Chronon corrections to the Dirac equation

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March 27, 2022

Abstract

The Dirac equation is not semisimple. We therefore regard it as a contraction of a simpler decontracted theory. The decontracted theory is necessarily purely algebraic and non-local. In one simple model the algebra is a Clifford algebra with \(6N\) generators. The quantum imaginary \(\bar{h}\) is the contraction of a dynamical variable whose back-reaction provides the Dirac mass. The simplified Dirac equation is exactly Lorentz invariant but its symmetry group is \(SO(3, 3)\), a decontraction of the Poincaré group, and it has a slight but fundamental non-locality beyond that of the usual Dirac equation. On operational grounds the non-locality is \(\sim 10^{-25}\) sec in size and the associated mass is about the Higgs mass.

There is a non-standard small but unique spin-orbit coupling \(\sim 1/N\), whose observation would be some evidence for the simpler theory. All the fields of the Standard Model call for similar non-local simplification.

1 Introduction

We begin with basic concepts:

A *simple* theory is one with simple (irreducible) dynamical and symmetry groups. What is not simple or semi-simple we call *compound*. A *contraction* of a theory is a deformation of the theory in which some physical scale parameter, called the *simplifier*, approaches a singular limit, taken to be 0 with no loss of generality. The contraction of a simple theory is in general compound [1, 2, 3]. By *simplification* we mean the more creative, non-unique inverse process, finding a simple theory that contracts to a given compound theory and agrees better with experiment. The main revolutions in physics of the twentieth century were simplifications with simplifiers \(c, G, \hbar\).

One sign of a compound theory is a breakdown of reciprocity, the principle that every coupling works both ways. The classic example is Galilean relativity. There reciprocity between space and time breaks down; boosts couple time into space and there is no reciprocal coupling. Special relativity established reciprocity by replacing the compound Galilean bundle of space fibers over the time base by the simple Minkowski space-time. Had Galileo insisted on simplicity and reciprocity he could have formulated special relativity in the 17th century (unless he were to choose \(SO(4)\) instead of \(SO(1, 3)\)). Every bundle theory violates reciprocity

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as much as Galileo’s. The bundle group couples the base to the fiber but not conversely. Every bundle theory cries out for simplification.

This now requires us to establish reciprocity between space-time (base coordinates) $x^\mu$ and energy-momenta (fiber coordinates) $p_\mu$. [Segal [1] postulated $x \leftrightarrow p$ symmetry exactly on grounds of algebraic simplicity; his work stimulated that of Inönü and Wigner, and ours. Born [4] postulated $x \leftrightarrow p$ reciprocity, on the grounds that it is impossible in principle to measure the usual four-dimensional interval of two events within an atom. We see no law against measuring space-time coordinates and intervals at that gross scale. We use his term “reciprocity” in a broader sense that includes his.]

Einstein’s gravity theory and the Standard Model of the other forces are bundle theories, with field space as fiber and space-time as base. Therefore these theories are ripe for simplification [3]. Here we simplify a spinor theory, guided by criteria of experimental adequacy, operationality, causality, and finity.

Classical field theory is but a singular limit of quantum field theory; it suffices to simplify the quantum field theory. Quantum field theory in turn we regard as many-quantum theory. Its field variables all arise from spin variables of single quanta. By quantification we mean the transition from the one-body to the many-body theory, converting yes-or-no predicates about an individual into how-many predicates about an aggregate of isomorphic individuals; as distinct from quantization. For example, a spinor field theory arises by quantifying the theory of a single quantum of spin $1/2$.

To unify field with space-time in quantum field theory, it suffices to unify spin with space-time in the one-quantum theory, and to quantify the resulting theory. We unify in this paper and quantify in a sequel.

Some unification programs concern themselves with simplifying just the internal symmetry group of the elementary particles, ignoring the fracture between the internal and external variables. They attempt to unify (say) the hypercharge, isospin and color variables, separate from the space-time variables. Here we close the greatest wound first, expecting that the internal variables will unite with each other when they unite with the external variables; as uniting space with time incidentally unified the electric and magnetic fields. We represent space-time variables $x^\mu$ and $p_\mu$ as approximate descriptions of many spin variables, in one quantum-spin-space-time structure described in a higher-dimensional spin algebra. This relativizes the split between field and space-time, as Einstein relativized the split between space and time.

The resulting quantum atomistic space-time consists of many small exactly Lorentz-invariant isomorphic quantum bits, qubits which we call chronons. [Feynman, Penrose and Weiszäcker attempted to atomize space or space-time into quantum spins. Feynman wrote a space-time vector as the sum of a great many Dirac spin-operator vectors [4], $x^\mu \sim \sum_n \gamma^\mu(n)$, Penrose dissected the sphere $S^2$ into a spin network [7]; his work inspired this program. Weiszäcker [8], attempted a cosmology of spin-1/2 urs. The respective groups are Feynman’s SO(3,1), Penrose’s SO(3), Weiszäcker’s SU(2) and our SO(3N,3N) ($N \gg 1$).]

Simplifying a physical theory generally detaches us from a supporting condensate. [For Galileo and Kepler, the condensate was the Earth’s crust, and to detach from it they moved in thought to a ship or the moon, respectively [9, 10].] In the present situation of physics the prime condensate is the ambient vacuum. Atomizing space-time enables us to present the vacuum as a condensate of a simple system, and to detach from it in thought by a phase transition, a space-time melt-down.
Chronons carry a fundamental time-unit $\chi$, one of our simplifiers. We have argued that $\chi$ is much greater than the Planck time and is on the order of the Higgs time $\bar{\hbar}/M_H c^2$. [In an earlier effort to dissect space-time, assuming multiple Fermi-Dirac statistics for the elements $[11, 12]$. This false start led us eventually to the Clifford-Wilczek statistics $[13, 14, 15, 16, 17]$; an example of Clifford-Wilczek statistics is unwittingly developed in chapter 16 of $[12]$.] We now replace the classical Maxwell-Boltzmann statistics of space-time events with the simple Clifford-Wilczek statistics appropriate for distinguishable isomorphic units. This enormously reduces the problem of forming a theory.

We single out two main quantifications in field theories like gravitation and the Standard Model:

A classical quantification assembles a space-time from individual space-time points.

A separate quantification constructs a many-quantum theory or quantum field theory from a one-quantum theory on that space-time.

In the standard physics the space-time quantification tacitly assumes Maxwell-Boltzmann statistics for the elements of space-time, and the field quantification uses Fermi-Dirac or Bose-Einstein statistics. The simplified theory we propose uses one Clifford quantification for all of these purposes.

In this paper we work only with one-quantum processes of $N \gg 1$ chronons. To describe several quanta and their interactions, getting closer to field theory and experiment, will require no further quantification, but only additional internal combinatorial structure that is readily accomodated within the one Clifford-Wilczek quantification.

Each physical theory defines at least three algebras that should be simple: the associative operator algebra of the system $[12, 13]$, the kinematical Lie algebra consisting of possible Hamiltonians, and the symmetry Lie algebra of one preferred Hamiltonian.

There is no second quantization. But there is a second simplification; and a third, and so on, all of different kinds with different simplifiers. Each of the historic revolutions that guide us now introduced a simplifier, small on the scale of previous experience and therefore long overlooked, into the multiplication table and basis elements of one or more of these algebras, and so deformed a compound algebra into a simpler algebra that works better. Among these simplifiers are $c, G,$ and $\hbar$.

Here we simplify the free Dirac equation and its underlying Dirac-Heisenberg (real unital associative) algebra

$$A_{DH} = A_D \otimes A_H,$$

the tensor product of the Dirac and the relativistic Heisenberg algebras, in turn defined as follows:

**Relativistic Heisenberg algebra** $A_H = A[i, \hat{p}, \hat{x}]$ is generated by the imaginary unit $i$ and the space-time and energy-momentum translation generators $\hat{p}_\nu := i p_\nu \equiv -\hbar \partial/\partial x^\nu$ and $\hat{x}^\mu := ix^\mu$, subject to the relations

$$[\hat{p}^\mu, \hat{x}^\nu] = -i \hbar g^{\mu\nu},$$

$$[\hat{p}^\mu, \hat{p}^\nu] = 0,$$

$$[\hat{x}^\mu, \hat{x}^\nu] = 0,$$

$$[i, \hat{p}^\mu] = 0,$$

$$[i, \hat{x}^\mu] = 0.$$
Here $g^{\mu\nu}$ is the Minkowski metric, held fixed in this paper. The hats (on $\hat{p}$, for example) indicate that a factor $i$ has been absorbed to make the operator anti-Hermitian $[19]$. The algebra $A_H$ has both the usual associative product and the Lie commutator product. As a real Lie algebra $A_H$ is compound, Segal emphasized, containing the non-trivial ideal generated by the unit $i$.

The orbital Lorentz-group generators are

$$\hat{O}^{\mu\nu} := iO^{\mu\nu} = -i (\hat{x}^{\mu} \hat{p}^{\nu} - \hat{x}^{\nu} \hat{p}^{\mu}).$$

These automatically obey the usual relations

$$\begin{aligned}
[\hat{O}^{\mu\nu}, \hat{O}^{\lambda\kappa}] &= \hbar \left( g^{\mu\lambda} \hat{O}^{\nu\kappa} - g^{\mu\kappa} \hat{O}^{\nu\lambda} - g^{\nu\kappa} \hat{O}^{\mu\lambda} + g^{\nu\lambda} \hat{O}^{\mu\kappa} \right), \\
[\hat{x}^{\mu}, \hat{O}^{\nu\lambda}] &= \hbar \left( g^{\mu\nu} \hat{x}^{\lambda} - g^{\mu\lambda} \hat{x}^{\nu} \right), \\
[\hat{p}^{\mu}, \hat{O}^{\nu\lambda}] &= \hbar \left( g^{\mu\nu} \hat{p}^{\lambda} - g^{\mu\lambda} \hat{p}^{\nu} \right), \\
[i, \hat{O}^{\mu\nu}] &= 0. 
\end{aligned}$$

**Dirac algebra** $A_D = A[\gamma_{\mu}]$ is generated by Dirac-Clifford units $\gamma_{\mu}$ subject to the familiar relations

$$\{\gamma_{\nu}, \gamma_{\mu}\} = 2g_{\nu\mu}.$$

As usual we write $\gamma_{\mu\nu...}$ for the anti-symmetric part of the tensor $\gamma_{\mu}\gamma_{\nu}...$.

**Statistics** One may define the statistics of an (actual, not virtual) aggregate by defining how the aggregate transforms under permutations of its units. That is, to describe $N$ units with given unit mode space $V_1$ we give, first, the mode space $V_N$ of the aggregate quantum system and, second, a simple representation $R_N : S_N \to \text{End} V_N$ of the permutation group $S_N$ on the given $N$ units by linear operators on $V_N$. This also defines the quantification that converts yes-or-no questions about the individual into how-many questions about a crowd.

In *Clifford statistics* $\text{End} V_N$ is a Clifford algebra $C = \text{Cliff}(V_1)$, and so $V_N$ is a spinor space for that Clifford algebra, with $C = \text{End} V_N$. We write $C_1$ for the first-degree subspace of $C$. A Clifford statistics is defined by a projective (double-valued) representation $R_C : S_N \to C_1 \subset C$ of the permutation group $S_N$ by first-grade Clifford elements over the unit mode space $V_1$. To define $R_C$ we associate with the $n$th unit (for all $n = 1, \ldots, N$) a Clifford unit $\gamma_n$, and we represent every swap (transposition or 2-cycle) $(mn)$ of two distinct units by the difference $\pm(\gamma_n - \gamma_m) \in C_1$ of the associated Clifford units.

For a free Clifford statistics the units $\gamma_n$ are independent and the metric $g_{mn}$ is Euclidean. This representation has dimension $2^{N/2}$ and is reducible. The irreducible representations have dimension $[21]$

$$\begin{aligned}
2^{\left\lceil \frac{N+1}{2} \right\rceil} &= 1, 1, 2, 2, 4, 4, 8, 8, 16, \ldots \\
\text{for } \text{Dim } V_1 = N &= 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots 
\end{aligned}$$

Some useful terms:
A **cliffordon** is a quantum with Clifford statistics.
A **squadron** is a quantum aggregate of cliffordons.
A **sib** is a quantum aggregate of bosons.
A **set** of quanta is an aggregate of fermions.
A **sequence** of quanta is a aggregate of Maxwell-Boltzmann quanta with a given sequential order [1].

*RC* can be extended to a spinor representation of SO(*N*) ⊃ SN on the same spinor space Σ(*N*).

The symmetry group *G*U of the quantum kinematics for a universe *U* of *N*U chronons is an orthogonal group

\[ G_U = \text{SO}(N_{U+}, N_{U-}), \]
\[ N_{U} = N_{U+} + N_{U-}. \]  (7)

The algebra of observables of *U* is the simple finite-dimensional real Clifford algebra

\[ C_U = \text{Cliff}(V_1) = \text{Cliff}[1, \gamma(1), \ldots, \gamma(N_U)] \]  (8)
generated by the *N*U Clifford units \( \gamma(n) \), \( n = 1, \ldots, N_U \) representing exchanges. The Clifford units \( \gamma(n) \) span a vector space \( V_1 \cong C_1 \) of first-grade elements of \( C_U \).

Within \( C_U \) we shall construct a simplified Dirac-Heisenberg algebra

\[ \tilde{A}_{DH} = A[\tilde{i}, \tilde{p}, \tilde{x}, \tilde{O}] \subset C(V_1) \]  (9)
whose commutator Lie algebra is simple and which contracts to the usual Dirac-Heisenberg algebra \( A_{DH} \) in the continuum limit. We factor \( \tilde{A}_{DH} \) into the Clifford product

\[ \tilde{A}_{DH} = \text{Cliff}(N\mathbf{6}_0) = \text{Cliff}((N - 1)\mathbf{6}_0) \sqcup \text{Cliff}(\mathbf{6}_0) \]  (10)
of two Clifford algebras, an “internal” algebra from the last hexad and an “external” algebra from all the others.

We designate our proposed simplifications of \( \gamma \) and \( i, \tilde{p}, \tilde{x} \), and \( \tilde{O} \) by \( \tilde{\gamma} \) and \( \hat{i}, \hat{p}, \hat{x} \) and \( \hat{O} \).

In the limit \( \chi \to 0 \) the tildes \( \sim \) disappear and the breves \( \sim \) become hats \( ^\wedge \).

We use the following quadratic spaces:

\( N\mathbb{R} := \mathbb{R} \oplus \ldots \oplus \mathbb{R} \) (with *N* terms) = \( \bigoplus_1^N \mathbb{R} \) is the positive-definite *N*-dimensional real quantum-mode space.

\( -N\mathbb{R} \) is the corresponding negative-definite space.

\( \mathbb{M} \) is Minkowski space with signature 1 - 3.

\( -\mathbb{M} \) is the same space with the opposite signature 3 - 1.

Also,

\[ X \ominus Y := X \oplus (-Y) \]
\[ 1 := \mathbb{R}, \]
\[ -1 := -\mathbb{R}, \]
\[ 3 := 3\mathbb{R}, \]
\[ -3 := -3\mathbb{R}, \]
\[ \mathbb{M} := 1 \ominus 3, \]

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\[ M := 3 \ominus 1, \]
\[ 6_0 := 3 \ominus 3. \] (11)

\( M \) and \( -M \) are tangent spaces to Minkowski space-times and support natural representations of the Lorentz group.

## 2 Simplification of the relativistic Heisenberg algebra

As already mentioned, field theory employs a compound field-space-time bundle with space-time for base and field-space for fiber; just as Galilean space-time is a four-dimensional bundle with \( \mathbb{R}^3 \) for base and \( \mathbb{R}^1 \) for fiber. The prototype is the covector field, where the fiber is the cotangent space to space-time, with coordinates that we designate by \( p_\mu \).

We assume that in experiments of sufficiently high resolution the space-time tangent bundle (or the Dirac-Heisenberg algebra) manifests itself as a simple quantum-field-space-time synthesis. The space-time variables \( x^\mu \) and the tangent space variables \( p_\mu \) unite into one simple construct, as space and time have already united. Now, however, the simplification requires an atomization, because the field variable actually derives from an atomic spin.

We first split the space-time tangent bundle into quantum cells. The minimum number of elements in a cell for our simplification is six: four for space-time and two for a complex or symplectic plane. We provisionally adopt the hexadic cell. Earlier work, done before our present stringent simplicity standard, assumed a pentadic cell [16]. This provided no natural correspondent for the energy-momentum operators.

\( N \) hexads define a unit mode space \( V_1 = N6_0 \). Each term \( 6_0 \) has a Clifford algebra \( \text{Cliff}(3, 3) \) whose spinors have eight real components, forming an \( 8_0 \). [Eight-component spinors have also been used in physics by Penrose [7], Robson and Staudte [21], and Lunsford [22]; though not to unify spin with space-time.] The spinors of \( V \) form the spinor space

\[ \bigotimes_1^N 8_0 = 8_0^N \]

We do not deal with empty space-time. We explore space-time with one relativistic quantum spin-\( \frac{1}{2} \) probe of rest mass \( m \sim 1/\chi \). We express the usual spin operators \( \gamma^\mu \), space-time position operators \( x^\mu \), and energy-momentum operators \( p_\mu \) of this probe as contractions of operators in the Clifford algebra \( \text{Cliff}(3N, 3N) \).

We write the dynamics of the usual contracted, compound Dirac theory in manifestly covariant form, with a Poincaré-scalar Dirac operator

\[ D = \gamma^\mu p_\mu - mc. \] (12)

\( D \) belongs to the algebra of operators on spinor-valued functions \( \psi(x^\mu) \) on space-time. Any physical spinor \( \psi(x^\mu) \) is to obey the dynamical equation

\[ D\psi = 0. \] (13)
We simplify the dynamical operator $D$, preserving the form of the dynamical equation (13).

The compound symmetry group for the Dirac equation is the covering group of the Poincaré group $\text{ISO}(\mathbb{M})$. We represent this as the contraction of a simple group $\text{SO}(3,3)$ acting on the spinor pseudo-Hilbert (ket) space of $6N$ Clifford generators $\gamma^\omega(n)$ $(\omega = 0, \ldots, 5; n = 1, \ldots, N)$ of the orthogonal group $\text{SO}(3N,3N)$. The size of the experiment fixes the parameter $N$.

We first simplify the anti-Hermitian space-time and energy-momentum translation generators $\hat{p}_\mu$ and $\hat{x}_\nu$, not the associated Hermitian observables $p_\mu$, $x_\nu$. Then we simplify the Hermitian operators by multiplying the anti-Hermitian ones by a suitably simplified $i$ and symmetrizing the product.

As in Dirac one-electron theory, we use the spinor representation $\text{Spin}(\mathbb{M})$ of $\text{SO}(\mathbb{M})$ to describe the contracted generators $\dot{S}^{\mu\nu}$ of rotations and boosts. The spin generators are represented by second-degree elements

$$\dot{S}^{\mu\nu} := \frac{\hbar}{4} [\gamma^\mu, \gamma^\nu] \equiv \frac{\hbar}{2} \gamma^{\mu\nu}, \; \mu, \nu = 0, \ldots, 3$$

(14)

of the Clifford algebra $\text{Cliff}(1,3)$.

We simplify $\text{ISO}(\mathbb{M})$ within $\text{Spin}(\mathbb{N}6_0)$ by representing the simplified space-time symmetry generators of the probe by second degree elements of $\text{Cliff}V_1$. We associate the position and momentum axes with the $\gamma^4$ and $\gamma^5$ elements of the hexad respectively, so that an infinitesimal orthogonal transformation in the 45-plane couples momentum into position. This accounts for the symplectic symmetry of classical mechanics and the $i$ of quantum mechanics.

We therefore define the simplified $\tilde{i}$, $\tilde{x}^\mu$, and $\tilde{p}^\nu$ of the probe by

$$\tilde{i} \equiv \frac{1}{N - 1} \sum_{n=1}^{N-1} \tilde{i}(n) := \frac{1}{N - 1} \sum_{n=1}^{N-1} \gamma^{45}(n),$$

$$\tilde{x}^\mu \equiv \sum_{n=1}^{N-1} \tilde{x}^\mu(n) := -\chi \sum_{n=1}^{N-1} \gamma^\mu(n),$$

$$\tilde{p}^\nu \equiv \sum_{n=1}^{N-1} \tilde{p}^\nu(n) := \phi \sum_{n=1}^{N-1} \gamma^\nu(n),$$

(15)

where $\chi$, $\phi$ and $N$ are simplifiers of our theory, and

$$\gamma^{\rho\sigma}(n) := \frac{1}{2}[\gamma^\rho(n), \gamma^\sigma(n)].$$

(16)

To support this choice for the expanded generators we form the following commutation relations among them (cf. [23], 1, 24):

$$[\tilde{p}^\mu, \tilde{x}^\nu] = -2\phi\chi(N - 1) \; g^{\mu\nu} \tilde{i},$$

$$[\tilde{p}^\mu, \tilde{p}^\nu] = -\frac{4\phi^2}{\hbar} \; \tilde{L}^{\mu\nu},$$

$$[\tilde{x}^\mu, \tilde{x}^\nu] = -\frac{4\chi^2}{\hbar} \; \tilde{L}^{\mu\nu},$$

$$[\tilde{i}, \tilde{p}^\nu] = -\frac{2\phi}{\chi(N - 1)} \; \tilde{x}^\nu.$$
\[ [i, \tilde{x}^\mu] = \frac{2\chi}{\phi(N - 1)} \tilde{j}^\mu. \]  

(17)

In (17),

\[ \tilde{L}^{\mu\nu} := \frac{\hbar}{2} \sum_{n=1}^{N-1} \gamma^{\mu\nu}(n), \]

\[ \tilde{j}^{\mu\nu} := \frac{\hbar}{2} \sum_{n=1}^{N} \gamma^{\mu\nu}(n) \equiv L^{\mu\nu} + S^{\mu\nu}. \]

(18)

where \( \tilde{S}^{\mu\nu} \) is the Dirac spin operator (cf. (29)),

(17) incorporates two decontractions: one leading to finite commutators between coordinates of the Snyder type, and one leading to finite commutators between \( \hbar i \) and the coordinates and momenta of the Segal type. Both are necessary for simplicity.

The Snyder decontraction makes the theory more non-local than the Dirac equation. In the contracted theory, the coordinates \( x, y, z \) commute. This means that in principle one can produce the single quantum at a definite place and register it at a definite place. To be sure, to do so will mix positive and negative energy levels. In the more physical many-quantum theory, a pair will be created in these processes. Nevertheless, in the standard interpretation of the quantum theory it is still possible in principle to precisely determine the operators \( x, y, z \) with arbitrary precision at one instant, before the pair separates.

In the decontracted theory, any one of the operators \( x, y, z \) can be determined with arbitrary precision, say \( z \). Its spectrum will then be discrete. The operators \( x, y \) will then have fundamental indeterminacies, depending on the magnitude of \( L_z \) and the constant \( \chi \). Thus the single quantum can no longer be localized in principle. This non-locality is intrinsic to the space-time-momentum-energy-spin unification that makes the theory simpler.

\( j^{\mu\nu}\) obeys the Lorentz-group commutation relations:

\[ [\tilde{j}^{\mu\nu}, \tilde{j}^{\lambda\kappa}] = \hbar \left( g^{\mu\lambda} \tilde{j}^{\nu\kappa} - g^{\nu\lambda} \tilde{j}^{\mu\kappa} - g^{\mu\kappa} \tilde{j}^{\nu\lambda} + g^{\nu\kappa} \tilde{j}^{\mu\lambda} \right), \]

(19)

and generates a total Lorentz transformation of the variables \( x^\mu, p^\mu, i \) and \( S^{\mu\nu} \):

\[ [\tilde{x}^\mu, \tilde{j}^{\nu\lambda}] = \hbar \left( g^{\mu\lambda} \tilde{x}^\nu - g^{\mu\nu} \tilde{x}^\lambda \right), \]

\[ [\tilde{p}^\mu, \tilde{j}^{\nu\lambda}] = \hbar \left( g^{\mu\lambda} \tilde{p}^\nu - g^{\mu\nu} \tilde{p}^\lambda \right), \]

\[ [i, \tilde{j}^{\mu\nu}] = 0, \]

\[ [\tilde{S}^{\mu\nu}, \tilde{j}^{\lambda\kappa}] = 0. \]

(20)

There is a mock orbital angular momentum generator of familiar appearance,

\[ \tilde{O}^{\mu\nu} := -\tilde{i} (\tilde{x}^\mu \tilde{p}^\nu - \tilde{x}^\nu \tilde{p}^\mu). \]

(21)

\( \tilde{O} \) too obeys the Lorentz group commutation relations. We relate \( \tilde{L}^{\mu\nu} \) and \( \tilde{O}^{\mu\nu} \) in Sec.3.

Since the usual complex unit \( i \) is central and the simplified \( \tilde{i} \) is not, we suppose that the contraction process includes a projection that restricts the probe to one of the two-dimensional invariant subspaces of \( \tilde{i} \), associated with the maximum negative eigenvalue \(-1\) of \( \tilde{i}^2 \). This
represents a condensation that aligns all the mutually commuting hexad spins $\gamma^{45}(n)$ with each other, so that

$$\gamma^{45}(n)\gamma^{45}(n') \longrightarrow -1,$$

for any $n$ and $n'$. We call this the condensation of $i$.

Projection onto a sharp value of $i$ kills $i$-changing variables like $x^\mu$ and $p_\mu$. Only SO(2)-invariant combinations like $\chi^2p^2 + \phi^2x^2$ should survive. Nonetheless one observes position and momentum separately. This is a spontaneous symmetry-breaking by the vacuum condensate, analogous to the fact that a crystal in its ground state, with spherically symmetric Hamiltonian, can have a non-zero internal magnetization.

Under Wigner time-reversal, $t \rightarrow -t$ and $i \rightarrow -i$. Since the variable $t$ is chosen by the experimenter, not the system, we must suppose that $i$ too is mainly fixed by the experimenter, not the system. But since the boundary between system and experimenter is somewhat arbitrary, we must therefore suppose that the entire universe contributes uniformly to $i$; it is simply that the system is much smaller than the experimenter, and influences $i$ less. This fits with an earlier theory of $i$ as a Stückelberg-Higgs variable that imparts mass to some otherwise massless gauge vector bosons [25, 28].

Then the momentum variables $p_\mu$ that we usually attribute to the system, for example, are actually $i$-invariant bilinear combinations $P_{\rho\sigma}[\mu]J_{\rho\sigma}$ of experimenter standards $P_{\rho\sigma}$ and the system tensor $J_{\rho\sigma}$. As creatures of the space-time condensate we do not experience the symmetry of the dynamics that produced it, but only its residual symmetries. The spontaneously broken symmetries reappear when the condensate melts down.

To recover the canonical commutation relations for $\hat{x}^\mu$ and $\hat{p}_\mu$ we must impose

$$\chi\phi(N - 1) = \frac{\hbar}{2}$$

and assume that

$$\chi \rightarrow 0,$$

$$\phi \rightarrow 0,$$

$$N \rightarrow \infty.$$

Then the relations (22) reduce to the commutation relations (2) of the relativistic Heisenberg algebra $A_H$ as required.

The three parameters $\chi, \phi, 1/N$ are subject to one constraint $\chi\phi(N - 1) = \hbar/2$ leaving two independent simplifiers. $N$ is not a physical constant like $\hbar$ and $c$, but depends on the scope of the experiment, and is under the experimenter’s control. $N$-dependent effects might appear as curious boundary effects. We set a cosmological limit $N \lesssim N_{\text{Max}}$ below.

This leaves one $N$-independent physical constant with the dimensions of time. We can consider two contractions, $\chi \rightarrow 0$ with $N$ constant, and $N \rightarrow \infty$ with $\chi$ constant. They combine into the continuum limit $\chi \rightarrow 0$, $N \rightarrow \infty$. We fix one simplifier $\chi$ in Sec.4 by supposing that the mass of a probe approaches a finite limit as $N \rightarrow \infty$.

3 Orbital, spin, and total angular momentum

As was shown in Sec 2, three sets of operators obeying Lorentz-group commutation relations appear in our theory. $\hat{L}_{\mu\nu}$ represents the simplified orbital angular momentum generators,
\( \mathcal{S}^{\mu\nu} \) represents the spin angular momentum, and \( \mathcal{J}^{\mu\nu} \) represents the simplified total angular momentum generators. There is a mock orbital angular momentum \( \hat{O}^{\mu\nu} \). \( \hat{O}^{\mu\nu} \) represents the simplified total angular momentum generators. There is a mock orbital angular momentum \( \hat{O}^{\mu\nu} \). In this section we show that \( \hat{O} \to \hat{L} \) in the contraction limit. Consider \( \hat{O}^{\mu\nu} \). By definition,

\[
\hat{O}^{\mu\nu} = - (\hat{x}^{\mu} \hat{p}^{\nu} - \hat{x}^{\nu} \hat{p}^{\mu}) \gamma^0
\]

\[
= + \frac{\chi}{N - 1} \left( \sum_{n=1}^{N-1} \gamma^{\mu_4}(n) \sum_{n'=1}^{N-1} \gamma^{\nu_5}(n') - \sum_{n=1}^{N-1} \gamma^{\nu_4}(n) \sum_{n'=1}^{N-1} \gamma^{\mu_5}(n') \right) \sum_{m=1}^{N-1} \gamma^{45}(m)
\]

\[
= + \frac{\chi}{N - 1} \sum_{n \neq n'} \left( \gamma^{\mu_4}(n) \gamma^{\nu_5}(n') - \gamma^{\nu_4}(n) \gamma^{\mu_5}(n') \right) \sum_{m=1}^{N-1} \gamma^{45}(m)
\]

\[
+ \frac{\chi}{N - 1} \sum_{n \neq n'} \left( \gamma^{\mu_4}(n) \gamma^{\nu_5}(n') - \gamma^{\nu_4}(n) \gamma^{\mu_5}(n') \right) \sum_{m=1}^{N-1} \gamma^{45}(m)
\]

\[
\sum_{m \neq n, m \neq n'} \gamma^{45}(m). \quad (25)
\]

Thus, in the contraction limit \( (22)-(24) \) when condensation singles out the eigenspace of \( \gamma^{45}(n) \gamma^{45}(n') \) with eigenvalue -1,

\[
\hat{O}^{\mu\nu} \to \hat{J}^{\mu\nu} - \hat{S}^{\mu\nu} \equiv \hat{L}^{\mu\nu}, \quad (26)
\]

as asserted.

4 Simplified Dirac dynamics

Dirac’s one-body theory in real (Majorana) form uses the operator algebra \( \mathcal{A}_{\text{DH}} \) acting on a vector space

\[
V_1 := \Sigma^{-\mathbb{M}}. \quad (27)
\]

of spinor-valued wavefunctions, mapping the space-time to the spinor space \( \Sigma = \Sigma(-\mathbb{M}) \) over the Minkowski space-time \( -\mathbb{M} \). This exhibits part of the compound structure we must simplify.
by decontraction, the split between spin space $\Sigma$ and space-time $M$. We construct the new space entirely from spins, replacing the infinite-dimensional function space $V_1$ by a spinor space of high but finite dimensionality.

To simplify Dirac’s spin-$\frac{1}{2}$ dynamics, we regard the position of the probe as the resultant of $N$ quantum steps, each represented by one hexad of chronons. We identify the spin variables of the probe with those of the last hexad in $[\Omega]$, the growing tip of the world line of the probe.

We thereby simplify the Dirac-Heisenberg algebra $\mathbb{A}_{DH}$ to $\hat{\mathbb{A}}_{DH} := \text{Cliff}(N \mathbb{6}_0)$, the Clifford algebra of a large squadron of cliffordons.

To construct the contraction from $\hat{\mathbb{A}}_{DH}$ to $\mathbb{A}_{DH}$, we group the generators of Cliff$(3N, 3N)$ into $N$ hexads $\gamma^\omega(n)$ ($\omega = 0, \ldots, 5; n = 1, \ldots, N$). Each hexad algebra acts on eight-component real spinors in $\mathbb{8}_0$. Hexad $N$ will be used for the spin of the quantum. The remaining $N - 1$ hexads provide the space-time variables.

We identify the usual Dirac gammas $\gamma^\mu$ for $\mu = 0, \ldots, 3$ of Cliff$(-\mathbb{M})$ with second-degree elements of the last hexad:

$$\gamma^\mu \cong \gamma^\mu := \gamma^{\mu5}(N) \quad (28)$$

Dirac’s spin generators $\hat{S}^{\mu\nu}$ simplify to the corresponding 16 components of the tensor

$$\hat{S}^{\omega\rho} := \frac{\hbar}{2} \gamma^{\omega\rho}(N), \quad (29)$$

where $\omega, \rho = 0, \ldots, 5$ and $\mu, \nu = 0, \ldots, 3$.

It is now straightforward to simplify the Dirac equation $D\psi = 0$ of (13). The internal degrees of freedom will be seen to contribute a rest mass term $m_\chi = \hbar/2\chi c$, and for simplicity we take this to be the entire rest mass of the Dirac equation, omitting any bare mass term in $\hat{D}$. We simplify $D \rightarrow \hat{D}$ and extend the internal symmetry group from SO$(1, 3)$ to the group SO$(3, 3)$ of a hexad by setting

$$\hat{D} := \frac{2\phi}{\hbar^2} \hat{S}^{\omega\rho} \hat{L}_{\omega\rho}, \quad (30)$$

where (cf. (18))

$$\hat{L}_{\omega\rho} := \frac{\hbar}{2} \sum_{n=1}^{N-1} \gamma_{\omega\rho}(n). \quad (31)$$

Our proposed dynamical operator is invariant under a conformal group SO$(3, 3)$ whose contraction includes the Poincaré group. [Our symmetry group SO$(3, 3)$ incorporates and extends the SO$(3, 2)$ symmetry possessed by Dirac’s dynamics for an electron in de-Sitter space-time [29]. That dynamics has the form

$$D' = \frac{1}{\hbar R} \hat{S}^{\omega\rho} \hat{O}_{\omega\rho} - mc,$$

where $\hat{S}^{\omega\rho}$ and $\hat{O}_{\omega\rho}$ are the five-dimensional spinorial and orbital angular momentum generators and $R$ is the radius of the de-Sitter universe. Its group is still compound, not simple, unifying translations, rotations and boosts, but not symplectic transformations.]

A complete set of commuting generators for the Poincaré group ISO$(1, 3)$ consists of the time translation generator $\hat{p}_0$, the rotation generator $\hat{L}_{12}$, and the boost generator $\hat{L}_{03}$. In the present context, we adjoin the imaginary unit $i$. In the proposed simplification ISO$(1, 3) \times
SO(2) ← SO(3, 3) these simplify according to $\hat{p}_0 \leftarrow \hat{L}_{04}$, $\hat{L}_{12} \leftarrow \hat{L}_{12}$, $\hat{L}_{03} \leftarrow \hat{L}_{03}$, $i \leftarrow \hat{L}_{45}$. A commuting set cannot contain both $\hat{L}_{04}$ and $\hat{L}_{45}$. Since varying energy is more familiar than varying $i$, in a first treatment we hold $\hat{L}_{45}$ constant and couple different masses in one representation.

5 Reduction to the Poincaré group

We now assume a condensation that reduces SO(3, 3) to its subgroup SO(1, 3) $\times$ SO(2). Relative to this reduction, the $\hat{D}$ of (30) breaks up into

$$\hat{D} = \frac{\phi}{2} \gamma^{\omega \rho}(N) \sum_n \gamma_{\omega \rho}(n)$$

$$(\omega, \rho = 0, 1, \ldots, 5)$$

$$= \phi \gamma^{\mu 5}(N) \sum_n \gamma_{\mu 5}(n) + \phi \gamma^{\mu 4}(N) \sum_n \gamma_{\mu 4}(n) + \phi \gamma^{\mu \nu}(N) \sum_n \gamma_{\mu \nu}(n)$$

$$+ \phi \gamma^{45}(N) \sum_n \gamma_{45}(n)$$

$$= \gamma^{\mu 5} \hat{p}_\mu - \frac{\phi}{\chi} \gamma^{\mu 4} \hat{x}_\mu + \frac{2\phi}{\hbar} \gamma^{\mu \nu} \hat{L}_{\mu \nu} + (N - 1)\phi \gamma^{45} \hat{i}. \quad (32)$$

In the condensate all the operators $\gamma^{45}(n)\gamma_{45}(n')$ attain their minimum eigenvalue $-1$. Then

$$(N - 1)\phi \gamma^{45} \hat{i} \longrightarrow -\frac{\hbar}{2\chi}. \quad (33)$$

and the dynamics becomes

$$\hat{D} = \gamma^{\mu 5} \hat{p}_\mu - \frac{\phi}{\chi} \gamma^{\mu 4} \hat{x}_\mu + \frac{2\phi}{\hbar} \gamma^{\mu \nu} \hat{L}_{\mu \nu} - m_\chi c, \quad (34)$$

with rest mass

$$m_\chi = \frac{\hbar}{2\chi c}. \quad (35)$$

For sufficiently large $N$ this reduces to the usual Dirac dynamics.

We identify the mass $m_\chi$ with the $N$-independent mass $m$ of the Dirac equation for the most massive individual quanta that the condensate can propagate without melt-down, on the order of the top quark or Higgs masses:

$$m_\chi \sim 10^2 \text{ GeV}, \quad \chi \sim 10^{-25} \text{ sec}. \quad (36)$$

The universe is $\sim 10^{10}$ years old. This leads to an upper bound

$$N_{\text{Max}} \sim 10^{41}. \quad (37)$$

This implies that $\chi$ is independent of $N$ as $N \to \infty$ and that $\phi \sim 1/N \to 0$ as $N \to \infty$ even for finite $\chi$. In experiments near the Higgs energy, $p \sim \hbar/\chi$. If we also determine $N$ by setting $x \sim N\chi$ then all four terms in (34) are of the same order of magnitude.

To estimate experimental effects, however, we must take gauge transformations into account. These transform the second term away in the continuum limit. This refinement of the theory is still in progress.
6 Conclusions

Like classical Newtonian mechanics, the Dirac equation has a compound (non-semisimple) invariance group. Its variables break up into three mutually commuting sets: the space-time-energy-momentum variables \( (x^\mu, p_\mu) \), the spin variables \( \gamma^\mu \), and the imaginary unit \( i \).

To unify them we replace the space-time continuum by an aggregate of \( M < \infty \) finite elements, chronons, described by spinors with \( \sim 2^{M/2} \) components. Chronons have Clifford-Wilczek statistics, whose simple operator algebra is generated by units \( \gamma^m, m = 1, \ldots, M \). We express all the variables \( x^\mu, p_\mu, \gamma^\mu \) and \( i \) as polynomials in the \( \gamma^m \). We group the \( M = 6N \) chronons into \( N \) hexads for this purpose, corresponding to tangent spaces; the hexad is the least cell that suffices for this simplification. There are three simplifiers \( \chi, \phi, 1/N \), all approaching 0 in the continuum limit, subject to the constraint \( \chi \phi (N - 1) = \hbar/2 \) for all \( N \).

In the continuum limit the Dirac mass becomes infinite. In our theory, the finite Dirac masses in nature are consequences of a finite atomistic quantum space-time structure with \( \chi > 0 \).

The theory predicts a certain spin-orbit coupling \( \gamma^{\mu\nu} L_{\mu\nu} \) not found in the Standard Model, and vanishing only in the continuum limit. The experimental observation of this spin-orbit coupling would further indicate the existence of a chronon.

In this theory, the spin we see in nature is a manifestation of the (Clifford) statistics of atomic elements of space-time, as Brownian motion is of the atomic elements of matter. As we improve our theory we will interpret better other indications of chronon structure that we already have, and as we improve our measuring techniques we shall meet more such signs.

ACKNOWLEDGMENTS

This work was aided by discussions with James Baugh, Heinrich Saller and Frank Wilczek. It was partially supported by the M. and H. Ferst Foundation.

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