On the non-integrability and dynamics of discrete models of threads

Valery Kozlov and Ivan Polekhin

Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia

E-mail: ivanpolekhin@mi-ras.ru

Received 28 March 2021, revised 3 July 2021
Accepted for publication 16 July 2021
Published 3 August 2021

Abstract
In the paper, we study the dynamics of planar \( n \)-gons, which can be considered as discrete models of threads. The main result of the paper is that, under some weak assumptions, these systems are not integrable in the sense of Liouville. This holds for both completely free threads and for threads with fixed points that are placed in external force fields. We present sufficient conditions for the positivity of topological entropy in such systems. We briefly consider other dynamical properties of discrete threads and we also consider discrete models of inextensible yet compressible threads.

Keywords: inextensible thread, non-integrability, topological entropy, discrete model, planar linkage

Mathematics Subject Classification numbers: 37J30.

1. Introduction
The study of the motion of a flexible inextensible thread has historically been a popular mechanical problem (see, for instance, the classical book by Appell [5]). The importance of this topic stems from the role that threads, tethers, ropes, chains and other string-like elements play in a wide range of industrial applications. During most of the twentieth century, these elements were commonly used in light and marine industry and in instrument engineering [3, 18, 25, 28, 42, 43, 54, 61]. Later, in the second half of the twentieth century, it became understood that tethers can be also useful in space applications [4, 7, 9, 10, 14, 17, 37, 44–46, 50]. Here, it is worth mentioning the monographs by one of the pioneers in this area, Beletsky with co-authors [11, 12, 60]. Recently, interest in studying thread dynamics has been revived by the so-called ‘chain fountain’ [13, 24, 41, 51].
There exist various approaches to the modeling of threads and in all of them it is usually assumed that the thread is a one-dimensional object, i.e. the thread has some length, yet all their cross sections have zero area.

The other properties of the model can vary. In particular, the thread can be inextensible (when each small element of the thread cannot be elongated) or extensible. One can also consider compressible and incompressible threads. In the first case the total length of the thread can be shorter than the initial length. In the latter case, the compression of the system is not allowed. As an example of an incompressible and inextensible thread one can consider a usual chain. The model of an incompressible thread could be also called ‘a Poinsot thread’ after Louis Poinsot, who proposed to consider rigid balls strung on a thread to give an interpretation for negative tension that may occur in some equilibrium configurations of a flexible thread [5].

From the dynamical point of view, the threads can be absolutely flexible (in this case the thread can be bent without any resistance) or can be elastic (there exists a moment of force trying to straighten the thread). One of the simplest models is the absolutely flexible inextensible incompressible thread. This model is usually considered when the classical catenary equation is derived (figure 1).

Another general property of the threads which is usually assumed to hold is that the shape of thread can be an arbitrary smooth curve, i.e. the thread is an infinite dimensional object. Even though this assumption is quite natural, it can complicate the study of the dynamics of the system: the rigorous study of thread dynamics inevitably leads to the consideration of generalized solutions, which is caused by the singularities (cusps) that may appear during the motion of a thread [35]. Therefore, its purely dynamical behavior is difficult to study and only stationary or quasi-stationary configurations can be considered analytically.

We will mainly consider the following model of an inextensible and incompressible thread: a finite number \( n \) of rigid segments of equal lengths \( l \); the segments form a broken line and are connected by planar hinges; and, masses \( m_1, \ldots, m_{n+1} \) are located in the endpoints of the segments. This model is finite dimensional, i.e. each configuration of the thread is defined by a finite number of real-valued parameters (see, for instance, figure 2).

It is also possible to consider various constraints imposed on a thread. For instance, one can fix some points of the system. The following five configurations can be considered as basic from the point of view of possible applications:

(a) A thread with fixed endpoints (broken line with fixed points),
(b) A closed thread with a fixed point (\( n \)-gon with a fixed point),
(c) A free closed thread (planar \( n \)-gon),
(d) A thread with one fixed endpoint (\( n \)-link pendulum),
(e) A free non-closed thread.

All these systems can be considered as threads moving by inertia, in the sense that there are no external forces acting on the system, and as threads in external potential force fields. One can also assume that there are internal forces acting between the masses or the segments of the thread. For instance, these forces can model various elastic properties of our mechanical system.

It is important to note that it is assumed that the motion of the thread is frictionless and all systems are Hamiltonian. Therefore, it is possible to consider the problem of integrability for such systems. The integrability means that there exists a sufficiently large number of the first integrals (constants of motion). In accordance to the Liouville–Arnold theorem [6], the dynamics of the integrable threads are relatively simple and, in appropriate coordinates, it is always a quasi-periodic motion. In contrast, if the system is not integrable, its dynamics can be extremely complex.

In most of our results on the non-integrability, we do not assume that the threads are free and move by inertia: we consider threads both in external force fields and threads with fixed points. Even though our models are finite, we do not impose any limitations on the number and the masses of the elements that make up our system.

The topology of the configuration space of the corresponding discrete system will play a key role in our considerations. For cases (d) and (e), the structure of the configuration space can be easily understood: it is either an $n$-dimensional torus, or a direct product of an $n$-dimensional torus and a group of parallel translations of the plane (which is isomorphic to $\mathbb{R}^2$). When these systems move by inertia (i.e., the only forces acting on the system are the forces of reaction), we have natural Noetherian first integrals. In case (d), this first integral is the kinetic moment w.r.t. the fixed point. The configuration space obtained after the corresponding reduction is an $(n - 1)$-dimensional torus. In case (e), the group of symmetries coincides with the symmetries of the Euclidean plane and we have three Noetherian integrals. After the reduction, we again obtain a system on an $(n - 1)$-dimensional torus. For the cases (a)–(c), that will be our main objects of study, the topology of the configuration space can be more complex.

For the threads moving by inertia, we will also consider the problem of positivity of the topological entropy of the system. The positivity of this parameter can also be interpreted as a measure of complexity of the system.

The rest of this paper is structured as follows. First, we recall some results on the integrability of Hamiltonian systems and we present auxiliary results concerning the topology of planar $n$-gons. In the next section, we show that threads described by models (a)–(c) are not integrable in the class of real analytical functions provided that some natural assumptions
hold. Then, we discuss the question of the positivity of the topological entropy for our models. We also present some geometrical results concerning the dynamics of these systems. In the conclusion, we briefly consider several related problems, including possible generalization of our results to higher dimensions and possible models for compressible threads and their properties.

2. Auxiliary results and definitions

2.1. Integrability and non-integrability

A triple $(S, \omega, H)$, where $S$ is a $2n$-dimensional smooth manifold, $\omega$ is a symplectic structure on $S$ and $H : S \to \mathbb{R}$ is a smooth function, is called a Hamiltonian system. A smooth function $F : S \to \mathbb{R}$ is called a first integral of the system $(S, \omega, H)$ if

$$\{F, H\} \equiv 0.$$ 

Here $\{\cdot, \cdot\}$ is the Poisson bracket corresponding to the symplectic structure. We say that system $(S, \omega, H)$ is Liouville integrable (or simply, integrable) if

(a) There are $n$ first integrals $F_1 = H, \ldots, F_n : S \to \mathbb{R}$;
(b) These functions are independent, that is, almost everywhere on $S$, one-forms $dF_1, \ldots, dF_n$ are linearly independent;
(c) $\{F_i, F_j\} \equiv 0$ for any $i$ and $j$.

Everywhere below we will consider analytic Hamiltonian systems: manifold $S$ is an analytic manifold, $H$ is a real analytic function. The system is analytically integrable if all of the functions $F_i$ are analytic.

In our considerations, we will assume that $S$ is a cotangent bundle of an $n$-dimensional manifold $M$, that is, $S = T^* M$. By $q$ we will denote local coordinates on $M$ and by $p$ we denote local coordinates on $T_q M$. In particular, in these coordinates the Poisson bracket has the standard form

$$\{F, G\} = \sum_{i=1}^{n} \left( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right).$$

A more detailed exposition of Hamiltonian mechanics can be found, for instance, in [1, 6]. The problem of integrability of Hamiltonian systems is discussed in detail in [34].

The main tool that will be used here to prove the non-integrability of our system is the following theorem of Taimanov on the non-integrability of geodesic flows [57, 58].

**Theorem 1.** Given a geodesic flow on an $n$-dimensional closed analytic manifold $M$ with an analytic Hamiltonian function $H : M \to \mathbb{R}$. If

$$\dim H_1(M, \mathbb{Q}) > n,$$  \hspace{1cm} (1)

then there are no functions $F_2, \ldots, F_n : M \to \mathbb{R}$ such that for some energy level $F_1 = H = \text{const.} > 0$ we have

(a) Functions $F_1, \ldots, F_n$ are analytic and $\{F_i, F_j\} \equiv 0$ for any $i, j$ in a neighborhood of the level set $H = \text{const.}$;

(b) Differentials $dF_1, \ldots, dF_n$ are linearly independent on $H = \text{const.}$.
As a corollary from this theorem, we obtain sufficient conditions for non-integrability of so-called natural Hamiltonian systems. Let us recall that the system is called natural if its Hamiltonian has the form

$$H(p, q) = H_2(p, q) + H_0(q) = \sum_{i,j=1}^{n} g^{ij}(q) p_i p_j + H_0(q),$$

where $H_2$ is a positive definite quadratic form in $p$ (kinetic energy). In accordance to the Maupertuis principle, projections of the solutions of this system onto $M$ can be considered as geodesics of the Jacobi metric. To be more precise, consider level set $H(p, q) = h$, where $h > \max H_0$. The trajectories of the system with the Hamiltonian function $H$ on the level $H = h$ then coincide with the trajectories of the system with the Hamiltonian function $\tilde{H}$

$$\tilde{H} = \sum_{i,j=1}^{n} \frac{g^{ij}(q)}{h - H_0(q)} p_i p_j$$
on the level set $\tilde{H} = 1$. Moreover, if the original Hamiltonian system has a first integral $F$ on a level set $H = h$, then the corresponding geodesic flow of the Jacobi metric has a first integral $\tilde{F}$ on $T^*M$ (possibly, except for the set $\tilde{H} = 0$) and

$$\tilde{F}(p, q) = F \left( \frac{p}{\sqrt{H(p, p)}}, q \right).$$

**Corollary 1.** Given an analytic manifold $M$ such that condition (1) holds and a natural Hamiltonian system on $T^*M$. Then, this system cannot be analytically integrable.

Indeed, if we have $n$ first analytic independent integrals $F_1, \ldots, F_n$ for the natural Hamiltonian system, then we obtain functions $\tilde{F}_1, \ldots, \tilde{F}_n$ satisfying the conditions of theorem 1.

Theorem 1 can also be applied to more general Hamiltonian systems. Let us have a system with an analytic Hamiltonian function $H$

$$H = H_2(p, q) + H_1(p, q) + H_0(q),$$

where $H_2$ and $H_0$ coincides with the corresponding terms in (2) and

$$H_1(p, q) = \sum_{i=1}^{n} b^i(q) p_i.$$

Here $b^i(q)$ are the components of a vector field globally defined on $M$. We will say that system (3) is integrable in the class of polynomial in $p$ first integrals with independent highest degree terms if there exist $n$ first integrals $F_1: T^*M \to \mathbb{R}$ ($F_1 = H$) of the form

$$F_1(p, q) = F_1^{m_1}(p, q) + F_1^{m_1-1}(p, q) + \ldots + F_1^{0}(q),$$

where $F_1^{m_1}, F_1^{m_1-1}, \ldots, F_1^{0}$ are homogeneous in $p$ analytic polynomials of degrees $m_1, m_1 - 1, \ldots, 0$ correspondingly, $\{F_1^{m_1}, F_1^{j}\} \equiv 0$ for any $i$ and $j$, and $dF_1^{m_1}, \ldots, dF_1^{m_n}$ are linearly independent almost everywhere.

**Corollary 2.** Given an analytic manifold $M$ such that condition (1) holds and a Hamiltonian system on $T^*M$ with the Hamiltonian function (3). Then, this system cannot be integrable in the class of polynomials in $p$ with independent highest degree terms.
The proof directly follows from the fact that the highest degree terms $F_m^1 = H_2, \ldots, F_m^n$ are first integrals for the geodesic flow with Hamiltonian $H = H_2$.

It is worth mentioning here that all of the known integrable mechanical systems are integrable in the class of polynomial first integrals with independent highest degree terms.

### 2.2. Topology of linkages

In this section, we present some results on the topological properties of planar linkages (see, for instance, [21, 22]).

Let us have $n$ planar segments with lengths $l_1, l_2, \ldots, l_n$, which form a closed polygon. In the following we assume that for the lengths the following condition holds

$$\sum_{i=1}^{n} l_i \nu_i \neq 0, \quad \text{for any } \nu_i = \pm 1.$$  \hfill (4)

We will denote the configuration space of the polygon, viewed up to isometries of the Euclidean plane, by $\tilde{M}$:

$$\tilde{M} = \left\{(u_1, \ldots, u_n) \in S^1 \times \cdots \times S^1 : \sum_{i=1}^{n} l_i u_i = 0 \right\} / SO(2). \hfill (5)$$

Equivalently, we can consider our $n$-gon with one of its sides fixed: the configuration space of this system naturally coincides with $\tilde{M}$.

**Theorem 2.** $\tilde{M}$ is an analytic closed orientable manifold of dimension $n - 3$.

**Remark 1.** When condition (4) does not hold, there are a finite number of singularities on $\tilde{M}$ that correspond to the collinear configurations of the linkage. Note that this condition holds for a generic set of lengths.

**Definition 1.** Given a polygon with lengths $l_1, l_2, \ldots, l_n$, we call a subset of its sides $J = \{i_1, i_2, \ldots, i_k\}$ short when

$$\sum_{i \in J} l_i < \sum_{i \notin J} l_i.$$

In the following, we will use the following result on the topology of the configuration space of a linkage.

**Theorem 3.** Given a planar polygon satisfying (4), let $l_i$ be a side of the maximal length (i.e., $l_i \geq l_j$ for any $j$), for every $k \in \{0, 1, \ldots, n - 3\}$, the homology group $H_k(\tilde{M}; \mathbb{Z})$ is a free Abelian group of rank $a_k + a_{n-3-k}$, where $a_k$ denotes the number of short subsets of $k+1$ elements containing $l_i$.

**Corollary 3.** Let $n = 2r + 1$ and for all $i$ we have $l_i = 1$. Then

$$b_k(\tilde{M}) = \begin{cases} 
C_{n-1}^{k}, & \text{for } k < r - 1, \\
2C_{n-1}^{r-1}, & \text{for } k = r - 1, \\
C_{n-1}^{k+2}, & \text{for } k > r - 1.
\end{cases} \hfill (6)$$
Here \( C_k^n \) stands for the usual binomial coefficient
\[
C_k^n = \frac{n!}{k!(n-k)!},
\]
and \( b_k(\tilde{M}) \) denotes the \( k \)th Betti number of manifold \( \tilde{M} \). Let us prove the first equality of the proposition. The second and the third cases can be considered in a similar way. Let \( l_1 \) be the side of the maximal length (by our assumption they all have the same length and we can consider the first one as the longest). Then \( a_k \) is the number of short subsets (of \( k+1 \) elements) containing \( l_1 \). Therefore, we have to choose \( k \) segments and the total number of available segments is \( n-1 \), so \( a_k = C_{n-1}^k \). At the same time, we have \( a_{n-3-k} = 0 \). Indeed, since \( k < r-1 \) and \( n = 2r+1 \), then \( n-3-k \geq r \) and we have no short subsets of \( r+1 \) elements.

\[\textbf{Theorem 4.} \quad \text{Given a polygon with sides } l_1 = \cdots = l_k = \varepsilon, l_{k+1}, l_{k+2},\ldots, l_{k+n}. \text{ Let } \tilde{M} \text{ be the configuration space of this polygon and } \tilde{M}_l \text{ be the configuration space of a polygon with sides } l_{k+1}, l_{k+2},\ldots, l_{k+n}. \text{ Then } \tilde{M} \text{ is diffeomorphic to } \mathbb{T}^k \times \tilde{M}_l, \text{ provided that } \varepsilon \text{ is small and condition (4) holds for the lengths } l_{k+1}, l_{k+2},\ldots, l_{k+n}.\]

The proof is based on the fact that the addition of a short side to a polygon leads to a new configuration space which is the direct product of \( S^1 \) and the original configuration space. More details can be found in [55].

It is important to note that the topology of the configuration space of a planar polygon does not depend on the order of its sides, but only on their lengths.

3. Non-integrability

We will mainly consider the following natural model for threads: a thread is a collection of \( n \) rigid planar segments of the same length, the segments are pairwise connected by joints at their endpoints and form a planar broken line. Without loss of the generality we can assume that all segments have unit length.

If it is a thread with fixed endpoints (model (a)), then we will denote the distance between the fixed points by \( l \) and assume that \( l < n \) and \( l \notin \mathbb{N} \). Then (4) holds for the closed \((n+1)\)-gon such that \( l_1 = \ldots = l_n = 1 \) and \( l_{n+1} = l \) (figure 2). For a closed thread with a fixed point and a free closed thread (models (b) and (c)) we have a planar \( n \)-gon and its sides have unit lengths (figure 3).

The above specifies the kinematics of the thread. Its dynamical behavior is determined by the distribution of mass of the thread and by the forces acting on the system.

We assume that all mass of the thread is concentrated in the joints, i.e., for model (a), there are \( n-1 \) masses \( m_i > 0 \) such that the \( i \)th mass is located in the joint between the \( i \)th and the \((i+1)\)th segments of the broken line. We do not consider mass points at the fixed points since they do not affect the dynamics of the system. For models (b) and (c), there are \( n \) masses \( m_i > 0 \). When the thread is homogeneous it is natural to assume that \( m_i = m_j \) for all \( i \) and \( j \).

Everywhere below we assume that threads move without friction and the corresponding system is Hamiltonian. In this section we study the non-integrability of models (a)–(c). For each of these models, we consider two classes of systems.

A natural Hamiltonian system: in this case the Hamiltonian of the corresponding thread has the form
\[
H = H_2(p, q) + H_0(q). \tag{7}
\]
where $H_2(p, q)$ is a quadratic positive definite form in $p$, i.e., $H_2$ is the kinetic energy; $p = (p_1, \ldots, p_k)$ are the generalized momenta and $q = (q_1, \ldots, q_k)$ are the local coordinates on the configuration space of dimension $k$ and $H_0(q)$ is a function on the configuration space which corresponds to the potential forces acting on the system. These forces include both external and internal forces, i.e., for instance, it can be an external force field of gravity or restoring forces caused by the springs located in the joints of the thread. A thread moving by inertia can be considered as a natural system such that $H_0 \equiv 0$.

A Hamiltonian system with gyroscopic forces: the Hamiltonian has the form

$$H = H_2(p, q) + H_1(p, q) + H_0(q), \quad (8)$$

where $H_1(p, q) = \sum_{i=1}^{k} b_i(q) p_i$ and $H_2$ and $H_0$ are defined as above. The term $H_1$ corresponds to so-called gyroscopic forces. For instance, it can be magnetic forces acting on the thread.

### 3.1. A thread with fixed endpoints

Let the system have $k$ degrees of freedom, that is, the dimension of the configuration space equals $k$.

Consider the closed polygon such that one of its sides has length $l$ and all other sides are of unit length. From theorem 2 we obtain that $k = n - 2$. Therefore, we assume that $n \geq 4$, since all one-dimensional cases are trivially integrable. Let us now calculate the first Betti number of this space.

**Proposition 1.** If $0 < l < 1$, then

$$b_1(M) = \begin{cases} 
    n + 4, & \text{if } n = 4, 5 \\
    n, & \text{if } n > 5.
\end{cases}$$
If $1 < l < n - 2$, $l \notin \mathbb{N}$, then

$$b_1(M) = \begin{cases} 8, & \text{if } n = 4 \\ n, & \text{if } n > 4. \end{cases}$$

If $n - 2 < l < n$ then $b_1(M) = 0$.

The proof is a direct calculation based on theorem 3. We see that (1) holds for all $n$ and for any $l < n - 2$. From corollaries 1 and 2 we have the following.

**Proposition 2.** Consider a thread with fixed endpoints (model (a)). Let $0 < l < n - 2$ and $l \notin \mathbb{N}$, $n \geq 4$, and the Hamiltonian function (7) of the system is an analytic function, then the system is not analytically integrable.

**Proof.** Indeed, $\dim(M) = n - 2$ and $b_1(M) \geq n$. \hfill $\Box$

**Proposition 3.** Consider a thread with fixed endpoints (model (a)). Let $0 < l < n - 2$ and $l \notin \mathbb{N}$, $n \geq 4$, and the Hamiltonian function (8) of the system is an analytic function, then the system is not integrable in the class of polynomials in $p$ with independent highest degree terms.

In other words, the system of a discrete thread between two fixed points cannot be analytically integrable for $l < n - 2$. This holds for a thread moving by inertia and for a thread in external or internal force fields. In particular, if we have a thread in a gravity field, then this system is not analytically integrable. If we add gyroscopic (e.g., magnetic) forces to the system, then this system cannot be integrated in the class of polynomials in $p$ with independent highest degree terms.

Note that for large $n$, that is, when the discrete model of a thread is relatively fine, the condition $l < n - 2$ holds for the most part of the distances between the fixed points. This statement should be understood in the following sense. We can assume that the total length of the thread is fixed and equals 1. In this case, we can rescale the lengths of the segments and assume that each segment has length $1/n$. Therefore, the system cannot be integrable provided that the distance between the fixed points is less than $(n - 2)/n$, i.e., the measure of distances for which the system can possibly be integrable tends to zero as $n$ tends to infinity.

For $n = 4$ we have a system with a two-dimensional configuration space and in this case the non-integrability follows directly from the result for natural Hamiltonian systems with two degrees of freedom [33]. The proof of this result is based on the existence of a large number of unstable periodic solutions. The asymptotic surfaces of these solutions intersect and form a complex net such that the additional first integral has a constant value at all points of this net. Therefore, taking into account the fact that this integral is analytic, we obtain that this function is a constant.

From the above result on the non-integrability of a thread with fixed endpoints we can obtain the non-integrability for more complex systems having this non-integrable thread as a subsystem. To be more precise, let us have a system such that its configuration space is a direct product of $M$, the configuration space of a non-integrable thread, and $K$, a $k$-dimensional analytic manifold. If $b_1(K) \geq k - 1$, then analytic Hamiltonian system with configuration space $M \times K$ cannot be integrable. Indeed, from the Künneth theorem, we have

$$b_1(M \times K) = b_1(M) + b_1(K) \geq n + k - 1.$$  

Obviously, $\dim(M \times K) = n + k - 2$ and condition (1) holds. As an example we can consider a non-integrable free thread with a $k$-link pendulum attached to one of the moving joints of the thread. The configuration space of the pendulum is a $k$-dimensional torus and $b_1(T^k) = k$. 

6406
3.2. Closed threads

In this section we will consider models (b) and (c). To a large degree they are similar and both these models will be shown to be non-integrable. However, for model (c), we will impose some additional conditions to prove the non-integrability.

Consider a closed \(n\)-gon assuming that all its sides have the same unit length and \(n\) is an odd number \(n \geq 5\). Then (4) obviously holds. We also assume that one of the points of the \(n\)-gon is fixed and there are \(n - 1\) masses \(m_i\) located in all non-fixed vertices of the \(n\)-gon (figure 3). Note that we allow self-intersections during the motion of the \(n\)-gon. Similarly to the case of a thread with two fixed points, we assume that all forces acting on the system are potential and the system is Hamiltonian with the Hamiltonian of the form (7).

**Proposition 4.** Consider a closed thread with a fixed point (model (b)). Let the Hamiltonian function (7) of the system be an analytic function. Then the system is not analytically integrable.

**Proof.** The configuration space \(M\) of the system is the direct product of a one-dimensional circle and \(\tilde{M}\) and has dimension \(n - 2\). From corollary 3 we obtain that \(b_1(\tilde{M}) = n - 1\). Therefore, \(b_1(M) = b_1(\tilde{M}) + 1 = n\) and we can apply theorem 1.

Similar result holds for the system with gyroscopic forces.

**Proposition 5.** Consider a closed thread with a fixed point (model (b)). Let the Hamiltonian function (8) of the system be an analytic function. Then the system is not integrable in the class of polynomials in \(p\) with independent highest degree terms.

Again, the system has the Hamiltonian function of the form (7) when we consider a thread such that there are no external or internal forces acting on the system, except for the forces of reaction. Also, we can consider a thread in an external force field or a thread with interactions between its elements.

Let us consider a closed \(n\)-gon moving on a plane without friction. The lengths of the sides of this \(n\)-gon equal 1 (\(n\) is an odd number greater than 3) and masses \(m_i\) are located in the vertices of the polygon. Suppose that the only forces acting on the system are the forces of reaction and internal potential forces acting between the elements of the thread.

The configuration space of this system is not compact and theorem 1 cannot be applied directly. However, if we assume that there are no external forces acting on the thread, we can consider the reduced system with a compact configuration space. To be more precise, let \(x\) and \(y\) be the Cartesian coordinates of some mass point of the thread and we consider these coordinates as a part of the set of generalized coordinates. Let the Hamiltonian function of the system has the form (7). Since there are no external forces acting on the system, we can conclude that \(H\) does not depend on \(x\) and \(y\). Clearly,

\[
\frac{\partial H}{\partial x} = c_x = \text{const}, \quad \frac{\partial H}{\partial y} = c_y = \text{const}.
\]

After the Routh reduction w.r.t. variables \(x\) and \(y\) we obtain a Hamiltonian system with the Hamiltonian of the form (8) where \(H_1 \equiv 0\) iff \(c_x = 0\) and \(c_y = 0\). Therefore, similarly to propositions 4 and 5, we obtain

**Proposition 6.** Consider a free closed thread (model (c)). Suppose that there are no external forces acting on the thread and \(c_x = 0, c_y = 0\) and the Hamiltonian function of the reduced system is an analytic function of the form (7). Then the reduced system is not analytically integrable.
If the initial system contains non-zero terms $H_1$ or at least one of the constants $c_x$ or $c_y$ does not equal zero, then the Hamiltonian of the reduced system takes the form (8). Therefore, we obtain the following result.

**Proposition 7.** Consider a free closed thread (model (c)). Suppose that the Hamiltonian function $H$ of the system is an analytic function and has the form (8) and $H$ does not depend on $x$ and $y$. Then the reduced system is not integrable in the class of polynomials in $p$ with independent highest degree terms.

### 3.3. Threads with segments of different length

Everywhere above we assumed that the segments of the discrete thread are of the same length. Taking into account possible internal forces acting between the segments, we can conclude that this setting allows one to model a broad range of real-life systems. However, for the sake of completeness, we will consider the cases when the segments have different length.

First, we will consider model (a). Let $l_i > 0$, $1 \leq i \leq n$ be the lengths of segments and $l > 0$ be the length between the fixed points. As above, we assume that (4) holds. Inequality $b_1(M) \geq n - 1$ plays the key role in the proofs of propositions 2 and 3. From theorem 3 we have that $b_1(M) \geq a_1$. Therefore, if $a_1 \geq n - 1$, then the system is not integrable. Let $l$ or $l_j$ (for some $1 \leq j \leq n$) be the side of the maximal length. If there are at least $n - 1$ lengths $l_k$, $1 \leq k \leq n - 1$ (different from the maximal length) such that the pair of lengths $l_k$ and $l_j$ (or $l$) is a short subset, then the corresponding system is not integrable in the sense of propositions 2 and 3.

Absolutely similar conditions can be formulated for models (b) and (c). For these cases we have to obtain $b_1(\tilde{M}) \geq n - 2$ where $n$ is the number of segments in the thread. Therefore, there should be at least $n - 2$ lengths $l_k$, $1 \leq k \leq n - 2$ (different from the maximal length $l_j$) such that the pair of lengths $l_k$ and $l_j$ is a short subset. If the above conditions hold, it is possible to prove results similar to propositions 2–7.

### 4. Topological entropy

In this section we will consider only threads moving by inertia, that is, everywhere below we assume that there are no external or internal forces acting on the system, except for the forces of reaction.

First, let us recall the definition of the topological entropy (see, for instance, [30]). Let $X$ be a compact metric space with a metric $d$ and $f : X \to X$ be a continuous map. Consider the following sequence of metrics

$$d_n(x, y) = \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)).$$

Consider an open ball $B(x, \varepsilon, n) = \{ y \in X : d_n(x, y) < \varepsilon \}$. A set $U \subset X$ is an $(n, \varepsilon)$-covering if $X \subset \bigcup_{x \in E} B(x, \varepsilon, n)$. Let $S(n, \varepsilon)$ be the minimal number of elements in an $(n, \varepsilon)$-covering. Put

$$h(f, \varepsilon) = \lim_{n \to \infty} \frac{1}{n} \log S(f, \varepsilon, n).$$

Then, the topological entropy of the map $f$ is defined as

$$h(f) = \lim_{\varepsilon \to 0} h(f, \varepsilon).$$
The definition of the topological entropy for flows can be expressed in terms of the topological entropy for maps: let us have a flow $\phi^t : \mathbb{R} \times X \to X$, then we put $f = \phi^1$.

**Remark 2.** This definition is based on a metric structure on $X$. However, it can be shown that this definition does not depend on the choice of the metric, provided that all metrics define the same topology on $X$. A definition that is not based on the metric structure has been given in [2]. The definition given above was first given in [19]. In addition, the first definition of entropy for a dynamical system has been formulated by Kolmogorov [32]. We also would like to note that the ($C^\infty$) integrability does not imply that the topological entropy vanishes [15].

For a geodesic flow on a Riemannian manifold the topological entropy can be defined as follows [38]:

$$h = \lim_{L \to \infty} \frac{1}{L} \log \int_{M \times M} n_L(x, y) \, dx \, dy,$$

where $n_L(x, y)$ is the number of geodesics of lengths no more than $L$ connecting points $x$ and $y$ of manifold $M$.

Positivity of the topological entropy usually corresponds to the complexity of the dynamics of a system. It can also imply the chaotic behavior of a system [20]. At the same time, the positivity of topological entropy is not equivalent to the ergodicity and there are non-ergodic systems with a positive topological entropy.

Let us have a geodesic flow on a closed Riemannian manifold. It is known that for some manifolds it is impossible to find a metric with zero topological entropy, that is, for any given smooth metric, the entropy is positive. For instance, if the fundamental group of the manifold is a group of exponential growth, then the topological entropy of the geodesic flow is positive. The details can be found in [19, 40], where the problem of existence of a metric with zero entropy is considered.

In addition, the following has been proven in [19].

**Theorem 5.** If there exists a metric of negative sectional curvature on a closed manifold, then the geodesic flow on this manifold has a positive topological entropy for any metric.

It is known that there exists a metric of negative curvature on any two-dimensional closed manifold of genus greater than one [56].

In particular, for the previous discrete models of threads, the topological entropy can be proven to be positive when the thread is moving by inertia. To be more precise, given a thread with two fixed endpoints and $n = 4$, the dynamics is described by the geodesic equation provided the only forces acting on the thread are the forces of reaction. The metric is given by the kinetic energy of the system and the genus of the configuration manifold is greater than one. Therefore, the topological entropy is strictly positive.

Similar result holds for model (c). However, it is worth mentioning that results about the positivity of geodesic flows can only be applied here for the cases where the constants of the Noetherian integrals equal zero.

To be more precise, we can conclude that the following results hold for two-dimensional configuration spaces.

**Proposition 8.** Consider a thread with fixed endpoints (model (a)). Let $n = 4$ and $l_i = 1$ for all $i$. Let the distance between the fixed points be $l < 2$ and condition (4) holds. Suppose that there are massive points with masses $m_i$ located in the joints of the thread and that the only forces acting on the system are the forces of reaction. Then, the topological entropy of this system is positive.
Proof. It is known that the genus $g$ of the surface equals $b_1/2$, that is, for our surface we have $g = 4$.

Proposition 9. Consider a free closed thread (model (c)). Let $n = 5$ and $l_i = 1$ for all $i$. Suppose that there are five massive points with masses $m_i$ located in the joints of the thread and that the only forces acting on the system are the forces of reaction. Also suppose that the constants of three Noetherian first integrals equal zero. Then, the topological entropy of the system (after the Routh reduction) is positive.

Here it is worth mentioning that two of three Noetherian first integrals correspond to the parallel translations of the plane and their values become zero in the appropriate inertial frame of references.

Some results on the existence of a metric corresponding to zero topological entropy for low dimensional manifolds can be found in [47–49].

In particular, it was proven in [49] that, given a four-dimensional closed manifold $M$ with an infinite fundamental group, it is only possible to find a metric on $M$ with zero topological entropy when the Euler characteristic of $M$ is zero.

As corollaries from this result, we obtain the following propositions for models (a) and (c). In the both cases, the proofs are obtained by direct calculations.

Proposition 10. Consider a thread with fixed endpoints (model (a)). Let $n = 6$ and $l_i = 1$ for all $i$. Let the distance between the fixed points is $l < 4$, $l \notin \mathbb{N}$. Suppose that there are massive points with masses $m_i$ located in the joints of the thread and the only forces acting on the system are the forces of reaction. Then, the topological entropy of this system is positive.

Proposition 11. Consider a free closed thread (model (c)). Let $n = 7$ and $l_i = 1$ for all $i$. Suppose that there are massive points with masses $m_i$ located in the joints of the thread and the only forces acting on the system are the forces of reaction. Also suppose that the constants of three Noetherian first integrals equal zero. Then, the topological entropy of the system (after the Routh reduction) is positive.

Note that the fundamental groups of these systems are clearly infinite because their abelianizations, the first homology groups, are infinite.

Note that a result similar to proposition 11 holds for model (b) for $n = 7$ if we assume that the only Noetherian first integral equals zero. However, if we do not want to consider the reduced system and, at the same time, we want to obtain a configuration space of dimension 4, then there should be six segments in the closed contour. If we assume that these segments are of the same length, then the configuration space will not be a smooth manifold.

Another interesting example of a low-dimensional linkage system was given by Hunt and MacKay [27] who proved that a certain family of triple linkages have Anosov dynamics.

For an arbitrarily large $n$ (i.e., for the cases when the thread is modeled by a large number of segments), it is also possible to prove that the entropy is positive based on theorem 4.

Proposition 12. Given a discrete thread with two fixed endpoints. Let the distance between the fixed points be $l$ and the lengths of the segments of the thread equal $l_1, l_2, l_3, l_4, \varepsilon, \ldots, \varepsilon$. Suppose that there are massive points with masses $m_i$ located in the joints of the thread and the only forces acting on the system are the forces of reaction. Then the topological entropy of the system is positive, provided that $\varepsilon$ is small and the configuration space of the polygon with sides $l, l_1, l_2, l_3, l_4$ is a smooth manifold of genus greater than 2.

Proof. First, from theorem 4 we have that the configuration space of our system is the direct product of the $k$-dimensional torus, where $k$ is the number of sides of length $\varepsilon$, and an oriented
smooth surface of genus greater than 2. The fundamental group of this surface is a group of exponential growth. Therefore, the fundamental group of the configuration space is also a group of exponential growth and the topological entropy of any geodesic flow on it is strictly positive.

For instance, if we put $l_1 = l_2 = l_3 = l_4 = 1$ and $l < 2$, then we can conclude that the topological entropy of the system is positive for small $\varepsilon$ (figure 4).

5. Geometric results

First, we present a geometrical result concerning the dynamics of the thread in the most general case, that is, in the presence of potential and gyroscopic forces.

Let us shortly recall the correspondence between the Hamiltonian and Lagrangian approaches to the dynamics of mechanical systems. Given a Hamiltonian function of the form (8), we can obtain a Lagrangian $L$ by means of the Legendre transformation:

$$L(q, \dot{q}) = \dot{q} \cdot p - H(p, q), \quad \dot{q} = \frac{\partial H}{\partial p}.$$ 

In the new variables $(q, \dot{q})$ we have

$$L(q, \dot{q}) = L_2(q, \dot{q}) + L_1(q, \dot{q}) + L_0(q),$$

where, again, $L_2(q, \dot{q})$ is a quadratic positive definite form in $\dot{q}$ and $L_1(q, \dot{q})$ is linear in $\dot{q}$, that is, $L_1(q, \dot{q})$ defined by a one-form. The dynamics on the tangent bundle $TM$ is defined by the corresponding Lagrange equations.

**Proposition 13.** Given a Lagrangian system with Lagrangian (9) and an energy level $h > \max_{M}(-L_0)$, then any two configurations $q_0, q_1 \in M$ of the thread can be connected by a solution with energy $h$ provided that

$$4(h + L_0)L_2 - L_1^2 > 0$$

for all $(q, \dot{q})$ on the corresponding energy level.
Proof. In accordance to the Maupertuis principle, a path $\gamma : [t_0, t_1] \rightarrow M$ is a trajectory of a solution of the Lagrangian system iff $\gamma(t)$ is a critical point for the functional $F$

$$F(\gamma) = \int_{t_1}^{t_2} (2\sqrt{(h + L_0(\gamma))L_2(\gamma, \dot{\gamma}) + L_1(\gamma, \dot{\gamma})}) \, dt$$

in the class of all paths of fixed energy $h$ [6]. If inequality (10) holds, then $F$ defines a Finsler length on $M$. From the Hopf–Rinow theorem for Finsler manifolds [8, 53], we have that any two points of $M$ can be connected by a Finsler geodesic. This geodesic corresponds to the desirable solution. □

It is also possible to consider another type of thread, a thread that is inextensible yet can be compressed. In this case one should assume that the distance between two consecutive mass points is not equal to $l_i$, but does not exceed this value. From the mechanical point of view, one can imagine that the mass points are connected not by rigid massless rods, but by inextensible ropes.

In the simplest case we have only one mass point connected to two fixed points. We assume that the thread is moving by inertia, that is, there are no external forces acting on the system. Let $L$ be the distance between the fixed points, $l_1$ and $l_2$ be the lengths of the two ropes connecting the mass point to the fixed points (figure 5).

This system can be considered as a billiard with a non-smooth boundary. One of the first works where this system was considered for $l_1 = l_2$ is [26]. Later this case was studied numerically [39, 52]. It was shown that for almost all distances between the fixed points, the system is not ergodic, since there exist stable periodic trajectories. Apparently, for $l_1 \neq l_2$, the ergodicity of the corresponding billiard systems is not exceptional. To be more precise, in the two-dimensional space of parameters $l_1/l_2, L$ there is a set of non-zero measure corresponding to the ergodic systems [16]. This set is a subset of all systems with the hyperbolic periodic trajectory of period 2 (this trajectory corresponds to the horizontal periodic motion in figure 5). Note, that the stability of the elliptic trajectory of period 2 has been rigorously established in [29], of course, these systems cannot be ergodic.
The dynamics of two and more mass points connected by inextensible ropes is even more complex and, to the best of our knowledge, has not been studied—at least numerically—before.

However, it is possible to obtain a geometric result concerning the dynamics of the compressible threads provided that we consider ‘almost inextensible’ threads. Let us have $n$ massive points moving on a plane without friction and the first and the last point are fixed. We assume that the point with number $2 \leq i \leq n-1$ interacts with points $i-1$ and $i+1$ and the potential energy of this interaction has the form $U(r_{i-1}, i) + U(r_i, i+1)$, where $r_{i-1}$ and $r_{i+1}$ are the distances between the corresponding points and $U(d)$ is a smooth monotonous function such that $U(d) \equiv 0$ for $d \leq 1$ and $U(d) \to +\infty$ as $d \to +\infty$. Let the Lagrangian of the system have the form (9), that is, we assume that there can be external potential and gyroscopic forces acting on the system. Since the total energy $L_2 - L_0 = h$, where $h \in \mathbb{R}$, does not change along the solutions of the considered system and $L_2 \geq 0$, then for any solution we have $-L_0(q) + h \geq 0$. Therefore, for a given energy $h$, the possible motion area $B_h$ is defined as follows

$$B_h = \{ q : -L_0(q) + h \geq 0 \}. \quad (11)$$

If $U(d)$ is a rapidly increasing function, then, for a fixed $h$, the maximum distance between any two consecutive points is close to 1, that is, the thread is ‘almost inextensible’. The following result is proved in [36].

**Theorem 6.** Let $B_h$ be a compact region and there are no critical points of $L_0$ at boundary $\partial B_h$. If the inequality $4(h + L_0)L_2 > L_1^2$ is true in $B_h \setminus \partial B_h$, for any $\dot{q} \neq 0$, then any point inside $B_h$ can be connected with the boundary $\partial B_h$ by a solution of energy $h$.

This result in some sense complements proposition 4: we obtain that any configuration of the thread in the possible motion area can be obtained if we start from the boundary $\partial B_h$. In particular, if $L_1 \equiv 0$, the potential energy of the external forces acting on the system is bounded and $h$ is relatively large, then we can conclude that any configuration in $B_h$ can be obtained from another configuration such that at least one pair of massive points are under tension (the corresponding distance is slightly greater than 1).

6. **Conclusion and final remarks**

To the best of our knowledge, the above propositions give the first non-trivial applications of the Taimanov’s theorem [57, 58]. Note that, apparently, models (d) and (e) are also non-integrable. However, their non-integrability does not follow from the topological properties of the configuration space ($n-1$-dimensional torus), but follows from the metric structure defined by the kinetic energy on this torus.

The next natural question that can be considered is the generalization of these results for non-integrability to the cases of spatial motion of the threads. The homology groups of spatial chains has been obtained in [31]. In particular, for a closed $n$-gon where $n = 2k + 1$ and all $l_i = 1$, odd Betti numbers of the configuration space (again, considered up to the symmetries of the Euclidean space) vanish. Therefore, theorem 1 cannot be applied and, similarly to models (d) and (e), non-integrability does not follow from these topological considerations. Here it is worth mentioning that there is a conjecture [59] generalizing theorem 1 that claims that the system is not integrable if for some $k$

$$\dim H_k(M, \mathbb{Q}) > C_n^k.$$

If this conjecture is true, that it is also possible to prove the non-integrability of spatial threads.
The positivity of topological entropy can be also proved for a more natural model than considered in proposition 12.

**Proposition 14.** Consider a thread with fixed endpoints (model (a)). Let \( n > 5 \) and \( l_i = 1 \) for all \( i \). Let the distance between the fixed points is \( l \) and \( n - 4 < l < n - 2 \), \( l \notin \mathbb{N} \). Suppose that there are massive points with masses \( m_i \), located in the joints of the thread and that the only forces acting on the system are the forces of reaction. Then, the topological entropy of this system is positive.

The proof is based on the following unpublished result by Dirk Schütz (see also [55], where the same technique has been used to calculate the fundamental groups for more complex types of planar linkages).

**Theorem 7.** Let us have a planar polygon and \( l_i = 1 \) for \( 1 \leq i \leq n - 1 \) and \( l_n = l \), where \( 1 \leq l < n - 1 \), \( n \geq 7 \). Then

\[
\pi_1(\tilde{M}) \cong \left\langle a_1, \ldots, a_{n-1} \middle| a_k, \text{ if } \{k,n\} \text{ is not short} \right. \\
\left. [a_i, a_j], \text{ if } \{i,j,n\} \text{ is short} \right\rangle, \tag{12}
\]

where \([a_i, a_j] = a_i^{-1}a_j^{-1}a_i a_j\).

In other words, we have a free group with generators \( a_1, \ldots, a_{n-1} \) and we put \( a_k = 1 \) if \( \{k,n\} \) is not short and we put \([a_i, a_j] = 1 \) if \( \{i,j,n\} \) is short.

Note, that for the closed thread with equal segments from theorem 7, we obtain that \( \pi_1(\tilde{M}) \) is commutative. Therefore, we cannot conclude that the entropy is positive. Nevertheless, one can expect the topological entropy to be positive for these systems as well, yet the proof of this fact should follow not from the topological properties of the configuration space, but from the metric properties defined by the distribution of mass of the thread.

In conclusion, returning to the question of non-integrability, we would like to mention an interesting parallel between the non-integrability of threads with inner interactions between the elements, which can be considered as various models for elastic properties of the system, and the classical wave equation describing the motion of an elastic string with fixed endpoints. In contrast to our model of the thread, this equation can be integrated explicitly and the general solution is a sum of the standing waves. The key difference between these two systems is that the wave equation describes the motion of an extensible string. Therefore, it may be useful to consider yet another model based on a planar or spatial polygon with extensible sides. The topology of such systems has been already studied in [23].

**Acknowledgment**

This work was supported by the Russian Science Foundation under Grant No. 19-71-30012.

**Conflict of interest**

The authors declare that they have no conflict of interest.

**ORCID iDs**

Ivan Polekhin https://orcid.org/0000-0002-6312-6511
References

[1] Abraham R and Marsden J E 1978 *Foundations of Mechanics* vol 36 (New York: Benjamin-Cummings)

[2] Adler R L, Konheim A G and McAndrew M H 1965 Topological entropy *Trans. Am. Math. Soc.* **114** 309

[3] Alekseev N I 1970 *Statics and Steady Motion of a Flexible String* (Moscow: Legkaja Industrija) [In Russian]

[4] Anderson L A and Haddock M H 1992 Tethered elevator design for space station *J. Spacecr. Rockets* **29** 233–8

[5] Appell P 1904 *Traité de Mécanique Rationnelle* vol 2 (Paris: Gauthier-Villars)

[6] Arnold V I, Kozlov V V and Neishtadt A I 2007 *Mathematical Aspects of Classical and Celestial Mechanics* vol 3 (Berlin: Springer)

[7] Bainum P M and Kumar V K 1980 Optimal control of the shuttle-tethered-subsatellite system *Acta Astronaut.* **7** 1333–48

[8] Bao D, Chern S-S and Shen Z 2012 *An Introduction to Riemann–Finsler Geometry* vol 200 (Berlin: Springer)

[9] Bekey I 1983 Tethers open new space options *Aeronaut. Astronaut.* **21** 33–40

[10] Beletskii V V and Levin E M 1985 Dynamics of the orbital cable system *Acta Astronaut.* **12** 285–91

[11] Beletsky V V and Levin E M 1993 *Dynamics of Space Tether Systems* vol 83 (San Diego, CA: Unievkt Incorporated)

[12] Beletsky V V 2012 *Essays on the Motion of Celestial Bodies* (Basel: Birkhäuser)

[13] Biggins J S and Warner M 2014 Understanding the chain fountain *Proc. R. Soc. A* **470** 2163

[14] Bolotina N E and Vilke V G 1979 Stability of the equilibrium positions of a flexible heavy fiber attached to a satellite in a circular orbit *Cosmic Res.* **16** 506–10

[15] Bolsinov A V and Taimanov I A 2000 Integrable geodesic flows with positive topological entropy *Invent. Math.* **140** 639–50

[16] Chen J, Mohr L, Zhang H-K and Zhang P 2013 Ergodicity of the generalized lemon billiards *Chaos* **23** 043137

[17] Cosmo M L and Lorenzini E C 1997 *Tethers in Space Handbook* (Cambridge, MA: Smithsonian Astrophysical Observatory)

[18] Costello G A 1997 *Theory of Wire Rope* (Berlin: Springer)

[19] Dinaburg E I 1971 On the relations among various entropy characteristics of dynamical systems *Math. USSR Izv.* **5** 337–78

[20] Downarowicz T 2014 Positive topological entropy implies chaos dc2 *Proc. Am. Math. Soc.* **142** 137–49

[21] Farber M and Schütz D 2007 Homology of planar polygon spaces *Geom. Ded.* **125** 75–92

[22] Farber M 2008 *Invitation to Topological Robotics* vol 8 (Zürich, Switzerland: European Mathematical Society)

[23] Farber M and Fromm V 2010 Homology of planar telescopic linkages *Algebr. Geom. Topol.* **10** 1063–87

[24] Gyulamirova N S and Kugushev E I 2018 Stationary form of a moving heavy flexible thread *Matematika, Mekhanika* vol 1 (Serija: Vestnik Moskovskogo Universiteta) pp 39–43

[25] Hearle J, Grosberg P and Backer S 1969 *Structural Mechanics of Fibers, Yarns, and Fabrics* (New York: Wiley-Interscience)

[26] Heller E J and Tomsovic S 1993 Postmodern quantum mechanics *Phys. Today* **46** 38–46

[27] Hunt T J and MacKay R S 2003 Anosov parameter values for the triple linkage and a physical system with a uniformly chaotic attractor *Nonlinearity* **16** 1499

[28] Irvine H M 1981 *Cable Structures* (Cambridge, MA: MIT Press)

[29] Kamphorst S O and Pinto-de-Carvalho S 2005 The first Birkhoff coefficient and the stability of two-periodic orbits on billiards *Exp. Math.* **14** 299–306

[30] Katok A and Hasselblatt B 1997 *Introduction to the Modern Theory of Dynamical Systems* vol 54 (Cambridge: Cambridge University Press)

[31] Klyachko A A 1994 Spatial polygons and stable configurations of points in the projective line *Algebraic Geometry and its Applications* (Berlin: Springer) pp 67–84

[32] Kolmogorov A N 1959 Entropy per unit time as a metric invariant of automorphisms *Dokl. Akad. Nauk SSSR* **124** 754–5
[33] Kozlov V V 1979 Topological obstructions to the integrability of natural mechanical systems Sov. Math. Dokl. 20 1413–5
[34] Kozlov V V 2012 Symmetries, Topology and Resonances in Hamiltonian Mechanics vol 31 (Berlin: Springer)
[35] Kozlov V V 2019 Isoperimetric inequalities for moments of inertia and stability of stationary motions of a flexible thread Russ. J. Nonlinear Dyn. 15 513–23
[36] Kozlov V and Polekhin I 2017 On the covering of a Hill’s region by solutions in systems with gyroscopic forces Nonlinear Anal. Theory Methods Appl. 148 138–46
[37] Levin E M 1994 Nonlinear oscillations of space tethers Acta Astronaut. 32 405–8
[38] Mane R 1997 On the topological entropy of geodesic flows J. Differ. Geom. 45 74–93
[39] Makino H, Harayama T and Aizawa Y 2001 Quantum-classical correspondences of the Berry–Robnik parameter through bifurcations in lemon billiard systems Phys. Rev. E 63 056203
[40] Manning A 1979 Topological entropy for geodesic flows Ann. Math. 110 567–73
[41] Martins R 2019 The (not so simple!) chain fountain Exp. Math. 28 398–403
[42] Merkin D R 1980 Introduction to the Mechanics of a Flexible Yarn (Moscow: Nauka) [In Russian]
[43] Minakov A P 1941 Fundamentals of the Thread Mechanics vol 9 (Moscow: The Research Work of the Moscow Textile Institute) pp 1–88 [In Russian]
[44] Modi V J, Bachmann S and Misra A K 1993 Dynamics and control of a space station based tethered elevator system Acta Astronaut. 29 429–49
[45] Modi V J, Chang-Fu G, Misra A K and Xu D M 1982 On the control of the space shuttle based tethered systems Acta Astronaut. 9 437–43
[46] Netzer E and Kane T R 1992 An alternate approach to space missions involving a long tether J. Astronaut. Sci. 40 131–27
[47] Paternain G 1991 Entropy and completely integrable Hamiltonian systems Proc. Am. Math. Soc. 113 871
[48] Paternain G P 2001 Differentiable structures with zero entropy on simply connected four-manifolds Boletim da Sociedade Brasileira de Matematica 31 1–8
[49] Paternain G and Petean J 2006 Zero entropy and bounded topology Comment. Math. Helv. 81 287–304
[50] Pearson J 1975 The orbital tower: a spacecraft launcher using the earth’s rotational energy Acta Astronaut. 2 785–99
[51] Pfeiffer F and Mayet J 2017 Stationary dynamics of a chain fountain Arch. Appl. Mech. 87 1411–26
[52] Ree S and Reichl L E 1999 Classical and quantum chaos in a circular billiard with a straight cut Phys. Rev. E 60 1607
[53] Shen Z 2001 Lectures on Finsler Geometry (Singapore: World Scientific)
[54] Schedrov V S 1961 Fundamentals of the Flexible Thread Mechanics (Moscow: Mashgiz) [In Russian]
[55] Schütz D 2010 The isomorphism problem for planar polygon spaces J. Topol. 3 713–42
[56] Spivak M D 1970 A Comprehensive Introduction to Differential Geometry (Boston, MA: Publish or perish)
[57] Taimanov I A 1987 Topological obstructions to integrability of geodesic flows on non-simply-connected manifolds Math. USSR Izv. 30 403–9
[58] Taimanov I A 1988 On topological properties of integrable geodesic flows Mat. Zametki 44 283–4
[59] Taimanov I A 1994 The topology of Riemannian manifolds with integrable geodesics flows Trudy Matematicheskogo Instituta imeni VA. Steklova vol 205 pp 150–63
[60] Troger H, Alpatov A P, Beletsky V V, Dranovskii V I, Khoroshilov V S, Pirozhenko A V and Zakrzhevskii A E 2010 Dynamics of Tethered Space Systems (Boca Raton, FL: CRC Press)
[61] Yakubovskii Y V, Zhivotov S V, Korytyskii Y I and Migushov I I 1973 Principles of the Yarn Mechanics (Moscow: Legkaya Industriya) [In Russian]