**Range entropy**: A bridge between signal complexity and self-similarity

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(Dated: September 19, 2018)

**Background**: Sample entropy (SampEn) has in recent years been accepted as an alternate, and sometimes a replacement, measure to approximate entropy (ApEn) for characterizing temporal complexity of time series. However, it still suffers from issues such as inconsistency over short-length signals and its tolerance parameter r, susceptibility to signal amplitude changes and insensitivity to self-similarity of time series.

**Purpose**: We propose modifications to the ApEn and SampEn measures. These new signal entropy measures which are defined for 0 < r ≤ 1, are more robust to signal amplitude changes and sensitive to self-similarity property of time series.

**Method**: We modified ApEn and SampEn by redefining the distance function used originally in their definitions. We then evaluated the new entropy measures, called range entropies (RangeEn) using different random processes and nonlinear deterministic signals. The aim was to study the effects of signal length, amplitude changes, and self-similarity on the new entropy measures. We further applied the proposed entropies to normal and epileptic electroencephalographic (EEG) signals under different states.

**Results**: Our results suggest that, unlike ApEn and SampEn, RangeEn measures are robust to stationary and nonstationary signal amplitude variations and that their trajectories in the tolerance r-plane are constrained between 0 (maximum entropy) and 1 (minimum entropy). We also showed that RangeEn have direct relationships with the Hurst exponent; suggesting that the new definitions are sensitive to self-similarity structures of signals. RangeEn analysis of epileptic EEG data showed distinct behaviours in the r-domain for extracranial versus intracranial recordings as well as different states of epileptic EEG data.

**Conclusions**: The present work shows that the new entropy measures are robust to signal amplitude changes and sensitive to in-built self-similarity of time series. Their constrained trajectory in the r-plane makes them a good candidate for studying complex biological signals such as EEG during seizure and non-seizure states.

**Code availability**: The Python package used to generate the results shown in this paper is publicly available at: https://github.com/omidvarnia/RangeEn

**Running title**: Range entropy (RangeEn) analysis of time series

PACS numbers: 05.45.-a, 89.70.Cf, 05.45.Tp, 87.19.xm, 87.19.le, 87.18.-h

Keywords: signal entropy, complexity, self-similarity, epilepsy, EEG

I. INTRODUCTION

In information theory, entropy of a random process is defined as the average rate of generation of new information [1]. Therefore, independent and identically distributed white noise is assumed to be the projection of thermodynamic equilibrium state into a signal with maximal entropy. This is because identically distributed white noise has a uniform probability density function where each upcomong time point contains new information. On the other hand, a completely regular signal with a repeating pattern of constant values will lead to minimal entropy, as there is no generation of new information. The most prominent types of signal entropy measures may include Shannon entropy [1], Renyi entropy [2], Kulumgorov entropy [3, 4], Kulumgorov-Sinai entropy [5], Eckmann-Ruelle entropy [6], approximate entropy (ApEn) [7], sample entropy (SampEn) [8] and multi-scale entropy [9].

Among these signal entropy measures, ApEn and SampEn are two of the most commonly used measures in contemporary science, especially in the analysis of biological signals [10]. Like ApEn, SampEn resembles a template-matching search throughout the input signal with two main parameters: embedding dimension m and tolerance r. The former governs the length of each segment (template) to be searched and the later controls the level of similarity between segments. In fact, SampEn stems from ApEn after addressing some of its limitations including inconsistency over the parameter r and strong dependency to the input signal length [8]. However, SampEn still suffers from few drawbacks such as: i) high sensitivity to signal amplitude changes, ii) lack of a lower bound and iii) insensitivity to self-similar properties of time series.

To address these issues, we developed two new sig-
nal entropy measures called range entropies ($RangeEn_A$ and $RangeEn_B$, respectively). These novel measures are based on the mathematical definitions of $ApEn$ and $SampEn$ but with a different normalized distance function in the phase space. We compared the proposed $RangeEn$ measures with $ApEn$ and $SampEn$ using multiple simulations. We showed that $RangeEn$: i) always reach to zero at $r=1$; ii) are robust to stationary and nonstationary signal amplitude changes; iii) have better consistency over signal length $N$ and tolerance $r$; iv) are related to the self-similarity properties of time series. Finally, we demonstrated the capacity of $RangeEn$ measures to differentiate different states of EEG signals.

II. SIGNAL ENTROPY ANALYSIS

There are several variants of information theory based entropy in the literature. Our focus in this study will be on $ApEn$ and $SampEn$. We first review the evolution from Shannon entropy to $ApEn$ and $SampEn$ via Kolmogorov-Sinai and Eckmann-Ruelle entropies.

A. Shannon entropy

Let $x = \{x_1, x_2, ..., x_N\}$ be a finite discrete random process with probability density functions $p(x_i) = Pr\{x = x_i\}$ satisfying the following condition:

$$\begin{align*}
\sum_{i=1}^{N} p(x_i) &= 1, \\
\sum_{i=1}^{N} p(x_i) &\geq 0,
\end{align*}$$

(1)

for $N \to \infty$. Shannon entropy $H_{Sh}(x)$ of this random process is defined as:

$$H_{Sh} = -K \sum_{i=1}^{N} p(x_i) \log p(x_i),$$

(2)

where $\log$ denotes the logarithm operator and $K$ is a positive constant. Here, Eq. (2) quantifies the predictability of observations: highly predictable events (i.e., $p(x_i) \to 1$) lead to small entropy (i.e., $H_{Sh} \to 0$), whilst unpredictable events (i.e., $p(x_i) \to 1/N$) lead to large entropy (i.e., $H_{Sh} \to K \log(N)$) or infinity for $N \to \infty$.

B. Kolmogorov-Sinai entropy

Many dynamical systems may go through some intervals during which their Shannon entropy linearly increases with time [5][10]. This rate of change in Shannon entropy is called Kolmogorov-Sinai entropy [5] and can be quantified by $K$ in Eq. (2). It reveals the deterministic or stochastic nature of $x$ during a time interval, as it is equal to the summation of all its positive Lyapunov exponents [5][10].

C. Eckmann-Ruelle entropy

In practice and when dealing with a discrete dynamic process $x$ such as a biosignal, the probability terms $p$ in Eq. (2) can be estimated from the reconstructed phase space [11]:

$$\begin{align*}
X_i^m,\tau &= \{x_i, x_{i+\tau}, ..., x_{i+(m-1)\tau}\}, \\
i &= 1, ..., N - (m - 1)\tau,
\end{align*}$$

(3)

where $m$ denotes the embedding dimension and $\tau$ is the delay time. $X_i^m,\tau$ represents a state vector in an $m$-dimensional phase space $V_x$. The parameter $\tau$ is usually refereed to as scale.

Given a reconstructed state vector $X_i^m,\tau$, it is possible to partition $V_x$ into small non-overlapping and equisized regions $\varepsilon_k$, so that $\bigcup_k \varepsilon_k = V_x$ and $\bigcap_k \varepsilon_k = \emptyset$. Shannon entropy can then be computed by assigning a probability value $p_k$ to each region as the probability of visiting the phase trajectory. Since the values of $p_k$ depend on the partition size ($\varepsilon$) [10], Shannon entropy $H_{Sh}$ will be a function of $\varepsilon$ as well as the embedding space parameters, i.e., $(x, \varepsilon, m, \tau)$.

Moving towards the definitions of $ApEn$ and $SampEn$, let’s now consider a spacial case of $V_x$ where $\tau = 1$. In this case, the state vector $X_i^m,\tau$ is reduced to a vector sequence of $x_i$ through to $x_{i+m-1}$, i.e.,:

$$X_i^m = \{x_i, x_{i+1}, ..., x_{i+m-1}\}, \quad i = 1, ..., N - m + 1.$$  

(4)

Let each $X_i$ be used as a template to search for ‘neighbouring’ samples in the reconstructed phase space. Two templates $X_i$ and $X_j$ are matching if their relative distance is less than a predefined tolerance $r$. The distance function used in both $ApEn$ and $SampEn$ is the Chebyshev distance defined as: $d_{chebyshev}(X_i, X_j) := \max_k |x_i - y_j|, \quad k = 0, ..., m - 1$. It leads to an ‘$r$-neighbourhood’ conditional probability function $C_i^m(r)$ for any vector $X_i^m$ in the phase space $V_x$:

$$C_i^m(r) = \frac{1}{N - m + 1} B_i^m(r), \quad i = 1, ..., N - m + 1,$$

(5)

where

$$B_i^m(r) = \left[\text{No. of } X_j \text{ \ s } | \ d_{chebyshev}(X_i, X_j) \leq r \right], \quad j = 1, ..., N - m + 1.$$  

(6)

Let $\Phi^m(r)$ be the sum of natural logarithms $C_i^m(r)$, that is:

$$\Phi^m(r) = \sum_{i=1}^{N-m+1} \ln C_i^m(r).$$

(7)

The rate of change in $\Phi^m(r)$ along the embedding dimension $m$ is called the Eckmann-Ruelle entropy and is defined as [6]:

$$H_{ER} = \lim_{r \to 0} \lim_{m \to 0} \lim_{N \to \infty} \Phi^m(r) - \Phi^{m+1}(r).$$

(8)
D. Approximate entropy

An 'approximation' of $H_{ER}$, proposed by Pincus through fixing $r$ and $m$ in Eq. [8], is called approximate entropy (ApEn) \[12\]:

$$\text{ApEn} = \lim_{N \to \infty} \Phi^m(r) - \Phi^{m+1}(r). \tag{9}$$

Put in words, ApEn quantifies the mean negative log probability that an $m$-dimensional state vector would repeat itself at dimension $(m+1)$. In contrast to Eckmann-Ruelle entropy, ApEn is not intended to approximate Kolmogorov-Sinai entropy, but should be considered as a family of statistics for each dynamical system over a range of $m$ and $r$ \[12\]. Also, it is recommended to correct the tolerance as $r \times SD$ (SD being standard deviation of $x$) to account for amplitude variations across different signals. The parameter set of $m=2$ and $r=0.2$ has been widely used for extracting ApEn in the literature.

E. Sample entropy

As Eqs. [5] and [6] suggest, ApEn allows self-matching of the templates $X'_m$ in the definition of $C^m_i(r)$ to avoid the occurrence of $\log(0)$ in its formulation \[12\]. However, this will result in an unwanted bias which occurs in particular for short signal lengths (small $N$). Inconsistency of ApEn over the tolerance parameter $r$ has also been reported \[8\] \[12\]. In order to address these issues, sample entropy was developed by updating $B'_m^r(r)$ in Eq. [6] \[8\]:

$$\{B'_m^r(r) = \text{No. of } X'_m^s \mid d_{\text{chebyshev}}(X'_m^s, X'_j^m) \leq r, j = 1, ..., N - m, j \neq i, \}$$

and averaging over time as:

$$B'_m^r = \frac{1}{N-m} \sum_{i=1}^{N-m} B'_m^r(r). \tag{10}$$

Sample entropy is then defined as:

$$\text{SampEn} = \lim_{N \to \infty} -\ln \frac{B'_m^{r+1}}{B'_m^r}. \tag{12}$$

There are three major differences between SampEn and ApEn:

1) Conditional probabilities of SampEn, i.e., $B'_m^r(r)$ in Eq. (10), are obtained without self-matching of the templates $X'_m$.

2) Unlike ApEn which takes the logarithm of each individual probability value (see Eq. (7)), SampEn considers the logarithm of the sum of probabilities in the phase space.

3) ApEn is defined under all circumstances due to its self-matching, while SampEn can be sometimes undefined, as $B'_m^r$ and $B'^{r+1}_m$ in Eq. (12) are allowed to be zero.

Since $d_{\text{chebyshev}}(X'_m^m, X'_m^{m+1})$ is always smaller than or equal to $d_{\text{chebyshev}}(X'_m^m+1, X'_m^{m+1})$, $B'^{m+1}_m$ is less than $B'^m_m$ for all values of $m$. So, SampEn is always non-negative \[8\].

III. SIGNAL ENTROPY AND SELF SIMILARITY

A time series is self-similar or scale-invariant, if it repeats the same statistical characteristics across many temporal scales \[13\]. Intuitively, the more a signal is self-similar, the more its long-term memory increases and the less it generates new information with passing time. Given the definition of signal entropy as ‘the average rate of generation of new information’ \[4\], one would expect to see a correspondence between the entropy of a typical time series and its self-similarity behaviour. This can be investigated by looking into the signal entropy values of time series with certain degrees of self-similarity. Fractional Brownian motion (fBm) is well suited for this purpose. It is a centred Gaussian process whose self-similarity level can be controlled by its so-called Hurst exponent \[14\]. An fBm process with a Hurst exponent of $H$ is usually denoted as $B_H(t)$ and has the following covariance function:

$$E\{B_H(t)B_H(s)\} = \frac{1}{2}(t^{2H} + s^{2H} - | t - s |^{2H}), \tag{13}$$

It can be shown that $B_H(t)$ and its time-scaled version $B_H(at)$ have the same covariance function \[15\]. The fBm process represents self-similarity (or long-term memory) for $H > 0.5$ and anti self-similarity (or short-term memory) for $H < 0.5$. The spacial case of $B_{0.5}(t)$ is a Gaussian random process whose frequency spectrum follows the $1/f^2$ pattern and its derivative leads to white noise with a uniform spectrum over all frequencies. $B_{0.5}(t)$ was given several names including ordinary Brownian motion, random walk, Brownian noise, Brown noise and Red noise.

A commonly used approach for estimating the Hurst exponent of an $N$-long time series $x$ is through rescaled range analysis \[14\]. It applies a multi-step procedure on $x = \{x_1, x_2, ..., x_N\}$ as follows:

1) Divide $x$ into $n$ equally sized non-overlapping segments $x^s_n$ with the length of $N/n$ where $s = 1, 2, 3, ..., n$ and $n = 1, 2, 4, ...$. This process is repeated as long as $x^s_n$ has more than 4 data points.

2) For each segment $x^s_n$:  

(a) Center it as \( y^*_n = x^*_n - m^*_n \) where \( m^*_n \) is the mean of \( x^*_n \). \( y^*_n \) shows the deviation of \( x^*_n \) from its mean.

(b) Compute the cumulative sum of centred segment \( y^*_n \) as \( z^*_n = \sum_{i=1}^{N/n} y^*_n(k) \). \( z^*_n \) shows the total sum of \( y^*_n \) as it proceeds in time.

(c) Calculate the largest difference within the cumulative sum \( z^*_n \), namely:
\[
R^*_n = \max_k z^*_n(k) - \min_k z^*_n(k).
\]

(d) Calculate the standard deviation of \( x^*_n \) as \( S^*_n \) and obtain its rescaled range as \( R^*_n/S^*_n \).

3) Compute the average rescaled range at \( n \) as \( R(n)/S(n) = \frac{\sum_{n=1}^{N} R^*_n}{N} \).

The asymptotic behaviour of the rescaled range \( R(n)/S(n) \) over \( n \) defines the Hurst exponent \( H \):
\[
\lim_{n \to \infty} E[R(n)/S(n)] = Cn^H.
\]

The exponent \( H \) can be estimated as the slope of the logarithmic plot of the rescaled ranges versus \( \log(n) \). The main idea behind rescaled range analysis is to quantify the dynamical characteristic of a signal at different scales. The main idea behind rescaled range analysis is to quantify the self-similarity behaviour of a signal at different scales and summarize it in the form of its Hurst exponent. In the following, we apply a similar idea on the distance function of \( ApEn \) and \( SampEn \) and show that it makes them more sensitive to the self-similarity behaviour of time series.

IV. PROPOSED RANGE ENTROPY MEASURES

A. A modification to ApEn and SampEn

Both ApEn and SampEn aim to extract the conditional probabilities of \( B^m_i(r) \) by computing the Chebyshev distance between two templates (or state vectors) \( X^m_i \) and \( X^m_j \) in the reconstructed \( m \)-dimensional phase space as shown in Eqs. (6) and (10). The idea here is to estimate the (logarithmic) likelihood that runs of patterns that are close remain close on next incremental comparisons \[12\]. The closer the two states stay together in the reconstructed phase space over time, the less 'change' they will introduce into the signal dynamics. The idea of quantifying the Chebyshev distance between two state vectors originated from the seminal paper by Takens \[16\].

Although \( d_{\text{chebyshev}}(X^m_i, X^m_j) \) can provide useful information about the variation of state vectors, it has two limitations. First, it is not normalized as it has no upper limit. It leads to an unbounded range for the tolerance parameter \( r \) in the conditional probabilities \( B^m_i(r) \) (Eqs. (6), (10)). Second, it only considers the maximum difference between two state vectors, so it is blind to the range of their element-wise differences. To address these issues, we adapt the general idea behind the rescaled range \( R(n)/S(n) \) in Eq. [15] and propose an updated version of distance function for \( ApEn \) and \( SampEn \) as follows:
\[
d_{\text{range}}(X^m_i, X^m_j) = \frac{\max|x_{i+k} - x_{j+k}| - \min|x_{i+k} - x_{j+k}|}{\max|x_{i+k} - x_{j+k}| + \min|x_{i+k} - x_{j+k}|},
\]
\[
k = 0, ..., m - 1.
\]

In the special case of a two-dimensional reconstructed phase space (\( m=2 \)), \( d_{\text{range}}(X^m_i, X^m_j) \) is reduced to the simple form of \((a - b)/(a + b)\) where \( a = \max\{(x_i - x_j), (x_{i+1} - x_{j+1})\} \) and \( b = \min\{(x_i - x_j), (x_{i+1} - x_{j+1})\} \). In fact, \( d_{\text{range}} \) normalizes the stretching of state vectors across time and dimension. In contrast to \( d_{\text{chebyshev}}(X^m_i, X^m_j) \), the proposed \( d_{\text{range}}(X^m_i, X^m_j) \) is normalized between 0 and 1. It also recognizes the range of element-wise differences between \( X^m_i \) and \( X^m_j \) by combining the absolute value, \( min \) and \( max \) operators. \( d_{\text{range}}(X^m_i, X^m_j) \) is defined over all values, except for equal \( m \)-dimensional segments where the denominator in Eq. [16] becomes zero.

Strictly speaking, \( d_{\text{range}}(X^m_i, X^m_j) \) is not a 'distance' because it doesn’t satisfy all conditions of a distance function. For any two equi-length vectors \( v_1 \) and \( v_2 \), these requirements are defined as follows \[17\]:
\[
\begin{align*}
1) \ dist(v_1, v_2) & \geq 0 \quad \text{(non-negativity)}, \\
2) \ dist(v_1, v_2) & = dist(v_2, v_1) \quad \text{(symmetry)}, \\
3) \ dist(v_1, v_1) & = 0 \quad \text{(reflexivity)}.
\end{align*}
\]

\( d_{\text{range}}(X^m_i, X^m_j) \) violates the third condition, as it is undefined for equal templates. However, we have loosely called \( d_{\text{range}}(X^m_i, X^m_j) \) as a distance function throughout this paper. By replacing \( d_{\text{chebyshev}}(X^m_i, X^m_j) \) in Eqs. (6) and (10) with \( d_{\text{range}}(X^m_i, X^m_j) \), we update \( ApEn \) and \( SampEn \) as two new range entropy measures, i.e., \( RangeEn_A \) and \( RangeEn_B \), respectively.

B. Properties of range entropies

Property 1: \( RangeEn \) is more stable over different signal lengths. In contrast to \( ApEn \) and \( SampEn \), \( RangeEn \) measures have less variations over different realizations of short-length signals. This will be shown through simulations of random processes with multiple durations in the next sections.

Property 2: \( RangeEn \) is robust against signal amplitude changes. Unlike \( SampEn \) and \( ApEn \) which are highly sensitive to signal amplitude changes, \( RangeEn \) is less affected by variation in the magnitude of signals. This originates from the in-built normalization step in \( d_{\text{range}}(X^m_i, X^m_j) \) which is directly applied on the
amplitude of all templates.

Property 3: In terms of $r$, RangeEn is constrained in the interval $[0, 1]$. This is mathematically expressed as (see Eq. (6) and Eq. (10)):

$$B_i^m(r) = \sum_{j=1}^{N-m+1} \Psi(r - d_{\text{range}}(X_i^m, X_j^m))$$

(18)

where $\Psi(.)$ is the Heaviside function defined as:

$$\Psi(a) = \begin{cases} 0 & a < 0 \\ 1 & a \geq 0. \end{cases}$$

Since $d_{\text{range}}(X_i^m, X_j^m)$ is normalized, we conclude from Eq. (18) that:

$$\begin{align*}
\text{RangeEn}_A & : B_i^m(r) = N - m + 1 \; \forall \; r \geq 1 \\
\text{RangeEn}_B & : B_i^m(r) = N - m \; \forall \; r \geq 1.
\end{align*}$$

(20)

This ensures that both conditional probability functions $C_i^m(r)$ in Eq. (3) and $B_i^m$ in Eq. (11) will always be equal to 1 for $r \geq 1$. This fact leads to the following property for RangeEn$_A$ and RangeEn$_B$:

$$\begin{align*}
\text{RangeEn}_A(x, m, r) & = 0 \; \forall \; r \geq 1 \\
\text{RangeEn}_B(x, m, r) & = 0 \; \forall \; r \geq 1.
\end{align*}$$

(21)

Property 4: RangeEn is linked to the Hurst exponent. The distance function $d_{\text{range}}(X_i^m, X_j^m)$ overlaps with rescaled range analysis [15] through its numerator which is equal to the largest difference between the two templates (resembling $R_n$ in Eq. (14)) and its denominator which is somewhat similar to the standard deviation $S_n$. This characteristic of $d_{\text{range}}(X_i^m, X_j^m)$ makes RangeEn sensitive to the self-similar structures of the underlying time series. We will show this link in Section VI through simulations of the fBm process over a range of values of the Hurst exponent.

Next, we describe several signal models with different dynamical behaviours to be used to highlight the desirable properties of RangeEn in contrast to ApEn and SampEn.

V. DATA

A. Simulated data

In this section, we describe three groups of dynamic processes that were used to characterize and compare the different signal entropy measures.

1. Stochastic processes

We simulated 100 realizations of Gaussian white noise ($N(0,1)$), pink ($1/f$) noise as well as Brown noise ($1/f^2$), as three well-known stochastic processes and extracted their different entropy estimates across a range of signal lengths and tolerance parameters $r$.

2. Self-similar processes

To investigate the behaviour of signal entropy measures in presence of self-similarity, we generated a range of fixed-length fBm processes ($N=1000$) covering the entire span of Hurst exponent (i.e., 99 incremental $\Delta H$ steps from 0.01 to 0.99).

3. Chaotic processes

We next simulated signal from the three well-studied nonlinear deterministic systems, namely the logistic map, the Henon map and the Roessler oscillator [12].

a) The logistic map is a 1-dimensional discrete-time dynamical system defined as:

$$x_{i+1} = Rx_i(1 - x_i).$$

(22)

We simulated the signal for $N=1000$ time steps and $R=3.8$, as this parameter leads to a chaotic behaviour.

b) The Henon map is a 2-dimensional discrete-time model described as:

$$\begin{align*}
x_{i+1} & = 1 - ax_i^2 + y_i \\
y_{i+1} & = bx_i.
\end{align*}$$

(23)

We simulated $x_i$ with $N=1000$ time points and the parameter set of $a=1.4$ and $b=0.3$ corresponding to a chaotic dynamics.

c) The Roessler oscillator is a 3-dimensional continuous-time system defined as:

$$\begin{align*}
dx/dt & = -z - y \\
dy/dt & = x + ay \\
dz/dt & = b + z(x - c).
\end{align*}$$

(24)

We used the parameter set of $a=0.15$, $b=0.20$ and $c=5$ for a chaotic dynamics of the model and generated $y(t)$ with the time step of $\Delta t=50$ msec and duration of 50 sec (i.e., $N=1000$ time points).

B. EEG datasets

We used five datasets of single-channel EEG recordings with the length of 23.6 sec and sampling frequency
of 173.61 Hz ($N=4097$) \cite{18}. Each dataset consisted of 100 EEG segments which were randomized with regards to their recording contact and subjects. The first two datasets (denoted as $A$ and $B$) included scalp-level EEG (sEEG) recordings from five healthy subjects according to the 10-20 international standard system with eyes open (dataset $A$) and eyes closed (dataset $B$). Datasets $C$, $D$ and $E$ are intracranial EEG (iEEG) of five epilepsy patients who had epilepsy surgery in the hippocampal area. All epilepsy patients became seizure-free after the surgery. Dataset $C$ was recorded from the hippocampal formation on the opposite side of seizure focus (contralateral), while dataset $D$ was recorded from the hippocampal area on the seizure side (ipsilateral). Both datasets $C$ and $D$ were obtained during interictal (seizure-free) intervals. In contrast, dataset $E$ covered ictal (seizure) intervals.

All datasets were obtained using a 128-channel EEG recorder with common average referencing. Also, eye movement artifacts and strong pathological activities were identified and removed from the signals through visual inspection. A band-pass filter of 0.53-40 Hz was applied to the data. See \cite{18} for more details regarding the datasets.

VI. RESULTS

A. Simulated data

1. Sensitivity to signal length

We simulated three noise types (white, pink and brown) at different lengths varying from 50 to 1000 samples increasing with 10-point increasing steps. 100 realizations of each noise type were generated. Four entropy measures ($ApEn$, $SampEn$, $RangeEn_A$ and $RangeEn_B$) were then computed from the simulated noise signals. For all entropy measures, we fixed the dimension $m$ to 2 and the tolerance $r$ to 0.2. FIG. 1 illustrates the errorbar plot of each entropy measure for the three noise types over different signal lengths.

As the figure suggests, the variations of $RangeEn_A$ and $RangeEn_B$ are smaller than both $ApEn$ and $SampEn$ over different lengths. Among the four measures, $SampEn$ had largest standard deviations (poor repeatability) at each signal length, especially for shorter signals. A common observation in all measures is that their standard deviation increases and their mean decreases by increasing the exponential decay in the frequency domain, given higher spectral exponent of brown noise compared to pink noise and pink noise compared to white noise. $ApEn$ is the most sensitive measure to signal length, as its mean tends to change (almost linearly) with the data length confirming previous findings \cite{8}. $RangeEn$ measures present both a more stable mean (in contrast to $ApEn$) with small variance (in contrast to $SampEn$).

![Signal entropy values of colour noise at different signal lengths](image)

**FIG. 1:** Variation of the entropy measures over different signal lengths ($N$ in time samples). Each noise type has been simulated for 100 times and errorbars represent the variation over noise realizations. $RangeEn_A$ (in black) and $RangeEn_B$ (in blue) show less deviation around their mean values compared $ApEn$ (in green) and $SampEn$ (in red), in particular over short signal lengths. In all panels, the x-axis is in logarithmic scale.

2. The role of tolerance $r$

To investigate the effect of tolerance $r$ on entropy measures, we again simulated three noise types, i.e., white, pink and brown noise at the fixed length of $N=1000$ time samples. We computed the measures at $m=2$, but over a range of tolerance values $r$ from 0.01 to 1 in increment of 0.01. FIG. 2 illustrates the entropy patterns in the $r$-plane for each noise type. Five observations can be drawn from this analysis. First, both $RangeEn_A$ and $RangeEn_B$ reach zero at $r=1$. This is not the case for $ApEn$ and $SampEn$. Second, $SampEn$ shows the high-
est standard deviation, in particular at low \( r \) values \( (r \leq 0.3) \). Third, \( \text{RangeEn}_A \) has the highest number of undefined values across the four measures (note the missing values of \( \text{RangeEn}_A \) as vacant points in the figures, especially in the white noise and pink noise results). Finally, the level of exponential decay in the frequency domain appears to be coded in the slope and starting point of the \( \text{RangeEn} \) trajectories in the \( r \)-plane. FIG. 2 suggests that brown noise \( (1/f^2 \), with the largest exponent among the three noise types) has the lowest entropy pattern, while white noise with no decay in the frequency domain has the steepest entropy trajectory with largest starting value of \( \geq 4 \) at \( r=0 \).

3. Dependency on signal amplitude

To evaluate the effect of signal amplitude on entropy, we simulated a white noise signal \( x_1(n) \) with \( N=1000 \) time points and its copy multiplied by 5, i.e., \( x_2(n) = 5x_1(n) \) (see first and second rows in the top panel of FIG. 3 respectively). We then computed \( \text{ApEn} \), \( \text{SampEn} \), \( \text{RangeEn}_A \) and \( \text{RangeEn}_B \) for \( m=2 \) and a range of tolerance values \( r \) from 0.01 to 1 with \( \Delta r=0.01 \). As the figure shows, \( \text{RangeEn}_A \) and \( \text{RangeEn}_B \) obtained from \( x_1(n) \) and \( x_2(n) \) are near identical, while \( \text{ApEn} \) and \( \text{SampEn} \) diverges. In most of the existing \( \text{ApEn} \) and \( \text{SampEn} \) studies in the literature, the input signal is divided by its standard deviation to reduce the dependency of the entropy on the signal gain factor. This solution is useful only for stationary changes of signal amplitude where the entire standard deviation of the whole signal is an accurate description of its variability. We also computed the entropies for the nonstationary signal \( x_3(n) \) whose variance is time-varying:

\[
x_3(n) = \begin{cases}  
x_1(n) & n = 1, \ldots, 200, 
3x_1(n) & n = 201, \ldots, 400, 
10x_1(n) & n = 401, \ldots, 600, 
4x_1(n) & n = 601, \ldots, 800, 
x_1(n) & n = 801, \ldots, 1000.
\end{cases} 
\] (25)

The signal \( x_3(n) \) (illustrated in third row in the top panel of FIG. 3) resembles a nonstationary random process which has been generated through a stationary process modelled by \( x_1(n) \), but has been affected by a time-varying amplitude change. In order to correct for the amplitude (gain) variation prior to computing the entropies \( \text{ApEn} \) and \( \text{SampEn} \), we replaced \( x_3(n) \) by \( x_3(n)/\sigma_{x_3} \) for these two entropy measures where \( \sigma_{x_3} \) is the standard deviation of \( x_3(n) \). As entropy patterns of FIG. 3 suggest, even after applying this amplitude correction, \( \text{ApEn} \) and \( \text{SampEn} \) are still sensitive to amplitude changes. This is, however, not the case for \( \text{RangeEn} \) measures that are much less affected by this nonstationary change.

4. Sensitivity to self-similarity

Next, we checked the behaviour of entropy measures in presence of self-similarity in random processes. To this end, we simulated a group of \( fBm \) processes with pre-defined Hurst exponents ranging from 0.01 (minimal self-similarity) to 0.99 (maximal self-similarity) with the increasing step of \( \Delta H=0.01 \). For each time series, we computed the different entropies for \( m=2 \) and over a range of tolerance values \( r \) from 0.01 to 1 with 0.01 increasing
steps. In this way, we investigated the relationship between a systematic increase in self-similarity (modelled by the Hurst exponent) and the tolerance parameter $r$ in the measures. The results are summarized in FIG. 4. Amongst the four measures, SampeEn shows no specific pattern related to changes in the Hurst exponent of the underlying signals. In contrast, ApEn, RangeEn$_A$ and RangeEn$_B$ show decreasing tendencies with increasing Hurst exponent (increasing self-similarity). RangeEn$_B$ offers the most straightforward and readable pattern compared to ApEn and RangeEn$_A$, as 1) it provides a monotonically changing spectrum of Hurst exponents, 2) it is defined over all $r$ values of [0,1] and 3) it is always finished at $r$=0. The patterns of RangeEn$_A$ and RangeEn$_B$ are comparable, except that RangeEn$_A$ has a considerable amount of missing (undefined) values, specially over low $r$ values. FIG. 4 also suggests that higher levels of self-similarity in a random process lead to lower initial entropy values and more flatness for the RangeEn trajectories in the $r$-plane.

FIG. 4: Sensitivity of the entropy measures to in-built self-similarity of time series. Panels A to D illustrate the entropy trajectories in the $r$-plane extracted from fractional Brownian motion (fBm) with pre-defined Hurst exponents ranging from 0.01 to 0.99 with the increasing steps of $\Delta H=0.01$. The $H$ values have been colour coded. The missing points in each plot have been left as blank. The RangeEn as well as ApEn trajectories (panels A, C and D) show a systematic relationship with $H$ exponents, while no specific pattern is observed for SampeEn (panel B). Again, RangeEn trajectories end up with 0 at $r$=1, while ApEn and SampeEn have no specific lower limit. In all panels, the x-axis is in logarithmic scale.

5. Signal entropy and deterministic behaviour

FIG. 5 illustrates the trajectories of signal entropy measures in the $r$-plane for the three deterministic dynamical systems described in section V A 3. In contrast to the RangeEn patterns of stochastic processes (FIGs 2 to 4), FIG. 5 suggests that low-dimensional deterministic nonlinear models such as the logistic map (FIG. 5-A) follow a very different trajectory in the $r$-plane. As the phase space dimensionality increases (e.g., from the chaotic map to the Henon map through to Roessler oscillator), the RangeEn trajectories become more similar to those of complex random processes. Note the similarity between the RangeEn pattern of pink noise with the Henon map (FIG. 2-B and FIG. 5-B, respectively) or the similarity between the patterns of brown noise and Roessler oscillator (FIG. 2-C and 5-C, respectively).

There is a significant difference between the ApEn and SampeEn trajectories for lower dimensions (FIG. 5-A,B) and higher dimensional systems (FIG. 5-B,C). However, the main difference between the two trajectories at the lower dimensions is associated with their lower limit in the $r$-plane.

B. RangeEn and Epileptic EEG

In this section, we demonstrate the capacity of RangeEn$_A$ and RangeEn$_B$ for characterizing EEG data.
during ictal and interictal intervals. FIG. 6 summarizes the $\text{Range}En_A$ and $\text{Range}En_B$ patterns in the $r$-plane for the five EEG datasets described in section VII B. Our assessment of the two entropy measures was based on visual comparison of the $r$-trajectories. The most revealing points can be listed as follows: 1) $\text{Range}En$ measures show very similar patterns for datasets $A$ and $B$ (sEEG of healthy subjects) as well as datasets $C$ and $D$ (interictal iEEG of epilepsy patients), while the $r$-trajectory of ictal iEEG shows an independent trajectory. 2) In both entropy measures, the ictal iEEG pattern constitutes transient state between the interictal state and normal state. 3) The $\text{Range}En_A$ values are undefined for small $r$-values, while $\text{Range}En_B$ is usually defined over all values of $r$.

FIG. 6: $\text{Range}En$ trajectories of five EEG datasets in the $r$-plane.

In the legend, sEEG stands for scalp-level EEG or extracranial recording and iEEG stands for intracranial EEG. Trajectories of the pathological states of the intracranial regions (datasets $C$, $D$ and $E$) have quite distinct morphology in contrast to normal and extracranial EEG states (datasets $A$ and $B$). The distinction between interictal and ictal states of intracranial EEG is pronounced more by $\text{Range}En_A$ compared to $\text{Range}En_B$ (note the bigger gap between the trajectory of dataset $E$ with the trajectories of datasets $C$ and $D$ in $\text{Range}En_A$ in contrast to $\text{Range}En_B$). Also, $\text{Range}En$ trajectories suggest that ictal state seems to be a transient bridge between interictal state and normal state.

VII. DISCUSSION

In this study, we introduced two new signal entropy measures as modified versions of $\text{ApEn}$ and $\text{SampEn}$ called $\text{Range}En_A$ and $\text{Range}En_B$, respectively. We showed that, compared to $\text{ApEn}$ and $\text{SampEn}$, $\text{Range}En$ measures have higher consistency over short signal lengths and are more robust to stationary and nonstationary signal amplitude changes. Also, they showed more sensitivity to self-similarity of time series. Application-wise, $\text{Range}En$ measures presented a good...
capacity for dealing with epileptic EEG signals as a real-world example of complex time series.

In brief, RangeEn utilizes reconstructed phase space representation of the signals, as it allows to relate the template-based definition of ApEn (and similarly, SampEn) with the concept of state vectors in the phase space [7][12][19]. From this perspective, ApEn and SampEn of a random process assess the dynamics in the phase space by quantifying the evolution of its state vectors over time. This is done through computing the Chebyshev distance $d_{chebyshev}$, as a measure of similarity between state vectors, and obtaining the conditional probability of space occupancy by the trajectory, as detailed in sections [1C][1D] and [1E]. However, $d_{chebyshev}$ only considers the largest component difference and, hence, is insensitive to the difference of the other components. In addition, it is not normalized, making it sensitive to changes in signal magnitude (gain) and defined for all values of $r$ (from 0 to $\infty$). This last issue leads to unbounded values of ApEn and SampEn as a function of the tolerance parameter $r$. In order to address these limitations, we replaced $d_{chebyshev}$ with a normalized distance (called range distance or $d_{range}$) defined in Eq. (16) prior to computing the entropies ApEn and SampEn. This led to modified forms of ApEn and SampEn, namely RangeEn$_A$ and RangeEn$_B$, respectively.

In this work, we showed that RangeEn$_A$ and RangeEn$_B$ offer a set of desirable characteristics when applied to simulated and biomedical signals. First, they are robust to signal amplitude changes. This property originates from the fact that the distance used in the definition of the proposed entropies is normalized between 0 and 1. This is unlike ApEn and SampEn measures that require an extra amplitude regulation step that involves multiplying the tolerance parameter $r$ by the input signal’s standard deviation [5], the RangeEn measures are needless of any amplitude correction. This is an important feature when analysing biosignals, which are usually affected by confounding amplitude changes such as artifacts. FIG. 3 illustrates two situations where ApEn and SampEn are highly sensitive to variations of signal amplitude, contrary to RangeEn measures. A practical example of such circumstances is when nonstationary movement artifacts contaminate biosignals such as scalp EEG. The second desirable property of the RangeEn measures is related to their robustness to signal length $N$ as demonstrated in FIG. 1. The consistency of RangeEn$_A$ and RangeEn$_B$ for small values of $N$ makes them useful for studying the dynamics of short duration events in biosignals, such as transient changes in heart rate variability or interictal epileptic discharges.

The third desirable property of RangeEn is that regardless of the dynamic nature of the signal, both RangeEn$_A$ and RangeEn$_B$ measures always reach 0 at the tolerance value $r$ of 1. The explanation of this property is straightforward: $r=1$ is the value where all $m$-long segments $X^m_1$ and $X^m_2$ match. This leads to the joint conditional probability being 1 (see Eqs. (18) to (21)). Therefore, the RangeEn signatures of dynamical systems over the interval of $0 \leq r \leq 1$ can be directly compared with each other. For instance, white noise has a unique linearly decreasing RangeEn pattern over $\log(r)$ (FIG. 2-A) which makes it distinct from deterministic systems (FIG. 5). Even within stochastic processes (FIGs. 2 and 1), trajectories of RangeEn over $\log(r)$ are distinctive. It suggests that the representation of RangeEn measures in the $r$-plane may be modelled in a general form denoted as:

$$RangeEn(x, m, r) = f_m(r)$$ (26)

where $f_m(r)$ is a characteristic function of the signal $x$ that relates RangeEn to $r$. The dimension $m$ of the phase space also seems to influence the shape of $f_m$, as reflected in different trajectories of three deterministic processes in FIG. 5, i.e., Roessler oscillator (3-dimensional), the Henon map (2-dimensional) and the logistic map (1-dimensional). We hypothesize that higher dimensional order systems produce $f_m(r)$ plots that are closer to a straight line. White noise may be considered as an extreme case with the phase space dimensionality approaching infinity [20] and an almost perfect linear $f_m(r)$ in the $r$-plane (see FIG. 2). This is justified by the link between ApEn and the correlation dimension in the phase space through the probability functions $C^n_m(r)$ defined in Eq. (5) [7][12][19]. The RangeEn measures inherit this feature of ApEn. In fact, $d_{range}$ may be interpreted as the degree of stretching of state vectors across time and embedding dimension. From this point of view, RangeEn of a nonlinear process has a direct relationship with its Lyapunov exponents and correlation dimension [7][12][19]. These observations and hypotheses open up new directions for future research.

This brings us to the forth property of the RangeEn measures, namely their ability to detect time series self-similarity in the $r$-plane. In fact, $d_{range}$ is designed so that it can detect localized stretching and contraction of time series over time which is later transferred into the reconstructed phase space through state vectors. We evaluated this property by extracting RangeEn measures from $fBm$ processes, as their level of self-similarity can be accurately controlled through their Hurst exponent [14]. We simulated $fBm$ signals for different values of the Hurst exponents ranging from 0.01 (very short memory or high anti self-similarity) to 0.99 (very long memory or high self-similarity). The simulation results (FIG. 4) reinforce this capacity of RangeEn measures by showing a regular pattern of monotonically decreasing Hurst exponents associated with the level of self-similarity in $fBm$ signals, reflected in the morphology of their RangeEn trajectories in the $r$-plane. In this case, $f_m(r)$ in Eq. (26) may be modelled as a fractional polynomial function:

$$f_m(r) = E_0(r - 1)^a$$ (27)

where the power $a \in \mathbb{R}$ is associated with the Hurst exponent of the signal and the constant coefficient $E_0$ denotes the initial value of RangeEn at $r \rightarrow 0$. The idea
of parametrizing entropy curves may be also applied to the ApEn patterns, as they show high sensitivity to self-similarity (see FIG. 4-A). In this case, one may model the ApEn trajectories in the r-plane using a mixture of Gaussian or gamma functions. SampEn, however, does not seem to reflect any specific pattern of self-similarity in the signals (FIG. 4-B). Amongst the four signal entropy measures investigated in our study, ApEn is the only measure which is always defined due to the self-matching of state vectors (or templates) in its definition [7]. SampEn and RangeEn may result in undefined values, as they compute the log of sum of conditional probabilities \( C^m(r) \) which could lead to \( \log(0) \) (see sections II E and IV for more details). This issue may also happen to \( \text{RangeEn}_A \), as it calculates the sum of log probabilities (i.e., \( \ln C^m(r) \)). However, the number of undefined values in \( \text{RangeEn}_A \) is usually much higher than SampEn and \( \text{RangeEn}_B \). This is because it is more likely that all joint conditional probabilities \( (C^m_i(r)) \) between a single state vector and the rest of state vectors in phase space become zero, in particular, at small tolerance values of \( r \) where the small partitions of phase space do not get visited by any trajectory. FIG. 4-C provides an exemplary situation where there are a lot of undefined \( \text{RangeEn}_A \) values for fBM.

A realization of brain complexity is reflected in EEG which conveys information about electrical activity of neuronal populations within cortical or sub-cortical structures. They can be recorded at the scalp level using surface electrodes (aka extracranial/scalp EEG or sEEG) or directly from inside the skull (referred to as intracranial EEG or iEEG). Either way, EEG signals have been shown to be highly informative about the brain dynamics in both health and disease [21]. Epilepsy research is a field that significantly benefited from EEG analysis, as the disease is associated with abnormal patterns in EEG such as seizures and interictal epileptiform discharges [22]. Therefore, characterization of abnormal events in epileptic EEG recordings is of great significance in diagnosis, prognosis and management of epilepsy [10, 23, 24]. Analysis of most existing entropy measures on epileptic EEG has mostly revealed lower entropy values for ictal and interictal EEG compared to non-epileptic EEG segments [23, 24]. This is in line with the decomplexification theory of illness where a functionally impaired biological system loses its dynamic complexity [25]. This pattern is also observed in the \( \text{RangeEn} \) trajectories in the \( r \) domain of FIG. 6 where epilepsy-related curves are dragged towards lower entropy values. Given the \( \text{RangeEn} \) patterns of fBM processes with predefined Hurst exponents (FIG. 4) as a ‘guideline’, FIG. 5 suggests that intracranial EEG is more self-similar than extracranial EEG and interictal EEG is more self-similar than ictal EEG. The difference between the \( \text{RangeEn} \) results of sEEG and iEEG may be due to the smoothing effect of brain volume conduction which destroys fine dynamical structures of EEG signals [26, 27] and may suppress the self-similarity characteristics of cortical electrical activity at the scalp level. The difference between normal EEG and epileptic EEG can also be speculated in the context of the decomplexification theory of illness. Decomplexified processes of epileptic EEG may follow quite different paths in the \( r \)-plane compared to normal processes. Due to the non-linear, nonstationary and complex nature of EEG signals, characterization of epileptic patterns could be very challenging. The EEG analysis results of this study suggest that \( \text{RangeEn} \) can be a good candidate to deal with these challenges for various reasons:

1. As shown in FIG. 3, the \( \text{RangeEn} \) measures are robust to stationary and nonstationary signal amplitude changes. Therefore, they are less influenced by artifacts.

2. \( \text{RangeEn} \) behaviour of epileptic EEG at different states (e.g., ictal versus interictal) can be parametrized in the \( r \)-plane, similar to what we discussed for the modelling of fBM in this section. It provides a systematic framework for not only comparing different EEG states in a single epilepsy patient, but also quantifying the dynamical signatures of different epilepsy types at the group level.

3. Relationship between \( \text{RangeEn} \) and the Hurst exponent (see FIG. 5) can be utilized to study the self-similar properties of epileptic EEG signals. To the best of our knowledge, self-similarity of EEG recordings in presence of epilepsy hasn’t been deeply investigated.

4. FIG. 6 clearly shows distinctive patterns of \( \text{RangeEn} \) in the \( r \)-plane for EEG at different epileptic states (interictal versus ictal) and different recording scenarios (extracranial versus intracranial) with relatively low variance over EEG segments. This implies that \( \text{RangeEn} \) behaviour (in particular, \( \text{RangeEn}_A \)) may be used as a discriminating feature of EEG for epileptic seizure detection and classification.

Multiscale entropy is a generalization of SampEn where the delay time (aka scale factor) \( \tau \) in Eq. (3) is expanded to an interval of successive integers starting from 1 through ‘coarse-graining’ of the input signal [9]. It is straightforward to extend this idea to the \( \text{RangeEn} \) measures. Exploring the properties and capacities of multiscale \( \text{RangeEn} \) is left for future research. In this study, we limited our analyses to the embedding dimension of \( m=2 \) for all entropy measures. The effect of this important parameter needs to be investigated in depth. Also, the systematic relationship between the \( \text{RangeEn} \) measures and the Hurst exponent as well as fractal dimensions in the phase space deserves more attention.

ACKNOWLEDGMENT

This work was supported by the National Health and Medical Research Council (NHMRC) of Australia (program grant 628952). G.J. was supported by an NHMRC.
practitioner fellowship (1060312). The Florey Institute of Neuroscience and Mental Health acknowledges the strong support from the Victorian Government and in particular the funding from the Operational Infrastructure Support Grant.

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