PALEY-WIENER PROPERTIES FOR SPACES OF POWER SERIES EXPANSIONS

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Abstract. We extend Paley-Wiener results in the Bargmann setting deduced in [5] to larger class of power series expansions. At the same time we deduce characterisations of all Pilipović spaces and their distributions (and not only of low orders as in [5]).

0. Introduction

Classical Paley-Wiener theorems characterize functions and distributions with certain restricted supports in terms of estimates of their Fourier-Laplace transforms. For example, let \( f \) be a distribution on \( \mathbb{R}^d \) and let \( B_{r_0}(0) \subseteq \mathbb{R}^d \) be the ball with center at origin and radius \( r_0 \). Then \( f \) is supported in \( B_{r_0}(0) \) if and only if

\[
|\hat{f}(\zeta)| \lesssim (\zeta)^N e^{r_0 |\text{Im}(\zeta)|}, \quad \zeta \in \mathbb{C}^d,
\]

for some \( N \geq 0 \).

In [13] a type of Paley-Wiener results were deduced for certain spaces of entire functions, where the usual Fourier transform were replaced by the reproducing kernel \( \Pi_A \) of the Bargmann transform. (See [6] and Section 1 for notations.) This reproducing kernel is an analytic global Fourier-Laplace transform with respect to a suitable Gaussian measure. For example, in [13] it is proved that if \( A(\mathbb{C}^d) \) is the set of all entire functions and \( A_{b_\sigma}(\mathbb{C}^d) \) (\( A_{b_{\sigma}}(\mathbb{C}^d) \)), \( \sigma > 0 \), is the set of all \( F \in A(\mathbb{C}^d) \) such that \( |F(z)| \lesssim e^{r |z|^\sigma} \) for some \( r > 0 \) (for every \( r > 0 \)), then

\[
A_{b_1}(\mathbb{C}^d) = \Pi_A(\delta'(\mathbb{C}^d)) = \Pi_A(\delta'(\mathbb{C}^d) \cap L^\infty(\mathbb{C}^d)).
\]

The latter equality in (0.1) is in [5] refined into the relation

\[
F_0 \in A_{b_1}(\mathbb{C}^d) \iff F_0 = \Pi_A(\chi \cdot F)
\]

for some characteristic function \( \chi \) of a polydisc \( D \), centered at origin, and a function \( F \) which is defined and analytic in a neighbourhood of \( D \). The spaces \( A_{b_{\sigma_1}}(\mathbb{C}^d) \) is also characterized in [5] by

\[
A_{b_{\sigma_1}}(\mathbb{C}^d) = \Pi_A(\chi \cdot A(\mathbb{C}^d))
\]

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when \( \chi \) above is fixed.

In fact, in [8], similar characterizations are deduced for all \( \mathcal{A}_\sigma(C^d) \) and \( \mathcal{A}_{0,\sigma}(C^d) \) with \( \sigma \leq 1 \) as well as for the set \( \mathcal{A}_s(C^d) \) when \( s \in [0, \frac{1}{2}) \) which consists of all \( F \in A(C^d) \) such that \( |F(z)| \lesssim e^{r(\log|z|)^{\frac{1}{1-2\sigma}}} \). For example, let \( \chi \) be the characteristic function of a polydisc, centered at origin in \( C^d \). Then it is proved in [8] that

\[
\mathcal{A}_{\sigma}(C^d) = \{ \Pi_A(F \cdot \chi); F \in \mathcal{A}_{\sigma_0}(C^d) \} \quad \text{when} \quad \sigma < \frac{1}{2}, \quad \sigma_0 = \frac{\sigma}{1 - 2\sigma}
\]

and

\[
\mathcal{A}_s(C^d) = \{ \Pi_A(F \cdot \chi); F \in \mathcal{A}_s(C^d) \} \quad \text{when} \quad s < \frac{1}{2}.
\]

In Section 2 we extend the set of characterizations given in [8] in different ways. Firstly we show that we can choose \( \chi \) above in a larger class of functions and measures. Secondly, we characterize strictly larger spaces than \( \mathcal{A}_{\frac{1}{2}}(C^d) \). More specifically, we characterize all the spaces

\[
\mathcal{A}_{0,\sigma}(C^d) \quad \text{and} \quad \mathcal{A}_s(C^d) \quad (\mathcal{A}_{0,\frac{1}{2}}(C^d) \quad \text{and} \quad \mathcal{A}_{0,s}(C^d)), \quad (0.2)
\]

for any \( s, \sigma \in \mathbb{R} \), as well as the larger spaces

\[
\mathcal{A}'_{0,\sigma}(C^d) \quad \text{and} \quad \mathcal{A}'_s(C^d) \quad (\mathcal{A}'_{0,\frac{1}{2}}(C^d) \quad \text{and} \quad \mathcal{A}'_{0,s}(C^d)), \quad (0.3)
\]

the sets of all formal power series expansions

\[
F(z) = \sum_{\alpha \in \mathbb{N}^d} c(F, \alpha)e_\alpha(z), \quad e_\alpha(z) = \frac{e^z}{\sqrt{\alpha!}}, \quad \alpha \in \mathbb{N}^d \quad (0.4)
\]

such that

\[
|c(F, \alpha)| \lesssim h^{\alpha_1} \alpha_1^{\frac{1}{2\sigma}} \quad \text{respective} \quad |c(F, \alpha)| \lesssim e^{r|\alpha|^{\frac{1}{2}}}
\]

for every \( h, r > 0 \) (for some \( h, r > 0 \)). We notice that the spaces (0.2) discussed above can also be described as the sets of all formal power series expansions (0.4) such that

\[
|c(F, \alpha)| \lesssim h^{\alpha_1} \alpha_1^{\frac{1}{2}} \quad \text{respective} \quad |c(F, \alpha)| \lesssim e^{-r|\alpha|^{\frac{1}{2}}}
\]

for some \( h, r > 0 \) (for every \( h, r > 0 \)) (cf. [4, 13] and Section 1). We remark that for \( s \geq \frac{1}{2} \), \( \mathcal{A}_s(C^d) \) and \( \mathcal{A}_{0,s}(C^d) \), defined in this way, are still spaces of entire functions, but with other types of estimates, compared to the case \( s < \frac{1}{2} \) considered above (cf. [13] and Section 1).

By the definitions we have \( \mathcal{A}'_{\frac{1}{2}}(C^d) = A(C^d) \), and by using the convention

\[
s < b_{\sigma_1} < b_{\sigma_2} < \frac{1}{2} \quad \text{when} \quad \sigma_1 < \sigma_2,
\]
it follows that
\[ A_0(C^d) \subseteq A_{0,s_1}(C^d) \subseteq A_{s_1}(C^d) \subseteq A_{0,s_2}(C^d) \]
\[ \subseteq A'_{0,s_2}(C^d) \subseteq A'_{s_2}(C^d) \subseteq A'_{0,s_1}(C^d), \]
when \( s_1, s_2 \in \mathbb{R} \cup \{ \sigma \} \), \( \sigma > 0, \) \( s_1 < s_2 \)
(cf. [13]). In particular, if \( \sigma > 1 \), then \( A'_{\sigma}(C^d) \) and \( A'_{0,\sigma}(C^d) \) are contained in \( A(C^d) \), and in [13], it is proved that the former spaces consists of all \( F \in A(C^d) \) such that \( |F(z)| \lesssim e^{r|z|^\frac{\sigma}{2}} \) for every \( r > 0 \) respective for some \( r > 0 \).

In Section 2 we show among others that if \( T_\chi \) is the map \( F \mapsto \Pi_A(\chi \cdot F) \) with \( \chi \) as above, then the mappings
\[ T_\chi : A'_{0,s}(C^d) \to A'_{0,s}(C^d), \quad T_\chi : A'_s(C^d) \to A'_s(C^d), \quad s < \frac{1}{2}, \]
\[ T_\chi : A'_{0,\sigma_0}(C^d) \to A'_{0,\sigma_0}(C^d), \quad T_\chi : A'_{\sigma_0}(C^d) \to A'_{\sigma_0}(C^d), \quad \sigma_0 = \frac{\sigma}{2\sigma + 1}, \]
when \( \sigma > 0 \),
\[ T_\chi : A'_{0,\sigma_0}(C^d) \to A_{\sigma}(C^d), \quad T_\chi : A'_{\sigma_0}(C^d) \to A_{0,\sigma}(C^d), \quad \sigma_0 = \frac{\sigma}{2\sigma - 1}, \]
when \( \sigma > \frac{1}{2}, \) and
\[ T_\chi : A_{0,\sigma_0}(C^d) \to A_{0,\sigma_0}(C^d), \quad T_\chi : A_{\sigma_0}(C^d) \to A_{\sigma_0}(C^d), \quad \sigma_0 = \frac{\sigma}{1-2\sigma}, \]
when \( \sigma < \frac{1}{2} \), are well-defined and bijective.

In Section 3 we apply these mapping properties to deduce characterizations of Pilipović spaces and their distribution spaces in terms of images of the adjoint of Gaussian windowed short-time Fourier transform, acting on suitable spaces which are strongly linked to the spaces in (0.2) and (0.3). These spaces are obtained by imposing the same type of estimates on their Hermite coefficients as for the power series coefficients of the spaces in (0.2) and (0.3). It turns out that Pilipović spaces and their distribution spaces are exactly the counter images of the spaces in (0.2) and (0.3) under the Bargmann transform (cf. [13]).

1. Preliminaries

In this section we recall some basic facts. We start by discussing Pilipović spaces and some of their properties. Then we recall some facts on modulation spaces. Finally we recall the Bargmann transform and some of its mapping properties, and introduce suitable classes of entire functions on \( C^d \).
1.1. Spaces of sequences. The definitions of Pilipović spaces and spaces of power series expansions are based on certain spaces of sequences on $\mathbb{N}^d$, indexed by the extended set

$$ R_0 = R_+ \bigcup \{ b_\sigma ; \sigma \in R_+ \}, $$

of $R_+$. We extend the inequality relations on $R_+$ to the set $R_0$, by letting

$$ s_1 < b_\sigma < s_2 \quad \text{and} \quad b_{\sigma_1} < b_{\sigma_2} $$

when $s_1 < \frac{1}{2} \leq s_2$ and $\sigma_1 < \sigma_2$. (Cf. [13].)

**Definition 1.1.** Let $s \in R_0$ and $\sigma \in R_+$.

1. The set $\ell'_0(\mathbb{N}^d)$ consists of all formal sequences $a = \{a(\alpha)\}_{\alpha \in \mathbb{N}^d} \subseteq \mathbb{C}$, and $\ell_0(\mathbb{N}^d)$ is the set of all $a \in \ell'_0(\mathbb{N}^d)$ such that $a(\alpha) \neq 0$ for at most finite numbers of $\alpha \in \mathbb{N}^d$;

2. If $r, s \in R_+$ and $a \in \ell'_0(\mathbb{N}^d)$, then the Banach spaces

$$ \ell^\infty_{r,s}(\mathbb{N}^d), \quad \ell^\infty_{r,s}^*(\mathbb{N}^d) \quad \text{and} \quad \ell^\infty_{r,s}(\mathbb{N}^d) $$

consists of all $a \in \ell'_0(\mathbb{N}^d)$ such that their corresponding norms

$$ \|a\|_{\ell^\infty_{r,s}} \equiv \sup_{\alpha \in \mathbb{N}^d} |a(\alpha)| e^{r|\alpha|^{\frac{1}{r}}} \quad \text{and} \quad \|a\|_{\ell^\infty_{r,s}^*} \equiv \sup_{\alpha \in \mathbb{N}^d} |a(\alpha)| e^{-r|\alpha|^{\frac{1}{r}}} $$

respectively, are finite;

3. The space $\ell_s(\mathbb{N}^d) \ (\ell_{0,s}(\mathbb{N}^d))$ is the inductive limit (projective limit) of $\ell^\infty_{r,s}(\mathbb{N}^d)$ with respect to $r > 0$, and $\ell'_s(\mathbb{N}^d) \ (\ell'_{0,s}(\mathbb{N}^d))$ is the projective limit (inductive limit) of $\ell^\infty_{r,s}^*(\mathbb{N}^d)$ with respect to $r > 0$.

We also let $\ell_{0,N}(\mathbb{N}^d)$ be the set of all $a \in \ell'_0(\mathbb{N}^d)$ such that $a(\alpha) = 0$ when $|\alpha| \geq N$. Then

$$ \ell_0(\mathbb{N}^d) = \bigcup_{N \geq 0} \ell_{0,N}(\mathbb{N}^d), $$

and $\ell_{0,N}(\mathbb{N}^d)$ is a Banach space under the norm

$$ \|a\|_{\ell_{0,N}} \equiv \sup_{|\alpha| \leq N} |a(\alpha)|. $$

We equip $\ell_0(\mathbb{N}^d)$ with the inductive limit topology of $\ell_{0,N}(\mathbb{N}^d)$, and supply $\ell'_0(\mathbb{N}^d)$ with Fréchet space topology through the semi-norms

$$ \| \cdot \|_{\ell_{0,N}}. $$

In what follows, $(\cdot, \cdot)_{\mathcal{H}}$ denotes the scalar product in the Hilbert space $\mathcal{H}$.

**Remark 1.2.** For the spaces in Definition 1.1, the following is true. We leave the verifications for the reader.
It follows that for some polynomial $p$ of order $H$ transform, and to the Harmonic oscillator $R$ functions and (ultra-)distributions defined on $f$ is the Hermite coefficient of $f$ is the Hermite series expansion of $f$ and an orthonormal basis for $L^{\infty}(N^d)$, it is well-known that the set of Hermite functions is a basis for $(L^{\infty}(N^d))$. We recall that the Hermite function of order $\alpha$.

1.2. Pilipović spaces and spaces of power series expansions on $C^d$. We recall that the Hermite function of order $\alpha \in N^d$ is defined by

$$h_{\alpha}(x) = \pi^{-\frac{d}{4}}(-1)^{|\alpha|}(2^{\frac{|\alpha|}{2}}\alpha!)^{-\frac{1}{2}}e^{-\frac{|x|^2}{2}}(\partial^\alpha e^{-|x|^2}).$$

It follows that

$$h_{\alpha}(x) = ((2\pi)^{\frac{d}{2}}\alpha!)^{-1}e^{-\frac{|x|^2}{2}}p_{\alpha}(x),$$

for some polynomial $p_{\alpha}$ of order $\alpha$ on $R^d$, called the Hermite polynomial of order $\alpha$. The Hermite functions are eigenfunctions to the Fourier transform, and to the Harmonic oscillator $H_d \equiv |x|^2 - \Delta$ which acts on functions and (ultra-)distributions defined on $R^d$. More precisely, we have

$$H_d h_{\alpha} = (2|\alpha| + d)h_{\alpha}, \quad H_d \equiv |x|^2 - \Delta.$$
We shall also consider formal power series expansions on \( \mathbb{C}^d \), centered at origin. That is, we shall consider formal expressions of the form

\[
F(z) = \sum_{\alpha \in \mathbb{N}^d} c(F, \alpha)e_{\alpha}(z), \quad e_{\alpha}(z) = \frac{z^\alpha}{\sqrt{\alpha!}}, \quad \alpha \in \mathbb{N}^d. \tag{1.3}
\]

**Definition 1.3.** The set \( \mathcal{H}_0'(\mathbb{R}^d) \) consists of all formal hermite series expansions \((1.1)\), and \( \mathcal{A}_0'(\mathbb{C}^d) \) consists of all formal power series expansions \((1.3)\). The set \( \mathcal{H}_0(\mathbb{R}^d) \) \( (A_0(\mathbb{C}^d)) \) consists of all \( f \in \mathcal{H}_0'(\mathbb{R}^d) \) \( (F \in \mathcal{A}_0'(\mathbb{C}^d)) \) such that \( c_h(f, \alpha) \neq 0 \) \( (c(F, \alpha) \neq 0) \) for at most finite numbers of \( \alpha \in \mathbb{N}^d \).

(1) If \( s_1 \in \mathbb{R}_+ \) and \( s_2 \in \mathbb{R}_+ \), then

\[
\mathcal{H}_{s,1}(\mathbb{R}^d), \quad \mathcal{H}_{0,s_1}(\mathbb{R}^d), \quad \mathcal{H}'_{0,s_1}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{H}'_{s_2}(\mathbb{R}^d) \tag{1.4}
\]

are the sets of all Hermite series expansions \((1.1)\) such that their coefficients \( \{c_h(f, \alpha)\}_{\alpha \in \mathbb{N}^d} \) belong to \( \ell_{s_2}(\mathbb{N}^d) \), \( \ell_{0,s_1}(\mathbb{N}^d) \), \( \ell'_{0,s_1}(\mathbb{N}^d) \) respective \( \ell'_{s_2}(\mathbb{N}^d) \);

(2) If \( s_1 \in \mathbb{R}_+ \) and \( s_2 \in \mathbb{R}_+ \), then

\[
\mathcal{A}_{s_2}(\mathbb{C}^d), \quad \mathcal{A}_{0,s_1}(\mathbb{C}^d), \quad \mathcal{A}'_{0,s_1}(\mathbb{C}^d) \quad \text{and} \quad \mathcal{A}'_{s_2}(\mathbb{C}^d) \tag{1.5}
\]

are the sets of all power series expansions \((1.3)\) such that their coefficients \( \{c(F, \alpha)\}_{\alpha \in \mathbb{N}^d} \) belong to \( \ell_{s_2}(\mathbb{N}^d) \), \( \ell_{0,s_1}(\mathbb{N}^d) \), \( \ell'_{0,s_1}(\mathbb{N}^d) \) respective \( \ell'_{s_2}(\mathbb{N}^d) \).

The spaces \( \mathcal{H}_{s}(\mathbb{R}^d) \) and \( \mathcal{H}_{0,s}(\mathbb{R}^d) \) in Definition \(1.3\) are called *Pilipović spaces of Roumieu* respectively *Beurling types* of order \( s \), and \( \mathcal{H}'_{s}(\mathbb{R}^d) \) and \( \mathcal{H}'_{0,s}(\mathbb{R}^d) \) are called *Pilipović distribution spaces of Roumieu* respectively *Beurling types* of order \( s \).

**Remark 1.4.** Let \( T_H \) be the map from \( \ell'_0(\mathbb{N}^d) \) to \( \mathcal{H}_0'(\mathbb{R}^d) \) which takes the sequence \( \{c_h(f, \alpha)\}_{\alpha \in \mathbb{N}^d} \) to the expansion \((1.1)\), and let \( T_A \) be the map from \( \ell'_0(\mathbb{N}^d) \) to \( \mathcal{A}_0'(\mathbb{C}^d) \) which takes the sequence \( \{c(F, \alpha)\}_{\alpha \in \mathbb{N}^d} \) to the expansion \((1.3)\). Then it is clear that \( T_H \) restricts to bijective mappings from

\[
\ell_{s_2}(\mathbb{N}^d), \quad \ell_{0,s_1}(\mathbb{N}^d), \quad \ell'_{0,s_1}(\mathbb{N}^d) \quad \text{and} \quad \ell'_{s_2}(\mathbb{N}^d) \tag{1.6}
\]

to respective spaces in \((1.4)\), and that \( T_A \) restricts to bijective mappings from the spaces in \((1.6)\) to respective spaces in \((1.5)\).

We let the topologies of the spaces in \((1.4)\) and \((1.5)\) be inherited from the topologies of respective spaces in \((1.6)\), through the mappings \( T_H \) and \( T_A \).

The following result shows that Pilipović spaces of order \( s \in \mathbb{R}_+ \) may in convenient ways be characterized by estimates of powers of harmonic oscillators applied on the involved functions. We omit the proof since the result follows in the case \( s \geq \frac{1}{2} \) from \([10]\) and from \([13]\) for general \( s \).
Proposition 1.5. Let \( s \geq 0 \) (\( s > 0 \)) be real, \( H_d = |x|^2 - \Delta \) be the harmonic oscillator on \( \mathbb{R}^d \) and set
\[
\| f \|(r,s) \equiv \sup_{N \in \mathbb{N}} \left( \frac{\| H_N^r f \|_{L^\infty(\mathbb{R}^d)}}{r^N N!^{2s}} \right), \quad f \in C^\infty(\mathbb{R}^d).
\]
Then \( f \in \mathcal{H}_s(\mathbb{R}^d) \) (\( f \in \mathcal{H}_{0,s}(\mathbb{R}^d) \)), if and only if \( \| f \|(r,s) < \infty \) for some \( r > 0 \) (for every \( r > 0 \)). The topologies of \( \mathcal{H}_s(\mathbb{R}^d) \) and \( \mathcal{H}_{0,s}(\mathbb{R}^d) \) agree with the inductive and projective limit topologies, respectively, induced by the semi-norms \( \| \cdot \|(r,s) \), \( r > 0 \).

Remark 1.6. Let \( \mathcal{S}_s(\mathbb{R}^d) \) and \( \Sigma_s(\mathbb{R}^d) \) be the Fourier invariant Gelfand-Shilov spaces of order \( s \in \mathbb{R}^+ \) and of Rourme and Beurling types respectively (see [13] for notations). Then it is proved in [9, 10] that
\[
\mathcal{H}_{0,s}(\mathbb{R}^d) = \Sigma_s(\mathbb{R}^d) \neq \{0\}, \quad s > \frac{1}{2},
\]
\[
\mathcal{H}_{0,s}(\mathbb{R}^d) \neq \Sigma_s(\mathbb{R}^d) = \{0\}, \quad s = \frac{1}{2},
\]
and
\[
\mathcal{H}_s(\mathbb{R}^d) = \mathcal{S}_s(\mathbb{R}^d) \neq \{0\}, \quad s \geq \frac{1}{2}.
\]

1.3. Spaces of entire functions and the Bargmann transform.
Let \( \Omega \subseteq \mathbb{C}^d \) be open and let \( \Omega_0 \subseteq \mathbb{C}^d \) be non-empty (but not necessary open). Then \( A(\Omega) \) is the set of all analytic functions in \( \Omega \), and
\[
A(\Omega_0) = \bigcup A(\Omega),
\]
where the union is taken over all open sets \( \Omega \subseteq \mathbb{C}^d \) such that \( \Omega_0 \subseteq \Omega \). In particular, if \( z_0 \in \mathbb{C}^d \) is fixed, then \( A(\{z_0\}) \) is the set of all complex-valued functions which are defined and analytic near \( z_0 \).

We shall now consider the Bargmann transform. We set
\[
\langle z, w \rangle = \sum_{j=1}^d z_j w_j \quad \text{and} \quad \langle z, w \rangle = \langle z, \overline{w} \rangle, \quad \text{when}
\]
\[
z = (z_1, \ldots, z_d) \in \mathbb{C}^d \quad \text{and} \quad w = (w_1, \ldots, w_d) \in \mathbb{C}^d,
\]
and otherwise \( \langle \cdot, \cdot \rangle \) denotes the duality between test function spaces and their corresponding duals. The Bargmann transform \( \mathfrak{B}_d f \) of \( f \in L^2(\mathbb{R}^d) \) is defined by the formula
\[
(\mathfrak{B}_d f)(z) = \pi^{-\frac{d}{2}} \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2} \left( \langle z, z \rangle + |y|^2 \right) + 2^{\frac{1}{2}} \langle z, y \rangle \right) f(y) \, dy \quad (1.7)
\]
(cf. [1]). We note that if \( f \in L^2(\mathbb{R}^d) \), then the Bargmann transform \( \mathfrak{B}_d f \) of \( f \) is the entire function on \( \mathbb{C}^d \), given by
\[
(\mathfrak{B}_d f)(z) = \int_{\mathbb{R}^d} \mathfrak{A}_d(z, y) f(y) \, dy,
\]
where the Bargmann kernel $\mathcal{A}_d$ is given by

$$\mathcal{A}_d(z, y) = \pi^{-\frac{d}{2}} \exp\left( -\frac{1}{2} \left( |z|^2 + |y|^2 \right) + 2\overline{z}y \right).$$

Evidently, the right-hand side in (1.8) makes sense when $f \in S_d^0(R^d)$ and defines an element in $A(C^d)$, since $y \mapsto \mathcal{A}_d(z, y)$ can be interpreted as an element in $S_d^0(R^d)$ with values in $A(C^d)$.

It was proved in [1] that $f \mapsto \mathcal{A}_df$ is a bijective and isometric map from $L^2(R^d)$ to the Hilbert space $A^2(C^d) \equiv B^2(C^d) \cap A(C^d)$, where $B^2(C^d)$ consists of all measurable functions $F$ on $C^d$ such that

$$\|F\|_{B^2} \equiv \left( \int_{C^d} |F(z)|^2 d\mu(z) \right)^{\frac{1}{2}} < \infty. \quad (1.9)$$

Here $d\mu(z) = \pi^{-d} e^{-|z|^2} d\lambda(z)$, where $d\lambda(z)$ is the Lebesgue measure on $C^d$. We recall that $A^2(C^d)$ and $B^2(C^d)$ are Hilbert spaces, where the scalar product are given by

$$(F, G)_{B^2} \equiv \int_{C^d} F(z)\overline{G(z)} \, d\mu(z), \quad F, G \in B^2(C^d). \quad (1.10)$$

If $F, G \in A^2(C^d)$, then we set $\|F\|_{A^2} = \|F\|_{B^2}$ and $(F, G)_{A^2} = (F, G)_{B^2}$.

Furthermore, Bargmann showed that there is a convenient reproducing formula on $A^2(C^d)$. More precisely, let

$$(\Pi_A F)(z) \equiv \int_{C^d} F(w)e^{z\overline{w}} \, d\mu(w), \quad (1.11)$$

when $z \mapsto F(z)e^{R|z|^2}$ belongs to $L^1(C^d)$ for every $R \geq 0$. Then it is proved in [1, 2] that $\Pi_A$ is the orthogonal projection of $B^2(C^d)$ onto $A^2(C^d)$. In particular, $\Pi_A F = F$ when $F \in A^2(C^d)$.

In [1] it is also proved that

$$\mathcal{A}_d h_\alpha = e_\alpha, \quad \text{where} \quad e_\alpha(z) \equiv \frac{z^\alpha}{\sqrt{\alpha!}}, \quad z \in C^d. \quad (1.12)$$

In particular, the Bargmann transform maps the orthonormal basis $\{h_\alpha\}_{\alpha \in N^d}$ in $L^2(R^d)$ bijectively into the orthonormal basis $\{e_\alpha\}_{\alpha \in N^d}$ of monomials in $A^2(C^d)$.

For general $f \in \mathcal{H}_0'(R^d)$ we now set

$$\mathfrak{J}_d f \equiv (T_A \circ T_{\mathcal{H}}^{-1}) f, \quad f \in \mathcal{H}_0'(R^d), \quad (1.13)$$

where $T_{\mathcal{H}}$ and $T_A$ are given by Remark [1.4]. It follows from [1.12] that $\mathfrak{J}_d f$ in (1.13) agrees with $\mathfrak{J}_d f$ in (1.7) when $f \in L^2(R^d)$, and that this is the only way to extend the Bargmann transform to the space $\mathcal{H}_0'(R^d)$. It follows that $\mathfrak{J}_d = T_A \circ T_{\mathcal{H}}^{-1}$ is a homeomorphism from $\mathcal{H}_0'(R^d)$ to $\mathcal{A}_0'(C^d)$, which restricts to homeomorphisms from the spaces in (1.4) to the spaces in (1.5), respectively. If $f \in \mathcal{H}_0'(R^d)$ and $F \in \mathcal{A}_0'(C^d)$ are
given by (1.1) and (1.3) with \( c_\mathcal{H}(f, \alpha) = c(F, \alpha) \) for all \( \alpha \in \mathbb{N}^d \), then it follows that \( \mathcal{H}_d f = F \).

It follows that if \( f, g \in L^2(\mathbb{R}^d) \) and \( F, G \in A^2(\mathbb{C}^d) \), then

\[
(f, g)_{L^2(\mathbb{R}^d)} = \sum_{\alpha \in \mathbb{N}^d} c_\mathcal{H}(f, \alpha) c_\mathcal{H}(g, \alpha),
\]

\[
(F, G)_{A^2(\mathbb{C}^d)} = \sum_{\alpha \in \mathbb{N}^d} c(F, \alpha) c(G, \alpha).
\]

By the definitions we get the following proposition on duality for Pilipović spaces and their Bargmann images. The details are left for the reader.

**Proposition 1.7.** Let \( s_1 \in \mathbb{R}_0 \) and \( s_2 \in \overline{\mathbb{R}}_0 \). Then the form \( (\cdot, \cdot)_{L^2(\mathbb{R}^d)} \) on \( \mathcal{H}_0(\mathbb{R}^d) \times \mathcal{H}_0(\mathbb{R}^d) \) is uniquely extendable to sesqui-linear forms on

\[
\mathcal{H}'_{s_2}(\mathbb{R}^d) \times \mathcal{H}_{s_2}(\mathbb{R}^d), \quad \mathcal{H}_{s_2}(\mathbb{R}^d) \times \mathcal{H}'_{s_2}(\mathbb{R}^d),
\]

\[
\mathcal{H}'_{0,s_1}(\mathbb{R}^d) \times \mathcal{H}_{0,s_1}(\mathbb{R}^d) \quad \text{and on} \quad \mathcal{H}_{0,s_1}(\mathbb{R}^d) \times \mathcal{H}'_{0,s_1}(\mathbb{R}^d).
\]

The duals of \( \mathcal{H}_{s_2}(\mathbb{R}^d) \) and \( \mathcal{H}_{0,s_1}(\mathbb{R}^d) \) are equal to \( \mathcal{H}'_{s_2}(\mathbb{R}^d) \) and \( \mathcal{H}'_{0,s_1}(\mathbb{R}^d) \), respectively, through the form \( (\cdot, \cdot)_{L^2(\mathbb{R}^d)} \).

The same holds true if the spaces in (1.4) and the form \( (\cdot, \cdot)_{L^2(\mathbb{R}^d)} \) are replaced by corresponding spaces in (1.5) and the form \( (\cdot, \cdot)_{A^2(\mathbb{C}^d)} \), at each occurrence.

If \( s \in \overline{\mathbb{R}}_0, f \in \mathcal{H}_s(\mathbb{R}^d), g \in \mathcal{H}'_s(\mathbb{R}^d), F \in \mathcal{A}_s(\mathbb{C}^d) \) and \( G \in \mathcal{A}'_s(\mathbb{C}^d) \), then \( (f, g)_{L^2(\mathbb{R}^d)} \) and \( (F, G)_{A^2(\mathbb{C}^d)} \) are defined by the formula (1.14). It follows that

\[
c_\mathcal{H}(f, \alpha) = c(F, \alpha) \quad \text{when} \quad F = \mathcal{H}_d f, \ G = \mathcal{H}_d g.
\]

(1.15) holds for such choices of \( f \) and \( g \). Furthermore, the duals of \( \mathcal{H}_s(\mathbb{R}^d) \) and \( \mathcal{A}_s(\mathbb{C}^d) \) can be identified with \( \mathcal{H}'_s(\mathbb{R}^d) \) and \( \mathcal{A}'_s(\mathbb{C}^d) \), respectively, through the forms in (1.14). The same holds true with

\[
\mathcal{H}_{0,s}, \quad \mathcal{H}'_{0,s}, \quad \mathcal{A}_{s}, \quad \text{and} \quad \mathcal{A}'_{0,s},
\]

in place of

\[
\mathcal{H}_s, \quad \mathcal{H}'_s, \quad \mathcal{A}_s, \quad \text{and} \quad \mathcal{A}'_s,
\]

respectively, at each occurrence.

**Remark 1.8.** In [13], the spaces in (1.5), contained in \( \mathcal{A}'_{0,s_1}(\mathbb{C}^d) \) are identified as follows as spaces of analytic functions:

1. if \( s \in [0, \frac{1}{2}) \) and \( \sigma > 0 \), then \( \mathcal{A}_s(\mathbb{C}^d) \) (\( \mathcal{A}_{0,s}(\mathbb{C}^d) \)) is equal to

\[
\{ F \in A(\mathbb{C}^d) ; |F(z)| \lesssim e^{|r(\log|z|)^{r/4}} \text{ for some (every) } r > 0 \}
\]
Lemma 4.9] justifies the separation. The result is essential when deducing the characterizations of Pilipović spaces in Section 3.

In the function

\[ S \]

function space, we recall some comparison results deduced in \[ 13 \], between a test

A test function space introduced by Gröchenig.

1.4. The space \( A \) is given as follows.

\[ A(C^d) = \bigcap_{r \in \mathbb{R}_+^d} A(D_{d,r}(z)) \quad \text{and} \quad A(\{0\}) = \bigcup_{r \in \mathbb{R}_+^d} A(D_{d,r}(z)). \]

Here and in what follows, \( D_{d,r}(z_0) \) is the (open) polydisc

\[ \{ z = (z_1, \ldots, z_d) \in C^d ; |z_j - z_{0,j}| < r_j, j = 1, \ldots, d \}; \]

with center and radii given by

\[ z_0 = (z_{0,1}, \ldots, z_{0,d}) \in C^d \quad \text{and} \quad r = (r_1, \ldots, r_d) \in [0, \infty)^d. \]

1.4. A test function space introduced by Gröchenig. In this section we recall some comparison results deduced in \[ 13 \], between a test function space, \( S_C(R^d) \), introduced by Gröchenig in \[ 5 \] to handle modulation spaces with elements in spaces of ultra-distributions.

The definition of \( S_C(R^d) \) is given as follows.

**Definition 1.9.** The space \( S_C(R^d) \) consists of all \( f \in \mathcal{S}(R^d) \) such that

\[ f(x) = \int_{R^{2d}} F(y, \eta)e^{-i \frac{1}{2}(x-y)^2 + i|y|^2 + i|\eta|^2)}e^{it \xi(y, \eta - (x, \eta))} dyd\eta, \quad (1.16) \]

for some \( F \in L^\infty(R^{2d}) \cap \mathcal{E}'(R^{2d}) \).

Evidently, we could have included the factors \( e^{-|y|^2 + |\eta|^2}} \) and \( e^{it \xi(y, \eta)} \) in the function \( F(y, \eta) \) in \[ (1.16) \]. The following reformulation of \[ 13 \], Lemma 4.9] justifies the separation. The result is essential when deducing the characterizations of Pilipović spaces in Section 3.
Lemma 1.10. Let $F \in L^\infty(C^d) \cup \mathcal{E}'(C^d)$. Then the Bargmann transform of $f$ in (1.16) is given by $\Pi_A F_0$, where

$$F_0(x + i \xi) = (8\pi^5)^{-d} F(\sqrt{2}x, \sqrt{2}\xi). \quad (1.17)$$

The first part of the next result follows from [13, Theorem 4.10] and the last part of the result follows [8, Theorem 2.2]. The proof is therefore omitted.

Proposition 1.11. The following is true:

1. $\mathcal{S}_C(R^d) = \mathcal{H}_{b_1}(R^d)$;
2. the image of $L^\infty(C^d) \cap \mathcal{E}'(C^d)$ under the map $\Pi_A$ equals $\mathcal{A}_{b_1}(C^d)$.

2. PALEY-WIENER PROPERTIES FOR BARGMANN-PILIPOVIC SPACES

In this section we consider spaces of compactly supported functions with interiors in $\mathcal{A}_s(C^d)$ or in $\mathcal{A}'_s(C^d)$. We show that the images of such functions under the reproducing kernel $\Pi_A$ are equal to $\mathcal{A}_s(C^d)$, for some other choice of $s \leq b_1$. In the first part we state the main results given in Theorems 2.7–2.8. They are straight-forward consequences of Propositions 2.11, where more detailed information concerning involved constants are given. Thereafter we deduce results which are needed for their proofs. Depending on the choice of $s$, there are several different situations for characterizing $\mathcal{A}_s(C^d)$. This gives rise to a quite large flora of main results, where each one takes care of one situation.

In order to present the main results, we need the following definition.

Definition 2.1. Let $t_1, t_2 \in R^d_+$ be such that $t_1 \leq t_2$. Then the set $\mathcal{R}_{t_1,t_2}^*(C^d)$ consists of all non-zero positive Borel measures $\nu$ on $C^d$ such that the following is true:

1. $d\nu(z_1, \ldots, z_d)$ is radial symmetric in each variable $z_j$;
2. the support of $\nu$ contains

$$\{ z = (z_1, \ldots, z_d) \in C^d; |z_j| = t_{1,j} \text{ for every } j = 1, \ldots, d \}$$

and is contained in

$$\{ z = (z_1, \ldots, z_d) \in C^d; |z_j| \leq t_{2,j} \text{ for every } j = 1, \ldots, d \}.$$

The set of compactly supported, positive, bounded and radial symmetric measures is given by

$$\mathcal{R}^*(C^d) \equiv \bigcup_{t_1 \leq t_2 \in R^d_+} \mathcal{R}_{t_1,t_2}^*(C^d).$$

Remark 2.2. Let $t_1, t_2 \in R^d_+$ be such that $t_1 \leq t_2$, $p \in [1, \infty]$, $\mathcal{R}_{t_1,t_2}^p(C^d)$ be the set of all non-negative $F \in L^p(C^d)$ such that (1) and (2) in Definition 2.1 holds with $F$ in place of $\nu$ and let

$$\mathcal{R}^p(C^d) \equiv \bigcup_{t_1 \leq t_2 \in R^d_+} \mathcal{R}_{t_1,t_2}^p(C^d).$$
Then it is clear that $R_{t_1,t_2}^p(C^d)$ and $R_{t_1,t_2}(C^d)$ decrease with $p$ and are contained in $R_{t_1,t_2}^*(C^d)$ and $R_{t_1,t_2}^*(C^d)$, respectively. In particular, these sets contain $R_{t_1,t_2}^\infty(C^d)$ and $R_{t_1,t_2}^\infty(C^d)$, respectively, in [8].

Remark 2.3. It is clear that the sets in Definition 2.1 and Remark 2.2 are invariant under multiplications with positive measurable, locally bounded functions on $C^d$ which are radial symmetric in each complex variable $z_j$ in $z = (z_1, \ldots, z_d) \in C^d$. In particular, they are invariant under multiplications with $e^{t|z|^2}$ for every $t \in \R$.

Remark 2.4. Let $t_1, t_2 \in \R_+^d$ be such that $t_1 \leq t_2$ and $\nu \in R_{t_1,t_2}^*$. By Riesz representation theorem it follows that

$$d\nu(z) = d\theta d\nu_0(r),$$

where

$$z_j = r_j e^{i\theta_j}, \quad \theta = (\theta_1, \ldots, \theta_d) \in [0, 2\pi)^d,$$

and some non-zero positive Borel measures $\nu_0$ on $\R_+^d$ such that the support of $\nu_0$ contains $t_1$ and is contained in

$$\{ r \in \R_+^d ; r \leq t_2 \}.$$

2.1. Main results. Our main investigations concern mapping properties of operators of the form

$$F \mapsto \Pi_A(F \cdot \nu) \quad (2.1)$$

when acting on the spaces given in Definition 1.3 (2).

Before stating the main results we need the following lemmas, which explain some properties of the map (2.1) when acting on the monomials $e_\alpha(z)$.

**Lemma 2.5.** Let $t_1, t_2 \in \R_+^d$ be such that $t_1 \leq t_2$, $\nu \in R_{t_1,t_2}^*$, and let $\nu_0$ be the same as in Remark 2.4. Then

$$\Pi_A(e_\alpha \cdot \nu) = \varsigma_\alpha \alpha!^{-1} e_\alpha, \quad \alpha \in \N^d, \quad (2.2)$$

where

$$\varsigma_\alpha = 2^d \int e^{-|r|^2} r^{2\alpha} d\nu_0(r) \quad (2.3)$$

satisfies

$$t_1^{2\alpha} e^{-|t_2|^2} \lesssim \varsigma_\alpha \lesssim t_2^{2\alpha}, \quad \alpha \in \N^d. \quad (2.4)$$
Proof. By using polar coordinates in each complex variable when integrating we get

\[(\Pi_A(e_\alpha \cdot \nu))(z) = \pi^{-d} \alpha!^{-\frac{1}{2}} \int_{C^d} w^{\alpha} e^{(z,w) - |w|^2} \, d\nu(w)\]

\[= \pi^{-d} \alpha!^{-\frac{1}{2}} \int_{\Delta_{t_2}} I_\alpha(r, z) r^{\alpha} e^{-|r|^2} \, dv_0(r), \quad (2.5)\]

where

\[I_\alpha(r, z) = \int_{[0,2\pi]^d} e^{i(\alpha,\theta)} \left( \prod_{j=1}^d e^{z_j r_j e^{-i\theta_j}} \right) \, d\theta = \prod_{j=1}^d I_{\alpha_j}(r_j, z_j) \quad (2.6)\]

with

\[I_{\alpha_j}(r_j, z_j) = \int_0^{2\pi} e^{i\alpha_j \theta_j} e^{z_j r_j e^{-i\theta_j}} \, d\theta_j.\]

By Taylor expansions we get

\[I_{\alpha_j}(r_j, z_j) = \int_0^{2\pi} e^{i\alpha_j \theta_j} \left( \sum_{k=0}^{\infty} \frac{z_j^k r_j^k e^{-i k \theta_j}}{k!} \right) \, d\theta_j = \sum_{k=0}^{\infty} \left( \int_0^{2\pi} e^{i(\alpha_j - k) \theta_j} \, d\theta_j \right) \frac{z_j^k r_j^k}{k!} = \frac{2\pi z_j^{\alpha_j} r_j^{\alpha_j}}{\alpha_j!},\]

where the second equality is justified by

\[\sum_{k=0}^{\infty} \left( \int_0^{2\pi} |e^{i(\alpha_j - k) \theta_j}| \, d\theta_j \right) \frac{z_j^k r_j^k}{k!} = e^{2\pi(|z_1 r_1| + \cdots + |z_d r_d|)} < \infty\]

and Weierstrass’ theorem.

By inserting this into (2.5) and (2.6) we get

\[(\Pi_A(e_\alpha \cdot \nu))(z) = \pi^{-d} \alpha!^{-\frac{1}{2}} \int_{\Delta_{t_2}} (2\pi)^d r^{\alpha} z^{\alpha} e^{-|r|^2} \, dv_0(r)\]

\[= \varsigma_\alpha \alpha!^{-1} e_\alpha(z),\]

and (2.2) follows.

The estimates in (2.4) are straight-forward consequences of (2.3) and the support properties of \(\nu_0\). The details are left for the reader. \(\square\)

By replacing \(\nu\) in the previous lemma with suitable radial symmetric compactly supported distributions we get the following.

Lemma 2.6. Let \(s > 1\), \(t_1, t_2 \in \mathbb{R}^d_+\) be such that \(t_1 \leq t_2\), \(\nu(z) \in \mathcal{E}'(\mathbb{C}^d)\) be radial symmetric in each \(z_j\) such that

\[\text{supp } \nu \subseteq \{z \in \mathbb{C}^d; t_{1,j} \leq |z_j| \leq t_{2,j}\}\]
Then $(2.2)$ holds with
\[ \varsigma_\alpha = 2^d \alpha^{-1} \langle \nu_0, \phi_\alpha \rangle, \quad \phi_\alpha(r) = e^{-|r|^2} r^{2\alpha} \cdots r_d, \ r \in \mathbb{R}^d, \ \alpha \in \mathbb{N}^d \]
for some $\nu_0 \in \mathcal{E}'_s(\mathbb{R}^d)$ with
\[ \text{supp} \nu_0 \subseteq \{ r \in \mathbb{R}^d : t_{1,j} \leq r_j \leq t_{2,j} \text{ for every } j \}. \]
Furthermore,
\[ \varsigma_\alpha \lesssim t_2^{2\alpha}. \]  

**Proof.** By using polar coordinates in each complex variable, the pullback formula [8, Theorem 6.1.2] and Fubini’s theorem for distributions and ultra-distributions, we get
\[ (\Pi_A(e_\alpha \cdot \nu))(z) = \pi^{-d} \langle \nu, e^{(z \cdot -): 1/2} e_\alpha \rangle = \pi^{-d} \alpha^{-1/2} \langle \nu_0 \otimes 1_{[0,2\pi]^d}, \Psi \rangle, \]
for some $\nu_0 \in \mathcal{E}'_s(\mathbb{R}^d)$, where
\[ \Psi(r, \theta) = e^{-|r|^2} \prod_{j=1}^d \left( e^{z_j r_j e^{-i\theta_j}} r_j^{a_j} e^{i\alpha_j j} \right). \]
By the same arguments as in the proof of Lemma 2.5 we get
\[ \langle 1_{[0,2\pi]^d}, \Psi(r, \cdot) \rangle = (2\pi)^d \alpha^{-1} e^{-|r|^2} r^{2\alpha} \cdots r_d \cdot z^\alpha = (2\pi)^d \alpha^{-1/2} \phi_\alpha(r) e_\alpha(z), \]
and $(2.2)$ follows with $\varsigma_\alpha$ given by $(2.7)$, by combining the latter identity with $(2.9)$.

The support assertions for $\nu_0$ follow from the support properties of $\nu$, and the estimate $(2.8)$ follows from the fact that $t_2^{2\alpha} \phi_\alpha$ is a bounded set in $\mathcal{E}_s(\mathbb{R}^d)$ with respect to $\alpha$ in the support of $\nu_0$. This gives the result. \( \square \)

Due to Lemma 2.6, we let $\mathcal{E}_{RS}(C^d)$ be the set of all
\[ \nu(z) \in \bigcup_{s > 1} \mathcal{E}'_s(C^d) \]
which are radial symmetric in each $z_j$ and such that $0 \notin \text{supp}(\nu)$.

**Theorem 2.7.** Let $s_1 \in (0, \frac{1}{d})$, $s_2 \in [0, \frac{1}{d})$ and $\nu \in \mathcal{R}^s(C^d) (\nu \in \mathcal{E}_{RS}(C^d))$. Then the map $(2.1)$ from $\mathcal{A}_0(C^d)$ to $A(C^d)$ is uniquely extendable to homeomorphisms (continuous mappings) on
\[ \mathcal{A}_{s_2}(C^d), \quad \mathcal{A}_{0,s_1}(C^d), \quad \mathcal{A}'_{0,s_1}(C^d) \text{ and on } \mathcal{A}'_{s_2}(C^d). \]

**Theorem 2.8.** Let $\sigma, \sigma_0 \in \mathbb{R}_+$ and $\nu \in \mathcal{R}^s(C^d) (\nu \in \mathcal{E}_{RS}(C^d))$. Then the following is true:

1. If $\sigma_0 = \frac{\sigma}{2^{\sigma+1}}$, then the map $(2.1)$ from $\mathcal{A}_0(C^d)$ to $A(C^d)$ is uniquely extendable to homeomorphisms (continuous mappings) from $\mathcal{A}'_{\sigma_0}(C^d)$ to $\mathcal{A}'_{\sigma_0}(C^d)$, and from $\mathcal{A}'_{0,\sigma_0}(C^d)$ to $\mathcal{A}'_{0,\sigma_0}(C^d)$;
Remark 2.10

Proposition 2.11. Let \( \sigma \) and let \( \mathcal{A}_0(C^d) \) to \( \mathcal{A}(C^d) \) is uniquely extendable to homeomorphisms (continuous mappings) from \( \mathcal{A}_{0,\beta_0}(C^d) \) to \( \mathcal{A}_{0,\beta}(C^d) \), and from \( \mathcal{A}_{\alpha_0}(C^d) \) to \( \mathcal{A}_{\alpha}(C^d) \);

Theorem 2.7 still holds true after \( E \) is extended.

The limit cases for the situations in the previous theorem are treated in the next result.

Theorem 2.9. Let \( \nu \in \mathcal{R}^*(C^d) \) (\( \nu \in \mathcal{E}_{RS}(C^d) \)) and \( s = \frac{1}{2} \). Then the following is true:

1. The map \((2.1)\) from \( \mathcal{A}_0(C^d) \) to \( \mathcal{A}(C^d) \) is uniquely extendable to homeomorphisms (continuous mappings) from \( \mathcal{A}_{0,\beta_0}(C^d) \) to \( \mathcal{A}_{0,\beta}(C^d) \), and from \( \mathcal{A}_{\alpha_0}(C^d) \) to \( \mathcal{A}_{\alpha}(C^d) \);

2. The map \((2.1)\) from \( \mathcal{A}_0(C^d) \) to \( \mathcal{A}(C^d) \) is uniquely extendable to homeomorphisms (continuous mappings) from \( \mathcal{A}'_{0,\alpha}(C^d) \) to \( \mathcal{A}_{0,\beta}(C^d) \), and from \( \mathcal{A}'_{\alpha_0}(C^d) \) to \( \mathcal{A}_{0,\beta}(C^d) \).

Remark 2.10. Since

\[ \mathcal{E}'(C^d) \cap L^\infty(C^d) \subseteq \mathcal{E}'(C^d) \subseteq \mathcal{E}'_s(C^d), \]

Theorem 2.7 still holds true after \( \mathcal{E}'_s \) has been replaces by \( \mathcal{E}' \) in (6).

The following result is an essential part of the proof of Theorem 2.7

Proposition 2.11. Let \( r_0 \in \mathbb{R}_+ \), \( s \in [0, \frac{1}{2}) \) and \( \nu \in \mathcal{R}^*(C^d) \) be fixed, and let \( \varsigma_\alpha \) be as in (2.3). Then the following is true:

1. The map \( F \mapsto \Pi_A(F \cdot \nu) \) from \( \mathcal{A}_0(C^d) \) to \( \mathcal{A}(C^d) \) is uniquely extendable to a homeomorphism on \( \mathcal{A}_0'(C^d) \), and

\[ c(\Pi_A(F \cdot \nu), \alpha) = \varsigma_\alpha \alpha!^{-1} c(F, \alpha), \quad F \in \mathcal{A}_0'(C^d), \quad \alpha \in \mathbb{N}^d; \quad (2.10) \]

2. it holds

\[ |c(F, \alpha)| \lesssim e^{-r|\alpha|^{\frac{1}{2}}} \quad (2.11) \]

for some \( r \in \mathbb{R}_+ \) such that \( r < r_0 \), if and only if

\[ |c(\Pi_A(F \cdot \nu), \alpha)| \lesssim e^{-r|\alpha|^{\frac{1}{2}}} \quad (2.12) \]

for some \( r \in \mathbb{R}_+ \) such that \( r < r_0 \);

3. it holds

\[ |c(F, \alpha)| \lesssim e^{r|\alpha|^{\frac{1}{2}}} \quad (2.13) \]
for some $r \in \mathbb{R}_+$ such that $r < r_0$, if and only if

$$|c(\Pi_A(F \cdot \nu), \alpha)| \lesssim e^{r |\alpha|^{1/2}}$$  \hspace{1cm} (2.14)

for some $r \in \mathbb{R}_+$ such that $r < r_0$.

Here it is understood that the signs in the exponents in (2.11) and (2.12) agree.

**Proof.** The assertion (1) is an immediate consequence of (2.2) in Lemma 2.5. In fact, by Lemma 2.5 and (2.2), the only possible extension of $F \mapsto \Pi_A(F \cdot \nu)$ is to let

$$\Pi_A(F \cdot \nu)(z) = \sum_{\alpha \in \mathbb{N}^d} (\varsigma_{\alpha} \alpha!^{-1} c(F, \alpha)) e_\alpha(z)$$  \hspace{1cm} (2.15)

when

$$F(z) = \sum_{\alpha \in \mathbb{N}^d} c(F, \alpha) e_\alpha(z),$$  \hspace{1cm} (2.16)

which obviously defines a continuous map on $\mathcal{A}'_0(\mathbb{C}^d)$.

Since

$$t_{\alpha} \alpha! \lesssim e^{r |\alpha|^{1/2}} \quad \text{and} \quad t_1^{2\alpha} \lesssim \varsigma_{\alpha} \lesssim t_2^{2\alpha},$$

for every $r \in \mathbb{R}_+$, it follows from (2.10) that

$$e^{-r |\alpha|^{1/2}} \lesssim \frac{|c(\Pi_A(F \cdot \nu), \alpha)|}{|c(F, \alpha)|} \lesssim e^{r |\alpha|^{1/2}}$$

for every $r \in \mathbb{R}_+$. This gives (2) and (3). \hfill \Box

In order to prove Theorem 2.8 we need the next proposition. The result is a straightforward consequence of (2.2), (2.4) in Lemma 2.5 and Proposition 2.11 (1). The details are left for the reader.

**Proposition 2.12.** Let $h \in \mathbb{R}_+$, $\tau \in \mathbb{R}$, $t_1, t_2 \in \mathbb{R}_+^d$, $\nu \in \mathcal{R}_{t_1, t_2}(\mathbb{C}^d)$ and $\varsigma_{\alpha}$ be as in (2.3). Then

$$|c(F, \alpha)| \lesssim h^{|\alpha|} \alpha!^{\tau}$$

$$\Rightarrow \quad |c(\Pi_A(F \cdot \nu), \alpha)| \lesssim h^{|\alpha|} t_2^{2\alpha} \alpha!^{\tau - 1}$$  \hspace{1cm} (2.17)

and

$$|c(\Pi_A(F \cdot \nu), \alpha)| \lesssim h^{|\alpha|} t_1^{2\alpha} \alpha!^{\tau - 1}$$

$$\Rightarrow \quad |c(F, \alpha)| \lesssim h^{|\alpha|} \alpha!^{\tau}$$  \hspace{1cm} (2.18)

**Proof of Theorems 2.7 and 2.8.** Theorem 2.7 is a straightforward consequence of Proposition 2.11. The details are left for the reader.
By letting $\sigma > 0$, $\sigma_0 = \frac{\sigma}{2\sigma_0 + 1}$ and $\tau = \frac{1}{2\sigma_0}$, then $\tau - 1 = \frac{1}{2\sigma}$. Hence (2.17) and (2.18) give

$$|c(F, \alpha)| \lesssim h^{\|\alpha\|_1}$$

for some (for every) $h > 0$ \(\Leftrightarrow\)

$$|c(\Pi_A(F \cdot \nu), \alpha)| \lesssim h^{\|\alpha\|_1}$$

for some (for every) $h > 0$.

Theorem 2.8 (1) now follows from Proposition 2.11 (1) and (2.19).

If instead $\sigma > \frac{1}{2}$, $\sigma_0 = \frac{\sigma^2}{2\sigma - 1}$ and $\tau = \frac{1}{2\sigma_0}$, then $\tau - 1 = \frac{1}{2\sigma}$. Hence (2.17) and (2.18) give

$$|c(F, \alpha)| \lesssim h^{\|\alpha\|_1}$$

for some (for every) $h > 0$ \(\Leftrightarrow\)

$$|c(\Pi_A(F \cdot \nu), \alpha)| \lesssim h^{\|\alpha\|_1}$$

for some (for every) $h > 0$.

Theorem 2.8 (2) now follows from Proposition 2.11 (1) and (2.20).

If instead $\sigma < \frac{1}{2}$, $\sigma_0 = \frac{\sigma}{\sigma - 1}$ and $\tau = \frac{1}{2\sigma_0}$, then $\tau - 1 = \frac{1}{2\sigma}$. Hence (2.17) and (2.18) give

$$|c(F, \alpha)| \lesssim h^{\|\alpha\|_1}$$

for some (for every) $h > 0$ \(\Leftrightarrow\)

$$|c(\Pi_A(F \cdot \nu), \alpha)| \lesssim h^{\|\alpha\|_1}$$

for some (for every) $h > 0$.

Theorem 2.8 (3) now follows from Proposition 2.11 (1) and (2.21).

Finally, Theorem 2.9 follows by similar arguments, letting $\tau = 1$ when proving (1) and letting $\tau = 0$ when proving (2) in Theorem 2.8. The details are left for the reader.

3. Characterizations of Pilipović spaces

In this section we combine Lemma 1.10 with Theorems 2.7–2.9 to get characterizations of Pilipović spaces.

We shall perform such characterizations by considering mapping properties of extensions of the map

$$F \mapsto \Theta_F = \Theta_{F,r}$$

from $A_0(\mathbb{C}^d)$ to $C^\infty(\mathbb{R}^d)$, where

$$\Theta_{F,r}(x) = \int\int_{D_r(0)} F(y + i\eta)e^{-\frac{1}{2}(|y|_2^2 + |\eta|_2^2)}e^{i\frac{1}{2}(y \cdot \eta) - (x \cdot \eta)} dy d\eta.$$  (3.1)

Here we have identified $D_r(0) \subseteq \mathbb{C}^d$ with the polydisc

$$\{ (x, \xi) \in \mathbb{R}^{2d} ; x_j^2 + \xi_j^2 < r_j^2, \quad j = 1, \ldots, d \}$$

in $\mathbb{R}^{2d}$ when $r = (r_1, \ldots, r_d) \in \mathbb{R}^d_+$. We notice that $\Theta_F$ equals $f$ in (1.16), if in addition $F \in L^\infty(\mathbb{R}^{2d})$.\]
We recall that the Bargmann transform is homeomorphic between the spaces in (1.4) and (1.5) when \( s_1 \in \mathbb{R}_+ \) and \( s_2 \in \mathbb{R}_+ \). The following results of Paley-Wiener types for Pilipović spaces, follow from these facts and by some straight-forward combinations of Lemma [4.10] and Theorems 2.7, 2.9. The details are left for the reader.

**Theorem 3.1.** Let \( s_1 \in [0, \frac{1}{2}) \), \( s_2 \in (0, \frac{1}{2}) \) and \( r \in \mathbb{R}^d \). Then the map \( (3.1) \) from \( \mathcal{A}_0(\mathbb{C}^d) \) to \( C^\infty(\mathbb{R}^d) \) is uniquely extendable to homeomorphisms (continuous mappings) from the spaces in (1.4) to corresponding spaces in (1.5).

**Theorem 3.2.** Let \( \sigma, \sigma_0 \in \mathbb{R}_+ \) and \( r \in \mathbb{R}^d_+ \). Then the following is true:

1. If \( \sigma_0 = \frac{\sigma}{2\sigma + 1} \), then the map \( (3.1) \) from \( \mathcal{A}_0(\mathbb{C}^d) \) to \( C^\infty(\mathbb{R}^d) \) is uniquely extendable to homeomorphisms from \( \mathcal{A}_{\beta s_0}(\mathbb{C}^d) \) to \( \mathcal{H}_{\beta s}(\mathbb{R}^d) \), and from \( \mathcal{A}'_{\beta s_0}(\mathbb{C}^d) \) to \( \mathcal{H}'_{\beta s}(\mathbb{R}^d) \);

2. if \( \sigma > \frac{1}{2} \) and \( \sigma_0 = \frac{\sigma}{2\sigma - 1} \), then the map \( (3.1) \) from \( \mathcal{A}_0(\mathbb{C}^d) \) to \( C^\infty(\mathbb{R}^d) \) is uniquely extendable to homeomorphisms from \( \mathcal{A}'_{0,\beta s_0}(\mathbb{C}^d) \) to \( \mathcal{H}_{\beta s}(\mathbb{R}^d) \), and from \( \mathcal{A}'_{\beta s_0}(\mathbb{C}^d) \) to \( \mathcal{H}'_{\beta s}(\mathbb{R}^d) \);

3. if \( \sigma < \frac{1}{2} \) and \( \sigma_0 = \frac{\sigma}{2\sigma} \), then the map \( (3.1) \) from \( \mathcal{A}_0(\mathbb{C}^d) \) to \( C^\infty(\mathbb{R}^d) \) is uniquely extendable to homeomorphisms from \( \mathcal{A}_{\beta s}(\mathbb{C}^d) \) to \( \mathcal{H}_{\beta s}(\mathbb{R}^d) \), and from \( \mathcal{A}_{0,\beta s}(\mathbb{C}^d) \) to \( \mathcal{H}'_{\beta s}(\mathbb{R}^d) \).

**Theorem 3.3.** Let \( s = \sigma = \frac{1}{2} \) and \( r \in \mathbb{R}^d_+ \). Then the following is true:

1. The map \( (3.1) \) from \( \mathcal{A}_0(\mathbb{C}^d) \) to \( C^\infty(\mathbb{R}^d) \) is uniquely extendable to homeomorphisms (continuous mappings) from \( \mathcal{A}'_{0,\beta s}(\mathbb{C}^d) \) to \( \mathcal{H}_{\beta s}(\mathbb{R}^d) \), and from \( \mathcal{A}'_{\beta s}(\mathbb{C}^d) \) to \( \mathcal{A}_{0,\beta s}(\mathbb{R}^d) \);

2. The map \( (3.1) \) from \( \mathcal{A}_0(\mathbb{C}^d) \) to \( C^\infty(\mathbb{R}^d) \) is uniquely extendable to homeomorphisms (continuous mappings) from \( \mathcal{A}'_{\beta s}(\mathbb{C}^d) \) to \( \mathcal{H}_{\beta s}(\mathbb{R}^d) \), and from \( \mathcal{A}_{0,\beta s}(\mathbb{C}^d) \) to \( \mathcal{H}'_{\beta s}(\mathbb{R}^d) \).

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