Oriented Bipartite Graphs and the Goldbach Graph

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Abstract

In this paper we study oriented bipartite graphs. In particular, several characterizations of bitournaments are obtained. We introduce the concept of odd-even graphs and show that any (oriented) bipartite graph can be represented by some (oriented) odd-even graph. We show that the famous Goldbach’s conjecture is equivalent to the connectedness of certain odd-even graphs.

Keywords: prime number, bipartite graph, directed bipartite graph, oriented bipartite graph, bitournament, Goldbach conjecture.

1 Introduction

Let $X$ and $Y$ be two nonempty disjoint sets and $E$ be a set (may be empty). If $E = \emptyset$, then $g = \emptyset$ and if $E \neq \emptyset$, then $g \colon E \to (X \times Y) \cup (Y \times X)$ be a function (called incidence function) from $E$ into $(X \times Y) \cup (Y \times X)$. Then the quadruple $D = (X, Y, E, g)$ is called a directed bipartite graph. Also we denote $X \cup Y$ by $V(D)$. Let $X$ and $Y$ be two nonempty disjoint sets and $E \subseteq (X \times Y) \cup (Y \times X)$. Then the triple $D_1 = (X, Y, E)$ is called a simple directed bipartite graph (in brief, SDBG).

Example 1.1. Consider the directed graph $D$ in Figure 1 (left). Here $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, $g(e_1) = g(e_2) = (x_1, y_1)$, $g(e_3) = (x_1, y_2)$, $g(e_4) = (y_1, x_2)$, $g(e_5) = (x_2, y_3)$ and $g(e_6) = (y_3, x_3)$. Then $D = (X, Y, E, g)$ is a directed bipartite graph.

Let $D_1$ be the directed graph in Figure 1 (right). Then $D_1 = (X, Y, E)$ is an SDBG with $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $E = \{(x_1, y_1), (y_1, x_1), (x_1, y_2), (y_1, x_2), (x_2, y_3), (y_3, x_3)\}$.

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In the sequel we write $xy$ and $yx$ instead of $(x,y)$ and $(y,x)$ in $E$ respectively.

The adjacency matrix $M(D)$ of an SDBG, $D = (X,Y,E)$ is of the following form:

$$M(D) = \begin{bmatrix} X & Y \\ X & 0 & A \\ Y & B & 0 \end{bmatrix}$$

where $A$ and $B$ are two $(0,1)$-matrices. Note that in the case of an undirected bipartite graph $B = A^T$, but here it is not true in general. Certainly $B = A^T$ if and only if for any arc $xy \in E$, we also have $yx \in E$.

For a directed graph $D$, the undirected graph $G(D)$ obtained from $D$ by disregarding directions of arcs is the underlying graph of $D$. Moreover two arcs $e,f$ of $D$ are adjacent if they have a common end point in $G(D)$.

2 Oriented bipartite graphs

A simple directed bipartite graph $D = (X,Y,E)$ is called oriented if for any $x \in X, y \in Y$, either $(x,y) \in E$ or $(y,x) \in E$ or $(x,y),(y,x) \notin E$. Oriented trees form an interesting subclass of the class of oriented graphs. Let $T$ be an oriented tree. Then a path in the underlying tree $G(T)$ of $T$ is called alternating if each pair of adjacent arc are of opposite direction in $T$. Let $D = (X,Y,E)$ be an SDBG. Then $D$ is called unidirectional if either $xy \notin E$ for all $x \in X, y \in Y$ or $yx \notin E$ for all $x \in X, y \in Y$. In this case either $A = 0$ or $B = 0$ in $M(D)$.

Observation 2.1. In an oriented tree $T$, there is an alternate path between any two vertices of $T$ if and only if for each vertex $v \in V(T)$, either $\text{indeg}(v) = 0$ or $\text{outdeg}(v) = 0$ (i.e., $T$ is unidirectional).
Definition 2.2. An oriented bipartite graph $D = (X,Y,E)$ is called bitransitive if for any $x_1,x_2 \in X$ and $y_1,y_2 \in Y$, $x_1y_1, y_1x_2, x_2y_2 \in E \implies x_1y_2 \in E$ (see Figure 2). An oriented bipartite graph $D = (X,Y,E)$ is called a bitournament if for all $x \in X$ and $y \in Y$, either $xy \in E$ or $yx \in E$.

![Figure 2: An illustration of bitransitive property](image)

Example 2.3. Let $S$ be a nonempty subset of natural numbers. Define a digraph $D_S$ with vertex set $S$ and $a \to b$ if and only if $b > a$ and $a$ and $b$ are of opposite parity. Then $D = (X,Y,E)$ is a bitournament with $X = \{u \in S \mid u$ is even$\}$, $Y = \{u \in S \mid u$ is odd$\}$ and $E = \{(a,b) \in (X \times Y) \cup (Y \times X) \mid b > a$ and $a$ and $b$ are of opposite parity$\}$.

The following theorem characterizes bitransitive bitournaments. A Ferrers digraph $D = (V,E)$ is a directed graph whose successor sets are linearly ordered by inclusion where the successor set of $v \in V$ is its set of out-neighbors $\{u \in V \mid vu \in E\}$. It is known that a directed graph $D$ is a Ferrers digraph if and only if its adjacency matrix does not contain any $2 \times 2$ permutation matrix (called a couple) $[5, 6]$:

$$
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
$$

Theorem 2.4. Let $D = (X,Y,E)$ be a bitournament. Then the following are equivalent:

1. $D$ is bitransitive.

2. $D$ has no directed 4-cycle.

3. $D$ has no directed cycle.

4.

$$
M(D) =
\begin{array}{cc}
X & Y \\
\hline
0 & A \\
\hline
\overline{A} & 0
\end{array}
$$

where $A$ is a Ferrer’s digraph and $\overline{A}$ is obtained from $A$ by interchanging 0’s and 1’s.

5. $D \cong D_S$ for some $\emptyset \neq S \subseteq \mathbb{N}$. 

Proof. 2 $\implies$ 1: Suppose there is no directed 4-cycle in a bitournament $D = (X,Y,E)$. Let $u_1u_2, u_2u_3, u_3u_4 \in E$ for some $u_1, u_2, u_3, u_4 \in V(D) = X \cup Y$. Then $u_4u_1 \notin E$. Since $D$ is a bitournament, we have $u_1u_4 \in E$. Hence it follows from Definition 2.2 that $D$ is bitransitive.

1 $\implies$ 3: Suppose $D = (X,Y,E)$ is bitransitive but has a directed cycle. Since $D$ is bipartite, there cannot be any odd cycle. Hence the cycle is even. Now let the cycle be $(u_1,u_2,\ldots,u_{2n})$. We prove by induction that $u_1u_{2k} \in E$ for all $k = 1,2,\ldots,n$. By induction hypothesis, $u_1u_{2k-1} \in E$. Now $u_{2k-1}u_{2k-2} \in E$. Hence $u_1u_{2k} \in E$. So by induction, $u_1u_{2k} \in E$ for all $k = 1,2,\ldots,n$. Hence $u_1u_{2n} \in E$. But we have already $u_{2n}u_1 \in E$. Since $D$ is a bitournament, both $u_1u_{2n}, u_{2n}u_1$ cannot be in $E$. Hence there is a contradiction.

3 $\implies$ 2: Obvious.

5 $\implies$ 2: Suppose $D \cong D_S$ for some $\emptyset \neq S \subseteq \mathbb{N}$. Suppose it has a directed 4-cycle $(u_1,u_2,u_3,u_4)$. So $u_1u_2, u_2u_3, u_3u_4,u_4u_1 \in E$. This implies $u_1 < u_2 < u_3 < u_4 < u_1$ which is a contradiction. So $D$ cannot have a directed 4-cycle.

2 $\iff$ 4: The adjacency matrix $A$ is not of a Ferrer’s digraph if and only if there is a couple in $A$ such that

|   | $y_r$ | $y_s$ |
|---|---|---|
| $x_i$ | 1 | 0 |
| $x_j$ | 0 | 1 |

Hence $A$ has the submatrix.

|   | $x_i$ | $x_j$ |
|---|---|---|
| $y_r$ | 0 | 1 |
| $y_s$ | 1 | 0 |

Thus, $x_i \rightarrow y_r$, $y_r \rightarrow x_j$, $x_j \rightarrow y_s$ and $y_s \rightarrow x_i$. Then we get a 4-cycle. Hence $A$ is not the adjacency matrix of a Ferrer’s digraph if and only if there is a directed 4-cycle. That is, $A$ is the adjacency matrix of a Ferrer’s digraph if and only if there is no directed 4-cycle.

3 $\implies$ 5: We prove this by induction on number of vertices of a bitournament $D = (X,Y,E)$. The result is trivially true for 2 vertices, one in each partite set. Now suppose there are $n+1 > 2$ vertices in $D$. Now we remove a vertex $v$ from $D$. Then by induction hypothesis, the result is true for the resultant graph, say $D_1$ which has $n$ vertices, i.e., $D_1 \cong D_S$ for some $\emptyset \neq S \subseteq \mathbb{N}$. Now, let $A$ be the set vertices $u$ of $D$ such that there is a directed path from $u$ to $v$. Let $B$ be the set of vertices $w$ of $D$ such that there is a directed path from $v$ to $w$. Since there is no directed cycle, $A$ and $B$ are disjoint. Now in $D_S$, any two vertices of opposite parity are adjacent so they are belonging to different partite sets in $D$. Thus $v$ cannot be adjacent to both of them. Let $v \in X$. Without loss of generality we may assume that other vertices of $X$ are labeled by even numbers in $D_1$ for otherwise we increase the label of each vertex in $D_1$ by 1.
Let \( m \) be an even number that is greater than all labels of vertices in \( D_1 \). We label \( v \) as \( m \) and for each \( w \in B \), we relabel \( w \) as \( w + m \). We first note that adding \( m \) does not change the parity for any \( w \) in \( B \). Next we prove that this relabeling does not violate the adjacency condition. Let there be an arc from \( w \in B \) to a vertex \( x \in D_1 \). Then by construction \( x \in B \). Hence all arcs from any \( w \in B \) go to vertices in \( B \) itself. Since the original labeling did not violate the adjacency condition, increasing each label by \( m \) also does not violate it for arcs from some vertex of \( B \) to another vertex of \( B \). Now for the arcs from some \( x \notin B \) to some \( w \in B \), the adjacency condition is not violated as we have increased the label of \( w \). All arcs from \( v \) go to some vertex of \( B \). Since \( v = m \) and \( w + m > m \), the adjacency condition is not violated for arcs from \( v \) to some vertex of \( B \). If there is an arc from a vertex \( x \) to \( v \), then \( x \in A \) and since the label of \( v \) is higher than any vertex of \( A \), the adjacency retains. In all other cases, labels are not changed. Hence the relabeling matches the adjacency condition of any arc in \( D \). This completes the proof. \( \square \)

3 Odd-even graphs

Let \( \mathcal{E} \) and \( \mathcal{O} \) be the set of all non-negative even numbers and positive odd numbers, respectively. For some \( A \subseteq \mathcal{E} \) and \( O \subseteq \mathcal{O} \) an oriented odd-even graph \( \overrightarrow{G}_A(O) \) is an oriented graph with set of vertices \( A \) and with set of arcs \( E = \left\{ ab \mid \frac{a+b}{2}, \frac{b-a}{2} \in O \right\} \) while an odd-even graph \( G_A(O) \) is its underlying (undirected) graph. Observe that \( \overrightarrow{G}_A(O) \) is an oriented bipartite graph with partite sets \( V_1 = \{ v \in A \mid v \equiv 0 \pmod{4} \} \) and \( V_2 = \{ v \in A \mid v \equiv 2 \pmod{4} \} \) as both \( \frac{a+b}{2} \) and \( \frac{b-a}{2} \) are even for any pair of \( a, b \in V_i \) and for each \( i \in \{1,2\} \). Interestingly, the converse is also true in the following sense:

**Theorem 3.1.** Let \( B \) be an oriented bipartite graph. Then there exist \( A \subseteq \mathcal{E} \) and \( O \subseteq \mathcal{O} \) such that \( \overrightarrow{G}_A(O) \) is isomorphic to \( B \).

**Proof.** Let \( B = (X,Y,E) \) be an oriented bipartite graph with the partite sets \( X \) and \( Y \). Let \( X = \{ b_0, b_2, \ldots, b_{2n} \} \), \( Y = \{ b_1, b_3, \ldots, b_{2n-1} \} \) and \( V = X \cup Y \). Now define a function \( f : V \rightarrow \mathcal{E} \) with \( f(b_i) = 10i+2+1+(-1)^{i+1} \). It is easy to check that the function \( f \) is well-defined and injective. Take the even set \( A \) to be the image of \( f \) and let the odd set \( O = \left\{ \frac{f(a)+f(b)}{2}, \frac{f(b)-f(a)}{2} \mid ab \in E(B) \right\} \). Now to show that \( B \) is isomorphic to \( G_A(O) \) it is enough to observe that \( f(x)+f(y) \neq f(x')+f(y') \), \( f(x)+f(y)-f(x')-f(y') \) and \( f(y)-f(x) \neq f(y')-f(x') \) for any \( xy \in E(B) \) and \( x'y' \notin E(B) \). \( \square \)

**Corollary 3.2.** Let \( B \) be a bipartite graph. Then there exist \( A \subseteq \mathcal{E} \) and \( O \subseteq \mathcal{O} \) such that \( G_A(O) \) is isomorphic to \( B \).

**Proof.** Consider any orientation \( B_1 \) of \( B \). Then by Theorem 3.1, \( B_1 \cong \overrightarrow{G}_A(O) \) for some \( A \subseteq \mathcal{E} \) and \( O \subseteq \mathcal{O} \). Then \( B \cong G_A(O) \). \( \square \)

Note that the above theorem and corollary can easily be extended to (oriented) bipartite graphs with countably infinite number of vertices. Therefore, the family of odd-even graphs is, in fact, the
family of all bipartite graphs with countable number of vertices. Now we will prove some conditions for finite odd even graphs to be connected. For any odd-even graph $G_A(O)$, let the relevant odd set be $O_{rel} = O \cap \{\frac{a+b}{2}, \frac{|a-b|}{2} \mid a, b \in E\}$. Note that $G_A(O)$ is isomorphic to $G_A(O_{rel})$.

**Theorem 3.3.** If $G_A(O)$ is connected with $|A| \geq 2$, then $|O_{rel}| > \sqrt{2/|A|}$.

**Proof.** Suppose $|A| = n$ and $|O_{rel}| = k$. Now, the number of edges in $G_A(O)$ is at least $n - 1$ (since $G_A(O)$ is connected). The number of edges is at most $\frac{k(k-1)}{2}$ as each edge $ab$ corresponds to a pair of odd numbers $\frac{a+b}{2}, \frac{|a-b|}{2} \in O_{rel}$. Thus

$$\frac{k(k-1)}{2} \geq n - 1 \implies (k - 1 + \sqrt{8n - 7})(k - 1 - \sqrt{8n - 7}) \geq 0$$

$$\implies k - 1 + \sqrt{8n - 7} \geq 0 \text{ (since } k - 1 - \sqrt{8n - 7} > 0)$$

$$\implies k \geq \frac{1 + \sqrt{8n - 7}}{2} > \sqrt{2n} \text{ (for } n \geq 2)$$

$$\implies |O_{rel}| > \sqrt{2/|A|}.$$

**Theorem 3.4.** Suppose $A = \{0, 2, 4, \ldots, 2(m - 1)\}$. If $|O_{rel}| > \frac{3|A|}{4}$, then $G_A(O)$ is connected.

**Proof.** Suppose $G_A(O)$ is disconnected with $|O_{rel}| > \frac{3|A|}{4}$. Then there exist at least 2 connected components. Let $X$ be a connected component. Let $Y$ be the union of the other connected components. Then at least one of $|X|$ and $|Y| \geq \frac{|A|}{2}$. Without loss of generality, $|Y| \geq \frac{|A|}{2}$. Take a vertex $a \in X$. Then $a$ does not have an edge with any vertex in $Y$. Define $S_b = \{\frac{a+b}{2}, \frac{|a-b|}{2}\}$ for $b \in Y$. Then at least one element from each $S_b$ does not belong to $O$. Let $T = \{t \mid t \in S_b \text{ for some } b \text{ and } t \notin O_{rel}\}$. Then the number of distinct elements in $T \geq \frac{|A|}{4}$ (Since at least one element from each $S_b$ gives at least $\frac{|A|}{2}$ elements and an element can be in at most 2 $S_b$’s). Hence, $|O_{rel}| \leq m - |T| = |A| - |T| \leq \frac{3|A|}{4}$. But $|O_{rel}| > \frac{3|A|}{4}$, a contradiction.

Now we will study odd-even graphs with odd sets of the following form where $\mathbb{N}$ is the set of all natural numbers:

$$O_{a,b} = \{ak + b \mid a \text{ is even, } b \text{ is odd, } k \in \mathbb{N}\}.$$

**Theorem 3.5.** The oriented graph $\vec{G} = \vec{G}_E(O_{a,b}) = (V, E)$ is unidirectional if and only if 4 divides $a$.

**Proof.** Let $V_1 = \{v \in E \mid v \equiv 0 \text{ (mod 4)}\}$ and $V_2 = \{v \in E \mid v \equiv 2 \text{ (mod 4)}\}$. Then $V = V_1 \cup V_2$. First assume that $a$ is divisible by 4. Let $u = 4x \in V_1$, $v = 4y + 2 \in V_2$ and $\vec{uv} \in E$. So that forces $u > v$. Then we must have $\frac{u+v}{2}, \frac{u-v}{2} \in O_{a,b}$. That is, we have $2(x + y) + 1, 2(x - y) - 1 \in O_{a,b}$. This implies

$$2x = a(n_1 + n_2)/2 + b$$
where \( n_1, n_2 \) are some positive integers. But this is a contradiction as \( a(n_1 + n_2)/2 + b \) is an odd number while \( 2x \) is even. So all the arcs in \( \overrightarrow{G} \) are from \( V_1 \) to \( V_2 \), i.e., \( \overrightarrow{G} \) is unidirectional.

For the converse part, assume that \( a \) is not divisible by 4. Let \( n_1 > n_2 \) be two positive even integers. Then \( u = a(n_1 - n_2) \in V_1 \) and \( v = a(n_1 + n_2) + 2b \in V_2 \). In this case, \( \frac{u+v}{2}, \frac{u-v}{2} \in O_{a,b} \) and we have the arc \( \overrightarrow{uv} \in E \). On the other hand, consider two positive integers \( m_1 > m_2 \) where \( m_1 \) is odd and \( m_2 \) is even. Then \( u' = a(m_1 + m_2) + 2b \in V_1 \) and \( v' = a(m_1 - m_2) \in V_2 \). In this case, \( \frac{u+v}{2}, \frac{u-v}{2} \in O_{a,b} \) and we have the arc \( \overrightarrow{v'v} \in E \). So the graph \( \overrightarrow{G} \) is not unidirectional when \( a \) is not divisible by 4. \( \square \)

We can easily generalize this result to the following.

**Theorem 3.6.** If the odd set can be expressed as \( O = \{a_i + 1| i \in I \subset \mathbb{N}\} \), then the oriented graph \( \overrightarrow{G}_E(O) \) is unidirectional if and only if 4 divides \( a_i \) for all \( i \in I \).

The adjacency matrix of the oriented graph \( \overrightarrow{G}_E(O_{4,1}) \) is of the form

\[
\begin{bmatrix}
0 & X \\
0 & 0
\end{bmatrix}
\]

where

\[
X = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & \ldots \ \\
0 & 1 & 0 & 1 & 0 & 1 & \ldots \\
0 & 0 & 1 & 0 & 1 & 0 & \ldots \\
0 & \ldots & & & & & \\
\end{bmatrix}
\]

and the adjacency matrix of \( \overrightarrow{G}_E(O_{6,1}) \) is

\[
\begin{bmatrix}
0 & \boldsymbol{0} \\
0 & \boldsymbol{0}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & \ldots & & & & & \\
\end{bmatrix}
& \boldsymbol{0} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & \ldots & & & & & \\
\end{bmatrix}
& \boldsymbol{0}
\end{bmatrix}
\]

Note that according to Theorem 3.5, \( \overrightarrow{G}_E(O_{4,1}) \) is unidirectional while \( \overrightarrow{G}_E(O_{6,1}) \) is not. From the above two examples one can observe the difference between the adjacency matrices of unidirectional and not unidirectional oriented odd-even graphs.

### 4 The Goldbach graph

Now we will focus on a particular odd-even graph \( \overrightarrow{G}_E(P) \) and \( G_E(P) \) where the odd set \( P \) is the set of all odd primes, and call it the **Goldbach graph** for reason that will become apparent in the first result of this section. Let \( E_n \) denote the set of all even numbers less than or equal to \( 2n \). Also, the graph \( G_{E_n}(P) \) will be denoted by \( G_n \) and the neighborhood \( N_{G_n}(v) \) (or, the out-neighbor \( N^+_n(v) \) or the in-neighbor \( N^-_n(v) \)) of a vertex \( v \) in \( G_n \) (or, in \( \overrightarrow{G}_n \)) will be denoted by \( N_n(v) \) (or \( N^+_n(v) \) or \( N^-_n(v) \), respectively) for the remainder of the section. Also the degree \( d_{G_n}(v) \) (or, the out-degree \( d^+_n(v) \) or the in-degree \( d^-_n(v) \)) of a vertex \( v \) in \( G_n \) will be denoted by \( d_n(v) \) (or \( d^+_n(v) \) or \( d^-_n(v) \) or...
\( d_n^-(v), \) respectively) for the remainder of the section. We denote \( \overrightarrow{G}_\epsilon(\mathcal{P}) \) and \( \overrightarrow{G}_\epsilon(\mathcal{P}) \) by \( \overrightarrow{G}_\infty \) and \( \overrightarrow{G}_\infty \) respectively and the out-degree and the in-degree of \( v \in \mathcal{E} \) in \( \overrightarrow{G}_\infty \) by \( d_\infty^+(v) \) and \( d_\infty^-(v) \) respectively.

Now we state the result that, by and large, motivated this work.

**Theorem 4.1.** The following statements are equivalent.

(i) (Goldbach’s conjecture) Every even integer greater than 5 can be written as sum of two odd primes.

(ii) \( \mathcal{G}_n \) is connected for all \( n \geq 7 \).

(iii) Every vertex of \( \overrightarrow{G}_\infty \) has non-zero in-degree, that is, \( d_\infty^-(v) > 0 \) for all \( v \geq 6 \).

**Proof.** (i) \( \Rightarrow \) (ii): Suppose that the Goldbach’s conjecture is true. Observe that \( \mathcal{G}_7 \) is connected. Now assume that \( \mathcal{G}_n \) is connected for all \( n \leq k \). By Goldbach Conjecture, \( 2(k+1) = p + q \) for some \( p, q \in \mathcal{P} \). Then \( |p - q| \) is even and \( |p - q| < p + q = 2(n + 1) \). Thus \( 2(k+1) \) is adjacent to \( |p - q| \) which is a vertex of \( \mathcal{G}_k \) as well. This implies that \( \mathcal{G}_{k+1} \) is connected.

(ii) \( \Rightarrow \) (iii): Suppose \( \mathcal{G}_n \) is connected for all \( n \geq 7 \). Let \( v \) be any even integer greater equal to 14. Then, as the graph \( \mathcal{G}_{v/2} \) is connected, the vertex \( v \) of the graph must be adjacent to some other vertex of the graph. Note that \( v \) is the greatest vertex in \( \mathcal{G}_{v/2} \). Hence \( d_\infty^-(v) > 0 \). Now it is a simple observation that for \( 6 \leq v \leq 12 \) indeed we have \( d_\infty^-(v) > 0 \). This completes the proof.

(iii) \( \Rightarrow \) (i): Suppose \( d_\infty^-(v) > 0 \) for all \( v \geq 6 \). Now for any even number \( a > 5 \) there exists \( b \) such that \( b \in \mathcal{N}_\infty(a) \). That means, there exists odd primes \( p, q \) such that we have \( p + q = a \). This is precisely the Goldbach’s conjecture. \( \square \)

The above result shows that the Goldbach’s conjecture can be formulated using graph theoretic notions. Note that in Theorem 3.3 and 3.4 we presented one necessary and another sufficient conditions for connectedness of finite odd-even graphs. Improved results of similar nature might give rise to an alternative way of digging into the Goldbach’s conjecture using graph theory due to Theorem 4.1. Having proved this equivalence, naturally we tried to explore more such equivalent formulations. Our observation which was integral in proving the above result is that, given an even integer \( 2n \), it is adjacent to a smaller integer implies that \( 2n \) can be expressed as the sum of two odd primes. Similarly, its adjacency with a greater integer implies that \( 2n \) can be expressed as difference of two odd primes. This readily provides graph theoretic formulation of another well-known conjecture in number theory.

**Theorem 4.2.** The following statements are equivalent.

(i) (A conjecture by Maillet [2]) Every even integer can be written as difference of two odd primes.

(ii) Every vertex of \( \overrightarrow{G}_\infty \) has non-zero out-degree, that is, \( d_\infty^+(v) > 0 \) for all \( v \geq 2 \).
After this the first thing that came to our notice is that the degree of the vertices of our graph is particularly interesting. As the graph is an infinite graph, the natural question about the degrees are, if they are finite or not. In particular, note that each vertex have finite in-degree, as its in-neighbors are smaller even numbers, while its out-degree can be unbounded. So the vertex 0 have no in-neighbors while its out-neighbors are precisely $2p$ for all $p \in \mathcal{P}$. We know that there are infinitely many odd primes due to Euclid’s theorem (which says, there are infinitely many prime numbers). Hence, $d^+_\infty (0)$ is infinite and this is equivalent to Euclid’s theorem.

**Observation 4.3.** The vertex 0 of $\vec{G}_\infty$ has infinitely many out-neighbors and hence, has infinitely many neighbors.

This observation naturally motivates us to wonder if the degree (or out-degree) of the other vertices are finite or not. It turns out to be a difficult question as it is equivalent to another well-known conjecture, the Kronecker’s conjecture.

**Theorem 4.4.** The following statements are equivalent.

(i) (Kronecker’s conjecture [2]) Given an even number $2k$, there are infinitely many pairs of primes of the form \{p, p + 2k\}.

(ii) For every vertex $v \in \mathcal{E}$ we have $d^+_\infty (v)$ is infinite in $\vec{G}_\infty$.

(iii) For every vertex $v \in \mathcal{E}$ we have $d_\infty (v)$ is infinite in $G_\infty$.

**Proof.** (i) $\Rightarrow$ (ii): Suppose that the conjecture is true. Let $2k$ be an even number for some $k \geq 1$. So, there are infinitely many pairs of primes of the form \{p, p + 2k\} by assumption. Note that for each such pair of primes the vertex $2k$ is adjacent to the vertex $2(p + k)$ in $G_\infty$.

(ii) $\Leftrightarrow$ (iii): Clearly follows from the fact that $d^+_\infty (v) \leq d^+_\infty (v) + d^-_\infty (v) = d_\infty (v)$ for all $v \in \mathcal{E}$ while $d^-_\infty (v)$ is finite.

(iii) $\Rightarrow$ (i): Suppose $d^+_\infty (v)$ is infinite for all $v \in \mathcal{E}$. Let $v = 2k$ be an even number for some $k \geq 1$. Now for each out-neighbor $u = 2n$ of v in $G_\infty$ we have $\frac{2n+2k}{2}, \frac{2n-2k}{2} \in \mathcal{P}$. Hence, both $(n - k)$ and $(n + k)$ are primes and there are infinitely such distincts pairs for each $k \geq 0$.

In particular, determining if degree (or out-degree) of 2 is finite or not will settle the twin prime conjecture [10] (positively if $d(2)$ is infinite). This implies an immediate corollary.

**Corollary 4.5.** The following statements are equivalent.

1. (Twin prime conjecture [10]) There are infinitely many pairs of primes of the form \{p, p + 2\}.

2. In $\vec{G}_\infty$, $d_\infty (2) = d^+_\infty (2)$ is infinite.
Next we will try to understand the significance of the degrees of the vertices in $G_\infty$. Given an even number $2n$, the in-degree $d^-_\infty(2n)$ is the number of ways $2n$ can be expressed as the sum of two odd primes. Similarly, the out-degree $d^+_\infty(2n)$ is the number of ways $2n$ can be expressed as the difference of two odd primes. Moreover, the degree of 0 in $G_n$ is the number of odd primes less than or equal to $n$. So, the graph parameter $d_n(0)$ can be regarded as a function similar to the prime counting function $\pi(n)$, which denotes the number of primes less than or equal to $n$. So, for $n \geq 2$ we have
\[ \pi(n) = d_n(0) + 1 \]
as the only even prime 2 is not adjacent to 0. As it turned out to be an interesting yet difficult problem to figure out what the degrees of the vertices are, we started to establish some relations between them. Hence the following result.

**Theorem 4.6.** For all $n \geq 2r$ and for $0 \leq m \leq 4$, in $\overrightarrow{G}_\infty$ we have
\[ \sum_{i=0}^{m} d^+_n(2i) \geq \sum_{i=0}^{m} d^-_n(2r-2i). \]

Sketch of the proof. Let $A_i = \{q \mid p + q = 2r - 2i \text{ and } q \leq p\}$ for $i \in \{0, 1, 2, 3, 4\}$ where $p, q$ are odd primes. Observe that $d^-_n(2r-2i) = |A_i|$. Note that for any $q \in \bigcup_{i=0}^{4} A_i$, we have $2q \in N_n^+(0)$. Thus, $d^+_n(0) \geq |\bigcup_{i=0}^{4} A_i|$.

Now suppose $q \in A_i \cap A_j$ for some $i, j \in \{0, 1, 2, 3, 4\}$ and $i < j$. Then there are primes $p_1, p_2 \geq q$ such that $p_1 + q = 2r - 2i$ and $p_2 + q = 2r - 2j$. So $p_2 = p_1 - 2(j - i)$. As both $2p_1 - 2(j - i) + 2(j - i) = p_1$ and $2p_1 - 2(j - i) - 2(j - i) = p_1 - 2(j - i) = p_2$ are odd primes we have
\[ 2p_1 - 2(j - i) \in N_n^+(2(j - i)). \] (4.1)

Let $S_i = \{(i, x) \mid x \in N_n^+(2i)\}$ for $i \in \{0, 1, 2, 3, 4\}$. Note that $S_i \cap S_j = \emptyset$ for $i \neq j$ and $|S_i| = d^+_n(2i)$ for all $i \in \{0, 1, 2, 3, 4\}$. Also let $S = \bigcup_{i=0}^{4} S_i$. Then we will construct a subset $T \subseteq S$ such that $|T| \geq \sum_{i=0}^{4} |A_i|$. This will complete the proof.

**Step 0:** We know that for each $q \in \bigcup_{i=0}^{4} A_i$, we have $(0, 2q) \in S_0$. Put all these $(0, 2q)$’s in the set $T$. Next we have to deal with the elements that are in more than one $A_i$’s.

**Step 1:** First we handle the case where an element $q \in A_0 \cap A_j$ for some $j \in \{1, 2, 3, 4\}$. For each such $q$ there is a prime $p$ such that $p + q = 2r$. By (4.1) we know that for each such $q$, there is an edge between $2j$ and $(2p - 2j)$. Put all these $(j, 2p - 2j)$’s in $T$. Observe that all these are new elements in $T$ as $j \geq 1$.

**Step 2:** Now consider an element $q \in A_1 \cap A_2$. Then there exists a prime $p$ such that $p + q = 2r - 2$ and by (4.1) we know that $(1, 2p - 2) \in S_1$. We will put all such $(1, 2p - 2)$’s in $T$ if they were already not in $T$. Let for some $q$, its corresponding $(1, 2p - 2)$ were already in $T$. That means that element was included to $T$ due to Step 1. Therefore, $p + (q + 2) = 2r$ where $(q + 2)$ is also a prime. Hence, $(1, 2q + 2) \in S_1$ as both $\frac{2(q+2) + 2}{2} = (q+2)$ and $\frac{2(q+2) - 2}{2} = q$ are primes. Note that, all the
There are four more steps, namely, for the assumption of the lemma. Hence are each of the form 6

Proof. Let 2r = 2r - 2 and by [4.1] we know that (2, 2p - 4) ∈ S₂. We will put all such (2, 2p - 4)'s in T if they were not already in T.

Let for some q, its corresponding (2, 2p - 4) were already in T. That means that element was included to T due to Step 1. An argument similar to Step 2 will show that there is an edge between (2q + 2) and 2. We will include all those (1, 2q + 2)'s to T which were not included to T before.

There may be some (1, 2q + 2) which was included to T before. Then that inclusion was due to Step 2. This implies p, (p - 2) and (p - 4) are all odd primes. The only such instance is when p = 7. Thus, 2r - 6 = (p - 4) + q = 3 + q. As (p - 4) ≥ q we have q = 3. Hence, 2r = 12. It is easy to check (manually) that the theorem holds for 2r = 12. Therefore, we can ignore this case.

There are four more steps, namely, for q ∈ A₁ ∩ A₃, q ∈ A₂ ∩ A₃, q ∈ A₂ ∩ A₄ and q ∈ A₃ ∩ A₄ in that order, that will conclude the proof. Those cases can be handled in a similar way like above. □

Our interest in the degree of the bipartite graph G∞ pronted us to study the complete bipartite subgraphs of G∞ from number theoretic point of view.

Proposition 4.7. If the complete bipartite graph Kₘₙ is a subgraph of G∞, then there exists a set \{p₁, p₂, ..., pₘ\} of m primes and a set \{r₁, r₂, ..., rₙ₋₁\} of (n - 1) positive integers such that pᵢ + rᵢ is a prime for all (i, j) ∈ \{1, 2, ..., m\} × \{1, 2, ..., n - 1\}.

Proof. Let X and Y be the two partite sets of Kₘₙ. Index the vertices of X = \{x₁, x₂, ..., xₘ\} and Y = \{y₁, y₂, ..., yₙ\} in increasing order. Let \(p_i = \frac{x_i + y_i}{2}\) and \(r_j = \frac{y_j + 1 - y_i}{2}\) for \((i, j) \in \{1, 2, ..., m\} \times \{1, 2, ..., n - 1\}\). Note that \(p_i + r_j = \frac{x_i + y_i + 1}{2}\) is a prime for each \((i, j) \in \{1, 2, ..., m\} \times \{1, 2, ..., n - 1\}\). □

Now we will prove some conditions for a complete bipartite subgraph of G∞ with the aid of the following two lemmas. Let us denote \(\mathbb{N} \cup \{0\}\) by \(\mathbb{N}_0\).

Lemma 4.8. Let \(a, b \notin 6\mathbb{N}_0\) and ab ∈ E(G∞), then |a - b| = 6.

Proof. Let \(a > b\). Then \(a = p + q\) and \(b = p - q\) for some odd primes \(p\) and \(q\). If \(p, q ≠ 3\), then \(p\) and \(q\) are each of the form \(6k + 1\) or \(6k - 1\). Then either \(6 \mid (p + q) = a\) and \(6 \mid (p - q) = b\) contradicting the assumption of the lemma. Hence \(q = 3\) (as \(p ≥ q ≥ 3\)) which implies \(a - b = 2q = 6\). □

Lemma 4.9. Let \(a, b \in 6\mathbb{N}_0\) and ab ∈ E(G∞) with \(a ≥ b\), then \(a = 6\) and \(b = 0\).

Proof. As both \(a\) and \(b\) are divisible by 6, both \(\frac{a+b}{2}\) and \(\frac{a-b}{2}\) are divisible by 3. Since they are primes, \(\frac{a+b}{2} = 3 = \frac{a-b}{2}\). Therefore, \(a = 6, b = 0\). □

Theorem 4.10. Let \(Kₘₙ\) be a subgraph of G∞ with partite sets \(X\) and \(Y\) such that \(m, n > 2\). Then either \(X \subseteq 6\mathbb{N}_0\) and \(Y \cap 6\mathbb{N}_0 = \emptyset\) or \(Y \subseteq 6\mathbb{N}_0\) and \(X \cap 6\mathbb{N}_0 = \emptyset\).
Proof. If neither $X \not\subseteq 6\mathbb{N}_0$ nor $X \cap 6\mathbb{N}_0 \neq \emptyset$, then either there exist $a \in X \cap 6\mathbb{N}_0$ and $\{b, c\} \subset X \cap (\mathbb{N}_0 \setminus 6\mathbb{N}_0)$ or there exist $\{b, c\} \subset X \cap 6\mathbb{N}_0$ and $a \in X \cap (\mathbb{N}_0 \setminus 6\mathbb{N}_0)$.

If $a \in X \cap 6\mathbb{N}_0$ and $\{b, c\} \subset X \cap (\mathbb{N}_0 \setminus 6\mathbb{N}_0)$, then $|Y \cap 6\mathbb{N}_0| \leq 1$ by Lemma 4.9. Again for any $d \in Y \cap (\mathbb{N}_0 \setminus 6\mathbb{N}_0)$, Lemma 4.8 forces $b - 6 = d = c + 6$ assuming $b > c$, without loss of generality. Hence $|Y \cap (\mathbb{N}_0 \setminus 6\mathbb{N}_0)| \leq 1$. Therefore, $|Y| = |Y \cap 6\mathbb{N}_0| + |Y \cap (\mathbb{N}_0 \setminus 6\mathbb{N}_0)| \leq 2$, a contradiction.

If $\{b, c\} \subset X \cap 6\mathbb{N}_0$ and $a \in X \cap (\mathbb{N}_0 \setminus 6\mathbb{N}_0)$, then $Y \cap 6\mathbb{N}_0 = \emptyset$ as otherwise each vertex of $Y \cap 6\mathbb{N}_0$ must be adjacent to both $b$ and $c$ forcing them to be the same vertex by Lemma 4.9. On the other hand, if $d \in Y \cap (\mathbb{N}_0 \setminus 6\mathbb{N}_0)$, then $d = a - 6$ or $d = a + 6$ by Lemma 4.8. This implies $|Y| = |Y \cap 6\mathbb{N}_0| + |Y \cap (\mathbb{N}_0 \setminus 6\mathbb{N}_0)| \leq 2$, a contradiction. So either $X \subset 6\mathbb{N}_0$ or $X \cap 6\mathbb{N}_0 = \emptyset$.

If $X \subset 6\mathbb{N}_0$, then $Y \cap 6\mathbb{N}_0 = \emptyset$ by Lemma 4.9. If $X \cap 6\mathbb{N}_0 = \emptyset$, then $Y \cap (\mathbb{N}_0 \setminus 6\mathbb{N}_0) = \emptyset$ by Lemma 4.8.

In the next result we will also capture the case where at least one of the partite sets have exactly two vertices while the other one has at least four of them.

**Theorem 4.11.** Let $K_{2,n}$ be a subgraph of $G_\infty$ with partite sets $X$ and $Y$ such that $|X| = 2$ and $|Y| = n > 3$. Then either $X \subset 6\mathbb{N}_0$ and $Y \cap 6\mathbb{N}_0 = \emptyset$, or $X \cap 6\mathbb{N}_0 = \emptyset$ and $|Y \cap (\mathbb{N}_0 \setminus 6\mathbb{N}_0)| \leq 1$.

**Proof.** Suppose $X = \{a, b\}$. Without loss of generality, let $a \in 6\mathbb{N}_0$ and $b \in (\mathbb{N}_0 \setminus 6\mathbb{N}_0)$. Then $|Y \cap 6\mathbb{N}_0| \leq 1$ and $|Y \cap (\mathbb{N}_0 \setminus 6\mathbb{N}_0)| \leq 2$ by Lemma 4.9 and 4.8. Therefore, $|Y| \leq 3$, a contradiction. Hence either $X \subset 6\mathbb{N}_0$ or $X \cap 6\mathbb{N}_0 = \emptyset$. If $X \subset 6\mathbb{N}_0$, then $Y \cap 6\mathbb{N}_0 = \emptyset$ by Lemma 4.9. If $X \cap 6\mathbb{N}_0 = \emptyset$, then $a$ and $b$ can have at most one common neighbour $c$ such that $c \in (\mathbb{N}_0 \setminus 6\mathbb{N}_0)$ by Lemma 4.8.

Now let us try to understand the structure of independent sets in $G_\infty$. Of course, as $G_\infty$ is a bipartite graph, there are at least two distinct (and disjoint) independent sets in the form of the two partite sets. But how big can an independent set consisting of only consecutive even numbers be? We answer this question in the following result.

**Theorem 4.12.** There exists arbitrarily large independent sets containing consecutive even numbers in $G_\infty$.

**Proof.** Given any $n$, the set $R = \{(2n + 2)! + 2, (2n + 2)! + 3, ..., (2n + 2)! + (2n + 2)\}$ is a set of consecutive composite numbers. Hence no two vertices in the set

$$S = \{(2n + 2)! + 2, (2n + 2)! + 4, ..., (2n + 2)! + 2(n + 1)\}$$

are adjacent to each other as $\frac{a+b}{2} \not\in R$ for all $a, b \in S$. Hence $S$ is an independent set containing $n$ consecutive even numbers in $G_\infty$. 

\[\square\]
5 Conclusions

We conclude the paper with an interesting observation that the graphs \( G_{E_n}^*(P_1) \) is Hamiltonian for all even \( n \) with \( 4 \leq n \leq 58 \), where \( E_n^* = E_n \setminus \{0\} \) and \( P_1 = P \cup \{1\} \) (see Appendix). Since the graph \( G_{E_n}^*(P_1) \) is bipartite, there cannot be any odd cycle in the graph. But it follows from the above observations that \( G_{E_n}^*(P_1) \) has a Hamiltonian path (i.e., a spanning path) for all odd \( n \) with \( 5 \leq n \leq 57 \). The following is an interesting Hamiltonian path of \( G_{E_{58}}^*(P_1) \) that starts with 2 and ends at 116:

\[
2, 4, 6, 8, 14, 12, 10, 16, 18, 20, 26, 32, 30, 28, 34, 24, 22, 36, 38, 44, 42, 40, 46, 48, 58, 60, 62, 56, 50, 72, 70, 64, 54, 52, 66, 68, 74, 84, 82, 76, 90, 88, 78, 80, 86, 92, 102, 100, 94, 108, 98, 96, 106, 112, 114, 104, 110, 116.
\]

Now this observations lead to the following questions. Let us call two even natural numbers conjugate to each other if they are adjacent in \( G_{E_\infty}^*(P_1) \), where \( E_\infty^* = E_\infty \setminus \{0\} \). We have seen that there is a sequence of even natural numbers up to 116 such that any two consecutive numbers in this sequence are conjugate to each other.

1. Does there exist a sequence of all even natural numbers such that any two consecutive numbers in this sequence are conjugate to each other?

2. If the answer to the above question is negative, then what is the least value of \( m \) such that \( G_{E_{2m}}^*(P_1) \) is not Hamiltonian?

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## Appendix

| Number of Vertices | Hamiltonian Cycle                                                                 |
|-------------------|----------------------------------------------------------------------------------|
| 4                 | (4, 2, 8, 6, 4)                                                                   |
| 6                 | (4, 6, 8, 2, 12, 10, 4)                                                           |
| 8                 | (4, 2, 8, 14, 12, 10, 16, 6, 4)                                                  |
| 10                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 4)                                          |
| 12                | (4, 2, 8, 6, 16, 10, 12, 22, 24, 14, 20, 18, 4)                                  |
| 14                | (4, 2, 8, 6, 28, 18, 16, 22, 12, 26, 20, 14, 24, 10, 4)                          |
| 16                | (4, 2, 8, 6, 16, 10, 12, 22, 24, 14, 20, 26, 32, 30, 28, 18, 4)                 |
| 18                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 34, 24, 22, 36, 26, 32, 30, 4)         |
| 20                | (4, 2, 8, 6, 16, 10, 12, 22, 36, 38, 24, 14, 20, 26, 32, 30, 28, 34, 40, 18, 4) |
| 22                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 32, 28, 34, 40, 42, 4, 4) |
| 24                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 22, 34, 40, 46, 48, 48, 4, 2, 4) |
| 30                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 26, 50, 44, 38, 46, 48, 32, 42, 4) |
| 32                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 32, 22, 34, 40, 46, 48, 48, 4, 2) |
| 34                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 32, 24, 34, 40, 46, 48, 48, 4, 2) |
| 36                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 32, 22, 34, 40, 46, 48, 48, 4, 2) |
| 40                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 36, 22, 34, 40, 42, 54, 48, 48, 4, 2) |
| 42                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 32, 22, 34, 40, 42, 44, 48, 48, 4, 2) |
| 44                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 32, 36, 22, 34, 40, 42, 54, 48, 48, 4, 2) |
| 46                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 36, 22, 34, 40, 42, 44, 48, 48, 4, 2) |
| 48                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 32, 22, 34, 40, 42, 44, 48, 48, 4, 2) |
| 50                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 36, 22, 24, 40, 46, 48, 48, 4, 2) |
| 52                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 36, 22, 24, 40, 46, 48, 48, 4, 2) |
| 54                | (4, 2, 8, 6, 16, 10, 12, 14, 20, 18, 28, 30, 32, 26, 32, 22, 24, 40, 46, 48, 48, 4, 2) |
| 56                | (4, 10, 6, 8, 14, 14, 20, 18, 28, 30, 32, 26, 36, 22, 24, 40, 46, 48, 48, 4, 2) |
| 58                | (6, 4, 2, 8, 14, 12, 10, 16, 18, 20, 26, 32, 30, 28, 34, 24, 22, 36, 38, 44, 42, 50, 48, 48, 4, 2) |