Matter fields with $c > 1$ coupled to $2d$ gravity

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Abstract

We solve a class of branched polymer models coupled to spin systems and show that they have no phase transition and are either always magnetized or never magnetized depending on the branching weights. By comparing these results with numerical simulations of two-dimensional quantum gravity coupled to matter fields with central charge $c$ we provide evidence that for $c$ sufficiently large ($c \geq 12$) these models are effectively described by branched polymers. Moreover, the numerical results indicate a remarkable universality in the influence on the geometry of surfaces due to the interaction with matter. For spin systems this influence only depends on the total central charge.

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1 Introduction

In the last couple of years there has been a significant progress in our understanding of two-dimensional gravity coupled to matter with central charge \( c \leq 1 \). The situation for \( c > 1 \) is unclear. The KPZ formula leads to complex critical exponents for \( 1 < c < 25 \) and the continuum formalism indicates the existence of tachyons for \( c \) in this region. On the other hand the discrete regularization of two-dimensional gravity coupled to matter \([1, 2, 3]\) is well defined for any \( c \). The discretized gravity models are (formally) reflection positive \( ^2 \) and consequently they cannot have tachyonic excitations. A folklore has evolved to the effect that the existence of tachyons in the spectrum of the continuum theory is reflected in dominance of branched polymer-like surfaces in the discretized models. It is possible to prove that the string tension in a class of discrete models of strings does not scale at the critical point \([1, 7]\). The fact that the string-tension does not scale and the consistent assumption of a scaling of the lowest mass excitation associated with the two-point function in target space, lead to the branched polymer picture: The surface of minimal area (depending on the boundary conditions) totally dominates in the scaling limit, and the only excitations allowed are spikes of essentially no area, branching out from this surface. Stated slightly differently, one can say that in these models the entropy of such surfaces completely dominates the critical behaviour for \( d = 2, 3, \ldots \).

While the above scenario is plausible, one should keep in mind that no rigorous proof of the branched polymer picture has been given in the case of dynamically triangulated surfaces (see, however, \([3]\)). Extensive numerical simulations have not provided a convincing support to the dominance of branched polymers for a small number of dimensions \( d \) whereas for \( d \) large (\( \geq 12 \)) there seems to be a reasonable agreement between the numerical results and the branched polymer picture.

The question we want to address in this article is that of universality for \( c > 1 \), i.e. do the critical properties only depend on the total central charge of the matter fields? Interesting observations in this direction can be found in \([11, 12, 13]\). In \([11, 13]\) Monte Carlo simulations were used to investigate various properties of multiple \( q \)-state Potts models coupled to two-dimensional quantum gravity, and an apparent universality, characterized by the total central charge \( c \) of the system, was observed. One surprising aspect was that the universality seemed to apply to quantities which one does not expect to be universal, like the average number of vertices of order three in the triangulations, calculated at the critical point of the multiple Potts model. In

\(^2\)The principle of reflection positivity can be extended from Euclidean field theory to random surfaces \([3]\). It is valid for hypercubic random surface theory, and we expect it to be satisfied in the scaling limit for triangulated random surfaces embedded in \( d \) dimensional space.
this paper we extend the analysis of [11, 13] and verify that that the distributions of orders of vertices seem to fall into classes characterized by the total central charge of the system. In [12] a strong coupling expansion is used for the calculation of $\gamma_{\text{string}}$ for various combinations of Ising spins and of Ising spins and gaussian fields. Again universality is observed within the accuracy of the strong coupling expansion.

We first consider multiple spin models on random trees and verify that they have no phase transition at finite temperature. We show by an explicit calculation that if the branching weights are sufficiently large, so that $\gamma_{\text{string}} < 0$, then a single Ising model on a branched polymer is in the magnetized phase at all finite temperatures. We emphasize that there is no choice of branching weights which leads to a finite temperature phase transition for spin systems on branched polymers. We then turn to a detailed discussion of the numerical results.

In summary, we show that there are good reasons to believe that the branched polymer picture (with some modifications to be discussed later) is true for sufficiently large values of $c$. We consider it unlikely to be true all the way down to $c = 1$ since a few spin systems couple almost as a single spin system.

2 Multiple systems coupled to gravity

In the case of a fixed triangulation (or any other fixed lattice) the consideration of $n$ independent copies of a spin system adds nothing interesting to the information already available in the partition function $Z(\beta)$ of a single spin system. The reason is that the partition function $Z_n(\beta)$ for the multiple system factorizes:

$$Z(\beta) = \sum_{\{\sigma_i\}} e^{\beta \sum_{i,j} \sigma_i \sigma_j}, \quad Z_n(\beta) = Z^n(\beta)$$

and the free energy $F(\beta) = -\ln Z(\beta)$, which determines the critical behaviour of the system will only differ by a factor from that of a single spin system.

When we couple the systems to two-dimensional quantum gravity the situation changes due to the back-reaction of matter on gravity. In the dynamical triangulation approach to quantum gravity one replaces the fixed lattice with a sum over all random lattices ($\text{triangulations } T$) with weight $Z_T^n(\beta)$:

$$Z_n(\mu, \beta) = \sum_T e^{-\mu N_T} Z^n_T(\beta).$$

In this formula $\mu$ denotes the bare cosmological constant, while $N_T$ denotes the number of triangles in the triangulation $T$. We assume that the topology of triangulations is that of $S^2$. When we discuss the effect of multiple spins on gravity we see that all they do is to change the weight of the triangulation from one to $Z^n_T(\beta)$. In
general this observation is not of much help, since it is difficult to calculate $Z_T(\beta)$ for an arbitrary triangulation $T$ and evaluating $Z_T(\beta)$ is not sufficient to judge its significance. We have to know the entropy factor associated with $T$ and nearby related triangulations. In the limit where $n \to \infty$ we expect the importance of the entropy factor to vanish relative to $Z_n^T(\beta)$ in accordance with standard mean field arguments, and for a given $\beta$ the triangulations with the largest partition functions is expected to dominate.

2.1 Gaussian multiple systems

The simplest systems with central charge $c \geq 1$ are multiple Gaussian systems. If we couple them to two-dimensional gravity we get Polyakovs formulation of string theory. The discretized version takes the form

$$Z(\mu, \tilde{\beta}) = \sum_T \rho(T)e^{-\mu N_T} \int_{x_i} \prod_{i \in T\setminus\{i_0\}} dx_i e^{-\tilde{\beta} \sum_{<i,j>} (x_i - x_j)^2}$$

where $N_T$ denote the number of triangles and $i_0$ is a vertex which is kept fixed in order to eliminate the translational mode. $Z(\mu, \tilde{\beta})$ is independent of the choice of $i_0$. The function $\rho(T)$ denotes a weight assigned to each triangulation $T$. In the above discussion of spin systems we tacitly assumed that $\rho(T) = 1$ for all $T$. This choice is the one which allows for an explicit solution in the case of the Ising model. Other weights have been considered in the past. One choice, which seems particularly well motivated assigns the weight

$$\rho(T) = \prod_i n_i^{-\alpha}$$

(2.4)

to the triangulation $T$. Here $n_i$ denotes the order of vertex $i$, i.e. the number of links to which $i$ belongs. The case $\alpha = 0$ corresponds to $\rho(T) = 1$, while $\alpha = -c/2$ corresponds to the so-called conformal weight for the path integral in quantum gravity. Since (2.3) is a discretized version of (two-dimensional) quantum gravity, which is not conformally invariant, there is no obvious reason for choosing one weight rather than another, as long as the weights satisfy a number of general requirements outlined in [1]. By universality the detailed choice of $\rho$ should not be important.

One difference between the Gaussian system and the discrete spin systems we are going to consider is that the coupling constant $\tilde{\beta}$ in (2.3) can be absorbed in a redefinition of the cosmological coupling constant $\mu$ by scaling the Gaussian variables $x_i$. This reflects that the Gaussian field on a regular lattice is automatically critical as soon as the infinite volume limit is taken. In the rest of this article we assume that such a rescaling has been performed so that $\tilde{\beta} = 1$. The coupling constant $\beta$ will be used only for the discrete spin systems.
Another difference between the Gaussian system and the discrete spin systems is that we can find the part of the partition function corresponding to a specific triangulation explicitly:

$$Z_T = \int \prod_{i \in T / \{i_0\}} dx_i e^{-\sum_{i,j > \{x_i - x_j\}}^{(x_i - x_j)^2} \pi^{(V_T - 1)/2} (\det(C_T(i_0)))^{-c/2}}$$  \hspace{1cm} (2.5)$$

where $V_T$ is the number of vertices in $T$ and $C_T(i_0)$ is the so-called incidence matrix of the triangulation. Its entries are labelled by pairs of vertices from $T \setminus \{i_0\}$, and equal -1 if $i$ and $j$ are neighbours, $n_i$ (the order of vertex $i$) if $i = j$ and 0 otherwise. Again $\det(C_T(i_0))$ is independent of $i_0$ and we suppress the index $i_0$. The partition function (2.3) can now be written

$$Z(\mu) = \sum_T \rho(T) (\det(C_T))^{-c/2} e^{-\mu N_T},$$  \hspace{1cm} (2.6)$$

with a redefinition of $\mu$. Much is known about the function $\det C_T$ and this allowed the determination of the class of triangulations which dominate the partition function for large $d$ \cite{8, 9, 10}. One has to go to large $d$ in order to be able to apply mean field theory, i.e. in order to be able to ignore the entropy factors associated with triangulations. A class of triangulations which leads to an explicit solvable model and which includes those with minimal determinants for a given number of triangles $N_T$ was found in \cite{8, 9}. The class is constructed by gluing together tetrahedra which are cut open along two non-adjacent links. There is a natural correspondence between surfaces of this class and branched polymers, such that each tetrahedron corresponds to a link in the branched polymer. Along the links cut open it is possible to glue many tetrahedra, each of which corresponds to a new branch in the associated polymer. In this way the order of the branching is not restricted. The determinant $\det C_T$ can be calculated for this class of triangulations and one finds:

$$\det C_T = \prod_{i \in T} n_i,$$  \hspace{1cm} (2.7)$$

where $n_i$ is the order of vertex $i$. We see that $\det C_T$ for this class of triangulations is of the same form as weight factors $\rho(T)$ considered in (2.4). It is now possible to rewrite the partition function for $d$ Gaussian fields on the class of random surfaces described above as

$$Z(\mu) = \sum_{BP} \rho(BP) e^{-\mu N_{BP}},$$  \hspace{1cm} (2.8)$$

where the sum is over all branched polymers, $N_{BP}$ is the number of links in the branched polymers and the weight function $\rho(BP)$ according to (2.4) and (2.7) is given by:

$$\rho(BP) = \prod_{i \in BP} w(n_i), \quad w(n) = \frac{1}{n^{c/2 + 2\alpha}}.$$  \hspace{1cm} (2.9)$$
The “cosmological constant” $\mu$ is trivially related to the original $\mu$ for the surfaces by a linear transformation, since various integration factors have been absorbed in $\mu$ for convenience. We refer to [8] for details.

The theory of branched polymers was solved in [8] for arbitrary positive weights $w(n)$, and for arbitrary weights in [17]. In the following we will only be interested in the situation where the weights are positive (the non-positive weights lead to a sequence of multi-critical models, by the mechanism which gave rise to the multi-critical models when discretized random surfaces were allowed to have negative weights). Let us briefly, for future reference, review how to solve for the critical exponents ([8, 17]). The self-consistent equation for the partition function for rooted branched polymers is shown in fig. 1 and reads

$$Z(\mu) = e^{-\mu} \left( 1 + \sum_n w(n) Z^n(\mu) \right).$$  \hspace{1cm} (2.10)$$

For future convenience let us introduce the two functions $f$ and $F$ defined by

$$f(z) = \sum_n w(n) z^n \quad \text{and} \quad F(z) = 1 + f(z).$$  \hspace{1cm} (2.11)$$

The functional relation between $\mu$ and $Z(\mu)$ is

$$e^\mu = F(Z).$$  \hspace{1cm} (2.12)$$

This equation allows us to determine $Z$ as a function of $\mu$ near the critical point $\mu_c$ which is the smallest value of $\mu$ for which (2.12) has a solution. Two situations can arise. Either $F(z)$ has a local minimum $z_c$, which is necessarily quadratic since all weights are positive, or the series which defines $f(z)$ has a finite radius of convergence $z_c$ and $F'(z_c) < 0$ exists, in which case $F(z_c)$ is still the absolute minimum. The critical behaviour is determined by expanding $F$ to the left of $z_c$. In the first case we get in the neighbourhood of the critical point $\mu_c$:

$$Z(\mu) \approx z_c - \frac{2}{F''(z_c)} \sqrt{\mu - \mu_c}.$$  \hspace{1cm} (2.13)$$

This leads to the well known result that the susceptibility exponent $\gamma_{\text{string}} = 1/2$ in the case where branched polymers of this kind can be considered a good approximation.

In the other case we have

$$\mu - \mu_c \approx c_1(z_c - Z) + c_2(z_c - Z)^2 + \cdots + c_n(z_c - Z)^{n+\lambda}$$  \hspace{1cm} (2.14)$$

where $n + \lambda$ is the first non-analytic power in the expansion, $0 < \lambda < 1$. By assumption we have $n \geq 1$, and by inverting the above relation we find:

$$Z \approx z_c + d_1(\mu - \mu_c) + d_2(\mu - \mu_c)^2 + \cdots + d_n(\mu - \mu_c)^{n+\lambda}.$$  \hspace{1cm} (2.15)$$
which leads to $\gamma_{\text{string}} = -n + 1 - \lambda < 0$.

In the case where $w(n) = n^{-\kappa}$ we have $\gamma_{\text{string}} = -\kappa + 2$ for $\kappa > 2$. The two cases ($\gamma_{\text{string}} = 1/2$ and $\gamma_{\text{string}} < 0$) are characterized by different patterns of branching when the number of links or vertices goes to infinity. For $\kappa < 2$ the model is the “ordinary” phase with $\gamma_{\text{string}} = 1/2$ and in which vertices of very high order are not too dominant. For $\kappa \to -\infty$ branching is increasingly suppressed, but for finite $\kappa$ the model always stays in the same universality class. For $\kappa > 2$ the ratio between the vertices of order one (ends) and the total number of vertices is large and goes to 1 for $\kappa \to \infty$. This means that there will be vertices of very large order present. Although this class of branched polymers might seem rather artificial, it is nevertheless precisely this class which will dominate if the weight of each triangulation is taken to be one and $d$ is large. If we on the other hand choose the so-called “conformal” weight $\rho(T) = \prod_i n_i^{c/2}$, we stay in the “ordinary” branched polymer phase for large $d$. This analysis was originally performed in [4] and is confirmed by numerical simulations.

### 2.2 Branched polymers coupled to Ising spins

It is most natural to couple Ising spins to branched polymers by associating the spin variables with the vertices of the branched polymers. If one wants to go back to the random surface picture and consider the polymers as coming from successive gluing of tetrahedra the spin assignment is slightly more complicated, but by universality the critical properties should be the same. The partition function can be written as:

$$Z(\mu, \beta, h) = \sum_{BP} e^{-\mu N_{BP}} \rho(BP) Z_{BP}(\beta, h)$$  \hspace{1cm} (2.16)

$$Z_{BP}(\beta, h) = \sum_{\{\sigma_i\}} e^{\beta \sum_{<i,j>} \sigma_i \sigma_j + h \sum_i \sigma_i}$$  \hspace{1cm} (2.17)

where the summation $\sum_{\{\sigma_i\}}$ is over all Ising spin configurations on the branched polymer, while $\sum_{<i,j>}$ is a sum over all neighbour pairs of vertices in a given branched polymer. Finally $h$ denotes an external magnetic field.

Let us denote by $Z_+(\mu, \beta, h)$ the partition function for rooted branched polymers with the spin fixed to be up at the root and let $Z_-(\mu, \beta, h)$ be the partition function with the spin down at the root. If we use the convention that no magnetic field is attached to the root we get the following equations:

$$Z_+ = e^{-\mu} \left[ 2 \cosh(\beta + h) + e^{\beta + h} f(Z_+) + e^{-(\beta + h)} f(Z_-) \right]$$  \hspace{1cm} (2.18)

$$Z_- = e^{-\mu} \left[ 2 \cosh(\beta - h) + e^{-(\beta - h)} f(Z_+) + e^{\beta - h} f(Z_-) \right]$$  \hspace{1cm} (2.19)
In the case where there is no external field we have by symmetry $Z_+ = Z_-$ and the solution to (2.18) and (2.19) is identical to the solution without Ising spins attached to the polymers except for a displacement of the critical point

$$\mu_c(\beta) = \mu_c(0) - \ln(2 \cosh \beta).$$

(2.20)

Until now we have considered the partition function $Z(\mu)$ where the “volume” $N$ is allowed to fluctuate (the grand canonical partition function). It is related to the canonical partition function $Z_N$ by

$$Z(\mu) = \sum_N Z_N e^{-\mu N},$$

(2.21)

and the asymptotic behaviour of $Z_N$ for large $N$ is

$$Z_N = N^{\gamma-2} e^{\mu c N} (1 + O(1/N)).$$

(2.22)

Since the bulk properties of the spin system are determined by the the infinite “volume” limit of the free energy $F(\beta)$ we get

$$F(\beta) = \lim_{N \to \infty} \frac{\ln Z_N(\beta)}{N} = \mu_c(\beta).$$

(2.23)

The infinite “volume” limit of the free energy of the spin system is just the critical value $\mu_c(\beta)$ and we see from (2.20) that the system of branched polymers never shows any critical behaviour when coupled to spin. The dependence is the same as in a linear chain of spins, a fact which reflects the tree-structure of the branched polymers we consider here. This result can easily be generalized to multiple Ising models coupled to branched polymers.

Although there is no critical behaviour for Ising spins coupled to branched polymers their behaviour is not necessarily identical to that of a linear chain of spins. In the last subsection we saw that there (in the case of positive weights) existed two classes of branched polymers characterized by $\gamma = 1/2$ and $\gamma < 0$, respectively. The latter class had a branching pattern dominated by a few vertices of very high order and it has an infinite Hausdorff dimension, both intrinsically and embedded in target space in agreement with the results of [15], while the $\gamma = 1/2$ branched polymers have intrinsic Hausdorff dimension two [17] and Hausdorff dimension four when embedded in target space. We will now show that the “ordinary” branched polymers with $\gamma = 1/2$ allow no spontaneous magnetization, in accordance with their tree-like structure which is much like a linear chain, while Ising spins on branched polymers with $\gamma < 0$ are always magnetized. The fact that the spins couple more
strongly for large $d_H$ is in accordance with intuition, but the fact that ordinary branched polymers with intrinsic Hausdorff dimension 2 have no phase transition shows that the intrinsic Hausdorff dimension is not always a reliable guide to the critical properties of systems with fractional dimension like the ones considered in this article.

In order to prove the above statements we expand equations (2.18) and (2.19) to first order in $h$ to determine the change $\Delta \mu(\beta, h)$ of the critical point due to a small change in $h$. According to (2.23) this allows us to calculate the spontaneous magnetization:

$$\langle M \rangle = \lim_{h \to 0^+} \frac{\partial F(\beta, h)}{\partial h}. \quad (2.24)$$

We treat the cases $\gamma = 1/2$ and $\gamma < 0$ separately. For $\gamma = 1/2$ we get (after some algebra) by expanding around the critical point (2.20) and using the notation $Z_{\pm}(h, \beta) = z_c(\beta) + \Delta Z_{\pm}$:

$$(\Delta Z_+ + \Delta Z_-) \left( e^{\mu_c(\beta)} - f'(z_c) \right) + 2z_c e^{\mu_c(\beta)} \Delta \mu = 0 \quad (2.25)$$

$$(\Delta Z_+ - \Delta Z_-) \left( e^{\mu_c(\beta)} - \tanh(\beta) f'(z_c) \right) = 4h(1 + f(z_c)) \sinh(\beta). \quad (2.26)$$

Since $e^{\mu_c(\beta)} = F(z_c(\beta))$ where $z_c(\beta)$ is determined by the requirement that

$$F'(z_c(\beta)) = 0 \quad \text{or} \quad f'(z_c(\beta)) = F(z_c(\beta)) = e^{\mu_c(\beta)} \quad (2.27)$$

we conclude that

$$\Delta \mu = O(h^2), \quad \Delta Z_+ - \Delta Z_- = h \left( e^{2\beta} - 1 \right) z_c(\beta). \quad (2.28)$$

In case where $\gamma < 0$, $z_c(\beta)$ coincides with the radius of convergence of $\sum_n w(n) z^n$ and $\Delta Z_+ = 0$ (since it is $\geq 0$ if we assume $h \geq 0$). If $z_c$ denotes the radius of convergence, the linearized equations for $\Delta Z_-$ and $\Delta \mu$ can now be solved and we get:

$$\Delta \mu(\beta, h) = h \tanh(\beta) \left[ \frac{F'(z_c)}{F'(z_c) + (1 + \tanh(\beta)) f'(z_c)/z_c} \right]. \quad (2.29)$$

Since $F'(z_c) \leq 0$ and $f'(z_c) > 0$ for $\gamma < 0$ the above expression is well defined and the system is magnetized for $\beta > 0$ except in the borderline case $F'(z_c) = 0$.

2.3 Multiple Ising spins coupled to two-dimensional gravity

The Ising model coupled to two-dimensional gravity was solved in the seminal papers [14, 15]. Unfortunately it seems impossible to use the same analytic technique for multiple Ising models coupled to gravity. This is why we in the next sections will
turn to numerical methods. It is, however, possible to give a few arguments which show how difficult it is to judge what will be the effective back-reaction of multiple spin systems. One could be tempted to argue as follows: Since the Ising model on a regular lattice, at the critical point, has a fermionic representation, it is natural to expect that one can just replace the bosonic determinant term $(\det C_T)^{-c/2}$ in (2.6) by $(\det C_T)^{n/2}$, where $n$ is the number of Ising spins. The effect of this would be drastic for large $n$ since it would be equivalent to taking $c \to -\infty$. This $c$-limit is known to correspond to dominance of regular or quasi-regular triangulations ([8]), quite contrary to the dominance of branched polymers for $c \to \infty$. Thanks to the exact solution of the Ising model coupled to two-dimensional gravity we know the above argument is incorrect for a single spin and the numerical results presented in the next section do not support it for multiple spin systems either.

3 Summary of results

Inspired by the observations in [11] we have coupled multiple $q$-state Potts models to two-dimensional quantum gravity for $q = 2, 3, 4$. These are the simplest discrete systems which on a regular lattice exhibit a second order phase transition and to which we can associate a central charge $c_q$. To be precise we place the spins at the (centers of) the triangles in the (abstract) triangulation and let them interact with the spins on neighbouring triangles. This is equivalent to placing the spins at the vertices in the dual $\phi^3$ graph.

The coupling to gravity will influence the critical properties of the system and the back-reaction of the spin system will modify the critical properties of two-dimensional gravity. For a single spin system the back-reaction is in a sense weak, since it is known that the critical properties of two-dimensional gravity are only changed at the critical point of the spin model. Whether the coupling remains weak in this sense for many spin systems coupled to gravity is unknown, but as we shall see the numerical results indicate that this is not the case.

We have verified numerically the existence of a transition between a magnetized phase and one where the average magnetization is zero, when many spin systems are coupled to two dimensional gravity. This transition seems to change its character when the number of systems increases as we describe below. From general considerations it appears most likely that there will always be a transition, no matter how many spin systems we couple to gravity. The reason is the following: For sufficiently small $\beta$ (high temperature) a single spin system is disordered for a whole range of temperatures and effectively the coupling of spins to gravity disappears when $\beta \to 0$. In this limit the multiple spins will consequently be independent. For
large $\beta$ we have a similar decoupling between spin degrees of freedom and gravity and we know that for a single spin system there is spontaneous magnetization, which for large $\beta$ goes to one. In this limit there is no interaction between gravity and spin. It is therefore consistent that the multiple spin system for large $\beta$ moves to a phase with magnetization. However, between the limits of small and large $\beta$ the coupling between gravity and spins can (and will) be strong and it depends on the number of spin systems seemingly parametrized to some extent by their central charge.

The back-reaction of the spin systems on gravity will manifests itself in a change of the average geometrical properties of the random surfaces. Whether such a change leads to a different critical behaviour is much more difficult to decide. We have already mentioned that a single Ising spin only influences the critical exponents of two-dimensional gravity when $\beta = \beta_c$, but it gives of course weights $\rho(T) = Z_T(\beta) \neq 1$ for each triangulation when $\beta \neq \beta_c$.

We have tried to estimate the back-reaction of the multiple spin system by measuring a number of quantities: The distribution of vertices of different orders, the intrinsic Hausdorff dimension and the average number of branches of a surface. For all these quantities we have compared the results with similar measurements made on multiple Gaussian systems coupled to gravity.

In order to define concepts like the average radius of a triangulation and the (intrinsic) Hausdorff dimension of the ensemble of triangulations we need the concept of geodesic distance on the triangulations. While the geodesic distance has a unique definition if we view the triangulation as a piecewise flat manifold glued together of equilateral triangles with the curvature assignment of Regge calculus, this definition is not convenient from a computational point of view. It is more convenient to define the geodesic distance between two vertices of a triangulation as the length of (one of) the shortest path within the triangulation which connects the two vertices. In addition we can talk about the shortest path between two triangles and define it to be the smallest number of triangles which constitute a connected path of triangles between the two. This length between two triangles obviously becomes the the shortest path between two vertices along links in the dual triangulation. With these definitions of geodesic length $r$ on a triangulation we can define the Hausdorff dimension $d_H$ in the usual way:

$$V(r) \propto r^{d_H},$$

where $V(r)$ denotes the number of triangles within geodesic distance $r$ of a given vertex or triangle. An average of $V(r)$ over the given manifold and over the ensemble of manifolds should now be performed in order to extract $d_H$ from (3.1). We can also define the (average) maximal radius $r_{\text{max}}$ of two-dimensional (piecewise flat)
manifolds in the following way: Take an arbitrary vertex (the “center”) and mark it. Mark all its neighbours. These are the vertices at distance one from the center. For all these points find their neighbours which are not already marked. These are the vertices a distance two from the center. In this way it is easy (from a numerical point of view) to compute $V(r)$ defined by (3.1). When all points are marked we have the maximal extension of the manifold with respect to the chosen center.

Let us now qualitatively summarise our observation and draw some conclusions. The detailed description of the numerical data will be given in the next section.

For the magnetic properties of the spin systems we have seen a transition from a phase with zero magnetization to a phase with magnetization. However with increased central charge the critical value of $\beta$, $\beta_c$, increases. It appears to be more difficult to magnetize the system. When the transition finally takes place, the crossover from the non-magnetized phase to the magnetized phase is increasingly rapid. At any given value of $\beta$ the magnetization is a decreasing function of the central charge for a given kind of spin systems. The phase transition point becomes increasingly difficult to localize by standard methods and the value of the magnetization at the critical point increases with increasing $c$. Many aspects of the systems (specific heat etc.) look for large $c$ quite similar to the results one obtains by coupling a single spin system and a multiple Gaussian system with the same central charge to gravity. One difference is that the latter system seems to have no phase transition in the spin variables.

These properties are what we would expect if we had a random surface system which degenerated to a branched polymer system with $\gamma = 1/2$ as the multiplicity of the spin system increases \[.\] In that case one would see a decreasing magnetization as a function of multiplicity as the random surface system changes towards branched polymers with $\gamma = 1/2$ which have no magnetization, as shown above. For sufficiently large $\beta$ the coupling will however weaken. The random surfaces return to their pure gravity phase, independent of the spin systems and the magnetization of the spin systems will be like that of a single system. If we however, as mentioned above, couple a single spin system and multiple Gaussian systems, the spin system is a small perturbation and the surfaces are forced to stay in the branched polymer phase where there is no transition. From this point of view there should be agree-

\[3\] In order to avoid confusion the following should be made clear: Since the spins live on the triangles it is appropriate to view them as located on the vertices of the dual lattice. The dominant configurations of branched polymers generated by the gaussian action discussed earlier are polymers which in the dual lattice look like long chains\[\] which should in a natural way be associated with polymers of $\gamma_{string} = 1/2$. 

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ment below the critical $\beta_c$ between multiple spin systems and multiple Gaussian plus one spin system. And this seems qualitatively to be the case.

The above observations raise the question at what temperature we should compare the spin systems to a Gaussian system. For large central charge the back-reaction from the spin systems on the geometry seems to be largest before we reach $\beta_c$. This is seen very clearly when we measure $r_{\text{max}}$, the maximal radius of the surfaces, as defined above. There is an extended region before $\beta_c$ where the influence of the spin systems on the geometry seems to consistent with, and at least as strong as that of a similar multiple Gaussian system. Intuitively this is in agreement with the hypothesis that surfaces in this region degenerate into branched polymers: The strong influence of the spin systems deforms the geometry to that of branched polymers and prevents thereby their own criticality since, as we have shown above, there can be no critical behaviour for branched polymers. It is only when $\beta$ becomes sufficiently large that the spins decouple (close to $\beta_c$) and the geometry returns to that of two-dimensional gravity.

From this point of view the situation is very different from the case of a single Ising system where we know that only at the critical point is the coupling between spin and gravity sufficiently strong to change $\gamma_{\text{string}}$. We conjecture that for the central charge $c$ sufficiently large ($c \geq 10 - 12$) there is a whole range of $\beta$ ending at $\beta_c$ where the geometry has undergone a transition to branched polymers like the ones induced by a multiple Gaussian system with the same central charge.

4 Data

The numerical results reported in the following sections is based on micro-canonical Monte Carlo simulations where lattice sizes ranged from 500 to 8000 triangles $N_T$. The standard “link flip” algorithm [3] was used to update the random triangulation, while the Swendsen-Wang cluster algorithm [16] was used to update the spins. This algorithm is known to be very efficient in eliminating critical slowing down close the spin phase transition.

Usually 40,000-100,000 sweeps were used, where one sweep consisted of updating the whole spin configuration and of performing $N_T$ link-flips.

As already mentioned we also looked at systems with Gaussian fields coupled to two-dimensional gravity. The Gaussian fields were placed on the triangles in analogue with the spin systems and a simple Metropolis algorithm was used to update them.

The measurements fall in two classes: Measurements of the “magnetic” properties of the spin systems when they are coupled to two-dimensional gravity and
measurements of quantities which characterize the intrinsic geometry of the triangulations, and thereby the back-reaction of the spins on two-dimensional gravity.

4.1 Magnetic properties of the spin systems

4.1.1 Multiple Ising spins

We have looked at 1, 2, 4, 8 and 16 Ising spin models coupled to gravity. In fig. 2 we have shown typical magnetization curves for one Ising system and sixteen Ising systems coupled to two-dimensional gravity. The value of the average magnetization $|\mathcal{M}|$ per spin per volume decreases with the multiplicity of the spins and the transition to a magnetized state is pushed to larger $\beta$.

In fig. 3 we show Binder’s cumulant

$$BC = \frac{\langle M^4 \rangle}{\langle M^2 \rangle^2} - 3 \quad (4.1)$$

for the two systems. It is well known that the use of Binder’s cumulant in the case of regular lattices offers an efficient way to decide whether we have a first or higher order transition and in addition, if the transition is of higher order, allows us to determine the critical point. $BC$ as a function of $\beta$ is much smoother for a higher order transition than it is in the case of a first order transition. In addition $BC$ for different volumes will intersect in a common point which is the critical point at infinite volume. In the case of dynamical triangulations the method works not quite as well, and one has to go to somewhat larger systems ($N_T > 1000$) than for regular lattices in order to see a clear picture. For a single or a few spin systems we eventually see curves corresponding to a higher order transition and curves of different (sufficiently large) volume intersect in the same point. As the number of spin systems increases the signal of a phase transition deteriorates. It becomes more difficult to identify a common intersection point and it (in case it really exists) moves to larger $\beta$-values and the value of $BC$ at the transition moves closer to $BC = -2$, which corresponds to magnetization close to one. From this point of view we are far from the nice situation of a single Ising spin where the intersection point for $BC$’s corresponds to $|\mathcal{M}| \approx 1/2$. We have illustrated this in fig. 4 in a curve which exhibits the average value $|\mathcal{M}|$ at the critical point as a function of the multiplicity of Ising spins.

In fig. 2 we also show for comparison the effect of coupling gravity to a multiple Gaussian systems. In this case it is known that the surfaces degenerate into branched polymers for sufficiently many gaussian fields. To this system we have now coupled a single Ising spin. The magnetization curve looks much like the one for a branched
polymer without constant magnetization or a one-dimensional chain. This similarity becomes more pronounced as the multiplicity of the Gaussian system increased from 8 to 16. Effectively one sees a magnetization $|M|$ for a finite value, but the slope is soft and gets softer with increasing volume. In addition it is impossible to find an intersection point for $BC$ (fig. 5). The main difference compared to the case of multiple Ising spins is that we have no phase transition to a magnetized state since the single spin system is only a small perturbation of the gaussian fields which, by their back reaction, force the geometry of the triangulations to be that of branched polymers. In fact the results of computer simulations of a single Ising spin directly on branched polymers look similar to the ones shown here.

4.1.2 Multiple q-state Potts models

For $q$-state Potts models ($q = 3, 4$) we have performed similar measurements to those reported above for the Ising model, which corresponds to $q = 2$. We have coupled 1,2,4,5,8,10 and 16 3-state Potts models to two-dimensional gravity and the same has been done for 1,2,4,8 and 16 4-state Potts models. The results for the magnetic properties are in agreement with those found for the multiple Ising models, i.e. the magnetization decreases with the multiplicity of the spin systems and the critical points, found as the intersection of the $BC$-curves, are located at a value of $BC$ closer and closer to -2.

The critical points change with the multiplicity, reflecting the interaction between the different spin systems mediated by two-dimensional gravity. One surprising aspect is that the displacement of the critical point seems to be a simple (almost linear) function of the multiplicity, identical for $q = 2, 3$ and 4. This is in contrast to the coupling to gravity which as we shall see later can be grouped according to the central charge $c$ of the multiple spin system.

4.1.3 Specific heat and susceptibility

The direct measurements of the magnetic susceptibility did not reveal any signals which were not compatible with it being the derivative of the magnetization curve. The peak moved to somewhat larger $\beta$ values and got steeper with increasing multiplicity.

The specific heat peak does not scale with lattice size for any of the systems. The peak gets less pronounced with increasing multiplicity. Again this happens relative smoothly as a function of multiplicity. When we compare the curves for sixteen Ising models coupled to gravity to that of one Ising model and eight Gaussian models coupled to gravity they look almost identical and qualitatively quite similar
to the curves one would get for a linear spin chain (or a branched polymer chain) where we know there is no transition, but where it is trivial to show that the specific heat nevertheless has a peak for a finite $\beta$ like the one shown in fig. 6 for sixteen Ising models and for one Ising model plus eight Gaussian models. For comparison we have shown the peak for one Ising model coupled to gravity.

From this observation it is tempting to guess that this reflects the linear structure (branched polymer structure) of the two-dimensional surfaces for such high value of the central charge.

### 4.1.4 Finite size scaling results

We have applied the methods of finite size scaling to the data obtained at the critical points $\beta_c$ for the various models in order to extract the critical behaviour of observables associated with the magnetic transition. In order to follow the usual philosophy of finite size scaling we must first extract a linear scale $L$ for the system. The only natural choice is

$$L \sim N_T^{1/d_H}$$

where $N_T$ is the number of triangles of the system and $d_H$ denotes the Hausdorff dimension. Finite size scaling then leads to the following relations at the critical point for the magnetization $\mathcal{M}$, the magnetic susceptibility $\chi$ and the specific heat $C_V$:

$$\mathcal{M} \sim N_T^{-\beta/\nu d_H}, \quad \chi \sim N_T^{\gamma/\nu d_H}, \quad C_V = B + C_0 N_T^{\alpha/\nu d_H}. \tag{4.3}$$

We have used lattices with $N_T = 500 - 4000$ and performed measurements for all spin models we have examined. For the magnetization and the susceptibility we get good linear fits to the log-log plots and the results are presented in table 1. We see reasonable (although not impressive) universality with respect to central charge. The trends are the same for all spins: $\gamma/\nu d_H$ increases with $c$ while $\beta/\nu d_H$ decreases.

It is more problematic to extract the specific heat exponents due to the constant term $B$ in (4.3). The fits we made indicated that $\alpha/\nu d_H$ approaches zero as the multiplicity is increased, but we do not consider this result reliable.

Let us end this section with a few remarks about the use of finite size scaling in quantum gravity. The standard arguments rely on the concept of a linear scale which characterizes the system of a given volume. In case there exists a Hausdorff dimension the identification (4.2) is sensible, but it is not clear whether there is any well defined Hausdorff dimension in two-dimensional quantum gravity. There are many indications that $d_H$ should be considered infinite \[8, 9\]. It is hard to believe that the interaction of a single Ising system with gravity should change this. In the same way it is hard to believe that the concept of a correlation length which
scales makes strict sense in such a space. Probably one should view $\nu$ as zero in this context and since the finite size scaling seems to give reasonable results it might be that the product $\nu d_H$ is well defined. It would be quite interesting to understand this better in case it is proven that $d_H = \infty$.

### 4.2 Back-reaction on gravity

We now turn to the measurements of quantities related to the geometry of the random surfaces.

#### 4.2.1 The maximal radius

The simplest quantity to measure is the maximal radius $r_{\text{max}}$ defined above. It turns out to be the most interesting, too. In fig. 7 we have plotted $r_{\text{max}}(\beta)$ for various multiple spin systems as a function of the coupling constant $\beta$. We see precisely the picture advocated during the discussion in the last section. For all spin systems $r_{\text{max}}$ decreases as $\beta$ increases from zero. This we take as a sign of an increased coupling between the dynamical triangulation and the spin systems. Close to the magnetic phase transition $r_{\text{max}}(\beta)$ rapidly increases to its value without spins: The spin systems are magnetized and effectively decouple from gravity. However, the important point is that $r_{\text{max}}$ seems to be lowered quite significantly with increasing multiplicity of the spin systems. Although there is not really any stringent universality with respect to central charge for these curves we can say that when the total central charge gets larger than 6-8, then $r_{\text{max}}$ seems to dive to its smallest values before $\beta$ reaches its critical value $\beta_c$. In the region where the system magnetizes there is an increasingly rapid change back to the size of random surfaces without spin systems. It is tempting to view the transition in geometry as one between a branched polymer phase (corresponding to that induced by the multiple Gaussian interaction) for $\beta$ in the $\beta$-region of low $r_{\text{max}}(\beta)$ to a phase where the geometry is like in the absence of spin systems. For comparison we show in fig. 8 a graph of $r_{\text{max}}$ as a function of $c$ for multiple Gaussian systems where there also is a rapid decrease of $r_{\text{max}}$ with increasing $c$.

#### 4.2.2 Fractal properties

Another obvious geometrical quantity to measure is the Hausdorff dimension. We have tried to extract it. As known it is quite problematic even in pure two-dimensional gravity \[13\], and this is unchanged when gravity is coupled to multiple spin systems. One observes the same phenomenon as for pure gravity, namely that the Hausdorff dimension seems to drift towards higher values with the system size.
This drift is seen for both definitions of the geodesic distance given above, but most clearly in the case where the “triangle distance” was used. Consequently it is difficult to extract a unique Hausdorff dimension and it makes better sense to compare the Hausdorff dimension for surfaces with different spin systems in order to reveal the general trend in change of geometry as the multiplicity of spins is varied. In fig. 9 we show the curves of radius versus volume for different spin systems, again at their tentative critical point. We observe universality with respect to central charge of the multiple spin systems, but no precise agreement if we compare with multiple gaussian systems of similar central charge. The fact that there is not agreement with the Gaussian systems for large values of $c$ is actually in agreement with the observation that the spin systems are increasingly magnetized at $\beta_c$ for large $c$ and therefore couple more weakly to gravity. In fact $r_{\text{max}}$ is almost constant at $\beta_c$ as a function of $c$, even if it decreases drastically for $\beta$ slightly smaller than $\beta_c$ as a function of $c$, cf. fig. 8. It is the region below $\beta_c$ which shares the characteristics of the multiple Gaussian systems.

It is important to keep in mind that the Hausdorff dimension is a very rough measure of geometrical properties of the system and it might not tell us much about the geometry. As examples we mentioned that both smooth surfaces and branched polymers with $\gamma = 1/2$ have intrinsic Hausdorff dimension two. We have therefore attempted to get a more detailed knowledge of the fractal properties of the geometry by measuring the number of connected components of the set of points at a certain distance from a fixed point (or fixed triangle). In the case of smooth surfaces one would measure few such components, while in the case of surfaces which resemble branched polymers we expect to see many components. The results of the measurements at the critical points are shown in fig. 10. We see an increase in the number of components as a function of $c$ for the Gaussian systems, but again the spin systems show a weak coupling to the geometry. The qualitative trend is towards many components and therefore increased branching and again there seems to be universality with respect to the central charge of the spin systems.

The branching is less pronounced than for Gaussian systems of the same central charge $c$. If we however go below the critical point we see the stronger coupling to gravity already observed for $r_{\text{max}}$. The effect is even slightly larger in a $\beta$ region below $\beta_c$ than the corresponding effect induced by a Gaussian system with the same central charge.
4.2.3 Distribution of orders of the vertices

The first observation is a confirmation and extension of the result found in [11]: If we consider the distribution of orders of vertices for multiple $q$-state Potts models ($q = 2, 3, 4$ corresponding to central charge $c_q = 1/2, 4/5, 1$) at their critical points, then combinations with identical central charge seem to lead to identical distributions. This is a strong argument in favor of the existence of some kind of universality for random surface theories even if $c > 1$. Of course the distributions are only identical within the errorbars of the simulations, but the errorbars are quite small and while (for instance) five $q = 4$ state Potts models lead to the same distributions as 8 Ising models coupled to gravity, there is a marked difference when we compare with the distribution of five Ising models. This is illustrated in figs. 11 and 12. It is interesting to show in addition the fraction of vertices of order three as a function of $\beta$ for the various multiple spin systems (fig. 13). It again reflects very clearly that the coupling to geometry is much stronger below $\beta_c$ than at the critical point, especially for multiple spin systems. As expected by now there is not any agreement if we compare the distributions of vertices at the critical $\beta_c$’s with similar distributions in the multiple Gaussian models of the same central charge. In order to get qualitative agreement one has to go below the critical $\beta$’s.

5 Discussion

We have tried to check the claim of universality put forward in [11, 12, 13]. We have verified and extended the remarkable universality in distributions of the orders of vertices of the multiple $q$-state Potts models as a function of the central charge. The universality seems also to extend to the fractal properties of triangulations coupled to multiple spin systems.

However, there is not universality when we allow Gaussian fields on the surfaces. This does not necessarily imply that the critical properties of two-dimensional gravity coupled to non-Gaussian fields are different from the critical properties of two-dimensional gravity coupled to Gaussian fields of the same central charge. There is no reason why the detailed distributions of the order of the vertices should be the same. The Hausdorff dimension is related to the critical exponents and should be the same if finite. Qualitatively the coupling of multiple spins and Gaussian fields to two-dimensional gravity lead to the same back-reaction on gravity. From the old grand canonical simulations of Gaussian fields coupled to random surfaces it is known that the system moves to a branched polymer phase for large values of the central charge $c$. This value is however not very precisely determined, but
is compatible with a value where the spin systems start to interact strongly with
gravity, where the radius of extension $r_{\text{max}}$ becomes small and the critical point
becomes difficult to identify by means of Binders cumulant. All this is consistent
with a branched polymer picture, but it is also clear from the data presented that
we cannot claim much more than consistency. The transition to branched polymers
is too soft to give a clear signal for the size of systems we are able to study by
numerical simulations, or maybe the transition is simply of a new kind and we
have just not found the right quantity to characterize it.

One could ask why we did not try to measure $\gamma_{\text{string}}$ directly for the multiple
spin systems, and in this way tested universality. The reason is that one would
have to use a grand canonical ensemble and the measurements of $\gamma_{\text{string}}$ for the
Gaussian models were never satisfactory [20]. Repeating those measurements would
at most have yielded a qualitative agreement with the Gaussian result, which we
have already been able to confirm without use of the grand canonical ensemble.

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Table 1 Numerical results for the critical exponents $\beta/\nu d_h$ and $\gamma/\nu d_h$ for the different number of Potts models coupled to two-dimensional gravity. The values are obtained using finite size scaling and the errors indicated are those of the 95% confidence levels.

| $N_{\text{models}}$ | $q = 2$  | $q = 3$  | $q = 4$  | $q = 2$  | $q = 3$  | $q = 4$  |
|---------------------|--------|--------|--------|--------|--------|--------|
| 1                   | 0.146(4) | 0.093(4) | 0.111(5) | 0.803(10) | 0.861(20) | 0.824(16) |
| 2                   | 0.095(3) | 0.077(3) | 0.084(4) | 0.888(10) | 0.856(13) | 0.860(12) |
| 4                   | 0.073(3) | 0.085(3) | 0.087(4) | 0.883(11) | 0.890(13) | 0.888(12) |
| 5                   | 0.061(3) |          |          | 0.894(13) |          |          |
| 8                   | 0.053(3) | 0.055(3) | 0.059(3) | 0.906(13) | 0.983(13) | 0.956(18) |
| 10                  |          | 0.042(2) |          |          | 0.933(15) |          |
| 16                  | 0.059(2) | 0.044(2) | 0.039(2) | 1.023(14) | 0.994(15) | 0.990(12) |
**Figure captions**

Fig.1 A graphical representation of the equation which determines the rooted partition function for branched polymers.

Fig.2 Magnetization curves for one and sixteen Ising models \((1 \ q = 2\text{ and }16 \ q = 2)\) coupled to two-dimensional gravity. Also shown are the curves for a single Ising model and, respectively, eight and sixteen Gaussian systems coupled to gravity \((1 \ q = 2\ & 8 \ G\text{ and }1 \ q = 2\ & 16 \ G)\).

Fig.3 Binder’s cumulant for one and sixteen Ising models coupled to gravity. The intersections of curves corresponding to the different lattice sizes yield the infinite volume critical points.

Fig.4 Average value of the magnetization \(|\mathcal{M}|\) in the critical points for 1,2,4,8 and 16 Ising system models coupled to gravity.

Fig.5 Binder’s cumulant for a single Ising model and sixteen Gaussian systems coupled to gravity. It is not possible to find any clear point of intersection for the different \(BC\) curves.

Fig.6 Specific heat curves for sixteen Ising models and a single Ising model plus eight Gaussian systems coupled to gravity. For comparison the specific heat curve for one Ising model coupled to gravity is also shown.

Fig.7 The maximal radius \(r_{\text{max}}\) as function of the reduced "temperature" \(\tau = (\beta - \beta_c)/\beta_c\). Measurements are shown for 1,8 and 16 Ising models, and 8 and 16 \(q = 4\) Potts models, coupled to two-dimensional gravity.

Fig.8 The changes in \(r_{\text{max}}\) as a function of \(c\) for one to sixteen Gaussian systems coupled to gravity. For comparison are shown the values of \(r_{\text{max}}\), measured in the critical points, for all the spin models we studied.

Fig.9 A log-log plot of the radius vs volume, measured at the critical points, for 8 and 16 \(q = 2\), 5 and 10 \(q = 3\) and 4 and 8 \(q = 4\) Potts models coupled to gravity. Also shown are the curves for 4 and 8 Gaussian systems coupled to gravity.

Fig.10 Number of connected components \(b\) (branching) as a function of the distance from a fixed point, for the same systems as considered in fig. 9.
Fig. 11 Fraction of vertices of order three as a function of the central charge $c$ for one to sixteen Gaussian systems coupled to gravity. This is also shown for all the spin models we studied, calculated at their critical points.

Fig. 12 Fraction of vertices as a function of their order $n_i$ for $16 \: q = 2$, $10 \: q = 3$ and $8 \: q = 4$ Potts models coupled to gravity, calculated at $\beta_c$. The corresponding curve for 8 Gaussian systems is also shown. In all cases the total central charge is 8.

Fig. 13 Fraction of vertices of order three as a function of the reduced "temperature" $\tau$ for the same systems as in fig. 7.