ON THE MINIMAL AFFINIZATIONS OF TYPE $F_4$

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Abstract. In this paper, we apply the theory of cluster algebras to study minimal affinizations over the quantum affine algebra of type $F_4$. We show that the $q$-characters of a large family of minimal affinizations of type $F_4$ satisfy some equations. Moreover, every minimal affinization these equations corresponds to some cluster variable in some cluster algebra $\mathcal{A}$. For the other minimal affinizations of type $F_4$ which are not these equations, we give some conjectural equations which contains these minimal affinizations. Furthermore, we introduce the concept of dominant monomial graphs to study the equations satisfied by $q$-characters of modules of quantum affine algebras.

Key words: quantum affine algebra of type $F_4$; minimal affinizations; cluster algebras; $q$-characters; Frenkel-Mukhin algorithm

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1. Introduction

The theory of cluster algebras are introduced by Fomin and Zelevinsky in [FZ02]. It has many applications to mathematics and physics including quiver representations, Teichmüller theory, tropical geometry, integrable systems, and Poisson geometry.

Let $\mathfrak{g}$ be a simple Lie algebra and $U_q\widehat{\mathfrak{g}}$ the corresponding quantum affine algebra. In [C95], V. Chari and A. Pressley introduced minimal affinizations of representations of quantum groups. The family of minimal affinizations is an important family of simple modules which contains the Kirillov-Reshetikhin modules. Minimal affinizations are studied intensively in recent years, see for examples, [CG11], [CMY13], [Her07], [LM13], [M10], [MP07], [MP11], [MY12a], [MY12b], [MY14], [Nao13], [Nao14], [QL14], [SSL14], [ZDLL15]. However, there is not much work of the minimal affinizations over the quantum affine algebra of type $F_4$ in the literature. The aim of this paper is to apply the theory of cluster algebras to study minimal affinizations over the quantum affine algebra of type $F_4$.

M-systems and dual M-systems of types $A_n$, $B_n$, $G_2$ are introduced in [ZDLL15], [QL14] to study the minimal affinizations of types $A_n$, $B_n$, $G_2$. The equations in these systems are satisfied by the $q$-characters of minimal affinizations of types $A_n$, $B_n$, $G_2$. It is shown that every equation in these systems corresponds to a mutation equation in some cluster algebra.

The minimal affinizations of type $F_4$ are much more complicated than the minimal affinizations of types $A_n$, $B_n$, $G_2$. In types $A_n$, $B_n$, $G_2$, all minimal affinizations are special or anti-special, [Her07], [LM13]. A $U_q\widehat{\mathfrak{g}}$-module $V$ is called special (resp. anti-special) if there is only one dominant (resp. anti-dominant) monomial in the $q$-character of $V$. In the case of type $F_4$, there are minimal affinizations which are neither special nor anti-special, [Her07].

In [ZDLL15], [QL14], the M-systems and dual M-systems of types $A_n$, $B_n$, $G_2$ contain all minimal affinizations and only contain minimal affinizations. The situation is different in the
case of type $F_4$. It is quite possible that a closed system which contains all minimal affinizations and only contains minimal affinizations of type $F_4$ does not exist. However, we are able to find two closed systems which contain a large family of minimal affinizations of type $F_4$, Theorem 3.4. We show that the equations these systems are satisfied by the $q$-characters of the minimal affinizations in the systems. We prove that the modules in one system are special, Theorem 3.3, and the modules in the other system are anti-special, Theorem 5.2. Moreover, we show that every minimal affinization in Theorem 3.4 (resp. Theorem 5.4) corresponds to a mutation equation in some cluster algebra $\mathcal{A}$ (resp. $\tilde{\mathcal{A}}$). The cluster algebra $\mathcal{A}$ is the same as the cluster algebra for the quantum affine algebra of type $F_4$ introduced in [HL13]. Moreover, every minimal affinization in Theorem 3.4 (resp. Theorem 5.4) corresponds to a cluster variable in the cluster algebra $\mathcal{A}$ (resp. $\tilde{\mathcal{A}}$), Theorem 4.1 (resp. Theorem 5.6).

The system of equations in Theorem 5.4 is dual to the system in Theorem 3.4. This system contains the modules which are dual to the modules in the system in Theorem 3.4.

For the minimal affinizations which are not in Theorem 3.4 and Theorem 5.4, we give some conjectural equations which contain these modules, Conjecture 9.1.

We introduce the concept of dominant monomial graphs to study the equations satisfied by $q$-characters of modules of quantum affine algebras. We draw dominant monomial graphs for the modules in the equivalence classes of the left hand side of some equations in Conjecture 9.1. In these graphs, we find that every graph can be divided into two parts. The vertices in the first (resp. second) part of the graph are dominant monomials in the first (resp. second) summand of the right hand side of the corresponding equation.

The paper is organized as follows. In Section 2, we give some background information about cluster algebras and representation theory of quantum affine algebras. In Section 3, we describe a closed system containing a large family of minimal affinizations of type $F_4$. In Section 4, we study relations between the system in Theorem 3.4 and cluster algebras. In Section 5, we study the dual system of Theorem 3.4. In Section 6, Section 7, and Section 8, we prove Theorem 3.3, Theorem 3.4, and Theorem 3.6 given in Section 3, respectively. In Section 9, we give a conjecture about the equations satisfied by the $q$-characters of the other minimal affinizations of type $F_4$ and introduce the concept of dominant monomial graphs to study the conjecture.

2. Background

2.1. Cluster algebras. We first recall the definition of cluster algebras introduced by Fomin and Zelevinsky in [FZ02]. Let $Q$ be the rational field and $\mathcal{F} = \mathbb{Q}(x_1, x_2, \cdots, x_n)$ the field of rational functions in $n$ indeterminates over $\mathbb{Q}$. A seed in $\mathcal{F}$ is a pair $\Sigma = (y, Q)$, where $y = (y_1, y_2, \cdots, y_n)$ is a free generating set of $\mathcal{F}$, and $Q$ is a quiver with vertices labeled by $\{1, 2, \cdots, n\}$. Assume that $Q$ has neither loops nor 2-cycles. For $k \in \{1, 2, \cdots, n\}$, one defines a new seed $\mu_k(y, Q) = (y', Q')$ by the mutation of $(y, Q)$ at $k$. Here $y' = (y'_1, \cdots, y'_n)$, $y'_i = y_i$, for $i \neq k$, and

$$y'_k = \prod_{i \to k} y_i + \prod_{k \to j} y_j y_k,$$  \hspace{1cm} (2.1)

where the first (resp. second) product in the right hand side is over all arrows of $Q$ with target (resp. source) $k$, and $Q'$ is obtained from $Q$ by the follow rule:

(i) Reverse the orientations of all arrow incident with $k$;

(ii) Add a new arrow $i \to j$ for every existing pair of arrow $i \to k$ and $k \to j$;
The quantum affine algebra \( U \) generated by all cluster variables. \( \mathcal{F} \) cluster is an infinite set and \( y \) untwisted affine algebra corresponding to \( g \). Let \( \alpha \) simple roots and fundamental weights respectively. Let \( q \) weights respectively. Let \( \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \) and scalar product \( \langle \cdot, \cdot \rangle \) such that

\[
\begin{align*}
(\alpha_1, \alpha_1) &= 2, \quad (\alpha_1, \alpha_2) = -1, \quad (\alpha_2, \alpha_2) = 2, \quad (\alpha_2, \alpha_3) = -2, \\
(\alpha_3, \alpha_3) &= 4, \quad (\alpha_3, \alpha_4) = -2, \quad (\alpha_4, \alpha_4) = 4.
\end{align*}
\]

Therefore \( \alpha_1, \alpha_2 \) are the short simple roots and \( \alpha_3, \alpha_4 \) are the long simple roots.

Let \( \{ \alpha_1', \alpha_2', \alpha_3', \alpha_4' \} \) and \( \{ \omega_1, \omega_2, \omega_3, \omega_4 \} \) be the sets of simple coroots and fundamental weights respectively. Let \( C = (C_{ij})_{i,j \in I} \) denote the Cartan matrix, where \( C_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{(\alpha_i, \alpha_i)}. \) Let \( d_1 = 1, d_2 = 1, d_3 = 2, d_4 = 2, D = \text{diag}(d_1, d_2, d_3, d_4) \) and \( B = DC = (b_{ij})_{i,j \in I}. \) Then

\[
C = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}, \quad B = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -2 & 4 & -2 \\
0 & 0 & -2 & 4
\end{pmatrix}.
\]

Let \( q_i = q^{d_i}, i \in I \). Let \( Q \) (resp. \( Q^+ \)) and \( P \) (resp. \( P^+ \)) denote the \( \mathbb{Z} \)-span (resp. \( \mathbb{Z}_{\geq 0} \)-span) of the simple roots and fundamental weights respectively. Let \( \leq \) be the partial order on \( P \) in which \( \lambda \leq \lambda' \) if and only if \( \lambda' - \lambda \in Q^+ \).

Quantum affine algebras are introduced independently by Jimbo [Jim85] and Drinfeld [Dri87]. Quantum affine algebras form a family of infinite-dimensional quantum groups. Let \( \hat{\mathfrak{g}} \) denote the untwisted affine algebra corresponding to \( \mathfrak{g} \). In this paper, we fix a \( q \in \mathbb{C}^* \), not a root of unity. The quantum affine algebra \( U_q\hat{\mathfrak{g}} \) in Drinfeld’s new realization, see [Dri88], is generated by \( x_i^{\pm \ell} \) \((i \in I, n \in \mathbb{Z})\), \( k_i^{\pm 1} \) \((i \in I)\), \( h_{i,n} \) \((i \in I, n \in \mathbb{Z}\setminus\{0\})\) and central elements \( c^{\pm 1/2} \), subject to certain relations.

The algebra \( U_q\mathfrak{g} \) is isomorphic to a subalgebra of \( U_q\hat{\mathfrak{g}} \). Therefore \( U_q\hat{\mathfrak{g}} \)-modules restrict to \( U_q\mathfrak{g} \)-modules.
2.3. Finite-dimensional $U_q\hat{\mathfrak{g}}$-modules and their $q$-characters. We recall some known results on finite-dimensional $U_q\hat{\mathfrak{g}}$-modules and their $q$-characters, [CP94], [CP95a], [FR98], [MY12a].

Let $\mathcal{P}$ be the free abelian multiplicative group of monomials in infinitely many formal variables $(Y_{i,a})_{i \in I, a \in \mathbb{C}^\times}$. Then $Z\mathcal{P} = Z[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$. For each $j \in I$, a monomial $m = \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{u_{i,a}}$, where $u_{i,a}$ are some integers, is said to be $j$-dominant (resp. $j$-anti-dominant) if and only if $u_{j,a} \geq 0$ (resp. $u_{j,a} \leq 0$) for all $a \in \mathbb{C}^\times$. A monomial is called dominant (resp. anti-dominant) if and only if it is $j$-dominant (resp. $j$-anti-dominant) for all $j \in I$.

Every finite-dimensional simple $U_q\hat{\mathfrak{g}}$-module is parametrized by a dominant monomial in $\mathcal{P}^+$, [CP94], [CP95a]. That is, for a dominant monomial $m = \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{u_{i,a}}$, there is a corresponding simple $U_q\hat{\mathfrak{g}}$-module $L(m)$. Let $\text{Rep}(U_q\hat{\mathfrak{g}})$ be the Grothendieck ring of finite-dimensional $U_q\hat{\mathfrak{g}}$-modules and $[V] \in \text{Rep}(U_q\hat{\mathfrak{g}})$ the class of a finite-dimensional $U_q\hat{\mathfrak{g}}$-module $V$.

The $q$-character of a $U_q\hat{\mathfrak{g}}$-module $V$ is given by

$$\chi_q(V) = \sum_{m \in \mathcal{P}} \dim(V_m) m \in Z\mathcal{P},$$

where $V_m$ is the $l$-weight space with $l$-weight $m$, see [FR98]. We use $\mathcal{M}(V)$ to denote the set of all monomials in $\chi_q(V)$ for a finite-dimensional $U_q\hat{\mathfrak{g}}$-module $V$. Let $\mathcal{P}^+ \subset \mathcal{P}$ denote the set of all dominant monomials. For $m_+ \in \mathcal{P}^+$, we use $\chi_q(m_+)$ to denote $\chi_q(L(m_+))$. We also write $m \in \chi_q(m_+)$ if $m \in \mathcal{M}(\chi_q(m_+))$.

The following lemma is well-known.

**Lemma 2.2.** Let $m_1, m_2$ be two monomials. Then $L(m_1 m_2)$ is a sub-quotient of $L(m_1) \otimes L(m_2)$. In particular, $\mathcal{M}(L(m_1 m_2)) \subseteq \mathcal{M}(L(m_1)) \cap \mathcal{M}(L(m_2))$. □

A finite-dimensional $U_q\hat{\mathfrak{g}}$-module $V$ is said to be special if and only if $\mathcal{M}(V)$ contains exactly one dominant monomial. It is called anti-special if and only if $\mathcal{M}(V)$ contains exactly one anti-dominant monomial. Clearly, if a module is special or anti-special, then it is simple.

Let $a \in \mathbb{C}^\times$ and

$$A_{1,a} = Y_{1,a} Y_{1,a}^{-1} Y_{2,a}^{-1},$$

$$A_{2,a} = Y_{2,a} Y_{2,a}^{-1} Y_{1,a}^{-1} Y_{3,a},$$

$$A_{3,a} = Y_{3,a} Y_{3,a}^{-1} Y_{4,a}^{-1} Y_{2,a} Y_{4,a}^{-1} Y_{2,a}^{-1},$$

$$A_{4,a} = Y_{4,a} Y_{4,a}^{-1} Y_{3,a}^{-1}.$$ (2.2)

Let $Q$ be the subgroup of $\mathcal{P}$ generated by $A_{i,a}, i \in I, a \in \mathbb{C}^\times$. Let $Q^\pm$ be the monoids generated by $A_{i,a}^{\pm 1}, i \in I, a \in \mathbb{C}^\times$. There is a partial order $\leq$ on $\mathcal{P}$ in which

$$m \leq m' \text{ if and only if } m' m^{-1} \in Q^+.$$ (2.3)

For all $m_+ \in \mathcal{P}^+, \mathcal{M}(L(m_+)) \subset m_+ Q^-$, see [FM01].

The concept of right negative is introduced in Section 6 of [FM01].

**Definition 2.3.** A monomial $m$ is called right negative if for all $a \in \mathbb{C}^\times$, for $L = \max\{l \in \mathbb{Z} | u_{i,a}(m) \neq 0 \text{ for some } i \in I\}$ we have $u_{j,a}^+(m) \leq 0$ for $j \in I$.

For $i \in I, a \in \mathbb{C}^\times$, $A_{i,a}^{-1}$ is right-negative. A product of right-negative monomials is right-negative. If $m$ is right-negative and $m' \leq m$, then $m'$ is right-negative, see [FM01], [Her06].
2.4. $q$-characters of $U_q\hat{sl}_2$-modules and the Frenkel-Mukhin algorithm. We recall the results of the $q$-characters of $U_q\hat{sl}_2$-modules which are well-understood, see [CP91], [FR98].

Let $W_k^{(a)}$ be the irreducible representation $U_q\hat{sl}_2$ with highest weight monomial
\[
X_k^{(a)} = \prod_{i=0}^{k-1} Y_{aq^{k-2i-1}},
\]
where $Y_a = Y_{1,a}$. Then the $q$-character of $W_k^{(a)}$ is given by
\[
\chi_q(W_k^{(a)}) = X_k^{(a)} \sum_{i=0}^{k-1} A_{aq^{k-2i}}^{-1},
\]
where $A_a = Y_{aq^{-1}Y_a}$.

For $a \in \mathbb{C}^\times$, $k \in \mathbb{Z}_{\geq 1}$, the set $\Sigma_k^{(a)} = \{aq^{k-2i-1}\}_{i=0,...,k-1}$ is called a string. Two strings $\Sigma_k^{(a)}$ and $\Sigma_k^{(a')} = \Sigma_k^{(a')} \cup \Sigma_k^{(a')} \cup \Sigma_k^{(a')}$ are said to be in general position if the union $\Sigma_k^{(a)} \cup \Sigma_k^{(a')} \cup \Sigma_k^{(a')}$ is not a string or $\Sigma_k^{(a')}$ or $\Sigma_k^{(a')}$ is called a string

Denote by $L(m_+)$ the irreducible $U_q\hat{sl}_2$-module with highest weight monomial $m_+$. Let $m_+ \neq 1$ and $\in \mathbb{Z}[Y_a \in \mathbb{C}^\times$ be a dominant monomial. Then $m_+$ can be uniquely (up to permutation) written in the form
\[
m_+ = \prod_{i=1}^{s} \left( \prod_{b \in \Sigma_{k_i}} Y_b \right),
\]
where $s$ is an integer, $\Sigma_{k_i} = \{a \in \mathbb{C}^\times\}$, $i = 1, \ldots, s$, are strings which are pairwise in general position and
\[
L(m_+) = \bigotimes_{i=1}^{s} W_k^{(a)}(a), \quad \chi_q(L(m_+)) = \prod_{i=1}^{s} \chi_q(W_k^{(a)}(a)).
\]

For $j \in I$, let
\[
\beta_j : \mathbb{Z}[Y_{i,a}^{(1)} \in I, a \in \mathbb{C}^\times \rightarrow \mathbb{Z}[Y_{i,a}^{(1)} \in I, a \in \mathbb{C}^\times]
\]
be the ring homomorphism such that for all $a \in \mathbb{C}^\times$, $Y_{k,a} \mapsto 1$ for $k \neq j$ and $Y_{j,a} \mapsto Y_a$.

Let $V$ be a $U_q\hat{g}$-module. Then $\beta_i(\chi_q(V))$, $i \in I$, is the $q$-character of $V$ considered as a $U_q\hat{g}$-module.

The Frenkel-Mukhin algorithm is introduced in Section 5 in [FM01] to compute the $q$-characters of $U_q\hat{g}$-modules. In Theorem 5.9 of [FM01], it is shown that the Frenkel-Mukhin algorithm works for modules which are special.

2.5. Truncated $q$-characters. In this paper, we need to use the concept truncated $q$-characters, see [HL10], [MY12a]. Given a set of monomials $\mathcal{R} \subset \mathcal{P}$, let $\mathbb{Z}\mathcal{R} \subset \mathbb{Z}\mathcal{P}$ denote the $\mathbb{Z}$-module of formal linear combinations of elements of $\mathcal{R}$ with integer coefficients. Define
\[
\text{trunc}_\mathcal{R} : \mathcal{P} \rightarrow \mathcal{R}; \quad m \mapsto \begin{cases} m & \text{if } m \in \mathcal{R}, \\
0 & \text{if } m \notin \mathcal{R}, \end{cases}
\]
and extend $\text{trunc}_\mathcal{R}$ as a $\mathbb{Z}$-module map $\mathbb{Z}\mathcal{P} \rightarrow \mathbb{Z}\mathcal{R}$.
Given a subset $U \subset I \times \mathbb{C}^\times$, let $Q_U$ be the subgroups of $Q$ generated by $A_{i,a}$ with $(i, a) \in U$. Let $Q_U^\subset$ be the monoid generated by $A_{i,a}^\subset$ with $(i, a) \in U$. The polynomial $\text{trunc}_{m_+}Q_U^\subset \chi_q(m_+)$ is called the $q$-character of $L(m_+)$ truncated to $U$.

The following theorem can be used to compute some truncated $q$-characters.

**Theorem 2.4** (MY12a, Theorem 2.1). Let $U \subset I \times \mathbb{C}^\times$ and $m_+ \in \mathcal{P}^+$. Suppose that $\mathcal{M} \subset \mathcal{P}$ is a finite set of distinct monomials such that

1. $\mathcal{M} \subset m_+ Q_U^\subset$,
2. $\mathcal{P}^+ \cap \mathcal{M} = \{ m_+ \}$,
3. for all $m \in \mathcal{M}$ and all $(i, a) \in U$, if $mA_{i,a}^{-1} \not\in \mathcal{M}$, then $mA_{i,a}^{-1} A_{j,b} \not\in \mathcal{M}$ unless $(j, b) = (i, a)$,
4. for all $m \in \mathcal{M}$ and all $i \in I$, there exists a unique $i$-dominant monomial $M \in \mathcal{M}$ such that

$$\text{trunc}_{\beta_i(M) Q_U^\subset} \chi_q(\beta_i(M)) = \sum_{m' \in m Q_U^\subset \cap \mathcal{M}} \beta_i(m').$$

Then

$$\text{trunc}_{m_+ Q_U^\subset} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m.$$

Here $\chi_q(\beta_i(M))$ is the $q$-character of the irreducible $U_q(\mathfrak{g})$-module with highest weight monomial $\beta_i(M)$ and $\text{trunc}_{\beta_i(M) Q_U^\subset}$ is the polynomial obtained from $\chi_q(\beta_i(M))$ by keeping only the monomials of $\chi_q(\beta_i(M))$ in the set $\beta_i(M) Q_U^\subset$.

### 2.6. Minimal affinizations of $U_q\mathfrak{g}$-modules

In what follows, we fix an $a \in \mathbb{C}^\times$ and denote $i_a = Y_{i,aq^s}, i \in I, s \in \mathbb{Z}$. Let $\lambda = k\omega_1 + l\omega_2 + m\omega_3 + n\omega_4, k, l, m, n \in \mathbb{Z}_{\geq 0}$ and $V(\lambda)$ the simple $U_q\mathfrak{g}$-module with highest weight $\lambda$. Without loss of generality, we may assume that a simple $U_q\mathfrak{g}$-module $L(m_+)$ is a minimal affinization of $V(\lambda)$ if and only if $m_+$ is one of the following monomials:

$$\tilde{T}^{(s)}_{n,m,l,k} = \left( \prod_{i=0}^{n-1} Y_{4,aq^{s+4i}} \right) \left( \prod_{i=0}^{m-1} Y_{3,aq^{+4n+4i+2}} \right) \left( \prod_{i=0}^{l-1} Y_{2,aq^{+4n+4m+2i+3}} \right) \left( \prod_{i=0}^{k-1} Y_{1,aq^{+4n+4m+2l+2i+4}} \right),$$

$$T^{(s)}_{n,m,l,k} = \left( \prod_{i=0}^{n-1} Y_{4,aq^{s-4i}} \right) \left( \prod_{i=0}^{m-1} Y_{3,aq^{-s-4n-4i-2}} \right) \left( \prod_{i=0}^{l-1} Y_{2,aq^{-s-4n-4m-2i-3}} \right) \left( \prod_{i=0}^{k-1} Y_{1,aq^{-s-4n-4m-2l-2i-4}} \right),$$

where $\prod_{0 \leq j \leq -1} = 1, s \in \mathbb{Z}$, see CP96a. We denote $A_{i,aq^s}$ by $A_{i,s}^{-1}$. We use $T^{(s)}_{k,l,m,n}$ (resp. $\tilde{T}^{(s)}_{k,l,m,n}$) to denote the irreducible finite-dimensional $U_q\mathfrak{g}$-module with highest $l$-weight $T^{(s)}_{k,l,m,n}$ (resp. $\tilde{T}^{(s)}_{k,l,m,n}$).
3. A closed system which contains a large family of minimal affinizations of type $F_4$

In this section, we introduce a closed system of type $F_4$ that contains a large family of minimal affinizations:

\[
\begin{align*}
\mathcal{F}_{0,0,l,k}^{(-2)}, & \quad \mathcal{F}_{n,0,l,0}^{(s)}, \quad \mathcal{F}_{0,m,l,0}^{(s)}, \quad \mathcal{F}_{n,m,l,0}^{(s)}, \quad \mathcal{F}_{n,0,0,k}^{(s)}, \quad \mathcal{F}_{0,m,0,k}^{(s)} (k \leq 2), \\
\mathcal{F}_{n,m,0,k}^{(s)} (k \leq 2), & \quad \mathcal{F}_{n,m,l,k}^{(s)} (k \leq 2), \quad \mathcal{F}_{n,0,l,k}^{(s)} (k \leq 2), \quad \mathcal{F}_{0,m,l,k}^{(s)} (k \leq 2).
\end{align*}
\]

Here $k, l, m, n \in \mathbb{Z}_{\geq 1}, s \in \mathbb{Z}$.

3.1. Special modules. Let $k, l, m, n \in \mathbb{Z}_{\geq 1}, s \in \mathbb{Z}$. For $k \leq 0$, let $\mathcal{F}_{0,0,0,k}^{(s)} = 1$. We define

\[
\begin{align*}
\mathcal{F}_{0,0,l,k}^{(s)} &= \mathcal{F}_{0,0,l,k}^{(s)}, \quad k \in \mathbb{Z}_{\leq 2}, \\
\mathcal{F}_{0,m,0,k}^{(s)} &= \mathcal{F}_{0,m,0,k}^{(s)}, \quad k \in \mathbb{Z}_{\leq 2}, \\
\mathcal{F}_{0,0,0,k}^{(s)} &= \mathcal{F}_{0,0,0,k}^{(s)}, \quad k \in \mathbb{Z}_{\geq 3}, \\
\mathcal{F}_{0,0,l,k}^{(s)} &= \mathcal{F}_{0,0,l,k}^{(s)}, \quad k \in \mathbb{Z}_{\leq 2}, \\
\mathcal{F}_{0,m,0,k}^{(s)} &= \mathcal{F}_{0,m,0,k}^{(s)}, \quad k \in \mathbb{Z}_{\geq 3},
\end{align*}
\]

We use $\mathcal{F}_{n,m,l,k}^{(s)}$ to denote the simple $U_q\mathfrak{g}$-module with the highest weight monomial $\mathcal{F}_{n,m,l,k}^{(s)}$.

Theorem 3.1 (Theorem 3.9, [Her07]). For $l, m, n \in \mathbb{Z}_{\geq 1}, s \in \mathbb{Z}$, the modules $\mathcal{F}_{n,m,0,0}^{(s)}, \mathcal{F}_{0,n,l,0}^{(s)}$, $\mathcal{F}_{0,m,0,k}^{(s)}, \mathcal{F}_{n,m,l,k}^{(s)}$ are special.

Remark 3.2. In the paper [Her07], $\alpha_1, \alpha_2$ are simple long roots and $\alpha_3, \alpha_4$ are simple short roots. In this paper, $\alpha_1, \alpha_2$ are simple short roots and $\alpha_3, \alpha_4$ are simple long roots.

Theorem 3.3. The modules

\[
\begin{align*}
\mathcal{F}_{0,0,l,k}^{(-2)}, & \quad \mathcal{F}_{n,0,l,0}^{(s)}, \quad \mathcal{F}_{0,m,l,0}^{(s)}, \quad \mathcal{F}_{n,m,l,0}^{(s)}, \quad \mathcal{F}_{n,0,0,k}^{(s)}, \quad \mathcal{F}_{0,m,0,k}^{(s)} (k \leq 2), \\
\mathcal{F}_{n,m,l,k}^{(s)} (k \leq 2), & \quad \mathcal{F}_{n,0,l,k}^{(s)} (k \leq 2), \quad \mathcal{F}_{0,m,l,k}^{(s)} (k \leq 2), \quad \mathcal{F}_{0,m,0,k}^{(s)} (k \leq 2), \quad \mathcal{F}_{0,m,l,k}^{(s)} (k \leq 2),
\end{align*}
\]

where $k, l, m, n \in \mathbb{Z}_{\geq 1}, s \in \mathbb{Z}$, are special.

We will prove Theorem 3.3 in Section 6.

Since the modules are special in Theorem 5.1 and Theorem 3.3, we can use the Frenkel-Mukhin algorithm to compute the $q$-characters of these modules.

3.2. A closed system of type $F_4$

Theorem 3.4. For $s \in \mathbb{Z}$ and $k, l, m, n \geq 1$, we have

\[
\begin{align*}
\mathcal{F}_{0,0,l-1,k}^{(-2)} & \mathcal{F}_{0,0,l,k}^{(-2)} = \mathcal{F}_{0,0,l,k-1}^{(-2)} \mathcal{F}_{0,0,l-1,k+1}^{(-2)} + \mathcal{F}_{0,0,k+l,0}^{(-2)} \mathcal{F}_{0,0,l-1,0}^{(-2)}, \\
\mathcal{F}_{n,m-1,0}^{(s+4)} & \mathcal{F}_{n,m,0}^{(s)} = \mathcal{F}_{n-1,m,0}^{(s+4)} \mathcal{F}_{n,m-1,0}^{(s)} + \mathcal{F}_{0,m-1,0}^{(s+4)} \mathcal{F}_{0,m,0}^{(s)}, \\
\mathcal{F}_{n,0,0}^{(s+4)} & \mathcal{F}_{n,0,0}^{(s)} = \mathcal{F}_{n-1,0,0}^{(s+4)} \mathcal{F}_{n,0,0}^{(s)} + \mathcal{F}_{0,n,0}^{(s+4)} \mathcal{F}_{0,0,0}^{(s)}.
\end{align*}
\]
\[\[\tilde{T}_{n,0,l-2,0}[\tilde{T}_{n,0,l,0}] = [\tilde{T}_{n-1,0,l,0}][\tilde{T}_{n+1,0,l-2,0}] + [\tilde{T}_{0,0,l-2,0}][\tilde{T}_{0,n,l,0}], l \geq 2, \quad (3.4)\]

\[\[\tilde{T}_{0,m,0,0}[\tilde{T}_{0,m,1,0}] = [\tilde{T}_{0,m-1,1,0}][\tilde{T}_{0,m+1,0,0}] + [\tilde{T}_{0,m,0,0}][\tilde{T}_{0,0,1+2m,0}], \quad (3.5)\]

\[\[\tilde{T}_{0,m,l-2,0}[\tilde{T}_{n,0,l,0}] = [\tilde{T}_{0,m-l-1,0}][\tilde{T}_{0,m+1,0,0}] + [\tilde{T}_{0,m,0,0}][\tilde{T}_{0,0,1+2m,0}], l \geq 2, \quad (3.6)\]

\[\[\tilde{T}_{n,m,l-2,0}[\tilde{T}_{n,0,l,0}] = [\tilde{T}_{n-1,m-l-1,0}][\tilde{T}_{n+1,m,0,0}] + [\tilde{T}_{n,m,0,0}][\tilde{T}_{n+l+1,0,0}], \quad (3.7)\]

\[\[\tilde{T}_{n,0,0,k}[\tilde{T}_{n,0,0,k}][\tilde{T}_{n,0,0,0,k}] = [\tilde{T}_{n-1,0,0,k}][\tilde{T}_{n+1,0,0,0,k}] + [\tilde{T}_{n,0,0,0,k}][\tilde{T}_{n+l+1,0,0,0,k}], k = 1, 2, \quad (3.8)\]

\[\[\tilde{T}_{0,m,0,0}[\tilde{T}_{n,0,0,m+k,0}] = [\tilde{T}_{0,m-1,0,0,m+k,0}][\tilde{T}_{0,m+1,0,0,m+k,0}] + [\tilde{T}_{0,m,0,0,m+k,0}][\tilde{T}_{0,0,2,0,m+k,0}], k = 1, 2, \quad (3.9)\]

\[\[\tilde{T}_{n,0,0,k-1}][\tilde{T}_{n,0,0,1,k}] = [\tilde{T}_{n-1,0,0,k-1}][\tilde{T}_{n+1,0,0,1,k}] + [\tilde{T}_{n,0,0,0,k-1}][\tilde{T}_{n+l+1,0,0,1,k}], k = 1, 2, \quad (3.10)\]

\[\[\tilde{T}_{n,0,l-2,k}[\tilde{T}_{n,0,l,k}] = [\tilde{T}_{n-1,0,l,k}][\tilde{T}_{n+1,0,l-2,k}] + [\tilde{T}_{n,0,0,l-2,k}][\tilde{T}_{n+l+1,0,l-2,k}], l \geq 2, \quad (3.11)\]

\[\[\tilde{T}_{0,m,0,k-1}[\tilde{T}_{0,m,0,k}] = [\tilde{T}_{0,m-1,0,k}][\tilde{T}_{0,m+1,0,k-1}] + [\tilde{T}_{0,m,0,k-1}][\tilde{T}_{0,0,1+2m,k}], \quad (3.12)\]

\[\[\tilde{T}_{0,m,l-2,k}[\tilde{T}_{0,m,l,k}] = [\tilde{T}_{0,m-l-1,k}][\tilde{T}_{0,m+1,l-2,k}] + [\tilde{T}_{0,m,0,l-2,k}][\tilde{T}_{0,0,1+2m,l-2,k}], l \geq 2, \quad (3.13)\]

\[\[\tilde{T}_{n,m-1,0,k}[\tilde{T}_{n,m,0,k}] = [\tilde{T}_{n-1,m,0,k}][\tilde{T}_{n+1,m-1,0,k}] + [\tilde{T}_{n,m,0,k}][\tilde{T}_{n+1,m-1,0,k}], k = 1, 2, \quad (3.14)\]

\[\[\tilde{T}_{n,m-1,0,k}[\tilde{T}_{n,m,0,k}] = [\tilde{T}_{n-1,m,0,k}][\tilde{T}_{n+1,m-1,0,k}] + [\tilde{T}_{n,m,0,k}][\tilde{T}_{n+1,m-1,0,k}], k = 1, 2, \quad (3.15)\]

\[\[\tilde{P}_{0,0,1,k-1}[\tilde{P}_{0,0,1,k}] = [\tilde{P}_{0,0,0,k-1}][\tilde{P}_{0,0,0,k}][\tilde{P}_{0,0,2,k-1}] + [\tilde{P}_{0,0,0,k}][\tilde{P}_{0,0,0,k-1}][\tilde{P}_{0,0,2,k-1}], k \geq 3, l = 1, \quad (3.16)\]

\[\[\tilde{P}_{0,0,l,k-1}[\tilde{P}_{0,0,l,k}] = [\tilde{P}_{0,0,l-1,k}][\tilde{P}_{0,0,l+1,k-1}] + [\tilde{P}_{0,0,0,k-5}][\tilde{P}_{0,0,0,k+5}][\tilde{P}_{0,0,0,k-1}][\tilde{P}_{0,0,0,k+1}], \quad (3.17)\]

where \( k \geq 3, l \text{ is odd, } l \geq 3, \]

\[\[\tilde{P}_{0,0,l,k-1}[\tilde{P}_{0,0,l,k}] = [\tilde{P}_{0,0,l-1,k}][\tilde{P}_{0,0,l+1,k-1}] + [\tilde{P}_{0,0,0,k-5}][\tilde{P}_{0,0,0,k+5}][\tilde{P}_{0,0,l-1,k}][\tilde{P}_{0,0,l+1,k}][\tilde{P}_{0,0,0,k-1}][\tilde{P}_{0,0,0,k+1}], \quad (3.18)\]
where $k \geq 3$, $l$ is even, $l \geq 2$,

\[ [\widetilde{T}_{0,1,0,k-2}] [\widetilde{T}_{0,1,0,k}] = [\widetilde{T}_{0,0,0,k-2}] [\widetilde{T}_{0,0,0,k}] + [\widetilde{T}_{0,0,0,k-2}] [\widetilde{T}_{1,0,0,k-2}], \quad k \geq 3, \tag{3.19} \]

\[ [\widetilde{T}_{0,0,m,0,k-2}] [\widetilde{T}_{0,m,0,k}] = [\widetilde{T}_{0,0,m-1,0,k-2}] [\widetilde{T}_{0,m-1,0,k}] + [\widetilde{T}_{0,0,0,m,k}][\widetilde{T}_{0,m,0,0,k-2}], \quad k \geq 3, \ m \geq 2, \tag{3.20} \]

\[ [\widetilde{T}_{n,0,0,k-2}] [\widetilde{T}_{n,0,0,k}] = [\widetilde{T}_{n-1,0,0,k}] [\widetilde{T}_{n+1,0,0,k-2}] + [\widetilde{T}_{0,n,0,k}], \quad k \geq 3. \tag{3.21} \]

Theorem 3.7 will be proved in Section 7. The system in Theorem 3.7 is a closed system in the sense that all modules in the system can be computed recursively using Kirillov-Reshetikhin modules.

**Example 3.5.** The following are some equations in the system in Theorem 3.7.

\[ [1,-2][1,-2]_1 = [1,-2][2,-1] + [2,-1], \]
\[ [1,-4][1,0,-2][1,0,-2] = [1,-4][1,0,-1,2] + [2,-5], \]
\[ [1,0][1,0,1,2][1,0,1,2] = [1,0][1,0,1,1,2] + [2,-5], \]
\[ [3,1][1,0,3,3,4] = [1,0][3,3,3,4] + [1,0]+ [3,3,3,4], \]
\[ [3,0][3,3,3,4] = [1,0][3,3,3,4] + [1,0]+ [3,3,3,4], \]
\[ [3,0][3,3,3,4] = [1,0][3,3,3,4] + [1,0]+ [3,3,3,4], \]
\[ [3,0][3,3,3,4] = [1,0][3,3,3,4] + [1,0]+ [3,3,3,4], \]
\[ [3,0][3,3,3,4] = [1,0][3,3,3,4] + [1,0]+ [3,3,3,4], \]
\[ [3,0][3,3,3,4] = [1,0][3,3,3,4] + [1,0]+ [3,3,3,4], \]

Moreover, we have the following theorem.

**Theorem 3.6.** For each equation in Theorem 3.7 all summands on the right hand side are simple.

Theorem 3.6 will be proved in Section 8.

### 3.3. A system corresponding to the system in Theorem 3.7

Given $k, l, m, n \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$, let

\[ m_{k,l,m,n} = \text{Res}(T_{k,l,m,n}^{(s)}) \quad (\text{resp. } m_{k,l,m,n} = \text{Res}(T_{k,l,m,n}^{(s)})) \]

be the restriction of $T_{k,l,m,n}^{(s)}$ (resp. $T_{k,l,m,n}^{(s)}$) to $U_q\mathfrak{g}$. It is clear that $\text{Res}(T_{k,l,m,n}^{(s)})$ and $\text{Res}(T_{k,l,m,n}^{(s)})$ do not depend on $s$. Let $\chi(M)$ (resp. $\chi(M)$) be the character of a $U_q\mathfrak{g}$-module $M$ (resp. $M$). By replacing each $T_{n,m,l,k}^{(s)}$ (resp. $T_{n,m,l,k}^{(s)}$) in the system in Theorem 3.3 with $\chi(m_{n,m,l,k})$ (resp. $\chi(m_{n,m,l,k})$), we obtain a system of equations consisting of the characters of $U_q\mathfrak{g}$-modules. The following are two equations in the system.

\[ \chi(m_{0,0,1,1,1})\chi(m_{0,0,1,1,1}) = \chi(m_{0,0,1,1,1})\chi(m_{0,0,1,1,1}) + \chi(m_{0,0,1,1,1})\chi(m_{0,0,1,1,1}), \]
\[ \chi(m_{n,m,1,0,0})\chi(m_{n,m,1,0,0}) = \chi(m_{n,m,1,0,0})\chi(m_{n,m,1,0,0}) + \chi(m_{n,m,1,0,0})\chi(m_{n,m,1,0,0}). \]
4. Relation between the system in Theorem 3.4 and cluster algebras

In this section, we will show that the equations in the system in Theorem 3.4 correspond to mutations in some cluster algebra $\mathcal{A}$. Moreover, every minimal affinization in the system in Theorem 3.4 corresponds to a cluster variable in the cluster algebra $\mathcal{A}$.

4.1. Definition of a cluster algebra $\mathcal{A}$. Let $I = \{1, 2, 3, 4\}$ and

\[
S = \{-2u \mid u \in \mathbb{Z}_{\geq 0}\},
\]

\[
S' = \{-2u - 1 \mid u \in \mathbb{Z}_{\geq 0}\}.
\]

Let

\[
V = (\{1\} \times S) \cup (\{2\} \times S') \cup (\{3\} \times S) \cup (\{4\} \times S).
\]

We define $Q$ with vertex set $V$ as follows. The arrows of $Q$ from the vertex $(i, r)$ to the vertex $(j, s)$ if and only if $b_{ij} \neq 0$ and $s = r - b_{ij} + d_i - d_j$. The quiver $Q$ is the same as the quiver $G^-$ of type $F_4$ defined in [HL13].

Let $t = t_1 \cup t_2$, where

\[
t_1 = \{t_{0,0,1,0}, t_{0,0,0,k} \mid k, l \in \mathbb{Z}_{\geq 1}\},
\]

\[
t_2 = \{\tilde{t}_{n,0,0,0}, \tilde{t}_{n,0,0,0}, \tilde{t}_{n,0,0,0}, \tilde{t}_{n,0,0,0}, \tilde{t}_{n,0,0,0}, \tilde{t}_{n,0,0,0} \mid k, l, m, n \in \mathbb{Z}_{\geq 1}\}.
\]

Let $\mathcal{A}$ be the cluster algebra defined by the initial seed $(t, Q)$. By Definition 2.4, $\mathcal{A}$ is the $\mathbb{Q}$-subalgebra of the field of rational functions $\mathbb{Q}(t)$ generated by all the elements obtained from some elements of $t$ via a finite sequence of seed mutations.

4.2. Mutation sequences. We use “$C_1$”, “$C_2$”, “$C_3$”, “$C_4$”, “$C_5$”, “$C_6$” to denote the column of vertices $(1, 0)$, $(1, 2)$, ... $(1, -2u)$, the column of vertices $(2, -1)$, $(2, 3)$, ... $(2, -2u - 1)$, the column of vertices $(3, 0)$, $(3, -4)$, ... $(3, -4u)$, the column of vertices $(4, -2)$, $(4, -4)$, ... $(4, -4u)$, respectively in $Q$.

By saying that we mutate at the column $C_i$, $i \in \{1, 2, 3, 4, 5, 6\}$, we mean that we mutate the vertices of $C_i$ as follows. First we mutate at the first vertex in this column, then the second vertex, and so on until the vertex at infinity. By saying that we mutate $(C_{i_0}, C_{i_1}, \ldots, C_{i_u})$, where $i_j \in \{1, 2, 3, 4, 5, 6\}$, $j = 0, 1, 2, \ldots, u$, we mean that we first mutate the column $C_{i_1}$, then the column $C_{i_2}$, and so on up to the column $C_{i_u}$.

For $k, l, m, n \in \mathbb{Z}_{\geq 1}$, we define some variables

\[
t_{0,0,0,0}^{(-2)}, t_{n,m,0,0}^{(-4n-4m)}, t_{n,0,0,0}^{(-4n-4m)}, t_{n,0,1,0}^{(-4n-2l-2)}, t_{n,0,m,0}^{(-4n-2l-2)}, t_{n,m,l,0}^{(-4n-4m-2l-2)},
\]

\[
\tilde{t}_{n,0,0,0}^{(-4n-2l-2)}, \tilde{t}_{n,0,0,0}^{(-4n-4m-2l-2)}, \tilde{t}_{n,0,0,0}^{(-4n-4m-2l-2)}, \tilde{t}_{n,0,0,0}^{(-4n-4m-2l-2)}
\]

recursively as follows. The variables in $t$ are already defined.
We mutate the first vertex of the first $C_1$ in $(C_1, C_1, \ldots, C_1)$ from the initial seed, and define $t_{0,0,0,1}^{(-2)} = \rho_{0,0,0,1}^{(-4)}$, we obtain a quiver $(Q_{11})$. Therefore

$$\frac{t_{0,0,0,1}^{(-2)} - \rho_{0,0,0,1}^{(-4)}}{t_{0,0,0,1}^{(-4)}} = \frac{t_{0,0,0,2}^{(-4)} + t_{0,0,1,0}^{(-2)}}{t_{0,0,0,1}^{(-4)}}. \tag{4.2}$$

We mutate the second vertex of the first $C_1$ in $(C_1, C_1, \ldots, C_1)$ and define $t_{0,0,0,2}^{(-2)} = \rho_{0,0,0,2}^{(-4)}$, the quiver $(Q_{11})$ becomes a quiver $(Q_{12})$. Therefore

$$t_{0,0,0,2}^{(-2)} = \frac{t_{0,0,0,2}^{(-4)} + t_{0,0,0,1}^{(-2)}}{t_{0,0,0,2}^{(-4)}}. \tag{4.3}$$

We continue this procedure and mutate the vertices of the first $C_1$ in $(C_1, C_1, \ldots, C_1)$ and define $t_{0,0,0,k}^{(-2)} = \rho_{0,0,0,k}^{(-4)} (k = 3, 4, \ldots)$ recursively. Therefore

$$t_{0,0,0,k}^{(-2)} = \frac{t_{0,0,0,k+1}^{(-4)} + t_{0,0,0,k-1}^{(-2)}}{t_{0,0,0,k}^{(-4)}}, \quad k = 3, 4, \ldots \tag{4.4}$$

Now we finish the mutation of the first $C_1$ in $(C_1, C_1, \ldots, C_1)$.

We start to mutate the second $C_1$ in $(C_1, C_1, \ldots, C_1)$. We mutate the first vertex of the second $C_1$ in $(C_1, C_1, \ldots, C_1)$ and define $t_{0,0,1,1}^{(-2)} = \rho_{0,0,1,1}^{(-4)}$, we obtain a quiver $(Q_{21})$. Therefore

$$t_{0,0,1,1}^{(-2)} = \frac{t_{0,0,1,1}^{(-4)} + t_{0,0,1,0}^{(-2)}}{t_{0,0,0,1}^{(-4)}}. \tag{4.5}$$

We mutate the second vertex of the second $C_1$ in $(C_1, C_1, \ldots, C_1)$ and define $t_{0,0,1,2}^{(-2)} = \rho_{0,0,1,2}^{(-4)}$, the quiver $(Q_{21})$ becomes a quiver $(Q_{22})$. Therefore

$$t_{0,0,1,2}^{(-2)} = \frac{t_{0,0,1,2}^{(-4)} + t_{0,0,1,0}^{(-2)}}{t_{0,0,0,2}^{(-4)}}. \tag{4.6}$$

We continue this procedure and mutate the vertices of the second $C_1$ in $(C_1, C_1, \ldots, C_1)$ and define $t_{0,0,1,k}^{(-2)} = \rho_{0,0,0,k}^{(-4)} (k = 3, 4, \ldots)$ recursively. Therefore

$$t_{0,0,1,k}^{(-2)} = \frac{t_{0,0,1,k+1}^{(-4)} + t_{0,0,1,k-1}^{(-2)}}{t_{0,0,0,k}^{(-4)}}, \quad k = 3, 4, \ldots \tag{4.7}$$

Now we finish the mutation of the second $C_1$ in $(C_1, C_1, \ldots, C_1)$.

We start to mutate the third $C_1$ in $(C_1, C_1, \ldots, C_1)$. We mutate the first vertex of the third $C_1$ in $(C_1, C_1, \ldots, C_1)$ and define $t_{0,0,2,1}^{(-2)} = \rho_{0,0,2,1}^{(-4)}$, we obtain a quiver $(Q_{31})$. Therefore

$$t_{0,0,2,1}^{(-2)} = \frac{t_{0,0,2,1}^{(-4)} + t_{0,0,1,1}^{(-2)}}{t_{0,0,1,1}^{(-4)}}. \tag{4.8}$$
We mutate the second vertex of the third $C_1$ in $(C_1, C_1, \ldots, C_1)$ and define $t_{0,0,2,2}^{(-2)} = t_{0,0,1,2}^{(-2)}$, the quiver $(Q_{31})$ becomes a quiver $(Q_{32})$. Therefore

$$t_{0,0,2,2}^{(-2)} = t_{0,0,1,2}^{(-2)} = \frac{t_{0,0,1,3}^{(-2)}t_{0,0,2,1}^{(-2)} + t_{0,0,1,0}^{(-2)}t_{0,0,4,0}^{(-2)}}{t_{0,0,1,2}^{(-2)}}. \tag{4.9}$$

We continue this procedure and mutate vertices of the third $C_1$ in $(C_1, C_1, \ldots, C_1)$ and define $t_{0,0,2,k}^{(-2)} = t_{0,0,1,k}^{(-2)}$ $(k = 3, 4, \ldots)$ recursively. Therefore

$$t_{0,0,2,k}^{(-2)} = t_{0,0,1,k}^{(-2)} = \frac{t_{0,0,1,k+1}^{(-2)}t_{0,0,2,k-1}^{(-2)} + t_{0,0,1,0}^{(-2)}t_{0,0,2,k+2,0}^{(-2)}}{t_{0,0,1,k}^{(-2)}}, \quad k = 3, 4, \ldots \tag{4.10}$$

Now we finish the mutation of the third $C_1$ in $(C_1, C_1, \ldots, C_1)$. We continue this procedure and mutate the $(d + 1)$-th $C_1$ $(d = 3, 5, \ldots, l)$ in order. We define $t_{0,0,d,k}^{(-2)} = t_{0,0,d-1,k}^{(-2)}$, where $(0, 0, d, k) = \{(0, 0, 3, 1), (0, 0, 3, 2), (0, 0, 3, 3), (0, 0, 3, 4), \ldots; (0, 0, 4, 1), (0, 0, 4, 2), (0, 0, 4, 3), (0, 0, 4, 4)\ldots; (0, 0, 5, 1), (0, 0, 5, 2), (0, 0, 5, 3), (0, 0, 5, 4), \ldots; (0, 0, l, 1), (0, 0, l, 2), (0, 0, l, 3), (0, 0, l, 4), \ldots\}$ recursively. Therefore

$$t_{0,0,d,k}^{(-2)} = t_{0,0,d-1,k}^{(-2)} = \frac{t_{0,0,d-1,k+1}^{(-2)}t_{0,0,d,k-1}^{(-2)} + t_{0,0,d-1,0}^{(-2)}t_{0,0,d,k+d,0}^{(-2)}}{t_{0,0,d-1,k}^{(-2)}}. \tag{4.11}$$

We write the definition of the variables in \(\text{(4.1)}\) and the corresponding mutation sequences in Table \[\text{I}\].

In Table \[\text{I}\] we use

$$C_{i_1}, C_{i_2}, \ldots, C_{i_u}, C_{i_1}, C_{i_2}, \ldots, C_{i_u}, \ldots, C_{i_1}, C_{i_2}, \ldots, C_{i_u},$$

where $m \in \mathbb{Z}_{\geq 1}$, $u \in \mathbb{Z}_{\geq 1}$, to denote the mutation sequences

$$C_{i_1}, C_{i_2}, \ldots, C_{i_u}, C_{i_1}, C_{i_2}, \ldots, C_{i_u}, \ldots, C_{i_1}, C_{i_2}, \ldots, C_{i_u},$$

where the number of $C_{i_j}$ $(j \in \{1, 2, \ldots, u\})$ is $m$. 
4.3. The equations in the system of type $F_4$ correspond to mutations in the cluster algebra $\mathcal{A}$. Equation (1) corresponds to Equation (3.1) in Theorem 3.1. Equations (2) and (3) correspond to Equation (3.2) in Theorem 3.1. Equations (4)–(22) correspond to Equations (3.3)–(3.21) in Theorem 3.1, respectively. Therefore, we have the following theorem.

| Mutation sequences | Definition of variables in 3.3 and mutation equations |
|--------------------|------------------------------------------------------|
| $\xi = (\xi_1, \xi_2, \xi_3)$ | $\xi_{i+1} = (\xi_1, \xi_2, \xi_3)$; $\xi_{i+2} = (\xi_1, \xi_2, \xi_3)$ | (1) |
| $\eta = (\eta_1, \eta_2, \eta_3)$ | $\eta_{i+1} = (\eta_1, \eta_2, \eta_3)$; $\eta_{i+2} = (\eta_1, \eta_2, \eta_3)$ | (2) |
| $\delta = (\delta_1, \delta_2, \delta_3)$ | $\delta_{i+1} = (\delta_1, \delta_2, \delta_3)$; $\delta_{i+2} = (\delta_1, \delta_2, \delta_3)$ | (3) |
| $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ | $\gamma_{i+1} = (\gamma_1, \gamma_2, \gamma_3)$; $\gamma_{i+2} = (\gamma_1, \gamma_2, \gamma_3)$ | (4) |
| $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ | $\alpha_{i+1} = (\alpha_1, \alpha_2, \alpha_3)$; $\alpha_{i+2} = (\alpha_1, \alpha_2, \alpha_3)$ | (5) |
| $\beta = (\beta_1, \beta_2, \beta_3)$ | $\beta_{i+1} = (\beta_1, \beta_2, \beta_3)$; $\beta_{i+2} = (\beta_1, \beta_2, \beta_3)$ | (6) |

Table 1. Mutation sequences.
Theorem 4.1. Each minimal affinizations in Theorem 3.4 corresponds to a cluster variable in $\mathcal{A}$ defined in Section 4.2.

5. The dual system of Theorem 3.4

In this section, we study the dual system of Theorem 3.4.

Theorem 5.1 (Theorem 3.9, [Her07]). For $l,m,n \in \mathbb{Z}_{\geq 1}$, $s \in \mathbb{Z}$, the modules $T_{n,m,0,0}^{(s)}$, $T_{n,0,l,0}^{(s)}$, $T_{0,m,l,0}^{(s)}$, $T_{n,m,l,k}^{(s)}$, $T_{n,m,0,k}^{(s)}$ are anti-special.

We have the following theorem.

Theorem 5.2. The modules

$$\widetilde{T}_{0,0,l,k}^{(s)} \quad T_{n,0,l,0}^{(s)} \quad T_{n,m,0,k}^{(s)} \quad T_{n,m,0,0}^{(s)} \quad T_{n,m,l,0}^{(s)} \quad T_{n,m,0,k}^{(s)} \quad T_{n,m,0,0}^{(s)} \quad T_{n,m,l,k}^{(s)} \quad T_{n,m,0,k}^{(s)}$$

and $k,l,m,n \in \mathbb{Z}_{\geq 1}$, $s \in \mathbb{Z}$, are anti-special.

Proof. The proof of the theorem follows from dual arguments in the proof of Theorem 5.4.

Lemma 5.3. Let $\iota : \mathbb{Z} \mathcal{P} \rightarrow \mathbb{Z} \mathcal{P}$ be a homomorphism of rings such that $Y_{1,aq}^{-1} \rightarrow Y_{1,aq}^{-1}$, $Y_{2,aq}^{-1} \rightarrow Y_{2,aq}^{-1}$, $Y_{3,aq}^{-1} \rightarrow Y_{3,aq}^{-1}$, $Y_{4,aq}^{-1} \rightarrow Y_{4,aq}^{-1}$ for all $a \in \mathbb{C}^{\times}$, $s \in \mathbb{Z}$. Then

$$\chi_\iota(\widetilde{T}_{k,l,m,n}^{(s)}) = \iota(\chi_\iota(T_{k,l,m,n}^{(s)})), \quad \chi_\iota(T_{k,l,m,n}^{(s)}) = \iota(\chi_\iota(T_{k,l,m,n}^{(s)})).$$

Proof. The proof is similar to Lemma 7.3 in [LM13].

Theorem 5.4. For $s \in \mathbb{Z}$, $\iota, l,m,n \in \mathbb{Z}_{\geq 0}$, we have the following system of equations.

\[
\begin{align*}
[\widetilde{T}_{0,0,l-1,k}^{(s)}][\widetilde{T}_{0,0,l,k}^{(s)}] &= [\widetilde{T}_{0,0,l,k-1}^{(s)}][\widetilde{T}_{0,0,l,k}^{(s)}] + [\widetilde{T}_{0,0,l-1,k}^{(s)}][\widetilde{T}_{0,0,l-1,k}^{(s)}], \quad (5.1) \\
[T_{n,m,0,0}^{(s)}][T_{n,m,0,0}^{(s)}] &= [T_{n-1,m,0,0}^{(s)}][T_{n+1,m-1,0,0}^{(s)}] + [T_{n,m-1,0,0}^{(s)}][T_{n,m,0,0}^{(s)}], \quad (5.2) \\
[T_{0,0,l,0}^{(s)}][T_{0,0,l,0}^{(s)}] &= [T_{0,0,l-2,0}^{(s)}][T_{0,0,l-2,0}^{(s)}] + [T_{0,0,l-1,0}^{(s)}][T_{0,0,l-1,0}^{(s)}], \quad l \geq 2, \quad (5.3) \\
[T_{0,m,0,0}^{(s)}][T_{0,m,0,0}^{(s)}] &= [T_{0,m-1,0,0}^{(s)}][T_{0,m+1,0,0}^{(s)}] + [T_{0,m,0,0}^{(s)}][T_{0,m+1,0,0}^{(s)}], \quad (5.4) \\
[T_{0,m,l,0}^{(s)}][T_{0,m,l,0}^{(s)}] &= [T_{0,m,l-1,0}^{(s)}][T_{0,m,l-1,0}^{(s)}] + [T_{0,m,l-2,0}^{(s)}][T_{0,m,l-2,0}^{(s)}], \quad l \geq 2, \quad (5.5) \\
[T_{n,m,l,0}^{(s)}][T_{n,m,l,0}^{(s)}] &= [T_{n,m,l-1,0}^{(s)}][T_{n,m,l-1,0}^{(s)}] + [T_{n,m,l-2,0}^{(s)}][T_{n,m,l-2,0}^{(s)}], \quad l \geq 2, \quad (5.6) \\
[T_{n,m,l,0}^{(s)}][T_{n,m,l,0}^{(s)}] &= [T_{n-1,m,l,0}^{(s)}][T_{n+1,m-1,l,0}^{(s)}] + [T_{n,m,l-1,0}^{(s)}][T_{n,m,l-1,0}^{(s)}], \quad (5.7) \\
[T_{n,0,m,0}^{(s)}][T_{n,0,m,0}^{(s)}] &= [T_{n-1,0,m,0}^{(s)}][T_{n+1,0,m-1,0}^{(s)}] + [T_{n,0,m-1,0}^{(s)}][T_{n,0,m-1,0}^{(s)}], \quad (5.8)
\end{align*}
\]
Moreover, every module in the summands on the right hand side of above equation is simple.
The lowest weight monomial of $\chi_q(\widehat{T}_{n,m,l,k})$ is obtained from the highest weight monomial of $\chi_q(\widehat{T}_{n,m,l,k})$ by the substitutions: $1_s \mapsto 1_{18+s}$, $2_s \mapsto 2_{18+s}$, $3_s \mapsto 3_{18+s}$, $4_s \mapsto 4_{18+s}$. After we apply $\iota$ to $\chi_q(\widehat{T}_{n,m,l,k})$, the lowest weight monomial of $\chi_q(\widehat{T}_{n,m,l,k})$ becomes the highest weight monomial of $\iota(\chi_q(\widehat{T}_{n,m,l,k}))$. Therefore the highest weight monomial of $\iota(\chi_q(\widehat{T}_{n,m,l,k}))$ is obtained from the lowest weight monomial of $\chi_q(\widehat{T}_{n,m,l,k})$ by the substitutions: $1_s \mapsto 1_{18-s}$, $2_s \mapsto 2_{18-s}$, $3_s \mapsto 3_{18-s}$, $4_s \mapsto 4_{18-s}$. It follows that the highest weight monomial of $\iota(\chi_q(\widehat{T}_{n,m,l,k}))$ is obtained from the highest weight monomial of $\chi_q(\widehat{T}_{n,m,l,k})$ by the substitutions: $1_s \mapsto 1_{-s}$, $2_s \mapsto 2_{-s}$, $3_s \mapsto 3_{-s}$, $4_s \mapsto 4_{-s}$.

The simplify of every module in the summands on the right hand side of every equation follows from Theorem 5.3 and Lemma 5.3.

**Example 5.5.** The following are some equations in the system in Theorem 5.4.

$$\begin{align*}
[12][1421] &= [1412][21] + [221], \\
[142][161421] &= [1421][1614] + [2221], \\
[16412][161421] &= [161421][161412] + [22221], \\
[31][10233] &= [1023][333] + [10322][46], \\
[333][1023312] &= [10233][333312] + [1023222][4610], \\
[333312][102333123] &= [10233312][333312316] + [10232222][46104141], \\
[346][1023410] &= [1023][46410] + [10233], \\
[4610][102341041] &= [1023410][46410414] + [1023332], \\
[46410414][102341041418] &= [1023410414][4641041448] + [10233312316].
\end{align*}$$

**5.1. The system in Theorem 5.4.** By replacing each $[\widehat{T}_{n,m,l,k}]$ (resp. $[T_{n,m,l,k}]$) in the system of Theorem 5.4 with $\chi(\widehat{m}_{n,m,l,k})$ (resp. $\chi(m_{n,m,l,k})$), we obtain a system of equations consisting of the characters of $U_q\mathfrak{g}$-modules. The following are two equations in the system.

$$\begin{align*}
\chi(\widehat{m}_{0,0,l-1,k})\chi(\widehat{m}_{0,0,l,k}) &= \chi(\widehat{m}_{0,0,l-1,l})\chi(\widehat{m}_{0,0,l-1,k+1}) + \chi(\widehat{m}_{0,0,l,k})\chi(\widehat{m}_{0,0,l-1,0}), \\
\chi(\widehat{m}_{n,m-1,0,0})\chi(\widehat{m}_{n,m,0,0}) &= \chi(\widehat{m}_{n-1,m,0,0})\chi(\widehat{m}_{n+1,m-1,0,0}) + \chi(\widehat{m}_{n-1,m,0,0})\chi(\widehat{m}_{n,m,1-m,0,0}).
\end{align*}$$

**5.2. Relation between the system in Theorem 5.4 and cluster algebras.** Let $I = \{1, 2, 3, 4\}$ and

$$\begin{align*}
S &= \{2u \mid u \in \mathbb{Z}_{\geq 0}\}, \\
S' &= \{2u + 1 \mid u \in \mathbb{Z}_{\geq 0}\}.
\end{align*}$$

Let

$$V = \{(1) \times S\} \cup \{(2) \times S'\} \cup \{(3) \times S\} \cup \{(4) \times S\}.$$
Let $\tilde{t} = \tilde{t}_1 \cup \tilde{t}_2$, where

$$\tilde{t}_1 = \left\{ \tilde{t}_{0,0,t}^{(-2)}, \tilde{t}_{0,0,0,k}^{(-4)} \mid k, l \in \mathbb{Z}_{\geq 1} \right\}$$

and

$$\tilde{t}_2 = \left\{ \tilde{t}_{n,0,0,0}^{(-4n+4)}, \tilde{t}_{0,m,0,0}^{(-4m)}, \tilde{t}_{n,0,0,0}^{(-4n+2)}, \tilde{t}_{0,m,0,0}^{(-4m+2)}, \tilde{t}_{0,0,0,0}^{(-2l-2)}, \tilde{t}_{0,0,0,k}^{(2k-2)} \mid k, l, m, n \in \mathbb{Z}_{\geq 1} \right\}.$$

Let $\tilde{A}$ be the cluster algebra defined by the initial seed $(\tilde{t}, \tilde{Q})$. By similar arguments in Section 4, we have the following theorem.

**Theorem 5.6.** Every equation in the system in Theorem 5.4 corresponds to a mutation equation in the cluster algebra $\tilde{A}$. Every minimal affinization in the system in Theorem 5.4 corresponds to a cluster variable of the cluster algebra $\tilde{A}$.

### 6. Proof of Theorem 3.3

In this section, we prove Theorem 3.3. Namely, we will prove that for $s \in \mathbb{Z}, k, l, m, n \in \mathbb{Z}_{\geq 1}$, the modules

$$\tilde{t}_{n,0,0,0}^{(-2)}, \tilde{t}_{0,m,0,0}^{(-4)}, \tilde{t}_{n,0,0,k}^{(-4)}, \tilde{t}_{0,0,0,k}^{(2k-2)} \quad (6.1)$$

are special. Since the modules

$$\tilde{t}_{0,0,0,k}^{(s)}, \tilde{t}_{0,0,0,0}^{(s)}, \tilde{t}_{m,0,0,0}^{(s)}$$

are Kirillov-Reshetikhin modules, they are special. By Theorem 3.1, the modules $\tilde{t}_{n,0,0,0}^{(s)}, \tilde{t}_{0,0,0,0}^{(s)}, \tilde{t}_{n,0,0,k}^{(s)}, \tilde{t}_{0,0,0,k}^{(s)}$ are special. In the following, we will prove that the other modules in (6.1) are special. Without loss of generality, we may assume that $s = 0$ in $\tilde{t}^{(s)}$, where $\tilde{t}$ is a module in (6.1).

#### 6.1. The cases of $\tilde{t}_{n,0,0,0}^{(0)}$, $\tilde{t}_{0,0,0,k}^{(-2)}$, $\tilde{t}_{0,0,0,0}^{(s)}$, and $\tilde{t}_{0,0,0,k}^{(s)}$.

Since the proof of each case is similar to each other, we give a detailed proof of the case of $\tilde{t}_{n,0,0,0}^{(0)}$ as follows.

Let $m_+ = \tilde{t}_{n,0,0,0}^{(0)}$ with $n, k \in \mathbb{Z}_{\geq 1}$. Then

$$m_+ = (4_04_1 \cdots 4_{n-4})(1_{4n+4}4_{n+6} \cdots 1_{2n+2k+2}).$$

Suppose that $n = 1$. Let

$$U = I \times \{ aq^s : s \in \mathbb{Z}, s \leq 2k + 6 \}.$$

Since all monomials in $\mathcal{M}(\chi_q(m_+) - \operatorname{trunc}_{m_+} \chi_q(m_+))$ are right-negative, it is sufficient to show that $\operatorname{trunc}_{m_+} \chi_q(m_+)$ is special.

Let

$$\mathcal{M} = \{ m_0 = m_+, m_1 = m_0A_{4,2}^{-1}, m_2 = m_1A_{3,4}^{-1}, m_3 = m_2A_{2,6}^{-1}, m_4 = m_3A_{1,8}^{-1}, m_5 = m_4A_{3,6}^{-1}, m_6 = m_5A_{4,8}^{-1} \}.$$

It is easy to see that $\mathcal{M}$ satisfies the conditions in Theorem 2.4. Therefore

$$\operatorname{trunc}_{m_+} \chi_q(m_+) = \sum_{m \in \mathcal{M}} m.$$
and hence \( \text{trunc}_{m_+ \mathbb{Q}_2} \chi_q(m_+) \) is special.

Suppose that \( n \geq 2 \). In the following, we write \( m_+ = m'_1 m'_2 = m''_1 m''_2 \) for some monomials \( m'_1, m'_2, m''_1, m''_2 \). We will show that the only dominant monomial in \( \mathcal{M}(\chi_q(m'_1) \chi_q(m''_1)) \cap \mathcal{M}(\chi_q(m'_2) \chi_q(m''_2)) \) is \( m_+ \) which implies that \( L(m_+) \) is special.

Let \( m_+ = m'_1 m'_2 \), where

\[
m'_1 = 4_0 4_1 \cdots 4_{n-8}, \quad m'_2 = 4_{4n-4} 4_{4n+4} 4_{4n+6} \cdots 4_{4n+2k+2}.
\]

We have shown that \( L(m'_2) \) is special. Therefore the Frenkel-Mukhin algorithm works for \( L(m'_2) \). We will use the Frenkel-Mukhin algorithm to compute \( \chi_q(L(m'_1)), \chi_q(L(m'_2)) \) and classify all dominant monomials in \( \chi_q(L(m'_1)) \chi_q(L(m'_2)) \). Let \( m = m_1 m_2 \) be a dominant monomial, where \( m_i \in \mathcal{M}(L(m'_i)), i = 1, 2 \).

Suppose that \( m_2 \neq m'_2 \). If \( m_2 \) is right-negative, then \( m \) is a right negative monomial and therefore \( m \) is not dominant, this is a contradiction. Hence \( m_2 \) is not right-negative. Through the above discussion, \( m_2 \) is one of the following monomials

\[
\begin{align*}
m_1 &= m'_1 4_{1,4n-2} = 4_{4n}^{-1} 3_{4n-2} 4_{4n+4} 4_{4n+6} \cdots 4_{4n+2k+2}, \\
m_2 &= m'_1 3_{1,4n} = 3_{4n+2}^{-1} 4_{4n-1} 4_{4n+4} 4_{4n+6} \cdots 4_{4n+2k+2}, \\
m_3 &= m'_2 4_{2,4n+3} = 2_{4n-1} 1_{4n+3} 4_{4n+4} 4_{4n+6} \cdots 4_{4n+2k+2}, \\
m_4 &= m'_2 4_{3,4n} = 4_{4n} 4_{4n+2} 4_{4n+4} 4_{4n+6} \cdots 4_{4n+2k+2}, \\
m_5 &= m'_4 4_{3,4n+2} = 4_{4n+2} 4_{4n+4} 4_{4n+6} \cdots 4_{4n+2k+2}, \\
m_6 &= m'_4 4_{4,4n+4} = 4_{4n} 4_{4n+2} 4_{4n+4} 4_{4n+6} \cdots 4_{4n+2k+2}.
\end{align*}
\]

We next discuss \( m_2 \) as follows:

**Case 1.** The factor 4\(_{4n}\) can only come from the monomials in \( \chi_q(4_{4n-8}) \), the monomials in \( \chi_q(4_{4n-8}) \) which contain a factor 4\(_{4n}\) are

\[
3_{4n-2} 4_{4n-4}, \quad 2_{4n+1} 2_{4n+3} 3_{4n-4} 2_{4n+4} 4_{4n}, \quad 1_{4n+1} 2_{4n+2} 4_{4n+5} 3_{4n+2} 4_{4n}, \quad 1_{4n+1} 4_{4n+6} 3_{4n+2} 4_{4n}.
\]

The negative factors in (6.2) can not be canceled by monomials in \( \chi_q(4_{4n-12}) \). However, the negative factor \( 1_{4n+1} 4_{4n+6} 3_{4n+2} 4_{4n} \) can be canceled by \( m_1 \). Therefore \( m_2 = m_1 \) in the situation.

**Case 2.** The factor 3\(_{4n+2}\) can only come from the monomials in \( \chi_q(4_{4n-8}) \), the monomials in \( \chi_q(4_{4n-8}) \) which contain a factor 3\(_{4n+2}\) are

\[
1_{4n+2} 4_{4n+4} 4_{4n+4} 3_{4n+2} 4_{4n+4} 4_{4n+4}, \quad 1_{4n+2} 4_{4n+4} 4_{4n+4} 3_{4n+2} 4_{4n+4} 4_{4n+4}.
\]

The negative factors in (6.3) can not be canceled by monomials in \( \chi_q(4_{4n-12}) \). The factors 2\(_{4n+3}, 3_{4n+4}, 4_{4n+6}\) can only come from the monomials in \( \chi_q(4_{4n-8}) \), the monomials in \( \chi_q(4_{4n-8}) \) which contain factors 2\(_{4n+3}, 3_{4n+4}, 4_{4n+6}\) are

\[
1_{4n+2} 4_{4n+4} 4_{4n+4} 3_{4n+2} 4_{4n+4} 4_{4n+4}, \quad 1_{4n+2} 4_{4n+4} 4_{4n+4} 3_{4n+2} 4_{4n+4} 4_{4n+4}.
\]

The negative factors in (6.4) can not be canceled by monomials in \( \chi_q(4_{4n-12}) \).
ON THE MINIMAL AFFINIZATIONS OF TYPE $F_4$

Since

$$\chi_q(4_04_4 \cdots 4_{4n-8}) \leq \chi_q(4_04_4 \cdots 4_{4n-12}) \chi_q(4_{4n-8}) \leq \chi_q(4_04_4 \cdots 4_{4n-16}) \chi_q(4_{4n-12}) \chi_q(4_{4n-8}),$$

and $m = m_1m_2$ is dominant, $m_2 = m'_2$ or $m_1$.

Suppose that $m_2 = m_1$. By the above discussion, $m$ is in

$$\chi_q(4_04_4 \cdots 4_{4n-12})3_{4n+8} \cdots 1_{4n+2k+2}. \quad (6.5)$$

Suppose that $m_2 = m'_2$. If $m_1 \neq m'_1$, then $m_1$ is right negative. Since $m$ is dominant, each factor with a negative power in $m_1$ needs to be canceled by a factor in $m'_2$. We have $\mathcal{M}(L(m'_1)) \subset \mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12})) \chi_q(L(4_{4n-8}))$. Only monomials in $\chi_q(L(4_{4n-8}))$ can cancel $4_{4n-4}$, $1_{4n+4}$, $1_{4n+6}$. Therefore $m_1$ is one of the sets

$$\begin{align*}
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))3_{4n-6}4_{4n-4}, \\
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n+4}1_{4n+6}4_{4n}, \\
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n+6}3_{4n+2}4_{4n}, \\
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n+1}1_{4n+2}4_{4n+6}, \\
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n+4}1_{4n+6}2_{4n+3}4_{4n+4}, \\
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n+4}2_{4n+2}4_{4n+7}, \\
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n+1}4_{4n+6}, \\
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n+1}2_{4n+6}2_{4n+3}4_{4n+3}, \\
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n+1}2_{4n+6}3_{4n+2}4_{4n+4}, \\
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n+1}2_{4n+6}3_{4n+2}4_{4n+4}, \\
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n+1}2_{4n+6}5_{4n+6}.
\end{align*}$$

By (6.3), we know that $m_1 \notin \mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n+6}2_{4n-1}3_{4n-2}$. The factors which contain $1_{4n-2}$ in $\chi_q(4_{4n-12})$ are

$$\begin{align*}
1_{4n-2}4_{4n}2_{4n-1}2_{4n+1}4_{4n-4}, & \quad 1_{4n-2}4_{4n}2_{4n-1}2_{4n+1}3_{4n-2}4_{4n-1}, & \quad 1_{4n-2}4_{4n}2_{4n+1}3_{4n-2}4_{4n-4}, \\
1_{4n-2}4_{4n}2_{4n-1}3_{4n-2}4_{4n-1}, & \quad 1_{4n-2}4_{4n}2_{4n-1}3_{4n-2}4_{4n-4}, & \quad 1_{4n-2}4_{4n}2_{4n+1}3_{4n-2}4_{4n-4} \quad (6.6)
\end{align*}$$

The negative factors in (6.6) can not be canceled by monomials in $\chi_q(4_{4n-16})$.

Since $\chi_q(4_04_4 \cdots 4_{4n-12}) \leq \chi_q(4_04_4 \cdots 4_{4n-16}) \chi_q(4_{4n-12})$ and $1_{4n+4}$, $1_{4n+6}$, $2_{4n+7} \notin \chi_q(4_{4n-12})$. Therefore $m_1$ is in one of the sets

$$\begin{align*}
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))3_{4n-6}4_{4n-4}, \\
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n}4_{4n+6}
\end{align*}$$

Therefore $m$ is in one of the sets

$$\begin{align*}
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))3_{4n-6}4_{4n+4}4_{4n+6} \cdots 1_{4n+2k+2}, & \quad (6.7) \\
\mathcal{M}(\chi_q(4_04_4 \cdots 4_{4n-12}))1_{4n}4_{4n-4}4_{4n+4}4_{4n+6} \cdots 1_{4n+2k+2}. & \quad (6.8)
\end{align*}$$
Let $m_+ = m_1''m_2''$, where
\[ m_1'' = 4044 \cdots 4_{4n-4}, \quad m_2'' = 1_{4n+1}1_{4n+6} \cdots 1_{4n+2k+2}. \]

If $m$ is the expression of (6.5), (6.7), (6.8), we know that $m \notin \mathcal{M}(\chi_q(m_1'')\chi_q(m_2''))$ by the Frenkel-Mukhin algorithm.

Therefore the only dominant monomial in $\mathcal{M}(\chi_q(m_1')\chi_q(m_2')) \cap \mathcal{M}(\chi_q(m_1'')\chi_q(m_2''))$ is $m_+$. Hence the only dominant monomial in $\chi_q(m_+)$ is $m_+$.

6.2. The case of $\tilde{T}_{n,m,0,k}^{(0)} (k \leq 2)$, $\tilde{T}_{n,0,l,k}^{(0)} (k \leq 2)$, and $\tilde{T}_{0,m,l,k}^{(0)} (k \leq 2)$. Since the proof of each case is similar to each other, we give a detailed proof of the case of $\tilde{T}_{n,m,0,k}^{(0)} (k \leq 2)$ as follows.

Let $m_+ = \tilde{T}_{n,m,0,k}^{(0)}$ with $n, m, k \in \mathbb{Z}_{\geq 1}$ and $k \leq 2$. Then
\[ m_+ = (4044 \cdots 4_{4n-4})(3_{4n+2}3_{4n+6} \cdots 3_{4n+4m-2})(1_{4n+4m+4} \cdots 1_{4n+4m+2k+2}). \]

Let
\[ m_1' = (4044 \cdots 4_{4n-4})(3_{4n+2}3_{4n+6} \cdots 3_{4n+4m-2}), \quad m_2' = (1_{4n+4m+4} \cdots 1_{4n+4m+2k+2}), \]
\[ m_1'' = (4044 \cdots 4_{4n-4}), \quad m_2'' = (3_{4n+2}3_{4n+6} \cdots 3_{4n+4m-2})(1_{4n+4m+4} \cdots 1_{4n+4m+2k+2}). \]

Then $\mathcal{M}(L(m_+)) \subset \mathcal{M}(\chi_q(m_1')\chi_q(m_2')) \cap \mathcal{M}(\chi_q(m_1'')\chi_q(m_2''))$.

By using similar arguments as in Subsection 6.1, we can show that the only possible dominant monomial in $\chi_q(m_1')\chi_q(m_2') \cap \chi_q(m_1'')\chi_q(m_2'')$ is $m_+$. Hence the only dominant monomial in $\chi_q(m_+)$ is $m_+$.

6.3. The case of $\tilde{T}_{n,m,l,k}^{(0)} (k \leq 2)$. Let $m_+ = \tilde{T}_{n,m,l,k}^{(0)}$ with $n, m, l, k \in \mathbb{Z}_{\geq 1}$ and $k \leq 2$. Then
\[ m_+ = (4044 \cdots 4_{4n-4})(3_{4n+2}3_{4n+6} \cdots 3_{4n+4m-2})(2_{4n+4m+3}2_{4n+4m+5} \cdots 2_{4n+4m+2l+1}) \cdots 1_{4n+4m+2l+2k+2}). \]

Let
\[ m_1' = (4044 \cdots 4_{4n-4})(3_{4n+2}3_{4n+6} \cdots 3_{4n+4m-2})(2_{4n+4m+3}2_{4n+4m+5} \cdots 2_{4n+4m+2l+1}), \]
\[ m_2' = (1_{4n+4m+2l+4}1_{4n+4m+2l+6} \cdots 1_{4n+4m+2l+2k+2}), \]
\[ m_1'' = (4044 \cdots 4_{4n-4}), \quad m_2'' = (3_{4n+2}3_{4n+6} \cdots 3_{4n+4m-2})(2_{4n+4m+3}2_{4n+4m+5} \cdots 2_{4n+4m+2l+1})(1_{4n+4m+2l+4}1_{4n+4m+2l+6} \cdots 1_{4n+4m+2l+2k+2}). \]

Then $\mathcal{M}(L(m_+)) \subset \mathcal{M}(\chi_q(m_1')\chi_q(m_2')) \cap \mathcal{M}(\chi_q(m_1'')\chi_q(m_2''))$.

By using similar arguments as in Subsection 6.1, we can show that the only possible dominant monomial in $\chi_q(m_1')\chi_q(m_2') \cap \chi_q(m_1'')\chi_q(m_2'')$ is $m_+$. Hence the only dominant monomial in $\chi_q(m_+)$ is $m_+$.

7. Proof of Theorem 3.4

In this section, we prove Theorem 3.4.
7.1. Classification of dominant monomials in the summands on both sides of the system. In Section 6, we have shown that for \( s \in \mathbb{Z}, k, l, m, n \in \mathbb{Z}_{\geq 1} \), the modules in Theorem 3.3 are special. Now we use the Frenkel-Mukhin algorithm to classify dominant monomials in the summands on both sides of the system in Theorem 3.4.

**Lemma 7.1.** The dominant monomials in each summand on the left and right hand sides of every equation in the system of Theorem 3.4 are given in Table 2 and 3.

Proof. We will prove the case of (3.15), the other cases are similar. Let \( m_1 = \tilde{T}_{n,m-1,l,k}^{(s+4)}, m_2 = \tilde{T}_{n,m,l,k}^{(s)} \). Without loss of generality, we may assume that \( s = 0 \).

Then

\[
m_1' = (4 \cdot 4n) (3^{n+6} + 10 \cdots 3^{n+4m-2}) (2^{4n+4m+3} 2^{4n+4m} \cdots 2^{4n+4m+2l+1})
\]

\[
(1^{4n+4m+2} 4^{4n+4m+2} \cdots 1^{4n+4m+2l+2}),
\]

\[
m_2' = (40^{4} \cdot 4n) (3^{n+3} + 10 \cdots 3^{n+4m-2}) (2^{4n+4m+3} 2^{4n+4m} \cdots 2^{4n+4m+2l+1})
\]

\[
(1^{4n+4m+2} 4^{4n+4m+2} \cdots 1^{4n+4m+2l+2}).
\]

Let \( m = m_1m_2 \) be a dominant monomial, where \( m_i \in \chi_q(m_i'), i = 1, 2 \). We denote

\[
m_3 = (3^{n+6} + 10 \cdots 3^{n+4m-2}) (2^{4n+4m+3} 2^{4n+4m} \cdots 2^{4n+4m+2l+1})
\]

\[
(1^{4n+4m+2} 4^{4n+4m+2} \cdots 1^{4n+4m+2l+2}),
\]

\[
m_4 = (3^{n+3} + 10 \cdots 3^{n+4m-2}) (2^{4n+4m+3} 2^{4n+4m} \cdots 2^{4n+4m+2l+1})
\]

\[
(1^{4n+4m+2} 4^{4n+4m+2} \cdots 1^{4n+4m+2l+2}).
\]

Suppose that \( m_1 \in \chi_q(4 \cdots 4n) (\chi_q(m_3) - m_3) \), then \( m = m_1m_2 \) is right negative and hence \( m \) is not dominant. Therefore \( m_1 \in \chi_q(4 \cdots 4n)m_3 \). Similarly, if \( m_2 \in \chi_q(4 \cdots 4n) (\chi_q(m_4) - m_4) \), then \( m = m_1m_2 \) is right negative and hence \( m \) is not dominant. This contradicts our assumption. Therefore \( m_2 \notin \chi_q(4 \cdots 4n)m_4 \).

Suppose that \( m_1 \in \mathcal{M}(L(m_1')) \cap \chi_q(4 \cdots 4n) (\chi_q(m_4) - 4n)m_3 \). By the Frenkel-Mukhin algorithm for \( L(m_1') \), \( m_1 \) must have the factor \( 4^{-1} \cdot 4^{4n+4} \). But by the Frenkel-Mukhin algorithm and the fact that \( m_2 \notin \chi_q(4 \cdots 4n) \), \( m_2 \) does not have the factor \( 4^{4n+4} \). Therefore \( m_1m_2 \) is not dominant. Hence \( m_1 \notin \chi_q(4 \cdots 4n)m_4 \). It follows that \( m_1 = m_1' \).

By the Frenkel-Mukhin algorithm and the fact that \( m_2 \notin \chi_q(4 \cdots 4n)m_4 \), \( m_2 \) must be one of the following monomials,

\[
v_1 = m_2' A_{4,4n-2}^{-1} = 4 0 4 \cdots 4^{4n-8} 4^{4n-3} m_2,
\]

\[
v_2 = m_2' A_{4,4n-2}^{-1} A_{4,4n-6}^{-1} = 4 0 4 \cdots 4^{4n-12} 4^{-1} 4^{4n-6} 3 m_2,
\]

\[
\vdots
\]

\[
v_n = m_2' A_{4,4n-2}^{-1} A_{4,4n-6}^{-1} \cdots A_{4,2}^{-1} = 4^{-1} \cdots 4^{-1} 4^{4n-4} 3 \cdots 3 \cdot 3 m_2.
\]

It follows that the dominant monomials in \( \chi_q(\tilde{T}_{n,m-1,l,k}^{(s+4)}) \chi_q(\tilde{T}_{n,m,l,k}^{(s)}) \) are...
Table 2. Classification of dominant monomials in the system in Theorem 3.4
By Lemma 7.1, the dominant monomials in the $q$-characters of the right hand side and of the left hand side of every equation in Theorem 3.4 are the same. Therefore the theorem is true.

### 7.2. Proof of Theorem 3.4

By Lemma 7.1, the dominant monomials in the $q$-characters of the right hand side and of the left hand side of every equation in Theorem 3.4 are the same. Therefore the theorem is true.

### 8. Proof of Theorem 3.6

In this section, we prove Theorem 3.6. By Lemma 7.1, we have known that the modules in the second summand on the right hand side of every equation in Theorem 3.4 are special, and hence they are simple. Therefore in order to prove Theorem 3.6 we only need to show that the modules in the first summand on the right hand side of every equation in Theorem 3.4 are simple.

### 8.1. Proof of Theorem 3.6

We will prove the case of $\chi_q(\widetilde{T}_{n-1,m,l,k})\chi_q(\widetilde{T}_{n,1,m-1,l,k})$. The other cases are similar. By definition, we have

\[
M = m_1 m_2, \quad M_1 = v_1 m_1 = MA_{4,n-2}^{-1}, \quad M_2 = v_2 m_1' = M \prod_{i=0}^{n-1} A_{4,4n-4i-2}^{-1}, \quad \ldots,
\]

\[
M_{n-1} = v_{n-1} m_1' = M \prod_{i=0}^{n-2} A_{4,4n-4i-2}^{-1}, \quad M_n = v_n m_1' = M \prod_{i=0}^{n-1} A_{4,4n-4i-2}^{-1}.
\]
\[ M_r = M \prod_{i=0}^{r-1} A_{4s+4n+2-2i}, 0 \leq r \leq n-1, \text{ where } M = \tilde{T}_{n-1,m,l,k}(s+4) \tilde{T}_{n+1,m-1,l,k}(s). \]

We need to show that \( \chi_q(M_r) \not\subseteq \chi_q(M) \) for \( 1 \leq r \leq n-1 \). We will prove the case of \( r = 1 \), the other cases are similar.
\[ M_1 = MA^{-1}_{4,aq+n-2} = MA^{-1}_{3,aq+n-2} = MA^{-1}_{s+4n-4}. \]

By \( U_q^2(\mathfrak{sl}_2) \) argument, it is clear that \( n_1 = MA^{-1}_{4,aq+n-2}A^{-1}_{3,aq+n} \) is in \( \chi_q(M_1) \).

If \( n_1 \) is in \( \chi_q(T_{n,m-1,l,k}^0)\chi_q(T_{n,m,l,k}^0) \), then \( T_{n,m-1,l,k}^0A^{-1}_{3,aq+n-2}A^{-1}_{3,aq+n+4} \) is in \( \chi_q(T_{n,m-1,l,k}^0) \) which is impossible by the Frenkel-Mukhin algorithm for \( T_{n,m-1,l,k}^0 \). Hence \( \chi_q(M_1) \not\subseteq \chi_q(M) \).

9. Conjectural equations satisfied by the \( q \)-characters of other minimal affinizations in type \( F_4 \)

In this section, we give some conjectural equations satisfied by the \( q \)-characters of the minimal affinizations in type \( F_4 \) which are not in Theorem 3.3 and Theorem 5.4. In order to study equations satisfied by \( q \)-characters, we introduce the concept of dominant monomial graphs for a tensor product of simple \( U_q\mathfrak{g} \)-modules.

9.1. Conjecture about the minimal affinizations which are not in Theorem 3.4 and Theorem 5.4. Let \( l, m, n \in \mathbb{Z}_{\geq 1}, k \in \mathbb{Z}_{\geq 3}, s \in \mathbb{Z} \). We define

\[ \tilde{S}_{0,m,0,k}(s) = \tilde{T}_{0,0,k}^{s+3}, \tilde{S}_{0,0,l,k}^{s+4} \]

\[ \tilde{S}_{n,0,l,k}(s) = \begin{cases} \tilde{T}_{0,0,l,k}^{s+4} & l = 1, \\ \tilde{T}_{0,0,l,k}^{s+4} & l \geq 2, \end{cases} \]

\[ \tilde{S}_{0,m,0,k}(s) = \begin{cases} \tilde{T}_{0,0,m,0,k}^{s+3} & l = 1, \\ \tilde{T}_{0,0,m,0,k}^{s+3} & l \geq 2, \end{cases} \]

\[ \tilde{S}_{n,m,0,k}(s) = \begin{cases} \tilde{T}_{0,m,0,k}^{s+4} & l = 1, \\ \tilde{T}_{0,m,0,k}^{s+4} & l \geq 2, \end{cases} \]

\[ \tilde{S}_{n,m,l,k}(s) = \begin{cases} \tilde{T}_{0,m,l,k}^{s+4} & l = 1, \\ \tilde{T}_{0,m,l,k}^{s+4} & l \geq 2. \end{cases} \]

We use \( \tilde{S}_{n,m,l,k}(s) \) to denote the simple \( U_q\mathfrak{g} \)-module with the highest weight monomial \( \tilde{S}_{n,m,l,k}(s) \).

Conjecture 9.1. For \( s \in \mathbb{Z}, n, m, l \in \mathbb{Z}_{\geq 1}, k \in \mathbb{Z}_{\geq 3} \), we have the following equations in \( \text{Rep}(U_q\mathfrak{g}) \).

\[ [\tilde{T}_{0,m,0,k}^{s+4}][\tilde{T}_{0,m,0,k}^{s+4}] = [\tilde{T}_{0,0,m,0,k}^{s+4}][\tilde{T}_{0,0,m,0,k}^{s+4} + \tilde{S}_{0,m,0,k}^{s+4}], \quad (9.1) \]

\[ [\tilde{T}_{n,m-1,l,k}^{s+4}][\tilde{T}_{n,m,l,k}^{s+4}] = [\tilde{T}_{n,0,l,k}^{s+4}][\tilde{T}_{n,0,l,k}^{s+4} + \tilde{S}_{n,m,0,k}^{s+4}], \quad (9.2) \]

\[ [\tilde{T}_{n,0,l,k}^{s+4}][\tilde{T}_{n,m,l,k}^{s+4}] = [\tilde{T}_{n,0,l,k}^{s+4}][\tilde{T}_{n,0,l,k}^{s+4} + \tilde{S}_{n,m,l,k}^{s+4}], \quad (9.3) \]

\[ [\tilde{T}_{n,0,l,k}^{s+4}][\tilde{T}_{n,0,l,k}^{s+4}] = [\tilde{T}_{n,0,l,k}^{s+4}][\tilde{T}_{n,0,l,k}^{s+4} + \tilde{S}_{n,0,l,k}^{s+4}], \quad l = 1, \quad (9.4) \]

\[ [\tilde{T}_{n,0,l,k}^{s+4}][\tilde{T}_{n,0,l,k}^{s+4}] = [\tilde{T}_{n,0,l,k}^{s+4}][\tilde{T}_{n,0,l,k}^{s+4} + \tilde{S}_{n,0,l,k}^{s+4}], \quad l \geq 2. \quad (9.5) \]
\[
\tilde{T}_{0,m,0,k-1}(s)\tilde{T}_{0,m,1,k}(s) = \tilde{T}_{0,m-1,1,k}(s)\tilde{T}_{0,m+1,0,k-1}(s) + \tilde{S}_{0,m,1,k}(s), \quad l = 1, \\
\tilde{T}_{0,m,l-2,k}(s)\tilde{T}_{0,m,l,k}(s) = \tilde{T}_{0,m-1,l,k}(s)\tilde{T}_{0,m+1,l-2,k}(s) + \tilde{S}_{0,m,l,k}(s), \quad l \geq 2.
\]

Example 9.2. The following are some equations of Equation (9.1) in Conjecture 9.1.
\[
[1_{t=3}]_{l=4} [1_{t=3}]_{l=10} = [1_{t=4}1_{t=6}]_{l=10}[1_{t=3}]_{l=10} + [1_{t=4}1_{t=2}]_{l=2} + [1_{t=4}1_{t=2}]_{l=4}.
\]

Example 9.3. The following are some equations of Equation (9.4) and Equation (9.5) in Conjecture 9.1.
\[
[1_{t=1}]_{l=4} [1_{t=1}]_{l=10} = [1_{t=1}]_{l=4} [1_{t=1}]_{l=10} + [1_{t=1}]_{l=4} [1_{t=1}]_{l=10} + [1_{t=1}]_{l=4} [1_{t=1}]_{l=10} + [1_{t=1}]_{l=4} [1_{t=1}]_{l=10}.
\]

9.2. Dominant monomial graphs. In order to study equations satisfied by \(q\)-characters, we introduce dominant monomial graphs for a tensor product of simple \(U_q\mathfrak{g}\)-modules.

Definition 9.4. Let \(T = T_1 \otimes \cdots \otimes T_k\) be a tensor product of simple \(U_q\mathfrak{g}\)-modules. We define the dominant monomial graph \(G(T)\) for \(T\) as follows. The vertices of \(G(T)\) are dominant monomials in \(\chi_q(T) = \chi_q(T_1) \cdots \chi_q(T_k)\). For two vertices \(v_1, v_2\) in \(G(T)\), there is an arrow from \(v_1\) to \(v_2\) if and only if \(v_2 < v_1\).

Let \(G\) be a dominant monomial graph. Suppose that \(a, b\) are two vertices in \(G\) and \(b < a\). Then \(b = ma\) for some \(m \in \mathbb{Q}^+\). We draw

\[
a \longrightarrow \quad b
\]

when we draw the graph \(G\).

In the following, we draw the dominant monomial graphs for the modules in the equivalence classes on the left hand side of the equations in Examples 9.2 and 9.3. Figure 1 – Figure 10 correspond Equation (9.8) – Equation (9.17) respectively.

In all examples of dominant monomial graphs, we find that every graph can be divided into two parts. The vertices in the first (resp. second) part of the graph are dominant monomials in the first (resp. second) summand of the right hand side of the corresponding equation. For
example, in Figure 2, the monomials $M$, $a_2$, $a_3$, $a_4$ (resp. $a_5$, $a_6$, $a_7$, $a_8$, $a_9$, $a_{10}$, $a_{11}$) are the dominant monomials of the first (resp. second) summand of the right hand side of Equation (1.9).

These graphs are also conjectural, since we are not able to show that the Frenkel-Mukhin algorithm works for the modules which are not special. If we can show that these graphs are indeed the dominant monomial graphs for the corresponding modules, then the corresponding conjectural equations are true.

For $k \in \mathbb{Z}$, let

$$ N_k^{(1)} = A_{4,k}^{-1} A_{3,k+2}^{-1} A_{2,k+4}^{-1} A_{1,k+5}^{-1}, $$

$$ N_k^{(2)} = A_{4,k}^{-1} A_{3,k+2}^{-1} A_{2,k+4}^{-1} A_{1,k+5}^{-1}, $$

$$ N_k^{(3)} = A_{4,k-2}^{-1} A_{3,k}, $$

$$ N_k^{(4)} = A_{4,k-2}^{-1} A_{3,k} A_{1,k-2}, $$

$$ M_k^{(1)} = A_{4,k-2}^{-1} A_{3,k} A_{1,k+2}, $$

where $A_{i,k} = A_{i,a_k}$ which is defined in 2.2.

We have the following relations

$$ A_{2,k} N_k^{(2)} = A_{4,k} N_k^{(1)}, \quad A_{2,k+2} A_{1,k+3} N_k^{(1)} = N_k^{(3)}, $$

$$ N_k^{(3)} = A_{2,k-2} N_k^{(4)}, \quad A_{4,k-2} N_k^{(4)} = A_{4,k+2} M_k^{(1)}. $$

\[\text{Figure 1. The dominant monomial graph for } L(m_1) \otimes L(m_2) \text{ (} M = m_1 m_2 \text{ where } m_1 = 1_0 3_{-6}, m_2 = 1_{-4} 1_{-2} 10_3 3_{-10})\]

\[\text{Figure 2. The dominant monomial graph for } L(m_1) \otimes L(m_2) \text{ (} M = m_1 m_2 \text{ where } m_1 = 1_0 3_{-10} 3_{-6}, m_2 = 1_{-4} 1_{-2} 10_3 3_{-14} 3_{-10}).\]
where $m = 1_{-41-2103_{-10}}, m_2 = 1_{-81-61-41-2103_{-14}}$.

Figure 3. The dominant monomial graph for $L(m_1) \otimes L(m_2)$ ($M = m_1 m_2$ where $m_1 = 1_{-41-2103_{-10}}, m_2 = 1_{-81-61-41-2103_{-14}}$).

Figure 4. The dominant monomial graph for $L(m_1) \otimes L(m_2)$ ($M = m_1 m_2$ where $m_1 = 1_{-2104_{-10}}, m_2 = 1_{-41-2102-74_{-14}}$).

Figure 5. The dominant monomial graph for $L(m_1) \otimes L(m_2)$ ($M = m_1 m_2$ where $m_1 = 1_{-2104_{-14}4_{-10}}, m_2 = 1_{-41-2102-74_{-18}4_{-14}}$).

Figure 6. The dominant monomial graph for $L(m_1) \otimes L(m_2)$ ($M = m_1 m_2$ where $m_1 = 1_{-41-2104_{-12}}, m_1 = 1_{-41-2102-92-74_{-16}}$).
Figure 7. The dominant monomial graph for $L(m_1) \otimes L(m_2)$ ($M = m_1 m_2$ where $m_1 = 1_{-4}1_{-2}1_{0}4_{-16}4_{-12}, m_2 = 1_{-4}1_{-2}1_{0}2_{-9}2_{-7}4_{-20}4_{-16}$).

Figure 8. The dominant monomial graph for $L(m_1) \otimes L(m_2)$ ($M = m_1 m_2$ where $m_1 = 1_{-4}1_{-2}1_{0}2_{-7}4_{-14}, m_2 = 1_{-4}1_{-2}1_{0}2_{-11}2_{-9}2_{-7}4_{-18}$).
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Figure 9. The dominant monomial graph for $L(m_1) \otimes L(m_2)$ ($M = m_1 m_2$ where $m_1 = 1_{-4} 1_{-2} 102_{-74} 1_{-14}$, $m_2 = 1_{-4} 1_{-2} 102_{-112} 9_{-2} 74_{-224} 1_{-18}$).

Figure 10. The dominant monomial graph for $L(m_1) \otimes L(m_2)$ ($M = m_1 m_2$ where $m_1 = 1_{-4} 1_{-2} 102_{-92} 74_{-16}$, $m_2 = 1_{-4} 1_{-2} 102_{-13} 10_{-112} 9_{-2} 74_{-20}$).
ON THE MINIMAL AFFINIZATIONS OF TYPE $F_4$

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