PROBLEMS AND CURES (PARTIAL) FOR HOLOGRAPHIC COSMOLOGY

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We analyze the validity of the generalized covariant entropy bound near the apparent horizon of isotropic expanding cosmological models. We encounter violations of the bound for cosmic times smaller than a threshold. By introducing an infrared cutoff we are able to maintain the bound for a radiation dominated universe. We study different physical mechanisms to restore the bound, as a non-additivity of the entropy at a fundamental level and/or a cosmological uncertainty relation.

1 Introduction

The very nature of the theory of quantum gravity is being revealed by consistent formulations of quantum gravitational phenomena as the string/M theory do, as well as by principles that underlie the precise formulation of the theory. It is believed that a primary principle of the theory of quantum gravity is the Holographic Principle, that states that a physical system can be described only by degrees of freedom living in its boundary. When we descend to a classical description the previous principle adopts the form of an entropy bound that admits different formulations depending on the strength of the gravitational interaction.

There was a proliferation of entropy bounds that made any prediction meaningless. Fortunately a covariant formulation for the maximum entropy allowed for a physical system was given\([1]\) and the possibilities drastically reduced. In order that different observers agree with the entropy of a system they must measure the entropy traversing a light sheet (LS), that is the locus spanned by the null congruence, generated with the area decreasing light rays orthogonal to a spacelike codimension two surface. The LS ends in a singularity, or a caustic where the area begins to increase. The covariant entropy bound (CEB) establishes that the amount of entropy measured in this way is bounded by one quarter of the area, in Planck units, of the spacelike codimension two surface where the congruence begins. All the entropy bounds were special relaxed cases of the covariant bound, except the Bekenstein bound\([2]\) (BB).

To put all the entropy bounds together we need to use the generalized co-
variant entropy bound\(^{3}\) (GCEB) that truncates the LS with a second spacelike codimension two surface and establish that the amount of entropy traversing the truncated LS is bounded by one quarter of the difference of area between both boundaries. It is evident that the GCEB implies the CEB, and it has been shown\(^{4}\) that the BB can also be deduced from the GCEB. So the GCEB is the stronger formulation of the bound and by imposing it on different physical systems we obtain further insights on the nature of the holographic principle and the theory of quantum gravity.

We are interested on the validity of the GCEB for expanding isotropic cosmological scenarios. It is the case that the GCEB is violated near the apparent horizon (AH)\(^{5}\). The reason is that on the AH the LS develops a maximum area; if we truncate the LS near the AH the difference of areas goes to zero as the second power of the affine parameter that parametrizes the LS, whereas the entropy traversing the LS goes to zero as the first power of the same parameter; so unavoidably we are facing a violation of the GCEB for small enough values of the affine parameter. In\(^{6}\) a possible resolution of the violation of the GCEB is addressed by admitting for the carriers of entropy, only particles with a wavelength smaller than the physical separation between the two spacelike surfaces bounding the LS.

In this article we study the difficulties that appear near the AH to satisfy the GCEB in Friedman-Lemaître-Robertson-Walker (FLRW) cosmologies. We can solve the discrepancies using an infrared cutoff to cut the modes with a wavelength larger than the size they traverse; we do it for a radiation dominated model. The use of different cutoffs for different parts of a given LS seems to be related with an intrinsic non additivity of the entropy that we comment. Because we use the cosmic time as the affine parameter, we are able to obtain an expression for the amount of time we must wait since the AH to satisfy the GCEB. Our result can be put in a suggestive form as a cosmological uncertainty relation.

2 The GCEB in a FLRW universe

2.1 The location of the AH in proper coordinates

Now we study the problems that appear near the AH in FLRW cosmologies where the LS develops the maximum area. The metric has its standard form

\[
\begin{align*}
\text{ds}^2 &= -dt^2 + R^2(t) \left( \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega_2^2 \right);
\end{align*}
\]

(1)
the coordinates are comoving with the cosmic fluid, \( t \) is the cosmic time and \( d\Omega^2 \) is the metric of the two dimensional unit sphere. \( \kappa = \pm 1, 0 \) is the spatial curvature and the velocity of light is \( c = 1 \). By the isotropy property the origin \( r = 0 \) is a generic point and all directions are equivalent; we fix the polar angles by fixing a radial direction. We can use conformal coordinates

\[
d\eta = \frac{dt}{R(t)} \quad d\chi = \frac{dr}{(1 - \kappa r^2)^{1/2}},
\]

and in this coordinates the metric is conformal to the flat metric,

\[
ds^2 = R^2(\eta)(-d\eta^2 + d\chi^2).
\]

When we analyze the causal structure in the unphysical flat metric, we do not appreciate nothing particular at the location of the AH; however, if we use proper coordinates\(^7\) to locate the events, a maximum of the proper distance appears for the flat case (see Figure 1). The proper distance to the origin \( D \) for an event at a given time, is relate to its radial coordinate by

\[
dD^2 = R^2(t) \frac{dr^2}{1 - \kappa r^2},
\]

and the relevant part of the metric adopts the form

\[
ds^2 = -dt^2 + (dD - HDdt)^2,
\]

where \( H = H(t) \) is the Hubble constant. The null geodesics, that would constitute the LS where we must measure the entropy, are given, in the conformal case by \( \eta = \pm \chi \); if we use proper coordinates, \( ds^2 = 0 \) translates into

\[
\dot{D} = HD \pm 1;
\]

\( \dot{D} \) is the derivative of \( D \) with respect to cosmic time. The two signs correspond to the outgoing/ingoing nature of the photon with respect to the origin. The AH appears for the ingoing light ray; the photon near the Big Bang, although it is directed toward the origin (its radial coordinate \( r \) is decreasing), due to the expansion of the Universe begins to recede in proper coordinates; as the expansion evolves the proper distance of the photon attains a maximum and subsequently begins to diminishes until the origin is reached\(^a\). The AH is locate where the area spanned by the ingoing photons is maximum. If the universe is flat maximum area implies maximum proper distance \( D \), so that \( \dot{D} = 0 \), and substituting in \( \Box \) the proper distance to the AH is given by \( D_{AH} = 1/H \). In Figure 1 the past lightcone of a fiducial observer placed at

\(^a\)It is possible that the photon never reaches the origin; this is a signal of the presence of event horizons.
2.2 Violation of the GCEB near the AH

The GCEB establish that the entropy traversing a truncated LS is bounded by the difference of the areas of the two limiting spacelike surfaces. In our analysis one of the boundaries will be the spacelike surface spanned by the AH. We extend our LS up to another isotropic spacelike surface. We assume a density of entropy $s(t)$ that depends only on cosmic time $t$, and use proper coordinates $(D(t), t)$. For simplicity we develop the flat case $\kappa = 0$, but the result is easily generalized. We use units with $G = c = \hbar = k_B = 1$.

The formulation of the GCEB is

$$S(t) \leq \frac{1}{4} \Delta A = \frac{1}{4} (A(t_{AH}) - A(t)), \quad (7)$$

where $A(t)$ is the area of the the isotropic surface bounding the LS, $t_{AH}$ is the cosmic time that locates the AH and $S(t)$ is the amount of entropy passing the truncated LS. In terms of the proper distance $D(t)$ we have

$$S(t) = \int_{t_{AH}}^{t} dt' 4\pi D^2(t') s(t'), \quad (8)$$
and
\[ \frac{1}{4} \Delta A = \pi (D^2(t_{AH}) - D^2(t)). \] (9)

Taking \( t \) near \( t_{AH} \) and Taylor expanding (8) and (9) near the AH gives
\[ S \simeq 4\pi D^2(t_{AH})s(t_{AH})\Delta t, \] (10)
and
\[ \frac{1}{4} \Delta A \simeq -\pi D(t_{AH}) \ddot{D}(t_{AH}) \Delta t^2, \] (11)
where \( \Delta t = |t_{AH} - t| \). It is then clear that for small enough values of \( \Delta t \), \( \frac{1}{4} \Delta A < S \) and the GCEB is violated.

3 Restricting \( \Delta t \) to satisfy the GCEB

In order to respect the GCEB we must limit the separation in cosmic time between both spacelike surfaces, so \( \Delta t \) would be greater than a minimum value \( \Delta t_m \) that is obtained saturating the bound, \( S(\Delta t_m) = \Delta A(\Delta t_m)/4 \),
\[ \Delta t_m = -4 \frac{D(t_{AH})}{\ddot{D}(t_{AH})} s(t_{AH}). \] (12)

The equation of the null geodesic (6) is now derived with respect to the cosmic time
\[ \ddot{D} = \dot{H} D + \dot{D} H, \] (13)
and on the AH, for a flat universe, the maximum area implies maximum distance \( \dot{D} = 0 \); substituting in (13) we obtain that \( \dot{H} = \ddot{D}/D \); then, using (12), the GCEB is satisfied if
\[ \Delta t \geq \Delta t_m = -4 \frac{s(t_{AH})}{\dot{H}(t_{AH})}. \] (14)
For \( \kappa \neq 0 \) the previous expression generalizes to
\[ \Delta t \geq -4 \frac{s(t_{AH})}{\dot{H}(t_{AH}) - \frac{\kappa}{R^2(t_{AH})}}. \] (15)

Our analysis has been purely kinematic, now we use the dynamics. The Friedman equations allows us to write
\[ \dot{H} - \frac{\kappa}{R^2} = -4\pi (\rho + p), \] (16)
where $\rho$ and $p$ are the density and pressure of the fluid that governs the cosmic evolution. The previous expression is valid also for non zero cosmological constant. If the equation of state of the fluid is $p = \omega \rho$, the restriction on the cosmic time can be put as

$$\Delta t \geq \frac{1}{\pi (1 + \omega)} \frac{s}{\rho},$$

(17)

where the variables are evaluated on the AH. If the expansion is adiabatic

$$s = \frac{\rho + p}{T},$$

(18)

where $T$ is the temperature of the fluid. Then (17) adopts the form

$$\Delta t \geq \frac{1}{\pi T_{AH}},$$

(19)

the temperature being evaluated on the AH. This result was obtained by Bousso for the particular example of Guedens (a closed radiation dominated universe). We see that (19) is a general result, only requiring an adiabatic expansion.

4 Interpreting the results

We have obtained the minimum value for the cosmic time that we must wait to satisfy the GCEB, that is $\Delta t_m \sim s/\rho$. It is important to know the behavior of this value as the universe evolves; so we study the quotient between $\Delta t_m$ and a cosmic time scale. We know that if the expansion is adiabatic and $\omega$ constant, $s \sim R^{-3}$ and $\rho \sim R^{-3(1+\omega)}$. On the other hand for a flat universe $R \sim t^{2(1+\omega)/3}$. Taking $t_{AH}$ as the scale we have

$$\frac{\Delta t_m}{t_{AH}} \sim \frac{1}{t_{AH}^{2(1+\omega)/3}};$$

(20)

it is clear that the previous quotient decreases with cosmic time if $\omega < \omega_c = 1$, the Fischler-Susskind limit.

Let us now focus on the expression (19). If $T_{AH}$ is the temperature of the cosmic fluid, its inverse will be a measure of the typical wavelength of the quanta that carries the entropy, $\Delta t \geq 1/\pi T_{AH} \equiv \lambda_M$. So, when we count the entropy that traverses the LS, it is natural to consider only those modes whose wavelength is smaller than the size of the LS; that is, $\lambda < \lambda_M$ and we must cut the modes using the previous infrared cutoff. If $t_0$ is the cosmic time where the LS, beginning in the AH, is truncated, when counting the amount of entropy passing by such LS we must use a thermal distribution...
with an IR cutoff $\lambda \leq \lambda_M = t_0 - t_{AH}$, the density of entropy at each time (or temperature) is obtained integrating the distribution with the appropriate limits,

$$s(T, \lambda_M) = \frac{2gT^3}{3\pi^2} \int_{\lambda_M^{-1}}^{\infty} \frac{dx}{x^2 e^x - 1}; \quad (21)$$

we suppose that the quanta is bosonic and that the degeneracy is $g$. Now we integrate the previous density between two temperatures (or times)

$$S(t_f, \lambda_M) = \int_{T_{AH}}^{T_0} dt(T) 4\pi D^2 s(T, \lambda_M), \quad (22)$$

where for the dependence of $D$ with $t$ we must integrate using the explicit function for the scale factor. The functional dependence of cosmic time with the temperature depends on the nature of the fluid that fills the universe; for radiation, $t \sim T^2$ and we can compute the previous integral; it only remains to compare with one quarter of the decrease of the area. In the Figure 2, we plot this relation and we see that the introduction of the physical IR cutoff restores the GCEB for all truncated LS on this sort of cosmological scenarios.

The introduction of a cutoff to satisfy the GCEB imply a non-additivity of the entropy. Consider a truncated LS $L$ made by two adjacent LS’s so that

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Figure 2. Validity and violation of the GCEB with and without an IR cutoff respectively, for a radiation dominated universe.
\[ L = L_1 + L_2; \] the entropy on \( L_i \) accounts for the modes with \( \lambda \leq \lambda_i, \ i = 1, 2; \) for the entropy on \( L \) we must consider the modes with \( \lambda \leq \lambda_1 + \lambda_2 \). It is clear that we lose the additivity (extensivity) of the entropy and
\[
S(L = L_1 + L_2) \geq S(L_1) + S(L_2). \tag{23}
\]
Wether this is a mathematical artifact or can be related with a fundamental non additivity of the entropy in string/M-theory is an open question.

To finish the discussion of our results, consider the expression \[17\]; near the AH a constant density is a good approximation and we have
\[
\Delta t \geq \frac{S}{M}, \tag{24}
\]
where \( S \) and \( M \) are the total entropy and mass traversing the small LS. There is an adjusting mechanism; if more entropy tries to pass the LS, for a given \( \Delta t \), more energy is used to carry the entropy and consequently the LS curves more and the bound is satisfied. In the extreme situation, when only one bit of mass \( M_1 \) passes through the LS, \( S \sim 1 \) and \[24\] has the form of a cosmological uncertainty relation, namely
\[
\Delta t M_1 \geq 1, \tag{25}
\]
suggesting a deep connection between quantum mechanics, general covariance and the GCEB, in this cosmological setup.

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