Different Estimation Procedures For Topp Leone Exponential And Topp Leone q Exponential Distribution

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Abstract

Topp–Leone Exponential Distribution is a continuous model distribution used for modelling lifetime phenomena. In this study, we introduce different estimation methods for the unknown parameters of Topp–Leone Exponential(TLE) distribution and Topp–Leone q Exponential(TLqE) distribution.

Key words: Topp–Leone exponential distribution, Topp–Leone q exponential distribution, estimation methods.

1. Introduction

In the survival, a number of continuous univariate distributions have been widely utilized for demonstrating information in numerous areas, for example, biology, medicine, engineering, public health, epidemiology and economics. In any case, applied areas, for example, lifetime analysis obviously require expanded types of these distributions. In this way, a few classes of distributions have been built by extending common families of continuous distributions. These generalized distributions give greater adaptability by including ‘at least one’ parameters to the standard model. The Topp-Leone(TL) distribution was introduced by Topp and Leone in 1955 (Topp and Leone, 1955). Topp-Leone Generalized(TLG) family of distributions was inferred by Rezaei et al. (2016). The distribution and density function of proposed family is known by

\[
F_{TLG}(x) = 2\alpha \int_0^{G(x)} t^{\alpha-1}(1-t)(2-t)^{\alpha-1} dt \\
= G(x)^\alpha(2 - G(x))^{\alpha} 
\]

differentiating, we get the corresponding pdf,

\[
f_{TLG}(x) = 2\alpha g(x)(1 - G(x))G((x)^{\alpha-1}(2 - G(x))^{\alpha-1}) 
\]
where $G(x)$ and $g(x)$ are cdf and pdf respectively, and $\alpha \geq 0$.

1.1. Topp Leone Exponential distribution (TLE)

In reliability analysis, a frequently used distribution is exponential distribution (Crowder et al., 1994). Its characterizing property is its constant hazard function. Due to this, exponential distribution is sometimes not suitable for analyzing data. This implies the need for more generalization. In such situations we use distribution called Topp-Leone Exponential distribution (TLE). TLE distribution comes as the combination of TL distribution and exponential distribution. Here TL distribution is the generator and exponential is the parent distribution.

For creating the TLE, we need cdf $G(x)$ and pdf $g(x)$ of exponential distribution

$$G(x) = 1 - \exp(-\lambda x); x \geq 0, \lambda \geq 0 \quad (3)$$

and

$$g(x) = \lambda \exp(-\lambda x) \quad (4)$$

The TLE distribution is obtained by taking $G(x)$ and $g(x)$ into

$$F_{TLE}(x) = (1 - \exp(-\lambda x))^{\alpha} (2 - (1 - \exp(-\lambda x)))^{\alpha} \quad (5)$$

and

$$f_{TLE}(x) = 2\alpha \lambda \exp(-2\lambda x)(1 - \exp(-2\lambda x))^{\alpha-1} \quad (7)$$

here $\alpha$ is the shape parameter and $\lambda$ is the scale parameter.

1.2. Topp Leone $q$ Exponential distribution (TL$q$E)

$q$-exponential distribution is a higher version of exponential distribution. It provides more flexibility regarding to its decay than exponential distribution. It does not have the limitation of constant hazard rate. And thus allows modeling system improvement and degeneration.

Here we generalize $q$-exponential distribution into Topp Leone $q$-Exponential (TL$q$E) distribution by exponential distribution. The TL distribution combined with $q$-exponential distribution gives TL$q$E distribution. Here $q$-exponential distribution is the parent distribution and TL distribution is the generator distribution. The distribution function and density function of $q$-exponential distribution is given as

$$F_{qE}(X) = 1 - [1 - (1 - q)\lambda x]^{\frac{2}{q-1}} \quad (8)$$

and

$$f_{qE}(x) = (2 - q)\lambda[1 - (1 - q)\lambda x]^{\frac{1}{q-1}}; x \geq 0, \lambda \geq 0, q \leq 2, q \neq 0 \quad (9)$$
respectively. Substituting these in the cdf and pdf of TLG distribution we get the cdf and pdf of TLqE distribution.

A random variable X has a TLqE distribution, if X has the cdf and pdf respectively as follows,

\[ F_{TLqE}(x) = [1 - [1 - (1 - q)\lambda x]^{\frac{2-q}{1-q}}]^\alpha [2 - [1 - (1 - q)\lambda x]^{\frac{2-q}{1-q}}]^\alpha \]  
\[ x \geq 0, \lambda, \alpha \geq 0, q \leq 2, q \neq 0 \]  

and

\[ f_{TLqE}(x) = 2\alpha(2-q)\lambda\left[1 - (1 - q)\lambda x\right]^{\frac{1}{1-q}}[1 - (1 - q)\lambda x]^{\frac{2-q}{1-q}-1} \]  
\[ = 2\alpha\lambda(2-q)[1 - (1 - q)\lambda x]^{\frac{1}{1-q}}[1 - (1 - q)\lambda x]^{\frac{2-q}{1-q}-1} \]  

2. Parameter Estimation

We now explore the statistical aspect of the TLE distribution and TLqE distribution and investigate the estimation of the unknown parameters \( \lambda, \alpha \) and \( q \) by five methods. Howafter \( x_1, x_2, \ldots, x_n \) denote realizations from a random sample of size \( n \) from X and \( x(1), x(2), \ldots, x(n) \) their ascending order.

2.1. Topp Leone Exponential Distribution

2.1.1 Method of Least Squares Estimation

Here we consider the least squares estimation introduced by in their joint work. The least square estimates (LSEs) of \( \lambda \) and \( \alpha \) can be determined by minimizing the least square function, with respect to \( \lambda \) and \( \alpha \). The least square function is defined by

\[ LS(\lambda, \alpha) = \sum_{i=1}^{n} \left[F(x(i); \lambda, \alpha) - \frac{i}{n+1}\right]^2 \]  
\[ = \sum_{i=1}^{n} \left[[1 - e^{-2\lambda x(i)}]^{\alpha} - \frac{i}{n+1}\right]^2 \]  

Thus, the least square estimates can be obtained by solving the following equations simultaneously: \( \frac{\partial LS(\lambda, \alpha)}{\partial \lambda} = 0 \) and \( \frac{\partial LS(\lambda, \alpha)}{\partial \alpha} = 0 \), where

\[ \frac{\partial LS(\lambda, \alpha)}{\partial \alpha} = 2\alpha \sum_{i=1}^{n} \left[[1 - e^{-2\lambda x(i)}]^{\alpha} - \frac{i}{n+1}\right][1 - e^{-2\lambda x(i)}] \alpha \log [1 - e^{-2\lambda x(i)}] \]  
\[ = 2\lambda \sum_{i=1}^{n} \left[[1 - e^{-2\lambda x(i)}]^{\alpha} - \frac{i}{n+1}\right][1 - e^{-2\lambda x(i)}] \alpha \log [1 - e^{-2\lambda x(i)}] \]  
\[ (14) \]
and
\[
\frac{\partial LSW(\lambda, \alpha)}{\partial \lambda} = 2 \sum_{i=1}^{n} \left[ (n+1)^2 \right] \left[ 1 - e^{-2\lambda x(i)} \right] \alpha - \frac{i}{n+1} \right] 2 \alpha x(i) \left[ 1 - e^{-2\lambda x(i)} \right]^{\alpha-1} e^{-2\lambda x(i)}
\] (15)

2.1.2. Method of Weighted Least Squares estimation

The weighted least square estimates (WLSEs) of \( \lambda \) and \( \alpha \) can be obtained by minimizing the weighted least square function, with respect to \( \lambda \) and \( \alpha \). The weighted least square function is defined by
\[
LSW(\lambda, \alpha) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[ F(x(i); \lambda, \alpha) - \frac{i}{n+1} \right]^2
\] (16)

Thus, the weighted least square estimates can be obtained by solving the following equations simultaneously: \( \frac{\partial LSW(\lambda, \alpha)}{\partial \lambda} = 0 \) and \( \frac{\partial LSW(\lambda, \alpha)}{\partial \alpha} = 0 \), where
\[
\frac{\partial LSW(\lambda, \alpha)}{\partial \alpha} = 2 \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[ 1 - e^{-2\lambda x(i)} \right] \alpha - \frac{i}{n+1} \left[ 1 - e^{-2\lambda x(i)} \right]^{\alpha-1} \log \left[ 1 - e^{-2\lambda x(i)} \right]
\] (17)

and
\[
\frac{\partial LSW(\lambda, \alpha)}{\partial \lambda} = 2 \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[ 1 - e^{-2\lambda x(i)} \right] \alpha - \frac{i}{n+1} \right] 2 \alpha x(i) \left[ 1 - e^{-2\lambda x(i)} \right]^{\alpha-1} e^{-2\lambda x(i)}
\] (18)

2.1.3 Method of Cramer-von Mises Distance Estimation

The Cramer-von Mises estimator (CME) is a type of minimum distance estimators (also called maximum goodness of fit estimators) which is based on the difference between the estimate of the cumulative distribution function and the empirical distribution function.
MacDonald motivates the choice of Cramer-von Mises type minimum distance estimators providing empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. The cramer-von Mises minimum distance estimates (CVEs) of \( \alpha \) and \( \lambda \) is determined by minimizing the cramer-von Mises minimum distance function, with respect to \( \lambda \) and \( \alpha \). The cramer-von Mises minimum distance function is defined by

\[
C(\lambda, \alpha) = \frac{1}{12n} + \sum_{i=1}^{n} \left[ F(x(i); \lambda, \alpha) - \frac{2i - 1}{2n} \right]^2 \tag{19}
\]

Thus, the cramer-von Mises minimum distance estimates can be obtained by solving the following equations simultaneously: \( \partial C(\lambda, \alpha)/\partial \lambda = 0 \) and \( \partial C(\lambda, \alpha)/\partial \alpha = 0 \), where

\[
\frac{\partial C(\lambda, \alpha)}{\partial \alpha} = \sum_{i=1}^{n} \left[ [1 - e^{-2\lambda x(i)}]^{\alpha} - \frac{2i - 1}{2n} \right] \left[ 1 - e^{-2\lambda x(i)} \right] \log \left[ 1 - e^{-2\lambda x(i)} \right] \tag{20}
\]

and

\[
\frac{\partial C(\lambda, \alpha)}{\partial \lambda} = \sum_{i=1}^{n} \left[ [1 - e^{-2\lambda x(i)}]^{\alpha} - \frac{2i - 1}{2n} \right] 2\alpha x(i) \left[ 1 - e^{-2\lambda x(i)} \right]^{\alpha-1} e^{-2\lambda x(i)} \tag{21}
\]

### 2.1.4 Method of Anderson-Darling Estimation

The method of Anderson-Darling estimation was introduced by Anderson-Darling in the context of statistical tests. In TLE distribution, the Anderson-Darling estimates (ADEs) of \( \lambda \) and \( \alpha \) can be determined by minimizing the Anderson-Darling function, with respect to \( \lambda \) and \( \alpha \). The Anderson-Darling function is defined by

\[
A(\lambda, \alpha) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i - 1) \{ \log[F(x(i); \lambda, \alpha)] + \log[1 - F(x(n+1-i); \lambda, \alpha)] \} \tag{22}
\]

\[
= -n - \frac{1}{n} \sum_{i=1}^{n} (2i - 1) \{ \log[1 - e^{-2\lambda x(i)}]^\alpha + \log[1 - e^{-2\lambda x(n+1-i)}]^\alpha \} \tag{23}
\]
Thus, the Anderson-Darling estimates can be obtained by solving the following equations simultaneously: 
\[ \frac{\partial A(\lambda, \alpha)}{\partial \lambda} = 0 \text{ and } \frac{\partial A(\lambda, \alpha)}{\partial \alpha} = 0, \]
where
\[ \frac{\partial A(\lambda, \alpha)}{\partial \lambda} = -\frac{1}{n} \sum_{i=1}^{n} (2i - 1) \frac{1}{[1 - e^{-2\lambda x(i)}]^\alpha} 2x(i) \alpha [1 - e^{-2\lambda x(i)}]^{\alpha-1} e^{-2\lambda x(i)} \]
\[ + \frac{1}{[1 - e^{-2\lambda x(n+1-i)}]^\alpha} 2x(i) \alpha [1 - e^{-2\lambda x(n+1-i)}]^{\alpha-1} e^{-2\lambda x(n+1-i)} \] \hspace{1cm} (24)

and
\[ \frac{\partial A(\lambda, \alpha)}{\partial \alpha} = -\frac{1}{n} \sum_{i=1}^{n} (2i - 1) \eta x(i) \log[1 - e^{-2\lambda x(i)}] + 2x(n+1-i) \log[1 - e^{-2\lambda x(n+1-i)}] \] \hspace{1cm} (25)

2.2. Topp Leone $q$ Exponential Distribution

2.2.1 Method of Least Squares Estimation

As in the previous case here we can estimate $\lambda$, $\alpha$, and $q$ by minimizing the least square function, with respect to $\lambda$, $\alpha$ and $q$. Therefore LSEs can be obtained by solving the following equations simultaneously:
\[ \frac{\partial LS(\lambda, \alpha, q)}{\partial \lambda} = 0, \] \[ \frac{\partial LS(\lambda, \alpha, q)}{\partial \alpha} = 0, \] \[ \frac{\partial LS(\lambda, \alpha, q)}{\partial q} = 0, \]
where
\[ \frac{\partial LS(\lambda, \alpha, q)}{\partial \lambda} = 2 \sum_{i=1}^{n} \eta x(i) \alpha (2 - q) \psi^{\frac{1}{1-q}} [\psi + 1]^{\frac{2-q}{1-q}} \] \hspace{1cm} (26)
\[ \frac{\partial LS(\lambda, \alpha, q)}{\partial \alpha} = 2 \sum_{i=1}^{n} \eta \ln [1 - \psi^{\frac{2-q}{1-q}}] [1 - \psi^{\frac{2-q}{1-q}}]^{\alpha} \] \hspace{1cm} (27)
\[ \frac{\partial LS(\lambda, \alpha, q)}{\partial q} = 2 \sum_{i=1}^{n} \eta \exp \left[ \ln \left[ \psi^{\frac{2-q}{1-q}} \right] \right] \alpha [\psi + 1]^{\alpha-1} \frac{\lambda x(i)(2 - q)}{(1 - q)\psi} + \frac{\ln [\psi]}{(1 - q)^2} \] \hspace{1cm} (28)

where \[ \psi = 1 - \lambda x(i)(1-q) \] and \[ \eta = (1 - (1 - \lambda x(i)(1-q))^{\frac{2-q}{1-q}})^{\alpha-1} - \frac{i}{n+1} \]

2.2.2. Method of Weighted Least Squares estimation

Here we can estimate $\lambda$, $\alpha$, and $q$ by minimizing the weighted least square function, with respect to $\lambda$, $\alpha$ and $q$. Therefore LSWEs can be obtained by solving the following equations simultaneously:
\[ \frac{\partial LSW(\lambda, \alpha, q)}{\partial \lambda} = 0, \] \[ \frac{\partial LSW(\lambda, \alpha, q)}{\partial \alpha} = 0, \] \[ \frac{\partial LSW(\lambda, \alpha, q)}{\partial q} = 0, \]
where
\[ \frac{\partial LSW(\lambda, \alpha, q)}{\partial \lambda} = 2 \sum_{i=0}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \eta x(i) \alpha (2 - q) \psi^{\frac{1}{1-q}} [1 - (1 - \psi^{\frac{1}{1-q}})]^{(\alpha-1)} \] \hspace{1cm} (29)
\[ \frac{\partial \text{LSW}(\lambda, \alpha, q)}{\partial \alpha} = 2 \sum_{i=0}^{\infty} \frac{(n+1)^2(n+2)}{i(n-i+1)} \eta x(i) \ln \left[ 1 - (1 - \psi)^{\frac{\alpha}{1-q}} \right] \alpha \left[ 1 - (1 - \psi)^{\frac{\alpha}{1-q}} \right] \alpha \] (30)

\[ \frac{\partial \text{LSW}(\lambda, \alpha, q)}{\partial q} = 2 \sum_{i=0}^{\infty} \frac{1}{(n-1)^2(1-\psi)} \left[ e^{\ln[1-\psi^{\frac{\alpha}{1-q}}]} \alpha \left( n+1 \right)^2 (n+2) \right] \eta x(i) \]
\[ \times (2-q)(1-q) + \ln(\psi)(\lambda x(i) - \lambda qx(i) - 1)(1 - (1 - \psi)^{\frac{\alpha}{1-q}})^{(a-1)} \] (31)

where \( \psi = 1 - \lambda x(i)(1-q) \) and \( \eta = (1 - (1 - \lambda x(i)(1-q))^{\frac{\alpha}{1-q}})^{\alpha - \frac{i}{n+1}} \)

### 2.2.3 Method of Cramer-von Mises Distance Estimation

In method of Cramer-von Mises Distance Estimation we can estimate \( \lambda, \alpha, \) and \( q \) by minimizing the Cramer-von Mises Distance function, with respect to \( \lambda, \alpha \) and \( q \).

Therefore CEs can be obtained by solving the following equations simultaneously:

\[ \frac{\partial C(\lambda, \alpha, q)}{\partial \lambda} = 0, \quad \frac{\partial C(\lambda, \alpha, q)}{\partial \alpha} = 0, \quad \frac{\partial C(\lambda, \alpha, q)}{\partial q} = 0 \]

where

\[ \frac{\partial C(\lambda, \alpha, q)}{\partial \lambda} = -2 \sum_{i=0}^{n} x(i) \alpha \left[ q \psi^{\frac{1}{1-q}} \right] \left[ (1 - \psi)^{2\alpha - 1} - 2(1 - \psi^{\frac{i}{n}}) - \frac{q(2i-1)(1-\psi^{\frac{2\alpha}{1-q}}) + 2(2i-1)(1-\psi^{\frac{2\alpha}{1-q}})}{2n} \right] \] (32)

\[ \frac{\partial C(\lambda, \alpha, q)}{\partial \alpha} = 2 \sum_{i=0}^{n} \ln(1 - \psi^{\frac{\alpha}{1-q}}) \left[ (1 - \psi^{\frac{2\alpha}{1-q}})^{2\alpha} - \frac{(2i-1)(1-\psi^{\frac{2\alpha}{1-q}})}{n} \right] \] (33)

\[ \frac{\partial C(\lambda, \alpha, q)}{\partial q} = 2 \sum_{i=0}^{n} (1 - \psi^{\frac{2\alpha}{1-q}} - \frac{2i-1}{2n}) \alpha \left[ (1 - \psi^{\frac{2\alpha}{1-q}})^{\alpha - 1} e^{\frac{2\alpha}{1-q} \ln \psi} \right] \]
\[ \left( \frac{1}{(1-q)^2} \ln \psi + \frac{\lambda x(i)}{\psi} \frac{2-q}{1-q} \right) \] (34)

where \( \psi = 1 - \lambda x(i)(1-q) \) and \( \eta = (1 - (1 - \lambda x(i)(1-q))^{\frac{\alpha}{1-q}})^{\alpha - \frac{i}{n+1}} \)

### 2.1.4 Method of Anderson-Darling Estimation

Here we can estimate \( \lambda, \alpha, \) and \( q \) of TLqE distribution by minimizing the Anderson-Darling function, with respect to \( \lambda, \alpha \) and \( q \). Therefore Anderson-Darling estimates can be obtained by solving the following equations simultaneously:
\[ \frac{\partial A(\lambda, \alpha, q)}{\partial \lambda} = 0, \quad \frac{\partial A(\lambda, \alpha, q)}{\partial \alpha} = 0, \quad \frac{\partial A(\lambda, \alpha, q)}{\partial q} = 0 \]

where

\[ \frac{\partial A(\lambda, \alpha, q)}{\partial \lambda} = -\frac{1}{n} \sum_{i=1}^{n} (2i - 1) \left[ \frac{1}{\xi(i)} \alpha \xi(i)^{-1} (2 - q) \psi \frac{2(2 - q)}{1 - q} \right] x(i) \]

\[ \frac{\partial A(\lambda, \alpha, q)}{\partial \alpha} = -\frac{1}{n} \sum_{i=1}^{n} (2i - 1) \left[ \log \xi(i) + \log \xi(n+1-i) \right] \] (35)

\[ \frac{\partial A(\lambda, \alpha, q)}{\partial q} = -\frac{1}{n} \sum_{i=1}^{n} 2(2i - 1) \left\{ \frac{\alpha^{2(2-q)} \psi \log(\xi)}{\xi^{(q-1)}(1-q)} \psi + (2 - q)(1 - q) \lambda x(i) \right\} \]

\[ \frac{\alpha^{2(2-q)} \psi \log(\xi)}{\xi^{(q-1)}(1-q)} \psi + (2 - q)(1 - q) \lambda x(n+1-i) \}

(37)

where \( \psi_i = 1 - \lambda x(i)(1 - q) \), \( \psi_{n+1-i} = 1 - \lambda x(i)(1 - q) \)

These are the different estimates of Topp Leone Exponential(TLE) distribution and Topp Leone q Exponential(TLqE) distribution.

References

[1] Al-Shomrani, A., Arif, O., Shawky, K., Hanif, S. and Shahbaz, M.Q. (2016). Topp-Leone family of distributions: Some properties and application. Pak. J. Stat. Oper. Res., 12, 443-451.

[2] Casella, G.and Berger, R.L. (1990). Statistical Inference; Brooks/Cole Publishing Company: Bel Air, CA, USA.

[3] David, H.A. and Nagaraja, H.N. (2003). Order Statistics; John Wiley and Sons: Hoboken, NJ, USA.

[4] Reyad, H.M., Alizadeh, M., Jamal, F., Othman, S. and Hamedani, G.G. (2019). The Exponentiated Generalized Topp Leone-G Family of Distributions: Properties and Applications. Pak. J. Stat. Oper. Res. 15, 1-24.
[5] Macdonald, P.D.M. (1971). Comment on ‘An estimation procedure for mixtures of distributions’ by Choi and Bulgren. *J. R. Stat. Soc. B*, 33, 326–329.

[6] Sangsanit, Yuwadee, and Winai Bodhisuwan. (2016). The Topp-Leone generator of distributions: properties and inferences. *Songklanakarin Journal of Science and Technology* 38.5

[7] Swain, J.J., Venkatraman, S. and Wilson, J.R.(1988), Least-squares estimation of distribution functions in johnson’s translation system. *J. Stat. Comput. Simul.* 29, 271–297.

[8] Ramadan A. ZeinEldin, Christophe Chesneau, Farrukh Jamal and Mohammed Elgarhy (2019). Different Estimation Methods for Type I Half-Logistic Topp-Leone Distribution. *Mathematics*, 7(10), 985; https://doi.org/10.3390/math7100985 985.