Stable spherically symmetric monopole field background in a pure QCD

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We consider a stationary spherically symmetric monopole-like solution with a finite energy density in a pure quantum chromodynamics (QCD). The solution can be treated as a static Wu--Yang monopole dressed in a time-dependent field corresponding to off-diagonal gluons. We have proved that such a stationary monopole field represents a background vacuum field of the QCD effective action which is stable against quantum gluon fluctuations. This resolves a long-standing problem of the existence of a stable vacuum field in QCD and opens a new avenue towards a microscopic theory of the vacuum.

1. Introduction

The non-perturbative structure of the vacuum and the confinement phenomenon represent two of the most important and closely related problems in the foundations of QCD, which is supposed to be a basic fundamental theory of strong interactions. Despite significant progress in lattice studies \cite{1} there is still no deep knowledge about the microscopic vacuum structure and origin of the color confinement from the first principles of QCD (see \cite{2} and references therein). One of the attractive mechanisms for quark confinement is based on the Meissner effect in a dual-color superconductor which assumes generation of the monopole condensation \cite{3–7}. The previously known Savvidy–Nielsen–Olesen QCD vacuum based on a homogeneous magnetic field configuration \cite{8,9} suffers instability against the quantum fluctuations. The most popular Copenhagen “spaghetti” vacuum model \cite{10–12} based on vortex domain structure does not provide a consistent microscopic description of the vacuum. Numerous attempts to construct a rigorous true QCD vacuum using various vacuum field configurations (instantons: \cite{13}; monopoles, dyons: \cite{14,15}; center vortices: \cite{16}; monopoles: \cite{17–20}; etc. \cite{21,22}) show that the existence of a stable vacuum represents a most serious long-standing problem in QCD. One of the principal obstacles in resolving that problem was the absence of a proper regular classical solution that must be stable against vacuum fluctuations at microscopic space scale and can serve as a structure element in the formation of the vacuum. Note that all previous vacuum models employ static vacuum field configurations, none of which possess quantum stability at microscopic scale. For instance, a single static vortex field in the “spaghetti” vacuum model is unstable, and the vacuum stability is restored only due to an averaging procedure.
over the statistical ensemble of the vortex domains. An advantage of the “spaghetti” vacuum is that it provides an approximate qualitative description of the vacuum structure; in particular, it was conjectured from quantum mechanical considerations that color magnetic vortices should vibrate at a small space-time scale. This gives a hint that time-dependent field configurations might play an important role in vacuum structure.

In the present paper we explore the idea that a stationary generalized monopole-like solution can serve as an initial structure element in the formation of the QCD vacuum. We consider a recently proposed classical stationary spherically symmetric monopole solution that can be treated as a system of a static Wu–Yang monopole interacting with a time-dependent off-diagonal gluon field [23]. We have proved that such a solution provides a monopole vacuum field that is stable against gluon fluctuations in a one-loop approximation. This opens a new way towards construction of a microscopic theory of the QCD vacuum through the condensation of monopoles and/or monopole–antimonopole pairs.

2. Stationary generalized Wu–Yang monopole solution

Let us describe the main properties of the spherically symmetric stationary monopole solution proposed in [23]. Such a solution will be used in the subsequent sections as a background field in the effective action functional, and its non-trivial properties will provide the stability under quantum gluon fluctuations. We start with a standard classical Lagrangian of a pure $SU(3)$ QCD,

$$\mathcal{L} = -\frac{1}{4} F_{a\mu\nu} F^{a\mu\nu}. \quad (1)$$

We introduce the following spherically symmetric ansatz for non-vanishing components of the gauge potential in spherical coordinates $(r, \theta, \phi)$:

$$A_\phi^1 = -\frac{1}{r} \psi_1(t, r), \quad A_\theta^2 = \frac{1}{r} \psi_1(t, r), \quad A_\phi^3 = \frac{1}{gr} \cot \theta,$$

$$A_\phi^4 = \frac{1}{r} \psi_2(t, r), \quad A_\phi^5 = \frac{1}{r} \psi_2(t, r), \quad A_\phi^8 = -\frac{\sqrt{3}}{gr} \cot \theta, \quad (2)$$

where the Abelian gauge potentials $A_\phi^3$ and $A_\phi^8$ describe a static Wu–Yang monopole with a total color magnetic charge of two, $g_{m}^{\text{tot}} = 2$ [17], and the functions $\psi_1(t, r)$ and $\psi_2(t, r)$ correspond to dynamical degrees of freedom of the off-diagonal components of the gluon field $A_\mu$. The ansatz in Eq. (2) describes two coupled monopole field configurations corresponding to $I$- and $U$-type $SU(2)$ subgroups. One can verify that the ansatz in Eq. (2) is consistent with all equations of motion of the pure $SU(3)$ QCD, and substitution of the ansatz into the equations of motion results in only two non-trivial independent partial differential equations:

$$\partial_t^2 \psi_1 - \partial_r^2 \psi_1 + \frac{g^2}{2r^2} \psi_1 \left(2\psi_1^2 - \psi_2^2 - \frac{2}{g^2}\right) = 0,$$

$$\partial_t^2 \psi_2 - \partial_r^2 \psi_2 + \frac{g^2}{2r^2} \psi_2 \left(2\psi_2^2 - \psi_1^2 - \frac{2}{g^2}\right) = 0. \quad (3)$$

In the special case when $A_\theta^5 = A_\phi^4 = A_\phi^8 = 0$ the ansatz describes an $SU(2)$ embedded field configuration which corresponds to a system of the Wu–Yang monopole with a unit monopole charge, $g_{m}^{\text{tot}} = 1$, interacting with an off-diagonal gluon field. With $\psi_1(t, r) \equiv \psi(t, r), \quad \psi_2(t, r) = 0$
Eqs. (3) reduce to one differential equation:

$$\partial_t^2 \psi - \partial_r^2 \psi + \frac{g^2}{r^2} \left( \psi^2 - \frac{1}{g^2} \right) = 0.$$  (4)

Equation (4) admits a wide class of time-dependent solutions including non-stationary soliton-like propagating solutions in the effective two-dimensional space-time \((r, t)\) [24–28]. We show that there is a subclass of generalized stationary Wu–Yang monopole solutions which possess a finite energy density. The most important issue is that such solutions are stable against quantum gluon fluctuations in pure QCD. For simplicity we consider first the vacuum stability problem in the case of \(SU(2)\) Yang–Mills theory. In that case, by performing an appropriate gauge transformation [29] one can rewrite the \(SU(2)\) part of the ansatz in Eq. (2) in a regular gauge as follows \((a = 1, 2, 3)\):

$$A^a_m = -e^{abc} \hat{n}^b \partial_m \hat{n}^c \left( \frac{1}{g} - \psi(t, r) \right),$$  (5)

where \(\hat{n} = \vec{r}/r\). It is clear that the ansatz in Eq. (5) describes a generalized time-dependent Wu–Yang monopole field configuration. A known static Wu–Yang monopole corresponds to the limiting case \(\psi(t, r) = 0\), and a trivial pure gauge vacuum configuration is described by \(\psi(t, r) = \pm 1/g\). We prefer the ansatz written in the so-called singular Abelian gauge [29], Eq. (2), since such a notation is more suitable for describing stationary monopole solutions in \(SU(N)\) Yang–Mills theory and in the description of multimonopole configurations.

For simplicity we consider first the vacuum stability problem in the case of \(SU(2)\) embedded solution. For an arbitrary function \(\psi(t, r)\) one has the following non-vanishing field strength components:

$$F^1_{t\phi} = -\frac{1}{r} \partial_t \psi, \quad F^1_{r\phi} = -\frac{1}{r} \partial_r \psi,$$

$$F^2_{t\theta} = \frac{1}{r} \partial_t \psi, \quad F^2_{r\theta} = \frac{1}{r} \partial_r \psi, \quad F^3_{\theta\phi} = \frac{1}{gr^2} (g^2 \psi^2 - 1).$$  (6)

The radial magnetic field component \(F^3_{\theta\phi}\) generates a non-zero magnetic flux through a sphere with a center at the origin, \(r = 0\), so that the color magnetic charge of the monopole depends on the time and distance from the center. Note that various generalized static Wu–Yang monopoles have been considered before, all of them having singularities in agreement with Derrick’s theorem [30]. Note that the presence of singularities in the expressions for the gauge potential in Eq. (2) represents coordinate singularities related to the chosen singular gauge. Such coordinate singularities disappear in the regular gauge, Eq. (5). By substitution of the ansatz in Eq. (2) into the energy functional one can verify that the energy density is finite everywhere:

$$E = 4\pi \int dr \left( \left( \partial_t \psi \right)^2 + \left( \partial_r \psi \right)^2 + \frac{1}{2g^2 r^2} (g^2 \psi^2 - 1)^2 \right).$$  (7)

Equation (4) admits a local non-static solution near the origin that removes the singularity of the monopole at the center:

$$\psi(t, r) = \frac{1}{g} + \sum_{n=1} c_{2n}(t) r^{2n},$$
\begin{align}
    c_4(t) &= \frac{1}{10}(3gc_2^2(t) + c_2(t)), \\
    c_6(t) &= \frac{1}{28}(c_4''(t) + 6gc_2(t)c_4(t) + g^2c_2^3(t)), \\
    \vdots
\end{align}

where the coefficient functions $c_{2n}(t) \ (n \geq 2)$ are determined in terms of one arbitrary function $c_2(t)$. In the asymptotic region, $r \approx \infty$, the non-linear equation in Eq. (4) reduces to a free D’Alembert equation which has a standing spherical wave solution,

\begin{equation}
    \psi(t, r) \approx a_0 + A_0 \cos(Mr) \sin(Mt) + \mathcal{O}(\frac{1}{r}),
\end{equation}

where $a_0$ and $A_0$ are integration constants, and the mass scale $M$ appears due to scaling invariance in the theory under the dilatations $r \to Mr, \ t \to Mt$.

To solve numerically Eq. (4) we use the local solution in Eq. (8) to impose initial Dirichlet conditions along the boundary $r = L_0$ in the numeric domain ($L_0 \leq r \leq L$, $0 \leq t \leq L$). The initial profile function $c_2(t)$ can be chosen arbitrarily as any regular periodic function; we set

\begin{equation}
    c_2(t) = c_0 + c_1 \sin(Mt),
\end{equation}

where $c_0$ and $c_1$ are numeric constants. Note that only one of two parameters in the local solution in Eq. (10) is free, the other is fixed by the requirement that a numeric solution matches the asymptotic solution in Eq. (9). Dimensional analysis implies that the energy of the solutions is proportional to $M$, so that the energy vanishes in the limit $M \to 0$. This might cause some doubts on the stability of the solution. However, one should stress that standard arguments on the existence of static solitonic solutions based on Derrick’s theorem [30] are not applicable to the case of stationary solutions which satisfy a condition of extremum value of the classical action, not the energy functional. In the case of Yang–Mills theory the action is conformal invariant and its first variational derivative with respect to the scale parameter $M$ vanishes identically. So, the parameter $M$ determines the scale of the space-time coordinates and can be set to one without loss of generality. With this, one can solve numerically Eq. (4), and the corresponding solution is depicted in Fig. 1. Note that one has a stiffness numeric problem near the origin, so we have checked the regular structure and convergence of the numeric solution in close vicinity of the origin up to $L_0 = 10^{-6}$ while keeping the radial size of the numeric domain up to $L = 64\pi$.

The solution has several surprising features. The field configuration of the solution includes a static Wu–Yang monopole immersed in a standing spherical wave made of off-diagonal gluons. The standing wave does not screen completely the color monopole charge at large distances. One can find that in the asymptotic region the function $\psi(t, r)$ oscillates around the value $a_0 \approx 0.65$. So the radial component of the color magnetic field $F_{1\theta\phi}$ has a non-zero averaged value which provides a non-vanishing total color magnetic charge. Another interesting feature of the solution is that a corresponding canonical spin density vanishes identically. This gives a hint that such a solution, treated as a quantum mechanical wave function, can lead to a stable condensate of massive spinless particles. The idea that particles (or pseudo-particles) can be described by stationary solutions was proposed a long time ago [30–32]. The main obstacle towards practical realization of this idea was the absence of known regular stationary solutions in real QCD. Even though our solution has an infinite total energy, the energy density falls as $1/r^2$ with increasing distance from the center of the
monopole. Note that our solution differs from the known non-linear spherical-wave-type solutions which have a singularity at the center and light-speed velocity. The most important question related to the properties of the solutions is whether the monopole solution is stable against the quantum gluon fluctuations.

3. Vacuum stability of the monopole field in $SU(2)$ QCD

3.1. Effective action approach to the vacuum stability problem

In order to prove the quantum stability of the monopole field we study the structure of the effective action of QCD using the standard functional integral methods. We will consider the stability of the monopole solution under small quantum fluctuations of the gluon field. An initial gauge potential $A_{\mu}^a$ is split into classical, $B_{\mu}^a$, and quantum, $Q_{\mu}^a$, parts:

$$A_{\mu}^a = B_{\mu}^a + Q_{\mu}^a.$$  \hspace{1cm} (11)

We choose a temporal gauge for the classical and the quantum field, $B_{t}^a = Q_{t}^a = 0$. The temporal gauge has a residual symmetry which can be fixed by imposing an additional covariant Coulomb gauge condition, $D_{i}Q^{ia} = 0$, with a covariant derivative $D_{i}$ containing the background field $B_{i}^a$. Hereafter we use Latin indices for the spatial components. A one-loop correction to the classical action is obtained by functional integration over the quantum fields $Q_{\mu}^a$, and it can be expressed in terms of functional logarithms in a compact form [18,33–39]:

$$S_{\text{1loop}}^{\text{eff}} = -\frac{1}{2} \text{Tr} \ln[K_{ij}^{ab}] + \text{Tr} \ln[M_{\text{FP}}^{ab}],$$

$$K_{ij}^{ab} = -\delta^{ab} \delta_{ij} \partial_{j}^{2} - \delta_{ij}(D_{k}D^{k})^{ab} - 2f^{aci}F_{ijc},$$

$$M_{\text{FP}}^{ab} = -(D_{k}D^{k})^{ab},$$ \hspace{1cm} (12)

where the operators $K_{ij}^{ab}$ and $M_{\text{FP}}^{ab}$ correspond to the contributions of gluons and Faddeev–Popov ghosts, respectively. The operator $K_{ij}^{ab}$ represents a well-defined second-order differential operator of elliptic type in four-dimensional Euclidean space $(t, r, \theta, \varphi)$. The question of vacuum stability is reduced to the question of whether the operator $K_{ij}^{ab}$ has negative eigenvalues for a given classical
background field $B^a_\mu$. To find the eigenvalues one has to solve the Schrödinger type equation with the operator $K_{ij}^{ab}$ treated as a Hamiltonian operator for a quantum mechanical system,

$$K_{ij}^{ab} \Psi^b_j = \lambda \Psi^a_i, \quad (13)$$

where the “wave functions” $\Psi^a_i(t,r,\theta,\phi)$ describe quantum gluon fluctuations. Note that the ghost operator originates from the interaction of spinless ghost fields with the color magnetic field. Such interaction does not produce negative tachyon modes [9], so it is sufficient to study the eigenvalue spectrum of the operator $K_{ij}^{ab}$ only.

One should note that the expression in Eq. (12) for the operator $K_{ij}^{ab}$ is valid for an arbitrary space-time-dependent background field $B^a_\mu$. Moreover, Eq. (12) is manifestly gauge invariant. This follows from the gauge-invariant background field method applied to one-loop quantization [40,41].

A key idea in the background field quantization scheme is that splitting the initial gauge potential into a sum of the classical, $B^a_\mu$, and quantum, $Q^a_\mu$, fields implies two types of symmetries originating from the original $SU(2)$ gauge transformation. The first one, (I), is defined by a quantum gauge transformation,

$$\delta Q^a_\mu = (D_\mu \bar{\alpha})^a, \quad \delta B^a_\mu = 0, \quad (14)$$

where $D_\mu = \partial_\mu + \bar{A}_\mu$ is a covariant derivative containing the full gauge potential, and $\alpha^a(x)$ is an infinitesimal gauge parameter. The second type of symmetry, (II), is defined by a background gauge transformation,

$$\delta Q^a_\mu = f^{abc} Q^b_\mu \alpha^c, \quad \delta B^a_\mu = (D_\mu \bar{\alpha})^a. \quad (15)$$

As a result, within the framework of the background quantization the classical field $B^a_\mu$ appears in the expression for the operator $K_{ij}^{ab}$ only in terms of background covariant derivatives and the gauge field strength $F^{ab}_\mu$. It is clear that the operator $K_{ij}^{ab}$ transforms as an adjoint vector under $SU(2)$ type (II) transformations, and the fluon fluctuation field $\Psi^a_i$ transforms as a vector in fundamental representation. Taking into account that the $SU(2)$ group is locally isomorphic to the orthogonal group $SO(3)$, one concludes that the eigenvalue spectrum is gauge invariant and does not depend on the chosen gauge of the background field.

### 3.2. Qualitative analysis of the stability of the monopole field

Before solving the Schrödinger equation in Eq. (13) numerically, let us perform a qualitative estimate of ground state eigenvalues to trace the origin of the positiveness of the operator $K_{ij}^{ab}$. The operator $K_{ij}^{ab}$ does not have explicit dependence on the azimuthal angle $\phi$, so that the ground state eigenfunctions $\Psi^a_i$ depend only on three coordinates $(t,r,\theta)$. First, we apply the variational method to reduce the three-dimensional equation in Eq. (13) to an effective equation in two-dimensional space-time $(t,r)$. Within the variational approach one has to minimize the following “Hamiltonian” functional:

$$\langle H \rangle = \int dr \, d\theta \, d\varphi \, r^2 \sin \theta \, \Psi^a_i K_{ij}^{ab} \Psi^b_j. \quad (16)$$
One can make qualitative estimates assuming that all ground state eigenfunctions $\Psi^a_1$ include angle dependence, which guarantee the finiteness of the Hamiltonian:

$$\Psi^a_1(t, r, \theta, \varphi) = f^a_1(t, r) \sin \theta. \quad (17)$$

With this, one can perform integration over the angle variables $(\theta, \varphi)$ in Eq. (16) and obtain an effective Schrödinger equation for the ground state:

$$\tilde{K}^{ab}_{ij} f^b_j(t, r) = \lambda f^a_i(t, r), \quad (18)$$

where the operator $\tilde{K}^{ab}_{ij}$ includes dependence only on two coordinates $(t, r)$,

$$\tilde{K}^{ab}_{ij} = \delta_{ij} \delta^{ab} \left( -\partial^2_t - \partial^2_r - \frac{2}{r} \partial_r \right) + V^{ab}_{ij}(t, r). \quad (19)$$

The quadratic form corresponding to the potential $V^{ab}_{ij}$ can be written in the form

$$f^a_i V^{ab}_{ij} f^b_j = \frac{1}{r^2} V_0[f] + \frac{1}{r^2} V_1[f] + \frac{\psi^2}{r^2} V_2[f] + \frac{\psi}{r^2} V_3[f] + \frac{\partial_r \psi}{r} V_4[f], \quad (20)$$

where the first term includes a contribution from a free vector Laplace operator, $V_1[f]$ corresponds to the contribution of the Wu–Yang monopole, and $V_{2,3,4}[f]$ contains contributions proportional to $\psi^2$, $\psi$, and $\partial_r \psi$, respectively. One can verify by using a variational minimization procedure that a total coefficient function in front of the centrifugal potential $1/r^2$ is positively defined for arbitrary fluctuating functions $f^a_i$, so that the effective Schrödinger equation in Eq. (18) contains a positive centrifugal potential and a Coulomb-type potential with the oscillating coefficient function $\partial_r \psi V_4[f]$. It is known that a quantum mechanical problem for a particle in a potential well with small enough depth does not admit bound states (in a space of dimension $d \geq 3$). Therefore, there should be a critical value of the amplitude $A_0$ of the monopole solution, Eq. (9), below which the eigenvalue spectrum becomes positive. Indeed, numerical study of solutions to Eq. (18) implies a critical value $a_{1\text{cr}}^{\text{bound}} = 1.3$. The monopole field $\psi(r, t)$ is approximated by a simple interpolating function,

$$\psi^{\text{int}} = 1 - \frac{(1 - a_0)r^2}{1 + r^2} + A_0(1 - e^{-d_0r^2}) \cos(Mr + b_0) \sin(Mt), \quad (21)$$

where $d_0$ and $b_0$ are fitting parameters. A typical field configuration of $f^a_i$ corresponding to the background monopole field $\psi(r, t)$ with the asymptotic parameters $a_0 = 0.895$ and $A_0 = 0.615$, and $g = 1, M = 1$, is shown in Fig. 2.

### 3.3. Exact numeric solution to the eigenvalue problem

The qualitative estimates obtained in the previous subsection provide only an upper bound for the critical parameter $a_{1\text{cr}}$. To prove rigorously that there are no negative eigenvalues one should solve numerically the original Schrödinger type equation in Eq. (13) without any approximation on the functional dependence of the eigenfunctions describing the gluon fluctuations. Equation (13) contains nine non-linear partial differential equations which should be solved on a three-dimensional numeric domain with sufficient numeric accuracy. An additional technical difficulty in numeric calculation on three-dimensional space-time is that one must solve the equations while changing the size of the numeric domain in the radial direction in the limit $r \to \infty$ to verify that all the eigenvalues remain...
positive. Fortunately, the numeric analysis of the solutions corresponding to the lowest eigenvalue is simplified drastically due to the factorization property of the original equation in Eq. (13) and special features of the ground state solutions, as we will see below.

The eigenvalue equation in Eq. (13) written in component form admits factorization, and it can be rewritten as two decoupled systems of partial differential equations as follows (for brevity of notation we set $g = 1$ since the coupling constant can be absorbed by the monopole function $\psi(r, t)$):

(I):

\[
\begin{align*}
(\hat{\Delta} \Psi)^3_1 - \frac{2}{r^2} \partial_r \Psi^2_1 + \frac{1}{r^2} \left( (\psi^2 - 1) \Psi^2_1 - 2 \psi^2 \Psi^3_1 + 2 \psi \partial \theta \Psi^3_1 + 2 \cot \psi \Psi^3_1 + 2 \cot \theta \Psi^3_1 \right) &= \lambda \Psi^2_1, \\
(\hat{\Delta} \Psi)^3_1 - \frac{2}{r^2} \Psi \partial_\theta \Psi^3_1 + \frac{1}{r^2} \left( \psi^2 (\Psi^2_1 - \Psi^1_2) - \Psi^1_2 + 2 \psi \partial \theta (\Psi^1_2 + \Psi^1_3) + 2 \cot \theta \Psi^1_3 \right) &= \lambda \Psi^1_3, \\
(\hat{\Delta} \Psi)^2_1 + \frac{2}{r^2} \partial_r \Psi^3_1 + \frac{1}{r^2} \left( \cot^2 \theta + \psi^2 \right) \Psi^2_1 + 2 \cot \theta \Psi^2_1 + 2 \psi \Psi^3_1 + 2 \psi^2 \right) &- \frac{2}{r} \partial_r \psi \Psi^3_1 = \lambda \Psi^2_1, \\
(\hat{\Delta} \Psi)^3_3 + \frac{2}{r^2} \partial_\theta \Psi^1_1 + \frac{1}{r^2} \left( 2 \psi \Psi^3_1 + 2 \cot \theta \Psi^2_1 + \psi \Psi^3_1 + 2 \psi \Psi^3_1 + 2 \psi^2 \right) &- \frac{2}{r} \partial_r \psi \Psi^3_1 = \lambda \Psi^3_3; \\
(\hat{\Delta} \Psi)^1_2 - \frac{2}{r^2} \partial_r \Psi^2_1 - \frac{2}{r^2} \Psi \partial_\theta \Psi^1_2 + \frac{1}{r^2} \left( 2 \cot \theta + \psi^2 \right) \Psi^2_1 + 2 \psi \Psi^2_1 - 2 \cot \theta (\Psi^2_2 - \Psi^2_1) &+ \frac{1}{r^2} \left( \cot^2 \theta + \psi^2 \right) \Psi^2_1 + 2 \cot \theta \Psi^1_3 + 2 \psi \Psi^3_1 + 2 \psi^2 \right) &- \frac{2}{r} \partial_r \psi \Psi^3_1 = \lambda \Psi^1_1, \\
(\hat{\Delta} \Psi)^1_2 - \frac{2}{r^2} \partial_\theta \Psi^1_2 + \frac{2}{r^2} \partial_r \Psi^1_2 + \frac{1}{r^2} \left( 2 \psi \Psi^1_1 + 2 \cot \theta \Psi^1_2 - \psi \Psi^1_3 + 2 \psi^2 \right) &+ \frac{1}{r^2} \left( 2 \psi \Psi^1_1 + 2 \cot \theta \Psi^1_2 - \psi \Psi^1_3 + 2 \psi^2 \right) &+ \frac{1}{r^2} \left( 2 \psi \Psi^1_1 + 2 \cot \theta \Psi^1_2 - \psi \Psi^1_3 + 2 \psi^2 \right) &- \frac{2}{r} \partial_r \psi \Psi^3_1 = \lambda \Psi^3_2, \\
(\hat{\Delta} \Psi)^1_2 - \frac{2}{r^2} \partial_\theta \Psi^1_2 + \frac{2}{r^2} \partial_r \Psi^1_2 + \frac{1}{r^2} \left( -2 \psi \Psi^1_1 + \psi^2 (\Psi^1_2 + \Psi^1_3) + 2 \psi \Psi^3_1 + 2 \psi^2 \right) &+ \frac{1}{r^2} \left( 2 \psi \Psi^1_1 + 2 \cot \theta \Psi^1_2 - \psi \Psi^1_3 + 2 \psi^2 \right) &+ \frac{1}{r^2} \left( 2 \psi \Psi^1_1 + 2 \cot \theta \Psi^1_2 - \psi \Psi^1_3 + 2 \psi^2 \right) &- \frac{2}{r} \partial_r \psi \Psi^3_1 = \lambda \Psi^1_1.
\end{align*}
\]

Fig. 2. Wave function $f^1_3$ for the ground state with eigenvalue $\lambda_0 = 0.0293, L = 6\pi$. 
\[ + \frac{2}{r} \partial_r \psi \Psi_3^1 = \lambda \Psi_1^1, \]  

(23)

\[ (\hat{\Delta} \Psi)^3_1 + \frac{2}{r^2} \partial_r \psi \Psi_2^3 + \frac{2}{r^2} \psi \partial_r \Psi_1^1 + \frac{1}{r^2} \left( 2 \cot \theta \psi \Psi_1^1 + 2(1 + \psi^2) \Psi_3^1 - 2 \psi (\Psi_2^1 + \Psi_3^2) \right) \]

\[ + \frac{2}{r} \partial_r \psi (\Psi_2^1 + \Psi_3^2) = \lambda \Psi_1^1, \]

(24)

\[ (\hat{\Delta} \Psi)^3_3 + \frac{1}{r^2} \left( 2 \cot \theta (\psi \Psi_3^2 - \Psi_1^1) - 2 \psi \Psi_1^1 + \psi^2 (2 \Psi_2^1 + \Psi_3^2) - 2 \csc^2 \theta (\Psi_2^1 - \Psi_3^2) - \Psi_3^2 \right) \]

\[ + \frac{2}{r} \partial_r \psi \Psi_1^1 = \lambda \Psi_3^2, \]

where

\[ \hat{\Delta} \Psi_m^a \equiv - \left( \partial_t^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_\theta + \frac{\cot \theta}{r^2} \partial_\theta \right) \Psi_m^a. \]

To solve numerically the systems of equations (I) and (II) we choose a rectangular three-dimensional domain \((0 \leq t \leq 2\pi, r_0 \leq r \leq L, 0 \leq \theta \leq \pi)\) and use the same interpolating function for the monopole solution \(\psi(r,t)\) as before. The numeric solution of the system of equations (I) in Eq. (22) implies that the lowest eigenvalue is positive, \(\lambda_1 = 0.0531\), and the corresponding eigenfunctions have the following properties: the functions \(\Psi_1^1\) and \(\Psi_3^3\) vanish identically, and the remaining two functions are related by the constraint \(\Psi_1^1 = -\Psi_3^2\). Thus there is only one independent non-vanishing function which can be chosen as \(\Psi_2^1\). An important feature of the solution corresponding to the lowest eigenvalue is that the functions \(\Psi_1^1\) and \(\Psi_3^3\) do not depend on the polar angle (Fig. 3).

This allows us to simplify the system of equations (I) in the case of the ground state solutions with the lowest eigenvalues. One can easily verify that the system of equations (I) in Eq. (22) reduces to one partial differential equation on two-dimensional space-time:

\[ \left( -\partial_t^2 - \frac{2}{r} \partial_r + \frac{1}{r^2} (3 \psi^2 - 1) \right) \Psi_2^3 = \lambda \Psi_3^2. \]

(25)

**Fig. 3.** Eigenfunction \(\Psi_2^2\) for the ground state with the lowest eigenvalue \(\lambda_1 = 0.0531, a_0 = 0.895, A_0 = 0.615, \)

\(0 \leq r \leq 6\pi, 0 \leq t \leq 2\pi, 0 \leq \theta \leq \pi.\)
The last equation represents a simple Schrödinger type equation for a quantum mechanical problem. The equation does not admit negative eigenvalues if the parameter $a_0$ of the monopole solution satisfies the condition $a_0 \geq 1/\sqrt{3} \approx 0.577\ldots$, which provides a totally repulsive quantum mechanical potential in this equation.

The structure of the system of equations (II) in Eq. (23) admits a similar factorization property on the space of ground state solutions. We have solved equations (II) numerically with the same background monopole function $\psi(r,t)$ for various values of the parameters $a_0, A_0,$ and $M$. In the special case of $a_0 = 0.895, A_0 = 0.615, g = 1, M = 1, 0 \leq r \leq 6\pi$ the numeric solution obtained for the ground state has a lowest eigenvalue $\lambda_{II} = 0.0142$, which is less than $\lambda_{I}$. All the components of the solution do not have a dependence on the polar angle and satisfy the following relationships: $\psi_2^1 = \psi_2^3$ and $\psi_1^2 = \psi_1^3 = 0$. There are two independent non-vanishing functions that can be chosen as $\psi_1^2$ and $\psi_3^2$. One can check that on the space of solutions corresponding to the lowest eigenvalue the system of equations (II) is reduced to two coupled partial differential equations for two functions $\psi_1^2(r,t)$ and $\psi_3^2(r,t)$,

$$
\begin{align*}
&\left(-\partial_t^2 - \partial_r^2 - \frac{2}{r} \partial_r\right) \psi_1^2 + \frac{2}{r^2}\left((1 + \psi^2)\psi_1^3 - 2\psi \psi_2^3\right) + \frac{4}{r} \partial_r \psi \psi_3^2 = \lambda \psi_1^1, \\
&\left(-\partial_t^2 - \partial_r^2 - \frac{2}{r} \partial_r\right) \psi_3^2 + \frac{1}{r^2}\left((3\psi^2 - 1)\psi_3^3 - 2\psi \psi_1^3\right) + \frac{2}{r} \partial_r \psi \psi_1^3 = \lambda \psi_3^2.
\end{align*}
$$

(26)

Exact numeric solution profiles for the functions $\psi_1^2$ and $\psi_3^2$ are shown in Fig. 4.

We have obtained that the lowest eigenvalue is positive when the asymptotic monopole amplitude is less than a critical value $a_{1\text{cr}} \simeq 0.625$. We conclude that since the system of equations (II) has a ground state solution with a lower eigenvalue than the ground state solution to system (I), it is enough to study the properties of the couple of equations in Eq. (26) defined on two-dimensional space-time.

Note that numeric solving of the original eigenvalue equations in Eq. (13) on a three-dimensional domain does not provide high numeric accuracy, especially in the limit of large values of the size $L$ of the numeric domain along the radial direction. Remember that one should prove the positiveness of the ground state eigenvalues in the limit of infinite space when the eigenvalues become very close.
to zero. Solving the reduced equations in Eq. (26) defined on two-dimensional space-time can be performed easily using standard numeric packages with sufficient accuracy. The obtained numeric accuracy for the eigenvalues \( \lambda(L) \) in solving the two-dimensional equations in Eq. (26) is \( 1.0 \times 10^{-5} \), which is enough to construct the eigenvalue dependence on the size \( L \) of the box in the range \( 6\pi \leq L \leq 64\pi \).

### 4. Vacuum stability of the monopole field in \( SU(3) \) QCD

The analysis of the quantum stability of the stationary monopole solution performed in the previous section can be generalized straightforwardly to the case of a pure \( SU(3) \) QCD. In general, the system of two partial differential equations in Eq. (3) admits non-stationary and quasi-stationary solutions. We are interested in stationary monopole solutions which can be obtained in a full analogy with \( SU(2) \) QCD by using a constraint \( \psi_1 = \psi_2 = \tilde{\psi} \). The Schrödinger type equation in Eq. (13) for possible unstable modes corresponding to quantum gluon fluctuations contains 24 partial differential equations. Since the initial ansatz in Eq. (2) describes a monopole solution corresponding to \( I \)- and \( U \)-type \( SU(2) \) subgroups, one expects that the factorization property and reduction of the equations on the space of ground state solutions will take place in a similar manner to the case of \( SU(2) \) QCD. Indeed, the numeric analysis of ground state solutions of the full \( SU(3) \) Schrödinger equation in Eq. (13) with a charge-two monopole background field \( \tilde{\psi} \) implies factorization of the equations in each sector of the \( I, U \) \( SU(2) \) subgroups. The obtained numeric solution with the lowest eigenvalue has a basis which contains six non-vanishing functions, \( \tilde{\psi}_1, \tilde{\psi}_3, \tilde{\psi}_2^1 = \tilde{\psi}_2^0 = \tilde{\psi}_3^1 = \tilde{\psi}_3^2, \tilde{\psi}_1^3 = \sqrt{3}\tilde{\psi}_1^3, \) and all other functions vanish identically. An important feature of the numeric solution is that all the solution profile functions do not depend on the polar angle \( \theta \). One can choose the functions \( \tilde{\psi}_1^3 \) and \( \tilde{\psi}_3^2 \) as independent ones and verify that the 24 equations in the Schrödinger type equation of Eq. (13) reduce to a simple system of two independent partial differential equations on two-dimensional space-time \((r, t)\) in a consistent manner:

\[
\begin{align*}
\partial_t^2 \tilde{\psi}_3^3 + \frac{2}{r} \partial_r \tilde{\psi}_3^3 + \tilde{\psi}_3^3 - \frac{1}{r^2} \left( (2\tilde{\psi}^2 + 1)\tilde{\psi}_1^3 - 2(\tilde{\psi} - r\tilde{\psi}_r)\tilde{\psi}_3^3 \right) &= 0, \\
\partial_t^2 \tilde{\psi}_1^3 + \frac{2}{r} \partial_r \tilde{\psi}_1^3 + \tilde{\psi}_1^3 - \frac{1}{r^2} \left( (3\tilde{\psi}^2 - 2)\tilde{\psi}_2^3 - 4(\tilde{\psi} - r\tilde{\psi}_r)\tilde{\psi}_1^3 \right) &= 0.
\end{align*}
\]

One can show that the structure of the reduced equations is equivalent to the structure of the equations (II) in Eq. (26) obtained by reduction in the case of \( SU(2) \) QCD. Simple rescaling of the function \( \tilde{\psi} = \sqrt{2}\tilde{\psi} \) representing the monopole background field and function renormalization \( \tilde{\psi}_3^2 = \sqrt{2}\tilde{\psi}_3^2 \) \((\tilde{\psi}_1^3 = \tilde{\psi}_1^3)\) in Eqs. (27) lead exactly to the system of equations (II) in Eq. (26).

We have verified that increasing the size of the space-time numeric domain in the radial direction, \( L \to \infty \), the corresponding eigenvalue \( \lambda(L) \) vanishes asymptotically from positive values. The eigenvalue dependencies \( \lambda(L) \) obtained by solving the approximate, Eq. (18), and exact Schrödinger equations, Eq. (13) in the cases of \( SU(2) \) and \( SU(3) \) QCD are presented in Fig. 5.

Note that in \( SU(3) \) QCD one has a stable monopole field configuration for both stationary monopole solutions, for the embedded \( SU(2) \) monopole with a unit magnetic charge, and for the \( SU(3) \) monopole with the magnetic charge two. The \( SU(3) \) symmetric monopole solution corresponding to two \( SU(2) \) subgroups is preferable since for constant-valued functions \( \psi_1, \psi_2 \) the corresponding classical potential has an absolute minimum at \( \psi_1 = \psi_2 = \sqrt{2} \). This is similar to the behavior of the one-loop effective potential for homogeneous color magnetic fields where the potential has an
Fig. 5. Lowest eigenvalue dependence, $\lambda(L)$, on the radial size $L$ of the numeric domain: approximate results in $SU(2)$ QCD (dotted line), exact numeric results in $SU(3)$ QCD (solid line), exact numeric results in $SU(2)$ QCD (dashed line).

absolute minimum for non-vanishing values of both magnetic fields $H_3$ and $H_8$ corresponding to a Cartan subalgebra of $su(3)$ [42].

5. Conclusion

In our consideration of the stationary monopole solutions we set the mass scale parameter $M$ to unity for simplicity. We would like to stress the importance of the presence of such a parameter and its relation to the microscopic structure of the vacuum. One of the main characteristics of the QCD vacuum is the vacuum gluon condensate, $\langle \vec{F}^2_{\mu\nu} \rangle$. In a first approximation the vacuum gluon condensate represents a constant parameter which describes a mass gap in the confinement phase. Within the framework of the confinement mechanism based on monopole condensation and the dual Meissner effect it is assumed that the vacuum gluon condensate is generated due to condensation of monopoles, i.e., the dominant contribution to the gauge-invariant quantity $\langle \vec{F}^2_{\mu\nu} \rangle$ is made of the color magnetic field. For the present moment an exact vacuum field configuration corresponding to the vacuum monopole condensate is unknown. We expect that the QCD vacuum is formed through the condensation of monopoles and/or monopole pairs since only these objects possess quantum stability locally in a small vicinity of each space point before any averaging procedure over the whole space-time region. Simple consideration shows that the mass scale parameter $M$ determines the microscopic scale of the confinement phase.

Let us consider the structure of the one-loop effective potential of a pure $SU(N)$ QCD with a color magnetic background field which is treated as a constant field [8,33–39]:

$$V_{\text{eff}} = \frac{1}{4} H^2 + \frac{11N g^2(\mu)}{48 \pi^2} H^2 \left( \ln \frac{g(\mu)}{\mu^2} - c \right),$$

where $g(\mu)$ is a renormalized coupling constant defined at the subtraction point $\mu^2 \simeq \Lambda_{\text{QCD}}$. The gauge-invariant quantity $H^2 \equiv \langle \vec{F}^2_{\mu\nu} \rangle$ represents a vacuum-averaged value of the gluon field operator which describes the monopole condensate. Note that in the standard QCD the notion of the vacuum gluon condensate differs from the notion of the electron pair condensate in a superconductor, which is described by a wave function of the Cooper electron pair (we do not consider the phenomenological approach to QCD based on Ginsburg–Landau type models). The effective potential has a non-trivial minimum at the non-zero value of the vacuum magnetic field $\langle H \rangle \simeq 0.14\mu^2$, which fixes the scale
of the microscopic structure of the vacuum monopole condensate. In the confinement phase one assumes that the periodic structure of the classical monopole field configuration is characterized by the length parameter \( \lambda_M = \frac{2\pi}{M} \), which should be less than the inverse deconfinement temperature parameter \( kT_{\text{dec}} \), i.e., \( 2\pi/M \leq \beta_{\text{dec}} = 1/kT_{\text{dec}} \). Under this condition one can perform space-time averaging of the classical monopole configuration and estimate a lower bound \( M^2_{\text{bound}} \approx 1.2 \mu^2 \), which is of the same order as the parameter \( \Lambda_{\text{QCD}} \). In the confinement phase, \( T \approx 0 \), the vacuum averaging value of the gluon field operator \( \langle 0 | A_\mu^a | 0 \rangle \) vanishes since the effective size \( \beta = 1/kT \) of the time interval in the Euclidean functional integral of the effective action becomes much larger than \( \lambda_M \). With increasing temperature the value of the parameter \( \beta \) becomes comparable, and less than the periodic scale \( \lambda_M \) of the vacuum monopole field configuration. This implies that the vacuum averaging value of the gluon field operator, \( \langle 0 | A_\mu^a | 0 \rangle \), becomes non-vanishing, which leads to the deconfinement phase with spontaneous symmetry breaking, and the gluon can be observed as a color object.

In conclusion, we have demonstrated that the stationary spherically symmetric monopole solution represents a stable quasi-classical vacuum field background under small gluon fluctuations. Certainly, our consideration is restricted by one-loop consideration within the formalism of the effective action. Note that the classical monopole solution is non-perturbative, and it is valid for arbitrary values of the coupling constant. Also, the one-loop effective action includes a non-perturbative contribution due to summation over all one-loop Feynman diagrams corresponding to interaction of the background monopole field with virtual gluons. This gives the hope that a modified monopole solution to quantum equations of motion obtained beyond one-loop approximation will admit the quantum stability as well.

It has recently been shown that the spherically symmetric monopole solution is rather classically unstable under axially symmetric deformations of the initial spherically symmetric ansatz [23]. This might cause some doubts as to whether spherically symmetric monopoles can serve as a structure element of a true vacuum. One should note that the classical stability of the multi-monopole system represents a non-trivial problem due to the presence of interaction between the monopoles. Another candidate for a stable structure element in the formation of a stable vacuum has been proposed in [43], where it has been shown that a monopole–antimonopole pair solution is stable under quantum gluon fluctuations. We expect that monopole and/or monopole–antimonopole pair condensation can be realized in QCD in analogy with the Cooper electron pair condensation in an ordinary superconductor, as was conjectured in the seminal papers a long time ago [3–6]. This issue will be considered in a separate paper.

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