Exact Equations and Scaling Relations for $\bar{f}_0$-avalanche in the Bak-Sneppen Evolution Model

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Infinite hierarchy of exact equations are derived for the newly observed $\bar{f}_0$-avalanche in the Bak-Sneppen model. By solving the first order exact equation, we find that the critical exponent $\gamma$, governing the divergence of the average avalanche size, is exactly 1 (for all dimensions), which has been confirmed by extensive simulations. Solution of the gap equation yields another universal result $\rho = 1$ ($\rho$ is the exponent of relaxation to attractor). Scaling relations are established among the critical exponents ($\gamma$, $\tau$, $D$, $\sigma$ and $\nu$) for $\bar{f}_0$-avalanche.

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In the Bak-Sneppen (BS) evolution model [1], random numbers, \( f_i \), chosen from a flat distribution between 0 and 1, \( p(f) \), are assigned independently to each species located on a \( d \)-dimensional lattice of linear size \( L \). At each time step, the extremal site, i.e., the species with the smallest random number, and its \( 2d \) nearest neighboring sites, are assigned \( 2d + 1 \) new random numbers also chosen from \( p(f) \). This updating continues indefinitely. After a long transient process the system reaches a statistically stationary state where the density of random numbers in the system is uniform above \( f_c \) (the self-organized threshold) and vanishes for \( f < f_c \).

Despite the fact that it is an oversimplification of real biological process, BS model exhibits such common interesting features observed by paleontologists [2,3] as punctuated equilibria, power-law probability distributions of lifetimes of species and of the sizes of extinction events. These behaviors suggest that the ecology of interacting species might have evolved to a self-organized critical state.

BS model displays spatial-temporal complexity, which also emerges from many natural phenomena, such as fractals [4], 1/f noise [5], etc. This strongly suggests that various complex behaviors may be attributed to a common underlying mechanism. Authors in Ref. [6] suggest that the relation of these different phenomena can be established on the basis of their unique models. It is even proposed by them that spatial-temporal complexity comes out as the direct results of avalanche dynamics in driven systems, and different complex phenomena are related via scaling relations to the fractal properties of the avalanches. It hence can be inferred that avalanche dynamics plays a key role in dealing with complex systems, especially when one needs to know the macroscopic features of the systems, since lingering on the inner structure of individuals will not be helpful [7].

Avalanche is a kind of macroscopic phenomenon driven by local interactions. The size of an avalanche may be extremely sensitive to the initial configuration of the system, while the distribution of the sizes (spatial and temporal) of avalanches, i.e. the "fingerprint" should be robust with respect to the modifications, due to the universality of complexity and the definition of self-organized criticality (SOC) [8]. In this sense, the extent that we know about avalanche will determine to what extent we do a complex system. Avalanche dynamics provides insight into complexity and enables one to further investigate the system studied.

Though avalanche dynamics may be a possible underlying mechanism of complexity, the definitions of avalanches can be vastly different for various complex systems, or for same sorts of systems, even for the same one. In BTW model [9], an avalanche is intrigued by the adding of a grain or several grains of sand into the system. The avalanche is considered over when the heights of all the sites are less than the critical value, say, 4. In BS model [1,6], several types of avalanches, for instance, \( f_0 \)-avalanche, \( G(s) \)-avalanche, forward avalanche and backward avalanche, etc, are presented. These different definitions of avalanches may show their unique hierarchal structures, while they manifest the common fractal feature of the complex system, that is, SOC. It can be inferred that various types of avalanches are equivalent in the sense that they imply complexity.

Since similar structures and common features evidently arise in different types of avalanches, it is straightforward that various avalanches differ each other only in the contexts from which one comprehends them. As known, the major aim of avalanche study is to investigate the universal rules possibly hidden behind the evolution of the systems or the models. Hence, the means of understanding the avalanches appear crucial. Better ways may enable one to know more about the system or the model and hence to have better comprehension of the features corresponding to complexity. From this point of view, when studying avalanches one should try to choose the easier ways instead of the more difficult ones.

The evolution of the highly sophisticated BS model shows a hierarchal structure specified by avalanches, which correspond to sequential mutations below certain threshold. It has been noted [10] that in BS model an avalanche is initiated when the fitness of the globally extremal site (the species with the least random number) is larger than the self-organized threshold. That is, the triggering event of an avalanche is directly related to the fitness, the feature of individuals. In other words, the avalanche is directly associated with the feature of individuals instead of general features of the ecosystem as a whole. Is it feasible that the avalanches are directly intrigued by the global feature of the whole system? Can such global feature be expressed in terms of the corresponding quantity? If such quantity found and such avalanches observed, may the new avalanches provide a new and easier way in investigating properties of the model?

One of our previous works [10] presents such a different hierarchy of avalanches (\( f_0 \)-avalanche) for BS model. We defined a global quantity, \( f \), which denotes the average fitness of the system. The new type of avalanches are directly related to \( f \). In this paper, we present a master equation for the hierarchal structure of \( f_0 \)-avalanches. It prescribes the cascade process of smaller avalanches merging into bigger avalanches when the critical parameter \( f_0 \) is changed. An infinite series of exact equations can be derived from this master equation. The first order exact equation, together with an scaling ansatz of the average sizes of avalanches, shows the exact result of \( \gamma \), the critical exponent governing self-organization, to be universally 1 for all dimensional BS models, which has been confirmed by extensive simulations of the model. We also establish scaling relations related to some critical exponents for \( f_0 \)-avalanche and make predictions on the values of some exponents.

The quantity \( f \) is a global one of the ecosystem and can be expected to involve some general information about the whole system. It may represent the average popu-
lation or living capability of the whole species system. Larger \( f \) shows that the average population is immense or the average living capability is great, and vice versa. \( f \) is defined as

\[
\bar{f} = \frac{1}{L_d} \sum_{i=1}^{L_d^d} f_i,
\]

where \( f_i \) is the fitness of the \( i \)th species of a system consists of \( L^d \) species. Let BS model start to evolve. At each time step of the evolution, apart from the random numbers of the globally extremal site and its 2d nearest neighboring sites, the signal \( f \) is also tracked. Initially, \( \bar{f} \) tends to increase step-wisely. As the evolution continues further, \( \bar{f} \) approaches a critical value \( \bar{f}_c \) and remain statistically stable around \( \bar{f}_c \). The plot of \( f \) versus time step \( s \) shows that the increasing signals of \( f \) follow a Devil’s staircase [8], which implies that punctuated equilibrium emerges. Denote \( F(s) \) the gap of the punctuated equilibrium. Actually, \( F(s) \) tracks the peaks in \( \bar{f} \). After some careful derivation one can write down an exact gap equation [6,10]

\[
\frac{dF(s)}{ds} = \frac{\bar{f}_c - F(s)}{L^d(S)P(s)},
\]

where \( \langle S \rangle_{F(s)} \) denotes the average size of avalanches occurred during the gap \( F(s) \) when \( \bar{f} < F(s) \). This exact gap equation will be exactly solved in this paper.

Signals \( f(s) \) play important roles in defining \( f_0 \)-avalanche. For any value of the auxiliary parameter \( f_0 \) (0.5 < \( f_0 \) < 1.0), an \( f_0 \)-avalanche of size \( S \) is defined as a sequence of \( S - 1 \) successive events when \( f(s) < f_0 \) confined between two events when \( f(s) > f_0 \). This definition ensures that the mutation events during an avalanche are spatially and temporally correlated. It can also guarantee the hierarchical structure of the avalanches: larger avalanches consists of smaller ones. As \( f_0 \) is raised, smaller avalanches gather together and form bigger ones. The statistics of \( f_0 \) will inevitably have a cutoff if \( f_0 \) is not chosen to be \( \bar{f}_c \). This will not affect the size distribution provided that \( f_0 \) approaches \( \bar{f}_c \). Extensive simulations show that exponents \( \tau \) of \( f_0 \)-avalanche size distribution are 1.800 and 1.725 for 1D and 2D BS models respectively, amazingly different from the counterparts of the \( f_0 \)-avalanche, 1.07 and 1.245 [6]. This strengthens the speculation that \( f_0 \)-avalanche is a different type of avalanche, distinguished from any types of avalanches found previously.

Denote \( P(S, f_0) \) the probability of acquiring a \( f_0 \)-avalanche of size \( S \). The signals \( f(s) \) \( \langle f(s) < f_0 + df_0 \rangle \) will stop the \( f_0 \)-avalanches and not \( \langle f_0 + df_0 \rangle \)-avalanches. That is \( f_0 \) is raised by an infinitesimal amount \( df_0 \) some of \( f_0 \)-avalanches merge together to form bigger \( f_0 + df_0 \)-avalanches. This exhibits a hierarchical structure of \( f_0 \)-avalanches and will be prescribed by the below exact master equation. In some sense, the master equation reflects the "flow" of probability of avalanche size distribution with respect to the change in \( f_0 \).

Simulations show that \( f \) approaches \( \bar{f}_c \) and remain statistically stable in the critical state. This feature is greatly different from the feature of \( f_{min} \) (fitness of globally extremal site), which can vary between 0 and 1. While \( f \) in the critical state fluctuates slightly around \( \bar{f}_c \). Therefore, the \( f_0 \)-avalanches will have no good statistics if \( f_0 \) is chosen as the value far less than \( \bar{f}_c \), since there only exists smaller avalanches in the model. To acquire a better and reasonable distribution of \( f_0 \)-avalanches sizes, one should choose the value of \( f_0 \) under the condition \( f_0 \to \bar{f}_c \). It should be emphasized that the master equation listed below is valid also for \( f_0 \to \bar{f}_c \).

Both theoretical analysis and extensive simulations suggest that the signals \( f(s) \) which terminate \( f_0 \)-avalanches are uncorrelated and evenly distributed between \( (f_0, \bar{f}_c) \) provided that \( f_0 \to \bar{f}_c \). The direct consequence of this observation is that the probability of an \( f_0 \)-avalanche merging to \( f_0 + df_0 \)-avalanche is prescribed by \( \frac{df_0}{f_0 - f_0} \). It is important to note that any two subsequent avalanches are mutually independent for the following arguments to be true. In other words, the probability distribution of \( f_0 \)-avalanches, initiated immediately after the termination of an \( f_0 \)-avalanche of size \( S \) is independent of \( S \). This is true because in BS model the dynamics within an \( f_0 \)-avalanche is completely independent of the particular value of the signals \( f(s) > f_0 \) in the background that were left by the previous avalanches.

Here present the master equation. As \( f_0 \) is raised by an infinitesimal amount \( df_0 \), the probability "flowing" out of the size distribution of \( f_0 \)-avalanches is given by \( P(S, f_0) \left( \frac{df_0}{f_0 - f_0} \right) \), while the probability "flowing" into is given by \( \sum_{S=1}^{S-1} P(S, f_0) + P(S - S_1, f_0) \). Let \( f_0 \to \bar{f}_c \) and \( df_0 \to 0 \), one can write down the master equation as

\[
(\bar{f}_c - f_0) \frac{\partial P(S, f_0)}{\partial f_0} = -P(S, f_0)
\]

\[
+ \sum_{S=1}^{S-1} P(S_1, f_0) P(S - S_1, f_0).
\]

The first term on the right hand of the equation expresses the loss of avalanches of size \( S \) due to the merging with the subsequent one, while the second one describes the gain in \( P(S, f_0) \) due to merging of avalanches of size \( S_1 \) with avalanches of size \( S - S_1 \).

In order to investigate the exact master equation it is convenient to make some variable changes. Define \( h = -\ln(f_0 - f_0) \). Therefore, \( f_0 = \bar{f}_c \) corresponds to \( h = +\infty \). Since in the master equation \( f_0 \) is chosen to be close to \( \bar{f}_c \), \( h \) varies from a very large number to +\( \infty \). Due to the variable change the variable \( h \) is chosen from the distribution \( P(h) = e^{-h} \), which seems to be more "natural". In the following part we will use the new variable \( h \) instead of \( f_0 \). The master equation can be rewritten, in terms of \( h \), as

\[
\frac{dt}{dh} = \frac{\partial P(S, h)}{\partial h},
\]

where \( t \) is the time, and the argument of the probability density function \( P(S, h) \) is the size of the avalanche and the variable \( h \).
\[
\frac{\partial P(S,h)}{\partial h} = -P(S,h) + \sum_{S_i=1}^{S-1} P(S,h)P(S-S_i,h).
\] (4)

Making Laplace transformation of Eq. (4), after some calculation, one obtains

\[
\frac{\partial \ln(1-p(\beta,h))}{\partial h} = p(\beta,h),
\] (5)

where \(p(\beta,h) = \sum_{S=1}^{\infty} P(S,h)e^{-\beta S}\). This exact equation is the key one of this work. Many interesting physical features can be derived from it. As \(h < +\infty\) avalanches size will have a cutoff. The normalization of \(P(S,h)\) can be expressed as \(p(0,h) = \sum_{S=1}^{\infty} P(S,h) = 1\). Expanding both sides of Eq. (5) as Taylor series throughout a neighborhood of the point \(\beta = 0\), one can immediately obtain

\[
\frac{\partial}{\partial h} \left[ 1 - \langle S \rangle_h \beta + \frac{1}{2!} \langle S^2 \rangle_h \beta^2 - \frac{1}{3!} \langle S^3 \rangle_h \beta^3 + ... \right] = \left[ \langle S \rangle_h \beta - \frac{1}{2!} \langle S^2 \rangle_h \beta^2 + \frac{1}{3!} \langle S^3 \rangle_h \beta^3 + ... \right] \times
\]

\[
-1 + \langle S \rangle_h \beta - \frac{1}{2!} \langle S^2 \rangle_h \beta^2 + \frac{1}{3!} \langle S^3 \rangle_h \beta^3 + ...
\] . (6)

Since the equation (6) holds for arbitrary \(\beta\), comparing the coefficients of different powers of \(\beta\) in the above Taylor series gives an infinite series of exact equations. Comparison of the coefficients of \(\beta^1\) results in

\[
\frac{\partial \ln \langle S \rangle_h}{\partial h} = 1.
\] (7)

Eq. (7) is extremely interesting. Changing variable \(h\) back into \(\bar{f}_0\), one can obtain the "gamma" equation [6,11]

\[
\frac{d \ln \langle S \rangle_{\bar{f}_0}}{d \bar{f}_0} = \frac{1}{\bar{f}_c - \bar{f}_0}.
\] (8)

Inserting the scaling ansatz [6] \(\langle S \rangle_{\bar{f}_0} \sim (\bar{f}_c - \bar{f}_0)^{-\gamma}\) into Eq. (8), one immediately obtain an interesting result

\[
\gamma = 1
\] (9)

It should be noted that \(\gamma = 1\) is universal, that is, independent of the dimension. The value of \(\gamma\) for \(\bar{f}_0\)-avalanches is different from those for \(f_0\)-avalanche found in Ref. [6], which are 2.70 and 1.70 for 1D and 2D BS models respectively. Extensive simulations show \(\gamma = 0.99 \pm 0.01\) and \(\gamma = 0.98 \pm 0.01\) for 1D and 2D BS models respectively. Fig. (1) shows our simulation results, which confirms the universal result \(\gamma = 1\).

Higher powers of \(\beta\) gives new exact equations. Here present the first two

\[
\frac{\partial}{\partial h} \langle S^2 \rangle_h = 2\langle S \rangle_h;
\] (10)

\[
\frac{\partial}{\partial h} \langle S^3 \rangle_h - \frac{\langle S^2 \rangle_h^2}{2\langle S \rangle_h^2} = \langle S^2 \rangle_h.
\] (11)

Next present the solution of the exact gap equation for \(f_0\)-avalanches. Inserting the scaling relation \(\langle S \rangle_{f(s)} \sim (\bar{f}_c - F(s))^{-1}\) into the equation and integrating, one obtains

\[
\Delta \bar{f}(s) = \bar{f}_c - F(s) \sim \left( \frac{s}{L^D} \right)^{-\rho} = \left( \frac{s}{L^D} \right)^{-1},
\] (12)

where \(\rho\) is the exponent of relaxation to attractor [6]. Thus, we obtain \(\rho = 1\). Interestingly, \(\rho\) is also a universal exponent for all dimensional BS models. It shows that the critical point (\(\Delta \bar{f} = 0\)) is approached algebraically with an exponent \(-1\).

Up to now, we have obtained some exponents of corresponding physical properties of \(f_0\)avalanches: \(\tau\), avalanche size distribution [10], \(D\), avalanche dimension [10], \(\gamma\), average avalanche size [10], and \(\rho\), relaxation to attractor [6]. Recall another two exponents [6]: \(\nu\), \(\sigma\), which are defined as \(r_{co} \sim (\bar{f}_c - \bar{f}_0)^{-\nu}\) and \(S_{co} = (\bar{f}_c - \bar{f}_0)^{-\sigma}\). Here \(r_{co}\) and \(S_{co}\) are referred to as the cut-off of the spatial extent of avalanches (due to the limit system size) and that of the avalanche size (due to the fact that \(\bar{f}_0\) is not chosen as \(\bar{f}_c\)) respectively. It is natural to establish some scaling relations of these exponents for \(f_0\)-avalanches similar to those found in Ref. [6,12] for \(f_0\)-avalanches. Nevertheless, these two types of avalanches manifest similar fractal properties. Hence some common features should be shared by them. Integrating of the equation \(\langle S \rangle = \int \langle \bar{f}_c - \bar{f}_0 \rangle^\nu \cdot S \cdot d\bar{f}_0\) and the scaling \(\langle S \rangle \sim (\bar{f}_c - \bar{f}_0)^{-\gamma}\) result in

\[
\gamma = \frac{2 - \tau}{\sigma} = 1.
\] (13)

Due to the compactness [6] of avalanches, we have \(S_{co} \sim d_{co}^{-\sigma D} = (\Delta f)^{-\nu D}\), thus

\[
\nu = \frac{1}{\sigma D} = \frac{1}{(2 - \tau) D}.
\] (14)

Eqs. (11)-(12) establish scaling relations among the critical exponents, and they imply that the self-organization time to reach the critical state is independent of the initial configuration of the system. A system of size \(L\) reaches the stationary state when \(\Delta f(s) \sim L\). It can be inferred from Eqs. (11)-(12) that, if one chooses \(\tau\) and \(D\) as two independent exponents other exponents can be expressed in terms of them. Among the six exponents mentioned above, \(\tau\) and \(D\) can be numerically measured [10], \(\gamma\) and \(\rho\) can be analytically obtained, while \(\nu\) and \(\sigma\) are difficult to explore despite some methods measuring the corresponding exponents for \(f_0\)-avalanches were introduced in Ref. [13]. Therefore, we can rely on the scaling relations and values of the exponents obtained to predict the values of \(\nu\) and \(\sigma\). We predict \(\sigma = 0.2\) (1D) and 0.275 (2D), \(\nu = 2.04\) (1D) and 1.17 (2D).

Comparing \(f_0\)-avalanche with \(f_0\)-avalanche we find that the former is more readily to be treated. Two critical exponents can be analytically obtained and are found
to be universal for all dimensional BS models. Furthermore, the infinite hierarchy of exact equations and the exact gap equations, together with their solutions, provide exclusive investigation of the new type of avalanches. Another asset of $f_0$-avalanche is that it involves some information concerning the whole system. It can be concluded that $f_0$-avalanche does enable us to comprehend the complex system from an effective and different context. The weak point of this avalanche is that it loses some knowledge of individuals. It is still unknown how these individual features will matter. It is worthwhile to investigate the avalanche dynamics further in the future.

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Figure Captions

Fig. 1: The average size of avalanches $\langle S \rangle$ vs $(f_c - f_0)$ for (a) 1D and (b) 2D Bak-Sneppen evolution models. The asymptotic slope yields $\gamma = 0.99 \pm 0.01$ and 0.98 $\pm 0.01$ respectively.
Fig. 1