COMBINED EFFECTS OF TWO NONLINEARITIES IN LIFESPAN OF SMALL SOLUTIONS TO SEMI-LINEAR WAVE EQUATIONS

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In memory of Rentaro Agemi

Abstract. This paper investigates combined effects of two nonlinearities in lifespan of small solutions to semi-linear wave equations. Using the method of vector fields in Klainerman (Comm. Pure Appl. Math. 38 (1985), 321–332) together with some of techniques developed in Li and Zhou (Indiana Univ. Math. J. 44 (1995), 1207–1248), we show that the upper bound on the lifespan recently obtained by Han and Zhou (Comm. Partial Differential Equations 39 (2014), 651–665) is sharp in general, at least in space dimensions $n = 2, 3$. Moreover, we also show that in the threshold case, the equation admits global (in time) solutions.

1. Introduction and Result

This paper is concerned with the Cauchy problem for the wave equation with its nonlinearity consisting of sum of the two power-type terms

$$\partial_t^2 u - \Delta u = |u|^q + |\partial_t u|^p, \quad t > 0, \quad x \in \mathbb{R}^n,$$

subject to initial data of the order of small $\varepsilon > 0$

$$u(0, x) = \varepsilon f(x), \quad \partial_t u(0, x) = \varepsilon g(x).$$

Our main concern lies in the estimate of lifespan of solutions to (1.1)-(1.2) when $f$ and $g$ are smooth, decaying fast at the spatial infinity, and $q > q_0(n)$, $p > p_0(n)$. Here, and in the following, we mean

$$q_0(n) := \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)}$$

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and
\[ p_0(n) := \frac{n + 1}{n - 1}. \]
Throughout this paper, we suppose \( n \geq 2 \). Actually, in our main theorem we will assume \( n = 2, 3 \).

It is well known that if \( q > q_0(n) \), then the Cauchy problem for the semi-linear wave equation
\[ \partial_t^2 v - \Delta v = |v|^q, \quad t > 0, \quad x \in \mathbb{R}^n \]
admits a unique global (in time) solution for small, smooth data with compact support. (See [12] for \( n = 3 \), [6] for \( n = 2 \), and [30] for \( n = 4 \). For \( n \geq 5 \) and radial data, see [16]-[17], [21]. For \( n \geq 5 \) and general data, see [4] and [27]. See also a recent paper of [18]). We also know that if \( p > p_0(n) \) and \( n = 2, 3 \), then the Cauchy problem for the semi-linear wave equation
\[ \partial_t^2 w - \Delta w = |\partial_t w|^p, \quad t > 0, \quad x \in \mathbb{R}^n \]
has a unique, global (in time) solution for small, smooth data decaying fast at the spatial infinity. (See [9], [28]. See also an earlier result of Sideris for \( n = 3 \) [25].) Recently, the present authors have succeeded in extending the result of [9], [28], [25] to the case of higher space dimensions \( n \geq 4 \) under the assumption that the initial data is spherically symmetric about the origin \( x = 0 \) [10]. We note that the assumptions \( q > q_0(n) \) and \( p > p_0(n) \) are sharp in general, because if \( 1 < q \leq q_0(n) \) (resp. \( 1 < p \leq p_0(n) \)), then there exist smooth, compactly supported functions \( v_0 \) and \( v_1 \) (resp. \( w_0 \) and \( w_1 \)) such that the Cauchy problem for (1.5) (resp. (1.6)) with data \((v(0), \partial_t v(0)) = (\varepsilon v_0, \varepsilon v_1) \) (resp. \((w(0), \partial_t w(0)) = (\varepsilon w_0, \varepsilon w_1) \)) has a solution which blows up in finite time no matter how small \( \varepsilon > 0 \) is. (See [12], [5], [26], [24], and [29] for (1.5), [1], [13], [24], [22], and [31] for (1.6).)

Recently, Han and Zhou [7] have studied the upper bound on lifespan of small solutions to (1.1). The feature of (1.1) lies in its “combined” nonlinearity, and they have found a new and surprising combined effect in the lifespan. Among others, they have proved that even if \( q > q_0(n) \), \( p > p_0(n) \) and \( f, g \in C_0^\infty(\mathbb{R}^n) \), the additional condition that
\[ q < \frac{4}{(n - 1)p - 2} + 1 \quad \text{and} \quad f, g \geq 0, g \text{ does not identically vanish} \]
leads to the blowup of solutions in finite time and
\[ T_\varepsilon \leq C\varepsilon^{-\frac{2p(1-q)}{(2p-q+2)(n-1)p-2q+1)}} \]
for all \( \varepsilon > 0 \); see Theorem 1.4 of [7].
The purpose of this paper is twofold. First, we intend to show that when
\( q > q_0(n) \), \( p > p_0(n) \) and \( q < 4/(n-1)p - 2 + 1 \), a unique solution to
(1.1)-(1.2) exists and its lifespan satisfies
\[
T_\varepsilon \geq c\varepsilon^{-\frac{2p(q-1)}{2q+2-(n-1)p(q-1)}}
\]
for small \( \varepsilon > 0 \), which means that the upper bound (1.8) is sharp in general.
Second, we explore the “opposite case” where \( q > q_0(n) \), \( p > p_0(n) \) and
\( q \geq 4/(n-1)p - 2 + 1 \), and intend to show that the Cauchy problem of
(1.1)-(1.2) has a unique global (in time) solution for small \( \varepsilon > 0 \). Actually,
for some technical reasons, it is the case of \( n = 2, 3 \) that we can handle.
Moreover, in the proof of our result of global existence, we like to focus on
the case of \( q = 4/(n-1)p - 2 + 1 \) so as to keep this paper to a moderate
length. Now we are in a position to state the main theorem precisely.

**Theorem 1.1.** Let \( n = 2, 3 \). Assume
\[
q > q_0(n), \ p > p_0(n) \text{ and } p \leq q \leq \frac{4}{(n-1)p - 2} + 1.
\]
Suppose that \( f \) and \( g \) satisfy for \( s(q) := 1/2 - 1/q \),
\[
\partial_x^\alpha f \in \dot{H}^1(\mathbb{R}^n) \cap \dot{H}^{s(q)}(\mathbb{R}^n), \ \partial_x^\alpha g \in L^2(\mathbb{R}^n) \cap \dot{H}^{s(q)-1}(\mathbb{R}^n)
\]
for \( |\alpha| \leq 2 \),
\[
x_i\partial_x^\beta f \in \dot{H}^1(\mathbb{R}^n) \cap \dot{H}^{s(q)}(\mathbb{R}^n), \ x_i\partial_x^\beta g \in L^2(\mathbb{R}^n) \cap \dot{H}^{s(q)-1}(\mathbb{R}^n)
\]
for \( 1 \leq i \leq n, 1 \leq |\beta| \leq 2 \), and
\[
x_i x_j \partial_x^\gamma f \in \dot{H}^1(\mathbb{R}^n) \cap \dot{H}^{s(q)}(\mathbb{R}^n), \ x_i x_j \partial_x^\gamma g \in L^2(\mathbb{R}^n) \cap \dot{H}^{s(q)-1}(\mathbb{R}^n)
\]
for \( 1 \leq i, j \leq n, |\gamma| = 2 \). Set
\[
\Lambda := \sum_{|\alpha| \leq 2} (\|\partial_x^\alpha f\|_{\dot{H}^1 \cap \dot{H}^{s(q)}} + \|\partial_x^\alpha g\|_{L^2 \cap \dot{H}^{s(q)-1}})
+ \sum_{1 \leq i \leq n} \left( \|x_i \partial_x^\beta f\|_{\dot{H}^1 \cap \dot{H}^{s(q)}} + \|x_i \partial_x^\beta g\|_{L^2 \cap \dot{H}^{s(q)-1}} \right)
+ \sum_{1 \leq i, j \leq n} \left( \|x_i x_j \partial_x^\gamma f\|_{\dot{H}^1 \cap \dot{H}^{s(q)}} + \|x_i x_j \partial_x^\gamma g\|_{L^2 \cap \dot{H}^{s(q)-1}} \right).
\]
Then, there exists an \( \varepsilon_0 \) depending on \( n, p, q, \) and \( \Lambda \) such that if \( 0 < \varepsilon < \varepsilon_0 \),
then the Cauchy problem (1.1)-(1.2) admits a unique solution satisfying
\[
\Gamma^\alpha u \in C([0, T); \dot{H}^{s(q)}(\mathbb{R}^n)), \ \partial_j \Gamma^\alpha u \in C([0, T); L^2(\mathbb{R}^n))
\]
for \( j = 0, \ldots, n, \) \(|\alpha| \leq 2,\)

\[
\sum_{|\alpha| \leq 2} (1 + t)^{-\gamma(p,q)} \| \Gamma^\alpha u(t) \|_{\dot{H}^s(q)} + \sum_{0 \leq j \leq n} \| \partial_j \Gamma^\alpha u(t) \|_{L^2} \leq C\varepsilon
\]

for \( 0 < t < T. \) Here

\[
\gamma(p,q) := \frac{q + 1}{q - 1} \left(\frac{1}{p} - \frac{1}{q}\right)
\]

and

\[
T = c\varepsilon^{-\frac{2p(q-1)}{n(q-1)p(q-1)}} \text{ if } q < \frac{4}{(n-1)p-2} + 1,
\]

\[
T = \infty \quad \text{if } q = \frac{4}{(n-1)p-2} + 1.
\]

Here, and in the following discussion as well, by \( \| \cdot \|_{\dot{H}^1 \cap \dot{H}^1(q)} \) we naturally mean

\[
\| \phi \|_{\dot{H}^1 \cap \dot{H}^1(q)} := \| \phi \|_{\dot{H}^1(\mathbb{R}^n)} + \| \phi \|_{\dot{H}^1(q)(\mathbb{R}^n)}.
\]

**Remark 1.2.** Since the assumption (1.10) is equivalently stated as

\[
p_0(n) < p < q_0(n) < q < 1 + \frac{4}{n-1} \quad \text{and} \quad (q-1)\left(p - \frac{2}{n-1}\right) \leq \frac{4}{n-1},
\]

the quantity \( \gamma(p,q) \) given in (1.17) is strictly positive.

**Remark 1.3.** If we assume \( q > q_0(n), \) \( p > p_0(n) \) and \( q > 4/((n-1)p-2) + 1 \) \((n = 2, 3)\) in place of (1.10), then we get a result of global existence of small solutions. Its proof naturally uses a norm similar to (1.16), with the factor \((1 + t)^{-\gamma(p,q)}\) chosen differently from (1.17). In a forthcoming paper, we wish to address such a result, together with sharp lower bounds on lifespan in the case \( q \leq q_0(n) \) or \( p \leq p_0(n) \).

**Remark 1.4.** As is obvious from its proof, Theorem 1.1 remains valid for the equation such as

\[
\partial_t^2 u - \Delta u = C|u|^q + \sum_{j=0}^n C_j|\partial_j u|^p, \quad (C, C_0, \ldots, C_n \in \mathbb{R}).
\]

In the proof of Theorem 1.1, we view the equation (1.1) as a “principal part” \( \partial_t^2 u - \Delta u = |u|^q \) plus a “forcing term” \( |\partial_t u|^p \). Hence our consideration naturally starts with recalling how to solve the Cauchy problem for (1.5). In the iteration argument, we reuse an old idea of one of the present authors [8] using the homogeneous Sobolev space \( \dot{H}^s(q) \). This method stemmed from
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the important paper of Li and Zhou [20], and the article [8] reported an idea of proving global existence of small solutions to (1.5) under the condition $q > q_0(n)$ ($n = 2, 3, 4$). The equation (1.1) has the “forcing term” $|\partial_t u|^p$, which involves a higher-order derivatives of $u$. This naturally leads us to a modification of the norm in the iteration scheme, and we allow for some growth of the $H^{s(a)}$ norm; see the growth factor $(1 + t)^{\gamma(p,q)}$ in (1.16). Also, we inevitably use the standard (generalized) energy norm to recover the loss of derivatives.

We close this section by explaining the notation in this paper. We employ the notation $\langle \tau \rangle := \sqrt{1 + \tau^2}$ for $\tau \in \mathbb{R}$. Following Klainerman [14]–[15], we introduce several partial differential operators as follows: $\partial_0 = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $L_J = t \partial_0 + x_j \partial_j$ ($j = 1, \ldots, n$), $L_0 = t \partial_0 + x_1 \partial_1 + \cdots + x_n \partial_n$. These operators $\partial_0, \ldots, \partial_n, L_1, \ldots, L_n, \Omega_{12}, \ldots, \Omega_{1n}, \Omega_{23}, \ldots, \Omega_{n-1n}$ and $L_0$ are denoted by $\Gamma_0, \ldots, \Gamma_\nu$ in this order, where $\nu := (n^2 + 3n + 2)/2$. For a multi-index $\alpha = (\alpha_0, \ldots, \alpha_\nu)$, $\Gamma^\alpha_0 \cdots \Gamma^\alpha_\nu$ is denoted by $\Gamma^\alpha$. Moreover we will use the operators $|D| := \sqrt{-\Delta}$.

It is necessary to define the norm for $1 \leq p, q < \infty$

\begin{equation}
\|v(\cdot)\|_{p,q} := \left( \int_0^\infty \left( \int_{S^{n-1}} |v(r\omega)|^q dS_\omega \right)^{p/q} r^{n-1} dr \right)^{1/p}
\end{equation}

with an obvious modification for $p = \infty$

\begin{equation}
\|v(\cdot)\|_{\infty,q} := \sup_{r > 0} \left( \int_{S^{n-1}} |v(r\omega)|^q dS_\omega \right)^{1/q}
\end{equation}

where $r = |x|$, $\omega \in S^{n-1}$. These types of norms have been effectively used for the existence theory of solutions to fully nonlinear wave equations in [19], [20]. Let $N$ be a nonnegative integer and $\Psi$ a characteristic function of a set of $\mathbb{R}^n$. We define the norm

\begin{equation}
\|u(t, \cdot)\|_{\Gamma, N, p, q, \Psi} := \sum_{|\alpha| \leq N} \|\Psi(\cdot)\Gamma^\alpha u(t, \cdot)\|_{p,q}.
\end{equation}

For $\Psi \equiv 1$ in (1.22), we omit the subscript $\Psi$. If $p = q$, then we omit $q$. If $N = 0$, then we omit both the subscripts $\Gamma$ and $N$. In sum, we abbreviate
the notation of the norm $\|u(t, \cdot)\|_{\Gamma, N, p, q, \Psi}$ to
\[\|u(t, \cdot)\|_{\Gamma, N, p, q, \Psi}, \text{ when } \Psi \equiv 1,\]
\[\|u(t, \cdot)\|_{\Gamma, N, p, \Psi}, \text{ when } p = q,\]
\[\|u(t, \cdot)\|_{\Gamma, N, p, q}, \text{ when } p = q \text{ and } \Psi \equiv 1,\]
\[\|u(t, \cdot)\|_{p, q, \Psi}, \text{ when } N = 0,\]
\[\|u(t, \cdot)\|_{p, q}, \text{ when } N = 0 \text{ and } \Psi \equiv 1,\]
\[\|u(t, \cdot)\|_{p}, \text{ when } N = 0, p = q, \text{ and } \Psi \equiv 1.\]

Throughout this paper, when denoting by $a+$ (or $a-$) for $a \in \mathbb{R}$, we mean that the relevant estimate holds for $a + \varepsilon$ (or $a - \varepsilon$) for sufficiently small $\varepsilon > 0$. Also, the notation $\infty -$, which appears in (3.36) and so on, means that the relevant estimate holds for sufficiently large values.

This paper is organized as follows. In the next section, we collect several key inequalities, and in Section 3 they will play an essential role in the proof of Theorem 1.1. Section 4 is devoted to the proof of the inequalities (2.2) and (2.9) which will be of independent interest. We also give a proof of (3.6).

2. Preliminaries

In this section, we give some preliminary results which we will use in the proof of Theorem 1.1.

**Proposition 2.1.** For any $0 < s < n/2$, there exists a constant $C > 0$ such that the inequality
\[\|v\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|v\|_{\dot{H}^s(\mathbb{R}^n)}\]
holds, where $p^*$ is defined as $1/p^* = 1/2 - s/n$.

*Proof.* This is the standard Sobolev embedding.

**Proposition 2.2.** For any $2 < q < \infty$, there exists a constant $C > 0$ such that the inequality
\[\left\|v(r\omega)\right\|_{L^2(S^{n-1})} \left\|L_{((\lambda, \infty); t^{n-1}, dr)}\right\| \leq C\lambda^{-(n-1)s(q)}\|v\|_{\dot{H}^s(\mathbb{R}^n)}\]
holds for all $\lambda > 0$, where $s(q) := 1/2 - 1/q$.

*Proof.* See Theorem 2.10 of Li and Zhou [20]. For the reader’s convenience, we give a different proof of (2.2) in Section 4 below, by using the end-point trace inequality and the real interpolation method.
Proposition 2.3. If $1 \leq p < \infty$ and $s > n/p$, then the inequality
\[(2.3) \quad (1 + |t| + |x|)^{(n-1)/p}(1 + ||t| - |x||)^{1/p}|v(t, x)| \leq C\|v(t, \cdot)\|_{\Gamma, s, p}\]
holds. If $1 \leq p < q < \infty$ and $1/q \geq 1/p - 1/n$, then the inequality
\[(2.4) \quad \|v(t, \cdot)\|_{q, \chi_1} \leq C(1 + |t|)^{-n(1/p - 1/q)}\|v(t, \cdot)\|_{\Gamma, 1, p}\]
holds. Here $\chi_1$ denotes the characteristic function of the set $\{x \in \mathbb{R}^n : |x| < (1 + |t|)/2\}$.

Proof. For the proof of (2.3), see Klainerman [15]. For (2.4), see Theorem 2.9 of Li and Zhou [20].

Proposition 2.4. Let $n \geq 2$. If $1/2 < s < n/2$, then the inequalities
\[(2.5) \quad \sup_{r > 0} r^{(n/2) - s}(1 - \Delta_{S^{n-1}})^{(2s-1)/4}v(r, \cdot)\|_{L^2(S^{n-1})} \leq C\|v\|_{\dot{H}^s(\mathbb{R}^n)},\]
\[(2.6) \quad \sup_{r > 0} r^{(n/2) - s}\|v(r, \cdot)\|_{L^p(S^{n-1})} \leq C\|v\|_{\dot{H}^s(\mathbb{R}^n)}\]
hold, where $p$ is defined as
\[(2.7) \quad \frac{1}{p} = \frac{1}{2} - \frac{s - \frac{1}{2}}{n - 1}.
\]
If
\[(2.8) \quad 2 \leq p \leq \min \left\{4, \frac{2(n - 1)}{n - 2}\right\} \quad \text{for } n = 2, 4, 5, \ldots,\]
\[2 \leq p < 4 \quad \text{for } n = 3,\]
then the inequality
\[(2.9) \quad \sup_{r > 0} r^{(n-1)/2}\|v(r, \cdot)\|_{L^p(S^{n-1})} \leq C\|\partial_r v\|_{L^2(\mathbb{R}^n)}^{1/2} \left(\sum_{|\alpha| \leq 1} \|\Omega^\alpha v\|_{L^2(\mathbb{R}^n)}\right)^{1/2}\]
holds.

Proof. For the proof of Trace Lemma (2.5), see [11] for $n \geq 3$ and [3] for $n \geq 2$. By the Sobolev embedding on the unit sphere $S^{n-1}$, we obtain (2.6) directly from (2.5). We will prove (2.9) in Section 4 below.

Proposition 2.5. Let $n \geq 2$ and $2 < q < \infty$. Set $s(q) := 1/2 - 1/q$. Suppose that $u$ is a solution to the inhomogeneous wave equation $\partial^2_t u - \Delta u = F$ in $\mathbb{R}^n \times (0, \infty)$ with data $(f, g)$ given at $t = 0$. Then, for $t > 0$ we have
\[(2.10) \quad \|u(t, \cdot)\|_{\dot{H}^s(q)} \leq \|f\|_{\dot{H}^s(q)} + \|g\|_{\dot{H}^{-1+s(q)}} + C \int_0^t \|F(\tau, \cdot)\|_{p_1, \chi_1} d\tau
\[+ C \int_0^t (\tau)^{-(n-1)/2 + 1/q}\|F(\tau, \cdot)\|_{1, p_2, \chi_2} d\tau\]
and, for \( j = 0, \ldots, n \)

\[
\| \partial_j u(t, \cdot) \|_{L^2(\mathbb{R}^n)} \leq \| f \|_{H^1} + \| g \|_{L^2} + \int_0^t \| F(\tau, \cdot) \|_{L^2(\mathbb{R}^n)} d\tau.
\]

Here \( p_1 \) and \( p_2 \) are defined as

\[
(2.12) \quad \frac{1}{p_1} = \frac{(n+1)q+2}{2nq}, \quad \frac{1}{p_2} = \frac{q(n-1)+2}{2q(n-1)}.
\]

Also, the functions \( \chi_1 \) and \( \chi_2 \) denote the characteristic functions of the sets \( \{ x \in \mathbb{R}^n : |x| < (1+\tau)/2 \} \) and \( \{ x \in \mathbb{R}^n : |x| > (1+\tau)/2 \} \), respectively.

**Proof.** The proof of (2.11) is standard. For the proof of (2.10), we employ (2.1) and (2.6) with \( s = 1 - (1/2 - 1/q) \), and follow the proof of Theorem 2.11 of Li and Zhou [20].

**Proposition 2.6.** The following commuting relations hold:

\[
(2.13) \quad [\Gamma_i, \Box] = 0 \quad \text{for } i = 0, \ldots, \nu - 1, \quad \text{and } [L_0, \Box] = -2\Box,
\]

\[
(2.14) \quad [\Gamma_j, \Gamma_k] = \sum_{l=0}^{\nu} C_{l}^{j,k} \Gamma_l, \quad j, k = 0, \ldots, \nu.
\]

Here \( C_{l}^{j,k} \) denotes a constant depending on \( j, k, \) and \( l \).

**Proof.** We can verify these relations by direct computations.

**Remark 2.7.** In particular, we see by this proposition the equivalence between \( \| \partial_t u(t, \cdot) \|_{\Gamma,2,2} \) and \( \sum_{|\alpha| \leq 2} \| \partial_\alpha \Gamma^\alpha u(t, \cdot) \|_2 \). This fact will be repeatedly employed in the next section.

### 3. Proof of Theorem 1.1

We find the solutions to (1.1)-(1.2) by iteration. For any \( T > 0, \varepsilon > 0, \) and \((f, g)\) satisfying (1.11)-(1.13), let us define the set of functions

\[
X_T := \{ u \in C([0, T]; H^{s(q)} \cap H^1) : \Gamma^\alpha u \in C([0, T]; H^{s(q)}) \quad (|\alpha| \leq 2),
\]

\[
\partial_j \Gamma^\alpha u \in C([0, T]; L^2) \quad (|\alpha| \leq 2, \ j = 0, \ldots, n),
\]

\[
u(0) = \varepsilon f, \ \partial_t u(0) = \varepsilon g \}.
\]

We will see in Section 4 that the solution of \( \partial_t^2 w - \Delta w = 0 \) with data \((\varepsilon f, \varepsilon g)\) given at \( t = 0 \) belongs to this set. Hence the set \( X_T \) is nonempty.

Set \( u_0(t) := w(t) \). We are going to define the sequence \( u_m \) \((m = 1, 2, \ldots)\) by solving

\[
(3.2) \quad \partial_t^2 u_m - \Delta u_m = |u_{m-1}|^q + |\partial_t u_{m-1}|^p, \quad 0 < t < T, \ x \in \mathbb{R}^n,
\]
subject to initial data

\[ u_m(0, x) = \varepsilon f(x), \quad \partial_t u_m(0, x) = \varepsilon g(x). \]

The following argument shows that this sequence is well-defined in \( X_T \). Using the quantity \( M \) defined in (3.7) below, we set

\[ X_T(2C_0\varepsilon) := \{ u \in X_T : N_T(u) \leq 2C_0\varepsilon \}, \]

where

\[ N_T(u) := \sum_{|\alpha| \leq 2} \sup_{0 < t < T} (t)^{-\gamma(p,q)} \| \Gamma^\alpha u(t) \|_{H^s(q)} \]

\[ + \sum_{0 \leq j \leq n} \sup_{0 < t < T} \| \partial^j \Gamma^\alpha u(t) \|_{L^2}. \]

(For \( C_0 \) and \( \gamma(p,q) \), see (3.6) and (1.17), respectively.) Replacing \([0, T] \) by \([0, \infty) \) in (3.1), we define the set of functions \( X(2C_0\varepsilon) \) similarly, when \( T = \infty \).

To begin with, in view of Propositions 2.5 and 2.6, we must consider the bound on \( (\Gamma^\alpha u_m, \partial_t \Gamma^\alpha u_m)_{t=0} \). Rewriting \( (\Gamma^\alpha u_m)(0) \) and \( (\partial_t \Gamma^\alpha u_m)(0) \) in terms of \( f \) and \( g \), we will later show for \( 0 < \varepsilon < 1 \)

\[ \sum_{|\alpha| \leq 2} \left( \| \Gamma^\alpha u_m(0) \|_{H^s(q)} + \| \partial_t \Gamma^\alpha u_m(0) \|_{H^{s-1}(q)} \right) \]

\[ + (n+1) \sum_{|\alpha| \leq 2} \left( \| \Gamma^\alpha u_m(0) \|_{H^1} + \| \partial_t \Gamma^\alpha u_m(0) \|_{L^2} \right) \leq C_0 M \varepsilon \]

with a constant \( C_0 > 0 \) depending only on \( n, p, q \), and independent of \( m \) and \( \varepsilon \). Here, we have defined

\[ M := \Lambda + \Lambda^p + \Lambda^q + \Lambda^{2p-1} + \Lambda^{p+q-1}. \]

(For \( \Lambda \), see (1.14).) We will put off the proof of this estimate (3.6) till the next section, and let us see how it is used in the proof of Theorem 1.1. We prove the following proposition.

**Proposition 3.1.** Suppose that

\[ q > q_0(n), \quad p > p_0(n), \quad p \leq q \leq \frac{4}{(n-1)p-2} + 1 \]
and that \((f, g)\) satisfies (1.11)-(1.13). If \(u_{m-1} \in X_T\), then \(u_m\), which denotes the unique solution to (3.2)-(3.3), satisfies \(u_m \in X_T\) and

\[
N_T(u_m) \\
\leq C_0 M \varepsilon + C_1 \left\{ (1 + T)^{1 - \frac{n-1}{2} + \frac{1}{q} - \frac{(n-1)/2 + \gamma(p,q)}{2} - \gamma(p,q)} N_T(u_{m-1})^{p-1} \\
+ (1 + T)^{1 - \frac{n-1}{2} + \frac{1}{q} - \frac{(n-1)/2 + \gamma(p,q)}{2} - \gamma(p,q)} N_T(u_{m-1})^{q-1} \right\} N_T(u_{m-1})
\]

when \(q < 4/((n-1)p - 2) + 1\), and

\[
N_T(u_m) \leq C_0 M \varepsilon + C_2 (N_T(u_{m-1})^{p-1} + N_T(u_{m-1})^{q-1}) N_T(u_{m-1})
\]

when \(q = 4/((n-1)p - 2) + 1\). In (3.8) and (3.9), the constants \(C_1\) and \(C_2\) are independent of \(T\).

By this proposition, we can define the sequence \(\{u_m\} \subset X_T\). For \(u \in X_T\), we set

\[
\tilde{N}_T(u) := \sum_{|\alpha| \leq 1} \sup_{0 < t < T} \langle t \rangle^{-\gamma(p,q)} \| \Gamma^\alpha u(t) \|_{\tilde{H}^{s(q)}} + \sum_{0 \leq j \leq n} \sup_{0 < t < T} \| \partial^j \Gamma^\alpha u(t) \|_{L^2}.
\]

We also prove that this sequence has the following property.

**Proposition 3.2.** Suppose that

\[
q > q_0(n), \quad p > p_0(n), \quad p \leq q \leq \frac{4}{(n-1)p - 2} + 1.
\]

(i) Let \(n = 2\). It holds that

\[
N_T(u_{m+1} - u_m) \\
\leq C_3 \left\{ \sum_{i=m-1}^{m} \left\{ (1 + T)^{1 - (1/2) + (1/q) - (p-2)/2} - \gamma(p,q) N_T(u_i)^{p-1} \\
+ (1 + T)^{1 - (1/2) + (1/q) - (1/2 - 1/q)q + \gamma(p,q)(q-1)} N_T(u_i)^{q-1} \right\} \right\}
\\
\times N_T(u_m - u_{m-1}) \quad \text{when } q < \frac{4}{p - 2} + 1,
\]

\[
N_T(u_{m+1} - u_m) \leq C_4 \left\{ \sum_{i=m-1}^{m} \left( N_T(u_i)^{p-1} + N_T(u_i)^{q-1} \right) \right\}
\\
\times N_T(u_m - u_{m-1}) \quad \text{when } q = \frac{4}{p - 2} + 1.
\]

(ii) Let \(n = 3\). It holds that
\[ \tilde{N}_T(u_{m+1} - u_m) \leq C_5 \left\{ \sum_{i=m-1}^{m} \left[ (1 + T)^{(1/q) - (p-2) - \gamma(p,q)} N_T(u_i)^{p-1} + (1 + T)^{(1/q) - 2(1/2-1/q)q + \gamma(p,q)q} N_T(u_i)^{q-1} \right] \right\} \times \tilde{N}_T(u_m - u_{m-1}) \quad \text{when } q < \frac{2}{p-1} + 1, \]

\[ \tilde{N}_T(u_{m+1} - u_m) \leq C_6 \left\{ \sum_{i=m-1}^{m} \left( N_T(u_i)^{p-1} + N_T(u_i)^{q-1} \right) \right\} \times \tilde{N}_T(u_m - u_{m-1}) \quad \text{when } q = \frac{2}{p-1} + 1. \]

Moreover, it also holds that

\[ N_T(u_{m+1} - u_m) \leq C_7 (1 + T)^{(1/q) - (p-2) - \gamma(p,q)} \left( \sum_{i=m-1}^{m} N_T(u_i) \right)^2 \tilde{N}_T(u_m - u_{m-1})^{p-2} \]

\[ + C_8 (1 + T)^{(1/q) - 2(1/2-1/q)q + \gamma(p,q)q} \left( \sum_{i=m-1}^{m} N_T(u_i) \right)^2 \tilde{N}_T(u_m - u_{m-1})^{q-2} \]

\[ + C_9 \left\{ \sum_{i=m-1}^{m} \left[ (1 + T)^{(1/q) - (p-2) - \gamma(p,q)} N_T(u_i)^{p-1} + (1 + T)^{(1/q) - 2(1/2-1/q)q + \gamma(p,q)q} N_T(u_i)^{q-1} \right] \right\} \times N_T(u_m - u_{m-1}) \quad \text{when } q < \frac{2}{p-1} + 1, \]

\[ N_T(u_{m+1} - u_m) \leq C_{10} \left( \sum_{i=m-1}^{m} N_T(u_i) \right)^2 \left( \tilde{N}_T(u_m - u_{m-1})^{p-2} + \tilde{N}_T(u_m - u_{m-1})^{q-2} \right) \]

\[ + C_{11} \left( \sum_{i=m-1}^{m} N_T(u_i)^{p-1} + \sum_{i=m-1}^{m} N_T(u_i)^{q-1} \right) \times N_T(u_m - u_{m-1}) \quad \text{when } q = \frac{2}{p-1} + 1. \]
Proof of Proposition 3.1. We drop the subscript \( m - 1 \) until the last step of the proof. Obviously by Proposition 2.5, our task is to bound the followings for any fixed \( \tau \in (0, T) \):

\[
\begin{align*}
(3.16) & \quad \|u(\tau, \cdot)\|_{\Gamma, 2, p_1, \chi_1}, \quad \|u(\tau, \cdot)\|_{\Gamma, 2, 1, p_2, \chi_2}, \\
(3.17) & \quad \|\partial_t u(\tau, \cdot)\|_{\Gamma, 2, p_1, \chi_1}, \quad \|\partial_t u(\tau, \cdot)\|_{\Gamma, 2, 1, p_2, \chi_2}, \\
(3.18) & \quad \|u(\tau, \cdot)\|_{\Gamma, 2, 2}, \quad \|\partial_t u(\tau, \cdot)\|_{\Gamma, 2, 2}.
\end{align*}
\]

Here

\[
(3.19) \quad p_1 = \frac{2nq}{(n + 1)q + 2}, \quad p_2 = \frac{2q(n - 1)}{q(n - 1) + 2}.
\]

In the following discussion, by \( s^*, p^*, p_*, \) and \( \gamma^* \), we mean the exponents defined as

\[
(3.20) \quad s^* := s(q) = \frac{1}{2} - \frac{1}{q}, \quad p^* = \frac{1}{2} - \frac{s^*}{n}, \quad 1 - \frac{1}{p*} = \frac{1}{2},
\]

\[
\gamma^* := \gamma(p, q) = \frac{q + 1}{q - 1} \left( \frac{1}{p} - \frac{1}{q} \right).
\]

The reader is advised not to confuse \( p^* \) with \( p_* \). For simplicity, we also employ the notation

\[
(3.21) \quad \rho(u) := \sum_{|\alpha| \leq 2} \sup_{0 < t < T} \langle t \rangle^{-\gamma^*} \|D\alpha\Gamma^\alpha u(t, \cdot)\|_{L^2(\mathbb{R}^n)},
\]

\[
\sigma(u) := \sum_{0 < j \leq n} \sup_{0 < t < T} \|\partial_j \Gamma^\alpha u(t, \cdot)\|_{L^2(\mathbb{R}^n)}.
\]

We start with the estimate of (3.16). Using (2.3)-(2.4) and (2.1), we get

\[
(3.22) \quad \sum_{|\alpha|, |\beta| \leq 1} \|u|q^2 \Gamma^\alpha u \Gamma^\beta u\|_{p_1, \chi_1} \leq \|u\|_{\infty, \chi_1}^q \|u\|_{\Gamma, 1, 2p_1, \chi_1}^2,
\]

\[
\leq C(\tau)^{-\left(\frac{n}{p^*}(q - 1) + 1 + \gamma^* \rho(u)\right)q}.
\]

Since \( 2p_1 < p^* \) for \( n = 2 \), we have actually employed the Hölder inequality to get \( \|u\|_{\Gamma, 1, 2p_1, \chi_1} \leq C(\tau)^{-\left(\frac{n}{p^*} - 1\right)\left(\frac{q}{(2p_1)}\right)\times 2} \|u\|_{\Gamma, 2, p^*}^q \). Using \( 1/p_1 = 1/n + 1/p^* \), we also get

\[
(3.23) \quad \sum_{|\alpha| = 2} \|u|^{q - 1} \Gamma^\alpha u\|_{p_1, \chi_1} \leq \|u\|_{n(q - 1), \chi_1}^{q - 1} \|u\|_{\Gamma, 2, p^*}^q,
\]

\[
\leq C(\tau)^{-\left(\frac{n}{p^*} - 1\right)\left(\frac{q}{(q - 1)}\right)\times 1} \|u\|_{\Gamma, 2, p^*}^q \leq C(\tau)^{-\left(\frac{n}{p^*} - 1 + 1 + \gamma^* \rho(u)\right)q}.
\]
Using \((q - 2)/q + 2/q = 1\), the Sobolev embedding on \(S^{n-1}\), and (2.2), we obtain
\[
(3.24) \quad \sum_{|\alpha|,|\beta| \leq 1} \|u|^{q-2} \Gamma^\alpha u \Gamma^\beta u\|_{1,p_2,\chi_2} \leq \|u|^{q-2}_{q,\infty,\chi_2}\|u\|_{1,1,q,2p_2,\chi_2}^2
\leq C\|u\|_{1,1,q,2,\chi_2}^q \leq C(\tau)^{-(n-1)(1/2-1/q)q+\gamma^*} \rho(u)^q.
\]
Using \((q - 1)/q + 1/q = 1\) and \(1/(q(n-1)) + 1/2 = 1/p_2\), we also obtain
\[
(3.25) \quad \sum_{|\alpha| = 2} \|u|^{q-1} \Gamma^\alpha u\|_{1,p_2,\chi_2} \leq \|u|^{q-1}_{q,(n-1)q,\chi_2}\|u\|_{1,1,q,2,\chi_2}
\leq C\|u\|_{1,1,q,2,\chi_2} \leq C(\tau)^{-(n-1)(1/2-1/q)q+\gamma^*} \rho(u)^q.
\]
The estimate of (3.16) has been finished. Next, we deal with (3.17). Using (2.3)–(2.4), we get
\[
(3.26) \quad \sum_{|\alpha|,|\beta| \leq 1} \|\partial_t u|^{p-2} \Gamma^\alpha \partial_t u \Gamma^\beta \partial_t u\|_{p_1,\chi_1} \leq \|\partial_t u\|_{\infty,\chi_1}^{p-2}\|\partial_t u\|_{1,1,p_1,\chi_1}^2
\leq C(\tau)^{-(n/2)(p-2)-(n/2-1/(2p_1))^q} \|\partial_t u\|_{p,2}^p
= C(\tau)^{-(n/2)(p-2)-(n-1)/2+1/q} \|\partial_t u\|_{p,2}^p.
\]
Using \(1/p_1 = (q + 2)/(2nq) + 1/2\), we also get
\[
(3.27) \quad \sum_{|\alpha| = 2} \|\partial_t u|^{p-1} \Gamma^\alpha \partial_t u\|_{p_1,\chi_1} \leq \|\partial_t u\|_{\infty,\chi_1}^{p-2}\|\partial_t u\|_{2nq/(q+2),\chi_1}^2 \|\partial_t u\|_{2,1,p_1,\chi_1}^2
\leq C(\tau)^{-(n/2)(p-2)-(n/2-(q+2)/(2nq))} \|\partial_t u\|_{p,2}^p
= C(\tau)^{-(n/2)(p-2)-(n-1)/2+1/q} \|\partial_t u\|_{p,2}^p.
\]
On the other hand, employing the Sobolev embedding on \(S^{n-1}\) and (2.9), we get
\[
(3.28) \quad \sum_{|\alpha|,|\beta| \leq 1} \|\partial_t u|^{p-2} \Gamma^\alpha \partial_t u \Gamma^\beta \partial_t u\|_{1,p_2,\chi_2} \leq \|\partial_t u\|_{\infty,\chi_2}^{p-2}\|\partial_t u\|_{1,1,2,p_2,\chi_2}^2
\leq C\|\partial_t u\|_{1,1,\infty,2+\chi_2} \|\partial_t u\|_{p,2,\chi_2}^2 \leq C(\tau)^{-(n-1)(p-2)/2} \|\partial_t u\|_{p,2,\chi_2}^p.
\]
Using \(1/p_2 = 1/2 + 1/(q(n - 1))\), we also obtain
\[
(3.29) \quad \sum_{|\alpha| = 2} \|\partial_t u|^{p-1} \Gamma^\alpha \partial_t u\|_{1,p_2,\chi_2} \leq \|\partial_t u\|_{\infty,\chi_2}^{p-2}\|\partial_t u\|_{2,q(n-1),\chi_2}^2 \|\partial_t u\|_{p,2,\chi_2}^2
\leq C(\tau)^{-(n-1)(p-2)/2} \|\partial_t u\|_{p,2,\chi_2}^p.
\]
It remains to bound (3.18). Let us divide \(\mathbb{R}^n\) into the “interior” region \(\{x \in \mathbb{R}^n : |x| < (1+\tau)/2\}\) and the “exterior” one \(\{x \in \mathbb{R}^n : |x| > (1+\tau)/2\}\).
For the estimate of $\|u(\tau, \cdot)\|^{q}_{\Gamma, 2, 2, \chi_1}$, we get by using (2.3)-(2.4) and (2.1)

\begin{align}
\sum_{|\alpha|, |\beta| \leq 1} \|u\|^{q-2}_{\Gamma, 2, \chi_1} \|u\|^{2}_{\Gamma, 1, 4, \chi_1} \leq \|u\|^{q}_{\infty, \chi_1} \|u\|^{2}_{\Gamma, 1, 4, \chi_1} \\
\leq C(\tau)^{-n/p^*} \|u\|^{q-2}_{\Gamma, 2, p^*} (\tau)^{-n(1/p^*-1/4)} \|u\|^{2}_{\Gamma, 2, p^*} \\
\leq C(\tau)^{-n/p^*}(n/2)+\gamma^q \rho(u)^q
\end{align}

and

\begin{align}
\sum_{|\alpha| = 2} \|u\|^{q-1}_{\Gamma, 2, \chi_1} \leq \|u\|^{q/2}_{\infty, \chi_1} \|u\|^{q/2-1}_{\Gamma, 2, p^*} \\
\leq C(\tau)^{-n/p^*} \|u\|^{q/2}_{\Gamma, 2, p^*} / (\tau)^{-n(1/p^*-1/(q/2-1))} \|u\|^{q/2-1}_{\Gamma, 1, p^*} \|u\|^{q/2-1}_{\Gamma, 2, p^*} \\
\leq C(\tau)^{-n/p^*}(n/2)+\gamma^q \rho(u)^q.
\end{align}

For the estimate of $\|u(\tau, \cdot)\|^{q}_{\Gamma, 2, 2, \chi_2}$, we proceed as follows. Noting $1/2 = (q-2)/(2q) + 1/q$ and $1/2 - 1/(2q) = s^*(1-\theta) + 1/\theta$ for $\theta := 1/(q+2)$, we obtain by using the Sobolev embedding on $S^{n-1}$ and (2.2)

\begin{align}
\sum_{|\alpha|, |\beta| \leq 1} \|u\|^{q-2}_{\Gamma, \chi_2} \|u\|^{2}_{\Gamma, \chi_2} \leq C(\tau)^{-n(1/2-1/(2q))} q \left( \sum_{|\alpha| \leq 2} \|D^{1/2-1/(2q)} \Gamma^\alpha \|^{2}_{\Gamma, \chi_2} \right)^{q/2} \\
\leq C(\tau)^{-n(1-1/2+\gamma^*(1-\theta))q} \rho(u)^{(1-\theta)q} \sigma(u)^{\theta q}.
\end{align}

Moreover, noting $1/2 = (q-1)/(2q) + 1/(2q)$ and using the Sobolev embedding on $S^{n-1}$ and (2.2), we obtain

\begin{align}
\sum_{|\alpha| = 2} \|u\|^{q-1}_{\Gamma, \chi_2} \leq C(\tau)^{-n(1-q)/2+\gamma^*(1-\theta)q} \rho(u)^{(1-\theta)q} \sigma(u)^{\theta q}.
\end{align}

What remains to be done is the estimate of $\|\partial_t u(\tau, \cdot)\|^{p}_{\Gamma, 2, 2}$. For the estimate of $\|\partial_t u(\tau, \cdot)\|^{p}_{\Gamma, 2, 2, \chi_1}$, we get by using (2.3)-(2.4)

\begin{align}
\sum_{|\alpha|, |\beta| \leq 1} \|\partial_t u\|^{p-2}_{\Gamma, \chi_1} \|\partial_t u\|^{2}_{\Gamma, 1, 4, \chi_1} \leq C(\tau)^{-(n/2)(r-1)} \|\partial_t u\|^{p}_{\Gamma, 2, 2}
\end{align}
and

\[ \sum_{|\alpha|=2} \left\| \partial_t u \right\|_{p-1}^{p-1} \partial_t u \right\|_{2,\chi_1} \leq \left\| \partial_t u \right\|_{\infty,\chi_1} \left\| \partial_t u \right\|_{\Gamma,2,2,\chi_2} \leq C(\tau)^{-\frac{(n/2)(p-1)}{p}} \left\| \partial_t u \right\|_{p,\Gamma,2,2}^p. \]

For the estimate of \( \left\| \partial_t u(\tau, \cdot) \right\|_{p,\Gamma,2,\chi_2} \), we proceed as follows. Using (2.3), (2.9), and the Sobolev embedding on \( S^{n-1} \), we get

\[ \sum_{|\alpha|,|\beta| \leq 1} \left\| \partial_t u \right\|_{p-2}^{p-2} \Gamma^\alpha \partial_t u \Gamma^\beta \partial_t u \right\|_{2,\chi_2} \leq \left\| \partial_t u \right\|_{\infty,\chi_2} \left\| \partial_t u \right\|_{\Gamma,1,\infty,2^{+},\chi_2} \left\| \partial_t u \right\|_{\Gamma,1,2,\infty} \leq C(\tau)^{-\frac{(n-1)(p-1)}{2}} \left\| \partial_t u \right\|_{p,\Gamma,2,2}^p \]

and

\[ \sum_{|\alpha|=2} \left\| \partial_t u \right\|_{p-1}^{p-1} \Gamma^\alpha \partial_t u \right\|_{2,\chi_2} \leq \left\| \partial_t u \right\|_{\infty,\chi_2} \left\| \partial_t u \right\|_{\Gamma,2,2} \leq C(\tau)^{-\frac{(n-1)(p-1)}{2}} \left\| \partial_t u \right\|_{p,\Gamma,2,2}^p. \]

To end the proof of Proposition 3.1, we must verify the following five claims:

\[ -\frac{n(q-1)}{p^*} + 1 + \gamma^* q = -\frac{n-1}{2} + \frac{1}{q} - (n-1)\left(\frac{1}{2} - \frac{1}{q}\right)q + \gamma^* q, \]

\[ -\frac{n-1}{2} + \frac{1}{q} - (n-1)\left(\frac{1}{2} - \frac{1}{q}\right)q + \gamma^* q \geq -1 + \gamma^*, \]

\[ -\frac{n-1}{2} + \frac{1}{q} - \frac{(n-1)(p-2)}{2} \geq -1 + \gamma^*, \]

\[ -\frac{pq}{p^*} + \frac{n}{2} + \gamma^* q < -\frac{(n-1)(q-1)}{2} + \gamma^*(1-\theta)q < -\frac{n-1}{2} + \frac{1}{q} - (n-1)\left(\frac{1}{2} - \frac{1}{q}\right)q + \gamma^* q - \gamma^*, \]

\[ -\frac{(n-1)(p-1)}{2} < -1. \]

Here, in (3.39)-(3.40) the equality holds if and only if \( q = 4/((n-1)p-2)+1 \).

A straightforward computation shows (3.38). We easily see that (3.42) is equivalent to the assumption \( p > p_0(n) \). To verify (3.39), we first see by a direct computation that the equality holds when \( q = 4/((n-1)p-2)+1 \).

Since

\[ \frac{\partial}{\partial q} \left( \frac{1}{q} - (n-1)\left(\frac{1}{2} - \frac{1}{q}\right)q + \gamma(p,q)(q-1) \right) = \frac{1}{p} - \frac{n-1}{2} < 0 \]

for each \( p > p_0(n) \), we then find that the strict inequality holds in (3.39) for any \( q \) with \( p \leq q < 4/((n-1)p-2)+1 \). In the same way, we can verify
The first inequality in (3.41) can be verified as follows:
\[
-\frac{nq}{p^*} + \frac{n}{2} + \gamma^*q + \frac{(n-1)(q-1)}{2} - \gamma^*(1-\theta)q = -\frac{1}{2} + \gamma^*\theta q
\]
\[
= -\frac{1}{2} + q + 1 \left(\frac{1}{p} - \frac{1}{q}\right) q\frac{q}{q+2} = -\frac{1}{2} + q + 1 \cdot q - p \cdot \frac{1}{q} - \frac{1}{p} < -\frac{1}{2} + \frac{1}{p} < 0.
\]
We can verify the second inequality in (3.41) as follows:
\[
-\frac{(n-1)(q-1)}{2} + \gamma^*(1-\theta)q
\]
\[
+ \frac{n-1}{2} - \frac{1}{q} + (n-1) \left(\frac{1}{2} - \frac{1}{q}\right) q - \gamma^*(q-1)
\]
\[
= -\frac{1}{q} + \gamma^*(1-\theta q) = -\frac{1}{q} \left(1 - \frac{q+1}{q+2} \cdot \frac{q-p}{q-1} \cdot \frac{2}{p}\right) < 0.
\]
Now that the five claims have been verified, we are ready to complete the proof of Proposition 3.1. It follows from (2.10), (3.6), (3.22)-(3.29), (3.38)-(3.40) that for all \(|\alpha| \leq 2\) and \(0 < t < T\)
\[
(t)^{-\gamma^*} \|D|^{\alpha}u_m(t, \cdot)\|_2
\]
\[
\leq \langle t \rangle^{-\gamma^*} \left(\|\Gamma^\alpha u_m(0, \cdot)\|_{\dot{H}^{s+\gamma}} + \|\partial_t \Gamma^\alpha u_m(0, \cdot)\|_{\dot{H}^{-1+s^*}}\right)
\]
\[
+ C \langle t \rangle^{-\gamma^*} \int_0^t \langle \tau \rangle^{-\gamma^*} \int_0^t \langle \tau \rangle^{-(n-1)/2+1/q-(n-1)(1/2-1/q)q+\gamma^* q} \rho(u_m-1(\tau))^q
\]
\[
+ \langle \tau \rangle^{-(n-1)/2+1/q-(n-1)(p-2)/2}\sigma(u_m-1(\tau))^p d\tau
\]
\[
\leq \|\Gamma^\alpha u_m(0, \cdot)\|_{\dot{H}^{s^*}} + \|\partial_t \Gamma^\alpha u_m(0, \cdot)\|_{\dot{H}^{-1+s^*}}
\]
\[
+ C \langle t \rangle^{-(n-1)/2+1/q-(n-1)(1/2-1/q)q+\gamma^* (q-1) N_T(u_{m-1})^q}
\]
\[
+ C \langle t \rangle^{-(n-1)/2+1/q-(n-1)(p-2)/2-\gamma^* N_T(u_{m-1})^p}.
\]
Similarly, we get by (2.11), (3.6), (3.30)-(3.37), (3.41)-(3.42) for all \(j = 0, \ldots, n, |\alpha| \leq 2, \) and \(0 < t < T\)
\[
\|\partial_j \Gamma^\alpha u_m(t, \cdot)\|_2
\]
\[
\leq \|\Gamma^\alpha u_m(0, \cdot)\|_{\dot{H}^1} + \|\partial_1 \Gamma^\alpha u_m(0, \cdot)\|_{L^2}
\]
\[
+C \int_0^t \langle \tau \rangle^{-\gamma^*} \|\partial_j \Gamma^\alpha u_m(0, \cdot)\|_{L^2}^q
\]
\[
+ \langle \tau \rangle^{-\gamma^*} \int_0^t \langle \tau \rangle^{-(n/p^*)q+n/2+\gamma^*} \rho(u_m-1(\tau))^q
\]
\[
+ \langle \tau \rangle^{-\gamma^*} \int_0^t \langle \tau \rangle^{-(n-1)(q-1)/2+\gamma^* (1-\theta) q} \rho(u_m-1(\tau))^{1-\theta} \sigma(u_m-1(\tau))^{\theta} q^q
\]
\[
+ \langle \tau \rangle^{-\gamma^*} \int_0^t \langle \tau \rangle^{-(n-1)(p-2)/2} \sigma(u_m-1(\tau))^p d\tau
\]
\[
\leq \|\Gamma^\alpha u_m(0, \cdot)\|_{\dot{H}^1} + \|\partial_1 \Gamma^\alpha u_m(0, \cdot)\|_{L^2}
\]
\[
+C \langle t \rangle^{-(n/p-1/2+1/q-(n-1)(1/2-1/q)q+\gamma^* (q-1) N_T(u_{m-1})^q}
\]
\[
+ C N_T(u_{m-1})^p.\]
We obtain (3.8) by summing (3.43)-(3.44) over all $|\alpha| \leq 2$ (also for $j = 0, \ldots, n$ with respect to (3.44)) and then using (3.6).

As we saw, the equality holds in (3.39)-(3.40) when $q = 4/((n-1)p-2)+1$. We therefore see that (3.9) holds in that case. The proof of Proposition 3.1 has been finished.

**Proof of Proposition 3.2.** We may focus on the case of $n = 3$. The proof for $n = 2$ is a little easier. Define

\begin{equation}
\rho(u) := \sum_{|\alpha| \leq 1} \sup_{0 \leq t < T} |D|^{\gamma} u(t, \cdot) \|L^2(\mathbb{R}^3)},
\end{equation}

\begin{equation}
\tilde{\sigma}(u) := \sum_{0 < j < 3 \leq |\alpha|} \sup_{0 \leq t < T} \|\partial_j \Gamma^\alpha u(t, \cdot) \|L^2(\mathbb{R}^3)}.
\end{equation}

Note that in the definition of $\tilde{\rho}(u)$ and $\tilde{\sigma}(u)$, the number of occurrences of $\Gamma$ is limited to at most 1. Let us start with the proof of (3.12)-(3.13). Taking account of the elementary inequality

\begin{equation}
|x_1|^\sigma y_1 - |x_2|^\sigma y_2 \leq (\sigma - 1)(|x_1|^{\sigma - 2} + |x_2|^{\sigma - 2})y_1|x_1 - x_2| + |x_2|^{\sigma - 1}|y_1 - y_2|
\end{equation}

($x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $2 < \sigma < 3$), we may focus on the estimate of $|u_i|^{q-2}\Gamma^\alpha u_m|u_m - u_{m-1}|$ and $|\partial_i u_i|^{p-2}\Gamma^\alpha \partial u_m|\partial_i u_m - \partial_i u_{m-1}|$ ($i = m-1, m, |\alpha| \leq 1$) in $\|\cdot\|_{p,1,\chi_1}$, $\|\cdot\|_{1,p_2,\chi_2}$, and $\|\cdot\|_2$. The estimate for $|u_m - u_{m-1}|^{q-1}|\Gamma^\alpha u_m - \Gamma^\alpha u_{m-1}|$ and $|\partial_i u_m - \Gamma^\alpha \partial_i u_{m-1}|$ ($|\alpha| \leq 1$) has a proof similar to that of Proposition 3.1.

We start with the estimate of $|u_i|^{q-2}\Gamma^\alpha u_m|u_m - u_{m-1}|$. Without loss of generality, we deal with only the case $i = m$. Just for simplicity, we set $v := u_m$, $w := u_{m-1}$. As in (3.23), we get by using $1/3 = (q-2)/(3(q-1)) + 1/(3(q-1))$

\begin{equation}
||v||^{q-2}\Gamma^\alpha v|v - w||_{p,1,\chi_1} \leq ||v||_{\chi_1}^{q-2}|v - w||_{\chi_1} \leq ||v||_{3(q-1),\chi_1}^{q-2}|v - w||_{3(q-1),\chi_1} \leq C(\gamma)^{q-1}\tilde{\rho}(v)^{q-1}\tilde{\rho}(v - w).
\end{equation}

We also get, recalling $1/(q(n-1)) + 1/2 = 1/p_2$ (hence $p_2 < 2$),

\begin{equation}
||v||^{q-2}\Gamma^\alpha v|v - w||_{p_2,1,\chi_2} \leq ||v||^{q-2}\Gamma^\alpha v|v - w||_{q/(q-1),p_2+1,\chi_2} \leq C||v||^{q-2}_{1,1,\chi_2}||v - w||_{q,\infty,\chi_2} \leq C(\gamma)^{q-2}\tilde{\rho}(v)^{q-1}\tilde{\rho}(v - w).
\end{equation}
We deal with $\|v\|^{q-2} \Gamma^\alpha v\|v - w\|_2$ by dividing it into the two pieces $\| \cdot \|_{2, \chi_1}$ and $\| \cdot \|_{2, \chi_2}$. The former is estimated as follows:

\begin{equation}
\|v\|^{q-2} \Gamma^\alpha v\|v - w\|_{2, \chi_1} \leq \|v\|^{q-2} \|v\|_{\infty, \chi_1} \|v\|_{1,4, \chi_1} \|v - w\|_{4, \chi_1}
\end{equation}

\begin{equation}
\leq C(\tau)^{-(3/p')(q-2)-(3/1'-1/4)\times2} \|v\|^{q-1} \|v\|_{1, \Gamma, 2,p^*} \|v - w\|_{1, \Gamma, 1,p^*}
\leq C(\tau)^{-(3/p')q+3/2+\gamma^* q} \rho(v)^{q-1} \tilde{\rho}(v - w),
\end{equation}

where (2.3), (2.4), and (2.1) have been used. Furthermore, we find

\begin{equation}
\|v\|^{q-2} \Gamma^\alpha v\|v - w\|_{2, \chi_2} \leq \|v\|^{q-2} \|v\|^{2q/(q-1),2+1, \chi_2} \|v - w\|_{2q, \infty, - \chi_2}
\leq \|v\|^{q-2} \|v\|_{1,2,q,2, \chi_2} \|v\|_{1,2,q,2, \chi_2} \|v - w\|_{1,2,q,2, \chi_2}
\leq C(\tau)^{-(3/2)(p-2)-1+1/q} \sigma(v)^{p-1} \tilde{\sigma}(v - w).
\end{equation}

Let us turn our attention to the estimate of $|\partial_v|^{p-2} \Gamma^\alpha \partial_t v|\partial_t v - \partial_t w|$. As in (3.27), we get

\begin{equation}
|\partial_v|^{p-2} \Gamma^\alpha \partial_t v|\partial_t v - \partial_t w|\|_{p_1, \chi_1}
\leq |\partial_v|^{p-2} \|v\|_{1, \chi_1} \|v\|_{3, \chi_1} \|\partial_t v - \partial_t w\|_{6q/(q+2), \chi_1}
\leq C(\tau)^{-(3/2)(p-2)-1+1/q} \sigma(v)^{p-1} \tilde{\sigma}(v - w).
\end{equation}

As in (3.29), we also obtain

\begin{equation}
|\partial_v|^{p-2} \Gamma^\alpha \partial_t v|\partial_t v - \partial_t w|\|_{1,p_2, \chi_2}
\leq |\partial_v|^{p-2} \|\partial_v\|_{\chi_2} \|v\|_{2, \chi_2} \|\partial_t v - \partial_t w\|_{2, \chi_2}
\leq C(\tau)^{-(p-2)} \sigma(v)^{p-1} \tilde{\sigma}(v - w).
\end{equation}

We divide $|\partial_v|^{p-2} \Gamma^\alpha \partial_t v|\partial_t v - \partial_t w|\|_2$ into the two pieces $\| \cdot \|_{2, \chi_1}$ and $\| \cdot \|_{2, \chi_2}$, as before. The former is estimated as

\begin{equation}
|\partial_v|^{p-2} \Gamma^\alpha \partial_t v|\partial_t v - \partial_t w|\|_{2, \chi_1}
\leq |\partial_v|^{p-2} \|v\|_{\infty, \chi_1} \|\Gamma^\alpha \partial_t v\|_{3, \chi_1} \|\partial_t v - \partial_t w\|_{6, \chi_1}
\leq C(\tau)^{-(p-1)/2} \sigma(v)^{p-1} \tilde{\sigma}(v - w),
\end{equation}

where we have used (2.3), (2.4). We estimate the latter by using (2.3) and (2.9)

\begin{equation}
|\partial_v|^{p-2} \Gamma^\alpha \partial_t v|\partial_t v - \partial_t w|\|_{2, \chi_2}
\leq |\partial_v|^{p-2} \|v\|_{\infty} \|\Gamma^\alpha \partial_t v\|_{2, \chi_2} \|\partial_t v - \partial_t w\|_{\infty, \chi_2}
\leq C(\tau)^{-(p-1)} \sigma(v)^{p-1} \tilde{\sigma}(v - w).
\end{equation}

Putting together (3.47)-(3.54) and repeating the same argument as in (3.38)-(3.44), we obtain (3.12)-(3.13).
Next, let us prove (3.14)-(3.15). In addition to (3.46), we use another elementary inequality

\[(3.55) \quad \|x_1\|^{\sigma-2}y_1 z_1 - |x_2|^{\sigma-2}y_2 z_2 \leq |x_1|^{\sigma-2}(|y_1| z_1 - z_2) + |y_1 - y_2||z_2| + |x_1 - x_2|^{\sigma-2}|y_2 z_2| \]

\[(2 < \sigma < 3, x_1, \ldots, z_2 \in \mathbb{R}).\] In view of (3.55), the proof of (3.14)-(3.15) may obviously center especially on how to estimate \(|u_m - u_{m-1}|^{q-2}(\Gamma^\alpha u_{m-1})/(\Gamma^\beta u_{m-1})\) and \(\partial_t u_m - \partial_t u_{m-1}\) for the former. Define \(p_3\) and \(p_4\) as

\[(3.56) \quad \frac{1}{p_3} = \frac{1}{p^*} - \frac{1}{3}, \quad \frac{1}{p_1} = \frac{q-2}{p_3} + \frac{1}{p_4}.\]

(We easily see that in fact, \(p_3 = 3q\) and \(p_4 = 3q/(q+3).\)) Using (2.4) and then (2.1), we get

\[(3.57) \quad \|v - w|^{q-2}(\Gamma^\alpha w)(\Gamma^\beta w)\|_{p_1,1} \leq \|v - w\|^{q-2}_{p_3,1} \|w\|_{\Gamma,1,2}^{2} \leq C(\tau)^{-3(1/p^*/1/p_3)(q-2)-6(1/p^*/1/(2p_4)+\gamma^*q)} \times (\tau)^{-\gamma^*q\|v - w\|_{\Gamma,1,2}^{q-2}\|w\|_{\Gamma,2,p^*}^{2}} \leq C(\tau)^{-(3/p^*)(q-1)+1+\gamma^*q\tilde{\rho}(v - w)^{q-2}\rho(w)^2}.

Here, we have employed the equality \(1/p_1 = 1/p^* + 1/n\) to show

\[-3\left(\frac{1}{p^*} - \frac{1}{p_3}\right)(q - 2) - 6\left(\frac{1}{p^*} - \frac{1}{2p_4}\right) = -3\left(\frac{1}{p^*} - \frac{1}{p_3}\right)(q - 2) - 6\left(\frac{1}{p^*} - \frac{1}{2}\left(\frac{1}{p_1} - \frac{q-2}{p_3}\right)\right)\]

\[= -\frac{3q}{p^*} + \frac{3}{p_1} = -\frac{3}{p^*}(q - 1) + 1.\]

On the other hand, we obtain by slightly modifying the argument in (3.24)

\[(3.58) \quad \|v - w|^{q-2}(\Gamma^\alpha w)(\Gamma^\beta w)\|_{1,p_2,2} \leq \|v - w\|^{q-2}_{q,\infty} \|w\|_{\Gamma,1,2}^{q} \|w\|_{\Gamma,2,q,2}^{2} \leq C\|v - w\|^{q-2}_{\Gamma,1,q,2} \|w\|_{\Gamma,2,q,2}^{2} \leq C(\tau)^{-2(1/2-1/q)+\gamma^*q\tilde{\rho}(v - w)^{q-2}\rho(w)^2}.

The estimate in the \(L^2(\mathbb{R}^3)\)-norm is carried out by decomposing it into the two pieces \(\|\cdots\|_{2,1}^2\) and \(\|\cdots\|_{2,2}^2\), as before. Define \(p_5\) as \(1/2 = (q-2)/p_3 + 1/p_5\); see (3.56) for \(p_3\). Since \(p_3 = 3q\), we easily see that \(p_5 = 6q/(q + 4).\)
Using (2.4) and then (2.1), we get
\begin{align*}
\|v - w\|_{2,\chi_1}^{q-2} (\Gamma^\alpha w)(\Gamma^\beta w) &\leq \|v - w\|_{\mathcal{L}^p,\chi_1}^{q-2} \|w\|_{1,2p_5,\chi_1}^2 \\
&\leq C(\tau)^{-(1/p^*/1/p_5)}(q-2)/(p-1)/(2p_5) \|v - w\|_{1,p}^{q-2} \|w\|_{1,2p^*}^2 \\
&\leq C(\tau)^{-(3/p^*)q+(3/2)+\gamma^*q} \rho(v - w)^{q-2} \rho(w)^2.
\end{align*}

On the other hand, by modifying the argument in (3.32), we get
\begin{align*}
\|v - w\|_{2,\chi_2} \leq \|v - w\|_{2,q,\chi_2}^{q-2} \|w\|_{2,1,2q,4,\chi_2}^2 \\
&\leq C\|v - w\|_{1,q,2q,2,\chi_2}^{q-2} \|w\|_{1,2q,2,\chi_2}^2 \\
&\leq C(\tau)^{-(q-1)+\gamma^*(1-q)} \rho(v - w)^{q-2} \rho(w)^2.
\end{align*}

Finally, we deal with \( \|\partial_t u_m - \partial_t u_{m-1}\|_{p-2}^2 (\Gamma^\alpha \partial_t u_{m-1})(\Gamma^\beta \partial_t u_{m-1}) \). Define \( p_6 \) as \( 1/p_1 = (p - 2)/6 + 1/p_6 \); we easily verify \( 2p_6 \leq 6 \). We get
\begin{align*}
\|\partial_t v - \partial_t w\|_{p-2}^2 \|\partial_t w\|_{1,p,\chi_1} \leq &\|\partial_t v - \partial_t w\|_{p-2}^2 \|\partial_t w\|_{1,1,2p_5,\chi_1}^2 \\
&\leq C(\tau)^{-(3/2)(p-2)-1+(1/q)} \rho(v - w)^{p-2} \sigma(w)^2.
\end{align*}

On the other hand, define \( p_7 \) as \( 1/p_2 = (p - 2)/2 + 1/p_7 \). By slightly modifying the argument in (3.28), we obtain
\begin{align*}
\|\partial_t v - \partial_t w\|_{p-2}^2 \|\partial_t w\|_{1,p_2,\chi_2} \leq &\|\partial_t v - \partial_t w\|_{p-2}^2 \|\partial_t w\|_{1,1,2p_7}^2 \\
&\leq C(\tau)^{-(p-2)} \|\partial_t v - \partial_t w\|_{1,1,2p_7}^2 \|\partial_t w\|_{1,2,2}^2 \\
&\leq C(\tau)^{-(p-2)} \rho(v - w)^{p-2} \sigma(w)^2.
\end{align*}

We next define \( p_8 \) as \( 1/2 = (p - 2)/6 + 1/p_8 \) to get by (2.4)
\begin{align*}
\|\partial_t v - \partial_t w\|_{2,\chi_1}^{p-2} \|\partial_t w\|_{2,\chi_1} \leq &\|\partial_t v - \partial_t w\|_{2,\chi_1}^{p-2} \|\partial_t w\|_{2,1,2p_8,\chi_1}^2 \\
&\leq C(\tau)^{-(3/2)(p-2)-1+(1/q)} \rho(v - w)^{p-2} \sigma(w)^2.
\end{align*}

Finally, define \( p_9 \) as \( 1/2 = ((p - 2)/(4-)) + (1/(4-)) + 1/p_9 \). Using (2.9), we get
\begin{align*}
\|\partial_t v - \partial_t w\|_{2,\chi_2}^{p-2} \|\partial_t w\|_{2,\chi_2} \leq &\|\partial_t v - \partial_t w\|_{\infty,4,\chi_2}^{p-2} \|\partial_t w\|_{1,\infty,4,\chi_2} \|\partial_t w\|_{1,1,2,p_9}^2 \\
&\leq C(\tau)^{-(p-1)} \rho(v - w)^{p-2} \sigma(w)^2.
\end{align*}
Now that we have obtained (3.57)-(3.64), the inequalities (3.14), (3.15) can be proved in analogy to the proof of (3.43)-(3.44). We have finished the proof of Proposition 3.2 (ii). Since it is easier, we omit the proof of (i). The proof has been finished.

Now we are in a position to carry out an iteration argument. Since the proof of Theorem 1.1 for \( n = 2 \) is more straightforward, we may focus on the case of \( n = 3 \). The proof uses the modified iteration method of John; see [12] on page 259. Let us start with the case of \( q = 2/(p - 1) + 1 \). Recall the definition of \( X(2C_0M\varepsilon) \), that is, \( X(2C_0M\varepsilon) := \{ u \in X : N(u) \leq 2C_0M\varepsilon \} \), where \( X \) and \( N(u) \) are defined by an obvious modification of \( X_T \) and \( N_T(u) \); we replace \([0, T)\) by \([0, \infty)\) in the definition of \( X_T \) and \( \sup_{0 < t < T} \) by \( \sup_{0 < t < \infty} \) in that of \( N_T(u) \), respectively. From (3.9) we then deduce by induction that if \( \varepsilon \) is small so that \( C_2((2C_0M\varepsilon)^p - 1 + (2C_0M\varepsilon)^q - 1) \leq 1/2 \) may hold, the sequence \( \{ u_m \} \) of Proposition 3.1 satisfies \( N(u_m) \leq 2C_0M\varepsilon \) for \( m \in \mathbb{N} \). Moreover, if \( \varepsilon \) is small so that \( 2C_6((2C_0M\varepsilon)^p - 1 + (2C_0M\varepsilon)^q - 1) \leq 1/2 \) may also hold, then we obtain by (3.13)

\[
(3.65) \quad \tilde{N}(u_{m+1} - u_m) \leq 2^{-1}\tilde{N}(u_m - u_{m-1}) \leq 2^{-m}\tilde{N}(u_1 - u_0), \quad m \in \mathbb{N}.
\]

Here, the definition of \( \tilde{N}(\cdots) \) is an obvious modification of that of \( \tilde{N}_T(\cdots) \). If \( \varepsilon \) is further small so that \( 2C_{11}((2C_0M\varepsilon)^p - 1 + (2C_0M\varepsilon)^q - 1) \leq 1/2 \) may also hold, then we finally get by (3.15)

\[
(3.66) \quad N(u_{m+1} - u_m)
\leq C(2^{-(p-2)(m-1)} + 2^{-(q-2)(m-1)}) + 2^{-1}N(u_m - u_{m-1})
\leq 2C(2^{-(p-2)(m-1)} + 2^{-(q-2)(m-1)}) + 2^{-2}N(u_{m-1} - u_{m-2})
\leq mC(2^{-(p-2)(m-1)} + 2^{-(q-2)(m-1)}) + 2^{-m}N(u_1 - u_0).
\]

Here we have repeatedly used the inequalities \( 1/2 < (1/2)^{p-2} \) and \( 1/2 < (1/2)^{q-2} \) for \( p, q < 3 \). This implies that \( \{ u_m \} \) is a Cauchy sequence in \( X(2C_0M\varepsilon) \). Its limit in \( X(2C_0M\varepsilon) \) is obviously the global (in time) solution we have been seeking for. Uniqueness of solutions in \( X(2C_0M\varepsilon) \) can be proved in analogy to the proof of (3.13). We now turn our attention to the case of \( q < 4/((n - 1)p - 2) + 1 \). From (3.8) we deduce by induction that the sequence \( \{ u_m \} \) of Proposition 3.1 satisfies \( N_T(u_m) \leq 2C_0M\varepsilon \) (\( m \in \mathbb{N} \)), provided that \( G(T, \varepsilon) \leq 1/(2C_1) \). Here, and in the following discussion, we mean by \( G(T, \varepsilon) \)

\[
G(T, \varepsilon) := \max\left\{(1 + T)^{1 - \frac{(n-1)(p-2)}{2}} - \frac{1}{2} + \gamma^*(2C_0M\varepsilon)^{p-1}, (1 + T)^{1 - \frac{n+1}{2} + \frac{1}{q} - (n-1)(\frac{4}{2} - \frac{1}{q})} + \gamma^*(2C_0M\varepsilon)^{q-1}\right\}.
\]
Moreover, we see from Proposition 3.2 that if $G(T, \varepsilon) \leq 1/(4C)_3$ for $n = 2$ and $G(T, \varepsilon) \leq \min\{1/(4C)_3, 1/(4C)_9\}$ for $n = 3$, then $\{u_m\}$ is a Cauchy sequence in $X_T(2C_0M\varepsilon)$ whose limit is obviously the solution to (1.1)-(1.2).

It remains to determine $T > 0$ properly. Set $k_0 := \min\{1/(2C_1), 1/(4C_3), 1\}$ when $n = 2$, and $k_0 := \min\{1/(2C_1), 1/(4C_5), 1/(4C_9), 1\}$ when $n = 3$, respectively. Naturally, we are going to choose $T > 0$ so that $G(T, \varepsilon) = k_0$, which will imply not only the local existence result but also the sharp lifespan estimate (1.18).

To complete the proof of Theorem 1.1, we must prove (2.9), (3.6), and Lemma 3.3. In fact, the first two are proved in the next section.

Lemma 3.3. Suppose that $q < 4/((n - 1)p - 2) + 1$ for $n \geq 2$. Then the equality

$$
(3.67) \quad (1 + T)^{1 - \frac{(n - 1)(p - 2)}{2} - \frac{n - 1}{2} + \frac{1}{q} - \gamma^*} (2C_0M\varepsilon)^{p - 1} = k_0,
$$

namely

$$
(3.68) \quad 1 + T = (2C_0Mk_0^{-1/(p - 1)}\varepsilon)^{\frac{2p(q - 1)}{2q + 2 - (n - 1)p(q - 1)}}.
$$

To proceed, we invoke the following key lemma.

**Lemma 3.3.** Suppose that $q < 4/((n - 1)p - 2) + 1$ for $n \geq 2$. Then the equality

$$
(3.69) \quad (D^{1 - \frac{(n - 1)(p - 2)}{2} - \frac{n - 1}{2} + \frac{1}{q} - \gamma^*} E^{p - 1})^q = \left(D^{1 - \frac{2q}{q + 2 - (n - 1)p(q - 1)}}\right)^q.
$$

holds when

$$
(3.70) \quad D = E^{-\frac{2p(q - 1)}{2q + 2 - (n - 1)p(q - 1)}}.
$$

Before showing this lemma, let us see how the equality (3.69) is helpful in our discussion. Choosing $D = 1 + T$ and $E = 2C_0Mk_0^{-1/(p - 1)}\varepsilon$, we obtain under the condition (3.67)-(3.68)

$$
(1 + T)^{1 - \frac{2q}{q + 2 - (n - 1)p(q - 1)}}(2C_0Mk_0^{-1/(p - 1)}\varepsilon)^q - 1 = 1,
$$

which means that $G(T, \varepsilon)$ turns out to be max$\{k_0, k_0^{(q - 1)/(p - 1)}\}$. Becuase $k_0 \leq 1$ and $p \leq q$ (see (1.10) above), we finally conclude that the equality $G(T, \varepsilon) = k_0$ actually holds, as desired.

To complete the proof of Theorem 1.1, we must prove (2.9), (3.6), and Lemma 3.3. In fact, the first two are proved in the next section.
Proof of Lemma 3.3. First of all, recall the definition of $\gamma^*$; see (3.20) above. Since a straightforward computation shows the useful expression

$$1 - \frac{(n-1)(p-2)}{2} - \frac{n-1}{2} + \frac{1}{q} - \gamma^* = (p-1) \left( \frac{q+1}{p(q-1)} - \frac{n-1}{2} \right),$$

$$1 - \frac{n-1}{2} + \frac{1}{q} - (n-1) \left( \frac{1}{2} - \frac{1}{q} \right) q + \gamma^*(q-1)$$

$$= (q-1) \left( \frac{q+1}{p(q-1)} - \frac{n-1}{2} \right),$$

we now easily see that (3.69) is actually equivalent to (3.70).

4. Proof of (2.2), (2.9), and (3.6)

Let us start with the proof of (2.2). We need the following lemma (see [3])

Lemma 4.1. Suppose $n \geq 2$. Then the inequality

$$(4.1) \quad \sup_{r > 0} r^{(n-1)/2} \| u(r \cdot) \|_{L^2(S^{n-1})} \leq C \| u \|_{\dot{B}^{1/2}_{2,1}}$$

holds.

Here, and in the following discussion, by $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$ we mean the homogeneous Besov space; see, e.g., Chapter 6 of [2]. We also need the obvious equality

$$(4.2) \quad \| r^{(n-1)/2} u \|_{L^2(\mathbb{R}^n)} = \| u \|_{L^2(\mathbb{R}^n)}$$

(here, and in the following, $\mathbb{R}^+ := (0, \infty)$). Recall the assumption $2 < q < \infty$. By (4.1)-(4.2), Theorems 3.1.2, 6.4.5, and 5.2.1 of [2] (see also Remark on page 41 of [2]), we then see that the sublinear operator

$$T : \dot{B}^{1/2}_{2,1}(\mathbb{R}^n) + L^2(\mathbb{R}^n) \to L^\infty(\mathbb{R}^+) + L^2(\mathbb{R}^+)$$

defined as $T(u) := r^{(n-1)/2} \| u(r \cdot) \|_{L^2(S^{n-1})}$ satisfies

$$T : [\dot{B}^{1/2}_{2,1}(\mathbb{R}^n), \dot{B}^0_{p,2}(\mathbb{R}^n)]_{2/q,q} \to [L^\infty(\mathbb{R}^+), L^2(\mathbb{R}^+)]_{2/q,q}$$

and the inequality

$$(4.3) \quad \| T(u) \|_{L^q(\mathbb{R}^+)} \leq C \| u \|_{\dot{B}^{1/2-1/q}_{2,1,q}(\mathbb{R}^n)}$$

holds. The last inequality immediately implies (2.2), because $\dot{H}^{1/2-1/q}$ is continuously embedded in $\dot{B}^{1/2-1/q}_{2,q}$.
We next prove (2.9). Obviously, it suffices to show it for \( u \in S(\mathbb{R}^n) \). We show the following inequality (4.5), which generalizes the well-known inequality
\[
(r^{(n-1)/2}) \left\| u(r) \right\|_{L^2(S^{n-1})} \leq \sqrt{2} \left\| \partial_r u \right\|_{L^2(\mathbb{R}^n)}^{1/2} \left\| u \right\|_{L^2(\mathbb{R}^n)}^{1/2} \quad (u \in S(\mathbb{R}^n)).
\]

**Lemma 4.2.** Let \( 2 < p \leq 4, q = 2p / (4 - p) \), and \( n \geq 2 \). Then, we have
\[
(r^{(n-1)/2}) \left\| u(r) \right\|_{L^p(S^{n-1})} \leq \sqrt{2} \left\| \partial_r u \right\|_{L^2(\mathbb{R}^n)}^{1/2} \left\| u \right\|_{L^2(\mathbb{R}^n)}^{1/2} \quad (u \in S(\mathbb{R}^n)).
\]

**Proof.** We use a natural modification of the proof of (4.4). We first note for any fixed \( R > 0 \)
\[
(R^{(n-1)/2}) \left\| u(R) \right\|_{L^p(S^{n-1})}^p = \int_{S^{n-1}} R^{(n-1)p/2} |u(R\omega)|^p d\omega
\]
\[
\leq p \int_{R}^\infty \int_{S^{n-1}} r^{(n-1)p/2} |u(r\omega)|^p |(\omega \cdot \nabla u)(r\omega)| dr d\omega
\]
\[
\leq p \| r^{(n-1)/2} u \|^p_{L^{(p-1)}(\mathbb{R}^n)} \left\| \partial_r u \right\|_{L^2(\mathbb{R}^n)},
\]
where \( \theta := (p - 2)/(p - 1) \). Using \( 1/(2(p - 1)) = \theta/p + (1 - \theta)/q \), we get
\[
\| r^{(n-1)/2} u \|^p_{L^{p(p-1)}(\mathbb{R}^n)}
\]
\[
\leq \left( \sup_{r > 0} r^{(n-1)/2} \left\| u(r \cdot) \right\|_{L^p(S^{n-1})} \left( \left\| u \right\|_{L^0(S^{n-1})} \right)^{1-\theta} \right),
\]
where \( (1 - \theta)(p - 1) = 1 \). Note \( \sup_{r > 0} r^{(n-1)/2} \left\| u(r \cdot) \right\|_{L^p(S^{n-1})} < \infty \) for \( u \in S(\mathbb{R}^n) \) by the inequality (4.4) together with the Sobolev embedding on \( S^{n-1} \). The inequalities (4.6) and (4.7) yield
\[
(R^{(n-1)/2}) \left\| u(R) \right\|_{L^p(S^{n-1})}^p \leq p \left( \sup_{r > 0} r^{(n-1)/2} \left\| u(r \cdot) \right\|_{L^p(S^{n-1})} \right)^{p-2}
\]
\[
\times \left\| u \right\|_{L^2(\mathbb{R}^n)} \left\| \partial_r u \right\|_{L^2(\mathbb{R}^n)},
\]
and the inequality (4.5) is an immediate consequence of (4.8).

Now we are ready to prove (2.9). Suppose \( 2 \leq p \leq 4 \) for \( n = 2 \), \( 2 \leq p < 4 \) for \( n = 3 \), and \( 2 \leq p \leq 2(n - 1)/(n - 2) \) for \( n \geq 4 \). Then, we see that the embedding \( H^1(S^{n-1}) \hookrightarrow L^q(S^{n-1}) \) holds for \( q = 2p / (4 - p) \), and thus the inequality (2.9) follows immediately from (4.5).

Lastly, we prove (3.6). By simple calculation, we can easily see that
\[
\Gamma^\alpha u_m(0) = \sum_{|\beta| \leq 2} \sum_{|\alpha| \leq |\beta|} C_{\alpha\beta} x^\alpha \partial^\beta u_m(0),
\]
\[
\partial_t \Gamma^\alpha u_m(0) = \sum_{1 \leq |\beta| \leq 2} \sum_{|\alpha| \leq |\beta| - 1} \tilde{C}_{\alpha\beta} x^\alpha \partial^\beta u_m(0) + \sum_{|\alpha| \leq |\beta| = 2} \tilde{C}_{\alpha\beta} x^\alpha \partial^\beta \partial_t u_m(0)
\]
for $|\alpha| \leq 2$, where $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$ and $\partial^\beta = \partial_1^{b_1} \cdots \partial_n^{b_n}$. Thus we have

\begin{equation}
(4.11) \sum_{|\alpha| \leq 2} (\|\Gamma^\alpha u_m(0)\|_{\dot{H}^{s*}} + \|\partial_t \Gamma^\alpha u_m(0)\|_{\dot{H}^{s* - 1}}) + (n + 1) \sum_{|\alpha| \leq 2} (\|\Gamma^\alpha u_m(0)\|_{\dot{H}^1} + \|\partial_t \Gamma^\alpha u_m(0)\|_{L^2}) \leq C \sum_{|\alpha| \leq 1} \|x^\alpha \partial^2_t u_m(0)\|_{\dot{H}^1 \cap \dot{H}^{s*}} + C \sum_{|\alpha| \leq 2} \|x^\alpha \partial^3_t u_m(0)\|_{L^2 \cap \dot{H}^{s* - 1}} + C \sum_{|\alpha| \leq 1} \|x^\alpha \Box u_m(0)\|_{L^2 \cap \dot{H}^{s* - 1}} + C \sum_{|\alpha| \leq 2} \|x^\alpha \partial_t \Box u_m(0)\|_{L^2 \cap \dot{H}^{s* - 1}}.
\end{equation}

which further yields

\begin{align*}
&\leq C \varepsilon \Lambda + C \sum_{|\alpha| \leq 2} \|x^\alpha \partial^2_t u_m(0)\|_{\dot{H}^1 \cap \dot{H}^{s*}} + C \sum_{|\alpha| \leq 1} \|x^\alpha \partial^2_t u_m(0)\|_{L^2 \cap \dot{H}^{s* - 1}} + C \sum_{|\alpha| \leq 2} \|x^\alpha \partial^3_t u_m(0)\|_{L^2 \cap \dot{H}^{s* - 1}} + C \sum_{|\alpha| \leq 1} \|x^\alpha \Box u_m(0)\|_{L^2 \cap \dot{H}^{s* - 1}} + C \sum_{|\alpha| \leq 2} \|x^\alpha \partial_t \Box u_m(0)\|_{L^2 \cap \dot{H}^{s* - 1}}.
\end{align*}

Since $L^p_1 \subset \dot{H}^{s* - 1}$ (recall (3.19), (3.20)), we arrive at

\begin{equation}
(4.12) \sum_{|\alpha| \leq 2} (\|\Gamma^\alpha u_m(0)\|_{\dot{H}^{s*}} + \|\partial_t \Gamma^\alpha u_m(0)\|_{\dot{H}^{s* - 1}}) + (n + 1) \sum_{|\alpha| \leq 2} (\|\Gamma^\alpha u_m(0)\|_{\dot{H}^1} + \|\partial_t \Gamma^\alpha u_m(0)\|_{L^2}) \leq C \varepsilon \Lambda + C \sum_{|\alpha| \leq 1} \|\partial_t (x^\alpha \Box u_m(0))\|_{L^2 \cap L^p_1} + C \sum_{|\alpha| \leq 1} \|x^\alpha \Box u_m(0)\|_{L^2 \cap L^p_1} + C \sum_{|\alpha| \leq 2} \|x^\alpha \partial_t \Box u_m(0)\|_{L^2 \cap L^p_1}.
\end{equation}

Therefore, the proof of (3.6) is reduced to the estimate

\begin{equation}
(4.13) \sum_{|\alpha| \leq |\beta| + 1} \|x^\alpha \partial^\beta \Box u_m(0)\|_{L^2 \cap L^p_1} \leq C \varepsilon \left( \Lambda^p + \Lambda^q + \Lambda^{2p - 1} + \Lambda^{p + q - 1} \right).
\end{equation}
Let us first show the estimate for \( \| \partial^h \Box u_m(0) \|_{L^2 \cap L^{p_1}} \). According to the definition of \( \{ u_m \} \), we have

\[
(4.14) \quad \sum_{|b| \leq 1} \| \partial^b \Box u_m(0) \|_{L^2 \cap L^{p_1}} \\
\leq \| \varepsilon f \|^{q_0} + \| \varepsilon g \|^{p_0} \|_{L^2 \cap L^{p_1}} \\
+ C \sum_{1 \leq i \leq n} \| \varepsilon f \|^{q_0} |\partial_i f| + \| \varepsilon g \|^{p_0} |\partial_i g| \|_{L^2 \cap L^{p_1}} \\
+ C \| \varepsilon f \|^{q_0} |\varepsilon f| + \| \varepsilon g \|^{p_0} \| \varepsilon f \|^{q_0} + \| \varepsilon g \|^{p_0} \|_{L^2 \cap L^{p_1}}.
\]

Note that the Sobolev embedding yields

\[
(4.15) \quad \| g \|_r \leq C \| g \|_2 \quad \text{for} \quad 2 \leq r \leq \infty, \\
(4.16) \quad \| f \|_r \leq C \| f \|_{H^s \cap H^2} \quad \text{for} \quad 0 \leq \frac{n}{r} \leq \frac{n - 1}{2} + \frac{1}{q}.
\]

Thus (4.14) leads to

\[
(4.17) \quad \sum_{|b| \leq 1} \| \partial^b \Box u_m(0) \|_{L^2 \cap L^{p_1}} \leq C \varepsilon \left( \Lambda^q + \Lambda^p + \Lambda^{p+q_0 - 1} + \Lambda^{2p_0 - 1} \right).
\]

We next consider the bounds for \( \| x^a \partial^h \Box u_m(0) \|_{L^2 \cap L^{p_1}} \) with \( |a| = 1 \):

\[
(4.18) \quad \sum_{|a|=1} \sum_{|b| \leq 1} \| x^a \partial^b \Box u_m(0) \|_{L^2 \cap L^{p_1}} \\
\leq C \varepsilon \sum_{1 \leq i, j \leq n} \left( \| x_i (|f|^{q_0} + |g|^{p_0}) \|_{L^2 \cap L^{p_1}} \\
+ \| x_i (|f|^{q_0} |\partial_j f| + |g|^{p_0} |\partial_j g|) \|_{L^2 \cap L^{p_1}} \\
+ \| x_i (|f|^{q_0} |g| + |g|^{p_0} (|\Delta f| + |f|^{q_0} + |g|^{p_0})) \|_{L^2 \cap L^{p_1}} \right).
\]

In addition to (4.15)–(4.16), we use

\[
(4.19) \quad \| x_i g \|_{L^{p_0}} \leq C \| x_i g \|_{H^{s_0}} \leq C \sum_{1 \leq j \leq n} \| \partial_j (x_i g) \|_{H^{s_0 - 1}}, \\
(4.20) \quad \| x_i \partial_j f \|_{L^{p_0}} \leq C \| x_i \partial_j f \|_{H^{s_0}},
\]
and

\[(4.21) \quad \|x_i f\|_{L^q} \leq C\|x_i f\|_{H^1_{p'}} \leq C\|f\|_{H^{s'}} + C \sum_{1 \leq j \leq n} \|x_i \partial_j f\|_{H^{s'}} \quad \text{for} \quad n = 3,\]

\[(4.22) \quad \|x_i f\|^2_{L^q} \leq C\|x_i f\|^2_{H^1_{p'/2}} \leq C\|f\|^2_{H^{s'}} + C \sum_{1 \leq j \leq n} \|x_i \partial_j f\|^2_{H^{s'}} \quad \text{for} \quad n = 2.\]

We need (4.21) and (4.22) to estimate \(\|x_i f\|^q_{L^{p_1} \cap L^2}\). Since \(q \geq 1 + \sqrt{2}\) when \(n = 3\), we have

\[(4.23) \quad \|x_i f\|^q_{L^{p_1} \cap L^2} \leq \|x_i f\|_{L^q} \left( \|f\|^{q-1}_{L^{3(q-1)/2}} + \|f\|^{q-1}_{L^{3(q-1)/(3q-2)}} \right) \leq C\|x_i f\|_{L^q} \|f\|^{q-1}_{H^{s'} \cap H^2};\]

by applying (4.16) and the Sobolev embedding. Similarly, since \(q \geq (3 + \sqrt{17})/2\) when \(n = 2\), we have

\[(4.24) \quad \|x_i f\|^q_{L^{p_1} \cap L^2} \leq \|x_i f\|^2_{L^q} \left( \|f\|^{q-2}_{L^{3(q-2)/(3q-2)}} + \|f\|^{q-2}_{L^{2q}} \right) \leq C\|x_i f\|^2_{L^q} \|f\|^{q-2}_{H^{s'} \cap H^2};\]

All the other terms in (4.18) can be treated by using the Hölder inequality, (4.19) and (4.20), together with (4.15)–(4.16). Thus we see that (4.18) is bounded by the right-hand side of (4.13).

Finally, let us consider the bounds for \(\|x^a \partial^b \Box u_m(0)\|_{L^2 \cap L^{p_1}}\) with \(|a| = 2\) and \(|b| = 1\):

\[(4.25) \quad \sum_{|a|=2} \sum_{|b|=1} \left\|x^a \partial^b \Box u_m(0)\right\|_{L^2 \cap L^{p_1}} \leq C\varepsilon \sum_{1 \leq i, j, k \leq n} \left( \|x_i x_j (|f|^{q-1} |\partial_k f| + |g|^{p-1} |\partial_k g|)\|_{L^2 \cap L^{p_1}} + \|x_i x_j (|f|^{q-1} |g| + |g|^{p-1} (|\Delta f| + |f|^q + |g|^p))\|_{L^2 \cap L^{p_1}} \right).

Corresponding to (4.19)–(4.22), we have

\[(4.26) \quad \|x_i x_j \partial_k g\|_{L^{p'}} \leq C\|x_i x_j \partial_k g\|_{H^{s'}},\]
\[(4.27) \quad \|x_i x_j \partial_k \partial_t f\|_{L^{p'}} \leq C\|x_i x_j \partial_k \partial_t f\|_{H^{s'}}.\]
\[(4.28) \quad \|x_i x_j \partial_k f\|_{L^q} + \|x_i x_j g\|_{L^q} \leq C \sum_{1 \leq \|a\| \leq 2} \|x^a \partial^b f\|_{\dot{H}^{s+}} + C \sum_{1 \leq \|a\| \leq 2, \|b\|=\|a\|-1} \|x^a \partial^b g\|_{\dot{H}^{s+}} \quad \text{for } n = 3,
\]

\[(4.29) \quad \|x_i x_j f \partial_k f\|_{L^q} + \|x_i x_j |g|^2\|_{L^q} \leq C \sum_{|a| \leq 2, |b|=|a|} \|x^a \partial^b f\|_{\dot{H}^{s+}} + C \sum_{|b| \leq |a|} \|x^a \partial^b g\|_{\dot{H}^{s+}}^2 \quad \text{for } n = 2.
\]

In the same manner as we have estimated (4.18), we can deduce from these estimates that (4.25) is bounded by the right-hand side of (4.13). Hence (4.13) is proved and the proof of (3.6) is completed.

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