PROBABILISTIC ASPECTS
OF AL-SALAM–CHIHARA POLYNOMIALS

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Abstract. We solve the connection coefficient problem between the Al-Salam–Chihara polynomials and the $q$-Hermite polynomials, and we use the resulting identity to answer a question from probability theory. We also derive the distribution of some Al-Salam–Chihara polynomials, and compute determinants of related Hankel matrices.

1. Introduction and main identity

The aim of the paper is to point out the connection of Al-Salam–Chihara polynomials with a regression problem in probability, and to use it to give a new simple derivation of their density. Our approach exploits identity (1.8) below, which connects the Al-Salam–Chihara polynomials to the continuous $q$-Hermite polynomials. This connection is more direct and elementary but less general than the technique of attachment exploited in [B196, Section 2]. We also compute determinants of Hankel matrices with entries that are linear combinations of the $q$-Hermite polynomials.

The Al-Salam–Chihara polynomials were introduced in [ASC76], and their weight function was found in [AI84]. We are interested in the renormalized Al-Salam–Chihara polynomials \( \{p_n(x|q,a,b)\} \), which are defined by the following three-term recurrence relation:

\[
p_{n+1}(x) = (x - aq^n)p_n(x) - (1 - bq^{n-1}) [n]_q p_{n-1}(x) \quad (n \geq 0),
\]

with the usual initial conditions \( p_{-1} = 0, \ p_0 = 1 \). Here, we use the standard notation

\[
[n]_q = 1 + q + \cdots + q^{n-1},
\]

\[
[n]_q! = [1]_q[2]_q \cdots [n]_q,
\]

\[
\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!},
\]

with the usual conventions \( [0]_q = 0, \ [0]_q! = 1 \).

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For $|q| < 1$, their generating function
\[ f(t, x|q, a, b) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} p_n(x|q, a, b) \]
is given by\n\[ f(t, x|q, a, b) = \prod_{k=0}^{\infty} \frac{1 - (1-q)atq^k + (1-q)bt^2q^{2k}}{1 - (1-q)xtq^k + (1-q)t^2q^{2k}}; \]
compare [AI84] (3.6) and (3.10).

The corresponding (renormalized) continuous $q$-Hermite polynomials $H_n(x|q) = p_n(x|q, 0, 0)$ satisfy the three-term recurrence relation\n\[ H_{n+1}(x) = xH_n(x) - [n]_q H_{n-1}(x). \]
For $|q| < 1$ their generating function $\phi(t, x|q) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} H_n(x|q)$ is\n\[ \phi(t, x|q) = \prod_{k=0}^{\infty} \left( 1 - (1-q)xtq^k + (1-q)t^2q^{2k} \right)^{-1}. \]

Of course, these are well-known special cases of (1.1) and (1.2); see [JSV87] (2.11) and (2.12)], which we state here for further reference.

We will also use polynomials $\{B_n(x|q)\}$ defined by the three-term recurrence relation\n\[ B_{n+1}(x) = -q^n xB_n(x) + q^{n-1}[n]_q B_{n-1}(x) \quad (n \geq 0) \]
with the usual initial conditions $B_{-1} = 0, B_0 = 1$. These polynomials are related to the $q$-Hermite polynomials by\n\[ B_n(x|q) = \begin{cases} \frac{[n]_q}{[n]_q!} B_n(x|q) & \text{if } q > 0, \\ (-1)^n q^{n(n-1)/2} H_n(-\sqrt{q} x) & \text{if } q < 0 \end{cases} \]
and have been studied in [Ask89], [LM94]. Their generating function $\psi(t, x|q) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} B_n(x|q)$ is given by\n\[ \psi(t, x|q) = \prod_{k=0}^{\infty} \left( 1 - (1-q)xtq^k + (1-q)t^2q^{2k} \right). \]

We now point out the mutual relationship between the Al-Salam–Chihara polynomials $\{p_n(x|q, a, b)\}$ and the polynomials $\{H_n(x|q)\}$ and $\{B_n(x|q)\}$.

**Theorem 1.** For all $a, c, q \in \mathbb{C}$, $c \neq 0$, and $n \geq 1$ we have\n\[ p_n(x|q, a, b) = \sum_{k=1}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q c^{n-k} B_{n-k}(\frac{a}{c}|q) \left( H_k(x|q) - c^k H_k(\frac{a}{c}|q) \right), \]
where $b = c^2$.

**Proof.** From the recurrence relations (1.1), (1.3), and (1.5), it is clear that $p_n(x|q, a, b)$, $H_n(x|q)$, and $B_n(x|q)$ are given by polynomial expressions in the variable $q$. The $q$-binomial coefficient $\left[ \begin{array}{c} n \\ k \end{array} \right]_q$ is also a polynomial in $q$. Therefore, we see that identity (1.5) is equivalent to a polynomial identity in variable $q \in \mathbb{C}$.
Hence it is enough to prove that (1.8) holds true for all \(|q| < 1\). When \(|q| < 1\), inspecting (1.12), (1.13), and (1.14) we notice that for \(b = c^2\) we have
\[
(1.9) \quad f(t, x|q, a, b) = \psi(tc, a/c|q)\phi(t, x|q)
\]
and
\[
(1.10) \quad \psi(t, x|q)\phi(t, x|q) = 1.
\]
Therefore,
\[
f(t, x|q, a, b) = 1 + \psi(tc, a/c|q) (\phi(t, x|q) - \phi(tc, a/c|q)),
\]
which is valid for all small enough \(|t|\). Comparing the coefficients at \(t^n\) for \(n \geq 1\) and taking into account that \(H_k(x|q) - c^kH_k(a/c|q) = 0\) for \(k = 0\), we get (1.8). □

Remark 1. One could split (1.8) into the following two separate identities, which are implied by (1.9) and (1.10) respectively:
\[
(1.11) \quad \forall n \geq 0: \quad p_n(x|q, a, c^2) = \sum_{k=0}^{n} \binom{n}{k}_q c^{n-k}B_{n-k}(a/c|q)H_k(x|q),
\]
\[
(1.12) \quad \forall n \geq 1: \quad \sum_{k=0}^{n} \binom{n}{k}_q B_{n-k}(x|q)H_k(x|q) = 0.
\]
Formula (1.11) is a renormalized inverse of formula [IRS99, (4.7)], which expresses the \(q\)-Hermite polynomials as linear combinations of Al-Salam–Chihara polynomials. Formula (1.12) resembles [Car56, (2.28)], which considers \(q\)-Hermite polynomials of the form \(h_n(x|q) = \sum_{k=0}^{n} \binom{n}{k}_q x^k\), paired with \(b_n(x|q) = h_n(x|1/q)\).

2. Probabilistic aspects

Quadratic regression questions in the paper [Bry01] lead to the problem of determining all probability distributions \(\mu\) which are defined indirectly by the relationships
\[
(2.1) \quad \int H_n(x|q)\mu(dx) = \rho^n H_n(y|q), \quad n = 1, 2, \ldots ,
\]
where \(y, \rho, q \in \mathbb{R}\) are fixed parameters, and \(\{H_n\}_{n \geq 0}\) is the family of the \(q\)-Hermite polynomials.

Our next result shows that this problem can be solved using the Al-Salam–Chihara polynomials.

Theorem 2. If \(\mu = \mu(dx|\rho, y)\) satisfies (2.1), then its orthogonal polynomials are Al-Salam–Chihara polynomials \(\{p_n(x|q,a,b)\}\) with \(a = \rho y\), \(b = \rho^2\).

Proof. Recall that \(H_n(x|q) = p_n(x|q,0,0)\). Thus if \(\rho = 0\), then (2.1) implies that
\[
\int p_n(x|q,a,b)\mu(dx) = 0 \quad \text{for all } n = 1, 2, \ldots .
\]
Suppose now that \(\rho \neq 0\). Combining (1.8) with (2.1) we get
\[
\int p_n(x|q,a,b)\mu(dx) = \sum_{k=1}^{n} \binom{n}{k}_q \rho^{n-k}B_{n-k}(y|q) \int (H_k(x|q) - \rho^kH_k(y|q)) \mu(dx) = 0
\]
for all \(n = 1, 2, \ldots .\) Since \(\{p_n\}\) satisfy a three-step recurrence, this implies
\[
\int p_k(x)p_n(x)\mu(dx) = 0 \quad \text{for all } 0 \leq k < n.
\]
□
Next we answer an unresolved case from [Bry01].

**Corollary 1.** Fix \( q > 1, y \in \mathbb{R} \). Let \( \mathcal{R}_q = \{1, 1/q, 1/q^2, \ldots, 1/q^n, \ldots, 0\} \).

(i) If \( \rho^2 \not\in \mathcal{R}_q \), then (2.1) has no probabilistic solution \( \mu \).

(ii) If \( \rho^2 \in \mathcal{R}_q \) is non-zero, then the probabilistic solution of (2.1) exists, and is a discrete measure supported on \( 1 + \log_q 1/\rho^2 \) points.

**Proof.** Suppose that \( \mu \) is positive and solves (2.1). Therefore its monic orthogonal polynomials satisfy the three-term recurrence relation

\[
\begin{equation}
\tag{2.2}
\rho_{n+1}(x) = (x - \rho y q^n) \rho_n(x) - (1 - \rho^2 q^{n-1}) \rho_{n-1}(x).
\end{equation}
\]

For a positive non-degenerate measure \( \mu_y(dx) \), and \( n \geq 1 \) we have

\[
\int \rho_n^2(x) \mu_y(dx) = (1 - \rho^2 q^{n-1}) \int \rho_{n-1}^2(x) \mu_y(dx).
\]

If \( \rho^2 \not\in \mathcal{R}_q \), then \( (1 - \rho^2 q^{n-1}) \neq 0 \) for all \( n \). Since \( \int \rho_n^2(x) \mu_y(dx) > 0 \), this shows that \( \int \rho_n^2(x) \mu_y(dx) = 0 \) for all \( n \geq 0 \). But then the coefficients \( 1 - \rho^2 q^{n-1} \) must be non-negative for all \( n \), which is false. This proves (i).

To conclude the proof it remains to notice that if \( \rho^2 = 1/q^m \), then from (2.2) and (the proof of) Favard’s theorem, see [Fre71, Theorem II.1.5], it follows that the solution of (2.1) is given by a measure supported on the roots of the polynomial \( p_{m+1} \). Indeed, (2.2) implies that the polynomial \( p_{m+2} \) is divisible by \( p_{m+1} \). Therefore, \( p_{m+1} \) is the common factor of all polynomials \( \{p_k : k \geq m + 1\} \). It is also known, see [Fre71, Theorem I.2.2], that \( p_{m+1} \) has exactly \( m + 1 \) distinct real roots \( x_1, \ldots, x_{m+1} \). Thus, any measure \( \mu(dx) = \sum \lambda_j \delta_{x_j} \) supported on the roots of the polynomial \( p_{m+1} \) satisfies \( \int p_{m+1} \mu(dx) = 0 \). Solving the remaining \( m + 1 \) equations \( \int p_k \mu(dx) = 1 \), and \( \int p_k(x) \mu(dx) = 0, k = 1, 2, \ldots, m \) for \( \lambda_j \), we get a measure that solves (2.1). This measure is non-negative since the coefficients at the third term in the recurrence (2.2) are non-negative for \( n = 1, \ldots, m \); see [Fre71, page 58].

From Theorem 2 it follows that if the solution of (2.1) exists, then it is given by the distribution of the Al-Salam–Chihara polynomials. The distribution of the Al-Salam–Chihara polynomials is derived in [Al84, Chapter 3]. However, in [Bry01, Proposition 8.1] we found the solution of (2.1) that relies solely on the facts about the \( q \)-Hermite polynomials. We repeat the latter argument here, and then use it to re-derive the distribution of the corresponding Al-Salam–Chihara polynomials.

**Corollary 2.** If \( \rho, q, y \in \mathbb{R} \) are such that \( |\rho| < 1, |q| < 1, \) and \( y^2(1-q) < 4 \), then the probabilistic solution of (2.1) is given by the absolutely continuous measure \( \mu \) with the density on \( x^2 < 4/(1-q) \) given by

\[
\frac{\sqrt{1-q}}{2\pi \sqrt{4-4(1-q)x^2}} \prod_{k=0}^\infty \frac{(1-\rho^2 q^k) (1-q^{k+1}) ((1+q^k) - (1-q)x^2 q^k)}{(1-\rho^2 q^{2k})^2 - (1-q) q x(1+q^{2k}) xy + (1-q)^2 (x^2 + y^2) q^{2k}}.
\]

**Proof.** The distribution of the \( q \)-Hermite polynomials \( H_n(x|q) \) is supported on \( x^2 < 4/(1-q) \) with the density

\[
f_H(x) = \frac{\sqrt{1-q}}{2\pi \sqrt{4-4(1-q)x^2}} \prod_{k=0}^\infty ((1+q^k)^2 - (1-q)x^2 q^k) \prod_{k=0}^\infty (1-q^{k+1});
\]
Corollary 3. Instead of our proof. By Theorem 2, the distribution of polynomials \( H_n(x) \) converges uniformly and defines the Poisson-Mehler kernel, which is given by

(2.5) \[ g_H(x, y, \rho) = \prod_{k=0}^{\infty} \frac{(1 - \rho^2 q^{2k})}{(1 - \rho^2 q^{2k})^2 - (1 - q) \rho q^k (1 + \rho^2 q^{2k}) x y + (1 - q) \rho^2 (x^2 + y^2) q^{2k}}; \]

this is the renormalized version of the well-known result; see e.g. [IS88 (2.2)], which considers the \( q \)-Hermite polynomials given by \( \{ (1 - q)^n/2 H_n(2x/\sqrt{1-q}) \} \) instead of our \( \{ H_n(x) \} \).

Since (1.3) implies that (2.6) implies that \( \mu(x|\rho_1, \rho_2, x) = \int \mu(x|\rho_1, y) \mu(dy|\rho_2, x) \).

For \( |q| < 1, |\rho| < 1 \) the density of \( \mu \) is given in Corollary 2 hence, after simplifying common factors and substituting \( x = 2\zeta/\sqrt{1-q}, y = 2\eta/\sqrt{1-q}, z = 2\zeta/\sqrt{1-q}, \) the relationship (2.6) takes the following form:

\[
\int_{-1}^{1} \prod_{k=0}^{\infty} \frac{(1 - \rho_1^2 q^{2k})}{(1 - \rho_1^2 q^{2k})^2 - 4\rho_1 q^k (1 + \rho_1^2 q^{2k}) \eta^2 + (1 + \rho_1^2) \eta^2 + (q^2 + \rho_1^2) q^{2k}} d\eta \\
\times \prod_{k=0}^{\infty} \frac{(1 - \rho_2^2 q^{2k})^2 - 4\rho_2 q^k (1 + \rho_2^2 q^{2k}) \xi^2 + 4\rho_2^2 (\xi^2 + \rho_2^2) q^{2k}} {2\pi \sqrt{1 - \eta^2}} \\
= \prod_{k=0}^{\infty} \frac{(1 - \rho_1^2 \rho_2^2 q^{2k})^2 - 4\rho_1 \rho_2 q^k (1 + \rho_1^2 \rho_2^2 q^{2k}) \xi^2 + 4\rho_1 \rho_2^2 (\xi^2 + \rho_2^2) q^{2k}} {2\pi \sqrt{1 - \eta^2}}.
\]
3. Determinants of Hankel matrices

In this section we are interested in calculating the determinants of the Hankel matrices

\[ M_n = [m_{i+j}]_{i,j=0,...,n-1}, \]

where \( m_i = \int x^i \mu(dx) \) are the moments of a certain (perhaps signed) measure \( \mu \). It is well known that for positive measures we must have \( \det M_n \geq 0 \), and that these determinants can be read out from the three-term recurrence for the corresponding monic orthogonal polynomials.

Consider first the moments \( m_k(y) = \int x^k \mu(dx) \) of the (perhaps signed) measure \( \mu = \mu_{y,q} \), which solves (2.1). Then \( m_k(y) \) are polynomials of degree \( k \) in the variable \( y \) and can be written as follows. Let \( a_{n,2i}, i \leq \lfloor n/2 \rfloor \) be the coefficients in the expansion of the monomial \( x^n \) into the \( q \)-Hermite polynomials,

\[ x^n = \sum_{i=0}^{\lfloor n/2 \rfloor} a_{n,2i} H_{n-2i}(x|q), \quad n \geq 0. \]

Then

\[ m_n(y) = \int x^n d\mu(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \rho^{n-2k} a_{n,2k} H_{n-2k}(y|q). \]

Let \( S_n \) be the Hankel matrix of moments \( m_k(y) \),

\[ S_n(y|q, \rho) = \begin{bmatrix}
    m_0(y) & m_1(y) & \cdots & m_{n-1}(y) \\
    m_1(y) & m_2(y) & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    m_{n-1}(y) & \cdots & m_{2n-2}(y)
\end{bmatrix}. \]

It is well known that \( \det S_n \) is the product of the coefficients at the third term of (2.2), which implies the following.

**Corollary 4.** \( \det S_{n+1}/\det S_n = [n]_q! \prod_{i=1}^{n} (1 - \rho^2 q^{i-1}) \).

Our second Hankel matrix has an even simpler form. As indicated in [IS97], [IS02] the \( q \)-Hermite polynomials can be viewed as moments of a signed measure, \( H_n(x|q) = \int u^n \mu(du|x,q) \). It turns out that if \( q \neq 0 \), the measure \( \mu(du|x,q) \) cannot be positive even for a single value of \( x \). To see this, consider the following \( n \times n \) matrices:

\[ M_n(x|q) = \begin{bmatrix}
    H_0(x|q) & H_1(x|q) & H_2(x|q) & \cdots & H_{n-1}(x|q) \\
    H_1(x|q) & H_2(x|q) & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    H_{n-1}(x|q) & H_n(x|q) & \cdots & H_{2n-2}(x|q)
\end{bmatrix}. \]

The following \( q \)-generalization of [Kra99, (3.55)] shows that the determinants \( \det M_n(x|q) \) are free of the variable \( x \) and take negative values.

**Theorem 3.**

\[ \frac{\det M_{n+1}}{\det M_n} = (-1)^n q^{n(n-1)/2} [n]_q! \]
Proof. Using (1.3), we row-reduce the first column of the matrix. Namely, from the second row of $M_{n+1}$, we subtract the first one multiplied by $x$. Similarly, for $i \geq 3$, we subtract $x$ times row $i-1$ and add the $(i-2)$-th row multiplied by $[i-1]_q$. Taking (1.3) into account, $\det M_{n+1}(x|q)$ becomes

$$
\begin{bmatrix}
H_0 & H_1 & H_2 & \cdots & H_{n-1} \\
0 & ([0] - [1])H_0 & ([0] - [2])H_1 & ([0] - [n])H_{n-1} \\
0 & ([1] - [2])H_1 & ([1] - [3])H_2 & ([1] - [n + 1])H_n \\
\vdots & \ddots & \ddots & \ddots \\
0 & ([n - 1] - [n])H_{n-1} & ([n - 1] - [n + 1])H_n & ([n - 1] - [2n - 1])H_{2n - 2}
\end{bmatrix}.
$$

Now, we use the fact that for $m \leq n$ we have $[n]_q - [m]_q = q^n [n - m]_q$. Thus $\det M_{n+1}(x|q)$ becomes

$$
\det M_{n+1} = (-1)^n q^{-\sum_{i=1}^{n} i} \prod_{j=1}^{n} [j]_q \det M_n = (-1)^n q^{n(n-1)/2} [n]_q \det M_n.
$$

The formula stated in Corollary 4 was originally discovered through symbolic computations and motivated this paper. We were unable to find a direct algebraic proof along the lines of the proof of Theorem 3 and our search for the explanation of why $\det S_n(y)$ does not depend on $y$ led us to Al-Salam–Chihara polynomials and identity (1.8).

The fact that Hankel determinants formed of certain linear combinations of the $q$-Hermite polynomials do not depend on the argument of these polynomials as exposed in Theorem 3 and Corollary 4 is striking and unexpected to us. A natural question arises whether other linear combinations have this property.

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