I. INTRODUCTION

In this paper we study some common properties of static spherically symmetric configurations of General Relativity with several non-linear scalar fields.

Scalar field models are widely used in relativistic gravitational physics as elements of alternative gravitational theories [1, 2], especially in cosmology within approaches to the dark energy problem [3, 4]. Therefore, it is natural to ask how the scalar field (SF) works in compact astrophysical objects. In particular, recent publications of the Event Horizon Telescope (EHT) results [7, 8] have increased interest in black hole (BH) mimickers, which differ from BH but can give similar images of the radiating material around these objects. Indeed, within the EHT angular resolution it is difficult to rule out a number of alternative astrophysical objects (see, e.g., [5, 6] and references therein). Theories with additional scalar fields, which model the ubiquitous "Dark Energy", create a suitable soil where such mimickers can grow [11, 12].

It is well known that introduction of SF in models of compact astrophysical objects may lead to important consequences. For example, the arbitrarily small free SF affects the space-time topology, e.g., leading to a naked singularity (NS). This is clearly seen in case of the Fisher analytic solution [13, 14]. This is closely related to the famous Bekenstein theorems [17, 18] (see, also [19] and references therein), which prohibit existence of horizons in presence of SF. Though, space-times with topological properties quite different from those of BH can have similar observational properties from the perspective of a distant observer [11, 12, 20, 22].

Note that models of spherically symmetric compact objects with linear SFs are best studied for massless linear SF, starting from the early publications [13, 16]; more complicated SF potentials have been considered elsewhere (see, e.g., [23, 24]). As we will see below, main asymptotic properties of the solutions are common for a fairly wide class of the SF potentials.

One of main issues of the present study concerns the domain of regular solutions and occurrence of singularities. Indeed, the singular behavior is very common in case of nonlinear differential equations and it is not evident that there is no singularities outside the center (that is, for \( r > 0 \) in the Schwarzschild (curvature) coordinates). In particular, equations of a static self-interacting SF in special relativity (i.e. on the fixed Minkowsky background) can have solutions with singularities at arbitrary spatial points (see Appendix A).

In this paper we study asymptotically flat space-times in presence of \( N \) nonlinear SFs, which represent static spherically symmetric configurations of General Relativity. The scalar fields are minimally coupled to gravity. In Section I we formulate general requirements for the SF potentials and the solutions and introduce various forms of the basic equations. The potentials are assumed to be exponentially bounded and to fulfill conditions analogous to that used in the proof of the Bekenstein [19]
II. BASIC RELATIONS

We consider \( N \) real scalar fields \( \Phi = \{ \phi_1, ..., \phi_N \} \) that are described by Lagrangian density

\[
L = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial \phi_i}{\partial \xi} \frac{\partial}{\partial \phi_i} \Phi - V(\Phi).
\]

Throughout the paper we assume that \( V(\Phi) \) is a twice continuously-differentiable function,

\[
V(\Phi) \geq 0,
\]

and

\[
\phi_i V'_i(\Phi) \geq 0, \quad i = 1, ..., N,
\]

where \( V'_i = \partial V/\partial \phi_i \).

Assumptions \([2, 3]\) can be fulfilled, e.g., in case of a polynomial potential

\[
V(\Phi) = \sum_{n_1 + ... + n_N \geq 1} w_{n_1, ..., n_N} \prod_{i=1}^{N} \phi_i^{2n_i}
\]

where \( w_{n_1, ..., n_N} \geq 0 \).

Also we assume that there exist positive constants \( C_0, C'_0, \kappa, \kappa' \) such that for all \( \Phi \):

\[
|V(\Phi)| < C_0 \exp(\kappa \| \Phi \|),
\]

and for all \( i = 1, 2, ..., N \)

\[
\left| \frac{\partial V}{\partial \phi_i} \right| < C'_0 \exp(\kappa' \| \Phi \|).
\]

Evidently, estimates \([5, 6]\) are fulfilled for any finite degree polynomial including example \([4]\). One can show, if \([6]\) is valid, then there exist some constants \( C_0, \kappa \) such that \([5]\) is also valid; i.e. in fact only \([6]\) is necessary.

The space-time endowed with the metric \( g_{\mu\nu} \) subject to the Einstein equations\(^1\)

\[
G^\nu_\mu = 8\pi T^\nu_\mu
\]

is assumed to be asymptotically flat. The field equations

\[
g^{\mu\nu} \nabla_\mu \nabla_\nu \phi_i = -V'_i(\Phi), \quad i = 1, ..., N;
\]

follow from \([1]\).

The energy-momentum tensor of the scalar fields is

\[
T_{\mu\nu} = \sum_{i=1}^{N} \partial_\mu \phi_i \partial_\nu \phi_i - g_{\mu\nu} L.
\]

We work with a static spherically symmetric space-time metric in curvature coordinates

\[
ds^2 = e^{\alpha(r)} dr^2 - e^{\beta(r)} d\varphi^2 - r^2 dO^2,
\]

where \( dO^2 = d\varphi^2 + (\sin \theta)^2 d\varphi^2 \); radial variable \( r > 0 \).

In case of metric \([10]\) the Einstein equations yield

\[
\frac{d}{dr} \left[ r (e^{-\beta} - 1) \right] = -8\pi r^2 T^0_0,
\]

where \( T^0_0 = e^{-\beta} \sum_{i=1}^{N} \phi_i^2/2 + V(\Phi), \quad \phi'_i = d\phi_i/dr \),

\[
re^{-\beta} \frac{d\alpha}{dr} + e^{-\beta} - 1 = -8\pi r^2 T^1_1,
\]

where \( T^1_1 = -e^{-\beta} \sum_{i=1}^{N} \phi_i^2/2 + V(\Phi) \).

Equations \([8]\) yield

\[
\frac{d}{dr} \left[ r^2 e^{\alpha - \frac{\beta}{2}} \phi'_i \right] = r^2 e^{\alpha + \frac{\beta}{2}} V'_i(\Phi),
\]

\( i = 1, ..., N \).

In view of the asymptotic flatness we assume

\[
\lim_{r \to \infty} \left[ r (e^{-\alpha} - 1) \right] = \lim_{r \to \infty} \left[ r (e^{-\beta} - 1) \right] = -r_g,
\]

where \( r_g = 2M \) and \( M > 0 \) is the configuration mass.

It can be assumed that at spatial infinity the SF components behave as independent fields in the flat space and tend to zero. We assume \( \Phi(r) \to 0 \) for \( r \to \infty \) and

\[
\exists K: \quad r^2 \| \Phi'(r) \| < K < \infty,
\]

whence also

\[
r\| \Phi(r) \| < K,
\]

where \( \| \Phi \| \) stands for the Euclidean norm of the \( N \)-component vector \( \Phi \). Stronger restrictions for SF can

\(^1\) Units: \( G=c=1 \)
be assumed for some potentials (see, Appendix A), but conditions (15, 16) are sufficient for our purposes.

Definition. Functions $\alpha(r), \beta(r) \in C^1$ and $\Phi(r) \in C^2$ will be said to be a solution of equations (11 - 13) on $(r_0, \infty)$, $r_0 \geq 0$, if they satisfy these equations on $(r_0, \infty)$ and conditions (14, 15, 16).

Equations (11, 12) are equivalent to

$$\alpha' + \beta' = 8\pi r \sum_{i=1}^{N} \phi_i^2, \quad (17)$$

$$\beta' - \alpha' = \frac{2}{r} + e^\beta \left[ 16\pi r V(\Phi) - \frac{2}{r} \right]. \quad (18)$$

Following 13, instead of $\alpha$ and $\beta$ we introduce new (positive) variables

$$X = e^{(\alpha+\beta)/2}, \quad Y = re^{(\alpha-\beta)/2}, \quad (19)$$

satisfying, in view of 14,

$$\lim_{r \to \infty} [r(X-1)] = 0, \quad \lim_{r \to \infty} (Y-r) = -r_g. \quad (20)$$

Also we introduce

$$Z_i = -rY \phi_i', \quad i = 1, ..., N. \quad (21)$$

Conditions (16, 15) and the second condition of (20) yield

$$|Z_i(r)| < K, \quad \lim_{r \to \infty} [\phi_i(r) Z_i(r)] = 0. \quad (22)$$

After simple transformations from (17, 18) we get equivalent system

$$\frac{dX}{dr} = 4\pi r \frac{X}{rY^2} \sum_{i=1}^{N} Z_i^2, \quad (23)$$

$$\frac{dY}{dr} = X \left[ 1 - 8\pi r^2 V(\Phi) \right]. \quad (24)$$

Equation (13) is reduced to a pair of the first-order equations

$$\frac{dZ_i}{dr} = -r^2 XV_i', \quad (25)$$

$$i = 1, ..., N,$$

$$\frac{d\phi_i}{dr} = -\frac{Z_i}{rY}. \quad (26)$$

### III. REGULARITY OF SOLUTIONS FOR $r > 0$

As we mentioned above, for a nonlinear SF it is necessary to be careful about the global behavior of solutions in connection with possible singularities that may arise when we continue the solutions from infinity to smaller values of the radial variable (see the example of Appendix A in case of the Minkowski space-time).

In this Section we analyze the joint system (11, 12) or equivalent system (23, 25) and state some general conditions guaranteeing that the scalar field and the metric (10) is regular for all $r > 0$.

Below we use monotonicity properties of solutions following from condition (3). Using (19, 21, 25, 26), we get

$$- \frac{d}{dr} (\phi_i Z_i) = \frac{Z_i^2}{rY} + r^2 X \phi_i V_i'. \quad (27)$$

**Lemma 1.** Let condition (3) is valid for all $\Phi$, $\alpha(r), \beta(r)$ are continuously differentiable on $(r_0, \infty)$, $r_0 \geq 0$, and satisfy (14); $\phi_i(r) \in C^2$ is a non-trivial solution of (13) on this interval with conditions (15, 16). Then functions $\phi_i(r)$, $Z_i(r)$ and $d\phi_i/dr$ do not change their signs, $\phi_i(r) Z_i(r) > 0$ and $\phi_i(r) d\phi_i/dr < 0$ on $(r_0, \infty)$.

**Proof.** The right-hand side of (27) is non-negative for a non-trivial $\phi_i$ and $\phi_i(r) Z_i(r)$ is monotonically decreasing. Taking into account the limiting value (22), in case of a non-trivial $\phi_i(r)$ this yields strict inequality $\phi_i(r) Z_i(r) > 0$. This means that $\phi_i(r)$ and $Z_i(r)$ cannot change their signs. This proves all the statements of this Lemma.

Further we assume that at least for one component of $\Phi$ is non-trivial: $\phi_i(r) \neq 0$.

**Lemma 2.** Let conditions (2, 3) are fulfilled, functions $\alpha(r), \beta(r), \Phi(r) \in C^1$ satisfy equations (13, 17, 18) and $\phi_i(r) \neq 0$ for $i = 1, ..., N$ in $(r_0, r_1)$, where $0 < r_0 < r_1 < \infty$. Then there exists $\eta_0 > 0 : Y(r) > \eta_0$ and $S_i Z_i(r) > S_i Z_i(r_1) > 0$, where $S_i = \text{sign } \phi_i$.

**Proof.** We use system (23, 25).

Let for some $i$ we have $\phi_i(r) \neq 0$. In view of Lemma 1 we can assume $\phi_i(r) > 0$, $Z_i(r) > 0$, $\phi_i(r) < 0$ without loss of generality. Then $\phi_i(r)$ is monotonically decreasing. In view of (25) and (3), $Z_i(r) > 0$ is decreasing and $Z_i(r) > Z_i(r_1)$ for $r < r_1$. Analogously, inequality $S_i Z_i(r) > S_i Z_i(r_1)$ is fulfilled for the other non-trivial SF components.

In view of (23), function $X(r)$ is monotonically increasing. From (23, 24) we have for $r < r_1$

$$\frac{1}{Y^2} \frac{dY}{dX} = \frac{r}{4\pi \sum_{i=1}^{N} Z_i^2} \left[ 1 - 8\pi r^2 V(\Phi) \right] \leq$$

(11)

using (2)

$$\leq \frac{r_1}{4\pi \sum_{i=1}^{N} Z_i^2(r_1)} \leq \frac{r_1}{4\pi \sum_{i=1}^{N} Z_i^2(r_1)},$$

where we take into account $X' > 0$, and we used the monotonicity properties of $S_i Z_i$. Integration of this in-
equality yields
\[ \frac{1}{Y(r)} \leq \frac{1}{Y_1} + \frac{r_1 X_1}{4\pi \sum_{i=1}^N Z_i^2(r_1)} \]

Therefore, \( 1/Y(r) > 0 \) is bounded and \( \exists \eta_0 > 0 : Y(r) > \eta_0 \). The Lemma 2 is proved.

On account of this Lemma we see if there is a non-trivial component \( \phi_i(r) \neq 0 \) that satisfies equations (13, 17, 18) in \( (r_0, r_1) \), where \( 0 < r_0 < r_1 < \infty \), then there exists \( \eta_0 > 0 : Y(r) > \eta_0 \).

Lemma 3. Let the conditions (3, 4) are fulfilled and functions \( \alpha(r), \beta(r) \in C^1, \Phi(r) \in C^2, \phi_i \neq 0 \) (at least for some \( i \)) satisfy equations (17, 18) and (13) on \( (r_0, r_1) \), where \( 0 < r_0 < r_1 < \infty \). Then there exist finite limits
\[ \bar{Y}(r_0) = \lim_{r \to r_0^+} Y(r) > 0, \quad \bar{Z}_i(r_0) = \lim_{r \to r_0^+} Z_i(r) > 0, \quad (28) \]
\[ \bar{X}(r_0) = \lim_{r \to r_0^+} X(r) > 0, \quad \bar{\phi}_i(r_0) = \lim_{r \to r_0^+} \phi_i(r) \neq 0. \quad (29) \]

Proof. According to the assumption of this Lemma, \( X(r_1) \) is finite. Let \( r_0 < r \leq r_1 \). Equation (17) on account of (19) yields
\[ X(r) = X(r_1) \exp \left\{ -4\pi \int_r^{r_1} x \sum_{i=1}^N \phi_i^2(x) dx \right\} \quad (30) \]
Using the Cauchy – Bunyakovsky – Schwarz inequality we have for \( r < r_1 \)
\[ |\phi_i(r) - \phi_i(r_1)| = \left| \int_r^{r_1} \phi_i'(x) \sqrt{x} \cdot \frac{1}{\sqrt{x}} \cdot dx \right| \leq \int_r^{r_1} |\phi_i'(x)| \sqrt{x} \cdot \frac{1}{\sqrt{x}} \cdot dx \leq \int_r^{r_1} x |\phi_i'(x)|^2 dx \ln(r_1/r) . \]
Then
\[ \int_r^{r_1} x |\phi_i'(x)|^2 dx \geq \frac{|\phi_i(r) - \phi_i(r_1)|^2}{\ln(r_1/r)}. \quad (31) \]
whence using (30) we have
\[ X(r) \leq X(r_1) \exp \left\{ -4\pi \sum_{i=1}^N \frac{|\phi_i(r) - \phi_i(r_1)|^2}{\ln(r_1/r)} \right\}, \quad (32) \]
and we strengthen this inequality by replacing \( \ln(r_1/r) \) by \( \ln(r_1/r_0) \):
\[ X(r) \leq X(r_1) \exp \left\{ -4\pi \sum_{i=1}^N \frac{|\phi_i(r) - \phi_i(r_1)|^2}{\ln(r_1/r_0)} \right\}. \quad (33) \]

Denote
\[ B(r) = X(r)|V(\Phi(r))|, \quad B'(r) = X(r)|V'(\Phi(r))|, \quad (34) \]
As \( \|\Phi\| \leq \sum_{i=1}^N |\phi_i| \), then according to (33) and (5) we obtain
\[ B(r) \leq \left\{ \sum_{i=1}^N \left[ -4\pi \frac{|\phi_i(r) - \phi_i(r_1)|^2}{\ln(r_1/r_0)} + k|\phi_i(r)| \right] \right\} \quad (35) \]
where \( C_1 = X(r_1)C_0 > 0 \). Term \( 4\pi |\phi(r)|^2/\ln(r_1/r_0) \) dominates the exponent for \( \phi \to \infty \), the expression in the exponent as a function of \( \phi \) has maximum, so \( B(r) \) is uniformly bounded for \( r \to r_0 + 0, r_0 > 0 \). Analogous consideration shows that \( B'(r) \) is also bounded (even if \( \phi_i \to \infty \)). Then expressions (34) and the right-hand sides of (24), (25) are bounded, integrable yielding the existence of limits \( \bar{Y}(r_0), \bar{Z}_i(r_0) \). Inequalities \( \bar{Y}(r_0) > 0, \quad S_i \bar{Z}_i(r_0) > 0 \) follow from considerations of Lemmas 1.2, from (26) in view of Lemma 1 follows that \( |d\phi_i/dr| \) and \( \phi_i(r) \) are bounded and have limits for \( r \to r_0 \). Existence of \( \bar{X}(r_0) > 0 \) follows either from (23) or directly from (17) in view of the previous results. The Lemma 3 is proved.

We summarize the above statements in the form of the following

Theorem. Let the SF potential satisfies conditions (2, 3) and (4, 5) for all \( \Phi \). Let \( \alpha(r), \beta(r), \in C^1, \Phi(r) \in C^2 \) represent a non-trivial \( \phi_i(r) \neq 0, i = 1, ..., N \) solution of equations (13, 17, 18) on open interval \( (r_0, \infty), r_0 > 0 \) with conditions (14, 15, 16). Then
(i) there exist finite limits of functions \( \alpha(r), \beta(r), \phi_i(r) \) and \( \phi_i'(r) \) for \( r \to r_0 \);
(ii) this solution can be regularly continued onto a left neighborhood of \( r_0 \);
(iii) this solution can be regularly continued for all \( r > 0 \) up to the center.

Proof. Statement (i) of the theorem is essentially the result of Lemma 3. The right hand sides of equations (25, 26) are analytic in the neighborhood of \( \bar{X}(r_0) > 0, \quad \bar{Y}(r_0) > 0, \quad S_i \bar{Z}_i(r_0) > 0, \quad S_i \bar{\phi}_i(r_0) > 0 \). Then statement (ii) follows from the existence-uniqueness theorem for ordinary differential equations. Application of the continuous induction in order to continue the solutions for all \( r > 0 \) completes the proof.

We note that the regularity for \( r > 0 \) does not exclude a singularity at the origin \( r = 0 \).

IV. ASYMPTOTICS AT THE CENTER

The next question concerns the behavior of the solutions in the vicinity of the center that can be studied using considerations similar to Lemma 3 with
some restrictions on $\kappa, \kappa'$ from (5) [6]. We take into account that signs $S_i \equiv \text{sign}(\phi_i) = \text{sign}(Z_i)$ do not change on $(0, \infty)$.

Lemma 4. Let conditions (2) [3] and (5) [6] are fulfilled with max $(\kappa^2, \kappa'^2) < 32\pi/N$. Let $\alpha(r), \beta(r), \phi_i(r) \neq 0$ ($i = 1, ..., N$) represent a solution of (11) [12] [13] on $(0, \infty)$ with conditions (14) [15] [16].

Then there exist finite nonzero limits

$$Z_{i,0} = \lim_{r \to 0+0} Z_i(r), \quad Y_0 = \lim_{r \to 0+0} Y(r)$$

(36)

such that $S_iZ_{i,0} > 0$, $Y_0 > 0$.

Proof. Let $0 < r < r_1 < \infty$. Now we repeat considerations of Lemma 3 leading to (35), but we can leave $L = \ln(r_1/r)$ in this inequality instead of $\ln(r_1/r_0)$:

$$D(r) \equiv r^2 B(r) \leq C_2 \exp(-2L) \cdot \exp \left\{ - \sum_{i=1}^{N} \left[ \frac{4\pi}{L} [\phi_i(r) - \phi_i\eta] - \kappa |\phi_i(r)| \right] \right\}$$

(37)

where $C_2 = r_1^2 X(r_1)C_0 > 0$, $\phi_{i,1} = \phi_i(r_1)$.

We strengthen the inequality by discarding some negative terms in the exponent. After simple calculations we get

$$D(r) \leq C_2 \exp \left\{ -2L + \frac{4\pi}{L} \sum_{i=1}^{N} \left[ |\phi_{i,1}| + \kappa L \frac{r_1}{8\pi} \right]^2 \right\}$$

(38)

If $\kappa^2 < 32\pi/N$, then this expression is bounded for $r \to 0$ ($L \to \infty$); then the right-hand side of (24) is integrable and the limit $Y_0 \geq 0$ exists.

Analogously, under suppositions of this Lemma we get that $r^2 X \eta''$ in the right hand side of (25) is integrable and limits $Z_{i,0}$ exist. After that, inequalities $S_iZ_{i,0} > 0$, $Y_0 > 0$ are obtained similarly to Lemma 3, Q.E.D.

Using Lemma 4 we can obtain the asymptotic behavior of SF for $r \to 0$. From (26) and using (36) we have

$$\frac{d\phi_i}{dr} \sim -\frac{\zeta_i}{r}, \quad \phi_i(r) \sim -\zeta_i \ln r,$$

(39)

where $\zeta_i = Z_{i,0}/Y_0$. The singularity of $\phi_i(r)$ for $r \to 0$ is a physical one; it takes place in any coordinate system and cannot be removed by a coordinate transformation.

From (17) [18] we obtain the leading terms of asymptotics for $r \to 0$:

$$\alpha(r) \sim (\eta - 1) \ln r, \quad \beta \sim (\eta + 1) \ln r,$$

(40)

where $\eta = 4\pi \sum_{i=1}^{N} \kappa_i^2$. 

Note that the asymptotics (38) [39] are similar to those of the generalized Fisher solution with $V \equiv 0$ (see Appendix B). Indeed, under conditions (5) [6] the terms containing $V(\phi)$ are asymptotically much smaller as compared with the other terms in (17) [18] for $r \to 0$.

V. TEST PARTICLE MOTION

The asymptotic relations (39) enable us to highlight main qualitative situations concerning the geodesic structure of the space-time around the spherically symmetric static configuration with $N$ SFs.

The integrals for trajectories of photons and test particles in the plane $\theta = \pi/2$ are

$$e^\alpha \left( \frac{dt}{d\tau} \right)^2 - e^\beta \left( \frac{dr}{d\tau} \right)^2 - r^2 \left( \frac{d\varphi}{d\tau} \right)^2 = S,$$

(41)

$$e^\alpha \left( \frac{dt}{d\tau} \right)^2 = E, \quad r^2 \left( \frac{d\varphi}{d\tau} \right)^2 = L,$$

(42)

where $S = 0$ in case of photons and $S = 1$ for test particles, $r$ is a canonical parameter, $E > 0$ and $L$ are constants of motion. This yields

$$e^{\alpha + \beta} \left( \frac{dr}{d\tau} \right)^2 = E^2 - U_{eff}(r, L, S),$$

(43)

where $U_{eff}(r, L, S) = e^\alpha (S + L^2/r^2)$.

Thus, we are dealing with one-dimensional particle motion in the field of effective potential $U_{eff}$.

In view of asymptotics (14), we have $U_{eff} \approx S + L^2/r^2$ for $r \to \infty$. In case of the radial motion of photons ($L = 0, S = 0$) using (39), for $r \to 0$ we have $r^4dr/d\tau \approx \pm E$, so photons can reach the singularity at the center for a finite value of $r$. For $L \neq 0$ both for $S = 0$ and $S = 1$, we have two main situations defined by the sign of $\eta - 3$. In Appendix C we consider an example showing that both signs may be indeed possible. For $r \to 0$ and $L \neq 0$ asymptotic relations (39) yield $U_{eff} \approx e^{\alpha}r^{-2} \sim r^{\eta - 3}$. Thus for $\eta > 3$ we have $U_{eff}(r, L, S) \to 0$ for $r \to 0$ and for sufficiently large $L$ there is a maximum of $U_{eff}(r, L, S)$ as a function of $r$. Otherwise, for $\eta < 3$ we have $U_{eff}(r, L, S) \to \infty$ for $r \to 0$.

Let us consider in more detail the motion of photons ($S = 0$). In this case by an appropriate choice of the canonical parameter we can put $E = 1$. Let $\eta > 3$. Then there exists a global maximum of the effective potential $\max U_{eff}(r, L, 0) = L^2M_0$, where $M_0 = \max e^\alpha/r^2$. The solutions $r(\tau)$ of (42) that describe incoming photons with $L^2M_0 < 1$, $d\tau/dr < 0$, can be continued to the values $r \to 0$, that is, these photons fall on the center. There is a non-zero capture cross section of the incident photons by the singularity (see Fig. 5 of Appendix C for the example). On the other hand, if the singularity does not radiate, the external observer will see a dark spot in the center surrounded by a luminous ring due to radiating substance around the configuration. The occurrence of the maximum of $\max U_{eff}(r, L, 0)$ means that there exists at least one "photon sphere" (cf., e.g. [11]) – the set of unstable circular photon trajectories. The situation with this configuration is very similar to
that for BH, and it will be difficult to distinguish images of these objects without additional independent information about the surrounding matter.

In case of $\eta < 3$, $L \neq 0$ the effective potential $U_{eff}(r, L, S)$ is unbounded for $r \to 0$. Therefore, photons falling from infinity with $L \neq 0$ are reflected back from the potential and do not reach the center. Nevertheless, in this case it is also possible to have a dark spot in the center; this is described in Appendix C where we present a detailed consideration of null geodesics by the example of one non-linear SF with the monomial potential.

VI. DISCUSSION

We considered static spherically symmetric configurations of General Relativity in presence of $N$ scalar fields minimally coupled to gravity, which obey conditions of asymptotic flatness. The SF potential is supposed to satisfy conditions guaranteeing a monotonic dependence of non-trivial SF modes upon radial variable $\phi$; also it must be exponentially bounded. These conditions are fulfilled for a number of widely used field-theoretic models, such as positively definite polynomial potentials. A superposition of independent SFs with monomial potentials $\phi_i^n$ can serve as a simple example.

Under these conditions, we proved that any asymptotically flat solution of the Einstein–SF equations cannot have singularities on a sphere of non-zero radius, i.e. the solutions are regular for all non-zero values $r > 0$ of radial variable $r$ in the Schwarzschild (curvature) coordinates outside the center. On the other hand, all non-trivial SF components $\phi_i$ have singularities for $r \to 0$. The results are illustrated by the numerical example for one SF with monomial potential (Appendix C).

We note that asymptotic properties of the solutions with $N$ non-linear scalar fields are remarkably similar to that of the case of one linear massless SF and its generalization for $N$ free SFs (Appendix B). All these cases have naked singularities at the center with the logarithmic asymptotic behavior of SF and power-law metric components.

It would be interesting to relax the restrictions on the potentials $\phi_i$. Though we note the examples of $\phi_i^n$ showing that violation of the conditions $\phi_i^n$ can lead to the black hole configurations with scalar hair. One can also suppose that non-linear SF potentials with a more strong dependence upon SF than in $\phi_i^n$ (like $V(\phi) \sim \sinh(\phi^2)$, $n > 2$) can lead to spherical singularities with a non-zero radius.

In view of the asymptotic properties of the metric we show that there exist two different types of the geodesic structure, depending on the strength of the SF components at infinity. They are related to different behaviors of isotropic geodesics. In case of the first one the incident photons with sufficiently small impact parameters $L$ are captured by the singularity. There exists a ”photon sphere” and spiral trajectories of photons passing near it that fall to the center after several revolutions. This case is similar to that of BH, and here one can also have the ring-like image of the accretion disk with a dark spot in the center. Different brightness distributions over the ring are possible, which, however, are also a function of complicated physical processes that are unattainable for independent observations. Therefore, there is a lot of configurations with naked singularity that can mimic the Schwarzschild BH.

In the other type, the photons and the test particles with non-zero angular momentum cannot reach the center. The examples show that in this case the incident photons can be strongly deflected by the center, which makes it impossible some regions near the singularity to be observed from some directions at infinity; in this case the dark spot in the center can be possible as well.

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Appendix A: One SF in the flat space

A1. Asymptotic behavior for $r \to \infty$. Here we consider one real SF $\phi$ with the power-law potential

$$V(\phi) = w|\phi|^p, \quad w > 0,$$

(A1)

The SF equation in Minkowski space-time is

$$\frac{d}{dr} \left[ r^2 \frac{d\phi}{dr} \right] = \mu w^2 |\phi|^{p-2}.$$

(A2)

We assume conditions $\mu, p > 0$ to be fulfilled for $\Phi(r) \equiv \phi(r)$. Due to considerations analogous to Lemma 1 of Section [11] we infer that $\phi(r)$ and $\phi'(r)$ do not change their signs and without loss of generality we further assume $\phi > 0$, $\phi'(r) < 0$.

In case of $p = 2$, $w = \mu^2/2$ (massive SF with mass $\mu$) there is the exact solution satisfying zero condition for $r \to \infty$:

$$\psi(r) = \frac{Q}{r} \exp(-\mu r), \quad Q = \text{const.}$$

(A3)

For $p > 2$, substitution

$$\phi = e^{-qt} \psi, \quad t = \ln r, \quad q = \frac{2}{p-2}$$

(A4)
leads to autonomous differential equation
\[ \frac{d^2 \psi}{dt^2} + (-2q + 1) \frac{d\psi}{dt} + q(q - 1)\psi = pw\psi^{p-1}. \]  
(A5)

Equation (A5) is equivalent to the dynamical system on the plane
\[ \frac{du}{dt} = (2q - 1) - q(q - 1)\psi + pw\psi^{p-1}, \quad \frac{d\psi}{dt} = u. \]  
(A6)

(i) First consider the case \( 2 < p < 4 \) when \( q(q - 1) = 2(4(p - 2) - 2 > 0 \). On the half-plane \( \psi \geq 0 \) the dynamical system (A6) has critical points \( (\psi = 0, u = 0) \) and \( (\psi = Q_0, u = 0) \), where
\[ Q_0 = \left[ \frac{q(q - 1)}{pw} \right]^{\frac{1}{p-2}}. \]  
(A7)

The point \((0,0)\) is a repeller (unstable node) with the eigenvalues of the linearized system
\[ \lambda_1^* = q = \frac{2}{p - 2}, \quad \lambda_2^* = q - 1 = \frac{4 - p}{p - 2}. \]  
(A8)

The point \((Q_0,0)\) is a saddle with eigenvalues
\[ \lambda_\pm = \frac{6 - p}{2(p - 2)} \left[ 1 \pm \sqrt{1 + \frac{8(4 - p)(p - 2)}{(6 - p)^2}} \right], \]  
(A9)

The separatix branches that enter the saddle correspond to asymptotic solutions of (A2) for \( r \to \infty \). For these branches we have
\[ \psi(t) \approx Q_0[1 + C \exp(-\lambda t)], \quad t \to \infty, \] where \( C \) is an arbitrary constant, \( \lambda = -\lambda_\pm > 0 \). Correspondingly, we have asymptotic solution of (A2) for \( r \to \infty \):
\[ \phi_1(r) \approx \phi_1(r) = \frac{Q_0}{r^q} \left( 1 + \frac{C}{r^\lambda} \right), \]  
(A10)

The other solutions near the saddle do not fit condition \( \phi(\infty) = 0 \).

(ii) For \( p > 4 \) there is the only critical point \((0,0)\) and this is a saddle (see (A5) for \( p < 4 \)). Condition \( \phi(\infty) = 0 \) leads to the separatix of the saddle yielding \( \psi(t) \sim Qe^{-\lambda_+ t}, t \to \infty \) and
\[ \phi(r) \approx Q/r, \quad r \to \infty, \] \( Q \) being an arbitrary constant.

(iii) For \( p = 4 \quad (q = 1) \) there is also the only critical point \((0,0)\), but it is not simple. Analogously to the case \( p > 4 \), there are solutions that tend to \((0,0)\) for \( t \to \infty \) yielding asymptotic solution for SF
\[ \phi(r) = \frac{Q}{r \sqrt{|\ln r|}} \left( 1 + \frac{3}{4} \frac{\ln |\ln r|}{|\ln r|} + \ldots \right) \]  
(A12)

where \( Q \) being an arbitrary constant.

**A2. Spherical singularities.** Typical phenomenon for some types of a non-linear equation is the occurrence of singularities for finite values of the independent variable. Here we present an example showing that this may be the case for certain solutions of equation (A2).

First of all, one can check directly that with \( p > 2 \) there exists a solution of equation (A2), which is singular at \( r = r_s > 0 \), which can be represented approximately near this point as
\[ \phi(r) \approx \left[ \frac{q(q + 1)}{pw(r - r_s)^2} \right]^{q/2} \] with arbitrary \( r_s > 0 \) and \( q = 2/(p - 2) \). However, here we do not know the behavior of this solution for \( r \to \infty \). In this view we present an example with more detailed consideration of the solutions satisfying asymptotic condition (A11) at infinity. We confine ourselves to the case \( p = 2n, \ n > 2 \).

Multiplying (A2) by \( \phi'(r) \) after some transformations we get
\[ \frac{dE}{dr} = -\frac{2}{r} \phi'(r)^2 \leq 0, \]  
(A13)

where \( E(r) = [\phi'(r)]^2/2 - w\phi^{2n} \). Then for \( r < r_0 \) we have \( E(r) \geq E(r_0) \). For a sufficiently large \( r_0 \) we have
\[ \phi(r_0) \approx Q/r_0 \ll 1, \quad \phi'(r) \approx -Q/r_0^2, \]  
(A14)

this yields \( E(r_0) > 0 \) and
\[ [\phi'(r)]^2 > w\phi^{2n}(r). \]

Taking into account the signs \( (\phi > 0, \phi' < 0) \)
\[ \phi'(r) < -\sqrt{w}\phi^n \quad \rightarrow \quad \frac{d}{dr} \left( \frac{1}{\phi^{n-1}} \right) > \sqrt{w(n - 1)}. \]

Integration of this inequality yields on \([r, r_0], \ (r < r_0)\)
\[ \phi(r) > \left[ \phi_0^{-(n-1)} - \sqrt{w(n - 1)(r_0 - r)} \right]^{1/(n-1)}. \]

If
\[ \phi_0^{-(n-1)} - \sqrt{w(n - 1)r_0} < 0, \]  
(A15)

then we necessarily have singularity of \( \phi \) for some \( r = r_s > 0 \). The inequality (A15) needs to be checked to be compatible with (A14). Both estimates will be satisfied for a sufficiently large \( r_0 \) and
\[ Q > r_0^{(n-2)/(n-1)} |\sqrt{w(n - 1)}|^{-1/(n-1)}. \]

This ensures the existence of the singularity for some \( r = r_s > 0 \). However, this seems to be a too tight assessment. Numerical integration shows that the singularity occurs for much lower \( Q \) (see Fig. [1]).
Appendix B: Generalized Fisher solution: $N$ free scalar fields

Here we consider the case of $N$ scalar fields with $V(\Phi) \equiv 0$. Our considerations closely follow the Fisher work \cite{13}.

From (25) we have $Z_i \equiv \text{const}$, and (23, 24) are separated yielding the second order equation

$$\frac{d^2 Y}{dr^2} = \frac{\Xi}{rY^2} \frac{dY}{dr},$$

where

$$\Xi = 4\pi \sum_{i=1}^{N} Z_i^2 = \text{const.} \quad \text{(B2)}$$

We assume non-trivial $\Xi > 0$.

Substitution $r = \exp(t)$ transforms (B1) into autonomous equation can be easily integrated yielding

$$\frac{dY}{dt} = Y - \frac{\Xi}{Y} + A,$$

where $A$ is an integration constant. Besides $A$, the result contains one more integration constant. Both are defined on account of (20), in particular $A = r_g = 2M$.

The final result is

$$[g_{-}(Y)]^{(1-\nu)/2} [g_{+}(Y)]^{(1+\nu)/2} = r,$$

where $g_{\pm}(Y) = Y + M \pm \sqrt{M^2 + \Xi}$. Here $Y(r)$ varies from $\sqrt{M^2 + \Xi} - M$ to infinity as $r$ varies from zero to infinity. This determines implicitly $Y(r) \geq \sqrt{M^2 + \Xi} - M$ as a function of $r > 0$.

The metric components are

$$e^\alpha = (g_{-}/g_{+})^\nu, \quad e^\beta = g_{+}g_{-}/Y^2,$$

and SF as a function of $Y$ is

$$\phi_i(Y) = \frac{Z_i}{2\sqrt{M^2 + \Xi}} \ln \left( \frac{g_{+}(Y)}{g_{-}(Y)} \right). \quad \text{(B6)}$$

This is the Fisher solution \cite{13}, the only difference is due to the presence of $N$ fields in (B2) and (B6).

Transition to a new radial variable $Y$ leads to the Janis-Newman-Winicour \cite{14} (see also \cite{15, 16}) representation of the metric

$$ds^2 = \left( \frac{g_{-}}{g_{+}} \right)^\nu dt^2 - \left( \frac{g_{+}}{g_{-}} \right)^\nu dY^2 - (g_{+})^{1+\nu}(g_{-})^{1-\nu} dO^2. \quad \text{(B7)}$$
Appendix C: Numerical solutions: one field, monomial potential

Here we consider one SF with monomial potential

\[ V(\phi) = \phi^{2n}, \]  

(C1)

where \( n = 2, 3, \ldots \). The case of the linear massive scalar field (\( n = 1 \)) has been considered in [27]. Obviously, assumptions (2, 3) and (5, 6) are fulfilled with appropriately chosen \( C_0, C'_0 \). Therefore all the results of Lemmas 1–4 and Theorem 1 are valid for solutions of equations (11 – 13) with the potential (C1).

The asymptotic formulas for sufficiently large \( r \) are obtained on account of conditions (15, 16, 20), assuming that the behavior of \( \phi(r) \) for \( r \rightarrow \infty \) must be the same as in the Minkowski space-time (see Appendix A). For \( n = 3, 4, \ldots \) the asymptotic formulas can be derived using expansions in powers of \( 1/r \). The leading terms of solutions are as follows:

\[ \phi(r) = \frac{Q}{r} \left[ 1 + \frac{r_g}{2r} + O \left( \frac{1}{r^3} \right) \right], \]  

(C2)

\[ e^\alpha = \left( 1 - \frac{r_g}{r} \right) \left[ 1 + O \left( \frac{1}{r^3} \right) \right], \]  

(C3)

\[ e^\beta = \left( 1 - \frac{r_g}{r} \right)^{-1} \left[ 1 - \frac{4\pi Q^2}{r^2} + O \left( \frac{1}{r^3} \right) \right], \]  

(C4)

where constants \( Q \) and \( M \) fix the solution uniquely. For \( n = 2 \) the asymptotic formula for SF involves logarithmic terms according to (A12). We note that in fact only the first terms of the asymptotic formulas are sufficient to obtain stable numerical results described below. Namely, one can use (A11) and (A12) to obtain initial conditions for the numerical integration of equations (23, 24, 25, 26), starting with a sufficiently large value of \( r \) towards the center \( r = 0 \).

We checked the asymptotic behavior near the center \( (r \rightarrow 0) \), which is described by the relations (38) for SF and (39) for the metric with \( \eta = 4\pi \zeta_0^2 > 0 \). Figs. 2, 3 illustrate typical behavior of the solutions. Fig. 4 shows the relationship \( \eta(Q) \) of asymptotics at infinity and near the center. We see that both signs of \( \eta - 3 \) are possible leading to different types of the photon trajectories (Figs. 3, 6).

The numerical solutions have been used to study the geodesic structure around the configuration with potential (C1) according to equations (41, 42). The configuration parameters were \( n = 2, Q = 0.2 \) (\( \eta > 3 \), Fig. 5) and \( n = 3, Q = 0.5 \) (\( \eta < 3 \), Fig. 6); the figures show the qualitative features of the incident photon trajectories, which are typically used for imaging the configuration by means of the inverse ray tracing.

To illustrate, we considered a simple model of an accretion disk (AD) observed face-on, which is formed by the planar distribution of the test body stable circular orbits (SCO). The SCO distribution have been studied by means of the technique of our papers [26], [27] where the key point is the occurrence and disposition of extrema of \( U_{eff}(r, L, 1) \).

In case of \( \eta > 3 \), we have the Schwarzschild-like SCO distribution of AD: there is the inner region where the test body circular orbits do not exist at all, then there exists a ring of unstable circular orbits with larger radii and then there is an outer region of SCO that extends to infinity. These regions are indicated by the empty section of the AD plane and creates an image of a point on the back of AD.

For \( \eta < 3 \), SCO densely fill the area near the center and there exist SCOs with arbitrarily small radii (bold
solid black line without gaps in Fig. [6]. Also in this case there may be discontinuous SCO distributions: in this case there is the region of SCO near the center, then there is a ring of unstable orbits and then there is an outer region of SCO that extends to infinity.

A detailed description of these results is beyond the scope of the present paper; however, we note that in case of the configurations with potential $\Omega_1$ fixed by parameters $M, Q$, possible SCO distributions turn out to be qualitatively similar to the case of the linear massive scalar field (see [27, 29]).

For $\eta < 3$, there is a strong light deflection near the singularity and due to this effect the light from the inner orbits of AD near the center avoids certain directions. The photons incident from infinity with a sufficiently small $L$ deviate strongly from the initial direction and do not reach the accretion disk that is located face-on; this means that the distant observer must see a dark "hole" in the center of the configuration. This does not mean that the innermost SCO will be invisible for all possible observers since the effect depends on the direction of the line of sight with respect to the AD plane.

In case of $\eta > 3$, the qualitative picture is as described in Section V in particular, the photons from infinity with sufficiently small $L$ fall to the singularity.

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