Borsuk-Ulam Theorems for Complements of Arrangements

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Abstract

In combinatorial problems it is sometimes possible to define a $G$-equivariant mapping from a space $X$ of configurations of a system to a Euclidean space $\mathbb{R}^m$ for which a coincidence of the image of this mapping with an arrangement $\mathcal{A}$ of linear subspaces insures a desired set of linear conditions on a configuration. Borsuk-Ulam type theorems give conditions under which no $G$-equivariant mapping of $X$ to the complement of the arrangement exist. In this paper, precise conditions are presented which lead to such theorems through a spectral sequence argument. We introduce a blow up of an arrangement whose complement has particularly nice cohomology making such arguments possible. Examples are presented that show that these conditions are best possible.

1 Borsuk-Ulam type results

Theorems of Borsuk-Ulam type prevent conditions preventing the existence of certain equivariant mappings between spaces. The classical Borsuk-Ulam theorem, for example, treats mappings of the form $f: S^n \to \mathbb{R}^n$ for which $f(-x) = -f(x)$, that is, $f$ is equivariant with respect to the antipodal action of $\mathbb{Z}/2$ on $S^n$ and the action of $\mathbb{Z}/2$ on $\mathbb{R}^n$. Such a map must meet the origin.

Generalizations of the Borsuk-Ulam theorem abound and their applications include some of the more striking results in some fields (see [10]). One of the more general formulations of Borsuk-Ulam type is the theorem of Dold [7]: For an $n$-connected $G$-space $X$ and $Y$ a free $G$-space of dimension at most $n$, there are no $G$-equivariant mappings $X \to Y$. In this paper we consider a theorem of this type for which the target space is the complement of an arrangement of linear subspaces in a Euclidean space. Such spaces have been intensely investigated in recent years and they represent natural test spaces for problems in combinatorics and geometry. The nonexistence of an equivariant mapping from a configuration space to the Euclidean space containing the arrangement must meet the arrangement, that is, the image must satisfy the linear conditions defining the arrangement.

To control the algebraic topology of the complement of an arrangement, we introduce the notion of a blow-up of a given arrangement whose cohomology is especially nice [9]. The argument for the main theorem is a novel use of the spectral sequence associated to the Borel construction on a $G$-space.

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1.1 Statement of the main result

A finite family of linear subspaces $\mathcal{A}$ in some Euclidean space $\mathbb{R}^m$ is known as an arrangement. Let $M_{\mathcal{A}}$ denote the complement of the arrangement $\mathbb{R}^m \setminus \bigcup \mathcal{A}$. Suppose a group $G$ acts on $\mathbb{R}^m$. The set of fixed points of the action of $G$ is denoted by $(\mathbb{R}^m)^G$. An arrangement $\mathcal{A}$ is a $G$-invariant arrangement if for all $g \in G$ and for all $L \in \mathcal{A}$, $gL \in \mathcal{A}$. In the statement of the following theorem of Borsuk-Ulam type we use notion of a blow up introduced in Section 2.1.

Theorem 1 Let $G$ denote a finite or a compact Lie group and $\mathbb{k}$ a field. Let $X$ be a $G$-space satisfying $H^i(X, \mathbb{k}) = 0$ for $1 \leq i \leq n$ for some $n \geq 2$. Consider a $G$-invariant arrangement $\mathcal{A}$ in (some subspace $V$ of) $\mathbb{R}^m$ and its $G$-invariant blow up $\mathcal{B}(\mathcal{A})$ such that

(A) the codimension of all maximal elements in $\mathcal{A}$ is less then $n + 2$.
(B) for all \( g \in G \) and all maximal elements \( L \) of the arrangement \( \mathcal{A} \), we have \( g \cdot L = L \),

(C) \( G \) acts trivially on the cohomology \( H^*(M_{\mathbb{B}(\mathcal{A})}, k) \),

(D) the map \( H^*(BG, k) \to H^*(EG \times_G M_{\mathbb{B}(\mathcal{A})}, k) \), induced by the natural projection \( EG \times_G M_{\mathbb{B}(\mathcal{A})} \to BG \), is not a monomorphism, and

(E) for all \( L \in \mathcal{A} \), \( L \supseteq (\mathbb{R}^m)^G \).

Then there is no \( G \)-map \( X \to M_{\mathcal{A}} \).

**Remark.** Condition (A) implies that the complement \( M_{\mathcal{A}} \) is \((n-1)\)-connected. Theorem 1 resembles Dold’s theorem [7, Remark on page 68] in which condition (C) follows when the action on the complement is free. The essential difference is not condition (C) (we formulated it a little bit more generally) but that the dimension of the space \( M_{\mathcal{A}} \) can be arbitrary large. Also, it should be noted that it is not generally possible to produce a \( G \)-invariant deformation of the complement \( M_{\mathcal{A}} \) to an \( n \)-dimensional subspace. Lemma 6.1 in [1] is only known example of an equivariant deformation of an arrangement.

**Remark.** Condition (A) can be substituted with the less restrictive condition that

\[
\min\{\text{codim } L \mid L \in \mathcal{A}\} = n + 1.
\]

Then in all the remaining conditions, replace the arrangement \( \mathcal{A} \) by a subarrangement \( \mathcal{A}' \) generated by all maximal elements of minimal codimension.

**Remark.** Conditions (B) and (C) are in many cases equivalent. This can be seen from the equivariant Goresky-MacPherson formula in [12, Theorem 2.5.(ii)].

**Remark.** Conditions (B) and (C) in some examples can be relaxed a little, but not dropped all together. We illustrate this in Section 3 with the construction of a \( G \)-map \( X \to M_{\mathcal{A}} \) from an \( n \)-connected \( G \)-space \( X \) to a \( n+1 \) arrangement \( \mathcal{A} \) complement. The arrangement \( \mathcal{A} \) satisfies conditions (A), (D), (E) but not (B) and (C). However, particular results can be obtained even when the conditions (B) and (C) are not satisfied.

**Remark.** When the group \( G \) is connected, all arrangements satisfy condition (C) since \( \pi_1(BG) = \pi_0(G) = 0 \). Moreover, conditions (B) and (C) are equivalent (from [12, Theorem 2.5.(ii)])). Condition (D) is satisfied when, for example,

- \( G \) acts freely on the complement \( M_{\mathbb{B}(\mathcal{A})} \), or
- \( G \) is a \( k \)-torus or an elementary abelian \( p \)-group acting without fixed points on \( M_{\mathcal{A}} \) and consequently on \( M_{\mathbb{B}(\mathcal{A})} \).

### 1.2 An Application: Antipodal cheese problem

Every theorem of general type is usually a product of successful or more often unsuccessful effort in solving some concrete problem. One of the motivating problems for the study of Borsuk Ulam type theorems for complements of arrangements is a class of mass partition problems discussed in [1], [2], [3], [4] and [5]. Particularly, we present a problem which solution, after careful combinatorial reformulation, presents a direct consequence of Theorem 1.

**Antipodal cheese problem**

Suppose that even number \( 2k \) of people is sitting around a circle table in such a way that every one has its antipodal friend. On the table there is pile of \( j \) (high dimensional) cheese pieces (in \( \mathbb{R}^d \)); all of different shape, mass, density and flavor. The line knife is available and the cheese can be cut only simultaneously, all \( j \) pieces at once. There are two types of cuts we allow (Figure 1):

- half-straight cut: pick a point on the table and make \( 2k \) straight (line) cuts starting at the chosen point and continuing in one direction,
- straight cut: pick a point on the table and make \( k \) straight (line) cuts through the chosen point in both directions.

The objective is to divide cheese (\( j \) pieces) in \( \mathbb{R}^d \) by \( 2k \) half-straight cuts or \( k \) straight cuts in such a way that every member of an antipodal pair get the same, non-negative, part of each piece of cheese.
Mathematical reformulation

The vocabulary for mathematical translation of the antipodal cheese problem is:

- half-straight cut $\to$ fan,
- straight cut $\to$ arrangement in fan position,
- piece of cheese $\to$ measure.

Let $H$ be a hyperplane in $\mathbb{R}^d$ and $L$ a codimension one subspace inside $H$. The connected components of a space $H \setminus L$ are half-hyperplanes determined by a pair $(H, L)$. If $F_1$ and $F_2$ are the half-hyperplane determined by the pair $(H, L)$, then $L$ is the boundary of both half-hyperplanes.

**Definition 2** A $k$-fan in $\mathbb{R}^d$ is a collection $(L; F_1, \ldots, F_k)$ consisting of
- (A) a $(d-2)$-dimensional oriented linear subspace $L$, and
- (B) different half-hyperplanes $F_1, \ldots, F_k$ with the common boundary $L$, oriented by a compatible orientation on the plane $L^\perp$.

Let $S(L^\perp)$ denote the unit circle lying in the plane $L^\perp$. Condition 7.(B) suggests that the intersection points $F_1 \cap S(L^\perp), \ldots, F_k \cap S(L^\perp)$ are consecutive points on the circle $S(L^\perp)$ oriented consistently with the given orientation on $L^\perp$. The $k$-fan $(L, l_1, \ldots, l_k)$ on the sphere $S^{d-1} \subset \mathbb{R}^d$ is the trace of a $k$-fan $(L; F_1, \ldots, F_k)$ in $\mathbb{R}^d$ obtained by slicing the sphere $S^{d-1}$ along half-hyperplanes, $l_i = S^{d-1} \cap F_i$.

Sometimes, instead of a sequence $l_1, \ldots, l_k$ of cuts for the model of a fan, we will prefer the sequence of open sets, called orthants, $O_i$, $i \in \{1, \ldots, k\}$ on the sphere $S^{d-1}$ lying between consecutive cuts $l_i, l_{i+1}$, $i \in \{1, \ldots, k\}$ (we assume $l_{k+1} = l_1$). A third model for a $k$-fan is the collection $(L; v_1, \ldots, v_k)$ of a codimension 2 subspace $L$ inside $\mathbb{R}^d$ and $k$ vectors $v_1, \ldots, v_k$ on the circle $S(L^\perp) \cong S^1$. Denote by $\phi_i$ the angle between $v_i$ and $v_{i+1}$ ($v_{k+1} = v_1$). Then $\phi_1 + \cdots + \phi_k = 2\pi$. The space of all $k$-fans in $\mathbb{R}^d$ or on the sphere $S^{d-1}$ is denoted by $\mathcal{F}_k$.

**Definition 3** A coordinate hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_k\}$ is in $k$-fan position if the intersection $H_1 \cap \cdots \cap H_k$ is a subspace of codimension one inside each $H_i$. In other words, $\mathcal{A}$ is in a fan position if there is a $2k$-fan $(L; F_1, \ldots, F_{2k})$ such that $H_1 \cap \cdots \cap H_k = L$ and $H_1 \cup \cdots \cap H_k = L \cup F_1 \cup \cdots \cup F_{2k}$.

As in the case of a fan, an arrangement $\mathcal{A} = \{H_1, \ldots, H_k\}$ in fan position inherits a natural orientation from $L^\perp$, $L = H_1 \cap \cdots \cap H_k$. Besides the natural ordering of hyperplanes, the orientation on $L^\perp$ induces an orientation of the connected components (orthants) of the complement $\mathbb{R}^d \setminus \bigcup H_i$. The orientation is determined up to a cyclic permutation. If $(H_1, \ldots, H_k)$ is the induced ordering and $H_{k+1} = H_1$, then we denote by

- $O_i^+$ the orthant between $H_i$ and $H_{i+1}$, and by
- $O_i^-$ the orthant between $H_{i+1}$ and $H_i$.

A measure $\mu$ on a sphere $S^{d-1}$ is a proper measure if for every hyperplane $H \subset \mathbb{R}^d$, $\mu(H \cap S^{d-1}) = 0$ and for every non-empty open set $U \subseteq S^{d-1}$, $\mu(U) > 0$. From now on a measure on a sphere $S^{d-1}$ will mean a proper Borel probability measure.
Figure 2: A 9-fan in $\mathbb{R}^3$

Rational vector $\alpha = (\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{2k}) \in \mathbb{Q}^{2k}$ satisfying

for all $i \in \{1, \ldots, k\}$, $\alpha_i > 0$, $\alpha_i = \alpha_{k+i}$,

and

$$\sum_{i=1}^{k} \alpha_i = \frac{1}{2}$$

is called ration.

**Definition 4** Let $M = \{\mu_1, \ldots, \mu_j\}$ be a collection of measures on $S^{d-1}$. A $2k$-fan $(L, O_1, \ldots, O_{2k})$ is a $\alpha$-partition of $M$ if there is $i \in \{1, \ldots, j\}$ such that

for all $t \in \{1, \ldots, 2k\}$, $\mu_i(O_t) = \alpha_t$

and

for all $t \in \{1, \ldots, k\}$ and all $l \in \{1, \ldots, j\}$, $\mu_l(O_t) = \mu_l(O_{k+t})$.

An arrangement $A = \{H_1, \ldots, H_k\}$ in $k$-fan position is an $\alpha$-partition of $M$ if there is $i \in \{1, \ldots, j\}$ such that

for all $t \in \{1, \ldots, 2k\}$, $\mu_i(O_t) = \alpha_t$

and

for all $t \in \{1, \ldots, k\}$ and all $l \in \{1, \ldots, j\}$, $\mu_l(O_t) = \mu_l(O_{k+t})$.

**A consequence of Theorem 1**

The solutions of anitpodal cheese problem implied by Theorem 1 are given in the following theorem.

**Theorem 5**

(A) Let $k > 0$ be an integer and $\alpha \in \mathbb{Q}^{2k}$ be a ration. If $kj(d-1) < d-1$, then for every collection of $j$ measures $M$ on a sphere $S^{d-1}$, there exists an $\alpha$-partition of $M$ by a $k$-fan

(B) Let $k > 0$ be an integer and $\alpha \in \mathbb{Q}^{2k}$ be a ration. If $kj < d-1$, then for every collection of $j$ measures $M$ on a sphere $S^{d-1}$, there exists an $\alpha$-partition of $M$ by an arrangement in a $k$-fan position.

**2 Proof of Theorem 1**

The proof is carried out using the Borel construction and its associated Serre spectral sequence. Given an equivariant mapping between $G$-spaces, $f: X \to Y$, there is an induced mapping of Borel constructions, $EG \times_G f: EG \times_G X \to EG \times_G Y$. The notion of a blow up of an arrangement is the key construction which leads to the proof of the main theorem.
2.1 Blow up of an arrangement

By the codimension of an arrangement \( \mathcal{A} \), denoted \( \text{codim}_{\mathbb{R}^m} \mathcal{A} \), we understand
\[
\text{codim}_{\mathbb{R}^m} \mathcal{A} = \min_{L \in \mathcal{A}} \{ \text{codim}_{\mathbb{R}^n} L \}.
\]

Following Definition 5.3 in [9], an arrangement \( \mathcal{A} \) is a c-arrangement if

- for every maximal element \( L \) in \( \mathcal{A} \), \( \text{codim}_{\mathbb{R}^m} L = c \)
- for all pairs \( L_1 \subset L_2 \) of elements in \( \mathcal{A} \), \( c \) divides \( \text{codim}_{L_2} L_1 \).

Recall that if \( X \) and \( Y \) are \( G \)-spaces, the diagonal action of \( G \) on the product \( X \times Y \) is given by \( g \cdot (x, y) = (g \cdot x, g \cdot y) \). By \( G \mathcal{A} \) we denote the minimal \( G \)-invariant arrangement containing the arrangement \( \mathcal{A} \), namely, \( G \mathcal{A} = \{ gL \mid g \in G \text{ and } L \in \mathcal{A} \} \). An arrangement \( \mathcal{A} \) is \( G \)-invariant if and only if \( G \mathcal{A} = \mathcal{A} \).

If \( L \subset \mathbb{R}^m \) is a linear subspace, recall that there exists a family of forms, \( \xi_1, \ldots, \xi_t \), given by
\[
\xi_i(x_1, \ldots, x_m) = a_{1i}x_1 + \cdots + a_{mi}x_m
\]
for which \( L = \{ x \in \mathbb{R}^m \mid \xi_i(x) = \cdots = \xi_t(x) = 0 \} \).

**Definition 6** Let \( \mathcal{A} \) be an arrangement of linear subspaces in \( \mathbb{R}^m \), \( \{ L_1, \ldots, L_w \} \) the set of maximal elements of \( \mathcal{A} \) and \( k_i = \text{codim}_{\mathbb{R}^m} L_i \), for \( i \in \{ 1, \ldots, w \} \). For each maximal element \( L_i \), there is a family \( \{ \xi_{i,1}, \ldots, \xi_{i,k_i} \} \) of (linearly independent) forms defining \( L_i \). The blow up of the arrangement \( \mathcal{A} \) is the arrangement \( \mathcal{B}(\mathcal{A}) \) in
\[
(\mathbb{R}^m)^{k_1+\cdots+k_w} = \big( (\mathbb{R}^m)^{k_1} \times \cdots \times (\mathbb{R}^m)^{k_w} \big) = E_1 \times \cdots \times E_w
\]
(where \( (\mathbb{R}^m)^{k_i} = E_i \)) is defined by \( w \) maximal elements \( \tilde{L}_1, \ldots, \tilde{L}_w \) introduced in the following way. The subspace \( \tilde{L}_i \), \( i = 1, \ldots, w \), is defined by forms:

\[
\begin{align*}
\xi_{i,1} &= 0 \text{ seen as a form on the 1-st copy of } \mathbb{R}^n \text{ in } E_i; \\
\xi_{i,2} &= 0 \text{ seen as a form on the 2-nd copy of } \mathbb{R}^n \text{ in } E_i; \\
&\vdots \\
\xi_{i,k_i} &= 0 \text{ seen as a form on the } k_i\text{-th copy of } \mathbb{R}^n \text{ in } E_i.
\end{align*}
\]

The blow up \( \mathcal{B}(\mathcal{A}) \) depends on the choice of the linear forms \( \xi_{*,*} \). Observe that we do not allow any extra dependent forms. Note also that the arrangement operations \( \mathcal{B}(\cdot) \) and \( G(\cdot) \) do not commute.

**Remark.** For an arrangement \( \mathcal{A} \) inside (an invariant \( G \)-) subspace \( V \subset \mathbb{R}^m \), the blow up is an arrangement inside \( (V)^{k_1+\cdots+k_w} \) defined as in Definition 11.

**Example 7** Let \( L \subset \mathbb{R}^2 \) denote the trivial subspace \( L = \{ (0,0) \} \), and \( \mathcal{A} = \{ L \} \). Then the blow up \( \mathcal{B}(\mathcal{A}) \) is an arrangement in \( \mathbb{R}^4 \) with one element defined by \( x_1 = x_4 = 0 \).

Here is a list of significant properties of the blow up of arrangement.

**Proposition 8** Let \( \mathcal{A} \) be an arrangement of linear subspaces in \( \mathbb{R}^m \) and \( \mathcal{B}(\mathcal{A}) \) its associated blow up in \( (\mathbb{R}^m)^{k_1+\cdots+k_w} \).

(A) \( \text{codim}_{\mathbb{R}^m} \mathcal{A} = \text{codim}_{(\mathbb{R}^m)^{k_1+\cdots+k_w}} \mathcal{B}(\mathcal{A}) \).

(B) \( \mathcal{B}(\mathcal{A}) \) is a \( (\text{codim}_{\mathbb{R}^m} \mathcal{A}) \)-arrangement.

(C) The identity map \( \mathbb{R}^m \to \mathbb{R}^m \) induces the diagonal map \( D: \mathbb{R}^m \to (\mathbb{R}^m)^{k_1+\cdots+k_w} \) which restricts to a map of complements
\[
D: \mathbb{R}^m \setminus \bigcup \mathcal{A} \to (\mathbb{R}^m)^{k_1+\cdots+k_w} \setminus \bigcup \mathcal{B}(\mathcal{A}).
\]

**Proof.** These statements are direct consequences of the blow up construction. □
Proposition 9 Consider a $G$-action on $\mathbb{R}^m$, which extends diagonally to the product $(\mathbb{R}^m)^{k_1+\cdots+k_w}$. Let $\mathcal{A}$ be a $G$-invariant arrangement in $\mathbb{R}^m$ such that conditions (B) and (C) of Theorem 1 are satisfied. Then the defining forms can be chosen in such a way that $\mathcal{B}(\mathcal{A})$ is also a $G$-invariant arrangement satisfying the same conditions, and more:

(A) The diagonal map $D: \mathbb{R}^m \to (\mathbb{R}^m)^{k_1+\cdots+k_w}$ is a $G$-map. Moreover, the diagonal map restricts to a $G$-map of complements

$$D: \mathbb{R}^m \setminus \bigcup \mathcal{A} = M_\mathcal{A} \to (\mathbb{R}^m)^{k_1+\cdots+k_w} \setminus \bigcup \mathcal{B}(\mathcal{A}) = M_{\mathcal{B}(\mathcal{A})};$$

(B) If $k_1 = \cdots = k_w = k$, then the blow up $\mathcal{B}(\mathcal{A})$ is a $k$-arrangement and the cohomology ring $H^*(M_{\mathcal{B}(\mathcal{A})}, k)$ is generated as an algebra by $H^{k-1}(M_{\mathcal{B}(\mathcal{A})}, k)$.

(C) If for all $L \in \mathcal{A}, L \supseteq (\mathbb{R}^m)^G$, then, for all $L \in \mathcal{B}(\mathcal{A})$, we have $L \supseteq \left( (\mathbb{R}^m)^{k_1+\cdots+k_w} \right)^G$.

(D) If the map $H^*(BG, k) \to H^*(EG \times_G M_{\mathcal{B}(\mathcal{A})}, k)$ is not a monomorphism, then the same will be true for the map $H^*(BG, k) \to H^*(EG \times_G M_{\mathcal{A}}, k)$.

Proof. Let $\mathcal{A}$ be a $G$-invariant arrangement satisfying conditions (B) and (C) of Theorem 1. For $L$ a maximal element of $\mathcal{A}$, we can choose defining forms $\{\xi_i, \ldots, \xi_k\}$ in such a way that for all $g \in G$ and all $j \in \{1, \ldots, k\}$ holds $g \cdot \xi_{i,j} = \pm \xi_{i,j}$. With this choice of defining forms and because we assumed condition (C) of Theorem 1 the blow up $\mathcal{B}(\mathcal{A})$ satisfies the required properties. Statement (A) is a consequence of the diagonal action on $(\mathbb{R}^m)^{k_1+\cdots+k_w}$. The first part of the property (B) follows from assuming condition (B) of Theorem 1. The second part of (B) follows from Corollary 5.6 in [9]. The equality $\left( (\mathbb{R}^m)^{k_1+\cdots+k_w} \right)^G = (\mathbb{R}^m)^G$ implies (C).

To prove Statement (D) we consider the mapping induced by the $G$-equivariant diagonal mapping $M_\mathcal{A} \to M_{\mathcal{B}(\mathcal{A})}$ on the Borel constructions,

$$D: EG \times_G M_\mathcal{A} \to EG \times_G M_{\mathcal{B}(\mathcal{A})}.$$

By assumption, the edge homomorphism $H^*(BG, k) \to H^*(EG \times_G M_{\mathcal{B}(\mathcal{A})}, k)$ is not a monomorphism and this is equivalent to the fact that there is a nonzero differential in the Serre spectral sequence for $EG \times_G M_{\mathcal{B}(\mathcal{A})} \to EG \times_G \{pt\} = BG$. By assumption, the $E_2$-term may be written $E_2^{p,q} \cong H^p(BG, k) \otimes H^q(M_{\mathcal{B}(\mathcal{A})}, k)$. By property (B), $H^*(M_{\mathcal{B}(\mathcal{A})}, k)$ is generated as an algebra in dimension $k-1$. Since the cohomology Serre spectral sequence is multiplicative, the first differential must be $d_k: \tilde{H}^{k-1}(M_{\mathcal{B}(\mathcal{A})}, k) \to H^k(BG, k)$. If $d_k = 0$, then $d_{k+l} = 0$ for $l \geq 1$, and the spectral sequence collapses at $E_2$, which contradicts the assumption that the edge homomorphism is not a monomorphism.

Suppose $1 \otimes v = d_k(u \otimes 1)$ for $v \in H^*(BG, k)$ and $u \in H^*(M_{\mathcal{B}(\mathcal{A})}, k)$. The diagonal mapping induces a mapping of spectral sequences that is given on the $E_2$-term by the identity on $E_2^{0,0}$ and the induced mapping on cohomology on $E_2^{0,*}$. Since the differential commutes with this induced mapping, we have

$$0 \neq 1 \otimes v = E_2(D)(1 \otimes v) = E_2(D)(d_k(u \otimes 1)) = d_k(D^*(u) \otimes 1).$$

If $D^*(u) = 0$, then $d_k(D^*(u) \otimes 1) = 0$, which contradicts the choice of $v$. Thus, $D^*(u) \neq 0$ and the differential $d_k \neq 0$ on the spectral sequence for $EG \times_G M_\mathcal{A} \to BG$. Statement (C) follows immediately.

2.2 Comparing Serre spectral sequences; proof of Theorem 1

For simplicity reasons let us assume that the codimension of all maximal elements of the arrangement $\mathcal{A}$ is $n+1$.

To prove Theorem 1 we made assumption (D), that the mapping

$$H^*(BG, k) \to H^*(EG \times_G M_{\mathcal{B}(\mathcal{A})}, k)$$

is not a monomorphism. By the assumption (C) and the choice of a field $k$ for coefficients, we can write the $E_2$-term of the spectral sequence for the Borel construction $EG \times_G M_{\mathcal{B}(\mathcal{A})}$ as

$$E_2^{0,*} \cong H^*(BG, k) \otimes H^*(M_{\mathcal{B}(\mathcal{A})}, k).$$
It follows from the assumption (A) that $M_{\mathcal{B}(A)}$ is $n$-connected, and so there is a nonzero differential $d_{n+1}: E_{n+1}^{0,n} \to E_{n+1}^{n+1,0}$ in this spectral sequence, as argued in the proof of Proposition 9. We apply this observation to study the existence of a $G$-map $f: X \to M_A$. As in Proposition 9, we have the $G$-map $D: M_A \to M_{\mathcal{B}(A)}$ and consequently a $G$-map

$$X \xrightarrow{\delta} M_A \xrightarrow{D} M_{\mathcal{B}(A)}.$$ (1)

Then there is an induced homomorphism $(f \circ D)^*$ in equivariant cohomology:

$$H^*(X \times_G EG; k) \xleftarrow{\ell} H^*(M_A \times_G EG; k) \xrightarrow{D^*} H^*(M_{\mathcal{B}(A)} \times_G EG; k),$$

as well as an induced map of Serre spectral sequences

$$E^*_v(X \times_G EG; k) \xleftarrow{\ell} E^*_v(M_A \times_G EG; k) \xrightarrow{D^*} E^*_v(M_{\mathcal{B}(A)} \times_G EG; k)$$

associated to fibrations

$$X \to X \times_G EG \quad M_{\mathcal{B}(A)} \to M_{\mathcal{B}(A)} \times_G EG$$

We use the naturality of the spectral sequence and the property that $E^2_{0,0}(f \circ D) = \text{id}$ on $H^*(BG,k)$. The induced map $E_2(f \circ D)$ and the $E_2$-term of the associated spectral sequences for $EG \times_G X$ and $EG \times_G M_{\mathcal{B}(A)}$ can be pictured:

![Diagram](image)

Figure 3:

The first nonzero differential is $d_{n+1}: E_{n+1}^{0,n} \to E_{n+1}^{n+1,0} = E_{2}^{n+1,0}$ in the spectral sequence for $EG \times_G M_{\mathcal{B}(A)}$. By the same argument in Proposition 9, there must be a nonzero class in $H^n(X,k)$ to be the image of a generator $u \in H^n(M_{\mathcal{B}(A)},k)$ which transgresses to $H^{n+1}(BG,k)$. However, the connectivity of $X$ presents no nonzero classes in this dimension, nor any below this dimension from which to launch a differential. Thus, the existence of the $G$-map leads to a contradiction.

The proof of the main theorem is now complete.

3 Is it possible to obtain more?

The following examples will illustrate that Theorem 1 is the best general theorem of Borsuk-Ulam type one can obtain for complements of arrangements one can obtain.

3.1 An example

Let $G$ be the cyclic group $\mathbb{Z}/n = \langle \omega \rangle$ where $n > 2$ is odd. Let $G$ act freely on $S^3$ and on $\mathbb{R}^n$ by the cyclic shift, $\omega \cdot (x_1, \ldots, x_n) = (x_2, \ldots, x_n, x_1)$. Let $W_n$ denote the $G$-invariant subspace $\{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 0\}$. Consider the minimal $G$-invariant arrangement $A$ containing the subspace $L \subset W_n^{\mathbb{R}^3}$ defined by

$$L = \{(x_{i,1}; x_{i,2}; x_{i,3})_{i \in \{1, \ldots, n\}} \mid x_{1,1} = x_{1,2} = x_{1,3} = 0\}.$$  

Endow $W_n^{\mathbb{R}^3}$ with the diagonal $G$-action. Then there is a $G$-map $S^3 \to W_n^{\mathbb{R}^3} \setminus \bigcup A = M_A$ and the $G$-action on $H^*(M_A,k)$ is nontrivial for any field $k$.  

7
To prove this statement we use equivariant obstruction theory (for background, consult \[13\] and for applications \[4\]). The obstruction theory can be applied because the action of \( \mathbb{Z}/n \) on the sphere \( S^3 \) is assumed to be free. Since \( \text{codim}_{\mathbb{Z}/n} \mathcal{A} = 3 \) the complement \( M_A \) is 1-connected, 2-simple and therefore the problem of the existence of a map \( S^3 \to M_A \) depends on the primary obstruction. The primary obstruction lives in \( H^3_G(S^3, \pi_2(M_A)) \).

**Lemma 10** \( H^3_G(S^3, \pi_2(M_A)) \cong \mathbb{Z} \).

**Proof.** The complement \( M_A \) is 1-connected and according to the Hurewicz theorem

\[
\pi_2(M_A) = H_2(M_A, \mathbb{Z}).
\]

Since \( \mathcal{A} \) is a 3-arrangement, the Goresky-MacPherson formula applied to \( \mathcal{A} \) implies that \( H_2(M_A, \mathbb{Z}) = \mathbb{Z}[\mathbb{Z}/n] \) as a \( \mathbb{Z}/n \)-module. The equivariant Poincaré duality isomorphism \([4\) Theorem 1.4\)] applied to the \( G \)-manifold \( S^3 \) yields an isomorphism

\[
H^3_G(S^3, H_2(M_A, \mathbb{Z})) \cong H^3_G(S^3, H_2(M_A, \mathbb{Z}) \otimes \mathbb{Z})
\]

where \( \mathbb{Z} \) is the \( G \)-module \( H_{n+1}(S^3, \mathbb{Z}) \cong \mathbb{Z} \). Since \( n \) is odd, the \( G \)-module \( \mathbb{Z} \) is trivial and therefore,

\[
H^3_G(S^3, H_2(M_A, \mathbb{Z})) \cong H^3_G(S^3, H_2(M_A, \mathbb{Z})).
\]

From homological algebra (for example, \([6\) (1.5), p.57\)] we have that

\[
H^3_G(S^3, H_2(M_A, \mathbb{Z})) \cong H^3_G(G, H_2(M_A, \mathbb{Z})) \cong H_2(M_A, \mathbb{Z}).
\]

Thus \( H^3_G(S^3, H_2(M_A, \mathbb{Z})) \cong (\mathbb{Z}[\mathbb{Z}/n])_{\mathbb{Z}/n} \cong \mathbb{Z} \). For general discussion of this type of argument see \([4\) Section 1\] .

**Lemma 11** The primary obstruction is a torsion element in \( H^3_G(S^3, \pi_2(M_A)) \).

**Proof.** Let \( H \) be a subgroup of \( G \). There is a natural restriction map

\[
r : H^3_G(S^3, H_2(M_A, \mathbb{Z})) \to H^3_H(S^3, H_2(M_A, \mathbb{Z})),
\]

which on the cochain level is just the forgetful map sending a \( G \)-cochain to the same cochain interpreted as a \( H \)-cochain. From the geometric definition of the obstruction cocycle \([4\) Section 1.2\] it follows that the restriction of the primary obstruction is the primary obstruction for the existence of a \( H \)-map \( H_2(M_A, \mathbb{Z}) \).

Furthermore there exists a natural map

\[
\tau : H^3_H(S^3, H_2(M_A, \mathbb{Z})) \to H^3_G(S^3, H_2(M_A, \mathbb{Z})).
\]

in the opposite direction given by the transfer. It is known \([6\) Section III.9. Proposition 9.5.(ii)\] that the composition of the restriction with the transfer is just multiplication by the index \([G : H]\):

\[
H^3_G(S^3, H_2(M_A, \mathbb{Z})) \xrightarrow{\tau} H^3_H(S^3, H_2(M_A, \mathbb{Z})) \xrightarrow{r} H^3_G(S^3, H_2(M_A, \mathbb{Z})).
\]

Consider \( H = \{e\} \), the trivial group. The \( H \) primary obstruction is zero, since there is an \( H \)-map \( S^3 \to M_A \). Therefore the primary \( G \)-obstruction multiplied by the index \([G : H]\) vanishes, that is, the primary obstruction is a torsion element in \( H^3_G(S^3, H_2(M_A, \mathbb{Z})) \).

The primary obstruction responsible for the existence of a \( G \)-map, \( S^3 \to M_A \), as a torsion element in \( \mathbb{Z} \), must vanish. Therefore, there exists a \( G \)-map \( S^3 \to M_A \).

### 3.2 Particular results

We present an example, originally appearing in \([11]\) as a fragment of the proof of the existence of a \((1,1,1,2)\) partition of two masses by a 4-fan on the sphere \( S^2 \).
Let $G = \mathbb{Z}/5 = \langle \omega \rangle$ acts on the sphere $S^3$ freely and on $\mathbb{R}^5$ by the cyclic shift. Again $W_5$ is the subspace of $\mathbb{R}^5$ given by $\{(x_1, \ldots, x_5) \mid x_1 + \cdots + x_5 = 0\}$. Let $\mathcal{A}$ be a minimal $G$-invariant arrangement containing subspace $L \subset W_5$ defined by

$$x_1 = x_2 = x_3 = x_4 + x_5 = 0.$$  

Then there are no $\mathbb{Z}/5$-map $S^3 \to W_5 \setminus \bigcup A = M_A$ and the $\mathbb{Z}/5$-action on $H^*(M_A, k)$ is nontrivial for any field $k$. The Hasse diagram of the intersection poset of the arrangement $\mathcal{A}$ is pictured as follows. The cohomology of the group $G$ and $E$-term of the Serre spectral sequence associated with the Borel construction of the fibration $EG \times_G M_A \to BG$ is given by

$$E_2^{p,q} = H^p(BG, H^q(M_A, \mathbb{F}_5)) = H^p(G, H^q(M_A, \mathbb{F}_5)).$$

The cohomology of the group $\mathbb{Z}/5$ with coefficients in the modules

$$\mathbb{F}_5[\mathbb{Z}/5], \quad (1 + \omega + \cdots + \omega^4)\mathbb{F}_5[\mathbb{Z}/5]$$

is well known. Therefore,

$$E_2^{p,q} = \begin{cases} 
\mathbb{F}_5 \oplus \mathbb{F}_5, & q = 2, p = 0 \\
\mathbb{F}_5, & q = 2, p \geq 1 \\
\mathbb{F}_5, & q = 0, p \geq 0 \\
0, & \text{otherwise}. 
\end{cases}$$

The only possibly nontrivial differential in this spectral sequence is $d_3$. The $G$-action on $M_A$ is free and therefore there is a homotopy equivalence

$$EG \times_G M_A \simeq M_A/G$$

and consequently a group isomorphism

$$H^*(EG \times_G M_A, \mathbb{F}_5) \cong H^*(M_A/G, \mathbb{F}_5).$$

Since $H^p(M_A/G, \mathbb{F}_5) = 0$ for $p \geq 4$, and the fact that the Serre spectral sequence converges to $H^*(EG \times_G M_A, \mathbb{F}_5)$ the only possibly nontrivial differential $d_3$ is indeed nontrivial. In particular, $d_3: E_3^{0,2} \to E_3^{3,0}$ is nontrivial and

$$E_3^{3,0}(EG \times_G M_A) \not\cong E_3^{3,0}(EG \times_G M_A).$$

Let us assume that there is a $G$-map $S^3 \to M_A$. The induced map on the Serre spectral sequences of Borel constructions given by

$$E_2^{3,0}(f): E_2^{3,0}(EG \times_G M_A) \to E_2^{3,0}(EG \times_G S^3)$$

is the identity. Since $E_2^{3,0}(EG \times_G S^3) = E_2^{3,0}(EG \times_G S^3)$ there is an element $0 \neq x \in E_2^{3,0}(EG \times_G M_A \times_G EG)$ such that

$$E_2^{3,0}(EG \times_G M_A) \ni x \xrightarrow{E_2^{3,0}(f)} x \in E_2^{3,0}(EG \times_G S^3)$$

and

$$E_2^{3,0}(EG \times_G M_A) \ni (0 = \text{class}(x)) \xrightarrow{E_2^{3,0}(f)} (x \neq 0) \in E_2^{3,0}(EG \times_G S^3).$$

This is a contradiction to the assumption of the existence of a $G$-map $S^3 \to M_A$. Thus, there is no $G$-map $S^3 \to M_A$.  

Figure 4: Hasse diagram.
4 Proof of Theorem 5

Motivated by ideas in [1] and [3], we consider questions of the existence of partitions and transform them to questions of the existence of equivariant maps.

Let \( k \in \mathbb{N} \) and \( \mathcal{M} = \{ \mu_1, \ldots, \mu_j \} \) be a collection of measures on \( S^{d-1} \). Let \( \alpha = (\frac{\alpha_1}{n}, \ldots, \frac{\alpha_k}{n}) \in \frac{1}{n}\mathbb{N}^{2k} \subset \mathbb{Q}^{2k} \) be a rational, that is,

\[
\begin{align*}
\alpha_1 + \cdots + \alpha_{2k} &= n \\
\text{for all } i \in \{1, \ldots, k\}, \text{ we have } \alpha_i &= \alpha_{k+i}.
\end{align*}
\]

These two conditions imply that \( n \) is even.

4.1 Configuration space

The configuration space associated with the measure \( \mu_1 \) is defined by

\[
X_{\mu_1,n} = \{ (L, \mathcal{O}_1, \ldots, \mathcal{O}_n) \in \mathcal{F}_n \mid \mu_1(\mathcal{O}_1) = \cdots = \mu_1(\mathcal{O}_n) = \frac{1}{n} \}.
\]

Any \( n \)-fan \( (L, \mathcal{O}_1, \ldots, \mathcal{O}_n) = (L; v_1, \ldots, v_n) \) in the configuration space \( X_{\mu_1,n} \) is completely determined by the vector \( v_1 \) and the orientation of the circle \( S(L^\perp) \). The orientation on \( S(L^\perp) \) is determined by an unit tangent vector \( v \) to the circle \( S(L^\perp) \) at a point \( S(L^\perp) \cap \text{span}\{v_1\} \) inside \( L^\perp \). Therefore, we can identify

\[
X_{\mu_1,n} \cong V_2(\mathbb{R}^d).
\]

To see this, let \((u, w) \in V_2(\mathbb{R}^d)\). Then the subspace \( L \) can be recovered as \((\text{span}\{u, w\})^\perp \) and the vector \( v_1 = u \). So far we have the subspace \( L \) and the first half-hyperplane \( F_1 \) determined by \( u \). The half-hyperplane \( F_2 \) is the one in the direction determined by \( w \) such that the measure \( \mu_1 \) of the open sphere sector determined by \( F_1 \) and \( F_2 \) is \( \frac{1}{n} \). The process continues until we recover all the half-hyperplanes of a fan. There is a natural action of the dihedral group \( D_{2n} = \langle \varepsilon, \sigma \mid \varepsilon^k = \sigma^2 = 1, \varepsilon^{k-1}\sigma = \sigma \varepsilon \rangle \) on \( \mathcal{F}_n \) given by

\[
\begin{align*}
\varepsilon \cdot (L; v_1, \ldots, v_n) &= (L; v_n, v_1, \ldots, v_{n-1}) \\
\sigma \cdot (L; v_1, \ldots, v_n) &= (L; v_n, v_{n-1}, \ldots, v_1).
\end{align*}
\]

4.2 Test Maps

Let \( W_n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0 \} \subset \mathbb{R}^n \). A \( D_{2n} \)-action on \( \mathbb{R}^n \) and on \( W_n \) is given by

\[
\begin{align*}
\omega \cdot (x_1, \ldots, x_n) &= (x_n, x_1, \ldots, x_{n-1}) \\
\sigma \cdot (x_1, \ldots, x_n) &= (x_n, x_{n-1}, \ldots, x_1).
\end{align*}
\]

The group \( D_{2n} \) acts diagonally on the sum \( (W_n)^{\oplus l} \).

(A) A test map \( F : X_{\mu_1,n} \to (W_n)^{\oplus (j-1)} \) associated with a \( \alpha \)-partition \( k \)-fan problem (Theorem 5 (A)) is defined by

\[
X_{\mu_1,n} \ni (L, \mathcal{O}_1, \ldots, \mathcal{O}_n) \mapsto (\mu_i(\mathcal{O}_1) - \frac{1}{n}, \ldots, \mu_i(\mathcal{O}_n) - \frac{1}{n})_{i=1}^j \in (W_n)^{\oplus (j-1)}.
\]

(B) A test map \( H : X_{\mu_1,n} \to W_n \oplus (W_n)^{\oplus (j-1)} \) associated with \( \alpha \)-partition \( k \)-fan position arrangement is given by

\[
X_{\mu_1,n} \ni (L, \mathcal{O}_1, \ldots, \mathcal{O}_n) \mapsto (\phi_r - \frac{2\pi}{n})_{r=1}^n \times (((\mu_i(\mathcal{O}_1) - \frac{1}{n})_{i=1}^j)_{i=2}^{j-1}) \in W_n \oplus (W_n)^{\oplus (j-1)}
\]

where \((L, \mathcal{O}_1, \ldots, \mathcal{O}_n) = (L; v_1, \ldots, v_n)\) and \( \phi_r \) denotes the angle between \( v_r \) and \( v_{r+1} \) (as introduced in Section 1.2). Both maps are defined in such a way that the following proposition holds:

**Proposition 12** The maps \( F : X_{\mu_1,n} \to (W_n)^{\oplus (j-1)} \) and \( H : X_{\mu_1,n} \to W_n \oplus (W_n)^{\oplus (j-1)} \) are \( D_{2n} \)-equivariant maps.
4.3 Test spaces

Natural test spaces for both statements of Theorem\[15\] are arrangements, introduced in the following way.

(A) Let $\mathcal{B}$ be the minimal $D_{2n}$-invariant arrangement in $(W_n)^{\oplus(j-1)}$ containing the subspace $L_\mathcal{B}$ given by following $k \times (j-1)$ equalities

\[
x_{1,i} + \cdots + x_{\alpha_1,i} = x_{\frac{1}{2}+1,i} + \cdots + x_{\frac{1}{2}+\alpha_1,i}
\]

\[
x_{\alpha_1+1,i} + \cdots + x_{\alpha_1+\alpha_2,i} = x_{\frac{1}{2}+\alpha_1+1,i} + \cdots + x_{\frac{1}{2}+\alpha_1+\alpha_2,i}
\]

\[
\vdots
\]

\[
x_{\alpha_1+\cdots+\alpha_{k-1}+1,i} + \cdots + x_{\frac{1}{2}+i} = x_{\frac{1}{2}+\alpha_1+\cdots+\alpha_{k-1}+1,i} + \cdots + x_{n,i}
\]

for all $i \in \{1, \ldots, j-1\}$. Here $x_{a,b}$ denotes the $a$-th coordinate in the $b$-th copy of $W_n$.

(B) Let $\mathcal{A}$ be the minimal $D_{2n}$-invariant arrangement in $W_n \oplus (W_n)^{\oplus(j-1)}$ containing the subspace $L_\mathcal{A}$ described by $k+k \times (j-1)$ equalities

\[
x_{1,1} + \cdots + x_{\frac{1}{2},1} = 0
\]

\[
x_{\alpha_1+1,1} + \cdots + x_{\alpha_1+\alpha_2,1} = 0
\]

\[
\vdots
\]

\[
x_{\alpha_1+\cdots+\alpha_{l-1},1} + \cdots + x_{\frac{1}{2}+1,1} = 0
\]

and

\[
x_{1,i} + \cdots + x_{\alpha_1,i} = x_{\frac{1}{2}+1,i} + \cdots + x_{\frac{1}{2}+\alpha_1,i}
\]

\[
x_{\alpha_1+1,i} + \cdots + x_{\alpha_1+\alpha_2,i} = x_{\frac{1}{2}+\alpha_1+1,i} + \cdots + x_{\frac{1}{2}+\alpha_1+\alpha_2,i}
\]

\[
\vdots
\]

\[
x_{\alpha_1+\cdots+\alpha_{l-1},i} + \cdots + x_{\frac{1}{2}+i} = x_{\frac{1}{2}+\alpha_1+\cdots+\alpha_{l-1}+1,i} + \cdots + x_{n,i}
\]

for all $i \in \{2, \ldots, j\}$. The test spaces are defined in such a way that the following basic proposition holds.

**Proposition 13**

(A) If there is no $D_{2n}$-equivariant map

\[V_2(\mathbb{R}^d) \rightarrow (W_n)^{\oplus(j-1)} \setminus \bigcup \mathcal{B},\]

then the statement of Theorem\[6\] (A) is true.

(B) If there is no $D_{2n}$-equivariant map

\[V_2(\mathbb{R}^d) \rightarrow W_n \oplus (W_n)^{\oplus(j-1)} \setminus \bigcup \mathcal{A},\]

then the statement of Theorem\[6\] (B) is true.

4.4 Applying Theorem\[1\]

Proposition\[13\] provides a chance to apply Theorem\[1\]. Unfortunately, conditions (B) and (C) for the group $D_{2n}$ with either of arrangements $\mathcal{A}$ and $\mathcal{B}$ are not satisfied. To overcome this difficulty we substitute the dihedral group $D_{2n}$ with its subgroup $G = \langle \epsilon_{\frac{1}{2}}^n \rangle \cong \mathbb{Z}/2$. The assumptions of Theorem\[1\] are satisfied:

- The Stiefel manifold $V_2(\mathbb{R}^d)$ is $d-3$ connected and so $H^i(V_2(\mathbb{R}^d), \mathbb{F}_2) = 0$ for $1 \leq i \leq d-3$;

- the codimension of the maximal elements of the arrangement $\mathcal{B}$ inside $(W_n)^{\oplus(j-1)}$ is $k(j-1)$ and the codimension of the maximal elements in $\mathcal{A}$ inside $W_n \oplus (W_n)^{\oplus(j-1)}$ is $kj$;

- since rations are symmetric, then $\epsilon_{\frac{1}{2}}^a : L_A = L_A$ and $\epsilon_{\frac{1}{2}}^a : L_B = L_B$; thus, the blow-ups $\mathfrak{B}(\mathcal{A})$ and $\mathfrak{B}(\mathcal{B})$ can be constructed in such a way that $G$ acts trivially on the $\mathbb{F}_2$ cohomology of the complements;
for all $g \in D_{2n}$, we have $g \cdot L_B \supseteq \left((W_n) \oplus (j-1)\right)^G$ and $g \cdot L_A \supseteq (W_n) \oplus (W_n) \oplus (j-1)^G$;

- the $G$-action on the complements $(W_n) \oplus (j-1) \backslash B$ and $(W_n) \oplus (j-1) \backslash A$ is free.

Theorem II implies that there are no $G$-equivariant maps

$$V_2(\mathbb{R}^d) \rightarrow (W_n) \oplus (j-1) \backslash \bigcup B$$

and

$$V_2(\mathbb{R}^d) \rightarrow W_n \oplus (W_n) \oplus (j-1) \backslash \bigcup A,$$

and consequently there are no $D_{2n}$-equivariant maps. This proves Theorem 5.

References

[1] I. Bárány, J. Matoušek, Simultaneous partitions of measures by $k$-fans, Discrete Comp. Geometry, 25 (2001), 317–334.

[2] I. Bárány, J. Matoušek, Equipartitions of two measures by a 4-fan, Discrete Comput. Geom., 27: 293-302, 2002.

[3] P. Blagojević, The partition of measures by 3-fans and computational obstruction theory, arXiv: math.CO/ 0402400, 2004.

[4] P. Blagojević, A. Dimitrijević Blagojević, Using Equivariant Obstruction Theory in Combinatorial Geometry, Topology and its Applications 154 (2007) 2635-2655, http://dx.doi.org/10.1016/j.topol.2007.04.007

[5] P. Blagojević, S. Vrećica, R. Živaljević, Computational Topology of Equivariant Maps from Spheres to Complement of Arrangements, arXiv:math.AT/0403161.

[6] K.S. Brown, Cohomology of groups, New York, Berlin, Springer-Verlag, 1982.

[7] A. Dold, Simple proofs of some Borsuk-Ulam results, Contemporary Mathematics, Vol 19, 65-69, 1983

[8] E. Fadell, S. Husseini, An ideal-valued cohomological index, theory with applications to Borsuk-Ulam and Bourgin-Yang theorems, Ergod. Th. and Dynam. Sys. 8*(1988), 73-85.

[9] M. de Longueville, C. A. Schultz, The cohomology rings of complements of subspace arrangements, Math, Ann, 319, 625-646 (2001).

[10] J. Matoušek, Topological methods in Combinatorics and Geometry, Lecture notes, Prague 1994. (updated version, February 2002, www.ms.mff.cuni.cz/matousek/lecturenotes.html).

[11] J. McCleary, A User’s Guide to Spectral Sequences, New York, NY, Cambridge University Press, second edition, 2000.

[12] S. Sundaram, V. Welker, Group actions on arrangements of linear subspaces and applications to configuration spaces, TAMS, Vol 349, No. 4, 1997, 1389-1420

[13] T. tom Dieck, Transformation groups, de Gruyter Studies in Math. 8, Berlin, 1987.

[14] C.T.C. Wall, Surgery on Compact Manifolds, Academic Press, 1970.