Perturbative matching of the staggered four-fermion operators for $\epsilon'/\epsilon$ *

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Using staggered fermions, we calculate the perturbative corrections to the bilinear and four-fermion operators that are used in the numerical study of weak matrix elements for $\epsilon'/\epsilon$. We present results for one-loop matching coefficients between continuum operators, calculated in the Naive Dimensional Regularization (NDR) scheme, and gauge invariant staggered fermion operators. Especially, we concentrate on Feynman diagrams of the current-current insertion type. We also present results for the tadpole improved operators. These results, combined with existing results for penguin diagrams, provide the complete one-loop renormalization of the staggered four-fermion operators. Therefore, using our results, it is possible to match a lattice calculation of $K^0 - \bar{K}^0$ mixing and $K \rightarrow \pi\pi$ decays to the continuum NDR results with all corrections of $O(g^2)$ included.

I. INTRODUCTION

The neutral kaon system has been much studied since it was observed that the strangeness states ($K^0$, $\bar{K}^0$) in the quark model mix to produce short-lived and long-lived kaons ($K_S$, $K_L$) in nature. The discovery of $K_L \rightarrow \pi\pi$ decays revealed that the weak interaction violates CP symmetry. In nature, CP asymmetry happens in two ways: indirect and direct CP violations. The dominant effect parameterized by $\epsilon$ (the indirect CP violation) comes from the fact that Nature’s neutral kaon mass eigenstates are not CP-symmetric. The phenomenon that the weak interaction itself allows $K_L \rightarrow \pi\pi$ decays directly is referred to as “direct CP violation” and it is parameterized by $\epsilon'$.

The recent results of $\text{Re}(\epsilon'/\epsilon)$ announced by the KTeV [1] and NA48 [2] collaborations supported the existence of the direct CP violation. Even though the Standard Model (SM) of strong and weak interactions provides a straight-forward qualitative understanding of CP violation in terms of a single phase ($\delta$), perturbation theory does not allow a reliable quantitative calculation of the size of CP violation. In particular, in $K \rightarrow \pi\pi$ decays, the direct CP violation is a result of destructive interference from various contributions. Theoretically, these come from strong interaction effects on hadronic matrix elements of the effective four-quark operators. Since the energy involved in these decays is at a scale of around 500 MeV, non-perturbative tools such as lattice QCD must be used to calculate them.

There have been a number of attempts to calculate $\epsilon'/\epsilon$ on the lattice using clover/Wilson fermions [3,4], staggered fermions [5], and domain-wall fermions [6–8]. In this paper, we study the method using staggered fermions [9]. Recently there has been a numerical study with controlled statistical errors in calculating the $\Delta I = 1/2$ amplitude, using staggered fermions [8]. However, this numerical study suffers from two major difficulties. The first is the puzzling crux in calculating $\langle \pi\pi \mid O_i \mid K \rangle$ directly in Euclidean space [10]. Therefore, one calculates hadronic matrix elements of $\langle \pi \mid O_i \mid K \rangle$ and $\langle 0 \mid O_i \mid K \rangle$ on the lattice, and reconstructs $\langle \pi\pi \mid O_i \mid K \rangle$ out of these amplitudes, using chiral perturbation at leading order [11]. Correspondingly, this method has uncertainties from the neglected higher orders in chiral perturbation such as final state interactions. The second difficulty is that the complete set of one-loop matching formulae for gauge invariant staggered fermion operators was not available. Only part of one-loop matching formulae for gauge invariant $B_K$ and a complete set of one-loop matching relationship for gauge non-invariant Landau gauge operators were available in Ref. [12–14,15]. Hence, even though the numerical simulation has reached a point to have statistical control over “eye” diagrams [8], it was not quite possible to do the complete one-loop matching of the gauge invariant lattice results to the continuum hadronic matrix elements.

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In this paper, we present the one-loop matching formulae between the continuum operators, defined in the NDR scheme, and the lattice gauge invariant operators of phenomenological interest for the current-current insertion type (corresponding to the “eight” contraction type in Ref. [14]). These results, combined with existing results for penguin diagrams in Ref. [12], allow matching lattice calculation of $K^0 - \bar{K}^0$ mixing and $K \to \pi\pi$ decays (including $\Delta I = 1/2$ amplitudes) to the continuum NDR scheme with all corrections of $O(g^2)$ included.

This paper is organized as follows. In Sec. II, we explain our notation for staggered fermions, bilinears and four-fermion operators on the lattice. In this method, we interpret 16 staggered fermion fields (lattice. We use the same notation for the staggered fermion action as in Ref. [15]: the Dirac fields $Q^{\alpha i}(y)$ where $\alpha$ is the Dirac spin index, $i$ is the flavor index and $N_f$ is the number of degenerate flavors ($N_f = 4$).

Define also

$$
\gamma_A = \gamma_1^{A_1} \gamma_2^{A_2} \gamma_3^{A_3} \gamma_4^{A_4} .
$$

In order to construct bilinear operators, we use a $\gamma$ matrix basis for both spin and flavor matrices. Using the Dirac fields $Q$, we can express a general form of quark bilinear operators:

$$
\overline{Q}^{\alpha i}(y)(\gamma_S^{n,\beta} \otimes \xi_F^{j,i})Q^{\beta j}(y) = \frac{1}{N_f} \overline{\chi}(y_A)(\gamma_S \otimes \xi_F)_{AB} \chi(y_B)
$$

Here, $\alpha, \beta$ are Dirac spin indices and $i, j$ are flavor indices. The matrix $(\gamma_S \otimes \xi_F)$ is defined as

$$
(\gamma_S \otimes \xi_F)_{AB} \equiv \frac{1}{N_f} \text{Tr}(\gamma_A^B \gamma_S \xi_F^B) .
$$

This interpretation is called the coordinate-space method.

We can translate this coordinate space method directly into the momentum space method [17] as follows:

$$
(\gamma_S \otimes \xi_F)_{AB} = \sum_{CD} \frac{1}{N_f} (-1)^{A-C}(\gamma_S \otimes \xi_F)_{CD} \frac{1}{N_f} (-1)^{D-B} .
$$

II. NOTATIONS AND TERMINOLOGY

In this section we will specify our notation for the action, fermion fields and composite operators on the lattice. We use the same notation for the staggered fermion action as in Ref. [15]:

$$
S = a^4 \sum_n \left[ \frac{1}{2a} \sum_{\mu} \eta_\mu(n) \left( \chi(n)U_\mu(n)\chi(n + \hat{\mu}) - \chi(n + \hat{\mu})U^\dagger_\mu(n)\chi(n) \right) + m\overline{\chi}(n)\chi(n) \right] ,
$$

where $n = (n_1, n_2, n_3, n_4)$ is the lattice coordinate and $\eta_\mu(n) = (-1)^{n_1+\cdots+n_{\mu-1}}$.

In order to construct the four spin component Dirac field, we adopt the coordinate space method suggested in Ref. [16]. In this method, we interpret 16 staggered fermion fields ($\chi$) of each hypercube in the coordinate space as 4 Dirac spin and 4 flavor components. The 4 flavor Dirac field $Q(y)$ is defined as

$$
Q^{\alpha i}(y) = \frac{1}{N_f \sqrt{N_f}} \sum_A (\gamma_A)^{\alpha i} \chi(y_A)
$$

where $\alpha$ is the Dirac spin index, $i$ is the flavor index and $N_f$ is the number of degenerate flavors ($N_f = 4$). Define also

$$
y_A \equiv 2y + A, \text{ with } A \in \{0, 1\}^4
$$

and

$$
\gamma_A = \gamma_1^{A_1} \gamma_2^{A_2} \gamma_3^{A_3} \gamma_4^{A_4} .
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$$
Note that the momentum-space matrices \((\gamma_S \otimes \xi_F)\) proposed in Ref. \[13\] are unitarily related to the coordinate-space matrices \((\gamma_S \otimes \xi_F)\).

The continuum limit of the staggered fermion action on the lattice corresponds to QCD with four degenerate flavors \((N_f = 4)\) \[13\]. In this limit, \(\gamma_S\) represents the Dirac spin matrix and \(\xi_F\) belongs to \(SU(4)\) flavor symmetry group. There are numerous choices to transcribe the lattice operator for a given continuum operator \[13,20,21\]. We adopt the following conventional choice of bilinear operator transcription:

\[
O_{SF}(y) = \frac{1}{N_f} \sum_{AB} \chi(y_A)(\gamma_S \otimes \xi_F)_{AB} \chi(y_B) \mathcal{U}(y_A,y_B) .
\]

Here, the continuum flavor indices are dropped out. For the four-fermion operators, we use the same notation \[13,18,19\], such that \(O_{SF}\) is as symmetric as possible:

\[
\mathcal{U}(y_A,y_B) = \frac{1}{4!} \sum_P U(y_A,y_A + \Delta P_1) \cdots
\]

\[
U(y_A + \Delta P_1 + \Delta P_2 + \Delta P_3 + y_B) ,
\]

where \(P\) is an element of the permutation group \((1234)\) and

\[
\Delta \mu = (B_\mu - A_\mu) \hat{\mu} .
\]

There are two kinds of four-fermion operators which have different color contraction structure. The general form of color two-trace operators is

\[
[\chi^f_A(y_A)(\gamma_S \otimes \xi_F)_{AB} \chi^f_B(y_B)] [\chi^f_C(y_C)(\gamma_S' \otimes \xi_F)_{CD} \chi^f_D(y_D)] \mathcal{U}_{ab}(y_A,y_B) \mathcal{U}_{cd}(y_C,y_D)
\]

and the general form of color one-trace operators is

\[
[\chi^f_A(y_A)(\gamma_S \otimes \xi_F)_{AB} \chi^f_B(y_B)] [\chi^f_C(y_C)(\gamma_S' \otimes \xi_F)_{CD} \chi^f_D(y_D)] \mathcal{U}_{ad}(y_A,y_D) \mathcal{U}_{bc}(y_B,y_C)
\]

where \(f_1, f_2, f_3, f_4\) label the continuum flavor (for example, u, d, s, \(\cdots\)). Note that we have introduced flavor in two ways: first, the 4 degenerate flavors hidden in the hypercubic index \(A\) of the staggered fermion field \(\chi^f(y_A)\), and second, the continuum flavors labeled by the index \(f\).

For later convenience, let us introduce a compact notation. We can represent the bilinear and four-fermion operators in terms of \(S\) (scalar), \(V\) (vector), \(T\) (tensor), \(A\) (axial) and \(P\) (pseudo-scalar). For example, the bilinears can be expressed as

\[
[V_\mu \times S] = \frac{1}{N_f} \sum_{A,B} [\chi(y_A)(\gamma_\mu \otimes I)_{AB} \chi(y_B)] \mathcal{U}(y_A,y_B)
\]

\[
[A_\mu \times P] = \frac{1}{N_f} \sum_{A,B} [\chi(y_A)(\gamma_\mu_5 \otimes \xi_5)_{AB} \chi(y_B)] \mathcal{U}(y_A,y_B)
\]

Here, the continuum flavor indices are dropped out. For the four-fermion operators, we use the same notation as the bilinears but need to distinguish between color one trace and color two trace operators. This notation is exemplified by the following:

\[
[S \times P][P \times S]_{II} = \frac{1}{N_f^2} \sum_{A,B,C,D} [\chi(y_A)(\gamma_5 \otimes I)_{AB} \chi(y_B)] [\chi(y_C)(\gamma_5 \otimes I)_{CD} \chi(y_D)] \mathcal{U}(y_A,y_B) \mathcal{U}(y_C,y_D)
\]

\[
[A_\mu \times S]_{I} = \frac{1}{N_f^2} \sum_{A,B,C,D} [\chi(y_A)(\gamma_\mu_5 \otimes \xi_5)_{AB} \chi(y_B)] [\chi(y_C)(\gamma_\mu_5 \otimes I)_{CD} \chi(y_D)] \mathcal{U}(y_A,y_B) \mathcal{U}(y_C,y_B)
\]

Here note that the sub-indices \(I, II\) represent the color one trace and color two trace operators respectively.

There are two completely independent methods to transcribe the lattice operators using Fierz transformation: one spin trace formalism and two spin trace formalism \[13\]. In this paper, we choose two spin trace formalism to construct the lattice operators and it is also adopted for our numerical study on \(\epsilon'/\epsilon\).
III. RENORMALIZATION AND MATCHING

In this section, we will review the renormalization of the continuum operators and explain the matching procedure between the continuum and the lattice operators at one loop level. We use the formulation of Ref. [18,19,22,14,12,13].

A. Feynman rules

We use the same Feynman rules as in Ref. [13]. There was a typo in the Feynman rule for the gluon propagator in Eq. (78) of Ref. [13]. Therefore, we provide the correct Feynman rule here.

\[ D_{IJ}^{\mu\nu}(k) = \frac{\delta_{IJ} \delta_{\mu\nu}}{\sum_{\beta} \frac{k_{\beta}}{k^2} \sin^2 \left( \frac{1}{2} ak_{\beta} \right)} - (1 - \alpha) \frac{\delta_{IJ} \frac{1}{2} \sin \left( \frac{1}{2} ak_{\mu} \right) \sin \left( \frac{1}{2} ak_{\nu} \right)}{\left[ \sum_{\beta} \frac{k_{\beta}}{k^2} \sin^2 \left( \frac{1}{2} ak_{\beta} \right) \right]^2}, \]  

(17)

where \( \alpha \) is a gauge fixing parameter in the general covariant gauge and superscripts \( I,J \) represent the color indices in the adjoint representation of SU(3).

All the Feynman diagrams are calculated at zero quark mass, which induces infrared divergences. We regulate the infrared divergences by adding a mass term to the gluon propagators both on the lattice and in the continuum. This dependence on the gluon mass cancels out, when we match the lattice and continuum renormalized operators.

B. Bilinear Operators

Let us consider a bilinear operator, \( O_{SF} \) with spin \( S \) and flavor \( F \). In the \( \overline{MS} \) scheme, there are three distinct sets of commutation rules regarding \( \gamma_5 \): Naive Dimensional Regularization (NDR), Dimensional Reduction (DRED, D\( \widehat{\text{E}} \)Z), and t’Hooft-Veltman prescription (HV) [23,25,22]. Here, we choose NDR, the most commonly used scheme. At one loop level, we can write a general form of the renormalized continuum operators as

\[ \langle O_{SF}^{\text{Cont}(1)} \rangle = \sum_{S'F'} \langle \delta_{SS'} \delta_{F F'} \rangle + \frac{g^2}{(4\pi)^2} Z_{SF;S'F'}^{\text{Cont}} \langle O_{SF}^{\text{Cont}(0)} \rangle. \]  

(18)

where \( \langle \cdot \cdot \cdot \rangle \) represent amputated Green’s functions between external quark states. Here, the superscript \( i \) \((i \in \{0,1,2,\cdots\})\) represents the number of loops and

\[ Z_{SF;S'F'}^{\text{Cont}} = \delta_{SS'} \delta_{F F'} \left[ \Gamma_S \log \left( \frac{\mu}{\kappa} \right) + C_S^{\text{Cont}} + R_S \right]. \]  

(19)

The \( \mu \) is the renormalization scale and \( \kappa \) is the gluon mass, which regulates the infra-red. \( \Gamma_S \) is the anomalous dimension matrix:

\[ \Gamma_S = \frac{8}{3}(\sigma_S - 1) \]  

(20)

\[ \sigma_S \gamma_S = \frac{1}{4} \sum_{\mu, \nu} \gamma_{\mu} \gamma_{\nu} \gamma_S \gamma_{\mu} \gamma_{\nu}. \]  

(21)

Note that the vector and axial currents have vanishing anomalous dimensions. \( C_S^{\text{Cont}} \) is a finite constant term which depends on the regularization and renormalization scheme. For the NDR scheme [23],

\[ \sigma_S = (4,1,0,1,4), \]  

(22)

\[ C_S^{\text{Cont}} = \left( \frac{10}{3},0,\frac{2}{3},0,\frac{10}{3} \right) \text{ for } \gamma_S = (I, \gamma_{\mu}, \sigma_{\mu}, \gamma_{\mu 5}, \gamma_5) \text{ respectively.} \]  

(23)

The \( R_S \) terms contain logarithmic dependence on the external quark momentum due to anomalous dimension. They are universal (i.e. independent of the regularization and renormalization scheme).
On the lattice, there are eight Feynman diagrams (Figure 2 in Ref. [13]) contributing to bilinear operators at one loop level. Note that only half of the self-energy diagrams (Figure 2 (e), (f), (g) and (h) in Ref. [13]) contribute since the other half are absorbed in the wave-function renormalization of the external quark fields. The explicit analytic results of these diagrams are given in Appendix A of Ref. [14]. The renormalized lattice bilinear operators at one loop level can be expressed in terms of tree level lattice operators as

\begin{equation}
\langle O_{SF}^{Latt(1)} \rangle = \sum_{SF'} (\delta_{SS'} \delta_{FF'} + \frac{g^2}{(4\pi)^2} Z_{SF,SS'}^{Latt} (\delta_{FF'} \langle O_{SF'}^{Latt(0)} \rangle)
\end{equation}

where

\begin{equation}
Z_{SF,SS'}^{Latt} = -\delta_{SS'} \delta_{FF'} \Gamma S \log (ak) + C_{SF,SS'}^{Latt} + \delta_{SS'} \delta_{FF'} R_S .
\end{equation}

Here, \( \Gamma S \) is the same as in the continuum (see Eq. (20)) since the anomalous dimension at one loop is independent of the regularization and renormalization schemes. \( C_{SF,SS'}^{Latt} \) is the finite constant term unique to the lattice regularization scheme. Numerical values of \( C_{SF,SS'}^{Latt} \) are summarized in Tables 3 and 4. In Ref. [12,13], it is proven that the \( U_A(1) \) symmetry of the staggered fermion action insures that for any spin-flavor structure the two bilinear operators \( \chi_s (\bar{c} S \otimes \bar{\xi}_F) \chi_d \) and \( \chi_s (\bar{c} S S' \otimes \bar{\xi}_F) \chi_d \) are renormalized identically regardless of their distance. Hence, in Tables 3 and 4, we provide the \( C_{SF,SS'}^{Latt} \) values for only one of these two bilinear operators. Our values of the \( C_{SF,SS'}^{Latt} \) constants are related to the \( c_{ij} \) \((i = SF \text{ and } j = S'F')\) in Tables 6 and 7 of Ref. [22] as follows:

\begin{equation}
c_{ij} = \frac{1}{C_F} \left[ \delta_{SS'} \delta_{FF'} \left( C_{SF,SS'}^{Cont,DREZ} + \Gamma S \log (\pi) \right) - C_{SF,SS'}^{Latt} \right]
\end{equation}

\begin{equation}
C_{SF,SS'}^{Latt} = -C_F \left[ c_{ij} - \delta_{ij} \left( (\sigma_S - 1) (2 \log (\pi) + 1) + t_S \right) \right],
\end{equation}

where

\begin{equation}
C_F = \frac{N_C^2 - 1}{2N_C} = \frac{4}{3}.
\end{equation}

Here, note that \( c_{ij} \) is calculated for the DREZ scheme and

\begin{equation}
C_{SF,SS'}^{Cont,DREZ} = \left( \frac{14}{3}, 0, \frac{2}{3}, 0, \frac{14}{3} \right) \text{ for } \gamma_S = (I, \gamma_I, \sigma_{\mu\nu}, \gamma_{\mu5}, \gamma_5) \text{ respectively.}
\end{equation}

The values of \( t_S \) for DREZ are the same as in Ref. [22].

\begin{equation}
t_S = \left( \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2} \right) \text{ for } \gamma_S = (I, \gamma_I, \sigma_{\mu\nu}, \gamma_{\mu5}, \gamma_5) \text{ respectively.}
\end{equation}

The \( C_{SF,SS'}^{Latt} \) values in this paper are in agreement with those of Ref. [22] within the statistical uncertainty of finite integrals obtained using the Monte Carlo integration method. However, our results (and those of Ref. [22]) for the scalar bilinears \( [S \times X] \ (X \in \{S, V, T, A, P\}) \) and the mixing of the bilinear \( [V_{\mu} \times V_{\nu}] \) with \( [V_{\mu} \times V_{\mu}] \) are in disagreement with those in Ref. [14], while the rest of our results agree. We have presented the one-loop renormalized operators in terms of tree level operators both in the continuum and on the lattice. Now we need to relate the lattice and continuum operators. The matching condition is based upon the observation that at tree level, the continuum operators are, by construction, the same as lattice operators in the limit of the zero lattice spacing \( (a \to 0) \). In other words, at tree level as \( a \to 0 \), the lattice operators are related to the continuum operators as follows:

\begin{equation}
O_{SF}^{Latt(0)} = O_{SF}^{Cont(0)} + O(a).
\end{equation}

Here, \( O(a) \) represents terms of order \( a \) or higher, which are supposed to vanish as \( a \to 0 \). At one loop (and higher) order, the lattice and the continuum operators will differ. A carefully constructed mixture of lattice operators is needed to reproduce the desired continuum operator. From Eqs. (18), (24), (11), we can connect the lattice operators with the continuum operators at one loop level via
Under this rescaling, the bilinear operators transform as

\[ O_{SF}^{Cont(1)} = \sum_{S'F'} \left[ \delta_{SS'} \delta_{FF'} + \frac{g^2}{(4\pi)^2} \left( Z_{Cont}^{SF} ; S'F' - Z_{Cont}^{SF} ; S'F' \right) \right] O_{SF}^{Cont(1)} \]

\[ = \sum_{S'F'} \left[ \delta_{SS'} \delta_{FF'} + \frac{g^2}{(4\pi)^2} \delta_{SS'} \delta_{FF'} \Gamma S \ln(\mu a) + \frac{g^2}{(4\pi)^2} \left( \delta_{SS'} \delta_{FF'} C_{Cont}^{SF} - C_{Cont}^{SF} ; S'F' \right) \right] O_{SF}^{Cont(1)} \]  

(32)

This is the matching formula between the lattice and continuum bilinear operators.

Lepage, Mackenzie, and Parisi observed that the tadpole diagrams cause the large difference between the bare gauge coupling \( g_0(a) \) and the renormalized coupling \( g_{\text{MS}}(\mu = 1/a) \) [26,27]. They proposed a mean field method for removing the dominant effect of tadpole diagrams. This method of tadpole improvement has been applied to quark mass [28] and composite operator renormalization [22,12,14,13] extensively. Here, let us explain the tadpole improvement procedure for the bilinear operators. The first step in this procedure is the rescaling of the staggered fermion fields and gauge link fields [13,28].

\[ \chi \rightarrow \psi = \sqrt{u_0} \chi \]
\[ \nabla \chi \rightarrow \nabla \psi = \sqrt{u_0} \nabla \chi \]
\[ U_\mu \rightarrow \tilde{U}_\mu = \frac{U_\mu}{u_0} . \]  

(33)

A gauge invariant choice of the mean field scaling factor \( u_0 \) is [27,29]

\[ u_0 = \left[ \frac{1}{3} \text{Re}(\text{Tr}U_0) \right]^{1/4} = 1 - \frac{1}{12} g^2 + O(g^4) . \]  

(34)

Under this rescaling, the bilinear operators transform as

\[ O_{SF}^{\text{Latt}(1)} \rightarrow \tilde{O}_{SF}^{\text{Latt}(1)} = \frac{1}{N_f} \bar{\psi}(y_A)(\gamma S \otimes \xi F)_{AB} \psi(y_B) \tilde{U}_{AB} \]
\[ = u_0^{1-\Delta_{SF}} \cdot \frac{1}{N_f} \bar{\chi}(y_A)(\gamma S \otimes \xi F)_{AB} \chi(y_B) U_{AB} \]
\[ = u_0^{1-\Delta_{SF}} \cdot O_{SF}^{\text{Latt}(1)} . \]  

(35)

Here, \( \Delta_{SF} = \sum_\mu (S_\mu - F_\mu) \) corresponds to the distance between the quark and antiquark fields of the bilinear operator. The point is that the tadpole improved operators \( \tilde{O}_{SF}^{\text{Latt}(1)} \) are better representations of the continuum operators \( O_{SF}^{\text{Cont}(1)} \). Now we can rewrite Eqs. (24,25), the one loop relationship on the lattice, in terms of the tadpole improved operators.

\[ \langle \tilde{O}_{SF}^{\text{Latt}(1)} \rangle = \sum_{S'F'} (\delta_{SS'} \delta_{FF'} + \frac{\tilde{g}^2}{(4\pi)^2} Z_{\text{Latt}}^{SF} ; S'F') \cdot u_0^{1-\Delta_{SF}} \cdot \langle O_{SF}^{\text{Latt}(0)} \rangle \]
\[ = \sum_{S'F'} (\delta_{SS'} \delta_{FF'} + \frac{\tilde{g}^2}{(4\pi)^2} Z_{\text{Latt}}^{SF} ; S'F') \langle O_{SF}^{\text{Latt}(0)} \rangle , \]  

(36)

where

\[ \tilde{Z}_{SF}^{SF} ; S'F' \equiv -\delta_{SS'} \delta_{FF'} \Gamma S \log(\alpha \kappa) + \tilde{C}_{SF}^{SF} ; S'F' + \delta_{SS'} \delta_{FF'} R_S , \]
\[ \tilde{C}_{SF}^{SF} ; S'F' \equiv C_{SF}^{SF} ; S'F' - (1 - \Delta_{SF}) \frac{(4\pi)^2}{12} \delta_{SS'} \delta_{FF'} . \]  

(37)

(38)

Here, note that the gauge coupling constant is also tadpole improved: \( g^2 \rightarrow \tilde{g}^2 \). Regarding the tadpole improvement of gauge coupling constants, refer to Refs. [27,28,29] for details. Correspondingly, we can rewrite Eq. (32), the relationship between the continuum and lattice operators, as follows:
Let us consider a general form of the staggered four-fermion operators on the lattice:

$$\left[\chi(\gamma_S \otimes \xi_F)\chi\right]$$

There are $16^4 = 65536$ four-fermion operators. Hence, the mixing matrices are naively $65536 \times 65536$. However, we are interested in only a subset of these huge mixing matrices. In this paper, we focus on operators of phenomenological interest, especially relevant to the direct CP violation, $\epsilon'/\epsilon$.

On the lattice, gauge non-invariant four-fermion operators such as Landau-gauge operators mix with gauge non-invariant lower dimension bilinear operators. This requires additional non-perturbative subtractions and it is significantly more difficult to extract the divergent mixing coefficients in a completely non-perturbative way. For this reason, we are not interested in gauge non-invariant operators. Therefore, we study gauge invariant operators in this paper, since the gauge invariance protects operators from mixing with lower dimension gauge non-invariant operators [12].

Let us first classify the Feynman diagrams into two categories. One is the current-current insertion type and the other is the penguin insertion type [2] [12]. In the case of the penguin diagrams, the perturbative corrections and the matching formula for the gauge invariant operators are given at one loop in Ref. [12]. However, for the current-current diagrams, the perturbative matching formulae at one loop level are not available for the complete set of the gauge invariant staggered fermion operators of phenomenological interest ($\epsilon'/\epsilon$). Therefore, in this paper, we calculate perturbatively the current-current diagrams of these operators and obtain the corresponding matching formulae at one loop.

We begin by reviewing the renormalization of the four-fermion operators for the current-current insertion type in the continuum. The continuum operators of phenomenological interest can be expressed in terms of the practical basis introduced in Ref. [30].

$$O_1 = (\bar{\psi}_1 \gamma_\mu L \psi_2)(\bar{\psi}_3 \gamma_\mu L \psi_4)$$
$$O_2 = (\bar{\psi}_1 \gamma_\mu L \psi_2)(\bar{\psi}_3 \gamma_\mu R \psi_4)$$
$$O_3 = -2(\bar{\psi}_1 L \psi_2)(\bar{\psi}_3 R \psi_4)$$

(40)

where $L = (1 - \gamma_5)$ and $R = (1 + \gamma_5)$. There are two more operators included in the practical basis for completeness in Ref. [30].

$$O_4 = (\bar{\psi}_1 L \psi_2)(\bar{\psi}_3 L \psi_4)$$
$$O_5 = -\frac{1}{8} \left(1 + \frac{3}{4} \varepsilon\right) \sum_{\mu, \nu} (\bar{\psi}_1 \gamma_\mu \gamma_\nu L \psi_2)(\bar{\psi}_3 \gamma_\mu \gamma_\nu L \psi_4)$$

(41)

The odd parity parts of these operators are not important for the following discussion, but they are included for generality. As in the bilinear operators, we choose NDR as our continuum renormalization scheme. In this scheme, there are many ways to extend the definition of four-fermion operators to $n = 4 - 2\varepsilon$ dimensions, which corresponds to different choices of evanescent operators. Here, we adopt the convention of Ref. [30]. In this convention, the perturbative corrections of four-fermion operators are derived from those of bilinear operators by imposing the Fierz symmetry and by loosening the charge conjugation. Using this convention, we can express a general form of the renormalized continuum four-fermion operators as
Here, note that \( \hat{Z}_i^{\text{Cont}} = \delta_{ij} \hat{Z}_i \) for \( i = \{1, 2, 3\} \), and \( \hat{Z}_i^{\text{Cont}} \) is a 2 × 2 matrix defined as:

\[
\begin{align*}
\hat{Z}_1^{\text{Cont}} &= z_{V+A} \hat{C}_a + z_{V+A} \hat{C}_b - (z_{S+P} + 6) \hat{C}_c \\
\hat{Z}_2^{\text{Cont}} &= z_{V+A} \hat{C}_a + z_{S+P} \hat{C}_b - (z_{V+A} + 6) \hat{C}_c \\
\hat{Z}_3^{\text{Cont}} &= z_{S+P} \hat{C}_a + z_{V+A} \hat{C}_b - (z_{V+A} + 6) \hat{C}_c
\end{align*}
\]

The \( \vec{O}_4 \) and \( \vec{O}_5 \) operators mix with each other.

\[
\begin{align*}
\hat{Z}_{44}^{\text{Cont}} &= z_{S+P} \hat{C}_a + 2z_T \hat{C}_b - \frac{1}{2}(z_{S+P} + 2z_T + 4) \hat{C}_c \\
\hat{Z}_{45}^{\text{Cont}} &= (-1) \hat{C}_a + \frac{1}{2}(2z_T - z_{S+P}) \hat{C}_b - \frac{1}{2}(z_{S+P} - 2z_T + 2) \hat{C}_c \\
\hat{Z}_{54}^{\text{Cont}} &= \frac{1}{2}(2z_T - z_{S+P}) \hat{C}_a + (-1) \hat{C}_b - \frac{1}{2}(z_{S+P} - 2z_T + 2) \hat{C}_c \\
\hat{Z}_{55}^{\text{Cont}} &= (2z_T) \hat{C}_a + z_{S+P} \hat{C}_b - \frac{1}{2}(z_{S+P} + 2z_T + 4) \hat{C}_c
\end{align*}
\]

Here, the color mixing matrices \( \hat{C} \) are

\[
\hat{C}_a = \frac{1}{6} \begin{pmatrix} -1 & 3 \\ 0 & 8 \end{pmatrix}, \quad \hat{C}_b = \frac{1}{6} \begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix}, \quad \hat{C}_c = \frac{1}{6} \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}
\]

The renormalization coefficients \( z_{V+A}, z_{S+P} \) and \( z_T \) can be expressed in terms of the bilinear corrections given in Eq. (11) as follows:

\[
\begin{align*}
z_{V+A} &\equiv z_V + z_A \\
z_{S+P} &\equiv z_S + z_P \\
z_X &\equiv \frac{1}{C_F} \left[ \Gamma_X \log(\frac{\mu}{\Lambda}) + C_X^{\text{Cont}} + R_X \right]
\end{align*}
\]

Here, \( X \in \{S, V, T, A, P\} \) represents the spin structure of bilinear operators and \( \Gamma_X, C_X^{\text{Cont}} \) are given in Eqs. (22) and (23) respectively. Note that \( R_X \) terms will cancel out when we match between the lattice and continuum operators. For the NDR and DRE \( \xi \) schemes, \( z_V = 0 \) and \( z_A = 0 \), and so \( z_{V+A} = 0 \). This simplifies the \( \hat{Z}_i^{\text{Cont}} \) further:

\[
\begin{align*}
\hat{Z}_1^{\text{Cont}} &= -(z_{S+P} + 6) \hat{C}_c \\
\hat{Z}_2^{\text{Cont}} &= z_{S+P} \hat{C}_b - 6 \hat{C}_c \\
\hat{Z}_3^{\text{Cont}} &= z_{S+P} \hat{C}_a - 6 \hat{C}_c.
\end{align*}
\]

As a summary, we present the results of the renormalization of the continuum four-fermion operators in the practical basis as follows:

\[
\langle \vec{O}_i^{\text{Cont}(1)} \rangle = \langle \vec{O}_i^{\text{Cont}(0)} \rangle + \frac{g^2}{(4\pi)^2} \left[ \hat{\Gamma}_{ij} \log(\frac{\mu}{\Lambda}) + \hat{C}_{ij}^{\text{Cont}} + \hat{R}_{ij} \right] \langle \vec{O}_j^{\text{Cont}(0)} \rangle
\]

Here, note that \( \hat{\Gamma}_{ij} = \delta_{ij} \hat{\Gamma} \) and \( \hat{C}_{ij}^{\text{Cont}} = \delta_{ij} \hat{C}_i^{\text{Cont}} \) for \( i \in \{1, 2, 3\} \). The anomalous dimension matrices are

\[
\begin{align*}
\hat{\Gamma}_1 &= \begin{pmatrix} +2 & -6 \\ -6 & +2 \end{pmatrix} \\
\hat{\Gamma}_2 &= \begin{pmatrix} +16 & 0 \\ +6 & -2 \end{pmatrix} \\
\hat{\Gamma}_3 &= \begin{pmatrix} -2 & +6 \\ 0 & +16 \end{pmatrix}
\end{align*}
\]
\[
\begin{aligned}
\hat{\Gamma}_{44} &= +\frac{4}{3} \begin{pmatrix} -5 & +3 \\ -3 & +13 \end{pmatrix} \\
\hat{\Gamma}_{45} &= -\frac{4}{3} \begin{pmatrix} +7 & +3 \\ +6 & -2 \end{pmatrix} \\
\hat{\Gamma}_{54} &= -\frac{4}{3} \begin{pmatrix} -2 & +6 \\ +3 & +7 \end{pmatrix} \\
\hat{\Gamma}_{55} &= +\frac{4}{3} \begin{pmatrix} +13 & -3 \\ +3 & -5 \end{pmatrix} .
\end{aligned}
\]

These are universal and independent of the renormalization scheme. For the NDR scheme, the finite constant matrices are

\[
\begin{aligned}
\hat{C}_{1}^{\text{Cont}} &= \hat{C}_{V+A}^{\text{Cont}} = \frac{11}{12} \hat{\Gamma}_{1} = \begin{pmatrix} +11/6 & -11/2 \\ -11/2 & +11/6 \end{pmatrix} \\
\hat{C}_{2}^{\text{Cont}} &= \hat{C}_{V-A}^{\text{Cont}} = \begin{pmatrix} +23/3 & -3 \\ -1/2 & +1/6 \end{pmatrix} \\
\hat{C}_{3}^{\text{Cont}} &= \hat{C}_{[P-S]}^{\text{Cont}} = \begin{pmatrix} +1/6 & -1/2 \\ -3 & +23/3 \end{pmatrix} \\
\hat{C}_{44}^{\text{Cont}} &= \frac{2}{3} \begin{pmatrix} +2 & 0 \\ -3 & +11 \end{pmatrix} \\
\hat{C}_{45}^{\text{Cont}} &= -\frac{1}{2} \begin{pmatrix} +4 & +4 \\ +5 & +1 \end{pmatrix} \\
\hat{C}_{54}^{\text{Cont}} &= -\frac{1}{2} \begin{pmatrix} -1 & +5 \\ +4 & +4 \end{pmatrix} \\
\hat{C}_{55}^{\text{Cont}} &= +\frac{2}{3} \begin{pmatrix} +11 & -3 \\ 0 & +2 \end{pmatrix} .
\end{aligned}
\]

For further details of other continuum renormalization schemes, refer to Ref. [30] and references in it.

This completes our review of the renormalization of the four-fermion operators of the current-current insertion type in the continuum. We now describe renormalization of staggered four-fermion operators of the same type on the lattice. For the purpose of matching, it is convenient to work with the practical basis as in the continuum. Hence, we choose the same practical basis on the lattice. Using the compact notation introduced in Section II in the two spin trace formalism [13], we can express the practical basis as follows:

\[
\begin{aligned}
\mathcal{O}_{1}^{\text{Latt}} &= [V_{\mu} \times P] [V_{\mu} \times P] + [A_{\mu} \times P] [A_{\mu} \times P] \\
\mathcal{O}_{2}^{\text{Latt}} &= [V_{\mu} \times P] [V_{\mu} \times P] - [A_{\mu} \times P] [A_{\mu} \times P] \\
\mathcal{O}_{3}^{\text{Latt}} &= -2 \left( [S \times P] [S \times P] - [P \times P] [P \times P] \right) \\
\mathcal{O}_{4}^{\text{Latt}} &= [S \times P] [S \times P] + [P \times P] [P \times P] \\
\mathcal{O}_{5}^{\text{Latt}} &= -\frac{1}{2} \left( [S \times P] [S \times P] + [P \times P] [P \times P] - \sum_{\mu < \nu} [T_{\mu \nu} \times P] [T_{\mu \nu} \times P] \right) .
\end{aligned}
\]

Here, note that we choose the same flavor structure as the pseudo-Goldstone mode of the conserved $U(1)_{A}$ symmetry.

The $SU(4)$ flavor symmetry of the staggered fermion action becomes exactly conserved only in the limit of $a = 0$. On the finite lattice of $a \neq 0$, only the discrete $SW_{4,\text{diag}}$ symmetry and $U(1)_{V} \otimes U(1)_{A}$ symmetry are conserved exactly, which rotates the spin and flavor simultaneously [1]. In calculating the perturbative corrections to the operators of Eq. (54), the operators with different flavors mix with one another, since the $SU(4)$ flavor symmetry is not conserved at $a \neq 0$. Therefore, we can express a general form of the renormalized staggered four-fermion operators on the lattice as follows:
\[ \mathcal{O}_{i,\text{Latt}}^{(1)} = \left( \begin{array}{c} \mathcal{O}_{i,\text{Latt}}^{(1)} \\ \mathcal{O}_{i,\text{II}}^{(1)} \end{array} \right) \]

\[ \langle \mathcal{O}_{i,\text{Latt}}^{(1)} \rangle = \langle \mathcal{O}_{i,\text{Latt}}^{(0)} \rangle + \frac{g^2}{(4\pi)^2} \left[ -\tilde{F}_{ij} \log(ak) + \tilde{C}_{ij}^{\text{Latt}} + \tilde{R}_{ij} \right] \langle \mathcal{O}_{j,\text{Latt}}^{(0)} \rangle \]

where \( \tilde{C}_{ij}^{\text{Latt}} \) may include operators with spin or flavor structure not included in the practical basis of Eq. (74). The anomalous dimension matrices \( \tilde{F}_{ij} \) are scheme-independent and the same as in Eq. (52). The \( \tilde{R}_{ij} \) contains the external momentum dependence and will cancel out when we perform the matching between the continuum and lattice operators. The scheme-dependent finite constant matrix terms \( \tilde{C}_{ij}^{\text{Latt}} \) are given in Tables III–XXII. The analytic formula to calculate these finite constant terms are provided in Ref. [14]. Regarding Eq. (C2) of Ref. [14], we have obtained a slightly different analytical result for the corresponding

\[ G^{3(e)} = \frac{g^2}{(4\pi)^2} \Gamma_{ab}^{\alpha} \delta_{ab} \left[ -2 Z_{0000} (\gamma^S \otimes \xi_F) CD (\gamma^{S'} \otimes \xi_F') CD' \right. \]

\[ + \frac{1}{4} Z_{0000} \sum_{\mu} \left[ (-1)^{(S+F)_\mu} + (-1)^{(S'+F')_{\mu}} \right] (\gamma_{\mu 5S} \otimes \xi_{\mu 5F}) CD (\gamma_{5S} \otimes \xi_{5F'}) CD' \]

\[ + (1 - \alpha) \frac{1}{2} Z_{0000} (\gamma^S \otimes \xi_F) CD (\gamma^S \otimes \xi_F') CD' \]

\[ - (1 - \alpha) \frac{1}{16} Z_{0000} \sum_{\mu} \left[ (-1)^{(S+F)_\mu} + (-1)^{(S'+F')_{\mu}} \right] (\gamma_{\mu 5S} \otimes \xi_{\mu 5F}) CD (\gamma_{5S} \otimes \xi_{5F'}) CD' \]

\[ + (1 - \alpha) \frac{1}{2} \sum_{\mu \neq \nu, M} T_{\mu\nu} \left[ - \left[ (-1)^{M \cdot (S+F)} + (-1)^{\tilde{M} \cdot (S'+F')} \right] (\gamma^{MS} \otimes \xi_{MF}) CD (\gamma^{MS'} \otimes \xi_{MF'}) CD' \right. \]

\[ + 2 \left[ (-1)^{(S+F)_{\mu} + \tilde{M} \cdot (S+F)} + (-1)^{(S'+F')_{\mu} + \tilde{M} \cdot (S'+F')} \right] \]

\[ \times (\gamma_{\mu 5MS} \otimes \xi_{\mu 5MF}) CD (\gamma_{5MS} \otimes \xi_{5MF'}) CD' \]

\[ - \left[ (-1)^{(S+F)_{\mu} + (S+F)_{\mu} + \tilde{M} \cdot (S+F)} + (-1)^{(S'+F')_{\mu} + (S'+F')_{\mu} + \tilde{M} \cdot (S'+F')} \right] \]

\[ \times (\gamma_{\mu MS} \otimes \xi_{\mu MF}) CD (\gamma_{\mu MS} \otimes \xi_{\mu MF}) CD' \left. \right] \]

The difference is localized only to the gauge-dependent terms proportional to \((1 - \alpha)\), the coefficient of the covariant gauge fixing. Using Eq. (77), we confirmed the gauge invariance by checking that the summation of all the gauge-dependent terms \((i.e. \alpha (1 - \alpha))\) of all the one-loop Feynman diagrams vanishes for each spin and flavor structure of the gauge invariant four-fermion operators.

We have checked the results of finite constant terms \( \tilde{C}_{ij}^{\text{Latt}} \) in three independent ways. First, we checked the conserved \( U(1)_A \) symmetry. For example, the following operators are connected under the \( U(1)_A \) transformation [13]:

\[ [V_\mu \times P][V_\mu \times P] \xrightarrow{U(1)_A} [V_\mu \times P][A_\mu \times S] \]

\[ [A_\mu \times P][A_\mu \times P] \xrightarrow{U(1)_A} [A_\mu \times P][V_\mu \times S] \]

\[ [P \times P][P \times P] \xrightarrow{U(1)_A} [P \times P][S \times S] \]

\[ [S \times P][S \times P] \xrightarrow{U(1)_A} [S \times P][P \times S] \]

Hence, we confirmed that all the finite renormalization constants of the operators in the right-hand side of Eq. (58) can be reproduced from those in the left-hand side, using a simple \( U(1)_A \) transformation.
we checked the gauge invariance. In other words, we confirmed that the summation of the gauge dependent terms proportional to $(1 - \alpha)$ vanishes for individual gauge invariant operators in the practical basis. Third, we checked the Fierz identities of Eq. (A4). Any color one trace operator can be expressed as a linear combination of color two trace operators, using Fierz transformation, which holds valid under renormalization on the lattice (but not in the NDR scheme in the continuum). We explain the details of these lattice Fierz identities in Appendix A and B.

The values of $\hat{C}_{ij}^{\text{Latt}}$ presented in Tables XI–XXII are new. The values of $\hat{C}_{ij}^{\text{Latt}}$ presented in Tables XI–XXII are new.

We have presented renormalized operators in the continuum and on the lattice. We now describe the matching formula. As in the case of bilinears, the tree level lattice operators are equal to the tree level continuum operators in the limit of $a = 0$.

$$\tilde{O}^{\text{Cont}(0)} = \tilde{O}^{\text{Cont}(0)} + O(a)$$ (59)

Using Eqs. (51), (56) and (59), we can obtain the matching relationship between continuum operators and lattice operators at one loop level.

$$\tilde{O}_i^{\text{Cont}(1)} = \sum_j \left[ \delta_{ij} + \frac{g^2}{(4\pi)^2} \tilde{\Gamma}_{ij} \log(\mu a) + \frac{g^2}{(4\pi)^2} \left( \hat{C}_{ij}^{\text{Cont}} - \hat{C}_{ij}^{\text{Latt}} \right) \right] \tilde{O}_j^{\text{Latt}(1)}$$ (60)

This is the one-loop matching relation between the lattice and the continuum four-fermion operators.

We now turn to the tadpole improvement of the staggered four-fermion operators. First, let us consider a general color two trace operator:

$$O_{II} = \frac{1}{N_f^2} \left[ \chi(y_A)(\gamma_S \otimes \xi_F)_{AB} U_{AB} \chi(y_B) \right] \left[ \chi(y_C)(\gamma_S \otimes \xi_F')_{CD} U_{CD} \chi(y_D) \right]$$ (61)

Under the rescaling transformation of Eq. (33), this operator transforms as

$$O_{II} \rightarrow \tilde{O}_{II} = u_0^{1-\Delta_{SF}} u_0^{1-\Delta_{SF'}} O_{II}$$ (62)

In case of color one trace operators, the distance between quark and anti-quark fields of the bilinears does not define the length of the gauge link. Therefore it is convenient to express the color one trace operator as a linear combination of color two trace operators, using Fierz transformations.

$$O_i^j = \sum_j H_{ij} O_{II}^j$$ (63)

where $H_{ij}$ is the Fierz transformation matrix. Under the rescaling transformation of Eq. (33), this operator transforms as

$$O_i^j \rightarrow \tilde{O}_i^j = \sum_j H_{ij} \tilde{O}_j^j = \sum_j H_{ij} u_0^{1-\Delta_{SF}} u_0^{1-\Delta_{SF'}} O_{II}^j$$ (64)

where $\Delta_{SF}$ and $\Delta_{SF'}$ correspond to distances between quark and anti-quark fields of the two bilinears in $O_{II}^j$ respectively. Using perturbation, we can rewrite $\tilde{O}_i^j$ as

$$\tilde{O}_i^j = \sum_j H_{ij} O_{II}^j - \frac{g^2}{(4\pi)^2} \frac{(4\pi)^2}{12} \sum_j H_{ij} \left[ 2 - \Delta_{SF}^j - \Delta_{SF'}^j \right] O_{II}^j$$

$$= O_i^j - \frac{g^2}{(4\pi)^2} \frac{(4\pi)^2}{12} \sum_{j,k} H_{ij} H_{jk} \left[ 2 - \Delta_{SF}^j - \Delta_{SF'}^j \right] O_{II}^k$$

$$= O_i^j + \frac{g^2}{(4\pi)^2} \frac{(4\pi)^2}{12} \sum_k M_{ik} O_{II}^k$$ (65)
where

$$M_{ik} \equiv -\frac{(4\pi)^2}{12} \sum_j H_{ij} H_{jk} \left[ 2 - \Delta_{SF}^{i} - \Delta_{SF'}^{i} \right]$$  \hspace{1cm} (66)$$

Here, note that the tadpole improvement for color two trace operators are diagonal ($\propto \delta_{ij}$), while the tadpole improvement for color one trace operators allows mixing with off-diagonal operators. As a summary, let us express the tadpole-improved four-fermion operators in terms of unimproved operators as follows:

$$\tilde{O}_i^{\text{Latt}} = \left( \begin{array}{c} \tilde{\tilde{O}}_{i,1}^{\text{Latt}} \\ \tilde{\tilde{O}}_{i,II}^{\text{Latt}} \end{array} \right) = \tilde{O}_i^{\text{Latt}} + \frac{g^2}{(4\pi)^2} \sum_j \tilde{T}_{ij} \cdot \tilde{O}_j^{\text{Latt}}$$  \hspace{1cm} (67)$$

where

$$\tilde{T}_{ij} = \left( \begin{array}{cc} M_{ij} & 0 \\ 0 & N_{ij} \end{array} \right)$$  \hspace{1cm} (68)$$
$$N_{ij} = -\delta_{ij} \frac{(4\pi)^2}{12} \left[ 2 - \Delta_{SF}^{i} - \Delta_{SF'}^{i} \right]$$  \hspace{1cm} (69)$$

The values of $M_{ij}$ and $N_{ij}$ are tabulated in Tables III–XXII. Using Eq. (67), one can rewrite the one-loop renormalization relationship of Eq. (56) on the lattice in terms of tadpole-improved operators.

$$\langle \tilde{O}_i^{\text{Latt}}(1) \rangle \equiv \langle \tilde{\tilde{O}}_{i,1}^{\text{Latt}(0)} \rangle + \frac{g^2}{(4\pi)^2} \left[ -\tilde{\Gamma}_{ij} \log(ak) + \tilde{C}_{ij}^{Latt} + \tilde{T}_{ij} + \tilde{R}_{ij} \right] \langle \tilde{\tilde{O}}_{j}^{\text{Latt}(0)} \rangle$$  \hspace{1cm} (70)$$

Here, note that the gauge coupling is also tadpole-improved. Correspondingly, we need to rewrite the one-loop matching formula of Eq. (50) in terms of tadpole-improved operators as follows:

$$\tilde{O}_i^{\text{Cont}(1)} = \sum_j \left[ \delta_{ij} + \frac{g^2}{(4\pi)^2} \tilde{\Gamma}_{ij} \log(ak) + \frac{g^2}{(4\pi)^2} \left( \tilde{C}_{ij}^{\text{Cont}} - \tilde{C}_{ij}^{Latt} - \tilde{T}_{ij} \right) \right] \tilde{O}_j^{\text{Latt}(1)}$$  \hspace{1cm} (71)$$

This is our final result for the tadpole-improved matching relationship of four-fermion operators.

IV. CONCLUSION

We have studied the one-loop perturbative matching formula between the lattice and continuum bilinear and four-fermion operators that are used in the numerical calculation of $\epsilon'/\epsilon$. Our main contribution is that the results presented in this paper, combined with the existing results for the penguin diagrams, make it possible to match a lattice calculation of $K \to \pi\pi$ decays to the continuum NDR results with all the $O(\alpha)$ corrections included.

We have studied all five four-fermion operators in the practical basis. The first three operators ($O_1$, $O_2$, and $O_3$) are relevant to the numerical evaluation of $\epsilon'/\epsilon$. These operators mix with a large number of off-diagonal operators. However, we found that the coefficients of the off-diagonal mixing are small compared with those of the diagonal mixing. After tadpole-improvement, the diagonal mixing terms are also under control except for the $([P \times P][P \times P])_{11}$ term, which receives a large correction at one loop. Various attempts to reduce this large perturbative correction of the $([P \times P][P \times P])_{11}$ term by improving the action and operators are under development [31].

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APPENDIX A: FIERZ TRANSFORMATION (I)

First, we define the notation for Fierz transformation:

\[
M_S = I \otimes I, \quad M_V = \sum_\mu \gamma_\mu \otimes \gamma_\mu, \quad M_T = \sum_{\mu < \nu} \sigma_{\mu\nu} \otimes \sigma_{\mu\nu}, \\
M_A = \sum_\mu \gamma_\mu \otimes \gamma_\mu, \quad M_P = \gamma_5 \otimes \gamma_5
\]  

(A1)

where \(\otimes\) represents the direct product and \(\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]\) with signature defined by \(\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\). We also introduce the following compact notation for spin indices:

\[
(\mathcal{M}_V)_{\alpha\beta;\alpha'\beta'} = \sum_\mu (\gamma_\mu)_{\alpha\beta} \otimes (\gamma_\mu)_{\alpha'\beta'}
\]

(A2)

The same notation also applies to the flavor structure \(\xi_F \otimes \xi_F\) by switching \(\gamma\) to \(\xi\). Now we can express the Fierz transformation of \(\mathcal{M}_X\) for \(X \in \{S, V, T, A, P\}\) as follows:

\[
\mathcal{M} = \begin{pmatrix} M_S & M_V & M_T & M_A & M_P \end{pmatrix}, \quad \mathcal{M}^\dagger_{\alpha\beta';\alpha'\beta} = \mathcal{F}\mathcal{M}^\dagger_{\alpha\beta';\alpha'\beta}, \quad (\mathcal{M}_i)_{\alpha\beta';\alpha'\beta} = [F_{ij}] (\mathcal{M}_j)_{\alpha\beta';\alpha'\beta}
\]

(A3)

where

\[
\mathcal{F} = [F_{ij}] = \frac{1}{4} \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\
4 & -2 & 0 & -2 & 4 \\
-6 & 0 & -2 & 0 & -6 \\
-4 & -2 & 0 & -2 & 4 \\
1 & -1 & -1 & 1 & 1 \end{pmatrix}
\]

(A4)

The spin and flavor matrices of the staggered four-fermion operators are Fierz-transformed separately using the same relationship of Eq. (A3). Let us define \(\Gamma_X\) as the spin part and \(\Xi_X\) as the flavor part. For example,

\[
[\Gamma_V \otimes \Xi_P] \equiv [V_\mu \times P][V_\mu \times P] \\
[\Gamma_A \otimes \Xi_P] \equiv [A_\mu \times P][A_\mu \times P] \\
[\Gamma_S \otimes \Xi_P] \equiv [S \times P][S \times P]
\]

Using this notation, we can express a general form of the Fierz transformation as:

\[
[\Gamma_i \otimes \Xi_m]_{IJ}^{f_1 f_2 f_3 f_4} = - F_{ij} F_{mn} [\Gamma_j \otimes \Xi_n]_{II}^{f_1 f_2 f_3 f_4} \\
[\Gamma_i \otimes \Xi_m]_{IJ}^{f_1 f_2 f_3 f_4} = - F_{ij} F_{mn} [\Gamma_j \otimes \Xi_n]_{II}^{f_1 f_2 f_3 f_4}
\]

(A5)

where \(I, II\) represent color one trace form and color two trace form respectively. Note that the negative sign on the right-hand side of Eq. (A5) comes from the anti-commuting characteristics of the fermion fields. Using these relationships of Eqs. (A3) and (A4), we obtain the Fierz transformation of the practical basis defined in Eq. (A4).
\[-\frac{1}{8} ([\Gamma_S + \Gamma_P - \Gamma_T] \otimes (\Xi_S - \Xi_V - \Xi_T + 3A + 3P))_{[f_1 f_2]}^{f_3 f_4}
\]
\[\frac{1}{2} ([\Gamma_S + \Gamma_P - \Gamma_T] \otimes \Xi_P)_{[f_1 f_2]}^{f_3 f_4} = \frac{1}{4} ([\Gamma_S + \Gamma_P] \otimes (\Xi_S - 3\Xi_V - 3\Xi_T + 3A + 3P))_{[f_1 f_2]}^{f_3 f_4}
\]

Here, note that these Fierz identities hold valid under a transformation of switching color-trace indices $(I \leftrightarrow II)$ between the left-hand and right-hand sides. Similarly, we can also derive another Fierz identity used to calculate $B_K$ in the one spin trace formalism [13]:

\[([\Gamma_V + \Gamma_A] \otimes (\Xi_P + \Xi_S))_{[f_1 f_2]}^{f_3 f_4} = \frac{1}{2} ([\Gamma_V + \Gamma_A] \times (\Xi_S - 3\Xi_T + 3A + 3P))_{[f_1 f_2]}^{f_3 f_4}
\]  

**APPENDIX B: FIERZ TRANSFORMATION (II)**

As shown in Eq. (A6), color one trace operators can be expressed as a linear combination of color two trace operators. These Fierz identities hold under renormalization on the lattice. The finite integrals for color one trace operators are completely different from those for color two trace operators. Therefore, it will provide a non-trivial check of our calculation if the results in this paper satisfies the Fierz identities of Eq. (A6).

Here, we describe how to check the Fierz identities step by step. First, we calculate the complete set of one-loop perturbative corrections for both color one trace and color two trace operators in Eq. (A6). Usually, the operators in the right-hand side of Eq. (A6) mix with hundreds of operators at one-loop level. In order to compare these results with the operators in the left-hand side, one needs to Fierz-transform the mixing operators in the right-hand side into the flavor combination $([f_1 f_4]; [f_3 f_2])$ as in the left-hand side. For this purpose of large scale Fierz transformation, there exists a handy formula [14]. Using the following group orthogonality relationship for the gamma matrices:

\[
\frac{1}{4} \sum_A (\gamma_A)_{\alpha\beta} \otimes (\gamma_A)^\dagger_{\rho\lambda} = \delta_{\alpha\lambda} \delta_{\beta\rho},
\]

one can derive a handy tool for the large scale Fierz transformation:

\[
(\gamma_S \otimes \xi_F)_{AB} (\gamma_{S'} \otimes \xi_{F'})_{A'B'} = \frac{1}{16} \sum_{KL} (\gamma_S)_{K} \otimes (\xi_L \xi_F)_{A'B'} (\gamma_S)_{K} \otimes (\xi_{L'} \xi_{F'})_{A'B'}.
\]

Using this tool, we confirmed that the lattice results in this paper satisfy the Fierz identities of Eq. (A6).

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| $O_{SF}$ | $\Gamma_S$ | $C_{\text{clatt}}^{SF;SF'}$ |
|------------|-----------|------------------|
| $1 \otimes 1$ | 8 | $+55.628(1)$ |
| $1 \otimes \xi_\mu$ | 8 | $+14.855(1)$ |
| $1 \otimes \xi_\mu\nu$ | 8 | $-10.578(2)$ |
| $1 \otimes \xi_\mu5$ | 8 | $-29.965(3)$ |
| $1 \otimes \xi_5$ | 8 | $-47.804(2)$ |
| $\gamma_\mu \otimes 1$ | 0 | $0.000(0)$ |
| $\gamma_\mu \otimes \xi_\mu$ | 0 | $+19.705(1)$ |
| $\gamma_\mu \otimes \xi_\nu$ | 0 | $-13.387(2)$ |
| $\gamma_\mu \otimes \xi_\mu\nu$ | 0 | $+4.527(1)$ |
| $\gamma_\mu \otimes \xi_\mu\rho$ | 0 | $-29.654(4)$ |
| $\gamma_\mu \otimes \xi_\mu5$ | 0 | $-45.998(2)$ |
| $\gamma_\mu \otimes \xi_5$ | 0 | $-13.411(3)$ |
| $\gamma_\mu \otimes \xi_5$ | 0 | $-30.010(3)$ |
| $\gamma_\mu \otimes 1$ | $-8/3$ | $-14.620(3)$ |
| $\gamma_\mu \otimes \xi_\mu$ | $-8/3$ | $-0.425(1)$ |
| $\gamma_\mu \otimes \xi_\rho$ | $-8/3$ | $-29.670(4)$ |
| $\gamma_\mu \otimes \xi_\mu\nu$ | $-8/3$ | $+7.731(1)$ |
| $\gamma_\mu \otimes \xi_\mu\rho$ | $-8/3$ | $-14.199(2)$ |
| $\gamma_\mu \otimes \xi_\mu\rho$ | $-8/3$ | $-45.396(2)$ |

**TABLE I.** Renormalization constants of bilinear operators at one loop. $O_{SF}$ represents bilinear operators. $\Gamma_S$ is the anomalous dimension defined in Eq. (20). $C_{\text{clatt}}^{SF;SF'}$ is the finite constant term defined in Eq. (25) for the diagonal case: $S' = S$ and $F' = F$.

| $O_{SF}$ | $O_{SF'}$ | $C_{\text{clatt}}^{SF;SF'}$ |
|------------|------------|------------------|
| $\gamma_\mu \otimes \xi_\nu$ | $\gamma_\mu \otimes \xi_\rho$ | $-4.055(0)$ |
| $\gamma_\mu \otimes \xi_\mu$ | $\gamma_\mu \otimes \xi_5$ | $+0.862(0)$ |
| $\gamma_\mu \otimes \xi_\mu\nu$ | $\gamma_\mu \otimes \xi_\mu\rho$ | $+1.982(0)$ |
| $\gamma_\mu \otimes \xi_\mu\rho$ | $\gamma_\mu \otimes \xi_\mu\nu$ | $-1.982(0)$ |
| $\gamma_\mu \otimes \xi_\rho$ | $\gamma_\mu \otimes \xi_\mu$ | $+0.902(0)$ |
| $\gamma_\mu \otimes \xi_5$ | $\gamma_\mu \otimes \xi_\rho$ | $+0.902(0)$ |

**TABLE II.** Renormalization constants of bilinear operators at one loop. $O_{SF}$ represents bilinear operators. $O_{SF'}$ corresponds to the mixing bilinear operators in Eq. (24). $C_{\text{clatt}}^{SF;SF'}$ is the finite constant term defined in Eq. (25) for the off-diagonal case: $S' \neq S$ and $F' \neq F$. 
| $O^{\text{lat}(4)}_{ij}$ | color trace | $\Gamma_{ij}$ | $C^{\text{lat}}_{ij}$ | $M_{ij}$ |
|-----------------|-------------|-------------|-------------------|---------|
| $[S \times V_\mu](S \times V_\mu]$ | I            | 0           | $-14.395(3)$      | $+c_{T_1}$ |
| $[S \times V_\mu](S \times V_\mu]$ | II           | 0           | $-3.992(1)$       | 0       |
| $[S \times V_\mu](S \times V_\mu]$ | I            | 0           | $-1.110(0)$       | 0       |
| $[S \times A_\mu](S \times A_\mu]$ | I            | 0           | $-0.313(0)$       | 0       |
| $[S \times A_\mu](S \times A_\mu]$ | II           | 0           | $-1.219(0)$       | 0       |
| $[S \times A_\mu](S \times A_\mu]$ | I            | 0           | $-0.110(0)$       | 0       |
| $V_\mu \times S[V_\mu \times S]$ | I            | 0           | $+0.342(0)$       | 0       |
| $V_\mu \times S[V_\mu \times S]$ | II           | 0           | $-0.539(0)$       | 0       |
| $V_\mu \times T_\mu[V_\mu \times T_\mu]$ | I            | 0           | $+1.397(0)$       | 0       |
| $V_\mu \times T_\mu[V_\mu \times T_\mu]$ | II           | 0           | $-2.170(0)$       | 0       |
| $V_\mu \times T_\mu[V_\mu \times T_\mu]$ | I            | 0           | $+0.565(0)$       | 0       |
| $V_\mu \times T_\mu[V_\mu \times T_\mu]$ | II           | 0           | $-0.499(0)$       | 0       |
| $V_\mu \times T_\mu[V_\mu \times T_\mu]$ | I            | 0           | $-0.051(0)$       | 0       |
| $V_\mu \times T_\mu[V_\mu \times T_\mu]$ | II           | 0           | $+0.154(0)$       | 0       |
| $V_\mu \times T_\mu[V_\mu \times T_\mu]$ | I            | 0           | $-0.051(0)$       | 0       |
| $V_\mu \times T_\mu[V_\mu \times T_\mu]$ | II           | 0           | $+0.154(0)$       | 0       |
| $V_\mu \times T_\mu[V_\mu \times T_\mu]$ | I            | 0           | $+1.205(0)$       | 0       |
| $V_\mu \times T_\mu[V_\mu \times T_\mu]$ | II           | 0           | $-1.970(0)$       | 0       |
| $V_\mu \times T_\mu[V_\mu \times T_\mu]$ | I            | 0           | $+0.592(0)$       | 0       |
| $V_\mu \times T_\mu[V_\mu \times T_\mu]$ | II           | 0           | $-0.397(0)$       | 0       |
| $V_\mu \times P[V_\mu \times P]$ | I            | $+2$        | $-24.167(2)$      | $+2c_{T_1}$ |
| $V_\mu \times P[V_\mu \times P]$ | II           | $-6$        | $-7.028(3)$       | 0       |
| $T_{\mu\nu}[V_\mu \times V_\mu]$ | I            | 0           | $+18.270(2)$      | $-c_{T_1}$ |
| $T_{\mu\nu}[V_\mu \times V_\mu]$ | II           | 0           | $-4.544(1)$       | 0       |
| $T_{\mu\nu}[V_\mu \times V_\mu]$ | I            | 0           | $-1.952(0)$       | 0       |
| $T_{\mu\nu}[V_\mu \times V_\mu]$ | II           | 0           | $+2.527(0)$       | 0       |
| $T_{\mu\nu}[V_\mu \times V_\mu]$ | I            | 0           | $-0.223(0)$       | 0       |
| $T_{\mu\nu}[V_\mu \times V_\mu]$ | II           | 0           | $+0.669(0)$       | 0       |
| $T_{\mu\nu}[V_\mu \times V_\mu]$ | I            | 0           | $-0.223(0)$       | 0       |
| $T_{\mu\nu}[V_\mu \times V_\mu]$ | II           | 0           | $+0.669(0)$       | 0       |
| $T_{\mu\nu}[V_\mu \times V_\mu]$ | I            | 0           | $-15.294(2)$      | $+c_{T_1}$ |
| $T_{\mu\nu}[V_\mu \times V_\mu]$ | II           | 0           | $-4.387(1)$       | 0       |
| $T_{\mu\nu}[V_\mu \times V_\mu]$ | I            | 0           | $+1.110(0)$       | 0       |

**TABLE III.** Renormalization constants of a color one trace four-fermion operator at one loop, $(\mathcal{O}^{\text{lat}}_{1})_{ij}$ defined in Eq. (56). The $\hat{C}_{ij}$ matrix represents an anomalous dimension defined in Eq. (54) and its values are given in Eq. (62). The $C_{ij}$ matrix is defined in Eq. (54). The $M_{ij}$ matrix represents the tadpole-improvement defined in Eq. (66). $c_{T_1} \equiv (4\pi)^2/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$. 


| $\mathcal{O}_{1}^{(\mu\nu)}$ | color trace | $\Gamma_{ij}$ | $C_{ij}^{\mu\nu}$ | $M_{ij}$ |
|---|---|---|---|---|
| $[T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}]$ | I | 0 | +0.018(0) | 0 |
| $[T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}]$ | II | 0 | −1.061(0) | 0 |
| $[T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}]$ | I | 0 | −0.055(0) | 0 |
| $[T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}]$ | II | 0 | −0.191(0) | 0 |
| $[T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}]$ | I | 0 | +0.032(0) | 0 |
| $[T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}]$ | II | 0 | −0.096(0) | 0 |
| $[T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}]$ | I | 0 | +0.032(0) | 0 |
| $[T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}]$ | II | 0 | −0.096(0) | 0 |
| $[T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}]$ | I | 0 | +0.742(0) | 0 |
| $[T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}]$ | II | 0 | −1.219(0) | 0 |
| $[T_{\mu\nu} \times A_{\mu}] [T_{\mu\nu} \times A_{\mu}]$ | I | 0 | +0.119(0) | 0 |
| $[A_{\mu} \times S] [A_{\mu} \times S]$ | I | 0 | +0.342(0) | 0 |
| $[A_{\mu} \times S] [A_{\mu} \times S]$ | II | 0 | −0.539(0) | 0 |
| $[A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}]$ | I | 0 | +1.397(0) | 0 |
| $[A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}]$ | II | 0 | −2.170(0) | 0 |
| $[A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}]$ | I | 0 | +0.399(0) | 0 |
| $[A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}]$ | II | 0 | −0.083(0) | 0 |
| $[A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}]$ | I | 0 | +0.249(0) | 0 |
| $[A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}]$ | II | 0 | −0.083(0) | 0 |
| $[A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}]$ | I | 0 | +0.249(0) | 0 |
| $[A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}]$ | II | 0 | −2.367(0) | 0 |
| $[A_{\mu} \times T_{\mu\nu}] [A_{\mu} \times T_{\mu\nu}]$ | I | 0 | −0.0399(0) | 0 |
| $[A_{\mu} \times P] [A_{\mu} \times P]$ | I | +2 | −24.768(2) | +2$c_{\mu}$ |
| $[A_{\mu} \times P] [A_{\mu} \times P]$ | II | −6 | −5.225(1) | 0 |
| $[P \times V_{\mu}] [P \times V_{\mu}]$ | I | 0 | +17.108(2) | −$c_{\mu}$ |
| $[P \times V_{\mu}] [P \times V_{\mu}]$ | II | 0 | −4.149(1) | 0 |
| $[P \times V_{\mu}] [P \times V_{\mu}]$ | I | 0 | +0.805(0) | 0 |
| $[P \times V_{\mu}] [P \times V_{\mu}]$ | II | 0 | +0.915(0) | 0 |
| $[P \times A_{\mu}] [P \times A_{\mu}]$ | I | 0 | +1.073(0) | 0 |
| $[P \times A_{\mu}] [P \times A_{\mu}]$ | II | 0 | −1.061(0) | 0 |
| $[P \times A_{\mu}] [P \times A_{\mu}]$ | I | 0 | +0.183(0) | 0 |
| $[P \times A_{\mu}] [P \times A_{\mu}]$ | II | 0 | −0.191(0) | 0 |

**TABLE IV.** Renormalization constants of a color one trace four-fermion operator at one loop, $(\mathcal{O}_{1}^{(\mu\nu)})_{ij}$ defined in Eq. (54). The $\tilde{C}_{ij}$ matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (57). The $C_{ij}$ matrix is defined in Eq. (44). The $M_{ij}$ matrix represents the tadpole-improvement defined in Eq. (60). $c_{T1} \equiv (4\pi)^2/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$. 


| $\mathcal{O}_{ij}^{\text{Latt}(0)}$ | color trace | $\Gamma_{ij}$ | $C_{ij}^{\text{Latt}}$ | $N_{ij}$ |
|--------------------------------|-------------|-------------|-----------------|---------|
| $[S \times V_\mu][S \times V_\mu]$ | I           | 0           | $-5.581(1)$     | 0       |
| $[S \times V_\mu][S \times V_\mu]$ | II          | 0           | $+1.860(0)$     | 0       |
| $[S \times V_\mu][S \times V_\nu]$ | I           | 0           | $+1.114(0)$     | 0       |
| $[S \times V_\mu][S \times V_\mu]$ | II          | 0           | $-0.371(0)$     | 0       |
| $[S \times A_\mu][S \times A_\mu]$ | I           | 0           | $-1.311(0)$     | 0       |
| $[S \times A_\mu][S \times A_\mu]$ | II          | 0           | $+0.437(0)$     | 0       |
| $[S \times A_\mu][S \times A_\mu]$ | I           | 0           | $-0.236(0)$     | 0       |
| $[S \times A_\mu][S \times A_\mu]$ | II          | 0           | $+0.079(0)$     | 0       |
| $V_\mu \times S[V_\nu \times S]$ | I           | 0           | $-1.107(0)$     | 0       |
| $V_\mu \times S[V_\nu \times S]$ | II          | 0           | $+0.369(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | I           | 0           | $-1.106(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | II          | 0           | $+0.369(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | I           | 0           | $-1.150(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | II          | 0           | $-0.505(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | I           | 0           | $+0.403(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | II          | 0           | $-0.134(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | I           | 0           | $+0.403(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | II          | 0           | $-0.134(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | I           | 0           | $-2.776(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | II          | 0           | $+0.925(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | I           | 0           | $+1.505(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | II          | 0           | $-0.505(0)$     | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | I           | 0           | $-6$           | 0       |
| $V_\mu \times T_{\mu\rho}[V_\nu \times T_{\mu\rho}]$ | II          | 0           | $-58.529(7)$    | $+4\pi T_I$ |
| $T_{\mu\nu} \times V_\mu[T_{\mu\nu} \times V_\mu]$ | I           | 0           | $-2.776(0)$     | 0       |
| $T_{\mu\nu} \times V_\mu[T_{\mu\nu} \times V_\mu]$ | II          | 0           | $+0.925(0)$     | 0       |
| $T_{\mu\nu} \times V_\mu[T_{\mu\nu} \times V_\mu]$ | I           | 0           | $-0.416(0)$     | 0       |
| $T_{\mu\nu} \times V_\mu[T_{\mu\nu} \times V_\mu]$ | II          | 0           | $+0.139(0)$     | 0       |
| $T_{\mu\nu} \times V_\mu[T_{\mu\nu} \times V_\mu]$ | I           | 0           | $-0.765(0)$     | 0       |
| $T_{\mu\nu} \times V_\mu[T_{\mu\nu} \times V_\mu]$ | II          | 0           | $-0.255(0)$     | 0       |
| $T_{\mu\nu} \times V_\mu[T_{\mu\nu} \times V_\mu]$ | I           | 0           | $+0.765(0)$     | 0       |
| $T_{\mu\nu} \times V_\mu[T_{\mu\nu} \times V_\mu]$ | II          | 0           | $-0.255(0)$     | 0       |
| $T_{\mu\nu} \times V_\mu[T_{\mu\nu} \times V_\mu]$ | I           | 0           | $-5.968(1)$     | 0       |
| $T_{\mu\nu} \times V_\mu[T_{\mu\nu} \times V_\mu]$ | II          | 0           | $+1.989(0)$     | 0       |
| $T_{\mu\nu} \times V_\mu[T_{\mu\nu} \times V_\mu]$ | I           | 0           | $+1.946(0)$     | 0       |
| $T_{\mu\nu} \times V_\mu[T_{\mu\nu} \times V_\mu]$ | II          | 0           | $-0.649(0)$     | 0       |

**TABLE V.** Renormalization constants of a color two trace four-fermion operator at one loop, $(O_{ij}^{\text{Latt}})_{ij}$ defined in Eq. [5]. The $\Gamma_{ij}$ matrix represents an anomalous dimension defined in Eq. [54] and its values are given in Eq. [52]. The $C_{ij}^{\text{Latt}}$ matrix is defined in Eq. [50]. The $N_{ij}$ matrix represents the tadpole-improvement defined in Eq. [60]. $cT_I \equiv (4\pi)^2/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$.  

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| $O_{ij}^{\text{Latt}(0)}$ | color trace | $\Gamma_{ij}$ | $C_{ij}^{\text{Latt}}$ | $N_{ij}$ |
|--------------------------|-------------|--------------|----------------|--------|
| $T_{\mu\nu} \times A_\mu [T_{\mu\nu} \times A_\mu]$ | I | 0 | -0.772(0) | 0 |
| $T_{\mu\nu} \times A_\mu [T_{\mu\nu} \times A_\mu]$ | II | 0 | +0.257(0) | 0 |
| $T_{\mu\nu} \times A_\mu [T_{\mu\nu} \times A_\mu]$ | I | 0 | -0.045(0) | 0 |
| $T_{\mu\nu} \times A_\mu [T_{\mu\nu} \times A_\mu]$ | II | 0 | +0.15(0) | 0 |
| $T_{\mu\nu} \times A_\mu [T_{\mu\nu} \times A_\mu]$ | I | 0 | -1.705(0) | 0 |
| $T_{\mu\nu} \times A_\mu [T_{\mu\nu} \times A_\mu]$ | II | 0 | +0.568(0) | 0 |
| $T_{\mu\nu} \times A_\mu [T_{\mu\nu} \times A_\mu]$ | I | 0 | -0.529(0) | 0 |
| $T_{\mu\nu} \times A_\mu [T_{\mu\nu} \times A_\mu]$ | II | 0 | +0.176(0) | 0 |

**TABLE VI.** Renormalization constants of a color two trace four-fermion operator at one loop, $(O_{ij}^{\text{Latt}})_{ij}$ defined in Eq. (54). The $\hat{\Gamma}_{ij}$ matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (52). The $C_{ij}^{\text{Latt}}$ matrix is defined in Eq. (60). The $N_{ij}$ matrix represents the tadpole-improvement defined in Eq. (69). $ct1 \equiv (4\pi)^2/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$. 
| $\mathcal{O}^\text{latt(1b)}$ | color trace | $\Gamma_{ij}$ | $C^\text{latt}_{ij}$ | $M_{ij}$ |
|-----------------|--------------|--------------|----------------|--------|
| $[S \times V_\mu][S \times V_\mu]$ | I | 0 | +23.853(3) | $-c_{TI}$ |
| $[S \times V_\mu][S \times V_\mu]$ | II | 0 | +0.839(1) | 0 |
| $[S \times V_\mu][S \times V_\mu]$ | I | 0 | −0.139(0) | 0 |
| $[S \times V_\mu][S \times V_\mu]$ | II | 0 | +0.416(0) | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | I | 0 | −1.443(0) | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | II | 0 | −0.842(0) | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | I | 0 | +0.015(0) | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | II | 0 | −0.045(0) | 0 |
| $V_\mu \times S][V_\mu \times S]$ | I | 0 | +0.515(0) | 0 |
| $V_\mu \times S][V_\mu \times S]$ | II | 0 | +0.254(0) | 0 |
| $V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | I | 0 | −3.977(0) | 0 |
| $V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | II | 0 | +1.064(0) | 0 |
| $V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | I | 0 | −0.216(0) | 0 |
| $V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | II | 0 | +0.648(0) | 0 |
| $V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | I | 0 | −0.051(0) | 0 |
| $V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | II | 0 | +0.154(0) | 0 |
| $V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | I | 0 | −0.051(0) | 0 |
| $V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | II | 0 | +0.154(0) | 0 |
| $V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | I | 0 | −3.486(0) | 0 |
| $V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | II | 0 | +0.053(0) | 0 |
| $V_\mu \times T_{\mu\nu}][V_\mu \times T_{\mu\nu}]$ | I | 0 | −0.158(0) | 0 |
| $V_\mu \times P][V_\mu \times P]$ | I | +16 | −13.668(2) | $+2c_{TI}$ |
| $V_\mu \times P][V_\mu \times P]$ | II | 0 | −2.529(3) | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I | 0 | −24.725(2) | $+c_{TI}$ |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II | 0 | +1.778(1) | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I | 0 | +0.981(0) | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II | 0 | −2.943(0) | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I | 0 | −0.223(0) | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II | 0 | +0.669(0) | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I | 0 | −0.223(0) | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II | 0 | +0.669(0) | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I | 0 | −23.594(2) | $+c_{TI}$ |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II | 0 | −1.621(1) | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | I | 0 | −0.139(0) | 0 |
| $T_{\mu\nu} \times V_\mu][T_{\mu\nu} \times V_\mu]$ | II | 0 | +0.416(0) | 0 |

**TABLE VII.** Renormalization constants of a color one trace four-fermion operator at one loop, ($\mathcal{O}^\text{latt}_2$) defined in Eq. (54), $\Gamma_{ij}$ matrix represents an anomalous dimension defined in Eq. (54) and its values are given in Eq. (52). The $\mathcal{C}^\text{latt}_{ij}$ matrix is defined in Eq. (54). The $M_{ij}$ matrix represents the tadpole-improvement defined in Eq. (60). $c_{TI} \equiv (4\pi)^2/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$.  

\[\frac{\partial}{\partial g^2} \left[ \frac{1}{\pi} \ln Z \right] = -2g^2 + \sum_{\mu,\nu} \Gamma_{\mu\nu} \frac{\partial}{\partial g^2} \left[ \frac{1}{\pi} \ln Z \right] + \sum_{\mu,\nu} C^\text{latt}_{\mu\nu} \frac{\partial}{\partial g^2} \left[ \frac{1}{\pi} \ln Z \right] + M_{\mu\nu} \frac{\partial}{\partial g^2} \left[ \frac{1}{\pi} \ln Z \right].\]
TABLE VIII. Renormalization constants of a color one trace four-fermion operator at one loop, $(O_1^{lat})_{I}$ defined in Eq. \(\text{[32]}\). The $\hat{\Gamma}_{ij}$ matrix represents an anomalous dimension defined in Eq. \(\text{[54]}\) and its values are given in Eq. \(\text{[52]}\). The $C_{ij}^{lat}$ matrix is defined in Eq. \(\text{[54]}\). The $M_{ij}$ matrix represents the tadpole-improvement defined in Eq. \(\text{[60]}\). $c_{Tl} \equiv (4\pi)^2/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$.

| $O_1^{lat}$ | color trace | $\Gamma_{ij}$ | $C_{ij}^{lat}$ | $M_{ij}$ |
|-------------|-------------|-------------|-------------|---------|
| $[T_{\mu\nu} \times A_\mu] [T_{\mu\nu} \times A_\mu]$ | I | 0 | +1.820(0) | 0 |
| $[T_{\mu\nu} \times A_\mu] [T_{\mu\nu} \times A_\mu]$ | I | 0 | +0.049(0) | 0 |
| $[T_{\mu\nu} \times A_\mu] [T_{\mu\nu} \times A_\mu]$ | I | 0 | -0.146(0) | 0 |
| $[T_{\mu\nu} \times A_\mu] [T_{\mu\nu} \times A_\mu]$ | I | 0 | +0.032(0) | 0 |
| $[T_{\mu\nu} \times A_\mu] [T_{\mu\nu} \times A_\mu]$ | I | 0 | -0.096(0) | 0 |
| $[T_{\mu\nu} \times A_\mu] [T_{\mu\nu} \times A_\mu]$ | I | 0 | +0.054(0) | 0 |
| $[T_{\mu\nu} \times A_\mu] [T_{\mu\nu} \times A_\mu]$ | I | 0 | +1.575(0) | 0 |
| $[A_\mu \times S][A_\mu \times S]$ | I | 0 | -0.515(0) | 0 |
| $[A_\mu \times S][A_\mu \times S]$ | I | 0 | -0.254(0) | 0 |
| $[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$ | I | 0 | +3.977(0) | 0 |
| $[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$ | I | 0 | -1.064(0) | 0 |
| $[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$ | I | 0 | +0.050(0) | 0 |
| $[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$ | I | 0 | -0.150(0) | 0 |
| $[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$ | I | 0 | -0.083(0) | 0 |
| $[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$ | I | 0 | +0.249(0) | 0 |
| $[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$ | I | 0 | -0.083(0) | 0 |
| $[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$ | I | 0 | +0.249(0) | 0 |
| $[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$ | I | 0 | +3.354(0) | 0 |
| $[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$ | I | 0 | +0.804(0) | 0 |
| $[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$ | I | 0 | +0.050(0) | 0 |
| $[A_\mu \times T_{\mu\nu}][A_\mu \times T_{\mu\nu}]$ | I | 0 | -0.150(0) | 0 |

\[ A_\mu \times P][A_\mu \times P] \]

| $A_\mu \times P][A_\mu \times P] $ | I | -16 | +14.269(2) \text{[54]} | -2c_{Tl} |

$P \times V_{\mu}] [P \times V_{\mu}]$ | I | 0 | +24.465(2) \text{[54]} | -c_{Tl} |

$P \times V_{\mu}] [P \times V_{\mu}]$ | I | 0 | -0.997(1) | 0 |

$P \times V_{\mu}] [P \times V_{\mu}]$ | I | 0 | -0.166(0) | 0 |

$P \times A_\mu] [P \times A_\mu]$ | I | 0 | +0.499(0) | 0 |

$P \times A_\mu] [P \times A_\mu]$ | I | 0 | -1.952(0) | 0 |

$P \times A_\mu] [P \times A_\mu]$ | I | 0 | +0.685(0) | 0 |

$P \times A_\mu] [P \times A_\mu]$ | I | 0 | -0.079(0) | 0 |

$P \times A_\mu] [P \times A_\mu]$ | I | 0 | +0.236(0) | 0 |
| $O^{\text{Lat}}_{ij}$ | color trace | $\Gamma_{ij}$ | $C^{\text{Lat}}_{ij}$ | $N_{ij}$ |
|---------------------|--------------|--------------|-----------------|---------|
| $[S \times V_\mu][S \times V_\mu]$ | I | 0 | $+5.581(1)$ | 0 |
| $[S \times V_\mu][S \times V_\mu][S \times V_\mu]$ | II | 0 | $-1.860(0)$ | 0 |
| $[S \times V_\mu][S \times V_\mu][S \times V_\mu][S \times V_\mu]$ | I | 0 | $-1.114(0)$ | 0 |
| $[S \times V_\mu][S \times V_\mu][S \times V_\mu][S \times V_\mu]$ | II | 0 | $+0.371(0)$ | 0 |
| $[S \times A_\mu][S \times A_\mu]$ | I | 0 | $-1.311(0)$ | 0 |
| $[S \times A_\mu][S \times A_\mu][S \times A_\mu]$ | II | 0 | $+0.437(0)$ | 0 |
| $[S \times A_\mu][S \times A_\mu][S \times A_\mu][S \times A_\mu]$ | I | 0 | $-0.236(0)$ | 0 |
| $[S \times A_\mu][S \times A_\mu][S \times A_\mu][S \times A_\mu]$ | II | 0 | $+0.079(0)$ | 0 |
| $V_\mu \times S[V_\mu \times S]$ | I | 0 | $+1.107(0)$ | 0 |
| $V_\mu \times S[V_\mu \times S]$ | II | 0 | $-0.369(0)$ | 0 |
| $V_\mu \times T_{\mu \nu}[V_\mu \times T_{\mu \nu}]$ | I | 0 | $-1.106(0)$ | 0 |
| $V_\mu \times T_{\mu \nu}[V_\mu \times T_{\mu \nu}]$ | II | 0 | $+0.369(0)$ | 0 |
| $V_\mu \times T_{\mu \nu}[V_\mu \times T_{\mu \nu}]$ | I | 0 | $+0.150(0)$ | 0 |
| $V_\mu \times T_{\mu \nu}[V_\mu \times T_{\mu \nu}]$ | II | 0 | $-0.50(0)$ | 0 |
| $V_\mu \times T_{\mu \nu}[V_\mu \times T_{\mu \nu}]$ | I | 0 | $+0.403(0)$ | 0 |
| $V_\mu \times T_{\mu \nu}[V_\mu \times T_{\mu \nu}]$ | II | 0 | $-0.134(0)$ | 0 |
| $V_\mu \times T_{\mu \nu}[V_\mu \times T_{\mu \nu}]$ | I | 0 | $+0.403(0)$ | 0 |
| $V_\mu \times T_{\mu \nu}[V_\mu \times T_{\mu \nu}]$ | II | 0 | $-0.134(0)$ | 0 |
| $V_\mu \times T_{\mu \nu}[V_\mu \times T_{\mu \nu}]$ | I | 0 | $-2.776(0)$ | 0 |
| $V_\mu \times T_{\mu \nu}[V_\mu \times T_{\mu \nu}]$ | II | 0 | $+0.925(0)$ | 0 |
| $V_\mu \times T_{\mu \nu}[V_\mu \times T_{\mu \nu}]$ | I | 0 | $+0.150(0)$ | 0 |
| $V_\mu \times T_{\mu \nu}[V_\mu \times T_{\mu \nu}]$ | II | 0 | $-0.050(0)$ | 0 |
| $V_\mu \times P[V_\mu \times P]$ | I | 6 | $+6.500(0)$ | 0 |
| $V_\mu \times P[V_\mu \times P]$ | II | $-2$ | $-61.539(7)$ | $+4\Gamma_{ij}$ |
| $T_{\mu \nu}[V_\mu \times V_\mu]$ | I | 0 | $-2.776(0)$ | 0 |
| $T_{\mu \nu}[V_\mu \times V_\mu]$ | II | 0 | $+0.922(0)$ | 0 |
| $T_{\mu \nu}[V_\mu \times V_\mu]$ | I | 0 | $-0.416(0)$ | 0 |
| $T_{\mu \nu}[V_\mu \times V_\mu]$ | II | 0 | $+0.139(0)$ | 0 |
| $T_{\mu \nu}[V_\mu \times V_\mu]$ | I | 0 | $+0.765(0)$ | 0 |
| $T_{\mu \nu}[V_\mu \times V_\mu]$ | II | 0 | $-0.255(0)$ | 0 |
| $T_{\mu \nu}[V_\mu \times V_\mu]$ | I | 0 | $+0.765(0)$ | 0 |
| $T_{\mu \nu}[V_\mu \times V_\mu]$ | II | 0 | $-0.255(0)$ | 0 |
| $T_{\mu \nu}[V_\mu \times V_\mu]$ | I | 0 | $-5.968(1)$ | 0 |
| $T_{\mu \nu}[V_\mu \times V_\mu]$ | II | 0 | $+1.989(0)$ | 0 |
| $T_{\mu \nu}[V_\mu \times V_\mu]$ | I | 0 | $+1.946(0)$ | 0 |
| $T_{\mu \nu}[V_\mu \times V_\mu]$ | II | 0 | $-0.649(0)$ | 0 |

**TABLE IX.** Renormalization constants of a color two trace four-fermion operator at one loop, $(O^{\text{Lat}}_{ij})_{ij}$ defined in Eq. (34). The $\Gamma_{ij}$ matrix represents an anomalous dimension defined in Eq. (64) and its values are given in Eq. (65). The $C^{\text{Lat}}_{ij}$ matrix is defined in Eq. (34). The $N_{ij}$ matrix represents the tadpole-improvement defined in Eq. (69). $c_{ij} \equiv (4\pi)^2/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$. 

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| $O_{2 \text{Lat}}^{\text{con}}$ | color trace | $\Gamma_{ij}$ | $C_{ij}^{\text{Lat}}$ | $N_{ij}$ |
|-----------------|-------------|-----------|----------------|--------|
| $\hat{T}_{\mu \nu} \times A_{\mu} | T_{\mu \nu} \times A_{\mu}$ | I | 0 | +0.772(0) | 0 |
| $\hat{T}_{\mu \nu} \times A_{\mu} | T_{\mu \nu} \times A_{\mu}$ | II | 0 | -0.257(0) | 0 |
| $\hat{T}_{\mu \nu} \times A_{\mu} | T_{\mu \nu} \times A_{\mu}$ | I | 0 | +0.455(0) | 0 |
| $\hat{T}_{\mu \nu} \times A_{\mu} | T_{\mu \nu} \times A_{\mu}$ | II | 0 | -0.015(0) | 0 |
| $\hat{T}_{\mu \nu} \times A_{\mu} | T_{\mu \nu} \times A_{\mu}$ | I | 0 | +1.705(0) | 0 |
| $\hat{T}_{\mu \nu} \times A_{\mu} | T_{\mu \nu} \times A_{\mu}$ | II | 0 | -0.568(0) | 0 |
| $\hat{T}_{\mu \nu} \times A_{\mu} | T_{\mu \nu} \times A_{\mu}$ | I | 0 | +0.529(0) | 0 |
| $\hat{T}_{\mu \nu} \times A_{\mu} | T_{\mu \nu} \times A_{\mu}$ | II | 0 | -0.176(0) | 0 |
| $A_{\mu} \times S | A_{\mu} \times S$ | I | 0 | -0.478(0) | 0 |
| $A_{\mu} \times S | A_{\mu} \times S$ | II | 0 | +0.159(0) | 0 |
| $A_{\mu} \times T_{\mu \nu} | A_{\mu} \times T_{\mu \nu}$ | I | 0 | +1.106(0) | 0 |
| $A_{\mu} \times T_{\mu \nu} | A_{\mu} \times T_{\mu \nu}$ | II | 0 | -0.369(0) | 0 |
| $A_{\mu} \times T_{\mu \nu} | A_{\mu} \times T_{\mu \nu}$ | I | 0 | -0.150(0) | 0 |
| $A_{\mu} \times T_{\mu \nu} | A_{\mu} \times T_{\mu \nu}$ | II | 0 | +0.050(0) | 0 |
| $A_{\mu} \times T_{\nu \nu} | A_{\mu} \times T_{\nu \nu}$ | I | 0 | +2.776(0) | 0 |
| $A_{\mu} \times T_{\nu \nu} | A_{\mu} \times T_{\nu \nu}$ | II | 0 | -0.925(0) | 0 |
| $A_{\mu} \times T_{\nu \nu} | A_{\mu} \times T_{\nu \nu}$ | I | 0 | +0.656(0) | 0 |
| $A_{\mu} \times T_{\nu \nu} | A_{\mu} \times T_{\nu \nu}$ | II | 0 | -0.219(0) | 0 |
| $A_{\mu} \times P | A_{\mu} \times P$ | I | -6 | -4.500(0) | 0 |
| $A_{\mu} \times P | A_{\mu} \times P$ | II | +2 | +1.498(2) | 0 |
| $P \times V_{\mu} | P \times V_{\mu}$ | I | 0 | +3.153(1) | 0 |
| $P \times V_{\mu} | P \times V_{\mu}$ | II | 0 | -1.051(0) | 0 |
| $P \times V_{\mu} | P \times V_{\mu}$ | I | 0 | -4.16(0) | 0 |
| $P \times V_{\mu} | P \times V_{\mu}$ | II | 0 | +1.39(0) | 0 |
| $P \times A_{\mu} | P \times A_{\mu}$ | I | 0 | -0.377(0) | 0 |
| $P \times A_{\mu} | P \times A_{\mu}$ | II | 0 | +0.126(0) | 0 |
| $P \times A_{\mu} | P \times A_{\mu}$ | I | 0 | +0.045(0) | 0 |
| $P \times A_{\mu} | P \times A_{\mu}$ | II | 0 | -0.015(0) | 0 |

**TABLE X.** Renormalization constants of a color two trace four-fermion operator at one loop, $(O_{2 \text{Lat}}^{\text{con}})_{ij}$ defined in Eq. (54). The $\Gamma_{ij}$ matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (52). The $C_{ij}^{\text{Lat}}$ matrix is defined in Eq. (56). The $N_{ij}$ matrix represents the tadpole-improvement defined in Eq. (69). $c_{T1} \equiv (4\pi)^2/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$. 
\begin{table}
\begin{tabular}{|c|c|c|c|c|}
\hline
$\mathcal{O}_{\mu}^{\text{Latt}(0)}$ & color trace & $\Gamma_{ij}$ & $c_{ij}^{\text{Latt}}$ & $M_{ij}$ \\
\hline
$[S \times S][S \times S]$ & I & 0 & $-0.155(0)$ & 0 \\
$[S \times S][S \times S]$ & II & 0 & $-0.507(0)$ & 0 \\
$[S \times T_{\mu\nu}][S \times T_{\mu\nu}]$ & I & 0 & $-1.068(0)$ & 0 \\
$[S \times T_{\mu\nu}][S \times T_{\mu\nu}]$ & II & 0 & $-0.458(0)$ & 0 \\
$[S \times T_{\mu\nu}][S \times T_{\mu\nu}]$ & I & 0 & $-0.349(0)$ & 0 \\
$[S \times T_{\mu\nu}][S \times T_{\mu\nu}]$ & II & 0 & $-0.150(0)$ & 0 \\
$[S \times P][S \times P]$ & I & $2 \times (+2)$ & $2 \times (+25.186(2))$ & $2 \times (-2c_{T1})$ \\
$[S \times P][S \times P]$ & II & $2 \times (-6)$ & $2 \times (+3.972(1))$ & 0 \\
$V_{\mu} \times V_{\rho}[V_{\nu} \times V_{\eta}]$ & I & 0 & $+33.748(1)$ & $-2c_{T1}$ \\
$V_{\mu} \times V_{\rho}[V_{\nu} \times V_{\eta}]$ & II & 0 & $-2.258(1)$ & 0 \\
$V_{\mu} \times V_{\rho}[V_{\nu} \times V_{\eta}]$ & I & 0 & $-0.581(1)$ & 0 \\
$V_{\mu} \times V_{\rho}[V_{\nu} \times V_{\eta}]$ & II & 0 & $+0.194(0)$ & 0 \\
$V_{\mu} \times V_{\rho}[V_{\nu} \times V_{\eta}]$ & I & 0 & $+1.249(0)$ & 0 \\
$V_{\mu} \times V_{\rho}[V_{\nu} \times V_{\eta}]$ & II & 0 & $-0.416(0)$ & 0 \\
$V_{\nu} \times A_{\mu}[V_{\rho} \times A_{\eta}]$ & I & 0 & $-0.402(0)$ & 0 \\
$V_{\nu} \times A_{\mu}[V_{\rho} \times A_{\eta}]$ & II & 0 & $+0.766(0)$ & 0 \\
$V_{\nu} \times A_{\mu}[V_{\rho} \times A_{\eta}]$ & I & 0 & $+0.461(0)$ & 0 \\
$V_{\nu} \times A_{\mu}[V_{\rho} \times A_{\eta}]$ & II & 0 & $+0.198(0)$ & 0 \\
$V_{\nu} \times A_{\mu}[V_{\rho} \times A_{\eta}]$ & I & 0 & $-0.064(0)$ & 0 \\
$V_{\nu} \times A_{\mu}[V_{\rho} \times A_{\eta}]$ & II & 0 & $+0.191(0)$ & 0 \\
$V_{\nu} \times A_{\mu}[V_{\rho} \times A_{\eta}]$ & I & 0 & $-0.064(0)$ & 0 \\
$V_{\nu} \times A_{\mu}[V_{\rho} \times A_{\eta}]$ & II & 0 & $+0.191(0)$ & 0 \\
$V_{\nu} \times A_{\mu}[V_{\rho} \times A_{\eta}]$ & I & 0 & $+0.104(0)$ & 0 \\
$V_{\nu} \times A_{\mu}[V_{\rho} \times A_{\eta}]$ & II & 0 & $+0.045(0)$ & 0 \\
$T_{\mu\nu} \times T_{\rho\sigma}[T_{\mu\nu} \times T_{\rho\sigma}]$ & I & 0 & $+0.815(0)$ & 0 \\
$T_{\mu\nu} \times T_{\rho\sigma}[T_{\mu\nu} \times T_{\rho\sigma}]$ & II & 0 & $+2.065(0)$ & 0 \\
$T_{\mu\nu} \times T_{\rho\sigma}[T_{\mu\nu} \times T_{\rho\sigma}]$ & I & 0 & $+0.449(0)$ & 0 \\
$T_{\mu\nu} \times T_{\rho\sigma}[T_{\mu\nu} \times T_{\rho\sigma}]$ & II & 0 & $-0.150(0)$ & 0 \\
$T_{\mu\nu} \times T_{\rho\sigma}[T_{\mu\nu} \times T_{\rho\sigma}]$ & I & 0 & $-0.449(0)$ & 0 \\
$T_{\mu\nu} \times T_{\rho\sigma}[T_{\mu\nu} \times T_{\rho\sigma}]$ & II & 0 & $+1.50(0)$ & 0 \\
$T_{\mu\nu} \times T_{\rho\sigma}[T_{\mu\nu} \times T_{\rho\sigma}]$ & I & 0 & $+1.02(0)$ & 0 \\
$T_{\mu\nu} \times T_{\rho\sigma}[T_{\mu\nu} \times T_{\rho\sigma}]$ & II & 0 & $-0.307(0)$ & 0 \\
$T_{\mu\nu} \times T_{\rho\sigma}[T_{\mu\nu} \times T_{\rho\sigma}]$ & I & 0 & $+0.102(0)$ & 0 \\
$T_{\mu\nu} \times T_{\rho\sigma}[T_{\mu\nu} \times T_{\rho\sigma}]$ & II & 0 & $-0.307(0)$ & 0 \\
$T_{\mu\nu} \times T_{\rho\sigma}[T_{\mu\nu} \times T_{\rho\sigma}]$ & I & 0 & $-0.552(0)$ & 0 \\
$T_{\mu\nu} \times T_{\rho\sigma}[T_{\mu\nu} \times T_{\rho\sigma}]$ & II & 0 & $+1.275(0)$ & 0 \\
\hline
\end{tabular}
\caption{Renormalization constants of a color one trace four-fermion operator at one loop, $(\mathcal{O}_{\mu}^{\text{Latt}})_{ij}$ defined in Eq. (54). The $\hat{\Gamma}_{ij}$ matrix represents an anomalous dimension defined in Eq. (52) and its values are given in Eq. (52). The $c_{ij}^{\text{Latt}}$ matrix is defined in Eq. (54). The $M_{ij}$ matrix represents the tadpole-improvement defined in Eq. (66). $c_{T1} \equiv (4\pi)^2/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$.}
\end{table}
\[
\begin{array}{|c|c|c|c|c|}
\hline
\mathcal{O}^{\text{Latt}(0)} & \text{color trace} & \hat{\Gamma}_{ij} & C_{ij}^{\text{Latt}} & M_{ij} \\
\hline
[A_{\mu} \times V_{\mu}] [A_{\mu} \times V_{\mu}] & I & 0 & +32.138(1) & -2c_{T I} \\
[A_{\mu} \times V_{\mu}] [A_{\mu} \times V_{\nu}] & I & 0 & +2.572(1) & 0 \\
[A_{\mu} \times V_{\nu}] [A_{\mu} \times V_{\mu}] & I & 0 & -0.452(1) & 0 \\
[A_{\mu} \times V_{\nu}] [A_{\mu} \times V_{\nu}] & I & 0 & -0.194(0) & 0 \\
[A_{\mu} \times V_{\mu}] [A_{\mu} \times V_{\nu}] & II & 0 & +0.305(0) & 0 \\
[A_{\mu} \times V_{\nu}] [A_{\mu} \times V_{\mu}] & II & 0 & -0.915(0) & 0 \\
[A_{\mu} \times V_{\nu}] [A_{\mu} \times V_{\nu}] & II & 0 & +0.305(0) & 0 \\
[A_{\mu} \times V_{\nu}] [A_{\mu} \times V_{\nu}] & II & 0 & +0.971(0) & 0 \\
[A_{\mu} \times V_{\nu}] [A_{\mu} \times V_{\nu}] & II & 0 & +0.416(0) & 0 \\
[A_{\mu} \times A_{\mu}] [A_{\mu} \times A_{\mu}] & I & 0 & +0.220(0) & 0 \\
[A_{\mu} \times A_{\mu}] [A_{\mu} \times A_{\mu}] & II & 0 & -1.090(0) & 0 \\
[A_{\mu} \times A_{\mu}] [A_{\mu} \times A_{\mu}] & I & 0 & +0.593(0) & 0 \\
[A_{\mu} \times A_{\mu}] [A_{\mu} \times A_{\mu}] & II & 0 & -0.198(0) & 0 \\
[A_{\mu} \times A_{\mu}] [A_{\mu} \times A_{\mu}] & I & 0 & +0.134(0) & 0 \\
[A_{\mu} \times A_{\mu}] [A_{\mu} \times A_{\mu}] & II & 0 & -0.045(0) & 0 \\
[P \times S] [P \times S] & I & 0 & +0.155(0) & 0 \\
[P \times S] [P \times S] & II & 0 & +0.507(0) & 0 \\
[P \times T_{\mu \nu}] [P \times T_{\mu \nu}] & I & 0 & +1.068(0) & 0 \\
[P \times T_{\mu \nu}] [P \times T_{\mu \nu}] & II & 0 & +0.458(0) & 0 \\
[P \times T_{\mu \nu}] [P \times T_{\mu \nu}] & I & 0 & +0.349(0) & 0 \\
[P \times T_{\mu \nu}] [P \times T_{\mu \nu}] & II & 0 & +0.150(0) & 0 \\
[P \times F] [P \times F] & I & 2 \times (-2) & 2 \times (-39.872(2)) & 2 \times (+2c_{T I}) \\
[P \times F] [P \times F] & II & 2 \times (+6) & 2 \times (+40.087(1)) & 0 \\
\hline
\end{array}
\]

TABLE XII. Renormalization constants of a color one trace four fermion operator at one loop, \((\mathcal{O}^{\text{Latt}(0)})_I\) defined in Eq. (54). The \(\hat{\Gamma}_{ij}\) matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (52). The \(C_{ij}^{\text{Latt}}\) matrix is defined in Eq. (56). The \(M_{ij}\) matrix represents the tadpole-improvement defined in Eq. (60). \(c_{T I} \equiv (4\pi)^2/12\). All the greek indices are summed under the condition of \(\mu \neq \nu \neq \rho \neq \eta\).
| $O_{ij}^{L_{\text{Latt}}(b)}$ | color trace | $\Gamma_{ij}$ | $C_{ij}^{L_{\text{Latt}}}$ | $N_{ij}$ |
|---------------------|-----------------|----------------|------------------|----------------|
| $[S \times T_{\mu\nu}] [S \times T_{\mu\nu}]$ | I | 0 | -0.915(0) | 0 |
| $[S \times T_{\mu\nu}] [S \times T_{\mu\nu}]$ | II | 0 | +0.305(0) | 0 |
| $[S \times T_{\mu\nu}] [S \times T_{\mu\nu}]$ | I | 0 | -0.299(0) | 0 |
| $[S \times T_{\mu\nu}] [S \times T_{\mu\nu}]$ | II | 0 | +0.100(0) | 0 |
| $[S \times P] [S \times P]$ | II | 2 × (-16) | 2 × (+95.6074(0)) | 2 × (-6cTJ) |
| $V_{\mu} \times A_{\mu} [V_{\mu} \times A_{\mu}]$ | I | 0 | +0.682(0) | 0 |
| $V_{\mu} \times A_{\mu} [V_{\mu} \times A_{\mu}]$ | II | 0 | -0.227(0) | 0 |
| $V_{\mu} \times A_{\mu} [V_{\mu} \times A_{\mu}]$ | I | 0 | +0.395(0) | 0 |
| $V_{\mu} \times A_{\mu} [V_{\mu} \times A_{\mu}]$ | II | 0 | -0.132(0) | 0 |
| $V_{\mu} \times A_{\mu} [V_{\mu} \times A_{\mu}]$ | I | 0 | +0.191(0) | 0 |
| $V_{\mu} \times A_{\mu} [V_{\mu} \times A_{\mu}]$ | II | 0 | -0.064(0) | 0 |
| $V_{\mu} \times A_{\mu} [V_{\mu} \times A_{\mu}]$ | I | 0 | +0.191(0) | 0 |
| $V_{\mu} \times A_{\mu} [V_{\mu} \times A_{\mu}]$ | II | 0 | -0.064(0) | 0 |
| $V_{\mu} \times A_{\mu} [V_{\mu} \times A_{\mu}]$ | I | 0 | +0.089(0) | 0 |
| $V_{\mu} \times A_{\mu} [V_{\mu} \times A_{\mu}]$ | II | 0 | -0.030(0) | 0 |
| $A_{\mu} \times V_{\mu} [A_{\mu} \times V_{\mu}]$ | I | 0 | +6.015(1) | 0 |
| $A_{\mu} \times V_{\mu} [A_{\mu} \times V_{\mu}]$ | II | 0 | -2.005(0) | 0 |
| $A_{\mu} \times V_{\mu} [A_{\mu} \times V_{\mu}]$ | I | 0 | -0.387(1) | 0 |
| $A_{\mu} \times V_{\mu} [A_{\mu} \times V_{\mu}]$ | II | 0 | +0.129(0) | 0 |
| $A_{\mu} \times V_{\mu} [A_{\mu} \times V_{\mu}]$ | I | 0 | -1.530(1) | 0 |
| $A_{\mu} \times V_{\mu} [A_{\mu} \times V_{\mu}]$ | II | 0 | +0.510(0) | 0 |
| $A_{\mu} \times V_{\mu} [A_{\mu} \times V_{\mu}]$ | I | 0 | -1.530(1) | 0 |
| $A_{\mu} \times V_{\mu} [A_{\mu} \times V_{\mu}]$ | II | 0 | +0.510(0) | 0 |
| $A_{\mu} \times V_{\mu} [A_{\mu} \times V_{\mu}]$ | I | 0 | +0.833(0) | 0 |
| $A_{\mu} \times V_{\mu} [A_{\mu} \times V_{\mu}]$ | II | 0 | -0.278(0) | 0 |
| $P \times T_{\mu\nu} [P \times T_{\mu\nu}]$ | I | 0 | +0.915(0) | 0 |
| $P \times T_{\mu\nu} [P \times T_{\mu\nu}]$ | II | 0 | -0.305(0) | 0 |
| $P \times T_{\mu\nu} [P \times T_{\mu\nu}]$ | I | 0 | +0.299(0) | 0 |
| $P \times T_{\mu\nu} [P \times T_{\mu\nu}]$ | II | 0 | -0.100(0) | 0 |
| $P \times P [P \times P]$ | II | 2 × (+16) | 2 × (+111.255(2)) | 2 × (-2cTJ) |

**TABLE XIII.** Renormalization constants of a color two trace four-fermion operator at one loop, $(O_{ij}^{L_{\text{Latt}}})_{ij}$ defined in Eq. (54). The $\Gamma_{ij}$ matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (57). The $C_{ij}^{L_{\text{Latt}}}$ matrix is defined in Eq. (58). The $N_{ij}$ matrix represents the tadpole-improvement defined in Eq. (59). $c_{TJ} \equiv (\pi)^2/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$. 


TABLE XIV. Renormalization constants of a color one trace four-fermion operator at one loop, \( C^{\text{Latt}}_{ij}(0) \) defined in Eq. (54). The \( \hat{\Gamma}_{ij} \) matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (57). The \( C^{\text{Latt}}_{ij} \) matrix is defined in Eq. (58). The \( M_{ij} \) matrix represents the tadpole-improvement defined in Eq. (60). \( c_{T1} \equiv (4\pi)^2/12 \). All the greek indices are summed under the condition of \( \mu \neq \nu \neq \rho \neq \eta \).
The $\hat{\Gamma}_{ij}$ matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (52). The $\hat{C}_{ij}^{\text{Latt}}$ matrix is defined in Eq. (56). The $M_{ij}$ matrix represents the tadpole-improvement defined in Eq. (66). $c_{T1} \equiv (4\pi)^2/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$. 

### TABLE XV. Renormalization constants of a color one trace four-fermion operator at one loop, $([O_{4}^{\text{Latt}}]_I$, defined in Eq. (54). The $\hat{\Gamma}_{ij}$ matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (52). The $\hat{C}_{ij}^{\text{Latt}}$ matrix is defined in Eq. (56). The $M_{ij}$ matrix represents the tadpole-improvement defined in Eq. (66). $c_{T1} \equiv (4\pi)^2/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$. 

| Color Trace | $\Gamma_{ij}$ | $\hat{C}_{ij}^{\text{Latt}}$ | $M_{ij}$ |
|-------------|---------------|---------------------|---------|
| $[A_\mu \times V_\mu][A_\mu \times V_\mu]$ | I | 0 | -16.069(1) | +$c_{T1}$ |
| $[A_\mu \times V_\mu][A_\mu \times V_\mu]$ | II | 0 | -1.286(0) | 0 |
| $[A_\mu \times A_\mu][A_\mu \times A_\mu]$ | I | 0 | -2.228(0) | 0 |
| $[A_\mu \times V_\nu][A_\mu \times V_\nu]$ | II | 0 | -0.955(0) | 0 |
| $[A_\mu \times V_\mu][A_\mu \times V_\mu]$ | I | 0 | -0.153(0) | 0 |
| $[A_\mu \times A_\mu][A_\mu \times A_\mu]$ | II | 0 | +0.458(0) | 0 |
| $[P \times P][P \times P]$ | I | 0 | +0.110(0) | 0 |
| $[P \times S][P \times S]$ | II | 0 | -0.545(0) | 0 |
| $[P \times T][P \times T]$ | I | 0 | +0.673(0) | 0 |
| $[P \times T][P \times T]$ | II | 0 | -0.224(0) | 0 |
| $[P \times T][P \times T]$ | I | -2 | -39.872(2) | +$2c_{T1}$ |
| $[P \times P][P \times P]$ | II | +6 | +40.087(1) | 0 |
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$O_{L}^{\text{Latt}(0)}$ & color trace & $\Gamma_{ij}$ & $C_{ij}^{\text{Latt}}$ & $N_{ij}$ \\
\hline
$[S \times T_{\mu\nu}][S \times T_{\mu\nu}]$ & I & 0 & $-0.837(0)$ & 0 \\
$[S \times T_{\mu\nu}][S \times T_{\mu\nu}]$ & II & 0 & $+0.279(0)$ & 0 \\
$[S \times T_{\mu\nu}][S \times T_{\mu\nu}]$ & I & 0 & $-0.085(0)$ & 0 \\
$[S \times T_{\mu\nu}][S \times T_{\mu\nu}]$ & II & 0 & $+0.028(0)$ & 0 \\
$[S \times P][S \times P]$ & II & 16 & $-95.607(4)$ & $+6c_{Tl}$ \\
\hline
$V_{\mu} \times A_{\mu}$ & I & 0 & $-0.341(0)$ & 0 \\
$V_{\mu} \times A_{\mu}$ & II & 0 & $+0.114(0)$ & 0 \\
$V_{\mu} \times A_{\mu}$ & I & 0 & $-0.449(0)$ & 0 \\
$V_{\mu} \times A_{\mu}$ & II & 0 & $+0.150(0)$ & 0 \\
$V_{\mu} \times A_{\mu}$ & I & 0 & $-0.096(0)$ & 0 \\
$V_{\mu} \times A_{\mu}$ & II & 0 & $+0.032(0)$ & 0 \\
$V_{\mu} \times A_{\mu}$ & I & 0 & $-0.096(0)$ & 0 \\
$V_{\mu} \times A_{\mu}$ & II & 0 & $+0.032(0)$ & 0 \\
\hline
$T_{\mu\nu} \times S[T_{\mu\nu} \times S]$ & I & 0 & $-0.190(0)$ & 0 \\
$T_{\mu\nu} \times S[T_{\mu\nu} \times S]$ & II & 0 & $+0.063(0)$ & 0 \\
$T_{\mu\nu} \times P[T_{\mu\nu} \times P]$ & I & $-4$ & $-3.000(0)$ & 0 \\
$T_{\mu\nu} \times P[T_{\mu\nu} \times P]$ & II & $+4/3$ & $+1.000(0)$ & 0 \\
\hline
$A_{\mu} \times V_{\mu}$ & I & 0 & $-0.007(1)$ & 0 \\
$A_{\mu} \times V_{\mu}$ & II & 0 & $+0.002(0)$ & 0 \\
$A_{\mu} \times V_{\mu}$ & I & 0 & $-1.909(0)$ & 0 \\
$A_{\mu} \times V_{\mu}$ & II & 0 & $+0.636(0)$ & 0 \\
$A_{\mu} \times V_{\mu}$ & I & 0 & $+0.765(0)$ & 0 \\
$A_{\mu} \times V_{\mu}$ & II & 0 & $-0.255(0)$ & 0 \\
$A_{\mu} \times V_{\mu}$ & I & 0 & $+0.765(0)$ & 0 \\
$A_{\mu} \times V_{\mu}$ & II & 0 & $-0.255(0)$ & 0 \\
\hline
$P \times T_{\mu\nu} [P \times T_{\mu\nu}]$ & I & 0 & $-0.587(0)$ & 0 \\
$P \times T_{\mu\nu} [P \times T_{\mu\nu}]$ & II & 0 & $+0.279(0)$ & 0 \\
$P \times T_{\mu\nu} [P \times T_{\mu\nu}]$ & I & 0 & $-0.085(0)$ & 0 \\
$P \times T_{\mu\nu} [P \times T_{\mu\nu}]$ & II & 0 & $+0.028(0)$ & 0 \\
$P \times P [P \times P]$ & II & 16 & $+111.255(2)$ & $-2c_{Tl}$ \\
\hline
\end{tabular}
\caption{Renormalization constants of a color two trace four-fermion operator at one loop, $O_{L}^{\text{Latt}(0)}$ defined in Eq. (54). The $\Gamma_{ij}$ matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (53). The $C_{ij}^{\text{Latt}}$ matrix is defined in Eq. (54). The $N_{ij}$ matrix represents the tadpole-improvement defined in Eq. (51), $c_{Tl} \equiv (4\pi)^{2}/12$. All the greek indices are summed under the condition of $\mu \neq \nu \neq \rho \neq \eta$.}
\end{table}

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### TABLE XVII. Renormalization constants of a color one trace four-fermion operator at one loop, \( \mathcal{O}_{L}^{\text{Latt}} \) defined in Eq. (53). The \( \hat{\Gamma}_{ij} \) matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (52). The \( \hat{C}_{ij}^{\text{Latt}} \) matrix is defined in Eq. (56). The \( M_{ij} \) matrix represents the tadpole-improvement defined in Eq. (66). \( c_{T1} \equiv \frac{(4\pi)^2}{12} \). All the greek indices are summed under the condition of \( \mu \neq \nu \neq \rho \neq \eta \).
| \( C_{ij}^{\text{Latt}} \) | color trace | \( \Gamma_{ij} \) | \( C_{ij}^{\text{Latt}} \) | \( M_{ij} \) |
|---|---|---|---|---|
| \( T_{\mu\nu} \times S \left[ T_{\mu\nu} \times S \right] \) | I | 0 | +0.358(0) | 0 |
| \( T_{\mu\nu} \times S \left[ T_{\mu\nu} \times S \right] \) | II | 0 | −0.174(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | I | 0 | −1.761(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | II | 0 | −0.151(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | I | 0 | −1.602(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | II | 0 | −0.627(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | I | 0 | +0.042(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | II | 0 | −0.125(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | I | 0 | +0.042(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | II | 0 | +0.014(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | I | 0 | −0.042(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | II | 0 | −0.097(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | I | 0 | −0.292(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | II | 0 | +0.026(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | I | 0 | +0.011(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | II | 0 | −0.077(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | I | 0 | +0.026(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | II | 0 | −0.077(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | I | 0 | −1.695(0) | 0 |
| \( T_{\mu\nu} \times T_{\mu\rho} \left[ T_{\mu\nu} \times T_{\mu\rho} \right] \) | II | 0 | −0.349(0) | 0 |
| \( T_{\mu\nu} \times P \left[ T_{\mu\nu} \times P \right] \) | I | +26/3 | −6.617(1) | +ctI |
| \( T_{\mu\nu} \times P \left[ T_{\mu\nu} \times P \right] \) | II | −2 | −1.915(1) | 0 |

**TABLE XVIII.** Renormalization constants of a color one trace four-fermion operator at one loop, \( (O_5^{\text{Latt}}) \) defined in Eq. (54). The \( \Gamma_{ij} \) matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (52). The \( \hat{C}_{ij}^{\text{Latt}} \) matrix is defined in Eq. (56). The \( M_{ij} \) matrix represents the tadpole-improvement defined in Eq. (66). \( c_{T I} \equiv (4\pi)^2/12 \). All the greek indices are summed under the condition of \( \mu \neq \nu \neq \rho \neq \eta \).
The Table XIX. Renormalization constants of a color one trace four-fermion operator at one loop, \( \mathcal{O}_i^{\text{Latt}} \), defined in Eq. (56). The \( \Gamma_{ij} \) matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (52). The \( C_{ij}^{\text{Latt}} \) matrix is defined in Eq. (56). The \( M_{ij} \) matrix represents the tadpole-improvement defined in Eq. (56). \( c_{T1} \equiv (4\pi)^2/12 \). All the Greek indices are summed under the condition of \( \mu \neq \nu \neq \rho \neq \eta \).
\( \mathcal{C}^{\text{Latt}}_{ij} \) is defined in Eq. (56). The \( \Gamma_{ij} \) matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (69). The \( \tilde{\mathcal{C}}^{\text{Latt}}_{ij} \) matrix is defined in Eq. (54). The \( N_{ij} \) matrix represents the tadpole-improvement defined in Eq. (54). \( c_{T1} \equiv (4\pi)^2/12 \). All the greek indices are summed under the condition of \( \mu \neq \nu \neq \rho \neq \eta \).
### Table XXI. Renormalization constants of a color two trace four-fermion operator at one loop, \((\mathcal{O}_L^{\text{lat}})_{11}\) defined in Eq. (54). The \(\hat{\Gamma}_{ij}\) matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (52). The \(\hat{C}_{L}^{\text{lat}}\) matrix is defined in Eq. (56). The \(N_{ij}\) matrix represents the tadpole-improvement defined in Eq. (69). \(c_T I \equiv (4\pi)^2/12\). All the greek indices are summed under the condition of \(\mu \neq \nu \neq \rho \neq \eta\).
TABLE XXII. Renormalization constants of a color two trace four-fermion operator at one loop, \((O_{L_{\text{lat}}}^{5})_{II}\) defined in Eq. (55). The \(\hat{\Gamma}_{ij}\) matrix represents an anomalous dimension defined in Eq. (56) and its values are given in Eq. (52). The \(\hat{C}_{L_{\text{lat}}}^{ij}\) matrix is defined in Eq. (56). The \(N_{ij}\) matrix represents the tadpole-improvement defined in Eq. (69). \(c_{TI} \equiv (4\pi)^{2}/12\). All the greek indices are summed under the condition of \(\mu \neq \nu \neq \rho \neq \eta\).