Renormalization Group Functions of the $\phi^4$ Theory from High-Temperature Expansions

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Abstract
It has been previously shown that calculation of renormalization group (RG) functions of scalar $\phi^4$ theory is reduced to thermodynamic properties of the Ising model. Using high temperature expansions for the latter, RG functions of the four-dimensional theory can be calculated for arbitrary coupling constant $g$, with an accuracy of $10^{-4}$ for the $\beta$-function and with an accuracy of $10^{-3} - 10^{-2}$ for anomalous dimensions. The expansions of the RG functions up to the 13th order in $g^{-1/2}$ have been obtained.

1. INTRODUCTION

As was recently shown in [1, 2], the Gell-Mann – Low function $\beta(g)$ and anomalous dimensions of the $\phi^4$ theory can be expressed in terms of the functional integrals, providing the representation

$$g = F(g_0, m_0, \Lambda), \quad \beta(g) = F_1(g_0, m_0, \Lambda),$$

where $g_0$ and $m_0$ are the bare charge and mass, respectively; $\Lambda$ is the momentum cutoff parameter, and $g$ is the renormalized charge. Large $g$ values are reached only near a zero of one of the functional integrals, where the right-hand sides of Eqs.1 are significantly simplified and the parametric representation is resolved in the explicit form. As a result, asymptotic expressions for the $\beta$ function and anomalous dimensions are obtained. A similar approach can also be implemented in QED [3].

Parametric representation (1) has the following general property. If $g_0$ is expressed in terms of $g$ using the first of Eqs.1 and the resulting expression is substituted into the second equation, the dependence on $m_0$ and $\Lambda$ disappears according to the general theorems [4], so that the $\beta$ function depends only on $g$. However, this property is not automatically satisfied in applied calculations. The reason is that the general theorems imply the continual limit $\Lambda \to \infty$, which physically means the condition

$$m \ll \Lambda \quad \text{or} \quad \xi \gg a,$$

where $\Lambda$ is the momentum cutoff parameter, $m$ is the mass, and $a$ is a typical scale of the physical process.
where \( m \) is the renormalized mass, \( \xi \) is the correlation radius, and \( a = \Lambda^{-1} \) is the constant of a lattice at which the functional integral is defined. Under condition (2) in the region of large \( g_0 \) values, the functional integrals of the \( \phi^4 \) theory are reduced to Ising sums; as a result, Eqs.1 have the form

\[
g = F(\kappa), \quad \beta(g) = F_1(\kappa)
\]

where \( \kappa \) has the meaning of inverse temperature in the Ising model, and it is obvious that the \( \beta \) function depends only on \( g \). Condition (2) formally corresponds to the inequality

\[-g_0^{-1}m_0^2/\Lambda^2 > 1, \]

but the reducing to the Ising model is really possible under the weaker condition

\[-g_0^{-1/2}m_0^2/\Lambda^2 \gg 1, \quad -g_0^{-1}m_0^2/\Lambda^2 = \kappa = \text{arbitrary}.
\]

For this reason, parametric representation (3) remains valid in the region of small \( \kappa \), where values of \( g \) are large and gradient expansions are applicable. At first glance, the condition \( g_0 \gg 1 \) corresponds to the strong coupling regime and parametric representation (3) is limited by only this condition. However, there is another view on this situation. Let us strengthen conditions (4) by passing to the limit

\[g_0 \to \infty, \quad -g_0^{-1/2}m_0^2/\Lambda^2 \to \infty, \quad -g_0^{-1}m_0^2/\Lambda^2 = \kappa = \text{const}
\]

In this case, the transition from Eqs.1 to Eqs.3 is valid without any approximations and conserves strict equivalence with the initial \( \phi^4 \) theory under a certain choice of its bare parameters; the last property ensures the conservation of the form of the Lagrangian under renormalizations. The passage to the limit \( g_0 \to \infty \) does not mean the same passage for the renormalized charge \( g \); in fact, according to gradient expansions, \( g \) varies from infinity to about unity when \( \kappa \) varies from zero to about unity. Since parametric representation (3) is exact and specifies the \( \beta \) function in the interval \( 1 \leq g < \infty \), it can be analytically continued and treated as a definition of \( \beta(g) \) at arbitrary \( g \) values. However, there is a question: Does this definition provide correct results in the weak-coupling region?

An answer to this question can be obtained using high-temperature series [5]. Such series are traditionally constructed for quantities \( \chi_2, \mu_2, \chi_4 \) (see Section 2), which completely specify the right-hand sides of Eqs.3. High-temperature expansions are formally applicable for small \( \kappa \), but their comparatively large length (up to 30 terms in some cases) allows a successful analysis of the vicinity of the phase transition point \( \kappa_c \) and leads to the results consistent with other methods. Consequently, good approximations for the indicated quantities can be obtained throughout the interval \( 0 \leq \kappa \leq \kappa_c \). The substitution of such results into the right-hand sides of Eqs.3 makes it possible to determine the renormalization group functions in the interval \( g^* \leq g < \infty \), where \( g^* \) is the fixed point of the renormalization group. In the four-dimensional case, \( g^* = 0 \) and the mentioned procedure completely determines the renormalization group functions. In many works [6 — 16], the high-temperature series were used to test logarithmic corrections to scaling [17]. Already those works provide the positive answer to the above question: parametric representation
(3) gives correct results in the weak-coupling region. Therefore, we can concentrate our efforts on constructing the renormalization group functions of the four-dimensional $\phi^4$ theory for arbitrary $g$ values. This can be done with an accuracy of $10^{-4}$ for the $\beta$ function and with a slightly lower accuracy for anomalous dimensions.

The determination of calculated renormalization group functions implies the use of a lattice regularization different from the usual Pauli–Villars regularization scheme, isotropic cutoff in the momentum space, dimensional regularization, etc. However, the $\beta$ function in the used scheme is determined in terms of the observed charge and mass [1, 2] and should be independent of the cut-off procedure. Such a dependence is possible for anomalous dimensions, because they are determined in terms of the unobservable $Z$ factors. In any case, the distinction of this way of regularization from the usual procedures is no more than difference between the latter procedures.

2. INITIAL RELATIONS

Let us consider the $n$ component $\phi^4$ theory with the action

$$S\{\varphi\} = \int d^d x \left\{ \frac{1}{2} \sum_{\alpha=1}^{n} (\nabla \varphi_\alpha)^2 + \frac{1}{2} m_0^2 \sum_{\alpha=1}^{n} \varphi_\alpha^2 + \frac{1}{8} u_0 \left( \sum_{\alpha=1}^{n} \varphi_\alpha^2 \right)^2 \right\} ,$$

$$u_0 = g_0 \Lambda^\epsilon, \quad \epsilon = 4 - d ,$$

where $g_0$ and $m_0$ are the bare charge and mass, respectively; $d$ is the dimensionality of space; and $\Lambda$ is the momentum cutoff parameter. The most general functional integral of this theory contains $M$ multipliers of the field $\varphi$ in the pre-exponential factor,

$$Z_{\alpha_1...\alpha_M}^{\{p_i\}}(x_1, \ldots, x_M) = \int D\varphi \, \varphi_{\alpha_1}(x_1)\varphi_{\alpha_2}(x_2)\ldots \varphi_{\alpha_M}(x_M) \exp \left( - S\{\varphi\} \right) ,$$

and will be denoted as $K_{M}\{p_i\}$ after the transition to the momentum representation and the separation of $\delta$ factors,

$$Z_{\alpha_1...\alpha_M}^{\{p_1, \ldots, p_M\}} = K_{M}\{p_i\} N \delta_{p_1+\ldots+p_M} I_{\alpha_1...\alpha_M} ,$$

where $I_{\alpha_1...\alpha_M}$ is the sum of terms $\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} \ldots$ with all possible pairings, and $N$ is the number of sites of the lattice on which the functional integral is defined. The integrals $K_{M}\{p_i\}$ are usually estimated at zero momenta and only one integral $K_2\{p\}$ is required for small $p$ values,

$$K_2(p) = K_2 - \tilde{K}_2 p^2 + \ldots$$

Below, the case with $d = 4$ and $n = 1$ is considered, but the general formulas are written for arbitrary $d$ and $n$ values.
The below consideration concerns the renormalization group functions $\beta(g)$, $\eta(g)$, and $\eta_2(g)$ entering into the Callan–Symanzik equation [4]

\[
\left[ \frac{\partial}{\partial \ln m} + \beta(g) \frac{\partial}{\partial g} + (L - N/2) \eta(g) - L \eta_2(g) \right] \Gamma^{(L,N)} = 0 ,
\]

for the vertex $\Gamma^{(L,N)}$ with $N$ external lines of the field $\phi$ and $L$ external interaction lines. The expression of these functions in terms of the functional integrals leads to the parametric representation [2]

\[
g = - \left( \frac{K_2}{\tilde{K}_2} \right)^{d/2} \frac{K_4 K_0}{K_2^2} ,
\]

\[
\beta(g) = - \left( \frac{K_2}{\tilde{K}_2} \right)^{d/2} \frac{K_4 K_0}{K_2^2} \left\{ d + 2 \frac{(\ln K_4 K_0 / K_2^2)'}{(\ln K_2 / \tilde{K}_2)'} \right\} ,
\]

\[
\eta(g) = 2 \frac{(\ln K_2 / K_0)' + (\ln K_2 / \tilde{K}_2)'}{(\ln K_2 / \tilde{K}_2)'} ,
\]

\[
\eta_2(g) = -2 \frac{(\ln K_0 / \tilde{K}_2)'' + [(\ln K_0 / K_2)']^2}{(\ln K_2 / K_0)'} ,
\]

where primes stand for derivatives with respect to $m_0^2$. Under condition (4), the functional integral of the scalar theory can be written in the form

\[
Z_M \{ x_i \} = (2\kappa)^{\frac{d+4}{2}} \int \left( \prod_x d\varphi_x \varphi_{x_1} \cdots \varphi_{x_N} \right) \exp \left\{ -\kappa \sum_{x,x'} J_{x-x'} \varphi_x \varphi_{x'} \right\} \prod_x \delta(\varphi_x^2 - 1)
\]

and is transformed to an Ising sum over the values $\varphi_x = \pm 1$. The quantities studied in high-temperature expansions are introduced as

\[
\chi_2 = \sum_x \langle \varphi_x \varphi_0 \rangle^c , \quad \mu_2 = \sum_x x^2 \langle \varphi_x \varphi_0 \rangle^c , \quad \chi_4 = \sum_{x,y,z} \langle \varphi_x \varphi_y \varphi_z \varphi_0 \rangle^c ,
\]

(\text{where superscript } c \text{ marks the connected diagrams}) and coincides up to factors with the ratios $K_2/K_0$, $\tilde{K}_2/K_0$, and $K_4/K_0$ of the functional integrals introduced above; more precisely,

\[
\frac{K_2}{\tilde{K}_2} = 2d \frac{\chi_2}{\mu_2} \equiv \frac{1}{\kappa} f_0(\kappa) ,
\]

\[
\frac{K_2}{K_0} = 2\kappa \chi_2 \equiv \kappa f_2(\kappa) ,
\]

\[
\frac{K_4 K_0}{K_2^2} = \frac{1}{3} \frac{\chi_4}{\chi_2^2} \equiv -f_4(\kappa) ,
\]
where the introduced functions $f_i(\kappa)$ will be used below. It was taken into account that there is no zeroth term in the expansion of $\mu_2$ in $\kappa$ (see Eq.20 below), so that all functions $f_0(\kappa)$, $f_2(\kappa)$, and $f_4(\kappa)$ are regular and their expansions begin with the zeroth term. The substitution of Eqs. 17 into Eqs. 11–14 gives

$$g = \left( \frac{f_0(\kappa)}{\kappa} \right)^{d/2} f_4(\kappa),$$

$$\beta(g) = d - 2\kappa \frac{[\ln f_4(\kappa)]'}{1 - \kappa [\ln f_0(\kappa)]'},$$

$$\eta(g) = -2\kappa \frac{[\ln f_0(\kappa)f_2(\kappa)]'}{1 - \kappa [\ln f_0(\kappa)]'},$$

$$\eta_2(g) = -2 \left( \frac{1 + \kappa [\ln f_2(\kappa)]'}{(1 - \kappa [\ln f_0(\kappa)]') (1 + \kappa [\ln f_2(\kappa)]')^2} \right) + 1 - \kappa^2 [\ln f_2(\kappa)]'' (1 - \kappa [\ln f_0(\kappa)]').$$

(18)

It is easy to obtain the strong coupling behavior for renormalization group functions taking limit $\kappa \to 0$ [2]:

$$\beta(g) = dg, \quad \eta(g) = 0, \quad \eta_2(g) = -4 \quad (g \to \infty).$$

(19)

For a simple hypercubic lattice with the interaction between the nearest neighbors, the first terms of the expansion of functions (16) for $d = 4$ and $n = 1$ have the form [18]

$$\chi_2 = 1 + 16\kappa + 224\kappa^2 + \ldots$$

$$\mu_2 = 16\kappa + 512\kappa^2 + 33920/3\kappa^3 + \ldots$$

$$\chi_4 = -2 - 128\kappa - 4672\kappa^2 - \ldots$$

(20)

The substitution into Eqs. 18 makes it possible to obtain the expansion of the renormalization group functions in $g^{-2/d}$ and, in particular, a more accurate asymptotic expression for $\eta(g)$

$$\eta(g) = \frac{16}{9} \frac{1}{g}, \quad g \to \infty.$$  

(21)

The universality of this asymptotics has not been tested and, strictly speaking, it refers to the indicated model. Below, 14 terms of expansion (20) presented for $n = 1$ in tables 5, 8, and 11 of the paper [18] are used.

3. VICINITY OF THE PHASE TRANSITION

3.1. General Strategy
The foundation of the application of high-temperature expansions for investigating the critical behavior is as follows. Let a certain quantity $F(\kappa)$ has a power-law behavior near the transition point $\kappa_c = 1/T_c$

$$F \propto (T - T_c)^{-\lambda} \propto (\kappa_c - \kappa)^{-\lambda}. \quad (22)$$

In this case, the convergence radius of the expansion in $\kappa$ is limited by the quantity $\kappa_c$. In actual cases, $\kappa_c$ is the nearest singularity to the coordinate origin; this circumstance facilitates its analysis. It is easily seen that the nearest singularity for the logarithmic derivative

$$(\ln F)' = \frac{F'}{F} \sim -\frac{\lambda}{\kappa - \kappa_c} \quad (23)$$

is a simple pole with a residue $-\lambda$ and can be investigated using the Pade-approximation. The Pade-approximant $[M/N]$ is defined as the ratio of the polynomials of the degrees $M$ and $N$,

$$(\ln F)' = \frac{P_M(\kappa)}{Q_N(\kappa)} = \frac{p_0 + p_1\kappa + \ldots + p_M\kappa^M}{1 + q_1\kappa + \ldots + q_N\kappa^N}, \quad (24)$$

whose coefficients are chosen such that the first $M + N + 1$ coefficients of the expansion of $(\ln F)'$ in $\kappa$ are reproduced. It is known that Pade-approximants successfully predict the nearest singularities of the approximated function if these singularities are simple poles [5, 19]. Diagonal ($M = N$) or quasidiagonal ($M \approx N$) approximants are usually used for which convergence to the corresponding function is proved under the most general assumptions. The use of this strategy in the four-dimensional case is complicated by the existence of logarithmic corrections to scaling [17, 4]:

$$\chi_2 \sim \tau^{-1}|\ln \tau|^p, \quad \xi^2 \sim \frac{\mu_2}{\chi_2} \sim \tau^{-1}|\ln \tau|^p, \quad p = -\frac{\zeta_1}{\beta_2} = \frac{n+2}{n+8}, \quad \xi_4 \sim \tau^{-4}|\ln \tau|^{4p-1}, \quad (25)$$

where $\tau \sim (\kappa_c - \kappa)$ is the distance to the transition and the exponent $p$ is determined by the first terms of the expansion of the renormalization group functions,

$$\beta(g) = \beta_2 g^2 + \beta_3 g^3 + \ldots, \quad \eta(g) = \delta_2 g^2 + \delta_3 g^3 + \ldots, \quad \eta_2(g) = \zeta_1 g + \zeta_2 g^2 + \ldots, \quad (26)$$

where

$$\beta_2 = S_4 \frac{n+8}{2}, \quad \beta_3 = -S_4 \frac{9n+42}{4}, \quad \delta_2 = S_4 \frac{n+2}{8}, \quad \zeta_1 = -S_4 \frac{n+2}{2} \quad (27)$$

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and $S_4 = 1/8\pi^2$. According to (25) we have for functions $f_i$

$$
f_0 \sim \tau |\ln \tau|^{-p}, \quad f_2 \sim \tau^{-1}|\ln \tau|^p, \quad f_4 \sim \tau^{-2}|\ln \tau|^{2p-1}.
$$

(28)

The behavior of the charge $g$ is given by the expression

$$
g = \frac{c_0}{|\ln \tau|}, \quad c_0 = 2/\beta_2 \quad (\tau \to 0),
$$

(29)

where the coefficient of the logarithmic factor is universal. When Eqs. (28) and (29) are valid, parametric representation (18) automatically ensures the results $\beta(g) = \beta_2 g^2$, $\eta(g) = 0 \cdot g$, and $\eta_2(g) = \zeta_1 g$, the correct behavior of the renormalization group functions at small $g$ values.

The objective test of Eqs.25) for lattice models were performed in many works [6–15]. In particular, it was convincingly shown in [6, 7] that high-temperature series for the Ising model allow reliable prediction of the exponent $p$. Expression (29) was confirmed with a satisfactorily accuracy in [7, 9]. Already these results provide the positive answer to the question formulated in the Introduction: parametric representation (18) gives correct results for the renormalization group functions in the weak-coupling region.

3.2. Zeroth Approximation

The Pade-analysis of Eqs. 28 is performed by the successive approximation method. In the zeroth approximation, the logarithmic factors are ignored and the functions $f_i$ are processed under the assumption of their power-law dependence on $\kappa$. The results of such an analysis presented in Table 1 show a significant difference of the obtained exponents from the exact values (see Eqs. 28) and provide a rough estimate of the critical point $\kappa_c = 0.07476 \div 0.07490$.

A more accurate estimate of $\kappa_c$ can be obtained taking into account that the ratio $\chi_4/\chi_2 \sim f_4/f_2$ in the scalar case (when $p = 1/3$) behaves as $\tau^{-3}$ and contains no logarithms [6]. As is seen in Table 2, the Pade-analysis of this quantity provides the exponent really close to the exact value and the corresponding estimate of $\kappa_c$

$$
\kappa_c = 0.07481 \div 0.07487
$$

(30)

is almost final and will be only slightly refined below. The central value of interval (30) almost coincides with the result $\kappa_c = 0.074834(15)$ obtained in [6] with a more sophisticated processing.

3.3. First Approximation
In this approximation, the following representation is used:

\[ f_0 = \tilde{f}_0 |\ln \tau|^{-p}, \quad f_2 = \tilde{f}_2 |\ln \tau|^p, \quad f_4 = \tilde{f}_4 |\ln \tau|^{2p-1} \quad (31) \]

and the Pade-analysis is applied to the functions \( \tilde{f}_i \). Since the relation \( \tau = A(\kappa_c - \kappa) \) includes the nonuniversal factor \( A \), it can be accepted that

\[ |\ln \tau| = A_0 - \ln(1 - \kappa/\bar{\kappa}_c), \quad (32) \]

**Table 1.** Position of the pole corresponding to the critical point \( \kappa_c \) and residue at it (in parentheses) for the Pade approximant \([N/N]\) of functions \([\ln f_i(\kappa)]’\).

| \( N \) | \([\ln f_0(\kappa)]’\) | \([\ln f_2(\kappa)]’\) | \([\ln f_4(\kappa)]’\) |
|---|---|---|---|
| 2 | 0.07519 (1.130) | 0.07510 (−1.113) | 0.07442 (−1.832) |
| 3 | 0.07521 (1.131)* | 0.07543 (−1.085) | 0.07419 (−1.814) |
| 4 | 0.07502 (1.116) | 0.07497 (−1.101) | 0.07476 (−1.879) |
| 5 | 0.07480 (1.063) | 0.07513 (−1.103) | 0.07477 (−1.881) |
| 6 | 0.07486 (1.082) | 0.07490 (−1.088) | 0.07476 (−1.879) |

**Table 2.** Position of the pole corresponding to the critical point \( \kappa_c \) and residue at it for the indicated Pade-approximants of function \([\ln f_2 f_4]’\).

| \( N \) | \([N + 1/N]\) | \([N/N]\) | \([N/N + 1]\) |
|---|---|---|---|
| 2 | 0.07418 (−2.871) | 0.07461 (−2.936) | 0.07558 (−2.963) |
| 3 | 0.07488 (−2.993) | 0.07450 (−2.923) | 0.07465 (−2.946) |
| 4 | 0.07486 (−2.988) | 0.07485 (−2.986) | 0.07486 (−2.988) |
| 5 | 0.07487 (−2.989) | 0.07486 (−2.987)* | 0.07491 (−2.998)* |
| 6 | 0.07481 (−2.970) | 0.07484 (−2.983) | 0.07483 (−2.978) |

where the free parameter \( A_0 \) and trial value for \( \bar{\kappa}_c \) the critical point are used to accurately fit the exponent and to obtain a self-consistent result for \( \kappa_c \). According to Table 3, such a

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1The asterisk in Tables 1–4 marks defective approximants. A "defect" in the Pade-analysis is the appearance of a pair of a pole and a root close to each other; as a result, the corresponding Pade-approximant is reduced to a lower order approximant. The defectiveness of the approximant can lead to loose of the accuracy and is a reason for its discrimination.
fit is easy and good results for the exponent are obtained in a wide range of the $A_0$ values.
The optimal $A_0$ lie in the interval $0.13 \div 0.63$ and a new estimate of the critical point

$$\kappa_c = 0.07483 \div 0.07489$$

**Table 3.** Padé analysis of the functions $\tilde{f}_i$, introduced according to Eqs.31.

| $A_0$  | $[\ln \tilde{f}_0(\kappa)]'$  | $[\ln \tilde{f}_2(\kappa)]'$  | $[\ln \tilde{f}_4(\kappa)]'$ |
|-------|-------------------------------|-------------------------------|-------------------------------|
| 2.0   | 0.07491 (1.037)*              | 0.07486 (-1.023)              | 0.07488 (-1.968)*             |
| 1.0   | 0.07487 (1.018)               | 0.07484 (-1.007)              | 0.07493 (-1.989)*             |
| 0.625 | 0.074855 (1.0085)             | 0.074834 (-1.00005)           |                               |
| 0.5   | 0.074846 (1.0029)             | 0.07475 (-0.973)              | 0.07477 (-1.960)              |
| 0.25  |                               | 0.07475 (-0.973)              | 0.07488 (-1.960)              |
| 0.2   |                               |                               | 0.074890 (-1.9994)            |
| 0.13  | 0.0748420 (1.00005)           |                               |                               |
| 0.1   | 0.074840 (0.9993)             | 0.07483 (-0.990)*             | 0.07490 (-2.0044)             |
| 0.06  | 0.07487 (1.0033)              | 0.07482 (-0.988)*             | 0.07491 (-2.0063)             |

is only slightly shifted as compared to Eq.30. The results for the constant $c_0$ in Eq.29 are shown in Fig. 1a; they are close to the theoretical value $c_0^{th} = 35.09$, but are systematically above it. Similar inaccuracies in the determination of $c_0$ were observed in other works. The use of constants $A$, $B$, and $D$ for a simple hypercubic lattice from Table 5 in [7] gives the estimate $c_0 = B/A^2D^4 = 142.8$ instead of a theoretical result of 105.2 referring to the used normalization. A worse estimate was obtained in [10]; very bad results (discrepancies of 9 and 18 times) were obtained for other lattices [7]. A satisfactory test of Eq.29 was declared in [9], where the tested relation was not Eq.29, but its consequence $dg^{-1}/d\ln \tau = 1/c_0$; in this case, the central value $c_0$ approximately corresponds to Fig. 1a and the agreement with the theory was achieved at the expense of an increase in the uncertainty of the results because of differentiation.

### 3.4. Second Approximation

Expressions (25) and (28) are obtained in the leading logarithmic approximation. In the next-to-leadinglogarithmic approximation (see Appendix A), they have the form

$$f_0 = h_0 \tau (f_{\text{sing}})^{-p}, \quad f_2 = h_2 \tau^{-1} (f_{\text{sing}})^p h_{\text{sing}}, \quad f_4 = h_4 \tau^{-2} (f_{\text{sing}})^{2p-1},$$

(33)

Here, the functions $h_i(\kappa)$ are regular at $\kappa \to \kappa_c$ and singular functions are chosen in the form

$$f_{\text{sing}}(\kappa) = 1 - \bar{g} \ln \tau + s\bar{g} \ln (1 - \bar{g} \ln \tau),$$

(34)
Figure 1: Constant $c_0$ in Eq.29 versus the parameter $A_0$ in the leading logarithmic approximation (a), and versus $\bar{g}$ in the next-to-leading logarithmic approximation (b).
where
\[ s = \frac{2\beta_3}{\beta_2} - \frac{\zeta_1}{\beta_2} = \frac{n^2 - 8n - 68}{(n+8)^2}, \quad q = \frac{2\delta_2}{\beta_2^2} = \frac{n+2}{(n+8)^2}. \] (36)

The main distinction from Eqs.28 is reduced to the replacement of \(|\ln \tau|\) by \(|\ln \tau| + s \ln |\ln \tau|\) with a known parameter \(s\); in view of the ambiguity of the normalization of \(\tau\), it is necessary to consider the combinations \(A + |\ln \tau| + s \ln(B + |\ln \tau|)\), where the constants \(A\) and \(B\) are different for different functions. Formally, these constants do not affect the character of a singularity, but their unsuccessful choice can strongly distort the results. To avoid a large number of fitting parameters, \(f_{\text{sing}}(\kappa)\) was taken in the functional form following from perturbation theory. A reason for such a choice is as follows. The parameter \(\bar{g}\) has the sense of the Ginzburg number and determines the size of the critical region, where logarithmic corrections are significant. It is of interest to estimate this parameter, because the Ginzburg number is often small even in the absence of theoretical reasons for this. The function \(f_{\text{sing}}(\kappa)\) at small values of \(\bar{g}\) is close to unity almost everywhere, but increases sharply near \(\kappa_c\). If the singularity is separated inappropriately, regular functions \(h_i(\kappa)\) in Eqs.33 are rapidly varying near \(\kappa_c\) and are poorly reproduced by Padé-approximants. However, for small values of \(\bar{g}\) the form of Eq.34 is practically exact, so that functions \(h_i(\kappa)\) are almost constant. For \(\bar{g} \gtrsim 1\), the form of Eq.34 is not exactly correct, but inaccuracy in the separation of singularities in this case is not so critical, because the function \(f_{\text{sing}}(\kappa)\) is rather slowly varying.

The universal choice \(f_{\text{sing}}(\kappa)\) for all functions is possible if the \(O(\bar{g})\) contributions are negligible as compared to unity (see Appendix A), so that the inclusion of factors of the \(h_{\text{sing}}(\kappa)\) type is strictly speaking beyond of accuracy. However, such factors are sometimes of qualitative importance. In Eqs.33 they are taken into account in the minimal manner: the product \(f_0 f_2\) in this form has the correct singularity and ensures the correct behavior of \(\eta(g)\) at small \(g\); similarly, the product \(f_4 f_2\) is incompletely free of logarithms and this property makes it possible to slightly correct deviations observed in Table 2.

Table 4 presents the Padé-analysis of the functions \(\tilde{f}_i\) introduced by the relations
\[ f_0 = \tilde{f}_0 (f_{\text{sing}})^{-p}, \quad f_2 = \tilde{f}_2 (f_{\text{sing}})^p h_{\text{sing}}, \quad f_4 = \tilde{f}_4 (f_{\text{sing}})^{2p-1}, \] (37)
rather than by Eqs.31; the estimate of the parameter \(c_0\) in Eq.29 is illustrated in Fig.1b. It is easily seen that the actual interval of \(\bar{g}\) values is much narrower than that in the leading logarithmic approximation (where the parameter \(1/A_0\) is similar to \(\bar{g}\)). The optimum values for various functions cover the range of 0.85 ÷ 1.06, which provides the estimate
\[ c_0 = 36.3 \pm 1.8 \] (38)
in good agreement with a theoretical value of 35.09. The exact \(c_0\) value is realized at \(\bar{g} \approx 1.02\) (see Fig.1b). Finally, Table 4 presents the maximally accurate estimate of the critical point
\[ \kappa_c = 0.074840 \div 0.074867, \] (39)
which is available with the existing information. The values accepted below are \( \kappa_c = 0.074850 \) from the middle of interval (39) and \( \bar{g} = 1.020385 \), which ensures the exact \( c_0 \) value for the \([3/3]\) approximant.

Table 4. Pade-analysis of the functions \( \tilde{f}_i \) introduced according to Eqs.37.

| \( \bar{g} \)   | \( [\ln \tilde{f}_0(\kappa)]' \) | \( [\ln \tilde{f}_2(\kappa)]' \) | \( [\ln \tilde{f}_4(\kappa)]' \) |
|-------------|-----------------|-----------------|-----------------|
|             | \([6/6]\), \( \bar{\kappa}_c = 0.074843 \) | \([6/6]\), \( \bar{\kappa}_c = 0.074840 \) | \([6/5]\), \( \bar{\kappa}_c = 0.074867 \) |
| 0.5         | 0.07492 (1.036)* | 0.07488 (−1.024) | 0.07487 (−1.968)* |
| 0.7         | 0.07488 (1.019)* | 0.07485 (−1.0096) | 0.07491 (−1.988)* |
| 0.85        | ——              | 0.074840 (−1.0008) | ——              |
| 0.9         | 0.07485 (1.0052) | 0.074836 (−0.998) | 0.074877 (−1.994) |
| 0.99        | 0.074843 (1.00005)| ——             | 0.074865 (−1.997) |
| 1.0         | 0.074842 (0.9995)| 0.07483 (−0.994) | 0.074867 (−2.0001)|
| 1.06        | ——              | ——             | 0.074867 (−2.010) |
| 1.2         | 0.07482 (0.988) | 0.07476 (−0.976)* | 0.07488 (−2.010) |

4. RESULTS FOR RENORMALIZATION GROUP FUNCTIONS

The derivatives of singular functions can be written in the form

\[
[\ln f_{\text{sing}}]' = \frac{u_1(\tau)}{\bar{\kappa}_c \tau}, \quad [\ln f_{\text{sing}}]'' = \frac{u_2(\tau)}{(\bar{\kappa}_c \tau)^2}, \quad [\ln h_{\text{sing}}]' = \frac{v_1(\tau)}{\bar{\kappa}_c \tau}, \quad [\ln h_{\text{sing}}]'' = \frac{v_2(\tau)}{(\bar{\kappa}_c \tau)^2},
\]

where

\[
u_1(\tau) = -\frac{q\bar{g}^2}{f_{\text{sing}}(f_{\text{sing}} + q\bar{g})} \left(1 + \frac{s\bar{g}}{1 - \bar{g} \ln \tau}\right),
\]

\[
u_2(\tau) = -\frac{q\bar{g}^2}{f_{\text{sing}}(f_{\text{sing}} + q\bar{g})} \left(1 + \frac{s\bar{g}}{1 - \bar{g} \ln \tau}\right) - \frac{s\bar{g}^2}{(1 - \bar{g} \ln \tau)^2} - \left(\frac{\bar{g}}{f_{\text{sing}}} + \frac{\bar{g}}{f_{\text{sing}} + q\bar{g}}\right) \left(1 + \frac{s\bar{g}}{1 - \bar{g} \ln \tau}\right)^2.
\]
Taking into account (40), the substitution of Eqs.33 into Eqs.18 provides the parametric representation for the renormalization group functions in the form

$$g = \frac{H(\kappa)}{\kappa^2 f_{\text{sing}}}, \quad H(\kappa) = h_4 h_0^2,$$

$$\beta(g) = \frac{2\kappa c \tau}{g} \left( 2 - \kappa \ln h_0 h_0' + 2\kappa u_1 \right) \kappa c \tau \left( 1 - \kappa \ln h_0' \right) + \kappa (1 + pu_1),$$

$$\eta(g) = \frac{-2\kappa c \tau}{g} \left( \ln h_0 h_2' - 2\kappa v_1 \right) \kappa c \tau \left( 1 - \kappa \ln h_0' \right) + \kappa (1 + pu_1).$$

$$\eta_2(g) = -2 \frac{(\kappa c \tau)^2 (1 - \kappa^2 \ln h_2')}{\{ \kappa c \tau \left( 1 - \kappa \ln h_0' \right) + \kappa (1 + pu_1) \}} \left\{ \kappa c \tau \left( 1 + \kappa \ln h_2' \right) + \kappa (1 + pu_1 + v_1) \right\}^2 - \kappa^2 (1 + pu_2 + v_2) \left\{ \kappa c \tau \left( 1 - \kappa \ln h_0' \right) + \kappa (1 + pu_1 + v_1) \right\}^2 - \kappa^2 (1 + pu_2 + v_2) \left\{ \kappa c \tau \left( 1 + \kappa \ln h_2' \right) + \kappa (1 + pu_1 + v_1) \right\}^2$$

Asymptotic expressions (19) are obtained at $\kappa \to 0$ irrespective of the form of regular functions, whereas at $\tau \to 0$ we have the results

$$g = \frac{2\bar{g}}{\beta_2 f_{\text{sing}}}, \quad \beta(g) = \frac{2\bar{g}}{f_{\text{sing}}} + \frac{2(s - p)\bar{g}^2}{f_{\text{sing}}^2}, \quad \eta(g) = \frac{2\bar{g}^2}{f_{\text{sing}}^2}, \quad \eta_2(g) = -\frac{2p\bar{g}}{f_{\text{sing}}},$$

which reproduce the first two terms of the expansion for $\beta(g)$ and the first terms of the expansions for $\eta(g)$ and $\eta_2(g)$ in Eqs.26. When the terms with $\tau$ are neglected, Eqs.42 provide the regular expansions of the renormalization group functions in $g$ (certainly without the reproduction of correct coefficients), whereas the terms with $\tau$ provide the $\exp(-\text{const}/g)$ singularity, which should exist owing to the factorial divergence of the perturbation series [20, 21]. Thus, the parametric representation is rather "intelligent" and ensures the correct analytical properties at $g \to 0$.

The accuracy of the entire construction is determined by the accuracy of the determination of the regular functions $h_i(\kappa)$. The expansions of these functions in $\kappa$ are obtained from Eqs.33 and are used to construct the Padé-approximants, which are regular in the interval $(0, \kappa_c)$, because all singularities have been separated. The obtained regular functions are shown in Fig. 2. For the functions $H(\kappa)$ and $[\ln h_0(\kappa)]'$, all approximants provide almost coinciding results; small distinctions are visible for the function $[\ln h_4(\kappa)]'$ near $\kappa_c$ (see Fig. 2). The situation is less satisfactory with the function $[\ln h_2(\kappa)]'$ for which an increase in the order of the Padé-approximation leads to an increase in the deviations from the regular behavior predicted by lower approximants. It is unclear whether the sequence of approximants converged sufficiently or such deviations will further increase.

Note that the coefficients $\beta_2$, $\beta_3$, $\beta_4$, $\zeta_1$ exhaust invariant (scheme-independent) information on the renormalization group functions and a further refinement of the procedure (the construction of the next-to-next-to-leading logarithmic approximation, etc.) requires the calculation of the subsequent coefficients for the corresponding lattice regularization.
Figure 2: Regular functions $H(\kappa)$ and $[\ln h_i(\kappa)]'$ obtained in the Pade-approximation.

these deviations can be artifact due to an incompletely consistent separation of singulari-
ties leading to a residual singularity in the function $[\ln h_2(\kappa)]'$ (in the used approximation),
which affects higher approximants. In the latter case, the behavior predicted by the $[3/3]$
$[2/3]$, and $[3/2]$ approximantscan be more authentic. Fortunately, this dilemma can be
resolved using the strong-coupling expansions (see Section 5), which certainly indicate that
the use of higher Pade-approximants is correct and the results obtained in this case are
satisfactory. Appendix B presents the parameters of the approximants used for $H(\kappa)$ and
$[\ln h_i(\kappa)]'$, which allow the application of parametric representation (42).

To represent the results, it is convenient to use the so called ”natural normalization” of
the charge, which is obtained by the change $g \rightarrow (16\pi^2/3)g$ and corresponds to the repre-
sentation of the interaction term in the form $(16\pi^2/4!)g_0\phi^4$; in this case, the parameter $a$
in the Lipatov asymptotic form $ca^N\Gamma(N + b)$ [20, 21] is unity and the nearest singularity
in the Borel plane lies at the unit distance from the coordinate origin [21]; this property
defines functions varying at an approximately unit scale. The solid lines in Fig.3 are the
resulting renormalization group functions, whereas the dashed lines are the strong- and
weak-coupling asymptotic behaviors. The approach to the strong-coupling asymptotics is

\footnote{The traditional representation $g_0\phi^4/8$ in the $n$-component case is motivated by the fact that the vertex $\Gamma^{(4)}_{\alpha\beta\gamma\delta} = gI_{\alpha\beta\gamma\delta}$ in the lowest order is $g_0I_{\alpha\beta\gamma\delta}$, which ensures the relation $g = g_0$ in the limit $g_0 \rightarrow 0$. In the scalar case, the tensor $I_{\alpha\beta\gamma\delta}$ is reduced to three and the interaction is represented as $g_0\phi^4/4!$. This motivation logical at first glance is in fact illusory, because the bare charge has no physical sense.}
strongly prolonged in agreement with the results reported in [22]. However, the prolongation of the one-loop behavior of the $\beta$ function pointed out in that work is not confirmed: it appears to be an artifact, conditioned by essential exceeding of the limiting value of $\beta(g)/g$ obtained in [22] in comparison with Fig. 3 [1].

To illustrate the accuracy of the construction, the dotted lines show the results obtained if the functions $h_i(\kappa)$ are changed to constants; in this case, the results contain no information on these functions, because $\ln h_i(\kappa)' = 0$ and a constant value of $H(\kappa)$ is fixed by Eq.29. It is easy to see that an accuracy of about 1% for $\beta(g)/g$ and $\eta_2(g)$ is reached even in the complete absence of information on regular functions.\footnote{The reason is that the terms $[\ln h_i]'$ in Eqs.42 has the factor $\kappa \kappa c \tau = \kappa (\kappa c - \kappa)$, which is small both for $\kappa \to 0$ and for $\kappa \to \kappa c$; this factor in the middle of the interval $\kappa = \kappa c/2$ is equal to $\kappa c^2/4$, whereas the other terms are on the order of $\kappa c$. In view of $\kappa c \approx 1/15$, the effect of regular functions on $\beta(g)/g$ and $\eta_2(g)$ is about 1%. The situation for $\eta(g)$ is different in view of the absence of the $\kappa c \tau$ term in the numerator.}

The real uncertainty of the construction is about two orders of magnitude smaller than the difference between the solid and dotted lines, because the regular functions (see Fig. 2) are specified better than 1% except for the region $\kappa > 0.8\kappa c$, where the error for the function $[\ln h_2(\kappa)]' = 0$ can reach 10%. However, this region corresponds to $g < 0.5$ (see Fig. 4), where the effect of regular functions is insignificant.

Table 5. Coefficients of the expansions in $g^{-2/d}$ for the functions $\beta(g)/g$, $\eta(g)$ and $\eta_2(g)$.

| N | $\beta(g)/g$ | $\eta(g)$ | $\eta_2(g)$ |
|---|-------------|-----------|-------------|
| 0 | 4.0000000000000000 | 0.0000000000000000 | -4.0000000000000000 |
| 1 | -26.127890589687201 | 0.0000000000000000 | 26.127890589687201 |
| 2 | 106.666666666666666 | 1.7777777777777778 | -60.444444444444444 |
| 3 | -557.394999246655315 | -11.612395817638043 | 81.286770723472347 |
| 4 | 3214.2222222222222 | 29.708641975308642 | -44.879012345695039 |
| 5 | -16396.702894504412 | 22.708651544477281 | -120.7213779957037 |
| 6 | 67356.444444444432 | -961.1312561239812 | 9071.1992161453812 |
| 7 | -139720.34647768043 | 7188.4949076856250 | -49662.87860424104 |
| 8 | 717634.3703724248 | -27680.89232384043 | 197619.3919150387 |
| 9 | 9878174.820924701 | -7609.77032773753 | -226822.0836412604 |
| 10 | -59767955.489704091 | 938732.278408470 | -3873286.846552091 |
| 11 | 186179701.36333458 | -7226487.636373591 | 41826925.334796871 |
| 12 | 355069103.58896850 | 27981910.625965901 | -249549251.3846024 |
| 13 | -8851453360.742100 | 7407298.571430670 | 794136522.5461800 |
Figure 3: Solid lines are the renormalization group functions. The dashed lines are the strong- and weak-coupling asymptotic behaviors. The dotted lines are the results obtained under the assumption of the constancy of regular functions $h_i(\kappa)$ under which Eqs.42 contain no information on them.
5. STRONG-COUPLING EXPANSIONS

Expanding the right-hand sides of Eqs.18 in $\kappa$ and expressing $\kappa$ in terms of $g$, it is easy to verify that the functions $\beta(g)/g$, $\eta(g)$, $\eta_2(g)$ are expanded in $g^{-2/d}$ as

$$\frac{\beta(g)}{g} = \sum_{N=0}^{\infty} B_N \left(-g^{-2/d}\right)^N \text{ etc.} \quad (44)$$

The expansion coefficients up to $N = 13$ recalculated from high-temperature series are given in Table 5.\footnote{Fourteen digits output by a computer are formally presented. The accuracy decreases beginning with $N = 3$ and the last four digits are unreliable at $N = 13$.}

It is easy to verify that the ratios $B_{N+1}/B_N$ are the same order of magnitude for all $N$, indicating the finite convergence radius. The Padé-analysis of series (44) reveals poles in the region $|g^{-1/2}| \sim 0.1$; these poles for most approximants do not lie on positive semiaxis in agreement with regularity of the renormalization group functions. To obtain the correct power-law behavior in the limit $g \to 0$, it is necessary to use the $[N/N + 2]$ approximants for $\beta(g)/g$ and $\eta_2(g)$ and the $[N/N + 4]$ approximants for $\eta(g)$. Such a procedure predicts $\delta_2$ with an accuracy of about 20%, whereas $\beta_2$ and $\zeta_1$ are estimated only by the order of magnitude. For this reason, the summation of series (44) in the region of small $g$ gives less accurate results than the procedure described above.

All approximants provide almost coinciding results in the region of large $g$; this coincidence holds to $g = 0.5$ with an accuracy of about 1%. Such estimates for the functions $\beta(g)/g$ and $\eta_2(g)$ are in agreement with the more accurate results obtained above. The estimates for the function $\eta(g)$ certainly indicate that the highest order approximants should be used for $[\ln h_2]'$ and the results are confirmed at a level of about 1%. Series (44) can apparently be used more efficiently, but analysis of this possibility is beyond the scope of this work.

6. DISCUSSION OF THE RESULTS

The resulting $\beta$ function is non-alternating and has the asymptotic behavior $\beta(g) = 4g$ in the limit $g \to \infty$. According to the classification proposed by Bogoliubov and Shirkov [23] (see discussion in [1]), this means the possibility of the construction of a continual theory with a finite interaction at large distances. The last conclusion contradicts the widespread opinion that the $\phi^4$ theory is "trivial" [24–28]. As was discussed in [1, 30], two definitions — Wilson triviality [24] and mathematical triviality [25, 26] — were confused in the literature. The first triviality is firmly established (it corresponds to positivity of the $\beta$ function), whereas pieces of evidence in favor of the second triviality are scarce [27] and allow another interpretation [1–30]. According to above analysis, we have no contradictions in the properties of the lattice $\phi^4$ theory with the works cited in [1, 30]. However, there is
a conceptual contradiction which we want to stress: it concerns the role and significance of
the lattice theory.

The usual point of view implies that the lattice φ4 theory provides a reasonable ap-
proximation for the actual field theory. This interpretation provides the natural condition
ξ ≫ a, according to which many sites of the lattice should be at the characteristic variation
scale of the field φ(x). This condition can be liberalized to ξ ∼ a or strengthen to ξ/a → ∞.
In the former case the restriction g < ∼ 1 for renormalized charge is obtained (for the natural
normalization) [28], while g = 0 in the latter case (corresponding to the phase transition
point). Thus, the usual statements are obtained: the theory is trivial in the continual limit
(Λ/m → ∞), whereas in the presence of a cutoff the interaction is limited from above and
cannot be strong. The latter circumstance is used to obtain an upper bound for the mass
of the Higgs boson [28, 29].

Our position is that the lattice theory should not be considered as any approximation
to the actual field theory (although this is possible at g0 ≪ 1). The continual theory
fundamentally involves no lattice; a lattice appears only in the bare theory, which is an
auxiliary construction and is completely eliminated later. The bare theory has no physical
sense and should not satisfy any physical requirements. Without restriction ξ ∼ a, the
renormalized charge can have any value (see Fig. 4). The proposed concept is completely
consistent with the “rules” accepted in mathematical works [25, 26] according to which the
continual limit a → 0 is taken at arbitrarily chosen dependences g0(a) and m0(a); in this
paper, they are taken under conditions (5).

The only alternative for the perturbative approach is that all quantities referring to the
continual theory are expressed in terms of functional integrals. These integrals depend on
g0, m0, and Λ and, with dimensionality taken into account, we have for the charge, mass,
and other physical quantities Ai (observables, renormalization group functions, etc.)

\[ g = F_g (g_0, m_0/\Lambda), \quad m = \Lambda F_m (g_0, m_0/\Lambda), \quad A_i = \Lambda^{d_i} \tilde{F}_i (g_0, m_0/\Lambda), \tag{45} \]

where \( d_i \) is the physical dimension of the quantity \( A_i \). According to Eqs.45, the real
designation of the bare theory is to ensure the representation of the physical quantities
in a parametric form. The relations between \( g, m, \) and \( A_i \) are of physical interest; the
parametric representation is of no deep sense in view of its ambiguity: it can be written in
various forms by changing \( g_0 \) and \( m_0/\Lambda \) to any other pair of variables. For this reason,
an attempt to give the physical sense to the bare theory faces the question: Why one of
numerous parametrizations is of particular significance?

Excluding \( g_0 \) and \( m_0/\Lambda \) in favor of \( g \) and \( m/\Lambda \), it is possible to arrive at the relation

\[ A_i = \Lambda^{d_i} \tilde{F}_i (g, m/\Lambda). \tag{46} \]

In the general case, the exclusion of the dependence on Λ requires the passage to the limit
m/Λ → 0, which corresponds to the critical point and returns us to the “zero charge”
situation. However, the central point is that the general-position situation does not occur
in Eq.46: after the transformation to the Ising model (valid under conditions (5)), all
functions in Eqs.45 depend on the single parameter $\kappa$; as a result, the dependence on $m/\Lambda$ is completely absent in Eq.(46)

$$A_i = m^d F_i(g).$$

The renormalization program is thereby completed and no additional passages to limits are required. This means that (a) the lattice can be retained in the bare theory (as a convenient technical tool for the representation of functional integrals) and (b) the relation between $m$ and $\Lambda$ can be assumed to be arbitrary, which ensures the attainability of any value of $g$ (see Fig. 4).

We consider the above procedure as a real scheme for constructing the continual $\phi^4$ theory with a finite interaction. In fact, dependence of $g$ and $m$ on bare parameters (Fig. 4), as well as the results for the renormalization group functions (Fig. 3), have been obtained in the present paper.

**APPENDIX A. Next-to-Leading Logarithmic Approximation**

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6 This is not surprising, because the passage to the continual limit was performed in the process of the transformation to the Ising model [2], which was required by the needs of renormalized, but not bare theory.
The basic formulas referring to the next-to-leading logarithmic approximation underlying representation (33) will be given below. The starting point is the Callan-Symanzik equation in the cutoff scheme\[7\]

\[
\left[ \frac{\partial}{\partial \ln \Lambda} + \beta(g_0) \frac{\partial}{\partial g_0} - \gamma(g_0) \right] F(g_0, \Lambda/m) = 0, \tag{A.1}
\]

where the function \(F\) satisfies the logarithmic expansion

\[
F(g_0, \Lambda/m) = \sum_{N=0}^{\infty} g_0^N \sum_{K=0}^{N} A_K^N \left( \ln \frac{\Lambda}{m} \right)^K.	ag{A.2}
\]

The substitution of (A.2) to (A.1) taking into account the expansions

\[
\beta(g_0) = \sum_{M=2}^{\infty} \beta_M g_0^M, \quad \gamma(g_0) = \sum_{M=1}^{\infty} \gamma_M g_0^M
\]
yields the system of recurrence relations for the coefficients \(A^K_N\):

\[
-K A^K_N = \sum_{M=1}^{N-1-K+1} [\beta_{M+1}(N - M) - \gamma_M] A^{K-1}_{N-M}, \quad K = 1, 2, \ldots, N \tag{A.3}
\]

In particular, for \(K\) close to \(N\)

\[
-N A^N_N = [\beta_2(N - 1) - \gamma_1] A^{N-1}_{N-1}, \quad \tag{A.4}
\]

\[
-(N - 1) A^{N-1}_N = [\beta_2(N - 1) - \gamma_1] A^{N-2}_{N-2} + [\beta_3(N - 2) - \gamma_2] A^{N-2}_{N-2},
\]

\[
-(N - 2) A^{N-2}_N = [\beta_2(N - 1) - \gamma_1] A^{N-3}_{N-3} + [\beta_3(N - 2) - \gamma_2] A^{N-3}_{N-3} + [\beta_4(N - 3) - \gamma_3] A^{N-3}_{N-3},
\]

e tc. The first equation in (A.4) is solved immediately; after that, the next equations can be solved one-by-one using the method of variation of constants.

**Vertex \(\Gamma^{(1,2)}\).** For this vertex, \(\gamma(g_0) = \eta_2(g_0)\), all coefficients are nonzero, and \(A^0_0 = 1\); the first two equations in Eqs. (A.4) give

\[
A^N_N = (-\beta_2)^N \frac{\Gamma(N + p)}{\Gamma(p)\Gamma(N + 1)}, \quad p = -\frac{\gamma_1}{\beta_2} = -\frac{\zeta_1}{\beta_2}, \tag{A.5}
\]

\[
A^{N-1}_{N-1} = (-\beta_2)^{N-1} \frac{\Gamma(N + p)}{\Gamma(1 + p)\Gamma(N)} \left\{ \frac{\beta_2}{\beta} \sum_{n=1}^{N-1} \frac{1}{n + p} + O(1) \right\}.
\]

The substitution of (A.5) into (A.2) and the summation of the corresponding series using the formulas

\[
(1 + x)^\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(n - \alpha)}{\Gamma(-\alpha)\Gamma(n + 1)} (-x)^n, \tag{A.6}
\]

\(^7\) Its difference from Eq.10 is of no significance at present context, because the first coefficients \(\beta_2, \beta_3, \delta_2, \zeta_1\) are independent of the renormalization scheme.
\[(1 + x)^\alpha \ln(1 + x) = \sum_{n=0}^{\infty} \frac{\Gamma(n - \alpha)}{\Gamma(-\alpha)\Gamma(n + 1)} (-x)^n \frac{1}{\alpha - k}\]

yield

\[\Gamma^{(1,2)} = \left\{1 + O(g_0) + \beta_2 g_0 \ln \frac{\Lambda}{m} + g_0 \frac{\beta_3}{\beta_2} \ln \left(1 + \beta_2 g_0 \ln \frac{\Lambda}{m}\right) \right\}^{-p} \quad (A.7)\]

The \(O(g_0)\) terms will be omitted below.

The renormalized charge \(g\) satisfies Eq. (A.1) with \(\gamma(g_0) \equiv 0\), whereas all coefficients \(A_N^0\) in expansion (A.2) are zero and \(A_0^1 = 1\). Similar to Eqs. (A.5) and (A.7) we have a result

\[A_N^{-1} = (-\beta_2)^{N-1}, \quad A_N^{N-2} = (-\beta_2)^{N-2} (N - 1) \left\{\frac{\beta_3}{\beta_2} \sum_{n=1}^{N-1} \frac{1}{n} + O(1)\right\} \quad (A.8)\]

and

\[g = g_0 \left\{1 + \beta_2 g_0 \ln \frac{\Lambda}{m} + g_0 \frac{\beta_3}{\beta_2} \ln \left(1 + \beta_2 g_0 \ln \frac{\Lambda}{m}\right) \right\}^{-1} \quad (A.9)\]

which can also be obtained directly from the GellMann-Low equation.

Renormalized mass. Neglecting the \(Z\) factor, the Ward identity

\[\Gamma^{(1,2)} = \frac{d}{dm_0^2} \Gamma^{(0,2)} = \frac{d}{dm_0^2} \frac{m^2}{Z} \quad (A.10)\]

can be written in the form \(dm_0^2/dm^2 = 1/\Gamma^{(1,2)}\); the integration with respect to \(m^2\) within the necessary accuracy is reduced to the multiplication by \(m^2\),

\[m^2 = (m_0^2 - m_e^2) \left\{1 + \beta_2 g_0 \ln \frac{\Lambda}{m} + g_0 \frac{\beta_3}{\beta_2} \ln \left(1 + \beta_2 g_0 \ln \frac{\Lambda}{m}\right) \right\}^{-p}, \quad (A.11)\]

where \(m_e^2\) is the value of \(m_0^2\) corresponding to the transition point. The introduction of the dimensionless distance to the transition \(\tau \propto (m_0^2 - m_e^2)\) and iterative exclusion of \(m\) from the righthand side give

\[m^2 = \tau \left[1 + \tilde{g} \ln 1/\tau + s \tilde{g} \ln (1 + \tilde{g} \ln 1/\tau)\right]^{-p}, \quad \tilde{g} = \beta_2 g_0/2 \quad (A.12)\]

where \(s\) is given in Eq.36. Similarly, (A.9) reduces to the form

\[g = \frac{2}{\beta_2} \frac{\tilde{g}}{1 + \tilde{g} \ln 1/\tau + s \tilde{g} \ln (1 + \tilde{g} \ln 1/\tau)}. \quad (A.13)\]

The \(Z\) factor satisfies Eq. (A.1) with \(\gamma(g_0) = -\eta(g_0)\), while \(A_0^0 = 1, A_1^0 = A_1^1 = 0\) and \(A_N^N = 0\) for \(N \geq 2\) in expansion (A.2). Similar to (A.8), we have for \(N \geq 2\)

\[A_N^{-1} = A_2^1 (-\beta_2)^{N-2}, \quad A_N^{N-2} = A_2^2 (-\beta_2)^{N-2} (N - 1) \left\{-\frac{\beta_3}{\beta_2} \sum_{n=2}^{N-1} \frac{1}{n} + O(1)\right\}, \quad (A.14)\]
and after summation

$$Z = 1 + \frac{A_1^2 g_0}{\beta_2} = 1 + \frac{A_1^2 g_0}{\beta_2} \left( 1 + \beta_2 g_0 \ln \frac{\Lambda}{m} + g_0 \frac{\beta_3}{\beta_2} \ln \left( 1 + \beta_2 g_0 \ln \frac{\Lambda}{m} \right) \right)^{-1}$$  \hspace{1cm} (A.15)

Taking into account the relation $A_1^2 = -\delta_2$, expressing $m$ in terms of $\tau$ and omitting an insignificant constant factor, one obtains with the necessary accuracy

$$Z = 1 + \frac{2\delta_2}{\beta_2^2} \frac{\bar{g}}{1 + \bar{g} \ln 1/\tau + s\bar{g} \ln (1 + \bar{g} \ln 1/\tau)}.$$  \hspace{1cm} (A.16)

The substitution of (A.12), (A.13), (A.16) into the relations

$$\frac{K_2}{K_2} = m^2, \quad \frac{K_2}{K_0} = \frac{Z}{m^2}, \quad \frac{K_4 K_0}{K_2} = -\frac{g}{m^4},$$  \hspace{1cm} (A.17)

yields Eqs.33 for $f_i(\kappa)$. The difference of the $Z$ factor from unity corresponds to the corrections of the order $g_0/\ln \tau$, which were neglected above, and strictly speaking is beyond the accuracy. However, without the inclusion of the $Z$ factor, the product $f_0 f_2$ would be a regular function and, correspondingly, the behavior of $\eta(g)$ at small $g$ values would be incorrect. For this reason, the function $h_{\text{sing}}$ corresponding to the $Z$ factor is introduced in Eqs.33 by the minimal manner to ensure the correct singularity in $f_0 f_2$.

### Table 6. Parameters of Pade-approximation (24) of regular functions

| $H(\kappa)$ | $[\ln h_0(\kappa)]'$ |
|-------------|----------------------|
| $n$ | $p_n$ | $q_n$ | $p_n$ | $q_n$ |
| 0 | 0.166666 | 1.000000 | -2.389114 | 1.000000 |
| 1 | 2.173343 | 12.28756 | 39.93594 | 1.218909 |
| 2 | -8.874246 | -6.056224 | 134.2565 | -14.76806 |
| 3 | 103.5876 | -124.8396 | -1759.943 | 498.1762 |
| 4 | 0 | 0 | 14434.97 | -2468.179 |

| $[\ln h_2(\kappa)]'$ | $[\ln h_4(\kappa)]'$ |
|------------------|------------------|
| $n$ | $p_n$ | $q_n$ | $p_n$ | $q_n$ |
| 0 | 2.416517 | 1.000000 | 5.530725 | 1.000000 |
| 1 | -50.63241 | -3.794992 | 13.37787 | 21.09480 |
| 2 | -345.9676 | -201.7335 | 630.6971 | 57.28333 |
| 3 | 9156.772 | 738.3887 | 3430.220 | 252.1934 |
| 4 | -1285.833 | 4787.275 | 0 | 10511.06 |
| 5 | -267488.9 | -26827.13 | 0 | 0 |
| 6 | 109199.7 | 363530.4 | 0 | 0 |
Table 6 shows the coefficients $p_n$ and $q_n$ in Eq.24 for the Pade-approximantion of the regular functions $H(\kappa)$ and $[\ln h_i(\kappa)]'$; the lowest order approximants having the complete accuracy are presented. The singularities were separated with the values $\kappa_c = 0.074850$ and $\bar{g} = 1.020385$.

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