All-electrical spectroscopy of topological phases in semiconductor-superconductor heterostructures

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Semiconductors in the proximity of superconductors have been proposed to support phases hosting Majorana bound states. When the systems undergo a topological phase transition towards the Majorana phase, the spectral gap closes, then reopens, and the quasiparticle band spin polarization is inverted. We focus on two paradigmatic semiconductor-superconductor heterostructures and propose an all-electrical spectroscopic probe sensitive to the spin inversion at the topological transition. Our proposal relies on the indirect coupling of a time-dependent electric field to the electronic spin due to the strong Rashba spin-orbit coupling in the semiconductor. We analyze within linear response theory the dynamical correlation functions and demonstrate that some components of the susceptibility can be used to detect the nontrivial topological phases.

I. INTRODUCTION

There has been a growing interest in the condensed matter scientific community in exploring topological superconducting phases supporting Majorana bound states, partially motivated by the prospects of a quantum computer. Majorana bound states are quasiparticles in condensed matter theory which are their own antiparticle and possess non-Abelian statistics. They may appear unpaired as zero-energy excitations, energetically separated from the quasiparticle continuum by a superconducting gap.1 Within the restricted subspace formed by a collection of such Majorana bound states, a quantum computer would perform calculations through braiding operations.2

So far, several physical platforms have been proposed to realize Majorana bound states: topological insulator-superconductor,3 semiconductor-superconductor (SM-SC) heterostructures,4–6 or magnetic atom chains.7–9 The growing number of theoretical proposals and avenues investigated in experiments was surveyed in several recent reviews.10–16

The focus in our paper is on the, maybe, most promising condensed matter candidate for the realization of Majorana bound states: the SM-SC heterostructures.17–19

The first experimental signatures of Majorana bound states were obtained by measuring a zero-bias peak in the tunneling conductance.20 Other proposed measurements are to detect Majoranas using fractional Josephson effect,1,21–24 and in current correlations.25–28 The detection of Majorana states remains still open to debate as the signal sought from them may be due to low-energy Andreev bound states trapped in the heterostructure due to smooth confining potentials.29–35

Another fruitful alternative is to detect the topological phases in SM-SC heterostructures by indirect means, using, for example, bulk measurements. Signatures of the topological transitions have been theoretically shown to arise in the electromagnetic response of the system to weak time-dependent magnetic fields36,37 or in its entanglement spectrum.38 More recently, it was realized that at the topological phase transition the spin polarization of electronic bands is inverted, a feature that might be exploited as a reliable marker to discriminate the topological phases.39 Further studies have sought to make use of this observation to devise detection methods using the local measurement of spin in the electronic bands at the transition,40 in the generation of supercurrents,41 or using spin-selective measurements via quantum dots connected to the heterostructure.42,43 In this paper, we propose an alternative detection method which relies on all-electrical probes of the system’s bulk electronic structure, coupled with optical detection.

Our analysis carries on two SM-SC heterostructures, where the existence of Majorana phases has been proposed—a one-dimensional model4,5 and a two-dimensional one6 (see Fig. 1). Both setups aim to realize
an effective spinless \(p\)-wave superconductor with topological properties\(^1\) in the low-energy sector, near the Fermi energy. The basic ingredients are a magnetic field, which removes the Kramers degeneracy of the electronic states, while a strong spin-orbit coupling breaks the spin conservation to allow tunneling of Cooper pairs from a neighboring \(s\)-wave superconductor into the semiconductor. This induces a superconducting gap in the semiconductor, creating an effective topological superconductor. The magnetic field modifies the spectrum, acting against the induced superconducting gap, allowing to close the spectral gap for a critical magnetic field, at zero wave vector in the proximitized semiconductor. Above the critical field, the gap reopens, and the system enters a topological nontrivial phase where Majorana bound states are expected to form.

This basic physical picture readily allows one to understand the band spin inversion at the topological transition.\(^39\) At the zero wave vector, near the topological transition, the spin-orbit term is dominated by the Zeeman field which sets the band spin orientation either parallel or antiparallel to it. Due to low-energy particle-hole symmetry, opposite-energy quasiparticle bands have opposite spin polarization. Since at the topological transition the gap closes and the bands cross each other, while remaining spin polarized, the spin polarization of these bands is inverted between the trivial and nontrivial phase (see e.g. Fig. 2). This picture is limited to a region around \(k = 0\), since spin-orbit coupling acts at finite momenta to rotate the electronic spins. The challenge of the present paper is to find signatures of the electronic spin inversion at the transition.

We propose an all-electrical detection of spin polarization in the electronic bands in semiconductor-superconductor heterostructures which is capable to discriminate the phases near the topological transition. Our proposal is to use techniques similar to the electronic spin resonance (ESR) spectroscopy, where spin relaxation is measured using microwave-frequency magnetic fields.\(^44\) However, in the SM-SC heterostructures, the proximity to a superconductor renders such methods not ideal. The present all-electrical scheme relies on the indirect coupling of the electric field to the electronic spin due to the strong spin-orbit coupling present in semiconductors such as InSb or InAs, which are regularly used in building the SM-SC heterostructures. The electric fields have been shown to control the Landé \(g\)-factor in semiconductor devices\(^15\) and, moreover, time-varying electric field may be used to dynamically modulate the \(g\)-factors as a means to control quantum spins.\(^46\)–\(^53\) Moreover, a time-dependent spin-orbit coupling has been predicted to generate spin currents.\(^54\)–\(^57\)

In our proposal, the electric fields modulate the strength of the Rashba spin-orbit coupling in the material. Then, under the electric field, resonant transitions are induced between the low-energy quasiparticle bands, leading to an increase in the spin polarization in either trivial or nontrivial phases. Nevertheless, since the spin polarization is opposite in the two phases, longitudinal spin-relaxation processes near \(k = 0\) are either favored or unfavored by the external magnetic field. We show that in the topological nontrivial phase the quasi-electrons have spins aligned with the magnetic field, and therefore they relax by emitting photons, while in the trivial phase, they relax by absorbing photons. This allows the use of optical spectroscopic probes to detect the topological phases. The associated response function \(\chi(\omega)\), defined below in Eq. (8) and in particular its imaginary part \(\chi''(\omega)\), which is related to spin relaxation processes, encodes these features and distinguishes on which side of the topological transition the system is. We call such measurement Rashba modulated resonance spectroscopy,\(^47\)

The paper is organized as follows. Sec. II introduces the two Hamiltonian models for the semiconductor-superconductor heterostructures, in one (1D) and two dimensions (2D). Sec. III discusses the detection of band spin polarization using electrical modulation of the Rashba spin-orbit coupling. The section defines a response function modeling the experiment and further determined within this paper. Sec. IV analyzes the system response at vanishing chemical potential, where a complete analytical solution is available. The results are extended in the next Section V in a perturbation theory in small spin-orbit coupling near the topological transition. The perturbation theory yields also the spin-polarization of the electronic bands. Sec. VI generalizes the above results for any system parameters in the low-frequency regime, while an arbitrary frequency formula for the response function is relegated to Appendix A. Finally, Sec. VII sums up the conclusions of our study.

**II. MODELS**

In this paper we investigate two paradigmatic models, originally proposed in Refs. 4–6, as condensed matter platforms for the realization of Majorana bound states. Because of its relative simplicity, the 1D model has been the subject of intense experimental scrutiny.\(^20\),\(^58\)–\(^64\) To treat the models on equal footing, we assume in both cases that the semiconductor is deposited on a superconductor in \(xy\) plane (see Fig. 1). The proximity to the superconductor induces superconducting correlations in the semiconductor, characterized by the order parameter \(\Delta\). The semiconductors are also characterized by a strong Rashba spin-orbit coupling due to broken inversion symmetry along the \(z\) axis. Finally, the time-reversal symmetry is broken by an (effective) magnetic field which gives rise to a Zeeman spin-splitting between the electronic bands in the semiconductor. In the 1D model, the external magnetic field is applied along the semiconducting wire. In the 2D setup there is only an effective Zeeman field perpendicular to the semiconducting plane induced by a magnetic insulator placed under the semiconductor.
The effective Hamiltonian for both semiconductors reads
\[ H = \frac{1}{2} \int d^2k \Psi^\dagger(k)H(k)\Psi(k), \]
with the Nambu field operator defined as \( \Psi^\dagger(k) = (\psi_{k\uparrow}, \psi_{k\downarrow}, \psi_{-k\downarrow}, -\psi_{-k\uparrow}) \). We use the convention that
\[ k = (k_x, 0, 0), \]
\[ k = (k_x, k_y, 0), \]
(1D),
\[ k = (k_x, k_y, 0), \]
(2D).

The Bogoliubov-de Gennes Hamiltonian for the 1D model reads
\[ H(k_x) = \left( \frac{\hbar k^2}{2m} - \mu \right) \tau_z - \alpha k_x \tau_z \sigma_y + E_Z \sigma_x + \Delta \tau_x, \]
where, without the loss of generality, we choose an uniform induced order parameter \( \Delta > 0 \).

In the two-dimensional model, the order parameter \( \Delta \) has a vortex structure, and goes to zero in the middle of the annular structure shown in Fig. 1(b). Our focus is on the bulk excitation spectrum which is determined far away from the vortex, where the order parameter is assumed to have an uniform amplitude \( \Delta > 0 \). The Hamiltonian for the system under the above approximation reads
\[ H(k) = \left( \frac{\hbar^2 k^2}{2m} - \mu \right) \tau_z + H_{so} + E_Z \sigma_x + \Delta \tau_x, \]
\[ H_{so}(k) = \alpha \tau_x (\sigma_x k_y - \sigma_y k_x), \]
(2D)

In both models, \( E_Z \) denotes the Zeeman energy, \( \alpha \), the Rashba spin-orbit coupling strength, and \( \mu \) the chemical potential. The Pauli matrices \( \sigma_i \) act in spin space, while \( \tau_i \), with \( i = x, y, z \), in particle-hole space. We use the convention that \( \tau_i \sigma_j \equiv \tau_i \otimes \sigma_j \), and absence of a Pauli matrix in the Hamiltonian implies the presence of the identity matrix in the respective space.

Despite the somewhat different physical realization, the models share many attributes, allowing throughout a parallel treatment and leading to similar conclusions. Formally, the 2D model reduces to the 1D model under a rotation in spin space and confinement of electron motion along the \( x \) axis. Since the Rashba spin-orbit vector is orthogonal to the effective magnetic field, the energy spectrum is determined analytically. In both models there are two positive-energy quasiparticle bands
\[ E_\pm(k) = \left[ \xi^2 + \alpha^2 k^2 + E_Z^2 + \Delta^2 \right]^{1/2}, \]
with two negative-energy bands \( -E_\pm(k) \), with \( \xi \) denoting the kinetic energy \( \xi = \hbar^2 k^2/2m - \mu \). The band structures undergo a topological transition when the topological gap at \( k = 0 \),
\[ \Delta_{\text{top}} = E_Z - \sqrt{\Delta^2 + \mu^2}, \]
closes and reopens under variation of system parameters. The topological nontrivial phases are realized for \( \Delta_{\text{top}} > 0 \) with zero energy Majorana bound states localized either at the 1D wire edges or, in 2D model, in the superconductor vortex.

Near the topological phase transition at \( k = 0 \), the spin-orbit coupling term is dominated by the Zeeman field which polarizes the quasiparticle bands parallel or antiparallel to it. Due to particle-hole symmetry, bands with opposite energies display opposite spin polarization. A more detailed discussion of the spin polarization near the transition is presented in Sec. V.

To get a sense of the units involved, we take throughout an InSb semiconductor with the material parameters: g-factor \( \sim 50 \), effective mass \( \sim 0.015 \), \( \alpha = 20 \text{ nm} \cdot \text{meV} \), and induced superconducting gap \( \Delta = 0.25 \text{ meV} \). We investigate systems that exhibit spectral gaps on the order of \( \Delta_{\text{top}} \sim 0.05 \text{ meV} \), which puts the frequency in the range \( \omega \sim 75 \text{ GHz} \). Therefore the systems could be probed in the microwave regime. For the sake of simplicity we take throughout a similar set of parameters in the 2D model and present all energies in units of \( \Delta \).

III. RASHBA SPECTROSCOPY AND THE RESPONSE FUNCTION

To probe the system, a time-dependent electric field is generated perpendicular to the superconductor \( \delta E(t) \hat{z} \), for example, by laser pulses and microwaves exciting a voltage gate connected to the proximitized semiconductor.\, Alternately, one can imagine modulating a perpendicular electric field applied directly to the system. The electric field generates an effective in-plane magnetic field which couples with the spins in the semiconductor. This yields a time-dependent modulation of the Rashba spin-orbit coupling of the form,
\[ \delta H_{so}(t) = \tau_z (\sigma \times k) \cdot \hat{z} \delta \alpha(t) = \frac{H_{so}}{\alpha} \delta \alpha(t). \]
In general, since the spin-orbit coupling strength depends linearly on the external electric field in Rashba nanowires, its time modulation remains linear in the electric field, \( \delta \delta E(t) \). This allows us to investigate the system using perturbative approaches.

Near the topological transition, i.e. \( \Delta_{\text{top}} = 0 \), the effect of spin-orbit coupling is small and the Zeeman field polarizes the quasiparticle bands along its direction. The time-dependent perturbation creates quasiparticle excitations, which change their spin polarization. In linear response theory, the change in the polarization

\[
\delta \langle \sigma_j(t) \rangle = \int_{t'}^t \chi_{jR}(t-t') \delta \epsilon(t') \, dt', \tag{7}
\]

is measured by the susceptibility \( \chi_{jR}(t) \), with \( j \in \{x, y, z\} \). More exactly, the response function \( \chi_{jR}(t) \) measures the indirect coupling of the electric field to the electronic spin \( \sigma_j \) due to the strong Rashba coupling present in the semiconductor. The expectation values are computed in a basis of the Hamiltonian eigenstates \( H(k) |n\rangle = E_n(k) |n\rangle \), with \( n \) a band index. The spin polarization of a \( k \) state is therefore denoted \( \langle \sigma_j \rangle \equiv \langle n\langle \sigma_j |n\rangle \rangle \).

The energy provided by the electric field causes resonant optical (momentum-conserved) transitions for electrons between the quasiparticle bands, and it should be in the microwave range, according to our estimates. These transitions are detected in what we call Rashba spectroscopy, by some of the components of \( \chi_{jR}(\omega) \), in analogy to the ESR spectroscopy. The latter involves measuring spin-spin correlations functions, due to direct coupling of external ac-magnetic fields to the electronic spin. In contrast, Rashba spectroscopy measures the response caused by the coupling between the modulated Rashba spin-orbit term to the electronic spin. Therefore the dynamical long-wavelength response function reads:

\[
\chi_{jR}(t-t') = -i \theta(t-t') \langle [\sigma_j(t), \tau_z(\sigma(t') \times k) \cdot \hat{z}] \rangle, \tag{8}
\]

with \( k \) defined accordingly either to the 1D or the 2D model, and \( [\cdot, \cdot] \) is a commutator. Similar ideas, in the context of quantum dot spin control in semiconductor quantum wells, have been experimentally put forward under the name of \( g \)-tensor modulation resonance spectroscopy.\(^{47}\)

The response function is invariant under time translations, and therefore a Fourier transform yields readily its frequency-dependent expression

\[
\chi_{jR}(\omega) = \sum_{m,n} \int \frac{d^d k}{(2\pi)^d} \left[ f_m(k) - f_n(k) \right] \times (m|\sigma_j|n\rangle \langle n|H_{\text{so}}/\alpha|m\rangle \langle n|H_{\text{so}}/\alpha|k\rangle, \tag{9}
\]

with \( \delta/\Delta \to 0^+ \). The summation is over the four quasiparticle bands, and the momentum integration carries over the available momentum states. The Fermi-Dirac function is \( f_n(k) = \left[ e^{\beta E_n(k) + 1} \right]^{-1} \), with \( \beta = 1/k_BT \), the inverse temperature.

Alternatively, the dynamical correlator \( \chi_{jR}(\omega) \) is calculated within the Matsubara Green’s function formalism

\[
\chi_{jR}(i\omega) = \frac{1}{\beta} \sum_k \int \frac{d^d k}{(2\pi)^d} \text{Tr}[\sigma_j G(k, i\nu) \frac{H_{\text{so}}}{\alpha} G(k, i\omega + i\nu)], \tag{10}
\]

with \( G(k, i\nu) = 1/(i\nu - H) \), the superconducting Green’s function at fermionic Matsubara frequencies \( \nu \).

The energy provided by the electric field causes resonant optical (momentum-conserved) transitions for electrons between the quasiparticle bands, and it should be in the microwave range, according to our estimates. These transitions are detected in what we call Rashba spectroscopy, by some of the components of \( \chi''_{jR}(\omega) \) along the Zeeman field discriminate between trivial and nontrivial superconducting phases at the topological transition. According to the effective magnetic field orientation chosen in Eqs. (2) and (3), they are denoted

\[
\chi_1(\omega) \equiv \chi_{xR}(\omega) \quad (1D), \quad \chi_2(\omega) \equiv \chi_{yR}(\omega) \quad (2D). \tag{11}
\]

The other components are vanishingly small near the phase transition, for frequencies on the scale of the topological gap \( \hbar \omega \sim |\Delta_{\text{top}}| \), since in this limit, the bands are almost completely polarized by the Zeeman field.

**IV. ANALYTICAL SOLUTION FOR \( \mu = 0 \)**

To analyze the response function \( \chi_d(\omega) \), it is useful to investigate the limit \( \mu = 0 \) where closed-form solutions are possible. Later, we demonstrate that the main features captured in this limit carry over to more general choices of parameters.

Let us focus on the low-energy physics near the Fermi energy at \( E = 0 \). The lower band \( E_{-}(k) \), given in Eq. (4), displays minima both at \( k = 0 \) and \( k \approx 2ma/\hbar^2 \) (for strong spin-orbit strength). The induced superconducting correlations open a superconducting gap \( \Delta \) at finite momenta \( k \). In contrast, the spectrum at \( k = 0 \) is defined by the topological gap Eq. (5). The closing and reopening of \( \Delta_{\text{top}} \) marks a transition from the topological trivial phase \( (\Delta_{\text{top}} < 0) \) to the nontrivial phase supporting Majorana bound states \( (\Delta_{\text{top}} > 0) \). Our analysis is concerned in the parameter regime around the phase transition point, where \( |\Delta_{\text{top}}| \ll \Delta \). Under this approximation only momenta near \( k = 0 \) are relevant and quartic terms in momentum are neglected. Moreover, we work at vanishing chemical potential, \( \mu = 0 \), and therefore \( \xi \approx 0 \) and

\[
\Delta_{\text{top}} = E_Z - \Delta. \tag{12}
\]

At \( k = 0 \), the lowest energy band and its particle-hole partner are eigenstates of \( \sigma_x \) in 1D, and \( \sigma_z \) in 2D. Due to particle-hole symmetry the two bands have opposite polarization \( \langle \sigma_j \rangle \) (see Fig. 2). At the transition point the
two eigenstates cross and there is a change in the polarization of the crossing bands. This change in polarization is detected by the imaginary part of the response function \( \chi_d(\omega) \), defined in Eq. (11).

In the zero-temperature limit, the sum over the Matsubara frequencies \( \nu \) in Eq. (10) may be replaced by an integral. The susceptibility \( \chi_d(\omega) \) for both 1D and 2D models follows after performing the trace over particle-hole and spin degrees of freedom,

\[
\chi_d(\omega) = \int \frac{d^4 k d\nu}{(2\pi)^2} \frac{4\alpha k^2 \Delta_{\text{top}}}{\left[ E_{-}(k)^2 + \nu^2 \right] \left[ E_{-}(k)^2 + (\omega + \nu)^2 \right]} + (\Delta \to -\Delta),
\]

(13)

with \( E_{-}(k) = (\alpha^2 k^2 + \Delta_{\text{top}}^2)^{1/2} \). The second term in \( \chi_d(\omega) \) contributes to the imaginary part of the susceptibility only at higher frequency, equal or larger than the separation between the lowest and highest bands \( \sim 2(E_{Z} + \Delta) \). Therefore it can be neglected when probing the system at smaller frequencies, \( \hbar \omega \sim 2|\Delta_{\text{top}}| \). The first term in Eq. (13) gives the low-frequency contribution which is, as expected, proportional to \( \Delta_{\text{top}} \), and, furthermore, is changing sign at the topological transition. We note that, in contrast, the static susceptibility \( \chi_d(\omega = 0) \propto \int dw \chi''(\omega')/\omega' \), is an unreliable marker of the topological transition, since it includes the information from the high-frequency transitions.

In 1D, the low-frequency dynamical susceptibility for transitions between the low-energy bands follows after performing the integral over the Matsubara frequency \( \nu \) and the analytical continuation \( i\omega \to \omega + i\delta/\hbar \):

\[
\chi_1(\omega) = \int \frac{dk k^2}{\pi E_{-}(k)} \left[ \frac{2\alpha \Delta_{\text{top}}}{4E_{-}(k)^2 - (\hbar \omega + i\delta)^2} \right],
\]

(14)

FIG. 3. Imaginary part of the susceptibility \( \chi''(\omega) \) as a function of frequency for (a) the 1D model and (b) the 2D model. The spectral gap has the same magnitude in all cases \( |\Delta_{\text{top}}| = 0.04\Delta \), but, according to the sign of \( \Delta_{\text{top}} \), the systems may be either in a topologically trivial phase (T, blue), or a nontrivial one (NT, red). The response shows different behavior depending on the phase. The solid lines are obtained by numerical integration in Eq. (10), while the dashed lines plot the analytical results in Eqs. (15) and (17). The panels share the legend.

\[
\chi''(\omega) \propto \frac{\Delta_{\text{top}}}{\alpha^2 \hbar \omega} \sqrt{\frac{\hbar^2 \omega^2}{4} - \Delta_{\text{top}}^2} \Theta \left( \frac{\hbar^2 \omega^2}{4} - \Delta_{\text{top}}^2 \right).
\]

(15)

where \( \Theta \) is the Heaviside step function.

In 2D, an additional trivial angular integration in the 2D plane is required, which yields

\[
\chi_2(\omega) = \int_0^\infty \frac{dk k^3}{\pi E_{-}(k)} \left[ \frac{2\alpha \Delta_{\text{top}}}{4E_{-}(k)^2 - (\hbar \omega + i\delta)^2} \right].
\]

(16)

Therefore the imaginary susceptibility in 2D reads

\[
\chi''_2(\omega) = \frac{\Delta_{\text{top}}}{2\alpha^3 \hbar \omega} \left( \frac{\hbar^2 \omega^2}{4} - \Delta_{\text{top}}^2 \right) \Theta \left( \frac{\hbar^2 \omega^2}{4} - \Delta_{\text{top}}^2 \right).
\]

(17)

As expected, \( \chi''(\omega) \) is odd in frequency and, due to vanishing density of states in the spectral gap, is zero below the gap. At a threshold \( 2|\Delta_{\text{top}}| \), which is the energy gap between the lowest bands \( E_{-}(k) \), the 1D response develops a square-root dependence on the frequency, while in 2D, the susceptibility displays a linear dependence. A comparison between the analytical predictions and numerical integration of \( \chi_d \) either using Eq. (9) or (10) is presented in Fig. 3.

The response functions change sign at the topological transition, an observation that can be validated experimentally. Moreover, in experiments it is also possible to keep the frequency fixed, but to vary the Zeeman field to bring the system across the topological transition. Near the transition at \( \Delta_{\text{top}} = 0 \) the response is linear in \( \Delta_{\text{top}} \) as indicated by Eqs. (15) and (17). At larger Zeeman field, above the fixed electric-field frequency \( \omega \), the transitions between the bands are energetically unfavored, leading to a decay of the signal. The dependence of \( \chi''(\omega) \) on \( \Delta_{\text{top}} \), when increasing \( E_Z \), is displayed in Fig. 4, showing the expected sign change at the transition. We also note that with increasing frequency, additional transitions to higher
energy bands are also possible, but the low-frequency response close to the topological transition is insensitive to them.

V. PERTURBATION THEORY IN THE SPIN-ORBIT COUPLING

The results of the previous section are extended here to finite chemical potential \( \mu \) using a perturbation theory in the spin-orbit coupling strength near the topological transition. This allows an intuitive understanding of the processes modeled in the Rashba spectroscopy response function.

The perturbation theory is justified close to the topological transition at \( k = 0 \) where the spin-orbit coupling term, which is linear in momentum, is dominated by the other terms in the Hamiltonian. The kinetic term \( \sim k^2/2m \) remains neglected, since it is quadratic in momentum.

To be more specific, in this section, we focus on the 1D model, described by a simplified Hamiltonian

\[
H = H_0 + H_{so}, \\
H_0 = -\mu \tau_z + E_Z \sigma_x + \Delta \tau_x,
\]

with the Rashba spin-orbit term \( H_{so} = -\alpha k_x \tau_y \sigma_y \) as a perturbation on \( H_0 \).

Our goal is to determine \( \chi_1(\omega) \equiv \chi_{XR}(\omega) \), proving that it changes sign at the topological transition \( \Delta_{top} = 0 \), with

\[
\Delta_{top} = E_Z - \sqrt{\Delta^2 + \mu^2}. \tag{19}
\]

The response function \( (9) \) in the zero-temperature limit follows readily using the eigenstates of the Hamiltonian, determined within the perturbation theory.

Let us perform a \( \pi/2 \)-rotation around \( x \) axis in particle-hole space

\[
\tau_x \mapsto \tau_y, \quad \tau_y \mapsto -\tau_x, \tag{20}
\]

only for notational simplicity. The Hamiltonian changes accordingly, \( H \mapsto \tilde{H} \), with tilde denoting the effect of the unitary transformation.

The \( H_0 \) eigenstates are momentum independent and may be indexed as \( |\tau\sigma\rangle \) with \( \tau = \pm \) and \( \sigma = \pm \). Since \( |\tau\sigma\rangle \) are eigenstates of \( \sigma_x \), it follows immediately that correlations \( \chi_{XR} = \chi_{YR} = 0 \) and only the response along the magnetic field may be relevant. The four energy bands of either \( H_0 \) or \( \tilde{H}_0 \) are

\[
\varepsilon_{\tau\sigma} = \sigma E_Z + \tau \sqrt{\Delta^2 + \mu^2}, \tag{21}
\]

with normalized eigenstates \( |\tau\sigma\rangle \):

\[
|\pm\rangle = \frac{1}{2} \left( e^{i\theta} \otimes \left( \begin{array}{c} 1 \\ \pm 1 \end{array} \right) \right),
|\mp\rangle = \frac{1}{2} \left( e^{-i\theta} \otimes \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \right), \tag{22}
\]

and

\[
e^{i\theta} = \frac{\Delta - i\mu}{\sqrt{\Delta^2 + \mu^2}}. \tag{23}
\]

At the topological transition \( \Delta_{top} = 0 \), the bands \(|++\rangle\) and \(|+-\rangle\) cross each other. Note that in the trivial phase \( \Delta_{top} < 0 \), the “conduction” bands are \(|+\pm\rangle\), while \(|-\pm\rangle\) are “valence” bands.

Let us analyze the matrix elements in the susceptibility from Eq. (9) using the first-order perturbed eigenstates, linear in \( \alpha \),

\[
|\tau\sigma(1)\rangle = |\tau\sigma\rangle - \sum_{\tau'\sigma' \neq \tau\sigma} \langle \tau'\sigma' | \tilde{H}_{so} | \tau\sigma \rangle |\tau'\sigma'\rangle. \tag{24}
\]

To first order, the only finite matrix elements of \( \tilde{H}_{so} \) are those between the valence and the conduction bands. The modulated Rashba term \( \tilde{H}_{so}/\alpha \), which couples to the time-varying electric field, excites quasiparticles from the lower to the upper bands. Its matrix elements, are to lowest order independent on \( \alpha \),

\[
\langle +1 | -k \tau_y \sigma_y | -1 \rangle \approx \alpha k e^{i\theta} \cos \theta, \tag{25}
\]

This leads in either topological phases to an increase in the spin polarization for the upper band.

Relaxation processes are determined by the matrix elements of spins along the Zeeman field. To linear order in the spin-orbit coupling, they are given by:

\[
\langle -\pm(1) | \sigma_z | +\pm(1) \rangle \approx 0, \tag{26a}
\]

\[
\langle -1 | \sigma_z | +1 \rangle \approx -\frac{\alpha k e^{i\theta} \cos \theta}{E_Z + \sqrt{\Delta^2 + \mu^2}}, \tag{26b}
\]

\[
\langle -1 | \sigma_z | -1 \rangle \approx \frac{\alpha k e^{i\theta} \cos \theta}{\Delta_{top}}. \tag{26c}
\]

The second matrix element in Eq. (26) describes transitions between the highest and lowest energy bands (\( \varepsilon_{++} \) and \( \varepsilon_{--} \)). The corresponding transition frequency is of order \( 2(E_Z + \Delta) \), which is much larger than the topological gap \( \Delta_{top} \), and it is therefore irrelevant for our analysis. Here we focus on the last matrix element in Eq. (26), which is relevant for transitions between the two quasiparticle bands closest to the Fermi energy since the associated transition frequency is on the order of \( \hbar \omega \sim 2|\Delta_{top}| \). Crucially, the matrix element behaves as \( 1/\Delta_{top} \), so it changes sign at the topological transition. This central result shows that the relaxation processes are dependent on whether the quasiparticles in the lowest conduction band are aligned to the magnetic field, as in the topological nontrivial phase (for \( \varepsilon_{--} \)), or antialigned, as in the trivial phase (for \( \varepsilon_{++} \)). Note also that the spin-spin correlation functions, which model the conventional ESR experiments, would have in the present case a dependence on the absolute value of the spectral gap \( |\Delta_{top}| \), and therefore cannot discriminate the topological phases.

The intraband terms in the susceptibility are neglected since they are all real and do not contribute to the imaginary susceptibility \( \chi''_1(\omega) \). The transitions between highest and lowest energy bands are also neglected since they occur for higher frequencies than the ones comparable to
the topological gap. Therefore $\chi''_d(\omega)$ at low frequencies is determined only by the energy difference between the two quasiparticle bands closest to the Fermi energy. To lowest order in $\alpha$, in a second-order perturbation theory, the energy difference reads

$$\varepsilon^{(2)}_{+-} - \varepsilon^{(2)}_{-+} \simeq 2\Delta_{\text{top}} + \frac{(\alpha k \cos \theta)^2}{\Delta_{\text{top}}} + \frac{(\alpha k \sin \theta)^2}{E_Z}. \quad (27)$$

The last term in Eq. (27) may also be neglected, since it barely shifts the transition frequency due to the large value of the Zeeman energy $E_Z \gg |\Delta_{\text{top}}|$ (and it vanishes at $\mu = 0$ or $\sin \theta = 0$). Then, the energy difference reads

$$\varepsilon^{(2)}_{+-} - \varepsilon^{(2)}_{-+} \simeq 2\text{sgn}(\Delta_{\text{top}})\sqrt{|\Delta_{\text{top}}|^2 + (\alpha k \cos \theta)^2}. \quad (28)$$

Using the Eqs. (25-27) in Eq. (9) yields the susceptibility

$$\chi''_1(\omega) = \int \frac{d\alpha}{2\Delta_{\text{top}}} \delta(h\omega - 2|\Delta_{\text{top}}|) - (\omega \rightarrow -\omega). \quad (29)$$

Again, the overall dependence on the sign of $\Delta_{\text{top}}$ indicates that the susceptibility is a reliable marker for the topological transition. This result translates the fact that in the nontrivial phase excited quasi-electrons relax by emitting photons at frequencies comparable to the topological gap since they are aligned with the effective magnetic field, while in the trivial region, they absorb photons, since they are antialigned with it. Integrating over the momentum in Eq. (29) and using the definition for $\cos \theta$ from Eq. (23) we obtain

$$\chi''_1(\omega) = \text{sgn}(\omega \Delta_{\text{top}}) \sqrt{\Delta_{\text{top}}^2 + \mu^2} \frac{2\alpha^2 |\Delta|}{h|\omega| - 2|\Delta_{\text{top}}|} \times \Theta(h^2\omega^2 - 4\Delta_{\text{top}}^2). \quad (30)$$

The approximation of Eq. (28) yields an alternative result for the susceptibility

$$\chi''_1(\omega) \simeq \frac{h\omega \Delta_{\text{top}}^2 + \mu^2}{2\alpha^2 |\Delta|} \frac{h^2\omega^2}{4} - \Delta_{\text{top}}^2 \times \Theta(h^2\omega^2 - 4\Delta_{\text{top}}^2). \quad (31)$$

which reduces to the previous one at small $\alpha$ and $h\omega \sim 2|\Delta_{\text{top}}|$. The susceptibility is odd in frequency, and changes sign with the topological gap $\Delta_{\text{top}}$. A quick check shows that Eq. (31) recovers the $\mu = 0$ case from Eq. (15), near the transition, with a frequency $h\omega \sim 2|\Delta_{\text{top}}|$. Both analytical and numerical calculations show again that in 1D the susceptibility has a square-root dependence on frequency $\chi''_1(\omega) \sim \sqrt{|\omega|}$ near the topological transition [see Fig. 3(a)].

Although the complete perturbation theory of the 2D case is not performed here, a simple scaling analysis shows that $\chi''_d(\omega)$ has in general a linear dependence on frequency, similar to the $\mu = 0$ case displayed in Fig. 3(b). This behavior is understood by noticing that the matrix elements of the spin $\sigma_z$ and spin-orbit term are linear in $k$. Moreover, the susceptibility has poles at momenta $k_0 \sim \sqrt{|\omega|}$, and therefore from Eq. (9) it follows that $\chi''_d(\omega) \sim k_0^{-1} k_0^d / k_0 = k_0^d$, where the dimensional effects $(d = 1$ or $2)$ enter only from the integral measure. Then indeed, in 2D, $\chi''_d \sim k_0^d \sim |\omega|$, as in the $\mu = 0$ case.

Finally, the perturbation theory also yields the spin polarization in the two low-energy bands near the topological transition:

$$(-1)^{\mu} |\sigma_x| = -(-1)^{\mu} \simeq 1 - \frac{(\alpha k \cos \theta)^2}{4\Delta_{\text{top}}^2} - \frac{(\alpha k \sin \theta)^2}{4E_Z^2}, \quad (32)$$

which generalizes at finite $\mu$ the results in Ref. 39. Due to particle-hole symmetry, the eigenstate with opposite energy and momentum have also opposite polarization, and, since the polarization is even in $\omega$,

$$(-1)^{\mu} |\sigma_x| = -(-1)^{\mu} |\sigma_x| - (-1)^{\mu}). \quad (33)$$

The energy scale where the spin polarization first vanishes in the band sets a natural scale for the frequencies that one may use to probe the system. As larger momenta, the spin-orbit starts to dominate and reverts the polarization, such that at higher frequencies, the susceptibility may show no sign change. In the approximation that $E_Z \gg \Delta_{\text{top}}$, we use Eqs. (28) and (32) to estimate that a reasonable frequency window to probe the system is $h|\omega| \lesssim 6|\Delta_{\text{top}}|.$

A few remarks are in order. As the spectral gap in the system increases, non-linear effects distort the band structure and the band minimum is no longer guaranteed at $k = 0$. The bending of the electronic bands lowers the energy of higher momentum states (of opposite spin polarization compared to the same-band $k \sim 0$ states), thus diminishing or reversing again the spin polarization of the band in a frequency window characteristic for $2E_{-}(k = 0)$. Therefore the detection method proposed here is expected generally to work whenever the minimum of the band is at the $\Gamma$ point, with frequencies tuned near the resonance condition, or, in particular, if the system is close to transition ($E_Z, \Delta \gg |\Delta_{\text{top}}|$), with frequencies $h|\omega| \lesssim 6|\Delta_{\text{top}}|.$

VI. THE GENERAL RESPONSE FUNCTION

In this section we present an analysis of the dynamical susceptibility valid for arbitrary driving frequency and choice of material parameters. The particular limits, discussed before, are recovered from the more general expression presented here.

The response functions follows from Eq. (10). The full Matsubara Green’s function is a $4 \times 4$ matrix that can be inverted analytically to give

$$G(k, i\nu) = \frac{(i\nu + H)^2 (i\nu - H)}{[(i\nu)^2 - E_k^\pm(k)][(i\nu)^2 - E_k^\pm(k)]}. \quad (34)$$
with the energies $E_±(k)$ from Eq. (4). The susceptibility follows from Eq. (10) after performing the trace over spin and particle-hole degrees of freedom and integrating over the Matsubara frequency. The general result is quite lengthy, and it is relegated to Appendix A. Nevertheless, it is further simplified near the transition by keeping in mind that the energy $E_+(k)$ is always much larger than the spectral gap [set by $E_-(k)$], namely $E_+(k) ≫ E_-(k)$. Considering dynamics on the scale of twice the topological gap $\Delta_{\text{top}}$, allows us to neglect terms from high-frequency transitions, corresponding to $\hbar|\omega| > 2E_+$ and $\hbar|\omega| > E_+ + E_-.

In the 2D model there is an additional angular integral which, due to the rotation symmetry of the Hamiltonian, is trivial and yields $2\pi$. Therefore in both the 1D and 2D models, the response function reduces to a simple form involving a single integral over momenta. After analytical continuation $\omega \rightarrow \omega + i\delta/\hbar$, it reads

$$\chi_1(\omega) = \int \frac{dk}{2\pi} F(k), \quad \chi_2(\omega) = \int_0^\infty \frac{dkk}{2\pi} F(k),$$

$$F(k) = \frac{\alpha E_2 \Delta^2 k^2 [E_2^2 - \sqrt{\alpha^2 k^2 \xi^2 + E_2^2(\Delta^2 + \xi^2)}]}{E_-(k)[\alpha^2 k^2 \xi^2 + E_2^2(\Delta^2 + \xi^2)]} \times \left[ \frac{1}{2E_-(k) - i\omega - i\delta} + \frac{1}{2E_-(k) + i\omega + i\delta} \right].$$

The remaining integral over momentum is performed numerically, usually with $\delta/\Delta = 0.004$.

The susceptibility recovers Eqs. (14) and (16), which were obtained in the approximation $\xi \rightarrow 0$. Therefore it recovers near the topological transition the square-root scaling with frequency for $\chi_0''(\omega)$ and, the linear one, for $\chi_0''(\omega)$. A density plot for the susceptibility is shown in Fig. 5 in the $(\mu, E_2)$ parameter space, with the frequencies tuned at the resonance condition. As expected the dynamical susceptibility maps exactly the position of the topological phase transition $\Delta_{\text{top}} = 0$ and changes sign across it. This confirms that the topological nontrivial phases could be identified by measuring $\chi_0''(\omega)$.

VII. CONCLUSION

In this paper, we have studied theoretically two model systems of semiconductor-superconductor heterostructures which support Majorana bound states. We have proposed an all-electric spectroscopic method to discriminate the topological phases in such materials by exploiting the bulk spin inversion at the topological transition. Our proposal uses time-varying electric fields, which dynamically modulate the Rashba spin-orbit coupling strength of the semiconductor, and cause resonant transitions between the electronic bands. Relaxation processes are then measured by optical spectroscopy at microwave frequencies using, for example, techniques developed in electron spin resonance spectroscopy. The above protocol is modeled within linear response theory by a modified susceptibility. We have shown that its imaginary part, $\chi_0''(\omega)$, can be used to discriminate the topological phases, since spin relaxation processes depend on the sign of the topological gap. Such measurements may be used to detect the topological nontrivial phases without the need to access information about the localized Majorana modes hosted in them.

ACKNOWLEDGMENTS

We thank P. Szumniak for drawing our attention to Ref. 39 on the subject of bulk spin polarization in the Rashba nanowire-superconductor heterostructures. D. S. was supported by the The Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, through the project No. PN-III-P1-1.2-PCCDI-2017-0338, and, additionally, for a mobility, through the project No. 32PFE/19.10.2018. B. D. acknowledges support from the National Research, Development and Innovation Office - NKFIH within the Quantum Technology National Excellence Program (Project No. 2017-1.2.1-NKP-2017-00001), K119442.

Appendix A: The high-frequency response function

Eq. (35) contains the general low-frequency Rashba response function for both 1D and 2D models, for arbitrary choice of material parameters. Here we present the susceptibility containing the information about transitions between all quasiparticle bands, at arbitrary frequency. After tracing out the particle-hole and spin degrees of freedom and performing the integral over the Matsubara frequencies in Eq. (10), it reads
After analytical continuation of $\omega$, we have checked that the above expression gives the same results as those of

$$\chi_1(\omega) = \int \frac{dk}{2\pi} N(k)$$

$$\chi_2(\omega) = \int_0^\infty \frac{d(k^2)}{2\pi} D(k),$$

$$N(k) = 8\alpha E_0 k^2 \left\{ E_0^2 E_+ (4E_0^2 + 3\hbar^2 \omega^2) + E_0^2 \left[ 4(E_0^2 + \alpha^4 k^4 - 2\Delta^2 \alpha^2 k^2 - 3\Delta^4 - 2(2\alpha^2 k^2 + \Delta^2) \xi^2 + \xi^4 

+ 2E_0^2(3\alpha^4 k^2 - \Delta^2 - 3\xi^2) + 2E_0^2(\alpha^2 k^2 + \Delta^2 - \xi^2) - [E_0^4 + 4(\Delta^2 + \xi^2 - 2\alpha^2 k^2 - E_0^2)]\hbar^2 \omega^2 \right] + E_0 E_+ \left[ 12E_0^2 + \alpha^4 k^4 - 3\Delta^4 - 2\Delta^2 \xi^2 + \xi^4 + 2E_0^2(\alpha^2 k^2 + \Delta^2 - \xi^2) - 2\alpha^2 k^2(\Delta^2 + \xi^2) \right] + [E_0^2 + 6(E_0^2 + \alpha^2 k^2 - \Delta^2 + \xi^2)]\hbar^2 \omega^2 + h^2 \omega^4 \right\} + \left[ 4E_0^4 + h^2 \omega^4 \right] \right\} \right\} \right\} \right\}

$$D(k) = E_0 E_+ (E_0^2 + h^2 \omega^2)(4E_0^2 + h^2 \omega^2)(4E_0^2 + h^2 \omega^2)(E_0^2 + E_0^2 + h^2 \omega^2)$$

with $k$ changing accordingly in the 1D and 2D system.
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