New consequences of prime-counting function

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Abstract: Our objective in this paper is to study a particular set of prime numbers, namely \( \{p \in \mathbb{P} \text{ and } \pi(p) \notin \mathbb{P} \} \). As a consequence, estimations of the form \( \sum f(p) \) with \( p \) being prime belonging to this set are derived.

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1 Introduction

As usual, let \( \mathbb{P} \) be the set of all primes, \( \pi(x) = \#\mathbb{P} \cap [2, x] \) and

\[
\text{Li}(x) = \int_2^x \frac{1}{\log t} \, dt = \frac{x}{\log x} \left( 1 + \sum_{k=1}^{N} \frac{k!}{\log^k x} + O\left( \frac{1}{\log^{N+1} x} \right) \right), \quad (x \to +\infty). \tag{1.1}
\]

The Prime Number Theorem states that

\[
\pi(x) \sim \text{Li}(x), \quad (x \to +\infty). \tag{1.2}
\]

The theorem was proved, independently, by Hadamard [1] and de la Vallée-Poussin [2] in 1896. Another paper of de la Vallée-Poussin is [3], where he estimated the error term in the Prime Number Theorem by showing existence of a zero-free region for the Riemann zeta-function \( \zeta(s) \) to the left of the line \( \Re(s) = 1 \). The error is given by

\[
\pi(x) = \text{Li}(x) + O \left( xe^{-a\sqrt{\log x}} \right) \text{ as } x \to \infty, \tag{1.3}
\]

where \( a \) is a positive absolute constant.
The aim of this paper is to use the Prime Number Theorem to give some estimations related to the following subset of primes
\[ \{ p \in \mathbb{P} \text{ and } \pi(p) \notin \mathbb{P} \}. \]

2 Preparatory lemmas

We will need several preparatory lemmas. The first one is a new version and extension of the result obtained in [4]. Let us use the denotations \( \pi_2(x) \) for \( \pi(\pi(x)) \), \( \text{Li}_2(x) \) for \( \text{Li}(\text{Li}(x)) \) and \( \text{Li}_c(x) = \text{Li}(x) - \text{Li}_2(x) \).

**Lemma 2.1.** Let \( x \) be a positive real number. Let us denote by \( \pi_c(x) \) (respectively, \( \pi_c(x) \)) the number of primes \( p \leq x \) such as \( \pi(p) \) is not a prime (respectively, \( \pi(p) \) is prime). Precisely,
\[
\pi_c(x) := \# \{ p \leq x | \pi(p) \text{ is not prime} \} = \sum_{p \leq x \atop \pi(p) \notin \mathbb{P}} 1,
\]
and
\[
\pi_c(x) := \# \{ p \leq x | \pi(p) \text{ is prime} \} = \sum_{p \leq x \atop \pi(p) \in \mathbb{P}} 1.
\]

Then,
1. \( \pi(x) = \pi_c(x) + \pi_c(x) \).
2. \( \pi_c(x) = \pi(x) - \pi(\pi(x)) \).
3. \( \pi_c(x) = \pi(\pi(x)) \).

**Proof.**
1. It is straightforward to see that the set of prime numbers less than or equal to \( x \) can be partitioned into two subsets as follows
\[
\{ p \leq x | p \text{ is prime} \} = \{ p \leq x | p \text{ is prime and } \pi(p) \text{ is prime} \}
\]
\[
\cup \{ p \leq x | p \text{ is prime and } \pi(p) \text{ is not prime} \}.
\]
(2.1)

By passage to cardinality, we get
\[
\# \{ p \leq x | p \text{ is prime} \} = \# \{ p \leq x | p \text{ is prime and } \pi(p) \text{ is prime} \}
\]
\[
+ \# \{ p \leq x | p \text{ is prime and } \pi(p) \text{ is not prime} \}
\]

or
\[
\pi(x) = \pi_c(x) + \pi_c(x).
\]
(2.2)

2. It is not difficult to see that \( \# \{ p \leq x | p \text{ is prime and } \pi(p) \text{ is not prime} \} \) is equal to the number of different equivalence classes \( \hat{p} \) which was denoted in [4] by \( \pi_c(x) \) (for more details, see [4]).

3. From the equation (2.2), we obtain
\[
\pi_c(x) = \pi(x) - \pi_c(x) = \pi(x) - (\pi(x) - \pi(\pi(x))) = \pi(\pi(x)).
\]
Lemma 2.2. We have the following estimations

\[
\frac{1}{\log \text{Li}(x)} = \frac{1}{\log x} + O\left(\frac{\log \log x}{\log^2 x}\right) \quad (x \to \infty). 
\] (2.3)

**Proof.** Using formula (1.1), we get

\[
\frac{1}{\log \text{Li}(x)} = \frac{1}{\log x - \log \log x + \log \left(1 + \sum_{k=1}^{N} \frac{k!}{\log^k x} + O\left(\frac{1}{\log^{N+1} x}\right)\right)}, 
\] (2.4)

and by using Taylor’s expansion, we acquire

\[
\log \left(1 + \sum_{k=1}^{N} \frac{k!}{\log^k x} + O\left(\frac{1}{\log^{N+1} x}\right)\right) = \frac{1}{\log x} + O\left(\frac{1}{\log^2 x}\right) \quad (x \to \infty). 
\] (2.5)

Next, we replace (2.5) in (2.4), we get

\[
\frac{1}{\log \text{Li}(x)} = \frac{1}{\log x \left(1 - \frac{\log \log x}{\log x} + \frac{1}{\log^2 x} + O\left(\frac{1}{\log^3 x}\right)\right)} = \frac{1}{\log x \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right)} \quad (x \to \infty).
\] \(\Box\)

Lemma 2.3. We have

\[
\pi_2(x) - \text{Li}_2(x) = O\left(\frac{x}{\log x} e^{-a\sqrt{\log \log x}}\right) \quad (x \to \infty), 
\] (2.6)

\[
\pi_c(x) - \text{Li}_c(x) = O\left(\frac{x}{\log x} e^{-a\sqrt{\log \log x}}\right) \quad (x \to \infty). 
\] (2.7)

**Proof.** From (1.3), we have on the one hand

\[
\pi(\pi(x)) = \text{Li}(\pi(x)) + O\left(\pi(x)e^{-a\sqrt{\log \pi(x)}}\right) = \text{Li}(\pi(x)) + O\left(\frac{x}{\log x} e^{-a\sqrt{\log \log x}}\right). 
\] (2.8)

And, on the other hand, by Taylor’s series, we acquire

\[
\text{Li}(\pi(x)) - \text{Li}_2(x) = \text{Li}\left(\text{Li}(x) + O(xe^{-a\sqrt{\log x}})\right) - \text{Li}(\text{Li}(x)) \\
= \frac{1}{\log \text{Li}(x)} O\left(x e^{-a\sqrt{\log x}}\right) \\
= O\left(\frac{x}{\log x} e^{-a\sqrt{\log x}}\right). 
\] (2.9)

Now, we can estimate \(\pi_2(x) - \text{Li}_2(x)\):

\[
\pi_2(x) - \text{Li}_2(x) = \pi_2(x) - \pi(\pi(x)) + \text{Li}(\pi(x)) - \text{Li}_2(x).
\]
Using (2.8) and (2.9), we get

\[
\pi_2(x) - \text{Li}_2(x) = O \left( \frac{x}{\log x} e^{-a\sqrt{\log x}} \right) + O \left( \frac{x}{\log x} e^{-a\sqrt{\log x}} \right) = O \left( \frac{x}{\log x} e^{-a\sqrt{\log x}} \right).
\]

For the formula (2.7), we have

\[
\pi_c(x) - \text{Li}_c(x) = \pi(x) - \pi_2(x) - (\text{Li}(x) - \text{Li}_2(x))
\]

\[
= \pi(x) - \text{Li}(x) - (\pi_2(x) - \text{Li}_2(x))
\]

\[
= O \left( xe^{-a\sqrt{\log x}} \right) - O \left( \frac{x}{\log x} e^{-a\sqrt{\log x}} \right)
\]

\[
= O \left( \frac{x}{\log x} e^{-a\sqrt{\log x}} \right).
\]

3 Main results

Our main result may be stated as follows.

**Theorem 3.1.** Let us have \( f(x) = Cx^{-b} \log^w x \) with \( C, b \geq 0 \) and \( w \geq 1 \). Then

\[
\sum_{\substack{p \leq x \\pi(p) \notin \mathbb{P}}} f(p) = \int_2^x \frac{f(y)}{\log y} \left( 1 - \frac{1}{\log y} \right) dy + O \left( x^{1-b} (\log x)^{w-3} \log \log x \right).
\] (3.1)

**Proof.** Using Stieltjes integral, we obtain

\[
\sum_{\substack{p \leq x \\pi(p) \notin \mathbb{P}}} f(p) = \int_2^x f(y) d\pi_c(y).
\]

Integration by parts gives

\[
\sum_{\substack{p \leq x \\pi(p) \notin \mathbb{P}}} f(p) = f(x)\pi_c(x) - f(2) - \int_2^x f'(y) \pi_c(y) dy
\]

\[
= f(x)\pi_c(x) - f(2) - \int_2^x f'(y) \text{Li}_c(y) dy - \int_2^x f'(y) (\pi_c(y) - \text{Li}_c(y)) dy.
\]

Then integration by parts gives

\[
\sum_{\substack{p \leq x \\pi(p) \notin \mathbb{P}}} f(p) = \int_2^x f(y) \frac{1}{\log y} \left( 1 - \frac{1}{\log \text{Li}(y)} \right) dy + f(2)\text{Li}_c(2) - f(2) + f(x)(\pi_c(x) - \text{Li}_c(x))
\]

\[
- \int_2^x f'(y) (\pi_c(y) - \text{Li}_c(y)) dy
\] (3.2)
Now, using estimation (2.3) of Lemma 2.2, we get
\[
\sum_{p \leq x \atop \pi(p) \notin \mathbb{P}} f(p) = \int_2^x \frac{f(y)}{\log y} \left( 1 - \frac{1}{\log y} - O \left( \frac{\log \log y}{\log^2 y} \right) \right) dy \\
+ f(2)\text{Li}(2) + f(x)(\pi_{c}(x) - \text{Li}_{c}(x)) - \int_2^x f'(y)(\pi_{c}(y) - \text{Li}_{c}(y)) dy. \tag{3.3}
\]

We have, on the one hand, by (2.7) of Lemma 2.3
\[
f(x)(\pi_{c}(x) - \text{Li}_{c}(x)) = O \left( x^{1-b} \log^{w} e \sqrt[2]{\frac{x}{\log x}} \right) \tag{3.4}
\]
and
\[
\int_2^x \frac{f(y)}{\log y} O \left( \frac{\log \log y}{\log^2 y} \right) dy = O \left( x^{1-b} \log^{w-3} x \log \log x \right). \tag{3.5}
\]
On the other hand, we have
\[
f'(x) = Cx^{1-b} \log^{w-1} x(-b \log x + w).
\]

Then, again by (2.7) of Lemma 2.3
\[
f'(y)(\pi_{c}(y) - \text{Li}_{c}(y)) = O \left( x^{-b}(\log x)^{w} e \sqrt[2]{\frac{y}{\log y}} \right).
\]

Consequently,
\[
\int_2^x f'(y)(\pi_{c}(y) - \text{Li}_{c}(y)) dy = \int_2^x O \left( y^{-b}(\log y)^{w} e \sqrt[2]{\frac{x}{\log x}} \right) dy \\
= O \left( x^{1-b} \log^{w-3} x \log \log x \right). \tag{3.6}
\]

Finally, by replacing estimations (3.4), (3.5) and (3.6) in (3.3), we find that
\[
\sum_{p \leq x \atop \pi(p) \notin \mathbb{P}} f(p) = \int_2^x \frac{f(y)}{\log y} \left( 1 - \frac{1}{\log y} \right) dy + O \left( x^{1-b} \log^{w-3} x \log \log x \right). \tag{3.7}
\]

We now present applications of Theorem 3.1.

**Corollary 1.** We have
\[
\sum_{p \leq x \atop \pi(p) \notin \mathbb{P}} \log p = x - \text{Li}(x) - 2 + O \left( x \log^{-2} x \log \log x \right) \quad (x \to \infty), \tag{3.7}
\]
\[
\sum_{p \leq x \atop \pi(p) \notin \mathbb{P}} \frac{\log p}{p} = \log x - \log \log x + \log \log \sqrt{2} + O \left( \log^{-2} x \log \log x \right) \quad (x \to \infty), \tag{3.8}
\]
\[
\sum_{\substack{p \leq x \\ \pi(p) \notin \mathbb{P}}} \frac{1}{p} = \log \log x - (\log \log 2 + (\log 2)^{-1}) + O \left( \log^{-3} x \log \log x \right) \quad (x \to \infty), \quad \text{(3.9)}
\]

\[
\sum_{\substack{p \leq x \\ \pi(p) \notin \mathbb{P}}} \frac{\log^n p}{p} = \frac{\log^n x}{n} - \frac{\log^{n-1} x}{n-1} + c_n(2) + O \left( \log^{n-3} x \log \log x \right) \quad (x \to \infty), \quad \text{(3.10)}
\]

with \(c_n(2) = - \left( \frac{\log^n 2}{n} - \frac{\log^{n-1} 2}{n-1} \right)\) and \(n \geq 2\).

\[
\sum_{i=2}^{n} \sum_{\substack{p \leq x \\ \pi(p) \notin \mathbb{P}}} \frac{\log^i p}{p} = \frac{\log^n x}{n} - \log x + C_n(2) + O \left( \log^{n-3} x \log \log x \right) \quad (x \to \infty), \quad \text{(3.11)}
\]

with \(C_n(2) = \sum_{i=2}^{n} c_i(2)\).

**Proof.** The first four estimations are immediate from formula (3.1). Now, for the latest, we have

\[
k = 2, \quad \sum_{\substack{p \leq x \\ \pi(p) \notin \mathbb{P}}} \frac{\log^2 p}{p} = \frac{\log^2 x}{2} - \log x + c_2(2) + O \left( \log^{-1} x \log \log x \right),
\]

\[
k = 3, \quad \sum_{\substack{p \leq x \\ \pi(p) \notin \mathbb{P}}} \frac{\log^3 p}{p} = \frac{\log^3 x}{3} - \frac{\log^2 x}{2} + c_3(2) + O \left( \log \log x \right),
\]

\[
\vdots
\]

\[
k = n, \quad \sum_{\substack{p \leq x \\ \pi(p) \notin \mathbb{P}}} \frac{\log^n p}{p} = \frac{\log^n x}{n} - \frac{\log^{n-1} x}{n-1} + c_n(2) + O \left( \log^{n-3} x \log \log x \right).
\]

These equations can be added to yield the desired formula. \(\square\)

**Remark.** The absolute error in (3.9) tends to zero as \(x\) tends to infinity, then

\[
\lambda_0 = \lim_{x \to \infty} \left( \sum_{\substack{p \leq x \\ \pi(p) \notin \mathbb{P}}} \frac{1}{p} - \log \log x \right)
\]

exists and has a finite value.

**Theorem 3.2.**

1. We have

\[
\pi_2(x) \leq k \frac{x}{\log^2 x}, \quad k \simeq 2.4919 \text{ and } x \geq 2.
\]  

\[
\text{(3.12)}
\]

2. The following sum is convergent

\[
\sum_{p, \pi(p) \text{ are primes}} \frac{1}{p}.
\]  

\[
\text{(3.13)}
\]
Proof. 1. As it is well-known

\[ \pi(x) < c \frac{x}{\ln x}, \quad x \geq 1 \]

with \( c = 1.25506 \). Then

\[ \pi(\pi(x)) < c \frac{\pi(x)}{\ln \pi(x)} < c \frac{x}{\ln x \ln \pi(x)} < c \frac{x}{\ln x (\ln x - \ln \ln x)} = c \frac{x}{\ln^2 x \left(1 - \frac{\ln \ln x}{\ln x}\right)}, \quad x \geq 2 \]

The function \( \frac{1}{1 - \frac{\ln \ln x}{\ln x}} \) has its maximum value \( \frac{e}{e - 1} \) at \( e \), then we get

\[ \pi(\pi(x)) < \frac{c^2 e}{e - 1} \frac{x}{\ln^2 x} \approx 2.4919 \frac{x}{\ln^2 x}, \quad x \geq 2. \]

2. Using Abel’s summation formula and since inequality (3.12) holds, we get

\[
\sum_{\substack{p \leq x \\text{p, } \pi(p) \text{ are primes}}} \frac{1}{p} = \frac{\pi_c(x)}{x} + \int_{\lambda_1}^{x} \frac{\pi_c(t)}{t^2} \, dt \\
\leq \frac{1}{\log^2 x} + k \int_{\lambda_1}^{x} \frac{1}{t \log^2 t} \, dt = \frac{k}{\log^2 x} + \frac{k}{\log x} + \frac{k}{\log \lambda_1} \\
\leq k \left( \frac{1}{\log^2 x} + \frac{1}{\log x} + \frac{1}{\log \lambda_1} \right) \leq k \left( \frac{1}{\log^2 3} + \frac{2}{\log 3} \right).
\]

This implies that \( \exists M > 0 \) such that for all sufficiently large \( x \),

\[
\sum_{\substack{p, \pi(p) \text{ are primes}}} \frac{1}{p} \leq M.
\]

This implies that the sum is convergent. \( \square \)

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