Synchronization interfaces and overlapping communities in complex networks

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We show that a complex network of phase oscillators may display interfaces between domains (clusters) of synchronized oscillations. The emergence and dynamics of these interfaces are studied in the general framework of interacting phase oscillators composed of either dynamical domains (influenced by different forcing processes), or structural domains (modular networks). The obtained results allow to give a functional definition of overlapping structures in modular networks, and suggest a practical method to identify them. As a result, our algorithm could detect information on both single overlapping nodes and overlapping clusters.

The functioning of many natural (biological, neural, chemical) or artificial (technological) networks displays coordination of parallel tasks. This phenomenon may be represented as the interplay between two simultaneous processes. The first (involving most of the network nodes) leads to the emergence of organized clusters (or moduli, or cohesive subgroups), where nodes in the same cluster adjust their dynamics into a common (synchronized) behavior to enhance the performance of a specific task. The second process (involving just few nodes of the graph) is to form interfaces (or overlapping structures) between the moduli, that are responsible for the coordination between the different tasks.

A practical example is the brain functioning, wherein the vision of the images from the right and left eye can be represented as the emergence of a collective behavior in different, close-by areas of neurons, whereas perception (as e.g. feature binding) implies the coordination of the two emergent dynamics. Another example in social science is, for instance, consensus formation in polarized elections (as e.g., between a left-wing and a right-wing candidate for the presidency), where usually most of citizens cast their votes following a consistent opinion (they always vote for the left-wing or the right-wing party), whereas usually a small fraction (mostly being responsible for the final election’s outcome) alternate in time their preference depending on their actual opinion.

In this Letter, we present the first evidence that, under the presence of different functional (synchronized) clusters, interfaces appear and show a novel specific dynamical behavior. This finding enables us to develop an algorithm for extracting information on the overlapping structure in a modular network. Indeed, the study of separate modular structures and synchronization in complex graphs has not so far unraveled the crucial point concerning the role of synchronization interfaces and its usefulness in detecting overlapping communities.

In the following, without lack of generality, we study networks consisting of two domains of interacting phase oscillators (each one synchronously evolving at a different frequency), where the nature of the two different frequency domains is the result of either a dynamical process (influenced by different forcing processes) or a structural design (modular network). Under these conditions, at each time, most of the oscillators will contribute to the synchronous behavior of the two clusters, whereas a few nodes will find themselves in a frustrating situation of having to decide how to behave as a consequence of the contrasting inputs received by the two clusters.

In our simulation, we study a network $G$ which consists of $N = 200$ phase oscillators arranged in an Erdős-Rényi configuration. Initially, we set $d_p = 0$ and $k_p = 0 \forall i$, and we choose $d = 0.1$ such that $G$ features an unsynchronized motion. Next, we arbitrarily divide the nodes into two groups: nodes from $i = 1, \cdots, 100$ (from $i = 101, \cdots, 200$) are assigned to the community $A$ (to the community $B$). We introduce two pacemakers of frequencies $\omega_{p_A}$ and $\omega_{p_B}$, and connect the nodes in the first group (in the second group) with the first (second) pacemaker. This implies in Eq. 1 to set $\phi_{p_i} = \phi_{p_A} \equiv \omega_{p_A} t$ ($\phi_{p_i} = \phi_{p_B} \equiv \omega_{p_B} t$) for all the nodes in $A$ ($B$). In order to assign to each node its $k_p$ links with the pacemaker, we start at $t_0 = 0$ the evolution of Eq. 1 from random ini-
entrained to any arbitrary frequency, this attachment over a given time interval, the resulting nodes in \( A \phi \) and the other 1,000 to entrain those in \( j \) that node \( \omega \) of the period \( t \). The dynamics displays two large communities of entrained with \( \Delta \), for \( d = 3.5 \). The solid line represents a linear fit with \( T_0 \sim 120/\omega_{\Delta} \). Each point is the average of 5 independent realizations. Other parameters reported in the text.

By selecting \( \omega_{\Delta} = 0.7 \), \( \omega_{\Delta B} = 0.3 \), \( \sum_{i=1}^{N} k_{pi} = 2,000 \) links to the pacemakers \( 1,000 \) to entrain the nodes in \( A \) and the other \( 1,000 \) to entrain those in \( B \), and \( d_p = 1 \), the dynamics displays two large communities of entrained oscillators, densely intermingled by the connectivity of \( G \).

The situation is depicted in Fig. 1(a) where we report the instantaneous frequency of each oscillator in \( G \) (averaged over a small window to smooth fluctuations) as a function of time. We observe that most of the nodes in \( A \) (in \( B \) have a constant frequency (that of the corresponding pacemaker), whereas the few nodes belonging to the synchronization interface feature a frequency which is oscillating around the mean value of that of the two communities \( \bar{\omega} = (\omega_{\Delta A} + \omega_{\Delta B})/2 \). Furthermore, Fig. 1(d) gives evidence that the period \( T_0 \) of the switching process in the interface is inversely proportional to \( \omega_{\Delta} = (\omega_{\Delta A} - \omega_{\Delta B})/2 \).

Notice that this switching mechanism is the result of the competition of two conflicting processes: the synchronization within \( G \) (controlled by the parameter \( d \)) that would lead the whole network to exhibit a unique frequency, and the forcing of the two pacemakers (controlled by \( d_p \)) that would try to separate the nodes into two clusters of entrained oscillators. In order to quantify this competition and to describe the size of the synchronization interface we fix \( d_p = 1 \) and gradually increase \( d \). The results are shown in Figs. 1(a-c) with (a) \( d = 3.25 \), (b) \( d = 5.75 \), and (c) \( d = 9.75 \). For intermediate coupling Fig. 1(b), it is observed that the interface grows in size, recruiting more and more nodes out of the two communities into the oscillating mode. Due to the presence of the two forcing pacemakers, this interface is organized in an oscillating mode rather than in a constant frequency mode, as it would occur in the classical Kuramoto regime \( \S \). As the coupling \( d \) is further increased, almost the entire system of oscillators eventually participates into this interface oscillating mode, as seen in Fig. 1(c). Notice that similar switching dynamics was observed before in the case of a chain of oscillators subjected to two forcing frequencies applied to the two ends of the chain \( \S \).

In order to give an analytical insight to our findings, let us consider the simple case of three interacting phase oscillators described by: \( \dot{\phi}_1 = \omega_1 + K_1 \sin(\phi_3 - \phi_1); \dot{\phi}_2 = \omega_2 + K_2 \sin(\phi_3 - \phi_2); \dot{\phi}_3 = \omega_3 + K \sin(\phi_3 - \phi_2) + \sin(\phi_2 - \phi_3) \).

Here, \( \phi_3 \) is the phase of an oscillator (with natural frequency \( \omega_3 \)) receiving a simultaneous coupling from two other oscillators at natural frequencies \( \omega_1 \neq \omega_2 \neq \omega_3 \), and \( K_1, K_2 \ll K \) are coupling constants. The oscillators 1 and 2 model the two functional (frequency) domains \( A \) and \( B \), where the nature of the two different frequency domains is from either a dynamical process or a structural design. These two domains of synchronous oscillators simultaneously interact with the small group of nodes in the interface (modeled by the third oscillator), so that we can reasonably assume \( K_1 = K_2 = 0 \), and consequently \( \phi_{1,2} = \omega_{1,2} \).

The resulting equation for \( \phi_3 \)

\[
\dot{\phi}_3 = \omega_3 + 2K \sin \left( \frac{\omega_1 + \omega_2}{2} - \phi_3 \right) \cos \left( \frac{\omega_1 - \omega_2}{2} \right) \tag{2}
\]

yields an analytic solution

\[
\dot{\theta}_3 = \frac{A \sin(\omega_{\Delta} t)}{1 + B e^{\frac{2K}{\omega_{\Delta}}} \sin(\omega_{\Delta} t)} \tag{3}
\]

for \( \omega_3 = \bar{\omega} \), and where \( \dot{\theta}_3 = \dot{\phi}_3 - \dot{\bar{\omega}} \), \( \bar{\omega} = (\omega_1 + \omega_2)/2 \), \( \omega_{\Delta} = (\omega_1 - \omega_2)/2 \), and \( A, B \) suitable parameters. In good agreement with the results of Fig. 1(d), Eq. 3 predicts that the instantaneous frequency \( \phi_3 \) (once \( \omega_3 \) is selected to be the mean frequency of the two forcing clusters) will oscillate around \( \bar{\omega} \) with a period \( T_0 \) inversely proportional to the difference in the frequencies of the two forcing clusters. Notice that, as \( K \) increases (i.e. in the strong coupling regime), \( \dot{\theta}_3 \) will progressively vanish, which means that \( \phi_3 = \bar{\omega} \) or, in other words, the...
frequency of the oscillator will be locked to the mean frequency of the two giant clusters.

While so far we have considered the behavior of interfaces as the result of the competition of dynamical domains, we now move to describe the case of synchronization interfaces under the competition of structural domains in modular graphs in the absence of the forcing \( d_p = 0 \) and \( k_{pi} = 0 \) \( \forall i \) in Eq. (11). For this purpose, we construct the adjacency matrix of \( G \) by considering two large communities \((A, B)\), each one formed by 50 densely and randomly connected nodes (the average degree in the same community is 16), which are overlapped by a small community \( O \) made of a complete graph of 5 nodes (3 links to \( A \) (\( B \))) that form symmetric connections to nodes in both \( A \) and \( B \). Each of the 105 nodes of \( G \) is associated with a phase oscillator obeying Eq. (11), which is integrated with a Runge-Kutta-Fehlberg method, for an initial distribution of frequencies such that nodes in \( A \) (\( B \)) [i.e. nodes from \( i = 1 \) (\( i = 51 \)) to \( i = 50 \) (\( i = 100 \))] have natural frequencies uniformly distributed in the range \( 0.25 \pm 0.25 \) (0.5 \( \pm 0.25 \)), while nodes in \( O \) have frequencies uniformly distributed in the range \( 0.375 \pm 0.05 \) (i.e. around the mean frequency of the two distributions).

Figure 2 shows that all oscillators in the communities \( A \) and \( B \) behave synchronously with an almost constant frequency in time (well approximating the mean of the original frequency distribution), while all oscillators in \( O \) constitute the synchronization interface and, as so, display an instantaneous frequency oscillating in time around the mean value of the two frequencies in the two clusters [Fig. 2(a)]. Similarly, Fig. 2(b) shows the good agreement between the analytical prediction of Eq. (3) and the numerical results concerning the scaling of the period of the frequency oscillations of \( O \) with the frequency difference between the two communities \( A \) and \( B \).

It is important to remark that all other analytical predictions of Eq. (3) concerning the behavior of the synchronization interface in the large coupling regime can be confirmed in our simulations. Indeed, looking at Fig. 3(a-b) one realizes that the effect of increasing the coupling strength \( d \) is to lock almost always the synchronized frequency of the interface to the mean of the frequencies of the two communities, with persistent events of short shootings to even larger (or lower) frequencies.

What described so far refers to the specific case in which initially the nodes of the interface were prepared to have the same number of links to nodes of the two communities. It is therefore interesting to ask what happens when the nodes in \( O \) are asymmetrically connected to \( A \) and \( B \). While the low coupling regime does not substantially differ from the symmetrical case, the high coupling regime [illustrated in Fig. 3(c)] features frequency oscillations of the nodes in \( O \) that are biased toward that community to which the nodes have more connections. In fact, it can be proved that if the nodes in \( O \) have \( k^A_i \) (\( k^B_j \)) connections to nodes in \( A \) (\( B \)), they tend to lock to the weighted mean frequency \( \bar{\omega}_w = \frac{k^A_i \omega^A_i + k^B_j \omega^B_j}{k^A_i + k^B_j} \), where \( \omega^A \) (\( \omega^B \)) is the average frequency of the nodes in \( A \) (\( B \)).

The overall scenario reported above suggests a practical way to detect overlapping communities (Fig. 4(a)) in generic modular networks. It is essential to remark that most of the definitions of network communities proposed so far led essentially to a graph partition into components, such that a given node belongs to and only to one of the components of the partition \( \Phi \). The possibility, instead, that two components of a partition can have an overlapping set of nodes has been recently investigated by means of topological arguments \( \Phi \). The novelty of our approach is in introducing a functional concept of overlapping structures, that are defined in relationship to dynamical response of the network as a whole. Namely, as far as synchronized behavior of phase oscillators is concerned, we define an overlapping structure as the set of nodes which, instead of following the constant frequency of one of the two domains, balance their instantaneous frequencies in between these two, and as so, they cannot be considered as a functional part of any single domain.
As we will see below, this definition allows detection of further information including single overlapping nodes, as compared to previous studies.

To illustrate this idea, we construct a network $G$ made of two large moduli ($A$ and $B$ of 100 nodes each), where the majority of nodes forms random connections (with average degree 15) with only elements of the same community, while only very few nodes form links with nodes of both communities. Precisely, we call $k_i^A$ ($k_i^B$) the total number of links these few nodes form with nodes in $A$ ($B$), and define a quantity to evaluate the degree of overlapping of these nodes as the ratio $D_i = k_i^A/k_i^B$. When the node is perfectly overlapped between two clusters, $D_i = 1$. Under these conditions, Eq. (1) is simulated for $d_p = 0$, $k_{pi} = 0 \forall i$, the set $\{\omega_i\}$ drawn from a Gaussian distribution with standard deviation 0.1 and mean value 0.3 (0.6) for nodes belonging to community $A$ ($B$). To identify those overlapping nodes, the quantity $C_i = sgn(\dot{\phi}_i(t) - \bar{\omega}) \min_i(\{\dot{\phi}_i(t) - \bar{\omega}\})$ ($\bar{\omega}$ is the mean of the two averaged frequencies assigned to the two communities) is introduced, which allows to monitor how close in time the dynamics of a node gets to $\bar{\omega}$. When the node is in the dynamics of fully synchronized interface, $C_i = 0$. Therefore, the closer $D_i$ gets to 1, the closer $C_i$ is expected to get to 0. The results are shown in Fig. 4(b,c) for two different arrangements of the overlapping community: overlapping nodes [Fig. 4(b)] and overlapping clusters [Fig. 4(c)] which have symmetrical connections to two clusters. In both cases, two large synchronized clusters are identified very far from the overlapping synchronization, corresponding to those nodes performing distinct tasks and already classified by the structural partition. At the same time, the dynamical evolution manifests a group of nodes whose dynamics is located significantly out of the two main clusters (thus identifying the overlapping community). In Fig. 4(b), each overlapping node gives rise to a different value of $C_i$ in correspondence to its specific degree of overlapping $D_i$. Namely, the node with $D_i = 1$ yields $C_i = 0$. On the contrary, in Fig. 4(c), all the nodes inside the overlapping cluster are identified as a whole and feature the same value of $C_i = 0$ due to symmetrical connection.

In conclusion, we have analytically and numerically described the behavior of synchronization interfaces in both random and modular complex networks of phase oscillators. We identify the overlapping structure (nodes or clusters) in the modular network, using for the first time a functional approach. Therefore, our study is of practical relevance for applications to large biological, neural, chemical, social and technological modular networks, where, besides detection of the structure of overlapping communities, one is interested in inspecting the coordinating role of such overlapping communities in the functional performance of the graph.

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