Symplectic geometry of Maxwell theory and the photon concept

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Abstract. For an arbitrary multiply connected cavity a solution theory for the Maxwell equations is developed under ideal conductor boundary conditions. The trajectories of the electromagnetic fields are decomposed into their Helmholtz–Hodge components, including so-called cohomological fields which correspond to the harmonic differential forms. A canonical formalism is formulated in the Coulomb gauge, which comprises only the transversal and cohomological components. The canonical vector potentials and their conjugate fields are smeared in terms of smooth test functions. The Poisson bracket for the smeared fields is based on a symplectic form in the test function space and the corresponding infinite dimensional symplectic Lie algebra is introduced. The conservation quantities of Maxwell theory arise in this manner as co–momentum images of the Lie algebra. By means of a symplectic transformation the transversal dynamical generator is diagonalized and defines a special complexification of the test function space. A diagonalization of the cohomological dynamical generator is proved impossible.

Since the Weyl quantization provides a symplectic equivariant map of the classical fields onto the quantum fields (affiliated with our C*–Weyl algebra), we have in quantum electrodynamics the same freedom of symplectic transformations and complexifications in the test function space (and not in the representing Hilbert space). Using the diagonalizing complexification we introduce the (smeared) transversal creation and annihilation operators and arrive at a unique particle structure for the transversal fields in a Fock representation. The symplectic Lie algebra elements, multiplied by $i\hbar$, constitute the one–photon observables, where especially the diagonal dynamical Maxwell generator generalizes Einstein’s photon energy to our general set up. The quantum co–momentum map acquires the form of a second quantization of the one–photon observables and leads to particle conserving multi–photon Hamiltonians for the complex linear Lie algebra elements, whereas the anti–linear Lie algebra elements generate particle non–conserving squeezing Hamiltonians. Since there is no particle structure for the quantized cohomological fields their collective nature is confirmed, illustrating the difference between quantization and particle discretization.

1. Introduction
As is well known, the photon concept has been introduced by Einstein 1905 in the framework of a nonrelativistic, statistical theory of light.
In the very first paper on canonical commutation relations (the latter not appearing in Heisenberg’s paper), namely in Zur Quantenmechanik by Born and Jordan (1925), it is stated that the infinitely many degrees of freedom of the electromagnetic field should be quantized analogously to the material ones. This point of view is sharpened by Dirac’s principle, formulated shortly afterwards, by which one has to replace the Poisson bracket by the scaled commutator (see also our Section on deformation quantization below), since then Poisson commuting classical variables lead to commuting quantum observables.
After the introduction of creation and annihilation operators for photons by Dirac (1927) (inappropriately in terms of number and phase variables), relativistic quantum electrodynamics (QED) started (1929/30) with two papers by Heisenberg and Pauli. In his book *Anschauliche Quantentheorie* (1936) Jordan writes that it be time to finally clarify the connection between electromagnetic fields and photons. Since then there have still been published many articles and books on this subject, where it is surprising that in several texts on Quantum Optics the authors explicitly renounce to define one-photon observables. Also in perturbation theoretic QED the photon states appear only in the degenerated form of plane waves.

In the present work we take up again the quantization of nonrelativistic ED making use of the mathematical methods which are at hand in our time, especially of infinite dimensional Poisson manifolds, group theory, and C*-algebraic Weyl quantization.

We smear the classical electromagnetic fields by test functions to increase the solution manifold of the Maxwell equations and to gain a greater similarity to the smeared quantized fields. In view of the recent developments on trapped photons and on sophisticated opto-electronic devices it is necessary to formulate the theory in arbitrary, open, possibly multiply connected regions $\Lambda \subset \mathbb{R}^3$ (cavities).

In this general setup we are going to demonstrate that the photon observables have their formal roots in a diagonalized form of deterministic Maxwell theory.

The (open, connected) $\Lambda$ is called an *interior domain* if it is bounded, and an *exterior domain* if its complement $\Lambda^c \equiv \mathbb{R}^3 \setminus \Lambda$ is bounded. By $\bar{\Lambda}$ we denote the closure of $\Lambda$, and by $\partial \Lambda$ its boundary. Often we require some smoothness conditions on the boundary of $\Lambda$. Note that also an exterior domain has a bounded boundary $\partial \Lambda$.

As is well known, the physical meaning of the *longitudinal* and *transversal* parts of the fields is very different, and for multiply connected cavities $\Lambda$ there arise also *additional types* of electric and magnetic fields, which combine properties of longitudinal and transversal fields. Since they depend on the cohomological features of $\Lambda$, especially on the first and second Betti numbers, we term them *cohomological fields*. (They correspond to the so-called harmonic differential forms of differential geometry.)

They got into focus rather recently, in spite of the following remark of Maxwell [1]:

*We are led here to considerations belonging to the Geometry of Position, a subject which, though its importance was pointed out by Leibniz and illustrated by Gauss, has been little studied.*

(Maxwell called the second Betti number $b_2(\Lambda)$ *perihractic number*.)

Nowadays the “Geometry of Position” is widely studied in calculational electrotechnics and is the basis for the *finite elements method*. There are computer programs to determine the cuts for making $\Lambda$ simply connected. The mathematical aspects are pursued in *topological electromagnetism* (cf. [2]), which mainly is concerned with quasi-stationary situations.

To formulate a photon theory we demonstrate that the evolution of the classical (and then of the quantized) fields is decomposable into *separated dynamical systems* associated with the different field types. By exploiting the reduction techniques for Hilbert space operators we reduce in the main text the topological considerations, connected with differentiable manifolds, to a minimum and gain the mathematical control of non-smooth boundaries at the same time. (Proofs which are missing in the present Article are elaborated in collaboration with R. Honegger in still unpublished material.)

To apply qualitative methods the connection with differential forms is however useful, what we describe in a forthcoming paper.
2. Fields and potentials

We assume at first finite electromagnetic energy in Λ and formulate the dynamics in $L^2$–Hilbert space, where we define the unbounded vector–differential operators with their domains of definition as follows:

$$
\text{grad}_0 \varphi := \nabla \varphi, \quad \forall \varphi \in \text{dom} (\text{grad}_0) := W^1_0(\Lambda, \mathbb{R}) , \\
\text{grad} \varphi := \nabla \varphi , \quad \forall \varphi \in \text{dom}(\text{grad}) := \text{W}^1(\Lambda, \mathbb{R}) , \\
\text{div}_0 \phi := \nabla \cdot \phi, \quad \forall \phi \in \text{dom}(\text{div}_0) := \text{W}^0(\text{div}; \Lambda, \mathbb{R}^3) , \\
\text{div} \phi := \nabla \cdot \phi, \quad \forall \phi \in \text{dom}(\text{div}) := \text{W}(\text{div}; \Lambda, \mathbb{R}^3) , \\
\text{curl}_0 \phi := \nabla \times \phi, \quad \forall \phi \in \text{dom}(\text{curl}_0) := \text{W}^0(\text{curl}; \Lambda, \mathbb{R}^3) , \\
\text{curl} \phi := \nabla \times \phi , \quad \forall \phi \in \text{dom}(\text{curl}) := \text{W}(\text{curl}; \Lambda, \mathbb{R}^3) .
$$

(2.1)

We make use of certain Sobolev spaces, which we characterize only for the special case that there exists a normal vector almost everywhere at the boundary. The derivatives are meant in the weak sense.

$$
\text{dom} (\text{grad}_0) = \text{W}^1_0(\Lambda, \mathbb{R}) = \{ \varphi \in L^2(\Lambda, \mathbb{R}) | \frac{\partial \varphi}{\partial x_i} \in L^2(\Lambda, \mathbb{R}), \varphi|_{\partial \Lambda} = 0 \} , \\
\text{dom} (\text{grad}) = \text{W}^1(\Lambda, \mathbb{R}) = \{ \varphi \in L^2(\Lambda, \mathbb{R}) | \frac{\partial \varphi}{\partial x_i} \in L^2(\Lambda, \mathbb{R}), 1 \leq i \leq 3 \} .
$$

(2.2)

$$
\text{dom}(\text{div}_0) = \text{W}^0(\text{div}; \Lambda, \mathbb{R}^3) = \{ \phi \in L^2(\Lambda, \mathbb{R}) | \nabla \cdot \phi \in L^2(\Lambda, \mathbb{R}), \phi \cdot n|_{\partial \Lambda} = 0 \} , \\
\text{dom}(\text{div}) = \text{W}(\text{div}; \Lambda, \mathbb{R}^3) = \{ \phi \in L^2(\Lambda, \mathbb{R}) | \nabla \cdot \phi \in L^2(\Lambda, \mathbb{R}) \} , \\
\text{dom}(\text{curl}_0) = \text{W}^0(\text{curl}; \Lambda, \mathbb{R}^3) = \{ \phi \in L^2(\Lambda, \mathbb{R}) | \nabla \times \phi \in L^2(\Lambda, \mathbb{R}), \phi \times n|_{\partial \Lambda} = 0 \} , \\
\text{dom}(\text{curl}) = \text{W}^0(\text{curl}; \Lambda, \mathbb{R}^3) = \{ \phi \in L^2(\Lambda, \mathbb{R}) | \nabla \times \phi \in L^2(\Lambda, \mathbb{R}) \} .
$$

(2.3)

As a first gratification for specifying the operator domains we have nice adjointing relations for the closed operators in the various $L^2$–spaces:

$$
\text{grad}_0^* = - \text{div} \quad \text{div}^* = - \text{grad}_0 , \\
\text{grad}^* = - \text{div}_0 \quad \text{div}_0^* = - \text{grad} , \\
\text{curl}_0^* = \text{curl} \quad \text{curl}^* = \text{curl}_0 .
$$

We work with the direct real Hilbert sum

$$
\mathcal{R} := L^2(\Lambda, \mathbb{R}^3) \oplus L^2(\Lambda, \mathbb{R}^3) ,
$$

(2.5)

in which we combine the electric and magnetic fields to the 6–tuples

$$
\psi := (\mathbf{E}, \mathbf{B}) \equiv \mathbf{E} \oplus \mathbf{B} \in \mathcal{R} .
$$

(We use $\oplus$ for the orthogonal decomposition into the electric and magnetic parts, whereas other orthogonal decompositions are indicated by $\otimes$.)
2.1. Boundary conditions and Maxwell operator

We arrange the Maxwell equations as follows. The first two equations

\begin{align}
\nabla \cdot E_t(x) &= \epsilon_0^{-1} \rho_t(x), \\
\nabla \cdot B_t(x) &= 0,
\end{align}

(2.6)

(2.7)

do not describe the dynamical evolution, which rather is expressed by the remaining two:

\begin{align}
\frac{\partial E_t(x)}{\partial t} &= c^2 \nabla \times B_t(x) - \frac{1}{\epsilon_0} j_t(x), \\
\frac{\partial B_t(x)}{\partial t} &= -\nabla \times E_t(x).
\end{align}

(2.8)

If we concentrate on the tuple of dynamical equations we supplement it by the continuity equation (derivable from (2.8) combined with (2.6))

\begin{align}
\frac{\partial \rho_t(x)}{\partial t} + \nabla \cdot j_t(x) &= 0, \quad \forall x \in \Lambda, \quad \forall t \in \mathbb{R}.
\end{align}

(2.9)

Throughout our exposition the boundary surface \( \partial \Lambda \) of \( \Lambda \) is supposed to consist physically of a perfect conductor. In a perfect conductor the skin region is zero and the surface charge and current densities accommodate without delay to the arbitrarily fast varying fields.

On the surface of a perfect conductor wall, enclosing vacuum, the continuous field components have to vanish. This yields the boundary conditions

\begin{align}
E \times n|_{\partial \Lambda} &= 0, \\
B \cdot n|_{\partial \Lambda} &= 0, \quad \forall t \in \mathbb{R}.
\end{align}

(2.10)

The extension of \( L^2(\Lambda, \mathbb{R}^3) \)-classes of fields to the boundary is achieved by means of a boundary operator. If the system of the Maxwell differential Eqs. (2.6) to (2.8) is solved under the boundary condition (2.10), leading to the solution trajectory \( \mathbb{R} \ni t \mapsto (E_t, B_t) \), then the surface current density \( j^{\partial \Lambda}_t : \partial \Lambda \to \mathbb{R}^3 \) and the surface charge density \( \rho^{\partial \Lambda}_t : \partial \Lambda \to \mathbb{R} \) may be calculated from the relations

\begin{align}
\rho^{\partial \Lambda}_t &= -\epsilon_0 E_t \cdot n|_{\partial \Lambda}, \\
j^{\partial \Lambda}_t &= -\mu_0^{-1} B_t \times n|_{\partial \Lambda}, \quad \forall t \in \mathbb{R}.
\end{align}

(2.11)

We assume now that \( j_t \in L^2(\Lambda, \mathbb{R}^3) \), for all \( t \in \mathbb{R} \). So we rewrite the dynamical Maxwell Eqs. (2.8) in the mathematically concise form

\begin{align}
\frac{d}{dt} \begin{pmatrix} E_t \\ B_t \end{pmatrix} &= \begin{pmatrix} 0 & \text{curl} \\ -\text{curl}_0 & 0 \end{pmatrix} \begin{pmatrix} E_t \\ B_t \end{pmatrix} + \begin{pmatrix} -j_t \\ 0 \end{pmatrix} \\
&=: \psi_t
\end{align}

(2.12)

(For convenience we have momentarily set \( \epsilon_0 = \mu_0 = c = 1 \).)

**Lemma 2-1** \( \mathbf{A} \) is a closed operator in the real Hilbert space \( \mathcal{R} \) and satisfies \( \mathbf{A}^* = -\mathbf{A} \). (This is equivalent to \( i \mathbf{A} \) being selfadjoint in the complexified Hilbert space \( \mathcal{R} + i\mathcal{R} \).)

Consequently, the free Maxwell dynamics \( \exp\{i t \mathbf{A}\}, \ t \in \mathbb{R} \), constitutes a strongly continuous orthogonal group in \( \mathcal{R} \).
2.2. Solution of the Maxwell Cauchy problem

A Cauchy problem is an initial–value problem, where in general – as in the present case – also spatial boundary conditions have to be taken into account, what then is called an initial boundary value problem (e.g. [3]).

**Theorem 2-2** Let $\Lambda$ be an arbitrary domain in $\mathbb{R}^3$ and suppose a current density $t \mapsto j_t$, which is weakly locally integrable. Then the Maxwell weak Cauchy problem, is uniquely solvable for every initial value $\psi_{t_0} \in \mathcal{R}$, which satisfies the boundary conditions. Its solution trajectory $\mathbb{R} \ni t \mapsto \psi_t \in \mathcal{R}$ is $\| \cdot \|_{\mathcal{R}}$–continuous and is uniquely given by

$$
\psi_t = \exp\{(t - t_0)\Lambda\} \psi_{t_0} + \int_{t_0}^{t} \exp\{(t - s)\Lambda\} \gamma_s \, ds, \quad \forall t \in \mathbb{R}.
$$

(2.13)

This is the first form of our magic formula, which discloses important features of the solution trajectory.

2.3. The Helmholtz–Hodge decomposition

The subsequent result is a modification of material in [4].

**Theorem 2-3** We have for an arbitrary (also unbounded) domain the following unique decomposition of quadrat integrable vector fields into mutually orthogonal components

$$
\mathcal{R} = L^2(\Lambda, \mathbb{R}^3) = \ker(\text{div}) = \ker(\text{div}_0)
\begin{cases}
\text{ran(\text{grad}_0)} \oplus \mathbb{H}_2 \oplus \text{ran(curl)} = \ker(\text{curl}_0)
\end{cases}
$$

$$
\psi = E \oplus B
\begin{cases}
\text{ran(\text{grad})} \oplus \mathbb{H}_1 \oplus \text{ran(curl)} = \ker(\text{curl})
\end{cases}
$$

$$
\gamma_t = -\text{\textstyle \hat{j}_t} \oplus \text{\textstyle \hat{j}_t}^\psi \oplus -\text{\textstyle \hat{j}_t}^\psi
\begin{cases}
0 \oplus 0^\psi \oplus 0^\psi
\end{cases}
$$

We now assume as our “Standard Assumption” that $\Lambda$ is an interior domain, satisfying the following conditions: The boundary $\partial \Lambda$ consists of a finite number of connected $C^2$–smooth surfaces, $\Lambda$ is locally situated on one side of $\partial \Lambda$, and $\Lambda$ is made simply connected by a finite number of regular, non–intersecting cuts, which are non–tangential to $\partial \Lambda$ (cf. [4], Vol. III). We may omit then the closure operations for defining the Helmholtz–Hodge subspaces, and obtain the following spaces of cohomological fields

$$
\mathbb{H}_1 = \{ \phi \in L^2(\Lambda, \mathbb{C}^3) \mid \nabla \cdot \phi = 0, \nabla \times \phi = 0, \phi \cdot n_{|\partial \Lambda} = 0 \} \subset W^1(\Lambda, \mathbb{C}^3)
$$

$$
\mathbb{H}_2 = \{ \psi \in L^2(\Lambda, \mathbb{C}^3) \mid \nabla \cdot \psi = 0, \nabla \times \psi = 0, \psi \times n_{|\partial \Lambda} = 0 \} \subset W^1(\Lambda, \mathbb{C}^3).
$$

For the connection with differential geometry we assume an interior domain $\Lambda$ such as described in the second part of the preceding Theorem. Its closure is a manifold with boundary as described in [5], where also vector analysis is translated into the language of differential forms. According to this, we associate the vector fields of $\mathbb{H}_1$ by means of Hilbert space duality with one–forms, and those of $\mathbb{H}_2$ at first with one–forms and then by Hodge duality with two–forms. In this manner we obtain the connection of $\mathbb{H}_1$ and $\mathbb{H}_2$ with the first resp. second cohomology group of de Rham (the quotients of the closed forms over the exact forms, of the first resp. second degree).
The geometric meaning is elucidated by using the duality for the transition of the \( n \)-differential forms (cochains) to \( n \)-dimensional objects (chains) (caused by integrating the \( n \)-cochains over the \( n \)-chains). The \( 1 \)-chains (modulo closed paths contractible to a point) determine the multiplicity of connectedness of \( \Lambda \). Thus \( \dim H_1 = b_1(\Lambda) \). The \( 2 \)-chains (modulo closed 2-dimensional surfaces contractible to a point) give the number of the disconnected components of \( \partial \Lambda \) without counting the surface of the infinite, connected component of \( \Lambda^c \). Thus \( \dim H_2 = b_2(\Lambda) \).

In this manner we get the rule

\begin{align*}
(1) \quad b_1(\Lambda) = 0 & \iff H_1 = \{0\} & \iff \Lambda \text{ is simply connected.} \\
(2) \quad b_2(\Lambda) = 0 & \iff H_2 = \{0\} & \iff \partial \Lambda \text{ is connected.}
\end{align*}

In the drawing of a cubic cavity \( \Lambda \), the two columns prevent two classes of closed paths to be contracted to a point and we have \( b_1(\Lambda) = 2 \). (And one needs in fact two cuts to make \( \Lambda \) simply connected.) The three inserted egg-shaped conductors prevent three classes of closed surfaces to be shrunk to a point. The surfaces of the conductors constitute the surfaces of the bounded components of \( \Lambda^c \) (whereas the interior regions of the columns is part of the infinite component of \( \Lambda^c \)). We see that here \( b_2(\Lambda) = 3 \).

From the Helmholtz–Hodge decomposition of the magnetic field it follows

\[ B \in \ker(\text{div}_0) \iff [B^\parallel = 0 \quad \& \quad B^{co}, B^\top \in \ker(\text{div}_0)] , \]

Thus the magnetic field (where magnetic monopoles are excluded by vanishing divergence) has in general not only a transversal but also a \textit{cohomological component} \( B^{co} \).
2.4. Decomposition of dynamical operators

A reducing subspace for $A$ is a closed subspace $K$ of $R$ satisfying

$$\exp\{tA\}(K) = K, \quad \forall t \in \mathbb{R}.$$  

The reduction $A^\top$ of $A$ from $R$ to the transversal sub–Hilbert space

$$R_i^\top = \text{ran}(\text{curl}) \oplus \text{ran}(\text{curl}_0) = \ker(A)^\perp = \text{ran}(A)$$

is the matrix operator

$$A^\top = \begin{pmatrix} 0 & \text{curl} \| \\ -\text{curl}_0 \| & 0 \end{pmatrix}. \quad (2.14)$$

curl$|$ and curl$_0|$ are the restrictions to the orthogonal complements of their kernels (and thus possess an inverse). We have

$$\begin{array}{c}
\mathcal{R} = (\text{ran}(\text{grad})_0 \oplus \text{ran}(\text{grad})) \oplus (H_2 \oplus H_1) = \text{ran}(\text{curl}) \oplus \text{ran}(\text{curl}_0)_0 \\
\exp\{tA\} = 1 \oplus 1 \oplus \exp\{tA^\top\}.
\end{array}$$

The material current being omitted, just the circulations of the transversal fields give rise to their temporal variations (and vice versa).

The injectivity of $A^\top$ ensures that the diagonal matrix–operator

$$(A^\top)^*A^\top = -(A^\top)^2 = \begin{pmatrix} \text{curl}^2_{0} & 0 \\ 0 & \text{curl}^2_{n0} \end{pmatrix}$$

is strictly positive, where

$$\text{curl}^2_{0} = \text{curl}|\text{curl}_0|, \quad \text{curl}^2_{n0} = \text{curl}_0|\text{curl}|.$$  

The two curlcurl operators consist each of a product of the restricted, unbounded curl operators (what improves the formal notation $\nabla \times \nabla \times$).

2.5. Helmholtz–Hodge decomposition of the field trajectories

**Theorem 2-4** A solution trajectory of Theorem 2-2 (with the magic formula for the total fields), is equivalent to the Helmholtz–Hodge decomposed system of trajectories

$$\begin{align*}
E^\parallel_t = E^\parallel_{t_0} - \int_{t_0}^{t} j^\parallel_s ds & \quad \in \text{ran}(\text{grad}_0), \\
E^\perp_t = E^\perp_{t_0} - \int_{t_0}^{t} j^\perp_s ds & \quad \in H_2, \\
B^\parallel_t = 0 \quad (\text{equivalently} \quad B_t \in \ker(\text{div}_0)), \\
B^\perp_t = B^\perp_{t_0} & \quad \in H_1 \quad \forall t, \text{ having no driving current},
\end{align*}$$

$$\begin{align*}
\begin{pmatrix} E^\top_t \\ B^\top_t \end{pmatrix} &= \exp\{(t-t_0)A^\top\} \begin{pmatrix} E^\top_{t_0} \\ B^\top_{t_0} \end{pmatrix} + \int_{t_0}^{t} \exp\{(t-s)A^\top\} \begin{pmatrix} -j^\top_s \\ 0 \end{pmatrix} ds \quad \in \mathcal{R}^\top_i.
\end{align*}$$

This is our transversally reduced magic formula.
3. Introduction of the potentials

As usual we look for potentials which satisfy

\[ E_t = -\nabla u_t - \frac{dA_t}{dt}, \quad E_t^\parallel = -\text{grad}_0 u_t - \dot{A}_t^\parallel, \quad (3.1) \]

\[ B_t = \nabla \times A_t, \text{ for all } t \in \mathbb{R}. \quad (3.2) \]

More precisely, we intend the following Helmholtz–Hodge decomposition of the vector potential, fitting to that for \( E \):

\[
\begin{align*}
L^2(\Lambda, \mathbb{R}^3) &= \text{ran(\text{grad}_0)} \oplus \text{ker(\text{div})} \\
&= \text{ker(curl}_0) \\
E &= E^\parallel \oplus E^{\text{co}} \oplus E^\top \\
A &= A^\parallel \oplus A^{\text{co}} \oplus A^\top 
\end{align*} \quad (3.3)
\]

Since Poincaré’s Lemma is globally valid in simply connected regions only, the existence of the potentials in more general cavities is not trivial. The more so, as we have to care about the boundary conditions and the Helmholtz–Hodge decomposition. Only part of the Maxwell equations are needed for the existence of the potentials.

**Assumption 3-1** We consider electromagnetic field trajectories \( [t \mapsto (E_t, B_t)] \) with values in the real Hilbert space \( \mathcal{R} = L^2(\Lambda, \mathbb{R}^3) \oplus L^2(\Lambda, \mathbb{R}^3) \), which split according to the Helmholtz–Hodge decomposition of Theorem 2-4, and which satisfy the following requirements:

(a) \( E_t^\parallel \in \text{ran(\text{grad}_0)} \) for all \( t \in \mathbb{R} \), and \( [t \mapsto \text{grad}_0^{-1} E_t^\parallel] \in C(\mathbb{R}, W^1_0(\Lambda, \mathbb{R})) \).

(b) \( [t \mapsto E_t^{\text{co}}] \in C(\mathbb{R}, H^2) \).

(c) \( B_t^\top \in \text{ran(curl}_0) \) for all \( t \in \mathbb{R} \), meaning the existence condition for the transversal vector potential \( A_t^\top := \text{curl}_0^{-1} B_t^\top \).

(d) Those Maxwell equations which do not involve sources

\[
B_t \in \text{ker(div}_0), \quad \frac{d B_t^{\text{co}}}{dt} = 0, \quad \frac{d B_t^\top}{dt} = -\text{curl}_0 E_t^\top; \quad (3.4)
\]

are fulfilled (in the weak formulation).

**Theorem 3-2** For an arbitrary (open connected) domain \( \Lambda \) let be given a force field trajectory \( [t \mapsto (E_t, B_t)] \) which satisfies Assumption 3-1.

Assume additionally an arbitrary scalar potential trajectory \( [t \mapsto u_t] \in C(\mathbb{R}, W^1_0(\Lambda, \mathbb{R})) \) (including the special case \( u_t \equiv 0 \)).

Then there exists a vector potential trajectory \( [t \mapsto A_t \in W^1_0(\text{curl}; \Lambda, \mathbb{R})] \) such that we have in terms of weak time derivatives

\[
\begin{align*}
\dot{A}_t^\parallel &= -(E_t^\parallel + \text{grad}_0 u_t), \quad \text{thus } A_t^\parallel = A_0^\parallel - \int_0^t (E_s^\parallel + \text{grad}_0 u_s)ds, \\
\dot{A}_t^{\text{co}} &= -E^{\text{co}}_t, \quad \text{thus } A_t^{\text{co}} = A_0^{\text{co}} - \int_0^t E^{\text{co}}_s ds, \\
\dot{A}_t^\top &= -E_t^\top = \frac{d}{dt} \text{curl}_0^{-1} B_t^\top, \quad \text{thus } A_t^\top = A_0^\top - \int_0^t E_s^\top ds. 
\end{align*} \quad (3.5)
\]
We generalize the standard Lagrangian point charges is found by reading system functions external data the system and are termed $A_M = (\phi, \ldots, \phi_M)$. For the total system “charged fluid plus field” we have the position and velocity variables: $\sum_{\mathbf{x}}$. The functions $m_{\mathbf{x}}$ are the material properties of the fluid elements are given by the matter density $\rho_{\mathbf{x}}$. The return to the discrete case is achieved by setting $d\mathbf{x}$. 

The time independent cohomological magnetic field $B^c_{\mathbf{x}}$ is not covered by the above Helmholtz–Hodge compatible vector potential $A_{\mathbf{x}}$. At least under the Standard Assumption there exists an additional cohomological vector potential $A_{\mathbf{co}}$, such that 

$$ B^c_{\mathbf{x}} = \text{curl} A_{\mathbf{co}} = \text{constant}. \quad (3.6) $$

Whereas $A^c_{\mathbf{x}}$ can be completely gauged away adding the gradient of a gauge function $[t \mapsto \lambda_t] \in C_{\text{fin}}(\mathbb{R}, W^1_0(\Lambda, \mathbb{R}))$, $A^c_{\mathbf{co}}$ is the gradient of a scalar function with different boundary conditions. $A_{\mathbf{co}}$ is the gradient only in the region $\Lambda$ minus the cuts which make $\Lambda$ simply connected. Thus it can maximally be gauged away only in the cutted region, leaving a singular expression for the gradient (not for $A_{\mathbf{co}}$) on the cuts. If $B$ vanishes, physical conditions may require nevertheless a non–vanishing $A_{\mathbf{co}}$ which is then for itself a cohomological vector field. Its couplings to the electromagnetic current (cf. the Lagrange and Hamilton functions) may be gauged away, but it influences the phases in the quantized theory in terms of non–vanishing line integrals around the columns of Figure 2.3 (generalized Aharanov–Bohm effects).

Note that the Dirac monopole requires a region $\Lambda$ with one (interior) point omitted, which does not satisfy the Standard Assumption.

4. Lagrange and Hamilton formalism

4.1. Lagrange theory of charged fluids

Consider $M$ charged point particles in $\Lambda$ with positions $q^k_{\mathbf{x}}$, $1 \leq k \leq M$. The transition to continuous matter, called charged fluid, is performed by $k \mapsto x$ at $t = 0$ such that $q_{\mathbf{x}}^k \mapsto q_{\mathbf{x}}(x)$ with $q_{\mathbf{x}}^0(x) = x, x \in \Lambda$. The position map (at time $t$) $q_t : \Lambda \rightarrow \Lambda$ is bijective and volume preserving, if the fluid is incompressible. In dependence on $t$ it is in general no group, but one always has $q_{-t} \circ q_t = q_0 = \text{id}$.

The material properties of the fluid elements are given by the matter density $m(\mathbf{x})$ and charge density $\varrho(\mathbf{x})$ at $t = 0$. As the time proceeds (forwards or backwards) these data of the fluid elements are at the positions $q_{\mathbf{x}}(t)$. Thus $q_t(x) = q(\mathbf{x}, t(x))$.

For a fluid element with index $\mathbf{x}$ at time $t$ (at position $q_{\mathbf{x}}(t)$) we have the velocity $\dot{q}_{\mathbf{x}}(t) = dq_{\mathbf{x}}(t)/dt$ and current $\dot{\mathbf{j}}_{\mathbf{x}}(t) = \varrho(\mathbf{x})\dot{q}_{\mathbf{x}}(t)$. (The current at $x$ is $\mathbf{j}(\mathbf{x})$.) The return to the discrete case is achieved by setting $m_t(\mathbf{x}) = \sum_{k=1}^M m_k \delta(q^k_{\mathbf{x}} - x)$, $\rho_t(\mathbf{x}) = \sum_{k=1}^M \epsilon_k \delta(q^k_{\mathbf{x}} - x)$. $\dot{\mathbf{j}}_{\mathbf{x}}(t) = \sum_{k=1}^M \epsilon_k \dot{q}^k_{\mathbf{x}} \delta(q^k_{\mathbf{x}} - x)$.

For the total system “charged fluid plus field” we have the position and velocity variables: $\phi = (\mathbf{q}, \mathbf{A}, u) \in \mathcal{Q}$, $\dot{\phi} = (\dot{\mathbf{q}}, \dot{\mathbf{A}}, \dot{u}) \in \mathcal{V}$, the values of which constitute the velocity phase space $\mathcal{M} = \mathcal{Q} \times \mathcal{V}$. $A = (A^\parallel, A^\text{co}, A^\top)$ is the Helmholtz–Hodge compatible vector potential and $u$ the scalar potential.

We assume here square integrability for all fields over $\Lambda$. The functions $m(\mathbf{x})$, $\varrho(\mathbf{x})$, and $A_{\mathbf{co}}(\mathbf{x})$, which are prepared at $t = 0$, characterize as a kind of external data the system and are termed system functions. The velocity phase space for $M$ point charges is found by reading $q_{\mathbf{x}} = (q^1_{\mathbf{x}}, \ldots, q^M_{\mathbf{x}})$.

We generalize the standard Lagrangian, as it is used for discrete charges especially in Quantum
We obtain for the canonical variables
\[ N \]
The momentum phase space is denoted \( \mathcal{P} \) and work in the temporal gauge condition

\[ A \]
Because of the longitudinal fields \( A^\parallel, Y^\parallel \). Thus we use the reduced phase space \( \mathcal{N}^r := \mathcal{Q}^r \times \mathcal{P}^r \) with the canonical variables

\[ \phi^r = (\mathbf{q}, \mathbf{A}^\perp, \mathbf{A}^\perp) \in \mathcal{Q}^r, \quad \pi^r = (\mathbf{p}, \mathbf{Y}^\perp, \mathbf{Y}^\perp, \mathbf{Y}^\perp) \in \mathcal{P}^r. \]
Because of the Coulomb gauge condition we have $A^0_t = A^0_{t_0}$. Since $Y^\parallel$ is proportional to $E^\parallel$ we may get hold on it by (2.6) and (3.1) and use the invertibility of the divergence operator $\text{div}$, when restricted to the longitudinal fields, to obtain

$$Y^\parallel := - \text{div} |^{-1} \rho =: \epsilon_0 \text{grad}_0 u_{\text{Coul}} * \rho,$$

(4.3)

where we have introduced the Green’s function of the Laplacian, namely the Coulomb potential $u_{\text{Coul}}$, arising from the Poisson equation

$$\rho = - \epsilon_0 \text{div} \text{grad}_0 u_{\text{Coul}} = - \epsilon_0 \Delta_\infty u_{\text{Coul}},$$

(4.4)

with the Dirichlet Laplacian $-\Delta_\infty = - \text{div} \text{grad}_0$ in $\Lambda$.

This gives $Y^\parallel_t$ as the convolution with the charge density $\rho_t$, without propagational delay.

We obtain the Coulomb Hamilton function $H_{\text{Coul}} = H_{\text{mat}} + H_{\text{rad}} + H_{\text{int}} : N^r \rightarrow \mathbb{R}$, including now the four system functions $m, \varrho, A_{\text{co}},$ and $u_{\text{Coul}}$:

$$H_{\text{mat}}[\phi, \pi 1] := (p) \frac{1}{2m} p + \frac{1}{2 \epsilon_0} (\varrho | u_{\text{Coul}} * \varrho)$$

(4.5)

$$H_{\text{rad}}[\phi, \pi] := \frac{1}{2\epsilon_0} \left[ \| Y^\text{co} \|^2 + \| Y^\top \|^2 \right] + \frac{1}{2 \mu_0} \left[ \| \text{curl}_0 A^\top \|^2 + \| \text{curl} A_{\text{co}} \|^2 \right],$$

(4.6)

$$H_{\text{int}}[\phi, \pi] := - \left( \frac{2}{m} p \left[ A^0 \circ \varrho + A_{\text{co}} \circ \varrho \right] + \frac{\varrho^2}{2m} \| A^0 \circ \varrho + A_{\text{co}} \circ \varrho \|^2 \right),$$

(4.7)

where we omit from now on the superscript $r$ in $(\phi, \pi) \in N^r \equiv N$.

The canonical equations from the Coulomb Hamiltonian yield indeed the Newton–Lorentz and Maxwell equations, expressed in terms of continuous force and current densities. They are highly non–linear in the position function $q$. They split into the Helmholtz–Hodge components. Especially, by calculating the functional derivatives of the Coulomb Hamiltonian to the canonical variables $A^\text{co}$ and to $Y^\text{co}_t$ one derives the dynamics for the cohomological canonical fields

$$\frac{d}{dt} \begin{pmatrix} A_t^\text{co} \\ Y_t^\text{co} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_t^\text{co} \\ Y_t^\text{co} \end{pmatrix} + \begin{pmatrix} 0^\text{co} \\ J_t^\text{co} \end{pmatrix},$$

(4.8)

(Observe the introduction of the cohomological phase space points $F_t^\text{co}$ and modified current $J_t^\text{co}$.)

By calculating the functional derivatives of the Coulomb Hamiltonian to $A_t^\top$ and to $Y_t^\top$ one derives the dynamics for the transversal canonical fields

$$\frac{d}{dt} \begin{pmatrix} A_t^\top \\ Y_t^\top \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ - \text{curl}_0^2 & 0 \end{pmatrix} \begin{pmatrix} A_t^\top \\ Y_t^\top \end{pmatrix} + \begin{pmatrix} 0^\top \\ J_t^\top \end{pmatrix},$$

(4.9)

(Observe the introduction of the transversal phases space points $F_t^\top$ and modified current $J_t^\top$.)

In the following we shall often write $F_t^\top \equiv F_t$ for shortness. The transformation $S'$ connects the transversal canonical fields with the transversal force fields

$$\begin{pmatrix} E_t^\top \\ B_t^\top \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \text{curl}_0 & 0 \end{pmatrix} \begin{pmatrix} A_t^\top \\ Y_t^\top \end{pmatrix},$$

$$\begin{pmatrix} E_t^\top \\ B_t^\top \end{pmatrix} = \psi^\top \quad S' = F,$$

(4.10)
where the inverse relation is given by
\[
\begin{pmatrix}
A^\top \\
Y^\top
\end{pmatrix} = \begin{pmatrix}
0 & \text{curl}\theta_0^{-1} \\
eg 1 & 0
\end{pmatrix} \begin{pmatrix}
E^\top \\
B^\top
\end{pmatrix} = F = (S')^{-1} = \psi^\top.
\tag{4.11}
\]

By this we gain directly the indicated relation (4.9) between the transversal force–field Maxwell operator $A'$ and the canonical Maxwell operator $B'$. The dash indicates duality with respect to the corresponding quantity in the test function space, which we introduce below.

This part of the canonical dynamical equations may suffice to illustrate, that we have for charged fluids, with square integrable sources, a well defined infinite dimensional Hamilton formalism. In this Hamilton formulation the original Maxwell theory is supplemented by the kinetic energy and the Coulomb interaction of the fluid elements.

In order to arrive at specific matter models one goes over to the singular point charge distribution. We have no finite field energy and the norms in (4.13) are not well defined!

From Eq. (4.9) we conclude the magic formula for canonical transversal fields:
\[
\begin{pmatrix}
A^\top \\
Y^\top
\end{pmatrix} = \exp\{(t - t_0)B'\} \begin{pmatrix}
A_0^\top \\
Y_0^\top
\end{pmatrix} + \int_{t_0}^t \exp\{(t - s)B'\} \begin{pmatrix}
0^\top \\
J_s^\top
\end{pmatrix} ds \in E^\top_{cl}.
\tag{4.15}
\]

We see immediately that the fields are not square integrable, if we insert the singular point current, since the free (canonical) Maxwell dynamics $\exp\{(t - s)B'\}$ does not spread the sharp current distribution. We have no finite field energy and the norms in (4.13) are not well defined!

On the other hand, fundamental models work with point charges. So we are forced to smear the current and the fields by test functions. For the canonical fields the test function space is denoted $E_{cl}$. It should be introduced as a locally convex topological vector space (LC–space), in a way that makes the operators of interest continuous.

5. Poisson formalism and phase space dynamics

We describe first the phase space in the Coulomb gauge, for which we have to introduce the test function space. We do this separately for each Helmholtz–Hodge component. We begin with the test function spaces $E^\kappa_{cl}$ for the $E$–fields and make the same choice for the test spaces $E^\kappa_{cl}$ and $E^\kappa_{cl}$ for the $A$– and $Y$–fields, where $\kappa$ assumes the values $\{\|,\text{co},\top\}$.

We introduce the total electric test function space, equipped with an LC–topology, according to the following scheme of a smooth Helmholtz–Hodge decomposition (without closures) into
gradients, cohomological fields, and curls

\[
E_c := \underbrace{\ker(\text{div})} \subseteq \ker(\text{div}) \oplus E^{\text{co}}_c \oplus E^\top_c := \text{grad}_0(\text{curl}_0) \subseteq \ker(\text{curl}_0) \oplus H^2 \oplus \text{curl}(E^\top_c)
\]

and observe that it is \(\|\cdot\|\)-dense in \(L^2(\Lambda, \mathbb{R}^3)\) and constitutes a Hilbert space core for the unrestricted operators \(\text{div}\) and \(\text{curl}_0\).

The test function space for the canonical fields is then \(E_{\text{cf}} = E_a \oplus E_y \ni f = (f_a, f_y)\). The dual spaces contain the respective \(L^2\)-spaces

\[
E_{\text{cf}} \subseteq \mathcal{R} \subseteq E_{\text{cf}}' = \left(E_a \subseteq L^2(\Lambda, \mathbb{R}^3) \subseteq E'_a\right) \oplus \left(E_y \subseteq L^2(\Lambda, \mathbb{R}^3) \subseteq E'_y\right).
\]

In the Coulomb gauge the longitudinal fields are not dynamical variables, rather may considered as indices for a foliation of the total phase space as Poisson space into nondegenerated leaves. Since we consider \(M\) point particles the material part of the phase space is finite dimensional, namely \(\mathbb{R}^{6M} \ni (\mathbf{q}, \mathbf{p})\). Thus each nondegenerated leave is isomorphic to the reduced phase space \(\mathcal{N} \ni (\phi, \pi) = (\mathbf{q}, \mathbf{A}^{\text{co}}, \mathbf{A}^\top, \mathbf{p}, \mathbf{Y}^{\text{co}}, \mathbf{Y}^\top)\).

The elements of the predual space

\[
E = (\mathbb{R}^{6M}, E_{\text{cf}}) = (\mathbb{R}^{6M}, E_{\text{co}}^{\text{cf}}, E^\top_{\text{cf}}), \quad \text{with} \quad E' = \mathcal{N},
\]

are written \((\phi, \pi) = (\mathbf{q}, f_a, \mathbf{p}, f_y) = (\mathbf{q}, f_a^{\text{co}}, f_y^{\text{co}}, f_y^{\top}, f_y^\top)\). The symplectic form on \(E\) is

\[
\sigma((\phi_1, \pi_1), (\phi_2, \pi_2)) :=
\]

\[
= \sum_{k=1}^{M} [\mathbf{q}_1^k \cdot \mathbf{p}_2^k - \mathbf{p}_1^k \cdot \mathbf{q}_2^k] + \left[(f_{a_1}^{\text{co}} | f_{a_2}^{\text{co}}) - (f_{a_1}^{\text{co}} | f_{a_2}^{\text{co}})\right] + \left[(f_{a_1}^{\top} | f_{a_2}^{\top}) - (f_{a_1}^{\top} | f_{a_2}^{\top})\right].
\]

\(\sigma\) is nondegenerated on \(E\), but would be degenerated on the predual of the non–reduced phase space. Note that \(E^\text{co}_{\text{cf}}\) and \(E^\top_{\text{cf}}\) are symplectically orthogonal.

Quite generally, let \((E, \sigma)\) denote an arbitrary pre–symplectic space, that is a real vector space equipped with a pre–symplectic form and with an \(\text{LC}–\text{topology}\) for which \(\sigma\) is separately continuous (and for which the relevant operators are continuous).

The topological dual \(E'\) of \(E\) is taken as the flat \textit{phase space manifold}, equipped with the weak*-\textit{topology}, or \(\sigma(E', E)–\text{topology},\) in which we have \(E'' = E\).

\section{5.1. Poisson bracket}

Because our phase space is flat, we have for the tangent and cotangent spaces

\[
T_F E' = E', \quad T^*_F E' = E'' = E
\]

at each \(F \in E'\). The total differential of \(A : E' \to \mathbb{R}\) is given by

\[
d_TAB[G] := \left. \frac{dA[F + tG]}{dt}\right|_{t=0} , \quad \forall G \in T_F E',
\]
if (required even for Banach Poisson manifolds)
\[ d_F A \in T^*_F E' = E, \quad \forall F \in E'. \]  
(5.6)
The complexified cotangent space is
\[ T^*_F E' \oplus iT^*_F E' = E \oplus iE. \]
The most relevant observables are the smeared fields
\[ \Phi^0(f) : E' \to \mathbb{R}, \quad \Phi^0(f)[F] := F(f), \quad \forall F \in E'. \]  
(5.7)
(The superscript 0 indicates \( \hbar = 0 \).) Since the smeared fields are unbounded functions on the phase space \( E' \) one uses often the classical Weyl elements
\[ W^0(f) := \exp\{i\Phi^0(f)\}, \quad W^0(f)[F] = \exp\{iF(f)\}, \quad \forall F \in E'. \]  
(5.8)
Their total differentials are calculated to
\[ d_F \Phi^0(f) = f \in T^*_F E' = E, \]
\[ d_F W^0(f) = i \exp\{iF(f)\} f = iW^0(f) f \in E \oplus iE. \]  
(5.9)
We define the Poisson bracket
\[ \{A, B\}[F] := \Sigma_F (d_F A, d_F B) \]
\[ = -\sigma(d_F A_1, d_F B_1) - i\sigma(d_F A_2, d_F B_2) - i\sigma(d_F A_2, d_F B_1) + \sigma(d_F A_2, d_F B_2), \]  
(5.10)
(5.11)
where \( A = A_1 + iA_2 \) and \( B = B_1 + iB_2 \) are smooth \( \mathbb{C} \)-valued functions, still called classical observables. One finds the fundamental Poisson brackets
\[ \{\Phi^0(f), \Phi^0(g)\} = \sigma(f, g)1, \]  
(5.12)
\[ \{W^0(f), W^0(g)\} = \sigma(f, g)W^0(f + g). \]  
(5.13)
(Equation (5.12) generalizes the usual Poisson bracket for canonical point fields involving \( \delta \)-functions.)
As a Poisson algebra \( (\mathcal{P}, \{., .\}) \) we take the following small commutative \( * \)-algebra (and not \( C^\infty(E') \), common for finite dimensional phase space):
\[ \Delta(E, 0) := LH\{W^0(f) \mid f \in E\}, \]  
(5.14)
in which one has besides the Poisson product the commutative point–wise multiplication and the complex conjugation. The supremum norm is a C*-norm in \( \Delta(E, 0) \), but we cannot form the norm–closure, since then the Poisson product would fail.

5.2. Hamiltonian phase space flow
For each \( f \in E \) the \( \mathbb{R} \)-linear mapping
\[ \sigma_f : E \to \mathbb{R}, \quad \text{where} \quad (\sigma_f)(g) := \sigma(f, g) \quad \forall g \in E, \]  
(5.15)
is continuous, implying \( \sigma_f(f) \in E' \). Thus we have an \( \mathbb{R} \)-linear mapping \( \sigma_1 : E \to E', f \mapsto \sigma_f, \) which is LC–weak* continuous.
If the form $\sigma$ is non–degenerate then $\sigma_2 : E \rightarrow E'$ is injective and the image $\sigma_2(E)$ is $\sigma(E',E')$–dense in $E'$ (a proper subspace for infinite dimensions).

The Hamiltonian vector field $F \mapsto X_H[F] \in T_F E' = E'$, associated with $H : E' \rightarrow \mathbb{R}$ is defined by

$$X_H[F] := -\sigma_2(d_F H), \quad \forall F \in E'. \tag{5.16}$$

(Recall that the vectors are in $E'$ and the allowed co–vectors in $E$.) Equivalently we may express this relation in terms of a Poisson bracket with the smeared field

$$X_H[F](f) = -\sigma(d_F H, f) = \{H, \Phi^0(f)\}[F], \quad f \in E. \tag{5.17}$$

The Hamiltonian flow satisfies the differential equation

$$\frac{d}{dt}(\varphi^H_t(F))(f) = -\sigma(d_{\varphi^H_t(F)} H, f), \quad \forall f \in E, \quad \forall F \in E'. \tag{5.18}$$

**Proposition 5-1** Provided a smooth Hamiltonian function $H$, then

$$\frac{d}{dt} A \circ \varphi^H_t = \{H, A\} \circ \varphi^H_t \tag{5.19}$$

for every smooth function $A : E' \rightarrow \mathbb{C}$. Furthermore, we have the energy conservation

$$H[\varphi^H_t F] = H[F], \quad \forall F \in E', \quad \forall t \in \mathbb{R}. \tag{5.20}$$

**Definition 5-2** Let $\sigma$ be jointly continuous symplectic form. Denote by $\text{symp}(E, \sigma)$ the set of symplectic transformations $T : E \xrightarrow{\text{bijective}} E$ satisfying

$$\sigma(Tf, Tg) = \sigma(f, g), \quad \forall f, g \in E. \tag{5.21}$$

Clearly $\text{symp}(E, \sigma)$ is a group.

A strongly LC–continuous symplectic one–parameter (semi–) group $\{T_t \mid t \in \mathbb{R}_+\} \subset \text{symp}(E, \sigma)$, for which by definition $\mathbb{R} \ni t \mapsto T_t = \exp(tG) \circ \varphi^H_t$ LC–continuous for each $f \in E$, is termed strongly LC–differentiable if it has a generator $G$ which is LC–continuous and defined on all of $E$. We denote by $\mathcal{G}_{\text{sym}}(E, \sigma)$ the set of all these strong symplectic generators $G$, endowed with the $\mathbb{R}$–linear operations and the commutator product, and term it the **symplectic Lie algebra**.

It seems natural to equip $\mathcal{G}_{\text{sym}}(E, \sigma)$ with the weak operator topology, where $G_t \rightarrow G$, if and only if $F(G_t f) \rightarrow F(G f), \forall F \in E', \forall f \in E$. The Lie product in $\mathcal{G}_{\text{sym}}(E, \sigma)$ is then separately continuous but in general not bi–continuous. $\mathcal{G}_{\text{sym}}(E, \sigma)$ has an isomorphic realization by the dual transformations acting in the phase space $E'$, that is by the set $\{G' \mid G \in \mathcal{G}_{\text{sym}}(E, \sigma)\}$.

**Theorem 5-3** For each $G \in \mathcal{G}_{\text{sym}}(E, \sigma)$ we introduce the quadratic Hamilton function $H_G$ by

$$H_G[\sigma_2 f] := \frac{1}{2} \sigma(f, G f), \quad \forall f \in E, \tag{5.22}$$

defined on $\sigma_2 E \subset E'$. The map $J : \mathcal{G}_{\text{sym}}(E, \sigma) \rightarrow C^\infty(\sigma_2 E), \quad J_G = H_G$, is called the co–momentum map.

We have for each $G \in \mathcal{G}_{\text{sym}}(E, \sigma)$ that the Hamiltonian vector field associated with $H_G$ is effectuated by the linear phase space map

$$X_{H_G} = G'. \tag{5.23}$$
Consequently, the global phase space flow $\varphi^H_G$ associated with $H_G$ exists, and is given by the weak*–continuous one–parameter group
\[ \varphi^H_G = \exp\{tG\}, \quad t \in \mathbb{R}, \text{ defined everywhere on } E'. \tag{5.24} \]
This is the dual group to the original $\{\exp\{tG\} \mid t \in \mathbb{R}\} \subset \text{symp}(E, \sigma)$ (the strongly LC–differentiable symplectic group).

Moreover, the dynamics for the classical field observable $\Phi_t^H$ (defined in the Eqs. (5.7) and (5.8)) may be written
\[ \mathbb{R} \ni t \mapsto \alpha^H_G(\Phi^H_t(f)) = \Phi^H_t(f) \circ \varphi^H_G = \Phi^H_0(\exp\{tG\}f), \tag{5.25} \]
\[ \mathbb{R} \ni t \mapsto \alpha^H_G(W^0(f)) = W^0(f) \circ \varphi^H_G = W^0(\exp\{tG\}f) \tag{5.26} \]
for each test function $f \in E$.

In terms of the co–momentum map we obtain the Lie homomorphism
\[ \{J_{G_1}, J_{G_2}\} = J_{[G_1, G_2]}, \quad G_1, G_2 \in \mathcal{G}_{\text{symp}}(E, \sigma). \tag{5.27} \]

Only as a side remark let us introduce also in our infinite dimensional case the *momentum map associated with the action of the Lie algebra $\mathcal{G}_{\text{symp}}(E, \sigma)$ on $E'$ by means of
\[ J : \sigma_\ast(E) \mapsto \mathcal{G}_{\text{symp}}(E, \sigma), \quad J[F](G) := J_G[F], \quad F \in \sigma_\ast(E), \quad G \in \mathcal{G}_{\text{symp}}(E, \sigma), \tag{5.28} \]
where $\mathcal{G}_{\text{symp}}(E, \sigma)^\ast$ denotes the continuous linear forms on $\mathcal{G}_{\text{symp}}(E, \sigma)$.

In other words, the momentum map is the evaluation of the Hamilton function $H_G$ at each $\sigma_\ast f$, viewed as a dual element to the Lie algebra in virtue of its linear dependence on $G \in \mathcal{G}_{\text{symp}}(E, \sigma)$.

The interpretation of this evaluation as an element in $\mathcal{G}_{\text{symp}}(E, \sigma)^\ast$ gives the discussion of group orbits a more abstract setting in form of the co–adjoint orbits.

A Lie–Poisson structure on $\mathcal{G}_{\text{symp}}(E, \sigma)^\ast$ is introduced by the Poisson bracket for the special functions $\tilde{G}(\eta) := \eta(G), \forall \eta \in \mathcal{G}_{\text{symp}}(E, \sigma)^\ast, G \in \mathcal{G}_{\text{symp}}(E, \sigma)$, as follows
\[ \{\tilde{G}_1, \tilde{G}_2\} := \tilde{\{G}_1, G_2\}. \tag{5.29} \]
Thus, the momentum map transforms (part of) the phase space $E'$ into the more abstract Poisson manifold $\mathcal{G}_{\text{symp}}(E, \sigma)^\ast$, preserving the Poisson structure. That means that the pullback $J^* \tilde{G}$ leads to the phase space function $J_G = H_G$ for all $G \in \mathcal{G}_{\text{symp}}(E, \sigma)$ and satisfies
\[ J^*\{\tilde{G}_1, \tilde{G}_2\} = \{J^*\tilde{G}_1, J^*\tilde{G}_2\}, \quad G_1, G_2 \in \mathcal{G}_{\text{symp}}(E, \sigma). \tag{5.30} \]

6. Maxwell Generators and their Hamilton Functions

We return to the phase space in the Coulomb gauge. The dynamical operators in the test function space are the Hermitian adjoints of the original operators. In order to possess a dual operator they should be defined on the whole test function space and be continuous in the LC–topology of the test function space.

By means of the Helmholtz–Hodge decomposition we have a separated cohomological free Maxwell dynamics which we deduce from Eq. (4.8) to be given by $\{\exp\{tI\} \mid t \in \mathbb{R}\} \subset \text{symp}(E^0_{cl}, \sigma)$, with the strong symplectic generator (inserting for the moment the physical constants)
\[ I = \begin{pmatrix} 0 & 0 \\ \frac{1}{\epsilon_0} & 0 \end{pmatrix} \in \mathcal{G}_{\text{symp}}(E^0_{cl}, \sigma). \tag{6.1} \]
It leads with Eq. (5.22) to the Hamilton function
\[ H_B[A^\top, Y^\top] = \frac{1}{2\epsilon_0} \|Y^\top\|^2 + \frac{1}{2\mu_0} \|\text{curl}_0 A^\top\|^2 \]
where we have denoted the test functions \( f \) embedded via \( \sigma f \) into \( E^{\text{cf}} \) by the original field symbols.

The transversal dynamical pre–dual operator is
\[ \mathbb{B} = \begin{pmatrix} 0 & -\frac{1}{\mu_0} \text{curl}^2 \nabla_0 \cr \frac{1}{\epsilon_0} & 0 \end{pmatrix} \]
and this gives rise with Eq. (5.22) to the strictly positive quadratic Hamiltonian function corresponding to the transversal field energy in the domain \( \Lambda \),
\[ H_B[A^\top, Y^\top] = \frac{1}{2\epsilon_0} \|Y^\top\|^2 + \frac{1}{2\mu_0} \|\text{curl}_0 A^\top\|^2 = \frac{\epsilon_0}{2} \|E^\top\|^2 + \frac{1}{2\mu_0} \|B^\top\|^2 = \frac{\epsilon_0}{2} \int_{\Lambda} (E^\top(x))^2 + c^2 B^\top(x)^2 \, d^3x. \] (6.3)

If \( \Lambda \) is invariant under rotations about a fixed axis, then these rotations constitute a symplectic transformation in the field phase space and their generator leads to a quadratic Hamiltonian, the angular momentum of the field about the axis. Similarly the invariance of the (unbounded) \( \Lambda \) under translations in a fixed direction provides the field momentum as quadratic Hamiltonian. These considerations disclose the symplectic invariant structure of Maxwell theory.

We make sure, that \( \mathbb{B}' \) indeed gives rise to a mathematically reasonable weak*–continuous one–parameter group \( t \mapsto \exp\{t\mathbb{B}\} \), if we prove that the predual transformation group
\[ t \mapsto \exp\{t\mathbb{B}\} = S \exp\{-tA^\top\} S^{-1}, \quad \text{acting on} \quad E^{\text{cf}}_t, \] (6.4)
is strongly continuous in the following sense: Each \( \exp\{t\mathbb{B}\} \) is an LC–homeomorphism on \( E^{\text{cf}}_t \).

By transposing the transversal magic formula Eq. (4.15) into the predual and applying it under use of Eq. (5.25) to the classical Weyl elements we gain the following form
\[ \alpha_{t_0 \to t}^{\text{curr}}(W^0(f)) = \exp\{i\int_{t_0}^{t} J_s^\top \exp\{\{t-s\mathbb{B}\} f\} ds\} W^0(\exp\{\{t-t_0\mathbb{B}\} f\}). \] (6.6)

Note that the influence of the current is expressed by means of a phase factor, mathematically a character on the additive group \( E^{\text{cf}} \). Since the time dependence of the current is rather arbitrary the dynamical operators in the classical Heisenberg picture do not form a group, but only a groupoid, in which the partial composition law
\[ \alpha_{t_0 \to t}^{\text{curr}} \circ \alpha_{t \to t'}^{\text{curr}} = \alpha_{t_0 \to t'}^{\text{curr}}, \quad \forall \, t_0, t, t' \in \mathbb{R}, \] (6.7)
requires the equality of intermediate time.
6.1. Diagonalization of the Transversal Free Maxwell Dynamics

For the following it is essential that \( \text{curl}^2_{t_0} = \text{curl}(\text{curl})_{t_0} \) is strictly positive. By spectral calculus the \( \sqrt{\text{curl}^2_{t_0}} \pm 1/2 \) are well defined.

**Definition 6-1** Let \((E,\sigma)\) be a symplectic space. A real–linear operator \(j : E \to E\) is called a complex structure in \((E,\sigma)\), if it satisfies

\[
\begin{align*}
j^2 f &= -f, \\
\sigma(jf, jg) &= \sigma(f, g), \\
\sigma(f, jf) &\geq 0,
\end{align*}
\]

for all \(f, g \in E\).

It follows that \(j \in \text{symp}(E, \sigma)\) with inverse \(j^{-1} = -j\). If \(j\) exists, it allows for the multiplication by complex numbers \(z = \text{Re}(z) + i\text{Im}(z)\), where \(if := jf\), so that

\[
zf := \text{Re}(z)f + \text{Im}(z)jf, \quad \forall z \in \mathbb{C}, \quad f \in E.
\]

In this manner \(E\) becomes a complex vector space. A complex inner product is introduced by

\[
(f|g)_j := \sigma(f, jg) + i\sigma(f, g), \quad \forall f, g \in E.
\]

The associated norm is denoted \(\|\cdot\|_j\).

If \(j\) is any complex structure in \((E,\sigma)\), then

\[
j_T := T^{-1}jT, \quad T \in \text{symp}(E, \sigma),
\]

is also a complex structure in \((E,\sigma)\), for which we have

\[
\|f\|_{j_T} = \|Tf\|_j, \quad \text{Re}(f|g)_{j_T} = \text{Re}(Tf|Tg)_j, \quad \forall f, g \in E.
\]

Our transversal test function space \((E^\top_{\text{cf}}, \sigma)\) of smooth canonical fields \((f_a, f_y)\) has as symplectic form \(\sigma\) the transversal part of (5.3):

\[
\sigma((f_a, f_y), (f'_a, f'_y)) = (f_a f'_y) - (f'_a f_y),
\]

which involves the \(L^2(\Lambda, \mathbb{R}^3)\)-scalar products. The most simple complex structure in \((E^\top_{\text{cf}}, \sigma)\), satisfying the requirements (6.8), is given by

\[
j(f_a, f_y) := (-f_y, f_a), \text{ expressed in matrix notation by } j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

In terms of the function realization of \((E^\top_{\text{cf}}, \sigma)\) we get the isomorphic complexification if we write \((f_a, f_y)\) as \(f_a + if_y\) and define the complex unit as the multiplication of the function values by \(i \in \mathbb{C}\). The symplectic form is then realized by \(\text{Im}(\cdot, \cdot)\), making use of the scalar product in \(L^2(\Lambda, \mathbb{C}^3)\). This complexified symplectic space is denoted by \((E^\top, \text{Im}(\cdot, \cdot))\), where \(E^\top\) is a dense subspace of \(\mathcal{H}^\top\), the transversal part of \(L^2(\Lambda, \mathbb{C}^3)\). So the most natural isomorphism \(v\) between the symplectic spaces is

\[
(E^\top_{\text{cf}}, \sigma) \ni (f_a, f_y) \to v(f_a, f_y) := f_a + if_y \in (E^\top, \text{Im}(\cdot, \cdot)),
\]

which gives

\[
j = v^{-1}i v, \quad \text{with } i \text{ the complex unit in } \mathcal{H}^\top.
\]
Remark 6-2 The qualification “natural” of the foregoing complexification of \((E_{cl}^\top, \sigma)\) is substantiated by the fact that the physically induced splitting \(E_{cl}^\top = E_a \oplus E_y\) (in a fixed inertial system) is expressible as \(E_a\) being the Lagrangian subspace of \((E_{cl}^\top, \sigma)\) corresponding to \(j\). That means that \(E_a\) is the \(\sigma\)-orthogonal complement of \(jE_a = E_y\). Changing the \(j\) would lead to other Lagrangian subspaces.

Let us now introduce the diagonalization transformation \(V\) as the \(\mathbb{R}\)-linear, injective mapping from the real space \(E_{cl}^\top \ni (f_a, f_y)\) into the complex Hilbert space \(\mathcal{H}^\top \ni f_a + if_y\), given by

\[
V(f_a, f_y) := w^{-1}f_a + iw f_y = V f \quad \text{with} \quad w := (\epsilon_0 c \sqrt{\text{curl}_{0}|0|})^{1/2},
\]

so that

\[
V = v \circ W \quad \text{with} \quad W = \begin{pmatrix} w^{-1} & 0 \\ 0 & w \end{pmatrix}.
\]

Theorem 6-3 Let again \(E^\top \subset \mathcal{H}^\top\) be the image of \(E_{cl}^\top\) under the above diagonalization transformation \(V\). Then it holds:

(a) The diagonalization \(V\) from Eq. (6.16) is a symplectic LC–homeomorphism from \((E_{cl}^\top, \sigma)\) onto \((E^\top, \text{Im}(\langle \cdot , \cdot \rangle))\), satisfying

\[
\text{Im}(Vf|Vg) = \sigma(f, g), \quad \forall f, g \in E_{cl}^\top.
\]

(b) \((\mathcal{H}^\top, E^\top, cv\sqrt{\text{curl}_{0}|0|}, V)\) is the unique (up to unitary equivalence) diagonalization of the transversal canonical Maxwell dynamics, given by the one–parameter unitary group \(\exp\{t\mathbb{B}\}, t \in \mathbb{R}\), on \((E_{cl}^\top, \sigma)\):

\[
V \exp\{t\mathbb{B}\}V^{-1} = \exp\{it c \sqrt{\text{curl}_{0}|0|}\}E^\top, \quad \forall t \in \mathbb{R}.
\]

That means that any other unitary group in another Hilbert space realizing \(\exp\{t\mathbb{B}\}\) is reached by a unitary transformation from the given one.

(c) \(V \mathbb{B} V^{-1} = ic \sqrt{\text{curl}_{0}|0|} =: i\mathbb{D}\) is an LC–homeomorphism in \(E^\top \subset \mathcal{H}^\top\).

(By (b) \(\mathbb{D}\) is a symmetric operator which has a unique extension to a selfadjoint operator in \(\mathcal{H}^\top\), and is thus a complex linear operator.)

(d) There is no diagonalization for the longitudinal and cohomological fields.

The form of \(\mathbb{D}\) is obtained by direct calculation. The domain of definition of \(\mathbb{D}\) is a subspace of the domain of \(\text{curl}_{0}|E^\top|\), that are “smooth” transversal complex fields with vanishing tangential component at the boundary \(\partial \Lambda\).

If for the nilpotent generator \(\mathbb{I}\) of the longitudinal and cohomological dynamics there would be any diagonalization map onto an \(\mathbb{J}\), with \(\mathbb{J}\) a bounded selfadjoint complex–linear operator in a complex Hilbert space, then \(\mathbb{J}\) would be also nilpotent, and thus equal to 0. (See however our Observation 8-7.)

It is remarkable that the quadratic Hamilton function Eq. (6.3) (the transversal Maxwell energy, originally of the type (5.22)) may by (6.18) be written in terms of a scalar product in the complex diagonalization Hilbert space, as reveals the following calculation

\[
H_{\mathbb{B}}[\sigma; f] = \frac{1}{2} \sigma(f, \mathbb{B}f) = \frac{1}{2} \text{Im}(Vf|\mathbb{D}Vf) = \frac{1}{2} \langle Vf|\mathbb{D}Vf \rangle, \quad \forall f \in E_{cl}^\top.
\]

In the last step the essential selfadjointness of \(\mathbb{D}\) has been observed, which makes the last scalar product real.
If we choose a real orthonormal basis \( \{ e_n \mid n \in \mathbb{N} \} \) in the complex \( E^+ \subset \mathcal{H}^+ \), then the real linear combination of \( \{ e_n, i e_n \mid n \in \mathbb{N} \} \) are dense in \( \mathcal{H}^+ \). Observe that \( \Phi^0(V^{-1}e)[\sigma_f] = (\sigma_f)(V^{-1}e) = \sigma(f, V^{-1}e) = \text{Im}(Vf|e) \), for all \( f \in E^+_{cl} \) and all \( e \in E^+ \). We then get the diagonal representation of the Hamiltonian in terms of quadratic field expressions

\[
H_{\mathbb{B}}[\sigma_f] = \frac{1}{2} \sum_{n \in \mathbb{N}} (Vf|e_n) (e_n|\mathbb{D}Vf) \\
= \frac{1}{2} \sum_{n \in \mathbb{N}} (\text{Im}(Vf|e_n) \text{Im}(Vf|\mathbb{D}e_n) + \text{Im}(Vf|ie_n) \text{Im}(Vf|i\mathbb{D}e_n)) \\
= \frac{1}{2} \sum_{n \in \mathbb{N}} (\Phi^0(V^{-1}e_n)\Phi^0(V^{-1}\mathbb{D}e_n) + \Phi^0(V^{-1}ie_n)\Phi^0(V^{-1}i\mathbb{D}e_n))[\sigma_f].
\]

(6.21)

Provided the orthonormal basis \( \{ e_n \mid n \in \mathbb{N} \} \) consists of eigenvectors of \( \mathbb{D} \) with eigenvalues \( \varepsilon_n \in \mathbb{R} \) (supposing a pure point spectrum for \( \mathbb{D} \), valid in bounded cavities), then \( H_{\mathbb{B}} \) admits the simpler diagonal representation on \( \sigma_f^*(E^+_{cl}) \) which has a similarity to a sum of harmonic oscillators \( \Phi^0(V^{-1}e_n) \) corresponds to the \( n \)-th position and \( \Phi^0(V^{-1}ie_n) \) to the \( n \)-th momentum coordinate:

\[
H_{\mathbb{B}} = \frac{1}{2} \sum_{n \in \mathbb{N}} \varepsilon_n(\Phi^0(V^{-1}e_n)^2 + \Phi^0(V^{-1}ie_n)^2).
\]
(6.22)

The \( V^{-1}e_n \) and \( V^{-1}ie_n \) are the back transforms of the complex eigen modes to the eigen value \( \varepsilon_n \) into the real field formalism (where we assume again a finite cavity). Since \( \text{curl}^2\omega = \text{curl}\text{curl}\omega \) is strictly positive we know \( \varepsilon_n > 0 \). One also knows that the \( \varepsilon_n \) converge to \( +\infty \) (and are thus at most finitely degenerate). We see that \( H_{\mathbb{B}}[\sigma_f(V^{-1}e_n)] + H_{\mathbb{B}}[\sigma_f(V^{-1}ie_n)] = \varepsilon_n \), so that the evaluation of the Hamilton function at these two special phase space points sums up to an eigen energy of \( \mathbb{D} \). \( \mathbb{D} \) describes the filtering of \( Vf \) into eigen modes, and \( \mathbb{B} \) does the same for \( f \in E^+_{cl} \). But neither \( \mathbb{D} \) nor \( \mathbb{B} \) are observables in the proper sense for the present classical theory, since one must strictly adhere to defining a classical observable as a phase space function. Note for further illustration that neither the frequency nor the polarization of a plane wave in a cubic box, which we write out below, are classical observables.

7. QED as Weyl quantized electrodynamics

7.1. The C*-Weyl algebra and field operators

Our developments employ the very flexible and universal construction of the Weyl algebra \( W(E, \hbar \sigma) \) for an arbitrary pre-symplectic test function space \( (E, \sigma) \) [7]. The C*-Weyl algebra \( W(E, \hbar \sigma) \) is generated by the Weyl elements \( W^\hbar(f), f \in E \), satisfying the Weyl relations

\[
W^\hbar(f)W^\hbar(g) = \exp\{-\frac{i}{\hbar}\sigma(f,g)\}W^\hbar(f + g), \quad \forall f, g \in E.
\]
(7.1)

We use the same test function space for all values of the Planck parameter \( \hbar \), where the latter varies in \( \mathbb{R}_+ \) (including \( \hbar = 0 \)).

The set of all finite linear combinations

\[
\Delta(E, \hbar \sigma) := \text{LH}\{W^\hbar(f) \mid f \in E\}
\]
(7.2)

is a *-algebra.
There exists a unique \( C^* \)-norm \( \| \| \) \([8], [9]\) in \( \Delta(E, \hbar \sigma) \), such that every representation and every state on \( \Delta(E, \hbar \sigma) \) is \( \| \| \)-continuous:

\[
\| A \| = \sup \{ \| \Pi(A) \| \mid \Pi \text{ representation of } \Delta(E, \hbar \sigma) \}, \quad A \in \Delta(E, \hbar \sigma),
\]

the so-called minimal regular norm.

**Definition 7-1** For all \( \hbar \in \mathbb{R}_+ \) the \( \| \| \)-completion of \( \Delta(E, \hbar \sigma) \) by means of the minimal regular norm

\[
W(E, \hbar \sigma) := \overline{\Delta(E, \hbar \sigma)}^{||}
\]

is called the Weyl algebra over the pre–symplectic space \( (E, \hbar \sigma) \) \([7]\).

**Definition 7-2** Suppose \( (\Pi, \mathcal{H}_\Pi) \) to be a regular representation of the \( C^* \)-Weyl algebra \( W(E, \hbar \sigma) \), in which by definition \( t \rightarrow W_\Pi(tf) \) is a strongly continuous unitary group. Then we associate with it the quantum field system \( (\Phi_\Pi, \mathcal{H}_\Pi) \)

\[
\Phi_\Pi(f) = -i \left. \frac{dW_\Pi(tf)}{dt} \right|_{t=0}, \quad f \in E.
\]

according to Stone’s Theorem.

The field operators satisfy the (smeared) canonical commutation relations

\[
[\Phi_\Pi(f), \Phi_\Pi(g)] \subseteq i\hbar \sigma(f, g) \mathbb{1}_\Pi.
\]

The most suggestive choice for a quantization map \( Q_\hbar \) is now given by

\[
Q_\hbar^\text{Weyl}(\mathbb{W}(f)) := \mathbb{W}(h(f)), \quad \forall f \in E,
\]

and its \( \mathbb{C} \)-linear resp. continuous extension to \( \Delta(E, 0 \cdot \sigma) \subset \mathbb{W}(E, 0) \).

But we may take also other prescriptions \( Q_\hbar = Q_\hbar^w \), namely

\[
Q_\hbar^w(\mathbb{W}(f)) := w(h, f)\mathbb{W}(f), \quad \forall f \in E,
\]

for certain quantization factors \( w(h, f) \in \mathbb{C} \). The choice of \( w \) determines the operator ordering of the field operators. For example \( w(h, f) = \exp\left( -\frac{\hbar(1+2)}{4} \| f \|_2^2 \right) \) leads to normal resp. anti–normal ordering of the creation and annihilation operators defined via the complexification \( j \) (see (8.1) below). Beside other things one has to postulate \( w(h, f) \rightarrow 1 \) for \( h \rightarrow 0 \) \([10]\).

**Theorem 7-3** The modified Weyl quantization maps of Eq. (7.7), decorated with an appropriate quantization factor \( w(h, f) \), lead to strict deformation quantizations of the Poisson algebra \( \Delta(E, 0) \), as characterized by the subsequent Definition below \([11]\).

These strict deformation quantizations all are equivalent to each other in the sense of \([12]\) (by approaching smoothly the same classical limit in the norm topology).

**Definition 7-4** Let \( I \subseteq \mathbb{R} \) be the range of values for \( h \), including 0 as accumulation point. A strict quantization \( (\mathbb{A}^h, Q_\hbar)_{h \in I} \) of a Poisson algebra \( (\mathcal{P}, \{.,.\}) \) (which owes also a commutative product \( . \)) consists for each value \( h \in I \) of a \( C^* \)-algebra \( \mathbb{A}^h \) with norm \( \| . \|_h \) and of a linear, \( * \)-preserving map

\[
Q_\hbar : \mathcal{P} \rightarrow \mathbb{A}^h,
\]

such that \( Q_0 \) is the identical embedding of \( \mathcal{P} \) into the commutative \( \mathbb{A}^0 \), and such that the following conditions are satisfied:
(a) [Dirac’s Condition] The $\hbar$-scaled commutator approaches the Poisson bracket in the norm topology as $I_0 \ni \hbar \to 0$, that means,
\[
\lim_{\hbar \to 0} \|[Q_\hbar(A), Q_\hbar(B)]/\hbar - Q_\hbar(\{A, B\})\|_\hbar = 0, \quad \forall A, B \in \mathcal{P}.
\]

(b) [von Neumann’s Condition] In the limit $\hbar \to 0$ one has the asymptotic product homomorphism
\[
\lim_{\hbar \to 0} \|[Q_\hbar(A)Q_\hbar(B) - Q_\hbar(AB)]\|_\hbar = 0, \quad \forall A, B \in \mathcal{P}.
\]

(c) [Rieffel’s Condition] $I \ni \hbar \mapsto \|Q_\hbar(A)\|_\hbar$ is continuous for each $A \in \mathcal{P}$.

The strict quantization $(A^\hbar, Q_\hbar)_{\hbar \in I}$ is called a strict deformation quantization, if

(d) [Deformation Condition] The map $Q_\hbar: \mathcal{P} \to A^\hbar$ is injective. And moreover, its image $Q_\hbar(\mathcal{P})$ is closed with respect to the product of $A^\hbar$ (i.e. $Q_\hbar(A)Q_\hbar(B) \in Q_\hbar(\mathcal{P})$ for all $A, B \in \mathcal{P}$), or equivalently, $Q_\hbar(\mathcal{P})$ is a sub-$\ast$–algebra of $A^\hbar$.

One gets the connection with the usual deformation quantization [13], [14] by equipping the $\ast$–algebra of classical observables $\mathcal{P}$ with the deformed, non–commutative product $\cdot_\hbar$ according to
\[
A \cdot_\hbar B := Q_\hbar^{-1}(Q_\hbar(A)Q_\hbar(B)), \quad \forall A, B \in \mathcal{P}.
\]

Theorem 7-3 tells us that the decorated Weyl quantizations satisfy all of the foregoing conditions and may especially be formulated in terms of a deformed product for phase space functions. From Eq. (7.1) we obtain the commutator
\[
[W^\hbar(f), W^\hbar(g)] = [\exp\{-i\hbar \sigma(f, g)/2\} - \exp\{i\hbar \sigma(f, g)/2\}]W^\hbar(f + g), \quad \forall f, g \in E,
\]
which approaches 0 (commutativity!) for small $\hbar$, whereas the scaled commutator is seen to approach the classical Poisson bracket Eq. (5.13). For the unbounded field operators we obtain the corresponding classical asymptotic behavior only in terms of certain expectation values. For a foundational discussion it is noteworthy that a quantized (field) theory, obtained by strict deformation quantization, may be formulated in terms of functions over the classical phase space.

We understand henceforth under Weyl quantization the quantization map Eq. (7.6), in which the trivial quantization factor leads to symmetric products for the field operators.

### 7.2. Symplectic–affine equivariance of Weyl quantization

The linear test function space determines not only the symplectic group $\text{symp}(E, \sigma)$ but also the additive group of (affine) translations. The Bohr compactification [15] of the latter, and also that of $E'_\tau$, is isomorphic to the group $\hat{E}$ of all characters on $E$, independently of the LC topology $\tau$.

The semi–direct product of $\text{symp}(E, \sigma)$ with $\hat{E}$ forms the symplectic–affine group.

Since the test function space is the same for all $\hbar$ one observes by inspection that we have the following commutative diagram, where $\beta^\hbar_T$ denotes the $\ast$–automorphism in the Weyl algebra $\mathcal{W}(E, \hbar \sigma)$, which is induced by the symplectic transformation $T \in \text{symp}(E, \sigma)$ (called Bogoliubov automorphism in [8]; note that the symplectic form in the Weyl relations is invariant under $T$):

| Classical Theory | Quantum Theory |
|------------------|---------------|
| $\mathcal{W}(E, 0)$ | $\mathcal{W}(E, \hbar \sigma)$ |
| $\downarrow \beta^\hbar_T$ | $\downarrow \beta^\hbar_T$ |
| $\mathcal{W}(E^\top, 0)$ | $\mathcal{W}(E^\top, \hbar \sigma)$ | (7.9)
As for the affine translations, they are given by the multiplication of the Weyl elements by the associated character and commute thus also with the linear quantization map.

The application of the Weyl quantization to ED proceeds by adapting the test function space. In Coulomb gauge the degenerate pre–symplectic form causes the longitudinal fields to stay classical. On the other side, not only the transversal fields but also the cohomological fields are transformed into canonical quantum fields.

Especially for the Weyl quantization of the transversal canonical fields we have the following commutative diagram, where $\beta^0_c$ denotes now the *–isomorphism between $W(E^{|\mathcal{c}|}, h \sigma)$ and $W(E^{|\mathcal{c}|}, h \Im(\{\cdot\})$, which is induced by the special symplectic diagonalization transformation $V$ (6.16) (leading from the real $(E^{|\mathcal{c}|}, h \sigma)$ to the complex $(E^{|\mathcal{c}|}, h \Im(\{\cdot\}))$:

\[
\begin{array}{ccc}
\text{Classical ED} & \xrightarrow{\text{quantization}} & \text{QED} \\
W(E^{|\mathcal{c}|}, 0) & \downarrow \beta^0_c & W(E^{|\mathcal{c}|}, h \sigma) \\
W(E^{|\mathcal{c}|}, 0) & \xrightarrow{\text{quantization}} & W(E^{|\mathcal{c}|}, h \Im(\{\cdot\})).
\end{array}
\]

We learn from the commutativity of the diagram that the complexification of the test function space, and thus of the dual phase space, is not a necessary pre–requisite for quantization. But it has, in fact, a special meaning in the Fock representation, as we discuss later on. As is common usage in physics [16], we employ the quantized ED only after diagonalization (and thus after complexification).

If $R \in E^{|\mathcal{c}|}$, we write for the “diagonaled” linear form $\hat{R}(V f) := R(f)$ resp. $\hat{R}(h) = R(V^{-1} h)$ for $f = (f_a, f_y) \in E^{|\mathcal{c}|}$ and $h = h_1 + i h_2 \in E^{|\mathcal{c}|}$. The diagonalized smeared current is therefore

\[
\hat{J}^{|\mathcal{c}|}_t (h) = (0, j^{|\mathcal{c}|}_t) (V^{-1} h) = j^{|\mathcal{c}|}_t (\epsilon_0 c \sqrt{\operatorname{curl} j^{|\mathcal{c}|}_0})^{-1/2} h_2.
\]

We may apply the symplectic–affine equivariance of quantization to the magic formula Eq. (6.6) under the condition that the current remains classical and obtain the quantized version.

**Theorem 7-5** The direct quantization of the transversal Maxwell dynamics (in Weyl form for the canonical fields) leads, under the condition that the current stays classical, to the algebraic, quantized magic formula in diagonal form

\[
\alpha^{|\mathcal{c}|}_{t_0 \rightarrow t} (\hat{W}^{|\mathcal{c}|}(h)) = \exp\{i \int_{t_0}^t j^{|\mathcal{c}|}_s \{\exp((t - s)i \square) h \} ds\} \hat{W}^{|\mathcal{c}|}(\exp\{(t - t_0)i \square\} h),
\]

where $\hat{J}^{|\mathcal{c}|}_t$ is the diagonalized current and $h \in E^{|\mathcal{c}|} \subset \mathcal{H}^{|\mathcal{c}|}$.

The expression (7.12) gives, in dependence on $t$, a trajectory in the abstract Weyl algebra $W(E^{|\mathcal{c}|}, h \Im(\{\cdot\}))$, provided $\hat{J}^{|\mathcal{c}|}_s \in E^{|\mathcal{c}|}$, $\forall s \in \mathbb{R}$.

By linear and norm continuous extension the $\alpha^{|\mathcal{c}|}_{t_0 \rightarrow t}$ constitute a groupoid of *–automorphisms in $W(E^{|\mathcal{c}|}, h \Im(\{\cdot\}))$.

Even for the free case $\hat{J}^{|\mathcal{c}|}_t \equiv 0$ the $\alpha^{|\mathcal{c}|}_{t_0 \rightarrow t}$ do not depend (point–wise norm) continuously on the time parameters (since the abstract Weyl operators do not depend continuously on the test functions).

In the following we work with the quantities only after having applied the diagonalization map and drop the hat over the symbols!

For a (mesoscopic) system of lasing two–level atoms with level splitting $\Delta \eta$ the current is classical, namely the polarization current of the phase–locked, oscillating atoms, and has the form $\hat{J}^{|\mathcal{c}|}_t (h) = \exp(it \Delta \eta / \hbar) \hat{J}^{|\mathcal{c}|}_0 (h)$. (Note that $\Delta \eta / \hbar = \omega_{\text{atom}}$ is independent of $\hbar$.) In this case $t \rightarrow \alpha^{|\mathcal{c}|}_{0 \rightarrow t}$ is a group of *–automorphisms in $W(E^{|\mathcal{c}|}, h \Im(\{\cdot\}))$. 

Quite generally, the current is determined in mesoscopic quantum mechanical radiation models by the coupling between matter and quantized field and becomes classical in the weak coupling limit ([17]). In this manner one gets the diagonalized current and one needs Eq. (7.11) to find the current for Maxwell theory. In this context the Heisenberg dynamics is always a group. For explicit discussions one looks for a regular representation \((\Pi, \mathcal{H}_\Pi)\) of \(\mathcal{W}(E^T_+), h \text{Im}(\.|.\))\), which contains – beside \(W^h(h) := \Pi(W^h(h))\) – also \(\exp\{i\int_{t_0}^t J^\sigma_i(h) ds\}\) \(W^h(h)\) for all \(h \in E^T\) and for all \(t, t_0 \in \mathbb{R}\), and in which the Heisenberg dynamics may be implemented by a strongly continuous group of unitaries. In the representation is then a selfadjoint operator \(H^\Pi\) such that

\[
\Pi \left( \alpha^h_{t_0 \rightarrow t} (W^h(h)) \right) = \exp\{i(t - t_0)H^\Pi/h\} W^h(f) \exp\{-i(t - t_0)H^\Pi/h\}. \tag{7.13}
\]

\(H^\Pi\) is not uniquely determined by this requirement and should be renormalized with respect to a reference state (ground or equilibrium state). By differentiation one formulates the dynamics for the quantized, representation dependent fields \(\Phi^h_i(h)\).

Even for the free case \(J_i^\sigma \equiv 0\) it is important to perform the discussion in the outlined manner, in order to obtain the correct form for \(H^\Pi\) and \(\Phi^h_i(h)\).

Note that up to now, we did not introduce a particle structure for the quantized fields.

8. Particle structure

Let us go back to the Weyl algebra over the real transversal test function space \(\mathcal{W}(E^T_+, h\sigma)\), where \(h\) may range in \(\mathbb{R}_+\). The first step for a particle structure is to define the annihilation and creation operators

\[
a^h_\Pi(g) := \frac{1}{\sqrt{2}}(\Phi^h_\Pi(g) + i\Phi^h_\Pi(jg)), \quad a^{h*}_\Pi(g) := \frac{1}{\sqrt{2}}(\Phi^h_\Pi(g) - i\Phi^h_\Pi(jg)), \tag{8.1}
\]

for any \(g \in E^T_+\) and for any regular representation \((\Pi, \mathcal{H}_\Pi)\) of \(\mathcal{W}(E^T_+, h\sigma)\). These are closed operators which are Hermitian adjoint to each other. The decisive point is that we need for their definition a complexification of the test function space with imaginary unit \(j\) in the sense of Def. 6-1. (whereas \(i\) is the imaginary unit of the representation Hilbert space \(\mathcal{H}_\Pi\). We read from (8.1) \(a^{h}_\Pi(jg) = -ia^{h*}_\Pi(g)\) (anti-linearity in the test function) and \(a^{h*}_\Pi(jg) = ia^{h}_\Pi(g)\) (linearity in the test function). These relations elucidate, by the way, the usual splitting of the Hermitian quantum field into “positive and negative frequency parts”.

From the field commutation relations we obtain, observing (6.10),

\[
[a^{h}_\Pi(g), a^{h*}_\Pi(g')] \subset \frac{1}{2}[ih\sigma(g, g') + i\sigma(jg, jg') - h\sigma(jg, g') + h\sigma(g, jg')]|\mathcal{H}_\Pi = h(g|g'), j|\mathcal{H}_\Pi, \quad g, g' \in E^T. \tag{8.2}
\]

But there are infinitely many complexifications (cf. (6.11)) , fitting to the given symplectic form. So it is important that the uniqueness of the diagonalization of the free Maxwell dynamics provides us with a distinguished \(j = i\), already used in \(E^T \subset \mathcal{H}^T\) after the diagonalization. (The two just mentioned \(i\)'s are discriminated by the vectors on which they act from the left.)

Given a complexification \(j\) of the test function space there is a distinguished representation \((\Pi_F, \mathcal{H}_F)\) of the Weyl algebra, which is called the Fock representation. For notational economy we describe it only for our special complexification \(j = i\) and use our Weyl algebra in the form \(\mathcal{W}(E^T, h\text{Im}(\.|.\))\).

The Fock representation \((\Pi_F, \mathcal{H}_F)\) is uniquely characterized by the existence of a cyclic vector \(\Omega_{vac}\) with \(a^i_F(h)|\Omega_{vac} := 0, \forall h \in E^T\) and by the family of products of creation operators \(\{a^i_F(h_1)\ldots a^i_F(h_n)|\Omega_{vac}, n \in \mathbb{N}_0\}\) applied to the vacuum, spanning the total space \(\mathcal{H}_F\).
The second quantized generator is introduced by second quantization the unitaries \( \Gamma(\exp \{ \it A \}) \), which implement the gauge transformations of the first kind in Fock space. They commute with all other \( \Gamma(\exp \{ \it A \}) \).

\[
[a_F(h), a_F^*(h')] \subset (h|h')1_F, \quad h, h' \in E^\top.
\]

(8.3)

Let us denote by \( \mathcal{H}_F^{(n)} \), \( n \in \mathbb{N}_0 \), the sub–Hilbert space, generated by just \( n \) creation operators. \( \mathcal{H}_F^{(0)} \) is then the one–dimensional \( \mathbb{C}\Omega_{\text{vac}} \). \( \mathcal{H}_F^{(1)} \) is spanned by the vectors \( a_F^*(h)\Omega_{\text{vac}}, h \in E^\top \), which satisfy by (8.3)

\[
(a_F^*(h)\Omega_{\text{vac}}|a_F^*(h')\Omega_{\text{vac}}) = (h|h'), \quad h, h' \in E^\top.
\]

(8.4)

Thus we obtain the same scalar product as in \( \mathcal{H}^\top \) and may identify the \( a_F^*(h)\Omega_{\text{vac}} \in \mathcal{H}_F^{(1)} \) with \( h \in \mathcal{H}^\top \) (and may identify the two \( \iota \)'s). This is a hint for the fact that we can smear the creation and annihilation operators actually by test functions from all of \( \mathcal{H}^\top \) and that the Weyl operators \( W^\beta(h) \) in the Fock representation depend continuously (in the strong operator topology) on the test functions (in the norm topology).

**Corollary 8-1** The quantized Maxwell dynamics (7.12) stays (for finite times) in Fock space, if and only if the diagonalized classical current \( \mathbf{J}_t^\top \) is square integrable.

If \( U \) is a unitary in \( \mathcal{H}^\top \) we define its second quantization \( \Gamma(U) \) by application on the total set of vectors as follows

\[
\Gamma(U)a_F^*(h_1) \ldots a_F^*(h_n)\Omega_{\text{vac}} := a_F^*(Uh_1) \ldots a_F^*(Uh_n)\Omega_{\text{vac}}, \quad h_i \in E^\top, n \in \mathbb{N}_0.
\]

(8.5)

For \( n = 0 \) Eq. (8.5) expresses the invariance of \( \Omega_{\text{vac}} \) under \( \Gamma(U) \).

**Observation 8-2** A unitary \( U \) in \( \mathcal{H}^\top \), which leaves \( E^\top \) invariant, is a special symplectic transformation in \( (E^\top, h \text{Im}(.|.)) \) and leads to the Bogoliubov isomorphism \( \beta^h_U \) in \( \mathcal{W}(E^\top, h \text{Im}(.|.) ) \). \( \Gamma(U) \) is the unique implementation of \( \beta_U \) in \( \mathcal{H}_F \) which leaves the vacuum vector invariant.

\( \Gamma(U) \) leaves each subspace \( \mathcal{H}_F^{(n)} \) invariant.

For a strongly continuous unitary group \( U_t = \exp \{ \it tA \} \) we define the second quantization of its selfadjoint generator by

\[
d\Gamma(A) := \frac{d \Gamma(\exp \{ \it tA \})}{dt}\bigg|_{t=0}, \quad \text{on a suitable domain in } \mathcal{H}_F.
\]

(8.6)

Observe that \( d\Gamma(A)\Omega_{\text{vac}} = 0 \), for all selfadjoint \( A \) in \( \mathcal{H}^\top \). Especially the diagonalized free Maxwell dynamics has the form \( \exp \{ \it tD \} \), so that the corresponding Bogoliubov automorphisms have in Fock space the implementing unitaries \( \Gamma(\exp \{ \it tD \}) \) with generator \( d\Gamma(D) \). Since in quantum theory we write the dynamics as \( \exp \{ \it ih\mathcal{D}/h \} \), the physical relevant generator is \( d\Gamma(h\mathcal{D}) =: H^F \). This is already the systematic introduction of the free electrodynamic Hamiltonian in Fock space (independently of any basis system). Since \( \mathcal{D} \) is strictly positive (and may have 0 as spectral value, but not as eigen value) the spectrum \( \sigma(H^F) \) is positive and \( \Omega_{\text{vac}} \) is the only eigen vector of \( H^F \) to the eigenvalue 0.

Starting from the unit operator \( 1 \) in \( \mathcal{H}^\top \), which generates the phase rotations \( \exp \{ \it t1 \} \), we introduce by second quantization the unitaries \( \Gamma(\exp \{ \it t1 \}) \), which implement the gauge transformations of the first kind in Fock space. They commute with all other \( \Gamma(\exp \{ \it A \}) \).

The second quantized generator is

\[
d\Gamma(1) =: N_F, \quad \text{the particle number operator in } \mathcal{H}_F.
\]

(8.7)
The selfadjoint $N_F$ has the eigen spaces $\mathcal{H}_F^{(n)}$ with the eigen values $n \in \mathbb{N}_0$. Thus the spectrum $\sigma(N_F)$ equals $\mathbb{N}_0$ and $\Omega_{\text{vac}}$ is also a ground state for the gauge automorphisms.

There are representations of $\mathcal{W}(E^\top, \hbar \text{Im}(.,.))$ in which the generators of the free Maxwell dynamics and of the first kind gauge transformations have different spectra from those of the Fock generators, or do not commute with each other, or do not exist at all.

**Theorem 8-3** The Fock representation with the special complexification $j = i$, defined by the diagonalization of the free Maxwell dynamics, is the only representation with a particle structure. By this we mean that it is the only representation in which the Bogoliubov automorphisms of the free Maxwell dynamics and of the first kind gauge transformations are unitarily implemented so that the following conditions are satisfied:

The unitaries for the time and gauge transformations commute with each other, and the corresponding selfadjoint generators have a positive spectrum, where the particle number spectrum equals $\mathbb{N}_0$.

They have a cyclic vector, namely $\Omega_{\text{vac}}$, as common lowest eigenvector with eigenvalue 0.

The requirements on a representation with particle structure, formulated in the foregoing Theorem, demand the existence of a vacuum, void of particles and energy, a so-called bare vacuum.

The fact that the cohomological quantized fields satisfy the canonical commutation relations, but do not possess a bare vacuum, illustrates the difference between quantum behavior and particle structure for fields. Thus it is a special feature of the Maxwell dynamics, that the transversal quantized fields exhibit a particle structure in a Fock representation, which is associated with a uniquely determined complexification of the theory.

### 8.1. Photon observables

In the distinguished Fock representation with particle structure, the corresponding quanta, the photons, display all features of quantum particles.

**Definition 8-4** The vectors $\psi^{(n)} \in \mathcal{H}_F^{(n)}$ are called $n$–photon wave functions, $n \in \mathbb{N}$, and the selfadjoint operators $A^{(n)}$ in $\mathcal{H}_F^{(n)}$ are called $n$–photon observables.

Especially $d\Gamma(h\mathcal{D})|_{\mathcal{H}_F^{(1)}} \equiv h\mathcal{D} = \hbar c \text{curl} |_{\partial_0}$ is the one–photon energy operator, and correspondingly $d\Gamma(h\mathcal{D})|_{\mathcal{H}_F^{(n)}}$ (without interaction terms!) the $n$–photon energy operator.

Since $\mathcal{H}_F^{(1)}$ is unitarily isomorphic to $\mathcal{H}^\top \supset E^\top = V E_{\text{cf}}^\top$ the “smooth” one–photon wave functions have the form $\psi^{(1)} \equiv \psi = w^{-1} f_a + iw f_y = V f$ with $w := (\epsilon_0 c \sqrt{\text{curl} |_{\partial_0}})^{1/2}$.

**Definition 8-5** If $i$ is the complex unit in $\mathcal{H}^\top$, then let $\iota$ be the “un–diagonalized” complex unit $V^{-1}iV$ (which equals according to (6.17) $W^{-1} j W$, with $j = v^{-1} iv$ the natural complex unit in the real $(E_{\text{cf}}^\top, \sigma)$).

Then we denote by $\mathcal{G}\text{sym}_\iota(E^\top, \sigma)$ the sub–Lie algebra of $\mathcal{G}\text{sym}(E^\top, \sigma)$ of all elements which are linear for $\iota$ (i.e. commute with $\iota$).

We denote by $\mathcal{G}\text{sym}_\iota(E^\top, \sigma)$ the subset of $\mathcal{G}\text{sym}(E^\top, \sigma)$ of all elements $G$ which are anti–linear for $\iota$ (i.e. which satisfy $G\iota = -\iota G$).

The operators in $\mathcal{G}\text{sym}_\iota(E^\top, \hbar \text{Im}(.,.))$ are linear resp. antilinear with respect to $i$.

**Corollary 8-6** Let $A^{(1)}$ be a selfadjoint one–photon observable, which leaves the dense domain $E^\top \subset \mathcal{H}_F^{(1)}$ invariant. Then the “un–diagonalized” symplectic generator $G := V^{-1}iA^{(1)}V$ is an element in the sub–Lie algebra $\mathcal{G}\text{sym}_\iota(E_{\text{cf}}^\top, \sigma)$ of complex linear elements (with respect to $\iota$).
Reversely each \( G \in \mathcal{G}_{\text{sym}}(E_{\text{cf}}^T, \sigma) \) leads via the diagonalization map \( V \) to a anti–symmetric operator \( i\hat{G} \) with domain \( E^T \subset \mathcal{H}^T \), so that \( \hat{G} \) determines a selfadjoint one–photon observable \( \hat{A}^{(1)} \).

Like in (6.20) we calculate the Hamilton function for \( G \in \mathcal{G}_{\text{sym}}(E_{\text{cf}}^T, \sigma) \)

\[
H_G[\sigma_j f] = \frac{1}{2} \sigma(f, Gf) = \frac{1}{2} \text{Im}(Vf|\hat{G}Vf) = \frac{1}{2} \|Vf\|^2(\psi_f|\hat{G}\psi_f), \quad \forall f \in E_{\text{cf}}^T,
\]

where in the last equality we used the reality of the scalar product with the essentially selfadjoint \( \hat{G} \) and defined \( \psi_f := Vf/\|Vf\| \). In the quantum interpretation the \( \psi_f \) are (smooth, complex, normalized) one–photon wave functions, that are probability amplitudes.

The preceding formula expresses the values of certain classical quadratic Hamilton functions in terms of the statistical expectation values of the corresponding one–photon observables. This will only say, that also in the presence of many photons, described by an arbitrary vector in Fock space, their energy may be expressed by a one–photon wave function with scaled amplitude.

Nevertheless we take this as an indication, that for detecting the quantum features of photons the measurement of \( n \)–photon observables, with \( n \geq 2 \), resp. of correlation functions, are required. Recall in this context that in EPR–experiments always correlations between one–photon observables of different particles, like spin (classically: angular momentum of a polarized beam) are measured.

Interesting is the counting of photons, which measures the eigenvalues of the particle number operator \( N_F \), and not of the corresponding trivial one–photon observable \( \hat{1} \). Taken as \( \hat{G} \), the latter leads to the undiagonalized Lie algebra element \( G = V^{-1}iV = i \), with classical Hamilton function \( H_i[\sigma_j f] = \frac{1}{2} \sigma(f, if) = \frac{1}{2}[(w^{-1}f_{a}|w^{-1}f_{a}) + (w_{f_{y}}|w_{f_{y}})] \). Thus we have also classically a Hamilton function for the Poisson generator of the first kind gauge transformations.

For illustration let us discuss the one–photon energy \( \hbar \mathbb{D} \) in the cubic box \( \Lambda_{a} \) of side \( a \), depicted in our figure (2.3). \( \Lambda_{a} \) is assumed to sit in the first quadrant, with lower left vertex at 0. If there are no conductors within \( \Lambda_{a} \) a complete system of transversal eigen functions \( e_{n} \), with vanishing tangential components at the boundary, is given by the formula

\[
e_{n,i}(x) = \varepsilon^{\alpha}(k) \sin(k_{i} \cdot x) \cos(k_{i} \cdot x) \sin(k_{3} \cdot x), \quad i = 1, 2, 3, \quad k = k(n), \quad \alpha = 1, 2,
\]

which indicates that the \( i \)–th factor is the cos–function. The vector \( k(n) \) has the components \( k_{j}(n) = \pi n_{j}/a \), where \( n_{j} = 0, 1, 2, \ldots, j = 1, 2, 3 \). (Actually \( n_{j} = 0 \) is only allowed for \( j = i \).)

The two polarization vectors \( \varepsilon^{\alpha}(k) \), which contain also the normalization constants, are orthogonal to \( k \), so that \( \nabla \cdot e_{n}(x) = -k \cdot \varepsilon^{\alpha}(k) \sin(k_{1} \cdot x) \sin(k_{2} \cdot x) \sin(k_{3} \cdot x) = 0 \). Therefore, on the linear hull of this basis, the operator \( \text{curl}_{0}^{2} \) coincides with \( -\Delta \), acting component–wise, and we recover Einstein’s photon energy

\[
\hbar \mathbb{D} e_{n}(x) = \hbar c \sqrt{-\Delta} e_{n}(x) = \hbar c |k(n)| e_{n}(x).
\]

Since the spacing between the eigen values tends to 0 for increasing box volumina, the spectrum is called in physics a quasi–continuum, and summation over the spectrum is often replaced by integration (as in the Planck formula).

Any kind of conducting obstacles within \( \Lambda_{a} \) makes the cavity a more complicated manifold and deforms the spectrum of \( \hbar \mathbb{D} \). It is a challenging question, which topological features of the cavity are encoded in the deformed spectrum of the one–photon Hamiltonian, the latter displaying some similarity to an elliptic Dirac operator (?).
8.2. General symplectic generators in Fock space

We have seen that each \( G \in \mathcal{G}_{\text{sym}}(E^*_\chi, \sigma) \), which generates the strongly differentiable one–parameter symplectic group \( \{ \exp \{ tG \} \mid t \in \mathbb{R} \} \subset \text{sym}(E^*_\chi, \sigma) \), is by the transformation \( V \) mapped onto \( i\hat{G} \in \mathcal{G}_{\text{sym}}(E^T, h \text{Im}(\cdot)) \). In the latter form it generates the complex symplectic transformations \( \{ \exp \{ it\hat{G} \} \mid t \in \mathbb{R} \} \subset \text{sym}(E^T, h \text{Im}(\cdot)) \).

So we have to analyze the general \( \mathbb{R} \)–linear part \( T_l \) and the antilinear part \( T_a \) according to

\[
T = T_l + T_a , \quad \text{where} \quad T_l := \frac{1}{2}(T - iTi) , \quad T_a := \frac{1}{2}(T + iTi) .
\]

(8.11)

By the embedding \( E^T \subset \mathcal{H}^T \) one knows [18] that there are a positive selfadjoint (linear) operator \( S \), a (linear) unitary \( U \), and an antilinear involution \( J \) (that is, \( J^* = J^{-1} \)), so that \( T_l = U \cosh(S)|_E \) and \( T_a = UJ \sinh(S)|_E \). We see that \( T \) is linear, iff \( S = 0 \) and iff it is unitary. Thus \( \exp \{ it\hat{G} \} \) is linear iff \( \hat{G} \) extends to a selfadjoint \( A \) in \( \mathcal{H}^T \).

The association \( \mathcal{G}_{\text{sym}}(E^T, h \text{Im}(\cdot)) \ni \hat{G} \rightarrow d\Gamma(A) \) is a Lie homomorphism and may be considered a quantum mechanical co–momentum map. In the end, each \( G \in \mathcal{G}_{\text{sym}}(E^T, \sigma) \) yields a second quantized \( d\Gamma(A) \) in Fock space, for which one knows the series representation

\[
d\Gamma(A) = \sum_{k=1}^{\infty} a^h_{F}^*(A u_k) a^h_F(u_k) ,
\]

(8.12)

where \( \{ u_k \mid k \in \mathbb{N} \} \) is a complete orthonormal system in the domain of \( A \), and the series converges in the strong resolvent sense. Note that \( \Omega_{\text{vac}} \) is annihilated in virtue of the normally ordered form of \( d\Gamma(A) \).

The connection with the classical diagonal representation (6.21) is achieved by noting that

\[
\Phi^h_F(h)^2 + \Phi^h_F(ih)^2 - ||h||^2 \mathbb{1}_F = 2a^h_F^*(h) a^h_F(h) .
\]

This allows for (8.12) the form

\[
d\Gamma(A) = \sum_{k=1}^{\infty} \lambda_k a^h_{F}^*(u_k) a^h_F(u_k) = \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k [\Phi^h_F(u_k)^2 + \Phi^h_F(iu_k)^2 - \mathbb{1}_F] ,
\]

(8.13)

if \( A u_k = \lambda_k u_k \) for all \( k \in \mathbb{N} \). A direct formal quantization of (6.22), the popular “harmonic oscillator quantization”, would not produce the infinite renormalization constant \( \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k \) and would not at all lead to a meaningful operator in Fock space.

More complicated are the anti–linear symplectic generators \( G \in \mathcal{G}_{\text{sym}}^a(E^T, \sigma) \). We sketch only an example. Assume

\[
T_l = \exp \{ -itD \} = \exp \{ -itJ |D| \} = \cosh(t |D|) - iJ \sinh(t |D|) , \quad t \in \mathbb{R} ,
\]

(8.14)

with the selfadjoint, \( \mathbb{C} \)–antilinear operator \( D \), the \( \mathbb{C} \)–linear positive selfadjoint operator \( |D| = \sqrt{D^*D} \) in \( \mathcal{H}^T \) and with an \( \mathbb{C} \)–antilinear involution \( J \) in \( \mathcal{H}^T \) (i.e., \( J = J^* = J^{-1} \)), which respectively are given in terms of an orthonormal system by

\[
D = \sum_{k=1}^{N} \zeta_k (|e_k| e_k) , \quad |D| = \sum_{k=1}^{N} |\zeta_k| (|e_k|)(e_k) ,
\]

(8.15)

\[
J e_k = \frac{\zeta_k}{|\zeta_k|} e_k , \quad \forall k \in \{1, \ldots, N\} ,
\]
(where we define $\frac{\partial}{\partial \zeta_k} := 1$ for $\zeta_k = 0$). $D = J[D]$ is a kind of polar decomposition of the antilinear selfadjoint $D$.

Since the $T_i$ are symplectic they lead to Bogoliubov transformations in $\mathcal{W}(E^\top, \mathcal{h} \text{Im}(\cdot))$. For finite $N$ the latter are implemented by unitaries of the form $\exp(itH_D)$ with

$$H_D = \frac{1}{2} \sum_{k=1}^{N} (\zeta_k a^*(e_k)^2 + \overline{\zeta_k} a(e_k)^2),$$

(8.16)

involving complex coefficients with

$$\zeta_{2m-1} = -\zeta_{2m}, \quad m \in \{1, \ldots, M\}, \quad N := 2M.$$

(8.17)

The Fock–implemented $T_i$ transform the field operator as

$$\exp\{itH_D\} \Phi_F(f) \exp\{-itH_D\} = \Phi(T_i f), \quad \forall f \in E^\top,$$

(8.18)

that is like the complex linear symplectic generators. But, since the $T_i$ have now also an antilinear part (the odd powers in the exponential series), we derive for the transformed annihilation operators

$$a_{T_i}(f) := \exp\{itH_D\} a_F(f) \exp\{-itH_D\} = a_F(\cosh(t|D|)f) - ia_F^*(J \sinh(t|D|)f),$$

(8.19)

for all $f \in E^\top$. The condition for a finite Hilbert–Schmidt norm

$$\|H_D \Omega_{\text{vac}}\|^2 = \frac{1}{2} \sum_{k=1}^{\infty} |\zeta_k|^2 = \frac{1}{2} \|D\|_{\text{HS}}^2 = \frac{1}{2} \|D\|_{\text{HS}}^2.$$

(8.20)

is necessary and sufficient that the $H_D$ generate the symplectic transformations in Fock space also for infinite $N$. Note that $D$ is then an antilinear Hilbert–Schmidt operator, a special bounded operator in $\mathcal{H}^\top$.

The physical interest for symplectic transformations of this kind, i.e. with generators $G \in G_{\text{sym}}(E^\top, \sigma)$, stems from the fact that the $\exp\{itG\}$ leave only the imaginary part of the scalar product invariant, but may decrease (or increase) the real part (cf. (6.12)). Since the real part of the scalar product shows up in the variance of the field operators in certain states (squeezed states) one may have reduced quantum noise in optical communication.

If condition (8.20) is not satisfied, then there are however other representations (GNS–representations over the squeezed vacuum), in which the infinite–mode squeezing Hamiltonians exist. Thus the quantum co–momentum map requires many Fock representations for representing general strong symplectic generators.

The physical realization of a squeezing Hamiltonian is based on a form, which arises by changing the basis system as follows

$$e_{2m-1} := \frac{1}{\sqrt{2}} (u_{s,m} + u_{i,m}), \quad e_{2m} := \frac{1}{\sqrt{2}} (u_{s,m} - u_{i,m}), \quad m \in \{1, \ldots, M\}.$$

The so–called signal and idler modes $\{u_{s,1}, \ldots, u_{s,M}, u_{i,1}, \ldots, u_{i,M}\}$ generate a new orthonormal system in the test function space $E^\top$. From the $\mathbb{C}$–linearity of $f \mapsto a^*(f)$ and $\mathbb{C}$–antilinearity of $f \mapsto a(f)$ we obtain the relations

$$a^#(u_{s,m}) a^#(u_{i,m}) = \frac{1}{2} \left( a^#(e_{2m-1})^2 - a^#(e_{2m})^2 \right),$$

which are valid for both the smeared creation
operators \( (a^\#(h) \equiv a^*(h)) \) and annihilation operators \( (a^\#(h) \equiv a(h)) \). This leads to the following “non–degenerate” form of the squeezing Hamiltonian (e.g. [19], [20], [21],)

\[
H_{\text{Dnd}} = \sum_{m=1}^{M} \left( \eta_m a^*(u_{s,m}) a^*(u_{i,m}) + \eta_m a(u_{s,m}) a(u_{i,m}) \right).
\] (8.21)

For each term \( m \) one has so–called four–wave mixing: A non–linear crystal is pumped by the idler mode \( u_{i,m} \) of a laser. The incoming laser mode interacts with the incoming signal mode in the first term, whereas the second term describes this matter mediated interaction for the two outgoing modes. The microscopically detailed description would require the raising and lowering operators of the material. The complex \( \eta_m \in \mathbb{C} \) result from averaging over the non–Hermitian matter operators and contain also the interaction strength, which is, for itself, dependent from the laser intensity. Therefore, the \( \eta_m \) are called pumping parameters. Only by this averaging procedure it appears as if one could have a direct interaction between different photon rays.

Altogether we see that the anti–linear symplectic generators \( G \in \mathcal{G}_\text{sym}^s(E^\top, \sigma) \) lead to quantum co–momenta, which do not commute with the particle number and they describe creation and annihilation processes.

**Observation 8-7** A simple but physically relevant generator with anti–linear part is given by \( \mathbb{I} \) of Eq. (6.1). We may perform the natural complexification \( v \) of the cohomological test function space \( E_{\mathbb{C}}^\sigma \) from (5.2) analogously as in \( E_{\mathbb{C}}^1 \) and obtain \( v^\top v^{-1} = i \mathbb{I} = i(J_1 + J_2) = \frac{1}{2}(1 + \mathcal{C}) \), where \( \mathbb{I} \) and \( \mathcal{C} \) are the unit and complex conjugation in the finite dimensional \( E_{\mathbb{C}}^\sigma \oplus iE_{\mathbb{C}}^\sigma \), the complexification of \( \mathbb{H}_2 \). Related to this complexification we may introduce creation and annihilation operators for the quantized cohomological canonical fields, as prescribed in (8.1), and then a Fock space and a particle number operator. Since \( J_1 \) and \( J_2 \) commute with each other, a Hamiltonian can easily be implemented in the Fock space. It is of the form \( d\Gamma(J_1) + H_D, D = J_1 \) (cf. (8.16)). Because of \( H_D \) the Hamiltonian is not lower bounded and and does not commute with the particle number operator. This illustrates the fact that the cohomological fields have no particle structure in the sense of Theorem 8-3.

The quantized cohomological canonical fields, which commute with all transversal fields, have nevertheless an interesting dynamics, which seems to describe collective excitations.

We end our discussion with a theorem on strongly continuous symplectic groups, adapted to the complexified and completed version \( (\mathcal{H}^\top, \text{Im}(\cdot)) \) of our test function space \( (E^1_{\mathbb{C}}, \sigma) \) for the transversal canonical electromagnetic fields.

**Theorem 8-8** The following assertions are valid:

(a) Let \( C = C^* \) be a linear operator in \( \mathcal{H}^\top \), and let \( D = D^* \) be an antilinear bounded operator in \( \mathcal{H}^\top \). Then

\[
\{ \exp\{it(C + D)\} \mid t \in \mathbb{R} \} \subset \text{sympl}(\mathcal{H}^\top, \text{Im}(\cdot))
\]

constitutes a strongly continuous symplectic one–parameter group with the \((c = 1)\)–growth

\[
\| \exp\{it(C + D)\} \| \leq \exp\{\|D\| |t|\}, \quad \forall t \in \mathbb{R},
\]

(meaning that the coefficient in front of the right hand exponential is unity).

(b) Let \( \{ T_t \mid t \in \mathbb{R} \} \subset \text{sympl}(\mathcal{H}^\top, \text{Im}(\cdot)) \) be a strongly continuous one–parameter group with \((c = 1)\)–growth, i.e. with \( \|T_t\| \leq \exp\{\delta |t|\} \) for all \( t \in \mathbb{R} \) for some \( \delta \geq 0 \). Then there exists a unique linear \( C = C^* \) and a unique antilinear bounded \( D = D^* \) in \( \mathcal{H}^\top \) such that

\[
T_t = \exp\{it(C + D)\}, \quad \forall t \in \mathbb{R}.
\]
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