Identifying sinks and sources in the potential equation with weighted sparsity regularization

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Abstract

We explore the possibility for using boundary measurements to recover several sources and sinks in the potential equation. This task arises in important applications, e.g., in connection with EEG and ECG data. Employing weighted sparsity regularization and standard results for subgradients, we derive simple-to-check criteria which assure that a number of sinks and sources can be identified. Furthermore, we present two cases for which these criteria always are fulfilled: a) well-separated sources and sinks, and b) many sources or sinks located at the boundary plus one interior source/sink. Both the regularized problem, as well as the basis pursuit version of it, are analyzed. Provided that suitable assumptions hold, we prove that a source can not be misinterpreted as a sink, or vice versa. The theory is developed in terms of Euclidean spaces, and it can therefore be applied to a wide range of problems. In particular, it can be applied to both isotropic and anisotropic cases. We present a series of numerical experiments. This work is motivated by the observation that standard methods typically suggest that internal sinks and sources are located close to the boundary.

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1 Introduction

The boundary value problem

\[
-\nabla \cdot \sigma \nabla u = f, \quad x \in \Omega,
\]

\[
n \cdot \sigma \nabla u = 0, \quad x \in \partial \Omega,
\]

\[
\int_{\partial \Omega} u = 0,
\]

appears in several inverse problems. For example, as a model for the electrical potential in the brain or the heart, i.e., in investigations of EEG and ECG data \[2, 8, 16, 17, 19\]. In such models, \( u \) represents the voltage, \( \sigma = \sigma(x) \) is a conductivity tensor or function, \( f = \nabla \cdot J_i \) incorporates sinks and sources, \( J_i \) is the impressed current, equation (2) assures electrical insulation at \( \partial \Omega \) and (3) is included to enforce uniqueness of the solution \( u \) of (1)-(2).

Typically, the state function \( u \), or some noisy version of it, is observed on the boundary \( \partial \Omega \). Using such a recording \( d \), the goal of the inverse problem is to recover \( f \) by solving

\[
\min_{f, u} \| u - d \|_{L^2(\partial \Omega)} \quad \text{subject to (1)-(3)}.
\]

This problem is ill-posed: The associated forward operator has a nontrivial null space and its Moore-Penrose inverse is unbounded.

In some applications one seeks sources and sinks which have small local support. For example, epileptic seizures (EEG) and ischemic regions (ECG) may only have rather limited extent – otherwise the patient wouldn’t survive. Hence, we will in this paper employ weighted sparsity regularization to a discrete version of (4). The weighting is needed because standard regularization methods fail to correctly identify sinks and sources which are far from the boundary (where the data is recorded), see, e.g., \[18, 4, 11\].

Problems similar to (4) have been studied by many mathematicians, especially the case with constant, or piecewise constant, conductivity \( \sigma \). Often one assumes that \( f \) is a sum of point sources/sinks, or dipoles, and one seeks to compute the positions of these localized functions and/or their strengths or moments, see, e.g., \[3, 6, 7\]. This approach typically yields nonlinear systems of algebraic equations, but reliable and fast methods have been developed. It is also possible to determine the number of point sources with this type of methodology \[7\]. Furthermore, see \[1\] for results for the anisotropic case.

The task of approximately determining the support of the right-hand-side \( f \) in (1)-(3) from boundary data has also been studied in detail, see, e.g., \[13, 14\]. This leads to involved analyzes since the problem is severely ill posed and due to the large null space of the associated forward operator. Roughly speaking, one can not expect to determine accurate information about both the position and the size of the sources and the sinks without imposing further apriori restrictions \[13\].

If one employs the ansatz that \( f \) is composed of a number of point sources and sinks, then the parameter-to-observation map, which, e.g., maps the positions of
the point sources to the potential at the surface $\partial \Omega$, typically becomes nonlinear. This is in contrast to the linearity of the forward operator $f \mapsto u|_{\partial \Omega}$ associated with (1)-(3). In this paper we do not impose any extra assumptions about the form or the strength of $f$, and the linearity is therefore preserved in the discrete forward problem. Consequently, the isotropic and anisotropic cases can be treated with the same methodology and almost by the same software.

Instead of requiring that $f$ has a particular form, we employ a tailored diagonal weight matrix and combine it with sparsity regularization. This weighting procedure was also studied in [12] for the case when apriori box-constraints $0 \leq f(x) \leq s$, $x \in \Omega$, are available and when the state equation is either the screened Poisson equation or the Helmholtz equation. Here, $s$ is a given upper bound. However, in the EEG and ECG applications one will typically have both sources ($f(x) > 0$) and sinks ($f(x) < 0$), and the right-hand-side $f$ in (1)-(3) must satisfy the complimentary condition

$$\int_{\Omega} f = 0.$$  \hspace{1cm} (5)

Also, there is no zero order term present in (1). The results published in [12] can therefore not be applied, and a separate investigation is needed.

Removing the box-constraints, and not requiring that $f$ is non-negative, makes the source-sink identification "harder". Nevertheless, we will in this paper prove that a number of well-separated sources and sinks can be recovered. Our analysis addresses both the basis pursuit version (section 3.1) and the regularized version (section 3.2) of the problem. We also prove that a source will not be misinterpreted as a sink, or vice versa, and we present results regarding the uniqueness of the solution of the inverse problem. The analysis is complemented with numerical experiments in section 4.

One may regard this paper to be follow-up work to [11]: In [11] the single-source-case is analyzed when the state equation has the form $\Delta u + \epsilon u$, $\epsilon \neq 0$, and in all the numerical experiments in that paper the true source function is non-negative, $f(x) \geq 0$, $x \in \Omega$.

We only consider finite dimensional problems in this paper, and some remarks concerning the discretization of (4) is presented in section 2. In fact, our analysis is presented in terms of Euclidean spaces and can therefore be applied in other contexts, not just to the present inverse problem.

## 2 Discretization

In the finite element method (FEM) one usually approximates $f$ with a function $f_h$ in some finite element space $F_h$. The basis functions of $F_h$ typically have small and local support, e.g., they are "hat-functions" or characteristic functions of the involved grid cells, making them a suitable choice for incorporating spatial sparsity in our inverse problem.

On the other hand, due to the complimentary condition (5), we must search
for a solution to our inverse problem in the subspace
\[ F_{h,\phi} = \left\{ f_h \in F_h \mid \int_{\Omega} f_h = 0 \right\} \]
of \( F_h \). This subspace is spanned by the functions
\[ \psi_j = \phi_j - \frac{1}{|\Omega|} \int_{\Omega} \phi_j, \]
where \( \{\phi_j\} \) is the set of basis functions for \( F_h \). The imposed restriction of zero integral reduces the degrees of freedom of \( F_h \) by one, compared with the dimension of \( F_h \). This makes the family \( \{\psi_j\} \) a so-called frame for \( F_{h,\phi} \). For an introduction to the use of frames in inverse problems, see, e.g., [5].

We end this section with the definition of the forward matrix \( A \) associated with (4). That is, \( A \) represents the following chain of operations
\[ x \in \mathbb{R}^n \rightarrow f_h = \sum x_j \psi_j \rightarrow u_h \rightarrow T u_h|_{\partial \Omega} \in \mathbb{R}^m, \]
where \( u_h = u_h(f_h) \) denotes the FEM solution of (1)-(3) generated by a given \( f_h \) and \( T \) yields the nodal values of \( u_h|_{\partial \Omega} \). We discretized both \( f_h \) and \( u_h \) with first order Lagrange elements. Typically, \( n \gg m \) and \( A \) will have a large null space. As mentioned above, for the sake of generality, we develop the theory in terms of Euclidean spaces.

### 3 Analysis

Consider the equation
\[ A x = b, \quad (6) \]
where \( A \in \mathbb{R}^{m \times n} \) has a nontrivial null space \( N(A) \). We can multiply (6) with the pseudo-inverse \( A^\dagger \) of \( A \) to obtain
\[ A^\dagger A x = A^\dagger b, \quad (7) \]
where \( A^\dagger A \) is the orthogonal projection
\[ P = A^\dagger A : \mathbb{R}^n \rightarrow N(A)^\perp. \quad (8) \]

In [9] we introduced the diagonal weight/regularization matrix \( W \in \mathbb{R}^{n \times n} \) defined by
\[ W e_i = \|Pe_i\|_2 e_i, \quad (9) \]
where we assume that \( w_i = \|Pe_i\|_2 \neq 0 \) for \( i = 1, 2, \ldots, n \), i.e., none of the standard basis vectors belong to the null space of \( A \). Some of the beneficial mathematical properties of \( W \) are explored in [10] [12]. In particular, let
\[ x^\dagger = A^\dagger A e_j = Pe_j \]
denote the minimum norm solution of
\[ Ax = Ae_j. \]

Then,
\[
j = \arg \max_{i \in \{1,2,\ldots,n\}} |\langle W^{-1}x_i \rangle| = \arg \max_{i \in \{1,2,\ldots,n\}} |\langle W^{-1}Pe_j \rangle|,
\]
\[
|\langle W^{-1}Pe_j \rangle| = |\langle Pe_j \rangle| \leq 1.
\] (10)

where, \([v]_i\) represents the \(i\)th component of the Euclidean vector \(v\). The property (10) shows that \(x_i^\dagger\) achieves its maximum for the "correct" index \(j\). We will use this result at several occasions below. The proof of (10) requires that the images under \(A\) of any two standard unit basis vectors are not parallel, i.e.,
\[
Ae_i \neq \eta Ae_j \quad \forall i, j \in \{1,2,\ldots,n\}, i \neq j, \forall \eta \in \mathbb{R}.
\] (11)

The property (10) is the basic observation reported for the sLORETA algorithm in the EEG literature\(^1\). As far as the authors know, the first mathematical proof of (10) was presented in connection with Theorem 4.1 in [9], see also Theorem 2.1 in [10].

3.1 Basis pursuit

This section is devoted to a study of the basis pursuit problem associated with (6), using the weight matrix (9):
\[
\min_{x} \|Wx\|_1 \quad \text{subject to} \quad Ax = b.
\] (12)

We want to analyze whether we can recover a sparse solution from exact data. That is, if the true data reads
\[
b = b^\dagger = A\left(\sum_{j} x_j^* e_j\right),
\]
where \(\mathcal{J} = \text{supp}(x^*)\), can we recover
\[
x^* = \sum_{j} x_j^* e_j
\] (13)
by solving (12)?

We first present requirements which guarantee that the true source-sink configuration \(x^*\) is a solution of the basis pursuit problem and that the support of any other solution of this problem is contained in the support of \(x^*\). This

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\(^1\)In the sLORETA algorithm the weights are not defined in terms of the projection \(P = A^\dagger A\), but instead one uses Tikhonov regularization, i.e., \((A^T A + \alpha I)^{-1} A^T A \approx A^\dagger A\), to compute the weights. Hence, strictly speaking, (10) may not hold for the sLORETA method unless \(\alpha > 0\) is very small.
result, and its proof, is rather similar to Theorem 2.1 in [12]. Nevertheless, for the sake of completeness, and since the requirements differ from those presented in [12], we present its proof in detail. Below we will use this theorem to prove that perfect recovery is possible in some cases.

**Theorem 3.1** (The support will not increase and \( x^\ast \) is a solution). Assume that there exists a vector \( c \) such that

\[
\frac{Pe_i}{\|Pe_i\|_2} \cdot c = \text{sgn}(x^*_i) \quad \forall i \in J,
\]

\[
\frac{Pe_i}{\|Pe_i\|_2} \cdot c < 1 \quad \forall i \in J^c.
\]

where \( J = \text{supp}(x^\ast) \) and \( J^c = \{1, 2, \ldots, n\} \setminus J \). Then any solution \( y \) of

\[
\min_x \|Wx\|_1 \quad \text{subject to} \quad Ax = A\sum_{j} x^*_je_j \tag{16}
\]

satisfies

\[
\text{supp}(y) \subseteq \text{supp}(x^\ast),
\]

and \( x^\ast \) is a solution of (16).

**Proof.** Let

\[
y = \sum_{J^c \cup J} y_i e_i
\]

denote a solution of (16). Then

\[
Ay = A\sum_{J} x^*_je_j
\]

or, because \( P = A^\dagger A \),

\[
P_y = P\sum_{J} x^*_je_j.
\]

Taking the inner product with \( c \) yields

\[
\sum_{J^c \cup J} y_i Pe_i \cdot c = \sum_{J} x^*_je_j Pe_j \cdot c \tag{17}
\]

where, using (14) and (9),

\[
\sum_{J} x^*_je_j Pe_j \cdot c = \sum_{J} x^*_j \text{sgn}(x^*_j) \|Pe_j\|_2 \\
= \sum_{J} |x^*_j| \|Pe_j\|_2 \\
= \|Wx^\ast\|_1 \tag{18}
\]
and, invoking (14), (15) and (9),
\[
\sum_{J^c \cup J} y_i P e_i \cdot c \leq \sum_{J^c \cup J} |y_i| |P e_i \cdot c| \\
\leq \sum_{J^c \cup J} |y_i| \|P e_i\|_2 \\
= \|W y\|_1.
\]  
(19)

From (17), (18) and (19) we can conclude that
\[
\|W y\|_1 \geq \|W x^*\|_1
\]
and it follows that \(x^*\) is a solution of \((16)\).

If there is \(i' \in J^c\) such that \(y_{i'} \neq 0\), then the strict inequality in (15) yields a strict inequality in (19):
\[
\sum_{J^c \cup J} y_i P e_i \cdot c \leq \sum_{J^c} |y_i| |P e_i \cdot c| + \sum_{J^c} |y_i| |P e_i\|_2 \\
< \sum_{J^c} |y_i| \|P e_i\|_2 + \sum_{J^c} |y_i| \|P e_i\|_2 \\
= \|W y\|_1,
\]
and this would imply that \(\|W y\|_1 > \|W x^*\|_1\). We conclude that any solution \(y\) of (16) must satisfy \(\text{supp}(y) \subseteq \text{supp}(x^*)\). \(\square\)

We now prove that well-separated sources and sinks can be recovered, provided that the images under \(A\) of any two basis functions aren’t parallel:

**Theorem 3.2** (Well-separated sources and sinks, i.e., projections with disjoint supports). Assume that (11) holds. If
\[
\text{supp}(P e_j) \cap \text{supp}(P e_k) = \emptyset \text{ for all } j, k \in J, j \neq k,
\]  
(20)
then
\[
x^* = \sum_{J} x^*_j e_j,
\]
where \(J = \text{supp}(x^*)\), is the unique solution of
\[
\min_{x \in \mathbb{R}^n} \|W x\|_1 \quad \text{subject to} \quad Ax = Ax^*.
\]  
(21)

**Proof.** We will first show that (14) and (15) hold for an appropriate choice of \(c\). Thereafter we prove the uniqueness.

Let
\[
c = \sum_{J} \text{sgn}(x^*_j) \frac{P e_j}{\|P e_j\|_2}
\]
and note that (20) assures that the projections \( \{ P_j \} \) are orthogonal. Therefore, if \( i \in J \), then
\[
\frac{P_e_i}{\|P_e_i\|_2} \cdot c = \frac{P_e_i}{\|P_e_i\|_2} \cdot \frac{P_e_i}{\|P_e_i\|_2} \text{sgn}(x_i^*) = \text{sgn}(x_i^*),
\]
and we conclude that (14) is fulfilled.

Assume that \( i \in J^c \). Due to (20) there can at most be one \( k \in J \) such that \( i \in \text{supp}(P_e_k) \). Consequently, provided that such a \( k \) exists, and using the fact that \( P \) is an orthogonal projection,
\[
\frac{P_e_i}{\|P_e_i\|_2} \cdot c = \sum_{j} \frac{\text{sgn}(x_j^*)}{\|P_e_i\|_2 \|P_e_j\|_2} (P_e_i, P_e_j)
\]
\[
= \sum_{j} \frac{\text{sgn}(x_j^*)}{\|P_e_i\|_2 \|P_e_j\|_2} \cdot (e_i, P_e_j)
\]
\[
= \sum_{j} \frac{\text{sgn}(x_j^*)}{\|P_e_i\|_2 \|P_e_j\|_2} \cdot [P_e_j],
\]
\[
= \frac{\text{sgn}(x_k^*)}{\|P_e_i\|_2 \|P_e_k\|_2} \cdot [P_e_k]_i
\]
\[
= \frac{\text{sgn}(x_k^*)}{\|P_e_i\|_2 \|P_e_k\|_2} \cdot (e_i, P_e_k)
\]
\[
= \frac{\text{sgn}(x_k^*)}{\|P_e_i\|_2 \|P_e_k\|_2} \cdot (P_e_i, P_e_k),
\]
where \( [P_e_j]_i \) denotes the \( i \)'th component of the (Euclidean) vector \( P_e_j \). Since (11) holds, it follows from the Cauchy-Schwarz inequality and the fact that \( i \in J^c \) and \( k \in J \), i.e., \( i \neq k \), that
\[
\left| \frac{P_e_i}{\|P_e_i\|_2} \cdot c \right| < 1,
\]
which is (15). On the other hand, if \( i \in J^c \) and \( i \notin \text{supp}(P_e_j) \) for all \( j \in J \), then
\[
[P_e_j]_i = 0 \quad \forall j \in J.
\]
We thus instead find, see (22), that
\[
\frac{P_e_i}{\|P_e_i\|_2} \cdot c = 0,
\]
and (15) also holds in this case.

Since both (14) and (15) are satisfied, Theorem 3.1 assures that \( x^* \) solves (21) and that any other solution \( y \) of this problem must be such that \( \text{supp}(y) \subseteq \text{supp}(x^*) = J \). Hence,
\[
Ay = Ax^*
\]
can be written in the form

\[ A \sum_{j} y_j e_j = A \sum_{j} x_j^* e_j, \]

or, multiplying with \( A^\dagger \),

\[ \sum_{j} y_j P e_j = \sum_{j} x_j^* P e_j. \]

Finally, the orthogonality of the projections \( \{P e_j\}_J \) implies that \( y_j = x_j^* \) for all \( j \in J \), and it follows that \( x^* \) is the unique solution of (21). \( \square \)

Note that this theorem also covers the single-source-case, i.e., the case \( J = \{j\} \). Then (20) automatically holds, and the true source \( x^* = x_j^* e_j \) can always be recovered. An alternative proof addressing the single-source-situation is presented in [11].

The projections will typically be (almost) disjoint if the distances between the sources and sinks are sufficiently large or if they are located in different "pockets" of the solution domain. For the above mentioned ECG and EEG applications, this roughly suggests that non-convex domains are preferable to convex regions.

For the PDE constrained optimization problem (4) it is plausible that a large number of sources or sinks located at the boundary, where data is recorded, can be recovered. This corresponds to the situation where the sources and sinks are close to, or in, the orthogonal complement of the null space of the associated forward operator/matrix. We will now not only prove that our methodology can handle such cases, but that also an additional interior source/sink can be detected.

**Corollary 3.2.1** (Several sources and sinks in \( \mathcal{N}(A)^\perp \), plus one more). Let \( J \) be an index subset of \( \{1, 2, \ldots, n\} \) and assume that \( e_j \in \mathcal{N}(A)^\perp \) for all \( j \in J \). If (11) holds, then

\[ x^* = \sum_{j \in J} x_j^* e_j + x_{\tilde{j}}^* e_{\tilde{j}} \]

is the unique solution of

\[ \min_{x \in \mathbb{R}^n} \|Wx\|_1 \quad \text{subject to} \quad Ax = Ax^*. \]

Here, \( \tilde{j} \) is an index outside \( J \), i.e., \( \tilde{j} \in J^c \).

**Proof.** First,

\[ e_j \in \mathcal{N}(A)^\perp \quad \forall j \in J \quad \Rightarrow \quad P e_j = e_j \quad \forall j \in J, \]

and it follows that

\[ \text{supp}(P e_j) \cap \text{supp}(P e_k) = \emptyset \quad \forall j, k \in J, \ j \neq k. \quad (23) \]
Second, for any \( j \in J \), keeping in mind that \( P^T = P \),

\[
[Pe_j]_j = (Pe_j, e_j) = (e_j, Pe_j) = (e_j, e_j) = 0.
\]

Consequently,

\[
\emptyset = \text{supp}(Pe_j) \cap \text{supp}(e_j) = \text{supp}(Pe_j) \cap \text{supp}(Pe_j) \quad \forall j \in J.
\]  

(24)

Finally, (23) and (24) imply that

\[
\text{supp}(Pe_j) \cap \text{supp}(Pe_k) = \emptyset \quad \forall j, k \in J \cup \{\tilde{j}\}, j \neq k,
\]

and the result therefore follows from Theorem 3.2.

In practice it seems important that a source is not misinterpreted as a sink or vice versa. The next result addresses this issue. We use the notation

\[
\|v\|_0 = \#\{i \mid v_i \neq 0\}.
\]

**Theorem 3.3** (All solutions have components with the same signs). Assume that \( b \in \text{col}(A) \) and that there exist

\[
\bar{x}^* \in \arg\min_x \|x\|_0 \quad \text{subject to} \quad Ax = b
\]

and a vector \( c \) such that (14) and (15) hold with \( J = \text{supp}(\bar{x}^*) \). Then any two solutions \( z \) and \( q \) of

\[
\min_x \|Wx\|_1 \quad \text{subject to} \quad Ax = b
\]

obey

\[
\text{sgn}(z_k) = \text{sgn}(q_k) \quad \forall k \in \{1, 2, \ldots, n\}.
\]

Moreover,

\[
\text{supp}(z) = \text{supp}(q) = \text{supp}(\bar{x}^*).
\]

**Proof.** Since \( A\bar{x}^* = b \), any solution \( y \) of (26) must also be a solution of (16) with \( x^* = \bar{x}^* \). Therefore, according to Theorem 3.1 \( y \) satisfies

\[
\text{supp}(y) \subseteq \text{supp}(\bar{x}^*). 
\]

(27)

Consequently,

\[
\|y\|_0 \leq \|\bar{x}^*\|_0,
\]

but from the definition (25) of \( \bar{x}^* \)

\[
\|y\|_0 \geq \|\bar{x}^*\|_0,
\]

and we conclude that \( \|y\|_0 = \|\bar{x}^*\|_0 \), which together with (27) show that

\[
\text{supp}(y) = \text{supp}(\bar{x}^*)
\]
for any solution $y$ of (26).

Consequently, any two solutions $z$ and $q$ of (26) satisfy

$$\text{supp}(z) = \text{supp}(q) = \text{supp}(\tilde{x}^*),$$

which implies that

$$\text{sgn}(z_k) = \text{sgn}(q_k) = 0 \quad \forall k \in J^c.$$  \hfill (29)

Since the cost-functional in (26) is convex and its constraint is linear, it follows that

$$(1 - t)z + tq$$

also solves (26) for all $t \in [0,1]$. Furthermore, from the argument presented in the first paragraph of this proof we therefore find that

$$\text{supp}((1 - t)z + tq) = J \quad \forall t \in [0,1].$$  \hfill (30)

Assume that there exists $k \in J$ such that $z_k \leq 0$ and $q_k > 0$, or $z_k > 0$ and $q_k \leq 0$, then there exists $\hat{t} \in [0,1]$ such that

$$(1 - \hat{t})z_k + \hat{t}q_k = 0.$$  \hfill (33)

This would lead to

$$\text{supp}((1 - \hat{t})z + \hat{t}q) \neq J,$$

which cannot be the case because of (30). We conclude that

$$\text{sgn}(z_k) = \text{sgn}(q_k) \quad \forall k \in J.$$  \hfill (31)

The theorem now follows from (28), (29) and (31). \hfill $\square$

In the next theorem we present a criterion which asserts that the true source-sink vector $x^*$ is the only solution to the basis pursuit problem. Note that we use the notation

$$\text{sgn}(v) = (\text{sgn}(v_1), \text{sgn}(v_2), \ldots, \text{sgn}(v_n))^T.$$  \hfill (32)

**Theorem 3.4** (Uniqueness). *Assume that $x^*$ is such that (14) and (15) hold. Then $x^*$ is the unique solution of (16) if, and only if,*

$$\text{sgn}(x^*)^T W q \neq 0 \quad \forall q \in \{z \in N(A) \mid \text{supp}(z) \subseteq J, z \neq 0\},$$  \hfill (32)

*where $J = \text{supp}(x^*)$.\*

*Proof.* According to Theorem 3.1, if (14) and (15) hold, then $x^*$ solves (16). Let $y \neq x^*$ be another solution of (16). Then $\text{supp}(y) \subseteq \text{supp}(x^*) = J$, see Theorem 3.1 and therefore

$$q = y - x^*$$

satisfies

$$\text{supp}(q) \subseteq J.$$  \hfill (33)
We also note that \( q \in \mathcal{N}(A) \), since both \( x^* \) and \( y \) solve (16), and hence
\[
q \in \{ z \in \mathcal{N}(A) \mid \text{supp}(z) \subseteq J, \ z \neq 0 \}.
\]

The cost-functional in (16) is convex and the constraint is defined in terms of an affine function. Consequently,
\[
(1 - t)x^* + ty = x^* + tq \quad \forall t \in [0, 1]
\]
also solves (16), and it follows that
\[
g(t) := \|W(x^* + tq)\|_1 = c \quad \text{(a constant)} \quad \forall t \in [0, 1].
\]
From (33), and the fact that \( W \) is a diagonal matrix with diagonal entries \( w_1, w_2, \ldots, w_n \), we find that
\[
g(t) = \sum_{J} |w_j x^*_j + w_j t q_j| = c \quad \forall t \in [0, 1].
\]
Since \( w_j > 0 \) and \( x^*_j \neq 0 \) for all \( j \in J \), it follows that there exists \( \hat{t} > 0 \) such that
\[
|w_j x^*_j + w_j t q_j| = \text{sgn}(x^*_j)(w_j x^*_j + w_j t q_j) \quad \forall t \in [0, \hat{t}], \forall j \in J.
\]
Consequently,
\[
g(t) = \sum_{J} \text{sgn}(x^*_j)(w_j x^*_j + w_j t q_j) = c \quad \forall t \in [0, \hat{t}],
\]
and differentiation wrt. \( t \) yields that
\[
\sum_{J} \text{sgn}(x^*_j)w_j q_j = 0
\]
or
\[
\text{sgn}(x^*)^T W q = 0.
\]

We have thus proven that
\[
\exists y \neq x^* \text{ which solves (16)}
\]

\[
\downarrow
\]
\[
\exists q \in \{ z \in \mathcal{N}(A) \mid \text{supp}(z) \subseteq J, \ z \neq 0 \} \text{ such that } \text{sgn}(x^*)^T W q = 0,
\]
which finishes the first part of this proof.

For the second part, assume that there exists
\[
q \in \{ z \in \mathcal{N}(A) \mid \text{supp}(z) \subseteq J, \ z \neq 0 \} \text{ such that } \text{sgn}(x^*)^T W q = 0, \quad (34)
\]
and recall that \( x^* \) solves (16). As in the first part of this proof, we find that there exists \( \hat{t} > 0 \) such that
\[
g(t) = \|W(x^* + tq)\|_1
\]
\[
= \sum_{J} \text{sgn}(x^*_j)(w_j x^*_j + w_j t q_j) \quad \forall t \in [0, \hat{t}],
\]
\[
\sum_{J} \text{sgn}(x^*_j)w_j q_j = 0
\]
or
\[
\text{sgn}(x^*)^T W q = 0.
\]
and we can use (34) to deduce that

\[ g(t) = \sum_J \text{sgn}(x_j^*)(w_j x_j^*) + t\text{sgn}(x^*)^TWq \]

\[ = \sum_J \text{sgn}(x_j^*)(w_j x_j^*) \quad \forall t \in [0, \hat{t}] . \]

Hence, \( g(t) \) equals a constant for \( t \in [0, \hat{t}] \) and therefore

\[ \|W(x^* + tq)\|_1 = \|W(x^*)\|_1 \quad \forall t \in [0, \hat{t}] . \]

Since \( q \in \mathcal{N}(A) \), it follows that \( x^* + tq \) satisfies the constraint in (16) for all \( t \in [0, \hat{t}] \). We can therefore conclude that \( x^* + tq \), for every \( t \in [0, \hat{t}] \), solves (16).

In summary,

\[ \exists q \in \{ z \in \mathcal{N}(A) \mid \text{supp}(z) \subseteq J, \ z \neq 0 \} \text{ such that } \text{sgn}(x^*)^TWq = 0 \]

\[ \Downarrow \]

\[ \exists \hat{t} > 0 \text{ such that } x^* + t_q \text{ solves (16) } \forall t \in [0, \hat{t}] . \]

This completes the second part of the proof.

Using modern software tools, it is "easy" to check whether (32) holds for a reasonable sized matrix \( A \) and a given true sink-source vector \( x^* \). Nevertheless, this result can also be employed to prove the following non-uniqueness result.

**Corollary 3.4.1** (Non-uniqueness). Let

\[ S = \{ z \in \mathcal{N}(A) \mid \text{supp}(z) \subseteq J \} \]

and assume that \( x^* \) is such that (14) and (15) hold, where \( J = \text{supp}(x^*) \). If \( \dim(S) \geq 2 \), then (16) has several solutions.

**Proof.** We will show that (32) can not hold if \( \dim(S) \geq 2 \). Let \( v_1 \) and \( v_2 \) be two linearly independent vectors in \( S \). If either \( \text{sgn}(x^*)^TWv_1 = 0 \) or \( \text{sgn}(x^*)^TWv_2 = 0 \), then Theorem 3.4 asserts the non-uniqueness.

On the hand, if \( \text{sgn}(x^*)^TWv_1 \neq 0 \) and \( \text{sgn}(x^*)^TWv_2 \neq 0 \), consider the linear combination \( q = sv_1 + tv_2 \in S \) of \( v_1 \) and \( v_2 \). Then there clearly exist scalars \( s, t \) such that \( q \neq 0 \) and such that

\[ \text{sgn}(x^*)^TWq = s \text{sgn}(x^*)^TWv_1 + t \text{sgn}(x^*)^TWv_2 = 0 . \]

Again the result follows from Theorem 3.4.

\[ \Box \]

### 3.2 Regularized problems

In this section we establish regularized counterparts to Theorem 3.2 and Corollary 3.2.1. This is important because in practice one needs to apply regularization in order to avoid disastrous amplification of noise in real world data.
If we apply weighted regularization to (7), using the weight matrix \( W \), we obtain

\[
\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \| A^\dagger A x - A^\dagger b \|_2^2 + \alpha \| W x \|_1 \right\}.
\] (35)

Since \( W \) is defined in terms of the projection \( P = A^\dagger A \), it turns out that it is convenient to consider (35) instead of the standard formulation which is based on (6). In fact, we have not succeeded in proving similar results, to those presented in this subsection, for the problem

\[
\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \| A x - b \|_2^2 + \alpha \| W x \|_1 \right\}.
\]

If the underlying problem is ill posed, then \( A \) will typically have very small nonzero singular values, and the use of \( A^\dagger \) is not recommendable. In practice we therefore replace \( A^\dagger \) with a more "well behaved" matrix. This can be accomplished by applying truncated SVD or standard Tikhonov regularization. Analogously to the investigation of the basis pursuit problem, we will analyze whether

\[
x^* = \sum_{j} x_j^* e_j
\]

can be approximately recovered from the true data

\[
b = b^\dagger = A \left( \sum_{j} x_j^* e_j \right).
\]

For this problem, using the fact that \( P = A^\dagger A \), the minimization problem (35) becomes

\[
y^*_\alpha = \arg \min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} \left\| P x - P \left( \sum_{j} x_j^* e_j \right) \right\|_2^2 + \alpha \| W x \|_1 \right\}.
\] (36)

Our first result yields two simple-to-check criteria for determining the identifiability of \( x^* \): It turns out that it is sufficient to explore whether a specific \( s \times s \) linear system, where \( s \) is the cardinality of \( J = \text{supp}(x^*) \), admits a solution and whether the product of the involved system matrix and this solution has components, associated with indexes in \( J^c = \{ 1, 2, \ldots, n \} \setminus J \), belonging to the interval \([-1, 1]\). Recall that we use the notation \( [v]_i \) for the \( i \)’th component of the Euclidean vector \( v \).

**Theorem 3.5 (Sufficient criteria).** Let \( P \) and \( W \) be the matrices defined in (8) and (9), respectively, and let \( J = \text{supp}(x^*) \). If the linear system

\[
\left[ \sum_{j} a_j W^{-1} P e_j \right]_i = \text{sgn}(x_i^*) \quad \forall i \in J
\] (C.1)
has a solution \( \{a_j\}_J \) which satisfies
\[
\left[ \sum_{j} a_j W^{-1} P e_j \right]_i \in [-1, 1] \quad \forall i \in J^c, \tag{C.2}
\]
then
\[
y^*_\alpha = \sum_{J} (x^*_j - \alpha a_j)e_j
\]
solves (36), provided that \( \alpha > 0 \) is chosen such that
\[
\text{sgn}(x^*_j - \alpha a_j) = \text{sgn}(x^*_j) \quad \forall j \in J. \tag{37}
\]

**Remark 3.6.** Before proving the theorem, we note that the conditions (C.1) and (C.2) only involve the sign of \( x^*_j, j \in J \). This leads to a rather interesting observation: Whether we are, in principle, able to recover an \( s \)-sparse solution is independent of the magnitudes \( \{|x^*_j|\}_J \) of the individual sources and sinks. Moreover, when \( \alpha > 0 \) is sufficiently small, (37) will always hold because we consider finite dimensional problems. We also mention that the linear system (C.1) typically will be small, unless the number \( s \) of sparse sinks and sources is large. In the corollaries below we explore situations where we can guarantee that both (C.1) and (C.2) hold.

This theorem provides us with a simple two step strategy for checking whether a configuration of sources and sinks can be recovered under ideal conditions:

- **Solve**, if possible, (C.1) for a given configuration of sinks and sources.
- **If** (C.1) is solvable, verify whether (C.2) holds.

For example, for the ECG and EEG applications mentioned in the previous sections, one can investigate whether certain configurations of sinks-sources/focal-epileptic-seizures are detectable, provided that the noise level is sufficiently small.

**Proof of Theorem 3.5.**

**Existence:** Let us define the cost-functional \( T_\alpha : \mathbb{R}^n \to \mathbb{R} \) associated with (36),
\[
T_\alpha(x) = \frac{1}{2} \left\| P x - P \left( \sum_{J} x^*_j e_j \right) \right\|^2 + \alpha \|Wx\|_1, \tag{38}
\]
where \( g(\cdot) \) and \( h(W \cdot) \) represent the fidelity and regularization terms, respectively. According to standard convex optimization theory, \( x \) is a minimizer of \( T_\alpha \) if and only if
\[
0 \in \partial T_\alpha(x) = \nabla g(x) + \alpha W^T \partial h(Wx),
\]

15
where "∂" denotes the subgradient. Since $W^T = W$, we can multiply with $W^{-1}$ to obtain

$$-W^{-1}\nabla g(x) \in \alpha \partial h(Wx),$$

and from the expression (38) for $g$ we find, keeping in mind that $P^T P = PP = P$,

$$W^{-1}P \left( \sum J x^*_j e_j - x \right) \in \alpha \partial h(Wx). \quad (39)$$

We also observe, using the fact that $h(z) = \|z\|_1$ and that $W$ is a diagonal matrix with positive entries at its diagonal,

$$[\partial h(Wx)]_i = [\partial h(W[x_1 x_2 \ldots x_n]^T)]_i = \begin{cases} \{1\}, & x_i > 0, \\ \{-1\}, & x_i < 0, \\ [-1,1], & x_i = 0. \end{cases}$$

We will now investigate whether there exist scalars $\{\tilde{\gamma}_j\}_J$ such that $x = \sum_J \tilde{\gamma}_j e_j$, $\tilde{\gamma}_j \neq 0$, satisfies the optimality criterion (39). Note that

$$[\partial h(W\sum_J \tilde{\gamma}_j e_j)]_i = \begin{cases} \{\text{sgn}(\tilde{\gamma}_j)\}, & i \in J, \\ [-1,1], & i \notin J, \end{cases} \quad (40)$$

so the condition (39), with $x = \sum_J \tilde{\gamma}_j e_j$, becomes

$$\left[ \sum_J (x^*_j - \tilde{\gamma}_j)W^{-1}Pe_j \right]_i \in \alpha \begin{cases} \{\text{sgn}(\tilde{\gamma}_j)\}, & i \in J, \\ [-1,1], & i \notin J. \end{cases} \quad (41)$$

Assume that (C.1)-(C.2) hold. Then, with

$$\tilde{\gamma}_j = x^*_j - \alpha a_j,$$

we can conclude from (C.1)-(C.2) that

$$\left[ \sum_J (x^*_j - \tilde{\gamma}_j)W^{-1}Pe_j \right]_i = \alpha \left[ \sum a_j W^{-1}Pe_j \right]_i \in \alpha \begin{cases} \{\text{sgn}(x^*_j)\}, & i \in J, \\ [-1,1], & i \notin J^c. \end{cases} \quad (42)$$

and (41) is satisfied when $\alpha > 0$ is chosen such that $\text{sgn}(\tilde{\gamma}_j) = \text{sgn}(x^*_j)$, $\forall j \in J$, i.e., when (37) holds.

The regularized counterpart to Theorem 3.2, i.e., the case with well-separated sinks and sources, reads:

**Corollary 3.6.1 (Well-separated sources and sinks, i.e., disjoint projections).** Let $x^* = \sum_J x^*_j e_j$, $J = \text{supp}(x^*)$, and assume that (11) holds. If

$$\text{supp}(Pe_j) \cap \text{supp}(Pe_k) = \emptyset \quad j \neq k, j, k \in J,$$
then the conditions (C.1)-(C.2) hold. Furthermore,

\[ y_\alpha^* = \sum_{j} (x_j^* - \alpha a_j) e_j \]

solves (36), provided that \( \alpha > 0 \) is chosen such that (37) is satisfied.

**Proof.** Since \( W \) is a diagonal matrix, it follows that \( \text{supp}(W^{-1}P e_j) = \text{supp}(P e_j) \) for all \( k \). Consequently,

\[ \text{supp}(W^{-1}P e_j) \cap \text{supp}(W^{-1}P e_k) = \emptyset \quad j \neq k, j, k \in J. \]  

(43)

As proven in [10], see (10), because (11) is assumed to hold, \( W^{-1}P e_j \) achieves its unique absolute maximum for index \( j \), i.e., \( [W^{-1}P e_j]_j \neq 0 \). Due to (43), condition (C.1) therefore simplifies to

\[ [a_j W^{-1}P e_j]_j = \text{sgn}(x_j^*) \quad \forall j \in J, \]  

(44)

which clearly can be solved for \( \{a_j\}_J \).

Furthermore, again because of the non-overlapping supports (43), there can for each \( i \in J^c \) be at most one element \( \iota(i) \in J \) such that \( [P e_{\iota(i)}]_i \neq 0 \). Consequently, condition (C.2) reads

\[ [a_{\iota(i)} W^{-1}P e_{\iota(i)}]_i \in (-1,1) \quad \forall i \in J^c. \]  

(45)

Equation (44) implies that \( [a_{\iota(i)} W^{-1}P e_{\iota(i)}]_i = 1 \), and the maximum property (10) therefore implies that (45) is satisfied.

The next result shows that a regularized version of Corollary 3.2.1 holds:

**Corollary 3.6.2** (Several sources and sinks in \( \mathcal{N}(A)^\perp \), plus one more). Assume that (11) holds. If \( e_j \in \mathcal{N}(A)^\perp, \forall j \in J \setminus \{\tilde{j}\} \), then the conditions (C.1)-(C.2) are satisfied, and

\[ y_\alpha^* = \sum_{j} (x_j^* - \alpha a_j) e_j \]

solves (36), provided that \( \alpha > 0 \) is chosen such that (37) is satisfied.

**Proof.** Using Theorem 3.5, we only need to show that (C.1)-(C.2) hold. First, note that for \( e_j \in \mathcal{N}(A)^\perp \) it follows from the definition (8) of the orthogonal projection \( P \) that \( Pe_j = e_j \). Therefore,

\[ W^{-1}P e_j = W^{-1}e_j = e_j, \]

where the last equality follows from (9):

\[ e_j \in \mathcal{N}(A)^\perp \Rightarrow \|Pe_j\|_2 = \|e_j\|_2 = 1. \]
Consequently, (C.1) simplifies to
\[
\sum_{J \setminus j} a_j e_j^i + a_j^i [W^{-1} P e_j]_i = \text{sgn}(x^*_i) \quad \forall i \in J.
\] (46)

This can be written as the matrix-vector equation
\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
[W^{-1} P e_j]_{j_1} \\
[W^{-1} P e_j]_{j_2} \\
\vdots \\
[W^{-1} P e_j]_{j_s}
\end{bmatrix}
= \begin{bmatrix}
\text{sgn}(x^*_{j_1}) \\
\text{sgn}(x^*_{j_2}) \\
\vdots \\
\text{sgn}(x^*_{j_s})
\end{bmatrix}
\]

From the maximum property (10) we know that
\[
||W^{-1} P e_j||_{j_\eta} < ||W^{-1} P e_j||_j \leq 1, \quad j_\eta \neq \tilde{j}.
\]

Consequently, the matrix above is strictly diagonally dominant, and this system therefore has a unique solution. Hence, condition (C.1) is satisfied.

To show that (C.2) also holds, we first note that for \(i = \tilde{j}\), (46) asserts that \(a_{\tilde{j}} \) must satisfy
\[
a_{\tilde{j}} [W^{-1} P e_j]_{\tilde{j}} = \text{sgn}(x^*_\tilde{j}).
\]

Again, by (10) we then have that for \(i \in J^c\),
\[
a_{\tilde{j}} [W^{-1} P e_j]_i \in (-1, 1).
\]

In this case, this is sufficient to conclude that condition (C.2) holds because \(P e_j = e_j \) for all \(j \in J \setminus \{\tilde{j}\}\). \(\square\)

### 4 Numerical examples

In this section we illuminate the theoretical results presented in section 3.2. More specifically, we will show experiments where the assumptions needed in corollaries 3.6.1 and 3.6.2 are almost satisfied in order to demonstrate the robustness of the method. (We do not present computations for problems satisfying the assumptions exactly because then we know that the true sinks and sources will be perfectly recovered.)

Concerning the numerical solution of (36), where we recall that \(P = A^t A\), we employed truncated SVD to obtain a "well-behaved" approximation of \(A^t\), using 50 and 20 singular values in examples 1 and 2, respectively. The conductivity \(\sigma\) in the first example equaled 1, cf. equation (1), whereas in the second example \(\sigma(x, y) = 2 + \sin(x) \cos(y)\). All domains were partitioned in terms of non-uniform grids, and the mesh parameter \(h\) in the inverse computations was twice as large as in the forward simulations. All code was written in Python, using the FEniCS and Scipy libraries.
4.1 Example 1: All but one in $\mathcal{N}(A)^\perp$

The first example concerns Corollary 3.6.2. All but one of the sources or sinks are in the orthogonal complement $\mathcal{N}(A)^\perp$ of the null space $\mathcal{N}(A)$ of the forward matrix $A$. To set up the experiment, we selected a number of basis vectors $\{e_j\}_{j \in J \setminus \{\tilde{j}\}}$ satisfying $\|Pe_j\|_2 \geq 0.95$, $j \in J \setminus \{\tilde{j}\}$, and arbitrarily set them to be sinks or sources with unit magnitude. Here, we recall that $\|Pe_j\|_2 = 1$ implies that $e_j \in \mathcal{N}(A)^\perp$, see (8). Thereafter we added one interior source, i.e., $e_{\tilde{j}}$. The outcome of this process is illustrated in figure 1(a), which shows the true sources and sinks.

Panels (b) and (c) in figure 1 display the regularized solutions when weighted and unweighted regularization are employed, respectively. We observe that all the sinks and sources located at the boundary are recovered with both methods, but the interior source is only detected when weighted regularization is employed.

4.2 Example 2: Well-separated sources and sinks

According to Corollary 3.6.1 a collection of well-separated sinks and sources can be recovered. That is, this corollary requires that the projections $\{Pe_j\}_{j \in J}$ are disjoint.

Panel (a) in figure 2 visualises the cross-shaped domain and the true sink-source configuration considered in our second test problem. Figure 3 shows that the supports of $\{W^{-1}Pe_j\}_{j \in J}$ are almost disjoint, where we note that the supports of $W^{-1}Pe_j$ and $Pe_j$ are identical because $W$ is a diagonal matrix with positive diagonal entries. (In figure 3 we visualise, for the sake of easy interpretation, the vector $|W^{-1}Pe_j|$ containing the absolute values of the components of $W^{-1}Pe_j$.)

The source-sink detection works well in this case, see panels (b), (c) and (d) in figure 2. More precisely, the positions of the individual sinks and sources are perfectly recovered, but the magnitude is underestimated when the noise level increases. We also mention that no source is misinterpreted as a sink, or vice versa.

Concerning the noise, we computed the synthetic observation data

$$b = Ax^* + \tau \rho,$$

where $\tau$ is a scalar, $\rho$ is a vector containing normally distributed numbers with zero mean and standard deviation 1, and $x^*$ represents the true sources and sinks. The noise level is then defined as the ratio

$$\frac{\tau}{\max(b) - \min(b)}.$$
procedure deteriorates when a square domain Ω = (−1, 1)^2 is used instead of a cross-shaped domain, see figure 5. In the square domain case, the supports of \{Pe_j\}_{j \in J} become much more overlapping (illustration not included) than with the cross-shaped geometry, cf. Corollary 3.6.1. Roughly speaking, source-sink detection is more difficult for convex domains than for non-convex geometries.

Figure 6 shows that the weighting procedure might not work very well for determining the size of composite sinks and sources. That is, sources and sinks that are not generated by a single basis vector e_j. The inverse solution correctly identifies the positions of the sources and sinks, but the magnitudes and extends are not correct. How to handle this, is an open problem. Will box constraints resolve this issue?
(a) True sinks and sources.

(b) The solution $y^\alpha_\alpha$ of (36), i.e., using weighted regularization.

(c) Inverse solution computed with unweighted/standard regularization.

Figure 1: Example 1. Comparison of the true sinks and sources and the inverse solutions. The regularization parameter was $\alpha = 10^{-4}$. 
Figure 2: Example 2. Comparison of the true sinks and sources and the inverse solutions computed with weighted regularization and different levels of noise.
Figure 3: Example 2. Plots of $|W^{-1} P e_j|$ for the four individual sources/sinks displayed in Figure 2a).

Figure 4: Example 2. The inverse solution computed with unweighted regularization. The regularization parameter was $\alpha = 10^{-4}$. Panel (a) in figure shows the true sinks and sources.
Figure 5: Example 2. Comparison of the true sources and sinks and the inverse solution computed with weighted regularization. The regularization parameter was $\alpha = 10^{-4}$.

Figure 6: Example 2. Comparison of the true composite sources and sinks and the inverse solution computed with weighted regularization. The regularization parameter was $\alpha = 10^{-4}$.

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