The Rate of Convergence of the Augmented Lagrangian Method for a Nonlinear Semidefinite Nuclear Norm Composite Optimization Problem*

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Abstract

We propose two basic assumptions, under which the rate of convergence of the augmented Lagrange method for a class of composite optimization problems is estimated. We analyze the rate of local convergence of the augmented Lagrangian method for a nonlinear semidefinite nuclear norm composite optimization problem by verifying these two basic assumptions. Without requiring strict complementarity, we prove that, under the constraint nondegeneracy condition and the strong second order sufficient condition, the rate of convergence is linear and the ratio constant is proportional to $1/c$, where $c$ is the penalty parameter that exceeds a threshold $\overline{c} > 0$. The analysis is based on variational analysis about the proximal mapping of the nuclear norm and the projection operator onto the cone of positively semidefinite symmetric matrices.

Key words: Composite optimization, nonlinear semidefinite nuclear norm composite optimization, rate of convergence, the augmented Lagrangian method, variational analysis.

1 Introduction

Nuclear norm optimization problems have seen many applications in engineering and science. They arise from the convex relaxation of a rank minimization problem with noisy data in many machine learning and compressed sensing applications such as dimensionality reduction, matrix classification, multi-task learning and matrix completion, as well as in theoretical applications from mathematics ([13],[1], [30],[6],[19]). A proximal point algorithmic framework was developed in [20] for solving convex nuclear norm optimization problems and numerical results show that the proposed proximal point algorithms perform quite well

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in comparison to several recently proposed state-of-the-art algorithms. For non-convex non-linear programming and non-convex semidefinite programming, related to proximal point algorithms, the augmented Lagrange method is regarded as an effective numerical method. It is quite natural to consider the augmented Lagrange method for the non-convex nuclear norm composite optimization problem and study its theoretical properties. In the general setting, the augmented Lagrangian method can be used to solve the following composite optimization problem

\[(\text{COP}) \quad \min f(x) + \theta(F(x)) \quad \text{s.t.} \quad h(x) = 0, \ g(x) \in K,\]

where \(f : \mathbb{R}^n \mapsto \mathbb{R}, F : \mathbb{R}^n \mapsto \mathbb{Z}, h : \mathbb{R}^n \mapsto \mathbb{R}^m\) and \(g : \mathbb{R}^n \mapsto \mathcal{Y}\) are twice continuously differentiable mappings, \(\theta : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}\) is a proper lower semicontinuous convex function, \(\mathbb{Z}\) and \(\mathcal{Y}\) are finite-dimensional real Hilbert spaces equipped with scalar product \(\langle \cdot, \cdot \rangle\) and induced norm \(\| \cdot \|\), and \(K\) is a closed convex cone in \(\mathcal{Y}\).

Let \(c > 0\) be a parameter. The augmented Lagrangian function with the penalty parameter \(c\) for problem \((\text{COP})\) is defined as (with no composite term, see [28, Section 11.K])

\[
L_c(x, Y, \mu, \lambda) := f(x) + \theta_c(F(x) + Y/c) - \frac{\|Y\|^2}{2c} + \langle \mu, h(x) \rangle + \frac{c}{2} ||h(x)||^2 + \frac{1}{2c} \left[ ||\Pi_{K^*}(\lambda - c g(x))||^2 - ||\lambda||^2 \right],
\]

where \((x, Y, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{Z} \times \mathbb{R}^m \times \mathcal{Y}\) and \(\Pi_{K^*}(\cdot)\) denotes the metric projection operator onto the set \(K^* (K^*\text{ is the dual cone of } K)\), \(\theta_c = e_1/c \theta\) and \([e_r \theta](\cdot)\) is the Moreau-Yosida regularization of \(\theta\) defined by

\[
[e_r \theta](Z) = \inf_{Z' \in \mathcal{Z}} \left\{ \theta(Z') + \frac{1}{r} ||Z' - Z||^2 \right\}.
\]

The augmented Lagrangian method for solving \((\text{COP})\) can be stated as follows. Let \(c_0 > 0\) be given. Let \((Y^0, \mu^0, \lambda^0) \in \mathcal{Z} \times \mathbb{R}^m \times K^*\) be the initial estimated Lagrange multiplier. At the \(k\)th iteration, determine \(x^k\) by minimizing \(L_{c_k}(x, Y^k, \mu^k, \lambda^k)\), compute \((Y^{k+1}, \mu^{k+1}, \lambda^{k+1})\) by

\[
\begin{cases}
Y^{k+1} := D\theta_c(F(x^k) + Y^k/c), \\
\mu^{k+1} := \mu^k + ch(x^k), \\
\lambda^{k+1} := \Pi_{K^*}(\lambda^k - c_k g(x^k)),
\end{cases}
\]

and update \(c_{k+1}\) by

\[
c_{k+1} := c_k \quad \text{or} \quad c_{k+1} := \kappa c_k
\]

according to certain rules, where \(\kappa > 1\) is a given positive number. In the case when the sequence of parameters \(\{c_k\}\) satisfies \(c_k \to +\infty\), the global convergence of the augmented Lagrangian method can be discussed similarly as in [2]. In this paper, instead of considering global convergence properties, we consider the rate of convergence of the augmented Lagrangian method for \((\text{COP})\) when \(c_k\) has a finite limit, namely the case in which \(c_k \equiv c\) for all sufficient large \(k\). For simplicity in our analysis, for \(k\) sufficiently large, we choose \(x^k\) as an exact local solution of \(L_c(\cdot, Y^k, \mu^k, \lambda^k)\).
The augmented Lagrangian method was proposed by Hestenes [15] and Powell [24] for solving equality constrained nonlinear programming problems and was generalized by Rockafellar [26] to nonlinear programming problems with both equality and inequality constraints. For convex programming, Rockafellar [26] established a saddle point theorem in terms of the augmented Lagrangian and Rockafellar [27] proved the global convergence of the augmented Lagrangian method for any positive penalty parameter.

For nonlinear programming, the study about the rate of convergence of the augmented Lagrangian method is quite complete. For the equality constrained problem, Powell offered a proof in [24] showing that if the linear independence constraint qualification and the second-order sufficient condition are satisfied, then the augmented Lagrangian method can converge locally at a linear rate. Bertsekas [2, Chapter 3] established an important result on the linear rate of convergence of the augmented Lagrangian method for nonlinear programming when the strict complementarity condition is assumed, in which the ratio constant is proportional to $1/c$. On the other hand, without assuming the strict complementarity condition, Conn et al. [9], Contesse-Becker [10], and Ito and Kunisch [17] derived linear convergence rate for the augmented Lagrangian method.

For nonlinear semidefinite programming, without requiring strict complementarity, Sun et al. [34] proved that, under the constraint nondegeneracy condition and the strong second-order sufficient condition, the rate of convergence of the augmented Lagrangian method is linear and the ratio constant is proportional to $1/c$, where $c$ is the penalty parameter that exceeds a threshold $\bar{c} > 0$. Moreover, Sun et al. [34] used a direct way to derive the same linear rate of convergence under the strict complementarity condition.

The main objective of this paper is to study, without assuming the strict complementarity, the rate of convergence of the augmented Lagrangian method for solving the nonlinear semidefinite nuclear norm composite optimization problem

\[
\text{(SDNOP)} \quad \min f(x) + \theta(F(x)) \quad \text{s.t.} \quad h(x) = 0, \ g(x) \in S^+_p,
\]

where $\theta(X) = \|X\|_*$ is the nuclear norm function of $X \in S^q$ (for simplicity, here we only consider the nuclear norm of a symmetric matrix), $S^+_p$ is the cone of all positive semidefinite matrices in $S^p$, the linear space of all $p$ by $p$ symmetric matrices in $\mathbb{R}^{p \times p}$.

The organization of this paper is as follows. In Section 2, we develop a general theory on the rate of convergence of the augmented Lagrangian method for a class of composite optimization problems under two basic assumptions. In Section 3, we discuss variational properties of the projection over the cone of symmetric positively semidefinite matrices and the proximal mapping of the nuclear norm, and the second-order optimality conditions for nonlinear semidefinite nuclear norm composite optimization problem. Section 4 is devoted to applying the theory developed in Section 2 to nonlinear semidefinite nuclear norm composite optimization problem. Finally, we give our conclusions in Section 5.

2 General discussions on the rate of convergence

In this section, we always assume that the cone $K$ presented in the optimization problem (COP) is a closed convex cone and that $\Pi_{K^*}(\cdot)$ is semismooth everywhere, where $K^*$ is the
dual cone of $K$, i.e.,

$$K^* := \{ v \in \mathcal{Y} \mid \langle v, z \rangle \geq 0, \ \forall z \in K \}.$$  

The cones $\mathcal{R}^n_+, \mathcal{S}^n_+$, epi $\| \cdot \|_2$ and epi $\| \cdot \|_*$ satisfy these assumptions, where $\| \cdot \|_2$ and $\| \cdot \|_*$ stand for the spectral norm of a matrix and the nuclear norm of a matrix, respectively. Moreover we always assume that $D\theta_c(\cdot)$ is semismooth everywhere, where $\theta_c(\cdot) = e_{1/c}\theta(\cdot)$ and $[e_c\theta](\cdot)$ is the Moreau-Yosida regularization of $\theta$ defined by (1.2).

A feasible point $x \in \mathbb{R}^n$ to (COP) is called a stationary point if there exists $(Y, \mu, \lambda) \in \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y}$ such that the following Karush-Kuhn-Tucker (KKT) condition is satisfied at $(x, Y, \mu, \lambda)$:

$$\nabla_x L(x, Y, \mu, \lambda) = 0, \ Y \in \partial \theta(F(x)), h(x) = 0, g(x) \in K, \ \lambda \in K^* \text{ and } \langle g(x), \lambda \rangle = 0, \quad (2.1)$$

where the Lagrangian function $L : \mathbb{R}^n \times \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y} \mapsto \mathbb{R}$ is defined as

$$L(x, Y, \mu, \lambda) := f(x) + \langle Y, F(x) \rangle + \langle \mu, h(x) \rangle - \langle \lambda, g(x) \rangle.$$  

Any point $(x, Y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y}$ satisfying (2.1) is named as a KKT point and the corresponding point $(Y, \mu, \lambda)$ is called a Lagrange multiplier at $x$. Let $\mathcal{M}(x)$ be the set of all Lagrange multipliers at $x$.

Let $c > 0$ and $\pi$ be a stationary point of (COP), namely $\mathcal{M}(\pi) \neq \emptyset$. Since $f, F, h, \text{ and } g$ are assumed to be twice continuously differentiable, we know from (1.1), [37] and Chapter 2 of [28] that the augmented Lagrangian function $L_c(\cdot)$ is continuously differentiable and for any $(x, Y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y}$,

$$\nabla_x L_c(x, Y, \mu, \lambda) = \nabla f(x) + D\Phi_c(x, Y, \mu, \lambda) = Df(x)^* D\theta_c(F(x) + Y/c)^* + \mathcal{J}h(x)^T(\mu + ch(x)) - Dg(x)^* \Pi_{K^*}(\lambda - cg(x)).$$  

(2.2)

Therefore, from (2.1), we have $\nabla_x L_c(\pi, Y, \mu, \lambda) = 0$ for any $(Y, \mu, \lambda) \in \mathcal{M}(\pi)$.

For any $(x, Y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y}$, let

$$\Phi_c(x, Y, \mu, \lambda) := Df(x)^* D\theta_c(F(x) + Y/c)^*, \quad \Psi_c(x, Y, \mu, \lambda) := Dg(x)^* \Pi_{K^*}(\lambda - cg(x)).$$

Let $(x, Y, \mu, \lambda) \in \mathbb{R}^n \times \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y}$. Then from the semismoothness of $D\theta_c(\cdot)$ and $\Pi_{K^*}(\cdot)$ we obtain that for any $(\Delta x, \Delta Y, \Delta \mu, \Delta \lambda) \in \mathbb{R}^n \times \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y},$

$$\partial_B \Phi_c(x, Y, \mu, \lambda)(\Delta x, \Delta Y, \Delta \mu, \Delta \lambda) = D^2f(x)(\Delta x) D\theta_c(F(x) + Y/c)^* + Df(x)^* \partial_B [D\theta_c]^*(F(x) + Y/c)(DF(x)\Delta x + \Delta Y/c),$$

$$\partial_B \Psi_c(x, Y, \mu, \lambda)(\Delta x, \Delta Y, \Delta \mu, \Delta \lambda) = D^2g(x)(\Delta x) \Pi_{K^*}(\lambda - cg(x)) + Dg(x)^* \partial_B \Pi_{K^*}(\lambda - cg(x))(\Delta \lambda - cDg(x)\Delta x).$$  

(2.3)

From (2.2) and the definition of $\Psi_c(\cdot)$ we know that

$$\partial_B(\nabla_x L_c)(x, Y, \mu, \lambda) = (\nabla^2 f(x), 0, 0, 0) + \left( \sum_{i=1}^m (\mu_i + ch_i(x)) \nabla^2 h_i(x) + c\mathcal{J}h(x)^T \mathcal{J}h(x), 0, \mathcal{J}h(x)^T, 0 \right) + \partial_B \Phi_c(x, Y, \mu, \lambda) \partial_B \Psi_c(x, Y, \mu, \lambda),$$

(2.4)
which implies that for any $\Delta x \in \mathbb{R}^n$,
\[
(\pi_x \partial_B(\nabla_x L_c)(x, Y, \mu, \lambda))(\Delta x)
= \nabla^2_{xx} L(x, D\theta_c(F(x) + Y/c)^*, \mu + ch(x), \Pi_{K^*}(\lambda - cg(x)))(\Delta x)
+ Df(x)^* \partial_B[D\theta_c]^*(F(x) + Y/c)DF(x)(\Delta x)
+ cJh(x)^T Jh(x)(\Delta x) + cDg(x)^* \partial_B\Pi_{K^*}(\lambda - cg(x))Dg(x)(\Delta x),
\]
where
\[
\nabla^2_{xx} L(x, D\theta_c(F(x) + Y/c)^*, \mu + ch(x), \Pi_{K^*}(\lambda - cg(x)))(\Delta x)
= \nabla^2 f(x)(\Delta x) + D^2 F(x)(\Delta x)D\theta_c(F(x) + Y/c)^*
+ D^2 h(x)(\Delta x)(\mu + ch(x)) - D^2 g(x)(\Delta x)\Pi_{K^*}(\lambda - cg(x)).
\]

Let $(\overline{Y}, \overline{\mu}, \overline{\lambda}) \in \mathcal{M}(\overline{x})$ be a Lagrange multiplier at $\overline{x}$. For any linear operators $W_1 : Z \mapsto Z$, $W_2 : \mathcal{Y} \mapsto \mathcal{Y}$, let
\[
\mathcal{A}_c(\overline{Y}, \overline{\mu}, \overline{\lambda}, W_1, W_2) := \nabla^2_{xx} L(\overline{x}, \overline{Y}, \overline{\mu}, \overline{\lambda}) + Df(\overline{x})^* W_1 Df(\overline{x})
+ cJh(\overline{x})^T Jh(\overline{x}) + cDg(\overline{x})^* W_2 Dg(\overline{x}).
\]

Then for any $\Delta x \in \mathbb{R}^n$,
\[
(\pi_x \partial_B(\nabla_x L_c)(\overline{x}, \overline{Y}, \overline{\mu}, \overline{\lambda}))(\Delta x)
= \begin{cases}
\mathcal{A}_c(\overline{Y}, \overline{\mu}, \overline{\lambda}, W_1, W_2)(\Delta x) : & W_1 \in \partial_B[D\theta_c]^*(F(\overline{x}) + \overline{Y}/c) \\
& W_2 \in \partial_B\Pi_{K^*}(\overline{\lambda} - cg(\overline{x}))
\end{cases}.
\]

Next, we make two basic assumptions for the constrained optimization composite optimization problem (COP). The first one is about the positive definiteness of $\mathcal{A}_c(\overline{Y}, \overline{\mu}, \overline{\lambda}, \cdot, \cdot)$.

**Assumption B1.** We assume that $(\overline{Y}, \overline{\mu}, \overline{\lambda})$ is the unique Lagrange multiplier at $\overline{x}$, i.e., $\mathcal{M}(\overline{x}) = \{(\overline{Y}, \overline{\mu}, \overline{\lambda})\}$ and that there exist two positive numbers $c_0$ and $\eta$ such that for any $c \geq c_0$ and any $W_1 \in \partial_B[D\theta_c]^*(F(\overline{x}) + \overline{Y}/c)$, $W_2 \in \partial_B\Pi_{K^*}(\overline{\lambda} - cg(\overline{x}))$,
\[
\langle d, \mathcal{A}_c(\overline{Y}, \overline{\mu}, \overline{\lambda}, W_1, W_2)d \rangle \geq \eta \langle d, d \rangle, \quad \forall d \in \mathbb{R}^n.
\]

Assumption B1 is related to the sufficient optimality conditions for the constrained composite optimization problem (COP). It will be shown in Proposition 4.1 that, under the constraint nondegeneracy condition and the strong second order sufficient condition (they will be clarified in Section 3), Assumption B1 is valid for (SDNOP).

Let $\overline{y} := (\overline{Y}, \overline{\mu}, \overline{\lambda})$. Then $\nabla_x L_c(\overline{x}, \overline{y}) = 0$. Let $c_0$ and $\eta$ be two positive numbers defined in Assumption B1 and $c \geq c_0$ be a positive number. Since by (2.6) and Assumption B1, every element in $\pi_x \partial_B(\nabla_x L_c)(\overline{x}, \overline{y})$ is positive definite, we know from the implicit function theorem for semismooth functions developed in [31], that there exist an open neighborhood $O_{\overline{y}}$ of $\overline{y}$ and a locally Lipschitz continuous function $x_c(\cdot)$ defined on $O_{\overline{y}}$ such that for any $y \in C_{\overline{y}}$, $\nabla_x L_c(x_c(y), y) = 0$. Furthermore, since $D\theta_c(\cdot)$ and $\Pi_{K^*}(\cdot)$ are assumed to be semismooth everywhere, $x_c(\cdot)$ is semismooth (strongly semismooth if $\nabla^2 f, D^2 F, D^2 g$, and
D^2h are locally Lipschitz continuous, and both D\theta_c(\cdot) and \Pi_K(\cdot) are strongly semismooth everywhere) at any point in \mathcal{O}_\overline{\mathcal{Y}}. Moreover, there exist two positive numbers \varepsilon > 0 and \delta_0 > 0 (both depending on c) such that for any x \in \mathbb{B}_{\varepsilon}(\mathcal{Y}) and y \in \mathbb{B}_{\delta_0}(\mathcal{Y}) := \{y \in \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y} \mid \|y - \overline{\mathcal{Y}}\| < \delta_0\} \subset \mathcal{O}_\overline{\mathcal{Y}}, every element in \pi_x\partial_B(\nabla_x L_c)(x, y) is positive definite. Thus, for any y \in \mathbb{B}_{\delta_0}(\mathcal{Y}), x_c(y) is the unique minimizer of L_c(\cdot, y) over \mathbb{B}_{\varepsilon}(\mathcal{Y}), i.e.,

\{x_c(y)\} = \arg\min \left\{L_c(x, y) \mid x \in \mathbb{B}_{\varepsilon}(\mathcal{Y}) \right\}. \quad (2.7)

Summarizing the above discussions, we obtain the following proposition.

**Proposition 2.1** Suppose that Assumption B1 is satisfied. Let c \geq c_0. Then there exist two positive numbers \varepsilon > 0 and \delta_0 > 0 (both depending on c) and a locally Lipschitz continuous function x_c(\cdot), given by (2.7), defined on the open ball \mathbb{B}_{\delta_0}(\mathcal{Y}) such that the following conclusions hold:

(i) The function x_c(\cdot) is semismooth at any point in \mathbb{B}_{\delta_0}(\mathcal{Y}).

(ii) If \nabla^2 f, D^2 F, D^2 g, and D^2 h are locally Lipschitz continuous, D\theta_c(\cdot) and \Pi_K(\cdot) are strongly semismooth everywhere, then x_c(\cdot) is strongly semismooth at any point in \mathbb{B}_{\delta_0}(\mathcal{Y}).

(iii) For any x \in \mathbb{B}_{\varepsilon}(\mathcal{Y}) and y \in \mathbb{B}_{\delta_0}(\mathcal{Y}), every element in \pi_x\partial_B(\nabla_x L_c)(x, y) is positive definite.

(iv) For any y \in \mathbb{B}_{\delta_0}(\mathcal{Y}), x_c(y) is the unique optimal solution to

\[
\min \ L_c(x, y) \quad \text{s.t.} \ x \in \mathbb{B}_{\varepsilon}(\mathcal{Y}).
\]

Let \vartheta_c : \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y} \mapsto \mathbb{R} be defined as

\[
\vartheta_c(Y, \mu, \lambda) := \min_{x \in \mathbb{B}_{\varepsilon}(\mathcal{Y})} L_c(x, Y, \mu, \lambda), \quad (Y, \mu, \lambda) \in \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y}. \quad (2.8)
\]

Since for each fixed x \in \mathcal{X}, L_c(x, \cdot) is a concave function, we have that \vartheta_c(\cdot) is also a concave function. By using the fact that for any y \in \mathbb{B}_{\delta_0}(\mathcal{Y}), x_c(y) is the unique minimizer of L_c(\cdot, y) over \mathbb{B}_{\varepsilon}(\mathcal{Y}), we have

\[
\vartheta_c(y) = L_c(x_c(y), y), \quad y \in \mathbb{B}_{\delta_0}(\mathcal{Y}).
\]

For any y \in \mathbb{B}_{\delta_0}(\mathcal{Y}) with y = (Y, \mu, \lambda) \in \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y}, let

\[
\left(\begin{array}{c}
Y_c(y) \\
\mu_c(y) \\
\lambda_c(y)
\end{array}\right) := \left(\begin{array}{c}
D\theta_c(F(x_c(y)) + Y/c)^* \\
\mu + c h(x_c(y)) \\
\Pi_K(\lambda - c g(x_c(y)))
\end{array}\right). \quad (2.9)
\]

Then we have

\[
\nabla_x L_c(x_c(y), Y_c(y), \mu_c(y), \lambda_c(y)) = \nabla_x L_c(x_c(y), y) = 0, \quad y \in \mathbb{B}_{\delta_0}(\mathcal{Y}). \quad (2.10)
\]
Proposition 2.2 Suppose that Assumption B1 is satisfied. Let \( c \geq c_0 \). Then the concave function \( \vartheta_c(\cdot) \) defined by (2.8) is continuously differentiable on \( \mathbb{B}_{\delta_0}(\overline{Y}) \) with

\[
D \vartheta_c(y)^* = \begin{pmatrix}
-c^{-1}Y + c^{-1}D\vartheta_c(F(x_c(y)) + Y/c)^* \\
-\lambda + c^{-1}\Pi_{K^*}(\lambda - cg(x_c(y)))
\end{pmatrix}, \quad y = (Y, \mu, \lambda) \in \mathbb{B}_{\delta_0}(\overline{Y}). \tag{2.11}
\]

Moreover, \( D \vartheta_c(\cdot) \) is semismooth at any point in \( \mathbb{B}_{\delta_0}(\overline{Y}) \). It is strongly semismooth at any point in \( \mathbb{B}_{\delta_0}(\overline{Y}) \) if \( \nabla^2 f, \nabla^2 F, \nabla^2 g, \) and \( \nabla^2 h \) are locally Lipschitz continuous, and \( D \vartheta_c(\cdot) \) and \( \Pi_{K^*}(\cdot) \) are strongly semismooth everywhere.

Proof. Let \( y = (Y, \mu, \lambda) \in \mathbb{B}_{\delta_0}(\overline{Y}) \). Then from (2.10) and [8, Theorem 2.6.6] we have for any \( (\Delta Y, \Delta \mu, \Delta \lambda) \in \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y} \) that

\[
\partial \vartheta_c(y)(\Delta Y, \Delta \mu, \Delta \lambda) = J_cL_c(x_c(y), y)(\partial x_c(y)(\Delta Y, \Delta \mu, \Delta \lambda))
+ D\Delta Y L_c(x_c(y), y)(\Delta Y) + J\Delta \mu L_c(x_c(y), y)(\Delta \mu) + D\Delta \lambda L_c(x_c(y), y)(\Delta \lambda)
= \left\langle -c^{-1}Y, \Delta Y \right\rangle + c^{-1}D\vartheta_c(F(x_c(y)) + Y/c)(\Delta Y)
+ \left\langle h(x_c(y)), \Delta \mu \right\rangle - c^{-1}\left\langle \lambda, \Delta \lambda \right\rangle + \left\langle c^{-1}\Pi_{K^*}(\lambda - cg(x_c(y))), \Delta \lambda \right\rangle.
\]

Thus, \( \partial \vartheta_c(y)(\Delta Y, \Delta \mu, \Delta \lambda) \) is a singleton for each \( (\Delta Y, \Delta \mu, \Delta \lambda) \in \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y} \). This implies that \( \partial \vartheta_c(y) \) is a singleton. Therefore, \( \vartheta_c(\cdot) \) is Fréchet-differentiable at \( y \) and \( D\vartheta_c(y) \) is given by (2.11). The continuity of \( D\vartheta_c(\cdot) \) follows from the continuity of \( x_c(\cdot) \).

The properties on the (strong) semismoothness of \( D\vartheta_c(\cdot) \) at \( y \) follows directly from (2.11) and Proposition 2.1. \(\square\)

For any \( c \geq c_0 \) and \( \Delta y := (\Delta Y, \Delta \mu, \Delta \lambda) \in \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y} \), define

\[
\nabla_c(\Delta y) := \begin{pmatrix}
-c^{-1}W_1DF(\overline{\pi}) \\
\mathcal{J}h(\overline{\pi}) \\
-W_2Dg(\overline{\pi})
\end{pmatrix} A_c(\overline{\pi}, W_1, W_2)^{-1} \begin{pmatrix}
-c^{-1}DF(\overline{\pi})^*W_1(\Delta Y) \\
-c^{-1}\Pi_{K^*}(\lambda - cg(\overline{\pi}))
\end{pmatrix}.
\]

(2.12)

Since by Assumption B1, \( A_c(\overline{\pi}, W_1, W_2) \) is positive definite for any \( W_1 \in \partial_B[DF(\overline{\pi}) + \overline{Y}/c] \), \( W_2 \in \partial_B\Pi_{K^*}(\overline{x} - cg(\overline{\pi})) \), \( \nabla_c(\cdot) \) is well defined. The next proposition shows that \( \nabla_c(\cdot) \) is an outer approximation to \( \partial_B[D\vartheta_c]^*(\overline{\pi})(\cdot) \).

Proposition 2.3 Suppose that Assumption B1 is satisfied. Let \( c \geq c_0 \). Then for any \( \Delta y := (\Delta Y, \Delta \mu, \Delta \lambda) \in \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y} \),

\[
\partial_B[D\vartheta_c]^*(\overline{\pi})(\Delta y) \subseteq \overline{\nabla_c}(\Delta y). \tag{2.13}
\]
Proof. Choose $\Delta y := (\Delta Y, \Delta \mu, \Delta \lambda) \in Z \times \mathbb{R}^m \times \mathcal{Y}$. From Proposition 2.2, we know that $D\theta_c(\cdot)$ is semismooth at any point $y \in \mathbb{B}_0(\overline{Y})$. Let $\mathcal{D}_{D\theta_c}$ denote the set of all Fréchet-differentiable points of $D\theta_c(\cdot)$ in $\mathbb{B}_0(\overline{Y})$. Then for any $y = (Y, \mu, \lambda) \in \mathcal{D}_{D\theta_c}$, we have

\[
D^2\theta_c(y)(\Delta y) = \begin{pmatrix}
-c^{-1}\Delta Y + c^{-1}[D\theta_c]^*(F(x_c(y)) + Y/c; DF(x_c(y))(x_c)'(y; \Delta y) + \Delta Y/c) \\
\mathcal{J}h(x_c(y))(x_c)'(y; \Delta y) \\
-c^{-1}\Delta \lambda + c^{-1}\Pi_K^*\left(\lambda - c\gamma(x_c(y)); \Delta \lambda - cDg(x_c(y))(x_c)'(y; \Delta y)\right)
\end{pmatrix}.
\]

Let $y \in \mathbb{B}_0(\overline{Y})$. Now, we derive the formula for $(x_c)'(y; \Delta y)$. From (2.10) and (2.9) we have

\[
0 = \nabla_x^2 L(x_c(y), Y_c(y), \mu_c(y), \lambda_c(y))(x_c)'(y; \Delta y) + c\mathcal{J}h(x_c(y))^T \mathcal{J}h(x_c(y))(x_c)'(y; \Delta y) + D\mathcal{J}h(x_c(y))^T \mathcal{J}h(x_c(y))(x_c)'(y; \Delta y)
\]

\[
+ D\mathcal{J}h(x_c(y))^T \mathcal{J}h(x_c(y))(x_c)'(y; \Delta y) + D\mathcal{J}h(x_c(y))^T \mathcal{J}h(x_c(y))(x_c)'(y; \Delta y)
\]

\[
+ D\mathcal{J}h(x_c(y))^T \mathcal{J}h(x_c(y))(x_c)'(y; \Delta y) + D\mathcal{J}h(x_c(y))^T \mathcal{J}h(x_c(y))(x_c)'(y; \Delta y)
\]

\[
\Pi_K^*\left(\lambda - c\gamma(x_c(y)); \Delta \lambda - cDg(x_c(y))(x_c)'(y; \Delta y)\right)
\]

Since $D\theta_c(\cdot)$ and $\Pi_K^*(\cdot)$ are semismooth everywhere, there exist $\widehat{W}_1 \in \partial_B [D\theta_c]^*(F(x_c(y)) + Y/c)$ and $\widehat{W}_2 \in \partial_B \Pi_K^*(\lambda - c\gamma(x_c(y)))$ such that

\[
[D\theta_c]^*(F(x_c(y)) + Y/c; DF(x_c(y))(x_c)'(y; \Delta y) + \Delta Y/c)
\]

\[
= \widehat{W}_1(DF(x_c(y))(x_c)'(y; \Delta y) + \Delta Y/c),
\]

\[
\Pi_K^*\left(\lambda - c\gamma(x_c(y)); \Delta \lambda - cDg(x_c(y))(x_c)'(y; \Delta y)\right)
\]

\[
= \widehat{W}_2(\Delta \lambda - cDg(x_c(y))(x_c)'(y; \Delta y)).
\]

For any $W_1 \in \partial_B [D\theta_c]^*(F(x_c(y)) + Y/c)$ and $W_2 \in \partial_B \Pi_K^*(\lambda - c\gamma(x_c(y)))$, let

\[
\mathcal{A}_c(y, W_1, W_2) := \nabla_x^2 L(x_c(y), Y_c(y), \mu_c(y), \lambda_c(y)) + DF(x_c(y))^* W_1 \mathcal{Y} + \mathcal{Y} + D\mathcal{J}h(x_c(y))^T \mathcal{J}h(x_c(y)) + cDg(x_c(y))^* W_2 Dg(x_c(y))
\]

From (2.4) and the definition of $\delta_0$, $\mathcal{A}_c(y, W_1, W_2)$ is positive definite for any $W_1 \in \partial_B [D\theta_c]^*(F(x_c(y)) + Y/c) \text{ and } W_2 \in \partial_B \Pi_K^*(\lambda - c\gamma(x_c(y)))$. Then from (2.15) and (2.16) we obtain that

\[
(x_c)'(y; \Delta y) = \mathcal{A}_c(y, \widehat{W}_1, \widehat{W}_2)^{-1}\left([-DF(x_c(y))^* \widehat{W}_1(\Delta Y/c) - \mathcal{J}h(x_c(y))^T(\Delta \mu) + Dg(x_c(y))^* \widehat{W}_2(\Delta \lambda)]\right).
\]

Therefore, we have from (2.17) and (2.14) that for any $y = (Y, \mu, \lambda) \in \mathcal{D}_{D\theta_c}$,
\[
D^2 \vartheta_c(y)(\Delta y) \in \left\{ \begin{bmatrix}
- c^{-1} W_1 D F(x_c(y)) \\
J h(x_c(y)) \\
-W_2 D g(x_c(y))
\end{bmatrix}
\begin{bmatrix}
A_c(y, W_1, W_2)^{-1} \left[ - c^{-1} D F(x_c(y))^* W_1(\Delta Y) \\
-J h(x_c(y))^T \Delta \mu + D g(x_c(y))^* W_2(\Delta \lambda) \right] \\
+ \begin{pmatrix}
- c^{-1} \Delta Y + c^{-2} W_1 \Delta Y \\
0 \\
- c^{-1} \Delta \lambda + c^{-1} W_2(\Delta \lambda)
\end{pmatrix}
\begin{bmatrix}
W_1 \in \partial_B[D \vartheta_c]^*(F(x_c(y)) + Y/c) \\
W_2 \in \partial_B \Pi_K(\lambda - c g(x_c(y)))
\end{bmatrix}
\right\},
\]

which, together with the continuity of \( x_c(\cdot) \) and the upper semicontinuity of \( \partial_B \Pi_K^*(\cdot) \), implies that \( V(\Delta y) \in \nabla_c(\Delta y) \) for any \( V \in \partial_B[D \vartheta_c]^*(y) \). Consequently, (2.13) holds.

The second basic assumption required in this section is stated as below.

**Assumption B2.** There exist positive numbers \( \tau \geq c_0, \mu_0 > 0, \varrho_0 > 0, \) and \( \gamma > 1 \) such that for any \( c \geq \tau \) and \( \Delta y \in \mathbb{Z} \times \mathbb{R}^m \times Y \),

\[
\| (x_c)'(y; \Delta y) \| \leq \varrho_0 \| \Delta y \| / c
\]

and

\[
\langle V(\Delta y) + c^{-1} \Delta y, \Delta y \rangle \in \mu_0 [-1, 1] \| \Delta y \|^2 / c^\gamma \quad \forall V(\Delta y) \in \nabla_c(\Delta y).
\]

It will be shown in Proposition 4.2 that Assumption B2 is valid for (SDNAP) when the constraint nondegeneracy condition and the strong second order sufficient condition are satisfied.

Let \( C \) be a closed convex set in \( Y \). It follows from [37] that the metric projector \( \Pi_C(\cdot) \) is Lipschitz continuous with the Lipschitz modulus 1. Then for any \( y \in Y \), \( \partial \Pi_C(y) \) is well defined and it has the following variational properties.

**Lemma 2.1** [21, Proposition 1] Let \( C \subseteq Y \) be a closed convex set. Then, for any \( y \in Y \) and \( V \in \partial \Pi_C(y) \), it holds that

(i) \( V \) is self-adjoint.

(ii) \( \langle d, V d \rangle \geq 0, \quad \forall d \in Y \).

(iii) \( \langle V d, d - V d \rangle \geq 0, \quad \forall d \in Y \).

Under Assumptions B1 and B2, we are ready to give the main result on the rate of convergence of the augmented Lagrangian method for the composite optimization problem (COP).

**Theorem 2.1** Suppose that \( K \) is an nonempty closed convex cone and that \( D \vartheta_c(\cdot) \) and \( \Pi_K^*(\cdot) \) are semismooth everywhere. Let Assumptions B1 and B2 be satisfied. Let \( c_0, \eta, \tau, \mu_0, \varrho_0, \) and \( \gamma \) be the positive numbers defined in these assumptions. Define

\[
\varrho_1 := 2 \varrho_0 \quad \text{and} \quad \varrho_2 := 4 \mu_0.
\]
Then for any $c \geq \overline{c}$, there exist two positive numbers $\varepsilon$ and $\delta$ (both depending on $c$) such that for any $(Y, \mu, \lambda) \in B_{\delta}(\overline{Y}, \overline{\mu}, \overline{\lambda})$, the problem

$$\min \ L_c(x, Y, \mu, \lambda) \quad \text{s.t.} \ x \in B_{\varepsilon}(\overline{x})$$

(2.20)

has a unique solution denoted $x_c(Y, \mu, \lambda)$. The function $x_c(\cdot, \cdot, \cdot)$ is locally Lipschitz continuous on $B_{\delta}(\overline{Y}, \overline{\mu}, \overline{\lambda})$ and is semismooth at any point in $B_{\delta}(\overline{Y}, \overline{\mu}, \overline{\lambda})$, and for any $(Y, \mu, \lambda) \in B_{\delta}(\overline{Y}, \overline{\mu}, \overline{\lambda})$, we have

$$\|x_c(Y, \mu, \lambda) - \overline{x}\| \leq \varrho_1\|Y, \mu, \lambda\|/c \quad \text{and}$$

$$\|Y_c(Y, \mu, \lambda), \mu_c(Y, \mu, \lambda), \lambda_c(Y, \mu, \lambda) - (\overline{Y}, \overline{\mu}, \overline{\lambda})\| \leq \varrho_2\|Y, \mu, \lambda\|/(c^{\gamma} - 1) \quad , \quad (2.21)$$

where $Y_c(Y, \mu, \lambda), \mu_c(Y, \mu, \lambda)$ and $\lambda_c(Y, \mu, \lambda)$ are defined by (2.9), i.e.,

$$Y_c(Y, \mu, \lambda) = D\theta_c(F(x_c(y)) + Y/c),$$

$$\mu_c(Y, \mu, \lambda) = \mu + c(\vartheta_c(y)),$$

$$\lambda_c(Y, \mu, \lambda) = \Pi_K^\lambda (\lambda - cg(x_c(y))) .$$

**Proof.** Let $c \geq \overline{c}$. From Proposition 2.1 we have already known that there exist two positive numbers $\varepsilon > 0$ and $\delta_0 > 0$ (both depending on $c$) and a locally Lipschitz continuous function $x_c(\cdot, \cdot, \cdot)$ defined on $B_{\delta_0}(\overline{Y}, \overline{\mu}, \overline{\lambda})$ such that the function $x_c(\cdot, \cdot, \cdot)$ is semismooth at any point in $B_{\delta_0}(\overline{Y}, \overline{\mu}, \overline{\lambda})$ and for any $(Y, \mu, \lambda) \in B_{\delta_0}(\overline{Y}, \overline{\mu}, \overline{\lambda})$, $x_c(Y, \mu, \lambda)$ is the unique solution to (2.20).

Denote $y := (Y, \mu, \lambda) \in Z \times \mathbb{R}^m \times \mathcal{V}$. Since $x_c(\cdot)$ is locally Lipschitz continuous on $B_{\delta_0}(\overline{y})$ and is directionally differentiable at $\overline{y}$, by [29] we know that $x_c(\cdot)$ is Bouligand-differentiable at $\overline{y}$, i.e., $x_c(\cdot)$ is directionally differentiable at $\overline{y}$ and

$$\lim_{y \to \overline{y}} \frac{\|x_c(y) - x_c(\overline{y}) - (x_c)(y; y - \overline{y})\|}{\|y - \overline{y}\|} = 0 .$$

By Proposition 2.2, $D\theta_c(\cdot)$ is semismooth at $\overline{y}$, and thus is also Bouligand-differentiable at $\overline{y}$. Then there exists $\delta \in (0, \delta_0]$ such that for any $y \in B_{\delta}(\overline{y})$, $\|x_c(y) - x_c(\overline{y}) - (x_c)(y; y - \overline{y})\| \leq \varrho_0\|y - \overline{y}\|/c \quad (2.23)$

and

$$\|D\theta_c(y) - D\theta_c(\overline{y}) - (D\theta_c)(y; y - \overline{y})\| \leq \mu_0\|y - \overline{y}\|/c^{\gamma} . \quad (2.24)$$

Let $y := (Y, \mu, \lambda) \in B_{\delta}(\overline{y})$ be an arbitrary point. From (2.18), (2.23), and the fact that $x_c(\overline{y}) = \overline{x}$, we have

$$\|x_c(y) - \overline{x}\| \leq \|(x_c)(\overline{y}; y - \overline{y})\| + \varrho_0\|y - \overline{y}\|c = \varrho_1\|y - \overline{y}\|/c,$$

which shows that (2.21) holds.

Since $D\theta_c(\cdot)$ is semismooth at $\overline{y}$, there exists an element $V \in \partial_B[D\theta_c]^*(\overline{y})$ such that $(D\theta_c)^*(\overline{y}; y - \overline{y}) = V(y - \overline{y})$. By using the fact that $V$ is self-adjoint (see Lemma 2.1), we know from (2.19) in Assumption B2 and Proposition 2.3 that

$$\|V(y - \overline{y}) + c^{-1}(y - \overline{y})\| \leq 3\mu_0\|y - \overline{y}\|/c^{\gamma} . \quad (2.25)$$
Therefore, we have from (2.24) and (2.25)
\[
\|y + c(D\vartheta_c)^*(y) - \gamma\| = c\|(D\vartheta_c)^*(y) - (D\vartheta_c)^*(\gamma) - (D\vartheta_c)^*(\gamma; y - \gamma) + (D\vartheta_c)^*(\gamma; y - \gamma) + c^{-1}(y - \gamma)\|
\leq c\|(D\vartheta_c)^*(y) - (D\vartheta_c)^*(\gamma) - (D\vartheta_c)^*(\gamma; y - \gamma)\| + c\|\gamma - \gamma\| + c^{-1}(y - \gamma)\|
\leq \mu_0\|y - \gamma\|/c^\gamma - 1 + 3\mu_0\gamma - \gamma\|/c^\gamma - 1 = \varphi_2\|y - \gamma\|/c^\gamma - 1,
\]
which, together with (2.11) and the definitions of \(Y_c(Y, \mu, \lambda), \mu_c(Y, \mu, \lambda)\) and \(\lambda_c(Y, \mu, \lambda),\) proves (2.22). The proof is completed.

Under Assumptions B1 and B2, Theorem 2.1 shows that if for all \(k\) sufficiently large with \(c_k \equiv c\) larger than a threshold and if \((x^k, Y^k, \mu^k, \lambda^k)\) is sufficiently close to \((\overline{x}, \overline{Y}, \overline{\mu}, \overline{\lambda})\), then the augmented Lagrangian method can locally be regarded as the gradient ascent method applied to the dual problem
\[
\max \vartheta_c(Y, \mu, \lambda) \quad \text{s.t.} \quad (Y, \mu, \lambda) \in \mathcal{Z} \times \mathbb{R}^m \times \mathcal{Y}
\]
with a constant step-length \(c\), i.e., for all \(k\) sufficiently large
\[
\begin{pmatrix}
  y^{k+1} \\
  \mu^{k+1} \\
  \lambda^{k+1}
\end{pmatrix}
= \begin{pmatrix}
  Y^k \\
  \mu^k \\
  \lambda^k
\end{pmatrix} + cD\vartheta_c(Y^k, \mu^k, \lambda^k)^*.
\]

In Section 4, we shall check, under what kind of conditions, Assumptions B1 and B2 imposed in this section can be satisfied by the nonlinear semidefinite nuclear norm composite optimization problem.

### 3 Variational analysis for SDNOP

For studying the rate of convergence of the augmented Lagrange method for the nonlinear semidefinite nuclear norm composite optimization problem (SDNOP), we have to provide some variational properties of \(\Pi_{S^p_+}^* (\cdot)\) and \(\| \cdot \|_\ast\), and the second-order optimality conditions for (SDNOP).

#### 3.1 Variational properties of \(\Pi_{S^p_+}^* (\cdot)\) and \(\| \cdot \|_\ast\)

Since there exists an nonlinear semidefinite constraint in Problem (SDNOP), we need more properties about the tangent cone of the cone \(S^p_+\) and the B-subdifferential of the metric projector \(\Pi_{S^p_+}^* (\cdot)\) over \(S^p_+\). Let \(\mathcal{O}^p\) be the set of all \(p \times p\) orthogonal matrices. For a given matrix \(M \in S^p\), there exists \(P \in \mathcal{O}^p\) such that
\[
M = P\Lambda(M)P^T, \quad \text{(3.1)}
\]
where \(\Lambda(M) = \text{diag}(\lambda_1(M), \lambda_2(M), \ldots, \lambda_p(M))\) and \(\lambda_1(M) \geq \lambda_2(M) \geq \ldots \geq \lambda_p(M)\) are eigenvalues of \(M\). We denote the set of such \(P\) in the eigenvalue decomposition by \(\mathcal{O}(M)\). Let \(\overline{M} \in S^p\) and \(\overline{M}_+ := \Pi_{S^p_+}^* (\overline{M})\). Suppose that \(\overline{M}\) has the following spectral decomposition
\[
\overline{M} = \overline{P}\Lambda\overline{P}^T, \quad \text{(3.2)}
\]
where $P \in \mathcal{O}(\overline{M})$ and $\Lambda$ is the diagonal matrix of eigenvalues of $\overline{Z}$. Then

$$\overline{M}_+ = P \Lambda_+ P^T,$$

where $\Lambda_+$ is the diagonal matrix whose diagonal entries are the nonnegative parts of the respective diagonal entries of $\Lambda$ [16, 35]. Define three index sets of positive, zero, and negative eigenvalues of $\overline{M}$, respectively, as

$$\alpha := \{i \mid \lambda_i > 0\}, \quad \beta := \{i \mid \lambda_i = 0\}, \quad \gamma := \{i \mid \lambda_i < 0\}.$$

Write

$$\Lambda = \begin{bmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda_\gamma \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} P_\alpha & P_\beta & P_\gamma \end{bmatrix},$$

with $P_\alpha \in \mathbb{R}^{p \times |\alpha|}$, $P_\beta \in \mathbb{R}^{p \times |\beta|}$, and $P_\gamma \in \mathbb{R}^{p \times |\gamma|}$. Let $\Theta$ be any matrix in $\mathcal{S}^p$ with entries

$$\Theta_{ij} = \begin{cases} \max\{\lambda_i, 0\} + \max\{\lambda_j, 0\} & \text{if } (i, j) \notin \beta \times \beta, \\ |\lambda_i| + |\lambda_j| & \text{if } (i, j) \in \beta \times \beta. \end{cases}$$

(3.3)

The projection operator $\Pi_{\mathcal{S}_+^p}(\cdot)$ is directionally differentiable everywhere in $\mathcal{S}_+^p$ [3] and is a strongly semismooth matrix-valued function [33]. For any $H \in \mathcal{S}_+^p$, we have

$$\Pi'_{\mathcal{S}_+^p}(\overline{M}; H) = P \begin{bmatrix} P_\alpha^T H P_\alpha & P_\alpha^T H P_\beta & \Theta_{\alpha \gamma} \circ P_\alpha^T H P_\gamma \\ P_\beta^T H P_\alpha & \Pi_{S_{\mathcal{S}_{+}^p}}(P_\beta^T H P_\beta) & 0 \\ P_\gamma^T H P_\alpha \circ \Theta_{\gamma \alpha} & 0 & 0 \end{bmatrix} P^T,$$

(3.4)

where "$\circ$" denotes the Hadamard product [33]. When $\beta = \emptyset$, $\Pi'_{\mathcal{S}_+^p}(\cdot)$ is Fréchet-differentiable at $\overline{M}$ and (3.4) reduces to the classical result:

$$\mathcal{J} \Pi_{\mathcal{S}_+^p}(\overline{M}) H = P \begin{bmatrix} P_\alpha^T H P_\alpha & \Theta_{\alpha \gamma} \circ P_\alpha^T H P_\gamma \\ P_\gamma^T H P_\alpha \circ \Theta_{\gamma \alpha} & 0 \end{bmatrix} P^T \quad \forall H \in \mathcal{S}_+^p.$$ 

(3.5)

The tangent cone of $\mathcal{S}_+^p$ at $\overline{M}_+$, denoted $\mathcal{T}_{\mathcal{S}_+^p}(\overline{M}_+)$, can be completely characterized as follows

$$\mathcal{T}_{\mathcal{S}_+^p}(\overline{M}_+) = \{B \in \mathcal{S}_+^p \mid B = \Pi'_{\mathcal{S}_+^p}(\overline{M}_+; B)\} = \{B \in \mathcal{S}_+^p \mid [P_\beta P_\gamma]^T B[P_\beta P_\gamma] \succeq 0\}.$$

The lineality space of $\mathcal{T}_{\mathcal{S}_+^p}(\overline{M}_+)$, i.e., the largest linear space in $\mathcal{T}_{\mathcal{S}_+^p}(\overline{M}_+)$, denoted by $\text{lin} \left( \mathcal{T}_{\mathcal{S}_+^p}(\overline{M}_+) \right)$, takes the following form:

$$\text{lin} \left( \mathcal{T}_{\mathcal{S}_+^p}(\overline{M}_+) \right) = \{B \in \mathcal{S}_+^p \mid [P_\beta P_\gamma]^T B[P_\beta P_\gamma] = 0\}.$$
The critical cone of \( S^+ \) at \( M \in S^+ \) associated with the problem of finding the metric projection of \( M \) onto \( S^+ \) (i.e., \( M^+ \)) is defined as \([5, \text{Section 5.3}]\)

\[
C_{S^+}(M) := T_{S^+}(M+) \cap \{ B \in S^p \mid \langle B, M+ - M \rangle = 0 \}.
\]

Thus, it holds that

\[
C_{S^+}(M) = \{ B \in S^p \mid P_{\beta}BP_{\gamma} = 0, \quad P_{\gamma}BP_{\gamma} = 0 \}.
\]

The affine hull of \( C_{S^+}(M) \), denoted by \( \text{aff}(C_{S^+}(M)) \), can then be written as

\[
\text{aff}(C_{S^+}(M)) = \{ B \in S^p \mid P_{\beta}BP_{\gamma} = 0, \quad P_{\gamma}BP_{\gamma} = 0 \}.
\] (3.6)

The following lemma on \( \partial B_{\Pi_{S^+}}(M) \) is part of \([32, \text{Proposition 4}]\), which is based on \([23, \text{Lemma 11}]\).

**Lemma 3.1** Let \( \Theta \in S^p \) satisfy (3.3). Then \( W \in \partial B_{\Pi_{S^+}}(M) \) if and only if there exists \( W_0 \in \partial B_{\Pi_{S^+}}(0) \) such that

\[
W(H) = P \begin{bmatrix}
P_THP_{\alpha} & P_THP_{\beta} & \Theta_{\alpha\gamma} \circ P_THP_{\gamma} \\
P_{\beta}HP_{\alpha} & W_0(P_{\beta}HP_{\beta}) & 0 \\
P_{\beta}HP_{\alpha} \circ \Theta_{\gamma\alpha} & 0 & 0
\end{bmatrix} PT \quad \forall H \in S^p.
\]

From the definition of \( \partial B_{\Pi_{S^+}}(0) \) and (3.5) we know that if \( W_0 \in \partial B_{\Pi_{S^+}}(0) \), then there exist matrices \( Q \in O^{[\beta]} \) and \( \Omega \in S^{[\beta]} \) with entries \( \Omega_{ij} \in [0, 1] \) such that

\[
W_0(D) = Q(\Omega \circ (Q^TDQ))Q^T \quad \forall D \in S^{[\beta]}.
\]

For an extension to the above result, see \([7, \text{Lemma 4.7}]\). By using Lemma 3.1 we obtain the following useful lemma, which does not need further explanation.

**Lemma 3.2** For any \( W \in \partial B_{\Pi_{S^+}}(M) \), there exist two matrices \( P \in O(M) \) and \( \Theta \in S^p \) satisfying (3.3) such that

\[
W(H) = P \left( \Theta \circ (P^THP) \right) P^T \quad \forall H \in S^p.
\]

For discussions on the nuclear norm function we need more properties about the first and second-order directional derivatives of \( \theta \) and the sub-differential of its proximal mapping. For a given matrix \( X \in S^q \), there exists \( Q \in O^q \) such that

\[
X = QA(X)Q^T, \quad (3.7)
\]

where \( A(X) = \text{diag}(\lambda_1(X), \lambda_2(X), \ldots, \lambda_q(X)) \) and \( \lambda_1(X) \geq \lambda_2(X) \geq \ldots \geq \lambda_q(X) \) are eigenvalues of \( X \). We denote the set of such \( Q \) in the eigenvalue decomposition by \( O(X) \).
Let $\varpi_1 > \varpi_2 > \ldots > \varpi_r$ be the distinct eigenvalues of $X$. Define

$$a_k := \{i \mid \lambda_i(X) = \varpi_k\}, \quad k = 1, \ldots, r.$$  

Partition $Q$ as $Q = [Q_{a_1} \ Q_{a_2} \ \ldots \ Q_{a_r}]$, where $Q_{a_k} = (q_i : i \in a_k)$ and $Q_{a_k} \in \mathbb{R}^{q \times |a_k|}$, $k = 1, \ldots, r$.

For a given $H \in S^q$ and $k \in \{1, \ldots, r\}$, suppose that $Q_{a_k}^T HQ_{a_k} \in \mathbb{R}^{|a_k| \times |a_k|}$ has the following spectral decomposition:

$$(Q^k)^T(Q_{a_k}^T HQ_{a_k})Q^k = \text{diag}(\xi^k_1, \ldots, \xi^k_{|a_k|}),$$

where $Q^k \in \mathcal{O}^{q \times q}(Q_{a_k}^T HQ_{a_k})$ and $\xi^k_i = \lambda_i(Q_{a_k}^T HQ_{a_k})$, $i = 1, \ldots, |a_k|$. Let $\eta^k_1, \ldots, \eta^k_{|a_k|}$ be the distinct eigenvalues of $Q_{a_k}^T HQ_{a_k}$ and define

$$b^k_j := \{i \mid \xi^k_i = \eta^k_j, \quad i = 1, \ldots, |a_k|\}, \quad j = 1, \ldots, n_k.$$

For simplicity, we denote $\kappa_i := \sum_{j=1}^i |a_j|$ ($\kappa_0 := 0$), $\kappa^{(k)}_i := \sum_{j=1}^i |b^k_j|$ ($\kappa^{(k)}_0 := 0$), and define the following mappings:

$$\nu : \{1, \ldots, n\} \rightarrow \{1, \ldots, r\}, \quad \nu(i) = k, \text{ if } i \in a_k,$$

$$l : \{1, \ldots, n\} \rightarrow \mathbb{N}, \quad l(i) = i - \kappa^{(k)}_{\nu(i)} - 1,$$

$$\omega : \{1, \ldots, n\} \rightarrow \mathbb{N}, \quad \omega(i) = j, \text{ if } l(i) \in b^k_j,$$

$$l' : \{1, \ldots, n\} \rightarrow \mathbb{N}, \quad l'(i) = l(i) - \kappa^{(k)}_{\nu(i)} - 1.$$  

Then, for $i' \in b^k_j$, its corresponding index $i \in \{1, \ldots, q\}$ is expressed as $i = \kappa^{(k)}_{j-1} + i' + \kappa_{k-1}$.

Define

$$\hat{Q}_{a_k} := Q_{a_k}^T HQ_{a_k}, \quad k = 1, \ldots, r,$$

$$\hat{V}_k(H, W) = Q_{a_k}^T [W - 2H(X - \varpi_k I)^\dagger H]Q_{a_k}, \quad k = 1, \ldots, r.$$  

Then we have from [38, Theorem 3.1] that

$$\lambda'_i(X; H) = \lambda'_{\nu(i)}(\hat{H}_{a\nu(i)\nu(i)}), \quad i = 1, \ldots, q,$$

$$\lambda''_i(X; H, W) = \lambda''_{\nu(i)} \left( \frac{Q_{\nu(i)}^T}{\omega(i)} \hat{V}_{\nu(i)}(H, W)Q_{\nu(i)}^T \right), \quad i = 1, \ldots, q.$$  

Assume that there exists an integer $s_0$ satisfying $1 \leq s_0 \leq N_s$ and $\eta^{s_0} = 0$. Let $b^s_+ = b^s_1 \cup \cdots \cup b^s_{s_0-1}$, $b^s_0 = b^s_0$ and $b^s_+ = b^s_{s_0+1} \cup \cdots \cup b^s_{N_s}$. Then we obtain the following proposition about the directional derivative and the second-order directional derivative of $\theta(X)$.

**Lemma 3.3** Under the above notations, one has he directional derivative of $\theta$ at $X$ along $H$ is expressed as

$$\theta'(X; H) = \sum_{i=1}^{s-1} \text{Tr}(\hat{H}_{a_i a_i}) - \sum_{i=s+1}^{r} \text{Tr}(\hat{H}_{a_i a_i}) + \|\hat{H}_{a_s a_s}\|_*.$$  

\(\text{(3.8)}\)
and the second-order directional derivative of \( \theta \) at \( X \) along \((H, W)\) is expressed as

\[
\theta''(X; H, W) = \sum_{i=1}^{s-1} \text{Tr}(\hat{V}_i(H, W)) - \sum_{i=s+1}^{r} \text{Tr}(\hat{V}_i(H, W)) + \text{Tr}(Q_{b_+}^s \hat{V}_s(H, W) Q_{b_+}^s) - \text{Tr}(Q_{b_-}^s \hat{V}_s(H, W) Q_{b_-}^s) + \|Q_{b_0}^s \hat{V}_s(H, W) Q_{b_0}^s\|_*.
\]

(3.9)

Proof. For \( \theta(X) = \|X\|_* \), the nuclear norm of a symmetric matrix in \( X \in S^q \), it is the spectral function corresponding to the symmetric function

\[
\zeta(z) = \sum_{j=1}^{q} |z_j|, z = (z_1, \ldots, z_q)^T \in \mathbb{R}^q,
\]

namely \( \theta(X) = \|X\|_* = [\zeta \circ \lambda](X) \).

Let \( \overline{z} \in \mathbb{R}^q \). We define

\[
I_+ (\overline{z}) = \{i: \overline{z}_i > 0\}, \quad I_0 (\overline{z}) = \{i: \overline{z}_i = 0\}, \quad I_- (\overline{z}) = \{i: \overline{z}_i < 0\}
\]

and

\[
I_{0+} (\overline{z}, \Delta z) = \{i \in I_0 (\overline{z}) : \Delta z_i > 0\}, \quad I_{00} (\overline{z}, \Delta z) = \{i \in I_0 (\overline{z}) : \Delta z_i = 0\}, \quad I_{0-} (\overline{z}, \Delta z) = \{i \in I_0 (\overline{z}) : \Delta z_i < 0\}.
\]

Then the directional derivative of \( \zeta \) at \( \overline{z} \) along \( \Delta z \) is

\[
\zeta'(\overline{z} ; \Delta z) = \sum_{i \in I_+ (\overline{z})} \Delta z_i - \sum_{i \in I_- (\overline{z})} \Delta z_i + \sum_{i \in I_0 (\overline{z})} |\Delta z_i|
\]

and the second-order parabolic directional derivative at \( \overline{z} \) along \( \Delta z \) and \( \Delta w \) is

\[
\zeta''(\overline{z} ; \Delta z, \Delta w) = \sum_{i \in I_+ (\overline{z}) \cup I_{0+} (\overline{z}, \Delta z)} \Delta w_i - \sum_{i \in I_- (\overline{z}) \cup I_{0-} (\overline{z}, \Delta z)} \Delta w_i + \sum_{i \in I_0 (\overline{z}, \Delta z)} |\Delta w_i|.
\]

Then, from the chain rules of directional derivatives (see Chapter 2 of [5]), we obtain

\[
\theta'(X; H) = \zeta'(\lambda(X); \lambda'(X; H)) = \sum_{i=1}^{s-1} \text{Tr}(\hat{H}_{a_i}) - \sum_{i=s+1}^{r} \text{Tr}(\hat{H}_{a_i}) + \|\hat{H}_{a_i}\|_*
\]

and

\[
\theta''(X; H, W) = \zeta'(\lambda(X); \lambda'(X; H), \lambda''(X; H, W))
\]

\[
= \sum_{i=1}^{s-1} \text{Tr}(\hat{V}_i(H, W)) - \sum_{i=s+1}^{r} \text{Tr}(\hat{V}_i(H, W)) + \text{Tr}(Q_{b_+}^s \hat{V}_s(H, W) Q_{b_+}^s) - \text{Tr}(Q_{b_-}^s \hat{V}_s(H, W) Q_{b_-}^s) + \|Q_{b_0}^s \hat{V}_s(H, W) Q_{b_0}^s\|_*.
\]

The proof is completed. \( \square \)

By direct calculation, we may obtain the following conclusion.
Proposition 3.1 Let $\psi(W) = \theta''(X; H, W)$, then

$$
\psi^*(Y) = \begin{cases}
2 \sum_{i=1}^{s-1} \text{Tr}(Q^T_{a_i} H(X - \omega_i I)^J H Q_{a_i}) \\
+ 2 \sum_{i=s}^{r} \text{Tr}(Q^T_{a_i} H(X - \omega_i I)^J H Q_{a_i}) \\
+ 2 \text{Tr}(Q^T_{b_+} T Q^T_{a} H X^J H Q_{a} Q^T_{b_+}) \\
+ 2 \text{Tr}(Q^T_{b_*} T Q^T_{a} H X^J H Q_{a} Q^T_{b_*}) \\
+ 2 \langle Q^T_{b_0} \hat{Y}_{a\alpha s} Q_{b_0}^*, Q^T_{b_0} T Q^T_{a} H X^J H Q_{a} Q_{b_0}^* \rangle \\
0
\end{cases}
$$

\[ \begin{align*}
\hat{Y}_{a\alpha i} &= I_{|a_i|}, \\
& \text{for } 1 \leq i \leq s-1, \\
\hat{Y}_{a\alpha i} &= -I_{|a_i|}, \\
& \text{for } s \leq i \leq r, \\
Q^T_{b_+} \hat{Y}_{a\alpha s} Q^*_{b_+} &= I_{|b_+|}, \\
Q^T_{b_*} \hat{Y}_{a\alpha s} Q^*_{b_*} &= I_{|b_*|}, \\
\|Q^T_{b_0} \hat{Y}_{a\alpha s} Q^*_{b_0}\|_2 &\leq 1,
\end{align*}\] otherwise.

Now we characterize elements in $\partial \theta(X)$ for $X \in S^3$. If follows from Page 121 of Borwin and Lewis (2006) [4], for the given $X \in S^3$ with the spectral decomposition (3.7), that $Y \in \partial \theta(X)$ if and only if there exists $w \in \partial \varsigma(\lambda(X))$

$$
Y = Q \text{Diag}(w) Q^T,
$$

where $X$ has the spectral decomposition $X = Q \text{Diag}(\lambda(X)) Q^T$. Define the following three index sets:

$$
a = \{i : \lambda_i(X) > 0\}, \ b = \{i : \lambda_i(X) = 0\}, \ c = \{i : \lambda_i(X) < 0\},
$$
or alternatively $a = a_1 \cup \cdots \cup a_{s-1}$, $b = a_s$ and $c = a_{s+1} \cup \cdots \cup a_r$. Then, $\omega \in \partial \varsigma(\lambda(X))$ has the following property

$$
w_a = I_{|a|}, \ w_c = -I_{|c|} \text{ and } -I_{|b|} \leq w_b \leq I_{|b|},
$$
and

$$
Y = Q \text{Diag}(w) Q^T = Q_a Q_a^T + Q_b \text{Diag}(w_b) Q_b^T - Q_c Q_c^T.
$$

For the index set $b$, we partition it as follows $b = b_L \cup b_S \cup b_U$:

$$
b_L = \{i \in b : w_i = -1\}, \ b_S = \{i \in b : -1 < w_i < 1\}, \ b_U = \{i \in b : w_i = 1\}.
$$

Then $Y \in \partial \theta(X)$ can be expressed as

$$
Y = Q \text{Diag}(w) Q^T = Q_{a\cup b_L} Q_{a\cup b_L}^T + Q_{b_S} \text{Diag}(w_{b_S}) Q_{b_S}^T - Q_{a\cup b_L} Q_{a\cup b_L}^T
$$
and for $Z = X + Y$,

$$
Z = [Q_a \ Q_{b_L} \ Q_{b_S} \ Q_{b_L} \ Q_c] \begin{bmatrix}
\Lambda_a + I_{|a|} & & & \\
& I_{|b_L|} & & \\
& & I_{|b_S|} & \\
& & & -I_{|b_L|} \\
\Lambda_c - I_{|c|} & & & \\
\end{bmatrix}
\begin{bmatrix}
Q_a^T \\
Q_{b_S}^T \\
Q_{b_L}^T \\
Q_{b_L}^T \\
Q_c^T
\end{bmatrix}. \tag{3.12}
$$
The critical cone of \(\theta\) at \(Z\) associated with \(Y \in \partial \theta(X)\) is defined by

\[
\mathcal{C}_\theta(Z) = \{ H \in S^q : \theta'(X; H) = \langle Y, H \rangle \}. \tag{3.13}
\]

The next lemma gives an characterization of the critical cone \(\mathcal{C}_\theta\).

**Lemma 3.4** Let \(X, Y, Z \in S^q\), \(Z = X + Y\) satisfies \(Y \in \partial \theta(X)\). Then \(H \in \mathcal{C}_\theta(Z)\) if and only if

\[
\mathcal{C}_\theta(Z) = \left\{ H \in S^q : \begin{array}{l}
Q^T_{bs} H[Q_b] = 0, Q^T_{bu} H[Q_{bL}] = 0 \\
Q^T_{bu} HQ_{b_L} \in S^{|b|}_+, Q^T_{bL} HQ_{b_L} \in S^{|b|}_- \end{array} \right\}. \tag{3.14}
\]

**Proof.** Noting that

\[
\partial \theta(X) = \{ Q_a Q^T_a + Q_b W_b Q^T_b - Q_c Q^T_c : W_b \in S^{|b|}, \|W_b\|_2 \leq 1 \},
\]

where \(\|W_b\|_2\) denotes the spectral norm of \(W_b\). And the directional derivative of \(\theta\) at \(X\) along \(H\) is

\[
\theta'(X; H) = \langle Q_a Q^T_a - Q_c Q^T_c, H \rangle + \|Q^T_b[H]Q_b\|_*.
\]

Noting that \(B\) has the expression

\[
B = Q_a Q^T_a + Q_b \text{Diag}(w_b) Q^T_b - Q_c Q^T_c
\]

where \(w_b \in \mathbb{R}^{|b|}\) satisfies \(\|w_b\|_\infty \leq 1\). Then \(\theta'(A; H) = \langle B, H \rangle\) is equivalent to

\[
\|Q^T_b[H]Q_b\|_* = \langle Q^T_b[H]Q_b, \text{Diag}(w_b) \rangle. \tag{3.15}
\]

From Fan’s inequality one has

\[
\langle Q^T_b[H]Q_b, \text{Diag}(w_b) \rangle \leq \lambda(Q^T_b[H]Q_b)^T w_b,
\]

which implies, from (3.15), for \(\lambda(Q^T_b[H]Q_b) = (\lambda_1(Q^T_b[H]Q_b), \ldots, \lambda_{|b|}(Q^T_b[H]Q_b))^T\) with \(\lambda_1(Q^T_b[H]Q_b) \geq \cdots \geq \lambda_{|b|}(Q^T_b[H]Q_b)\), that

\[
\langle Q^T_b[H]Q_b, \text{Diag}(w_b) \rangle = \lambda(Q^T_b[H]Q_b)^T w_b = \|Q^T_b[H]Q_b\|_*.
\]

Then \(Q^T_b[H]Q_b\) and \(\text{Diag}(w_b)\) admit a simultaneous ordered eigenvalue decomposition, and thus we can check that \(H\) satisfies

\[
Q^T_{bs} H[Q_b] = 0, Q^T_{bu} H[Q_{bL}] = 0, Q^T_{bu} HQ_{b_L} \in S^{|b|}_+, Q^T_{bL} HQ_{b_L} \in S^{|b|}_-.
\]

The proof is completed. \(\square\)

**Corollary 3.1** Let \(X, Y, Z \in S^q\), \(Z = X + Y\) satisfies \(Y \in \partial \theta(X)\). Then \(H \in \mathcal{C}_\theta(Z)\) if and only if

\[
\|Q^T_a H Q_a\|_* = \langle Q^T_{as} Y Q_{as}, Q^T_{as} H Q_{as} \rangle.
\]
Proposition 3.2 Let $X, Y, Z, H \in S^q$, $Z = X + Y$ satisfies $Y \in \partial \theta(X)$ and $H \in C_\theta(Z)$. Then

$$
\psi^*(Y) = 2 \sum_{i=1}^{s-1} \text{Tr}(Q^T_{a_i} H(X - \varpi_i I)^\dagger H Q_{a_i}) + 2 \sum_{i=s+1}^{r} \text{Tr}(Q^T_{a_i} H(X - \varpi_i I)^\dagger H Q_{a_i})
+ 2 \text{Tr}([Q^T_{b_i} H X^\dagger H Q_{b_i}]) + 2 \langle Q^T_{b_0} Y Q_{b_0}, Q^T_{b_0} H X^\dagger H Q_{b_0}\rangle.
$$

(3.16)

Proof. Since $H \in C_\theta(Z)$, we have from (3.14) that there exist $\hat{Q}^*_U \in O(Q^T_{b_u} H Q_{b_u})$ and $\hat{Q}^*_L \in O(Q^T_{b_l} H Q_{b_l})$ such that

$$
Q^s = \begin{bmatrix}
\hat{Q}^*_U & 0 & 0 \\
0 & I_{|b_s|} & 0 \\
0 & 0 & \hat{Q}^*_L
\end{bmatrix}.
$$

Then $Q^s_{b_k^+}$ and $Q^s_{b_k^-}$ can be expressed as

$$
Q^s_{b_k^+} = \begin{bmatrix}
\hat{Q}^*_U \\
0 \\
0
\end{bmatrix}
\text{ and } Q^s_{b_k^-} = \begin{bmatrix}
0 \\
0 \\
\hat{Q}^*_L
\end{bmatrix}.
$$

Then we obtain (3.16) from (3.10). \qed

Corollary 3.2 Let $X, Y, Z, H \in S^q$, $Z = X + Y$ satisfies $Y \in \partial \theta(X)$ and $H \in C_\theta(Z)$. Then

$$
\psi^*(Y) = 2 \sum_{i=1}^{r} \langle Q^T_{a_i} Y Q_{a_i}, Q^T_{a_i} H(X - \varpi_i I)^\dagger H Q_{a_i}\rangle,
$$

(3.17)

or alternatively

$$
\psi^*(Y) = 2 \sum_{i \neq s} \frac{1}{\varpi_i} (Q^T_{b_s} Y Q_{b_s} - I_{|b_s|}, Q^T_{b_s} H Q_{a_s} Q^T_{a_s} H Q_{b_s}).
$$

(3.18)

We now discuss the differential of $[e_{\tau}\theta](X)$ for $\theta(X) = \|X\|_*$, where $[e_{\tau}\theta](X)$ is the Moreau-Yosida regularization defined by (1.2). Let proximal mapping of $\theta$ be defined by

$$
[P_{\tau}\theta](X) = \arg\min_{X' \in S^q} \left\{ \theta(X') + \frac{1}{\tau} \|X' - X\|^2 \right\}.
$$

For simplicity, we use $P\theta$ to denote $P_1\theta$. Then $[e_{\tau}\theta](X)$ is the spectral function corresponding to the Moreau-Yosida regularization $e_{\tau\varsigma}$, namely

$$
[e_{\tau}\theta](X) = [e_{\tau\varsigma} \circ \lambda](X).
$$

It follows from [18] or [36] that

$$
[P_{\tau}\theta](X) = P\text{Diag}([P_{\tau\varsigma} \circ \lambda(X)]) P^T
$$

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or \([P_t\theta](X)\) is the Löwner operator associated with \(p_r(t) = [t - \tau]_+ - [-t - \tau]_+\), namely
\[
[P_t\theta](X) = Q\text{Diag} (p_r(\lambda_1(X)), \cdots, p_r(\lambda_q(X))) Q^T.
\]
Let \(X\) have \(r\) distinct eigenvalues, among them there are \(r_1\) positive distinct eigenvalues and \(r - r_1\) negative distinct eigenvalues and zero eigenvalues:
\[
\varpi_1 > \varpi_2 > \cdots > \varpi_{r_1} > \varpi_{r_1+1} = 0 > \varpi_{r_1+2} > \cdots > \varpi_r.
\]
Define
\[
a_k = \{i : \lambda_i(X) = \varpi_k\}, \quad k = 1, \ldots, r
\]
and the first divided difference matrix at \(X\) along \(H \in S^q\) as follows for \(k, l = 1, \ldots, r\),
\[
(p_r^{[1]}(\Lambda(X), Q^T H Q))_{a_k a_l} := \begin{cases}
p_r(\varpi_k) - p_r(\varpi_l) & \text{if } k \neq l,
\frac{Q^T H Q_{a_k}}{\varpi_k - \varpi_l} & \text{if } k = l.
\end{cases}
\]
where \(\Psi_{k}(\cdot)\) is the Löwner operator with respect to \(\psi_{k}(\cdot) = p_r^{[1]}(\varpi_k; \cdot)\). Then the directional derivative of \(P_t\theta\) at \(X\) along \(H \in S^q\) is expressed as
\[
[P_t\theta]'(X; H) = Q[p_r^{[1]}(\Lambda(X), Q^T H Q)]Q^T.
\]

**Proposition 3.3** Let \(X, Y, Z, H \in S^q\), \(Z_\tau = X + \tau Y\) satisfies \(Y \in \partial \theta(X)\). Then \(P_t\theta\) is strongly semismooth at \(Z_\tau\) and for \(W \in \partial P_t\theta(Z_\tau)\), there exist \(W_{bU} \in \partial \Pi_{S^{bU}_+}(0)\) and \(W_{bL} \in \partial \Pi_{S^{bL}_-}(0)\) such that
\[
W(H) = 
\begin{bmatrix}
\tilde{H}_{aa} & \tilde{H}_{abu} & \tilde{H}_{abs} \circ (\Omega_{\tau})_{abs} & \tilde{H}_{abl} \circ (\Omega_{\tau})_{abl} & \tilde{H}_{ac} \circ (\Omega_{\tau})_{ac} \\
\tilde{H}_{bua} & W_{bu}(\tilde{H}_{bubu}) & 0 & 0 & \tilde{H}_{buc} \circ (\Omega_{\tau})_{buc} \\
\tilde{H}_{bua} & 0 & 0 & 0 & \tilde{H}_{buc} \circ (\Omega_{\tau})_{buc} \\
\tilde{H}_{ca} \circ (\Omega_{\tau})_{ca} & \tilde{H}_{cbu} \circ (\Omega_{\tau})_{cbu} & \tilde{H}_{cbu} \circ (\Omega_{\tau})_{cbu} & \tilde{H}_{cbu} \circ (\Omega_{\tau})_{cbu} & \tilde{H}_{cc} \\
W_{bL}(\tilde{H}_{bLbL}) & \tilde{H}_{bLc} & \tilde{H}_{bLc} & \tilde{H}_{bLc} & \tilde{H}_{cc}
\end{bmatrix} Q^T,
\]
where \(\tilde{H} = Q^T H Q\),
\[
(\Omega_{\tau})_{ij} = [p_r^{[1]}(\Lambda(Z_\tau))]_{ij}, (i, j) \in a \times [b_S \cup b_L \cup c] \text{ or } (i, j) \in c \times [b_U \cup b_S].
\]
In other words, \(\nabla e_{t\theta}\) is strongly semismooth at \(Z_\tau\) and for \(V \in \partial \nabla e_{t\theta}(Z_\tau)\), there exist
$$V_{b_L} \in \partial \Pi_{S_+^L}(0) \text{ and } V_{b_L} \in \partial \Pi_{S_+^L}(0)$$ such that

$$\tau V(H) = \begin{bmatrix} 0 & 0 & \tilde{H}_{abS} \circ (\Delta_T)_{abS} & \tilde{H}_{abL} \circ (\Delta_T)_{abL} & \tilde{H}_{ac} \circ (\Delta_T)_{ac} \\ 0 & V_{b_L}(\tilde{H}_{b_Lb_L}) & \tilde{H}_{b_Lb_S} & \tilde{H}_{b_Lb_L} & \tilde{H}_{b_Lc} \circ (\Delta_T)_{b_Lc} \\ \tilde{H}_{b_Sa} \circ (\Delta_T)_{b_Sa} & \tilde{H}_{b_Sb_U} & \tilde{H}_{b_Sb_S} & \tilde{H}_{b_Sb_L} & \tilde{H}_{b_SC} \circ (\Delta_T)_{b_SC} \\ \tilde{H}_{ca} \circ (\Delta_T)_{ca} & \tilde{H}_{cb_U} \circ (\Delta_T)_{cb_U} & \tilde{H}_{cb_S} \circ (\Delta_T)_{cb_S} & 0 & 0 \end{bmatrix} Q^T,$$

where $$\tilde{H} = Q^T HQ,$$

$$(\Delta_T)_{ij} = 1 - [p^{[1]}(A(Z_r))]_{ij}, (i, j) \in a \times [b_S \cup b_L \cup c] \text{ or } (i, j) \in c \times [b_U \cup b_S].$$

### 3.2 Optimality conditions for (SDNOP)

This subsection is devoted to studying optimality conditions for the following nonlinear semidefinite nuclear norm composite optimization problem (SDNOP)

$$\min f(x) + \theta(F(x)) \text{ s.t. } h(x) = 0, g(x) \in S^p_+,$$

where $$\theta(X) = \|X\|_*$$ is the nuclear norm function of $$X \in S^q$$, $$f : \mathbb{R}^n \mapsto \mathbb{R}$$, $$F : \mathbb{R}^n \mapsto S^q$$, $$h : \mathbb{R}^n \mapsto \mathbb{R}^m$$ and $$g : \mathbb{R}^n \mapsto S^p$$ are twice continuously differentiable functions. Obviously, Problem (SDNOP) is a special case of (COP) with $$Z := S^q$$, $$\theta(X) := \|X\|_*$$, $$Y := S^p$$ and $$K := S^p_+$$. The Lagrange function for (SDNOP) is

$$L(x, Y, \mu, \Gamma) = f(x) + \langle Y, F(x) \rangle + \langle \mu, h(x) \rangle - \langle \Gamma, g(x) \rangle (x, Y, \mu, \Gamma) \in \mathbb{R}^n \times S^q \times \mathbb{R}^m \times S^p.$$

Then for any $$(x, Y, \mu, \Gamma) \in \mathbb{R}^n \times S^q \times \mathbb{R}^m \times S^p$$,

$$\nabla_x L(x, Y, \mu, \Gamma) = \nabla f(x) + DF(x)^*Y + Jh(x)^T\mu - Dg(x)^*\Gamma.$$

If $$x$$ is a stationary point, the set of Lagrange multipliers at $$x$$ is defined by

$$\Lambda(x) = \left\{ (Y, \mu, \Gamma) \in S^q \times \mathbb{R}^m \times S^p : \nabla_x L(x, Y, \mu, \Gamma) = 0, Y \in \partial \theta(F(x)), \Gamma \in -N_{S^p_+}(g(x)) \right\}.$$

When discussing optimality conditions, we need some constraint qualifications. We say that Robinson constraint qualification holds at $$\bar{\tau}$$ if

$$\begin{bmatrix} Jh(\bar{\tau}) \\ Dg(\bar{\tau}) \end{bmatrix} \in \mathbb{R}^n + \left\{ 0 \right\}_{T_{S^p_+}(g(\bar{\tau}))} = \mathbb{R}^n \times S^p.$$

The critical cone of Problem (SDNOP) at $$x$$ is defined by

$$C(x) = \left\{ d \in T_{\mathcal{S}_0}(x) : \nabla f(x)^Td + \theta'(F(x); D\nabla(F(x))d) \leq 0 \right\}.$$

We can easily derive the following necessary optimality conditions and second-order sufficient optimality conditions.
Proposition 3.4 If $\overline{x} \in \Phi$ is a local minimizer around which $f, F, h$ and $g$ are twice continuously differentiable and Robinson constraint qualification holds at $\overline{x}$. Then

1. $\Lambda(\overline{x})$ is non-empty, compact and convex.
2. For any $d \in C(\overline{x})$,
   \[
   \sup_{y \in \Lambda(\overline{x})} \left\{ \langle d, \nabla^2_{xx}L(\overline{x}, y)d \rangle - \psi^*(Y) + 2 \left\langle \nabla F(\overline{x}) d|g(\overline{x})\rangle \nabla^2 g(\overline{x})d \right\} \geq 0,
   \]
   where $\psi(W) = \theta''(F(\overline{x}); DF(\overline{x})d, W)$.

Proposition 3.5 Let $\overline{x}$ be a feasible point around which $f, F, h$ and $g$ are twice continuously differentiable. Suppose the following conditions hold:

1. $\Lambda(\overline{x})$ is non-empty;
2. For any $d \in C(\overline{x}) \setminus \{0\}$,
   \[
   \sup_{y \in \Lambda(\overline{x})} \left\{ \langle d, \nabla^2_{xx}L(\overline{x}, y)d \rangle - \psi^*(Y) + 2 \left\langle \nabla F(\overline{x}) d|g(\overline{x})\rangle \nabla^2 g(\overline{x})d \right\} > 0,
   \]
   where $\psi(W) = \theta''(F(\overline{x}); DF(\overline{x})d, W)$.

Then the second-order growth condition holds at $\overline{x}$.

Now we list our two assumptions for Problem (SDNOP), which will be used in the next section to derive Assumptions B1 and B2.

Assumption (sdnop-A1)[12]. The constraint nondegeneracy condition holds at $\overline{x}$:

\[
\begin{pmatrix}
DF(\overline{x}) \\
Jh(\overline{x}) \\
Dg(\overline{x})
\end{pmatrix}
\mathbb{R}^n + \begin{pmatrix}
\mathcal{T}_{\text{lin}}(F(\overline{x})) \\
\{0\} \\
\text{lin}(\mathcal{T}_{S^0}^+(g(\overline{x})))
\end{pmatrix} = \begin{pmatrix}
S^q \\
\mathbb{R}_m^m \\
S_p
\end{pmatrix},
\]

where
\[
\mathcal{T}_{\text{lin}}(\overline{x}) = \{ H \in S^q : \theta'(\overline{x}; H) = -\theta'(\overline{x}; -H) \} = \{ H \in S^q : Q^T_b HQ_b = 0 \}. 
\]

Assumption (sdnop-A1) is the analogue to the linear independence constraint qualification for nonlinear programming, which implies that $M(\overline{x})$ is a singleton [5, Proposition 4.50].

Assumption (sdnop-A2) The strong second order sufficient condition holds at $\overline{x}$:

\[
\langle d, \nabla^2_{xx}L(\overline{x}, \overline{Y}, \overline{\mu}, \overline{\Gamma})d \rangle - \psi^*(\overline{Y}) + 2 \left\langle \nabla F(\overline{x}) d|g(\overline{x})\rangle \nabla^2 g(\overline{x})d \right\} > 0,
\]
where $\psi(W) = \theta''(F(\overline{x}); DF(\overline{x})d, W)$ and
\[
\text{app}(\overline{Y}, \overline{x}, \overline{\Gamma}) := \left\{ d \in \mathbb{R}^n : Jh(\overline{x})d = 0, DF(\overline{x})d \in \text{aff}(C_\theta(F(\overline{x}) + \overline{Y})) \right\}. 
\]
From the expressions $C_\theta$ and $C_{S^p}$, we obtain the following expression of $app(Y, \mu, \Gamma)$:

$$app(Y, \mu, \Gamma) = \left\{ d \in \mathbb{R}^n : \begin{align*}
Q_b^T(DF(x)d)Q_b &= 0, Q_{b_L}^T(DF(x)d)Q_{b_L} = 0 \\
F_\alpha^T(Jg(x)d)F_\alpha &= 0, \ F_\alpha^T(Jg(x)d)F_\beta = 0, \ Jh(x)d = 0
\end{align*} \right\}.$$ (3.27)

At the end of this subsection, we list two technical results coming from [34], which will be used in the next section.

**Lemma 3.5** [34, Lemma 7] Let $\phi : \mathcal{X} \mapsto \mathbb{R}$ be continuous and positive homogeneous of degree two:

$$\phi(td) = t^2 \phi(d), \ \forall \ t \geq 0 \ \text{and} \ \ d \in \mathcal{X}.$$ 

Suppose that there exists a positive number $\eta_0 > 0$ such that for any $d$ satisfying $Ld = 0$, one has $\phi(d) \geq \eta_0 \|d\|^2$, where $L : \mathcal{X} \mapsto \mathcal{Y}$ is a given linear operator. Then there exist positive numbers $\eta \in (0, \eta_0]$ and $c_0 > 0$ such that

$$\phi(d) + c_0 \langle Ld, Ld \rangle \geq \eta \langle d, d \rangle, \ \forall \ d \in \mathcal{X}.$$ 

**Lemma 3.6** [34, Lemma 8] Let $a, b, c, \text{ and } c_0$ be four positive scalars with $c \geq c_0$. Let

$$\psi(t; c, a, b, c_0) := a - \frac{1}{c} t + \frac{t^2}{b + (c - c_0)t}, \ \ t \in [0, 1].$$ (3.28)

Then, for any $c \geq \max \{c_0, (b - c_0)^2/c_0\}$, $\psi(\cdot; c, a, b, c_0)$ is a convex function on $[0, 1],

$$\min_{t \in [0, 1]} \psi(t; c, a, b, c_0) = a - \frac{1}{c} \left( \frac{b}{\sqrt{c} + \sqrt{c_0}} \right)^2,$$ (3.29)

and

$$\max_{t \in [0, 1]} \psi(t; c, a, b, c_0) = \max \left\{ \psi(0; c, a, b, c_0), \psi(1; c, a, b, c_0) \right\}.$$ (3.30)

4 On the augmented Lagrange method for SDNOP

This section is devoted to studying the rate of convergence of the augmented Lagrange method for Problem (SDNOP). Let $(\bar{x}, \bar{Y}, \bar{\mu}, \bar{\Gamma}) \in \mathbb{R}^n \times \mathcal{S}^q \times \mathbb{R}^m \times \mathcal{S}^p$ be a given KKT point. Then, $L(\bar{x}, \bar{Y}, \bar{\mu}, \bar{\Gamma})$ satisfies

$$\nabla_x L(\bar{x}, \bar{Y}, \bar{\mu}, \bar{\Gamma}) = 0, \ \bar{Y} \in \partial \theta(F(\bar{x})), \ h(\bar{x}) = 0, \ \bar{\Gamma} \succeq 0, \ g(\bar{x}) \succeq 0 \ \text{and} \ \langle \bar{\Gamma}, g(\bar{x}) \rangle = 0.$$ (4.1)

Let $\bar{X} = F(\bar{x})$ and $\bar{Y} \in \partial \theta(\bar{X})$. Define the following three index sets:

$$a = \{ i : \lambda_i(\bar{X}) > 0 \}, \ b = \{ i : \lambda_i(\bar{X}) = 0 \}, \ c = \{ i : \lambda_i(\bar{X}) < 0 \},$$

then

$$\bar{X} = Q \begin{bmatrix} \Lambda_a & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \Lambda_c \end{bmatrix} Q^T \text{ and } Q \in \mathcal{O}(\bar{X}) \text{ with } Q = \begin{bmatrix} Q_a & Q_b & Q_c \end{bmatrix}$$
with $Q_a \in \mathbb{R}^{q \times |a|}$, $Q_b \in \mathbb{R}^{q \times |b|}$, and $Q_c \in \mathbb{R}^{q \times |c|}$. Then there exists $w \in \partial \kappa(\lambda(X))$ satisfying

$$Y = Q \text{Diag}(w)Q^T$$

and $w$ has the following relations

$$w_a = 1_{|a|}, w_c = -1_{|c|} \text{ and } -1_{|b|} \leq w_b \leq 1_{|b|}.$$

For the index set $b$, we partition it as follows

$$b = b_L \cup b_S \cup b_U:$$

$$b_L = \{i \in b : w_i = -1\}, b_S = \{i \in b : -1 < w_i < 1\}, b_U = \{i \in b : w_i = 1\}.$$

Then $Y$ can be expressed as follows:

$$Y = (Q_{a \cup b_U} \quad Q_{b_S} \quad Q_{c \cup b_L}) \begin{bmatrix} I_{|a \cup b_U|} & 0 & 0 \\ 0 & \text{Diag}(w_{b_S}) & 0 \\ 0 & 0 & I_{|c \cup b_L|} \end{bmatrix} \begin{pmatrix} Q_{a \cup b_U}^T \\ Q_{b_S}^T \\ Q_{c \cup b_L}^T \end{pmatrix}$$

(4.2)

with $Q_{a \cup b_U} \in \mathbb{R}^{q \times |a \cup b_U|}$, $Q_{b_S} \in \mathbb{R}^{q \times |b_S|}$, and $Q_{c \cup b_L} \in \mathbb{R}^{q \times |c \cup b_L|}$.

Let $\overline{M} := \Gamma - g(\overline{x})$. Suppose that $\overline{M}$ has the spectral decomposition as in (3.2), i.e., $\overline{M} = P \Lambda P^T$. Define three index sets of positive, zero, and negative eigenvalues of $\overline{M}$, respectively, as

$$\alpha := \{i \mid \lambda_i > 0\}, \quad \beta := \{i \mid \lambda_i = 0\}, \quad \gamma := \{i \mid \lambda_i < 0\}.$$

Write

$$\Lambda = \begin{bmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda_\gamma \end{bmatrix} \quad \text{and} \quad P = [P_\alpha \quad P_\beta \quad P_\gamma]$$

with $P_\alpha \in \mathbb{R}^{p \times |\alpha|}$, $P_\beta \in \mathbb{R}^{p \times |\beta|}$, and $P_\gamma \in \mathbb{R}^{p \times |\gamma|}$. From (4.1), we know that $\overline{\Gamma}g(\overline{x}) = g(\overline{x})\overline{\Gamma} = 0$. Thus, we have

$$\overline{\Gamma} = P \begin{bmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^T, \quad g(\overline{x}) = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\Lambda_\gamma \end{bmatrix} P^T$$

$$\overline{\Gamma} - tg(\overline{x}) = P \begin{bmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t\Lambda_\gamma \end{bmatrix} P^T.$$  

(4.3)
For $\overline{X} = F(\overline{x})$, let

\[
\nu_{a,b} := \min_{i \in a, 1 \leq j \leq |b|} \frac{1 - (w_b)_j}{\lambda_i(\overline{X})} \quad \nu_{a,c} := \max_{i \in a, 1 \leq j \leq |c|} \frac{1 - (w_b)_j}{\lambda_i(\overline{X})} ;
\]

\[
\nu_{a,b_L} := \min_{i \in a} \frac{2}{\lambda_i(\overline{X})} \quad \nu_{a,b_c} := \max_{i \in a} \frac{2}{\lambda_i(\overline{X})} ;
\]

\[
\nu_{a,c} := \min_{i \in a, j \in c} \frac{2}{\lambda_i(\overline{X}) - \lambda_j(\overline{X})} \quad \nu_{a,c} := \max_{i \in a, j \in c} \frac{2}{\lambda_i(\overline{X}) - \lambda_j(\overline{X})} ;
\]

\[
\nu_{b,c} := \min_{i \in c} \frac{2}{-\lambda_i(\overline{X})} \quad \nu_{b,c} := \max_{i \in c} \frac{2}{-\lambda_i(\overline{X})} ;
\]

\[
\nu_{a,\gamma} := \min_{i \in a, j \in \gamma} \lambda_i/|\lambda_j| \quad \nu_{a,\gamma} := \max_{i \in a, j \in \gamma} \lambda_i/|\lambda_j|
\]

and

\[
\nu_0 = \min\{\nu_{a,b}, \nu_{a,b_L}, \nu_{a,c}, \nu_{b,c}, \nu_{b,c}, \nu_{a,\gamma}\}; \quad \nu_0 = \min\{\nu_{a,b}, \nu_{a,b_L}, \nu_{a,c}, \nu_{b,c}, \nu_{b,c}, \nu_{a,\gamma}\}.
\]

(4.4)

(4.5)

For a given symmetric matrix $M$, we use $\text{vec}(M)$ to denote the vector obtained by stacking up all the columns of a given matrix $M$ and $\text{svec}(M)$ to denote the vector obtained by stacking up all the columns of the upper triangular part of $M$.

Let $Q \in \mathcal{O}(\overline{X})$ with $Q = \{Q_a, Q_{b_L}, Q_{b_S}, Q_{b_U}, Q_{c}\}$. For index sets $\chi, \chi' \in \{a, b_U, b_S, b_L, c\}$, let

\[
B_{(\chi, \chi')}(Q) := \left( \text{vec}(Q_{\chi}^T J_{x_1} F(\overline{x}) Q_{\chi'}) \cdots \text{vec}(Q_{\chi}^T J_{x_n} F(\overline{x}) Q_{\chi'}) \right)
\]

and

\[
\hat{B}_{(\chi, \chi)}(Q) := \left( \text{svec}(Q_{\chi}^T J_{x_1} F(\overline{x}) Q_{\chi}) \cdots \text{svec}(Q_{\chi}^T J_{x_n} F(\overline{x}) Q_{\chi}) \right).
\]

Let $P \in \mathcal{O}(g(\overline{x}))$ with $P = [P_\alpha P_\beta P_\gamma]$. For index sets $\chi, \chi' \in \{\alpha, \beta, \gamma\}$, let

\[
C_{(\chi, \chi')}(P) := \left( \text{vec}(P_{\chi}^T J_{x_1} g(\overline{x}) P_{\chi'}) \cdots \text{vec}(P_{\chi}^T J_{x_n} g(\overline{x}) P_{\chi'}) \right)
\]

and

\[
\hat{C}_{(\chi, \chi)}(P) := \left( \text{svec}(P_{\chi}^T J_{x_1} g(\overline{x}) P_{\chi}) \cdots \text{svec}(P_{\chi}^T J_{x_n} g(\overline{x}) P_{\chi}) \right).
\]

Define

\[
n_1 := m + |b|(|b| + 1)/2, \quad n_2 := n_1 + (|\alpha| + |\beta|)(|\alpha| + |\beta| + 1)/2, \quad n_3 := n - n_2,
\]
Thus there exist numbers $\nu$ and $\sigma_n$ such that $A(Q,P)$ is of full row rank. Let $A(Q,P)$ have the following singular value decomposition:

$$A(Q,P) = U[\Sigma(Q,P) 0]R^T,$$

(4.6)

where $U \in \mathbb{R}^{n_2 \times n_2}$ and $R \in \mathbb{R}^{n \times n}$ are orthogonal matrices, $\Sigma(Q,P) = \text{Diag}(\sigma_1(A(Q,P)), \cdots, \sigma_{n_2}(A(Q,P)))$, and $\sigma_1(A(Q,P)) \geq \sigma_2(A(Q,P)) \geq \cdots \geq \sigma_{n_2}(A(Q,P)) > 0$ are the singular values of $A(Q,P)$. It should be pointed out here that $U$ and $R$ also depend on $(Q,P)$. But for the sake of notational simplification, we drop the argument $(Q,P)$ from $U$ and $R$ in our analysis below.

Let

$$\sigma := \min \left\{1, \min_{Q \in \mathcal{O}(X), P \in \mathcal{O}(M)} \min_{1 \leq i \leq n_2} \sigma_i^{-2}(A(Q,P)) \right\}$$

and

$$\sigma := \max \left\{1, \max_{Q \in \mathcal{O}(X), P \in \mathcal{O}(M)} \max_{1 \leq i \leq n_2} \sigma_i^{-2}(A(Q,P)) \right\}.$$ 

Then, since $\mathcal{O}(X)$ and $\mathcal{O}(M)$ are compact sets and $\Sigma(Q,P)$ changes continuously with respect to $(Q,P)$, both $\underline{\sigma}$ and $\bar{\sigma}$ are finite positive numbers. Define

$$C = \begin{bmatrix}
B_{(a,b_S)} \\
B_{(a,b_L)} \\
B_{(a,c)} \\
B_{(c,b_S)} \\
-C_{(\alpha,\gamma)}
\end{bmatrix}.$$

Thus there exist numbers $\nu \geq 0$ and $\overline{\sigma} > 0$ such that for any $Q \in \mathcal{O}(X)$, $P \in \mathcal{O}(M)$ and $s \in \mathbb{R}^{\|a\|b_S+\|a\|b_L+\|a\|c+\|c\|b_S+\|a\|\gamma}$,

$$\nu \|s\|^2 \leq \max \left\{ \langle s, \tilde{C}(Q,P)(\tilde{C}^T(Q,P))s \rangle, \langle s, C \tilde{C}^T_s \rangle \right\} \leq \overline{\sigma} \|s\|^2,$$

(4.7)

where

$$\tilde{C}(Q,P) := CR$$

and

$$\tilde{R} := R \begin{bmatrix}
\Sigma(Q,P)^{-1}U^T & 0 \\
0 & I_{n_3}
\end{bmatrix}.$$
When no ambiguity arises, we often drop $Q$ and $P$ from $A(Q, P)$, $B_{(x', x)}(Q)$, $\hat{B}_{(a, \gamma)}(Q)$, $C_{(x, x')}(Q)$, and $\hat{C}_{(a, \gamma)}(P)$. Let $c > 0$ and $W_1 \in \partial B[\mathbf{0}^T_{c}](\hat{F}(\mathbf{y}) + \mathbf{y}/c)$, there exist matrices $Q \in \mathcal{O}(F(\mathbf{y}))$ and $\Delta_{1/c} \in \mathcal{S}^q$ such that

$$W_1(H_1) = Q \left( \Delta_{1/c} \circ (Q^T H_1(Q)) \right) Q^T, \quad \forall H_1 \in \mathcal{S}^q. \quad (4.8)$$

with the entries of $\Delta_{r}$ being given by

$$[\Delta_{1/c}]_{ij} = \begin{cases} 0 & (i, j) \in (a \times a \cup b_U \cup c) \cup (c \times b_L \cup c), \\ \frac{c^{-1}(1 - (w_{b_S})_j)}{\lambda_i(\mathbf{X}) + c^{-1}(1 - (w_{b_S})_j)} & (i, j) \in a \times \{1, \ldots, |b_S|\}, \\ \frac{2c^{-1}}{\lambda_i(\mathbf{X}) + 2c^{-1}} & (i, j) \in a \times b_L, \\ \frac{2c^{-1}}{\lambda_i(\mathbf{X}) - \lambda_j(\mathbf{X}) + 2c^{-1}} & (i, j) \in a \times c, \\ \frac{2c^{-1}}{-\lambda_j(\mathbf{X}) + 2c^{-1}} & (i, j) \in b_U \times c, \\ \frac{c^{-1}((w_{b_S})_i + 1)}{c^{-1}((w_{b_S})_i + 1) - \lambda_j(\mathbf{X})} & (i, j) \in \{1, \ldots, |b_S|\} \times c. \\ \end{cases} \quad (4.10)$$

Let $c > 0$ and $W_2 \in \partial B \Pi_{S^p}(\Gamma - cg(\mathbf{y}))$. Define $\lambda_c \in \mathbb{R}^p$ as

$$(\lambda_c)_i := \begin{cases} \lambda_i & \text{if } i \in \alpha \cup \beta, \\ c \lambda_i & \text{if } i \in \gamma. \end{cases}$$

Then it follows from Lemma 3.2 that there exist two matrices $Q \in \mathcal{O}(\mathbf{Y})$ and $\Theta_c \in \mathcal{S}^p$ such that

$$W_2(H_2) = P \left( \Theta_c \circ (P^T H_2 P) \right) P^T, \quad \forall H_2 \in \mathcal{S}^p \quad (4.11)$$

with the entries of $\Theta_c$ being given by

$$[\Theta_c]_{ij} = \begin{cases} \max\{(\lambda_c)_i, 0\} + \max\{(\lambda_c)_j, 0\} & \text{if } (i, j) \notin \beta \times \beta, \\ |(\lambda_c)_i| + |(\lambda_c)_j| & \text{if } (i, j) \in \beta \times \beta. \end{cases} \quad (4.12)$$

For index sets $x, x' \in \{a, b_U, b_S, b_L, c\}$, we introduce the following notation:

$$(\Delta_{r})_{(x, x')} = \text{Diag} \left( \text{vec}((\Delta_{r})_{xx'}) \right), \quad (\hat{\Delta}_{r})_{(x, x')} = \text{Diag} \left( \text{svec}((\Delta_{r})_{xx} \circ E_{xx}) \right),$$

where $\text{vec}$ and $\text{svec}$ denote the vectorization and skew-vectorization, respectively.
where “•” is the Hadamard product and $E$ is a matrix in $S^q$ with entries being given by

$$
E_{ij} := \begin{cases} 
1 & \text{if } i = j, \\
2 & \text{if } i \neq j.
\end{cases}
$$

For index sets $\chi, \chi' \in \{\alpha, \beta, \gamma\}$, we introduce the following notation:

$$(\Theta_c)_{(\chi, \chi')} = \text{Diag} \left( \text{vec}(\Theta_c)_{\chi\chi'} \right), \quad (\hat{\Theta}_c)_{(\chi, \chi)} = \text{Diag} \left( \text{vec}(\Theta_c)_{\chi\chi} \circ E'_{\chi\chi} \right),$$

where $E'$ is a matrix in $S^p$ with entries being given by

$$
E'_{ij} := \begin{cases} 
1 & \text{if } i = j, \\
2 & \text{if } i \neq j.
\end{cases}
$$

Let

$$
D_c := \begin{bmatrix}
I_m & 0 & 0 & 0 & 0 \\
0 & \Sigma_c & 0 & 0 & 0 \\
0 & 0 & (\hat{\Theta}_c)_{(\alpha, \alpha)} & 0 & 0 \\
0 & 0 & 0 & (\hat{\Theta}_c)_{(\beta, \beta)} & 0 \\
0 & 0 & 0 & 0 & 2I_{[\alpha||\beta]} \\
\end{bmatrix},
$$

where

$$
\Sigma_c = \begin{bmatrix}
(\hat{\Delta}_1/c)_{bU,bU} & 0 & 0 \\
0 & 2I_m & 0 \\
0 & 0 & (\hat{\Delta}_1/c)_{bL,bL}
\end{bmatrix}
$$

with $m_0 = |b_U|(|b_S| + |b_L|) + |b_S|((|b_S| + 1)/2 + |b_L|)$.

Let $A_c(\bar{\gamma}, \bar{\mu}, \bar{\Gamma}, W_1, W_2)$ be defined as (2.5) for the semidefinite nuclear norm composite optimization problem (SDNOP), i.e.,

$$
A_c(\bar{\gamma}, \bar{\mu}, \bar{\Gamma}, W_1, W_2) = \nabla^2_{xx} L(\bar{\gamma}, \bar{\mu}, \bar{\Gamma}) + c Jh(\bar{\gamma})^T Jh(\bar{\gamma}) + DF(\bar{\gamma})^* W_1 DF(\bar{\gamma}) + c Dg(\bar{\gamma})^* W_2 Dg(\bar{\gamma}).
$$

A compact formula for $A_c(\bar{\gamma}, \bar{\mu}, \bar{\Gamma}, W_1, W_2)$ is given in the next lemma.

**Lemma 4.1** The matrix $A_c(\bar{\gamma}, \bar{\mu}, \bar{\Gamma}, W_1, W_2)$ can be expressed equivalently as

$$
A_c(\bar{\gamma}, \bar{\mu}, \bar{\Gamma}, W_1, W_2) = \nabla^2_{xx} L(\bar{\gamma}, \bar{\mu}, \bar{\Gamma}) + c \left( Jh(\bar{\gamma})^T Jh(\bar{\gamma}) + 2b_{bU,bS}^{T} B_{bU,bS} + 2b_{bL,bS}^{T} B_{bL,bS} + 2b_{bU,bL}^{T} B_{bU,bL} + \hat{\Delta}_{1/c}(bU,bU) \hat{B}_{bU,bU} + \hat{\Delta}_{1/c}(bL,bL) \hat{B}_{bL,bL} + 2\hat{\Delta}_{1/c}(a,bS) B_{a,bS} + 2\hat{\Delta}_{1/c}(a,bL) B_{a,bL} + 2\hat{\Delta}_{1/c}(b,c) B_{b,c} + \hat{C}_{a,a}^{T} \hat{\Theta}_{c}(a,a) \hat{C}_{a,a} + 2\hat{\Theta}_{c}^{T}(\Theta_{c})(a,a) C_{a,a} + \hat{\Theta}_{c}^{T}(\Theta_{c})(\beta,\beta) \hat{\Theta}_{c}^{T}(\Theta_{c})(\beta,\beta) \right).
$$

(4.13)
Lemma 4.1 shows that $A_c(\bar{Y}, \bar{\mu}, \bar{\Gamma}, W_1, W_2)$ can be written as

$$A_c(\bar{Y}, \bar{\mu}, \bar{\Gamma}, W_1, W_2) = \nabla^2_{xx} L(\bar{\pi}, \bar{Y}, \bar{\mu}, \bar{\Gamma}) + c^T D_c A$$

$$+ 2cB^T_{a,b}(\Delta_{1/c}(a,b))B(a,b) + 2cB^T_{a,b,L}(\Delta_{1/c}(a,b))B(a,b_L)$$

$$+ 2cB^T_{a,c}(\Delta_{1/c}(a,c))B(a,c) + 2cB^T_{c,b}(\Delta_{1/c}(c,b))B(c,b)$$

$$+ 2cB^T_{c,b}(\Delta_{1/c}(c,b))B(c,b) + 2cC^T_{(\alpha,\gamma)}(\Theta(c)\alpha, \gamma)C(\alpha, \gamma) \cdot$$

(4.14)

For any $\epsilon, c > 0$, let

$$B_{\epsilon,c}(\bar{Y}, \bar{\mu}, \bar{\Gamma}, W_1, W_2) = \nabla^2_{xx} L(\bar{\pi}, \bar{Y}, \bar{\mu}, \bar{\Gamma}) + \epsilon^T D_c A$$

$$+ 2cB^T_{a,b}(\Delta_{1/c}(a,b))B(a,b) + 2cB^T_{a,b,L}(\Delta_{1/c}(a,b))B(a,b_L)$$

$$+ 2cB^T_{a,c}(\Delta_{1/c}(a,c))B(a,c) + 2cB^T_{c,b}(\Delta_{1/c}(c,b))B(c,b)$$

$$+ 2cB^T_{c,b}(\Delta_{1/c}(c,b))B(c,b) + 2cC^T_{(\alpha,\gamma)}(\Theta(c)\alpha, \gamma)C(\alpha, \gamma) \cdot$$

(4.15)

The following proposition shows that, under Assumptions (sdnop-A1) and (sdnop-A2), the basic Assumption B1 made in Section 2 is satisfied by nonlinear semidefinite nuclear norm composite optimization problem.

**Proposition 4.1** Suppose that Assumptions (sdnop-A1) and (sdnop-A2) are satisfied. Then there exist two positive numbers $c_0$ and $\eta$ such that for any $c \geq c_0$ and $W_1 \in \partial_B[\theta_c]^*(F(\bar{\pi}) + \bar{Y}/c)$, $W_2 \in \partial_B \Pi_{S^p}(\bar{\Gamma} - cg(\bar{\pi}))$,

$$\langle d, A_c(\bar{Y}, \bar{\mu}, \bar{\Gamma}, W_1, W_2)d \rangle \geq \langle d, B_{\epsilon_0,c}(\bar{Y}, \bar{\mu}, \bar{\Gamma}, W_1, W_2)d \rangle \geq \eta \langle d, d \rangle, \quad \forall d \in \mathbb{R}^n.$$

**Proof.** It follows from Assumption (sdnop-A2) that there exists $\eta_0 > 0$ such that

$$\langle d, \nabla^2_{xx} L(\bar{\pi}, \bar{Y}, \bar{\mu}, \bar{\Gamma})d \rangle - \psi^*(\bar{Y}) + 2 \langle \bar{\Gamma}, [D g(\bar{\pi})d]g(\bar{\pi})^T [D g(\bar{\pi})d] \rangle \geq \eta_0 \|d\|^2 \quad (4.16)$$

for all $d \in \text{app}(\bar{Y}, \bar{\mu}, \bar{\Gamma}) \setminus \{0\}$. By (3.26), we obtain

$$\text{app}(\bar{Y}, \bar{\mu}, \bar{\Gamma}) = \left\{ d \in \mathbb{R}^n : \begin{array}{l}
Jh(\bar{\pi})d = 0, B_{(b_U,b_S)}(Q)d = 0, B_{(b_L)}(Q)d = 0 \\
\hat{B}_{(b_S,b_S)}(Q)d = 0, B_{(b_S,b_L)}(Q)d = 0 \\
\check{C}_{(\alpha,\alpha)}(P)d = 0, C_{(\alpha,\beta)}(P)d = 0
\end{array} \right\}.$$

(4.17)

Since (4.16) and (4.17) hold, by using Lemma 3.5 with $\phi$ and $\mathcal{L}$ being defined by

$$\phi(d) := \langle d, \nabla^2_{xx} L(\bar{\pi}, \bar{Y}, \bar{\mu}, \bar{\Gamma})d \rangle - \psi^*(\bar{Y}) + 2 \langle \bar{\Gamma}, [D g(\bar{\pi})d]g(\bar{\pi})^T [D g(\bar{\pi})d] \rangle$$

and

$$\mathcal{L}(d) := (Jh(\bar{\pi})d; B_{(b_U,b_S)}(Q)d; B_{(b_L)}(Q)d; \hat{B}_{(b_S,b_S)}(Q)d; B_{(b_S,b_L)}(Q)d; \check{C}_{(\alpha,\alpha)}(P)d; C_{(\alpha,\beta)}(P)d),$$

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for any $d \in \mathbb{R}^n$, respectively, we know that there exist two positive numbers $c_1$ and $\eta \in (0, \eta_0/2]$ such that for any $c \geq c_1$,  

$$
\langle d, \nabla^2_L(L(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{t})); d \rangle - \psi^*(\mathbf{y}) + 2 \langle \mathbf{y}, [Dg(\mathbf{x})]d \rangle \nabla g(\mathbf{x}) \rangle + 2 \langle \mathbf{y}, [Dg(\mathbf{x})]d \rangle [Dg(\mathbf{x})]d \rangle
$$

$$
+ c\|B_{(b_u, b_s)}(Q)d\|^2 + c\|B_{(b_u, b_s)}(Q)d\|^2 + c\|\hat{B}_{(b_s, b_s)}(Q)d\|^2 + c\|B_{(b_s, b_s)}(Q)d\|^2 (4.18)$$

$$
+ c\|Jh(\mathbf{x})\|^2 + c\|\tilde{C}_{(a, a)}(P)d\|^2 + c\|C_{(a, b)}(P)d\|^2 \geq 2\eta\|d\|^2, \quad \forall d \in \mathbb{R}^n.
$$

Let $c_0 \geq c_1$ be such that for any $c \geq c_0,$

$$
\max_{1 \leq i \leq n} \|J_{x_i}F(\mathbf{x})\|^2 \sum_{i \in a, 1 \leq j \leq b_s} \frac{c^{-1}(1-(w_{b_s})^j)^2}{\lambda_i(\mathbf{x})(\mathbf{y}) + c^{-1}(1-(w_{b_s})^j))} \leq \eta/4, \quad (4.19)
$$

$$
\max_{1 \leq i \leq n} \|J_{x_i}g(\mathbf{x})\|^2 \sum_{i \in \gamma, j \in \alpha} \frac{\lambda_i^2}{\lambda_i(\mathbf{x})(\mathbf{y}) + c}\leq \eta/4.
$$

Let $c \geq c_0$ and $W_1 \in \partial g_B[\partial \mathbf{y}^*](F(\mathbf{x}) + \mathbf{y}/c), W_2 \in \partial g_B \nabla (\mathbf{x} - cg(\mathbf{x}))$. Then there exist two matrices $Q \in \mathcal{O}(F(\mathbf{x}))$ and $P \in \mathcal{O}(g(\mathbf{x}))$ and $\Delta_{1/c} \in S^q$ satisfying (4.9) and $\Theta_{c} \in S^p$ satisfying (4.12) such that

$$
W_1(\mathbf{H}_1) = Q \left( \Delta_{1/c} \circ (Q^T H_1 Q) \right) Q^T, \quad \forall \mathbf{H}_1 \in S^q.
$$

and

$$
W_2(\mathbf{H}_2) = P \left( \Theta_{c} \circ (P^T H_2 P) \right) P^T, \quad \forall \mathbf{H}_2 \in S^p.
$$

It is easy to see from (4.19) that for any $c \geq c_0$ and $d \in \mathbb{R}^n$ we have for $H_1 = DF(\mathbf{x})d$ and $\tilde{H}_1 = Q^T H_1 Q$ that

$$
-\psi^*(\mathbf{y}) - 2c \langle d, [B_{(a,b_s)}(\Delta_{1/c})(a,b_s)]B_{(a,b_s)} \rangle d
$$

$$
= 2 \sum_{i \neq s} \frac{1}{\omega_i} \left| b_j \right| (I_{\left| b_j \right|} - Q_{b_s} Y_{b_s} Q_{b_s}^T H_1 Q_{a_i} Q_{a_i}^T H Q_{b_s}) - 2c \langle d_{1}, [B_{(a,b_s)}(\Delta_{1/c})(a,b_s)]B_{(a,b_s)} \rangle d)
$$

$$
\leq 2 \sum_{i \neq s} \frac{1}{\omega_i} \left| b_j \right| \|Q_{a_i}^T H_1 Q_{b_s}\|^2 - 2c \sum_{i = 1}^{s-1} (Q_{a_i}^T H_1 Q_{b_s} \circ \Delta_{a_i S}, Q_{a_i}^T H Q_{b_s})
$$

$$
\leq 2 \sum_{i = 1}^{s-1} \frac{1}{\omega_i} \left| b_j \right| \|Q_{a_i}^T H_1 Q_{b_s}\|^2 - 2c \sum_{i = 1}^{s-1} (Q_{a_i}^T H_1 Q_{b_s} \circ \Delta_{a_i S}, Q_{a_i}^T H Q_{b_s})
$$

$$
\leq 2 \sum_{i = 1}^{s-1} \frac{1}{\omega_i} \left| b_j \right| \|Q_{a_i}^T H_1 Q_{b_s}\|^2 - 2c \sum_{i = 1}^{s-1} \frac{1}{\omega_i + c^{-1}(1-(w_{b_s})^j))} \|Q_{a_i}^T H_1 Q_{b_s}\|^2
$$

$$
\leq 2 \sum_{i = 1}^{s-1} \frac{1}{\omega_i} \left| b_j \right| \|Q_{a_i}^T H_1 Q_{b_s}\|^2 - 2c \sum_{i = 1}^{s-1} \frac{1}{\omega_i + c^{-1}(1-(w_{b_s})^j))} \|Q_{a_i}^T H_1 Q_{b_s}\|^2
$$

$$
\leq 2 \sum_{i \neq a} \sum_{j=1}^n \lambda_i(\mathbf{x}) \left| b_j \right| (\mathbf{y}) + c^{-1}(1-(w_{b_s})^j)) \|Q_{a_i}^T D(\mathbf{x})d(Q_{b_s})\|^2
$$

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\[
\begin{align*}
&= 2 \sum_{i \in a} \sum_{j=1}^{[b]_i} c^{-1}(1 - (w_{b_j})_j^2) \lambda_i(F(\vec{r})) \langle \lambda_i(F(\vec{r})) + c^{-1}(1 - (w_{b_j})_j) \rangle \sum_{i=1}^{n} (Q_{i}^T [J_{x_i} F(\vec{r})] (Q_{b_j})_j d_i)^2 \\
&\leq 2 \sum_{i \in a} \sum_{j=1}^{[b]_i} c^{-1}(1 - (w_{b_j})_j^2) \lambda_i(F(\vec{r})) \langle \lambda_i(F(\vec{r})) + c^{-1}(1 - (w_{b_j})_j) \rangle \sum_{i=1}^{n} \|Q_i\|^2 \|Q^T [J_{x_i} F(\vec{r})]\|^2 (Q_{b_j})_j d_i^2 \\
&\leq \max_{1 \leq i \leq n} \|J_{x_i} F(\vec{r})\|^2 \sum_{i \in a} \sum_{j=1}^{[b]_i} c^{-1}(1 - (w_{b_j})_j^2) \lambda_i(F(\vec{r})) \langle \lambda_i(F(\vec{r})) + c^{-1}(1 - (w_{b_j})_j) \rangle \|d\|^2 \\
&\leq \eta \|d\|^2 / 2.
\end{align*}
\]

Similarly, we have from (4.19) that for any \( c \geq c_0 \) and \( d \in \mathbb{R}^n \) that
\[
2 \langle \vec{r}, [D g(\vec{r})] d g(\vec{r})^\dagger [D g(\vec{r})] d \rangle - 2c \langle d, C^T_{(a,\gamma)}(\Theta_c)(a,\gamma) C_{(a,\gamma)} d \rangle \\
= 2 \langle \vec{r}, [D g(\vec{r})] d g(\vec{r})^\dagger [D g(\vec{r})] d \rangle - 2c \langle d, C^T_{(a,\gamma)}(\Theta_c)(a,\gamma) C_{(a,\gamma)} d \rangle \\
= 2 \sum_{i \in a} \lambda_i \left( \sum_{i=1}^{n} P_i^T J_{x_i} g(\vec{r}) P_j d_i \right)^2 - 2c \sum_{i \in a} \lambda_i \left( \sum_{i=1}^{n} P_i^T J_{x_i} g(\vec{r}) P_j d_i \right)^2 \\
\leq 2 \sum_{i \in a} \lambda_i \left( \sum_{i=1}^{n} \|J_{x_i} g(\vec{r})\|^2 \|P_i\|^2 \|P_j\|^2 d_i^2 \right) \\
\leq 2 \sum_{i \in a} \lambda_i \left( \sum_{i=1}^{n} \|J_{x_i} g(\vec{r})\|^2 \|P_i\|^2 \|P_j\|^2 d_i^2 \right) \\
\leq \eta \|d\|^2 / 2.
\]

Therefore, we have from (4.18), for any \( c \geq c_0 \), that
\[
\langle d, \nabla_x^2 L(\vec{r}, \vec{\pi}, \vec{\tau}, \vec{\Gamma}) d \rangle + 2c \langle d, [B^T_{(a,b)}(\Delta_{1/c})(a,b)] d \rangle \\
+ c_0 \|B_{(b_U,b_L)}(Q) d\|^2 + c_0 \|B_{(b_U,b_L)}(Q) d\|^2 + c_0 \|\hat{B}_{(b_U,b_L)}(Q) d\|^2 \\
+ c_0 \|\hat{B}_{(b_U,b_L)}(Q) d\|^2 + 2c \langle d, C^T_{(a,\gamma)}(\Theta_c)(a,\gamma) C_{(a,\gamma)} d \rangle \\
+ c_0 \|\hat{C}_{(a,\gamma)}(P) d\|^2 + c_0 \|\hat{C}_{(a,\gamma)}(P) d\|^2 \geq \eta \|d\|^2, \quad \forall d \in \mathbb{R}^n.
\]

In view of the expression \((\Delta_{1/c})_{ij}\) from (4.10) for \((i, j) \in (a \times b_L) \cup (a \times \{1, \ldots, [b_S]\}) \cup (a \times c) \cup (b_U \times c) \cup \{\{1, \ldots, [b_S]\} \times c\}, \) we obtain
\[
B^T_{(a,b_L)}(\Delta_{1/c})(a,b_L) B_{(a,b_L)} \succeq 0, \quad B^T_{(a,c)}(\Delta_{1/c})(a,c) B_{(a,c)} \succeq 0, \\
B^T_{(c,b_U)}(\Delta_{1/c})(c,b_U) B_{(c,b_U)} \succeq 0, \quad B^T_{(c,b_S)}(\Delta_{1/c})(c,b_S) B_{(c,b_S)} \succeq 0, \\
\hat{B}^T_{(b_U,b_L)}(\Delta_{1/c})(b_U,b_L) \hat{B}_{(b_U,b_L)} \succeq 0, \quad \hat{B}^T_{(b_L,b_L)}(\Delta_{1/c})(b_L,b_L) \hat{B}_{(b_L,b_L)} \succeq 0.
\]

From this and the fact that \(\hat{C}^T_{(a,\gamma)}(\hat{\Theta}_{c})(a,\gamma) \hat{C}_{(a,\gamma)} \succeq 0,\) we can see that for any \( c \geq c_0,\)
\[
\langle d, B_{c_0,c}(\vec{Y}, \vec{\pi}, \vec{\tau}, W_1, W_2) d \rangle \geq \eta \|d\|^2, \quad \forall d \in \mathbb{R}^n.
\]

By noting the fact that
\[
A_{c}(\vec{Y}, \vec{\pi}, \vec{\tau}, W_1, W_2) = B_{c_0,c}(\vec{Y}, \vec{\pi}, \vec{\tau}, W_1, W_2) + (c - c_0) A^T D_c A,
\]

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we complete the proof.

Let Assumptions (sdnop-A1) and (sdnop-A2) be satisfied. Let the two positive numbers \( c_0 \) and \( \eta \) be defined as in Proposition 4.1. Let \( c \geq c_0 \). Then, by Propositions 2.1 and 4.1 and the fact that \( D\theta_c(\cdot) \) and \( \Pi S^p_\theta(\cdot) \) are strongly semismooth everywhere, there exist two positive numbers \( \varepsilon > 0 \) and \( \delta_0 > 0 \) (both depending on \( c \)) and a locally Lipschitz continuous function \( x_c(\cdot,\cdot,\cdot) \) defined on \( \mathbb{B}_{\delta_0}(Y,\pi,\Gamma) \) such that for any \((Y,\mu,\Gamma) \in \mathbb{B}_{\delta_0}(Y,\pi,\Gamma), x_c(Y,\mu,\Gamma) \) is the unique minimizer of \( L_c(\cdot,\cdot,\cdot) \) over \( \mathbb{B}_\varepsilon(\pi) \) and \( x_c(\cdot,\cdot,\cdot) \) is semismooth at \((Y,\mu,\Gamma)\). Let \( \vartheta_c : \mathcal{S} \times \mathbb{R}^m \times \mathcal{S}^p \mapsto \mathbb{R} \) be defined as (2.8), i.e.,

\[
\vartheta_c(Y,\mu,\Gamma) := \min_{x \in \mathbb{B}_\varepsilon(\pi)} L_c(x,Y,\mu,\Gamma), \quad (Y,\mu,\Gamma) \in \mathcal{S} \times \mathbb{R}^m \times \mathcal{S}^p.
\]

Then it holds that

\[
\vartheta_c(Y,\mu,\Gamma) = L_c(x_c(Y,\mu,\Gamma),Y,\mu,\Gamma), \quad (Y,\mu,\Gamma) \in \mathbb{B}_{\delta_0}(Y,\pi,\Gamma).
\]

Furthermore, it follows from Propositions 2.2 and 4.1 that the concave function \( \vartheta_c(\cdot,\cdot,\cdot) \) is continuously differentiable on \( \mathbb{B}_{\delta_0}(Y,\pi,\Gamma) \) with

\[
D\vartheta_c(Y,\mu,\Gamma) = \begin{pmatrix} -c^{-1}Y + c^{-1}D\theta_c(F(x_c(Y,\mu,\Gamma)) + Y/c) \varepsilon^{-1} \theta_c(X,\mu,\Gamma) \varepsilon^{-1} \Pi S^p_\theta(\Gamma - cg(x_c(Y,\mu,\Gamma))) \end{pmatrix}, \quad \gamma = (Y,\mu,\Gamma) \in \mathbb{B}_{\delta_0}(Y,\pi,\Gamma).
\]

For any \((\Delta Y,\Delta \mu,\Delta \Gamma) \in \mathcal{S} \times \mathbb{R}^m \times \mathcal{S}^p\), let \( \nabla \vartheta_c(\Delta Y,\Delta \mu,\Delta \Gamma) \) be defined as in (2.12). By Propositions 2.3 and 4.1, we have for any \((\Delta Y,\Delta \mu,\Delta \Gamma) \in \mathcal{S} \times \mathbb{R}^m \times \mathcal{S}^p\) that

\[
\vartheta_B(\nabla \vartheta_c)(Y,\mu,\Gamma)(\Delta Y,\Delta \mu,\Delta \Gamma) \subseteq \nabla \vartheta_c(\Delta Y,\Delta \mu,\Delta \Gamma).
\]

Since when \( c \to \infty \),

\[
\begin{pmatrix}
0 \\
0 \\
\frac{1 - (w_{bs})_j}{\lambda_i(\underline{X}) + c^{-1}(1 - (w_{bs})_j)} \\
\frac{2}{\lambda_i(\underline{X}) + 2c^{-1}} \\
\frac{2}{\lambda_i(\underline{X}) - \lambda_j(\underline{X}) + 2c^{-1}} \\
\frac{2}{(w_{bs})_i + 1 - \lambda_j(\underline{X}) + c^{-1}(w_{bs})_i + 1 - \lambda_j(\underline{X})} \\
\end{pmatrix} \to \begin{pmatrix}
0 \\
0 \\
\frac{1 - (w_{bs})_j}{\lambda_i(\underline{X})} \\
\frac{2}{\lambda_i(\underline{X})} \\
\frac{2}{\lambda_i(\underline{X}) - \lambda_j(\underline{X})} \\
\frac{2}{(w_{bs})_i + 1 - \lambda_j(\underline{X})} \\
\end{pmatrix}
\]

\((i,j) \in (a \times a \cup b_U),
\)

\((i,j) \in (c \times b_L \cup c),
\)

\((i,j) \in (a \times (1,\ldots,|b_S|),
\)

\((i,j) \in (c \times a),
\)

\((i,j) \in (a \times b_L),
\)

\((i,j) \in (a \times c),
\)

\((i,j) \in (b_U \times c),
\)

\((i,j) \in (1,\ldots,|b_S| \times c),
\)

where \( \underline{X} = F(\pi) \), and

\[
\lim_{c \to \infty} c(\Theta_c)_{ij} = \lim_{c \to \infty} c \frac{\lambda_i}{\lambda_i + c|\lambda_j|} = \frac{\lambda_i}{|\lambda_j|}, \forall (i,j) \in \alpha \times \gamma.
\]

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we know that there exists a positive number \( \overline{\eta} \) such that

\[
\langle d, \mathcal{B}_{c,0}(\overline{Y}, \overline{\pi}, \overline{\Gamma}, W_1, W_2) d \rangle \leq \overline{\eta} d(d) \quad \forall d \in \mathbb{R}^n, \ c \geq c_0 \ 	ext{and} \ W_1 \in \partial B[D\theta_c]^*(F(\overline{x}) + \overline{Y}/c), \quad W_2 \in \partial B \Pi_{S^p} (\overline{\Gamma} - c g(\overline{x})). \tag{4.21}
\]

Let \( c \geq c_0, \ W_1 \in \partial B[D\theta_c]^*(F(\overline{x}) + \overline{Y}/c) \) and \( W_2 \in \partial B \Pi_{S^p} (\overline{\Gamma} - c g(\overline{x})) \). Then there exist two matrices \( Q \in \mathcal{O}(F(\overline{x})) \) with \( P \in \mathcal{O}(g(\overline{x})) \) and \( \Delta_{1/c} \) satisfying (4.9) such that (4.8) holds, \( \Theta_c \in S^p \) satisfying (4.12) such that (4.11) holds. Let \( A(Q, P) \) have the singular value decomposition as in (4.6), i.e.,

\[
A(Q, P) = U[\Sigma(Q, P) \ 0] R^T. \tag{4.22}
\]

Let \( \overline{\eta} := (\overline{Y}, \overline{\pi}, \overline{\Gamma}) \). Then we have the following result for \( A_c(\overline{\eta}, W_1, W_2) \).

**Lemma 4.2** Let \( c > c_0 \) and \( W_1 \in \partial B[D\theta_c]^*(F(\overline{x}) + \overline{Y}/c) \) and \( W_2 \in \partial B \Pi_{S^p} (\overline{\Gamma} - c g(\overline{x})) \). Suppose that Assumptions (sdnop-A1) and (sdnop-A2) are satisfied. Then we have

\[
A_c(\overline{\eta}, W_1, W_2)^{-1} \leq R \begin{bmatrix}
\Sigma^{-1} U^T \left( \frac{\overline{\eta}}{n} I_{n_2} + (c - c_0) D_c \right)^{-1} U \Sigma^{-1} & 0 \\
0 & \overline{\sigma}^{-1} \overline{\eta}^{-1} I_{n_3}
\end{bmatrix} R^T, \tag{4.23}
\]

\[
\| A_c(\overline{\eta}, W_1, W_2)^{-1} A^T D_c u \| \leq \sqrt{2} \left( \overline{\pi} + (\overline{\eta} \overline{\pi})^{-2} (\overline{\sigma} \overline{\eta})^2 \right) \| u \|/(c - c_0), \quad \forall u \in \mathbb{R}^{n_2}, \tag{4.25}
\]

where \( \Sigma := \Sigma(Q, P) \).

**Proof.** Let \( \hat{c} := c - c_0 \). By (4.14), (4.15), and the singular value decomposition (4.22) of \( A := A(P) \), we have

\[
A_c(\overline{\eta}, W_1, W_2)^{-1} = \left( B_{c,0,c}(\overline{\eta}, W_1, W_2) + \hat{c} A^T D_c A \right)^{-1}
\]

\[
= B_{c,0,c}(\overline{\eta}, W_1, W_2) + \hat{c} R[\Sigma \ 0]^T U^T D_c U[\Sigma \ 0] R^T)^{-1}
\]

\[
= R \left( R^T B_{c,0,c}(\overline{\eta}, W_1, W_2) R + \hat{c} \left[ \Sigma^{-1} 0 \\
0 I_{n_3}
\right] \left[ U^T D_c U \ 0 \right] \left[ \Sigma^{-1} 0 \\
0 I_{n_3}
\right] \right)^{-1} R^T
\]

\[
= R \left[ \Sigma^{-1} 0 \\
0 I_{n_3}
\right] \left( G_{c,0,c}(\overline{\eta}, W_1, W_2) + \hat{c} \left[ U^T D_c U \ 0 \right] \left[ \Sigma^{-1} 0 \\
0 I_{n_3}
\right] \right)^{-1} R^T, \tag{4.26}
\]

where

\[
G_{c,0,c}(\overline{\eta}, W_1, W_2) := \left[ \Sigma^{-1} 0 \\
0 I_{n_3}
\right] R^T B_{c,0,c}(\overline{\eta}, W_1, W_2) R \left[ \Sigma^{-1} 0 \\
0 I_{n_3}
\right].
\]

It follows from Proposition 4.1, the definitions of \( \overline{\sigma} \) and \( \overline{\pi} \), and (4.21) that

\[
G_{c,0,c}(\overline{\eta}, W_1, W_2) \geq \overline{\eta} \left[ \Sigma^{-1} 0 \\
0 I_{n_3}
\right]^2 \geq \overline{\sigma} \overline{\eta} I_n \tag{4.27}
\]

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from (4.27) and (4.28) that $H$ with
Therefore, (4.23) and (4.24) follow from (4.26).

Now we turn to the proof of (4.25). Let
\[
\phi_{c,\varepsilon}(\mathcal{G}, W_1, W_2) := \left[ \begin{array}{cc} U & 0 \\ 0 & I_{n_3} \end{array} \right] \mathcal{G}_{c,\varepsilon}(\mathcal{G}, W_1, W_2) \left[ \begin{array}{cc} U^T & 0 \\ 0 & I_{n_3} \end{array} \right]
\]
and
\[
\phi_{c,\varepsilon}(\mathcal{H}, W_1, W_2) := \phi_{c,\varepsilon}(\mathcal{G}, W_1, W_2)^{-1}.
\]
Partition $\phi_{c,\varepsilon}(\mathcal{H}, W_1, W_2)$ as
\[
\phi_{c,\varepsilon}(\mathcal{H}, W_1, W_2) = \left[ \begin{array}{cc} H_1(W_1, W_2) & H_2(W_1, W_2)^T \\ H_2(W_1, W_2) & H_3(W_1, W_2) \end{array} \right]
\]
with $H_1(W_1, W_2) \in \mathcal{S}^{n_2}$, $H_2(W_1, W_2) \in \mathbb{R}^{n_2 \times n_2}$, and $H_3(W_1, W_2) \in \mathcal{S}^{n_3}$. Then, it follows from (4.27) and (4.28) that
\[
\|H_1(W_1, W_2)\|_2 \leq (\xi)\gamma^{-1}, \quad \|H_1(W_1, W_2)^{-1}\|_2 \leq \overline{\sigma} \eta
\]
and
\[
\|H_2(W_1, W_2)H_1(W_1, W_2)^{-1}\|_2 \leq (\xi)\gamma^{-1} \sigma \eta.
\]

For any $\varepsilon > 0$, let
\[
D_{c,\varepsilon} := D_c + \varepsilon I_{n_2}, \quad A_{c,\varepsilon}(\mathcal{G}, W_1, W_2) := B_{c,\varepsilon}(\mathcal{G}, W_1, W_2) + \hat{c} A^T D_{c,\varepsilon} A.
\]
Let $\varepsilon > 0$. By referring to (4.26), we obtain
\[
A_{c,\varepsilon}(\mathcal{G}, W_1, W_2)^{-1} = R \left[ \begin{array}{cc} \Sigma^{-1} U^T & 0 \\ 0 & I_{n_3} \end{array} \right] \left( \phi_{c,\varepsilon}(\mathcal{G}, W_1, W_2) + \hat{c} \left[ \begin{array}{cc} D_{c,\varepsilon} & 0 \\ 0 & 0 \end{array} \right] \right)^{-1} \left[ \begin{array}{cc} U \Sigma^{-1} & 0 \\ 0 & I_{n_3} \end{array} \right] R^T,
\]
which, together with (4.22) and the Sherman-Morrison-Woodbury formula (cf. [14, Section 2.1]), implies
\[
A_{c,\varepsilon}(\mathcal{G}, W_1, W_2)^{-1} A^T D_{c,\varepsilon} = R \left[ \begin{array}{cc} \Sigma^{-1} U^T & 0 \\ 0 & I_{n_3} \end{array} \right] \left[ \begin{array}{cc} (H_1(W_1, W_2)^{-1} + \hat{c} D_{c,\varepsilon})^{-1} & D_{c,\varepsilon} \\ H_2(W_1, W_2)H_1(W_1, W_2)^{-1}(H_1(W_1, W_2)^{-1} + \hat{c} D_{c,\varepsilon})^{-1} & D_{c,\varepsilon} \end{array} \right].
\]
Since, it follows from the Sherman-Morrison-Woodbury formula that
\[
(H_1(W_1, W_2)^{-1} + \hat{c} D_{c,\varepsilon})^{-1} D_{c,\varepsilon} = (\hat{c} I_{n_2} + D_{c,\varepsilon}^{-1} H_1(W_1, W_2)^{-1})^{-1}
\]
\[
= \hat{c}^{-1} I_{n_2} - \hat{c}^{-2} D_{c,\varepsilon}^{-1} (I_{n_2} + \hat{c}^{-1} H_1(W_1, W_2)^{-1} D_{c,\varepsilon}^{-1})^{-1} H_1(W_1, W_2)^{-1}
\]
\[
= \hat{c}^{-1} I_{n_2} - \hat{c}^{-1} (\hat{c} D_{c,\varepsilon} + H_1(W_1, W_2)^{-1})^{-1} H_1(W_1, W_2)^{-1},
\]

we have

\[ A_c(\overline{r}, W_1, W_2)^{-1}A^T D_c = \lim_{\varepsilon \downarrow 0} A_{c,\varepsilon}(\overline{r}, W_1, W_2)^{-1} A^T D_{c,\varepsilon} \]

\[ = R \begin{bmatrix} \Sigma^{-1} U^T \\ H_2(W_1, W_2)H_1(W_1, W_2)^{-1} \end{bmatrix} \left( \hat{e}^{-1} I_{n_2} - \hat{e}^{-1} (\hat{c}D_c + H_1(W_1, W_2)^{-1}) \right) H_1(W_1, W_2)^{-1}. \]

Therefore, from the definition of $\overline{r}$ and (4.29) we have for any $u \in \mathbb{R}^{n_2}$ that

\[ \| A_c(\overline{r}, W_1, W_2)^{-1} A^T D_c u \|^2 \leq (\overline{r} + (\sigma \eta)^{-2}(\overline{r}\eta)^2) \left( \| \hat{e}^{-1} I_{n_2} - \hat{e}^{-1} (\hat{c}D_c + H_1(W_1, W_2)^{-1}) \| H_1(W_1, W_2)^{-1} \| u \| \right)^2 \]

which, together with the fact that $\overline{r} \geq 1$, proves (4.25).

Let

\[ \overline{\nu} := \max \left\{ (2 + \sqrt{2})c_0, (\overline{r}\eta - c_0)^2/c_0, (\sigma \eta/2 - c_0)^2/c_0 \right\} \]  

(4.30)

and

\[ \nu_0 := (\sigma \overline{\nu}^{-2} - \eta)^{-2} \max \left\{ 8\overline{r}_1^2, 16\overline{r}_2^2, 32\overline{r}_3^2, 64\overline{r}_4^2, 128\overline{r}_5^2, 128\overline{r}_0^2, 4\kappa_0^2 \right\}^{1/2}. \]

(4.31)

where $\nu = F(\overline{\nu})$

\[ \begin{align*}
\overline{r}_1 &= \left( \max_{i \in a, j \in \{1, \ldots, |b_5|\}} \frac{1 - (w_{b_5})_{ij}}{\lambda_i(\nu)} \right) \\
\overline{r}_2 &= \left( \max_{i \in a, j \in b_4} \frac{2}{\lambda_j(\nu)} \right) \\
\overline{r}_3 &= \left( \max_{i \in a, j \in c} \frac{2}{\lambda_i(\nu) - \lambda_j(\nu)} \right) \\
\overline{r}_4 &= \left( \max_{i \in b_5, j \in c} \frac{2}{-\lambda_j(\nu)} \right) \\
\overline{r}_5 &= \left( \max_{i \in \{1, \ldots, |b_5|\}, j \in c} \frac{(w_{b_5})_{ij} + 1}{-\lambda_j(\nu)} \right)
\end{align*} \]

and

\[ \kappa_0 := \sqrt{2} \left( \overline{r} + (\sigma \eta)^{-2}(\overline{r}\eta)^2 \right). \]

**Proposition 4.2** Suppose that Assumptions (sdn-op-A1) and (sdn-op-A2) are satisfied. Then there exists a positive number $\mu_0$ such that for any $c \geq \overline{r}$ and $\Delta y \in \mathcal{S}^q \times \mathbb{R}^m \times \mathcal{S}^p$,

\[ \|(x_c)'(\overline{r}; \Delta y)\| \leq \mu_0 \|\Delta y\|/c \]  

(4.32)

and

\[ \langle V(\Delta y) + c^{-1}\Delta y, \Delta y \rangle \leq \mu_0[-1, 1]\|\Delta y\|^2/c^2, \quad \forall V(\Delta y) \in \nu_c(\Delta y). \]

(4.33)
Proof. Let \( c \geq \tau \). Let \( \Delta y := (\Delta Y, \Delta \mu, \Delta \Gamma) \in S^q \times \mathbb{R}^m \times S^p \). From the proof of Proposition 2.3 we know that there exist \( W_1 \in \partial_B [D\theta_c]^*(F(\bar{\pi}) + \bar{\gamma}/c) \) and \( W_2 \in \partial_{B^+} \Pi (\Gamma - cg(\bar{\pi})) \) such that

\[
(x_c)'(\bar{\eta}; \Delta y) = A_c(\bar{\eta}, W_1, W_2)^{-1} \left( -DF(\bar{\pi}) W_1(\Delta Y/c) - \mathcal{J} h(\bar{\pi})^T (\Delta \mu) + Dg(\bar{\pi})^* W_2(\Delta \Gamma) \right).
\]

(4.34)

For this \( W_1 \in \partial_B [D\theta_c]^*(F(\bar{\pi}) + \bar{\gamma}/c) \), there exist matrices \( Q \in \mathcal{O}(F(\bar{\pi})) \) and \( \Delta_{1/c} \in S^q \) satisfying (4.10) such that

\[
W_1(H_1) = Q (\Delta_{1/c} \circ (Q^T H_1 Q)) Q^T, \quad \forall H_1 \in S^q.
\]

For this \( W_2 \in \partial_{B^+} \Pi (\Gamma - cg(\bar{\pi})) \), there exist two matrices \( P \in O(\bar{\pi}) \) and \( \Theta_c \in S^p \) satisfying (4.12) such that

\[
W_2(H_2) = P (\Theta_c \circ (P^T H_2 P)) P^T, \quad \forall H_2 \in S^p.
\]

Let \( A := A(Q, P) \) have the singular value decomposition as in (4.6), i.e.,

\[
A = U[\Sigma \ 0] R^T,
\]

where \( \Sigma := \Sigma(Q, P) \).

For any two index sets \( \chi, \chi' \in \{b_U, b_S, b_L\} \), let

\[
\xi_{(\chi, \chi')} := \text{vec}(Q_{\chi}^T \Delta Y Q_{\chi'}), \quad \hat{\xi}_{(\chi, \chi')} := \text{svec}(Q_{\chi}^T \Delta Y Q_{\chi}).
\]

For any two index sets \( \chi, \chi' \in \{\alpha, \beta, \gamma\} \), let

\[
\omega_{(\chi, \chi')} := \text{vec}(P_{\chi}^T \Delta \Gamma P_{\chi'}), \quad \hat{\omega}_{(\chi, \chi')} := \text{svec}(P_{\chi}^T \Delta \Gamma P_{\chi}).
\]

Define

\[
\Delta d_0 := \begin{pmatrix} \Delta \mu \\ \hat{\xi}_{(b_U, b_U)} \\ \xi_{(b_U, b_S)} \\ \xi_{(b_U, b_L)} \\ \hat{\xi}_{(b_S, b_S)} \\ \xi_{(b_S, b_L)} \\ \hat{\xi}_{(b_L, b_L)} \\ \hat{\omega}_{(a, a)} \\ \hat{\omega}_{(b, b)} \\ \omega_{(a, \gamma)} \end{pmatrix}, \quad \Delta d := \begin{pmatrix} \Delta d_0 \\ \xi_{(a, b_S)} \\ \xi_{(a, b_L)} \\ \xi_{(a, c)} \\ \xi_{(c, b_U)} \\ \xi_{(c, b_S)} \\ \omega_{(a, \gamma)} \end{pmatrix}.
\]

Then, from (4.34), we have

\[
(x_c)'(\bar{\eta}; \Delta y) = -A_c(\bar{\eta}, W_1, W_2)^{-1} [A^T D_c \Delta d_0 + 2B_{(a, b_S)}^T (\Delta_{1/c})_{(a, b_S)} \xi_{(a, b_S)} \\
+ 2B_{(a, b_L)}^T (\Delta_{1/c})_{(a, b_L)} \xi_{(a, b_L)} + 2B_{(a, c)}^T (\Delta_{1/c})_{(a, c)} \xi_{(a, c)} \\
+ 2B_{(c, b_U)}^T (\Delta_{1/c})_{(c, b_U)} \xi_{(c, b_U)} + 2B_{(c, b_S)}^T (\Delta_{1/c})_{(c, b_S)} \xi_{(c, b_S)} \\
- 2C_{(a, \gamma)}^T (\Theta_c)_{(a, \gamma)} \omega_{(a, \gamma)}] \]

35
and

\[
\begin{align*}
\langle (x_c)'(\overline{y}; \Delta y), (x_c)'(\overline{y}; \Delta y) \rangle & \\
& \leq 2 \left\langle A^T D_c \Delta d_0, A_c(\overline{y}, W_1, W_2)^{-2} A^T D_c \Delta d_0 \right\rangle \\
& + 16 \left\langle B^T_{(a,b_3)}(\Delta_{1/c}(a,b_3)\xi(a,b_3), A_c(\overline{y}, W_1, W_2)^{-2} B^T_{(a,b_3)}(\Delta_{1/c}(a,b_3)\xi(a,b_3)) \right\rangle \\
& + 32 \left\langle B^T_{(a,b_L)}(\Delta_{1/c}(a,b_L)\xi(a,b_L), A_c(\overline{y}, W_1, W_2)^{-2} B^T_{(a,b_L)}(\Delta_{1/c}(a,b_L)\xi(a,b_L)) \right\rangle \\
& + 64 \left\langle B^T_{(a,c)}(\Delta_{1/c}(a,c)\xi(a,c), A_c(\overline{y}, W_1, W_2)^{-2} B^T_{(a,c)}(\Delta_{1/c}(a,c)\xi(a,c)) \right\rangle \\
& + 128 \left\langle B^T_{(c,b_U)}(\Delta_{1/c}(c,b_U)\xi(c,b_U), A_c(\overline{y}, W_1, W_2)^{-2} B^T_{(c,b_U)}(\Delta_{1/c}(c,b_U)\xi(c,b_U)) \right\rangle \\
& + 256 \left\langle B^T_{(c,b_S)}(\Delta_{1/c}(c,b_S)\xi(c,b_S), A_c(\overline{y}, W_1, W_2)^{-2} B^T_{(c,b_S)}(\Delta_{1/c}(c,b_S)\xi(c,b_S)) \right\rangle \\
& + 256 \left\langle C^T_{(\alpha,\gamma)}(\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma), A_c(\overline{y}, W_1, W_2)^{-2} C^T_{(\alpha,\gamma)}(\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma) \right\rangle.
\end{align*}
\]

From (4.25), we have for \(c \geq \overline{c} \geq (2 + \sqrt{2})c_0\) that

\[
\begin{align*}
\left\langle A^T D_c \Delta d_0, A_c(\overline{y}, W_1, W_2)^{-1} A^T D_c \Delta d_0 \right\rangle & \\
& \leq \kappa_0^{-2} c^{-2} \left\| \Delta d_0 \right\|^2 \\
& \leq \kappa_0^{-2} c^{-2} \left\| (\Delta \mu, \xi(b_U,b_U), \xi(b_S,b_S), \xi(b_L,b_L), \omega(\alpha,\alpha), \omega(\beta,\beta)) \right\|^2
\end{align*}
\]

(4.37)

Let

\[
\begin{align*}
\xi_c := (\overline{c} - b_0) D_c & \quad \xi_c := (\overline{c} - b_0) D_c \\
\overline{c} := (\overline{c} - b_0) D_c & \quad \overline{c} := (\overline{c} - b_0) D_c
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{H}_c := \begin{bmatrix}
\xi_c & 0 \\
0 & \overline{c}^{-1} \overline{y}^{-1} I_{n_3}
\end{bmatrix}, \quad \mathcal{H}_c := \begin{bmatrix}
\xi_c & 0 \\
0 & \overline{c}^{-1} \overline{y}^{-1} I_{n_3}
\end{bmatrix}.
\end{align*}
\]

We know from Lemma 4.2, (4.38), (4.7), and (4.4) that

\[
\left\langle C^T_{(\alpha,\gamma)}(\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma), A_c(\overline{y}, W_1, W_2)^{-2} C^T_{(\alpha,\gamma)}(\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma) \right\rangle
\]

\[
= \left\langle A_c(\overline{y}, W_1, W_2)^{-1/2} C^T_{(\alpha,\gamma)}(\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma), A_c(\overline{y}, W_1, W_2)^{-1/2} A_c(\overline{y}, W_1, W_2)^{-1/2} C^T_{(\alpha,\gamma)}(\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma) \right\rangle
\]

\[
\leq \left\langle A_c(\overline{y}, W_1, W_2)^{-1/2} C^T_{(\alpha,\gamma)}(\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma), A_c(\overline{y}, W_1, W_2)^{-1/2} A_c(\overline{y}, W_1, W_2)^{-1/2} C^T_{(\alpha,\gamma)}(\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma) \right\rangle
\]

\[
\leq \overline{c}^{-1} \overline{y}^{-1} \left\langle C^T_{(\alpha,\gamma)}(\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma), A_c(\overline{y}, W_1, W_2)^{-1} C^T_{(\alpha,\gamma)}(\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma) \right\rangle
\]

\[
\leq \overline{c}^{-2} \overline{y}^{-2} \left\langle C^T_{(\alpha,\gamma)}(\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma), C^T_{(\alpha,\gamma)}(\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma) \right\rangle
\]

\[
\leq \overline{c}^{-2} \overline{y}^{-2} \left\langle (\Theta_c)(\alpha,\gamma)\omega(\alpha,\gamma) \right\|^2
\]

\[
\leq \overline{c}^{-2} \overline{y}^{-2} \left( \max_{i \in \alpha, j \in \gamma} \lambda_i/\lambda_j \right)^{2} \|\omega(\alpha,\gamma)\|^2
\]
\[
\begin{align*}
&\leq \nu \sigma^{-2} \eta^{-2} \nu_0^2 (\nu_0 + c)^2 \| \omega_{(\alpha, \gamma)} \|^2 \\
&\leq \nu \sigma^{-2} \eta^{-2} \nu_0^2 c^{-2} \| \omega_{(\alpha, \gamma)} \|^2 \\
&\leq \frac{1}{256} \phi_0^2 c^{-2} (2 \| \omega_{(\alpha, \gamma)} \|^2).
\end{align*}
\]

Similarly, we obtain
\[
\begin{align*}
\left< B_{(a,b)}^{T}(\Delta_{1/c})_{(a,b)} \xi_{(a,b)}, A_{c}(\bar{\eta}, W_{1}, W_{2})^{-2} B_{(a,b)}^{T}(\Delta_{1/c})_{(a,b)} \xi_{(a,b)} \right>
&\leq \nu \sigma^{-2} \eta^{-2} \| (\Delta_{1/c})_{(a,b)} \xi_{(a,b)} \|^2 \\
&\leq \nu \sigma^{-2} \eta^{-2} \nu_1 = \left( \max_{j \in \{1, \ldots, b_{s}\}} 1 - \frac{(w_{b_{b}})_{j}}{c \lambda_{i}(X)} \right)^2 \| \xi_{(a,b)} \|^2 \\
&\leq \frac{1}{10} \phi_0^2 c^{-2} (2 \| \xi_{(a,b)} \|^2); \\
\left< B_{(a,b)}^{T}(\Delta_{1/c})_{(a,b)} \xi_{(a,b)}, A_{c}(\bar{\eta}, W_{1}, W_{2})^{-2} B_{(a,b)}^{T}(\Delta_{1/c})_{(a,b)} \xi_{(a,b)} \right>
&\leq \nu \sigma^{-2} \eta^{-2} \| (\Delta_{1/c})_{(a,b)} \xi_{(a,b)} \|^2 \\
&\leq \nu \sigma^{-2} \eta^{-2} \nu_2 = \left( \max_{i \in \{a,b\}} \frac{2}{c \lambda_{i}(X)} \right)^2 \| \xi_{(a,b)} \|^2 \\
&\leq \frac{1}{32} \phi_0^2 c^{-2} (2 \| \xi_{(a,b)} \|^2); \\
\left< B_{(c,b_{s})}^{T}(\Delta_{1/c})_{(c,b_{s})} \xi_{(c,b_{s})}, A_{c}(\bar{\eta}, W_{1}, W_{2})^{-2} B_{(c,b_{s})}^{T}(\Delta_{1/c})_{(c,b_{s})} \xi_{(c,b_{s})} \right>
&\leq \nu \sigma^{-2} \eta^{-2} \| (\Delta_{1/c})_{(c,b_{s})} \xi_{(c,b_{s})} \|^2 \\
&\leq \nu \sigma^{-2} \eta^{-2} \nu_3 = \left( \max_{j \in \{b_{u}, c \}} \frac{2}{c \lambda_{j}(X)} \right)^2 \| \xi_{(b_{u}, c)} \|^2 \\
&\leq \frac{1}{128} \phi_0^2 c^{-2} (2 \| \xi_{(b_{u}, c)} \|^2)
\end{align*}
\]
and
\[
\begin{align*}
\left< B_{(c,b_{s})}^{T}(\Delta_{1/c})_{(c,b_{s})} \xi_{(c,b_{s})}, A_{c}(\bar{\eta}, W_{1}, W_{2})^{-2} B_{(c,b_{s})}^{T}(\Delta_{1/c})_{(c,b_{s})} \xi_{(c,b_{s})} \right>
&\leq \nu \sigma^{-2} \eta^{-2} \| (\Delta_{1/c})_{(c,b_{s})} \xi_{(c,b_{s})} \|^2 \\
&\leq \nu \sigma^{-2} \eta^{-2} \nu_4 = \left( \max_{i \in \{1, \ldots, b_{s}\}, j \in \{c \}} \frac{(w_{b_{s}})_{i} + 1}{c \lambda_{j}(X)} \right)^2 \| \xi_{(b_{s}, c)} \|^2 \\
&\leq \frac{1}{256} \phi_0^2 c^{-2} (2 \| \xi_{(b_{s}, c)} \|^2).
\end{align*}
\]
Combining (4.40)-(4.44) with (4.36) and (4.37), we obtain
\[
\langle x_c'((\overline{y}, \Delta y), (x_c')((\overline{y}, \Delta y)) \leq 2 \sigma_0^2 \|\Delta y\|^2 / c^2.
\]
Thus (4.32) holds for \( \mu_0 \geq \sigma_0 \).

Now we prove (4.33) for some \( \mu_0 \geq \sigma_0 \). Let \( V(\Delta y) \in \overline{\mathcal{V}}(\Delta y) \). Then from the definition of \( \overline{\mathcal{V}}(\Delta y) \), there exist \( W_1 \in \partial_B [D\theta_c]^∗(F(\overline{x}) + \overline{y} / c) \) and \( W_2 \in \partial_B \Pi_{S^t}(T - cg(\overline{x})) \) such that
\[
V(\Delta y) = \left[ \begin{array}{c} c^{-1} W_1 DF(\overline{x}) \\ \mathcal{J} h(\overline{x}) \\ -W_2 Dg(\overline{x}) \end{array} \right] A_c(\overline{y}, W_1, W_2)^{-1} \left[ -c^{-1} DF(\overline{x})^\ast W_1(\Delta Y) - \mathcal{J} h(\overline{x})^T \Delta \mu + D g(\overline{x})^\ast W_2(\Delta \Gamma) \right] \\
+ \left( \begin{array}{c} -c^{-1} \Delta Y + c^{-2} W_1 \Delta Y \\ 0 \\ -c^{-1} \Delta \Gamma + c^{-1} W_2(\Delta \Gamma) \end{array} \right).
\]
For notational convenience, we assume that \( (W_1, W_2) \in \partial_B [D\theta_c]^∗(F(\overline{x}) + \overline{y} / c) \times \partial_B \Pi_{S^t}(T - cg(\overline{x})) \) is the same as in (4.34). After direct calculations, we obtain
\[
- \langle V(\Delta y), \Delta y \rangle = [A^T D_c \Delta d_0 + 2B^T_{(a,b,s)}(\Delta_{1/c}(a,b,s))\xi(a,b,s)] \\
+ 2B^T_{(a,b,L)}(\Delta_{1/c}(a,b,L))\xi(a,b,L) + 2B^T_{(a,c)}(\Delta_{1/c}(a,c))\xi(a,c) \\
+ 2B^T_{(c,b,l)}(\Delta_{1/c}(c,b,l))\xi(c,b,l) + 2B^T_{(c,b,s)}(\Delta_{1/c}(c,b,s))\xi(c,b,s) \\
- 2C^T_{(a,\gamma)}(\Theta_{(a,\gamma)})\omega(\alpha,\gamma)\xi(a,c) + 2B^T_{(a,b,L)}(\Delta_{1/c}(a,b,L))\xi(a,c) \\
+ 2B^T_{(a,b,L)}(\Delta_{1/c}(a,b,L))\xi(a,b,s) + 2B^T_{(a,c)}(\Delta_{1/c}(a,c))\xi(a,c) \\
+ 2B^T_{(c,b,l)}(\Delta_{1/c}(c,b,l))\xi(c,b,l) + 2B^T_{(c,b,s)}(\Delta_{1/c}(c,b,s))\xi(c,b,s) - 2C^T_{(a,\gamma)}(\Theta_{c,\gamma})\omega(\alpha,\gamma) \\
+ c^{-1}\|\Delta Y\|^2 - c^{-2} \langle W_1 \Delta Y, \Delta Y \rangle + c^{-1}\|\Delta \Gamma\|^2 - c^{-1} \langle \Delta \Gamma, W_2(\Delta \Gamma) \rangle
\]
(4.45)

Next, we estimate the lower and upper bounds of the right hand side of (4.45). By using (4.35) and Lemma 4.2 we obtain
\[
\overline{\mathcal{E}}_c \preceq A A_c(\overline{y}, W_1, W_2)^{-1}A^T \preceq \overline{\mathcal{E}}_c.
\]
Thus, for \( l_U = |b_U|/(|b_U| + 1)/2, l_L = |b_L|/(|b_L| + 1)/2 \) and \( l_\beta = |\beta|(|\beta| + 1)/2 \), we have
\[
\langle A^T D_c \Delta d_0, A_c(\overline{y}, W_1, W_2)^{-1} A^T D_c \Delta d_0 \rangle \geq \langle D_c \Delta d_0, \overline{\mathcal{E}}_c D_c \Delta d_0 \rangle \\
\geq (\sigma_0 + (c - c_0))^{-1} ||(\Delta \mu, \xi(b_0, b_0), \omega(\alpha, \alpha))||^2 \\
+ 4(\sigma_0 + 2(c - c_0))^{-1} ||(\xi(b_0, b_0), \xi(b_0, b_L), \xi(b_0, b_L))||^2 \\
+ \left( \langle \Delta_{1/c}(b_0, b_L)\xi(b_0, b_L), (\overline{\mathcal{E}}_c I_U + (c - c_0)(\Delta_{1/c}(b_0, b_L)))^{-1} (\Delta_{1/c}(b_0, b_L)\xi(b_0, b_L)) \rangle \\
+ \langle \Delta_{1/c}(b_0, b_L)\xi(b_0, b_L), (\overline{\mathcal{E}}_c I_L + (c - c_0)(\Delta_{1/c}(b_0, b_L)))^{-1} (\Delta_{1/c}(b_0, b_L)\xi(b_0, b_L)) \rangle \\
+ \langle (\Delta_{e})(\beta, \beta)\omega(\beta, \beta), (\overline{\mathcal{E}}_c I_\beta + (c - c_0)(\Delta_{e})(\beta, \beta))^{-1} (\Delta_{e})(\beta, \beta)\omega(\beta, \beta) \rangle \right)
\]

\[
\begin{align*}
&\geq (\sigma \eta + (c - c_0))^{-1} \| (\Delta \mu, \xi_{(b_s,b_s)}, \omega_{(\alpha,\alpha)}) \|^2 \\
&+ 4 (\sigma \eta + 2(c - c_0))^{-1} \| (\xi_{(b_s,b_s)}, \xi_{(b_s,b_L)}), \xi_{(b_s,b_L)}, \omega_{(\alpha,\beta)}) \|^2 \\
&+ \left( \left( \Delta_1/c \right)(b_s,b_L) \xi_{(b_s,b_L)}, (\sigma \eta I_{[b_s]} + (c - c_0)(\Delta_1/c)(b_s,b_L))^{-1} \left( \Delta_1/c \right)(b_s,b_L) \xi_{(b_s,b_L)} \right) \\
&+ \left( \left( \Delta_1/c \right)(b_s,b_L) \xi_{(b_s,b_L)}, (\sigma \eta I_{[b_s]} + (c - c_0)(\Delta_1/c)(b_s,b_L))^{-1} \left( \Delta_1/c \right)(b_s,b_L) \xi_{(b_s,b_L)} \right) \\
&+ \left( \left( \Delta_1/c \right)(b_s,b_L) \xi_{(b_s,b_L)}, (\sigma \eta I_{[b_s]} + (c - c_0)(\Theta_c)_{(\beta,\beta)} \xi_{(b_s,b_L)})^{-1} \left( \Theta_c \right)_{(\beta,\beta)} \xi_{(b_s,b_L)} \right)
\end{align*}
\]

and
\[
\begin{align*}
&\left\langle ATD_0 \Delta d_0, A_c(\mathbb{F}, W_1, W_2)A^T \right\rangle \leq \left( D_0 \Delta d_0, \mathcal{F}_c D_c \Delta d_0 \right) \\
&\leq (\sigma \eta/2 + (c - c_0))^{-1} \| (\Delta \mu, \xi_{(b_s,b_s)}, \omega_{(\alpha,\alpha)}) \|^2 \\
&+ 4 (\sigma \eta/2 + (c - c_0))^{-1} \| (\xi_{(b_s,b_s)}, \xi_{(b_s,b_L)}), \xi_{(b_s,b_L)}, \omega_{(\alpha,\beta)}) \|^2 \\
&+ \left( \left( \Delta_1/c \right)(b_s,b_L) \xi_{(b_s,b_L)}, (\sigma \eta I_{[b_s]} + (c - c_0)(\Delta_1/c)(b_s,b_L))^{-1} \left( \Delta_1/c \right)(b_s,b_L) \xi_{(b_s,b_L)} \right) \\
&+ \left( \left( \Delta_1/c \right)(b_s,b_L) \xi_{(b_s,b_L)}, (\sigma \eta I_{[b_s]} + (c - c_0)(\Delta_1/c)(b_s,b_L))^{-1} \left( \Delta_1/c \right)(b_s,b_L) \xi_{(b_s,b_L)} \right) \\
&+ \left( \left( \Delta_1/c \right)(b_s,b_L) \xi_{(b_s,b_L)}, (\sigma \eta I_{[b_s]} + (c - c_0)(\Theta_c)_{(\beta,\beta)} \xi_{(b_s,b_L)})^{-1} \left( \Theta_c \right)_{(\beta,\beta)} \xi_{(b_s,b_L)} \right)
\end{align*}
\]

By recalling that
\[
\bar{C} = CR \quad \text{and} \quad \bar{R} = R \left[ \Sigma^{-1}U^T \quad 0 \right] \quad \text{and} \]
\[
\nu \| s \|^2 \leq \max \left\{ \left\langle s, \bar{C} \bar{C}^T s \right\rangle, \left\langle s, C C^T s \right\rangle \right\} \leq \nu \| s \|^2, \quad \forall s,
\]
from Lemma 4.2, (4.38), (4.7), and (4.4) we know that
\[
\left\langle \left[ B_{(a,b_2)}(\Delta_1/c)(a,b_2) \xi_{(a,b_2)} + B_{(a,b_L)}(\Delta_1/c)(a,b_L) \xi_{(a,b_L)}, (\Delta_1/c)(a,b_2) \xi_{(a,b_2)} + B_{(a,b_L)}(\Delta_1/c)(a,b_L) \xi_{(a,b_L)} \right] \\
+ \left[ B_{(b,b_2)}(\Delta_1/c)(b_2,b_2) \xi_{(b_2,b_2)} + B_{(a,b_2)}(\Delta_1/c)(a,b_2) \xi_{(a,b_2)} - C^T_{(a,c)}(\Theta_c)_{(a,c)} \xi_{(a,c)} \right]
\right\rangle
\]
\[
\begin{align*}
&\geq \left( |B_{(a,b_s)}^T| (\Delta_1/c)(a,b_s) \xi(a,b_s) + B_{(a,b_L)}^T (\Delta_1/c)(a,b_L) \xi(a,b_L) + \bar{B}_{(a,c)}^T (\Delta_1/c)(a,c) \xi(a,c)
+ B_{(c,b_u)}^T (\Delta_1/c)(c,b_u) \xi(c,b_u) + \bar{B}_{(c,b_L)}^T (\Delta_1/c)(c,b_L) \xi(c,b_L) - \tilde{C}_{(a,c)} (\Theta_c)(a,c) \nu(a,c) \right),

&\mathcal{H}_c \left[ B_{(a,b_s)}^T (\Delta_1/c)(a,b_s) \xi(a,b_s) + B_{(a,b_L)}^T (\Delta_1/c)(a,b_L) \xi(a,b_L) + \bar{B}_{(a,c)}^T (\Delta_1/c)(a,c) \xi(a,c)
+ B_{(c,b_u)}^T (\Delta_1/c)(c,b_u) \xi(c,b_u) + \bar{B}_{(c,b_L)}^T (\Delta_1/c)(c,b_L) \xi(c,b_L) - \tilde{C}_{(a,c)} (\Theta_c)(a,c) \nu(a,c) \right]

&\geq (\overline{\sigma} \eta + 2(c - c_0))^{-1} \left( |B_{(a,b_s)}^T| (\Delta_1/c)(a,b_s) \xi(a,b_s) + B_{(a,b_L)}^T (\Delta_1/c)(a,b_L) \xi(a,b_L) + B_{(a,c)}^T (\Delta_1/c)(a,c) \xi(a,c)
+ B_{(c,b_u)}^T (\Delta_1/c)(c,b_u) \xi(c,b_u) + \bar{B}_{(c,b_L)}^T (\Delta_1/c)(c,b_L) \xi(c,b_L) - \tilde{C}_{(a,c)} (\Theta_c)(a,c) \nu(a,c) \right)

&\geq \nu (\overline{\sigma} \eta + 2(c - c_0))^{-1} \left( |B_{(a,b_s)}^T| (\Delta_1/c)(a,b_s) \xi(a,b_s) \nu(a,b_s), (\Delta_1/c)(a,b_L) \xi(a,b_L), (\Delta_1/c)(a,c) \xi(a,c),
(\Delta_1/c)(c,b_u) \nu(c,b_u), (\Delta_1/c)(c,b_L) \xi(c,b_L), (\Theta_c)(a,c) \nu(a,c) \right)
\end{align*}
\]

Similarly, we get

\[
\begin{align*}
&\left( |B_{(a,b_s)}^T| (\Delta_1/c)(a,b_s) \xi(a,b_s) + B_{(a,b_L)}^T (\Delta_1/c)(a,b_L) \xi(a,b_L) + B_{(a,c)}^T (\Delta_1/c)(a,c) \xi(a,c)
+ B_{(c,b_u)}^T (\Delta_1/c)(c,b_u) \xi(c,b_u) + B_{(c,b_L)}^T (\Delta_1/c)(c,b_L) \xi(c,b_L) - C_{(a,c)} (\Theta_c)(a,c) \nu(a,c) \right),

&\mathcal{A}_c (\overline{\eta}, W_1, W_2)^{-1} \left[ B_{(a,b_s)}^T (\Delta_1/c)(a,b_s) \xi(a,b_s) + B_{(a,b_L)}^T (\Delta_1/c)(a,b_L) \xi(a,b_L) + B_{(a,c)}^T (\Delta_1/c)(a,c) \xi(a,c)
+ B_{(c,b_u)}^T (\Delta_1/c)(c,b_u) \xi(c,b_u) + B_{(c,b_L)}^T (\Delta_1/c)(c,b_L) \xi(c,b_L) - C_{(a,c)} (\Theta_c)(a,c) \nu(a,c) \right]
\end{align*}
\]
\[
\begin{align*}
&\leq \langle \tilde{B}_{(a,b)}^T (\Delta_{1/c})(a,b) \xi(a,b) \rangle + \tilde{B}_{(a,b)}^T (\Delta_{1/c})(a,b) \xi(a,b) + \tilde{B}_{(a,c)}^T (\Delta_{1/c})(a,c) \xi(a,c) \\
&\quad + \tilde{B}_{(c,b)}^T (\Delta_{1/c})(c,b) \xi(c,b) + \tilde{B}_{(c,b)}^T (\Delta_{1/c})(c,b) \xi(c,b) - \tilde{C}_{(a,\gamma)}^T (\Theta_{(c,\gamma)} \omega(\gamma)) \\
&\quad + \tilde{B}_{(c,b)}^T (\Delta_{1/c})(c,b) \xi(c,b) + \tilde{B}_{(c,b)}^T (\Delta_{1/c})(c,b) \xi(c,b) - \tilde{C}_{(a,\gamma)}^T (\Theta_{(c,\gamma)} \omega(\gamma))
\end{align*}
\]

\[
\begin{align*}
&\leq \nu^{-1} \nu^{-1} \langle \tilde{B}_{(a,b)}^T (\Delta_{1/c})(a,b) \xi(a,b) \rangle + \tilde{B}_{(a,b)}^T (\Delta_{1/c})(a,b) \xi(a,b) \\
&\quad + \tilde{B}_{(a,c)}^T (\Delta_{1/c})(a,c) \xi(a,c) + \tilde{B}_{(a,c)}^T (\Delta_{1/c})(a,c) \xi(a,c) \\
&\quad - \tilde{C}_{(a,\gamma)}^T (\Theta_{(c,\gamma)} \omega(\gamma)) \\
&\quad + \tilde{B}_{(a,c)}^T (\Delta_{1/c})(a,c) \xi(a,c) + \tilde{B}_{(a,c)}^T (\Delta_{1/c})(a,c) \xi(a,c) \\
&\quad - \tilde{C}_{(a,\gamma)}^T (\Theta_{(c,\gamma)} \omega(\gamma))
\end{align*}
\]

\[
\begin{align*}
&\leq \nu^{-1} \nu^{-1} \left( \max_{i \in a, 1 \leq j \leq |b|} \frac{1 - \nu_{b,j}}{c \lambda_i(F(\overline{a}))} + (1 - \nu_{b,j}) \right)^2 \|\xi(a,b)\|^2 \\
&\quad + \nu^{-1} \nu^{-1} \left( \max_{i \in a} \frac{2}{c \lambda_i(F(\overline{a})) + 2} \right)^2 \|\xi(a,b)\|^2 \\
&\quad + \nu^{-1} \nu^{-1} \left( \max_{i \in a, j \in c} \frac{2}{c \lambda_i(F(\overline{a})) - \lambda_i(F(\overline{a})) + 2} \right)^2 \|\xi(a,c)\|^2 \\
&\quad + \nu^{-1} \nu^{-1} \left( \max_{i \in c} \frac{2}{-c \lambda_i(F(\overline{a})) + 2} \right)^2 \|\xi(c,b)\|^2 \\
&\quad + \nu^{-1} \nu^{-1} \left( \max_{i \in c} \frac{2}{(w_{b,j} + 1) - c \lambda_i(F(\overline{a}))} \right)^2 \|\xi(c,b)\|^2 \\
&\leq \nu^{-1} \nu^{-1} \nu_a^{2} \nu_{b, a, b} + \nu^{-1} \nu^{-1} \nu_{c, b, c} + \nu^{-1} \nu^{-1} \nu_{a, c} + \nu^{-1} \nu^{-1} \nu_{c, a} + \nu^{-1} \nu^{-1} \nu_{b, c} + \nu^{-1} \nu^{-1} \nu_{b, c} + \nu^{-1} \nu^{-1} \nu_{a, \gamma}
\end{align*}
\]
\[ +\nu \left( \max_{i \in a} -\frac{2}{c^2 \lambda_i(F(\bar{T})) + 2} \right)^2 \|\xi(a,b_L)\|^2 \\
+\nu \left( \max_{i \in a,j \in c} -\frac{2}{c^2 [\lambda_i(F(\bar{T})) - \lambda_i(F(\bar{T}))] + 2} \right)^2 \|\xi(a,c)\|^2 \\
+\nu \left( \max_{i \in c} -\frac{2}{c^2 \lambda_i(F(\bar{T})) + 2} \right)^2 \|\xi(c,b_U)\|^2 \\
+\nu \left( \max_{i \in c} \left( \frac{u_{b_S}}{(u_{b_S})_j + 1} \right)^2 \right)^2 \|\xi(c,b_S)\|^2 \]  
(4.50)

By using (4.37) and (4.50) we have
\[
\left( A^T D_c \Delta d_0, A_c(\bar{Y}, W_1, W_2) \right)^{-1} B_{(a,b_S)}(\Delta_1/c)(a,b_S) \xi(a,b_S) \\
+ B_{(a,b_L)}(\Delta_1/c)(a,b_L) \xi(a,b_L) + B_{(a,c)}(\Delta_1/c)(a,c) \xi(a,c) + B_{(c,b_U)}(\Delta_1/c)(c,b_U) \xi(c,b_U) \\
+ B_{(c,b_S)}(\Delta_1/c)(c,b_S) \xi(c,b_S) - C_{(a,b_S)}(\Theta(c)(a,b_S)) \right) \\
\leq \| A_c(\bar{Y}, W_1, W_2)^{-1} A^T D_c \Delta d_0 \| \left( B_{(a,b_S)}(\Delta_1/c)(a,b_S) \xi(a,b_S) \\
+ B_{(a,b_L)}(\Delta_1/c)(a,b_L) \xi(a,b_L) + B_{(a,c)}(\Delta_1/c)(a,c) \xi(a,c) \\
+ B_{(c,b_U)}(\Delta_1/c)(c,b_U) \xi(c,b_U) + B_{(c,b_S)}(\Delta_1/c)(c,b_S) \xi(c,b_S) - C_{(a,b_S)}(\Theta(c)(a,b_S)) \right) \\
\leq g_0 \frac{1}{\sqrt{2}} c^{-1} \left( \|\Delta \mu, \xi_{(b_S,b_U)}, \xi_{(b_S,b_L)}, \xi_{(b_L,b_U)}, \omega(a,c), \omega(\beta,\beta)\| \right)^2 \\
+ 2\|\xi_{(b_S,b_U)}, \xi_{(b_S,b_L)}, \xi_{(b_L,b_U)}, \omega(a,\beta)\|^2 \right)^{1/2} \\
\times \left( \nu_0 \sqrt{\nu} c^{-1} \right) \left( \|\xi(a,b_S), \xi(a,b_L), \xi(a,c), \xi(c,b_U), \xi(c,b_S), \omega(a,\gamma)\| \right) \right) \\
\leq g_0 \frac{\nu_0 \sqrt{\nu}}{4} c^{-2} \left( \|\Delta \mu, \xi_{(b_S,b_U)}, \xi_{(b_S,b_L)}, \xi_{(b_L,b_U)}, \omega(a,c), \omega(\beta,\beta)\| \right)^2 \\
+ 2\|\xi_{(b_S,b_U)}, \xi_{(b_S,b_L)}, \xi_{(b_L,b_U)}, \omega(a,\beta)\|^2 \\
+ 2\|\xi(a,b_S), \xi(a,b_L), \xi(a,c), \xi(c,b_U), \xi(c,b_S), \omega(a,\gamma)\|^2 \right). \]  
(4.51)

By direct calculations we have
\[
\|\Delta Y\|^2 - c^{-1} \langle W_1 \Delta Y, \Delta Y \rangle \\
= \|\xi_{(a,b)}\|^2 + 2\|\xi_{(a,b_U)}\|^2 + 2\|\xi_{(b_L,b_U)}\|^2 + 2\|\xi_{(c,b)}\|^2 \\
+ 2\|\xi_{(a,b_S)}\|^2 - \langle \xi_{(a,b_S)}, (\Delta_1/c)(a,b_S) \xi_{(a,b_S)} \rangle \]
\begin{equation}
\begin{aligned}
+2(||\xi(a,bl)||^2 - \langle \xi(a,bl), (\Delta_1/c)(a,bl)\xi(a,bl) \rangle) \\
+2(||\xi(c,bl)||^2 - \langle \xi(c,bl), (\Delta_1/c)(c,bl)\xi(c,bl) \rangle) \\
+2(||\xi(c,bu)||^2 - \langle \xi(c,bu), (\Delta_1/c)(c,bu)\xi(c,bu) \rangle) \\
+2(||\xi(c,bs)||^2 - \langle \xi(c,bs), (\Delta_1/c)(c,bs)\xi(c,bs) \rangle) \\
+||\xi(bu,bu)||^2 - \langle \xi(bu,bu), (\Delta_1/c)(bu,bu)\xi(bu,bu) \rangle \\
+||\xi(bl,bl)||^2 - \langle \xi(bl,bl), (\Delta_1/c)(bl,bl)\xi(bl,bl) \rangle)
\end{aligned}
\end{equation}

and
\begin{equation}
\begin{aligned}
||\Delta \Gamma||^2 - \langle \Delta \Gamma, W_2(\Delta \Gamma) \rangle \\
= (||\omega_{(\gamma, \gamma)}||^2 + 2||\omega_{(\beta, \gamma)}||^2) + 2(||\omega_{(\alpha, \gamma)}||^2 - \langle \omega_{(\alpha, \gamma)}, (\Theta_c)(\alpha, \gamma)\omega_{(\alpha, \gamma)} \rangle) \\
+ (||\omega_{(\beta, \beta)}||^2 - \langle \omega_{(\beta, \beta)}, (\Theta_c)(\beta, \beta)\omega_{(\beta, \beta)} \rangle).
\end{aligned}
\end{equation}

Now we are ready to estimate the lower and upper bounds of \(-\langle V(\Delta y), \Delta y \rangle\). In light of (4.45), (4.46), (4.48), (4.51), (4.52) and (4.53), we have
\begin{equation}
\begin{aligned}
-\langle V(\Delta y), \Delta y \rangle \geq \ & c^{-1}(||\xi(a,\alpha)||^2 + 2||\xi(a,bl)||^2 + 2||\xi(c,bl)||^2 + ||\xi(c,\gamma)||^2) \\
+ c^{-1}(||\omega_{(\gamma, \gamma)}||^2 + 2||\omega_{(\beta, \gamma)}||^2) \\
+ \kappa_1(c)||\Delta \mu, \xi(b_s, b_s), \omega_{(\gamma, \alpha)}||^2 + \kappa_2(c)||\xi(a,bs)||^2 \\
+ \kappa_3(c)||\xi(a,bl)||^2 + \kappa_4(c)||\xi(a,\gamma)||^2 + \kappa_5(c)||\xi(bu, bl)||^2 \\
+ \kappa_6(c)||\xi(bu, bl)||^2 + \kappa_7(c)||\xi(bu, bl)||^2 + \kappa_8(c)||\xi(bu, bl)||^2 \\
+ \kappa_9(c)||\xi(bu, bl)||^2 + \kappa_{10}(c)||\xi(bu, bl)||^2 + \kappa_{11}(c)||\xi(bl, bl)||^2 \\
+ \kappa_{12}(c)||\omega_{(\gamma, \beta)}||^2 + \kappa_{13}(c)||\omega_{(\alpha, \gamma)}||^2 + \kappa_{14}(c)||\omega_{(\beta, \beta)}||^2,
\end{aligned}
\end{equation}

where
\begin{align*}
\kappa_1(c) &:= (\sigma \bar{\mu} + (c - c_0))^{-1} - \vartheta_0 \bar{V}_0 \sqrt{\bar{V}} c^{-2} \\
\kappa_2(c) &:= \nu (\sigma \bar{\mu} + 2(c - c_0))^{-1} \bar{V}_0^2 (\nu_0 + c)^{-2} - \vartheta_0 \bar{V}_0 \sqrt{\bar{V}} c^{-2} + \frac{\min_{i \in a} \lambda_i(F(\bar{v}))}{c \min_{i \in a} \lambda_i(F(\bar{v})) + \max_{j \in c} \lambda_j(F(\bar{v}))} \\
\kappa_3(c) &:= \nu (\sigma \bar{\mu} + 2(c - c_0))^{-1} \bar{V}_0^2 (\nu_0 + c)^{-2} - \vartheta_0 \bar{V}_0 \sqrt{\bar{V}} c^{-2} + \frac{\min_{i \in a} \lambda_i(F(\bar{v}))}{c \min_{i \in a} \lambda_i(F(\bar{v})) + \max_{j \in c} \lambda_j(F(\bar{v})) + 2} \\
\kappa_4(c) &:= \nu (\sigma \bar{\mu} + 2(c - c_0))^{-1} \bar{V}_0^2 (\nu_0 + c)^{-2} - \vartheta_0 \bar{V}_0 \sqrt{\bar{V}} c^{-2} \\
&\quad + \frac{(\min_{i \in a} \lambda_i(F(\bar{v})) - \max_{j \in c} \lambda_j(F(\bar{v})))}{c (\min_{i \in a} \lambda_i(F(\bar{v})) - \max_{j \in c} \lambda_j(F(\bar{v}))) + 2} \\
\kappa_5(c) &:= 2(\sigma \bar{\mu}/2 + (c - c_0))^{-1} - \vartheta_0 \bar{V}_0 \sqrt{\bar{V}} c^{-2} \\
\kappa_6(c) &:= \kappa_4(c) \\
\kappa_7(c) &:= \kappa_4(c) \\
\kappa_8(c) &:= -\vartheta_0 \bar{V}_0 \sqrt{\bar{V}} c^{-2} + \frac{\max_{j \in c} \lambda_j(F(\bar{v}))}{\bar{V}_0 \sqrt{\bar{V}} c^{-2}} \\
\kappa_9(c) &:= \kappa_4(c) \\
\kappa_{10}(c) &:= \kappa_4(c) \\
\kappa_{11}(c) &:= \kappa_4(c) \\
\kappa_{12}(c) &:= \kappa_4(c) \\
\kappa_{13}(c) &:= \kappa_4(c) \\
\kappa_{14}(c) &:= \kappa_4(c)
\end{align*}
\( K_{10}(c) := \frac{\nu}{\sigma\eta} + 2(c - c_0) c^{-2} - \varrho_0 \nu_0 \sqrt{\nu} c^{-2} + \max_{j \in c} \frac{\lambda_j(F(x))}{\left[-c \max_{j \in c} \lambda_j(F(x)) + 2\right]}
\)
\( K_{12}(c) := K_{01}(c)
\)
\( K_{13}(c) := 2c^{-1} \left[1 - \nu_0(\nu_0 + c)^{-1}\right] + 2\nu (\sigma\eta + 2(c - c_0))^{-1} \nu_0 (\nu_0 + c)^{-2} - \varrho_0 \nu_0 \sqrt{\nu} c^{-2},
\)
and
\( K_5(c) = K_{11}(c) = K_{14}(c) := \min_{t \in [a, b]} \psi(t; c, a_c, b_c, c_0)
\)
with \( \psi(\cdot; \cdot) \) being defined as (3.28) in Lemma 3.6 and
\( a_c := c^{-1} - \varrho_0 \nu_0 \sqrt{\nu} c^{-2}, \quad b_c := \sigma\eta.
\)
It follows from (3.29) in Lemma 3.6 that for \( c \geq \sigma\eta \),
\[
K_{14}(c) = c^{-1} - \varrho_0 \nu_0 \sqrt{\nu} c^{-2} - \frac{\sigma\eta}{c(\sqrt{c} + \sqrt{c_0})^2}.
\]
Thus, there exists a positive number \( \xi_1 \) such that for \( c \geq \sigma\eta \) we have
\[
\min \left\{ \frac{1}{2} \min_{i \in (1, 5, 11, 14)} \left\{ K_i(c) \right\}, \min_{i \in (1, 5, 11, 14)} \left\{ K_i(c) \right\} \right\} \geq c^{-1} - \xi_1 c^{-2}.
\]
Therefore, from (4.54) we have
\[
-\langle V(\Delta y), \Delta y \rangle \geq (c^{-1} - \xi_1 c^{-2}) \| \Delta y \|^2.
\] (4.55)
On the other hand, in light of (4.45), (4.47), (4.49), (4.51), (4.52) and (4.53), we have
\[
-\langle V(\Delta y), \Delta y \rangle \leq c^{-1} \left\| \xi_{(a,a)} \right\|^2 + 2 \left\| \xi_{(a,b)} \right\|^2 + 2 \left\| \xi_{(b,b)} \right\|^2 + \left\| \xi_{(c,c)} \right\|^2
\]
\[
+ c^{-1} \left\| \omega_{(\gamma,\gamma)} \right\|^2 + 2 \left\| \omega_{(\beta,\beta)} \right\|^2
\]
\[
+ \sum_{j=1}^{11} \left\| \omega_{(b_j,b_j)} \right\|^2 + \sum_{j=1}^{11} \left\| \omega_{(b_j,b_j)} \right\|^2 + \sum_{j=1}^{11} \left\| \omega_{(b_j,b_j)} \right\|^2
\]
\[
+ \sum_{j=1}^{11} \left\| \omega_{(b_j,b_j)} \right\|^2 + \sum_{j=1}^{11} \left\| \omega_{(b_j,b_j)} \right\|^2 + \sum_{j=1}^{11} \left\| \omega_{(b_j,b_j)} \right\|^2
\]
\[
+ \sum_{j=1}^{11} \left\| \omega_{(b_j,b_j)} \right\|^2 + \sum_{j=1}^{11} \left\| \omega_{(b_j,b_j)} \right\|^2 + \sum_{j=1}^{11} \left\| \omega_{(b_j,b_j)} \right\|^2
\]
\[
+ \sum_{j=1}^{11} \left\| \omega_{(b_j,b_j)} \right\|^2 + \sum_{j=1}^{11} \left\| \omega_{(b_j,b_j)} \right\|^2 + \sum_{j=1}^{11} \left\| \omega_{(b_j,b_j)} \right\|^2,
\]
where
\[
\begin{align*}
\bar{\kappa}_1(c) &:= \left( \frac{\sigma\eta}{2} + (c - c_0) \right)^{-1} + \varrho_0 \nu_0 \sqrt{\nu} c^{-2} \\
\bar{\kappa}_2(c) &:= 2c^{-1} + \varrho_0 \nu_0 \sqrt{\nu} c^{-2} \\
\bar{\kappa}_3(c) &:= \bar{\kappa}_4(c) := \bar{\kappa}_2(c) \\
\bar{\kappa}_6(c) &:= 2 \left( \frac{\sigma\eta}{2} + (c - c_0) \right)^{-1} + \varrho_0 \nu_0 \sqrt{\nu} c^{-2} \\
\bar{\kappa}_7(c) &:= \bar{\kappa}_6(c) \\
\bar{\kappa}_8(c) &:= 2c^{-1} + \varrho_0 \nu_0 \sqrt{\nu} c^{-2} \\
\bar{\kappa}_9(c) &:= \bar{\kappa}_6(c) \\
\bar{\kappa}_{10}(c) &:= \varrho_0 \nu_0 \sqrt{\nu} c^{-2} \\
\bar{\kappa}_{12}(c) &:= \bar{\kappa}_6(c) \\
\bar{\kappa}_{13}(c) &:= \bar{\kappa}_2(c)
\end{align*}
\]
κ, c, η be two positive numbers obtained by µ, c, c µ, be two positive numbers obtained by 

\[ \kappa_5(c) = \kappa_{11}(c) = \kappa_{14}(c) := \max_{t \in [0,1]} \psi(t; c, a'_c, b'_c, c_0) \]

with
\[ a'_c := c^{-1} + g_0 \bar{c}^2, \quad b'_c := c\eta/2. \]

It follows from (3.30) in Lemma 3.6 that for \( c \geq \bar{c}, \)
\[ \kappa_{14}(c) = \max \{ \psi(0; c, a'_c, b'_c, c_0), \psi(1; c, a'_c, b'_c, c_0) \} \]
\[ = g_0 \bar{c}^2 + \max \{ c^{-1}, (\eta/2 + (c - c_0))^{-1} \}. \] (4.57)

Thus, there exists a positive number \( \mu_0 \geq \max \{ \xi_0, \xi_1 \} \) such that for \( c \geq \bar{c} \) we have
\[ \max \left\{ \frac{1}{2} \max_{i \in \{1,5,11,14\}} \{ \kappa_i(c) \}, \min_{i \in \{1,5,11,14\}} \{ \kappa_i(c) \} \right\} \leq c^{-1} + \mu_0 c^{-2}. \]

Therefore, from (4.56) we have
\[ \langle V(\Delta y), \Delta y \rangle \leq (c^{-1} + \mu_0 c^{-2}) \| \Delta y \|^2. \] (4.58)

By (4.55) and (4.58), noting that \( \mu_0 \geq \xi_1, \) we obtain that
\[ \mu_0 c^{-2} \| \Delta y \|^2 \geq \langle V(\Delta y) + c^{-1} \Delta y, \Delta y \rangle \geq -\mu_0 c^{-2} \| \Delta y \|^2. \]

This shows that (4.33) holds. The proof is completed. \( \square \)

Now we are ready to state our main result on the rate of convergence of the augmented Lagrangian method for nonlinear semidefinite nuclear norm composite optimization.

**Theorem 4.1** Suppose that Assumptions (sdnop-A1) and (sdnop-A2) are satisfied. Let \( c_0 \) and \( \eta \) be two positive numbers obtained by Proposition 4.1. Let \( \eta, \bar{c}, \) and \( \varrho_0 \) be defined as in (4.21), (4.30), and (4.31), respectively. Let \( \mu_0 \) be obtained by Proposition 4.2. Define
\[ \varrho_1 := 2\varrho_0 \quad \text{and} \quad \varrho_2 := 4\mu_0. \]

Then for any \( c \geq \bar{c}, \) there exist two positive numbers \( \varepsilon \) and \( \delta \) (both depending on \( c \)) such that for any \( (Y, \mu, \Gamma) \in \mathcal{B}_\delta(Y, \mu, \Gamma), \) the problem
\[ \min L_c(x; Y, \mu, \Gamma) \quad \text{s.t.} \quad x \in \mathcal{B}_\varepsilon(\Gamma) \]

has a unique solution denoted \( x_c(Y, \mu, \Gamma). \) The function \( x_c(\cdot; \cdot, \cdot) \) is locally Lipschitz continuous on \( \mathcal{B}_\delta(Y, \mu, \Gamma) \) and is semismooth at any point in \( \mathcal{B}_\delta(Y, \mu, \Gamma), \) and for any \( (\zeta, \Xi) \in \mathcal{B}_\delta(Y, \mu, \Gamma), \) we have
\[ \| x_c(Y, \mu, \Gamma) - \Xi \| \leq \varrho_1 \left\| (Y, \mu, \Gamma) - (\Xi, \mu, \Gamma) \right\| / \sqrt{c} \]
and
\[ \| (Y_c(Y, \mu, \Gamma), \mu_c(Y, \mu, \Gamma), \Gamma_c(Y, \mu, \Gamma)) - (\Xi, \mu, \Gamma) \| \leq \varrho_2 \left\| (Y, \mu, \Gamma) - (\Xi, \mu, \Gamma) \right\| / \sqrt{c}, \]

where \( Y_c(Y, \mu, \Gamma), \mu_c(Y, \mu, \Gamma) \) and \( \Gamma_c(Y, \mu, \Gamma) \) are defined as
\[ Y_c(Y, \mu, \Gamma) := D\theta_c(F(x_c(Y, \mu, \Gamma)) + Y/c)^*, \]
\[ \mu_c(Y, \mu, \Gamma) := \mu + ch(x_c(Y, \mu, \Gamma)) \quad \text{and} \]
\[ \Gamma_c(Y, \mu, \Gamma) := \Pi_{S^p_+}(\Gamma - cg(x_c(Y, \mu, \Gamma))). \]
Proof. If Assumptions (sdnop-A1) and (sdnop-A2) are satisfied, then from Propositions 4.1 and 4.2 we know that both Assumption B1 and Assumption B2 (with \( \gamma = 2 \)) made in Section 2 are satisfied. Then the conclusions in this theorem follow from Theorem 2.1.

5 Conclusions

This paper provides an analysis on the rate of convergence of the augmented Lagrangian method for solving the nonlinear semidefinite nuclear norm optimization problem. By assuming that \( K \) is a closed convex cone, and that \( D\theta_c(\cdot) \) and \( \Pi_{K^*}(\cdot) \) are semismooth everywhere, we first establish a general result on the rate of convergence of the augmented Lagrangian method for a class of general composite optimization problems. Then we apply this general result to the nonlinear semidefinite nuclear norm optimization problem under the constraint nondegeneracy condition and the strong second order sufficient condition. The methodology suggests us that we may verify Assumptions B1 and B2 to obtain the rate of convergence of the augmented Lagrange method for other optimization problems.

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