LOCAL BRUNELLA’S ALTERNATIVE I. RICH FOLIATIONS

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To Marco Brunella, in memoriam.

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Abstract. This paper is devoted to studying the structure of codimension one singular holomorphic foliations on \((\mathbb{C}^3, 0)\) without invariant germs of analytic surface. We focus on the so-called CH-foliations, that is, foliations without saddle nodes in two dimensional sections. Considering a reduction of singularities, we detect the possible existence of “nodal components”, which are a higher dimensional version of the nodal separators in dimension two. If the foliation is without nodal components, we prove that all the leaves in a neighborhood of the origin contain at least one germ of analytic curve at the origin. We also study the structure of nodal components for the case of “Relatively Isolated CH-foliations” and we show that they cut the dicritical components or they exit the origin through a non compact invariant curve. This allows us to give a precise statement of a local version of Brunella’s alternative: if we do not have an invariant surface, all the leaves contain a germ of analytic curve or it is possible to detect the nodal components in the generic points of the singular curves before doing the reduction of singularities.

1. Introduction

It is a question of M. Brunella to decide if the following alternative is true:

Let \(\mathcal{F}\) be a singular holomorphic foliation of codimension one in the projective space \(\mathbb{P}^n_\mathbb{C}\). If there is no projective algebraic surface invariant by \(\mathcal{F}\), then each leaf of \(\mathcal{F}\) is a union of algebraic curves.

The answer to this question is known [10] to be positive in the case of generic foliations in a pencil of foliations. For degree \(d = 0, 1, 2\) all the irreducible components of the space of foliations \(\mathcal{F}(3,d)\) are known but, for \(d \geq 3\), although several irreducible components have been recognized, it is not known if this list is exhaustive. What is known is that some irreducible components of \(\mathcal{F}(3,d)\) admit such pencils, hence the positive answer to the alternative in these cases.

This paper is the first one concerning a local version of the above alternative for complex hyperbolic foliations on \((\mathbb{C}^3, 0)\). As we state in Definition 4, a germ \(\mathcal{F}\) of singular holomorphic foliation of codimension one in \((\mathbb{C}^n, 0)\) is a complex hyperbolic foliation (for short, a CH-Foliation) if for every holomorphic map germ \(\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)\) generically transversal to \(\mathcal{F}\), the transformed foliation \(\phi^* \mathcal{F}\) is a generalized curve on \((\mathbb{C}^2, 0)\) in the sense of [2]: that is, there are no saddle nodes in its reduction of singularities. As there are dicritical CH-foliations without invariant surfaces, this phenomenon warns against the use of the terminology “generalized surface” for dicritical situations. In contrast with this, the authors in [16] use the term “generalized surface” in the non-dicritical case, since the reduction of singularities of the set of invariant surfaces provides a reduction of singularities for the foliation.

The first result we prove in this paper is the following one

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Theorem 1. Let $\mathcal{F}$ be a CH-foliation on $(C^3, 0)$ without germ of invariant surface. Assume that there is a reduction of singularities of $\mathcal{F}$ without nodal components. There is a neighborhood $U$ of the origin $0 \in C^3$ such that, for each leaf $L \subset U$ of $\mathcal{F}$ in $U$ there is a germ of analytic curve $\gamma$ at the origin such that $\gamma \subset L \cup \{0\}$.

We know by [4] that there is a reduction of singularities for any codimension one foliation $\mathcal{F}$ on $(C^3, 0)$. That is, there is a morphism

$$\pi : (M, \pi^{-1}(0)) \rightarrow (C^3, 0)$$

which is a composition of blow-ups with invariant centers that produces a normal crossings exceptional divisor $E \subset M$, in such a way that all the points $p \in \pi^{-1}(0)$ are simple points for the pair $\pi^*\mathcal{F}, E$. The simple points for CH-foliations are of a special type that we call simple CH-points (in dimension two this corresponds exactly to avoiding saddle nodes in the reduction of singularities as in [2]). A natural generalization of the “nodal separators” of Mattei and Marín [17] is given by our definition of nodal point for a codimension one foliation in any dimension; these are the points where the foliation is locally given, in local coordinates $x_1, x_2, \ldots, x_n$, by $\omega = 0$ where

$$\omega = \sum_{i=1}^{\tau} \lambda_i \frac{dx_i}{x_i}; \quad \lambda_i \in \mathbb{C}^*, i = 1, 2, \ldots, \tau$$

with $\lambda_i/\lambda_j \in \mathbb{R}$, for any $i, j$, and $\lambda_s/\lambda_j \in \mathbb{R}_{<0}$ for at least two indices $s, j$. It is known that the singular locus $\text{Sing}(\pi^*\mathcal{F})$ is a union of nonsingular curves. One such curve is generically nodal provided its generic point is a nodal point. A nodal component $\mathcal{N}$ of the pair $\pi^*\mathcal{F}, E$ is a connected component of the union of generically nodal curves such that all the points in $\mathcal{N}$ are nodal points (and not only the generic points of the curves).

A key remark for the understanding of germs of foliations without an invariant germ of surface is that they must be dicritical. In a general way, we say that $\mathcal{F}$ is dicritical if there is a holomorphic map germ

$$\phi : (C^2, 0) \rightarrow (C^3, 0); \quad (x, y) \mapsto \phi(x, y) = (\phi_1(x, y), \phi_2(x, y), \phi_3(x, y))$$

such that $\phi\{y = 0\}$ is invariant by $\mathcal{F}$ and the pullback $\phi^*\mathcal{F}$ coincides with the foliation $dx = 0$ in $(C^2, 0)$. In the paper [5] it is proved that any non dicritical foliation in $(C^3, 0)$ has an invariant germ of analytic hypersurface, this is also true in any ambient dimension $\mathbb{C}^n$. In fact, the arguments of [5] may be extended to the case where all compact components of the exceptional divisor are invariant; note that an irreducible component $D$ of $E$ is compact if and only if $D \subset \pi^{-1}(0)$. Thus, if $\mathcal{F}$ is without invariant surfaces, there is at least one compact component $D$ of $E$ that is generically transversal (dicritical component).

The main idea for Theorem [11] is that all the leaves of $\pi^*\mathcal{F}$ must intersect the union of compact dicritical components. At the intersection points we detect a germ of analytic curve contained in the leaf, which projects over the desired germ of analytic curve in $(C^3, 0)$. The obstruction to having this property is the possible existence of nodal components, that could “attract the leaves”.

The second result in this paper concerns the structure of the nodal components for a particular type of foliations that we call RICH-foliations. The idea is that we will be able to detect the possible existence of a nodal component $\mathcal{N}$ before doing the reduction of singularities, in the sense that $\mathcal{N}$ should project onto at least one of the curves $\Gamma \subset (C^3, 0)$ of the singular locus and the transversally generic behavior of $\Gamma$ is either dicritical or has a nodal separator in the sense of Mattei-Marín. To be precise, we prove the following result

Theorem 2. Let $\mathcal{F}$ be a RICH-foliation in $(C^3, 0)$. Assume that there is no germ of invariant analytic surface for $\mathcal{F}$. Then one of the two properties holds

(i) There is a neighborhood $U$ of the origin $0 \in C^3$ such that, for each leaf $L \subset U$ of $\mathcal{F}$ in $U$ there is an analytic curve $\gamma \subset L$ with $0 \in \gamma$.

(ii) There is an analytic curve $\Gamma$ contained in the singular locus $\text{Sing} \mathcal{F}$ such that, $\mathcal{F}$ is generically dicritical or it has a nodal separator along $\Gamma$.

Let us explain the concepts appearing in Theorem 2. The term RICH-foliation stands for Relatively Isolated Complex Hyperbolic Foliation. A germ $\mathcal{F}$ of singular holomorphic foliation of codimension one in $(C^3, 0)$ is a RICH-foliation if it is a CH-foliation and, furthermore, there is a reduction of singularities for $\mathcal{F}$

$$\pi : (M, \pi^{-1}(0)) \rightarrow (C^3, 0)$$

where $\pi$ is a composition of blow-ups $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_N$ such that for any index $0 \leq k \leq N - 1$ the blow-up $\pi_{k+1} : M_{k+1} \rightarrow M_k$ satisfies

- The center $Y_k \subset M_k$ of $\pi_{k+1}$ is non singular, has normal crossings with the total exceptional divisor $E^k \subset M_k$ and is contained in the adapted singular locus $\text{Sing}(\mathcal{F}_k, E^k)$, where $\mathcal{F}_k$ is the transform of $\mathcal{F}$.
- The intersection $Y_k \cap (\pi_1 \circ \pi_2 \circ \cdots \circ \pi_k)^{-1}(0)$ is a single point.

(The adapted singular locus $\text{Sing}(\mathcal{F}_k, E^k)$ is the locus of points where $\mathcal{F}_k$ is singular or it does not have normal crossings with $E^k$, in particular $Y_k$ is invariant by $\mathcal{F}_k$. For more details, see [5, 4]). This kind of reduction of singularities will be called a RI-reduction of singularities of $\mathcal{F}$.}
The condition “relatively isolated” is less restrictive than “absolutely isolated”. It contains as examples the case of equireduction along a curve and the foliations of the type $df = 0$ where $f = 0$ defines a germ of surface with absolutely isolated singularity. There are also examples without invariant surface, for instance the classical conic foliation given by Jouanolou [13]. Absolutely isolated singularities of vector fields have been studied in [1], whereas for codimension one foliations on $(\mathbb{C}^3, 0)$ the singular locus has codimension two unless we have a holomorphic first integral [15]. Also, in the paper [7] the authors consider foliations desingularized essentially by punctual blow-ups, which is a condition stronger than being relatively isolated.

Following [17], we say that a germ of foliation $\mathcal{G}$ on $(\mathbb{C}^2, 0)$ contains a nodal separator if, in the reduction of singularities, there is a singularity analytically equivalent to $xy - \lambda ydx = 0$ were $\lambda$ is a non rational positive real number. Now, take a germ of curve $\Gamma$ contained in the singular locus of a foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$. We say that $\mathcal{F}$ is generically dicritical along $\Gamma$ if it is dicritical at a generic point of $\Gamma$. We can verify this fact at the equireduction points of $\Gamma$ (see [4]). If $\mathcal{F}$ is not generically dicritical along $\Gamma$, it is known [4] that the equireduction along $\Gamma$ is given by the (non-dicritical) reduction of singularities of the restriction $\mathcal{G}$ of $\mathcal{F}$ to a plane section transversal to $\Gamma$ at a generic point. In this case, we say that $\mathcal{F}$ has a nodal separator along $\Gamma$ if this is true for such plane transversal sections $\mathcal{G}$.

Finally, the condition (ii) of Theorem 2 is equivalent to the fact that any nodal component intersects the dicritical components or it contains a non compact curve. To be precise, Theorem 2 is a consequence of Theorem 1 and the following result of structure for the nodal components.

**Theorem 3.** Let $\mathcal{F}$ be a RICH-foliation in $(\mathbb{C}^3, 0)$ and let $\pi : (M, \pi^{-1}(0)) \to (\mathbb{C}^3, 0)$ be an RI-reduction of singularities for $\mathcal{F}$ with total exceptional divisor $E \subset M$. Any compact nodal component $\mathcal{N}$ of $\pi^* \mathcal{F}, E$ intersects the union of the dicritical components of $E$.

It is an open question if the analogous of Theorem 3 is true for CH-foliations.

In some sense the global alternative of Brunella may be interpreted as a property concerning the “concentration-diffusion” of the non-transcendency of the leaves of a foliation: either we concentrate the non-transcendency in an algebraic leaf, or all the leaves are not completely transcendent in the sense that they are foliated by algebraic curves. In our local situation we have an analogous of this phenomenon based on the concept of “end of a leaf”. In a forthcoming paper we will study the ends of the leaves for CH-foliations without invariant surface. All these ends will be “semi-transcendental” in the sense that, either they contain an analytic curve, or they are of a “valuative type” that admits bifurcation at all the accumulation points after blow-up. Moreover, the leaves in a neighborhood will have at least one end and in this sense we can reformulate a local version of Brunella’s alternative by saying that, either we have an invariant germ of surface, or there is a neighborhood of the origin such that all the leaves are “semi-transcendental”.

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2. **Nodal and Saddle Simple Complex Hyperbolic Points**

We introduce here the simple complex hyperbolic points, which are the higher-dimensional version of the simple singularities in the sense of Seidenberg (see [6] [21]) given by vector fields with two non-null eigenvalues.

Let $\mathcal{F}$ be a germ of singular holomorphic foliation of codimension one on $(\mathbb{C}^n, 0)$. We say that $\mathcal{F}$ has *dimensional type $\tau$ at the origin* if there is a submersion

$$\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^\tau, 0)$$

and a codimension one foliation $\mathcal{G}$ on $(\mathbb{C}^\tau, 0)$ such that $\mathcal{F} = \phi^* \mathcal{G}$ and moreover there is no such submersion $(\mathbb{C}^n, 0) \to (\mathbb{C}^{\tau-1}, 0)$. In other words, there are local coordinates $x_1, x_2, \ldots, x_n$ at the origin $0 \in \mathbb{C}^n$ and an integrable 1-form $\omega$ such that $\mathcal{F}$ is given by $\omega = 0$, where $\omega$ can be written down as follows

$$\omega = \sum_{i=1}^{\tau} a_i(x_1, x_2, \ldots, x_\tau) dx_i$$

and $\tau$ is the minimum integer with this property. We have that $\tau = 1$ if and only if $\mathcal{F}$ is non-singular. Note also that if there are $k$ germs of non-singular vector fields $\xi_1, \xi_2, \ldots, \xi_k$ tangent to $\mathcal{F}$ such that $\xi_1(0), \xi_2(0), \ldots, \xi_k(0)$ are $C$-linearly independent tangent vectors, then $\tau \leq n-k$ and conversely.

**Definition 1** ([4] [5]). Let $\mathcal{F}$ be a germ of codimension one singular holomorphic foliation on $(\mathbb{C}^n, 0)$ of dimensional type $\tau$. We say that $\mathcal{F}$ is a simple complex hyperbolic point at the origin if and only there are local coordinates $x_1, x_2, \ldots, x_n$ and a meromorphic integrable 1-form $\omega$ such that $\mathcal{F}$ is given by $\omega = 0$ and $\omega$ can be written down as follows

$$\omega = \sum_{i=1}^{\tau} (\lambda_i + b_i(x_1, x_2, \ldots, x_\tau)) \frac{dx_i}{x_i}, \quad b_i \in \mathbb{C}\{x_1, x_2, \ldots, x_\tau\}, \quad b_i(0) = 0$$
where the residual vector \( \lambda = (\lambda_i)_{i=1}^n \in \mathbb{C}^r \) satisfies the non-resonance property:

“For any \( m = (m_i)_{i=1}^n \in \mathbb{Z}^n_0 \) we have \( \sum_{i=1}^n m_i \lambda_i \neq 0 \) if \( m \neq 0 \).”

Let \( E \subset (\mathbb{C}^n, 0) \) be a normal crossings divisor. We decompose \( E = E_{\text{inv}} \cup E_{\text{dic}} \), where \( E_{\text{inv}} \) is the union of the irreducible components of \( E \) invariant by \( \mathcal{F} \) and \( E_{\text{dic}} \) is the union of those that are generically transversal to \( \mathcal{F} \) (dicritical components). The origin is a simple complex hyperbolic point for \( \mathcal{F} \) adapted to \( E \) if it is a simple complex hyperbolic point for \( \mathcal{F} \) and the coordinates in Definition 1 may be chosen in such a way that

\[
(\prod_{i=1}^{\tau-1} x_i = 0) \subset E_{\text{inv}} \subset (\prod_{i=1}^{\tau} x_i = 0); \quad E_{\text{dic}} \subset (\prod_{i=\tau+1}^n x_i = 0).
\]

We adopt the following terminology:

- If \( E_{\text{inv}} = (\prod_{i=1}^{\tau-1} x_i = 0) \), we have a simple complex hyperbolic corner.
- If \( E_{\text{inv}} = (\prod_{i=1}^{\tau} x_i = 0) \), we have a simple complex hyperbolic trace point.

Notation 1. We denote simple CH-point, simple CH-corner or simple CH-trace point the above types of points.

**Remark 1.** When the origin is a simple CH-point as in Definition 1 it is known (\cite{5}) that the coordinate hyperplanes \( x_i = 0 \), where \( i = 1, 2, \ldots, \tau \) are the only invariant hypersurfaces of \( \mathcal{F} \). The singular locus \( \text{Sing} \mathcal{F} \) is given by

\[
\text{Sing} \mathcal{F} = \cup_{1 \leq i < j \leq \tau} (x_i = x_j = 0).
\]

**Remark 2.** [Formal normal forms] In the paper \cite{3} it is shown that there are formal coordinates \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n \) such that \( \mathcal{F} \) is given at a CH-simple point by an integrable formal 1-form \( \hat{\omega} \) of one of the following types:

1. \( \hat{\omega} = \sum_{i=1}^{\tau} \lambda_i (d\hat{x}_i/\hat{x}_i), \) (\( \lambda \) non resonant).
2. \( \hat{\omega} = \sum_{i=1}^{\tau} p_i (d\hat{x}_i/\hat{x}_i) + \psi (\hat{x}_1^2 \hat{x}_2^s \cdots \hat{x}_n^t), \) where \( \psi(0) = 0. \)

**Definition 2.** We say that a vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\tau) \in \mathbb{C}^\tau \) is of saddle type if it is of one of the following types

- Complex-saddle case: There are two indices \( i, j \) such that \( \lambda_i/\lambda_j \notin \mathbb{R}. \)
- Real-saddle case: \( \lambda_i/\lambda_j \in \mathbb{R}_{>0} \), for any \( i, j. \)

Otherwise we say that \( \lambda \) is of nodal type, that is \( \lambda_i/\lambda_j \in \mathbb{R} \), for any \( i, j \) and there are two indices \( s, j \) such that \( \lambda_s/\lambda_j \in \mathbb{R}_{<0}. \)

**Definition 3.** Let \( \mathcal{F} \) be a germ of codimension one foliation in \( (\mathbb{C}^n, 0) \) of dimensional type \( \tau \) having a simple CH-point at the origin. The origin is of saddle type (complex or real saddle), respectively of nodal type if the residual vector \( \lambda \) is so.

**Remark 3.** By a result of Cerveau-Lins Neto \cite{11} (see also \cite{12}), we know that nodal singularities may be normalized in a convergent way. That is, if the residual vector is of nodal type, there are local coordinates \( x_1, x_2, \ldots, x_n \) around the origin such that \( \mathcal{F} \) is given by \( \omega = 0 \) where

\[
\omega = \sum_{i=1}^{k} r_i \frac{dx_i}{x_i} - \sum_{i=k+1}^{\tau} r_i \frac{dx_i}{x_i}; \quad r_i \in \mathbb{R}_{>0}, \ 1 \leq k < \tau.
\]

Note that the multi-valuated function \( x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k} x_{k+1}^{r_{k+1}} x_{k+2}^{r_{k+2}} \cdots x_\tau^{r_\tau} \) is a first integral of the foliation.

**Remark 4.** Complex-saddle singularities may also be normalized in a convergent way as a consequence of the results in \cite{11} and thus they are expressed in convergent coordinates as \( \omega = 0 \) where

\[
\omega = \sum_{i=1}^{\tau} \lambda_i \frac{dx_i}{x_i},
\]

and there are two indices \( i, j \) such that \( \lambda_i/\lambda_j \notin \mathbb{R}. \) On the other hand, real-saddle singularities are not necessarily given by a 1-form expressed in convergent coordinates as in Equation 2. This is due to two possible facts: the existence of “small denominators” or a formal normal for of type (2) as in Remark 2

3. Leaves Around a Saddle Point

In this section we give a description of the behavior of the leaves of a foliation \( \mathcal{F} \) that has a simple CH-point at the origin of \( \mathbb{C}^n \) of saddle type and which is of dimensional type \( \tau. \) More precisely, we are interested in the saturation by \( \mathcal{F} \) of small transversal curves to the coordinate hyperplanes \( x_i = 0, \) for \( i = 1, 2, \ldots, \tau. \) In dimension two, computations of this nature may be found in \cite{17}.
Lemma 1. Let
\[ \omega = \sum_{i=1}^{r} (\lambda_i + b_i(x_1, x_2, \ldots, x_r)) \frac{dx_i}{x_i}, \quad b_i(0) = 0, |b_i| < |\lambda_i|. \]
Denote \( E_0 = (x_i = 0), \ E = (\bigcap_{i=1}^{r} x_i = 0) \) and \( E_0^\tau = E \setminus \bigcup_{i=1, i \neq j} E_i \). Consider a small nonsingular curve \( \Delta \) transversal to \( E \) at a point \( Q \in E_0^\tau \). We are interested in the saturation \( \text{Sat}_{F,U}(\Delta) \) of \( \Delta \) by the leaves of \( F \) in \( U \).

More precisely, this section is devoted to giving a proof of the following result

**Proposition 1.** If the origin of \((\mathbb{C}^n, 0)\) is a simple CH-point of saddle type for \( F \), then \( \text{Sat}_{F,U}(\Delta) \cup E \) is a neighborhood of the origin, where \( \Delta \) is a small curve transversal to \( E \).

**Remark 5.** The situation in the case of a nodal type point is different from the one described in Proposition 1. Let \( F \) be given by \( \omega = 0 \) as in Equation 1. For any positive constant \( C \in \mathbb{R}_{>0} \), the sets
\[ S_C = \{(z_1, z_2, \ldots, z_n); \frac{|z_1|^{r_1}|z_2|^{r_2} \cdots |z_n|^{r_n}}{|z_{k+1}|^{r_{k+1}}|z_{k+2}|^{r_{k+2}} \cdots |z_r|^{r_r}} = C \} \]
are invariant sets for \( F \). If we take \( t \leq k \) the curve \( \Delta \) cuts only the sets \( S_C \) with \( 0 < C < c \), for some \( c \). Noting that \( S_{t+1} \) is adherent to the origin, we see that \( \text{Sat}_{F,U}(\Delta) \cup E \) is not a neighborhood of the origin. See [17] for a description of this situation in dimension two.

Let us make another remark for nodal singularities of dimensional type three

**Remark 6.** Consider the foliation \( F \) defined in \( \mathbb{B}_1 \subset \mathbb{C}^1 \) by \( \omega = 0 \) with
\[ \omega = \frac{dx}{x} - \frac{dy}{y} - \frac{dz}{z}, \]
where \( \lambda, \mu \in \mathbb{R}_{>0} \). Up to a local coordinate change, all the nodal singularities of dimensional type three are of this form. Note that there are exactly two curves \( x = y = 0 \) and \( x = z = 0 \) formed by nodal points and the curve \( y = z = 0 \) has a generic point that is of real-saddle type. Moreover, by a direct integration the saturation of a small neighborhood of any point in \((x = y = 0) \cup (x = z = 0)\) is a neighborhood of \( xyz = 0 \).

Proposition 1 comes by induction on the dimension and standard computations of holonomy as in [19] or [14]. Before starting the proof, let us precise the notations. For any \( 1 \leq i, j \leq r \), write \( \lambda_{ij} = \lambda_i/\lambda_j \) and \( (\lambda_i + b_i)/(\lambda_j + b_j) = \lambda_{ij} + f_{ij} \).

Then \( F \) is defined in \( U \) by \( \omega_j = 0 \) where
\[ \omega_j = \frac{1}{\lambda_j + b_j} \omega = \sum_{i=1}^{r} (\lambda_{ij} + f_{ij}) \frac{dx_i}{x_i}. \]
Note that \( \lambda_{ij} = 1, f_{ij} = 0 \) and \( f_{ij}(0) = 0 \). By taking \( \delta \) small enough, we assume that the following two properties are satisfied
\[
(*) \text{ If we are in the complex-saddle case, then } b_i = 0, \text{ for all } i.
\]
\[
(**) \text{ If we are in the real-saddle case, there is } \rho < 0 \text{ such that } |f_{ij}| + \rho < \lambda_{ij} \text{ for all } i, j.
\]
Take \( \ell \in \{1, 2, \ldots, r\} \) and \( \mu : \{1, 2, \ldots, n\} \rightarrow \mathbb{B}_\delta \) such that \( \mu(t) = 0 \) and \( \mu_i \neq 0 \) for \( i \in \{1, 2, \ldots, r\} \setminus \{\ell\} \). Denote by \( Q_\mu \) the point defined by \( x_\ell(Q_\mu) = \mu_\ell. \) Given a radius \( 0 < \epsilon \leq \delta \), we consider the curve \( \Delta_\ell(\mu; \epsilon) \) over \( Q_\mu \) defined by
\[ \Delta_\ell(\mu; \epsilon) = \{(x_1, x_2, \ldots, x_n); x_i = \mu_i, \text{ for } i \neq \ell, 0 \leq |x_\ell| < \epsilon \}. \]
Now, we reformulate Proposition 1 as follows
\[
\text{"Sat}_{F,U}(\Delta_\ell(\mu; \epsilon) \cap \bigcap_{i=1}^r x_i = 0) \text{ is a neighborhood of the origin."
}

We start the proof of Proposition 1 with the case that the origin is of complex-saddle type. That is, we have \( \omega_j = \sum_{i=1}^{r} \lambda_{ij} dx_i/x_i, \) where there is some \( \lambda_{ij} \notin \mathbb{R}. \)

**Lemma 1.** Let \( \Delta = (\lambda_1, \lambda_2, \ldots, \lambda_r) \subset \mathbb{C}^r \) be a vector of complex-saddle type and assume that \( r \geq 3 \). There are two indices \( u,v, u \neq v \), such that the vectors
\[ \Delta^u = (\lambda_1, \lambda_2, \ldots, \lambda_{u-1}, \lambda_{u+1}, \ldots, \lambda_r), \quad \Delta^v = (\lambda_1, \lambda_2, \ldots, \lambda_{v-1}, \lambda_{v+1}, \ldots, \lambda_r) \]
are of complex-saddle type.

**Proof.** Let \( L_s, s = 1, 2, \ldots, k \) be the real rays (half real lines starting at the origin of \( \mathbb{C} \)) that contain all the \( \lambda_s, s = 1, 2, \ldots, r \). We know that \( k \geq 2 \). If \( k \geq 3 \), we take three distinct rays \( L_1, L_2, L_3 \) such that \( L_1 \) is not aligned (opposite ray to) with \( L_2 \) or \( L_3 \) and we consider \( \lambda_u \in L_1, \lambda_v \in L_2 \); then \( L_2 \) and \( L_3 \) are rays for \( \Delta^u \) and \( L_1, L_3 \) are rays for \( \Delta^v \). If \( k = 2 \), there are exactly two rays \( L_1, L_2 \) that are not opposite; one of these, say \( L_1 \) has at least two \( \lambda_u, \lambda_v \in L_1; \) now, \( L_1 \) and \( L_2 \) are still rays for \( \Delta^u \) and for \( \Delta^v \).
Now, Proposition 4 for complex-saddles is a consequence of Lemma 2.

**Lemma 2.** Assume that \( \mathcal{F} \) has a simple CH-point at the origin of \( \mathbb{C}^n \) of complex-saddle type. We have \( \text{Sat}_{\mathcal{F}, \mathcal{U}} \Delta_{\ell}(\mu; \epsilon) \cup (\prod_{i=1}^{\ell} x_i = 0) = \mathbb{D}_3^\alpha \).

**Proof.** Given an index \( u \in \{1, 2, \ldots, \tau\} \) and \( \alpha \in \mathbb{D}_3^\tau \), we consider the hyperplane

\[
\Sigma^u_\alpha = \{ x_u = \alpha \} \cap \mathbb{D}_3^\alpha.
\]

The section \( \mathcal{F}^u_{\alpha} \) of \( \mathcal{F} \) by \( \Sigma^u_\alpha \) has a CH-simple singularity at the points \( Q_\nu \), where \( \nu : \{1, 2, \ldots, n\} \to \mathbb{D}_3 \) is such that \( \nu_u = \alpha \) and \( \nu_i = 0 \) for \( i \in \{1, 2, \ldots, \tau\} \setminus \{u\} \). Moreover, \( \mathcal{F}^u_{\alpha} \) is locally given at \( Q_\nu \) by \( \omega|_{\Sigma^u_\alpha} = 0 \) where

\[
\omega|_{\Sigma^u_\alpha} = \sum_{i \in \{1, 2, \ldots, \tau\} \setminus \{u\}} \lambda_i \frac{dy_i}{y_i}; \ y_i = x_i|_{\Sigma^u_\alpha}.
\]

In particular, the residual vector is \( \Delta^u \) defined as in Lemma 4.

Let us do the induction step. Assume that the result is true for dimensional type \( \tau' \) with \( 2 \leq \tau' < \tau \). We have to show that it is true for dimensional type \( \tau \). Choose indices \( u, v \) as in Lemma 4. We reduce first the problem to the case where \( \ell = u, v \). Applying induction hypothesis to the section \( \mathcal{F}^u_{\alpha} \), we deduce that \( \Sigma^\mu_\alpha \) is contained in the saturation of \( \Delta_{\ell}(\mu; \epsilon) \). Take now \( \ell' \in \{1, 2, \ldots, \tau\} \setminus \{u, v\} \) and \( \mu' \) defined by \( \mu'_i = \mu_i \), for \( i \in \{1, 2, \ldots, n\} \setminus \{\ell', u\} \) and \( \mu'_u = 0 \). We have that \( \Delta_{\ell'}(\mu'; \epsilon) \subset \Sigma^\mu_\alpha \). The saturation of \( \Delta_{\ell'}(\mu'; \epsilon) \) is then contained in the saturation of \( \Delta_{\ell}(\mu; \epsilon) \) and we are done. Thus we assume that \( \ell \neq u, v \). Consider \( \mathcal{F}^u_{\alpha} \), applying induction hypothesis, we obtain that \( \Sigma^\mu_\alpha \) is contained in \( \text{Sat}_{\mathcal{F}, \mathcal{U}} \Delta_{\ell}(\mu; \epsilon) \). Now, for any \( \alpha \in \mathbb{D}_3^\tau \) we have that

\[
\Delta_{\ell}(\mu'(\alpha); \epsilon) \subset \Sigma^\mu_\alpha,
\]

where \( \mu'_i(\alpha) = \mu_i \), for \( i \in \{1, 2, \ldots, \tau\} \setminus \{v\} \) and \( \mu_u(\alpha) = \alpha \). Now, applying induction to \( \mathcal{F}^u_{\alpha} \) we obtain that \( \text{Sat}_{\mathcal{F}, \mathcal{U}} \Delta_{\ell}(\mu'(\alpha); \epsilon) \supset \Sigma^\mu_\alpha \) and taking the union over all the \( \alpha \), we have

\[
\text{Sat}_{\mathcal{F}, \mathcal{U}} \Delta_{\ell}(\mu; \epsilon) \supset \text{Sat}_{\mathcal{F}, \mathcal{U}} \Sigma^\mu_\alpha \supset \bigcup_{\alpha \in \mathbb{D}_3^\tau} \text{Sat}_{\mathcal{F}, \mathcal{U}} \Delta_{\ell}(\mu'(\alpha); \epsilon) \supset \bigcup_{\alpha \in \mathbb{D}_3^\tau} \Sigma^\mu_\alpha \supset \mathbb{D}_3^\tau.
\]

This ends the induction step.

We end the proof by considering the case \( \tau = 2 \). It is enough to consider the case \( \tau = n = 2 \) where

\[
\omega = \frac{dx}{x} + \lambda \frac{dy}{y}; \ \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

We assume also that \( \ell = 2 \) and \( \mu_1 = \alpha \in \mathbb{D}_3^\tau \). Thus we take

\[
\Delta(\alpha; \epsilon) = \{(x, y); x = \alpha, 0 < |y| < \epsilon\}
\]

and we have to show that \( \text{Sat}_{\mathcal{F}, \mathcal{U}} \Delta(\alpha; \epsilon) \supset \mathbb{D}_3^\tau \). Take another \( \alpha' \in \mathbb{D}_3^\tau \). We can connect the points \( (\alpha, 0) \) and \( (\alpha', 0) \) by a path contained in \( \mathbb{D}_3 \times \{0\} \). By doing the holonomy along this path, we deduce that there is \( 0 < \epsilon' \leq \delta \) such that \( \Delta(\alpha'; \epsilon') \) is contained in \( \text{Sat}_{\mathcal{F}, \mathcal{U}} \Delta(\alpha; \epsilon) \). Then it is enough to prove that

\[
\text{Sat}_{\mathcal{F}, \mathcal{U}} \Delta(\alpha; \epsilon) \supset \Delta(\alpha; \delta),
\]

since we would have

\[
\text{Sat}_{\mathcal{F}, \mathcal{U}} \Delta(\alpha; \epsilon) \supset \bigcup_{\alpha' \in \mathbb{D}_3^\tau} \text{Sat}_{\mathcal{F}, \mathcal{U}} \Delta(\alpha'; \epsilon') \supset \bigcup_{\alpha' \in \mathbb{D}_3^\tau} \Delta(\alpha'; \delta) = \mathbb{D}_3^\tau.
\]

Let us show that (4) holds. To see this, we compute the holonomy at \( (\alpha, 0) \) with respect to the loop \( \sigma(t) = (\alpha \exp(it), 0) \) starting at a point \( (\alpha, \beta) \). Let \( \gamma(t) \) be the lifted path, where \( \gamma(t) = (\alpha \exp(it), g(t)) \), \( t \in \mathbb{R} \). The condition \( \omega(\gamma'(t)) = 0 \) means that \( g'(t) = -i/\lambda g(t) \). Then \( g(t) \) is explicitly given by

\[
g(t) = \beta \exp(-it/\lambda) = \beta \exp(-\text{Im}(\lambda)/|\lambda|^2 t) \exp(-i \text{Re}(\lambda) t/|\lambda|^2).
\]

Since \( \text{Im}(\lambda) \neq 0 \) we have a contraction or an expansion. If it is a contraction, we take the positive time to reach \( \Delta(\alpha; \epsilon) \) from \( \Delta(\alpha; \delta) \); if it is an expansion, taking the negative time we have a contraction. \( \square \)

We give now a proof of Proposition 4 in the case we have a real-saddle point. We do it by means of several lemmas. Proposition 4 for real-saddles is a direct consequence of Lemma 5.

**Lemma 3.** Take two elements \( i, j \in \{1, 2, \ldots, \tau\} \) and a point \( Q \in \{x_i = x_j = 0\} \), then \( f_{ij}(Q) = 0 \).
Proof. It is a direct consequence of the integrability property. For any \( s = 1, 2, \ldots, n \), the integrability property \( \omega_j \wedge d\omega_j = 0 \) implies that

\[
0 = A_s \left( x_i \frac{\partial A_j}{x_j} - x_j \frac{\partial A_i}{x_i} \right) - A_i \left( x_j \frac{\partial A_s}{x_s} - x_s \frac{\partial A_j}{x_j} \right) + A_j \left( x_s \frac{\partial A_i}{x_i} - x_i \frac{\partial A_s}{x_s} \right)
\]

where \( A_s = (g_{ij} + f_{ij}) \). Let \( \bar{f}_{ij} \) be the restriction of \( f_{ij} \) to \( x_i = x_j = 0 \). By applying the integrability property along \( x_i = x_j = 0 \), we find that

\[
\frac{\partial \bar{f}_{ij}}{\partial x_s} = 0, \quad s \neq i, j.
\]

In particular the value \( f_{ij}(Q) \) does not depend on the point \( Q \in \{ x_i = x_j = 0 \} \) and thus \( f_{ij}(Q) = 0. \)

The above Lemma 3 is useful for an argument of induction on \( \tau \). More precisely, if we take \( \Sigma = \Sigma^\alpha \) as in Equation 3, the section \( F^u_\mu \) gives also a real-saddle at any point \( Q \) in \( x_u = \mu_u, x_i = 0, i \in \{ 1, 2, \ldots, \tau \} \setminus \{ u \} \), given by

\[
\omega_j|_{\Sigma} = \frac{dy_j}{y_j} + \sum_{i \notin \{ j, u \}} (\lambda_{ij} + f_{ij}|_{\Sigma}) \frac{dy_i}{y_i}; \quad y_i = x_i|_{\Sigma}, y_j = x_j|_{\Sigma}; \quad i, j \neq u.
\]

Moreover, the property (***) is satisfied at \( Q \), with same \( \rho \) as for \( F \).

Lemma 4. Assume that \( F \) is given at the origin of \( \mathbb{C}^2 \) by \( \omega = 0 \), where \( \omega \) is the 1-form \( \omega = dx/x + (\lambda + f)dy/y \) defined in \( U = \mathbb{D}^2_\delta \) such that \( \lambda \in \mathbb{R}_{>0} \) with \( |f| + |\rho| < \lambda \). Take \( 0 < \epsilon < \delta \), a complex number \( \alpha \in \mathbb{D}_\delta \setminus \mathbb{D}_\delta/2 \) and consider the curve \( \Delta(\alpha; \epsilon) \) as in \( (\mathbb{3}) \). There is a constant \( 0 < c < \delta \), depending only on \( \epsilon, \lambda, \rho \) and \( \delta \) such that \( \text{Sat}_{\mathbb{U}}(\Delta(\alpha; \epsilon)) \subseteq \mathbb{D}_\delta \times \mathbb{D}_c^\epsilon \).

Proof. Consider a point \((\alpha', \beta') \in (\mathbb{D}^+)^2\). We will show that if \(|\beta'| < c\), then the holonomy over a path \( \sigma(t) \) allows us to reach \( \Delta(\epsilon; \alpha) \) from \((\alpha', \beta') \). We consider two particular cases, the general situation is a combination of both:

First case: \( \alpha = \alpha' \exp(\i \theta) \), where \( 0 \leq \theta < 2\pi \). Consider the path \( \sigma(t) = (\alpha' \exp(\i t), 0), 0 \leq t < 2\pi \) and let \( \gamma(t) = (\alpha' \exp(\i t), \gamma(t)) \) be the lifted path such that \( \gamma(0) = (\alpha', \beta') \). The condition \( \omega(\gamma(t)) = 0 \) gives that \( g'(t) = -i\gamma(t)/(\lambda + f) \) and if \( F(t) = g(t)\overline{\gamma(t)} \), we have that \( F'(t) = -(2\overline{\epsilon}(\lambda + f)/|\lambda + f|^2)F(t) \).

Note that

\[
\frac{2\overline{\epsilon}(\lambda + f)}{|\lambda + f|^2} \leq \frac{2}{|\lambda + f|} < 2/\rho.
\]

We conclude that \( F(t) \leq |\beta'|^2 \exp(2t/\rho) \), for \( 0 \leq t < 2\pi \). Put \( \epsilon = \epsilon \exp(-2\pi/\rho) \), then \((\alpha', \beta') \in \Delta(\alpha'; \epsilon) \) is in the saturation of \( \Delta(\alpha; \epsilon) \).

Second case: \( \alpha' = \rho \alpha \), for a positive real number \( 0 < \rho < 2 \). Take the path \( \phi(t) = (\rho \alpha, 0), 0 < t < 2 \) and denote \( \psi(t) = (\rho \alpha, u(t)) \) the lifted path, such that \( \psi(r) = (\alpha', \beta') \). The condition \( \omega(\psi(t)) = 0 \) gives that \( u'(t) = -u(t)/\rho(\lambda + f) \). Put \( U(t) = u(t)|_{\Sigma}(t) \). We have that

\[
U'(t) = -\frac{2\text{Re}(\lambda + f)}{|\lambda + f|^2} \frac{U(t)}{t}.
\]

Let us note that

\[
-\frac{2\text{Re}(\lambda + f)}{|\lambda + f|^2} < -\frac{2\rho}{|\lambda + f|^2} < -2\rho/|\lambda + f| < 0.
\]

Moreover, \( 1/t > 1/2 \) if \( 0 < t < 2 \). Consider the case \( r < 1 \), hence the function \( U(t), r \leq t \leq 1 \) satisfies \( U(t) < V(t) \) where \( V(t) \) is a solution of \( V'(t) = -\rho/(\lambda + f) \), with \( U(r) = V(r) = |\beta'|^2 \). That is, we have

\[
U(t) \leq |\beta'|^2 \exp(-\rho/(\lambda + f)(1-t)).
\]

We deduce that \( U(1) \geq |\beta'|^2 \exp(-\rho/(\lambda + f)(1-r)) \leq |\beta'|^2 \). In this case, if we take \( \epsilon' = \epsilon \), we obtain that \( \Delta(\alpha'; \epsilon) \) is in the saturation of \( \Delta(\alpha; \epsilon) \).

Consider the case \( 1 < r < 2 \). Put \( t(s) = r + s(1-r) \) and let us define \( \tilde{\phi}(s) = \phi(t(s)), \tilde{\psi}(s) = \psi(t(s)), \tilde{u}(s) = u(t(s)), \tilde{U}(s) = U(t(s)) \). We have

\[
\tilde{U}'(s) = (r-1)\frac{2\text{Re}(\lambda + f)}{|\lambda + f|^2} \tilde{U}(s) \quad \text{and} \quad 0 < (r-1)\frac{2\text{Re}(\lambda + f)}{|\lambda + f|^2} \leq \frac{2(\lambda + f)}{\rho^2}.
\]

Hence the function \( \tilde{U}(s), 0 \leq s \leq 1 \) satisfies \( \tilde{U}(s) < \tilde{V}(s) \) where \( \tilde{V}(s) \) is a solution of \( \tilde{V}'(t) = (2(\lambda + f)/\rho^2) \tilde{V}(t) \), with \( \tilde{U}(0) = \tilde{V}(0) = |\beta'|^2 \). That is, we have

\[
\tilde{U}(s) \leq |\beta'|^2 \exp((2(\lambda + f)/\rho^2)s).
\]

We deduce that \( \tilde{U}(1) \leq |\beta'|^2 \exp(2(\lambda + f)/\rho^2) \). If we take \( \epsilon' = \epsilon \exp(-2(\lambda + f)/\rho^2) \), we obtain that \( \Delta(\alpha'; \epsilon') \) is in the saturation of \( \Delta(\alpha; \epsilon) \).
Combining the two situations above, we can go in a holonomic way from $\Delta(\alpha'; e')$ to $\Delta(\alpha''; e'')$, where $\alpha'' = \alpha \exp(i\theta)$ is such that $e' = re''$ and from $\Delta(\alpha'; e')$ to $\Delta(\alpha; e)$. If we take $e' = e'' \exp(-2(\lambda + \rho)/\rho^2)$ and $e'' = \epsilon \exp(-2(\lambda + \rho)/\rho^2)$, we obtain that $\Delta(\alpha'; e')$ is in the saturation of $\Delta(\alpha; e)$. Hence it is enough to select the constant $c = \epsilon \exp(-2(\tau + 1)\rho + \lambda)/\rho^2$.

\[\Box\]

**Lemma 5.** There is a constant $0 < c < \delta$ depending only on $\lambda$, $\rho$, $\epsilon$ and $\delta$ such that if $\delta/2 < |\mu_i| < \delta$ for all $i \neq \ell$, we have $\text{Sat}_\mathcal{F}_\mathcal{U} \Delta_i(\mu; \epsilon) \cap \{(x_1, x_2, \ldots, x_n); 0 < |x_i| < \delta, \text{ for } i \neq \ell \text{ and } 0 < |x_\ell| < c\}$.

**Proof.** We proceed by induction on $\tau$. The case $\tau = 2$ is given by Lemma 5. Assume that $\tau \geq 3$ and take two indices $u, v \neq \ell$. We consider $\Sigma_{\mu_u}$ as in Equation 3. By induction hypothesis applied to the section $\mathcal{F}_{\mu_u}$, we have that $\text{Sat}_\mathcal{F}_\mathcal{U} \Delta_i(\mu; \epsilon) \cap \{0 < |x_i| < \delta, \text{ for } i \neq \ell \text{ and } 0 < |x_\ell| < c\}$ for a certain $0 < e' < \delta$. Take an $\alpha \in \mathbb{D}^*_\delta$, we have that $\Delta_i(\mu'_{\epsilon}; e')$ is contained in $\text{Sat}_\mathcal{F}_\mathcal{U} \Delta_i(\mu; \epsilon)$, where $\mu'_\epsilon = \mu_i$, $i \neq v$ and $\mu'_\ell = \alpha$. Now, we apply induction to $\mathcal{F}_\alpha$ to conclude that there is a constant $c$ such that

$$\bigcup_{\alpha \in \mathbb{D}^*_\delta} \Sigma_{\mu_{\alpha}} \cap \{0 < |x_i| < \delta, \text{ for } i \neq \ell \text{ and } 0 < |x_\ell| < c\}$$

is contained in $\text{Sat}_\mathcal{F}_\mathcal{U} \Delta_i(\mu; \epsilon)$. We are done. \[\Box\]

4. **Complex Hyperbolic Foliations**

Here we define the class of complex hyperbolic foliations. It is the higher dimensional version of the generalized curves in dimension two (the reader may look at [10] for more details in the non dicritical case and ambient dimension three).

**Definition 4.** Let $\mathcal{F}$ be a germ of singular holomorphic foliation of codimension one on $(\mathbb{C}^n, 0)$. We say that $\mathcal{F}$ is a complex hyperbolic foliation (for short, a "CH-foliation") at the origin if for any map $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)$ generically transversal to $\mathcal{F}$, the foliation $\mathcal{G} = \phi^* \mathcal{F}$ has no saddle-nodes in its reduction of singularities (it is a generalized curve in the sense of [2]).

**Remark 7.** By performing a two-dimensional reduction of singularities of $\mathcal{G}$ (see [21], or [6]) we see that $\mathcal{F}$ is a CH-foliation on $(\mathbb{C}^n, 0)$ if and only if there is no map $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)$ generically transversal to $\mathcal{F}$ such that $\mathcal{G} = \phi^* \mathcal{F}$ has a saddle-node at the origin.

**Lemma 6.** Let $\mathcal{F}$ be a germ of singular holomorphic foliation of codimension one on $(\mathbb{C}^n, 0)$ having a simple CH-point at the origin. Then $\mathcal{F}$ is a CH-foliation.

**Proof.** If $\mathcal{F}$ is not a CH-foliation, there is a map $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)$ such that $\mathcal{G} = \phi^* \mathcal{F}$ has a saddle-node at the origin. Let us show that this is not possible. By performing finitely many local blow-ups of $(\mathbb{C}^2, 0)$ we obtain $\pi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that if $\psi = \phi \circ \pi$ we have that $\psi^* \mathcal{F}$ has a saddle-node at the origin and

$$\psi_i = U_i(z_1, z_2)z_1^{a_i}z_2^{b_i}, \quad a_i, b_i \in \mathbb{Z}_{\geq 0}, \quad U_i(0) \neq 0,$$

for $i = 1, 2, \ldots, n$, where we write $\psi(z_1, z_2) = (\psi_1(z_1, z_2), \psi_2(z_1, z_2), \ldots, \psi_n(z_1, z_2))$. Recall that $\mathcal{F}$ is given by a 1-form of the type

$$\omega = \sum_{i=1}^{\tau} (f_i(x_1, x_2, \ldots, x_\tau)) \frac{dz_i}{x_i}, \quad f_i \in \mathbb{C}\{x_1, x_2, \ldots, x_\tau\}, \quad f_i(0) = 0.$$

Put $\tilde{f}_i = f_i \circ \psi$. Then we have

$$\psi^* \omega = \sum_{i=1}^{\tau} (\lambda_i + \tilde{f}_i) \frac{dz_i}{x_i} = \sum_{i=1}^{\tau} (\lambda_i + \tilde{f}_i)(a_i \frac{dz_1}{z_1} + b_i \frac{dz_2}{z_2} + dU_i) =$$

$$= \sum_{i=1}^{\tau} a_i(\lambda_i + \tilde{f}_i) \frac{dz_1}{z_1} + \sum_{i=1}^{\tau} b_i(\lambda_i + \tilde{f}_i) \frac{dz_2}{z_2} + \sum_{i=1}^{\tau} (\lambda_i + \tilde{f}_i) dU_i.$$
Proof. Consider a map \( \phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0) \) generically transversal to \( F \). By the universal property of the blow-up, there is a map \( \sigma : \Delta \to (\mathbb{C}^2, 0) \) composition of a sequence of blow-ups that lifts \( \phi \) to \( \pi \). That is, there is \( \psi : \Delta \to M \) such that \( \pi \circ \psi = \phi \circ \sigma \). By Lemma \[6\], the foliation \( \tilde{G} = \psi^* F = \sigma^* \phi^* F \) is a generalized curve at the points of \( \Delta \). “A fortiori” \( \tilde{G} = \phi F \) is a generalized curve. \( \square \)

As a direct consequence of Proposition \[2\], a RICH-foliation is also a CH-foliation.

It is not excluded for a CH-foliation to be dicritical. For instance, the foliation in \((\mathbb{C}^3, 0)\) given by the integrable 1-form

\[
\omega = (y^{m+1} - z x^m)dx + (z^{m+1} - x y^m)dy + (x^{m+1} - y z^m)dz
\]

is a dicritical CH-foliation. This foliation has no invariant surface \([13]\).

Another example of CH-foliations is provided by the logarithmic foliations given by a 1-form

\[
\omega = \sum_{i=1}^k \lambda_i df_i / f_i, \quad \lambda_i \in \mathbb{C}, \quad i = 1, 2, \ldots, k,
\]

which correspond to the “levels” of the multivaluated function

\[
f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_k^{\lambda_k}.
\]

Remark 8. In ambient dimension three, there is a reduction of singularities for any germ of codimension one foliation \([4]\). A reduction of singularities \( \pi : M \to (\mathbb{C}^3, 0) \) of \( F \) is complex hyperbolic if all the points of \( M \) are simple CH-points for \( \pi^* F \). By Proposition \[2\], we see that if \( F \) has a complex hyperbolic reduction of singularities then \( F \) is a CH-foliation and thus all the reduction of singularities of \( F \) are also complex hyperbolic. This property has been taken as a definition in \([16]\), where the authors consider the so-called generalized surfaces that are the non-dicritical CH-foliations in ambient dimension three. In this situation they prove that the reduction of singularities of the invariant surfaces automatically gives the reduction of singularities of the generalized surface. Next we state a result of this type in any ambient dimension that can be proved as in the three dimensional case.

Proposition 3. Let \( F \) be a germ of non-dicritical CH-foliation on \((\mathbb{C}^n, 0)\) of dimensional type \( n \). Assume that the invariant hypersurfaces of \( F \) are exactly the coordinate hyperplanes \( \prod_{i=1}^n x_i = 0 \). Then the origin is a simple CH-point.

Proof. (See also \([16]\).) We give a sketch of the proof. Assume that \( F \) is locally given by \( \omega = \sum_{i=1}^n f_i dx_i / x_i \) and put \( \Omega = \prod_{i=1}^n x_i \). Take a general linear plane section \( \phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0) \). By the transversality theorem of Mattei-Moussu \([18]\) and taking into account that the foliation is non dicritical, we have that \( \phi^* \Omega \) has isolated singularity. Moreover, \( \phi^* F \) is a non dicritical generalized curve. The only invariant curves of \( \phi^* \Omega \) are the lines \( \phi^{-1}(x_i = 0), \quad i = 1, 2, \ldots, n \). In fact if there were another invariant curve \( \Gamma \) for \( \phi^* F \), we could use the arguments in \([5] \) and \([8]\) to find an hypersurface for \( F \) different from the coordinate hyperplanes. Now, in dimension two, it is known (see \([2]\)) that a non dicritical generalized curve having exactly \( n \) invariant lines has multiplicity equal to \( n - 1 \). That is, \( \phi^* \Omega \) has multiplicity \( n - 1 \). Thus \( \Omega \) also has multiplicity \( n - 1 \). Now, up to a reordering and to multiplying \( \omega \) by a unit, we have that

\[
\omega = \frac{dx_1}{x_1} + \sum_{i=2}^n f_i \frac{dx_i}{x_i}.
\]

Let \( \pi : M \to (\mathbb{C}^n, 0) \) and \( \sigma : N \to (\mathbb{C}^2, 0) \) be the blowing-ups of the respective origins. Then \( \phi \) lifts to \( \tilde{\phi} : N \to M \).

Because of the multiplicity \( n - 1 \) of \( F \) and \( \phi^* F \) and the fact that both foliations are non dicritical, we see that \( \phi \) is transversal to \( \pi^* F \), moreover, all the points in \( N \) are simple CH-points for \( \sigma^* \phi^* F = \sigma^* \pi^* F \), in particular they are not saddle nodes. A direct computation allows us to deduce from this that the order of each \( f_i \) is equal to 0, that is, we can write

\[
\omega = \frac{dx_1}{x_1} + \sum_{i=2}^n (\lambda_i + b_i) \frac{dx_i}{x_i}, \quad b_i(0) = 0, \quad \lambda_i \neq 0, \quad i = 2, 3, \ldots, n.
\]

The vector \( \lambda = (1, \lambda_2, \lambda_3, \ldots, \lambda_n) \) is not resonant, otherwise we find a dicritical component by doing an appropriate sequence of blowing-ups. \( \square \)

5. Pre-Simple CH-Corners

Let \( F \) be a germ of CH-foliation on \((\mathbb{C}^n, 0)\) and let us consider a normal crossings divisor \( E \subset (\mathbb{C}^n, 0) \). In this section we prove that blowing-up pre-simple CH-corners produces only adapted singularities that are pre-simple CH-corners. Moreover this is a characteristic property, that is, if we blow-up a non pre-simple corner, we will also find a singular point in the transformed space which is not a pre-simple CH-corner. It is important to remark that pre-simple CH-corners may be dicritical. Let us precise the statements and definitions.

Let us recall the definition of the adapted singular locus \( \text{Sing}(F, E) \) (see \([4][5]\)). We say that \( F \) and \( E \) have normal crossings at \( p \) if \( F \) is non singular at \( p \) and \( H \cup E \) defines locally a normal crossings divisor, where \( H \)
is the invariant hypersurface of $F$ through $p$. Then $\text{Sing}(F, E)$ is the set of points $p$ such that $F$ and $E$ do not have normal crossings at $p$.

**Definition 5**. Let $F$ be a germ of codimension one singular holomorphic foliation on $(\mathbb{C}^n, 0)$ of dimensional type $\tau$. We say that the pair $F, E$ has a pre-simple complex hyperbolic corner at the origin if and only if there are local coordinates $x_1, x_2, \ldots, x_n$ such that $E_{\text{inv}} = \prod_{i=1}^n x_i = 0$ and $F$ is given by $\omega = 0$ where

$$\omega = \sum_{i=1}^n (\lambda_i + b_i(x_1, x_2, \ldots, x_\tau)) \frac{dx_i}{x_i}, \quad b_i \in \mathbb{C}\{x_1, x_2, \ldots, x_\tau\}, \quad b_i(0) = 0,$$

with $\prod_{i=1}^n \lambda_i \neq 0$.

Let $Y \subset (\mathbb{C}^n, 0)$ be a nonsingular subspace of codimension $\geq 2$ having normal crossings with $E$ and invariant by $F$. Let us do the blowing-up $\pi : M \rightarrow (\mathbb{C}^n, 0)$ with center $Y$ and consider the normal crossings divisor $E' = \pi^{-1}(E \cup Y) \subset M$. Denote $F' = \pi^*F$ the transformed foliation of $F$ by $\pi$.

**Proposition 4**. If the origin is a pre-simple CH-corner for $F, E$, then any point $p \in \pi^{-1}(0) \cap \text{Sing}(F', E')$ is a pre-simple CH-corner for $F', E'$.

Proof. For simplicity, we assume that $E = E_{\text{inv}}$, the general case can be done with similar computations. Let $\tau$ be the dimensional type of $F$ and choose local coordinates $x_1, x_2, \ldots, x_n$ as in Definition 4, where $E = (\prod_{i=1}^n x_i = 0)$ and

$$Y = (x_1 = x_2 = \cdots = x_s = 0, x_\tau+1 = x_\tau+2 = \cdots = x_{\tau+t} = 0)$$

with $1 \leq s \leq \tau$, $0 \leq t \leq n - \tau$ and $s + t \geq 2$ (note that $s \geq 1$, otherwise $Y$ is not invariant). Up to a reordering of the indices, it is enough to verify the cases with local coordinates $x'_1, x'_2, \ldots, x'_n$ at $p \in \pi^{-1}(0) \cap E_{\text{inv}}$ given by (1) or (2) as follows

1. There are scalars $\mu_2, \mu_3, \ldots, \mu_s, \nu_{\tau+1}, \nu_{\tau+2}, \ldots, \nu_t$ such that
   a. $x_1 = x'_1$,
   b. $x_i = (x'_i + \mu_i)x'_1$, for $i \in \{2, 3, \ldots, s\}$,
   c. $x_i = (x'_i + \nu_i)x'_1$, for $i \in \{\tau+1, \tau+2, \ldots, \tau+t\}$,
   d. $x_i = x'_i$ for $i \notin \{1, 2, \ldots, s\} \cup \{\tau+1, \tau+2, \ldots, \tau+t\}$.

2. There are scalars $\nu_{\tau+2}, \nu_{\tau+3}, \ldots, \nu_{\tau+t}$ such that
   a. $x_1 = x'_1x'_{\tau+1}$, for $i \in \{1, 2, \ldots, s\}$,
   b. $x_\tau+1 = x'_{\tau+1}$,
   c. $x_i = (x'_i + \nu_i)x'_{\tau+1}$, for $i \in \{\tau+2, \tau+3, \ldots, \tau+t\}$,
   d. $x_i = x'_i$ for $i \notin \{1, 2, \ldots, s\} \cup \{\tau+1, \tau+2, \ldots, \tau+t\}$.

The second case occurs only if $t \geq 1$. Put $r = 1$ if we are in the first case and $r = \tau + 1$ if we are in the second one. In both cases $\pi^{-1}(Y)$ is locally given at $p$ by $x'_1 = 0$ and the divisor $E'$ is given by

$$E' = \left\{ \begin{array}{ll} \prod_{i \in \{1\} \cup \{i \in \{2, \ldots, \tau\}; \mu_i = 0\} x_i = 0, & \text{first case} \\ \prod_{i \in \{1, 2, \ldots, \tau, \tau+1\} x_i = 0, & \text{second case} \end{array} \right.$$ 

We recall that the foliation $F$ is locally given at the origin by $\omega$ as in Equation 5 of Definition 5.

**Case (1).** The 1-form $\omega$ is locally given at $p$ as

$$\omega = (\tilde{\lambda}_1 + \tilde{b}_1) \frac{dx'_1}{x'_1} + \sum_{i \in B} (\lambda_i + \tilde{b}_i) \frac{dx'_i}{x'_i} + \sum_{i \in C} \lambda_i + \tilde{b}_i \frac{dx'_i}{x'_i},$$

where $\tilde{\lambda}_1 = \sum_{i=1}^r \lambda_i$, the germs $\tilde{b}_j \in \mathbb{C}\{x'_1, x'_2, \ldots, x'_\tau\}$ are in the ideal generated by $x'_1, x'_{\tau+1}, x'_{\tau+2}, \ldots, x'_t$ and $B = \{i \in \{2, 3, \ldots, \tau\}; \mu_i = 0\}; \quad C = \{i \in \{2, 3, \ldots, \tau\}; \mu_i \neq 0\}$.

If $\tilde{\lambda}_1 \neq 0$, the nonsingular vector fields

$$\xi_i = \frac{\lambda_i + \tilde{b}_i}{\mu_i + x'_i} x'_i \frac{\partial}{\partial x'_i} - (\tilde{\lambda}_1 + \tilde{b}_1) \frac{\partial}{\partial x'_1}; \quad i \in C,$$

trivialize the foliation and up to an appropriate coordinate change we may assume that $F'$ is given by a form of the type

$$\omega = (\tilde{\lambda}_1 + \tilde{b}_1) \frac{dx'_1}{x'_1} + \sum_{i \in B} (\lambda_i + \tilde{b}_i) \frac{dx'_i}{x'_i}.$$ 

Thus, the point $p$ is a pre-simple CH-corner with dimensional type $\tau' = \tau - 2\mathbb{R}$. Assume now that $\tilde{\lambda}_1 = 0$, in particular $s \geq 2$. Then $x'_1$ divides $\tilde{b}_1$ or not.
If \( x'_1 \) divides \( b_1 \), with \( f = b_1/x'_1 \), the component \( x'_1 = 0 \) is dicritical and thus we have \( E'_{\text{inv}} = (\prod_{i \in B} x_i = 0) \). In particular \( B \neq \emptyset \). Write
\[
\omega = f dx_1 + \sum_{i \in B} \left( \lambda_i + \hat{b}_i \right) \frac{dx'_i}{x'_i} + \sum_{i \in C} \frac{\lambda_i + \hat{b}_i}{\mu_i + x'_i} dx'_i.
\]
Take an index \( j \in B \cup C \) and consider the non singular vector field \( \xi \) where
\[
\xi = (\lambda_j + \hat{b}_j) \frac{\partial}{\partial x'_j} - f x_j \frac{\partial}{\partial x_j}, \quad \text{if } j \in B; \quad \xi = \frac{\lambda_j + \hat{b}_j}{\mu_j + x'_j} \frac{\partial}{\partial x_j} - f \frac{\partial}{\partial x_j}, \quad \text{if } j \in C.
\]
Now \( \xi \) trivializes the foliation and we can assume that \( f \) is identically zero, that is
\[
\omega = \sum_{i \in B} \left( \lambda_i + \hat{b}_i \right) \frac{dx'_i}{x'_i} + \sum_{i \in C} \frac{\lambda_i + \hat{b}_i}{\mu_i + x'_i} dx'_i.
\]
If \( B = \emptyset \), then \( s \geq 2 \) we have that \( C \neq \emptyset \) and \( \omega \) is non singular and has normal crossings with \( E' = (x'_1 = 0) \) at \( p \), then we are done, since \( p \notin \text{Sing}(F', E') \). If \( B \neq \emptyset \), take \( i \in B \) and consider the trivializing nonsingular vector fields
\[
\xi_j = \frac{\lambda_j + \hat{b}_j}{\mu_j + x'_j} \frac{\partial}{\partial x_j} - (\lambda_i + \hat{b}_i) \frac{\partial}{\partial x_j}, \quad j \in C.
\]
Then, we get that \( p \) is a pre-simple CH-corner with dimensional type \( \tau' = \tau - 1 - C \).

Now, we assume that \( \lambda_0 = 0 \) and \( x'_1 \) does not divide \( b_1 \). We can write \( b_1 = x'_1 f + g(x'_{s+1}, x'_{s+2}, \ldots, x'_s) \), where \( g(0) = 0 \) and \( g \neq 0 \). Note in particular that \( s < \tau \). Let us choose integers \( \alpha_{s+1}, \alpha_{s+2}, \ldots, \alpha_{r} \in \mathbb{Z}_{\geq 1} \) such that \( \psi(v') \neq 0 \) where \( \psi(v') = g(v^{\alpha_{s+1}}, v^{\alpha_{s+2}}, \ldots, v^{\alpha_{r}}) \) and \( \lambda \neq 0 \) where \( \lambda = \sum_{i=s+1}^{r} \alpha_i \lambda_i \). Let us consider the map \( \delta: (\mathbb{C}^2, 0) \to (M, p) \) given by \( x'_1 = u, x'_j = v^{\alpha_j}, j = s + 1, s + 2, \ldots, \tau \) and \( x'_j = 0 \) otherwise. We have
\[
\delta^* \omega = (aw(u, v) + \psi(v)) \frac{du}{u} + (\lambda + \rho(u, v)) \frac{dv}{v}; \quad \psi(0) = 0 = \rho(0).
\]
This is a saddle node in contradiction with the hypothesis that \( \mathcal{F} \) is a CH-foliation.

Case (2). We have
\[
\omega = \sum_{i=1}^{s} \left( \lambda_i + \hat{b}_i \right) \frac{dx'_i}{x'_i} + (\hat{\lambda}_{r+1} + \hat{b}_{r+1}) \frac{dx_{r+1}}{x_{r+1}}; \quad \hat{\lambda}_{r+1} = \sum_{i=1}^{s} \lambda_i,
\]
where the \( \hat{b}_i \) are in the ideal generated by \( x'_{r+1} \) and \( x'_j \), \( j \in \{s + 1, s + 2, \ldots, \tau \} \). If \( \hat{\lambda}_{r+1} \neq 0 \), we see that \( p \) is a pre-simple CH-corner for \( F', E' \) of dimensional type \( \tau' = \tau + 1 \). Assume that \( \hat{\lambda}_{r+1} = 0 \). As before, if \( x'_{r+1} \) does not divide \( \hat{b}_{r+1} \) we find a contradiction with the fact that \( \mathcal{F} \) is a CH foliation. If \( x'_{r+1} \) divides \( \hat{b}_{r+1} \), with \( \hat{b}_{r+1} = x'_{r+1} f' \), the created exceptional divisor \( x'_{r+1} = 0 \) is dicritical, the foliation is given by
\[
\omega = \sum_{i=1}^{s} \left( \lambda_i + \hat{b}_i \right) \frac{dx'_i}{x'_i} + fd x'_{r+1}
\]
and we can trivialize it to find a pre-simple CH-corner with \( \tau' = \tau \).

**Lemma 7.** Let \( \mathcal{G} \) be a codimension one singular foliation in the projective space \( \mathbb{P}^n \) and consider the normal crossings divisor \( D \subset \mathbb{P}^n \) given in homogeneous coordinates by \( D = (\prod_{i=0}^{s} X_i = 0) \). Assume that all the components of \( D \) are invariant by \( \mathcal{G} \) and any point in \( \text{Sing}(\mathcal{G}, D) \) is a pre-simple CH-corner for \( \mathcal{G}, D \). Let \( \mathcal{G} \) be defined by a non-null homogeneous integrable 1-form
\[
W = \sum_{i=0}^{n} A_i(X_0, X_1, \ldots, X_n) \frac{dX_i}{X_i}; \quad \sum_{i=0}^{n} A_i = 0,
\]
where the (non-null) coefficients \( A_i \) are homogeneous polynomials of degree \( r \) without common factor. Then \( r = 0 \) and \( A_{e+1} = A_{e+2} = \cdots = A_n = 0 \). More precisely, \( W \) has the form \( W = \sum_{i=0}^{e} \lambda_i dX_i/X_i \) with \( \sum_{i=0}^{e} \lambda_i = 0 \) and \( \prod_{i=0}^{e} \lambda_i \neq 0 \).

**Proof.** It is known that \( \text{Sing} \mathcal{G} \) is a nonempty subset of \( \mathbb{P}^n \). Thus \( D \neq \emptyset \) and more precisely \( e \geq 2 \), otherwise we find singular points that are not pre-simple CHorners. Let us show now that \( X_i \) divides \( A_i \) for each \( i = c + 1, c + 2, \ldots, n \). If \( X_{c+1} \) does not divide \( A_{c+1} \), the hyperplane \( X_{c+1} = 0 \) is invariant for \( \mathcal{F} \); since \( X_0 = 0 \) is also invariant, any point
\[
P \in (X_0 = X_{c+1} = 0) \setminus (\prod_{i=1}^{c} X_i = 0)
\]
is a singular point \( P \in \text{Sing} \mathcal{G} \) that cannot be a pre-simple CH-corner for \( \mathcal{G}, D \). Thus we have \( A_i = X_iB_i \), for \( i = c + 1, c + 2, \ldots, n \), where either \( A_i = 0 \) or \( B_i \) is a homogeneous polynomial of degree \( r - 1 \).
If $r \geq 1$, the set $Z = (A_0 = A_1 = \cdots = A_n = 0) = (A_1 = A_2 = \cdots = A_n = 0)$ is a non-empty Zariski-closed subset of $\mathbb{P}_n^C$ and no point $P \in Z$ can be a pre-simple CH-corner. Thus $r = 0$. This implies that the degree of $B_i$ is equal to $-1$ for $i = e + 1, e + 2, \ldots, n$, that is $A_{e+1} = A_{e+2} = \cdots = A_n = 0$ and moreover $A_j = \lambda_j \in C$ for $j = 0, 1, \ldots, e$. We have that $\lambda_j \neq 0$ for each $j = 0, 1, \ldots, e$ because $X_j = 0$ is invariant by $G$. 

**Proposition 5.** If all $p \in \pi^{-1}(0) \cap \text{Sing}(F', E')$ are pre-simple CH-corners for $F', E'$, the origin is also a pre-simple CH-corner for $F, E$.

**Proof.** Choose local coordinates $x_1, x_2, \ldots, x_n$ and subsets $A, B, C \subset \{1, 2, \ldots, n\}$ such that

$$E_{\text{inv}} = \left( \prod_{i \in A} x_i = 0 \right); \quad Y = (x_i = 0; i \in B); \quad E_{\text{dic}} = \left( \prod_{i \in C} x_i = 0 \right).$$

Then $F$ is given by a meromorphic 1-form $\omega = \sum_{i \in A} a_i dx_i/x_i + \sum_{i \notin A} b_i dx_i$ where the coefficients do not have a common factor and $x_i$ does not divide $a_i$, for $i \in A$. Let us put $a_i = x_i b_i$ for $i \notin A$ and $\omega = \sum_{i=1}^n a_i dx_i/x_i$. We first show that at least one of the coefficients $a_i$, $i \in A$ is a unit. Let $r$ be the generic order $r = \nu_Y(a_i; i = 1, 2, \ldots, n)$ of the coefficients $a_i$ along $Y$. Let us write $a_i = A_i, r + A_{i,r+1} + \cdots$, the decomposition of $a_i$ in homogeneous components with respect to the variables $x_i, i \in B$, where one of the $A_i, r$ is not identically zero. We also have that

$$\bar{a}_i = a_i|_{(x_i, 0 \in \pi B)} = \bar{A}_{i,r} + \bar{A}_{i,r+1} + \cdots$$

is the decomposition in homogeneous components of $\bar{a}_i \in C\{x_i; i \in B\}$. It is enough to show that $r = 0$ and there is an index $i$ such that $\bar{A}_{i,0} \neq 0$ (note that $i \in A$).

Let us denote $f = \sum_{i \in B} a_i$. We decompose $f = F_0 + F_1 + \cdots$ as before. Consider an affine chart of the blow-up by taking $i_0 \in B$ and coordinates $x'_i = x_i$ for $i \notin B \setminus \{i_0\}, x'_j = x_j/x_{i_0}$ for $i \in B \setminus \{i_0\}$. The transformed foliation in this chart is given by the 1-form

$$\omega' = \frac{1}{x'_{i_0}} \left( f dx'_{i_0}/x'_{i_0} + \sum_{i \neq i_0} a'_i dx'/x'_i \right) = f' dx'_{i_0}/x'_{i_0} + \sum_{i \neq i_0} a'_i dx'/x'_i,$$

where $f' = f/x'_{i_0}$ and $a'_i = a_i/x'_{i_0}$ for $i \neq i_0$. We note that the blow-up is dicritical if and only if $x_{i_0}$ divides $f'$, this is equivalent to saying that $F_0 = 0$.

Let us consider points $p' \in \pi^{-1}(0)$ belonging to the selected affine chart; that is

$$x'_i(p') = 0; i \notin B \setminus \{i_0\}, \quad x'_j(p') = \mu_i; i \in B \setminus \{i_0\}.$$ 

**First case: the blow-up is non dicritical.** Since $p'$ is a pre-simple CH-corner, we have that $f'(p') \neq 0$. Repeating this argument for other affine charts, we deduce that $r = 0$ and $F_0 = 0$. This implies that some $\bar{A}_{i,0} \neq 0$ and hence $a_i$ is a unit.

**Second case: the blow-up is dicritical.** We have two possibilities, either $A \subset B$ or there is $j \in A \setminus B$. In the second case, we have $a'_j(p') \neq 0$ for all $p' \in \pi^{-1}(0)$, this implies that $r = 0$ and $\bar{A}_{j,0} \neq 0$ and hence $a_j$ is a unit. Thus we suppose that $A \subset B$. By the hypothesis, there is some $i \neq i_0$ such that $a'_i(p') \neq 0$, but if $i \notin A$ we have $a'_i(p') = 0$. In a more precise way, the restriction $\omega|_{\pi^{-1}(0)}$ is given in homogeneous coordinates by

$$\omega|_{\pi^{-1}(0)} = \sum_{i \in A} \bar{A}_{i,r} \frac{dx'_i}{x'_i}$$

and all the points in $\pi^{-1}(0)$ are pre-simple CH-corners for $F'|_{\pi^{-1}(0)}$, $E'_{\text{inv}} \cap \pi^{-1}(0)$. Now, we apply Lemma 4 to deduce that $r = 0$ and $\bar{A}_{i,0} \neq 0$ for all $i \in A$. In this case we deduce already that the origin is a pre-simple CH-corner.

Now, let us end the proof. We know that there is an index $s \in A$ such that $a_s$ is a unit. Up to dividing by this unit, we can write

$$\omega = \frac{dx_s}{x_s} + \sum_{i \in A \setminus \{s\}} a_i \frac{dx_i}{x_i} + \sum_{j \notin A} b_j dx_j.$$ 

We can trivialize the foliation by the tangent vector fields $\xi_j = b_j x_s \partial/\partial x_s - \partial/\partial x_j$ for $j \notin A$. This allows us to suppose that $b_j = 0$ for all $j \notin A$. The integrability condition also gives in this situation that $a_s \in C\{x_j; j \in A\}$. Now, it is enough to prove that $a_j(0) \neq 0$ for all $j \in A \setminus \{s\}$. Assume that $a_j(0) = 0$, we blow-up the axis $x_s = x_j = 0$ and we find a saddle node, contradiction with the fact that $F$ is a CH-foliation. □
6. COMPACT DICRITICAL COMPONENTS AND PARTIAL SEPARATRICES

In this section we extend to the dicritical case some features of the argument in [5] to find invariant germs of surfaces for a germ of foliation \( F \) in \( (\mathbb{C}^3, 0) \). We are focusing on the case of CH-foliations, although the results are of a wider scope. For other extensions of this argument, the reader may see [20].

Let us consider a germ of CH-foliation \( F \) in \( (\mathbb{C}^3, 0) \) and a reduction of singularities

\[
\pi : (M, \pi^{-1}(0)) \to (\mathbb{C}^3, 0)
\]

of \( F \) which is a composition of blow-ups with invariant nonsingular centers, where the exceptional divisor \( E \) has normal crossings and each point \( p \in \pi^{-1}(0) \) is a CH-simple point for \( \pi^* F \) adapted to \( E \). The existence of such a reduction of singularities is guaranteed by the main result in [4]. Note that since \( F \) may be a dicritical foliation, the morphism \( \pi \) is not necessarily obtained from any reduction of singularities of the invariant surfaces as in [16]; it is even possible that there are no invariant surfaces.

The fiber \( \pi^{-1}(0) \) is a connected closed analytic subset of \( M \) whose irreducible components have dimension two or dimension one. The irreducible components of dimension two of the fiber coincide with the compact irreducible components of the exceptional divisor \( E \). Each irreducible component \( \Gamma \) of dimension one of \( \pi^{-1}(0) \) is contained in at least one non-compact irreducible component of \( E \) and never contained in a compact irreducible component of \( E \) (otherwise it coincides with it).

Let us briefly recall the argument of construction of invariant surfaces in [5]. Take a point \( p \in \text{Sing}(\pi^* F, E) \); it can be a simple CH-corner or a simple CH-trace point and the dimensional type \( \tau \) is 2 or 3. Assume that it is a simple CH-trace point. Then there is a unique germ of invariant surface \( (S_p, p) \) in \( p \) different from the invariant components of \( E \) through \( p \). Moreover \( (S_p, p) \) has normal crossings with \( E \). In the case \( \tau = 2 \), the adapted singular locus \( \text{Sing}(\pi^* F, E) \) is a nonsingular curve contained in the unique invariant component \( E_j \) of \( E \) through \( p \). More precisely

\[
\text{Sing}(\pi^* F, E) = S_p \cap E_j
\]

locally at \( p \) and the foliation is analytically equivalent to the germ of \( \pi^* F \) at \( p \) in the points of \( S_p \cap E_j \) close to \( p \). In the case \( \tau = 3 \), there are exactly two invariant components \( E_i, E_j \) of \( E \) through \( p \) and the two lines \( S_p \cap E_i \) and \( S_p \cap E_j \) correspond locally at \( p \) to the singular simple CH-trace points; all of them, except \( p \) itself, are of dimensional type two. Thus, the set

\[
\text{STr}(\pi^* F, E) = \{ p \in \text{Sing}(\pi^* F, E) ; p \text{ is a simple CH-trace point } \}
\]

is the union of curves of \( \text{Sing}(\pi^* F, E) \) that are generically contained in only one irreducible component of \( E \). We call these curves the s-trace curves, to be generalized in Section 8.

Let us note that \( \text{STr}(\pi^* F, E) \) defines a germ of analytic set along \( \text{STr}(\pi^* F, E) \cap \pi^{-1}(0) \), hence the irreducible components of \( \text{Sing}(\pi^* F, E) \) are either compact curves contained in \( \pi^{-1}(0) \) or germs of curves.

Let \( C \) be a connected component of \( \text{STr}(\pi^* F, E) \); then \( C \cap \pi^{-1}(0) \) is also connected, and it is either reduced to one point or it is a finite union of compact curves. The germs of invariant surface \( S_p \), for \( p \in C \), can be glued together (see [3]) to obtain a unique invariant surface \( S_C \) that is a germ along \( C \cap \pi^{-1}(0) \). An important remark is that the immersion

\[
(S_C, C \cap \pi^{-1}(0)) \subset (M_N, \pi^{-1}(0))
\]

is a closed immersion (that is \( S_C \cap \pi^{-1}(0) = C \cap \pi^{-1}(0) \)) if and only if \( C \) does not intersect any compact dicritical component of \( E \). If this is the case, we can produce an invariant surface \( \pi(S_C) \) for \( E \) by properness of the morphism \( \pi \). This is the main argument in [5].

We can extend the same type of construction by considering also non singular trace points as follows. Define the set \( \text{Inv}(\pi^{-1}(0)) \) to be the union of the irreducible components of \( \pi^{-1}(0) \) that are invariant by \( \pi^* F \) and consider the closed analytic set

\[
\text{ITr}(\pi^* F, E) = \{ p \in \text{Inv}(\pi^{-1}(0)) ; p \text{ is a simple CH-trace point for } \pi^* F, E \).
\]

(Compare with Equation 5). The irreducible components of \( \text{ITr}(\pi^* F, E) \) are points or compact curves contained in the fiber, but not necessarily contained in the adapted singular locus.

**Lemma 8.** Given a point \( p \in \text{ITr}(\pi^* F, E) \) there is exactly one germ of invariant surface \( S_p \) at \( p \) not included in \( E \) and moreover \( S_p \) has normal crossings with \( E \).

**Proof.** If \( p \) is a singular point we have seen this property before. If \( p \) is a simple CH-trace point not in \( \text{Sing}(\pi^* F, E) \), there are no invariant components of \( E \) at \( p \) and \( S_p \) is the only leaf through \( p \).

Take a connected component \( \tilde{C} \) of \( \text{ITr}(\pi^* F, E) \). We can glue together the invariant surfaces \( S_p \) to obtain a unique germ \( (S_{\tilde{C}}, \tilde{C}) \) of invariant surface. Exactly as before, the immersion

\[
(S_{\tilde{C}}, \tilde{C}) \subset (M_N, \pi^{-1}(0))
\]
is a closed immersion if and only if \( \mathcal{C} \) does not intersect any compact dicritical component of the exceptional divisor \( E \).

The above two constructions are related as follows. Given a connected component \( C \) of \( \text{STr}(\pi^*F, E) \), we have that \( C \cap \text{ITr}(\pi^*F, E) \) is nonempty and connected. In particular, the germ \((S_C, C \cap \pi^{-1}(0))\) is contained in the germ \((S_{\mathcal{C}}, \mathcal{C})\) where \( \mathcal{C} \) is the connected component of \( \text{ITr}(\pi^*F, E) \) that contains \( C \cap \text{ITr}(\pi^*F, E) \). The inclusion of germs of surfaces \((S_C, C \cap \pi^{-1}(0)) \subset (S_{\mathcal{C}}, \mathcal{C})\) is not necessarily a closed immersion. Moreover, due to the possible existence of curves \( \Gamma \subset \text{ITr}(\pi^*F, E) \) whose points are nonsingular for \( \pi^*F \), it is possible to have two connected components \( C_1 \) and \( C_2 \) of \( \text{STr}(\pi^*F, E) \) such that \( \mathcal{C} \) is a common connected component of \( \text{ITr}(\pi^*F, E) \) that contains \( C_1 \cap \text{ITr}(\pi^*F, E) \) and \( C_2 \cap \text{ITr}(\pi^*F, E) \). Hence we can have two non closed inclusions of germs

\[
(S_{C_1}, C_1 \cap \pi^{-1}(0)) \subset (S_{\mathcal{C}}, \mathcal{C}) \supset (S_{C_2}, C_2 \cap \pi^{-1}(0)).
\]

**Definition 6.** Given a connected component \( \mathcal{C} \) of \( \text{ITr}(\pi^*F, E) \), the partial separatrix over \( \mathcal{C} \) is the germ of invariant surface \((S_{\mathcal{C}}, \mathcal{C})\).

Now, let us give some results for the case that \( F \) has no germ of invariant surface.

**Proposition 6.** Assume that \( F \) has no germ of invariant surface. Then we have

1. Any one dimensional irreducible component of \( \pi^{-1}(0) \) is invariant by \( \pi^*F \).
2. Any connected component \( \mathcal{C} \) of \( \text{ITr}(\pi^*F, E) \) intersects at least one compact dicritical component of \( E \).
3. Any connected component \( C \) of \( \text{STr}(\pi^*F, E) \) intersects at least one compact dicritical component of \( E \).
4. There is at least one compact dicritical component of \( E \).

**Proof.**

1. Let \( A \) be a one dimensional irreducible component of \( \pi^{-1}(0) \). Assume that \( A \) is not invariant by \( \pi^*F \). At a generic point \( p \in A \), we have that \( \pi^*F \) is transversal to \( A \) and \( \pi^{-1}(0) \) locally coincides with \( A \). Then there is a germ \((\mathcal{S}, p)\) of invariant surface for \( \pi^*F \) transversal to \( \pi^{-1}(0) \) and such that \( \mathcal{S} \cap \pi^{-1}(0) = \{p\} \). Thus we have a closed inclusion of germs \((\mathcal{S}, p) \subset (M, \pi^{-1}(0))\). Since \( \pi \) is a proper morphism, the image \( S = \pi(\mathcal{S}) \) is a germ of invariant surface for \( F \) at \( 0 \in \mathbb{C}^4 \). This is the desired contradiction.

2. If \( \mathcal{C} \) does not intersect any compact dicritical component of \( E \), the partial separatrix \((S_{\mathcal{C}}, \mathcal{C})\) is closed in \((M, \pi^{-1}(0))\) and hence \( S = \pi(S_{\mathcal{C}}) \) is an invariant surface for \( F \).

3. Same argument as in (2).

4. Assume that there are no compact dicritical components to find a contradiction. By taking enough two dimensional sections \( F|_\Delta \) where \( \Delta \subset (\mathbb{C}^4, 0) \) is non-singular and transversal to \( F \) in the sense of Mattei-Moussu [18], we find an invariant curve \( \gamma \) that is not included in any center of the sequence of blowing-ups. Let \( \gamma' \subset M \) be the strict transform of \( \gamma \) and put \( \{p\} = \gamma \cap E \). We know that \( p \in \pi^{-1}(0) \). Let us see that \( p \in \text{ITr}(\pi^*F, E) \). If \( p \) is in an invariant component \( D \) of \( E \), we are done, since \( \gamma' \not\subset E \) thus \( p \) is singular and it cannot be a corner point (at the corner points the only invariant curves are contained in the divisor). If \( p \) is not in an invariant component of \( E \), it is a trace point and it belongs to a one dimensional component \( \Gamma \) of the fiber, but \( \Gamma \) is invariant and thus \( p \in \text{ITr}(\pi^*F, E) \). Now it is enough to consider the connected component \( \mathcal{C} \) of \( \text{ITr}(\pi^*F, E) \) that contains \( p \) and apply (2). \( \square \)

7. Complex Hyperbolic Foliations Without Nodal Components

We devote this section to giving a proof of Theorem 1. Consider a CH-foliation \( F \) in \((\mathbb{C}^4, 0)\) and fix a reduction of singularities \( \pi \) as in Equation 7. Let us denote \( F' = \pi^*F \) and let \( E \) be the exceptional divisor of \( \pi \). Since all the points of \( M \) are CH-simple for \( F', E \), the singular locus \( \text{Sing}(F') \) is equal to the adapted singular locus \( \text{Sing}(F'^t, E) \) and it is a union of nonsingular connected curves

\[
\text{Sing}(F') = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_s.
\]

**Definition 7.** An irreducible component \( \Gamma_i \) of \( \text{Sing}(F') \) is of generic nodal type if and only if \( \Gamma_i \) contains a nodal point of dimensional type two.

Note that this is equivalent to saying that all the points of \( \Gamma_i \) of dimensional type two are of nodal type.

**Definition 8.** The generic nodal set \( \text{GN}(F', E) \) is the union of the irreducible components of \( \text{Sing}(F') \) of generic nodal type. A connected component \( \mathcal{N} \) of \( \text{GN}(F', E) \) is a nodal component for \( F', E \) if and only if all the points of \( \mathcal{N} \) are of nodal type. We say that \( F \) is without nodal components if there is a reduction of singularities \( \pi \) such that \( F' \), \( E \) is without nodal components.

**Remark 9.** Let \( p \in \text{Sing}(F') \) be of dimensional type three. We recall that \( \text{Sing}(F') \) is locally given at \( p \) by three curves \( \Gamma_1, \Gamma_2, \Gamma_3 \). One of these curves, say \( \Gamma_1 \), is never of generic nodal type. Concerning the other ones, we have that \( p \) is of nodal type if and only if both \( \Gamma_2 \) and \( \Gamma_3 \) are of generic nodal type (see Remark 8). In particular, a connected component \( \mathcal{N} \) of \( \text{GN}(F', E) \) is a nodal component if and only if there are two irreducible components \( \Gamma_2 \) and \( \Gamma_3 \) of \( \mathcal{N} \) through any given point \( p \in \mathcal{N} \) of dimensional type three.
Before starting the proof of Theorem 1, let us give some combinatorial results concerning the irreducible components of the exceptional divisor $E$ and the partial separatrices $(S_C, C)$, where $C$ runs over the connected components of $\text{ITr}(F', E)$. The elements of the set $E(F', E)$ of exceptional components for $F'$, $E$ are by definition the irreducible components of $E$ and the partial separatrices $(S_C, C)$ (identified one to one with the connected components $\tilde{C}$ of $\text{ITr}(F', E)$). Given two exceptional components $A_1$ and $A_2$, the intersection $A_1 \cap A_2$ is the corresponding intersection as germs; it is either the empty set or a finite union of disjoint nonsingular curves. To be precise, we have the following types of exceptional components $A \in E(F', E)$

1. $A$ is a compact irreducible component of $E$, it can be dicritical or invariant.
2. $A$ is a non-compact irreducible component of $E$. In this case it is a germ over a finite union of compact curves $A \cap \pi^{-1}(0)$. It can be dicritical or invariant.
3. $A$ is a partial separatrix. It is a germ $A = (S_C, \tilde{C})$ over a finite union $\tilde{C}$ of curves that is a connected component of $\text{ITr}(F', E)$. By construction $A$ is invariant by $F'$.

**Definition 9.** An exceptional component $A \in E(F', E)$ is called regular if and only if it is invariant or a compact dicritical component of $E$. (The non-regular exceptional components are the non-compact dicritical components of $E$).

**Remark 10.** Assume that $F$ has no germ of invariant surface. Each irreducible component $\Delta$ of the fiber $\pi^{-1}(0)$ is contained in at least one regular exceptional component $B_\Delta$. If $\Delta$ has dimension two, we are done, since $\Delta$ itself is a compact component of $E$. If $\Delta$ has dimension one, it is invariant by Proposition 6 and thus $\Delta$ is contained in the partial separatrix $(S_C, C)$, where $C$ is the connected component of $\text{ITr}(F', E)$ that contains $\Delta$. In particular, since the fiber $\pi^{-1}(0)$ is connected, given two irreducible components $\Delta_1$ and $\Delta_2$ of $\pi^{-1}(0)$, we can find a finite chain of regular exceptional components $B_0, B_1, \ldots, B_t$ such that $\Delta_1 \cap B_0 \neq \emptyset \neq B_t \cap \Delta_2$, and $B_{i-1} \cap B_i \neq \emptyset$ for $i = 1, 2, \ldots, t$.

**Definition 10.** Two regular exceptional components $A_1$ and $A_2$ are nodally-free connected if and only if $A_1 = A_2$ or there is a finite chain of regular exceptional components

$$A_1 = B_0, B_1, \ldots, B_k = A_2$$

such that $B_{i-1} \cap B_i$ contains a not generically nodal curve, for $i = 1, 2, \ldots, k$.

**Lemma 9.** Assume that the pair $F', E$ is without nodal components and $F$ has no germ of invariant surface. Any given pair $A_1, A_2$ of regular exceptional components is nodally-free connected.

**Proof.** Let us first reduce the problem to the case that $A_1 \cap A_2 \neq \emptyset$. We know that the exceptional divisor $E$ is connected. Thus we can find a finite chain

$$A_1 = B_0, B_1, \ldots, B_s, B_{s+1} = A_2$$

such that $B_i \cap B_{i+1} \neq \emptyset$ for $i = 0, 1, \ldots, s$ and $B_1', B_2', \ldots, B'_s$ are irreducible components of $E$. Now, suppose that $B_i'$ is the last non-compact dicritical component in the list. We know that $B_i'$ is a germ along the connected finite union of compact curves $B_i' \cap \pi^{-1}(0)$. Thus, there are irreducible components $\Delta_1$ and $\Delta_2$ of the fiber such that

$$B_i' \cap \Delta_1 \neq \emptyset, \Delta_1 \cap B_i' \neq \emptyset, B_i' \cap \Delta_2 \neq \emptyset, \Delta_2 \cap B_{i+1}' \neq \emptyset.$$ 

We can substitute $B_i'$ by the sequence $B_0, B_1, \ldots, B_t$ of regular exceptional components given in Remark 10. Applying finite induction in this way, we can suppose that all the $B_i'$ are regular exceptional components. This reduces the problem to the case that $A_1 \cap A_2 \neq \emptyset$.

Now, assume that $A_1 \cap A_2$ intersect only at generically nodal curves (note that $A_1$ and $A_2$ are necessarily invariant by $F'$). Take a point $p \in A_1 \cap A_2 \cap \pi^{-1}(0)$. The intersection $A_1 \cap A_2$ locally at $p$ coincides with a generically nodal curve $\Gamma$. Since there are no nodal components, we can find generically nodal curves $\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_k$ and points $p = p_0, p_1, \ldots, p_k, p_{k+1}$ such that

$$p_0 \in \Gamma_0, p_1 \in \Gamma_0 \cap \Gamma_1, p_2 \in \Gamma_1 \cap \Gamma_2, \ldots, p_k \in \Gamma_{k-1} \cap \Gamma_k, p_{k+1} \in \Gamma_k,$$

where the points $p_i$ are nodal points for $i = 1, 2, \ldots, k$ and $p_{k+1}$ is not a nodal point. We shall proceed by induction on this length $k$. If $k = 0$, then $p_1 \in \Gamma$ is not a nodal point. Moreover, the curve $\Gamma$ is locally given at $p_1$ by $A_1 \cup A_2$. Since the dimensional type of $p_1$ is three, there is an invariant exceptional component $B$ transversal to $\Gamma$ at $p_1$ such that the intersections $\Delta_1 = A_1 \cap B$ and $\Delta_2 \cap B$ are curves not generically nodal. Then $A_1$ and $A_2$ are nodally-free connected through $B$.

Assume now that $k \geq 1$. The curve $\Gamma_1$ is locally at $p_1$ the intersection of two invariant exceptional components $B_1$ and $B_2$. Moreover one of them, say $B_1$, is equal to $A_1$ or $A_2$, say that $B_1 = A_1$; then, $B_2 \cap A_2$ defines at $p_1$ a curve $\Delta$ that is not generically nodal (see Remark 9). We have that $A_2$ is nodally-free connected with $B_2$, by induction on $k$, we also have that $B_2$ is nodally-free connected with $B_1 = A_1$ and we are done. □
Let us suppose that the pair $F', E$ is without nodal components and $F$ has no germ of invariant surface. In order to give a proof of Theorem 11 we need to find a neighborhood $U$ of the origin and hence we start by representing our objects and morphisms in appropriate sets. We consider an open neighborhood $U_0$ of the origin of $\mathbb{C}^3$ such that the following properties hold:

1. The foliation $F$ is represented in $U_0$. We denote by $\tilde{F}$ the corresponding foliation on $U_0$.
2. The morphism of germs $\pi : (M, \pi^{-1}(0)) \to (\mathbb{C}^3, 0)$ is represented in $U_0$ by
   \[ \tilde{\pi} : \tilde{M} = \pi^{-1}(U_0) \to U_0. \]

Moreover, we ask $\tilde{\pi}$ to be a composition of blowing-ups with connected nonsingular curves, in such a way that the centers of $\pi$ are the corresponding germs of subvarieties.

3. The total exceptional divisor $\tilde{E}$ of $\tilde{\pi}$ is a normal crossings divisor. Note of course that $E$ coincides with the germ $(\tilde{E}, \tilde{\pi}^{-1}(0))$.
4. The points in $\tilde{M}$ are CH-simple points for $\tilde{\pi}$. By Remark 10 each irreducible component $\Delta$ of the fiber $\pi^{-1}(0)$ is contained in at least one regular exceptional component $B_\Delta$. In this way, we can include $\pi^{-1}(0)$ in the union of the regular exceptional components. By Lemma 9 each two regular exceptional components are nodally-free connected. Now, we proceed as follows:

1. We show that $H \cup \tilde{E}$ is a neighborhood of $B \setminus \text{GN}(\tilde{F}', \tilde{E})$, for each regular exceptional component $B$.
2. We show that $H \cup E^N$ is a neighborhood of $\text{GN}(\tilde{F}', \tilde{E})$.

Let us prove the first assertion. If $B$ is a compact dicritical component, we are done. Otherwise, by Lemma 9 and by Proposition 6 there is a finite chain of regular exceptional components that connects $B$ with a compact dicritical component $B'$ through non-nodal curves. Now we do a holonomic transport from $B'$ to $B$ that allows us to cover with the leaves arriving to $B'$ the part of $B$ outside the generically nodal curves. To do this we invoke the behavior of the leaves at the regular points and at the simple points that are not nodal ones, described in Proposition 1.

In order to prove the second assertion, given a connected component $\mathcal{N}$ of $\text{GN}(\tilde{F}', \tilde{E})$, we find a non-nodal point $p \in \mathcal{N}$. Then $H \cup \tilde{E}$ is a neighborhood of this point and by saturation along $\mathcal{N}$, applying Remark 5 we cover $\mathcal{N}$. $\square$

Now, Proposition 7 implies Theorem 1 as follows. The set $V = \tilde{\pi}(H \cup \tilde{E})$ is a neighborhood of the origin in $U_0$ by $\tilde{F}$. Let $W \subset \mathbb{C}^3$ be an open neighborhood of the origin with $W \subset V$. The saturation $\text{Sat}_{\tilde{\pi}} W$ is open and contained in $V$. We take $U = \text{Sat}_{\tilde{\pi}} W$. Let $L$ be a leaf in $U$. Now $\tilde{\pi}^{-1}(L) \setminus \tilde{E}$ is connected and invariant by $\tilde{F}$. Let $\tilde{L}$ be the leaf of $\tilde{F}$ containing $\tilde{\pi}^{-1}(L) \setminus \tilde{E}$.

**Lemma 10.** $\tilde{L} \setminus \tilde{E} = \tilde{\pi}^{-1}(L) \setminus \tilde{E}$.

**Proof.** Take points $p \in \tilde{\pi}^{-1}(L) \setminus \tilde{E}$ and $q \in \tilde{L} \setminus \tilde{E}$. Since the two points are in $\tilde{L}$, there is a compact path $\delta(t)$, $\delta(0) = p$, $\delta(1) = q$ such that $\delta(t) \in \tilde{L}$ for $t \in [0, 1]$. By a local study at the points of the dicritical components, the set of the $t \in [0, 1]$ such that

\[ \delta(t) \in (\tilde{\pi}^{-1}(L) \setminus \tilde{E}) \cup \bigcup \{ E_j^N : E_j^N \text{ is a dicritical component of } E^N \} \]

is closed and open in $[0, 1]$. Thus, $q = \delta(1) \in \tilde{\pi}^{-1}(L)$. $\square$

Now, since $\tilde{L} \subset H$, there is a compact dicritical component $E_j$ of $E$ such that $\tilde{L} \cap E_j \neq \emptyset$. We find a germ of non singular analytic curve $\tilde{\gamma} \subset \tilde{L}$ transversal to $E_j$ in a point $p \in E_j$ with $e(E, p) = 1$. The projection $\gamma = \pi(\tilde{\gamma})$ is a germ of curve contained in $L$. 
8. SINGULAR LOCUS OF A RICH-FOLIATION

In this section we describe some features of the singular locus of a RICH-foliation $F$ at the intermediate steps of a fixed RI-reduction of singularities

\[(
\pi : (M, \pi^{-1}(0)) \to (\mathbb{C}^3, 0).
\]

We recall that $\pi$ is a composition $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_N$ of blow-ups $\pi_k : M_k \to M_{k-1}$, $k = 1, 2, \ldots, N$ such that for any $0 \le k \le N-1$ we have

1. The center $Y_k \subset M_k$ of the blow-up $\pi_{k+1}$ is non-singular, has normal crossings with the total exceptional divisor $E^k \subset M_k$ and it is contained in the adapted singular locus $\text{Sing}(F_k, E^k)$, where $F_k$ is the transform of $F$ (in particular it is invariant by $F_k$).

2. The intersection $Y_k \cap (\pi_1 \circ \pi_2 \circ \cdots \circ \pi_k)^{-1}(0)$ is a single point.

3. All the points of $M_N = M$ are CH-simple for $F, E$, where $E = E^N$ is the total exceptional divisor.

Given $0 \le k \le k' \le N$ we denote $\pi_{k,k} = \text{id}_{M_k}$ and $\pi_{k,k'} = \pi_{k+1} \circ \pi_{k+2} \circ \cdots \circ \pi_{k'}$ if $k < k'$. We take special notations for some particular cases $\rho_k = \pi_{N,k}$, $\sigma_k = \pi_{1,k}$, $\pi = \sigma_k \circ \rho_k$, with $\rho_k : M = M_N \to M_k$ and $\sigma_k : M_k \to (\mathbb{C}^3, 0)$.

We decompose the exceptional divisor $E^k$ into irreducible components

\[E^k = E^k_1 \cup E^k_2 \cup \cdots \cup E^k_{k'} \]

where $E^k_j$ is the strict transform by $\pi_k$ of $E^k_{j-1}$ for $j < k$ and $E^1_k = \pi^{-1}_k(Y_k-1)$. We write $E^k_{\text{dic}} \subset E^k$, the union of the irreducible components of $E^k$ invariant by $F_k$, respectively the generically transversal (dircital) components of $E^k$.

Remark 11. The $\pi_{k,k'}$ are morphisms of germs $\pi_{k,k'} : (M_{k'}, \sigma_{k'}^{-1}(0)) \to (M_k, \sigma_k^{-1}(0))$ around the compact subsets $\sigma_{k'}^{-1}(0) \subset M_{k'}$ and $\sigma_k^{-1}(0) \subset M_k$. An irreducible component $E^k_j$ of the exceptional divisor $E^k$ is compact if and only $E^k_j \subset \sigma_{k'}^{-1}(0)$ and this is equivalent to saying that $Y_k-1 \subset \sigma_{k'}^{-1}(0)$. In view of Property 2 of the reduction of singularities, this is also equivalent to saying that $Y_k-1$ is a single point. Conversely, the irreducible component $E^k_j$ is non-compact if and only if the center $Y_k-1$ is a germ of curve. Moreover, in this case $Y_k-1$ is a germ of curve not contained in $\sigma_{k'}^{-1}(0)$, in particular, it projects by $\sigma_{k'}^{-1}$ onto a curve in $M_0 = (\mathbb{C}^3, 0)$.

Remark 12. Let $\Gamma \subset M_k$ be a curve contained in the adapted singular locus of $F_k, E^k$. By Property 2, we see that only finitely many points of $\Gamma \cap \sigma_k^{-1}(0)$ are not simple points for $F_k, E^k$. In particular, if $\Gamma$ is a compact curve, that is $\Gamma \subset \sigma_k^{-1}(0)$, all points in $\Gamma$, except maybe finitely many, are simple points for $F_k, E^k$; moreover, up to eliminating finitely many other points of dimensional type three, the foliation has dimensional type two along $\Gamma$.

Remark 13. If $E^k_j$ is a compact dicritical component of $F_k$, there are only finitely many points in $\text{Sing}(F_k, E^k) \cap E^k_j$. That is, there is no curve $\Gamma$ contained in $\text{Sing}(F_k, E^k) \cap E^k_j$. If such a curve exists, by Remark 12 all points in $\Gamma$, except may be finitely many of them, are simple points for $F_k, E^k$; now, if $q \in \Gamma \cap E^k_j$ is a simple point for $F_k, E^k$, the dimensional type of $F_k$ in $q$ is two and $\Gamma$ is transversal to $E^k_j$, contradiction with the fact that $\Gamma \subset E^k_j$. This property has the following consequence in terms of local equations. Take a point $q \in E^k_j$ and suppose that $F_k$ is locally given at $q$ by $\omega = 0$ where

\[(a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)dz; \quad E^k_j = (z = 0))\]

and $a, b, c$ do not have common factors in $\mathbb{C}[x, y, z]$. Then the restriction $\mathcal{G}$ of $F_k$ to $E^k_j$ is locally given at $q$ by $\eta = a(x, y, 0)dx + b(x, y, 0)dy$ and moreover $a(x, y, 0), b(x, y, 0)$ do not have common factors in $\mathbb{C}[x, y]$.

Definition 11. An $\eta \in M_k$ is a trace point for $F_k, E^k$ if and only if it is not a pre-simple CH-corner for $F_k, E^k$. If in addition we have that $p \in \text{Sing}(F_k, E^k)$, then we say that $p$ is an s-trace point. An irreducible curve $\Gamma \subset \text{Sing}(F_k, E^k)$ is an s-trace curve for $F_k, E^k$ if and only if all the points of $\Gamma$ are s-trace points.

Remark 14. By the local description of pre-simple CH-corners, all the singular points around a pre-simple CH-corner are also pre-simple CH-corners. More precisely, the set of s-trace points

\[\text{Str}(F_k, E^k) = \text{Sing}(F_k, E^k) \setminus \{\text{pre-simple CH-corners}\}\]

is a closed analytic subset of $\text{Sing}(F_k, E^k)$. Hence the s-trace curves are the irreducible components of dimension one of $\text{Str}(F_k, E^k)$.

Notation 2. Given a point $p \in M$ and a normal crossings divisor $D \subset M$ we denote by $\epsilon(D; p)$ the number of irreducible components of $D$ passing through $p$. In the same way, if $\Gamma \subset M$ is an irreducible curve, we denote by $\epsilon(D; \Gamma)$ the number of irreducible components $D_j$ of $D$ such that $\Gamma \subset D_j$.

Remark 15. Let $\Gamma \subset \text{Sing}(F_k, E^k)$ be an irreducible curve. We have the following properties
(1) The curve \( \Gamma \) is not an s-trace curve if and only if all but finitely many points in \( \Gamma \) are pre-simple CH-corners of dimensional type two. In this case \( \Gamma \) is the intersection of two invariant components \( \Gamma = E^{b}_{k} \cap E^{b}_{f} \). We also say that \( \Gamma \) is generically pre-simple CH-corner.

(2) Assume that the generic point of \( \Gamma \) is simple for \( F_{k}, E^{b} \); note that this is always the case if \( \Gamma \) is a compact curve \( \Gamma \subset \sigma^{-1}_{k}(0) \). Then \( \Gamma \) is an s-trace curve if and only if \( e(E^{b}_{\text{inv}}, \Gamma) \leq 1 \).

Moreover, in the case that \( \Gamma \) is a compact curve, we have exactly the following two possibilities

a) \( e(E^{b}, \Gamma) = e(E^{b}_{\text{inv}}, \Gamma) = 2 \) and \( \Gamma \) is generically simple CH-corner.

b) \( e(E^{b}, \Gamma) = e(E^{b}_{\text{inv}}, \Gamma) = 1 \) and \( \Gamma \) is an s-trace curve.

Let us do some considerations about the fiber \( \sigma^{-1}_{k}(0) \). It decomposes into irreducible components

\[ \sigma^{-1}_{k}(0) = A^{b}_{1} \cup A^{b}_{2} \cup \cdots \cup A^{b}_{s_{b}} \]

where each one is either a compact irreducible component of \( E^{b} \) or a compact nonsingular curve. More precisely, \( \sigma^{-1}_{k}(0) \) has normal crossings with \( E^{b} \). If \( A^{b}_{i} \) is an irreducible component of \( \sigma^{-1}_{k}(0) \) of dimension one, the intersection of \( A^{b}_{i} \) with a compact component of \( E^{b} \) is at most one point and there is at least one non compact component \( E^{b}_{j} \) such that \( A^{b}_{i} \subset E^{b}_{j} \). Denote by Inv(\( \sigma^{-1}_{k}(0) \)) \( \subset \sigma^{-1}_{k}(0) \) the union of the invariant irreducible components of \( \sigma^{-1}_{k}(0) \).

Let us recall that if \( F \) is a foliation on \( M \) and \( p \in M \), the multiplicity \( \nu_{p} F \) is defined as the minimum of the multiplicities \( \nu_{p}(a_{i}), i = 1, 2, \ldots, n \), where \( F \) is locally given by \( \omega = 0 \) with \( \omega = \sum_{i=1}^{n} a_{i} dx_{i} \) and the coefficients \( a_{i} \in \mathbb{C}[x_{1}, x_{2}, \ldots, x_{n}] \) are without common factor.

**Remark 16.** If \( F \) is a germ of foliation in \( (\mathbb{C}^{2}, 0) \) and \( L = (x = 0) \) is an invariant line, then \( \mathcal{G} \) is given by \( \alpha = 0 \) where \( \alpha = a(x, y)dx + xb(x, y)dy \) and \( a, xb \) are without common factors. The restricted multiplicity \( \mu(\mathcal{G}, L; 0) \) is the order \( \nu_{y}(a(0, y)) \). Moreover, when \( \mathcal{G} \) is a foliation on the projective plane \( \mathbb{P}^{2}_{\mathbb{C}} \) and \( L \) is a straight line invariant by \( \mathcal{G} \) we have that

\[ d + 1 = \sum_{q \in L} \mu(\mathcal{G}, L; q). \]

where \( d \) is the degree of \( \mathcal{G} \) (see [6] for more details).

We will do frequently arguments by induction on the height \( h(p) \) of a point \( p \in \sigma^{-1}_{k}(0) \) with respect to the sequence \( S \). This number is defined by

\[ h(p) = \# \{ k' \geq k; p \in \pi_{k'}(Y_{k'}) \}. \]

**Lemma 11.** Assume that \( Y_{k-1} = \{ p \} \) is a single point and \( \pi_{b} \) is a dicritical blowing-up. Let \( \mathcal{G} \) be the restriction of \( F_{b} \) to the projective plane \( E^{b}_{b} \) and let \( d \) be the degree of \( \mathcal{G} \). We have \( d + 1 = \nu_{p} F \) and moreover

1. A point \( q \in E^{b}_{k} \) is a pre-simple CH-corner for \( F_{b} \) if and only if it is a pre-simple CH-corner for \( \mathcal{G}, D \), where \( D = \cup_{i=1}^{b-1} E^{b}_{i} \cap E^{b}_{i} \).

2. If \( p \) is an s-trace point that belongs to a non dicritical component \( E^{b}_{k-1} \) of \( E^{b-1} \), there is an s-trace point \( q \in E^{b}_{k} \cap E^{b}_{b} \).

**Proof.** Let \( F_{b-1} \) be locally given at \( p \) by \( \omega = 0 \) with \( \omega = adx + bdy + cdz \), where \( a, b, c \in \mathbb{C}\{x, y, z\} \) are without common factor. Put \( r = \nu_{p} F_{b-1} \) and write

\[ a = A_{r} + A_{r+1} + \cdots; b = B_{r} + B_{r+1} + \cdots; c = C_{r} + C_{r+1} + \cdots \]

the decomposition into homogeneous components. Since \( \pi_{b} \) is dicritical, we have \( XA_{r} + YB_{r} + ZC_{r} = 0 \). The foliation \( \mathcal{G} \) is defined in the projective space \( E^{b}_{b} \) by the global 1-form

\[ W = A_{r} dx + B_{r} dy + C_{r} dz \]

and in view of Remark [13] the coefficients \( A_{r}, B_{r}, C_{r} \) do not have common factor. This means that \( d = r - 1 \).

Now, let \( q \in E^{b}_{k} \) be a pre-simple CH-corner for \( F_{b}, E^{b} \). There are local coordinates \( x, y, z \) at \( q \) such that \( E^{b}_{k} = (z = 0) \) and we have one of the following cases

1. (1) The point \( q \) is non singular and the coordinates may be chosen such that \( F = (dx = 0), E^{b}_{\text{inv}} = (x = 0) \) and \( (z = 0) \subset E^{b}_{\text{dic}} \subset (yz = 0) \) locally at \( q \).

2. (2) The point \( q \) is singular and the coordinates may be chosen such that

\[ F = \{(\lambda + f(x, y))gdx + (\mu + g(x, y))xydy = 0\}, \lambda \mu \neq 0, \nu_{y}(f, g) \geq 1, \]

with \( E^{b}_{\text{inv}} = (xy = 0) \) and \( (z = 0) = E^{b}_{\text{dic}} \) locally at \( q \).
In both cases, we see that \( q \) is a pre-simple corner for \( G, D \). Conversely, assume that \( q \) is a pre-simple corner for \( G, D \). Take notations as in Equation 10 with \( E^b_p = (z = 0) \). If \( q \) is non singular for \( G \), it is also non singular for \( F_b \) and we deduce that it is a pre-simple CH-corner just by looking at the positions of the divisors. If it is singular, then \( E^b = (xyz = 0) \) locally at \( q \) and
\[
G = \{ a(x, y, 0)dx + b(x, y, 0)dy = (\lambda + f(x, y))ydx + (\mu + g(x, y))xdy = 0 \}.
\]
The vector field \( \xi = c(x, y, z)x\partial / \partial x - a(x, y, z)\partial / \partial z \) trivialis the foliation \( F_b \) and we get a pre-simple CH-corner for \( F_b, E^b \).

Finally, assume that \( p \) is an s-trace point belonging to a non dicritical component \( E^{b-1}_i \) of \( E^{b-1} \). We have three cases to consider:

**Case 1** \( e(E^{b-1}_{inv}, p) = 1 \). The straight line \( L = E^{b-1}_1 \cap E^{b-1}_{inv} = E^{b-1}_1 \cap E^{b-1}_3 \) is invariant by \( G \). By Equation 11 we have that \( L \) contains \( d + 1 = r \geq 1 \) singular points of \( G \). Moreover, any point \( q \in \text{Sing}G \cap L \) is an s-trace point. If \( d = 0 \), the intersection point \( q_0 \in L_1 \cap L_2 \) is the only singular point of \( G \), moreover it is a pre-simple CH-corner; this implies in view of Proposition 5 that \( p \) is a pre-simple CH-corner, contradiction. Hence \( d \geq 1 \), in this case by Equation 11 we find at least one singular point in \( L_1 \) (and also in \( L_2 \)) that is an s-trace point. To be precise, if \( q_0 \) is a pre-simple CH-corner, we have that \( \mu(G, L_1; q_0) = 1 \) and hence there is another singular point \( q' \in L_1 \) that must be an s-trace point.

**Case 2** \( e(E^{b-1}_{inv}, p) = 2 \). We have \( E^{b-1}_2 \cap E^{b-1}_{inv} = L_1 \cup L_2 \) with \( L_i = E^{b-1}_2 \cap E^{b-1}_i \), where \( E^{b-1}_i, E^{b-1}_j \) and \( E^{b-1}_k \) are the two non dicritical components of \( E^{b-1} \) containing \( p \). Note that \( L_1 \) and \( L_2 \) are invariant lines for \( G \). If \( d = 0 \), the intersection point \( q_0 \in L_1 \cap L_2 \) is the only singular point of \( G \), moreover it is a pre-simple CH-corner; this implies in view of Proposition 5 that \( p \) is a pre-simple CH-corner, contradiction. Hence \( d \geq 1 \), in this case by Equation 11 we find at least one singular point in \( L_i \) (and also in \( L_j \)) that is an s-trace point.

**Case 3** \( e(E^{b-1}_{inv}, p) = 3 \). We have \( E^{b-1}_3 \cap E^{b-1}_{inv} = L_1 \cup L_2 \cup L_3 \) with \( L_i = E^{b-1}_3 \cap E^{b-1}_i \), \( L_k = E^{b-1}_3 \cap E^{b-1}_k \), where \( E^{b-1}_i, E^{b-1}_j \) and \( E^{b-1}_k \) are the three non dicritical components of \( E^{b-1} \) that contain \( p \). Note as before that \( L_1, L_2 \) and \( L_3 \) are invariant lines for \( G \) and there are three singular points \( q_{ij}, q_{ik}, q_{jk} \) corresponding to the respective intersections of two lines. This implies that \( d \geq 1 \). If \( d = 1 \) the points \( q_{ij}, q_{ik}, q_{jk} \) are the only singular points of \( G \) and they are pre-simple CH corners; thus \( p \) must be a pre-simple CH-corner, contradiction. If \( d \geq 2 \), we find as before at least a point in \( L_i \) (and also in \( L_j, L_k \)) that is an s-trace point.

**Lemma 12.** Assume that \( Y_{b-1} = E^{b-1}_1 \cap E^{b-1}_j \) is a generically pre-simple CH-corner curve and let \( p \) be the intersection point \( Y_{b-1} \cap \sigma_{b-1}^{-1}(0) \). Let us consider the point \( \{ q' \} = \pi_{b-1}(p) \cap E^{b-1}_i \). If \( p \) is an s-trace point, then \( q' \) is also a s-trace point.

**Proof.** Take local coordinates at \( p \) such that \( E^{b-1}_i = \{ y = 0 \} \) and \( E^{b-1}_j = \{ x = 0 \} \) and suppose that \( F_{b-1} \) is locally given at \( p \) by \( \omega = 0 \) where \( \omega \) is the 1-form
\[
\omega = a(x, y, z)\frac{dx}{x} + b(x, y, z)\frac{dy}{y} + c(x, y, z)\frac{dz}{z}.
\]
If \( e(E^{b-1}_{inv}, p) = 2 \) we put \( \epsilon = 0 \) and if \( e(E^{b-1}_{inv}, p) = 3 \) we put \( \epsilon = 1 \). Either way, \( a, b, c \) are without common factors and in view of the hypothesis we have that
\[
a = \phi(z) + x f_1 + y f_2; \quad b = \psi(z) + x g_1 + y g_2; \quad \phi(z) \psi(z) \neq 0.
\]
Moreover, since \( p \) is not a pre-simple CH-corner, we deduce that \( \phi(0) = \psi(0) = 0 \), otherwise, we should contradict the CH character of the foliation \( F_{b-1} \) as in the proof of Proposition 5. In local coordinates \( x, y' = y/x, z \) the foliation \( F_{b-1} \) is given at \( q' \) by
\[
\omega = a'\frac{dx}{x} + b'\frac{dy'}{y'} + c'\frac{dz}{z},
\]
where \( \delta = 0 \) is the dicritical case and \( \delta = 1 \) if the blow-up is non dicritical. The coefficients \( a', b', c' \in \mathbb{C}\{x, y', z\} \) are without common factor and given by
\[
a' = x^{d-1}(a(x, xy', z) + b(x, xy', z)); \quad b' = b(x, xy', z); \quad c' = c(x, xy', z).
\]
We have that \( b'(0, y', 0) = 0 \) and thus \( q' \) is not a pre-simple CH-point. Moreover, it is a singular point since the foliation \( F_{b-1} \) is locally given at \( q' \) by the holomorphic form \( \Omega = y'x^d z^\omega \). Hence \( q' \) is an s-trace point.

**Proposition 8.** Let \( p \in \sigma_{b-1}^{-1}(0) \) be an s-trace point for \( F_b, E^k \) that belongs to a non dicritical component \( E^k_i \) of \( E^k \). There is a s-trace curve \( \Gamma \) such that \( p \in \Gamma \subset E^k_i \).

**Proof.** We do induction on the height \( h(p) \) of \( p \). If \( h(p) = 0 \), we are done, since \( p \) is a simple CH-corner point. Assume that \( h(p) \geq 1 \). Let \( b > k \) be the first index such that \( p \in \pi_{k(b-1)}(Y_{b-1}) \). We consider several cases.

**First case:** the center \( Y_{b-1} \) is a point \( Y_{b-1} = \{ p' \} \) and \( \pi_k \) is non-dicritical. We do the blow-up \( \pi_b \) and by Proposition 5 there is a trace point \( q \in \pi_{b-1}^{-1}(p') = E^b_b \). Since \( E^b_b \) is compact (it is isomorphic to \( E^b_2 \)) and invariant, we can apply induction hypothesis to \( q \in E^b_b \) to find a trace compact curve \( \Gamma \subset E^b_b \). The curve \( \Gamma \) intersects the projective line \( E^b_b \cap E^b_b \) at least in a point \( q' \) that must be a trace point. We apply induction hypothesis to \( q' \in E^b_b \) to find a trace curve \( \Gamma' \subset E^b_b \) such that \( q' \in \Gamma' \) and we take \( \Gamma = \pi_{kb}(\Gamma') \). (See Remark 114)
Second case: $Y_{b-1}$ is a point $Y_{b-1} = \{p'\}$ and $\pi_b$ is a dicritical blow-up. By Lemma[11] we find a point $q' \in E_i^b$ that is a trace point and we proceed by induction as before.

Third case: $Y_{b-1}$ is a curve transversal to $E_i^{b-1}$. By Proposition[5] there is a trace point $q$ in $\pi_b^{-1}(p') = E_b^b \cap E_i^b$, where $p'$ is the (only) point over $p$ such that $\pi_{b-1}(p') = p$. We apply induction hypothesis to $q \in E_i^b$ at $q$ to obtain a trace curve $\Gamma' \subset E_i^b$ and we put $\Gamma = \pi_{b \ell}(\Gamma')$.

The remaining situation is that $Y_{b-1}$ is a curve contained in $E_i^{b-1}$. If $Y_{b-1}$ is an s-trace curve, we are done by taking $\Gamma = \pi_{b(b-1)}(Y_{b-1})$. Otherwise we apply Lemma[12] to proceed by induction. □

Proposition 9. Assume that the center $Y_{b-1}$ of $\pi_b$ is an s-trace curve. Then there is an s-trace curve $\Gamma \subset E_i^b$ such that $\pi_b(\Gamma) = Y_{b-1}$.

Proof. It is enough to look at the generic point of $Y_{b-1}$ and to apply Proposition[4]. In this way we find at least one s-trace point over each generic point of $Y_{b-1}$ and thus we necessarily have at least an s-trace curve as stated. □

9. Nodal components for RICH-Foliations

Consider a RICH-foliation $\mathcal{F}$ in $(\mathbb{C}^3,0)$ and fix an RI-reduction of singularities $\pi$ as in Equation[2]. Take a nodal component $\mathcal{N}$ of $\pi^*\mathcal{F}, E$, where $E = E_0^{\mathcal{N}}$ is the exceptional divisor of $\pi$. In this section we prepare the proof of Theorem[3] by giving a list of structural properties of $\mathcal{N}$ at intermediate steps of the reduction of singularities assuming that $\mathcal{N}$ is compact and does not intersect the union $E_{\text{dic}}$ of the dicritical components of $E$. In the next section we will find a contradiction with these properties.

For any $0 \leq k \leq N$, let us denote $N_k = \rho_k(\mathcal{N})$. We have that $N_k \subset \sigma_k^{-1}(0)$ and hence $N_k$ is a connected and compact analytic subset of $E^k$. We have two possibilities: either $N_k$ is a single point (in this case we put $s_k = 0$) or it is a finite union of $s_k \geq 1$ compact irreducible analytic curves $N_k = \Gamma_1^k \cup \Gamma_2^k \cup \cdots \Gamma_{s_k}^k$. Let us remark that the curves $\Gamma_j^k \subset \sigma_k^{-1}(0)$ will never be used as a center of blow-up in the reduction of singularities. This implies that the generic points of $\Gamma_j^k$ are CH-simple for $\mathcal{F}_k, E^k$ and only finitely many points in $\Gamma_j^k$ will be modified by subsequent blow-ups. In particular $N_{k+1}$ has the form

$$N_{k+1} = \Gamma_{1}^{k+1} \cup \Gamma_{2}^{k+1} \cup \cdots \Gamma_{s_{k+1}}^{k+1}$$

where $s_{k+1} \geq s_k$ and for each $1 \leq j \leq s_k$ the curve $\Gamma_j^{k+1}$ is the strict transform of $\Gamma_j^k$ by $\pi_{k+1}$. The date of birth $b(\mathcal{N})$ of $\mathcal{N}$ is the index such that $s_k = 0$ if $k < b(\mathcal{N})$ and $s_k(\mathcal{N}) \geq 1$. Note that $1 \leq b(\mathcal{N}) \leq N$.

We will give a list of results about the local behavior of $N_k$ when $k \geq b(N)$. The following Lemma[13] shows that $N_k$ has a behavior similar to $\mathcal{N} = N_N$ concerning the corners of the exceptional divisor.

Lemma 13 (Non dicriticality and nodality at corners). Assume that $k \geq b(N)$. Let $p \in N_k$ be locally the intersection of three components $E_i^k, E_j^k$ and $E_b^k$ of $E^k$ and suppose that $E_i^k \cap E_j^k \subset N^k$.

Then $E_i^k, E_j^k$ and $E_b^k$ are invariant for $\mathcal{F}_k$ and

$$E_i^k \cap E_j^k \subset N_k \Rightarrow E_i^k \cap E_j^k \subset N_k.$$

(Equivalently, we have $E_i^k \cap E_j^k \not\subset N_k \Rightarrow E_i^k \cap E_j^k \not\subset N_k$).

Proof. We do induction on the height $h(p)$ of $p$. If $h(p) = 0$, the point $p$ is a simple CH-corner for $\mathcal{F}_k, E^k$ of nodal type that will not be modified by further blow-ups, so we can think locally at $p$ as in the case $k = N$. Since $\mathcal{N}$ does not intersect $E^{\text{dic}}$, the three components $E_i^k, E_j^k$ and $E_b^k$ are invariant for $\mathcal{F}_k$ and $p$ is a simple CH-corner of dimensional type three. The second assertion of the Lemma is a direct consequence of the observations in Remark[9].

Now, assume that $h(p) \geq 1$. Let $b \geq k$ be the first index such that $p_b \in Y_b$, where $\pi_{b b}^{-1}(p_b) = p$. Note that the local situation at $p_b$ is exactly the same one as the local situation at $p$. Let us consider the blow-up $\pi_{b+1}$. Let us note that $Y_b \not\subset E_i^k \cap E_j^k$, since the fact that $E_i^k \cap E_j^k \subset N_k$ implies that $E_i^k \cap E_j^k$ is a compact curve and we do not use compact curves as centers in view of Remark[11].

First, we see that $\pi_{b+1}$ is a non dicritical blow-up (that is, the component $E_{b+1}^k$ is invariant); otherwise, we apply induction hypothesis to the intersection point $p'$ in $E_i^{b+1} \cap E_j^{b+1} \cap E_b^{b+1}$ that is a point in $N_{b+1}$ such that $E_i^{b+1} \cap E_j^{b+1} \subset N_{b+1}$. Then we have the next possibilities to consider

1. The center of $\pi_{b+1}$ is a point $Y_b = \{p_b\}$.

2. The center of $\pi_{b+1}$ is a point $Y_b = E_i^k \cap E_j^k$.

3. The center of $\pi_{b+1}$ is a point $Y_b = E_i^k \cap E_b^k$. 


The case (3) is like the case (2) just by interchanging the roles of the indices $i, j$. Consider the case (1). Define the points $p'_i$, $p'_j$ and $p'_k$ by
\[
\{p'_i\} = E_i^{b+1} \cap E_j^{b+1} \cap E_{b+1}^{b+1};
\]
\[
\{p'_j\} = E_i^{b+1} \cap E_k^{b+1} \cap E_{b+1}^{b+1};
\]
\[
\{p'_k\} = E_j^{b+1} \cap E_k^{b+1} \cap E_{b+1}^{b+1}.
\]
We apply induction hypothesis at $p'_i$ to see that $E_i^{b+1}$ and $E_j^{b+1}$ are invariant components of $E_{b+1}^{b+1}$, hence $E_i^b$ and $E_j^b$ are invariant components of $E^b$. Also by induction hypothesis applied at $p'_k$ one of the following properties holds
\[
\text{i)} \ E_i^{b+1} \cap E_{b+1}^{b+1} \subset N_{b+1} \text{ and } E_j^{b+1} \cap E_{b+1}^{b+1} \not\subset N_{b+1},
\]
\[
\text{ii)} \ E_i^{b+1} \cap E_{b+1}^{b+1} \not\subset N_{b+1} \text{ and } E_j^{b+1} \cap E_{b+1}^{b+1} \subset N_{b+1}.
\]
Assume we have i). We apply induction hypothesis at $p'_j$ to see that $E_j^{b+1}$ is invariant, hence $E_j^b$ is also invariant, and moreover one of the following properties holds
\[
a) \ E_i^{b+1} \cap E_{b+1}^{b+1} \subset N_{b+1} \text{ and } E_j^{b+1} \cap E_{b+1}^{b+1} \not\subset N_{b+1},
\]
\[
b) \ E_i^{b+1} \cap E_{b+1}^{b+1} \not\subset N_{b+1} \text{ and } E_j^{b+1} \cap E_{b+1}^{b+1} \subset N_{b+1}.
\]
In case a) we have $E_i^b \cap E_j^b \subset N_b$. It remains to prove that $E_j^b \cap E_i^b \not\subset N_b$. But if $E_i^b \cap E_j^b \subset N_b$, we apply induction hypothesis at $p'_j$ and then either $E_i^{b+1} \cap E_{b+1}^{b+1} \subset N_{b+1}$ or $E_j^{b+1} \cap E_{b+1}^{b+1} \subset N_{b+1}$; in the first case we find a contradiction at the point $p'_i$ and in the second one we find a contradiction at $p'_j$.

In case b) we have $E_i^b \cap E_j^b \not\subset N_b$. Moreover, we can apply induction hypothesis at $p'_j$ to deduce that either $E_i^{b+1} \cap E_{b+1}^{b+1} \subset N_{b+1}$ or $E_j^{b+1} \cap E_{b+1}^{b+1} \subset N_{b+1}$, but in the second case we find a contradiction at $p'_i$ and hence we have that $E_i^b \cap E_j^b \not\subset N_b$.

Finally, if we have ii) we do the same arguments as in i) by interchanging the indices $i, j$.

Now, we consider the case (2) where the center of $\pi_{b+1}$ is the curve $Y_b = E_i^b \cup E_j^b$. Note that $Y_b$ is non compact and hence $Y_b \not\subset N_b$. Consider the points
\[
\{q'_i\} = E_i^{b+1} \cap E_j^{b+1} \cap E_{b+1}^{b+1};
\]
\[
\{q'_j\} = E_i^{b+1} \cap E_k^{b+1} \cap E_{b+1}^{b+1}.
\]
Applying induction at $q'_i$ we see that $E_i^{b+1}$, $E_j^{b+1}$, and $E_{b+1}^{b+1}$ are invariant components of $E_{b+1}^{b+1}$. Hence $E_i^b$, $E_j^b$ are invariant. Also by induction at $q'_j$ and since $E_i^{b+1} \cap E_{b+1}^{b+1}$ is non compact, we have that $E_k^{b+1} \cap E_{b+1}^{b+1} \subset N_{b+1}$.

Now, we apply induction at $q'_i$ to see that $E_j^{b+1}$ is invariant and that $E_j^{b+1} \cap E_{b+1}^{b+1} \subset N_{b+1}$ since $E_{b+1}^{b+1}$ is non compact. Hence $E_j^b$ is invariant and $E_j^b \cap E_i^b \subset N_b$.

**Remark 17.** In view of the terminology introduced in Remark [15] and considering the fact that a compact curve of the adapted singular locus is not contained in a dicritical component by Remark [13] we can reformulate Lemma [13] as follows:

"Let $\Gamma$ be a curve that is an irreducible component of $N_b$ with $e(E^k, \Gamma) = 2$. Then $\Gamma$ is generically a simple CH-corner and if $p$ is a point of intersection of $\Gamma$ with a component $E^k_i$ of $E^k$ transversal to $\Gamma$, we have that $E^k_i$ is invariant and there is exactly one curve $\Gamma' \subset N_b$ such that $\Gamma' \not= \Gamma$, $p \in \Gamma \cap \Gamma'$ and $\Gamma'$ is generically simple CH-corner."

**Definition 12.** Let $\Gamma$ be an s-trace curve for $F_k$, $E^k$ with $e(E^k, \Gamma) \geq 1$. We say that $\Gamma$ is interrupted by an irreducible component $E^k_i$ of $E^k$ at a point $p$ if $p \in \Gamma \cap E^k_i$ and $\Gamma \not\subset E^k_i$. The interruption is of nodal type if $E^k_i \cap E^k_j \subset N_b$ (locally at $p$), for any $E^k_i$ such that $\Gamma \subset E^k_i$.

**Remark 18.** In the case $e(E^k, \Gamma) = 2$ with $\Gamma \subset E^k_i \cap E^k_j$ the curve $\Gamma$ is not compact since it is supposed to be an s-trace curve, see Remark [15] In particular $\Gamma \not\subset N_b$. In this case, the following statements are equivalent
\[
a) \ \text{The interruption of } \Gamma \text{ by } E^k_i \text{ at } p \text{ is a nodal interruption.}
\]
\[
b) \ E^k_i \cap E^k_j \subset N_b;
\]
\[
c) \ E^k_i \cap E^k_j \subset N_b.
\]
This is because by Lemma [13] we have that $E^k_i \cap E^k_j \subset N_b$ implies that $E^k_j \cap E^k_i \subset N_b$ and conversely. Also by Lemma [13] in the case of a nodal interruption, all the concerned components $E^k_i, E^k_j, E^k_k$ are invariant ones.

**Proposition 10** (Non dicriticalness and nodality at trace points). Consider a point $p \in N_b$ with $k \geq b(N)$. Then all the components of $E^k$ through $p$ are non dicritical. Moreover, let $\Gamma$ be an s-trace curve for $F_k$, $E^k$ interrupted by $E^k_i$ at $p$ such that $e(E^k, \Gamma) \geq 1$. There is an s-trace curve $\Gamma' \subset E^k_i$ with $p \in \Gamma'$ such that
\[
\text{(1)} \ \text{If the interruption is of nodal type we have } \Gamma \subset N_b \Rightarrow \Gamma' \not\subset N_b.
\]
(2) If the interruption is not of nodal type we have \( \Gamma \subset N_k \Leftrightarrow \Gamma' \subset N_k \).

Proof. We proceed by induction on the height \( h(p) \). If \( h(p) = 0 \) we are done in view of the description of the singular locus at simple points and the hypothesis that \( \mathcal{N} \) has only compact irreducible components and does not intersect the dicritical components of the divisor.

Assume that \( h(p) \geq 1 \). In order to simplify the notation we can assume that \( p \in Y_k \), otherwise we consider the first index where this holds as in the proof of Lemma [12]. Also in order to simplify the writing, let us denote \( b = k + 1 \). Let us note that since \( k \geq b(\mathcal{N}) \), there is at least one curve \( \Delta \subset N_k \) such that \( p \in \Delta \). Note that \( \Delta \subset E^k \) and \( \Delta \neq Y_k \), because \( \Delta \) is a compact curve. We will denote by \( \Delta \) such curves if no confusion arises.

First we prove that \( \pi_b \) is a non-dicritical blow-up. We apply the induction hypothesis at a point \( p' \in E^k_b \cap \Delta' \) where \( \Delta' \) is the strict transform of \( \Delta \) by \( \pi_b \), this implies that \( E^k_b \) is invariant, that is \( \pi_b \) is a non dicritical blow-up.

A) Let us now prove that all the components of \( E^k \) through \( p \) are non dicritical. We have to consider the cases that \( Y_k = \{ p \} \) and \( Y_k \) is a germ of curve at \( p \).

Case A-1: \( Y_k = \{ p \} \). We recall that the divisor \( E^k_b \) is invariant and isomorphic to a projective plane \( \mathbb{P}^2 \). We have two possibilities

a) There is an s-trace curve \( \Delta \).

b) All the curves \( \Delta \) are generically simple CH-corners.

If we are in case \( a \), take a point \( p' \in \Delta' \cap E^k_b \), where \( \Delta' \) is the strict transform of \( \Delta \) by \( \pi_b \). We can apply induction hypothesis at \( p' \), since \( \Delta' \) is an s-trace curve and \( \Delta' \) is interrupted by \( E^k_b \) at \( p' \). The interruption may be of nodal type or not, in both cases we find a curve \( \Delta'' \subset E^k_b \cap N_b \) with \( p' \in \Delta'' \). Now, given any component \( E^k_i \) with \( p \in E^k_i \), there is a point \( \pi'' \) belonging to the intersection of the projective line \( E^k_i \cap E^k_b \) and \( \Delta'' \). By induction hypothesis at \( \pi'' \) we conclude that \( E^k_i \) is invariant. If we are in case \( b \), we take a generically simple CH-corner curve \( \Delta \) with \( p \in \Delta \) and we apply Lemma [13] at the point \( p' \in \Delta' \cap E^k_b \) to find a curve \( \Delta'' \subset E^k_b \cap N_b \), as before. We conclude as in case \( a \) that any component of \( E^k \) through \( p \) is invariant.

Case A-2: \( Y_k \) is a germ of curve at \( p \). Note that \( e(E^k, p) \geq 1 \) since \( \Delta \subset E^k \) (this is also valid for the previous case). Suppose that \( \Delta \subset E^k \), taking a point \( p' \in E^k \cap \Delta' \) we have that \( p' \in N_b \cap E^k \). By induction \( E^k \) and hence \( E^k_i \) are invariant components. Thus, it is enough to look at the components \( E^k_i \) such that \( p \in E^k_i \) and \( \Delta \not\subset E^k_i \) for any \( \Delta \), in particular, we can suppose that \( e(E^k, p) \geq 1 \).

We have several possibilities

1. \( e(E^k, Y_k) = 1, e(E^k, p) = 2 \). Put \( Y_k \subset E^k_b \) and \( p \in E^k_b \cap E^k_i \). If \( \Delta \subset E^k_b \), the point \( p' \in \Delta' \cap E^k_b \) belongs to \( E^k_i \cap E^k_b \) and both \( E^k_i \) and \( E^k_b \) are invariant components. Assume now that \( \Delta \subset E^k_b \) but \( \Delta \not\subset E^k_i \), in particular \( \Delta \) is an s-trace curve. We consider a point \( p' \in \Delta' \cap E^k_b \) and we are going to apply induction at \( p' \). We have two possible situations

a) \( p' \in E^k_i \). Then \( E^k_b \) and \( E^k_i \) are invariant components as above.

b) \( p' \not\in E^k_i \). Since \( p' \in \Delta' \subset E^k_b \), we have that \( E^k_b \) is an invariant component. Moreover, the s-trace curve \( \Delta' \) is interrupted at \( p' \) by \( E^k_i \) and we can apply induction. The only compact curve through \( p' \) contained in \( E^k_b \) is \( E^k_i \cap E^k_b \). If the interruption is not a nodal one, we should have a compact curve \( \Gamma' \subset N_b \cap E^k_b \) different from \( E^k_i \cap E^k_b \). This is not possible, then we have a nodal interruption and thus \( \Delta'' \subset N_b \), where \( \Delta'' = E^k_b \cap E^k_i \). Consider the point \( \pi'' \) belonging to \( \Delta'' \cap E^k_b \). By induction hypothesis at \( \pi'' \) we deduce that \( E^k_i \) and hence \( E^k_b \) are invariant components.

2. \( e(E^k, Y_k) = e(E^k, p) = 2 \). Put \( Y_k \subset E^k_b \cap E^k_i \). Note that \( \Delta \neq E^k_b \cap E^k_i \) and thus, up to a reordering of the indices we have a curve \( \Delta \subset E^k_b \) and \( \Delta \not\subset E^k_i \). We deduce as above that \( E^k_i \) is invariant. Let \( p' \) be the point given by \( p' \in \Delta' \cap E^k_b \), that is \( \{ p' \} = \pi_b^{-1}(p) \cap E^k_b \). The only compact curve contained in \( E^k_b \) through \( p' \) is \( \Delta'' = \pi_b^{-1}(p) \). By applying induction at \( p' \) we conclude that \( \Delta'' \subset N_b \). Moreover \( \Delta'' \) is interrupted in a non nodal way by \( E^k_i \) at \( p'' \in E^k_i \cap \Delta'' \). In particular \( p'' \in N_b \) and by induction \( E^k_i \) is an invariant component. (By the way, we find an s-trace curve \( \Delta'' \subset E^k_i \cap N_b \) and thus there is also a compact curve \( \Delta_1 \subset N_b \cap E^k_i \).)

3. \( e(E^k, Y_k) = 2, e(E^k, p) = 3 \). Put \( Y_k \subset E^k_b \cap E^k_i \) and \( p \in E^k_b \cap E^k_i \). Denote \( \Delta'' = \pi_b^{-1}(p) \). We are going to show that \( \Delta'' \subset N_b \), then we can conclude by induction applied at \( \Delta'' \cap E^k_i \) and \( \Delta'' \cap E^k_b \) that \( E^k_i \) and \( E^k_b \) are invariant components, moreover since \( \Delta'' \subset E^k_b \) we also conclude that \( E^k_i \) is an invariant component. Now, if \( \Delta \) is an s-trace curve, we conclude as in the previous cases that \( \Delta'' \subset N_b \). Finally, if \( \Delta \subset E^k_b \cap E^k_i \), we apply Proposition [5] to conclude that \( \Delta'' \subset N_b \).

B) Now, let \( \Gamma \) be an s-trace curve interrupted by \( E^k_b \) at \( p \) such that \( e(E^k, \Gamma) \geq 1 \). If \( e(E^k, \Gamma) = 1 \) we put \( \Gamma \subset E^k_b \) and if \( e(E^k, \Gamma) = 2 \) we put \( \Gamma \subset E^k_i \) and hence \( \Gamma = E^k_i \cap E^k_b \), locally at \( p \). We denote by \( \Gamma \) the strict
transform of $\Gamma$ (in the cases that $\Gamma \not\subset Y_k$) and by $\tilde{p}$ a point in $\tilde{\Gamma} \cap E^b_k$. Let us also consider a curve $\Delta \subset N_k$ with $p \in \Delta$, denote by $\Delta'$ the strict transform of $\Delta$ and take a point $q' \in \Delta'$.

**B-0) Case** $Y_k = \Gamma$. Note that in this case we have that $\Gamma \not\subset N_k$. If all the points in $\pi_b^{-1}(p) \cap \text{Sing}(F_b, E^b_k)$ are pre-simple CH-corners, we deduce that $p$ is also a pre-simple CH-corner by Proposition 8, but this is not possible since $\Gamma = Y_k$ is an s-trace curve. Otherwise, there is at least one s-trace point $r' \in \pi_b^{-1}(p)$. We apply Proposition 8 at $r'$ to find a trace curve $\tilde{\Gamma} \subset E^b_k$ and $\tilde{\Gamma}' \subset E^b_k$ with $r' \in \tilde{\Gamma}$. Now, $\tilde{\Gamma}$ is not compact and hence $\tilde{\Gamma} \not\subset N_k$. Moreover, if the interruption of $\Gamma = Y_k$ at $p$ is nodal, we find by induction that $\pi_b^{-1}(p) \subset N_b$ and thus the interruption of $\Gamma$ at $r'$ by $E^b_k$ is also a nodal interruption. By induction, we find a trace curve $\tilde{\Gamma}' \subset E^b_k$ with $r' \in \tilde{\Gamma}'$ such that $\tilde{\Gamma}' \subset N_b$. By projection of $\tilde{\Gamma}'$ we find $\Gamma' \subset N_b$. If the interruption of $\Gamma$ at $p$ is not nodal, we also find that $\pi_b^{-1}(p) \not\subset N_b$ and thus the interruption of $\Gamma$ at $r'$ by $E^b_k$ is also a nodal interruption. By induction we find $\Gamma' \not\subset N_b$ as above.

**B-1) Case** $Y_k = \{p\}$. Assume first that $e(E^k, p) = 2$ with $p \in E^k_i \cap E^k_j$. If $\tilde{p} \in E^b_i \cap E^b_j \cap E^b_k$ we can apply simultaneously induction at $\tilde{p}$ and Proposition 8 to see that there is an s-trace curve $\tilde{\Gamma}'$ with $\tilde{p} \in \tilde{\Gamma}' \subset E^b_k$ such that

$$
\text{If } E^b_i \cap E^b_j \subset N_b \text{ then } (\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}' \subset N_b)
$$

$$
\text{If } E^b_i \cap E^b_j \not\subset N_b \text{ then } (\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}' \not\subset N_b)
$$

If $\tilde{p} \in E^b_i \cap E^b_j \setminus E^b_k$, by induction applied at $\tilde{p}$ we find an s-trace curve $\tilde{\Gamma}_1$ with $\tilde{p} \in \tilde{\Gamma}_1 \subset E^b_k$ that cuts $E^b_i \cap E^b_j$ in a point $\tilde{p}_1$ and hence it is interrupted at $\tilde{p}_1$ by $E^b_k$, this implies the existence of an s-trace curve $\tilde{\Gamma}'$ with $\tilde{p}_1 \in \tilde{\Gamma}' \subset E^b_k$ such that

1. If $E^b_i \cap E^b_j \subset N_b$ then $(\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}' \subset N_b)$ and, doing an argument through $\tilde{\Gamma}_1$ we have $(\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}' \subset N_b)$.

2. If $E^b_i \cap E^b_j \not\subset N_b$ then $(\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}' \subset N_b)$ and, doing an argument through $\tilde{\Gamma}_1$ we have $(\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}' \subset N_b)$.

We end by projecting $\tilde{\Gamma}'$ by $\pi_b$ to obtain $\Gamma'$ and noting that the interruption is a nodal one if and only if $E^b_i \cap E^b_j \subset N_b$. Finally, the case that $e(E^k, p) = 3$ is done with the same arguments as in the situation with $e(E^k, p) = 2$.

**B-2) Case** $Y_k$ is a curve with $Y_k \not\subset \Gamma$. Since $e(E^k, p) \geq 2$ and $Y_k$ has normal crossings with $E^k$ we have that $Y_k \subset E^b_k$ or $Y_k \subset E^k_i$. If $Y_k \subset E^b_k$ but $Y_k \not\subset E^b_k$ we have that $\pi_b^{-1}(p) \subset E^b_k$ and we can apply induction at $\tilde{p}$ to see that there is an s-trace curve $\tilde{\Gamma}'$ with $\tilde{p} \in \tilde{\Gamma}' \subset E^b_k$ such that

$$
\text{If } E^b_i \cap E^b_j \subset N_b \text{ then } (\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}' \subset N_b)
$$

$$
\text{If } E^b_i \cap E^b_j \not\subset N_b \text{ then } (\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}' \subset N_b)
$$

We end as in the previous cases of B-1). Assume that $Y_k \subset E^b_k$ but $Y_k \not\subset E^b_k$. Now we have that $\pi_b^{-1}(p) = E^b_k \cap E^b_b$. By induction at $\tilde{p}$ there is an s-trace curve $\tilde{\Gamma}'$ with $\tilde{p} \in \tilde{\Gamma}' \subset E^b_k$ such that

$$
\text{If } \pi_b^{-1}(p) \subset N_b \text{ then } (\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}' \subset N_b)
$$

$$
\text{If } \pi_b^{-1}(p) \not\subset N_b \text{ then } (\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}' \subset N_b)
$$

Let $q$ be the point of intersection of $E^b_k$ and $\pi_b^{-1}(p)$. Note that $E^b_k \cap E^k_i$ is not compact and thus $E^b_k \cap E^k_i \not\subset N_b$. Thus, by Proposition 8 we have that

$$
\pi_b^{-1}(p) \subset N_b \Leftrightarrow E^b_i \cap E^b_k \subset N_b.
$$

We conclude that

$$
\text{If } E^b_i \cap E^b_j \subset N_b \text{ then } (\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}' \not\subset N_b)
$$

$$
\text{If } E^b_i \cap E^b_j \not\subset N_b \text{ then } (\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}' \subset N_b).
$$

We end this case by noting that the interruption is nodal if and only if $E^b_i \cap E^b_j \not\subset N_b$.

It remains to consider the case that $Y_k = E^b_k \cap E^b_k$ locally at $p$. The interruption then is a non nodal one, since $Y_k \not\subset N_k$. If $Y_k$ is an s-trace curve, we are done by taking $\Gamma' = Y_k$. Assume that $Y_k$ is generically pre-simple CH-corner. By induction at $\tilde{p}$, there is an s-trace curve $\tilde{\Gamma}_1 \subset E^b_k$ such that

$$
\tilde{\Gamma}_1 \subset N_b \Leftrightarrow \tilde{\Gamma} \subset N_b.
$$

We necessarily have that $\tilde{\Gamma}_1 = \pi_b^{-1}(p)$ since there are not other possible singular curves over a generically pre-simple CH-corner like $Y_k$ after a nondicritical blow-up. We apply induction at the point of intersection of $\pi_b^{-1}(p)$ with $E^b_k$ to obtain $\tilde{\Gamma}' \subset E^b_k$ such that

$$
\tilde{\Gamma} \subset N_b \Leftrightarrow \tilde{\Gamma}_1 \subset N_b \Leftrightarrow \tilde{\Gamma}' \subset N_b.
$$
since the interruption of $\pi^{-1}(p) = \tilde{\Gamma}_1$ by $E^k_1$ is not nodal because $E^k_1 \cap E^b_1$ is not compact and thus $E^k_1 \cap E^b_1 \not\subset \mathcal{N}_b$.

This ends the proof.

\begin{remark}
It is not necessary to work at a point $p \in \mathcal{N}_b$ to obtain conclusion (2) of Proposition 10.

To be precise, the following statement is also true as a direct consequence of Proposition 8.

"Let $\Gamma$ be an s-trace curve for $F_k, E^k$ and suppose that there is an invariant component $E^k_1$ with $\Gamma \not\subset E^k_1$ and $p$ is a point $p \not\in \mathcal{N}_b$ with $p \in \Gamma \cap E^k_1$. There is an s-trace curve $\Gamma' \subset E^k_1$ with $p \in \Gamma'$ such that $\Gamma' \not\subset \mathcal{N}_b$".
\end{remark}

\begin{proposition}[Incompatibility of trace curves]
Consider two s-trace curves $\Gamma_1$ and $\Gamma_2$ having a common point $p \in \Gamma_1 \cap \Gamma_2$ and contained in a common component $E^k_1$ of $E^k$. Then $\Gamma_1 \subset \mathcal{N}_b$ if and only if $\Gamma_2 \subset \mathcal{N}_b$.

In an equivalent way, it is not possible that $\Gamma_1 \subset \mathcal{N}_b$ and $\Gamma_2 \not\subset \mathcal{N}_b$.
\end{proposition}

\begin{proof}
Induction on the height $h(p)$. If $h(p) = 0$ we are done, since there is at most one s-trace curve through $p$ contained in a component of the divisor. Assume that $h(p) \geq 1$ and suppose that $\Gamma_1 \subset \mathcal{N}_b$ and $\Gamma_2 \not\subset \mathcal{N}_b$ in order to find a contradiction. Take notations and conventions as in the proof of Proposition 10. If $Y_b = \{p\}$, we have that $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are interrupted by $E^k_2$ and the interruption given by $E^k_1 \cap E^k_2$ is simultaneously nodal or not nodal for $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$. By Proposition 10 and Remark 19 we obtain two trace curves $\tilde{\Gamma}_1'$ and $\tilde{\Gamma}_2'$ contained in $E^k_1$ that must have a common point $p' \in \tilde{\Gamma}_1' \cap \tilde{\Gamma}_2'$, where we find a contradiction by induction hypothesis.

In the case $Y_b$ is a germ of curve $p \in Y_b$ and $Y_b \subset E^k_i$, with $Y_b \neq \Gamma_b$, (note that we know that $Y_b \neq \Gamma_b$) we find directly a contradiction by looking at $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ that are contained in $E^k_1$ and have the common point $p' = \pi^{-1}(p) \cap E^k_1$.

Assume that $Y_b = \Gamma_b$. We know that $E^k_1 \cap E^b_1 \not\subset \mathcal{N}_b$. Take the point $q$ such that $q \in \pi^{-1}_b(p) \cap E^b_1$. The strict transform $\tilde{\Gamma}_2$ is interrupted in a not nodal way by $E^b_1$ at $q$ and applying Proposition 10 we find a compact s-trace curve $\Delta \subset E^b_1 \cap \mathcal{N}_b$. The only possibility is that $\Delta = \pi^{-1}_b(p)$. Moreover, by Proposition 8 there is an s-trace curve $\gamma \subset E^k_1$ that must intersect $\Delta$ in at least one point $r$. We find a contradiction at the point $r$ by applying induction hypothesis.

Assume that $Y_b$ is a germ of curve $p \in Y_b$ and $Y_b \subset E^b_i$. If $\pi^{-1}(p) \subset \mathcal{N}_b$ we find a contradiction since the nodal interruption of $\tilde{\Gamma}_2$ produces a trace curve of $\mathcal{N}_b$ contained in $E^b_1$ different from the only compact curve $\pi^{-1}(0)$. If $\pi^{-1}(p) \not\subset \mathcal{N}_b$ we find a contradiction since the nodal interruption of $\tilde{\Gamma}_1$ produces a trace curve of $\mathcal{N}_b$ contained in $E^b_1$ different from the only compact curve $\pi^{-1}(0)$. This ends the proof. \hfill \blacksquare
\end{proof}

\section{Conclusion}

In this section we end the proof of Theorem 3. As in Section 9 we consider a RICH-foliation $F$ in $(\mathbb{C}^3, 0)$ jointly with an RI-reduction of singularities $\pi : (M, \pi^{-1}(0)) \to (\mathbb{C}^3, 0)$. We will find a contradiction with the existence of a nodal component $\mathcal{N}$ which is compact and does not intersect the union of the dicritical components of the exceptional divisor $E$.

Let us go to the date of birth $b = b(\mathcal{N})$. Put $k = b - 1$. Note that $b \geq 1$. Consider first the situation that $b = 1$. We consider the possibilities that $Y_0 = \{0\}$ or $Y_0$ is a germ of curve.

Assume that $Y_0 = \{0\}$. We have an s-trace compact curve $\Gamma \subset E^1_1$ such that $\Gamma \subset \mathcal{N}_1$. In particular $E^1_1$ is invariant by Proposition 10. Now, take a plane $(\Sigma, 0) \subset (\mathbb{C}^3, 0)$ that induces a transversal section of $F$ in the sense of Mattei-Moussu 18. We know that $\Sigma$ may be chosen generic enough to assure that the strict transform $\tilde{\Sigma}$ of $\Sigma$ by $\pi_1$ cuts transversely $\text{Sing}(F_1, E^1)$ only at simple points (the set of non simple points in $E^1_1$ is finite).

Thus, the restriction
\[ \sigma : (\tilde{\Sigma}, \tilde{\Sigma} \cap E^1_1) \to (\Sigma, 0) \]
of $\pi_1$ provides a reduction of singularities of $\tilde{G} = F|_{\tilde{\Sigma}}$. Let $p \in \Gamma \cap \tilde{\Sigma}$. The Camacho-Sad index (see 3) of $\tilde{G}$ at $p$ is a positive real number, since $p$ is a simple point of nodal type for $\tilde{F}$ and hence for $\tilde{G}$. Since the sum of indices is $-1$, there is another point $q \in \tilde{\Sigma} \cap E^1_1$ which is a simple not nodal point for $\tilde{G}$ and hence for $\tilde{F}$. This implies the existence of an s-trace curve $\Gamma_1 \subset E^1_1$ that is not generically nodal and hence $\Gamma_1 \not\subset \mathcal{N}_1$. Since $E^1_1$ is isomorphic to a projective plane, there is a common point $r \in \Gamma \cap \Gamma_1$ in contradiction with Proposition 11.

Assume now that $Y_0$ is a germ of curve. Then $\Gamma = \pi^{-1}_1(0)$ is the only possible compact curve in $\mathcal{N}_1$ and we also have that $E^1_1$ is invariant. Looking at two dimensional transversal sections of the center $Y_0$ at generic points and recalling that a non-dicritical blow-up in dimension two produces at least one singular point (given for instance by Camacho-Sad separatrix, see 3, 4), we find at least one singular curve $\Gamma_1 \subset E^1_1$ which projects onto $Y_0$. Then $\Gamma_1$ and $\Gamma$ also give a contradiction with Proposition 11.

Assume that $b > 1$. Let us suppose first that the center of $\pi_b$ is a point $Y_b = \{p\}$. We have a compact curve $\Delta \subset E^b_1$ such that $\Delta \subset \mathcal{N}_b$. By Proposition 10 and Lemma 13 we conclude that $E^k_b$ is invariant (that is the blow-up $\pi_k$ is non-dicritical) and also all the components $E^k_i$ through $p$ are invariant, since $\Delta$ cuts each $E^k_i$ such that $p \in E^k_i$. Let us note that $\varepsilon(E^k_b, p) \geq 1$, since $b > 1$ in particular there is at least one $E^k_b$ with $p \in E^k_1$. Let us prove the following statement
“For all $E_i^h$ with $p \in E_i^h$ we have that $\Delta' = E_i^h \cap E_j^h \subset N_b$.”

In the case that $\Delta = E_i^h \cap E_j^h$ we are done. Otherwise $\Delta$ cuts in at least one point $q$ the intersection $E_i^h \cap E_k^h$.

If $\Delta$ is not an s-trace curve, we have $\Delta = E_l^h \cap E_k^h$ and we can apply Lemma 13 to see that either $E_i^h \cap E_j^h$ or $E_i^h \cap E_k^h$ is a curve in $N_b$. But in this case, we can assume that all of them intersect at least one compact dicritical component $E_i$. When we have a nodal component $E_i$ that is a simple not nodal point for $\Gamma$, we can assume that all of them intersect at least one compact dicritical component $E_i$.

Now, we take a transversal two dimensional section $\Sigma$ at $p$ as in the case $b = 1$. We find a point $q \in \Sigma \cap E_k^h$ that is a simple not nodal point for $\Gamma$; moreover, the point $q$ is outside the sections $\Sigma \cap E_i^h \cap E_j^h$ for each $E_i^h$, since these points are nodal ones. In this way we discover an s-trace curve $\Gamma_1 \subset E_i^h$ which is not generically nodal and hence $\Gamma_1 \not\subset N_b$. Take a point $r \in \Gamma_1 \cap E_i^h$. We apply Proposition 10 at $r$, since the interruption of $\Gamma_1$ by $E_i^h$ at $r$ is a nodal one, there is a s-trace curve $\Gamma_2 \subset N_b \cap E_i^h$ that must project onto an s-trace curve $\Gamma_2 \subset N_b \cap E_i^h$. We obtain in this way a contradiction as above.

In order to end the proof, let us suppose that $b > 1$ and $Y_b$ is a germ of curve with $\{p\} = Y_b \cap E_i^h$. The only new compact curve after blow-up is $\Delta = \pi_b^{-1}(p)$ and hence we have that $\Delta \subset N_b$. By a direct computation as in the case $b = 1$, we find a singular curve $\gamma \subset E_i^h$ that projects onto $Y_b$. Note that $\gamma \not\subset N_b$, since it is not a compact curve. Let $q$ be a point $q \in \gamma \cap \Delta$. Looking at the point $q$, we obtain a contradiction as follows

1. If $\gamma$, $\Delta$ are both s-trace curves, we apply the incomparability result of Proposition 11.
2. If $\Delta$ is an s-trace curve, but $\gamma$ is not, then $\gamma = E_l^h \cap E_k^h$ and we find by Proposition 10 an s-trace curve $\Gamma \subset N_b \cap E_i^h$ that projects onto an s-trace curve $\Gamma \subset N_b \cap E_i^h$, contradiction.
3. If $\Delta$ is not an s-trace curve, but $\gamma$ is, then $\Delta = E_l^h \cap E_k^h$ and we find a contradiction as in the preceding case.
4. If $\Delta$ and $\gamma$ are both generically pre-simple corner curves then $\Delta = E_l^h \cap E_k^h$ and $\gamma = E_l^h \cap E_j^h$ and by Lemma 13, we deduce that $E_l^h \cap E_j^h \subset N_b$ and hence $E_l^h \cap E_j^h \subset N_b$, contradiction.

This finishes the proof of Theorem 3.

Now, Theorem 2 is a consequence of Theorem 1 and Theorem 3 as follows. Property (ii) of Theorem 2 occurs when we have a nodal component $\mathcal{N}$ which is neither compact nor cuts a non compact dicritical component. So, if there are nodal components, we can assume that all of them intersect at least one compact dicritical component of the exceptional divisor. In this situation we can cover the nodal components with leaves containing germs of analytic curves at the origin and the arguments of Theorem 1 apply.

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