AN ALGEBRAIC INTERPRETATION OF
THE WHEELER-DEWITT EQUATION

JOHN W. BARRETT
LOUIS CRANE

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Abstract. We make a direct connection between the construction of three dimen-
sional topological state sums from tensor categories and three dimensional quantum
gravity by noting that the discrete version of the Wheeler-DeWitt equation is ex-
actly the pentagon for the associator of the tensor category, the Biedenharn-Elliott
identity. A crucial role is played by an asymptotic formula relating 6j-symbols to
rotation matrices given by Edmonds.

INTRODUCTION

It has been known since the work of Ponzano and Regge [1] that the quantum
theory of gravity in three Euclidean dimensions can be described by means of the
evaluation of spin networks, which are combinations of representations of the Lie
group SU(2) under the tensor product in the category of representations of SU(2).
The ideas were elaborated by Hasslacher and Perry [2].

The evaluation of a spin network on a sphere can be calculated by triangulating
the three-dimensional ball bounded by the diagram and labelling all interior edges
with representations. Ponzano and Regge gave a formula as a sum over all such
labellings of the product of the 6J symbols associated to the tetrahedra of the
triangulation. The edge labels $j$ are identified with half-integers.

$$
\Psi(\text{boundary edge labels}) = \sum_{\text{interior edge labels}} (-1)^\chi \left( \prod_{\text{tetrahedra}} 6j\text{-symbol} \right) \prod_{\text{interior edges}} (2j + 1) \quad (1)
$$

where the integer \( \chi \) is a function of the edge labels. If the sum is finite, then it
is independent of the triangulation in the interior of the ball. A more detailed
account of this has appeared in [3]. The formula can also be applied to obtain a
wavefunction for other 3-manifolds more general than the ball.

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Subsequent developments have put this formula into a broader context. Turaev and Viro [4] showed that the analogous formula, using the truncated category of representations of a quantum group at a root of unity gives rise to a three dimensional topological quantum field theory (TQFT), closely related to the Chern Simons theory of Witten [5].

Barrett and Westbury [6,7], have demonstrated that the natural generalisation of this construction is a tensor category with unit and duals and only a very weak and natural assumption on the left and right traces. In particular the braiding or symmetry of the tensor product is unnecessary. The useful generalization of 6J symbols is the associator isomorphism of the tensor category.

Thus, we have what seems to be a rather mysterious connection between a very abstract branch of mathematics, namely the theory of tensor categories, and the very deep physical problem of quantising general relativity.

The purpose of this paper is to make this connection more direct, and possibly somewhat clearer. What we show is that the Wheeler-DeWitt equation for 3d general relativity reduces directly to the pentagon relation for the tensor category we are using, in the presence of an asymptotic formula for thin tetrahedra. Since the pentagon [8] is just the coherence condition for the associator of the tensor product of the category, we are identifying the laws of motion of general relativity directly with the most fundamental part of the theory of tensor categories.

This allows us to focus our efforts to make physical interpretations of theories based on other tensor categories on the discovery of the thin tetrahedra asymptotic forms of the associator isomorphisms.

This paper should be thought of as shedding some light on the algebraic program for quantising general relativity outlined in [9]. The great desideratum, of course, would be to find an analog of the categorical formula which worked in four dimensions. We have not yet worked out the four dimensional argument analogous to the one given in this paper but we believe it should be possible to do so, and that it might help to recognize whether an algebraic construction of a four dimensional state sum is related to general relativity.

The Ponzano-Regge wavefunction is defined for triangulations of a 3-manifold with boundary where the state-sum (1) is a finite sum. The wavefunction is then the Turaev-Viro partition function with the parameter q set to 1, up to a normalisation. This shows that when the Ponzano-Regge wavefunction is defined it depends only on the boundary data and the topology of the 3-manifold, being independent of the particular triangulation. However the Ponzano-Regge formula does not define a TQFT because for an arbitrary triangulation the formula (1) may be infinite.

In any TQFT defined by a state sum in a similar manner to the Turaev-Viro state sum, there is a vector space $V$ determined by a triangulation of a surface. The partition functions, or wavefunctions, all lie in a particular subspace $S \subset V$. While the space $V$ depends on the triangulation, the subspace $S$ is independent of the triangulation.

One can see on general grounds that $S$ is determined by a set of projectors $\pi_v$, one associated to each vertex $v$ in the surface, and that $S$ is determined by the
equations
\[ \pi_v \psi = \psi \]  
being simultaneously satisfied. The equations are obviously candidates for the analogues in this theory of the Hamiltonian constraint equations which are often supposed to characterise diffeomorphism invariant quantum field theories.

In this paper, we present constraint equations in the form of (2) for the Ponzano-Regge wavefunction. However the argument is independent of the general formalism of TQFT. We give an argument that the equations are a quantisation of the appropriate discrete version of the Wheeler-DeWitt equations, thus identifying the Ponzano-Regge wavefunction as belonging to a quantisation of a specific action, namely Einstein gravity.

**The constraint equations**

The Einstein equation for a metric in three dimensions is that it is locally flat. On the boundary surface \( \Sigma \) of the manifold, this implies that there are constraints between the extrinsic curvature \( K_{ab} \) and the metric \( g_{ab} \) of \( \Sigma \). Assuming that the three-dimensional metric is positive definite, these are

\[ \nabla_a K^a_b - \nabla_b K = 0 \]

and

\[ K^2 - K^{ab} K_{ab} - R = 0, \]  
where \( K = K^{ab} g_{ab} \), \( \nabla \) is the connection for \( g \), and \( R \) its scalar curvature. The conventions are those of [10].

The Einstein action can be used to cast these equations in the framework of Hamiltonian mechanics [11], using the momentum density

\[ \pi^{ab} = \left( K g^{ab} - K^{ab} \right) \Omega \]  
where \( \Omega \) is the metric volume measure. Then the constraints are

\[ \left( \pi^2 - \pi^{ab} \pi_{ab} \right) \Omega^{-1} - R \Omega = 0 \]

and

\[ \nabla_a \pi^a_b = 0. \]

For each tangent \( h_{ab} \) to the space of metrics on \( \Sigma \), canonical quantisation associates to the momentum

\[ \phi = \int_{\Sigma} \pi^{ab} h_{ab} \]

the first order differential operator corresponding to \( h_{ab} \).

Making this substitution in the Hamiltonians determined by (5) and (6) does not generally make sense in this infinite-dimensional configuration space. However this
system can be quantised by changing variables and fixing the gauge in a particular way [12].

Alternatively, one can make progress by replacing the space of metrics on $\Sigma$ by a finite-dimensional approximating space. Then canonical quantisation gives partial differential equations, with some ambiguity due to the different possible orderings of the operators.

Our quantisation procedure is to use the finite-dimensional space of metrics determined by Regge calculus [13] and regard quantisation as giving an approximation to the equations satisfied by the wavefunction, which is already defined. Thus we are not using quantisation to define a quantum theory, rather to compare a quantum theory with a classical mechanical one.

The momentum $\phi$ survives the discretisation procedure, and for a metric fluctuation given by changing one particular edge length $l$, $\phi$ is just the parameter for the extrinsic curvature at the edge discussed by Hartle and Sorkin [14]. This identity is specific to three dimensions, the momentum and the extrinsic curvature being logically distinct quantities. If $\Sigma$ is the boundary of a 3-manifold $M$ with a Euclidean signature metric, this is the angle of rotation of the outward unit normal from the triangle at one side of an edge to the normal to the triangle at the other side. It is positive if the edge is convex and negative if the edge is concave.

Consider a vertex in $\Sigma$ which belongs to exactly three triangles, such as appears on the boundary of a tetrahedron (figure 1). The connection in $M$ gives a map $TM_p \to TM_{p'}$ if a path from $p$ to $p'$ in $\Sigma$ is given which avoids vertices. If an orthonormal frame is chosen at $p$ and $p'$, then this map determines a rotation in $SO(3)$.

\[
\begin{array}{c}
\mathbb{R}^3 \xrightarrow{\text{rotation}} \mathbb{R}^3 \\
\text{frame at } p \downarrow \quad \downarrow \text{frame at } p' \\
TM_p \longrightarrow TM_{p'}
\end{array}
\]
Consider a path which circulates the vertex via six points at which canonical frames are chosen, as shown in figure 2. At each point, vector $z$ is chosen to lie along the adjacent edge tangent to $\Sigma$ and $x$ is chosen rotated by a local orientation of the surface. The frame is completed by vector $y$ which is the outward normal to $\Sigma$. The six rotation matrices are $R_{12}, R_{23}, R_{31}$, which are rotations through $\theta_{12}, \theta_{23}, \theta_{31}$ about the $y$-axis, and $K_1, K_2, K_3$, which are rotations by $\phi_1, \phi_2, \phi_3$ about the $z$-axis. The $R$’s can be described as the rotations which take one edge into the next and the $K$’s are the rotations which make one face parallel to the next.

As for the smooth case, the Einstein equations in three dimensions are that the metric is flat. The condition that $M$ is flat is that the holonomy in $M$ is the identity,

$$h = R_{23}K_3R_{31}K_1R_{12}K_2 = I \in \text{SO}(3).$$

(8)

Figure 2. Path in $\Sigma$ and choice of frames

This gives three equations per vertex of $\Sigma$ which constrain the metric of $\Sigma$ and its extrinsic curvature. These are the key equations which we wish to quantise, to give our discrete version of the Wheeler-DeWitt equation.

For the rest of this section there is an argument to demonstrate that the equations (8) are the discrete analogues of the Hamiltonian constraint equations. However the development of the rest of the paper is independent of this argument.

This can be seen by considering small perturbations of the geometry around a flat configuration $\phi_1 = \phi_2 = \phi_3 = 0$. One gets three equations, corresponding to the three standard basis elements $X, Y, Z$ of the Lie algebra of $\text{SO}(3)$, which generate rotations about the $x, y$ and $z$ axes. At the flat configuration, the two-dimensional holonomy is the identity,

$$R_{23}R_{31}R_{12} = I \in \text{SO}(3),$$

and a small perturbation results in a small deficit angle $\delta$. Each edge has associated a Lie algebra element

$$L_1 = (\cos \theta_{12})Z + (\sin \theta_{12})X$$
$$L_2 = Z$$
$$L_3 = (\cos \theta_{23})Z - (\sin \theta_{23})X$$
Expanding (8) for small $\phi_1$, $\phi_2$, $\phi_3$, and keeping only the lowest-order non-zero terms, one gets

$$\phi_1 L_1 + \phi_2 L_2 + \phi_3 L_3 = 0$$

and

$$\delta = \frac{1}{2} \phi_1 \phi_2 \sin \theta_{12}.$$ 

The first of these is two equations, linear in the momenta, given by the coefficients of $X$ and $Z$. It is a divergence equation, and corresponds to (6). The second equation, quadratic in the momenta, corresponds to (5), and is symmetrical in the three edges by virtue of the first equation, giving the area of a triangle with edges $\phi_1$, $\phi_2$, $\phi_3$. It is the coefficient of $Y$ on expanding (8).

**Quantisation**

The Ponzano-Regge wavefunction is a function of the edge labels on the boundary $\Sigma$ of $M$. The length of an edge is defined to be the value of the label plus one half. Thus the edge lengths are discrete, taking the values $j + \frac{1}{2}$, where $j \in \frac{1}{2}\mathbb{Z}$, $j \geq 0$, indexes the representations of $\text{SU}(2)$. These lengths determine a Euclidean signature metric for $\Sigma$. We consider the equations satisfied when the edge lengths $l_1$, $l_2$, $l_3$ around a vertex at which the three edges meet are varied, the other edges in $\Sigma$ having fixed lengths. The simplest example is for a single tetrahedron, when

$$\psi(l_1, l_2, l_3) = \begin{vmatrix} a & b & c \\ l_1 & l_2 & l_3 \end{vmatrix},$$

(9)

the 6j-symbol, for fixed $a,b,c$.

![Figure 3. Crossing move](image)

Actually the case where three edges meet at a vertex suffices to handle the general case, as the cases where more than three edges meet can be reduced to this by the crossing move, shown in figure 3, which is an invertible linear map $V \rightarrow V'$ and so
generates no new equations [3]. Also, any vertex at which three edges meet reduces to the case (9) because the Ponzano-Regge wavefunction factorises as

\[ \psi(l_1, l_2, l_3) = \left\{ \begin{array}{ccc} a & b & c \\ l_1 & l_2 & l_3 \end{array} \right\} \psi'(a, b, c, \text{other edges}) \]

by the excision move, shown in figure 4 [3]. Alternatively, this can be seen as an expression of the invariance of the wavefunction under a change of triangulation of the interior of \( M \).

![Figure 4. Excision move](image)

Our result for the Ponzano-Regge wavefunction is that for each representation \( J \) it satisfies an equation

\[ D^{(23)}_{m_2m_3} \psi(l_1, l_2, l_3) = (-1)^{2J} \sum_{m_1=-J}^{J} D^{(31)}_{m_3m_1} D^{(12)}_{m_1m_2} \psi(l_1 + m_1, l_2 + m_2, l_3 + m_3) \]  

In this formula, there are three square matrices of numerical coefficients \( D_{mn} \), of dimension \( 2J + 1 \). These matrices depend on the edge lengths \( (l_1, l_2, l_3, a, b, c) \). More specifically, \( D^{(23)} \) depends on \((l_2, l_3, a)\), \( D^{(31)} \) depends on \((l_1, l_3, b)\), and \( D^{(12)} \) depends on \((l_1, l_2, c)\), so that each matrix is associated to one of the triangles in Figure 1.

**Proof of formula (10).** Apply the Biedenharn-Elliott relation to the complex shown in figure 5. This gives

\[
\begin{align*}
\left\{ \begin{array}{ccc} a & l_2 + m_2 & l_3 + m_3 \\
J & l_1 & l_3 \\
J & l_3 + m_3 & l_1 + m_1 \end{array} \right\} & \left\{ \begin{array}{ccc} a & b & c \\
l_1 & l_2 & l_3 \end{array} \right\} = \sum_{m_1=-J}^{J} (-1)^{\chi(2l_1 + 2m_1 + 1)} \\
\left\{ \begin{array}{ccc} b & l_1 & l_3 \\
J & l_3 + m_3 & l_1 + m_1 \end{array} \right\} & \left\{ \begin{array}{ccc} c & l_2 & l_1 \\
J & l_1 + m_1 & l_2 + m_2 \end{array} \right\} \left\{ \begin{array}{ccc} a & b & c \\
l_1 + m_1 & l_2 + m_2 & l_3 + m_3 \end{array} \right\}
\end{align*}
\]
with $\chi = a + b + c + l_1 + l_2 + l_3 + J + l_1 + m_1 + l_2 + m_2 + l_3 + m_3$. This is equation (10), with

$$D_{m_2m_3}^{(23)} = (-1)^{l_2 + l_3 + a + J + m_3}(2l_2 + 2m_2 + 1)^{\frac{3}{2}}(2l_3 + 2m_3 + 1)^{\frac{3}{2}} \left\{ \begin{array}{ccc} a & l_3 & l_2 \\ J & l_2 + m_2 & l_3 + m_3 \end{array} \right\}$$

(and likewise for $D^{(31)}$, $D^{(12)}$).

This equation can now be compared to (8). We claim that it is a quantisation of (8). First, (8) expressed in any irreducible representation $J$. It can be written very explicitly using the weight basis in which the $K$’s are diagonal.

$$K_n = \begin{pmatrix} e^{iJ\phi_n} & e^{i(J-1)\phi_n} & \cdots & e^{-iJ\phi_n} \end{pmatrix}$$

and the $R$’s then have the real matrix elements denoted $d^J_{m'm'}(\theta)$ in the angular momentum literature, related to the Jacobi polynomials. For example,

$$d^{\frac{3}{2}}(\theta) = \begin{pmatrix} \cos \theta/2 & \sin \theta/2 \\ -\sin \theta/2 & \cos \theta/2 \end{pmatrix},$$

$$d^1(\theta) = \begin{pmatrix} \frac{1}{2}(1 + \cos \theta) & \frac{1}{\sqrt{2}} \sin \theta & \frac{1}{2}(1 - \cos \theta) \\ -\frac{1}{\sqrt{2}} \sin \theta & \cos \theta & \frac{1}{\sqrt{2}} \sin \theta \\ \frac{1}{2}(1 - \cos \theta) & -\frac{1}{\sqrt{2}} \sin \theta & \frac{1}{2}(1 + \cos \theta) \end{pmatrix}.$$

Equation (8) becomes

$$d^J_{m_2m_3}(\theta_{23}) = \sum_{m_1=-J}^{J} e^{im_3\phi_3} d^J_{m_3m_1}(\theta_{31}) e^{im_1\phi_1} d^J_{m_1m_2}(\theta_{12}) e^{im_2\phi_2}. \quad (11)$$
According to the formula given by Edmonds [15], $D^{(23)}_{m_2m_3}$ is asymptotic to

$$d^J_{m_2m_3}(\theta_{23}) = d^J_{m_3m_2}(-\theta_{23})$$

for fixed $m_2, m_3, J$ but $a, l_2, l_3 \to \infty$. Similar formulae hold for $D^{(12)}, D^{(31)}$. For this approximation to be valid, it is therefore necessary to assume $J$ is much smaller than the six edge labels appearing in Figure 1.

Then the equation (10) is a quantisation of (11) in which $e^{im\phi}$ is replaced by the shift operator $T$

$$T\psi(l) = \psi(l + m),$$

which was suggested as a heuristic by Ponzano and Regge.

The other noteworthy feature is the factor $(-1)^{2J}$. This is $-1$ for odd spin representations and gives the extension of (8) to SU(2) as the equation

$$h^{SU(2)} = -\mathbb{I}.$$

This is the natural extension as the product of the matrices in (8) gives a rotation which is exactly one full turn, which lifts to $-\mathbb{I}$ in SU(2). This phenomenon is discussed in [16].

**Remarks.** Our quantisation of the discrete Wheeler-DeWitt equation gives an appealing physical explanation of the Ponzano-Regge wavefunction by an analogy with other quantised systems such as particle mechanics. But it is not clear exactly what the mathematical ‘point’ of our work is; can anything now be proved that was not known before?

The connection between spin networks and 2+1 dimensional gravity has already been made in [1,2]. However this proceeded in an indirect manner from the Biedenharn-Elliott relation to form a 3-term recursion relation for 6j-symbols, and then to a semi-classical limit. We have shown here that in the Hamiltonian picture the analog of the Wheeler-DeWitt equation is exactly the Biedenharn-Elliott relation itself. This has two advantages: firstly it makes a more direct connection with the ideas of topological quantum field theory, since it focuses on the consideration of a manifold with boundary, and secondly it suggests an easier and more direct approach to making physical interpretations of other topological state sums. We note that the recursion relation occurs as a special case of our formula, namely when $m_2 = m_3 = 0$ and $J = 1$.

Our quantisation appears to give more information than the semiclassical formula for the 6j symbols given by Ponzano and Regge. For example, it gives an explanation for the discreteness of the edge length labels, which is not required by the semiclassical formula. Normally one would quantise the variable conjugate to the length $l$ by $\phi \to d/dl$. Then $l$ would have to have a continuous spectrum. However, we are interested in quantising $\exp(im\phi)$, which, making the above substitution, gives the shift operator, given by the action $l \to l + m$. As $m$ is a half-integer, this operator naturally acts on functions defined only on the discrete set of points $\frac{1}{2} \mathbb{Z}$.

Although we have considered quantising the formulae obtained from 3-dimensional metrics with a Euclidean signature, one can also start with the corresponding
Lorentzian formulae. Then there are some sign differences in (3) to (5), which are reflected in the fact that (8) becomes an equation in SO(2,1) with the $K$’s boosts and the $R$’s unchanged, c.f. [17]. The quantisation replaces the matrix element $e^{i\phi}$ with a shift operator, and one obtains the same asymptotic approximation to (10).

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**Department of Mathematics, University of Nottingham, University Park, Nottingham, NG7 2RD, UK**

*E-mail address: jwb@maths.nott.ac.uk*

**Mathematics Department, Kansas State University, Manhattan KS, 66502, USA**

*E-mail address: crane@math.ksu.edu*