THE CHERN AND EULER COEFFICIENTS OF MODULES

L. GHEZZI, S. GOTO, J. HONG, K. OZEKI, T.T. PHUONG, AND W. V. VASCONCELOS

Abstract. The set of the first Hilbert coefficients of parameter ideals relative to a module—its Chern coefficients—over a local Noetherian ring codes for considerable information about its structure—noteworthy properties such as that of Cohen-Macaulayness, Buchsbaumness, and of having finitely generated local cohomology. The authors have previously studied the ring case. By developing a robust setting to treat these coefficients for unmixed rings and modules, the case of modules is analyzed in a more transparent manner. Another series of integers arise from partial Euler characteristics and are shown to carry similar properties of the module. The technology of homological degree theory is also introduced in order to derive bounds for these two sets of numbers.

Dedicated to Professor G. Valla for his groundbreaking contributions to the theory of Hilbert functions.

1. Introduction

Let \( R \) be a Noetherian local ring with maximal ideal \( m \) and let \( I \) be an \( m \)-primary ideal. There is a great deal of interest on the set of \( I \)-good filtrations of \( R \). More concretely, on the set of multiplicative, decreasing filtrations

\[
\mathcal{A} = \{ I_n \mid I_0 = R, I_{n+1} = II_n, n \gg 0 \}
\]

of \( R \) ideals which are integral over the \( I \)-adic filtration, conveniently coded in the corresponding Rees algebra and its associated graded ring

\[
\mathcal{R}(\mathcal{A}) = \sum_{n \geq 0} I_n t^n, \quad \text{gr}_A(R) = \sum_{n \geq 0} I_n/I_{n+1}.
\]

Our focus here is on a set of filtrations both broader and more narrowly defined. Let \( M \) be a finitely generated \( R \)-module. The Hilbert polynomial of the associated graded module

\[
\text{gr}_I(M) = \bigoplus_{n \geq 0} I^n M/I^{n+1} M,
\]

more precisely the values of the length \( \lambda(M/I^{n+1} M) \) of \( M/I^{n+1} M \) for large \( n \), can be assembled as

\[
P_M(n) = \sum_{i=0}^{r} (-1)^i e_i(I, M) \binom{n + r - i}{r - i},
\]

AMS 2000 Mathematics Subject Classification: 13H10, 13H15, 13A30.

The first author is partially supported by a grant from the City University of New York PSC-CUNY Research Award Program-41. The second author is partially supported by Grant-in-Aid for Scientific Researches (C) in Japan (19540054). The fourth author is supported by a grant from MIMS (Meiji Institute for Advanced Study of Mathematical Sciences). The fifth author is supported by JSPS Ronpaku (Dissertation of PhD) Program. The last author is partially supported by the NSF.
where \( r = \dim M > 0 \). In most of our discussion, either \( I \) or \( M \) is fixed, and by simplicity we set \( e_1(I, M) = e_1(M) \) or \( e_1(I, M) = e_1(I) \), accordingly. Occasionally the first Hilbert coefficient \( e_1(I, M) \) is referred to as the Chern coefficient of \( I \) relative to \( M \) \((33)\).

The authors have examined \([8], [9], [14], [33]\) how the values of \( e_1(Q, R) \) codes for structural information about the ring \( R \) itself. More explicitly one defines the set

\[
\Lambda(M) = \{ e_1(Q, M) \mid Q \text{ is a parameter ideal for } M \}
\]

and examines what its structure expresses about \( M \). In case \( M = R \), this set was analyzed for the following extremal properties:

(a) \( 0 \in \Lambda(R) \);
(b) \( \Lambda(R) \) contains a single element;
(c) \( \Lambda(R) \) is bounded.

The following describes some of the main results:

**Theorem.** Let \( R \) be an unmixed Noetherian local ring.

(a) \((8, \text{Theorem 2.1})\) \( R \) is Cohen-Macaulay if and only if \( e_1(Q) = 0 \) for some parameter ideal \( Q \).

(b) \((14, \text{Theorem 4.1})\) \( R \) is a Buchsbaum ring if and only if \( e_1(Q) \) is constant for all parameter ideals \( Q \).

(c) \((8, \text{Proposition 4.2})\) \( R \) is a generalized Cohen-Macaulay ring if and only if the set \( \Lambda(R) \) is finite.

The task of determining the elements of \( \Lambda(M) \) has turned out to be rather daunting, along with various versions of these results without the unmixedness hypothesis. More amenable has been the approach to obtain specialized bounds using cohomological techniques. An unresolved issue has been to describe the character of the set \( \Lambda(M) \), in particular the role of its extrema and the gap structure of the set itself.

The other invariant of the module \( M \) in our investigation is the following. Let \( Q = (x_1, x_2, \ldots, x_r) \) be a parameter ideal for \( M \). We denote by \( H_i(Q; M) \) \((i \in \mathbb{Z})\) the \( i \)-th homology module of the Koszul complex \( K_\bullet(Q; M) \) generated by the system \( x = \{x_1, x_2, \ldots, x_r\} \) of parameters of \( M \). We put

\[
\chi_1(Q; M) = \sum_{i \geq 1} (-1)^{i-1} \lambda_R(H_i(Q; M))
\]

and call it the first Euler characteristic of \( M \) relative to \( Q \); hence

\[
\chi_1(Q; M) = \lambda_R(M/QM) - e_0(Q, M)
\]

by a classical result of Serre (see \([1], [27]\)).

In analogy to \( \Lambda(M) \), one defines the set

\[
\Xi(M) = \{ \chi_1(Q; M) \mid Q \text{ is a parameter ideal for } M \}
\]

and examines again what its structure expresses about \( M \). Most of the properties of this set can be assembled from a diverse literature, particularly from \([27, \text{Appendice II}]\). The outcome is a listing that mirrors, step-by-step, all the properties of the set \( \Lambda(M) \) that we study.

We shall now describe more precisely our results. Section 2 starts with a review of some elementary computation rules for \( e_1(Q, M) \) under hyperplane sections, more properly modulo superficial elements. Since part of our goal is to extend to modules the results on rings above,
given the ubiquity of the unmixedness hypothesis, we develop a fresh setting to treat the module case. It made for more transparent proofs. These are carried out in Sections 3–5.

Section 6 introduces homological degree techniques to obtain special bounds for the set \( \Lambda(M) \). The treatment here is more general and sharper than in [33]. Thus in Corollary 6.8 it is proved that the set

\[
\Lambda_Q(M) = \{ e_1(q, M) : q \text{ is a parameter ideal for } M \text{ with the same integral closure as that of } Q \},
\]

is finite. In Section 7, we treat the sets \( \Xi(M) \) and \( \Xi_Q(M) \), focusing on the properties that have analogs in \( \Lambda(M) \) (see Table 1). In particular, we prove that Euler characteristics can be uniformly bounded by homological degrees (Theorem 7.2).

In the brief Section 8, we consider the numerical function, which we call the Hilbert characteristic of \( M \) with respect to \( (x) \):

\[
h(x; M) = \sum_{i=0}^{r} (-1)^i e_i(x; M).
\]

If the system \( x = \{x_1, x_2, \ldots, x_r\} \) of parameters of \( M \) forms a \( d \)-sequence for \( M \), \( h(x; M) \) has some properties of a homological degree. They are enough to bound the Betti numbers \( \beta_i(M) \) in terms of \( \beta_i(R/\mathfrak{m}) \), a well-known property of cohomological degrees. Finally, in Section 9, we recast in the context of Buchsbaum-Rim coefficients several questions treated in this paper.

A street view of our results for the convenience of the reader is given in the following table. Let \( R \) be a Noetherian local ring with infinite residue class field and \( M \) a finitely generated \( R \)-module with \( r = \dim_R M \geq 2 \). Let \( \mathcal{P}(M) \) be the collection of systems \( x = \{x_1, x_2, \ldots, x_r\} \) of parameters of \( M \). In [8] and in this paper, the authors study multiplicity derived numerical functions

\[
f : \mathcal{P}(M) \rightarrow \mathbb{N}
\]

on emphasis on the nature of its range

\[
X_f(M) = \{ f(x) \mid x \in \mathcal{P}(M) \}.
\]

For the two functions \( e_1(x, M) \) and \( \chi_1(x; M) \), more properly \( f_1(x) = -e_1(x, M) \), and \( f_2(x) = \chi_1(x; M) \), respectively, identical assertions about the character of \( X_f(M) \) are expressed in the following grid:

| \( X_f(M) \subseteq [0, \infty) \) | \( M \) | [22] | [27, Appendix II] |
|---|---|---|---|
| 0 \in X_f(M) | \( M \) Cohen-Macaulay | [8*, Theorem 3.1] | [27, Appendix II] |
| \( |X_f(M)| < \infty \) | \( M \) generalized Cohen-Macaulay | [8*, Theorem 4.5] | [6] |
| \( |X_f(M)| = 1 \) | \( M \) Buchsbaum | [14*, Theorem 5.1] | [29] |
| \( |f(x) \mid Q = (x)| < \infty \) | \( Q \) | Corollary 6.8 | Corollary 7.2 |

**TABLE 1. Properties of a finitely generated module \( M \) carried by the values of either function.** An adorned reference \([XY]\) requires that the module \( M \) be unmixed. The third and fourth columns refer to the functions \( f_1(x) \) and \( f_2(x) \) respectively.
2. Preliminaries

Throughout this section let $R$ be a Noetherian local ring with maximal ideal $m$ and let $M$ be a finitely generated $R$–module. For basic terminology and properties of Noetherian rings and Cohen–Macaulay rings and modules we make use of [3] and [23]. For convenience of exposition we treat briefly the role of hyperplane sections in Hilbert functions and examine unmixed modules. We add further clarifications when we define homological degrees.

Hyperplane sections and Hilbert polynomials. We need rules to compute these coefficients. Typically they involve so called superficial elements or filter regular elements. We keep the terminology of generic hyperplane section, even when dealing with Samuel’s multiplicity with respect to an $m$–primary ideal $I$ and its Hilbert coefficients $e_i(M) = e_i(I, M)$. Hopefully this usage will not lead to undue confusion. We say that $h \in I$ is a parameter for $M$, if $\dim_R M/hM < \dim_R M$.

Let us begin with the following.

**Lemma 2.1.** Let $(R, m)$ be a Noetherian local ring, $I$ an $m$–primary ideal of $R$, and $M$ a finitely generated $R$–module. Let $h \in I$ and suppose that $\lambda(0 :_M h) < \infty$. Then we have the following.

(a) $h$ is a parameter for $M$, if $\dim_R M > 0$.

(b) $\lambda(0 :_M h) \leq \lambda(\Ext^0_M(M/hM))$.

(c) If $\dim_R M > 1$ and $M/hM$ is Cohen–Macaulay, then $M$ is Cohen–Macaulay.

**Proof.** Suppose that $\dim_R M > 0$ and let $p \in \Supp\, \p$ with $\dim_R \p = \dim_R M$. Then

$$\lambda(0 :_M h) = \lambda(\Ext^0_M(M/hM)) = \lambda(\Ext^0_M(M/hM)) - \lambda(0 :_M h) \leq \lambda(\Ext^0_M(M/hM)).$$

Therefore, if $\dim_R M > 1$ and $M/hM$ is Cohen–Macaulay, then $h$ is $M$–regular and hence $M$ is Cohen–Macaulay as well.

We will make repeated use of [24, (22.6)] and [22, Section 3].

**Proposition 2.2.** Let $(R, m)$ be a Noetherian local ring, $I$ an $m$–primary ideal of $R$, and $M$ a finitely generated $R$–module with $r = \dim_R M > 0$. Let $h \in I$ and assume that $h$ is superficial for $M$ with respect to $I$ (in particular $h \in I \setminus mI$).

(a) The Hilbert coefficients of $M$ and $M/hM$ satisfy

$$e_i(M) = e_i(M/hM) \quad \text{for } 0 \leq i < r - 1 \quad \text{and}$$

$$e_{r-1}(M) = e_{r-1}(M/hM) + (-1)^r \lambda(0 :_M h).$$
(b) Let $0 \to A \to B \to C \to 0$ be an exact sequence of finitely generated $R$–modules. If $t = \dim_R A < s = \dim_R B$, then $e_i(B) = e_i(C)$ for $0 \leq i < s - t$. In particular, if $t = 0$ and $s \geq 2$, then $e_1(B) = e_1(C)$.

(c) If $M$ is a module of dimension 1 and $I$ is a parameter ideal for $M$, then
$$e_1(M) = -\lambda(H^0_m(M)).$$

(d) If $M$ is a module of dimension 2 and $I$ is a parameter ideal for $M$, then
$$e_1(M) = e_1(M/hM) + \lambda(0 : M h) = -\lambda(H^0_m(M/hM)) + \lambda(0 : M h) = -\lambda((0 : \hat{h})(M) h).$$

Proof. See Proof of Lemma 2.1 for assertion (d).

The following Corollary was previously observed in [22]. By induction on $r = \dim_R M$, it also can be achieved independently, using Proposition 2.2.

**Corollary 2.3.** If $M$ is a module of positive dimension and $I$ is a parameter ideal for $M$, then $e_1(I, M) \leq 0$.

**Unmixed modules.** We recall the notion of unmixed local rings and modules and develop a setting to study their Hilbert coefficients.

**Definition 2.4.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d$. Then we say that $R$ is unmixed, if $\dim \hat{R}/p = d$ for every $p \in \text{Ass} \hat{R}$, where $\hat{R}$ is the $\mathfrak{m}$–adic completion of $R$. Similarly, let $M$ be a finitely generated $R$–module of dimension $r$. Then we say that $M$ is unmixed, if $\dim \hat{R}/p = r$ for every $p \in \text{Ass} \hat{M}$, where $\hat{M}$ denotes the $\mathfrak{m}$–adic completion of $M$.

Our formulation of unmixedness is the following.

**Theorem 2.5.** Let $R$ be a Noetherian local ring and $M$ a finitely generated $R$–module with $\dim_R M = \dim R$. Then the following conditions are equivalent:

1. $M$ is an unmixed $R$–module;
2. There exists a surjective homomorphism $S \to \hat{R}$ of rings together with an embedding $\hat{M} \to S^n$ as an $S$–module for some $n > 0$, where $S$ is a Gorenstein local ring with $\dim S = \dim R$.

Proof. We have only to prove (1) $\Rightarrow$ (2). We may assume $R$ is complete. Thanks to Cohen’s structure theorem of complete local rings, we can choose a surjective homomorphism $S \to R$ of rings such that $S$ is a Gorenstein local ring with $\dim S = \dim R$. Then, because $\text{Ass}_S M \subseteq \text{Ass}_S S$ and the $S_p$–module $M_p$ is reflexive for all $p \in \text{Ass}_S M$, the canonical map
$$M \to \text{Hom}_S(\text{Hom}_S(M, S), S)$$
is injective, while we get an embedding
$$\text{Hom}_S(\text{Hom}_S(M, S), S) \to S^n$$for some $n > 0$, because $\text{Hom}_S(M, S)$ is a finitely generated $S$–module. Hence the result.

**Corollary 2.6 ([11]).** Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M$ a finitely generated $R$–module with $\dim_R M = \dim R \geq 2$. If $M$ is an unmixed $R$–module, then $H^0_\mathfrak{m}(M)$ is finitely generated.
Proof. We may assume \( R \) is complete. We maintain the notation in Proof of Theorem 2.5 and let \( n \) denote the maximal ideal of \( S \). Then, applying the functors \( H^i_n(*) \) to the exact sequence

\[
0 \to M \to S^n \to C \to 0
\]

of \( S \)-modules, we get \( H^1_m(M) \cong H^0_n(C) \), because depth \( S \geq 2 \). Hence \( H^1_m(M) \) is finitely generated.

3. Vanishing of \( e_1(Q, M) \)

Let \( R \) be a Noetherian local ring with maximal ideal \( m \) and \( M \) a finitely generated \( R \)-module with \( r = \dim_R M \). Recall that a parameter ideal for \( M \) is an ideal \( Q = (x_1, x_2, \ldots, x_r) \subseteq m \) in \( R \) with \( \lambda(M/QM) < \infty \). We then have the following, which is a generalization of [8, Theorem 2.1] to the module case. For the sake of completeness let us note a proof based on Theorem 2.5

**Theorem 3.1.** Let \( R \) be a Noetherian local ring and \( M \) a finitely generated \( R \)-module with \( \dim_R M \geq 2 \). Suppose that \( M \) is unmixed and let \( Q \) be a parameter ideal for \( M \). Then the following conditions are equivalent:

1. \( M \) is a Cohen–Macaulay \( R \)-module;
2. \( e_1(Q, M) = 0 \).

**Proof.** We have only to prove \( (2) \Rightarrow (1) \). To do this, passing to the ring \( \hat{R}/[(0) : \hat{R} M] \), we may assume that \( R \) is a Gorenstein ring with \( \dim_R = \dim_R M = r \). Hence, without loss of generality, we may assume \( Q \) is also a parameter ideal for \( R \). Enlarging the residue class field \( R/m \) if necessary, we may assume the field \( R/m \) is infinite. By Theorem 2.5 we get an exact sequence

\[
(\sharp) \quad 0 \to M \to R^n \to C \to 0
\]

of \( R \)-modules for some \( n > 0 \). Let us choose an element \( h \in Q \setminus mQ \) so that \( h \) is superficial both for \( M \) and \( C \) with respect to \( Q \). Hence \( h \) is \( R \)-regular. If \( r = 2 \), then by Proposition 2.2 (d) \( h \) is a non-zerodivisor for \( H^1_m(M) \), so that \( H^1_m(M) = (0) \). Therefore \( M \) is Cohen–Macaulay.

Suppose that \( r \geq 3 \) and that our assertion holds true for \( r - 1 \). We look at the exact sequence

\[
0 \to (0) : C h \to M/hM \xrightarrow{\varphi} (R/hR)^n \to C/hC \to 0
\]

of \( R/hR \)-modules derived from exact sequence (\( \sharp \)) above and put \( L = \text{Im} \varphi \). Then \( Q/hR \) is a parameter ideal for the unmixed \( R/hR \)-module \( L \) of dimension \( r - 1 \geq 2 \). We get \( e_1(Q/hR, L) = e_1(Q/hR, M/hM) \), because

\[
L = (M/hM)/[(0) : C h] \quad \text{and} \quad \lambda((0) : C h) < \infty.
\]

Consequently, since \( e_1(Q/hR, L) = e_1(Q/hR, M/hM) = e_1(Q, M) = 0 \) by Proposition 2.2 (a), by the hypothesis of induction on \( r \) the \( R \)-module \( L \) is Cohen-Macaulay. Hence

\[
H^i_m(M/hM) = H^i_m(L) = (0) \quad \text{for} \quad 1 \leq i < r - 1.
\]

We now apply the functors \( H^i_m(*) \) to the exact sequence

\[
0 \to M \xrightarrow{h} M \to M/hM \to 0
\]

and get a long exact sequence

\[
0 \to H^0_m(M/hM) \to H^1_m(M) \xrightarrow{h} H^1_m(M) \to H^1_m(M/hM) \to \cdots
\]
of local cohomology modules. Then $H^1_m(M/hM) = H^1_m(L) = (0)$, so that $H^1_m(M) = (0)$, since $H^1_m(M)$ is finitely generated by Corollary 2.6 and $H^1_m(M) = hH^1_m(M)$. Consequently, $H^1_m(M/hM) = (0)$, so that $M/hM$ is a Cohen–Macaulay $R$–module. Thus $M$ is a Cohen–Macaulay $R$–module as well, because $h$ is $M$–regular. 

Let us list some consequences of Theorem 3.1. Let $R$ be a Noetherian local ring and $M$ a finitely generated $R$–module. We put

$$\text{Ass}_{hR} M = \{ p \in \text{Supp}_R M \mid \dim R/p = \dim R M \}.$$ 

Let $(0) = \bigcap_{p \in \text{Ass}_{hR} M} M(p)$ be a primary decomposition of $(0)$ in $M$, where $M(p)$ is a $p$–primary submodule of $M$ for each $p \in \text{Ass}_{hR} M$. We put

$$U_M(0) = \bigcap_{p \in \text{Ass}_{hR} M} M(p)$$

and call it the unmixed component of $M$. We then have the following.

**Lemma 3.2.** Let $R$ be a Noetherian local ring and $M$ a finitely generated $R$–module with $r = \dim_R M > 0$. Let $Q$ be a parameter ideal for $M$. Let $U = U_M(0)$ and suppose that $U \neq (0)$. We put $N = M/U$. Then the following assertions hold.

(a) $\dim_R U < \dim_R M$.

(b) $e_1(Q, M) = \begin{cases} e_1(Q, N) & \text{if } \dim_R U \leq r - 2, \\ e_1(Q, N) - s_0 & \text{if } \dim_R U = r - 1, \end{cases}$

where $s_0 \geq 1$ denotes the multiplicity of the graded $\text{gr}_Q(R)$–module $\bigoplus_{n \geq 0} U/(Q^{n+1}M \cap U)$.

(c) $e_1(Q, M) \leq e_1(Q, N)$ and the equality $e_1(Q, M) = e_1(Q, N)$ holds if and only if $\dim_R U \leq r - 2$.

**Proof.** (a) This is clear, since $U_p = (0)$ for all $p \in \text{Ass}_{hR} M$.

(b) We write

$$\lambda_R(U/(Q^{n+1}M \cap U)) = s_0 \binom{n + t}{t} - s_1 \binom{n + t - 1}{t - 1} + \cdots + (-1)^t s_t$$

for $n \gg 0$ with integers $\{s_i\}_{0 \leq i \leq t}$, where $t = \dim U$. Then the claim follows from the exact sequence $0 \to U \to M \to N \to 0$ of $R$–modules, which gives

$$\lambda_R(M/Q^{n+1}M) = \lambda_R(N/Q^{n+1}N) + \lambda_R(U/(Q^{n+1}M \cap U)), \quad \forall n \geq 0.$$ 

(c) This follows from (b) and the fact that $s_0 \geq 1$. 

We can now state [8, Theorem 2.6] for modules. Thanks to Lemma 3.2, one can prove it similarly as in the ring case. Notice that the implication $(1) \Rightarrow (2)$ in Theorem 3.3 is not true in general without the assumption that $R$ is a homomorphic image of a Cohen–Macaulay ring. See [8, Remark 2] for an example.
Theorem 3.3. Let $\mathcal{R}$ be a Noetherian local ring and $M$ a finitely generated $\mathcal{R}$–module of dimension $r \geq 2$. Suppose that $\mathcal{R}$ is a homomorphic image of a Cohen–Macaulay ring. Let $U = U_M(0)$ and let $Q$ be a parameter ideal for $M$. Then the following conditions are equivalent:

1. $e_1(Q, M) = 0$;
2. $M/U$ is a Cohen–Macaulay $\mathcal{R}$–module and $\dim_\mathcal{R} U \leq r - 2$.

The following corollary gives a characterization of Cohen–Macaulayness.

Corollary 3.4. Let $\mathcal{R}$ be a Noetherian local ring, $M$ a finitely generated $\mathcal{R}$–module with $r = \dim_{\mathcal{R}} M > 0$, and $Q$ a parameter ideal for $M$. Let $1 \leq k \leq r$ be an integer and assume that $e_i(Q, M) = 0$ for all $1 \leq i \leq k$. Then

$$\dim_\mathcal{R} U_{\hat{M}}^{-}(0) \leq r - (k + 1) \quad \text{and} \quad H_{m}^{r-j}(M) = (0)$$

for all $1 \leq j \leq k$. In particular, if $k = r$, then $M$ is a Cohen–Macaulay $\mathcal{R}$–module.

Proof. We may assume that $\mathcal{R}$ is complete. We put $U = U_M(0)$ and $N = M/U$. Then $e_0(Q, M) = e_0(Q, N)$, since $\dim_\mathcal{R} U < r$. Therefore, by Theorem 3.3, $N$ is a Cohen–Macaulay $\mathcal{R}$–module, so that we have exact sequences

$$0 \to U/Q^{n+1}U \to M/Q^{n+1}M \to N/Q^{n+1}N \to 0$$

of $\mathcal{R}$–modules for all $n \geq 0$. Hence, computing Hilbert polynomials, we get $\dim_\mathcal{R} U \leq r -(k+1)$. Let $1 \leq j \leq k$. Then $H_{m}^{r-j}(U) = (0)$, since $\dim_\mathcal{R} U < r - j$, while $H_{m}^{r-j}(N) = (0)$, as $N$ is a Cohen–Macaulay $\mathcal{R}$–module with $\dim_\mathcal{R} N = r$. Thus $H_{m}^{r-j}(M) = (0)$ as claimed. \hfill $\square$

Let $\mathcal{R}$ be a Noetherian local ring and $M$ a finitely generated $\mathcal{R}$–module. In [8] the authors examined the rings with $e_1(Q, \mathcal{R})$ vanishing. Here we briefly extend this theory to modules. Let us begin with the definition.

Definition 3.5. A finitely generated $\mathcal{R}$–module $M$ is called a Vasconcelos module, if either $\dim_\mathcal{R} M = 0$, or $\dim_\mathcal{R} M > 0$ and $e_1(Q, M) = 0$ for some parameter ideal $Q$ for $M$.

Every Cohen–Macaulay module is by definition Vasconcelos. Here is a basic characterization. We skip the proof, since it is exactly the same as in the ring case.

Theorem 3.6. Let $(\mathcal{R}, m)$ be a Noetherian local ring and $M$ a finitely generated $\mathcal{R}$–module with $r = \dim_\mathcal{R} M \geq 2$. Let $U = U_{\hat{M}}(0)$ be the unmixed component of $(0)$ in the $m$–adic completion $\hat{M}$ of $M$. Then the following conditions are equivalent.

1. $M$ is a Vasconcelos $\mathcal{R}$–module.
2. $e_1(Q, M) = 0$ for every parameter ideal $Q$ for $M$.
3. $\hat{M}/U_{\hat{M}}(0) \cong M$ is a Cohen–Macaulay $\mathcal{R}$–module and $\dim_\mathcal{R} U_{\hat{M}}(0) \leq r - 2$.
4. There exists a proper $\mathcal{R}$–submodule $L$ of $\hat{M}$ such that $\hat{M}/L$ is a Cohen–Macaulay $\mathcal{R}$–module with $\dim_\mathcal{R} L \leq r - 2$.

When this is the case, $\hat{M}$ is a Vasconcelos $\mathcal{R}$–module and $H_{m}^{r-1}(M) = (0)$.

Remark 3.7. Several properties of Vasconcelos rings such as [8, 3.5, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.15, 3.16, 3.17] can be all extended to Vasconcelos modules.

\footnote{The terminology is due to the other five authors.}
4. Generalized Cohen–Macaulayness of modules with $\Lambda(M)$ finite

Let $R$ be a Noetherian local ring with maximal ideal $m$ and $M$ a finitely generated $R$–module with $r = \dim_R M > 0$. In this section we study the problem of when the set

$$\Lambda(M) = \{e_1(Q, M) \mid Q \text{ is a parameter ideal for } M\}$$

is finite. Part of the motivation comes from the fact that generalized Cohen–Macaulay modules have this property. Recall that $M$ is said to be generalized Cohen–Macaulay, if all the local cohomology modules $\{H^i_m(M)\}_{0 \leq i < r}$ are finitely generated (see [9] where these modules originated).

Assume that $M$ is a generalized Cohen–Macaulay $R$–module with $r = \dim_R M \geq 2$ and put

$$s = \sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M),$$

where $h^i(M) = \lambda_R(H^i_m(M))$ for each $i \in \mathbb{Z}$. If $Q$ is a parameter ideal for $M$, by the proof of [13] Lemma 2.4 we have that $e_1(Q, M) \geq -s$. Since $e_1(Q, M) \leq 0$ by Corollary 2.3 it follows that $\Lambda(M)$ is a finite set.

Let us establish here that if $M$ is unmixed and $\Lambda(M)$ is finite, then $M$ is indeed a generalized Cohen–Macaulay $R$–module (Proposition 4.1). See Section 4 of [8] for the ring case.

Assume now that $R$ is a homomorphic image of a Gorenstein local ring and that $\text{Ass}_R M = \text{Ass}_{\text{Sh}} R M$. Then $R$ contains a system $x_1, x_2, \ldots, x_r$ of parameters of $M$ which forms a strong $d$-sequence for $M$, that is, the sequence $x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r}$ is a $d$-sequence for $M$ for all integers $n_1, n_2, \ldots, n_r \geq 1$ (see [5] Theorem 2.6 or [19] Theorem 4.2 for the existence of such systems of parameters). For each integer $q \geq 1$ let $\Lambda_q(M)$ be the set of values $e_1(Q, M)$ where $Q$ runs over the parameter ideals for $M$ such that $Q \subseteq m^q$ and $Q = (x_1, x_2, \ldots, x_r)$ with $x_1, x_2, \ldots, x_r$ a $d$-sequence for $M$. We then have $\Lambda_q(M) \neq \emptyset$, $\Lambda_q(M) \subseteq \Lambda_q(M)$ for all $q \geq 1$, and $\alpha \leq 0$ for every $\alpha \in \Lambda_q(M)$ (Corollary 2.3).

The following result plays a key role in our argument. We note a proof for the sake of completeness. The proof which we present here is based on Theorem 2.5 and slightly different from that of the ring case.

**Lemma 4.1.** Let $(R, m)$ be a Noetherian local ring and assume that $R$ is a homomorphic image of a Gorenstein ring. Let $M$ be a finitely generated $R$–module with $r = \dim_R M \geq 2$ and $\text{Ass}_R M = \text{Ass}_{\text{Sh}} R M$. Assume that $\Lambda_q(M)$ is a finite set for some integer $q \geq 1$ and put $\ell = -\min \Lambda_q(M)$. Then $m^i H^i_m(M) = (0)$ for all $i \neq r$ and hence all the local cohomology modules $\{H^i_m(M)\}_{0 \leq i < r}$ are finitely generated.

**Proof.** Passing to the ring $R/((0) :_R M)$, we may assume that $R$ is a Gorenstein ring with $\dim R = \dim_R M = r$. Enlarging the residue class field $R/m$ of $R$ if necessary, we may assume the field $R/m$ is infinite. By Corollary 2.4 $H^1_m(M)$ is finitely generated, since $M$ is unmixed.

Suppose that $r = 2$. We put $\ell' = \lambda(H^1_m(M))$. Let $Q = (x, y) \subseteq m^q$ be a system of parameters for $M$ such that $Q H^1_m(M) = (0)$ and $x, y$ is a $d$-sequence for $M$. Then $x$ is superficial for $M$ with respect to $Q$. Hence by Proposition 2.2 (d) we get $e_1(Q, M) = -\lambda(H^1_m(M)) = -\ell'$. Thus $\ell \geq \ell'$, as $-\ell' = e_1(Q, M) \in \Lambda_q(M)$. Hence $m^i H^1_m(M) = (0)$, because $m^{\ell'} H^1_m(M) = (0)$.

Suppose that $r \geq 3$ and that our assertion holds true for $r - 1$. We have an exact sequence

$$\begin{array}{c}
\{\#\} & 0 \to M \to R^n \to C \to 0
\end{array}$$

...
of $R$–modules by Theorem 2.5. Choose an $R$–regular element $x \in R$ so that $x$ is superficial both for $M$ and $C$ with respect to $m$. Let us fix an integer $m \geq 1$. We put $y = x^m$, $N = M/yM$, and look at the exact sequence

$$0 \rightarrow (0) \rightarrow C \rightarrow N \cong (R/yR)^n \rightarrow C/yC \rightarrow 0$$

of $R$–modules obtained by sequence (2). Let $L = \text{Im} \varphi$. Then $\dim_R L = r - 1$, $\text{Ass}_R L = \text{Assh}_R L$, and $H^q_m(N) \cong (0) : C$, because $L$ is an $R$–submodule of $(R/yR)^n$ and $\lambda((0) : C y) < \infty$. Hence $L \cong N/H^q_m(N)$.

Let $q' \geq q$ be an integer such that $m^{q'} N \cap H^q_m(N) = (0)$. Let $y_2, y_3, \ldots, y_r \in m^{q'}$ be a system of parameters for $L$ and assume that $y_2, y_3, \ldots, y_r$ is a $d$-sequence for $L$. Then, since $(y_2, y_3, \ldots, y_r) N \cap H^q_m(N) = (0)$, we get $y_2, y_3, \ldots, y_r$ forms a $d$-sequence for $N$ also. Therefore, since $y$ is $M$-regular, the sequence $y_1, y_2, \ldots, y_r$ forms a $d$-sequence for $M$, whence $y_1$ is superficial for $M$ with respect to $Q = (y_1, y_2, \ldots, y_r)$. Consequently

$$e_1((y_2, y_3, \ldots, y_r), L) = e_1((y_2, y_3, \ldots, y_r), N) = e_1(Q, M) \in \Lambda_q(M),$$

so that $\Lambda_{q'}(L) \subseteq \Lambda_q(M)$. Hence, because the set $\Lambda_{q'}(L)$ is finite, the hypothesis of induction on $r$ yields that $m^{\ell''} H^q_m(L) = (0)$ for all $i \neq r - 1$, where $\ell'' = -\min \Lambda_{q'}(L)$. Thus, because $\ell'' \leq \ell$, $m^{\ell''} H^q_i(L) = (0)$ for all $i \neq r - 1$. Hence $m^\ell H^q_i(N) = (0)$ for all $1 \leq i < r - 1$, because $H^q_{i}(N) \cong H^q_i(L)$ for $i \geq 1$.

Look now at the exact sequence

$$HM^1_m(M) \xrightarrow{x^m} HM^1_m(N) \rightarrow \cdots \rightarrow HM^1_m(N) \rightarrow H^{i+1}_m(M) \xrightarrow{x^m} H^{i+1}_m(N) \rightarrow \cdots$$

of local cohomology modules. We then have

$$m^\ell \left[ (0) : H^{i+1}_m(M) x^m \right] = (0)$$

for all integers $1 \leq i \leq r - 2$ and $m \geq 1$, since $m^\ell H^i_m(N) = (0)$ for all $1 \leq i \leq r - 2$. Thus

$$H^{i+1}_m(M) = \bigcup_{m \geq 1} \left[ (0) : H^{i+1}_m(M) \right] m^m.$$

On the other hand, from sequence (2) we get the embedding $H^1_m(M) \subseteq H^1_m(N)$, choosing the integer $m \geq 1$ so that $x^m H^1_m(N) = (0)$. Hence $m^\ell H^1_m(M) = (0)$, which completes the proof of Lemma 4.1.

Since $\Lambda(M) = \Lambda(M)$, passing to the completion $\widehat{M}$ of $M$ and applying Lemma 4.1, we readily get the following.

**Proposition 4.2.** Let $(R, m)$ be a Noetherian local ring and $M$ a finitely generated unmixed $R$–module with $r = \dim_R M \geq 2$. Assume that $\Lambda(M)$ is a finite set and put $\ell = -\min \Lambda(M)$. Then $m^\ell H^i_m(M) = (0)$ for every $i \neq r$, so that $M$ is a generalized Cohen–Macaulay $R$–module.

We conclude this section with a characterization of $R$–modules for which $\Lambda(M)$ is finite. Let us note the following with a brief proof.

**Lemma 4.3.** Let $R$ be a Noetherian local ring and $M$ a finitely generated $R$–module with $r = \dim_R M \geq 2$. Assume that there exists an integer $t \geq 0$ such that $e_1(Q, M) \geq -t$ for every parameter ideal $Q$ for $M$. Then $\dim_R U_M(0) \leq r - 2$. 

Proof. Let $U = U_M(0)$ and $N = M/U$. Assume that $\dim_R U = r - 1$. Choose a system $x_1, x_2, \ldots, x_r$ of parameters of $M$ such that $x_r U = (0)$. Let $\ell > t$ be an integer and put $Q = (x_1^t, x_2, \ldots, x_r)$. Then we get exact sequences

$$0 \to U/(Q^{n+1}M \cap U) \to M/Q^{n+1}M \to N/Q^{n+1}N \to 0$$

of $R$–modules for all $n \geq 0$. Let us take an integer $k \geq 0$ so that

$$Q^n M \cap U = Q^{n-k}(Q^k M \cap U)$$

for $n \geq k$ and consider $U' = Q^k M \cap U$. Let $q = (x_1^t, x_2, \ldots, x_{r-1})$. We then have

$$Q^{n-k}U' = q^{n-k}U',$$

as $x_r U = (0)$. Hence for all $n \geq k$

$$\lambda_R(M/Q^{n+1}M) = \lambda_R(N/Q^{n+1}N) + \lambda_R(U'/q^{n-k+1}U') + \lambda_R(U/U'),$$

which yields $-t \leq e_1(Q, M) = e_1(Q, N) - e_0(q, U')$. Hence

$$-t \leq e_1(Q, M) = e_1(Q, N) - e_0(q, U'),$$

because $e_0(q, U) = e_0(q, U')$ (remember that $\lambda(U/U') < \infty$). Therefore, since $e_1(Q, N) \leq 0$ by Corollary 2.3, we get

$$\ell \leq \ell e_0((x_1, x_2, \ldots, x_{r-1}), U) = e_0(q, U) \leq e_1(Q, N) + t \leq t,$$

which is impossible. Thus $\dim_R U \leq r - 2$. \hfill $\Box$

Remark 4.4. Let $R$ be a Noetherian local ring and $M$ a finitely generated $R$–module with $r = \dim_R M \geq 2$. Assume that $\dim_R U_M(0) \leq r - 2$. Let $q$ be a parameter ideal for $N = M/U_M(0)$. Then one can find a parameter ideal $Q$ for $M$ with $QN = qN$, so that $e_1(q, N) = e_1(q, M)$ by Lemma 3.2. Hence $\Lambda(M) = \Lambda(N)$.

The goal of this section is the following.

Theorem 4.5. Let $(R, m)$ be a Noetherian local ring and $M$ a finitely generated $R$–module with $r = \dim_R M \geq 2$. Let $U = U_M(0)$ denote the unmixed component of $(0)$ in the $m$–adic completion $\widehat{M}$ of $M$. Then the following conditions are equivalent:

1. $\Lambda(M)$ is a finite set;
2. $\dim_R : U \leq r - 2$ and $\widehat{M}/U$ is a generalized Cohen–Macaulay $\widehat{R}$–module.

When this is the case, one has the estimation

$$0 \geq e_1(Q, M) \geq -\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(\widehat{M}/U)$$

for every parameter ideal $Q$ for $M$.

Proof. We may assume that $R$ is complete.

1. $\Rightarrow$ (2) Since the set $\Lambda(M)$ is finite, by Proposition 4.3, we get $\dim_R U \leq r - 2$. By Remark 4.3, the set $\Lambda(M/U)$ is finite, so that $M/U$ is a generalized Cohen–Macaulay $R$–module by Proposition 4.2.

(2) $\Rightarrow$ (1) By [13, Lemma 2.4] the set $\Lambda(M/U)$ is finite and hence the set $\Lambda(M)$ is also finite by Lemma 3.2.

See [13, Lemma 2.4] for the last assertion. \hfill $\Box$
5. Buchsbaumness of modules possessing constant first Hilbert coefficients of parameters

Let $R$ be a Noetherian local ring with maximal ideal $m$ and $M$ a finitely generated $R$-module with $r = \dim_R M > 0$. In this section we study the problem of when $e_1(Q, M)$ is independent of the choice of parameter ideals $Q$ for $M$. Part of the motivation comes from the fact that Buchsbaum modules have this property. We establish here that if $e_1(Q, M)$ is constant and $M$ is unmixed, then $M$ is indeed a Buchsbaum $R$-module (Theorem 5.4). See [14] for the ring case.

First of all let us recall some definitions. A system $x_1, x_2, \ldots, x_r$ of parameters of $M$ is said to be standard, if it forms a $d^r$-sequence for $M$, that is, $x_1, x_2, \ldots, x_r$ forms a strong $d$-sequence for $M$ in any order. Remember that $M$ possesses a standard system of parameters if and only if $M$ is a generalized Cohen–Macaulay $R$-module ([30]).

Let $Q$ be a parameter ideal for $M$. Then we say that $Q$ is standard, if it is generated by a standard system of parameters of $M$. Remember that $Q$ is standard if and only if the equality

$$\lambda_R(M/QM) - e_0(Q, M) = \sum_{i=0}^{r-1} \binom{r-1}{i} h^i(M) := \mathbb{I}(M)$$

holds ([30, Theorem 2.1]). It is known that every system of parameters of $M$ contained in a standard parameter ideal for $M$ is standard ([30]).

Suppose that $M$ is a generalized Cohen–Macaulay $R$-module with $r = \dim_R M \geq 2$ and $s = \sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M)$. If $Q$ is a parameter ideal for $M$, then by [13, Lemma 2.4] we get $e_1(Q, M) \geq -s$, where the equality holds if $Q$ is standard ([26, Korollar 3.2]).

We say that our $R$-module $M$ is Buchsbaum, if every parameter ideal for $M$ is standard. Hence, if $M$ is a Buchsbaum $R$-module with $r = \dim_R M \geq 2$, then $M$ is a generalized Cohen–Macaulay $R$-module with

$$e_1(Q, M) = -\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M)$$

for every parameter ideal $Q$. See [29] for a detailed theory of Buchsbaum rings and modules.

We begin with the following two results, whose proofs are exactly the same as in the ring case (see [8, Lemma 4.5] and [14, Proposition 2.3]).

**Lemma 5.1.** Let $(R, m)$ be a Noetherian local ring and $M$ a generalized Cohen–Macaulay $R$-module with $\dim_R M = r \geq 2$ and $\depth_R M > 0$. Let $Q$ be a parameter ideal for $M$ such that $e_1(Q, M) = -\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M)$. Then $QH^i_{m}(M) = (0)$ for all $1 \leq i \leq r - 1$.

For each $x \in m$, we put $U_M(x) := \bigcup_{n \geq 0} \langle x M : M \rangle$.  

**Proposition 5.2.** Let $(R, m)$ be a Noetherian local ring and $M$ a generalized Cohen–Macaulay $R$-module with $r = \dim_R M \geq 3$ and $\depth_R M > 0$. Let $Q = (x_1, x_2, \ldots, x_r)$ be a parameter ideal for $M$. Assume that $(x_1, x_r)H^1_{m}(M) = (0)$ and that the parameter ideal $(x_1, x_2, \ldots, x_{r-1})$ for the generalized Cohen–Macaulay $R$-module $M/U_M(x_r)$ is standard. Then $U_M(x_1) \cap QM = x_1 M$. 

We then have the following, which is the key in our argument. The proof is similar to the ring case [14, Theorem 2.1] but let us note a brief proof in order to see how we use the previous results Lemma 5.1 and Proposition 5.2.

**Theorem 5.3.** Let \((R, m)\) be a Noetherian local ring and let \(M\) be a generalized Cohen–Macaulay \(R\)-module with \(r = \dim R M \geq 2\) and \(\text{depth}_R M > 0\). Let \(Q\) be a parameter ideal for \(M\). Then the following conditions are equivalent:

1. \(Q\) is a standard parameter ideal for \(M\);
2. \(e_1(Q, M) = -\sum_{i=1}^{r-2} \binom{r-3}{i} h^i(M)\).

**Proof.** We have only to show the implication (2) \(\Rightarrow\) (1). To do this, we may assume that the residue class field \(R/m\) of \(R\) is infinite. We write \(Q = (x_1, x_2, \ldots, x_r)\), where each \(x_j\) is superficial for \(M\) with respect to \(Q\). Remember that by Lemma 5.1 \(QH_m^r(M) = (0)\) for all \(i \neq r\). Hence \(Q\) is standard, if \(r = 2\) ([30, Corollary 3.7]).

Assume that \(r \geq 3\) and that our assertion holds true for \(r - 1\). Let \(1 \leq j \leq r\) be an integer. We put \(N = M/x_j M\), \(\overline{M} = N/H_m^r(N) (= M/U_M(x_j))\), and \(Q_j = (x_i | 1 \leq i \leq r, i \neq j)\). Then \(H_m^i(N) \cong H_m^i(\overline{M})\) for all \(i \geq 1\). On the other hand, since \(x_j H_m^i(M) = (0)\) for \(i \neq r\) and \(x_j\) is \(M\)-regular, for each \(0 \leq i \leq r - 2\) we have the short exact sequence

\[
0 \to H_m^i(M) \to H_m^i(N) \to H_m^{i+1}(M) \to 0
\]

of local cohomology modules. Hence \(\mathbb{I}(M) = \mathbb{I}(N)\) and

\[
e_1(Q, M) = e_1(Q_j, N) = e_1(Q_j, \overline{M})
\]

\[
\geq - \sum_{i=1}^{r-2} \binom{r-3}{i} h^i(\overline{M})
\]

\[
= - \sum_{i=1}^{r-2} \binom{r-3}{i} h^i(N)
\]

\[
= - \sum_{i=1}^{r-2} \binom{r-3}{i} [h^i(M) + h^{i+1}(M)]
\]

\[
= - \sum_{i=1}^{r-1} \binom{r-2}{i} h^i(M)
\]

\[
e_1(Q, M),
\]

so that the equality

\[
e_1(Q_j, \overline{M}) = - \sum_{i=1}^{r-2} \binom{r-3}{i} h^i(\overline{M})
\]

holds for the parameter ideal \(Q_j\) for the generalized Cohen–Macaulay \(R\)-module \(\overline{M} = M/U_M(x_j)\).

Thus by the hypothesis of induction on \(r = \dim R M\), \(Q_j\) is a standard parameter ideal for \(M/U_M(x_j)\) for every \(1 \leq j \leq r\). Hence \(U_M(x_1) \cap QM = x_1 M\) by Proposition 5.2. Thus \(Q_1\) is standard parameter ideal for \(M/x_1 M\) ([30, Corollary 2.3]). Therefore \(Q\) is a standard parameter ideal for \(M\), because \(\mathbb{I}(M) = \mathbb{I}(M/x_1 M)\) ([30, Corollary 2.4]).

We are now ready to prove the main result of this section.
Theorem 5.4. Let \((R, \mathfrak{m})\) be a Noetherian local ring and \(M\) an unmixed \(R\)-module with \(r = \dim_R M \geq 2\). Then the following conditions are equivalent:

1. \(M\) is a Buchsbaum \(R\)-module;
2. The first Hilbert coefficient \(e_1(Q, M)\) of \(M\) is constant and independent of the choice of parameter ideals \(Q\) for \(M\).

When this is the case, one has the equality

\[ e_1(Q, M) = -\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M) \]

for every parameter ideal \(Q\) for \(M\), where \(h^i(M) = \lambda(H^i_m(M))\) for each \(1 \leq i \leq r-1\).

Proof. (1) \(\Rightarrow\) (2) This is due to Schenzel [26].

(2) \(\Rightarrow\) (1) Since \(\sharp \Lambda(M) = 1\), by Proposition 4.2 \(M\) is a generalized Cohen–Macaulay \(R\)-module. Hence \(\Lambda(M) = \{-\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(M)\}\) by [26, Korollar 3.2], so that by Theorem 5.3 every parameter ideal \(Q\) for \(M\) is standard. Thus \(M\) is, by definition, a Buchsbaum \(R\)-module (28).

See [26] for the last assertion.

We now in a position to conclude this section with a characterization of \(R\)-modules possessing \(\sharp \Lambda(M) = 1\).

Theorem 5.5. Let \((R, \mathfrak{m})\) be a Noetherian local ring and \(M\) a finitely generated \(R\)-module with \(r = \dim_R M \geq 2\). Let \(U = U^\wedge_M(0)\) be the unmixed component of \((0)\) in the \(\mathfrak{m}\)-adic completion \(\widehat{M}\) of \(M\). Then the following conditions are equivalent:

1. \(\sharp \Lambda(M) = 1\);
2. \(\dim_{\widehat{M}} U \leq r - 2\) and \(\widehat{M}/U\) is a Buchsbaum \(\widehat{R}\)-module.

When this is the case, one has the equality

\[ e_1(Q, M) = -\sum_{i=1}^{r-1} \binom{r-2}{i-1} h^i(\widehat{M}/U) \]

for every parameter ideal \(Q\) for \(M\).

Proof. We may assume \(R\) is complete.

(1) \(\Rightarrow\) (2) Since \(\sharp \Lambda(M) = 1\), \(\dim_{\widehat{R}} U \leq r - 2\) by Proposition 4.3. We get \(\sharp \Lambda(M/U) = 1\) by Remark 4.4 so that by Theorem 5.4 \(M/U\) is a Buchsbaum \(R\)-module.

(2) \(\Rightarrow\) (1) We get by Theorem 5.4 that \(\sharp \Lambda(M/U) = 1\) and hence \(\sharp \Lambda(M) = 1\) by Lemma 3.2.

See Theorem 5.4 for the last assertion.

6. Homological degrees

In this section we deal with the variation of the extended degree function \(hdeg (\text{[17, 31]}),\) labeled \(hdeg_I\) (see [21, p. 142]). We recall the basic properties of these functions. These techniques and their relationships to \(e_1(I)\) have been mentioned in [33] but the treatment here is more focused. It will lead to slightly sharper bounds in the case of \(e_1(I, M)\).
Cohomological degrees. Let \( R \) be a Noetherian local ring with maximal ideal \( m \) and infinite residue class field. Let \( \mathcal{M}(R) \) denote the category of finitely generated \( R \)-modules and let \( I \) be an \( m \)-primary ideal of \( R \). Then one has the following extension of the classical multiplicity.

**Definition 6.1.** A cohomological degree, or extended multiplicity function with respect to \( I \), is a function

\[
\text{Deg}(\cdot) : \mathcal{M}(R) \to \mathbb{N}
\]

that satisfies the following conditions. Let \( M \in \mathcal{M}(R) \).

(a) If \( L = \Gamma_m(M) \) is the \( R \)-submodule of elements of \( M \) that are annihilated by a power of the maximal ideal \( m \) and \( \overline{M} = M/L \), then

\[
\text{Deg}(M) = \text{Deg}(\overline{M}) + \lambda(L),
\]

where \( \lambda(\cdot) \) is the ordinary length function.

(b) (Bertini’s rule) If \( M \) has positive depth, then

\[
\text{Deg}(M) \geq \text{Deg}(M/hM)
\]

for every generic hyperplane section \( h \in I \setminus mI \).

(c) (The calibration rule) If \( M \) is a Cohen-Macaulay \( R \)-module, then

\[
\text{Deg}(M) = \deg(M),
\]

where \( \deg(M) \) is the ordinary multiplicity of \( M \) with respect \( I \).

The existence of cohomological degrees in arbitrary dimensions was established in [31]. Let us formulate it for the case where the ring \( R \) is complete.

For the rest of this section suppose that \( R \) is complete. For each finitely generated \( R \)-module \( X \) and \( j \in \mathbb{Z} \) let

\[
X_j = \text{Hom}_R(H^j_m(X), E),
\]

where \( E = E_R(R/m) \) denotes the injective envelope of the residue class field. Then, thanks to the local duality theorem, one gets \( \dim_R X_j \leq j \) for all \( j \in \mathbb{Z} \).

**Definition 6.2.** Let \( M \) be a finitely generated \( R \)-module with \( r = \dim_R M > 0 \). Then the homological degree of \( M \) is the integer

\[
h\text{deg}(M) = \deg(M) + \sum_{j=0}^{r-1} \binom{r-1}{j} \cdot \text{hdeg}(M_j).
\]

We call attention to the fact (see [31] for details) that the notion of generic hyperplane section used for \( h\text{deg}(M) \) are superficial elements for \( M \) and for all \( M_j \), but also for the iterated ones of these modules (there are only a finite number of them).

The definition of homological degree morphs easily into an extended degree, noted \( h\text{deg}_I \), where Samuel multiplicities relative to \( I \) are used ([21]). In particular \( \deg(M) \) stands for \( e_0(I, M) \). We will employ \( h\text{deg}_I \) to derive lower bounds for \( e_1(I, M) \).
**Homological torsion.** There are other combinatorial expressions of the terms $h_{\deg_I}(M_j)$ that behave well under hyperplane sections.

**Definition 6.3.** Let $M$ be an $R$-module with $r = \dim_R M \geq 2$. For each integer $1 \leq i \leq r - 1$ we put

$$T_I^{(i)}(M) = \sum_{j=1}^{r-i} \binom{r-i-1}{j-1} \cdot h_{\deg_I}(M_j).$$

Hence

$$h_{\deg_I}(M) > T_I^{(1)}(M) \geq T_I^{(2)}(M) \geq \cdots \geq T_I^{(r-1)}(M).$$

If $M$ is a generalized Cohen-Macaulay $R$-module, then

$$T_I^{(i)}(M) = \sum_{j=1}^{r-i} \binom{r-i-1}{j-1} \lambda(H^i_{\mathfrak{m}}(M))$$

which is independent of $I$.

We then have the following.

**Theorem 6.4 (31 Theorem 2.13).** Let $M$ be a finitely generated $R$-module with $r = \dim_R M$ and let $h$ be a generic hyperplane section. Then $T_I^{(i)}(M/hM) \leq T_I^{(i)}(M)$ for all $1 \leq i \leq r - 2$.

**Remark 6.5.** There are a number of rigidity questions about the values of the Hilbert coefficients $e_i(I)$. Typically they have the form

$$|e_1(I)| \geq |e_2(I)| \geq \cdots \geq |e_d(I)|.$$

Two of these cases are (i) parameter ideals and (ii) normal ideals, or more generally the case of the normalized filtration. Because values of $T_I^{(i)}(R)$ have been used, in a few cases, to bound the $e_i(I)$, the descending chain in Definition 6.3 argues for an underlying rigidity. This runs counter to the known formulas for the values of the $e_i(I)$ for general ideals (e.g., (25 Theorem 4.1)).

We now turn this into a uniform bound for the first Hilbert coefficient of a module $M$ relative to an ideal $I$ generated by a system $x = \{x_1, x_2, \ldots, x_r\}$ of parameters of $M$.

**Theorem 6.6.** Let $M$ be a finitely generated $R$-module with $\dim_R M = \dim R \geq 2$ and let $Q$ be a parameter ideal of $R$. Then

$$-e_1(Q, M) \leq T_Q^{(1)}(M).$$

**Proof.** Let $d = \dim R$ and let $h \in Q \setminus \mathfrak{m}Q$ be a generic hyperplane section used for $h_{\deg_Q}(M)$. Since

$$-e_1(Q, M) = -e_1(Q, M/H^0_{\mathfrak{m}}(M))$$

and $T_Q^{(1)}(M/H^0_{\mathfrak{m}}(M)) \leq T_Q^{(1)}(M)$, replacing $M$ with $M/H^0_{\mathfrak{m}}(M)$ if necessary, we may assume depth $R_M \geq 1$. We may also assume that $h$ is superficial for $M$ and for all $M_j \ (0 \leq j \leq d - 1)$ with respect to $Q$. Hence $h$ is $M$-regular and $\lambda(M_1/hM_1) < \infty$ (remember that $\dim_R M_1 \leq 1$). Suppose $d = 2$. Then $T_Q^{(1)}(M) = h_{\deg_Q}(M_1)$ and $-e_1(Q, M) = \lambda(0 : H^1_{\mathfrak{m}}(M))$ by Proposition 2.2(d). On the other hand, from the exact sequence

$$0 \rightarrow M \xrightarrow{h} M \rightarrow M/hM \rightarrow 0$$
of $R$–modules, we obtain the exact sequence

$$0 \to (0) : H^1_{h}(M) \to H^1_{m}(M) \to H^1_{m}(M).$$

Then, taking the Matlis dual, we have an epimorphism

$$M_1/hM_1 \to \text{Hom}_R((0) : H^1_{m}(M), E) \to 0,$$

so that

$$\lambda((0) : H^1_{m}(M)) = \text{hdeg}(\text{Hom}_R((0) : H^1_{m}(M), E)).$$

by Proposition 6.4. Thus

$$-e_1((0) : H^1_{m}(M)) \leq \text{T}^{(1)}_{Q}(M).$$

Hence the result follows, since $-e_1((0) : H^1_{m}(M)) = -e_1(Q/hM, M/hM)$ by Proposition 2.2 (a).

We now derive from the previous estimates bounds for the Hilbert coefficients of parameter ideals.

**Corollary 6.7.** If $M$ is a generalized Cohen-Macaulay $R$–module with $\text{dim}_R M \geq 1$, then the Hilbert coefficients $e_1(Q, M)$ are bounded for all parameter ideals $Q$ for $M$.

**Proof.** Passing to the ring $R/((0) : R M)$, we may assume that $\text{dim} R = \text{dim}_R M$ and that $Q$ is a parameter ideal of $R$. Then $e_1(Q, M) \leq 0$ by Corollary 2.3. We get by Theorem 6.6

$$-e_1(Q, M) \leq \text{T}^{(1)}_{Q}(M),$$

while $\text{T}^{(1)}_{Q}(M) = \sum_{j=1}^{d-1} \binom{d-2}{j-1} \lambda(H^j_m(M))$ is independent of the choice of $Q$. Hence the result. □

**Corollary 6.8.** Suppose that $\text{dim}_R M \geq 1$. Then the set

$$\{e_1(Q, M) \mid Q \text{ are parameter ideals of } M \text{ with the same integral closure}\}$$

is finite.

**Proof.** For each parameter ideal $Q$ of $M$ we get $e_1(Q, M) \leq 0$, while Corollary 6.7 asserts that $e_1(Q, M) \geq -\text{T}^{(1)}_{Q}(M)$. Hence the result follows, because $\text{T}^{(1)}_{Q}(M)$ depends only on $\overline{Q}$, the integral closure of $Q$. □

### 7. Bounding Euler characteristics

The relationship between partial Euler characteristics and superficial elements make for a straightforward comparison with extended degree functions. Unless otherwise specified, throughout it is assumed that $R$ is a Noetherian complete local ring with infinite residue class field. We will prove that Euler characteristics can be uniformly bounded by homological degrees. The basic tool is the following observation, which is found in the proof of [3 Theorem 4.6.10 (a)].

**Proposition 7.1.** Let $M$ be a finitely generated $R$–module with $r = \text{dim}_R M \geq 2$. Let $x = \{x_1, x_2, \ldots, x_r\}$ be a system of parameters for $M$ and set $x' = \{x_2, \ldots, x_n\}$. Then

$$\chi_1(x; M) = \chi_1(x'; M/x_1 M) + \chi_1(x'; 0 :_M x_1).$$
Theorem 7.2. Let $M$ be a finitely generated $R$-module with $d = \dim_R M = \dim R \geq 1$. Then for every system $\mathbf{x} = \{x_1, x_2, \ldots, x_d\}$ of parameters of $R$, one has
\[
\chi_1(\mathbf{x}; M) \leq \text{hdeg}_Q(M) - \deg_Q(M),
\]
where $Q = (\mathbf{x})$.

Proof. As $\lambda(M/QM) = \chi_1(\mathbf{x}; M) + \deg_Q(M)$, we have only to show $\lambda(M/QM) \leq \text{hdeg}_Q(M)$.

Let $\lambda \in Q \setminus mQ$ be a generic hyperplane section used for $\text{hdeg}_Q(M)$. Then, since $\lambda(M/QM) = \lambda((M/hM)/Q(M/hM))$ and $\text{hdeg}_{Q/(\lambda)}(M/hM) \leq \text{hdeg}_Q(M)$, by induction on $d$ we may assume $d = 1$. When $d = 1$, $\chi_1(\mathbf{x}; M) = \lambda(0 :_M x_1)$ and hence $\lambda(M/QM) = \chi_1(\mathbf{x}; M) + \deg_Q(M) \leq (\lambda(H^0_m(M)) + \deg_Q(M) = \text{hdeg}_Q(M)$, as wanted. $\square$

Corollary 7.3. Suppose that $\dim_R M \geq 1$. Then for every primary ideal $I$ of $M$, the set
\[
\Xi_1(I, M) = \{\chi_1(\mathbf{x}, M) \mid \mathbf{x} \text{ are systems of parameters of } M \text{ with } (\mathbf{x}) = \mathbf{T}\}
\]
is finite.

Proof. Both $\text{hdeg}_Q(M)$ and $\deg_Q(M)$ depend only on the integral closure $\mathbf{T} = \overline{\mathbf{Q}}$ of $Q = (\mathbf{x})$. $\square$

8. Quasi–cohomological degree

Let $R$ be a Noetherian local ring with maximal ideal $m$ and let $M$ be a finitely generated $R$-module with $r = \dim_R M \geq 1$. We are going to define a numerical function on systems $\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ of parameters which are akin to homological degrees.

Definition 8.1. A function $h(\mathbf{x}; M) : \mathcal{P}(M) \to \mathbb{N}$ is a quasi-cohomological degree for $M$, if the following conditions are met.

(a) Set $\mathbf{x}' = \{x_2, \ldots, x_r\}$. If $x_1$ is a superficial element and $\text{depth}_R M \geq 1$,
\[
h(\mathbf{x}; M) = h(\mathbf{x'}; M/x_1M).
\]

(b) If $M_0 = H^0_m(M)$ and $M' = M/M_0$, then
\[
h(\mathbf{x}; M) = h(\mathbf{x}; M') + \lambda(M_0).
\]

The following characterization of $d$-sequences via approximation complexes is given in [16, Theorem 9.1].

Theorem 8.2. Let $M$ be a finitely generated $R$-module with $r = \dim_R M \geq 1$. Assume that $\mathbf{x} = x_1, x_2, \ldots, x_r$ is a system of parameters of $M$ which is also a $d$-sequence for $M$. Let $S = R[T_1, T_2, \ldots, T_r]$, $Q = (\mathbf{x})$, and $G = \text{gr}_Q(M)$. Then the approximation complex $M(\mathbf{x}; M)$
\[
0 \to H_r(\mathbf{x}; M) \otimes S[-r] \to \cdots \to H_1(\mathbf{x}) \otimes S[-1] \to H_0(\mathbf{x}; M) \otimes S \to G \to 0
\]
of $M$ associated to $\mathbf{x}$ is acyclic.

Out of this complex one can extract information about the Hilbert series of $G$ ([10] has a fuller listing). For instance:

(a) $\text{depth}_R M = \text{depth} \text{gr}_Q(M);

(b) $e_r(\mathbf{x}; M) = \lambda(M_0)$.

Basic properties of $d$-sequences, useful in induction arguments, are described in [17], and see also [18] and [20] for part (e) in the following proposition.
Proposition 8.3. Let $R$ be a commutative ring and $M$ an $R$–module. Suppose that $x = \{x_1, x_2, \ldots, x_r\}$ is a d-sequence for $M$. Then we have the following.

(a) $x_2, x_3, \ldots, x_r$ is a d-sequence for $M/x_1M$.
(b) $[(0): Mx_1] \cap (x_1, x_2, \ldots, x_r)M = (0)$.
(c) $x_1, x_2, \ldots, x_r$ is a d-sequence for $M/[(0): Mx_1]$.
(d) $x$ is a regular sequence for $M$ if and only if $(x_1, \ldots, x_{r-1})M :_M x_r = (x_1, \ldots, x_{r-1})M$.
(e) The sequence $x^* = \{x_1^*, x_2^*, \ldots, x_r^*\}$ of $\text{gr}_Q(R)$ is a d-sequence for $\text{gr}_Q(M)$, where $Q = (x)$ and $x_i^* = x_i \mod Q^2$ denotes the initial form of $x_i$ for each $1 \leq i \leq r$.

We now define a very special function and restrict it to some systems of parameters. Let $R$ be a Noetherian local ring and $M$ a finitely generated $R$–module $M$ with $r = \dim_R M \geq 1$. For each system $x = \{x_1, x_2, \ldots, x_r\}$ of parameters of $M$, write the Hilbert characteristic of $M$ with respect to $(x)$:

$$h(x; M) = \sum_{i=0}^{r} (-1)^i e_i(x, M).$$

We then have the following.

Theorem 8.4. Over systems of parameters which are d-sequences for $M$, this function satisfies the rules of a quasi-cohomological degree.

Proof. They are based on three observations about d–sequences.

(i) $e_r(x; M) = (-1)^r \lambda(M_0)$, according to comment above. In particular $e_r(x; M) = 0$ if and only if $\text{depth } M \geq 1$.

(ii) To verify the condition 8.1 (1), first note that according to Proposition 8.3, $x'$ is a d–sequence on $M/x_1M$ while by the previous item $e_r(x; M) = 0$. Thus to prove the equality $h(x; M) = h(x'; M/x_1M)$, it suffices to show that for all $i \geq 0$

$$e_i(x; M) = e_i(x'; M/x_1M).$$

(iii) This will follow from Proposition 2.2 for $i < r$, once we recognize $x_1$ as a superficial element relative to $M$ (we already have that $e_r(x; M) = 0$). For that we appeal to a general assertion of [18] (see also [20]) that if $x$ is a d-sequence on $M$, then the sequence $x_1^*, x_2^*, \ldots, x_r^*$ of 1–forms of gr$_Q(R)$ is a d-sequence for the associated graded module gr$_Q(M)$ of $M$, where $Q = (x)$. On the other hand, the approximation complex of $M$ associated to $x$ is exact and of length $< r$, in particular depth gr$_Q(M) \geq 1$, which implies that $x_1$ is a superficial element for $M$ with respect to $Q$.

Corollary 8.5. If $x$ is a system of parameters of $M$ which is a d–sequence for $M$ also, then $h(x; M) = \lambda(M/QM)$, where $Q = (x)$. In particular, $h(x; M) \geq e_0(Q, M)$ with equality if and only if $M$ is Cohen–Macaulay.

Proof. This follows from the telescoping in the formulas for the coefficients $e_i(Q, M)$ in Theorem 8.2.

Corollary 8.6. Let $x$ be a system of parameters of $M$ which is a d–sequence for $M$. Suppose that $x$ is a part of a minimal basis of the maximal ideal $m$. Then the Betti numbers $\beta_i^R(M)$ satisfy

$$\beta_i^R(M) \leq \lambda(M/(x)M) \cdot \beta_i^R(k).$$
Proof. It follows from the argument of [32, Theorem 2.94], where we use the properties of $h(x; M)$ in the induction part.

Remark 8.7. Note that the condition that $x$ is a part of a minimal basis of the maximal ideal $m$ in Corollary 8.6 is needed in the induction argument which requires the inequality of Betti numbers $\beta_i^{R/(x_1)}(k) \leq \beta_i^R(k)$ ([32, Corollary 3.4.2]).

9. Buchsbaum-Rim coefficients

In this section let us note another set of related questions, concerned about the vanishing and the negativity of the Buchsbaum-Rim coefficients of modules.

Let $R$ be a Noetherian local ring with maximal ideal $m$ and $d = \dim R \geq 1$. The Buchsbaum-Rim multiplicity ([4]) arises in the context of an embedding

$$0 \to E \longrightarrow F = R' \longrightarrow C \to 0$$

of $R$–modules, where $E \subseteq mF$ and $C$ has finite length. Let

$$\varphi : R^m \longrightarrow F = R'$$

be an $R$–linear map represented by a matrix with entries in $m$ such that $\text{Im} \varphi = E$. We then have a homomorphism

$$S(\varphi) : S(R^m) \longrightarrow S(R')$$

of symmetric algebras, whose image is the Rees algebra $R(E)$ of $E$, and whose cokernel we denote by $C(\varphi)$. Hence

$$0 \to R(E) \longrightarrow S(R') = R[T_1, T_2, \ldots, T_r] \longrightarrow C(\varphi) \to 0.$$ 

This exact sequence (with a different notation) is studied in [4] in great detail. Of significance for us is the fact that $C(\varphi)$, with the grading induced by the homogeneous homomorphism $S(\varphi)$, has components of finite length, for which we have the following. Let $E^n = [R(E)]_n$ and $F^n = [S(F)]_n$ for $n \geq 0$, where $F = R'$.

**Theorem 9.1.** $\lambda(F^n/E^n)$ is a polynomial in $n$ of degree $d + r - 1$ for $n \gg 0$:

$$\lambda(F^n/E^n) = \text{br}(E) \binom{n + d + r - 2}{d + r - 1} - \text{br}_1(E) \binom{n + d + r - 3}{d + r - 2} + \text{lower terms}.$$ 

This polynomial is called the Buchsbaum-Rim polynomial of $E$. The leading coefficient $\text{br}(E)$ is the Buchsbaum-Rim multiplicity of $\varphi$; if the homomorphism $\varphi$ is understood, we shall simply denote it by $\text{br}(E)$. This number is determined by an Euler characteristic of the Buchsbaum-Rim complex ([4]).

Assume now the residue class field of $R$ is infinite. The minimal reductions $U$ of $E$ are generated by $d + r - 1$ elements. We refer to $U$ as a parameter module of $F$. The corresponding coefficients are $\text{br}(U) = \text{br}(E)$ but $\text{br}_1(U) \leq \text{br}_1(E)$. It is not clear what the possible values of $\text{br}_1(U)$ are, and in similarity to the case of ideals, we can ask the following.

(a) $\text{br}_1(U) \leq 0$?

(b) Suppose that $R$ is unmixed. Then is $R$ Cohen-Macaulay, if $\text{br}_1(U) = 0$?

(c) The values of $\text{br}_1(U)$ are bounded?

(d) What happens in low dimensions?

As for question (a), a surprising result of Hayasaka and Hyry shows the negativity of $\text{br}_1(U)$ in the following way. It gives an eminent proof of Corollary 2.
Theorem 9.2 ([15] Theorem 1.1). \( \lambda(F^n/U^n) \geq \text{br}(U) \binom{n+d+r-2}{d+r-1} \) for all \( n \geq 0 \). Hence \( \text{br}_1(U) \leq 0 \).

They also proved that \( R \) is a Cohen-Macaulay ring, once \( \lambda(F^n/U^n) = \text{br}(U) \binom{n+d+r-2}{d+r-1} \) for some \( n \geq 1 \). When this is the case, one has the equality \( \lambda(F^n/U^n) = \text{br}(U) \binom{n+d+r-2}{d+r-1} \) for all \( n \geq 0 \), whence \( \text{br}_1(U) = 0 \) ([2] Theorem 3.4).

Note that question (c) is answered affirmatively for \( r = 1 \) in Corollary 6.8.

We close this paper with the following.

Conjecture 9.3. Let \( (R, \mathfrak{m}) \) be a Noetherian local ring with \( \text{dim } R \geq 2 \) and let \( U \subseteq \mathfrak{m}R^r \) be a parameter module of \( R^r \) \( (r > 0) \). Then \( R \) is a Cohen-Macaulay ring if and only if \( R \) is unmixed and \( \text{br}_1(U) = 0 \).

References

[1] M. Auslander and D. Buchsbaum, Codimension and multiplicity, Ann. Math. 68 (1958), 625–657.
[2] J. Brennan, B. Ulrich and W. V. Vasconcelos, The Buchsbaum–Rim polynomial of a module, J. Algebra 241 (2001), 379–392.
[3] W. Bruns and J. Herzog, Cohen-Macaulay Rings, Cambridge University Press, 1993.
[4] D. Buchsbaum and D. S. Rim, A generalized Koszul complex II. Depth and multiplicity, Trans. Amer. Math. Soc. 111 (1965), 197–224.
[5] N. T. Cuong, \( p \)-standard system of parameters and \( p \)-standard ideals in local rings, Acta Mathematica Vietnamica 20 (1995), 145–161.
[6] N. T. Cuong, P. Schenzel and N. V. Trung, Verallgemeinerte Cohen-Macaulay-Moduln, Math. Nachr. 85 (1978), 57–73.
[7] L. R. Doering, T. Gunston and W. V. Vasconcelos, Cohomological degrees and Hilbert functions of graded modules, Amer. J. Math. 120 (1998), 493–504.
[8] L. Ghezzi, S. Goto, J. Hong, K. Ozeki, T. T. Phuong and W. V. Vasconcelos, Cohen–Macaulayness versus the vanishing of the first Hilbert coefficient of parameter ideals, J. London Math. Soc. 81 (2010), 679–695.
[9] L. Ghezzi, J. Hong and W. V. Vasconcelos, The signature of the Chern coefficients of local rings, Math. Research Letters 16 (2009), 279–289.
[10] S. Goto, J. Hong and W. V. Vasconcelos, The homology of parameter ideals, Preprint, 2011.
[11] S. Goto and Y. Nakamura, Multiplicities and tight closures of parameters, J. Algebra 244 (2001), 302–311.
[12] T. Gulliksen and G. Levin, Homology of local rings, Queen’s Paper in Pure and Applied Mathematics, No. 20 Queen’s University, Kingston, Ont. 1969.
[13] S. Goto and K. Nishida, Hilbert coefficients and Buchsbaumness of associated graded rings, J. Pure and Appl. Algebra 181 (2003), 61–74.
[14] S. Goto and K. Ozeki, Buchsbaumness in local rings possessing constant first Hilbert coefficient of parameters, Nagoya Math. J. 199 (2010), 95–105.
[15] F. Hayasaka and E. Hyry, On the Buchsbaum-Rim function of a parameter module, J. Algebra 327 (2011), 307–315.
[16] J. Herzog, A. Simis and W. V. Vasconcelos, Koszul homology and blowing-up rings, in Commutative Algebra, Proceedings: Trento 1981 (S. Greco and G. Valla, Eds.), Lecture Notes in Pure and Applied Mathematics 84, Marcel Dekker, New York, 1983, 79–169.
[17] C. Huneke, The theory of \( d \)-sequences and powers of ideals, Adv. in Math. 46 (1982), 249–279.
[18] C. Huneke, Symbolic powers of prime ideals and special graded algebras, Comm. in Algebra 9 (1981), 339–366.
[19] T. Kawasaki, On Cohen-Macaulayfication of certain quasi-projective schemes, J. Math. Soc. Japan 50 (1998), 969–991.
[20] M. Kühl, Über die symmetrische Algebra eines Ideals, Dissertation, Universität Essen, 1981.
[21] C. H. Linh, Upper bound for the Castelnuovo-Mumford regularity of associated graded modules, *Comm. Algebra* **33** (2005), 1817–1831.
[22] M. Mandal, B. Singh and J. K. Verma, On some conjectures about the Chern numbers of filtrations, *J. of Algebra* **325** (2011), 147–162.
[23] H. Matsumura, *Commutative Algebra*, Benjamin/Cummings, Reading, 1980.
[24] M. Nagata, *Local Rings*, Interscience, New York, 1962.
[25] M. E. Rossi, N. V. Trung and G. Valla, Castelnuovo-Mumford regularity and extended degree, *Trans. Amer. Math. Soc.* **355** (2003), 1773–1786.
[26] P. Schenzel, Multiplizitäten in verallgemeinerten Cohen–Macaulay–Moduln, *Math. Nachr.* **88** (1979), 295–306.
[27] J.-P. Serre, *Algèbre Locale. Multiplicités*, Lecture Notes in Mathematics **11**, Springer, Berlin, 1965.
[28] J. Stückrad and W. Vogel, Toward a theory of Buchsbaum singularities, *Amer. J. Math.* **100** (1978), 727–746.
[29] J. Stückrad and W. Vogel, *Buchsbaum Rings and Applications*, Springer, Berlin, 1986.
[30] N. V. Trung, Toward a theory of generalized Cohen–Macaulay modules, *Nagoya Math. J.* **102** (1986), 1–49.
[31] W. V. Vasconcelos, The homological degree of a module, *Trans. Amer. Math. Soc.* **350** (1998), 1167–1179.
[32] W. V. Vasconcelos, *Integral Closure*, Springer Monographs in Mathematics, New York, 2005.
[33] W. V. Vasconcelos, The Chern coefficients of local rings, *Michigan Math. J.* **57** (2008), 725–743.

Department of Mathematics, New York City College of Technology-Cuny, 300 Jay Street, Brooklyn, NY 11201, U. S. A.
*E-mail address: lghezzi@citytech.cuny.edu*

Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashimita, Tama-ku, Kawasaki 214-8571, Japan
*E-mail address: goto@math.meiji.ac.jp*

Department of Mathematics, Southern Connecticut State University, 501 Crescent Street, New Haven, CT 06515-1533, U. S. A.
*E-mail address: hongj2@southernct.edu*

Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashimita, Tama-ku, Kawasaki 214-8571, Japan
*E-mail address: kozeki@math.meiji.ac.jp*

Department of Information Technology and Applied Mathematics, Ton Duc Thang University, 98 Ngo Tat To Street, Ward 19, Binh Thanh District, Ho Chi Minh City, Vietnam
*E-mail address: sugarphuong@gmail.com*

Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd, Piscataway, NJ 08854-8019, U. S. A.
*E-mail address: vasconce@math.rutgers.edu*