Defects in Modified Axial Gauge QCD$^{3+1}$

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Abstract

Magnetically charged vortex defects are shown to arise in canonically quantised modified axial gauge QCD$^{3+1}$ on a torus. This gauge – being an Abelian projection – keeps only the eigenphases as dynamical variables of the Wilson line in $x_3$-direction. A mismatch between the identification of large gauge transformations before and after gauge fixing is indicated.

1Talk presented at the “Workshop on QCD” at the American University of Paris, June 3rd – 8th, 1996, and the conference “Quark Confinement and the Hadron Spectrum II” at Villa Olmo, Como, Italy, June 26th – 29th, 1996. Email: hgrie@theorie3.physik.uni-erlangen.de
One of the main issues in non-Abelian gauge theories is the presence of redundant variables. Eliminating them by “gauge fixing”, one hopes to identify the relevant degrees of freedom, the non-perturbative part of which may solve the outstanding questions in the low energy régime of these theories. Monopole field configurations in the Abelian projection gauges \[1, 2\] seem to be a useful device to explain confinement by the dual Meissner effect \[3, 4\].

In this context, a Hamiltonian formulation of QCD is especially useful since it allows one to bear in mind all intuition and techniques of ordinary quantum mechanics; formulating the theory in terms of unconstrained, “physical” variables is the easiest way to render gauge invariant results in approximations. The choice of a compact base manifold softens the infrared problem and allows the definition of zero modes.

Here, the formulation by Lenz et al. \[5\] of Hamiltonian QCD in the modified axial gauge \[6\] on a torus \(T^3\) as spatial manifold is used, in which – in contradistinction to the naive axial gauge \(A_3 = 0\) – the eigenphases of the Wilson line/Polyakov loop in \(x_3\)-direction are kept as dynamical variables. This gauge belongs to the Abelian projection gauges, but the magnetically charged configurations found do not have particle character. On the contrary, their appearance indicates a failure of gauge fixing. The outline of these results is the goal of the present paper; a more rigorous treatment may be found in references \[7, 8\] and a forthcoming publication.

As is well known, the Hamiltonian of pure QCD in the Weyl gauge \(A_0 = 0\), quantised by imposing the canonical commutation relations between fields \(\vec{A}\) and momenta \(\vec{\Pi}\), allows for time independent gauge transformations whose infinitesimal generator is Gauß’ law,

\[
G^a(\vec{x}) = \partial_i \vec{\Pi}^a(\vec{x}) + gf^{abc} \vec{A}_b(\vec{x}) \cdot \vec{\Pi}_c(\vec{x}).
\]

It cannot be derived as an equation of motion in the Hamiltonian formalism but has to be imposed on states, defining the physical Hilbert space \(\mathcal{H}_{\text{phys}}\) as

\[
\forall |\text{phys}\rangle \in \mathcal{H}_{\text{phys}} : G^a(\vec{x}) |\text{phys}\rangle = 0 \quad \forall a, \vec{x}.
\]

(1)

On a torus \(T^3\) with length of the edge \(L\), one imposes periodic boundary conditions for all fields and derivatives as well as for the gauge transformations \[9\]

\[
\vec{A}(\vec{x}) = \vec{A}(\vec{x} + L\vec{e}_i), \quad \vec{\Pi}(\vec{x}) = \vec{\Pi}(\vec{x} + L\vec{e}_i), \quad V(\vec{x}) = V(\vec{x} + L\vec{e}_i).
\]

(2)

The functional space \(\mathcal{H}\) whose subspace \(\mathcal{H}_{\text{phys}}\) is consists hence of the space of periodic functionals. Eq. (2) is closely related to the vanishing of all total colour charges in the box and to translation invariance \[8, 10\].

“Gauge fixing” corresponds to a coordinate transformation in field space

\[
\vec{A}(\vec{x}) = \vec{U}(\vec{x})[\vec{A}^{\prime}(\vec{x}) + \frac{i}{g} \vec{\partial}]\vec{U}^{\dagger}(\vec{x})
\]

(3)

to a basis splitting explicitly in unconstrained \((\vec{A}^{\prime}, \vec{\Pi}^{\prime})\) and constrained \((\vec{U}, \vec{\rho})\) variables and respective conjugate momenta. Any operator \(\mathcal{O}'\) commuting with Gauß’ law does not contain redundant degrees of freedom and hence can be written as (possibly complicated)

\[\text{Fermions will not affect the arguments given here. Their sole trace is not to allow for twisted boundary conditions}\ [1].\]
function of $\vec{A}'$ and $\vec{\Pi}'$ only. Operators for which $[\vec{\mathcal{O}}, G^a(\vec{x})] \neq 0$ must contain variables $\vec{U}$, $\vec{\Pi}$ constrained by Gauß’ law.

It is also well known that not all gauge transformations are generated by (1) but that the unitary operator $\Omega[V]$ of arbitrary, time-independent gauge transformations $V(\vec{x})$ leaves physical states invariant only up to a phase into which—besides the winding number $\nu(V)$—the vacuum-θ-angle enters as a new, hidden parameter:

$$\Omega[V] | \text{phys} \rangle = e^{i\nu(V)} | \text{phys} \rangle \quad (4)$$

The winding number detector is the integral over the Chern–Simons three-form,

$$W[A] := \frac{g^2}{16\pi^2} \int d^3 x \ \epsilon^{ijk} tr[F_{ij}A_k + \frac{2i}{3}gA_iA_jA_k] \quad (5)$$

since it commutes with Gauß’ law and therefore is invariant under small gauge transformations, but changes under large ones

$$\Omega[V]W[A]\Omega^\dagger[V] = W[A] + \nu(V) + \frac{i g}{8\pi^2} \int d^3 x \ \epsilon^{ijk} \partial_i tr[A_j(\partial_k V^\dagger)V] \quad (6)$$

by the winding number of the mapping $V(\vec{x}) : T^3 \to SU(N)$,

$$\nu(V) := \frac{1}{24\pi^2} \int d^3 x \ \epsilon^{ijk} tr[(V \partial_i V^\dagger)(V \partial_j V^\dagger)(V \partial_k V^\dagger)] \in \mathbb{Z} \quad (7)$$

The surface term in (6) vanishes because of the periodicity condition (2). Every (and hence every physical) configuration has mirror configurations from which it differs by large gauge transformations.

Since (3) behaves as a gauge transformation, it acts on $W[A]$ by

$$W[A] \xrightarrow{g-\text{fix}} W[A'] + \nu(\vec{U}) + \frac{i g}{8\pi^2} \epsilon^{ijk} \int d^3 x \ \partial_i tr[A'_j(\partial_k \vec{U}^\dagger)\vec{U}] \quad (8)$$

The operator $\nu(\vec{U})$ and the surface term, which depends explicitly on the physical field, must vanish since $W[A]$ commutes with Gauß’ law and hence contains only physical variables,

$$W[A] \xrightarrow{g-\text{fix}} W[A'] \quad (9)$$

so that the freedom to perform large gauge transformations remains and the vacuum-θ-angle is kept track of.

The modified axial gauge [5, 6] has often been chosen since for it, the splitting into $\vec{U}$ and $\vec{A}'$ for every configuration $\vec{A}$ can be given concretely, enabling the construction of the Hamiltonian in terms of primed variables [5]. The “zero mode” fields $A'_3(\vec{x}_\perp)$ obey the modified axial “gauge condition”

$$A'_3(\vec{x}_\perp) \text{ diagonal } , \ \partial_3 A'_3(\vec{x}_\perp) = 0 \quad (10)$$

and must remain relevant degrees of freedom since $A'_3(\vec{x}_\perp)$ are the phases of the gauge invariant eigenvalues $\exp igLA'_3(\vec{x}_\perp)$ of the Wilson line in $x_3$-direction. This is also expressed in the fact that the solution to (3) cannot be given for $A'_3(\vec{x}) = 0$ since then $\vec{U}$
would not be periodic in $x_3$-direction and hence one would leave the space $\mathcal{H}$ of periodic functionals. Allowing for a colour diagonal zero mode, one finds as $x_3$-periodic solution to (3) and (10)
\[ \tilde{U}(\vec{x}) = P e^{ig \int_0^{x_3} dy_3 A_3(\vec{x}_\perp,y_3)} \Delta(\vec{x}_\perp) e^{-ig x_3 A'_3(\vec{x}_\perp)}, \]
where $\Delta(\vec{x}_\perp)$ diagonalises the Wilson line,
\[ \Delta(\vec{x}_\perp) e^{ig L A'_3(\vec{x}_\perp)} \Delta^\dagger(\vec{x}_\perp) = P \exp \int_0^L dx_3 A_3(\vec{x}). \]

The residual gauge transformations can either be constructed by explicit transformation of $\Omega[V]$ to the transformed physical Hilbert space [3] or by an inspection [5] of the freedoms in the solution [10] of equation (3) defining $A'_3(\vec{x}_\perp)$. The displacements of the colour neutral fields
\[ \Omega'[V] \vec{A}'_p(\vec{x}) \Omega'^\dagger[V] = \vec{A}'_p(\vec{x}) + \frac{2\pi}{gL} \vec{n}_p \]
are of importance in what follows. The pp-entry of the matrices $\vec{A}'$ is denoted by $\vec{A}'_p$, and $(\vec{n})_{pq} \in \delta_{pq} \mathbb{Z}^3$ is a vector consisting of diagonal, traceless matrices in colour space with integer entries, fixed in this form by the boundary conditions (2).

A word of caution is in order here. It is clear that (11) is a local solution to (3), i.e. that for a given $\vec{x}_\perp$, one can expect the eigenvalues of the Wilson line to be the only physical variables. Still, that $\tilde{U}(\vec{x})$ is also a solution globally, i.e. that it can be chosen regular and periodic in all directions for all possible configurations and simultaneously for all points on $T^3$ is not self-understood. This is the crucial point in the following.

As seen above, if the modified axial gauge choice resolves only Gauß’ law – namely if (9) holds – one can start from the winding number operator $W[A']$ in order to identify all large gauge transformations for all (physical) configurations. With (6/13) one obtains
\[ W[A'] \xrightarrow{\Omega'[V]} W[A'] + \frac{1}{2} \sum_p \vec{n}_{i,p} \left[ \frac{g}{2\pi L} \varepsilon^{ijk} \int d^3x \partial_j A'_{k,p}(\vec{x}) \right]. \]

The surface term in brackets defines the total magnetic Abelian fluxes $\Phi_i$ in the $i$-th direction through the box and is nonzero only if the diagonal gauge fields are non-periodic. This suggests that the allegedly unconstrained variables of the modified axial gauge are not operators on the Hilbert space of periodic functionals. But they must be if $\vec{A}'$ acts solely inside $\mathcal{H}_{\text{phys}} \subset \mathcal{H}$. On the other hand, if $\vec{A}'$ is sometimes (or even always) periodic, as (2) may suggest, no large gauge transformations would exist in the allegedly physical Hilbert space of the modified axial gauge representation for certain configurations, in contradistinction to the above argument.
It turns out that the mismatch is due to the presence of magnetically charged defects [8]. Note from (3) and the periodicity of $\vec{A}$ that $\vec{A}'$ is periodic only if $\vec{U}$ is. Although the latter is by construction periodic in the $x_3$-direction, and its leftmost term in (11) is periodic in the transverse directions $\vec{x}_\perp$, the two other terms are not [5, 8]. Firstly, the definition of the phases $A'_{3,p}(\vec{x}_\perp)$ necessary to define the rightmost term in (11) involves a logarithm so that the eigenphases might lie on different Riemann sheets at opposite boundaries,

$$A'_3(\vec{x} + L\vec{e}_i) - A'_3(\vec{x}) = \frac{2\pi}{gL} \varepsilon^{ij} m_j \quad \text{for } i = 1, 2 \ ,$$

(15)

where $m_i$ is a diagonal, traceless matrix with integer entries. The mapping $\exp igL A'_3(\vec{x}_\perp) : T^2 \to [U(1)]^{N-1}$ splits into topologically distinct classes labelled by the winding numbers $m_i \in \mathbb{Z}^{N-1}$. On the other hand, assuming a lattice regularisation of the Jacobian of the coordinate transformation (3) [12], $A'_{3,p}$ is interpreted as azimuthal angle in $SU(N)$ and other Riemann sheets are inaccessible. Then, such defects will play no rôle for dynamical reasons.

Secondly, the diagonalisation matrix $\Delta$ (12) is determined only up to right multiplication with a diagonal matrix. Drawing from the technique of Gross et al. [11], one can show [8] that the diagonalisation can be chosen continuous on $T^3$, but in general not periodic because the mapping $\Delta(\vec{x}_\perp) : T^2 \to SU(N)/[U(1)]^{N-1}$ decomposes into topologically distinct classes which are labelled by a diagonal, traceless matrix with winding numbers as entries,

$$m_{3,p} := \frac{i}{2\pi} \int d^2\vec{x}_\perp \tilde{\partial}_\perp \times (\Delta^\dagger(\vec{x}_\perp)\tilde{\partial}_\perp \Delta(\vec{x}_\perp)) \in \mathbb{Z} \ .$$

(16)

Here, the Jacobian plays no rôle and $m_3 \neq 0$ even when $A'_3$ is periodic. The occurrence of this defect is completely analogous to the introduction of a point singularity on each sphere $S^2$ about a ’t Hooft–Polyakov monopole when one tries to diagonalise the Higgs field, i.e. to transform it to the unitary gauge.

Like this Dirac string, the magnetic vortex defects which yield nonzero $\Phi_i$ in the modified axial gauge are not physical particles, having neither mass nor position, but are artifacts of the gauge chosen. These objects can therefore not condense, and the ’t Hooft–Mandelstam mechanism for confinement [1, 3, 4] cannot be associated with them (in fact, Lenz et al. [12] showed that gluons are confined in the strong coupling limit due to the Jacobian). The magnetic fluxes $\Phi_i$ obey the Dirac quantisation condition and have no relation to ’t Hooft’s magnetic twist configurations [9]: The defects occur even when fermions are included since one started with strictly periodic boundary conditions (2) in $\mathcal{H}$.

As a result, $W[A']$ changes under residual gauge transformations by the field-dependent number $\text{tr}[\vec{n} \cdot \vec{m}] / 2$. Therefore, the local gauge fixing condition (10) and its global counterpart (9) do not match. The variables $\vec{A}'$ are not operators in the original Hilbert space of periodic functionals, showing an inconsistency of this gauge choice. At this moment, a gauge choice in which the Wilson line is not diagonalised is under investigation with the expectations that this concretely solvable gauge allows for large gauge transformations after the elimination of redundant degrees of freedom and that it may clarify the relevance of the above defects for physical questions. Nonetheless, the modified axial gauge
may serve as a starting point for an effective theory in which the defects have a “long lifetime”. A special rôle will then be played by the diagonalisation defect \( m_3 \neq 0 \) since it is – in twofold contradistinction to \( \vec{m}_1 \neq 0 \) – only present in two transversal dimensions even when one chooses another compact transverse manifold like \( S^2 \) instead of \( T^2 \). They share this property with the large gauge transformations which are also special to 3+1 dimensions.

On the other hand, since only sparse concrete information on the non-perturbative sector of QCD exists, the presence of large gauge transformations and the associated vacuum–ϑ–angle should be carefully kept accounted for.

**Acknowledgments**

I am grateful for intense discussions with F. Lenz and A.C. Kalloniatis.

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