Analytical study of the velocity and pressure of the electrovortex flow in hemispherical bowl in a Stokes approximation

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Abstract. The problem of an electrovortex flow in a hemispherical bowl with an internal electrode is considered. In the framework of the Stokes approximation, analytical expressions for the velocity and pressure fields are obtained. Two types of boundary conditions were used: non-leakage on the entire boundary (with no-slip condition on the free surface and partly slip on the electrodes) and an approximate adhesion condition on the electrodes. The non-leakage condition corresponds to the homogeneous Dirichlet boundary condition for the vector potential and the vorticity on the entire boundary. An approximate adhesion condition is posed using the Thom difference approximation. The analytical expressions for the velocity and pressure fields are written uniformly for both types of boundary conditions.

1. Introduction

Electrovortex flow occurs when an inhomogeneous current flowing through a liquid interacts with its own magnetic field [1]. It is quite important effect which takes place in different fundamental problems and technical applications. For example, it takes place in processes connected with the electrical melting. The electrovortex flow has been studied for several decades by different groups in Russia [1], Latvia [2], Ukraine [3], United Kingdom [4], Germany [5], Austria [6] and other countries. There are different examples of experimental researches which have been done in different laboratories of liquid metal technology.

One of the most interesting problems is connected with the electrovortex flow in the hemispherical vessel [1] (fig. 1). It is filled by the liquid metal, and the surface of the vessel is the outer electrode. The inner electrode has the same form and it is located in the centre of the bowl. The current passes from the inner electrode to the outer and the lowering density causes the azimuthal magnetic field which is the reason of the vortex flow. It is necessary to mention a lot of different works which have been done in the Joint Institute for High Temperatures which were devoted to experimental and numerical study of this problem.

Progress in numerical methods and the growth of the computational resources increased the role of the computational research, and now most of the results are obtained using standard program complexes [6]. Although they are very effective in some cases, numerical modelling does not give us
opportunity to study some basic laws for the electrovortex flows and the typical dependencies of the velocities on different parameters. So it is still important to construct the analytical solutions, including series expansions.

From the theoretical point of view, these processes were first analytically investigated for special forms of central electrodes: point source [4] and plane source [5]. As for the analytical research for case finite size hemispherical central electrode and non-leakage condition, the standard spectral decomposition of the solution has been obtained [7]. However, it uses the a double eigenfunctions series, which is a bit inconvenient. Also we need to take a lot of terms to have proper results. So, the time for this solution can be even more than the time for direct numerical simulation of this problem.

In this work we present another approach which is based on another principles and gives us the opportunity to solve the equations using single series expansion. It has a lot of advantages and even the first few terms in the decomposition gives us the results which are close to the solution of our problem.

![Figure 1](image1.png)

**Figure 1.** The electrovortex flow in a hemispherical bowl. 1 – small electrode; 2- big electrode; 3 – conductive liquid.

### 2. Basic equations

The current from the inner electrode passes to the outer boundary and has the density \( \vec{j} \), which becomes smaller for increasing distance from the centre. It produces the magnetic field \( \vec{B} \). They jointly generate the Ampere force which has the density \( \vec{f} = \vec{j} \times \vec{B} \). If the current has the value of \( I \), they will be the following (full description can be found in [8]):

\[
\vec{j} = \frac{I}{2\pi r^2} \vec{e}_r;
\]

\[
\vec{B} = \frac{\mu_0 I}{2\pi} \frac{1 - \cos \theta}{\sin \theta} \vec{e}_\phi;
\]

\[
\vec{f} = -\frac{\mu_0 I^2}{4\pi r^3} \frac{1 - \cos \theta}{\sin \theta} \vec{e}_\phi;
\]
where \( \vec{e}_r \), \( \vec{e}_\theta \) and \( \vec{e}_\phi \) are the unit orts of spherical coordinates (fig. 1).

It is useful to take the dimensionless units, measuring the distances in units of radius of the outer electrode \( R \).

The current can be measured in units of [8]:
\[
j_0 = \frac{I}{R^2}.
\]

The unit for the magnetic field will be the following [8]:
\[
B_0 = \frac{\mu_0 I}{R}.
\]

The Ampere force should be measured in units of [8]:
\[
f_0 = \frac{\mu_0 I^2}{R^3}.
\]

The unit for the speed is [8]:
\[
\nu_0 = \sqrt{\frac{f_0 R}{\rho}},
\]
where \( \rho \) is the density of the liquid.

The unit for the time is [8]:
\[
t_0 = \frac{R}{\nu_0}.
\]

The Reynolds number in this case will be the described by the formulae [8]:
\[
Re = \frac{1}{\nu} \sqrt{\frac{f_0 R}{\rho}},
\]
where \( \nu \) is kinematic viscosity coefficient. It is also necessary to introduce the parameter of the electrovortex flow which is often used in this field:
\[
S = Re^2.
\]

In the dimensionless units the Navier – Stokes equation and incompressibility condition can be written in the form [8]:
\[
\frac{\partial \vec{u}}{\partial t} + (\vec{u}, \vec{V}) \vec{u} = -\vec{\nabla} p + \frac{1}{\sqrt{S}} \Delta \vec{u} + \frac{\cos \theta - 1}{4\pi^2 r^3 \sin \theta} \vec{e}_\theta;
\]
\[
(\vec{V}, \vec{u}) = 0.
\]

It is useful to take the Stokes approximation, which assumes zero the left part of the equation (so, we take the time-independent linear case, which is possible for \( S < 10^3 \)):\[
\frac{1}{\sqrt{S}} \Delta \vec{u} = \vec{\nabla} p - \frac{\cos \theta - 1}{4\pi^2 r^3 \sin \theta} \vec{e}_\theta, \quad (r, \theta) \in D,
\]
where \( D \) is the following region:
\[
a < r < 1;
\]
\[
0 < \theta < \frac{\pi}{2}.
\]
For the electrodes, we have the adhesion condition in the form:
\[ \vec{v} \Big|_{r=a} = \vec{v} \Big|_{r=1} = 0. \]

For low currents, we have on the free surface the non-leakage condition:
\[ v_\theta \Big|_{\phi=\frac{\pi}{2}} = 0. \]

Also, we need a slip boundary condition on the free surface:
\[ \frac{\partial}{\partial \theta} v_\theta \Big|_{\phi=\frac{\pi}{2}} = 0. \]

It is useful to solve the equations using the vorticity \( \vec{\omega} \) and the vector potential \( \vec{\Psi} \) which are defined as:
\[ \vec{v} = \text{rot} \vec{\Psi}; \]
\[ \vec{\omega} = \text{rot} \vec{v}. \]

They give us the opportunity to exclude the pressure if we take the rotor of both parts of the equation. Because of the axial symmetry of the problem, the vorticity vector and the vector potential will have only azimuthal components, so we can rewrite the equation in the scalar form. It will be convenient to use the notations \( \omega \) and \( \Psi \) for them [8].

As for the boundary condition for these functions, it is not very trivial. For the simplest cases the Dirichlet condition on all boundaries are used, which gives an opportunity to use standard series expansions [8]. However, from the physical point of view it is better to take another boundary conditions which satisfy the no-slip condition at the electrodes. We use the setting of boundary conditions used in the practice of numerical calculations. In [9], the no-slip condition was approximated using the Thom’s formula [11, 12], containing the parameter \( h \) - the step of the difference grid. We construct an analytical solution to the obtained differential-difference problem for an arbitrary value of the parameter \( h \).

### 3. Vector potential and vorticity of the flow

For the studied functions we obtain the following equations [8]:
\[
\begin{align*}
\Delta \omega - \frac{\omega}{r^2 \sin^2 \theta} &= \sqrt{S} \frac{\cos \theta - 1}{2 \pi^2 r^4 \sin \theta}, \quad M \in D; \\
\Delta \Psi - \frac{\Psi}{r^2 \sin^2 \theta} &= -\omega, \quad M \in D.
\end{align*}
\]

The vector potential should satisfy the Dirichlet condition on the all boundary:
\[ \Psi \big|_{\partial D} = 0. \]

As for the vorticity, there are two different types of the conditions:

A. Dirichlet conditions on the entire border [8]:
\[ \omega \big|_{\partial D} = 0. \]

B. Dirichlet conditions on the free surface and Thom’s conditions with parameter \( h \) on electrodes [9]:

\[ \omega \big|_{\phi=\frac{\pi}{2}} = 0. \]
\[ \omega_{\theta} = 0; \]
\[ \omega_{r} = \frac{2}{h^2} \Psi |_{r=a+h}; \]
\[ \omega_{t} = -\frac{2}{h^2} \Psi |_{r=1-h}. \]

For both types of the boundary conditions the solution can be the next:
\[ \omega = \sum_{l=1}^{\infty} R_{l}(r) P_{2l}^{(1)}(\cos \theta); \]
\[ \Psi = \sum_{l=1}^{\infty} G_{l}(r) P_{2l}^{(1)}(\cos \theta); \]

where
\[ P_{2l}^{(1)}(\cos \theta) \] - the associated Legendre polynomials [10];
\[ R_{l}(r) = a_{l} r^{2l} + b_{l} r^{-2l+1} - \sqrt{S} \frac{(-1)^l (2l)!}{4\pi^2 \cdot 2^{2l}(l!)^2} \frac{4l+1}{16l^4 + 16l^2 - 4l^2 - 4l^2} r^{-2}; \]
\[ G_{l}(r) = A_{l} r^{2l} + B_{l} r^{-(2l+1)} - \frac{a_{l}}{8l+6} r^{2l+2} + \frac{b_{l}}{8l-2} r^{-(2l-1)} - \sqrt{S} \frac{(-1)^l (2l)!}{4\pi^2 \cdot 2^{2l}(l!)^2} \frac{4l+1}{64l^6 + 96l^5 + 16l^4 - 24l^3 - 8l^2}; \]

and the constants \( a_{l}, b_{l}, A_{l}, B_{l} \) can be obtained from the boundary conditions.

It can be shown that conditions A for the vorticity correspond to the non-leakage and the slip condition on the free surface \( \theta = \frac{\pi}{2} \) and give partly slip on the electrodes. Conditions B give us an opportunity to satisfy the non-leakage and the slip conditions on the free surface and approximate satisfy the adhesion condition on the electrodes.

4. Velocity field structure

For the velocity field components, we obtain the formulas:
\[ \nu_{r} = \frac{1}{r} \sum_{l=1}^{\infty} 2l(2l+1) G_{l}(r) P_{2l}^{(1)}(\cos \theta); \]
\[ \nu_{\theta} = -\frac{1}{r} \sum_{l=1}^{\infty} \frac{d}{dr} (r G_{l}(r)) P_{2l}^{(1)}(\cos \theta), \]

where \( P_{2l}^{(1)}(\cos \theta) \) - the Legendre polynomials [10], \( G_{l}(r) \) is described above.

The formulas for the senior mode for the type A boundary conditions for \( a = 0.1 \) are the following
\[ \left( \frac{\sqrt{S}}{2\pi^2} = 1 \right): \]
\[ \nu_{r} = (0.04017r^3 - 0.13393r + 0.1041(6)r^{-1} - 0.01041r^{-2} + 1.3 \cdot 10^{-5} r^{-4}) \frac{3\cos^2 \theta - 1}{2}, \]
\[ \nu_{\theta} = (-0.06808r^3 + 0.08543r - 0.01736(1)r^{-1} + 1.6 \cdot 10^{-5} r^{-4}) \frac{3\sin 2\theta}{2}. \]
The formulas for the senior mode for the type B boundary conditions for \( a = 0.1 \), \( h = 0.01 \) are the following:

\[
\begin{align*}
\nu_r &= (0.0817 r^3 - 0.1708 r + 0.1041(6) r^{-1} - 0.015045 r^{-2} + 4.8 \cdot 10^{-5} r^{-4}) \frac{3\cos^2 \theta - 1}{2}, \\
\nu_\theta &= (-0.06808 r^3 + 0.08543 r - 0.01736(1) r^{-1} + 1.6 \cdot 10^{-5} r^{-4}) \frac{3\sin 2\theta}{2}.
\end{align*}
\]

The formulas for the senior mode for the parameters \( a = 0.05 \), \( h = 0.0001 \) are:

\[
\begin{align*}
\nu_r &= (0.0815 r^3 - 0.1705 r + 0.1041(6) r^{-1} - 0.01520 r^{-2} + 5.0 \cdot 10^{-6} r^{-4}) \frac{3\cos^2 \theta - 1}{2}, \\
\nu_\theta &= (-0.067907 r^3 + 0.085251 r - 0.01736(1) r^{-1} + 1.65 \cdot 10^{-6} r^{-4}) \frac{3\sin 2\theta}{2}.
\end{align*}
\]

The formulas for the senior mode for the parameters \( a = 0.05 \), \( h = 0.0001 \) are:

\[
\begin{align*}
\nu_r &= (0.092552 r^3 - 0.188972 r + 0.1041(6) r^{-1} - 0.00775350 r^{-2} + 6.42 \cdot 10^{-6} r^{-4}) \frac{3\cos^2 \theta - 1}{2}, \\
\nu_\theta &= (-0.07712 r^3 + 0.094485 r - 0.01736(1) r^{-1} + 2.14 \cdot 10^{-6} r^{-4}) \frac{3\sin 2\theta}{2}.
\end{align*}
\]

Axial velocity for different condition is shown on the figure 2.

![Axial velocity](image)

Figure 2. Axial velocity along axis. 1 - \( a=0.1 \), Dirichlet boundary conditions; 2 - Thom’s boundary conditions, \( h=0.01 \); 3 - Thom’s boundary conditions, \( h=0.0001 \); 4 - Thom’s boundary conditions, \( a=0.05 \), \( h=0.0001 \).

### 5. Pressure field

To obtain the pressure field, we can obtain the gradient for it:

\[
\begin{align*}
\nabla p_r &= U(r, \theta) = -\frac{1}{r \sqrt{S}} \sum_{l=1}^{\infty} R_l(r) 2l(2l+1) P_{2l}(\cos \theta), \\
\nabla p_\theta &= V(r, \theta) = \frac{1-\cos \theta}{r^2 \sin \theta} + \frac{1}{r \sqrt{S}} \sum_{l=1}^{\infty} \frac{d}{dr} (r R_l(r)) P_{2l}(\cos \theta),
\end{align*}
\]

where \( R_l(r) \) is described above.
The pressure can be found by integrating:

\[
p(r, \theta) - p(a, 0) = \int_a^r U(\xi, \theta) d\xi + \int_0^\theta V(r, \zeta) d\zeta = \sum_{l=1}^\infty p_l(r) P_l(\cos \theta) - 2r^2 \ln \cos \frac{\theta}{2} + \frac{1}{\sqrt{S}} \sum_{l=1}^\infty \frac{d[R_l(r)]}{dr},
\]

where

\[
p_l(r) = \frac{1}{\sqrt{S}} \left( \frac{d[R_l(r)]}{dr} + 2l(2l+1) \int_a^r \xi^{-1} R_l(\xi) d\xi \right).
\]

If \( \sqrt{\frac{S}{2\pi^2}} = 1 \) the senior mode of pressure for the parameters \( a = 0.1, \ h = 0.01 \) are:

\[
\langle \bar{\nabla} p \rangle_r = (-1.143r + 6.25 \cdot 10^{-1} r^3 - 9.03 \cdot 10^{-2} r^4) \frac{3\cos^2 \theta - 1}{8\pi^2};
\]

\[
\langle \bar{\nabla} p \rangle_\theta = \frac{1 - \cos \theta}{r^3 \sin \theta} + (0.5719r^{-1} + 0.1042r^{-3} - 0.0301r^{-4}) \frac{3\sin 2\theta}{8\pi^2};
\]

\[
p_l(r) = -\frac{1}{4\pi^2} (0.5719r^{-1} - 0.0782r^{-3} - 0.0301r^{-4} + 0.1875r^{-2} + 3.423r^2 + 163.616).
\]

The formulas for the senior mode for the parameters \( a = 0.1, \ h = 0.0001 \) are:

\[
\langle \bar{\nabla} p \rangle_r = (-1.141r + 6.25 \cdot 10^{-1} r^3 - 9.12 \cdot 10^{-2} r^4) \frac{3\cos^2 \theta - 1}{8\pi^2};
\]

\[
\langle \bar{\nabla} p \rangle_\theta = \frac{1 - \cos \theta}{r^3 \sin \theta} + (0.5704r^{-1} + 0.1042r^{-3} - 0.0304r^{-4}) \frac{3\sin 2\theta}{8\pi^2}.
\]

Conclusions

For the problem of the electravortex flow in a hemispherical bowl, which was previously studied numerically and experimentally in a series of papers [8, 9], analytical expressions of the velocity and pressure fields obtained using the Stokes approximation [8] are presented. One of the main advantages is the extension of single series. Our results can be useful for elucidating the limits of applicability of the Stokes approximation in the problem under consideration. A qualitative study of the flow is already possible using only one senior mode; quantitative agreement with numerical calculations occurs when 4-5 modes are taken into account.

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