Generic properties of Lagrangians on surfaces: the Kupka-Smale theorem

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Abstract

We consider generic properties of Lagrangians. Our main result is the Theorem of Kupka-Smale, in the Lagrangian setting, claiming that, for a convex and superlinear Lagrangian defined in a compact surface, for each $k \in \mathbb{R}$, generically, in Mané’s sense, the energy level, $k$, is regular and all periodic orbits, in this level, are nondegenerate at all orders, that is, the linearized Poincaré map, restricted to this energy level, does not have roots of the unity as eigenvalues. Moreover, all heteroclinic intersections in this level are transversal. All the results that we present here are true in dimension $n \geq 2$, except one (Theorem 18), whose proof we are able to obtain just for dimension 2.

1 Introduction

Our main purpose here is to obtain generic properties, in the sense of Mané (see [8], [13]), for a convex and superlinear Lagrangian, in a fixed smooth and compact surface $M$ without boundary.

Our main result is the Kupka-Smale Theorem, claiming that, for each, $k \in \mathbb{R}$, generically, this level is regular, all periodic orbits in this level are nondegenerate at all orders and all heteroclinic intersections in this level are transversal. Where, Nondegeneracy of order $m$, means that, the linearized Poincaré map, restricted to this energy level, does not have $m$-roots of the unity as eigenvalues.

In the proof of the Kupka-Smale Theorem, we will use the Nondegeneracy Lemma (Lemma 4) and a Perturbation Lemma (Lemma 9) for

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Lagrangian submanifolds in order to get the transversality of heteroclinic intersections.

The Kupka-Smale Theorem, in this formulation, resembles the Bumpy Metrics Theorem, for geodesic flows, formulated by R. Abraham in 1968, and proved by D. V. Anosov in 1983 (Anosov, [4]).

The work of W. Klingenberg and F. Takens [12] in the Bumpy Metrics Theorem proof was corrected by Anosov [4] using an induction method similar to the one used by M. Peixoto [16] in the proof of the classical Kupka-Smale Theorem.

In this work, we will employ the same techniques used by Anosov [4], adapted to perturbations by potentials. In order to apply to the case of perturbation by potentials, it is necessary to introduce a modification of the standard argument in the Control Theory for differential equations, initially used by Klingenberg [11], for geodesic flow perturbation setting, by J. A. Miranda [14] for magnetic flows on surfaces and by Contreras [7] for the proof of Franks’ Lemma for geodesic flows. In this case we do not have special coordinates, like Fermi coordinates, as in [11], [14] and [7], thus we introduce a new method without the use of tubular neighborhoods, that solves this trouble. In the beginning of the proof we use an argument similar to the one used by Robinson [17], Lemma 19, Pg. 592, but the proof is quite different.

Observe that the generic properties in Mañé’s sense cannot be obtained from the pioneering work of Robinson in the general Hamiltonian setting (see [17] and [18]) because the set of all Hamiltonians is bigger than the set of all potentials in $M$.

The transversality is the easy part. Here we follow the approach of Contreras & Paternain [9], Lemma 2.6 or J.A. Miranda [14], Lemma 3.9. The problem in this case is to construct an explicit potential that represents the perturbation.

The main obstacle in the proof of the Kupka-Smale Theorem in dimension $n > 2$ is the nature of the perturbation constructed. As in Contreras [7], Lemma 7.3 and 7.4, we need to solve a matrix equation in the Lie algebra of the symplectic group $S_p(n) = \{\text{Symplectic matrices } 2n \times 2n\}$. The solubility of this equation is strongly related with the existence of repeated eigenvalues of the matrix $H_{pp}$ in local coordinates. The problem is that we cannot change this characteristic by adding a potential. Moreover, the equation involved is very complex too.

However, we point out that the Kupka-Smale Theorem in dimension 2 is a strong result in the study of generic Lagrangians because it works below the critical level. More than that, it can be combined with other results on the structure of Aubry-Mather sets in surfaces, like Haefliger theorem for Mather measures with rational homology in an orientable surface, claiming that such measures are supported in periodic orbits, and results on the nonexistence of conjugated points in the Aubry-Mather sets from Contreras
and Iturriaga [6], in order to guarantee the hyperbolicity of the periodic orbits in this set.

2 The Kupka-Smale Theorem

We consider \((M; g)\) a, \(n\)-dimensional, smooth and compact, Riemannian manifold without boundary, \(L : TM \to \mathbb{R}\), a Lagrangian in \(M\), convex and fiberwise superlinear (see [8] to definitions) and \(H : T^*M \to \mathbb{R}\) the associated Hamiltonian obtained by Legendre transform.

In the study of generic properties of Lagrangians we use the concept of genericity due to Mané. The idea is that, the properties studied in the Aubry-Mather theory become much more strongest in this generic setting. For more details and applications see [6], [8], [14] and [13].

We will say that a property \(P\) is generic, in Mané’s sense, for \(L\), if there exists a generic set \(O \subset C^\infty(M; \mathbb{R})\), in \(C^\infty\) topology, such that, for all \(f \in O\), \(L + f\) has the property \(P\).

Consider \(E_L(x, v) = \partial L(x, v) \partial v - L(x, v)\) the energy function associated to \(L\) and \(\varepsilon^k_L = \{(x, v) \in TM \mid E_L(x, v) = k\}\) the set of all points in the energy level \(k\).

Let \(\theta \in TM\) be a periodic point of positive period, \(T_{\min}\) of the Euler-Lagrange flow \(\phi^L_t : TM \to TM\). Fixed a local section transversal to this flow, \(\Sigma\) contained in the energy level of \(\theta\), there exists a smooth function \(\tau : U \subset \Sigma \to \mathbb{R}\), such that, \(\tau(\theta) = T_{\min}\) which is the time of first return to \(\Sigma\), such that the map \(P(\Sigma, \theta) : U \to \Sigma\) given by

\[P(\Sigma, \theta)(\theta) = \phi^L_{\tau(\theta)}(\theta)\]

is a local diffeomorphism and \(\theta\) is a fix point of \(P(\Sigma, \theta)\). This map is called Poincaré first return map. We will say that \(\theta\) (or the orbit of \(\theta\)) is a nondegenerate orbit of order \(m \geq 1\) for \(L\) if

\[\text{Ker}((d\theta P(\Sigma, \theta))^m - Id) = 0.\]

The property of Nondegeneracy of order \(m\) means that \(d\theta P(\Sigma, \theta)\) does not have \(m\)-roots of the unity as eigenvalues.

If we are interested in the Hamiltonian viewpoint of the described Lagrangian dynamics, then we consider the Hamiltonian \(H\) associated to \(L\) by the Legendre transform in the speed, that is,

\[H(x, p) = \max_{v \in T_xM} \{pv - L(x, v)\}.\]

Let \(X^H\) be the Hamiltonian field, which is the unique field \(X^H\) in \(T^*M\) such that \(\omega_\theta(X^H(\dot{\psi}), \xi) = d\psi H\xi\) for all \(\xi \in T_\theta T^*M\). In the local coordinates \((x, p)\), \(X^H = H_p \partial \partial x - H_x \partial \partial p\). We denote by \(\psi^H_t : T^*M \to T^*M\) the
flow in $T^*M$ associated with the Hamiltonian field $X^H : T^*M \to TT^*M$. This flow preserves the canonical symplectic form $\omega$. Since $L$ is a convex and superlinear Lagrangian we have that $H$ is a convex and superlinear Hamiltonian. Using the Legendre transform

$$p = L_v(x, v) \text{ and } v = H_p(x, p),$$

we have that, $H_{pp}(x, p)$ is positive defined in $T^*_x M$, uniformly in $x \in M$. Observe that the Legendre transform associates the energy level $e^L_k$ with the level set $H^{-1}(k)$ of $H$. From the conjugation property between Lagrangian and Hamiltonian viewpoint, the nondegeneracy of an orbit is equivalent in both senses.

One can prove that the restriction of the symplectic form $\omega$ to $T\vartheta \Sigma$ is nondegenerate and closed form, therefore the Poincaré map is symplectic. Moreover,

$$d_\vartheta \psi^H_{T_{min}}(\xi) = -d_\vartheta \tau(\xi)X^H + d_\vartheta P(\Sigma, \vartheta)(\xi), \forall \xi \in T\vartheta \Sigma.$$ 

Therefore we have that

$$d_\vartheta \psi^H_{T_{min}} |_{T\vartheta H^{-1}(k)} = \begin{bmatrix} 1 & d_\vartheta \tau \\ 0 & d_\vartheta P(\Sigma, \vartheta) \end{bmatrix},$$

in general for $T = mT_{min}$

$$d_\vartheta \psi^H_T(\xi) = -d_\vartheta \tau(\sum_{i=0}^{M-1} d_\vartheta P(\Sigma, \vartheta)^i)(\xi)X^H + d_\vartheta P(\Sigma, \vartheta)^m(\xi), \forall \xi \in T\vartheta \Sigma.$$

So, the condition of that $\vartheta$ is nondegenerate of order $m \geq 1$ is equivalent to say that the algebraic multiplicity of $\lambda = 1$ as eigenvalue of $d_\vartheta \psi^H_T |_{T\vartheta H^{-1}(k)}$ is equal to 1, because the characteristic polynomials are related by $p_{d_\vartheta \psi^H_T}(\lambda) = (1 - \lambda) \cdot p_{d_\vartheta P(\Sigma, \vartheta)^m}(\lambda)$.

Our main result is the Kupka-Smale Theorem that relies the Bumpy Metrics Theorem proved by Anosov \cite{4}, but here for the Lagrangian setting.

We state our main result just when $\text{dim}(M) = 2$ because we do not know the proof for Theorem \cite{18} when $\text{dim}(M) = n \geq 3$. This problem remains an open question.

**Theorem 1.** (Kupka-Smale Theorem) Suppose $\text{dim}(M) = 2$. Let $L : TM \to \mathbb{R}$, be a Lagrangian in $M$, convex and fiberwise superlinear. Then, for each $k \in \mathbb{R}$, the property

i) $e_k^L$ is regular;

ii) Any periodic orbit in the level $e_k^L$ is nondegenerate for all orders;

iii) All heteroclinic intersections, in this level, are transversal.

is generic for $L$.

### 3 Proofs of the main results

Given $k \in \mathbb{R}$, we define the set of the regular potentials for $k$, as being

$$\mathcal{R}(k) = \{ f \in C^\infty(M; \mathbb{R}) \mid e^k_f := (H + f)^{-1}(k) \text{ is regular} \}.$$
where $H$ is the associated Hamiltonian.

**Lemma 2.** Consider $k \in \mathbb{R}$ and $f_0 \in C^\infty(M; \mathbb{R})$. For each sequence $f_n \to f_0$ in $C^\infty(M; \mathbb{R})$ topology and points $\vartheta_n = (x_n, p_n) \in \varepsilon_{f_n}^k$, there exists a subsequence $\vartheta_{n_i} \to \vartheta_0 \in \varepsilon_{f_0}^k$.

In fact, this lemma is an easy consequence of the compactness of the energy level.

**Theorem 3.** (Regularity of the energy level) Given $k \in \mathbb{R}$, the subset $\mathcal{R}(k)$ is open and dense in $C^\infty(M; \mathbb{R})$.

**Proof.** The openness of the set $\mathcal{R}(k)$ follows directly of the Lemma 2. In order to obtain the density of $\mathcal{R}(k)$ in $C^\infty(M; \mathbb{R})$, consider $f_0 \in C^\infty(M; \mathbb{R})$ and $\mathcal{U}$, a neighborhood that contains a ball of radius $\varepsilon > 0$, and center, $f_0$. We claim that $\mathcal{U} \cap \mathcal{R}(k) \neq \emptyset$. In fact, if it is not the case, we can achieve a contradiction by considering the Hamiltonian $H_\delta := H + (f_0 + \delta)$, with $\delta \in (0, \varepsilon)$.

The Nondegeneracy Lemma

Given $k \in \mathbb{R}$ and $0 < a \leq b \in \mathbb{R}$, we define the set $\mathcal{G}^a_b(k)$ as $\mathcal{G}^a_b(k) = \{ f \in \mathcal{R}(k) \mid \text{all periodic points } \vartheta \in (H + f)^{-1}(k), \text{ with } T_{\min}(\vartheta) \leq a \text{ are nondegenerate of order } m \text{ for } H + f, \forall m \leq \frac{b}{T_{\min}} \}$. Observe that, if we have $a, a', b, b' \in \mathbb{R}$, such that, $0 < a, a' < \infty$, $a \leq a'$ and $b \leq b'$, then $\mathcal{G}^{a'}_{b'}(k) \subseteq \mathcal{G}^{a}_b(k)$. We define $\mathcal{G}(k) = \bigcap_{n=1}^{+\infty} \mathcal{G}^{a_n}_b(k)$. Then $\mathcal{G}(k)$ is the set of all potentials $f \in \mathcal{R}(k)$ such that, all periodic orbits with positive period in the energy level $(H + f)^{-1}(k)$ are nondegenerate of all orders for $H + f$.

**Lemma 4.** (Nondegeneracy Lemma) Given $k \in \mathbb{R}$ and $0 < c \in \mathbb{R}$, the set $\mathcal{G}^{c,c}_k$ is open and dense in $C^\infty(M; \mathbb{R})$.

If $\mathcal{G}_k$ is generic, then generically in $L$, the energy level $k$ is regular and all periodic orbits in this level are nondegenerate of all orders for $H + f$. Thus, we must to prove that $\mathcal{G}^{c,c}_k$ is open in $C^\infty(M; \mathbb{R})$, $\forall c \in \mathbb{R}_+$ and dense in $\mathcal{R}(k)$, since Theorem 3 implies that $\mathcal{R}(k)$ is dense in $C^\infty(M; \mathbb{R})$. The proof of this lemma requires a sequence of technical constructions.

**Lemma 5.** Given $k \in \mathbb{R}$ and $f_0 \in \mathcal{R}(k)$ there exists a neighborhood, $\mathcal{U}$, of $f_0$ in $C^\infty(M; \mathbb{R})$ and $0 < \alpha := \alpha(\mathcal{U}, f_0)$ such that, for all $f \in \mathcal{U}$, the period of all periodic orbits of $H + f$, in the level $(H + f)^{-1}(k)$, is bounded below by $\alpha$. 

Proof. If we suppose that our claiming is false, we get the existence of sequences, \( U \ni f_n \to f_0 \), \( T_n > 0 \) with \( T_n \to 0 \) and \( \vartheta_n \in (H + f_n)^{-1}(k) \) such that \( \psi_{T_n}^{H + f_n}(\vartheta_n) = \vartheta_n \).

From Lemma 2 we can choose a subsequence such that
\[
d_{T^*M}(\psi_t^{H + f_n}(\vartheta_0), \vartheta_0) = 0, \quad \forall t > 0
\]
that is, \( \vartheta_0 \in (H + f_0)^{-1}(k) \) which is a fix point, contradicting the fact of \( f_0 \in \mathcal{R}(k) \).

Lemma 6. Given \( k \in \mathbb{R}, \ a, b \in \mathbb{R} \) with \( 0 < a \leq b < \infty \), the set \( \mathcal{G}_{k}^{a,b} \) is open in \( C^{\infty}(M; \mathbb{R}) \).

Proof. If \( \mathcal{G}_{k}^{a,b} \neq \emptyset \), take \( f_0 \in \mathcal{G}_{k}^{a,b} \). If \( f_0 \) is not an interior point we get the existence of a sequence \( f_n \to f_0 \) where \( f_n \notin \mathcal{G}_{k}^{a,b} \). Therefore, there exists \( \vartheta_n \in (H + f_n)^{-1}(k) \), \( T_n = T_{\min}(\vartheta_n) \in (0; a] \) and natural numbers \( \ell_n \geq 1 \) such that, \( \ell_n T_n \leq b, \psi_{\ell_n T_n}^{H + f_n}(\vartheta_n) = \vartheta_n \) and \( d_{\vartheta_n} \psi_{\ell_n T_n}^{H + f_n} \) do not have 1 as eigenvalue with algebraic multiplicity bigger than 1. Consider \( U_0 \) and \( 0 < \alpha := \alpha(U_0, f_0) < a \) as in Lemma 5. Choosing a subsequence we can assume that \( f_n \in U_0 \) and therefore \( T_n \in [\alpha; a] \), with \( \vartheta_n \to \vartheta_0 \in \varepsilon f_0 \), \( T_n \to T_0 \), \( \ell_n = \ell_0 \), \( 0 < \alpha \leq T_0 \leq a \) and \( \ell_0 T_0 \leq b \). Then \( \psi_{\ell_0 T_0}^{L + f_0}(\vartheta_0) = \vartheta_0 \), and \( d_{\vartheta_0} \psi_{\ell_0 T_0}^{H + f_0} \) has 1 as eigenvalue with algebraic multiplicity bigger than 1, that is, \( \vartheta_0 \) is a periodic orbit with minimal period \( \leq a \), degenerate of order \( \ell_0 \leq \frac{b}{T_0} \), contradicting the fact of \( f_0 \in \mathcal{G}_{k}^{a,b} \).

In order to prove that \( \mathcal{G}_{k}^{c,c} \) is dense in \( C^{\infty}(M; \mathbb{R}) \) \( \forall c \in \mathbb{R}_+ \), we observe that is enough show that, \( \mathcal{G}_{k}^{c,c} \) is dense in \( \mathcal{R}(k) \). So, we can reduce this proof to a local approach. More precisely, the claim is a direct consequence of the following Reduction Lemma, whose proof we will present in the Section 4.

Lemma 7. (Reduction Lemma) For each \( c \in \mathbb{R}_+ \), and any \( f_0 \in \mathcal{R}(k) \), there exists an open neighborhood \( U_{f_0} \) of \( f_0 \), such that, \( \mathcal{G}_{k}^{c,c} \cap U_{f_0} \) is dense in \( U_{f_0} \).

Thus the Nondegeneracy Lemma is proven.

Proof of the Kupka-Smale Theorem

Consider a periodic orbit \( \gamma = \{ \phi_t^{L}(\theta_0), \ 0 \leq t \leq T \} \subseteq H^{-1}(k) \), in \( T^*M \), where \( H \) is the Hamiltonian associated to \( L \) by the Legendre transform. We will say that this orbit is hyperbolic if the Poincaré map associated does not have eigenvalue of norm 1. It is clear that the hyperbolicity implies in the nondegeneracy property. The converse is not true. There exists nondegenerate orbits such that all eigenvalues has norm 1. Such orbits will...
be called elliptic orbits. We define the strong stable and strong unstable manifolds, of $\gamma$ in $\theta_0 = \gamma(0)$, as

$$W^{ss}(\theta_0) = \{ \theta \in H^{-1}(k) \mid \lim_{t \to \pm \infty} d(\phi_t^L(\theta_0), \phi_t^L(\theta)) = 0 \}$$

and

$$W^{uu}(\theta_0) = \{ \theta \in H^{-1}(k) \mid \lim_{t \to -\infty} d(\phi_t^L(\theta_0), \phi_t^L(\theta)) = 0 \}.$$

Respectively we define the stable and unstable manifolds (weak) of $\gamma$ as

$$W^s(\gamma) = \bigcup_{t \in \mathbb{R}} \phi_t^L(W^{ss}(\theta_0)) \quad \text{and} \quad W^u(\gamma) = \bigcup_{t \in \mathbb{R}} \phi_t^L(W^{uu}(\theta_0)).$$

From the general theory of the Lagrangians systems we know that, $W^s(\gamma)$, $W^u(\gamma) \subset H^{-1}(k)$ are Lagrangians submanifolds of $TM$, with the symplectic twist form, given by $\omega(\xi, \zeta) = \langle (\xi_0, \zeta_0)^*, J(\xi_0, \zeta_0) \rangle$ in local coordinates.

A point $\theta \in H^{-1}(k)$, is heteroclinic if $\theta \in W^s(\gamma_1) \cap W^u(\gamma_2)$, where $\gamma_1, \gamma_2 \subset H^{-1}(k)$ are hyperbolic periodic orbits. Additionally, if $T_0 W^s(\gamma_1) + T_0 W^u(\gamma_2) = T_0 H^{-1}(k)$, that is, if $W^s(\gamma_1) \cap W^u(\gamma_2)$ then $\theta$ will be called a transversal heteroclinic point, same thing for homoclinics.

A fundamental domain for $W^s(\gamma)$ (or $W^u(\gamma)$) is a compact subset $\mathcal{D} \subset W^s(\gamma)$, such that, all orbits in $W^s(\gamma)$ intersects $\mathcal{D}$ in one point, at least. One can show that there exists fundamental domains arbitrarily small and arbitrarily close to $\gamma$. Fixed $a > 0$ we define the local stable and local unstable submanifolds of $\gamma$ as being

$$W^s_a(\gamma) = \{ \theta \in W^s(\gamma) \mid d(\theta, \gamma) < a \}$$

and

$$W^u_a(\gamma) = \{ \theta \in W^u(\gamma) \mid d(\theta, \gamma) < a \}.$$

They are Lagrangians submanifolds of $TM$.

In order to prove the Kupka-Smale Theorem, we define $\mathcal{K}_k^a = \{ f \in \mathcal{G}_k^{a,a} \mid \forall \gamma_1, \gamma_2 \subset (H + f)^{-1}(k), \text{hyperbolic periodic orbits for } L + f, \text{with period } \leq a \text{ we have } W^s_a(\gamma_1) \cap W^u_a(\gamma_2) \} \text{ and } \mathcal{K}(k) = \bigcap_{n \in \mathbb{N}} \mathcal{K}_k^n.$$

It is clear that the properties (i), (ii) and (iii) of the Kupka-Smale Theorem are valid for all $f \in \mathcal{K}(k)$. Thus, in order to prove the Kupka-Smale Theorem, we must to show that $\mathcal{K}(k)$ is generic, or equivalent, that each, $\mathcal{K}_k^n$ is an open and dense set (in $C^\infty$ topology). Since, the local stable and unstable manifolds depends $C^1$ continuously on compact parts, of the Lagrangian field, we get the openness of $\mathcal{K}_k^n$, because the transversality is an open property.

The next lemma can be found in Paternain [15], Proposition 2.11, Pg.34, for the geodesic case, but here we present a Lagrangian version.
Lemma 8. (Twist Property of the vertical bundle) Let $L$ be a smooth, convex and superlinear, Lagrangian in $M$, $\theta \in TM$ and $F \subset T\gamma TM$ an Lagrangian subspace for the twist form in $T^*M$. Then, the set,

$$Z_F = \{ t \in \mathbb{R} \mid d_\theta \phi^L_t(E) \cap V(\phi^L_t(\theta)) \neq \emptyset \}$$

is discrete, where $V$ is the vertical bundle in $M$.

The next lemma allow us to make a local perturbation of a potential $f$ in such way that the correspondent stable and unstable manifolds become transversal in a certain heteroclinic point $\theta$. The density of $\mathcal{K}^u_1$ follows from Lemma 10.

Lemma 9. Let $L$ be a Lagrangian, and $f \in C^\infty(M, \mathbb{R})$. Given $\gamma_1, \gamma_2 \subset (H - f)^{-1}(k)$ hyperbolic periodic orbits with period $\leq a$ and $\theta \in W^a_u(\gamma_2)$, such that, the canonic projection $\pi |_{W^u_2(\gamma_2)}$ is a local diffeomorphism in $\theta$ and $U, V$, are neighborhoods of $\theta$ in $TM$ such that $\theta \in V \subset \bar{V} \subset U$. Then, there exists $\bar{f} \in C^\infty(M, \mathbb{R})$, such that,

i) $\bar{f}$ is $C^\infty$ close to $f$;

ii) $\text{supp}(f - \bar{f}) \subset \pi(U)$;

iii) $\gamma_1, \gamma_2 \subset (H - \bar{f})^{-1}(k)$ are hyperbolic periodic orbits to $\bar{f}$, with the same period as to $f$;

iv) The connected component of $W^a_u(\gamma_2) \cap V$ that contains $\theta$ is transversal to $W^s(\gamma_1)$.

Proof. Initially we consider the Hamiltonian $H - f$ associated to the Lagrangian $L + f$ by the Legendre transform $\mathcal{L}$:

$$H - f(x, p) = \sup_{v \in T_x M} \{ p(v) - (L + f)(x, v) \}$$

with the canonic symplectic form of $T^*M$, $\omega = \sum dx_i \wedge dp_i$.

We know, from the general theory of the Hamiltonian systems, that $\gamma_1, \gamma_2$ are in correspondence, by Legendre transform with hyperbolic periodic orbits of same period, $\gamma_1, \gamma_2 \subset (H - f)^{-1}(k)$ for the Hamiltonian flow $\psi^H_{t-f}$. Consider $W_u^u(\gamma_2)$ and $W_u^s(\gamma_1)$, respectively, the invariant submanifolds, they will be Lagrangian submanifolds of $T^*M$, and $\vartheta = \mathcal{L}(\theta) \in W^u_u(\gamma_2)$. If $\pi : TM \to M$ and $\pi^* : T^*M \to M$ are the canonic projections, then $\partial \pi^* = d_\theta \pi \circ (d_\theta \mathcal{L})^{-1}$. Therefore the canonic projection $\pi^* |_{W^u_u(\gamma_2)}$ is a local diffeomorphism in $\vartheta$. Moreover, $X^{H-f}(\vartheta) = d_\theta \mathcal{L} \circ X^{H+f}(\theta) \neq 0$. Thus we can prove the lemma in the Hamiltonian setting. By [9], Lemma A3, we can find a neighborhood $U$ of $\vartheta$, and $V \subset U$, such that, $V \subset \bar{V} \subset U$, and a Lagrangian submanifold, $\mathcal{N}$, $C^\infty$ close to $\bar{W}^u(\gamma_2)$, satisfying the following conditions

1) $\vartheta \in \{ U \setminus \bar{V} \}$
2) \( \mathcal{N} \cap \{ U \setminus V \} = \tilde{W}^u(\gamma_2) \cap \{ U \setminus V \} \subset (H - f)^{-1}(k); \)
3) \( \mathcal{N} \cap V \ni \tilde{W}^s(\gamma_1) \cap V. \)

As, \( \mathcal{N} \) is \( C^\infty \) close to \( \tilde{W}^u(\gamma_2) \), we have that the canonnic projection \( \pi^* |_{\mathcal{N}} \) is a local diffeomorphism in \( \vartheta \). If \( U \) is small enough, then \( \mathcal{N} \cap U = \{(x, p(x)) \mid x \in \pi^*(U)\} \) that is, \( \mathcal{N} |_U \) is a \( C^\infty(M, \mathbb{R}) \) graph. We define the following potential, \( \bar{f} \in C^\infty(M, \mathbb{R}) \),

\[
\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in \pi^*(U)^C \\ H(x, p(x)) - k & \text{if } x \in \pi^*(U) \end{cases}
\]

Observe that, \( \text{supp}(\bar{f} - \tilde{f}) \subset \pi^*(U) \) and \( \vartheta \not\in \text{supp}(\bar{f} - \tilde{f}) \), moreover, choosing \( U \) small enough, we will have that \( \pi^*(U) \cap \{ \gamma_1, \gamma_2 \} = \emptyset \) and therefore \( \gamma_1, \gamma_2 \) still, hyperbolic periodic orbits of period \( \leq t \), contained in \( (H - f)^{-1}(k) \). We denote \( \tilde{W}^u(\gamma_2) \) and \( \tilde{W}^s(\gamma_1) \), the invariant manifolds for the new flow \( \psi_{\bar{f}}^{H - f} \). Clearly \( (H - \tilde{f})(\mathcal{N}) = k. \)

By [9], Lemma A1, we have that \( \mathcal{N} \) is \( \psi_{\bar{f}}^{H - f} \) invariant. Since, \( \tilde{W}^u(\gamma_2) \) depends only of the negative times and, the connected component of \( \tilde{W}^u(\gamma_2) \cap U \) that contains \( \vartheta \) and \( \mathcal{N} \) are coincident in a neighborhood of \( \gamma_2 \) disjoint of \( \text{supp}(\bar{f} - \tilde{f}) \), we have \( \mathcal{N} = \tilde{W}^u(\gamma_2) \). On the other hand, as \( \tilde{W}^s(\gamma_1) \) depends only of the positive times and \( \bar{f} = \tilde{f} \) in \( \{ U \setminus V \} \), we have \( \tilde{W}^s(\gamma_1) = \tilde{W}^s(\gamma_1) \).

Since \( \mathcal{N} \cap V \ni \tilde{W}^s(\gamma_1) \cap V \), we have \( \tilde{W}^u(\gamma_2) \cap V \ni \tilde{W}^s(\gamma_1) \cap V \). From the initial considerations we choose \( L + \tilde{f} \). The lemma is proven. ■

**Lemma 10.** The set \( \mathcal{K}_n^k \), is dense in \( C^\infty(M, \mathbb{R}) \), for all \( n \in \mathbb{N} \).

**Proof.** Take \( f_0 \in C^\infty(M, \mathbb{R}) \), by the Nondegeneracy Lemma we can find \( f_0 \) arbitrarily close to \( f' \in \mathcal{G}_{k}^{n,n} \), which is open and dense. Thus, is enough to find \( f \) arbitrarily close to \( f' \), such that, for any \( \gamma_1, \gamma_2 \subset (H - f)^{-1}(k) \), hyperbolic periodic orbits of period \( \leq n \), is valid \( W^u(\gamma_1) \cap W^u(\gamma_2) \). Then, \( f \in \mathcal{K}_n^k \) and \( f \) is arbitrarily close to \( f_0 \). Given \( \gamma_1, \gamma_2 \subset (H + f')^{-1}(k) \) hyperbolic periodic orbits of period \( \leq n \), in order to conclude that \( W^u(\gamma_1) \cap W^u(\gamma_2) \) we should to prove that \( W^s(\gamma_1) \ni W^u(\gamma_2) \) where \( D \) is a fundamental domain of \( W^u(\gamma_2) \), because if \( W^s(\gamma_1) \ni W^u(\gamma_2, \theta) \) then \( W^s(\gamma_1) \ni \phi_{\theta,\tilde{f} + f'} W^u(\gamma_2, \theta) \), \( \forall \theta \).

Take \( D \) a fundamental domain of \( W^u(\gamma_2) \) and \( \theta \in D \). By the inverse function theorem we know that \( \pi |_{W^u(\gamma_2)} \) is a local diffeomorphism in \( \theta \) if, and only if, \( T_\theta W^u(\gamma_2) \cap \text{Kerd}_\theta \pi = 0 \). As \( W^u(\gamma_2) \) is a Lagrangian submanifold we have, from Lemma 9, that \( \{ t \in \mathbb{R} \mid \text{d}_\theta \phi^t \mid_{T_\theta W^u(\gamma_2)} \cap \text{Kerd}_{\phi^t \theta, \tilde{f} + f'} \pi \neq \emptyset \} \), is discrete. Then there exists \( t(\theta) \) arbitrarily close to 0, such that, \( \pi |_{W^u(\gamma_2)} \) is a local diffeomorphism in \( \tilde{\theta} = \phi^t_{\theta(\theta)} \). As, \( f' \in \mathcal{G}_{k}^{n,n} \), we can choose, \( t(\theta) \), such that, \( \pi(\tilde{\theta}) \) does not intercept any periodic orbit of period \( \leq n \).

Fix a neighborhood \( U \), of \( \tilde{\theta} \), arbitrarily small, such that, \( \pi(U) \) does not intercept any periodic orbit of period \( \leq n \). Taking \( V \), a neighborhood of \( \tilde{\theta} \),
such that $V \subset \bar{V} \subset U$, from Lemma\(^{[6]}\) we can find $f_1 = f'$ in $\pi(U)^{C}$, such that, the connected component of $W^{u}_{n}(\gamma_2) \cap \bar{V}$ (to the new flow) contain $\hat{\vartheta}$, and is transversal to $W^{s}(\gamma_1)$ (to the new flow). Taking $\hat{V}_1 = \phi^{L+T}_\vartheta (\bar{V})$ we will have that $W^{u}_{n}(\gamma_2) \pitchfork W^{s}(\gamma_1)$. We can cover the fundamental domain $D$ with a finite number of neighborhoods like $\hat{V}_1$, that is, $W_1, \ldots, W_s$. Since the transversality is an open condition and the local stable (unstable) manifold depends continuously on compact parts, we can choose successively $W_{i+1}$ such that the transversality in $W_j, j \leq i$, is preserved. Thus, $W^{u}_{n}(\gamma_2) \pitchfork W^{s}(\gamma_1)$.

\section{Proof of the Reduction Lemma}

For the proof of the Reduction Lemma (Lemma\(^{[7]}\)) we will use an induction method similar to the one used by Anosov\(^{[4]}\), using transversality arguments as described in Abraham\(^{[1]}\) and \(^{[2]}\). In this way, we remember a useful theorem, the Parametric Transversality Theorem of Abraham.

Remember that, if $X$ is a topological space. A subset $\mathcal{R} \subseteq X$ is said generic if $\mathcal{R}$ is a countable intersection of open and dense sets. The space $X$ will be a Baire Space if all generic subsets are dense. For additional results and definitions of Differential Topology, see \(^{[2]}\), \(^{[3]}\) or \(^{[5]}\).

**Theorem 11.** (\(^{[2]}\), pg. 48, Abraham’s Parametric Transversality Theorem) Consider $\mathcal{X}$ a submanifold finite dimensional (with boundary or boundaryless), $\mathcal{Y}$ a boundaryless manifold and $S \subseteq \mathcal{Y}$ a submanifold with finite codimension. Consider $\mathcal{B}$ boundaryless manifold, $\rho : \mathcal{B} \to C^{\infty}(\mathcal{X}; \mathcal{Y})$ a smooth representation and your evaluation $ev_{\rho} : \mathcal{B} \times \mathcal{X} \to \mathcal{Y}$. If $\mathcal{X}$ and $\mathcal{B}$ are Baire spaces and $ev_{\rho} \cap \mathcal{S}$ then the set $\mathcal{R} = \{ \varphi \in \mathcal{B} \mid \rho_{\varphi} \cap \mathcal{S} \}$ is a generic subset (and obviously dense) of $\mathcal{B}$.

Given a Hamiltonian $H$ we define the normal field associated, as being the gradient field, $Y^{H} = \nabla H = H_{x} \partial_{x} + H_{p} \partial_{p}$ in $T^{*}M$. Observe that $JY^{H} = X^{H}$, where $J$ is the canonic sympletic matrix. We denote $\psi^{H \pm}_{s} : T^{*}M \times (-\varepsilon, \varepsilon) \to T^{*}M$ the flow in $T^{*}M$ generated by the normal field. Let us briefly describe the properties of the normal field. Initially observe that $\omega \equiv (Y^{H}, X^{H}) = H_{x}^{2} + H_{p}^{2}, \forall \vartheta \in T^{*}M$. If $X^{H}(\vartheta) \neq 0$ then $0 \neq Y^{H}(\vartheta) \not\in T_{\vartheta}H^{-1}(k)$ where $k = H(\vartheta)$, that is $Y^{H}$ points to the outside of the energy level. From the compactness of the energy level $H^{-1}(k)$ we have that the flow of the normal field, restricted to $H^{-1}(k)$ is defined in $H^{-1}(k) \times (-\varepsilon(H), \varepsilon(H))$ where $\varepsilon(H) > 0$ is uniformly defined in $H^{-1}(k)$. Then the flow of the normal field is defined in a neighborhood of the energy level $H^{-1}(k)$. The action of the differential of the normal flow through an orbit is given by

$$
\begin{align*}
\dot{Z}^{H}(s) &= H(\gamma(s))Z^{H}(s) \\
Z^{H}(0) &= Y^{H}(\vartheta)
\end{align*}
$$
where $\mathcal{H}$ is the hessian matrix of $H$. The main property of $Y^H$ is to establish a symplectic decomposition of $T_0T^*M$ given by the next proposition.

**Proposition 12.** Consider the normal field $Y^H$ associated to $H$ in the regular energy level, $H^{-1}(k)$. Then, for each $\vartheta \in H^{-1}(k)$, periodic of period $T > 0$, there exists a symplectic base $\{u_1, \ldots, u_n, \alpha^1, \ldots, \alpha^n\}$ of $T_0T^*M$ verifying

1. $u_1 = X^H$ and $u_1^* = -\frac{1}{1+H^*}Y^H$;
2. $\mathcal{W}_1 = \langle u_1, u_1^* \rangle^\perp \subset T_0H^{-1}(k)$ in particular, $T_0H^{-1}(k) = \langle u_1 \rangle \oplus \mathcal{W}_1$;
3. If $\Sigma \subset H^{-1}(k)$ is a section transversal to the flow, such that, $T_0\Sigma = \mathcal{W}_1$, we have that $d_\vartheta P(\Sigma, \vartheta)\mathcal{W}_1 \subseteq \mathcal{W}_1$;
4. If $T = mT_{\min}(\vartheta)$, then $d_\vartheta(Y^H)u_1 = u_1$ and $d_\vartheta(Y^H)u_1^* = cu_1 + u_1^* + \xi$, $\xi \in \mathcal{W}_1$. In particular, $(d_\vartheta(Y^H) - Id)(T_0T^*M) \subseteq \langle u_1 \rangle \oplus \mathcal{W}_1 = T_0H^{-1}(k)$.
5. There exists, $\varepsilon > 0$ uniform in $\vartheta \in H^{-1}(k)$, such that, the map $e_{\vartheta} : (-\varepsilon, \varepsilon) \to \mathbb{R}$ given by $e_{\vartheta}(s) = H \circ \vartheta^H(\vartheta)$ is injective with $e_{\vartheta}(0) = k$.

Using the normal field we are able to construct a representation, in order to apply the Parametric Transversality Theorem.

**Proposition 13.** Given $k \in \mathbb{R}$, $0 < a < b < +\infty$, and $f_0 \in \mathcal{R}(k)$, consider the normal field $Y^{H+f_0}$ as described before, $\varepsilon = \varepsilon(H + f_0) > 0$ as in the Proposition 12. (v), and the sets, $\mathcal{U}_{f_0} \subset \mathcal{R}(k)$ a $C^\infty$ neighborhood of $f_0$, $\alpha = \alpha(\mathcal{U}_{f_0}) > 0$ as in Lemma 2. $\mathcal{X} = T^*M \times (a, b) \times (-\varepsilon, \varepsilon)$ and $\mathcal{Y} = T^*M \times T^*M \times \mathbb{R}$. Then the map $\rho : \mathcal{U}_{f_0} \to C^\infty(\mathcal{X}, \mathcal{Y})$ given by $\rho(f) := \rho_f$, where

$$\rho_f(\vartheta, t, s) = (\psi^H f^\perp(\vartheta), \psi^L f(\vartheta), (H + f)(\vartheta) - k)$$

is an injective representation (see [1] or [2]).

**Proof.** Initially we point out that $\rho$ is well defined, therefore $\mathcal{Y}$ has the structure of a product manifold. Writing $\rho_f = (\rho_1^f, \rho_2^f, \rho_3^f)$, where $\rho_1^f(\vartheta, t, s) = \psi^H f^\perp(\vartheta)$, $\rho_2^f(\vartheta, t, s) = \psi^L f(\vartheta)$, $\rho_3^f(\vartheta, t, s) = (H + f)(\vartheta) - k$, we can see that each coordinate is a smooth function. Thus $\rho_f \in C^\infty(\mathcal{X}, \mathcal{Y})$. Observe that $\rho$ is injective. Indeed, if $\rho_{f_1} = \rho_{f_2}$ then $(H + f_1)(\vartheta) - k = (H + f_2)(\vartheta) - k$, for all $\vartheta \in T^*M$, so $f_1(x) = f_2(x)$, for all $x \in M$. Thus $f_1 = f_2$. We must to verify that $cv_{\rho} : \mathcal{U}_{f_0} \times \mathcal{X} \to \mathcal{Y}$ is smooth. Since $\mathcal{U}_{f_0} \times \mathcal{X}$ have the structure of product manifold, we can write

$$d_{(f, x)}cv_{\rho} := \frac{\partial cv_{\rho}}{\partial f}(f, x) + \frac{\partial cv_{\rho}}{\partial x}(f, x), \forall (f, x) \in \mathcal{U}_{f_0} \times \mathcal{X}$$

It is clear that $\frac{\partial cv_{\rho}}{\partial x}(f, x)$ is always defined as $\frac{\partial cv_{\rho}}{\partial x}(f, x) = d_x\rho_f$. More precisely, given $(\xi, \dot{t}, \dot{s}) \in T_{x=(\vartheta, T, S)} \mathcal{X}$ we have

$$\frac{\partial cv_{\rho}}{\partial x}(f, x)(\xi, \dot{t}, \dot{s}) = d_x\rho_f(\xi, \dot{t}, \dot{s}) = \frac{d}{dr} \rho_f(\vartheta(r), t(r), s(r)) \mid_{r=0}=\,$$
Calculations (see [10], pg.46), we get

\( \forall \)

Combining, Propositions 12 and 13 we get

\( h \)

for any \( \mathcal{C} \) of

Thus

Observe that, if \( S = 0 \) and \( \psi_H^f(\vartheta) = \vartheta \), then

\( \frac{\partial ev_\rho}{\partial x}(f, x)(\xi, \dot{i}, \dot{h}) = (\xi + sY^H(\vartheta), d_\vartheta \psi_H^f(\xi) + iX^H(\psi_H^f(\vartheta)), \)

\( d_\vartheta(H + f)(\xi) \).

However we must to show that \( \frac{\partial ev_\rho}{\partial f}(f, x) \) is always defined. By the structure of \( C^\infty(M; \mathbb{R}) \) we know that this fact is equivalent to show that there exists

\( \frac{d}{dr}\psi_S^H(\vartheta^H + rh)(\vartheta) \mid_{r=0} \), \( \frac{d}{dr}\psi_T^H(\vartheta^H + rh)(\vartheta) \mid_{r=0} \) and \( \frac{d}{dr}(H + f + rh)(\vartheta) \mid_{r=0} \)

for any \( h \in C^\infty(M; \mathbb{R}) \) and \( x = (\vartheta, T, S) \in X \). From some straightforward calculations (see [10], pg.46), we get

\( \frac{d}{dr}(H + f + rh)(\vartheta) \mid_{r=0} = h \circ \pi(\vartheta), \)

\( \frac{d}{dr}\psi_T^H(\vartheta^H + rh)(\vartheta) \mid_{r=0} = Z_h(T) = d_\vartheta \psi_T^H(\vartheta^H + f) \int_0^T (d_\vartheta \psi_T^H) \int_{-1}^1 b_h(t) dt \) and

\( \frac{d}{dr}\psi_S^H(\vartheta^H + rh)(\vartheta) \mid_{r=0} = Z^h(S) = d_\vartheta \psi_S^H(\vartheta^H + f) \int_0^S (d_\vartheta \psi_S^H) \int_{-1}^1 b_h(s) ds \).

Thus

\( \frac{\partial ev_\rho}{\partial f}(f, x)(h) = \)

\( (d_\vartheta \psi_S^H(\vartheta^H + f) \int_0^S (d_\vartheta \psi_S^H)(s) ds, d_\vartheta \psi_T^H(\vartheta^H + f) \int_0^T (d_\vartheta \psi_T^H)(t) dt, \)

\( h \circ \pi(\vartheta)). \)

If \( S = 0 \) and \( \psi_H^f(\vartheta) = \vartheta \), then

\( \frac{\partial ev_\rho}{\partial f}(f, x)(h) = (0, d_\vartheta \psi_T^H(\vartheta^H + f) \int_0^T (d_\vartheta \psi_T^H)(t) dt, h \circ \pi(\vartheta)). \)

Thus \( ev_\rho \) is smooth and therefore \( \rho \) is a representation.

Define the null diagonal \( \Delta_0 \subseteq Y \) given by \( \Delta_0 = \{(\vartheta, 0) \mid \vartheta \in T^*M \} \). Combining, Propositions [12] and [13] we get

**Lemma 14.** With the same notations of the Proposition [13], we have that, \( \forall f \in \mathcal{U}_f \), with \( T \in (a, b) \) and \( S \in (-\varepsilon, \varepsilon) \),

i) If \( \vartheta \) is a periodic orbit of positive period \( T \) for \( H + f \) in the level \( (H + f)^{-1}(k) \) then, \( \rho_f(\vartheta, T, 0) \in \Delta_0 \). Reciprocally, if \( \rho_f(\vartheta, T, S) \in \Delta_0 \) then, \( S = 0 \) and \( \vartheta \) is a periodic orbit of positive period \( T \) for \( H + f \) in the level \( (H + f)^{-1}(k) \).
ii) If $\vartheta$ is a periodic orbit of positive period, for $H + f$ in the level $(H + f)^{-1}(k)$. Then, $\vartheta$ is nondegenerate of order $m = \frac{T}{T_{\min}(\vartheta)}$ if, and only if, $\vartheta^* \cap (\vartheta, T, 0) \Delta_0$.

The next corollary it is an easy consequence of the Lemma 14.

**Corollary 15.** With the same notations of the Lemma 14 we have that, given $f \in \mathcal{U}_{\vartheta_0}$, all periodic orbits $\vartheta$, with positive period, $T_{\min}(\vartheta) \in (a, b)$, in $(H + f)^{-1}(k)$, are nondegenerate for $H + f$ of order $m$, $\forall m \leq \frac{b}{T_{\min}}$ if, and only if, $\varrho_f \cap \Delta_0$.

The previous corollary shows that, the nondegeneracy of the periodic orbits of positive period in an interval $(a, b)$, for a given energy level $(H + f)^{-1}(k)$, is equivalent to the transversality of the map $\varrho_f$ in relation to the diagonal $\Delta_0$. The key element for the proof of the Lemma 14 is the nest lemma.

**Lemma 16.** Consider the representation $\varrho$ as in the Proposition 13 and its evaluation in $\mathcal{U}_{\vartheta_0}$, that is, $ev : \mathcal{U}_{\vartheta_0} \times X \rightarrow Y$, given by $ev(f, \vartheta, t, s) = \varrho_\vartheta(\vartheta, t, s)$. Suppose that $ev(f, \vartheta, T, S) \in \Delta_0$ then,

i) If $\vartheta$ is nondegenerate of order $m = \frac{T}{T_{\min}}$ for $H + f$ then $ev \cap (\vartheta, T, S) \Delta_0$;

ii) If $T = T_{\min}(\vartheta)$ then, $ev \cap (\vartheta, T, S) \Delta_0$.

**Proof.**

i) We know that $ev(f, \vartheta, T, S) = \varrho_\vartheta(\vartheta, T, S)$ therefore $\varrho_\vartheta(\vartheta, T, S) \in \Delta_0$, and $S = 0$. If $\vartheta$ is nondegenerate of order $m = \frac{T}{T_{\min}}$ for $H + f$, then, from the Lemma 14 (ii), $\varrho_\vartheta \cap (\vartheta, T, 0) \Delta_0$, in particular $ev \cap (\vartheta, T, 0) \Delta_0$.

ii) As $ev(f, \vartheta, T, S) \in \Delta_0$ we must to show that

$$d_{(f, \vartheta, T, 0)}ev_{T, (f, \vartheta, T, 0)}(\mathcal{U}_{\vartheta_0} \times X) + T_{(\vartheta, \vartheta_0)}\Delta_0 = T_{(\vartheta, \vartheta_0)}Y.$$

Take any $(u, v, w) \in T_{(\vartheta, \vartheta_0)}Y$, $(\zeta, \zeta, 0) \in T_{(\vartheta, \vartheta_0)}\Delta_0$ and $(h, \xi, \tilde{s}) \in T_{(f, \vartheta, T, 0)}(\mathcal{U}_{\vartheta_0} \times X)$. From Proposition 13 we have that

$$d_{(f, \vartheta, T, 0)}ev_{\rho}(h, \xi, \tilde{s}) =$$

$$(0, d\vartheta \psi_T^{H+f} f^0 d\vartheta \psi_T^{H+f} - 1 b_h(t) dt, h \circ \pi(\vartheta)) +$$

$$(\xi + \dot{s}Y^{H+f}(\vartheta), d\vartheta \psi_T^{H+f}(\xi) + \dot{i}X^{H+f}(\vartheta), d\vartheta (H + f)(\xi))$$

$$= (\xi + \dot{s}Y^{H+f}(\vartheta), d\vartheta \psi_T^{H+f}(\xi) + \dot{i}X^{H+f}(\vartheta) + d\vartheta \psi_T^{H+f} f^0 (d\vartheta \psi_t^{H+f} - 1 b_h(t) dt, h \circ \pi(\vartheta)) + d\vartheta (H + f)(\xi))$$

Therefore $ev_{\rho} \cap (f, \vartheta, T, 0) \Delta_0$, if and only if, the system

$$\begin{cases}
u = \xi + \dot{s}Y^{H+f}(\vartheta) + \zeta & (1) \\
u = d\vartheta \psi_T^{H+f}(\xi) + \dot{i}X^{H+f}(\vartheta) + d\vartheta \psi_T^{H+f} f^0 (d\vartheta \psi_t^{H+f} - 1 b_h(t) dt + \zeta & (2) \\
u = h \circ \pi(\vartheta) + d\vartheta (H + f)(\xi) & (3)
\end{cases}$$
has a solution. Using the coordinates of the Proposition 12 and taking \( \zeta = u - \xi - \dot{s}Y^{H+f}(\vartheta) \) we have that the equation (2) restricted to the set of the solutions of (3),

\[
V_w = \left\{ h \in C^\infty(M, \mathbb{R}), \xi = aX^{H+f}(\vartheta) + b_0Y^{H+f}(\vartheta) + U \right\}
\]

where \( b_0 = \frac{w-h_{\varphi \sigma}(\vartheta)}{d_\varphi(H+f)Y^{H+f}(\vartheta)} \), will have the expression

\[
(t + b_0c + \tau_0)X^{H+f}(\vartheta) + (\epsilon^* - \dot{s})Y^{H+f}(\vartheta) + (d_\varphi P(\Sigma, \vartheta) - Id)(U) + b_0u_0 + d_\varphi \psi_T H+f \int_0^T (d_\varphi \psi_T H+f)^{-1}b_h(t)dt = \tilde{a}X^{L+f}(\vartheta) + \tilde{b}Y^{H+f}(\vartheta) + \tilde{U},
\]

where

\[
v - u = \tilde{a}X^{L+f}(\vartheta) + \tilde{b}Y^{H+f}(\vartheta) + \tilde{U},
\]

\[
(d_\varphi \psi_T^{H+f} - Id)(Y^{H+f}(\vartheta)) = cX^{H+f}(\vartheta) + c^*Y^{H+f}(\vartheta) + U_0
\]

and

\[
(d_\varphi \psi_T^{H+f} - Id)(U) = \tau_0X^{H+f}(\vartheta) + (d_\varphi P(\Sigma, \vartheta) - Id)(U).
\]

That is, the system always has a solution, if the expression,

\[
(t + b_0c + \tau_0)X^{H+f}(\vartheta) + (\epsilon^* - \dot{s})Y^{H+f}(\vartheta) + (d_\varphi P(\Sigma, \vartheta) - Id)(U) + b_0u_0 + d_\varphi \psi_T H+f \int_0^T (d_\varphi \psi_T^{H+f})^{-1}b_h(t)dt
\]

is surjective in \( T_\vartheta T^*M \). So, we must to show that

\[
d_\varphi \psi_T^{H+f} \int_0^T \left(d_\varphi \psi_T^{H+f}\right)^{-1}b_h(t)dt
\]

generates a \( 2n - 2 \) dimensional space complementary to the space generated by \( X^{H+f}(\vartheta) \) and \( Y^{H+f}(\vartheta) \), in \( T_\vartheta T^*M \), which is the claim of the next lemma.

\[\blacksquare\]

**Lemma 17.** With the same notations as in Lemma 12, the map \( \mathcal{B} : C^\infty(M; \mathbb{R}) \rightarrow T_\vartheta T^*M \),

\[
\mathcal{B}(h) = -d_\varphi \psi_T^{H+f} \int_0^T \left(d_\varphi \psi_T^{H+f}\right)^{-1}b_h(t)dt
\]

generates a space complementary to \( \langle X^{H+f}(\vartheta), Y^{H+f}(\vartheta) \rangle \).

**Proof.** In order to prove this claim is enough to restrict the map \( \mathcal{B} \) to a subspace chosen in \( C^\infty(M; \mathbb{R}) \). Consider \( t_0 \in (0, T), \varepsilon > 0 \), and denote \( \mathcal{A}_{t_0} \), the subspace of the smooth functions

\[
\mathcal{A}_{t_0} = \{ \alpha : \mathbb{R} \rightarrow \mathbb{R}^{n-1} \mid \alpha(t) = (a_1(t), ..., a_{n-1}(t)) \neq 0, \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon) \}.
\]
We assume that, \( x(t) = \pi(\gamma(t)) \), where \( \gamma(t) = \psi_{t}^{H+f}(\vartheta) \), does not contain autointersections for \( t \in (t_0 - \varepsilon, t_0 + \varepsilon) \), that is, that \( H_p(\gamma(t)) = d\pi X^{H+f}(\gamma(t)) \neq 0 \). Then there exists a system of tubular coordinates \( \mathcal{V} \), in a neighborhood of \( \pi(\gamma(t_0)) \), \( F : \mathcal{V} \to \mathbb{R}^n \), such that,

i) \( F(x) = (t, z_1, \ldots, z_{n-1}) \);

ii) \( F(x(t)) = (t, 0, \ldots, 0) \).

Observe that, by construction, \( d_{x(t)} F H_p(\gamma(t)) = (1,0,\ldots,0) \). Consider a bump function \( \sigma : M \to \mathbb{R} \), such that, \( \text{supp}(\sigma) \subset \mathcal{V} \), \( \sigma|_{\mathcal{V}_0} \equiv 1 \), with \( x(t_0) \in \mathcal{V}_0 \subset \mathcal{V} \). Define the perturbation space \( \mathcal{F}_{t_0} \subset C^\infty(M; \mathbb{R}) \) as being

\[
\mathcal{F}_{t_0} = \{ h_{\alpha,\beta}(x) = \tilde{h}_{\alpha,\beta}(x) \cdot \sigma(x) \mid \alpha, \beta \in \mathcal{A}_{t_0} \}
\]

where, \( \tilde{h}_{\alpha,\beta}(x) = \langle \alpha(t)\delta_{t_0}(t) + \beta(t)\delta_{t_0}(t), z \rangle, F(x) = (t, z) \) and \( \delta_{t_0} \) is a smooth approximation of the delta of Dirac in the point \( t = t_0 \). Given \( h_{\alpha,\beta} \in \mathcal{F}_{t_0} \) we get \( d_x h_{\alpha,\beta} = d_x \tilde{h}_{\alpha,\beta} \cdot \sigma(x) + \tilde{h}_{\alpha,\beta} \cdot d_x \sigma(x) \). On the other hand

\[
d_x \tilde{h}_{\alpha,\beta} = \left( \frac{d}{dt} \langle \alpha(t)\delta_{t_0}(t) + \beta(t)\delta_{t_0}(t), z \rangle, \alpha(t)\delta_{t_0}(t) + \beta(t)\delta_{t_0}(t) \rangle d_x F
\]

Evaluating \( x(t) \) and using that \( h_{\alpha,\beta}(x(t)) = 0 \) and \( \sigma(x(t)) = 1 \), we get

\[
dx(t) h_{\alpha,\beta} = (0, \alpha(t)\delta_{t_0}(t) + \beta(t)\delta_{t_0}(t))d_x F.
\]

In particular,

\[
dx(t) h_{\alpha,\beta} H_p(\gamma(t)) = (0, \alpha(t)\delta_{t_0}(t) + \beta(t)\delta_{t_0}(t))d_x F H_p(\gamma(t)) = 0
\]

for any, \( h_{\alpha,\beta} \in \mathcal{F}_{t_0} \).

We claim that,

1) \( \mathcal{B}(\mathcal{F}_{t_0}) \subset T(H + f)^{-1}(k) \);

2) \( X^{H+f}(\vartheta) \notin \mathcal{B}(\mathcal{F}_{t_0}) ;

3) \( \dim(\mathcal{B}(\mathcal{F}_{t_0})) = 2n - 2 \);

4) In particular, \( \mathcal{B}(\mathcal{F}_{t_0}) \) generates a space complementary to \( \langle X^{H+f}(\vartheta), Y^{H+f}(\vartheta) \rangle \).

In order to get (1) consider, \( \alpha_0 = d_{x(t)} h_{\alpha,0} = (0, \alpha(t)\delta_{t_0}(t))d_x F = \alpha_1 \delta_{t_0}(t) \) and \( \beta_0 = d_{x(t)} h_{0,\beta} = (0, \beta(t)\delta_{t_0}(t))d_x F = \beta_1 \delta_{t_0}(t) \), then,

\[
\mathcal{B}(h_0) = d_{\vartheta} \psi_{t}^{H+f} \int_{0}^{T} (d_{\vartheta} \psi_{t}^{H+f})^{-1} \begin{bmatrix} 0 \\ \alpha_0 \end{bmatrix} dt
\]

and

\[
\mathcal{B}(h_0) = d_{\vartheta} \psi_{t}^{H+f} \int_{0}^{T} (d_{\vartheta} \psi_{t}^{H+f})^{-1} \begin{bmatrix} 0 \\ \beta_0 \end{bmatrix} dt.
\]

Observe that, \( \omega(\begin{bmatrix} 0 \\ \alpha_0 \end{bmatrix}, X^{H+f}(\gamma(t))) = \alpha_0 H_p(\gamma(t)) = 0 \), and

\[
\omega(\begin{bmatrix} 0 \\ \beta_0 \end{bmatrix}, X^{H+f}(\gamma(t))) = \beta_0 H_p(\gamma(t)) = 0,
\]

therefore \( \begin{bmatrix} 0 \\ \alpha_0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ \beta_0 \end{bmatrix} \) are in \( T(H + f)^{-1}(k) \). Thus, \( \mathcal{B}(\mathcal{F}_{t_0}) \subset T(H + f)^{-1}(k) \).
In order to get (2), we will make $\delta_{t_0} \to \delta_{\text{Dirac}}$ and will write $\mathcal{B}(h_\alpha)$ and $\mathcal{B}(h_\beta)$ as

$$\mathcal{B}(h_\alpha) = d_\phi \psi_T^{H+f}(d_\phi \psi_t^{H+f})^{-1} \begin{bmatrix} 0 \\ \alpha_1(t_0) \end{bmatrix}.$$  

Analogously,

$$\mathcal{B}(h_\beta) = d_\phi \psi_T^{H+f}(d_\phi \psi_t^{H+f})^{-1} \left\{ J \mathcal{H}^{H+f}(t_0) \begin{bmatrix} 0 \\ \beta_1(t_0) \end{bmatrix} - \begin{bmatrix} 0 \\ \hat{\beta}_1(t_0) \end{bmatrix} \right\}.$$  

If we assume, by contradiction, that $X^{H+f}(\gamma(t)) = \mathcal{B}(h_\alpha) + \mathcal{B}(h_\beta)$ then

$$X^{H+f}(\gamma(t_0)) = \left\{ \begin{bmatrix} 0 \\ \alpha_1(t_0) \end{bmatrix} + J \mathcal{H}^{H+f}(t_0) \begin{bmatrix} 0 \\ \beta_1(t_0) \end{bmatrix} - \begin{bmatrix} 0 \\ \hat{\beta}_1(t_0) \end{bmatrix} \right\}.$$  

From this equality we have $H_p(\gamma(t_0)) = H_{pp}(\gamma(t_0))\beta_1(t_0)$. Since $H_p(\gamma(t_0)) \neq 0$ we have $n-1$ choices, linearly independent, for $\beta_1(t_0)$. Indeed, $dF$ is an isomorphism and for all $\beta(t_0) \in \mathbb{R}^{n-1}$ we have $\beta_1(t_0)H_p(\gamma(t_0)) = (0, \beta(t_0))dF H_p(\gamma(t_0)) = 0$. Thus, $0 = \beta_1(t_0)H_p(\gamma(t_0)) = \beta_1(t_0)H_{pp}(\gamma(t_0))\beta_1(t_0)$, contradicting the superlinearity of $H$. For (3) observe that, in (2) we got the limit representation

$$\mathcal{B}(h_\alpha) + \mathcal{B}(h_\beta) = d_\phi \psi_T^{H+f}(d_\phi \psi_t^{H+f})^{-1} \left\{ \begin{bmatrix} 0 \\ \alpha_1(t_0) \end{bmatrix} + J \mathcal{H}^{H+f}(t_0) \begin{bmatrix} 0 \\ \beta_1(t_0) \end{bmatrix} - \begin{bmatrix} 0 \\ \hat{\beta}_1(t_0) \end{bmatrix} \right\}.$$  

From this equation we get $\dim(\{\mathcal{B}(h_\alpha) + \mathcal{B}(h_\beta)\}) = \dim(\{\alpha_1(t_0), \beta_1(t_0)\}) = 2n-2$, since $\begin{bmatrix} 0 \\ \alpha_1(t_0) \end{bmatrix}$ is an isomorphism.

Finally, we observe that, the claim (1) is true independently of the approximation $\delta_{t_0}$ of the delta of Dirac in the point $t = t_0$. Moreover the claims (2) and (3) still true for $\delta_{t_0}$, close enough to the delta of Dirac.

The next theorem allow us to make a local perturbation of a periodic orbit nondegenerate of order $\leq m$ in such way that it becomes nondegenerate of order $\leq 2m$. The proof is just for dimension 2 and the $n$-dimensional case is still open. Almost all th parts of the argument are true in the $n$-dimensional case, but we do not know how to show the surjectivity of the representation in this case.

**Theorem 18.** (Local perturbation of periodic orbits) Let $\dim(M) = 2$, $H : T^*M \to \mathbb{R}$ be a smooth, convex and superlinear Hamiltonian and $\gamma = \{ \psi_t^{\mathbb{H}}(\partial_0) \mid 0 \leq t \leq T \} \subseteq \mathbb{H}^{-1}(k)$, where $\mathbb{H}^{-1}(k)$ is a regular energy level, $T$ is the minimal period of $\gamma$, and $\gamma$ is isolated in this energy level, nondegenerate of order $\leq m \in \mathbb{N}$. Then there exists a potential $f_0 \in C^\infty(M, \mathbb{R})$ arbitraily close to zero, with $\text{supp}(f_0) \subset U \subset M$ such that, $\gamma$ is nondegenerate of order $\leq 2m$ to $\mathbb{H} + f_0$. Moreover, $U$ can be chosen arbitrally small.
Proof. Choose \( t_0 \in (0, T) \) and \( E(0) = \{e_1(0), e_2(0), e_1^*(0), e_2^*(0)\} \) a symplectic frame in \( \gamma(t_0) \) with \( e_1(0) = X_H^\ast(\gamma(t_0)) \). Consider
\[
E(t) = \{e_1(t), e_2(t), e_1^*(t), e_2^*(t)\}
\]
where \( \xi(t) = d\gamma_{(t_0)}^{\ast}H - \xi, \quad \forall \xi \in E(0) \), for \( t \in (0, r) \) with \( r > 0 \) arbitrarily small.

Then we can decompose the matrix of the differential of the flow, in the base \( E(0) \), \( [d\gamma_{(t_0)}^{\ast}H]_{E(0)} \in Sp(2) \), as \( [d\gamma_{(t_0)}^{\ast}H]_{E(0)} = [d\gamma_{(t_0-r)}^{\ast}H]_{E(0)} \cdot [d\gamma_{(t_0)}^{\ast}H]_{E(0)} \). By construction we have that \( [d\gamma_{(t_0-r)}^{\ast}H]_{E(0)} = I_4 \), therefore
\[
[d\gamma_{(t_0)}^{\ast}H]_{E(0)} = [d\gamma_{(t_0-r)}^{\ast}H]_{E(0)} \cdot [d\gamma_{(t_0)}^{\ast}H]_{E(0)}
\]
Consider \( U \) an arbitrarily small neighborhood of \( \gamma(t_0) \) in \( T^*M \) and \( r \) small enough, in such way that, \( \gamma = \{\psi_t^H(0) \mid t \in (t_0-r, t_0)\} \subseteq \mathbb{H}^\ast \cap U \). Fix \( t_1 \in (t_0 - r, t_0) \) and \( V \) a neighborhood of \( \gamma(t_1) \) in \( T^*M \), small enough, in such way that, \( V \subset U \) and that \( \gamma(t_0), \gamma(t_0-r) \notin \nabla \). Suppose that we have \( \mathbb{H} : T^*M \rightarrow \mathbb{R} \) a smooth Hamiltonian representing a smooth perturbation of \( H \), such that, \( \text{supp}(\mathbb{H} - H) \subset V \) and that \( \text{jet}_{t_1}(\mathbb{H}) |_{\gamma(t)} = \text{jet}_{t_1}(H) |_{\gamma(t)}, \) and \( [d\gamma_{(t_0)}^{\ast}H]_{E(0)} = [d\gamma_{(t_0-r)}^{\ast}H]_{E(0)} \cdot [d\gamma_{(t_0)}^{\ast}H]_{E(0)} \cdot [d\gamma_{(t_0)}^{\ast}H]_{E(0)} \). Since \( \text{supp}(\mathbb{H} - H) \subset V \), we have \( [d\gamma_{(t_0)}^{\ast}H]_{E(0)} = [d\gamma_{(t_0-r)}^{\ast}H]_{E(0)} \cdot [d\gamma_{(t_0)}^{\ast}H]_{E(0)} \cdot [d\gamma_{(t_0)}^{\ast}H]_{E(0)} \). By (1), \( [d\gamma_{(t_0)}^{\ast}H]_{E(0)} = [d\gamma_{(t_0-r)}^{\ast}H]_{E(0)} \cdot [d\gamma_{(t_0)}^{\ast}H]_{E(0)} \), so
\[
[d\gamma_{(t_0)}^{\ast}H]_{E(0)} = [d\gamma_{(t_0-r)}^{\ast}H]_{E(0)} \cdot [d\gamma_{(t_0)}^{\ast}H]_{E(0)} \]
From the construction of the perturbation described above we have that \( [d\gamma_{(t_0)}^{\ast}H]_{E(0)} \) has the expression
\[
[d\gamma_{(t_0)}^{\ast}H]_{E(0)} = \begin{bmatrix} 1 & \alpha & \sigma & \beta \\ 0 & A & \hat{\alpha} & B \\ 0 & 0 & 1 & 0 \\ 0 & C & \beta & D \end{bmatrix} \in Sp(2),
\]
because, the energy level in \( \gamma(t_0) \) and \( \gamma(t_0-r) \) is the same for \( \mathbb{H} \) and \( H \). Thus it is invariant by the action of the flow of both Hamiltonians. Let \( Sp(2) \) be the following subgroup of \( Sp(2) \),
\[
Sp(2) = \left\{ \begin{bmatrix} 1 & \alpha & \sigma & \beta \\ 0 & A & \hat{\alpha} & B \\ 0 & 0 & 1 & 0 \\ 0 & C & \beta & D \end{bmatrix} \in SL(4) \mid \begin{bmatrix} \hat{\alpha} \\ \beta \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} J [\begin{bmatrix} \alpha \\ \beta \end{bmatrix}]^* \right\}
\]
and consider the projection π : \(Sp(2) \to Sp(1)\) given by

\[
\pi \left( \begin{bmatrix} 1 & \sigma & \beta \\ 0 & A & B \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} A & B \\ C & D \end{bmatrix},
\]

which is a homomorphism of Lie groups. Observe that

\[
[d_{\gamma((t_0)\psi)}^{\hat{H}}|_{E(0)}, [d_{\gamma((t_0)\psi)}^{H}|_{E(0)}] \in Sp(2)
\]

and det\(\left([d_{\gamma((t_0)\psi)}^{\hat{H}}|_{E(0)}] - \lambda I_4\right) = (\lambda - 1)^2 \det(\pi([d_{\gamma((t_0)\psi)}^{H}|_{E(0)}) - \lambda I_2).\) Thus \(\gamma\) will be a nondegenerate orbit of order \(\leq 2m\), to the perturbed Hamiltonian, if \(\pi([d_{\gamma((t_0)\psi)}^{\hat{H}}|_{E(0)}) does not have roots of the unity of order \(\leq 2m\) as eigenvalues. Since, the symplectic matrices that are 2m-elementary \(\mathbb{H}\) (in particular, does not have roots of the unity of order \(\leq 2m\) as eigenvalues), forms an open and dense subset of \(Sp(1)\), we must to show that, for a choice of the perturbation space, the correspondence \(\mathbb{H} \to \pi([d_{\gamma((t_0)\psi)}^{\hat{H}}|_{E(0)}) applied to a neighborhood of \(\mathbb{H}\), generate an open neighborhood of \(\pi([d_{\gamma((t_0)\psi)}^{\hat{H}}|_{E(0)})\) in \(Sp(1)\). Using the homomorphism property

\[
\pi([d_{\gamma((t_0)\psi)}^{\hat{H}}|_{E(0)}) = \pi([d_{\gamma((t_0-r)\psi)}^{\hat{H}}|_{E(0)}]) \cdot \pi([d_{\gamma((t_0)\psi)}^{H}|_{E(0)})
\]

We define \(\chi_0 = \pi([d_{\gamma((t_0)\psi)}^{\hat{H}}|_{E(0)}]) and \(\hat{S}(\mathbb{H}) = \pi([d_{\gamma((t_0-r)\psi)}^{\hat{H}}|_{E(0)}])\). Since the translation \(\chi \to \chi \cdot \chi_0\) is an isomorphism of the of the Lie group \(Sp(1)\), we need to show that the map \(\mathbb{H} \to \hat{S}(\mathbb{H})\) applied to a neighborhood of \(\mathbb{H}\) generates an open neighborhood of \(I_2\) in \(Sp(1)\). In order to construct the perturbation space we will consider \(\mathcal{N} \subset \mathbb{H}^{-1}(k)\) a local Lagrangian submanifold in \(\gamma(t_0)\). We can reduce, if necessary, the size of the neighborhood \(U\) of \(\gamma(t_0)\) chosen previously in such way that \(U\) admits the parameterization \((x = (x_1, x_2), p = (p_1, p_2)) : U \to \mathbb{R}^{2+2}\) as in [9], Lemma A3, that is,

a) \(\mathcal{N} \cap U = \{(x, 0)\}\);

b) \(\omega = dx \wedge dp\);

c) \(X_t|_{\mathcal{N} \cap U} = 1_{\partial X_t}\).

In these coordinates we can see that \(\hat{\gamma} = \{(t, 0, 0, 0) \mid t \in (t_0 - r, t_0)\}\). Consider, the perturbation space

\[
\hat{\mathcal{F}} = \{f : T^*M \to \mathbb{R} \mid \text{supp}(f) \subset \hat{W} \subset W\}
\]

where \(W = \mathcal{N} \cap V\) and \(\hat{W}\) is a compact set contained in \(W\) that contains \(\gamma(t_1)\) in its interior. Observe that, \(\hat{\mathcal{F}}\) can be identified with \(C^\infty(\hat{W}, \mathbb{R})\),

\[\text{A symplectic matrix is N-elementary if its principal eigenvalues (the eigenvalues } \lambda \text{ such that } \|\lambda\| < 1 \text{ or } \text{Re}(\lambda) \geq 0 \text{ are multiplicatively independent over the integer, that is, if } A\mathbb{N} = 1, \text{ where } \sum p_i = N \text{ then } p_i = 0, \forall i.\]
therefore we can think $\hat{F}$ as a vectorial space. Consider the following finite dimensional subspace $F \subset \hat{F}$

$$F = \{ f \mid f(x, p) = \eta(x)(a\delta_t(x_1) + b\delta'_t(x_1) + c\delta''_t(x_1)) + \frac{1}{2}x_2^2, \quad a, b, c \in \mathbb{R} \}$$

where $\eta$ is a fix function with supp($\eta$) $\subset \hat{W}$ and $\eta \equiv 1$ in some neighborhood of $\gamma(t_1)$, in $\mathcal{N}$. Moreover, $\delta_t$ is a smooth approximation of the delta of Dirac in the point $t_1$. Now we are able to define the differentiable map $S : F \rightarrow Sp(1)$ given by $S(f) = \hat{S}(\mathbb{H} + f) = \pi([d_{\gamma(t_0-r)}^{H+h}\psi_r^{E(r)}]_{E(0)})$. Observe that $\dim(F) = 3 = \dim(Sp(1))$ and $S(0) = \hat{S}(\mathbb{H} + 0) = \pi([d_{\gamma(t_0-r)}^{\mathbb{H}+h\mathbb{I}}\psi_r^{E(r)}]_{E(0)}) = I_2$. Thus we must to show that,

$$d_0f : T_0F \cong F \rightarrow T_{id_2x2}Sp(1) \cong sp(1)$$

is surjective. Given $h \in F$ we have $d_0f(h) = \pi(\frac{d}{dt}[d_{\gamma(t_0-r)}^{\mathbb{H}+h\mathbb{I}}\psi_r^{E(r)}]_{E(0)})t=0)$. Consider $\xi \in T_{\gamma(t_0-r)}T^*M$, where $t \in (0, r)$ and define

$$\xi(t, l) = d_{\gamma(t_0-r)}^{H+h}\psi_r^{H}\xi.$$ 

For a fix $l$ we define a field through $\gamma$ that verifies the equation

$$\begin{cases} \dot{\xi}(t, l) = JHess(\mathbb{H} + lh)(\gamma(t))\xi(t, l) \\ \xi(0, l) = \xi. \end{cases}$$

Taking the derivative of the equation above with respect to $l$ and using the commutativity of the derivatives we get

$$\frac{d}{dt}(d\xi(t, l)|_{t=0}) = JHess(h)\xi(t, l)|_{t=0} + JHess(\mathbb{H})\frac{d}{dt}\xi(t, l)|_{t=0}$$

Denote $\mathcal{H} = Hess(\mathbb{H})$, $\xi(t) = \xi(t, l)|_{l=0}$ e $\mathcal{Y}(t) = \frac{d}{dt}\xi(t, l)|_{l=0}$, then

$$\begin{cases} \dot{\mathcal{Y}}(t) = J\mathcal{H}\mathcal{Y}(t) + JHess(h)\xi(t) \\ \mathcal{Y}(0) = 0. \end{cases}$$

Applying the method of variation of constants and using

$$\begin{cases} \dot{\xi}(t) = J\mathcal{H}(\gamma(t))\xi(t) \\ \xi(0) = \xi, \end{cases}$$

we get

$$\mathcal{Y}(t) = d_{\gamma(t_0-r)}^{H}\psi_r^{H}\int_0^r d_{\gamma(t_0-r)}^{H}JHess(h)d_{\gamma(t_0-r)}^{H}\psi_r^{H}\xi dt.$$

Remember that $\mathcal{Y}(r) = \frac{d}{dt}\xi(r, l)|_{l=0} = \frac{d}{dt}d_{\gamma(t_0-r)}^{H+h}\psi_r^{H}(\xi)|_{l=0}$, so

$$\frac{d}{dt}d_{\gamma(t_0-r)}^{H+h}\psi_r^{H}|_{l=0} = d_{\gamma(t_0-r)}^{H+h}\psi_r^{H}\int_0^r d_{\gamma(t_0-r)}^{H}JHess(h)d_{\gamma(t_0-r)}^{H}\psi_r^{H}dt.$$
From this calculation
\[
d_0 \mathcal{F}(h) = \pi \left( [d_{\gamma(t_0-r)} \psi_r^E]_0^r d_{\gamma(t_0-r)} \psi_t^E JHess(h) d_{\gamma(t_0-r)} \psi_t^E dt ]_{E(0)}^{E(r)} \right).
\]

In order to obtain the expression (3) we need to calculate \(JHess(h)\). All the integrals will be calculated with the delta of Dirac and not with the approximations, however the same conclusions are true for an approximation, good enough. Consider \(\tilde{h}(x) = (a \delta_{t_1}(x_1) + b \delta'_{t_1}(x_1) + c \delta''_{t_1}(x_1))\frac{1}{2}x_2^2\) and \(h(x) = \eta(x) \tilde{h}(x)\) then \(dh = \eta dh + \hat{h} d\eta\) and \(d^2h = \eta d^2\hat{h} + dh^* d\eta + d\eta^* dh + \hat{h} d^2\eta\). As, \(Hess(h)(\gamma) = d^2_h h\) and \(jet_1(h)\), \(d\eta\gamma = 0\) we have that \(Hess(h)(\gamma) = \eta(\gamma)d^2_h h\).

On the other hand, \((d^2_h \tilde{h})_{ij} = a \delta_{t_1}(t_0-r+t) + b \delta'_{t_1}(t_0-r+t) + c \delta''_{t_1}(t_0-r+t)\) if \(ij = 22\) and, and equal to 0 otherwise. Taking the \(x_1\)-support of \(\delta_{t_1}\), small enough, we can assume that \(JHess(h)(\gamma) = \hat{A} \delta_{t_1}(t_0-r+t) + \hat{B} \delta'_{t_1}(t_0-r+t) + \hat{C} \delta''_{t_1}(t_0-r+t)\) where \((\hat{A})_{ij} = -a\) if \(ij = 42\), and equal to 0 otherwise, \((\hat{B})_{ij} = -b\) if \(ij = 42\), and equal to 0 otherwise and \((\hat{C})_{ij} = -c\) if \(ij = 42\), and equal to 0 otherwise.

Define,
\[
\hat{I}_1 = d_{\gamma(t_0-r)} \psi_r^E \int_0^r d_{\gamma(t_0-r)} \psi_t^E \delta_{t_1}(t_0-r+t) dt,
\]
\[
\hat{I}_2 = d_{\gamma(t_0-r)} \psi_r^E \int_0^r \bar{d}_{\gamma(t_0-r)} \psi_t^E \delta'_{t_1}(t_0-r+t) dt,
\]
\[
\hat{I}_3 = d_{\gamma(t_0-r)} \psi_r^E \int_0^r \gamma_{\gamma(t_0-r)} \psi_t^E \delta''_{t_1}(t_0-r+t) dt.
\]

Thus
\[
\hat{I}_1 = d_{\gamma(t_0-r)} \psi_r^E d_{\gamma(t_0-r)} \psi_r^{(t_1-t_0+r)} \hat{A} d_{\gamma(t_0-r)} \psi_t^{(t_1-t_0+r)},
\]
\[
\hat{I}_2 = -d_{\gamma(t_0-r)} \psi_r^E d_{\gamma(t_0-r)} \psi_r^{(t_1-t_0+r)} [\hat{B}, \bar{J}H] d_{\gamma(t_0-r)} \psi_t^{(t_1-t_0+r)},
\]
and
\[
\hat{I}_3 = d_{\gamma(t_0-r)} \psi_r^E d_{\gamma(t_0-r)} \psi_r^{(t_1-t_0+r)} \gamma_{\gamma(t_0-r)} \psi_t^{(t_1-t_0+r)} \gamma_{[\hat{C}, JH]} + [\hat{C}, \gamma_{JH}].
\]

Define \(Z = A - [\hat{B}, \bar{J}H] + [\hat{C}, \gamma_{JH}] + [\hat{C}, \gamma_{JH}]. \) Then,
\[
d_0 \mathcal{F}(h) = \pi \left( [d_{\gamma(t_0-r)} \psi_r^E]_0^r d_{\gamma(t_0-r)} \psi_t^E \int_0^r Z d_{\gamma(t_0-r)} \psi_t^E [E(r)] \right)_{E(0)}^{E(r)}.
\]

Writing this matrix in the bases \(E(0)\) and \(E(r)\), in each point of the curve, we get,
\[
[d_{\gamma(t_0-r)} \psi_r^E]_0^r d_{\gamma(t_0-r)} \psi_t^E \int_0^r Z d_{\gamma(t_0-r)} \psi_t^E [E(r)]_{E(0)}^{E(r)} = [d_{\gamma(t_0-r)} \psi_r^E]_{E(0)} d_{\gamma(t_0-r)} \psi_t^E \int_0^r Z d_{\gamma(t_0-r)} \psi_t^E [E(r)]_{E(0)}^{E(r)}.
\]
Moreover \( [d_{\gamma(t_0-r)}\hat{\psi}_E^{(0)}]_{E(0)} = I_4 \) and there exists a symplectic conjugation \( G \in Sp(2) \) between the base \( E(t_0 - t_1) \) and the canonic symplectic base, 
\[
\left\{ \frac{\partial}{\partial z_1}(\gamma(t_1)), \frac{\partial}{\partial z_2}(\gamma(t_1)), \frac{\partial}{\partial p_1}(\gamma(t_1)), \frac{\partial}{\partial p_2}(\gamma(t_1)) \right\}
\]
such that, 
\[
[Z]_{E(t_0-t_1)}^{E(t_0-t_1)} = G^{-1} Z G. \text{ Thus } d_0 F(h) = \]

\[
\pi \left( G[d_{\gamma(t_0-r)}\hat{\psi}_E^{(0)}]_{E(t_0-t_1)}^{E(t_0-t_1)} \right)^{-1} \pi(Z) \pi \left( G[d_{\gamma(t_0-r)}\hat{\psi}_E^{(t_0-t_1)}]_{E(t_0-t_1)}^{E(t_0-t_1)} \right) \]

that is, we need to show that \( \pi(Z) \) is surjective in \( sp(1) \). A simple calculation shows that, \( \pi(Z) = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \) where,

\[
\begin{align*}
z_{11} &= -b H_{p_2 p_2} + 2c H_{p_1 p_2} \mathbb{H}_{x_2 x_2} + \mathbb{H}_{p_2 p_2} \\
z_{12} &= 2c(H_{p_2 p_2})^2 \\
z_{21} &= -a + 2b H_{x_1 x_2} + 2c H_{p_1 p_2} \mathbb{H}_{x_1 x_2} + 2c H_{p_2 p_2} \mathbb{H}_{x_2 x_2} - 2c H_{x_2 p_1} \mathbb{H}_{x_1 p_2} - 4c(H_{x_2 x_2})^2 - 2c \mathbb{H}_{x_2 x_2} \\
z_{22} &= b H_{p_2 p_2} - 2c H_{p_1 p_2} \mathbb{H}_{x_2 x_2} - \mathbb{H}_{p_2 p_2}
\end{align*}
\]

Remember that, \( sp(1) = \left\{ \begin{bmatrix} B & C \\ A & -B \end{bmatrix} | A, B, C, D \in \mathbb{R}, \right\} \) and \( \mathbb{H}_{p_2 p_2} \neq 0 \), thus we have the surjectivity. In order to conclude the proof we must to find a potential in \( M \) adapted to this perturbation. Consider \( f \in \mathcal{F} \) arbitrarily close to zero such that \( \pi([d_{\gamma(t_0-r)}\hat{\psi}_E^{(0)}]_{E(0)}) \) is nondegenerate of order \( \leq 2m \). Let us remember that the \( x \)-support of \( f \) is contained in \( W \) which is an arbitrarily small neighborhood of \( \gamma(t_1) \) in \( N \). Consider \( (\hat{x}, \hat{p}) \) the canonic symplectic coordinates in \( \gamma(t_1) \), and \( \hat{\pi} : T^*M \to \mathbb{R} \) given by \( \hat{\pi}(\hat{x}, \hat{p}) = \hat{x} \). As we are free to dislocate the point \( t_1 \) by a \( \varepsilon \) arbitrarily small, we can use the twist property of the vertical fiber bundle as in Lemma \( \S \) to conclude that \( \hat{\pi}|_N \) is a local diffeomorphism in \( \gamma(t_1) \). Take a diffeomorphism \( q : W \subset N \to M \) given by \( q(x) = \hat{x} \), where \((x, 0) \equiv (\hat{x}, \hat{p}) \) in \( N \). Choose the potential
\[
f_0(\hat{x}) = \begin{cases} f(q^{-1}(\hat{x})) & x \in \hat{\pi}(W) \\ 0 & x \notin \hat{\pi}(W), \end{cases}
\]
by construction, \( \mathbb{H}(\hat{x}, \hat{p}) + f_0(\hat{x}) \) has the desired property. The lemma is proven. \( \Box \)

**Conjecture:**
If \( \dim(M) = n \) then \( \pi(Z) \) is surjective in \( sp(n-1) \).

The main obstruction to prove this conjecture is that if \( Z = \tilde{A} - [\hat{B}, J\mathcal{H}] + [\hat{C}, J\mathcal{H}] + [[\hat{C}, J\mathcal{H}], J\mathcal{H}] \) then we need to solve equations like \( UX + XU = D, \)
in the space of symmetric $n-1 \times n-1$. But it is well known (see [7]) that, in our case, the solving of this type of equations requires additional hypothesis on the eigenvalues of $H_{pp}$, which are not generic in Mané’s sense. On the other hand, our approach it is essentially the only way to construct perturbations by potentials, thus we hope that in the future, we will be capable to solve this equation in higher dimensions.

**Lemma 19.** Given $k \in \mathbb{R}$, $f_0 \in \mathcal{R}(k)$, $\mathcal{U}_{f_0} \subseteq \mathcal{R}$ a $C^\infty$ neighborhood of $f_0$, $\alpha = \alpha(\mathcal{U}_{f_0}) > 0$, as in the Lemma 14, $a \in \mathcal{R}$ such that $0 < a < \infty$ and $\mathcal{G}_{k}^{a,a} \cap \mathcal{U}_{f_0} \neq \emptyset$, we have that $\mathcal{G}_{k}^{a,2a} \cap \mathcal{U}_{f_0}$ is dense in $\mathcal{G}_{k}^{a,a} \cap \mathcal{U}_{f_0}$.

**Proof.** Take $f \in \mathcal{G}_{k}^{a,a} \cap \mathcal{U}_{f_0}$ and $\mathcal{U}$ an arbitrary neighborhood of $f$. We must show that $\mathcal{U} \cap (\mathcal{G}_{k}^{a,2a} \cap \mathcal{U}_{f_0}) \neq \emptyset$. From the definition of $\mathcal{G}_{k}^{a,a}$ we have that all periodic orbits of $H + f$ in the level $k$ with minimal period $a$ are nondegenerate of order $m \leq \frac{a}{\min{T}}$. Take $\rho_f : T^*M \times (0, a) \times (-\varepsilon, \varepsilon) \rightarrow T^*M \times T^*M \times \mathbb{R}$. From Corollary 13 we have that $\rho_f \cap \Delta_0$. Moreover, by Lemma 14 (i),

$$\rho_f^{-1}(\Delta_0) = \{ (\vartheta, T, 0) \mid \vartheta \in (H + f)^{-1}(k), T \in (0, a), \psi_T^{H+I}(\vartheta) = \vartheta \}$$

Observe that $\rho_f^{-1}(\Delta_0) \subset (H + f)^{-1}(k) \times [0, a] \times \{0\}$ which is a compact set. As $\Delta_0$ is closed we have that $\rho_f^{-1}(\Delta_0)$ is a submanifold of dimension 1, with a finite number of connected components. Since each periodic orbit, $\{\psi_t^{H+I}(\vartheta) \mid t \in [0, T_i], (\vartheta, T_i, 0) \in \rho_f^{-1}(\Delta_0)\}$, is a connected component of dimension 1, the number of periodic orbits for $H + f$ in the level $k$ with minimal period $a$, distinct, is finite. Denote, $\{\psi_t^{H+I}(\vartheta_i) \mid t \in [0, T_i = \min{T}(\vartheta_i)], (\vartheta_i, T_i, 0) \in \rho_f^{-1}(\Delta_0)\}$, for $i = 1, \ldots, N$, the $N$ periodic orbits for $H + f$ in the level $k$, with its respective minimal periods. From Theorem 15 we can find a sum of $N$ potentials $f_0 = f_1 + \ldots + f_N$ arbitrarily close to 0, such that, all orbits are nondegenerate of order $\leq 2m$ for $(H + f) + f_0$. The claim is proven because $f + f_0 \in \mathcal{U} \cap (\mathcal{G}_{k}^{a,2a} \cap \mathcal{U}_{f_0})$.

**Lemma 20.** With the same notation of the Lemma 14, if $\mathcal{G}_{k}^{a,a} \cap \mathcal{U}_{f_0} \neq \emptyset$, then $ev_\rho : \mathcal{G}_{k}^{a,2a} \cap \mathcal{U}_{f_0} \times T^*M \times (0, 2a) \times (-\varepsilon, \varepsilon) \rightarrow T^*M \times T^*M \times \mathbb{R}$ is transversal to $\Delta_0$.

**Proof.** Indeed, given $(f, \vartheta, T, S) \in \mathcal{G}_{k}^{a,2a} \cap \mathcal{U}_{f_0} \times T^*M \times (0, 2a) \times (-\varepsilon, \varepsilon)$, if $ev(f, \vartheta, T, S) \notin \Delta_0$, is done. So we can assume that $ev(f, \vartheta, T, S) \in \Delta_0$, that is, $\vartheta$ is a periodic orbit of $H + f$ in the level $k$ with minimal period, $\min{T} = \min{T}(\vartheta)$ and $S = 0$.

If $T = \min{T}$ then $ev \cap (f, \vartheta, T, 0) \Delta_0$ by Lemma 16 (ii). On the other hand, if $T = m\min{T}$, $m \geq 2$ we have that $m \leq 2a/\min{T}$, that is, $\min{T} \leq 2a/m \leq a$ so $\vartheta$ is nondegenerate of order $m$, because $f \in \mathcal{G}_{k}^{a,2a} \cap \mathcal{U}_{f_0}$. Thus $ev \cap (f, \vartheta, T, S) \Delta_0$ by Lemma 16 (i).
Lemma 21. With the same notation of the Lemma 19, if \( G_k^{a,\alpha} \cap U_{f_0} \neq \emptyset \), then \( (G_k^{3a/2,3a/2} \cap U_{f_0}) \cap (G_k^{a,\alpha} \cap U_{f_0}) \) is dense in \( G_k^{a,\alpha} \cap U_{f_0} \).

Proof. Consider \( B = G_k^{a,\alpha} \cap U_{f_0} \), which is a submanifold of \( C^\infty(M; \mathbb{R}) \) because it is open. From the Lemma 20 we have that \( ev_p : G_k^{a,\alpha} \cap U_{f_0} \times T^\ast M \times (0,2\alpha) \times (-\varepsilon,\varepsilon) \rightarrow T^\ast M \times T^\ast M \times \mathbb{R} \) is transversal to \( \Delta_0 \). Then, Theorem 11 implies that \( \mathcal{R} = \{ f \in G_k^{a,\alpha} \ | \ \rho_f \pitchfork \Delta_0 \} \) is a generic subset of \( G_k^{a,\alpha} \cap U_{f_0} \). In particular \( \mathcal{R} \) is dense in \( G_k^{a,\alpha} \cap U_{f_0} \). We claim that \( \mathcal{R} \subset (G_k^{3a/2,3a/2} \cap U_{f_0}) \cap (G_k^{a,\alpha} \cap U_{f_0}) \). Indeed, take \( f \in \mathcal{R} \), from Corollary 15 all periodic orbits of the flow defined by \( H + f \) in the energy level \( k \), with minimal period \( T_{min} \) are nondegenerate of order \( m \leq \frac{2a}{T_{min}} \) because \( \rho_f \pitchfork \Delta_0 \). If we have a periodic orbit for \( H + f \) in the level \( k \), with minimal period \( T_{min} \leq 3a/2 \) take \( m' \leq \frac{3a/2}{T_{min}} = \frac{2a}{T_{min}} \leq \frac{2a}{3a/2} \) then this orbit is nondegenerate of order \( m' \) in particular \( f \in G_k^{3a/2,3a/2} \cap U_{f_0} \). Therefore \( (G_k^{3a/2,3a/2} \cap U_{f_0}) \cap (G_k^{a,\alpha} \cap U_{f_0}) \) is dense in \( G_k^{a,\alpha} \cap U_{f_0} \).

Lemma 22. With the same notation of the Lemma 19, if \( G_k^{a,\alpha} \cap U_{f_0} \neq \emptyset \), we have that \( (G_k^{3a/2,3a/2} \cap U_{f_0}) \cap (G_k^{a,\alpha} \cap U_{f_0}) \) is dense in \( G_k^{a,\alpha} \cap U_{f_0} \).

Proof. From Lemma 21 we have that \( (G_k^{3a/2,3a/2} \cap U_{f_0}) \cap (G_k^{a,\alpha} \cap U_{f_0}) \) is dense in \( G_k^{a,\alpha} \cap U_{f_0} \) and from Lemma 19 \( G_k^{a,\alpha} \cap U_{f_0} \) is dense in \( G_k^{a,\alpha} \cap U_{f_0} \) therefore \( G_k^{3a/2,3a/2} \cap U_{f_0} \) is dense in \( G_k^{a,\alpha} \cap U_{f_0} \).

Proof of the Lemma 17:

Proof. Given \( k \in \mathbb{R}, f_0 \in \mathcal{R}(k), \ U_{f_0} \subset \mathcal{R} \) a \( C^\infty \) neighborhood of \( f_0 \), \( \alpha = \alpha(U_{f_0}) > 0 \) as in Lemma 3. Take \( c \in \mathbb{R}_+, \) if \( c < \alpha \) then \( G_k^{c,c} \cap U_{f_0} = U_{f_0} \) by Lemma 3. So we can assume that \( c \in \mathbb{R}_+, \) with \( c \geq \alpha > a > 0 \).

We claim that, \( G_k^{(\frac{3}{2})^\ell a, (\frac{3}{2})^\ell a} \cap U_{f_0} \) is dense in \( G_k^{a,\alpha} \cap U_{f_0} \), \( \forall \ell \in \mathbb{N} \). The proof is by induction in \( \ell \).

For \( \ell = 1 \) observe that, \( G_k^{a,\alpha} \cap U_{f_0} = U_{f_0} \neq \emptyset \), because \( \alpha > a > 0 \). Therefore \( G_k^{\frac{3}{2}a, \frac{3}{2}a} \cap U_{f_0} \) is dense in \( G_k^{a,\alpha} \cap U_{f_0} \) by Lemma 22.

Suppose that, \( G_k^{(\frac{3}{2})^\ell a, (\frac{3}{2})^\ell a} \cap U_{f_0} \) is dense in \( G_k^{a,\alpha} \cap U_{f_0} \), with \( \ell \geq 1 \). Then \( G_k^{(\frac{3}{2})^\ell a, (\frac{3}{2})^\ell a} \cap U_{f_0} \neq \emptyset \), from the density, and taking \( a' = (\frac{3}{2})^\ell a \), we have that \( G_k^{a', \frac{3}{2}a'} \cap U_{f_0} \) is dense in \( G_k^{a', \frac{3}{2}a'} \cap U_{f_0} \) by Lemma 22. So \( G_k^{(\frac{3}{2})^{\ell+1}a, (\frac{3}{2})^{\ell+1}a} \cap U_{f_0} \) is dense in \( G_k^{a', \frac{3}{2}a'} \cap U_{f_0} \), concluding the proof of the claim.

Consider \( \ell_0 \), such that, \( (\frac{3}{2})^\ell_0 a > a \). Then, \( G_k^{(\frac{3}{2})^{\ell_0}a, (\frac{3}{2})^{\ell_0}a} \cap U_{f_0} \subset G_k^{c,c} \cap U_{f_0} \subset G_k^{a,\alpha} \cap U_{f_0} = U_{f_0} \). Since \( G_k^{(\frac{3}{2})^{\ell_0}a, (\frac{3}{2})^{\ell_0}a} \cap U_{f_0} \) is dense in \( G_k^{a,\alpha} \cap U_{f_0} \), we conclude that \( G_k^{c,c} \cap U_{f_0} \) is dense in \( U_{f_0} \). The lemma is proven.
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