Families of surfaces lying in a null set

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1 Introduction

In this note we generalise the following result of Sawyer [5]:

**Theorem 1.** There is a function \( \psi \) on \( \mathbb{R} \) such that whenever \( g \) is a real-valued Borel measurable function on (a subset of) \( \mathbb{R} \times \mathbb{R}^{n-1} \) with the property that \( y \mapsto g(y, t) \) is \( C^1 \) for a.e. \( t \), the set

\[
E_f := \bigcup_y \{ (x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} : x = g(y, t) - \psi(y) \}
\]

has measure zero.

That is, a smooth one-parameter family of measurable hypersurfaces may be translated to lie in a null set. Moreover, the translations may be taken parallel to \( \mathbb{R} \) and need not depend on \( g \).

Our motivation came from studying curved analogues of the Kakeya and Nikodym problems: we wished to show that a null set could contain a translate of every member of a specified family of curves, or in the Nikodym case, a curve from a specified family through every point. So we generalise Sawyer’s result in two ways. First, we will allow any codimension since curves of course have codimension \( n-1 \). Second, we do not want to be restricted to using translations, since in the Nikodym case it is the “directions” or “shapes” we are allowed to vary while the positions are kept fixed. So we remove all distinction between “shape parameters” and “position parameters”, simply denoting those that are “given” by \( y \) and those we are free to choose by \( \omega \).

Consider objects of the following form

\[
\Gamma(y, \omega) := \left\{ \left( f(y, \omega, t) \right)_t : t \in \mathbb{R}^d \right\}
\]

where \( f : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^d \to \mathbb{R}^{n-d} \). So \( \Gamma(y, \omega) \) can be thought of as a \( d \)-dimensional surface in \( \mathbb{R}^n \), and the family of them has \( p + q \) parameters in total. In the case above, \( f(y, \omega, t) = g(y, t) - \omega \).

Our aim is to show that under certain hypotheses, a null set may include a representative of every combination of the first \( p \) parameters provided that the remaining \( q \) parameters can be chosen to depend on them. That is, there exists a set of measure zero that includes a \( \Gamma(y, \omega(y)) \) for every \( y \). In fact, this function of \( y \) will be the obvious generalisation of Sawyer’s universal translation function \( \psi \), and will not depend on \( f \).

More precisely, our theorem is the following

**Theorem 2.** There is a function \( \psi : \mathbb{R}^p \to \mathbb{R}^q \) with the following property: Let \( f : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^d \to \mathbb{R}^{n-d} \) where \( p \leq n - d \leq q \) and \( d < n \). Suppose that \( f \) is measurable, and for almost every fixed \( t \) that the map \( (y, \omega) \mapsto f(y, \omega, t) \) is \( C^1 \), that the Jacobian \( \frac{\partial f}{\partial \omega} \) has full rank (namely \( n-d \)) and that this Jacobian is Lipschitz. Then the set

\[
E_f := \bigcup_{y \in \mathbb{R}^p} \Gamma(y, \psi(y))
\]

has measure zero.
Thus Theorem 11 is the special case obtained by setting $d = n - 1$, $p = q = 1$ and $f(y, \omega, t) = g(y, t) - \omega$.

The proof will have three parts. First, we define the universal transformation function $\psi$, which will be the obvious higher dimensional analogue of that used by Sawyer. Then, we show that all of the slices through the set at fixed $t$ have zero measure, which is where the conditions on the Jacobian and the $C^1$ assumption are used. Finally we show that the whole set is measurable, using the $C^1$ condition again. This allows us to apply Fubini’s theorem to obtain the result.

2 Definition of $\psi$

We begin with a few easily verified facts needed for the proof.

(i) Factorial Expansion: Every $a \in (0, 1]$ has a unique expansion of the form

$$a = \sum_{n=2}^{\infty} \frac{a_n}{n!}$$

where the $a_n$ are integers, $0 \leq a_n \leq n - 1$ and infinitely many of the coefficients are non-zero.

(ii) There are countably many numbers in $(0, 1]$ that also have a finite factorial expansion.

(iii) $\sum_{N}^{\infty} \frac{n-1}{n!} = \frac{1}{(N - 1)!}$

All norms, whether of matrices or vectors, will denote the largest absolute value of the entries—rather than their components. Thus for $y \in (0, 1]^q$ we can write $y = \sum_{n=2}^{\infty} \frac{y_n}{n!}$ in the natural way.

Our aim is to construct a kind of “universal transformation function” $\psi : \mathbb{R}^p \to \mathbb{R}^q$ by generalising the approach in [5]. The plan is that $\psi$ will be a series similar to the factorial expansion of $y$, and we hope to make $f(y, \psi(y), t)$ close to that value of $f$ where the series for both of the first two arguments are truncated—a finite set of values. We choose the coefficients in the series to get rid of the main error term; it turns out that the coefficients therefore must correspond to the values of $\frac{\partial f}{\partial y}^{-1} \frac{\partial f}{\partial \omega}$. (The inverse here means a right inverse of the $(n - d) \times q$ matrix $\frac{\partial f}{\partial \omega}$.) So we need to devise a sequence of $q \times p$ real matrices that is in some sense ‘dense’ and takes on arbitrarily large values, but does not grow too quickly. This is what we shall do now.

For $k \geq 3$ set

$$D_k = \left\{ (y_2, \ldots, y_{k-1}) \in (k-2)^p : 0 \leq y_i \leq n - 1 \right\},$$

that is, a set of $(k-2)$-tuples of those $p$-dimensional vectors that can form the first $k-2$ coefficients in a factorial expansion. Let $\Omega_k$ be the set of all maps $D_k \to \mathbb{R}^{[-\log \log k, \log \log k]^p}$, that is, $q \times p$ matrices whose elements are bounded by $\log \log k$. Next let $\{s_j^{k}\}_{j=1}^{m_k}$ be a finite $1/k$-dense subset of $\Omega_k$, meaning that

$$\forall s \in \Omega_k \exists j \forall (y_2, \ldots, y_{k-1}) \in D_k \| s(y_2, \ldots, y_{k-1}) - s_j^{k}(y_2, \ldots, y_{k-1}) \| < \frac{1}{k}.$$ 

At this point it will be helpful to notice that $m_k \sim (k \log \log k)^{pq(k-1)q}$, by taking the number of possible matrices and raising it to the power of the number of arguments in the function.

Next we define the sequence of maps to use as coefficients in the definition of $\psi$. Call $r \in \Omega_l$ an extension of $s \in \Omega_k$ if $l \geq k$ and for all $(y_2, \ldots, y_{k-1}) \in D_l$ we have $r(y_2, \ldots, y_{k-1}) = s(y_2, \ldots, y_{k-1})$. Set $r_2 \equiv 1$ and for each $n \geq 3$ choose $r_n \in \Omega_n$ so that for all $k \geq 3$ and $1 \leq j \leq m_k$ there is an $r_n$ that is an extension of $s_j^{k}$.
Now for \( y \in (0, 1]^p \) define \( \psi \) by

\[
\psi(y) = \sum_{n=2}^{\infty} r_n(y_2, \ldots, y_{n-1}) \frac{y_n}{n!}
\]

where each summand contains a matrix multiplication. Finally, extend \( \psi \) to all of \( \mathbb{R}^p \) by periodicity.

We observe some continuity properties of \( \psi \).

**Lemma 3.** Suppose that \( y \) and \( \bar{y} \) have the same factorial expansion up to the \( N \)th term (meaning that \( y_n = \bar{y}_n \) for \( 2 \leq n \leq N \)). Then \( |y - \bar{y}| \leq 1/N! \) and \( |\psi(y) - \psi(\bar{y})| \leq C \log \log N/N! \).

**Proof:** 

\[
|\psi(y) - \psi(\bar{y})| \leq \sum_{n=N+1}^{\infty} \frac{(n-1) \log n}{n!} 
\]

\[
\leq \frac{N \log \log (N+1)}{(N+1)!} + \sum_{n=N+2}^{\infty} \frac{(n-1) \log n}{n!} 
\]

\[
\leq \frac{C \log \log N}{N!} + C \sum_{n=N+2}^{\infty} \frac{n-2}{(n-1)!} 
\]

\[
= \frac{C \log \log N}{N!} \quad \square
\]

In particular, this shows that \( \psi \) is continuous except at points where one of the components can also have a terminating factorial expansion. At such points there is left continuity in the “bad components” and the right limits exist. Also, \( \psi((0, 1]^p) \) is a bounded set.

3 Slices have measure zero

We now need to show that for suitable values of \( n, d, p, q \) this \( \psi \) has the property claimed, that is, the set

\[ E_f := \bigcup_y \Gamma(y, \psi(y)) \]

has measure zero. In this section we show that almost all of the slices through the set at fixed \( t \) have measure zero; since \( t \) is fixed we suppress it and just prove the following:

**Lemma 4.** Let \( f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{n-d} \), with \( p \leq n - d \leq q \) and \( d < n \). Then if \( f \) is \( C^1 \) and \( \frac{\partial f}{\partial x_2} \) always has highest possible rank (namely \( n - d \)) and is Lipschitz, then the range of \( f(\cdot, \psi(\cdot)) \) is of measure zero.

These hypotheses are very natural: \( d < n \) is merely to avoid trying to pack \( n \)-dimensional objects in \( \mathbb{R}^n \), and the other inequalities mean that we should not try to include too large a family of surfaces, and we must be free to choose many of the parameters. The condition about the rank simply says that the surface must actually depend on the parameters that we are free to vary.

**Proof:** By periodicity it is enough to consider only the image of \((0, 1]^p\). For a vector \( y \) and natural number \( k \geq 3 \) write

\[
y^{(k)} = \sum_{n=2}^{k-1} \frac{y_n}{n!} \quad \psi^{(k)}(y) = \sum_{n=2}^{k-1} r_n(y_2, \ldots, y_{n-1}) \frac{y_n}{n!}.
\]

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Then for natural numbers $k$ and $N$ we have
\[
f(y, \psi(y)) = \left[ f(y, \psi(y)) - f(y^{(N)}, \psi(y)) - \frac{\partial f}{\partial y}(y^{(k)}, \psi^{(k)}(y))(y - y^{(N)}) \right]
\]
\[
+ \left[ f(y^{(N)}, \psi(y)) - f(y^{(N)}, \psi^{(N)}(y)) - \frac{\partial f}{\partial \omega}(y^{(k)}, \psi^{(k)}(y))(\psi(y) - \psi^{(N)}(y)) \right]
\]
\[
+ \left[ \frac{\partial f}{\partial y}(y^{(k)}, \psi^{(k)}(y)) + \frac{\partial f}{\partial \omega}(y^{(k)}, \psi^{(k)}(y))r(y_2, \ldots, y_{N-1}) \right] \frac{y_N}{N!}
\]
\[
+ \sum_{n=N+1}^{\infty} \left[ \frac{\partial f}{\partial y}(y^{(k)}, \psi^{(k)}(y)) + \frac{\partial f}{\partial \omega}(y^{(k)}, \psi^{(k)}(y))r(y_2, \ldots, y_{N-1}) \right] \frac{y_n}{n!}
\]
\[
+f(y^{(N)}, \psi^{(N)}(y)) =: I(y) + II(y) + III(y) + IV(y) + V(y).
\]

The final term takes a very large, but finite, number of values, so our task is to show that the other terms are correspondingly extremely small.

Let $\varepsilon > 0$ be given. Using the hypothesis that $f$ is $C^1$ together with the fact that $y$ and $\omega$ lie in the bounded sets $(0, 1]^p$ and $\psi((0, 1]^p)$ respectively, choose $k$ so large that the following hold:

(i) If both $|y - \bar{y}| < \frac{1}{(k-1)!}$ and $|\omega - \bar{\omega}| < \frac{\log \log k}{(k-1)!}$, then $\left\| \frac{\partial f}{\partial y}(y, \omega) - \frac{\partial f}{\partial y}(\bar{y}, \bar{\omega}) \right\| < \varepsilon$. This is possible since the Jacobian is continuous.

(ii) If both $|y - \bar{y}| < \frac{1}{k!}$ and $|\omega - \bar{\omega}| < \frac{\log \log k}{k!}$, then $\left\| \frac{\partial f}{\partial \omega}(y, \omega) - \frac{\partial f}{\partial \omega}(\bar{y}, \bar{\omega}) \right\| < \varepsilon/(k \log k)$. This is possible since the Jacobian is Lipschitz.

(iii) $\left\| \frac{\partial f}{\partial \omega}(y, \omega) \right\| < \log \log k$ and $\left\| \frac{\partial f}{\partial y}(y, \omega) \right\| < \log \log k$.

(iv) $\left\| \frac{\partial f}{\partial \omega}(y, \omega) \right\| < \log \log k$ where $\frac{\partial f}{\partial \omega}$ is a right inverse of the $(n - d) \times q$ matrix $\frac{\partial f}{\partial \omega}$. Here we are using the assumptions that $q \geq n - d$ and that the matrix has full rank.

(v) $\frac{(\log \log k)^2}{k} < \varepsilon$

Next, find an $s^k_j$ within $1/k$ of the matrix $-\frac{\partial f}{\partial \omega}^{-1}(y^{(k)}, \psi^{(k)}(y))$. Then find $N$ such that $r_N$ is an extension of $s^k_j$. We show that parts I–IV above are smaller than $\frac{C\varepsilon}{(N-1)!}$.

Part I is handled using the mean value theorem. The $i$th component of $I(y)$ is
\[
f^i(y, \psi(y)) - f^i(y^{(N)}, \psi(y)) - \nabla_y f^i(y^{(k)}, \psi^{(k)}(y)) \cdot (y - y^{(N)})
\]
which, by the one-dimensional mean value theorem in the direction $y - y^{(N)}$, equals
\[
\left( \nabla f^i(\xi, \psi(y)) - \nabla f^i(y^{(k)}, \psi^{(k)}(y)) \right) \cdot (y - y^{(N)})
\]
for some $\xi \in [y^{(N)}, y]$. But then $|\xi - y^{(k)}| < \frac{1}{(k-1)!}$, and $|\psi(y) - \psi^{(k)}(y)| < \frac{\log \log k}{(k-1)!}$ so that by applying (i) to this and all the other components we eventually get
\[
|I(y)| \leq \varepsilon|y - y^{(N)}|
\]
\[
\leq \varepsilon \sum_{n=N}^{\infty} \frac{1}{n!}|y_n|
\]
\[
\leq \varepsilon \frac{1}{(N-1)!}.
\]
II works similarly, except that we end up with

\[ |II(y)| \leq \frac{\varepsilon}{k \log k} |\psi(y) - \psi(N)(y)| \leq C \frac{\varepsilon}{k \log k} \log \log N. \]

But note that \( N \) was chosen to make \( r_N \) an extension of \( s_j \), so that provided we ordered the sequence \( (r_n) \) sensibly, we have

\[ N \leq \sum_{l<k} m_l + j \leq Ck \log \log k \log \log N (N-1)! \]

and hence \( \log \log N \lesssim k \log k \). So the estimate of \( C \varepsilon \frac{1}{k \log k} \log \log N (N-1)! \) for II follows. (This is the step for which we need the rather unlikely-looking double log—in Sawyer’s proof this issue does not arise, because \( \frac{\partial}{\partial x} \) is minus the identity matrix and so this whole term is zero.)

For III, our choice of \( N \) gives us cancellation.

\[ |III(y)| \leq 0 + \frac{1}{k} \left\| \frac{\partial f}{\partial \omega}(y^{(k)}, \psi^{(k)}(y)) \right\| \frac{|y_N|}{N!} \leq \frac{\log \log k}{k} \frac{N-1}{N!} \leq \frac{C \varepsilon}{(N-1)!}. \]

Finally,

\[ IV(y) \leq \sum_{n=N+1}^{\infty} \frac{\log \log k (1 + \log \log n)(n-1)}{n!} \leq \log \log k \left[ \frac{CN \log \log N}{(N+1)!} + \sum_{n=2}^{N+1} \frac{(n-1) \log \log n}{n!} \right] \leq \frac{C \log \log k \log \log N}{N!} \leq \frac{C(\log \log k)^2}{k(N-1)!} \quad \text{since } N > k \]

\[ \leq \frac{C \varepsilon}{(N-1)!}. \]

Combining these estimates we see that

\[ \text{range } \left( f(\cdot, \psi(\cdot)) \right) \subseteq \bigcup_{z \in \text{range}(V)} B \left( z, \frac{C \varepsilon}{(N-1)!} \right). \]

But \( V(y) \) depends only on \( y_2, \ldots, y_{N-1} \), so \( \text{range}(V) \) has at most \( (N-1)!^p \) elements. Hence

\[ \left| \text{range } \left( f(\cdot, \psi(\cdot)) \right) \right| \leq (N-1)!^p \left( \frac{C \varepsilon}{(N-1)!} \right)^{n-d} = C \varepsilon^{n-d} \frac{1}{(N-1)!^{n-d-p}} \]

which, since \( \varepsilon \) is arbitrary, proves the result since \( p \leq n - d \) and \( d < n \).

\[ \square \]

**Proof of Theorem 2.** To conclude the proof of the theorem we must show that the entire set \( E_f \) is measurable. Consider the set

\[ E := \bigcup_{(y_2, y_3, \ldots)} \bigcap_{k=3}^{\infty} \left\{ \left( x, t \right) : \left| x - f(y^{(k)}, \psi^{(k)}(y), t) \right| \leq \frac{1}{k} \right\}. \]
where the union is taken over all infinite sequences \((y_2, y_3, y_4, \ldots)\) with each vector \(y_m\) belonging to \(\{0, 1, \ldots, m-1\}\). The sets intersected are measurable sets depending only on the first \(k - 2\) terms of the sequence; therefore \(E\) is the result of applying the Souslin operation to a class of measurable sets and hence (see for example [4, page 45]) is measurable. Since \((y, \omega) \mapsto f(y, \omega, t)\) is \(C^1\) for a.e. \(t\), the set \(E\) is just the union of the surfaces \(\Gamma(y, \psi(y))\) except at those \(t\) for which \(f\) is not \(C^1\). That is, \(E\) differs from \(E_f\) only on a set of measure zero. Therefore \(E_f\) is measurable.

4 Discussion

We remark that our hypotheses are stronger than needed: The Lipschitz condition on \(\frac{\partial f}{\partial \omega}\) was only used to show that given \(\varepsilon > 0\) we can find \(k\) such that \((ii)\) is true: this would still hold with a weaker condition on the modulus of continuity of the Jacobian. Moreover, by replacing \(\log \log\) throughout the proof by three or more logs, we could weaken the condition further. In fact, we could do without any such condition if we sacrificed the universality of \(\psi\) and allowed it to depend on the rate of growth of the derivatives of \(f\). It may also be possible to relax the \(C^1\) hypothesis slightly, although Sawyer shows that it cannot be replaced by a Lipschitz condition of any order less than 1.

Our theorem sheds some light on other known results on curve-packing. For example, a null set in the plane can be constructed so as to include a circle of every radius (Besicovitch and Rado, Kinney 1968), but if a set has a circle centred at every point in the plane then it must have positive measure (Bourgain 1986, Marstrand 1987). However, with circles centred at all points on a curve the set can still be null (Talagrand 1980). These examples illustrate the numerology of the theorem and suggest that the conditions on the parameters might in fact be necessary as well as sufficient. Higher dimensional examples include the \(k\)-plane problem: A set in \(\mathbb{R}^3\) that includes a plane in every direction must have positive measure (Marstrand 1979, Falconer 1980)—what can be said about packing \(k\)-planes in \(\mathbb{R}^n\)? This problem has been studied by Falconer, Bourgain and others but remains unsolved. In this case we would have \(d = k\), \(p = k(n-k)\) and \(q = n-k\), so that if the numerology of Theorem\(^2\) was found to be sharp, then \(k\)-planes could be packed into a null set only when \(k = 1\). (Towards this, Mitsis \(^3\) has recently shown that a set in \(\mathbb{R}^n\) containing a translate of every 2-plane must have full dimension.) References for these and similar results can be found in \([1],[2]\) and \([7]\).

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References

[1] K. J. Falconer. The geometry of fractal sets. No. 85 in Cambridge Tracts in Mathematics. Cambridge University Press, 1985.

[2] P. Mattila. Geometry of Sets and Measures in Euclidean Spaces. No. 44 in Studies in Advanced Mathematics. Cambridge, 1995.

[3] T. Mitsis. \((n, 2)\)-sets have full Hausdorff dimension. To appear in Revista Matemática Iberoamericana, 2003.

[4] C. A. Rogers. Hausdorff measures. Cambridge University Press, Cambridge, 1998. Reprint of the 1970 original, with a foreword by K. J. Falconer.

[5] E. Sawyer. Families of plane curves having translates in a set of measure zero. Mathematika, 34: 69–76, 1987.
[6] L. Wisewell. *Oscillatory Integrals and Curved Kakeya Sets*. Ph.D. thesis, University of Edinburgh, 2003.

[7] T. Wolff. Recent work connected with the Kakeya problem. In H. Rossi, ed., *Prospects in Mathematics (Princeton, New Jersey, 1996)*, pp. 129–162. American Mathematical Society, Providence, RI, 1999.

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