Wilson loop of the heterotic sigma model and the sv-map

Wei Fan$^{1,b}$

$^a$Department of Physics, Northeastern University, Boston, MA 02115, USA
$^b$Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, Poland

Abstract
The single-valued projection (sv) is a relation between scattering amplitudes of gauge bosons in heterotic and open superstring theories. Recently we have studied sv from the aspect of nonlinear sigma models [1], where the gauge physics of open string sigma model is under the Wilson loop representation but the gauge physics of heterotic string sigma model is under the fermionic representation since the Wilson loop representation is absent in the heterotic case. There we showed that the sv comes from a sum of six radial orderings of heterotic vertices on the complex plane. In this paper, we propose a Wilson loop representation for the heterotic case and using the Wilson loop representation to show that sv comes from a sum of two opposite-directed contours of the heterotic sigma model. We firstly prove that the Wilson loop is the exact propagator of the fermion field that carry the gauge physics of the heterotic string in the fermionic representation. Then we construct the action of the heterotic string sigma model in terms of the Wilson loop, by exploring the geometry of the Wilson loop and by generalizing the nonabelian Stokes’s theorem [2, 3, 4] to the fermionic case. After that, we compute some three loop and four loop diagrams as an example, to show how the sv for $\zeta_2$ and $\zeta_3$ arises from a sum of the contours of the Wilson loop. Finally we conjecture that this sum of contours of the Wilson loop is the mechanism behind the sv for general cases.

Keywords: superstring theory, sigma models, scattering amplitudes, multiple zeta values, wilson loop

1. Introduction
For tree-level string amplitudes, the single-valued projection (sv) [5] is a map between gluon amplitudes of the open superstring and gluon amplitudes of heterotic string [6, 7, 8]. For some recent proof see [9, 10, 11]. These tree-level string amplitudes are expressed in terms of multiple Gaussian hypergeometric functions, which contains the parameter of $\alpha' [12]$. The Taylor expansion of them in terms of small $\alpha'$ contains coefficients of multiple zeta values (MZV) at each order. Focusing on the single-trace part of the gluon amplitudes, when we take the expansion for the open superstring case and perform the sv on the MZV of the coefficients, the result is directly the expansion of amplitudes of the heterotic string case.

Generically, in the $\alpha'$-expansion of open superstring tree-level amplitudes the whole space of MZVs enters [13, 14], while closed superstring tree-level amplitudes exhibit only the subset of SVMZVs in their $\alpha'$-expansion [15, 6]. The relation between open and closed
string amplitudes through the the single-valued projection has been observed in [6] and established in [7].

Since all the information are encoded in the hypergeometric functions once for all, it’s hard to see the detailed origin or mechanism behind this sv. We need to go to the nonlinear sigma model approach to investigate its origin for each MZV at each order of $\alpha'$ expansion. We need to compute Feynman diagrams corresponding to single-trace gauge terms on the world-sheet, where the loop number of diagrams of the sigma model corresponds to the order of $\alpha'$ expansion of string amplitudes. We have studied sv from this aspect in the previous paper [1], and proposed a sv-map which states that the sv works on the Feynman diagram level for the corresponding sigma models.

The gauge physics of the open string sigma model can be studied in terms of the Wilson loop representation [15, 16]. The Wilson loop is directly gauge covariant, but is hard for the perturbation calculation because the path ordering nature of the Wilson loop is highly nontrivial [17, 18]. Since the gauge physics comes from the Chan-Paton factor on the boundary of the open string world-sheets, it is a purely 1D problem and the Wilson loop can be rewritten as a functional integral of a pair of auxiliary Grassmann fields [19, 20]. The Wilson line between two points is just the exact propagator of the Grassmann fields evaluated at those two points. The perturbation calculation is straightforward in this fermionic representation, but gauge covariance becomes nontrivial and hard to deal with.

The gauge physics of the heterotic string sigma model lives on the whole 2D world-sheets, and currently can only be described by the fermionic representation. The Wilson loop representation is still missing. Although the perturbation calculation is straightforward in the fermionic representation [21], the result of each Feynman diagram is not gauge covariant. The gauge covariance is hard to deal with, because by definition the model only has superconformal symmetry and does not have the gauge symmetry. A gauge covariant perturbation process was given in [22], but it involves a very specific nonlocal field redefinition procedure. In [1], we proposed a method of reorganized perturbation for the fermionic representation to put the perturbation in a gauge covariant manner.

However, the absence of a Wilson loop representation for the heterotic string sigma model is still a very bothering fact, because the open string sigma model has it. So in this paper, we will propose a Wilson loop representation for the gauge physics of the heterotic sigma model. It turns out that the Wilson line between two points on the 2D world-sheets is also the exact propagator of the fermionic field, just like the case of the open string. So now we have a complete correspondence of description of gauge physics between the open string and the heterotic string, as shown in figure [1].

Furthermore, after the gauge physics of both the open and the heterotic string sigma model are put under the Wilson loop representation, the sv-map [1] between them turns out to have a very simple geometric origin: the sv-map comes from the sum of two path-ordered integrals of opposite directions. The Wilson loop is path-ordered, so the Feynman integral of gauge physics is a path-order integral. For the open string case, the integration contour is just the boundary itself and we can compute the Feynman integral along this contour. For the heterotic case, we will show that the integration contour is a sum of two opposite-directed contours. When we sum the Feynman integral along these two opposite-directed contours, we will obtain a result which is the sv-map of the corresponding result of the open string case. In the previous paper [1], we showed that the sv-map comes
from a sum of six radial orderings of heterotic vertices on the complex plane when the
gauge physics of the heterotic sigma model is not under the Wilson loop representation.
In this paper, we show that the sv-map comes from a sum of two opposite-directed
contours, using the Wilson loop representation. So the geometric origin is simpler. To
show this perturbative calculation, we firstly need to compute the functional derivative
of the Wilson loop for the heterotic string case, which is just an analog of the open string
case [17]. Then we get the background field expansion of the Wilson loop for the heterotic
string case, which corresponds consistently (in the sense of exact propagators) to the
background field expansion of the fermionic representation obtained using reorganized
perturbation in our previous paper [1]. Finally we will compute several diagrams as an
example to show this geometric picture.

This paper is organized as following. In section 2, we will briefly review the open
string sigma model in the fermionic representation and in the Wilson loop representa-
tion. In section 3, we will study the heterotic string sigma model. Firstly, we will
briefly review the reorganized perturbation method in our previous paper [1] for the
fermionic representation. Then we will construct the Wilson loop, compute its func-
tional derivatives and prove that it is the exact propagator of the fermion field of the
fermionic representation. After that, we will investigate the geometry of the Wilson loop
and generalize the nonabelian Stokes’s theorem to the fermionic case, from which we
construct the heterotic sigma model action using the Wilson loop. In section 4, we will
explore the connection between the geometry of the Wilson loop and the sv-map. We
will compute several diagrams of three loop and four loop as an example and give the
conjecture for the general case.

2. Open superstring

In this section, we briefly review the fermionic representation and the Wilson loop
representation of the gauge physics of the open string sigma model. This is just for the

| gauge physics | open string | heterotic string |
|--------------|-------------|------------------|
|               | Chan-Paton factor | left-movers |
| stringy description | nonfermionic representation | fermionic representation |
| nonlinear sigma model | exact propagator | Wilson loop representation |
| Wilson loop representation | Wilson loop representation |
purpose of completeness, to be compared with the heterotic sigma model in section [9]. We are not going to talk about any perturbation calculations here.

2.1. Wilson loop representation

For the open string, the gauge degrees of freedom is manifested via the Chan-Paton factor and only lives on the boundary $\partial \Sigma$ of the open string world-sheet $\Sigma$, which is just a 1-dim curve (either the unit circle or the real axis depending on the choice) parameterized by $\tau$. The Wilson loop is directly parametrized by this 1-dim curve itself. The sigma model action can be written in a covariant manner on the Euclidean world-sheet as

$$S = S_{\Sigma} + S_{\partial \Sigma}$$

$$S_{\Sigma} = \frac{1}{4\pi \alpha'} \int d^2 \sigma_E (\partial X^\mu \partial X_\mu + \Phi^\mu \partial \Phi_\mu + \bar{\Phi}^\mu \partial \bar{\Phi}_\mu)$$

$$S_{\partial \Sigma} = \ln \text{Tr} \mathcal{P} \exp \{i \oint_{\partial \Sigma} d\tau (A_\mu(X) \partial_\tau X^\mu - \frac{1}{2} \phi^\mu \phi_\nu F_{\mu \nu})\}$$

$$= \ln \text{Tr} \mathcal{P} W[A],$$

(1)

where $\phi = \Phi|_{\Sigma} = \tilde{\Phi}|_{\Sigma}$ is the fermionic string on the boundary and the open superstring part $S_{\Sigma}$ follows from [23, Chapter 12.3] and the Chan-Paton part $S_{\partial \Sigma}$ is defined in [24].

The Wilson loop is $W[A] = \text{Tr} V[X, \phi, \tau, \tau]$ where $V[X, \phi, \tau_2, \tau_1]$ is the Wilson line defined as

$$V[X, \phi, \tau_2, \tau_1] := \mathcal{P} e^{i \int_1^{\tau_2} d\tau (A_\mu(X) \partial_\tau X^\mu - \frac{1}{2} \phi^\mu \phi_\nu F_{\mu \nu})}$$

(2)

For convenience, we will omit the path ordering symbol $\mathcal{P}$ from now on. Whenever we deal with the Wilson loop, the path ordering is always implicitly there.

The functional variation [17] of the bosonic part of the Wilson loop under $X \rightarrow X + \delta X$ is

$$V[X + \delta X, \tau_0, \tau_0] = e^{i \int d\tau (A_\mu(X) \partial_\tau X^\mu + \partial X^\nu [F_{\mu \nu} + \sum_{n=2}^1 \frac{1}{n!} D_{\mu_1} \cdots D_{\mu_n} F_{\mu_1 \cdots \mu_n} \xi^{\mu_1} \cdots \xi^{\mu_n}])}$$

$$= V[X] - i \delta X^\mu(\tau_0) A_\mu(X(\tau_0)) V[X] + i V[X] A_\mu(X(\tau_0)) \delta X^\mu(\tau_0)$$

$$+ i \int d\tau V[X, \tau_0, \tau] F_{\mu \nu} \partial_\tau X^\nu(\tau) \delta X^\mu(\tau) V[X, \tau, \tau].$$

(3)

Combined with the fermionic part it gives [18] the background field expansion of the action under $X \rightarrow X + \xi$

$$S_{\partial \Sigma}[X + \xi, \phi] = i \int d\tau \text{Tr} V[X] \{(A_\mu(X) \partial_\tau X^\mu + \partial X^\nu [F_{\mu \nu} \xi^{\mu_1} + \sum_{n=2}^1 \frac{1}{n!} D_{\mu_1} \cdots D_{\mu_n} F_{\mu_1 \cdots \mu_n} \xi^{\mu_1} \cdots \xi^{\mu_n}])$$

$$+ \frac{1}{2} F_{\mu_1 \mu_2} \xi^{\mu_1} \partial \xi^{\mu_2} + \sum_{n=3}^1 \frac{n-1}{n!} D_{\mu_1} \cdots D_{\mu_n} F_{\mu_1 \cdots \mu_n} \xi^{\mu_1} \cdots \xi^{\mu_n} \partial \xi^{\mu_n}]$$

$$- \frac{1}{2} [F_{\nu_1 \nu_2} \xi^{\nu_1} \phi^{\nu_2} + \sum_{n=1}^1 \frac{1}{n!} D_{\mu_1} \cdots D_{\mu_n} F_{\nu_1 \nu_2}(X) \phi^{\nu_1} \phi^{\nu_2} \xi^{\mu_1} \cdots \xi^{\mu_n}],$$

(4)

where $\xi$ and $\phi$ are treated as quantum fields.
2.2. Fermionic representation

The path ordering nature of the Wilson loop can be written in terms of the Heaviside step function. Since this is a 1D problem, the Heaviside step function can be viewed as the free propagator of a pair of fermionic coordinates \([15, 25]\). Then the Wilson loop of the open string can be viewed as coming from the following ordinary action

\[
S_{\partial \Sigma} = \int d\tau \bar{\psi}(\tau) \left( \frac{d}{d\tau} - i A_\mu(X) \partial_\tau X^\mu - \frac{1}{2} \phi^\mu \phi^\nu F_{\mu\nu} \right) \psi(\tau) \tag{5}
\]

where the pair of fermion coordinates \(\psi(\tau), \bar{\psi}(\tau)\) live on the boundary\([15]\). The Wilson line \(V[A]\) is just the exact propagator of this fermion coordinate \([19]\)

\[
V[X, \phi, \tau_2, \tau_1] = \langle \psi(\tau_2) \bar{\psi}(\tau_1) \rangle. \tag{6}
\]

3. Heterotic string

In this section, we will study the gauge physics of the heterotic string sigma model. For the heterotic string \([26]\), the gauge physics are generated by the 16 left-movers. By analog of the bosonization of two fermion fields, the gauge physics can be described by 32 real, anticommuting, left-moving, right-handed coordinates \(\psi^j\) which transform under the fundamental representation of \(SO(32)\). These coordinates are Majorana-Weyl spinors on the world-sheet and we will just call them fermion fields for simplicity. This is the fermionic representation of the heterotic sigma model and the action is given as \([23, \text{Chapter 12. 3}]\)

\[
S_E = \frac{1}{2\pi\alpha'} \int d^2z \{ \partial X^\mu \bar{\partial} X_\mu + \phi^\mu \bar{\partial} \phi_\mu + \psi^j \bar{\partial} \psi^j - i \psi \left( \partial X^\nu A_\nu - \frac{1}{2} F_{\mu_1 \nu_2} \phi^{\mu_1} \phi^{\nu_2} \right) \psi \}, \tag{7}
\]

where the equal time contour on the complex plane is the circle and \(\phi^\mu\) is the super partner of \(X^\mu\). This corresponds to the fermionic representation of the open string sigma model eq. \((5)\), except that the gauge terms here is a 2D integral while in the open string case it is a 1D integral.

3.1. Reorganized perturbation method

The advantage of the fermionic representation is that the perturbation calculation is straightforward. However, this perturbation process is not gauge invariant for each diagram, because of the presence of the term \(\partial A_\mu\) in the action. Only after combining all the diagrams at each loop can we get a gauge invariant result. If one is doing the complete renormalization, then this does not affect the final result. But if one just want to compute a subset of all the diagrams at each loop, then the lack of gauge invariant would be a trouble. In this case, a perturbation method that is gauge invariant at each diagram level is needed. In \([22]\), a specially chosen nonlocal field transformation is used to put the perturbation into a gauge invariant form at each term. In \([1]\), we use a reorganized perturbation method to achieve a gauge invariant perturbation without going through the nonlocal field transformation. We briefly recall this reorganized perturbation here, for the purpose of completeness and to be compared with the result of Wilson loop construction in next section \(3.2\).
Usually in the perturbation calculation

\[ e^{-S(F_{\mu\nu})} = \int DX D\phi D\psi e^{-S_E[X,A,\phi,\psi]}, \]

we compute diagrams involving all the propagators of \( X, \phi, \psi \) and combine all the diagrams at each loop to get a gauge invariant result. On the other hand, if we firstly integrate out the fermion fields \( \psi \)

\[ e^{-S_{eff}(X,F,\phi)} = \int D\psi e^{-S_E[X,A,\phi,\psi]}, \]

de we would get a gauge invariant effective action of the gauge field strength \( F_{\mu\nu} \). Then we can do the remaining functional integral perturbatively in a gauge invariant manner. But in practice, it is impossible to obtain a closed form for \( S_{eff} \) perturbatively, since we can not integrating out the \( \psi \) field in this brute force manner. Actually this effective action \( S_{eff}(X,F,\phi) \) is exactly the Wilson loop representation we want to construct, which corresponds to the open string case eq. (1). In the next section 3.2, we will use an indirect way to get the Wilson loop representation, and show that eq. (29) the Wilson line is the exact propagator of the fermion field, like the open string case eq. (6).

Our reorganized perturbation method is as following: firstly, we do the background field expansion \( X \to X + \xi \) and treat \( \xi, \phi, \psi \) as the quantum field; then we pick all the tree-level diagrams at each order of \( \alpha' \); finally we integrate out all the internal \( \psi \) propagators and get a gauge invariant results \( S(X + \xi, \phi, \psi, F_{\mu\nu}) \). Now the perturbation calculation is gauge invariant at each diagram.

The background field expansion of the gauge physics using the reorganized perturbation method [1] is as follows

\[ S_E = \frac{2}{4\pi\alpha'} \int d^2\sigma_E \left\{ \partial X^\mu \partial X_\mu + \phi^\mu \partial \phi_\mu + \psi^j \partial \psi^j \right. \\
- i\psi \left( \partial X^\nu A_\nu + \partial X^\nu \left[ F_{\mu_1 \nu_1} \xi^{\mu_1} + \sum_{n=2}^{1} \frac{1}{n!} D_{\mu_n} \ldots D_{\mu_2} F_{\mu_1 \nu_2} \xi^{\mu_1} \ldots \xi^{\mu_n} \right] \right) \\
+ \frac{1}{2} F_{\mu_1 \nu_2} \xi^{\mu_1} \partial \xi^{\nu_2} + \sum_{n=3}^{1} \frac{n-1}{n!} D_{\mu_{n-1}} \ldots D_{\mu_2} F_{\mu_1 \mu_n} \xi^{\mu_1} \ldots \xi^{\mu_n} \partial \xi^{\mu_n} \right\}. \]

The gauge terms here have the same structure as background field expansion of the Wilson loop in the open string case eq. (1), except that for the heterotic string the gauge terms are carried by the fermion fields \( \psi \).

3.2. Construct the Wilson loop

In this section, we will build up the Wilson loop of the gauge terms of the heterotic string sigma model, and show that it satisfies all the requirements. However, this is just a single quantity of Wilson loop, not the complete action. (In the next section 3.3 we will propose a way to rewrite the action of the heterotic sigma model using this Wilson loop, based on analyzing the geometry of the Wilson loop.)
So in this section we will only consider gauge terms of the heterotic sigma model

\[ L^f_E[A, \psi] = \psi Dz \psi + \frac{i}{2} \psi F_{\nu \mu \nu} \phi^\mu \phi^\nu \psi \]

\[ = \psi \partial_z \psi - i \psi (A_\mu \partial_z X^\mu - \frac{1}{2} F_{\nu \mu \nu} \phi^\mu \phi^\nu) \psi. \] (11)

The notation for the gauge fields are

\[ A_\mu = A_\mu^a T^a, \quad D_\mu = \partial_\mu - i [A_\mu, \cdot], \quad D_z = \partial_z - i [A_z, \cdot]. \]

Classically this Lagrangian has the gauge symmetry under the following transformation

\[ A_\mu \rightarrow UA_\mu U^\dagger + i U \partial_\mu U^\dagger \]

\[ \psi \rightarrow U \psi, \] (12)

which leads to \( F_{\mu \nu} \rightarrow UF_{\mu \nu} U^\dagger, \quad D_z \psi \rightarrow UD_z \psi \) and \( L^f_E[A, \psi] \rightarrow L^f_E[U, \psi]. \) Notice that the heterotic string sigma model is only required to have superconformal symmetry. The gauge symmetry defined above is just a field redefinition from the point of view of the world-sheet. That’s why we did not have gauge invariant results for each diagram in the perturbation calculation. Only after integrating out all the world-sheet, we get the space-time effective action of the gauge field with the true gauge symmetry. Here the purpose of constructing the Wilson loop is to get a gauge invariant result for each diagram, so the perturbation result is easier to see and to compare with the open string case.

To describe this gauge symmetry geometrically, firstly we need to build the Wilson line for an infinitesimal distance, then extend it to a finite length via path ordering and finally obtain the Wilson loop which is gauge invariant. See Peskin and Schroeder [27, Chapter 15] for the case of ordinary quantum field theory.

Here we define the Wilson line for an infinitesimal separation \( \epsilon \) in the same way as the open string case eq. (2)

\[ V[z_1 + \epsilon, z_1] := \exp\{i \int_{z_1}^{z_1 + \epsilon} dz [\partial_z X^\mu A_\mu - \frac{1}{2} F_{\nu \mu \nu} \phi^\mu \phi^\nu] \}, \] (13)

except that here \( \epsilon \) is on the complex plane while in the open string case it is on the boundary. The parametrization of the Wilson line is actually \( V[z_2, z_1] = V[X, \phi, z_2, z_1], \) but for convenience we will not explicitly distinguish them and will use whatever is convenient. Under the gauge transformation, this Wilson line transforms as

\[ V[z_1 + \epsilon, z_1] \rightarrow U[z_1 + \epsilon] V[z_1 + \epsilon, z_1] U[z_1]^\dagger, \] (14)

which is exactly the property we expect for the Wilson line. Now we define the Wilson loop by taking the trace for a path ordered loop of this Wilson line

\[ W[X, \phi, C] := \text{Tr} V[X, \phi, C, z, z], \] (15)

where \( C \) is a loop starting from \( z \) and ending at \( z \) and the dependence on the bosonic string \( X \) and the fermionic string \( \phi \) is written explicitly. Compared with the Wilson loop of the open string in section [2.1], the only difference is the contour on the world-sheet of the integral. For the open string, the contour is the 1-dim boundary. For the heterotic string, it can be any loop on the complex plane.
This definition of the Wilson line is simply an analog of the open string case and is not enough to justify itself. In the case of open string, the key property is that the Wilson line is the exact propagator of the fermion field eq. (6). To justify the definition for the heterotic case, we need to prove an analog of this property. To do this, we firstly need to investigate the background field expansion of the Wilson loop.

3.2.1. The functional variation of the Wilson loop

The perturbation calculation using the Wilson loop is usually done in the background field expansion. The background field expansion of the bosonic string is $X \rightarrow X + \delta X$, where $\delta X$ is a quantum field. For the fermionic string, it is taken as a quantum field itself $\phi = 0 + \phi$. In this way, the fermionic string contribution is an ordinary derivative

$$V[X+\delta X, \phi, z_0, z_0] = V[X, z_0, z_0] + \oint dz_1 V[X, z_0, z_1] \{-i \frac{1}{2} F_{\nu_1 \nu_2} (X(z_1)) \phi^{\nu_1} \phi^{\nu_2}\} V[X, z_1, z_0].$$

On the other hand, the functional derivative of the bosonic string contribution is highly nontrivial. We will follow the reference [17] to explain how to compute $V[X + \delta X, z_0, z_0]$.

For the Wilson loop, the background field expansion $X + \delta X$ is just a variation of the contour of the loop integral and the world-sheet coordinate serves as a parametrization of this contour. To do the functional derivative, we discretized the contour by $z_{j+1} - z_j = \epsilon$ and compute the variation, then take the limit $\epsilon \rightarrow 0$. The discretized loop is shown in figure 2. Then we define the following quantities.
\[a_j = 1 + i A_\mu(X(z_j))\delta X^\mu(z_j)\]
\[a_{j+1} = 1 + i A_\mu(X(z_{j+1}))\delta X^\mu(z_{j+1})\]
\[b_j = 1 + i A_\mu(X(z_j))(X^\mu(z_{j+1}) - X^\mu(z_j))\]
\[b'_j = 1 + i A_\mu(X(z_j) + \delta X(z_j))(X^\mu(z_{j+1}) + \delta X^\mu(z_{j+1}) - X^\mu(z_j) - \delta X^\mu(z_j)), \quad (16)\]

where \(b_j\) represents the infinitesimal segment of \(V[X + \delta X, z_0, z_0]\) and \(a_j\) characterizes the change of this infinitesimal segment under the variation \(X \to X + \delta X\). After a few algebra and just keeping the leading order of variation, we have the following result
\[a_j^{-1} b_j a_{j+1} b_j = 1 - i F_{\mu\nu}(X(z_j))(X^\nu(z_{j+1}) - X^\nu(z_j))\delta X^\mu(z_j). \quad (17)\]

Taking its inverse, we get the infinitesimal segment of the Wilson loop after the variation
\[b'_j = a_j^{-1} b_j (1 + i F_{\mu\nu}(X(z_j))(X^\nu(z_{j+1}) - X^\nu(z_j))\delta X^\mu(z_j)) a_j^{-1}.\]

Now the whole Wilson line after the variation is
\[\prod_{j=0}^{N} b'_j = b_N b_{N-1} \ldots b_0\]
\[= a_N+1 b_N(1 + i F_{\mu\nu}(X(z_N))(X^\nu(z_{N+1}) - X^\nu(z_N))\delta X^\mu(z_N)) a_N^{-1}\]
\[\times a_N b_{N-1}(1 + i F_{\mu\nu}(X(z_{N-1}))(X^\nu(z_N) - X^\nu(z_{N-1}))\delta X^\mu(z_{N-1})) a_N^{-1}\]
\[\ldots a_0 b_0(1 + i F_{\mu\nu}(X(z_0))(X^\nu(z_1) - X^\nu(z_0))\delta X^\mu(z_0)) a_0^{-1}\]
\[= a_0 b_N b_{N-1} \ldots b_0 a_0^{-1} + \sum_{j=0}^{N} a_0 b_N \ldots b_{j+1} [i F_{\mu\nu}(X(z_j)) \frac{X^\nu(z_{j+1}) - X^\nu(z_j)}{z_{j+1} - z_j} \delta X^\mu(z_j)] b_j \ldots b_0 a_0^{-1}. \quad (18)\]

Take the continuous limit \(\epsilon \to 0\), we obtain the functional variation of the bosonic part of the Wilson line
\[V[X + \delta X, z_0, z_0] = a_0 V[X, z_0, z_0] a_0^{-1} + a_0 \int dz V[X, z, z_0] [i F_{\mu\nu}(X(z)) \delta X^\nu(z) \delta X^\mu(z)] V[X, z, z_0] a_0^{-1}\]
\[= V[X, z_0, z_0] + i \{A_\mu(X(z_0)) V[X, z_0, z_0] - V[X, z_0, z_0] A_\mu(X(z_0))\} \delta X^\mu(z_0)\]
\[+ \int dz V[X, z, z_0] [i \delta X^\nu(z) F_{\mu\nu}(X(z)) \delta X^\mu(z)] V[X, z, z_0]. \quad (19)\]

Combine with the fermionic part, we get the complete variation
\[V[X + \delta X, \phi, z_0, z_0] = V[X, z_0, z_0] + i \delta X^\mu(z_0) A_\mu V[X, z_0, z_0] - i V[X, z_0, z_0] A_\mu \delta X^\mu(z_0)\]
\[+ i \int dz V[X, z_0, z_1] [\partial_z \delta X^\nu(z_1) F_{\mu\nu}(X(z_1)) \delta X^\mu(z_1)]\]
\[- \frac{1}{2} F_{\nu_1 \nu_2}(X(z_1)) \delta \phi^{\nu_1} \delta \phi^{\nu_2}] V[X, z_1, z_0]. \quad (20)\]

Taking its trace we get the function variation of the Wilson loop
\[W[X + \delta X, \phi] = W[X] + i \int dz_1 \text{Tr} \{V[X] [\partial_z \delta X^\nu(z_1) F_{\mu\nu}(X(z_1)) \delta X^\mu(z_1)]\}
\[\quad - \frac{1}{2} F_{\nu_1 \nu_2}(X(z_1)) \delta \phi^{\nu_1} \delta \phi^{\nu_2}\}, \quad (21)\]
The first part in the integrand is in the contribution from the bosonic string $X$ as in reference \[17\] and the second part is the contribution from the fermionic string $\phi$. From this functional variation, we can obtain the background field expansion of the Wilson loop

$$W[X + \xi, \phi] = i \int dz \text{Tr} V[X] \left\{ \partial_z X^\mu A_\mu + \partial X^\nu [F_{\mu\nu} + \sum_{n=2}^{\infty} \frac{1}{n!} D_{\mu_n} \ldots D_{\mu_1} F_{\mu_1} \xi^{\mu_1} \ldots \xi^{\mu_n}] + \left[ \frac{1}{2} F_{\mu_1} \xi^{\mu_1} \right] + \sum_{n=2}^{\infty} \frac{1}{n!} D_{\mu_n} \ldots D_{\mu_2} F_{\mu_1} \xi^{\mu_1} \ldots \xi^{\mu_2} \right\}. \tag{22}$$

This result is the same as eq. \[10\], except the difference between parameters $z$ and $\tau$. Now the Wilson loop between the open string case and the heterotic case corresponds to each other very well. In section 3.3, we will construct the action of the gauge physics using the Wilson loop for the heterotic sigma model, so the open string sigma model and the heterotic sigma model can correspond to each other in the level of action.

3.2.2. The exact propagator of $\psi$

Now let’s prove that the Wilson loop is the exact propagator of the fermion field $\psi$. Firstly, from the path ordering of the Wilson loop, we would have the following differential equation

$$\frac{d}{dz_2} V[z_2, z_1] = i(\partial_{z_2} X^\mu A_\mu(X(z_2)) - \frac{1}{2} F_{\mu_1} \phi^{\mu_1} \phi^{\mu_2})V[z_2, z_1], \tag{23}$$

which is just an analog of the differential equation of the time evolution operator in quantum field theory. Integrating out this differential equation gives us the path ordered Wilson line for a curve of finite length. This equation is essentially equivalent to the following variation

$$V[z_2 + \epsilon_2, z_1] = V[z_2, z_1] + \epsilon_2(\partial_{\epsilon_2} X^\mu A_\mu(z_2)) = V[z_2, z_1] + \frac{\partial V[z_2, z_1]}{\partial z_2} \epsilon_2$$

Secondly, we can obtain this variation in a different way using eq. \[20\]

$$V[z_2 + \epsilon_2, z_1] = V[X + \delta X, \phi + \delta \phi, z_2 + \epsilon_2, z_1]$$

$$= V[X, \phi, z_2, z_1] + \frac{\partial V[z_2, z_1]}{\partial z_2} \epsilon_2$$

$$+ i A_\mu(z_2) \delta X(z_2) V[z_2, z_1] - i V[z_2, z_1] A_\mu(z_1) \delta X(z_1)$$

$$+ i \int dz V[z_2, z] \left[ (\partial X^\nu(z) F_{\mu\nu} \delta X^\mu(z) - F_{\mu_1} \phi^{\mu_1} \phi^{\mu_2}) \right] V[z, z_1]. \tag{25}$$
Use $\delta X(z) = \partial X(z)\delta(z - z_2)\epsilon_2$ and $\delta \phi = \partial_\phi \delta(z - z_2)\epsilon_2 = 0$, the above equation becomes

$$
V[z_2 + \epsilon_2, z_1] = V[X, \phi, z_2, z_1] + \frac{\partial V[z_2, z_1]}{\partial z_2}\epsilon_2 + iA_\mu(z_2)\partial X^\mu(z_2)\delta(0)V[z_2, z_1]\epsilon_2
$$

$$
- iV[z_2, z_1]A_\mu(z_1)\partial X^\mu(z_1)\delta(z_2 - z_1)\epsilon_2 + i\partial X^\mu(z_2)\partial X^\nu(z_2)F_{\mu\nu}V[z_2, z_1]\epsilon_2
$$

$$
= V[X, \phi, z_2, z_1] + \frac{\partial V[z_2, z_1]}{\partial z_2}\epsilon_2 + iA_\mu(z_2)\partial X^\mu(z_2)\delta(0)V[z_2, z_1]\epsilon_2
$$

$$
- iV[z_2, z_1]A_\mu(z_1)\partial X^\mu(z_1)\delta(z_2 - z_1)\epsilon_2.
$$

Combining the two different ways of doing the variation, eq. (24) and eq. (26), we get

$$
\left[ \frac{\partial}{\partial z_2} - i(\partial_\nu X^\mu A_\mu(z_2) - \frac{1}{2}F_{\nu_1\nu_2}(z_2)\phi^{\nu_1^*}\phi^{\nu_2^*}) \right] V[z_2, z_1]
$$

$$
= -iA_\mu(z_2)\partial X^\mu(z_2)V[z_2, z_1]\delta(0) + iV[z_2, z_1]A_\mu(z_1)\partial X^\mu(z_1)\delta(z_2 - z_1).
$$

Except the $\delta(0)$ term, the Wilson line $V[z_2, z_1]$ is the inverse of the differential operator

$$
\frac{\partial}{\partial z_2} - i(\partial_\nu X^\mu A_\mu(z_2) - \frac{1}{2}F_{\nu_1\nu_2}(z_2)\phi^{\nu_1^*}\phi^{\nu_2^*}).
$$

Compared with eq. (11), we see that $V[z_2, z_1]$ is the exact propagator of the fermion field $\psi$,

$$
V[z_2, z_1] = \langle \psi(z_2)\psi(z_1) \rangle_{L_E}.
$$

For the $\delta(0)$ term, it can be incorporated into the normalization of the partition function of $\psi$, so we can just throw away this infinity term from the partition function.

Now this definition of the Wilson loop for the heterotic sigma model is justified. The relation between the Wilson loop and the fermionic representation for the heterotic sigma model, is exactly the same as that relation for the open string sigma model. The contour integral of the Wilson loop is equivalent to the ordinary perturbation in terms of the fermion field $\psi$. For the heterotic string, this relation is highly nontrivial, because the fermion field $\psi$ lives on the whole complex plane. For the open string, this relation is a trivial one, because its fermionic field $\psi$ just lives on the boundary and its propagator is just the Heaviside step function.

### 3.3. Geometry of the Wilson loop

The Wilson loop is a geometrical object that carry the gauge physics. To build up the action of the heterotic string sigma model using the Wilson loop, we need to explore its geometry. Firstly we will look at how we arrive at the classical Yang-Mills action using the Wilson loop and this will serve as a protocol. Then we will discuss the open string case and the heterotic string case. In all of these theories, the gauge invariant classical action is obtained from a sum over the loop contours, which is equivalent to a sum of all the gauge content over the spacetime.
3.3.1. Yang-Mills case

For the Yang-Mills case, we will use the lattice theory as a convenient illustration. In the lattice theory, the sum of Wilson loop over all the loops will generate the Yang-Mills action, as shown in the following equation (See Srednicki [28, Chapter 81] for details)

\[ S \approx \sum_{\text{loops}} W[\text{plaquette}], \]  

(30)

where \( W[\text{plaquette}] \) is the Wilson loop associated with a specific plaquette. \textit{This equivalence of the gauge invariant action and the sum over all the Wilson loops comes from the geometric nature of the Wilson loop.}

The line integral of the gauge field \( A_\mu \) in the Wilson loop is connected with the area integral of the field strength \( F_{\mu\nu} \) via

\[ \text{Tr} \mathcal{P} \exp \{i \oint_C dx A_\mu (x) \} = \text{Tr} \mathcal{P} \exp \{i \int_\Sigma d\sigma^{\mu\nu} (x) V[x_0, x] F_{\mu\nu} (x) V[x, x_0] \}, \]  

(31)

where \( \sigma^{\mu\nu} (x) \) is the area element on the surface \( \Sigma \) bounded by the closed loop \( C \) and \( V[x_0, x] = \mathcal{P} \exp \{i \int_{x_0}^x dy A_\mu (y) \} \). In the abelian case, this is simply the Stokes’s theorem. In the nonabelian case, this is called the nonabelian Stokes’s theorem and is highly nontrivial [2, 3, 4].

Now let’s go to a lattice theory to see how this Stokes’s theorem leads to the sum over loops. For simplicity (to be able to draw the figure), let’s assume a 3D spacetime lattice. And we choose the right-hand rule to associate the direction of the area with the direction of the loop. The gauge contribution to the Yang-Mills action should be a volume integral over the 3D spacetime. However, on the lattice, the gauge content is only defined on the 1D loops (the boundary of the plaquette). By the Stokes’s theorem, we extend the gauge content from the 1D loop to the 2D area (plaquette) bounded by the loop. In this way, the volume integral of the gauge content over the unit cube becomes the sum of the area integral of the gauge content over all the plaquettes of the cube. Let’s look at figure 3 for illustration. The gauge content \( \exp \{i \oint_{C_{1,2}} dX^\mu A_\mu \} \) is defined by the Wilson loop integral over the boundary of the plaquette \( \Sigma \), where there are two opposite directions \( C_1 \) and \( C_2 \) for the 1D loop. By Stokes’s theorem, the gauge content is extended to two area integrals over the plaquette (face of the cube) \( \exp \{i \int_{\vec{n}_1,2} d\sigma^{\mu\nu} F_{\mu\nu} \} \), with opposite normal directions \( \vec{n}_1 \) and \( \vec{n}_2 \) of the area. The area integral with normal direction \( \vec{n}_1 \) is associated with the gauge content of the left unit cube and the area integral with normal direction \( \vec{n}_2 \) is associated with the gauge content of the right unit cube.

In this way, the classical action \( S(\Sigma) \) of this plaquette \( \Sigma \), which is just the sum over both the directions of the Wilson loop \( W[C_{1,2}] \), turns out to be a sum of the gauge physics from the left cube and the right cube, which are all the unit cubes that are adjacent to the plaquette

\[ S(\Sigma) = \sum_{j=1}^2 \exp \{i \int_{C_{1,2}} dX^\mu A_\mu \} = \sum_{j=1}^2 \exp \{i \int_{\vec{n}_1,2} d\sigma^{\mu\nu} F_{\mu\nu} \} \]

= gauge content from the left cube + gauge content from the right cube

= sum of all the gauge content around \( \Sigma \).  

(32)
Figure 3: The sum of the Wilson loop over both the directions. In the lattice, each plaquette $\Sigma$ has two loops of opposite directions $C_1$ and $C_2$. By the Stokes’s theorem, the line integrals over $C_1$ and $C_2$ are connected with area integrals over the two area (faces) that have opposite normal directions $\vec{n}_1$ and $\vec{n}_2$. In this way, the Wilson loop over $C_1$ and $C_2$ are connected with the gauge contribution to the left cube and the right cube respectively.
3.3.2. Open string case

For the open string sigma model, the functional derivative of the Wilson loop eq. (33) is

\[ W[X + \delta X, \phi] = W[X] + i \int d\tau V[X] \{ \partial X^\mu F_{\mu\nu} \delta X^\nu - \frac{1}{2} F_{\nu\nu_2} \phi^{\nu_2} \phi^{\nu_2} \} + O(\delta X)^2. \]  

(33)

If we just look at the bosonic string part (set \( \phi = 0 \)), this equation is just the nonabelian Stokes’s theorem investigated in [3, 4, 15]. The area element is \( \delta \sigma_{\mu\nu}^{\text{bosonic}} = \delta \tau \partial X^\nu \delta X^\mu \) and the functional variation is an area integral of \( F_{\mu\nu} \). We will obtain the nonabelian Stokes’s theorem of the bosonic open string [15 9]

\[ \text{Tr} \mathcal{P} \exp \{ i \int d\tau \partial X^\mu (\tau) A_\mu (X) \} = \text{Tr} \mathcal{P} \exp \{ i \int d\sigma_{\mu\nu}^{\text{bosonic}} (X) V[X_0, X] F_{\mu\nu} (X(\tau)) V[X, X_0] \}, \]

(34)

where \( V[X_0(\tau_0), X(\tau)] = \mathcal{P} \exp \{ i \int_{\tau_0}^{\tau} d\tau' \partial X^\mu (\tau') A_\mu (X) \} \).

Now let’s turn on the fermionic string \( \phi \neq 0 \) and treat \( \phi \) itself as the variation (like \( \delta X \)). By analog of the bosonic area element, we define the area element of the fermionic string to be \( \delta \sigma_{\mu\nu}^{\text{fermionic}} = \delta \tau \phi^{\mu} \phi^{\nu} \) in the Grassmann space. Now the fermionic part of the functional derivative also becomes an area integral of \( F_{\mu\nu} \). Like the bosonic case, we can integrate out the functional derivative and obtain a fermionic contribution to the nonabelian Stokes’s theorem. So we obtain a generalization of the nonabelian Stokes’s theorem to the superstring

\[ W[C] = \text{Tr} \mathcal{P} \exp \{ i \int d\tau [ \partial X^\mu (\tau) A_\mu (X) - \frac{1}{2} F_{\nu\nu_2} (X) \phi^{\nu_2} \phi^{\nu_2} ] \} \]

\[ = \text{Tr} \mathcal{P} \exp \{ i \int [d\sigma_{\mu\nu}^{\text{bosonic}} - \frac{1}{2} d\sigma_{\mu\nu}^{\text{fermionic}}] V[X_0, X] F_{\mu\nu} V[X, X_0] \}, \]  

(35)

where \( V[X_0(\tau_0), X(\tau)] = \mathcal{P} \exp \{ i \int_{\tau_0}^{\tau} d\tau' [ \partial X^\mu (\tau') A_\mu (X) - \frac{1}{2} F_{\nu\nu_2} (X) \phi^{\nu_2} \phi^{\nu_2} ] \} \).

From this generalized nonabelian Stokes’s theorem, we can see the geometry of the Wilson loop in the open string case and then obtain the classical action from a sum over loops like the Yang-Mills case. Let’s look at figure 4 for illustration. For open string, the gauge field only lives on the boundary \( C \) of the string via the Chan-Paton factors. So the only loop we have is the boundary itself. After conformal transformation into the unit disk, the area \( \Sigma \) bounded by \( C \) is the disk itself. So by the Stokes’s theorem, the
3.3.3. Heterotic string case

For the heterotic string, the generalized nonabelian Stokes’s theorem can be obtained straightforwardly following the discussion of the open string case

\[ W[C] = \text{Tr} \mathcal{P} \exp \{ i \int dz [\partial X^\mu (z) A_\mu (X) - \frac{1}{2} F_{\nu_1 \nu_2} (X) \phi^{\nu_1} \phi^{\nu_2}] \} \]

\[ = \text{Tr} \mathcal{P} \exp \{ i \int \sigma^\mu [d\sigma^\mu_{\text{bosonic}} - \frac{1}{2} d\sigma^\mu_{\text{fermionic}}] V[X_0, X] F_{\mu \nu} (X) V[X, X_0] \} \tag{36} \]

where \( V[X_0(z_0), X(z)] = \mathcal{P} \exp \{ i \int z_0 dz' [\partial X^\mu (z') A_\mu - \frac{1}{2} F_{\nu_1 \nu_2} \phi^{\nu_1} \phi^{\nu_2}] \}. \) This is nearly the same as the open string one eq. (35), except that here the parametrization is \( z. \)

Let’s look at figure 5 for the geometry of the Wilson loop of the heterotic string. There are two types of loops on the closed string, the longitudinal one \( C_1 \) and the transversal one \( C_2 \). The conformal transformation maps the closed string into the whole complex plane and \( C_{1,2} \) are mapped into the real axis and the circle respectively. Let’s focus on \( C_1 \) first. The area bounded by \( C_1 \) is the upper half-plane \( \Sigma_1 \). By the nonabelian Stokes’s theorem, the Wilson loop of \( C_1 \) would give the gauge physics of the upper half-plane. If we revert the direction \(-C_1\), the area bounded will be the lower half-plane and the
nonabelian Stokes’s theorem would give the gauge physics of the lower half-plane. So if we sum the Wilson loop over the loop $C_1$ and $-C_1$, we will have the gauge physics of the whole complex plane, thus of the whole closed string. Now look at $C_2$. It is straightforward to see that the sum of the Wilson loop over $C_2$ and $-C_2$ will also give the gauge physics of the whole complex plane.

This result can be generalized to an arbitrary loop $C$. Because of the 2D nature of the closed string, the two areas bounded by $C$ and $-C$ are complementary and their sum is the whole complex plane. So by nonabelian Stokes’s theorem, we arrive at the following proposal for the Wilson loop approach of the heterotic string

$$e^{-S_{eff}[F_{\mu\nu}] = \sum_{\pm} \int_{C} D\phi DX e^{-S_{E}[X,A,\phi,C]}$$

$$= \sum_{\pm} \int_{C} D\phi DX \text{Tr} \mathcal{P} \exp\left\{-\frac{1}{2\pi\alpha'} \int d^2z [\partial X^\mu \partial X_\mu + \phi^\mu \partial \phi^\mu]\right\}$$

$$+ i \oint_{C} dz [\partial X^\mu(z) A_\mu(X) - \frac{1}{2} F_{\nu_1\nu_2}(X) \phi^{\mu_1} \phi^{\mu_2}].$$  \hspace{1cm} (37)$$

This is similar to the open string case eq. (1), except that now we have to sum over two directions of the contour. So the background field expansion of this action parallels that of the open string case, just replace $\tau$ in eq. (4) with $z$. Unlike the Yang-Mills case where the sum over Wilson loops is only calculable in lattice theory, here for the heterotic string, the sum over Wilson loops is practical: pick an arbitrary loop $C$, sum the Wilson loop over both directions $\pm C$, then the result is the classical action of the gauge field in spacetime.

3.4. Path ordering and contour direction

Since there are two directions of the contour, we need to give a comment about its relation with the path ordering of the gauge factors. We will take the convention of distinguishing the path ordering of the gauge field and the direction of the contour, i.e., we treat them as two different kinds of ordering. Firstly, we define the direction $C_+$ and $C_-$ for a contour loop. Since we are using the upper half plane for the open string world-sheet, we define $C_+$ to be from $-\infty$ to $\infty$ on the real axis and $C_-$ is just its inverse. Then, for a given vertex structure of a Feynman diagram, the gauge factors of the vertices are defined to be along the direction of $C_+$. Finally, when calculate this Feynman diagram, we just compute the integrand along $C_+$ in the open string case, and compute the integrand along both $C_+$ and $C_-$ in the heterotic string case. By this convention, we can just focus on the computation of the integrand, and leave the vertex structure of gauge factors aside.

4. Single-valued map

Now we can explore the sv-map using the Wilson loop representation for both the open string and the heterotic string. We will show how sv-map arise in three loop and four loop level for $\zeta_2$ and $\zeta_3$ respectively. Our purpose is to find the mechanism of the sv-map, rather than compute the complete beta function. So instead of pursuing the complete renormalization, we only compute the single pole (single logarithmic divergence) and will
show that for a given Feynman diagram of single-trace terms, the single poles of the open and the heterotic string integrals satisfy the sv-map. We will just focus on bosonic loops and compute a few representative diagrams of three loop and four loop, and show that sv-map connects the open case and the heterotic case in each diagram. (Representative here means they corresponds to permutations of the vertex structure at a given loop level).

For the computation we use the following setup (this is a recall of what we get)

\[
\text{open: } = \int \mathcal{D}X \mathcal{T} \mathcal{P} \exp \left\{ -\frac{1}{4\pi \alpha'} \int d^2z \left( \partial X^\mu \partial X_\mu + \Phi^\mu \partial \Phi_\mu + \Phi^\mu \partial \Phi_\mu \right) \right\} + \int_{C_+} dt [\partial X^\mu(t)A_\mu(X) - \frac{1}{2} F_{\mu \nu} (X) \phi^\nu \phi^\nu]
\]

\[
\text{hete: } = \sum_{C_+} \int \mathcal{D}X \mathcal{T} \mathcal{P} \exp \left\{ -\frac{2}{4\pi \alpha'} \int d^2z \left( \partial X^\mu \bar{\partial} X_\mu + \phi^\mu \bar{\partial} \phi_\mu \right) \right\}
\]

\[+ \int_{C_+} dz [\partial X^\mu(z)A_\mu(X) - \frac{1}{2} F_{\mu \nu} (X) \phi^\nu \phi^\nu], \tag{38}\]

where \(C_+\) is the real axis, \(C_-\) is its inverse and we will use variable \(t\) for the real axis from now on. The background field expansion is given in eq. (4). Both the propagators are \(\langle X^\mu(t_1)X^\nu(t_2)\rangle = -\eta^{\mu \nu} \alpha' \ln (t_1 - t_2)^2\) on the real axis. The bosonic propagators are represented by wavy lines and the contour of loop (the real axis) is represented by a solid line. A slash on the wavy line represents a derivative of the propagator, with respect to the most close vertex coordinate.

4.1. The zeta(2) case

Now, let's look at how the sv-map of \(\zeta(2)\) arises at three loop level. The mathematical sv of \(\zeta_3\) is \(sv(\zeta_2) = 0\) [9]. We will focus on the diagram shown in figure [6]. It contributes to the sigma model an ultra-violet divergent Lagrangian term of the form \(\partial X^\mu D_{\mu} F_{\nu} \phi^\nu F_{\mu \nu} \phi^\mu\). The open string integral associated to this diagram is

\[
I_{3,C_+} = (-2\alpha')^3 \int_{-\infty < t_1 < t_2 < t_3 < \infty} dt_1 dt_2 dt_3 f(V(t_1), V(t_2), V(t_3)) \frac{\ln t_{21}}{t_{31} t_{32}}
\]

\[= (-2\alpha')^3 \int_{-\infty}^\infty dt_3 f(V(t_3), V(t_3), V(t_3)) \int_{-\infty < t_1 < t_2 < t_3} dt_1 dt_2 \frac{\ln t_{21}}{t_{31} t_{32}}, \tag{39}\]

where \(t_{jk} = t_j - t_k\) and \(f(V(t_1), V(t_2), V(t_3)) = \partial X^\mu(t_1)D_{\mu} F_{\nu}(t_3)F_{\mu \nu}(t_3)\) is the vertex structure. In the last step we hide the constant factors \((-2\alpha')^3\) and the vertex structure, and just focus on the computation of the integrals. We will hide such constant factors and vertex structures in all the following computations.

To compute it, do the following change of variables

\[
w = t_{31}
\]

\[
u = \frac{t_{32}}{t_{31}} = \frac{t_{32}}{w}, \quad 0 < u < 1, \tag{40}\]
Figure 6: The Feynman diagram corresponding to structure $\partial X^\mu D_{\mu_1} F_{\nu_3}^{\mu_\nu_4} F_{\nu_3}^{\mu_4}$. Wavy lines are bosonic propagators and the solid line is the contour. The slash on the wavy line represents a derivative of the propagator, with respect to the most close vertex coordinate.

which changes the original integral as following

$$I_{3,C_+} = \int_{-\infty}^{\infty} dt_3 \int_{-\infty}^{0} dt_{13} \int_{t_{13}}^{0} dt_{23} \frac{\ln (t_{31} - t_{32})}{t_{31} t_{32}}$$

$$= \int_{-\infty}^{\infty} dt_3 \int_{0}^{\infty} dw \int_{0}^{1} du \frac{\ln w + \ln (1 - u)}{u}$$

$$= \int_{-\infty}^{\infty} dt_3 \left( -\zeta_2 (\ln \lambda - \ln \epsilon) \right), \quad (41)$$

where we use the brute force cutoff with $\epsilon$ the UV cutoff and $\lambda$ the IR cutoff. Only the single poles (single logarithmic divergence) are kept and higher order divergences are thrown away, the same to all the following computations.

For the heterotic integral we also need the other contour

$$I_{3,C_-} = \int_{-\infty}^{\infty} dt_1 \int_{t_3 > t_2 > t_1} dt_2 dt_3 \frac{\ln t_{21}}{t_{31} t_{32}}, \quad (42)$$

which is obtained in a similar manner as eq. (39). Notice that the contour $C_-$ means that we start from $\infty$ and to $-\infty$ (we will not explicitly mention this from now on). Here we do the following change of variables

$$w = t_{31}$$

$$u = \frac{t_{21}}{t_{31}} = \frac{t_{21}}{w}, \quad 0 < u < 1, \quad (43)$$
which changes the original integral as following

\[
I_{3,C^-} = -\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{0} dt_{31} \int_{t_{31}}^{0} dt_{21} \frac{\ln t_{21}}{t_{31} t_{42}}
\]

\[
= -\int_{-\infty}^{\infty} dt_1 \int_{0}^{\infty} \frac{dw}{w} \int_{0}^{1} du \frac{\ln w + \ln u}{1 - u}
\]

\[
= -\int_{-\infty}^{\infty} dt_1 (\zeta_2 (\ln \lambda - \ln \epsilon)). \tag{44}
\]

The heterotic integral is zero, which is just a sum of \(C_+\) and \(C_-\) given in eq. (41) and eq. (44). So we have the sv-map at three loop sv(\(\zeta_2\)) = 0.

### 4.2. The \(\zeta(3)\) case

Now let’s see how the sv-map of \(\zeta(3)\) arises at four loop. The mathematical sv of \(\zeta_3\) is \(sv(\zeta_3) = 2\zeta_3\) \[6\]. We choose three representative diagrams figure. 7, figure. 8 and figure. 9.

#### 4.2.1. Case 1

Firstly we compute diagram of figure. 7, which has the vertex structure \(\partial X^\nu D_{\mu_1} F_{\nu} \mu_3 F_{\nu_3} \mu_4 F_{\mu_4} \mu_5 F_{\mu_5} \mu_1\).

The open string integral is

\[
I_{41,C_+} = \int_{-\infty}^{\infty} dt_4 \int_{-\infty}^{\infty} dt_4 dt_3 dt_4 \frac{\ln t_{21}}{t_{41} t_{31} t_{43}}. \tag{45}
\]

Using the following change of variables

\[
w = t_{41}
\]

\[
v = \frac{t_{42}}{t_{41}} = \frac{t_{42}}{w}
\]

\[
u = \frac{t_{43}}{t_{42}}, 0 < u, v < 1. \tag{46}
\]
we get
\[
I_{41,C_+} = \int_{-\infty}^{\infty} dt_4 \int_{-\infty}^{0} dt_{14} \int_{t_{14}}^{0} dt_{24} \int_{t_{24}}^{0} dt_{34} \frac{\ln (t_{41} - t_{42})}{t_{41}(t_{42} - t_{43})t_{43}}
\]
\[
= \int_{-\infty}^{\infty} dt_4 \int_{0}^{\infty} dw \int_{0}^{1} dv \int_{0}^{1} du \frac{\ln w + \ln (1 - v)}{uv(1 - u)}
\]
\[
= 0.
\] (47)

There is no single pole. This is consistent with the results of [29], where no effective action terms were found corresponding to this respective part of the beta function.

The heterotic string integral needs the other contour
\[
I_{41,C_-} = \int_{-\infty}^{\infty} dt_4 \int_{t_{41}}^{\infty} dt_{41} \int_{t_{41}}^{0} dt_{31} \int_{t_{31}}^{0} dt_{21} \frac{\ln (t_{21})}{t_{41}(t_{31} - t_{21})(t_{41} - t_{31})}
\]
\[
= \int_{-\infty}^{\infty} dt_4 \int_{0}^{\infty} dw \int_{0}^{1} dv \int_{0}^{1} du \frac{\ln w + \ln v + \ln u}{(1 - u)(1 - v)}
\]
\[
= 0.
\] (50)

The heterotic integral is the sum of eq. (47) and eq. (50), while the open string integral is just eq. (47). So we see that \(sv(0) = 0\).

4.2.2. Case 2

Firstly we compute diagram of figure 8, which has the vertex structure \(\partial X^{\nu} D_{\mu_1} F_{\nu} \mu_3 F_{\mu_3} \mu_4 F_{\mu_4} \mu_5 F_{\mu_5}\).

The open string integral is
\[
I_{42,C_+} = \int_{-\infty}^{\infty} dt_4 \int_{-\infty}^{\infty} dt_{14} \int_{t_{14}}^{0} dt_{24} \int_{t_{24}}^{0} dt_{34} \frac{\ln t_{41} - t_{42}}{t_{41}(t_{41} - t_{43})t_{42}t_{43}}
\] (51)

Using the change of variables eq. (46), we get
\[
I_{42,C_+} = \int_{-\infty}^{\infty} dt_4 \int_{0}^{\infty} dw \int_{0}^{1} dv \int_{0}^{1} du \frac{\ln w + \ln (1 - v)}{uv(1 - uv)}
\]
\[
= \int_{-\infty}^{\infty} dt_4 (-2\zeta(3)(\ln \lambda - \ln \epsilon)).
\] (52)
Figure 8: The Feynman diagram corresponding to structure $\partial X^\nu D_\mu \ F_\nu \ ^\mu_3 F_\mu \ ^\mu_4 F_\mu \ ^\mu_5 F_\mu \ ^\mu_5$.

The heterotic string integral needs the other contour

$$I_{42,C-} = \int_{-\infty}^{\infty} dt_1 \int_{t_1 > t_2 > t_3 > t_4} dt_2 dt_3 dt_4 \frac{\ln t_{21}}{t_{31} t_{42} t_{43}}. \quad (53)$$

Using the change of variables eq. (49) we get

$$I_{42,C-} = -\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{0} dt_{14} \int_{t_{14}}^{0} dt_{24} \int_{t_{24}}^{0} dt_{34} \frac{\ln t_{21}}{t_{31} (t_{41} - t_{21})(t_{41} - t_{31})}$$

$$= \int_{-\infty}^{\infty} dt_1 \int_{0}^{\infty} dw \int_{0}^{1} dv \int_{0}^{1} du \ln w + \ln v + \ln u$$

$$= \int_{-\infty}^{\infty} dt_1 (-\zeta_3 (\ln \lambda - \ln \epsilon)). \quad (54)$$

The heterotic integral is the sum of eq. (52) and eq. (54), while the open string integral is just eq. (52). We see that the sv-map for $\zeta_3$ is satisfied $sv(\zeta_3) = (\frac{3}{4} \times 2\zeta_3).

4.2.3. Case 3

Firstly we compute diagram of figure 9 which has the vertex structure $\partial X^\nu D_\mu \ F_\nu \ ^\mu_3 F_\mu \ ^\mu_4 F_\mu \ ^\mu_5 F_\mu \ ^\mu_5$.

The open string integral is

$$I_{43,C+} = \int_{-\infty}^{\infty} dt_4 \int_{-\infty}^{\infty} dt_1 dt_2 dt_3 \frac{\ln t_{31}}{t_{41} t_{42} t_{32}}. \quad (55)$$

Using the change of variable eq. (46) we get

$$I_{43,C+} = \int_{-\infty}^{\infty} dt_4 \int_{-\infty}^{0} dt_{14} \int_{t_{14}}^{0} dt_{24} \int_{t_{24}}^{0} dt_{34} \frac{\ln t_{41} - t_{43}}{t_{41} t_{42} (t_{42} - t_{43})}$$

$$= \int_{-\infty}^{\infty} dt_4 \int_{0}^{\infty} dw \int_{0}^{1} dv \int_{0}^{1} du \ln w + \ln w (1 - uv)$$

$$= \int_{-\infty}^{\infty} dt_4 (2\zeta_3 (\ln \lambda - \ln \epsilon)). \quad (56)$$

The heterotic string integral needs the other contour

$$I_{43,C-} = \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_4 dt_3 dt_2 \frac{\ln t_{31}}{t_{41} t_{42} t_{32}}. \quad (57)$$
Using the change of variables eq. (49) we get

\[ I_{43,C_+} = -\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{0} dt_{41} \int_{0}^{t_{41}} dt_{31} \int_{t_{31}}^{0} dt_{21} \frac{\ln t_{31}}{t_{41}(t_{41} - t_{21})(t_{31} - t_{21})} \]

\[ = \int_{-\infty}^{\infty} dt_1 \int_{0}^{\infty} dw \int_{0}^{1} dv \int_{0}^{1} du \frac{\ln w + \ln v}{(1 - u)(1 - uv)} \]

\[ = \int_{-\infty}^{\infty} dt_1 (\zeta_3(\ln \lambda - \ln \epsilon)). \tag{58} \]

The heterotic integral is the sum of eq. (56) and eq. (58), while the open string integral is just eq. (56). We see that the sv-map for \( \zeta_3 \) is satisfied \( sv(\zeta_3) = \left( \frac{3}{4} \right) \times 2\zeta_3 \).

So from our computation, we have \( sv(\zeta_3) = \left( \frac{3}{4} \right) \times 2\zeta_3 \). There is a total factor of \( 3/4 \) here, compared with the mathematical result. We argue that this factor is just a total factor for all the diagrams at four loop level, so it can be incorporated into the action, since we found this same factor in both section 4.2.2 and section 4.2.3.

5. Conclusion

In this paper, we address the sv-map from the nonlinear sigma model approach. We show that the sv-map comes from a sum of two opposite-directed integral contours, when the gauge physics of both the open and the heterotic string sigma models are under the Wilson loop representation. In reference [1], the sv-map is shown to come from a sum of six radial orderings of heterotic vertices on the complex plane, when the gauge physics of the heterotic sigma model is not under the Wilson loop representation. So the Wilson loop representation gives sv-map a simpler geometric origin. To do that, we build a Wilson loop for the heterotic string sigma model and prove that it is the exact propagator of the fermion field that carry the gauge physics of the heterotic string in the fermionic representation. Then we construct the action of the heterotic sigma model.
using this Wilson loop, by studying the geometry of Wilson loop and generalizing the
nonabelian Stokes’s theorem into the fermionic case. We have shown how the sv-map
arises for $\zeta_2$ and $\zeta_3$ at three loop and four loop level, from the sum of contours of the
Wilson loop representation. Based on these, we finally conjecture that the sv-map of a
general MZV comes from this sum of contours of the Wilson loop representation.

6. Acknowledgments

The author is grateful to Stephan Stieberger for helpful suggestions and conversations.
The author is also grateful to Angelos Fotopoulos, Zygmunt Lalak, and Tomasz R. Taylor
for communications. This material is based in part upon work supported by the National
Science Foundation under Grant Number PHY1620575. Any opinions, findings, and
conclusions or recommendations expressed in this material are those of the author and
do not necessarily reflect the views of the National Science Foundation.

References

References

[1] W. Fan, A. Fotopoulos, S. Stieberger, T. Taylor, Sv-map between type i and heterotic sigma models,
Nuclear Physics B 930 (2018) 195 – 218. doi:https://doi.org/10.1016/j.nuclphysb.2018.02.024
URL http://www.sciencedirect.com/science/article/pii/S0550321318300690
[2] I. Arefeva, NonAbelian Stokes formula, Theor. Math. Phys. 43 (1980) 353, [Teor. Mat.
Fiz.43,111(1980)]. doi:10.1007/BF01018669
[3] P. M. Fishbane, S. Gasiorowicz, P. Kaus, Stokes’s theorems for non-abelian fields, Phys. Rev. D 24
(1981) 2324–2329. doi:10.1103/PhysRevD.24.2324
URL https://link.aps.org/doi/10.1103/PhysRevD.24.2324
[4] N. E. Bračić, Exact computation of loop averages in two-dimensional yang-mills theory
Phys. Rev. D 22 (1980) 3090–3103. doi:10.1103/PhysRevD.22.3090
URL https://link.aps.org/doi/10.1103/PhysRevD.22.3090
[5] F. Brown, Single-valued Motivic Periods and Multiple Zeta Values, SIGMA 2 (2014) e25. arXiv:
1309.5309 doi:10.1088/1121-8956/2/11/011
[6] S. Stieberger, Closed superstring amplitudes, single-valued multiple zeta values and the Deligne
associator, J. Phys. A47 (2014) 155401. arXiv:1310.3259 doi:10.1088/1751-8113/47/15/155401
[7] S. Stieberger, T. R. Taylor, Closed String Amplitudes as Single-Valued Open String Amplitudes,
Nucl. Phys. B881 (2014) 269–287. arXiv:1401.1218 doi:10.1016/j.nuclphysb.2014.02.005
[8] S. Stieberger, Periods and Superstring Amplitudes arXiv:1605.03630
[9] F. Brown, C. Dupont, Superstring amplitudes in genus 0 and 1, talk given by F. Brown in String
Math, Sendai, June 18, 2018.
[10] O. Schlotterer, O. Schnetz, Closed strings as single-valued open strings: A genus-zero deriva-
tion arXiv:1808.00713
[11] S. Stieberger, T. R. Taylor, Strings on Celestial Sphere arXiv:1806.05688
[12] D. Oprisa, S. Stieberger, Six gluon open superstring disk amplitude, multiple hypergeometric series
and Euler-Zagier sums arXiv:hep-th/0509042
[13] S. Stieberger, Constraints on tree-level higher order gravitational couplings in superstring theory
Phys. Rev. Lett. 106 (2011) 111601. doi:10.1103/PhysRevLett.106.111601
URL https://link.aps.org/doi/10.1103/PhysRevLett.106.111601
[14] O. Schlotterer, S. Stieberger, Motivic Multiple Zeta Values and Superstring Amplitudes, J. Phys.
A46 (2013) 475401. arXiv:1205.1516 doi:10.1088/1751-8113/46/47/475401
[15] H. Dorn, Renormalization of Path Ordered Phase Factors and Related Hadron Operators in Gauge
Field Theories, Fortsch. Phys. 34 (1986) 11–56. doi:10.1002/prop.19860340104
[16] E. Fradkin, A. Tseytlin, Non-linear electrodynamics from quantized strings, Physics Letters B
163 (1) (1985) 123 – 130. doi:http://dx.doi.org/10.1016/0370-2693(85)90205-9
URL http://www.sciencedirect.com/science/article/pii/0370269385902059

23
[17] E. Corrigan, B. Hasslacher, A functional equation for exponential loop integrals in gauge theories, Physics Letters B 81 (2) (1979) 181 – 184. doi:http://dx.doi.org/10.1016/0370-2693(79)90518-5
URL http://www.sciencedirect.com/science/article/pii/0370269379905185

[18] D. Brecher, M. Perry, Bound states of d-branes and the non-abelian born-infield action, Nuclear Physics B 527 (1) (1998) 121 – 141. doi:http://dx.doi.org/10.1016/S0550-3213(98)00297-1
URL http://www.sciencedirect.com/science/article/pii/S0550321398002971

[19] J. Gervais, A. Neveu, The slope of the leading Regge trajectory in quantum chromodynamics, Nuclear Physics B 163 (Supplement C) (1980) 189 – 216. doi:https://doi.org/10.1016/0550-3213(80)90397-1
URL http://www.sciencedirect.com/science/article/pii/0550321380903971

[20] R. A. Brandt, F. Neri, D. Zwanziger, Lorentz invariance from classical particle paths in quantum field theory of electric and magnetic charge, Phys. Rev. D 19 (1979) 1153–1167. doi:10.1103/PhysRevD.19.1153
URL https://link.aps.org/doi/10.1103/PhysRevD.19.1153

[21] A. Sen, The Heterotic String in Arbitrary Background Field, Phys. Rev. D32 (1985) 2102. doi:10.1103/PhysRevD.32.2102

[22] U. Ellwanger, J. Fuchs, M. G. Schmidt, The Heterotic $\sigma$ Model With Background Gauge Fields, Nucl. Phys. B314 (1989) 175. doi:10.1016/0550-3213(89)90117-X

[23] J. Polchinski, String theory, Cambridge University Press, Cambridge, UK New York, 1998.

[24] A. A. Tseytlin, On nonAbelian generalization of Born-Infeld action in string theory, Nucl. Phys. B501 (1997) 41–52. arXiv:hep-th/9701125, doi:10.1016/S0550-3213(97)90036-4

[25] I. Ya. Arefeva, QUANTUM CONTOUR FIELD EQUATIONS, Phys. Lett. 93B (1980) 347–353. doi:10.1016/0370-2693(80)90529-8

[26] D. J. Gross, J. A. Harvey, E. J. Martinec, R. Rohm, Heterotic String Theory. 1. The Free Heterotic String, Nucl. Phys. B256 (1985) 253. doi:10.1016/0550-3213(85)90394-3

[27] M. E. Peskin, D. V. Schroeder, An Introduction to quantum field theory, 1995.
URL http://www.slac.stanford.edu/spires/find/books/www?cl=QC174.45%3AP4

[28] M. Srednicki, Quantum field theory, Cambridge University Press, 2007.

[29] R. Medina, F. T. Brandt, F. R. Machado, The Open superstring five point amplitude revisited, JHEP 07 (2002) 071. arXiv:hep-th/0208121, doi:10.1088/1126-6708/2002/07/071

24