A Modal Logic Framework for Multi-agent Belief Fusion

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Abstract

This paper is aimed at providing a uniform framework for reasoning about beliefs of multiple agents and their fusion. In the first part of the paper, we develop logics for reasoning about cautiously merged beliefs of agents with different degrees of reliability. The logics are obtained by combining the multi-agent epistemic logic and multi-sources reasoning systems. Every ordering for the reliability of the agents is represented by a modal operator, so we can reason with the merged results under different situations. The fusion is cautious in the sense that if an agent’s belief is in conflict with those of higher priorities, then his belief is completely discarded from the merged result. We consider two strategies for the cautious merging of beliefs. In the first one, if inconsistency occurs at some level, then all beliefs at the lower levels are discarded simultaneously, so it is called level cutting strategy. For the second one, only the level at which the inconsistency occurs is skipped, so it is called level skipping strategy. The formal semantics and axiomatic systems for these two strategies are presented. In the second part, we extend the logics both syntactically and semantically to cover some more sophisticated belief fusion and revision operators. While most existing approaches treat belief fusion operators as meta-level constructs, these operators are directly incorporated into our object logic language. Thus it is possible to reason not only with the merged results but also about the fusion process in our logics. The relationship of our extended logics with the conditional logics of belief revision is also discussed.

Key Words: Epistemic logic, multi-sources reasoning, database merging, belief fusion, belief revision, multi-agent systems.

1 Introduction

Recently, there has been much attention on the infoglut problem in information retrieval research due to the rapid growth of internet information. If a keyword is input to a commonly-used search engine, it is not unusual to get back a list of thousands of web pages, so the real difficulty is not how to find information, but how to find useful information. To circumvent the problem, many software agents have been designed to do the information search works. The agents can search through the web and try to find and filter out information for matching the user’s need. However, not all internet information sources are reliable. Some web sites are out-of-date, some news provide wrong information, and someone even intentionally spreads rumor or deceives by anonymity. Thus an important task of information search agents is how to merge so much information coming from different sources according to their degrees of reliability.

In [65], an agent is characterized by mental attitudes, such as knowledge, belief, obligation, and commitment. This view of agent, in accordance with the intentional stance proposed in [22], has been widely accepted as a convenient way for the analysis and description of complex systems[71]. From this viewpoint, each information provider can be considered as an agent and the information provided by the agent corresponds to his belief, so our problem is also that of merging beliefs from different agents.

The philosophical analysis of these mental attitudes has motivated the development of many non-classical logical systems[34]. In particular, the analysis of informational attitudes, such as knowledge and belief, has been a traditional concern of epistemology, a very important branch of philosophy since the ancient times. To answer the basic questions such as “What is knowledge?” “What can we know” and ”What are the characteristic properties of knowledge?”, some formalism more rigorous than natural language is needed. This results in the development of

* A preliminary version of the paper has appeared in [45].
the so-called epistemic logic[36]. This kind of logic has attracted much attention of researchers from diverse fields such as artificial intelligence (AI), economics, linguistics, and theoretical computer science. Among them, the AI researchers and computer scientist have elaborated some technically sophisticated formalisms and applied them to the analysis of distributed and multi-agent systems[21, 22].

Though the original epistemic logic in philosophy is mainly about the single-agent case, the application to AI and computer science put its emphasis on the interaction of agents, so multi-agent epistemic logic is urgently needed. One representative example of such logic is proposed by Fagin et al.[31]. In their logic, the knowledge of each agent is represented by a normal modal operator[4], so if no interactions between agents occur, this is not more than a multi-modal logic. However, the most novel feature of their logic is the consideration of common knowledge and distributed knowledge among a group of agents. While common knowledge is the facts that everyone knows, everyone knows that everyone knows, everyone knows that everyone knows that everyone knows, and so on, distributed knowledge is that can be deduced by pooling together the knowledge of everyone, so it is the latter that really concerns the fusion of knowledge among agents. However, the term “knowledge” is used in a broad sense in [31] to cover the cases of belief and information[4]. Though it is required that proper knowledge must be true, the belief of an agent may be wrong, so there will be conflicts in general in the beliefs to be merged. In this case, everything can be deduced from the distributed beliefs due to the notorious omniscience property of epistemic logic, so the merged result will be useless to further reasoning.

Instead of directly put all the agents together, there are also many sophisticated techniques for knowledge base merging[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Most of the approaches treats belief fusion operators as meta-level constructs, so given a set of knowledge bases, this kind of fusion operators will return the merged results. Some of the works propose concrete operators which can be used directly in the fusion process, while the others stipulate the desirable properties of reasonable belief fusion operators by postulates. However, few of the approaches provides the capability of reasoning about the fusion process. One of the few exceptions is the work of multi-source reasoning[17].

Multi-source reasoning is to model the fusion process of multiple databases in a modal logic. The context of the work is to merge a set of databases according to a total ordering on the set to be merged. Each database is a finite and satisfiable set of literals. Two attitudes for merging are considered. According to the suspicious attitude, if a database contains a literal inconsistent with those in the databases of higher reliability, then the database is completely discarded in the merged result. On the other hand, according to the trusting attitude, if a literal in a database is inconsistent with those in the databases of higher reliability, only the literal is discarded, and other literals in the database will be still considered if they are consistent with those in the databases of higher reliability.

Since multi-source reasoning is modelled in a modal logic framework, it is very suitable for the integration with epistemic logic. The restriction here is that each database must be a set of literals in multi-source reasoning, however, in the multi-agent epistemic logic, it is expected that more complex compound formulas will be believed by agents. Therefore, we have to extend the multi-sources reasoning to the more general case. To achieve the purpose, the distributed knowledge operators in multi-agent epistemic logic may help. What we have to do is to adapt the multi-agent epistemic logic so that the distributed knowledge among a group of agents with reliability ordering can also be defined. However, since the set of facts believed by an agent is at least closed under classical logical equivalence, the trusting attitude does not work here. For example, if \( p \) and \( q \) are both believed by an agent and \( \neg p \lor \neg q \) is believed by another agent with higher reliability, then by trusting attitude, one of \( p \) or \( q \) should be in the merged result (assume there do not exist other conflicts), however, it is obvious that the belief of the first agent is equivalent to \( p \land q \) and if it is expressed in this way, then no belief of the first agent (except the obvious tautology) should be included in the merged belief. Thus, we will only consider the merging of beliefs according to the suspicious attitude, so this approach is very cautious from the viewpoint of belief fusion. However, we will show that the fusion according to the trusting attitude can also be simulated in our logic, though the simulation is syntax-dependent. We consider two strategies for the cautious merging of beliefs. In the first one, if inconsistency occurs at some level, then all beliefs at the lower levels are discarded simultaneously, so it is called level cutting strategy. For the second one, only the level at which the inconsistency occurs is skipped, so it is called level skipping strategy.

The logics integrate multi-source reasoning into multi-agent epistemic logic, so it enhance the reasoning capability of the latter. However, since the fusion technique used in the logics is essentially the so-called base revision in [10] it is too cautious in some cases. Thus we would also like to consider the extension of the logics with some more

1More precisely, the logic for belief is called doxastic logic. However, here we will use the three terms knowledge, belief, and information interchangeably, so epistemic logic is assumed to cover all these notions.
sophisticated fusion operators proposed in the literatures. We show that the multi-agent epistemic logic framework can accommodate these belief fusion operators to a large extent both syntactically and semantically. This means that the belief fusion operators as a standard add-on of multi-agent epistemic logic should be expectable.

The rest of the paper is organized as follows. In the next section, the multi-agent epistemic logic and mult-sources reasoning are reviewed. Then the logics integrating cautious fusion into multi-agent epistemic logic are presented. The level cutting and skipping strategies are presented respectively in section 3 and 4. The syntax, semantics, and axiomatic systems of the logics will be given. In section 5, the basic logics are compared with their ancestors and another cautious inconsistency handling technique. In section 6 and 7, accompanied by the brief introductions of some of the most important belief fusion or revision techniques, the possible extensions of our basic logics for accommodating them are presented. Finally, some further research directions are discussed in the concluding section.

2 Logical Preliminary

In this section, we review the syntax, semantics and some notations for multi-agent epistemic logic and multi-sources reasoning.

2.1 Multi-agent epistemic logic

In [31], some variants of epistemic logic systems are presented. The most basic one with distributed belief is called $K^D_n$ by following the naming convention in [14], with $n$ being the number of agents and $D$ denoting the distributed belief operators. In the system, no properties except logical omniscience are imposed on the agents’ beliefs. Nevertheless, in the following, we will assume the belief of each individual agent is consistent though the collective ones of several agents may be not, so the system will be $K^D_n$ where the additional axiom D is added to $K^D_n$ for ensuring the consistency of each agent’s belief.

Assume we have $n$ agents and a set $\Phi_0$ of countably many atomic propositions, then the set of well-formed formulas(wff) for the logic $K^D_n$ is the least set containing $\Phi_0$ and closed under the following formation rules:

- if $\varphi$ is a wff, so are $\neg \varphi$, $B_i \varphi$, and $D_G \varphi$ for all $1 \leq i, j \leq n$ and nonempty $G \subseteq \{1, \ldots, n\}$, and
- if $\varphi$ and $\psi$ are wffs, then $\varphi \lor \psi$ is, too.

As usual, other classical Boolean connectives $\land$ (and), $\supset$ (implication), $\equiv$ (equivalence), $\top$ (tautology), and $\bot$ (contradiction) can be defined as abbreviations.

The intuitive meaning of $B_i \varphi$ is “The agent $i$ believes $\varphi$.”, whereas that for $D_G \varphi$ is “The group of agents $G$ has distributed belief $\varphi$.”. The possible-worlds semantics provides a general framework for the modeling of knowledge and belief. In the semantics, an agent’s belief state corresponds to the extent to which he can determine what world he is in. In a given world, the belief state determines the set of worlds that the agent considers possible. Then an agent is said to believe a fact $\varphi$ if $\varphi$ is true in all worlds in this set. Since the distributed belief of a group is the result of pooling together the individual beliefs of its members, this can be achieved by intersecting the sets of worlds that each agent in the groups considers possible.

Formally, a $K^D_n$ model is a tuple $(W, (B_i)_{1 \leq i \leq n}, V)$, where

- $W$ is a set of possible worlds,
- $B_i \subseteq W \times W$ is a serial binary relation on $W$ for $1 \leq i \leq n$,
- $V : \Phi_0 \rightarrow 2^W$ is a truth assignment mapping each atomic proposition to the set of worlds in which it is true.

In the following, we will use some standard notations for binary relations. If $R \subseteq A \times B$ is a binary relation between $A$ and $B$, we will write $R(a, b)$ for $(a, b) \in R$ and $R(a)$ for the subset $\{b \in B \mid R(a, b)\}$. Thus for

$^2$Though it is well accepted that $K^D_{45} n$ is more appropriate for modeling of belief with positive and negative introspection (axioms 4 and 5), we adopt the $K^D_n$ system for emphasizing the agents may represent databases and their beliefs may be just the facts stored in the databases and their consequences.

$^3$In [31], the modal operators are denoted by $K_1$ instead of $B_i$.

$^4$A relation $R$ on $W$ is serial if $\forall w \exists u R(w, u)$. 


any \( w \in W \), \( B_i(w) \) is a subset of \( W \). Informally, \( B_i(w) \) is the set of worlds that agent \( i \) considers possible under \( w \) according to his belief. The informal intuition is reflected in the definition of satisfaction relation. Let \( M = (W, (B_i)_{1 \leq i \leq n}, V) \) be a \( KD_n^L \) model and \( \Phi \) be the set of wffs, then the satisfaction relation \( \models_M \subseteq W \times \Phi \) is defined by the following inductive rules (we will use the infix notation for the relation and omit the subscript \( M \) for convenience):

1. \( w \models p \) iff \( w \in V(p) \) for any \( p \in \Phi_0 \),
2. \( w \models \neg \varphi \) iff \( w \not\models \varphi \),
3. \( w \models \varphi \lor \psi \) iff \( w \models \varphi \) or \( w \models \psi \),
4. \( w \models B_i \varphi \) iff for all \( u \in B_i(w) \), \( u \models \varphi \),
5. \( w \models D_G \varphi \) iff for all \( u \in \bigcap_{i \in G} B_i(w) \), \( u \models \varphi \).

The notion of validity is defined from the satisfaction relation. A wff \( \varphi \) is valid in \( M \), denoted by \( \models_M \varphi \), if for every \( w \in W \), \( w \models_M \varphi \), and valid in a class of models \( M \), written as \( \models_M \varphi \), if for all \( M \in M \), \( \models_M \varphi \).

### 2.2 Multi-sources reasoning

The context of multi-sources reasoning is the merging of \( n \) databases. To encode the degrees of reliability of these databases, the total ordering on a subset of \( \{1, \ldots, n\} \) is used. Let \( TO_n \) denote the set of all possible total orders on the subsets of \( \{1, \ldots, n\} \). \( \Phi_0 \) denote a finite set of atomic propositions and \( L(\Phi_0) \) be the classical propositional language formed from \( \Phi_0 \), then the set of wffs for logic \( FU_n \) (originally called FUSION in \( [F] \)) is the least set containing \( \Phi_0 \) and \( \{[O] \varphi : \varphi \in L(\Phi_0), O \in TO_n \} \) and being closed under Boolean connectives. If \( O \) is the ordering \( i_1 > i_2 > \cdots > i_m \) for some \( \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} \), then the wff \( [O] \varphi \) means that \( \varphi \) holds after merging the databases \( i_1, \ldots, i_m \) according to the specified ordering. In this case, \( O > i_{m+1} \) denotes \( i_1 > i_2 > \cdots > i_m > i_{m+1} \). Furthermore, the set \( \{i_1, i_2, \ldots, i_m\} \) is called the domain of \( O \) and is denoted by \( \delta(O) \).

Let \( Lit(\Phi_0) \) denote the set of literals in \( L(\Phi_0) \). In the context of multi-sources reasoning, assume \( DB_1, \ldots, DB_n \) are \( n \) databases, where each \( DB_i \) is a finite satisfiable subset of \( Lit(\Phi_0) \), then the informal semantics for the merging databases can be given according to two attitudes. For the suspicious attitude, only the case of \( n = 2 \) is given in \( [F] \), where the definition of \( DB_{1>2} \) is defined by

\[
DB_{1>2} = \begin{cases} 
DB_1 \cup DB_2 & \text{if } DB_1 \cup DB_2 \text{ is consistent}, \\
DB_1 & \text{otherwise}.
\end{cases}
\]

On the other hand, for the trusting attitude, the definition of \( DB_O \) is given in the following recursive formula

\[
DB_{O>1} = DB_O \cup \{l \in DB_i : \overline{l} \not\in DB_O \},
\]

where \( \overline{l} \) is the complementary of \( l \). Then the intended meaning of \( [O] \varphi \) is \( DB_O \models_M \varphi \), where \( CL \) denotes classical propositional reasoning.

Thus an \( FU_n \) model is a tuple \( (W, (R_i)_{1 \leq i \leq n}, V) \), where \( W \) and \( V \) are as defined in \( KD_n^L \) models, and each \( R_i \) is a serial binary relation on \( W \). The clause for satisfaction of the formula \( [O] \varphi \) is then

\[
w \models [O] \varphi \iff \text{for all } u \in R_O(w), \ u \models \varphi,
\]

where \( R_O \) is defined from \( R_i \)’s according to two attitudes. For the suspicious attitude,

\[
R_{1>2}(w) = \begin{cases} 
R_1(w) & \text{if } R_1(w) \cap R_2(w) = \emptyset, \\
R_1(w) \cap R_2(w) & \text{otherwise},
\end{cases}
\]

for all \( w \in W \). For the trusting attitude, we need some auxiliary notations. Let \( f : 2^W \times 2^W \to 2^{Lit(\Phi_0)} \) be defined as

\[
f(S, T) = \{l \in Lit(\Phi_0) : \forall w \in S(w \models l) \land \exists w \in T(w \models l) \},
\]

\[
\text{A literal is an atom or a negated atom.}
\]

\[
\text{In } [F], \text{ it is assumed that each } R_i \text{ is an equivalence relation. However, since nested modalities are not allowed in } FU_n, \text{ the difference is inessential.}
\]
i.e., \( f(S,T) \) is the set of literals true in all worlds of \( S \) and some worlds of \( T \). Then for any \( w \in W \),

\[
\mathcal{R}_{O>i}(w) = \mathcal{R}_O(w) \cap \{ u \in W : u \models \bigwedge f(\mathcal{R}_i(w), \mathcal{R}_O(w)) \}.
\]

Note that if each \( \mathcal{R}_i(w) \) denotes the set of possible worlds in which the literals in \( DB_i \) are all true, then \( f(\mathcal{R}_i(w), \mathcal{R}_O(w)) \) is just the set \( \{ l \in DB_i : T \not\subseteq DB_O \} \), so \( \mathcal{R}_{O>i}(w) \) is exactly the set of possible worlds satisfying all literals in \( DB_{O>i} \).

An axiomatic system for \( FU_n \) based on trusting attitude semantics is proposed in a recent paper\cite{[16]}. One key axiom of that system is as follows

\[
[i]l \land \neg[O]\neg l \supset [O > i]l,
\]

where \( l \) is a literal. Thus a severe restriction of \( FU_n \) is the background databases \( DB_i \)'s can contain only literals which may not be the case in general practice. Though from the semantic viewpoint, there is no essential difficulty to lift the restriction, however the key axiom is no longer valid when the databases contain general formulas. On the other hand, for the suspicious semantics, the merged database in fact contains the distributed belief of the two databases if they are consistent. However, since distributed belief operator is not in the language of \( FU_n \), the modal operator \([O]\) can only be characterized by the modal operators \([i]\) for \( i \in \delta(O) \). Nevertheless, unless \( \varphi \) is a literal, it seems difficult (if not impossible) to define \( D_{1,2}\varphi \) in terms of the two individual agents' belief. Thus, a natural solution to merge general databases in the suspicious semantics is to introduce the distributed belief operators into the language of \( FU_n \). This is exactly what we will do in the following.

### 3 Level Cutting Strategy

To unify the notations from multi-agent epistemic logic and multi-sources reasoning, we will use the language \( DBF_n^c \) (for distributed belief fusion and cutting strategy) defined as follows. The wffs of \( DBF_n^c \) is the least set containing \( \Phi_0 \) and being closed under Boolean connectives and the following rule:

- if \( \varphi \) is a wff, so are \( [G]\varphi \) and \( [O]\varphi \) for any nonempty \( G \subseteq \{1, \ldots, n\} \) and \( O \in TO_n \).

When \( G \) is a singleton \( \{ i \} \) and \( O \) is the unique total order on \( \{ i \} \), we will use \( [i]\varphi \) to denote both \( [G]\varphi \) and \( [O]\varphi \). Thus \([i]\varphi \) and \([G]\varphi \) correspond respectively to \( B_i\varphi \) and \( D_G\varphi \) in \( KD_n^D \), so \( DBF_n^c \) is an extension of the multi-agent epistemic logic with distributed belief operators. On the other hand, \([O]\varphi \) and \([i]\varphi \) are precisely those in \( FU_n \), so \( DBF_n^c \) is also a generalization of multi-sources reasoning system. However, note that nested modalities are not allowed in \( FU_n \), whereas this is not restricted in \( DBF_n^c \) any more. Thus, for example, we can include a wff \([j]\varphi \) in a database \( DB_i \) which means that \( DB_i \) has the information that \( \varphi \) is in \( j \).

Let \( Q \) be a partial order on \( \{1, 2, \ldots, k\} \) for some \( k \leq n \) and \( O_Q \) be the set of all total orders on \( \{1, 2, \ldots, k\} \) containing \( Q \), then define \([Q]\varphi \) as the abbreviation of \( \bigwedge_{O \in O_Q} [O]\varphi \). Thus the restriction of the modalities to total orders is not essential since a partial order can be replaced by the set of total orders compatible with it.

For the semantics, a \( DBF_n^c \) model is just a \( FU_n \) model \( (W, (R_i)_{1 \leq i \leq n}, V) \). The clauses for the satisfaction of wffs are defined exactly as in \( FU_n \) model in addition to a clause for the \([G]\) operator which is the one for distributed knowledge in \( KD_n^D \). However, the relation \( \mathcal{R}_O \) is now defined in an inductive way:

\[
\mathcal{R}_{O>i}(w) = \begin{cases} 
\mathcal{R}_O(w) & \text{if } \bigcap_{j \in \delta(O>i)} \mathcal{R}_j(w) = \emptyset, \\
\mathcal{R}_O(w) \cap \mathcal{R}_i(w) & \text{otherwise},
\end{cases}
\]

for any \( w \in W \). Let \( O = (i_1 > i_2 > \cdots > i_m) \) and define \( G_j = \{i_1, i_2, \ldots, i_j\} \) for \( 1 \leq j \leq m \) and assume \( k \) is the largest \( j \) such that \( \bigcap_{i \in G_k} \mathcal{R}_i(w) \neq \emptyset \), then we have

\[
\mathcal{R}_O(w) = \bigcap_{i \in G_k} \mathcal{R}_i(w).
\]

In other words, the beliefs from the agents after the level \( k \) are completely discarded in the merged result. The rationale behind this is if belief in level \( k + 1 \) is not acceptable, neither any belief in a less reliable level, so this is a very cautious attitude to belief fusion.
1. Axioms:

P: all tautologies of the propositional calculus

G1: \([G]\phi \land [G](\phi \supset \psi) \supset [G]\psi\)

G2: \([-[i] \bot\]

G3: \([G_1] \phi \supset [G_2] \phi\) if \(G_1 \subset G_2\)

O1: \([-[O > i] \bot ] \supset ([O > i] \phi \equiv [\delta (O > i)] \phi)\)

O2: \([\delta (O > i)] \bot \supset ([O > i] \phi \equiv [O] \phi)\)

2. Rules of Inference:

R1 (Modus ponens, MP):

\[
\phi, \phi \supset \psi \quad \frac{\phi, \phi \supset \psi}{\psi}
\]

R2 (Generalization, Gen):

\[
\phi \quad \frac{\phi}{[G] \phi}
\]

Figure 1: The axiomatic system for DBF^n

The notion of validity in DBF^n is defined just as that for KD^n. The notation \(\models_{DBF^n} \phi\) denotes that \(\phi\) is valid in all DBF^n model and the subscript is usually omitted if there is no confusion. The valid wffs of DBF^n can be captured by the axiomatic system in Fig 1.

The axioms G1-G3 and rule R2 are those for KD^n. G1 and rule R2 are properties of knowledge for perfect reasoners. They also are the causes of the notorious logical omniscience problem. However, it is appropriate to describe implicit information in this way. G2 is the requirement that the belief of each individual agent is consistent. G3 is a characteristic property of distributed knowledge. The larger the subgroup, the more knowledge it possesses.

In [31], another axiom related distributed knowledge and individual ones is added. That is, \(D\{i\} \phi \equiv B_i \phi\), however, we do not need this because we identify \([i] \phi\) and \([\{i\}] \phi\) which respectively correspond to \(B_i \phi\) and \(D\{i\} \phi\) in KD^n. The two axioms O1 and O2 define the merged belief in terms of distributed belief in a recursive way. O1 is the case when \(\bigcap_{j \in \delta (O > i)} R_\phi (w) \neq \emptyset\), whereas O2 is the opposite case.

The derivability in the system is defined as follows. Let \(\Sigma \cup \{\phi\}\) be a subset of wffs, then \(\phi\) is derivable from \(\Sigma\) in the system DBF^n, written as \(\Sigma \vdash_{DBF^n} \phi\), if there is a finite sequence \(\varphi_1, \ldots, \varphi_m\) such that every \(\varphi_i\) is an instance of an axiom schema, a wff in \(\Sigma\), or obtainable from earlier \(\varphi_j\)’s by application of an inference rule. When \(\Sigma = \emptyset\), we simply write \(\vdash_{DBF^n} \phi\). We will drop the subscript when no confusion occurs. We have the soundness and completeness results for the system DBF^n.

**Theorem 1** For any wff of DBF^n, \(\models \phi \iff \vdash \phi\).

**Proof:** The proof of all theorems and propositions can be found in the appendix. □

Some basic theorems can be derived from the system.

**Proposition 1** For any \(O = (i_1 > i_2 > \cdots > i_m)\) and \(G_j = \{i_1, i_2, \ldots, i_j\}\) \((1 \leq j \leq m)\), we have

1. \(\vdash \neg[G_j] \bot \land [G_{j+1}] \bot \supset ([O] \phi \equiv [G_j] \phi)\), where the wff \([G_{j+1}] \bot\) is deleted from the antecedent when \(j = m\).
2. \(\vdash ([O] \phi \land [O](\phi \supset \psi)) \supset [O] \psi\),
3. \(\vdash \neg [O] \bot\),
Proposition 1.1 shows that any total order can be cut into a head and a tail according to some consistency level, and the merged belief according to the ordering is just the distributed belief of the agents from the head part. Proposition 1.2 and 1.4 show that merged belief inherits the properties of the distributed one since the former is equivalent to the latter for the head part of the ordering. Furthermore, Proposition 1.3 shows that belief fusion keeps consistency.

4 Level Skipping Strategy

Though level cutting strategy is useful in practice, it is sometimes too cautious from the viewpoint of information fusion. A less cautious strategy is to skip only the agent causing inconsistency and continue to consider the next level. The strategy corresponds to the suspicious attitude of multi-sources reasoning and has been used in belief revision by Nebel [54]. This strategy is easily obtained by modifying the inductive definition of $R^O_i(w)$ as follows.

$$R^O_i(w) = \begin{cases} R^O(w) & \text{if } R^O(w) \cap R_i(w) = \emptyset, \\ R^O(w) \cap R_i(w) & \text{otherwise}, \end{cases}$$

for any $w \in W$.

According to the definition, $[O > i]\varphi$ will be equivalent to the distributed fusion of $[O]\varphi$ and $[i]\varphi$ when the belief of $i$ is consistent with the merged belief of $O$, so to axiomatize reasoning under the strategy, we must view $O$ as a virtual agent and consider the distributed belief between $O$ and $i$. However, to get a bit more general, we will consider the distributed belief among a group of virtual agents. Thus, we define the wffs of the logic $\text{DBF}^s_n$ (for skipping strategy) as the least set containing $\Phi_0$ and being closed under Boolean connectives and the following rule:

- if $\varphi$ is a wff, so are $[\Omega]\varphi$ for any nonempty $\Omega \subseteq \mathcal{TO}_n$.

When $\Omega$ is a singleton $\{O\}$, we will write $[O]\varphi$ instead $[\{O\}]\varphi$. If $\Omega = \{O_1, \ldots, O_m\}$ is such that $|\delta(O_i)| = 1$ for all $i$'s, then $[\Omega]$ is the distributed belief operator among ordinary agents. Therefore, the language is more general than that of $\text{DBF}^c_n$.

For the semantics, a $\text{DBF}^s_n$ model is still a $\text{DBF}^c_n$ model, however, the satisfaction clauses for $[O]$ and $[G]$ operators are replaced by the following

$$w \models [\Omega]\varphi \iff \text{for all } u \in R_{\Omega}(w), u \models \varphi,$$

where $R_{\Omega}(w) = \bigcap_{O \in \Omega} R_O(w)$ and $R_O$ is defined inductively at the beginning of the section. Given this language and semantics, the valid wffs of $\text{DBF}^s_n$ are capture by the axiomatic system in Fig 2. The axioms V1-V3 and rule R2' correspond to G1-G3 and R2 for distributed belief, but now for virtual agents instead of ordinary agents. Nevertheless, since an ordinary agent is a special case of the virtual one, these in fact also cover G1-G3 and R2. $O_1'$ and $O_2'$ are axioms for describing the level skipping strategy and correspond exactly to the inductive definition of $R^O_{O > i}$, where $\Omega$ in these two axioms denote any subset (empty or not) of $\mathcal{TO}_n$. We can still have the soundness and completeness theorem.

Theorem 2 For any wff of $\text{DBF}^s_n$, $\models \varphi$ iff $\varphi$.

Since operator $[O]$ is a special case of $[\Omega]$, the properties 1.2 and 1.4 hold trivially for $\text{DBF}^s_n$. The property 1.3 can be easily proved by using V2, O1' and O2'. However, it is unclear whether a counterpart of property 1.1 can be given.

5 Related works

In this section, some important works related to the above-mentioned logical systems will be investigated. In the preceding sections, the strong dependence of our logics on multi-sources reasoning and multi-agent epistemic logic has been emphasized, so we will start from the comparison with them. Then we also compare $\text{DBF}^c_n$ with the possibilistic logic approach to inconsistency handling which is known to be very cautious in belief fusion [5].
1. Axioms:

P: all tautologies of the propositional calculus

V1: \((\Omega \varphi \land (\Omega (\varphi \supset \psi)) \supset (\Omega \psi)\) 

V2: \([-i] \bot \]

V3: \((\Omega_1 \varphi \supset \Omega_2 \varphi) \text{ if } \Omega_1 \subset \Omega_2\)

O1': \([-\{O,i\}] \bot \supset (\Omega \cup \{O \succ i\} \varphi \equiv (\Omega \cup \{O\}) \varphi)\)

O2': \([\{O,i\}] \bot \supset (\Omega \cup \{O \succ i\} \varphi \equiv (\Omega \cup \{O\}) \varphi)\)

2. Rules of Inference:

R1 (Modus ponens, MP):

\[
\begin{array}{c}
\varphi \\
\varphi \supset \psi \\
\hline
\psi
\end{array}
\]

R2' (Generalization, Gen):

\[
\begin{array}{c}
\varphi
\hline
[\Omega] \varphi
\end{array}
\]

Figure 2: The axiomatic system for DBF^*_n

5.1 Multi-sources reasoning

Since the original motivation of multi-sources reasoning is to model database merging, we will also consider the relationship of our logic to multi-sources reasoning in this context. In section 2.2, it is assumed that \(\Phi_0\) is finite and each \(DB_i\) is a finite satisfiable subset of \(Lit(\Phi_0)\). Let \(CLS(\Phi_0)\) be the set of clauses in \(L(\Phi_0)\), then in \(FU_n\), each \(DB_i\) is characterized by a wff

\[
\psi_i = \bigwedge\{|i|\varphi : l \in DB_i\} \land \bigwedge\{|-i|\varphi : c \in CLS(\Phi_0), DB_i \not\vdash c\}, \tag{1}
\]

and the reasoning problem is to decide whether the following holds:

\[
|= \bigwedge_{i=1}^n \psi_i \supset [O] \varphi,
\]

for some given \(O\) and \(\varphi \in L(\Phi_0)\). The formula \(\psi_i\) asserts not only the explicit information in \(DB_i\) but also the default negative information about it. However, since in our logic, no restrictions are put on the wffs in databases, this kind of default wffs are potentially infinite, so we will only assert a weaker form of wff. Let \(G\) be a subset of \(\{1, 2, \ldots, n\}\), then \(G\) is consistent if \(\bigcup_{i \in G} DB_i\) is classically consistent, otherwise, it is inconsistent. A subset \(G\) is a maximal consistent agent group if \(G\) is consistent and for any \(i \notin G\), \(G \cup \{i\}\) is inconsistent. Let \(MCAG\) denote the class of all maximal consistent agent groups and redefine \(\psi_i = \bigwedge\{|i|\varphi : \varphi \in DB_i\}\), then we can define the formula \(\psi\) representing the databases as

\[
\psi = \bigwedge_{i=1}^n \psi_i \land \bigwedge_{G \in MCAG} [-G] \bot.
\]

Thus the reasoning problem in our logic is to decide whether \(\vdash \psi \supset [O] \varphi\) holds in our system for some given \(O\) and \(\varphi\). Let us use an example to illustrate the application.

\[7\text{A clause is a disjunction of literals.}\]
Example 1 Assume there are four databases $DB_1 = \{p\}$, $DB_2 = \{q\}$, $DB_3 = \{\neg p \lor \neg q\}$, and $DB_4 = \{r, s\}$, where $p, q, r,$ and $s$ are propositional symbols, then according to the above discussion,
\[
\psi = [1]p \land [2]q \land [3](\neg p \lor \neg q) \land [4]((r \land s) \land \neg[\{1, 2, 4\}] \land \neg[\{1, 3, 4\}] \land \neg[\{2, 3, 4\}] \land .
\]

By using level cutting strategy, we have the following reasoning steps:
\[
1. \psi \vdash \neg[\{1, 2\}] \land \neg[\{1, 2, 3\}] \land \neg[\{1, 2, 4\}] \land \neg[\{1, 3, 4\}] \land \neg[\{2, 3, 4\}] \land G1, G3, P, MP
\]
\[
2. \neg[\{1, 2\}] \land \neg[\{1, 2, 3\}] \land \neg[\{1, 2, 4\}] \land \neg[\{1, 3, 4\}] \land \neg[\{2, 3, 4\}] \land (1 > 2 > 3 > 4) \varphi \equiv [\{1, 2\}] \varphi \land 1.2, MP
\]
\[
3. \psi \vdash (1 > 2 > 3 > 4) \varphi \equiv [\{1, 2\}] \varphi \land 1.2, MP
\]
Thus, by epistemic reasoning in $KD^n_Q$, we have the results $\vdash \psi \supset [O] (p \land q)$ but $\not\vdash \psi \supset [O] (r \land s)$ when $O = 1 > 2 > 3 > 4$. This means that both databases $DB_3$ and $DB_4$ are discarded according to the ordering even only $DB_3$ is in conflict with $DB_1$ and $DB_2$.

On the other hand, if the level skipping strategy is adopted. Then we have the following proof.
\[
1. \psi \vdash \neg[\{1, 2\}] \land \neg[\{1, 2, 3\}] \land \neg[\{1, 2, 4\}] \land \neg[\{1, 3, 4\}] \land \neg[\{2, 3, 4\}] \land V3
\]
\[
2. \psi \vdash (1 > 2 > 3 > 4) \varphi \equiv [\{1, 2\}] \varphi \land O1', 1, P, MP
\]
\[
3. \psi \vdash (1 > 2 > 3 > 4) \varphi \equiv [\{1, 2\}] \varphi \land O1', 1, P, MP
\]
\[
4. \psi \vdash (1 > 2 > 3 > 4) \varphi \equiv [\{1, 2, 4\}] \varphi \land V1, V3, 2, P, MP
\]
\[
5. \psi \vdash \neg[\{1, 2, 4\}] \land \neg[\{1, 2, 3\}] \land 3, 2, P, MP
\]
\[
6. [1 > 2 > 3 > 4] \land \neg[\{1, 2, 3\}] \land \neg[\{1, 2, 4\}] \land \neg[\{1, 3, 4\}] \land \neg[\{2, 3, 4\}] \land O2'
\]
\[
7. \psi \vdash \neg[\{1, 2 > 3 > 4\}] \land \neg[\{1, 2, 4\}] \land \neg[\{1, 3, 4\}] \land \neg[\{2, 3, 4\}] \land 4, 5, 6, P, MP
\]
\[
8. \psi \vdash (1 > 2 > 3 > 4) \varphi \equiv [\{1, 2, 4\}] \varphi \land 1, O1', P, MP
\]
\[
9. \psi \vdash \neg[\{1, 2 > 3 > 4\}] \land \neg[\{1, 2, 4\}] \land \neg[\{1, 3, 4\}] \land \neg[\{2, 3, 4\}] \land 4, 5, 6, P, MP
\]
\[
10. \psi \vdash (1 > 2 > 3 > 4) \varphi \equiv [\{1, 2 > 3 > 4\}] \varphi \land 7, O1', P, MP
\]
\[
11. \psi \vdash (1 > 2 > 3 > 4) \varphi \equiv [\{1, 2, 4\}] \varphi \land 8, 9, 10, P, MP
\]
Thus we have $\vdash \psi \supset [O] (p \land q \land r \land s)$ by epistemic logic, i.e. only $DB_3$ is discarded for its conflict with $DB_1$ and $DB_2$. $\Box$

The reasoning in the above example corresponds to the suspicious attitude in merging databases. In [45], it is shown that the trusting attitude merging can also be simulated in the system $DBF^n$, though the simulation is somewhat awkward. While the simulation in [45] is restricted to the databases containing only literals, here we consider the general case.

To simulate the trusting attitude merging, recall that for a partial order $Q$ and the set $O_Q$ of all total orders compatible with it, $[Q]\varphi$ is the abbreviation of $\land_{O \in O_Q} [O] \varphi$. The basic idea of the simulation is to split each database containing $m$ wffs into $m$ sub-databases, so we have in total $\sum_{i=1}^{n} |DB_i|$ sub-databases. Let $DB_{ij}$ denote the $j$-th sub-database obtained from the $i$-th database, then a total ordering $O \in T O_{\varphi}$ is transformed into a partial ordering $Q$ on the set $TD = \{ij \mid DB_{ij} \text{ is a sub-database}\}$ such that $i_1j_1 > i_2j_2$ in $Q$ iff $i_1 > i_2$ in $O$. Then the databases are represented by the following wff
\[
\psi' = \bigwedge_{i \in TD} \psi_{ij} \wedge \bigwedge_{G \in MCAG'} \neg[G] \land
\]
where $\psi_{ij} = \bigwedge \{ij \varphi \mid \varphi \in DB_{ij}\}$ and $MCAG'$ is the class of all maximal consistent agent subgroups of $TD$. Thus, to decide whether $\varphi$ is derivable from the merging of $DB_1, DB_2, \ldots, DB_n$ according to a total order $O$ by the trusting attitude, we only have to do the following deduction in $DBF^n$.
\[
\vdash \psi' \supset [Q] \varphi
\]

The idea is illustrated in the following example.

Example 2 Assume there are two databases $DB_1 = \{p \lor q\}$ and $DB_2 = \{\neg p, \neg q\}$, then according to the reasoning in $DBF^n$ or $DBF^n$, we have $\vdash \psi \supset (1 > 2) \varphi \equiv [1]\varphi$, where $\psi = [1]([p \lor q] \land [2]\neg p \land [2]\neg q$. Thus $DB_2$ is completely discarded in the merging process. However, if we first split $DB_2$ into two sub-databases $DB_{21} = \{\neg p\}$ and $DB_{22} = \{\neg q\}$ and let $DB_{11} = DB_1$, then we have $\vdash \psi' \supset ([O_1]([p \lor q] \land [O_2](\neg p \land \neg q))$ where $O_1 = 11 > 21 > 22,$
The next example shows that the belief about belief may play a role in the fusion process.

**Example 4** Assume there are two agents whose beliefs are described by the following set:

\[
\{(1)\lnot \{(1, 2)\} \land \{1\} \land \{1\}, \{1\} \land \{2\}, \{1\} \land \{3\}, \{2\} \land \{3\}, \{2\} \land \{1\}, \{3\} \land \{1\}
\]

Then it can be shown that \(1 > 2\) if and only if \(1\) and \(2\) are compatible, so the level skipping strategy will either accept the belief of agent 2 or skip it and consequently accept that of agent 3. This example shows that we can reason with the compatibility of the agents’ beliefs in the uniform framework of epistemic reasoning and information fusion. □

Sometimes, it is possible to infer the beliefs of individual agents from their merged beliefs. The next example shows a very simple case.

**Example 5** Assume it is known that two premises \(1 > 2\) if and only if \(1\) and \(2\) are compatible, so the level skipping strategy will either accept the belief of agent 2 or skip it and consequently accept that of agent 3. This example shows that we can reason with the compatibility of the agents’ beliefs in the uniform framework of epistemic reasoning and information fusion. □
A number of K45 belief operators hold. Since conflict between default rules is not unusual, it is possible to derive some individual or partially merged beliefs from the totally merged ones. When there are more than two agents, the situation would become more complicated. However, it is still possible to derive some individual or partially merged beliefs from the totally merged ones. □

A research area related to both epistemic logic and belief fusion is the modal logics for representing inconsistent beliefs. In [55], an epistemic default logic is proposed for the representation of inconsistent beliefs caused by default reasoning. The logic is based on S5 developed in [51, 52, 53] for modelling the monotonic part of default reasoning that deals with plausible assumptions. The basic modalities of S5 consist of an S5 epistemic operator K and a number of K45 belief operators \( P_i \) (1 \( \leq \) i \( \leq \) n). A wff \( P_i \varphi \) means that \( \varphi \) is a plausible working belief according to some context or default rules. Since conflict between default rules is not unusual, it is possible that \( P_i \varphi \wedge P_j \neg \varphi \) holds. Though \( P_i \) corresponds to an application context of some default rules, it can also be seen as the belief operator of some agent, so in this regard, the logic is like a multi-agent epistemic logic with an S5-based epistemic operator for the authority. However, instead of reasoning about the merging of different working beliefs in the logic directly, a downward reflection approach is adopted in [55]. Since the \( P_i \) operators are only applied to objective wffs in [52], the downward reflection function maps a set of S5 wffs (especially wffs of the form \( P_i \varphi \)) into a set of non-modal formulas. Some downward reflection mechanisms are employed to resolve the inconsistency between working beliefs of different contexts. The one based on the explicit ordering on frames is essentially similar to our cautious merging. The main difference is that we take the orderings as modal operators and reason about the fusion results directly in the object language, while the downward reflection approach consider the fusion in a meta-level.

5.3 Inconsistency handling in possibilistic logic

In [8], it is shown that the possibilistic logic approach to database fusion is very cautious, so a natural question is how the level cutting strategy is related with it. Here, we shown that the inconsistency handling technique of possibilistic logic can be modelled in the strategy.

Possibilistic logic(PL) is proposed by Dubois and Prade for uncertainty reasoning[28, 26, 27]. The semantic basis of PL is the possibility theory developed by Zadeh from fuzzy set theory[72]. Given a universe \( W \), a possibility distribution on \( W \) is a function \( \pi : W \rightarrow [0, 1] \). Obviously, \( \pi \) is a characteristic function of a fuzzy subset of \( W \). Two measures on \( W \) can be derived from \( \pi \). They are called possibility and necessity measures and denoted by \( \Pi \) and \( N \) respectively. Formally, \( \Pi, N : 2^W \rightarrow [0, 1] \) are defined as

\[
\Pi(A) = \sup_{w \in A} \pi(w),
\]

\[
N(A) = 1 - \Pi(A),
\]

where \( \overline{A} \) is the complement of \( A \) with respect to \( W \).

In [27], a fragment for necessity-valued formula in PL, called PL1, is introduced. Each wff of PL1 is of the form \( (\varphi, \alpha) \), where \( \varphi \in \mathcal{L}(\Phi_0) \) and \( \alpha \in (0, 1] \) is a real number. The number \( \alpha \) is called the valuation or weight of the formula. \( (\varphi, \alpha) \) expresses that \( \varphi \) is certain at least to degree \( \alpha \). Formally, a model for PL1 is given by a possibility distribution \( \pi \) on the set \( W \) of classical truth assignments for \( \mathcal{L}(\Phi_0) \). For any \( \varphi \in \mathcal{L}(\Phi_0) \), we can define \( |\varphi| \) as the set of truth assignments satisfying \( \varphi \). Then, by identifying \( \varphi \) and its truth set \( |\varphi| \), a PL1 model \( \pi \) satisfies \( (\varphi, \alpha) \), denoted by \( \pi \models (\varphi, \alpha) \), if \( N(\varphi) \geq \alpha \). Let \( \Sigma = \{(\varphi_i, \alpha_i) : 1 \leq i \leq m\} \) be a finite set of PL1 wffs, then \( \Sigma \models_{PL1} (\varphi, \alpha) \)
if for each $\pi_i$ for all $1 \leq i \leq m$ implies $\pi \models (\varphi, \alpha)$. It is shown that the consequence relation in PL1 can be determined completely by the least specific model satisfying $\Sigma$. Thus, if $\pi_\Sigma : W \rightarrow [0,1]$ is defined by

$$\pi_\Sigma(w) = \min\{1 - \alpha_i \mid w \models \neg \varphi, 1 \leq i \leq m\},$$

where $\min \emptyset = 1$, then $\Sigma \models_{PL1} (\varphi, \alpha)$ iff $\pi_\Sigma \models (\varphi, \alpha)$.

A special feature of PL1 is its capability to cope with partial inconsistency. For $\Sigma$ defined as above, let $\Sigma^*$ denote the set of classical formulas $\{\varphi \mid 1 \leq i \leq m\}$. Then the set $\Sigma$ is said to be partially inconsistent when $\Sigma^*$ is classically inconsistent. It can be easily shown that $\Sigma$ is partially inconsistent iff $\sup_{w \in W} \pi_\Sigma(w) < 1$. Thus $\sup_{w \in W} \pi_\Sigma(w)$ is called the consistency degree of $\Sigma$, denoted by $\text{Cons}(\Sigma)$, and $1 - \text{Cons}(\Sigma)$ is called the inconsistency degree of $\Sigma$, denoted by $\text{Incons}(\Sigma)$. When $\Sigma$ is partially inconsistent, it can be shown that $\Sigma \models_{PL1} (\bot, \text{Incons}(\Sigma))$, so for any classical wff $(\varphi, \text{Incons}(\Sigma))$ is a trivial logical consequence of $\Sigma$. On the contrary, if $\Sigma \models_{PL1} (\varphi, \alpha)$ for some $\alpha > \text{Incons}(\Sigma)$, then $\varphi$ is called a nontrivial consequence of $\Sigma$.

To model the nontrivial deduction of PL1, we assume that the weights of the wffs are drawn from a finite subset $\mathcal{V} = \{\alpha_1, \ldots, \alpha_n\}$ of $(0,1)$. Without loss of generality, we can assume $\alpha_1 > \cdots > \alpha_n$. Let us define $n$ databases from $\Sigma$ as $\mathcal{DB}_k = \{\varphi \mid (\varphi, \alpha_i) \in \Sigma\}$ for $1 \leq i \leq n$. It can easily be seen that when each $\mathcal{DB}_k$ is classically consistent, then for any $\varphi \in \mathcal{L}(\mathcal{P}_0)$, $\varphi$ is a nontrivial consequence of $\Sigma$ iff $\vdash_{\mathcal{DB}_{\psi}} \psi \supset [1 > 2 > \cdots > n] \varphi$, where $\psi$ is the formula representing the databases.

### 6 Incorporating Other Fusion Operators

While we adopt a modal logic approach to belief fusion, there have been also a lot of works on knowledge merging by using meta-level operators. In [1, 2, 4, 11, 12, 13, 18, 49, 55, 68], the meta-level approach is in general used to combine a set of knowledge bases $T_1, T_2, \cdots, T_k$, where each knowledge base is a theory in some logical language. The main difference between our approach and theirs is that the belief fusion operators are incorporated into the object language in our logic, so we can reason not only with the merged results but also about the fusion process. However, the suspicious attitude used in our logic may be too cautious in some cases. Thus our logic should also be extended to accommodate these more sophisticated knowledge merging operators both syntactically and semantically. In the following, we will describe these operators briefly and discuss some possible extensions of our logic for incorporating them into the modal language.

In the presentation below, we will extensively use the notions of pre-order. Let $S$ be a set, then a pre-order over $S$ is a reflexive and transitive binary relation $\leq$ on $S$. A pre-order over $S$ is called total (or connected) if for all $x, y \in S$, either $x \leq y$ or $y \leq x$ holds. We will write $x < y$ as the abbreviation of $x \leq y$ and $y \not\leq x$. For a subset $S'$ of $S$, $\min(S', \leq)$ is defined as the set $\{x \in S' \mid \forall y \in S', y \not< x\}$.

### 6.1 Combination by maximal consistency

One of the earliest approaches to knowledge merging is to manipulate the maximal consistent subsets of the union of the component databases. In [1, 2, 4], knowledge bases with integrity constraints are combined by a meta-level combination operator to form a new knowledge base. While in [1, 2, 4], logic programs and default logic theories are considered which have different semantics than the classical logic, the basic idea for combining first-order theories in [1] can be carried out in our logic.

In [1], a combination operator $C$ maps a set of knowledge bases $\{T_1, \cdots, T_k\}$ and a set of integrity constraints $\mathcal{IC}$ into a new knowledge base $C(T_1, \cdots, T_k, \mathcal{IC})$ which can be roughly considered as the disjunction of maximally consistent subsets of $T_1 \cup T_2 \cup \cdots \cup T_k$ with respect to $\mathcal{IC}$.

Unlike our fusion operators which correspond to total orders on the agents, the combination operator assumes all knowledge bases are equally important, so there are no priorities among them, though the priority is obviously given to the integrity constraints. Therefore, by using the partial order fusion operators, we can analogously model the combination operator in our logic. Let us consider $n$ agents where the belief of agent 1 is the set $\mathcal{IC}$ and each sentence in $T_1 \cup T_2 \cup \cdots \cup T_k$ is exactly represented as the belief of one agent in $\{2, \cdots, n\}$, then for the partial order $Q = \{1 > 2, 1 > 3, \cdots, 1 > n\}$, the modal operator $[Q]$ can produce the same result as the combination operator $C$.

Note that just like the simulation of trusting attitude multi-sources reasoning in our logic, the maximally consistent combination is also syntax-dependent.

In [11], it is argued that the maximally consistent combination lacks many desirable properties of knowledge merging. This is due to the fact that the source of information is lost in the combination process. Some improvements based on the selection of some maximally consistent subsets instead of all ones are then proposed to
circumvent the problem. Three approaches are suggested according to the difference of the selection functions. The first selects from the set of maximally consistent subsets those consistent with the most knowledge bases, the second selects those that have least difference (in terms of number of sentences) with the knowledge bases, and the third selects those that fit the knowledge bases on a maximum number of sentences. Though these improvements indeed satisfy the desirable logical properties argued by the author, they are all syntactical operator and lack a model-theoretic semantic characterization. Furthermore, since the second and the third improvements are based on the comparison of cardinalities of sets of wffs, they works only for finite knowledge bases. This makes it difficult to incorporate these improved combination operators into our logic where each agent’s beliefs are closed under logical consequence. Fortunately, there are other elegant merging operators with the desirable logical properties which can be incorporated into our framework, so we will consider some of them in the following sections.

Yet another syntax-based approach is to remove the wffs causing inconsistency. In [58], this approach is explored when only local ordering between the wffs is given. However, the approach is more algorithmic and it seems not appropriate to incorporate it into our framework.

6.2 Combination by meta-information

In the combination by maximal consistency, it is assumed that no information about how to combine the knowledge bases is available. However, sometimes the users can provide valuable meta-information about the combination process, such as the reliability of the component databases, the user’s preference, or the interaction of different databases, etc. In [58], a kind of priorities between sets of propositional atoms is represented and the combination is made according to the prioritized information. In fact, our fusion operators (either total orders or partial ones) also encode a kind of priorities. The main difference is that our priorities are between agents while theirs are between the sets of propositions believed by the agents. However, since transitivity is not required for the priority relation in [58], there may exist cyclic priorities (i.e., $x > y$ and $y > x$ holds simultaneously). In such cases, there would not be combined knowledge bases satisfying the priorities. Furthermore, since the knowledge bases in [58] are just sets of propositional atoms, the approach applies only to deduction-free relational databases and lacks the capability of reasoning about the inter-relationship between the knowledge bases.

A more flexible way for specifying the meta-information is proposed in [58]. In that work, a set of local databases $DB_1, \cdots, DB_n$ is combined with a supervisory knowledge base $S$. Intuitively, $S$ contains conflict resolution information. Since the databases are expressed in a very rich language, the supervisory knowledge base can specify complex relations between local databases. The language is called annotated logic and is constructed from some base language and a set of annotations. These annotations can denote the truth values for many-valued logic, complex relations between local databases. The language is called annotated logic and is constructed from some information. Since the databases are expressed in a very rich language, the supervisory knowledge base can specify complex relations between local databases. The language is called annotated logic and is constructed from some base language and a set of annotations. These annotations can denote the truth values for many-valued logic, timestamps, uncertainties, etc., so the expressive power of annotated logic is quite rich. In the framework, the local databases are just sets of sentences in the annotated logic, whereas the supervisory knowledge base contains sentences in another annotated logic where each atom is indexed by a subset of $\{1, 2, \cdots, n, s\}$.

To compare the framework in [58] with our logic, let us assume the only annotations are the classical truth values $\{t, f\}$, so the annotated logic reduces to the classical one. In this simplified case, the annotations can be simply removed and each local database contains logic program clauses of the form

$$p_0 \leftarrow p_1, \cdots, p_m, \not p_{m+1}, \cdots, \not p_{m+k}$$

where for all $0 \leq i \leq m + k$, $p_i$ is an atomic formula in classical logic, whereas the supervisory knowledge base $S$ contains indexed clauses of the form

$$p_0 : \{s\} \leftarrow p_1 : D_1, \cdots, p_m : D_m, \not p_{m+1} : D_{m+1}, \cdots, \not p_{m+k} : D_{m+k}$$

where for all $1 \leq i \leq m + k$, $D_i \subseteq \{1, 2, \cdots, n, s\}$. The intended meaning of $p : D_i$ is that the databases in $D_i$ jointly say that $p$ is true. The meaning is specified by a combination axiom scheme which is equivalent to

$$p : D \leftarrow \bigvee_{0 \leq D' \leq D} p : D'$$

in our simplified case. For each local database $DB_i$, the amalgamation transform of $DB_i$, $AT(DB_i)$, is defined as the result of replacing each clause $p_0 \leftarrow p_1, \cdots, p_m, \not p_{m+1}, \cdots, \not p_{m+k}$ in $DB_i$ by

$$p_0 : \{i\} \leftarrow p_1 : \{i\}, \cdots, p_m : \{i\}, \not p_{m+1} : \{i\}, \cdots, \not p_{m+k} : \{i\}$$

(4)
 Consequently, the amalgam of \((DB_1, \ldots, DB_n, S)\) is defined as the amalgamated knowledge base

\[
S \cup \bigcup_{i=1}^{n} AT(DB_i) \cup \text{Combination axioms}
\]

. Though the semantics of the amalgamated knowledge base is given according to that of logic program, so not comparable with that of classical logic. However, the idea of supervisory knowledge base can be easily realized in the multi-agent epistemic logic (and so in our logic). In fact, the clause in (4) can be translated into our logic as

\[
\bigwedge_{1 \leq i \leq m} [D_i]p_i \land \bigwedge_{m+1 \leq i \leq m+k} [D_i] \neg p_i \supset [s]p_0,
\]

whereas the combination axiom (3) is a special case of the axioms G3 or V3 in our systems. Though the annotated logic provides a far richer expressive power in the representation of objective knowledge than our systems and the supervisory knowledge base can express conflict resolution information among local databases, the framework in [8] still lacks the capability of reasoning about mutual information. Contrarily, each agent can easily reason about the beliefs of other agents in our logic. For example, it is possible to say that agent \(i\) believes that if agent \(j\) believes \(\varphi\), then agent \(k\) would also do. This somewhat reflects the essential difference between the modal logic approach and the meta-level ones to the belief fusion.

### 6.3 Merging by majority

Though the maximal consistent combination resolves the conflicts between knowledge bases, it does not reflect the view of the majority. For example, if three knowledge bases \(T_1 = \{\varphi\}, T_2 = \{\varphi\}, \text{ and } T_3 = \{\neg \varphi\}\) are combined by the maximal consistent combination rule, the result would be just a knowledge base containing the tautology. However, if the majority view is taken into account, then the result would be \(\{\varphi\}\). In [4], a merging operator reflecting the views of majority is proposed for knowledge bases consisting of finite propositional sentences. Since the propositional language is assumed finite there, the so-called Dalal distance between two interpretations of the language is used [9]. It is defined as the number of atoms whose valuations differs in the two interpretations. Let \(dist(w, w')\) denote the Dalal distance between two interpretations \(w\) and \(w'\), then the distance from \(w\) to a theory \(T\), denoted by \(dist(w, T)\), is defined as

\[
dist(w, T) = \min\{dist(w, w') \mid w' \models T\}. \tag{5}
\]

Given a set of knowledge bases \(T_1, T_2, \ldots, T_k\) to be merged, a total pre-order \(\preceq_{\{T_1, T_2, \ldots, T_k\}}\) is defined on the set of interpretations by

\[
w \preceq_{\{T_1, T_2, \ldots, T_k\}} w' \text{ iff } \sum_{i=1}^{k} dist(w, T_i) \leq \sum_{i=1}^{k} dist(w', T_i) \tag{6}
\]

Then the merged result \(Merge(T_1, T_2, \ldots, T_k)\) is the theory whose models are all interpretations minimal with respect to the order \(\preceq_{\{T_1, T_2, \ldots, T_k\}}\).

In [9], a set of postulates for characterizing the merging function is presented and its corresponding model-theoretic characterization is also given. It is then shown that \(Merge\) is indeed a function satisfying the postulates. In [9], the function \(Merge\) is further generalized for the application in weighted knowledge bases. Let \(wt : \{T_1, T_2, \ldots, T_k\} \rightarrow R^+\) is a weight function which assigns to each component knowledge base a positive real number, then the total pre-order in (6) is changed into

\[
w \preceq_{\{(T_1, T_2, \ldots, T_k), wt\}} w' \text{ iff } \sum_{i=1}^{k} dist(w, T_i) \cdot wt(T_i) \leq \sum_{i=1}^{k} dist(w', T_i) \cdot wt(T_i) \tag{7}
\]

Then the merged result \(Merge(T_1, T_2, \ldots, T_k, wt)\) is the theory whose models are all interpretations minimal with respect to the order \(\preceq_{\{(T_1, T_2, \ldots, T_k), wt\}}\).

Since the weighted version of the merging function is more general, we will consider the extension of our logic for the weighted merging operator. First, a new class of modal operators \([M(G, wt)]\) for any nonempty \(G \subseteq \{1, 2, \ldots, n\}\).
and weight function \( wt : G \rightarrow R^+ \) is added to our logic language. Then the semantics for the new modal operators is defined by extending a possible world model to \((W, (R_i)_{1 \leq i \leq n}, V, \mu)\), where \((W, (R_i)_{1 \leq i \leq n}, V)\) is a \( \text{DBF}^n \) (or \( \text{DBF}^\infty \)) model, whereas \( \mu : W \times W \rightarrow R^+ \cup \{0\} \) is a distance metric function between possible worlds satisfying \( \mu(w, w) = 0 \) and \( \mu(w, w') = \mu(w', w) \).

It must be noted that our possible worlds are more than the truth assignments of the propositional symbols, so it is inappropriate to define the distance between two possible worlds by merely enumerating the number of atoms whose valuations differs in the two worlds. However, it is assumed a distance metric between possible worlds can be defined just as in the semantics of conditional logic \([57, 62]\). To give the semantics of the new operators, we first define the distance from a possible world \( w \) to the belief state of an agent \( i \) in the possible world \( u \) by

\[
\text{dist}_u(w, i) = \inf \{ \mu(w, w') \mid (u, w') \in R_i \}. \tag{8}
\]

Then a total pre-order \( \preceq_{(G, wt)} \) on the possible worlds is defined for each possible world \( u \) and modal operator \([M(G, wt)]\)

\[
w \preceq_{(G, wt)} w' \iff \sum_{i \in G} \text{dist}_u(w, i) \cdot wt(i) \leq \sum_{i \in G} \text{dist}_u(w', i) \cdot wt(i). \tag{9}
\]

The most straightforward definition for the satisfaction of the wff \([M(G, wt)] \varphi \) is

\[
u \models [M(G, wt)] \varphi \iff \text{for all } w \in R_{M(G, wt)}(u), w \models \varphi,
\]

where \( R_{M(G, wt)} \) is a binary relation over the possible worlds such that \( R_{M(G, wt)}(u) = \min(W, \preceq_{(G, wt)}) \). However, since for infinite \( W \), the set \( \min(W, \preceq_{(G, wt)}) \) may be empty, the definition may result in \( u \models [M(G, wt)] \bot \) in some cases. Alternatively, since \( \preceq_{(G, wt)} \) is a total pre-order, it is just like the a system-of-spheres in the semantics of conditional logic \([57]\), so we can define the satisfaction of the wff \([M(G, wt)] \varphi \) by

\[
u \models [M(G, wt)] \varphi \iff \text{there exists } w_0 \text{ such that for all } w \preceq_{(G, wt)} w_0, w \models \varphi.
\]

An alternative approach to do majority merging is to employ the graded modal logic in \([3, 8]\). In the logic, a set of modal operators \( K_m \), where \( m \) is a natural number, is in place of the ordinary epistemic or doxastic operators. In the single agent case, a modal formula \( K_m \varphi \) means that in all possible worlds the agent considers possible, there are at most \( m \) worlds at which \( \varphi \) is false. By the abbreviations, \( M_m \varphi \equiv \neg K_m \neg \varphi, M_0 \varphi \equiv K_0 \neg \varphi, \) and \( M_m \neg \varphi \equiv (M_{m-1} \varphi \land M_m \varphi) \), it can be seen that \( M_m 1 \top \land K_{\frac{r}{0}} \varphi \) means the wff \( \varphi \) is true at more than half of the worlds. By generalizing this kind of graded modal operators for distributed belief fusion, we can consider the modal operators \( \{G\}_r \) for any real number \( r \) and subset of agents \( G \). The semantics for \( \{G\}_r \varphi \) is interpreted in the multi-agent epistemic logic model such that

\[
w \models \{G\}_r \varphi \iff \frac{\{u \models \varphi, \forall \cup_{i \in G} R_{(i)}(w)\}}{\cup_{i \in G} R_{(i)}(w)} > r.
\]

Then \( \{G\}_r \) for some threshold \( c \geq 0.5 \) can be taken as the fusion operator which merges the beliefs of agents in \( G \). However, it must be noted that the majority considered in the graded modal logic is the majority of possible worlds instead of that of agents. Furthermore, by using the cardinality of sets of possible worlds in the definition, the semantic models are restricted to finite ones. To lift the restriction, some numerical measures, such as the probability, should be added to the models.

### 6.4 Arbitration

The notion of distance measure between possible worlds is also used in another type of merging operator, called arbitration \([3, 8] \). Arbitration is the process of settling a conflict between two or more persons. The first version of arbitration operator between knowledge bases is proposed in \([57]\) via the so-called model-fitting operators. The postulates for model-fitting operators and its semantic characterization are given and then arbitration is defined as a special kind of model-fitting operators.

In \([8]\), the arbitration operator is further generalized so that it is applicable to the weighted knowledge bases. A set of postulates is also directly used in characterizing the arbitration between a weighted knowledge base and a regular one. A weighted knowledge base in \([8]\) is defined as a mapping \( \hat{K} \) from model sets to nonnegative real number and a regular knowledge base is just a finite set of propositional sentences. A generalized loyal assignment
is then defined as a function that assigns for each weighted knowledge base \( \tilde{K} \) a pre-order \( \leq_{\tilde{K}} \) between propositional sentences such that some conditions are satisfied for the pre-orders. Finally, the arbitration of a weighted knowledge base \( \tilde{K} \) by a regular knowledge base \( K' \) is defined as

\[
\tilde{K} \triangle K' = \min(K', \leq_{\tilde{K}}),
\]

where \( \min(K', \leq_{\tilde{K}}) \) is the set of sentences in \( K' \) which is minimal according to the ordering \( \leq_{\tilde{K}} \). However, this kind of arbitration is obviously syntax-dependent. For example, if \( \varphi_1 \) and \( \varphi_2 \) is two propositional sentences such that \( \varphi_1 <_{\tilde{K}} \varphi_2 \), then \( \tilde{K} \triangle \{ \varphi_1, \varphi_2 \} = \{ \varphi_1 \} \neq \tilde{K} \triangle \{ \varphi_1 \land \varphi_2 \} = \{ \varphi_1 \land \varphi_2 \} \) though the two knowledge bases \( \{ \varphi_1, \varphi_2 \} \) and \( \{ \varphi_1 \land \varphi_2 \} \) are semantically equivalent.

An alternative, seemingly more natural, characterization for arbitration is given in \([46]\) without resorting to the model-fitting operators. A knowledge base in that work is identified with the set of propositional models for it, thus the semantic characterization for this kind of arbitration is given by assigning to each subset of models \( A \) a binary relation \( \leq_A \) over the set of model sets satisfying the following conditions (the subscript is omitted when it means all binary relations of the form \( \leq_A \))

1. transitivity: if \( A \leq B \) and \( B \leq C \) then \( A \leq C \)
2. if \( A \subseteq B \) then \( B \leq A \)
3. \( A \leq A \cup B \) and \( B \leq A \cup B \)
4. \( B \leq_A C \) for every \( C \) iff \( A \cap B \neq \emptyset \)
5. \( A \leq_{C \cup_D B} \Leftrightarrow \begin{cases} C \leq_{A \cup_D B} D & \text{and} \ A \leq_C B \quad \text{or} \\ D \leq_{A \cup_D B} C & \text{and} \ A \leq_D B \end{cases} \)

Then the arbitration between two sets of models \( A \) and \( B \) is defined as

\[
A \triangle B = \min(A, \leq_B) \cup \min(B, \leq_A)
\]

(10)

Note that though the relation \( \leq_A \) is defined between sets of models, in the definition of the arbitration, only \( \leq_A \) between singletons is used. Thus by slightly abusing the notation, \( \leq_A \) may also denote an ordering between models.

To incorporate the arbitration operator of \([46]\) into our language, we must first note that according to (10), the arbitration operator should be a binary one between two agents. We can add a class of modal operators for arbitration into our logic just as in the case of majority merging. However, to be more expressive, we will also consider the interaction between arbitration and other epistemic operators, so we define the set of arbitration expressions over the agents recursively as the smallest set containing \( \{ 1, 2, \ldots, n \} \) and closed under the binary operators +, ·, and \( \triangle \). Here + and · correspond respectively to the distributed belief and the so-called “everybody knows” operators in multi-agent epistemic logic \([31]\). Then our language can be extended to include a new class of modal operators \( [a] \) where \( a \) is an arbitration expressions. Note that it has been shown that the only associative arbitration satisfying postulates 7 and 8 of \([46]\) is \( A \triangle B = A \cup B \), so if \( \triangle \) is an associative arbitration satisfying those postulates, then \( [a \cdot b] \varphi \) is reduced to \( [a] \varphi \land [b] \varphi \) which is in turn equivalent to \([a] \varphi \land [b] \varphi \).

For the semantics, a model is extended to \((W, (\mathcal{R}_i)_{1 \leq i \leq n}, V, \leq)\), where \((W, (\mathcal{R}_i)_{1 \leq i \leq n}, V)\) is a DBF* (or DBFc) model, whereas \( \leq \) is a function assigning to each subset of possible worlds \( A \) a binary relation \( \leq_A \subseteq 2^W \times 2^W \) satisfying the above-mentioned five conditions. Note that the first two conditions imply that \( \leq_A \) is a pre-order over \( 2^W \). Then for each arbitration expression, we can define the binary relations \( \mathcal{R}_{a\triangle b}, \mathcal{R}_{a\cdot b} \) and \( \mathcal{R}_{a+b} \) over \( W \) recursively by

\[
\mathcal{R}_{a\triangle b}(w) = \min(\mathcal{R}_a(w), \leq_{\mathcal{R}_b}(w)) \cup \min(\mathcal{R}_b(w), \leq_{\mathcal{R}_a}(w))
\]

(11)

\[
\mathcal{R}_{a+b} = \mathcal{R}_a \cap \mathcal{R}_b
\]

(12)

\[
\mathcal{R}_{a\cdot b} = \mathcal{R}_a \cup \mathcal{R}_b
\]

(13)

Thus the satisfaction for the wff \([a] \varphi \) is defined as

\[
u \models [a] \varphi \iff \text{for all } w \in \mathcal{R}_a(u), w \models \varphi.
\]

Note that the distributed belief operator \([G] \) can be equivalently defined as an abbreviation of \([i_1 + (i_2 + \cdots + (i_{k-1} + i_k))]\) if \( G = \{ i_1, i_2, \ldots, i_k \} \).

By this kind of modal operators, the postulates 2-8 of \([46]\) can be translated into the following axioms:
1. \([a \triangle b] \varphi \equiv [b \triangle a] \varphi\)
2. \([a \triangle b] \varphi \supset [a + b] \varphi\)
3. \(\neg[a + b] \bot \supset ([a + b] \varphi \supset [a \triangle b] \varphi)\)
4. \([a \triangle b] \bot \supset [a] \bot \wedge [b] \bot\)
5. \(((a \triangle (b \cdot c)) \varphi \equiv [a \triangle b] \varphi) \lor ([a \triangle (b \cdot c)] \varphi \equiv [a \triangle c] \varphi) \lor ([a \triangle (b \cdot c)] \varphi \equiv [(a \triangle b) \cdot (a \triangle c)] \varphi)\)
6. \([a] \varphi \wedge [b] \varphi \supset [a \triangle b] \varphi\)
7. \(\neg[a] \bot \supset \neg[a + (a \triangle b)] \bot\)

However, since the set of possible worlds \(W\) may be infinite in our logic, the minimal models in (11) may not exist, so the axioms 4 and 7 are not sound with respect to the semantics. To make them sound, we must add the following limit assumption to the binary relations \(\leq_A\) for any \(A \subseteq W\):

\[
\text{for any nonempty } U \subseteq W, \min(U, \leq_A) \text{ is nonempty.}
\]

It must be noted that the axioms listed above are not yet complete for the logic with arbitration operators. In fact, the search of a complete axiomatization for the modal logic of arbitration expressions is of independent interest by itself and can be the further research direction. The brief presentation here just shows that the modal logic approach can provide a uniform framework for integrating the epistemic reasoning and different knowledge merging operators into the object logic level.

### 6.5 General merging

In [4], an axiomatic framework unifying the majority merging and arbitration operators is presented. A set of postulates common to majority and arbitration operators is first proposed to characterize the general merging operators and then additional postulates for differentiating these two are considered respectively. In the framework, a knowledge base is also a finite set of propositional sentences. The general merging operator is defined as a mapping from a multi-set(also called a bag) of belief states, then \(U\) for the naming convenience, we will call a subset of possible worlds a belief state. Let \(U = \{U_1, U_2, \ldots, U_k\}\) denote a multi-set of belief states, then \(\bigcap U = U_1 \cap \cdots U_k\). For the semantics, a possible world model is extended to \((W, (R_i)_{1 \leq i \leq n}, V, \leq)\), where \((W, (R_i)_{1 \leq i \leq n}, V)\) is a DBF model, whereas \(\leq\) is an assignment mapping each multi-set of belief states \(U\) to a total pre-order \(\leq_U\) over \(W\) satisfying the following conditions:

\[\text{A multi-set (or bag) is a collection of elements over some domain which allows multiple occurrences of elements.}\]
1. If \( w, w' \in \mathcal{U} \), then \( w \leq u w' \)
2. If \( w \in \mathcal{U} \) and \( w' \notin \mathcal{U} \), then \( w < u w' \)
3. For any \( w \in U_1 \), there exists \( w' \in U_2 \), such that \( w' \leq_{\{U_1, U_2\}} w \), where \( U_1 \) and \( U_2 \) are two belief states
4. If \( w \leq_{U_1} w' \) and \( w \leq_{U_2} w' \), then \( w \leq_{U_1 \cup U_2} w' \), where \( \cup \) denotes the union of two multi-sets
5. If \( w <_{U_1} w' \) and \( w \leq_{U_2} w' \), then \( w <_{U_1 \cup U_2} w' \)

For a sub-group of agents \( G \) and a possible world \( u \), let us define a total pre-order \( \leq^n_G \) over \( W \) as follows:

\[
w \leq^n_G w' \iff w \leq \bigcup_{\{i \in G\}} w'.
\]

Then a possible world \( u \) satisfies the wff \([\Delta_\psi(G)]\psi\) in the model, i.e. \( u \models [\Delta_\psi(G)]\psi\), iff

(i) there are no possible worlds in \( W \) satisfying \( \varphi \), or
(ii) there exists \( w_0 \in W \) such that \( w_0 \models \varphi \) and for any \( w \leq^n_G w_0 \), \( w \models \varphi \supset \psi \).

Though we can incorporate the general merging operator into the modal logic framework, we should not overlook the difference between the meta-level merging operators and the modal ones. First, in the meta-level approach, the knowledge set consists of a multi-set of objective sentences, whereas in the modal operator \([\Delta_\psi(G)]\), \( G \) is a set of agents whose belief may contains subjective sentences or beliefs of other agents. Second, the integrity constraint can only be the objective sentences in \([43]\) whereas \( \varphi \) may be arbitrary complex wff of our extended language. Finally, instead of selecting the minimal models from those of \( \varphi \), since the set of possible worlds may be infinite in our case, we adopt the system-of-spheres semantics as that in section \([13]\) for the modal operator \([\Delta_\psi(G)]\).

7 Belief Change and Conditional Logic

7.1 Incorporating belief revision operators

Unlike knowledge merging, where the component knowledge bases are equally important, belief change is a kind of asymmetry operators, where the new information always outweighs the old one. The main belief change operators are belief revision and update. They are characterized by different postulates\([1, 38, 39]\). In \([38]\), a uniform model-theoretic framework is provided for the semantic characterization of the revision and update operators. In their works, a knowledge base is a finite set of propositional sentences, so it can also be represented by a single sentence(i.e., the conjunction of all sentences in the knowledge base).

For the revision operator, it is assumed that there is a total pre-order \( \leq_\psi \) over the propositional interpretations for each knowledge base \( \psi \). The revision operators satisfying the AGM postulates in \([3]\) are exactly those that select from the models of the new information \( \varphi \) the minimal ones with respect to the ordering \( \leq_\psi \). More precisely, let \( \psi \) be a knowledge base and \( \varphi \) denote the new information, then the result of revising \( \psi \) by \( \varphi \), denoted by \( \psi \circ \varphi \), will have the set of models

\[
Mod(\psi \circ \varphi) = \min(\text{Mod}(\varphi), \leq_\psi).
\]

As for the update operator, assume for each propositional interpretation \( w \), there exists some partial pre-order \( \leq_w \) over the interpretations for closeness to \( w \), then update operators select for each model \( w \) in \( \text{Mod}(\psi) \) the set of models from \( \text{Mod}(\varphi) \) that are closest to \( w \). The updated theory is characterized by the union of all such models. That is,

\[
\text{Mod}(\psi \circ \varphi) = \bigcup_{w \in \text{Mod}(\psi)} \min(\text{Mod}(\varphi), \leq_w)
\]

where \( \psi \circ \varphi \) is the result of updating the knowledge base \( \psi \) by \( \varphi \).

Both belief revision and update may occur in the observation of the new information \( \varphi \). For the belief revision, it is assumed that the world is static, so if the new information is incompatible with the agent’s original beliefs, then the agent may have wrong belief about the world. Thus he will try to accommodate the new information by minimally changing his original beliefs. However, for the belief update, it is assumed that the observation may be due to the dynamic change of the outside world, so the agent’s belief may be out-of-date, though it may be totally
correct for the original world. Thus the agent will assume the possible worlds are those resulting from the minimal change of the original world. In [3], a generalized update model is proposed which combines aspects of revision and update. It is shown that a belief update model will be inadequate without modelling the dynamic aspect (i.e. the events causing the update) in the same time. Since the dynamic change of the external worlds does not play a role in the belief fusion process, we would not try to model the belief update in our logic, so in what follows, we will concentrate on the belief revision operator.

Let us now consider the possibility of incorporating the belief revision operator into our logic. In addition to the original meaning of revising a knowledge base \( \psi \) by new information \( \varphi \), there is an alternative reading for the revision operator. That is, we can consider \( \circ \) as a prioritized belief fusion operator which gives the priority to its second argument [32]. In the context of knowledge base revision, these two interpretations are essentially equivalent. However, from the perspective of our logic in multi-agents systems, they may be quite different. Roughly speaking, \( i \circ \varphi \) will denote the result of revising the beliefs of agent \( i \) by new information \( \varphi \), whereas \( i \circ j \) is the result of merging the beliefs of agents \( i \) and \( j \) by giving priority to \( j \). More formally, an revision expression will be defined inductively as follows:

- If \( 1 \leq i, j \leq n \) and \( \varphi \) is a wff, then \( i \circ j \) and \( i \circ \varphi \) are revision expressions.
- If \( r \) is a revision expression, \( 1 \leq i \leq n \) and \( \varphi \) is a wff, then \( r \circ i \) and \( r \circ \varphi \) are revision expressions.

The syntactic rule is extended to include the modal operators [\( \rho \)] for any revision expression \( r \), so [\( \rho \)]\( \varphi \) would be a wff if \( \varphi \) is. To interpret the modal operator in our semantic framework, a possible world model is extended to (\( W, (R_i)_{1 \leq i \leq n}, V, \leq \)) where (\( W, (R_i)_{1 \leq i \leq n}, V \)) is a DBF\(_n\) (or DBF\(_n^*\)) model, whereas \( \leq \) is an assignment mapping each belief state (i.e. subset of possible worlds) \( U \) to a total pre-order \( \leq_U \) over \( W \) such that (i) if \( w, w' \in U \), then \( w \leq_U w' \) and (ii) if \( w \in U \) and \( w' \not\in U \), then \( w <_U w' \). Let \( S \cdot U \) denote the sequence \((U_1, U_2, \ldots, U_k, U)\) if \( S = (U_1, U_2, \ldots, U_k) \) is a sequence of belief states, then the assignment \( \leq \) is extended to sequences of belief states in the following way (we assume \( \leq(U) = \leq_U \)):

1. \( w <_{S \cdot U} w' \) if \( w \in U \) and \( w' \not\in U \)
2. \( w \leq_{S \cdot U} w' \) iff \( w \leq U \) or \( w, w' \in U \) and \( w, w' \not\in U \)

For each wff \( \varphi \), let the truth set of \( \varphi \), denoted by \( \{ \varphi \} \), be defined as \( \{ w \in W \mid w \models \varphi \} \). For each possible world \( u \), define a function mapping any agent \( i \) and revision expression \( r \) into a sequence of belief states \( u(i) \) and \( u(r) \) as follows:

1. \( u(i) = (R_i(u)) \)
2. \( u(r \circ i) = u(r) \cdot R_i(u) \)
3. \( u(r \circ \varphi) = u(r) \cdot \{ \varphi \} \)

Then the truth condition for the wff \([r \circ \varphi] \) is \( u \models [r \circ \varphi] \psi \) iff

(i) there are no possible worlds in \( W \) satisfying \( \varphi \), or
(ii) there exists \( w_0 \in W \) such that \( w_0 \models \varphi \) and for any \( w \leq u(r) w_0, w \models \varphi \lor \psi \).

Analogously, \( u \models [r \circ i] \psi \) iff there exists \( w_0 \in R_i(u) \) such that for any \( w \leq u(r) w_0, w \models \varphi \). It can be seen that \([i \circ \varphi] \psi \) is equivalent to \([\Delta_\varphi(\{i\})] \psi \) in section 6.3 according to the semantics.

### 7.2 Relationship with conditional logic

There have been also various attempts in formalizing the belief change process by modal logic or conditional logic systems [4, [1, 11, 12, 33, 61, 63]. For example, in [11], a modal logic CO\(^*\) is proposed for modelling the belief revision. CO\(^*\) is an extension of the logic CO proposed in [3]. In CO\(^*\), revision of a theory by a sentence is represented using a conditional connective. The connective is not primitive, but rather defined using two unary modal operators \( \Box \) and \( \Diamond \). The modal operators are interpreted with respect to a total pre-order \( R \) over the possible worlds which is assumed to rely on a background theory \( K \). Thus \( w \models \Box \varphi \) iff \( \varphi \) is true in all possible worlds which
are as plausible as \( w \) given the theory \( K \) and \( w \models \varphi \) iff \( \varphi \) is true in all possible worlds which are less plausible than \( w \) given \( K \). By defining \( \Box \varphi \) as \( \square \varphi \wedge \Box \varphi \) and \( \diamond \varphi \) as \( \neg \Box \neg \varphi \), the conditional \( \varphi^{KB} \psi \) is defined as

\[
\Box \neg \varphi \lor \diamond (\varphi \wedge \Box (\varphi \supset \psi)),
\]

where \( KB \) is a finite representation of the theory \( K \). Since there is only one global ordering \( R \) in CO* model which is associated with the background theory, the logic is appropriate only for reasoning about the revision of a single theory \( K \). On the other hand, our logic allows the reasoning about revisions of many agents' belief states.

Furthermore, since the ordering \( R \) in CO* model is global, \( \varphi^{KB} \psi \) is true in a world iff it is true in all worlds, thus no iterated revisions are allowed in the model. In [12], this restriction is lifted by allowing the revision of the ordering \( R \) to \( R' \) at the same time. The idea is to move the most plausible \( \varphi \)-models with respect to \( R \) to the most plausible level of \( R' \) and keep the other parts of \( R \) unchanged. Our assignment of a total pre-order to a sequence of belief states is basically based on the same idea. However, while the definition of [12] presumes the existence of the minimal models for any propositional formulas, we do not need such assumption.

In [33], a logic with conditional and epistemic operators is used in the reasoning of belief revision. The conditional and epistemic sentences are interpreted in an abstract belief change system (BCS). The basic components of a BCS are a set of belief states and a belief change function mapping each belief state and sentence of some base language into a new belief state. In a more concrete preferential interpretation, each belief state \( s \) is interpreted as a subset of possible world \( K(s) \) and a pre-order \( \preceq_s \) over the possible worlds is associated with it. In this regard, \( \preceq_s \) corresponds to \( \preceq_{K(s)} \) in our semantic models and the conditional wff \( \varphi \supset \psi \) in the logic \( \mathcal{L}^> \) of [33] is roughly equivalent to our wff \( \iota \varphi \psi \) for some fixed agent \( i \). However, since in \( \mathcal{L}^> \), the antecedent \( \varphi \) of a conditional is restricted to a wff in the base language \( L \), it does not allow the epistemic wffs of the form \( \Box \varphi \). It is argued that the antecedent must be observable whereas conditional wffs are unobservable, so we should not allow conditional wffs in the antecedent. However, from the multi-agent systems perspective, one agent may learn the beliefs of other agents by communication, so we should not exclude the flexibility. Another significant difference is that the BCS allows only revision of a belief state by a sentence, while in our system, the prioritized fusion of two belief states held by two agents are also incorporated.

A dynamic doxastic logic for belief revision is proposed in [33] and further developed in [63]. By using the notations of [12], the doxastic operator \( \Box \) and two kinds of dynamic modal operators \( [+\varphi] \) and \([\neg \varphi] \) for propositional wff \( \varphi \) are taken as the basic constructs of the language. The operators \( [+\varphi] \) and \([\neg \varphi]\) corresponds respectively to the expansion and contraction operators in AGM theory. Thus the revision operator \( [\circ \varphi] \) is defined as \([\neg (\neg \varphi)] \) according to the so-called Levi’s identity [1]. The wffs of the language are interpreted with respect to a hypertheory. A hypertheory \( H \) is a set of subsets of possible worlds linearly ordered by inclusion. A hypertheory is similar to the widening ranked model defined in [14]. However, the latter assumes that the subsets of models are indexed by natural numbers. A hypertheory is said to be replete if the set of all possible worlds \( W \) is in \( H \). From the hypertheory \( H \), a pre-order \( \leq_H \) over \( W \) can be defined as follows:

\[
w' \leq_H w \text{ iff for any } U \in H, \text{ if } w \in U, \text{ then there exists } U' \in H \text{ such that } U' \subseteq U \text{ and } w' \in U'.
\]

When \( H \) is replete, the pre-order defined in this way is total. When the wffs \([+\varphi]\psi\) and \([\neg \varphi]\psi\) are evaluated with respect to a hypertheory \( H \), it causes evaluation of \( \psi \) in some revised hypertheory \( H' \), so the semantics are essentially equivalent to that proposed in [12], though the revisions of the corresponding pre-order are somewhat different in the two approaches. Therefore, as the proposal of [12], the logic allows only reasoning about the belief revision of a single agent by some new information and the prioritized fusion of multi-agent beliefs can not be represented in such logic.

### 7.3 Alternative representations of belief states

In our presentation above, we assume an agent’s belief states are represented as a subset of possible worlds, i.e. \( \mathcal{R}_i(w) \) is the belief state of agent \( i \) in world \( w \). However, some more fine-grained representations have been also proposed, such as total pre-orders over the set of possible worlds \([12, 20, 14, 33]\), ordinal conditional functions \([13, 72, 70]\), possibility distributions \([6, 29, 20]\), belief functions \([97]\), and pedigreed belief states \([37]\). Perhaps, the most popular representation among them is an ordering of the possible worlds. While a set of possible worlds can be seen as the minimal worlds with respect to a given ordering, it is claimed that the fusion of two orderings is more general than the revision of an ordering by a set of possible worlds \([31]\). Thus it is shown that AGM revision
is in fact a special case of the fusion operator in [37]. Indeed, in our extended models, the assignment \( \leq \) has mapped each subset of possible worlds to a total pre-orders between worlds. However, to fully utilize the semantic power of an ordering, the logic language should be extended further to cover the conditional connectives. Since the purpose of the present paper is to integrate the belief fusion operators into the epistemic reasoning framework, this extension is beyond its scope. Nevertheless, the further development of logical systems incorporating the fusion operators based on fine-grained representations of belief states should be a very interesting research direction.

8 Conclusions and Further Researches

The main contribution of the paper is the integration of belief fusion operators into the multi-agent epistemic logic. We first propose two basic logical systems for reasoning about the cautiously merged beliefs of multiple agents and then extend them to cover more sophisticated and adventurous fusion and revision operators.

The basic systems are cautious in the sense that if an information source is in conflict with other more reliable ones, then the information from that source is totally discarded. The two systems correspond to two different strategies of discarding the information sources. For level cutting strategy, if an information source is to be discarded, then all those less reliable than it are also discarded without further examination. On the other hand, for level skipping strategy, only the level under conflict is skipped, and the next level will be considered independent of those discarded before it. Thus, level skipping strategy is relatively less cautious than the level cutting one and indeed, we can simulate the trusting attitude multi-sources reasoning in [15] by level skipping strategy.

Then some of the most important knowledge base merging approaches are reviewed and it is shown that many fusion operators proposed in those approaches can be incorporated into our logic. While most of the knowledge base merging research takes the fusion process as a meta-level operator, our approach incorporate them into the object logic directly. Therefore, it is possible to integrate the belief fusion operators into the multi-agent epistemic logic. What we can benefit from the epistemic logic is the capability to reasoning with not only the beliefs about the objective world but also the beliefs about beliefs.

In the discussion of the belief fusion logic, for simplicity, we do not distinguish belief and information. However, in a genuine agent systems, an agent’s belief may be different than the information he sends to or receives from other agents. Thus, in general, we should have a set of modal operators \([j]\) such that \([j]_i\phi\) means that agent \(i\) receives the information \(\phi\) from \(j\). In particular, \([i]\_\phi\) may represent the observation of agent \(i\) himself, which should be the most reliable information source for \(i\). Then agent \(i\) may form his belief by fusing the information he received from different agents according to the degrees of trust he has on other agents. The fusion may be represented by the operators \([O]\). If we consider \([j]_i\phi\) as the communication of message \(\phi\) from \(j\) to \(i\), then we have a general framework for reasoning about agent’s belief and communication. In such a framework, we can discuss the problems like deception of agent. For example, \([O]_i\phi \land [i]_j\neg \phi\) may mean that agent \(i\) deceives to agent \(j\) by telling \(j\) the negation of what he believes. In [24], an application of our basic systems to reasoning about beliefs and trusts of multiple agents has been proposed along this direction. However, more works remain to be done for the real applications. These applications may also benefit from some fundamental works on multi-agent belief revision [23, 24, 25, 35, 40].

From a more foundational viewpoint, though we have proposed some extensions of the basic systems for accommodating the belief fusion operators both syntactically and semantically, the complete axiomatization of these extended logics remain to be found. To be practically useful, other proof methods more suitable for automated theorem proving should be also developed.

In a recent paper, it is shown that the multi-sources reasoning can be applied to deontic logic under conflicting regulations [17]. Essentially, this is to merge conflicting regulations according to the priorities of them analogously to the fusion of information. However, inherited from the restriction of FU, \(\perp\) it is also required that each regulation to be merged must be a set of deontic literals. Now, by the systems developed here, it is expected that the general forms of regulations can also be merged.

A real difficulty in the application of our logic to model the database merging reasoning is the representational problem of the databases. In the discussion of section 3, we suggest to find all maximal consistent agent groups in advance and add the wff \(\bigwedge_{G \in MCAG} \neg [G]_\perp\) to the representation. This is a rather time-consuming work. In practice, we can omit this part and check the consistency of some agent groups when it is necessary during the course of proof. Even further, we can consider the implementation of the logic with some non-monotonic reasoning techniques [4] so that only the explicit information in the databases have to be represented by the wffs. This will
be investigated in the further research.

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A Proof of the Proposition and Theorems

A.1 Proof of Proposition 1

1. By induction on \( m = |\delta(O)| \):

   - \( m = 1 \): this is trivial since we identify \([i_1]|\varphi\) and \(\{i_1\}|\varphi\) in our language.
   - Assume this result holds for all \( m \leq k \).
   - \( m = k + 1 \): there are two cases:

      [1] For the proposition and theorems, the reader should refer to the original sources for the detailed proofs and discussions. The reference list includes various authors and publications, each contributing to the understanding of conditional logic, belief revision, and other related topics in logic, information systems, and artificial intelligence.
$j = m$: $\vdash \neg [G_j] \varphi \supset ([O] \varphi \equiv [G_j] \varphi)$ is just an instance of axiom O1, $j < m$: let $O$ be written as $O' = i_m$ where $O' = i_1 \ldots > i_k$, then this proof is as follows:

1. $\neg [G_j] \bot \land [G_{j+1}] \bot \supset [G_m] \bot$  
2. $[G_m] \bot \supset ([O] \varphi \equiv [O'] \varphi)$  
3. $\neg [G_j] \bot \land [G_{j+1}] \bot \supset ([O] \varphi \equiv [O'] \varphi)$  
4. $\neg [G_j] \bot \land [G_{j+1}] \bot \supset ([O] \varphi \equiv [G_j] \varphi)$  
5. $\neg [G_j] \bot \land [G_{j+1}] \bot \supset ([O] \varphi \equiv [G_j] \varphi)$ 

$G3, m \geq j + 1$

2. By induction on $|\delta(O)|$: if $|\delta(O)| = 1$, this is an instance of G1. Assume this holds for modal operator $[O]$, let us consider the proof for $[O > i]$. Let $p$ and $G$ denote respectively $[\delta(O > i)] \bot$ and $\delta(O > i)$

1. $[O > i] \varphi \supset (\neg p \supset [G] \varphi)$  
2. $[O > i] (\varphi \supset \psi) \supset (\neg p \supset (\varphi \supset \psi))$  
3. $[O > i] \varphi \supset (p \supset [O] \varphi)$  
4. $[O > i] (\varphi \supset \psi) \supset (p \supset [O] (\varphi \supset \psi))$  
5. $[O > i] \varphi \land [O > i] (\varphi \supset \psi) \supset (\neg p \supset [G] \psi)$  
6. $[O > i] \varphi \land [O > i] (\varphi \supset \psi) \supset (p \supset [O] \psi)$  
7. $[O > i] \varphi \land [O > i] (\varphi \supset \psi) \supset (\neg p \supset [O > i] \psi)$  
8. $[O > i] \varphi \land [O > i] (\varphi \supset \psi) \supset (p \supset [O > i] \psi)$  
9. $[O > i] \varphi \land [O > i] (\varphi \supset \psi) \supset [O > i] \psi$  

$O1, G1, P, MP$

3. By induction on $|\delta(O)|$: if $|\delta(O)| = 1$, this is an instance of G2. Assume we have $\vdash \neg [O] \bot$, then the proof of $\vdash \neg [O > i] \bot$ is as follows:

1. $\neg [\delta(O > i)] \bot \supset ([O > i] \bot \supset [\delta(O > i)] \bot)$  
2. $[\delta(O > i)] \bot \supset ([O > i] \bot \supset [O] \bot)$  
3. $[O > i] \bot \supset [\delta(O > i)] \bot$  
4. $[O > i] \bot \supset ([\delta(O > i)] \bot \supset [O] \bot)$  
5. $[O > i] \bot \supset [O] \bot$  
6. $\neg [O] \bot$  
7. $\neg [O > i] \bot$ 

$O2, P, MP$

4. By induction on $|\delta(O)|$: if $|\delta(O)| = 1$, this is an instance of Gen rule. Assume it is the case for modal operator $[O]$, let us consider the proof for $[O > i]$

1. $\varphi$  
2. $[O] \varphi$  
3. $[\delta(O > i)] \varphi$  
4. $[O > i] \varphi \equiv ([\delta(O > i)] \varphi \lor [O] \varphi)$  
5. $[O > i] \varphi$ 

$Assumption, \text{Ind. Hyp.}, \text{Gen}, O1, O2, P, MP$

$\square$

A.2 Proof of Theorem 1

The proof of the theorem is based on that for $S55^{\nu}$ in [11, 12]. As usual, the verification of the soundness part is a routine checking, so we focus on the completeness part. Let $L$ denote a logical system. A wff $\varphi$ is $L$-inconsistent if its negation $\neg \varphi$ can be proved in $L$. Otherwise, $\varphi$ is $L$-consistent. A set $\Sigma$ of wffs is said to be $L$-inconsistent if there is a finite subset $\{ \varphi_1, \ldots, \varphi_k \} \subseteq \Sigma$ such that the wff $\varphi_1 \land \cdots \land \varphi_k$ is $L$-inconsistent; otherwise, $\Sigma$ is $L$-consistent. A maximal $L$-consistent set of wffs ($L$-MCS) is a consistent set $\chi$ of wffs such that whenever $\psi$ is a wff not in $\chi$, then $\chi \cup \{ \psi \}$ is $L$-inconsistent.

On the other hand, $\varphi$ is $L$-satisfiable iff there exists a $L$ model $M$ and a possible world $w$ such that $w \models_M \varphi$, otherwise $\varphi$ is $L$-unsatisfiable. Sometimes the prefix $L$ will be omitted without confusion. To prove the completeness, we will show that every DBF$_n$-consistent wff is DBF$_n$-satisfiable.
Let $\mathcal{I} = T\mathcal{O}_n \cup 2^{\{1, 2, \ldots, n\}} - \{\emptyset\}$ be the set of all modal operators for the language $DBF_n^c$. A pseudo $DBF_n^c$ structure is a tuple $(W, \{R^*_i\}_{i \in \mathcal{I}}, V)$ where $W$ and $V$ are defined as in $DBF_n^c$ models and each $R^*_i$ is a binary relation on $W$. Furthermore, it is required that $R^*_i$ is a serial relation for each $1 \leq i \leq n$. The satisfaction clauses for $DBF_n^c$ wffs in pseudo structures are defined as usual, so for example, we have $w \models \phi$ iff $u \models R^*_i(w)$, $u \models \phi$ when $i \models w$. What makes difference is that in a pseudo structures, each $R^*_i$ is considered as an independent relation instead of the intersection of other individual ones. A pseudo $DBF_n^c$ structure is a tuple $(W, \{R^*_i\}_{i \in \mathcal{I}}, V)$ where $W$ and $V$ are defined as in $DBF_n^c$ models and each $R^*_i$ is a binary relation on $W$. Furthermore, it is required that $R^*_i$ is a serial relation for each $1 \leq i \leq n$. The satisfaction clauses for $DBF_n^c$ wffs in pseudo structures are defined as usual, so for example, we have $w \models \phi$ iff $u \models R^*_i(w)$, $u \models \phi$ when $i \models w$. What makes difference is that in a pseudo structures, each $R^*_i$ is considered as an independent relation instead of the intersection of other individual ones. A pseudo $DBF_n^c$ structure is a pseudo model if all wffs provable in $DBF_n^c$ are valid in $M^*$. A $DBF_n^c$ wff $\phi$ is pseudo satisfiable if there exists a pseudo model $M^*$ and a possible world $w$ such that $w \models_{M^*} \phi$.

The following two results will be proved:

**Lemma A.1**

1. If $\phi$ is $DBF_n^c$-consistent, then $\phi$ is pseudo $DBF_n^c$-satisfiable.

2. If $\phi$ is pseudo $DBF_n^c$-satisfiable, then it is $DBF_n^c$-satisfiable.

The first result is proved by a standard canonical model construction procedure. A canonical pseudo structure $M^* = (W, \{R^*_i\}_{i \in \mathcal{I}}, V)$ is defined as follows

- $W = \{w_\chi \mid \chi \text{ is a DBF}_n^c\text{-MCS}\}$, in other words, each possible world corresponds precisely to a DBF$^c_n$-MCS.
- $R^*_i(w_\chi, w_\psi)$ iff $\chi / I \subseteq \psi$ for all $I \in \mathcal{I}$, where $\chi / I = \{\phi \mid [I]\phi \in \chi\}$.
- $V : \Phi_0 \rightarrow 2^W$ is defined by $V(p) = \{w_\chi \mid p \in \chi\}$.

The most important result for such construction is the truth lemma.

**Lemma A.2 (Truth lemma)** For any wff $\varphi$ and $DBF_n^c$-MCS $\chi$, we have $w_\chi \models_{M^*} \varphi$ iff $\varphi \in \chi$.

**Proof:** By induction on the structure of the wff, the only interesting case is the wff of the form $[I]\psi$ for some $I \in \mathcal{I}$. By definition, $w_\chi \models_{M^*} [I]\psi$ iff for all $w_\chi' \in R^*_i(w_\chi)$, $w_\chi' \models_{M^*} \psi$ iff for all $\chi / I \subseteq \chi'$, $\psi \in \chi'$ (by induction hypothesis) iff $\chi / I \cup \{-\psi\}$ is inconsistent iff $[I]\psi \in \chi$ when $[I]$ is a normal modal operator [4]. However, by the axioms P and G1, rules MP and Gen, and propositions [12] and [14], both kinds of modal operators $[O]$ and $[G]$ are normal ones.

Since every $DBF_n^c$-MCS contains all wffs provable in $DBF_n^c$, by the truth lemma, all provable wffs are valid in $M^*$. Furthermore, by axiom G2, each $R^*_i$ is serial for $1 \leq i \leq n$. Thus $M^*$ is indeed a pseudo model. If $\varphi$ is $DBF_n^c$-consistent, then there exists an MCS $\chi$ containing $\varphi$, so by the truth lemma, $w_\chi \models_{M^*} \varphi$, i.e., $\varphi$ is pseudo $DBF_n^c$-satisfiable. This proves the first part of lemma A.1.

Note that if $[I] = m$, then a pseudo model is in fact a model for the multi-agent epistemic logic $K_m$ [3]. The logic $K_m$ has $m$ modal operators corresponding to the knowledge or belief of $m$ independent agents. Admittedly, $m$ may be a very large number, however, it does not matter for the current purpose. What is important is that it can be shown that without loss of generality, we can assume a pseudo model is tree-like. The detail definition of a tree-like model and the proof that each pseudo model can be “unwound” into a tree-like one verifying the same set of valid wffs are rather technical and can be found in [12, pp.354] and [31, Exercise 3.30]. What is needed here is the property that in a tree-like model, if $I \neq J$, then $R^*_i \cap R^*_j = \emptyset$.

Thus, from now on, we can assume that if $\varphi$ is $DBF_n^c$-consistent, then $\varphi$ is pseudo $DBF_n^c$-satisfiable in a tree-like model $M^* = (W, \{R^*_i\}_{i \in \mathcal{I}}, V)$. The next step is to construct a $DBF_n^c$ model $M = (W, \{R_i\}_{1 \leq i \leq n}, V)$ from $M^*$ by defining $R_i = \bigcup_{g \in G} R^*_i$. Note that $R_i$ is serial since $R_i \supseteq R^*_i$ which is serial by the definition of pseudo models. From the definition, we can prove the following lemma.

**Lemma A.3** For any $w \in W$ and wff $\varphi$, $w \models_{M^*} \varphi$ iff $w \models_{M} \varphi$.

**Proof:** By induction on the structure of $\varphi$, the basis and classical cases are easy since both models have the same truth assignment function $V$. For the modal cases, if $\varphi = [G]\psi$, then $w \models_{M} [G]\psi$ iff for all $u \in \bigcap_{i \in G} R_i(u)$, $u \models_{M} \psi$ when $i \models w$. By definition of $R_i$, the tree-likeness of $M^*$, and the induction hypothesis, if $w \models_{M^*} [G]\psi$ (by definition of satisfaction in pseudo model $M^*$) iff $w \models_{M^*} [G]\psi$ (since axiom G3 is valid in $M^*$).

If $\varphi = [O]\psi$, then since proposition [4] is both valid in $M^*$ (by definition of pseudo models) and $M$ (by soundness), and by the case for modal operators $[G]$, we can find a $j$ such that $w$ satisfies the wff $\lnot [G_j] \perp \lnot [G_{j+1}] \perp \lnot$
(or just \(\neg [G_j] \perp\) in case of \(j = |\delta(O)|\)) in both \(M\) and \(M^*\), so it can be shown that \(w \models_M [O] \psi\) iff \(w \models_{M^*} [G_j] \psi\) iff \(w \models_{M^*} [G_j] \psi\) iff \(w \models_{M^*} [O] \psi\). □

This finishes the proof for the second part of lemma \(\text{A.1}\) and by combining the two parts, we have proved the completeness theorem for \(\text{DBF}_n^\omega\).

### A.3 Proof of Theorem \[2\]

The proof is very analogous to the previous one. What is different is that we do not have a counterpart for proposition \[1\] in the system \(\text{DBF}_n^\omega\). First, a pseudo \(\text{DBF}_n^\omega\) structure is analogously defined as a tuple \((W, (R_\Omega)_{\Omega \not\in \mathcal{T}C_n}, V)\) and it is required that \(R_{\{i\}}\) is serial for all \(1 \leq i \leq n\). Then a pseudo \(\text{DBF}_n^\omega\) model is a pseudo \(\text{DBF}_n^\omega\) structure in which all wffs provable in \(\text{DBF}_n^\omega\) are valid.

We still have to prove the following lemma.

**Lemma A.4** 1. If \(\varphi\) is \(\text{DBF}_n^\omega\)-consistent, then \(\varphi\) is pseudo \(\text{DBF}_n^\omega\)-satisfiable.

2. If \(\varphi\) is pseudo \(\text{DBF}_n^\omega\)-satisfiable, then it is \(\text{DBF}_n^\omega\)-satisfiable.

The first part of the lemma is proved exactly in the same way as in lemma \(\text{A.1}\). It can also be obtained that if \(\varphi\) is \(\text{DBF}_n^\omega\)-consistent, then \(\varphi\) is pseudo \(\text{DBF}_n^\omega\)-satisfiable in a tree-like model \(M^* = (W, (R_\Omega)_{\Omega \not\in \mathcal{T}C_n}, V)\).

However, the proof of the second part is somewhat different. Let us define the level of a modal operator \(\Omega\) as \(l(\Omega) = \max_{\delta|O} |\delta(O)|\) and the length of \(\Omega\) as \(z(\Omega) = \text{the number of elements } O \text{ in } \Omega \text{ such that } |\delta(O)| = l(\Omega)\). Then we define a function \(Ag^* : W \times (2^{TO} - \{\emptyset\}) \rightarrow (2^{\{1,2,...,n\}} - \emptyset)\) from the model \(M^*\) by

\[
Ag^*(w, \Omega) = \begin{cases}
\bigcup_{\Omega \subseteq \Omega} Ag^*(w, \{\Omega\}) & \text{if } |\Omega| > 1, \\
Ag^*(w, \{\Omega\}) \cup \{i\} & \text{if } \Omega = \{O > i\} \text{ and } w \models_{M^*} \neg \{(O, i)\} \perp, \\
Ag^*(w, \{\Omega\}) & \text{if } \Omega = \{O > i\} \text{ and } w \models_{M^*} \{(O, i)\} \perp, \\
\{i\} & \text{if } \Omega = \{i\}.
\end{cases}
\]

Note that since \(Ag^*(w, \Omega)\) is a subset of agents, it can also be used as a modal operator of level 1. We can now construct a \(\text{DBF}_n^\omega\) model \(M = (W, (R_i)_{1 \leq i \leq n}, V)\) from \(M^*\) such that \(R_i = \bigcup_{l(\Omega) = 1, i \in \Omega} R_\Omega\).

**Lemma A.5** 1. For all \(w \in W\), modal operators \(\Omega\), and wffs \(\varphi\), we have \(w \models_{M^*} [\Omega] \varphi \equiv [Ag^*(w, \Omega)] \varphi\).

2. \(R_\Omega(w) = \bigcup\{R^*_{\Omega'}(w) \mid l(\Omega') = 1 \wedge Ag^*(w, \Omega) \subseteq \Omega'\}\) for all \(w \in W\) and modal operators \(\Omega\), where \(R_\Omega\) is defined in section \[3\].

Proof:

1. By induction on the level of \(\Omega\):

   The basis case \(l(\Omega) = 1\): then by definition, \(Ag^*(w, \Omega) = \Omega\), so the result holds trivially.

   Assume the result holds for all \(\Omega\) such that \(l(\Omega) \leq k\),

   \[l(\Omega) = k + 1\]  by induction on the length of \(\Omega\):

   \(z(\Omega) = 1\): let \(\Omega = \{O > i\} \cup \Omega_1\), where \(l(\Omega_1) \leq k\), then \(Ag^*(w, \Omega) = Ag^*(w, \{O > i\}) \cup Ag^*(w, \Omega_1) = Ag^*(w, \{O > i\}) \cup Ag^*(w, \Omega_1) \cup \{i\} = Ag^*(w, \Omega_2)\) if \(w \models_{M^*} \neg \{(O, i)\} \perp\) and \(Ag^*(w, \{O, i\}) \cup Ag^*(w, \Omega_1) = Ag^*(w, \Omega_3)\) if \(w \models_{M^*} \{(O, i)\} \perp\), where \(\Omega_2 = \{O, i\} \cup \Omega_1\) and \(\Omega_3 = \{O\} \cup \Omega_1\). Since \(l(\Omega_2) = l(\Omega_3) = k\), then by induction hypothesis, we have \(w \models_{M^*} [\Omega_1] \varphi \equiv [Ag^*(w, \Omega_1)] \varphi\) for \(i = 2, 3\), so by axioms O1' and O2' (recall that all axioms are valid in a pseudo model), \(w \models_{M^*} [\Omega_2] \varphi \equiv [Ag^*(w, \Omega)] \varphi\) no matter whether \(w \models_{M^*} \{(O, i)\} \perp\) or not.

   Assume the result holds for all \(\Omega\) such that \(z(\Omega) \leq t\):

   \(z(\Omega) = t + 1\): the induction step is completely the same as in the basis case except that \(l(\Omega_2) = l(\Omega_3) = k + 1\) but \(z(\Omega_2) = z(\Omega_3) = t\).

2. By induction on \(l(\Omega)\):
Lemma A.6

For any $\omega \in W$ and wff $\phi$, $w \models_{M^*} \psi \iff w \models_M \phi$.

Proof: By induction on the structure of $\phi$, the only interesting case is $\phi = [\Omega]_w$.

\[
\begin{align*}
  w \models_{M^*} [\Omega]_w & \iff w \models_{M^*} [Ag^*(w, \Omega)]_w \quad \text{(lemma A.3.1)} \\
  & \iff w \models_{M^*} [\Omega']_w \quad \text{for all } \Omega' \text{ such that } l(\Omega') = 1 \text{ and } Ag^*(w, \Omega) \subseteq \Omega' \quad \text{(V3)} \\
  & \iff u \models_{M^*} \psi \forall u \in \bigcup \{R^*_{\Omega'}(w) \mid l(\Omega') = 1 \land Ag^*(w, \Omega) \subseteq \Omega'\} \\
  & \iff u \models_{M^*} \psi \quad \forall u \in R_{\Omega'}(w) \quad \text{(induction hypothesis and lemma A.3.2)} \\
  & \iff w \models_{M} [\Omega]_w
\end{align*}
\]

This completes the proof of the second part of lemma A.4 and the completeness theorem for DBF* n.