Zastavnyi Operators and Positive Definite Radial Functions

Tarik Faouzi\textsuperscript{a}, Emilio Porcu\textsuperscript{b,∗}, Moreno Bevilacqua\textsuperscript{c}, Igor Kondrashuk\textsuperscript{d}

\textsuperscript{a}Department of Statistics, Applied Mathematics Research Group, University of Bio Bio, Chile.
\textsuperscript{b}School of Mathematics and Statistics, University of Newcastle, UK. Department of Mathematics and University of Atacama, Copiapó, and Millennium Nucleus Center for the Discovery of Structures in Complex Data, Chile.
\textsuperscript{c}University of Valparaiso, Department of Statistics. Millennium Nucleus Center for the Discovery of Structures in Complex Data, Chile.
\textsuperscript{d}Grupo de Matemática Aplicada, Departamento de Ciencias Básicas, Universidad del Bío-Bío, Campus Fernando May, Av. Andres Bello 720, Casilla 447, Chillán, Chile.

Abstract
Positive definite functions are fundamental to many areas of applied mathematics, probability theory, spatial statistics and machine learning, amongst others. Motivated by a problem coming from the maximum likelihood estimation under fixed domain asymptotics, we consider a new operator acting on rescaled weighted differences between two members of the class $\Phi_d$ of positive definite radial functions. In particular, we study the positive definiteness of the operator for the Matérn, Generalized Cauchy and Generalized Wendland families. It turns out that proposed operator allows to govern differentiability at the origin, and to attain negative correlations.

Keywords: Completely Monotonic, Fourier Transforms, Positive Definite, Radial Functions

1. Introduction
Positive definite functions are fundamental to many branches of mathematics as well as probability theory, statistics and machine learning amongst others. There has been an increasing interest in positive definite functions in $d$-dimensional Euclidean spaces, and the reader is referred to [8], [18], [17], [21] and [20].

Let $d$ be a positive integer. This paper is concerned with the class $\Phi_d$ of continuous functions $\phi : [0, \infty) \mapsto \mathbb{R}$ such that $\phi(0) = 1$ and the function $x \mapsto \phi(||x||)$ is positive definite in $\mathbb{R}^d$. The class $\Phi_d$ is nested, with the strict inclusion relation:

$$\Phi_1 \supset \Phi_2 \supset \cdots \supset \Phi_d \supset \cdots \supset \Phi_\infty := \bigcap_{d \geq 1} \Phi_d.$$
The classes $\Phi_d$ are convex cones that are closed under product, non-negative linear combinations, and pointwise convergence. Further, for a given member $\phi$ in $\Phi_d$, the rescaled function $\phi(\cdot/\alpha)$ is still in $\Phi_d$ for any given $\alpha$. We make explicit emphasis on this fact because it will be repeatedly used subsequently.

For any nonempty set $A \subseteq \mathbb{R}^d$, we call $C(A)$ the set of continuous functions from $A$ into $\mathbb{R}$. For $p$ a positive integer, let $\theta \in \Theta \subset \mathbb{R}^p$ and let $\phi(\cdot; \theta)$ be a parametric family belonging to the class $\Phi_d$. For $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$ and $0 < \beta_1 < \beta_2$, we define the Zastavnyi operator $K_{\varepsilon; \theta; \beta_2, \beta_1}[\phi] : \Phi_d \mapsto C(\mathbb{R})$ by

$$
K_{\varepsilon; \theta; \beta_2, \beta_1}[\phi](t) = \frac{\beta_2 \phi\left(\frac{t}{\beta_2}; \theta\right) - \beta_1 \phi\left(\frac{t}{\beta_1}; \theta\right)}{\beta_2^2 - \beta_1^2}, \quad t \geq 0,
$$

with $K_{\varepsilon; \theta; \beta_2, \beta_1}[\phi](0) = 1$. Motivation for studying positive definiteness of the radial functions $\mathbb{R}^d \ni x \mapsto K_{\varepsilon; \theta; \beta_2, \beta_1}[\phi](||x||)$ arises from the study of the monotonicity of the so called microergodic parameter of specific parametric families $[5, 6]$ when studying the asymptotic properties of the maximum likelihood estimation under fixed domain asymptotics. The operator (1.1) is a generalization of the operator proposed in [16] where $\varepsilon$ is assumed to be positive. Our problem can be formulated as:

**Problem 1.1.** Let $d$ and $q$ be positive integers. Let $\phi(\cdot; \theta) \in \Phi_d$ with $\phi(0; \cdot) = 1$ and let $\theta \in \Theta \subset \mathbb{R}^q$ a parameter vector. Find the conditions on $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$ and $\theta$ such that $K_{\varepsilon; \theta; \beta_2, \beta_1}[\phi]$ belongs to the class $\Phi_n$ for some $n = 1, 2, \ldots$ for given $0 < \beta_1 < \beta_2$.

We first note that Problem 1.1 has at least two possible solutions. Indeed, direct inspection shows

$$
\lim_{\varepsilon \to +\infty} K_{\varepsilon; \theta; \beta_2, \beta_1}[\phi](t) = \phi\left(\frac{t}{\beta_1}; \theta\right), \quad \lim_{\varepsilon \to -\infty} K_{\varepsilon; \theta; \beta_2, \beta_1}[\phi](t) = \phi\left(\frac{t}{\beta_2}; \theta\right), \quad t \geq 0,
$$

where the convergence is pointwise in $t$.

The positive definiteness of (1.1), assuming $\varepsilon > 0$, has been studied in [16] when $\phi$ belongs to the Buhmann class [7]. An important special case of the Buhmann class is the the Generalized Wendland family [11]. For $\kappa > 0$, we define the class $\mathcal{GW} : [0, \infty) \to \mathbb{R}$ as:

$$
\mathcal{GW}(t; \kappa, \mu) = \begin{cases}
\int_{0}^{t} u^{(d-2)\kappa - 1}(1-u)^{\mu} du, & 0 \leq t < 1, \\
0, & t \geq 1,
\end{cases}
$$

and, for $\kappa = 0$, by continuity we have

$$
\mathcal{GW}(t; 0, \mu) = \begin{cases}
(1 - t)^{\mu}, & 0 \leq t < 1, \\
0, & t \geq 1,
\end{cases}
$$

The function $\mathcal{GW}(t; \kappa, \mu)$ is a member of the class $\Phi_d$ if and only if $\mu \geq 0.5(d + 1) + \kappa$ [22]. [16] found that if $\phi(\cdot; \theta) = \mathcal{GW}(\cdot; \kappa, \mu)$ and $\varepsilon > 0$ then $K_{\varepsilon; \kappa, \mu; \beta_2, \beta_1}[\mathcal{GW}](t)$ is positive definite if $\mu \geq (d + 7)/2 + \kappa$ and $\varepsilon \geq 2\kappa + 1$.

This paper is especially interested to the solution of Problem 1.1 when considering two celebrated parametric families:
The Matérn family: in this case \( \phi(\cdot; \theta) = \mathcal{M}(\cdot; \nu) \), with
\[
\mathcal{M}(t; \nu) = \frac{2^{1-\nu}}{\Gamma(\nu)} t^\nu \mathcal{K}_\nu(t), \quad t \geq 0,
\]
where \( \mathcal{K}_\nu \) is the modified Bessel function of the second kind of order \( \nu > 0 \) \([1]\). It is a member of the class \( \Phi_\infty \) \([19]\).

The Generalized Cauchy family: in this case \( \phi(\cdot; \theta) = \mathcal{C}(\cdot; \delta, \lambda) \), with
\[
\mathcal{C}(t; \delta, \lambda) = (1 + t^\delta)^{-\lambda/\delta}, \quad t \geq 0,
\]
where \( 0 < \delta < 2 \) and \( \lambda > 0 \). It is a member of the class \( \Phi_\infty \) \([12]\).

Additionally, we provide a solution when \( \phi(\cdot; \theta) = t \mathcal{W}(\cdot; \kappa, \mu)^\top \), \( \theta = (\kappa, \mu) \), assuming \( \varepsilon < 0 \).

To give an idea of how the operator \( K_{x; \theta; \beta_1, \beta_2}[\phi](t) \) acts on \( \phi(t, \theta) \) for given \( 0 < \beta_1 < \beta_2 \)
when \( \phi \) is the Matérn family, Figure 1(A) compares \( K_{x; 0.5, \beta_2, \beta_1}[\mathcal{M}](t) \) with \( \mathcal{M}(t/\beta_1; 0.5) \) and \( \mathcal{M}(t/\beta_2; 0.5) \) when \( \beta_1 = 0.075, \beta_2 = 0.15, \varepsilon = 1 \) (red line) and \( \varepsilon = -2 \) (blue line). We note that the behaviour at the origin of \( K_{1; 0.5, \beta_2, \beta_1}[\mathcal{M}](t) \) changes drastically with respect to the behaviour at the origin of \( \mathcal{M}(\cdot; 0.5) \). Moreover, \( K_{-2; 0.5, \beta_2, \beta_1}[\mathcal{M}](t) \) can attain negative values. It turns out from Theorem 3.1 that \( K_{1; 0.5, \beta_2, \beta_1}[\mathcal{M}](t) \in \Phi_\infty \) and \( K_{-2; 0.5, \beta_2, \beta_1}[\mathcal{M}](t) \in \Phi_2 \).

A similar graphical representation is given in Figure 1(B) when \( \phi \equiv \mathcal{C} \), the Cauchy family in \([15]\). In this case, we consider \( \varepsilon = -0.7, 1.25, \delta = 0.6, \lambda = 2.5, \beta_1 = 0.2, \beta_2 = 0.3 \).

Note that, under this setting, \( K_{-1.25, 0.6, 2.5, \beta_2, \beta_1}[\mathcal{C}](t) \) attains negative values as well. It turns out from Theorem 3.3 that \( K_{1; 2.5, 0.6, \beta_2, \beta_1}[\mathcal{C}](t) \in \Phi_\infty \) for \( \varepsilon = -0.7 \) and 1.25.

Finally, Figure 1(C) compares \( K_{x; 0.45, \beta_2, \beta_1}[\mathcal{W}](t) \) with \( \mathcal{W}(t/\beta_1; 0, 4.5) \) and \( \mathcal{W}(t/\beta_2; 0, 4.5) \) with \( \beta_2 = 0.6, \beta_1 = 0.4 \) when \( \varepsilon = 1 \) (red line) and \( \varepsilon = -2 \) (blue line). As for the Matérn case, the behaviour at the origin of \( K_{x; 0.45, \beta_2, \beta_1}[\mathcal{W}](t) \) changes neatly with respect to the behaviour at the origin of \( \mathcal{W}(\cdot; 0, 4.5) \). Moreover \( K_{-2; 0.45, \beta_2, \beta_1}[\mathcal{W}](t) \) can reach negative values. It turns out from Theorem 3.2 that \( K_{-2; 0.45, \beta_2, \beta_1}[\mathcal{W}](t) \) belongs to \( \Phi_2 \).

These three examples show that operator \( 1.1 \) can change substantially the features of given families \( \phi \) in terms of both differentiability at the origin and negative correlations.

The solution of Problem 1.1 for the Matérn and Cauchy families, passes necessarily through the specification of the properties of the radial Fourier transforms of the radially symmetric functions \( \mathcal{M}(||\cdot||; \nu) \) and \( \mathcal{C}(||\cdot||; \delta, \lambda) \) in \( \mathbb{R}^d \). For the Matérn family, such a Fourier transform is available in closed form. For the Generalized Cauchy family we obtain the Fourier transform as series expansions generalizing a recent result obtained by \([14]\). The plan of the paper is the following: Section 2 contains the necessary preliminaries and background. Section 3 gives the main results of this paper.

2. Preliminaries

We start with some expository material. A function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is called positive definite if, for any collection \( \{a_k\}_{k=1}^N \subset \mathbb{R} \) and points \( x_1, \ldots, x_N \in \mathbb{R}^d \), the following holds:
\[
\sum_{k=1}^N \sum_{h=1}^N a_k f(x_k - x_h) a_h \geq 0.
\]

November 26, 2018
By Bochner’s theorem, continuous positive definite functions are the Fourier transforms of positive bounded measures, that is

\[ f(x) = \int_{\mathbb{R}^d} e^{i\langle x, z \rangle} F(dz), \quad x \in \mathbb{R}^d, \tag{2.1} \]

where \( \langle \cdot, \cdot \rangle \) denotes inner product and where \( i \) is the complex number such that \( i^2 = -1 \). Additionally, if \( f(x) = \phi(\|x\|) \) for some continuous function defined on the positive real line, Schoenberg’s theorem [8] with the references therein shows that \( f \) is positive definite if and only if its radial part \( \phi \) can be written as

\[ \phi(t) = \int_{[0, \infty)} \Omega_d(t\xi)G_d(d\xi), \quad t \geq 0, \tag{2.2} \]

where \( G_d \) is a positive and bounded measure, and where

\[ \Omega_d(t) = t^{-(d-2)/2} J_{(d-2)/2}(t), \quad t \geq 0, \]

where \( J_\nu \) defines a Bessel function of order \( \nu \). If \( \phi(0) = 1 \), then \( G_d \) is a probability measure [8]. Classical Fourier inversion in concert with Bochner’s theorem shows that the function \( \phi \) belongs to the class \( \Phi_d \) if and only if it admits the representation (2.2), and this in turn happens if and only if the function \( \hat{\phi}_d : [0, \infty) \to [0, \infty) \), defined through

\[ \hat{\phi}_d(z) := \mathcal{F}_d[\phi(t)](z) = \frac{z^{1-d/2}}{(2\pi)^d} \int_0^\infty u^{d/2} J_{d/2-1}(uz)\phi(u)du, \quad z \geq 0, \tag{2.3} \]

where \( \mathcal{F}_d[\cdot](z) \) denotes the Fourier transform of the function \( \phi \).
is nonnegative and such that \( \int_{[0,\infty)} \hat{\phi}_d(t)t^{d-1}dt < \infty \). Note that we intentionally put a subscript \( d \) into \( G_d \) and \( \hat{\phi}_d \) to emphasize the dependence on the dimension \( d \) corresponding to the class \( \Phi_d \) where \( \phi \) is defined. This is explicitly stated in \cite{8}, where it is explained that for any member of the class \( \Phi_d \) there exists at least \( G_1, \ldots, G_d \) nonnegative bounded measures in the representation (2.2). Hence the term \( d \)-Schoenberg measures proposed therein. Finally, a convergence argument as much as in Schoenberg \cite{18} shows that the series representation given in in \cite{14} is valid without any restriction on the parameters.

Theorem 2.1. Let \( C(\cdot; \delta, \lambda) \) be the Generalized Cauchy covariance function as defined in Equation (1.5). Then, it is true that

\[
\hat{C}_{d,\beta}(z; \delta, \lambda) = \frac{z^{-d}}{\pi^{d/2}\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda+n)\Gamma(d/2-(\lambda+n)\delta/2)}{n! \Gamma((\lambda+n)\delta/2)} \left( \frac{z\beta}{2} \right)^{2n+d},
\]

(2.7)

with \( z > 0 \), where \( \delta \in (0,2) \) and \( \lambda > 0 \).
Proof. We use the Mellin-Barnes transformation, which is given by

\[
\frac{1}{(1 + x)^\alpha} = \frac{1}{2\pi i} \frac{1}{\Gamma(\alpha)} \int_C du \; x^u \Gamma(-u) \Gamma(\alpha + u),
\]  

(2.8)

here \(\Gamma(\cdot)\) is a Gamma function. This representation is valid for any \(x \in \mathbb{R}\). The contour \(C\) contains the vertical line which passes between left and right poles in the complex plane \(u\) from negative to positive imaginary infinity, and should be closed to the left in case \(x > 1\), and to the right complex infinity if \(0 < x < 1\).

We now proceed to compute \(\hat{C}_{d,\beta}\) as follows,

\[
\hat{C}_{d,\beta}(\|z\|; \delta, \lambda) = \beta^d \mathcal{F}_d[C_{d,\beta}(t; \delta, \lambda)](\|z\|) = \beta^d \mathcal{F}_d[C_d(t; \delta, \lambda)](\beta \|z\|).
\]

Applying Equation (2.8), we obtain

\[
\hat{C}_{d,\beta}(\|z\|; \delta, \lambda) = \frac{\beta^d}{(2\pi)^d} \frac{1}{\Gamma(\lambda)} \int_{\mathbb{R}^d} e^{i\beta(z,x)} \left( \int_C du \; \Gamma(-u) \Gamma(u + \lambda) \|x\|^{u\delta} \right) dx
\]

\[
= \frac{\beta^d}{(2\pi)^d} \frac{1}{\Gamma(\lambda)} \int_C du \; \Gamma(-u) \Gamma(u + \lambda) \int_{\mathbb{R}^d} e^{i\beta(z,x)} \|x\|^{u\delta} dx.
\]

We now invoke the well known relationship [2],

\[
\int_{\mathbb{R}^d} e^{i(z,x)} \|x\|^{u\delta} dx = \frac{2^{d+u\delta} \pi^{d/2} \Gamma(d/2 + u\delta/2)}{\Gamma(-u\delta/2) \|z\|^{d+u\delta}},
\]

and, by abuse of notation, we now write \(z = \|z\|\). We have

\[
\hat{C}_{d,\beta}(z; \delta, \lambda) = \frac{\beta^d}{(2\pi)^d} \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_C du \; \Gamma(-u) \Gamma(u + \lambda) \frac{2^{d+u\delta} \pi^{d/2} \Gamma(d/2 + u\delta/2)}{\Gamma(-u\delta/2)} \frac{1}{(\beta z)^{d+u\delta}}
\]

\[
= z^{-d} \frac{1}{\pi^{d/2} \Gamma(\lambda)} \frac{1}{2\pi i} \int_C du \; \Gamma(-u) \Gamma(u + \lambda) \Gamma(d/2 + u\delta/2) \left( \frac{2}{z\beta} \right)^{u\delta}.
\]

(2.9)

For any given value of \(|2/z\beta|\), it does not matter whether it is smaller or greater than 1. In fact, we may close the contour to the left complex infinity. This is a convergent series for any values of the variable \(z\) because we have a situation when denominator supresses numerator in the coefficient infront of powers of \(z\). We now observe that the functions \(u \mapsto \Gamma(\lambda + u)\) and \(u \mapsto \Gamma(\lambda + u)\) contain poles in the complex plane, respectively when \(\lambda + u = -n\), and when \(d/2 + u\delta/2 = -n\), \(n \in \mathbb{N}\). Using this fact and through direct inspection we obtain that the right hand side in (2.9) matches with (2.7). The proof is completed.

\[\square\]

3. Main Results

We start by providing a solution to Problem 1.1 when \(\phi(\cdot; \theta) = \mathcal{M}(\cdot; \nu), \nu > 0\).
**Theorem 3.1.** Let $\mathcal{M}(\cdot; \nu)$ be the Matérn function as defined in Equation (1.4). Let $K_{\beta_1, \beta_2, \beta_1}[\mathcal{M}]$ with $0 < \beta_1 < \beta_2$, be the Zastavnyi operator (1.1) related to the function $\mathcal{M}(\cdot; \nu)$.

1. Let $\varepsilon > 0$. Then, $K_{\varepsilon, \beta_2, \beta_1}[\mathcal{M}] \in \Phi_\infty$ if and only if $\varepsilon \geq 2\nu > 0$.
2. For a given $d \in \mathbb{N}$, let $\varepsilon < 0$. Then, $K_{\varepsilon, \beta_2, \beta_1}[\mathcal{M}] \in \Phi_d$ if and only if $\varepsilon \leq -d < 0$.

**Proof.** We give a constructive proof. The operator $K_{\varepsilon, \beta_2, \beta_1}$ maps $\mathcal{M}$ into $\Phi_d$ if and only if

$$F_d[K_{\varepsilon, \beta_2, \beta_1}[\mathcal{M}](t)](z) \geq 0, \quad z \in [0, \infty).$$

If the last is true for any $d \in \mathbb{N}$, then $K_{\varepsilon, \beta_2, \beta_1}[\mathcal{M}]$ belongs to $\Phi_\infty$ and this fact comes by definition of the class $\Phi_\infty$. Direct inspection shows that

$$F_d[K_{\varepsilon, \beta_2, \beta_1}[\mathcal{M}](t)](z) = \frac{\beta_2 \tilde{M}_{d, \beta_2}(z; \nu) - \beta_1 \tilde{M}_{d, \beta_1}(z; \nu)}{\beta_2 - \beta_1} \geq 0, \quad \forall z > 0. \quad (3.1)$$

Let $\varepsilon > 0$. Then, $\beta_2 - \beta_1 > 0$ and (3.1) is true if and only if the mapping $0 \leq x \mapsto x^\varepsilon \tilde{M}_{d, \nu}(z; \nu)$ is nondecreasing. Direct inspection of the derivative shows that it is true if and only if $\varepsilon \geq 2\nu$. Now let $\varepsilon < 0$. Then $\beta_2 - \beta_1 < 0$ and (3.1) is true if and only if $x \mapsto x^\varepsilon \tilde{M}_{d, \nu}(z; \nu)$ is nonincreasing. Direct inspection of the derivative shows that it is true if and only if $\varepsilon \leq -d$. \hfill \Box

The following result gives some conditions for the solution of Problem 1.1 when $\phi(\cdot; \theta) = \mathcal{GW}(\cdot; \mu, \kappa)$.

**Theorem 3.2.** Let $d$ be a positive integer. Let $\mathcal{GW}(\cdot; \mu, \kappa)$ be the GW function as defined in Equations (1.2) and (1.3) for $\kappa > 0$ and $\kappa = 0$ respectively. Let $K_{\varepsilon, \mu, \kappa, \beta_2, \beta_1}[\mathcal{GW}]$ with $0 < \beta_1 < \beta_2$, be the Zastavnyi operator (1.1) related to the function $\mathcal{GW}(\cdot; \mu, \kappa)$. Then:

1. If $\varepsilon \geq 2\kappa + 1 > 0$ and $\mu \geq (d + 1)/2 + \kappa$, then $K_{\varepsilon, \mu, \kappa, \beta_2, \beta_1}[\mathcal{GW}] \in \Phi_d$.
2. $K_{\varepsilon, \mu, \kappa, \beta_2, \beta_1}[\mathcal{GW}] \in \Phi_d$ if and only if $\varepsilon = 2\kappa + 1$ and $\mu \geq (d + 7)/2 + \kappa$.
3. If $\varepsilon \leq -d < 0$ and $\mu \geq (d + 7)/2 + \kappa$, then $K_{\varepsilon, \mu, \kappa, \beta_2, \beta_1}[\mathcal{GW}] \in \Phi_d$.
4. $K_{-d, \mu, \kappa, \beta_2, \beta_1}[\mathcal{GW}] \in \Phi_d$ if and only if $\mu \geq (d + 1)/2 + \kappa$.

**Proof.** Points 1 and 2 are a corollary of Theorem 3.1 in [16]. To prove Point 3, let $\varepsilon < 0$. From Proposition 1 in [16] $K_{\varepsilon, \mu, \kappa, \beta_2, \beta_1}[\mathcal{GW}] \in \Phi_d$ if and only if the mapping

$$0 \leq x \mapsto \frac{2\kappa - \varepsilon - 1 - (d + \varepsilon)x^2}{x^\mu(1 + x^2)^{\frac{d+1}{2}+\kappa+1}} \quad (3.2)$$

is completely monotonic on the positive real line. From (10) we have that $x^{-\mu}(1+x^2)^{-\nu}$ is completely monotonic if and only if $\mu \geq \nu$, for $\nu \geq 1$. Then, (3.2) is true if $\varepsilon \leq -d < 0$ and $\mu \geq (d + 7)/2 + \kappa$. Point 4 can be verified easily by applying (3.2) and assuming $\varepsilon = -d$. The proof is completed. \hfill \Box
We now assume that for a fixed $d$, $\delta \lambda > d$. This condition is necessary to ensure integrability of Generalized Cauchy covariance functions, and hence the related spectral density to be bounded. The following result provides a solution to Problem 1.1 when $\phi(\cdot; \theta) = C(\cdot; \delta, \lambda)$ for $0 < \delta < 2$.

**Theorem 3.3.** Let $C(\cdot; \delta, \lambda)$ for $0 < \delta < 2$ be the Generalized Cauchy function as defined in Equation (1.5). Let $K_{x; \delta; \lambda; \beta_2; \beta_1}[C]$ with $0 < \beta_1 < \beta_2$, be the Zastavnyi operator (1.1) related to the function $C(\cdot; \delta, \lambda)$. Then,

1. let $\varepsilon > 0$. If $\varepsilon \geq \delta > 0$ and $\delta < 1$, then $K_{x; \delta; \lambda; \beta_2; \beta_1}[C] \in \Phi_{\infty}$;
2. let $\varepsilon < 0$. $K_{x; \delta; \lambda; \beta_2; \beta_1}[C] \in \Phi_{\infty}$ if and only if $\varepsilon \leq -\delta \lambda < 0$.

**Proof.** We give a constructive proof. We need to find conditions on $\theta = (\delta, \lambda)$ such that $K_{\theta}[C] \in \Phi_{\infty}$. This is equivalent to the fact that

$$0 \leq z \mapsto \beta_2\hat{\mathcal{C}}_{d, \beta_2}(z; \delta, \lambda_2) - \beta_1\hat{\mathcal{C}}_{d, \beta_1}(z; \delta, \lambda_1)$$

has the same sign of $\varepsilon$.

Point 2. is equivalent to find the condition for which $\beta^d\hat{\mathcal{C}}_{d, \beta}(z; \delta, \lambda)$ is nondecreasing with respect to $\beta$. Using (2.6), we write

$$\beta^d\hat{\mathcal{C}}_{d, \beta}(z; \delta, \lambda) = \frac{\beta^d}{2^d-1\frac{\pi}{2}+1} \Gamma \left(\int_0^\infty \frac{K_{(d-2)/2}(\beta z t^{2/3})(1+it)^{\lambda}}{(1+it)^{\lambda}} dt\right)
= \frac{\beta^d}{2^d-1\frac{\pi}{2}+1} \Gamma \left(\int_0^\infty \frac{K_{(d-2)/2}(\beta z t^{2/3})(1+it)^{\lambda}}{(1+it)^{\lambda}} dt\right)
= \beta^d\hat{\mathcal{C}}_{d, 1}(z; \delta, \lambda).$$

Then, $\beta^d\hat{\mathcal{C}}_{d, 1}(z; \delta, \lambda)$ is nondecreasing with respect to $\beta$ if and only if $(z\beta)^+d\hat{\mathcal{C}}_{d, 1}(z; \delta, \lambda)$ is nondecreasing with respect to $\beta$ for all $z > 0$. Thus, an equivalent condition is that $z \mapsto z^{+d}\hat{\mathcal{C}}_{d, 1}(z; \delta, \lambda)$ is nondecreasing. Using Theorem 2.1, under the condition $\delta \lambda > d$, we obtain

$$z^{+d}\hat{\mathcal{C}}_{d, 1}(z; \delta, \lambda) = \frac{z^\varepsilon}{\pi^d/2 \Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\lambda+n)\Gamma(d/2-\lambda+n)\delta/2)}{n!} \left(\frac{\lambda+n}{2}\right) \Gamma \left(\frac{2n+d}{\delta}\right)$$

Taking derivative on both sides of the last equation with respect to $z$, we get the following formula:

$$\frac{d}{dz} \left( z^{+d}\hat{\mathcal{C}}_{d, 1}(z; \delta, \lambda) \right) = \frac{z^{\varepsilon-1}}{\pi^d/2 \Gamma(\lambda)} \sum_{n=0}^{\infty} (\varepsilon+\lambda+n)\delta \left(\frac{-1)^n \Gamma(\lambda+n)\Gamma(d-\lambda+n)\delta/2)}{n! (\lambda+n)\delta/2} \right) z^{\lambda+n}\delta
+ 2z^{\varepsilon-1} \frac{\pi^d/2 \Gamma(\lambda)}{\Gamma(\lambda)} \sum_{n=0}^{\infty} (\varepsilon+d+2n) \left(\frac{-1)^n \Gamma(d+2n)\Gamma(\lambda-(d+2n)\delta/2}{n! (d+2n)\delta/2} \right) \frac{1}{2^{d+2n} z^{2n+d}}.$$
Simple yet tedious calculus leads to the following equality:

\[
\frac{d}{dz} \left( z^{\varepsilon + \delta} \tilde{C}_{d,1}(z; \delta, \lambda) \right)^{(1)} = z^{\varepsilon - 1} \left( (\varepsilon + \delta \lambda) \tilde{C}_{d,1}(z; \delta, \lambda) - \delta \lambda \tilde{C}_{d,1}(z; \delta, \lambda + 1) \right).
\]

Then, if \( \varepsilon \leq -\delta \lambda \), we have

\[
\frac{d}{dz} \left( z^{\varepsilon + \delta} \tilde{C}_{d,1}(z; \delta, \lambda) \right) \leq 0,
\]

which implies that \( K_{\varepsilon, \delta, \lambda; 0, \varepsilon} [C] \) is nondecreasing.

Showing Point 1. is equivalent to show that \((\varepsilon + \delta \lambda) \tilde{C}_{d,1}(z; \delta, \lambda) - \delta \lambda \tilde{C}_{d,1}(z; \delta, \lambda + 1)\) is nonnegative for all \( z > 0 \). This is in turn equivalent to the fact that its Fourier transform is positive definite (because integrability of \((\varepsilon + \delta \lambda) \tilde{C}_{d,1}(z; \delta, \lambda) - \delta \lambda \tilde{C}_{d,1}(z; \delta, \lambda + 1)\) is trivially attained), that is

\[
\mathcal{F}_d \left( (\varepsilon + \delta \lambda) \tilde{C}_{d,1}(z; \delta, \lambda) - \delta \lambda \tilde{C}_{d,1}(z; \delta, \lambda + 1) \right)(r) = \frac{\varepsilon + \delta \lambda}{(1 + r^\delta)^\lambda} - \frac{\delta \lambda}{(1 + r^\delta)^{\lambda + 1}} =: f(r; \theta'),
\]

where \( \theta' = (\varepsilon, \delta, \lambda) \).

Now, we discuss the range of the parameter \( \varepsilon \) for which the function \( f(r; \theta') \) is positive definite. A simple calculation leads to the following result:

\[
f(r; \theta') = \frac{\varepsilon + \delta \lambda}{(1 + r^\delta)^\lambda} - \frac{\delta \lambda}{(1 + r^\delta)^{\lambda + 1}} = \frac{(\varepsilon + \delta \lambda) r^\delta + \delta \lambda}{(1 + r^\delta)^{\lambda + 1}}, \quad r > 0.
\]

Applying Bernstein Theorem [9], we have that the right hand side is positive definite on \( \mathbb{R}^d \) if and only if \( f(r^{1/2}) \) is a completely monotonic function in Equation (2.4). Furthermore, \( f(r^{1/2}; \theta') \) is positive and \( C^\infty \)-function, where \( C^\infty \)-function is a set of functions being infinitely differentiable on \( \mathbb{R} \). Thus, it is completely monotone if and only if \(-f'(r^{1/2}; \theta') \) is completely monotone on the positive real line.

Direct inspection shows that

\[
-f^{(1)}(r^{1/2}; \theta') = \delta \lambda \left\{ \frac{(\varepsilon - \delta) r^{\delta/2 - 1}}{(1 + r^{\delta/2})^{\lambda + 1}} + \frac{(\varepsilon + \delta \lambda) r^{\delta - 1}}{(1 + r^{\delta/2})^{\lambda + 1}} \right\}, \quad r > 0.
\]

To show that the right hand side of Equation (3.4) is completely monotone, we need theorem 3 of [9]: a positive function \( g_{\mu, \lambda, \delta} : [0, \infty) \rightarrow \mathbb{R}_+ \) defined as

\[
g_{\mu, \lambda, \delta}(r) = \frac{1}{r^\mu (1 + r^{\delta/2})^\lambda}, \quad r > 0,
\]

is completely monotone if \( \mu \geq 0, \lambda > 0 \) and \( 0 \leq \delta < 2 \).

Invoking (3.5), we can thus conclude that \( f(r^{1/2}; \theta') \) is completely monotone if \( \delta \leq 1 \) and \( \varepsilon \geq \delta \).

November 26, 2018
Two technical lemmas are needed prove our last result.

**Lemma 3.1.** Let $\mathcal{K}_\nu : [0, \infty) \to \mathbb{R}$ be the function defined through (1.4). Let $\nu > 0$. Then, for all $z > 0$,

1. $\lim_{z \to +\infty} z \frac{\mathcal{K}'_\nu(z)}{\mathcal{K}_\nu(z)} = -\infty$, for all $\nu \in (-\infty, +\infty)$;
2. $\lim_{z \to +0} z \frac{\mathcal{K}'_\nu(z)}{\mathcal{K}_\nu(z)} = -\nu$, for $\nu > 1$.

**Proof.** To prove the two assertions, it is enough to use the following result [3],

$$\sqrt{\frac{\nu}{\nu - 1}}z^2 + \nu^2 < z \frac{\mathcal{K}'_\nu(z)}{\mathcal{K}_\nu(z)} < -\sqrt{z^2 + \nu^2}, \quad (3.6)$$

where the left hand side of (3.6) is true for all $\nu > 1$, and the right hand side holds for all $\nu \in \mathbb{R}$. \hfill \Box

**Lemma 3.2.** Let $\mathcal{C}(\cdot; 2, \lambda)$ be the Cauchy correlation function as defined at (1.5). Let $\beta > 0$. Then, for $d > \lambda + 2$ and $\epsilon < -\lambda$ the following assertions are equivalent:

1. $\beta \hat{\mathcal{C}}_{d, \beta}(z; 2, \lambda)$ is decreasing with respect to $\beta$ on $[0, +\infty)$;
2. $\beta^{\epsilon + d/2 + \lambda/2} \mathcal{K}_{d-\lambda}(\beta)$ is decreasing with respect to $\beta$ on $[0, +\infty)$;
3. $(\epsilon + \frac{d + \lambda}{2}) + \beta \frac{\mathcal{K}'_{d-\lambda}(\beta)}{\mathcal{K}_{d-\lambda}(\beta)} < 0$, $\beta \in [0, \infty)$.

**Proof.** Showing that $\beta^{\epsilon} \hat{\mathcal{C}}_{d, \beta}(z; 2, \lambda)$ is decreasing with respect to $\beta$ is the same as showing that $\beta^{\epsilon + \lambda/2 + d/2} \mathcal{K}_{d-\lambda}(\beta)$ is decreasing. Point 2 of Lemma 3.2 holds if and only if

$$\beta^{\epsilon + \frac{d + \lambda}{2} - 1} \left( (\epsilon + \frac{d + \lambda}{2}) + \beta \frac{\mathcal{K}'_{d-\lambda}(\beta)}{\mathcal{K}_{d-\lambda}(\beta)} \right) < 0.$$

Applying point 2 of Lemma 3.1 and the fact that $\beta \mathcal{K}'_{d-\lambda}(\beta)/\mathcal{K}_{d-\lambda}(\beta)$ is decreasing with respect to $\beta$, the three assertions of Lemma 3.2 are true if $d > \lambda + 2$ and $\epsilon < -\lambda$. The proof is completed. \hfill \Box

We are now able to fix a solution to Problem 1.1 when $\phi(\cdot; \nu) = \mathcal{C}(\cdot; 2, \lambda)$.

**Theorem 3.4.** Let $\mathcal{C}(\cdot; 2, \lambda)$ be the Cauchy function as defined in Equation (1.5) and let $K_{\epsilon; 2, \lambda; \beta_2, \beta_1}[\mathcal{C}]$ with $0 < \beta_1 < \beta_2$ be the Zastavnyi operator (1.1) related to the function $\mathcal{C}(\cdot; 2, \lambda)$. Then, for $\lambda < d - 2$, $K_{\epsilon; 2, \lambda; \beta_2, \beta_1}[\mathcal{C}] \in \Phi_d$ provided $\epsilon < -\lambda$.

**Proof.** We need to find conditions such that $K_{\epsilon; 2, \lambda; \beta_2, \beta_1}[\mathcal{C}] \in \Phi_d$. This is equivalent to the following condition:

$$\beta_{\epsilon 1}^{\beta} \hat{\mathcal{C}}_{d, \beta_1}(z; 2, \lambda) - \beta_{\epsilon 2}^{\beta} \hat{\mathcal{C}}_{d, \beta_2}(z; 2, \lambda) \geq 0.$$

Thus, we need to prove that the function $\beta^{\epsilon} \hat{\mathcal{C}}_{d, \beta}(z; 2, \lambda)$ is decreasing with respect to $\beta$. Using Lemma 3.2, we have that $K_{\epsilon; 2, \lambda; \beta_2, \beta_1}[\mathcal{C}] \in \Phi_d$. \hfill \Box

November 26, 2018
Acknowledgements

Partial support was provided by Millennium Science Initiative of the Ministry of Economy, Development, and Tourism, grant "Millenium Nucleus Center for the Discovery of Structures in Complex Data" for Moreno Bevilacqua and Emilio Porcu, by FONDECYT grant 1160280, Chile for Moreno Bevilacqua and by FONDECYT grant 1130647, Chile for Emilio Porcu and by grant Diubb 170308 3/I from the university of Bio Bio for Tarik Faouzi. Tarik Faouzi and Igor Kondrashuk thank the support of project DIUBB 172409 GI/C at University of Bio-Bio. The work of I.K. was supported in part by Fondecyt (Chile) Grants No. 1121030 and by DIUBB (Chile) Grant No. 1814093/R.

References

References

[1] Abramowitz, M. and Stegun, I. A., editors (1970). Handbook of Mathematical Functions. Dover, New York.

[2] Allendes, P., Kniehl, B. A., Kondrashuk, I., Notte-Cuello, E. A., and Rojas-Medar, M. (2013). Solution to bethe–salpeter equation via mellin–barnes transform. Nuclear Physics B, 870(1):243–277.

[3] Baricz, Á., Ponnusamy, S., and Vuorinen, M. (2011). Functional inequalities for modified Bessel functions. Expositiones Mathematicae, 29(4):399–414.

[4] Berg, C., Mateu, J., and Porcu, E. (2008). The Dagum family of isotropic correlation functions. Bernoulli, 14(4):1134–1149.

[5] Bevilacqua, M. and Faouzi, T. (2018). Estimation and prediction of Gaussian processes using Generalized Cauchy covariance model under fixed domain asymptotics. ArXiv e-prints.

[6] Bevilacqua, M., Faouzi, T., Furrer, R., and Porcu, E. (2018). Estimation and prediction using Generalized Wendland functions under fixed domain asymptotics. The Annals of Statistics. Forthcoming.

[7] Buhmann, M. D. (2001). A new class of radial basis functions with compact support. Math. Comput., 70(233):307–318.

[8] Daley, D. J. and Porcu, E. (2014). Dimension walks and Schoenberg spectral measures. Proc. Amer. Math. Soc., 142:1813–1824.

[9] Feller, W. (1968). An introduction to Probability Theory and its Applications, volume 1. Wiley, New York.

[10] Fields, J. L. and Ismail, M. E. H. (1975). On the positivity of some \( \Gamma_2 \)'s. SIAM Journal on Mathematical Analysis, 6:551–559.

[11] Gneiting, T. (2002). Compactly supported correlation functions. Journal of Multivariate Analysis, 83:493–508.

[12] Gneiting, T. and Schlather, M. (2004). Stochastic Models that Separate Fractal Dimension and Hurst Effect. SIAM Rev., 46(2):269–282.

[13] Lim, S. and Teo, L. (2009). Gaussian fields and Gaussian sheets with Generalized Cauchy covariance structure. Stochastic Processes and Their Applications, 119(4):1325–1356.

[14] Lim, S. C. and Teo, L. P. (2010). Analytic and Asymptotic Properties of Multivariate Generalized Linnik’s Probability Densities. Journal of Fourier Analysis and Applications, 16(5):715–747.

[15] Moak, D. S. (1987). Completely monotonic functions of the form \( s^{-b} (s^2 + 1)^{-a} \). The Rocky Mountain Journal of Mathematics, pages 719–725.

[16] Porcu, E., Zastavnyi, P., and Bevilacqua, M. (2017). Buhmann covariance functions, their compact supports, and their smoothness. Dolomites Research Notes on Approximation, 10:33–42.

[17] Schaback, R. (2011). The missing Wendland functions. Advances in Computational Mathematics, 34(1):67–81.

November 26, 2018
[18] Schoenberg, I. J. (1938). Metric spaces and completely monotone functions. *Annals of Mathematics*, pages 811–841.

[19] Stein, M. L. (1999). *Interpolation of Spatial Data. Some Theory of Kriging*. Springer, New York.

[20] Wendland, H. (1995). Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree. *Advances in Computational Mathematics*, 4:389–396.

[21] Wu, Z. (1995). Compactly supported positive definite radial functions. *Advances in Computational Mathematics*, 18(4):283–292.

[22] Zastavnyi, V. and Trigub, R. (2002). Positive definite splines of special form. *English transl. in Sb. Math.*, 193:1771–1800.