z\) acts as multiplication by \(\chi(z)\) on \(V\). It is routine to verify that \(\chi\) defines a \(\mathbb{C}\)-algebra homomorphism \(Z_{0} \rightarrow \mathbb{C}\).

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Let $\chi$ be as in the lemma. Define $I_{\chi}$ (resp. $I^0_{\chi}$) to be the two-sided ideal of $U_{\eta}(g_0)$ generated by the central elements

$$x_{ij} - \chi(x_{ij}), y_{ij} - \chi(y_{ij}), z_s^{\pm 1} - \chi(z_s^{\pm 1}), (i, j) \in I_0, s \in [1, m + n].$$

Define the superalgebras

$$u_{\eta, \chi} := U_{\eta}/I_{\chi}, u_{\eta, \chi}(g_0) = U_{\eta}(g_0)/I^0_{\chi}.$$

**Lemma 7.7.** $I_{\chi} = N_{-1, \eta} I^0_{\chi} N_{1, \eta}.$

**Proof.** Since the elements $x - \chi(x), x = x_{ij}, y_{ij}, z_s^{\pm 1}$ are central in $U_{\eta}$ and all contained in $U_{\eta}(g_0)$, we have

$$I_{\chi} = \sum_x U_{\eta}(x - \chi(x)) = N_{-1, \eta} \sum_x U_{\eta}(g_0)(x - \chi(x)) N_{1, \eta} = N_{-1, \eta} I^0_{\chi} N_{1, \eta}.$$

**Corollary 7.8.** There is a $C$-algebra isomorphism: $u_{\eta, \chi} \cong N_{-1, \eta} \otimes u_{\eta, \chi}(g_0) \otimes N_{1, \eta}$.

**Proof.** By the lemma above, we have

$$u_{\eta, \chi} = U_{\eta}/I_{\chi} \cong N_{-1, \eta} \otimes U_{\eta}(g_0) \otimes N_{-1, \eta} / N_{-1, \eta} \otimes I^0_{\chi} \otimes N_{-1, \eta} \cong N_{-1, \eta} \otimes (U_{\eta}(g_0)/I^0_{\chi}) \otimes N_{1, \eta} = N_{-1, \eta} \otimes u_{\eta, \chi}(g_0) \otimes N_{1, \eta}.$$

By Lemma 7.6, each simple $U_{\eta}$-module is a simple $u_{\eta, \chi}$-module for some $\chi$. As in [2], one can define derivations $e_{\alpha_i}, f_{\alpha_i}, k_{\pm \alpha_j}, i \in [1, m + n) \setminus m, j \in [1, m + n]$ of the superalgebra $U_{\eta}$ by

$$e_{\alpha_i} = [E^{(l)}_{\alpha_i}, -], f_{\alpha_i} = [F^{(l)}_{\alpha_i}, -], k_{\pm \alpha_j} = [K^{(l)}_{\pm \alpha_j}, -].$$

These derivations induce derivations on $U_{\eta}$. By applying automorphisms of $U_{\eta}$ as that in [1, 3.5,3.6], [2, Th.6.1], one can assume $\chi(x_{ij}) = 0$ for any $(i, j) \in I_0$ in studying simple $U_{\eta}$-modules or simple $U_{\eta}(g_0)$-modules.
Assume $\chi(x_{ij}) = 0$ in the following. Denote by $B_\chi$ (resp. $B_\chi^{-}; U^0$) the image of $B_\eta$ (resp. $B_\eta^{-}; U^0$) in $u_{\eta,\chi}$. Since

$$I_\chi^0 = \sum_x U_\eta(g_0)(x - \chi(x))$$

$$= (\sum_{x = y_{ij}} B_\eta(x - \chi(x))U_\eta B_\eta$$

$$+ B_\eta^{-}\left(\sum_{x = y_{ij}} U_\eta^0(x - \chi(x))B_\eta$$

$$+ B_\eta^{-}U_\eta^0(\sum_{x = y_{ij}} B_\eta(x - \chi(x)))\right).$$

By a proof similar to that in Corollary 7.8, we get

$$u_{\eta,\chi} \cong B_\chi^{-} \otimes U_\chi^0 \otimes B_\chi.$$ 

In addition, $B_\chi$ is the quotient of $B_\eta$ by the ideal generated by the central elements $E_{ij}, (i, j) \in \mathcal{I}_0$. It follows that $B_\chi$ is the algebra generated by the elements $E_{i\chi}, i \neq m$ and relations (R5), (R6) with $q$ replaced by $\eta$, together with $E_{ij}^t = 0, (i, j) \in \mathcal{I}_0$.

**Corollary 7.9.** The C-algebra $B_\chi$ is nilpotent.

**Proof.** Let $G_m$ be the one dimensional multiplicative group([4]). By the description of $B_\chi$ above, there is a well-defined $G_m$-action on $B_\chi$ defined by $t \cdot E_{ij} = t^{j-i}E_{ij}$, $(i, j) \in \mathcal{I}_0$. Then $B_\chi$ becomes a rational $G_m$-module. Since $B_\chi$ is finite dimensional, there is a largest $G_m$-weight $N \in \mathbb{N}$. It follows that any finite product $E_{i_1,j_1} \cdots E_{i_t,j_t} \in B_\chi$ is equal to zero, if $t > N$, since otherwise it has a $G_m$-weight $\sum_{s=1}^t (j_s - i_s) > N$. Thus, $B_\chi$ is nilpotent. \qed

### 7.2 The simplicity of Kac modules for $u_{\eta,\chi}$

In this section, we study $u_{\eta,\chi}$-modules. For the elements in $U_\eta$, we denote the images in $u_{\eta,\chi}$ by the same notation. $\chi$ is assumed to satisfy $\chi(x_{ij}) = 0$ for all $(i, j) \in \mathcal{I}_0$. Let $M = M_0 \oplus M_1$ be a simple $u_{\eta,\chi}(g_0)\mathcal{N}_{1,\eta}$-module. Then since $u_{\eta,\chi}(g_0)\mathcal{N}_{1,\eta}^+$ is a nilpotent ideal of $u_{\eta,\chi}(g_0)\mathcal{N}_{1,\eta}$, $M$ is annihilated by $u_{\eta,\chi}(g_0)\mathcal{N}_{1,\eta}^+$. Since

$$u_{\eta,\chi}(g_0)\mathcal{N}_{1,\eta}/u_{\eta,\chi}(g_0)\mathcal{N}_{1,\eta}^+ \cong u_{\eta,\chi}(g_0),$$

$M$ is a simple $u_{\eta,\chi}(g_0)$-module. Conversely, each $u_{\eta,\chi}(g_0)$-module can be viewed as a $u_{\eta,\chi}(g_0)\mathcal{N}_{1,\eta}$-module annihilated by $u_{\eta,\chi}(g_0)\mathcal{N}_{1,\eta}^+$.

Let $M$ be a simple $u_{\eta,\chi}(g_0)$-module annihilated by $u_{\eta,\chi}(g_0)\mathcal{N}_{1,\eta}^+$. Define the Kac module

$$K(M) = u_{\eta,\chi} \otimes u_{\eta,\chi}(g_0)\mathcal{N}_{1,\eta} M.$$ 

Then we have $K(M) \cong \mathcal{N}_{-1,\eta} \otimes_{\mathbb{C}} M$ as $\mathcal{N}_{-1,\eta}$-modules.
Let \( M' \subseteq M \) be a simple \( U_\lambda^0 B_\lambda \)-submodule. Then \( B_\lambda M' \) is a \( U_\lambda^0 B_\lambda \)-submodule. Since \( B_\lambda \) is nilpotent, \( B_\lambda M' = 0 \), and hence \( M' \) is a simple \( U_\lambda^0 \)-module. Since \( U_\lambda^0 \) is commutative, we have that \( M' \) is 1-dimensional. Assume \( M' = \mathbb{C}v \). Then there is a \( \mathbb{C} \)-algebra homomorphism \( \lambda \) from \( U_\lambda^0 \) to \( \mathbb{C} \) such that \( hv = \lambda(h)v \) for all \( h \in U_\lambda^0 \). Such an element \( v \in M \) is referred to as a primitive vector of weight \( \lambda \). We denote \( X(U_\lambda^0) = \text{Hom}_{\mathbb{C}-alg}(U_\lambda^0, \mathbb{C}) \).

Let \( M \) be simple \( u_{\eta,\chi}(g_0) \)-module containing a primitive vector \( v_\lambda \) of weight \( \lambda \). Then \( M \) is spanned by elements in the form \( F_I F_0^\psi v_\lambda \) with \( \psi \in [0, I] \), \( I \subseteq \mathcal{I}_1 \). It follows that \( M = \sum_{\mu \in X(U_\lambda^0)} M_\mu \). Each \( x \in M_\mu \) is called a weight vector of weight \( \mu \).

In the superalgebra \( u_{\eta,\chi} \), from Sec. 5 we may assume
\[
E_{I_1} F_{I_1} = f(K) + \sum u_i^- u_i^0 u_i^+,
\]
where \( u_i^\pm \) are in the images of \( U_q^\pm \) in \( u_{\eta,\chi} \), \( f(K) \), \( u_i^0 \in U_\lambda^0 \). Then
\[
E_{I_1} F_{I_1} v_\lambda = f(K)v_\lambda = f(K)(\lambda)v_\lambda.
\]
Denote \( f(K)(\lambda) \) by \( f(\lambda) \).

Note that all the lemmas in Sec. 5 hold in \( u_{\eta,\chi} \) (with \( \eta \) in place of \( q \)) as well. By a similar argument as that in Prop. 5.9, we have

**Proposition 7.10.** \( K(M) \) is a simple \( u_{\eta,\chi} \)-module if and only if \( f(\lambda) \neq 0 \).

A weight \( \lambda \in X(U_\lambda^0) \) is called integral if \( \lambda(K^i_{\pm 1}) = \eta^i \lambda_i \) with \( \lambda_1, \ldots, \lambda_{m+n} \in \mathbb{Z} \). In this case, we have \( \lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_{m+n} \epsilon_{m+n} \in \Lambda \). For each \( \alpha = \epsilon_i - \epsilon_j \in \Phi^+ \), set \( K_\alpha = K_i K_j^{-1} \). It is then easy to check that \( \lambda(K_\alpha) = \eta^{(\lambda, \alpha)} \) for any \( \alpha \). Moreover, for any \( K^\mu, \mu \in \Lambda \), we have \( \lambda(K^\mu) = \eta^{(\lambda, \mu)} \). Then by a similar argument as that for Prop. 6.2, we have \( f(\lambda) = \Pi_{(i,j) \in \mathcal{I}_1} [(\lambda + \rho)(K_{ij})] \), where
\[
[(\lambda + \rho)(K_{ij})] = \frac{(\lambda + \rho)(K_{ij}) - (\lambda + \rho)(K_{ij}^{-1})}{\eta - \eta^{-1}}.
\]

Let \( M \) be a \( u_{\eta,\chi}(g_0) \)-module. Regard \( M \) as a \( u_{\eta,\chi}(g_0)N_{1,\eta} \)-module annihilated by \( u_{\eta,\chi}(g_0)N_{1,\eta}^+ \). Define the induced functor from the categories of \( u_{\eta,\chi}(g_0) \)-modules to the categories of \( u_{\eta,\chi} \)-modules by
\[
\text{Ind}(M) = u_{\eta,\chi} \otimes_{u_{\eta,\chi}(g_0)N_{1,\eta}} M.
\]
Clearly \( \text{Ind} \) is an exact functor and \( \text{Ind}(M) = K(M) \) in case \( M \) is a simple \( u_{\eta,\chi}(g_0) \)-module.

For any \( N_{1,\eta} \)-module \( N = N_0 \oplus N_1 \), denote
\[
N^{N_{1,\eta}^+} = \{ x \in N \mid gx = 0 \quad \text{for any} \quad g \in N_{1,\eta}^+ \}.
\]
If \( N \) is a \( u_{\eta,\chi} \)-module, it is easy to check that \( N^{N_{1,\eta}^+} \) is a \((\mathbb{Z}_2\text{-graded})\) \( u_{\eta,\chi}(g_0)N_{1,\eta}^+ \)-submodule.
**Lemma 7.11.** Let $\mathcal{N}_{1,\eta}$ be the left-regular $\mathcal{N}_{1,\eta}$-module. Then $\mathcal{N}_{1,\eta}^{\mathcal{N}_{1,\eta}^+} = \mathbb{C}E_{\mathcal{I}_1}$.

**Proof.** Using the anti-automorphism $\Omega$, we need only show that

$$\mathcal{N}_{-1,\eta}^{\mathcal{N}_{-1,\eta}^+} = \mathbb{C}F_{\mathcal{I}_1},$$

for the right-regular $\mathcal{N}_{-1,\eta}$-module $\mathcal{N}_{-1,\eta}$. Recall that $\mathcal{N}_{-1,\eta}$ has a basis $F_I$, $I \subseteq \mathcal{I}_1$. By Lemma 5.3, $F_{I_1}F_{ij} = 0$ for all $(i, j) \in \mathcal{I}_1$, so that $F_{I_1} \in \mathcal{N}_{-1,\eta}^{\mathcal{N}_{-1,\eta}^+}$. Let $x = \sum_{I \subseteq \mathcal{I}_1} c_I F_I \in \mathcal{N}_{-1,\eta}$. Suppose there is $I \not\subseteq \mathcal{I}_1$ with $c_I \neq 0$. Let $(i, j)$ be the largest(w.r.t the order in $\mathcal{I}_1$) pair not contained in some $I$ with $c_I \neq 0$. Then by Lemma 5.3 and 5.4 we have $xF_{ij} \neq 0$. Thus $\mathcal{N}_{-1,\eta}^{\mathcal{N}_{-1,\eta}^+} = \mathbb{C}F_{\mathcal{I}_1}$. \qed

**Lemma 7.12.** If $\chi(z_i z_j^{-1})^2 \neq 1$ for all $(i, j) \in \mathcal{I}_1$, then $K(M)$ is simple for any simple $u_{\eta,\chi}(\mathfrak{g}_0)$-module.

**Proof.** Let $v_{\lambda} \in M$ be a primitive vector of weight $\lambda$, and let $N = N_0 \oplus N_1$ be a nonzero submodule of $K(M)$. By a similar proof as that in Lemma 5.6 we have $F_{I_1} \otimes x \in N$ for some $0 \neq x \in M$. We may assume $x$ is a weight vector of weight $\mu$. Since $M$ is a simple $u_{\eta,\chi}(\mathfrak{g}_0)$-module, we have $u_{\eta,\chi}(\mathfrak{g}_0)x = M$. Hence, there is an element

$$f = \sum c_i u_i^0 u_i^+ \in u_{\eta,\chi}(\mathfrak{g}_0)$$

such that $fx = v_{\lambda}$, where $u_i^0$ (resp. $u_i^+$; $u_i^0$) is the product of $F_{ij}$ (resp. $E_{ij}$; $K_{\pm 1}$), $(i, j) \in \mathcal{I}_0, 1 \leq s \leq m + n, c_i \in \mathbb{C}$.

Since $x$ is a weight vector, we may assume $f = \sum c_i u_i^- u_i^+$ using Lemma 5.7 and 5.8, by a minor modification of the coefficients of $f$, we get $f' = \sum c_i' u_i^- u_i^+$, which applied to $F_{I_1} \otimes x \in N$ to get $F_{I_1} \otimes v_{\lambda} \in N$. Applying $E_{I_1}$ to which we get

$$\Pi_{(i, j) \in \mathcal{I}_1}[(\lambda + \rho)(K_{ij})] v_{\lambda} \in N.$$

Note that $K_{ij}^l = \chi(z_i z_j^{-1})$ in $u_{\eta,\chi}$, which implies that $[(\lambda + \rho)(K_{ij})] \neq 0$ for any $(i, j) \in \mathcal{I}_1$. Suppose otherwise $[(\lambda + \rho)(K_{ij})] = 0$ for some $(i, j) \in \mathcal{I}_1$. Then we have

$$\lambda(K_{ij}^2) = \rho(K_{ij}^{l-2}) = \eta^{-2(\rho \varepsilon_i - \varepsilon_j)},$$

which gives $\chi(z_i z_j^{-1})^2 = \lambda(K_{ij}^2)^l = 1$, a contradiction. Then we have $v_{\lambda} \in N$. Therefore $N = K(M)$, and hence $K(M)$ is simple. \qed

**Theorem 7.13.** If $\chi(z_i z_j^{-1})^2 \neq 1$ for all $(i, j) \in \mathcal{I}_1$, then $u_{\eta,\chi}(\mathfrak{g}_0)$ and $u_{\eta,\chi}$ are Morita equivalent.

**Proof.** We show that $K(M)^{\mathcal{N}_{-1,\eta}^+} = M$. Note that the subspace $F_{I_1} \otimes M \subseteq K(\lambda)$ is annihilated by $\mathcal{N}_{1,\eta}^{\mathcal{N}_{1,\eta}^+}$. Since $E_{ij}$, $F_{ij}, (i, j) \in \mathcal{I}_0$ commutes with $F_{I_1}$ up to scalar multiple, the subspace is a simple $\mathcal{N}_{-1,\eta} u_{\eta,\chi}(\mathfrak{g}_0)$-submodule of $K(M)$. Since $K(M)$ is simple, we have

$$K(M) = u_{\eta,\chi} F_{I_1} \otimes M = \mathcal{N}_{1,\eta} F_{I_1} \otimes M.$$
Set
\[ K^-(F_{\mathcal{I}_1} \otimes M) = u_{\eta,\chi} \otimes_{\mathcal{N}_{-1,\eta}u_{\eta,\chi}(\mathfrak{g}_0)} (F_{\mathcal{I}_1} \otimes M), \]

where \( F_{\mathcal{I}_1} \otimes M \) is viewed as a \( \mathcal{N}_{-1,\eta}u_{\eta,\chi}(\mathfrak{g}_0) \)-module annihilated by \( \mathcal{N}_{-1,\eta}u_{\eta,\chi}(\mathfrak{g}_0) \). By the comparison of dimensions we have that \( K(M) \) is isomorphic to \( K^-(F_{\mathcal{I}_1} \otimes M) \) as \( u_{\eta,\chi} \)-modules. Thus, as \( \mathcal{N}_{1,\eta} \)-modules, we have
\[ K(M) \cong \mathcal{N}_{1,\eta} \otimes_{F_{\mathcal{I}_1}} F_{\mathcal{I}_1} \otimes M, \]

from which it follows that
\[ K(M)^{\mathcal{N}_{1,\eta}^+} \cong (\mathcal{N}_{1,\eta} \mathcal{N}_{1,\eta}^+) \otimes F_{\mathcal{I}_1} \otimes M \]
\[ \cong E_{\mathcal{I}_1} F_{\mathcal{I}_1} \otimes M \]
\[ = M, \]
where the last equality is given by the fact that \( E_{\mathcal{I}_1} E_{\mathcal{I}_1} v_{\lambda} \neq 0 \).

From above discussion, we have that the functor \( (\cdot)^{\mathcal{N}_{1,\eta}^+} \) is right adjoint to \( \text{Ind} \). By a similar argument as that for [3, Th. 3.2], \( u_{\eta,\chi}(\mathfrak{g}_0) \) and \( u_{\eta,\chi} \) are Morita equivalent. \( \square \)

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