WZNW Models from Non-Standard Bilinear Forms

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Abstract

We study the WZNW models based on nonstandard bilinear forms. We approach the problem from algebraic, perturbative and functional exact methods. It is shown that even in the case of integer $k$ we can find irrational CFT’s. We prove that when the base group is noncompact with nonabelian maximal compact subgroup, the Kac-Moody representations are nonunitary.
1 Introduction:

Certain WZNW models do not allow the standard formulation based on the Killing form as the bilinear form in their algebra used to define the action. These models are based on a group $G$ whose Killing form is singular (so $G$ is a non-semisimple lie group). In some cases such as centrally extended non-semisimple groups existence of another invariant bilinear form \cite{1} comes to our help, enabling us to construct a nonsingular invariant bilinear form to be used in the definition of the action \cite{2}. Such WZNW models have been used to construct cosmological models. A non-semisimple group $E_{2}^{c}$ in this manner is used to describe a homogeneous space with central charge four.

These models were approached in another way in \cite{3} and were generalized to include $E_{d}^{c}$ (centrally extended Euclidean group in $d$ dimensions) \cite{4} and double extension of the lie groups \cite{5}. Also it was shown that the Sugawara construction can be extended to such models with an arbitrary invariant nondegenerate bilinear form \cite{6}.

In this paper we consider several aspects of these non standard models. We show that conformal invariance is achieved in a similar way to the standard case. In the Sugawara construction of the Virasoro generators one needs to know the inner product of two currents. This is given by a bilinear form of the group which is subject to the Affine-Virasoro Master Equation (AVME) \cite{7}. We show that certain symmetry requirements on the Sugawara construction restrict the solutions of the AVME to a unique form in agreement with \cite{6}. From this solution one can find the Virasoro’s central charge \cite{6}.

As a special case we consider $SO(3, 1)$ group to exemplify the general arguments. $SO(3, 1)$ has more than one bilinear form with the desired properties. We find central charges cor-
responding to these nonstandard bilinear forms, then look for representations of the corresponding Kac-Moody algebras. We also find the conformal weights. Since the Kac-Moody level must be an integer for the Lorentz group, the standard WZNW models based on it can have only rational conformal central charges and weights, in contrast to the present case with real central charges and weights which is due to extra parameter in the nonstandard bilinear form. This is interesting that despite the fact that $k$ must be an integer we can obtain irrational central charges and noninteger effective level.

On the other hand, WZNW models based on non-compact groups have also been considered. In [8] it is shown that $SU(1, 1)$ theory is nonunitary, but latter works [9, 10] showed that string models for this group is indeed unitary thanks to Virasoro conditions (no-ghost theorem). There is an argument in [12] for nonunitarity of the Kac-Moody of $SO(3, 1)$ in large limit of the level. In this paper we generalize these results to the representations of any Kac-Moody algebra based on a non-compact group with nonabelian maximal compact subgroup. Whether despite nonunitarity of Kac-Moody representations unitary Virasoro representations can be extracted shall be addressed elsewhere.

In the second part of this paper we treat the model perturbatively, finding the $\beta$-function and generating functional to all orders of perturbations. A non trivial question about the path integral is whether path integral measure receives any changes due to the change of bilinear form, indeed we show in the appendix that the choice of bilinear form in the path integral measure is independent of the choice of bilinear form in the action. Finally in this approach we calculate the trace anomaly. We also analyze the model by functional method obtaining the interesting result of the renormalization of the bilinear form which corresponds to $k$ renormalization in ordinary WZNW models.
2 General Properties; Conformal and Affine Structures:

Let us consider a WZNW action on a Riemann surface $\Sigma$ whose topology is $S^2$:

$$ S = \frac{1}{4\lambda^2} \int_{\Sigma} d^2x Tr(\partial_{\mu} U \partial_{\mu} U^{-1}) + \frac{k}{12\pi} \int_{B} d^3x \epsilon^{\mu\nu\rho} Tr(\partial_{\mu} U U^{-1} \partial_{\nu} U U^{-1} \partial_{\rho} U U^{-1}). $$  \hfill (2.1)

$B$ is a three-dimensional manifold whose boundary is $\Sigma$, and $U$ is a map from $\Sigma$ (or its extension $B$) to the group $G$. The path integral should be independent of the extension of $U$ to $B$. When $\Pi_3(G)$, the third homotopy group of $G$ is nontrivial, parameter $k$ in the action is restricted. On the other hand, if $\Pi_3(G) = 0$ there is no restriction on $k$ and it can be any real number. The topology of a non-compact group $G$ is the same as the topology of $H \times \mathbb{R}^{d_c - d_H}$ in which $H$ is maximal compact subgroup, implying $\Pi_3(G) = \Pi_3(H)$. When $H$ is abelian, $\Pi_3(H) = 0$ and $k$ can be any real number, and when $\Pi_3(H) = \mathbb{Z}$ it is restricted to be an integer.

If we take $U^{-1} \partial_{\mu} U = A_{\mu}^a T_a$ in which $T_a$’s are generators of the group $G$; the WZNW action can be written as:

$$ S = \frac{1}{2\lambda^2} \int_{\Sigma} d^2x \Omega_{ab} A_{\mu}^a A_{\mu}^b + \frac{k}{12\pi} \int_{B} d^3x \epsilon^{\mu\nu\rho} \Omega_{cd} f_{ab}^d A_{\mu}^a A_{\nu}^b A_{\rho}^c, $$  \hfill (2.2)

where $f_{ab}^c$ is structural constant and $\Omega_{ab}$ is Cartan-Killing metric of the group $G$. $\Omega_{ab}$ is a bilinear form which can be constructed from

$$ - c_v \Omega_{ab} = f_{ac}^d f_{bd}^c, $$  \hfill (2.3)

(or if $T$’s are taken in the fundamental representation $\Omega_{ab} = -2 Tr(T_a T_b)$ ) in which $c_v$ is the second Casimir of the group.

In a more compact notation one can represent the action as:

$$ S = \frac{1}{2\lambda^2} \int_{\Sigma} d^2x < A_{\mu}, A_{\mu}>_\Omega + \frac{k}{12\pi} \int_{B} d^3x \epsilon^{\mu\nu\rho} < [A_{\mu}, A_{\nu}], A_{\rho}>_\Omega, $$  \hfill (2.4)
where \( \langle x, y \rangle_\Omega = \Omega_{ij} x^i y^j \) is an inner product of \( x \) and \( y \), two vectors in the Lie algebra of the group.

We can generalize the standard WZNW model by considering a bilinear form \( M \) instead of \( \Omega \) which has to have three properties: Firstly, \( M \) is a symmetric bilinear form:

\[
\langle x, y \rangle_M = \langle y, x \rangle_M, \tag{2.5}
\]

which is a property of real inner products; secondly, it is an invariant of the group which will guarantee the affine symmetry of the model: \( \langle U x U^{-1}, U y U^{-1} \rangle_M = \langle x, y \rangle_M \) (for all \( U \) in \( G \)), or equivalently \( \langle [x, y], z \rangle_M + \langle y, [x, z] \rangle_M = 0 \). This condition can be expressed as:

\[
f_{ab}^d M_{cd} + f_{ac}^d M_{bd} = 0, \tag{2.6}
\]

where, \( \langle T_i, T_j \rangle_M = M_{ij} \); and finally, it has an inverse \( M^{ab} \), i.e.:

\[
M^{ab} M_{bc} = \delta^a_c. \tag{2.7}
\]

For several simple and semi-simple groups \( M \) has a unique form which is just \( \Omega \) the Cartan-Killing form, however, there are some groups with more than one bilinear form with the above properties. Example is \( SO(3,1) \) or \( SL(2,\mathbb{C}) \) group.

Consider the algebras of \( SO(3,1) \):

\[
[J_a, J_b] = \epsilon_{abc} J_c \tag{2.8}
\]

\[
[J_a, K_b] = \epsilon_{abc} K_c \tag{2.9}
\]

\[
[K_a, K_b] = -\epsilon_{abc} J_c, \tag{2.10}
\]

\( \Omega \) can be derived to be:

\[
\Omega = \begin{pmatrix} 1_3 & 0 \\ 0 & -1_3 \end{pmatrix} \tag{2.11}
\]
here $1_3$ is the $3 \times 3$ unit matrix; and $c_v = 4$. The first(second) three values of $i$ and $j$ in $\Omega_{ij}$ are corresponding to $J$’s ($K$’s). This Killing-form corresponds to the first Casimir of the group:

$$J^2 - K^2 = \Omega_{ij} T^i T^j. \tag{2.12}$$

Another invariant bilinear form is introduced via the other Casimir:

$$2J \cdot K = \Omega'_{ij} T^i T^j, \tag{2.13}$$

where

$$\Omega' = \begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix} \tag{2.14}$$

One could take $M$ to be any linear combination of these two bilinear forms:

$$M = \Omega + \alpha \Omega', \tag{2.15}$$

where $\alpha$ is a constant.

Furthermore, for a non-semi-simple group the Killing-form is degenerate and the corresponding WZNW action is ill-defined, nonetheless for the most general known cases of double extended groups in [3], it is possible to find another bilinear form with required properties (2.3, 2.6, 2.7). $M$ can be introduced as a linear combination of these two bilinear forms that is non-degenerate, invariant and symmetric.

Using $M$ the WZNW action could be written as:

$$S = \frac{1}{2\lambda^2} \int_{\Sigma} d^2x M_{ab} A^a_{\mu} A^b_{\mu} + \frac{k}{12\pi} \int_B d^3x \epsilon^{|\mu|\nu|\rho} M_{cd} f_{ab}^d A^a_{\mu} A^b_{\nu} A^c_{\rho}. \tag{2.16}$$

Here topological nature of WZ term is important. As in the standard WZNW models (2.2), we need to find the third homotopy group of $H$ the maximal compact subgroup,
because the change of bilinear form on $G$ can not change its topological properties. The topological term restricts the values of $k$ to integers when $\Pi_3(G) = \Pi_3(H)$ is nontrivial. For example in the case of $SL(2,\mathbb{C})$ where $H$ is $SU(2)$ restriction of $M$ to $H$ is just $\Omega$ and hence $k$ is integer, however, there is no restriction on $\alpha$.

Now we are ready to derive algebras from symmetries of the model. First, let us introduce conserved currents;

$$J_{+a} = k < T_a, U^{-1} \partial_+ U >_M .$$

(2.17)

Following [13], one can find the Kac-Moody algebra from the action (2.16) by calculating Poisson brackets of currents. The resulting algebra can be stated as an OPE;

$$J^a_+(z)J^b_+(w) = \frac{kM_{ab}}{(z-w)^2} + \frac{f^c_{ab}J^c(w)}{z-w} + \cdots .$$

(2.18)

In a traditional ansatz energy-momentum tensor could be constructed from normal ordering of two currents:

$$T(z) = L^{ab} : J_a J_b : (z) .$$

(2.19)

In order to have a conformal field theory energy-momentum tensor and currents have to satisfy the following OPE:

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \partial J^a(w) - \cdots .$$

(2.20)

From the OPE’s (2.18) and (2.20), one finds that the coefficients $L^{ab}$ are restricted to the solutions of the Affine-Virasoro Master-Equation (AVME):

$$L^{ab} = kL^{ac}M_{cd}L^{db} - L^{cd}L^{ef}f^a_{ce}f^b_{df} - (L^{cd}f^a_{ce}f^b_{df}L^{be} + (a \leftrightarrow b)).$$

(2.21)
find Virasoro’s central charge from OPE’s of the energy-momentum tensors:

\[ T(z)T(w) = \frac{c/2}{(z-w)} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots. \]  

(2.22)

to be:

\[ c = kM_{ab}L^{ab}. \]  

(2.23)

Now we solve the general AVME (2.21) exactly by supposing that \( L \) is a symmetric and group invariant form (i.e. equation (2.6) holds when replacing \( M \) by \( L \)). The latter assumption is legitimate because the energy-momentum tensor does not take any change when one changes the currents by the action of the group\(^1\). Using this invariance (2.6 for \( L \)) to AVME:

\[ L^{ab} = kL^{ac}M_{cd}L^{db} + L^{cd}L^{ef}f^f_{ce}f^b_{df} - (L^{cd}f^f_{ce}f^a_{df}L^{be} + (a \leftrightarrow b)) \]

\[ = kL^{ac}M_{cd}L^{db} - L^{cd}f^f_{ce}f^a_{df}L^{be} \]

\[ = kL^{ac}M_{cd}L^{db} - L^{ad}f^f_{ce}f^c_{df}L^{be}, \]

and using (2.3), we get:

\[ L = (kM + c_v\Omega)^{-1}. \]  

(2.24)

This result is very pleasant. It can be considered as renormalization of \( kM \) corresponding to renormalization of \( k \) in the ordinary WZNW models. As a special case for semi-simple groups one can choose \( M \) to be \( \Omega \) then from (2.24):

\[ L = \frac{\Omega^{-1}}{k + c_v} \]  

(2.25)

\(^1\)This assumption is violated when one considers only the action of some subgroup in coset-constructions.
which is just the Sugawara construction. More interesting is the $SO(3, 1)$ group with $M$
defined in (2.13):

$$L_{SO_{M}(3, 1)} = \frac{1}{k + c_v} \frac{1}{1 + \eta^2} \begin{pmatrix} 1_3 & \eta 1_3 \\ \eta 1_3 & -1_3 \end{pmatrix}$$  \hspace{1cm} (2.26)

where $\eta = \alpha k / (k + c_v)$, and by (2.23) the Virasoro central charge is:

$$c = 6 \frac{k(k + c_v)}{(k + c_v)^2 + \alpha^2 k^2} + \alpha^2 k^2.$$  \hspace{1cm} (2.27)

Notice that $k$ is an integer but there is no restriction on $\alpha$, thus the above expression
indicates that we can find both irrational and rational central charges which were impossible
for $SO_{\Omega}(3, 1)$ models.

The solution (2.24) also recovers the results of [14] and is in agreement with [8].

The next step is to find the conformal weights of the Kac-Moody primary fields. First
let us construct the Virasoro operator $L_0$ for $SO_M(3, 1)$:

$$L_0 = L^{ab} J_{0a} J_{0b}$$  \hspace{1cm} (2.28)

$$= L^{ab} T_{0a} T_{0b}$$  \hspace{1cm} (2.29)

$$= \frac{k + c_v}{(k + c_v)^2 + k^2 \alpha^2} (J^2 - K^2) + \frac{2k \alpha}{(k + c_v)^2 + k^2 \alpha^2} (J \cdot K).$$  \hspace{1cm} (2.30)

In an irreducible representation of the group, Casimirs of the group are just $c$ numbers,
so any state in an irreducible representation of the group is an eigenstate for $L_0$. As in [13]
an irreducible representation is characterized by two parameters $j_0$ and $j_1$, where $j_0$
is an integer or half integer number which denotes the lowest spin in this representation and $j_1$
is an arbitrary complex number. In terms of these parameters the Casimirs of the group are:

$$J^2 - K^2 = 1 - (j_0^2 + j_1^2),$$  \hspace{1cm} (2.31)
\[ J \cdot K = i j_0 j_1. \] (2.32)

In this notation, conformal weights for \((j_0, j_1)\) representations are:

\[ \Delta = \frac{k + c_v}{(k + c_v)^2 + k^2 \alpha^2} \left\{ 1 - (j_0^2 + j_1^2) \right\} + \frac{2k\alpha}{(k + c_v)^2 + k^2 \alpha^2} (i j_0 j_1). \] (2.33)

Let us look at unitary representations of the Lorentz group. First consider the main series in which \(j_0\) is arbitrary and \(j_1\) is purely imaginary. Putting \(j_1 = is\):

\[ \Delta_1 = \frac{k + c_v}{(k + c_v)^2 + k^2 \alpha^2} (1 + s^2 - j_0^2) - \frac{2k\alpha}{(k + c_v)^2 + k^2 \alpha^2} (j_0 s). \] (2.34)

Other unitary representations of the Lorentz group are supplementary representations in which \(j_0 = 0\) and \(j_1\) is a real number such that \(0 < |j_1| \leq 1\), in this case conformal weights are:

\[ \Delta_2 = \frac{k + c_v}{(k + c_v)^2 + k^2 \alpha^2} (1 - j_1^2). \] (2.35)

In the supplementary representations one can introduce the following parameter which we call it effective \(k\):

\[ k_{\text{eff}} = k + \frac{k^2 \alpha^2}{k + c_v}. \] (2.36)

In terms of this new parameter conformal weights for supplementary representations and central charges can be expressed as follows:

\[ \Delta_2 = \frac{1}{k_{\text{eff}} + c_v} (1 - j_1^2), \] (2.37)

\[ c = \frac{6k_{\text{eff}}}{k_{\text{eff}} + c_v}. \] (2.38)

These are just the expressions for \(\Delta\) and \(c\) in the absence of \(\alpha\) which is the ordinary Virasoro algebra based on the \(SO_\Omega(3, 1)\). It means that the conformal structures of \(SO_\Omega(3, 1)\) and \(SO_M(3, 1)\) are partially homeomorphic; ”partially” because these effective \(k_{\text{eff}}\) could
be introduced only in supplementary representations. In the main representations one can not find such a reparameterization to transform central charges and conformal weights simultaneously to their values in the standard $SO_{\Omega}(3,1)$. This shows that these theories are generally different from standard models.

Another aspect of the expressions in (2.27) and (2.33) is degeneracy of central charges and conformal weights which means that for a large set of $k$ and $\alpha$ one can obtain a common central charges and conformal weights. More explicitly, by transforming $(k, \alpha)$ to $(k', \alpha')$ as:

$$k' = mk + c_v(m - 1),$$
$$\alpha'^2 = m\alpha^2 - m(m - 1)\frac{(k + c_v)^2}{k^2},$$

for a non-zero integer parameter $m$, the same central charges can be obtained for $(k', \alpha')$ as for $(k, \alpha)$, and again for the supplementary representations the conformal weights are the same for $(k', \alpha')$ and $(k, \alpha)$.

Non-Unitarity in Representations:

Here we shall prove a theorem on the unitarity of representations of the Kac-Moody algebra based on a non-compact group.

We shall see that unlike the case of compact base groups where we have a finite number of unitary representations, in general we have no unitary representation of the Kac-Moody algebra based on non-compact Lie groups.

Let $G$ be a non-compact group, and $H$ be its maximal compact subgroup. Suppose Kac-Moody algebra based on the group $G$ has a highest weight unitary representation on a vector space $V$.

The vectors in $V$ form a unitary representation $\Lambda$ of $G$ for the lowest value of $L_0$ which
is infinite dimensional since $G$ is non-compact. $V$ must also carry a unitary representation of $\hat{H}$ level $k$. It is well known that unitary representations of $\hat{H}$ are specified by highest weights of $H$ belonging to:

$$\Gamma_H = \{ \lambda | \lambda \cdot \alpha_H \leq k \}$$

(2.41)

where $\alpha_H$'s are the positive roots of $H$ which for the moment are assumed to be non-zero.

As it can be seen $\Gamma_H$ is a finite set. This contradicts the well known fact that $\Lambda$ includes an infinite set of inequivalent representations of $H$. Hence the unitarity of representations of $\hat{G}$ and $\hat{H}$ are contradictory.

On the other hand, when $\alpha_H = 0$ there is no restriction on the representation of $H$ which is now an abelian subgroup. The above statements can be summarized in the following theorem:

For a Kac-Moody algebra based on a noncompact group, with a non-abelian maximal compact subgroup, highest weight unitary representations do not exist.

According to this theorem the commutative nature of the maximal compact subgroup is a necessary condition for the existence of a unitary representation for the Kac-Moody algebra\(^2\).

\(^2\)The sufficient condition is not obvious. However, there are some well known examples such as $SU(1, 1)$ and $E_6^\ast$ which are non compact with an abelian maximal compact subgroup and admit unitary representations \([1, 14]\).
3 Effective Action, Perturbative Results and Functional Approach:

In the last section we had an algebraic approach to the problem. In this section we reproduce some of our results in the previous section among other things using a path integral formulation. This will shed light on the intricacies involved in the nonstandard bilinear form. In particular it will make clear why the master equation has the solution (2.24). For a path integral formulation of WZNW models, firstly consider the WZNW action on a curved manifold $\Sigma$:

$$S\{U|\gamma\} = \frac{1}{4\lambda^2} \int_{\Sigma} d^2x \sqrt{\gamma} \gamma^{\mu\nu} \partial_{\mu}U\partial_{\nu}U^{-1} >_{\text{M}} + k\Gamma_{WZ}$$

(3.1)

where $\Gamma_{WZ}$ is Wess-Zumino term (the three dimensional integral in (2.4)) and $\gamma^{\mu\nu}$ and $\gamma$ are the inverse and the determinant of the metric tensor, respectively.

For this chiral model let us look at the holomorphic part of the theory. To find correlation functions for the holomorphic currents $J$, one can introduces sources $l_{\mu}^a(x)$, and add a source term to the action as:

$$S\{U|\gamma, l\} = S\{U|\gamma\} + \int d^2x \sqrt{\gamma} l_{\mu}^a(x) J_{a}^{\mu}(x).$$

(3.2)

The effective action of the model can be introduced by the following path integral:

$$e^{-S_{eff}(\gamma,l)} = \int DU e^{-S(U|\gamma,l)}.$$  

(3.3)

All the connected correlation functions for currents and energy-momentum tensor could be derived by taking derivatives of $S_{eff}$ with respect to $l$ and/or $\gamma^{\mu\nu}$.

To study quantum theory it is convenient to set:

$$U(x) = U_{cl}(x)U_q(x),$$

(3.4)
in the action, where $U_q$ is quantum fluctuation around the classical field $U_{cl}$ which satisfies classical equations of motion. By applying generalization of Polyakov-Wiegmann equation \[16\], we obtain:

$$S\{U|\gamma,l\} = S\{U_{cl}|\gamma,l\} + S\{U_q|\gamma,0\} + 2i \int d^2x \gamma^\mu \gamma^\nu < R_\mu, \partial_\nu U_q U^{-1} >_M$$

(3.5)

where $R_\mu$ is the classical antiholomorphic current which equals to zero by using the equation of motion \[17\]. Therefore, we can write:

$$S_{eff}\{\gamma,l\} = S_{cl}\{\gamma,l\} + S_q\{\gamma\},$$

(3.6)

in which $S_{cl}$ depends on both metric and sources explicitly, and $S_q\{\gamma\}$ is quantum effective action defined by:

$$e^{S_q}\{\gamma\} = \int DU_q e^{S_q\{U|\gamma\}},$$

(3.7)

and does not depend on sources at all, so all the correlation functions can be obtained from the classical action. This feature has a crucial role in solubility of the WZNW models which also exists in our extended case.

Despite this fact we still need to calculate $S_q$ since it gives the energy momentum tensor and conformal anomaly.

Since current-current correlation functions could be found by taking derivatives of $S_{eff}$ with respect to $l_\mu(x)$, the quantum part $S_q$ does not contribute to these correlation functions, i.e. all loop corrections will cancel out \[17\]. Firstly, let us calculate these current-current correlation functions when the external sources vanish:

$$< J^a_\mu(x) J^b_\nu(y) >_{conn.} = -2\pi k M^{ab} \partial_\mu \partial_\nu G(x,y).$$

(3.8)

Note that this is a connected correlation function.
Turning on the external sources we find the correlation function:

\[
<J^a_\mu(x)J^b_\nu(y)> = \frac{\delta S_{\text{eff}}}{\delta b^\mu_a(x)} \frac{\delta S_{\text{eff}}}{\delta b^\nu_b(y)} - \frac{\delta^2 S_{\text{eff}}}{\delta b^\mu_a(x)\delta b^\nu_b(y)}.
\]  

(3.9)

It consists of disconnected (first term) and connected (second term) parts of the correlation function. After some calculation one finds:

\[
<J^a_+(x)J^a_+(y)>_{\text{disc.}} = 2k^2 <\partial_+ U^1_{cl}(y), \partial_+ U^{-1}_{cl}(y)>_M, \]  

(3.10)

\[
<J^a_+(x)J^a_+(y)>_{\text{conn.}} = -2\pi kM^{ab} M_{ab} \partial^\mu U(x, y) + 2kc_v <\partial_+ U, \partial_+ U^{-1}>_\Omega. \]  

(3.11)

The first term in the second equation is just the connected contribution in the absence of the external fields, and the last term renormalizes the disconnected part by renormalizing \(kM\) in the disconnected part to \(kM + c_v\Omega\). It means that turning on the external field only renormalizes the disconnected part. This renormalization is a reflection of the Sugawara construction and the solubility of the model.

In order to derive the \(\beta\)-function and the energy-momentum tensor, we try to find quantum corrections via the quantum generating functional \(S_q\). We put \(U_q = \exp(i\lambda \pi)\) in which \(\pi = \pi^a(x)T_a\) and expand it around unity:

\[
U_q = 1 + i\lambda \pi - \frac{\lambda^2}{2}\pi^2 - i\frac{\lambda^3}{6}\pi^3 + \cdots. \quad (3.12)
\]

Using \(A^a_\mu T_a = U^{-1}\partial_\mu U\) and the above expression in the action (2.16) to the fourth order in \(\pi\) one finds the following lagrangian density:

\[
L = -\frac{1}{2}M_{ab}\partial_\mu \pi^a \partial_\mu \pi^b - \frac{\lambda^2}{24}M_{ab}f^a_{mn} f^b_{kl} \partial_\mu \pi^m \partial_\nu \pi^n \partial_\rho \pi^k \partial_\sigma \pi^l + \frac{k\lambda^3}{12\pi} \epsilon^{\mu\nu} M_{cd} f^d_{ab} \partial_\mu \pi^a \partial_\nu \pi^b \pi^c. \]  

(3.13)

The three dimensional integration on manifold \(B\) is converted to a two dimensional term (last term in (3.13)) by divergence theorem.
From the above lagrangian we can find Feynman rules for perturbative calculations. These rules include two factors. The ordinary part which includes the effects of numerical constants and derivative operators, and a second part which consists of group theoretic tensorial structure.

The main difference between the two WZNW models in (2.2) and (2.16) is their tensorial properties which will be manifested in the second part of the Feynman rules. So, let us consider the group theoretic or the tensorial structural of the Feynman rules:

\[
\begin{array}{c}
\text{a} \\
\text{m} \\
\text{n} \\
\text{b}
\end{array}
\begin{array}{c}
\text{M}^{ab} \\
M_{ab} f^{a}_{mn} f^{b}_{kl} \\
M_{cd} f^{d}_{ab}
\end{array}
\]

The β-function could be derived considering the quantum corrections to the two point propagator. For first order of approximation, consider one loop corrections in fig.1.a and b. Corresponding expressions for each graphs are:

\[
\Pi^{ab}_{1}(x_1, x_2) = \left(\frac{k\lambda^3}{12\pi}\right)^2 c_v M^{ac} \Omega^{bd} \int d^2 x d^2 y \sqrt{\gamma(x)} \sqrt{\gamma(y)} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \\
\times \partial_{\mu} G(x, y) \partial_{\rho} G(x, y) G(x_1, x) G(y, x_2),
\]

\[
\Pi^{ab}_{2}(x_1, x_2) = -\frac{\lambda^2}{24} c_v M^{ac} \Omega^{bd} \int d^2 x \sqrt{\gamma(x)} \partial_{\mu} G(x_1, x) \partial_{\rho} G(x, x_2) G(x_1, x) G(x, x_2).
\]

From which after regularization the β-function could be derived as:

\[
\beta^{ab} = -\frac{\lambda^2 c_v}{4\pi} M^{ac} \Omega^{bd} \left\{ 1 - \left(\frac{k\lambda^2}{2\pi}\right)^2 \right\}.
\]

\[\text{Notice that for complete expressions we consider permutations of partial derivatives on Green functions, but for brevity we write here only one form of these permutations.}\]

\[\text{Regularizations of such integrals including two dimensional } \epsilon \text{ tensors are fully developed in [18].}\]
Its fixed point will be at \( k = \frac{2\pi}{\lambda} \) the same as the ordinary WZNW.

\[ a \quad b \quad c \]

Figure 1. One Loop (Propagators and Generating Functional).

In order to calculate \( S_q \), let us restrict ourselves to this fixed point. In the first order of perturbation we obtain:

\[ e^{S_q} = \int d\pi e^{\int d^2 \sqrt{\gamma} \langle \partial_\mu \pi, \partial^\nu \pi \rangle M}, \quad (3.18) \]

from which, it is evident that:

\[ S_q^{(1)} \{ \gamma \} = \text{dim}(G) D\{ \gamma \} \quad (3.19) \]

where:

\[ D\{ \gamma \} = \frac{1}{2}(\text{Indet}'(-\Delta) - \ln V), \quad (3.20) \]

in which \( \Delta \) and \( V \) are the laplacian and volume of the manifold \( \Sigma \), respectively, and prime denotes that the zero modes of the laplacian are to be omitted when evaluating the determinant. This one loop result corresponds to the graph in fig.1.c.

For two loops, we need to consider graphs in fig.2. For each graph we have a contribution to the generating functional:

\[ S_a = \frac{\pi c_v}{3 k} Tr(M^{-1}\Omega) \int d^2 x \gamma [\partial^\nu G(x,y) \partial_\nu G(x,y) - G(x,y) \partial^\nu G(x,y) \partial_\nu G(x,y)]_{x=y}, \quad (3.21) \]

\[ S_b = \frac{2\pi c_v}{3 k} Tr(M^{-1}\Omega) \int d^2 x d^2 y \gamma e^{\mu\nu} e^{\rho\sigma} \partial_\mu \partial_\nu G(x,y) \partial_\rho \partial_\sigma G(x,y) G(x,y) \quad (3.22) \]
which make sense after regularization. Moreover, we ought to add a contribution from the path integral measure due to changing variables from $U$ to $\pi$. The measure contribution $S_m$ to the quantum effective action has been calculated in the appendix to be:

$$S_m = \frac{\pi c_v}{3k} \text{Tr}(M^{-1}\Omega) \int d^2x \gamma G(x, y)[\delta(x, x) + \frac{2}{V}].$$  \hspace{1cm} (3.23)

As stated in the appendix, the path integral measure can be constructed by any non-degenerate bilinear form $\Xi$, however, $S_m$ and hence $S_q$ and its derivatives do not depend on the choice of $\Xi$.

\hspace{1cm}

Figure 2. Two Loops.

To regularize the above expressions for $S_a$, $S_b$ and $S_m$ we use the method of [17] with new bilinear form $M$. Putting all together we find the generating functional up to two loops order:

$$S_q^{(2)} = S_q^{(1)} + S_a + S_b + S_m$$ \hspace{1cm} (3.24)

$$= D\{\gamma\} \text{Tr}[(M^{-1} - \frac{c_v}{k}M^{-1}\Omega)M].$$

In higher loops, by using the three properties of $M$ (2.5, 2.6, 2.7), it is possible to transform the tensorial part of each graph at each loop order into a unique form. This means that the choice of different bilinear form consistent with (2.5, 2.6, 2.7) and addition of the tensorial part to the Feynman rules does not effect the solubility of the model. This is manifested in (3.24) when we find the two loop generating functional to be the same as one loop result up
to a multiplicative factor.

Figure 3. Three Loops.

Let us look at the three loops as an illustration. There are eight 3-loops graphs as in fig.3. One can easily check by using the properties of $M$ and the Jacobi identity for structural constants that all of these graphs could be converted to a unique tensorial form as follows:

$$\left(\frac{c_v}{k}\right)^2 Tr[M^{-1}\Omega M^{-1}\Omega M].$$

In an obvious manner, we generalize this expression to N-loop:

$$\left(\frac{c_v}{k}\right)^{N-1} Tr[(M^{-1}\Omega)^{N-1} M].$$

(3.25)

Taking account of the numerical coefficients, this is manifestly the trace of Nth-term in the expansion of $(kM + c_v\Omega)^{-1}$ times $M$, so:

$$S_q = D\{\gamma\} Tr[k(kM + c_v\Omega)^{-1}M].$$

(3.26)

In each order for $S_q$, the results of [17] could be recovered by replacing $M$ by $\Omega$ in $S_q$ for a semi-simple group. By using (2.23) we obtain:

$$S_q = c D\{\gamma\}. $$

(3.27)
Thus, in this category of WZNW, we have found the same expression for $S_q$ as the ordinary WZNW models in terms of the central charge $c$ and $D\{\gamma\}$, from which we find the trace anomaly to all orders of perturbation to be [17]:

$$T_\mu^\mu = \frac{c}{24\pi} R,$$

(3.29)

where $R$ is the two dimensional scalar curvature. This result is the same as that the ordinary WZNW models based on semi-simple groups.

The renormalization of $kM$ can also be obtained by the functional method. Consider the partition function of the model:

$$Z = \int dU e^{kS_M\{U|\gamma\}}.$$  

(3.30)

Change the variables of integration from $U$ to $A$:

$$A^a_+ T_a = U^{-1} \partial_+ U$$

(3.31)

which introduces a Jacobian in the path integral measure:

$$dU = dA det\left(\frac{\partial U}{\partial A_+}\right).$$

(3.32)

From (3.31) the determinant can be found to be [19]:

$$det\left(\frac{\partial U}{\partial A_+}\right) = det[\Delta_+(A)]$$

(3.33)

where $\Delta_+(A) = \partial_+ + [A_+, ]$ and can be calculated using ghost fields integration technique:

$$det[\Delta_+(A)] = \int db dce^{<b,\Delta_+ c>z}$$

(3.34)

$$= \int db dce^{<b,\partial_+ c>_z + <b,[A_+,c]>z}.$$  

(3.35)
We shall apply bosonization techniques \[19\] to calculate the above determinant.

From the $bc$ fields one can find corresponding conserved current to be:

$$J^{(gh)}_{ij} = f^{i}_{m}b^{n}c^{m}\Xi_{ij}. \quad (3.36)$$

These currents will obey the Kac-Moody algebra with $c_{v}$ as the central charge by the following OPE:

$$J^{(gh)}_{+a}(z)J^{(gh)}_{+b}(w) = \frac{c_{v}\Omega_{ab}}{(z-w)^{2}} + \frac{f^{c}_{ab}J^{(gh)}_{++c}(w)}{z-w} + \cdots. \quad (3.37)$$

So, the effective action for the ghost fields will be the WZNW action for gauge fields $A$, i.e.:

$$det[\Delta_{+}(A)] = e^{c_{v}S_{\Omega}(A|\gamma)} \quad (3.38)$$

where,

$$S_{\Omega}(A|\gamma) = \int d^{2}\gamma \gamma^{\mu\nu} < A_{\mu}, A_{\nu} >_{\Omega} + \Gamma_{WZ} \quad (3.39)$$

whose current algebra is just (3.37). It is worth noting that as long as $\Xi$ is nondegenerate, (3.34) is independent of $\Xi$.

Putting these results into the path integral one find the following partition function:

$$Z = \int dA e^{kS_{M}(A)+c_{v}S_{\Omega}(A)}. \quad (3.40)$$

It manifestly demonstrates the $kM$ renormalization to $kM + c_{v}\Omega$.

4 Conclusions

In this work WZNW models for noncompact and non-semisimple lie groups are developed.

Firstly for non-compact groups we proved an important theorem that the highest weight
representations of Kac-Moody algebras for these groups are nonunitary when maximal compact subgroup of the base group is nonabelian. On the other hand, in such cases the third homotopy group of the lie group is nontrivial which leads to integer Kac-Moody central charges \( k \). However, using a nonstandard bilinear form which couples compact and non-compact subspaces as in the case of \( SO(3, 1) \), we obtained irrational conformal weights and central charges.

Hence, introducing nonstandard bilinear forms is useful in developing new conformal theories in the case of non-compact groups and is necessary in the case of non-semisimple groups. In this relation we treated the model perturbatively and showed that \( \beta \)-function vanishes at the same points as in standard models.

Renormalization of \( kM \) was investigated in different approaches. Firstly, it is due to the renormalization of disconnected graphs in current-current correlations. Secondly, by calculating quantum generating functional of two dimensional metric to all orders of perturbation we found the renormalized \( kM \) in the Sugawara form. Finally, this renormalization can be seen from functional determinant. From the generating functional it is easy to obtain the trace anomaly.

These approaches show that the nonstandard WZNW theories can serve as novel conformal field theories. There are still open questions such as existence of unitary representations of the Virasoro algebra which shall be addressed elsewhere.

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Here, we treat the path integral measure in terms of $\pi$ fields. To define a measure we need to define distance notion in field configuration space. It can be done by using some bilinear form $\Xi$ as follows:

$$d^2 = \int d^2 x \sqrt{\gamma} < dU(x), dU^{-1}(x) >_{\Xi}.$$  \hspace{1cm} (5.1)

Now decompose $U$ as $U = U_0 U'$, where $U_0$ and $U'$ describe zero modes and nonzero modes of the Laplace operator, respectively. Then the distance becomes:

$$d^2 = \int d^2 x \sqrt{\gamma} \{ < dU_0, dU_0^{-1} >_{\Xi} - 2 < U_0^{-1} dU_0, dU' U'^{-1} >_{\Xi} + < dU', dU'^{-1} >_{\Xi} \}. \hspace{1cm} (5.2)$$

We can arrange it as follows:

$$d^2 = V \int d^2 x \sqrt{\gamma} < (i U_0^{-1} dU_0 + u)^2 >_{\Xi} + d'_{\Xi}, \hspace{1cm} (5.3)$$

where:

$$d^2 = \int d^2 x \sqrt{\gamma} < dU', dU'^{-1} >_{\Xi} - u^2 >_{\Xi}, \hspace{1cm} (5.4)$$

$$u = \frac{i}{V} \int d^2 x \sqrt{\gamma} dU' U'^{-1}. \hspace{1cm} (5.5)$$

Since $d^2$ is independent of collective variable $U_0$, the measure factorize to:

$$dU = V^{1/2 \dim G} d\mu(U_0)[dU']. \hspace{1cm} (5.6)$$

Now putting $U' = exp(i \lambda \pi)$ and expand in powers of $\lambda$ one obtains:

$$< dU', dU'^{-1} >_{\Xi} = \lambda^2 < d\pi, d\pi >_{\Xi} + \frac{\lambda^2}{12} < [d\pi, \pi]^2 >_{\Xi} + \cdots, \hspace{1cm} (5.7)$$

$$u = \frac{i \lambda^2}{2V} \int d^2 x \sqrt{\gamma} [d\pi, \pi] + \cdots. \hspace{1cm} (5.8)$$
\( \pi \) fields could be parameterized in terms of hermitian matrix functions \( g_r \) which span the space of non-zero modes as follows:

\[
\pi(x) = \Sigma_r g_r(x)q^r. \tag{5.9}
\]

Then the distance \( d'^2 \) will be:

\[
d'^2 = \Sigma G_{rs}(q)dq^r dq^s. \tag{5.10}
\]

From this expression the volume element can be read as:

\[
dU' = (\det G)^{1/2} \Pi_r dq^r. \tag{5.11}
\]

Expanding the metric \( G \) in powers of \( \lambda \), we find:

\[
G_{rs} = \lambda^2 G_{rs}^{(0)} + \lambda^4 G_{rs}^{(1)}(q) + \lambda^6 G_{rs}^{(2)}(q) + \cdots, \tag{5.12}
\]

in which \( G_{rs}^{(0)} \) does not depend on the variables \( q \). From the last equation \( \det G \) can be determined as follows:

\[
(\det G)^{1/2} \sim (\det G^{(0)})^{1/2}(1 + \lambda^2 m_1 + \lambda^4 m_2 + O(\lambda^6)) \tag{5.13}
\]

where:

\[
m_1 = \frac{1}{2} Tr(G^{(0)-1}G^{(1)}), \tag{5.14}
\]

\[
m_2 = \frac{1}{2} Tr(G^{(0)-1}G^{(2)}) + \frac{1}{8} [Tr(G^{(0)-1}G^{(1)})]^2 - \frac{1}{4} Tr(G^{(0)-1}G^{(1)})^2. \tag{5.15}
\]

By substituting (5.14) and (5.15) in (5.4) for \( d'^2 \), \( G \) can be read from (5.10):

\[
G_{rs} = \lambda^2 \int d^2 x \sqrt{\gamma} < g_r, g_s >_\Xi + \frac{\lambda^4}{12} \int d^2 x \sqrt{\gamma} < [\pi, g_r], [\pi, g_s] >_\Xi \\
+ \frac{\lambda^4}{4V} \int d^2 x d^2 y \sqrt{\gamma} < [\pi_x, g_r], [\pi_y, g_s] >_\Xi. \tag{5.16}
\]
Using the completeness of \( g_r \) matrices which means that:

\[
\left(G^{(0)^{-1}}\right)^{rs} g_r(x) \otimes g_s(y) = \left(\delta(x,y) - \frac{1}{V}\right),
\]

it can be seen that:

\[
m_1 = \frac{1}{2} Tr(G^{(0)^{-1}}G^{(1)}) = \frac{-c_v}{12} \int d^2 x \sqrt{\gamma} < \pi, \pi > \Omega \left(\delta(x,x) + 2 \frac{V}{V}\right).
\]

Contribution of this term to the action can be easily found to be :

\[
S_{m_1} = \frac{c_v}{6k} Tr(M^{-1}\Omega) \int d^2 x \sqrt{\gamma} G(x,x) (\delta(x,x) + 2 \frac{V}{V}).
\]

It is worth mentioning that from (5.18) \( m_1 \) is independent of \( \Xi \) and this is also true for \( m_2 \) and higher order terms.

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