Essential Self-adjointness of Differential Operators on Riemannian Manifolds

Hany Atia (✉ h_a_atia@hotmail.com)
Zagazig University Faculty of Science

Hassan Abu Donia
Zagazig University Faculty of Science

Hala Emam
Higher Colleges of Technology

Research Article

Keywords: Essential self-adjointness, differential operators, Riemannian manifolds.

DOI: https://doi.org/10.21203/rs.3.rs-277783/v1

License: ☒ This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License
Essential Self-adjointness of Differential Operators on Riemannian Manifolds

H. A. Atia$^{1*}$, H. M. Abu Donia$^1$, and Hala. H. Emam$^2$.

$^1$Department Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt.
$^2$Department of Basic Science, Higher Institute for Engineering & Technology, Al-Obour, Egypt.
Email: $^1*$h_a_atia@hotmail.com, Tel: $^1*$00201147723122.

Abstract
In this paper we have studied the essential self-adjointness for the differential operator of the form:

\[ T = \Delta^8 + V, \]

on sections of a Hermitian vector bundle over a complete Riemannian manifold, with the potential $V$ satisfying a bound from below by a non-positive function depending on the distance from a point. We give sufficient condition for the essential self-adjointness of such differential operator on Riemannian Manifolds.

Keywords: Essential self-adjointness, differential operators, Riemannian manifolds.

AMS Subject Classification: 47F05, 58J99.

1 Introduction

The study of self-adjointness of differential operators on Euclidean spaces has many works, such as [8] and [13]. M. Gaffney initiated this problem on Riemannian manifolds in [8]. This work focused on the essential self-adjointness of the scalar Laplacian and the Hodge Laplacian. H. Cordes proved the case of positive integer powers of the scalar Laplacian and Hodge Laplacian in [5]. Generalisations to the case of essential self-adjointness of positive integer powers of the first order differential operators was proved by P. Chernoff in [4]. After these works many studied of the essential self-adjointness problem on Riemannian manifolds were done such as [6], [8] and [13]. Perturbations were considered of a biharmonic operator, $\Delta^2 + V$, over a complete Riemannian manifold $M$ by Milatovic in [11], where $V \in L^\infty_\text{loc}(\text{End } E)$ is the potential, $\text{End } E$ denotes the endomorphism bundle associated to $E$ and $V$ satisfies a bound from below by a
non-positive function depending on the distance from a point. Milatovic was obtained suitable localised derivative estimates, which was important for the proof of the essential self-adjointness of operators on $C_c^\infty (E)$. For a study of separation in the context of a perturbation of the magnetic Bi-Laplacian on $L^2 (M)$, see the paper [1]. Atia studied the separation problem on Riemannian manifolds in [2] and [3]. In this paper, we consider the operator, $\Delta^8 + V$, acting on sections of a Hermitian vector bundle $E$ over a complete Riemannian manifold $M$, where $\Delta = \nabla^+ \nabla$ denotes a Bochner Laplacian associated to a Hermitian connection $\nabla$, $V$ is a potential that satisfies $V \in L_{loc}^1 (\text{End} \ E)$, and satisfies a bound from below by a non-positive function depending on the distance from a point. Let $(M, g)$ be a smooth connected Riemannian n-manifold without boundary, where $g$ is the Riemannian metric on $M$. We define the following components: $\Delta_M$ denotes the Laplace Beltrami operator on functions on $M$, the associated Riemannian volume form by $d\mu$, also $\nabla_M$ denotes the canonical Levi-Civita connection on $M$, and the associated curvature tensor by $R_m$. The pair $(E, h)$ is the smooth Hermitian vector bundle over $M$, where $h$ is Hermitian metric. $\nabla$ denotes the metric connection on $E$, this connection gives a curvature tensor $F$. The formal adjoint of $\nabla$ will be denoted by $\nabla^\ast$, with the associated Bochner Laplacian being given by $\Delta := \nabla^\ast \nabla$. We define $C^\infty (M)$ and $C_c^\infty (M)$ are the smooth functions and smooth functions with compact support on $M$ respectively. Similarly, We define $C^\infty (E)$ and $C_c^\infty (E)$ are the smooth sections and smooth sections with compact support of $E$ respectively. We will use the notation $L^2 (E)$ to denote the Hilbert space of square integrable sections of $E$, with the inner product

$$\langle u, v \rangle := \int_M h(u, v) \, d\mu.$$ 

We will denote the associated $L^2$-norm by

$$\|u\| := \left( \int_M |u|^2 \, d\mu \right)^{1/2},$$

where $|u|^2 = h(u, u)$. We denote local Sobolev spaces of sections in $L^2 (E)$, by $W^{k,2}_{loc} (E)$, with $k$ indicating the highest order of derivatives. Let the distance from a point, which we denote by $r$, there exist a fixing point $x \in M$ we let

$$r(x) := d(x_0, x)$$

(1)

where $d$ is the distance function induced from the Riemannian metric $g$ on $M$, for all $x \in M$.

## 2 Bounded Geometry

**Definition 1** Let $(M, g)$ be a smooth non-compact Riemannian manifold. We say $(M, g)$ admits bounded geometry if the following conditions are satisfied.

1. $r_m > 0$, 

(2) $\sup_{x \in M} |\nabla^k R_m(x)| \leq C_k$ for $k \geq 0$, and $C_k > 0$, where $r_m$ denotes the injectivity radius of $M$, $\nabla$ is the Levi-Civita connection, and $R_m$ denotes the curvature tensor.

**Definition 2** Let $M$ be a smooth manifold and $(E, h, \nabla)$ a Hermitian vector bundle over $M$, with Hermitian metric $h$ and connection $\nabla$, we say the triple $(E, h, \nabla)$ admits $k$-bounded geometry if the following condition is satisfied:

$$\sup_{x \in M} |\nabla^j F(x)| \leq C_j \text{ for } 0 \leq j \leq k, \text{ and } C_j > 0,$$

where $F$ is the curvature tensor associated to $\nabla$. We say $(E, h, \nabla)$ admits bounded geometry if it admits $k$-bounded geometry for all $k \geq 0$.

**Proposition 3** Let $(M, g)$ be a Riemannian manifold of bounded geometry. Then there exists a $\delta > 0$ such that the metric and the Christoffel symbols are bounded in normal coordinates of radius $\delta$ around each $x \in M$ and the bounds are uniform in $x$. For the proof, the reader may consult theorem 2.4 and corollary 2.5 in [7].

In this paper we let two conditions on the geometry of our Riemannian manifolds and the vector bundles over them.

All Riemannian manifolds $(M, g)$ are admit bounded geometry.

All Hermitian vector bundles $(E, h, \nabla)$ are admit 1- bounded geometry.

### 3 Distance functions

In this paper we will be using distance functions.

**Lemma 4** see lemma 2.1 in [15], let $M$ be a smooth Riemannian manifold of bounded geometry. Then there exists a smooth function $d : M \times M \to [0, \infty)$ satisfying the following conditions:

1. There exists $\rho > 0$ such that $|d(x, y) - d(x, y)| < \rho$ for every $x, y \in M$.
2. For every multi-index $\alpha$ with $|\alpha| > 0$ there exists a constant $C_\alpha > 0$ such that $|\partial^\alpha \tilde{d}(x, y)| \leq C_\alpha$, $x, y \in M$, where the derivative $\partial^\alpha \tilde{d}$ is taken with respect to normal coordinates.

Moreover for every $\epsilon > 0$, there exists a smooth function $\tilde{d}_\epsilon : M \times M \to [0, \infty)$ satisfying (1) with $\rho < \epsilon$.

By using the previous lemma. For all fixed point $x_0 \in M$, and we assume $\tilde{d}_{1/\epsilon}(y) := \tilde{d}_{1/\epsilon}(x_0, y)$, where $\tilde{d}_{1/\epsilon}$ denotes the smooth distance function given by the previous lemma since $\tilde{d}_{1/\epsilon}$ is smooth on $M$. Let $F : R \to R$ be a smooth function since

$$F(x) = \begin{cases} 1 & \text{for all } x \leq 1 \\ 0 & \text{for all } x \geq 8 \end{cases}$$

monotonically decreasing on $[1, 8]$. 

3
We define $x_\epsilon (y) = F \left( \epsilon d_{1/\epsilon} (y) \right)$ and $x_\epsilon = 1$ on $B_{1/\epsilon} (x_0)$ and that $\text{Supp} (x_\epsilon) \subseteq B_{3/\epsilon} (x_0)$. From the second condition of the previous lemma we have

$$|\partial^\alpha_y x_\epsilon (y)| \leq C_\alpha \epsilon,$$

(2)

where $\alpha$ is a multi-index, $C_\alpha > 0$ is a constant, and the derivative $\partial^\alpha_y$ is taken with respect to normal coordinates. In particular, this implies to

$$|\Delta^k_M x_\epsilon| \leq C_k \epsilon, \text{ for } k \geq 1$$

(3)

$$|dx_\epsilon| \leq C_0 \epsilon$$

(4)

Let $(M, g)$ be a Riemannian manifold and $(E, h, \nabla)$ be a Hermitian vector bundle over $M$, with Hermitian metric $h$ and metric connection $\nabla$. Let $u \in W^{k,1}_0 (E)$ and $f \in C^\infty_c (M)$. We have

$$\nabla^+ (f \nabla u) = f \nabla^+ \nabla u - \nabla (df \# u).$$

(5)

Also we have for the product

$$\Delta (f u) = f \Delta u - 2 \nabla (df \# u + \Delta_M (f) u).$$

(6)

We will be iterating, we obtain

$$\Delta^2 (f u) = \Delta (f \Delta u - 2 \nabla (df \# u + \Delta_M (f) u)$$

$$= f \Delta^2 u - 2 \nabla (df \# u + 2 \Delta_M (f) \Delta u - 2 \nabla (df \# u + \Delta^2 (f) u).$$

(7)

Finally, we have

$$\Delta (f \circ u) = f'' (u) |du|^2 + f' (v) \Delta (u).$$

(8)

We will be using lemma 5.15 in [9].

**Lemma 5** Let $E$ be a Hermitian vector bundle over a Riemannian manifold $(M, g)$, with metric compatible connection $\nabla$, let $\Delta = \nabla^+ \nabla$ denote the Bochner Laplacian, and let $u$ be a section of $E$. We have

$$\nabla^{(n)} \Delta^{(k)} u = \Delta^{(k)} \nabla^{(n)} u + \sum_{j=0}^{2k+n-2} \left( \nabla^{(j)} R_m + \nabla^{(j)} F \right) \ast \nabla^{(2k+n-2-j)} u).$$

We will be applying the previous lemma for $n = k = 1$.

**Corollary 6** Let $E$ be a Hermitian vector bundle over a Riemannian manifold $(M, g)$, with metric compatible connection $\nabla$, let $\Delta = \nabla^+ \nabla$ denote the Bochner Laplacian, and let $u$ be a section of $E$. We have

$$\nabla \Delta u = \Delta \nabla u + (R_m + F) \ast \nabla u + \nabla (R_m + F) \ast u.$$
We will use localised derivative estimates needed for the proof of the next theorem. We will need the following result of Saratchandran, H.

**Proposition 7** For $u \in W^{2,2}_{loc}(E)$ and $\epsilon > 0$ sufficiently small, we have the following estimate

$$\|x_\epsilon^k \nabla u\|^2 \leq \frac{1}{2(1-k\epsilon)} \|x_\epsilon^{k+1} \Delta u\|^2 + \frac{1+2k\epsilon}{2(1-k\epsilon)} \|x_\epsilon^{k-1} u\|^2.$$

**(Proof.** see proposition 4.1 in [14]. ■

**Proposition 8** For $u \in W^{2,2}_{loc}(E)$ and $\epsilon > 0$ sufficiently small, we have the following estimate

$$\|x_\epsilon^k \nabla^4 u\|^2 \leq \left(\frac{1}{1-2k^2C^2\epsilon^2}\right) \left(\frac{1}{4(1-k\epsilon)}\right) \|x_\epsilon^{k+1} \Delta^2 u\|^2$$

$$+ \left(\frac{1}{1-2k^2C^2\epsilon^2}\right) \left(\frac{1+2k\epsilon}{2(1-k\epsilon)}\right) \|x_\epsilon^{k+1} \Delta^3 u\|^2$$

$$+ \left(\frac{1}{1-2k^2C^2\epsilon^2}\right) \left(\frac{1}{4(1-(k-1)\epsilon)}\right) \|x_\epsilon^{k} \Delta^3 u\|^2$$

$$+ \left(\frac{1}{1-2k^2C^2\epsilon^2}\right) \left(\frac{1+2k\epsilon}{2(1-k\epsilon)}\right) \|x_\epsilon^{k-1} \Delta u\|^2$$

$$+ \left(\frac{1}{1-2k^2C^2\epsilon^2}\right) \left(\frac{\epsilon^2}{2}\right) \|x_\epsilon^k u\|^2$$

$$+ \left(\frac{1}{1-2k^2C^2\epsilon^2}\right) \left(\frac{1+2k\epsilon}{2(1-k\epsilon)}\right) \left(\frac{1+2k\epsilon}{4(1-k\epsilon)}\right) \|x_\epsilon^{k-1} u\|^2$$

$$+ \left(\frac{1}{1-2k^2C^2\epsilon^2}\right) \left(\frac{1+2(k-1)\epsilon}{4(1-(k-1)\epsilon)}\right) \|x_\epsilon^{k-2} u\|^2.$$

**(Proof.** We have

$$\langle x_\epsilon^{2k} \nabla^4 u, \nabla^4 u \rangle = \langle \nabla^+ (x_\epsilon^{2k} \nabla^4 u), \nabla^3 u \rangle$$

$$= \langle x_\epsilon^{2k} \nabla^+ \nabla^4 u - \nabla (dx)^{2k} \# \nabla^3 u, \nabla^3 u \rangle$$

$$= \langle x_\epsilon^{2k} \nabla^+ \nabla^4 u, \nabla^3 u \rangle - \langle \nabla (dx)^{2k} \# \nabla^3 u, \nabla^3 u \rangle$$

$$= \langle x_\epsilon^{2k} \Delta \nabla^3 u, \nabla^3 u \rangle - \langle \nabla (dx)^{2k} \# \nabla^3 u, \nabla^3 u \rangle$$

$$= \langle x_\epsilon^{2k} (\nabla^3 \Delta u - (R_m + F) * \nabla^3 u - \nabla^3 (R_m + F) * u), \nabla^3 u \rangle$$

$$- \langle \nabla (dx)^{2k} \# \nabla^3 u, \nabla^3 u \rangle$$

$$= \langle x_\epsilon^{2k} (\nabla^3 \Delta u), \nabla^3 u \rangle - \langle x_\epsilon^{2k} (R_m + F) * \nabla^3 u, \nabla^3 u \rangle$$

$$- \langle x_\epsilon^{2k} \nabla^3 (R_m + F) * u, \nabla^3 u \rangle$$

$$- 2kx_\epsilon^{2k-1} \nabla (dx)^{2} \# \nabla^3 u, \nabla^3 u \rangle,$$
so we obtain

\[
\| x^k \nabla^4 u \| \leq |\langle x^k (\nabla^3 \Delta u), \nabla^3 u \rangle | + |\langle x^k (R_m + F) \ast \nabla^3 u, \nabla^3 u \rangle | + |\langle 2k x^{2k-1} \nabla_{(dx, \cdot)} \nabla^3 u, \nabla^3 u \rangle |
\]

applying Cauchy-Schwarz inequality, we get

\[
\| x^k \nabla^4 u \|^2 \leq \frac{1}{2} \| x^k (\nabla^3 \Delta u) \|^2 + \frac{1}{2} \| x^k \nabla^3 u \|^2 + C^2 \| x^k \nabla^3 u \|^2 + \frac{C^2}{2} \| x^k \nabla^3 u \|^2 + \frac{C^2}{2} \| x^k \nabla^3 u \|^2 \]

\[
+ \frac{C^2}{2} \| x^k \nabla^3 u \|^2 + 2k^2 C^2 \| x^k \nabla^3 u \|^2 + \frac{1}{2} \| x^{k-1} \nabla^3 u \|^2
\]

\[
= \frac{1}{2} \| x^k (\nabla^3 \Delta u) \|^2 + \left( \frac{1}{2} + C^2 + \frac{C^2}{2} \right) \| x^k \nabla^3 u \|^2 + \frac{1}{2} \| x^{k-1} \nabla^3 u \|^2
\]

\[
+ \frac{C^2}{2} \| x^k \nabla^3 u \|^2 + 2k^2 C^2 \| x^k \nabla^3 u \|^2 ,
\]

where to get the first inequality, we have used our bounded geometry conditions 4 and 5. We can estimate the term \( \| x^k (\nabla^3 \Delta u) \|^2 \) by using proposition 9.

\[
\| x^k (\nabla^3 \Delta u) \|^2 \leq \frac{1}{2} \| x^{k+1} \Delta^2 u \|^2 + \frac{1 + 2k \epsilon}{2 (1 - k \epsilon)} \| x^{k-1} \Delta u \|^2 .
\]

We also estimate the term \( \| x^{k-1} \nabla^3 u \|^2 \) by using proposition 9.

\[
\| x^{k-1} \nabla^3 u \|^2 \leq \frac{1}{2} \| x^k \Delta^3 u \|^2 + \frac{1 + 2 (k - 1) \epsilon}{2 (1 - (k - 1) \epsilon)} \| x^{k-2} u \|^2 .
\]

Then

\[
\| x^k \nabla^4 u \|^2 \leq \frac{1}{4} \| x^{k+1} \Delta^2 u \|^2 + \frac{1 + 2k \epsilon}{4 (1 - k \epsilon)} \| x^{k-1} \Delta u \|^2
\]

\[
+ \left( \frac{1}{2} + C^2 + \frac{C^2}{2} \right) \left( \frac{1}{2} (1 - k \epsilon) \right) \| x^{k+1} \Delta^3 u \|^2
\]

\[
+ \left( \frac{1}{2} + C^2 + \frac{C^2}{2} \right) \left( \frac{1 + 2k \epsilon}{2 (1 - k \epsilon)} \right) \| x^{k-1} u \|^2
\]

\[
+ \frac{1}{4} \| x^k \Delta^3 u \|^2 + \frac{1 + 2 (k - 1) \epsilon}{4 (1 - (k - 1) \epsilon)} \| x^{k-2} u \|^2
\]

\[
+ \frac{C^2}{2} \| x^k u \|^2 + 2k^2 C^2 \| x^k \nabla^3 u \|^2 ,
\]

6
which gives

\[
(1 - 2k^2C^2e^2) \left\| x_\varepsilon \nabla^4 u \right\|^2 \leq \frac{1}{4(1 - k\varepsilon)} \left\| x_\varepsilon^{k+1} \Delta^2 u \right\|^2 + \frac{1 + 2k\varepsilon}{4(1 - k\varepsilon)} \left\| x_\varepsilon^{k-1} \Delta u \right\|^2 \\
+ \left( \frac{1}{2} + C^2 + \frac{C^2}{2} \right) \left( \frac{1}{2(1 - k\varepsilon)} \right) \left\| x_\varepsilon^{k+1} \Delta^3 u \right\|^2 \\
+ \left( \frac{1}{2} + C^2 + \frac{C^2}{2} \right) \left( \frac{1 + 2k\varepsilon}{2(1 - k\varepsilon)} \right) \left\| x_\varepsilon^{k-1} u \right\|^2 \\
+ \frac{1}{4(1 - (k - 1)\varepsilon)} \left\| x_\varepsilon^k \Delta^3 u \right\|^2 + \frac{1 + 2(k - 1)\varepsilon}{4(1 - (k - 1)\varepsilon)} \left\| x_\varepsilon^{k-2} u \right\|^2 \\
+ \frac{C^2}{2} \left\| x_\varepsilon^k u \right\|^2.
\]

Choosing \( \varepsilon \) small enough so that \((1 - 2k^2C^2e^2) > 0\), we obtain

\[
\left\| x_\varepsilon \nabla^4 u \right\|^2 \leq \left( \frac{1}{1 - 2k^2C^2e^2} \right) \left( \frac{1}{4(1 - k\varepsilon)} \right) \left\| x_\varepsilon^{k+1} \Delta^2 u \right\|^2 \\
+ \left( \frac{1}{1 - 2k^2C^2e^2} \right) \left( \frac{1 + 2k\varepsilon}{4(1 - k\varepsilon)} \right) \left\| x_\varepsilon^{k-1} \Delta u \right\|^2 \\
+ \left( \frac{1}{1 - 2k^2C^2e^2} \right) \left( \frac{1}{2} + C^2 + \frac{C^2}{2} \right) \left( \frac{1}{2(1 - k\varepsilon)} \right) \left\| x_\varepsilon^{k+1} \Delta^3 u \right\|^2 \\
+ \left( \frac{1}{1 - 2k^2C^2e^2} \right) \left( \frac{1}{2} + C^2 + \frac{C^2}{2} \right) \left( \frac{1 + 2k\varepsilon}{2(1 - k\varepsilon)} \right) \left\| x_\varepsilon^{k-1} u \right\|^2 \\
+ \frac{1}{4(1 - (k - 1)\varepsilon)} \left\| x_\varepsilon^k \Delta^3 u \right\|^2 \\
+ \frac{1}{4(1 - (k - 1)\varepsilon)} \left\| x_\varepsilon^{k-1} u \right\|^2 \\
+ \frac{1}{2} \left\| x_\varepsilon^k u \right\|^2.
\]

Which proves the result.  

From (8), we have the following formula for the Laplacian of the cut-off function \( x_\varepsilon \)

\[
\Delta_M \left( x_\varepsilon^{2k} \right) = 2k(2k - 1)x_\varepsilon^{2k-2}\|dx_\varepsilon\|^2 + 2kx_\varepsilon^{2k-1}\Delta_M x_\varepsilon \\
= x_\varepsilon^{2k-2} \left( 2k(2k - 1)\|dx_\varepsilon\|^2 + 2k\Delta_M x_\varepsilon \right) \\
= x_\varepsilon^{2k-2}G_1 \left( \|dx_\varepsilon\|, \Delta_M x_\varepsilon, x_\varepsilon \right). 
\]

**Corollary 9** We have the following estimate \( |\Delta_M \left( x_\varepsilon^{2k} \right) | \leq Cx_\varepsilon^{2k-2} \) for some constant \( C > 0 \).
Proof. We applied $\Delta_M$ to (10) and using (6) and (8), we obtain

$$\Delta_M^2 x^{2k}_e = 2k (2k - 1) \Delta_M \left( x^{2k-2}_e |dx_e|^2 \right) + 2k \Delta_M \left( x^{2k-1}_e \Delta_M x_e \right)$$

$$= 2k (2k - 1) x^{2k-2}_e \Delta_M \left( |dx_e|^2 \right) - 2k (2k - 1) \nabla_{(dx^{2k-2}_e)\#} |dx_e|^2$$

$$+ 2k (2k - 1) |dx_e|^2 \delta (x^{2k-2}_e)$$

$$+ 2k x^{2k-1}_e \Delta_M^2 x_e - 4k \nabla_{(dx^{2k-2}_e)\#} \Delta_M x_e + 2k \Delta_M x_e \Delta_M (x^{2k-1}_e)$$

$$= 2k (2k - 1) x^{2k-2}_e \Delta_M |dx_e|^2 - 2k (2k - 1) (2k - 2) x^{2k-3}_e \nabla_{(dx_e)\#} |dx_e|^2$$

$$+ 2k (2k - 1) |dx_e|^2 x^{2k-4}_e G_1 (|dx_e|, \Delta_M x_e, x_e)$$

$$+ 2k x^{2k-1}_e \Delta_M^2 x_e - 4k (2k - 1) x^{2k-2}_e \nabla_{(dx_e)\#} \Delta_M x_e$$

$$+ x^{2k-3}_e \Delta_M x_e G_1 (|dx_e|, \Delta_M x_e, x_e)$$

$$= x^{2k-4}_e \left( 2k (2k - 1) x^{2}_e \Delta_M |dx_e|^2 - 2k (2k - 1) (2k - 2) x^{2}_e \nabla_{(dx_e)\#} |dx_e|^2 \right)$$

$$+ 2k (2k - 1) |dx_e|^2 G_1 (|dx_e|, \Delta_M x_e, x_e)$$

$$+ 2k x^{2k-3}_e \Delta_M^2 x_e - 4k (2k - 1) x^{2k-2}_e \nabla_{(dx_e)\#} \Delta_M x_e$$

$$+ x^{2k-3}_e \Delta_M x_e G_1 (|dx_e|, \Delta_M x_e, x_e)$$

$$= x^{2k-4}_e G_2 \left( |dx_e|^2 , \Delta_M x_e, x_e, \Delta_M |dx_e|^2 , \nabla_{(dx_e)\#} |dx_e|^2 \right). \quad (11)$$

Now we use lemma (5.6) in [14]

$$|G_2| \leq C e,$$

then the corollary is satisfied. $\blacksquare$

In order to get $G_3$, applying $d$ to the above formula for $\Delta_M (x^{2k}_e)$ we get

$$d \Delta_M (x^{2k}_e) = (2k - 2) x^{2k-3}_e (dx_e) \left( 2k (2k - 1) |dx_e|^2 + 2k x_e \Delta_M x_e \right)$$

$$+ x^{2k-2}_e \left( 2k (2k - 1) d |dx_e|^2 + 2k (dx_e) (\Delta_M x_e) \right)$$

$$+ 2k x^{2k-4}_e \Delta_M x_e$$

$$= x^{2k-3}_e (2k (2k - 1) (2k - 2) (dx_e) |dx_e|^2$$

$$+ 2k (2k - 2) x_e (dx_e) (\Delta_M x_e)$$

$$+ 2k (2k - 1) x_e d |dx_e|^2 + 2k x_e (dx_e) (\Delta_M x_e)$$

$$+ 2k x^{2k-4}_e \Delta_M x_e$$

$$= x^{2k-3}_e G_3 \left( x_e dx_e, \Delta_M x_e, d |dx_e|^2, d \Delta_M x_e \right). \quad (12)$$

We will define the minimal operator associated to $T$ by $T_{\min} u := Tu$ with domain $D_{\min} := C^\infty (E)$. We also define the maximal operator associated to $T$ as the adjoint of the minimal operator, $T_{\max} := (T_{\min})^*$, since for a linear densely defined operator $L$, since $L^*$ denote the adjoint. We can be defined the domain of the operator $T_{\max}$ as

$$D_{\max} = \{ u \in L^2 (E) : Tu \in L^2 (E) \}, \quad (13)$$

8
where \(T_{\max} u := Tu\) for \(u \in D_{\max}\).

The following lemma can be existed as a Bilaplacian version of Milatovic’s lemma 4.1 in [11].

**Lemma 10** Let \(V\) satisfies the hypotheses of the following theorem, assume \(u \in \text{Dom} (T_{\max})\) and \(T_{\max} u = i\lambda u\), for any \(\lambda \in \mathbb{R}\). Then given \(\epsilon > 0\) sufficiently small, we get the following estimate

\[
\|\Delta^4 (x^k_e u)\|^2 \leq \frac{C_2(\epsilon)}{1 - 2\epsilon C_1(\epsilon)} \|u\|^2 + 2 \langle \langle q \circ r \rangle (u), x^k_e u \rangle,
\]

where \(C_1(\epsilon)\) and \(C_2(\epsilon)\) are constants depending on \(\epsilon\) such that \(\lim_{\epsilon \to 0} C_1(\epsilon) < \infty\) and \(\lim_{\epsilon \to 0} C_2(\epsilon) < \infty\).

**Proof.** Since \(T_{\max} u = i\lambda u\) thus \(\Delta^8 u + Vu = i\lambda u\). As \(V \in L^\infty(\text{End} E)\) and \(u \in L^2(E)\), elliptic regularity, see theorem 10.3.6 in [13], so \(u \in W^{8,2}_{\text{loc}}(E)\).

By using integrating by parts we get

\[
i\lambda \langle u, x^k_e u \rangle = \langle \Delta^8 u + Vu, x^k_e u \rangle = \langle \Delta^8 u, x^k_e u \rangle + \langle Vu, x^k_e u \rangle = \langle \Delta^4 u, \Delta^4 (x^k_e u) \rangle + \langle Vu, x^k_e u \rangle.
\]

By using (7) we get

\[
\Delta^4 (x^k_e u) = x^k_e \Delta^4 u - 2\nabla (dx^k_e) \# \Delta^2 u + 2\Delta^2_M (x^k_e) \Delta^2 u - 2\Delta^2 \nabla (dx^k_e) \# u - 2\nabla (d\Delta^2_M (x^k_e) \# u + (\Delta^4_M x^k_e) u).
\]

Substituting this into the above equation, we get

\[
i\lambda \langle u, x^k_e u \rangle = \langle \Delta^4 u, \Delta^4 (x^k_e u) \rangle - 2 \langle \Delta^4 u, \Delta^2 \nabla (dx^k_e) \# u \rangle - 2 \langle \Delta^4 u, \Delta^2_M (x^k_e) \Delta^2 u \rangle + 2 \langle \Delta^4 u, \Delta^2_M (x^k_e) \Delta^2 u \rangle + 2 \langle \Delta^4 u, (\Delta^4_M x^k_e) u \rangle + \langle Vu, x^k_e u \rangle.
\]

Taking real parts of the above equation, we get

\[
0 = \|x^k_e \Delta^4 u\|^2 - 2 \Re \langle \Delta^4 u, \Delta^2 \nabla (dx^k_e) \# u \rangle - 2 \Re \langle \Delta^4 u, \Delta^2_M (x^k_e) \Delta^2 u \rangle + 2 \Re \langle \Delta^4 u, \Delta^2_M (x^k_e) \Delta^2 u \rangle - 2 \Re \langle \Delta^4 u, \Delta^2_M (x^k_e) \# u \rangle + \Re \langle \Delta^4 u, (\Delta^4_M x^k_e) u \rangle + \langle Vu, x^k_e u \rangle,
\]

then

\[
\|x^k_e \Delta^4 u\|^2 = 2 \Re \langle \Delta^4 u, \Delta^2 \nabla (dx^k_e) \# u \rangle + 2 \Re \langle \Delta^4 u, \nabla (dx^k_e) \# \Delta^2 u \rangle - 2 \Re \langle \Delta^4 u, \Delta^2_M (x^k_e) \Delta^2 u \rangle + 2 \Re \langle \Delta^4 u, \nabla (d\Delta^2_M (x^k_e) \# u \rangle - \Re \langle \Delta^4 u, (\Delta^4_M x^k_e) u \rangle - \langle Vu, x^k_e u \rangle.
\]
Using corollary 8, we get

\[
\|x_e^{4k} \Delta^4 u\|^2 = 2 \text{Re} \langle \Delta^4 u, \nabla_{(dx_e^{2k})} \Delta^2 u \rangle \\
+ 2 \text{Re} \langle \Delta^4 u, \nabla_{(dx_e^{2k})} (R_m + F) \ast u \rangle \\
+ 2 \text{Re} \langle \Delta^4 u, (R_m + F) \ast \nabla_{(dx_e^{2k})} \rangle + 2 \text{Re} \langle \Delta^4 u, \nabla_{(dx_e^{2k})} \Delta^2 u \rangle \\
- 2 \text{Re} \langle \Delta^4 u, \Delta^2_M (x_e^{2k}) \Delta^2 u \rangle + 2 \text{Re} \langle \Delta^4 u, \nabla_{(dx_e^{2k})} \Delta^2 u \rangle \\
- \text{Re} \langle \Delta^4 u, (\Delta^4_M x_e^{2k}) u \rangle - \langle V u, x_e^{2k} u \rangle.
\] (15)

Now, the first seven terms in the above equation can be bounded above by \(\epsilon C_1 (\epsilon) \|x_e^{4k} \Delta^4 u\|^2 + C_2 (\epsilon) \|u\|^2\), since \(C_1 (\epsilon)\) and \(C_2 (\epsilon)\) are constants depending on \(\epsilon\), such that \(\lim_{\epsilon \to 0} C_1 (\epsilon) < \infty\) and \(\lim_{\epsilon \to 0} C_2 (\epsilon) < \infty\). See lemmas 6.2 to 6.10, and corollaries 6.3 to 6.11 in [14]. Briefly, by using (12) we can write \(d \Delta^2_M (x_e^{2k}) = x_e^{2k-3} G_3\), then

\[
\langle \Delta^4 u, \nabla_{(dx_e^{2k})} \Delta^2 u \rangle = \langle \Delta^4 u, x_e^{2k-3} \nabla_{(G_3)^e} \rangle \\
= \langle x_e^{4k} \Delta^4 u, x_e^{4k-3} \nabla_{(G_3)^e} \rangle.
\]

Applying Cauchy-Schwarz and Young's inequality, we get

\[
\left| \langle \Delta^4 u, \nabla_{(dx_e^{2k})} \Delta^2 u \rangle \right| \leq \frac{\epsilon}{2} \left\| x_e^{4k} \Delta^4 u \right\|^2 + \frac{1}{2\epsilon} \left\| x_e^{2k-3} \nabla_{(G_3)^e} \right\|^2.
\]

After that

\[
\langle x_e^{2k} \Delta^2 u, x_e^{2k} \Delta^2 u \rangle = \langle \Delta^2 u, x_e^{4k} \Delta^2 u \rangle = \langle \Delta^2 (x_e^{2k} u), \Delta^2 (x_e^{2k} u) \rangle,
\]
we can rewrite equation (15) to obtain

\[
\langle \Delta^4 (x_e^{4k} u), \Delta^4 (x_e^{4k} u) \rangle = -4 \text{Re} \langle x_e^{4k} \Delta^4 u, \nabla_{(dx_e^{4k})} \Delta^2 u \rangle \\
- 4 \text{Re} \langle x_e^{4k} \Delta^4 u, \Delta^2 \nabla_{(dx_e^{4k})} u \rangle \\
- 4 \text{Re} \langle x_e^{4k} \Delta^4 u, \nabla_{(dx_e^{4k})} \Delta^2 u \rangle \\
- 4 \text{Re} \langle x_e^{4k} \Delta^4 u, \Delta^2_M (x_e^{4k}) \Delta^2 u \rangle \\
+ 2 \text{Re} \langle x_e^{4k} \Delta^4 u, u \Delta^2_M x_e^{4k} \rangle \\
+ 4 \langle \nabla_{(dx_e^{4k})} \Delta^2 u, \nabla_{(dx_e^{4k})} \Delta^2 u \rangle \\
+ 8 \text{Re} \langle \nabla_{(dx_e^{4k})} \Delta^2 u, \Delta^2 \nabla_{(dx_e^{4k})} u \rangle \\
+ 8 \text{Re} \langle \nabla_{(dx_e^{4k})} \Delta^2 u, \nabla_{(dx_e^{4k})} \Delta^2 u \rangle \\
+ 8 \text{Re} \langle \nabla_{(dx_e^{4k})} \Delta^2 u, \Delta^2_M (x_e^{4k}) \Delta^2 u \rangle.
\]
\[-4 \text{Re} \left\langle \nabla_{(dx^k)^\#} \Delta^2 u, u \Delta^1_{x^k} x^k \right\rangle + 4 \text{Re} \left\langle \Delta^2 \nabla_{(dx^k)^\#} u, \Delta^2 \nabla_{(dx^k)^\#} u \right\rangle + 8 \text{Re} \left\langle \Delta^2 \nabla_{(dx^k)^\#} u, \nabla_{(d\Delta^2 x^4_k(x^i)\#)} u \right\rangle + 8 \text{Re} \left\langle \Delta^2 \nabla_{(dx^k)^\#} u, \Delta^2_{x^k} (x^k_x) \Delta^2 u \right\rangle - 4 \text{Re} \left\langle \Delta^2 \nabla_{(dx^k)^\#} u, u \Delta^1_{x^k} x^k \right\rangle + 4 \left\langle \nabla_{(d\Delta^2_{x^k} x^4_k)^\#} u, \nabla_{(d\Delta^2_{x^k} x^4_k)^\#} u \right\rangle + 8 \text{Re} \left\langle \nabla_{(d\Delta^2_{x^k} x^4_k)^\#} u, \Delta^2_{x^k} (x^k_x) \Delta^2 u \right\rangle - 4 \text{Re} \left\langle \nabla_{(d\Delta^2_{x^k} x^4_k)^\#} u, u \Delta^1_{x^k} x^k \right\rangle + 4 \left\langle \Delta^2_{x^k} (x^k_x) \Delta^2 u, \Delta^2_{x^k} (x^k_x) \Delta^2 u \right\rangle - 4 \text{Re} \left\langle \Delta^1_{x^k} x^k, u \Delta^1_{x^k} x^k \right\rangle + 2 \text{Re} \left\langle \Delta^4 u, \nabla_{(dx^k)^\#} \Delta^2 u \right\rangle + 2 \text{Re} \left\langle \Delta^4 u, \nabla_{(dx^k)^\#} (R_m + F) \ast u \right\rangle + 2 \text{Re} \left\langle \Delta^4 u, (R_m + F) \ast \nabla_{(dx^k)^\#} u \right\rangle + 2 \text{Re} \left\langle \Delta^4 u, \nabla_{(dx^k)^\#} \Delta^2 u \right\rangle - 2 \text{Re} \left\langle \Delta^4 u, \Delta^2_{x^k} (x^k_x) \Delta^2 u \right\rangle + 2 \text{Re} \left\langle \Delta^4 u, \nabla_{(d\Delta^2_{x^k} x^4_k)^\#} u \right\rangle - \text{Re} \left\langle \Delta^4 u, (\Delta^1_{x^k} x^k) u \right\rangle - \langle V, u \rangle.
\]

All terms can be bounded above by \(cC_1(\epsilon) \|x^4_{x^k} \Delta^4 u\|^2 + C_2(\epsilon) \|u\|^2\), this was shown by details in corollaries 6.3 to 6.11 in [14]. \(C_1(\epsilon)\) and \(C_2(\epsilon)\) depend on \(\epsilon\) satisfy the following limit conditions, \(\lim_{\epsilon \to 0} C_1(\epsilon) < \infty\) and \(\lim_{\epsilon \to 0} C_2(\epsilon) < \infty\). Then we get

\[
\|\Delta^4 (x^4_{x^k} u)\|^2 \leq cC_1(\epsilon) \|x^4_{x^k} \Delta^4 u\|^2 + C_2(\epsilon) \|u\|^2 - \langle V, u x^{sk} u \rangle.
\]

Using our assumption on the potential \(V\), we get

\[
- \langle V, u x^{sk} u \rangle \leq \langle (q \circ r)(u), x^{sk} u \rangle,
\]

then we obtain

\[
\|\Delta^4 (x^4_{x^k} u)\|^2 \leq cC_1(\epsilon) \|x^4_{x^k} \Delta^4 u\|^2 + C_2(\epsilon) \|u\|^2 + \langle (q \circ r)(u), x^{sk} u \rangle.
\]

Applying proposition 6.12 in [14],

\[
\|x^{sk} \Delta^2 u\|^2 \leq \left( \frac{1}{1 - cC_1(\epsilon)} \right) \|\Delta^2 (x^{sk} u)\|^2 + C_1(\epsilon) C_2(\epsilon) \frac{1}{1 - cC_1(\epsilon)} \|u\|^2.
\]
We can estimate the term \( \|x^4_k \Delta u\|^2 \) by using 6.12 in [14], we obtain
\[
\|x^4_k \Delta u\|^2 \leq \left( \frac{1}{1 - \epsilon C_1 (\epsilon)} \right) \|\Delta^4 (x^4_k u)\|^2 + \frac{C_1 (\epsilon) C_2 (\epsilon)}{1 - \epsilon C_1 (\epsilon)} \|u\|^2.
\]
Which implies
\[
\|\Delta^4 (x^4_k u)\|^2 \leq \frac{\epsilon C_1 (\epsilon)}{1 - \epsilon C_1 (\epsilon)} \|\Delta^4 (x^4_k u)\|^2 + \frac{\epsilon C_1 (\epsilon) C_2 (\epsilon)}{1 - \epsilon C_1 (\epsilon)} \|u\|^2 + C_2 (\epsilon) \|u\|^2 + \langle (q \circ r) (u), x^{8k} u \rangle,
\]
then
\[
\left( 1 - \frac{\epsilon C_1 (\epsilon)}{1 - \epsilon C_1 (\epsilon)} \right) \|\Delta^4 (x^4_k u)\|^2 \leq \frac{C_2 (\epsilon)}{1 - 2\epsilon C_1 (\epsilon)} \|u\|^2 + \frac{1 - \epsilon C_1 (\epsilon)}{1 - 2\epsilon C_1 (\epsilon)} \langle (q \circ r) (u), x^{8k} u \rangle.
\]
By choosing \( \epsilon \) very small, then \( \left( 1 - \frac{\epsilon C_1 (\epsilon)}{1 - \epsilon C_1 (\epsilon)} \right) = \frac{1 - 2\epsilon C_1 (\epsilon)}{1 - \epsilon C_1 (\epsilon)} > 0 \), dividing the above equation by this, we get
\[
\|\Delta^4 (x^4_k u)\|^2 \leq \frac{C_2 (\epsilon)}{1 - 2\epsilon C_1 (\epsilon)} \|u\|^2 + 2 \langle (q \circ r) (u), x^{8k} u \rangle.
\]
Where we can \( \frac{1 - \epsilon C_1 (\epsilon)}{1 - 2\epsilon C_1 (\epsilon)} = 1 + \frac{\epsilon C_1 (\epsilon)}{1 - 2\epsilon C_1 (\epsilon)} \), since \( \epsilon \) sufficiently small. \( \frac{1 - \epsilon C_1 (\epsilon)}{1 - 2\epsilon C_1 (\epsilon)} < 2 \), as \( \lim_{\epsilon \to 0} C_1 (\epsilon) < \infty \) and \( \lim_{\epsilon \to 0} C_2 (\epsilon) < \infty \), then
\[
\|\Delta^4 (x^4_k u)\|^2 \leq \frac{C_2 (\epsilon)}{1 - 2\epsilon C_1 (\epsilon)} \|u\|^2 + 2 \langle (q \circ r) (u), x^{8k} u \rangle.
\]

Theorem 11 Let \((M, g)\) be a complete connected Riemannian manifold, we assume that \((E, h)\) be a Hermitian vector bundle over \(M\) with metric connection \(\nabla\), assume \(M\) and \(E\) satisfy the conditions 4 and 5. Let a potential \(V \in \mathcal{L}_{\text{loc}}^\infty (\text{End} E)\) that is self-adjoint and such that
\[
V (x) \geq -q (r (x)) I_x \quad \text{for} \quad x \in M,
\]
where \(I_x : E_x \to E_x\) is the identity endomorphism, \(r (x)\) is as in (1) and \(q : [0, \infty) \to [0, \infty)\) is a non-decreasing function such that \(q (x) = O (x)\) as \(x \to \infty\). Then the operator \(T : = \Delta^4 + V\), with domain \(\mathcal{C}^\infty (E)\) is essentially self-adjoint.

Proof. We will use the strategy Milatovich employs in [11]. Suppose \(u \in \text{Dom} (T_{\text{max}})\) satisfies \(T_{\text{max}} u = i\lambda u\), for \(\lambda \in \mathbb{R}\). The essential self-adjointness of the operator \(T \mid \mathcal{C}^\infty (E)\) will satisfied if we show that \(u = 0\). For \(\epsilon > 0\) we define
\[
G_\epsilon = \{x \in M : r (x) \leq \frac{8}{\epsilon} \},
\]
Then we get
\[ \langle (q \circ r) u, x^k_r u \rangle \leq \int \frac{q(r(x))}{C_r} |u(x)|^2 \, dx \]
\[ \leq q \left( \frac{8}{r} \right) \|u\|^2. \]

Using the previous lemma, and let \( q(s) = O(s) \), we have
\[ \| \Delta^4 \langle x^k_r u \rangle \|^2 \leq \left( \frac{C_2 (\epsilon)}{1 - 2C_1 (\epsilon)} \right) \|u\|^2 + \frac{C}{\epsilon} \|u\|^2 \]
\[ = \left( \frac{C_2 (\epsilon)}{1 - 2C_1 (\epsilon)} + \frac{C}{\epsilon} \right) \|u\|^2. \]

Since \( C > 0 \) is a constant, taking imaginary parts in equation (14), we obtain
\[ \lambda \langle u, x^k_r u \rangle = -2 \text{Im} \left( \langle \delta^4 u, \delta^2 \nabla (\delta x^k_r u) \rangle \right) - 2 \text{Im} \left( \langle \delta^4 u, \nabla (\delta x^k_r u) \delta^2 u \rangle \right) \]
\[ + 2 \text{Im} \left( \langle \delta^4 u, \delta^2 M (\delta x^k_r u) \rangle \right) - 2 \text{Im} \left( \langle \delta^4 u, \nabla (\delta^2 M (\delta x^k_r u)) \rangle \right) \]
\[ + \text{Im} \left( \langle \delta^4 u, \delta^2 M (\delta x^k_r u) \rangle \right). \]

By using (10), (11) and (12), we can write
\[ \lambda \langle u, x^k_r u \rangle = -2 \text{Im} \left( 2x^k_r \delta^4 u, \delta^2 \nabla (\delta x^k_r u) \right) \]
\[ - 2 \text{Im} \left( 2x^k_r \delta^4 u, \nabla (\delta x^k_r u) \delta^2 u \rangle \right) \]
\[ + 2 \text{Im} \left( x^k_r \delta^4 u, x^k_r \delta^2 G_1 \delta^2 u \rangle \right) \]
\[ - 2 \text{Im} \left( x^k_r \delta^4 u, x^k_r \delta^2 G_2 u \rangle \right) \]
\[ + \text{Im} \left( x^k_r \delta^4 u, x^k_r \delta^2 G_2 u \right). \]

In the previous lemma, we explained the bounds each of the terms on the right hand side of the previous equation by \( \varepsilon C_1 (\epsilon) \| x^k_r \delta^4 u \|^2 + C_2 (\epsilon) \|u\|^2 \), since \( \lim_{\varepsilon \to 0} C_1 (\epsilon) < \infty \) and \( \lim_{\varepsilon \to 0} C_2 (\epsilon) < \infty \). Using proposition (6.12) in [14], then each terms on the right hand side of the previous equation will be bounded above by
\[ \left( \frac{\varepsilon C_1 (\epsilon)}{1 - \varepsilon C_1 (\epsilon)} \right) \| \Delta^4 (x^k_r u) \|^2 + \left( \frac{\varepsilon C_1 (\epsilon) C_2 (\epsilon)}{1 - \varepsilon C_1 (\epsilon)} + C_2 (\epsilon) \right) \|u\|^2. \]

Then we get
\[ |\lambda| \| x^k_r u \|^2 \leq \left( \frac{\varepsilon C_1 (\epsilon)}{1 - \varepsilon C_1 (\epsilon)} \right) \| \Delta^4 (x^k_r u) \|^2 + \left( \frac{\varepsilon C_1 (\epsilon) C_2 (\epsilon)}{1 - \varepsilon C_1 (\epsilon)} + C_2 (\epsilon) \right) \|u\|^2 \]

Then
\[ |\lambda| \| x^k_r u \|^2 \leq \left( \frac{\varepsilon C_1 (\epsilon) C_2 (\epsilon)}{1 - \varepsilon C_1 (\epsilon) (1 - 2\varepsilon C_1 (\epsilon))} + \frac{CC_1 (\epsilon)}{1 - \varepsilon C_1 (\epsilon)} + \frac{\varepsilon C_1 (\epsilon) C_2 (\epsilon)}{1 - \varepsilon C_1 (\epsilon)} + C_2 (\epsilon) \right) \|u\|^2. \]
Let $\epsilon \to 0$ in the previous inequality, we get
\[
\lim_{\epsilon \to 0} |\lambda| \|x_\epsilon^{4k} u\|^2 \leq C \|u\|^2,
\]
since $C = \lim_{\epsilon \to 0} (C_1(\epsilon) + C_2(\epsilon)) < \infty$. Using dominated convergence theorem, we have $\lim_{\epsilon \to 0} |\lambda| \|x_\epsilon^{4k} u\|^2 = |\lambda| \|u\|^2$. Then
\[
|\lambda| \|u\|^2 \leq C \|u\|^2.
\]
As $|\lambda|$ arbitrary number, this implies $u = 0$. ■

4 Conflict of interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

5 author contribution

All authors contributed to the study conception, design, material preparation, data collection and analysis. The first draft of the manuscript was written by Hala. H. Emam and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

References

[1] Atia, H. A, Alsaedi, R. S, Ramady, R, separation of bi-harmonic differential operators on Riemannian manifolds, Forum Math. 26 (2014) 953-966.

[2] Atia, H. A. "Magnetic Bi-Harmonic Differential operators on Riemannian manifolds and the Separation problem" Journal of Contemporary Mathematical Analysis 51 (5) (2016), 222 – 226.

[3] Atia, H. A. "Separation problem for Bi-harmonic Differential operators in $L_p$-spaces on Manifolds" Journal of the Egyptian Mathematical Society. https://doi.org/10.1186/s42787-019-0029-6, (2019).

[4] Chernoff, P. Essential self adjoinness of powers of generators of hyperbolic equations, J. Funct. Anal. 12 (1973), 401 – 414.

[5] Cordes, H. O. Self-adjointness of powers of elliptic operators on non-compact manifolds, Math. Ann. 195 (1972), 257 – 272.

[6] Cycon, H. L, Froese, R. G., Kirsch. W. and Simon, B. Schrodmger Operators with Applications to Quantum Mechanics and Global Geometry, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1987.
[7] Eichhorn, J. The Banach manifold structure of the space of metrics on non-compact manifolds, Differential Geometry and its Applications, 1 (1991), 89 - 108.

[8] Gaffney, M. A special Stokes’s theorem for complete Riemannian manifolds, Ann. of Math. 60 (1954), 140-145.

[9] Kato, T. Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1980.

[10] Kelleher, C. Higher order Yang-Mills flow. arXiv: 1505.07092 [math.DG].

[11] Milatovic, O. Self-adjointness of perturbed biharmonic operators on Riemannian manifolds, Mathematische Nachrichten, 290 (2017), 2948-2960.

[12] Nicolaescu, L. Lectures on the Geometry of Manifolds. World Scientific Publishing Co., Hackensack, NJ, 2007.

[13] Reed. M. and Simon, B. Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness, Academic Press, New York, 1975.

[14] Saratchandran, H. Essential Self-Adjointness of perturbed Quadharmonic operators on Riemannian manifolds. arXiv: 1904.07210 V1 [math.DG].

[15] Shubin, M. A. Spectral theory of elliptic operators on non-compact manifolds, Asterisque, 2017 (1992), 37-108.