ON HERMITE–HADAMARD TYPE INEQUALITIES FOR HARMONICAL $h$–CONVEX INTERVAL–VALUED FUNCTIONS

DAFANG ZHAO, TIANQING AN, GUOJU YE AND DELFIM F. M. TORRES

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Abstract. We introduce and investigate the concept of harmonical $h$-convexity for interval-valued functions. Under this new concept, we prove some new Hermite–Hadamard type inequalities for the interval Riemann integral.

1. Introduction

The following inequality is known in the literature as the Hermite–Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2},$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval $I$ and $a, b \in I$ with $a < b$. For various interesting extensions and generalizations of this inequality, see [7, 23, 25]. In 2014, İşcan introduced the concept of harmonical convexity and established some Hermite–Hadamard type inequalities for this class of functions [9]. Some further refinements of such inequalities, for harmonical convex functions, have been studied in [10, 12, 22]. In 2015, Noor et al. introduced the class of harmonical $h$-convex functions and established some Hermite–Hadamard type inequalities [21]. For some recent investigations on harmonical $h$-convexity, we refer the interested readers to [1, 15, 16].

On the other hand, interval analysis and interval-valued functions were initially introduced in numerical analysis by Moore in the celebrated book [18]. Because of its wide applications in various fields, interval analysis has emerged as a very useful research area over the last fifty years: see, e.g., [4, 5, 19] and references therein. Recently, several classical integral inequalities have been extended not only to the context of interval-valued functions by Chalco-Cano et al. [2, 3], Román-Flores et al. [24], Flores-Franulič et al. [8], Costa and Román-Flores [6], but also to more general set-valued maps by Klaričić Bakula and Nikodem [11], Matkowski and Nikodem [14], Mitroi et al. [17], and Nikodem et al. [20].

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Our research is mainly motivated by the results of İlşan [9] and Noor et al. [21]. We begin by introducing the notion of harmonical $h$-convexity for interval-valued functions. Then we prove some new Hermite–Hadamard type inequalities for the introduced class of functions. Our inequalities are interval-valued counterparts of the results from [9, 21].

The paper is organized as follows. After Section 2 of preliminaries, in Section 3 the harmonical $h$-convexity concept for interval-valued functions is given and new Hermite–Hadamard type inequalities are proved. We end with Section 4 of conclusions and future work.

2. Preliminaries

We begin by recalling some basic definitions, notation and properties, which are used throughout the paper. A real interval $[u]$ is the bounded, closed subset of $\mathbb{R}$ defined by

$$[u] = [\underline{u}, \overline{u}] = \{x \in \mathbb{R} | \underline{u} \leq x \leq \overline{u}\},$$

where $\underline{u}, \overline{u} \in \mathbb{R}$ and $\underline{u} \leq \overline{u}$. The numbers $\underline{u}$ and $\overline{u}$ are called the left and right endpoints of $[\underline{u}, \overline{u}]$, respectively. When $\underline{u}$ and $\overline{u}$ are equal, the interval $[\underline{u}]$ is said to be degenerated. In this paper, the term interval will mean a nonempty interval.\[u\] is called positive if $\underline{u} > 0$ or negative if $\overline{u} < 0$. The inclusion “$\subseteq$” is defined by

$$[\underline{u}, \overline{u}] \subseteq [\underline{v}, \overline{v}] \iff \underline{v} \leq \underline{u}, \overline{u} \leq \overline{v}.$$

For an arbitrary real number $\lambda$ and $[u]$, the interval $\lambda [u]$ is given by

$$\lambda [u, \overline{u}] = \begin{cases} \lambda \underline{u}, \lambda \overline{u} & \text{if } \lambda > 0, \\ \{0\} & \text{if } \lambda = 0, \\ \lambda \overline{u}, \lambda \underline{u} & \text{if } \lambda < 0. \end{cases}$$

For $[u] = [\underline{u}, \overline{u}]$ and $[v] = [\underline{v}, \overline{v}]$, the four arithmetic operators are defined by

$$[u] + [v] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}],$$

$$[u] - [v] = [\underline{u} - \overline{v}, \overline{u} - \underline{v}],$$

$$[u] \cdot [v] = [\min \{\underline{u} \underline{v}, \underline{u} \overline{v}, \overline{u} \underline{v}, \overline{u} \overline{v}\}, \max \{\underline{u} \underline{v}, \underline{u} \overline{v}, \overline{u} \underline{v}, \overline{u} \overline{v}\}],$$

$$[u]/[v] = [\min \{\underline{u}/\underline{v}, \underline{u}/\overline{v}, \overline{u}/\underline{v}, \overline{u}/\overline{v}\}, \max \{\underline{u}/\underline{v}, \underline{u}/\overline{v}, \overline{u}/\underline{v}, \overline{u}/\overline{v}\}],$$

where $0 \notin [\underline{v}, \overline{v}]$.

We denote by $\mathbb{R}_\geq$ the set of all intervals of $\mathbb{R}$, and by $\mathbb{R}_+ \geq$ and $\mathbb{R}_- \geq$ the set of all positive intervals and negative intervals of $\mathbb{R}$, respectively. The Hausdorff–Pompeiu distance between intervals $[\underline{u}, \overline{u}]$ and $[\underline{v}, \overline{v}]$ is defined by

$$d([\underline{u}, \overline{u}], [\underline{v}, \overline{v}]) = \max \left\{|\underline{u} - \underline{v}|, |\overline{u} - \overline{v}|\right\}.$$

It is well known that $(\mathbb{R}_\geq, d)$ is a complete metric space.
DEFINITION 1. (See [13]) Let \( f : [a, b] \to \mathbb{R} \) be such that \( f(t) = [\overline{f(t)}, \underline{f(t)}] \) for each \( t \in [a, b] \), and \( \overline{f}, \underline{f} \) are Riemann integrable on \([a, b]\). Then we say that \( f \) is Riemann integrable on \([a, b]\) and denote
\[
\int_{a}^{b} f(t) dt = \left[ \int_{a}^{b} \overline{f(t)} dt, \int_{a}^{b} \underline{f(t)} dt \right].
\]
The collection of all interval-valued functions that are \( R \)-integrable on \([a, b]\) will be denoted by \( \mathcal{I}_R([a, b]) \).

We end this section of preliminaries by recalling some useful known concepts.

DEFINITION 2. (See [9]) We say that \( K_h \subset \mathbb{R} \setminus \{0\} \) is a harmonical convex set if
\[
\frac{xy}{tx + (1-t)y} \in K_h
\]
for all \( x, y \in K_h \) and \( t \in [0, 1] \).

DEFINITION 3. (See [21]) Let \( h : [0, 1] \subset J \to \mathbb{R} \) be a non-negative function with \( h \neq 0 \), and \( K_h \) a harmonical convex set. We say that \( f : K_h \to \mathbb{R} \) is a harmonical \( h \)-convex function if
\[
f \left( \frac{xy}{tx + (1-t)y} \right) \leq h(t)f(x) + h(1-t)f(y)
\]
for all \( x, y \in K_h \) and \( t \in [0, 1] \).

Note that if \( h(t) = t \), then function \( f \) is called a harmonical convex function [9]; if \( h(t) = 1 \), then \( f \) is called a harmonical \( P \)-convex function [21]; while, if \( h(t) = t' \), then \( f \) is called a harmonical \( s \)-convex function [21].

3. Main results: new Hermite–Hadamard type inequalities

In this section, we prove new Hermite–Hadamard type inequalities for harmonical \( h \)-convex interval-valued functions.

DEFINITION 4. Let \( h : [0, 1] \subset J \to \mathbb{R} \) be a non-negative function such that \( h \neq 0 \), and \( K_h \) a harmonical convex set. We say that \( f : K_h \to \mathbb{R}_+ \) is a harmonical \( h \)-convex interval-valued function if
\[
h(t)f(x) + h(1-t)f(y) \leq f \left( \frac{xy}{tx + (1-t)y} \right)
\]
for all \( x, y \in K_h \) and \( t \in [0, 1] \). If the set inclusion (1) is reversed, then \( f \) is said to be a harmonical \( h \)-concave interval-valued function. The set of all harmonical \( h \)-convex and harmonical \( h \)-concave interval-valued functions are denoted by \( SX(h, K_h, \mathbb{R}_+) \) and \( SV(h, K_h, \mathbb{R}_+) \), respectively.
The next theorem is an interval-valued counterpart of \[21\] Theorem 3.2.

**Theorem 1.** Let \( f : K_h \to \mathbb{R}_+^I \) be an interval-valued function with \( a < b \) and \( a, b \in K_h \), \( f \in \mathcal{F}_{[a,b]} \), and let \( h : [0,1] \to (0,\infty) \) be a continuous function. If \( f \in SX(h,K_h,\mathbb{R}_+^I) \), then

\[
\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2ab}{a+b}\right) \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \supseteq \left[ f(a) + f(b) \right] \int_0^1 h(t) dt.
\]

If \( f \in SV(h,K_h,\mathbb{R}_+^I) \), then

\[
\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2ab}{a+b}\right) \subset \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \subseteq \left[ f(a) + f(b) \right] \int_0^1 h(t) dt.
\]

**Proof.** We first assume that \( f \in SX(h,K_h,\mathbb{R}_+^I) \). Then one has

\[
h\left(\frac{1}{2}\right) f(x) + h\left(\frac{1}{2}\right) f(y) \subseteq f\left(\frac{xy}{\frac{x}{a} + \frac{y}{b}}\right) = f\left(\frac{2xy}{x+y}\right).
\]

Let

\[
x = \frac{ab}{ta+(1-t)b}, \quad y = \frac{ab}{tb+(1-t)a}.
\]

Then,

\[
h\left(\frac{1}{2}\right) \left[ f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{tb+(1-t)a}\right) \right] \subseteq f\left(\frac{2ab}{a+b}\right).
\]

 Integrating both sides of inequality (2) over \([0,1]\), we have

\[
\int_0^1 f\left(\frac{2ab}{a+b}\right) dt = \left[ \int_0^1 f\left(\frac{2ab}{a+b}\right) dt, \int_0^1 \mathcal{F}\left(\frac{2ab}{a+b}\right) dt \right]
\]

\[
= f\left(\frac{2ab}{a+b}\right)
\]

\[
\supseteq h\left(\frac{1}{2}\right) \int_0^1 \left[ f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{tb+(1-t)a}\right) \right] dt
\]

\[
= h\left(\frac{1}{2}\right) \left[ \int_0^1 \left( f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{tb+(1-t)a}\right) \right) dt, \int_0^1 \left( \mathcal{F}\left(\frac{ab}{ta+(1-t)b}\right) + \mathcal{F}\left(\frac{ab}{tb+(1-t)a}\right) \right) dt \right]
\]

\[
= h\left(\frac{1}{2}\right) \left[ \frac{2ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx, \frac{2ab}{b-a} \int_a^b \mathcal{F}\left(\frac{f(x)}{x^2}\right) dx \right]
\]

\[
= 2h\left(\frac{1}{2}\right) \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx.
\]
This implies that
\[
\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2ab}{a+b}\right) \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx.
\]

The proof of the second relation follows by using (1) with \(x = a\) and \(y = b\) and integrating with respect to \(t\) over \([0,1]\), that is,
\[
\frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \supseteq \left[ f(a) + f(b) \right] \int_0^1 h(t) \, dt.
\]

The intended result follows. If \(f \in SV(h, K_h, \mathbb{R}^+_p)\), then the proof is similar and is left to the reader. □

**Remark 1.** If \(h(t) = t^s\), then Theorem 1 gives a result for harmonical \(s\)-functions:
\[
2^{s-1} f\left(\frac{2ab}{a+b}\right) \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \supseteq \frac{1}{s+1} \left[ f(a) + f(b) \right]. \tag{3}
\]

If \(h(t) = t\), then Theorem 1 gives a result for harmonical convex functions:
\[
f\left(\frac{2ab}{a+b}\right) \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \supseteq \frac{f(a) + f(b)}{2}. \tag{4}
\]

If \(h(t) = 1\), then Theorem 1 gives a result for harmonical \(P\)-functions:
\[
\frac{1}{2} f\left(\frac{2ab}{a+b}\right) \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \supseteq f(a) + f(b). \tag{5}
\]

**Theorem 2.** Let \(f : K_h \to \mathbb{R}^+_p\) be an interval-valued function with \(a < b\) and \(a, b \in K_h\), \(f \in \mathcal{I}R(a,b)\), and let \(h : [0,1] \to (0,\infty)\) be a continuous function. If \(f \in SV(h, K_h, \mathbb{R}^+_p)\), then
\[
\frac{1}{4h\left(\frac{1}{2}\right)^2} f\left(\frac{2ab}{a+b}\right) \supseteq \Delta_1 \supseteq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \supseteq \Delta_2
\]
\[
\supseteq \left[ f(a) + f(b) \right] \left[ \frac{1}{2} + h\left(\frac{1}{2}\right) \right] \int_0^1 h(t) \, dt,
\]
where
\[
\Delta_1 = \frac{1}{4h\left(\frac{1}{2}\right)^2} \left[ f\left(\frac{4ab}{a+3b}\right) + f\left(\frac{4ab}{3a+b}\right) \right],
\]
and
\[
\Delta_2 = \left[ \frac{f(a) + f(b)}{2} + f\left(\frac{2ab}{a+b}\right) \right] \int_0^1 h(t) \, dt.
\]

If \(f \in SV(h, K_h, \mathbb{R}^+_p)\), then the opposite signs of inclusion are valid in the above formulas.
Proof. We only give the proof of the first part of Theorem 2. Since $f \in \mathcal{S}(h, K_h, \mathbb{R}_+^2)$, we have

$$f\left(\frac{2xy}{x+y}\right) \supseteq h\left(\frac{1}{2}\right)[f(x) + f(y)]$$

for all $x, y \in K_h$ and $t = \frac{1}{2}$. Choosing

$$x = \frac{a\frac{2ab}{a+b}}{ta + (1-t)\frac{2ab}{a+b}}, \quad y = \frac{a\frac{2ab}{a+b}}{t\frac{2ab}{a+b} + (1-t)a},$$

we get

$$f\left(\frac{4ab}{a+3b}\right) \supseteq h\left(\frac{1}{2}\right)\left[ f\left(\frac{a\frac{2ab}{a+b}}{ta + (1-t)\frac{2ab}{a+b}}\right) + f\left(\frac{a\frac{2ab}{a+b}}{t\frac{2ab}{a+b} + (1-t)a}\right) \right].$$

Integrating both sides of the above inequality over $[0, 1]$, we have

$$f\left(\frac{4ab}{a+3b}\right) \supseteq h\left(\frac{1}{2}\right)\left[ \int_0^1 f\left(\frac{a\frac{2ab}{a+b}}{ta + (1-t)\frac{2ab}{a+b}}\right) + f\left(\frac{a\frac{2ab}{a+b}}{t\frac{2ab}{a+b} + (1-t)a}\right) dt, \right.\
\left. \int_0^1 \int_0^{\frac{2ab}{a+b}} f(x) x^2 dx, \int_0^{\frac{2ab}{a+b}} \int_0^{\frac{2ab}{a+b}} f(x) x^2 dx \right]$$

$$= h\left(\frac{1}{2}\right)\frac{4ab}{b-a}\left[ \int_0^{\frac{2ab}{a+b}} f(x) x^2 dx, \int_0^{\frac{2ab}{a+b}} \int_0^{\frac{2ab}{a+b}} f(x) x^2 dx \right]$$

$$= h\left(\frac{1}{2}\right)\frac{4ab}{b-a}\int_0^{\frac{2ab}{a+b}} f(x) x^2 dx.$$

Similarly, we have

$$f\left(\frac{4ab}{3a+b}\right) \supseteq h\left(\frac{1}{2}\right)\frac{4ab}{b-a}\int_0^{\frac{2ab}{a+b}} f(x) x^2 dx.$$  

Consequently, we get

$$f\left(\frac{2ab}{a+b}\right) \supseteq h\left(\frac{1}{2}\right)\left[ f\left(\frac{4ab}{a+3b}\right) + f\left(\frac{4ab}{3a+b}\right) \right] = 4 \left[ h\left(\frac{1}{2}\right) \Delta_1 \right]^2$$

$$\supseteq 4 \left[ h\left(\frac{1}{2}\right) \Delta_1 \right] \frac{ab}{b-a}\int_0^b f(x) x^2 dx.$$
Thanks to Theorem 1, 

\[
\frac{1}{4} \left[ \frac{(a+b)^2}{2} \right]^2 \int_a^b \frac{f(x)}{x^2} \, dx \geq \Delta_1 \geq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx
\]

\[
\geq \frac{1}{2} \left[ \frac{2ab}{b-a} \int_a^{\frac{ab}{a+b}} \frac{f(x)}{x^2} \, dx + \frac{2ab}{b-a} \int_{\frac{ab}{a+b}}^b \frac{f(x)}{x^2} \, dx \right]
\]

\[
\geq \frac{1}{2} \left[ f(a) + f(b) + 2f\left(\frac{2ab}{a+b}\right) \right] \int_0^1 h(t) \, dt = \Delta_2
\]

\[
\geq \left[ \frac{f(a) + f(b)}{2} + h\left(\frac{1}{2}\right) f(a) + h\left(\frac{1}{2}\right) f(b) \right] \int_0^1 h(t) \, dt
\]

\[
= [f(a) + f(b)] \left[ \frac{1}{2} + h\left(\frac{1}{2}\right) \right] \int_0^1 h(t) \, dt
\]

and the result follows. \( \square \)

**Remark 2.** Like in Remark 1 from Theorem 2, we obtain particular results for harmonical convex, harmonical \( P \)-convex, and harmonical \( s \)-convex functions.

**Theorem 3.** Let \( f, g : K_h \rightarrow \mathbb{R}^+ \) be interval-valued functions with \( a < b \) and \( a, b \in K_h \), \( f, g \in \mathcal{J}(a, b) \), and \( h_1, h_2 : [0, 1] \rightarrow (0, \infty) \) be continuous functions. If \( f \in SX(h_1, K_h, \mathbb{R}^+) \), \( g \in SX(h_2, K_h, \mathbb{R}^+), \), then

\[
\frac{ab}{b-a} \int_a^b \frac{f(x)g(x)}{x^2} \, dx \subseteq M(a, b) \int_0^1 h_1(t)h_2(t) \, dt + N(a, b) \int_0^1 h_1(t)h_2(1-t) \, dt,
\]

where

\[
M(a, b) = f(a)g(a) + f(b)g(b), \quad N(a, b) = f(a)g(b) + f(b)g(a).
\]

If \( f \in SV(h_1, K_h, \mathbb{R}^+), \) \( g \in SV(h_2, K_h, \mathbb{R}^+), \) then

\[
\frac{ab}{b-a} \int_a^b \frac{f(x)g(x)}{x^2} \, dx \subseteq M(a, b) \int_0^1 h_1(t)h_2(t) \, dt + N(a, b) \int_0^1 h_1(t)h_2(1-t) \, dt.
\]

**Proof.** By hypothesis, one has

\[
f\left(\frac{ab}{ta + (1-t)b}\right) \geq h_1(t)f(a) + h_1(1-t)f(b),
\]

\[
g\left(\frac{ab}{ta + (1-t)b}\right) \geq h_2(t)g(a) + h_2(1-t)g(b).
\]

Then,

\[
f\left(\frac{ab}{ta + (1-t)b}\right)g\left(\frac{ab}{ta + (1-t)b}\right)
\]

\[
\geq h_1(t)h_2(t)f(a)g(a) + h_1(t)h_2(1-t)f(a)g(b)
\]

\[
+ h_1(1-t)h_2(t)f(b)g(a) + h_1(1-t)h_2(1-t)f(b)g(b).
\]

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Integrating both sides of the above inequality over $[0, 1]$, we have
\[
\int_0^1 f\left(\frac{ab}{ta + (1-t)b}\right) g\left(\frac{ab}{ta + (1-t)b}\right) dt
= \int_0^1 \left[ f\left(\frac{ab}{ta + (1-t)b}\right) g\left(\frac{ab}{ta + (1-t)b}\right) dt, \int_0^1 \frac{g\left(\frac{ab}{ta + (1-t)b}\right)}{f\left(\frac{ab}{ta + (1-t)b}\right)} dt \right]
= \left[ \frac{ab}{b-a} \int_a^b f(x)g(x) dx, \frac{ab}{b-a} \int_a^b \frac{g(x)}{x^2} dx \right]
= \frac{ab}{b-a} \int_a^b f(x)g(x) dx
\geq f(a)g(a) \int_0^1 h_1(t)h_2(t) dt + f(a)g(b) \int_0^1 h_1(t)h_2(1-t) dt
+ f(b)g(a) \int_0^1 h_1(1-t)h_2(t) dt + f(b)g(b) \int_0^1 h_1(1-t)h_2(1-t) dt
= M(a,b) \int_0^1 h_1(t)h_2(t) dt + N(a,b) \int_0^1 h_1(t)h_2(1-t) dt.
\]
This concludes the proof. □

**Remark 3.** Similarly as before, from Theorem 4 we obtain particular results for harmonical convex, harmonical $P$-convex, and harmonical $s$-convex functions.

**Theorem 4.** Let $f, g : K_h \to \mathbb{R}_x^+$ be interval-valued functions with $a < b$, where $a, b \in K_h$ and $f, g \in \mathcal{R}_{\mathcal{R}}(a, b)$, and $h_1, h_2 : [0, 1] \to (0, \infty)$ be continuous functions. If $f \in SX(h_1, K_h, \mathbb{R}_x^+$) and $g \in SX(h_2, K_h, \mathbb{R}_x^+$), then
\[
\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} \int_a^b f(x)g(x) dx \geq \frac{ab}{b-a} \int_a^b f(x)g(x) dx
+ M(a,b) \int_0^1 h_1(t)h_2(1-t) dt + N(a,b) \int_0^1 h_1(t)h_2(t) dt. \quad (6)
\]
If $f \in SX(h_1, K_h, \mathbb{R}_x^+$) and $g \in SX(h_2, K_h, \mathbb{R}_x^+$), then previous formula (6) holds with the opposite sign of inclusion.

**Proof.** Let $\xi = \frac{2ab}{a+b}$. By hypothesis, one has
\[
f(\xi) \geq h_1\left(\frac{1}{2}\right) f\left(\frac{ab}{ta + (1-t)b}\right) + h_1\left(\frac{1}{2}\right) f\left(\frac{ab}{ib + (1-t)a}\right),
g(\xi) \geq h_2\left(\frac{1}{2}\right) g\left(\frac{ab}{ta + (1-t)b}\right) + h_2\left(\frac{1}{2}\right) g\left(\frac{ab}{ib + (1-t)a}\right).
\]
Then,
\[ f(\xi)g(\xi) \]
\[ \geq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \left[ f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right) \right] \]
\[ + h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \left[ f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{ta+(1-t)b}\right) \right] \]
\[ \geq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \left[ f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right) \right] \]
\[ + h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \left[ (h_1(t)f(a) + h_1(1-t)f(b))(h_2(1-t)g(a) + h_2(t)g(b)) \right] \]
\[ + (h_1(1-t)f(a) + h_1(t)f(b))(h_2(t)g(a) + h_2(1-t)g(b)) \]
\[ = h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \left[ f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right) \right] \]
\[ + h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \left[ (h_1(t)h_2(1-t) + h_1(1-t)h_2(t))M(a,b) \right] \]
\[ + (h_1(t)h_2(t) + h_1(1-t)h_2(1-t))N(a,b) \].

Integrating over \([0,1]\), we have
\[
\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f(\xi)g(\xi) \geq \frac{ab}{b-a} \int_a^b \frac{f(x)g(x)}{x^2} dx \]
\[ + M(a,b) \int_0^1 h_1(t)h_2(1-t)dt + N(a,b) \int_0^1 h_1(t)h_2(t)dt. \]

This concludes the proof. \(\square\)

**Remark 4.** From Theorem 4, we obtain particular results for harmonical convex, harmonical \(P\)-convex, and harmonical \(s\)-convex functions.

### 4. Conclusions

We introduced the new concept of harmonical \(h\)-convexity for interval-valued functions. Some interesting Hermite–Hadamard type inequalities for harmonical \(h\)-convex interval-valued functions have then been proved. Our results give interval-valued counterparts of the inequalities presented by İşcan and Noor et al., respectively in [9] and [21].

Further developments are possible. As a future research direction, we intend to investigate Hermite–Hadamard type inequalities for harmonical \(h\)-convex interval-valued functions on arbitrary time scales.
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Dafang Zhao  
College of Science, Hohai University, Nanjing, Jiangsu 210098, China  
School of Mathematics and Statistics, Hubei Normal University, Huangshi, Hubei 435002, China  
e-mail: dafangzhao@163.com

Tianqing An  
College of Science, Hohai University, Nanjing, Jiangsu 210098, China  
e-mail: antq@hhu.edu.cn

Guojia Ye  
College of Science, Hohai University, Nanjing, Jiangsu 210098, China  
e-mail: yegj@hhu.edu.cn

Delfim F. M. Torres  
Center for Research and Development in Mathematics and Applications (CIDMA),  
Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal  
e-mail: delfim@ua.pt

Corresponding Author: Delfim F. M. Torres