Ramsey expansions of metrically homogeneous graphs

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Abstract

We investigate Ramsey expansions, the coherent extension property for partial isometries (EPPA), and the existence of a stationary independence relation for all classes of metrically homogeneous graphs from Cherlin’s catalogue. We show that, with the exception of tree-like graphs, all metric spaces in the catalogue have precompact Ramsey expansions (or lifts) with the expansion property. With two exceptions we can also characterise the existence of a stationary independence relation and coherent EPPA.

Our results are a contribution to Nešetřil’s classification programme of Ramsey classes and can be seen as empirical evidence of the recent convergence in techniques employed to establish the Ramsey property, the expansion property, EPPA and the existence of a stationary independence relation. At the heart of our proof is a canonical way of completing edge-labelled graphs to metric spaces in Cherlin’s classes. The existence of such a “completion algorithm” then allows us to apply several strong results in the areas that imply EPPA or the Ramsey property.

The main results have numerous consequences for the automorphism groups of the Fraïssé limits of the classes. As corollaries, we prove amenability, unique ergodicity, existence of universal minimal flows, ample generics, small index property, 21-Bergman property and Serre’s property (FA).

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1. Introduction

Given a graph, its associated metric space has its vertices as points, and the distance between two points is the length of the shortest path connecting them. A (countable) structure is homogeneous if every isomorphism between finite substructures is induced by an automorphism. A metrically homogeneous graph is a connected graph with the property that the associated metric space is a homogeneous metric space.

As a new contribution to the well known classification programme of homogeneous structures (of Lachlan and Cherlin [Lac84, LW80, Che98]), Cherlin [Che22] has recently provided a catalogue of metrically homogeneous graphs. While originally this catalogue was only conjectured to be complete, there is now a purported (yet unpublished) proof.3

This paper investigates properties of classes of finite substructures of structures from Cherlin’s catalogue. To state our main results we introduce some terminology first. A graph is bipartite if it contains no odd cycles and it is regular if there exists $0 \leq k \leq \infty$ such that the degree of every vertex is $k$. The diameter of a graph is the maximal distance in its associated metric space. A graph of diameter $\delta$ is antipodal if in its associated metric space for every vertex there exists at most one vertex in distance $\delta$ (and it is antipodally closed if every vertex has a unique such vertex). A graph is tree-like if it is isomorphic to a graph $T_{m,n}$, $2 \leq m, n \leq \infty$, defined as follows:

**Definition 1.1.** Given $2 \leq m, n \leq \infty$, the graph $T_{m,n}$ is defined to be the (regular) graph in which the blocks (two-connected components) are cliques of order $n$ and every vertex is a cut vertex, lying in precisely $m$ blocks.

Our main results can be summarised as follows (all the remaining notions will be introduced below):

**Theorem 1.1.** Let $\Gamma$ be the associated metric space of a countably infinite metrically homogeneous graph $G$ of diameter $\delta$ from Cherlin’s catalogue. Then one of the following applies:

1. If $G$ is a tree-like graph, then $\text{Age}(\Gamma)$ has no precompact Ramsey expansion.

2. If $G$ is not tree-like, then one of the following holds:

   (a) If $G$ is primitive and 3-constrained (i.e. neither antipodal nor bipartite), then the class of free orderings of $\text{Age}(\Gamma)$ is Ramsey.

   (b) If $G$ is bipartite, 3-constrained and not antipodal then $\text{Age}(\Gamma)$ is Ramsey when extended by a unary predicate denoting the bipartition and by convex linear orderings.

   (c) If $G$ is antipodal, denote by $A$ the subclass of $\text{Age}(\Gamma)$ of antipodally closed metric spaces. Then:

      i. If $G$ is not bipartite or has odd diameter, then $A$ is Ramsey when extended by a linear ordering convex with respect to the podes.

      ii. If $G$ is bipartite of even diameter, then $A$ is Ramsey when extended by a unary predicate denoting the bipartition and a linear ordering convex with respect to the bipartition and the podes.

In both these cases, the linear order on one pode in $A$ uniquely extends to the other pode so that it is (linearly) isomorphic. For precise definitions and discussion about non-antipodally closed metric spaces, see Section 7.

By “a predicate denoting the bipartition” we mean a unary predicate defining two equivalence classes of vertices, such that all distances in the same equivalence class are even.

All the Ramsey expansions above have the expansion property.

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3This is claimed on Cherlin’s website: https://sites.math.rutgers.edu/~cherlin/Paper/inprep.html
We thus completely characterise Ramsey expansions of all currently known infinite metrically homogeneous graphs. In addition to that we show two related properties, arriving at an almost complete characterisation.

**Theorem 1.2.** Let $\Gamma$ be the associated metric space of a countably infinite metrically homogeneous graph $G$ from Cherlin’s catalogue.

1. If $G$ is a tree-like graph, then $\text{Age}(\Gamma)$ does not have EPPA.

2. If $G$ is not tree-like then:
   (a) If $G$ is antipodal, and:
      i. If $G$ is non-bipartite of even diameter or bipartite of odd diameter, then $\text{Age}(\Gamma)$ has coherent EPPA.
      ii. If $G$ is bipartite of even diameter or non-bipartite of odd diameter, then $\text{Age}(\Gamma)$ extended by a unary predicate denoting podality has coherent EPPA.
   (b) If $G$ is not antipodal then it has coherent EPPA.

By “a predicate denoting podality” we mean a unary predicate defining two equivalence classes of vertices, such that no pair of vertices in distance $\delta$ are in the same equivalence class.

In the submitted version of this paper, we asked whether the statement of Theorem 1.2 is best possible, that is, whether one needs to extend the even-diameter bipartite and odd-diameter non-bipartite antipodal classes by a unary predicates denoting podality (it was stated as Problem 1.3). Since then this was resolved by Evans, Hubička, Konečný and Nešetřil [EHKN20] who proved that the predicated are not necessary for the diameter 3 non-bipartite case and subsequently by Konečný [Kon20] who extended their techniques to all classes in question.

**Theorem 1.3.** Let $\Gamma$ be the associated metric space of a countably infinite metrically homogeneous graph $G$ from Cherlin’s catalogue.

1. If $G$ is isomorphic to $T_{m,n}$ then $\Gamma$ has no stationary independence relation and it has a local stationary independence relation if and only if $m = \infty$ and $n \in \{2, 3, \infty\}$.

2. If $G$ is not tree-like then:
   (a) If $G$ is antipodal, and:
      i. If $G$ is not bipartite and has even diameter, then there exists a stationary independence relation on $\Gamma$.
      ii. If $G$ is bipartite and has odd diameter, then there exists a local stationary independence relation, but there is no stationary independence relation on $\Gamma$.
      iii. Otherwise there is no local stationary independence relation on $\Gamma$.
   (b) If $G$ is not antipodal:
      i. If $G$ is not bipartite, then there exists a stationary independence relation on $\Gamma$.
      ii. If $G$ is bipartite, then there exists a local stationary independence relation, but there is no stationary independence relation on $\Gamma$.

We remark that for structures with closures it makes sense to consider an additional axiom as discussed by Evans, Gha)dernezhad and Tent [EGT16, Definition 2.2]. Due to the simple nature of closures considered in this paper this does not make a practical difference in our results.
1.1. Edge-labelled graphs and metric completions

Our arguments are based on an analysis of an algorithm to fill holes in incomplete metric spaces presented in Section 4. This algorithm works for a significant (namely the primitive) part of Cherlin’s catalogue and leads to a strong notion of “canonical amalgamation”. When constraints on the amalgamation classes are pushed to the extreme, new phenomena (such as antipodality or bipartiteness) appear. We show that even in this case our algorithm is useful, provided that we apply some additional techniques. First, we introduce the necessary notation to speak about metric spaces with holes.

**Definition 1.2.** An edge-labelled graph \( G \) is a pair \((G, d)\) where \( G \) is the vertex set and \( d \) is a partial function from \( G^2 \) to \( \mathbb{N} \) such that \( d(u, v) = 0 \) if and only if \( u = v \), and \( d(u, v) = d(v, u) \) whenever either number is defined. A pair of vertices \( u, v \) on which \( d(u, v) \) is defined is called an edge of \( G \). We also call \( d(u, v) \) the length of the edge \( u, v \).

The standard graph-theoretic notions of homomorphism, embedding, and isomorphism extend naturally to edge-labelled graphs. Our subgraphs will always be induced (details are given in Section 2). An edge-labelled graph can also be seen as a relational structure (as discussed in Section 2.1). We find it convenient to use notation that resembles the standard notation of metric spaces.

We denote by \( G^\infty \) the class of all finite edge-labelled graphs and by \( G^\delta \) the subclass of \( G^\infty \) of those graphs containing no edge of length greater than \( \delta \).

An (edge-labelled) graph \( G \) is complete if every pair of vertices forms an edge: a complete edge-labelled graph \( G \) is called a metric space if the triangle inequality holds, that is \( d(u, w) \leq d(u, v) + d(v, w) \) for every \( u, v, w \in G \). An edge-labelled graph \( G = (G, d) \) is metric if there exists a metric space \( M = (G, d') \) such that \( d(u, v) = d'(u, v) \) for every edge \( u, v \) of \( G \). Such a metric space \( M \) is also called a (strong) metric completion of \( G \).

Given \( G = (G, d) \in G^\infty \) the path distance \( d^+(u, v) \) of \( u \) and \( v \) is the minimum

\[
\ell = \sum_{1 \leq i \leq n-1} d(u_i, u_{i+1})
\]

taken over all possible sequences of vertices for which \( u_1 = u, u_2, \ldots u_n = v \) and \( d(u_i, u_{i+1}) \) is defined for every \( 1 \leq i \leq n-1 \). If there is no such sequence we put \( \ell = \infty \). It is a well-known fact that a connected edge-labelled graph \( G = (G, d) \) is metric if and only if \( d(u, v) = d^+(u, v) \) for every edge of \( G \). In this case \( (G, d^+) \) is a metric completion of \( G \) which we refer to as the shortest path completion. This completion algorithm also leads to an easy characterisation of metric graphs: \( G \) is metric if and only if it does not contain a non-metric cycle, by which we mean an edge-labelled graph corresponding to a graph-theoretic cycle such that one distance in the cycle is greater than sum of the remaining distances. See e.g. [HN19] for details.

Our algorithm is a generalisation of the shortest path completion algorithm, tailored to preserve some other properties (e.g., forbidding long cycles whereas the shortest path completion permits forbidding only short cycles).

1.2. Extension property for partial automorphisms (EPPA)

A partial automorphism of the structure \( A \) is an isomorphism \( f : B \to B' \) where \( B \) and \( B' \) are substructures of \( A \). We say that a class of finite structures \( K \) has the extension property for partial automorphisms (EPPA, sometimes called the Hrushovski extension property or in the context of metric spaces the extension property for partial isometries) if for all \( A \in K \) there is \( B \in K \) such that \( A \) is a substructure of \( B \) and every partial automorphism of \( A \) extends to an automorphism of \( B \). We call \( B \) with such a property an EPPA-witness of \( A \).

In addition to being a non-trivial and beautiful combinatorial property, classes with EPPA have further interesting properties. For example, Kechris and Rosendal [KR07] have shown that the automorphism groups of their Fraissé limits are amenable.

In 1992 Hrushovski [Hru92] showed that the class \( G \) of all finite graphs has EPPA. A simple combinatorial argument for Hrushovski’s result was given by Herwig and Lascar [HL00] along with...
a non-trivial strengthening for certain, more restricted, classes of structures described by forbidden homomorphisms. This result was independently used by Solecki [Sol05] and Vershik [Ver08] to prove EPPA for the class of all finite metric spaces with integer, rational or real distances. (Because EPPA is a property of finite structures, there is no difference between integer and rational distances. The techniques also naturally extend to real numbers. For our presentation we will consider integer distances only.) These results were further strengthened by Rosendal [Ros11b, Ros11a]. Recently Conant developed this argument to generalised metric spaces [Con19], where the distances are elements of a certain classes of distance monoid. Siniora and Solecki [SS19, Sin17] introduced the stronger notion of coherent EPPA (where the extensions to automorphisms compose whenever the partial automorphisms do, see Definition 2.8), and proved a coherent strengthening of the Herwig–Lascar theorem.4

Hrushovski's result was a key ingredient for a paper by Hodges, Hodkinson, Lascar and Shelah [HHLS93] which proved the small index property for the random graph. This line of research has since expanded and the concept of ample generics (which follows from the combination of EPPA and the amalgamation property for automorphisms, where the automorphism group of the amalgam is the same as the automorphism group of the free amalgam) has been isolated. This is further outlined in Section 9.1.

1.3. Ramsey classes

The notion of Ramsey classes was isolated in the 1970s and, being a strong combinatorial property, it has found numerous applications, for example in topological dynamics [KPT05]. It was independently proved by Nešetřil–Rödl [NR76] and Abramson–Harrington [AH78] that the class of all finite linearly ordered hypergraphs is a Ramsey class. Several new classes followed.

For structures \( A, B \) denote the set of all substructures of \( B \) that are isomorphic to \( A \) by \( (B)_A \).

**Definition 1.3.** A class \( C \) of structures is a Ramsey class if for every two objects \( A \) and \( B \) in \( C \) and for every positive integer \( k \) there exists a structure \( C \) in \( C \) such that the following holds: For every partition of \( (A)_A \) into \( k \) classes there exists a \( \tilde{B} \in (C)_B \) such that \( (\tilde{B})_A \) is contained in a single class of the partition. In short we then write

\[
C \to (B)_A^k.
\]

We now briefly outline the results related to Ramsey classes of metric spaces, see [Nes95, NVT15, Bod15, HN19] for further references.

It was observed by Nešetřil [Nes89] that under mild assumptions every Ramsey class is an amalgamation class. The Nešetřil classification programme of Ramsey classes [Nes05] asks which amalgamation classes provided by the classification programme of homogeneous structures yield a Ramsey class. Amalgamation classes are often not Ramsey for simple reasons, such as the lack of a linear order on the vertices of its structures which is known to be present in every Ramsey class ([KPT05], see e.g. [Bod15, Proposition 2.22]). For this reason we consider enriched classes, where the language of an amalgamation class is expanded by additional relations. We refer to these classes as Ramsey expansions (or lifts), see Section 2.2.

In 2005 Nešetřil [Nes07] showed that the class of all finite metric spaces is a Ramsey class when enriched by free linear ordering of the vertices (see also [Mas18] for an alternative proof). This result was extended to some subclasses \( A_S \) of finite metric spaces where all distances belong to a given set \( S \) by Nguyen Van Thé [NVT10]. Recently, Hubička and Nešetřil further generalised this result to \( A_S \) for all feasible choices of \( S \) [HN19], earlier identified by Sauer [Sau13], as well as to the class of metric spaces omitting cycles of odd perimeter. Ramsey property of one additional case was shown by Sokić [Sok20]. Hubička, Konečný and Nešetřil study and further generalise Conant’s generalised monoid metric spaces and prove both the Ramsey property and EPPA for a much broader family of them [HKN21, HKN18, Kon19].

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4Since the submission of the paper, there has been a lot of development. For example, Hubička, Konečný, and Nešetřil gave a self-contained combinatorial proof of Solecki and Vershik’s result [HKN19], and strengthened the Herwig–Lascar and the Siniora–Solecki results [HKN22].
1.4. Stationary independence relations

A stationary independence relation (Definition 1.4) is a generalisation of the Katětov constructions recently developed by Tent and Ziegler [TZ13] to show simplicity of the automorphism group of the Urysohn space. Müller [Müll16] used the same property to prove universality of the same automorphism group. Examples of structures with a stationary independence relation include free amalgamation classes, metric spaces [TZ13] and the Hrushovski predimension construction (given by Evans, Ghadernezhad and Tent [EGT16]).

Given a structure $M$ and finite substructures $A$ and $B$, the substructure generated by their union is denoted by $\langle AB \rangle$. A stationary independence relation is a ternary relation on finite substructures of a given homogeneous structure $M$. The substructures $A$, $B$, $C$ are in the relation, roughly speaking, when $M$ induces a “canonical amalgamation” of $\langle AC \rangle$ and $\langle BC \rangle$ over $C$. In this case we write $A \downarrow C B$.

Stationary independence relations are axiomatised as follows:

**Definition 1.4 ((Local) Stationary Independence Relation).** Assume $M$ to be a homogeneous structure. A ternary relation $\downarrow C$ on the finite substructures of $M$ is called a stationary independence relation (SIR) if the following conditions are satisfied:

**SIR1 (Invariance).** The independence of finitely generated substructures in $M$ only depends on their type. In particular, for any automorphism $f$ of $M$, we have $A \downarrow C B$ if and only if $f(A) \downarrow f(C) f(B)$.

**SIR2 (Symmetry).** If $A \downarrow C B$, then $B \downarrow C A$.

**SIR3 (Monotonicity).** If $A \downarrow C (BD)$, then $A \downarrow C B$ and $A \downarrow (BC) D$.

**SIR4 (Existence).** For any $A$, $B$ and $C$ in $M$, there is some $A' \models \text{tp}(A/C)$ with $A' \downarrow C B$.

**SIR5 (Stationarity).** If $A$ and $A'$ have the same type over $C$ and are both independent over $C$ from some set $B$, then they also have the same type over $\langle BC \rangle$.

If the relation $A \downarrow C B$ is only defined for nonempty $C$, we call $\downarrow C$ a local stationary independence relation.

Here $\text{tp}(A/C)$ denotes the type of $A$ over $C$, see Definition 2.1.

1.5. Obstacles to completion

The list of subclasses of metric spaces with Ramsey expansions corresponds closely to the list of classes with EPPA. The similarity of these results is not a coincidence. All of the proofs proceed from a given metric space and, by a non-trivial construction, build an edge-labelled graph with either the desired Ramsey property or EPPA. However, these edge-labelled graphs might not be complete. Using some detailed information about when and how they can be completed to the given class of metric spaces $\mathcal{A}$ it is then possible to find an actual witness for the Ramsey property or EPPA in $\mathcal{A}$. The actual “amalgamation engines” have been isolated (see Theorems 2.1 and 2.2) and are based on a characterisation of each class by a set of obstacles in the sense of the definition below. Given a set $\mathcal{O}$ of edge-labelled graphs, let $\text{Forb}(\mathcal{O})$ denote the class of all finite edge-labelled graphs $G$ such that there is no $O \in \mathcal{O}$ with a homomorphism $O \rightarrow G$.

**Definition 1.5.** Given a class of metric spaces $\mathcal{A}$, we say that $\mathcal{O}$ is the set of obstacles of $\mathcal{A}$ if $\mathcal{A} \subseteq \text{Forb}(\mathcal{O})$ and moreover every $G \in \text{Forb}(\mathcal{O})$ has a metric completion into $\mathcal{A}$. 

1.6. Outline of the paper

Cherlin’s catalogue, given in Section 3, provides a rich spectrum of structures. We follow the catalogue in an order corresponding to the proof techniques (or main properties of the amalgamation classes), rather than strictly following the order of the catalogue as presented by Cherlin [Che22]. Several classes are refined and individual special cases are considered separately.

An essential part of the characterisation is the following description of triangle constraints by means of five numerical parameters. A triangle is a triple of distinct vertices \( u, v, w \in M \) and its perimeter is \( d(u, v) + d(v, w) + d(w, u) \). Classes described by constraints on triangles form an essential part of the catalogue; we call them 3-constrained classes. These classes can be described by means of five numerical parameters:

**Definition 1.6 (Triangle constraints).** Given integers \( \delta, K_1, K_2, C_0 \) and \( C_1 \) we consider the class \( \mathcal{A}_{K_1,K_2,C_0,C_1}^\delta \) of all finite metric spaces \( M = (M, d) \) with integer distances such that \( d(u, v) \leq \delta \) (we call \( \delta \) the diameter of \( \mathcal{A}_{K_1,K_2,C_0,C_1}^\delta \)) for every \( u, v \in M \) and for every triangle \( u, v, w \in M \) with perimeter \( p = d(u, v) + d(u, w) + d(v, w) \), the following are true: (\( m = \min\{d(u, v), d(u, w), d(v, w)\} \)) is the length of the shortest edge of \( u, v, w \)

- if \( p \) is odd then \( 2K_1 < p < 2K_2 + 2m \),
- if \( p \) is even then \( p < C_1 \), and
- if \( p \) is even then \( p < C_0 \).

Intuitively, the parameter \( K_1 \) forbids all odd cycles shorter than \( 2K_1 + 1 \), while \( K_2 \) ensures that the difference in length between even- and odd-distance paths connecting any pair of vertices is less than \( 2K_2 + 1 \). The parameters \( C_0 \) and \( C_1 \) forbid induced long even and odd cycles respectively. Not every combination of numerical parameters makes sense and leads to an amalgamation class. Those that do are characterised by Cherlin’s Admissibility Theorem 3.2.

We consider the following classes:

**Spaces of diameter 2:** Classes with diameter 2 are not discussed at all in this paper, because all the relevant results have already been established; these are just the homogeneous graphs from Lachlan and Woodrow’s catalogue [LW80]. The Ramsey properties of homogeneous graphs were first studied by Nešetřil [Neš89]. See also [JLNVTW14], which states the relevant results in more modern language and also shows the ordering properties in all classes from the Lachlan–Woodrow catalogue.

The EPPA of the class of all finite graphs was shown by Hrushovski [Hru92], EPPA of the class of all finite graphs omitting a given complete graph \( K_n \) by Herwig [Her95] and Hodkinson and Otto [HO03], and the remaining cases are particularly easy. Stationary independence relation for all those classes is an easy exercise.

**Primitive 3-constrained spaces:** A metrically homogeneous graph \( G \) is primitive if there are no non-trivial \( \text{Aut}(G) \)-invariant equivalence relations on its vertices.

In Section 4 we consider the richest regular family of amalgamation classes in the catalogue. This case contains the class of all finite metric spaces of finite diameter \( \delta \) and, more generally, most classes \( \mathcal{A}_{K_1,K_2,C_0,C_1}^\delta \) with \( \delta < \infty \), with parameters satisfying Case (II) or (III) of Theorem 3.2.

These classes are closed under strong amalgamation. We describe a generalised completion algorithm which allows us to show the coherent EPPA and present a Ramsey expansion.

**Bipartite 3-constrained spaces:** In Section 6 we consider the 3-constrained spaces \( \mathcal{A}_{K_0,0,C_0,2\delta+1}^\delta \), \( \delta < \infty \) that contain no odd triangles, but triangles with edge length \( \delta, 2 \) are allowed (and thus \( C_0 > 2\delta + 2 \)). This corresponds to the non-antipodal Case (I) and is similar to primitive 3-constrained spaces where the bipartitions form definable equivalence relations and thus introduce imaginary elements. The existence of imaginaries has some consequences on the completion algorithm, Ramsey expansions, and coherent EPPA.
Spaces with Henson constraints: As discussed in Section 5, many 3-constrained classes can be further restricted by Henson constraints (spaces with distances 1 and $\delta$ only). These constraints cannot be represented in the form of forbidden triangles. We show that they have little effect on our constructions because the completion algorithm never introduces new Henson constraints in any class where Henson constraints are admissible (Definition 3.3).

Antipodal spaces: Here we consider 3-constrained classes where edges of length $\delta$ form a matching. Every amalgamation class closed under forming antipodal companions (see Definition 7.2) can be extended to an antipodal metric space and these are special cases of our constructions because the matching implies non-trivial algebraic closure in the Fraïssé limits (in other words the class is not closed for strong amalgamation). We show how to carry the Ramsey and EPPA results from the underlying class to its antipodal variant in Section 7 and also discuss antipodal Henson constraints.

Analysis of antipodal spaces is surprisingly subtle and breaks down to 4 different sub-cases which needs to be considered separately. For two of the cases we can not show optimality of our EPPA argument.

Classes with infinite diameter: So far we have only discussed classes of finite diameter. However, we can derive EPPA and the Ramsey property in the remaining infinite diameter cases from what we already know from the classes with finite diameter, as we will show in Section 8.1. In Section 8.2, we deal with ages of tree-like graphs $T_{m,n}$. These graphs generalise $Z$ seen as a metric space, and we show that their basic properties are similar—there is no pre-compact Ramsey lift and no EPPA. Both conclusions follow from the fact that the algebraic closure of any vertex is infinite.

2. Preliminaries

We first review the standard model-theoretic notions of relational structures and amalgamation classes (see, for example [Hod93]). Next, we introduce the relevant “amalgamation engines” used to build Ramsey objects and EPPA-witnesses throughout this paper.

A language $L$ is a set of relational symbols $R \in L$, each associated with a natural number $a(R)$ called arity. A (relational) $L$-structure $A$ is a pair $(A, (R^A; R \in L))$ where $R^A \subseteq A^{a(R)}$ (i.e. $R^A$ is a $a(R)$-ary relation on $A$). The set $A$ is called the vertex set or the domain of $A$ and elements of $A$ are vertices. The language is usually fixed and understood from the context (and it is in most cases denoted by $L$). If $A$ is a finite set, we call $A$ a finite structure. We consider only structures with countably many vertices. The class of all (countable) relational $L$-structures will be denoted by Rel($L$).

A homomorphism $f : A \to B = (B, (R^B; R \in L))$ is a mapping $f : A \to B$ satisfying for every $R \in L$ the implication $(x_1, x_2, \ldots, x_{a(R)}) \in R^A \implies (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R^B$. (For a subset $A' \subseteq A$ we denote by $f(A')$ the set $\{f(x) : x \in A'\}$ and by $f(A)$ the homomorphic image of a structure.) If $f$ is injective, then $f$ is called a monomorphism. A monomorphism is called embedding if the above implication is an equivalence, i.e. if for every $R \in L$ we have $(x_1, x_2, \ldots, x_{a(R)}) \in R^A \iff (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R^B$. If $f$ is an embedding which is an inclusion then $A$ is a substructure (or subobject) of $B$. By the age of a structure $A$, or Age($A$) we denote the class of all finite substructures of $A$. For an embedding $f : A \to B$ we say that $A$ is isomorphic to $f(A)$ and $f(A)$ is also called a copy of $A$ in $B$. We define $\binom{B}{A}$ as the set of all copies of $A$ in $B$.

Let $A$, $B_1$ and $B_2$ be relational structures and $\alpha_1$ an embedding of $A$ into $B_1$, $\alpha_2$ an embedding of $A$ into $B_2$, then every structure $C$ with embeddings $\beta_1 : B_1 \to C$ and $\beta_2 : B_2 \to C$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ is called an amalgamation of $B_1$ and $B_2$ over $A$ with respect to $\alpha_1$ and $\alpha_2$. See Figure 1. We will call $C$ simply an amalgamation of $B_1$ and $B_2$ over $A$ (as in most cases $\alpha_1$ and $\alpha_2$ can be chosen to be inclusion embeddings).

We say that an amalgamation is strong when $\beta_1(x_1) = \beta_2(x_2)$ if and only if $x_1 \in \alpha_1(A)$ and $x_2 \in \alpha_2(A)$. Less formally, a strong amalgamation glues together $B_1$ and $B_2$ with an overlap no
greater than the copy of $A$ itself. A strong amalgamation is free if there are no tuples in any relations of $C$ spanning vertices from both $\beta_1(B_1 \setminus \alpha_1(A))$ and $\beta_2(B_2 \setminus \alpha_2(A))$.

An amalgamation class is a class $K$ of finite structures satisfying the following three conditions:

**Hereditary property:** For every $A \in K$ and a substructure $B$ of $A$ we have $B \in K$;

**Joint embedding property:** For every $A, B \in K$ there exists $C \in K$ such that $C$ contains both $A$ and $B$ as substructures;

**Amalgamation property:** For $A, B_1, B_2 \in K$ and $\alpha_1$ an embedding of $A$ into $B_1$, $\alpha_2$ an embedding of $A$ into $B_2$, there is a $C \in K$ which is an amalgamation of $B_1$ and $B_2$ over $A$ with respect to $\alpha_1$ and $\alpha_2$.

**Definition 2.1 (Type).** Let $L(X)$ denote the language $L \cup \{c_x : x \in X\}$, where each $c_x$ is a constant symbol interpreted as the vertex $x$. By a type over $X$, we mean a maximal satisfiable set of $L(X)$ formulas $p(x)$ with free variables $x$. The type $tp(a/X)$ of a tuple $a$ over $X$ is the set of all $L(X)$ formulas which are satisfied by $a$.

2.1. Edge-labelled graphs as relational structures

For notational convenience we introduced edge-labelled graphs (Definition 1.2) and we will view them, equivalently, also as relational structures in the language $L$ consisting of binary relations $R^1, R^2, \ldots$, which denote the distances. The notions of homomorphisms, embeddings and substructures in edge-labelled graphs correspond to the same notions for relational structures.

Given $G = (G, d) \in G^\infty$ and $G' = (G', d') \in G^\infty$ a homomorphism $G \to G'$ is a function $f: G \to G'$ such that $d(x, y) = d'(f(x), f(y))$ whenever $d(x, y)$ is defined. A homomorphism $f$ is an embedding (or isometry) if $f$ is one-to-one and $d(x, y) = d'(f(x), f(y))$ whenever either side of the equality makes sense. A surjective embedding is an isomorphism and an automorphism is an isomorphism $G \to G$. A graph $G$ is an (induced) subgraph of $H$ if the identity mapping is an embedding $G \to H$.

2.2. Ramsey expansions

Let $L^+$ be a language containing the language $L$. By this we mean $L \subseteq L^+$ and $L^+$ assigns the same arity as $L$ to all $R \in L$. Under these conditions, every structure $X = (X, (R_X; R \in L^+)) \in \text{Rel}(L^+)$ may be viewed as a structure $A = (X, (R_X; R \in L)) \in \text{Rel}(L)$ with some additional relations $R_X$ for $R \in L^+ \setminus L$. We call $X$ a expansion (or lift) of $A$ and $A$ is called the reduct (or shadow) of $X$. In this sense the class $\text{Rel}(L^+)$ is the class of all expansions of $\text{Rel}(L)$, or, conversely, $\text{Rel}(L)$ is the class of all shadows of $\text{Rel}(L^+)$.

The question about existence of an Ramsey expansion has been put into more precise setting by means of the following two definitions:
Definition 2.2 (Precompact expansion [NVT15]). Let $\mathcal{K}^+$ be a class of lifts to $L^+$ of $L$-structures in $\mathcal{K}$. We say that $\mathcal{K}^+$ is a precompact expansion (or lift) of $\mathcal{K}$ if for every structure $A \in \mathcal{K}$ there are only finitely many structures $A^+ \in \mathcal{K}^+$ such that $A^+$ is a lift of $A$ (i.e. the shadow of $A^+$ obtained by forgetting the relations in $L^+ \setminus L$ is isomorphic to $A$).

Definition 2.3 (Expansion property [NVT15]). Let a class $\mathcal{K}^+$ be a lift of $\mathcal{K}$. For $A, B \in \mathcal{K}$ we say that $\mathcal{K}^+$ has the expansion property for $A, B$ if for every lift $B^+ \in \mathcal{K}^+$ of $B$ there is an embedding of every lift $A^+ \in \mathcal{K}^+$ of $A$ into $B^+$.

$\mathcal{K}^+$ has the expansion property with respect to $\mathcal{K}$ if for every $A \in \mathcal{K}$ there is $B \in \mathcal{K}$ such that $\mathcal{K}^+$ has the expansion property for $A, B$.

It can be shown (see [NVT15]) that for every homogeneous class there is up to bi-definability at most one precompact Ramsey expansion with expansion property.

Our main tool for giving Ramsey property will be Theorem 2.1 of [HN19] which we introduce now after some additional definitions. An $L$-structure $A$ is an irreducible structure if every pair of vertices from $A$ is in some relation in $L$ (the relation need not be binary: for example, a complete $k$-hypergraph is irreducible).

Definition 2.4 (Homomorphism-embedding [HN19]). A homomorphism $f : A \rightarrow B$ is a homomorphism-embedding if for every irreducible substructure $C$ of $A$, the restriction of $f$ to $C$ is an embedding into $B$.

While for (undirected) graphs the homomorphism and homomorphism-embedding coincide, for general relational structures they may differ.

Definition 2.5 (Completion [HN19]). Let $C$ be a structure. An irreducible structure $C'$ is a (strong) completion of $C$ if there exists a one-to-one homomorphism-embedding $C \rightarrow C'$.

Of particular interest is the question of whether a completion of a given structure exists, such that the completed structure lies in a given class $\mathcal{K}$. If it does, we speak of its $\mathcal{K}$-completion.

Note that in [HN19] the definition of completion is weaker than the definition of strong completion. In this paper all completions will be implicitly strong.

Definition 2.6 (Locally finite subclass [HN19]). Let $\mathcal{R}$ be a class of finite irreducible structures and $\mathcal{K}$ a subclass of $\mathcal{R}$. We say that the class $\mathcal{K}$ is a locally finite subclass of $\mathcal{R}$ if for every $C_0 \in \mathcal{R}$ there is a finite integer $n = n(C_0)$ such that every structure $C$ has a strong $\mathcal{K}$-completion (i.e. there exists $C' \in \mathcal{K}$ that is a strong completion of $C$), provided that the following conditions are satisfied:

1. there is a homomorphism-embedding from $C$ to $C_0$ (in other words, $C_0$ is a, not necessarily strong, $\mathcal{R}$-completion of $C$), and,

2. every substructure of $C$ with at most $n$ vertices has a strong $\mathcal{K}$-completion.

Theorem 2.1 (Hubička–Nešetřil [HN19]). Let $\mathcal{R}$ be a Ramsey class of irreducible finite structures and let $\mathcal{K}$ be a hereditary locally finite subclass of $\mathcal{R}$ with strong amalgamation. Then $\mathcal{K}$ is Ramsey.

Explicitly: For every pair of structures $A, B$ in $\mathcal{K}$ there exists a structure $C \in \mathcal{K}$ such that

$C \rightarrow (B)^A_2$.

2.3. Coherent EPPA

The following is a strengthening of the Herwig–Lascar Theorem [HL00, Theorem 2] for coherent EPPA which will be our main tool to prove EPPA in this paper.

Definition 2.7 (Coherent maps [Sol09, SS19]). Let $X$ be a set and $\mathcal{P}$ be a family of partial bijections between subsets of $X$. A triple $(f, g, h)$ from $\mathcal{P}$ is called a coherent triple if

$\text{Dom}(f) = \text{Dom}(h), \text{Range}(f) = \text{Dom}(g), \text{Range}(g) = \text{Range}(h)$
and \[ h = g \circ f. \]

Let \( X \) and \( Y \) be sets, and \( \mathcal{P} \) and \( \mathcal{Q} \) be families of partial bijections between subsets of \( X \) and between subsets of \( Y \), respectively. A function \( \varphi: \mathcal{P} \to \mathcal{Q} \) is said to be a coherent map if for each coherent triple \( (f, g, h) \) from \( \mathcal{P} \), its image \( \varphi(f), \varphi(g), \varphi(h) \) in \( \mathcal{Q} \) is coherent.

**Definition 2.8** (Coherent EPPA [Sol09, SS19]). A class \( \mathcal{K} \) of finite \( L \)-structures is said to have coherent EPPA if \( \mathcal{K} \) has EPPA and moreover the extension of partial automorphisms is coherent. That is, for every \( A \in \mathcal{K} \), there exists \( B \in \mathcal{K} \) such that \( A \subseteq B \) and every partial automorphism \( f \) of \( A \) extends to some \( \tilde{f} \in \text{Aut}(B) \) with the property that the map \( \varphi \) from the partial automorphisms of \( A \) to the automorphism of \( B \) given by \( \varphi(f) = \tilde{f} \) is coherent.

**Theorem 2.2** (Solecki–Siniora [Sol09, SS19]). Let \( \mathcal{O} \) be a finite family of structures, \( A \in \text{Forb}(\mathcal{O}) \), and \( P \) be a set of partial isomorphisms of \( A \). If there exists a structure \( M \) containing \( A \) such that each element of \( P \) extends to an automorphism of \( M \) and moreover there is no \( O \in \mathcal{O} \) with a homomorphism \( O \to M \), then there exists a finite structure \( B \in \text{Forb}(\mathcal{O}) \) and \( \phi: P \to \text{Aut}(B) \) such that

1. \( \phi(p) \) is an extension of \( p \), and
2. \( \phi \) is coherent.

We will call a \( B \) with the properties as stated in Theorem 2.2 a coherent EPPA-witness of \( A \). Observe that while formulated differently, in our setting of strong amalgamation classes the conditions of Theorem 2.2 very similar to ones of Theorem [HN19]. The infinite structure extending all partial isomorphisms is the Fraïssé limit and thus in both cases we only need to show a bound on the size of obstacles for the completion algorithm. To show EPPA we additionally need to have a completion algorithm which preserve all symmetries.\(^5\)

3. **Cherlin’s catalogue of metrically homogeneous graphs**

Now we present Cherlin’s catalogue of metrically homogeneous graphs [Che22] and the relevant definitions. There is nothing new in this section and any divergence from [Che22] is a mistake.

**Conjecture 3.1** (Cherlin’s Metric Homogeneity Classification Conjecture [Che22]). The countable metrically homogeneous graphs are the following.

1. In diameter \( \delta \leq 2 \): the homogeneous graphs, classified by Lachlan and Woodrow [LW80].
2. In diameter \( \delta \geq 3 \):
   
   (a) The finite ones, classified by Cameron [Cam80].
   
   (b) Macpherson’s regular tree-like graphs \( T_{m,n} \) with \( 2 \leq m, n \leq \infty \),
   
   (c) The Fraïssé limits of amalgamation classes of the form \( A_3 \cap A_H \) with \( A_3 \) 3-constrained and \( A_H \) of Henson type or antipodal Henson type.

Let us remark that there is now a purported (yet unpublished) proof of this conjecture.\(^6\)

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\(^{5}\)This similarity has been formalized by a recent strengthening of the Solecki–Siniora theorem by Hubička, Konečný, and Nešetřil [HKN22].

\(^{6}\)This is claimed on Cherlin’s website: https://sites.math.rutgers.edu/~cherlin/Paper/inprep.html
3.1. 3-constrained spaces $A_3$

Recall the numerical parameters $\delta, K_1, K_2, C_0$ and $C_1$ introduced in Definition 1.6 to describe the 3-constrained spaces. Different numerical parameters can be used to describe the same classes of structures. To avoid this redundancy we will assume the following constraints which describe meaningful sets of parameters.

**Definition 3.1** (Acceptable numerical parameters). A sequence of parameters $(\delta, K_1, K_2, C_0, C_1)$ is acceptable if it satisfies the following conditions:

- $3 \leq \delta \leq \infty$;
- $1 \leq K_1 \leq K_2 \leq 2\delta$ or $K_1 = \infty$ and $K_2 = 0$;
- $2\delta < C_0, C_1$ and $C_0, C_1 \leq 3\delta + 2$. Here $C_0$ is even and $C_1$ is odd;
- If $K_1 = \infty$ (the bipartite case) then $C_1 = 2\delta + 1$.

**Theorem 3.2** (Cherlin’s Admissibility Theorem [Che22]). Let $(\delta, K_1, K_2, C_0, C_1)$ be an acceptable sequence of parameters (in particular, $\delta \geq 3$). Then the associated class $A^\delta_{K_1,K_2,C_0,C_1}$ is an amalgamation class if and only if one of the following three groups of conditions is satisfied, where we write $C$ for $\min(C_0,C_1)$ and $C'$ for $\max(C_0,C_1)$:

(I) $K_1 = \infty$ (the bipartite case; so $K_2 = 0$ and $C_1 = 2\delta + 1$).

(II) $K_1 < \infty, C \leq 2\delta + K_1$, and

- $C = 2K_1 + 2K_2 + 1$;
- $K_1 + K_2 \geq \delta$;
- $K_1 + 2K_2 \leq 2\delta - 1$, and:

(IIA) $C' = C + 1$, or

(IIIB) $C' > C + 1, K_1 = K_2, \text{ and } 3K_2 = 2\delta - 1$.

(III) $K_1 < \infty, C > 2\delta + K_1$, and:

- $K_1 + 2K_2 \geq 2\delta - 1 \text{ and } 3K_2 \geq 2\delta$;
- If $K_1 + 2K_2 = 2\delta - 1$ then $C \geq 2\delta + K_1 + 2$;
- If $C' > C + 1$ then $C \geq 2\delta + K_2$.

A sequence of parameters $(\delta, K_1, K_2, C_0, C_1)$ is called admissible if and only if it satisfies one of the three sets of conditions in Theorem 3.2.

**Example.** All admissible parameters with $\delta = 3$ are listed in Table 1.

3.2. Henson constraints

Suppose $\delta \geq 3$. A $(1, \delta)$-space is a metric space in which all distances are 1 or $\delta$; thus the relation $d(x,y) \leq 1$ is an equivalence relation, and the classes lie at mutual distance $\delta$. A $(1, \delta)$-space will also be called a Henson constraint.

A clique is a metric space $K = (K, d)$ where $d(u, v) = 1$ for every $u \neq v$. $K^* = (K, d^*)$ is an antipodal companion of the clique $K$ if there exists $S \subset K$ such that for distinct vertices $u, v$, $d^*(u, v) = \delta - 1$ if $u \in S, v \not\in S$ or vice versa and $d^*(u, v) = 1$ otherwise.

**Definition 3.2** (Acceptable parameters with Henson constraints). The sequence of parameters $(\delta, K_1, K_2, C_0, C_1, S)$ is acceptable if

1. $(\delta, K_1, K_2, C_0, C_1)$ is an acceptable sequence of numerical parameters.

2. $S$ is a set of $(\delta)$-Henson constraints, i.e., a set of $(1, \delta)$-spaces if $C_1 > 2\delta + 1$ or $C_0 > 2\delta + 2$, and a set of cliques and their antipodal companions if $C_1 = 2\delta + 1$ and $C_0 = 2\delta + 2$ (cf. Definition 7.2).
Table 1: All admissible parameters for $\delta = 3$ with the set $S$ of Henson constraints limited to meaningful choices [ACM21, Table 2] and parameter $M$ satisfying Definitions 4.4 and 6.1. $K_n$ denotes the $n$-clique (i.e. the metric space with $n$ vertices and all distances 1) and $I_n$ is the $n$-anticlique (all distances $\delta$). The second column lists the possible choices for magic distances (see Definition 6.1).

| $K_1$ | $K_2$ | $C_0$ | $C_1$ | $M$ | Case | $S$ | Structure |
|-------|-------|-------|-------|-----|------|----|-----------|
| $\infty$ | 0     | 8     | 7     | –   | (I)  | $\emptyset$ | Bipartite antipodal |
| $\infty$ | 0     | 10    | 7     | –   | (I)  | $\emptyset$ | Bipartite |
| 1      | 2     | 8     | 7     | –   | (IIA)| $\emptyset$ | Antipodal |
| 1      | 2     | 10    | 9     | 2   | (III)| $K_n$ | No $\delta\delta\delta, 1\delta\delta$ triangles |
| 1      | 2     | 10    | 11    | 2   | (III)| $K_n$ and/or $I_m$ | No $1\delta\delta$ triangles |
| 1      | 3     | 8     | 9     | 2   | (III)| $\emptyset$ | No 5-anticycle |
| 1      | 3     | 10    | 9     | 2   | (III)| Any w/o $I_3$ | No $\delta\delta\delta$ triangles |
| 1      | 3     | 10    | 11    | 2, 3| (III)| Any | All metric spaces |
| 2      | 2     | 10    | 9     | 2   | (III)| $\emptyset$ | No $\delta\delta\delta, 1\delta, 111$ |
| 2      | 2     | 10    | 11    | 2   | (III)| $I_m$ | No $1\delta\delta, 111$ triangles |
| 2      | 3     | 10    | 9     | 2   | (III)| Any w/o $K_3,I_3$ | No $\delta\delta\delta, 111$ triangles |
| 2      | 3     | 10    | 11    | 2, 3| (III)| Any w/o $K_3$ | No $111$ triangles |
| 3      | 3     | 10    | 11    | 3   | (III)| $\emptyset$ | No 5-cycle |

3. $S$ is *irredundant* in the sense that no constraint in $S$ contains triangles forbidden in $A_{K_1,K_2,C_0,C_1}^\delta$, and every constraint in $S$ consists of at least 4 vertices.

**Definition 3.3 (Admissible parameters with Henson constraints).** Let $(\delta, K_1, K_2, C_0, C_1, S)$ be an acceptable sequence of parameters, where $S$ is a set of $\delta$-Henson constraints. Let $C = \min(C_0, C_1)$ and $C' = \max(C_0, C_1)$. The sequence is *admissible* if

1. The sequence $(\delta, K_1, K_2, C_0, C_1)$ is admissible.
2. If $C = 2\delta + 1$, $K_1 < \infty$, and $S$ is nonempty, then $\delta \geq 4$ and $S$ consists of a clique and all its antipodal companions.
3. If $K_1 \leq \infty$ and $C > 2\delta + K_1$ (Case (III)), then
   - If $K_1 = \delta$ then $S$ is empty;
   - If $C = 2\delta + 2$ then $S$ is empty.

4. **Primitive 3-constrained spaces**

In this section we will work with fixed admissible parameters $\delta < \infty$, $K_1$, $K_2$, $C_0$ and $C_1$, such that the associated homogeneous metric space $\Gamma_{K_1,K_2,C_0,C_1}^\delta$ is primitive, i.e. it has no non-trivial definable equivalence relation on it. As one can verify, those are exactly the parameters where the class $A_{K_1,K_2,C_0,C_1}^\delta$ has strong amalgamation and contains at least one triangle with odd perimeter. Recall that we denote $C = \min(C_0, C_1)$, and $C' = \max(C_0, C_1)$. Then we can characterise the parameters as follows:

**Definition 4.1.** The admissible parameters $\delta < \infty$, $K_1$, $K_2$, $C_0$ and $C_1$ are *primitive* when they satisfy case (II) or (III) of Theorem 3.2 and moreover do not form an antipodal space.

In other words, the triangle $\delta\delta a$ for some $a \geq 1$ is permitted and

$$C_0,C_1 \geq 2\delta + 2.$$  

For many of the lemmas in this section, the inequalities $C_0,C_1 \geq 2\delta + 2$ will be an essential assumption.

Recall that $A_{K_1,K_2,C_0,C_1}^\delta$ was defined in Definition 1.6 as the class of all metric spaces that satisfy some constraints on its triangles, i.e. its 3-element subspaces. In the following it will often
be more convenient to think of $A_{K_1,K_2,c_0,c_1}^d$ as the class of metric spaces that do not embed any triangle violating those constraints — we are going to refer to such triangles as forbidden triangles.

We also will slightly abuse notation and use the term triangle for both triplets of vertices $u,v,w$ and for triplets of edges $a,b,c$ with $a = d(u,v), b = d(v,w), c = d(u,w)$. By Definition 1.6 a triangle $abc$ is forbidden if it satisfies one of the following conditions:

Non-metric: $a,b,c$ is forbidden if $a+b < c$,

$K_1$-bound: $a+b+c < 2K_1 + 1$ and $a+b+c$ is odd,

$K_2$-bound: $b+c \geq 2K_2 + a$ and $a+b+c$ is odd and $a \leq b,c$,

$C_1$-bound: $a+b+c \geq C_1$ and $a+b+c$ is odd,

$C_0$-bound: $a+b+c \geq C_0$ and $a+b+c$ is even,

$C$-bound: If $|C_0-C_1| = 1$, the $C_1$-bound and $C_0$-bound can be expressed together as $a+b+c \geq C$, where $C = \min(C_0, C_1)$.

Triangles that are not forbidden will be called allowed.

4.1. Generalised completion algorithm for 3-constrained classes

Let $D = \{1, 2, \ldots, \delta\}$ be a collection of (ordered) pairs. It is more natural to consider unordered pairs, but notationally easier to consider ordered pairs. We will refer to elements of $D$ as forks.

Consider a $\delta$-bounded variant of the shortest path completion, where we assume that the input graphs contain no distances greater than $\delta$ and in the output all edges longer than $\delta$ are replaced by an edge of that length. There is an alternative formulation of this completion: For a fork $\vec{f} = (a,b)$, define $d^+(\vec{f}) = \min(a+b,\delta)$. In the $i$-th step look at all incomplete forks $\vec{f}$ (i.e. triples of vertices $u,v,w$ such that exactly two edges are present) such that $d^+(\vec{f}) = i$ and define the length of the missing edge to be $i$.

This algorithm proceeds by first adding edges of length 2, then edges of length 3 and so on up to edges of length $\delta$ and has the property that out of all metric completions of a given graph, every edge of the completion yielded by this algorithm is as close to $\delta$ as possible.

It makes sense to ask what happens if, instead of trying to make each edge as close to $\delta$ as possible, one would try to make each edge as close to some parameter $M$ as possible. For $M$ in a certain range, such an algorithm exists. For each fork $\vec{f} = (a,b)$ one can define $d^+(\vec{f}) = a+b$ and $d^-(\vec{f}) = |a-b|$, i.e. the largest and the smallest possible distance that can metrically complete the fork $\vec{f}$. The generalised algorithm will complete $\vec{f}$ by $d^+(\vec{f})$ if $d^+(\vec{f}) < M$, by $d^-(\vec{f})$ if $d^-(\vec{f}) > M$ and by $M$ otherwise. It turns out that there is a good permutation $\pi$ of $\{1,\ldots,\delta\}$, such that if one adds the distances in the order prescribed by the permutation, this generalised algorithm will produce a correct completion whenever one exists. It is easy to check that the choice $M = \delta$ and $\pi = \text{id}_\delta$ corresponds to the shortest path completion algorithm.

Definition 4.2 (Completion algorithm). Given $c \geq 1$, $F \subseteq D$, and a graph $G = (G,d) \in G^\delta$, we say that $G' = (G,d')$ is the $(F,c)$-completion of $G$ if $d'(u,v) = d(u,v)$ whenever $u,v$ is an edge of $G$ and $d'(u,v) = c$ if $u,v$ is not an edge of $G$ and there exist $(a,b) \in F, w \in G$ such that $(d(u,w),d(v,w)) = \{a,b\}$. There are no other edges in $G'$.

Given $1 \leq M \leq \delta$, a one-to-one function $t: \{1,2,\ldots,\delta\} \setminus M \to \mathbb{N}$ and a function $F$ from $\{1,2,\ldots,\delta\} \setminus M$ to the power set of $D$, we define the $(F,t,M)$-completion of $G$ as the limit of a sequence of edge-labelled graphs $G_1,G_2,\ldots$ such that $G_1 = G$ and $G_{k+1} = G_k$ if $t^{-1}(k)$ is undefined and $G_{k+1}$ is the $(F(t^{-1}(k)),t^{-1}(k))$-completion of $G_k$ otherwise, with every pair of vertices not forming an edge in this limit set to distance $M$.

We will call the vertex $w$ from Definition 4.2 the witness of the edge $u,v$. The function $t$ is called the time function of the completion because edges of length $a$ are inserted to $G_{t(a)}$ the $t(a)$-th step of the completion. If for a $(F,t,M)$-completion and distances $a,c$ there is a distance $b$ such that $(a,b) \in F(c)$ (i.e. the algorithm might complete a fork $(a,b)$ with distance $c$), we say that $c$ depends on $a$. 

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Definition 4.3 (Magic distances). Let $M \in \{1, 2, \ldots, \delta\}$ be a distance. We say that $M$ is magic (with respect to $A_{K_1, K_2, C_0, C_1}$) if

$$\max \left( K_1, \left\lceil \frac{\delta}{2} \right\rceil \right) \leq M \leq \min \left( K_2, \left\lfloor \frac{C - \delta - 1}{2} \right\rfloor \right).$$

Note that for primitive admissible parameters $(\delta, K_1, K_2, C_0, C_1)$ such an $M$ always exists.

Observation 4.1. The set of magic distances (with respect to $A_{K_1, K_2, C_0, C_1}$) is

$$S = \{1 \leq a \leq \delta : aab \text{ is allowed for all } 1 \leq b \leq \delta\}.$$

Proof. If a distance $a$ is in $S$, then $a \geq K_1$ (otherwise the triangle $aa1$ has perimeter $2a+1$, which is odd and smaller than $2K_1+1$, hence forbidden by the $K_1$ bound), $a \geq \left\lfloor \frac{\delta}{2} \right\rfloor$ (otherwise the triangle $a\delta$ is non-metric), $a \leq \left\lceil \frac{C - \delta - 1}{2} \right\rceil$ (otherwise the triangle $aab$ has perimeter $C$ for $b = C - 2a \leq \delta$), and $a \leq K_2$ (otherwise the triangle $aa1$ has odd perimeter and $2a \geq 2K_2 + 1$, hence is forbidden by the $K_2$ bound). The other implication follows from the definition of $A_{K_1, K_2, C_0, C_1}$. □

Remark 4.1. A Sage-implementation of the following completion algorithm is available at [Paw17]. It also contains all the examples discussed in this paper.

The same algorithm is used in our earlier paper [ABWH+21], although there we restrict to a special set of parameters. The analysis here is significantly more detailed and it is necessary to consider the cases where $K_2 < \delta$ or $C' > C + 1$. It is somewhat surprising that considering these cases does not make the algorithm significantly more complicated.

Let $M$ be a magic distance and $x \in \{1, \ldots, \delta\} \setminus \{M\}$. Define

$$\mathcal{F}_x^+ = \{(a, b) \in \mathcal{D} : a + b = x\},$$

$$\mathcal{F}_x^- = \{(a, b) \in \mathcal{D} : |a - b| = x\},$$

$$\mathcal{F}_x^C = \{(a, b) \in \mathcal{D} : C - 1 - a - b = x\}.$$

We further denote

$$\mathcal{F}_M(x) = \begin{cases} \mathcal{F}_x^+ \cup \mathcal{F}_x^C & \text{if } x < M \\ \mathcal{F}_x^- & \text{if } x > M. \end{cases}$$

For a magic distance $M$, we also define the function $t_M : \{1, \ldots, \delta\} \setminus \{M\} \to \mathbb{N}$ as

$$t_M(x) = \begin{cases} 2x - 1 & \text{if } x < M \\ 2(\delta - x) & \text{if } x > M. \end{cases}$$

Forks and how they are completed according to $\mathcal{F}_M$ are schematically depicted in Figure 2.

Definition 4.4 (Completion with magic parameter $M$). Let $M$ be a magic distance satisfying the following extra conditions:
1. If the parameters satisfy Case (III) with $K_1 + 2K_2 = 2\delta - 1$, then $M > K_1$;

2. if the parameters satisfy Case (III) and further $C' > C + 1$ and $C = 2\delta + K_2$, then $M < K_2$.

We then call the $(\mathbb{F}, t_M, M)$-completion (of $G$) the completion (of $G$) with magic parameter $M$.

**Remark 4.2.** The completion with magic parameter $M$ can be equivalently stated as a shortest path completion, but using a different ordered monoid (see Section 10, paragraph 4).

Our main goal of the following section is the proof of Theorem 4.9 that shows that the completion of $G$ with magic parameter $M$ lies in $A^{\delta}_{K_1, K_2, C_0, C_1}$ if and only if $G$ has some completion in $A^{\delta}_{K_1, K_2, C_0, C_1}$.

The two extra conditions in Definition 4.4 are a way to deal with certain extremal choices of admissible primitive parameters $(\delta, K_1, K_2, C_0, C_1)$. We are going to check that also with those extra conditions, there will always exist a suitable magic distance:

**Lemma 4.2.** For primitive parameters $(\delta, K_1, K_2, C_0, C_1)$ there is always an $M$ satisfying Definitions 4.3 and 4.4.

**Proof.** For Case (III) with $K_1 + 2K_2 = 2\delta - 1$, we proceed as follows: From admissibility, we have

$$K_1 + 2K_2 = 2\delta - 1$$

$$3K_2 \geq 2\delta$$

so we conclude that $K_1 < K_2$. From this information and $K_1 + 2K_2 = 2\delta - 1$, we derive $K_1 < \frac{3}{2} \delta$.

We know that $\delta - 1 \geq \frac{3}{2} \delta$ for $\delta \geq 3$ and $K_1 \leq \delta - 2$, so $\left\lceil \frac{C - \delta - 1}{2} \right\rceil \geq \left\lceil \frac{\delta + K_1 + 1}{2} \right\rceil \geq \frac{\delta + K_1}{2}$. Hence, $K_1 < \left\lfloor \frac{C - \delta - 1}{2} \right\rfloor$ and there is always a magic number greater than $K_1$.

In Case (III) with $C' > C + 1$ and $C = 2\delta + K_2$, we know from admissibility that $C > 2\delta + K_1$, so $K_2 > K_1$. Now we need $\left\lfloor \frac{3}{2} \right\rfloor < K_2$. For $\delta \geq 3$, the inequality $\left\lfloor \frac{3}{2} \right\rfloor \leq \frac{3}{2} \delta$ holds with equality only for $\delta = 3$. Admissibility tells us $3K_2 \geq 2\delta$. Now, if $\delta > 3$ or $K_2 \neq \frac{3}{2} \delta$, it follows that $\left\lfloor \frac{3}{2} \right\rfloor < K_2$.

The only remaining possibility is $\delta = 3$ and $K_2 = 2$, which implies $C = 8$ and $K_1 = 1$, which gives us $2K_2 + K_1 = 5 = 2\delta - 1$. The admissibility condition $C \geq 2\delta + K_1 + 2$ then yields $C \geq 9$, a contradiction. Hence there always is a magic number smaller than $K_2$.

If both these situations occur simultaneously, then we further require $M$ with $K_1 < M < K_2$. But that follows as $C = 2\delta + K_2$ and whenever $K_1 + 2K_2 = 2\delta - 1$, from admissibility we have $C \geq 2\delta + K_1 + 2$, hence $K_2 \geq K_1 + 2$.

Observe that the algorithm only makes use of $C$, $\delta$ and $M$. The interplay of individual parameters of algorithm is schematically depicted in Figure 3.

**Example (Case (IIB)).** In our proofs, Case (IIB) will often form a special case. The smallest (in terms of diameter) set of acceptable parameters that is in Case (IIB) is:

$$\delta = 5, C = C_1 = 13, C' = C_0 = 16, K_1 = K_2 = \frac{2\delta - 1}{3} = 3.$$  

Here $M = 3$, and it is the only choice for a magic number.

Forbidden triangles are those that are non-metric (113, 114, 115, 124, 125, 135, 225), or rejected for the $K_1$-bound (111, 122), the $K_2$-bound (144, 155, 245), or the $C_1$-bound (355, 445, 555). There are no triangles forbidden by $C_0$. Table 2 lists all possible completions of forks, with the completion preferred by our algorithm in bold type. Completions for forks in this class depicted in Figure 4. Notably, the magic number $M = 3$ is chosen for all forks except (1, 1, 1), which is completed by $d^+((1, 1, 1)) = 2$, (1, 1, 5), which is completed by $d^-((1, 1, 5)) = 4$, and (5, 5), which is a $C$-bound case. Those cases are the only forks where $M = 3$ cannot be chosen, so instead the algorithm chooses the nearest possible completion. What makes Case (IIB) special is the situation where one can choose $M - 1$ or $M + 1$ but not $M$ when completing a fork (for $\delta = 5$ it is the
fork (5,5), as both the triangles 5,5,2 and 5,5,4 are allowed, while 5,5,3 is forbidden by the $C_1$ bound; this behaviour is going to force us to deal with some corner cases later).

The algorithm will thus effectively run in three steps. First (at time 2) it will complete all forks $(1,5)$ with distance 4, next (at time 3) it will complete all forks $(1,1)$ and $(5,5)$ with distance 2 and finally it will turn all non-edges into edges of distance 3. Examples of runs of this algorithm are given later, see Figures 6 and 7.

See also [ABWH+21] for an additional example of a run of the algorithm for the space $A_{3,3,16,13}$, or see the Sage implementation in [Paw17].

4.2. What do forbidden triangles look like?

The majority of the proofs in the following sections assume that the completion algorithm with magic parameter $M$ introduces some forbidden triangle and then we argue that the triangle must be forbidden in any completion, hence the input structure has no completion into $A_{K_1,K_2,C_0,C_1}^M$. In such an argument it will be helpful to know how the different types of forbidden triangles relate to the magic parameter $M$. Therefore, in the following paragraphs we will study how triangles forbidden by different bounds are related to the magic parameter $M$. We will use $a,b,c$ for the lengths of the sides of the triangle and without loss of generality assume $a \leq b \leq c$. All conclusions are summarised in Figure 5.

**non-metric:** If $a + b < c$, then $a < M$, because otherwise $a + b \geq 2M \geq \delta$.

**$K_1$-bound:** If $a + b + c < 2K_1 + 1$, $a + b + c$ is odd and $abc$ is metric, then $a,b,c < K_1 \leq M$, because if $c \geq K_1$, then from the metric condition $a + b \geq c \geq K_1$ and hence $a + b + c \geq 2K_1$, for odd $a + b + c$ this means $a + b + c \geq 2K_1 + 1$.

**$C$-bound:** If $a + b + c \geq C$ then $b,c > M$. Suppose for a contradiction that $a,b \leq M$. We then have $a + b \geq C - c \geq C - \delta$, but on the other hand $a + b \leq 2M \leq 2\lfloor \frac{C-\delta-1}{2} \rfloor \leq C - \delta - 1$,
Table 2: Possible ways to complete \((i, j)\) forks, the bold number is the completion with magic parameter \(M = 3\).

| \(i = 1\) | \(j = 1\) | \(j = 2\) | \(j = 3\) | \(j = 4\) | \(j = 5\) |
|---|---|---|---|---|---|
| 2 | 1, 3 | 2, 3, 4 | 3, 5 | 4 | 3, 5 |
| 1, 2, 3, 4, 5 | 1, 2, 3, 4, 5 | 2, 3, 4 | 1, 3, 5 | 2, 4 |

which together yield \(C - \delta - 1 \geq C - \delta\), a contradiction. Note that in some cases \(C' \neq C + 1\), but this observation still holds, as it only uses \(a + b + c \geq C\).

**K2-bound:** If \(abc\) is a metric triangle with odd perimeter, then \(abc\) breaks the \(K2\) condition if and only if \(b + c \geq 2K2 + a + 1\) (the 1 on the right side comes from \(a + b + c\) being odd and all distances being integers). Then \(b, c > K2\), because if \(b \leq K2\), from metricity we have \(c \leq a + b\), hence \(a + 2K2 \geq (a + b) + b \geq c + b \geq 2K2 + a + 1\), a contradiction.

Moreover, in Case (III) of Theorem 3.2 we have \(a \leq K1\) because if \(a > K1\), we have \(b + c \geq 2K2 + a + 1 > 2K2 + K1 + 1\) and from admissibility conditions for Case (III) we have \(2K2 + K1 \geq 2\delta - 1\), which gives \(b + c > 2\delta\), a contradiction. (Note that if \(2K2 + K1 > 2\delta - 1\), we have \(a < K1\).)

Finally if \(a + b + c < C\) (which is stronger than not being forbidden by the \(C\) bound, as it also includes the \(C' > C + 1\) cases) and we are in Case (II) (where \(C = 2K1 + 2K2 + 1\)), we get \(a < K1\), because if \(a \geq K1\), we would get \(a + b + c \geq 2K2 + 2a + 1 \geq 2K2 + 2K1 + 1 = C\).

Note that later we shall refer to all the corner cases mentioned in these paragraphs.

**4.3. Basic properties of the completion algorithm**

In this section we develop several technical observations about the algorithm which will be used in Section 4.4 to show the main result about the correctness of the algorithm.

Recall the definition of \(t_M\) and \(F_M\):

\[
t_M(x) = \begin{cases} 
2x - 1 & \text{if } x < M \\
2(\delta - x) & \text{if } x > M. 
\end{cases}
\]

\[
F_M(x) = \begin{cases} 
F_x^+ \cup F_x^C & \text{if } x < M \\
F_x^- & \text{if } x > M. 
\end{cases}
\]

Intuitively, the function \(F_M\) selects the forks that will be completed to triangles with an edge of type \(t_M^{-1}(x)\) at time \(x\). At time 0 it looks for forks that can be completed with distance \(\delta\), then
with distance 1, jumping back and forth on the distance set and approaching \( M \) (cf. Figure 3). Observe that all forks that cannot be completed with \( M \) are in some \( F_M(x) \).

Now we shall precisely state and prove that \( t_M \) gives a suitable injection for the algorithm, as claimed before Definition 4.2.

**Lemma 4.3 (Time Consistency Lemma).** Let \( a, b \) be distances different from \( M \). If \( a \) depends on \( b \), then \( t_M(a) > t_M(b) \).

**Proof.** We consider three types of forks used by the algorithm:

\( \mathcal{F}^+ \): If \( a < M \) and \( \mathcal{F}_a^+ \neq \emptyset \), then \( b < a < M \), hence \( t_M(b) < t_M(a) \).

\( \mathcal{F}^C \): If \( a < M \) and \( \mathcal{F}_a^C \neq \emptyset \), then we must have \( b, c > M \) \( \mathcal{F}_a^C \) with \( a = C - 1 - b - c \).

Otherwise, if for instance \( b \leq M \), then \( C - \delta - 1 < C - 1 - c = a + b < 2M \leq 2 \left\lfloor \frac{C - \delta - 1}{2} \right\rfloor \), a contradiction. As \( C \geq 2\delta + 2 \) (we are dealing with the primitive case), we obtain the inequality \( b = (C - 1) - c - a \geq (2\delta + 1) - \delta - a = \delta + 1 - a \). Hence \( t_M(b) \leq 2(a - 1) < 2a - 1 = t_M(a) \).

\( \mathcal{F}^- \): Finally, we consider the case where \( a > M \) and \( \mathcal{F}_a^- \neq \emptyset \). Then either \( a = b - c \), which implies \( b > a > M \) and thus \( t_M(b) < t_M(a) \), or \( a = c - b \), which means \( b = c - a \leq \delta - a \). Because \( a > M \geq \left\lfloor \frac{\delta}{2} \right\rfloor \), we have \( b < M \). So \( t_M(b) \leq 2(\delta - a) - 1 < 2(\delta - a) = t_M(a) \). \( \square \)

**Lemma 4.4 (\( F_M \) Completeness Lemma).** Let \( G \in \mathcal{G}^\delta \) and \( \mathcal{G} \) be its completion with magic parameter \( M \). If there is a forbidden triangle (w.r.t. \( \mathcal{A}^\delta_{K_1,K_2,C_0,C_1} \)) or a triangle with perimeter at least \( C \) in \( \mathcal{G} \) with an edge of length \( M \), then this edge is also present in \( G \).

Observe that for \( C' \neq C + 1 \), this lemma is talking not only about forbidden triangles, but about all triangles with perimeter at least \( C \).

**Proof.** By Observation 4.1 no triangle of type \( aMM \) is forbidden, so suppose that there is a forbidden triangle \( abM \) in \( \mathcal{G} \) such that the edge of length \( M \) is not in \( G \). For convenience define \( t_M(M) = \infty \), which corresponds to the fact that edges of length \( M \) are added in the last step.

**non-metric:** If \( abM \) is non-metric then either \( a + b < M \) or \( |a - b| > M \). By Lemma 4.3 we have in both cases that \( t_M(a + b) \) (respectively, \( t_M(|a - b|) \)) is greater than both \( t_M(a) \) and \( t_M(b) \). Therefore the completion algorithm would chose \( a + b \) (resp. \( |a - b| \)) as the length of the edge instead of \( M \).

**K_1-bound:** Now that we know that \( abM \) is metric, we also know that it is not forbidden by the \( K_1 \) bound, because \( M \geq K_1 \).

**C-bound:** If \( a + b + M \geq C \) (which includes all the triangles forbidden by \( C_0 \) or \( C_1 \) bounds), then \( t_M(C - 1 - a - b) > t_M(a), t_M(b) \) by Lemma 4.3, so the algorithm would set \( C - 1 - a - b \) instead of \( M \) as the length of the third edge.

**K_2-bound:** Finally we deal with the \( K_2 \) bound. Suppose that \( abM \) is metric, its perimeter is less than \( C \), and it is forbidden by the \( K_2 \) bound. From Section 4.2 we have that the two long edges have to be longer than \( K_2 \), and the shortest edge is at most \( K_1 \) with equality only in Case (III) with \( K_1 + 2K_2 = 2\delta - 1 \).

As \( M \leq K_2 \), we know that \( M \) is the shortest edge. But also \( M \geq K_1 \), hence this situation can happen only when \( K_1 \) is the length of the shortest edge, which is only in Case (III) with \( K_1 + 2K_2 = 2\delta - 1 \). But from Definition 4.4 we have in this case \( M > K_1 \). Hence this situation never occurs. \( \square \)
It may seem strange that the algorithm does not differentiate between \( C_0 \) and \( C_1 \). The following observation justifies this by showing that in the case where \( C' > C + 1 \), these bounds have a relatively limited effect on the run of the algorithm.

**Observation 4.5.** If \( C' > C + 1 \), then either \( \mathcal{F}_x^C \) is empty for all \( x < M \) or the parameters satisfy (IIB). In the latter case, only \( \mathcal{F}_{M-1}^C = \{ (\delta, \delta) \} \) is non-empty. Furthermore, in this case \( t_M(M-1) \) is the maximum of the time-function. This implies that \( (\delta, \delta) \)-forks are completed to \( M - 1 \) in the penultimate step of the completion algorithm.

**Proof.** Consider a fork \( (a, b) \in \mathcal{F}_x^C \) and the cases where \( C' > C + 1 \) is allowed.

In Case (III) with \( C' > C + 1 \) we have (by admissibility) \( C \geq 2\delta + K_2 \), so \( x = C - a - b \geq K_2 - 1 \) with equality only for \( C = 2\delta + K_2 \). From the extra condition for a magic parameter in Definition 4.4, we get that \( M < K_2 \).

In Case (IIB) we have \( M = K_2 = K_1 = \frac{2\delta - 1}{2} \), hence \( C = 2K_1 + 2K_2 + 1 = 2\delta + K_2 \), thus again we have \( C - 1 - a - b \geq K_2 - 1 \). This means that the only fork in \( \mathcal{F}_{M-1}^C \) is going to be \( (\delta, \delta) \), which will be completed by \( K_2 - 1 = M - 1 \).

In order to see that \( t_M(M-1) \) is maximal, it is enough to check \( t_M(M+1) < t_M(M-1) \). We have \( 3M = 3K_2 = 2\delta - 1 \), so by definition \( t_M(M-1) = 2M - 3 \) and \( t_M(M+1) = 2\delta - 2M - 2 \), so we want \( 2M - 3 > 2\delta - 2M - 2 \), or \( 4M > 2\delta + 1 \) which is true for \( \delta \geq 5 \) and this always holds in Case (IIB). So \( t_M(M-1) > t_M(M+1) \) and therefore \( t_M(M-1) > t_M(a) \) for any \( a \) different from \( M \) and \( M - 1 \).

**Lemma 4.6** (Optimality Lemma). Let \( G = (G, d) \in \mathcal{G}_d^\delta \) such that it has a completion in \( \mathcal{A}_d^{K_1, K_2, C_0, C_1} \). Denote by \( \overline{G} = (G, \overline{d}) \) the completion of \( G \) with magic parameter \( M \) and let \( G' = (G, d') \in \mathcal{A}_d^{K_1, K_2, C_0, C_1} \) be an arbitrary completion of \( G \). Then for every pair of vertices \( u, v \in G \) one of the following holds:

1. \( d'(u,v) \geq \overline{d}(u,v) \geq M \),
2. \( d'(u,v) \leq \overline{d}(u,v) \leq M \),
3. the parameters \( (\delta, K_1, K_2, C_0, C_1) \) satisfy Case (IIB), \( \overline{d}(u,v) = M - 1 \), \( d'(u,v) > M \) and \( d'(u,v) \) has the same parity as \( d(u,v) \).

Note that for \( \overline{d}(u,v) = M \) the statement trivially holds.

**Proof.** Suppose that the statement is not true and take any witness \( G' = (G, d') \) (i.e. a completion of \( G \)) into \( \mathcal{A}_d^{K_1, K_2, C_0, C_1} \) such that there is a pair of vertices violating the statement). Recall that the completion with magic parameter \( M \) is defined as a limit of a sequence \( G_1, G_2, \ldots \) of edge-labelled graphs such that \( G_1 = G \) and each two subsequent graphs differ at most by adding edges of a single distance.

Take the smallest \( i \) such that in the graph \( G_i = (G, d_i) \) there are vertices \( u, v \in G \) with \( d_i(u,v) > M \) and \( d_i(u,v) > d'(u,v) \) or \( d_i(u,v) < M \) and \( d_i(u,v) < d'(u,v) \). Let \( w \in G \) be the witness of \( d_i(u,v) \). In Case (IIB), by Observation 4.5 edges of length \( M - 1 \) are added in the last step of our completion algorithm. Therefore we know that the distances \( d_{i-1}(u,w) \) and \( d_{i-1}(v,w) \) satisfy the optimality conditions in point 1 or 2.

We shall distinguish three cases, based on whether \( d_i(u,v) \) was introduced by \( \mathcal{F}^- \), \( \mathcal{F}^+ \) or \( \mathcal{F}^C \):

**\( \mathcal{F}^- \) case.** We have \( M < d_i(u,v) = |d_{i-1}(u,w) - d_{i-1}(v,w)| \). Without loss of generality let us assume \( d_{i-1}(u,w) > d_{i-1}(v,w) \), which means that \( d_{i-1}(u,w) > M \) and \( d_{i-1}(v,w) < M \) (as \( M \geq \left\lceil \frac{\delta}{2} \right\rceil \)). From the minimality of \( i \), it follows that \( d'(u,w) \geq d_{i-1}(u,w) \) and \( d'(v,w) \leq d_{i-1}(v,w) \). Since \( G' \) is metric we have \( d_i(u,v) = d_{i-1}(u,w) - d_{i-1}(v,w) \leq d'(u,w) - d'(v,w) \leq d'(u,v) \), which is a contradiction.

**\( \mathcal{F}^+ \) case.** We have \( M > d_i(u,v) = d_{i-1}(u,w) + d_{i-1}(v,w) \), hence \( d_{i-1}(u,w), d_{i-1}(v,w) < M \). By the minimality of \( i \) we have \( d'(u,w) \leq d_{i-1}(u,w) \) and \( d'(v,w) \leq d_{i-1}(v,w) \). Since \( G' \) is metric, we get \( d'(u,v) \leq d_i(u,v) \), which contradicts our assumptions.
We have $M > d_i(u, v) = C - 1 - d_{i-1}(u, w) - d_{i-1}(v, w)$.

First suppose that $C' = C + 1$. Recall that, by the admissibility of $C$, we have $C - 1 \geq 2\delta + 1$ and $M \leq \lceil \frac{C - \delta + 1}{2} \rceil$. Thus we get $d_{i-1}(u, w), d_{i-1}(v, w) > M$ (otherwise, if, say, $d_{i-1}(u, w) \leq M$, we obtain the contradiction $C - \delta - 1 \geq 2M > d_{i-1}(u, w) + d_i(u, v) = C - 1 - d_{i-1}(v, w) \geq C - \delta - 1$). So again $d'(u, v) \geq d_{i-1}(u, w)$ and $d'(v, w) \geq d_{i-1}(v, w)$, which means that the triangle $u, v, w$ in $G'$ is forbidden by the $C$ bound, which is absurd as $G'$ is a completion of $G$ in $A_{K_1, K_2, C_0, C_1}^+$. It remains to discuss the case where $C' > C + 1$. By Observation 4.5, we only need to consider Case (IIb), $d_i(u, v) = K_2 - 1 = M - 1$ and $d_{i-1}(u, w) = d_{i-1}(v, w) = \delta$. By our assumption we have $d'(u, v) > d_i(u, v)$. Hence if $d'(u, v) \geq M$ it has to have the same parity as $d_i(u, v)$ (otherwise the triangle $u, v, w$ would be forbidden in $G'$ by the $C$ bound).

Next we show that the algorithm initially runs in a way that preserves the parity of completions to $A_{K_1, K_2, C_0, C_1}^+$.

**Lemma 4.7** (Parity Lemma). Let $G$, $\overline{G}$ and $G'$ be as in Lemma 4.6. Then for every pair of vertices $u, v \in G$ such that either $\overline{d}(u, v) \leq \min(K_1, M - 1)$ or $d(u, v) \geq \max(K_2, M + 1)$, at least one of the following holds:

1. The parity of $\overline{d}(u, v)$ is the same as the parity of $d'(u, v)$;
2. the parameters come from Case (IIb), $C = 2\delta + K_1 + 1$, $C \neq 2K_1 + 2K_2 + 1$, $M > K_1 > 1$ and $\overline{d}(u, v) = K_1$.

Note that we are only interested in distances not equal to $M$.

**Proof.** Suppose that the statement is not true, and let $G' = (G, d')$ be a counterexample. Recall that the completion with magic parameter $M$ is defined as a limit of a sequence $G_1, G_2, \ldots$ of edge-labelled graphs such that $G_1 = G$ and each two subsequent edge-labelled graphs differ at most by adding edges of a single distance.

Take the smallest $i$ such that in $G_i = (G, d_i)$ there are vertices $u, v \in G$ with $d_i(u, v)$ and $d'(u, v)$ not satisfying the lemma. Denote by $w$ a witness of the distance $d_i(u, v)$. As in the proof Lemma 4.6, we can argue that $d_{i-1}(u, w)$ respectively $d_{i-1}(v, w)$ satisfy the optimality conditions 1 or 2 in Lemma 4.6.

First we will show that the exceptional case 2 from the statement only happens at the very end of the induction, hence when using the induction hypothesis (or minimality of $i$), we can work only with the first part of the statement.

Suppose that the parameters satisfy Case (IIb) and further $C = 2\delta + K_1 + 1$, $C \neq 2K_1 + 2K_2 + 1$ and $M > K_1 > 1$. We have $t_M(K_1) > t_M(a)$ for any distance $a < K_1$ and also, by admissibility, $t_M(K_1) > t_M(b)$ for any distance $b \geq K_2$ and $b > M$: since $t_M(K_1) = 2K_1 - 1$ and $t_M(b) \leq 2\delta - 2K_2$, we need to verify that $2K_1 - 1 > 2\delta - 2K_2$ and thus $2K_1 + 2K_2 > 2\delta + 1$ by admissibility it follows $2K_2 + K_1 \geq 2\delta$ (when $2K_2 + K_1 = 2\delta - 1$, admissibility implies $C \geq 2\delta + K_1 + 2$), which give the desired bound.

Next observe that if $K_1 = 1$ then from Lemma 4.6 we have that whenever $\overline{d}(u, v) = 1$ for some vertices $u, v$, then in any completion the edge has also length 1, hence also fixed parity.

As in the proof of Lemma 4.6, we will now distinguish three cases based on whether $d_i(u, v)$ was introduced due to $\mathcal{F}^+$, $\mathcal{F}^-$ or $\mathcal{F}^C$:

**$\mathcal{F}^+$ case.** In this case $d_i(u, v) < M$ and $d_i(u, v) = d_{i-1}(u, w) + d_{i-1}(v, w)$. Because of our assumption $d_i(u, v) \leq K_1$, the perimeter of the triangle $uvw$ in $G_i$ is even and at most $2K_1$. By Lemma 4.6 either the third possibility happened, hence $\overline{d}(u, v)$ has the same parity as $d'(u, v)$, or we have $d'(u, v) \leq d_i(u, v), d'(u, v) \leq d_i(u, w)$ and $d'(v, w) \leq d_{i-1}(v, w)$, hence $d'(u, v) + d'(u, w) + d'(w, v)$ is odd and smaller than $2K_1 + 1$. Thus the triangle $uvw$ is forbidden by the $K_1$ bound in $G'$, a contradiction.
\( F^- \) case. Here \( d_i(u, v) > M \) and (without loss of generality) \( d_i(u, v) = d_{i-1}(u, w) - d_{i-1}(w, v) \). Then the triangle \( uvw \) has even perimeter with respect to \( d_i \). By our assumption we have \( d_i(u, v) \geq K_2 \) and thus \( d_{i-1}(u, w) > d_i(u, v) \geq K_2 \) and \( d_{i-1}(v, w) < M \).

This implies \( d_i(u, v) + d_{i-1}(u, w) = 2d_i(u, v) + d_{i-1}(v, w) \geq 2K_2 + d_{i-1}(v, w) \). From Lemma 4.6 we get that \( d_i'(u, v) \geq d_i(u, v), d_i'(u, v) \geq d_{i-1}(u, w) \) and \( d_i'(v, w) \leq d_{i-1}(v, w) \), hence also \( d_i'(u, v) + d_i'(v, w) \geq 2K_2 + d_i'(v, w) \) holds. Thus the triangle \( uvw \) is forbidden by the \( K_2 \) bound in \( G' \), a contradiction.

\( F^- \) case. Here \( d_i(u, v) < M \) and \( d_i(u, v) = C - d_{i-1}(u, w) - d_{i-1}(w, v) \). From our assumption it follows that \( d_i(u, v) \leq K_1 \).

In Case (III) we have \( C \geq 2\delta + K_1 + 1 \), hence \( d_i(u, v) = K_1 \) if and only if \( C = 2\delta + K_1 + 1 \), \( M > K_1 \) and \( d(u, w) = d(v, w) = \delta \); this case is treated in point 2.

It remains to consider Case (II). Hence we can assume that \( d_i'(u, v) < d(u, v) \) and these edges have different parity. Note that then the triangle \( uvw \) has even perimeter \( C - 1 \). By Lemma 4.6 we have \( d_i'(u, v) + d_i'(v, w) \geq d_{i-1}(u, w) + d_{i-1}(v, w) = C - 1 - d(u, v) = 2K_1 + 2K_2 - d_i(u, v) \). But as \( d_i'(u, v) \leq d_i(u, v) \leq K_1 \) we have \( d_i'(u, v) + d_i'(v, w) \geq 2K_2 + d_i'(v, w) \), so the triangle \( uvw \) is forbidden by the \( K_2 \) bound in \( G' \), a contradiction.

\[ \text{Lemma 4.8} \] (Automorphism Preservation Lemma). Let \( G \in G^\delta \) and let \( \bar{G} \) be its completion with magic parameter \( M \). Then every automorphism of \( G \) is also an automorphism of \( \bar{G} \).

\begin{proof}
Given \( G \) and an automorphism \( f : G \to \bar{G} \), it can be verified by induction that for every \( k > 0 \), \( f \) is also an automorphism graph \( G_k \) as in Definition 4.2. For every edge \( x, y \) of \( G_k \), which is not an edge of \( G_{k-1} \), it is true that \( f(x), f(y) \) is also an edge of \( G_k \) which is not an edge of \( G_{k-1} \), and moreover the edges \( x, y \) and \( f(x), f(y) \) are of the same length. This follows directly from the definition of \( G_k \).
\end{proof}

4.4. Correctness of the completion algorithm

In this section we prove:

\[ \text{Theorem 4.9} \] Let \( \delta, K_1, K_2, C_0 \) and \( C_1 \) be primitive admissible parameters. Suppose that \( G = (G, d) \in G^\delta \) has a completion into \( \mathcal{A}^{\delta}_{K_1, K_2, C_0, C_1} \), and let \( \bar{G} = (G, \bar{d}) \) be its completion with magic parameter \( M \). Then \( \bar{G} \in \mathcal{A}^{\delta}_{K_1, K_2, C_0, C_1} \).

\( \bar{G} \) is optimal in the following sense: Let \( G' = (G, d') \in \mathcal{A}^{\delta}_{K_1, K_2, C_0, C_1} \) be an arbitrary completion of \( G \) in \( \mathcal{A}^{\delta}_{K_1, K_2, C_0, C_1} \), then for every pair of vertices \( u, v \in G \) one of the following holds:

1. \( d'(u, v) \geq \bar{d}(u, v) \geq M \),
2. \( d'(u, v) \leq \bar{d}(u, v) \leq M \),
3. the parameters \( \delta, K_1, K_2, C_0 \) and \( C_1 \) satisfy Case (IIb), \( d'(u, v) \neq M \) and \( d(u, v) = M - 1 \).

Finally, every automorphism of \( G \) is also an automorphism of \( \bar{G} \).

In the next five lemmas we will use Lemmas 4.6 and 4.7 to show that \( G \in G^\delta \) has a completion into \( \mathcal{A}^{\delta}_{K_1, K_2, C_0, C_1} \), if and only if the algorithm with a magic parameter \( M \) yields such a completion. We will deal with each type of forbidden triangle separately, and in doing that, we will implicitly use the results of Section 4.2.

\[ \text{Lemma 4.10} \] (C-bound Lemma). Suppose \( C' = C + 1 \), and let \( G = (G, d) \in G^\delta \) be such that there is a completion of \( G \) into \( \mathcal{A}^{\delta}_{K_1, K_2, C_0, C_1} \); let \( \bar{G} = (G, \bar{d}) \) be its completion with magic parameter \( M \). Then there is no triangle forbidden by the \( C \) bound in \( \bar{G} \).

Proof. Suppose for a contradiction that there is a triangle with vertices \( u, v, w \) in \( G \) such that \( \bar{d}(u, v) + \bar{d}(v, w) + \bar{d}(u, w) \geq C \). For brevity let \( a = \bar{d}(u, v), b = \bar{d}(v, w) \) and \( c = \bar{d}(u, w) \). Assume without loss of generality that \( a \leq b \leq c \). Let \( a', b', c' \) be the corresponding edge lengths in an arbitrary completion of \( G \) into \( \mathcal{A}^{\delta}_{K_1, K_2, C_0, C_1} \). Then two cases can appear.
Either $a, b, c > M$, and then by Lemma 4.6 we have $a' \geq a$, $b' \geq b$ and $c' \geq c$, so we get the contradiction $a' + b' + c' \geq C$; or $a \leq M$, $c \geq b > M$ and $a + b + c \geq C$. In this case Lemma 4.6 implies $b' \geq b$ and $c' \geq c$ and $a' \leq a$. If the edge $(u, v)$ was already in $G$, then clearly $a' + b' + c' \geq a + b + c \geq C$, which is a contradiction. If $(u, v)$ was not already an edge in $G$, then it was added by the completion algorithm with magic parameter $M$ in step $t_M(a)$. Let $\bar{a} = C - 1 - b - c$. Then clearly $\bar{a} < a$, which means that $t_M(\bar{a}) < t_M(a)$, and as $\bar{a}$ depends on $b, c$, we have $t_M(b), t_M(c) < t_M(\bar{a})$. But then the completion with magic parameter $M$ actually sets the length of the edge $u, v$ to be $\bar{a}$ in step $t_M(\bar{a})$, which is a contradiction. \hfill $\Box$

**Lemma 4.11** (Metric Lemma). Let $G$ and $G'$ be as in Lemma 4.10. Then there are no non-metric triangles in $G$.

**Proof.** Suppose for a contradiction that there is a triangle with vertices $u, v, w$ in $G$ such that $d(u, v) + d(v, w) < d(u, w)$. Denote $a = d(u, v)$, $b = d(v, w)$ and $c = d(u, w)$ and assume without loss of generality that $a \leq b < c$. Let $a', b', c'$ be the corresponding edge lengths in an arbitrary completion of $G$ into $A_{K_1, K_2, C_0, C_1}$. We shall distinguish three cases based on Section 4.2:

1. First suppose $a, b, c < M$. Then $t_M(a) \leq t_M(b) < t_M(a + b) < t_M(c)$, which means that $c$ must be already in $G$. Note that in Case (IIB) if $b = K_1 - 1 = M - 1$, then $c \geq M$, hence we use Lemma 4.6 for $a$ and $b$, which gives us that $a' + b' \leq a + b < c = c'$, which is a contradiction.

2. Another possibility is $a < M$ and $b, c \geq M$ (actually $c > M$, since $abc$ is non-metric).

Suppose $a' \leq a$ and $c' \geq c$ (the first possibility of Lemma 4.6). If $b$ was already in $G$, then $G$ has no completion which is a contradiction. Otherwise clearly $c - a > b > M$, so $t_M(c - a) < t_M(b)$. But as $c - a$ depends on $c$ and $a$, we get $t_M(c - a) > t_M(c), t_M(a)$, which means that the completion algorithm with magic parameter $M$ would complete the edge $v, w$ with the length $c - a$ and not with $b$.

If the previous paragraph does not apply we have Case (IIB) and $a = K_1 - 1 = K_2 - 1$. But then as $M = K_2$, we have $b \geq K_2$, which means $a + b \geq 2K_2 - 1 = \frac{25}{3} - 1 > \delta$ for $\delta \geq 5$, which holds in (IIB), but that means that $abc$ is actually metric, a contradiction.\hfill \Box

3. The last possibility is $a, b < M$ and $c \geq M$. Then either (by Lemma 4.6 and Lemma 4.4 if $c = M$) we have $a' \leq a$, $b' \leq b$ and $c' \geq c$, hence the triangle $a', b', c'$ is again non-metric, or we have Case (IIB), $b = K_1 - 1, a \leq K_1 - 1$. The rest of proof of this lemma consists of a verification of this special case.

From admissibility of (IIB) we have $M = K_1 = K_2 = \frac{25 - 1}{3}$ and $\delta \geq 5$. Note that $c - a \geq b + 1 = K_1 = M$ from non-metricity of $abc$, hence $c > M$.

If both $a$ and $b$ were already in $G$, then $abc$ is non-metric in any completion by Lemma 4.6. The same thing is true if $b$ was already in $G$ and $a' \leq a$ in any completion (i.e. either $a < K_1 - 1$ or $a$ was not introduced by $F^C$ due to a $(\delta, \delta)$ fork).

Note that for $\delta \geq 8$ it cannot happen that $a = b = K_1 - 1$, as then $a + b = 2K_1 - 2 = \frac{25 - 1}{3} - 2 \geq \delta$, hence $a + b < c$ is absurd. So the only case when $a = b = K_1 - 1$ is $\delta = 5$ (because from (IIB) it follows that $\delta = 3m + 2$ for some $m \geq 1$). In that case we have triangle 5, 2, 2 and each of the two either was in $G$ or is supported by a fork (1,1) or by a fork (5,5). And it can be shown that none of these structures has a strong completion into $A_{K_1, K_2, C_0, C_1}$.

Hence $b$ was not in the input graph and $a < K_1 - 1$.

Observe that $c - a = M$. From non-metricity of $abc$ we have $c - a \geq b + 1 = K_1 = M$. And if $c - a \geq M + 1$, then $t_M(c - a) \leq t_M(M + 1) = 2\delta - 2M - 2$. And this is strictly less than $t_M(M - 1) = 2M - 3$ since $M = \frac{25 - 1}{3}$ and $\delta \geq 5$. Further as $M = \frac{25 - 1}{3}$ is odd, we see that $a$ and $c$ have different parities.

From Lemma 4.6 we have that in any completion $c' \geq c$ and $a' \leq a$. So the only way that the triangle $u, v, w$ can be metric is to have $b' > b$. Note that $c' - a' \geq c - a = M = K_2$, hence $c' \geq a' + K_2$. And from Lemma 4.7 we have that the parities of $a, b, c$ are preserved.
Note that as $M$ is odd, $b'$ is even. And since the parities of $c'$ and $a'$ are different, we have that $a' + b' + c'$ is odd. Also note that $c' + b' \geq a' + K_2 + K_2 + 1 \geq 2K_2 + a'$. Hence $u, v, w$ is forbidden by the $K_2$ bound in $G'$, which is a contradiction. \hfill $\Box$

**Lemma 4.12 (K₁-bound Lemma).** Let $G, \overline{G}$ be as in Lemma 4.10. Then there are no triangles forbidden by the $K_1$-bound in $\overline{G}$.

**Proof.** Suppose for a contradiction that there is a triangle in $G$ (from Lemma 4.11 we already know that all triangles in $G$ are metric) with vertices $u, v, w$ in $G$ such that $\overline{d}(u, v) + \overline{d}(v, w) + \overline{d}(u, w)$ is odd and less than $2K_1 + 1$. Denote $a = \overline{d}(u, v), b = \overline{d}(v, w)$ and $c = \overline{d}(u, w)$ From Section 4.2 we get $a, b, c < K_1 \leq M$.

First suppose that Lemma 4.6 gives us that for any completion $a', b', c'$ that $a' \leq a, b' \leq b$ and $c' \leq c$. Also $a$ has the same parity as $a', b$ as $b'$ and $c$ as $c'$ by Lemma 4.7, hence $a' + b' + c' \leq a + b + c$ and those two expressions have the same parity, hence $a', b', c'$ is also forbidden by the $K_1$ bound, a contradiction.

Otherwise we have Case (IIB) and $c = K_1 - 1$. But then from metricity of $abc$ either $a + b = c$ (but then $a + b + c$ is even, a contradiction), or $a + b = c + 1$ (if $a + b \geq c + 2$, then the perimeter of the triangle is too large to be forbidden by the $K_1$ bound). But again in any completion $a' \leq a$ and $b' \leq b$, so either $c' \leq c$ or $c' = c + 1$ (from metricity). From Lemma 4.7 we know that the parity of $c$ is preserved, hence $c' = c + 1$ is absurd, so $a' \leq a, b' \leq b$ and $c' \leq c$, and we can apply the same argument as in the preceding paragraph. \hfill $\Box$

**Lemma 4.13 (C₀, C₁-bound Lemma).** Let $C' > C + 1$ and let $G, \overline{G}$ be as in Lemma 4.10. Then there are no triangles forbidden by either of the $C_0$ and $C_1$ bounds in $\overline{G}$.

**Proof.** Suppose for a contradiction that there is a triangle with vertices $u, v, w$ in $\overline{G}$, such that $a + b + c \geq C$ and has parity such that it is forbidden by one of the $C$ bounds, where $a = \overline{d}(u, v), b = \overline{d}(v, w)$ and $c = \overline{d}(u, w)$.

In Case (IIB), we have $K_1 = K_2, C = 2K_1 + 2K_2 + 1 = 4K_2 + 1$ and $3K_2 = 2\delta - 1$, hence $C = 2\delta + K_2$. For parameters from Case (III) we have $C \geq 2\delta + K_2$, which means that we always have $C \geq 2\delta + K_2$. This implies that $b, c > K_2 \geq M$ and $a \geq K_2$. If $a$ was already present in $G$, then by Lemmas 4.4, 4.6 and 4.7 we have that any completion $a', b', c'$ has $a' = a, b' \geq b$ and $c' \geq c$ and the parities are preserved, hence $a', b', c'$ is forbidden by the $C$ bound as well, a contradiction to $G$ having a completion. If $a$ is not in $G$, we have $a \neq M$ (by Lemma 4.4) and actually $a > M$ as $a \geq K_2 \geq M$. Thus we can again use Lemmas 4.6 and 4.7 to get a contradiction. \hfill $\Box$

**Lemma 4.14 (K₂-bound Lemma).** Let $G, \overline{G}$ be as in Lemma 4.10. Then there are no triangles forbidden by the $K_2$-bound in $\overline{G}$.

**Proof.** We know that all triangles are metric and not forbidden by the $C$ bounds. Suppose for a contradiction that there is a triangle with vertices $u, v, w$ in $\overline{G}$ such that $a + b + c \geq 2K_2 + a + 1$, where $a = \overline{d}(u, v), b = \overline{d}(v, w)$ and $c = \overline{d}(u, w)$. We know that $b, c > K_2$ and $a \leq K_1$ by Section 4.2, where equality can occur only in Case (III) when $2K_2 + K_1 = 2\delta - 1$ and furthermore $M > a$ (because of Definition 4.4).

Note that from the conditions for Case (III), we know that if $2K_2 + K_1 = 2\delta - 1$, then $C \geq 2\delta + K_1 + 2$, which means that for edges $a, b, c$ Lemma 4.7 guarantees that the parity is preserved.

Unless $a = K_1 - 1$ and Case (IIB), Lemmas 4.6 and 4.7 yield that $a' + b' + c'$ has the same parity as $a + b + c$ for any completion $a', b', c'$ and $b' \geq b, c' \geq c$ and $a' \leq a$, hence triangle $u, v, w$ is forbidden by the $K_2$ bound in any completion of $G$, which is a contradiction.

The last case remaining is $a = K_1 - 1 = K_2 - 1$, Case (IIB). But then $b + c \geq 2K_2 + a + 1 = 3K_2 = 2\delta - 1$, so either $b + c = 2\delta - 1$, or $b + c = 2\delta$. But from being forbidden by the $K_2$-bound we know that $a + b + c$ is odd, hence $b + c$ has different parity than $a$. And we know that $a = K_2 - 1 = \frac{2\delta - 1}{3}$, which is even, hence $b + c = 2\delta - 1$. We also know that parities are preserved, so if $a' \geq K_2 + 1$, then $a' + b' + c' \geq 2\delta + K_2$ and it is thus forbidden by one of the $C$ bounds. \hfill $\Box$
Proof of Theorem 4.9. From the Lemmas 4.10, 4.11, 4.12, 4.13, 4.14 we conclude that the algorithm will correctly complete every graph $G$ which has completion into $\mathcal{A}_M^{K_1,K_2,C_0,C_1}$. The optimality statement follows by Lemma 4.6. Automorphisms are preserved according to Lemma 4.8.

4.5. Stationary independence relation

In this section we show a corollary to Theorem 4.9 proving that the completion with magic parameter $M$ has the right properties needed to define a stationary independence relation.

As pointed out in [TZ13], certain “canonical” ways of amalgamation give rise to stationary independence relations. We are going to show that also the opposite holds and every stationary independence relation on a homogeneous structure induces a canonical amalgamation operator. We start with some useful observations on stationary independence relations.

Lemma 4.15. Let $\downarrow$ be a (local) SIR. Then the following properties hold:

1. $A \downarrow_C B \leftrightarrow (AC) \downarrow_C B \leftrightarrow A \downarrow_C (BC)$,
2. (Transitivity). $A \downarrow_C B \land A \downarrow_C (BC) \rightarrow A \downarrow_C D$,
3. $A \downarrow_C (BD) \leftrightarrow A \downarrow_C B \land A \downarrow_C (BC) D$.

Proof. Point (1) and (2) are shown in [TZ13] and [Mü16] respectively. In order to show (3), i.e. that the inverse direction of the implication in the Monotonicity axiom holds, note that by (1) we have $A \downarrow_C (AC) D \leftrightarrow A \downarrow_C (BC) (BD)$. Then, by (2) $A \downarrow_C B \land A \downarrow_C (BC) (BD) \rightarrow A \downarrow_C (BD)$.

Definition 4.5. Let $\mathcal{C}$ be an amalgamation class. We say that $\oplus$ is an amalgamation operator, if it assigns to every triple of structures $A, B_1, B_2 \in \mathcal{C}$ with embeddings $e_1: A \rightarrow B_1$ and $e_2: A \rightarrow B_2$ a unique amalgam, i.e. a structure $D \in \mathcal{C}$ and embeddings $f_1: B_1 \rightarrow D, f_2: B_2 \rightarrow D$, such that $f_1 \circ e_1 = f_2 \circ e_2$. In short, we write $D = B_1 \oplus_A B_2$. We call $\oplus$ a local amalgamation operator if it is only defined for non-empty $A$, and $\oplus$ is canonical on $\mathcal{C}$ if additionally the following hold:

1. $B_1 \oplus_A B_2$ has minimal domain, i.e. it is generated by the union of $f_1(B_1)$ and $f_2(B_2)$
2. Monotonicity: If $B_1 \oplus_A B_2 = \langle f_1(B_1) \cup f_2(B_2) \rangle$, then $B_1 \oplus_B B_2 = \langle f_1(B_1) \cup f_2(B_2) \rangle$ for all substructures $e_2(A) \subseteq B_2$.
3. Associativity: $A \oplus_B (B_1 \oplus_B B_2) = A \oplus_B (B \oplus_B D)$.

Then the following holds:

Theorem 4.16. A homogeneous structure $M$ admits a (local) stationary independence relation if and only if $\text{Age}(M)$ has a (local) canonical amalgamation operator. Moreover, there is a one-to-one correspondence between (local) stationary independence relations $\downarrow$ and (local) canonical amalgamation operators $\oplus$ by: $A \downarrow_C B$ if and only if $\langle ABC \rangle$ is isomorphic to $\langle AC \rangle \oplus_C \langle BC \rangle$.

Proof. We first show that every canonical amalgamation operator gives rise to a stationary independence relation. Examples of this fact were already given in [TZ13]. Let $\oplus$ be a canonical amalgamation operator on $\text{Age}(M)$. Then we define a stationary independence relation by setting $A \downarrow_C B$ if and only if $\langle ABC \rangle$ is isomorphic to $\langle AC \rangle \oplus_C \langle BC \rangle$ under an isomorphism commuting with the embeddings. The axioms SIR1, SIR2, SIR4 and SIR5 follow straightforwardly from the fact that $\oplus$ is an amalgamation operator.

For (SIR3), observe first that by the minimality of $\oplus$ we have that $\langle XY \rangle = X \oplus_X (XY)$ for all $X, \langle XY \rangle \in \text{Age}(M)$. Let $A \downarrow_C (BD)$; by our observation this is equivalent to $\langle ABDC \rangle = \langle AC \rangle \oplus_C \langle BDC \rangle = \langle AC \rangle \oplus_C \langle (BC) \oplus_C (BD) \rangle$.

Since $\oplus$ is associative, this is equivalent to $\langle ABDC \rangle = \langle (AC) \oplus_C (BC) \rangle \oplus_C (BD)$. Since $\oplus$ is monotone, this implies $\langle (AC) \oplus_C (BC) \rangle = \langle ABC \rangle$, hence $A \downarrow_C B$ and $A \downarrow_C (BD)$. This concludes the proof that $\downarrow$ is a SIR.
For the opposite direction, let \( \perp \) be a stationary independence relation on the substructures of \( M \). Let \( A, B \) and \( C \) be in the age of \( M \) and let \( e_1 : C \rightarrow A \) and \( e_2 : C \rightarrow B \) be embeddings. By the homogeneity of \( M \) there are embeddings \( f_1 : A \rightarrow M \) and \( f_2 : B \rightarrow M \) such that \( f_1 e_1 = f_2 e_2 \).

By SIR4, there is an \( A' \) such that \( A' \perp f_1 e_1(C) \perp f_2(B) \) and such that there is an automorphism \( \alpha \) of \( M \) fixing \( f_1 e_1(C) \) that maps \( f_1(A) \) to \( A' \). We then define \( A \oplus_C B \) as the amalgam \( (A' \cup f_2(B)) \) with respect to the embeddings \( \alpha f_1 \) and \( f_2 \). By definition \( \oplus \) is an amalgamation operator, such that \( A \oplus_C B \) has minimal domain.

For showing associativity, let \( A' \) and \( B' \) be such that \( B \oplus_{C_2} D = \langle B/D \rangle \) and \( A \oplus_{C_1} (B \oplus_{C_2} D) = \langle A'B'D \rangle \). Thus we have \( B' \downarrow_{C_2} D \) and \( A' \downarrow_{C_1} B' \). By Lemma 4.15 (3) this is equivalent to

\[
A' \downarrow_{C_1} B' \wedge A' \downarrow_{C_1} D \wedge B' \downarrow_{C_2} D.
\]

Again by Lemma 4.15 (3) and \( C_2 \subseteq B' \) this is equivalent to

\[
A' \downarrow_{C_1} B' \wedge D \downarrow_{C_2} \langle A'B' \rangle.
\]

By our definition of \( \oplus \) we then have

\[
\langle A'B'D \rangle = A \oplus_{C_1} (B \oplus_{C_2} D) = (A \oplus_{C_1} B) \oplus_{C_2} D,
\]

proving that \( \oplus \) is associative.

**Corollary 4.17.** Let \( \delta, K_1, K_2, C_0 \) and \( C_1 \) be primitive admissible parameters. For every magic parameter \( M \) we can define a stationary independence relation with magic parameter \( M \) on \( \Gamma_{K_1,K_2,C_0,C_1}^\delta \) as follows: \( A \perp_{M} B \) if and only if \( \langle ABC \rangle \) is isomorphic to the completion with magic parameter \( M \) of the free amalgamation of \( \langle AC \rangle \) and \( \langle BC \rangle \) over \( C \).

**Proof.** For structures \( A, B, C \) in \( \mathcal{A}_{K_1,K_2,C_0,C_1}^\delta \), such that \( C \) embeds into \( A \) and \( B \) we define \( A \oplus_C B \) to be the completion with magic parameter \( M \) of the free amalgam of \( A \) and \( B \) over \( C \). If we can show that \( \oplus \) is a canonical amalgamation operator, then we are done by Theorem 4.16. By Theorem 4.9, \( A \oplus_C B \) is an element of \( \mathcal{A}_{K_1,K_2,C_0,C_1}^\delta \); hence \( \oplus \) is an amalgamation operator. Monotonicity and associativity of \( \oplus \) follow straightforwardly from the optimality property of the completion with magic parameter \( M \) (Theorem 4.9 (1),(2)). \( \square \)

### 4.6. Ramsey property and EPPA

Theorem 4.9 implies the following lemma which is crucial when applying Theorems 2.2 and 2.1.

**Lemma 4.18** (Finite Obstacles Lemma). Let \( \delta, K_1, K_2, C_0 \) and \( C_1 \) be primitive admissible parameters. Then the class \( \mathcal{A}_{K_1,K_2,C_0,C_1}^\delta \) has a finite set of obstacles which are all cycles of diameter at most \( 2^d \cdot 3 \).

**Example.** Consider \( \mathcal{A}_{3,3,16,13}^5 \) discussed in Section 4.1. The set of obstacles of this class contains all the forbidden triangles listed earlier, but in addition to that it also contains some cycles with 4 or more vertices. A complete list of those can be obtained by running the algorithm backwards from the forbidden triangles.

All such cycles with 4 vertices can be constructed from the triangles by substituting distances by the forks depicted at Figure 4. This means substituting 2 for 11 or 55, and 4 for 15 or 51. With equivalent cycles removed this give the following list:

- **non-metric:** 124 \( \Rightarrow \) 1114, 1554, 1215, 1251
- 125 \( \Rightarrow \) 1115, 1555**
- 114 \( \Rightarrow \) 1115*
- 225 \( \Rightarrow \) 1125*, 5525
- \( K_1 \)-bound: 122 \( \Rightarrow \) 1112, 1552
- \( K_2 \)-bound: 144 \( \Rightarrow \) 1154, 1514, 1415
- 245 \( \Rightarrow \) 1145, 5545, 2155*, 2515
- \( C \)-bound: 445 \( \Rightarrow \) 1545, 5145, 4155

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Observe that running the algorithm may produce multiple forbidden triangles which leads to duplicated cycles in the list. Such duplicates are denoted by $. For example, 125 was expanded to $1115$. The algorithm will first notice the fork $(1, 5)$ and produce $114$. This is also a forbidden triangle, but a different one. In the case of $1555$ (another expansion of 125) the algorithm will again use the fork $(1, 5)$ first and produce the triangles 455 and 145, which are valid triangles, see Figure 6. Not all expansions here are necessarily forbidden, because not all of them correspond to a valid run of the algorithm. However with the exception of cases denoted by $**$ all the above 4-cycles are forbidden.

Repeating the procedure one obtains the following cycles with five edges that cannot be completed into this class of metric graphs:

$$11111, 11115, 11155, 11515, 15155, 11555, 15555.$$  

An example of a failed run of algorithm trying to complete one of the forbidden cycles is depicted in Figure 7. Because there are no distances of 2 or 4 in the cycles with five edges that cannot be completed into this class, it follows that all cycles with at least six edges can be completed.

**Proof of Lemma 4.18.** Let $G = (G, d) \in \mathcal{G}^5$ be an edge-labelled graph no completion in $A_{K_1, K_2, C_0, C_1}^\delta$. We seek a subgraph of $G$ of bounded size which has also no completion into $A_{K_1, K_2, C_0, C_1}^\delta$.

Consider the sequence of graphs $G_0, G_1, \ldots, G_{2M+1}$ as given by Definition 4.4 when completing $G$ with magic parameter $M$. Set $G_{2M+2}$ to be the actual completion.

Because $G_{2M+2} \notin A_{K_1, K_2, C_0, C_1}^\delta$ we know it contains a forbidden triangle $O$. This triangle always exists, because $A_{K_1, K_2, C_0, C_1}^\delta$ is 3-constrained. By backward induction on $k = 2M + 1, 2M, \ldots, 0$ we obtain cycles $O_k$ of $G_k$ such that $O_k$ has no completion in $A_{K_1, K_2, C_0, C_1}^\delta$ and there exists a homomorphism $f: O_k \rightarrow G_k$.
Put $O_{2M+1} = O$. By Lemma 4.4 we know that this triangle is already in $G_{2M+1}$. At step $k$ consider every edge $u, v$ of $O_{k+1}$ which is not an edge of $G_k$ considering its witness $w$ (i.e. vertex $w$ such that the edges $u, w$ and $v, w$ implied the addition of the edge $u, v$ and extending $O_k$ by a new vertex $w'$ and edges $d(u, w') = d(u, w) = d(v, w') = d(v, w)$. One can verify that the completion algorithm will fail to complete $O_k$ the same way as it failed to complete $O_{k+1}$ and moreover there is a homomorphic image $O_{k+1} \to G_k$.

At the end of this procedure we obtain $O_0$, a subgraph of $G$, that has no completion into $A_{K_1, K_2, C_0, C_1}^\delta$. The bound on the size of the cycle follows from the fact that only $\delta$ steps of the algorithm are actually changing the graph and each time every edge may introduce at most one additional vertex.

Let $O$ consist of all edge-labelled cycles with at most $2^\delta 3$ vertices that are not completable in $A_{K_1, K_2, C_0, C_1}^\delta$. Clearly $O$ is finite. To check that $O$ is a set of obstacles it remains to verify that there is no $O \in O$ with a homomorphism to some $M \in A_{K_1, K_2, C_0, C_1}^\delta$. Denote by $O'$ the set of all homomorphic images of structures in $O$ that are not completable in $A_{K_1, K_2, C_0, C_1}^\delta$. Assume, to the contrary, the existence of such an $O = (O, d) \in O'$ and $M = (M, d')$ and a homomorphism $f : O \to M$ and among all those choose one minimising the difference of $|O|$ and $|M|$. It follows that $|O| - |M| = 1$. Denote by $x, y$ the pair of vertices identified by $f$. Let $O' = (O, d'')$ be a metric graph such that $d''((x', y'), (f(x), f(y)))$ for every pair $(x', y') \neq (x, y')$. It follows that because $A_{K_1, K_2, C_0, C_1}^\delta$ has the strong amalgamation property, and also $O' = (O, d'')$ has a completion in $A_{K_1, K_2, C_0, C_1}^\delta$.

To demonstrate the use of these results, we show the following theorems which we later strengthen in Section 3.2 for classes with Henson constraints.

**Theorem 4.19.** For every choice of admissible primitive parameters $\delta, K_1, K_2, C_0$ and $C_1$, the class $A_{K_1, K_2, C_0, C_1}^\delta$ of free orderings of $A_{K_1, K_2, C_0, C_1}^\delta$ is Ramsey and has the expansion property.

**Proof.** Ramsey property follows by a combination of Theorem 2.1 and Lemma 4.18.

To show the expansion property we use now standard argument that edge-Ramsey implies ordering property [Nes95, JLNVTW14]: Given metric space $A \in A_{K_1, K_2, C_0, C_1}^\delta$ construct ordered metric space $\hat{B}_0 \in \hat{A}_{K_1, K_2, C_0, C_1}^\delta$, as a disjoint union of all possible orderings of $A$. Now consider every pair of vertices $a < b$, $d(a, b) \neq M$ and add third vertex $c$ in distance $M$ from both $a$ and $b$ with order extended in a way so $a < c < b$ holds. Because $M$ is magic, by Observation 4.1, all new triangles are allowed and thus it is possible to complete this structure to ordered metric space $\hat{B}_1 \in \hat{A}_{K_1, K_2, C_0, C_1}^\delta$. Now denote by $\hat{E}$ an ordered metric space consisting of two vertices in distance $M$ and construct

$$\hat{B} \rightarrow (\hat{B}_1)^{\hat{E}}.$$ 

We claim that $B$ (the unordered reduct of $\hat{B}$) has the property that every ordering of $B$ contains every ordering of $A$. Denote by $\leq$ the order of $\hat{B}$ and chose arbitrary linear order order $\leq'$ of vertices of $B$. $\leq'$ implies two-coloring of copies of $E$ in $\hat{B}$: color copy red if both orders agree and blue otherwise. Because $\hat{B}$ is Ramsey, we obtain a monochromatic copy of $\hat{B}_1$ which contains a copy of $\hat{B}_0$ with the property that $\leq'$ restricted to this copy either agrees either with $\leq$ or with $\geq$. In both cases we obtain a copy of every ordering of $A$ within this copy of $\hat{B}_0$.

**Theorem 4.20.** For every choice of admissible primitive parameters $\delta, K_1, K_2, C_0$ and $C_1$, the class $A_{K_1, K_2, C_0, C_1}^\delta$ has coherent EPPA.

**Proof.** This follows by a combination of Theorem 2.2, Lemma 4.18 and Lemma 4.8.

5. Primitive spaces with Henson constraints

In this section we extend the results of Section 4 to classes with Henson constraints. An important fact about spaces with Henson constraints is the following [ACM21, Section 1.3]:
## Remark 5.1
Let $O$ be a set of $(1,\delta)$-spaces. Then there exists a finite set $S \subset O$, such that $\text{Forb}(S) = \text{Forb}(O)$.

A reflexive and transitive relation on a set $X$ is said to be a quasi order. It is a well quasi order if every non-empty subset has at least one, but only finitely many, minimal elements. Now, up to isometry, $(1,\delta)$-spaces are characterised by the sizes of their cliques i.e., maximal subsets of vertices at mutual distance 1. If we express every such space as a finite non-decreasing sequence of non-negative integers we see that a $(1,\delta)$-space is embeddable in another if and only if the latter contains a subsequence which majorises the former term by term. It is a well-known fact that the set of finite sequences of a well quasi ordered set is a well quasi ordered set with respect to the above relation, see [Hig52].

## Lemma 5.1
Let $(\delta, K_1, K_2, C_0, C_1, S)$ be a primitive admissible sequence of parameters and let $S$ be a non-empty class of Henson constraints. There is an $M$ such that the completion with magic parameter $M$ does not introduce distances 1 or $\delta$.

**Proof.** For $\delta > 3$ the completion algorithm with magic parameter $M \geq \lceil \frac{\delta}{2} \rceil$ introduces distances $\delta$ or 1 either when $M = \delta$ or when 1 is the only possible completion of some fork.

For $K_1 < \delta$ we can choose $M$ to be smaller than $\delta$, hence the first case only appears if $K_1 = \delta$. However then by Definition 3.3, $S = \emptyset$. The second case only appears if $C = 2\delta + 2$ and $C' = C + 1$, since then $(\delta, \delta)$-forks can only be completed by distance 1. Then again, by Definition 3.3, $S = \emptyset$.

## Theorem 5.2
Let $(\delta, K_1, K_2, C_0, C_1, S)$ be a primitive admissible sequence of parameters and let $S$ be a non-empty class of Henson constraints. For every magic parameter $M$ we can define a stationary independence relation with magic parameter $M$ on $\Gamma_{K_1, K_2, C_0, C_1, S}$ as follows: 

$A \perp_C B$ if and only if $\langle ABC \rangle$ is isomorphic to the completion with magic parameter $M$ of the free amalgamation of $\langle AC \rangle$ and $\langle BC \rangle$ over $C$.

**Proof.** This result can be shown as Corollary 4.17, the correctness of the completion algorithm can be verified by a combination of Theorem 4.9 and Lemma 5.1.

## Theorem 5.3
Let $(\delta, K_1, K_2, C_0, C_1, S)$ be a primitive admissible sequence of parameters such that $S$ is a non-empty class of Henson constraints. Then the class of $\Gamma$ of free linear orderings of $K = A_{K_1, K_2, C_0, C_1}^\delta \cap \mathcal{A}_S$ is Ramsey and has the expansion property.

**Proof.** Recall the definition of locally finite subclasses in Definition 2.6. We show that $\Gamma$ is a locally finite subclass of $A_{K_1, K_2, C_0, C_1}^{\delta}$. Given $C_0 \in A_{K_1, K_2, C_0, C_1}^{\delta}$ we put $n = |C_0|$. Consider any $C$ with a homomorphism to $C_0$ such that every substructure with at most $n$ vertices can be completed to $C' \in \Gamma$. Because Henson constrains are complete edge-labelled graphs, we know that every Henson constraint in $C$ is mapped to an isomorphic copy of itself by any homomorphism $C \to C_0$. Therefore there there is no $H \in S$, $|H| \geq n$ such that $H \to C$. Because every subgraph with at most $n$ vertices can be completed to $\Gamma$ we also know that there is no $H \in S$, $|H| \leq n$ such that $H \to C$.

By Lemma 5.1 we know that the magic completion $C'$ of $C$ (which exists by Theorem 4.9) will not introduce any edges of distance 1 and $\delta$ and thus also $C'$ contains no forbidden Henson constraints. The Ramsey property follows by Theorem 2.1.

## Theorem 5.4
Let $(\delta, K_1, K_2, C_0, C_1, S)$ be a primitive admissible sequence of parameters such that $S$ is a non-empty class of Henson constraints. Then $K = A_{K_1, K_2, C_0, C_1}^{\delta} \cap \mathcal{A}_S$ has coherent EPPA.

**Proof.** By Lemma 4.18 there is a finite set of obstacles $O$ for $A_{K_1, K_2, C_0, C_1}^{\delta}$. By Remark 5.1 $S$ is finite. Given $A \in K$, apply Theorem 2.2 to obtain an EPPA-witness $B \in \text{Forb}(O \cup S)$. Denote by $C$ its completion with magic parameter $M$. By Lemma 5.1 we know that $C \in \text{Forb}(S)$ and by Lemma 4.9 we know that the automorphism group is unaffected. It follows that $C$ is the desired completion.
6. Bipartite 3-constrained spaces

In this section we discuss the bipartite classes of finite diameter in Cherlin’s catalogue (Case (I) in Theorem 3.2). These are classes of metric spaces $\mathcal{A}_{K_1,K_2,C_0,C_1}^\delta$ with parameters

$$\delta < \infty, K_1 = \infty, K_2 = 0, C_1 = 2\delta + 1.$$ 

Furthermore we assume that

$$C_0 > 2\delta + 3.$$ 

The antipodal case where $C_0 = 2\delta + 2$ will be treated in Section 7. The parameter $C_0$ has to be even, so $2\delta + 3$ is not an acceptable value for $C_0$. We also discuss the Henson constraints for bipartite graphs.

By the condition $K_1 = \infty$, the metric spaces in $\mathcal{A}_{\infty,0,C_0,2\delta+1}^\delta$ contain no triangles of odd perimeter. As a direct consequence the relation consisting of all pairs $(x,y)$ such that $d(x,y)$ is even, is an equivalence relation with two equivalence classes; this fact also motivates the name “bipartite 3-constrained spaces”.

6.1. Generalised completion algorithm for bipartite 3-constrained classes

Our aim in this section is to again describe a procedure that completes a given edge-labelled graph $G$ to metric spaces in $\mathcal{A}_{\infty,0,C_0,2\delta+1}^\delta$ whenever possible. The completion algorithm constructed in Section 4.1 fails in general in the bipartite setting, since it might introduce triangles with odd perimeter (for instance when adding the magic distance in the final step). Hence Theorem 4.9 cannot be applied here.

In the following we show how we can slightly adapt the algorithm to ensure that no new odd cycles are generated. The basic idea for our completion algorithm is again to optimize the length of the newly introduced edges towards a magic parameter $M$, respectively its successor $M + 1$. The length of the remaining edges are then set to $M$ or $M + 1$ depending on the parity prescribed by the bipartition.

**Definition 6.1** (Bipartite magic distances). Let $M \in \{1,2,\ldots,\delta\}$ be a distance. We say that $M$ is magic (with respect to $\mathcal{A}_{\infty,0,C_0,2\delta+1}^\delta$) if

$$\left\lfloor \frac{\delta}{2} \right\rfloor \leq M < M + 1 \leq \left\lfloor \frac{C_0 - \delta - 1}{2} \right\rfloor.$$ 

Compare this with Definition 4.3. By the assumption $C_0 > 2\delta + 3$ a magic distance always exists.

**Observation 6.1.** $M$ is magic if and only if it has the following property: for all even $1 < b \leq \delta$ the triangles $MMb$ and $(M + 1)(M + 1)b$ are in $\mathcal{A}_{\infty,0,C_0,2\delta+1}^\delta$; and for all odd $1 \leq b \leq \delta$ the triangle $M(M + 1)b$ is in $\mathcal{A}_{\infty,0,C_0,2\delta+1}^\delta$.

**Proof.** If $M$ has this property then $M \geq \left\lfloor \frac{\delta}{2} \right\rfloor$ (otherwise for even $\delta$ the triangle $MM\delta$ would be non-metric; similarly for odd $\delta$ the triangle $M(M+1)\delta$ would be non-metric). Also, $M \leq \left\lfloor \frac{C_0 - \delta - 1}{2} \right\rfloor$ (otherwise if $\delta$ is even the triangle $(M+1)(M+1)\delta$ has perimeter $C_0$, hence is forbidden by the $C_0$ bound; and if $\delta$ is odd the triangle $M(M+1)\delta$ has perimeter $C_0$). The other implication follows from the definition of $\mathcal{A}_{\infty,0,C_0,2\delta+1}^\delta$. 

With the following simple lemma we reduce our discussion to completions of connected edge-labelled graphs, i.e. edge-labelled graphs such that there exists a path connecting each pair of vertices.

**Lemma 6.2.** $G \in \mathcal{G}^\delta$ has a completion to $\mathcal{A}_{\infty,0,C_0,2\delta+1}^\delta$ if and only if all of its connected components have a completion to $\mathcal{A}_{\infty,0,C_0,2\delta+1}^\delta$.

See Remark 6.1 for more discussion about the disconnected case.
Proof. It suffices to show that every $G = (G, d)$ that is the disjoint union of two graphs $A, B$ from $A_{\infty,0,C_0,2\delta+1}^{\delta}$ has a completion $(G, d') \in A_{\infty,0,C_0,2\delta+1}^{\delta}$.

Fix some $x \in A$ and $y \in B$. Then, for every non-edge $(x', y')$ with $x' \in A$ and $y' \in B$ let $d'(x', y') = M$ if $d(x, x') + d(y, y')$ is even and $d'(x', y') = M + 1$ otherwise. It is not hard to verify that all the newly introduced triangles are of the form $MMb, (M + 1)(M + 1)b$ where $b$ is even, or $(M(M + 1)b$ where $b$ is odd. Hence $(G, d') \in A_{\infty,0,C_0,2\delta+1}^{\delta}$.  

For connected graphs we now give the following definition of a completion algorithm:

**Definition 6.2** (Bipartite completion algorithm). Given $1 \leq M \leq M + 1 \leq \delta$, a one-to-one function $t : \{1, 2, \ldots, \delta\} \setminus \{M, M + 1\} \to \mathbb{N}$ and a function $F$ from $\{1, 2, \ldots, \delta\} \setminus \{M, M + 1\}$ to the power set of $D$, then we define the $(F, t, M, M + 1)$-completion of a connected edge-labelled graph $G = (G, d)$ as the limit of the sequence $G_1, G_2, \ldots$ that is constructed as in Definition 4.2: the length of all remaining non-edges $(u, v)$ in this limit is then set to $M$ if $d^+(u, v)$ has the same parity as $M$ and to $M + 1$ otherwise. In addition we will stick to the other notational conventions introduced in Section 4.1 (time function, $t$ function or $M$ function or $\delta$ function).

Let $M$ be a magic distance and let $1 \leq x \leq \delta$ with $x \neq M, M + 1$. Then we define the fork sets $F^+_x$ and $F^-_x$ as in the last section and $F^C_M = \{(a, b) \in D : C_0 - 2 - a - b = x\}$, i.e. $(a, b) \in F^C_M$ if and only if $a + b + x$ is equal to $C_0 - 2$ and hence also even. Forks used by the bipartite algorithm are schematically depicted in Figure 8.

We further define

$$F_M(x) = \begin{cases} F^+_x \cup F^C_M & x < M \\ F^-_x & x > M + 1. \end{cases}$$

For a magic distance $M$, we define the function $t_M : \{1, \ldots, \delta\} \setminus \{M, M + 1\} \to \mathbb{N}$ as

$$t_M(x) = \begin{cases} 2x - 1 & x < M \\ 2(\delta - x) & x > M + 1. \end{cases}$$

We then call the $(F_M, t_M, M, M + 1)$-completion of $G$ the **bipartite completion of $G$ with magic parameter $M$**.

**Lemma 6.3** (Bipartite Time Consistency Lemma). Let $a, b$ be distances different from $M$ and $M + 1$. If $a$ depends on $b$, then $t_M(a) > t_M(b)$.

**Proof.** Analogously to Lemma 4.3 we consider three types of forks.

$F^+$: This follows in complete analogy to Lemma 4.3: If $a < M$ and $F^+_a \neq \emptyset$, then $b < a < M$, hence $t_M(b) < t_M(a)$.

$F^C_M$: If $a < M$ and $F^C_M \neq \emptyset$, then we must have $b, c > M + 1$. Otherwise, if for instance $b \leq M + 1$, then $C_0 - \delta + 2 \leq C_0 - 2 - c = a + b < 2M + 1 \leq 2 \left\lfloor \frac{C_0 - \delta + 1}{2} \right\rfloor - 2 + 1$, a contradiction. As $C_0 \geq 2\delta + 4$, we obtain the inequality $b = (C_0 - 2) - c - a \geq (2\delta + 2) - \delta - a = \delta + 2 - a$. Hence $t_M(b) \leq 2(a - 2) < 2a - 1 = t_M(a)$. 

Figure 8: Forks used by $F$ by the bipartite algorithm.
\( \mathcal{F}^- \): Finally, we consider the case where \( a > M + 1 \) and \( \mathcal{F}^-_a \neq \emptyset \). Then either \( a = b - c \), which implies \( b > a > M + 1 \) and thus \( t_M(b) < t_M(a) \), or \( a = c - b \), which means \( b = c - a < \delta - a \). Because of \( a > M + 1 \geq \left\lfloor \frac{\delta}{2} \right\rfloor \), we have \( b < M \). So \( t_M(b) \leq 2(\delta - a) - 1 < 2(\delta - a) = t_M(a) \).

\[ \]
Bipartite Optimality and the Parity Lemma 6.5 \( d'(u, w) \geq d_{i-1}(u, w) \) and \( d'(v, w) \geq d_{i-1}(v, w) \), which means that the triangle \( u, v, w \) in \( G' \) is forbidden by the \( C_0 \) bound, which is absurd as \( G' \) is a completion of \( G \) in \( A_{\infty,0,C_0,2\delta+1}^{\delta} \).

For the second part (about parities), observe first that the parity of an edge \( d'(u, v) \) in \( G' \) has to be equal to \( d'(u, w) + d'(w, v) \) for every other vertex \( w \), since there are no triangles of odd perimeter in \( A_{\infty,0,C_0,2\delta+1}^{\delta} \). Since \( G \) is connected, the parity of \( d'(u, v) \) is equal to the parity of the path distance of \( (u, v) \) in \( G \).

By the definition of the bipartite completion algorithm as a limit of graphs \( G_1, G_2, \ldots \), if the statement is not true, then there has to be a smallest \( i \) such that in the graph \( G_i = (G, d_i) \) there are vertices \( u, v, w \in G \) where the parity of \( d_i(u, v) \) differs from the parity of \( d'(u, v) \). Let \( w \) be a witness for the edge \( (u, v) \). Then three cases can appear (corresponding to \( F^-, F^+, F^{C_0} \)).

\( F^- \): In the first case we have \( d_i(u, v) = |d_{i-1}(u, w) - d_{i-1}(v, w)| \), which has the same parity as \( d_{i-1}(u, w) + d_{i-1}(v, w) \). By minimality of \( i \) this value has the same parity as \( d'(u, w) + d'(v, w) \), hence this case cannot appear.

\( F^+ \): In the second case we have \( d_i(u, v) = d_{i-1}(u, w) + d_{i-1}(v, w) \). Analogously to the first case then \( d_i(u, v) \) has to have the same parity as \( d'(u, w) + d'(v, w) \), which is a contradiction.

\( F^{C_0} \): In the third case \( d_i(u, v) = C_0 - 2 - d_{i-1}(u, w) - d_{i-1}(v, w) \). Since \( C_0 - 2 \) is even, this distance has again the same parity \( d_{i-1}(u, w) + d_{i-1}(v, w) \) and hence \( d'(u, w) + d'(v, w) \), which is a contradiction.

In the last step, the distances \( M \) and \( M + 1 \) are added according to the parity of the path distance, hence also in this step the parity of edges is preserved.

Note that Lemma 6.3 implies that any magic completion of a graph \( G \) contains a triangle with odd perimeter if and only if every completion of \( G \) contains such a triangle. In order to verify the correctness of our completion algorithm it is therefore only left to verify the analogous statement for the two other types of forbidden triangles: non-metric triangles, and even triangles that are forbidden due to the \( C_0 \)-bound. But for those triangles the result can be shown just by following the corresponding proofs in the primitive case.

Lemma 6.6 (Bipartite Metric Lemma). Let \( G = (G, d) \in \mathcal{G}^d \) be a connected edge-labelled graph such that there is a completion of \( G \) into \( A_{\infty,0,C_0,2\delta+1}^{\delta} \); let \( \overline{G} = (G, \overline{d}) \) be its bipartite completion with magic parameter \( M \). Then there are no non-metric triangles in \( \overline{G} \).

\[ \text{Proof.} \] We proceed in analogy to the proof of Lemma 4.11.

Suppose for a contradiction that there is a triangle with vertices \( u, v, w \in \overline{G} \) such that \( \overline{d}(u, v) + \overline{d}(v, w) < \overline{d}(u, w) \). Denote \( a = \overline{d}(u, v), \ b = \overline{d}(v, w) \) and \( c = \overline{d}(u, w) \) and assume without loss of generality that \( a \leq b < c \). By Lemma 6.5 we know that \( a + b + c \) is even. Let \( a', b', c' \) be the corresponding edge lengths in an arbitrary completion of \( G \) into \( A_{\infty,0,C_0,2\delta+1}^{\delta} \). We shall distinguish three cases:

1. First suppose \( a, b, c < M \). Then \( t_M(a) \leq t_M(b) < t_M(a + b) < t_M(c) \), which means that \( c \) must be already in \( G \). By Lemma 6.5 \( a' + b' < a + b < c = c' \), which is a contradiction.
2. Another possibility is \( a < M \) and \( b, c \geq M \) (actually \( c > M + 1 \), since \( abc \) is non-metric). By Lemma 6.5 we know that \( a' \leq a \) and \( c' \geq c \). If \( b \) was already in \( G \), then \( G \) has no completion \(-\) a contradiction. Otherwise clearly \( c - a > b \geq M \), so \( \bar{t}(c - a) < \bar{t}(b) \) (define \( \bar{t}(M) = t(M + 1) = \infty \)). But as \( c - a \) depends on \( c \) and \( a \), we get \( \bar{t}(c - a) > \bar{t}(c) = \bar{t}(a) \), which means that the bipartite completion algorithm with magic parameter \( M \) would complete the edge \( v, w \) with the length \( c - a \) and not with \( b \).

3. The last possibility is \( a, b < M \) and \( c \geq M \). Then (by Lemma 6.5 and Lemma 6.4 if \( M \leq c \leq M + 1 \)) we have \( a' \leq a, b' \leq b \) and \( c' \geq c \), hence the triangle \( a', b', c' \) is again non-metric.

\[ \square \]

**Lemma 6.7 (Bipartite \( C_0 \)-bound Lemma).** Let \( \mathcal{G}, G \) be as in Lemma 6.6. Then there are no triangles forbidden by the \( C_0 \)-bound in \( \mathcal{G} \).

**Proof.** Again we proceed analogously to Lemma 4.10. Suppose for contradiction that there is a triangle with vertices \( u, v, w \) in \( \mathcal{G} \) such that \( \bar{d}(u, v) + \bar{d}(v, w) + \bar{d}(u, w) \geq C_0 \). For brevity let \( a = \bar{d}(u, v), b = \bar{d}(v, w) \) and \( c = \bar{d}(u, w) \). Assume without loss of generality \( a \leq b \leq c \). Let \( a', b', c' \) be the corresponding edge lengths in an arbitrary completion of \( G \) into \( \mathcal{A}_{\infty, 0, C_0, 2^{\delta + 1}} \). Then two cases can appear.

Either \( a, b, c > M + 1 \), and then by Lemma 6.5 we have \( a' \geq a, b' \geq b \) and \( c' \geq c \), so we get the contradiction \( a' + b' + c' \geq C_0 \); or \( a \leq M + 1 \), \( c \geq b > M + 1 \) and \( a + b + c \geq C_0 \). In this case Lemmas 6.4 and 6.5 imply \( b' \geq b \) and \( c' \geq c \) and \( a' \leq a \). If the edge \((u, v)\) was already in \( G \), then clearly \( a' + b' + c' \geq a + b + c \geq C_0 \), which is a contradiction. If \((u, v)\) was not already an edge in \( G \), then it was added by the bipartite completion algorithm with magic parameter \( M \) in step \( t(a) \). Let \( \bar{a} = C_0 - 2 - b - c \). Then clearly \( \bar{a} < a \), which means that \( t(M(\bar{a})) < t(M(a)) \), and as \( \bar{a} \) depends on \( b, c \), we have \( t(M(b)), t(M(c)) < t(M(\bar{a})) \). But then the bipartite completion with magic parameter \( M \) actually sets the length of the edge \( u, v \) to be \( \bar{a} \) in step \( t(M(\bar{a})) \), which is a contradiction. \[ \square \]

**Lemma 6.8 (Bipartite automorphism Preservation Lemma).** Let \( G \in \Delta^\delta \) be connected and \( \mathcal{G} \) be its bipartite completion with magic parameter \( M \). Then every automorphism of \( G \) is also an automorphism of \( \mathcal{G} \).

**Proof.** Cf. proof of Lemma 4.8. Observe that since \( G \) is assumed to be connected, the final step of the algorithm that includes edges of length \( M \) and \( M + 1 \) is canonical. Hence automorphisms are preserved. \[ \square \]

**Theorem 6.9.** Let \( \delta \geq 3 \), \( C_0 > 2\delta + 3 \) and \( S \) be an admissible set of Henson constraints for \( \mathcal{A}_{\infty, 0, C_0, 2^{\delta + 1}} \). Let \( G = (G, d) \) be a connected edge-labelled graph such that there is a completion of \( G \) into \( \mathcal{A}_{\infty, 0, C_0, 2^{\delta + 1}} \) and let \( \mathcal{G} = (G, \bar{d}) \) be its bipartite completion with magic parameter \( M \). Then \( \mathcal{G} \in \mathcal{A}_{\infty, 0, C_0, 2^{\delta + 1}} \cap \mathcal{A}_S \).

\( \mathcal{G} \) is optimal in the following sense: Let \( G' = (G, d') \in \mathcal{A}_{\infty, 0, C_0, 2^{\delta + 1}} \) be an arbitrary completion of \( G \) in \( \mathcal{A}_{\infty, 0, C_0, 2^{\delta + 1}} \), then for every pair of vertices \( u, v \in G \) one of the following holds:

1. \( d'(u, v) \geq \bar{d}(u, v) \geq M + 1 \),
2. \( d'(u, v) \leq \bar{d}(u, v) \leq M \),
3. \( M \leq \bar{d}(u, v) \leq M + 1 \).

Furthermore the parity of every distance in \( G' \) is the same as the parity of the corresponding distance in \( \mathcal{G} \) and every automorphism of \( G \) is also an automorphism of \( \mathcal{G} \).

**Proof.** For \( S = \emptyset \) the statement follows from Lemmas 6.6, 6.7, 6.5 and 6.8.

Observe that the only non-empty admissible set \( S \) consists of a single anti-clique (that is a metric space with all distances \( \delta \)) and in this case \( \delta \) is even. Then we can use the fact that \( \delta \geq 4 \) and thus \( M \) can be always chosen to be at most \( \delta - 2 \). In this case the bipartite completion with magic parameter \( M \) will never introduce an edge of distance \( \delta \). \[ \square \]
Remark 6.1. Note that if we drop the condition of $G$ being connected in Theorem 6.9, we can still compute a completion $\overline{G}$ of $G$ by first completing all connected components according to Theorem 6.9 and then adding edges $M$ and $M + 1$ as described in the proof of Lemma 6.2. This completion $\overline{G}$ still satisfies the optimality conditions 1, 2 and 3; however we will lose other important features:

1. The constructed completion is not uniquely determined by $G$, but also depends on how we connect the different components of $G$. (The proof of Lemma 6.2 used a non canonical choice of $x \in A$ and $y \in B$, two connected components.)

2. The automorphism group of $\overline{G}$ can be a proper subgroup of $\text{Aut}(G)$,

3. Edges in $G'$ and $\overline{G}$ may have different parities.

The above observations have an impact on the results of the following section: Point 1 corresponds to the fact that we only have local, but not global stationary independence relation (see Corollary 6.10).

6.2. Local stationary independence relation

Corollary 6.10. Let $\delta \geq 3$, $C_0 > 2\delta + 3$ and $S$ be an admissible set of Henson constraints for $A^{\delta}_{\infty,0, C_0, 2\delta + 1}$. Then there is no stationary independence relation on $\Gamma^{\delta}_{\infty,0, C_0, 2\delta + 1,S}$. However, for every magic parameter $M$ there is a local stationary independence relation on $\Gamma^{\delta}_{\infty,0, C_0, 2\delta + 1,S}$ as follows: $A \downarrow C B$ if and only if $(ABC)$ is isomorphic to the completion of the completion with magic parameter $M$ of the free amalgamation of $(AC)$ and $(BC)$ over $C$.

Proof. First, suppose for a contradiction that there is a stationary independence relation $\downarrow$ in $\Gamma^{\delta}_{\infty,0, C_0, 2\delta + 1}$. By Theorem 4.16 this is equivalent to the existence of a canonical amalgamation operator $\oplus$ on $A^{\delta}_{\infty,0, C_0, 2\delta + 1}$. Let $A, B \in A^{\delta}_{\infty,0, C_0, 2\delta + 1}$ such that $A \oplus B$ contains only one vertex $w$. In the canonical amalgam over the empty set $A \oplus B$, monotonicity implies that both $\{u, w\}, \{u, v\}$ and $\{u, v\}, d$ are canonical amalgams of two points over the empty set. By the uniqueness of $\oplus$ we have that $d(u, w) = d(v, w)$. By the triangle inequality we have that $d(u, v) = d(u, w) + d(w, v)$. Then the triangle $(u, v, w)$ has odd perimeter, which contradicts that $A \oplus B \in A^{\delta}_{\infty,0, C_0, 2\delta + 1}$.

The local stationary independence relation follows by same argument as in proof of Corollary 4.17. By locality of the stationary independence relation we note that all graphs that need to be completed in the proof are already connected, and thus Theorem 6.9 applies.

6.3. Ramsey property and EPPA

We follow the general direction of Section 4.6. The extra difficulty is that the bipartiteness cannot be expressed by means of a finite set of obstacles, because such a set must contain odd cycles of unbounded length. However we can obtain the following variant of Lemma 4.18.

Lemma 6.11 (Bipartite Finite Obstacles Lemma). Let $\delta \geq 3$, $C_0 > 2\delta + 3$. Then there is finite set $O$ of edge-labelled cycles such that every edge-labelled graph $G \in \text{Forb}(O)$ without odd cycles is in $A^{\delta}_{\infty,0, C_0, 2\delta + 1}$.

Proof. Cf. proof of Lemma 4.18. To verify that the final step of algorithm will succeed we use the fact that there are no odd cycles in $G$.

Theorem 6.12. Let $\delta \geq 3$, $C_0 > 2\delta + 3$ and $S$ be an admissible set of Henson constraints for $A^{\delta}_{\infty,0, C_0, 2\delta + 1}$. The class $\overline{A}^\delta_{\infty,0, C_0, 2\delta + 1} \cap \overline{A}S$ of convex orderings of $A^{\delta}_{\infty,0, C_0, 2\delta + 1} \cap AS$ with an additional unary predicate determining the bipartition is Ramsey and has the expansion property.

Here a convex ordering is any ordering such that vertices in first bipartition (denoted by the unary predicate) form an initial segment.
Proof. Let $O$ be given by Lemma 6.11. Denote by $O'$ the family of all possible expansions of $O$ by a unary predicate determining the bipartition with additional structure on two vertices which forbids vertices from the same bipartition from being connected by an edge of odd length and vertices from different bipartitions from being connected by an edge of even length. Observe that structures in Forb($O'$) have no odd cycles and thus Theorem 2.1 and Lemma 6.11 apply.

If $S$ is non-empty, then observe that $M$ can be chosen to be at most $\delta - 2$ and one can use the same argument as in proof of Theorem 5.3.

The expansion property again follows by the standard argument.  

\[\Box\]

Theorem 6.13. Let $\delta \geq 3$, $C_0 > 2\delta + 3$ and $S$ be an admissible set of Henson constraints for $A^{\delta}_{\infty,0,C_0,2\delta+1}$. Then the class $A^{\delta}_{\infty,0,C_0,2\delta+1} \cap A_S$ has coherent EPPA.

Proof. Let $O$ be given by Lemma 6.11. By an application of Theorem 2.2 obtain $B \in \text{Forb}(O)$ which is a coherent EPPA-witness of $A$. Without loss of generality we can assume that $B$ is connected (otherwise the connected component of $B$ containing $A$ is a coherent EPPA-witness, too). If $B$ has no odd cycles, apply Theorem 6.9 to obtain the desired EPPA-witness of $A$.

If $B$ contains odd cycles, construct $C = (C, d')$ as follows. The vertex set $C$ is $B \times \{0, 1\}$, and

\[
d'((u, i), (v, j)) = \begin{cases} 
0 & \text{if } (u, i) = (v, j) \\
d(u, v) & \text{if } d(u, v) \text{ is odd and } i \neq j \\
& \text{if } d(u, v) \text{ is even and } i = j. 
\end{cases}
\]

Observe that $C$ is connected and contains no odd cycles. Let $p: A \to \{0, 1\}$ be a function determining the bipartition of $A$. We verify that $C$ is an extension of $\phi(A)$ for the following embedding $\phi(v) = (v, p(v))$. Denote by $\pi(v, j) = v$ the projection, which is a homomorphism of $C \to B$ and also embedding $\phi(A) \to A$.

Every partial isometry $\psi$ of $\phi(A)$ induces a partial isometry $\psi \circ \pi$ of $A$ which extends to automorphism $\psi$ of $B$. This automorphism induces an automorphism $\psi'$ of $C$ by mapping $(u, i) \mapsto (\psi(u), i)$. It however might not be an extension of $\psi$ because $\psi'$ may change the values of the function $p$. In this case combine it with an automorphism mapping $(v, 0) \mapsto (v, 1)$ and $(v, 1) \mapsto (v, 0)$. It is easy to see that this construction preserves coherence of the extensions in $B$.

If $S$ is non-empty, again observe that $M$ can be chosen to be at most $\delta - 2$ and use same argument as in proof of Theorem 5.4. \[\Box\]

7. Antipodal spaces

In this section we discuss the antipodal classes in Cherlin’s catalogue. We say that an amalgamation class $K$ of metric spaces of diameter $\delta$ is antipodal if the edges of distance $\delta$ form a matching in the Fraïssé limit of $K$. In particular then there are no triangles with more than one edge of length $\delta$ in $K$. Note that this also implies that $K$ has no strong amalgamation. For admissible parameters $(\delta, K_1, K_2, C_0, C_1, S)$ the class $A^{\delta}_{K_1, K_2, C_0, C_1, S}$ is antipodal if and only if $C = 2\delta + 1$. More precisely only the two cases can appear:

**Definition 7.1.** The admissible parameters $3 \leq \delta < \infty$, $K_1$, $K_2$, $C_0$ and $C_1$ are antipodal when

1. $K_1 = \infty$, $C_0 = 2\delta + 2$ (the bipartite case; so $K_2 = 0$ and $C_1 = 2\delta + 1$), or
2. $1 \leq K_1 \leq \delta$, $K_2 = \delta - K_1$, $C_0 = 2\delta + 2$, $C_1 = 2\delta + 1$.

The parameters in this case are pushed to the extreme situation where, either there is no magic parameter, or $M = \lfloor \frac{\delta}{2} \rfloor$ is the only parameter satisfying Definition 4.4, respectively Definition 6.1.

**Definition 7.2.** Let $\delta \geq 3$. For $A = (A, d) \in G^{\delta-1}$, an antipodal companion of $A$ is any $A^* = (A, d^*) \in G^{\delta-1}$ such that there exists $B \subseteq A$ and:

\[
d^*(u, v) = \begin{cases} 
d(u, v) & \text{if } d(u, v) \text{ is defined and } u, v \in B \text{ or } u, v \notin B \\
\delta - d(u, v) & \text{if } d(u, v) \text{ is defined and } u \in B, v \notin B \text{ or vice versa.}
\end{cases}
\]
Figure 10: Antipodal extension of the triangle 122 for δ = 3, the matching formed by edges of distance 3 is denoted by dashed lines. The second and third picture highlight the original triangle and one of its antipodal companions.

Observe that for every admissible antipodal parameters, the class $\mathcal{A}_{d_1}^{K_1,K_2,C_0,C_1,S_1}$, and also the subclass of metric spaces of diameter at most $\delta - 1$ form amalgamation classes. This subclass is equal to $\mathcal{A}_{d_1}^{K_1,K_2,C_0,C_1,S_1}$ and corresponds to either a primitive case (Case (IIA) for $K_1 > 2$ or (III) for $1 \leq K_1 \leq 2$) or bipartite case (Case (I)) of Cherlin’s catalogue. The following describes a reverse way to produce an antipodal space of diameter $\delta$ from a space of diameter $\delta - 1$.

**Definition 7.3** (Antipodal extensions). Given an edge-labelled graph $M = (M,d) \in G^{d-1}$ its antipodal extension is the edge-labelled graph $M^{\oplus} = (M \times \{0,1\}, d') \in G^{d-1}$ such that $d'((u,i),(v,i)) = d(u,v)$ and $d'((u,i),(v,1-i)) = \delta - d(u,v)$ for $u,v \in M$ and $i \in \{0,1\}$.

For an ordered edge-labelled graph $M = (M,d,\leq_M)$, its ordered antipodal extension $\overrightarrow{M}^{\oplus} = (M \times \{0,1\}, d',\leq_{M^\oplus})$ where $(M,d,\leq_M) = (M \times \{0,1\}, d')$ and $(u,i) \leq_{M^\oplus} (v,j)$ if and only if $i < j$ or $i = j$ and $u \leq_M v$.

For an ordered bipartite edge-labelled graph $\overrightarrow{M} = (M,d,\leq_M,B)$ where $B$ is unary predicate determining the bipartition we define $\overrightarrow{M}^{\oplus} = (M \times \{0,1\}, d',\leq_{M^\oplus},B^\oplus)$ analogously and we put $(v,0) \in B^\oplus \iff (v) \in B$ and if $\delta$ is even then $(v,1) \in B^\oplus \iff (v) \in B$; if $\delta$ is odd then $(v,1) \in B^\oplus \iff (v) \notin B$.

**7.1. Ramsey property**

The Ramsey property follows from the above correspondence in full generality. Recall that we use arrows to indicate that the structures in a class are ordered (a requirement for any Ramsey class).

**Theorem 7.1.** Let $\overrightarrow{K}$ be a Ramsey class of expansions of metric spaces. Then the class $\overrightarrow{K}^{\oplus}$ of all antipodal expansions of $\overrightarrow{K}$ is Ramsey.

Here $\overrightarrow{K}$ can be either a Ramsey class of ordered metric spaces (given by Theorems 4.19 and 5.3) or ordered metric spaces with predicate denoting bipartition (given by Theorem 6.12).

**Proof.** Given $A^{\oplus},B^{\oplus} \in \overrightarrow{K}^{\oplus}$, apply the Ramsey property in $\overrightarrow{K}$ to obtain $C \rightarrow (B)^A_2$. It is easy to check that $C^{\oplus} \rightarrow (B^{\oplus})^A_2$. \qed

**Remark 7.1.** Observe that $\overrightarrow{K}^{\oplus}$ is not a hereditary class. It consists only of metric spaces where for every vertex there is precisely one vertex in a distance $\delta$. Thus one can color only structures $A$ having this property. The Ramsey property for antipodal expansions can be, equivalently, stated for the hereditary class of all subspaces of spaces in $\overrightarrow{K}^{\oplus}$ with a unary predicate determining the podality.

This unary predicate becomes necessary because in $\overrightarrow{K}^{\oplus}$ there is a definable equivalence relation on vertices which is not in $K^{\oplus}$, namely $u \sim v$ whenever both $\{w : w \leq u\}$ and $\{w : w \leq v\}$ span an edge of length $\delta$ or none does. As shown in [HN19] an equivalence relation on vertices implies the necessity for unary predicates in the Ramsey lift. It is interesting to observe that in showing EPPA (see Section 7.3) there is sometimes no need for further expansion of the language. To maintain that, it is necessary to assume that $\overrightarrow{K}$ is closed under the operation of forming antipodal companions (Definition 7.2).
7.2. Generalised completion algorithms for antipodal classes

To show the extension property for partial automorphisms we need a way to complete the spaces symmetrically.

Our completion algorithm for the antipodal classes \( K = A^d_{K_1, K_2, C_0, C_1, S} \) will be based on the completion algorithm for \( K^{d-1} = A^{d-1}_{K_1, K_2, C_0, C_1, S} \) with magic parameter \( M = \lceil \frac{d}{2} \rceil \) for non-bipartite spaces and \( K^{d-1} = A^{d-1}_{\infty, C_0, C_1, S} \) for bipartite spaces. More precisely, we are not considering the completion to \( A^d_{K_1, K_2, C_0, C_1, S} \), but to its expansion \( B^4_{K_1, K_2, C_0, C_1, S} \) by an additional unary predicate \( P \) determining the podality (so every edge of length \( \delta \) has precisely one vertex \( v \in P \)). Roughly speaking our completion algorithm for an input structure \((G, d, P)\) will first complete the pone \((P, d)\) in \( A^{d-1}_{K_1, K_2, C_0, C_1, S} \) and then forms its antipodal extension.

We are going to consider four separate cases:

1. Antipodal classes in Case (IIA) with even \( \delta \) (Corollary 7.4),

   **Example.** An example of such a class is \( A^{4}_1, 1, 3, 10, 9 \). The class of all metric spaces in \( A^{4}_1, 1, 3, 10, 9 \) of diameter 3 (the underlying class of metric space appearing on each pone) is \( A^{1}_1, 3, 10, 9 \) which is the class of all finite metric spaces of diameter 3 omitting triangle 333 (the primitive case covered by Section 4). All metric spaces in \( A^{1}_1, 3, 10, 9 \) are thus subspaces of antipodal extensions of spaces in \( A^{1}_1, 3, 10, 9 \).

   Another choice of such a class is \( A^{4}_2, 2, 10, 9 \) with underlying class of metric spaces \( A^{3}_2, 2, 10, 9 \) which forbids triangles 333 (by the \( C \)-bound), 111 (by the \( K_1 \)-bound) and 133 (by the \( K_2 \)-bound). It is useful to observe that from the conditions \( C = 2\delta + 1 \) and \( K_1 = \delta - K_2 \) it follows that the underlying metric spaces are always closed for antipodal companions: the companion of the triangle 333 is 311 which is non-metric and the companion of 111 is 133. This property is needed for the class of antipodal metric spaces to form an amalgamation class.

2. Antipodal bipartite classes in Case (I) with odd \( \delta \) (Corollary 7.6),

   **Example.** An example of such a class is \( A^{3}_0, 8, 7 \). This is a special case, because the underlying metric space is of diameter 2 and thus not analyzed in this paper. The underlying class of metric spaces consists of the complete bipartite graphs where non-edge is represented by distance 2. All metric spaces in \( A^{3}_0, 8, 7 \) are thus antipodal extensions of complete bipartite graphs.

   The first standard case is \( A^{3}_\infty, 8, 11 \) with underlying class of metric spaces \( A^{3}_\infty, 0, 12, 9 \) which is the class of all bipartite metric spaces omitting triangle 444. This is a bipartite case covered in Section 6.

   Observe that because \( \delta \) is odd, all edges in distance \( \delta \) cross the bipartition.

3. Antipodal classes in Case (IIA) with odd \( \delta \) (Theorem 7.3), and

   **Example.** In this category lies the class \( A^{3}_1, 2, 8, 7 \) which is again a special case, because the underlying class has diameter 2. It is the class of all metric spaces of diameter 2 (that is, a representation of graphs where non-edge corresponds to distance 2). Structures in \( A^{3}_1, 2, 8, 7 \) can equivalently be seen as double-covers of complete graphs which are further discussed in Section 10.

   An example of a standard case is the class \( A^{3}_4, 12, 11 \) with underlying space \( A^{3}_4, 4, 12, 11 \) which is the class of all metric spaces of diameter 4 with triangle 444 forbidden. Again this is a primitive case covered by Section 4.

   The main difference with the first case is that the graph consisting of two disjoint edges of length \( \delta \) (and no other edges or vertices) requires two different distances to be used in its completion, while in the first case all remaining edges can be completed by \( \frac{\delta}{2} \).
4. Antipodal bipartite classes in Case (I) with even $\delta$ (Theorem 7.5).

**Example.** An example of such a class is $A^4_{\infty,0,10.9}$ with underlying class $A^3_{\infty,0,10.7}$. This is the class of all bipartite metric spaces of diameter 3 covered in Section 6.

The main difference with the second case is that the edges of distance $\delta$ connect pairs of vertices in the same bipartition and thus one can define equivalence both by use of the bipartition and by the pairing of $\delta$ edges.

In the first two cases we will show the perhaps surprising fact that the antipodal completion of $(G,d,P)$ described as above does not depend on the choice of $P$, hence we obtain a unique completion $(G,\bar{d})$ to the original class of metric spaces $A^d_{K_1,K_2,C_0,C_1,S}$. In the latter two cases this however does not hold and similarly to the bipartite case (cf. Remark 6.1) we are confronted with different completions, depending on the choice of $P$. This ambiguity cannot be eliminated and is reflected in the fact that there is no canonical amalgamation on $A^d_{K_1,K_2,C_0,C_1,S}$ for those cases (Theorem 7.8).

In order to simplify our proof we first show that it is enough to only consider edge-labelled graphs $G$ that are symmetric according to the following definition:

**Definition 7.4** (Antipodal quadruple). A quadruple of distinct vertices $(u,v,u',v')$ in $G \in G^\delta$ is antipodal if $d(u,v) = d(u',v') = \delta$ and $d(u,u') = d(v,v') = d(v,u') = d(u,v') = \delta - d(u,u')$. (In particular all those distances are defined.) See Figure 11.

Let us call $G \in G^\delta$ antipodally symmetric, if for every vertex $x$ there is a unique $x^*$ with $d(x,x^*) = \delta$ and if every edge of length $< \delta$ is part of an antipodal quadruple.

**Lemma 7.2.** Let $G = (G,d) \in G^\delta$ and suppose that $G$ has a completion to $A^d_{K_1,K_2,C_0,C_1,S}$. Then there is an edge-labelled graph $(G^*,d^*)$ with $G^* \supseteq G$ and $d^* \supseteq d$ that is antipodally symmetric and every completion $(G,d') \in A^d_{K_1,K_2,C_0,C_1,S}$ of $G$ has a unique extension to a completion of $(G^*,d^*)$.

Furthermore every automorphism of $(G,d)$ extends uniquely to an automorphism of $(G^*,d^*)$.

**Proof.** First observe that if $G$ has a completion to $A^d_{K_1,K_2,C_0,C_1,S}$ then there are no $(\delta,\delta)$-forks in $G$. Moreover quadruples of vertices $(u,v,u',v')$ such that $d(u,v) = d(u',v') = \delta$ can only be completed to antipodal quadruple. Instead of completing $(G,d)$ we can therefore consider the antipodally symmetric graph $(G^*,d^*)$ that is constructed by adding a unique vertex $x^*$ for every $x$ and with $d^*(x,x^*) = \delta$ (if $x^*$ is not already present in $G$) and completing all quadruples of vertices $(u,v,u',v')$ such that $d^*(u,v) = d^*(u',v') = \delta$ and $d(u,u')$ is defined to their unique completion as antipodal quadruple. It is not hard to see that a completion $(G,d') \in A^d_{K_1,K_2,C_0,C_1,S}$ of $(G,d)$ extends to a unique automorphism $f \in \text{Aut}(G^*,d^*)$ defined by $f(x^*) = f(x^*)$.

Analogously, a structure $(G,d,P)$ with $(G,d) \in G^\delta$ and a unary predicate $P$ has a completion to $(G,d',P) \in B^\delta_{K_1,K_2,C_0,C_1,S}$ if and only if its extension to an antipodally symmetric space has a completion to $B^\delta_{K_1,K_2,C_0,C_1,S}$. Therefore, in the following we always assume $(G,d,P)$ to be antipodally symmetric.

Note that, for antipodally symmetric $(G,d,P)$, every completion of $(P,d|_{P_2})$ to $A^{d-1}_{K_1,K_2,C_0,C_1,S}$ extends to a unique completion of $(G,d,P)$ to $B^\delta_{K_1,K_2,C_0,C_1,S}$, namely its antipodal expansion.
**Definition 7.5 (Non-bipartite antipodal completion algorithm).** Let $\delta \geq 3$, $K_1 \leq \frac{\delta}{2}$, $M = \lfloor \frac{\delta}{2} \rfloor$, $C = 2\delta + 1$ and let $(G, d, P)$ be an antipodally symmetric edge-labelled graph with a predicate $P$ for a pode.

Then we define the **antipodal completion** $(G, \tilde{d}, P)$ of $(G, d, P)$ with parameter $M$ as follows: restricted to the pode, $(P, d)$ is the completion of $(P, d)$ with magic parameter $M$ and diameter $\delta - 1$ (cf. Definition 4.4) and $(G, \tilde{d})$ is its antipodal expansion. Since $(G, d, P)$ is antipodally symmetric, $(G, \tilde{d}, P)$ is in fact a completion of $(G, d, P)$, i.e. $d(x, y) = \tilde{d}(x, y)$, whenever $d(x, y)$ is defined.

Then the following holds:

**Theorem 7.3.** Let $3 \leq \delta < \infty$ and $K_1 \leq \frac{\delta}{2}$ and $M = \lfloor \frac{\delta}{2} \rfloor$. Let $(G, d, P)$ be antipodally symmetric. Suppose that $(G, d, P)$ has a completion into $B^\delta_{K_1, \delta - K_1, 2\delta + 2, 2\delta + 1, S}$ and let $(G, \tilde{d}, P)$ be its antipodal completion with magic parameter $M$. Then $(G, \tilde{d}, P) \in B^\delta_{K_1, \delta - K_1, 2\delta + 2, 2\delta + 1, S}$ and it is optimal in the following sense: Let $(G, d', P) \in B^\delta_{K_1, \delta - K_1, 2\delta + 2, 2\delta + 1, S}$ be an arbitrary completion of $(G, d, P)$, then, for all $u, v \in G$:

1. $d'(u, v) \geq \tilde{d}(u, v) \geq M$ or $d'(u, v) \leq \tilde{d}(u, v) \leq M$ if both $u, v \in P$ or if both $u, v \in G \setminus P$

2. $d'(u, v) \geq \tilde{d}(u, v) \geq \delta - M$ or $d'(u, v) \leq \tilde{d}(u, v) \leq \delta - M$ else.

Furthermore, $\text{Aut}(G, d, P) = \text{Aut}(G, \tilde{d}, P)$.

**Proof.** An antipodally symmetric graph $(G, d, P)$ has a completion to $B^\delta_{K_1, \delta - K_1, 2\delta + 2, 2\delta + 1, S}$ if and only if its pode $(P, d)$ has a completion to $A^\delta_{K_1, \delta - K_1, 2\delta + 2, 2\delta + 1, S}$.

By Theorem 4.9 the optimality statement $d'(u, v) \geq \tilde{d}(u, v) \geq M$ or $d'(u, v) \leq \tilde{d}(u, v) \leq M$ is true for all $u, v \in P$. In general, every pair of vertices $u, v$ in $G$ is part of an antipodal quadruple. This implies the optimality statement in its general form.

The fact that $\text{Aut}(G, d, P) = \text{Aut}(G, \tilde{d}, P)$ follows directly from the automorphism preservation Lemma 4.8 for the pode $(P, d)$ and the observation that every automorphism of $(G, d, P)$ is uniquely determined by its restriction to $P$.

**Corollary 7.4.** Let $3 \leq \delta < \infty$ be even, $K_1 \leq \frac{\delta}{2}$ and $M = \lfloor \frac{\delta}{2} \rfloor$. Let $(G, d)$ be antipodally symmetric. Suppose that $G = (G, d)$ has a completion into $A^\delta_{K_1, \delta - K_1, 2\delta + 2, 2\delta + 1, S}$. Then there is a unique completion $(G, \tilde{d}) \in A^\delta_{K_1, \delta - K_1, 2\delta + 2, 2\delta + 1, S}$ that is optimal in the following sense: Let $G' = (G, d') \in A^\delta_{K_1, \delta - K_1, 2\delta + 2, 2\delta + 1, S}$ be an arbitrary completion of $G$, then, for all $u, v \in G$:

$$d'(u, v) \geq \tilde{d}(u, v) \geq M$$

Furthermore $\text{Aut}(G, d) = \text{Aut}(G, \tilde{d})$.

**Proof.** We define $(G, \tilde{d})$ as the underlying metric space of the completion of $(G, d, P)$ for an arbitrary predicate $P$ for aisode, i.e. for some $P$ such that $|\{x, x^*\} \cap P| = 1$ for all $x, x^*$ with $d(x, x^*) = \delta$. By Theorem 7.3, $(G, \tilde{d}) \in A^\delta_{K_1, \delta - K_1, 2\delta + 2, 2\delta + 1, S}$ if and only if $(G, d)$ has a completion to $A^\delta_{K_1, \delta - K_1, 2\delta + 2, 2\delta + 1, S}$.

Since $\delta$ is even, we have that $M = \delta - M$. Hence, the optimality part of Theorem 7.3 states that actually for all $u, v \in G$ and every arbitrary completion $G' = (G, d') \in A^\delta_{K_1, \delta - K_1, 2\delta + 2, 2\delta + 1, S}$:

$$d'(u, v) \geq \tilde{d}(u, v) \geq M$$

Hence $(G, \tilde{d})$ does not depend on the choice of the pode $P$.

It remains to show that automorphisms of $(G, d)$ (that do not necessarily fix a pode $P$) are preserved by the completion. So let $f \in \text{Aut}(G, d)$ and $P$ be some predicate for a pode. Then $(G, d, P)$ and $(G, d, f(P))$ are isomorphic under $f$. Hence their completions are isomorphic, so $f \in \text{Aut}(G, \tilde{d})$. 

\[\Box\]
With the following example we would like to illustrate out that the assumption of $\delta$ to be even cannot be dropped in Corollary 7.4 and also the original completion algorithm fails for odd $\delta$.

**Example.** Consider $\mathcal{A}^{3,2,8,7}$ which is an example of a class of antipodal metric spaces. Completing a graph consisting of two edges of length 3 and no other edges using Definition 4.4 will result in a non-antipodal metric space where every non-edge will be completed by $M$. Because triangles 322 and 311 are forbidden neither $M = 1$ or $M = 2$ will make the completion algorithm from Definition 4.4 give the correct answer. Using the completion in Definition 7.5 for some choice of $P$ we obtain a completion, which depends on the choice of the pole $P$ and has fewer automorphisms then the input graph.

Next we are going to consider the bipartite cases. Again Lemma 7.2 allows us to only consider antipodally symmetric edge-labelled graphs. Then we define the bipartite antipodal completion of $(G, d, P)$ as follows:

**Definition 7.6 (Bipartite antipodal completion algorithm).** Given $\delta \geq 3$, $M = \lfloor \frac{\delta}{2} \rfloor$ and $C = 2\delta + 1$ and an antipodally symmetric, connected edge-labelled graph $(G, d, P)$ with a predicate $P$ for a pole.

Then we define the *antipodal completion* $(G, \bar{d}, P)$ of $(G, d, P)$ with parameter $M$ by setting $(P, \bar{d})$ to be the bipartite completion of $(P, d)$ with magic parameter $M$ and diameter $\delta - 1$ (according to Definition 6.2) and $(G, d)$ to be its antipodal expansion. Since $(G, d, P)$ is antipodally symmetric, $(G, \bar{d}, P)$ is in fact a completion of $(G, d, P)$.

**Theorem 7.5.** Let $3 \leq \delta$ and $M = \lfloor \frac{\delta}{2} \rfloor$. Let $(G, d, P)$ be such that $(G, d)$ is antipodally symmetric and connected and suppose that $(G, d, P)$ has a completion into $\mathcal{B}_{\infty, 0,2\delta+2,2\delta+1, S}^{\delta}$. Let $(G, \bar{d}, P)$ be its bipartite completion with magic parameter $M$. Then $(G, d, P) \in \mathcal{B}_{\infty, 0,2\delta+2,2\delta+1, S}^{\delta}$.

The completion is optimal in the following sense: Let $G' = (G', d', P) \in \mathcal{B}_{\infty, 0,2\delta+2,2\delta+1, S}^{\delta}$ be an arbitrary completion of $(G, d, P)$ in $\mathcal{B}_{\infty, 0,2\delta+2,2\delta+1, S}^{\delta}$ then for every pair of vertices $u, v \in G$, $d'(u, v)$ has the same parity as $\bar{d}(u, v)$. Furthermore the completion is optimal in the following sense: For all $u, v \in P$ such that $\bar{d}(u, v) \neq M, M + 1$ we have

$$d'(u, v) \geq \bar{d}(u, v) \geq M + 1 \text{ or } d'(u, v) \leq \bar{d}(u, v) \leq M.$$  

Furthermore every automorphism of $(G, d, P)$ is also an automorphism of $(G, \bar{d}, P)$.

**Proof.** As in Theorem 7.3 this follows directly from the properties of the bipartite completion algorithm with parameter $M$ for the non-antipodal class $\mathcal{A}_{\infty, 0,2\delta+2,2\delta-1}^{\delta-1}$.

As a direct consequence we obtain the following for odd diameters $\delta$:

**Corollary 7.6.** Let $3 \leq \delta < \infty$ be odd and $M = \frac{\delta-1}{2}$. Let $(G, d, P)$ be antipodally symmetric and connected. Suppose that $G = (G, d)$ has a completion into $\mathcal{A}_{\infty, 0,2\delta+2,2\delta-1}^{\delta-1}$. Then there is a unique completion $(G, \bar{d}) \in \mathcal{A}_{\infty, 0,2\delta+2,2\delta+1}^{\delta-1}$ that is optimal in the following sense: Let $G' = (G', d') \in \mathcal{A}_{\infty, 0,2\delta+2,2\delta-1}^{\delta-1}$ be an arbitrary completion of $G$, then, for all $u, v \in G$ one of the following holds:

1. $d'(u, v) \geq \bar{d}(u, v) \geq M + 1$ or
2. $d'(u, v) \leq \bar{d}(u, v) \leq M$ or
3. $M \leq \bar{d}(u, v) \leq M + 1$

Furthermore $\text{Aut}(G, d) = \text{Aut}(G, \bar{d})$.

**Proof.** Since $\delta$ is odd, we have that $M + (M + 1) = \delta$. Hence the optimality property for pairs $u, v \in P$, described in Theorem 7.5 transfers to all pairs $u, v \in G$ (by considering antipodal quadruples).

As in the proof of Corollary 7.4, this optimality properties imply that the completion does not depend of the choice of $P$. Furthermore analogously to Corollary 7.4 one can show that automorphisms of $(G, d)$ are also automorphisms of $(G, \bar{d})$. 

\[42\]
We first give a proof for the even diameter non-bipartite case with

Figure 12: An example of graph which fails to be completed to \( A_{4,0,10,9}^4 \) using the algorithm given by Definition 6.1.

We remark that in the last remaining case — bipartite antipodal spaces of even diameter — again the standard algorithm for bipartite spaces fails, as shown in the following example:

**Example.** Consider \( A_{4,0,10,9}^4 \) which is an example of a class of bipartite antipodal metric spaces. Completing the graph depicted in Figure 12 using Definition 6.1 and \( M = 2 \) will result in non-antipodal metric space where every non-edge will be completed by 3.

### 7.3. Antipodality-preserving EPPA

In the following we add extra layer to the construction of Herwig and Lascar which makes sure that vertex closures are preserved. This idea is the same as in [EHN19, EHN21].

**Theorem 7.7.** Let \((\delta, K_1, K_2, C_0, C_1, S)\) be an antipodal sequence of admissible parameters.

1. If \( K_1 < \infty \) and \( \delta \) is even then the class \( A_{K_1, K_2, C_0, C_1}^\delta \cap S \) has coherent EPPA.
2. If \( K_1 = \infty \) and \( \delta \) is odd then the class \( A_{K_1, K_2, C_0, C_1}^\delta \cap S \) has coherent EPPA.
3. If \( K_1 < \infty \) and \( \delta \) is odd then \( S = \emptyset \) and the class \( B_{K_1, K_2, C_0, C_1}^\delta \) expanding \( A_{K_1, K_2, C_0, C_1}^\delta \) by an additional unary predicate \( P \) determining the podality (so every edge of length \( \delta \) has precisely one vertex \( v \in P \) ) has coherent EPPA.
4. If \( K_1 = \infty \) and \( \delta \) is even then \( S = \emptyset \) and the class \( B_{K_1, K_2, C_0, C_1}^\delta \) expanding \( A_{K_1, K_2, C_0, C_1}^\delta \) by an additional unary predicate \( P \) determining the podality has coherent EPPA.

**Proof.** We first give a proof for the even diameter non-bipartite case with \( S = \emptyset \) and then discuss how to extend the construction for remaining cases.

1. Given \( A = (A, d) \in A_{K_1, K_2, C_0, C_1}^\delta \) without loss of generality we can assume that for every vertex \( u \in A \) there exists unique vertex \( v \in A \), with \( d(u, v) = \delta \). Denote by \( O \) the set of obstacles of \( A_{K_1, K_2, C_0, C_1}^{\delta-1} \) given by Lemma 4.18. Apply Theorem 2.2 to obtain \( B \in \text{Forb}(O) \) which is a coherent EPPA-witness of \( A \) and has no homomorphic image of any structure in \( O \). Note that \( B = (B, d) \) contains edges of type \( \delta \) and thus it can not be completed to a metric space in \( A_{K_1, K_2, C_0, C_1}^{\delta-1} \).

Now we describe edge-labelled graph \( C = (C, d') \). \( C \) is the set of all ordered pairs \((u, v)\) where \( u \neq v \in B \), \( d(u, v) = \delta \). We set

\[
d'(u, v) = \begin{cases} 
0 & \text{if } (u, v) = (u', v') \\
\delta & \text{if } u = v', v = u' \\
d(u, u') & \text{if } (u, v, u', v') \text{ is an antipodal quadruple.}
\end{cases}
\]

Clearly edges of distance \( \delta \) form a (complete) matching in \( C \).

Given \( u \in A \) put \( \varphi(u) = (u, v) \) where \( v \) is the unique vertex of \( A \) such that \( d(u, v) = \delta \). It is easy to check that \( \varphi \) is an embedding \( A \to C \). We show that \( C \) is an coherent EPPA-witness of \( \varphi(A) \).
Denote by $\pi: C \to B$ the projection assigning $(u, v) \mapsto u$. Let $\psi$ be a partial isometry of $\varphi(A)$. Then $\psi \circ \pi$ is a partial isometry of $A$. Now extend this isometry to $\tilde{\psi}: B \to B$. It follows that the mapping $\tilde{\psi}' : C \to C$ defined by $\tilde{\psi}'(u, v) = (\psi(u), \psi(v))$ is an automorphism of $C$ extending $\psi$.

It remains to show that there is a completion of $C$ to a metric space $C' \in A_{K_1, K_2, C_0, C_1}$. For every pair of vertices $(u, v)$, such that $d'(u, v) = \delta$ choose arbitrarily one of the vertices and denote by $C_{\delta-1}$ the subgraph induced by $C$ on all those vertices. As the edges of length $\delta$ form a complete matching, we know that $C_{\delta-1}$ is precisely half of the graph $C$ and whenever a distance is defined in $C \setminus C_{\delta-1}$ it is part of a unique antipodal quadruple.

Note that $\pi(C_{\delta-1})$ is an isomorphic copy of $C_{\delta-1}$ to $B$ and $C_{\delta-1}$ does not contain any edges of length $\delta$. Because $K$ is closed for antipodal companions, we then know that $C_{\delta-1} \in \Forb(O)$. Denote by $C'_{\delta-1} \in K$ its completion with magic parameter $M = \frac{\delta}{2}$ and by $C' \in \mathcal{K}^\oplus$ the completion of $C$ which extends $C'_{\delta-1}$ antipodally to the remaining vertices.

We proved that there is a completion of $C$ into $A_{K_1, K_2, C_0, C_1}$. Now we can look at the completion with magic parameter $M$ of $C$ given by Corollary 7.4. This completion (which in fact is equivalent to one described above) preserves all automorphisms of $C$, which is what we wanted.

Now consider the case when $S \neq \emptyset$. Again we use the fact that the completion algorithm will never introduce new Henson substructures and that the set of obstacles can be extended by $S$.

2. Now we adjust the construction for the bipartite case of odd diameter. Consider again $A = (A, d) \in A_{K_1, K_2, C_0, C_1}$, such that for every vertex $v \in A$ there exists unique vertex $v' \in A$, $d(u, v) = \delta$.

Denote by $O$ the set of obstacles given by Lemma 6.11 for $A_{K_1, K_2, C_0, C_1}$ (which is a non-antipodal bipartite amalgamation class). By the same construction as we used to build the EPPA-witness $C$ in the proof of Theorem 6.13 we obtain a coherent EPPA-witness $B \in \Forb(O)$ of $A$ without odd cycles. Now we use the construction as above to obtain a coherent EPPA-witness $C \in \Forb(O)$ where edges of length $\delta$ form a matching. Note that, since $B$ can be chosen to be connected, we can also assume that $C$ is connected. Finally we apply Corollary 7.6 to complete it to the bipartite metric space in $A_{K_1, K_2, C_0, C_1}$.

3. In the third case of odd diameter non-bipartite spaces we proceed similarly to the first case, however with the explicit predicate $P$. Consider a structure $A = (A, d, p) \in B_{C_1} \in A_{K_1, K_2, C_0, C_1}$, such that for every vertex $u \in A$ there exists a unique vertex $v \in A$ with $d(u, v) = \delta$. Denote by $O$ the set of obstacles given by Lemma 4.18 for $A_{\delta-1}$. We apply Theorem 2.2 to obtain a structure $B \in \Forb(O)$ which is a coherent EPPA-witness of $A$. Without loss of generality we can assume that the predicate $P$ in $B$ picks precisely one vertex of every edge of length $\delta$ (all other edges of length $\delta$ can be removed from $B$ because they are never contained in a copy of $A$). Then we construct $C$ in the same way as before, with $\pi$ being the projection of $C$ onto the $P$-part of $B$.

In the last step we complete $C$ according to the completion algorithm described in Definition 7.5. By Theorem 7.3 this completion preserves automorphisms and gives the desired coherent EPPA-witness in $B_{C_1} \in A_{K_1, K_2, C_0, C_1}$.

4. The last case can be dealt with by the combination of case 2 and 3. 

7.4. Stationary independence relation

In this section we discuss the existence of a stationary independence relation on antipodal spaces. As one might suspect the answer is related to the question if there is a deterministic completion algorithm with magic parameter. Our results show that if $K_1 < \infty$ and $\delta$ is odd, or if $K_1 = \infty$ and $\delta$ is even, there is no local SIR.

**Theorem 7.8.** Let $(\delta, K_1, K_2, C_0, C_1, S)$ be a sequence of admissible antipodal parameters. Then the following holds in $\Gamma_{K_1, K_2, C_0, C_1}$:

1. If $K_1 < \infty$ and $\delta$ is even, there is a stationary independence relation.
2. If $K_1 = \infty$ and $\delta$ is odd, there is no stationary independence relation, but a local one.
3. If $K_1 < \infty$ and $\delta$ is odd, there is no local stationary independence relation.

4. If $K_1 = \infty$ and $\delta$ is even, there is no local stationary independence relation.

Proof. First observe that we may consider stationary independence relation over antipodally closed structures only because the antipodal completion is unique.

1. For $K_1 < \infty$ and even diameter $\delta$, Corollary 7.4 gives us a completion algorithm. For structures $A, B, C \in A_{K_1,K_2,C_0,C_1,S}$ we define $A \oplus C B$ to be the space obtained by first forming the free amalgam of $A$ and $B$ over $C$ and then forming the completion with magic parameter $M = \frac{\delta}{2}$. This operator is a canonical amalgamation operator, hence there is a stationary independence relation on $\Gamma_{K_1,K_2,C_0,C_1,S}$ (cf. Corollary 4.17).

2. For $K_1 = \infty$ and odd diameter $\delta$, use Corollary 7.6 for connected edge-labelled graphs. Then, as in the proof of Corollary 6.10 one can show, that there is a local stationary independence relation. In order to see that there is no stationary independence relation, we show that there is no canonical amalgamation in $A_{K_1,K_2,C_0,C_1,S}$ over the empty structure. For that let $A$ consist of an antipodal pair $a_1, a_2$ and let $B$ be consist of a single vertex $b$. If there was a canonical amalgamation operator then in $A \oplus B$ we have that $d(b, a_1) = \delta - d(b, a_2)$, and since $\delta$ is odd $d(b, a_1) \neq d(b, a_2)$. Hence $\{a_1\} \oplus B$ and $\{a_2\} \oplus B$ are non-isomorphic, which contradicts the assumption that $\oplus$ is a canonical amalgamation operator.

3. Next assume that $K_1 < \infty$, $\delta$ is odd, and let $M = \lfloor \frac{\delta}{2} \rfloor$. We are going to show that there is no local canonical amalgamation operation on $A_{K_1,K_2,C_0,C_1,S}$. Suppose for a contradiction that there is such an operator $\oplus$. Let $C = \{c_0, c_1\}$, $d$ with $d(c_0, c_1) = M$, let $A = \{c_0, c_1, a_0, a_1\}$, $d$ with $d(a_0, a_1) = \delta$, $d(a_0, c_1) = M$ and $d(a_1, c_1) = M + 1$, where $i \in \{0, 1\}$. Note that the triangles $a_0, c_0, c_1$ and $a_1, c_1, c_0$ are isomorphic. Also let $B = \{c_0, c_1, b\}$, $d$ with $d(b, c_0) = d(b, c_1) = M$. Let us consider the canonical amalgam $A \oplus_C B$. Note that in this amalgam $b$ cannot be identified with $a_0$ or $a_1$, so we can write $A \oplus_C B$ as $\{a_0, a_1, b, c_0, c_1\}$, $d$ for some distance function $d$. By the antipodality we have that $d(a_0, b) = \delta - d(a_1, b)$, hence $d(a_0, b) \neq d(a_1, b)$. By monotonicity of $\oplus$, we have $\{a_0, b, c_0, c_1\} = \{a_0, b, c_0, c_1\} \oplus_C B$ and $\{a_1, b, c_1, c_0\} = \{a_1, c_1, c_0\} \oplus_C B$. But $\{a_0, b, c_0, c_1\}$ and $\{a_1, b, c_1, c_0\}$ are not isomorphic, which contradicts the assumption that $\oplus$ is an amalgamation operation.

4. Finally let us assume that $K_1 = \infty$ and $\delta$ is even. Then let us consider structures $A, B, C \in A_{K_1,K_2,C_0,C_1,S}$ such that $C = \{c\}$, $d$, $B = \{b_1, b_2, c\}$, $d$ with $d(b_1, b_2) = \delta$ and $d(b_1, c) = d(b_2, c) = \frac{\delta}{2}$ and $A = \{a, c\}$, $d$ with $d(a, c) = 1$. Assume that there is a local canonical amalgamation operation $\oplus$, then let us consider the canonical amalgam $A \oplus_C B$. By monotonicity of $\oplus$ we must have $d(a, b_1) = d(a, b_2)$ in this amalgam. Since $(b_1, b_2)$ is an antipodal pair this implies that $d(a, b_1) = d(a, b_2) = \frac{\delta}{2}$. However then $A \oplus_C B$ cannot be a bipartite distance graph, since then $(a, c, b_1)$ is a triangle of odd perimeter $\delta + 1$, a contradiction.

8. Classes with infinite diameter

Classes of infinite diameter have two basic types—the 3-constrained cases and tree-like graphs.

8.1. 3-constrained spaces of infinite diameter

The admissible numerical parameters with $\delta = \infty$ are

1. (I) $K_1, C_0, C_1 = \infty$, $K_2 = 0$ (the generic bipartite metric space)

2. (II) $1 \leq K_1 < \infty$, $C_0, C_1, K_2 = \infty$ (the generic metric spaces without short odd cycles)

Theorem 8.1. The class $\hat{A}_{\infty,0,\infty,\infty}$ of convex orderings of $A_{\infty,0,\infty,\infty}$ with an additional unary predicate determining the bipartition is Ramsey and has the expansion property.
Proof. For every choice of finite \( A, B \) denote by \( \delta \) the maximal distance in \( B \). Then apply Theorem 6.12 for \( \overrightarrow{A}_{\infty,0,C_0,2\delta+1} \) where \( C_0 > 3\delta + 1 \) even to obtain \( C \rightarrow (B)^A_2 \). It is easy to see that \( C \in \overrightarrow{A}_{\infty,0,\infty,\infty} \).

The expansion property follows again by the standard argument. \( \Box \)

Theorem 8.2. For every \( 1 \leq K_1 \leq \infty \) the class \( \overrightarrow{A}_{K_1,\infty,\infty,\infty} \) of free orderings of \( A_{K_1,\infty,\infty,\infty} \) is Ramsey and has the expansion property.

Proof. Analogously to the previous proof, for every finite \( A, B \) we choose \( \delta \) to be the maximal distance in \( B \) and reduce to Theorem 4.19. \( \Box \)

Theorem 8.3. The classes \( A_{K_1,\infty,\infty,\infty}^\infty, 1 \leq K_1 \leq \infty, \) and \( A_{\infty,0,\infty,\infty}^\infty \) have coherent EPPA.

Proof. Again, this follows by a reduction to Theorems 4.20 and 6.13. \( \Box \)

The stationary independence relation follows from the existence of completion algorithm which adds the shortest path distances, as discussed in Section 1.1. From this we immediately get:

Theorem 8.4. The classes \( A_{K_1,\infty,\infty,\infty}^\infty, 1 \leq K_1 \leq \infty, \) and \( A_{\infty,0,\infty,\infty}^\infty \) have no stationary independence relations but local stationary independence relations that can be defined by means of the shortest path completion algorithm.

8.2. Tree-like graphs

Recall Definition 1.1 of tree-like graphs and denote by \( \mathcal{A}_{T_{m,n}} \) the age of metric space associated with the tree-like graph \( T_{m,n} \).

Theorem 8.5. For every \( 2 \leq m, n \leq \infty \) the class \( \mathcal{A}_{T_{m,n}} \) has no EPPA.

Proof. For every \( F \in \mathcal{A}_{T_{m,n}} \), let us call a vertex of \( F \) a non-leaf if it is contained in at least two blocks (maximal cliques) of \( F \) and leaf otherwise. Note that every \( F \in \mathcal{A}_{T_{m,n}} \) contains a leaf. Now suppose for a contradiction that \( \mathcal{A}_{T_{m,n}} \) has EPPA, and consider any structure \( A \in \mathcal{A}_{T_{m,n}} \) which contains a non-leaf. Given a finite EPPA-witness \( B \in \mathcal{A}_{T_{m,n}} \) of \( A \), we can assume without loss of generality that for every vertex \( v \) of \( B \) there exists \( \phi \in \text{Aut}(B) \) such that \( \phi(v) \in A \). By the extension property it then follows that the graph \( B \) is vertex transitive and thus either every vertex of \( B \) is a leaf or every vertex of \( B \) is a non-leaf, which contradicts to the assumption that \( A \) contains both leaves and non-leaves as soon as \( A \) is not a clique (e.g. when \( A \) has more than \( n \) vertices). \( \Box \)

Theorem 8.6. For every \( 2 \leq m, n \leq \infty \) the class \( \mathcal{A}_{T_{m,n}} \) has no precompact Ramsey expansion.

Proof. Assume for a contradiction that there is a precompact Ramsey expansion \( K \) of \( \mathcal{A}_{T_{m,n}} \). Denote by \( A_1^+, A_2^+, \ldots, A_k^+ \) all structures in \( K \) with one vertex. Let \( B \) be a path of edges in distance 2 with \( k+1 \) vertices completed to a metric space in \( \text{Age}(T_{m,n}) \). Denote by \( B^+ \) its expansion in \( K \). There is some \( A_i, 1 \leq i \leq k \), with at least two copies in \( B^+ \). Denote by \( B_i^+ \) the substructure induced by \( B^+ \) on those two copies and by \( \ell \) their distance.

By the Ramsey property, there is \( C^+ \in K \) such that \( C^+ \rightarrow (B_i^+)^{A_i^+}_{\ell+1} \) in \( K \). Without loss of generality we can assume that \( C^+ \) is an expansion of a connected fragment of \( T_{m,n} \). Fix a vertex \( v \in C^+ \) and colour the vertices \( u \in C^+ \) with \( d(u,v) \mod \ell + 1 \). Denote by \( u_1, u_2 \) the vertices of the monochromatic copy of \( A_1^+ \) in \( C^+ \), ensuring that \( u_1 \) is closer to \( v \) than \( u_2 \). Notice that between any pair of vertices at distance greater than \( 1 \) there is precisely one shortest path, so it follows that either \( d(v,u_2) = d(v,u_1) + \ell \) or \( d(v,u_1) + d(v,u_1) = \ell \). By the choice of the colouring this pair is not monochromatic, contradicting the choice of \( C^+ \). \( \Box \)

Lemma 8.7. For all \( 2 \leq m, n \leq \infty \) with \( m < \infty \) or \( n \neq 2, 3, \infty \) there is no local stationary independence relation on the metric space associated with the tree-like graph \( T_{m,n} \)
Proof. For a contradiction, assume that there is a local stationary independence relation $\perp$. By Theorem 4.16 this is equivalent to the existence of a local canonical amalgamation $\oplus$ on the $\mathcal{A}_{T_{m,n}}$.

First let us look at the case where $m < \infty$. Then let $C$ consist of a single vertex $c$ and let $A = \{c, a_1, a_2, \ldots, a_m\}$ such that $d(c, a_i) = 1$ and $d(a_i, a_j) = 2$ for all $i \neq j$. And let $B = \{b, c\}$ with $d(b, c) = 1$. Then, by the definition of $(T_{m,n}, d)$, in the amalgam $A \oplus_C B$ there has to be an index $i$, such that $d(b, a_i)$ is 1 or 0 and $d(b, a_j) = 2$ for all $j \neq i$. Thus $\{a_i, c\} \oplus_C B$ and $\{a_j, c\} \oplus_C B$ are non-isomorphic for $j \neq i$. But this contradicts the assumption that $\oplus$ is an amalgamation operator.

Next assume that $n \notin \{2, 3, \infty\}$. Let $C = \{c_1, c_2\}$ be such that $d(c_1, c_2) = 1$, let $B = \{a_1, a_2, b\}$ be a clique of size 3, and let $A = \{c_1, c_2, a_1, \ldots, a_{n-2}\}$ be a clique of size $n$. Then, by definition of $\mathcal{A}_{T_{m,n}}$, in the amalgam $A \oplus_C B$ the vertex $b$ has to be identified with one of the vertices $a_i$ in $A$. By monotonicity of $\oplus$ we have $\{a_i, c_1, c_2\} \oplus_C B$ is a clique of size 3, but for every $j \neq i$, $\{a_i, c_1, c_2\} \oplus_C B$ is a 4-clique. This contradicts the assumption that $\oplus$ is an amalgamation operator.

Surprisingly, for the remaining cases there exists no SIR, but there is a local SIR. In order to show this, we first need a better understanding of how the metric space $(T_{n,m}, d)$ relates to the underlying tree-like graph.

Definition 8.1. Let $(G, E)$ be a graph, inducing the graph metric $(G, d)$, and let $A = (A, d)$ be a subspace of $(G, d)$. Let $a, b \in A$, then we say that a path $a = y_0, y_1, \ldots, y_k = b$ in $(G, E)$ witnesses the distance $d(a, b)$, if $d(a, b) = k$.

Lemma 8.8. Let $A$ be a subspace of $(T_{n,m}, d)$ and let $(G_A, E)$ be a subgraph of $T_{n,m}$, consisting of paths witnessing the distances in $A$. Then, $(G_A, E)$ is uniquely determined by $A$. Also, if $A \cong B$, then $(G_A, E) \cong (G_B, E)$.

Proof. It is enough to show that for every pair $u, v$ in $A$, there is exactly one path in $T_{n,m}$ witnessing its distance $d(u, v) = c$. Then clearly $(G_A, E)$ is uniquely determined by $A$; the second statement follows from the homogeneity of $(T_{n,m}, d)$.

For a contradiction, assume that $c \geq 2$ and that there are two paths $u = x_0, x_1, x_2, \ldots, x_c = v$ and $u = y_0, y_1, y_2, \ldots, y_c = v$. Since the two paths are not equal, there have to be indices $i < k < c$ such that $x_i = y_i$, $x_{i+1} \neq y_{i+1}$, $x_k \neq y_k$ and $x_{k+1} = y_{k+1}$. Then, by definition of $T_{n,m}$, the set $\{x_i, x_{i+1}, \ldots, x_k\} \cup \{y_{i+1}, \ldots, y_k\}$ has to be a clique. But this implies that $d(u, v) < c$, which contradicts our assumptions.

Theorem 8.9. For every $m, n$, the metric space associated with tree-like graph $T_{m,n}$ has no stationary independence relation. It has a local stationary independence relation if and only if $m = \infty$ and $n \in \{2, 3, \infty\}$.

Proof. By Lemma 8.7 the statement about local SIR holds for all cases where $m \neq \infty$ or $n \notin \{2, 3, \infty\}$.

In order to prove that $(T_{m,n}, d)$ has no SIR we show that there is no canonical amalgamation operator on $\mathcal{A}_{T_{m,n}}$ (cf. Theorem 4.16). Assume there is such an operator $\oplus$. Let $A = \{a_1, a_2, a_3\}$ with $d(a_1, a_2) = d(a_2, a_3) = 1$ and $d(a_1, a_3) = 2$ and let $B$ consist of a single point $b$. We claim that then, no matter how the amalgam $A \oplus B$ is formed, we have $d(b, a_1) \neq d(b, a_2)$ or $d(b, a_1) \neq d(b, a_3)$. In order to prove this claim, let us assume that $d(b, a_1) = d(b, a_2)$. By Lemma 8.8 the graph witnessing the distances in $A \oplus B$ has to consist of a path from $b$ to $a_2$ and the two edges $d(a_2, a_3) = d(a_2, a_1) = 1$. Hence $d(b, a_2) = d(b, a_1) - 1$, which proves our claim. By the claim and monotonicity of $\oplus$, $\{a_1\} \oplus B$ is non-isomorphic to $\{a_i\} \oplus B$ for some $i \in \{2, 3\}$. But this contradicts the assumption that $\oplus$ is an amalgamation operator.

It is only left to show that for $m = \infty$ and $n \in \{2, 3, \infty\}$, there is a local stationary independence relation on $(T_{n,m}, d)$. We do so again by finding a local canonical amalgamation operator. So let $A, B, C$ be non-empty substructures of $(T_{m,n}, d)$ such that $e_1: C \rightarrow A$ and $e_2: C \rightarrow B$ are embeddings. By Lemma 8.8 we can assume that every distance in the structures $A, B, C$ is
witnessed by a path (so informally we can think about $A, B, C$ as the underlying tree-like graphs instead of the metric spaces).

If $m = \infty$ and $n = 2$ we define $A \oplus_C B$ to be the free amalgam of the trees $A$ and $B$ over $C$. This graph is again a tree and thus in $T_{2, \infty}$. It is easy to see that this amalgamation operator is associative and monotone.

If $m = \infty$ and $n = 3$ we construct $A \oplus_C B$ again by first forming the free amalgam of the graphs $A$ and $B$ over $C$. This free amalgam might contain $c_1, c_2, a, b$, such that $a, c_1, c_2$ is a 3-clique and $b, c_1, c_2$ is a 3-clique, but there is no edge between $a$ and $b$. In this case we identify $a$ and $b$. Again this amalgamation operator is associative and monotone.

If $m = \infty$ and $n = \infty$ we construct $A \oplus_C B$ by first forming the free amalgam of the graphs $A$ and $B$ over $C$. Then we add edges to complete all the subgraphs $A' \cup B'$ with $A' \subseteq A$ and $B' \subseteq B$, such that $A'$ and $B'$ are cliques of size $\geq 3$ and share at least one edge. Again this amalgamation operator is associative and monotone.

\[ \square \]

9. Corollaries

While the main focus of our work is on the combinatorial properties of metrically homogeneous graphs, let us briefly discuss the corollaries of our results for the automorphism group of the Fraïssé limits. This section is not meant to be exhaustive and we refer the reader to a recent survey [NVT15] and thesis [Sin17] which discuss many of the topics in a greater detail.

9.1. Ample generics

The automorphism group of the Fraïssé limits of amalgamation classes in the catalogue can be seen as Polish groups when endowed with the pointwise convergence topology. A Polish group has \textit{generic automorphisms} if it contains a comeagre conjugacy class. A Polish group has \textit{ample generics} if it has a comeagre diagonal conjugacy class in every dimension.

Known examples of structures with ample generics include $\omega$-stable $\omega$-categorical structures [HHLS93], the random graph [HHLS93, Hrn92], the homogeneous $K_n$-free graph [Her98], the rational Urysohn space [Sol05, Ver08], free homogeneous structures over finite relational languages [SS19] and Philip Hall's locally finite universal group [Sin17].

Our study of coherent EPPA is directly motivated by the following result:

\textbf{Theorem 9.1} (Hodges–Hodkinson–Lascar–Shelah [HHLS93], see also Siniora–Solecki [Sin17]). \textit{Suppose that $M$ is a homogeneous structure such that $\text{Age}(M)$ has both EPPA and APA. Then $M$ has ample generics.}

Where we say a class $\mathcal{K}$ of finite structures has the \textit{amalgamation property with automorphisms (APA)} if whenever $A, B_1, B_2 \in \mathcal{K}$ with embeddings $\alpha_1 : A \to B_1$ and $\alpha_2 : A \to B_2$ then there is a structure $C \in \mathcal{K}$ with embeddings $\beta_1 : B_1 \to C$ and $\beta_2 : B_2 \to C$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ and whenever $f \in \text{Aut}(B_1)$ and $g \in \text{Aut}(B_2)$ such that $f \circ \alpha_1(A) = \alpha_1(A)$, $g \circ \alpha_2(A) = \alpha_2(A)$, and for every $a \in A$ we have $\alpha_1^{-1} \circ f \circ \alpha_1(a) = \alpha_2^{-1} \circ g \circ \alpha_2(a)$, then there exists $h \in \text{Aut}(C)$ which extends $\beta_1 \circ f \circ \beta_1^{-1} \cup \beta_2 \circ g \circ \beta_2^{-1}$. Observe that APA follows from our canonical amalgamation and the automorphism preservation lemma for all classes where we shown the coherent EPPA. As a corollary of Theorem 1.2 we thus extended the list of known structures with ample generics by many new examples.

Ample generics are a powerful tool to prove additional properties. Among multiple consequences of the ample generic we list the following:

\textbf{Theorem 9.2} (Kechris–Rosendal [KR07]). \textit{Suppose that $G$ is a Polish group with ample generics. Then $G$ has the small index property.}

\textbf{Theorem 9.3} (Kechris–Rosendal [KR07]). \textit{Suppose that $M$ is an $\omega$-categorical structure with ample generics. Then $\text{Aut}(M)$ has uncountable cofinality and 21-Bergman property.}

\textbf{Theorem 9.4} (Kechris–Rosendal [KR07]). \textit{Suppose that $M$ is an $\omega$-categorical structure with ample generics. Then $\text{Aut}(M)$ has Serre’s property (FA).}
9.2. Amenability and unique ergodicity

In this section, let \( G = \text{Aut}(\text{Flim}(K)) \), be the automorphism group of the Fraïssé limit of the class \( K \) endowed with the topology of pointwise convergence. In the following we set up the machinery needed to discuss when \( G \) is amenable (its universal minimal flow has a \( G \)-invariant measure) or uniquely ergodic (a unique such measure). See \cite{NVT15} and \cite{AKL14} for a more in depth discussion of the definitions and their history.

We now state precise definitions and lemmas which capture the idea that “amenability and unique ergodicity can sometimes be shown by counting finite quantities related to the number of Ramsey expansions of some finite structures.” We have in mind the example that arbitrary unique ergodicity can sometimes be shown by counting finite quantities related to the number \( |\{A \subseteq B \text{ and the relations induced by } B \text{ on } A \text{ give precisely } A\} | \) be structures in \( K \) and let \( (A * A^+) \in K^+ \). Then we write:

\[
\text{Nexp}_K(A^+, B) := |\{B^+ : (A * A^+) \leq (B * B^+) \in K^+\}|,
\]

which is the number of expansions of \( B \) in \( K^+ \) that extend \( (A * A^+) \). If there is no confusion then we write \( \text{Nexp}(A^+, B) \).

The following black-box lemma reduces checking amenability to a counting argument.

**Lemma 9.5** (Angel–Kečkić–Lyons [AKL14], Pawliuk–Sokić [PS18]). Let \( K^+ \) be a precompact Ramsey expansion of \( K \). If for every \( A \leq B \) in \( K \) and every \( (A * A'), (A * A'') \in K^+ \) we have

\[
\text{Nexp}(A', B) = \text{Nexp}(A'', B)
\]

then \( \text{Aut}(\text{Flim}(K)) \) is amenable.

In particular, this will happen if the values in equation (1) only depend on the isomorphism type of \( A \) and \( B \).

For fixed structures \( A \) and \( B \) with expansions \((A * A^+)\) and \((B * B^+)\), define

\[
\text{Nemb}(A, B) := |\{A : A \leq B\}|.
\]

Let \( H \in K \) with \(|H| = k \) and \( G \) is a random element of \( K \) with \(|G| = n \) (typically \( k << n \)). Let

\[
f(G) := \frac{\text{Nemb}(H, G)}{\mathbb{E}[\text{Nemb}(H, G)]}, \quad f^+(G) := \frac{\text{Nexp}(H^+, G)}{\mathbb{E}[\text{Nemb}(H, G)]},
\]

where \( \mathbb{E} \) is the expected value. See section 7.2 of [PS18] for a more in depth discussion of these notions.

The following black-box lemma reduces checking unique ergodicity to counting three quantities.

**Lemma 9.6** (Angel–Kečkić–Lyons [AKL14], Pawliuk–Sokić [PS18]). Using the notation defined above, suppose that \( \text{Aut}(\text{Flim}(K)) \) is amenable, that changing \( G \) by a single edge changes \( f \) and \( f^+ \) by no more than \( O\left(\frac{1}{n^2}\right) \), and that the number of expansions of \( G \leq O((nl)^k) \). Then \( \text{Aut}(\text{Flim}(K)) \) is uniquely ergodic.

Note that this counting trick goes back to Nešetřil and Rödl [NR78] where the strong ordering property is established for classes of graphs without short cycles, a key lemma from that paper is then used in [AKL14]. See also [NR17].
In general these two lemmas are proved by explicit computations. While the computations asked for by Lemma 9.6 are straightforward, it is often tedious to set up and the quantities are somewhat opaque. So instead of writing down these computations for metric graphs, we will leverage computations that have already been made for unlabelled directed graphs [PS18]. Despite the differences in the relations of the underlying structures (there is a single asymmetric relation, here there are many symmetric relations) the precompact expansions are nearly identical. These expansions only care about the definable equivalence classes of the structures. For example, in [PS18] the class \( D_2 \) of all complete bipartite directed graphs have convex (with respect to the bipartition) linear orders as expansions, and here we investigate various classes of metric graphs that form a bipartition and who also have convex (with respect to the bipartition) linear orders as expansions. In both cases, when showing amenability by counting, once the underlying structures \( \mathbf{A} \leq \mathbf{B} \) are chosen, all counting about the expansions becomes identical. In this sense, it is immaterial for our purposes that the computations in [PS18] are about directed graphs.

The next lemma makes this precise.

**Lemma 9.7.** Let \( \mathcal{K} \) and \( \mathcal{F} \) be Fraïssé classes in signatures \( L \subseteq F \) respectively. Let \( \pi \) be the map that forgets the extra structure in \( F \setminus L \). Suppose that \( \mathcal{K}^+ \) and \( \mathcal{F}^+ \) are precompact Ramsey expansions of \( \mathcal{K} \) and \( \mathcal{F} \) respectively, such that \( \pi[\mathcal{F}^+] = \mathcal{K}^+ \).

1. If amenability of \( \text{Aut}(\text{Flim}(\mathcal{K})) \) was shown by counting (i.e. Lemma 9.5), then \( \text{Aut}(\text{Flim}(\mathcal{F})) \) is amenable by counting.

2. If unique ergodicity of \( \text{Aut}(\text{Flim}(\mathcal{K})) \) was shown by counting (i.e. Lemma 9.6), then \( \text{Aut}(\text{Flim}(\mathcal{F})) \) is uniquely ergodic by counting.

The relevant computations from [PS18] will all be applicable in our case with at most minor modifications.

The case of tree-like graphs is exceptional because these classes do not have precompact expansions. Clearly, \( \text{Aut}(T_{2,2}) \) is amenable because \( T_{2,2} \) is just the infinite path with no endpoints. We conjecture that in all the other cases, \( \text{Aut}(T_{m,n}) \) is not amenable, see Conjecture 10.4 for more discussion.

The non-tree-like classes will be dealt with combinatorially:

**Theorem 9.8.** If \( \mathcal{K} \) is a primitive 3-constrained class of metric graphs, then \( G \) is amenable and uniquely ergodic.

**Proof.** Since we have shown that \( \overrightarrow{\mathcal{K}} \) is a precompact Ramsey expansion of \( \mathcal{K} \) (see Theorem 4.19), a direct, counting proof is given by Angel, Kechris and Lyons [AKL14]. Furthermore, this expansion guarantees that \( G \) is uniquely ergodic.

Alternatively, amenability of \( G \) will follow from the fact that \( \mathcal{K} \) has EPPA, although this is not enough for unique ergodicity. See Kechris and Rosendal [KR07].

**Theorem 9.9.** If \( \mathcal{K} \) is a bipartite 3-constrained class of metric graphs, then \( G \) is amenable and uniquely ergodic.

**Proof.** Amenability of \( G \) follows from the fact that \( \mathcal{K} \) has EPPA, although this is not enough for unique ergodicity. See Kechris and Rosendal [KR07].

Alternatively, we have shown that adding linear orders that are convex with respect to the equivalence classes of even distances and unary marks to determine the bipartition is a precompact Ramsey expansion of \( \mathcal{K} \) (see Theorem 6.12). A direct counting argument for Lemma 9.5, is

\[
\text{Nexp}(\mathbf{A}', \mathbf{B}) = 2! \cdot \frac{|B_1|||B_2|}{|A_1|||A_2|}
\]

where \( B_1, B_2 \) are the equivalence classes of even distances of \( \mathbf{B} \), and \( A_1, A_2 \) are the equivalence classes of even distances of \( \mathbf{A} \), and \( \mathbf{A} \) contains vertices from both bipartitions (otherwise the coefficient 2! changes). This computation appears as Theorem 4.2 in [PS18] (there it is discussed in the context of generic multipartite digraphs).

The computation for unique ergodicity and Lemma 9.7 appears as Theorem 8.2 in [PS18].
Theorem 9.10. If $K$ is an antipodal, non-bipartite class of metric graphs, then $G$ is amenable and uniquely ergodic.

Proof. When $\delta$ is even, amenability of $G$ follows from the fact that $K$ has EPPA (see Theorem 7.7).

We have shown that adding linear orders that are convex with respect to the podes and a unary marks denoting the podes is a precompact Ramsey expansion of $K$ (see Theorem 1.1 and Remark 7.1). Observe that this expansion is bi-definable with the expansion by an order and unary marks for the podes, where in the order each edge of length $\delta$ forms an interval and the smaller elements from each $\delta$ pair are in the same pode.

This alternative expansion corresponds to what is discussed in [PS18] for double-covers of the generic tournaments. Suppose, for simplicity, that both structures $A$ and $B$ are antipodally closed. Then for amenability, the direct counting argument for Lemma 9.5, is

$$\text{Nexp}(A', B) = \frac{b!}{a!} \cdot 2^{b-a},$$

where $b$ is the number of $\delta$-pairs in $B$ and $a$ is the number of $\delta$-pairs $A$. This computation appears as Theorem 4.3 in [PS18].

The computation for unique ergodicity and Lemma 9.7 appears as Theorem 8.3 in [PS18].

Theorem 9.11. If $K$ is an antipodal, bipartite class of metric graphs, then $G$ is amenable and uniquely ergodic.

Proof. For some values of $\delta$ amenability follows from the fact that $K$ has EPPA (see Theorem 7.7).

There are two cases to consider: $\delta$ even and $\delta$ odd. When $\delta$ is even, then each $\delta$-pair lives inside a bipartite equivalence class.

In both cases, for simplicity and to keep this section concise, we assume that $A$ and $B$ are antipodally closed and that $A$ contains elements from each part of the bipartition. If this is not true, the computations go in a similar manner with some further technicalities.

For $\delta$ even: Similarly as for antipodal non-bipartite, one can show that adding unary marks for the bipartitions and the podes and linear orders that are convex with respect to the bipartitions and where each edge of length $\delta$ forms an interval is a precompact Ramsey expansion of $K$ bi-definable with the one presented in Theorem 1.1 and Remark 7.1. A direct counting argument for Lemma 9.5, is

$$\text{Nexp}(A', B) = \frac{b_1!}{a_1!} \cdot \frac{b_2!}{a_2!} \cdot 2^{b-a},$$

where $b$ is the number of $\delta$-pairs in $B$, $a$ is the number of $\delta$-pairs in $A$, $b_i$ (for $i = 1, 2$) is the number of $\delta$-edges in the $i$-th part of the bipartition (and $a_i$ is defined similarly).

For $\delta$ odd: We have shown that adding unary marks for the bipartitions and linear orders that are convex with respect to the bipartition, and when restricted to the bipartitions (or equivalently in this case the podes) are isomorphic, is a precompact Ramsey expansion of $K$ (See Theorem 1.1). A direct counting argument for Lemma 9.5, is

$$\text{Nexp}(A', B) = \frac{|B|!}{|A|!}.$$

For unique ergodicity, it suffices to notice that defining the order on one node in a $\delta$-pair uniquely determines the order for the second node. In this way, unique ergodicity of these classes follows from unique ergodicity of the corresponding bipartite and non-bipartite 3-constrained classes.

10. Conclusion and open problems

Between the submission of this paper (2017) and its final revision (2023) there have been many developments, and in particular some of the open problems have been solved (and others have arisen). We decided to update this section and mention whenever we are aware of some progress, while keeping the original numbering.
The EPPA results of this paper need an extra expansion for the odd-diameter antipodal classes and the even-diameter bipartite antipodal classes. In the original version we asked whether this is necessary and argued that this is closely connected to EPPA for two-graphs:

The class \( A_{0,10,7,2} \) of all antipodal metric spaces of diameter 3, the metric spaces can be equivalently interpreted as double-covers of complete graphs. By double-cover of \( K_n \) we mean a graph \( G \) with a 2-to-1 covering map from the vertices of \( G \) to those of \( K_n \), so that each edge of \( K_n \) is covered by two edges of \( G \). There are two double-covers of \( K_3 \), a pair of triangles and a hexagon, see Figure 13.

It is a well known fact that double-covers of complete graphs correspond to two-graphs (3-regular hypergraphs where every subhypergraph with 4 vertices contains an even number of hyperedges): given a double-cover \( G \) of \( K_n \) put vertices \( a,b,c \in K_n \) to a hyper-edge if and only if the corresponding double-cover is a hexagon.

Two-graphs have been extensively studied since 1960s [Sei73, GR01] and many of those results are relevant to our problem. Because the correspondence between antipodal metric graphs and the underlying two-graphs is automorphism-preserving, by constructing an EPPA-witness in \( A \in A_{0,10,7,2} \) we would also construct an EPPA-witness of its associated two-graph. Extensions can be assumed to be vertex transitive (because every vertex may be assumed to be contained in the copy) and construction of vertex-transitive (or regular) two-graphs is related to construction of strongly regular graphs and Taylor graphs. This is a well established topic of algebraic graph theory and those graphs are rare. Having a positive answer to the problem of EPPA of the odd non-bipartite antipodal graphs would thus require a construction of strongly regular graphs with even stronger symmetry assumptions and thus seems difficult.

Removing the need for the unary expansion for the class \( A \in A_{0,10,7,2} \) would closely correspond to proving EPPA for two-graphs, a problem posed by Macpherson which also appears in [Sin17].

There has been progress and the expansion was shown not to be necessary by Evans, Hubička, Konečný, and Nešetřil [EHKN20] for the antipodal metric space of diameter 3 (and for two-graphs) and by Konečný [Kon20]. Curiously, while all the antipodal metric spaces do have coherent EPPA, coherence gets lost when applying the arguments to two-graphs. Hence the question whether the class of all finite two-graphs has EPPA still remains open:

**Question 10.1** ([EHKN20]). Does the class of all finite two-graphs have coherent EPPA?

The classification of metrically homogeneous graphs is not a classification of metric spaces with integer distances, because it includes the additional requirement of containing all geodesics, i.e all triangles with edge lengths \( a, b \) and \( |a - b| \). Our completion algorithm does not really rely on this requirement (though it is implicitly used in the definition of the time function). It would be interesting to give a more general characterisation of classes where such an approach works, possibly including non-binary relations in the sense of homogenizations defined in [HN19].

As already mentioned in the introduction, Sauer [Sau13] identified for which subsets \( S \subseteq \mathbb{R}_0^+ \) the class of all \( S \)-valued metric spaces is an amalgamation class. Hubička and Nešetřil [HN19] later proved the Ramsey property for all these classes. Conant [Con19] studied EPPA generalised metric spaces (that is metric spaces with values from some ordered distance monoid). Finally,
Braunfeld [Bra17] proved that Λ-ultrametric spaces, where Λ is a finite distributive lattice, also have the Ramsey property.

In [HKN18], Hubička, Konečný and Nešetřil generalise all these results and discuss the Ramsey property and EPPA of a broad family of classes for which there is a commutative ordered semigroup \( M \) satisfying some further axioms, such that the incomplete structures with completion to these classes can be completed by the shortest path algorithm, which uses the semigroup operation instead of + and the semigroup order instead of the standard order of the reals. See also [Kon19].

It turns out that for a primitive 3-constrained class and a magic parameter \( M \), one can define a commutative and associative operation \( \oplus^M \colon [\delta]^2 \to [\delta] \) as

\[
\begin{align*}
  x \oplus^M y &= \begin{cases} 
  |x - y| & \text{if } |x - y| > M \\
  \min (x + y, C - 1 - x - y) & \text{if } \min (\ldots) < M \\
  M & \text{otherwise}
  \end{cases}
\end{align*}
\]

and the partial order \( \preceq^M \) as \( x \preceq^M y \) if and only if \( x = y \) or there is \( z \) such that \( x \oplus z = y \) (this is called the natural order of \( \oplus^M \)).

One can observe that the order induced by \( t_M(x) \) is an extension of \( \preceq \). And in this setting, we basically proved in this paper the following proposition (cf. Theorem 4.9):

**Proposition 10.2.** Let \( G = (G, d) \in \mathcal{G}^d \) such that it has a completion in \( \mathcal{A}_{K_1,K_2,C_0,C_1}^\delta \). Let \( M \) be a magic distance. Define \( \bar{d} \colon [\delta]^2 \to [\delta] \) as

\[
\bar{d}(x,y) = \begin{cases} 
  d(x,y) & \text{if } d(x,y) \text{ defined} \\
  \min_{P \text{ path from } x \text{ to } y} \bigoplus_i^M d(P_i,P_{i+1}) & \text{otherwise}
  \end{cases}
\]

(minimum is taken in the \( \preceq^M \) order).

Then \( \mathcal{G} = (V, \bar{d}) \) is in \( \mathcal{A}_{K_1,K_2,C_0,C_1}^\delta \).

In [HKN18] the interpretation of some metrically homogeneous graphs as generalised semigroup-valued metric spaces is discussed and their Ramsey property and EPPA then follow from a more general result (which, though, still depends on the fact that the magic completion works for these classes). See also [Kon19].

5. One of our original motivations for investigating the problem was the problem stated by Lionel Nguyen Van Thé about the Ramsey expansion of the class of affinely independent Euclidean metric spaces [NVT10]. Our techniques do not seem to generalise to this setting, however it seems more clear that the lack of (local) canonical amalgamation is one of the main obstacles. It would be interesting to identify classes of a more combinatorial nature which also expose such problems as an additional step in this direction.

6. Stationary independence relations have been an ingredient for showing simplicity of automorphism groups of some homogeneous structures [TZ13, EGT16]. After the submission of this paper, Evans, Hubička, Konečný, Li, and Ziegler [EHK+21] showed that the existence of a stationary independence relation satisfying some extra (which are indeed satisfied by the stationary independence relations defined in this paper for the finite-diameter primitive 3-constrained case with Henson constraints) implies simplicity of the automorphism group. However, for the other cases the following question remains open:

**Question 10.3.** What are the normal subgroups of the automorphism groups of the non-tree-like countably infinite metrically homogeneous graphs from Cherlin’s catalogue?

In fact, we believe that a solution to this question is within reach by refining the analysis from [EHK+21].
7. Since the tree-like graphs do not have a precompact Ramsey expansion, the KPT correspondence cannot be used as to argue about amenability of the automorphism groups of the tree-like graphs. $T_{2,2}$ is the infinite path with no endpoints and Aut($T_{2,2}$) is isomorphic to $(\mathbb{Z}, +)$ which is amenable. We conjecture that this is the only amenable case:

**Conjecture 10.4.** If $(m, n) \neq (2, 2)$ then Aut($T_{m,n}$) is not amenable.

Our intuition for conjecturing this is the following: If $m = 2$ and $n < \infty$, one can consider a graph $T'$ whose vertices are the blocks of $T_{2,n}$ and two vertices are connected by an edge if and only if the blobs have a non-empty intersection. Then $T'$ is the regular infinite $n$-ary tree and the action of Aut($T_{2,n}$) on the blocks of $T_{2,n}$ should give a continuous surjection onto Aut($T'$). In general, if $m, n \leq \infty$, one can do a similar construction with the only complication being that $T'$ is additionally equipped with an equivalence relation on the neighbourhood of every vertex corresponding to several blocks intersecting in the same vertex. Regarding the infinite parameter cases, $T_{\infty,2}$ is just the infinitely branching tree and one should be able to use arguments from [EHN19] to prove non-amenability of its automorphism group. (Or maybe it is a known result?)

8. We have proved various results for the class of finite metric subspaces of the associated metric spaces of metrically homogeneous graphs. However, one might want to ask these questions in terms of finite graphs and their isometric embeddings. For example, for every finite metric space $A$ with integer distances there exists a finite graph $G$ whose associated metric space contains $A$ as a substructure. Moreover, this finite graph can be constructed canonically so that every automorphism of $A$ extends to an automorphism of the associated metric space of $G$ (one simply adds a disjoint path of length $k$ for every edge of distance $k$).

This becomes more interesting once one considers $A$ from some $A^{\delta}_{K_1,K_2,C_0,C_1}$ and also requires that the associated metric space of $G$ belongs to $A^{\delta}_{K_1,K_2,C_0,C_1}$:

**Question 10.5.** Given a choice of admissible primitive parameters $\delta$, $K_1$, $K_2$, $C_0$ and $C_1$, and $A \in A^{\delta}_{K_1,K_2,C_0,C_1}$, does there exist a finite graph $G$ with associated metric space $B \in A^{\delta}_{K_1,K_2,C_0,C_1}$ such that $B$ contains $A$ as a substructure?

**Question 10.6.** In the setting of Question 10.5, can one require that every automorphism of $A$ extends to an automorphism of $B$?

A positive answer to Question 10.5 was sketched by Cherlin for the unconstrained cases for every $\delta > 1$ (i.e., only non-metric triangles are forbidden) and also for all cases with $\delta = 3$.

This question can be traced back to Moss [Mos92] who studied *distanced graphs* (that is, graphs with isometric embeddings) and in particular asked which countable distanced graphs $G$ are *distance finite* (that is, for every finite metric subspace $(X, d)$ of the associated metric space of $G$ there is a finite subgraph $Y$ of $G$ which embeds isometrically into $G$ such that $X \subseteq Y$). He proved that the metrically homogeneous graph corresponding to the class $A^{\infty}_{\infty,\infty,\infty,\infty}$ (the class of all finite metric spaces with integer distances) is distance finite (the argument was sketched above) and that there exists a non-distance-finite countable distanced graph which, however, is not (metrically) homogeneous. On page 299 he remarks that distance finiteness is open even for *distance homogeneous graphs* (metrically homogeneous graphs in our language) – Question 10.5 is a special case of this remark. Moss’ question can be phrased in today’s terms as the first part of the following question:

**Question 10.7** (Moss [Mos92]). Given a countable metrically homogeneous graph $\Gamma$ and a finite subset $X \subseteq \Gamma$, does there exist a finite $Y \subseteq \Gamma$ such that $X \subseteq \Gamma$ and the graph induced by $\Gamma$ on $Y$ embeds isometrically to $\Gamma$? If this is the case, can one construct $Y$ canonically so that every isometry $X \to X$ extends to an isometry $Y \to Y$? 

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Gregory Cherlin, personal communication.
Let us remark that Question 10.5 is related to the finite model property. More precisely, this is a special case of it for formulas describing that a given finite graph is isometrically realised and constraints of the ambient class are satisfied in the associated metric space. Even for triangle-free graphs (which would correspond to the class \(A_{5,4,8,7}\) if \(\delta = 2\) wasn’t explicitly excluded in Definition 3.1 for practical reasons) the finite model property is wide open. See e.g. [Che11] for an overview and [EZL15] for the presently strongest result in this direction.

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Shortly before this paper was submitted we were informed that similar results on finite set of obstacles, Ramsey property and ample generics were also obtained by Rebecca Coulson and Gregory Cherlin which later appeared in [Cou19]. The Ramsey expansion of space \(A_{3,1,3,8,9}\) was also independently obtained by Miodrag Sokić [Sok20].

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