Approximating Subset Sum is equivalent to Min-Plus-Convolution

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Abstract

Approximating SUBSETSUM is a classic and fundamental problem in computer science and mathematical optimization. The state-of-the-art approximation scheme for SUBSETSUM computes a $(1 - \varepsilon)$-approximation in time $\tilde{O}(\min\{n/\varepsilon, n+1/\varepsilon^2\})$ [Gens, Levner’78, Kellerer et al.’97]. In particular, a $(1 - 1/n)$-approximation can be computed in time $O(n^2)$.

We establish a connection to the Min-Plus-Convolution problem, which is of particular interest in fine-grained complexity theory and can be solved naively in time $O(n^2)$. Our main result is that computing a $(1 - 1/n)$-approximation for SUBSETSUM is subquadratically equivalent to Min-Plus-Convolution. Thus, assuming the Min-Plus-Convolution conjecture from fine-grained complexity theory, there are no approximation schemes for SUBSETSUM with strongly subquadratic dependence on $n$ and $1/\varepsilon$. In the other direction, our reduction allows us to transfer known lower order improvements from Min-Plus-Convolution to SUBSETSUM, which yields a mildly subquadratic approximation scheme. This adds the first approximation problem to the list of Min-Plus-Convolution-equivalent problems.

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1 Introduction

In the SUBSETSUM problem the task is to decide, given a set $X$ of $n$ positive integers and a target number $t$, whether some subset of $X$ sums to $t$. This is a fundamental problem at the intersection of computer science, mathematical optimization, and operations research. It belongs to Karp’s initial list of 21 NP-complete problems [32], is a cornerstone of algorithm design [36, 43], and a special case of many other problems like KNAPSACK or INTEGER PROGRAMMING.

Since the problem is NP-hard, it is natural and imperative to study approximation algorithms. To this end, consider the following classic optimization version of SUBSETSUM: Given a set $X$ of $n$ positive integers and a target $t$, compute the maximum sum among all subsets summing to at most $t$. Formally, the task is to compute $\text{OPT} := \max\{\Sigma(Y) \mid Y \subseteq X, \Sigma(Y) \leq t\}$, where $\Sigma(Y)$ denotes the sum of all elements of $Y$. This optimization version is still a special case of KNAPSACK and INTEGER PROGRAMMING, and it naturally gives rise to the classic notion of an approximation scheme for SUBSETSUM: Given an instance of size $n$ and a parameter $\varepsilon > 0$, compute a number $R$ such that $(1 - \varepsilon)\text{OPT} \leq R \leq \text{OPT}$.

In 1975 Ibarra and Kim [28] designed the first approximation scheme for SUBSETSUM, running in time $O(n/\varepsilon^2)$. As claimed in [36] Section 4.6], a similar algorithm was found by Karp [33]. Lawler then presented an $O(n+1/\varepsilon^4)$-time algorithm [41]. These three algorithms in fact even work for the more general KNAPSACK problem. For SUBSETSUM, Gens and Levner designed further improved algorithms running in time $O(n/\varepsilon)$ [24, 25] and in time $O(\min\{n/\varepsilon, n + 1/\varepsilon^3\})$ [27]. Finally, Kellerer et al. presented an approximation scheme running in time $O\left(\min\{n/\varepsilon, n + 1/\varepsilon^2 \log(1/\varepsilon)\}\right)$ [35, 37]. This remains the state of the art for over 20 years. In particular, expressing the running time in the form $O((n + 1/\varepsilon)^c)$, the exponent $c = 2$ was achieved over 40 years ago [24, 25], but an exponent $c < 2$ remains elusive. This gives rise to the main question driving this paper:

Does SUBSETSUM admit an approximation scheme in time $\tilde{O}\left((n + 1/\varepsilon)^{2-\delta}\right)$ for some $\delta > 0$?

Pseudopolynomial-time Algorithms  Observe that an approximation scheme with a setting of $\varepsilon < 1/t$ solves SUBSETSUM exactly. For this reason, approximation algorithms are closely related to exact algorithms running in pseudopolynomial time. The classic pseudopolynomial time algorithm for SUBSETSUM is Bellman’s dynamic programming algorithm that runs in time $O(nt)$ [10]. This running time was recently improved, first by Kolliaris and Xu to $O\left(\tilde{O}\left(\min\{\sqrt{n} \cdot t, t^{1/3}\}\right)\right)$ [38], then further to a randomized $\tilde{O}(n + t)$ algorithm [12], see also [31] for further improvements in terms of logfactors. The new running time is optimal up to lower order factors; specifically there is no algorithm running in time $\text{poly}(n) t^{0.999}$ assuming the Strong Exponential Time Hypothesis [1] or the Set Cover Conjecture [21].

On the algorithmic side, these developments raise the hope that by generalizing the new algorithms one can design better approximation schemes, improving time $O(n/\varepsilon)$ to $\tilde{O}(n + 1/\varepsilon)$, analogously to the improvement from $O(nt)$ to $\tilde{O}(n + t)$.

On the lower bound side, from the connection of approximation algorithms and pseudopolynomial algorithms it directly follows that SUBSETSUM has no approximation scheme in time $\text{poly}(n)/\varepsilon^{0.999}$, assuming the Strong Exponential Time Hypothesis or the Set Cover Hypothesis. However, going beyond this lower bound seems difficult. Indeed, so far all conditional lower bounds for approximations schemes for SUBSETSUM, KNAPSACK, and related problems used the connection to pseudopolynomial time algorithms [44]. Since SUBSETSUM is in pseudopolynomial time $\tilde{O}(n + t)$, this connection cannot prove any higher lower bounds than $n + 1/\varepsilon$, up to lower order factors. In

\footnote{By $\tilde{O}(f)$ we hide factors of the form $\text{polylog}(nM)$, where $n$ is the number of input numbers and $M$ is the largest input number.}
some sense, proving a higher lower bound would thus be the first non-trivial lower bound for an approximation scheme.

**Min-Plus-Convolution** In this work we connect SubsetSum to the MinConv problem, in which we are given integer sequences $A, B \in \mathbb{Z}^n$ and the goal is to compute the sequence $C \in \mathbb{Z}^{2n}$ with $C[k] = \min_{i+j=k} A[i] + B[j]$. The naive running time of $O(n^2)$ can be improved to $n^2/2^{O(\sqrt{\log n})}$ by a reduction to All Pairs Shortest Path [11] and using Williams' algorithm for the latter [46]. Despite considerable attention [6, 7, 9, 11, 13, 14, 17, 22, 29, 39, 40], no $O(n^{2-\delta})$-time algorithm has been found, which was formalized as a hardness conjecture in fine-grained complexity theory [22, 39]. Many conditional lower bounds from the MinConv conjecture as well as several MinConv-equivalent problems are known, see, e.g., [7, 22, 29, 39, 40]. In particular, the Knapsack problem with weight budget $W$ can be solved in time $O((n + W)^{2-\delta})$ for some $\delta > 0$ if and only if MinConv can be solved in time $O(n^{2-\delta'})$ for some $\delta' > 0$ [22, 39].

**Our Contribution** We prove that computing a $(1 - 1/n)$-approximation for SubsetSum is equivalent to the MinConv problem, thus adding the first approximation problem to the list of known MinConv-equivalent problems. This negatively answers our main question for SubsetSum approximation schemes running in strongly subquadratic time, conditional on the MinConv conjecture. Moreover, our reductions allow us to transfer the known lower order improvements from MinConv to approximating SubsetSum, which yields the first algorithmic improvement in over 20 years. Finally, since we prove an equivalence, we precisely identify the two problems in terms of their subquadratic solvability, so one of the problems can be considered as closed.

2 Formal Statement of Results

**Problem Variants** Recall that an approximation scheme for SubsetSum asks to compute a number $R$ with $(1 - \varepsilon)OPT \leq R \leq OPT$, where $OPT = \max\{\Sigma(Y) \mid Y \subseteq X, \Sigma(Y) \leq t\}$. Beyond this standard approximation goal, one can define many different variants of approximating SubsetSum. For instance, computing such a number $R$ is not necessarily equivalent to computing a subset $Y \subseteq X$ summing to $R$. To avoid these details in the problem definition, in this paper we study the following two variants, which are in some sense the hardest and the simplest possible variants (subject to the strict constraint $\Sigma(Y) \leq t$).

- **HAPxSubsetSum**: Given $X, t$ and $\varepsilon > 0$, return any subset $Y \subseteq X$ satisfying $\Sigma(Y) \leq t$ and $\Sigma(Y) \geq \min\{OPT, (1 - \varepsilon)t\}$.

- **SAPxSubsetSum**: Given $X, t$ and $\varepsilon > 0$, distinguish whether $OPT = t$ or $OPT < (1 - \varepsilon)t$. If $OPT \in [(1 - \varepsilon)t, t)$ the output can be arbitrary.

Note that HAPxSubsetSum asks to compute $OPT$ exactly in case $OPT \leq (1 - \varepsilon)t$. The decision problem SAPxSubsetSum is the simplest formulation to work with for reducing the problem of approximating SubsetSum to further problems. Any algorithm for HAPxSubsetSum also solves SAPxSubsetSum, since if $OPT = t$ then the algorithm returns a set with sum in $[(1 - \varepsilon)t, t]$, while if $OPT < (1 - \varepsilon)t$ then the algorithm returns a set with sum $OPT < (1 - \varepsilon)t$.

Intuitively, HAPxSubsetSum is the hardest and SAPxSubsetSum is the simplest variant of approximating SubsetSum. There are several further variants that are intermediate between HAPxSubsetSum and SAPxSubsetSum, in the sense that any algorithm for HAPxSubsetSum also solves the intermediate variant and any algorithm for the intermediate variant also solves
SApXSubsetSum. Since we will prove HApXSubsetSum and SApXSubsetSum to be equivalent, all intermediate variants are also equivalent. Therefore, for the purpose of this paper we may concentrate on HApXSubsetSum and SApXSubsetSum. Examples of intermediate problem variants are as follows (we note that some of the reductions among these problem variants change \( \varepsilon \) by a constant factor):

- Return any value in \([(1 - \varepsilon)OPT, OPT]\).
- Return any subset \( Y \subseteq X \) with \((1 - \varepsilon)OPT \leq \Sigma(Y) \leq t\).
- If \( OPT = t \), compute a subset \( Y \subseteq X \) with \((1 - \varepsilon)t \leq \Sigma(Y) \leq t\), otherwise the output can be arbitrary.
- Distinguish whether \( OPT \geq (1 - \varepsilon/2)t \) or \( OPT < (1 - \varepsilon)t \). If \( OPT \in [(1 - \varepsilon)t, (1 - \varepsilon/2)t] \) the output can be arbitrary.

**Our Results** We prove an equivalence of approximating SUBSETSUM and MINCONV, by designing a reduction from HApXSubsetSum to MINCONV as well as a reduction from MINCONV to SApXSubsetSum. (The remaining reduction from SApXSubsetSum to HApXSubsetSum is trivial as discussed above.) The first reduction is as follows:

**Theorem 2.1** (Reduction: Approximation Algorithm). If MINCONV can be solved in time \( \tilde{O}(n^{2-\delta}) \) for some \( \delta > 0 \), then HApXSubsetSum can be solved in time \( \tilde{O}(n + 1/\varepsilon^{2-\delta}) \) by a randomized algorithm that is correct with high probability\(^2\).

We will show that our reduction even transfers the lower order improvements of the MINCONV algorithm that runs in time \( n^{2/2^{\Omega(\sqrt{\log n})}} \) \([11, 46]\). This yields the first improved approximation scheme for SUBSETSUM in over 20 years.

**Corollary 2.2** (Improved Approximation Algorithm). HApXSubsetSum can be solved by a randomized algorithm that is correct with high probability in time

\[
\tilde{O}\left(\left(n + \frac{(1/\varepsilon)^2}{2^{\Omega(\sqrt{\log(1/\varepsilon)})}}\right) \log^8 n\right).
\]

The second reduction is as follows:

**Theorem 2.3** (Reduction: Lower Bound). If SApXSubsetSum can be solved in time \( \tilde{O}((n + 1/\varepsilon)^{2-\delta}) \) for some \( \delta > 0 \), then MINCONV can be solved in time \( \tilde{O}(n^{2-\delta'}) \) for some \( \delta' > 0 \).

Under the MINCONV conjecture this rules out any significant further improvements for SUBSETSUM, specifically there are no approximation schemes for SUBSETSUM running in strongly subquadratic time \( \tilde{O}((n + 1/\varepsilon)^{2-\delta}) \) for any \( \delta > 0 \).

Our equivalence can be interpreted as closing the problem of approximating SUBSETSUM: We identify the subquadratic solvability of approximating SUBSETSUM and MINCONV, so further work on SUBSETSUM approximation schemes is not necessary, as it can be replaced by work on MINCONV (at least until a breakthrough for MINCONV is found, which might never happen). Moreover, our equivalence covers many different problem variants.

\(^2\)In this paper, “with high probability” means success probability \( 1 - (\varepsilon/n)^c \) for a constant \( c > 0 \) that can be freely chosen by adapting the constants in the algorithm.
Discussion of Weak Approximation and Partition In this paper we consider variants of approximating \textsc{SubsetSum} that keep the strict constraint $\Sigma(Y) \leq t$ intact. Mucha et al. \cite{Mucha} introduced a weaker variant of approximating \textsc{SubsetSum}, where they also relax this constraint from $\Sigma(Y) \leq t$ to $\Sigma(Y) \leq (1 + \varepsilon)t$. They showed that a weak approximation can be computed in time $\tilde{O}(n + 1/\varepsilon^{5/3})$, i.e., in strongly subquadratic time. This shows that the strict upper bound $\Sigma(Y) \leq t$ is crucial for our results. Indeed, their algorithm and our conditional lower bound separate their weak approximation from the variants of approximation studied in this paper.\footnote{The lower bound that we present in this paper was mentioned as private communication with Bringmann in \cite{Mucha}.}

Moreover, the \textsc{Partition} problem is the special case of \textsc{SubsetSum} where the target is half of the total input sum, i.e., $t = \Sigma(X)/2$. Any weak approximation scheme for \textsc{SubsetSum} yields a (standard) approximation scheme for \textsc{Partition} \cite{Mucha}. Therefore, we also separate the classic problems \textsc{SubsetSum} and \textsc{Partition} with respect to their approximability, conditional on the MinConv conjecture. This is the only separation of these problems that the author is aware of.

Technical Overview In our lower bound for SAPx\textsc{SubsetSum}, we start from the known reduction from MinConv to Knapsack \cite{12,31,38}, and design a surprisingly simple reduction from (exact) Knapsack to SAPx\textsc{SubsetSum}. In this reduction, we use the strict condition $\Sigma(Y) \leq t$ in an interesting way, to simulate exact inequality checks on sums of very large numbers, despite being in an approximate setting. This allows us to embed the potentially very large values of Knapsack items into an instance of SAPx\textsc{SubsetSum}.

The other reduction is essentially an approximation scheme for \textsc{SubsetSum}, using as black box an algorithm for MinConv with subquadratic running time $T_{\text{MinConv}}(n)$.

It seems difficult to adapt the known approximation schemes \cite{24,25,27,35,37} to make use of a MinConv algorithm, since they all in some way follow Bellman’s iteration. That is, writing $X = \{x_1, \ldots, x_n\}$, they compute an approximation of the set of all subset sums of $\{x_1, \ldots, x_i\}$ from an approximation of the set of all subset sums of $\{x_1, \ldots, x_{i-1}\}$, for $i = 1, \ldots, n$. To obtain total time $\tilde{O}(n/\varepsilon)$, each iteration must run in time $\tilde{O}(1/\varepsilon)$, so there is no point at which a subquadratic algorithm for MinConv seems useful. (The $\tilde{O}(n + 1/\varepsilon^2)$-time approximation schemes follow the same iteration, but start with a preprocessing step that removes all but $\tilde{O}(1/\varepsilon)$ items.)

In contrast, the recent pseudopolynomial algorithms for \textsc{SubsetSum} \cite{12,31,38} use convolution methods, so in principle their structure allows to plug in a MinConv algorithm. Moreover, the running time $\tilde{O}(n + t)$ \cite{12,31} suggests to replace standard convolution in time $\tilde{O}(t)$ by MinConv in time $T_{\text{MinConv}}(1/\varepsilon)$, to obtain the desired running time of $\tilde{O}(n + T_{\text{MinConv}}(1/\varepsilon))$. However, all previous algorithms along this line of research heavily assume an exact setting, specifically that we have computed exact solutions to subproblems.

Here, we bring these two approaches together, by using ideas from the known approximation schemes to define the right notion of approximation, and then following the high-level structure of the pseudopolynomial algorithms with adaptations for our notion of approximation.

In our notion of approximation, we say that a set $A$ approximates a set $B$ if for any $b \in B$ there are lower and upper approximations $a^-, a^+ \in A$ with $a^- \leq b \leq a^+ \leq a^- + \varepsilon t$. To avoid having to solve \textsc{SubsetSum} exactly, we need to relax this notion further, by allowing $a^+$ to take the value $t + 1$, for details see Definition 5.1. We establish that this notion satisfies several natural properties, e.g., it is transitive and behaves nicely under unions and sumsets.

The connection to MinConv is then as follows. The main subroutine used by the recent pseudopolynomial algorithms is sumset computation: Given $A, B \subseteq \mathbb{N}$, compute $A + B = \{a + b \mid a \in A, b \in B\}$. Since the output-size of sumset computation can be quadratic in the input-size, here we relax the goal and design a subroutine for approximate sumset computation, computing a
set \( R \) that approximates \( A + B \). To implement this subroutine, we first define a rasterization of \( A \) and \( B \) as vectors \( A', B' \) with

\[
A'[i] := \min (A \cap [i\varepsilon t/2, (i+1)\varepsilon t/2]) \quad B'[j] := \min (B \cap [j\varepsilon t/2, (j+1)\varepsilon t/2]).
\]

We then compute the vector \( C' \) as the \( \text{MinConv} \) of \( A' \) and \( B' \), that is,

\[
C'[k] = \min_{i+j=k} A'[i] + B'[j].
\]

Note that we used the operation \( \min \) at three positions. By replacing some of them by \( \max \), we obtain \( 2^3 = 8 \) similar expressions, giving rise to vectors \( C'_1, \ldots, C'_8 \). We show that the set of all entries of \( C'_1, \ldots, C'_8 \) approximates \( A + B \) according to our notion of approximation. Since all involved vectors have length \( O(1/\varepsilon) \), we can approximate sumsets in time \( O(T_{\text{MinConv}}(1/\varepsilon)) \).

Finally, we use this approximate subset computation as a subroutine and follow (a simplified variant of) the pseudopolynomial algorithm from [12]. The pseudocode is not changed much compared to [12], but the correctness proofs are significantly more complicated.

### 2.1 Related Work

Our reduction from \( \text{MinConv} \) to approximating \( \text{SUBSETSUM} \) follows a recent trend in fine-grained complexity theory to show hardness of approximation. The first such result was presented by Abboud et al. [2], who proved a PCP-like theorem for problems in \( P \) and obtained hardness of approximation for Orthogonal Vectors. Their results were extended to an equivalence and improved quantitatively (see, e.g., [18–20]) and generalized to parameterized complexity (see, e.g., [15, 34]).

A similar approach was used on All Pairs Shortest Path [13]. While this line of research developed techniques to prove conditional lower bounds for constant-factor approximation algorithms (and higher approximation ratios), in this paper for the first time we obtain a conditional lower bound for an approximation scheme, which does not already follow from a lower bound for a constant-factor approximation.

Approximations schemes for the related \( \text{KNAPSACK} \) and \( \text{PARTITION} \) problems have also been widely studied, see, e.g., [16, 26, 28, 36, 41]. For \( \text{KNAPSACK} \), the state-of-the-art approximation scheme runs in time \( \tilde{O}(n + 1/\varepsilon^{9/4}) \) [30], see also [16]. A lower bound of \( (n + 1/\varepsilon)^{2-o(1)} \) holds assuming the \( \text{MinConv} \) conjecture [44]; our conditional lower bound in this paper can be seen as an extension of this result to \( \text{SUBSETSUM} \). For \( \text{PARTITION} \), the state-of-the-art approximation scheme runs in time \( \tilde{O}(n + 1/\varepsilon^{5/3}) \) [44], and the connection of approximation schemes and pseudopolynomial algorithms shows that there is no \( \text{poly}(n)/\varepsilon^{0.999} \)-time algorithm assuming the Strong Exponential Time Hypothesis [11] or the Set Cover Conjecture [21]. The remaining gaps are interesting open problems to work on. Note that our results in this paper yield the first matching upper and lower bounds for approximation schemes for one of the classic \( \text{KNAPSACK} \)-type problems \( \text{PARTITION} \), \( \text{SUBSETSUM} \), and \( \text{KNAPSACK} \).

Further related work on \( \text{SUBSETSUM} \) includes an improved pseudopolynomial time algorithm for \( \text{MODULARSUBSETSUM} \) [5], see also [9, 11, 23, 42, 45] for more recent results on \( \text{SUBSETSUM} \).

### 3 Preliminaries

We write \( \mathbb{N} = \{0, 1, 2, \ldots\} \). For \( t \in \mathbb{N} \) we let \( [t] = \{0, 1, \ldots, t\} \). For sets \( A, B \subseteq \mathbb{N} \) we define their \emph{sumset} as \( A + B = \{a + b \mid a \in A, b \in B\} \) and their \emph{capped sumset} as \( A +_t B = (A + B) \cap [t] \).
We use $\Sigma(Y)$ as shorthand notation for $\sum_{y \in Y} y$, and we denote the set of all subset sums of $X$ below $t$ by $S(X; t) := \{\Sigma(Y) \mid Y \subseteq X, \Sigma(Y) \leq t\}$.

Recall that in MinConv we are given integer sequences $A = (A[0], \ldots, A[n-1])$ and $B = (B[0], \ldots, B[n-1])$ and the goal is to compute the sequence $C = (C[0], \ldots, C[2n-1])$ with $C[k] = \min_{i+j=k} A[i] + B[j]$, where the minimum ranges over all pairs $(i,j)$ with $0 \leq i, j < n$ and $i + j = k$. We assume that all entries of $A$ and $B$ are from some range $\{1, \ldots, M\}$ and that arithmetic operations on $O(\log M + \log n)$-bit numbers can be performed in constant time. This holds, e.g., in the RAM model with $\Theta(\log n)$-bit memory cells if $M = n^{O(1)}$.

Throughout the paper, by $\mathcal{O}$-notation we hide factors of the form $\text{polylog}(n, M)$, where $n$ is the number of input numbers and $M$ is the largest input number. For technical reasons, we assume all time bounds $T(\cdot)$ to satisfy $T(\mathcal{O}(n)) = \mathcal{O}(T(n))$ (and similarly for multivariate functions). This is a natural assumption in the polynomial time world as well as in the pseudopolynomial setting.

4 Lower Bounds

Our reduction from MinConv to SAPXSubsetSum (Theorem 2.3) goes via the Knapsack problem: Given $n$ items with weights $w_1, \ldots, w_n$ and values $v_1, \ldots, v_n$, and given a weight budget $W$ and a value goal $V$, decide whether for some $S \subseteq [n]$ we have $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i \geq V$. We again denote by $M$ the largest input number, that is $w_i, v_i, W, V \in \{1, \ldots, M\}$.

Bellman’s classic dynamic programming algorithm solves Knapsack in time $O(nW)$ [10]. In the regime $W \approx n$, this running time is $O(n^2)$ and any improvement to time $\mathcal{O}(n^{2-\varepsilon})$ would violate the MinConv conjecture, as was shown independently by Cygan et al. [22] Theorem 2 and Künnemann et al. [39] Theorem 4.8]. Specifically, both references contain the following reduction.

**Theorem 4.1.** If Knapsack can be solved in time $T(n, W)$, then MinConv can be solved in time $\tilde{O}(T(\sqrt{n}, \sqrt{n}) \cdot n)$.

We next show a simple reduction from (exact) Knapsack to SAPXSubsetSum.

**Theorem 4.2.** If SAPXSubsetSum can be solved in time $T(n, 1/\varepsilon)$, then Knapsack can be solved in time $\tilde{O}(T(n, W) + W)$.

Since any algorithm for HAPXSubsetSum also solves SAPXSubsetSum (as described in the introduction), the same statement also holds for HAPXSubsetSum. In combination with Theorem 4.1 we obtain that if SAPXSubsetSum can be solved in time $\tilde{O}((n + 1/\varepsilon)^{2-\delta})$ for some $\delta > 0$, then MinConv can be solved in time $\tilde{O}((n^{2-\delta}) \cdot n) = \tilde{O}(n^{2-\delta/2})$.

**Proof.** Given a Knapsack instance, if $n \leq \log M$ then we run Bellman’s algorithm to solve the instance in time $\mathcal{O}(nW) = \mathcal{O}(W \log M) = \tilde{O}(W)$. Therefore, from now on we assume $n \geq \log M$.

We construct an intermediate Knapsack instance by adding items of weight $2^i$ and value 0 for each $0 \leq i \leq \log(W)$, and adding items of weight 0 and value $-2^i$ for each $0 \leq i \leq \log(V)$. Since items of value less than or equal to 0 can be ignored, these additional items do not change whether the Knapsack instance has a solution. However, in case there is a solution, we can use these additional items to fill up the total weight to exactly $W$ and decrease the total value to exactly $V$. In other words, if the Knapsack instance has a solution, then it also has a solution of total weight $W$ and total value $V$. This increases the number of items by an additional $\mathcal{O}(\log M) = \mathcal{O}(n)$, since as discussed above we can assume $n \geq \log M$. As this is only an increase by a constant factor, with slight abuse of notation we still use $n$ to denote the number of items, and we denote the weights and values of the resulting items by $w_1, \ldots, w_n$ and $v_1, \ldots, v_n$, respectively.
We note that negative values are only used in the intermediate Knapsack instance, and that all weights and values are still bounded by $M$, now in absolute value. The constructed SubsetSum instance does not contain any negative numbers, as can be checked from the following construction.

We set $M' := 4nM$ and let $X$ consist of the numbers $x_i := w_i \cdot M' - v_i$ for $1 \leq i \leq n$. For $t := W \cdot M' - V$ and $\varepsilon := 1/(2W)$ we construct the $\text{SAPxSubsetSum}$ instance $(X, t, \varepsilon)$.

To argue correctness of this reduction, first suppose that the Knapsack instance has a solution. Using the added items, there is a set $S \subseteq [n]$ such that $\sum_{i \in S} w_i = W$ and $\sum_{i \in S} v_i = V$. The corresponding subset $\{x_i \mid i \in S\} \subseteq X$ thus sums to $W \cdot M' - V = t$. Hence, if the Knapsack instance has a solution, then the constructed SubsetSum instance satisfies $\text{OPT} = t$.

For the other direction, suppose that the Knapsack instance has no solution, and consider any set $S \subseteq [n]$. Then we are in one of the following three cases:

- **Case 1:** $\sum_{i \in S} w_i \geq W + 1$. Then we have $\sum_{i \in S} x_i \geq (W + 1)M' - \sum_{i \in S} v_i$. Since $\sum_{i \in S} v_i \leq nM = M'/4$ we obtain $\sum_{i \in S} x_i > WM' > t$. Hence, $S$ does not correspond to a feasible solution of the constructed SubsetSum instance.

- **Case 2:** $\sum_{i \in S} w_i = W$. Since the Knapsack instance has no solution, we have $\sum_{i \in S} v_i < V$. This yields $\sum_{i \in S} x_i = WM' - \sum_{i \in S} v_i > WM' - V = t$. Hence, $S$ does not correspond to a feasible solution of the constructed SubsetSum instance.

- **Case 3:** $\sum_{i \in S} w_i \leq W - 1$. Then we have $\sum_{i \in S} x_i \leq (W - 1)M' - \sum_{i \in S} v_i$. Since $|\sum_{i \in S} v_i| \leq nM = M'/4$, we obtain $\sum_{i \in S} x_i \leq WM' - 0.75M'$. Using $V \leq M \leq M'/4$ yields $\sum_{i \in S} x_i \leq WM' - V - 0.5M'$. Finally, since $0.5M' = \varepsilon WM' > \varepsilon (WM' - V)$, we obtain $\sum_{i \in S} x_i < (1 - \varepsilon)(WM' - V) = (1 - \varepsilon)t$.

From this case distinction we obtain that any subset $Y \subseteq X$ is either infeasible, i.e., $\Sigma(Y) > t$, or sums to less than $(1 - \varepsilon)t$. Hence, if the Knapsack instance has no solution, then the constructed SubsetSum instance satisfies $\text{OPT} < (1 - \varepsilon)t$.

It follows that solving $\text{SAPxSubsetSum}$ on the constructed instance decides whether the given Knapsack instance has a solution, which shows correctness.

Finally, we analyze the running time. Since $\varepsilon = 1/(2W)$, invoking a $T(n, 1/\varepsilon)$-time $\text{SAPxSubsetSum}$ algorithm on the constructed instance takes time $T(n, 2W) = O(T(n, W))$. Together with the first paragraph, we obtain total time $O(T(n, W) + W \log M)$ for Knapsack.

**Remark 4.3.** Knapsack can be solved in time $\tilde{O}(n + t^2)$ as follows. Note that any solution contains at most $t/w$ items of weight exactly $w$. We may therefore remove all but the $t/w$ most profitable items of any weight $w$. The number of remaining items is at most $\sum_{w=1}^t t/w = \tilde{O}(t)$. After this $\tilde{O}(n)$-time preprocessing, the classic $O(nt)$-time algorithm runs in time $\tilde{O}(t^2)$.

In particular, Knapsack is in time $\tilde{O}(\min\{nt, n + t^2\})$. This nicely corresponds to the best-known running time of $\tilde{O}(\min\{n/\varepsilon, n + 1/\varepsilon^2\})$ for $\text{HAPxSubsetSum}$ [24, 25, 35, 37]. Our reduction indeed transforms the latter time bounds into the former.

## 5 Algorithm for Approximating Subset Sum

Throughout this section we assume to have access to an algorithm for $\text{MinConv}$ running in time $T_{\text{MinConv}}(n)$ on sequences of length $n$. We will use this as a black box to approximate $\text{SubsetSum}$. 


5.1 Preparations

We start by defining and discussing our notion of approximation. We will set $\Delta = \varepsilon t$ in the end.

**Definition 5.1 (Approximation).** Let $t, \Delta \in \mathbb{N}$. For any $A \subseteq \{t\}$ and $b \in \mathbb{N}$, we define the lower and upper approximation of $b$ in $A$ (with respect to universe $\{t\}$) as

$$\text{apx}_-^t(b, A) := \max\{a \in A \cup \{t + 1\} \mid a \leq b\} \quad \text{and} \quad \text{apx}_+^t(b, A) := \min\{a \in A \cup \{t + 1\} \mid a \geq b\}.$$

For $A, B \subseteq \mathbb{N}$, we say that $A$ $(t, \Delta)$-approximates $B$ if $A \subseteq B \subseteq \{t\}$ and for any $b \in B$ we have

$$\text{apx}_+^t(b, A) - \text{apx}_-^t(b, A) \leq \Delta.$$

Note that the approximations of $b$ in $A$ are not necessarily elements of $A$, since we add $t + 1$. We will sometimes informally say that “$b$ has good approximations in $A$”, with the meaning that $\text{apx}_+^t(b, A) - \text{apx}_-^t(b, A) \leq \Delta$ instead of the more usual $\text{apx}_+^t(b, A) - b \leq \Delta$ and $b - \text{apx}_-^t(b, A) \leq \Delta$. For an example were this detail is crucially used see the proof of Lemma 5.2 below. This aspect of our definition is inspired by the approximation algorithm of Kellerer et al. [35, 37].

Second, we change the upper end by adding $t + 1$ to $A$. This relaxation is necessary because our goal will be to compute a set $A$ that $(t, \Delta)$-approximates the set of all subset sums of $X$ below $t$, or more precisely the set $\mathcal{S}(X; t) := \{\Sigma(Y) \mid Y \subseteq X, \Sigma(Y) \leq t\}$. Computing $\max(\mathcal{S}(X; t))$ means to solve SUBSETSUM exactly and is thus NP-hard. Therefore, we need a notion of approximation that does not force us to determine $\max(\mathcal{S}(X; t))$, which is achieved by relaxing the upper end.

We start by establishing some basic properties of our notion of approximation.

**Lemma 5.2 (Transitivity).** If $A$ $(t, \Delta)$-approximates $B$ and $B$ $(t, \Delta)$-approximates $C$, then $A$ $(t, \Delta)$-approximates $C$.

*Proof.* Since $A \subseteq B$ and $B \subseteq C$ we obtain $A \subseteq C$. For any $c \in C$, let $b^-$ and $b^+$ be the lower and upper approximations of $c$ in $B$. Note that $b^-, b^+ \in B \cup \{t + 1\}$. For any $b \in B$, since $A$ $(t, \Delta)$-approximates $B$ we find good approximations of $b$ in $A$. Additionally, for $b = t + 1$ one can check that $\text{apx}_+^t(b, A) = \text{apx}_-^t(b, A) = t + 1$, and thus we also have $\text{apx}_+^t(b, A) - \text{apx}_-^t(b, A) \leq \Delta$. Therefore, $b^-$ and $b^+$ both have good approximations in $A$. So let $a^-$ and $a^+$ be the lower and upper approximations of $b^-$ in $A$, and similarly let $a^{++}$ and $a^{++}$ be the lower and upper approximations of $b^+$ in $A$. If $c \leq a^-$, then $a^- \leq c \leq a^+$ form approximations of $c$ in $A$ within distance $\Delta$. Similarly, if $a^+ \leq c$, then $a^+ \leq c \leq a^{++}$ form approximations of $c$ in $A$ within distance $\Delta$. In the remaining case we have

$$a^- \leq b^- \leq a^- \leq c \leq a^+ \leq b^+ \leq a^{++}.$$  

It follows that $a^{++} \leq c \leq a^+$ form approximations of $c$ in $A$ that are within distance $\Delta$, since they are sandwiched between $b^-$ and $b^+$. \hfill $\square$

**Lemma 5.3.** If $A \subseteq B \subseteq C$ and $A$ $(t, \Delta)$-approximates $C$, then $B$ $(t, \Delta)$-approximates $C$.

*Proof.* We have $B \subseteq C$, and for any $c \in C$ its approximations in $B$ are at least as good as in $A$. \hfill $\square$

Our notion of approximation also behaves nicely under unions and sumsets, as shown by the following two lemmas.
Lemma 5.4 (Union Property). If $A_1$ $(t, \Delta)$-approximates $B_1$ and $A_2$ $(t, \Delta)$-approximates $B_2$, then $A_1 \cup A_2$ $(t, \Delta)$-approximates $B_1 \cup B_2$.

Proof. Let $r \in \{1, 2\}$, $b \in B_r$. The approximations of $b$ in $A_1 \cup A_2$ are at least as good as in $A_r$. \qed

Lemma 5.5 (Sumset Property). If $A_1$ $(t, \Delta)$-approximates $B_1$ and $A_2$ $(t, \Delta)$-approximates $B_2$, then $A_1 + A_2$ $(t, \Delta)$-approximates $B_1 + B_2$.

Proof. Since $A_r \subseteq B_r$ we obtain $A_1 + A_2 \subseteq B_1 + B_2$. So consider any $b_1 \in B_1, b_2 \in B_2$ and set $b := b_1 + b_2$. Let $a_1^-, a_1^+$ be the lower and upper approximations of $b_1$ in $A_1$, and let $a_2^-, a_2^+$ be the lower and upper approximations of $b_2$ in $A_2$. Consider the intervals

$$L := [a_1^- + a_2^-, a_1^+ + a_2^-] \cap [t + 1] \quad \text{and} \quad R := (a_1^+ + a_2^- + a_1^- + a_2^+) \cap [t + 1].$$

Note that the endpoints of $L$ and $R$ are contained in $(A_1 + A_2) \cup \{t + 1\}$. Moreover, the interval $L$ has length at most $a_1^- - a_1^- \leq \Delta$, and similarly $R$ has length at most $a_2^+ - a_2^- \leq \Delta$. Finally, since

$$a_1^- + a_2^- \leq b_1 + b_2 \leq a_1^+ + a_2^+,$$

we have $b \in L \cup R$, and thus either $b \in L$ or $b \in R$. The endpoints of the respective interval containing $b$ thus form lower and upper approximations of $b$ in $A_1 + A_2$ within distance $\Delta$. \qed

We next show that we can always assume approximation sets to have small size, or more precisely to be locally sparse.

Definition 5.6 (Sparsity). Let $A \subseteq \mathbb{N}$ and $\Delta \in \mathbb{N}$. We say that $A$ is $\Delta$-sparse if $|A \cap [x, x + \Delta]| \leq 2$ holds for any $x \in \mathbb{N}$. If $A$ is $\Delta$-sparse and $A$ $(t, \Delta)$-approximates $B$, we say that $A$ sparsely $(t, \Delta)$-approximates $B$.

Lemma 5.7 (Sparsification). Given $t, \Delta \in \mathbb{N}$ and a set $B \subseteq [t]$, in time $O(|B|)$ we can compute a set $A$ such that $A$ sparsely $(t, \Delta)$-approximates $B$.

Proof. Recall that our notion of approximation requires $A$ to be a subset of $B$. We inductively argue as follows. Initially, for $A := B$ it holds that $A$ $(t, \Delta)$-approximates $B$. If there exist $a_1, a_2, a_3 \in A$ with $a_1 < a_2 < a_3 \leq a_1 + \Delta$, remove $a_2$ from $A$. We claim that the resulting set $A$ still $(t, \Delta)$-approximates $B$. Indeed, consider any $b \in B$. If $b \leq a_1$ we have $\text{apx}_2^- (b, A) \leq \text{apx}_1^+ (b, A) \leq a_1$, and thus $a_2$ is irrelevant. Similarly, $a_2$ is also irrelevant for any $b \geq a_3$. Finally, for any $a_1 < b < a_3$, after removing $a_2$ we have $\text{apx}_2^- (b, A) \geq a_1$ and $\text{apx}_1^+ (b, A) \leq a_3$, and $a_3 - a_1 \leq \Delta$. Thus, after removing $a_2$ the set $A$ still $(t, \Delta)$-approximates $B$. Repeating this rule until it is no longer applicable yields a subset $A \subseteq B$ that contains at most two numbers in any interval $[x, x + \Delta]$.

Finally, it is easy to compute $A$ in time $O(|B|)$ by one sweep from left to right, assuming that $B$ is given in sorted order. Pseudocode for this is given in Algorithm 7\textsuperscript{4}.

\textsuperscript{4}Here and in the following, we assume that input sets such as $B$ are given as a sorted list of their elements.

Proof. Let $r \in \{1, 2\}$, $b \in B_r$. The approximations of $b$ in $A_1 \cup A_2$ are at least as good as in $A_r$. \qed

We next show that we can always assume approximation sets to have small size, or more precisely to be locally sparse.

Definition 5.6 (Sparsity). Let $A \subseteq \mathbb{N}$ and $\Delta \in \mathbb{N}$. We say that $A$ is $\Delta$-sparse if $|A \cap [x, x + \Delta]| \leq 2$ holds for any $x \in \mathbb{N}$. If $A$ is $\Delta$-sparse and $A$ $(t, \Delta)$-approximates $B$, we say that $A$ sparsely $(t, \Delta)$-approximates $B$.

Lemma 5.7 (Sparsification). Given $t, \Delta \in \mathbb{N}$ and a set $B \subseteq [t]$, in time $O(|B|)$ we can compute a set $A$ such that $A$ sparsely $(t, \Delta)$-approximates $B$.

Proof. Recall that our notion of approximation requires $A$ to be a subset of $B$. We inductively argue as follows. Initially, for $A := B$ it holds that $A$ $(t, \Delta)$-approximates $B$. If there exist $a_1, a_2, a_3 \in A$ with $a_1 < a_2 < a_3 \leq a_1 + \Delta$, remove $a_2$ from $A$. We claim that the resulting set $A$ still $(t, \Delta)$-approximates $B$. Indeed, consider any $b \in B$. If $b \leq a_1$ we have $\text{apx}_2^- (b, A) \leq \text{apx}_1^+ (b, A) \leq a_1$, and thus $a_2$ is irrelevant. Similarly, $a_2$ is also irrelevant for any $b \geq a_3$. Finally, for any $a_1 < b < a_3$, after removing $a_2$ we have $\text{apx}_2^- (b, A) \geq a_1$ and $\text{apx}_1^+ (b, A) \leq a_3$, and $a_3 - a_1 \leq \Delta$. Thus, after removing $a_2$ the set $A$ still $(t, \Delta)$-approximates $B$. Repeating this rule until it is no longer applicable yields a subset $A \subseteq B$ that contains at most two numbers in any interval $[x, x + \Delta]$.

Finally, it is easy to compute $A$ in time $O(|B|)$ by one sweep from left to right, assuming that $B$ is given in sorted order. Pseudocode for this is given in Algorithm 7\textsuperscript{4}.

\textsuperscript{4}Here and in the following, we assume that input sets such as $B$ are given as a sorted list of their elements.
Algorithm 1 Sparsification($B, t, \Delta$): Given $t, \Delta > 0$ and a set $B \subseteq [t]$ in sorted order, compute a set $A$ that sparsely $(t, \Delta)$-approximates $B$. We denote the elements of $B$ by $B[1], \ldots, B[m]$.

1: Initialize $A := \emptyset$ and $n := 0$
2: for $i = 1, \ldots, m$ do
3: \hspace{1cm} $n := n + 1$
4: \hspace{1cm} $A[n] := B[i]$
5: \hspace{1cm} if $n \geq 3$ and $A[n] - A[n - 2] \leq \Delta$ then
6: \hspace{2cm} $A[n - 1] := A[n]$
7: \hspace{1cm} $n := n - 1$
8: return $\{A[1], \ldots, A[n]\}$

Proof. We write $A' := A \cap [t']$ and $B' := B \cap [t']$. Clearly we have $A' \subseteq B'$. For any $b \in B'$, note that

$$\text{apx}_t^-(b, A') = \text{apx}_t^-(b, A),$$

since only elements larger than $b$ are removed from $A$ and $t + 1 \geq t' + 1 > b$. Moreover, if $\text{apx}_t^+(b, A) \leq t'$ then $\text{apx}_t^+(b, A) \in A'$, and thus $\text{apx}_t^+(b, A') = \text{apx}_t^+(b, A)$. Otherwise, we have $\text{apx}_t^+(b, A') \leq t' + 1 \leq \text{apx}_t^+(b, A)$. In any case, we have $\text{apx}_t^+(b, A') \leq \text{apx}_t^+(b, A)$, and therefore

$$\text{apx}_t^+(b, A') - \text{apx}_t^-(b, A') \leq \text{apx}_t^+(b, A) - \text{apx}_t^-(b, A) \leq \Delta.$$

Lemma 5.9 (Up Shifting). Let $t, t', \Delta \in \mathbb{N}$ with $t \leq t'$. If $A (t, \Delta)$-approximates $B$, then there exists $u \in (t - \Delta, t]$ such that $A (t', \Delta)$-approximates $B \cap [u]$.

Proof. Note that $A \subseteq B \subseteq [t]$. We consider two cases.

Case 1: If $\max(A) \leq t - \Delta$ then we set $u := t$, so that $B \cap [u] = B$. For any $b \in B$ by assumption we have $\text{apx}_t^+(b, A) - \text{apx}_t^-(b, A) \leq \Delta$. Since $\max(A) \leq t - \Delta$, we have $\text{apx}_t^-(b, A) \leq t - \Delta$ and thus $\text{apx}_t^+(b, A) \leq t$. It follows that $\text{apx}_t^+(b, A) \in A$ (instead of being $t+1$). Therefore, the same elements $\text{apx}_t^-(b, A), \text{apx}_t^+(b, A) \in A$ also form good approximations of $b$ in $A$ with respect to universe $[t']$. Thus, $A (t', \Delta)$-approximates $B \cap [u]$.

Case 2: If $\max(A) \in (t - \Delta, t]$ then we set $u := \max(A)$. For any $b \in B \cap [u]$ we have $b \leq \max(A)$, and thus the upper approximation of $b$ in $A$ is simply the smallest element of $A$ that is at least $b$. As this is independent of the universe $[t]$, we obtain $\text{apx}_t^+(b, A) = \text{apx}_t^+(b, A)$. We also clearly have $\text{apx}_t^-(b, A) = \text{apx}_t^-(b, A)$. Hence, $\text{apx}_t^+(b, A) - \text{apx}_t^-(b, A) = \text{apx}_t^+(b, A) - \text{apx}_t^-(b, A) \leq \Delta$, so $A (t', \Delta)$-approximates $B \cap [u]$.

5.2 Algorithm for Sunset Computation

We now present the main connection to MINCONV: We show how to compute for given $A_1, A_2$ a set $A$ that approximates $A_1 + A_2$, by performing two calls to MINCONV. At first we set $t := \infty$, so that we do not have to worry about the upper end. This will be fixed in Lemma 5.11 below.

Lemma 5.10 (Unbounded Sunset Computation). Given $t, \Delta \in \mathbb{N}$ with $t \geq \Delta$ and $\Delta$-sparse sets $A_1, A_2 \subseteq [t]$, in time $O(T_{\text{MINCONV}}(t/\Delta))$ we can compute a set $A$ that $(\infty, \Delta)$-approximates $A_1 + A_2$.

Proof. To simplify notation, for this proof we introduce the symbol $\bot$ indicating an undefined value. We let $\min \emptyset = \max \emptyset = \bot$. Furthermore, we let $x + \bot = \bot$ and $\min \{x, \bot\} = \max \{x, \bot\} = x$. This gives rise to natural generalizations of MINCONV and MAXCONV to sequences over $\mathbb{Z} \cup \{\bot\}$. We call an entry of such a sequence defined if it is not $\bot$. Note that since $\bot$ acts as a neutral
element for the min and max operations, we can think of \( \perp \) being \( \infty \) for \( \text{MinConv} \), and \( -\infty \) for \( \text{MaxConv} \). The fact that these neutral elements, \( \infty \) and \( -\infty \), are different is the reason why we introduce \( \perp \).

Observe that if \( \text{MinConv} \) on sequences over \( \{-M, \ldots, M\} \) is in time \( T_{\text{MinConv}}(n) \), then also \( \text{MinConv} \) on sequences over \( \{-M/4, \ldots, M/4\} \cup \{\perp\} \) is in time \( \mathcal{O}(T_{\text{MinConv}}(n)) \). Indeed, replacing \( \perp \) by \( M \), any output value in \( [-M/2, M/2] \) is computed correctly, while any output value in \( [3M/4, 2M] \) corresponds to \( \perp \). Also observe that \( \text{MaxConv} \) is equivalent to \( \text{MinConv} \) by negating all input and output values, and therefore \( \text{MaxConv} \) is also in time \( \mathcal{O}(T_{\text{MinConv}}(n)) \).

Our algorithm is as follows. Set \( n := 4[t/\Delta] \). We consider intervals \( I_i := [i\Delta/2, (i + 1)\Delta/2] \) for \( 0 \leq i < n \). Since \( A_1, A_2 \) are \( \Delta \)-sparse, they contain at most two elements in any interval \( I_i \). We may therefore “unfold” the sets \( A_1, A_2 \) into vectors \( X_1, X_2 \) of length \( 2n \) as follows. For \( r \in \{1, 2\} \) and \( 0 \leq i < n \) we set

\[
X_r[2i] := \min(I_i \cap A_r), \quad X_r[2i + 1] := \max(I_i \cap A_r).
\]

We then compute the sequences

\[
C^- := \text{MinConv}(X_1, X_2), \quad C^+ := \text{MaxConv}(X_1, X_2),
\]

that is, \( C^-[k] = \min_{i+j=k} X_1[i] + X_2[j] \), \( C^+[k] = \max_{i+j=k} X_1[i] + X_2[j] \), for \( 0 \leq k < 4n \). Finally, we return the set \( A \) containing all defined entries of \( C^- \) and \( C^+ \).

Clearly, this algorithm runs in time \( \mathcal{O}(T_{\text{MinConv}}(t/\Delta)) \). It remains to prove correctness. Since every defined entry of \( X_r \) corresponds to an element of \( A_r \), it follows that every defined entry of \( C^- \) and \( C^+ \) corresponds to a sum in \( A_1 + A_2 \). Hence, we have \( A \subseteq A_1 + A_2 \).

It remains to prove that for any \( a_1 \in A_1, a_2 \in A_2 \) their sum \( a_1 + a_2 \) has good approximations in \( A \). Let \( 0 \leq i^*, j^* < 2n \) be such that \( X_1[i^*] = a_1 \) and \( X_2[j^*] = a_2 \) and let \( k^* := i^* + j^* \). Then by definition of \( \text{MinConv} \) and \( \text{MaxConv} \) we have

\[
C^-[k^*] \leq a_1 + a_2 \leq C^+[k^*].
\]

It remains to prove that \( C^+[k^*] - C^-[k^*] \leq \Delta \). From the construction of \( X_r[2i] \) and \( X_r[2i + 1] \) it follows that any defined entry satisfies \( X_r[i] \in [(i - 1)\Delta/4, (i + 1)\Delta/4] \). In particular, the sum of two defined entries satisfies \( X_1[i] + X_2[j] \in [(i + j - 2)\Delta/4, (i + j + 2)\Delta/4] \). This yields

\[
C^-[k^*], C^+[k^*] \in [(k^* - 2)\Delta/4, (k^* + 2)\Delta/4].
\]

Moreover, at least one summand, \( X_1[i^*] + X_2[j^*] = a_1 + a_2 \), is defined, and thus \( C^-[k^*], C^+[k^*] \neq \perp \). This yields \( C^+[k^*] - C^-[k^*] \leq \Delta \). Together with (1) we see that any sum \( a_1 + a_2 \in A_1 + A_2 \) has good approximations in \( A \), which finishes the proof.

\[\text{Lemma 5.11 (Capped Sumset Computation).} \]

Let \( t, \Delta \in \mathbb{N} \) and \( B_1, B_2 \subseteq [t] \). Set \( B := B_1 + t B_2 \) and suppose that \( A_1 \) sparsely \((t, \Delta)\)-approximates \( B_1 \) and \( A_2 \) sparsely \((t, \Delta)\)-approximates \( B_2 \). In this situation, given \( A_1, A_2, t, \Delta \), we can compute a set \( A \) that sparsely \((t, \Delta)\)-approximates \( B \) in time \( \mathcal{O}(T_{\text{MinConv}}(t/\Delta)) \). We refer to this algorithm as \text{CappedSumset}(A_1, A_2, t, \Delta).

\[\text{Proof.} \] By Lemma 5.5 \( A_1 + t A_2 \) \((t, \Delta)\)-approximates \( B \). Using Lemma 5.10 we can compute a set \( A' \) that \((\infty, \Delta)\)-approximates \( A_1 + A_2 \). By Lemma 5.8 \text{(Down Shifting)}, \( A'' := A' \cap [t] \) \((t, \Delta)\)-approximates \((A_1 + A_2) \cap [t] = A_1 + t A_2 \). Using Lemma 5.7 given \( A'' \) we can compute a set \( A \) that sparsely \((t, \Delta)\)-approximates \( A'' \). By Lemma 5.2 \text{(Transitivity)}, these three steps imply that \( A \) \((t, \Delta)\)-approximates \( B \). Since \( A \) is \( \Delta \)-sparse, \( A \) also sparsely \((t, \Delta)\)-approximates \( B \).\[\]
5.3 Algorithms for Subset Sum

With the above preparations we are now ready to present our approximation algorithm for SubsetSum. It is an adaptation of a pseudopolynomial algorithm for SubsetSum [12], mainly in that we use Lemma 5.11 instead of the usual sumset computation by Fast Fourier Transform, but significant changes are required to make this work.

Given \((X, t, \Delta)\) where \(X\) has size \(n\), our goal is to compute a set \(A\) that sparsely \((t, \Delta)\)-approximates the set \(S(X; t) = \{\Sigma(Y) \mid Y \subseteq X, \Sigma(Y) \leq t\}\).

**Definition 5.12.** We say that an event happens with high probability if its probability is at least \(1 - \min\{1/n, \Delta/t\}^c\) for some constant \(c > 0\) that we are free to choose as any large constant. We say that \(A\) w.h.p. \((t, \Delta)\)-approximates \(B\) if we have
- \(A \subseteq B\), and
- with high probability \(A\) \((t, \Delta)\)-approximates \(B\).

It can be checked that the properties established in Section 5.1 still hold for this new notion of “w.h.p. \((t, \Delta)\)-approximates”.

5.3.1 Color Coding

In this section, we loosely follow [12, Section 3.1]. We present an algorithm ColorCoding (Algorithm 2) which solves SubsetSum in case all items are large, that is, \(X \subseteq [t/k, t]\) for a parameter \(k\).

**Lemma 5.13 (Color Coding).** Given \(t, \Delta, k \in \mathbb{N}\) with \(t \geq \Delta\) and a set \(X \subseteq [t/k, t]\) of size \(n\), we can compute a set \(A\) that w.h.p. \((t, \Delta)\)-approximates \(S(X; t)\), in time
\[
O\left((n + k^2 \cdot T_{\text{MinConv}}(t/\Delta)) \log(nt/\Delta)\right).
\]

**Proof.** Denote by \(X_1, \ldots, X_{k^2}\) a random partitioning of \(X\), that is, for every \(x \in X\) we choose a number \(j\) uniformly and independently at random and we put \(x\) into \(X_j\). For any subset \(Y \subseteq X\) with \(\Sigma(Y) \leq t\), note that \(|Y| \leq k\) since \(X \subseteq [t/k, t]\), and consider how the random partitioning acts on \(Y\). We say that the partitioning splits \(Y\) if we have \(|Y \cap X_j| \leq 1\) for any \(1 \leq j \leq k^2\). By the birthday paradox, \(Y\) is split with constant probability. More precisely, we can view the partitioning restricted to \(Y\) as throwing \(|Y| \leq k\) balls into \(k^2\) bins. Thus, the probability that \(Y\) is split is equal to the probability that the second ball falls into a different bin than the first, the third ball falls into a different bin than the first two, and so on, which has probability
\[
\frac{k^2 - 1}{k^2} \cdot \frac{k^2 - 2}{k^2} \cdots \frac{k^2 - (|Y| - 1)}{k^2} \geq \left(\frac{k^2 - (|Y| - 1)}{k^2}\right)^{|Y|} \geq \left(1 - \frac{1}{k}\right)^k \geq \left(\frac{1}{2}\right)^2 = \frac{1}{4}.
\]

We make use of this splitting property as follows. Let \(X'_j := X_j \cup \{0\}\) and
\[
T := X'_1 + \cdots + X'_{k^2}.
\]
Observe that \(T \subseteq S(X; t)\), since each sum appearing in \(T\) uses any item \(x \in X\) at most once. We claim that if \(Y\) is split, then \(T\) contains \(\Sigma(Y)\). Indeed, in any part \(X_j\) with \(|Y \cap X_j| = 1\) we pick this element of \(Y\), and in any other part we pick \(0 \in X'_j\), to form \(\Sigma(Y)\) as a sum appearing in \(T = X'_1 + \cdots + X'_{k^2}\).

Hence, we have \(\Sigma(Y) \in T\) with probability at least \(1/4\). To boost the success probability, we repeat the above random experiment several times. More precisely, for \(r = 1, \ldots, C \log(nt/\Delta)\)
\footnote{Here \(C\) is a large constant that governs the “with high probability” bound.}
we sample a random partitioning \( X = X_{r, 1} \cup \ldots \cup X_{r, k^2} \), set \( X'_{r,i} := X_{r,i} \cup \{0\} \), and consider \( T_r := X'_{r,1} + \ldots + t \ X'_{r,k^2} \). Since we have \( \Sigma(Y) \in T_r \) with probability at least 1/4, we obtain \( \Sigma(Y) \in \bigcup_r T_r \) with high probability. Moreover, we have \( \bigcup_r T_r \subseteq S(X; t) \).

Let \( S_{\Delta}\text{-sp}(X; t) \) be the sparsification of \( S(X; t) \) given by Lemma 5.7 and note that it has size \( |S_{\Delta}\text{-sp}(X; t)| = O(t/\Delta) \). Since we use “with high probability” to denote a probability of at least \( 1 - \min\{1/n, \Delta/t\}^c \) for large constant \( c \), we can afford a union bound over the \( O(t/\Delta) \) elements of \( S_{\Delta}\text{-sp}(X; t) \) to infer that with high probability

\[
S_{\Delta}\text{-sp}(X; t) \subseteq \bigcup_r T_r \subseteq S(X; t).
\]

Since \( S_{\Delta}\text{-sp}(X; t) \) \((t, \Delta)\)-approximates \( S(X; t) \), Lemma 5.3 implies that

\[
\bigcup_r T_r \text{  w.h.p.  } (t, \Delta)\text{-approximates } S(X; t).
\]  

(2)

We cannot afford to compute any \( T_r \) explicitly, but we can compute approximations of these sets. To this end, let \( Z_{r,j} \) be the sparsification of \( X'_{r,j} \) given by Lemma 5.7. We start with \( A_{r,0} := \{0\} \) and repeatedly compute the capped sumset with \( Z_{r,j} \), setting \( A_{r,j} := \text{CappedSumset}(A_{r,j-1}, Z_{r,j}, t, \Delta) \) for \( 1 \leq j \leq k^2 \). It now follows inductively from Lemma 5.11 that \( A_{r,j} \) sparsely \((t, \Delta)\)-approximates \( X'_{r,1} + t \ldots + t X'_{r,j} \).

Hence, \( A_{r,k^2} \) sparsely \((t, \Delta)\)-approximates \( T_r \). Let \( A' := \bigcup_r A_{r,k^2} \). By Lemma 5.4 \( A' \) \((t, \Delta)\)-approximates \( \bigcup_r T_r \). With (2) and transitivity, \( A' \) w.h.p. \((t, \Delta)\)-approximates \( S(X; t) \). Finally, we sparsify \( A' \) using Lemma 5.7 to obtain a subset \( A \) that sparsely \((t, \Delta)\)-approximates \( A' \). By transitivity, \( A \) w.h.p. \((t, \Delta)\)-approximates \( S(X; t) \). For pseudocode of this, see Algorithm 2.

The running time is immediate from Lemmas 5.7 and 5.11.

\[ \square \]

Algorithm 2 ColorCoding\((X, t, \Delta, k)\): Given \( t, \Delta \in \mathbb{N} \) and a set \( X \subseteq [t/k, t] \) in sorted order, we compute a set \( A \) that w.h.p. sparsely \((t, \Delta)\)-approximates \( S(X; t) \).

1: for \( r = 1, \ldots, C \log(nt/\Delta) \) do
2: \quad randomly partition \( X = X_{r, 1} \cup \ldots \cup X_{r, k^2} \)
3: \quad \( A_{r,0} := \{0\} \)
4: \quad for \( j = 1, \ldots, k^2 \) do
5: \quad \quad \( X'_{r,j} := X_{r,j} \cup \{0\} \)
6: \quad \quad \( Z_{r,j} := \text{Sparsification}(X'_{r,j}, t, \Delta) \)
7: \quad \quad \( A_{r,j} := \text{CappedSumset}(A_{r,j-1}, Z_{r,j}, t, \Delta) \)
8: \quad return \( \text{Sparsification}(\bigcup_r A_{r,k^2}, t, \Delta) \)

5.3.2 Greedy

We also need a special treatment of the case that all items are small, that is, \( \max(X) \leq \Delta \). In this case, we pick any ordering of \( X = \{x_1, \ldots, x_n\} \) and let \( P \) denote the set of all prefix sums \( 0, x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots \) that are bounded by \( t \), that is, \( P = \{\sum_{i=1}^j x_i \mid 0 \leq j \leq n\} \cap [t] \). We return a sparsification \( A \) of \( P \). See Algorithm 3 for pseudocode.

Claim 5.14. \( P \) \((t, \Delta)\)-approximates \( S(X; t) \).
Proof. Clearly $P \subseteq S(X; t)$. Moreover, any $s \in [0, \max(P)]$ falls into some interval between two consecutive prefix sums, and such an interval has length $x_i$ for some $i$. Hence, we have

$$\text{apx}_t^+(s, P) - \text{apx}_t^-(s, P) \leq x_i \leq \max(X) \leq \Delta.$$  

We now do a case distinction on $\Sigma(X)$. If $\Sigma(X) < t$, then observe that $\max(P) = \Sigma(X) = \max(S(X; t))$. Therefore, the interval $[0, \max(P)]$ already covers all $s \in S(X; t)$ and we are done.

Otherwise, if $\Sigma(X) \geq t$, then observe that $\max(P) > t - \Delta$, as otherwise we could add the next prefix sum to $P$. In this case, for any $s \in [\max(P), t]$,

$$\text{apx}_t^+(s, P) - \text{apx}_t^-(s, P) \leq t + 1 - \max(P) \leq \Delta.$$  

In total, every $s \in S(X; t)$ has good approximations in $P$.  

\begin{algorithm}[h]
\caption{Greedy($X, t, \Delta$): Given $t, \Delta \in \mathbb{N}$ and a set $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{N}$ with $\max(X) \leq \Delta$, we compute a set $A$ that sparsely $(t, \Delta)$-approximates $S(X; t)$.}
\begin{algorithmic}[1]
\State $P := \{0\}$, $s := 0$, $i := 1$
\While{$i \leq n$ and $s + x_i \leq t$}
\State $s := s + x_i$
\State $P := P \cup \{s\}$
\State $i := i + 1$
\EndWhile
\State $A := \text{Sparsification}(P, t, \Delta)$
\State \textbf{return} $A$
\end{algorithmic}
\end{algorithm}

From Claim \[5.14\] and transitivity it follows for $A = \text{Sparsification}(P, t, \Delta)$ that $A$ sparsely $(t, \Delta)$-approximates $S(X; t)$. We thus proved the following lemma.

**Lemma 5.15 (Greedy).** Given integers $t, \Delta > 0$ and a set $X \subseteq \mathbb{N}$ of size $n$ satisfying $\max(X) \leq \Delta$, we can compute a set $A$ that sparsely $(t, \Delta)$-approximates $S(X; t)$ in time $O(n)$.

### 5.3.3 Recursive Splitting

We now present a recursive algorithm making use of ColorCoding and Greedy.

Given a set $X \subseteq \mathbb{N}$ of size $n$ and numbers $t, \Delta > 0$, our goal is to compute a set $A$ that sparsely $(t, \Delta)$-approximates $S(X; t)$. We assume that initially $t \geq 8\Delta$. We will use parameters $k$ and $\eta$, which are set before the first call of the algorithm to$^6$ $k := \max\{8, C \log^3(nt/\Delta)\}$ and $\eta := 1/(2\log(t/\Delta))$. We can assume that $X \subseteq \mathbb{N}$, since larger numbers cannot be picked for subset sums in $[t]$.

We partition $X$ into the large numbers $X_L := X \cap [t/k, t]$ and the small numbers $X_S := X \setminus X_L$. On the large numbers we compute $A_L := \text{ColorCoding}(X_L, t, \Delta, k)$, so that $A_L$ w.h.p. sparsely $(t, \Delta)$-approximates $S(X_L; t)$. We then randomly partition the small numbers $X_S$ into subsets $X_1, X_2$, that is, for any $x \in X_S$ we choose a number $j \in \{1, 2\}$ uniformly at random and we put $x$ into $X_j$. We recursively call the same algorithm on $(X_1, t', \Delta)$ and on $(X_2, t', \Delta)$ for the new target bound $t' := (1 + \eta)t/2 + \Delta$. Call the results of these recursive calls $A_1, A_2$. Finally, we combine $A_1, A_2$ to $A_S$, and $A_S, A_L$ to $A$, by capped subset computations. We return $A$. The base case happens when $\max(X) \leq \Delta$, where we run Greedy. See Algorithm $4$ for pseudocode.

In the following we analyze this algorithm, proving the following lemma.

$^6$ Here $C$ is a large constant that governs the “with high probability” bounds.
Algorithm 4 \textit{RecursiveSplitting}(X, t, \Delta): Given \( t, \Delta \in \mathbb{N} \) and a set \( X \subseteq [t] \) in sorted order, we compute a set \( A \) that sparsely \((t, \Delta)\)-approximates \( S(X; t) \). The parameters \( k, \eta \) are set before the first call of the algorithm to \( k := \max\{8, C \log^2(nt/\Delta)\} \) and \( \eta := 1/(2 \log(t/\Delta)) \).

1: if max\( (X) \leq \Delta \) then return \textit{Greedy}(X, t, \Delta)
2: \( X_L := X \cap [t/k, t] \), \( X_S := X \setminus X_L \)
3: randomly partition \( X_S = X_1 \cup X_2 \)
4: \( t' := (1 + \eta)t/2 + \Delta \)
5: \( A_L := \text{ColorCoding}(X_L, t, \Delta, k) \)
6: \( A_1 := \text{RecursiveSplitting}(X_1, t', \Delta) \)
7: \( A_2 := \text{RecursiveSplitting}(X_2, t', \Delta) \)
8: \( A_S := \text{CappedSumset}(A_1, A_2, t, \Delta) \)
9: \( A := \text{CappedSumset}(A_L, A_S, t, \Delta) \)
10: return \( A \)

Lemma 5.16 (Recursive Splitting). Given integers \( t, \Delta > 0 \) with \( t \geq 8\Delta \) and a set \( X \subseteq [t] \) of size \( n \), we can compute a set \( A \) that sparsely \((t, \Delta)\)-approximates \( S(X; t) \) in time

\[ O((n + T_{\text{MinConv}}(t/\Delta)) \log^8(nt/\Delta)). \]

Recursion Depth Denote by \( t_i \) the target bound on the \( i \)-th level of recursion. Let us first check that \( t_i \) is monotonically decreasing. Initially we assume \( t_0 \geq 8\Delta \). On any level with \( t_i \geq 8\Delta \), the new target bound satisfies \( t_{i+1} = (1 + \eta)t_i/2 + \Delta \leq \frac{3}{4}t_i + \Delta < t_i \), where we used our choice of \( \eta \leq 1/2 \). Since we also have \( t_{i+1} \geq t_i/2 \), at some point we reach \( t_i \in [4\Delta, 8\Delta] \). The small items on this level are bounded by \( t_i/k \leq \Delta \), since \( k \geq 8 \). Hence, on the next level we will apply \textit{Greedy} and the recursion stops. In particular, \( t_i \) is monotonically decreasing throughout.

Note that

\[ t_i = \left(1 + \eta \right)^i t + \sum_{0 \leq j < i} \left(1 + \eta \right)^j \Delta. \]

Using \( \sum_{0 \leq j \leq i} q^j \leq \sum_{j \geq 0} q^j = 1/(1 - q) \) yields

\[ t_i \leq \left(1 + \eta \right)^i t + \frac{2}{1 - \eta} \Delta \leq \left(1 + \eta \right)^i t + 4\Delta, \]

where we used our choice of \( \eta = 1/(2 \log(t/\Delta)) \leq 1/2 \). Note that for \( 0 \leq i \leq \log(t/\Delta) \) we have \( (1 + \eta)^i \leq \exp(\eta i) \leq \exp(1/2) < 2 \). Hence, for any \( 0 \leq i \leq \log(t/\Delta) \) the target bound satisfies

\[ t_i \leq \frac{2t}{2^i} + 4\Delta. \]

(3)

It follows that \( t_{\log(t/\Delta)} = 1 \leq 8\Delta \), so the above argument shows that the recursion stops at the latest on level \( \log(t/\Delta) \). We have therefore shown that the recursion depth of \textit{RecursiveSplitting} is at most \( \log(t/\Delta) \). In particular, inequality (3) is applicable in each recursive call.

Correctness We inductively prove that with high probability for any recursive call of method \textit{RecursiveSplitting}(X, t, \Delta) the output \( A \) sparsely \((t, \Delta)\)-approximates \( S(X; t) \). Note that, as an output of \textit{CappedSumset}, \( A \) is clearly \( \Delta \)-sparse, and thus we only need to show that \( A \) \((t, \Delta)\)-approximates \( S(X; t) \), see Lemma 5.20. Since the recursion tree has total size \( O(t/\Delta) \), we can afford a union bound over all recursive calls. In particular, if we prove correctness of one recursive
Lemma 5.17. With high probability, there exist \( t_1, t_2 \in [(1 + \eta)t/2, (1 + \eta)t/2 + \Delta] \) such that \( A(t, \Delta) \)-approximates \( S(X_L; t) + tS(X_1; t_1) + tS(X_2; t_2) \).

Proof. We suppress the phrase “with high probability” throughout the proof. Inductively, \( A_i(t', \Delta) \)-approximates \( S(X_i; t') \) for \( t' = (1 + \eta)t/2 + \Delta \) and any \( i \in \{1, 2\} \). By Lemma 5.9 there exists a value \( t_i \in [t' - \Delta, t'] \) such that \( A_i(t, \Delta) \)-approximates \( S(X_i; t') \cap [t_i] = S(X_i; t_i) \). Now Lemma 5.11 (Capped Sunset Computation) shows that \( A_S(t, \Delta) \)-approximates \( S(X_1; t_1) + tS(X_2; t_2) \). Moreover, Lemma 5.13 (Color Coding) yields that \( A_L(t, \Delta) \)-approximates \( S(X_L; t) \). It follows that the set \( A(t, \Delta) \)-approximates \( S(X_L; t) + tS(X_1; t_1) + tS(X_2; t_2) \). \( \square \)

Lemma 5.18. Let \( S_{\Delta\text{-sp}}(X_S; t) \) be the sparsification of \( S(X_S; t) \) given by Lemma 5.7 and let \( \tilde{t} = (1 + \eta)t/2 \). With high probability we have \( S_{\Delta\text{-sp}}(X_S; t) \subseteq S(X_1; t) + tS(X_2; \tilde{t}) \).

Proof. For any \( s \in S_{\Delta\text{-sp}}(X_S; t) \), fix a subset \( Y \subseteq X_S \) with \( \Sigma(Y) = s \) and write \( Y = \{y_1, \ldots, y_t\} \). Let \( Y_r := Y \cap X_r \) for \( r \in \{1, 2\} \). Consider independent random variables \( Z_1, \ldots, Z_t \) where \( Z_i \) is uniformly distributed in \( \{0, y_i\} \), and set \( Z := Z_1 + \ldots + Z_t \). Note that \( Z \) has the same distribution as \( \Sigma(Y_1) \) and \( \Sigma(Y_2) \). Also note that \( \mathbb{E}[Z] = \Sigma(Y)/2 \). We use Hoeffding’s inequality on \( Z \) to obtain

\[
\Pr\left[Z - \mathbb{E}[Z] \geq \lambda\right] \leq \exp\left(-\frac{2\lambda^2}{\sum_i y_i^2}\right).
\]

Since \( Y \) is a subset of the small items \( X_S \), we have \( y_i \leq t/k \) for all \( i \), and thus \( \sum_i y_i^2 \leq \sum_i y_i \cdot t/k \leq t^2/k \). Setting \( \lambda := \frac{4}{t} \), we thus obtain

\[
\Pr\left[Z \geq \mathbb{E}[Z] + \frac{\eta}{2}\right] \leq \exp\left(-\frac{k\eta^2}{2}\right).
\]

By our choice of \( \eta := 1/(2\log(t/\Delta)) \) and \( k \geq C \log^3(nt/\Delta) \) we have \( k\eta^2/2 \geq \frac{C}{\log^2} \log(nt/\Delta) \). Moreover, since \( \mathbb{E}[Z] = \Sigma(Y)/2 \leq t/2 \), we obtain

\[
\Pr\left[Z \geq (1 + \eta)\frac{t}{2}\right] \leq \Pr\left[Z \geq \mathbb{E}[Z] + \frac{\eta}{2}\right] \leq \left(\frac{\Delta}{tn}\right)^{C/8}.
\]

For large \( C \), this shows that with high probability \( \Sigma(Y_1), \Sigma(Y_2) \leq (1 + \eta)\frac{t}{2} = \tilde{t} \), and hence \( s = \Sigma(Y) \in S(X_1; \tilde{t}) + tS(X_2; \tilde{t}) \). Since \( S_{\Delta\text{-sp}}(X_S; t) \) has size \( O(t/\Delta) \), we can afford a union bound over all \( s \in S_{\Delta\text{-sp}}(X_S; t) \) to obtain that with high probability \( S_{\Delta\text{-sp}}(X_S; t) \subseteq S(X_1; \tilde{t}) + tS(X_2; \tilde{t}) \). \( \square \)

Observation 5.19. For any partitioning \( Z = Z_1 \cup Z_2 \) we have \( S(Z_1, t) + tS(Z_2, t) = S(Z; t) \).

Proof. Follows from the fact that any subset sum of \( Z \) can be uniquely written as a sum of a subset sum of \( Z_1 \) and a subset sum of \( Z_2 \). \( \square \)

Lemma 5.20. A w.h.p. \((t, \Delta)\)-approximates \( S(X; t) \).

Proof. Let \( t_1, t_2 \) be as in Lemma 5.17 in particular \( \tilde{t} = (1 + \eta)t/2 \leq t_1, t_2 \leq (1 + \eta)t/2 + \Delta \leq t \). Using these bounds, Lemma 5.18 and Observation 5.19 we obtain with high probability

\[
S_{\Delta\text{-sp}}(X_S; t) \subseteq S(X_1; \tilde{t}) + tS(X_2; \tilde{t}) \subseteq S(X_1; t_1) + tS(X_2; t_2) \subseteq S(X_1; t) + tS(X_2; t) = S(X_S; t).
\]
Since \( S_{\Delta^p}(X_S; t) \) \((t, \Delta)\)-approximates \( S(X_S; t) \), it now follows from Lemma 5.3 that with high probability the sumset \( S(X_1; t_1) +_t S(X_2; t_2) \) \((t, \Delta)\)-approximates \( S(X_S; t) \).

Using Lemma 5.5 (Sumset Property) and Observation 5.19, we obtain that with high probability

\[
S(X_L; t) +_t (S(X_1; t_1) +_t S(X_2; t_2)) \quad (t, \Delta)\)-approximates \( S(X_S; t) = S(X; t) \).
\]

Lemma 5.17 and transitivity now imply that with high probability \( A \) \((t, \Delta)\)-approximates \( S(X; t) \).

It is easy to see that the inclusion \( A \subseteq S(X; t) \) holds deterministically (i.e., with probability 1), and thus we even have that \( A \) w.h.p. \((t, \Delta)\)-approximates \( S(X; t) \).

This finishes the proof of correctness.

**Running Time** Lines 1-4 of Algorithm 4 take time \( O(|\times|) \), which sums to \( O(n) \) on each level of recursion, or \( O(n \log(t/\Delta)) \) overall. On the \( i \)-th level of recursion, calling \texttt{ColorCoding} takes time

\[
O((|\times_L| + k^2 \cdot T_{MINCONV}(t_i/\Delta)) \log(nt/\Delta)),
\]

and calling \texttt{CappedSumset} is dominated by this running time. Since every item is large in exactly one recursive call, the terms \(|\times_L| \) simply sum up to \( n \). For the remainder, we have \( 2^i \) instances, each with target bound \( t_i \leq 2t/2^i + 4\Delta \). Note that \( 2^i \cdot t_i \leq 6t \), since \( i \leq \log(t/\Delta) \). Hence, by Lemma 5.21 below, we can solve \( 2^i \) \texttt{MINConv} instances, each of size at most \( t_i/\Delta \), in total time \( O(T_{MINCONV}(t/\Delta)) \). We can thus bound the time \( O(k^2 \cdot T_{MINCONV}(t/\Delta) \log(nt/\Delta)) \) summed over all recursive calls on level \( i \) by \( O(k^2 \cdot T_{MINCONV}(t/\Delta) \log(nt/\Delta)) \). Over all levels, there is an additional factor \( \log(t/\Delta) \). It follows that the total running time is

\[
O((n + k^2 \log(t/\Delta) \cdot T_{MINCONV}(t/\Delta)) \log(nt/\Delta)).
\]

Plugging in \( k = O(\log^3(nt/\Delta)) \) yields a running time of \( O((n + T_{MINCONV}(t/\Delta)) \log^8(nt/\Delta)) \).

**Lemma 5.21.** There is an algorithm that solves \( m \) given \texttt{MINConv} instances, each of size \( n \), in total time \( O(T_{MINCONV}(nm)) \).

**Proof.** Given \( A_0, B_0, \ldots, A_{m-1}, B_{m-1} \in \mathbb{N}^n \) our goal is to compute \( C_0, \ldots, C_{m-1} \in \mathbb{N}^n \) satisfying \( C_r[k] = \min_{0 \leq i \leq k} A_r[i] + B_r[k - i] \) for any \( 0 \leq k < n \) and \( 0 \leq r < m \). We assume that all entries of the input sequences are bounded by \( M \). We construct sequences \( A, B \in \mathbb{N}^{4nm} \) by setting for any \( 0 \leq i < n \) and \( 0 \leq r < m \):

\[
A[2rn + i] := r^2 \cdot 2M + A_r[i], \quad B[2rn + j] = r^2 \cdot 2M + B_r[j],
\]

and setting all remaining entries to \( \infty \) (we remark that a large finite number is sufficient). Then we compute \( C = \text{MINConv}(A, B) \), that is, \( C[k] = \min_{0 \leq i \leq k} A[i] + B[k - i] \) for any \( 0 \leq k < 4nm \).

We claim that for any \( 0 \leq k < n \) and \( 0 \leq r < m \) we have

\[
C[4rn + k] = r^2 \cdot 4M + C_r[k].
\]

The lemma follows from this claim, as we can infer \( C_0, \ldots, C_{m-1} \) from \( C \). In one direction, observe

\[
C[4rn + k] \leq \min_{0 \leq i \leq k} A[2rn + i] + B[2rn + k - i] = r^2 \cdot 4M + \min_{0 \leq i \leq k} A_r[i] + B_r[k - i] = r^2 \cdot 4M + C_r[k].
\]

In the other direction, consider \( C[4rn + k] = A[2rn + i] + B[2yn + j] \), where \( 0 \leq i, j < 2n \), \( 0 \leq x, y < 2m \), and \( 2xn + i + 2yn + j = 4rn + k \). We can assume that \( 0 \leq i, j < n \) and
0 \leq x, y < m$, since otherwise $A[2xn + i] + B[2yn + j] = \infty$. For this range of values, the equation $2xn + i + 2yn + j = 4rn + k$ implies that $x + y = 2r$ and $i + j = k$. If $x = y = r$, as in the first direction we obtain $C[4rn + k] = r^2 \cdot 4M + C_r[k]$. Otherwise, if $x \neq y$, note that $A[2xn + i] \geq x^2 \cdot 2M$ and $B[2yn + j] \geq y^2 \cdot 2M = (2r - x)^2 \cdot 2M$. Since the function $x^2 + (2r - x)^2$ is concave, we obtain for $x \neq y$

\begin{align*}
A[2xn + i] + B[2yn + j] \geq (\{(r - 1)^2 + (r + 1)^2\} \cdot 2M = r^2 \cdot 4M + 4M > r^2 \cdot 4M + C_r[k].
\end{align*}

Together with the first direction, this contradicts $C[4rn + k] = A[2xn + i] + B[2yn + j]$, and thus proves the claim.

### 5.3.4 Finishing the Proof

We show how to use \texttt{RecursiveSplitting} to solve \texttt{HApxSubsetSum} in time $\tilde{O}(n + T_{\text{MinConv}}(1/\varepsilon))$.

Note that this proves Theorem 2.1 as well as Corollary 2.2.

We show how to use \texttt{RecursiveSplitting} to solve \texttt{HApxSubsetSum} in time $\tilde{O}(n + T_{\text{MinConv}}(1/\varepsilon))$. Given $X, t$ and $\varepsilon > 0$, let $\text{OPT} := \max(S(X; t))$. Set $\Delta := \min\{\varepsilon t, t/8\}$ and call the procedure \texttt{RecursiveSplitting}(X, t, \Delta) to obtain a set $A$ that w.h.p. $(t, \Delta)$-approximates $S(X; t)$.

Claim 5.22. With high probability, we have $\max(A) \geq \min\{\text{OPT}, (1 - \varepsilon)t\}$.

**Proof.** Consider $\text{apx}_t^+(\text{OPT}, A)$ and $\text{apx}_t^-(\text{OPT}, A)$. Since $S(X; t)$ does not contain any numbers in $(\text{OPT}, t]$, and $A \subseteq S(X; t)$, we have $\text{apx}_t^+(\text{OPT}, A) \in \{\text{OPT}, t + 1\}$. If $\text{apx}_t^+(\text{OPT}, A) = \text{OPT}$, then $A$ contains $\text{OPT}$, so $\max(A) \geq \text{OPT}$. Otherwise, if $\text{apx}_t^+(\text{OPT}, A) = t + 1$, then $\text{apx}_t^-(\text{OPT}, A) \geq \text{apx}_t^+(\text{OPT}, A) - \Delta > t - \varepsilon t$. In particular, $\max(A) \geq (1 - \varepsilon)t$.

We have thus shown how to compute a subset sum $\max(A)$ with $\max(A) \geq \min\{\text{OPT}, (1 - \varepsilon)t\}$. It remains to determine a subset $Y \subseteq X$ summing to $\max(A)$. To this end, we retrace the steps of the algorithm, using the following idea. If $a \in \text{CappedSumset}(A_1, A_2, t, \Delta)$, then $a \in A_1 + A_2$, and thus we can simply iterate over all $a_1 \in A_1$ and check whether $a - a_1 \in A_2$, to reconstruct a pair $a_1 \in A_1, a_2 \in A_2$ with $a = a_1 + a_2$ in linear time. Starting with $\max(A)$, we perform this trick in each recursive call of the algorithm, to reconstruct a subset summing to $\max(A)$.

The total running time of this algorithm is $O((n + T_{\text{MinConv}}(1/\varepsilon)) \log^8 (n/\varepsilon))$.

### 6 Open Problems

We leave it as an open problem to derandomize our reduction and the resulting approximation scheme.

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