AFFINE CELLULARITY OF QUANTUM AFFINE ALGEBRAS

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This is an appendix to Cui’s paper [3] showing that the modified quantum affine algebra $\tilde{U} = \tilde{U}_q(g)$ of level 0 (more precisely its quotients, BLN algebras) is affine cellular in the sense of Koenig and Xi [5]. The proof is based on the structure of cells of $\tilde{U}$, studied previously in [1], the author’s joint work with Beck. We here give a proof based on [1, Lemma 6.17], together with a property of the bilinear form introduced in [11]. Note that [1, Lemma 6.17] and the bilinear form are crucial ingredients for the study of the structure of cells in [1]. In this sense the following proof is more direct and fundamental than one in [3].

We also prove that cell ideals are idempotent, and hence [5, Th. 4.4] is applicable. Therefore BLN algebras are of finite global dimension, and its derived category admits a stratification whose sections are equivalent to derived categories of representation rings of products of general linear groups. The proof is, more or less, a simple observation once we remember that $\tilde{U}$ has the highest weight theory.

We also give a remark, which explain why it is natural to expect that $\tilde{U}$ is affine cellular, in view of geometry of quiver varieties, when the underlying affine Lie algebra $g$ is symmetric, i.e., an untwisted affine Lie algebra of type $ADE$. However, it should be emphasized that convolution algebras and $\tilde{U}$ (or its quotients) are possibly different, and we do not know whether convolution algebras are affine cellular or not unfortunately.

The existence of the affine cellular structure on $\tilde{U}$ is a simple consequence of [1, 11]. However a point is its usefulness and generality [5], and hence it is worthwhile to note that $\tilde{U}$ is affine cellular. It is this reason why we write this short note to emphasize this observation again after [3], and to clarify where the affine cellular structure come from for $\tilde{U}$. 
One of applications of the theory of affine cellular algebras is a classification of simple modules \([5, \S 3]\). For \(\widetilde{U}\), it reproduces a well-known classification, namely simple modules are parametrized by Drinfeld polynomials.

On the other hand, it is probably not previously known that BLN algebras are of finite global dimension and their derived categories admit stratification. Therefore it is really useful to point out that the theory of affine cellular algebras is applicable to \(\widetilde{U}\).

Our notation follows Cui’s paper \([3]\) and the author’s previous ones \([1, 11]\), as well as \([10]\) for geometric objects.

Acknowledgment. The author thanks Dr. Ryosuke Kodera for comments on an earlier version of this paper, e.g., the observation that \(i_*\) below is injective.

A(i). Laurent polynomial valued bilinear form. Let \(\lambda = \sum_{i \in I_0} m_i \varpi_i\) be a level 0 dominant weight, and \(V(\lambda)\) be the corresponding extremal weight module.

In \([1, \S 4]\) we introduced a \(U\)-homomorphism

\[ \Phi_\lambda : V(\lambda) \to \widetilde{V}(\lambda) = \bigotimes_{i} V(\varpi_i)^{\otimes m_i}, \]

sending \(u_\lambda\) to the tensor product \(\tilde{u}_\lambda \overset{\text{def}}{=} \bigotimes_{i} u_i^{\otimes m_i}\) of extremal weight vectors, and then analyzed the structure of \(V(\lambda)\) via \(\Phi_\lambda\). Each factor \(V(\varpi_i)\) has a \(U'\)-linear automorphism \(z_i\) of weight \(d_i \delta\). We introduce variables \(z_{i,\mu} (\mu = 1, \ldots, m_i)\) as the automorphism for the \(\mu^{th}\)-factor \(V(\varpi_i)^{\otimes m_i}\), and regard them as automorphisms of \(\widetilde{V}(\lambda)\).

Let \((,\,\,\,\,\,)\) be the bilinear form on \(V(\lambda)\) introduced in \([11, \S 4]\). Recall \((11, \S 4)\) that we have defined \(Q(q_s)[z_{i,\mu}^{\pm}]_{i \in I_0, \mu = 1, \ldots, m_i}\)-valued bilinear form \((,\,\,\,\,)\) on \(\widetilde{V}(\lambda)\), which is related to \((,\,\,\,\,)\) on \(V(\lambda)\) via \(\Phi_\lambda\) by

\[ (u,v) = \left[ \frac{1}{m_i!} \prod_{\nu \neq \mu} \left( 1 - z_{i,\mu} z_{i,\nu}^{-1} \right) \right]_1. \]

Here \([\,\,\,\,\,]\) denotes the constant term. From its definition, we also have

\[ (f(z)u,v) = f(z)((u,v)), \quad ((u,g(z)v)) = g(z^{-1})((u,v)), \]

where \(f, g\) are Laurent polynomials in \(z_{i,\mu}\), and \(g(z^{-1})\) means that we replace all variables \(z_{i,\mu}\) by \(z_{i,\mu}^{-1}\).

Lemma A.3. \((,\,\,\,\,)\) takes values in \(\bigotimes_{i \in I_0} Q(q_s)[z_{i,\mu}^{\pm}]_{\mu = 1, \ldots, m_i}\) on \(V(\lambda)\), where \(S_{m_i}\) is the symmetric group permuting \(z_{i,\mu}\) (\(\mu = 1, \ldots, m_i\)).
Proof. It is enough to show that \((G(b), G(b'))\) is symmetric for \(b, b' \in \mathcal{B}(\lambda)\). The assertion is clear from [1, Th. 4.16]. □

Remark A.4. When \(g\) is symmetric, the bilinear form \((, )\) is coming from the intersection pairing on the equivariant \(K\)-theory of quiver varieties by [13]. Therefore the existence of \((, )\) and Lemma A.3 are apparent in this context.

A(ii). **Affine cellularity.** Let \(\tilde{\mathcal{U}}\) be the modified quantum affine algebra of level 0. For a \(\lambda\) as above, let \(\tilde{\mathcal{U}}[\geq \lambda]\) be the two sided ideal consisting of all elements \(x \in \tilde{\mathcal{U}}\) acting on \(V(\lambda')\) by 0 for any \(\lambda' \not\geq \lambda\). We define \(\tilde{\mathcal{U}}[\lambda] = \tilde{\mathcal{U}}[\geq \lambda]/\tilde{\mathcal{U}}[\lambda]\) in the same way. We thus have a chain of two sided ideals in \(\tilde{\mathcal{U}}\) for various \(\lambda\)’s.

In [1, §6] we showed that \(\tilde{\mathcal{U}}[\geq \lambda], \tilde{\mathcal{U}}[\lambda]\) are compatible with the global crystal base of \(\tilde{\mathcal{U}}\), and give a description of the induced base of \(\tilde{\mathcal{U}}[\lambda] = \tilde{\mathcal{U}}[\geq \lambda]/\tilde{\mathcal{U}}[\lambda]\). In particular, it was shown that the induced base (or the underlying abstract crystal) is parameterized by

\[(A.5) \quad \mathcal{B}(\lambda) \times \text{Irr} \, G_{\lambda} \times \mathcal{B}(\lambda),\]

where \(\mathcal{B}(\lambda)\) is a certain finite set, \(G_{\lambda} = \prod_i GL(m_i)\), and \(\text{Irr} \, G_{\lambda}\) is the set of irreducible representations of \(G_{\lambda}\). The corresponding global base elements are of the form

\[G_{\lambda}(b, s, b') = G(b)S \, G(b')^{\#} \mod \tilde{\mathcal{U}}[\lambda],\]

where \(G(b)\) (resp. \(S\)) is the global base element of \(\tilde{\mathcal{U}}\) corresponding to \(b \in \mathcal{B}(\lambda)\) (resp. \(s \in \text{Irr} \, G_{\lambda}\)), and \(\#\) is a certain anti-involution of \(\tilde{\mathcal{U}}\).

Moreover the product \(\mathcal{B}(\lambda) \times \text{Irr} \, G_{\lambda}\) of the first and second factor in (A.5) is identified with the underlying set \(\mathcal{B}(\lambda)\) of the global base of \(V(\lambda)\) by

\[(b, s) \mapsto G(b)Su_{\lambda} \in V(\lambda).\]

Similarly the product of the second and third factor gives also the global base of \(V(\lambda)\) by \((s, b') \mapsto (S \, G(b')^{\#})^{\#} = G(b')^{\#}S^{\#}\), and \(S^{\#}\) corresponds to the dual representation of \(s\).

Multiplication of two global base elements are expressed by the bilinear form \((, )\) on \(V(\lambda)\) by

\[(A.6) \quad G_{\lambda}(b_1, s_1, b'_1)G_{\lambda}(b_2, s_2, b'_2) = q^n \sum_{s'' \in \text{Irr} \, G_{\lambda}} (G(b_2)S_2 u_{\lambda}, G(b'_1)S'_{u_{\lambda}}G(b_1)S_1S''u_{\lambda})G(b_1)S_1S''G(b'_2)^{\#} \mod \tilde{\mathcal{U}}[\lambda],\]
where \( n = (\text{wt } b'_1, 2\lambda + \text{wt } b'_1)/2 \). See [1, Lemma 6.17]. Our goal is to show that this immediately implies the affine cellularity thanks to a reformulation of \((\ , \ )\) in the previous subsection.

From (A.1), we have

\[
(G(b_2)S_2u_\lambda, G(b'_1)S''u_\lambda) = \left[ \frac{1}{m_i!} \prod_{\nu \neq \mu} (1 - z_{i,\mu}z_{i,\nu}^{-1}) \right]
\]

where we have used (A.2) in the second equality.

By [6, Chap. VI, §9],

\[
\left[ f(z) g(z^{-1}) \prod_{\mu \neq \nu} (1 - z_{\mu}z_{\nu}^{-1}) \right]
\]

is the standard inner product on the symmetric polynomials \( f, g \) of \( m \)-variables \( z = (z_1, \ldots, z_m) \). Since Schur functions gives an orthonormal base, we have

\[
\sum_{s'' \in \text{Irr } G_\lambda} (G(b_2)S_2u_\lambda, G(b'_1)S''u_\lambda)s''(z) = (G(b_2)u_\lambda, G(b'_1)u_\lambda)s_2(z).
\]

Therefore the right hand side of (A.6) is

\[
q^n G(b_1)S_1S_2((G(b_2)u_\lambda, G(b'_1)u_\lambda))G(b'_2)^# \mod \bar{U}[\rangle \lambda].
\]

This equality means that \( \bar{U}[\lambda] \) is a generalized algebra over \( R(G_\lambda) \), the representation ring of \( G_\lambda \), where the bilinear form \( \psi \) (appeared in [5, Prop. 2.2]) is \( (\bullet u_\lambda, \bullet u_\lambda) \). The other ingredients, the anti-involution \( i \) on \( \bar{U} \) is \( # \), and \( \sigma \) is the induced involution on \( R(G_\lambda) \), given by the dual representation.

Therefore \( \bar{U} \) satisfies the axioms of affine cellular algebras from [5] except that the chain of two-sided ideals has the infinite lengths. If we want to cut out to a finite chain, we just need to consider quotients of \( \bar{U} \), called BLN algebras as in [7].

A(iii). Idempotents. By [5, Th. 3.12] the affine cellularity of \( \bar{U} \) gives us a classification of its simple modules. More precisely, isomorphism classes of simple modules of \( \bar{U} \) are parametrized by the open subset of the set of maximal ideals \( m \in \text{MaxSpec } R(G_\lambda) \) such that \( (\ , \ ) \) is not identically zero on \( R(G_\lambda)/m \).
On the other hand, a classification of simple modules of $\tilde{\mathcal{U}}$ is well-known: it is the same as those of the usual quantum affine algebra $\mathcal{U}$, and is given by Drinfeld polynomials. It means that simple modules correspond to the whole MaxSpec $R(G_\lambda)$, not its proper open subset. We directly check this assertion in this section.

It is clear that the key is the value of $\langle \langle u, u \rangle \rangle$ at the extremal vector $u_\lambda$, as the Drinfeld polynomial is given by eigenvalues of a commuting family of elements in $\mathcal{U}$.

One of the defining property of $\langle \langle u, u \rangle \rangle$ is $\langle \langle u_\lambda, u_\lambda \rangle \rangle = 1$. From the definition of $\langle \langle u, u \rangle \rangle$, we have $\langle \langle u_\lambda, u_\lambda \rangle \rangle = 1$. Therefore the pairing $\langle \langle u, u \rangle \rangle$ is never zero, hence the condition $\langle \langle , \rangle \rangle$ is nonzero on $R(G_\lambda)/m$ is vacuous.

Thanks to [5, Th. 4.1(1)], this condition is equivalent to that all cell ideals $\tilde{\mathcal{U}}[\succ \lambda]$ are idempotent.

There is the distinguished element in $\mathcal{B}(\lambda)$, corresponding to $u_\lambda$. (We may assume that the $\text{Irr} G_\lambda$-component is the trivial representation 1 in the description $\mathcal{B}(\lambda) = \mathcal{B}_W(\lambda) \times \text{Irr} G_\lambda$.) As the global base element in $\tilde{\mathcal{U}}$, it is the projector $a_\lambda$ to the weight $\lambda$-space. In particular, it is an idempotent.

**Remark A.7.** In the geometric picture, $a_\lambda$ is the class of the diagonal $\Delta \mathcal{M}(\lambda, \lambda) \subset Z(\lambda)$, where $\mathcal{M}(\lambda, \lambda)$ is a distinguished component of $\mathcal{M}(\lambda)$, consisting of a single point.

Therefore two conditions required in [5, Th. 4.4] are satisfied for $\tilde{\mathcal{U}}$, or more precisely for its quotients, BLN algebras. Hence

**Theorem A.8.** A BLN algebra is of finite global dimension, and its derived category admits a stratification whose sections are equivalent to derived categories of $R(G_\lambda)$.

A(iv). **Approach via quiver varieties.** Suppose again that $\mathfrak{g}$ is symmetric. We follow the notation in [10]. Let $\mathcal{A}\tilde{\mathcal{U}}$ be the $\mathbb{Z}[q, q^{-1}]$-form of $\tilde{\mathcal{U}}$. Let $Z(\lambda)$ denote the analog of the Steinberg variety for quiver varieties, which is the fiber product $\mathcal{M}(\lambda) \times_{\mathcal{M}_0(\lambda)} \mathcal{M}(\lambda)$. The algebra homomorphism $\Phi_\lambda : \mathcal{A}\tilde{\mathcal{U}} \to K^{C^\times \times G_\lambda}(Z(\lambda))$, constructed in [8] factors through $\mathcal{A}\tilde{\mathcal{U}}/\mathcal{A}\tilde{\mathcal{U}}[\succ \lambda]$. (The notation $\Phi_\lambda$ has been used already above, but it should be clear from the context.) Remark that it has been shown that $\Phi$ is an algebra homomorphism to $K^{C^\times \times G_\lambda}(Z(\lambda))/\text{torsion}$ in [8]. The proof is a reduction to the case $\mathfrak{g}_0 = \mathfrak{sl}_2$. The reduction argument works without dividing by torsion. Therefore it is enough to
check the relation holds for $g_0 = \mathfrak{sl}_2$. In this case, the corresponding quiver varieties are cotangent bundles to Grassmannians (of various dimensions). It is known that $K^{C^* \times G_\lambda}(Z(\lambda))$ is free [2, Cor. 6.2.6]. Therefore it is unnecessary to divide by torsion.

Let us consider the open embedding $j: Z^0(\lambda) \to Z(\lambda)$, the inverse image of $M_0(\lambda) \setminus \{0\}$ in the fiber product. The complement $Z(\lambda) \setminus Z^0(\lambda)$ is $\mathfrak{L}(\lambda) \times \mathfrak{L}(\lambda)$, where $\mathfrak{L}(\lambda)$ is the lagrangian subvariety in $M(\lambda)$, the inverse image of $0$ under $M(\lambda) \to M_0(\lambda)$. Let $i: \mathfrak{L}(\lambda) \times \mathfrak{L}(\lambda) \to Z(\lambda)$ be the closed immersion. We have pull-back $j^*$ and push-forward $i_*$ homomorphisms, which fits in the commutative diagram such that both horizontal sequences are exact:

\[
\begin{array}{cccccc}
K^{C^* \times G_\lambda}(\mathfrak{L}(\lambda) \times \mathfrak{L}(\lambda)) & \xrightarrow{i_*} & K^{C^* \times G_\lambda}(Z(\lambda)) & \xrightarrow{j^*} & K^{C^* \times G_\lambda}(Z^0(\lambda)) & \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{A}\tilde{U}[\lambda] = \mathcal{A}\tilde{U}[\geq\lambda] / \mathcal{A}\tilde{U}[>\lambda] & \to & \mathcal{A}\tilde{U} / \mathcal{A}\tilde{U}[>\lambda] & \to & \mathcal{A}\tilde{U} / \mathcal{A}\tilde{U}[\geq\lambda] & \to 0.
\end{array}
\]

The statement that the restriction of $\Phi_\lambda$ to $\mathcal{A}\tilde{U}[\lambda]$ is proved as follows: As we have explained above, $\mathcal{A}\tilde{U}[\lambda]$ has a base consisting of elements $G_\lambda(b, s, b') = G(b)S\lambda(b')^\# \mod \mathcal{U}[>\lambda]$. The element $S$ is identified with an irreducible representation of $G_\lambda$, and is sent to the class $R(C^* \times G_\lambda) = K^{C^* \times G_\lambda}(\mathfrak{L}(\lambda, \lambda) \times \mathfrak{L}(\lambda, \lambda))$, where $\mathfrak{L}(\lambda, \lambda) = M(\lambda, \lambda)$ is the distinguished component of $M(\lambda)$, consisting of a single point, mentioned above. Then from the definition of the convolution product, $K^{C^* \times G_\lambda}(\mathfrak{L}(\lambda) \times \mathfrak{L}(\lambda))$ is a bimodule, and hence the assertion follows.

And it also follows that $j^* \circ \Phi$ factors through $\mathcal{A}\tilde{U} / \mathcal{A}\tilde{U}[>\lambda]$.

Moreover, Künneth formula holds for $\mathfrak{L}(\lambda)$ [9, Th. 3.4], hence

\[
K^{C^* \times G_\lambda}(\mathfrak{L}(\lambda) \times \mathfrak{L}(\lambda)) \cong K^{C^* \times G_\lambda}(\mathfrak{L}(\lambda)) \otimes_{R(C^* \times G_\lambda)} K^{C^* \times G_\lambda}(\mathfrak{L}(\lambda)).
\]

Since $K^{C^* \times G_\lambda}(\mathfrak{L}(\lambda))$ is isomorphic to the extremal weight module $V(\lambda)$, it follows that $\clubsuit$ is an isomorphism. Therefore $\mathcal{A}\tilde{U}[\lambda]$ has a structure of a generalized matrix algebra, and hence we see why $\tilde{U}$ is affine cellular.

Let us turn to the convolution algebra $K^{C^* \times G_\lambda}(Z(\lambda))$. We have a stratification $M_0(\lambda) = \bigsqcup \mathcal{M}_0^{\text{reg}}(\mu, \lambda)$, where $\mu$ runs the set of (level 0) dominant weights with $\mu \leq \lambda$. The closure order is the opposite of the dominance order. Let $Z(\lambda) = \bigsqcup Z(\lambda)_\mu$ be the corresponding decomposition of $Z(\lambda)$, and let $Z(\lambda)_{\geq\mu} = \bigsqcup Z(\lambda)_{\mu' \geq \mu}$, $Z(\lambda)_{>\mu} = \bigsqcup Z(\lambda)_{\mu' > \mu}$. Both are closed subvarieties in $Z(\lambda)$. And $Z(\lambda)_\mu = \bigsqcup Z(\lambda)_{\mu' \leq \mu}$.
$Z(\lambda)_{\geq \mu} \setminus Z(\lambda)_{>\mu}$ is open in $Z(\lambda)_{\geq \mu}$. We consider a variant of (A.9):

\[
K^{C^* \times G\lambda}(Z(\lambda)_{\geq \mu}) \xrightarrow{i_*} K^{C^* \times G\lambda}(Z(\lambda)_{\geq \mu}) \rightarrow K^{C^* \times G\lambda}(Z(\lambda)_{\mu}) \rightarrow 0
\]

\[
0 \rightarrow \mathcal{A}\tilde{U}[>\mu]/\mathcal{A}\tilde{U}[\geq \mu] \rightarrow \mathcal{A}\tilde{U}[>\mu]/\mathcal{A}\tilde{U}[>\mu] \rightarrow \mathcal{A}\tilde{U}[\mu] \rightarrow 0.
\]

Lemma A.10. $i_*$ becomes injective, if we divide domain and target by $R(\mathbb{C}^*)$-torsion.

Proof. If we consider the fixed point $Z(\lambda)^{C^*}$, it is contained in $Z(\lambda)_{\lambda} = (\mathcal{L}(\lambda) \times \mathcal{L}(\lambda))^{C^*}$. It is because the action of $\mathbb{C}^*$ on an affine space containing $\mathcal{M}_0(\lambda)$ has only positive weights, and hence 0 is the only fixed point in $\mathcal{M}_0(\lambda)$. Since the projective morphism $\pi: \mathcal{M}(\lambda) \rightarrow \mathcal{M}_0(\lambda)$ is $\mathbb{C}^*$-equivariant, the assertion follows.

Now the localization theorem in the equivariant $K$-theory implies that $i_*$ becomes an isomorphism if we localize the equivariant $K$-group at $\text{Frac}(R(\mathbb{C}^*))$, the fractional field of $R(\mathbb{C}^*) = \mathbb{Z}[q, q^{-1}]$. We are done, as the kernel of $\bullet \rightarrow \bullet \otimes \text{Frac}(R(\mathbb{C}^*))$ is the torison part. \(\square\)

We ignore the torsion part hereafter.

The homomorphism $K^{C^* \times G\lambda}(Z(\lambda)_{\geq \mu}) \rightarrow K^{C^* \times G\lambda}(Z(\lambda))$, which is injective by above, is compatible with the convolution product. Therefore $K^{C^* \times G\lambda}(Z(\lambda)_{>\mu})$ is a two-sided ideal. The same holds for $K^{C^* \times G\lambda}(Z(\lambda)_{>\mu})$.

Therefore we need to analyze $K^{C^* \times G\lambda}(Z(\lambda)_{\mu})$ to show that $K^{C^* \times G\lambda}(Z(\lambda))$ is affine cellular.

For the original Steinberg variety of type $A$, we consider a short exact sequence

\[
0 \rightarrow K^{C^* \times G}(Z_{\overline{\mathcal{O}}}) \rightarrow K^{C^* \times G}(\mathcal{O}) \rightarrow K^{C^* \times G}(Z_{\mathcal{O}}) \rightarrow 0,
\]

where $O$ is a nilpotent orbit and $\overline{O}$ is its closure. (See [12] for the relevance of the above short exact sequence for the structure of cells of affine Hecke algebras of type $A$.) In this case, $Z_{\mathcal{O}}$ is a fiber bundle over $O$ whose fiber at $e$ is $\mathcal{B}_e \times \mathcal{B}_e$, where $\mathcal{B}_e$ is the Springer fiber at $e$. In the quiver variety, $Z(\lambda)_{\mu}$ is a fiber bundle whose fibers are isomorphic to $\mathcal{L}(\mu) \times \mathcal{L}(\mu)$. However the base $\mathcal{M}_0^{\text{reg}}(\mu, \lambda)$ is not an orbit of $G\lambda$, and hence we need a further study.

If we replace equivariant $K$-group by equivariant homology groups, we still have a similar diagram, where the bottom row is replaced by Yangian. But we still need to analyze $H^{C^* \times G\lambda}_* (Z(\lambda)_{\mu})$. S. Kato has studies this problem in his study of extension algebra [4]. In his case, $Z^*o(\lambda)$ is replaced by an union of orbits, and hence the picture is similar to the case of the original Steinberg variety. We will come back to this problem in near future.
It is also clear from our perspective that we need to care extra stratum, not of a form $\mathcal{M}_0^\text{reg}(\mu, \lambda)$ in $\mathcal{M}_0(\lambda)$ for quiver varieties of infinite types. For example, symmetric powers of $\mathbb{C}^2/\Gamma$ appear for affine types. Here $\Gamma$ is a finite subgroup of $SU(2)$.

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