Quadratic Differential Systems on $\mathbb{R}^3$ Having a Semisimple Derivation with one-Dimensional Kernel

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Abstract

The classification, up to a center-affinity, of the homogeneous quadratic differential systems defined on $\mathbb{R}^3$ that have at least a semisimple derivation with one-dimensional kernel, is achieved. It is proved that there exist 35 families of affine equivalence classes of such systems.

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1 Introduction

Let us recall that any homogeneous quadratic differential system (shortly, HQDS) on an $\mathbb{R}$-Banach space is congenitally connected with a commutative binary algebra. Indeed, each HQDS is defined by means of a covariant symmetric (1,2)-tensor which, in its turn, is the structure tensor of a commutative algebra. Consequently, the study of any HQDS could be achieved by means of its associated commutative algebra. In particular, the algebra of derivations of each HQDS is the same with derivation algebra of its associated algebra. Of course, it is more appropriate to study the properties of algebras having a derivation instead to make a direct study of the corresponding HQDS. Such a study is the object of our present paper.

Let $k$ be a field of characteristic 0 and $A(\cdot)$ be a finite dimensional $k$-algebra. Recall that the derivation $D \in \text{Der } A$ is said to be semisimple if it is diagonalisable in an extension of $k$, i.e. there exists a basis in $A$ consisting of eigenvectors of $D$.

In [5] were classified, up to an isomorphism, the real 3-dimensional commutative algebras having a semisimple nonsingular derivation. It was shown that the existence of such a derivation acts as a very strong constraint compelling each such algebra to be isomorphic to one of the 4 algebras listed in [5]. Accordingly, were classified...
the corresponding homogeneous quadratic differential systems up to a center-affine equivalence.

The aim of this paper is to classify, up to an isomorphism, the real 3-dimensional commutative algebras having at least a semisimple derivation with one-dimensional kernel. The main result is: there exist 35 families of isomorphism classes of real 3-dimensional commutative algebras having at least a semisimple derivation with one-dimensional kernel. For each of them are exhibited their main properties which allow to decide on the problem of their mutual isomorphism. Everyone of these 35 families is either a singleton or consists of a set of algebras that have the same lists of main properties and can be indexed by one- or two-parameters such that two algebras in family corresponding to different parameters are non-isomorphic.

In fact, we shall get the subalgebra lattices, the derivation algebras and the group of automorphisms for each class of algebras, as they are the most important invariants of binary algebras. Especially, we are interested in finding the set Ann $A$ of annulator elements, the set $N(A)$ of all nilpotent elements and the set $I(A)$ of all idempotent elements of algebra $A$. Further, the corresponding homogeneous quadratic dynamical systems are classified up to a center-affine equivalence. Recall that (see [5]) the subalgebra lattice of $A(·)$ allows to identify a natural partition $P_A$ of the ground space $A$ which, in its turn, defines a partition of the set of all integral curves of its associated HQDS.

COMMENT. Really, there exists infinite many isomorphism classes of such algebras. In order to identify them we need to find a partition of this set of isomorphism classes consisting of a finite number of sets. To this end we consider a list of "main properties" which allows to define an equivalence, the so-called MP-equivalence, on the set of isomorphism classes: two isomorphism classes are MP-equivalent if and only if their algebras have the same "main properties". The partition associated with this equivalence has a finite number of elements (here, 35 sets). Certainly, we can exclude in the list of main properties the part concerning the two partitions induced by subalgebra lattice on the ground space of analyzed algebra as well as on the set of all integral curves of the HQDS assigned to this algebra. We keep this part because it work like a conformity test for automorphism group of algebra.

2 Algebras having a semisimple derivation with one-dimensional kernel

Let $A(·)$ be the real 3-dimensional (nontrivial) commutative algebra associated with a HQDS on $\mathbb{R}^3$. Suppose that $D$ is a nonzero semisimple derivation of $A(·)$ with one-dimensional kernel. Then, algebra $A(·)$ has a semisimple derivation $D$ having the spectrum of the form $\text{Spec } D = (1, \omega, 0)$ with $\omega \neq 0$; this notation for spectrum is preferred because the eigenvalues $1, \omega, 0$ are not necessarily distinct each other. Moreover, there exists a basis $B = (e_1, e_2, e_3)$ of $A$ such that

$$D(e_1) = e_1, \quad D(e_2) = \omega e_2, \quad D(e_3) = 0.$$ 

In this case, $A$ decomposes into a direct vector sum of invariant subspaces with respect to $D$. 
In order to give the analytical expression for the existence of a derivation \( D \) of algebra \( A(\cdot) \), we define - as it is usual - the structure constants of \( A \) in basis \( B \) by equations
\[
e_i \cdot e_j = a^k_{ij} e_k.
\] (2.1)

For convenience, we shall denote
\[
\begin{align*}
a^1_{11} &= a, & a^2_{11} &= b, & a^3_{11} &= c, & a^1_{12} &= k, & a^2_{12} &= m, & a^3_{12} &= n, \\
a^1_{22} &= d, & a^2_{22} &= e, & a^3_{22} &= f, & a^1_{31} &= p, & a^2_{31} &= q, & a^3_{31} &= r, \\
a^1_{33} &= g, & a^2_{33} &= h, & a^3_{33} &= j, & a^1_{23} &= s, & a^2_{23} &= t, & a^3_{23} &= v.
\end{align*}
\] (2.2)

Then the endomorphism \( D \) is a derivation for \( A \) if and only if the next conditions are fulfilled:
\[
\begin{align*}
a &= c = e = f = g = h = k = m = r = v = 0 \\
(\omega - 2)b &= 0 \\
(1 - 2\omega)d &= 0 \\
(1 + \omega)n &= 0 \\
(\omega - 1)q &= 0 \\
(1 - \omega)s &= 0.
\end{align*}
\] (2.3)

Here \( j, p, t \) range free over \( \mathbb{R} \) while \( b, d, n, q, s \) take values depending on \( \omega \). Equations (2.3) impose to take into account of the natural decomposition of \( \mathbb{R}^* \) as range of \( \omega \) defined by means of sets: \( \{-1, \frac{1}{2}, 1, 2\} \) and \( \mathbb{R} \setminus \{-1, 0, \frac{1}{2}, 1, 2\} \).

Consequently, we have to analyze only algebras having a semisimple derivation \( D \) with \( \text{Spec } D \) in the following list:

1) (1, −1, 0), 2) (1, \( \frac{1}{2}, 0 \)), 3) (1, 1, 0), 4) (1, 2, 0), 5) (1, \( \omega \), 0) with \( \omega \notin \{-1, 0, \frac{1}{2}, 1, 2\} \).

Since when \( D \in \text{Der } A \) has \( \text{Spec } D = (1, \frac{1}{2}, 0) \) then derivation \( D' = 2D \) has \( \text{Spec } D' = (1, 2, 0) \), it follows the next result.

**Proposition 2.1** If a real 3-dimensional (nontrivial) commutative algebra \( A(\cdot) \) has a semisimple derivation with one-dimensional kernel, it has at least a derivation \( D \) with \( \text{Spec } D \) of one of the following forms:

1) (1, −1, 0), 2) (1, 1, 0), 3) (1, 2, 0), 4) (1, \( \omega \), 0) with \( \omega \notin \{-1, 0, \frac{1}{2}, 1, 2\} \).

**1) Case** \( \text{Spec } D = (1, −1, 0) \)

In basis \( B \) algebra \( A(\cdot) \) has the next multiplication table:

| Table T | \( e_1^2 = 0 \) | \( e_2^2 = 0 \) | \( e_3^2 = j e_3 \) |
|---------|-----------------|-----------------|-----------------|
|         | \( e_1 e_2 = n e_3 \) | \( e_1 e_3 = p e_1 \) | \( e_2 e_3 = t e_2 \) |

with \( j, n, p, t \in \mathbb{R} \). Algebra corresponding to \( j = n = p = t = 0 \) is just the null algebra that is of no interest in general. We have to consider the case when at least one of the parameter \( j, n, p, t \) is not zero.
Further we shall consider the next two mutually exclusive cases:

\[
I) \ jn \neq 0, \quad II) \ jn = 0.
\]

**Case I.** $jn \neq 0$

In this case, each algebra with multiplication table $T$ is isomorphic to algebra:

**Table T1**

\[
\begin{array}{c}
e_1^2 = 0 \quad e_2^2 = 0 \quad e_3^2 = e_3 \\
e_1e_2 = e_3 \quad e_1e_3 = ae_1 \quad e_2e_3 = \beta e_2
\end{array}
\]

with $\alpha, \beta \in \mathbb{R}$.

This time, $e_3$ is an idempotent whose left multiplication $L_{e_3}$ has the spectrum $(\alpha, \beta, 1)$. Consequently, as long as $e_3$ is the only idempotent of such an algebra, its eigenvalues $\alpha, \beta$ have to be the most important invariants characterizing algebra.

For convenience, let us denote by $A_1(\alpha, \beta)$ any algebra having the multiplication table $T_1$.

**Proposition 2.2** The algebras $A_1(\alpha, \beta)$ and $A_1(\beta, \alpha)$ are isomorphic.

Consequently, in the following we deal with algebras $A_1(\alpha, \beta)$ with $\alpha \leq \beta$, only.

Further, by a straightforward computation, it is proved the next proposition.

**Proposition 2.3** The algebras $A_1(\alpha, \beta)$ (with $\alpha \leq \beta$) and $A_1(\alpha_1, \beta_1)$ (with $\alpha_1 \leq \beta_1$) are isomorphic if and only if $\alpha = \alpha_1, \beta = \beta_1$.

**Proposition 2.4** Every algebra $A$ of type $A_1(\alpha, \beta)$ has:

\[(i) \ Ann \ A = \{0\}, \quad (ii) \ N(A) = \left\{ \begin{array}{l}
\mathbb{R}e_1 \cup \mathbb{R}e_2 \quad \text{if } \alpha^2 + \beta^2 \neq 0 \\
\{xe_1 - \frac{z^2}{2}e_2 + ze_3 \mid x, z \in \mathbb{R}, \ x \neq 0\} \cup \mathbb{R}e_2 \quad \text{if } \alpha = \beta = 0 \\
\{e_3\} \quad \text{if } \alpha = \beta = 0 \\
\{e_3\} \quad \text{if } \alpha \neq 0, \beta \neq 0 \\
\{xe_1 + e_3 \mid x \in \mathbb{R}\} \quad \text{if } \alpha = 0, \beta = 0 \\
\{e_3\} \quad \text{if } \alpha = 0, \beta \neq 0 \\
\{ye_2 + e_3 \mid y \in \mathbb{R}\} \cup \{xe_1 + e_3 \mid x \in \mathbb{R}\} \cup \{ye_2 + e_3 \mid y \in \mathbb{R}\} \quad \text{if } \alpha = 0, \beta = 0 \\
\{e_3\} \quad \text{if } \alpha = 0, \beta \neq 0 \\
\end{array} \right., \quad (iii) \ I(A) = \left\{ \begin{array}{l}
\{xe_1 - \frac{z^2}{2}e_2 + ze_3 \mid x, z \in \mathbb{R}, \ x \neq 0\} \cup \mathbb{R}e_2, \quad (i) \ N(A) = \{xe_1 - \frac{z^2}{2}e_2 + ze_3 \mid x, z \in \mathbb{R}, z \neq 0\} \cup \mathbb{R}e_2, \quad \mathbb{R}u \text{ for } u \in N(A) \cup I(A),
\end{array} \right. \]

We start the study of each kind of algebra exhibited in previous Proposition, taking into account that algebras $A_1(0, 0)$ and $A_1(0, \alpha)$ are isomorphic.

A1) Properties of algebra $A = A_1(0, 0)$

- $Ann \ A = \{0\}, \ N(A) = \{xe_1 - \frac{z^2}{2}e_2 + ze_3 \mid x, z \in \mathbb{R}, \ z \neq 0\} \cup \mathbb{R}e_2, \ I(A) = \{e_3\}$,
- 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in N(A) \cup I(A)$,
- 2-dimensional subalgebras: $\text{Span}_\mathbb{R}\{e_3, ae_1 + be_2\}$ ($a^2 + b^2 \neq 0$),
H is a normal divisor of \( \text{Aut} A \) and \( \text{Aut} A/H \cong \mathbb{Z}_2 \) (in fact, \( \text{Aut} A \cong \mathbb{R}^* \times \{-1,1\} \)), where \( \mathbb{R}^* \) denotes the multiplicative group of nonzero real numbers \( \mathbb{R}^*(\cdot) \); \( H \) is a normal divisor of \( \text{Aut} A \).

- the partition \( \mathcal{P}_A \) of \( \mathbb{R}^3 \), defined by the lattice of subalgebras of \( A \), consists of:
  - the singletons consisting of singular solutions that cover the cone \( 2x^1x^2 + (x^3)^2 = 0 \),
  - the half-lines of axis \( Ox^3 \) delimited by \( O \),
  - the connected components of each plane passing through \( Ox^3 \), delimited by axis \( Ox^3 \) and the generatrices of cone \( 2x^1x^2 + (x^3)^2 = 0 \) (whenever it is the case, i.e. \( x^1x^2 < 0 \)),

- the partition \( \mathcal{P}_A \) of \( A \) induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the cone \( 2x^1x^2 + (x^3)^2 = 0 \),
  - the families of ray-solutions contained in each half-line of \( Ox^3 \) delimited by \( O \),
  - the integral curves contained in the connected components of each plane passing through \( Ox^3 \), delimited by axis \( Ox^3 \) and the generatrices of cone \( 2x^1x^2 + (x^3)^2 = 0 \) (whenever it is the case, i.e. \( x^1x^2 < 0 \)).

Since each nonsingular integral curves lies into a plane passing through \( Ox^3 \) it has a null torsion tensor. Moreover, \( A/A^2 \) is a null algebra what implies that the curvature tensor of nonsingular integral curves vanishes too, so that each non-singular integral curves lies on a line parallel to \( Ox^3 \).

### A2) Properties of algebra \( A = A_1(0, \frac{1}{2}) \)

- \( \text{Ann} A = \{0\} \), \( \mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2 \), \( \mathcal{I}(A) = \{ye_2 + e_3 \mid y \in \mathbb{R} \} \),
- 1-dimensional subalgebras: \( \mathbb{R}u \) for \( u \in \mathcal{N}(A) \cup \mathcal{I}(A) \),
- 2-dimensional subalgebras: \( \text{Span}_\mathbb{R}\{e_1,e_3\}, \text{Span}_\mathbb{R}\{e_2,e_3\} \),
- ideals: \( \text{Span}_\mathbb{R}\{e_2,e_3\} \),
- \( A^2 = \text{Span}_\mathbb{R}\{e_2,e_3\} \); \( A/A^2 \) is the null 1-dimensional algebra,
- \( \text{Der} A = \mathbb{R}D \),
- \( \text{Aut} A = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x \in \mathbb{R}^* \right\} \cong \mathbb{R}^*(\cdot) \),

- the partition \( \mathcal{P}_A \) of \( \mathbb{R}^3 \), defined by the lattice of subalgebras of \( A \), consists of:
  - the singletons covering the axes \( \mathbb{R}e_1 \) and \( \mathbb{R}e_2 \),
the half-lines delimited by $O$ on lines $Ru$ for $u \in \mathcal{I}(A)$ (i.e. these lines cover the plane $x^2Ox^3$ less $x^2$-axis),

- the half-spaces $x^1 < 0$ and $x^1 > 0$ (delimited by plane $x^2Ox^3$) less the points of axis $Ox^1$,

- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:

  - the singletons consisting of singular solutions that cover the axes $Re_1$ and $Re_2$,

  - the families of ray-solutions contained in each half-line delimited by $O$ on each line $Ru$ for $u \in \mathcal{I}(A)$,

  - the integral curves contained in the half-spaces $x^1 < 0$ and $x^1 > 0$ (delimited by plane $x^2Ox^3$) less $x^1$-axis.

Let us remark that each nonsingular integral curve has a null torsion tensor (indeed, $A^2$ is a proper ideal of $A$).

**Remark 2.1** All idempotents have the same spectrum $(0, 1/2, 1)$. Then $A$ has the decomposition $A(0) \oplus A(1/2) \oplus A(1)$ consisting of the eigenspaces corresponding to any idempotent used in basis.

**A3)** Properties of algebras $A$ of type $A_1(1/2, 1/2)$

- $Ann A = \{0\}$, $\mathcal{N}(A) = Re_1 \cup Re_2$, $\mathcal{I}(A) = \{xe_1 + e_3 \mid x \in \mathbb{R}\} \cup \{ye_2 + e_3 \mid y \in \mathbb{R}^*\}$,

- 1-dimensional subalgebras: $Ru$ for $u \in \mathcal{N}(A) \cup \mathcal{I}(A)$,

- 2-dimensional subalgebras: $Span_{\mathbb{R}}\{e_3, pe_1 + qe_2\}$ with $p, q \in \mathbb{R}$,

- ideals: none,

- $A^2 = A$,

- $Der A = \mathbb{R}D$,

- $Aut A = H \cup JH$ where

$$H = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x \in \mathbb{R}^* \right\}, \quad J = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $J^2 = id$,

(i.e. $Aut A \cong \mathbb{R}^* \times \{-1, 1\}$) and $H$ is a normal divisor of $Aut A$,

- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:

  - the singletons covering the axes $Ox^1$ and $Ox^2$,

  - the half-lines of $x^1Ox^3$ passing through $O$, delimited by $O$, less the axis $Ox^1$,

  - the half-lines of $x^2Ox^3$ passing through $O$, delimited by $O$, less the axes $Ox^2$ and $Ox^3$,

  - the half-planes delimited by axis $Ox^3$ on each plane containing axis $Ox^3$, except the planes $x^1Ox^3$ and $x^2Ox^3$,

- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:

  - the singletons consisting of singular solutions that cover the axes $Ox^1$ and $Ox^2$, 
the families of ray-solutions contained in each half-line of $x^1 Ox^3$ passing through $O$ and delimited by $O$, less the axis $Ox^1$,

the families of ray-solutions contained in each half-line of $x^2 Ox^3$ passing through $O$ and delimited by $O$, less the axes $Ox^2$ and $Ox^3$,

the integral curves contained in each half-plane delimited by axis $Ox^3$ on each plan containing axis $Ox^3$ without planes $x^1 Ox^3$ and $x^2 Ox^3$.

Note that each nonsingular solution is a torsion-free curve. Moreover, all idempotents have the same spectrum $(\frac{1}{2}, \frac{1}{2}, 1)$.

A4) Properties of algebra $A$ of type $A_1(\alpha, 0)$ with $\alpha \notin \{0, \frac{1}{2}\}$

- the algebras $A_1(\alpha, 0)$ and $A_1(\alpha', 0)$ are isomorphic if and only if $\alpha = \alpha'$,
- Ann $A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2$, $\mathcal{I}(A) = \{e_3\}$,
- 1-dimensional subalgebras: $\mathbb{R}e_1$, $\mathbb{R}e_2$, $\mathbb{R}e_3$,
- 2-dimensional subalgebras: $\text{Span}_\mathbb{R}\{e_1, e_3\}$, $\text{Span}_\mathbb{R}\{e_2, e_3\}$,
- ideals: $\text{Span}_\mathbb{R}\{e_1, e_3\}$, $\text{Span}_\mathbb{R}\{e_2, e_3\}$,
- $A^2 = \text{Span}_\mathbb{R}\{e_1, e_3\}$; $A/A^2$ is the null 1-dimensional algebra,
- $\text{Der } A = \mathbb{R}D$,

- $\text{Aut } A = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}^* \right\} \cong \mathbb{R}^*$. (1)

- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  - the singletons covering the axes $\mathbb{R}e_1$ and $\mathbb{R}e_2$,
  - the half-lines of axis $Ox^3$ delimited by $O$,
  - the quarters of plane $x^1 Ox^3$ delimited by axes $Ox^1$ and $Ox^3$,
  - the quarters of plane $x^2 Ox^3$ delimited by axes $Ox^2$ and $Ox^3$,
  - the quarters of space delimited by planes $x^1 Ox^3$ and $Ox^2 Ox^3$,

- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the axes $\mathbb{R}e_1$ and $\mathbb{R}e_2$,
  - the families of ray-solutions contained in each half-line of $Ox^3$ delimited by $O$,
  - the integral curves contained in the quarters of plane $x^1 Ox^3$ delimited by axes $Ox^1$ and $Ox^3$,
  - the integral curves contained in the quarters of plane $x^2 Ox^3$ delimited by axes $Ox^2$ and $Ox^3$,
  - the integral curves contained in the quarters of space delimited by planes $x^1 Ox^3$ and $Ox^2 Ox^3$.

Note that each nonsingular integral curve has a null torsion tensor.

Note. In order to bring together the classes $A_1(\alpha, 0)$ with $\alpha < 0$ and $A_1(0, \beta)$ with $\beta > 0$ we have ignored, in case A4), the convention to consider increasing parameters in algebras of type $A_1(a, b)$. Such a behavior will be used whenever is possible, in order to save the space.
A5) Properties of algebras $A$ of type $A_1(\alpha, \frac{1}{2})$ with $\alpha \notin \{0, \frac{1}{2}\}$

- algebras $A_1(\alpha, \frac{1}{2})$ ($\alpha \notin \{0, \frac{1}{2}\}$) and $A_1(\alpha', \frac{1}{2})$ ($\alpha' \notin \{0, \frac{1}{2}\}$) are isomorphic if and only if $\alpha = \alpha'$,
- $\text{Ann } A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2$, $\mathcal{I}(A) = \{ye_2 + e_3 \mid y \in \mathbb{R}\}$,
- 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in \mathcal{N}(A) \cup \mathcal{I}(A)$,
- 2-dimensional subalgebras: $\text{Span}_\mathbb{R}\{e_1, e_3\}$, $\text{Span}_\mathbb{R}\{e_2, e_3\}$,
- ideals: none,
- $A^2 = A$,
- $\text{Der } A = \mathbb{R}D$,
- $\text{Aut } A = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x \in \mathbb{R}^* \right\} \cong \mathbb{R}^*(\cdot)$,
- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  - the singletons covering the axes $Ox^1$ and $Ox^2$,
  - the half-lines delimited by $O$ on lines of $x^2Ox^3$ passing through $O$, less the axis $Ox^2$,
  - the quarters of plane $x^1Ox^3$, delimited by axis $Ox^1$ and $Ox^3$,
  - the quarter-spaces delimited by planes $x^1Ox^3$ and $x^2Ox^3$,
- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the axes $Ox^1$ and $Ox^2$,
  - the families of ray-solutions contained in each half-line delimited by $O$ on each line in $x^2Ox^3$ passing through $O$, less the axis $Ox^2$,
  - the integral curves contained in each quarter of plane $x^1Ox^3$, delimited by axes $Ox^1$ and $Ox^3$,
  - the integral curves contained in each quarter-space delimited by planes $x^1Ox^3$ and $x^2Ox^3$.

Moreover, all idempotents have the same spectrum $(\alpha, \frac{1}{2}, 1)$.

Note. In order to bring together the classes $A_1(\alpha, \frac{1}{2})$ and $A_1(\frac{1}{2}, \beta)$ we have ignored, in case A5, the convention to consider increasing parameters in algebras of type $A_1(\alpha, b)$.

A6) Properties of algebras $A$ of type $A_1(\alpha, \alpha)$ with $\alpha \notin \{0, \frac{1}{2}\}$

- algebras $A_1(\alpha, \alpha)$ ($\alpha \notin \{0, \frac{1}{2}\}$) and $A_1(\alpha', \alpha')$ ($\alpha' \notin \{0, \frac{1}{2}\}$) are isomorphic if and only if $\alpha = \alpha'$,
- $\text{Ann } A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2$, $\mathcal{I}(A) = \{xe_1 + \frac{2\alpha-1}{\alpha\alpha'}e_2 + \frac{1}{\alpha'}e_3 \mid x \in \mathbb{R}^*\}$ (each idempotent lies on the cone $2x^1x^2 + (1-2\alpha)(x^3)^2 = 0$),
- 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in \mathcal{N}(A) \cup \mathcal{I}(A)$,
- 2-dimensional subalgebras: $\text{Span}_\mathbb{R}\{e_3, ae_1 + be_2\}$ for $a^2 + b^2 \neq 0$,
- ideals: none,
- $A^2 = A$,
- $\text{Der } A = \mathbb{R}D$,
\begin{itemize}
\item \( \text{Aut } A = H \cup JH \) where
\[
H = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x \in \mathbb{R}^* \right\}, \quad J = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and } J^2 = \text{id},
\]
i.e. \( \text{Aut } A \cong \mathbb{R}^* \times \{ -1, 1 \} \) and \( H \) is a normal divisor of \( \text{Aut } A \),
\item the partition \( \mathcal{P}_A \) of \( \mathbb{R}^3 \), defined by the lattice of subalgebras of \( A \), consists of:
\begin{itemize}
\item the singletons covering the axes \( Ox^1 \) and \( Ox^2 \),
\item the half-lines of axis \( Ox^3 \) delimited by \( O \),
\item the half-lines delimited by \( O \) on each generatrix of cone \( 2x^1x^2 + (1 - 2\alpha)(x^3)^2 = 0 \),
\item the quarters of plane \( x^1Ox^3 \) delimited by axes \( Ox^1 \) and \( Ox^3 \),
\item the quarters of plane \( x^2Ox^3 \) delimited by axes \( Ox^2 \) and \( Ox^3 \),
\item the connected components of planes passing through \( Ox^3 \), without planes \( x^1Ox^3 \) and \( x^2Ox^3 \), delimited by axis \( Ox^3 \) and the cone \( 2x^1x^2 + (1 - 2\alpha)(x^3)^2 = 0 \) (whenever it is the case, i.e. if \( (1 - 2\alpha)x^1x^2 < 0 \)),
\end{itemize}
\item the partition \( \mathcal{P}_A \) of \( A \) induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
\begin{itemize}
\item the singletons consisting of singular solutions that cover the axes \( Ox^1 \) and \( Ox^2 \),
\item the families of ray solutions contained in the half-lines of axis \( Ox^3 \) delimited by \( O \),
\item the families of ray solutions contained in the half-lines delimited by \( O \) on each generatrix of cone \( 2x^1x^2 + (1 - 2\alpha)(x^3)^2 = 0 \),
\item the integral curves contained in each quarter of plane \( x^1Ox^3 \) delimited by axes \( Ox^1 \) and \( Ox^2 \),
\item the integral curves contained in each quarter of plane \( x^2Ox^3 \) delimited by axis \( Ox^2 \),
\item the integral curves contained in the connected components of planes passing through \( Ox^3 \), without planes \( x^1Ox^3 \) and \( x^2Ox^3 \), delimited by axis \( Ox^3 \) and the cone \( 2x^1x^2 + (1 - 2\alpha)(x^3)^2 = 0 \) (whenever it is the case, i.e. if \( (1 - 2\alpha)x^1x^2 < 0 \)).
\end{itemize}
\end{itemize}

Note that, each non-singular integral curve has torsion zero. Moreover, idempotent \( e_3 \) has the spectrum \( \{ \alpha, \alpha, 1 \} \) while all other idempotents have the same spectrum \( \{ \frac{1}{2}, 1, \frac{1 - \alpha}{2} \} \) what could be connected with the presence of two connected components of \( \text{Aut } A \).

\textbf{A7) Properties of algebras } A \text{ of type } A_1(\alpha, \beta) \text{ with } \alpha < \beta \text{ and } \alpha, \beta \notin \{ 0, \frac{1}{2} \}

\begin{itemize}
\item algebras \( A_1(\alpha, \beta) \) (\( \alpha < \beta \)) and \( A_1(\alpha', \beta') \) (\( \alpha' < \beta' \)) are isomorphic if and only if \( \alpha = \alpha' \), \( \beta = \beta' \),
\item \( \text{Ann } A = \{ 0 \} \), \( \mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2 \), \( \mathcal{I}(A) = \{ e_3 \} \),
\item 1-dimensional subalgebras: \( \mathbb{R}e_1, \mathbb{R}e_2, \mathbb{R}e_3 \),
\item 2-dimensional subalgebras: \( \text{Span}_\mathbb{R}\{ e_1, e_3 \}, \text{Span}_\mathbb{R}\{ e_2, e_3 \} \),
\item ideals: none,
\item \( A^2 = A \),
\item \( \text{Der } A = \mathbb{R}D \),
\end{itemize}
• \( \text{Aut } A = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x \in \mathbb{R}^* \right\} \cong \mathbb{R}^*(\cdot) \).

• the partition \( \mathcal{P}_A \) of \( \mathbb{R}^3 \), defined by the lattice of subalgebras of \( A \), consists of:
  - the singletons covering the axes \( Ox^1 \) and \( Ox^2 \),
  - the half-axes delimited by \( O \) on \( Ox^3 \),
  - the quarters delimited by axes \( Ox^1 \) and \( Ox^3 \) on plane \( x^1Ox^3 \),
  - the quarter-spaces delimited by planes \( x^1Ox^3 \) and \( x^2Ox^3 \).

• the partition \( \mathcal{P}_A \) of \( A \) induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the axes \( Ox^1 \) and \( Ox^2 \),
  - the ray solutions lying on semi-axes of \( Ox^3 \), delimited by \( O \),
  - the integral curves contained in each quarters of plane of \( x^1Ox^3 \), delimited by axes \( Ox^1 \) and \( Ox^3 \),
  - the integral curves contained in each quarter of plane of \( x^2Ox^3 \), delimited by axis \( Ox^2 \) and \( Ox^3 \),
  - the integral curves contained in each quarter-space delimited by planes \( x^1Ox^3 \) and \( x^2Ox^3 \).

By comparing the lists of properties of algebras in classes \( A_1-A_7 \) and taking into account of remarks concerning the spectrum of their idempotents it follows the next result.

**Theorem 2.1** Each algebra of type \( A_i \) is not isomorphic to any algebra of type \( A_j \) for \( i, j \in \{1, 2, \ldots, 7\} \) and \( i \neq j \).

**Case II** \( jn = 0 \)

This case decomposes in:

(i) \( j \neq 0, \ n = 0 \),
(ii) \( j = 0, \ n \neq 0 \),
(iii) \( j = 0, \ n = 0 \).

**Subcase (i)** \( j \neq 0, \ n=0 \)

There exists a basis such that the multiplication table of algebra becomes

| Table T2 | \( e_1^2 = 0 \) | \( e_2^2 = 0 \) | \( e_3^2 = e_3 \) |
|----------|----------------|----------------|----------------|
|          | \( e_1e_2 = 0 \) | \( e_1e_3 = \alpha e_1 \) | \( e_2e_3 = \beta e_2 \) |

with \( \alpha, \beta \in \mathbb{R} \). Let us denote by \( A_2(\alpha, \beta) \) any algebra of type \( T2 \).

**Proposition 2.5** The algebras \( A_2(\alpha, \beta) \) and \( A_2(\beta, \alpha) \) are isomorphic.

Consequently, in the following we restrict our interest to the case \( \alpha \leq \beta \).
Proposition 2.6 The algebras \( A_2(\alpha, \beta) \) with \( \alpha \leq \beta \) and \( A_2(\alpha', \beta') \) with \( \alpha' \leq \beta' \) are isomorphic if and only if \( \alpha = \alpha' \) and \( \beta = \beta' \).

The next result holds true for any algebra \( A \) of type \( A_2(\alpha, \beta) \).

Proposition 2.7 Every algebra \( A \) of type \( A_2(\alpha, \beta) \) has:

\[
A \text{ is a direct sum of two ideals (this is like a Wedderburn-Artin decomposition by means of a maximal nilpotent ideal and a complementary subalgebra)}.
\]

\[\text{Ann } A = \begin{cases} 
(1) \text{ Span}_{\mathbb{R}}\{e_1, e_2\} & \text{if } \alpha = \beta = 0 \\
(2) \mathbb{R}e_1 & \text{if } \alpha = 0, \beta \neq 0 \\
(3) \mathbb{R}e_2 & \text{if } \alpha \neq 0, \beta = 0 \\
(4) \{0\} & \text{if } \alpha \beta \neq 0.
\end{cases}\]

\[\mathcal{N}(A) = \text{Span}_{\mathbb{R}}\{e_1, e_2\}\]

\[\mathcal{I}(A) = \begin{cases} 
\{e_3\} & \text{if } \alpha \neq \frac{1}{2}, \beta \neq \frac{1}{2} \\
\{xe_1 + e_3 \mid x \in \mathbb{R}\} & \text{if } \alpha = \frac{1}{2}, \beta \neq \frac{1}{2} \\
\{ye_2 + e_3 \mid y \in \mathbb{R}\} & \text{if } \alpha \neq \frac{1}{2}, \beta = \frac{1}{2} \\
\{xe_1 + ye_2 + e_3 \mid x, y \in \mathbb{R}\} & \text{if } \alpha = \beta = \frac{1}{2}.
\end{cases}\]

A8) Properties of algebras \( A = A_2(0, 0) \)

- \( \text{Ann } A = \text{Span}_{\mathbb{R}}\{e_1, e_2\} \), \( \mathcal{N}(A) = \text{Span}_{\mathbb{R}}\{e_1, e_2\} \), \( \mathcal{I}(A) = \{e_3\} \),
- \( 1 \)-dimensional subalgebras: \( \mathbb{R}u \) for \( u \in \mathcal{N}(A) \cup \mathcal{I}(A) \),
- \( 2 \)-dimensional subalgebras: \( \text{Span}_{\mathbb{R}}\{e_3, pe_1 + qe_2\} \), \( \text{Span}_{\mathbb{R}}\{e_1, e_2\} \),
- ideals: \( \mathbb{R}e_3 \), \( \mathbb{R}(pe_1 + qe_2) \), \( \text{Span}_{\mathbb{R}}\{e_1, e_2\} \), \( \text{Span}_{\mathbb{R}}\{e_3, pe_1 + qe_2\} \) \((p^2 + q^2 \neq 0)\),
- \( A^2 = \mathbb{R}e_3 \): \( A/A^2 \) is the null \( 2 \)-dimensional algebra and \( A = \text{Span}_{\mathbb{R}}\{e_1, e_2\} \oplus \mathbb{R}e_3 \) is a direct sum of two ideals (this is like a Wedderburn-Artin decomposition by means of a maximal nilpotent ideal and a complementary subalgebra): \( A \) is an associative algebra,

- \( \text{Der } A = \left\{ \begin{bmatrix} x & u & 0 \\ z & u & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x, y, z, u \in \mathbb{R} \right\} \) (i.e. \( \text{Der } A \cong gl(2, \mathbb{R}) \)),
- \( \text{Aut } A = \left\{ \begin{bmatrix} x & y & 0 \\ z & u & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z, u \in \mathbb{R}, \ xu - yz \neq 0 \right\} \) (i.e. \( \text{Aut } A \cong GL(2, \mathbb{R}) \)),

- the partition \( \mathcal{P}_A \) of \( \mathbb{R}^3 \), defined by the lattice of subalgebras of \( A \), consists of:
  - the singletons covering the plane \( x^1Ox^2 \),
  - the half-axes of \( Ox^3 \), delimited by \( O \),
  - the quarters of planes delimited by axis \( Ox^3 \) and plane \( x^1Ox^2 \) on each plane containing axis \( Ox^3 \),
- the partition \( \mathcal{P}_A \) of \( A \) induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the plane \( x^1Ox^2 \),
  - the families of ray solutions lying on semi-axes of \( Ox^3 \) delimited by \( O \),
  - the integral curves contained in each quarters of each plane passing through axis \( Ox^1 \), delimited by axis \( Ox^3 \) and plane \( x^1Ox^2 \).
Note that each integral curve lies on a half-line parallel to $Ox^3$ delimited by plane $x^1Ox^2$. It means that each integral curve has both curvature and torsion tensors zero.

**A9) Properties of algebras $A = A_2(0, \frac{1}{2})$**
- $Ann A = \mathbb{R}e_1$, $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_1, e_2\}$, $I(A) = \{ye_2 + e_3 \mid y \in \mathbb{R}\}$.
- 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in \mathcal{N}(A) \cup I(A)$.
- 2-dimensional subalgebras: $Span_{\mathbb{R}}\{e_1, be_2 + ce_3\}$ ($b^2 + c^2 \neq 0$), $Span_{\mathbb{R}}\{e_2, e_3\}$.
- Ideals: $Re_1$, $Span_{\mathbb{R}}\{e_2, e_3\}$.
- $A^2 = Span_{\mathbb{R}}\{e_2, e_3\}$; $A/A^2$ is the null 1-dimensional algebra; $A$ is a vector direct sum of ideals.
- $Der A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & 0 & 0 \end{bmatrix}$, $x, y, z \in \mathbb{R}$.
- $Aut A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & z \\ 0 & 0 & 1 \end{bmatrix}$, $x, y, z \in \mathbb{R}$, $xy \neq 0$.
- The partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  - The singletons covering the plane $x^1Ox^2$.
  - The half-axes delimited by $O$ on the lines $\mathbb{R}u$ for $u \in I(A)$ (they cover plane $x^2Ox^3$ less axis $Ox^2$).
  - The quarters of plane $x^2Ox^3$, delimited by axes $Ox^2$ and $Ox^3$.
  - The quarters of planes passing through $Ox^1$ (less the plane $x^1Ox^2$), delimited by plane $x^2Ox^3$ and axis $Ox^1$.
- The partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - The singletons consisting of singular solutions that cover the plane $x^1Ox^2$.
  - The families of ray solutions lying on semi-axes of $Ox^3$ delimited by $O$.
  - The integral curves contained in each quarter of plane of $x^2Ox^3$, delimited by axes $Ox^2$ and $Ox^3$.
  - The integral curves contained in each quarter of planes passing through $Ox^1$ (less the plane $x^1Ox^2$), delimited by plane $x^2Ox^3$ and axis $Ox^1$.

Note that each nonsingular integral curve is torsion-free.

**10) Properties of algebras $A$ of type $A_2(\frac{1}{2}, \frac{1}{2})$**
- $Ann A = \{0\}$, $\mathcal{N}(A) = Span_{\mathbb{R}}\{e_1, e_2\}$, $I(A) = \{xe_1 + ye_2 + e_3 \mid x, y \in \mathbb{R}\}$.
- 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in \mathcal{N}(A) \cup I(A)$.
- 2-dimensional subalgebras: $Span_{\mathbb{R}}\{pe_1 + qe_2 + re_3, ae_1 + be_2 + ce_3\}$ when $\begin{vmatrix} p & q & r \\ a & b & c \end{vmatrix} = 2$ (i.e. each 2-dimensional subspace of $A$ is a subalgebra).
- Ideals: $Re_1$, $Re_2$, $Span_{\mathbb{R}}\{e_1, e_2\}$.
- $A^2 = A$. 

\begin{itemize}
  \item \(\text{Der} \; A = \left\{ \begin{array}{ccc} x & y & z \\
                          u & v & w \\
                          0 & 0 & 0 \end{array} \right\} | x, y, z, u, v, w \in \mathbb{R} \cong \text{aff} (2, \mathbb{R})\)
  \item \(\text{Aut} \; A = \left\{ \begin{array}{ccc} x & y & z \\
                          u & v & w \\
                          0 & 0 & 1 \end{array} \right\} | x, y, z, u, v, w \in \mathbb{R}^* \cong \text{aff} (2, \mathbb{R}),\)
\end{itemize}

\begin{itemize}
  \item the partition \(\mathcal{P}_A\) of \(\mathbb{R}^3\), defined by the lattice of subalgebras of \(A\), consists of:
    \begin{itemize}
      \item the singletons covering the plane \(x^1Ox^2\),
      \item the half-axes delimiting by \(O\) on each line \(Ru\) for \(u \in \mathcal{I}(A)\) (these lines cover the space less the plane \(x^1Ox^2\)),
    \end{itemize}
  \item the partition \(\mathcal{P}_A\) of \(A\) induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
    \begin{itemize}
      \item the singletons consisting of singular solutions that cover the plane \(x^1Ox^2\),
      \item the ray solutions lying on semi-lines delimited by \(O\) on each line \(Ru\) for \(u \in \mathcal{I}(A)\) (these lines cover the space less the plane \(x^1Ox^2\)),
    \end{itemize}
\end{itemize}

Each nonsingular integral curve lies on a line \(Ru\) for \(u \in \mathcal{I}(A)\), so that both its curvature and torsion tensors vanish.

Note that all idempotents have the same spectrum \(\left(\frac{1}{2}, \frac{1}{2}, 1\right)\).

A11) \textit{Properties of algebras \(A\) of type } \(A_2(0, \beta)\) \textit{with } \(\beta \notin \{0, \frac{1}{2}\}\)

\begin{itemize}
  \item \(A_2(0, \beta)\) \((\beta \notin \{0, \frac{1}{2}\})\) and \(A_2(0, \beta')\) \((\beta' \notin \{0, \frac{1}{2}\})\) are isomorphic if and only if \(\beta = \beta'\),
  \item \(\text{Ann} \; A = \mathbb{R}e_1, \; \mathcal{N}(A) = \text{Span}_\mathbb{R}\{e_1, e_2\}, \; \mathcal{I}(A) = \{e_3\}\),
  \item 1-dimensional subalgebras: \(\mathbb{R}u\) for \(u \in \mathcal{N}(A) \cup \mathcal{I}(A)\),
  \item 2-dimensional subalgebras: \(\text{Span}_\mathbb{R}\{e_1, e_2\}, \; \text{Span}_\mathbb{R}\{e_1, e_3\}, \; \text{Span}_\mathbb{R}\{e_2, e_3\}\),
  \item ideals: \(\mathbb{R}e_1, \; \mathbb{R}e_2, \; \text{Span}_\mathbb{R}\{e_1, e_2\}, \; \text{Span}_\mathbb{R}\{e_2, e_3\}\),
  \item \(A^2 = \text{Span}_\mathbb{R}\{e_2, e_3\}\); \(A/\mathcal{A}^2\) is the null 1-dimensional algebra; \(A\) is a vector direct sum of ideals,
  \item \(\text{Der} \; A = \left\{ \begin{array}{ccc} x & 0 & 0 \\
                          y & 0 & 0 \\
                          0 & 0 & 0 \end{array} \right\} | x, y \in \mathbb{R}\),
  \item \(\text{Aut} \; A = \left\{ \begin{array}{ccc} x & 0 & 0 \\
                          0 & y & 0 \\
                          0 & 0 & 1 \end{array} \right\} | x, y \in \mathbb{R}^* \cong \mathbb{R}^*(\cdot) \times \mathbb{R}^*(\cdot)\).
\end{itemize}

\begin{itemize}
  \item the partition \(\mathcal{P}_A\) of \(\mathbb{R}^3\), defined by the lattice of subalgebras of \(A\), consists of:
    \begin{itemize}
      \item the singletons covering the plane \(x^1Ox^2\),
      \item the half-axes of \(Ox^3\), delimited by \(O\),
      \item the quarters of plane \(x^1Ox^3\), delimited by axes \(Ox^1\) and \(Ox^3\),
      \item the quarters of plane \(x^2Ox^3\), delimited by axes \(Ox^2\) and \(Ox^3\),
      \item the connected sets in space delimited by planes \(x^1Ox^2\), \(x^1Ox^3\), \(x^2Ox^3\),
    \end{itemize}
  \item the partition \(\mathcal{P}_A\) of \(A\) induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
    \begin{itemize}
      \item the singletons consisting of singular solutions that cover the plane \(x^1Ox^2\),
the families of ray solutions lying on semi-axes of Ox³, delimited by O,
the integral curves contained in each quarters of plane x¹Ox³, delimited by axes Ox¹ and Ox³,
the integral curves contained in each quarters of plane x²Ox³, delimited by axes Ox² and Ox³,
the integral curves contained in each connected sets in space delimited by planes x¹Ox², x¹Ox³ and x²Ox³.

Note. In order to bring together the classes A₂(α, 0) and A₂(0, β) we have ignored, in case 11), the convention to consider increasing parameters in algebras of type A₂(a, b).

A12) Properties of algebras A of type A₂(½, β) with β ≠ {0, ½}

- algebras A₂(½, β) (β ≠ {0, ½}) and A₂(½, β') (β' ≠ {0, ½}) are isomorphic if and only if β = β',
- Ann A = {0}, N(A) = Span R{e₁, e₂}, I(A) = {xe₁ + e₃ | x ∈ R},
- 1-dimensional subalgebras: Ru for u ∈ N(A) ∪ I(A),
- 2-dimensional subalgebras: Span R{e₁, e₃}, Span R{e₂, ae₁ + ce₃} (a² + c² ≠ 0),
- ideals: Re₁, Re₂, Span R{e₁, e₂},
- A² = A,
- Der A = \{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix} | x, y ∈ R \},
- Aut A = \{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix} | x, y ∈ R^* \} ≅ R^* × R^*,
- the partition Pₐ of R³, defined by the lattice of subalgebras of A, consists of:
  - the singletons covering the plane x¹Ox²,
  - the half-axes delimited by O on each Ru for u ∈ I(A) (these lines cover x¹Ox³ less axis Ox¹),
  - the quarters of space delimited by planes x¹Ox³ and x²Ox³,
- the partition Pₐ of A induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the plane x¹Ox²,
  - the families of ray solutions lying on semi-axes delimited by O on each Ru for u ∈ I(A)
  - the integral curves contained in each quarter of space delimited by planes x¹Ox² and x¹Ox³.

All idempotents have the same spectrum: {β, ½, 1}.

A13) Properties of algebras A of type A₂(α, α) with α ≠ {0, ½}

- algebras A₂(α, α) (α ≠ {0, ½}) and A₂(α', α') (α' ≠ {0, ½}) are isomorphic if and only if α = α',
- Ann A = {0}, N(A) = Span R{e₁, e₂}, I(A) = {e₃},
• 1-dimensional subalgebras: \( \mathbb{R} u \) for \( u \in \mathcal{N}(A) \cup \mathcal{I}(A) \),
• 2-dimensional subalgebras: \( \text{Span}_\mathbb{R} \{e_1, e_2\} \), \( \text{Span}_\mathbb{R} \{e_3, pe_1 + qe_2\} \) \((p^2 + q^2 \neq 0)\),
• ideals: \( \mathcal{R}_1 \), \( \mathcal{R}_2 \), \( \text{Span}_\mathbb{R} \{e_1, e_2\} \),
• \( A^2 = A \),
• \( \text{Der} \ A = \left\{ \begin{array}{l}
x & y & 0 \\
z & v & 0 \\
0 & 0 & 0 \end{array} \right\} \mid x, y, z, v \in \mathbb{R} \cong \mathfrak{gl}(2, \mathbb{R}) \),
• \( \text{Aut} \ A = \left\{ \begin{array}{l}
x & y & 0 \\
z & v & 0 \\
0 & 0 & 1 \end{array} \right\} \mid x, y, z, v \in \mathbb{R}, \ xv - yz \neq 0 \cong GL(2, \mathbb{R}) \),

• the partition \( \mathcal{P}_A \) of \( \mathbb{R}^3 \), defined by the lattice of subalgebras of \( A \), consists of:
  ◦ the singletons covering the plane \( x^1 Ox^2 \),
  ◦ the half-lines delimited by \( O \) on \( Ox^3 \),
  ◦ the quarters delimited by axis \( Ox^3 \) and plane \( x^1 Ox^2 \) on each plane passing through \( Ox^3 \),
• the partition \( \mathcal{P}_A \) of \( A \) induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  ◦ the singletons consisting of singular solutions that cover the plane \( x^1 Ox^2 \),
  ◦ the families of ray solutions lying on semi-axes delimited by \( O \) on axis \( Ox^3 \),
  ◦ the integral curves contained in the quarters delimited by axis \( Ox^3 \) and plane \( x^1 Ox^2 \) on each plane passing through \( Ox^3 \).

\textbf{A14)} \textit{Properties of algebras} \( A \) \textit{of type} \( A_2(\alpha, \beta) \) \textit{with} \( \alpha, \beta \notin \{0, \frac{1}{2}\} \) \textit{and} \( \alpha < \beta \)

• algebras \( A_2(\alpha, \beta) \) \((\alpha < \beta \text{ and } \alpha, \beta \notin \{0, \frac{1}{2}\})\) and \( A_2(\alpha', \beta') \) \((\alpha' < \beta' \text{ and } \alpha', \beta' \notin \{0, \frac{1}{2}\})\) are isomorphic if and only if \( \alpha = \alpha' \) \text{ and } \( \beta = \beta' \),
• \( \text{Ann} \ A = \{0\}, \mathcal{N}(A) = \text{Span}_\mathbb{R} \{e_1, e_2\}, \mathcal{I}(A) = \{e_3\} \),
• 1-dimensional subalgebras: \( \mathbb{R} u \) for \( u \in \mathcal{N}(A) \cup \mathcal{I}(A) \),
• 2-dimensional subalgebras: \( \text{Span}_\mathbb{R} \{e_1, e_2\} \), \( \text{Span}_\mathbb{R} \{e_1, e_3\} \), \( \text{Span}_\mathbb{R} \{e_2, e_3\} \),
• ideals: \( \mathcal{R}_1 \), \( \mathcal{R}_2 \), \( \text{Span}_\mathbb{R} \{e_1, e_2\} \),
• \( A^2 = A \),
• \( \text{Der} \ A = \left\{ \begin{array}{l}
x & 0 & 0 \\
0 & y & 0 \\
0 & 0 & 0 \end{array} \right\} \mid x, y \in \mathbb{R} \),
• \( \text{Aut} \ A = \left\{ \begin{array}{l}
x & 0 & 0 \\
0 & y & 0 \\
0 & 0 & 1 \end{array} \right\} \mid x, y \in \mathbb{R}^* \cong \mathbb{R}^* \times \mathbb{R}^* \),

• the partition \( \mathcal{P}_A \) of \( \mathbb{R}^3 \), defined by the lattice of subalgebras of \( A \), consists of:
  ◦ the singletons covering the plane \( x^1 Ox^2 \),
  ◦ the half-axes delimited by \( O \) on \( Ox^3 \),
  ◦ the quarters of plane \( x^1 Ox^3 \), delimited by axes \( Ox^1 \) and \( Ox^3 \),
  ◦ the quarters of plane \( x^2 Ox^3 \), delimited by axes \( Ox^2 \) and \( Ox^3 \),
  ◦ the connected components of space delimited by planes \( x^1 Ox^2 \), \( x^1 Ox^3 \) and \( x^2 Ox^3 \).
• the partition $\mathcal{P}_A$ of $A$ induces a partition of the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  ◦ the singletons consisting of singular solutions that cover the axes $Ox^1$ and $Ox^2$,
  ◦ the ray solutions lying on semi-axes of $Ox^3$, delimited by $O$,
  ◦ the integral curves contained in each quarter of the plane of $x^1Ox^3$, delimited by axes $Ox^1$ and $Ox^3$,
  ◦ the integral curves contained in each quarter of the plane of $Ox^2$, delimited by axes $Ox^2$ and $Ox^3$,
  ◦ the integral curves contained in each connected component of space delimited by planes $x^1Ox^2$, $x^1Ox^3$ and $x^2Ox^3$.

Subcase (ii) $j=0$, $n \neq 0$

There exists a basis such that the multiplication table of algebra becomes:

**Table T3**

|          | $e_1^2 = 0$ | $e_2^2 = 0$ | $e_3^2 = 0$ |
|----------|--------------|--------------|--------------|
| $e_1 e_2$ | $e_3$        | $e_1 e_3 = \alpha e_1$ | $e_2 e_3 = \beta e_2$ |

with $\alpha, \beta \in \mathbb{R}$. Let us denote by $A_3(\alpha, \beta)$ any algebra of type T3.

**Proposition 2.8** The algebras $A_3(\alpha, \beta)$ and $A_3(\beta, \alpha)$ are isomorphic.

Consequently, in the following we restrict our interest to the case $\alpha \leq \beta$.

**Proposition 2.9** Every algebra $A$ of type $A_3(\alpha, \beta)$ has:

| Ann $A$ | $\mathbb{R}e_3$ if $\alpha = \beta = 0$ | $\{0\}$ if $\alpha^2 + \beta^2 \neq 0$ |
|----------|---------------------------------|---------------------------------|
| $\mathcal{N}(A)$ | $\mathbb{R}e_1 \cup \mathbb{R}e_2\{e_1, e_3\}$ if $\alpha = \beta = 0$ | $\mathbb{R}e_1 \cup \mathbb{R}e_2\{e_2, e_3\}$ if $\alpha \neq 0, \beta = 0$ |
|          | $\mathbb{R}[e_1, e_3]$ if $\alpha = 0, \beta \neq 0$ |

| $\mathcal{I}(A)$ | $\{xe_1 + \frac{1}{\alpha^2}e_2 + \frac{1}{\beta^2}e_3 \mid x \in \mathbb{R}^*\}$ if $\alpha = 0, \beta = 0$ | $\alpha \neq \beta$. |

**A15) Properties of algebras $A$ of type $A_3(0, 0)$**

It is suitable to use the change of bases $(e_1, e_2, e_3) \to (e_3, e_2, e_1)$.

- Ann $A = \mathbb{R}e_1$, $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2\{e_1, e_3\}$, $\mathcal{I}(A) = \emptyset$,
- 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in \mathcal{N}(A)$,
- 2-dimensional subalgebras: $\mathbb{R}e_1 \cup \mathbb{R}e_2$ (if $b^2 + c^2 = 0$),
- ideals: $\mathbb{R}e_1, \mathbb{R}e_2\{e_1, e_2 + ce_3\}$ (if $b^2 + c^2 = 0$),
- $A^2 = \mathbb{R}e_1; A/A^2$ is a 2-dimensional null algebra,
• Der $A = \left\{ \begin{bmatrix} x + y & z & u \\ 0 & y & 0 \\ 0 & 0 & x \end{bmatrix} \mid x, y, z, u \in \mathbb{R} \right\}$.

• $\text{Aut } A = H \cup J \cdot H$, where

$$H = \left\{ \begin{bmatrix} xy & z & v \\ 0 & y & 0 \\ 0 & 0 & x \end{bmatrix} \mid x, y, z, v \in \mathbb{R}, xy \neq 0 \right\} \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix};$$

$H$ is a normal divisor of $\text{Aut } A$.

• the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  - the singletons covering the planes $x^1Ox^2$ and $x^1Ox^3$,
  - the half-planes delimited by axis $Ox^1$ on each plane passing through $Ox^1$ less the planes $x^1Ox^2$ and $x^1Ox^3$,

• the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the planes $x^1Ox^2$ and $x^1Ox^3$,
  - the nonsingular integral curves contained in each half-plane delimited by axis $Ox^1$ on each plane passing through $Ox^1$ less the planes $x^1Ox^2$ and $x^1Ox^3$.

Since each nonsingular integral curve lies on a plane passing through $Ox^1$ it has a zero torsion tensor. In addition, $A^2 = \mathbb{R}e_1$ assures that each nonsingular integral curve has the curvature tensors zero, so that each nonsingular integral curve lies on a line parallel to $Ox^1$.

**Case $\alpha^2 + \beta^2 \neq 0$**

The class of algebras $A(\alpha, \beta)$ with $\alpha^2 + \beta^2 \neq 0$ decomposes naturally in accordance with the next conditions:

1. $\alpha \neq 0, \beta = 0$
2. $\alpha = 0, \beta \neq 0$
3. $\alpha \beta \neq 0 \quad \alpha = \beta$
4. $\alpha \beta \neq 0 \quad \alpha \neq \beta, \alpha < \beta$.

**Proposition 2.10** The following assertions hold:

1. Each algebra $A_3(\alpha, 0)$ with $\alpha \neq 0$ is isomorphic to algebra $A_3(1, 0) \cong A_3(0, 1)$,
2. Each algebra $A_3(0, \beta)$ with $\beta \neq 0$ is isomorphic to algebra $A_3(0, 1) \cong A_3(1, 0)$,
3. Each algebra $A_3(\alpha, \beta)$ with $\alpha = \beta$ is isomorphic to algebra $A_3(1, 1)$,
4. Each algebra $A_3(\alpha, \beta)$ with $\alpha \beta \neq 0$ and $\alpha < \beta$ is isomorphic to algebra $A_3(1, \beta)$.

**A16** Properties of algebra $A = A_3(0, 1)$

• $\text{Ann } A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_2 \cup \text{Span}_\mathbb{R}\{e_1, e_3\}$, $\mathcal{I}(A) = \emptyset$,

• 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in \mathcal{N}(A)$,
• 2-dimensional subalgebras: $\text{Span}_\mathbb{R}\{e_1, e_3\}$, $\text{Span}_\mathbb{R}\{e_2, e_3\}$,
• ideals: $\text{Span}_\mathbb{R}\{e_2, e_3\}$,
• $A^2 = \text{Span}_\mathbb{R}\{e_2, e_3\}$; $A/A^2$ is a 1-dimensional null algebra,
• $\text{Der } A = \mathbb{R}D$,
• $\text{Aut } A = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & \frac{1}{x} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x \in \mathbb{R}^* \right\} \cong \mathbb{R}^*$(*)

• the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  ◦ the singletons covering the axis $Ox^2$ and the plane $x^1Ox^3$,
  ◦ the connected sets delimited by axis $Ox^2$ and the plane $x^1Ox^3$,
• the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  ◦ the singletons consisting of singular solutions that cover the axes $Ox^1$, $Ox^2$ and $Ox^3$,
  ◦ the integral curves contained in each connected sets delimited by axis $Ox^2$ and the planes $x^1Ox^3$.

Since algebra $A$ is solvable, the nonsingular integral curves are torsion-free.

A17) Properties of algebra $A = A_3(1, 1)$
• $\text{Ann } A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2 \cup \mathbb{R}e_3$, $\mathcal{I}(A) = \{xe_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 \mid x \in \mathbb{R}^*\}$,
• 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in \mathcal{N}(A) \cup \mathcal{I}(A)$,
• 2-dimensional subalgebras: $\text{Span}_\mathbb{R}\{e_3, ae_1 + be_2\}$,
• ideals: none,
• $A^2 = A$,
• $\text{Der } A = \mathbb{R}D$,
• $\text{Aut } A = H \cup JH$ where
\[ H = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & \frac{1}{x} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x \in \mathbb{R}^* \right\} , \quad J = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \]
i.e. $\text{Aut } A \cong \mathbb{R}^* \times \{-1, 1\}$,
• the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  ◦ the singletons covering the axes $Ox^1$, $Ox^2$ and $Ox^3$,
  ◦ the half-lines delimited by $O$ on each line passing through $O$ and directed by an idempotent,
  ◦ the connected sets delimited by axis $Ox^2$ and the line $\mathbb{R}E$ directed by an idempotent $E = \omega e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3$ on the plane parallel to $E$ passing through $Ox^3$,
  ◦ the half-planes delimited by axis $Ox^3$ on any plane passing through $Ox^3$ which is not parallel to any idempotent (i.e. with $x^1x^2 < 0$),
• the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  ◦ the singletons consisting of singular solutions that cover the axes $Ox^1$, $Ox^2$ and $Ox^3$, 
  ◦
⋄ the families of ray solutions contained in each half-line delimited by $O$ on each line passing through $O$ and directed by an idempotent,

⋄ the integral curves contained in the connected sets delimited by axis $Ox^3$ and the line $RE$ directed by an idempotent $E = \omega e_1 + \frac{1}{2\omega} e_2 + \frac{1}{4} e_3$ on the plane parallel to $E$ passing through $Ox^3$,

⋄ the integral curves contained in the half-planes delimited by axis $Ox^3$ on any plane passing through $Ox^3$ which is not parallel to any idempotent (i.e. with $x^1 x^2 < 0$).

All nonsingular integral curves are torsion-free.

**A18) Properties of algebra $A$ of type $A_3(1, \beta)$ with $\beta > 1$**

- the algebras $A_3(1, \beta) \ (\beta > 1)$ and $A_3(1, \beta') \ (\beta' > 1)$ are isomorphic if and only if $\beta = \beta'$,

- $\text{Ann } A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1 \cup \mathbb{R}e_2 \cup \mathbb{R}e_3$, $\mathcal{I}(A) = \emptyset$,

- 1-dimensional subalgebras: $\mathbb{R}e_1$, $\mathbb{R}e_2$, $\mathbb{R}e_3$,

- 2-dimensional subalgebras: $\text{Span}_\mathbb{R}\{e_1, e_3\}$, $\text{Span}_\mathbb{R}\{e_2, e_3\}$,

- ideals: none,

- $A^2 = A$,

- $\text{Der } A = \mathbb{R}D$,

- $\text{Aut } A = H \cup JH$ where

$$H = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x \in \mathbb{R}^* \right\}, \quad J = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

i.e. $\text{Aut } A \cong \mathbb{R}^* \times \{-1, 1\}$,

- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:

  ⋄ the singletons covering the axes $Ox^1$, $Ox^2$ and $Ox^3$,

  ⋄ the quarters of plane $x^1 Ox^3$ delimited by axes $Ox^1$ and $Ox^3$,

  ⋄ the quarters of plane $x^2 Ox^3$ delimited by axes $Ox^2$ and $Ox^3$,

  ⋄ the quarters of space delimited by planes $x^1 Ox^3$ and $x^2 Ox^3$,

- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:

  ⋄ the singletons consisting of singular solutions that cover the axes $Ox^1$, $Ox^2$ and $Ox^3$,

  ⋄ the integral curves contained in the quarters of plane $x^1 Ox^3$ delimited by axes $Ox^1$ and $Ox^3$,

  ⋄ the integral curves contained in the quarters of plane $x^2 Ox^3$ delimited by axes $Ox^2$ and $Ox^3$,

  ⋄ the integral curves contained in the quarters of space delimited by planes $x^1 Ox^3$ and $x^2 Ox^3$.

**Subcase (iii) $j = n = 0$**

There exists a basis such that the multiplication table of algebra becomes
Table T4

\[
\begin{array}{cccc}
  e_1^2 = 0 & e_2^2 = 0 & e_3^2 = 0 \\
  e_1 e_2 = 0 & e_1 e_3 = \alpha e_1 & e_2 e_3 = \beta e_2
\end{array}
\]

with \( \alpha, \beta \in \mathbb{R} \). Let us denote by \( A_4(\alpha, \beta) \) each algebra defined by multiplication table T4.

**Proposition 2.11** Algebra \( A_4(\alpha, \beta) \) and \( A_4(\beta, \alpha) \) are isomorphic.

Consequently, in the following we will be interested only in the case \( \alpha \leq \beta \).

We have to consider the next mutually exclusive situations:

(i) \( \alpha \neq 0, \beta = 0 \),
(ii) \( \alpha = 0, \beta \neq 0 \),
(iii) \( \alpha \beta \neq 0, \alpha = \beta \),
(iv) \( \alpha \beta \neq 0 \) and \( \alpha \neq \beta \).

**Proposition 2.12** The following assertions hold:

(i) each algebra \( A_4(\alpha, 0) \) with \( \alpha \neq 0 \) is isomorphic to algebra \( A_4(1, 0) \),
(ii) each algebra \( A_4(0, \beta) \) with \( \beta \neq 0 \) is isomorphic to algebra \( A_4(0, 1) \cong A_4(1, 0) \),
(iii) each algebra \( A_4(\alpha, \beta) \) with \( \alpha = \beta \) is isomorphic to algebra \( A_4(1, 1) \),
(iv) each algebra \( A_4(\alpha, \beta) \) with \( \alpha \beta \neq 0 \) and \( \alpha \neq \beta \) is isomorphic to algebra \( A_4(1, \beta) \).

**A19** Properties of algebra \( A = A_4(0, 1) \)

- \( \text{Ann} A = \mathbb{R} e_1, \ N(A) = \text{Span}_{\mathbb{R}} \{ e_1, e_2 \} \cup \text{Span}_{\mathbb{R}} \{ e_1, e_3 \}, \ I(A) = \emptyset \),
- 1-dimensional subalgebras: \( \mathbb{R} u \) for \( u \in N(A) \),
- 2-dimensional subalgebras: \( \text{Span}_{\mathbb{R}} \{ e_1, e_3 \}, \text{Span}_{\mathbb{R}} \{ e_2, a e_1 + b e_3 \} (a^2 + b^2 \neq 0) \),
- ideals: \( \mathbb{R} e_1, \mathbb{R} e_2, \text{Span}_{\mathbb{R}} \{ e_2, a e_1 + b e_3 \} (a^2 + b^2 \neq 0) \),
- \( A^2 = \mathbb{R} e_2; A/A^2 \) is a 2-dimensional null algebra; \( A/A^2 \) is a 2-dimensional null algebra,

\[
\text{Der } A = \left\{ \begin{bmatrix} x & 0 & y \\ 0 & z & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\},
\]

\[
\text{Aut } A = \left\{ \begin{bmatrix} x & 0 & y \\ 0 & z & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R}, xz \neq 0 \right\},
\]

- the partition \( \mathcal{P}_A \) of \( \mathbb{R}^3 \), defined by the lattice of subalgebras of \( A \), consists of:
  - the singletons covering the planes \( x^1 O x^2 \) and \( x^1 O x^3 \),
  - the quarters of space delimited by planes \( x^1 O x^2 \) and \( x^1 O x^3 \),
- the partition \( \mathcal{P}_A \) of \( A \) induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the planes \( x^1 O x^2 \) and \( x^1 O x^3 \),
  - the integral curves contained in the quarters of space delimited by planes \( x^1 O x^2 \) and \( x^1 O x^3 \).
A20) Properties of algebra $A = A_4(1, 1)$

- $\text{Ann } A = \{0\}$, $\mathcal{N}(A) = \text{Span}_\mathbb{R}\{e_1, e_2\} \cup \mathbb{R}e_3$, $\mathcal{I}(A) = \emptyset$,
- 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in \mathcal{N}(A)$,
- 2-dimensional subalgebras: $\text{Span}_\mathbb{R}\{e_1, e_2\}$, $\text{Span}_\mathbb{R}\{e_3, ae_1 + be_2\} (a^2 + b^2 \neq 0)$,
- ideals: $\mathbb{R}e_1$, $\mathbb{R}e_2$, $\text{Span}_\mathbb{R}\{e_1, e_2\}$,
- $A^2 = \text{Span}_\mathbb{R}\{e_1, e_2\}$; $A/A^2$ is a 1-dimensional null algebra,
- $\text{Der } A = \begin{bmatrix} x & y & 0 \\ z & u & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $| x, y, z, u \in \mathbb{R}$ $\cong \mathfrak{gl}(2, \mathbb{R})$,
- $\text{Aut } A = \begin{bmatrix} x & y \\ z & u \\ 0 & 0 & 1 \end{bmatrix}$ $| x, y, z, u \in \mathbb{R}, xu - yz \neq 0$ $\cong \text{GL}(2, \mathbb{R})$,
- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  - the singletons covering the plane $x^1 Ox^2$ and axis $Ox^3$,
  - the quarters of plane delimited by axis $Ox^3$ and $x^1 Ox^2$ on each plane passing through $Ox^3$,
- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the plane $x^1 Ox^2$ and the axis $Ox^3$,
  - the integral curves contained in the quarters of plane delimited by axis $Ox^3$ and $x^1 Ox^2$ on each plane passing through $Ox^3$.

Consequently, each nonsingular integral curve is torsion-free.

A21) Properties of algebra $A$ of type $A_4(1, \beta)$ ($\beta \notin \{0, 1\}$)

- the algebras $A_4(1, \beta)$ ($\beta \notin \{0, 1\}$) and $A_4(1, \beta')$ ($\beta \notin \{0, 1\}$) are isomorphic if and only if $\beta = \beta'$,
- $\text{Ann } A = \{0\}$, $\mathcal{N}(A) = \text{Span}_\mathbb{R}\{e_1, e_2\} \cup \mathbb{R}e_3$, $\mathcal{I}(A) = \emptyset$,
- 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in \mathcal{N}(A)$,
- 2-dimensional subalgebras: $\text{Span}_\mathbb{R}\{e_1, e_2\}$, $\text{Span}_\mathbb{R}\{e_3, e_1, e_2\}$, $\text{Span}_\mathbb{R}\{e_2, e_3\}$,
- ideals: $\mathbb{R}e_1$, $\mathbb{R}e_2$, $\text{Span}_\mathbb{R}\{e_1, e_2\}$,
- $A^2 = \text{Span}_\mathbb{R}\{e_1, e_2\}$; $A/A^2$ is a 1-dimensional null algebra,
- $\text{Der } A = \begin{bmatrix} x & 0 \\ 0 & y \\ 0 & 0 \end{bmatrix}$ $| x, y \in \mathbb{R}$
- $\text{Aut } A = \begin{bmatrix} x & 0 \\ 0 & y \\ 0 & 0 \end{bmatrix}$ $| x, y \in \mathbb{R}^*$ $\cong \mathbb{R}^* \times \mathbb{R}^*$,
- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  - the singletons covering the plane $x^1 Ox^2$ and axis $Ox^3$,
  - the quarters delimited by axes $Ox^1$ and $Ox^3$ on the plane $x^1 Ox^3$,
  - the quarters delimited by axes $Ox^2$ and $Ox^3$ on the plane $x^2 Ox^3$. 
the connected sets of space $\mathbb{R}^3$ delimited by coordinate planes $x^1Ox^2$, $x^1Ox^3$ and $x^2Ox^3$,

- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the plane $x^1Ox^2$ and the axis $Ox^3$,
  - the integral curves contained in the quarters of plane $x^1Ox^3$ delimited by axes $Ox^1$ and $Ox^3$,
  - the integral curves contained in the quarters of plane $x^2Ox^3$ delimited by axes $Ox^2$ and $Ox^3$,
  - the integral curves contained in the connected sets of space $\mathbb{R}^3$ delimited by coordinate planes $x^1Ox^2$, $x^1Ox^3$ and $x^2Ox^3$.

2) Case $Spec D = (1, 1, 0)$

Algebra $\mathcal{A}(\cdot)$ has the next multiplication table:

| Table T | $e_1^2 = 0$ | $e_2^2 = 0$ | $e_3^2 = e_3$ | $e_1e_2 = 0$ | $e_1e_3 = \alpha e_1 + \beta e_2$ | $e_2e_3 = \gamma e_1 + \delta e_2$ |
|---------|-------------|-------------|--------------|-------------|----------------|----------------|

with $\alpha$, $\beta$, $\gamma$, $\delta \in \mathbb{R}$. It is natural to consider the next complementary subcases:

I) $\alpha \delta - \beta \gamma = 0$,

II) $\alpha \delta - \beta \gamma \neq 0$.

In its turn, Subcase I is naturally divided into two disjoint parts:

I$_1$ $\alpha \delta - \beta \gamma = 0$, $\alpha \beta \gamma \delta \neq 0$,

I$_2$ $\alpha \delta - \beta \gamma = 0$, $\alpha \beta \gamma \delta = 0$.

I$_1$. $\alpha \delta - \beta \gamma = 0$, $\alpha \beta \gamma \delta \neq 0$

The algebra $\mathcal{A}(\cdot)$ has $Ann \mathcal{A} = \mathbb{R}(\gamma e_1 - \alpha e_2)$. We have to consider the next complementary situations:

I$_{11}$. $\alpha^2 + \beta \gamma \neq 0$, I$_{12}$ $\alpha^2 + \beta \gamma = 0$.

I$_{11}$ By using basis $(\gamma e_1 - \alpha e_2, \alpha e_1 + \beta e_2, e_3)$ with $\alpha^2 + \beta \gamma \neq 0$ the multiplication table of algebra becomes:

| Table $TI_{11}$ | $e_1^2 = 0$ | $e_2^2 = 0$ | $e_3^2 = e_3$ | $e_1e_2 = 0$ | $e_1e_3 = 0$ | $e_2e_3 = \alpha e_2$ |
|------------------|-------------|-------------|--------------|-------------|----------------|----------------|

with $\alpha \neq 0$. Let us denote by $B(\alpha)$ the algebra defined by means of Table $TI_{11}$. Consequently, algebra $B(\alpha)$ is isomorphic to an algebra of type $A_4$, i.e. $B(\alpha) \cong A_1(0, \alpha) \cong A_1(\alpha, 0)$ for $\alpha \notin \{0, \frac{1}{2}\}$. Similarly, algebra $B(\frac{1}{2})$ is isomorphic to an algebra of type $A_2$, i.e. $B(\frac{1}{2}) \cong A_1(0, \frac{1}{2})$.

Case I$_{12}$. $\delta = -\alpha$
By taking basis $(\gamma e_1 - \alpha e_2, e_2, e_3)$ the multiplication table for $A$ becomes

$$
\begin{array}{ccc}
\text{Table } T_{12} & e_1^2 = 0 & e_2^2 = 0 & e_3^2 = e_3 \\
e_1 e_2 = 0 & e_1 e_3 = 0 & e_2 e_3 = e_1 \\
\end{array}
$$

A22) Properties of algebra $A_{22}$

- $\text{Ann } A = \mathbb{R} e_1$, $N(A) = \text{Span} \mathbb{R} \{e_1, e_2\}$, $I(A) = \{e_3\}$,
- 1-dimensional subalgebras: $\mathbb{R} u$ for $u \in N(A) \cup I(A)$,
- 2-dimensional subalgebras: $\text{Span} \mathbb{R} \{e_1, e_2\}$, $\text{Span} \mathbb{R} \{e_1, e_3\}$,
- ideals: $\mathbb{R} e_1$, $\text{Span} \mathbb{R} \{e_1, e_2\}$, $\text{Span} \mathbb{R} \{e_1, e_3\}$,
- $A^2 = \text{Span} \mathbb{R} \{e_1, e_3\}$; $A/A^2$ is a 1-dimensional non-null algebra,
- $\text{Der } A = \left\{ \begin{pmatrix} x & y & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$,
- $\text{Aut } A = \left\{ \begin{pmatrix} x & y & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y \in \mathbb{R}, x \neq 0 \right\}$,
- the partition $P_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  - the singletons covering the plane $x^1 Ox^2$,
  - the half-axes of axis $Ox^3$ delimited by $O$,
  - the quarters of plane delimited by axes $Ox^1$ and $Ox^3$ on the plane $x^1 Ox^3$,
  - the quarters of space $\mathbb{R}^3$ delimited by planes $x^1 Ox^2$ and $x^1 Ox^3$,
- the partition $P_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the plane $x^1 Ox^2$,
  - the families of ray-solutions lying on the half-axes of $Ox^3$ delimited by $O$,
  - the integral curves contained in the quarters of plane $x^1 Ox^3$ delimited by axes $Ox^1$ and $Ox^3$,
  - the integral curves contained in the quarters of space $\mathbb{R}^3$ delimited by planes $x^1 Ox^2$ and $x^1 Ox^3$.

Ideal $\text{Span} \mathbb{R} \{e_1, e_3\}$ compels each nonsingular integral curve to be torsion-free.

Subcase $I_2 \alpha \delta - \beta \gamma = 0$, $\alpha \beta \gamma \delta = 0$

We have to consider the cases:

1) $\alpha \neq 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$,
2) $\alpha \neq 0$, $\beta = 0$, $\gamma \neq 0$, $\delta = 0$,
3) $\alpha = 0$, $\beta \neq 0$, $\gamma = 0$, $\delta = 0$,
4) $\alpha \neq 0$, $\beta \neq 0$, $\gamma = 0$, $\delta = 0$,
5) $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta \neq 0$,
6) $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta \neq 0$,
7) $\alpha = 0$, $\beta = 0$, $\gamma \neq 0$, $\delta = 0$,
8) $\alpha = 0$, $\beta = 0$, $\gamma \neq 0$, $\delta \neq 0$.

**Proposition 2.13** The algebras $1), 2), 4), 6), 7), 8)$ are isomorphic to $A_{11}$, while algebras $3)$ and $5)$ are respectively isomorphic to algebras $A_{22}$ and $A_{8}$.

**Subcase II**: $\alpha \delta - \beta \gamma \neq 0$

We shall look for a new basis such that the corresponding multiplication table of algebra should contain a minimal number of parameters. This is in fact the same with solving the problem: is or is not semisimple the endomorphism $L_{e_3}$? In fact, this problem is equivalent to the problem: is or is not diagonalisable the matrix

$$P = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}.$$ 

We have to analyze the next two mutually exclusive cases:

(i) $P$ has complex eigenvalues (i.e. $(\alpha + \delta)^2 - 4(\alpha \delta - \beta \gamma) < 0$),

(ii) $P$ has real eigenvalues (i.e. $(\alpha + \delta)^2 - 4(\alpha \delta - \beta \gamma) \geq 0$).

**Case (i)**

There exists a basis such that the multiplication table of algebra has the form:

**Table T5**

| $e_1^2$ = 0 | $e_2^2$ = 0 | $e_3^2 = e_3$ |
|-------------|-------------|---------------|
| $e_1e_2 = 0$| $e_1e_3 = ae_1 - be_2$| $e_2e_3 = be_1 + ae_2$ |

with $b > 0$. Let us denote by $A_5(a, b)$ any algebra having multiplication table $T5$.

**Proposition 2.14** The algebras $A_5(a, b) \ (b > 0)$ and $A_5(a', b') \ (b' > 0)$ are isomorphic if and only if $a = a'$, $b = b'$.

**Proof.** Suppose that the multiplication table of $A_5(a', b')$ is

**Table T'5**

| $f_1^2 = 0$ | $f_2^2 = 0$ | $f_3^2 = f_3$ |
|-------------|-------------|---------------|
| $f_1f_2 = 0$| $f_1f_3 = a'f_1 - b'f_2$| $f_2f_3 = b'f_1 + a'f_2$ |

and $T(e_i) = \sum_{i=1}^{3} s_{ij}f_i$ is an automorphism of $A(a, b) \ (b > 0)$ with $A(a', b') \ (b' > 0)$. Then, $s_{31} = s_{32} = 0$, $s_{33} = 1$ and

$$s_{13} = 2a's_{13} + 2b's_{23}, \ s_{23} = 2a's_{23} - 2b's_{13} \Leftrightarrow s_{13} = s_{23} = 0.$$ 

The conditions $T(e_1e_3) = T(e_1)T(e_3)$ and $T(e_2e_3) = T(e_2)T(e_3)$ are equivalent to next equations in unknown entries $s_{11}, \ s_{21}, \ s_{12}, \ s_{22}$:

$$\begin{cases}
(a - a')s_{11} - b's_{21} - bs_{12} = 0 \\
b's_{11} + (a - a')s_{21} - bs_{22} = 0 \\
bs_{11} + (a - a')s_{12} - b's_{22} = 0 \\
bs_{21} + b's_{12} + (a - a')s_{22} = 0.
\end{cases}$$
This system has nonzero solution if and only if \( a = a' \) and \( b = b' \). Indeed, the determinant \( (b^2 - b'^2)^2 + (a - a')^2[(a - a')^2 + 2b^2 + 2b'^2] \) of matrix of coefficients of this homogeneous system is zero if and only if \( a = a' \) and \( b = b' \).

\[ \square \]

**A23) Properties of algebra A of type \( A_5(a,b) \) \( (b > 0) \)**

- \( \text{Ann} \ A = \{0\}, \ N(A) = \text{Span}_R\{e_1, e_2\}, \ I(A) = \{e_3\}, \)
- 1-dimensional subalgebras: \( \mathbb{R}u \) for \( u \in N(A) \cup I(A) \),
- 2-dimensional subalgebras: \( \text{Span}_R\{e_1, e_2\} \),
- ideals: \( \text{Span}_R\{e_1, e_2\} \)
- \( A^2 = \mathbb{A} \)
- \( \text{Der} \ A = \left\{ \begin{bmatrix} x & y & 0 \\ -y & x & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\} \cong \mathbb{C}(\cdot), \)
- \( \text{Aut} \ A = \left\{ \begin{bmatrix} x & y & 0 \\ -y & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x, y \in \mathbb{R}^* \right\} \cong \mathbb{C}^*(\cdot), \)
- the partition \( \mathcal{P}_A \) of \( \mathbb{R}^3 \), defined by the lattice of subalgebras of \( A \), consists of:
  - the singletons covering the plane \( x^1Ox^2 \),
  - the half-axes of axis \( Ox^3 \) delimited by \( O \),
  - the half-spaces delimited by plane \( x^1Ox^2 \) less the points of axis \( Ox^3 \),
- the partition \( \mathcal{P}_A \) of \( A \) induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the plane \( x^1Ox^2 \),
  - the families of ray-solutions lying on the half-axes of \( Ox^3 \) delimited by \( O \),
  - the integral curves contained in half-spaces delimited by plane \( x^1Ox^2 \) less the points of axis \( Ox^3 \).

**Case (ii)**

There exists a basis, consisting of the eigenvectors of endomorphism \( L_{e_3} \), such that the multiplication table of algebra has the form:

**Table Tii**

\[
\begin{array}{ccc}
  e_1^2 & = 0 & e_2^2 = 0 & e_3^2 = e_3 \\
  e_1e_2 & = 0 & e_1e_3 = ae_1 & e_2e_3 = be_2 \\
\end{array}
\]

with \( a, b \in \mathbb{R} \). Let us remark that algebras of type \( \text{Tii} \) are necessarily algebras of type \( \text{A2} \).

**Case Spec \( D=(1,2,0) \)**

There exists a basis \( B = (e_1, e_2, e_3) \) such that the multiplication table of algebra is

**Table T’**

\[
\begin{array}{ccc}
  e_1^2 & = be_2 & e_2^2 = 0 & e_3^2 = je_3 \\
  e_1e_2 & = 0 & e_1e_3 = pe_1 & e_2e_3 = le_2 \\
\end{array}
\]
with \( b, j, p, t \in \mathbb{R} \). Will be suitable to use the change of bases \((e_1, e_2, e_3) \rightarrow (e_2, e_1, e_3)\). The corresponding multiplication table is:

**Table T**

\[
\begin{array}{ccc}
  e_1^2 &=& 0 \\
  e_1 e_2 &=& 0 \\
  e_2^2 &=& be_1 \\
  e_1 e_3 &=& te_1 \\
  e_2 e_3 &=& pe_2 \\
  e_3^2 &=& je_3
\end{array}
\]

with \( b, j, p, t \in \mathbb{R} \).

We have to distinguish two complementary situations:

**I)** \( bj \neq 0 \)

**II)** \( bj = 0 \)

1) **Case \( bj \neq 0 \)**

There exists a basis \( B = (e_1, e_2, e_3) \) such that the multiplication table of algebra is

**Table T6**

\[
\begin{array}{ccc}
  e_1^2 &=& 0 \\
  e_1 e_2 &=& 0 \\
  e_2^2 &=& e_1 \\
  e_1 e_3 &=& \alpha e_1 \\
  e_2 e_3 &=& \beta e_2 \\
  e_3^2 &=& e_3
\end{array}
\]

with \( \alpha, \beta \in \mathbb{R} \). Let us denote by \( A_6(\alpha, \beta) \) any algebra having the multiplication table \( T_6 \).

**Proposition 2.15** The algebras \( A_6(\alpha, \beta) \) and \( A_6(\alpha', \beta') \) are isomorphic if and only if \( \alpha = \alpha' \), \( \beta = \beta' \).

**Proposition 2.16** Every algebra \( A_6(\alpha, \beta) \) has:

\[
\begin{align*}
  \text{Ann } A &= \begin{cases}
    \mathbb{R} e_1 & \text{if } \alpha = 0 \\
    \{0\} & \text{if } \alpha \neq 0.
  \end{cases} \\
  N(A) &= \mathbb{R} e_1 \\

  \mathcal{I}(A) &= \begin{cases}
    \{e_3\} & \text{if } \alpha \neq \frac{1}{2}, \beta \neq \frac{1}{2} \\
    xe_1 + e_3 & \text{if } \alpha = \frac{1}{2} \\
    \{\frac{-1}{2e_2}e_1 + ye_1 + e_3 \mid y \in \mathbb{R}\} & \text{if } \alpha \neq \frac{1}{2}, \beta = \frac{1}{2}
  \end{cases}
\end{align*}
\]

Accordingly, we have to consider the classes of algebras:

(i) \( A_6(0, \beta) \) and (ii) \( A_6(\alpha, \beta) \) with \( \alpha \neq 0 \)

**Case (i)**

Algebras \( A_6(0, \beta) \) with \( \beta \notin \{\frac{1}{2}, 0\} \)

**Proposition 2.15** implies:

**Corollary 2.1** The algebras \( A_6(0, \beta) \) and \( A_6(0, \beta') \) are isomorphic if and only if \( \beta = \beta' \).
A24) Properties of algebra $A = A_6(0, \beta)$ with $\beta \notin \{0, \frac{1}{3}\}$

- $Ann A = {e_1}, \mathcal{N}(A) = \mathbb{R}e_1, \mathcal{I}(A) = \{e_2\},$
- 1-dimensional subalgebras: $\mathbb{R}e_1, \mathbb{R}e_3,$
- 2-dimensional subalgebras: $Span_\mathbb{R}\{e_1, e_2\}, \ Span_\mathbb{R}\{e_1, e_3\},$
- ideals: $\mathbb{R}e_1, \ Span_\mathbb{R}\{e_1, e_2\},$
- $A^2 = A$
- $Der A = \left\{ \begin{bmatrix} 2x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in \mathbb{R} \right\} \cong \mathbb{R}D,$
- $Aut A = \left\{ \begin{bmatrix} x^2 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x \in \mathbb{R}^* \right\} \cong \mathbb{R}^*,$
- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  - the singletons covering the axis $Ox^1,$
  - the half-axes of axis $Ox^3$ delimited by $O,$
  - the quarters delimited by axes $Ox^1$ and $Ox^3$ on plane $x^1Ox^3,$
  - the half-planes delimited by axis $Ox^1$ on plane $x^1Ox^2,$
  - the quarters of space delimited by planes $x^1Ox^2$ and $x^1Ox^3,$
- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the axis $Ox^1,$
  - the families of ray-solutions lying on the half-axes of $Ox^3$ delimited by $O,$
  - the integral curves contained in the quarters delimited by axes $Ox^1$ and $Ox^3$ on plane $x^1Ox^3,$
  - the integral curves contained in the half-planes delimited by axis $Ox^1$ on plane $x^1Ox^2,$
  - the integral curves contained in the quarters of space delimited by planes $x^1Ox^2$ and $x^1Ox^3.$

A25) Properties of algebra $A = A_6(0, 0)$

- $Ann A = \mathbb{R}e_1, \mathcal{N}(A) = \mathbb{R}e_1, \mathcal{I}(A) = \{e_3\},$
- 1-dimensional subalgebras: $\mathbb{R}e_1, \mathbb{R}e_3,$
- 2-dimensional subalgebras: $Span_\mathbb{R}\{e_1, e_2\}, \ Span_\mathbb{R}\{e_1, e_3\},$
- ideals: $\mathbb{R}e_1, \ Span_\mathbb{R}\{e_1, e_2\}, \ Span_\mathbb{R}\{e_1, e_3\},$
- $A^2 = Span_\mathbb{R}\{e_1, e_3\}$
- $Der A = \left\{ \begin{bmatrix} 2x & y & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\},$
- $Aut A = \left\{ \begin{bmatrix} x^2 & y & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x, y \in \mathbb{R}, \ x \neq 0 \right\},$
- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  - the singletons covering the axis $Ox^1,$
the half-axes of axis $Ox^3$ delimited by $O$,  
the quarters delimited by axes $Ox^1$ and $Ox^3$ on plane $x^1Ox^3$,  
the half-planes delimited by axis $Ox^1$ on plane $x^1Ox^2$,  
the quarters of space delimited by planes $x^1Ox^2$ and $x^1Ox^3$,  
the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:  
the singletons consisting of singular solutions that cover the axis $Ox^1$,  
the families of ray-solutions lying on the half-axes of $Ox^3$ delimited by $O$,  
the integral curves contained in the quarters delimited by axes $Ox^1$ and $Ox^3$ on plane $x^1Ox^3$,  
the integral curves contained in the half-planes delimited by axis $Ox^1$ on plane $x^1Ox^2$,  
the integral curves contained in the quarters of space delimited by planes $x^1Ox^2$ and $x^1Ox^3$.

A26) Properties of algebra $A = A_6(0, \frac{1}{2})$  
- $\text{Ann } A = \mathbb{R}e_1$, $\mathcal{N}(A) = \mathbb{R}e_1$, $\mathcal{I}(A) = \{x^2 e_1 + xe_2 + e_3 \mid x \in \mathbb{R}\}$,  
- 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in \mathcal{N}(A) \cup \mathcal{I}(A)$,  
- 2-dimensional subalgebras: $\text{Span}_{\mathbb{R}}\{e_1, be_2 + ce_3\}$, $(b^2 + c^2 \neq 0)$,  
- ideals: $\mathbb{R}e_1$, $\text{Span}_{\mathbb{R}}\{e_1, e_2\}$,  
- $A^2 = A$,  
- $\text{Der } A = \left\{ \begin{bmatrix} 2x & 2y & 0 \\ 0 & x & y \\ 0 & 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$,  
- $\text{Aut } A = \left\{ \begin{bmatrix} 2x^2 & 2xy & y^2 \\ 0 & x & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y \in \mathbb{R}, x \neq 0 \right\}$,  
- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:  
  the singletons covering the axis $Ox^1$,  
  the half-lines of $\mathbb{R}u$ for $u \in \mathcal{I}(A)$ delimited by $O$ (these lines cover the cone $(x^2)^2 - x^1x^3 = 0$),  
  the connected components delimited by axis $Ox^1$ and the cone $(x^2)^2 - x^1x^3 = 0$ on each plane containing $Ox^1$ (of course, this cone has a real contribution whenever $x^1x^3 > 0$),  
- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:  
  the singletons consisting of singular solutions that cover the axis $Ox^1$,  
  the families of ray-solutions lying on the half-lines of $\mathbb{R}u$ for $u \in \mathcal{I}(A)$ delimited by $O$,  
  the integral curves contained in the connected components delimited by axis $Ox^1$ and the cone $(x^2)^2 - x^1x^3 = 0$ on each plane containing $Ox^1$.

Case (ii)
Algebras $A_6(\alpha, \beta)$ with $\alpha \neq 0$

Proposition 2.15 implies:

**Corollary 2.2** The algebras $A_6(\alpha, \beta)$ ($\alpha \neq 0$) and $A_6(\alpha', \beta')$ ($\alpha' \neq 0$) are isomorphic if and only if $\alpha = \alpha'$, $\beta = \beta'$.

**Proposition 2.17** Every algebra $A_6(\alpha, \beta)$ has:

\[ \text{Ann } A = \{0\} \]

\[ \mathcal{N}(A) = R e_1 \]

\[ \mathcal{I}(A) = \begin{cases} 
\{e_3\} & \text{if } \alpha \neq \frac{1}{2}, \beta \neq \frac{1}{2} \\
\{xe_1 + e_3 \mid x \in \mathbb{R}\} & \text{if } \alpha = \frac{1}{2}, \beta \neq \frac{1}{2} \\
\left\{\frac{y^2}{1-2\alpha}e_1 + ye_2 + e_3 \mid y \in \mathbb{R}\right\} & \text{if } \alpha \neq \frac{1}{2}, \beta = \frac{1}{2} \\
\{xe_1 + e_3 \mid x \in \mathbb{R}\} & \text{if } \alpha = \frac{1}{2}, \beta = \frac{1}{2}.
\end{cases} \]

**A27** Properties of algebras $A$ of type $A_6(\alpha, \beta)$ with $\alpha, \beta \notin \{0, \frac{1}{2}\}$ and $\alpha \neq \beta$

- Ann $A = \{0\}$, $\mathcal{N}(A) = R e_1$, $\mathcal{I}(A) = \{e_3\}$,
- 1-dimensional subalgebras: $R e_1$, $R e_3$,
- 2-dimensional subalgebras: $\text{Span}_R\{e_1, e_2\}$, $\text{Span}_R\{e_1, e_3\}$,
- ideals: $R e_1$, $\text{Span}_R\{e_1, e_2\}$,
- $A^2 = A$
- Der $A = \begin{bmatrix} 2x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in \mathbb{R} \cong \mathbb{R} D$,
- Aut $A = \begin{bmatrix} 2x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x \in \mathbb{R}^* \cong \mathbb{R}^* (\cdot)$,
- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  - the singletons covering the axis $Ox^1$,
  - the half-axes of $Ox^3$ delimited by $O$,
  - the half-planes delimited by axis $Ox^1$ on plane $x^1Ox^2$,
  - the quarters of plane delimited by axes $Ox^1$ and $Ox^3$ on plane $x^1Ox^3$,
  - the quarters of space delimited by planes $x^1Ox^2$ and $x^1Ox^3$,
- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the axis $Ox^1$,
  - the families of ray-solutions lying on the half-axes of $Ox^3$ delimited by $O$,
  - the integral curves contained in the half-planes delimited by axis $Ox^1$ on plane $x^1Ox^2$,
  - the integral curves contained in the quarters of plane delimited by axes $Ox^1$ and $Ox^3$ on plane $x^1Ox^3$. 


the integral curves contained in the quarters of space delimited by planes $x^1Ox^2$ and $x^1Ox^3$.

A28) Properties of algebras $A$ of type $A_6(\alpha, \alpha)$ with $\alpha \notin \{0, \frac{1}{2}\}$

- the algebras $A_6(\alpha, \alpha)$ ($\alpha \neq 0$) and $A_6(\alpha', \alpha')$ ($\alpha' \neq 0$) are isomorphic if and only if $\alpha = \alpha'$
  - $\text{Ann } A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1$, $\mathcal{I}(A) = \{e_3\}$,
  - 1-dimensional subalgebras: $\mathbb{R}e_1$, $\mathbb{R}e_3$,
  - 2-dimensional subalgebras: $\text{Span}_{\mathbb{R}}\{e_1, e_2\}$, $\text{Span}_{\mathbb{R}}\{e_1, e_3\}$,
  - ideals: $\mathbb{R}e_1$, $\text{Span}_{\mathbb{R}}\{e_1, e_2\}$,
  - $A^2 = A$
  - $\text{Der } A = \begin{cases} 2x & 0 & 0 \\ y & x & 0 \\ 0 & 0 & 0 \end{cases} \mid x, y \in \mathbb{R}$,
  - $\text{Aut } A = \begin{cases} x^2 & 0 & 0 \\ y & x & 0 \\ 0 & 0 & 1 \end{cases} \mid x, y \in \mathbb{R}, x \neq 0$,
- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  - the singletons covering the axis $Ox^1$,
  - the half-axes of $Ox^3$ delimited by $O$,
  - the half-planes delimited by axis $Ox^1$ on plane $x^1Ox^2$,
  - the quarters of plane delimited by axes $Ox^1$ and $Ox^3$ on plane $x^1Ox^2$,
  - the quarters of space delimited by planes $x^1Ox^2$ and $x^1Ox^3$.
- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the axis $Ox^1$,
  - the families of ray-solutions lying on the half-axes of $Ox^3$ delimited by $O$,
  - the integral curves contained in the half-planes delimited by axis $Ox^1$ on plane $x^1Ox^2$,
  - the integral curves contained in the quarters of plane delimited by axes $Ox^1$ and $Ox^3$ on plane $x^1Ox^2$,
  - the integral curves contained in the quarters of space delimited by planes $x^1Ox^2$ and $x^1Ox^3$.

A29) Properties of algebras $A$ of type $A_6(\alpha, \frac{1}{2})$ with $\alpha \notin \{0, \frac{1}{2}\}$

- the algebras $A_6(\alpha, \frac{1}{2})$ ($\alpha \notin \{0, \frac{1}{2}\}$) and $A_6(\alpha', \frac{1}{2})$ ($\alpha' \notin \{0, \frac{1}{2}\}$) are isomorphic if and only if $\alpha = \alpha'$
  - $\text{Ann } A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1$, $\mathcal{I}(A) = \{ \frac{y^2}{25}e_1 + ye_2 + e_3 \mid y \in \mathbb{R} \}$,
  - 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in \mathcal{N}(A) \cup \mathcal{I}(A)$,
  - 2-dimensional subalgebras: $\text{Span}_{\mathbb{R}}\{e_1, be_2 + ce_3\}$, $b^2 + c^2 \neq 0$,
  - ideals: $\mathbb{R}e_1$, $\text{Span}_{\mathbb{R}}\{e_1, e_2\}$,
  - $A^2 = A$
Properties of algebras delimited by $Ox$ associated homogeneous quadratic differential system (HQDS) consisting of:

- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  
  - the singletons covering the axis $Ox^1$,
  - the half-lines of lines $\mathbb{R}u$ for $u \in \mathcal{I}(A)$ delimited by $O$ (these lines cover the cone $(x^2)^2 - (1 - 2\alpha)x^1x^3 = 0$),
  - the connected components delimited by axis $Ox^1$ and the cone $(x^2)^2 - (1 - 2\alpha)x^1x^3 = 0$ on each plane containing $Ox^1$,

- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  
  - the singletons consisting of singular solutions that cover the axis $Ox^1$,
  - the families of ray-solutions lying on the half-lines of lines $\mathbb{R}u$ for $u \in \mathcal{I}(A)$ delimited by $O$ (these lines cover the cone $(x^2)^2 - (1 - 2\alpha)x^1x^3 = 0$).
  - the integral curves contained in the connected components delimited by axis $Ox^1$ and the cone $(x^2)^2 - (1 - 2\alpha)x^1x^3 = 0$ on each plane containing $Ox^1$.

**A30)** Properties of algebras $A$ of type $A_6(1, \frac{1}{2}, \frac{1}{2})$

- $\text{Ann } A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_1$, $\mathcal{I}(A) = \{xe_1 + e_3 \mid x \in \mathbb{R}\}$,
- 1-dimensional subalgebras: $\mathbb{R}u$ for $u \in \mathcal{N}(A) \cup \mathcal{I}(A)$,
- 2-dimensional subalgebras: $\text{Span}_\mathbb{R}\{e_1, be_2 + ce_3\}$ ($b^2 + c^2 \neq 0$),
- ideals: $\mathbb{R}e_1$, $\text{Span}_\mathbb{R}\{e_1, e_2\}$,
- $A^2 = A$,

- $\text{Der } A = \left\{ \begin{bmatrix} 2x & 0 & 0 \\ 0 & x & y \\ 0 & 0 & 0 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$,

- $\text{Aut } A = \left\{ \begin{bmatrix} x^2 & 0 & 0 \\ 0 & x & z \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R}, x \neq 0 \right\}$,

- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  
  - the singletons covering the axis $Ox^1$,
  - the half-planes delimited by axis $Ox^1$ on each plane passing through $Ox^1$ without $x^1Ox^3$,

- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  
  - the singletons consisting of singular solutions that cover the axis $Ox^1$,
  - the families of ray-solutions lying on the half-planes delimited by axis $Ox^1$.
the integral curves contained in the half-planes delimited by axis \( Ox^1 \) on each plane passing through \( Ox^1 \) without \( x^1Ox^3 \).

Note that each nonsingular integral curve has zero torsion.

31) Properties of algebras \( A \) of type \( A_6(\frac{1}{2}, \beta) \) with \( \beta \notin \{0, \frac{1}{2}\} \)

- the algebras \( A_6(\frac{1}{2}, \beta) (\beta \neq \frac{1}{2}) \) and \( A_6(\frac{1}{2}, \beta') (\beta' \neq \frac{1}{2}) \) are isomorphic if and only if \( \beta = \beta' \),
- \( \text{Ann} \ A = \{0\}, \mathcal{N}(A) = \mathbb{R}e_1, \mathcal{I}(A) = \{xe_1 + e_3 \mid x \in \mathbb{R}\} \),
- 1-dimensional subalgebras: \( \mathbb{R}u \) for \( u \in \mathcal{N}(A) \cup \mathcal{I}(A) \),
- 2-dimensional subalgebras: \( \text{Span}_\mathbb{R}\{e_1, e_2\}, \text{Span}_\mathbb{R}\{e_1, e_3\} \),
- ideals: \( \mathbb{R}e_1, \text{Span}_\mathbb{R}\{e_1, e_2\} \),
- \( A^2 = A \)
- \( \text{Der} \ A = \left\{ \begin{bmatrix} 2x & 0 & (\frac{1}{2} - \beta)y \\ y & x & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\} \),
- \( \text{Aut} \ A = \left\{ \begin{bmatrix} x^2 & 0 & y \\ \frac{2xy}{1 - 2\beta} & x & \frac{y^2}{1 - 2\beta} \\ 0 & 0 & 1 \end{bmatrix} \mid x, y \in \mathbb{R}, x \neq 0 \right\} \),
- the partition \( \mathcal{P}_A \) of \( \mathbb{R}^3 \), defined by the lattice of subalgebras of \( A \), consists of:
  - the singletons covering the axis \( Ox^1 \),
  - the half-lines of lines \( \mathbb{R}u \) for \( u \in \mathcal{I}(A) \) delimited by \( O \) (these lines cover the plane \( x^1Ox^3 \) less the axis \( Ox^1 \)),
  - the half-planes delimited by axis \( Ox^1 \) on the plane \( x^1Ox^2 \),
  - the connected components of space delimited by planes \( x^1Ox^2 \) and \( x^1Ox^3 \),
- the partition \( \mathcal{P}_A \) of \( A \) induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the axis \( Ox^1 \),
  - the families of ray-solutions lying on the half-lines of lines \( \mathbb{R}u \) for \( u \in \mathcal{I}(A) \) delimited by \( O \) (these lines cover the plane \( x^1Ox^3 \) less the axis \( Ox^1 \)),
  - the integral curves contained in the connected components of space delimited by planes \( x^1Ox^2 \) and \( x^1Ox^3 \).

Note that each nonsingular integral curve has zero torsion.

Case II: \( bj=0 \)

We have to consider the subcases:

(i) \( b = 0, j \neq 0 \),  (ii) \( b \neq 0, j = 0 \),  (iii) \( b = j = 0 \).

Subcase (i)

There exists a basis \( \mathcal{B} = (e_1, e_2, e_3) \) such that the multiplication table of algebra becomes
\[ e_1^2 = 0 \quad e_2^2 = 0 \quad e_3^2 = e_3 \]
\[ e_1 e_2 = 0 \quad e_1 e_3 = \alpha e_1 \quad e_2 e_3 = \beta e_2 \]

with \( \alpha, \beta \in \mathbb{R} \). Consequently, this algebra is isomorphic to an algebra of type A14.

**Subcase (ii)**

There exists a basis \( \mathcal{B} = (e_1, e_2, e_3) \) such that the multiplication table of algebra has form

**Table T7**

\[
\begin{array}{ccc}
   e_1^2 = e_2 & e_2^2 = e_1 & e_3^2 = 0 \\
   e_1 e_2 = 0 & e_1 e_3 = \alpha e_1 & e_2 e_3 = \beta e_2 \\
\end{array}
\]

with \( \alpha, \beta \in \mathbb{R} \). The algebra having the multiplication table T7 is denoted by \( A_7(\alpha, \beta) \).

It is proved that only the following three multiplication tables are of interest:

(i1) \[
\begin{array}{ccc}
   e_1^2 = e_2 & e_2^2 = e_1 & e_3^2 = 0 \\
   e_1 e_2 = 0 & e_1 e_3 = \alpha e_1 & e_2 e_3 = \beta e_2 \\
\end{array}
\]

(ii) \[
\begin{array}{ccc}
   e_1^2 = e_2 & e_2^2 = e_1 & e_3^2 = 0 \\
   e_1 e_2 = 0 & e_1 e_3 = \alpha e_1 & e_2 e_3 = \beta e_2 \\
\end{array}
\]

(iii) \[
\begin{array}{ccc}
   e_1^2 = e_2 & e_2^2 = e_1 & e_3^2 = 0 \\
   e_1 e_2 = 0 & e_1 e_3 = \alpha e_1 & e_2 e_3 = e_2 \\
\end{array}
\]

**A32) Properties of algebra** \( A = A_7(0, 0) \) (of type (ii1))

- \( \text{Ann } A = \text{Span}_\mathbb{R} \{e_2, e_3\} \), \( \mathcal{N}(A) = \text{Span}_\mathbb{R} \{e_2, e_3\} \), \( \mathcal{I}(A) = \emptyset \),
- 1-dimensional subalgebras: \( \mathbb{R} u \) for \( u \in \mathcal{N}(A) \),
- 2-dimensional subalgebras: \( \text{Span}_\mathbb{R} \{e_2, ae_1 + ce_3\} \) \( (a^2 + c^2 \neq 0) \),
- ideals: \( \mathbb{R} u \) for \( u \in \text{Ann } A \), \( \text{Span}_\mathbb{R} \{e_2, ae_1 + ce_3\} \) \( (a^2 + c^2 \neq 0) \),
- \( A^2 = \mathbb{R} e_2 \): \( A \) is nilpotent; \( A/A^2 \) is a 2-dimensional null algebra,
- \( A \) is a nilpotent commutative associative algebra,
- \( \text{Der } A = \left\{ \begin{pmatrix} x & 0 & 0 \\
                 y & 2x & u \\
                 z & 0 & v \end{pmatrix} \mid x, y, z, u, v \in \mathbb{R} \right\} \),
- \( \text{Aut } A = \left\{ \begin{pmatrix} y & x^2 & u \\
                 x & 0 & 0 \\
                 0 & v & 0 \end{pmatrix} \mid x, y, z, u, v \in \mathbb{R}, \ xv \neq 0 \right\} \),
- the partition \( \mathcal{P}_A \) of \( \mathbb{R}^3 \), defined by the lattice of subalgebras of \( A \), consists of:
  - the singletons covering the plane \( x^2 O x^3 \),
  - the half-planes delimited by axis \( Ox^2 \) on each plane passing through axis \( Ox^2 \),
- the partition \( \mathcal{P}_A \) of \( A \) induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  - the singletons consisting of singular solutions that cover the plane \( x^2 O x^3 \),
○ the integral curves contained in the half-planes delimited by axis $Ox^2$ on each plane passing through axis $Ox^2$.

Note that each nonsingular integral curve has both torsion and curvature tensors zero; indeed, each of them is lying on a line of the form $x^1 = k_1$, $x^3 = k_3$.

A33) Properties of algebras $A = A_7(1, \beta)$ (of type (ii$_2$)) when $\beta \notin \{0, 1\}$

- the algebras $A_6(1, \beta)$ ($\beta \notin \{0, 1\}$) and $A_6(1, \beta')$ ($\beta' \notin \{0, 1\}$) are isomorphic if and only if $\beta = \beta'$,
- $\text{Ann } A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_2 \cup \mathbb{R}e_3$, $\mathcal{I}(A) = \emptyset$,
- 1-dimensional subalgebras: $\mathbb{R}e_2$, $\mathbb{R}e_3$,
- 2-dimensional subalgebras: $\text{Span}_\mathbb{R}\{e_1, e_2\}$, $\text{Span}_\mathbb{R}\{e_2, e_3\}$,
- ideals: $\mathbb{R}e_2$, $\text{Span}_\mathbb{R}\{e_1, e_2\}$,
- $A^2 = \text{Span}_\mathbb{R}\{e_1, e_2\}$: $A$ is solvable; $A/A^2$ is a 1-dimensional null algebra,
- $\text{Der } A = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & 2x & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}$,
- $\text{Aut } A = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & x^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x \in \mathbb{R}^* \right\} \cong \mathbb{R}^*$,
- the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:
  ○ the singletons covering the axes $Ox^2$ and $Ox^3$,
  ○ the half-planes delimited by axis $Ox^2$ on $x^1Ox^2$,
  ○ the quarters of plane delimited by axis $Ox^2$ and $Ox^3$ on $x^2Ox^3$,
  ○ the quarters of space delimited by planes $x^1Ox^2$ and $x^2Ox^3$,
- the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:
  ○ the singletons consisting of singular solutions that cover the axes $Ox^2$ and $Ox^3$,
  ○ the integral curves contained in the half-planes delimited by axis $Ox^2$ on $x^1Ox^2$,
  ○ the integral curves contained in the quarters of plane delimited by axis $Ox^2$ and $Ox^3$ on $x^2Ox^3$,
  ○ the integral curves contained in the quarters of space delimited by planes $x^1Ox^2$ and $x^2Ox^3$.

Note that each nonsingular integral curve are torsion-free.

A34) Properties of algebras $A = A_7(1, 1)$ (of type (ii$_2$))

- $\text{Ann } A = \{0\}$, $\mathcal{N}(A) = \mathbb{R}e_2 \cup \mathbb{R}e_3$, $\mathcal{I}(A) = \emptyset$,
- 1-dimensional subalgebras: $\mathbb{R}e_2$, $\mathbb{R}e_3$,
- 2-dimensional subalgebras: $\text{Span}_\mathbb{R}\{e_1, e_2\}$, $\text{Span}_\mathbb{R}\{e_2, e_3\}$,
- ideals: $\mathbb{R}e_2$, $\text{Span}_\mathbb{R}\{e_1, e_2\}$,
- $A^2 = \text{Span}_\mathbb{R}\{e_1, e_2\}$: $A$ is solvable; $A/A^2$ is a 1-dimensional null algebra,
$\text{Der } A = \left\{ \begin{bmatrix} x & 0 & 0 \\ y & 2x & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\},$

$\text{Aut } A = \left\{ \begin{bmatrix} x & 0 & 0 \\ y & x^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid x, y \in \mathbb{R}, \ x \neq 0 \right\},$

the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:

- the singletons covering the axes $Ox^2$ and $Ox^3$,
- the half-planes delimited by axis $Ox^2$ on $x^1Ox^2$,
- the quarters of plane delimited by axis $Ox^2$ and $Ox^3$ on $x^2Ox^3$,
- the quarters of space delimited by planes $x^1Ox^2$ and $x^2Ox^3$,

the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:

- the singletons consisting of singular solutions that cover the axes $Ox^2$ and $Ox^3$,
- the integral curves contained in the half-planes delimited by axis $Ox^2$ on $x^1Ox^2$,
- the integral curves contained in the quarters of plane delimited by axis $Ox^2$ and $Ox^3$ on $x^2Ox^3$,
- the integral curves contained in the quarters of space delimited by planes $x^1Ox^2$ and $x^2Ox^3$.

Note that each nonsingular integral curve are torsion-free.

A35) Properties of algebras $A = A_7(1,0)$ (of type $(ii_2)$)

- $\text{Ann } A = \mathbb{R} e_2, \mathcal{N}(A) = \text{Span}_\mathbb{R} \{e_2, e_3\}, \mathcal{I}(A) = \emptyset$,
- 1-dimensional subalgebras: $\mathbb{R} u$ for $u \in \mathcal{N}(A)$,
- 2-dimensional subalgebras: $\text{Span}_\mathbb{R} \{e_1, e_2\}, \text{Span}_\mathbb{R} \{e_2, e_3\}$,
- ideals: $\mathbb{R} e_2, \text{Span}_\mathbb{R} \{e_1, e_2\}$,
- $A^2 = \text{Span}_\mathbb{R} \{e_1, e_2\}$: $A$ is solvable; $A/A^2$ is the null 1-dimensional algebra,

$\text{Der } A = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & 2x & y \\ 0 & 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\},$

$\text{Aut } A = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & x^2 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y \in \mathbb{R}, \ x \neq 0 \right\},$

the partition $\mathcal{P}_A$ of $\mathbb{R}^3$, defined by the lattice of subalgebras of $A$, consists of:

- the singletons covering the plane $x^2Ox^3$,
- the half-planes delimited by axis $Ox^2$ on $x^1Ox^2$,
- the quarters of space delimited by planes $x^1Ox^2$ and $x^2Ox^3$,

the partition $\mathcal{P}_A$ of $A$ induces a partition on the set of integral curves of the associated homogeneous quadratic differential system (HQDS) consisting of:

- the singletons consisting of singular solutions that cover the plane $x^2Ox^3$,
the integral curves contained in the half-planes delimited by axis $Ox^2$ on $x^1Ox^2$,

⋄ the integral curves contained in the quarters of space delimited by planes $x^1Ox^2$ and $x^2Ox^3$.

Note that each nonsingular integral curve are torsion-free.

Properties of algebras $(ii_3)$

**Proposition 2.18** Each algebra of type $A_7(\alpha, 1)$ for a given $\alpha \neq 0$ is isomorphic to the algebra of type $A_7(1, \beta)$ for $\beta = \frac{1}{\alpha}$.

Subcase $(iii)$ $b=j=0$

There exists a basis $B = (e_1, e_2, e_3)$ such that the multiplication table of algebra has one of the following two multiplication tables:

- $(iii_1)$ $e_1^2 = 0$ $e_2^2 = 0$ $e_3^2 = 0$
  
  $e_1e_2 = 0$ $e_1e_3 = e_1$ $e_2e_3 = \beta e_2$

- $(iii_2)$ $e_1^2 = 0$ $e_2^2 = 0$ $e_3^2 = 0$
  
  $e_1e_2 = 0$ $e_1e_3 = \alpha e_1$ $e_2e_3 = e_2$

Properties of algebras $(iii_1)$

There exists a basis $B = (e_1, e_2, e_3)$ such that the multiplication table of algebra becomes:

- $(ii_1)$ $e_1^2 = 0$ $e_2^2 = 0$ $e_3^2 = 0$
  
  $e_1e_2 = 0$ $e_1e_3 = e_1$ $e_2e_3 = \beta e_2$

Consequently, this algebra is isomorphic to algebra $A_{22}$.

Moreover, the following result is true.

**Proposition 2.19** Each algebra of type $(iii_2)$ for a given $\alpha \neq 0$ is isomorphic to the algebra of type $(iii_1)$ for $\beta = \alpha$.

1) Case $Spec D = (1, \omega, 0)$ for $\omega \notin \{-1, 0, \frac{1}{2}, 1, 2\}$

Algebra $A(\cdot)$ has the next multiplication table:

| Table T | $e_1^2 = 0$ | $e_2^2 = 0$ | $e_3^2 = j e_3$ |
|---------|-------------|-------------|-----------------|
|         | $e_1e_2 = 0$ | $e_1e_3 = pe_1$ | $e_2e_3 = te_2$ |

with $j, p, t \in \mathbb{R}$. Then, in basis $(e_1, e_2, \frac{1}{j} e_3)$, the multiplication table of algebra becomes:

| Table T | $e_1^2 = 0$ | $e_2^2 = 0$ | $e_3^2 = e_3$ |
|---------|-------------|-------------|---------------|
|         | $e_1e_2 = 0$ | $e_1e_3 = \alpha e_1$ | $e_2e_3 = \beta e_2$ |
with \( \alpha, \beta \in \mathbb{R} \). Consequently, each such algebra is isomorphic to algebras of type \textbf{A14}).

Theorem 2.1 stated that any algebra in class \textbf{Ai} is not isomorphic to any algebra of class \textbf{Aj} for \( i, j \in \{1, 2, \ldots, 7\} \). This result can be now improved by a straight comparison of lists of properties for isomorphism classes \textbf{Ai} for \( i \in \{1, 2, \ldots, 35\} \).

\textbf{Theorem 2.2} Each algebra of type \textbf{Ai} is not isomorphic to any algebra of type \textbf{Aj} for \( i, j \in \{1, 2, \ldots, 35\} \) and \( i \neq j \).

This result induces a classification result for the corresponding HQDSs up to an affine-equivalence.

\textbf{Theorem 2.3} For any nontrivial HQDS on \( \mathbb{R}^3 \), having at least a semisimple derivation with 1-dimensional kernel, there exists a center-affinity such that it turns into one of the following 35 HQDSs:

1°. \[
\begin{align*}
\frac{dx_1}{dt} &= 0 \\
\frac{dx_2}{dt} &= 0 \\
\frac{dx_3}{dt} &= 2x_1x^2 + (x^3)^2,
\end{align*}
\]

2°. \[
\begin{align*}
\frac{dx_1}{dt} &= 0 \\
\frac{dx_2}{dt} &= x^2x^3 \\
\frac{dx_3}{dt} &= 2x_1x^2 + (x^3)^2,
\end{align*}
\]

3°. \[
\begin{align*}
\frac{dx_1}{dt} &= x^1x^3 \\
\frac{dx_2}{dt} &= x^2x^3 \\
\frac{dx_3}{dt} &= 2x_1x^2 + (x^3)^2,
\end{align*}
\]

4°. \[
\begin{align*}
\frac{dx_1}{dt} &= 2\alpha x^1x^3 \\
\frac{dx_2}{dt} &= 0 \\
\frac{dx_3}{dt} &= 2x_1x^2 + (x^3)^2,
\end{align*}
\]

(\( \alpha \notin \{0, \frac{1}{2}\} \))

5°. \[
\begin{align*}
\frac{dx_1}{dt} &= 2\alpha x^1x^3 \\
\frac{dx_2}{dt} &= x^2x^3 \\
\frac{dx_3}{dt} &= 2x_1x^2 + (x^3)^2,
\end{align*}
\]

(\( \alpha \notin \{0, \frac{1}{2}\} \))

6°. \[
\begin{align*}
\frac{dx_1}{dt} &= 2\alpha x^1x^3 \\
\frac{dx_2}{dt} &= 2\alpha x^2x^3 \\
\frac{dx_3}{dt} &= 2x_1x^2 + (x^3)^2,
\end{align*}
\]

(\( \alpha \notin \{0, \frac{1}{2}\} \))

7°. \[
\begin{align*}
\frac{dx_1}{dt} &= 2\alpha x^1x^3 \\
\frac{dx_2}{dt} &= 2\beta x^2x^3 \\
\frac{dx_3}{dt} &= 2x_1x^2 + (x^3)^2,
\end{align*}
\]

(\( \alpha, \beta \notin \{0, \frac{1}{2}\}, \alpha < \beta \))

8°. \[
\begin{align*}
\frac{dx_1}{dt} &= 0 \\
\frac{dx_2}{dt} &= 0 \\
\frac{dx_3}{dt} &= (x^3)^2,
\end{align*}
\]
9°. \[
\begin{align*}
\frac{dx^1}{dt} &= 0 \\
\frac{dx^2}{dt} &= x^2x^3 \\
\frac{dx^3}{dt} &= (x^3)^2,
\end{align*}
\]
10°. \[
\begin{align*}
\frac{dx^1}{dt} &= x^1x^3 \\
\frac{dx^2}{dt} &= x^2x^3 \\
\frac{dx^3}{dt} &= (x^3)^2,
\end{align*}
\]
11°. \[
\begin{align*}
\frac{dx^1}{dt} &= 0 \\
\frac{dx^2}{dt} &= x^2x^3 \\
\frac{dx^3}{dt} &= (x^3)^2,
\end{align*}
\]
12°. \[
\begin{align*}
\frac{dx^1}{dt} &= x^1x^3 \\
\frac{dx^2}{dt} &= 2\beta x^2x^3 \\
\frac{dx^3}{dt} &= (x^3)^2, \\
(\beta \notin \{0, \frac{1}{2}\})
\end{align*}
\]
13°. \[
\begin{align*}
\frac{dx^1}{dt} &= 2\alpha x^1x^3 \\
\frac{dx^2}{dt} &= 2\alpha x^2x^3 \\
\frac{dx^3}{dt} &= (x^3)^2, \\
(\alpha \notin \{0, \frac{1}{2}\})
\end{align*}
\]
14°. \[
\begin{align*}
\frac{dx^1}{dt} &= 2\alpha x^1x^3 \\
\frac{dx^2}{dt} &= 2\beta x^2x^3 \\
\frac{dx^3}{dt} &= (x^3)^2, \\
(\alpha, \beta \notin \{0, \frac{1}{2}\}, \alpha < \beta)
\end{align*}
\]
15°. \[
\begin{align*}
\frac{dx^1}{dt} &= 2x^2x^3 \\
\frac{dx^2}{dt} &= 0 \\
\frac{dx^3}{dt} &= 0,
\end{align*}
\]
16°. \[
\begin{align*}
\frac{dx^1}{dt} &= 0 \\
\frac{dx^2}{dt} &= 2x^2x^3 \\
\frac{dx^3}{dt} &= 2x^1x^2,
\end{align*}
\]
17°. \[
\begin{align*}
\frac{dx^1}{dt} &= 2x^1x^3 \\
\frac{dx^2}{dt} &= 2x^2x^3 \\
\frac{dx^3}{dt} &= 2x^1x^2, \\
(\beta > 1)
\end{align*}
\]
18°. \[
\begin{align*}
\frac{dx^1}{dt} &= 2x^1x^3 \\
\frac{dx^2}{dt} &= 2\beta x^2x^3 \\
\frac{dx^3}{dt} &= 2x^1x^2, \\
(\beta \notin \{0, 1\})
\end{align*}
\]
19°. \[
\begin{align*}
\frac{dx^1}{dt} &= 0 \\
\frac{dx^2}{dt} &= 2x^2x^3 \\
\frac{dx^3}{dt} &= 0,
\end{align*}
\]
20°. \[
\begin{align*}
\frac{dx^1}{dt} &= 2x^1x^3 \\
\frac{dx^2}{dt} &= 2x^2x^3 \\
\frac{dx^3}{dt} &= 0,
\end{align*}
\]
21°. \[
\begin{align*}
\frac{dx^1}{dt} &= 2x^1x^3 \\
\frac{dx^2}{dt} &= 2\beta x^2x^3 \\
\frac{dx^3}{dt} &= 0, \\
(\beta \notin \{0, 1\})
\end{align*}
\]
22°. \[
\begin{align*}
\frac{dx^1}{dt} &= 2x^2x^3 \\
\frac{dx^2}{dt} &= 0 \\
\frac{dx^3}{dt} &= (x^3)^2,
\end{align*}
\]
23°. \[
\begin{align*}
\frac{dx^1}{dt} &= 2ax^1x^2 + 2bx^2x^3 \\
\frac{dx^2}{dt} &= -2bx^1x^3 + 2ax^2x^3 \\
\frac{dx^3}{dt} &= (x^3)^2, \\
(b > 0)
\end{align*}
\]
24°. \[
\begin{align*}
\frac{dx^1}{dt} &= x^2 \\
\frac{dx^2}{dt} &= 2ax^2x^3 \\
\frac{dx^3}{dt} &= (x^3)^2, \\
(\alpha \notin \{0, \frac{1}{2}\})
\end{align*}
\]
\[ \begin{align*}
25^\circ. & \begin{cases} 
\frac{dx}{dt} = (x^2)^2 \\
\frac{dx^2}{dt} = 0 \\
\frac{dx^3}{dt} = (x^3)^2,
\end{cases} & 26^\circ. & \begin{cases} 
\frac{dx}{dt} = (x^2)^2 \\
\frac{dx^2}{dt} = x^2 x^3 \\
\frac{dx^3}{dt} = (x^3)^2,
\end{cases} \\
27^\circ. & \begin{cases} 
\frac{dx}{dt} = 2\alpha x^1 x^3 + (x^2)^2 \\
\frac{dx^2}{dt} = 2\beta x^2 x^3 \\
\frac{dx^3}{dt} = (x^3)^2, (\alpha, \beta \notin \{0, \frac{1}{2}\}, \alpha \neq \beta)
\end{cases} & 28^\circ. & \begin{cases} 
\frac{dx}{dt} = 2\alpha x^1 x^3 + (x^2)^2 \\
\frac{dx^2}{dt} = 2\alpha x^2 x^3 \\
\frac{dx^3}{dt} = (x^3)^2, (\alpha \notin \{0, \frac{1}{2}\})
\end{cases} \\
29^\circ. & \begin{cases} 
\frac{dx}{dt} = 2\alpha x^1 x^3 + (x^2)^2 \\
\frac{dx^2}{dt} = x^2 x^3 \\
\frac{dx^3}{dt} = (x^3)^2, (\alpha \notin \{0, \frac{1}{2}\})
\end{cases} & 30^\circ. & \begin{cases} 
\frac{dx}{dt} = x^1 x^3 + (x^2)^2 \\
\frac{dx^2}{dt} = x^2 x^3 \\
\frac{dx^3}{dt} = (x^3)^2,
\end{cases} \\
31^\circ. & \begin{cases} 
\frac{dx}{dt} = x^1 x^3 + (x^2)^2 \\
\frac{dx^2}{dt} = 2\beta x^2 x^3 \\
\frac{dx^3}{dt} = (x^3)^2, (\beta \notin \{0, \frac{1}{2}\})
\end{cases} & 32^\circ. & \begin{cases} 
\frac{dx}{dt} = (x^1)^2 \\
\frac{dx^2}{dt} = 0 \\
\frac{dx^3}{dt} = 0,
\end{cases} \\
33^\circ. & \begin{cases} 
\frac{dx}{dt} = 2x^1 x^3 \\
\frac{dx^2}{dt} = (x^1)^2 + 2\beta x^2 x^3 \\
\frac{dx^3}{dt} = 0, (\beta \notin \{0, 1\})
\end{cases} & 34^\circ. & \begin{cases} 
\frac{dx}{dt} = 2x^1 x^3 \\
\frac{dx^2}{dt} = (x^1)^2 + 2x^2 x^3 \\
\frac{dx^3}{dt} = 0
\end{cases} \\
35^\circ. & \begin{cases} 
\frac{dx}{dt} = 2x^1 x^3 \\
\frac{dx^2}{dt} = (x^1)^2 \\
\frac{dx^3}{dt} = 0.
\end{cases}
\end{align*} \]
3 Conclusions

The existence of a semisimple derivation for a HQDS is a strong constraint implying the existence of a center-affine equivalent system having a lot of coefficients either 0 or small integers in suitable bases. The classification result of this family of HQDSs up to an affine equivalence, just exhibited in Theorem 2.3, is a corollary of the classification of corresponding commutative algebras up to an isomorphism. There were identified 13 classes of algebras \( A \) having \( \text{Der} \ A = \mathbb{R} D \) while their automorphism groups are isomorphic either with \( \mathbb{R}^* (-) \) or to \( \mathbb{R}^* \times \{-1, 1\} \). The other 21 classes of algebras have larger derivations algebras and automorphism groups. Note that, for 3 algebras the automorphism group is \( \mathbb{R}^* (-) \times \mathbb{R}^* (-) \), for 3 algebras the automorphism group is \( GL(2, \mathbb{R}) \), one algebra has \( \text{Aff}(2, \mathbb{R}) \) as its automorphism group and one algebra has \( \mathbb{C}^* \) as its automorphism group; the other 14 algebras have more complex automorphism groups. In order to decide on the mutually non-isomorphism of pairs of algebras in different classes \( A_i \) for \( i \in \{1, 2, ..., 35\} \) we have supplied the lists of main properties assigned to each such a class. Note that the algebras having the same list of main properties were collected together by means of a label consisting in one or two parameters running over a specific range. In fact, this classification seems to be the finest under the isomorphism criterion. In these lists were also included the information concerning the partitions of ground space \( A \) as well as of the set of all integral curves of the associated systems. These partitions are invariant under the action of automorphism groups so that they can be used as tests of correctness of results. On the other hand we hope that these partitions will give important information about the stability of steady state solutions. Finally, let us remark that some algebraic properties, like the solvability or the existence of special ideals of algebras, attract important geometric properties of nonsingular integral curves of the corresponding HQDS like the vanishing of torsion or curvature tensor.

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