Confining properties of QCD at finite temperature and density

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A disorder parameter detecting dual superconductivity of the vacuum is used as a probe to characterize the confining properties of the phase diagram of two color QCD at finite temperature and density. We obtain evidence for the disappearing of dual superconductivity (deconfinement) induced by a finite density of baryonic matter, as well as for a coincidence of this phenomenon with the restoration of chiral symmetry both at zero and finite density. The saturation transition induced by Pauli blocking is studied as well, and a general warning is given about the possible effects that this unphysical transition could have on the study of the QCD phase diagram at strong values of the gauge coupling.

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I. INTRODUCTION

Color confinement emerges as an absolute property of strongly interacting matter from experimental facts, but is not yet fully understood starting from the QCD first principles. Lattice QCD simulations, however, have given some evidence about confinement and have even predicted the nature of confining properties of the QCD vacuum in the region of low densities and high temperatures, which is relevant for heavy ion experiments, and in the region of low densities and high temperatures, where confinement is expected for the confining properties of the theory when going from \(N_c = 2\) to \(N_c = 3\), where \(N_c\) is the number of colors: for that reason we believe that our study could be relevant also for real QCD.

In Section I we recall the general properties of lattice QCD at finite baryon density as well as the specific features of the two color model. In Section II we review the definition of the disorder parameter \(\langle M \rangle\) and

\[ \langle M \rangle \]

We change the usual notation for the disorder parameter, \(\langle \mu \rangle\), in order to avoid confusion with the notation for the chemical potential.
II. QCD AT FINITE DENSITY AND THE TWO-COLOR MODEL

We will consider a discretized lattice action for two-color QCD at finite chemical potential defined as follows:

\[ S = S_G + \sum_{i,j} \bar{\psi}_i M[U]_{i,j} \psi_i \tag{1} \]

where \( S_G \) is the pure gauge Wilson action,

\[ S_G = \beta \sum_\Box \left( 1 - \frac{1}{2} \text{Tr} \Box \right), \tag{2} \]

the sum being over all plaquettes, while the fermion matrix is defined, in the case of standard staggered fermions, as

\[ M_{i,j} = a_m \delta_{i,j} + \frac{1}{2} \sum_{\nu=1}^{3} \eta_{\nu,\nu} \left( U_{i,\nu} \delta_{i,j-\nu} - U_{i,-\nu,\nu}^\dagger \delta_{i,j+\nu} \right) + \eta_{\nu,4} \left( e^{a_\nu U_{i,4}} \delta_{i,j-4} - e^{-a_\nu U_{i,-4}} \delta_{i,j+4} \right). \tag{3} \]

Here \( i \) and \( j \) refer to lattice sites, \( \nu \)s are a unit vector on the lattice, \( \eta_{\nu,\nu} \) are staggered phases and \( U \) are gauge link variables; \( a_\nu \) and \( a_m \) are respectively the chemical potential and the quark mass in lattice units. The grand-canonical partition function can be written, after integrating out fermions, as:

\[ Z = \int \mathcal{D}U e^{-S_G[U]} \det M[U]. \tag{4} \]

In ordinary QCD the fermion determinant is complex for generic values of the chemical potential, thus hindering the use of numerical Monte-Carlo simulations. Various possibilities have been explored to circumvent the problem, like for instance reweighting techniques \[17, 18\], the use of an imaginary chemical potential either for analytic continuation \[2, 19, 20, 21, 22, 23, 24\] or for reconstructing the canonical partition function \[25\], Taylor expansion techniques \[26, 27\] and non-relativistic expansions \[28, 29, 30\].

The problem is absent in QCD with two colors, since the gauge group is real: indeed the fermion determinant, being expressible like any other gauge invariant observable in terms of traces over closed loops, is real as well, and numerical simulations are feasible. For this reason two-color QCD has been widely studied in the past as a laboratory for real QCD at finite density \[1, 3, 31, 32, 33, 34, 35, 36, 37\]. Despite some peculiar features of the model, like the fact that baryons and mesons are degenerate, one still expects to learn relevant information about specific questions, like for instance the fate of topology \( \mathbb{R}^4 \) or confinement at finite density.

III. THE DISORDER PARAMETER \( \langle \mathcal{M} \rangle \)

The magnetically charged operator \( \mathcal{M}(\vec{x}, t) \), whose expectation value detects dual superconductivity, is defined in the continuum as the operator which creates a magnetic monopole in \( \vec{x}, t \) by shifting the quantum field by the classical vector potential of a monopole, \( \vec{b}_\perp \), and can be written (see Ref. \[16\] for details) as

\[ \mathcal{M}(\vec{x}, t) = \exp \left[ \frac{i}{e} \int d^3 y \vec{E}_\perp(\vec{y}, t) \vec{b}_\perp(\vec{y} - \vec{x}) \right], \tag{5} \]

with the electric field \( \vec{E}_\perp(\vec{y}, t) \) being the momentum conjugate to the quantum vector potential. Its expectation value, when discretized on the lattice, can be expressed as the ratio of two different partition functions,

\[ \langle \mathcal{M} \rangle = \tilde{Z}/Z, \tag{6} \]

where \( Z \) is the usual QCD partition function, while \( \tilde{Z} \) is obtained from \( Z \) by a change in the pure gauge action \( S_G \rightarrow S_G \), consisting in the addition of the monopole field to the temporal plaquettes at a given timeslice where the monopole is created.

Being expressed as the ratio of two different partition functions, the numerical study of \( \langle \mathcal{M} \rangle \) is a highly non-trivial task, since \( \mathcal{M} \) gets significant contributions only on those configurations having very small statistical weight. While numerical methods have been recently developed which permit a direct determination of \( \langle \mathcal{M} \rangle \) \[38\], we shall not use them in the present study since they involve the combination of different Monte Carlo simulations, a task which in presence of dynamical fermions could be unpractical. We will instead study, as usual, susceptibilities of the disorder parameter, from which the behaviour of \( \langle \mathcal{M} \rangle \) at the phase transition can be inferred.

For instance, being interested in \( \langle \mathcal{M} \rangle \) as a function of \( \beta \), as for the \( \mu = 0 \) phase transition, one usually measures \[7, 8, 9\]

\[ \rho = \frac{\partial}{\partial \beta} \ln \langle \mathcal{M} \rangle = \frac{\partial}{\partial \beta} \ln \tilde{Z} - \frac{\partial}{\partial \beta} \ln Z = \langle S \rangle_S - \langle \tilde{S} \rangle_\tilde{S}, \tag{7} \]
where the subscript indicates the pure gauge action used for Monte Carlo sampling. The disorder parameter can be reconstructed from the susceptibility $\rho$, exploiting the fact that one has exactly $\langle M \rangle = 1$ at $\beta = 0$

$$\langle M \rangle (\beta) = \exp \left( \int_0^\beta \rho(\beta')d\beta' \right).$$ (8)

In particular $\rho \approx 0$ in the confined phase means $\langle M \rangle \neq 0$, a sharp negative peak of $\rho$ implies a sudden drop of $\langle M \rangle$. In particular $\rho \langle M \rangle$ fact that one has exactly related to the inverse temporal extension, the integral formulation of QCD, the physical temperature is increased. Indeed, in the Euclidean path space ($T \sim 1/\langle M \rangle$), where $a$ is the lattice spacing which for an asymptotically free field theory is a decreasing function of the inverse gauge coupling $\beta$. For that reason the inverse coupling $\beta$ is usually adopted in place of $T$ when studying the QCD phase diagram, the latter being an increasing function of the former.

At finite temperature and density we are interested in studying the behaviour of $\langle M \rangle$ in the two parameter space $(\beta, \rho)$, where $\rho = a\mu$ is the chemical potential in lattice units. For that reason we introduce the new susceptibility

$$\rho_D \equiv \frac{\partial}{\partial \rho} \ln \langle M \rangle = \frac{\partial \ln Z}{\partial \rho} - \frac{\partial \ln Z}{\partial \rho} = \langle N_q \rangle_S - \langle N_q \rangle_S$$ (9)

where $N_q$ is the quark number operator, i.e. according to the definition of $Z$ given in Eq. (1):

$$\langle N_q \rangle = \left( \text{Tr} \left( \frac{\partial M}{\partial \rho} \cdot M^{-1} \right) \right);$$ (10)

(an additional factor $2$ is actually needed for the case studied in the present paper, which deals with 8 staggered flavors, see Eq. (12)). The dependence of $\langle M \rangle$ on the chemical potential $\rho$ can then be reconstructed as follows:

$$\langle M \rangle (\beta, \rho) = \langle M \rangle (\beta, 0) \exp \left( \int_0^{\rho} \rho_D (\rho') d\rho' \right),$$

so that, if the starting point at $\rho = 0$ is in the confined phase ($\langle M \rangle (\beta, 0) \neq 0$), the behaviour expected for $\rho_D (\rho)$ in correspondence of a possible finite density deconfinement transition will be the same shown by $\rho$ across the finite temperature transition.

Assuming the presence of a (pseudo)critical line in the $T - \mu$ plane where the disorder parameter drops to zero and dual superconductivity disappears, the two susceptibilities $\rho$ and $\rho_D$ can be used not only to locate the position of the line, but also to compute its slope, thus providing a more comprehensive information about the QCD phase diagram. Indeed, it is quite natural to assume that the gradient of the disorder parameter,

$$\tilde{\nabla} \langle M \rangle = \left( \frac{\partial \langle M \rangle}{\partial \rho} , \frac{\partial \langle M \rangle}{\partial \mu} \right) = (\rho, \rho_D) / \langle M \rangle,$$ (11)

be orthogonal, in the $\beta - \mu$ plane, to the critical line, whose slope is then equal to $-\rho_D / \rho$. In the following we shall directly check this property on our numerical data and also make use of it to obtain testable predictions.

### IV. NUMERICAL RESULTS

In order to perform numerical simulations we have adopted the usual Hybrid Monte Carlo algorithm. The partition function in Eq. (1) can be rewritten, introducing pseudo-fermionic fields $\Phi$ as

$$Z = \int \mathcal{D} U \mathcal{D} \Phi e^{-S_q[U] - \Phi^\dagger [M^\dagger M]^{-1} \Phi}$$

$$= \int \mathcal{D} U e^{-S_q[U]} (\text{det } M[U])^2.$$ (12)

In presence of a real chemical potential the usual even-odd factorization trick for reducing the number of flavors cannot be performed, so that Eq. (12) actually describes a theory with 8 (degenerate in the continuum limit) flavours. The standard exact $\phi$ algorithm described in Ref. [39] has been used.

We have performed simulations on lattices $L_s^3 \times L_t$ with $L_t = 6$ and different values of the spatial size ranging from $L_s = 8$ to $L_s = 16$. The bare quark mass has been fixed to $am = 0.07$.

Simulations on the smallest lattice ($L_s = 8$) have been performed on a PC farm, making use of a numerical code obtained by adapting the publicly available MILC code for two colors and for the inclusion of a finite chemical potential. Simulations on larger lattices have been performed instead on the INFN apeNEXT facility in Rome.

The observables we look at are, apart from the susceptibilities of the disorder operator introduced in Section III, the average Polyakov loop, the average plaquette and the chiral condensate:

$$\langle L \rangle = \frac{1}{L_s^3} \sum_n \frac{1}{N_c} \langle \text{Tr } L(n) \rangle,$$ (13)

$$\langle P \rangle = \frac{1}{6L_s^3} \sum_{n, \mu < \nu} \frac{1}{N_c} \langle \text{Tr } \Pi_{\mu\nu}(n) \rangle,$$ (14)

$$\langle \bar{\psi} \psi \rangle = \frac{1}{L_t L_s^3} (\langle \bar{\psi} \psi \rangle)^2,$$ (15)

as well as their susceptibilities

$$\chi_\epsilon = L_t^3 L_s^3 \langle (\bar{\psi} \psi - \langle \bar{\psi} \psi \rangle)^2 \rangle,$$ (16)
\[ \chi_L \equiv L_s^3 \langle (L - \langle L \rangle)^2 \rangle, \quad (17) \]

\[ \chi_P \equiv L_s^3 L_t \langle (P - \langle P \rangle)^2 \rangle. \quad (18) \]

Notice that in the case of the chiral susceptibility we have explicitly considered only the disconnected contribution.

### A. The deconfining transition at zero chemical potential

It is a well known fact that in ordinary full QCD at zero baryon density, chiral symmetry restoration takes place at the same critical temperature as deconfinement, with the latter identified with the disappearance of dual superconductivity \[11, 12\]. We will check again this fact for the theory with two colors, since this will be an important reference information for our following analysis at finite density.

We show in Fig. 1 the peaks of the three susceptibilities \( \chi_c, \chi_L \), and \( \chi_P \) defined above, obtained on a \( 16^3 \times 6 \) lattice, together with curves corresponding to best fits to the location of their peaks. Our estimate for the location of the transition, obtained through a fit to the chiral susceptibility, is \( \beta_c = 1.582(2) \), to be compared to those obtained by fitting the Polyakov loop susceptibility (\( \beta_L = 1.587(4) \)) and the plaquette susceptibility (\( \beta_P = 1.575(5) \)). A clear drop of the chiral condensate and a rise of the Polyakov loop are also observed at \( \beta_c \), as shown in Fig. 2. The dependence of \( \beta_c \) on the spatial size is not significant, as can be appreciated from Table I where we report a summary of the pseudo-critical couplings (and chemical potentials) obtained from our simulations.

Let us now consider the fate of dual superconductivity. In Fig. 3 we show the behaviour of the susceptibility \( \rho \) as a function of \( \beta \) for three different lattice sizes. A clear peak can be appreciated, which deepens when increasing the lattice size and whose location is clearly coincident with that of the chiral transition. Moreover it is also apparent from the figure that \( \rho \) is practically independent of the lattice size in the low coupling region, confirming that \( \langle \mathcal{M} \rangle \neq 0 \) in the thermodynamical limit in that
in the weak coupling ($\beta = 0$ and $\hat{\mu}$ region, showing that $\rho$ is linear with it, as shown in Fig. 4, in the weak coupling region as a function of $L_s$ for $\hat{\mu} = 0$ and $\hat{\mu} = 0.15$. $\rho$ stays constant and close to zero in the thermodynamical limit at strong coupling, while it diverges linearly with $L_s$ at weak coupling.

Phase, while $\rho$ strongly depends on $L_s$, and in particular is linear with it, as shown in Fig. 4 in the weak coupling region, showing that $\langle M \rangle$ is exactly equal to zero in the thermodynamical limit beyond the transition (magnetic charge superselection [40]). Therefore $\beta_c$ seems to separate two phases characterized by a different realization of the $U(1)$ magnetic symmetry.

To better appreciate the coincidence of the chiral transition with the disappearance of dual superconductivity, we have tried a finite size scaling (f.s.s.) analysis of the critical behaviour of $\langle M \rangle$ around the transition temperature. We can assume for $\langle M \rangle$ the following f.s.s. ansatz:

$$\langle M \rangle = L_s^{-\frac{1}{\nu}} \tilde{\phi} \left( (\beta_c - \beta)L_s^{1/\nu} \right)$$

(19)

from which it can be easily derived

$$\rho = L_s^{1/\nu} \tilde{\phi} (\beta_c - \beta)L_s^{1/\nu}.$$  

(20)

We have checked this ansatz on our data, obtaining the best possible agreement for $\nu \simeq 0.63$ and $\beta_c \simeq 1.584$: a reasonable scaling is obtained, with deviations observed on the smaller lattice. In particular we estimate $\beta_c = 1.584(2)$ in good agreement with the location of the chiral transition given above. The fitted critical index $\nu$ seems to indicate an Ising 3D critical behaviour, to be compared to that taking place in the quenched limit (3D Ising) and the renormalization group prediction for the critical behaviour in the chiral limit (first order [41]). However a similar finite size scaling is not observed for the other susceptibilities and we believe that a definite answer about the universality class of the transition cannot be given in the present context, also due to the relatively small spatial volume used (our largest aspect ratio is slightly less than 3). A more careful investigation should be performed and we consider the present analysis, as well as that presented later for the finite density case, as only aimed at a quantitative estimate of the critical coupling where superconductivity disappears.

B. The deconfining transition at non zero chemical potential

The two susceptibilities $\rho$ and $\rho_D$ permit to study $\langle M \rangle$ either as a function of temperature at a fixed value of the chemical potential $\hat{\mu}$, or as a function of $\hat{\mu}$ at fixed temperature. Both strategies can be used to investigate the fate of dual superconductivity in presence of a finite density of baryonic matter: the first could be more effective at small chemical potentials, where the possible transition line starting at $\hat{\mu} = 0$ should be almost parallel to the $\hat{\mu}$ axis, the second could be more convenient at larger chemical potentials. Actually a proper combination of $\rho$ and $\rho_D$ could be used to study the behaviour of $\langle M \rangle$ along any given path in the $\beta - \hat{\mu}$ plane so that one could even choose an optimal combination corresponding to a relevant direction around a critical point: however we shall limit ourselves in the present context to the simpler cases of either fixed temperature or fixed chemical potential. The study at fixed temperature has a particular interest, since it may show how the disappearance of confinement (dual superconductivity) can be induced by simply increasing the density of baryonic matter.

We shall first consider the case of a fixed chemical potential, $\hat{\mu} = 0.15$. In Fig. 6 we show the chiral susceptibility obtained on a $16^3 \times 6$ lattice and compared to the same quantity computed at $\hat{\mu} = 0$. A clear shift of the pseudocritical coupling can be appreciated, in particular we obtain $\beta_c(\hat{\mu} = 0.15) = 1.568(2)$, showing that the (pseudo)critical temperature lowers as the chemical po-

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**FIG. 4:** Behaviour of $\rho$ in the strong coupling ($\beta = 0.5$) and in the weak coupling ($\beta = 2.5$) region as a function of $L_s$ for $\hat{\mu} = 0$ and $\hat{\mu} = 0.15$. $\rho$ stays constant and close to zero in the thermodynamical limit at strong coupling, while it diverges linearly with $L_s$ at weak coupling.

**FIG. 5:** Finite size scaling analysis of $\rho$ around the transition. The best possible scaling is obtained for $\nu \simeq 0.63$ and $\beta_c \simeq 1.584$. 

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the susceptibility of the chiral susceptibility can be understood in terms of the different algorithm used, which in the case of Ref. [3] was a non-exact molecular dynamics algorithm.

In Fig. 6 we show the chiral susceptibility determined for the case \( \beta = 1.50 \), where three different spatial sizes were available \( (L_s = 8, 12, 16) \), according to the ansatz

\[
\langle M \rangle = L_s^{-\frac{d}{2}} \phi \left( (\mu_c - \mu) L_s^{1/\nu} \right)
\]

hence

\[
\rho_D = L_s^{1/\nu} \tilde{\phi}((\mu_c - \mu) L_s^{1/\nu}) .
\]

A reasonable scaling is obtained for \( \nu \sim 0.55 \) and \( \mu_c \sim 0.31 \), in particular we estimate \( \beta_c = 0.315(15) \), marginally compatible with the location of the chiral transition.

We can therefore draw two important conclusions: dual superconductivity (confinement) disappears in presence of a critical density of baryonic matter; moreover the critical line in the \( T-\mu \) plane corresponding to deconfinement coincides, at least within our present uncertainties, of the susceptibility \( \rho \) on the same lattice are shown in Fig. 4 and compared to those obtained at zero density: the peak of \( \rho \) shifts consistently by an amount comparable to that of the chiral susceptibility. Notice that in both cases the actual position of the peak is at a slightly larger than \( \beta_c \). That is expected since \( \rho \) is a logarithmic derivative: assuming that \( \langle M \rangle \equiv \partial \langle M \rangle / \partial \beta \) has a minimum at \( \beta_c \), it follows that \( \partial \rho / \partial \beta = \langle M \rangle^\nu / \langle M \rangle - (\langle M \rangle / \langle M \rangle)^2 \) is still negative at the same point.

Data reported in Fig. 4 show that, also in the case \( \tilde{\mu} = 0.15 \), \( \rho \) is independent of the lattice size and practically equal to zero in the strong coupling region, while it diverges linearly with \( L_s \) in the weak coupling region. Therefore we can conclude that, also in presence of a finite density of baryonic matter, dual superconductivity disappears as the temperature is increased at the same point where chiral symmetry is restored.

Next we turn to the behaviour of \( \langle M \rangle \) as a function of \( \tilde{\mu} \) at fixed temperature \( (\beta) \), determined by means of the susceptibility \( \rho_D \). We have considered only values of \( \beta \) below the (pseudo)critical coupling \( \beta_c \) computed at \( \tilde{\mu} = 0 \), in particular \( \beta = 1.50 \) and \( \beta = 1.55 \): in this case we know that \( \langle M \rangle \neq 0 \) at \( \tilde{\mu} = 0 \), so that \( \rho_D \) may signal a possible disappearance of dual superconductivity induced by finite baryon density. Notice that the lowest value of \( \beta \), on the basis of a rough two-loop estimate of the \( \beta \)-function, corresponds to a physical temperature \( T/T_c \sim a(\beta = 1.582)/a(\beta = 1.5) \sim 0.4 \), where \( T_c \) is the critical temperature at zero chemical potential.

In Fig. 5 we show the chiral susceptibility determined on a \( 16^3 \times 6 \) lattice at \( \beta = 1.55 \) and \( \beta = 1.50 \). A best fit permits to locate the peak positions, hence the (pseudo)critical values of \( \tilde{\mu} \) corresponding to chiral restoration. We obtain \( \tilde{\mu}_c(\beta = 1.50) = 0.340(10)^2 \) and \( \tilde{\mu}_c(\beta = 1.55) = 0.215(10) \), as also reported in Table I.

In Fig. 6 we show instead the results obtained for \( \rho_D \) as a function of \( \tilde{\mu} \) at the same values of \( \beta \) and on various lattice sizes. It clearly appears that while \( \rho_D \) is independent of the lattice size and practically vanishing for small chemical potentials, it has a sharp negative peak in correspondence of the chiral transition which deepens as the spatial size is increased. In order to be more quantitative about the coincidence of chiral restoration and deconfinement, we have performed a f.s.s. analysis for the case \( \beta = 1.50 \), where three different spatial sizes were available \( (L_s = 8, 12, 16) \), according to the ansatz

\[
\langle M \rangle = L_s^{-\frac{d}{2}} \phi \left( (\mu_c - \mu) L_s^{1/\nu} \right)
\]

hence

\[
\rho_D = L_s^{1/\nu} \tilde{\phi}((\mu_c - \mu) L_s^{1/\nu}) .
\]

A reasonable scaling is obtained for \( \nu \sim 0.55 \) and \( \mu_c \sim 0.31 \), in particular we estimate \( \beta_c = 0.315(15) \), marginally compatible with the location of the chiral transition.

We can therefore draw two important conclusions: dual superconductivity (confinement) disappears in presence of a critical density of baryonic matter; moreover the critical line in the \( T-\mu \) plane corresponding to deconfinement coincides, at least within our present uncertainties,
FIG. 8: Chiral susceptibility on a $16^3 \times 6$ lattice as a function of $\tilde{\mu}$ for various values of $\beta$. Dotted curves correspond to best fit to the peak values.

FIG. 9: $\rho_D$ as a function of $\tilde{\mu}$ and for two lattice sizes for various values of $\beta$. Vertical bands correspond to the pseudo-critical chemical potential fitted according to the chiral susceptibility.

with the chiral transition line. These results concern that part of the phase diagram including temperatures down to $T/T_c \sim 0.4$, where $T_c$ is the critical temperature at zero chemical potential: we shall discuss their relevance for the $T \sim 0$ region of the phase diagram later in this paper.

C. The transition line

Having obtained four different locations of the transition line, in particular $\beta_c(\tilde{\mu} = 0) = 1.582(2)$, $\beta_c(\tilde{\mu} = 0.15) = 1.568(2)$, $\tilde{\mu}_c(\beta = 1.55) = 0.215(10)$ and $\tilde{\mu}_c(\beta = 1.50) = 0.340(10)$, as obtained on our larger lattices (see Table I), we can perform a fit of the dependence $\beta_c(\mu)$ in the whole $\beta - \tilde{\mu}$ plane, which will then be used in the following. We are also interested in testing what stated in Section III, i.e. that the ratio $-\rho_D/\rho$ at the transition point can be used as an estimate of the slope of the critical line: we give an example of a common plot of the two susceptibilities in Fig. 11, from which the ratio at $\beta_c, \tilde{\mu}_c$ can be inferred.

FIG. 10: Finite size scaling analysis for $\rho_D$. A critical index $\nu \sim 0.55$ has been used, the best value for the critical chemical potential being $\beta_c \simeq 0.315(15)$.

The good value of $\chi^2/d.o.f.$ shows that a quadratic dependence well describes the critical line down to $T/T_c \sim 0.5$; indeed a fit with a quartic term gives a coefficient for $\tilde{\mu}^4$ compatible with zero. Our estimates for the loca-
as a good estimator of the slope of the line in the $\beta - \mu$ plane. The chiral line has been fitted to a quadratic dependence on $\mu$. The slope of the critical line, has inferred from the disorder parameter for dual superconductivity, has been reported in the figure: a nice agreement (within one standard deviation) can be appreciated, showing that a nice agreement (within one standard deviation) can be appreciated.

The transition of the (pseudo)critical points are reported in Fig. 12 together with the fitted transition line.

In correspondence of our direct locations of the transition line we also show the estimates for the slope of the line obtained from the ratio $-\rho_D/\rho$: in particular we have drawn angles corresponding to one standard deviation from the average values. A good agreement can be appreciated, showing that $-\rho_D/\rho$ can indeed be taken as a good estimator of the slope of the line in the $\beta - \mu$ plane.

Finally, in Fig. 13 we report again the chiral transition line fitted above and compared to a quadratic fit in $\tilde{\mu}$ for the critical line corresponding to the disappearance of dual superconductivity (deconfinement). The plot supports our previous statement, i.e. that the chiral transition coincides with deconfinement in the range of $\beta$ values (temperatures) explored.

D. A few remarks on saturation

It is a well known fact that, even in absence of the sign problem, the study of lattice gauge theories in presence of a finite density of fermions cannot be pushed to arbitrarily high densities, i.e. to arbitrarily high values of the chemical potential. Indeed the number of available energy levels is limited by the presence of the UV cutoff, which places an upper limit to the possible values of the Fermi energy. Stated otherwise, we cannot place, because of the Pauli exclusion principle, more than one fermion with given quantum numbers per lattice site. Apart from the upper limit that this places on the densities reachable on the lattice, a much worse problem comes from the fact that, as saturation sets in, the absence of available fermion levels quenches fermion dynamics, modifying the field theory at the ultraviolet scale. As a matter of fact, the theory becomes equivalent to a pure gauge theory in the large $\tilde{\mu}$ limit.

Saturation is therefore an unphysical lattice artifact which may in principle invalidate numerical results, one should therefore be extremely careful in locating its onset. Indeed, while saturation effects are generically expected to appear for $\tilde{\mu} = a\mu$ of order 1, the exact value of $\tilde{\mu}$ where they start to be important may depend on the dynamics of the theory. In the following we will briefly explore the transition to saturation in the two color model under consideration, arriving to some interesting conclusions which may sound as a general warning.

We have explored saturation effects in some detail at $\beta = 1.55$. In Fig. 14 we show the behaviour of some observables as a function of $\tilde{\mu}$ in a wide range going up to $\tilde{\mu} = 1.6$. For small values of the chemical potential above $\tilde{\mu}$, the fermion density rises roughly with a cubic dependence in $\tilde{\mu}$, as expected for a gas of free fermions, but then saturates to a value which in the figure is normalized to two fermions per site: the departure from the cubic behaviour starts at $\tilde{\mu} \sim 0.6 - 0.8$. Also the rise of the Polyakov loop suddenly stops at a similar value of $\tilde{\mu}$, followed by a drop; in the same region the plaquette suddenly drops towards its quenched value. Complete saturation is reached for $\tilde{\mu} \sim 1.4 - 1.6$.

Much is learned by looking at the behaviour of the susceptibilities of the disorder parameter in the same range, which is shown in Fig. 15: the negative peak of $\rho_D$ at $\tilde{\mu} \sim 0.3$, corresponding to the physical deconfinement transition, is followed by a positive unphysical peak at $\tilde{\mu} \sim 0.7$. That means that the disorder parameter $\langle M \rangle$, which at first drops to zero thus signalling deconfinement, then rises again as an effect of saturation: indeed the “saturation transition” leads to a $SU(2)$ pure gauge theory, which at $\beta = 1.55$ and $L_t = 6$ is deep in the confined

FIG. 12: Phase diagram in the $\beta - \mu$ plane. The chiral line has been fitted to a quadratic dependence on $\mu$. The slope of the critical line, has inferred from the disorder parameter for dual superconductivity, has been reported in the figure: a nice agreement (within one standard deviation) can be appreciated.

FIG. 13: Comparison of the chiral pseudocritical line (continuous) and of that corresponding to the disappearance of dual superconductivity (dotted), as fitted from our data.
phase, implying $\langle M \rangle \neq 0$. To verify that we have explicitly reconstructed $\langle M \rangle(\hat{\mu})/\langle M \rangle(\hat{\mu} = 0)$ (see Eq. 11) and reported it in Fig. 15 in the same figure we have reported the location of the saturation transition as obtained by a fit to the peak of the plaquette susceptibility.

We should be satisfied, since the saturation transition at $\hat{\mu} \sim 0.7$ is well separated from the physical transition at $\hat{\mu} \sim 0.3$. However we notice that, defining a “saturation line” in the $\beta - \hat{\mu}$ plane corresponding to the onset of saturation effects, we can predict, according to what stated in the previous paragraph, its slope from the ratio $-\rho_D/\rho$. We see from Fig. 15 that in correspondence of the positive saturation peak for $\rho_D$, the other susceptibility $\rho$ has a negative peak, hence we expect a positive slope for the saturation line. That means that at lower values of $\beta$ the onset of saturation could take place at lower values of $\hat{\mu}$; that combined with the fact that the physical critical $\hat{\mu}_c$ instead increases as $\beta$ decreased, could lead to the unfortunate situation in which the two transition, physical and unphysical, merge at lower values of $\beta$, thus hindering, at least in the present case, the study of the strong coupling (low temperature) region of the phase diagram.

In order to further explore this possibility we have decided to make an estimate of the location of the saturation transition, through a fit to the plaquette susceptibilities (which are reported in Fig. 16), performing simulations also at a different value of the gauge coupling, $\beta = 1.675$. Our estimate for the pseudocritical saturation chemical potential $\hat{\mu}_{SC}$ are reported in Table I and are $\hat{\mu}_{SC}(\beta = 1.55) = 0.68(3)$ and $\hat{\mu}_{SC}(\beta = 1.675) = 0.79(3)$. In Fig. 17 we report our estimate for the location of the saturation line together with a rough linear extrapolation suggesting that the saturation line could meet the physical line, whose estimate given in previous paragraph is reported in the figure as well, for $\beta \sim 1.4$. Notice that the linear extrapolation adopted is supported by the slope of the line obtained through the ratio $-\rho_D/\rho$, whose estimates are reported in the figure as well.

We therefore give a general warning about the possible effects of saturation on the study of finite density QCD at low values of the gauge coupling. The situation may of course be quite different depending on the temporal extent $L_t$ of the lattice, on the number of flavors, of colors and on the lattice discretization (staggered or Wilson...
It is natural to ask whether our present following and well separated from the onset of a bosonic at
made, based on the analysis of the Polyakov loop, that could help understanding the nature of compact as-
ferrions) adopted. We plan to make a more extensive study of this problem in the future.

E. Did we catch the physics of the low temperature region of the phase diagram?

One of our starting questions was about the fate of confinement at high densities and low temperatures, since
that could help understanding the nature of compact astrophysical objects. In Ref. [1] the hypothesis has been
made, based on the analysis of the Polyakov loop, that at $T \sim 0$ deconfinement could occur at a critical density
following and well separated from the onset of a bosonic superfluid phase. It is natural to ask whether our present
results can be of any relevance regarding this specific issue, i.e. how close we have got to the low temperature region of the QCD phase diagram.

Since we have not included an explicit diquark source term in our model, we cannot obtain direct information about that observable; however we shall try to sketch a qualitative picture based on the distribution of the eigen-
values of the fermionic matrix. At zero density that can be written as $M = a m \text{Id} + D$ where $D$ is antihermitian, hence it has purely imaginary eigenvalues, therefore the eigenvalues of $M$ lie on a segment in the complex plane orthogonal to the real axis.

As a real chemical potential is switched on, $D$ ceases to be antihermitian and the eigenvalues get scattered in the whole complex plane: that is evident in the first inset of Fig. [13] where we show the distribution of eigenvalues on a typical configuration obtained at $\beta = 1.55$ and a small chemical potential, $\tilde{\mu} = 0.10$. The eigenvalues occupy a narrow vertical band and the finite density of eigenvalues in correspondence of the real axis is strictly linked to the presence of chiral symmetry breaking (Banks-Casher relation [42]). The width of the distribution on the real axis grows as $\tilde{\mu}$ increases, roughly proportionally to $\tilde{\mu}^2$, till the distribution touches the imaginary axis: at this point the chiral condensate is expected to rotate into a diquark condensate (see for instance Ref. [43] for a review): as it is clear from the second inset in Fig. [13] at $\beta = 1.55$ this happens roughly at $\tilde{\mu} \sim 0.3$, a value which actually turns out to be almost independent of the gauge coupling in the range of $\beta$ values explored in our simulations and is in agreement with the values found for diquark condensation in similar works using the same quark mass [31, 44]: we show as an example in Fig. [19] the eigenvalue distribution projected onto the real axis for three values of $\tilde{\mu}$ at $\beta = 1.45$.

However if we are in the high temperature region, i.e. slightly below $T_c(\mu = 0)$, chiral symmetry will be restored quite soon as $\tilde{\mu}$ is increased because of the transition to the Quark-Gluon Plasma. Therefore there will be actually no chiral condensate to be rotated into a diquark condensate at the point where the distribution touches the imaginary axis. Indeed we see from the second inset in Fig. [18] that the region around the real axis is quite depleted of eigenvalues for $\beta = 1.55$ at $\tilde{\mu} = 0.3$. We can easily understand this in terms of the chiral line we have drawn in Fig. [12] chiral symmetry gets restored already below $\tilde{\mu} = 0.3$ at $\beta = 1.55$.

Following this line of reasoning, the region relevant for low temperature physics on our lattices with $L_t = 6$ should be that below $\beta \sim 1.5$, where our fitted (pseudo)critical line passes beyond $\tilde{\mu} \sim 0.3$. In this region one could for instance observe, among other different possibilities, two different transitions, the first corresponding to the onset of diquark condensation, the second roughly being the continuation of the line in Fig. [12] thus corresponding to deconfinement: this is indeed the scenario suggested by Ref. [1].

We have therefore performed numerical simulations on

![FIG. 17: Saturation transition line and its relation with the physical transition line.](image)

| $L_t \times b_s$ | $\beta_c$ | $\mu_c$ |
|-----------------|-----------|--------|
| $8^3 \times 6$  | 1.584(2)  | 0      |
| $12^3 \times 6$ | 1.587(2)  | 0      |
| $16^3 \times 6$ | 1.582(2)  | 0      |
| $16^3 \times 6$ | 1.568(2)  | 0.15   |
| $8^3 \times 6$  | 1.55      | 0.222(10) |
| $16^3 \times 6$ | 1.55      | 0.215(10) |
| $8^3 \times 6$  | 1.5       | 0.325(10) |
| $12^3 \times 6$ | 1.5       | 0.349(15) |
| $16^3 \times 6$ | 1.5       | 0.342(10) |

TABLE I: Collection of pseudocritical couplings as determined from our numerical data. Physical critical couplings have been determined through the chiral susceptibility, while the unphysical saturation transitions have been located by means of the plaquette susceptibility.
a $8^3 \times 6$ lattice at $\beta = 1.45$. In this case a finite density of eigenvalues around the real axis is still present at $\mu \sim 0.3$, as can be better appreciated in Fig. 20 where we plot the distribution projected onto the imaginary axis at $\beta = 1.45$ and $\mu = 0.3$, compared to that obtained at higher temperatures.

In Fig. 21 we report the chiral susceptibility, compared to that measured on the same lattices at different gauge couplings. We notice that the peak is strongly reduced and its position is not much different from what obtained at $\beta = 1.5$ and in clear disagreement with what expected from the continuation of the chiral line in Fig. 12.

The first peak could indeed correspond to the onset of a bosonic superfluid phase. Nothing seems to happen thereafter.

In Fig. 22 we report instead data obtained for the susceptibility $\rho_D$ of the disorder parameter. In this case the negative peak has almost completely disappeared and a very small peak at $\mu \sim 0.3$ is followed by a region $\mu \geq 0.4$ where $\rho_D$ clearly changes its sign: on the basis of what we have discussed in Section IV D and comparing this behaviour with that observed at $\beta = 1.55$, a possible interpretation is that of an early onset of saturation effects in this case, preventing the observation of any further physical transition. We expected saturation effects to obscure the physical transition at $\beta \sim 1.4$, but we are not surprised that the situation may be worse.

This conclusion is supported by looking at the behaviour of the Polyakov loop (see Fig. 23), in this case saturation effects are signalled by an inversion in the growth of $\langle L \rangle$ as a function of $\mu$. 

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**FIG. 18:** Distribution of eigenvalues on typical configurations of the $8^3 \times 6$ lattice obtained respectively at ($\beta = 1.55$, $\mu = 0.10$), ($\beta = 1.55$, $\mu = 0.30$) and ($\beta = 1.45$, $\mu = 0.30$).

**FIG. 19:** Distribution of eigenvalues projected onto the real axis for various $\mu$ on a $8^3 \times 6$ lattice at $\beta = 1.45$.

**FIG. 20:** Distribution of eigenvalues projected onto the imaginary axis for various values of $\beta$ on a $8^3 \times 6$ lattice at $\mu = 0.3$. 

**FIG. 21:** Distribution of eigenvalues projected onto the imaginary axis for various values of $\beta$ on a $8^3 \times 6$ lattice at $\mu = 0.3$. 

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We conclude therefore that we are not be able to clarify the onset of deconfinement at $T \sim 0$, at least on the present lattice size. We could of course further decrease the temperature without decreasing $\beta$ by going to larger values of $L_s$. However that would imply a numerical effort which is not affordable with our present algorithmic and computational resources.

V. CONCLUSIONS

We have investigated the phase diagram of two-color QCD at finite temperature and density by means of a disorder parameter for color confinement detecting dual superconductivity of the QCD vacuum.

We have obtained evidence for deconfinement induced by a finite density of baryonic matter. Moreover the transition line corresponding to the disappearance of dual superconductivity (deconfinement) appears to coincide, in the range of temperature explored ($0.4 T_c < T < T_c$, where $T_c$ is the critical temperature at zero density), with that corresponding to chiral symmetry restoration, as it happens in the zero density case. We have also shown that the susceptibilities of the disorder parameter can be used in order to compute the slope of the critical line in the $\beta - \mu$ plane, obtaining consistent results.

We have investigated in some detail the unphysical transition corresponding to the onset of saturation and shown that it moves at lower values of $\mu$ as $\beta$ is decreased with a possible intersection with the physical transition line, thus giving a general warning about the possible effects of saturation on the study of finite density QCD at strong values of the gauge coupling. This phenomenon of course may be quite different depending on the fermion discretization, on the number of flavors and on other parameters of the system ($L_t$, quark masses); for this reason we plan to make a more systematic study in the future.

We have also verified that in our case saturation actually prevents us from obtaining results relevant for the $T \sim 0$ region of the phase diagram. For this reason we plan to extend our study in the future by adopting different lattice sizes and/or fermion discretizations. Results relevant for the high temperature region of real QCD could also be obtained within the imaginary chemical potential approach.

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