Constrained reinforcement learning is to maximize the expected reward subject to constraints on utilities/costs. However, the training environment may not be the same as the test one, due to, e.g., modeling error, adversarial attack, non-stationarity, resulting in severe performance degradation and more importantly constraint violation. We propose a framework of robust constrained reinforcement learning under model uncertainty, where the MDP is not fixed but lies in some uncertainty set, the goal is to guarantee that constraints on utilities/costs are satisfied for all MDPs in the uncertainty set, and to maximize the worst-case reward performance over the uncertainty set. We design a robust primal-dual approach, and further theoretically develop guarantee on its convergence, complexity and robust feasibility. We then investigate a concrete example of $\delta$-contamination uncertainty set, design an online and model-free algorithm and theoretically characterize its sample complexity.

1 Introduction

In many practical reinforcement learning (RL) applications, it is critical for an agent to meet certain constraints on utilities and costs while maximizing the reward. However, in practice, it is often the case that the evaluation environment deviates from the training one, due to, e.g., modeling error of the simulator, adversarial attack, and non-stationarity. This could lead to a significant performance degradation in reward, and more importantly, constraints may not be satisfied anymore, which is severe in safety-critical applications. For example, a drone may run out of battery due to model deviation between the training and test environments, resulting in a crash. To solve these issues, we propose a framework of robust constrained RL under model uncertainty. Specifically, the Markov decision process (MDP) is not fixed and lies in an uncertainty set [Nilim and El Ghaoui, 2004], [Iyengar, 2005], [Bagnell et al., 2001], and the goal is to maximize the worst-case accumulative discounted reward over the uncertainty set while guaranteeing that constraints are satisfied for all MDPs in the uncertainty set at the same time.

Despite of its practical importance, studies on the problem of robust constrained RL are limited in the literature. Two closely related topics are robust RL [Bagnell et al., 2001], [Nilim and El Ghaoui, 2004], [Iyengar, 2005] and constrained RL [Altman, 1999]. The problem of constrained RL [Altman, 1999] aims to find a policy that optimizes an objective reward while satisfying certain constraints on costs/utilities. For the problem of robust RL [Bagnell et al., 2001], [Nilim and El Ghaoui, 2004], [Iyengar, 2005], the MDP is not fixed but lies in some uncertainty set, and the goal is to find a policy that optimizes the robust value function, which measures the worst-case accumulative reward over the uncertainty set. The problem of robust constrained RL was investigated in Russel et al. [2020], Mankowitz et al. [2020], where two heuristic approaches were proposed. The basic idea in Russel et al. [2020], Mankowitz et al. [2020] is to first evaluate the worst-case performance of the policy over the uncertainty set, and then use that together with classical policy improvement methods, e.g., policy gradient [Sutton et al., 1999], to update the policy. However, as will be discussed in more details later, these approaches may not necessarily lead to an improved policy, and thus may not perform well in practice.

In this paper, we design the robust primal-dual algorithm for the problem of robust constrained RL. Our approach employs the true gradient of the Lagrangian function, which is the weighted sum of two robust value functions, instead of approximating the gradient heuristically as in Russel et al. [2020]. We theoretically characterize the convergence
and complexity of our robust primal-dual method, and prove the robust feasibility of our solution for all MDPs in the uncertainty set. We further present a concrete example of δ-contamination uncertainty set [Huber 1965, Du et al. 2018, Huber and Ronchetti 2009, Nishimura and Ozaki 2004, 2006, Prasad et al. 2020a,b, Wang and Zou 2021, 2022], for which we extend our algorithm to the online and model-free setting, and theoretically characterize its finite-time error bound. To the best of the authors’ knowledge, our work is the first in the literature of robust constrained RL that comes with model-free algorithms, theoretical convergence guarantee, complexity analyses, and robust feasibility guarantee. In particular, the technical challenges and our major contributions are summarized as follows.

• In the non-robust setting, the sum of two value functions is actually a value function of the combined reward. However, this does not hold in the robust setting, since the worst-case transition kernels for the two robust value functions are not necessarily the same. Therefore, the geometry of our Lagrangian function is much more complicated. In this paper, we formulate the dual problem of the robust constrained RL problem as a minimax linear-nonconcave optimization problem, and show that the optimal dual variable is bounded. We then construct a robust primal-dual algorithm by alternatively updating the primal and dual variables. We theoretically prove the convergence to stationary points, and characterize its complexity.

• In general, convergence to stationary points of the Lagrangian function does not necessarily imply that the solution is feasible. We design a novel proof to show that the gradient belongs to the normal cone of the feasible set, based on which we further prove the feasibility of the obtained policy.

• Based on existing literature on constrained MDP [Ding et al. 2020, 2021, Li et al. 2021a, Liu et al. 2021, Ying et al. 2021] and robust MDP [Wang and Zou 2022], at first, we expect that the robust constrained RL also has zero duality gap, and further global optimum can be achieved. However, this is not necessarily true. Note that the set of visitation distribution being convex is one key property to show zero duality gap of constrained MDP [Altman 1999, Paternain et al. 2019]. In this paper, we constructed a novel counter example showing that the set of robust visitation distributions for our robust problem is non-convex.

• We further apply and extend our results on an important uncertainty set referred as δ-contamination model [Huber 1965]. Under this model, the robust value functions are not differentiable and we hence propose a smoothed approximation of the robust value function towards a better geometry. We further investigate the practical online and model-free setting and design an actor-critic type algorithm. We also establish its convergence, sample complexity, and robust feasibility.

We then discuss works related to robust constrained RL.

Robust constrained RL. In Russel et al. 2020, the robust constrained RL problem was studied, and a heuristic approach was developed. The basic idea is to estimate the robust value functions, and then to use the vanilla policy gradient method [Sutton et al. 1999] with the vanilla value function replaced by the robust value function. However, this approach did not take into consideration the fact that the worst-case transition kernel is also a function of the policy (see Section 3.1 in Russel et al. 2020), and therefore the "gradient" therein is not actually the gradient of the robust value function. Thus, its performance and convergence cannot be theoretically guaranteed. The other work Mankowitz et al. 2020 studied the same robust constrained RL problem under the continuous control setting, and proposed a similar heuristic algorithm. They first proposed a robust Bellman operator and used it to estimate the robust value function, which is further combined with some non-robust continuous control algorithm to update the policy. Both approaches in Russel et al. 2020 and Mankowitz et al. 2020 inherit the heuristic structure of "robust policy evaluation" + "non-robust vanilla policy improvement", which may not necessarily guarantee an improved policy in general. In this paper, we employ a "robust policy evaluation" + "robust policy improvement" approach, which guarantees an improvement in the policy, and more importantly, we provide theoretical convergence guarantee, robust feasibility guarantee, and complexity analysis for our algorithms.

Constrained RL. The most commonly used method for constrained RL is the primal-dual method [Altman 1999, Paternain et al. 2019, 2022, Liang et al. 2018, Stooke et al. 2020, Tessler et al. 2018, Yu et al. 2019, Zheng and Ratliff 2020, Efroni et al. 2020, Auer et al. 2008], which augments the objective with a sum of constraints weighted by their corresponding Lagrange multipliers, and then alternatively updates the primal and dual variables. It was shown that the strong duality holds for constrained RL, and hence the primal-dual method has zero duality gap [Paternain et al. 2019, Altman 1999]. The convergence rate of the primal-dual method was investigated in Ding et al. 2020, 2021, Li et al. 2021a, Liu et al. 2021, Ying et al. 2021. Another class of method is the primal method, which is to enforce the constraints without resorting to the Lagrangian formulation [Achiam et al. 2017, Liu et al. 2020, Chow et al. 2018, Dalal et al. 2018, Xu et al. 2021, Yang et al. 2020]. The above studies, when directly applied to robust constrained RL, cannot guarantee the constraints when there is model deviation. Moreover, the objective and constraints in this paper take min over the uncertainty set (see (2)), and therefore have much more complicated geometry than the non-robust case.
Robust RL under model uncertainty. Model-based robust RL was firstly introduced and studied in [Iyengar, 2005]. Nilim and El Ghaoui [2004], Bagnell et al. [2001], Satia and Lave Jr. [1973], Wiesemann et al. [2013], Lim and Aute [2019], Xu and Mannor [2010], Yu and Xu [2015], Lim et al. [2013], Tamar et al. [2014], where the uncertainty set is assumed to be known, and only samples from its centroid can be collected by Roy et al. [2017], Wang and Zou [2021, 2022], Zhou et al. [2021], Yang et al. [2021], Pananganti and Kalathil [2021], Ho et al. [2018, 2021]. There are also empirical studies on robust RL, e.g., Vintzileos et al. [2020], Pinto et al. [2017], Abdullah et al. [2019], Hou et al. [2020], Rajeswaran et al. [2017], Huang et al. [2017], Kos and Song [2017], Lin et al. [2017], Pattanaik et al. [2018], Mandelkar et al. [2017]. These works focus on robust RL without constraints, whereas in this paper we investigate robust RL with constraints, which is more challenging.

There is a related line of works on (robust) imitation learning: Ho and Ermon [2016], Fu et al. [2017], Torabi et al. [2018], Viano et al. [2022], which can be formulated as a constrained problem. But their problem settings and approaches are fundamentally different from ours.

2 Preliminaries

Constrained MDP. A constrained MDP (CMDP) can be specified by a tuple \((\mathcal{S},\mathcal{A}, P, r, c_1, \ldots, c_m, \gamma)\), where \(\mathcal{S}\) and \(\mathcal{A}\) denote the state and action spaces, \(P = \{p^s_{a} \in \Delta_S | a \in \mathcal{A}, s \in \mathcal{S}\}\) is the transition kernel, \(r : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]\) is the reward function, \(c_i : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1], i = 1, \ldots, m\) are utility functions in the constraint, and \(\gamma \in [0, 1]\) is the discount factor. A stationary policy \(\pi\) is a mapping \(\pi : \mathcal{S} \rightarrow \Delta_{\mathcal{A}},\) where \(\pi(a|s)\) denotes the probability of taking action \(a\) when the agent is at state \(s\). The set of all the stationary policies is denoted by \(\Pi\).

The non-robust value function of reward \(r\) and a policy \(\pi\) is defined as the expected accumulative discounted reward if the agent follows policy \(\pi\): 

\[
\mathbb{E}_{\pi,p}[\sum_{t=0}^{\infty} \gamma^t r(S_t, A_t)|S_0 = s],
\]

where \(\mathbb{E}_{\pi,p}\) denotes the expectation when the policy is \(\pi\) and the transition kernel is \(P\). Similarly, the non-robust value function of \(c\) is defined as 

\[
\mathbb{E}_{\pi,p}[\sum_{t=0}^{\infty} \gamma^t c_i(S_t, A_t)|S_0 = s].
\]

The goal of CMDP is to find a policy that maximizes the expected reward subject to constraints on the expected utility:

\[
\max_{\pi \in \Pi} \mathbb{E}_{\pi,p}[\sum_{t=0}^{\infty} \gamma^t r(S_t, A_t)|S_0 = s] \geq \rho,
\]

where \(\rho\) is some positive threshold and \(\rho\) is the initial state distribution.

Define the visitation distribution induced by policy \(\pi\) and transition kernel \(P\): 

\[
d_{\pi,P}^s(a) = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t p_{S_t = s, A_t = a|S_0 = s}.\]

It can be shown that the set of the visitation distributions of all policies \(\{d_{\pi,P}^s \in \Delta_{\mathcal{S} \times \mathcal{A}} : \pi \in \Pi\}\) is convex [Paternain et al., 2022, Altman 1999]. A standard assumption in the literature is the Slater’s condition [Bertsekas, 2014, Ding et al., 2021]. There exists a constant \(\zeta > 0\) and a policy \(\pi \in \Pi\) s.t. \(\forall i, \mathbb{E}_{\pi,p}[\sum_{t=0}^{\infty} \gamma^t c_i(S_t, A_t)|S_0 = s] - b_i \geq \zeta\). Based on the convexity of the set of all visitation distributions and Slater’s condition, strong duality can be established [Altman 1999, Paternain et al. 2019].

Robust MDP. A robust MDP can be specified by a tuple \(\{\mathcal{S}, \mathcal{A}, P, r, \gamma\}\). In this paper, we focus on the \((s, a)\)-rectangular uncertainty set [Nilim and El Ghaoui 2004, Iyengar 2005], i.e., \(P = \bigotimes_{s,a} \Delta_{p_s^a}\), where \(\Delta_{p_s^a}\) is the probability simplex supported on \(S\). Denote the transition kernel at time \(t\) by \(P_t\), and let \(\kappa = \{P_0, P_1, \ldots\}\) be the dynamic model, where \(P_t \in P, \forall t \geq 0\). We then define the robust value function of a policy \(\pi\) as the worst-case expected accumulative discounted reward following policy \(\pi\) over all MDPs in the uncertainty set [Nilim and El Ghaoui 2004, Iyengar 2005]:

\[
V_{\pi}^r(s) \triangleq \min_{\kappa \in \mathbb{K}} \mathbb{E}_{\kappa} \mathbb{E}_{\kappa} \sum_{t=0}^{\infty} \gamma^t r(S_t, A_t)|S_0 = s, \pi,
\]

where \(\mathbb{E}_{\kappa}\) denotes the expectation when the state transits according to \(\kappa\). It has been shown that the robust value function is the fixed point of the robust Bellman operator [Nilim and El Ghaoui 2004, Iyengar 2005].

Puterman 2014: 

\[
\pi^* V(s) \triangleq \sum_{a \in \mathcal{A}} \pi(a|s) (r(s,a) + \gamma \sigma_{p^a_\pi}(V)), \quad \sigma_{p^a_\pi}(V) \triangleq \min_{p \in \Delta_{p_s^a}} p^\top V
\]

is the support function of \(V\) on \(p_s^a\). Similarly, we can define the robust action-value function for a policy \(\pi\): 

\[
Q_{\pi}^r(s, a) = \min_{\kappa} \mathbb{E}_{\kappa} \mathbb{E}_{\kappa} \sum_{t=0}^{\infty} \gamma^t r(S_t, A_t)|S_0 = s, A_0 = a, \pi.
\]

Note that the minimizer of \(Q_{\pi}^r(s, a)\) is stationary in time [Iyengar 2005], which we denote by \(\kappa^* = \{P^\pi, P^\pi, \ldots\}\), and refer to \(P^\pi\) as the worst-case transition kernel. Then the robust value function \(V^r_{\pi}\) is actually the value function under policy \(\pi\) and transition kernel \(P^\pi\). The goal of robust RL is to find the optimal robust policy \(\pi^*\) that maximizes the worst-case accumulative discounted reward: 

\[
\pi^* = \arg \max_{\pi} V^r_{\pi}(s), \forall s \in S.
\]
3 Robust Constrained RL

The motivation for the problem of robust constrained MDP has two folds. The first is to guarantee that the constraints are always satisfied even if there is a mismatch between the training and evaluation environments. The second one is that among those feasible policies, we want to find one that optimizes the worst-case reward performance in the uncertainty set. In the following, we formulate the robust constrained problem.

$$\max_{\theta \in \Theta} \min_{\lambda \geq 0} \sum_{i=1}^{m} \lambda_{i}(V_{c_{i}}^{\pi_{\theta}}(\rho) - b_{i}).$$

(4)

Unlike non-robust CMDP, strong duality for robust constrained RL may not hold. For robust RL, the robust value function can be viewed as the value function for policy \( \pi \) under its worst-case transition kernel \( P^{c} \), and therefore can be written as the inner product between the reward (utility) function and the visitation distribution induced by \( \pi \) and \( P^{c} \) (referred to as robust visitation distribution of \( \pi \)). The following lemma shows that the set of robust visitation distributions may not be convex, and therefore, the approach used in [Altman 1999, Paternain et al. 2019] to show strong duality cannot be applied here.

**Lemma 1.** There exists a robust MDP, such that the set of robust visitation distributions is non-convex.

In the following, we focus on the dual problem of (4). Due to the weak duality, the optimal solution of the dual problem is a sub-optimal solution of the (4). For simplicity, we investigate the case with one constraint, and extension to the case with multiple constraints is straightforward:

$$\min_{\lambda \geq 0} \max_{\theta \in \Theta} \lambda(V_{c}^{\pi_{\theta}}(\rho) - b).$$

(5)

We make an assumption of Slater’s condition, assuming there exists at least one strictly feasible policy [Bertsekas 2014, Ding et al. 2021], under which, we further show that the optimal dual variable of (5) is bounded.

**Assumption 2.** There exists \( \zeta > 0 \) and a policy \( \pi \in \Pi_{\Theta} \), s.t. \( V_{c}^{\pi}(\rho) - b \geq \zeta \).

**Lemma 2.** Denote the optimal solution of (5) by \((\lambda^{*}, \pi_{\theta^{*}})\). Then, \( \lambda^{*} \in \left[ 0, \frac{2}{2\zeta(1-\gamma)} \right] \).

Lemma 2 suggests that the dual problem (5) is equivalent to a bounded min-max problem:

$$\min_{\lambda \in \left[ 0, \frac{2}{2\zeta(1-\gamma)} \right]} \max_{\theta \in \Theta} \lambda(V_{c}^{\pi_{\theta}}(\rho) - b).$$

(6)
The problem (6) is a bounded linear-nonconcave optimization problem. We then propose our robust primal-dual algorithm for robust constrained RL in Algorithm 1.

**Algorithm 1 Robust Primal-Dual algorithm (RPD)**

**Input:** $T$, $\alpha_t$, $\beta_t$, $b_t$

**Initialization:** $\lambda_0$, $\theta_0$

**for** $t = 0, 1, ..., T - 1$ **do**

$\lambda_{t+1} \leftarrow \prod_{[0, A^t]} \left( \lambda_t - \frac{1}{\beta_t} \left( V^\pi_{\theta_t}(\rho) - b_t \right) \right)$

$\theta_{t+1} \leftarrow \prod_{\Theta} \left( \theta_t + \frac{1}{\alpha_t} \left( \nabla_\theta V^\pi_{\theta_t}(\rho) + \lambda_{t+1} \nabla_\theta V^\pi_{\theta_t}(\rho) \right) \right)$

**end for**

**Output:** $\theta_T$

The basic idea of Algorithm 1 is to perform gradient descent-ascent w.r.t. $\lambda$ and $\theta$ alternatively. When the policy $\pi$ violates the constraint, the dual variable $\lambda$ increases such that $\lambda V^\pi_{\theta}$ dominates $V^\pi_{\theta}$. Then the gradient ascent will update $\theta$ until the policy satisfies the constraint. Therefore, this approach is expected to find a feasible policy (as will be shown in Lemma 5).

Here, $\prod_X(x)$ denotes the projection of $x$ to the set $X$, and $\{b_t\}$ is a non-negative monotone decreasing sequence, which will be specified later. Algorithm 1 reduces to the vanilla gradient descent-ascent algorithm in Lin et al. [2020] if $b_t = 0$. However, $b_t$ is critical to the convergence of Algorithm 1 [Xu et al. 2020]. The outer problem of (6) is actually linear, and after introducing $b_t$, the update of $\lambda_t$ can be viewed as a gradient descent of a strongly-convex function $\lambda(V_{\theta} - b) + \frac{b_t}{2} \lambda^2$, which converges more stable and faster.

Denote that Lagrangian function by $V^L(\theta, \lambda) \triangleq V^\pi_{\theta}(\rho) + \lambda(V^\pi_{\theta}(\rho) - b)$, and further denote the gradient mapping of Algorithm 1 by

$$G_t \triangleq \begin{bmatrix}
\beta_t \left( \lambda_t - \prod_{[0, A^t]} \left( \lambda_t - \frac{1}{\beta_t} \left( \nabla_L V^L(\theta_t, \lambda_t) \right) \right) \right)

\alpha_t \left( \theta_t - \prod_{\Theta} \left( \theta_t + \frac{1}{\alpha_t} \left( \nabla_\theta V^L(\theta_t, \lambda_t) \right) \right) \right)
\end{bmatrix}. \tag{7}$$

The gradient mapping is a standard measure of convergence for projected optimization approaches Beck [2017]. Intuitively, it reduces to the gradient $(\nabla_L V^L, \nabla_\theta V^L)$, when $A^t = \infty$ and $\Theta = \mathbb{R}^d$, and it measures the updates of $\theta$ and $\lambda$ at time step $t$. If $\|G_t\| \to 0$, the updates of both variables are small, and hence the algorithm converges to a stationary solution.

To show the convergence of Algorithm 1, we make the following Lipschitz smoothness assumption.

**Assumption 3.** The gradients of the Lagrangian function are Lipschitz:

$$\| \nabla_L V^L(\theta, \lambda)|_{\theta_1} - \nabla_L V^L(\theta, \lambda)|_{\theta_2} \| \leq L_1 |\theta_1 - \theta_2|, \tag{8}$$

$$\| \nabla_\lambda V^L(\theta, \lambda)|_{\lambda_1} - \nabla_\lambda V^L(\theta, \lambda)|_{\lambda_2} \| \leq L_2 |\lambda_1 - \lambda_2|, \tag{9}$$

$$\| \nabla_\theta V^L(\theta, \lambda)|_{\theta_1} - \nabla_\theta V^L(\theta, \lambda)|_{\theta_2} \| \leq L_3 |\theta_1 - \theta_2|, \tag{10}$$

$$\| \nabla_\lambda V^L(\theta, \lambda)|_{\lambda_1} - \nabla_\lambda V^L(\theta, \lambda)|_{\lambda_2} \| \leq L_4 |\lambda_1 - \lambda_2|. \tag{11}$$

In general, Assumption 3 may or may not hold depending on the uncertainty set model. As will be shown in Section 4, even if Assumption 3 does not hold, we can design a smoothed approximation of the robust value function, so that the assumption holds for the smoothed problem.

In the following theorem, we show that our robust primal-dual algorithm converges to a stationary point of the min-max problem (14), with a complexity of $O(\epsilon^{-4}).$

**Theorem 1.** Under Assumption 3 if we set step sizes $\alpha_t$, $\beta_t$, and $b_t$ as in Section 2 and $T = O(\epsilon^{-4})$, then $\min_{1 \leq t \leq T} \|G_t\| \leq 2\epsilon$.}

The next proposition characterizes the feasibility of the obtained policy.

**Proposition 1.** Denote by $W \triangleq \arg \min_{1 \leq t \leq T} \|G_t\|$. If $\lambda_W - \frac{1}{\beta_W} \left( \nabla_\lambda V^L_W(\theta_W, \lambda_W) \right) \in [0, A^*)$, then $\pi_W$ satisfies the constraint with a $2\epsilon$-violation.
In general, convergence to stationary points of the Lagrangian function does not necessarily imply that the solution is feasible. Proposition 1 shows that Algorithm 1 always return a policy that is robust feasible, i.e., satisfying the constraints in (3). Intuitively, if we set \( \Lambda^* \) larger so that the optimal solution \( \lambda^* \in [0, \Lambda^*] \), then Algorithm 1 is expected to converge to an interior point of \([0, \Lambda^*]\) and therefore, \( \pi_{\lambda^*} \) is feasible. On the other hand, \( \Lambda^* \) can’t be set too large. Note that the complexity in Theorem 1 depends on \( \Lambda^* \) (see (52) in the appendix), and a larger \( \Lambda^* \) means a higher complexity.

4 \( \delta \)-Contamination Uncertainty Set

In this section, we investigate a concrete example of robust constrained RL with \( \delta \)-contamination uncertainty set. The method we developed here can be similarly extended to other type of uncertainty sets like KL-divergence or total variation. The \( \delta \)-contamination uncertainty set models the scenario where the state transition of the MDP could be arbitrarily perturbed with a small probability \( \delta \). This model is widely used to model distributional uncertainty in the literature of robust learning and optimization, e.g., Huber [1965], Du et al. [2018], Huber and Ronchetti [2009], Nishimura and Ozaki [2004, 2006], Prasad et al. [2020a, b], Wang and Zou [2021, 2022]. Specifically, let \( \mathcal{P} = \{ p_\sigma \mid s \in \mathcal{S}, a \in \mathcal{A} \} \) be the centroid transition kernel, then the \( \delta \)-contamination uncertainty set centered at \( \mathcal{P} \) is defined as \( \mathcal{P} \triangleq \bigotimes_{s \in \mathcal{S}, a \in \mathcal{A}} \mathcal{P}_s^a \), where \( \mathcal{P}_s^a \triangleq \{(1 - \delta)p_s^a + \delta|q| \in \Delta\} \), \( s \in \mathcal{S}, a \in \mathcal{A} \). Under the \( \delta \)-contamination setting, the robust Bellman operator can be explicitly computed: \( \mathcal{T}_\pi V(s) = \sum_{a \in \mathcal{A}} \pi(a|s)(r(s, a) + \gamma(\delta \min_{s'} V(s') + (1 - \delta) \sum_{s' \in \mathcal{S}} p_{s,s'}^a V(s'))) \). In this case, the robust value function is non-differentiable due to the min term, and hence Assumption 3 does not hold. One possible approach is to use sub-gradient [Clarke 1990], Wang and Zou [2022], which, however, is less stable, and its convergence is difficult to characterize. In the following, we design a differentiable and smooth approximation of the robust value function.

Specifically, consider a smoothed robust Bellman operator \( \mathcal{T}_\pi^\sigma \) using the LSE function [Wang and Zou 2021, 2022]:

\[
\mathcal{T}_\pi^\sigma V(s) = \mathbb{E}_{A \sim \pi(s)} \left[ r(s, A) + \gamma(1 - \delta) \sum_{s' \in \mathcal{S}} p_{s,s'}^A V(s') + \gamma \delta \text{LSE}(\sigma, V) \right],
\]

where LSE(\( \sigma, V \)) = \( \frac{\log(\sum_{s \in \mathcal{S}} e^{V(s)})}{\sum_{s \in \mathcal{S}} e^{V(s)}} \) for \( V \in \mathbb{R}^d \) and some \( \sigma < 0 \). The approximation error \( |\text{LSE}(\sigma, V) - \min V| \rightarrow 0 \) as \( \sigma \rightarrow -\infty \), and hence the fixed point of \( \mathcal{T}_\pi^\sigma \), denoted by \( V_\pi^\sigma \), is an approximation of the robust value function \( V_\pi \) [Wang and Zou 2022]. We refer to \( V_\pi^\sigma \) as the smoothed robust value function and define the smoothed robust action-value function as \( Q_\pi^\sigma(s, a) \triangleq r(s, a) + \gamma(1 - \delta) \sum_{s' \in \mathcal{S}} p_{s,s'}^a V_\pi^\sigma(s') + \gamma \delta \text{LSE}(\sigma, V_\pi^\sigma) \). It can be shown that for any \( \pi \), as \( \sigma \rightarrow -\infty \), \( \| V_\pi^\sigma - V_\pi \| \rightarrow 0 \) and \( \| V_\pi^\sigma - V_{\pi,\sigma,c}^\pi \| \rightarrow 0 \) [Wang and Zou 2021].

The gradient of \( V_\pi^\sigma \) can be computed explicitly [Wang and Zou 2022]:

\[
\nabla V_\pi^\sigma(s) = B(s, \theta) + \gamma \delta \sum_{s' \in \mathcal{S}} \theta(s') \sum_{s \in \mathcal{S}} \nabla \pi_\theta(a|s) Q_\pi^\sigma(s', a) d_{\pi,\theta}^s(\cdot),
\]

where \( B(s, \theta) \triangleq \frac{1}{1 - \gamma \theta_0} \sum_{s' \in \mathcal{S}} d_{\pi,\theta}^s(s') \sum_{s \in \mathcal{S}} \nabla \pi_\theta(a|s) Q_\pi^\sigma(s', a) \), and \( d_{\pi,\theta}^s(\cdot) \) is the visitation distribution of \( \pi_\theta \) under \( \mathcal{P} \) starting from \( s \). Denote the smoothed Lagrangian function by \( V_{\pi,\lambda}(\theta, \lambda) \triangleq V_{\pi,\sigma,c}(\rho) + \lambda(\| V_{\pi,\sigma,c}(\rho) - b \|) \). The following lemma shows that \( \nabla V_{\pi,\lambda} \) is Lipschitz.

**Lemma 3.** \( \nabla V_{\pi,\lambda}^L \) is Lipschitz in \( \theta \) and \( \lambda \). And hence Assumption 3 holds for \( V_{\pi,\lambda}^L \).

A natural idea is to use the smoothed robust value functions to replace the ones in (5):

\[
\min_{\lambda \geq 0} \max_{\pi \in \mathcal{P}} V_{\pi,\sigma,c}(\rho) + \lambda(\| V_{\pi,\sigma,c}(\rho) - b \|).
\]

As will be shown below in Lemma 6, this approximation can be arbitrarily close to the original problem in (5) as \( \sigma \rightarrow -\infty \). We first show that under Assumption 2, the following Slater’s condition holds for the smoothed problem in (13).

**Lemma 4.** Let \( \sigma \) be sufficiently small such that \( \| V_{\pi,\sigma,c}^\pi - V_{\pi,\sigma,c}^\sigma \| < \zeta \) for any \( \pi \), then there exists \( \zeta' > 0 \) and a policy \( \pi' \in \Pi(\Theta) \) s.t. \( V_{\pi',\sigma,c}(\rho) - b \geq \zeta' \).

The following lemma shows that the optimal dual variable for (13) is also bounded.

**Lemma 5.** Denote the optimal solution of (13) by \((\lambda^*, \pi_{\lambda^*})\). Then \( \lambda^* \in \left[ 0, \frac{2C_\sigma}{\zeta} \right] \), where \( C_\sigma \) is the upper bound of smoothed robust value functions \( V_{\pi,\sigma,c}^\pi \).

Denote by \( \Lambda^* = \max \left\{ \frac{2C_\sigma}{\zeta}, \frac{2}{\zeta(1 + \gamma)} \right\} \), then problems (6) and (13) are equivalent to the following bounded ones:

\[
\min_{\lambda \in [0, \Lambda^*]} \max_{\pi \in \mathcal{P}} V_{\pi,\sigma,c}(\rho) + \lambda(\| V_{\pi,\sigma,c}(\rho) - b \|),
\]

\[
\min_{\lambda \in [0, \Lambda^*]} \max_{\pi \in \mathcal{P}} V_{\pi,\lambda}(\theta, \lambda) + \lambda(\| V_{\pi,\lambda}(\theta, \lambda) - b \|).
\]

(14)
The following lemma shows that the two problems are within a gap of $O(\epsilon)$.

**Lemma 6.** Choose a small enough $\sigma$ such that $\|V^* - \tilde{V}_\sigma\| \leq \epsilon$ and $\|V^* - V_{\sigma,c}^\pi\| \leq \epsilon$. Then

$$\min_{\lambda \in [0,\Lambda^*]} \max_{\pi \in \Pi_{\theta}} V^{\pi}\sigma(\rho) + \lambda(V^*_{\sigma,c}(\rho) - b) - \min_{\lambda \in [0,\Lambda^*]} \max_{\pi \in \Pi_{\theta}} \tilde{V}^\pi_{\sigma,c}(\rho) + \lambda(V^*_{\sigma,c}(\rho) - b) \leq (1 + \Lambda^*) \epsilon.$$ 

In the following, we hence focus on the smoothed dual problem in (14), which is an accurate approximation of the original problem (6). Denote the gradient mapping of the smoothed Lagrangian function $V^\pi_{\sigma,c}$ by

$$G_t = \left[ \begin{array}{c} \beta_t \left( \lambda_t - \prod_{[0,\Lambda^*]} \left( \lambda_t - \frac{1}{\beta_t} \left( \nabla \lambda V^\pi_{\sigma,c}(\theta_t, \lambda_t) \right) \right) \right) \\ \alpha_t \left( \theta_t - \prod_{[0,\Lambda^*]} \left( \theta_t + \frac{1}{\alpha_t} \left( \nabla \theta V^\pi_{\sigma,c}(\theta_t, \lambda_t) \right) \right) \right) \end{array} \right]. \tag{15}$$

Applying our RPD algorithm in (14), we have the following convergence guarantee.

**Corollary 1.** If we set step sizes $\alpha_t$, $\beta_t$, and $b_t$ as in Section and set $T = O(\epsilon^{-4})$, then $\min_{1 \leq t \leq T} \|G_t\| \leq 2\epsilon$.

This corollary implies that our robust primal-dual algorithm converges to a stationary point of the min-max problem (14) under the $\delta$-contamination model, with a complexity of $O(\epsilon^{-4})$.

**Algorithm 2** Smoothened Robust TD [Wang and Zou 2022]

**Input:** $T_{inner}, \pi, \sigma, c$

**Initialization:** $Q_{0}, s_0$

for $t = 0, 1, ..., T_{inner} - 1$ do

Choose $\xi_t \sim \pi(\cdot|s_t)$ and observe $c_t, s_{t+1}$

$V_t(s_t) \leftarrow \sum_{a \in A} \pi(a|s_t)Q_{t}(s_t, a)$ for all $s \in S$

$Q_{t+1}(s_{t+1}, a_t) \leftarrow Q_{t}(s_t, a_t) + \alpha_t(c_t + \gamma(1 - \delta))$.

$V_t(s_{t+1}) + \gamma \delta \cdot \text{LSE}(\sigma, V_t) - Q_{t}(s_t, a_t))$

end for

**Output:** $Q_{T_{inner}, c}$, $\tilde{Q}_{T_{inner}}$

Note that Algorithm [1] assumes knowledge of the smoothened robust value functions which may not be available in practice. Different from the non-robust value function which can be estimated using Monte Carlo, robust value functions are the value function corresponding to the worst-case transition kernel from which no samples are directly taken. To solve this issue, we adopt the smoothed robust TD algorithm (Algorithm [2]) from Wang and Zou [2022] to estimate the smoothened robust value functions.

It was shown that the smoothed robust TD algorithm converges to the smoothened robust value function with a sample complexity of $O(\epsilon^{-2})$ [Wang and Zou 2022] under the tabular case. We then construct our online and model-free RPD algorithm as in Algorithm [3]. We note that Algorithm [3] is for the tabular setting with finite $S$ and $A$. It can be easily extended to the case with large/continuous $S$ and $A$ using function approximation.

**Algorithm 3** Online Robust Primal-Dual algorithm

**Input:** $T, \sigma, \epsilon_{est}, \beta_t, \alpha_t, b_t, r, c$

**Initialization:** $\lambda_0, \theta_0$

for $t = 0, 1, ..., T - 1$ do

Set $T_{inner} = O\left( \left( \frac{(t+1)^\frac{5}{2}}{\epsilon_{est}} \right) \right)$ and run Algorithm [2] for $r$ and $c$, output $Q_{T_{inner}, r}, Q_{T_{inner}, c}$

$\tilde{V}^\pi_{\sigma,r}(s) \leftarrow \sum_{a \in A} \pi(a|s)Q_{T_{inner}, r}(s, a), \tilde{V}^\pi_{\sigma,c}(s) \leftarrow \sum_{a \in A} \pi(a|s)Q_{T_{inner}, c}(s, a)$

$\tilde{V}^\pi_{\sigma,c}(\rho) \leftarrow \sum_a \rho(s)\tilde{V}^\pi_{\sigma,c}(s), \tilde{V}^\pi_{\sigma,c}(\rho) \leftarrow \sum_s \rho(s)\tilde{V}^\pi_{\sigma,c}(s)$

$\lambda_{t+1} \leftarrow \prod_{[0,\Lambda^*]} \left( \lambda_t - \frac{1}{\beta_t} \left( \tilde{V}^\pi_{\sigma,c}(\rho) - b \right) - \frac{b_t}{\beta_t} \lambda_t \right)$

$\theta_{t+1} \leftarrow \prod_{[0,\Lambda^*]} \left( \theta_t + \frac{1}{\alpha_t} \left( \nabla \theta \tilde{V}^\pi_{\sigma,c}(\rho) + \lambda_{t+1} \nabla \theta \tilde{V}^\pi_{\sigma,c}(\rho) \right) \right)$

end for

**Output:** $\theta_T$

Algorithm [3] can be viewed as a biased stochastic gradient descent ascent algorithm. It is a sample-based algorithm without assuming any knowledge of robust value functions, and can be performed in an online fashion. We further extend the convergence results in Theorem [1] to the model-free setting, and characterize the following finite-time error bound of Algorithm [3]. Similarly, Algorithm [3] can be shown to achieve a $2\epsilon$-feasible policy almost surely.

Under the online model-free setting, the estimation of the robust value functions is biased. Therefore, the analysis is more challenging than the existing literature, where it is usually assumed that the gradients are exact. We develop a new method to bound the bias accumulated in every iteration of the algorithm, and establish the final convergence results.
Theorem 2. Consider the same conditions as in Theorem 1. Let $\epsilon_{\text{est}} = O(\epsilon^2)$ and $T = O(\epsilon^{-4})$, then $\min_{1 \leq t \leq T} \|G_t\| \leq (1 + \sqrt{2})\epsilon$.

5 Numerical Results

In this section, we numerically demonstrate the robustness of our algorithm in terms of both maximizing robust reward value function and satisfying constraints under model uncertainty. We compare our RPD algorithm with the heuristic algorithms in [Russel et al. 2021], [Mankowitz et al. 2020] and the vanilla non-robust primal-dual method. Based on the idea of "robust policy evaluation + non-robust policy improvement" in [Russel et al. 2021], [Mankowitz et al. 2020], we combine the robust TD algorithm with non-robust vanilla policy gradient method [Sutton et al. 1999], which we refer to as the heuristic primal-dual algorithm. Several environments, including Garnet [Archibald et al. 1995], 8 × 8 Frozen-Lake and Taxi environments from OpenAI [Brockman et al. 2016], are investigated.

We first run the algorithm and store the obtained policies $\pi_t$ at each time step. At each time step, we run robust TD with a sample size 200 for 30 times to estimate the objective $V_c(\rho)$ and the constraint $V_c(\rho)$. We then plot them v.s. the number of iterations $t$. The upper and lower envelopes of the curves correspond to the 95 and 5 percentiles of the 30 curves, respectively. We repeat the experiment for two different values of $\delta = 0.2, 0.3$.

Garnet problem. A Garnet problem can be specified by $G(S_n, A_n)$, where the state space $S$ has $S_n$ states $(s_1, \ldots, s_S)$ and action space has $A_n$ actions $(a_1, \ldots, a_A)$. The agent can take any actions in any state, and receives a randomly generated reward/utility signal generated from the uniform distribution on [0,1]. The transition kernels are also randomly generated. The comparison results are shown in Fig.1.

8 × 8 Frozen-Lake problem. We then compare the three algorithms under the 8 × 8 Frozen-lake problem setting in Fig.2. The Frozen-Lake problem involves a frozen lake of size 8 × 8 which contains several "holes". The agent aims to cross the lake from the start point to the end point without falling into any holes. The agent receives $r = -10$ and $c = 0$ when falling in a hole, receives $r = 20$ and $c = 1$ when arrive at the end point; At other times, the agent receives $r = 0$ and a randomly generated utility $c$ according to the uniform distribution on [0,1].

Taxi problem. We then compare the three algorithms under the Taxi problem environment. The taxi problem simulates a taxi driver in a 5 × 5 map. There are four designated locations in the grid world and a passenger occurs at a random location of the designated four locations at the start of each episode. The goal of the driver is to first pick up the passenger and then to drop off at another specific location. At each time step, we run robust TD with a sample size 200 for 30 times to estimate the objective $V_c(\rho)$ and the constraint $V_c(\rho)$. We then plot them v.s. the number of iterations $t$. The upper and lower envelopes of the curves correspond to the 95 and 5 percentiles of the 30 curves, respectively. We repeat the experiment for two different values of $\delta = 0.2, 0.3$.

From the experiment results above, it can be seen that: (1) Both our RPD algorithm and the heuristic primal-dual approach find feasible policies satisfying the constraint under the worst-case scenario, i.e., $V_c^\pi \geq b$. However, the...
non-robust primal-dual method fails to find a feasible solution that satisfy the constraint under the worst-case scenario. 

(2) Compared to the heuristic PD method, our RPD method can obtain more reward and can find a more robust policy while satisfying the robust constraint. Note that the non-robust PD method obtain more reward, but this is because the policy it finds violates the robust constraint. Our experiments demonstrate that among the three algorithms, our RPD algorithm is the best one which optimizes the worst-case reward performance while satisfying the robust constraints on the utility.

6 Conclusion

In this paper, we formulate the problem of robust constrained reinforcement learning under model uncertainty, where the goal is to guarantee that constraints are satisfied for all MDPs in the uncertainty set, and to maximize the worst-case reward performance over the uncertainty set. We propose a robust primal-dual algorithm, and theoretically characterize its convergence, complexity and robust feasibility. Our algorithm guarantees convergence to a feasible solution, and outperforms the other two heuristic algorithms. We further investigate a concrete example with $\delta$-contamination uncertainty set, and construct online and model-free robust primal-dual algorithm. Our methodology can also be readily extended to problems with other uncertainty sets like KL-divergence, total variation and Wasserstein distance. The major challenge lies in deriving the robust policy gradient, and further designing model-free algorithm to estimate the robust value function. We also expect that Assumption 3 and further results can be derived if some proper smoothing technique is used.

Limitations: It is of future interest to generalize our results to other types of uncertainty sets, e.g., ones defined by KL divergence, total variation, Wasserstein distance. Negative societal impact: This work is a theoretical study. To the best of the authors’ knowledge, it does not have any potential negative impact on the society.
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Appendix

A Additional Experiments

4 × 4 Frozen Lake problem. The 4 × 4 frozen lake is similar to the 8 × 8 one but with a smaller map. Similarly, we randomly generate the utility signal for each state-action pair. The results are shown in Fig. 4.

![Figure 4: Comparison on 4 × 4 Frozen-Lake Problem.](image)

N-Chain problem. We then compare three algorithms under the N-Chain problem environment. The N-chain problem involves a chain contains N nodes. The agent can either move to its left or right node. When it goes to left, it receives a reward-utility signal (1, 0); When it goes right, it receives a reward-utility signal (0, 2), and if the agent arrives the N-th node, it receives a bonus reward of 40. There is also a small probability that the agent slips to the different direction of its action. In this experiment, we set N = 40. The results are shown in Fig. 5.

![Figure 5: Comparison on N-Chain Problem.](image)

B Proof of Lemma 1

Denote by \( P^\pi = \{ (p^\pi)_{s,a}^s \in S : s \in S, a \in A \} \) the worst-case transition kernel corresponding to the policy \( \pi \). We consider the \( \delta \)-contamination uncertainty set defined in Section 4. We then show that under \( \delta \)-contamination model, the set of visitation distributions is non-convex. The robust visitation distribution set can be written as follows:

\[
\begin{align*}
\{ d \in \Delta_{S \times A} : \exists \pi \in \Pi, \forall (s, a), & \quad d(s, a) = \pi(a|s) \sum_b d(s, b), \\
& \quad \gamma \sum_{s', a'} (p^\pi)_{s',a'}^a d(s', a') + (1 - \gamma)\rho(s) = \sum_a d(s, a). \}
\end{align*}
\]

Under the \( \delta \)-contamination model, \( P^\pi \) can be explicped as \( (p^\pi)_{s,s'}^a = (1 - \delta)p^a_{s,s'} + \delta \mathbb{1}_{s'=\arg \min \nu \pi} \). Hence the set in (16) can be rewritten as

\[
\begin{align*}
\{ d \in \Delta_{S \times A} : \exists \pi, \forall (s, a), & \quad d(s, a) = \pi(a|s) \left( \sum_b d(s, b) \right), \\
& \quad \gamma(1 - \delta) \sum_{s', a'} p^a_{s,s'} d(s', a') + \gamma \delta \mathbb{1}_{s=\arg \min \nu \pi} + (1 - \gamma)\rho(s) = \sum_a d(s, a). \}
\end{align*}
\]
Now consider any two pairs \((\pi_1, d_1), (\pi_2, d_2)\) of policy and their worst-case visitation distribution, to show that the set is convex, we need to find a pair \((\pi', d')\) such that \(\forall \lambda \in [0, 1] \text{ and } \forall s, a,
\[
\lambda d_1(s, a) + (1 - \lambda)d_2(s, a) = d'(s, a),
\]
\[
d'(s, a) = \pi'(a|s) \left( \sum_b d'(s, b) \right),
\]
\[
\sum_{a'} d'(s, a') = \gamma(1 - \delta) \sum_{s', a'} \pi'_{s', a'} d'(s', a') + \gamma \delta \mathbb{I}_{\{s = \arg \min V^{\pi'}\}} + (1 - \gamma)\rho(s).
\]

Equation (20) firstly implies that \(\forall s,\)
\[
\lambda \mathbb{I}_{\{s = \arg \min V^{\pi_1}\}} + (1 - \lambda)\mathbb{I}_{\{s = \arg \min V^{\pi_2}\}} = \mathbb{I}_{\{s = \arg \min V^{\pi'}\}},
\]
where from (18) and (19), \(\pi'\) should be
\[
\pi'(a|s) = \frac{d'(s, a)}{\sum_b d'(s, b)} = \frac{\lambda d_1(s, a) + (1 - \lambda)d_2(s, a)}{\sum_b (\lambda d_1(s, b) + (1 - \lambda)d_2(s, b))}.
\]

We then construct the following counterexample, which shows that there exists a robust MDP, two policy-distribution pairs \((\pi_1, d_1), (\pi_2, d_2)\), and \(\lambda \in (0, 1)\), such that \(\lambda \mathbb{I}_{\{s = \arg \min V^{\pi_1}\}} + (1 - \lambda)\mathbb{I}_{\{s = \arg \min V^{\pi_2}\}} \neq \mathbb{I}_{\{s = \arg \min V^{\pi'}\}}\), and therefore the set of robust visitation distribution is non-convex.

Consider the following Robust MDP. It has three states 1, 2, 3 and two actions a, b. When the agent is at state 1, if it takes action a, the system will transit to state 2 and receive reward \(r = 0\); if it takes action b, the system will transit to state 3 and receive reward \(r = 2\). When the agent is at state 2/3, it can only take action a/b, the system can only transits back to state 1 and the agent will receive reward \(r = 1\). The initial distribution is \(\mathbb{I}_{s=1}\).

![Robust MDP Diagram](diagram.png)

Clearly all policy can be written as \(\pi = (p, 1 - p)\), where \(p\) is the probability of taking action a at state 1. We consider two policies, \(\pi_1 = (1, 0)\) and \(\pi_2 = (0, 1)\).

It can be verified that \(\arg \min V^{\pi_1} = 1\), and its robust visitation distribution, denoted by \(d_1\), is
\[
d_1(1, a) = \frac{1 - \gamma}{1 - \gamma^2}, \quad \text{and } d_1(1, b) = 0,
\]
\[
d_1(2, a) = 0, \quad d_1(2, b) = 0,
\]
\[
d_1(3, a) = 0, \quad d_1(3, b) = 0.
\]

Similarly, \(\arg \min V^{\pi_2} = 2\), and and its robust visitation distribution, denoted by \(d_2\), is
\[
d_2(1, a) = 0, \quad d_2(1, b) = \frac{1 - \gamma}{1 - \gamma^2},
\]
\[
d_2(2, a) = 0, \quad d_2(2, b) = 0,
\]
\[
d_2(3, a) = 0, \quad d_2(3, b) = 0.
\]
We then show that there exists $\lambda$, where
\begin{equation}
\lambda \in [0, 1],
\end{equation}
which completes the proof.

\begin{proof}
We first set
\begin{equation}
C = \max_{\pi \in \Pi} V_{r}^{\pi}(\rho) + \lambda^{*}(V_{c}^{\pi}(\rho) - b),
\end{equation}
and hence
\begin{equation}
C = \max_{\pi \in \Pi} V_{r}^{\pi}(\rho) + \lambda^{*}(V_{c}^{\pi}(\rho) - b) \geq V_{c}^{\pi^{*}}(\rho) + \lambda^{*}(V_{c}^{\pi^{*}}(\rho) - b) \geq V_{r}^{\pi^{*}}(\rho) + \lambda^{*}\zeta.
\end{equation}

Thus we have that
\begin{equation}
\lambda^{*} \leq \frac{C - V_{r}^{\pi^{*}}(\rho)}{\zeta}.
\end{equation}

Note that
\begin{equation}
C = \min_{\lambda \geq 0} \max_{\pi \in \Pi} V_{r}^{\pi}(\rho) + \lambda(V_{c}^{\pi}(\rho) - b) \leq \max_{\pi \in \Pi} V_{r}^{\pi}(\rho) \leq \frac{1}{1 - \gamma},
\end{equation}
where $(a)$ is because $\min_{\lambda \geq 0} \max_{\pi \in \Pi} V_{r}^{\pi}(\rho) + \lambda(V_{c}^{\pi}(\rho) - b)$ is less than the optimal value of inner problem when $\lambda = 0$, i.e., $\max_{\pi \in \Pi} V_{c}^{\pi}(\rho)$, and $\frac{1}{1 - \gamma}$ is the upper bound of robust value functions. Hence we have that
\begin{equation}
\lambda^{*} \leq \frac{1}{(1 - \gamma)\zeta},
\end{equation}
which completes the proof.
\end{proof}

\section{Proof of Lemmas 2 and 5}

\subsection{Proof of Lemma 2}

\begin{proof}
We first set $C = V_{r}^{\pi^{*}}(\rho) + \lambda^{*}(V_{c}^{\pi^{*}}(\rho) - b)$, clearly $
\max_{\pi \in \Pi} V_{r}^{\pi}(\rho) + \lambda^{*}(V_{c}^{\pi}(\rho) - b) = C$, and hence
\begin{equation}
C = \max_{\pi \in \Pi} V_{r}^{\pi}(\rho) + \lambda^{*}(V_{c}^{\pi}(\rho) - b) \geq V_{r}^{\pi^{*}}(\rho) + \lambda^{*}(V_{c}^{\pi^{*}}(\rho) - b) \geq V_{r}^{\pi^{*}}(\rho) + \lambda^{*}\zeta.
\end{equation}

Thus we have that
\begin{equation}
\lambda^{*} \leq \frac{C - V_{r}^{\pi^{*}}(\rho)}{\zeta}.
\end{equation}

Note that
\begin{equation}
C = \min_{\lambda \geq 0} \max_{\pi \in \Pi} V_{r}^{\pi}(\rho) + \lambda(V_{c}^{\pi}(\rho) - b) \leq \max_{\pi \in \Pi} V_{r}^{\pi}(\rho) \leq \frac{1}{1 - \gamma},
\end{equation}
where $(a)$ is because $\min_{\lambda \geq 0} \max_{\pi \in \Pi} V_{r}^{\pi}(\rho) + \lambda(V_{c}^{\pi}(\rho) - b)$ is less than the optimal value of inner problem when $\lambda = 0$, i.e., $\max_{\pi \in \Pi} V_{c}^{\pi}(\rho)$, and $\frac{1}{1 - \gamma}$ is the upper bound of robust value functions. Hence we have that
\begin{equation}
\lambda^{*} \leq \frac{1}{(1 - \gamma)\zeta},
\end{equation}
which completes the proof.
\end{proof}

\subsection{Proof of Lemma 5}

\begin{proof}
Set $C = V_{r}^{\pi^{*}}(\rho) + \lambda^{*}(V_{c}^{\pi^{*}}(\rho) - b)$, then
\begin{equation}
C = \max_{\pi \in \Pi} V_{r}^{\pi}(\rho) + \lambda^{*}(V_{c}^{\pi}(\rho) - b) \geq V_{r}^{\pi^{*}}(\rho) + \lambda^{*}(V_{c}^{\pi^{*}}(\rho) - b) \geq V_{r}^{\pi^{*}}(\rho) + \lambda^{*}\zeta.
\end{equation}

\end{proof}
Thus we have that
\[ C \geq V_{\sigma,r}^\pi(\rho) + \lambda^* \zeta', \]  
(46)

hence
\[ \lambda^* \leq \frac{C - V_{\sigma,r}^\pi(\rho)}{\zeta'}. \]  
(47)

Note that
\[ C = \min_{\lambda \geq 0} \max_{\pi \in \Pi^\theta} V_{\sigma,r}^\pi(\rho) + \lambda(V_{\sigma,c}^\pi(\rho) - b) \leq \max_{\pi \in \Pi^\theta} V_{\sigma,r}^\pi(\rho) \leq C_{\sigma}, \]  
(48)

where \( C_{\sigma} \) is the upper bound of smoothed robust value functions \[\text{Wang and Zou [2022]}\]: \( C_{\sigma} = \frac{1}{1 - \gamma} (1 + 2\gamma R^{\log |S|}_\sigma) \).

Hence we have that
\[ \lambda^* \leq \frac{C_{\sigma}}{\zeta'}, \]  
(49)

which completes the proof.

\[ \square \]

### D Proof of Lemma 6

**Proof.** For any \( \lambda \), denote the optimal value of the inner problems \( \max_{\pi \in \Pi} V_{\sigma,r}^\pi(\rho) + \lambda(V_{\sigma,c}^\pi(\rho) - b) \) and \( \max_{\pi \in \Pi} V_{\sigma}^\pi(\rho) + \lambda(V_{\sigma}^\pi(\rho) - b) \) by \( V^D(\lambda) \) and \( V^D_{\sigma}(\lambda) \). It is then easy to verify that
\[ |V^D(\lambda) - V^D_{\sigma}(\lambda)| \leq (1 + \lambda) \epsilon \leq (1 + \lambda^*) \epsilon. \]  
(50)

Denote the optimal solutions of \( \min_{\lambda \in [0, A]} V^D(\lambda) \) and \( \min_{\lambda \in [0, A]} V^D_{\sigma}(\lambda) \) by \( \lambda^D \) and \( \lambda^D_{\sigma} \). We thus conclude that
\[ |V^D_{\sigma}(\lambda^D_{\sigma}) - V^D(\lambda^D)| \leq (1 + \lambda^*) \epsilon, \]  
and this thus completes the proof.

\[ \square \]

### E Proof of Theorem 1

We restate Theorem 1 with all the specific step sizes as follows.

Set \( h_t = \frac{19}{2 \mu_t \sigma^2} \), \( \mu_t = \xi (C_V')^2 + \frac{16 \tau (C_V')^2}{\xi (\beta_{t+1})^2} - 2 \nu, \beta_t = \frac{1}{\xi}, \alpha_t = \nu + \mu_t, \) where \( \xi > \frac{2(1 + \lambda^*) \nu R}{(C_V')^2}, \) \( \nu \) is any positive number and \( \tau \) is any number greater than 2, then
\[ \min_{1 \leq t \leq T} \| G_t \|^2 \leq 2 \epsilon, \]  
(51)

when
\[ T = \max \left\{ \frac{7 (A^*)^4}{\xi^4 \epsilon^4}, \left( 2 + \frac{9(\tau - 2)(C_V')^2 u K}{\epsilon^2} \right)^2 \right\} = O(\epsilon^{-4}). \]  
(52)

The definitions of \( u, K \) can be found in Section 1.
Theorem 1 can be proved similarly as Theorem 2 and hence the proof is omitted here.

### F Proof of Lemma 3

**Proof.** Recall that \( V^L_{\sigma}(\theta, \lambda) = V_{\sigma,r}^{\pi_{\theta}}(\rho) + \lambda(V_{\sigma,c}^{\pi_{\theta}}(\rho) - b) \), hence we have that
\[ \nabla_{\lambda} V^L_{\sigma}(\theta, \lambda) = V_{\sigma,c}^{\pi_{\theta}}(\rho) - b, \]  
(53)

\[ \nabla_{\theta} V^L_{\sigma}(\theta, \lambda) = \nabla_{\theta} V_{\sigma,r}^{\pi_{\theta}}(\rho) + \lambda \nabla_{\theta} V_{\sigma,c}^{\pi_{\theta}}(\rho). \]  
(54)

Note that in \[\text{Wang and Zou [2022]}\], it has been shown that
\[ \| V_{\sigma,r}^{\pi_{\theta_1}} - V_{\sigma,r}^{\pi_{\theta_2}} \| \leq C^V_{\sigma} \| \theta_1 - \theta_2 \|, \]  
(55)
\[ \| \nabla_{\theta} V_{\sigma,r}^{\pi_{\theta_1}} - \nabla_{\theta} V_{\sigma,r}^{\pi_{\theta_2}} \| \leq L_{\sigma} \| \theta_1 - \theta_2 \|. \]  
(56)
where the definition of constants $C_{\sigma}^{V}$ and $L_{\sigma}$ can be found in Section I. Hence
\[
\|\nabla_{\lambda}V_{\sigma}^{L}(\theta,\lambda)|_{\theta_{1}} - \nabla_{\lambda}V_{\sigma}^{L}(\theta,\lambda)|_{\theta_{2}}\| = \|V_{\sigma,\rho}^{\pi_{t+1}}(\rho) - V_{\sigma,\rho}^{\pi_{t+2}}(\rho)\| \leq C_{\sigma}^{V} \|\theta_{1} - \theta_{2}\|, \tag{57}
\]
\[
\|\nabla_{\lambda}V_{\sigma}^{L}(\theta,\lambda)|_{\lambda_{1}} - \nabla_{\lambda}V_{\sigma}^{L}(\theta,\lambda)|_{\lambda_{2}}\| = 0. \tag{58}
\]
Similarly, we have that
\[
\|\nabla_{\theta}V_{\sigma}^{L}(\theta,\lambda)|_{\theta_{1}} - \nabla_{\theta}V_{\sigma}^{L}(\theta,\lambda)|_{\theta_{2}}\| \leq (1 + \lambda)\|\theta_{1} - \theta_{2}\| \leq (1 + A^{*})L_{\sigma}\|\theta_{1} - \theta_{2}\|, \tag{59}
\]
\[
\|\nabla_{\theta}V_{\sigma}^{L}(\theta,\lambda)|_{\lambda_{1}} - \nabla_{\theta}V_{\sigma}^{L}(\theta,\lambda)|_{\lambda_{2}}\| \leq |(\lambda_{1} - \lambda_{2})| \max_{\theta \in \Theta} \|\nabla_{\theta}V_{\sigma,\rho}^{\pi_{t+1}}(\rho)\| \leq C_{\sigma}^{V} |\lambda_{1} - \lambda_{2}|. \tag{60}
\]
This completes the proof. \qed

\section{Proof of Proposition \[1\]}

\textbf{Proof.} The $\lambda$-entry of $G_{W}$ is smaller than $2\epsilon$, i.e.,
\[
|(G_{W})_{\lambda}| = \left| \beta_{W} \left( \lambda_{W} - \prod_{\theta \in [0, A^{*}]} \left( \lambda_{W} - \frac{1}{\beta_{W}} (\nabla_{\lambda}V_{\sigma}^{L}(\theta_{W}, \lambda_{W})) \right) \right) \right| < 2\epsilon. \tag{61}
\]
Denote $\lambda^{+} \triangleq \prod_{\theta \in [0, A^{*}]} \left( \lambda_{W} - \frac{1}{\beta_{W}} (\nabla_{\lambda}V_{\sigma}^{L}(\theta_{W}, \lambda_{W})) \right)$. From Lemma 3 in Ghadimi and Lan [2016], $-\nabla_{\lambda}V_{\sigma}^{L}(\theta_{W}, \lambda^{+})$ can be rewritten as the sum of two parts: $-\nabla_{\lambda}V_{\sigma}^{L}(\theta_{W}, \lambda^{+}) \in N_{[0, A^{*}]}(\lambda^{+}) + 4\epsilon B$, where $N_{K}(x) \triangleq \{ g \in \mathbb{R}^{d} : \langle g, y - x \rangle \leq 0 \forall y \in K \}$ is the normal cone, and $B$ is the unit ball.

This hence implies that for any $\lambda \in [0, A^{*}]$, $(\lambda - \lambda^{+})V_{c}^{W} - b \geq -4(\lambda - \lambda^{+})\epsilon$. By setting $\lambda = A^{*}$, we have $V_{c}^{W} + 4\epsilon \geq b$, which means $\pi_{W}$ is feasible with a $4\epsilon$-violation. \qed

\section{Proof of Theorem \[2\]}

We then prove Theorem 2. Our proof extends the one in Xu et al. [2020] to the biased setting.

To simplify notations, we denote the updates in Algorithm 1 by $f(\theta_{t}) \triangleq \hat{V}_{\sigma,\rho}^{\pi_{t+1}}(\rho) - b$, and $g(\theta_{t}, \lambda_{t+1}) \triangleq \nabla_{\theta}V_{\sigma,\rho}^{\pi_{t+1}}(\rho) + \lambda_{t+1}\nabla_{\theta}V_{\sigma,\rho}^{\pi_{t+1}}(\rho)$, and denote the update functions in Algorithm 1 by $f(\theta_{t}) \triangleq \hat{V}_{\sigma,\rho}^{\pi_{t+1}}(\rho) - b$, and $g(\theta_{t}, \lambda_{t+1}) \triangleq \nabla_{\theta}V_{\sigma,\rho}^{\pi_{t+1}}(\rho) + \lambda_{t+1}\nabla_{\theta}V_{\sigma,\rho}^{\pi_{t+1}}(\rho)$. Here $\hat{f}$ and $\hat{g}$ can be viewed as biased estimations of $f$ and $g$.

In the following, we will first show several technical lemmas that will be useful in the proof of Theorem 2.

\textbf{Lemma 7.} Recall that the step size $\alpha_{t} = \nu + \mu_{t}$. If $\mu_{t} > (1 + A^{*})L_{\sigma}, \forall t \geq 0$, then
\[
V_{\sigma}^{L}(\theta_{t+1}, \lambda_{t+1}) - V_{\sigma}^{L}(\theta_{t}, \lambda_{t+1}) \geq (\theta_{t+1} - \theta_{t}, -\hat{g}(\theta_{t}, \lambda_{t+1}) + g(\theta_{t}, \lambda_{t+1})) + \left( \frac{\mu_{t}}{2} + \nu \right) \|\theta_{t+1} - \theta_{t}\|^{2}. \tag{62}
\]
\textbf{Proof.} Note that from the update of $\theta_{t}$ and proposition of projection, it implies that
\[
\left\langle \theta_{t} + \frac{1}{\alpha_{t}} \hat{g}(\theta_{t}, \lambda_{t+1}) - \theta_{t+1}, \theta_{t} - \theta_{t+1} \right\rangle \leq 0. \tag{63}
\]
Hence
\[
\langle \hat{g}(\theta_{t}, \lambda_{t+1}) - \alpha_{t}(\theta_{t+1} - \theta_{t}), \theta_{t} - \theta_{t+1} \rangle \leq 0. \tag{64}
\]
From Lemma 8, we have that
\[
V_{\sigma}^{L}(\theta_{t+1}, \lambda_{t+1}) - V_{\sigma}^{L}(\theta_{t}, \lambda_{t+1}) \geq \langle \theta_{t+1} - \theta_{t}, g(\theta_{t}, \lambda_{t+1}) \rangle - \frac{(1 + A^{*})L_{\sigma}}{2}\|\theta_{t+1} - \theta_{t}\|^{2}. \tag{65}
\]
Summing up the two inequalities implies
\[
V_{\sigma}^{L}(\theta_{t+1}, \lambda_{t+1}) - V_{\sigma}^{L}(\theta_{t}, \lambda_{t+1}) \geq \langle \theta_{t+1} - \theta_{t}, -\hat{g}(\theta_{t}, \lambda_{t+1}) + g(\theta_{t}, \lambda_{t+1}) + \alpha_{t}(\theta_{t+1} - \theta_{t}) \rangle - \frac{(1 + A^{*})L_{\sigma}}{2}\|\theta_{t+1} - \theta_{t}\|^{2}
\]
where the last inequality is from (68).

Recall that the step size \( \beta_t = \frac{1}{t} \), and set \( \xi \leq \frac{1}{b_0} \), then

\[
V_{\sigma}^L(\theta_{t+1}, \lambda_{t+1}) - V_{\sigma}^L(\theta_t, \lambda_t)
\geq (f(\theta_{t-1}) - \hat{f}(\theta_{t-1}))(\lambda_{t+1} - \lambda_t) + (\theta_{t+1} - \theta_t, -\hat{g}(\theta_t, \lambda_{t+1}) + g(\theta_t, \lambda_{t+1})) + \left(\frac{\mu_t}{2} + \nu\right) \|\theta_{t+1} - \theta_t\|^2,
\]

(66)

and hence completes the proof. \( \square \)

**Lemma 8.** Recall that the step size \( \beta_t = \frac{1}{t} \), and set \( \xi \leq \frac{1}{b_0} \), then

\[
V_{\sigma}^L(\theta_{t+1}, \lambda_{t+1}) - V_{\sigma}^L(\theta, \lambda_t)
\geq (f(\theta_{t-1}) - \hat{f}(\theta_{t-1}))(\lambda_{t+1} - \lambda_t) + (\theta_{t+1} - \theta_t, -\hat{g}(\theta_t, \lambda_{t+1}) + g(\theta_t, \lambda_{t+1})) + \left(\frac{\mu_t}{2} + \nu\right) \|\theta_{t+1} - \theta_t\|^2
\]

(67)

**Proof.** For any \( t > 1 \), define \( \bar{V}_t(\theta, \lambda) \triangleq V_{\sigma}^L(\theta, \lambda) + \frac{b_{t-1}}{2} \lambda^2 \). Thus we have

\[
\|\nabla_\lambda \bar{V}_t(\theta, \lambda_{t+1}) - \nabla_\lambda \bar{V}_t(\theta, \lambda_t)\| = b_{t-1}|\lambda_{t+1} - \lambda_t| \leq b_0|\lambda_{t+1} - \lambda_t|,
\]

(68)

where that last inequality is due to \( b_{t-1} \leq b_0 \). Note that \( \bar{V}_t(\theta, \lambda) \) is \( b_{t-1} \)-strongly convex in \( \lambda \), hence we have

\[
(\nabla_\lambda \bar{V}_t(\theta, \lambda_{t+1}) - \nabla_\lambda \bar{V}_t(\theta, \lambda_t))(\lambda_{t+1} - \lambda_t)
\geq b_{t-1}(\lambda_{t+1} - \lambda_t)^2
\]

\[
\geq b_{t-1} \frac{b_{t-1} + b_0}{b_{t-1} + b_0} (\lambda_{t+1} - \lambda_t)^2
\]

\[
= \frac{b_{t-1}b_0}{b_{t-1} + b_0} (\lambda_{t+1} - \lambda_t)^2 + \frac{b_{t-1}^2}{b_{t-1} + b_0} (\lambda_{t+1} - \lambda_t)^2
\]

\[
\geq \frac{b_{t-1}b_0}{b_{t-1} + b_0} (\lambda_{t+1} - \lambda_t)^2 + \frac{1}{b_{t-1} + b_0} (\nabla_\lambda \bar{V}_t(\theta, \lambda_{t+1}) - \nabla_\lambda \bar{V}_t(\theta, \lambda_t))^2,
\]

(69)

where the last inequality is from (68).

Recall the update of \( \lambda_t \) in Algorithm 3 which can be rewritten as

\[
\lambda_{t+1} = \prod_{\tau=0}^{t} \left( \lambda_t - \frac{1}{\beta_t} \nabla_\lambda \bar{V}_{t+1}(\theta_t, \lambda_t) + \frac{1}{\beta_t}(f(\theta_t) - \hat{f}(\theta_t)) \right),
\]

(70)

This further implies that \( \forall \lambda \in [0, \Lambda^*] \):

\[
(\beta_t(\lambda_{t+1} - \lambda_t) + \nabla_\lambda \bar{V}_{t+1}(\theta_t, \lambda_t) - f(\theta_t) + \hat{f}(\theta_t))(\lambda - \lambda_{t+1}) \geq 0.
\]

Hence setting \( \lambda = \lambda_k \) implies that

\[
(\beta_t(\lambda_{t+1} - \lambda_t) + \nabla_\lambda \bar{V}_{t+1}(\theta_t, \lambda_t) - f(\theta_t) + \hat{f}(\theta_t))(\lambda_t - \lambda_{t+1}) \geq 0.
\]

Similarly, we have that

\[
(\beta_t(\lambda_t - \lambda_{t-1}) + \nabla_\lambda \bar{V}_t(\theta_{t-1}, \lambda_{t-1}) - f(\theta_{t-1}) + \hat{f}(\theta_{t-1}))(\lambda_{t-1} - \lambda_t) \geq 0.
\]

(71)

(72)

(73)

Note that \( \bar{V}_t \) is convex, we hence have that

\[
\bar{V}_t(\theta, \lambda_{t+1}) - \bar{V}_t(\theta, \lambda_t)
\geq (\nabla_\lambda \bar{V}_t(\theta_t, \lambda_t))(\lambda_{t+1} - \lambda_t)
\]

\[
= (\nabla_\lambda \bar{V}_t(\theta_t, \lambda_t) - \nabla_\lambda \bar{V}_t(\theta_{t-1}, \lambda_{t-1}))(\lambda_{t+1} - \lambda_t) + (\nabla_\lambda \bar{V}_t(\theta_{t-1}, \lambda_{t-1}))(\lambda_{t+1} - \lambda_t)
\]

\[
\geq (\nabla_\lambda \bar{V}_t(\theta_{t-1}, \lambda_{t-1}) - \nabla_\lambda \bar{V}_t(\theta_{t-1}))(\lambda_{t+1} - \lambda_t)
\]

\[
+ (f(\theta_{t-1}) - \hat{f}(\theta_{t-1}) - \beta_t(\lambda_t - \lambda_{t-1}))(\lambda_{t+1} - \lambda_t),
\]

(74)

where (a) is from (72). The first term in the RHS of (74) can be further bounded as follows.

\[
(\nabla_\lambda \bar{V}_t(\theta_{t-1}, \lambda_{t-1}) - \nabla_\lambda \bar{V}_t(\theta_{t-1}, \lambda_{t-1}))(\lambda_{t+1} - \lambda_t)
\]
We then provide bounds for each term in (76) as follows.

**Term (a)** can be bounded as follows:

\[
\begin{align*}
&= \langle \nabla \hat{V}_t(\theta_t, \lambda_t) - \nabla \bar{V}_t(\theta_{t-1}, \lambda_t) \rangle (\lambda_{t+1} - \lambda_t) \\
&= \langle \nabla \lambda V^L_\sigma(\theta_t, \lambda_t) - \nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_t) \rangle (\lambda_{t+1} - \lambda_t) \\
&\geq -\frac{(\lambda_{t+1} - \lambda_t)^2}{2\xi} - \frac{\xi}{2} \langle \nabla \lambda V^L_\sigma(\theta_t, \lambda_t) - \nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_t) \rangle^2 \\
&\geq -\frac{(\lambda_{t+1} - \lambda_t)^2}{2\xi} - \frac{\xi(C^V)^2}{2} \|\nabla \lambda V^L_\sigma\|^2,
\end{align*}
\]

which is from Cauchy–Schwarz inequality and $C^V$-smoothness of $V^L_\sigma(\theta, \lambda)$.

**Term (b)** can be bounded as follows:

\[
\begin{align*}
&= \frac{1}{b_t + b_0} \langle \nabla \lambda V^L_\sigma(\theta_t, \lambda_t) - \nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_{t-1}) \rangle \\
&\geq -\frac{\xi}{2} \langle \nabla \lambda V^L_\sigma(\theta_t, \lambda_t) - \nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_{t-1}) \rangle^2,
\end{align*}
\]

which is from (69).

**Term (c)** can be bounded as follows by Cauchy–Schwarz inequality:

\[
\begin{align*}
&= m_{t+1} \langle \nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_t) - \nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_{t-1}) \rangle \\
&\geq -\frac{\xi}{2} \langle \nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_t) - \nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_{t-1}) \rangle^2 - \frac{1}{2\xi} m_{t+1}^2
\end{align*}
\]

Moreover, it can be shown that

\[
\frac{1}{\xi}(\lambda_{t+1} - \lambda_t)(\lambda_t - \lambda_{t-1}) = \frac{1}{2\xi}(\lambda_{t+1} - \lambda_t)^2 + \frac{1}{2\xi}(\lambda_t - \lambda_{t-1})^2 - \frac{1}{2\xi} m_{t+1}^2.
\]

Plug (77) to (80) in (76) and we have that

\[
\begin{align*}
\hat{V}_t(\theta_t, \lambda_{t+1}) &\geq (f(\theta_t) - \hat{f}(\theta_{t-1}))(\lambda_{t+1} - \lambda_t) - \beta_t(\lambda_t - \lambda_{t-1})(\lambda_{t+1} - \lambda_t) \\
&+ (\nabla \lambda V^L_\sigma(\theta_t, \lambda_t) - \nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_{t-1})) (\lambda_{t+1} - \lambda_t) + (\nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_t) - \nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_{t-1})) (\lambda_t - \lambda_{t-1}) \\
&+ m_{t+1} (\nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_t) - \nabla \lambda V^L_\sigma(\theta_{t-1}, \lambda_{t-1})).
\end{align*}
\]
Then we have that
\[ \theta_t \]
where
\[ \text{Proof.} \]
From the definition of $\tilde{V}_t$, we have that
\[ \tilde{V}_t(\theta_t, \lambda_{t+1}) - \tilde{V}_t(\theta_t, \lambda_t) = V^L_\sigma(\theta_t, \lambda_{t+1}) + \frac{b_{t-1}}{2} \lambda_{t+1}^2 - V^L_\sigma(\theta_t, \lambda_t) - \frac{b_{t-1}}{2} \lambda_t^2. \]  
(82)
Then we have that
\[ V^L_\sigma(\theta_{t+1}, \lambda_{t+1}) - V^L_\sigma(\theta_t, \lambda_t) \geq \frac{b_{t-1}}{2} (\lambda_t^2 - \lambda_{t+1}^2) + (f(\theta_{t-1}) - f(\theta_{t-1})) (\lambda_{t+1} - \lambda_t) \]
\[ - \frac{1}{\xi} (\lambda_{t+1} - \lambda_t)^2 - \frac{1}{2\xi} (\lambda_t - \lambda_{t-1}^2)^2 - \frac{\xi (C^V_\sigma)^2}{2} \|\theta_t - \theta_{t-1}\|^2. \]
(83)
Combining with Lemma 7 if $\forall t, \mu_t > (1 + \lambda^*) L_\sigma$, we then have that
\[ V^L_\sigma(\theta_{t+1}, \lambda_{t+1}) - V^L_\sigma(\theta_t, \lambda_t) \geq (f(\theta_{t-1}) - f(\theta_{t-1})) (\lambda_{t+1} - \lambda_t) + \langle \theta_{t+1}, \theta_t - \tilde{g}(\theta_t, \lambda_{t+1}) + f(\theta_t, \lambda_{t+1}) \rangle - \frac{\xi (C^V_\sigma)^2}{2} \|\theta_t - \theta_{t-1}\|^2 \]
\[ + \left( \frac{\mu_t}{2} + \nu \right) \|\theta_{t+1} - \theta_t\|^2 + \frac{b_{t-1}}{2} (\lambda_t^2 - \lambda_{t+1}^2) - \frac{1}{\xi} (\lambda_{t+1} - \lambda_t)^2 - \frac{1}{2\xi} (\lambda_t - \lambda_{t-1}^2)^2. \]
(84)
Lemma 9. Define
\[ F_{t+1} \equiv -\frac{8}{\xi b_{t+1}} (\lambda_t - \lambda_{t+1})^2 - \frac{8}{\xi} \left( 1 - \frac{b_t}{b_{t+1}} \right) \lambda_t^2 + V^L_\sigma(\theta_{t+1}, \lambda_{t+1}) + \frac{b_{t-1}}{2} \lambda_{t+1}^2 \]
\[ + \left( \frac{16 (C^V_\sigma)^2}{\xi b_{t+1}^2} - \frac{\xi (C^V_\sigma)^2}{2} \right) \|\theta_{t+1} - \theta_t\|^2 + \left( \frac{8}{\xi} - \frac{1}{2\xi} \right) (\lambda_{t+1} - \lambda_t)^2, \]
(85)
and if $\frac{1}{b_{t+1}} - \frac{1}{b_t} \leq \frac{\xi}{2}$, then
\[ F_{t+1} - F_t \geq S_t + \left( \frac{\mu_t}{2} + \nu - \frac{16 (C^V_\sigma)^2}{\xi b_{t+1}^2} - \frac{\xi (C^V_\sigma)^2}{2} \right) \|\theta_{t+1} - \theta_t\|^2 + \frac{b_t - b_{t-1}}{2} \lambda_t^2 \]
\[ + \frac{9}{10\xi} (\lambda_{t+1} - \lambda_t)^2 + \frac{8}{\xi} \left( \frac{b_t - b_{t-1}}{b_t} \right) \lambda_{t+1}^2, \]
(86)
where
\[ S_t \equiv \frac{16}{\xi^2} (f(\theta_{t-1}) - f(\theta_t) - f(\theta_t) + f(\theta_t)) (-\lambda_t + \lambda_{t+1}) + (f(\theta_{t-1}) - f(\theta_{t-1})) (\lambda_{t+1} - \lambda_t) + (\theta_{t+1} - \theta_t - \tilde{g}(\theta_t, \lambda_{t+1}) + g(\theta_t, \lambda_{t+1})). \]
Proof. From (72) and (73), we have that
\[ \beta_t m_{t+1} (\lambda_t - \lambda_{t+1}) \geq (\nabla_\lambda \tilde{V}_{t+1}(\theta_t, \lambda_t) - \nabla_\lambda \tilde{V}_t(\theta_{t-1}, \lambda_{t-1})) (-\lambda_t + \lambda_{t+1}) \]
\[ + (f(\theta_{t-1}) - f(\theta_{t-1}) - f(\theta_t) + f(\theta_t)) (-\lambda_t + \lambda_{t+1}). \]
(87)
The first term can be rewritten as
\[ (\nabla_\lambda \tilde{V}_{t+1}(\theta_t, \lambda_t) - \nabla_\lambda \tilde{V}_t(\theta_{t-1}, \lambda_{t-1}))(\lambda_{t+1} - \lambda_t) \]
\[
= (\nabla \tilde{V}_{t+1}(\theta_t, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_t))(\lambda_{t+1} - \lambda_t) + (\nabla \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_{t-1}))(\lambda_t - \lambda_{t-1}) + m_{t+1}(\nabla \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_{t-1})) \tag{88}
\]

The first term in (88) can be bounded as
\[
= (\nabla \tilde{V}_{t+1}(\theta_t, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_t))(\lambda_{t+1} - \lambda_t) + (\nabla \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_{t-1}))(\lambda_t - \lambda_{t-1}) + m_{t+1}(\nabla \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_{t-1})) \tag{89}
\]

where (a) is from the Cauchy–Schwarz inequality and (b) is from the \(C^V\)-smoothness of \(V^L\), for any \(h > 0\).

Similar to (89), the second term in (88) can be bounded as
\[
= (\nabla \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_{t-1}))(\lambda_t - \lambda_{t-1}) \geq \frac{b_{t-1} - b_0}{b_t - 1 + b_0} (\lambda_t - \lambda_{t-1}^2) + \frac{1}{b_{t-1} + b_0}(\nabla \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_{t-1}))^2. \tag{90}
\]

The third term in (88) can be bounded as
\[
m_{t+1}(\nabla \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_{t-1})) \geq -\frac{\epsilon}{\lambda_t}(\nabla \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_{t-1}))^2 - \frac{1}{2\epsilon}m_{t+1}^2. \tag{91}
\]

Hence combine (89) to (90), and plug in (88), we have that
\[
= (\nabla \tilde{V}_{t+1}(\theta_t, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_t))(\lambda_{t+1} - \lambda_t) \geq -\frac{(C^V)^2}{2h} ||\theta_t - \theta_{t-1}||^2 - \frac{h}{2}(\lambda_{t+1} - \lambda_t)^2 + \frac{1}{2}(b_{t-1} - b_0)(\lambda_t - \lambda_{t-1}^2) + \frac{1}{b_{t-1} + b_0}(\nabla \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_{t-1}))^2 - \frac{\epsilon}{\lambda_t}(\nabla \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_{t-1}))^2 - \frac{1}{2\epsilon}m_{t+1}^2. \tag{92}
\]

Hence (87) can be further bounded as
\[
= (\beta_t m_{t+1})(\lambda_t - \lambda_{t+1}) \geq (f(\lambda_{t-1}) - f(\lambda_t) + \beta_t m_{t+1})(\lambda_t - \lambda_{t+1}) \geq (f(\lambda_{t-1}) - f(\lambda_t) + \beta_t m_{t+1})(\lambda_t - \lambda_{t+1}) \geq -\frac{(C^V)^2}{2h} ||\theta_t - \theta_{t-1}||^2 - \frac{h}{2}(\lambda_{t+1} - \lambda_t)^2 + \frac{1}{2}(b_{t-1} - b_0)(\lambda_t - \lambda_{t-1}^2) + \frac{1}{b_{t-1} + b_0}(\nabla \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_{t-1}))^2 - \frac{\epsilon}{\lambda_t}(\nabla \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla \tilde{V}_t(\theta_{t-1}, \lambda_{t-1}))^2 - \frac{1}{2\epsilon}m_{t+1}^2. \tag{93}
\]
It can be directly verified that
\[ m_{t+1}(\lambda_t - \lambda_{t+1}) = \frac{1}{2}(\lambda_t - \lambda_{t-1})^2 - \frac{1}{2}(\lambda_t - \lambda_{t+1})^2 - \frac{m_{t+1}^2}{2}. \]  

(94)

Recall that \( \beta_t = \frac{1}{\xi} \), hence
\[
\frac{1}{2\xi}(\lambda_t - \lambda_{t-1})^2 - \frac{1}{2\xi}(\lambda_t - \lambda_{t+1})^2 - \frac{m_{t+1}^2}{2\xi} \\
\geq (f(\theta_{t-1}) - \tilde{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1}) \\
- \frac{(C_V^\sigma)^2}{2h} \|\theta_t - \theta_{t-1}\|^2 - \frac{h}{2}(\lambda_{t+1} - \lambda_t)^2 \\
+ \frac{(b_t - b_{t-1})(\lambda_{t+1} - \lambda_t)^2}{2} - \frac{(b_t - b_{t-1})}{2}(\lambda_{t+1} - \lambda_t)^2 \\
+ \frac{b_{t-1}b_0}{b_t - b_{t-1} + b_0} (\lambda_t - \lambda_{t-1})^2 + \frac{1}{b_t - b_{t-1} + b_0} (\nabla_{\lambda} \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla_{\lambda} \hat{V}_t(\theta_{t-1}, \lambda_{t-1}))^2 \\
- \frac{\xi}{2}(\nabla_{\lambda} \hat{V}_t(\theta_{t-1}, \lambda_t) - \nabla_{\lambda} \tilde{V}_t(\theta_{t-1}, \lambda_{t-1}))^2 - \frac{1}{2\xi} m_{t+1}. \]

(95)

From \( \xi \leq \frac{1}{\xi_0} \leq \frac{2}{\xi_0 + \xi + \xi_0} \), we have \( \frac{1}{b_t + b_{t-1}} (\nabla_{\lambda} \tilde{V}_t(\theta_{t-1}, \lambda_t) - \nabla_{\lambda} \hat{V}_t(\theta_{t-1}, \lambda_{t-1}))^2 - \frac{\xi}{2}(\nabla_{\lambda} \hat{V}_t(\theta_{t-1}, \lambda_t) - \nabla_{\lambda} \tilde{V}_t(\theta_{t-1}, \lambda_{t-1}))^2 \geq 0 \). Also, it can be shown that \( \frac{b_{t-1}b_0}{b_t - b_{t-1} + b_0} \geq \frac{b_{t-1}b_0}{2b_0} = \frac{b_{t-1}}{2} \). Thus, it follows that
\[
\frac{1}{2\xi}(\lambda_t - \lambda_{t-1})^2 - \frac{1}{2\xi}(\lambda_t - \lambda_{t+1})^2 - \frac{m_{t+1}^2}{2\xi} \\
\geq (f(\theta_{t-1}) - \tilde{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1}) \\
- \frac{(C_V^\sigma)^2}{2h} \|\theta_t - \theta_{t-1}\|^2 - \frac{h}{2}(\lambda_{t+1} - \lambda_t)^2 \\
+ \frac{(b_t - b_{t-1})(\lambda_{t+1} - \lambda_t)^2}{2} - \frac{(b_t - b_{t-1})}{2}(\lambda_{t+1} - \lambda_t)^2 \\
+ \frac{b_{t-1}(\lambda_t - \lambda_{t-1})^2}{2} - \frac{1}{2\xi} m_{t+1}. \]

(96)

Re-arrange the terms, it follows that
\[
- \frac{1}{2\xi}(\lambda_t - \lambda_{t+1})^2 - \frac{b_t - b_{t-1}}{2} \lambda_{t+1}^2 \\
\geq (f(\theta_{t-1}) - \tilde{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1}) - \frac{(C_V^\sigma)^2}{2h} \|\theta_t - \theta_{t-1}\|^2 - \frac{h}{2}(\lambda_{t+1} - \lambda_t)^2 \\
+ \frac{(b_t - b_{t-1})}{2}(\lambda_{t+1} - \lambda_t)^2 \frac{b_{t-1}}{2}(\lambda_t - \lambda_{t-1})^2 - \frac{1}{2\xi}(\lambda_t - \lambda_{t-1})^2 \\
\geq -\frac{1}{2\xi}(\lambda_t - \lambda_{t-1})^2 - \frac{(b_t - b_{t-1})}{2} \lambda_{t-1}^2 + (f(\theta_{t-1}) - \tilde{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1}) \\
- \frac{(C_V^\sigma)^2}{2h} \|\theta_t - \theta_{t-1}\|^2 - \frac{h}{2}(\lambda_{t+1} - \lambda_t)^2 + \frac{b_{t-1}}{2}(\lambda_t - \lambda_{t-1})^2, \]

(97)

where the last inequality is from the fact that \( b_t \) is decreasing.

Now multiply \( \frac{2}{\lambda_t} \) on both sides, we further have that
\[
- \frac{1}{\xi^2b_t}(\lambda_t - \lambda_{t+1})^2 - \frac{1}{\xi} \left( 1 - \frac{b_{t-1}}{b_t} \right) \lambda_{t+1}^2 \\
\geq - \frac{1}{\xi^2b_t}(\lambda_t - \lambda_{t-1})^2 - \frac{1}{\xi} \left( 1 - \frac{b_{t-1}}{b_t} \right) \lambda_{t-1}^2 + \frac{2}{\xi b_t} (f(\theta_{t-1}) - \tilde{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1}) \\
- \frac{(C_V^\sigma)^2}{h \xi b_t} \|\theta_t - \theta_{t-1}\|^2 - \frac{h}{\xi b_t}(\lambda_{t+1} - \lambda_t)^2 \frac{1}{\xi} (\lambda_t - \lambda_{t-1})^2. \]

(98)
If we set $h = \frac{b_t}{\xi}$, (98) can be rewritten as

$$- \frac{1}{\xi^2 b_t} (\lambda_t - \lambda_{t+1})^2 - \frac{1}{\xi} \left( 1 - \frac{b_{t-1}}{b_t} \right) \lambda_{t+1}^2$$

$$\geq - \frac{1}{\xi^2 b_t} (\lambda_t - \lambda_{t-1})^2 - \frac{1}{\xi} \left( 1 - \frac{b_{t-1}}{b_t} \right) \lambda_{t+1}^2 + \frac{2}{\xi b_t} (f(\theta_{t-1}) - \hat{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1})$$

$$- \frac{2(C^V_0)^2}{\xi b_t^2} \|\theta_t - \theta_{t-1}\|^2 - \frac{1}{2\xi} (\lambda_{t+1} - \lambda_t)^2 + \frac{1}{\xi} (\lambda_t - \lambda_{t-1})^2.$$  \hspace{1cm} (99)

Further we have that

$$- \frac{1}{\xi^2 b_{t+1}} (\lambda_t - \lambda_{t-1})^2 + \left( \frac{1}{\xi^2 b_{t+1}} - \frac{1}{\xi^2 b_t} \right) (\lambda_t - \lambda_{t-1})^2 - \frac{1}{\xi} \left( 1 - \frac{b_{t-1}}{b_t} \right) \lambda_{t+1}^2 + \frac{2}{\xi^2 b_t} (f(\theta_{t-1}) - \hat{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1})$$

$$- \frac{2(C^V_0)^2}{\xi b_t^2} \|\theta_t - \theta_{t-1}\|^2 - \frac{1}{2\xi} (\lambda_{t+1} - \lambda_t)^2 + \frac{1}{\xi} (\lambda_t - \lambda_{t-1})^2.$$  \hspace{1cm} (100)

Re-arranging the terms in (100) implies that

$$- \frac{1}{\xi^2 b_{t+1}} (\lambda_t - \lambda_{t-1})^2 - \frac{1}{\xi} \left( 1 - \frac{b_t}{b_{t+1}} \right) \lambda_{t+1}^2 - \left( \frac{1}{\xi^2 b_{t+1}} - \frac{1}{\xi^2 b_t} \right) (\lambda_t - \lambda_{t-1})^2 - \frac{1}{\xi} \left( 1 - \frac{b_{t-1}}{b_t} \right) \lambda_{t+1}^2$$

$$\geq - \frac{1}{\xi^2 b_{t+1}} (\lambda_t - \lambda_{t-1})^2 - \frac{1}{\xi} \left( 1 - \frac{b_{t-1}}{b_t} \right) \lambda_{t+1}^2 + \frac{2}{\xi b_t} (f(\theta_{t-1}) - \hat{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1})$$

$$- \frac{2(C^V_0)^2}{\xi b_t^2} \|\theta_t - \theta_{t-1}\|^2 - \frac{1}{2\xi} (\lambda_{t+1} - \lambda_t)^2 + \frac{1}{\xi} (\lambda_t - \lambda_{t-1})^2$$

$$\geq - \frac{1}{\xi^2 b_{t+1}} (\lambda_t - \lambda_{t-1})^2 + \frac{2}{\xi b_t} (f(\theta_{t-1}) - \hat{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1})$$

$$\geq - \frac{1}{\xi^2 b_{t+1}} (\lambda_t - \lambda_{t-1})^2 + \frac{2}{\xi b_t} (f(\theta_{t-1}) - \hat{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1})$$

$$\geq - \frac{1}{\xi^2 b_{t+1}} (\lambda_t - \lambda_{t-1})^2 + \frac{2}{\xi b_t} (f(\theta_{t-1}) - \hat{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1})$$

$$\geq - \frac{1}{\xi^2 b_{t+1}} (\lambda_t - \lambda_{t-1})^2 + \frac{2}{\xi b_t} (f(\theta_{t-1}) - \hat{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1})$$

where the last inequality is from $\frac{1}{\xi^2 b_{t+1}} - \frac{1}{\xi b_t} \leq \frac{\lambda}{\xi}$. Recall in Lemma 8 we showed that

$$V^\sigma_\theta^L(\theta_{t+1}, \lambda_{t+1}) - V^\sigma_L(\theta_t, \lambda_t)$$

$$\geq (f(\theta_{t-1}) - \hat{f}(\theta_{t-1})) (\lambda_{t+1} - \lambda_t) + (\theta_{t+1} - \theta_t, -\hat{g}(\theta_t, \lambda_{t+1}) + g(\theta_t, \lambda_{t+1})) - \lambda \left( \frac{2(C^V_0)^2}{\xi b_t^2} \right) \|\theta_t - \theta_{t-1}\|^2$$

$$\geq \left( \frac{\mu_t}{2} + \nu \right) \|\theta_{t+1} - \theta_t\|^2 + \frac{b_{t-1}}{2} (\lambda_{t+1} - \lambda_t)^2 - \frac{1}{2\xi} (\lambda_{t+1} - \lambda_t)^2.$$  \hspace{1cm} (102)

Combine both inequality together, and we further have that

$$- \frac{8}{\xi^2 b_{t+1}} (\lambda_t - \lambda_{t-1})^2 - \frac{8}{\xi} \left( 1 - \frac{b_t}{b_{t+1}} \right) \lambda_{t+1}^2 - \left( \frac{8}{\xi^2 b_t} (\lambda_t - \lambda_{t-1})^2 - \frac{8}{\xi} \left( 1 - \frac{b_{t-1}}{b_t} \right) \lambda_{t+1}^2 \right)$$

$$+ V^L_\theta(\theta_{t+1}, \lambda_{t+1}) - V^L_\theta(\theta_t, \lambda_t)$$

$$\geq - \frac{8}{\xi^2 b_{t+1}} (\lambda_t - \lambda_{t-1})^2 - \frac{16(C^V_0)^2}{\xi b_t^2} \|\theta_t - \theta_{t-1}\|^2 + \frac{8}{\xi} (\lambda_t - \lambda_{t-1})^2 + \frac{8}{\xi} \left( \frac{b_t}{b_{t+1}} - \frac{b_{t-1}}{b_t} \right) \lambda_{t+1}^2$$

$$+ \left( \frac{\mu_t}{2} + \nu \right) \|\theta_{t+1} - \theta_t\|^2 + \frac{b_{t-1}}{2} (\lambda_{t+1}^2 - \lambda_t^2) - \frac{1}{2\xi} (\lambda_{t+1}^2 - \lambda_t^2) - \frac{1}{2\xi} (\lambda_{t+1} - \lambda_t)^2.$$  \hspace{1cm} (103)
We now restate Theorem 2 with all the specific step sizes. The definitions of these constants can also be found in
which then completes the proof.

\[ S_t = \left( -\frac{16(C_V)^2}{\xi b_t^2} - \frac{\xi(C_V)^2}{2} \right) \| \theta_t - \theta_{t-1} \|^2 + \left( -\frac{28}{5\xi} - \frac{1}{\xi} \right) (-\lambda_t + \lambda_{t+1})^2 + \frac{b_t - b_{t-1}}{2} \lambda_t^2, \]

where \( S_t \triangleq \frac{16}{b_t} \left( f(\theta_{t-1}) - \hat{f}(\theta_t) - f(\theta_t) + \hat{f}(\theta_t) \right) (-\lambda_t + \lambda_{t+1}) + (f(\theta_{t-1}) - \hat{f}(\theta_t)) (\lambda_{t+1} - \lambda_t) + (\theta_t + \theta_{t-1} - \hat{g}(\theta_t, \lambda_{t+1}) + g(\theta_t, \lambda_{t+1})). \) Now

\[ -\frac{8}{\xi^2 b_t} (\lambda_t - \lambda_{t-1})^2 \geq \frac{1}{\xi b_t} \left( \lambda_t - \lambda_{t-1} \right)^2, \]

\[ + V_t \left( \theta_{t-1}, \tau \right) - V_t \left( \theta_t, \lambda_t \right) + \frac{b_t}{2} \lambda_{t+1}^2 - \frac{b_{t-1}}{2} \lambda_{t-1}^2, \]

\[ + \left( -\frac{16(C_V)^2}{\xi b_t^2} - \frac{\xi(C_V)^2}{2} \right) \| \theta_t - \theta_{t-1} \|^2, \]

\[ + \left( -\frac{28}{5\xi} - \frac{1}{\xi} \right) (-\lambda_t + \lambda_{t-1})^2 - \frac{8}{\xi^2 b_t} (\lambda_t - \lambda_{t-1})^2 \]

\[ \geq S_t + \left( \frac{\mu_t}{2} + \nu - \frac{16(C_V)^2}{\xi b_t^2} - \frac{\xi(C_V)^2}{2} \right) \| \theta_{t+1} - \theta_t \|^2 + \frac{b_t - b_{t-1}}{2} \lambda_{t+1}^2. \]

which then completes the proof.

We now restate Theorem 2 with all the specific step sizes. The definitions of these constants can also be found in Section II.

**Theorem 3.** (Restatement of Theorem 2) Set \( b_t = \frac{19}{20 \xi f_{\theta_t}}, \mu_t = \xi(C_V)^2 + \frac{16 \xi(C_V)^2}{\xi (b_t + 1)} - 2\nu \), \( \beta_t = \frac{1}{\xi}, \alpha_t = \nu + \mu_t \),

where \( \xi > \frac{2n(1+\lambda^*) L_{\nu}}{(1+\lambda^*) C_V^2} \) is any positive number and \( \tau \) is any number greater than 2. Moreover, set \( \epsilon_{\text{est}} = \frac{1}{(\frac{2n}{\tau} + 1) C_V} \), \( \frac{1}{3200 (\tau - 2)(C_V)^2 u L_{\nu}} \), then

\[ \min_{1 \leq t \leq T} \| G_t \|^2 \leq (1 + \sqrt{2}) \epsilon, \]

when \( T = O(\epsilon^{-4}). \)

**Proof.** Denote by \( p_t \triangleq \frac{1}{2}(\sigma - 2)(C_V)^2 \) and \( M_1 \triangleq \frac{16}{(\xi - 2)} + \frac{\xi(C_V)^2}{2} \frac{\xi(C_V)^2}{2} - \frac{16(C_V)^2}{\xi b_t^2} = p_t. \) Then [104] can be rewritten as

\[ F_{t+1} - F_t \geq S_t + p_t \| \theta_{t+1} - \theta_t \|^2 + \frac{b_t - b_{t-1}}{2} \lambda_{t+1}^2, \]

\[ \frac{9}{10 \xi} (\lambda_{t+1} - \lambda_t)^2 + \frac{8}{\xi b_{t+1}} \left( \frac{b_t}{b_{t+1}} - \frac{b_{t-1}}{b_t} \right) \lambda_{t+1}^2. \]

From the definition, we have that

\[ G_t = \left[ \begin{array}{c} \beta_t \left( \lambda_t - \Pi_{[0, \lambda^*]} \left( \lambda_t - \frac{1}{\beta_t} \left( V^e_{\theta_t} (\theta_t, \lambda_t) \right) \right) \right) \\ \alpha_t \left( \theta_t - \Pi_{\Lambda_\alpha} \left( \theta_t + \frac{1}{\alpha_t} \left( V_{g^e} (\theta_t, \lambda_t) \right) \right) \right) \end{array} \right], \]
and denote by
\[
\tilde{G}_t \triangleq \left[ \beta_t \left( \lambda_t - \prod_{0 \neq \theta} \left( \lambda_t - \frac{1}{\alpha_t} \left( \nabla_{\lambda} V_t(\theta_t, \lambda_t) \right) \right) \right) \right].
\] (108)

It can be verified that
\[
\|G_t\| - \|\tilde{G}_t\| \leq \beta_{t-1} |\lambda_t|.
\] (109)

From Theorem 4.2 in [Xu et al. 2020], it can be shown that
\[
\|\tilde{G}_t\|^2 \leq 2(\mu_t + \nu)^2 \|\theta_{t+1} - \theta_t\|^2 + \left( 2(C_{\sigma}^2)^2 + \frac{1}{\xi^2} \right) (\lambda_{t+1} - \lambda_t)^2,
\] (110)

and
\[
M_1 \geq \frac{(2\mu_t + \nu)^2}{p_t^2}.
\] (111)

Hence
\[
\|\tilde{G}_t\|^2 \leq M_1 p_t^2 \|\theta_{t+1} - \theta_t\|^2 + \left( 2(C_{\sigma}^2)^2 + \frac{1}{\xi^2} \right) (\lambda_{t+1} - \lambda_t)^2.
\] (112)

Set \( u_t \triangleq \max \left\{ \frac{1}{M_1 p_t^2}, \frac{10+20(2(C_{\sigma}^2)^2)}{\xi} \right\} \), then from (106), we have that
\[
u_t \|\tilde{G}_t\|^2 \leq F_{t+1} - F_t - S_t \quad - \frac{b_t}{2} \lambda_{t+1} - \frac{8}{\xi} \left( \frac{b_t}{b_{t+1}} - \frac{b_{t-1}}{b_t} \right) \lambda_{t+1}^2.
\] (113)

Summing the inequality above from \( t = 1 \) to \( T \), then
\[
\sum_{t=1}^T u_t \|\tilde{G}_t\|^2 \leq F_{T+1} - F_1 - \sum_{t=1}^T S_t + \frac{8}{\xi} \left( \frac{b_t}{b_{t+1}} \lambda_{t+1}^2 - \frac{b_{t+1}}{b_{t+1}} \lambda_{T+1}^2 \right) + \left( \frac{b_0}{\xi} - \frac{b_T}{\xi} \right) (A^*)^2,
\] (114)

which is from \( b_t \) is decreasing and \( \lambda_t < A^* \). Note that
\[
\max_{t \geq 1} \max_{\theta \in \Theta, \lambda \in [0, \Lambda]} F_t = \max \left\{ \frac{-8}{\xi^2 b_{t+1}} (\lambda_t - \lambda_{t+1})^2 - \frac{8}{\xi} \left( 1 - \frac{b_t}{b_{t+1}} \right) \lambda_{t+1}^2 + V_{\lambda}^* (\theta_{t+1}, \lambda_{t+1}) + \frac{b_t}{2} \lambda_{t+1}^2 + \frac{15}{2\xi} (A^*)^2 \right\},
\] (115)

which is from the definition of \( b_t \), and \( 8(\frac{b_t}{b_{t+1}} - 1) \leq 8(\frac{(t+1)}{p_t^2} - 1) \leq 8(\frac{20.25}{1} - 1) < 1.6 \). Then plugging in the definition of \( b_t \) implies that
\[
\sum_{t=1}^T u_t \|\tilde{G}_t\|^2 \leq F^* - F_1 - \sum_{t=1}^T S_t + \frac{8}{\xi} (A^*)^2 + \left( \frac{b_0}{\xi} (A^*)^2 \right).
\] (116)

If moreover set \( u \triangleq \max \left\{ M_1, \frac{10+20(2(C_{\sigma}^2)^2)}{9\xi^2} \right\} \), then \( u_t \geq \frac{1}{u_{p_t^2}} \), and hence
\[
\frac{\sum_{t=1}^T \frac{1}{p_t} \|\tilde{G}_t\|^2}{\sum_{t=1}^T \frac{1}{p_t}} \leq \frac{u}{\sum_{t=1}^T \frac{1}{p_t}} \left( F^* - F_1 - \sum_{t=1}^T S_t + \frac{8}{\xi} (A^*)^2 + \left( \frac{b_0}{\xi} (A^*)^2 \right) \right).
\] (117)

26
Plug in the definition of $p_t$ then we have that
\[
\sum_{t=1}^{T} \frac{1}{p_t} \| \hat{G}_t \|^2 \leq \frac{3200\xi (\tau - 2)(C^V_{\sigma})^2 d}{19^2 (\sqrt{T} - 2)} \left( F^* - F_1 - \sum_{t=1}^{T} S_t + \frac{8}{\xi} (A^*)^2 + \left( \frac{b_0}{2} (A^*)^2 \right) \right). \tag{118}
\]
We moreover have that
\[
|S_t| = \frac{16}{b_t\xi} (f(\theta_{t-1}) - \hat{f}(\theta_{t-1}) - f(\theta_t) + \hat{f}(\theta_t))(-\lambda_t + \lambda_{t+1}) + (f(\theta_{t-1}) - \hat{f}(\theta_{t-1}))(\lambda_{t+1} - \lambda_t)
\]

\[+ (\theta_{t+1} - \theta_t, -\hat{g}(\theta_t, \lambda_{t+1}) + g(\theta_t, \lambda_{t+1})) \]
\[\leq 32t^{0.25} A^*(\Omega_{t-1} + \Omega_t) + 2A^*\Omega_{t-1} + \frac{1}{\alpha_t} (1 + A^*) C^V_{\sigma} \Omega_t, \tag{119}\]

where $\Omega_t = \max \left\{ \| g(\theta_t, \lambda_{t+1}) - \hat{g}(\theta_t, \lambda_{t+1}) \|, \| f(\theta_t) - \hat{f}(\theta_t) \| \right\}$. Note that it has been shown in Wang and Zou [2022] that $\Omega_t \leq L_{\Omega} \max \left\{ \| Q_{\sigma,r} - \hat{Q}_{\sigma,r} \|, \| Q_{\sigma,c} - \hat{Q}_{\sigma,c} \| \right\} = L_{\Omega} \varepsilon_{\text{est}}$, and hence $\Omega_t$ can be controlled by setting $\varepsilon_{\text{est}}$.

Note that $\alpha_t = \nu + \mu_t$ is increasing, hence $\frac{1}{\alpha_t} \leq \frac{1}{\alpha_1}$. Hence if we set $\varepsilon_{\text{est}} = \frac{1}{80 L_{\Omega}} \frac{3200 \xi (\tau - 2)(C^V_{\sigma})^2 u L_{\Omega}}{19^2 \xi^2}$, then
\[
|S_t| \leq \frac{1}{\xi} \frac{19^2 \xi^2}{3200 \xi (\tau - 2)(C^V_{\sigma})^2 u L_{\Omega}}, \tag{120}\]

and hence
\[
\sum_{t=1}^{T} S_t \leq \sqrt{T} \frac{19^2 \xi^2}{3200 \xi (\tau - 2)(C^V_{\sigma})^2 u L_{\Omega}}. \tag{121}\]

Thus plug in (118) and we have that
\[
\sum_{t=1}^{T} \frac{1}{p_t} \| \hat{G}_t \|^2 \leq \frac{3200\xi (\tau - 2)(C^V_{\sigma})^2 u K + \varepsilon^2}{19^2 (\sqrt{T} - 2)}, \tag{122}\]

where $K = F^* - F_1 + \frac{8}{\xi} (A^*)^2 + \left( \frac{b_0}{2} (A^*)^2 \right)$. When $T = \left( 2 + \frac{3200\xi (\tau - 2)(C^V_{\sigma})^2 u K}{19^2 \xi^2} \right)^2$, we have that
\[
\sum_{t=1}^{T} \frac{1}{p_t} \| \hat{G}_t \|^2 \leq 2\varepsilon^2. \tag{123}\]

Similarly to Theorem 4.2 in Xu et al. [2020], if $t > \frac{19^4 (A^*)^4}{2 \xi^2 10^2 \xi^2 \varepsilon^2}$, then $b_{t-1} < \frac{\varepsilon}{4T}$ and $b_{t-1} \lambda_t < \varepsilon$. Hence combine with (109) we finally have that
\[
\min_{1 \leq t \leq T} \| G_t \| \leq (1 + \sqrt{2})\varepsilon, \tag{124}\]

when $T = \max \left\{ \frac{7(A^*)^4}{\xi^2 T^4}, \left( 2 + \frac{9\xi (\tau - 2)(C^V_{\sigma})^2 u K}{\xi^2 T^4} \right)^2 \right\} = O(\varepsilon^{-4}).$ \hfill \qed

Remark 1. Note that the sample complexity of robust TD algorithm to achieve an $\epsilon_{\text{est}}$-error bound is $O(\epsilon_{\text{est}}^2)$, hence the sample complexity at the time step $t$ is $O(\epsilon_{\text{est}}^2) = O(\frac{\xi^2 \nu^2}{\epsilon^2})$. Thus the total sample complexity to find an $\epsilon$-stationary solution is $\sum_{t=1}^{T} \frac{1}{p_t} = O(\epsilon^{-14})$. This great increasing of complexity is due to the estimation of robust value functions.

I Constants

In this section, we summarize the definitions of all the constants we used in this paper.

$$L_V = \frac{k |A|}{(1 - \gamma)^2},$$
\[ C_\sigma = \frac{1}{1-\gamma} \left( 1 + 2\gamma\delta \frac{\log |S|}{\sigma} \right), \]

\[ C^V_\sigma = \frac{1}{1-\gamma} |A| kC_\sigma, \]

\[ k_B = \frac{1}{1-\gamma + \gamma\delta} \left( |A|C_\sigma I + |A| kC^V_\sigma \right) + \frac{2|A|^2 \gamma (1-\delta)}{(1-\gamma + \gamma\delta)^2} k^2 C_\sigma, \]

\[ L_\sigma = k_B + \frac{\gamma\delta}{1-\gamma} \left( \sqrt{|S|k_B + 2\sigma |S| C^V_\sigma} \frac{1}{1-\gamma + \gamma\delta} k |A| C_\sigma \right), \]

\[ b_t = \frac{19}{20} \xi^{0.25}, \]

\[ M_1 = \frac{16\tau^2}{(\tau - 2)^2} + \frac{(\xi (C^V_\sigma)^2 - \nu)^2}{64(\tau - 2)^2 (C^V_\sigma)^2 \xi^2}, \]

\[ u = \max \left\{ M_1, \frac{10 + 20 \xi^2 (C^V_\sigma)^2}{9\xi p_2} \right\}, \]

\[ F^* = \frac{1.6}{\xi} (A^*)^2 + (1 + A^*)(2C_\sigma) + \frac{b_1}{2} (A^*)^2 + \frac{15}{2\xi} (A^*)^2, \]

\[ K = F^* - F_1 + \frac{8}{\xi} (A^*)^2 + \left( \frac{b_1}{2} (A^*)^2 \right), \]

\[ \mu_t = \xi (C^V_\sigma)^2 + \frac{16\tau (C^V_\sigma)^2}{\xi (b_{t+1})^2} - 2\nu, \]

\[ \beta_t = \frac{1}{\xi}, \]

\[ \alpha_t = \nu + \mu_t. \]