Quark-hadron phase transition with surface fluctuation

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Abstract

The effect of surface fluctuation on the observables of quark-hadron phase transition is studied. The Ginzburg-Landau formalism is extended by the inclusion of an extra term in the free energy that depends on the vertical displacements from a flat surface. The probability that a bin has a particular net displacement is determined by lattice simulation, where the physics input is color confinement. The surface fluctuation from bin to bin is related to multiplicity fluctuation, which in turn is measured by the factorial moments. It is found that both the $F$-scaling behavior and the scaling exponent are essentially unaffected by the inclusion of surface fluctuation.

1 Introduction

Scaling behaviors of hadronic observables in quark-hadron transition (PT) have been studied in the framework of Ginzburg-Landau (GL) theory [1]-[4]. Since it is a mean-field theory, it cannot account for the dynamical fluctuations that can occur in heavy-ion collisions. An attempt to amend that defect has been made by treating the problem in the Ising model, so that by lattice simulation the effect of spatial fluctuation of hadron density at the surface of quark-gluon plasma can be considered [5]. In this paper we treat the problem of spatial fluctuation in a very different way. We generalize the GL description of second-order PT in 2D by including a term that accounts for surface fluctuation. The deviation from a flat surface is simulated by use of a term in the free energy that represents the confining potential between quarks. By associating bulges on the surface to regions where hadronization can take place more readily, we can determine the effect of spatial fluctuations on the hadronic observables. Thus this work represents a significant step toward making the GL formalism a more realistic description of the PT problem in heavy-ion collisions.

In Refs. [1]-[4] it has been found that the scaling behavior characteristic of the PT is associated with the normalized factorial moments $F_q$. Actually, that scaling behavior does not exhibit intermittency [6], which is the power law $F_q \propto M^{\nu_q}$, where
\( M \) is the number of bins in a fixed volume of phase space. Instead of \( M \) scaling, which we use to refer to the \( M^{2q} \) behavior, we have found that pure GL leads to \( F \) scaling, which is

\[
F_q \propto F_2^{2q},
\]

(1)

when \( M \) is varied. Furthermore, \( \beta_q \) satisfies the simple formula (1)

\[
\beta_q = (q - 1)^\nu.
\]

(2)

It is this \( \nu \) that has been referred to as the scaling exponent. It is not to be confused with the conventional critical indices in statistical physics, to which \( \nu \) bears absolutely no relationship. In problems where the temperature \( T \) is not observable, but the multiplicities of particles produced are (such as photons in lasers \( \mathbb{L} \) and pions in heavy-ion collisions \( \mathbb{L} \)), \( F \) scaling has been found to be the only scaling behavior that can be put to experimental tests. Thus far the value of \( \nu \) at 1.3 has proven to be remarkably universal theoretically, and has been verified experimentally to be accurate in quantum optics \( \mathbb{L} \). In this paper we shall check whether \( F \) scaling remains valid when spatial fluctuations are taken into account, and if so, whether \( \nu = 1.3 \) is still true.

## 2 Formulation of the problem

We give first a very brief review of the usual method in the GL approach to hadronic observables \( \mathbb{L} \). The multiplicity distribution \( P_n \) is expressed as a functional integral

\[
P_n = \frac{1}{Z} \int D\phi \frac{1}{n!} \left( \int d^2x |\phi|^2 \right)^n e^{-\int d^2x |\phi|^2} e^{-F}
\]

(3)

where \( Z = \int D\phi e^{-F} \), and the GL free energy is

\[
F = \int d^2x \left( a|\phi|^2 + b|\phi|^4 + c|\nabla\phi|^2 \right).
\]

(4)

Here the spatial integration is over a 2\( D \) space, which may be taken to be the surface of the cylinder containing quark-gluon plasma. When the surface temperature \( T \), on which the parameters \( a, b, \) and \( c \) in (4) depend, is low enough, hadrons are formed on that surface and are removed from the confining medium that exists within. If the integration in \( \vec{x} \) is limited to a small bin of size \( A \), then the \( A \)-dependence of \( P_n \) can best be determined in terms of \( F_q \), where

\[
F_q = \frac{\langle n(n-1) \cdots (n-q+1) \rangle}{\langle n \rangle^q}
\]

(5)

the average being defined by use of the distribution \( P_n \). It is this \( F_q \) that has the scaling properties \( \mathbb{L} \) and \( \mathbb{L} \).
In most treatments [1]-[4] the third term in (4) is neglected. Its inclusion is considered by Hwa and Pan [2], and is found to lead to only a small change in the value of $\nu$. Our approach here is to replace that term by another that has explicit spatial dependence with dynamical origin. But before embarking on that, let us rewrite the formalism in a more compact way.

If we write (5) in the form

$$F_q = f_q/f_1^q,$$

then we have

$$f_q = \sum_{n=q}^{\infty} \frac{n!}{(n-q)!} P_n = \int dm \ m^q D_q(m),$$

where $D_q(m)$ is proportional to $e^{-F(m)}$, and

$$m = \int_{A} d^2x |\phi(x)|^2 = A|\phi|^2.\hspace{1cm}(8)$$

Without the last term of (4), $\phi(x)$ may be regarded as constant inside $A$ so the last expression in (8) follows. With the interpretation of $|\phi|^2$ as the hadron density, an order parameter, we may regard $m$ as the “dynamical” multiplicity in a bin of area $A$. The statistical fluctuation represented by the Poissonian factors in front of $e^{-F}$ in (3) is removed by taking the factorial moment, so that the series in terms of the “experimental” multiplicity $n$ in (3) is replaced by the integral over the “dynamical” multiplicity $m$, which need not be an integer. $f_q$ has multiplicative factors that are cancelled by the same factors in $f_1^q$, when the ratio in (6) is taken. Leaving out those factors, we can write $f_q$ more transparently as

$$f_q = \int_{0}^{\infty} dm \ m^q \exp[-(m - m_a)^2/(A/b)],\hspace{1cm}(9)$$

where

$$m_a = -Aa/2b.\hspace{1cm}(10)$$

PT occurs when $a < 0$, so $m_a$ is always positive in the following consideration. Note that $a$ is dimensionless, while $b$ has the dimension of area. Thus the physical $A$ always appears in the ratio $A/b$, i.e., in the GL description the area is measured in units of $b$. In [1] it is shown that $F_q$ is a function of only one composite variable, $Aa^2/2b$. In the following because of our way of introducing spatial fluctuations, there will be separate dependences on $a$ and $A/b$. Hence, (3) will be our starting point, when there is no spatial fluctuation.

To introduce surface fluctuations, let us first map the 2D surface to a square lattice of size $L^2$, on which we impose periodic boundary condition. With bins of size $\delta^2$, there are $M = (L/\delta)^2$ bins on the lattice. At each lattice site we assign a variable $z_i$ that represents the vertical displacement from the flat surface at site $i$. We restrict $z_i$ to be in the range $-1 \leq z_i \leq 1$, deferring the physical scale of perpendicular
displacement to another parameter to be introduced below. Zero displacement is defined by the global constraint
\[ \sum_{i \in L^2} z_i = 0, \]  
(11)
where the sum is over all sites on the lattice.

The physics governing the \( z \)-displacement is color confinement. In the discretization of the physical space we do not mean that the lattice spacing is to be identified with the actual average distance between partons on the surface of the plasma. The lattice is a coarse-grain representation of the surface, and we let each site carry the net color charge of the basic cell (of area lattice-spacing squared) surrounding the site. For brevity we call such a color charge at the site a quark. Our aim is to simulate vertical fluctuations of the surface. With the lattice spacing being fixed, the vertical displacement of a site relative to its neighbors represents further separation among the quarks. Since energy is required to increase the quark separation, we add a term to the free energy to inhibit vertical displacement, calling it \( F_s \) to represent spatial contribution,
\[ F_s = C \sum_{\langle ij \rangle} |z_i - z_j|, \]  
(12)
where the sum is over the nearest neighbors \( j \) of each site \( i \), and then over all \( i \). The confinement parameter \( C \) includes the \( (k_B T)^{-1} \) factor, as in (4), and the scale factor for perpendicular displacement mentioned earlier. The \( |z_i - z_j| \) dependence represents the linearly increasing portion of the confinement potential. The \(-1/r\) Coulombic portion is omitted partly because the lattice spacing imposes a minimum distance between quarks that prevents \( r \) from becoming small, and partly because we want to avoid introducing another parameter. In place of the last term of (4), \( F_s \) as given in (12) provides a sensible description of spatial dependence of the energy of the system that incorporates the color dynamics into the GL formalism. A key link yet to be specified is the way in which \( z_i \) is related to the order parameter \( \phi \) or the dynamical multiplicity \( m \).

Our procedure is to use the Boltzmann factor \( \exp(-F_s) \) to simulate a configuration of \( z_i \) throughout the lattice. Clearly, because of thermal fluctuation the vertical displacement need not be \( z_i = 0 \) for all \( i \), even though that is the most favored configuration. Because of (11) there are as many net positive displacements as there are negatives on the lattice. However, as we focus on a particular bin of area \( \delta^2 \), there may be a nonvanishing displacement for the bin
\[ \zeta = \sum_{i \in \delta^2} z_i. \]  
(13)
We identify a positive \( \zeta \) as having a net outward protrusion of the surface at the bin, and a negative \( \zeta \) as having a net inward indentation. Nucleation dynamics suggests that hadronization is more likely to occur at the location of protruding surface and less likely at where the surface is indented. Thus given a value of \( \phi \) for a bin, the
dynamical multiplicity should deviate from $m$ in a way that is proportional to $\zeta$. That is, we should consider a convolution $G \otimes S$, where $G$ is the GL term given by the exponential in (9), and $S$ represents the spatial fluctuation specified by $S(\zeta)$, yet to be determined, which describes the probability that a bin has a net vertical displacement $\zeta$.

More precisely, we upgrade (9) to include spatial fluctuations by writing

$$f_q = \int_0^\infty d\mu \mu^q D(\mu),$$

where the dynamical distribution $D(\mu)$ is

$$D(\mu) = G \otimes S = \int_0^\infty dm G(m)S(\mu - m)$$

and

$$G(m) = \exp[-(m - m_a)^2/(A/b)].$$

In the next section we shall determine the distribution in $\zeta$. What remains to be specified is the relationship between $\zeta$ and hadron multiplicity. Without getting involved with the details of nucleation, surface tension, etc., let us introduce a new parameter $\alpha$ that summarizes all the dynamical details relating vertical surface displacement to multiplicity fluctuation, $\mu - m$, so that

$$\mu = m + \alpha \zeta.$$ 

Recall that $m$ is the mean multiplicity in the GL theory. The spatial fluctuation of the multiplicity distribution, $S(\alpha \zeta)$, to be determined in the next section is what appears in (15) for the net dynamical distribution $D(\mu)$.

### 3 Simulation on the lattice

Our first task is to reproduce to GL result [1] (without surface fluctuation) by lattice simulation. In the analytical method used in [1], there is only one composite variable, $x = Aa^2/2b$, that $F_q$ depends on. $F$ scaling is found to hold in the range $0.1 < \ln F_2 < 0.44$, which corresponds roughly to $3 < \ln F_{10} < 9$. From Fig. 1 of [1] one can see that such ranges of $F_q$ are for $x$ in the range $-2 < -\ln x < 4$. Since $x$ is a composite variable, it does not matter how $a$, $b$ and $A$ are separately varied. Phenomenologically, $a$ and $b$ are not under experimental control, so $A$ must be varied in the appropriate range such that the data on $F_q$ are in the range that exhibits $F$ scaling. Now, on the lattice we must work with specific bin sizes $\delta$. That necessity destroys the advantage of dealing with just one $x$ variable in the analytic method. We make the identification

$$A/b = \delta^2$$

and choose a range of $\delta$ that can cover the scaling range in $x$ mentioned above, i.e., $0.018 < x < 7.5$. The ratio of the extrema of that $x$ range corresponds, for $a$ held
constant, to the ratio of the extrema of \( \delta \) being very nearly 20 : 1. Thus, if we let the smallest bin to consist of just two sites in each direction, then the range of \( \delta \) should be \( 2 \leq \delta \leq 40 \). Choosing the lattice size to be \( L = 120 \), we have for the number of bins, \( M \), on the lattice to vary from \( 60^2 \) down to \( 3^2 \), a range that is wide enough to test scaling. The corresponding value of \( a \) is roughly \(-0.1\).

Applying the above parameters to (9) and (10), and using Monte Carlo method to calculate \( f_q \) and then \( F_q \), we find the result in Fig. 1 shown in log-log plot. The linear dependences for \( 3 \leq q \leq 6 \) confirm the \( F \)-scaling behavior. Moreover, the scaling exponent \( \nu \) defined in (2) is found to be \( \nu = 1.308 \), as shown in Fig. 2, in essential agreement with the result obtained in [1], as it should.

Proceeding now to the main task of including spatial fluctuations, we start by setting \( C = 1 \) in (12) initially. Other values of \( C \) will be considered later. We use the Metropolis algorithm to simulate the configurations of \( z_i \) on each site of the lattice, with the Boltzmann factor \( e^{-F_s} \) as the weight. We discard the first 800 sweeps for initialization. From \( 10^3 \) such configurations we calculate the distribution in \( \zeta \). However, owing to (17) it is more useful to plot the distribution as a function of \( \alpha \zeta \). The arbitrarily chosen value of \( \alpha \) affects only the scale of the abscissa and the normalization of \( S(\alpha \zeta) \). Fig. 3 shows the results for \( \delta = 2 \) and 40, and for \( \alpha = 1/4 \), which is chosen so that for the smallest bin the maximum multiplicity due to surface fluctuation is \( \alpha \zeta = 1 \). As Fig. 3(a) shows, the distribution is very nearly Gaussian with half-width around 0.3, plus a sharp peak at \( \alpha \zeta = 0 \), corresponding to the preponderant probability that \( z_i = 0 \) for all four sites in a bin. On the other hand, for the largest bin \( \delta = 40 \), the distribution shown in Fig. 3(b) is negligible for \( \alpha \zeta > 50 \), although the maximum possible value, for all \( z_i = +1 \), is \( \alpha \zeta = 40^2/4 = 400 \). Apart from the peak at \( \alpha \zeta = 0 \) the half-width is around 20. The statistics is poorer in this case because the lattice has far fewer bins per configuration when \( \delta \) is increased twenty-fold. As the area \( \delta^2 \) is increased by a factor of 400, the mean multiplicity fluctuation is increased by a factor of only \( 20/0.3 \approx 60 \). Thus the deviation from a flat surface does not have long wavelength modes, which would have resulted in large \( \zeta \) configurations in large bins. On the other hand, the surface fluctuations are not all local, lest Fig. 3(b) would not be as broad as it is.

Putting these \( S(\alpha \zeta) \) distributions for various values of \( \delta \) between 2 and 40 in (15) and performing the convolution, we obtain the dynamical multiplicity distribution \( D(\mu) \). Fig. 4 shows the results for a range of values of \( \delta \) with \( C = 1 \) and \( \alpha = \frac{1}{4} \). From the \( D(\mu) \) distribution we can use (3) and (14) to calculate \( F_q \), which we present in the log\( F_q \) vs log\( F_2 \) form in Fig. 5. We see that the straightline behavior supports \( F \) scaling. As \( \alpha \) is varied from \( 1/4 \) to 2, the ranges of \( F_q \) values are reduced but the slopes of the straightlines are essentially unchanged. Thus the first part of our objective has been accomplished; that is, \( F \) scaling persists even when surface fluctuations are considered.

Before we investigate the second part, \textit{viz.}, the effect on scaling exponent, we examine the dependence on \( C \). The foregoing analysis is done for \( C = 1 \), a value that has been arbitrarily chosen. We now vary \( C \) to see the effect on \( S(\zeta) \). From the simulated result on \( S(\zeta) \) for every set of values of \( C \) and \( \alpha \), we calculate the
width $\Gamma_\zeta = \langle \zeta^2 \rangle^{1/2}$ and show in Fig. 6 the dependence of $\Gamma_\zeta$ on $C$ for the two extreme bin-sizes, $\delta = 2$ and 40. We see that for $\delta = 40$ there is a maximum at $C = 2$, so the previous result on $C = 1$ is not far from maximum surface fluctuation. $\Gamma_\zeta$ is lower at both higher and lower values of $C$ because high $C$ stiffens the surface and suppresses $\zeta$, while low $C$ leads to random $z_i$ and therefore also reduces $\zeta$. The same is not true for $\delta = 2$, since even in the limit of $C \to 0$ the smallness of the bin size makes the cancellation of random $z_i$ ineffective, resulting in a Possionian $S(\zeta)$ that has nonvanishing width. As $C$ increases, the surface becomes smoother, and $\Gamma_\zeta$ continues to decrease monotonically. To summarize, we need only consider the range $0 < C < 2$ for maximum effect due to surface fluctuation.

We now can repeat the previous calculations for other values of $C$ and produce figures similar to those shown in Figs. 4 and 5. However, since the results are very similar, we omit those figures.

To determine the slopes $\beta_q$ of the $F$-scaling behavior, we fit all the points in Fig. 5 for a given $q$ value, including all values of $\alpha$, by a straight line. The resultant $\beta_q$ is then shown in Fig. 7 for a particular value of $C$ ($C = 1$ for Fig. 5). The error bars in Fig. 7 correspond to the deviations from the straightline fits of the points in Fig. 5 for $C = 1$ only. For other values of $C$ the results from similar calculations are also shown in Fig. 6. No corresponding error bars are shown, since they are similar. The $\beta_q$ values are determined only for integer values of $q$, but since the dependence on $C$ is so insignificant, we have for clarity exhibited the $\beta_q$ values by straightline fits according to (19). Even so, the overlap obscures the individual lines for different values of $C$, but successfully demonstrates that the scaling exponent $\nu$ is insensitive to $C$. The net value of $\nu$ that can be concluded from Fig. 7 is

$$\nu = 1.306 \pm 0.035,$$

(19)

when surface fluctuations are taken into account. Compared to the result of $\nu = 1.308$ obtained from Fig. 2 for pure GL, we see that the change in $\nu$ is well within the errors.

### 4 Conclusion

To study the effect of surface fluctuation on the observables of quark-hadron PT, we have incorporated into the GL description an additional term in the free energy that depends on the vertical displacements from a flat surface. We have used lattice simulation to determine the probability that a bin has a net displacement $\zeta$. The physics that controls such displacements is color confinement. We connect the surface fluctuations to multiplicity fluctuations, which in turn are quantified by normalized factorial moments $F_q$. The $F$ scaling of $F_q$, when the bin size is varied, results in the determination of the scaling exponent $\nu$. The aim of this work has been to study the dependence of $\nu$ on surface fluctuations, after verifying that the scaling behavior is not destroyed.

There are two parameters in the calculation, on which we have no a priori information. One is $C$; the other is $\alpha$. $C$ depends on the strength of confinement, and relates the vertical displacement $z_i$ (arbitrarily normalized) to the free energy (normalized by
the transition temperature). The parameter $\alpha$ relates the net displacement $\zeta$ of a cell to the deviation of the cell multiplicity from the mean. We have studied the variation of $\nu$, when $C$ and $\alpha$ are varied over the important ranges $0 < C < 2$ and $0 < \alpha < 2$. When $C \simeq 2$ or $\alpha \simeq 2$, we do see significant fluctuations. Our result, however, shows that $\nu$ is essentially unchanged when $C$ and $\alpha$ are varied in the ranges considered. There is no doubt that when $\alpha$ is large enough, the scaling behavior will break down. But then our method of calculation should become invalid in that case, since large multiplicity fluctuations cannot be treated in the framework of the mean-field theory. Besides, there is no physical reason to expect that small surface fluctuation can lead to large multiplicity fluctuations.

The significance of this work is therefore in finding that the scaling behavior of pure GL is stable against the perturbation introduced by surface fluctuations. Thus this result improves the Ginzburg-Landau description of quark-hadron phase transition and provides a more physical connection to the realistic situation in heavy-ion collisions. The end result is that $F$ scaling persists even when surface fluctuations are taken into account, and that the scaling exponent, which is experimentally measurable, remains at $\nu \simeq 1.3$.

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References

[1] R. C. Hwa and M. T. Nazirov, Phys. Rev. Lett. 69, 741 (1992).

[2] R. C. Hwa and J. Pan, Phys. Lett. B297, 35 (1992). R. C. Hwa, Phys. Rev. D47, 2773 (1993); R. C. Hwa, Phys. Rev. C50, 383 (1994).

[3] See A. K. Mohanty and S. K. Kataria, Phys. Rev. Lett. 73, 2672 (1994); A. K. Mohanty and S. K. Kataria, BARC-preprint 1995; L. F. Babichev, D. V. Klenitsky, and V. I. Kuvshinov, Phys. Lett. B 345, 269 (1995); I. A. Lebedev and M. T. Nazirov, Mod. Phys. Lett. A9, 2999 (1994).

[4] For a review see R. C. Hwa, in Quark-Gluon Plasma 2, edited by R. C. Hwa (World Scientific, Singapore, 1995).

[5] Z. Cao, Y. Gao, and R. C. Hwa, OITS-587 (1995).

[6] A. Bialas and R. Peschanski, Nucl. Phys. B273, 703 (1986).

[7] M. R. Young, Y. Qu, S. Singh and R. C. Hwa, Optics Comm. 105, 325 (1994).
Figure Captions

Fig. 1  $F$ scaling in Ginzberg-Landau theory without spatial inhomogeneity, i.e., $c = 0$ in Eq. (4).

Fig. 2  Log-log plot verifying Eq. (2), using the data of Fig. 1.

Fig. 3  Distribution of surface fluctuation for $C = 1$ and $\alpha = 1/4$. The bin sizes are different in (a) and (b) as indicated.

Fig. 4  Distribution of dynamical multiplicity when the GL description of PT is supplemented by surface fluctuations.

Fig. 5  $F$-scaling plot of $F_q$ for $C = 1$ and a range of $\alpha$.

Fig. 6  Width of $S(\zeta)$ vs $C$ for $\delta = 2$ (right scale) and 40 (left scale).

Fig. 7  The slopes $\beta_q$ for a range of values of $C$. Although $\beta_q$ are determined for integer values of $q$, they are plotted as straight lines to avoid overlap of points. The error bars are for $C = 1$ only, the ones for other values of $C$ being similar.
Fig 1.

$\log F_q$ vs. $\log F_2$

GL (c = 0)

$q = 6$

5

4

3
Fig 2.

\[ v = 1.308 \]
Fig 3.
Fig 4.

\[ \delta = 2 \]

\[ \alpha = 1/4 \]

\[ C = 1.0 \]
Fig 5.

$C = 1$
$\alpha = 1/4$
$1/2$
$1$
$2$

$q = 6$
$5$
$4$
$3$

$\log_{10} F_2$

$\log_{10} F_q$
Fig 6.
Fig 7.