Hardy and Hardy-Sobolev inequalities on Riemannian manifolds

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Abstract. Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(N \geq 3\) without boundary. Given \(p_0 \in M, \lambda, \sigma \in \mathbb{R}\) and \(\sigma \in (0, 2]\), we study existence and non existence of minimizers of the following quotient:

\[
\mu_{\lambda, \sigma} = \inf_{u \in H^1(M) \setminus \{0\}} \frac{\int_M |\nabla u|^2 dv_g - \lambda \int_M u^2 dv_g}{\left(\int_M \rho^{-\sigma} |u|^2(\sigma) dv_g\right)^{2/2^*}},
\]

where \(\rho(p) = \text{dist}(p, p_0)\) denoted the geodesic distance from \(p\) to \(p_0\). In particular for \(\sigma = 2\), we provide sufficient and necessary conditions of existence of minimizers in terms of \(\lambda\). For \(\sigma \in (0, 2)\) we prove existence of minimizers under scalar curvature pinching.

Key Words: Hardy inequality, Hardy-Sobolev inequality, scalar curvature.

1 Introduction

For \(N \geq 3\), the Hardy inequality states that

\[
\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} |x|^{-2} |u|^2 dx, \quad \forall u \in D^{1,2}(\mathbb{R}^N).
\]

The constant \(\left(\frac{N-2}{2}\right)^2\) is sharp and never achieved in \(D^{1,2}(\mathbb{R}^N)\). In contrast to the Sobolev inequality

\[
\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq S_{N,0} \left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}, \quad \forall u \in D^{1,2}(\mathbb{R}^N),
\]

the best constant \(S_{N,0} = \frac{N(N-2)}{4} \omega_N^{2/N}\) is achieved in \(D^{1,2}(\mathbb{R}^N)\). Here \(\omega_N = |S^{N-1}|\) is the volume of the N-sphere and \(2^* = \frac{2N}{N-2}\) is the critical Sobolev exponent.

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For more details related to Hardy and Sobolev inequalities you can refer to the works of Brezis-Vasquez [5], Davila-Dupaign [13], D’Ambrosio [9, 10], Brezis-Marcus-Safrir [4], Musina [36], a nice exposition book in Druet-Hebey-Robert [15] and references therein. There is also a detailed history related to Hardy inequality type in the book of Kufner-Persson [28].

We observe that inequalities (1.1) and (1.2) are scale invariant while the Sobolev inequality is additionally translation invariant. They have many applications in physics, spectral theory, differential geometry mathematical physics, analysis of linear and non-linear PDEs, harmonic analysis, quantum mechanic, stochastic analysis etc... For more details see the books of Lieb-Loss [33], Struwe [39] and Evans [16], the works of Grigor’yan [25], Gkikas [24], Grigor’yan Saloff [26] and references therein.

Using Hölder’s inequality, we get the interpolation between the above two inequalities called Hardy-Sobolev inequality

\[
\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq S_{N,\sigma} \left( \int_{\mathbb{R}^N} |x|^{-\sigma} |u|^{2^*(\sigma)} dx \right)^{2/2^*(\sigma)}, \quad \forall u \in \mathcal{D}^{1,2}(\mathbb{R}^N),
\]

where \(\sigma \in [0, 2]\) and \(2^*(\sigma) = \frac{2(N - \sigma)}{N - 2}\) is the Hardy-Sobolev critical exponent. See [22] for more details about Hardy-Sobolev inequality. We also remark that (1.3) is a particular case of the Caffarelli-Kohn-Nirenberg inequality, see [6]. The value of the best constant is

\[
S_{N,\sigma} := (N - 2)(N - \sigma) \left[ \frac{w_{N-1}}{2 - \sigma} \frac{\Gamma^2(N - \frac{\sigma}{2})}{\Gamma(\frac{2(N - \sigma)}{2 - \sigma})} \right]^{\frac{2 - \sigma}{N - \sigma}},
\]

where \(w_{N-1}\) is the volume of the \(N\)-sphere and \(\Gamma\) is the Euler function. It was computed by Lieb [32] when \(\sigma \in (0, 2)\). The ground state solution is given by

\[
\omega(x) = \left( (N - \sigma)(N - 2) \right)^{\frac{N-2}{2-\sigma}} (1 + |x|^{-\sigma})^{\frac{2-N}{2-\sigma}}.
\]

As said previously for \(\sigma = 2\) the optimal constant in (1.1) is not attained. However there exists a "virtual ground state" \(u(x) = |x|^{\frac{4-N}{2}}\) which satisfies

\[
\Delta u + \left( \frac{N - 2}{2} \right)^2 |x|^{-2} u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\]

Note also that (1.3) is scale invariant. Our interest in this paper is to study existence of minimizers of the Hardy and Hardy-Sobolev inequalities in a Riemannian manifold. Inequality (1.3) is not in general valid on a Riemannian manifold. For a compact Riemannian manifold \((\mathcal{M}^N, g)\), with metric \(g\) and \(N \geq 3\) and letting \(p_0 \in \mathcal{M}\), we have

\[
\lambda \int_{\mathcal{M}} u^2 dv_g + \int_{\mathcal{M}} |\nabla u|^2 dv_g \geq \mu \left( \int_{\mathcal{M}} \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)}, \quad \forall u \in H^1(\mathcal{M}),
\]

where \(\rho(p) = dist_g(p, p_0)\) is the geodesic distance between \(p\) and \(p_0\) and \(\lambda, \mu \in \mathbb{R}\) are constants depending on \(\mathcal{M}\). The above inequality can be obtained by a simple argument of the partition of
unity, see Lemma 4.1 below. We then propose to study existence and non existence of minimizers of the following quotient

\[(1.7) \quad \mu_{\lambda,\sigma,p_0} = \inf_{u \in H^1(M) \setminus \{0\}} \frac{\int_M |\nabla u|^2 dv_g - \lambda \int_M u^2 dv_g}{\left( \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)}}.\]

with $\lambda \in \mathbb{R}$, $\sigma \in (0,2]$. If there is no ambiguity, we will write $\mu_{\lambda,\sigma}$ instead of $\mu_{\lambda,\sigma,p_0}$.

In our first main result we deal with the pure Hardy problem $\sigma = 2$. We get the following

**Theorem 1.1** Let $(M,g)$ be a smooth compact Riemannian manifold of dimension $N \geq 3$ and $p_0 \in M$. Then there exists $\lambda^* = \lambda^*(M,p_0) \in \mathbb{R}$ such that $\mu_{\lambda,2,p_0}(M)$ is attained if and only if $\lambda > \lambda^*(M)$.

To explain our result and emphasize the differences between Hardy and Hardy-Sobolev inequalities in Riemannian manifolds, some definitions are in order. For an open set $\Omega \subset M$, we put

\[(1.8) \quad \mu_{\lambda,\sigma} = \inf_{u \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla u|^2 dv_g - \lambda \int_\Omega u^2 dv_g}{\left( \int_\Omega \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)}}.\]

The existence of $\lambda^* \in \mathbb{R}$ is a consequence of the local Hardy:

\[(1.9) \quad \mu_{0,2}(B_g(p_0, r)) = \left( \frac{N-2}{2} \right)^2 = S_{N,2} \]

which holds for small $r$. The existence and non existence of solution are based on the construction of appropriate super and sub-solution for the linear operator

\[L_\lambda := -\Delta_g - \frac{(N-2)^2}{4} \rho^{-2} - \lambda \rho^{-2}.\]

For that we consider the geodesic normal coordinates

\[x \in B(0, r) \subset \mathbb{R}^N \mapsto F(x) = \text{Exp}_{p_0} \left( \sum_i x^i E_i \right)\]

where $\text{Exp}_{p_0}$ is the exponential map on $M$. Using these local coordinates we perturb the mapping

\[p \mapsto \rho^{\frac{2-N}{4}}(p)\]

to obtain

\[v_a(p) = \rho^{\frac{2-N}{4}} |\log \rho|^a\]

for $a \in \mathbb{R}$. The function $|\log \rho|^a$ allows to control the lowered order terms of the linear operator $L_\lambda$. Hence a careful choice of the parameter $a$ yields super and sub-solutions to prove Theorem 1.1. However when $\sigma \in (0,2)$, the situation changes due to the effect of the local geometry of $M$. Indeed we have

\[\mu_{\lambda,\sigma} < S_{N,\sigma} \]
provided the scalar curvature is lower bounded by some positive constant we will precise in Theorem 1.2. Note that when $N \geq 5$ the positivity of the scalar curvature is enough to prove existence of minimizer. This then allows us to prove the following

**Theorem 1.2** Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $N \geq 4$, $\sigma \in (0, 2)$, $p_0 \in M$ and $\lambda < 0$. We suppose that

\begin{align*}
\begin{cases}
S_g(p_0) > 0 & \text{for } N \geq 5 \\
S_g(p_0) > -8\lambda \frac{N - 2}{N - \sigma} & \text{for } N = 4.
\end{cases}
\end{align*}

Then $\mu_{\lambda, \sigma} < S_{N, \sigma}$ and it’s achieved.

Note that for $\lambda < 0$ the operator $L_\lambda^g := \Delta_g - \lambda$ is coercive: there exists $C > 0$ such that $\forall u \in H^1(M)$

\begin{align*}
||u||^2_{H^1(M)} \leq \int_M (L_\lambda^g u) udv_g.
\end{align*}

That is: the $H^1$ norm of the function is controlled by the energy of $u$ with respect to $L_\lambda^g$. Furthermore the existence of a solution $u > 0$ of $\mu_{\lambda, \sigma}$ when $\sigma \in [0, 2)$ implies that $L_\lambda^g$ is coercive. Thanks to P.L. Lions [34], the inequality $\mu_{\lambda, \sigma} < S_{N, \sigma}$ is necessary and sufficient condition for the relative compactness of all minimizing sequences for $\mu_{\lambda, \sigma}$ in

\begin{align*}
S = \{ u \in H^1(M) \text{ such that } \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g = 1 \}.
\end{align*}

For more details and a nice book you can refer to [39]. As mentioned above, this inequality holds provided the scalar curvature at $p_0$ is positive and $N \geq 4$.

We should mention that when $\sigma = 0$ the above problem is related to the well know Yamabe problem, solved by Aubin in [2], Schoen in [38] and Trudinger in [42]. For an exposition book of such problem you can refer to the book of Druet-Hebey-Robert [15]. When $\sigma = 2$, we are dealing with an eigenvalue problem for the operator $-\Delta_g + \mu \rho^{-2}$. A problem of this kind was first studied by Brezis-Marcus in [3]. See also the work of Fall [18], Fall-Musina [20] and Fal-Mahmoudi [19].

In the afore mentioned paper the singularity is placed at the boundary. Hardy and Hardy-Sobolev inequalities on Riemannian manifolds have been also studied by Carron in [7], Adriano-Xia in [1], Shihshu-Li in [44], D’Ambrosio-Dipierro in [12], E. Mitidieri in [35] and references therein. The Hardy-Sobolev inequality with boundary singularities was first studied by Ghoussoub-Kang in [22] who discovered the local influence of the mean curvature of the boundary in order to get a minimizer. Further related problems, extensions and generalizations can be found in the works of Ghoussoub-Robert [21], Y. Li and Lin in [31], Chern-Lin in [8], Demyanov-Nazarov in [14].

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The paper is organized as follows: in Section 2 we give some preliminaries and notations, in Section 3 we study the linear case $\sigma = 2$ and in Section 4 we study the nonlinear case $\sigma \in (0, 2)$.

2 Preliminaries and notations

For $p_0 \in M$, we denote by $\{E_1, E_2, ..., E_N\}$ the standard orthonormal basis of $T_{p_0}M$; $S^N$ the unit sphere of $\mathbb{R}^{N+1}$ and $\omega_N$ the volume of the unit $N$-sphere, $S_g(p_0)$ the scalar curvature at $p_0$, $\mathcal{D}^{1,2}(\mathbb{R}^N)$ the space of functions for which their gradient are square integrable in $\mathbb{R}^N$. Let

$$B_g(p_0, r) = \{ p \in M : dist_g(p, p_0) < r \}$$

be the ball centered at $p_0$, of radius $r$, where $dist_g$ is the geodesic distance function of $M$. We recall the exponential mapping

$$\text{Exp}_{p_0} : T_{p_0}M \to M,$$

on $M$. A neighborhood of $p_0$ can be parametrized by the map:

$$x \ni B(0, r_0) \mapsto F(x) = \text{Exp}_{p_0} \left( \sum_{i=1}^{N} x_i E_i \right).$$

We notice that

$$\rho(F(x)) = |x|.$$

In the following we will choose $r_0$ small enough that $F$ is smooth in $B(0, r_0)$. In these normal coordinates, the Laplace-Beltrami operator is given by:

$$\Delta_g = -g^{ij} \left( \frac{\partial^2}{\partial x_i \partial x_j} - \Gamma^{k}_{ij} \frac{\partial}{\partial x_k} \right).$$

Where $\{\Gamma^k_{ij}\}_{1 \leq i, j, k \leq N}$ are the Christoffel symbols, $\{g_{ij}\}_{1 \leq i, j \leq N}$ are the components of the metric $g$ and $g^{ij} = (g^{-1})_{ij}$ are the elements of the inverse matrix of $g$. We have that

$$\Gamma^k_{ij}(x) = O(|x|),$$

$$g_{ij} = \delta_{ij} - \frac{R_{ij}(p_0)x_i x_j}{3} + O(|x|^3)$$

and

$$\int_{S^{N-1} \setminus \{0\}} \sqrt{|g|} |x| \theta d\theta = w_{N-1} \left( 1 - \frac{S_g(p_0)}{6N} |x|^2 + O(|x|^3) \right)$$

where $R_{ij}(p_0)$ is the Ricci curvature at $p_0$, $S_g(p_0)$ is the scalar curvature at $p_0$, $w_{N-1}$ the volume of the unit $N$-sphere $S^{N-1}$. Often in some demonstrations, we use the notation $o(1)$ which is a function of $n$ for which its limit is zero as $n \to +\infty$. 

5
3 Linear case: $\sigma = 2$

In this section, we will consider the case $\sigma = 2$. Recall the "virtual" ground state $\omega(x) = |x|^{\frac{2-N}{2}}$ which satisfies

\[ -\Delta \omega = \left( \frac{N-2}{2} \right)^2 |x|^{-2} \omega \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}. \]  

Using the geodesic normal coordinates, we will perturb the mapping $p \mapsto \omega \circ F^{-1} = \rho^{\frac{2-N}{2}}(p)$ to build super-solution to get the existence of $\lambda^*$. Moreover, with similar arguments, we will construct a subsolution which allows us to prove non existence of minimizer for $\lambda \leq \lambda^*$.

**Lemma 3.1** Let

\[ \omega_a(x) = |x|^{\frac{2-N}{2}}|\log(|x|)|^a. \]

Then setting $v_a(F(x)) = \omega_a(x)$, we have

\[ Lv_a = -a(a-1)\rho^{-2}(\log \rho)^{-2}v_a - \lambda \rho^{-2}v_a + O(\rho^{\frac{2-N}{2}}(-\log \rho)^a) \quad \text{in} \quad B_\rho(p_0, r_0) \]

where

\[ L = -\Delta_g - \left( \frac{N-2}{2} \right)^2 \rho^{-2} - \lambda \rho^{-2}. \]

**Proof.** Let

\[ \varphi(t) = t^{\frac{2-N}{2}}(-\log t)^a. \]

Then we have

\[ \varphi'(t) = \left( \frac{2-N}{2} \right)t^{\frac{N-2}{2}}(-\log t)^a - at^{\frac{2-N}{2}}(-\log t)^{a-1}, \]

so that

\[ \varphi'(t) = \varphi(t) \left( \frac{2-N}{2} + \frac{a}{t(\log t)} \right). \]

We also have

\[ \varphi''(t) = \varphi(t) \left( \frac{N(N-2)}{4t^2} + \frac{a(a-1)}{t^2(\log t)^2} + \frac{a(N-1)}{t^2(\log t)} \right). \]

Or

\[ \Delta \omega_a = \varphi'' + \frac{N-1}{t} \varphi' \]

so

\[ \Delta \omega_a(x) = -\left( \frac{N-2}{2} \right)^2 \omega_a(x)|x|^{-2} + a(a-1)\omega_a(x)|x|^{-2}(\log |x|)^{-2} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}. \]

Using the formula of the Laplace-Beltrami operator (2.4), (2.5) and (2.6), we have:

\[ \Delta_y v_a = -\Delta_g \omega_a + O_{ij}(|x|^2)\partial^2_{ij} \omega_a + O_k(|x|)\frac{\partial \omega_a}{\partial x_k}. \]

Therefore

\[ Lv_a = -a(a-1)\rho^{-2}(\log \rho)^{-2}v_a - \lambda \rho^{-2}v_a + O(\rho^{\frac{2-N}{2}}(-\log \rho)^a). \]

This end the proof of the lemma.
Lemma 3.2 Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(N \geq 3\) and \(p_0 \in M\). Then there exists \(r_0 > 0\) such that:

\[
\int_{B_g(p_0, r_0)} |\nabla u|^2 dv_g \geq \left( \frac{N - 2}{2} \right)^2 \int_{B_g(p_0, r_0)} \rho^{-2} |u|^2 dv_g + \int_{B_g(p_0, r_0)} \rho^{-2} (\log \rho)^{-2} u^2 dv_g, \quad \forall u \in H^1_0(B_g(p_0, r_0)).
\]

Proof. Using (3.5), we get that, for \(a = -1\), we let \(v_{-1}=V\). So

\[
-\Delta_V V \geq \left( \frac{N - 2}{2} \right)^2 \rho^{-2} + \rho^{-2} (\log \rho)^{-2} \quad \text{in} \quad B_g(p_0, r_0),
\]

for \(r_0\) small enough. Let \(u \in C^\infty_c(B_g(p_0, r_0))\) and consider \(\psi = \frac{u}{V}\). Then we get

\[
|\nabla u|^2 = |V \nabla \psi|^2 + \nabla V \nabla (V \psi).
\]

Using integration by parts, we get

\[
\int_{B_g(p_0, r_0)} |\nabla u|^2 dv_g = \int_{B_g(p_0, r_0)} |V \nabla \psi|^2 dv_g - \int_{B_g(p_0, r_0)} \frac{\Delta V}{V} u^2 dv_g.
\]

Therefore

\[
\int_{B_g(p_0, r_0)} |\nabla u|^2 dv_g \geq \left( \frac{N - 2}{2} \right)^2 \int_{B_g(p_0, r_0)} \rho^{-2} u^2 dv_g + \int_{B_g(p_0, r_0)} \rho^{-2} (\log \rho)^{-2} u^2 dv_g.
\]

The fact that \(C^\infty_c(B_g(p_0, r_0))\) is dense in \(H^1_0(B_g(p_0, r_0))\) ends the proof. \(\square\)

3.1 Existence of \(\lambda^*\)

Proposition 3.3 Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(N \geq 3\). Then there exists \(\lambda^* \in \mathbb{R}\) such that \(\mu_{\lambda^*, 2}(M) = \left( \frac{N - 2}{2} \right)^2\) and \(\mu_{\lambda, 2}(M) < \left( \frac{N - 2}{2} \right)^2\), \(\forall \lambda > \lambda^*\).

Proof.

Claim:

\[
\mu_{\lambda, 2}(M) \leq \left( \frac{N - 2}{2} \right)^2, \quad \forall \lambda \in \mathbb{R}.
\]

Indeed, we recall that

\[
\mu_{0, 2}(\mathbb{R}^N) = \left( \frac{N - 2}{2} \right)^2.
\]

Therefore for any \(\delta > 0\), we can find \(u_\delta \in C^\infty_c(\mathbb{R}^N)\) such that:

\[
\int_{\mathbb{R}^N} |\nabla u_\delta|^2 dy \leq \left( \frac{N - 2}{2} \right)^2 + \delta \int_{\mathbb{R}^N} |y|^{-2} u_\delta^2 dy.
\]

Let

\[
y = \epsilon^{-1} F^{-1}(p)
\]
\[ v_\delta(p) = \epsilon^{-N/2} u_\delta(\epsilon^{-1} F^{-1}(p)) \]

For \( \epsilon \) small enough, \( v_\delta \in C^\infty_c(M) \). By applying the change of variable formula and using (4.14) we get:

\[ \mu_{\lambda,2}(M) \leq \int_M |\nabla v_\delta|^2 \ dv_g - \lambda \int_M |v_\delta|^2 \ dv_g \leq (1 + \alpha \epsilon) \int_{\mathbb{R}^N} |\nabla u_\delta|^2 \ dy + \alpha \epsilon^2 |\lambda|. \]

Hence

\[ \mu_{\lambda,2}(M) \leq (1 + \alpha \epsilon) \left( \left( \frac{N-2}{2} \right)^2 + \delta \right) + \alpha \epsilon^2 |\lambda|. \]

As \( \epsilon, \delta \to 0 \) respectively, we get:

\[ \mu_{\lambda,2}(M) \leq \left( \frac{N-2}{2} \right)^2, \quad \forall \lambda \in \mathbb{R}. \]

Claim: there exist \( \overline{\lambda} \) such that:

\[ \left( \frac{N-2}{2} \right)^2 \int_M \rho^2 |u|^2 \ dv_g \leq \int_M |\nabla u|^2 \ dv_g - \overline{\lambda} \int_M |u|^2 \ dv_g, \quad \forall u \in H^1(M). \]

Indeed, let \( \varphi \in C^\infty_c(M) \) such that:

\[ \varphi = \begin{cases} 
1, & \text{in } B(p_0, r_0) \\
0, & \text{in } M \setminus B(p_0, 2r_0). 
\end{cases} \]

For \( u \in H^1(M) \), we write

\[ u = u \varphi + (1 - \varphi)u \]

and notice that

\[ u \varphi \in H^1_0(B_{\rho}(p_0, 2r_0)). \]

We then have

\[ \int_M |u|^2 \rho^{-2} \ dv_g = \int_M |u \varphi + (1 - \varphi)u|^2 \rho^{-2} \ dv_g = \int_M |u \varphi|^2 \rho^{-2} \ dv_g + \int_M (1 - \varphi)u|^2 \rho^{-2} \ dv_g + 2 \int_M |u \varphi (1 - \varphi)|^2 \rho^{-2} \ dv_g \leq \int_{B_{\rho}(p_0,2r)} |u \varphi|^2 \rho^{-2} \ dv_g + 3 \int_{B_{\rho}(p_0,2r)} (1 - \varphi)u|^2 \rho^{-2} \ dv_g \leq \left( \frac{N-2}{2} \right)^2 \int_{B_{\rho}(p_0,2r)} |\nabla (u \varphi)|^2 \ dv_g + 3 \int_{B_{\rho}(p_0,2r)} |u|^2 \rho^{-2} \ dv_g. \]
Furthermore we have
\[
\int_{B_g(p_0, 2r_0)} |\nabla (u\varphi)|^2 dv_g \leq \int_M |\varphi \nabla u + u \nabla \varphi|^2 dv_g \\
\leq \int_M |\nabla u|^2 dv_g + C(p_0, r_0, N) \int_M |u|^2 dv_g + \frac{1}{2} \int_M |\nabla u|^2 |\varphi|^2 dv_g \\
\leq \int_M |\nabla u|^2 dv_g + C(p_0, r_0, N) \int_M |u|^2 dv_g - \frac{1}{2} \int_M |\Delta u|^2 dv_g \\
\leq \int_M |\nabla u|^2 dv_g + C(p_0, r_0, N) \int_M |u|^2 dv_g.
\]

We conclude that
\[
(N - \frac{2}{N})^2 \int_M |u|^2 \rho^{-2} dv_g \leq \int_M |\nabla u|^2 dv_g + C(p_0, r_0, N) \int_M |u|^2 dv_g.
\]

This implies that there exist \( \bar{\lambda} \) such that:
\[
\mu_{\lambda, 2}(M) \geq \left( \frac{N - 2}{2} \right)^2.
\]

Since \( \lambda \rightarrow \mu_{\lambda} \) is decreasing, we can define \( \lambda^* \) as
\[
\lambda^* = \sup \left\{ \lambda \in \mathbb{R} : \mu_{\lambda, 2}(M) = \left( \frac{N - 2}{2} \right)^2 \right\}.
\]

The proof of the following result is similar to one in [3]. We expose it here for the reader’s convenience.

**Proposition 3.4** Let \( M \) be a smooth compact manifold of dimension \( N \geq 3 \). Then

\[
\mu_{\lambda, 2}(M) = \inf_{u \in H^1(M) \setminus \{0\}} \frac{\int_M |\nabla u|^2 dv_g - \lambda \int_M u^2 dv_g}{\int_M \rho^{-2} u^2 dv_g}
\]

is achieved for every \( \lambda > \lambda^* \).

**Proof.** Let \( \{u_n\}_{n \geq 0} \) be a minimizing sequence of (3.16) normalized so that:

\[
\int_M \frac{u_n^2}{\rho^2} dv_g = 1.
\]

So

\[
\mu_{\lambda, 2}(M) = \int_M |\nabla u_n|^2 dv_g - \lambda \int_M |u_n|^2 dv_g.
\]

Thus \( \{u_n\}_{n \geq 0} \) is bounded in \( H^1(M) \). After passing to a subsequence, we assume that there exists \( u \in H^1(M) \) such that

\[
u_n \rightharpoonup u \text{ weakly in } H^1(M).
\]
Let
\begin{equation}
(3.20) \quad v_n = u_n - u.
\end{equation}

Then we have:
\begin{equation}
(3.21) \quad v_n \to 0 \text{ in } L^2(M), \quad v_n \to 0 \text{ in } H^1(M) \text{ and } \frac{v_n}{\rho} \to 0 \text{ in } L^2(M).
\end{equation}

Using (3.18) and (3.21) we obtain:
\begin{equation}
(3.22) \quad \mu_{\lambda, 2} = \int_M |\nabla u_n|^2 dv_g - \lambda \int_M |u_n|^2 dv_g = \int_M |\nabla u|^2 dv_g + \int_M |\nabla v_n|^2 dv_g - \lambda \int_M |u|^2 dv_g + o(1).
\end{equation}

and that
\begin{equation}
(3.23) \quad 1 = \int_M \frac{|u_n|^2}{\rho^2} dv_g = \int_M \frac{|u|^2}{\rho^2} dv_g + \int_M \frac{|v_n|^2}{\rho^2} dv_g + o(1).
\end{equation}

Hence
\begin{equation}
(3.24) \quad \int_M |\nabla v_n|^2 dv_g \geq \left( \frac{N-2}{2} \right)^2 \left( 1 - \int_M \frac{|u|^2}{\rho^2} dv_g \right) + o(1).
\end{equation}

By (3.24) and (3.22) we obtain:
\begin{equation}
(3.25) \quad \int_M |\nabla u|^2 dv_g + \left( \frac{N-2}{2} \right)^2 \left( 1 - \int_M \frac{|u|^2}{\rho^2} dv_g \right) \geq \lambda \int_M |u|^2 dv_g - \mu_{\lambda, 2}.
\end{equation}

But
\[ \int_M |\nabla u|^2 dv_g - \lambda \int_M |u|^2 dv_g \geq \mu_{\lambda, 2} \int_M \frac{|u|^2}{\rho^2} dv_g \]
so that:
\[ \left( \mu_{\lambda, 2} - \left( \frac{N-2}{2} \right)^2 \right) \left( \int_M \frac{|u|^2}{\rho^2} dv_g - 1 \right) \leq 0. \]

Since $\mu_{\lambda, 2} < \left( \frac{N-2}{2} \right)^2$, we get that $1 \leq \int_M \frac{|u|^2}{\rho^2} dv_g$.

Therefore $\int_M \frac{|u|^2}{\rho^2} dv_g = 1$ so $u$ is a minimizer for
\[ \mu_{\lambda, 2}(M) := \inf_{u \in H^1(M) \setminus \{0\}} \frac{\int_M |\nabla u|^2 dv_g - \int_M |u|^2 dv_g}{\int_M \rho^{-2} |u|^2 dv_g} \]
and
\[ \int_M |\nabla v_n|^2 dv_g \to 0. \]

Thus $u_n \to u$ in $H^1(M)$ and the proof is complete. \qed
Proposition 3.5 Let $\mathcal{M}$ be a smooth compact manifold of dimension $N \geq 3$. Then $\mu_{\lambda,2}(\mathcal{M})$ is not achieved for every $\lambda \leq \lambda^*$. 

Proof. We study separately the case $\lambda = \lambda^*$ and the case $\lambda < \lambda^*$. For every $\lambda < \lambda^*$ the statement is verified. Indeed suppose that for some $\bar{\lambda} < \lambda^*$ the infimum is attained at an element $\bar{u} \in H^1(\mathcal{M})$. We suppose that $\bar{u}$ is normalized so that:

$$\int_{\mathcal{M}} \frac{|\bar{u}|^2}{\rho^2} dv_g = 1$$

and

$$\int_{\mathcal{M}} |\nabla \bar{u}|^2 dv_g - \bar{\lambda} \int_{\mathcal{M}} |\bar{u}|^2 dv_g = \left( \frac{N-2}{2} \right)^2.$$

Then, for $\bar{\lambda} < \lambda < \lambda^*$ we have,

$$\left( \frac{N-2}{2} \right)^2 = \mu_{\lambda} \leq \int_{\mathcal{M}} |\nabla \bar{u}|^2 dv_g - \lambda \int_{\mathcal{M}} |\bar{u}|^2 dv_g < \left( \frac{N-2}{2} \right)^2.$$

That means

$$\left( \frac{N-2}{2} \right)^2 < \left( \frac{N-2}{2} \right)^2$$

which is impossible. So for $\lambda < \lambda^*$, $\mu_{\lambda,2}$ is not achieved.

Case $\lambda = \lambda^*$:

We suppose by contradiction that for $\lambda = \lambda^*$, there exists $u \in H^1(\mathcal{M})$ such that $\mu_{\lambda^*,2}$ is achieved. Recall that for $u \in H^1(\mathcal{M})$, $|u| \in H^1(\mathcal{M})$ and $|\nabla u| = |\nabla|u||$ almost everywhere, see [15]. So we may assume that $u > 0$. Let

$$L := -\Delta_g - \left( \frac{N-2}{2} \right)^2 \rho^{-2} - \lambda \rho^{-2}.$$

By standard regularity theory, see [23] and thanks to the maximum principle $u$ is smooth and positive in $\mathcal{M} \setminus \{p_0\}$. Recall from Lemma 3.1 that

$$(3.27) \quad L v_a = -a(a-1)\rho^{-2}(-\log \rho)^{-2} v_a - \lambda \rho^{-2} v_a + O\left( \rho^{-\frac{aN}{2}}(\log \rho)^a \right).$$

The dominant term in the right hand side of equation (3.27) is $-a(a-1)\rho^{-2}(-\log \rho)^{-2} v_a$. So for $r$ small enough, we have for $a < -\frac{1}{2}$

$$L v_a \leq 0 \quad \text{in} \quad B(p_0,r)$$

and also $v_a \in H^1(B(p_0,r))$. Now let $\epsilon > 0$ such that :

$$\epsilon v_a \leq u \quad \text{on} \quad \Sigma_r = \{p \in \mathcal{M} : \rho(p) = r\}.$$

And let

$$W_a = \epsilon v_a - u.$$

Then $W_a^+ \in H^1_0(B(p_0,r))$ for all $a \in (-1, -\frac{1}{2})$. Furthermore

$$Lu \geq 0.$$
Therefore
\[ LW_a \leq 0 \quad \text{in} \quad B(p_0, r) \quad \forall \ a \in \left(-1, -\frac{1}{2}\right). \]

Using (3.7), we deduce that
\[ (3.28) \]
\[ \int_{B(p_0, r)} \left( |\nabla W_a^+|^2 - \left(\frac{N-2}{2}\right)^2 \rho^{-2} (W_a^+)^2 \right) \geq 0. \]

Therefore, the fact that
\[ \int_{B_g(p_0, r)} \left( |\nabla W_a^+|^2 - \left(\frac{N-2}{2}\right)^2 \rho^{-2} (W_a^+)^2 - \lambda \rho^{-2} (W_a^+)^2 \right) \leq 0 \]
implies
\[ \epsilon v_a \leq u \quad \text{in} \quad B_g(p_0, r). \]

Hence
\[ \epsilon \left( \frac{\rho^{\frac{2-N}{2}} \log \rho}{\sigma^2} \right)^{\frac{1}{2}} \leq u \quad \text{in} \quad B_g(p_0, r) \]
and consequently
\[ \frac{u}{\rho} \notin L^2(B(p_0, r)). \]

This contradicts the assumption that \( u \in H^1(M) \).

\[ \Box \]

### 3.2 Proof of Theorem 1.1

The existence of \( \lambda^* \) is given by the Proposition 3.3. The proof of the "if" part is done in Proposition 3.4 and the "only if" part is done in Proposition 3.5. \[ \Box \]

### 4 Nonlinear case: \( \sigma \in (0, 2) \)

We recall that
\[ (4.1) \]
\[ S_{N, \sigma} = \inf_{u \in D^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\left( \int_{\mathbb{R}^N} |x|^{-\sigma} |u|^{2^*(\sigma)} \, dx \right)^{2/2^*(\sigma)}} \]
and \( \lambda < 0 \). We will need the following approximate Hardy-Sobolev inequality on a Riemannian manifold \( M \).

**Lemma 4.1** Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \( N \geq 3 \). For all \( \epsilon > 0 \) small, there exist \( K(\epsilon, M) \) positive constants such that for all \( u \in H^1(M), \)
\[ (4.2) \]
\[ S_{N, \sigma} \left( \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} \, dv_g \right)^{2/2^*(\sigma)} \leq (1+\epsilon) \int_M |\nabla u|^2 \, dv_g + K(\epsilon, M) \left[ \int_M |u|^2 \, dv_g + \left( \int_M |u|^{2^*(\sigma)} \, dv_g \right)^{2/2^*(\sigma)} \right]. \]

12
Proof. Let $\epsilon > 0$ and $\varphi \in C_c^\infty(\mathcal{M})$ such that $0 \leq \varphi \leq 1$ and
\[
\varphi = \begin{cases} 
1 & \text{in } B(p_0, \epsilon) \\
0 & \text{in } \mathcal{M} \setminus B(p_0, 2\epsilon).
\end{cases}
\]
Clearly $u = u\varphi + (1 - \varphi)u$, so
\[
\int_\mathcal{M} |u|^{2^*(\sigma)} \rho^{-\sigma} dv_y = \int_\mathcal{M} |u\varphi + (1 - \varphi)u|^{2^*(\sigma)} \rho^{-\sigma} dv_y.
\]
But $2^*(\sigma) > 1$, so there exists $C(\epsilon) > 0$ such that
\[
|u\varphi + (1 - \varphi)u|^{2^*(\sigma)} \leq (1 + \epsilon)|u \varphi|^{2^*(\sigma)} + C(\epsilon)(1 - \varphi)|u|^{2^*(\sigma)}.
\]
Therefore
\[
\left(\int_\mathcal{M} |u|^{2^*(\sigma)} \rho^{-\sigma} dv_y\right)^{2/2^*(\sigma)} \leq (1 + \epsilon)\left(\int_{B(p_0, 2\epsilon)} |u\varphi|^{2^*(\sigma)} \rho^{-\sigma} dv_y\right)^{2/2^*(\sigma)} + C(\epsilon)\left(\int_\mathcal{M} |u\varphi|^{2^*(\sigma)} \rho^{-\sigma} dv_y\right)^{2/2^*(\sigma)}.
\]
Using (2.2), (2.6) and (4.14) we have that: there exists a constant $C > 0$ such that
\[
\left(\int_{B(p_0, 2\epsilon)} |u\varphi(p)|^{2^*(\sigma)} \rho^{-\sigma} (p) dv_y\right)^{2/2^*(\sigma)} \leq (1 + C\epsilon)\left(\int_{B(0, 2\epsilon)} |u\varphi(F(x))|^{2^*(\sigma)} |x|^{-\sigma} dx\right)^{2/2^*(\sigma)}
\]
and by (4.1), we have that
\[
S_{N,\sigma} \left(\int_{B(p_0, 2\epsilon)} |(u\varphi(p))^{2^*(\sigma)} \rho^{-\sigma} (p) dv_y\right)^{2/2^*(\sigma)} \leq (1 + C\epsilon) \int_{\mathbb{R}^N} |\nabla (u\varphi)(F(x))|^2 dx.
\]
Since
\[
|\nabla (u\varphi)|^2 = |\varphi \nabla u|^2 + |u \nabla \varphi|^2 + 2u\varphi\nabla u \nabla \varphi,
\]
using the fact that $p = F(x)$ and by integration by parts, there exists $C'(\epsilon, \mathcal{M}) > 0$ such that
\[
\int_{\mathbb{R}^N} |\nabla (u\varphi)(F(x))|^2 dx \leq \int_{B(p_0, 2\epsilon)} |\nabla u|^2 dv_y + C'(\epsilon, \mathcal{M}) \int_{B(p_0, 2\epsilon)} |u|^2 dv_y,
\]
Hence using (4.5), (4.6), (4.7) and (4.9), we get the result
\[
S_{N,\sigma} \left(\int_{\mathcal{M}} \rho^{-\sigma} |u|^{2^*(\sigma)} dv_y\right)^{2/2^*(\sigma)} \leq (1 + \epsilon) \int_{\mathcal{M}} |\nabla u|^2 dv_y + K(\epsilon) \left[\int_{\mathcal{M}} |u|^2 dv_y + \left(\int_{\mathcal{M}} |u|^{2^*(\sigma)} dv_y\right)^{2/2^*(\sigma)}\right],
\]
where $K(\epsilon) = \text{Max}(C'(\epsilon, \mathcal{M}), C(\epsilon))$. \hfill \Box

Remark

For all $u \in C^1(\mathcal{M})$, there exists a constant $C(\mathcal{M}, N)$ such that
\[
C(\mathcal{M}, N) \left(\int_{\mathcal{M}} \rho^{-\sigma} |u|^{2^*(\sigma)} dv_y\right)^{2/2^*} \leq \int_{\mathcal{M}} |\nabla u|^2 dv_y + \int_{\mathcal{M}} u^2 dv_y.
\]
Indeed we use the fact that there exists a constant $K(M, N)$ such that

$$K(M, N) \left( \int_M |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)} \leq \int_M |\nabla u|^2 dv_g + \int_M u^2 dv_g$$

and inequality (4.2). In particular $\mu_{\lambda, \sigma}$ is well defined for all $\lambda < 0$.

### 4.1 Existence of minimizer

We recall that

$$\mu_{\lambda, \sigma} = \inf_{u \in H^1(M)} \frac{\int_M |\nabla u|^2 dv_g - \lambda \int_M u^2 dv_g}{\left( \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)}}$$

$$g_{ij} = \delta_{ij} - \frac{R_{ij}(p_0)x_i x_j}{3} + O(|x|^3).$$

$$\int_{S_N-1} \sqrt{|\sigma|} |x|^\theta d\theta = w_N^{-1} \left( 1 - \frac{S_N(p_0)}{6N} |x|^2 + O(|x|^3) \right)$$

and that the ground state solution $w$ which achieved the Hardy-Sobolev best constant $S_{N, \sigma}$ verifies

$$\begin{cases} -\Delta w = S_{N, \sigma} w^{2^*(\sigma)-1} |x|^{-\sigma} & \text{in } \mathbb{R}^N \\ w > 0 \\ w \in D^{1,2}(\mathbb{R}^N). \end{cases}$$

We will need the following

**Proposition 4.2** Let $(M, g)$ be a smooth compact Riemannian manifold of dimension $N \geq 3$. If $\mu_{\lambda, \sigma} < S_{N, \sigma}$ then $\mu_{\lambda, \sigma}$ is attained.

**Proof.** Let $\{u_n\}$ be a minimizing sequence normalized so that

$$\int_M \rho^{-\sigma} u_n^{2^*(\sigma)} dv_g = 1.$$  

Then $\{u_n\}$ is bounded in $H^1(M)$ and we assume that, for a subsequence,

$$u_n \rightharpoonup u \text{ in } H^1(M), u_n \rightarrow u \text{ in } L^2(M), u_n \rightarrow u \text{ in } L^{2^*(\sigma)}(M).$$

The last assertion is a consequence of the fact that $2^*(\sigma) < 2^*$ which is the critical Sobolev exponent. Using (4.17) and (4.16), we get

$$\mu_{\lambda, \sigma} + o(1) = \int_M |\nabla u_n|^2 dv_g - \lambda \int_M u_n^2 dv_g = \int_M |\nabla u|^2 dv_g + \int_M |\nabla (u_n - u)|^2 dv_g - \lambda \int_M u^2 dv_g + o(1)$$

and by Brezis-Lieb Lemma

$$1 = \int_M \rho^{-\sigma} u_n^{2^*(\sigma)} dv_g = \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g + \int_M |u_n - u|^{2^*(\sigma)} \rho^{-\sigma} dv_g + o(1).$$
From lemma 4.1 and (4.17), we get

\[
S_{N,\sigma} \left( \int_M \rho^{-\sigma} |u_n - u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)} \leq (1 + \epsilon) \| \nabla (u_n - u) \|^2 + o(1).
\]

Therefore

\[
S_{N,\sigma} \left( 1 - \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)} \leq (1 + \epsilon) \| \nabla (u_n - u) \|^2 + o(1).
\]

From (4.18) and (4.20), we get

\[
\int_M |\nabla u|^2 dv_g + \frac{S_{N,\sigma}}{1 + \epsilon} \left( 1 - \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)} - \lambda \int_M u^2 dv_g \leq \mu_{\lambda,\sigma}.
\]

Since

\[
\mu_{\lambda,\sigma} \left( \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)} \leq \int_M |\nabla u|^2 dv_g - \lambda \int_M u^2 dv_g,
\]

we get

\[
\frac{S_{N,\sigma}}{1 + \epsilon} \left( 1 - \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)} \leq \mu_{\lambda,\sigma} \left( 1 - \left( \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)} \right).
\]

Or

\[
1 - \left( \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)} \leq \left( 1 - \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)}.
\]

Taking \( \epsilon \to 0 \) we get

\[
(S_{N,\sigma} - \mu_{\lambda,\sigma}) \left( 1 - \left( \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)} \right) \leq 0.
\]

Since \( S_{N,\sigma} < \mu_{\lambda,\sigma} \) and \( \int_M \rho^{-\sigma} |u|^{2^*(\sigma)} dv_g \leq 1 \) so we get \( \int_M \rho^{-\sigma} |u|^2 dv_g = 1 \). Therefore \( u_n \to u \) in \( H^1(M) \). In particular \( u \) is a minimizer for \( \mu_{\lambda,\sigma} \). \( \square \)

In our next result, we will give necessary condition so that \( \mu_{\lambda,\sigma} < S_{N,\sigma} \).

**Proposition 4.3** Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \( N \geq 4 \), \( \sigma \in (0, 2) \), \( p_0 \in M \) and \( \lambda < 0 \). We suppose that

\[
\begin{cases}
S_p(p_0) > 0 & \text{for } N \geq 5 \\
S_p(p_0) > -8 \lambda \frac{(N-2)}{N-\sigma} & \text{for } N = 4.
\end{cases}
\]

Then

\[
\mu_{\lambda,\sigma, p_0} = \mu_{\lambda,\sigma} < S_{N,\sigma}.
\]
Proof. Let
\[ w(|x|) = \left( (N - \sigma)(N - 2) \right)^{-\frac{N-2}{2}} (1 + |x|^{2-\sigma})^{\frac{\sigma-N}{2}} \]
defined on \( \mathbb{R}^n_+ \) the ground-state solution of the best Hardy-Sobolev constant
\[ S_{N,\sigma} = \frac{\int_{\mathbb{R}^N} |\nabla w|^2 dx}{\left( \int_{\mathbb{R}^N} |x|^{-\sigma} |w|^{2^*(\sigma)} dx \right)^{2/2^*(\sigma)}}. \]
The metric \( g \) is given by
\[ g_{ij}(x) = \delta_{ij} - \frac{R_{ij} x_i x_j}{3} + O(|x|^3). \]
Let \( n \in \mathbb{N}^* \) and consider the test function
\[ u_n(p) = n^{\frac{N-2}{2}} w(np). \]
It’s not difficult to verify that
\[ \int_{M \setminus B(p_0, \delta)} |\nabla u_n(p)|^2 dv_g = O(n^{2-N}). \]

Therefore using (4.30) and the change of variable formula we get
\[ \int_{M} |\nabla u_n(p)|^2 dv_g = \int_{\mathbb{R}^N} |\nabla w|^2 dx \times \left\{ \begin{array}{ll}
1 - \frac{S_g(p_0)}{6Nn^2} & \mbox{if } N \geq 4 \\
1 - \frac{S_g(p_0)}{6Nn^2} & \mbox{if } N = 4.
\end{array} \right. \]

We have also that
\[ \int_{M} u_n^2 dv_g = \left\{ \begin{array}{ll}
\frac{\sigma}{n^2} & \mbox{if } N \geq 5 \\
\frac{1}{n^2} \int_{\mathbb{R}^N} w^2(x) dx + O\left( \frac{1}{n^2} \right) & \mbox{if } N = 4.
\end{array} \right. \]

Next we use Taylor expansion to get for \( N \geq 4 \)
\[ \left( \int_{M} \rho^{-\sigma} |u_n(p)|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)} = \left( \int_{B_g(p_0, \delta)} \rho^{-\sigma} |u_n|^{2^*(\sigma)} + o\left( \frac{1}{n^2} \right) \right) \]
\[ = \left( \int_{\mathbb{R}^N} |x|^{-\sigma} |w|^{2^*(\sigma)} dx \right)^{2/2^*(\sigma)} \left( 1 - \frac{2}{2^*(\sigma)} \frac{S_g(p_0)}{6Nn^2} \int_{\mathbb{R}^N} |x|^{-\sigma} |w|^{2^*(\sigma)} dx + o\left( \frac{1}{n^2} \right) \right). \]
• Case $N \geq 5$: Using these three above inequalities we get

\begin{equation}
\mu_{\lambda, \sigma}(\mathcal{M}, \Sigma) \leq \frac{\int_M |\nabla u_n|^2 dv_g - \lambda \int_M u_n^2 dv_g}{\left( \int_M \rho^{-\sigma} |u_n|^{2^*(\sigma)} dv_g \right)^{2/2^*(\sigma)}} \\
= S_{N, \sigma} \left[ 1 - \frac{S_k(p_0)}{6N^2} \left( \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx - \frac{2}{2^*(\sigma)} \int_{\mathbb{R}^N} |x|^{-\sigma} |w|^{2^*(\sigma)} dx \right) \right] + o\left( \frac{1}{n^2} \right) \text{ if } N \geq 5.
\end{equation}

Next we show the following

\begin{equation}
\int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx \leq 2 \int_{\mathbb{R}^N} |x|^{-\sigma} |w|^{2^*(\sigma)} dx > 0 \text{ for } N \geq 5.
\end{equation}

We multiply (4.15) by $|x|^2 w$ and we integrate by parts to get

\begin{equation}
S_{N, \sigma} \int_{\mathbb{R}^N} |x|^{-\sigma} w^{2^*(\sigma)} dx = - \int_{\mathbb{R}^N} \Delta w |x|^2 dx
\end{equation}

\[= \int_{\mathbb{R}^N} |\nabla w|^2 |x|^2 dx + \int_{\mathbb{R}^N} w \nabla w (|x|^2) dx
\]

\[= \int_{\mathbb{R}^N} |\nabla w|^2 |x|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \nabla w^2 (|x|^2) dx
\]

\[= \int_{\mathbb{R}^N} |\nabla w|^2 |x|^2 dx - 2 \int_{\mathbb{R}^N} w^2 |x|^2 dx.
\]

Therefore

\begin{equation}
S_{N, \sigma} \int_{\mathbb{R}^N} |x|^{-\sigma} w^{2^*(\sigma)} dx = \int_{\mathbb{R}^N} |\nabla w|^2 |x|^2 dx - (N - 1) \int_{\mathbb{R}^N} w^2 dx \leq \int_{\mathbb{R}^N} |\nabla w|^2 |x|^2 dx.
\end{equation}

Moreover $\frac{2}{2^*(\sigma)} < 1$ because $\sigma \in (0, 2)$.

Hence

\begin{equation}
\int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx > \frac{2}{2^*(\sigma)} S_{N, \sigma} \int_{\mathbb{R}^N} |x|^{-\sigma} w^{2^*(\sigma)} \text{ for } N \geq 5.
\end{equation}

We have also that

\begin{equation}
\int_{\mathbb{R}^N} |\nabla w|^2 dx = S_{N, \sigma} \int_{\mathbb{R}^N} |x|^{-\sigma} w^{2^*(\sigma)} dx.
\end{equation}

Therefore

\begin{equation}
\int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 dx - \frac{2}{2^*(\sigma)} \int_{\mathbb{R}^N} |x|^{-\sigma} |w|^{2^*(\sigma)} dx > 0 \text{ for } N \geq 5.
\end{equation}

Hence $\mu_{\lambda, \sigma} < S_{N, \sigma}$ for $N \geq 5$ provided the scalar curvature at $p_0$ is positive.
The case \( N=4 \): Thanks to (4.32), (4.33), (4.34) and (4.39) we have that
\[
\mu_{\lambda, \sigma} < S_{N, \sigma} \left[ 1 - \frac{S_g(p_0)}{6Nn^2} \left( \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 \, dx - \frac{2S_{N, \sigma}}{2^*(\sigma)} \int_{\mathbb{R}^N} |x|^{2-\sigma} |w|^{2^*(\sigma)} \, dx + \frac{6N\lambda}{S_g(p_0)} \int_{\mathbb{R}^N} w^2 \, dx \right) \right] + O \left( \frac{1}{n^2} \right).
\]
We let
\[
S = \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 \, dx - \frac{2}{2^*(\sigma)} S_{N, \sigma} \int_{\mathbb{R}^N} |x|^{2-\sigma} |w|^{2^*(\sigma)} \, dx + 6N \frac{\lambda}{S_g(p_0)} \int_{\mathbb{R}^N} w^2 \, dx
\]
Using the equality part in (4.37) we get
\[
(4.40) \quad S = \left( 1 + \frac{8\lambda}{S_g(p_0)} \right) \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 \, dx - S_{N, \sigma} \left( \frac{2}{2^*(\sigma)} \frac{8\lambda}{S_g(p_0)} \right) \int_{\mathbb{R}^N} |x|^{2-\sigma} |w|^{2^*(\sigma)} \, dx
\]
\[
S > \left( \frac{2}{2^*(\sigma)} + \frac{8\lambda}{S_g(p_0)} \right) \int_{\mathbb{R}^N} |x|^2 |\nabla w|^2 \, dx - S_{N, \sigma} \int_{\mathbb{R}^N} |x|^{2-\sigma} |w|^{2^*(\sigma)} \, dx.
\]
Therefore a sufficient condition for \( S \) to be positive is
\[
\frac{2}{2^*(\sigma)} + \frac{8\lambda}{S_g(p_0)} > 0.
\]
Therefore for \( N=4 \), \( \mu_{\lambda, \sigma} < S_{N, \sigma} \) provided the scalar curvature is lower bounded by the constant
\[
C := -8\lambda \frac{N - \sigma}{N - 2}
\]
This ends the proof of the proposition. \( \square \)

4.2 Proof Of Theorem 2.1

Lemma 4.1 asserts that for \( \epsilon > 0 \), there exists \( C(\epsilon, r) > 0 \) such that
\[
S_{N, \sigma} \left( \int_{\mathcal{M}} \rho^{-2} |u|^{2^*(\sigma)} \, dv_{\gamma} \right)^{2/2^*(\sigma)} \leq (1 + \epsilon) \int_{\mathcal{M}} |u|^2 \, dv_{\gamma} + C(\epsilon) \int_{\mathcal{M}} |u|^2 \, dv_{\gamma}, \forall u \in H^1(\mathcal{M}).
\]
Using this inequality, we proof that for \( \mu_{\lambda, \sigma} < S_{N, \sigma} \), \( \mu_{\lambda, \sigma} \) is attained (Proposition 4.2). The last inequality follows provided condition (4.26) holds (Proposition 4.3). This ends the proof of the theorem. \( \square \)

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