ASYMPTOTICS FOR SOLUTION TO THE CAUCHY PROBLEM FOR VOLterra LATTICE WITH STEP-LIKE INITIAL VALUES

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Abstract
The connection between modulated Riemann surface of genus one and solution to Volterra lattice that tends to constants at infinity is studied. The main term of asymptotics for large time of solution to the mentioned Cauchy problem is written out.

1 Introduction

The Volterra lattice

\[ D_t c_n = c_n(c_{n+1} - c_{n-1}), \quad c_n = c_n(t), \quad n \in \mathbb{Z} \quad (1.1) \]

is known as an interesting example of integrable difference-differential model with various applications to plasma physics, crystallography, zoology. The Inverse Scattering Data Method (ISDM) allows to investigate in details Cauchy problem for the Volterra lattice in case of quickly decreasing initial data (see [1]) and in periodical case ([2], [3]). The further interest is stimulated by problems with more complex behavior of solutions to (1.1) as \( n \to \pm \infty \).

Examine the Cauchy problem for eq (1.1) with initial data (i.d.) \( c_n(0) \) which quickly tend to constant values as \( n \to \pm \infty \):
The goal of this paper is to construct asymptotics as $t \to \infty$ for solution of problem (1.1)-(1.2). Some similar problems were studied for Korteweg-de Vries equation (KdV) ([4]), Nonlinear Schrodinger equation ([5]), Modified KdV ([6]). The problem (1.1)-(1.2) was posed in [7] where the author formulated hypothesis to be proved lower.

The object of study of the next paragraph is a linear equation associated with Volterra lattice together with appropriate initial data for using the ISDM. In the third paragraph we formulate Riemann problem corresponding to initial conditions (1.2) as $u^\pm = v^\pm = c^\pm$. The last restriction makes calculations much less tedious whereas all the results can be easily expanded to the general case $u^\pm \neq v^\pm$.

In the fourth paragraph we describe one-gap solutions to the Volterra lattice, modulated Riemann surface and its connection with the one-gap solutions.

The last paragraph contains asymptotic as $t \to \infty$ investigation of the formulated above Riemann problem. We prove a statement on representation of the main term of asymptotic solution to the Cauchy problem (1.1)-(1.2) by modulated one-gap solution to the Volterra lattice.

## 2 Inverse Scattering Data Method

Examine a linear problem

\[
\begin{align*}
c_{n-1}^{1/2} \psi_{n-1} + c_n^{1/2} \psi_{n+1} &= \lambda \psi_n, \quad c_n \to c_\pm, \quad n \to \pm \infty, \quad c_+ > c_- > 0, \quad (2.1)
\end{align*}
\]

$c_n$ are solutions to Volterra lattice, $\lambda$ is a spectral parameter. Define Riemann surface of genus 0 $\Gamma_\pm(z_\pm, \lambda)$: \(2\lambda c_\pm^{-1/2} = z_\pm + z_\pm^{-1}\). Involution $\sigma(z_\pm) = z_\pm^{-1}$ near circle $|z| = 1$ induces change of sheet of the surface $\Gamma_\pm$ near the cut $[0, c_\pm] = E_\pm$.

Fix solutions of problem (2.1) by asymptotics

\[
\psi_n^\pm(\lambda) \sim e_n^\pm(z_\pm) = z_\pm^{n\pm} \exp \left( \frac{1}{8} \frac{t}{c_\pm} \left( z_\pm^2 - z_\pm^{-2} \right) \right), \quad n \to \pm \infty, \quad (2.2)
\]
\( \psi_n^+(P) \) is analytic on the lower sheet \( \Gamma_{-} \setminus \infty^- \) while \( \psi_n^-(P) \) is analytic on the upper sheet \( \Gamma_{+} \setminus \infty^+ \). In the spectral domain \( E_+ \) there is a scattering correlation

\[
\psi_n^+(P) = a(P)\psi_n^-(P) + b(P)\psi_n^-(\sigma P), \; P \in E_+.
\] (2.3)

Quantities \( a(P), b(P) \) are connected by condition

\[
a(P)a(\sigma P) - b(P)b(\sigma P) = \frac{c_+^{1/2}W(\psi_n^+(P), \psi_n^+(\sigma P))}{c_+^{1/2}W(\psi_n^-(P), \psi_n^-(\sigma P))} = -\frac{z_+ - z_+^{-1}}{z_- - z_-^{-1}},
\] (2.4)

\( W(\psi_n, \phi_n) = c_+^{1/2}(\psi_n \phi_{n+1} - \psi_{n+1} \phi_n) \) is a Wronskian. There is also equality

\[
a(P) = \frac{W(\psi_n^+(P), \psi_n^-(\sigma P))}{c_-^{1/2}(z_- - z_-^{-1})}, \quad b(P) = \frac{W(\psi_n^+(P), \psi_n^-(P))}{c_-^{1/2}(z_- - z_-^{-1})},
\] (2.5)

which signifies that function \( a(P) \) can be analytically expanded to the lower sheet \( \Gamma_- \).

We suppose the spectrum to be solitonless, i.e. \( a(P) \) has no zeroes on \( \Gamma_- \). One easily verifies that under \( P \in E_* = E_+ \setminus E_- \) the condition \( a(P) = b(\sigma P) \) holds whence (as well as from (2.4)) one obtains:

\[
r(P)r(\sigma P) = 1, \; P \in E_*,
\] (2.6)

where \( r(P) = b(P)/a(P) \) is reflection coefficient.

### 3 Riemann problem.

There is a general scheme for investigation of integrable systems via solving the inverse scattering problem in the form of matrix Riemann problem (MRP).

1. Let \( \Psi_n(P) \) be \( 2\times2 \)-matrix piecewise analytic in \( \mathbb{C} \) function with asymptotics under \( P \to \infty_{\pm} \).
\[
\Psi_n(P) = A_\pm \left( I + \lambda^{-1}V^\pm_n + O(\lambda^{-2}) \right) \begin{pmatrix}
\left( \lambda c_{\pm}^{-1/2} \right)^{\mp n} & 0 \\
0 & \left( \lambda c_{\pm}^{-1/2} \right)^{\pm n}
\end{pmatrix}, \quad \lambda \to \pm \infty
\]

where

\[
A_\pm = \begin{pmatrix} 0 & 1 \\ \frac{1}{\sqrt{c_{\pm}^{-1}}} & 0 \end{pmatrix}.
\]

2. Let function \( \Psi_n(P) \) be analytic outside of the spectrum \( E_+ \) and on cut \( E_+ \) have a jump

\[
\Psi_n(P - i0) = \Psi_n(P + i0)G(P), \quad P \in E_+, \text{where}
\]

\[
G(P) = -\begin{pmatrix}
\frac{r^-(P)}{r(P)r(\sigma P)^{-1}} & \Delta^- \\
\Delta^+ & r^-(\sigma P)
\end{pmatrix}, \quad \Delta = c_\pm^{1/2}(z_+ - z_-^{-1}),
\]

the upper index \((-\)) signifies value on lower side of cut \( E_+ \). It follows from (2.6) that matrix \( G(P) \) becomes triangular in spectral branch \( E_+ \).

3. Non-degenerate matrix \( \Psi_n(P) \) is a regular function in \( C \).

Statement 3.1. Solution \( c_n \) of Cauchy problem (1.1) - (1.2) for Volterra lattice is determined by solution \( \Psi_n(P) \) of formulated above MRP (3.1) - (3.2) via the following formula:

\[
c_n = \left[ \frac{(V_n^+)^{21}}{(V_n^+_1)^{21}} \right]^2, \quad (3.4)
\]

where \( V^+ \) is matrix coefficient of expansion (3.1).

Proof. One must just verify that exact solution to MRP (3.1) - (3.2) is given by matrix

\[
\Psi_n(P) = \begin{pmatrix}
\psi^+_n(P) & \psi^-_n(\sigma P) \\
\psi^+_n(\sigma P) & \psi^-_n(P)
\end{pmatrix} \begin{pmatrix}
W^{-1}(\psi^+_n(P), \psi^-_n(\sigma P)) & 0 \\
0 & 1
\end{pmatrix},
\]

(3.5)
Statement 3.2. Solution of the MRP (3.1) - (3.2) is unique.

Proof. Let Ψ and ˜Ψ be two solutions of MRP (3.1) - (3.2). According to Liouville theorem the matrix Ψ(P) ˜Ψ⁻¹(P) does not depend on P. Thus, the quantity V₂¹ is derived from the MRP solution uniquely.

Investigation of solution to Volterra lattice as \( t \to \infty \) implies asymptotic analysis of MRP (3.1) - (3.2).

4 One-gap solutions of Volterra lattice and modulation equations

Exact formulas for real solutions to Volterra lattice were written out in paper [3]:

\[
c_{2n} = u_n = u(\tau) = \zeta(2\omega \tau) - \zeta(2\omega \tau - a_-) - \zeta(a_+ - a_-),
\]
\[
c_{2n+1} = v_n = v(\tau) = \zeta(2\omega \tau - a_+) - \zeta(2\omega \tau) + \zeta(a_-) - \zeta(a_+ - a_-),
\]

(4.1)

where \( \tau = \frac{1}{2}(a_- - a_+)n + t + \omega' \); \( \zeta(x) \) is Weierstrass zeta-function:

\[
\zeta'(x) = -\varphi(x), \ x = \int_{\infty}^{\varphi(x)} \frac{d\nu}{\sqrt{4\nu^3 - g_2 \nu - g_3}}; \varphi(x + 2\omega) = \varphi(x); \varphi(x + 2\omega') = \varphi(x); a_\pm, \omega, i\omega' \) are four real parameters completely determining formulas (4.1). Define the following four quantities:

\[
r_j = 2\zeta\left(\frac{1}{2}(a_- - a_+) + \omega_j\right) - 2\eta_j + \zeta(a_+) - \zeta(a_-), \ j = 1, 2, 3, 4,
\]

(4.2)

where \( \omega_1 = 0, \omega_2 = \omega, \omega_3 = \omega + \omega', \omega_4 = \omega' \); \( \eta_j = \zeta(\omega_j) \).

The four parameters \( \overline{\varphi} \) depend on variables \( x \in \mathbb{R}, t : \overline{\varphi} = \varphi(x, t) \).

Definition. On Riemann surface \( \overline{\Gamma}(w, \lambda) : w^2 = R_4(\lambda) = (\lambda - r_1)(\lambda - r_2)(\lambda - r_3)(\lambda - r_4) \) we define a couple of Abel differentials \( \Omega_0, \Omega_1 \):

\[
\Omega_0 = \frac{1}{2}(\lambda + d_0)w^{-1}d\lambda, \quad \Omega_1 = \frac{1}{2}\left(\lambda^2 - \frac{1}{2}\Lambda \lambda + d_1\right)w^{-1}d\lambda,
\]

(4.3)

where \( \Lambda = r_1 + r_2 + r_3 + r_4 \); \( d_0, d_1 \) are constants fixed by conditions
\[
\int_{r_2}^{r_3} \Omega_0 = \int_{r_2}^{r_3} \Omega_1 = 0 \quad (4.4)
\]

Equation

\[
D_t \Omega_0 = D_x \Omega_1 \quad (4.5)
\]

is called Whitham equation or modulation equation.

Formula (4.5) can be rewritten in the form of system of quasilinear differential equations for which the quantities \(r_j, j = 1, 2, 3, 4\) are Riemann invariants:

\[
D_t r_j = W_j(\eta)D_x r_j, \quad j = 1, 2, 3, 4, \quad (4.6)
\]

where \(W_j(\eta) = \Omega_1/\Omega_0(r_j)\). In paper [7] characteristic velocities \(W_j\) were written out explicitly in terms of complete elliptic integrals. In the same paper self-similar solutions to system (4.6) were computed.

Here we are interested in solutions that depend only on unique variable \(\xi = x/t : \eta = \eta(\xi)\).

Define Riemann surface \(\Gamma(w, \lambda; \xi)\) in the following way:

\[
\Gamma(w, \lambda; \xi) = \begin{cases} 
  w^2 = (\lambda - r_1)(\lambda - r_2^-), & \xi \leq \xi^-, \\
  w^2 = (\lambda - r_1)(\lambda - r_2^-(\xi))(\lambda - r_4), & \xi^- < \xi \leq \xi_0^-, \\
  w^2 = (\lambda - r_1)(\lambda - r_2^{-}(\xi))(\lambda - r_4)(\lambda - r_4), & \xi_0^- < \xi \leq \xi^+, \\
  w^2 = (\lambda - r_3^+)(\lambda - r_4), & \xi > \xi^+,
\end{cases} \quad (4.7)
\]

where \(r_1 \leq r_2^- (\xi) \leq r_3^+ \leq r_4^0 (\xi) \leq r_2^- \leq r_3^- (\xi) \leq r_4; r_1, r_3^+, r_2^-, r_4\) are constants,

\[
\xi^- = -\frac{[(r_4 - r_1)(r_4 - r_2^-)]^{1/2}}{\log \left(\frac{\sqrt{r_4 - r_1} + \sqrt{r_4 - r_2^-}}{\sqrt{r_4 - r_1} - \sqrt{r_4 - r_2^-}}\right)},
\]

\[
\xi^+ = \frac{1}{2}(r_3^+ + r_4 - 2r_1) - 2\left(\frac{r_3^+ - r_1}{r_4 + r_3^+ - 2r_1}\right) + \frac{r_3^+ - r_1}{\log \left(\frac{1}{2}\frac{r_4 - r_3^+}{r_4 - r_1}\right)},
\]

\[
\xi_0 = \frac{1}{2}(r_1 + r_4 - r_2^-), \quad \xi_0^+ = \frac{1}{2}(r_1 + r_4) - r_3^+.
\]
Dependence of quantities $r_3^-(\xi), r_2^0(\xi), r_2^+(\xi)$ on variable $\xi$ is determined by self-similar solutions to system (4.5):

$$
\begin{align*}
\xi + W_3(r_1, r_2^-, r_3^-(\xi), r_4) &= 0, \quad \xi^- < \xi \leq \xi_0^- , \\
\xi + W_3(r_1, r_2^0(\xi), r_2^0(\xi), r_4) &= 0, \quad \xi_0^- < \xi \leq \xi_0^+ , \\
\xi + W_2(r_1, r_2^+(\xi), r_3^+(\xi), r_4) &= 0, \quad \xi_0^+ < \xi \leq \xi^+ .
\end{align*}
$$

(4.8)

Quantities $\mathbf{r} = (r_1, r_2, r_3, r_4)$ as functions of variable $\xi$ are depicted in Fig.1.

The Riemann curve combined in such a way is a generalization of that obtained in paper [7]. Define basic cycles as it is shown in Fig.2.

Holomorphic differential $\Omega = Dw^{-1}d\lambda$, $D$ is a constant fixed by condition $\oint_a \Omega = 1$.

Examine the following ansatz for solutions of Volterra lattice:

$$
\begin{pmatrix}
u(\tau) \\
u(\tau)
\end{pmatrix} = \tau(\tau | \Gamma(w, \lambda; \xi))_{x=n} + O(t^{-\delta}), \quad \delta > 0 ,
$$

$$
\tau = \begin{cases}
\begin{pmatrix} u \\ v \end{pmatrix}^\pm, & \xi > (<) \xi^\pm \\
\begin{pmatrix} u(\tau, \tau(\xi)) \\ v(\tau, \tau(\xi)) \end{pmatrix}, & \xi^- \leq \xi \leq \xi^+, 
\end{cases}
$$

$\tau(\xi)$ are branch points of curve $\Gamma(w, \lambda; \xi)$. Quantities $u^\pm, v^\pm, \tau$ are connected in the following way:

$$
4r_1 = u^- + v^- - 2\sqrt{u^-v^-}, \quad 4r_2^- = u^- + v^- + 2\sqrt{u^-v^-}, \\
4r_3^+ = u^+ + v^+ - 2\sqrt{u^+v^+}, \quad 4r_4 = u^+ + v^+ + 2\sqrt{u^+v^+},
$$

(4.10)

whereas one suppose that $r_1 \leq r_3^+$.

Investigations in this and the next paragraphs deal with case $u^\pm = v^\pm = c^\pm$, $r_1 = r_3^+ = 0$, $a_+ = -a_-$ which does not change principal scheme but considerably simplifies calculations.

5 Asymptotic solving the Riemann problem.

In paper [9] explicit formulas for the case of one-gap spectrum of linear problem [2.1] were obtained. These formulas represent Baker-Akhiezer
function in terms of elliptic functions:

\[
\tilde{e}_n^\pm(t, z) = \gamma_n^\pm \exp \left[ \pm \pi i \left( tV(z) + nP(z) \right) \right] \frac{\theta(\omega'\tau \pm z + P)}{\theta(\pm z + P)\theta(\omega'\tau - a + P)},
\]

(5.1)

where \( a = a_+ = -a_-, P = \omega + a, \theta(\tau) = \sigma(\tau + \omega) \exp \left( -\tau \eta - \frac{\tau^2 \eta'}{2\sigma} \right), \tau = an + Vt, \frac{\sigma'(\tau)}{\sigma(\tau)} = \zeta(\tau) \); variable \( z \) is connected with parameter \( \lambda \) via uniformization

\[
\lambda(z) = \zeta(z - a) - \zeta(z + a) + 2\zeta(a)
\]

\[
V(z) = \int_{r_1}^{\lambda(z)} \Omega_1, \quad P(z) = \int_{r_1}^{\lambda(z)} \Omega_0,
\]

(5.2)

\[
\gamma_n = \beta^n \left[ \frac{\sigma(\omega'\tau + \omega)(2\omega' + \omega)\sigma(\omega)}{\sigma(\omega'\tau + 2a + \omega)\sigma(\omega)} \right]^{1/2}, \quad \beta = \sigma(2\omega' + \omega)/V
\]

Examine function

\[
\Phi_n(P) = \begin{cases} 
    a^{-1}(P)\psi^+_n(P), & P \in \Gamma^- \\
    \psi^-_n(P), & P \in \Gamma^+.
\end{cases}
\]

(5.3)

It is analytic on each sheet of the curve and has a jump on contour \( \partial\Gamma^+ = E_+ \) equal to \( r(P)\psi^-_n(\sigma P) \). So the function \( \Phi_n(P) \) can be restored by formula

\[
\Phi_n(P) = e^-_n(P) + \frac{1}{2\pi i} \int_{\partial\Gamma^+} M(P, Q) r(Q)\psi^-_n(\sigma Q),
\]

(5.4)

\( M(P, Q) \) is Cauchy kernel:

\[
M(P, Q) = \frac{W(e^-_n(P), e^-_n(\sigma Q))}{W(e^-_n(Q), e^-_n(\sigma Q))} \frac{d\lambda(Q)}{\lambda(P) - \lambda(Q)}.
\]

(5.5)

Suppose now that \( \xi \in \left[ \xi^-, \xi_0 \right] \) and dynamics of the branch point \( r^-_3(\xi) \) is unknown. Then function \( \Phi_n(P) \) is specified by explicit formula via Cauchy integral on curve \( \tilde{\Gamma} \):

\[
\Phi_n(P) = e^-_n(P) + \frac{1}{2\pi i} \int_{\partial\Gamma^+ \cup L} \tilde{M}(P, Q)\tilde{f}(Q),
\]

(5.6)
where $\partial \tilde{\Gamma}^+ = \left[ r_1, r_2 \right] \cup \left[ r_3^-(\xi), r_4 \right] ; \quad L = \left[ r_2^2, r_3^- (\xi) \right] \cup \sigma \left( \left[ r_3^- (\xi), r_2^2 \right] \right)$. $\sigma$ is an involution that changes sheet on $\tilde{\Gamma}$,

$$
\tilde{M}(P, Q) = \frac{W(\tilde{e}_n^- (P), \tilde{e}_n^- (\sigma Q))}{W(\tilde{e}_n (Q), \tilde{e}_n^- (\sigma Q))} \frac{d\lambda(Q)}{\lambda(P) - \lambda(Q)} \quad (5.7)
$$

Choice of function $\tilde{f}$ is inspired by analogy with Zakharov-Manakov ansatz from paper [9]:

$$
\tilde{\psi}_n^- (P) = A(P)\tilde{e}_n^- (P) + B(P)\tilde{e}_n^- (\sigma P). \quad (5.8)
$$

Substitute this ansatz into the linear equation and get:

$$
A(P)\tilde{e}_n^- (P) + B(P)\tilde{e}_n^- (\sigma P) = 
\tilde{e}_n^- (P) + \frac{1}{2\pi i} \int_{\partial \tilde{\Gamma}^+ \cup L} \tilde{M}(P, Q) r(Q) [A(\sigma Q)\tilde{e}_n^- (\sigma Q) + B(\sigma Q)\tilde{e}_n^- (Q)] \quad (5.9)
$$

Examine equation (5.9) as $n \to \infty$ and use the known formula ($t \to \infty$)

$$
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\eta)}{\eta - \mu + i0} \exp(it\gamma(\eta)) d\eta = -f(\mu) \exp(it\gamma(\mu)) H(-\gamma'(\mu)) + O \left( \frac{1}{\sqrt{t(\mu - \mu_{CT})}} \right),
$$

where $\gamma'(\mu_{CT}) = 0$, $H(x) = (1 + \text{sign}x)/2$ is Heaviside function. The role of $\gamma(\mu)$ is played by differential $\Omega_0(P)\xi - \Omega_1(P)$.

Therefore one obtains:

$$
B(P) = -r(P)A(\sigma P)H(\lambda(P) - \lambda(P_{CT})), \quad P \in \partial \tilde{\Gamma}^+, \quad (5.10)
$$

where point $P_{CT}$ is determined by condition

$$
\Omega_0(P_{CT})\xi = \Omega_1(P_{CT}). \quad (5.11)
$$

Examine now (5.9) as $n \to -\infty$ and get integral equations on $A(P)$:

$$
A(P)e_n^- (P) = e_n^- (P) + \frac{1}{2\pi i} \int_{\partial \tilde{\Gamma}^+ \cup L} \tilde{M}(P, Q) r(Q) r(\sigma Q) A(Q)e_n^- (Q)H(\lambda(P) - \lambda(P_{CT})), \quad P \in \partial \tilde{\Gamma}^+, \quad (5.12)
$$

$$
A(P)e_n^- (P) = e_n^- (P) + \frac{1}{2\pi i} \int_{\partial \tilde{\Gamma}^+ \cup L} \tilde{M}(P, \sigma Q) r(\sigma Q) A(Q)e_n^- (Q), \quad P \in L \quad (5.13)
$$
Thus the function \( \tilde{f} \) in (5.6) must look in the following way:

\[
\tilde{f}(P) = \begin{cases} 
-r(P) [A(\sigma P)\tilde{e}_n(\sigma P) + B(\sigma P)\tilde{e}_n^-(P)], & P \in \partial \tilde{\Gamma}^+, \\
(1 + r^+(\sigma P))A(P)\tilde{e}_n^+(P), & P \in \left[r_2^-, r_3^-(\xi)\right], \\
(1 - r^+(P))A(P)\tilde{e}_n^-(P), & P \in \sigma(\left[r_2^-, r_3^-(\xi)\right]).
\end{cases}
\]

The quantity \( A(P) \) should be searched from eqs (5.12) - (5.13). They indicate that function \( A(P) \) is analytic on sheets \( \Gamma_\pm \) and has a jump on contour \( \partial \tilde{\Gamma}^+ \cup L \):

\[ A^-_n(P) = A^+_n(P)g(P), \quad P \in \partial \tilde{\Gamma}^+ \cup L, \quad (5.15) \]

where conjugation matrix \( g(P) \) has the form:

\[
g(P) = \begin{cases} 
1 - r(P)r(\sigma P)H(\lambda(P) - \lambda(P_{CT})), & P \in \partial \tilde{\Gamma}^+, \\
-r^+(\sigma P), & P \in \left[r_2^-, r_3^-(\xi)\right], \\
r^+(P), & P \in \sigma(\left[r_2^-, r_3^-(\xi)\right]).
\end{cases}
\]

Solution to such a problem is given by the following formula:

\[ A(P) = R(P)\exp\left(-\frac{1}{2\pi i} \int_{\partial \tilde{\Gamma}^+ \cup L} \tilde{M}(P,Q) \log g(Q)\right), \quad (5.17) \]

where \( \tilde{M}(P,Q) \) is an analogue of Cauchy kernel:

\[ \tilde{M}(P,Q) = \int_{P}^{P'} m(P',Q), \quad (5.18) \]

\( m(P,Q) \) is a meromorphic bidifferential fixed by conditions

\[ \oint m(P,Q) = 0, \quad m(P,Q) \sim \frac{dq(P) dq(Q)}{(q(P) - q(Q))^2}, \quad P \sim Q, \quad (5.19) \]

\( q \) is a local parameter. It ensues from norm conditions (5.19) that \( \oint m(P,Q) = 2\pi i \Omega(P) \).

Multiplier \( R(P) \) is specified by condition of uniqueness of \( A(P) \) and asymptotics \( A(P) \to e_\pm^{1/2}, \quad P \to \infty_\pm \):
\[ R(P) = \text{const} \frac{\theta \left( \int_{P_{\infty}^+}^P \Omega + D \right)}{\theta \left( \int_{P_{\infty}^+}^P \Omega + D_0 \right)}, \]

(5.20)

where \( D_0 = - \left( \int_{P_{\infty}^+}^{r_{1}^+} + \int_{P_{\infty}^+}^{r_{4}^+} \right) \Omega - K, \) \( K \) is a vector of Riemann constants (see [3]).

The conjugation conditions (5.16) provide a formula for phase shift:

\[ D = D_0 - \frac{1}{2\pi i} \int_{\partial \tilde{\Gamma}^+ \cup L} \Omega(Q) \log g(Q). \]

(5.21)

So we constructed function \( \Psi_n(P) \) (3.5) which in case \( r_3^-(\xi) = \lambda(P_{CT}) \) asymptotically satisfy the conjugation condition (3.2) and estimate (3.1).

Statement 5.1. Under \( r_3^-(\xi) = \lambda(P_{CT}) \) the following condition holds:

\[ \Psi_n(P + i0)G(P)\Psi_n(P - i0) = I + \delta(P, \xi, t), \]

where small parameter \( \delta \) has the form

\[ \delta(P, \xi, t) = \begin{cases} 
O \left( (t^{-\epsilon}(P - P_{CT})) \right), & |P - P_{CT}| \geq t^{-\epsilon}, \\
O(1), & |P - P_{CT}| < t^{-\epsilon}, \end{cases} \]

\( \epsilon > 0. \)

Analogously one studies the Riemann problem (3.2) for all values of parameter \( \xi. \)

Statement 5.2. The main term of asymptotics as \( t \to \infty \) of solution to Cauchy problem (1.1) - (1.2) for Volterra lattice under condition of solitonless spectrum is described by Whitham-modulated one-gap solution \( c(n, t \mid \Gamma(\xi), D(\xi)) \) (4.1), (4.2), where Riemann curve \( \Gamma(\xi) \) is determined by formulas (4.7), phase shift \( D(\xi) \) - by formula (5.21).

The remaining terms of the asymptotic series decrease as powers in \( t. \)

11
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