Absolute quantum energy inequalities

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Abstract

Quantum Energy Inequalities (QEIs) are results which limit the extent to which the smeared renormalised energy density of the quantum field can be negative, when averaged along a timelike curve or over a more general timelike submanifold in spacetime. On globally hyperbolic spacetimes the minimally-coupled massive quantum Klein–Gordon field is known to obey a ‘difference’ QEI that depends on a reference state chosen arbitrarily from the class of Hadamard states. In many spacetimes of interest this bound cannot be evaluated explicitly. In this paper we obtain the first ‘absolute’ QEI for the minimally-coupled massive quantum Klein–Gordon field on four dimensional globally hyperbolic spacetimes; that is, a bound which depends only on the local geometry. The argument is an adaptation of that used to prove the difference QEI and utilises the Sobolev wave-front set to give a complete characterisation of the singularities of the Hadamard series. Moreover, the bound is explicit and can be formulated covariantly under additional (general) conditions. We also generalise our results to incorporate adiabatic states.

Dedicated to Klaus Fredenhagen on the occasion of his 60th birthday.

1 Introduction

The classical minimally coupled scalar field, like most matter models studied in classical general relativity, obeys the weak energy condition (WEC). That is, the stress energy

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tensor \( T_{ab} \) obeys the inequality \( T_{ab} v^a v^b \geq 0 \) for all timelike vector fields \( v^a \), which entails that observers encounter only non-negative energy densities. However, it has been known since 1965 that no Wightman quantum field theory can obey the weak energy condition [8], (see [16, 17] for simple arguments as to why this is true). Moreover, under many circumstances there is no lower bound to the energy densities available in quantum field theory (QFT). This surprising feature of QFT has often been used to support proposals for exotic spacetimes, such as warp drive and traversable wormholes, which require WEC-violating matter distributions. In addition, the validity of the second law of thermodynamics is called into question by WEC violations in QFT [26].

With these concerns in mind, substantial effort has been directed to understand the magnitude and extent of negative energy densities permitted by QFT, starting with the work of Ford in 1978 [26]. Given the failure of pointwise energy inequalities, attention has been focussed on averages of the stress tensor along timelike worldlines or over spacetime regions. In many QFT models such averages turn out to obey Quantum Energy Inequalities (QEIs); that is, their expectation values are bounded from below as the state varies within the class of physically reasonable states [9, 10, 11, 14, 26, 27, 28, 41]. Since their inception QEIs have been applied to a variety of physical problems; they form the basis of the arguments constraining exotic spacetimes such as the warp drive [2, 28] and traversable wormholes [27, 19].

In their most common form, QEIs place lower bounds on the expectation value of the averaged stress energy tensor relative to that obtained in a reference state. For this reason they are called difference QEIs [20]. To give a specific example [10], consider the minimally coupled scalar field of mass \( \mu \geq 0 \) in a globally hyperbolic spacetime \((M, g)\) of dimension \( m \geq 2 \), and let \( \gamma : \mathbb{R} \to \mathcal{M} \) be a smooth timelike curve (not necessarily a geodesic) with velocity \( v^a \). Then for any real-valued \( f \in C_0^\infty(\mathbb{R}) \) the QEI

\[
\int_{\mathbb{R}} dt f^2(t) \left( \langle v^a v^b T_{\text{ren}}^{ab} \rangle_\omega(\gamma(t)) - \langle v^a v^b T_{\text{ren}}^{ab} \rangle_{\omega_0}(\gamma(t)) \right) \geq -B_D \tag{1}
\]

holds for all Hadamard states \( \omega \), where the bound

\[
B_D = \int_0^\infty \frac{d\xi}{\pi} \left[ f \otimes f \partial^\ast \langle v^a v^b T_{\text{split}}^{ab} \rangle_{\omega_0} \right] \wedge (-\xi, \xi) \tag{2}
\]

depends only on \( f, \gamma \), and the reference state \( \omega_0 \) (which may be any Hadamard state); note that it does not depend on the state of interest \( \omega \). Here the hat denotes the Fourier transform given, in our conventions, by \( \hat{f}(\xi) = \int_{\mathbb{R}^n} dx f(x) e^{i\xi \cdot x} \). The quantity \( \langle v^a v^b T_{\text{split}}^{ab} \rangle_{\omega_0}(x, x') \) is the (unrenormalised) point split energy density defined, in a neighbourhood of \( \gamma \), by

\[
\langle v^a v^b T_{\text{split}}^{ab} \rangle_{\omega_0}(x, x') = \left( \frac{1}{2} \sum_{\alpha=0}^{3} e_\alpha \nabla_a \otimes e_\alpha \nabla_b + \frac{1}{2} \mu^2 \mathbb{1} \otimes \mathbb{1} \right) \Lambda_{\omega_0}(x, x') \tag{3}
\]

where \( \{e_\alpha\}_{\alpha=0,1,2,3} \) is a tetrad field satisfying \( \varepsilon_0^a \varepsilon^a_\gamma = v^a \) (see Section 3 of [10] for a more detailed discussion) and \( \Lambda_{\omega_0} \) is the two point function of the state \( \omega_0 \). Finally,
\[ \vartheta^* \langle v^a v^b T^{\text{split}}_{ab'} \rangle_{\omega_0} \] denotes the pull-back
\[ \vartheta^* \langle v^a v^b T^{\text{split}}_{ab'} \rangle_{\omega_0}(\tau, \tau') = \langle v^a v^b T^{\text{split}}_{ab'} \rangle_{\omega_0}(\gamma(\tau), \gamma(\tau')) \] (4)
which may be defined rigorously as a distribution on \( \mathbb{R}^2 \) using the techniques of microlocal analysis, which also guarantee that the bound (2) is finite. Similar bounds also hold for the free spin-1/2 and spin-1 field in comparable generality and rigour [5, 12, 13, 14]. Recently, quantum energy inequalities have also been proven for free spin-3/2 fields in Minkowski spacetime [49, 37].

As already mentioned, one application of the QEIs is to place constraints on exotic spacetimes. However, the bound given above, while valid in any globally hyperbolic curved spacetime, depends crucially on the choice of a reference state. Although Hadamard states exist on any globally hyperbolic spacetimes [29, 30], closed form expressions for two point functions are known only in very special circumstances. For instance, no such expression is available for any Hadamard state on the warp drive spacetime. Typically these problems have been avoided by heuristic appeals to the equivalence principle to justify the use of Minkowski spacetime QEIs on sufficiently small scales. To date, this approach, while physically reasonable, lacks full mathematical justification and control over the scales on which it is valid. It would clearly be preferable to employ a lower bound which did not require the specification of a reference state and placed constraints directly on \( \langle T^{\text{ren}}_{ab} \rangle_\omega \). Bound of this type, known as absolute QEIs, have been established in flat spacetimes [24] but the only curved spacetime absolute QEI is that of Flanagan [25] (see also [15, 47]) which applies to massless free fields in two-dimensional globally hyperbolic spacetimes. This approach relies on the conformal invariance of the theory and does not generalise to higher dimensions or non-zero mass. However, it does provide the basis for a QEI on arbitrary positive energy conformal field theories in Minkowski spacetime, including interacting examples [18].

In this paper we present the first absolute QEI applicable to the scalar field of mass \( \mu \geq 0 \) in four-dimensions, by refining and modifying the argument presented in [10]. For averaging along timelike worldline \( \gamma \), our result takes the form
\[ \int_{\mathbb{R}} dt \, f^2(t) \langle v^a v^b T^{\text{ren}}_{ab} \rangle_\omega(t) \geq -B_A, \] (5)
where
\[ B_A = \int_{\mathbb{R}^+} \frac{d\xi}{\pi} \left[ f \otimes f \vartheta^* \langle T^{\text{split}} \rangle_\omega \right] (\xi, -\xi) + \text{“local curvature terms”} \] (6)
and \( T^{\text{split}} H \) is constructed in the same fashion as \( \langle v^a v^b T^{\text{split}}_{ab'} \rangle_{\omega_0} \) but with (essentially) the first few terms of the Hadamard series replacing the reference two-point function. At the technical level, we invoke a refined version of microlocal analysis which keeps track of the order of singularities.

The structure of this paper is as follows. In Section 2 we review the algebraic formulation of quantum field theory in curved spacetimes, review two (equivalent) formulations
of the Hadamard condition and give a detailed analysis of the singularity structure of the Hadamard series in terms of Sobolev wave-front sets. Section 3 contains our main result, theorem 3.1, which is then used to give a number of examples of absolute QEIs. Although our bound depends on a choice of coordinates, we describe how the dependence can be eliminated by restricting the choice of smearing tensor, thus providing a covariant formulation of our bounds.

2 Quantum field theory in curved spacetime

2.1 The algebra of observables and the first definition of Hadamard states

We shall employ the algebraic framework for describing the scalar quantum field in a classical curved four-dimensional spacetime \((\mathcal{M}, g)\). Here \(\mathcal{M}\) is a four-dimensional smooth manifold (assumed Hausdorff, paracompact and without boundary) with a Lorentz metric \(g_{ab}\) of signature \((+−−−)\). Furthermore, we require \((\mathcal{M}, g)\) to be globally hyperbolic, that is \(\mathcal{M}\) contains a Cauchy surface. Where index notation is used, Latin indices will run over the range \(0, 1, 2, 3\) unless explicitly stated otherwise, while Greek characters will denote frame indices and also run over \(0, 1, 2, 3\) unless explicitly stated otherwise.

We employ units in which \(c = \hbar = 1\).

The minimally coupled scalar field \(φ\) obeys the Klein–Gordon equation \((\nabla^2 + \mu^2)φ = 0\), where \(\nabla^2 = g^{ab}\nabla_a\nabla_b\) and \(\mu \geq 0\) is the mass of the field quanta. Global hyperbolicity entails the existence of unique global advanced \((E^-)\) and retarded \((E^+)\) Green functions \(E^\pm : C^\infty_0(\mathcal{M}) \to C^\infty(\mathcal{M})\) for the Klein–Gordon equation obeying

\[(\nabla^2 + \mu^2)E^\pm f = E^\pm(\nabla^2 + \mu^2)f = f,\] (7)

and

\[\text{supp } E^\pm f \subset J^\pm(\text{supp } f)\] (8)

for all \(f \in C^\infty_0(\mathcal{M})\), where \(J^\pm(S)\) denote the causal future (+) and past (−) of a set \(S\). One may use the set of smooth functions having compact support in \(\mathcal{M}\), \(C^\infty_0(\mathcal{M})\), to label a set of abstract objects \(\{φ(f) \mid f \in C^\infty_0(\mathcal{M})\}\) which generate a free unital ∗-algebra \(\mathfrak{A}\) over \(\mathbb{C}\). The algebra of smeared fields \(\mathfrak{A}(\mathcal{M}, g)\) is defined to be the quotient of \(\mathfrak{A}\) by the following relations:

i) Hermiticity, \(φ(f)^* = φ(\overline{f}) \forall f \in C^\infty_0(\mathcal{M})\);

ii) Linearity, \(φ(αf + βf') = αφ(f) + βφ(f') \forall α, β \in \mathbb{C} \text{ and } ∀f, f' \in C^\infty_0(\mathcal{M})\);

iii) Field equation, \(φ((\nabla^2 + \mu^2)f) = 0 \forall f \in C^\infty_0(\mathcal{M})\);

iv) Canonical commutation relations, \([φ(f), φ(f')] = iE(f, f')1 \forall f, f' \in C^\infty_0(\mathcal{M})\).

Here, \(E = E^- - E^+\) is the advanced-minus-retarded Green’s function for the Klein–Gordon operator and by \(E(f, f')\) we mean

\[E(f, f') = \int_{\mathcal{M}} \text{dvol}(x) f(x)(Ef')(x).\] (9)
It is relation (iv) that quantises the field theory.

In this framework, a state is a linear functional $\omega$ on $\mathcal{A}(\mathcal{M}, g)$ which is normalised so that $\omega(1) = 1$ and is positive in the sense that $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}(\mathcal{M}, g)$. The two point function associated with the state $\omega$ is a bilinear map $\Lambda_\omega : C^\infty_0(\mathcal{M}) \otimes C^\infty_0(\mathcal{M}) \to \mathbb{C}$ given by $\Lambda_\omega(f, f') = \omega(\phi(f)\phi(f'))$. We will only consider states for which $\Lambda_\omega$ is a distribution, i.e., $\Lambda_\omega \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$. It is clear from (iv) that the antisymmetric part of $\Lambda_\omega$,

$$\frac{1}{2} \left( \Lambda_\omega(f, f') - \Lambda_\omega(f', f) \right) = \frac{i}{2} E(f, f'),$$

is state independent.

As already mentioned, we will largely be concerned with Hadamard states. There are two equivalent formulations of the Hadamard condition, both of which will be used in the sequel. The original definition, given in a precise form by Kay & Wald [39], involves a local series expansion of the two point function $\Lambda_\omega$ associated with a state $\omega$ and is based upon Hadamard’s work on the fundamental solution for hyperbolic operators.

In order to give the precise formulation of the Hadamard series construction we first introduce some geometrical structures, following [39, 42, 44]. We denote by $\mathcal{X} \subset \mathcal{M} \times \mathcal{M}$ the set

$$\mathcal{X} = \{(x, x') \in \mathcal{M} \times \mathcal{M} \mid x, x' \text{ causally related and}$$

$$J^+(x) \cap J^-(x') \cup J^+(x) \cap J^-(x') \text{ are contained}$$

$$\text{within a convex normal neighbourhood}\}.$$  \hspace{1cm} (11)

For each $(x, x') \in \mathcal{X}$ let $U_{x,x'}$ be any convex normal neighbourhood containing $J^+(x) \cap J^-(x')$ and $J^+(x) \cap J^-(x')$. Then (cf. Lemma 3.1 in [42]) $X = \bigcup_{(x, x') \in \mathcal{X}} U_{x,x'} \times U_{x,x'}$ is an open neighbourhood of $\mathcal{X}$ in $\mathcal{M} \times \mathcal{M}$ on which the signed squared geodesic separation of points $\sigma$ is well-defined and smooth, and on which the Hadamard construction (to be described shortly) can be carried out. Any open neighbourhood of $\mathcal{X}$ defined in this way will be called a regular domain.

For each $k = 0, 1, 2, \ldots$, we may define a distribution $H_k \in \mathcal{D}'(X)$ by

$$H_k(x, x') = \frac{1}{4\pi^2} \left\{ \frac{\Delta^+(x, x')}{\sigma_+(x, x')} + \sum_{j=0}^k v_j(x, x') \frac{\sigma_j(x, x')}{\ell^{2(j+1)}} \ln \left( \frac{\sigma_+(x, x')}{\ell^2} \right) \right. $$

$$\left. + \sum_{j=0}^k w_j(x, x') \frac{\sigma_j(x, x')}{\ell^{2(j+1)}} \right\}, \tag{12}$$

where we have introduced a length scale $\ell$ to make $\sigma/\ell^2$ dimensionless and the coefficient functions $\Delta$, $v_j$ and $w_j$ will be explained below. We also set $H_{-1} = \Delta^{1/2}/(4\pi^2\sigma_+)$. By

1We adopt the convention that $\sigma(x, x') > 0$ if $x, x'$ are spacelike separated, $\sigma(x, x') < 0$ if $x, x'$ are timelike separated and $\sigma(x, x') = 0$ if they are null separated. In Minkowski spacetime, $(\mathbb{R}^4, \eta)$, for example $\sigma(x, x') = -\eta_{ab}(x - x^a)(x - x'^b)$.

2A different choice of length scale $\ell'$ can be absorbed into a redefinition of the local curvature terms $C_{ab}$ appearing in the renormalisation of the stress-energy tensor; see Section 2.3.
\( F(\sigma_+) \), for some function \( F \), we mean the distributional limit

\[
F(\sigma_+) = \lim_{\epsilon \to 0^+} F(\sigma_+),
\]

where \( \sigma_+(x, x') = \sigma(x, x') + 2i(t(x) - t(x')) + \epsilon^2 \) and \( t \) is a time function; that is, \( \nabla^a t \) is a normalised future directed timelike vector field on \( X \). We shall occasionally use the notation \( t = t(x) \) and \( t' = t(x') \). The function \( \Delta \in C^\infty(X) \) is the van Vleck-Morette determinant bi-scalar and is given by

\[
\Delta(x, x') = -\frac{\det (-\nabla_a \otimes \nabla_{b'} \sigma(x, x'))}{\sqrt{-g(x)} \sqrt{-g(x')}}.
\]

The functions \( v_j \) and \( w_j \) are found by fixing \( x' \) and applying \( (\nabla^2 + \mu^2) \otimes BD \) to \( H_k \) and equating all the coefficients of \( 1/\sigma_+, 1/\sigma_+^2, \ln \sigma_+ \) etc to zero. This determines a system of equations (known as the Hadamard recursion relations, given in appendix A) which can be solved uniquely (in \( X \)) for the \( v_j \) series. The \( w_j \) series is specified once the value of \( w_0 \) is fixed; we adopt Wald’s prescription that \( w_0 = 0 \) \[48\]. We remark that the \( k \to \infty \) limit of the right-hand side of (12) has a nonzero radius of convergence in analytic spacetimes, but not in general \[32\].

Let \( N \) be a causal normal neighbourhood of a Cauchy surface \( C \) \[39\]; that is, \( C \) is a Cauchy surface for \( N \) and every double-cone \( J^+(x) \cap J^-(y) \) with \( x, y \in N \) is contained in a convex normal neighbourhood of \( (M, g) \) (see Lemma 2.2 in \[39\] for the existence of causal normal neighbourhoods). We may further choose an open neighbourhood \( X_* \) of the set of pairs of causally related points in \( N \times N \) whose closure is contained in \( N \cap (N \times N) \) and a cut-off function \( \chi : N \times N \to [0, 1] \) so that

\[
\chi|_{X_*} = 1 \quad \text{and} \quad \chi|_{(N \times N) \setminus X} = 0.
\]

See Lemma 3.3 in \[42\] for the existence of \( X_* \) and \( \chi \) with these properties.

Given the above, a state \( \omega \) on \( \mathfrak{A}(M, g) \) is said to be Hadamard if for each \( k \in \mathbb{N} \) there exists a \( F_k \in C^k(N \times N) \) such that

\[
\Lambda_\omega = \chi H_k + F_k
\]

in \( N \times N \). We remark that this definition can be shown to be independent of the choices of \( C, N, t, \chi, X \) and \( X_* \) \[39\] \[44\].

In the special case in which \( M \) is a convex normal neighbourhood, we note that \( M \) would be a causal normal neighbourhood of any of its Cauchy surfaces and we could take \( X_* = X = M \times M \) and \( \chi \equiv 1 \), so (16) becomes \( \Lambda_\omega = H_k + F_k \) and holds on the whole of \( M \times M \). In the general case, it is easy to see (e.g., using the microlocal characterisation of the Hadamard condition) that if \( \omega \) is Hadamard then so is its restriction to any open globally hyperbolic subset of \( M \), considered as a spacetime in its own right. Thus \( \Lambda_\omega - H_k \) is \( C^k \) for all \( k \) on any set of the form \( U \times U \) where \( U \) is a globally hyperbolic convex normal neighbourhood. As every point \( x \in M \) has such a neighbourhood \( U_x \) we
may conclude that $\Lambda_\omega - H_k$ is $C^k$ for all $k$ in an open neighbourhood of the diagonal of the form $\bigcup_{x \in \mathcal{M}} U_x \times U_x$; we will refer to any such open neighbourhood as an ultra-regular domain.

Note: The need to introduce the notion of an ultra-regular domain only came to light as the final version of this paper was prepared for publication, and after [45] had gone to press. One of us (CJS) would like to warn the reader that some results in [45] hold on ultra-regular domains as opposed to the stated regular domain; with this modification, the results of [45] are unchanged.

2.2 Renormalisation of the stress tensor

The Hadamard series construction forms the basis for the renormalisation of the stress-energy tensor in curved spacetimes, to which we now turn. The classical stress tensor of the real scalar field

$$T_{ab}(x) = \left( \nabla_a \otimes \nabla_b - \frac{1}{2}g_{ab}g^{cd}\nabla_c \otimes \nabla_d + \frac{1}{2}\mu^2 g_{ab} \mathbb{1} \otimes \mathbb{1} \right)(\varphi \otimes \varphi)(x, x)$$  \hspace{1cm} (17)

must be renormalised in QFT owing to the divergent behaviour of the two point function. Define the point-split stress-energy operator (which should not be confused with the stress-energy tensor itself) by

$$T_{ab}' = \nabla_a \otimes \nabla_b - \frac{1}{2}g_{ab}g^{cd}\nabla_c \otimes \nabla_d + \frac{1}{2}\mu^2 g_{ab} \mathbb{1} \otimes \mathbb{1}$$  \hspace{1cm} (18)

near the diagonal in $\mathcal{M} \times \mathcal{M}$, where $g_{ab}'(x, x')$ is the parallel propagator. If $\omega$ is a Hadamard state we may define $\langle T_{ab}^{\text{ren}} \rangle_\omega(x)$ at any point $x \in \mathcal{M}$ by the following procedure:

a) note that $\Lambda_\omega - H_k \in C^2(X)$ for $k \geq 2$ and any ultra-regular domain $X$, so $T_{ab}'(\Lambda_\omega - H_k)$ is defined and continuous near the diagonal in $\mathcal{M} \times \mathcal{M}$;

b) define

$$\langle T_{ab}^{\text{fin}} \rangle_\omega(x) = \lim_{x' \to x} g_{ab}'(x, x')T_{ab}'(\Lambda_\omega - H_k)(x, x')$$  \hspace{1cm} (19)

for $k \geq 2$;

c) make finite corrections to $\langle T_{ab}^{\text{fin}} \rangle_\omega$ in order to obtain a conserved tensor $\langle T_{ab}^{\text{ren}} \rangle_\omega(x)$ with the correct properties in Minkowski space.

Step (c) is needed because the tensor $\langle T_{ab}^{\text{fin}} \rangle_\omega$ is not covariantly conserved and cannot be considered as an appropriate stress-energy tensor (it could not be inserted on the right hand side of the Einstein equations, for example). However, it turns out that $\nabla^a \langle T_{ab}^{\text{fin}} \rangle_\omega$ is of the form $\nabla_a Q$ where $Q$ is a local quantity, determined up to a constant; subtracting $Qg_{ab}$ by hand from $\langle T_{ab}^{\text{fin}} \rangle_\omega$ we therefore obtain a conserved quantity. The undetermined constant in $Q$ is fixed by the requirement that in Minkowski spacetime the

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\[\text{Thus, if } v^{\alpha'} \in T_{\alpha'} \mathcal{M}, g_{ab'}(x, x')v^{\alpha'} \text{ is its parallel transport to } T_x \mathcal{M} \text{ along the unique geodesic joining } x' \text{ to } x, \text{ which is well-defined sufficiently close to the diagonal in } \mathcal{M} \times \mathcal{M}.\]
vacuum expectation value vanishes. If we require that the difference $\langle T^{\text{ren}}_{ab} \rangle_\omega - \langle T^{\text{ren}}_{ab} \rangle_{\omega_0}$ should be given by

$$\langle T^{\text{ren}}_{ab} \rangle_\omega - \langle T^{\text{ren}}_{ab} \rangle_{\omega_0} = \lim_{x' \to x} g_b^{b'}(x, x') T^{\text{split}}_{ab'}(\Lambda_\omega - \Lambda_{\omega_0})(x, x')$$

(20)

then any remaining finite renormalisation must take the form of a state-independent conserved local curvature term $C_{ab}$ that vanishes in Minkowski space, and the finite renormalised expectation value of the quantum stress energy is given by

$$\langle T^{\text{ren}}_{ab} \rangle_\omega(x) = \langle T^{\text{fin}}_{ab} \rangle_\omega(x) - Q(x)g_{ab}(x) + C_{ab}(x).$$

(21)

We take the view that the tensor $C_{ab}$ is a necessary part of the specification of a given species of scalar field, alongside the mass and curvature coupling. Given sufficient experimental accuracy $C_{ab}$ should, in principle, be measurable.

The renormalisation prescription we have outlined above is vulnerable to the criticism that the $Qg_{ab}$ term needed to restore conservation of $\langle T^{\text{ren}}_{ab} \rangle_\omega$ is only found to be a local curvature term a posteriori. Moretti [40] has shown that this problem can be circumvented by an alternative construction of the quantum stress tensor. The basic idea is to modify the classical stress energy tensor $T_{ab}$ by adding a term of the form $\alpha \phi (\nabla^2 + \mu^2) \phi$ for constant $\alpha$. While this addition does not affect the classical physics, it has a non-trivial quantisation. A judicious choice of $\alpha$ ensures that the quantised stress energy tensor is conserved a priori (see theorem 2.1 of [40]) and agrees with the usual quantisation up to conserved local curvature terms. Although Moretti’s approach is certainly elegant, it turns out that the usual quantisation is better adapted to the derivation of QEIIs. In particular, our argument relies crucially on being able to write $T^{\text{split}}_{ab'}$ in a symmetric form which is not possible with a term of the form $1 \otimes (\nabla^2 + \mu^2)$. A similar problem arises for the non-minimally coupled scalar field. In this case one must smear the stress-energy tensor even to obtain an inequality on the classical field [21], which necessitates a more complicated analysis at the quantum level [22].

### 2.3 The wave-front set and second definition of Hadamard state

The above discussion shows that Hadamard states are characterised by their singularity structure. For this reason the techniques of microlocal analysis, which focus attention on singular behaviour, are ideally suited to this theory. This realisation has led to a number of important developments in the theory of quantum fields in curved backgrounds, following initial work of Radzikowski [42], particularly in regard to renormalisation [3, 33, 34]. In addition, the theory of (smooth) wave-front sets is a key tool in the proof of general difference QEIIs [10, 12, 14, 6] in general globally hyperbolic spacetimes. Our absolute QEIIs require the finer control on singularities of distributions afforded by the Sobolev wave-front set. In this subsection we briefly review the definition of the smooth and Sobolev wave-front sets and explain how they may be used to give a purely microlocal definition of the Hadamard condition, as first identified by Radzikowski [42]. In addition, we will state a result of Junker and Schrohe [38] on the Sobolev wave-front set of
the two-point functions of Hadamard states. This will form the basis of our analysis of the Sobolev wave-front sets of individual terms in the Hadamard series in the next subsection.

To begin, let \( u \in \mathcal{D}'(\mathbb{R}^m) \) be any distribution. We say that \( u \) is smooth at \( x' \) if there exists an open neighbourhood \( O \subset \mathbb{R}^m \) of \( x' \) and a smooth function \( \varphi \in \mathcal{C}^\infty(\mathcal{O}) \) such that \( u(f) = \int_{\mathbb{R}^m} d^m x \varphi(x)f(x) \) for all test functions \( f \in C^\infty_0(\mathcal{O}) \). The singular support, \( \text{singsupp} \ u \), of a distribution \( u \in \mathcal{D}'(\mathbb{R}^m) \) is the complement in \( \mathbb{R}^m \) of the set of all points at which \( u \) is smooth. In particular, a distribution is smooth if and only if its singular support is empty.

While the singular support tells us ‘where’ a distribution \( u \) fails to be smooth, Fourier transforms of localisations of \( u \) contain additional information. A covector \( \zeta \in \mathbb{R}^m \setminus \{0\} \) is a direction of rapid decay for \( u \) at \( x \) if there exists a conic neighbourhood \( \Gamma \subset \mathbb{R}^m \setminus \{0\} \) of \( \zeta \) and a localiser \( \chi \in \mathcal{C}^\infty_0(\mathbb{R}^m) \) which does not vanish at \( x \) such that
\[
(1 + |\xi|^N) |\hat{\chi}(\xi)| \longrightarrow 0 \quad \text{as} \quad \xi \rightarrow \infty \quad \text{in} \quad \Gamma, \quad \forall N \in \mathbb{N}.
\]
(22)
The set of singular directions of \( u \in \mathcal{D}'(\mathbb{R}^m) \) at \( x \), \( \Sigma_x(u) \), is the complement in \( \mathbb{R}^m \setminus \{0\} \) of the set of directions of rapid decay of \( u \) at \( x \). The wave-front set of \( u \) assembles this information in a convenient way (see Section 8.1 of \[35\] for more detail).

**Definition.** The (smooth) wave-front set \( WF(u) \) of a distribution \( u \in \mathcal{D}'(\mathbb{R}^m) \) is
\[
WF(u) = \{(x, \xi) \in \mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\}) \mid \xi \in \Sigma_x(u)\}
\]
(23)
As an example, it is easy to verify that the Dirac \( \delta \) and Heaviside \( \theta \) distributions have the following wave-front sets.
\[
WF(\delta) = WF(\theta) = \{(0, \xi) \mid \xi \in \mathbb{R} \setminus \{0\}\}.
\]
(24)
The wave-front set is a closed cone in \( \mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\}) \), whose elements transform as covectors under coordinate transformations (see Theorem 8.2.4 in \[35\]). Accordingly, the definition of wave-front set may be extended to distributions on a smooth manifold \( \mathcal{M} \) in the following way: We say \( (x, \xi) \in WF(u) \subset T^*\mathcal{M} \setminus \{0\} \) if and only if there exists a chart neighbourhood \( (\kappa, \mathcal{U}) \) of \( x \) such that the corresponding coordinate expression of \( (x, \xi) \) belongs to \( WF(u \circ \kappa^{-1}) \subset \mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\}) \), where \( m \) is the dimension of \( \mathcal{M} \).

The Sobolev wave-front set provides greater structure on the information in the wave-front set. Recall that the Sobolev space \( H^s(\mathbb{R}^m) \), \( s \in \mathbb{R} \), is the set of all tempered distributions \( u \) on \( \mathbb{R}^m \) such that
\[
\int_{\mathbb{R}^m} d^m \xi \ (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 < \infty.
\]
(25)
We summarise some relevant properties of Sobolev spaces for convenience.
Proposition 2.1. The Sobolev spaces $H^s(\mathbb{R}^m)$ have the following properties:

1. If $s > k + m/2$ then $H^s(\mathbb{R}^m) \subset C^k(\mathbb{R}^m)$ for $k \in \mathbb{N}$;
2. $H^s(\mathbb{R}^m) \subset H^{s'}(\mathbb{R}^m)$ for all $s \geq s'$;
3. $H^s(\mathbb{R}^m)$ is closed under multiplication by smooth functions.

Associated with the scale of Sobolev spaces, there is a refined notion of the wave-front set. Just as $WF(u)$ informs us where a distribution $u$ fails to be smooth, the Sobolev wave-front set $WF^s(u)$ contains information about where in phase space the distribution fails to be $H^s$.

Definition. The distribution $u \in \mathcal{D}'(\mathbb{R}^m)$ is said to be microlocally $H^s$ at $(x, \xi) \in \mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$ if there exists a conic neighbourhood $\Gamma$ of $\xi$ and a smooth function $\varphi \in C_0^\infty(\mathbb{R}^m)$, $\varphi(x) \neq 0$, such that

$$\int_{\Gamma} d^m z (1 + |\zeta|^2)^s |[\varphi u]^{\wedge}(|\zeta|)|^2 < \infty. \quad (26)$$

The Sobolev wave-front set $WF^s(u)$ of a distribution $u \in \mathcal{D}'(\mathbb{R}^m)$ is the complement, in $T^*\mathbb{R}^m \setminus \{0\}$, of the set of all pairs $(x, \xi)$ at which $u$ is microlocally $H^s$.

It is easy to verify, for example, that

$$WF^{s+1}(\theta) = WF^s(\delta) = \begin{cases} \{(0, \xi) \mid x \in \mathbb{R} \setminus \{0\}\} & s \geq -1/2 \\ \emptyset & s < -1/2 \end{cases} \quad (27)$$

and we see how this refines the information in (24). Like the wave-front set, $WF^s(u)$ is a closed cone in $T^*\mathbb{R}^n \setminus \{0\}$. Furthermore, part (ii) of Proposition 2.1 entails that $WF^s(u) \subset WF^{s'}(u) \subset WF(u)$ for all $s \leq s'$. In fact, one may show that $WF(u) = \bigcup_{s \in \mathbb{R}} WF^s(u)$. Additionally, if $\varphi \in C_0^\infty(\mathbb{R}^n)$ does not vanish in a neighbourhood of $x$ then $(x, \xi) \in WF^s(u)$ if and only if $(x, \xi) \in WF^s(\varphi u)$; we also have $WF^s(u + w) \subset WF^s(u) \cup WF^s(w)$. One may show that $WF^s(u)$ can be characterised in a coordinate-independent way as a subset of the cotangent bundle which then permits the definition to be extended to distributions on a manifold by referring back to (any choice of) local coordinates (see, e.g., remark (i) following Prop. B.3 in [38]). We shall occasionally use the notation $u \in H^s_{loc}(\mathcal{M})$ if $WF^s(u) = \emptyset$ for a distribution $u \in \mathcal{D}'(\mathcal{M})$ (see also the remarks following definition 8.2.5 of [38]).

In [42], Radzikowski proved the remarkable result that the definition of a Hadamard state in terms of the series construction previously given is equivalent to a condition on the wave-front set of the two-point function. Namely, the wave-front set is required to lie in a particular subset of the bicharacteristic set of the Klein–Gordon operator, which we now define.

Denote by $\mathcal{R} = \{(x, \xi) \in T^*\mathcal{M} \mid g^{ab}(x)\xi_a\xi_b = 0, \xi \neq 0\}$ the set of nonzero null covectors over $\mathcal{M}$. Since $(\mathcal{M}, g)$ is time orientable we may decompose $\mathcal{R}$ into two disjoint sets $\mathcal{R}^\pm$ defined by $\mathcal{R}^\pm = \{(x, \xi) \in \mathcal{R} \mid \pm \xi \succ 0\}$ where by $\xi \succ 0$ ($\xi \in T^*_x\mathcal{M}$) we mean...
that $\xi_a$ is in the dual of the future light cone at $x$. We define the notation $(x, \xi) \sim (x', \xi')$ to mean that there exists a null geodesic $\gamma : [0, 1] \to \mathcal{M}$ such that $\gamma(0) = x$, $\gamma(1) = x'$ and $\xi_a = \dot{\gamma}^b(0) g_{ab}(x)$, $\xi'_a = \dot{\gamma}^b(1) g_{ab}(x')$. In the instance where $x = x'$, $(x, \xi) \sim (x, \xi')$ shall mean that $\xi = \xi'$ is null. Then, the set
\[ C = \{(x, \xi; x', \xi') \in \mathbb{R} \times \mathbb{R} \mid (x, \xi) \sim (x', \xi')\} \tag{28} \]
is the bicharacteristic relation for the Klein–Gordon operator. We also define the related sets
\[ C^+ - = \{(x, \xi; x', -\xi') \in C \mid \xi > 0\} \tag{29} \]
and
\[ C^- + = \{(x, -\xi; x', \xi') \in C \mid \xi > 0\}. \tag{30} \]

We may now state the relevant portion of Radzikowski’s equivalence theorem [42]:

**Theorem 2.2.** Let $(\mathcal{M}, g)$ be a four-dimensional globally hyperbolic spacetime and suppose $\Lambda \in D'((\mathcal{M} \times \mathcal{M}))$ satisfies the Klein–Gordon equation and has antisymmetric part $iE/2$ modulo smooth functions. Choose a Cauchy hypersurface $\mathcal{C}$, a causal normal neighbourhood $\mathcal{N}$ of $\mathcal{C}$ and a time function $t$. Then, the following two conditions are equivalent:

i) $\Lambda$ has the Hadamard series structure given by (16) on $\mathcal{N} \times \mathcal{N}$,

ii) $WF(\Lambda) = C^+ -$.

As a consequence of this equivalence theorem we may now adopt the condition that $WF(\Lambda_\omega) = C^+ -$ as the second definition of the state $\omega$, on $\mathfrak{A}(\mathcal{M}, g)$, being Hadamard.

Junker and Schrohe [38] applied the theory of Sobolev wave-front sets to study both Hadamard states and the larger class of adiabatic states (see §3.2.3). In particular, lemma 5.2 of [38] gives the Sobolev singularity structure of the two point function of Hadamard states.

**Theorem 2.3.** Let $\omega$ be a Hadamard state on $\mathfrak{A}(\mathcal{M}, g)$ where $\mathcal{M}$ is a smooth four-dimensional globally hyperbolic spacetime. Then, the two point function, $\Lambda_\omega \in D'((\mathcal{M} \times \mathcal{M}))$, associated to $\omega$ has the following Sobolev wave-front set:
\[ WF^s(\Lambda_\omega) = \begin{cases} C^+ - & s \geq -1/2 \\ \emptyset & s < -1/2 \end{cases}. \tag{31} \]

**2.4 Sobolev microlocal analysis of the Hadamard series**

We will now employ theorem 2.3 to study the Sobolev wave-front sets of the individual terms in the Hadamard series, working within an ultra-regular domain $X$ on which the distributions $1/\sigma_+$ and $\sigma^j \ln \sigma_+$ featuring in (12) may be defined. (The length scale $\ell$ will be suppressed from now on.) We shall establish the following statement:
\[ WF^{s+j+1}(\sigma^j \ln \sigma_+) \subset WF^s(1/\sigma_+) = \begin{cases} C^+ - & s \geq -1/2 \\ \emptyset & s < -1/2 \end{cases}. \tag{32} \]
In order to do this we observe that the terms appearing in the Hadamard series are (loosely) related to one another via differentiation. If $P$ is any partial differential operator of order $r$ on a smooth manifold $\mathcal{M}$, i.e. in local coordinates

$$P = \sum_{|\alpha| \leq r} p_\alpha(x)(-i\partial)^\alpha$$  \hspace{1cm} (33)$$

where $\alpha$ is a multi-index and $p_\alpha$ are smooth functions, then the principal symbol, $p_r(x, \xi)$, of $P$ is

$$p_r(x, \xi) = \sum_{|\alpha| = r} p_\alpha(x)\xi^\alpha.$$  \hspace{1cm} (34)$$

The characteristic set of $P$, $\text{Char} P$, is the set of $(x, \xi) \in T^*\mathcal{M} \setminus \{0\}$ at which the principal symbol vanishes. Corollaries 8.4.9-10 of [36] encapsulate the effect of partial differential operators on the Sobolev wave-front set of a distribution:

**Lemma 2.4.** Let $\mathcal{M}$ be a smooth manifold. For $u \in \mathcal{D}'(\mathcal{M})$ and any partial differential operator $P$ of order $r$ with smooth coefficients then $WF^s(Pu) \subset WF^{s+r}(u)$ and $WF^{s+r}(u) \subset WF^s(Pu) \cup \text{Char} P$.

Lemma 2.4 enables us to quantify our earlier observation about the relationship between $1/\sigma_+$ and $\sigma^j \ln \sigma_+$:

**Proposition 2.5.** Within an ultra-regular domain we have

$$WF^{s+1+j}(\sigma^j \ln \sigma_+) \subset WF^s(1/\sigma_+) \hspace{1cm} \forall s \in \mathbb{R} \text{ and } \forall j \in \{0\} \cup \mathbb{N}. \hspace{1cm} (35)$$

**Proof.** We employ induction on $j$. If $v \in C^\infty(T\mathcal{M})$ is a smooth vector field, then

$$(v \cdot \nabla \otimes 1) \ln \sigma_+ = [(v \cdot \nabla \otimes 1)\sigma]/\sigma_+.$$  Hence, by lemma 2.4 we have

$$WF^{s+1}(\ln \sigma_+) \subset WF^s(1/\sigma_+) \cup \text{Char} (v \cdot \nabla \otimes 1). \hspace{1cm} (36)$$

As $v$ is arbitrary,

$$WF^{s+1}(\ln \sigma_+) \subset WF^s(1/\sigma_+) \cup \left( \bigcap_{v \in C^\infty(T\mathcal{M})} \text{Char} (v \cdot \nabla \otimes 1) \right) \hspace{1cm} (37)$$

and since

$$\text{Char} (v \cdot \nabla \otimes 1) = \{(x, \xi; x', \xi') \in T^*X \setminus \{0\} \mid v(x) \cdot \xi = 0\} \hspace{1cm} (38)$$

it is clear that the intersection is empty and the statement holds for $j = 0$. Now suppose it holds for some $j \in \{0\} \cup \mathbb{N}$. The identity

$$(v \cdot \nabla \otimes 1)\sigma^{j+1} \ln \sigma_+ = [(v \cdot \nabla \otimes 1)\sigma] ((j + 1)\sigma^j \ln \sigma_+ + \sigma^j) \hspace{1cm} (39)$$

and the inductive hypothesis give

$$WF^{s+2+j}(\sigma^{j+1} \ln \sigma_+) \subset WF^{s+1+j}(\sigma^j \ln \sigma_+) \cup \text{Char} (v \cdot \nabla \otimes 1) \hspace{1cm} (40)$$

\text{12}
and taking the intersection over all \( v \in C^\infty(TM) \) as before, we establish the result for \( j + 1 \) and hence all \( j \in \{0\} \cup \mathbb{N} \) by induction.

We next prove the intuitively reasonable result that \( \Lambda_\omega \) is as singular as the leading term in the Hadamard series.

**Proposition 2.6.** Let \( \omega \) be a Hadamard state. Then, within any ultra-regular domain \( X \), we have

\[
WF^s(\Lambda_\omega) = WF^s(1/\sigma_+) \quad \forall s \in \mathbb{R}.
\]  

**Proof.** Recall that for every \( k \in \mathbb{N} \) there exists a \( F_k \in C^k(X) \) such that \( \Lambda_\omega = H_k + F_k \). Hence, as \( WF^s(\sigma^j \ln \sigma_+) \subset WF^{s+j+1}(\sigma^j \ln \sigma_+) \subset WF^s(1/\sigma_+) \),

\[
WF^s(\Lambda_\omega) \subset WF^s(\Delta^1/\sigma_+) \cup WF^s(F_k).
\]  

We remark that it is known (from, say, [42]) that \( \Delta \) does not vanish where \( x, x' \) are null related and as such \( WF^s(\Lambda_\omega) \subset WF^s(1/\sigma_+) \cup WF^s(F_k) \). Moreover, given any particular \( s \) we can always find a \( k \) sufficiently large such that \( WF^s(F_k) = \emptyset \) and it remains to prove \( WF^s(1/\sigma_+) \subset WF^s(\Lambda_\omega) \). Let \( (x, \xi; x', \xi') \in WF^s(1/\sigma_+) \) such that \( WF^s-\epsilon(1/\sigma_+) = \emptyset \) for any \( \epsilon > 0 \). Hence, by proposition 2.5, \( WF^s(\sigma^j \ln \sigma_+) \subset WF^s-j-1(1/\sigma_+) = \emptyset \) for \( j \geq 0 \) and we have \( H_k - \Delta^1/4\pi^2\sigma_+ \in H^s_{loc}(X) \). Therefore, \( (x, \xi; x', \xi') \in WF^s(H_k) \) and by the nesting property \( (x, \xi; x', \xi') \in WF^{s'}(H_k) \) for all \( s' \geq s \).

As a consequence of theorem 2.3 we now have the Sobolev wave-front sets of the constituent distributions in the Hadamard series.

**Corollary 2.7.** The distributions \( 1/\sigma_+, \sigma^j \ln \sigma_+ \in \mathcal{D}'(X) \), where \( X \) is an ultra-regular domain, have the following Sobolev wave-front sets:

\[
WF^s(1/\sigma_+) = \begin{cases} 
C^{--} & s \geq -1/2 \\
\emptyset & s < -1/2 
\end{cases}
\]

\[
WF^s(\sigma^j \ln \sigma_+) \subset \begin{cases} 
C^{--} & s \geq j + 1/2 \\
\emptyset & s < j + 1/2
\end{cases}
\]  

In consequence, we also have, for arbitrary \( j \geq -1 \),

\[
WF^s(H_j) \subset \begin{cases} 
C^{--} & s \geq -1/2 \\
\emptyset & s < -1/2
\end{cases}
\]

and

\[
WF^s(H_{j+j'} - H_j) \subset \begin{cases} 
C^{--} & s \geq j + 3/2 \\
\emptyset & s < j + 3/2
\end{cases}
\]  

for \( j' > 0 \).

\[4\]That is, \( \Delta^{1/2} \) restricted to \( \text{singsupp } 1/\sigma_+ \) is non-vanishing.
Proof. As we have already established, \(1/\sigma_+\) possesses the lowest order singularity which lemma \ref{lemma:lowest_order_singularity} states is precisely that of \(\Lambda_\omega\). The remaining results follow from proposition \ref{proposition:some_results_follow}.

The remainder of this section is devoted to calculating the Sobolev wave-front sets of the advanced-minus-retarded fundamental solution \(E\) and a quantity \(\tilde{H}_k \in \mathcal{D}'(X)\) (\(X\) an ultra-regular domain) defined by

\[
\tilde{H}_k(x, x') = \frac{1}{2} \left(H_k(x, x') + H_k(x', x) + iE(x, x')\right),
\]

which plays an important role in our main result theorem \ref{theorem:main_result}. As \(iE\) is the antisymmetric part of \(\Lambda_\omega\), for all Hadamard \(\omega\), theorem \ref{theorem:wave_front_sets} implies that

\[
W F^s(iE) \subset \bigcup C^+ C^- s \geq 1/2 \cup\{C^+ s \leq -1/2\} \cup\{C^- s \leq -1/2\}.
\]

Proposition 2.8. Within an ultra-regular domain, the Sobolev wave-front set of \(\tilde{H}_k\) satisfies

\[
W F^s(\tilde{H}_k) \subset \bigcup C^+ C^- s \geq k + 3/2 \cup\{C^+ s \leq -1/2\} \cup\{C^- s \leq -1/2\} \cup\{\emptyset s \leq -1/2\}.
\]

Proof. Suppose first that \(s < k + 3/2\). It follows from the covariant commutation relations that, within an ultra-regular domain \(X\), \(iE(x, x') = H_{k+2}(x, x') - H_{k+2}(x', x)\) modulo \(C^{k+2}(X)\). Hence there exists \(F \in C^{k+2}(X)\) such that

\[
[H_k - H_{k+2}](x, x') = [H_k - H_{k+2}](x, x') - [H_k - H_{k+2}](x', x) + F(x, x').
\]

As \(k + 2 \geq s\), we have \(C^{k+2}(X) \subset H^s_{\text{loc}}(X)\), so all three terms on the right-hand side will belong to \(H^s_{\text{loc}}(X)\), using Eq. \ref{equation:some_important_thing} as well. Thus \(\tilde{H}_k = H_k\) modulo \(H^s_{\text{loc}}(X)\) for all \(s < k + 3/2\), which establishes \ref{equation:wave_front_sets_2} for \(s\) in this range. For \(s \geq k + 3/2\) the result follows from the definition \ref{equation:wave_front_sets} of \(\tilde{H}_k\), the wave-front set of \(H_k\) (and its behaviour under interchange of the arguments \(x\) and \(x')\) together with the rule for wave-front sets of sums of distributions. \hfill \(\square\)

Finally, we end this discussion of the microlocal properties with the following result concerning the singularities of \(\Lambda_\omega - \tilde{H}_k\) which follows directly from the proof of corollary \ref{corollary:relation_between_waves}.

Proposition 2.9. Within an ultra-regular domain, the Sobolev wave-front set of \(\Lambda_\omega - \tilde{H}_k\) is given by

\[
W F^s(\Lambda_\omega - \tilde{H}_k) \subset \bigcup C^+ C^- s \geq k + 3/2 \cup\{C^+ s \leq -1/2\} \cup\{C^- s \leq -1/2\} \cup\{\emptyset s \leq -1/2\}.
\]
2.5 Restriction results and a point-splitting lemma

In addition to the results of the previous subsection, our main result will make use of three additional technical results. The first, Beals’ restriction theorem, enables us to restrict \( \Lambda_\omega, H_k \) and their derivatives to certain submanifolds of \( M \times M \). The second, taken from [10], shows that positive type is preserved under such restrictions, while the third result is a technical tool that enables us to write integrals over the diagonal on product manifolds in terms of their ‘point-split’ Fourier transforms.

Our QEI results will encompass averages of the stress-energy tensor smeared over timelike submanifolds, e.g., timelike curves or hyperplanes, as well as averages over spacetime volumes. For these purposes, it is necessary to understand how restricting distributions (such as \( \Lambda_\omega, H_k \) and their derivatives) to a submanifold alters the Sobolev wave-front set. A theorem due to Beals (see Lemma 11.6.1 of [36]) tells us that, for suitably well behaved restrictions, the Sobolev order of the wave-front set is reduced by an amount proportional to the codimension of the restriction, while its elements are transformed according to the associated pull-back mapping. We will state a specialisation of Beals’ result to the case we will need, in which we restrict from a product manifold \( M \times M \) to a submanifold \( \Sigma \times \Sigma \), where \( \Sigma \) is a submanifold of \( M \).

Writing the embedding of \( \Sigma \) in \( M \) as a map \( \iota : \Sigma \to M \) and defining \( \vartheta = \iota \otimes \iota : \Sigma \times \Sigma \to M \times M \), the restriction of \( u \in D'(M \times M) \) to \( \Sigma \times \Sigma \) may also be regarded as the formation of a pull-back \( \vartheta^* u \). Beals’ result hinges on the relationship between the Sobolev wave-front set of \( u \) and the conormal bundle \( N^* \Sigma \) of \( \Sigma \) defined by

\[
N^* \Sigma = \{ (\iota(x), \xi) \in T^* M; x \in \Sigma, \iota^*(\xi) = 0 \}.
\]

(52)

**Theorem 2.10** (Beals’ restriction theorem). Let \( u \in D'(M \times M) \) and \( \vartheta \in C^\infty(\Sigma \times \Sigma, M \times M) \) be defined as above, and suppose \( M \) and \( \Sigma \) have dimensions \( m \) and \( n \) respectively. If \( (N^* \Sigma \times N^* \Sigma) \cap WF^s(u) = \emptyset \) for some \( s > m - n \) then \( \vartheta^* u \) is a well defined distribution in \( D'(\Sigma \times \Sigma) \). Moreover,

\[
WF^{s-(m-n)}(\vartheta^* u) \subset \vartheta^* WF^s(u)
\]

(53)

where the set \( \vartheta^* WF^s(u) \) is defined to be

\[
\vartheta^* WF^s(u) = \{ (t, \iota^*(\xi); t', \iota^*(\xi')) \in (T^* \Sigma \times T^* \Sigma) | (\iota(t), \xi; \iota(t'), \xi') \in WF^s(u) \}.
\]

(54)

The next result, theorem 2.2 of [10], asserts that the positive type condition is preserved under the restrictions carried out by Beals’ theorem.

**Lemma 2.11.** If, in addition to the hypotheses of Theorem 2.10, \( u \in D'(M \times M) \) is of positive type, then \( \vartheta^* u \) is of positive type on \( \Sigma \times \Sigma \).

Finally, we present a point-splitting identity for distributions of sufficient regularity. Beginning in \( \mathbb{R}^n \times \mathbb{R}^n \), we have the following.
**Lemma 2.12.** For all \( u \in C_0(\mathbb{R}^n \times \mathbb{R}^n) \), we have the identity

\[
\int_{\mathbb{R}^n} d^n t \, u(t, t) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} \frac{d^n \xi}{(2\pi)^n} e^{-\epsilon |\xi|^2} \hat{u}(-\xi, \xi).
\]

(55)

In particular, this holds if \( u \in H^s(\mathbb{R}^n \times \mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n \times \mathbb{R}^n) \) for \( s > n \), by virtue of the Sobolev embedding of \( H^s(\mathbb{R}^n \times \mathbb{R}^n) \) in \( C(\mathbb{R}^n \times \mathbb{R}^n) \).

**Proof.** By definition of the Fourier transform, we have

\[
\int_{\mathbb{R}^n} \frac{d^n \xi}{(2\pi)^n} e^{-\epsilon |\xi|^2} \hat{u}(-\xi, \xi) = \int_{\mathbb{R}^n} \frac{d^n \xi}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} d^n \tau \, d^n \tau' e^{-\epsilon |\xi|^2 - i\xi \cdot (\tau - \tau')} u(\tau, \tau')
\]

(56)

As the integrand is absolutely integrable on \( \mathbb{R}^{3n} \), Fubini’s theorem permits us to reorder the integrations and perform the \( \xi \) integral first, thus obtaining

\[
\int_{\mathbb{R}^n} \frac{d^n \xi}{(2\pi)^n} e^{-\epsilon |\xi|^2} \hat{u}(-\xi, \xi) = \int_{\mathbb{R}^n} \frac{d^n \xi}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} d^n \tau \, d^n \tau' \varphi_\epsilon(\tau - \tau') u(\tau, \tau')
\]

\[
= \int_{\mathbb{R}^n \times \mathbb{R}^n} d^n \tau d^n \tau' \varphi_\epsilon(t') u(t + t'/2, t - t'/2)
\]

\[
= \int_{\mathbb{R}^n} d^n t' \varphi_\epsilon(t') \int_{\mathbb{R}^n} d^n t u(t + t'/2, t - t'/2)
\]

(57)

where \( \varphi_\epsilon(t) = (4\pi \epsilon)^{-n/2} e^{-|t|^2/(4 \epsilon)} \). We have made the change of variables \( t = (\tau + \tau')/2 \), \( t' = \tau - \tau' \) (for which the Jacobian is unity) and reordered integrals using Fubini’s theorem again. As \( u \) is continuous and compactly supported, the inner integral exists for each \( t' \) and defines a continuous compactly supported function. The limit \( \epsilon \to 0^+ \) exists and yields the value of this function at \( t' = 0 \) because \( \varphi_\epsilon \) is an approximate identity. This is the required result. \( \square \)

Note that if \( \xi \mapsto \hat{u}(-\xi, \xi) \) is absolutely integrable on \( \mathbb{R}^n \) then the dominated convergence theorem permits us to dispense with the limiting procedure on the right-hand side\(^5\). For our application, we will need a straightforward generalisation of the above to distributions on manifolds.

**Lemma 2.13.** Let \((\Sigma, h)\) be a \( n \)-dimensional pseudo-Riemannian manifold and \( u \in \mathcal{E}'(\Sigma \times \Sigma) \cap H^s(\Sigma \times \Sigma) \) for \( s > n \). Suppose the support of \( u \) is contained within \( \mathbb{U} \times \mathbb{U} \) where \( \mathbb{U} \) is a single coordinate chart of \( \Sigma \) with associated coordinate map \( \kappa : \mathbb{U} \to \mathbb{R}^n \). Then

\[
\int_{\Sigma} d\text{vol}(x) \, u(x, x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} \frac{d^n \xi}{(2\pi)^n} e^{-\epsilon |\xi|^2} \hat{U}(-\xi, \xi)
\]

(58)

\(^5\)Our hypotheses are strong enough to guarantee that \( \hat{u} \) is absolutely integrable on \( \mathbb{R}^n \times \mathbb{R}^n \), and hence \( \xi \mapsto \hat{u}(\xi + \eta, \xi + \eta) \) is absolutely integrable a.e. in \( \eta \) by Fubini’s theorem. However, a simple proof of integrability for \( \eta = 0 \) was not forthcoming.
where $U : \kappa(\mathcal{U}) \times \kappa(\mathcal{U}) \to \mathbb{C}$ is defined by

$$U(x, x') = (|h_\kappa|^2 \otimes |h_\kappa|^2 u_\kappa)(x, x'), \quad (59)$$

where $u_\kappa = u \circ (\kappa^{-1} \otimes \kappa^{-1})$ and $h_\kappa$ is the determinant of the metric in coordinate chart $\kappa$.

Proof. We have

$$\int_{\Sigma} \mathrm{dvol}(x) u(x, x) = \int_{\mathbb{R}^n} \mathrm{d}^n x \left| h_\kappa(x) \right|^{1/2} u_\kappa(x, x) = \int_{\mathbb{R}^n} \mathrm{d}^n x U(x, x). \quad (60)$$

As $h_\kappa$ is a positive smooth function bounded away from zero and therefore has smooth fractional powers, we may apply lemma 2.12 to the function $U$ to obtain the desired result. \qed

3 An absolute quantum inequality

3.1 Main result

We now come to the statement and proof of our main result. Let $\Sigma$ be any $n$-dimensional timelike submanifold of $(\mathcal{M}, g)$ for $1 \leq n \leq 4$, that is, $h = \iota^* g$ is a Lorentzian metric on $\Sigma$, where $\iota : \Sigma \to \mathcal{M}$ embeds the submanifold $\Sigma$ in $\mathcal{M}$. We also equip $\Sigma$ with the time orientation induced from $\mathcal{M}$, so that non-zero future-directed causal covectors on $(\mathcal{M}, g)$ pull back to non-zero future-directed causal covectors on $(\Sigma, h)$. In our conventions a positive definite metric on a one-dimensional manifold is regarded as Lorentzian. As $\Sigma$ is timelike, its tangent space $T\Sigma$ can be annihilated only by covectors which can annihilate at least one nonzero timelike vector; in particular, all covectors in the conormal bundle $N^*\Sigma$ are spacelike.

Our aim is to obtain lower bounds, as $\omega$ varies among Hadamard states, on quantities of the form

$$\int_{\Sigma} \mathrm{dvol}(x) f^2(x) \left( Q \otimes Q(\Lambda_\omega - H_2) \right)(x, x) \quad (61)$$

where $Q = q^a \nabla_a + b$ is a partial differential operator with smooth real-valued coefficients $q^a$ and $b$ defined on a neighbourhood of $\Sigma$ and $f \in C_0^\infty(\Sigma)$ is real valued. Note that $\Lambda_\omega - H_2$ is $C^2$ so the coincidence limit is well defined. For simplicity it is convenient to assume in addition that $\Sigma$ may be covered by a single coordinate chart with certain properties.

Definition. A small sampling domain is an $n$-dimensional timelike submanifold $\Sigma$ of $(\mathcal{M}, g)$ such that (i) $\Sigma$ is contained in a globally hyperbolic convex normal neighbourhood in $\mathcal{M}$; (ii) $\Sigma$ may be covered by a single hyperbolic coordinate chart, i.e., a coordinate
system $x^0, \ldots, x^{n-1}$ on $\Sigma$ with $\partial/\partial x^0$ future-pointing and timelike, and for which there exists a constant $c > 0$ such that all causal covectors $u_\alpha$ on $\Sigma$ obey

$$c|u_0| \geq \sqrt{\sum_{j=0}^{n-1} u_j^2}$$

(i.e. the coordinate speed of light is bounded from above).

A sufficient condition for the existence of a maximum coordinate speed of light is that $h_{00} > \epsilon$ and $|\det(h_{ij})|_{i,j=1}^{n-1} > \epsilon$ for some $\epsilon > 0$.

It is easy to verify (e.g. by using suitable normal coordinates) that every point of a general timelike submanifold $\Sigma$ has a neighbourhood (in $\Sigma$) which is a small sampling domain. Thus any integral over a compact subset of a timelike submanifold may be decomposed into finitely many integrals over small sampling domains by a partition of unity.

Suppose then, that $\Sigma$ is a small sampling domain in $(\mathcal{M}, g)$ with hyperbolic chart $\{x^a\}_{a=0, \ldots, n-1}$. We may express these coordinates by a map $\kappa : \Sigma \to \mathbb{R}^n$, $\kappa(p) = (x^0(p), \ldots, x^{n-1}(p))$ and write $\Sigma_\kappa = \kappa(\Sigma)$. Any function $F$ on $\Sigma$ determines a function $F_\kappa = F \circ \kappa^{-1}$ on $\Sigma_\kappa$; in particular, we have a smooth map $\iota_\kappa : \Sigma_\kappa \to \mathcal{M}$. The significance of $\kappa$ being hyperbolic is that the bundle $\mathbb{R}^+$ of (non-zero) future pointing null covectors on $(\mathcal{M}, g)$ pulls back under $\iota_\kappa$ so that

$$\iota_\kappa^* \mathbb{R}^+ \subset \Sigma_\kappa \times \Gamma$$

where $\Gamma \subset \mathbb{R}^n$ is the set of all $u_\alpha$ with $u_0 > 0$ and satisfying (62), which means that $\Gamma$ is a proper subset of the upper half-space $\mathbb{R}^+ \times \mathbb{R}^{n-1}$ of $\mathbb{R}^n$ (here, we regard $\mathbb{R}^+ = [0, \infty)$).

We now state our main result:

**Theorem 3.1.** Let $\Sigma$ be a small sampling domain of dimension $n$ in $(\mathcal{M}, g)$ with hyperbolic coordinate map $\kappa$ and suppose $Q$ is a partial differential operator of order at most one with smooth real-valued coefficients in a neighbourhood of $\Sigma$. Set $k = \max\{n+3, 5\}$. For any real-valued $f \in C_0^\infty(\Sigma)$ and any Hadamard state $\omega$ we have the inequality

$$\int_\Sigma \text{dvol}(x) f^2(x)(Q \otimes Q(\Lambda_\omega - H_2))(x, x) \geq -B > -\infty$$

where

$$B = 2 \int_{\mathbb{R}^+ \times \mathbb{R}^{n-1}} \frac{d^n \xi}{(2\pi)^n} \left[ h_{\kappa} \frac{1}{2} f_{\kappa} \otimes |h_{\kappa}| \frac{1}{2} f_{\kappa} \vartheta_{\kappa} \right] \left((Q \otimes Q \tilde{H}_k)\right)^\wedge (-\xi, \xi),$$

$h_{\kappa}$ is the determinant of the matrix $\kappa^* h$ and $\vartheta : \Sigma \times \Sigma \to \mathcal{M} \times \mathcal{M}$ is the map $\vartheta(x, x') = (\iota \otimes \iota)(x, x')$.

**Remarks:** This bound depends nontrivially on the coordinates (and on any partition of unity used to reduce a general timelike submanifold into small sampling domains). In
we will discuss some classes of QEI averages which, in a sense, determine a natural choice of coordinates. Note that although $H_2$ is sufficient to renormalise the left-hand side, the bound is given in terms of $\tilde{H}_k$ for $k \geq 5$. Similar results hold when $Q$ has higher order, for suitably modified values of $k$; we have restricted attention to the cases relevant to QEIs.

Our strategy will be to mimic the proof of the general worldline quantum energy inequality presented in [10] but with Sobolev wave-front sets, as opposed to the smooth wave-front sets used in that paper.

**Proof of theorem 3.1.** We will break the proof into three parts. Part one will establish
\[
\int_{\Sigma} \text{dvol}(x) f^2(x) Q \otimes Q (\Lambda_\omega - H_2)(x, x)
\]
\[
= 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^+ \times \mathbb{R}^{n-1}} \frac{\text{d}^n \xi}{(2\pi)^n} e^{-\epsilon|\xi|^2} \left[ |h_{\kappa}| \frac{1}{\xi} f_{\kappa} \otimes |h_{\kappa}| \frac{1}{\xi} f_{\kappa} \vartheta^* \right] \left( Q \otimes Q (\Lambda_\omega - \tilde{H}_k) \right)(-\xi, \xi) \tag{66}
\]
for the given values of $k$. Part two contains a positivity result which enables us to discard the state dependent contribution to the right hand side of (66) to obtain the inequality
\[
\int_{\Sigma} \text{dvol}(x) f^2(x) Q \otimes Q (\Lambda_\omega - H_2)(x, x)
\]
\[
\geq -2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^+ \times \mathbb{R}^{n-1}} \frac{\text{d}^n \xi}{(2\pi)^n} e^{-\epsilon|\xi|^2} \left[ |h_{\kappa}| \frac{1}{\xi} f_{\kappa} \otimes |h_{\kappa}| \frac{1}{\xi} f_{\kappa} \vartheta^* \right] \left( Q \otimes Q \tilde{H}_k \right)(-\xi, \xi) \tag{67}
\]
Then, in part three, we show that the right-hand side of this expression is finite and equal to the required lower bound $-B$.

**PART ONE:** We begin by observing that $\Lambda_\omega - H_2$ and $\Lambda_\omega - \tilde{H}_k$ coincide on the diagonal in $\Sigma \times \Sigma$, so we may write
\[
\int_{\Sigma} \text{dvol}(x) f^2(x) \left( Q \otimes Q (\Lambda_\omega - H_2) \right)(x, x)
\]
\[
= \int_{\Sigma} \text{dvol}(x) f^2(x) \vartheta^* \left( Q \otimes Q (\Lambda_\omega - \tilde{H}_k) \right)(x, x). \tag{68}
\]
The latter form has the merit that $\Lambda_\omega - \tilde{H}_k$ is symmetric and more regular than $\Lambda_\omega - H_2$. We have also written in the restriction map $\vartheta^*$ explicitly, anticipating later steps in the proof. By hypothesis on $k$, we may choose $s \in (6, k + 3/2)$, and Proposition 2.9 tells us that within an ultra-regular domain $X \subset \mathcal{M} \times \mathcal{M}$
\[
\Lambda_\omega - \tilde{H}_k \in H^s_{\text{loc}}(X). \tag{69}
\]
As $Q \otimes Q$ is at most second order, lemma 2.4 entails that
\[
Q \otimes Q (\Lambda_\omega - \tilde{H}_k) \in H^{s-2}_{\text{loc}}(X). \tag{70}
\]
As the wave-front set $WF^{s-2}(Q \otimes Q(\Lambda_\omega - \tilde{H}_k))$ is therefore empty and $s - 2 > 4 - n$, Beals’ restriction theorem, theorem 2.11 yields

$$\vartheta^*Q \otimes Q(\Lambda_\omega - \tilde{H}_k) \in H^{n+s-6}_{\text{loc}}(\Sigma \times \Sigma),$$

(71)

so the point-splitting identity, lemma 2.13 may be applied to give

$$\int_\Sigma \text{dvol}(x) f^2(x) \vartheta^*Q \otimes Q(\Lambda_\omega - \tilde{H}_k)(x, x)$$

$$= \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} \frac{d^n \xi}{(2\pi)^n} e^{-\epsilon|\xi|^2} \left[|h_\kappa|^\frac{1}{4} f_\kappa \otimes |h_\kappa|^\frac{1}{4} f_\kappa \vartheta^*_\kappa Q \otimes Q(\Lambda_\omega - \tilde{H}_k)\right]^\wedge (-\xi, \xi).$$

(72)

Then, as $\Lambda_\omega - \tilde{H}_k$ is symmetric and $C^k$, the integrand of (72) is invariant under $\xi \to -\xi$, so we may replace the integration over $\mathbb{R}^n$ with that over $\mathbb{R}^+ \times \mathbb{R}^{n-1}$ at the expense of a factor of two, thus obtaining (66). Note that we are now integrating over those $\xi$ with $\xi_0 \geq 0$. This particular half-space of $\mathbb{R}^n$ is chosen because it contains the cone $\Gamma$ defined after (63).

**PART TWO:** We now assert that the pull-backs of $(Q \otimes Q)\Lambda_\omega$ and $(Q \otimes Q)\tilde{H}_k$ exist separately, which enables us to split the integrand on the right-hand side of (66) into two parts. Moreover, we will show that the first of these, namely $[|h_\kappa|^\frac{1}{4} f_\kappa \otimes |h_\kappa|^\frac{1}{4} f_\kappa \vartheta^*_\kappa Q \otimes Q\Lambda_\omega]^\wedge (-\xi, \xi)$, is nonnegative for all $\xi \in \mathbb{R}^n$.

The existence of the required pull-backs follows because we have already observed that non-zero covectors in $N^*\Sigma$ must be spacelike. As the covectors in the wave-front set of $(Q \otimes Q)\Lambda_\omega$ and $(Q \otimes Q)\tilde{H}_k$ are null, it follows that there is no intersection between their wave-front sets (at any Sobolev order) and $N^*\Sigma \times N^*\Sigma$. Therefore the pull-backs exist.

To establish that $[|h_\kappa|^\frac{1}{4} f_\kappa \otimes |h_\kappa|^\frac{1}{4} f_\kappa \vartheta^*_\kappa Q \otimes Q\Lambda_\omega]^\wedge (-\xi, \xi) \geq 0$, we define a one parameter family of functions $f_\kappa^\xi(x) = [h_\kappa(x)]^\frac{1}{4} f_\kappa(x) e^{i\xi \cdot x}$ then, as $f_\kappa \in C_0^\infty(\mathbb{R}^n)$, $[|h_\kappa|^\frac{1}{4} f_\kappa \otimes |h_\kappa|^\frac{1}{4} f_\kappa \vartheta^*_\kappa Q \otimes Q\Lambda_\omega] \in \mathcal{E}'(\Sigma_\kappa \times \Sigma_\kappa)$ where the Fourier transform is defined by $\hat{u}(\xi, \xi') = u(e^{i\xi}, e^{i\xi'}) \forall u \in \mathcal{E}'(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore,

$$\left[|h_\kappa|^\frac{1}{4} f_\kappa \otimes |h_\kappa|^\frac{1}{4} f_\kappa \vartheta^*_\kappa Q \otimes Q\Lambda_\omega\right]^\wedge (-\xi, \xi)$$

$$= \left[|h_\kappa|^\frac{1}{4} f_\kappa \otimes |h_\kappa|^\frac{1}{4} f_\kappa \vartheta^*_\kappa Q \otimes Q\Lambda_\omega\right](e^{-i\xi}, e^{i\xi'})$$

(73)

$$= \left[\vartheta^*_\kappa Q \otimes Q\Lambda_\omega\right](\hat{f}_\kappa^\xi, f_\kappa^\xi)$$

(74)

where we have exploited the fact that $f$ is real valued. It is clear that if $u$ is a distribution of positive type then $Q \otimes Qu$ is also of positive type because $Q$ has real coefficients. Accordingly, lemma 2.11 establishes that $\vartheta^*Q \otimes Q\Lambda_\omega$ is a distribution of positive type and the assertion is justified. Hence, we obtain the inequality (67) provided that the limit on the right-hand side exists and is finite, which is the remaining step in the proof.
Note that the integral converges for each \( \epsilon > 0 \) because the Fourier transform of any compactly supported distribution is polynomially bounded.

**Part three:** Our aim is to show that \( I(\xi) := \left[ |h_\kappa|^{\frac{1}{2}} f_\kappa \otimes |h_\kappa|^{\frac{1}{2}} f_\kappa \vartheta_\kappa Q \otimes Q \tilde{H}_k \right] (\xi, \xi) \) is absolutely integrable on the integration region \( \mathbb{R}^+ \times \mathbb{R}^{n-1} \), for then we may conclude that the limit on the right-hand side of (67) exists by dominated convergence and equals \( \mathcal{B} \) (which is thereby finite). To do this, we introduce an arbitrary Hadamard state \( \omega_0 \) and use the Hadamard series definition of Hadamard states to write \( \tilde{H}_k = \Lambda_{\omega_0} + F_k \) for some \( F_k \in C^k(X) \). We consider the contributions of these terms to \( I(\xi) \) in turn.

First, the results of Radzikowski and Beals entail that

\[
WF(\vartheta_\kappa^* Q \otimes Q \Lambda_{\omega_0}) \subset \vartheta_\kappa^* WF(Q \otimes Q \Lambda_{\omega_0}) \subset \vartheta_\kappa^* (\mathcal{R}^+ \times \mathcal{R}^-),
\]

(75)

where \( \mathcal{R}^\pm \) are the bundles of future- and past-directed null covectors defined earlier. Thus, we have

\[
WF(\vartheta_\kappa^* Q \otimes Q \Lambda_{\omega_0}) \subset \iota_\kappa^* \mathcal{R}^+ \times \iota_\kappa^* \mathcal{R}^- \subset (\Sigma_\kappa \times \Gamma) \times (\Sigma_\kappa \times (-\Gamma))
\]

(76)

using equation (63) and its obvious analogue for \( \mathcal{R}^- \).

By Prop. 8.1.3 in [35], it follows that the Fourier transform of localisations of \( \vartheta_\kappa^* Q \otimes Q \Lambda_{\omega_0} \) is of rapid decay outside the cone \( \Gamma \times (-\Gamma) \); in particular we have rapid decay in the cone \((-\mathbb{R}^+ \times \mathbb{R}^{n-1}) \times (\mathbb{R}^+ \times \mathbb{R}^{n-1})\) (here we have used the assumption that \( \Gamma \) is a proper subset of \( \mathbb{R}^+ \times \mathbb{R}^{n-1} \) because \( \kappa \) is hyperbolic). Accordingly we find that

\[
||h_\kappa|^{\frac{1}{2}} f_\kappa \otimes |h_\kappa|^{\frac{1}{2}} f_\kappa \vartheta_\kappa^* Q \otimes Q \Lambda_{\omega_0}|^*(\xi, \xi) \text{ is rapidly decaying in the integration region } \mathbb{R}^+ \times \mathbb{R}^{n-1} \text{ and is therefore absolutely integrable there.}
\]

It remains to show that the \( F_k \) dependent contribution to \( I(\xi) \) is also absolutely integrable. As \( F_k \in C^k(X) \), we have \( (Q \otimes Q) F_k \in C^{k-2}(X) \). Hence there is a constant \( c \) such that

\[
|\left\langle |h_\kappa|^{\frac{1}{2}} f_\kappa \otimes |h_\kappa|^{\frac{1}{2}} f_\kappa \vartheta_\kappa^* Q \otimes Q F_k \right\rangle(\xi, \xi')| \leq \frac{c}{(1 + |\xi|^2 + |\xi'|^2)^{(k-2)/2}}
\]

(77)

for all \( (\xi, \xi') \in \mathbb{R}^n \times \mathbb{R}^n \) because \( f_\kappa \) is compactly supported. As \( (k-2) > n \) for the values of \( k \) given in the hypotheses, it follows in particular that left-hand side is absolutely integrable on \( \mathbb{R}^+ \times \mathbb{R}^{n-1} \).

Accordingly, we have shown that \( I \in L^1(\mathbb{R}^+ \times \mathbb{R}^{n-1}) \), and the dominated convergence argument mentioned above completes the proof. \( \square \)

Two points should noted about the foregoing proof. First, the state \( \omega_0 \) was introduced purely as a convenient way of showing that our bound is finite; the bound itself does not depend on any reference state. Second, in the difference QELs studied in [10] the Gaussian cut-off was not necessary, because the point-splitting lemma was applied to a smooth compactly supported function. Moreover, the place of \( \tilde{H}_k \) was taken by the
two-point function of a reference state $\Lambda_{\omega}$ and the fact that $WF(\Lambda_{\omega}) = C^{+-}$ was used to show that the integrand decays rapidly in the integration region. This line of argument was not available to us here, because $\hat{H}_k$ (in contrast to $H_k$) has a portion of its wave-front set lying in $C^{-+}$ (see Proposition 2.8).

In the next subsection, we will show how Theorem 3.1 may be used to obtain QEI bounds, by appropriate choices of the operator $Q$.

### 3.2 Examples

#### 3.2.1 Worldvolume absolute quantum null energy inequality

Our first example is a quantum null energy inequality (QNEI), that is, a lower bound on quantities of the form $\int_M dvol \langle F_{ab} T^{\text{ren}}_{ab} \rangle_\omega$, where $F_{ab} = n^a n^b$ and $n^a$ is a smooth, compactly supported null vector field on $(\mathcal{M}, g)$ that is future-directed where it is nonzero. We will show how Theorem 3.1 allows us to obtain an absolute QEI on $\int_M dvol \langle F_{ab} T^{\text{ren}}_{ab} \rangle_\omega$. To do this, we suppose that $F_{ab}$ is supported within an open subset $\Sigma$ that is a four-dimensional small sampling domain in $(\mathcal{M}, g)$ with hyperbolic chart $\kappa$.

Noting that 

$$\langle F_{ab} T^{\text{ren}}_{ab} \rangle_\omega(x) = \lim_{x' \to x} \left( n^a \nabla_a \otimes n^b \nabla_{b'} \right) \left( (\Lambda_{\omega} - H_2)(x, x') + C_{ab} F_{ab}(x) \right),$$

we apply Theorem 3.1 with $Q = n \cdot \nabla$ and $f \in C^\infty_0(\Sigma)$ chosen to be real-valued and to equal unity on the support of $F_{ab}$. This yields the absolute QNEI:

$$\int_M dvol \langle F_{ab} T^{\text{ren}}_{ab} \rangle_\omega \geq -2 \int_{\mathbb{R}^+ \times \mathbb{R}^3} \frac{d^4 \xi}{(2\pi)^4} \left[ |g_{\kappa}|^{\frac{1}{2}} \otimes |g_{\kappa}|^{\frac{1}{2}} \left( (n \cdot \nabla \otimes n \cdot \nabla) \hat{H}_T \right) \right] (-\xi, \xi) + \int_M dvol F_{ab} C_{ab}$$

for all Hadamard states $\omega$.

Clearly the right-hand side of this inequality depends explicitly on the choice of coordinates $\kappa$. In Section 3.3 we will explain how this problem may be removed by restricting the class of sampling tensors $F_{ab}$ in such a way that there is a canonical class of coordinate systems, all of which give the same lower bound.

#### 3.2.2 Worldline absolute quantum weak energy inequality

Our second example applies Theorem 3.1 to the energy density sampled along a smooth timelike worldline $\gamma$. This was the situation studied in [10], where a difference QEI was obtained. We assume $\gamma$ is given in a proper time parameterisation as a smooth function $\gamma : I \to \mathbb{R}$, where $I$ is a possibly unbounded open interval of $\mathbb{R}$ and denote the four-velocity of the curve by $u = \dot{\gamma}$. The curve forms a small sampling domain, with
the proper time parameterisation as a hyperbolic coordinate system, provided that the 
track of \( \gamma \) can be contained in a globally hyperbolic convex normal neighbourhood in 
\( \mathcal{M} \).

The classical energy density of a field \( \varphi(x) \) along \( \gamma \) may be written in the form 
\[
 u^a u^b T_{ab}(x) = (T^{\text{split}}(\varphi \otimes \varphi))(x, x) \tag{80}
\]
where the point split energy density operator is defined within a suitable neighbourhood 
\( \mathcal{U} \) of \( \gamma \) by 
\[
 T^{\text{split}} = \frac{1}{2} \sum_{\alpha=0}^{3} \epsilon_\alpha^a \nabla_\alpha \otimes \epsilon_\alpha^b \nabla_{\nu'} + \frac{1}{2} u^2 \mathbb{1} \otimes \mathbb{1}, \tag{81}
\]
and \( \{\epsilon_\alpha^a\}_{\alpha=0,1,2,3} \) is any smooth tetrad defined in a neighbourhood of \( \gamma \) such that \( \epsilon_0^a = u^a \) 
on \( \gamma \). This operator may be used to define the renormalised energy density in the usual 
fashion. Given any real-valued \( f \in C^\infty_0(I) \), we may apply Theorem 3.1 in turn to the 
operators \( Q_\alpha = \epsilon_\alpha \cdot \nabla \) to obtain the absolute quantum weak energy inequality 
\[
 \int_{\mathbb{R}} d\tau f^2(\tau) \langle u^a u^b T^{\text{ren}}_{ab} \rangle_\omega(\gamma(\tau)) \geq -\mathcal{B} \overset{\text{def}}{=} \int_{\mathbb{R}^+} \frac{d\xi}{\pi} \left[ f \otimes f \vartheta^* T^{\text{split}} H_{-1} \right]^\wedge (-\xi, \xi) \tag{82}
\]
where \( \vartheta : (\tau, \tau') \mapsto (\gamma(\tau), \gamma(\tau')) \).

For the purposes of comparison with existing QEI results, let us consider this bound 
for the massless Klein–Gordon field in Minkowski spacetime \( (\mathbb{R}^4, \eta) \) for a worldline along 
the time axis. In this case, the bound simplifies because the full Hadamard series is given 
by the leading term; that is, \( \Lambda_\omega(x, x') - 1/(4\pi^2 \sigma_+(x, x')) \) is smooth and symmetric for 
any Hadamard state \( \omega \). Of course, \( \sigma_+ \) is globally defined in Minkowski space. Now 
consider the example above, applied to the case where \( \gamma \) is an inertial curve, 
\[
 \int_{\mathbb{R}} d\tau f^2(\tau) \langle u^a u^b T^{\text{ren}}_{ab} \rangle_\omega(\gamma(\tau)) \geq -\mathcal{B} \overset{\text{def}}{=} \int_{\mathbb{R}^+} \frac{d\xi}{\pi} \left[ f \otimes f \vartheta^* T^{\text{split}} H_{-1} \right]^\wedge (-\xi, \xi) \tag{83}
\]
where we have denoted \( H_{-1} = 1/(4\pi^2 \sigma_+) = \tilde{H}_{-1} \). Then, 
\[
 \mathcal{B} = \frac{3}{2\pi^2} \int_{\mathbb{R}^+} \frac{d\xi}{\pi} \lim_{\epsilon \to 0^+} \int_{\mathbb{R} \times \mathbb{R}} dt \, dt' \frac{f(t) f(t')}{(t - t' - i\epsilon)^4} e^{-i(t-t')} \tag{84}
\]
\[
 = \frac{3}{2\pi^2} \int_{\mathbb{R}^+} \frac{d\xi}{\pi} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} dt \, F(-t) \frac{1}{(t - i\epsilon)^4} e^{-i\xi t} \tag{85}
\]
where \( F(t) = \int_{\mathbb{R}} dt' f(t' - t) f(t') \) has Fourier transform \( \hat{F}(\xi) = |\hat{f}(\xi)|^2 \). Thus 
\[
 \mathcal{B} = \frac{1}{4\pi^2} \int_{\mathbb{R}^+} \frac{d\xi}{\pi} \int_{\mathbb{R}^+} d\zeta |\hat{f}(\xi + \zeta)|^2 \zeta^3 \tag{86}
\]
\[
 = \frac{1}{4\pi^2} \int_{\mathbb{R}^+} d\eta \int_{\eta}^\infty d\zeta |\hat{f}(\eta)|^2 \zeta^3 \tag{87}
\]
where we have utilised the fact that the Fourier transform of $1/(t - i0^+)^4$ is $\pi \xi^3 \theta(\xi)/3$ and changed variables to $\eta = \xi + \zeta$. Hence,

$$\int_{\mathbb{R}} dt \, f^2(t) \langle u^a u^b T_{\text{ren}}^{ab}(\gamma(t)) \rangle \geq -\frac{1}{16\pi^3} \int_{\mathbb{R}^+} d\eta |\hat{f}(\eta)|^2 \eta^4$$  \hspace{1cm} (88)

which is the same as the QEI for the massless field in Minkowski spacetime obtained in [9].

This example is of particular importance as on small length scales one expects the massive quantum field in a curved background to behave like its massless counterpart in flat spacetime. We expect that the same should hold for the quantum inequalities, i.e., on small length scales the dominant contribution to the bound arises from the $1/\sigma^+$ contribution to the Hadamard series. This will be investigated in a future work.

### 3.2.3 QEIs for adiabatic states

Finally, we show how our analysis of the Hadamard series using the Sobolev wavefront set allows us to establish QEIs for adiabatic states. We refer the reader to [38] for a detailed study of adiabatic states and further references. Adiabatic states, like Hadamard states, are defined in terms of their singular structure: following [38] a state $\omega$ on $\mathfrak{A}(\mathcal{M}, g)$ is an adiabatic state of order $N$ if its associated two point function $\Lambda_\omega$ satisfies $WF^s(\Lambda_\omega) \subset C^{+,+}$ for all $s < N + 3/2$. From this definition, we see that any Hadamard state is adiabatic to all orders. In what follows the following lemma, taken from [38], is essential:

**Lemma 3.2.** Let $\omega$ be a Hadamard state and $\omega'$ be an adiabatic state of order $N$, with associated two point functions $\Lambda_\omega, \Lambda_{\omega'}$ respectively, on $\mathfrak{A}(\mathcal{M}, g)$. Then

$$WF^s(\Lambda_\omega - \Lambda_{\omega'}) = \emptyset$$  \hspace{1cm} (89)

for all $s < N + 3/2$.

An immediate corollary is that [38] also holds for all $s < N + 3/2$ if $\omega$ and $\omega'$ are any two adiabatic states of order $N$. By the Sobolev embedding theorem (Proposition 2.1 (i)), differences of this type will be in $C^2(\mathcal{M} \times \mathcal{M})$ provided $N > 9/2$, thus permitting the construction of a normal ordered stress-energy tensor. Similarly, if $\omega$ is adiabatic of order $N > 9/2$ and $k \geq 2$, a difference of the form $\Lambda_\omega - H_k$ will be twice continuously differentiable on an ultra-regular domain, permitting the computation of the renormalised stress-energy tensor. It is straightforward to modify the proof of Theorem 3.1 to obtain the following.

---

In [38] the definition of adiabatic states was given, as for Hadamard states, only for quasi-free states, so the present usage is a slight extension.
Theorem 3.3. (a) Theorem 3.1 continues to hold (with the same lower bound) under the weaker hypothesis that \( \omega \) is an adiabatic state of order \( N > 9/2 \). (b) Using the assumptions and notation of Theorem 3.1, except that \( \omega \) is assumed only to be an adiabatic state of order \( N > 9/2 \), there is a difference inequality

\[
\int_{\Sigma} \text{dvol}(x) f^2(x) Q \otimes Q (\Lambda_\omega - \Lambda_\omega') (x, x) \geq -2 \int_{\mathbb{R}^+ \times \mathbb{R}^{n-1}} \frac{d^n \xi}{(2\pi)^n} \left[ |h_\kappa|^k f_\kappa \otimes |h_\kappa|^k f'_\kappa \right] \wedge (-\xi, \xi), \tag{90}
\]

for any reference state \( \omega' \) which is adiabatic of order \( N' > n + 11/2 \).

Proof. We sketch the main points only. For (a), note that the hypotheses on \( N \) and \( k \) entail that we may choose \( s \in (6, \min\{N, k\} + 3/2) \). Part one of the proof of Theorem 3.1 will continue to hold provided that \( \Lambda_\omega - \tilde{H}_k \in H^s_{\text{loc}} \). To see this, we introduce an arbitrary Hadamard state \( \omega_0 \) and note that

\[\nabla F^s(\Lambda_\omega - \Lambda_\omega) \subset W F^s(\Lambda_\omega - \Lambda_\omega_0) \cup W F^s(\Lambda_\omega_0 - \tilde{H}_k) = \emptyset \tag{91}\]

using Lemma 3.2 together with the fact that \( s < k + 3/2 \), and Proposition 2.9 together with \( s < k + 3/2 \). Part two of the proof holds because \( \Lambda_\omega \) is a bisolution to the Klein-Gordon equation, so all covectors in its wave-front set are null, from which it follows that the required pull-back exists. The third part is identical to the original argument. For (b), the hypotheses on \( N \) and \( N' \) permit us to choose \( s \in (6, \min\{N, N'\} + 3/2) \), and we have \( W F^s(\Lambda_\omega - \Lambda_\omega_0) = \emptyset \) by the remark following Lemma 3.2. Part one of the argument then goes through, as does the second part (with \( \Lambda_\omega' \) replacing \( \tilde{H}_k \)). The remaining issue is to check that the bound is finite. Introducing a reference Hadamard state \( \omega_0 \) as before, we note that \( \Lambda_\omega' - \Lambda_\omega_0 \in H^s_{\text{loc}}(\mathcal{M} \times \mathcal{M}) \) for some \( s' \in (n + 7, N' + 3/2) \), and hence \( \Lambda_\omega' - \Lambda_\omega_0 \in C^{n+3}(\mathcal{M} \times \mathcal{M}) \) (this is the reason for the constraint \( N' > n + 11/2 \)). This is sufficient for Part three to apply to \( \Lambda_\omega' \) in place of \( \tilde{H}_k \).

3.3 Covariance

The QEIs obtained from theorem 3.1 are not covariant in full generality because they depend non-trivially on the coordinates used, and, in some cases, on a choice of tetrad near \( \Sigma \). However, covariance may be rescued if we can restrict the freedom to choose coordinates and the tetrad in a covariant fashion so that the bound is independent of any residual choice. This strategy was successfully employed in [20] for worldline difference QEIs, and the same techniques would also apply to our worldline bounds; here we show how this may be accomplished for worldvolume averages (timelike submanifolds of other dimensions could be handled in an analogous fashion, but we will not do this here for brevity).

Consider the quantum null energy inequality studied in section 3.2.1. We will show that if the sampling tensor \( F^{ab} \) picks out a preferred smooth timelike curve \( \gamma \) in a
covariant way and we employ a system of Fermi normal coordinates near $\gamma$, then residual choices in our construction cannot affect the bound. With these restrictions, our absolute QEI would be locally covariant in the sense of $[20]$; see also $[23]$ for a more abstract discussion of these ideas in the formulation of locally covariant quantum field theory developed by Brunetti, Fredenhagen & Verch $[4]$ in terms of category theory.

The requirement that $F^{ab}$ should pick out a unique timelike curve may be addressed in various ways. For example, if we restrict to sampling tensors for which there exists a (necessarily unique) pair of points $x, x' \in \mathcal{M}$ such that the support of the sampling tensor obeys

$$\text{supp } F^{ab} = J^-(x) \cap J^+(x')$$

(92)

and is contained within a convex normal neighbourhood then the unique timelike geodesic between $x'$ and $x$ may be used as our choice of $\gamma$. From now on we assume that the sampling tensor does indeed select a preferred timelike curve, and that $\gamma$ is given in a proper time parameterisation.

As already mentioned, we will restrict our coordinate system to belong to the class of Fermi normal coordinates about $\gamma$. For completeness, we briefly summarise the salient features of Fermi–Walker transport and Fermi normal coordinates, mainly following chapters 1 §4 and 2 §10 of $[46]$. Recall that a vector field $\xi$ defined on $\gamma$ is said to be Fermi–Walker transported along it if $D_{FW} \xi = 0$, where

$$D_{FW} \xi^a = (\dot{\gamma} \cdot \nabla) \xi^a - g_{bc} (\dot{\gamma}^c \alpha^a - \alpha^c \dot{\gamma}^a) \xi^b$$

(93)

and $\alpha^a = \dot{\gamma} \cdot \nabla \dot{\gamma}^a$. Since $\alpha \cdot \dot{\gamma} = 0$ and $\dot{\gamma}^2 = 1$ it is easy to see that $D_{FW} \dot{\gamma} = 0$; hence, the velocity vector is preserved under Fermi–Walker transport. Moreover, it is possible to show that Fermi–Walker transport of two vectors along $\gamma$ preserves their inner-product. Therefore, a tetrad remains an orthogonal frame along $\gamma$ under Fermi–Walker transport. If $\gamma$ is a timelike geodesic, then Fermi–Walker and parallel transport coincide.

The construction of Fermi normal coordinates near $\gamma$ proceeds as follows. Let $y$ lie on $\gamma$ and construct an oriented and time-oriented orthonormal frame $\{e^a_\alpha\}_{\alpha=0,1,2,3}$ at $y$ with $e^a_0 = \dot{\gamma}^a|_y$. Fermi–Walker transport yields a tetrad along the whole of $\gamma$. In a convex normal neighbourhood $\mathcal{U}$ of $\gamma$ each point $x \in \mathcal{U}$ will be joined to $\gamma$ by a unique spacelike geodesic segment $c$ which is orthogonal to $\gamma$ and which meets it at some $\gamma(t)$. Assuming that $c$ is parameterised by proper length, the Fermi normal coordinates $x^a$ of $x$ are

$$x^0 = t; \quad x^i = s \dot{c} \cdot e_i|_{\gamma(t)},$$

(94)

where $s$ is the proper length of $c$.

This construction has two important features. First, the metric takes the Minkowski form in these coordinates everywhere on $\gamma$. By continuity, this guarantees that the Fermi normal coordinates form a hyperbolic chart in a neighbourhood of $\gamma$. Second, the only freedom in the construction is the choice of origin on $\gamma$ (which amounts to the freedom to add a constant to $x^0$) and the choice of the spatial tetrad vectors $e_i (i = 1, 2, 3)$ at $y$, which are determined only up to a rotation. Owing to the angle-preserving nature of
Fermi–Walker transport, any two coordinate systems obtained by the construction are therefore globally related by $x^0 = x^0 + \lambda$, $x^i = S^i_j x^j$ for constant scalar $\lambda$ and constant rotation matrix $S \in SO(3)$.

If that the sampling tensor is supported within the neighbourhood of $\gamma$ in which the Fermi normal coordinates are hyperbolic, it is easy to see that the absolute QEI (79) is independent of the particular system of Fermi normal coordinates chosen. The key point is that the Jacobian determinant for a change of coordinates between two Fermi normal coordinate systems is identically unity by the remarks given above. (Note also that $F^{ab}$ may be written uniquely as $F^{ab} = n^a n^b$ under the constraint that $n^a$ is future-pointing and null where it is nonzero.)

For more general QEIs one also needs to construct a tetrad throughout the support of the sampling tensor. This may be done by taking the tetrad formed along $\gamma$ and propagating it by parallel transport along spacelike geodesics which meet $\gamma$ orthogonally.

4 Conclusion

We have given the first explicit absolute quantum energy inequalities for the massive minimally coupled Klein–Gordon field in arbitrary four-dimensional globally hyperbolic backgrounds, by refining the argument of [10] to make use of the theory of the Sobolev wave-front set, and analysing microlocal properties of the components of the Hadamard series. The lower bounds are given in terms of partial sums of the Hadamard series, which are computed locally. Previously explicit absolute quantum energy inequalities were known only for the massless field in two dimensions [25] (a similar argument could be used to extend this to general positive energy unitary conformal field theories, based on [18]). Although the bounds make use of coordinate systems, we have shown that by restricting the class of sampling tensors, there are circumstances in which the bound is covariant.

Absolute QEIs may also be found for higher spin fields. In the case of the Dirac field, which will be reported elsewhere [45], one adapts the difference QEI obtained in [6] in a similar fashion to the way in which [10] has been adapted here. Moreover, it is expected that one should be able to employ our method to prove an absolute QEI for the spin-1 vector bosons, using the formulation of the Hadamard condition for the Maxwell and Proca fields given in [14].

One possibility which is opened up by our work is to obtain control over the size of spacetime region in which the absolute QEI bound can be approximated to a good degree by the QEIs obtained in Minkowski space for massless fields. This would potentially result in very simple bounds of wide applicability, and is the subject of ongoing work.
A Hadamard recursion relations

In this appendix we briefly summarise the method for generating the coefficient functions, \( \{v_j\}_{j=0,...,k} \) and \( \{w_j\}_{j=0,...,k} \), featuring in (12). These are obtained as the coefficients in the formal power series solution

\[
H(x, x') = \frac{1}{4\pi^2} \left\{ \frac{\Delta \hat{\gamma}(x, x')}{\sigma_+(x, x')} + \sum_{j=0}^{\infty} v_j(x, x') \frac{\sigma_j(x, x')}{\ell^{2(j+1)}} \ln \left( \frac{\sigma_+(x, x')}{\ell^2} \right) \right. \\
\left. + \sum_{j=0}^{\infty} w_j(x, x') \frac{\sigma_j(x, x')}{\ell^{2(j+1)}} \right\}. 
\]

(95)

to

\[
\left( (\nabla^2 + \mu^2) \otimes 1 \right) H_k(x, x') = 0 \quad \text{subject to} \quad w_0 = 0.
\]

(96)

The series does not actually converge except in analytic spacetimes, which is why one makes use of the partial sums \( H_k \). The recursion relations for the \( v_j \) for the massive field in a curved background are:

\[
0 = \ell^2 (\nabla^2 + \mu^2) \Delta \hat{\gamma} + 2 \nabla v_0 \cdot \nabla \sigma + 4 v_0 + v_0 \nabla^2 \sigma
\]

(97)

\[
0 = \ell^2 (\nabla^2 + \mu^2) v_j + 2 (j+1) \nabla v_{j+1} \cdot \nabla \sigma - 4 j (j+1) v_{j+1} + (j+1) v_{j+1} \nabla^2 \sigma
\]

(98)

where \( j \in \{0\} \cup \mathbb{N} \). In a regular domain \( X \) the system of differential equations uniquely determines the series of \( v_j \)'s. The \( w_j \) series is specified once the value of \( w_0 \) is fixed; we have adopted Wald’s prescription that \( w_0 = 0 \) \([48]\) and with this boundary condition the recursion relations are:

\[
0 = 2 \nabla w_1 \cdot \nabla \sigma + w_1 \nabla^2 \sigma + 2 \nabla v_1 \cdot \nabla \sigma - 4 v_1 + v_1 \nabla^2 \sigma
\]

(99)

\[
0 = \ell^2 (\nabla^2 + \mu^2) w_k + 2 (k+1) \nabla w_{k+1} \cdot \nabla \sigma - 4 k(k+1) w_{k+1} + (k+1) w_{k+1} \nabla^2 \sigma + 2 \nabla v_{k+1} \cdot \nabla \sigma - 4 (2k+1) v_{k+1} + v_{k+1} \nabla^2 \sigma
\]

(100)

where \( k \in \mathbb{N} \). The system of equations \([97,98,99,100]\) are known as the Hadamard recursion relations; these relations for the massless field may be found in \([1,7]\) where the dependency on a choice of length scale \( \ell \) is suppressed.

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References

[1] Adler S.L., Lieberman J. & Ng Y.J., 1976, *Regularization of the stress energy tensor for vector and scalar particles propagating in a general background metric*, Ann. Phys. 106, 279-321

[2] Alcubierre M., 1994, *The warp drive: hyper-fast travel within general relativity*, Class. Quantum Grav. 11, L73-L77

[3] Brunetti R. & Fredenhagen K., 2000, *Microlocal Analysis and Interacting Quantum Field Theories: renormalization on Physical Backgrounds*, Commun. Math. Phys. 208, 623-661

[4] Brunetti R., Fredenhagen K. & Verch R., 2003, *The generally covariant locality principle - A new paradigm for local quantum field theory*, Commun. Math. Phys. 237, 31-68

[5] Dawson S.P., 2006, *A quantum weak energy inequality for the Dirac field in two-dimensional flat spacetime*, Class. Quantum Grav. 23, 287-293

[6] Dawson S.P. & Fewster C.J., 2006, *An explicit quantum weak energy inequality for Dirac fields in curved spacetimes*, Class. Quantum Grav. 23, 6659-6681

[7] DeWitt B.S. & Brehme R.W., 1960, *Radiation damping in a gravitational field*, Ann. Phys. 9, 220-259

[8] Epstein H., Jaffe A. & Glaser V., 1965, *Nonpositivity in the energy density in quantised field theories*, Nuovo Cimento 36, 1016-1022

[9] Fewster C.J. & Eveson S.P., 1998, *Bounds on negative energy densities in flat spacetime*, Phys. Rev. D 58, 084010

[10] Fewster C.J., 2000, *A general worldline quantum inequality*, Class. Quantum Grav. 17, 1897-1911

[11] Fewster C.J. & Teo E., 1999, *Bounds on negative energy densities in static spacetimes*, Phys. Rev D 59, 104016

[12] Fewster C.J. & Verch R., 2002, *A quantum weak energy inequality for Dirac fields in curved spacetime*, Commun. Math. Phys. 225, 331-359

[13] Fewster C.J. & Mistry B., 2003, *Quantum weak energy inequalities for the Dirac field in flat spacetime*, Phys. Rev. D 68, 105010

[14] Fewster C.J. & Pfenning M.J., 2003, *A weak quantum energy inequality for spin-one fields in curved spacetime*, J. Math. Phys. 44, 4480-4513

[15] Fewster C.J., 2002, *Quantum energy inequalities in two dimensions*, Phys. Rev. D 70, 127501

[16] Fewster C.J., 2005, *Energy inequalities in quantum field theory*, in XIVth International Congress on Mathematical Physics, ed. J.C. Zambrini (World Scientific, Singapore, 2005). See [math-ph/0501073](http://arxiv.org/abs/math-ph/0501073) for an expanded and updated version.

[17] Fewster C.J., 2005, *Quantum energy inequalities and stability conditions in quantum field theory*, in Rigorous Quantum Field Theory: A Festschrift for Jacques Bros,
[18] Fewster C.J. & Hollands S., 2005, Quantum energy inequalities in two-dimensional conformal field theory, Rev. Math. Phys. 17, 577-612
[19] Fewster C.J. & Roman T.A., 2005, On wormholes with arbitrarily small quantities of exotic matter, Phys. Rev. D 72, 044023
[20] Fewster C.J. & Pfenning M.J., 2006, Quantum energy inequalities and local covariance I: Globally hyperbolic spacetimes, J. Math. Phys. 47, 082303
[21] Fewster C.J. & Osterbrink L.W., 2006, Averaged inequalities for the non-minimally coupled classical scalar field, Phys. Rev. D 74, 044021
[22] Fewster C.J. & Osterbrink L.W., 2007, Quantum Energy Inequalities for the Non-Minimally Coupled Scalar Field, J. Phys. A: Math. Theor. 41, 025402
[23] Fewster C.J., 2006, Quantum energy inequalities and local covariance II: Categorical formulation, Gen. Rel. and Grav. 39, 1855-1890
[24] Flanagan É.E., 1997, Quantum inequalities in two-dimensional Minkowski spacetime, Phys. Rev. D 56, 4922-4926
[25] Flanagan É.E., 2002, Quantum inequalities in two-dimensional curved spacetimes, Phys. Rev. D 66, 104007
[26] Ford L.H., 1978, Quantum coherence effects and the second law of thermodynamics, Proc. R. Soc. Lond. A. 364, 227-236
[27] Ford L.H. & Roman T.A., 1996, Quantum field theory constrains traversable wormhole geometries, Phys. Rev. D 53, 5496-5507
[28] Ford L.H. & Pfenning M.J., 1997, The unphysical nature of “warp drive”, Class. Quantum Grav. 14, 1743-1751
[29] Fulling S.A., Sweeny M. & Wald R.M., 1978, Singularity structure of the two-point function in quantum field theory in curved spacetime, Commun. Math. Phys. 65, 257-264
[30] Fulling S.A., Narcowich F.J. & Wald R.M., 1981, Singularity structure of the two-point function in quantum field theory in curved spacetime II, Ann. Phys. (N.Y.) 136, 243-272
[31] Gel’fand I.M. & Shilov G.E., 1964, Generalised functions, Academic Press, New York and London
[32] Günther P., 1988, Huygen’s principle and hyperbolic equations, Academic Press Inc, New York
[33] Hollands S. & Wald R.M., 2001, Local Wick Polynomials and Time Ordered Products of Quantum Fields in Curved Spacetime, Commun. Math. Phys. 223, 289-326
[34] Hollands S. & Wald R.M., 2002, Existence of Local Covariant Time Ordered Products of Quantum Fields in Curved Spacetime, Commun. Math. Phys. 231, 309-345
[35] Hörmander L., 1989, The analysis of linear partial differential operators I, second edition, Springer-Verlag, New York
[36] Hörmander L., 1996, Lectures on nonlinear hyperbolic differential equations, Springer, New York
[37] Hu B., Ling Y. & Zhang H., 2006, Quantum inequalities for massless spin-3/2 field in Minkowski spacetime, Phys. Rev. D 73, 045015
[38] Junker W. & Schrohe E., 2002, Adiabatic vacuum states on general spacetime manifolds: Definition, construction, and physical properties, Annales Poincaré Phys. Theor. 3, 1113-1182
[39] Kay B.S. & Wald R.M., 1991, Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon, Phys. Rep. 207, 49-136
[40] Moretti V., 2003, Comments on the stress-energy tensor operator in curved spacetime, Commun. Math. Phys. 232, 189-221
[41] Pfenning M., 1998, PhD Thesis Quantum inequality restrictions on negative energy densities on curved spacetimes, pre-print gr-qc/9805037
[42] Radzikowski M., 1996, Micro-local approach to the Hadamard condition in quantum field theory on curved space-time, Commun. Math. Phys. 179, 529-553
[43] Reed M. & Simon B., 1975, Methods of modern mathematical physics, Vol II, Fourier analysis and self adjointness, Academic Press, New York
[44] Sahlmann H. & Verch R., 2001, Microlocal spectral condition and Hadamard form for vector-valued quantum fields in curved spacetime, Rev. Math. Phys. 13, 1203-1246
[45] Smith C.J., 2007, An absolute quantum energy inequality for the Dirac field in curved spacetime, Class. Quantum Grav. 24, 4733-4750
[46] Synge J., 1960, Relativity: The general theory, North Holland, Amsterdam
[47] Vollick D.N., 2000, Quantum inequalities in curved two dimensional spacetimes, Phys. Rev. D 61, 084022
[48] Wald R.M., 1978, Trace anomaly of a conformally invariant quantum field in curved spacetime, Phys. Rev. D 17, 1477-1484
[49] Yu H. & Wu P., 2004, Quantum inequalities for the free Rarita-Schwinger fields in flat spacetime, Phys. Rev. D 69, 064008