Entropic uncertainty assisted by temporal memory

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The uncertainty principle brings out intrinsic quantum bounds on the precision of measuring non-commuting observables. The uncertainty principle marks an astounding departure from classical determinism by setting fundamental limits on the precision achievable in knowing non-commuting observables of a particle. To put it in simple terms, if one attempts to determine the position of a particle, prediction of its momentum gets inaccurate. Heisenberg quantified this intrinsic quantum indeterminacy encrypted in the measurements of position and momentum in terms of standard deviations $(\Delta Q)_\rho(\Delta P)_\rho \geq \frac{\hbar}{2}$ in any quantum state $\rho$ of the particle. Robertson generalized it to arbitrary pairs of non-commuting observables $X, Z$ as

$$(\Delta X)_\rho(\Delta Z)_\rho \geq \frac{1}{2}(|\langle X, Z \rangle_\rho|).$$

(1)

Uncertainty relation constraining the product of standard deviations suffers from the drawback that the right hand side of depends on the quantum state. In the specific example of a state $\rho$ prepared in an eigenstate of $X$, the standard deviation $(\Delta X)_\rho$ as well as the commutator $|[\langle X, Z \rangle_\rho]|$ vanish and in turn, the uncertainty relation doesn’t reveal any constraint on the spread $(\Delta Z)_\rho$ of the observable $Z$. It has been identified subsequently that Shannon entropies of the probabilities of measurement outcomes of the observables $X, Z$ given by $H_p(X) = -\sum_x P(x) \log_2 P(x)$, $H_p(Z) = -\sum_z P(z) \log_2 P(z)$ offer a more general framework to quantify the intrinsic ignorance associated with incompatible measurements. Here, $x (z)$ are the measurement outcomes of the observable $X (Z)$ and $P(x) = |\langle x|\rho\rangle|^2$ ($P(z) = |\langle z|\rho\rangle|^2$) denote the probability of outcomes $x$ ($z$): $\{|x\rangle\}$ ($\{|z\rangle\}$) is the set of eigenvectors of $X (Z)$. Trade-off between the entropies of a pair of discrete non-commuting observables $X$ and $Z$ was formulated by Deutsch and was subsequently improved by Maassen and Uffink:

$$H_p(X) + H_p(Z) \geq -2 \log_2 c(X, Z),$$

(2)

where $c(X, Z) = \max_{x,y} |\langle x|z \rangle|$. The lower bound limiting the sum of entropies is independent of the state $\rho$. The term $c(X, Z)$ can assume a maximum value $\frac{1}{d}$ resulting in the maximum entropic bound of $\log_2 d$, where $d$ denotes the dimension of the system. When $\rho$ is an eigenstate of one of the observables, say $X$, the entropy of measurement $H_p(X)$ vanishes, but in turn the entropy of the observable $Z$ gets constrained by $H_p(Z) \geq -2 \log_2 c(X, Z)$.

A recent uplifting happened with the extension of entropic uncertainty relation assisted by the presence of a quantum memory, which refined the lower bound of. Here an observer Bob, whose task is to minimize the uncertainty of Alice’s measurement of the observables $X$, $Z$, is allowed to share an entangled quantum state $\rho_{AB}$ with that in Alice’s possession. The uncertainty principle, when Bob possesses a quantum memory, is given by

$$S(X|B) + S(Z|B) \geq -2 \log_2 c(X, Z) + S(A|B),$$

(3)

where $S(X|B) = S(\rho_{AB}(X)) - S(\rho_B)$, $S(Z|B) = S(\rho_{AB}(Z)) - S(\rho_B)$ are the conditional von Neumann entropies of the post-measured states $\rho_{AB}(X) = \sum_x (\Pi_x \otimes I_B)\rho_{AB}(\Pi_x \otimes I_B)$, $\rho_{AB}(Z) = \sum_z (\Pi_z \otimes I_B)\rho_{AB}(\Pi_z \otimes I_B)$ obtained after the measurements of $X, Z$ performed by Alice on the system $A$: $\Pi_x = |x\rangle\langle x|$, $\Pi_z = |z\rangle\langle z|$; and $S(A|B) = S(\rho_{AB}) - S(\rho_B)$ is the conditional von Neumann entropy. 

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of the state $\rho_{AB}$. When Alice’s system is in a maximally entangled state with Bob’s quantum memory, the second term on the right hand side of (3) takes negative value: $S(AB) = -\log_2 d$ and as $-2\log_2 c(X, Z) \leq \log_2 d$, one can achieve a trivial lower bound of zero. Thus, with the help of a quantum memory maximally entangled with Alice’s state, Bob can beat the uncertainty bound and can predict the outcomes of incompatible observables $X$, $Z$ precisely.

Statistics of quantum correlations between the outcomes of spatially separated systems get mimicked in an interesting fashion by that of temporally separated observables, measured sequentially in a single quantum system [10–14]. Non-classicality of temporal correlations between outcomes of sequentially measured observables is reflected by the violation of Leggett-Garg inequality [10] (also termed as temporal Bell inequality [15]), experimental verification of which has gained momentum recently [16–21]. Sequential measurements on the same quantum system result in the transmission of temporal information. Temporal correlations resulting from consecutive observations on a single quantum system (in contrast to measurements on spatially separated systems) draw a surge of interest in foundational investigations on quantum vs classical world view [22]. Further, information gained from correlations between the outcomes of subsequent measurements on the same quantum system is shown to be advantageous in quantum communication tasks involving state discrimination [24] and in quantum cryptography [23, 26].

In this letter, we raise the question, ‘analogous to spatial correlations, do temporal correlations arising in sequential measurement of observables, play a distinct role in reducing the uncertainty of incompatible observables?’ This boils down to explore if the sum of conditional entropies $H_p(X|X_0) + H_p(Z|Z_0)$ is always smaller than the Massen-Uffink bound of $-2\log_2 c(X, Z)$ so, that measurements of incompatible observables $X$, $Z$, conditioned by outcomes of prior time measurements of $X_0$, $Z_0$ respectively, lead to better precision. We show that uncertainty does get reduced in the presence of a quantum temporal memory due to correlations between the outcomes of $X_0$ ($Z_0$) and $X$ ($Z$) — whereas it is impossible to beat the uncertainty bound if the temporal correlations are classical.

Let us consider a qubit prepared in a completely random mixture given by $\rho = I/2$ (I denotes $2 \times 2$ identity matrix). Measurements of the observables $X = \sigma_x$ and $Z = \sigma_z$ in this state leads to Shannon entropies of measurement $H_p(X) = H_p(Z) = 1/2$, $c(X, Z) = \sqrt{2}$ and the uncertainty bound [2] is $-2\log_2 c(X, Z) = 1$; the Massen-Uffink relation is satisfied. Let us envisage the following scenario: A dichotomic observable $X_0 = \cos \theta \sigma_x + \sin \theta \sigma_x$ is measured in the quantum state followed by which $X = \sigma_x$ is sequentially measured; the probabilities of realizing the outcomes $x_0 = \pm 1$ for $X_0$ and $x = \pm 1$ for $X$ in the sequential measurement is given by $P(x_0, x) = \text{Tr}[\Pi_{x_0}\rho\Pi_x\Pi_z] = \frac{1}{4}[1 + x_0 x \cos \theta]$, where the projectors associated with dichotomic observables $X_0$ and $X$ are given by $\Pi_{x_0} = \frac{1}{2}[I + x_0 X_0]$, $\Pi_z = \frac{1}{2}[I + X Z]$. Further, measurement of $Z = \sigma_z$ is preceded by that of another dichotomic observable $Z_0 = \cos \phi \sigma_z + \sin \phi \sigma_z$ results in the probabilities $P(z_0, z) = \text{Tr}[\Pi_{z_0}\rho\Pi_z\Pi_{z_0}] = \frac{1}{4}[1 + z_0 z \cos \phi]$. The conditional Shannon entropy associated with the sequential measurement of $X_0$ and $Z$ is given by $H_p(X|X_0) = - \sum_{x_0, x = \pm 1} P(x_0, x) \log_2[P(x|x_0)] = H[\cos^2(\theta/2)]$ (where the conditional probability $P(x|x_0) = P(x_0, x)/P(x_0)$; $H(p) = -p \log_2 p - (1 - p) \log_2(1 - p)$; $0 \leq p \leq 1$ denotes the binary entropy, which is bounded by $0 \leq H(p) \leq 1$. Similarly, one gets the conditional Shannon entropy $H_p(Z|Z_0) = H[\cos^2(\phi/2)]$ associated with the sequential measurement of $Z_0$, $Z$. Clearly, the sum of conditional entropies $H_p(X|X_0) + H_p(Z|Z_0) = H[\cos^2(\theta/2)] + H[\cos^2(\phi/2)]$ beats the uncertainty bound $-2\log_2 c(X, Z) = 1$. More specifically, the uncertainty relation [2] no longer holds for entropies of $X$ and $Z$ conditioned in general reduces the information entropy i.e., $H_p(X|X_0) \leq H_p(X)$, $H_p(Z|Z_0) \leq H_p(Z)$, we prove here that the temporal correlations between the sequential measurement outcomes of $X$, $X_0$ and $Z$, $Z_0$ must necessarily be non-classical in order to beat the uncertainty bound of [2], which operates in the absence of any temporal side information.

Conditioned entropic uncertainty relation: We proceed to prove the entropic uncertainty relation assisted by temporal correlations. Consider a single quantum system prepared in the state $\rho$. In the absence of any other assisting information, the uncertainty in the observables $X$ and $Z$ is bounded by [2]. A temporal memory is created by first noting down the outcome $x_0$ ($z_0$) of an observable $X_0$ ($Z_0$) at an earlier time before recording the measurement outcomes $x$ ($z$) of $X$ ($Z$). Then, the ignorance about the measurement outcome of $X$ conditioned on the information about $X_0$ stored in temporal memory is quantified in terms of the conditional Shannon entropy $H_p(X|X_0)$:

$$H_p(X|X_0) = H_p(X_0, X) - H_p(X_0)$$

$$= H_p(X) - H_p(X_0 : X)$$

(4)

which is expressed in terms of the mutual information entropies $H_p(X_0 : X) = H_p(X) + H_p(X_0) - H_p(X, X_0)$ and the unconditioned entropies $H_p(X)$. Similarly, entropy of $Z$, conditioned by the outcomes of $Z_0$ is given by:

$$H_p(Z|Z_0) = H_p(Z) - H_p(Z : Z_0).$$

(5)

The entropic uncertainty relation in the presence of temporal memory is then obtained by identifying the lower bound on the sum of conditional entropies $H_p(X|X_0) + H_p(Z|Z_0)$. Combining [4], [5], using the minimum value $[H_p(X) + H_p(Z)]_{\text{min}} =
conditional entropies obey a correlation probabilities are of the form (7), the sum of $X$ that in a prior measurement not temporal memory requires that the correlation measurement outcomes of the observable $X$ run on the quantum state are sure [min($H_p(X)$, $H_p(Z)$)] $= H_p^{(min)}(X)$, $H_p^{(min)}(Z)$, $H_p^{(min)}(X) - H_p^{(min)}(Z)$.

We now prove the following theorem.

**Theorem:** If temporal correlations of the outcomes of $X_0$, $X$ and those of $Z_0$, $Z$ obtained from sequential measurement runs on the quantum state are classical (the correlation probabilities are of the form 7), the sum of conditional entropies obey a temporal entropic steering inequality 23

$$H_p(X|X_0) + H_p(Z|Z_0) \geq -2 \log_2 c(X, Z).$$

**Proof:** Let us consider the conditional information for the measurement outcomes of the observable $X$, given that in a prior measurement $X_0$ has taken the value $x_0$:

$$H_p(X|X_0 = x_0) = -\sum_x P(x|x_0) \log_2 P(x|x_0)$$

-2 \log_2 c(X, Z) (as given by 2) and the maximum values $[H_p(X_0 : X)]_{\max}$, $[H_p(Z_0 : Z)]_{\max}$ of the mutual information entropies, we obtain,

$$H_p(X|X_0) + H_p(Z|Z_0) \geq \max \{0, -2 \log_2 c(X, Z) - \{H_p(X_0 : X)\}_{\max} - \{H_p(Z_0 : Z)\}_{\max}\}$$

$$\geq \max \{0, -2 \log_2 c(X, Z) - H_p^{(min)}(X) - H_p^{(min)}(Z)\}.$$ (6)

The conditional probability $P(x|x_0) = P(x_0, x)/P(x_0)$ corresponding to classical temporal correlations (see 7) is given by,

$$P(x|x_0) = \sum_\lambda p_\lambda P_\lambda(x_0) Q_\lambda(x)$$

$$= \sum_\lambda p_\lambda, x_0 Q_\lambda(x) (11)$$

where we have denoted $p_\lambda, x_0 = \sum_{\lambda'} p_\lambda P_\lambda(x_0)$. Note that $\sum_\lambda p_\lambda, x_0 = 1$, and $0 \leq p_\lambda, x_0 \leq 1$.

Consider the relative entropy $D(P_{x_0}||Q_{x_0})$ of the probability distributions $P_{x_0}(\lambda, x) = p_\lambda, x_0 Q_\lambda(x)$ and $Q_{x_0}(\lambda, x) = p_\lambda, x_0 P(x|x_0)$. Positivity of the relative entropy leads to the following identification 30:

$$D(P_{x_0}||Q_{x_0}) = \sum_\lambda \sum_x p_\lambda, x_0 Q_\lambda(x) \log_2 \left[\frac{Q_\lambda(x)}{P(x|x_0)}\right] \geq 0$$

$$\Rightarrow H_p(X|x_0) \geq \sum_\lambda p_\lambda, x_0 H_p^{(\lambda)}(X) (12)$$

where $H_p^{(\lambda)}(X) = -\sum_x Q_\lambda(x) \log_2 Q_\lambda(x)$. Thus, the average conditional information $H_p(X|X_0) = -\sum_\lambda p_\lambda H_p^{(\lambda)}(X|x_0)$; $p(x_0) = \sum_x P(x|x_0) = \sum_\lambda p_\lambda P_\lambda(x_0)$ should obey the constraint

$$H_p(X|X_0) \geq \sum_\lambda p_\lambda \sum p_\lambda, x_0 H_p^{(\lambda)}(X)$$

$$\geq \sum_\lambda p_\lambda H_p^{(\lambda)}(X),$$ (13)

Similarly, we obtain

$$H_p(Z|Z_0) \geq \sum_\lambda p_\lambda H_p^{(\lambda)}(Z).$$ (14)

Thus, the sum of conditional entropies are constrained by

$$H_p(X|X_0) + H_p(Z|Z_0) \geq \sum_\lambda p_\lambda [H_p^{(\lambda)}(X) + H_p^{(\lambda)}(Z)]$$

$$= -2 \log_2 c(X, Z).$$ (15)

In the second line of (15) we have employed the Massen-Uffink relation $H_p^{(\lambda)}(X) + H_p^{(\lambda)}(Z) \geq -2 \log_2 c(X, Z)$. 

Quantum temporal memory requires that the correlation outcomes of the observables at different time instants are not governed by the joint probabilities of the form 7.
This identification reveals the crucial significance of quantum temporal memory to achieve sharpened predictions of incompatible observables.

We illustrate how temporal correlations assist in reducing the entropic spread of non-commuting observables by considering an example of a spin-$s$ quantum rotor prepared initially in a state $\rho = \frac{1}{2s+1} \sum_{m_z=-s}^{s} |s,m_z\rangle \langle s,m_z| = I_{2s+1}$. (Here $|s,m_z\rangle$ are the simultaneous eigenstates of the squared spin operator $S^2 = S_x^2 + S_y^2 + S_z^2$ and the $z$-component of spin $S_z$ (with respective eigenvalues $s(s+1)$ and $m_z$); $I_{2s+1}$ is the $(2s+1) \times (2s+1)$ identity matrix.) Measurement of non-commuting observables $X = S_x$ and $Z = S_z$ results in the probabilities of outcomes $-s \leq m_x, m_z \leq s$ as $P(m_x) = \text{Tr}[\rho \Pi_{m_x}] = \frac{1}{2s+1}; P(m_z) = \text{Tr}[\rho \Pi_{m_z}] = \frac{1}{2s+1}$, where $\Pi_{m}$ denotes the projection operator of the corresponding observable. The spread in the completely random measurement outcomes is revealed in terms of the corresponding Shannon entropies of measurement $H_\rho(X) = \log_2(2s+1)$ and $H_\rho(Z) = \log_2(2s+1)$, which obey the trade-off relation [2] – the largest value of the uncertainty bound on the right hand side being $\log_2(2s+1)$.

In order to identify how entropic uncertainty relation for $S_x$ and $S_z$, assisted by prior conditioning, can reveal enhanced precision of the observables, we consider dynamical evolution of the system governed by the Hamiltonian $\mathcal{H} = \omega \mathcal{S}_y$. Under the Hamiltonian dynamics, the evolution of $z$ component of spin is given by $S_z(t) = e^{i S_{z}\omega t} S_z e^{-i S_{z} \omega t} = S_z \cos(\omega t) + S_x \sin(\omega t)$. We consider sequential measurement of $S_z(t)$ at different times as follows. In the first run, the observable $S_z(t)$ is measured at time $t = t_{x0}$ and consequently at $t_x = \pi/2\omega$ (this corresponds to sequential measurement of observables $X_0 = S_x \cos(\omega t_{x0}) + S_x \sin(\omega t_{x0})$ and $X = S_z$) with a dimensionless time separation $\theta = \omega t_{x0} - \pi/2$. The sequential measurements enable the observer to record the temporal correlation probabilities $P(m_{z0}, m_{z}; \theta)$ of the outcomes $-s \leq m_{z0}, m_{z} \leq s$ of the observables $X_0 = S_z(t_{x0})$ and $X = S_z$. Then, in one more round of observations, $S_z$ is measured sequentially at two different time instants $t_{z0}$ and $t_z = \pi/2\omega$ (i.e., a measurement of $Z_0 = S_z \cos(\omega t_{z0}) + S_z \sin(\omega t_{z0})$ and then consequently $Z = S_z$), separated by a dimensionless time parameter $\phi = \omega t_{z0} - \pi$ is performed and the correlation probabilities $P(m_{z0}, m_{z}; \phi)$ of the $(2s+1)^2$ outcomes $-s \leq m_{z0}, m_{z} \leq s$ are noted down.

The probabilities of sequential measurement outcomes of $S_z(t)$ at two different times $t_{x0}$ and $t_x = \pi/2\omega$ are given by [13].

$$P(m_{z0}, m_{z}; \theta) = P(m_{z0}; t_{x0}) P(m_{z}; t_{x} | m_{z0}; t_{x0})$$

$$= \frac{\text{Tr}[\rho \Pi_{m_{z0}(t_{x0})}] \text{Tr}[\Pi_{m_{z}(t_{x})} \rho \Pi_{m_{z0}(t_{x0})}] \Pi_{m_{z}(t_{x})}}{P(m_{z0}; t_{x0})}$$

$$= \frac{1}{2s+1} \frac{\text{Tr}[\Pi_{m_{z0}(t_{x0})} \Pi_{m_{z}(t_{x})}]}{\text{Tr}[\Pi_{m_{z0}(t_{x0})}]}$$

$$= \frac{1}{2s+1} |(s, m_{z0}) e^{-i\omega(t_{x0}-t_{x})} S_y |(s, m_{z0})|^2$$

$$= \frac{1}{2s+1} |d_{m_{z0}}^m(\theta)|^2$$

where $\Pi_{m}(t) = e^{i\omega t} S_y |s, m\rangle \langle s, m| e^{-i\omega t} S_y$ is the projection operator measuring the outcome $m$ of the spin component $S_z(t)$; and $d_{m_{z0}}^m(\theta) = |(s, m_{z0}) e^{-i\phi} S_y |(s, m_{z0})|^2$ are the matrix elements of the $(2s+1)$ dimensional irreducible representation of rotation [31] about $y$-axis by an angle $\theta = \omega(t_{x0}-t_{x})$. The marginal probability associated with measuring $S_z(t_{x0})$ is readily obtained as $P(m_{z0}; t_{x0}) = \text{Tr}[\rho \Pi_{m_{z0}(t_{x0})}] = \frac{1}{2s+1}$, Similarly, the correlation probabilities in the second run of sequential measurements are obtained as, $P(m_{z0}, m_{z}; \phi) = \frac{1}{2s+1} |d_{m_{z0}}^m(\phi)|^2$, and the marginal probabilities $P(m_{z0}; t_{x0}) = 1/(2s+1)$. The conditional entropies of measurement (which depend only on the time separations $\theta, \phi$) $H_\rho(X|X_0) = \mathcal{H}(\theta)$ and
\[ H_p(Z|Z_0) = \mathcal{H}(\phi) \] are given by,
\[
\mathcal{H}(\theta) = -\frac{1}{2s+1} \sum_{m_{i_1}, m_s} |d_{m_{i_1}, m_s}^1(\theta)|^2 \log_2 |d_{m_{i_1}, m_s}^1(\theta)|^2
\]
\[
\mathcal{H}(\phi) = -\frac{1}{2s+1} \sum_{m_{i_1}, m_s} |d_{m_{i_1}, m_s}^2(\phi)|^2 \log_2 |d_{m_{i_1}, m_s}^2(\phi)|^2.
\]

We define a quantity \(\mathcal{M}_s(\theta, \phi)\) as the difference between the sum of conditional entropies and the Massen-Uffink uncertainty bound \(-2\log_2 c(X, Z)\)
\[
\mathcal{M}_s(\theta, \phi) = H_p(X|X_0) + H_p(Z|Z_0) + 2 \log_2 c(X, Z)
\]
\[
= \mathcal{H}(\theta) + \mathcal{H}(\phi) + 2 \log_2 c(X, Z)
\]
in order to demonstrate improved precision in the measurement of the spin components \(X = S_x\) and \(Z = S_z\). While a classical temporal side information results in \(\mathcal{M}_s(\theta, \phi)\) being necessarily positive, presence of a quantum temporal memory, created by appropriate sequential measurements, can reveal itself in non-positive values of \(\mathcal{M}_s(\theta, \phi)\). In Fig. 1, we have plotted the quantity \(\mathcal{M}_s(\theta, \phi)\) as a function of \(\theta\) and \(\phi\) for spin values \(s = 1/2, 1, 3/2\) and 2. The results clearly demonstrate reduction in the uncertainties of the non-commuting spin components \(S_x, S_z\) (in the region where \(\mathcal{M}_s\) is negative) – in the presence of a quantum temporal memory. We note that the range of time-separation \(\theta\) and \(\phi\), over which \(\mathcal{M}_s\) assumes negative values, reduces with the increase of spin \(s\) – thus indicating a quantum to classical transition of the temporal memory in the limit of large spin \(s\).

**Conclusions:** Uncertainty principle reflects the inevitability built within the quantum framework in realizing deterministic outcomes for non-commuting observables of a particle. Entropic uncertainty relation\([5]\) captures the trade-off in the spread of the outcomes of incompatible observables. However, a deterministic prediction is ensured when the particle is entangled maximally with another party. Berta et. al.,\([3]\) brought out the subtle interplay between uncertainty and entanglement by extending the entropic uncertainty principle in the presence of quantum side information. In this work, we have explored the interesting association between temporal correlations and uncertainty. Our entropic uncertainty relation reveals that the presence of quantum temporal side information too plays a significant role in beating the uncertainty bound. More specifically, our results offer a unified view that a prior quantum knowledge, achieved with the help of suitable spatially/temporally separated observations, empower a deterministic prediction of non-commuting observables.

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