FINITE DIFFERENCE/ELEMENT METHOD FOR TIME-FRACTIONAL NAVIER-STOKES EQUATIONS

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Abstract. We apply a composite idea of semi-discrete finite difference approximation in time and Galerkin finite element method in space to solve the Navier-Stokes equations with Caputo derivative of order $0 < \alpha < 1$. The stability properties and convergence error estimates for both the semi-discrete and fully discrete schemes are obtained. Numerical example is provided to illustrate the validity of theoretical results.

Keywords: Time-fractional Navier-Stokes equations; finite difference approximation; finite element method; error estimates; numerical examples

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1. Introduction

In this paper, we study the following Navier-Stokes equations with time-fractional derivative in a bounded subset of $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial \Omega$:

$$
\begin{aligned}
C^D_t^\alpha u + (u \cdot \nabla)u - \nu \Delta u + \nabla p &= f, \quad \text{div } u = 0, \quad \forall (x, t) \in \Omega \times (0, T], \\
u u|_{\partial \Omega} &= 0, \quad t \in (0, T], \\
u(x, 0) &= u_0, \quad x \in \Omega,
\end{aligned}
$$

(1.1)

where $C^D_t^\alpha$ represents the Caputo-type fractional derivative of order $\alpha \in (0, 1)$, $u = (u_1(x, t), u_2(x, t))$ denotes the velocity field at a point $x \in \Omega$ and $t \in [0, T]$, $\nu > 0$ is viscosity coefficient, $p = p(x, t)$ represents the pressure field, $f = f(x, t)$ is the external force and $u_0 = u_0(x)$ is the initial velocity.

Notice that the problem (1.1) reduces to the classical Navier-Stokes equations (NSEs) for $\alpha = 1$. The existence and non-existence of solutions for the NSEs have been discussed in [1]. Chemin et al.[2] studied the global regularity for the large solutions to the NSEs. Miura [3] focused on the uniqueness of mild solutions to the NSEs. Germain [4] presented the uniqueness criteria for the solutions of the Cauchy problem associated to the NSEs. The existence of global weak solutions for supercritical NSEs was discussed [5]. The lower bounds on blow up solutions for the NSEs in homogeneous Sobolev spaces were studied in [6]. The numerical methods for solving the NSEs have been investigated by many authors [7,8,9,10,11,12,23]. The study of time-fractional Navier-Stokes equations (TFNSEs) has become a hot topic of research due to its significant role in simulating the anomalous diffusion in fractal media. There are also some analytical methods available for solving the TFNSEs. Momani and Odibat [13] applied Adomian decomposition method to obtain the analytical solution of the TFNSEs. In [14, 15], the homotopy perturbation (transform) method was used to find the analytical solution of the TFNSEs. Wang and Liu [16] solved TFNSEs by applying the transform methods. Concerning...
the existence of global and local mild solutions to TFNSEs, see Carvalho-Neto and Gabriela [17], Zhou and Peng [18]. Moreover, Zhou and Peng [19] investigated the existence of weak solutions and optimal control for TFNSEs, while Peng et al.[20] presented the rigorous exposition of local solutions of TFNSEs in Sobolev space. However, one can notice that there are only a few works related to the numerical solution of the TFNSEs. The details of meshless local Petrov-Galerkin method based on moving Kriging interpolation for solving the TFNSEs can be found in the literature [21]. The purpose of this paper is to present finite difference/element method to obtain the numerical solution of TFNSEs.

The rest of the paper is arranged as follows. In Section 2, we give some notations and preliminaries. Section 3 deals with a semi-discrete scheme for the TFNSEs, which is based on a mixed finite element method in space. We also discuss the stability and error estimates of this semi-discrete scheme. In Section 4, we use a finite difference approximation to discrete time direction to obtain the fully discrete scheme. The stability and error estimates for the discrete schemes are also found. In Section 5, numerical results are discussed to confirm our theoretical analysis. Conclusions are given in the final section.

2. Notations and preliminaries

In this section, we present some preliminary concepts of the functional spaces. Firstly, we introduce the following Hilbert spaces:

\[ X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L_0^2(\Omega) = \{v \in L^2(\Omega); \int_\Omega vdx = 0\}, \]

where the space \( L^2(\Omega) \) is associated with the usual inner product \((\cdot, \cdot)\) and the norm \(\| \cdot \|\). The space \( X \) is associated with the following inner product and equivalent norm:

\[(u, v) = (\nabla u, \nabla v), \quad \|u\|_X = \|\nabla u\|_0 = \|u\|_1.\]

Denote by \( V \) and \( H \) the closed subsets of \( X \) and \( Y \) respectively, which are given by

\[ V = \{v \in X; \text{div} v = 0\}, \quad H = \{v \in Y; \text{div} v = 0, v \cdot n|_{\partial\Omega} = 0\}. \]

We denote the Stokes operator by \( A = -P\triangle \), in which \( P \) is the \( L^2 \)-orthogonal projection of \( Y \) onto \( H \). The domain of \( A \) is \( \mathcal{D}(A) = H^2(\Omega)^2 \cap V \) and let \( H^s = \mathcal{D}(A^s) \) with the norm \( \|v\|_s = \|A^s v\| \). Observe that \( H^2 = \mathcal{D}(A), \quad H^1 = V \) and \( H^0 = H \).

Next, we define the Riemann-Liouville fractional integral operator of order \( \beta (\beta \geq 0) \) as (see [22])

\[ I_0^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s)ds, \quad t > 0, \]

with \( I_0^\beta g(t) = g(t) \).

The Caputo-type derivative of order \( \alpha \in (0, 1] \), \( C D_t^\alpha \) in (1.1) is defined by

\[ C D_t^\alpha g(t) = \frac{d}{dt} \{ I_t^{1-\alpha} [g(t) - g(0)] \} = \frac{d}{dt} \left\{ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} [g(t) - g(0)]ds \right\}. \]

Further, the operator \( C D_t^{-\alpha} \) is defined as

\[ C D_t^{-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds, \quad t > 0. \]

where \( \Gamma(\cdot) \) stands for the gamma function \( \Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \).

Next we introduce the following continuous bilinear forms \( a(\cdot, \cdot) \) and \( d(\cdot, \cdot) \) on \( X \times X \) and \( X \times M \) respectively as follows:

\[ a(u, v) = \nu(\nabla u, \nabla v), \quad u, v \in X, \quad d(v, q) = (q, \text{div} v), \quad v \in X, q \in M, \]
and the trilinear form $b(u, v, w)$ on $X \times X \times X$ is given by

$$b(u, v, w) = ((u \cdot \nabla)v + \frac{1}{2}(\text{div} u, v) + \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v), \ u, v, w \in X.$$ 

It is well-known that the trilinear form $b(u, v, w)$ has the following properties:

$$b(u, v, w) = -b(u, w, v), \ b(u, v, v) = 0, \ u, v, w \in X,$$

where $|b(u, v, w)| \leq \mu_0 \|u\|_1 \|v\|_1 \|w\|_1, \ u, v, w \in X.$

In terms of the above notations, the weak formulation of problem (1.1) is as follows: find $(u, p) \in (X, M)$ for all $t \in [0, T]$ such that for all $(v, q) \in (X, M)$:

$$\begin{cases}
C D_\tau^\alpha (u, v) + a(u, v) + b(u, u, v) - d(v, p) + d(u, q) = (f, v), \\
u(h(0)) = u_0.
\end{cases} \tag{2.4}$$

In [19], Zhou and Peng discussed the existence and uniqueness of weak solutions for the problem (2.4). The objective of the present work is to obtain the numerical solution of the problem at hand.

3. Finite element method for space discretization

Let $T^h(\Omega) = \{K\}$ be a mesh of $\Omega$ with a mesh size function $h(x)$, which is the diameter $h_K$ of element $K$ containing $x$. Assuming $h = h_\Omega = \max_{x \in \Omega} h(x)$ be the largest mesh size of $T^h(\Omega)$, we introduce the mixed finite element subspace $(X_h, M_h)$ of $(X, M)$ and define the subspace $V_h$ of $X_h$ as

$$V_h = \{v_h \in X_h; d(v_h, q_h) = 0, \ \forall q_h \in M_h\}.$$ 

Let $P_h : Y \rightarrow V_h$ denote the $L^2$-orthogonal projection defined by

$$(P_h v, v_h) = (v, v_h), \ \forall v \in Y, \ v_h \in V_h.$$ 

With the above notations, we need some further basic assumptions on the mixed finite element spaces (Refs.[8,10,12]).

(A1) Approximation. For each $(v, q) \in (D(A), M \cap H^1(\Omega))$, there exist approximations $(\pi_h v, \rho_h q) \in (X_h, M_h)$ such that

$$\|v - \pi_h v\|_1 \leq Ch\|Av\|, \ |q - \rho_h q| \leq Ch\|q\|_1. \tag{3.1}$$

(A2) Inverse estimate. For any $(v, q) \in (X_h, M_h)$, the following relations hold:

$$\|\nabla v\| \leq Ch^{-1}\|v\|, \ |q| \leq Ch^{-1}\|q\|_1. \tag{3.2}$$

(A3) Stability property. For any $(v, q) \in (X_h, M_h)$, the well-known inf-sup condition holds:

$$\sup_{v \in X_h} \frac{d(v, q)}{\|v\|_1} \geq \lambda\|q\|, \tag{3.3}$$

where $\lambda > 0$ is a constant.

Further, the following classical properties hold:

$$\|v - P_h v\| + h\|\nabla(v - P_h v)\| \leq Ch^2\|Av\|, \ v \in D(A); \tag{3.4}$$

$$\|v - P_h v\| \leq Ch\|\nabla(v - P_h v)\|, \ v \in X. \tag{3.5}$$

The standard finite element Galerkin approximation for (2.4) holds as follows: Find $(u_h, p_h) \in (X_h, M_h)$ for all $t \in [0, T]$ such that for all $(v_h, q_h) \in (X_h, M_h)$, we have

$$\begin{cases}
C D_\tau^\alpha (u_h, v_h) + a(u_h, v_h) + b(u_h, u_h, v_h) - d(v_h, p_h) + d(u_h, q_h) = (f, v_h), \\
u_h(0) = u_{0h} = P_h u_0.
\end{cases} \tag{3.6}$$
Then there exists a positive constant \( C \) such that

\[
\| u_h \|^2 + \beta \| u_h \|^2 = (f, u_h) \leq \| f \|_{-1} \| u_h \| \leq \frac{1}{2\nu} \| f \|_{-1}^2 + \frac{\nu}{2} \| u_h \|^2,
\]

where \( \beta = \frac{C_D}{2\nu} > 0 \) is constant.

**Proof.** Taking \( v_h = u_h, q_h = p_h \) in (3.6) and using the Young’s inequality, we get

\[
C D_t^\alpha \| u_h \|^2 + \nu \| u_h \|^2 = (f, u_h) \leq \| f \|_{-1} \| u_h \| \leq \frac{1}{2\nu} \| f \|_{-1}^2 + \frac{\nu}{2} \| u_h \|^2,
\]

Applying the integral operator (2.3) to both sides of (3.9) and using the Young’s inequality, we obtain

\[
\| u_h \|^2 + \nu \int_0^t (t-s)^{\alpha-1} \| u_h \|^2 ds \leq \| u_0 \|^2 + \frac{1}{2\nu} \int_0^t (t-s)^{\alpha-1} \| f \|_{-1}^2 ds
\]

\[
\leq \| u_0 \|^2 + \frac{1}{2\nu} \int_0^t (t-s)^{\alpha-1} ds \leq \| u_0 \|^2 + \frac{\alpha_1 T^{1+\beta}}{2\nu(1+\beta)\Gamma(\alpha)} \| f \|_{-1}^2
\]

where \( \alpha_1 > 0 \) with \( 0 < \alpha_1 < 1 \).

In view of the inequality

\[
\frac{\nu}{2\nu} \int_0^t (t-s)^{\alpha-1} \| u_h \|^2 ds \geq \frac{\nu T^{\alpha-1}}{2\nu} \int_0^t \| u_h \|^2 ds,
\]

it follows that

\[
\| u_h \|^2 + \frac{\nu T^{\alpha-1}}{2\nu} \int_0^t \| u_h \|^2 ds \leq \| u_0 \|^2 + \frac{\alpha_1 T^{1+\beta}}{2\nu(1+\beta)\Gamma(\alpha)} \| f \|_{-1}^2
\]

This completes the proof.

**Theorem 3.2.** For any \( t \in [0, T] \), \( 0 < \alpha < 1 \), let \((u, p)\) and \((u_h, p_h)\) be the solutions of equations (2.4) and (3.6) respectively. Then there exists a positive constant \( C \) such that

\[
\| u - u_h \| \leq C h^2, \quad \| p - p_h \| \leq C h.
\]

**Proof.** Setting \((\xi, \eta) = (u - u_h, p - p_h)\), we deduce from (2.4) and (3.6) that

\[
\begin{aligned}
C D_t^\alpha (\xi, v) + \alpha (\xi, v) + b(\xi, u_h, v) + b(u_h, \xi, v) + b(\xi, \xi, v) - d(v, \eta) + d(\xi, q) &= 0, \\
\xi_0 &= u_0 - P_h u_0.
\end{aligned}
\]

Taking \( v = \xi \) and \( q = \eta \) in (3.11), we get

\[
C D_t^\alpha \| \xi \|^2 + \nu \| \xi \|^2 + b(\xi, u_h, \xi) = 0.
\]
Using the properties of $b(u_h, v_h, w_h)$ together with Young’s inequality, we obtain
\[
C D^\alpha \|\xi\|^2 + \nu\|\xi\|^2 = b(\xi, \xi, u_h) \\
\leq C_0\|\xi\|_1\|u_h\|_1 \\
\leq \nu\|\xi\|^2 + C_1\|\xi\|^2\|u_h\|^2.
\] (3.12)

Applying (2.3) to both sides of (3.12), we have
\[
\|\xi\|^2 \leq \|\xi_0\|_1^2 + \frac{C_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|u_h\|^2 ds.
\] (3.13)

By means of the generalized integral version of Gronwall’s lemma [24], we get
\[
\|\xi\|^2 \leq \|\xi_0\|_1^2 \exp\left(\frac{C_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|u_h\|^2 ds\right) \\
\leq \|u_0 - P_h u_0\|^2 \exp\left(C_2\|u_0\|^2 + \frac{(1 - \alpha_1)T^{1+\beta}}{2\nu(1+\beta)\Gamma(\alpha)} + \frac{\alpha_1 T}{2\nu\Gamma(\alpha)} \max_{t\in[0,T]} \|f\|_{-1}^2\right) \\
\leq C h^4. 
\] (3.14)

Furthermore, setting $v = \xi$ and $q = 0$ in (3.11), and using inf-sup condition (3.3), combining (3.1)-(3.5), (3.8), (3.14) and using the integral operator (2.3) in (3.15), we conclude that
\[
\|\eta\| \leq \sup_{V_h} \frac{|d(\xi, \eta)|}{\lambda\|\xi\|_1} \\
= \sup_{V_h} \frac{|C D^\alpha \|\xi\|^2 + \nu\|\xi\|^2 + b(\xi, u_h, \xi)|}{\lambda\|\xi\|_1} \\
\leq C h.
\] (3.15)

This completes the proof.

4. Finite difference method for time discretization

The discretization of time-fractional derivative can be found in [25-32] and references therein. Here, we will introduce a uniform grid by discretizing the temporal domain $[0, T]$ given by the points: $t_n = n\tau$ for $n = 0, 1, \ldots, N$, with the time-step size $\tau = T/N$. Hence, the Riemann-Liouville fractional integral operator of order $\alpha$ can be discretized as follows:
\[
I^\alpha_t g(t_n) = \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{\alpha-1} g(s) ds \\
= \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{n} \left(\int_{t_{k-1}}^{t_k} (t_n - s)^{\alpha-1} g(t_k) ds + \gamma_n^\alpha\right) \\
= \frac{T^\alpha}{\Gamma(\alpha + 1)} \sum_{k=1}^{n} g(t_k)((n - k + 1)^\alpha - (n - k)^\alpha) + \gamma_n^\alpha \\
= \frac{T^\alpha}{\Gamma(\alpha + 1)} \sum_{k=0}^{n-1} u_k^\alpha g(t_{n-k}) + \gamma_n^\alpha,
\] (4.1)

where $u_k^\alpha = (k + 1)^\alpha - k^\alpha$ and the truncation error $\gamma_n^\alpha$ is given by
\[
\gamma_n^\alpha = \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{\alpha-1} [g(s) - g(t_k)] ds \\
= \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{\alpha-1} g'(\zeta)(s - t_k) ds, \ s < \zeta < t_k.
\]
Therefore, we have
\[
|\gamma^n_0| \leq \frac{\tau}{\Gamma(\alpha)} \max_{0 \leq t \leq t_h} |g'(t)| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - s)^{\alpha - 1} ds
\]
\[
\leq N^\alpha \tau^{\alpha + 1} \max_{0 \leq t \leq t_h} |g'(t)|.
\]

**Lemma 4.1.** (see [?]) If \( g(t) \in C^1[0, T] \), then
\[
I_\alpha^n g(t_n) = \frac{\tau^\alpha}{\Gamma(\alpha + 1)} \sum_{k=0}^{n-1} w_k^\alpha g(t_{n-k}) + \gamma^n_0,
\]
(4.2)
where \( |\gamma^n_0| \leq C \tau^{\alpha + 1}, n = 0, 1, \ldots, N \).

**Lemma 4.2.** (see [?]) For \( 0 < t_n \leq t_N = T \) and \( \alpha > 0 \), let the coefficient \( w_k^\alpha \) be given by (4.1). Then
(i) \( w_0^\alpha = 1, w_k^\alpha > 0, k = 0, 1, 2, \ldots \);
(ii) \( w_k^\alpha > w_{k+1}^\alpha, k = 0, 1, 2, \ldots \);
(iii) \( \sum_{k=0}^{n-1} w_k^\alpha = n^\alpha \leq N^\alpha \).

Applying the integral operator (2.3) to both sides of (3.6), we obtain
\[
(u_h, v_h) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1}[\alpha(u_h, v_h) + b(u_h, u_h, v_h) - d(v_h, p_h) + d(u_h, q_h)] ds
\]
\[
= (u_0h, v_h) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1}(f, v_h) ds.
\]
(4.3)

Let \( u_h^n \) and \( p_h^n \) be the numerical solutions of \( u_h(t) \) and \( p_h(t) \) at \( t = t_n \) respectively. By (4.2) and (4.3), our full discrete scheme of equation (2.4) can be defined by seeking \( (u_h^n, p_h^n) \in (X_h, M_h) \) such that for all \( (v_h, q_h) \in (X_h, M_h) \):
\[
(u_h^n, v_h) + \beta_0 \sum_{k=0}^{n-1} w_k^\alpha (a(u_h^{n-k}, v_h) + b(u_h^{n-k}, u_h^{n-k}, v_h) - d(v_h, p_h^{n-k}) + d(u_h^{n-k}, q_h))
\]
\[
= (u_0^n, v_h) + \beta_0 \sum_{k=0}^{n-1} w_k^\alpha (f^{n-k}, v_h),
\]
(4.4)
where \( \beta_0 = \frac{\tau^\alpha}{\Gamma(\alpha + 1)} \).

**Theorem 4.1.** For any \( 0 < \tau < T \), the full discrete scheme (4.4) is unconditionally stable, and that
\[
\|u_h^n\|^2 + \beta_1 \|u_h^n\|^2_1 \leq C^4(\|u_0^n\|^2 + \sum_{k=0}^{n} \|f^k\|^2_{-1}).
\]

**Proof.** Setting \( n = 1 \) in (4.4), we get
\[
(u_1^h, v_h) + \beta_0 [a(u_1^h, v_h) + b(u_1^h, u_1^h, v_h) - d(v_h, p_1^h) + d(u_1^h, q_h)] = (u_0^h, v_h) + \beta_0 (f^1, v_h).
\]
(4.5)
Taking \( v_h = u_h^1 \) and \( q_h = p_1^h \) in (4.5), we have
\[
(u_1^h, u_1^h) + \beta_0 [a(u_1^h, u_1^h) + b(u_1^h, u_1^h, v_h) - d(v_h, p_1^h) + d(u_1^h, q_h)] = (u_0^h, u_1^h) + \beta_0 (f^1, u_1^h).
\]
Making use of Young’s inequality, we obtain
\[
\|u_1^h\|^2 + \beta_0 \nu \|u_1^h\|^2_1 \leq \frac{1}{2} \|u_0^h\|^2 + \|u_1^h\|^2 + [\frac{\beta_0}{2\nu} \|f^1\|^2_{-1} + \frac{\beta_0 \nu}{2} \|u_1^h\|^2_1],
\]
that is,

\[ \| u_h^1 \|^2 + \beta_1 \| u_h^1 \|^2 \leq \frac{1}{2} \| u_0^0 \|^2 + \frac{\beta_0}{2 \nu} \| f^1 \|_{2,1}^2. \]

Assuming \( v_h = u_h^j \in X_h \) and \( q_h = p_h^n \in M_h \), the following inequality holds

\[ \| u_h^j \|^2 + \beta_1 \| u_h^j \|^2 \leq C(\| u_0^0 \|^2 + \sum_{k=1}^j \| f^k \|_{2,1}^2), \quad j = 2, 3, \ldots, n - 1. \]

Setting \( v_h = u_h^{n-k} \) and \( q_h = p_h^{n-k} \) in (4.4), we get

\[
(u_h^n, u_h^{n-k}) + \beta_0 \sum_{k=0}^{n-1} w_k^a (u_h^{n-k}, u_h^{n-k}) = (u_0^0, u_h^{n-k}) + \beta_0 \sum_{k=0}^{n-1} w_k^a (f^{n-k}, u_h^{n-k}).
\]

By the elementary identity \( ab = \frac{1}{4} (a^2 + b^2) - \frac{1}{4} (a - b)^2 \) and the Young’s inequality, we have

\[
\frac{1}{2} \| u_h^n \|^2 + \| u_h^{n-k} \|^2 + \beta_0 \sum_{k=0}^{n-1} w_k^a \| u_h^{n-k} \|^2 \leq (u_h^n, u_h^{n-k}) + \frac{1}{2} \| u_h^n - u_h^{n-k} \|^2 + \beta_0 \sum_{k=0}^{n-1} w_k^a (f^{n-k}, u_h^{n-k}) \leq \frac{1}{2} \| u_h^n \|^2 + \| u_h^{n-k} \|^2 + \frac{1}{2} \| u_h^n - u_h^{n-k} \|^2 + \sum_{k=0}^{n-1} w_k^a \| f^{n-k} \|_{1,1}^2 + \frac{\beta_0}{2} \| u_h^{n-k} \|^2.
\]

Together with Lemma 4.2 (ii) (that is, \( 0 < w_k^{a+1} < w_k^a < 1 \)) and \( \frac{\beta_0}{2} \sum_{k=1}^{n-1} w_k^a \| u_h^{n-k} \|^2 \geq 0 \), we obtain

\[
\| u_h^n \|^2 + \beta_1 \| u_h^n \|^2 \leq \| u_0^0 \|^2 + \| u_h^n - u_h^{n-k} \|^2 + \frac{\beta_0}{2 \nu} \sum_{k=0}^{n-1} w_k^a \| f^{n-k} \|_{1,1}^2 \leq C^\dagger (\| u_0^0 \| + \sum_{k=0}^n \| f^k \|_{1,1}^2). \tag{4.6}
\]

The proof is completed.

**Lemma 4.3.** Let \( \nu > 0 \) be the viscosity coefficient and that

\[
\| u_h^n \|^2 + \| f^n \|_{1,1}^2 \leq \frac{\beta_1 \nu}{C^\dagger \mu_0}. \tag{4.7}
\]

Then

\[
\| u_h^n \|^2 \leq \frac{\nu}{\mu_0},
\]

that is,

\[
\nu - \mu_0 \| u_h^n \|^2 \geq 0.
\]

By the property of \( b(v_h, u_h^n, v_h) \), we get

\[
a(v_h, v_h) + b(v_h, u_h^n, v_h) \geq (\nu - \mu_0 \| u_h^n \|^2) \| v_h \|_1^2 \geq 0.
\]

The proof of the lemma is completed.
Theorem 4.2. For $0 < \alpha < 1$, let $(u_h(t_n), p_h(t_n))$ and $(u^n_n, p^n_n)$ be the solutions of equations (3.6) and (4.4) respectively. There exists a constant $C$ such that

$$
\|u_h(t_n) - u^n_n\| \leq C\tau^{\alpha+1}, \|p_h(t_n) - p^n_n\| \leq C\tau^{\alpha+1}.
$$

Proof. Let $\xi^n = u_h(t_n) - u^n_n$ and $\eta^n = p_h(t_n) - p^n_n$. Using (4.2)-(4.4) and noting $\xi^0 = 0$, we deduce

$$
\langle \xi^n, v_h \rangle + \beta_0 \sum_{k=0}^{n-1} w^n_k [a(\xi^{n-k}, v_h) + b(\xi^{n-k}, u^n_h, v_h) - d(v_h, \eta^{n-k}) + d(\xi^{n-k}, q_h)]
$$

$$
= (\gamma^n_\alpha, v_h).
$$

For $n = 1$, taking $v_h = \xi^1$ and $q_h = \eta^1$ in (4.9), we have

$$
(\xi^1, \xi^1) + \beta_0 [a(\xi^1, \xi^1) + b(\xi^1, u^n_h, \xi^1)] = (\gamma^n_\alpha, \xi^1).
$$

By Cauchy-Schwarz inequality and Lemma 4.3, we get

$$
\|\xi^1\| \leq \|\gamma^n_\alpha\| \leq C\tau^{\alpha+1}.
$$

Let us assume that $\|\xi^m\| \leq C\tau^{\alpha+1}$ for $m = 2, 3, \ldots, n-1$. In order to show that the first inequality in (4.8) holds for $m = n$, we set $v_h = \xi^{n-k}$ and $q_h = \eta^{n-k}$ in (4.9). Then

$$
(\xi^n, \xi^{n-k}) + \beta_0 \sum_{k=0}^{n-1} w^n_k [a(\xi^{n-k}, \xi^{n-k}) + b(\xi^{n-k}, u^n_h, \xi^{n-k})] = (\gamma^n_\alpha, \xi^{n-k}).
$$

In view of the elementary identity $ab = \frac{1}{2}(a^2 + b^2) - \frac{1}{2}(a-b)^2$, Young’s inequality and Lemma 4.3, we get

$$
\frac{1}{2} [\|\xi^n\|^2 + \|\xi^{n-k}\|^2] \leq \frac{1}{2} \|\xi^n - \xi^{n-k}\|^2 + \frac{1}{2} [\|\xi^{n-k}\|^2 + \|\gamma^n_\alpha\|^2],
$$

which implies that

$$
\|\xi^n\|^2 \leq C\tau^{2\alpha+2}.
$$

By inverse estimate (3.2) together with (4.12), we have

$$
\|\xi^n\|_1 \leq C^{-1} \|\xi^n\| \leq C\tau^{\alpha+1}.
$$

On the other hand, setting $v_h = \xi^1$ and $q_h = 0$ for $n = 1$ in (4.9) and using Cauchy-Schwarz inequality together with (3.1)-(3.4), (3.8) and (4.13), we get

$$
\|\eta^1\| \leq \sup_{V_h} \frac{|d(\xi^1, \xi^1)|}{\lambda \|\xi^1\|_1} = \sup_{V_h} \frac{|\|\xi^1\|^2 + \beta_0 [a(\xi^1, \xi^1) + b(\xi^1, u^n_h, \xi^1)] - (\gamma^n_\alpha, \xi^1)|}{\lambda \|\xi^1\|_1}
$$

$$
\leq C\tau^{\alpha+1}.
$$

Using the assumption $\|\eta^m\| \leq C\tau^{\alpha+1}$ for $m = 2, 3, \ldots, n-1$ and taking $v_h = \xi^{n-k}$ and $q_h = 0$ in (4.9), by (4.12) and (4.13), similar to the derivation of (4.14) for $m = n$, we obtain

$$
\|\eta^n\| \leq \sup_{V_h} \frac{|d(\xi^n, \eta^n)|}{\lambda \|\xi^n\|_1} \leq \sup_{V_h} \frac{|\sum_{k=0}^{n-1} w^n_k d(\xi^{n-k}, \eta^{n-k})|}{\lambda \|\xi^n\|_1}
$$

$$
= \sup_{V_h} \frac{|(\xi^n, \xi^{n-k}) + \beta_0 \sum_{k=0}^{n-1} w^n_k [a(\xi^{n-k}, \xi^{n-k}) + b(\xi^{n-k}, u^n_h, \xi^{n-k})] - (\gamma^n_\alpha, \xi^{n-k})|}{\lambda \|\xi^n\|_1}
$$

$$
\leq C\tau^{\alpha+1}.
$$

This completes the proof.

Next we give the error estimate for fully discrete scheme.
Theorem 4.3. For $0 < \alpha < 1$, let $(u(t_n), p(t_n))$ and $(u_h^n, p_h^n)$ be the solutions of equations (2.4) and (4.4) respectively. Then there exists a positive constant $C$ such that

$$
\|u(t_n) - u_h^n\| \leq C(h^2 + \tau^{\alpha+1}), \quad \|p(t_n) - p_h^n\| \leq C(h + \tau^{\alpha+1}).
$$

Proof. It is easy to show that (4.15) follows from Theorem 3.2 and Theorem 4.2 via triangle inequality.

5. Numerical example

In this section, we demonstrate the effectiveness of our numerical methods with the aid of examples. We use mixed finite element method for the discretization of spatial direction and finite difference approximation for time discretization. The convergence rates of numerical solutions with respect to space step $h$ and time step $\tau$ are discussed. We consider the regular (uniform) domain $\Omega = (0,1) \times (0,1)$ and the time interval is chosen to be $[0,1]$ with the viscosity coefficient $\nu = 1.5$. For an appropriate body force $f$, the analytical solution $(u, p) = ((u_1, u_2), p)$ of the unstable flow problem with homogeneous boundary conditions becomes

$$
u = 2x^2(x - 1)^2 \frac{y(y - 1)(2y - 1)}{2}, \quad u_2 = -2y^2(y - 1)^2 x(x - 1)(2x - 1)e^{-t},
$$

$$p = (x^2 - y^2)e^{-t},$$

which automatically satisfy the initial and boundary conditions.

The errors $\|e^n\|$ are computed in $L^2$-discrete norm. The results of numerical experiments are compared with analytical solution by the rates of the convergence, which are approximately by

$$\text{Rate} = \left| \frac{\ln(\|e_f^n\|/\|e_c^n\|)}{\ln(N_f/N_c)} \right|,$$

where $\|e_f^n\|$ and $\|e_c^n\|$ denote the error on finer grid and coarser grid, $N_f$ and $N_c$ represent the numbers of meshes on finer grid and coarser grid, respectively.

The spatial convergence rates for the components of velocity $(u_1, u_2)$ and pressure $p$ with fixed time step $\tau = 1/8$ with different values of $\alpha$ are shown in Fig.1. The convergence rates of velocity $(u_1, u_2)$ are in accordance with spatial convergence order $O(h^2)$ and the pressure $p$ are closer to order $O(h)$. Fig.2 give the temporal convergence rates for the components of velocity and pressure with fixed spatial step $h = 1/15$ with different values of $\alpha$. We can see that the rates of convergence are closer to the theoretical convergence order $O(\tau^{\alpha+1})$.

Fig.3 depicts the numerical solutions of the components of velocity $(u_1, u_2)$ and pressure $p$, with $h = 1/15$ and $\tau = 0.1$, when $\alpha = 0.4$ and $\alpha = 0.8$, respectively. It is not difficult to find that a pair of warm- and cold-core eddies emerge in the velocity field.

6. Conclusion

In this study, the finite difference/element method is presented to solve the TFNSEs and the convergence error estimates for the discrete schemes in $L^2$-norm are obtained. We present the numerical experiment to illustrate the accuracy of schemes, and the result fully verify the convergence theory. The numerical examples also confirm the thesis [18,19,20] that in procedure of citing and novelty of the obtained results. Furthermore, the presented methods and analytical techniques in this work can also be extended to other nonlinear time-fractional partial differential equations.

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The components of velocity $u_1$
The components of velocity $u_2$

The rate of convergence

Spatial step $h$

$\alpha = 0.3$

$\alpha = 0.6$

$\alpha = 0.9$
The rate of convergence

The pressure $p$

$\alpha = 0.3$

$\alpha = 0.6$

$\alpha = 0.9$
\[ p = \left( \frac{1}{x} \right)^2 \times 10^{-3} \]
The rate of convergence $\alpha = 0.3$ for different time steps $\tau$. The chart shows the convergence rate for $u_1$, $u_2$, and $p$ as $\tau$ decreases from $1/2$ to $1/64$. The rate of convergence increases as $\tau$ decreases, indicating a higher precision for smaller time steps.
The rate of convergence $\alpha = 0.6$

- $u_1$
- $u_2$
- $p$

Time step $\tau$
The rate of convergence \( \alpha = 0.9 \)
