Phase Structure of a 3D Nonlocal U(1) Gauge Theory: Deconfinement by Gapless Matter Fields

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Abstract

In this paper, we study a 3D compact U(1) lattice gauge theory with a variety of nonlocal interactions that simulates the effects of gapless/gapful matter fields. We restrict the nonlocal interactions among gauge variables only to those along the temporal direction and adjust their coupling constants optimally to simulate the isotropic nonlocal couplings of the original model. This theory is quite important to investigate the phase structures of QED$_3$ and strongly-correlated electron systems like the 2D quantum spin models, the fractional quantum Hall effect, the t-J model of high-temperature superconductivity. We perform numerical studies of this theory to find that, for a certain class of power-decaying couplings, there appears a second-order phase transition to the deconfinement phase as the gauge coupling constant is decreased. On the other hand, for the exponentially-decaying coupling, there are no signals for second-order phase transition. These results indicate the possibility that introduction of sufficient number of massless matter fields destabilizes the permanent confinement in the 3D compact U(1) pure gauge theory due to instantons.
1 Introduction

Gauge theories and their associated concepts have played important roles not only in elementary particle physics but also in condensed matter physics. For example, the conventional superconducting phase transition is characterized as a change of the gauge dynamics of the electromagnetic U(1) gauge symmetry. Also, for a variety of strongly-correlated electron systems, it has been recognized that their phase structures and the properties of low-energy excitations are naturally described by using the terminology of gauge theories[1]. As such low-energy “quasi-particles”, the composite fermions/bosons in the fractional quantum Hall states[2] and the holons and spinons in the t-J model of high-$T_c$ cuprates[3] have been proposed. It was argued[4] that their unconventional properties like fractionality and their existence itself may be explained by a confinement-deconfinement phenomenon of the gauge dynamics of the effective gauge theories derived from the original models. This interesting idea is still controversial[5], but certainly warrants further investigation.

Most of the problems in gauge-theoretical studies on the strongly-correlated electron systems reduces to studying the phase structures of gauge theories coupled to gapless relativistic/nonrelativistic matter fields. In the elementary particle physics, it is generally believed that the phase structures of gauge systems coupled to matter fields are not easy to study analytically because introduction of matter fields results in the lack of simple order parameters such as Wilson loops for pure gauge systems[6]. Furthermore, inclusion of gapless fermions make numerical simulations a difficult task since one must face quite nonlocal interactions generated by integrating over fermion variables. Sometimes couplings to gapless matter fields change the universality class of the gauge system under consideration from that of the pure gauge model. A good example is the four-dimensional (4D) QCD coupled with light quarks in which the number of light quarks strongly influences its phase structure[7].

In this paper, we address this problem of gauge dynamics of coupled systems. Specifically, we are interested in the U(1) lattice gauge theory (LGT) with gapless/gapful and relativistic/nonrelativistic matter fields in three dimensions (two spatial dimensions at zero temperature). This theory of course covers the important model QED$_3$[8]. It appears also as a main part of the effective gauge theory of the strongly-correlated electron systems mentioned above, and so plays an important role in studying these systems[4, 9]. In the ordinary 3D compact U(1) pure gauge system (i.e., without matter fields) with local interactions, it is established that only the confinement phase is realized because of instanton condensation[11]. For the case with additional massless (gapless) matter fields coupled to the U(1) gauge field, recent studies give controversial results on the possibility of a deconfinement phase; it is supported in Ref.[4, 12] whereas it is denied in Ref.[13].
Our approach to this problem is by (i) introducing an effective theory of the original theory and (ii) studying its phase structure numerically. As the effective theory we use a 3D $U(1)$ pure LGT with nonlocal interactions among gauge variables. These nonlocal interactions are along the temporal direction and mimic the effect of matter fields. We consider exponentially-decaying interactions for massive matter fields (i.e., fields with gaps) and power-decaying interactions for massless (gapless) fields. We shall see that certain cases of power-decaying interactions exhibit second-order phase transitions which separate the confinement phase and the deconfinement phase\cite{10}. The existence of the deconfinement phase in the effective theory indicates that it is realized in the original model if the number of gapless matter fields is sufficiently large. This result is in agreement with the results of Ref.\cite{4, 12}.

The rest of the paper is organized as follows. In section 2, we briefly survey the effect of matter fields in several aspects. In section 3, the original model and its effective nonlocal gauge model are explained. Section 4 is devoted for numerical calculations. We calculate the internal energy, specific heat, expectation values of Polyakov loops, Wilson loops, and density of instantons. All these quantities indicate a second-order confinement-deconfinement phase transition (CDPT) for the gauge models with sufficiently long-range correlations among the gauge variables. In section 5, we study tractable low-dimensional spin models, which are obtained by simple reduction of the gauge degrees of freedom in the nonlocal effective model. Then we obtain an intuitive picture of the CDPT of the present long-range $U(1)$ gauge theories. Section 6 is devoted for conclusion. In Appendix, the effective nonlocal model is studied by the low- and high-temperature expansions. The analytic expressions of the internal energy and the specific heat are in good agreement with the numerical calculations in Sect.4.

2 Effects of Matter Fields on Gauge Dynamics

In this section, we briefly review the effects of matter fields upon the $U(1)$ gauge dynamics in several aspects.

2.1 Weak-coupling regime in noncompact/compact $U(1)$ gauge theory

One may generally expect that inclusion of massless matter fields to a system of gauge field may drastically change the gauge dynamics at long wavelengths. For the case that relativistic and massless matter fields are coupled to a $U(1)$ gauge field $A_\mu(x)$ ($\mu = 0, 1, 2$) in 3D, the one-loop radiative correction of matter fields at weak gauge couplings generates the following nonlocal term in the
effective action of $A_\mu(x)$;
\[
\Delta A \propto e^2 \int d^3x \int d^3y \sum_{\mu,\nu} F_{\mu\nu}(x) \frac{1}{|x-y|^2} F_{\mu\nu}(y), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.1}
\]

It is obvious that the above term strongly suppresses fluctuations of $A_\mu(x)$ at long distances due to the factor $|x-y|^{-2}$. The above nonlocal terms are leading at low energies and momenta if its effective coupling constant does not vanish at the infrared limit. Then the potential energy $V(r)$ between two charges separated by distance $r$ changes drastically,
\[
V(r) \propto \log r \rightarrow V(r) \propto \frac{1}{r}. \tag{2.2}
\]

In the compact U(1) gauge theory, topologically nontrivial excitations (e.g. instantons) of $A_\mu(x)$ appear whose effect at small gauge coupling $e$ can be estimated by replacing the field strength $F_{\mu\nu}(x)$ in Eq.(2.1) by $F_{\mu\nu}(x) - 2\pi n_{\mu\nu}(x)$, where $n_{\mu\nu}$ is an integer field whose rotation measures the instanton number(density) $\rho(x) = \epsilon_{\mu\nu\lambda}\partial_\mu n_{\nu\lambda}(x)$. In the gauge model with the usual Maxwell term, the potential energy between a pair of instantons at distance $r$ is Coulombic, $V_{\text{ins}}(r) \sim 1/r$. However, the long-range action (2.1) modifies $V_{\text{ins}}(r)$ to a long-range one\[12\],
\[
V_{\text{ins}}(r) \propto \frac{1}{r} \rightarrow V_{\text{ins}}(r) \propto \log r. \tag{2.3}
\]

Recently, it was argued that a gas of charged particles with the long-range interaction like log $r$ exhibits a phase transition between a dilute gas of dipoles and a plasma[14]. This result implies that instantons in the long-range gauge theories form dipole pairs and do not condense in the weak-coupling region. Since the confinement phase of gauge dynamics requires a condensation of isolated instantons (a plasma phase), this result leads to a deconfinement phase of gauge dynamics there.

### 2.2 The 3D $\mathbb{C}P^{N-1}$ model on the criticality

As an example of the nonperturbative effect of matter fields, let us consider the $\mathbb{C}P^{N-1}$ model in a 3D continuum. The action of the model is given as
\[
A_{\mathbb{C}P} = \int d^3x \left[ \frac{1}{f} |(\partial_\mu + iA_\mu)z|^2 + \sigma \left( |z|^2 - 1 \right) \right]. \tag{2.4}
\]

where $z_a(x) (a = 1, 2, \cdots, N)$ is the $\mathbb{C}P^{N-1}$ field satisfying the local constraint, $\sum_{a=1}^N |z_a(x)|^2 = 1$ for each $x$ via the Lagrange multiplier field $\sigma(x)$, and $A_\mu(x)$ is the auxiliary U(1) gauge field.

At large $N$, the $1/N$ expansion is reliable and predicts a second-order phase transition at the critical coupling $f = f_c$ [15]. For $f > f_c$, the model is in the disordered-confinement phase in which $\langle \sigma(x) \rangle \neq 0$, whereas for $f < f_c$, the model is in the ordered-Higgs phase in which $\langle \sigma(x) \rangle = 0$ and the
$CP^{N-1}$ field has a nonvanishing expectation value like $\langle z_N(x) \rangle = v_0 \neq 0$. As a result, the gauge field $A_\mu$ acquires a finite mass ($\propto v_0$) by the Anderson-Higgs mechanism, and the low-energy excitations are gapless $z_a(x)$ ($a \neq N$) fields.

Recently, considerable interests have been paid on the question how the gauge field behaves just at the critical point $f = f_c$[16]. In the leading order of the $1/N$ expansion, it is shown that all the components $z_a(x)$ ($a = 1, 2, \cdots, N$) are massless and the gauge field $A_\mu$ acquires the nonlocal “kinetic term” of Eq.(2.1). This implies that, at the critical point, the nonperturbative fluctuations of $A_\mu$ like instantons are suppressed, so the gauge dynamics at $f = f_c$ is in the Coulomb phase with the potential $V(r) \sim 1/r$ as Eq.(2.2).

Then we have numerically studied the 3D $CP^1 + U(1)$ LGT from the above point of view[17]. The numerical results of the $CP^1$ model show a second-order transition and the confinement phase is realized at the critical point. However, calculations of $CP^{N-1}$ model with $N = 3, 4, 5$ show that the topologically nontrivial configurations are suppressed more as $N$ increases. We expect that there exists a critical value of $N = N_c$, and the deconfinement phase is realized at the critical point for $N_c < N$ in compatible with the large-$N$ analysis. We note that a similar phenomenon of inducing a deconfinement phase by a plenty of massless matter fields has been established in lattice QCD in 4D. For a sufficiently large number of light flavors $N_f > 7$, the model stays in the deconfinement phase even the pure gauge term is missing[7].

### 2.3 Nonrelativistic fermions in strongly-correlated electron systems

Nonrelativistic fermions are distinguished from relativistic fermions by the properties: (i) they propagate only in the positive direction of the imaginary time in path-integral formulation, and (ii) they form a Fermi surface (line). In strongly-correlated electron systems, one faces nonrelativistic fermions not only in the original models but also in their effective gauge models. In studying the fractionalization phenomena of electrons like the charge-spin separation (CSS) [3] in high-temperature superconductivity and the particle-flux separation (PFS) in fractional quantum Hall systems [2], we regard an electron $C_x$ (we suppress the spin index for simplicity) at the site $x$ of a 2D spatial lattice as a composite of a fermion $A_x$ and a boson $B_x$ [4] as

$$C_x = A_x^\dagger B_x.$$  \hspace{1cm} (2.5)

To assure the correct physical space composed of $C_x$, one imposes the local constraint for the physical states $|\text{phys}\rangle$ as

$$(A_x^\dagger A_x + B_x^\dagger B_x - 1)|\text{phys}\rangle = 0 \text{ for each } x.$$ \hspace{1cm} (2.6)
The hopping term of electrons (e.g., the $t$-term of the t-J model) may be rewritten via a “decoupling” as

$$C_{x+i}^\dagger C_x + \text{H.c.} = B_{x+i}^\dagger A_{x+i} A_x^\dagger B_x + \text{H.c.}$$

$$\Rightarrow (B_{x+i}^\dagger W_{x+i} W_x + A_{x+i}^\dagger W_x A_{x+i} + \text{H.c.}) - |W_{x+i}|^2 + \cdots.$$  \hspace{1cm} (2.7)

$W_{x+i}$ is an auxiliary complex field defined on the link $(x,x+i)$ where $i = 1,2$ is the direction index of the lattice. Its phase degree of freedom $U_{x+i} \equiv W_{x+i}/|W_{x+i}|$ behaves as a spatial component of a compact U(1) gauge field $U_{x+i} = \exp(i\theta_{x+i})$, $U_{x+i}$ represents the binding force of the constituents $A_x$ and $B_x$. In the path-integral formalism, the partition function $\text{Tr}_C \equiv \exp[-\beta H(C)]$ ($\beta \equiv T^{-1}$) has the following representation:

$$Z = \int \prod_{x,\tau} d\tilde{A}_x(\tau) dA_x(\tau) dB_x(\tau) \prod_{\mu=0,1,2} dU_{x\mu}(\tau) \exp(A),$$

$$A = \int_0^\beta d\tau \sum_x \left[ -i\theta_{x0} - \tilde{A}_x(\partial_0 - i\theta_{x0} + \mu_A)A_x - \tilde{B}_x(\partial_0 - i\theta_{x0} + \mu_B)B_x - t\sum_i \left( \tilde{A}_{x+i} U_{x+i}(\tau) B_x + \text{H.c.} \right) + A_{\text{int}} \right].$$  \hspace{1cm} (2.8)

where $\tau \in [0,\beta]$ is the continuum imaginary time, $A_x(\tau)$ is a Grassmann number expressing fermions, $\mu_{A(B)}$ is the chemical potential, and $t$ is the hopping amplitude. The field $\theta_{x0}$ is a Lagrange multiplier field to enforce the constraint (2.6). It may be viewed as the time-component of gauge field. In fact, in the discrete-time formulation on the 3D lattice it appears as the exponent of gauge variable in the $\tau$-direction, $U_{x0} = \exp(i\theta_{x0})$.

When the gauge dynamics of $U_{x+i}$ is realized in the confinement phase, the constituents $A_x$ and $B_x$ are bound within electrons and the relevant low-energy quasi-particles are described by the electron operators $C_x$ themselves. On the other hand, if the gauge dynamics is realized in the deconfinement phase, the binding force is weak and these constituents are no more bound in each electron but dissociate each other. This is just the the fractionalization (separation) phenomena of electrons.

In Ref.[4] we have studied the system of Eq.(2.8) at finite temperatures ($T$) by the hopping expansion in the temporal gauge $\theta_{x0} = 0$. After integrating over $A_x$ and $B_x$ in powers of $t$, one obtains the effective interactions. Up to $O(t^2)$, by restoring the temporal components $U_{x0}$, one gets the following effective interaction among $U_{x\mu}$,

$$\Delta A \propto \delta(1-\delta) \cdot t^2 \sum_{x,i} \int_0^\beta d\tau \int_0^\beta d\tau' V_{x,i}(\tau,\tau') + \text{c.c.},$$

$$V_{x,i}(\tau,\tau') = U_{x,i}(\tau') \exp\left( i \int_{\tau}^{\tau'} d\tau'' \left[ \theta_{x,i,0}(\tau'') - \theta_{x,i,0}(\tau') \right] \right) U_{x,i}(\tau).$$  \hspace{1cm} (2.9)

where $\delta = \langle A_x^\dagger A_x \rangle$ is the concentration of fermions. The corresponding terms are illustrated in Fig.1.
Figure 1: Illustration of the nonlocal interactions in electron-fractionalization phenomena. $P_{x_\perp} = \exp(i \int d\tau \theta_{x_0})$ is the Polyakov line at the transverse coordinate $x_\perp = (x_1, x_2)$, $Q_{x_\perp,i}(\tau_1, \tau_2)$ is given by the product of propagators $\langle B_{x_\perp}(\tau_1) B_{x_\perp}^\dagger(\tau_2) \rangle$ and $\langle A_{x_\perp}(\tau_2) A_{x_\perp}^\dagger(\tau_1) \rangle$. The product of these $P_{x_\perp}$ and $Q_{x_\perp,i}(\tau_1, \tau_2)$ results an effective interaction $V_{x_\perp,i}(\tau_1, \tau_2)$ along a closed loop, which has the same form as a vacuum polarization of relativistic matter fields.

The interactions among $U_{x\mu}(\tau)$ in $\Delta A$ are quite nonlocal in the $\tau$-direction, and $\Delta A$ favors the ordered configurations of $U_{x\mu}$, so the deconfinement phase. In the previous papers[4], we argued that this is the essence of the mechanism of fractionalization phenomena like CSS and PFS. By mapping these gauge models approximately to a spin model, we concluded that the U(1) gauge dynamics is realized in the deconfinement phase at the low-$T$ region below certain critical line $T_c(\delta)$. Thus the possible deconfinement phase in the model similar to Eq.(2.9) is to support the electron fractionalization phenomena.

2.4 Chern-Simons term by Dirac fermions

On considering the effects of matter fields upon 3D gauge dynamics, there is an important difference between fermionic and bosonic matter fields. That is, relativistic fermions have a possibility to generate the Chern-Simons (CS) term $\Delta A_{\text{CS}}$ via radiative corrections:

$$\Delta A_{\text{CS}} = c \int d^3 x \, \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho,$$  \hspace{1cm} (2.10)

which violates the parity symmetry. In particular, as the mass term $m_D \bar{\psi} \psi$ of 3D two-component spinor Dirac fermion field $\psi$ violates the parity invariance, it generates the CS term with the coef-
icient $c \propto \text{sgn}(m_D)$ [18]. On the other hand, scalar fields do not renormalize the coefficient of the Chern-Simons term. In the perturbation theory, the CS term is the leading term at long distances and low energies. So it may change the phase structure of the gauge system, in particular that of the compact gauge systems. One can intuitively expect appearance of a deconfinement phase since the CS term suppresses fluctuations of the gauge field.

However in most of the effective gauge theories of strongly-correlated electron systems, the CS term is not spontaneously generated, because the most of these systems including the t-J model preserves the parity invariance. For example, in the flux state of the Heisenberg antiferromagnetic spin model and the t-J model in the slave fermion representation, relativistic Dirac fermions appear as low-energy excitations, but the parity invariance is preserved in the effective theory because the lattice fermions appear in doublets with opposite signatures of masses[9], which correspond to four-component spinor Dirac field in the continuum. Then the CS coefficient cancels with each other in the radiative correction as $c \propto \sum m_D = \pm m \text{sgn}(m_D) = 0$.

Though nonperturbative investigation of the Chern-Simons gauge theory is very important by itself¹, we shall focus on parity-invariant lattice gauge theory in the rest of discussions in this paper.

3 Nonlocal U(1) Lattice Gauge Theory

In the previous section, we have seen various approaches to study the effect of matter fields upon U(1) gauge dynamics. To confirm these results, one must examine the validity of the approximations employed there. In particular, to check the validity of the hopping expansion at $T = 0$ is quite important for studies on strongly-correlated electron systems.

Keeping this problem in mind, we start this section with a U(1) LGT coupled with matter fields and introduce its effective nonlocal LGT. Let us consider a 3D cubic lattice (i.e., a 2D lattice with a discrete imaginary time). The gauge field $U_{x\mu}$ ($\mu = 0, 1, 2$) is defined on the link $(x, x + \hat{\mu})$ between the pair of nearest-neighbor sites $x$ and $x + \hat{\mu}$. The partition function $Z$ is given by the following functional integral,

$$Z = \int \prod_x d\bar{\phi}_x d\phi_x \prod_{x\mu} dU_{x\mu} \exp(A),$$

$$A = -\sum_{x,y} \bar{\phi}_x \Gamma_{xy}(U)\phi_y + A_U,$$

$$A_U = q \sum_{x,\mu<\nu} (\bar{U}_{x\nu} U_{x+\hat{\mu},\nu} U_{x+\hat{\mu}\nu} U_{x\mu} + \text{c.c.}),$$

(3.1)

¹For example, phase structure of SU(N) Maxwell-CS gauge theory has been studied via frustrated Heisenberg spin model without parity invariance, which is a low-energy effective model of strongly-coupled SU(N) gauge theory of fermions [19]. Existence of a deconfinement phase transition is suggested there.
where \( x = (x_0, x_1, x_2) \) is the site-index of the 3D lattice of the size \( V = N_0 N_1 N_2 \) with the periodic boundary condition, \( \mu \) is the imaginary-time index \( (\mu = 0) \) and spatial direction indices \( (\mu = 1, 2) \), \( \phi_x \) is the matter field on \( x \), \( U_{x\mu} = \exp(i\theta_{x\mu}) (-\pi < \theta_{x\mu} \leq \pi) \) is the \( U(1) \) gauge variable on the link \((x, x + \hat{\mu})\), and \( q \) is inverse gauge coupling constant. \( \Gamma_{xy}(U) \) represents the local minimal couplings of \( \phi_x \) to \( U_{x\mu} \). For example, for a bosonic matter field,

\[
\sum_{x,y} \bar{\phi}_x \Gamma_{xy}(U) \phi_y = t \sum_{x,\mu} \left[ \bar{\phi}_{x+\hat{\mu}} U_{x\mu} \phi_x + \text{H.c.} \right] + M^2 \sum_x \bar{\phi}_x \phi_x, \quad M^2 = 6 + m^2,
\]

(3.2)

where \( m^2 \) is the mass in unit of the lattice spacing and 6 in \( M^2 \) is the number of links emanating from each site.

After integrating over the matter field \( \phi_x \), effective gauge model is obtained, which includes all contributions from \( \phi_x \) to the gauge dynamics,

\[
Z = \int \prod_{x,\mu} dU_{x\mu} \exp \left[ f \, \text{Tr} \log \Gamma_{xy}(U) + A_T \right],
\]

(3.3)

where \( f \) is a parameter counting the statistics and internal degrees of freedom of \( \phi_x \). Due to the \( (\text{Tr} \log \Gamma_{xy}(U)) \) term, the effective gauge theory becomes nonlocal. For relativistic matter fields, a formal expression of the effective gauge theory action is obtained by the hopping expansion and it is expanded as a sum over all the closed random walks \( \mathcal{R} \) (loops including backtracings) on the 3D lattice, which represent world lines of particles and antiparticles as

\[
\text{Tr} \log \Gamma_{xy}(U) = \sum_{\mathcal{R}} \frac{\gamma}{{L[\mathcal{R}]}} \prod_{(x,\mu) \in \mathcal{R}} U_{x\mu}.
\]

(3.4)

\( L[\mathcal{R}] \) is the length of \( \mathcal{R} \), and \( \gamma = (6 + m^2)^{-1} \) is the hopping parameter. There are many different random walks that have the same shape of a closed loop on the lattice. Each random walk in such a family may have a different starting point and/or backtracings. This degeneracy cancels out the denominator \( L[\mathcal{R}] \) in Eq.(3.4). For the constant gauge-field configuration \( U_{x\mu} = 1 \), the expansion in (3.4) is logarithmically divergent \( \sim \log m \) as \( m \to 0 \) due to the lowest-energy zero-momentum mode.

Below we shall study a slightly more tractable model than that given by Eq.(3.3). It is suggested by the hopping expansion (3.4), and obtained by retaining only the rectangular loops extending in the \( \tau \)-direction in the loop sum and choosing their expansion coefficients as follows:

\[
Z_T = \int \prod_{x,\mu} dU_{x\mu} \exp(A_T), \quad A_T = g \sum_{x} \sum_{i=1}^{N_0} \sum_{\tau=1}^{2} c_{\tau}(V_{x,i,\tau} + \bar{V}_{x,i,\tau}) + A_S, \quad V_{x,i,\tau} = \bar{U}_{x+i,0,0} \prod_{k=0}^{\tau-1} \left[ \bar{U}_{x+k,0,0} U_{x+i+k,0,0} \right] U_{x,i},
\]

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\[ A_S = g^\Lambda \sum_x (\bar{U}_{x+1,1}^x U_{x+2,1}^x + c.c.), \]  

(3.5)

where \( g \) is the (inverse) gauge coupling constant, and \( V_{x,i,\tau} \) is the product of \( U_{x\mu} \) along the rectangular \( (x, x + \hat{i}, x + \hat{i} + \hat{\tau}, x + \tau \hat{0}) \) of size \( (1 \times \tau) \) in the \((i - 0)\) plane. See Fig.2.

![Figure 2: Illustration of the nonlocal interaction in the action (3.5). Each rectangle with thick lines represents \( V_{x,i,\tau} \).](image)

In \( A_S \), we have retained only the single-plaquette coupling with the coefficient \((g\lambda)\). For the nonlocal coupling constant \( c_\tau \), we consider the following three cases:

\[
c_\tau = \begin{cases} 
\tau^{-\alpha}, & \text{power-law decay (PD-\(\alpha\)) (\(\alpha = 1, 2, 3\)),} \\
e^{-m\tau}, & \text{exponential decay (ED),} \\
1, & \text{no decay (ND).}
\end{cases}
\]  

(3.6)

The power \( \alpha = 1 \) in the PD model in (3.6) reflects the effect of the relativistic massless excitations without dimensional parameters. In fact, this \( c_\tau \) generates a logarithmically divergent action for \( U_{x\mu} = 1 \) explained below Eq.(3.4) as one can see from the relation, \( \sum_\tau \exp(-m\tau)\tau^{-1} \simeq \log(1/m) \). The action for \( m = 0 \) is then proportional to \( \sum_\tau \tau^{-1} \simeq \log N_0 \) for finite \( N_0 \). On the other hand, the ED model contains the parameter \( m \) with mass dimension and simulates the case of massive matter fields. (We used Eq.(3.6) instead of \( \exp(-m\tau)/\tau \) to make the comparison with the PD case more definitive.) The ND model corresponds to gauge model coupled to nonrelativistic fermions with a Fermi surface (or Fermi line) as it is seen from Eq.(2.9).
4 Numerical Results

In this section, we report our numerical calculations of the internal energy, the specific heat, the Polyakov lines, etc., and determine the phase structure of the models. Most of our interest concerns a possible CDPT in the 3D compact U(1) gauge models with the nonlocal interactions.

In the MC simulations, we consider the isotropic lattice, $N_\mu = N$ ($\mu = 0, 1, 2$), with the periodic boundary condition up to $N = 32$, where the limit $N \to \infty$ corresponds to the system on a 2D spatial lattice at $T = 0$. For the mass of the ED model, we set $m = 1$. For the spatial coupling $\lambda$ scaled by $g$, we consider the two typical cases $\lambda = 0$ (i.e., no spatial coupling) and $\lambda = 1$.

4.1 Internal energy and specific heat

First we calculate the following “internal energy” $E$ and the “specific heat” $C$ per site;

\[
Z_\tau = \int [dU] \exp(A_\tau) \equiv \exp(-FV),
\]

\[
E \equiv -\frac{1}{V} \langle A_\tau \rangle = -\frac{1}{V} \frac{dZ_\tau}{dg} = g \frac{dF}{dg},
\]

\[
C \equiv \frac{1}{V} \langle (A_\tau - \langle A_\tau \rangle)^2 \rangle = -g^2 \frac{d^2F}{dg^2}.
\]

(4.1)

Here we note that the definition of $C$ (4.1) is the response of $E$ under the variation of $g$ and is different from the conventional specific heat that measures the response under the variation of temperature itself. The latter contains extra terms associated with the change of $N_0$. Because these terms behave less singular than the variance of $E$, they are irrelevant in searching for phase transitions.

In Fig.3, we present $E$ and $C$ for $\lambda = 1$ vs the nonlocal coupling $g$. In Appendix we calculate $E$ and $C$ by the high-temperature expansion (HTE) for small $g$ and by the low-temperature expansion (LTE) for large $g$. The MC results of $E$ and $C$ are consistent with these expansions. For comparison with HTE (e.g., for $0 < g < 0.1$ in the PD-1 model ($\alpha = 1$)), see Fig.15 in Appendix. For comparison with LTE, the leading result of LTE is shown by the straight lines in Fig.3(a,b) and $C = 1$ in Fig.3(c,d).

First, let us see the PD-1 model ($\alpha = 1$) in detail. $E$ of Fig.3(a) connects the results of HTE and LTE. Since the LTE is an expansion around $V_{x,i,\tau} = 1$, this behavior implies $V_{x,i,\tau} \sim 1$ for large $g$. $C$ of Fig.3(c) shows that its peak develops as the system size $N$ increases. These two points indicate that the PD-1 model exhibits a second-order phase transition separating the disordered (confinement) phase and the ordered (deconfinement) phase at $g = g_c \approx 0.17$ which is determined from the data of $N = 24$. The existence and the nature of the phase transition will be confirmed later by the measurement of the Polyakov lines and the instanton density as we show in the following subsections.
Figure 3: Internal energy $E$ and its fluctuation $C$ of the action with $\lambda = 1$ vs non-local coupling $g$ for $N = 8(\bullet), 16(★), 24(■)$; (a,c) PD-1 model, (b,d) ED model. The solid lines in (a) and (b) are the leading order in the large-$g$ expansion (LTE). In the PD-1 model, strong $N$ dependence is observed in $E$ at large $g$ and in the developing peak of $C$. They indicate a second-order phase transition in the PD-1 model. In the ED model, on the other hand, $C$ does not develop and therefore the observed peak indicates not a phase transition but a crossover.
We notice that the location of the peak in $C$ of Fig.3(c) shifts to smaller $g$ as the system size $N$ increases in the direction opposite to the usual second-order phase transitions. This behavior reflects the fact that the couplings among $U_{x\mu}$ increases effectively as $N$ increases due to the additional terms in the summation over $N_0$ in the action even if one fixes the overall constant $g$. This is the characteristic nature of nonlocal interactions in strong contrast to local interactions.

On the contrary, in the ED model of Fig.3(d), the peak of $C$ does not develop as $N$ increases, showing no signals of a second-order transition. It may have a higher-order transition or just a crossover. Similar behavior of $C$ is observed in the ordinary U(1) gauge systems with local actions which have only the confinement phase. The physical meaning of the above “crossover” in the ED model shall become clear by studying instantons in Sect.4.3.

It is quite interesting to see whether the data of $C$ for $N = 8, 16$ and 24 in Fig.3(c) exhibit the finite-size scaling behavior[20]. To this end, let us introduce a parameter

$$\epsilon \equiv (g - g_{\infty})/g_{\infty}$$

(4.2)

where $g_{\infty}$ is the critical gauge coupling of the infinite system at $N \to \infty$. Then let us assume that the correlation length $\xi$ scales as $\xi \propto \epsilon^{-\nu}$ with a critical exponent $\nu$. We also expect that the peak of $C$ diverges as $C_{\text{peak}} \propto \epsilon^{-\sigma}$ as $N \to \infty$ with another critical exponent $\sigma$. The finite-size scaling hypothesis predicts that the specific heat $C(\epsilon, N)$ for sufficiently large $N$ scales as

$$C(\epsilon, N) = N^{\sigma/\nu} \phi(N^{1/\nu} \epsilon),$$

(4.3)

where $\phi(x)$ is a certain scaling function. In Fig.4, we present $\phi(x)$ determined by using the data in Fig.3(c) with $\nu = 1.2, \sigma/\nu = 0.25$ and $g_{\infty} = 0.11$. This result indicates that the finite-size scaling law holds quite well. Considering the errors in the data we estimate the values of scaling parameters and $g_{\infty}$ as

$$\nu = 1.2 \sim 1.3, \sigma/\nu = 0.25 \sim 0.26, g_{\infty} = 0.10 \sim 0.12.$$  

(4.4)

The simulations of the PD-1 and ED models with $\lambda = 0$ give similar behaviors of $E$ and $C$ as the $\lambda = 1$ case, preserving the above phase structure for $\lambda = 1$. That is, the PD-1 model ($\lambda = 0$) has a CDPT, while the ED model ($\lambda = 0$) has no transition.

Let us next consider the ND model. In Fig.5, we present $C$ for $\lambda = 0$ and 1, which show strong signals of a second-order phase transition. However, the value of the critical coupling for $\lambda = 1$, $g_c \sim 0.1(N = 8), 0.045(N = 16), 0.03(N = 24)$, decreases very rapidly as the system size increases. This behavior is explained by the increase of effective coupling explained above. Then one may think that $g_c \to 0$ as $N \to \infty$, i.e., only the deconfinement phase survives in the
Figure 4: Scaling function $\phi(x)$ of (4.3) obtained from the data $C$ of the PD-1 model in Fig.3(c) for $N = 8, 16$ and 24. The result obviously shows that the finite-size scaling law holds and the observed phase transition in the PD-1 model is of second-order.

ND model. This expectation is consistent with the fact that the coefficient $Q_2 \equiv \sum_{\tau} c_\tau^2$ of HTE, $C = 2(2Q_2 + \lambda^2)g^2 + O(g^3)$ in Eq.(A.8) diverges as $Q_2 \propto N \to \infty$ for the ND model, i.e., the radius of convergence in the HTE is zero. On the contrary, $Q_2$ is finite for the PD-1 model, assuring us $g_c \neq 0$. This is supported also by the scaling analysis given above.

Figure 5: $C$ vs $g$ in the ND model with (a) $\lambda = 0$ and (b) $\lambda = 1$ for $N = 8(\blacktriangle), 16(\blacksquare), 24(\bigstar)$. The peaks develops as the system-size increases.

From these results for the various cases of long-range interaction, it seems that there exists a critical power $\alpha = \alpha_c$ below which the CDPT takes place. In Fig.6 we show $C$ of the PD model with $\alpha = 2$ and $\alpha = 3$.

We obtain a very interesting result, i.e., for the PD-2 system ($\alpha = 2$) with the nonvanishing spatial coupling $\lambda = 1$, there exists a second-order CDPT as in the PD-1 case, whereas in the PD-3 case ($\alpha = 3$) the peak in $C$ does not develop as $N$ increases, hence no signals of CDPT. Furthermore, careful study of the PD-2 case shows that the second-order CDPT disappears for vanishing spatial
Figure 6: \( C \) vs \( g \) in the PD-2 and PD-3 models for \( N = 8(\uparrow), 16(\blacksquare), 24(\star) \). (a) PD-2(\( \lambda = 1 \)), (b) PD-2(\( \lambda = 0 \)), (c) PD-3(\( \lambda = 1 \)), (d) PD-3(\( \lambda = 0 \)). The peak in \( C \) develops in (a) as \( N \) increases, whereas it does not in (b), (c) and (d). This indicates that the CDPT starts to appear between the two cases, (a) PD-2(\( \lambda = 1 \)) and (b) PD-2(\( \lambda = 0 \)).

coupling \( \lambda = 0 \). Thus we conclude that the critical power is \( \alpha_c = 2 \) and the spatial coupling \( \lambda \) controls the existence of the CDPT. The above results will be confirmed by the study of the Polyakov line in the following subsection.

4.2 Polyakov lines

We have argued the possible CDPT by measuring the thermodynamic quantities like \( E \) and \( C \). In order to study the CDPT in more details and further the nature of gauge dynamics in each phase, it is useful to calculate order parameters in gauge theory like Polyakov lines and Wilson loops.

First, let us introduce the Polyakov lines \( P_{x_\perp} \) for each spatial site \( x_\perp \equiv (x_1, x_2) \) and study their spatial correlations \( f_P(x_\perp) \),

\[
P_{x_\perp} = \prod_{x_0 = 1}^{N_0} U_{x_\perp,x_0,0}, \quad x_\perp = (x_1, x_2),
\]

\[
f_P(x_\perp) = \langle \bar{\Phi}_{x_\perp} \Phi_0 \rangle.
\] (4.5)

Since the present model (3.5) contains no long-range interactions in the spatial directions, \( f_P(x_\perp) \) is expected to supply a good order parameter to detect a possible CDPT. In the deconfinement phase, the fluctuations of \( U_{x_0} \) are small, which implies an order in \( f_P(x_\perp) \), i.e., we expect \( f_P(x_\perp) \neq 0 \) as
$x_\perp \to \text{large}$ in the deconfinement phase.

In Fig.7, we present $f_P(x_\perp)$ for the PD-1 and ED models. The PD-1 model of Fig.7(a) clearly exhibits an off-diagonal long-range order, i.e., $\lim_{x_\perp \to \infty} f_P(x_\perp) \neq 0$ for $g \geq 0.20$, whereas the ED model of Fig.7(b) *does not* for all $g$’s. To see this explicitly, we plot in Figs.7(c) and (d) the order parameter $p \equiv (f_P(x^{\text{MAX}}_\perp))^{1/2}$ for the PD-1 model, where $x^{\text{MAX}}_\perp \equiv N/\sqrt{2}$ is the distance at which $f_P$ becomes minimum due to the periodic boundary condition. $p$ of the PD-1 model($\lambda = 0$) starts to develop continuously from zero at $g = g_c \simeq 0.15$. The size dependence of $p$ shows a typical behavior of a second-order phase transition. Thus the gauge dynamics of the PD-1 model is realized in the deconfinement phase for $g > g_c$, whereas it is in the confinement phase for $g < g_c$. In contrast, the ED model stays always in the confinement phase. These results including the value of $g_c$ are in good agreement with those derived from the data of $E$ and $C$ given in Fig.3.

![Figure 7: Correlations of Polyakov lines, $f_P(x_\perp)$, vs $|x_\perp|$. (a) PD-1 ($\lambda = 0$, N=16) with $g = 0.4, 0.3, 0.2, 0.1, 0.02$ from above. (b) ED ($\lambda = 0$, N=16) with $g = 2.5, 2.0, 1.5, 1.0, 0.5$ from above. In (c) and (d) the order parameter $p \equiv (f_P(x^{\text{MAX}}_\perp))^{1/2}$ vs $g$ is plotted for the PD-1 model; (c) PD-1 $\lambda = 0$ and (d) PD-1 $\lambda = 1$. They exhibit long-range orders for $g > g_c \simeq 0.15$ in the PD-1 model for both $\lambda$.](image)

Let us turn to the PD-2 model with and without the spatial coupling. In Fig.8, we show the result of $p$ for $\lambda = 1$ and $\lambda = 0$. We observe that the model with $\lambda = 1$ shows a typical behavior of the second-order phase transition as the system size is increased, whereas the case of $\lambda = 0$ does not. From this result and the observation of $C$ in the previous subsection, we conclude that the CDPT exists in the PD-2 model ($\lambda = 1$) whereas it disappears in the PD-2 model ($\lambda = 0$).

In Fig.9 we present $f_P(x_\perp)$ of the PD-3 model. It is obvious that there is no long-range order in
Figure 8: Off-diagonal long-range order of Polyakov lines, $p$ vs $|x_\perp|$ for the PD-2 model with (a) $\lambda = 1$ and (b) $\lambda = 0$. In the $\lambda = 1$ case, $p$ exhibits a steeper jump for larger $N$, whereas it does not in the $\lambda = 0$ case.

the PD-3 model both for $\lambda = 0, 1$ and only the confinement phase is realized for all $g$.

Figure 9: $f_P(x_\perp)$ vs $g$ for the PD-3 model with $N = 16$; (a) $\lambda = 1$ for $g = 0.4 \sim 0.7$, (b) $\lambda = 0$ for $g = 0.56 \sim 0.78$. The upper curves have larger $g$’s. It is obvious that there is no long-range order in the PD-3 model regardless of the values of $g$ and $\lambda$.

4.3 Wilson loops

Let us turn to study of the Wilson loops. For ordinary pure and local gauge systems, the Wilson loop $W[C]$ along a closed loop $C$ on the lattice is a good order parameter to study the gauge dynamics; $W[C]$ obeys the area law in the confinement phase and the perimeter law in the deconfinement phase:

$$W[C] \equiv \langle \prod_C U_{x\mu} \rangle \sim \begin{cases} \exp(-\alpha S[C]), & \text{area law,} \\ \exp(-\alpha'L[C]), & \text{perimeter law,} \end{cases}$$ (4.6)
where $S[C]$ is the minimum area of a surface, the boundary of which is $C$, and $a$ and $a'$ are constants. For a (local) gauge theory containing matter fields of the fundamental charge, $W[C]$ cannot be an order parameter because the matter fields generate the terms $\prod_c U_{x\mu}$ with coefficients $\sim \exp(-bL[C])$ in the effective action. However, in the present model (3.5), the nonlocal terms are restricted only along the temporal direction, so it is interesting to measure $W[C]$ for the loops lying in the spatial (1-2) plane. If $W[C]$ obeys the perimeter law, fluctuations of the spatial component of gauge field is small.

Figure 10: Wilson loops ($N = 32$) in the 1-2 plane at large $g$ vs $L[C]$ or $S[C]$. (a) PD($\lambda = 1, g = 0.25$), (b) ED($\lambda = 1, g = 1.5$). The PD model seems to prefer the perimeter law, whereas the ED model prefers the area law.

In Fig.10, we plot $W[C]$. For the PD-1 model in Fig.10(a), the data at $g = 0.25$ seem to prefer the perimeter law. For the ED model in Fig.10(b), the area law fits $W[C]$ better than the perimeter law at $g = 1.5$; a considerably larger value than $g \simeq 1.0$ at the peak of $C$. This suggests that the area law holds in the ED model at all $g$. These observations are consistent with the previous results on the (non)existence of the CDPT. We conclude that the Wilson loops in the spatial plane can be used as an order parameter of gauge dynamics in the present model.

The result that the PD-1 case of the simplified model (3.5) exhibits the CDPT strongly suggests that the original model (3.3) of massless matter fields also has the deconfinement phase, because the isotropic distribution of the nonlocal gauge couplings in the original model should give the similar effect of suppression of fluctuations of $U_{x\mu}$ as those in the temporal direction in Eq.(3.5). This expectation is supported by the measurement of the spatial Wilson loop given above.
4.4 Instantons

It is well known in a 3D continuum space-time that instanton (monopole) configurations of $U(1)$ gauge field carry nontrivial topological numbers, and the instanton density serves as an index to express the disorderness of gauge field\cite{11}. To see the details of the gauge dynamics of the present model and to support the conclusions obtained in the previous subsections, let us study instantons on the lattice. We employ the definition of the instanton density $\rho_x$ at the site $x$ in $U(1)$ lattice gauge theories by DeGrand and Toussaint\cite{21}. We introduce the “vector potential” $\theta_{x\mu}$ as the exponent of $U_{x\mu} = \exp(i\theta_{x\mu})$ [$\theta_{x\mu} \in (-\pi, \pi)$]. Then, the magnetic flux $\Theta_{x,\mu\nu}$ penetrating plaquette $(x, x+\mu, x+\mu+\nu, x+\nu)$ is expressed as

$$\Theta_{x,\mu\nu} \equiv \theta_{x\mu} + \theta_{x+\mu,\nu} - \theta_{x+\nu,\mu} - \theta_{x\nu}, \quad (-4\pi < \Theta_{x,\mu\nu} < 4\pi). \quad (4.7)$$

We decompose $\Theta_{x,\mu\nu}$ into its integer part $2\pi n_{x,\mu\nu}$ ($n_{x,\mu\nu}$ is an integer) and the remaining part $\tilde{\Theta}_{x,\mu\nu} \equiv \Theta_{x,\mu\nu} \pmod{2\pi}$ uniquely,

$$\Theta_{x,\mu\nu} = 2\pi n_{x,\mu\nu} + \tilde{\Theta}_{x,\mu\nu}, \quad (-\pi < \tilde{\Theta}_{x,\mu\nu} < \pi). \quad (4.8)$$

Physically speaking, $n_{x,\mu\nu}$ describes the Dirac string whereas $\tilde{\Theta}_{x,\mu\nu}$ describes the fluctuations around it. The quantized instanton charge $\rho_x$ at the cube around the site $\hat{x} = x + (\hat{1} + \hat{2} + \hat{3})/2$ of the dual lattice is defined as

$$\rho_x = \frac{1}{2} \sum_{\mu,\nu,\rho} \epsilon_{\mu\nu\rho}(n_{x+\mu,\nu,\rho} - n_{x,\nu,\rho})$$

$$= \frac{1}{4\pi} \sum_{\mu,\nu,\rho} \epsilon_{\mu\nu\rho}(\tilde{\Theta}_{x+\mu,\nu,\rho} - \tilde{\Theta}_{x,\nu,\rho}), \quad (4.9)$$

where $\epsilon_{\mu\nu\rho}$ is the complete antisymmetric tensor. $\rho_x$ measures the total flux emanating from the monopole(instanton) sitting at $\hat{x}$. Roughly speaking, $\rho_x$ measures the strength of nonperturbative gauge fluctuations around $\hat{x}$. For the local 3D $U(1)$ compact lattice gauge theory without matter fields, the average density

$$\rho \equiv \frac{1}{V} \sum_x \langle |\rho_x| \rangle, \quad (4.10)$$

is known to behave as $\rho \propto \exp(-cg)$ ($c$ is a constant) if the instanton action $cg$ is large; The instanton gas stabilizes the confinement phase for all the gauge coupling\cite{11}.

In Fig.11 we present $\rho$ as a function of $g$ in the PD-1 and ED models. It decreases as $g$ increases more rapidly in the PD-1 model than in the ED model. This difference in behavior is consistent with the result that the PD-1 model exhibits a second-order transition, while the ED model does not. The $\lambda$ coupling enhances the rate of decrease in $\rho$ as one expects since the spatial coupling enhances
the ordered deconfinement phase. In the ED model with $\lambda = 1$, $\rho$ is fitted by $\exp(-cg)$ in the dilute (large $g$) region, and the smooth increase for smaller $g$ indicates a crossover from the dilute gas of instantons to the dense gas, just the behavior similar to the case of pure and local lattice gauge theory[11, 22].

In Ref.[10] we presented the figure (Fig.3) which is very similar to Fig.11. However, the former was plotted by using the definition $\rho \equiv V^{-1} \sum_x (1 - \delta_0, \rho_x)$, the average of occupation number of instantons (1 for $\rho_x \neq 0$ and 0 for $\rho_x = 0$) per site instead of the instanton density of Eq.(4.10) itself. The curves in two figures are almost indistinguishable, because the configurations with $|\rho_x| \geq 2$ are rather rare.

Figure 11: Average instanton density $\rho$ vs $g$. (a)PD-1($\lambda = 0, N = 8$), (b)PD-1($\lambda = 0, N = 16$), (c)PD-1($\lambda = 1, N = 8$), (d)PD-1($\lambda = 1, N = 16$), (e)ED($\lambda = 0, N = 8,16$), (f)ED($\lambda = 1, N = 8,16$). The solid curve is $\propto \exp(-cg)$ and fits (f) at large $g$.

In Fig.12 we present snapshots of $\rho_x$ for the PD-1 model with $\lambda = 1$. Fig.12(a) is a dense gas whereas Fig.12(b) is a dilute gas. They are separated at $g_c \simeq 0.20$, the location of the peak of $C$ for $N = 16$. In Fig.12(b), instantons mostly appear in dipole pairs at nearest-neighbor sites, $\rho_x = \pm 1, \rho_{x+\mu} = \mp 1$, while in Fig.12(a), they appear densely and it is hard to determine their partners. In both cases, the distributions $\rho_x$ have no apparent anisotropies like column structures.
However, the orientations of dipoles in Fig.12(b) are mostly (~92%) in the temporal direction as expected from the nonlocal interactions of Eq.(3.5).

Next, let us examine the difference of gauge dynamics in various cases of nonlocal interactions in detail. To this end, it is useful to measure new observables made out of gauge-field configurations; nonlocal instantons elongated in the temporal direction with the length \( t = 2, 3, \cdots, N - 1 \). The density of these instantons is defined by

\[
\rho_{x,t} = \sum_{\tau=0}^{t-1} \rho_{x+\tau\hat{0}},
\]
\[
\rho_t = \langle \rho_{x,t} \rangle.
\]

In Fig.13 we plot \( \rho_t \) vs \( g \) for various cases of the exponent \( \alpha \) and \( \lambda \).

We notice that \( \rho_t \)'s in Fig.13(a,b,c) behave quite differently from \( \rho_t \)'s in Fig.13(d,e). For the former cases, all the curves of \( \rho_t \) with different \( t \)'s decrease similarly as \( g \) increases [except for \( t = 1 \) in (c)], whereas for the latter cases, each \( \rho_t \) decreases in a different manner. The present nonlocal instanton density \( \rho_t \) is capable to distinguish the cases (a,b,c) exhibiting a CDPT and the cases (d,e) without transition (cross over) through its \( t \) dependence. This is consistent with our common understanding that the second-order phase transition is a collective phenomenon at which wild fluctuations of all the variables in the system in the disordered phase start to be reduced coherently and also scale-invariantly near the transition point.
Figure 13: Density of nonlocal instantons $\rho_t$ vs $g$ for $N = 12$. (a) PD-1($\lambda = 1$), (b) PD-1($\lambda = 0$), (c) PD-2($\lambda = 1$), (d) PD-2($\lambda = 0$), (e) PD-3($\lambda = 1$). $\rho_t$ at a fixed $g$ increases as $t$ increases.
5 Effective Models at Large $g$ and $\lambda = 0$

In this section, we study a 1D XY spin model for the spatial gauge variables $U_{xi}$ and also a 2D XY spin model for the temporal ones $U_{x0}$, both of which are regarded as effective models for the present nonlocal model at large $g$ and $\lambda = 0$ obtained by a simple reduction of the dynamical degrees of freedom. Two models are complementary each other and study of them helps us to understand the properties of the present nonlocal gauge model studied by the numerical simulations in section 4. In particular, these models are capable to explain the existence of the deconfinement (ordered) phase in the PD-1 model at large $g$ and the nonexistence of it in the other PD and ED models with $\lambda = 0$.

5.1 The 1D XY model with spatial variables $U_{xi}$

Let us imagine the situation in which fluctuations of the temporal gauge fields $U_{x0}$ are small so that one may replace $U_{x0} = \exp(i\theta_{x0})$ by its average as follows;

$$U_{x0} \rightarrow u \ (0 < u < 1). \quad (5.1)$$

This is expected for sufficiently large $g$. Furthermore we put $\lambda = 0$, i.e., there are no direct interactions among spatial gauge variables $U_{xi} = \exp(i\theta_{xi})$. Then the system is decoupled to 1D subsystems defined at each spatial link $(x, x+i)$. The subsystem at $(x, x+i)$ is described by the U(1) angles $\theta_j (\equiv \theta_{xi})$ (we write $x_0 \equiv j = 1, \cdots, N_0$ and suppress the suffix $x_1, x_2, i$). Its energy takes the form of a 1D XY spin model with nonlocal interactions with couplings $c_\tau$. The partition functions of the subsystem and the total system are given as follows;

$$Z_{1DXY} = \prod_{j=1}^{N_0} \int_{-\pi}^{\pi} \frac{d\theta_j}{2\pi} \exp \left[ g \sum_{j=1}^{N_0} \sum_{\tau=1}^{N_0} u^{2\tau} c_\tau \cos(\theta_{j+\tau} - \theta_j) \right],$$

$$Z_T (g : \text{large}, \lambda = 0) \simeq (Z_{1DXY})^{2N_1N_2}. \quad (5.2)$$

To study the correlation function $\langle \exp[i(\theta_r - \theta_0)] \rangle$, we make the harmonic approximation for large $g$: (i) expand the cosine term up to the quadratic term, (ii) extend the range of $\theta_j$ to $(-\infty, \infty)$ and (iii) neglect topologically nontrivial configurations. We expect that the third approximation in the above is justified by the long-range ferromagnetic interactions in Eq.(5.2). Then we have

$$Z_{1DXY} \simeq \prod_{j=1}^{N_0} \int_{-\infty}^{\infty} \frac{d\theta_j}{2\pi} \exp \left[ -\frac{g}{2} \sum_{j=1}^{N_0} \sum_{\tau=1}^{N_0} u^{2\tau} c_\tau (\theta_{j+\tau} - \theta_j)^2 \right]$$

$$\propto \prod_{k=1}^{N_0} \int_{-\infty}^{\infty} \frac{d\tilde{\theta}_k}{2\pi} \exp \left[ -N_0 \sum_{k=1}^{N_0} G_k \tilde{\theta}_k^2 \right],$$

$$G_k = g \sum_{\tau} u^{2\tau} c_\tau [1 - \cos(k\tau)].$$
\[ f(r) \equiv \langle \exp[i(\theta_r - \theta_0)] \rangle = \exp[-\frac{1}{2}((\theta_r - \theta_0)^2)] = \exp[-\sum_k G^{-1}(k)(1 - \cos(kr))], \quad (5.3) \]

where \( \hat{\theta}_k \) is the Fourier-transformed variable of \( \theta_j \). For the standard local coupling \( c_\tau = \delta_\tau \), the \( k \)-sum in the exponent of \( f(r) \) gives \( \int dk \cos(kr)/k^2 \sim r \), which implies \( f(r) \sim \exp[-(gu^2)^{-1}r] \to 0 \) as \( r \to \infty \) due to the severe infrared fluctuations. For the nonlocal cases we have

\[ f(r) \sim \exp \left[ -\sum_k \frac{1 - \cos(kr)}{g \sum_{\tau} u^{2\tau} c_\tau r^2 k^2} \right] \sim \exp(-Mr), \quad (5.4) \]

Namely, the order in the correlation function survives as \( N_0 \to \infty \) for the PD-1 case if \( u \neq 0 \), whereas the order is destroyed in the other cases even for \( u \neq 0 \). This result is just consistent with the numerical results for specific heat and the Polyakov lines studied in section 4. For \( \lambda = 0 \) case, the ordered-deconfinement phase is realized only in the PD-1 model at large \( g \).

### 5.2 The 2D XY model with temporal variables \( U_{x,0} \)

The discussion in the previous subsection is supported by considering another effective model that focuses on the dynamics of the temporal gauge field \( U_{x,0} \). To obtain it, we first replace the spatial gauge field \( U_{xi} \) by its average,

\[ U_{x,0} \to v \quad (0 < v < 1). \quad (5.5) \]

This replacement is supported for the PD-1 model by the result of the 1D XY model (5.4) in the previous subsection. Then we fix the gauge to the temporal gauge by setting \( U_{x,0} = 1 \) except for \( x_0 = N_0 \) because of the periodic boundary condition. By keeping only the terms in the energy involving \( U_{x,N_0,0} \), we have

\[ U_{x,N_0,0} = \exp(i\varphi_{x,N_0}), \quad Z_{2DXY} = \prod_{x_\perp} \int_{-\pi}^{\pi} \frac{d\varphi_{x_\perp}}{2\pi} \exp(A_{2DXY}), \quad (5.6) \]

This is just the standard 2D XY spin model having the nearest-neighbor couplings among the XY spins \( \exp(i\varphi_{x_\perp}) \). Here the XY spin correlation function corresponds to the correlation function of
the Polyakov lines studied in section 4,

$$\langle \cos(\varphi_{x_\perp} - \varphi_{0_\perp}) \rangle = f_P(x_\perp).$$  \hspace{1cm} (5.7)

The spin stiffness in the effective 2D XY model is given as

$$gv^2Q_1 \to \begin{cases} 
\infty, & \text{PD - 1} \\
\frac{2e^2\pi^2}{6}, & \text{PD - 2} \\
\frac{2e^2\pi^4}{360}, & \text{PD - 3} \\
\frac{2e^2}{25}, & \text{ED} 
\end{cases}$$  \hspace{1cm} (5.8)

It is well known that the 2D XY model with finite spin stiffness has no long-range order, although the Kosterlitz-Thouless transition is possible. Only in the PD-1 case, the spin stiffness diverges for finite $v$ and the model may have the order $\langle U_{x_\perp,N_0,0} \rangle \neq 0$. This result (5.8) supports the procedure (5.1) for the 1D XY model in the previous subsection for the PD-1 case. On the other hand, Eq.(5.4) supports the replacement (5.5) as we mentioned before. Thus studies of these two XY models give us the conclusion that only the PD-1 model at large $g$ has the ordered (Coulomb) phase for the $\lambda = 0$ case.

Studies of the two effective XY models in this section predict that the cases of PD-2, PD-3 and ED with $\lambda = 0$ have no Coulomb phase, as it was verified by the previous numerical calculations. Inclusion of $\lambda$ term, however, generates direct interactions between the spatial variables $U_{x_i}$. For small $\lambda$, $U_{x_i}$’s ($i = 1, 2$) can be integrated out perturbatively in powers of $\lambda$. The obtained effective model contains nonlocal and multi-body interactions of $\varphi_{x_\perp}$’s which prefer the ferromagnetic order of $\varphi_{x_\perp}$. Therefore the effective model for $\lambda \sim 1$ may not belong to the same universality class of the 2DXY model with local interactions. Then the above result does not contradict the numerical result in the previous section, which shows the deconfinement phase exists in the PD-2 model with $\lambda = 1$.

6 Conclusion

In this paper, we studied the nonlocal compact U(1) gauge theory on the 3D lattice, which “simulates” gauge models coupled with massless/massive matter fields. The main contributions of the present paper may be the following two points: (i) MC simulations are feasible within reasonable computer time even for 3D lattice gauge theories with nonlocal interactions along one direction, and (ii) the measurements of $E,C$, Polyakov lines, Wilson loops, and local and nonlocal instantons give rise to clear and consistent results on the phase structure of the model. In particular, they distinguish the cases of ND($\lambda = 1, 0$), PD-1($\lambda = 1, 0$), PD-2($\lambda = 1$) with a CDPT from the other cases of PD-2($\lambda = 0$), PD-3,ED without CDPT in a definitive manner.
As explained in Sect.2.3, the results obtained in this paper are quite important for studies of the strongly-correlated electron systems like the high-${T_c}$ cuprates, the fractional quantum Hall effect, quantum spin models, etc. For example, in the t-J model of high-${T_c}$ superconductivity, by using the hopping expansion of holons and spinons at finite $T$ with the continuous imaginary time, we derived an effective gauge theory, which is highly nonlocal in the temporal direction. The obtained effective theory has a similar action as Eq.(3.5) with $c_\tau =$constant and $g \propto n$ where $n$ is the density of matter fields(holons and spinons)[4]. This corresponds to the ND model. Although the above effective gauge model is obtained for the system at finite $T$, we expect that a similar gauge model appears as an effective model at $T = 0$. The result that the ND model has a CDPT strongly suggests that the t-J model has the corresponding phase transition into the deconfinement phase, which is nothing but the charge-spin separated phase.

In the deconfinement phase, all the three variables $U_{x\mu}$ ($\mu = 0, 1, 2$) are stable, having small fluctuations. The stability of the Lagrange multiplier $U_{x0}$ means that the constraint (2.6) is not respected by the holons and spinons, the low-energy excitations of the system, whereas the stability of $U_{xi}$ indicates that the gauge interaction between the holons and spinons can be treated perturbatively. Therefore, quasi-particles in the CSS state are the holons, spinons and weakly interacting gauge bosons[4]. Of course, in order to present a definite “proof” of the CSS, it is necessary to investigate a gauge system with full isotropic nonlocal interaction, because the integration over holons and spinons generates nonlocal interactions not only in the temporal direction but also in the spatial directions and their combinations.

Another interesting model related with the present one is the U(1) Higgs model coupled with the nonlocal gauge field. At present, it is believed that there is no phase transition in the 3D U(1) gauge-Higgs model with the ordinary local action if the Higgs field has the fundamental charge and its radial fluctuations are suppressed[23]. However, the situation may be changed by nonlocal gauge interactions. The existence of the deconfinement phase in the present nonlocal gauge system without Higgs fields suggests that all the three phases, i.e., the confinement, Coulomb and Higgs phases, may be realized in the 3D nonlocal gauge model with a local coupling to a Higgs field. The deconfinement phase of the present model corresponds to the Coulomb phase. This problem is closely related with “doped holes” in the algebraic spin liquid which may be realized in certain antiferromagnetic spin models and materials in the spatial 2D lattice. We shall report on these problems in a separate publication[24].

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A High- and Low-Temperature Expansions

In this appendix we study the behavior of the nonlocal gauge theory (3.5) in two regions of \( g \) by analytic methods; the region of small \( g(<<1) \) by high-temperature expansion (HTE) in Sect.A.1 and the region of large \( g(>>1) \) by low-temperature expansion (LTE) in Sect.A.2. Once one obtains an approximate expression for the partition function \( Z_T \), one can calculate various thermodynamic quantities like \( E \) and \( C \) in Eq.(4.1).

A.1 High-temperature expansion (HTE) for small \( g \)

Let us consider the case of small \( g \). Since the action \( A_T \) is proportional to \( g \), one may expand the partition function \( Z_T \) of (3.5) in powers of \( g \) as

\[
Z_T = \int [dU] \exp(A_T) = \int [dU] \sum_{n=0}^{\infty} \frac{(A_T)^n}{n!}.
\]

We obtain the expansion up to \( O(g^4) \) as

\[
Z_T = 1 + (B_{2T} + B_{2T}A\lambda^2) g^2 + B_{3T}g^3 + (B_{4T} + B_{4T}A\lambda^2 + B_{4T}A\lambda^4) g^4 + O(g^5),
\]

where each coefficient is expressed as

\[
B_{nT} = \frac{1}{n!} \int [dU] \left[ \sum_q c_r(V_q + \bar{V}_q) \right]^n, \quad q \equiv (x,i,\tau),
\]

\[
B_{nS} = \frac{1}{n!} \int [dU] \left[ \sum_p (U_p + \bar{U}_p) \right]^n, \quad U_p \equiv \bar{U}_{x+1}U_{x+1}U_xU_{x+2};
\]

\[
B_{4TS} = \frac{4C_2}{4!} \int [dU] \left[ \sum_p (U_p + \bar{U}_p) \cdot \sum_q c_r(V_q + \bar{V}_q) \right]^2.
\]

To evaluate the above coefficients, we use the following formula for U(1) integral,

\[
\int dU dU^{\mu} U^{\mu\nu} = \delta_{xy}\delta_{\mu\nu}\delta_{mn}.
\]

Then we obtain the following result;

\[
B_{2T} = \sum_q c_r^2 = 2VQ_2, \quad Q_2 = \sum_r c_r^2,
\]

\[
B_{2S} = \sum_p 1 = V,
\]

\[
B_{3T} = \frac{3\cdot2}{3!} \sum_x \sum_i \sum_{\tau_1} \sum_{\tau_2} c_r c_{r_2} c_{r_3} V_{x,i,\tau_1} V_{x+\tau_1,\cdot,\tau_2} V_{x+\tau_1,\cdot,\tau_2} \delta_{\tau_3,\tau_1+\tau_2} + \text{c.c.}
\]

\[
= 4V\tilde{Q}_3, \quad \tilde{Q}_3 = \sum_{\tau_1=1}^{N} \sum_{\tau_2=1}^{N} \sum_{\tau_3=1}^{N} c_{r_1} c_{r_2} c_{r_3} \delta_{\tau_3,\tau_1+\tau_2},
\]

\[28\]
By using $Z$ as $B_{4T}$, we obtain the internal energy $U$ of Eq. (A.6), we obtain the internal energy and the specific heat $C$ as follows,

\begin{align*}
B_{4T} &= B_{4T}^a + B_{4T}^b + B_{4T}^c, \\
B_{4T}^a &= \frac{1}{4!} \sum_{q_1, q_2, q_3, q_4} \prod_{\ell=1}^{4} c_{q_{\ell}} \cdot 4C_2 [\delta_{q_1 q_2} \delta_{q_3 q_4} + \delta_{q_1 q_3} \delta_{q_2 q_4} - \delta_{q_1 q_2} \delta_{q_3 q_4}]
= 2V^2 Q_2^2 - \frac{1}{2} V Q_4, \quad Q_4 \equiv \sum_{\tau} c_{\tau}^4,
B_{4T}^b &= \frac{4 \cdot 3 \cdot 2}{4!} \sum_{x,i} \sum_{\tau_1, \tau_2, \tau_3, \tau_4} c_{\tau_1} c_{\tau_2} c_{\tau_3} c_{\tau_4} (1 - \delta_{\tau_1 \tau_3}) \delta_{\tau_1 + \tau_2 - \tau_3} V_{x,i,\tau_1} V_{x+i,\tau_1,\tau_2} \tilde{V}_{x,i,\tau_1} \tilde{V}_{x+i,\tau_1,\tau_2} \tilde{V}_{x+i,\tau_1,\tau_2} \tilde{V}_{x+i,\tau_1,\tau_2}
= 2V (\tilde{Q}_4 - Q_2^2), \quad \tilde{Q}_4 \equiv \sum_{\tau_1, \tau_2, \tau_3, \tau_4} c_{\tau_1} c_{\tau_2} c_{\tau_3} c_{\tau_4} \delta_{\tau_1 + \tau_2 - \tau_3},
B_{4T}^c &= \frac{4!}{4!} \sum_{x,i} \sum_{\tau_1, \tau_2, \tau_3, \tau_4} c_{\tau_1} c_{\tau_2} c_{\tau_3} c_{\tau_4} V_{x,i,\tau_1} V_{x+i,\tau_1,\tau_2} V_{x+i,\tau_1,\tau_2,\tau_3} V_{x+i,\tau_1,\tau_2,\tau_3,\tau_4} \tilde{V}_{x,i,\tau_1} \tilde{V}_{x+i,\tau_1,\tau_2} \tilde{V}_{x+i,\tau_1,\tau_2,\tau_3} \tilde{V}_{x+i,\tau_1,\tau_2,\tau_3,\tau_4}
= 4V \tilde{Q}_4, \quad \tilde{Q}_4 \equiv \sum_{\tau_1, \tau_2, \tau_3, \tau_4} c_{\tau_1} c_{\tau_2} c_{\tau_3} c_{\tau_4} \delta_{\tau_1 + \tau_2 - \tau_3},
B_{4TS} &= \frac{1}{4} \sum_{p} \sum_{q} 2c_{\tau}^2 = 2V^2 Q_2,
B_{4S} &= \frac{4!}{4!} \sum_{p_1, p_2, p_3, p_4} 4C_2 [\delta_{p_1 p_2} \delta_{p_3 p_4} + \delta_{p_1 p_3} \delta_{p_2 p_4} - \delta_{p_1 p_2} \delta_{p_3 p_4}]
= \frac{1}{2} V^2 - \frac{1}{4} V.
\end{align*}

Some of them are illustrated in Fig.14.

Finally, we obtain the expression of the partition function and the free energy as follows,

\begin{align*}
Z_T &= \exp(-FV),
F &= -2(2Q_2 + \lambda^2)g^2 - 4\tilde{Q}_3 g^3 + \left(\frac{Q_4}{2} + \frac{\lambda^4}{4} - 4\tilde{Q}_4 - 2\tilde{Q}_4^+\right) g^4 + O(g^5). \quad \text{(A.6)}
\end{align*}

In the following, we list up the values of $Q_2$ and $Q_4$ for $N \rightarrow \infty$,

|      | $Q_2$     | $Q_4$     |
|------|-----------|-----------|
| PD-1 | $\pi^2/6 = 1.64493$ | $\pi^4/90 = 1.08232$ |
| PD-2 | $\pi^2/90 = 1.00408$ | $\pi^6/935 = 1.01734$ |
| PD-3 | $\pi^6/935 = 1.00025$ | 1.018657 |
| ED   | 0.15652   | 0.018657  |

By using $Z_T$ of Eq. (A.6), we obtain the internal energy $E$ and the specific heat $C$ defined by Eq.(4.1) as

\begin{align*}
E &= -2(2Q_2 + \lambda^2)g^2 - 12\tilde{Q}_3 g^3 + 4\left(\frac{Q_4}{2} + \frac{\lambda^4}{4} - 4\tilde{Q}_4 - 2\tilde{Q}_4^+\right) g^4 + O(g^5), \\
C &= 2(2Q_2 + \lambda^2)g^2 + 24\tilde{Q}_3 g^3 - 12\left(\frac{Q_4}{2} + \frac{\lambda^4}{4} - 4\tilde{Q}_4 - 2\tilde{Q}_4^+\right) g^4 + O(g^5). \quad \text{(A.8)}
\end{align*}
Figure 14: Illustration of the terms (A.5) in HTE. Each rectangle represents $V_{x,i,\tau}$. The terms $B_{3T}, B_{4T}^b,c$ reflect the nonlocal nature; they are absent in models of local interaction characterized by $c_\tau = 0$ ($\tau > 1$).

Figure 15: Comparison of HTE and the MC simulation. We compare the HTE results and the MC results of $E$ and $C$ for the PD-1 model with $N = 8$ and $\lambda = 1$. The curves indicated by $E_n$ and $C_n$ are the HTE results with the terms up to $O(g^n)$. The dots are the MC data.
Fig. 15 shows that, as one includes the higher-order terms, the HTE results approaches the MC result systematically. However, the approach is rather slow compared with the related models of local interactions like the 3D XY spin model or the 3D U(1) pure LGT. This is because the present nonlocal interactions generate various important higher-order terms in the HTE that are absent from the local models.

Let us comment on the convergence of the HTE. As usual, the HTE is an expansion in the disordered (confinement) phase in which $U_{x\mu}$ fluctuates wildly. Equation (A.6) shows that the convergence radius $g_{\text{HTE}}$ of the expansion is finite $g_{\text{HTE}} \neq 0$, because both the harmonic numbers $Q_2$ and $Q_4$ appearing in the coefficients are finite. This means that there exists certainly the finite region $0 \leq g^2 < g_{\text{HTE}}^2$ of the confinement phase. We notice that if the long-range interaction $c_\tau$ is very strong such that $Q_2 = \infty$, the confinement phase may disappear.

A.2 Low-Temperature Expansion (LTE) for large $g$

For large $g$, we evaluate $Z_T$ by the LTE. The LTE is an expansion in powers of $g^{-1}$ around a fixed “lowest-energy configuration” of $U_{x\mu}$ like $U_{x\mu} = 1$, which gives rise to the global maximum of $A_T$. Let us expand $U_{x\mu}$ as

$$U_{x\mu} \equiv \exp(i\theta_{x\mu}) = 1 + i\theta_{x\mu} - \frac{1}{2}\theta_{x\mu}^2 + O(\theta_{x\mu}^3),$$

(A.9)

where $\theta_{x\mu}$ is treated as $O(g^{-1/2})$ as we shall see. $A_T$ is expanded up to the second order in $\theta_{x\mu}$ in the following quadratic form;

$$A_T = 4gQ_1V + 2g\lambda V - g \sum_{x,\mu, y, \nu} \theta_{x\mu} G_{x\mu,y\nu}(\lambda) \theta_{y\nu} + O(\theta^4),$$

(A.10)

where the first two terms $4gQ_1V + 2g\lambda V$ come from the first term, unity, of R.H.S. of Eq.(A.9). Due to the gauge invariance, one may extend the region of $\theta_{x\mu}$ from $\theta_{x\mu} \in (-\pi, \pi)$ to $\theta_{x\mu} \in (-\infty, \infty)$ together with a gauge fixing. We take the temporal gauge $\theta_{x0} = 0$ in the following calculation. Then we evaluate $Z_T$ by rescaling $\theta'_{x1} = g^{1/2} \theta_{x1}$ and performing Gaussian integration as

$$Z_T \simeq e^{(4Q_1 + 2\lambda)gV} \prod_x \left[ \int_{-\infty}^{\infty} d\theta'_{x1} \int_{-\infty}^{\infty} d\theta'_{x2} \right] \exp \left( -g \sum_{x,\mu, y, \nu} \theta_{x\mu} \theta_{y\nu} \right)$$

$$= e^{(4Q_1 + 2\lambda)gV} \prod_x \left[ g^{-1} \int_{-\infty}^{\infty} d\theta'_{x1} \int_{-\infty}^{\infty} d\theta'_{x2} \right] \exp \left( -\sum_{x,\mu, y, \nu} \theta'_{x\mu} \theta'_{y\nu} \right)$$

$$= \exp \left( (4Q_1 + 2\lambda)g + \ln g \right) V - \frac{1}{2} \text{Tr} \ln G(\lambda),$$

$$F = -(4Q_1 + 2\lambda)g + \ln g + O(g^0).$$

(A.11)
This gives

\[ E = -(4Q_1 + 2\lambda)g + 1 + O(g^{-1}), \]
\[ C = 1 + O(g^{-1}). \]  

\hfill (A.12)

The higher-order terms in \( F \) are \( O(g^{-n}) \) \((n \geq 0)\) which may be calculated by the usual perturbation theory.
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