Phase transition and Quasinormal modes for Charged black holes in 4D Einstein-Gauss-Bonnet gravity

Ming Zhang$^1$, Chao-Ming Zhang$^2$, De-Cheng Zou$^2$ and Rui-Hong Yue$^2$

$^1$Faculty of Science, Xi’an Aeronautical University, Xi’an 710077 China
$^2$Center for Gravitation and Cosmology, College of Physical Science and Technology, Yangzhou University, Yangzhou 225009, China

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In this paper, we study the quasinormal modes (QNMs) of massless scalar perturbations to probe the Van der Waals like small and large black holes (SBH/LBH) phase transition of (charged) AdS black holes in the 4-dimensional Einstein Gauss-Bonnet gravity. We find that the signature of this SBH/LBH phase transition in the isobaric process can be detected since the slopes of the QNMs frequencies change drastically different in small and large black holes near the critical point. The obtained results further support that the QNMs can be a dynamic probe of the thermodynamic properties in black holes.

I. INTRODUCTION

During the past decades, higher order derivative curvature gravities, as the effective models of gravity in their low-energy limit string theories, have attracted considerable interest. Among these higher order derivative curvature gravities, the most extensively studied theory is the so-called Gauss-Bonnet gravity ([1]-[18]), which naturally emerges when we want to generalize Einstein’s theory in higher dimensions by keeping all characteristics of usual general relativity excepting the linear dependence of the Riemann tensor. However, it is well known that the Gauss-Bonnet (GB) term’s variation is a total derivative in 4 dimensional spacetime, which has no contribution to the gravitational dynamics. Therefore, one requires $D \geq 5$ for non-trivial gravitational dynamics. Recently, Glavan and Lin [2] suggested a novel theory of gravity in 4-dimensional spacetime called “4D Einstein Gauss-Bonnet gravity” (EGB). By rescaling the GB coupling constant $\alpha \rightarrow \alpha/(D-4)$ with $D$ the number of spacetime dimensions, and defining the 4-dimensional theory as the limit $D \rightarrow 4$, the GB term gives rise to non-trivial dynamics. Furthermore, the spherically symmetric

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$^*$ e-mail: mingzhang0807@126.com; zhangming@xaau.edu.cn
$^†$ e-mail: 843395448@qq.com
$^‡$ e-mail: dczou@yzu.edu.cn
$^§$ e-mail: rhyue@yzu.edu.cn
black hole solutions have been also constructed in this paper. The generalization to other black holes has also appeared in refs. [3]-[11].

In the black hole physics, the thermodynamical phase transition of black hole is always a hot topic. Due to the AdS/CFT correspondence [19]-[21], lots of attentions have been attracted to study the phase transition of black holes in anti-de Sitter(AdS) space mainly. Recently thermodynamics of AdS black holes has been studied in the extended phase space where the cosmological constant is treated as the pressure of the system [22], where it was found that a first order small and large black holes phase transition is allowed and the $P-V$ isotherms are analogous to the Van der Waals fluid. More discussions in this direction can be found as well in, including reentrant phase transitions and more general Van der Waals behavior [23]-[30]. On the other hand, the quasinormal modes (QNMs) of dynamical perturbations are considered as the characteristic sounds of black holes. The QNMs of the dynamical perturbations are expected to reflect the black hole phase transitions in their surrounding geometries through frequencies and damping times of the oscillations. In fact, the thermodynamic phase transition of the AdS black hole in the dual field theory corresponds to the onset of instability of a black hole. With lots of researches on this issue, more and more evidences of the connections between the QNMs of black holes and the thermodynamic phase transitions were found [31]-[52].

Until now, this Van der Waals-like (SBH/LBH) phase transition for charged and neutral black hole was also recovered in Gauss-Bonnet gravity [53]-[57]. Motivated by these results and the extensive importance of AdS/CFT correspondence, the aim of this paper is to study whether signature of Van der Waals like SBH/LBH phase transition of AdS black holes in 4D EGB gravity can be reflected by the dynamical QNMs behavior with the massless scalar perturbation.

The paper is organized as follows: in Sect. [III] we firstly review the Van der Waals like phase transition of (charged) AdS black holes in 4-dimensional EGB gravity in the extended phase space. Then, we give discussions for QNM frequencies under test scalar field perturbations in Sect. [IV] and disclose the phase transition can be reflected by the QNM frequencies of dynamical perturbations. We end the paper with closing remakes in the last section.
II. THERMODYNAMICS AND PHASE TRANSITION OF CHARGED ADS BLACK HOLES

The action of $D$-dimensional charged EGB gravity in the presence of a negative cosmological constant $\Lambda \equiv -\frac{(D-1)(D-2)}{2l^2}$ is given by

$$S = \frac{1}{16\pi} \int d^Dx \sqrt{-g} \left[ R + \frac{(D-1)(D-2)}{l^2} + \frac{\alpha}{D-4} G - F_{\mu\nu} F^{\mu\nu} \right], \quad (1)$$

where the Gauss-Bonnet term is $G = R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$, the Gauss-Bonnet coefficient $\alpha$ with dimension $(\text{length})^2$ is positive in the heterotic string theory. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual Maxwell tensor.

Taking the limit $D \to 4$ [2], four dimensional static and spherically symmetric charged AdS black hole solution in EGB gravity are obtained as [5]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{D-2}, \quad (2)$$

$$f(r) = 1 + \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 + 4\alpha \left( \frac{2M}{r^3} - \frac{Q^2}{r^4} - \frac{1}{l^2} \right)} \right), \quad (3)$$

with the nonvanishing electrostatic vector potential $A_t = \frac{Q}{r}$. Here $M$ and $Q$ are the mass and charge of black hole. In the extended phase space, the cosmological constant $\Lambda$ is regarded as a variable and also identified with the thermodynamic pressure $P = -\frac{\Lambda}{8\pi}$ in the geometric units $G_N = \hbar = c = k = 1$. We will only consider the positive GB coefficient $\alpha$ in the following discussion. If we take $\alpha \to 0$, the solution $f(r)$ reduces to RN AdS case in general relativity.

In terms of the horizon radius $r_+$, mass $M$, Hawking temperature $T$ and entropy $S$ of 4D charged EGB-AdS black holes can be written as [5]

$$M = \frac{Q^2}{2r_+} + \frac{\alpha}{2r_+} + \frac{r_+}{2} + \frac{4\pi P}{3} r_+^3, \quad (4)$$

$$T = \frac{2r_+^3}{\alpha} - P \frac{Q^2 - r_+^2 + \alpha}{4\pi r_+(r_+^2 + 2\alpha)}, \quad (5)$$

$$S = \pi r_+^2 + 4\pi \alpha \ln r_+. \quad (6)$$

In the extended phase space, the black hole mass $M$ is considered as the enthalpy $H$ rather than the internal energy of the gravitational system.

From the Hawking temperature [6], we can obtain the equation of state

$$P = \left( \frac{\alpha}{r_+^3} + \frac{1}{2r_+} \right) T + \frac{Q^2 + \alpha - r_+^2}{8\pi r_+^4} \quad (7)$$
As usual, a critical point occurs when $P$ has an inflection point,

$$
\frac{\partial P}{\partial r_+} \bigg|_{T=T_c, r_+ = r_c} = \frac{\partial^2 P}{\partial r_+^2} \bigg|_{T=T_c, r_+ = r_c} = 0.
$$

(8)

Then we can obtain corresponding critical values

$$
T_c = \frac{r_c^2 - 2Q^2 - 2\alpha}{2\pi r_c(r_c^2 + 6\alpha)},
$$

(9)

$$
P_c = -\frac{Q^2(3r_c^2 + 2\alpha)}{8\pi r_c^4(r_c^2 + 6\alpha)} + \frac{r_c^4 - 5\alpha r_c^2 - 2\alpha^2}{8\pi r_c^4(r_c^2 + 6\alpha)},
$$

(10)

where

$$
r_c = \sqrt{3(Q^2 + 2\alpha) + \sqrt{3(3Q^2 + 4\alpha)(Q^2 + 4\alpha)}}.
$$

(11)

Here the subscript “c” represents the critical values of the physical quantities. For instance, we can obtain a critical point with $r_c = 0.4387$, $T_c = 0.219$ and $P_c = 0.0904$ by fixed $\alpha = 0.01$ and $Q = 0.1$. We plot the $P - r_+$ isotherm diagram around the critical temperature $T_c$ for this charged AdS black hole, see Fig. 1. The dotted line with $T > T_c$ corresponds to the “idea gas” phase behavior, and the Van der Waals like small/large black hole phase transition appears in the system when $T < T_c$.

![FIG. 1: The $P - r_+$ diagram of charged and uncharged AdS black holes.](image)

The behavior of Gibbs free energy $G$ is important to determine the thermodynamic phase transition. The free energy $G$ obeys the following thermodynamic relation $G = H - TS$ with

$$
G = 2\pi r_+^3 P \left( \frac{2}{3} - \frac{r_+^2 + 4\alpha \ln r_+}{r_+^2 + 2\alpha} \right) + \frac{(Q^2 - r_+^2 + \alpha)(r_+^2 + 4\alpha \ln r_+)}{4r_+(r_+^2 + 2\alpha)} + \frac{Q^2 + r_+^2 + \alpha}{2r_+}.
$$

(12)

Here $r_+$ is understood as a function of pressure and temperature, $r_+ = r_+(P, T)$, via equation of state (7).

In left panel of Fig 2 we see that the $G$ surface demonstrates the characteristic “swallow tail” behavior, which shows that there is a Van der Waals like first order phase transition in the system.
The right panel of Fig. 2 shows the coexistence line in the \((P, T)\) plane by finding a curve where the Gibbs free energy and temperature coincide for small and large black holes. The coexistence line is very similar to that in the Van der Waals fluid. The critical point is shown by a small circle at the end of the coexistence line. The small-large black hole phase transition occurs for \(T < T_c\). Moreover, in the uncharged case, Eq.(9)-Eq.(11) can be written as

\[
T_c = \frac{1 + \sqrt{3}}{2(3 + \sqrt{3})\sqrt{6 + 4\sqrt{3}\pi\sqrt{\alpha}}},
\]

\[
P_c = \frac{13 + 7\sqrt{3}}{3168\pi\alpha + 1824\sqrt{3}\pi\alpha},
\]

\[
r_c = \sqrt{2}\sqrt{3\alpha + 2\sqrt{3}\alpha}.
\]

For instance, we can get a critical point with \(r_c = 0.3596\), \(T_c = 0.2556\) and \(P_c = 0.1264\) by fixed \(\alpha = 0.01\). The corresponding “\(P - r_+\)” and “\(G - T\)” diagrams of black hole are also qualitatively similar to the charged case.

For Van der Waals liquid-gas system, the liquid-gas structure does not change suddenly but undergoes the second order phase transition at the critical point \((V = V_c, T = T_c, P = P_c)\). This is described by the Ehrenfest’s description \[58, 59\]. In conventional thermodynamics, Ehrenfest’s description consists of the first and second Ehrenfest’s equations \[60, 61\]

\[
\left. \frac{\partial P}{\partial T} \right|_S = \frac{C_{P2} - C_{P1}}{TV(\zeta_2 - \zeta_1)} = \frac{\Delta C_P}{TV\Delta \zeta},
\]

\[
\left. \frac{\partial P}{\partial T} \right|_V = \frac{\zeta_2 - \zeta_1}{\kappa_{T2} - \kappa_{T1}} = \frac{\Delta \zeta}{\Delta \kappa_T}.
\]

For a genuine second order phase transition, both of these equations have to be satisfied simultaneously. Here \(\zeta\) and \(\kappa_T\) denote the volume expansion and isothermal compressibility coefficients of
the system respectively

$$\zeta = \frac{1}{V} \frac{\partial V}{\partial T} \bigg|_P, \quad \kappa_T = -\frac{1}{V} \frac{\partial V}{\partial P} \bigg|_T.$$  \hspace{1cm} (18)

Using the similar method as Ref.\[54\], we can find that this phase transition at the critical point in the 4-dimensional EGB-AdS black hole is of the second order (the Ehrenfests equations is satisfied in this case). This result is consistent with the nature of the liquid-gas phase transition at the critical point.

III. PERTURBATION OF ADS BLACK HOLE IN 4D EINSTEIN-GAUSS-BONNET GRAVITY

In order to reflect the thermodynamical stabilities in dynamical perturbations, we can study the evolution of a massless scalar field perturbation around this 4-dimensional EGB-AdS black hole. A massless scalar field \( \Phi(r, t, \Omega) = \phi(r) e^{-i\omega t} Y_{lm}(\Omega) \), obeys Klein-Gordon equation

$$\nabla^2 \Phi(t, r, \Omega) = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g}g^{\mu\nu} \partial_{\nu} \Phi(t, r, \Omega)) = 0.$$  \hspace{1cm} (19)

Then radial equation for the function \( \phi(r) \) is obtained as

$$\phi''(r) + \frac{f'(r)}{f(r)} \phi'(r) + \left( \frac{\omega^2}{f(r)^2} - \frac{l(l+1)}{r^2 f(r)} - \frac{f'(r)}{r f(r)} \right) \phi(r) = 0,$$  \hspace{1cm} (20)

where \( \omega \) are complex numbers \( \omega = \omega_r + i\omega_{im} \), corresponding to the QNM frequencies of the oscillations describing the perturbation.

Near the horizon \( r_+ \), we can impose the boundary condition of the scalar field, \( \phi(r) \rightarrow (r - r_+) \frac{\omega}{i f(r)} \). Then we define \( \phi(r) \) as \( \varphi(r) e^{i \int \frac{\omega}{f(r)} dr} \), where the \( e^{i \int \frac{\omega}{f(r)} dr} \) asymptotically approaches to ingoing wave near horizon, we can rewrite Eq. (20) into

$$\varphi''(r) + \left( \frac{f'(r)}{f(r)} - \frac{2i\omega}{f(r)} \right) \varphi'(r) - \left( \frac{f'(r)}{r f(r)} + \frac{l(l+1)}{r^2 f(r)} \right) \varphi(r) = 0.$$  \hspace{1cm} (21)

For Eq. (21), we have \( \varphi(r) = 1 \) in the limit of \( r \rightarrow r_+ \). At the AdS boundary \( (r \rightarrow \infty) \), we need \( \varphi(r) = 0 \). Under these boundary conditions, we will numerically solve Eq. (21) to find QNM frequencies by adopting the shooting method.

In the left panel of Fig. 3 we plot \( T - r_+ \) diagram of charged AdS black holes with fixed pressure \( P = 0.06 < P_c = 0.219 \) in four dimensional EGB gravity. For \( P < P_c \) there is an inflection point and the behavior is reminiscent of the Van der Waals system. The critical point can be got from

$$\left. \frac{\partial T}{\partial r_+} \right|_{P=P_c, r_+=r_c} = \left. \frac{\partial^2 T}{\partial r_+^2} \right|_{P=P_c, r_+=r_c} = 0.$$  \hspace{1cm} (22)
The behavior of the Gibbs free energy is plotted in the right panel of Fig. 3. The cross point “5” between the solid line marked as “1-5” and the solid line denoted as “4-5” shows that the Gibbs free energy and P coincide for small and large black holes. In the left panel of Fig. 3, the point “5” is separated into “L5” and “R5” for the same Gibbs free energy and the chosen $T_* \approx 0.18946$ where the small and large black hole can coexist. Moreover, the physical phase marked between points “1-5” or “1-L5” corresponds to the small black hole, while physical phase indicated between points “5-4” or “R5-4” denotes the large black hole.

![Fig. 3: $T - r_+$ (left panel) and $G - T$ (right panel) diagrams of 4D EGB AdS black holes with $\alpha = 0.01, Q = 0.1$ and $P \approx 0.06$.](image)

In TABLE I, we further list the QNM frequencies of massless scalar perturbation (for $l = 0$ and 1) for small and large charged black holes near SBH/LBH phase transition point. With regard to small black hole phase, the radius of black hole becomes smaller and smaller when the temperature decreases from phase transition temperature $T_*$. In this process the absolute values of imaginary part of QNM frequencies decrease, while the real part frequencies change very little. On the other hand, when temperature for large black hole phase increases from the phase transition temperature $T_*$, the black hole gets bigger. The QNM frequencies increase in both the real part and the absolute value of imaginary part. It means that the massless scalar perturbation outside the black hole gets more oscillations but it decays faster. These results are consistent with the overall discussions reported in [51, 52]. Fig. 4 and Fig. 5 respectively illustrates the QNM frequencies with $l = 0$ and $l = 1$ for small and large black hole phases. Increase in the black hole’s size is indicated by the arrows.

In addition, at the critical position $P = P_c$, with $P_c \approx 0.06$, a second-order phase transition occurs. The QNM frequencies of the small and large black hole phases (for $l = 0$ and $l = 1$) are plotted in Fig. 6. We can see that QNM frequencies of two black hole phases possess the same
TABLE I: The QNM frequencies of massless scalar perturbation with the change of black hole temperature with $\alpha = 0.01$ and $Q = 0.1$. The upper part, above the horizontal line, is for the small black hole phase, while the lower part is for the large black hole phase.

| $T$     | $r_+$  | $\omega(l = 0)$                  | $\omega(l = 1)$                  |
|---------|--------|----------------------------------|----------------------------------|
| 0.1855  | 0.24719| 1.67171-0.342286I                | 3.30238-0.488428I                |
| 0.186   | 0.24833| 1.67141-0.343580I                | 3.30097-0.491701I                |
| 0.187   | 0.25072| 1.67077-0.346294I                | 3.29793-0.498354I                |
| 0.188   | 0.25325| 1.67009-0.349273I                | 3.29450-0.505050I                |
| 0.189   | 0.25594| 1.66937-0.352404I                | 3.29115-0.512268I                |
| 0.190   | 0.91129| 2.72028-0.982813I                | 4.02092-0.993442I                |
| 0.191   | 0.93516| 2.74446-0.992160I                | 4.05526-1.003787I                |
| 0.192   | 0.95733| 2.76751-1.001339I                | 4.08825-1.013420I                |
| 0.193   | 0.97819| 2.78981-1.010033I                | 4.11994-1.022668I                |
| 0.194   | 0.99800| 2.81150-1.018480I                | 4.15075-1.031497I                |

Behavior as the black hole horizon increases at the critical point.

FIG. 4: The behavior of QNMs for large and small black holes in the complex-$\omega$ with $Q = 0.1$ and $l = 0$. The arrow indicates the increase of black hole horizon.

In the neutral case, we can obtain similar $T - r_+$ and $G - T$ diagrams to the charged case. For instance, the coexistence temperature $T_c$ equals to 0.21860 when taking pressure $P = 0.08 < P_c = 0.1264$. The QNM frequencies of massless scalar perturbation (for $l = 0$ and 1) around small and large uncharged black holes for first order SBH/LBH phase transition are listed in TABLE II, which also exhibit a similar behavior to that of the charged case. The QNM frequencies with $l = 0$ and $l = 1$ for small and large uncharged black hole phases are shown in Fig. 7 and Fig. 8. Moreover, at the critical position, the corresponding QNM frequencies of the small and large uncharged black
FIG. 5: The behavior of QNMs for large and small black holes in the complex-$\omega$ with $Q = 0.1$ and $l = 1$. The arrow indicates the increase of black hole horizon.

FIG. 6: The behavior of QNM frequencies for large (dashed) and small (solid) black holes in the complex-$\omega$ with $Q = 0.1$. The arrow indicates the increase of black hole horizon.

hole phases are also qualitatively similar to the charged case.

**IV. CLOSING REMARKS**

In the 4-dimensional Einstein Gauss-Bonnet gravity, we have studied the $P - V$ criticality and phase transition of AdS black holes in the extended phase space. The VdW-like SBH/LBH phase transition could happen both in the charged and neutral cases. Then, we further calculated the QNMs of massless scalar perturbations under 4 situations (charged/uncharged and $l = 0$ or 1). These results reveal that the slopes of the QNM frequency change drastically different in the small and large black hole phases as increasing of the horizon radius $r_+$, when the Van der Waals
TABLE II: The QNM frequencies of massless scalar perturbation with the change of black hole temperature with $\alpha = 0.01$ and $Q = 0$. The upper part, above the horizontal line, is for the small black hole phase, while the lower part is for the large black hole phase.

| $T$   | $r_+$  | $\omega(l = 0)$          | $\omega(l = 1)$          |
|-------|--------|--------------------------|--------------------------|
| 0.214 | 0.18938| 1.97357-0.340401I        | 2.86372-0.557742I        |
| 0.215 | 0.19088| 1.97273-0.342787I        | 2.86116-0.573858I        |
| 0.216 | 0.19245| 1.97185-0.345297I        | 2.85852-0.591034I        |
| 0.217 | 0.19408| 1.97093-0.347917I        | 2.85572-0.609143I        |
| 0.218 | 0.19579| 1.96997-0.350686I        | 2.85233-0.626936I        |
| 0.219 | 0.79874| 2.42426-1.100808I        | 3.21754-1.100784I        |
| 0.220 | 0.81586| 2.44275-1.111345I        | 3.24017-1.110794I        |
| 0.221 | 0.83201| 2.46057-1.121425I        | 3.26214-1.120424I        |
| 0.222 | 0.84739| 2.47780-1.131115I        | 3.28349-1.129773I        |
| 0.223 | 0.86211| 2.49461-1.140502I        | 3.30429-1.138959I        |

FIG. 7: The behavior of QNMs for large and small black holes in the complex-$\omega$. The arrow indicates the increase of black hole horizon. $Q = 0, l = 0$

analogy SBH/LBH phase transition happens in the extended space. This clearly demonstrates the signature of the phase transition between small and large black holes. In addition, at the critical isobaric phase transitions, the QNM frequencies for both small and large black holes share the same behavior, which showing that QNMs are not appropriate to probe the black hole phase transition in the second order.

This is one more example exhibits that the QNM can provide the dynamical physical phenomenon of the thermodynamic phase transition of black holes in 4D EGB gravity. Since the QNM is expected to be detected and has strong astrophysical interest. The ability of QNMs to
FIG. 8: The behavior of QNMs for large and small black holes in the complex-$\omega$. The arrow indicates the increase of black hole horizon.$Q = 0, l = 1$

reflect the thermodynamic phase transition is interesting, which is expected to disclose the observational signature of the thermodynamic phase transition.

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