On the Hofer-Zehnder conjecture for non-contractible periodic orbits in Hamiltonian dynamics

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Abstract
In this paper, we treat non-contractible periodic orbits in Hamiltonian dynamics on symplectic manifolds. We prove that any Hamiltonian diffeomorphism has infinitely many simple non-contractible periodic orbits provided that the Hamiltonian diffeomorphism has at least one periodic orbit of infinite order in the first homology group. Our proof is an application of the equivariant Hamiltonian Floer cohomology.

1 Introduction and main results
In this section, we briefly explain the main theme of this paper. The precise definitions and notations are given in the next section. In this paper, we treat periodic orbits in Hamiltonian dynamics. The Hofer-Zehnder conjecture is a conjecture about the number of contractible periodic orbits in Hamiltonian dynamics and it states that “every Hamiltonian map on a compact symplectic manifold \((M, \omega)\) possessing more fixed points than necessarily required by the V. Arnold conjecture possesses always infinitely many periodic points” ([25] p. 263). This is a generalization of Franks’ theorem ([12], [13]), which states that any area-preserving homeomorphism on the sphere \(S^2\) with more than two fixed points has infinitely many periodic points. We have to clarify the meaning of “more fixed points than necessarily required by the V. Arnold conjecture”. The Floer cohomology group of a Hamiltonian function is generated by its contractible periodic orbits and it is isomorphic to the singular homology of the underlying manifold. So, the number of contractible periodic orbits required by the Arnold conjecture is the sum of the Betti numbers. In summary, the Hofer-Zehnder conjecture can be stated in the following form:

**Conjecture 1 ((Hofer-Zehnder conjecture))** Let \(\phi \in \text{Ham}(M, \omega)\) be a Hamiltonian diffeomorphism with more simple contractible periodic orbits than the total Betti number of \(M\). Then, \(\phi\) has infinitely many simple contractible periodic orbits.

Shelukhin proved this conjecture for spherically monotone symplectic manifolds with semi-simple quantum cohomology ring under the assumption that
periodic orbits are counted homologically (34). Readers may consider the relationship between the Conley conjecture and the Hofer-Zehnder conjecture. We say that the Conley conjecture holds on a closed symplectic manifold \((M, \omega)\) if every Hamiltonian diffeomorphism on \((M, \omega)\) has infinitely many simple contractible periodic orbits. Roughly speaking, the Hofer-Zehnder conjecture states that any Hamiltonian diffeomorphism with finitely many contractible periodic orbits is a so-called pseudo-rotation. A pseudo-rotation is a Hamiltonian diffeomorphism with the minimum number of contractible periodic orbits. The Conley conjecture and the Hofer-Zehnder conjecture imply that it is important to know the necessary and sufficient conditions for the existence of a pseudo-rotation on a symplectic manifold. The Conley conjecture implies that a symplectic manifold with a pseudo-rotation is very rare. In 4, 5, 21, 22, 35, 36, the importance of the existence of non-trivial pseudo-holomorphic curves was pointed out. They proved that the quantum Steenrod square is deformed if the symplectic manifold is monotone and has a pseudo-rotation. This is strong evidence of Chance-McDuff conjecture which states that the Conley conjecture holds if some Gromov-Witten invariants vanish or if the quantum cohomology ring is undeformed (18 19).

We can state an analogue of the Hofer-Zehnder conjecture for non-contractible periodic orbits. The set of non-contractible periodic orbits may be empty and the Floer cohomology of non-contractible periodic orbits is always trivial. This implies that the necessary number of non-contractible periodic orbits is zero. So, one is tempted to conjecture that the existence of one non-contractible periodic orbit implies the existence of infinitely many simple non-contractible periodic orbits. Gürel, Ginzburg-Gürel and Orita proved that any Hamiltonian diffeomorphism has infinitely many simple non-contractible periodic orbits provided that the Hamiltonian diffeomorphism has at least one periodic orbit of infinite order in the first homology group for some closed simplectic manifolds 17 23 30 31. In this paper, we apply equivariant Floer theory 33 34 37 and a key idea of Gürel 17. We prove this conjecture for very wide classes of symplectic manifolds.

Our main result is stated as follows: A symplectic manifold \((M, \omega)\) is called a weakly monotone symplectic manifold if it satisfies one of the following conditions. We explain the precise meaning of terminologies in the next section.

1. \((M, \omega)\) is monotone
2. \(c_1(A) = 0\) for every \(A \in \pi_2(M)\)
3. The minimal Chern number \(N > 0\) is greater than or equal to \(n - 2\)

Note that weakly monotone symplectic manifolds cover wide classes of symplectic manifolds. For example, every symplectic manifold whose dimension is less than or equal to 6 is a weakly monotone symplectic manifold. We prove the following result.

**Theorem 1.1** Let \((M, \omega)\) be a closed weakly monotone symplectic manifold, and let \(\phi \in \text{Ham}(M, \omega)\) be a Hamiltonian diffeomorphism generated by a periodic
Hamiltonian function $H \in C^\infty(S^1 \times M)$. Suppose that the number of 1-periodic orbits of $\phi$ in a non-trivial class $\gamma \neq 0 \in H_1(M; \mathbb{Z})/\text{Tor}$ is finite and non-zero. Further suppose that the local Floer cohomology $HF^{\text{loc}}(\phi, x)$ of at least one of these orbits is non-trivial. Then, for every sufficiently large prime $p$, $\phi$ must have at least one $p$-periodic or $p'$-periodic simple orbit in the class $p \cdot \gamma$. Here $p'$ is the smallest prime greater than $p$. In particular, there are infinitely many simple periodic orbits of $\phi$ with classes contained in the set of classes $\mathbb{N} \cdot \gamma$.

The assumption “weakly monotone” is purely technical. Our proof is based on the $\mathbb{Z}_p$-equivariant Floer cohomology theory and we need $\mathbb{Z}_p$-coefficient Floer theory. For general closed symplectic manifolds, we need the so-called virtual technique to define Floer theory ([16, 24]), but the virtual technique works over $\mathbb{Q}$-coefficient in general (see also [15]). We are not sure we can construct an equivariant theory on general closed symplectic manifolds.

2 Preliminaries

In this section, we explain notations and terminologies used in this paper.

2.1 Elementary notations

Let $(M, \omega)$ be a symplectic manifold, so $M$ is a finite-dimensional $C^\infty$-manifold and $\omega \in \Omega^2(M)$ is a symplectic form on $M$. In this paper, we always assume that $M$ is a closed manifold.

For any $C^\infty$-function $H \in C^\infty(M)$, we define the Hamiltonian vector field $X_H$ by the following relation:

$$\omega(X_H, \cdot) = -dH$$

We can also consider a $S^1$-dependent (in other words, 1-periodic) Hamiltonian function $H$ and a Hamiltonian vector field $X_H$ by the same formula. The time-one map of the flow of $X_H$ is called a Hamiltonian diffeomorphism generated by $H$ and is denoted by $\phi_H$. The set of all Hamiltonian diffeomorphisms is called the Hamiltonian diffeomorphism group and we denote the Hamiltonian diffeomorphism group of $(M, \omega)$ by $\text{Ham}(M, \omega)$. That is,

$$\text{Ham}(M, \omega) = \{ \phi_H \mid H \in C^\infty(S^1 \times M) \}.$$

We also consider “iterations” of $H$ and $\phi_H$. For any integer $k \in \mathbb{N}$, we define $H^{(k)}$ as follows:

$$H^{(k)} = kH(kt, x)$$

It is straightforward to see that $\phi_{H^{(k)}} = (\phi_H)^k$. Let $P^l(H)$ be the space of $l$-periodic orbits of $X_H$. Set $S^1_l = \mathbb{R}/l \cdot \mathbb{Z}$. Then

$$P^l(H) = \{ x : S^1_l \to M \mid \dot{x}(t) = X_{H_t(x(t))} \}.$$
It is also straightforward to see that there is a one-to-one correspondence between $P^k(H)$ and $P^l(H^{(k)})$. We abbreviate $P^l(H)$ to $P(H)$. An $l$-periodic orbit $x \in P^l(H)$ is called simple if there is no $l'$-periodic orbit $y \in P^{l'}(H)$ which satisfies the following conditions:

$$l = l' \cdot m \quad (l', m \in \mathbb{N})$$

$$x(t) = y(\pi_{l,l'}(t))$$

Here $\pi_{l,l'} : S_l \to S_{l'}$ is the natural projection. So a periodic orbit is simple if and only if it is not an iteration of a periodic orbit of a lower period.

Next, we explain the definition of the minimal Chern number $N$. $(M, \omega)$ becomes an almost complex manifold after a choice of a compatible almost complex structure $J$ on the tangent bundle $TM$. This gives the tangent bundle a structure of a complex vector bundle, hence allowing us to define its first Chern class $c_1(TM) \in H^2(M : \mathbb{Z})$. Moreover, the first Chern class does not depend on the choice of $J$ because the space of compatible almost complex structures on $M$ is contractible. The minimal Chern number $N \in \mathbb{N} \cup \{+\infty\}$ is the positive generator of the image of $c_1(TM)|_{\pi_2(M)}$. Note that if the image is zero, $N$ is defined by $N = +\infty$. A symplectic manifold $(M, \omega)$ is called monotone if the cohomology class of the symplectic form $[\omega]$ over $\pi_2(M)$ is a non-negative multiple of the first Chern class. In other words, there is a constant $\lambda \geq 0$ such that

$$\int_{S^2} v^* \omega = \lambda \int_{S^2} v^* c_1$$

holds for any smooth $v : S^2 \to M$. As we mentioned in the previous section, a weakly monotone symplectic manifold is defined as follows:

**Definition 2.1** A symplectic manifold $(M, \omega)$ is called weakly monotone if and only if it satisfies one of the following conditions:

1. $(M, \omega)$ is a monotone symplectic manifold.
2. $c_1(A) = 0$ holds for any $A \in \pi_2(M)$.
3. The minimal Chern number $N$ is greater than or equal to $n - 2$.

### 2.2 Floer cohomology theory

In this subsection, we explain Floer cohomology for non-contractible periodic orbits. References of this section are [7, 8, 9, 10, 11, 28, 16, 37, 1]. Essentially, there is nothing new in the non-contractible case, but in this paper, we need a non-contractible Floer cochain complex over the universal Novikov ring. Let $K$ be the ground field (In this paper, we consider the case of $K = \mathbb{F}_p$ where $\mathbb{F}_p$ is any field of characteristic $p$). We also assume that $(M, \omega)$ is a weakly monotone symplectic manifold. The universal Novikov ring $A$ is defined as follows:
\[ \Lambda = \left\{ \sum_{i=1}^{\infty} a_i \cdot T^{\lambda_i} \middle| a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}, \lambda_i \to +\infty \right\} \]

We need the subring \( \Lambda_0 \subset \Lambda \) as follows:

\[ \Lambda_0 = \left\{ \sum a_i \cdot T^{\lambda} \in \Lambda \middle| \lambda_i \geq 0 \right\} \]

We need non-contractible Floer theory over \( \Lambda \) and \( \Lambda_0 \). We fix \( \gamma \neq 0 \in H_1(M : \mathbb{Z})/\text{Tor} \) and we denote the set of 1-periodic orbits of \( H \in C^\infty(S^1 \times M) \) in \( \gamma \) by \( P(H, \gamma) \).

\[ P(H, \gamma) = \{ x \in P(H) \mid [x] = \gamma \} \]

The Floer cochain complex over \( \Lambda \) and \( \Lambda_0 \) is defined as follows. We assume that every periodic orbit in \( P(H, \gamma) \) is non-degenerate and set:

\[ CF(H, \gamma : \Lambda) = \bigoplus_{x \in P(H, \gamma)} \Lambda \cdot x \]

\[ CF(H, \gamma : \Lambda_0) = \bigoplus_{x \in P(H, \gamma)} \Lambda_0 \cdot x \]

Note that above \( CF(H, \gamma : \Lambda) \) and \( CF(H, \gamma : \Lambda_0) \) are not graded over \( \mathbb{Z} \) (It is possible to give a grading over \( \mathbb{Z}_2 \)). The differential operator \( d_F \) is defined as follows:

\[ d_F(x) = \sum_{y \in P(H, \gamma), \lambda \geq 0} n_\lambda(x, y) T^\lambda \cdot y, \]

where \( n_\lambda(x, y) \in \mathbb{K} \) is the number of the solutions of the following Floer equation modulo the natural \( \mathbb{R} \)-action:

\[ u : \mathbb{R} \times S^1 \to M \]

\[ \partial_su(s, t) + J_t(u(s, t)) (\partial_t u(s, t) - X_H(u(s, t))) = 0 \]

\[ \lim_{s \to -\infty} u(s, t) = x(t), \quad \lim_{s \to +\infty} u(s, t) = y(t) \]

\[ \lambda = \int_{\mathbb{R} \times S^1} u^*\omega + \int_0^1 H(t, x(t)) - H(t, y(t)) \, dt \]

**Remark 2.1** We have to achieve transversality of the linearized operator of this equation and determine an orientation of the moduli space to count the number of the solutions. They are very complex issues, but they are standard today. See the above references.

The Floer cohomology \( HF(H, \gamma : \Lambda) \) is defined to be the cohomology of \( (CF(H, \gamma : \Lambda), d_F) \). Similarly, \( HF(H, \gamma : \Lambda_0) \) is given by the cohomology of
(CF(H, γ : Λ₀), d_F). The Floer cohomology over Λ does not depend on the choice of Hamiltonian function H because we have the continuation map

\[ HF(H₁, γ : Λ) → HF(H₂, γ : Λ) \]

for any Hamiltonian functions H₁ and H₂ and this is an isomorphism. This is a very standard fact in the Floer cohomology theory (see for instance [24]). This implies that HF(H, γ : Λ) equals to zero because P(f, γ) is empty for C¹-small function f and hence HF(f, γ : Λ) is trivial. However, HF(H, γ : Λ₀) does not equal to zero in general. In fact, there is a sequence 0 < β₁ ≤ ⋅⋅⋅ ≤ βₖ such that

\[ HF(H, γ : Λ₀) ≅ \bigoplus_{i=1}^k Λ₀/T^β_i \cdot Λ₀ \]

holds [38] (see also [14]). We have the following filtration on (CF(H, γ : Λ), d_F).

For any c ∈ ℝ, we have the following subcomplex of (CF(H, γ : Λ), d_F):

\[ CF^c(H, γ : Λ) = \{ \sum_{x ∈ P(H; γ)} \left( \sum_{λ_i ≥ c} a_i \cdot T^λ_i \right) \cdot x ∈ CF(H, γ : Λ) \} \]

Then for any c < d, we define the following Floer complex with action window [c, d):

\[ CF^{[c,d]}(H, γ : Λ) = CF^d(H, γ : Λ)/CF^c(H, γ : Λ) \]

We define the Floer cohomology with action window [c, d) as follows:

\[ HF^{[c,d)}(H, γ : Λ) = H(CF^{[c,d)}(H, γ : Λ), d_F)) \]

2.3 Local Floer cohomology

We need the local version of the Floer cohomology theory, the so-called local Floer cohomology (see [20]). Let x ∈ P(H, γ) be an isolated periodic orbit of a Hamiltonian function H ∈ C∞(S¹ × M). In this subsection, we explain the local Floer cohomology HF_{loc}(H, x). Note that x is not necessarily a non-degenerate periodic orbit. Let \( U ⊂ S^1 \times M \) be a sufficiently small open neighborhood of the embedded circle \{ (t, x(t)) \} ⊂ S¹ × M and let \( \tilde{H} \) be a non-degenerate C∞-small perturbation of H. Then x splits into non-degenerate periodic orbits \{ x₁, ⋅⋅⋅, xₖ \} of \( \tilde{H} \) in U. The local Floer cochain complex \( CF^{loc}(H, x) \) is generated by these perturbed periodic orbits.

\[ CF^{loc}(H, x) = \bigoplus_{i=1}^k \mathbb{K} \cdot x_i \]

The boundary operator \( d_F^{loc} \) on \( CF^{loc}(H, x) \) is defined by counting solutions of the Floer equation in U. So, \( d_F^{loc} \) can be written by \( d_F^{loc}(x_i) = \sum n(x_i, x_j)x_j \)
where the coefficient \( n(x_i, x_j) \in \mathbb{K} \) is determined by the number of solutions of the following equation modulo the natural \( \mathbb{R} \)-action:

\[
\begin{align*}
    u : \mathbb{R} \times S^1 & \rightarrow M \\
    \partial_s u(s, t) + J_t(u(s, t))(\partial_t u(s, t) - X_{\tilde{H}_t}(u(s, t))) &= 0 \\
    \lim_{s \to -\infty} u(s, t) &= x_i(t), \quad \lim_{s \to +\infty} u(s, t) = x_j(t) \\
    (t, u(s, t)) &\in \mathcal{U}
\end{align*}
\]

**Remark 2.2** Any solution of the above Floer equation with small energy does not escape the isolating neighborhood of periodic orbits if the perturbation \( \tilde{H} \) is sufficiently small. This fact follows from the following argument. If this is false, we can choose a family of perturbations \( \tilde{H}_i \) which converges to \( H \) and a family of solutions \( u_i : \mathbb{R} \times S^1 \rightarrow M \) as follows:

- \( \partial_s u_i(s, t) + J_t(u_i(s, t))(\partial_t u_i(s, t) - X_{\tilde{H}_i}(u_i(s, t))) = 0 \)
- The energy \( E(u_i) = \int_{\mathbb{R} \times S^1} |\partial_s u_i(s, t)|^2 dsdt \) converges to zero.
- \( u_i \) escapes the isolating neighborhoods of periodic orbits.

By taking the limit of these \( u_i \) as \( i \to \infty \), we get a non-constant pseudo-holomorphic sphere

\[
v : \mathbb{C}P^1 \rightarrow (M, J_t)
\]

for some \( t \in S^1 \) or a non-constant Floer solution

\[
u : \mathbb{R} \times S^1 \rightarrow M \\
\partial_s u(s, t) + J_t(u(s, t))(\partial_t u(s, t) - X_{H_t}(u(s, t))) = 0
\]

This contradicts the assumption that the energy \( E(u_i) \) converges to zero because the energy of the above pseudo-holomorphic sphere \( v \) or the Floer solution \( u \) is greater than some positive constant \( \epsilon \) (the existence of such constant \( \epsilon \) follows from the Gromov compactness).

The local Floer cohomology \( HF^{loc}(H, x) \) is the cohomology of the cochain complex \((CF^{loc}(H), d_F^{loc})\). The following properties hold (see [20]).

1. If \( x \) is a non-degenerate periodic orbit of \( H \), then \( HF^{loc}(H, x) \cong \mathbb{K} \) holds.

2. \( HF^{loc}(H, x) \) does not depend on the choice of the perturbation \( \tilde{H} \) if it is sufficiently small.

3. Let \( \{\theta_1, \cdots, \theta_l\} \subset S^1 \setminus \{1\} \) be the set of eigenvalues of the differential map

\[
d\phi_H : T_x(0)M \rightarrow T_x(0)M
\]
on $S^1 \setminus \{1\} \subset \mathbb{C}$. An integer $k \in \mathbb{N}$ is called admissible if $\theta_i^k \neq 1$ holds for any $1 \leq i \leq l$. If $k$ is admissible, the local Floer cohomology of $(H, x)$ and $(H^{(k)}, x^{(k)})$ are isomorphic. In other words,

$$HF^{\text{loc}}(H^{(k)}, x^{(k)}) \cong HF^{\text{loc}}(H, x).$$

Note that the local Floer cochain complex $(CF^{\text{loc}}(H, x), d_F^{\text{loc}})$ is a cochain complex over a finite-dimensional vector space over $\mathbb{K}$ (not over $\Lambda$ nor $\Lambda_0$).

3 \quad \mathbb{Z}_p$-equivariant Floer theory

Our proof of Theorem 1.1 is an application of the $\mathbb{Z}_p$-equivariant Floer cohomology. In this section, we briefly review constructions and basic properties of the equivariant theory which we will use in our proof of Theorem 1.1. The readers can find the detailed constructions of the equivariant Floer theory in [37] (see also [33]). The first attempt in this direction is due to Seidel in [33], where he constructed $\mathbb{Z}_2$-equivariant Floer theory. After that, Seidel’s construction was generalized to $\mathbb{Z}_p$-equivariant Floer theory by Shelukhin-Zhao and Shelukhin in [37, 34]. In these papers, they gave various applications of the equivariant Floer cohomology. In particular, the equivariant Floer theory is very useful when we study the relationship between $HF(\phi)$ and the Floer cohomology $HF(\phi^p)$ associated to a prime iterate of $\phi$. As in [34], we will focus on the behavior of the torsion of $HF(\phi^p)$ as $p$ becomes bigger and bigger.

We fix a prime number $p$ and we assume that $H \in C^\infty(S^1 \times M)$ is a Hamiltonian function such that $H^{(p)}$ is non-degenerate. In this section, we assume that $(M, \omega)$ is a toroidally monotone symplectic manifold.

**Definition 3.1** A symplectic manifold $(M, \omega)$ is called toroidally monotone if there is a constant $\lambda \geq 0$ such that

$$\int_{T^2} v^* \omega = \lambda \int_{T^2} v^* c_1$$

holds for any smooth map $v : T^2 \to M$.

We will give a construction of the equivariant theory for weakly monotone symplectic manifolds in the last section, where we explain what is the technical difficulty in the weakly monotone case and how we can overcome this problem. The equivariant Floer cohomology is a mixture of the Floer theory on $(M, \omega)$ and the Morse theory on the classifying space.

Let $\mathbb{C}^\infty$ be the infinite-dimensional complex vector space

$$\mathbb{C}^\infty = \{ z = (z_k)_{k \in \mathbb{Z} \geq 0} \mid z_k \in \mathbb{C}, z_k = 0 \text{ for sufficiently large } k \}$$

and let $S^\infty \subset \mathbb{C}^\infty$ be the infinite-dimensional sphere defined by

$$S^\infty = \{ z \in \mathbb{C}^\infty \mid \sum |z_k|^2 = 1 \}.$$
There is a natural $\mathbb{Z}_p$-action and a shift operator $\tau$ on $S^{\infty}$ as follows:

$$(m \cdot z)_k = \exp\left(\frac{2\pi im}{p}\right) \cdot z_k \quad (m \in \mathbb{Z}_p)$$

$$(\tau(z))_k = \begin{cases} 0 & (k = 0) \\ z_{k-1} & (k \geq 1) \end{cases}$$

Let $f : S^{\infty} \to \mathbb{R}$ be the following Morse-Bott function:

$$f(z) = \sum_{k=0}^{\infty} k|z_k|^2$$

This $f$ satisfies $\tau^*f = f + 1$ and $f(z) = f(m \cdot z)$ for $m \in \mathbb{Z}_p$. The critical submanifolds are $S^l_1 = \{z \in S^{\infty} \mid |z| = 1\}$ and their indexes is $2l$. Next, we perturb $f$ to a Morse function with the above properties. For example, we can use the following explicit perturbation (see [37]):

$$\tilde{F} = f(z) + \epsilon \cdot \sum_k \text{Re}(z_k^p)$$

Then $\tilde{F}$ is a Morse function and $S^l_1$ splits into critical points $\{Z^l_{i+1} \cdot \tilde{F} = \tilde{F} + 1\}$ and $\{Z^l_{i+2} \cdot \tilde{F} = \tilde{F}(m \cdot z)\}$ hold. We also fix a Riemannian metric $\tilde{g}$ such that $\tilde{g}$ is invariant under $\mathbb{Z}_p$-action and $\tau$ and $(\tilde{F}, \tilde{g})$ is a Morse-Smale pair.

The Morse boundary operator $d_M$ of the pair $(\tilde{F}, \tilde{g})$ can be written in the following form:

$$d_M(Z^l_{i+1}) = Z^l_{i+2} - Z^l_{2i+1}$$

$$d_M(Z^l_{i+2}) = Z^l_{2i+2} + Z^l_{2i+1} + \cdots + Z^l_{2(p-i)+2} \quad (i \in \mathbb{Z}_p)$$

This $d_M$ is equivariant under the $\mathbb{Z}_p$ action $m \cdot Z^l_i = Z^l_{i+m(\text{mod} p)}$. Let $(H^p, J_t)$ be a pair consisting of a Hamiltonian function $H^p$ and a $S^1$-dependent almost complex structure on $(M, \omega)$. We want to extend $J_t$ to a family of $S^1$-dependent almost complex structures parametrized by $w \in S^{\infty}$. So, we consider a family of almost complex structures $\{J_{w,t}\}_{w \in S^{\infty}, t \in S^1}$ which satisfies the following conditions.

- (locally constant at critical points) For all $w$ in a small neighborhood of $Z^m_1$, $J_{w,t} = J_t - \frac{w}{p}$.

- ($\mathbb{Z}_p$-equivariance) $J_{m \cdot w, t} = J_{w, t} - \frac{w}{p}$ holds for any $m \in \mathbb{Z}_p$ and $w \in S^{\infty}$. 

9
• (invariance under the shift $\tau$) $J_{(w), \tau} = J_{w, \tau}$ holds.

We consider the following equation, which is a mixture of the Floer equation and the Morse equation for $x, y \in P(H^{(p)}), m \in \mathbb{Z}_p, \lambda \geq 0, \alpha \in \{0, 1\}, i \in \mathbb{Z}_{\geq 0}$.

\[(u, v) \in C^\infty(\mathbb{R} \times S^1, M) \times C^\infty(\mathbb{R}, S^\infty)\]
\[\partial_s u(s, t) + J_{v(s), t}(u(s, t))(\partial_t u(s, t) - X_{H^{(p)}}(u(s, t))) = 0\]
\[\frac{d}{ds} v(s) - \text{grad} \tilde{F} = 0\]
\[\lim_{s \to -\infty} v(s) = Z^{0}_{\alpha, i} \quad \lim_{s \to +\infty} v(s) = Z^{m}_{i} \quad \lim_{s \to -\infty} u(s, t) = x(t) \quad \lim_{s \to +\infty} u(s, t) = y(t - \frac{m}{p})\]
\[\int_{\mathbb{R} \times S^1} u^* \omega + \int_{0}^{1} H^{(p)}(t, x(t)) - H^{(p)}(t, y(t)) dt = \lambda\]

We denote the space of solutions of this equation by $M_{\alpha, i, m}^{\lambda}(x, y)$.

$M_{\alpha, i, m}^{\lambda}(x, y) = \{(u, v) \mid (u, v) \text{ satisfies the above equation}\} / \sim$

The equivalence relation $\sim$ is given by the natural $\mathbb{R}$-action on the solution space.

**Remark 3.1** We have to perturb the above equation to achieve transversality of the linearized operator. This can be achieved by a perturbation in a compact region $S^N \subset S^\infty$ because we assumed that $(M, \omega)$ is toroidally monotone. See \[\text{[33][34][77]}\]

We define $d_{\alpha}^{i, m} (\alpha \in \{0, 1\}, i \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_p)$ as follows:

\[d_{\alpha}^{i, m} : CF(H^{(p)}, \gamma : \Lambda_0) \longrightarrow CF(H^{(p)}, \gamma : \Lambda_0)\]
\[x \mapsto \sum_{y \in P(H^{(p)}), \lambda \geq 0} #M_{\alpha, i, m}^\lambda (x, y) \cdot T^\lambda y\]

Let $d_{\alpha}^{i}$ be the sum $d_{\alpha}^{i} = \sum_{m \in \mathbb{Z}_p} d_{\alpha}^{i, m}$. Note that $d_{0}^{0} = d_{1}^{1} = d_F$ holds. We define the $\mathbb{Z}_p$-equivariant Floer cochain complex as follows:

\[CF_{\mathbb{Z}_p}(H^{(p)}, \gamma) = CF(H^{(p)}, \gamma : \Lambda_0) \otimes \Lambda_0[[u]]\langle \theta \rangle\]

Here, $\langle \theta \rangle$ is the exterior algebra on the formal variable $\theta$ and $\Lambda_0[[u]]$ is the ring of formal power series of the formal variable $u$. So any element of $CF_{\mathbb{Z}_p}(H^{(p)}, \gamma)$ is written in the following form for some $k \in \mathbb{Z}$:

\[(x_k + y_k \theta) u^k + \sum_{i=k+1}^{\infty} (x_i + y_i \theta) u^i \quad (x_j, y_j \in CF(H^{(p)}, \gamma : \Lambda_0))\]
The equivariant Floer boundary operator $d_{eq}$ is a $\Lambda_0[[u]]$-linear map defined as follows:

$$d_{eq}(x \otimes 1) = \sum_{i=0}^{\infty} d_i^0(x) \otimes u^i + \sum_{i=0}^{\infty} d_i^{2i+1}(x) \otimes u^i \theta$$

$$d_{eq}(x \otimes \theta) = \sum_{i=0}^{\infty} d_i^{2i+1}(x) \otimes u^i \theta + \sum_{i=1}^{\infty} d_i^0(x) \otimes u^i$$

$d_0^0 = d_1^1 = d_F$ implies that $d_{eq}$ is a sum of $d_F$ and higher terms. The $\mathbb{Z}_p$-equivariant Floer cohomology $HF_{\mathbb{Z}_p}(H(p), \gamma)$ is defined by

$$HF_{\mathbb{Z}_p}(H(p), \gamma) = H((CF_{\mathbb{Z}_p}(H(p), \gamma), d_{eq}))$$

We also consider the coefficient extension of the $\mathbb{Z}_p$-equivariant Floer cochain complex. Let $\Lambda_0[u^{-1}, u]$ be the ring of the completion of Laurent polynomials. The $\mathbb{Z}_p$-equivariant Tate Floer cochain complex of $H(p)$ is defined as follows:

$$\hat{CF}_{\mathbb{Z}_p}(H(p), \gamma) = CF(H(p), \gamma) \otimes \Lambda_0[u^{-1}, u][\theta]$$

$$\hat{d}_{eq} : \hat{CF}_{\mathbb{Z}_p}(H(p), \gamma) \rightarrow \hat{CF}_{\mathbb{Z}_p}(H(p), \gamma)$$

Here, $\hat{d}_{eq}$ is the natural extension of $d_{eq}$. The $\mathbb{Z}_p$-equivariant Tate Floer cohomology of $H(p)$ is the cohomology of this cochain complex $(\hat{CF}_{\mathbb{Z}_p}(H(p), \gamma), \hat{d}_{eq})$ as follows:

$$\hat{HF}_{\mathbb{Z}_p}(H(p), \gamma) = H((\hat{CF}_{\mathbb{Z}_p}(H(p), \gamma), \hat{d}_{eq}))$$

**Remark 3.2** We consider coefficient extensions (in other words, Tate complexes) because the $\mathbb{Z}_p$-equivariant pair of pants product gives a local isomorphism between the local Floer cohomology and the local $\mathbb{Z}_p$-equivariant Tate Floer cohomology (see [33, 37]). We will explain this below and we will apply this local isomorphism.

**4 Local ($\mathbb{Z}_p$-equivariant Tate) Floer cohomology and homological perturbation theory**

In the proof of Theorem 1.1, we have to treat Hamiltonian diffeomorphisms with finitely many periodic orbits, which are not necessarily non-degenerate. One natural way to overcome this difficulty is to just perturb the original Hamiltonian diffeomorphism $\phi$ to a non-degenerate Hamiltonian diffeomorphism $\tilde{\phi}$ and consider $HF(\tilde{\phi} : \Lambda_0)$ instead of $HF(\phi : \Lambda_0)$. However, it is not sufficient because the structure of $HF(\tilde{\phi} : \Lambda_0)$ does depend on the perturbation $\tilde{\phi}$. We have to construct a cochain complex and a cohomology theory in a homologically canonical way. In this section, we explain the construction of Floer theory
for possibly degenerate Hamiltonian diffeomorphisms (see also [34]). This is an application of the homological perturbation theory in [27].

Assume that $H$ is a possibly degenerate Hamiltonian function and $P(H, \gamma)$ is finite for $\gamma \neq 0 \in H_1(M; \mathbb{Z})/$Tor. Let $H$ be a small perturbation of $H$ so that $\phi_H$ is non-degenerate. Then every element $x_i \in P(H, \gamma)$ splits into finitely many non-degenerate periodic orbits $Q(H, x_i) = \{x^1_i, \ldots, x^l_i\} \subset P(H, \gamma)$. We also fix a small isolating neighborhood of each $x_i \in P(H, \gamma)$. Let $U_i \subset S^1 \times M$ be a sufficiently small open neighborhood of $(t, x_i(t))$. First, we deform the complex $(CF(\tilde{H}, \gamma : \Lambda_0), d_F)$ in a canonical way. For any $x_i \in P(H, \gamma)$ and $x^j_i \in Q(\tilde{H}, x^j)$, we fix a sufficiently small connecting cylinder between $x_i$ and $x^j_i$ which is contained in the isolating neighborhood $U_i$ as follows:

$$v^j_i : [0, 1] \times S^1 \to U_i$$

$$v^j_i(0, t) = x_i(t), v^j_i(1, t) = x^j_i(t)$$

The “gap” of the action functional

$$c(x_i, x^j_i) = \int_{[0, 1] \times S^1} (v^j_i)^* \omega + \int_0^1 H(t, x_i(t)) - \tilde{H}(t, x^j_i(t))dt$$

does not depend on the choice of $v^j_i$. We define a correction map $\tau$ as follows:

$$\tau : CF(\tilde{H}, \gamma : \Lambda) \to CF(\tilde{H}, \gamma : \Lambda)$$

$$x^j_i \mapsto T^{v(x_i, x^j_i)} \cdot x^j_i$$

Note that $\tau$ is defined over $\Lambda$ (not over $\Lambda_0$). Then we can define the modified differential operator $\tilde{d}_F$ by $\tilde{d}_F = \tau^{-1} \circ d_F \circ \tau$. It is easy to see that $\tilde{d}_F$ can be defined over $\Lambda_0$ (see Lemma 4.1). Note that $\tau$ gives an isomorphism between Floer cochain complexes over $\Lambda$ (but not over $\Lambda_0$) as follows:

$$\tau : (CF(\tilde{H}, \gamma : \Lambda), \tilde{d}_F) \to (CF(\tilde{H}, \gamma : \Lambda), d_F)$$

$$x \mapsto \tau(x)$$

Let $\tilde{H}'$ be another non-degenerate perturbation of $H$ and we also denote the modified differential operator on $CF(\tilde{H}', \gamma : \Lambda)$ by $\tilde{d}_{\tilde{F}}$. It is straightforward to see that the natural continuation map between the Floer cochain complexes

$$\Phi : (CF(\tilde{H}, \gamma : \Lambda), d_F) \to (CF(\tilde{H}', \gamma : \Lambda), d_F)$$

descends to a continuation map over $\Lambda_0$ (not only over $\Lambda$) as follows:

$$\tilde{\Phi} : (CF(\tilde{H}, \gamma : \Lambda_0), \tilde{d}_F) \to (CF(\tilde{H}', \gamma : \Lambda_0), \tilde{d}_{\tilde{F}})$$

$$x \mapsto (\tau_{\tilde{H}'})^{-1} \circ \Phi \circ \tau_{\tilde{H}}$$

12
Lemma 4.1  The modified differential operator \( \tilde{d}_F \) is defined over \( \Lambda_0 \) if \( \tilde{H} \) is a sufficiently small perturbation of \( H \). Moreover, the modified continuation map \( \tilde{\Phi} \) is defined over \( \Lambda_0 \) if \( H \) and \( \tilde{H}' \) are sufficiently close to \( H \).

**proof** (Lemma 4.1): Let \( \tilde{H}_i \) be a series of perturbations of \( H \) which satisfies the following conditions.

- \( \tilde{H}_i \) converges to \( H \) in \( C^\infty \)-topology
- \( \tilde{d}_F \) is not defined over \( \Lambda_0 \) for each \( \tilde{H}_i \)

The second assumption implies that we can choose a sequence of cylinders \( u_i \) as follows:

\[
u_i : \mathbb{R} \times S^1 \rightarrow M \quad \partial_s u_i(s,t) + J_t(\partial_t u_i(s,t) - X_{\tilde{H}_i}(u_i(s,t))) = 0 \quad \exists (s_i, t_i) \in \mathbb{R} \times S^1 \text{ s.t. } u_i(s_i, t_i) \notin \cup_i U_i \]

\[
E(u_i) = \int_{\mathbb{R} \times S^1} |\partial_s u_i(s,t)|^2 dsdt \leq 2 \max \{ |c(x_i, x_j^i)| \mid x_j^i \in Q(\tilde{H}, x_i) \}
\]

So, the Gromov compactness implies that we get a cylinder \( u \) as follows:

\[
u : \mathbb{R} \times S^1 \rightarrow M \quad \partial_s u(s,t) + J_t(\partial_t u(s,t) - X_{\tilde{H}}(u(s,t))) = 0 \quad \exists (a, b) \in \mathbb{R} \times S^1 \text{ s.t. } u(a, b) \notin \cup_i U_i \]

\[
E(u) = \int_{\mathbb{R} \times S^1} |\partial_s u(s,t)|^2 dsdt = 0
\]

This is a contradiction because \( E(u) = 0 \) implies that \( u \) is a “constant” solution \((\partial_s u(s,t) \equiv 0)\), but the image of \( u \) is not contained in the union of isolating neighborhoods \( U_i \). A similar argument can be used to prove that \( \tilde{\Phi} \) is defined over \( \Lambda_0 \) if perturbations are sufficiently close to \( H \).

\( \square \)

The continuation map of the inverse direction

\[
\Psi : (CF(\tilde{H}', \gamma : \Lambda), d_F) \rightarrow (CF(\tilde{H}, \gamma : \Lambda), \tilde{d}_F)
\]

also descends to a continuation map

\[
\tilde{\Psi} : (CF(\tilde{H}', \gamma : \Lambda_0), \tilde{d}_F) \rightarrow (CF(\tilde{H}, \gamma : \Lambda_0), \tilde{d}_F)
\]
in the same way. The cochain homotopy maps between the identity and \( \Psi \circ \Phi \), \( \Phi \circ \Psi \) also descend to cochain homotopy maps between the identity and \( \tilde{\Psi} \circ \tilde{\Phi} \), \( \tilde{\Phi} \circ \tilde{\Psi} \). This implies that the perturbed and modified Floer cochain complex \((CF(\tilde{H}, \gamma : \Lambda_0), \tilde{d}_F)\) is unique up to cochain homotopy equivalence. So we can
define the Floer cohomology of $H$ by the cohomology of this cochain complex and this is well defined.

$$HF(H, \gamma : \Lambda_0) = H(CF(\tilde{H}, \gamma : \Lambda_0), \tilde{d}_F)$$

However, this construction is not sufficient for our purpose. The cochain complex $(CF(\tilde{H}, \gamma : \Lambda_0), \tilde{d}_F)$ is not strict in the following sense. “Strict” means that there is $\epsilon > 0$ such that

$$\tilde{d}_F(z) \in CF^\epsilon(\tilde{H}, \gamma : \Lambda_0)$$

holds for any $z \in CF(\tilde{H}, \gamma : \Lambda_0)$. Note that we can identify $CF(\tilde{H}, \gamma : \Lambda_0)$ and $\bigoplus_{x \in \mathcal{P}(H, \gamma)} CF^\text{loc}(H, x) \otimes \Lambda_0$. Then $\tilde{d}_F$ can be decomposed into the sum of $\tilde{d}_F = d^\text{loc}_F + D$ where $D$ is a higher energy term.

$$D(CF(\tilde{H}, \gamma : \Lambda_0)) \subset CF^\epsilon(\tilde{H}, \gamma : \Lambda_0) \quad (\epsilon > 0)$$

The existence of such decomposition for sufficiently small perturbation $\tilde{H}$ follows from the following arguments. Let $\delta_1 > 0$ be the smallest energy of non-constant solutions of the Floer equations and let $\delta_2$ be the smallest energy of non-constant pseudo-holomorphic spheres as follows.

$$\delta_1 = \min \left\{ E(u) > 0 \left| u : \mathbb{R} \times S^1 \to M \text{ s.t. } \partial_s u + J_t(\partial_t u(s, t) - X_{\tilde{H}}(s, t)) = 0 \right. \right\}$$

$$\delta_2 = \min \left\{ E(v) = \int_{\mathbb{C}P^1} v^* \omega > 0 \left| v : \mathbb{C}P^1 \to M \text{ s.t. } J_t \circ dv = dv \circ j_{\mathbb{C}P^1} \right. \right\}$$

We fix a constant $\epsilon < \min\{\delta_1, \delta_2\}$. If the existence of such decomposition is false, we can find a sequence of perturbations $\tilde{H}_i$ and a sequence of solutions of the Floer equations $\{u_i\}$ as follows:

- $\tilde{H}_i$ converges to $H$ in $C^\infty$-topology
- $\partial_s u_i(s, t) + J_t(\partial_t u_i(s, t) - X_{\tilde{H}_i})(u_i(s, t)) = 0$
- $E(u_i) \leq \epsilon$
- $\exists (s_i, t_i) \in \mathbb{R} \times S^1 \text{ s.t. } u_i(s_i, t_i) \not\in \cup_i U_i$

The Gromov compactness implies that there is a solution of the Floer equation as follows:

$$u : \mathbb{R} \times S^1 \to M$$

$$\partial_s u(s, t) + J_t(\partial_t u(s, t) - X_H(u(s, t))) = 0$$

$$\exists (a, b) \in \mathbb{R} \times S^1 \text{ s.t. } u(a, b) \not\in \cup_i U_i$$

$$E(u) = \int_{\mathbb{R} \times S^1} |\partial_s u(s, t)|^2 ds dt \leq \epsilon$$
This is a contradiction because the above conditions implies that $u$ is a non-constant solution whose energy is smaller than $\delta_1$.

Note that $(CF(\bar{H}, \gamma : \Lambda_0), \partial_F)$ is strict if and only if $d_F^\text{loc} = 0$ holds. Our aim in the rest of the section is to construct a homologically canonical boundary operator on $\bigoplus_{x \in P(\bar{H}, \gamma)} HF^{\text{loc}}(H, x) \otimes \Lambda_0$ instead on $\bigoplus_{x \in P(\bar{H}, \gamma)} CF^{\text{loc}}(H, x) \otimes \Lambda_0$ and prove that the cochain complex is cochain homotopy equivalent to the original cochain complex $(CH(\bar{H}, \gamma : \Lambda_0), \partial_F)$. We define the Floer cochain complex of $H$ by

$$CF(H, \gamma : \Lambda_0) = \bigoplus_{x \in P(H, \gamma)} HF^{\text{loc}}(H, x) \otimes \Lambda_0.$$ 

We choose a basis $\{X_i^a, Y_i^b, Z_i^c\}$ of $CF^{\text{loc}}(H, x_i) = \bigoplus F_p \cdot x_i^j$ which satisfies the following relations:

$$d_F^\text{loc}(X_i^a) = 0 \quad (1 \leq a \leq \dim_{\mathbb{F}_p} HF^{\text{loc}}(H, x_i))$$

$$d_F^\text{loc}(Y_i^b) = 0 \quad (1 \leq b \leq \frac{1}{2}(l_i - \dim_{\mathbb{F}_p} HF^{\text{loc}}(H, x_i)))$$

$$d_F^\text{loc}(Z_i^c) = Y_i^b \quad (1 \leq c \leq \frac{1}{2}(l_i - \dim_{\mathbb{F}_p} HF^{\text{loc}}(H, x_i)))$$

Here, $l_i$ is the number of the perturbed periodic orbits $(Q(\bar{H}, x_i) = \{x_1^1, \cdots, x_i^1\})$. Then $HF^{\text{loc}}(H, x_i)$ can be identified with $\text{span}_{\mathbb{F}_p}\{X_1^1, \cdots, X_i^{\dim_{\mathbb{F}_p} HF^{\text{loc}}(H, x_i)}\}$.

We define two operators $\Pi_i$ and $\Theta_i$ on $CF^{\text{loc}}(H, x_i)$ as follows.

$$\Pi_i(\sum \alpha_a X_i^a + \sum \beta_b Y_i^b + \sum \gamma_c Z_i^c) = \sum \alpha_a X_i^a$$

$$\Theta_i(\sum \alpha_a X_i^a + \sum \beta_b Y_i^b + \sum \gamma_c Z_i^c) = \sum \beta_c Z_i^c$$

Let $\Pi = \sum \Pi_i$ and $\Theta = \sum \Theta_i$ be the sums of the above operators. Next, we apply the perturbation theory in [27]. First, we explain what is the perturbation theory and how we can apply it to our case.

Let $M = (M, d_M)$ be a chain complex with a decreasing filtration $\{F^p M\}_{p \in \mathbb{Z}_{\geq 0}}$. $M = F^0 M \supset F^1 M \supset F^2 M \supset \cdots$ We assume that this filtration is complete. In other words, $M$ is complete with respect to the $F^p$-adic topology and an infinite sum $\sum_p m_p$ such that $m_p \in F^p M$ holds converges uniquely to an element of $M$. Let $(N, d_N)$ be another chain complex with a complete and decreasing filtration $\{F^p N\}$. Morphisms between filtered chain complexes are maps that preserve filtrations. Let $f : M \rightarrow N$ be a morphism. Another morphism $g : N \rightarrow M$ is called a perturbation of $f$ if $$(f - g)F^p M \subset F^{p+1} N$$ holds for any $p \in \mathbb{Z}_{\geq 0}$. Perturbations of boundary operators $d_M$ and $d_N$ are defined in the same way. In [27], Markl studied the following problem. Let

$$F : (M, d_M) \rightarrow (N, d_N)$$

$$G : (N, d_N) \rightarrow (M, d_M)$$

$$15$$
be chain maps that preserve filtrations and assume that we also have chain homotopies between the identity and $GF$, $FG$ as follows:

$$H : M \rightarrow M, \quad L : N \rightarrow N$$

(1)

$$GF - \text{Id}_M = d_M H + Hd_M$$

(2)

$$FG - \text{Id}_N = d_N L + Ld_N$$

(3)

Next, we perturb the original boundary operator $d_M$ to a new boundary operator $\tilde{d}_M = d_M + D_M (D_M(F^p M) \subset F^{p+1}M$ holds for any $p$). Then, can we perturb $d_N,F,G,H$ and $L$ so that the above equations (4.1) ~ (4.3) hold? Markl gave a complete answer to this problem. It is always possible if the “obstruction class” vanishes (Ideal perturbation lemma). However, we do not need the full generality of Markl’s theorem. We only treat the simplest case of these perturbation problems. We treat the case that $(N,d_N)$ is the strong deformation retract of $(M,d_M)$. Let $(M,d_M)$ and $(N,d_N)$ be two chain complexes with complete and decreasing filtrations. Assume that chain maps

$$F : (M,d_M) \rightarrow (N,d_N)$$

$$G : (N,d_N) \rightarrow (M,d_M)$$

preserve filtrations and there is a morphism $H : M \rightarrow N$ which satisfies the following conditions:

$$GF - \text{Id}_M = d_M H + Hd_M$$

$$FG = \text{Id}_N$$

$$HH = 0, HG = 0, FH = 0$$

(annihilation properties)

Let $\tilde{d}_M = d_M + D_M$ be a perturbation of $d_M$. Then the Basic perturbation lemma (27) states that there are perturbations $\tilde{d}_N, \tilde{F}, \tilde{G}$ and $\tilde{H}$ which satisfy the following conditions:

$$\tilde{GF} - \text{Id}_M = \tilde{d}_M \tilde{H} + \tilde{H} \tilde{d}_M$$

$$\tilde{FG} = \text{Id}_N$$

So we can perturb $(N,d_N)$ so that $(M,\tilde{d}_M)$ and $(N,\tilde{d}_N)$ are also chain homotopy equivalent. We also have explicit formulas for these perturbations as follows:

$$\tilde{d}_N = d_N + F \sum_{l \geq 0} (D_M H)^l D_M G$$

$$\tilde{F} = F \sum_{l \geq 0} (D_M H)^l$$

$$\tilde{G} = \sum_{l \geq 0} (HD_M)^l G$$

$$\tilde{H} = H \sum_{l \geq 0} (D_M H)^l$$
Next, we apply the Basic perturbation lemma to construct a strict Floer cochain complex of $H$. In our case, $(M, d_M)$, $(N, d_N)$ and $d_M$ correspond to the following:

\[
(M, d_M) = \bigoplus_{s \in P(H, \gamma)} CF^{loc}(H, x) \otimes \Lambda_0, d_F^{loc}
\]

\[
(N, d_N) = \bigoplus_{s \in P(H, \gamma)} HF^{loc}(H, x) \otimes \Lambda_0, 0
\]

\[
\tilde{d}_M = \tilde{d}_F = d_F^{loc} + D
\]

\[
F = \Pi, H = \Theta
\]

Filtrations on $M$ and $N$ are defined as follows:

\[
F^p M = T^{\text{pp}} M
\]

\[
F^p N = T^{\text{pp}} N
\]

The “inclusion”

\[
G : N \rightarrow M
\]

is defined as follows.

\[
HF^{loc}(H, x) \rightarrow CF^{loc}(H, x)
\]

\[
\left[ \sum \alpha_a X^a_i \right] \mapsto \sum \alpha_a X^a_i
\]

Then according to the basic perturbation lemma and the explicit construction of $d_N$, we can define the boundary operator $d_F : CF(H, \gamma : \Lambda_0) \rightarrow CF(H, \gamma : \Lambda_0)$ by the following formula:

\[
d_F = \Pi \circ \sum_{l=0}^{\infty} (D\Theta)^l D
\]

Note that we identify $HF^{loc}(H, x)$ with span of $\langle X^1_i, \cdots, X_i^\dim_p, HF^{loc}(H, x) \rangle$ in this formula. Then, $(CF(H, \gamma : \Lambda_0), d_F)$ is a strict cochain complex which is also cochain homotopy equivalent to the original cochain complex $(CF(\tilde{H}, \gamma : \Lambda_0), d_{\tilde{F}})$. So we defined the Floer cochain complex $(CF(H, \gamma : \Lambda_0), d_F)$ in a homologically canonical way.

Next, we introduce the notion of the local equivariant (Tate) Floer cohomology. Assume that $x \in P(H^{\langle p \rangle}, \gamma)$ is an isolated $p$-periodic orbit of $H^{\langle p \rangle}$. Recall that there is a $\mathbb{Z}_p$-action on $P(H^{\langle p \rangle}, \gamma)$ as follows:

\[
(m \cdot x)(t) = x(t + \frac{m}{p}) \quad (m \in \mathbb{Z}_p)
\]

The local $\mathbb{Z}_p$-equivariant (Tate) Floer cohomology is defined for $\mathbb{Z}_p x$. Let $\tilde{H}$ be a small perturbation of $H$ so that $\tilde{H}^{\langle p \rangle}$ is non-degenerate. Each $m \cdot x$ splits...
into non-degenerate $p$-periodic orbits of $\widetilde{H}(p)$. So we have the local Floer cochain complex $CF_{\text{loc}}^{\text{eq}}(H^{(p)}, m \cdot x)$ for each $m \cdot x$. The local $\mathbb{Z}_p$-equivariant Floer cochain complex of $\mathbb{Z}_p x$ is defined as follows:

$$CF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \mathbb{Z}_p x) = \bigoplus_{y \in \mathbb{Z}_p x} CF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, y) \otimes \mathbb{F}_p[[u]](\theta)$$

Let $\mathcal{U} \subset S^1 \times M$ be a union of small isolating neighborhoods of $y \in \mathbb{Z}_p x$. By counting the equivariant Floer equation contained in $\mathcal{U}$, we can define the local $\mathbb{Z}_p$-equivariant Floer boundary operator $d_{\text{loc}}^{\text{eq}}$. The local $\mathbb{Z}_p$-equivariant Floer cohomology of $\mathbb{Z}_p x$ is defined as the cohomology of this cochain complex.

$$HF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \mathbb{Z}_p x) = H(CF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \mathbb{Z}_p x), d_{\text{eq}}^{\text{loc}})$$

The local $\mathbb{Z}_p$-equivariant Tate Floer cochain complex is defined as follows:

$$CF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \mathbb{Z}_p x) = CF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \mathbb{Z}_p x) \otimes \mathbb{F}_p[u^{-1}, u]$$

The local $\mathbb{Z}_p$-equivariant Tate Floer cohomology is the cohomology of this cochain complex.

$$HF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \mathbb{Z}_p x) = H(CF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \mathbb{Z}_p x), d_{\text{eq}}^{\text{loc}})$$

Next, assume that $(M, \omega)$ is a toroidally monotone symplectic manifold and $P(H^{(p)}, \gamma)$ is finite. Then, we can construct a strict boundary operator $\tilde{d}_{\text{eq}}$ on the $\mathbb{Z}_p$-equivariant Tate Floer cochain complex

$$CF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \gamma) = \bigoplus_{\mathbb{Z}_p x \in P(H^{(p)}, \gamma)/\mathbb{Z}_p} HF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \mathbb{Z}_p x) \otimes \Lambda_0$$

by the same formula as in the definition of $(CF(H, \gamma : \Lambda_0), d_F)$. So we can also define the $\mathbb{Z}_p$-equivariant (Tate) Floer cochain complex for possibly degenerate Hamiltonian diffeomorphisms. For the later purpose, we specify the formula for $\mathbb{Z}_p$-equivariant Tate Floer cohomology of $\phi_H^p$. For every $\mathbb{Z}_p x_i \in P(H^{(p)}, \gamma)/\mathbb{Z}_p$, we choose a basis $\{U^a_i, V^b_i, W^c_i\}$ of $CF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \mathbb{Z}_p x_i)$ over $\mathbb{F}_p[u^{-1}, u]$ as follows:

$$\tilde{d}_{\text{eq}}^{\text{loc}}(U^a_i) = 0 \quad (1 \leq a \leq \dim_{\mathbb{F}_p[u^{-1}, u]} HF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \mathbb{Z}_p x_i))$$

$$\tilde{d}_{\text{eq}}^{\text{loc}}(V^b_i) = 0 \quad (1 \leq b \leq \frac{1}{2}(\bar{l}_i - \dim_{\mathbb{F}_p[u^{-1}, u]} HF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \mathbb{Z}_p x_i))$$

$$\tilde{d}_{\text{eq}}^{\text{loc}}(W^c_i) = V^c_i \quad (1 \leq c \leq \frac{1}{2}(\bar{l}_i - \dim_{\mathbb{F}_p[u^{-1}, u]} HF_{\mathbb{Z}_p}^{\text{loc}}(H^{(p)}, \mathbb{Z}_p x_i))$$
Here, $\tilde{l}_i$ is the number of perturbed periodic orbits of $Z_p x_i$. We define $\Pi_{i,eq}$, $\Theta_{i,eq}$, $\Pi_{eq}$ and $\Theta_{eq}$ as follows:

\[
\Pi_{i,eq} \left( \sum_a \alpha_a U_i^a + \sum_b \beta_b V_i^b + \sum_c \gamma_c W_i^c \right) = \sum_a \alpha_a U_i^a \\
\Theta_{i,eq} \left( \sum_a \alpha_a U_i^a + \sum_b \beta_b V_i^b + \sum_c \gamma_c W_i^c \right) = \sum_c \gamma_c W_i^c \\
\Pi_{eq} = \sum_i \Pi_{i,eq} \\
\Theta_{eq} = \sum_i \Theta_{i,eq}
\]

Let $\tilde{d}_{eq}$ be a differential operator on

\[
\bigoplus_{Z_p x \in P(H^{(p)}_\gamma)} \hat{CF}^{loc}_{Z_p}(H^{(p)}, Z_p x) \otimes \Lambda_0.
\]

**Remark 4.1** Note that this $\tilde{d}_{eq}$ is deformed by

\[
\tau_{eq} : \left( \bigoplus \hat{CF}^{loc}_{Z_p}(H^{(p)}, Z_p x) \otimes \Lambda_0 \right) \longrightarrow \left( \bigoplus \hat{CF}^{loc}_{Z_p}(H^{(p)}, Z_p x) \otimes \Lambda_0 \right)
\]

\[
x_i^j \mapsto T^{c(x_i^j, x_i^j)} x_i^j
\]

as in the case of the Floer cohomology. Since this is repetitive, we omit it here.

This $\tilde{d}_{eq}$ is decomposed as follows:

\[
\tilde{d}_{eq} = \hat{d}_{eq}^{loc} + D_{eq}
\]

\[
D_{eq} \left( \bigoplus \hat{CF}^{loc}_{Z_p}(H^{(p)}, Z_p x) \otimes \Lambda_0 \right) \subset T^e \left( \bigoplus \hat{CF}^{loc}_{Z_p}(H^{(p)}, Z_p x) \otimes \Lambda_0 \right)
\]

The existence of such $\epsilon > 0$ is trivial because we assumed that $(M, \omega)$ is toroidally monotone. Then, the boundary operator $\hat{d}_{eq}$ is defined as follows:

\[
\hat{d}_{eq} : \hat{CF}_{Z_p}(H^{(p)}, \gamma) \longrightarrow \hat{CF}_{Z_p}(H^{(p)}, \gamma)
\]

\[
\hat{d}_{eq} = \Pi_{eq} \circ \sum_{l=0}^{\infty} (D_{eq} \Theta_{eq})^l D_{eq}
\]

We can see that this sequence converges because

\[
\text{Im}(\Pi_{eq}(D_{eq} \Theta_{eq})^l D_{eq}) \subset T^{(l+1)\epsilon} \cdot \hat{CF}^{loc}_{Z_p}(H^{(p)}, \gamma)
\]

holds (Of course, it also follows from the Basic perturbation lemma.).
5 Proof of the main theorem: toroidally monotone case

In section 5 and section 6, we prove Theorem 1.1. We divide the proof into two parts, toroidally monotone cases, and weakly monotone cases. The reason for this is that we have not constructed $\mathbb{Z}_p$-equivariant Floer theory for weakly monotone symplectic manifolds yet. As we mentioned before, we have to overcome some technical difficulties in the weakly monotone case. Once we establish $\mathbb{Z}_p$-equivariant Floer theory on weakly monotone symplectic manifolds, the rest of the proof is almost the same as in the toroidally monotone case. So the essential part of our proof is given in the toroidally monotone case.

In this section, we prove Theorem 1.1 for toroidally monotone symplectic manifolds. Let $(M, \omega)$ be a closed toroidally monotone symplectic manifold. We fix $\hat{H} \in C^\infty(S^1 \times M)$ and $\gamma \neq 0 \in H_1(M; \mathbb{Z})/\text{Tor}$. We also assume that $P(H, \gamma) = \{x_1, \cdots, x_k\}$ and $P(H^{(p)}, \gamma) = \{y_1, \cdots, y_k\}$ and $y_i = x_i^{(p)}$ holds. In other words, every $p$-periodic orbit of $H$ in $P$ is not simple. Our purpose is to prove that there is a simple $p'$-periodic orbit in $P^\gamma$. Here $p'$ is the smallest prime greater than $p$.

In the previous section, we defined the $\mathbb{Z}_p$-equivariant Tate Floer cochain complex $(\hat{CF}(H^{(p)}, \gamma), \hat{d}_{eq})$ and the $\mathbb{Z}_p$-equivariant Tate Floer cohomology $\hat{H}F\hat{Z}_p(H^{(p)}, \gamma)$. Recall that we applied the perturbation theory to make $(\hat{CF}(H^{(p)}, \gamma), \hat{d}_{eq})$ a strict differential complex.

In the definition of $\hat{d}_{eq}$, we have seen that $\hat{d}_{eq}$ is written as $d_F + \text{higher terms}$ if $H^{(p)}$ is non-degenerate as follows:

$$\hat{d}_{eq}(x \otimes 1) = d_F(x) \otimes 1 + x_1 \otimes \theta + \sum_{k=1}^{k'} \sum_{k'=0,1} x_{2k+k'} \otimes u^k \theta^{k'}$$

$$\hat{d}_{eq}(x \otimes \theta) = d_F(x) \otimes \theta + \sum_{k=1}^{k'} \sum_{k'=0,1} y_{2k+k'} \otimes u^k \theta^{k'}$$

Next we prove that this is also true when $H^{(p)}$ is possibly degenerate. Let $\tilde{H}$ be a perturbation of $H$ such that $\tilde{H}^{(p)}$ is non-degenerate. In this case, each $y_i \in P(H^{(p)}, \gamma)$ splits into non-degenerate periodic orbits $\{y_{1, i}, \cdots, y_{k, i}\} \subset P(\tilde{H}^{(p)}, \gamma)$. We prove the following lemma.

**Lemma 5.1** Assume that $y_i = x_i^{(p)}$ holds and $p$ is an admissible prime number. We can choose a basis $\{\hat{X}_i^a, \hat{Y}_i^b, \hat{Z}_i^c, \hat{X}_{i, 0}, \hat{Y}_{i, \theta}, \hat{Z}_{i, \theta}\}$ of $\hat{CF}_{\text{loc}}^{\hat{Z}_p}(H^{(p)}, y_i) = CF_{\text{loc}}(H^{(p)}, y_i) \otimes \mathbb{F}_p[u^{-1}, u][\theta]$ over $\mathbb{F}_p[u^{-1}, u]$ and a basis $\{X_i^a, Y_i^b, Z_i^c\}$ of $CF_{\text{loc}}(H^{(p)}, y_i)$ which satisfies the
The following conditions:

\[
\widetilde{X}_i^a = X_i^a \otimes 1 + (a_1^{(a)} \otimes \theta + \sum_{k=1}^{\infty} \sum_{k'=0}^{1} a_{2k+k'}^{(a)} \otimes u^k \theta^{k'})
\]

\[
\widetilde{Y}_i^b = Y_i^b \otimes 1 + (b_1^{(b)} \otimes \theta + \sum_{k=1}^{\infty} \sum_{k'=0}^{1} b_{2k+k'}^{(b)} \otimes u^k \theta^{k'})
\]

\[
\widetilde{Z}_i^c = Z_i^c \otimes 1 + (c_1^{(c)} \otimes \theta + \sum_{k=1}^{\infty} \sum_{k'=0}^{1} c_{2k+k'}^{(c)} \otimes u^k \theta^{k'})
\]

\[
\tilde{X}_{i,\theta}^a = X_i^a \otimes \theta + \sum_{k=1}^{\infty} \sum_{k'=0}^{1} d_{k}^{(a)} \otimes u^k \theta^{k'}
\]

\[
\tilde{Y}_{i,\theta}^b = Y_i^b \otimes \theta + \sum_{k=1}^{\infty} \sum_{k'=0}^{1} e_{k}^{(b)} \otimes u^k \theta^{k'}
\]

\[
\tilde{Z}_{i,\theta}^c = Z_i^c \otimes \theta + \sum_{k=1}^{\infty} \sum_{k'=0}^{1} f_{k}^{(c)} \otimes u^k \theta^{k'}
\]

\[
d_F^{loc} (X_i^a) = d_F^{loc} (Y_i^b) = 0, \quad d_F^{loc} (Z_i^c) = Y_i^c, \quad d_{eq}^{loc} (\tilde{X}_{i,\theta}^a) = d_{eq}^{loc} (\tilde{Y}_{i,\theta}^b) = 0, \quad d_{eq}^{loc} (\tilde{Z}_{i,\theta}^c) = \tilde{Y}_{i,\theta}^c
\]

\[
1 \leq a \leq \text{dim}_F H F^{loc}(H^{(p)}, y_i), \quad 1 \leq b, c \leq \frac{1}{2} (\text{dim}_F H F^{loc}(H^{(p)}, y_i))
\]

\[
\{ a_1^{(a)}, b_1^{(b)}, c_1^{(c)}, d_1^{(a)}, e_1^{(b)}, f_1^{(c)} \} \subset CF^{loc}(H^{(p)}, y_i)
\]

**Proof (Lemma 5.1):** Note that $CF^{loc}(H^{(p)}, y_i) = \bigoplus_j F_p \cdot y_i^{j}$ holds. The existence of a basis $\{ X_i^a, Y_i^b, Z_i^c \}$ is trivial and we can define $\tilde{Y}_i^b, \tilde{Z}_i^c, \tilde{Y}_i^b, \tilde{Z}_i^c$, by $\tilde{Z}_i^c = Z_i^c \otimes 1$, $\tilde{Y}_i^b = Z_i^c \otimes \theta$, $\tilde{Y}_i^b = d_{eq}^{loc} (\tilde{Z}_i^c)$ and $\tilde{Y}_i^b = d_{eq}^{loc} (\tilde{Z}_i^c, \tilde{Z}_i^c, \tilde{Y}_i^b)$. So what we have to prove is the existence of $\tilde{X}_{i,\theta}^a$ and $\tilde{X}_{i,\theta}^a$.

Let $C$ be the set of all cycles in $(CF_{\mathbb{Z}_p}(H^{(p)}, y_i), d_F^{loc})$. We consider two projections

\[
\pi_1, \pi_2 : C \rightarrow \text{span}_{F_p} \{ X_i^1, \ldots, X_i^d \}, \quad (d = \text{dim}_F H F^{loc}(H^{(p)}, y_i)).
\]

For $z = (a_k \otimes 1 + b_k \otimes \theta) u^k + \sum_{\ell=k+1}^d (a_{\ell} \otimes 1 + b_{\ell} \otimes \theta) u^\ell$, we define $\pi_1(z)$ and $\pi_2(z)$ by the following formula. Let $\Pi_1 : CF(H^{(p)}, y_i) \rightarrow \text{span}_{F_p} \{ X_i^1, \ldots, X_i^d \}$ be the projection as before.

\[
\pi_1(z) = \Pi_1(a_k)
\]

\[
\pi_2(z) = \begin{cases} 
\Pi_1(b_k) & a_k = 0 \\
0 & a_k \neq 0
\end{cases}
\]
Note that $\pi_1$ and $\pi_2$ are not linear maps. However, their images are vector subspaces of $\text{span}_{p} (X_1^1, \ldots, X_1^d)$. This follows from the following arguments.

Let $c_1$ and $c_2$ be elements of $\text{Im}(\pi_1)$. Then we can choose cycles $\{z^{(1)}, z^{(2)}\} \subset C$ as follows.

\[
\begin{align*}
z^{(1)} &= a_0^{(1)} \otimes 1 + b_0^{(1)} \otimes \theta + \sum_{l \geq 1} (a_l^{(1)} \otimes 1 + b_l^{(1)} \otimes \theta) u^l \\
z^{(2)} &= a_0^{(2)} \otimes 1 + b_0^{(2)} \otimes \theta + \sum_{l \geq 1} (a_l^{(2)} \otimes 1 + b_l^{(1)} \otimes \theta) u^l
\end{align*}
\]

Then, for any $\alpha_1, \alpha_2 \in \mathbb{F}_p$,

\[
\pi_1(\alpha_1 z^{(1)} + \alpha_2 z^{(2)}) = \Pi_i (\alpha_1 a_0^{(1)} + \alpha_2 a_0^{(2)}) = \alpha_1 c_1 + \alpha_2 c_2
\]

holds if $\alpha_1 a_0^{(1)} + \alpha_2 a_0^{(2)} \neq 0$ holds. This implies that $\text{Im}(\pi_1)$ is a vector subspace. A similar argument can be used to prove that $\text{Im}(\pi_2)$ is a vector subspace.

Assume that $\text{Im}(\pi_1)$ is generated by $\tilde{V}_j$ and $\text{Im}(\pi_2)$ is generated by $\tilde{W}_j$.

\[
\begin{align*}
\text{Im}(\pi_1) &= \text{span}_{p} (\tilde{V}_1, \cdots, \tilde{V}_\alpha) \\
\text{Im}(\pi_2) &= \text{span}_{p} (\tilde{W}_1, \cdots, \tilde{W}_\beta)
\end{align*}
\]

We choose $\{V_1, \cdots, V_\alpha, W_1, \cdots, W_\beta\} \subset C$ so that $\pi_1(V_j) = \tilde{V}_j$ and $\pi_2(W_j) = \tilde{W}_j$ hold. Next we prove that $\{V_1, \cdots, V_\alpha, W_1, \cdots, W_\beta\} \subset C$ generates $\overline{HF^{\text{loc}}_{\mathbb{F}_p} (H^p, y_i)}$ over $\mathbb{F}_p[u^{-1}, u]$. We fix $z \in C$ as follows:

\[
z = \sum_{l \geq m} (a_l \otimes 1 + b_l \otimes \theta) u^l
\]

It suffices to find $z_m = (c_m \otimes 1 + d_m \otimes \theta) u^m$ and $\zeta_m^{(i)}, \eta_m^{(j)} \in \mathbb{F}_p$ so that

\[
z - d_{eq}^{\text{loc}} (z_m) - \sum_{i \leq \alpha} (c_m^{(i)} u^m) V_i - \sum_{j \leq \beta} (\eta_m^{(j)} u^m) W_j = \sum_{l \geq m+1} (a_l' \otimes 1 + b_l' \otimes \theta) u^l
\]

holds. If this is true, we can construct $z'$ and $\zeta^{(i)}, \eta^{(j)} \in \mathbb{F}_p[u^{-1}, u]$ so that

\[
z - d_{eq}^{\text{loc}} (z') = \sum_{i \leq \alpha} \zeta^{(i)} V_i + \sum_{j \leq \beta} \eta^{(j)} W_j
\]

holds. We choose $c_m \in CF^{\text{loc}}(H^p, y_i)$ so that

\[
a_m - d^{\text{loc}}_F (c_m) \in \text{span}_{\mathbb{F}_p} (X_1^1, \cdots, X_1^d)
\]

22
holds. Note that \( a_m - d_F^{loc}(c_m) \in \text{span}_{\mathbb{F}_p}(\tilde{V}_1, \cdots, \tilde{V}_a) \) holds because

\[
z - \tilde{d}_c^{\text{loc}}(c_m \otimes u^m) = \left( (a_m - d_F^{loc}(c_m)) \otimes 1 + \tilde{b}_m \otimes \theta \right) u^m + \sum_{l \geq m+1} (\tilde{a}_l \otimes 1 + \tilde{b}_l \otimes \theta) u^l
\]

implies that \( \pi_1(z - \tilde{d}_c^{\text{loc}}(c_m \otimes u^m)) = a_m - d_F^{loc}(c_m) \) holds. So we can choose \( \{\zeta_m(i)\}_{i \leq \alpha} \) so that

\[
a_m - d_F^{loc}(c_m) = \sum_{i \leq \alpha} \zeta_m(i) \tilde{V}_i
\]

holds. This implies that

\[
z - \tilde{d}_c^{\text{loc}}(z_m) - \sum_{i \leq \alpha} (\zeta_m(i) u^m) V_i = \sum_{i \geq \alpha} (a_l'' \otimes 1 + b_l'' \otimes \theta) u^l
\]

holds. A similar argument can be used to find \( d_m \in CF^{loc}(H^{(p)}, y_i) \) and \( \{\eta_m(j)\}_{j \leq \beta} \) so that

\[
z - \tilde{d}_c^{\text{loc}}(z_m) - \sum_{i \leq \alpha} (\zeta_m(i) u^m) V_i - \sum_{j \leq \beta} (\eta_m(j) u^m) W_j = \sum_{l \geq m+1} (a_l' \otimes 1 + b_l' \otimes \theta) u^l
\]

holds. So we proved that \( \tilde{HF}^{loc}_{\mathbb{Z}_p}(H^{(p)}, y_i) \) is generated by \( \{V_1, \cdots, V_\alpha, W_1, \cdots, W_\beta\} \).

Note that there is an isomorphism between the local Floer cohomology and the local \( \mathbb{Z}_p \)-equivariant Tate Floer cohomology (\cite{37}).

\[
\tilde{HF}^{loc}_{\mathbb{Z}_p}(H^{(p)}, y_i) \cong HF^{loc}(H^{(p)}, y_i) \otimes \mathbb{F}_p[u^{-1}, u][\theta]
\]

Here we use the fact that \( p \) is admissible and \( HF^{loc}(H^{(p)}, y_i) \cong HF^{loc}(H, x_i) \) holds. This implies that \( \alpha = \beta = \dim_{\mathbb{Q}}HF^{loc}(H^{(p)}, y_i) \) and \( \text{Im}(\pi_i) = \text{span}_{\mathbb{F}_p}(X_1, \cdots, X_d) \).

So we can choose \( \tilde{X}_i^a \) and \( \tilde{X}_i^a \) and we proved Lemma 5.1.

\[\square\]

**Corollary 5.1** Assume that \( P(H^{(p)}, p\gamma) \) consists of finitely many non-simple periodic orbits. We fix a basis \( \{X_i^a, Y_i^b, Z_i^c\} \) of \( HF^{loc}(H^{(p)}, y_i) \) and a basis \( \{\tilde{X}_i^a, \tilde{Y}_i^b, \tilde{Z}_i^c, \tilde{X}_i^a, \tilde{Y}_i^b, \tilde{Z}_i^c\} \) of \( C\tilde{HF}^{loc}_{\mathbb{Z}_p}(H^{(p)}, y_i) \). Let \( (CF(H^{(p)}, p\gamma), d_F) \) be the Floer cochain complex as in (4.4) and let \( C\tilde{HF}^{loc}_{\mathbb{Z}_p}(H^{(p)}, p\gamma) \) be the \( \mathbb{Z}_p \)-equivariant Tate Floer cochain complex of \( H^{(p)} \) as in (4.5). Let \( \iota \) and \( \iota_0 \) be the following injections:

\[
\iota : \bigoplus_i CF^{loc}(H^{(p)}, y_i) \otimes \Lambda_0 \longrightarrow \bigoplus_i C\tilde{HF}^{loc}_{\mathbb{Z}_p}(H^{(p)}, y_i) \otimes \Lambda_0
\]

\[
\sum_{i,a} \alpha_{i,a}X_i^a + \sum_{i,b} \beta_{i,b}Y_i^b + \sum_{i,c} \gamma_{i,c}Z_i^c \mapsto \sum_{i,a} \alpha_{i,a} \tilde{X}_i^a + \sum_{i,b} \beta_{i,b} \tilde{Y}_i^b + \sum_{i,c} \gamma_{i,c} \tilde{Z}_i^c
\]

23
\[
\begin{align*}
\iota_\iota : & \bigoplus_i C F^{loc}(H^{(p)}, y_i) \otimes \Lambda_0 \rightarrow \bigoplus_i \overline{C F}^{loc}_{Z_p}(H^{(p)}, y_i) \otimes \Lambda_0 \\
\sum_{i,a} \alpha_{i,a} X_i^a + \sum_{i,b} \beta_{i,b} Y_i^b + \sum_{i,c} \gamma_{i,c} Z_i^c \rightarrow & \sum_{i,a} \alpha_{i,a} \tilde{X}_i^a + \sum_{i,b} \beta_{i,b} \tilde{Y}_i^b + \sum_{i,c} \gamma_{i,c} \tilde{Z}_i^c \\
& (\alpha_{i,a}, \beta_{i,b}, \gamma_{i,c} \in \Lambda_0)
\end{align*}
\]

Then for every \(\{\alpha_{i,a}\} \subset \Lambda_0\),

\[
\begin{align*}
\tilde{d}_{eq}(\sum_{i,a} \alpha_{i,a} \tilde{X}_i^a) &= \iota(d_F(\sum_{i,a} \alpha_{i,a} X_i^a)) \otimes 1 + \sum_{i,a} \beta_{i,a} \tilde{X}_i^a + \sum_{i,a} \sum_{l \geq 1} (\delta_{i,a} \tilde{X}_i^a + \beta_{i,a} \tilde{X}_i^a) u^l \\
\tilde{d}_{eq}(\sum_{i,a} \alpha_{i,a} \tilde{X}_i^a) &= \iota(d_F(\sum_{i,a} \alpha_{i,a} X_i^a)) + \sum_{i,a} \sum_{l \geq 1} (\delta_{i,a} \tilde{X}_i^a + \beta_{i,a} \tilde{X}_i^a) u^l
\end{align*}
\]

holds for some \(\{\beta_{i,a}\}, \{\delta_{i,a}\}, \{\delta_{i,a}^\theta\} \subset \Lambda_0\). In this sense, \(\tilde{d}_{eq}\) is written as \(d_F + \) higher terms.

**proof** (Corollary 5.1): We can define \(\Pi\) and \(\Theta\) as follows:

\[
\begin{align*}
\Pi_{i,eq}(\sum_a \alpha_a \tilde{X}_i^a + \sum_b \beta_b \tilde{Y}_i^b + \sum_c \gamma_c \tilde{Z}_i^c + \sum_a \alpha_{a,\theta} \tilde{X}_i^a + \sum_b \beta_{b,\theta} \tilde{Y}_i^b + \sum_c \gamma_{c,\theta} \tilde{Z}_i^c) &= \sum_a \alpha_a \tilde{X}_i^a + \sum_a \alpha_{a,\theta} \tilde{X}_i^a \\
\Theta_{i,eq}(\sum_a \alpha_a \tilde{X}_i^a + \sum_b \beta_b \tilde{Y}_i^b + \sum_c \gamma_c \tilde{Z}_i^c + \sum_a \alpha_{a,\theta} \tilde{X}_i^a + \sum_b \beta_{b,\theta} \tilde{Y}_i^b + \sum_c \gamma_{c,\theta} \tilde{Z}_i^c) &= \sum_c \gamma_c \tilde{Z}_i^c + \sum_c \gamma_{c,\theta} \tilde{Z}_i^c
\end{align*}
\]

\[
\begin{align*}
\Pi_{eq} &= \sum_i \Pi_{i,eq} \\
\Theta_{eq} &= \sum_i \Theta_{i,eq}
\end{align*}
\]

We abbreviate \(\{X_i^a, Y_i^b, Z_i^c\}\) to \(\{W_j\}\) and \(\{\tilde{X}_i^a, \tilde{Y}_i^b, \tilde{Z}_i^c, \tilde{X}_{i,\theta}^a, \tilde{Y}_{i,\theta}^b, \tilde{Z}_{i,\theta}^c\}\) to \(\{\tilde{W}_j, \tilde{W}_{i,\theta}\}\).

Let \(\Gamma\) be one of the maps \(\{\Pi, D, \Theta\}\) in (4.4). Then, \(\Gamma_{eq}\) is one of the maps \(\{\Pi_{eq}, D_{eq}, \Theta_{eq}\}\) in (4.5). Lemma 5.1 implies that \(\Gamma_{eq}\) has the following form:

\[
\begin{align*}
\Gamma_{eq}(\sum_j \alpha_j \tilde{W}_j) &= \iota(\Gamma(\sum_j \alpha_j W_j)) + \sum_j \beta_j \tilde{W}_{j,\theta} + \sum_{l \geq 1} \sum_j (\delta_{j,l} \tilde{W}_j + \beta_{j,l} \tilde{W}_{j,\theta}) u^l \\
\Gamma_{eq}(\sum_j \alpha_j \tilde{W}_{j,\theta}) &= \iota(\Gamma(\sum_j \alpha_j W_{j,\theta})) + \sum_j \sum_{l \geq 1} (\delta_{j,l} \tilde{W}_j + \beta_{j,l} \tilde{W}_{j,\theta}) u^l
\end{align*}
\]

So, \(\tilde{d}_{eq}\) can be written as (5.1) and (5.2) because \(d_F\) is generated by \(\{\Pi, D, \Theta\}\) as in (4.4) and \(\tilde{d}_{eq}\) is generated by \(\{\Pi_{eq}, D_{eq}, \Theta_{eq}\}\) as in (4.5).
Our next purpose is to calculate $\hat{HF}_{Z_p}(H^{(p)}, \gamma)$. For this purpose, we introduce the $Z_p$-equivariant (Tate) cohomology for $(H, \gamma)$ (see [37]). The $Z_p$-equivariant cochain complex is defined as follows:

$$C(Z_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) = CF(H, \gamma : \Lambda_0)^{\otimes p} \otimes \Lambda_0[[u]](\theta)$$

The Floer differential $d_F$ on $CF(H, \gamma : \Lambda_0)$ naturally extends to a differential $d_F^{(p)}$ on $CF(H, \gamma : \Lambda_0)^{\otimes p}$. There is a natural $Z_p$ action $\tau$ on $CF(H, \gamma : \Lambda_0)^{\otimes p}$:

$$\tau(x_0 \otimes x_1 \otimes \cdots \otimes x_{p-1}) = (-1)^{|x_{p-1}|(|x_0| + \cdots + |x_{p-1}|)} x_{p-1} \otimes x_0 \otimes \cdots \otimes x_{p-2}.$$  

Let $N$ be the sum $N = 1 + \tau + \tau^2 + \cdots + \tau^{p-1}$. Then the $Z_p$-equivariant differential $d_{Z_p}$ is a $\Lambda_0[[u]]$-linear map defined as follows:

$$d_{Z_p}(x \otimes 1) = d_F^{(p)}(x) \otimes 1 + (1 - \tau)(x) \otimes \theta$$

$$d_{Z_p}(x \otimes \theta) = d_F^{(p)}(x) \otimes \theta + N(x) \otimes u\theta$$

The $Z_p$-equivariant cohomology is defined by the cohomology of this complex.

$$H(Z_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) = H(C(Z_p, CF(H, \gamma : \Lambda_0)^{\otimes p}), d_{Z_p})$$

The $Z_p$-equivariant Tate cochain complex is a coefficient extension of the $Z_p$-equivariant cochain complex as follows:

$$\hat{C}(Z_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) = CF(H, \gamma : \Lambda_0)^{\otimes p} \otimes \Lambda_0[u^{-1}, u]([u])\langle\theta\rangle$$

$$\hat{d}_{Z_p} : \hat{C}(Z_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) \to \hat{C}(Z_p, CF(H, \gamma : \Lambda_0)^{\otimes p})$$

Here, $\hat{d}_{Z_p}$ is the natural extension of $d_{Z_p}$. The $Z_p$-equivariant Tate cohomology is the cohomology of this cochain complex as follows:

$$\hat{H}(Z_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) = H(\hat{C}(Z_p, CF(H, \gamma : \Lambda_0)^{\otimes p}), \hat{d}_{Z_p})$$

$$\hat{H}(Z_p, CF(H, \gamma : \Lambda_0)^{\otimes p})$$ is determined by $HF(H, \gamma : \Lambda_0)$ from the following lemma.

**Lemma 5.2 ([34])** There is a so-called quasi-Frobenius isomorphism as follows:

$$r_p^*HF(H, \gamma : \Lambda_0) \otimes \Lambda_0[u^{-1}, u]([u])\langle\theta\rangle \cong \hat{H}(Z_p, CF(H, \gamma : \Lambda_0)^{\otimes p})$$

Here, $r_p : \Lambda_0 \to \Lambda_0$ is a the homomorphism defined by $T \to T^p$. 

25
Assume that there is an isomorphism
\[ HF(H, \gamma : \Lambda_0) \cong \bigoplus_{i=1}^{m} \Lambda_0/T^j \Lambda_0 \quad (\beta_j > 0). \]
Then, Lemma 5.2 implies that there is the following isomorphism:
\[
\hat{H}(\mathbb{Z}_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) \cong \left( \bigoplus_{i=1}^{m} \Lambda_0/T^{p\beta} \Lambda_0 \right) \otimes \Lambda_0[u^{-1}, u]|(\theta)
\]
So, the module structure of the \( \mathbb{Z}_p \)-equivariant Tate cohomology is completely determined by \( HF(H, \gamma : \Lambda_0) \).

There is another important operation we consider, the so-called \( \mathbb{Z}_p \)-equivariant pair of pants product.

\[ \mathcal{P} : H(\mathbb{Z}_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) \rightarrow HF_{\mathbb{Z}_p}(H^{(p)}, p\gamma) \]
\[ \hat{\mathcal{P}} : \hat{H}(\mathbb{Z}_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) \rightarrow \hat{HF}_{\mathbb{Z}_p}(H^{(p)}, p\gamma) \]

The construction of \( \mathcal{P} \) (and \( \hat{\mathcal{P}} \)) is just counting the solutions of Floer equations on a \( p \)-branched cover of the cylinder \( \mathbb{R} \times S^1 \) \((p\)-legged pants\) parametrized by \( S^{\infty} \) (see section 8 in [37], see also [33] for the \( \mathbb{Z}_2 \)-equivariant case). We give a detailed construction of \( \mathcal{P} \) and \( \hat{\mathcal{P}} \) in the last section for weakly monotone case. The important point is that \( \hat{\mathcal{P}} \) gives a local isomorphism between their local cohomologies in the following sense ([37]).

We define an action filtration on \( \check{C}(\mathbb{Z}_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) \) and \( \check{CF}_{\mathbb{Z}_p}(H^{(p)}, p\gamma) \).

We fix a sufficiently small positive real number \( \epsilon > 0 \). We define the filtration \( F^q\check{C}(\mathbb{Z}_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) \) and \( F^q\check{CF}_{\mathbb{Z}_p}(H^{(p)}, p\gamma) \) \((q \in \mathbb{Z}_{\geq 0})\) as follows:

\[ F^q\check{C}(\mathbb{Z}_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) = T^q\check{C}(\mathbb{Z}_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) \]
\[ F^q\check{CF}_{\mathbb{Z}_p}(H^{(p)}, p\gamma) = T^q\check{CF}_{\mathbb{Z}_p}(H^{(p)}, p\gamma) \]

Let \( \hat{d}^{oc}_{z_p} \) be a differential map defined as follows:

\[ \hat{d}^{oc}_{z_p}(x \otimes 1) = (d^{oc}_F)^{(p)}(x) \otimes 1 + (1 - \tau) \otimes \theta \]
\[ \hat{d}^{oc}_{z_p}(x \otimes \theta) = (d^{oc}_F)^{(p)}(x) \otimes \theta + N(x) \otimes u\theta \]

If we divide \( \hat{d}^{oc}_{z_p} \) into \( \hat{d}^{oc}_{z_p} + D_{z_p} \),
\[ D_{z_p}(F^q\check{C}(\mathbb{Z}_p, CF(H, \gamma : \Lambda_0)^{\otimes p})) \subset F^{q+1}\check{C}(\mathbb{Z}_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) \]
holds for any \( q \in \mathbb{Z}_{\geq 0} \). So the \( E_1 \) term of the associated spectral sequences of \( F^q\check{C} \) and \( F^q\check{CF}_{\mathbb{Z}_p} \) are given by local cohomologies. This fact and the local isomorphism theorem of \( \check{\mathcal{P}} \) imply that \( \check{\mathcal{P}} \) gives an isomorphism between the \( E_1 \) pages of their spectral sequences. So, to prove that \( \check{\mathcal{P}} \) is an isomorphism, it suffices to prove the following strong convergences of each spectral sequence (Theorem 3.2 in Chapter 15 in [3]).
Lemma 5.3 ((strong convergence)) Let \((A, d)\) be a filtered differential module so that \((A, d) = (\hat{C}, \hat{d})\) or \((A, d) = (\hat{CF}_p, \hat{d})\) holds.

(1) (weakly convergence) For each \(q \in \mathbb{Z}_{\geq 0}\), the intersection of the images of the homomorphisms

\[
H(F^q A/F^{q+r} A) \rightarrow H(F^{q+1} A) \quad r \geq 1
\]

\([z] \mapsto [dz]

is zero.

(2) The natural map

\[
u : H(A) \rightarrow \lim_{\leftarrow} H(A)/F^q H(A)
\]

is an isomorphism. Here, \(F^q H(A)\) is the image of

\[
H(F^q A) \rightarrow H(A).
\]

proof (Lemma 5.3) : (1) Let \(K\) be another Hamiltonian function and let \((B, d)\) be either the Tate chain complex \((\hat{C}(\mathbb{Z}_p, CF(K, \gamma : \Lambda_0) \otimes p), \hat{d})\) or \((CF_p(K^{(p)}, p\gamma : \Lambda_0), \hat{d})\) respectively. Then, there are continuation chain maps over \(\Lambda\)

\[
F : (A \otimes \Lambda, d) \rightarrow (B \otimes \Lambda, d)
\]

\[
G : (B \otimes \Lambda, d) \rightarrow (A \otimes \Lambda, d)
\]

and a chain homotopy \(\mathcal{H}\) between \(Id\) and \(GF\) as follows. Let \(D\) be the constant

\[
D = p \times ||H - K|| = p \times \int_0^1 \left\{ \max(H_t - K_t) - \min(H_t - K_t) \right\} dt.
\]

Then, \(\mathcal{H}(A^\alpha) \subset A^{\alpha - D}\) holds for any \(\alpha \in \mathbb{R}\) \((A^\alpha = T^\alpha A)\). If \(K\) is a \(C^\infty\)-small function, \(P(K, \gamma) = P(K^{(p)}, p\gamma) = \emptyset\) and \((B, d) = (0, 0)\) hold. This implies that there is a map

\[
\mathcal{L} : A \otimes \Lambda \rightarrow A \otimes \Lambda
\]

such that

\[
Id_{A \otimes \Lambda} = d\mathcal{L} + \mathcal{L}d
\]

\[
\mathcal{L}(A^\alpha) \subset A^{\alpha - p||H||}
\]

This implies that for \(r > \frac{p||H||}{\varepsilon} + 1\) and any cycle \(C \in F^{q+r} A\), \(C' = \mathcal{L}(C)\) satisfies \(C' \in F^{q+1} A\) and \(d(C') = C\). So the map

\[
H(F^q A/F^{q+r} A) \rightarrow H(F^{q+1} A)
\]

\([z] \mapsto [dz]

27
is zero for $r > \frac{p|H|}{\epsilon} + 1$ because $dz$ is a cycle in $F^{q+r}A$ and hence we can find $C' \in F^{q+1}A$ so that $d(C') = dz$ holds. So we proved (1).

(2) Let $z \in F^qA$ be a cycle for $q > \frac{p|H|}{\epsilon}$. Then, the above argument implies that we can choose $z' \in A$ so that $d(z') = z$ holds. So, $F^qH(A)$ is zero for $q > \frac{p|H|}{\epsilon}$ and hence

$$H(A)/F^qH(A) = H(A)$$

holds. In particular, the natural map

$$u : H(A) \longrightarrow \lim_{\leftarrow} H(A)/F^qH(A)$$

is an isomorphism.

So we proved that

$$r^*HF(H, \gamma : \Lambda_0) \otimes \Lambda_0[u^{-1}, u][\theta] \cong \widehat{H}(\mathbb{Z}_p, CF(H, \gamma : \Lambda_0)^{\otimes p}) \cong HF_{\mathbb{Z}_p}(H^{(p)}, p\gamma)$$

holds.

Next, we prove the following lemma (see also section 3.2 in [4]).

**Lemma 5.4** Assume that $P(H, \gamma) = \{x_1, \cdots, x_k\}$ and $P(H^{(p)}, p\gamma) = \{y_1, \cdots, y_k\}$ and $y_i = x_i^{(p)}$. We also assume that $p$ is an admissible prime number. Assume that there is an isomorphism

$$HF(H, \gamma : \Lambda_0) \cong \bigoplus_{j=1}^m \Lambda_0/T^{\beta_j} \Lambda_0$$

$$0 < \beta_1 \leq \beta_2 \cdots \leq \beta_m$$

and there is an isomorphism

$$HF(H^{(p)}, p\gamma : \Lambda_0) \cong \bigoplus_{j=1}^m \Lambda_0/T^{\delta_j} \Lambda_0$$

$$0 < \delta_1 \leq \delta_2 \cdots \leq \delta_m.$$

Then, $\delta_1 \geq p\beta_1$ holds.

**proof** (Lemma 5.4) : The above isomorphism implies that the following isomorphism holds.

$$\widehat{HF}_{\mathbb{Z}_p}(H^{(p)}, p\gamma) \cong \left( \bigoplus_{j=1}^m \Lambda_0/T^{p\beta_j} \Lambda_0 \right) \otimes \Lambda_0[u^{-1}, u][\theta]$$

We define spectral numbers $\sigma(z)$ and $\tau(z)$ for $z \in CF(H^{(p)}, p\gamma : \Lambda_0)$ as follows:
\[
\sigma(z) = \sup\{\alpha \in \mathbb{R} \mid z \in T^\alpha CF(H^{(p)}, p\gamma : \Lambda_0)\}
\]
\[
\tau(z) = \sigma(d_F(z)) - \sigma(z)
\]

We also define \(\sigma_{eq}(z)\) and \(\tau_{eq}(z)\) for \(z \in \widehat{CF}_\mathbb{Z}_p(H^{(p)}, p\gamma)\) as follows:

\[
\sigma_{eq}(z) = \sup\{\alpha \in \mathbb{R} \mid z \in T^\alpha \widehat{CF}_\mathbb{Z}_p(H^{(p)}, p\gamma)\}
\]
\[
\tau_{eq}(z) = \sigma_{eq}(d_{eq}(z)) - \sigma_{eq}(z)
\]

First, we prove the following claim.

**Claim 5.1** Let \(\beta_1\) and \(\delta_1\) be positive real numbers defined in the statement of Lemma 5.4. Then the following equalities hold:

\[
\delta_1 = \inf\{\tau(z) \mid z \in CF(H^{(p)}, p\gamma : \Lambda_0)\}
\]
\[
p\beta_1 = \inf\{\tau_{eq}(z) \mid \widehat{CF}_\mathbb{Z}_p(H^{(p)}, p\gamma)\}
\]

We prove only the first equality (the proof for the second equality is the same). We can choose a cycle \(z \in CF(H^{(p)}, p\gamma : \Lambda_0)\) such that \(\sigma(z) = 0\) holds and \(T^{\delta_1}z\) is a boundary. So we can choose \(w \in CF(H^{(p)}, p\gamma : \Lambda_0)\) such that \(d_F(w) = T^{\delta_1}z\) holds. This implies that \(\tau(w) \leq \delta_1\) and

\[
\delta_1 \geq \inf\{\tau(z) \mid z \in CF(H^{(p)}, p\gamma : \Lambda_0)\}
\]

holds. Assume that \(\delta_1 > \text{RHS}\) holds. Then there is \(z \in CF(H^{(p)}, p\gamma : \Lambda_0)\) such that

\[
\sigma(z) = 0, \quad \alpha = \tau(z) = \sigma(d_F(z)) < \delta_1
\]

holds. So, \(w = T^{-\alpha}d_F(z)\) satisfies \(\sigma(w) = 0\) and \(T^{\alpha}w = d_F(z)\) is a boundary. Let \(\{C_1, \cdots, C_m\}\) be the generators of

\[
HF(H^{(p)}, p\gamma : \Lambda_0) \cong \bigoplus_{j=1}^m \Lambda_0/T^{\delta_j}\Lambda_0.
\]

\([T^\alpha w] = 0 \in HF(H^{(p)}, p\gamma : \Lambda_0)\) implies that \([w] \in HF(H^{(p)}, p\gamma : \Lambda_0)\) is written in the following form.

\[
[w] = \sum_{\delta_1 - \alpha \leq \lambda_1, j < \delta_1} a_{1,\lambda_1,j} T^{\lambda_1,j} [C_1] + \cdots + \sum_{\delta_m - \alpha \leq \lambda_m, j < \delta_m} a_{m,\lambda_m,j} T^{\lambda_m,j} [C_m]
\]

Note that \([w] \neq 0\) holds because our chain complex is strict and any boundary \(b \in CF(H^{(p)}, p\gamma : \Lambda_0)\) satisfies \(\sigma(b) > 0\). We choose a chain \(v \in CF(H^{(p)}, p\gamma : \Lambda_0)\) so that

\[
C = \sum_{\delta_1 - \alpha \leq \lambda_1, j < \delta_1} a_{1,\lambda_1,j} T^{\lambda_1,j} C_1 + \cdots + \sum_{\delta_m - \alpha \leq \lambda_m, j < \delta_m} a_{m,\lambda_m,j} T^{\lambda_m,j} C_m
\]
\[
C = w + d_F(v)
\]

29
holds. However $\sigma(d_F(v)) > 0$ implies that

$$0 < \delta_1 - \alpha \leq \sigma(C) = \sigma(w + d_F(v)) = \sigma(w) = 0$$

holds. This is a contradiction and we proved Claim 5.1.

In Corollary 5.1 we proved that $\hat{d}_{eq}$ has the following form:

$$\hat{d}_{eq}(\sum_{i,a} \alpha_{i,a} \tilde{X}_i^a) = \iota(d_F(\sum_{i,a} \alpha_{i,a} X_i^a)) \otimes 1 + \sum_{i,a} \beta_{i,a} \tilde{X}_i^a + \sum_{l \geq 1} \sum_{i,a} (\delta_{i,a} \tilde{X}_i^a + \beta_{i,a} \tilde{X}_i^a) u^l$$

This implies that

$$\sigma(d_F(\sum_{i,a} \alpha_{i,a} X_i^a)) \geq \sigma_{eq}(\hat{d}_{eq}(\sum_{i,a} \alpha_{i,a} \tilde{X}_i^a))$$

$$\tau(\sum_{i,a} \alpha_{i,a} X_i^a) \geq \tau_{eq}(\sum_{i,a} \alpha_{i,a} \tilde{X}_i^a)$$

holds. So

$$\delta_1 = \inf\{\tau(x) \mid x \in CF(H^{(p)}, p\gamma : \Lambda_0)\} \geq \inf\{\tau_{eq}(z) \mid z \in CF_{\mathbb{Z}}(H^{(p)}, p\gamma)\} = p\beta_1$$

holds and we finished the proof of Lemma 5.4.

Next we apply Lemma 5.4 to prove Theorem 1.1. Recall that we assumed

$$P(H, \gamma) = \{x_1, \ldots, x_k\}$$

$$P(H^{(p)}, p\gamma) = \{x_1^{(p)}, \ldots, x_k^{(p)}\}$$

holds and $p$ is admissible in the beginning of this subsection. Our purpose is to prove that there is a simple $p'$ periodic orbit in $p\gamma$. This is equivalent to proving that $P(H^{(p')}, p\gamma) \neq \emptyset$ because any periodic orbit in $P(H^{(p')}, p\gamma)$ is simple if $p$ (hence $p'$) is sufficiently large. This follows from the following argument. Let \(\{e_1, \ldots, e_m\}\) be a basis of $H_1(M : \mathbb{Z})/\text{Tor}$. Then $\gamma$ is a sum as follows:

$$\gamma = \alpha_1 e_1 + \cdots + \alpha_m e_m \quad (\alpha_i \in \mathbb{Z})$$

If there is non-simple periodic orbit in $P(H^{(p')}, p\gamma)$, there is $\gamma' \in H_1(M : \mathbb{Z})/\text{Tor}$ so that

$$p'\gamma' = p\gamma$$

holds. This implies that $p'$ is a common divisor of $\{\alpha_1, \ldots, \alpha_m\}$. In particular, any periodic orbit in $P(H^{(p')}, p\gamma)$ is simple if $p > \max\{\alpha_1, \ldots, \alpha_m\}$ holds. We fix $C > 0$ independent of $p$ so that

$$HF(H, \gamma : \Lambda_0^{p\gamma}) \cong \bigoplus_{j=1}^m \Lambda_0^{p\gamma_j} / T^{\beta_{p,j}} \Lambda_0^{p\gamma_j} \quad (0 < \beta_{p,1} \leq \cdots \leq \beta_{p,m})$$

$$C < \frac{1}{2} \beta_{p,1}$$

30
holds. Here $\Lambda^F_p$ is the universal Novikov ring of the ground field $\mathbb{F}_p$. This is always possible because $\beta_{p,1}$ is greater than the minimum energy of a solution to the Floer equation, which is independent of $p$. Lemma 5.4 implies that

$$\tau(z) \geq p\beta_{p,1} \geq 2pC$$

holds for any $z \in CF(H^{(p)}, p\gamma: \Lambda^F_p)$. This implies that any element $x \neq 0 \in HF^{loc}(H^{(p)}, x^{(p)})$ determines a cycle in $CF[\mathbb{F}_p]^{(p)}(H^{(p)}, p\gamma: \Lambda^F_p)$ and they are not boundaries ($\epsilon > 0$ is sufficiently small) because $\tau(z) \geq 2pC$ holds for any $z$. This implies that the natural homomorphism

$$\iota: HF[-\epsilon, p\gamma: \Lambda^F_p] \rightarrow HF[-p\gamma, p\gamma: \Lambda^F_p]$$

is not zero ($\iota([x]) \neq 0$). Let $p'$ be the first prime number greater than $p$. We assume $p$ is sufficiently large prime so that

$$2(p' - p)||H|| < pC$$

holds. This is possible because $p' - p = o(p)$ holds (see [2]). We have two continuation homomorphisms as follows.

$$F: HF[-\epsilon, p\gamma: \Lambda^F_p] \rightarrow HF[-p\gamma, p\gamma: \Lambda^F_p]$$

$$G: HF[-(p' - p)||H||, p\gamma: \Lambda^F_p] \rightarrow HF[-p\gamma, p\gamma: \Lambda^F_p]$$

The composition of $F$ and $G$ satisfies $GF = \iota \neq 0$. So, $P(H^{(p')}, p\gamma) \neq \emptyset$ holds and we proved the theorem.

6 Weakly monotone case

6.1 Preliminary

In this section, we prove Theorem 1.1 for weakly monotone symplectic manifolds. The difference between toroidally monotone cases and weakly monotone cases is that we have not constructed $\mathbb{Z}_p$-equivariant Floer homology and $\mathbb{Z}_p$-equivariant pair of pants product in the weakly monotone case. So to prove Theorem 1.1 for the weakly monotone case, it suffices to construct these theories. The rest of the proof is the same as in the toroidally monotone case.

In the toroidally monotone case, we can exclude sphere bubbles easily because the Maslov index of a holomorphic sphere is greater than or equal to 2. In the weakly monotone case, we have to exclude sphere bubbles much more carefully. Recall the construction of Floer homology theory for weakly monotone symplectic manifolds [24, 29]. For a generic choice of an almost complex structure $J$, there are no holomorphic spheres with negative Chern numbers. Let $H$ be a Hamiltonian function. The pair $(H, J)$ is not necessary a Floer regular pair. However, we can perturb $H$ to $\tilde{H}$ so that $(\tilde{H}, J)$ is Floer regular and sphere bubbles do not appear in the definition of the Floer boundary operator $d_F$. 31
Recall that in the definition of $\mathbb{Z}_p$-equivariant Floer homology for toroidally monotone symplectic manifolds, we considered a family of almost complex structures $\{J_{w,t}\}$ parametrized by $S^\infty$ and $S^1$ while fixing a Hamiltonian function $H^{(p)}$. So one possible modification for the weakly monotone case is to consider a family of Hamiltonian functions parametrized by $S^\infty$ and $S^1$ while fixing an almost complex structure $J$. Let $K \in C^\infty(S^1 \times M)$ be a perturbation of $H^{(p)}$ so that $(K, J)$ is a Floer regular pair. Note that $K$ is not necessarily a $\frac{1}{p}$-periodic Hamiltonian function. We also consider a family of Hamiltonian functions $K_{w,t}$ parametrized by $(w, t) \in S^\infty \times S^1$ which satisfies the following conditions (compare it to the definition of $J_{w,t}$ in section 3).

- (locally constant at critical points) For all $w$ in a small neighborhood of $Z_i^m \in S^\infty$,
  $$K_{w,t} = K_{t-\frac{m}{p}}$$
- ($\mathbb{Z}_p$-equivariance) $K_{m \cdot w, t} = K_{w, t}$ holds for any $m \in \mathbb{Z}_p$ and $w \in S^\infty$.
- (invariance under the shift $\tau$) $K_{\tau(w), t} = K_{w, t}$ holds.

We consider the following equation for $x, y \in P(K), m \in \mathbb{Z}_p$ and $i \in \mathbb{Z}$.

$$(u, v) \in C^\infty(\mathbb{R} \times S^1, M) \times C^\infty(\mathbb{R}, S^\infty)$$
$$\partial_s u(s, t) + J(u(s, t)) (\partial_t u(s, t) - X_{K_m(t, x)}(u(s, t))) = 0$$
$$\frac{d}{ds} v(s) - \text{grad} \tilde{F} = 0$$
$$\lim_{s \to -\infty} v(s) = Z_i^0, \lim_{s \to +\infty} v(s) = Z_i^m, \lim_{s \to -\infty} u(s, t) = x(t), \lim_{s \to +\infty} u(s, t) = y(t - \frac{m}{p})$$

One might try to define
$$d_{i, m}^\alpha : CF(K, \gamma : \Lambda_0) \to CF(K, \gamma : \Lambda_0)$$
by counting above solutions and define $d_{eq}$. However, this attempt contains the following difficulty. The action gap of the solution $(u, v)$ of the above equation
$$\int_{\mathbb{R} \times S^1} u^* \omega + \int_0^1 K(t, x(t)) - K(t, y(t)) dt$$
is not necessarily non-negative. This problem happens when $i \in \mathbb{Z}$ becomes sufficiently large. As $i$ becomes bigger and bigger, the effect of the perturbation $H^{(p)} \to K_{w,t}$ becomes bigger and we cannot define a differential operator $d_{eq}$ over $\Lambda_0$. What we can do is to fix some $N \in \mathbb{Z}$ and define a finite operators $\{d_{i,m}^\alpha\}$ for $i \leq N$.

This difficulty is very similar to the difficulty in [14]. In [14], Fukaya, Oh, Ohta and Ono constructed an $A_\infty$-algebra associated to a Lagrangian submanifold in a symplectic manifold. They constructed $A_\infty$-operators $\{m_k\}_{k \in \mathbb{Z} \geq 0}$ by
using moduli spaces of holomorphic discs bounding the Lagrangian submanifold. To determine \( \{m_k\}_{k \in \mathbb{Z}_{\geq 0}} \), they had to achieve transversality of infinitely many interrelated moduli spaces by perturbing multisections of Kuranishi structures. Unfortunately, this is impossible because perturbations of lower-order operators influence perturbations of higher-order operators, and the higher perturbations become bigger and bigger. So what they could do is to construct finite operators \( \{m_0, m_1, \cdots, m_n, K\} \) \((A_{n, K}\text{-algebra})\) by one perturbation. They constructed \( A_\infty\text{-operators} \( \{m_k\}_{k \in \mathbb{Z}_{\geq 0}} \) by “gluing” infinitely many \( A_{n, K}\text{-algebras} \((n, K) \to (+\infty, +\infty)\)) by applying homological algebra developed in [14]. Our situation is much simpler because we do not have to consider higher algebraic operators (we only need differential operator) and we do not have to consider the space of infinitely many singular chains. So we can mimic the construction of Lagrangian \( A_\infty\text{-algebra} \) by applying the algebraic machinery developed in [14]. In the rest of this section, we explain how we can apply [14] in our cases.

6.2 \( X_K\)-module, \( X_K\)-morphism and \( X_K\)-homotopy

We consider the following situation. Let \( C \) be a \( \mathbb{Z}_2\)-graded \( \Lambda_0\)-module. We want to construct a family of degree 1 maps \( \{d_i : C \to C\}_{i \in \mathbb{Z}_{\geq 0}} \) so that \( \sum_{i+j=k} d_id_j = 0 \) holds for any \( k \in \mathbb{Z}_{\geq 0} \). Then the infinite sum \( d = d_0 + d_1u + d_2u^2 + d_3u^3 \cdots \) becomes a boundary operator on \( C \otimes \Lambda_0[[u]] \) as follows:

\[
d : C \otimes \Lambda_0[[u]] \to C \otimes \Lambda_0[[u]]
\]

\[
x \otimes u^m \mapsto \sum_{i \geq 0} d_i(x) \otimes u^{m+i}
\]

**Definition 6.1**

1. Let \( f = \{f_i : C \to D\}_{i \in \mathbb{Z}_{\geq 0}} \) be a family of maps between \( \Lambda_0\)-modules. We define \( f^{(L)} \in \text{Hom}(C, D) \otimes \Lambda_0[[u]] \) for some \( L \in \mathbb{Z}_{\geq 0} \) as follows:

\[
f^{(L)} = \begin{cases} 
  f_0 + f_1 \otimes u + \cdots + f_L \otimes u^L & L \leq K \\
  f_0 + f_1 \otimes u + \cdots + f_K \otimes u^K & L \geq K 
\end{cases}
\]

2. Let \( \vartheta = \{d_i : C \to C\}_{i \in \mathbb{Z}_{\geq 0}} \) be a family of degree 1 maps \((0 \leq K \leq \infty)\). We call \( (C, \vartheta) \) an \( X_K\)-module if \( \vartheta \) satisfies

\[
\vartheta^{(K)} \circ \vartheta^{(K)} \equiv 0 \mod(u^{K+1}).
\]

3. Let \( (C, \vartheta) \) and \( (D, \imath) \) be \( X_K\)-modules. An \( X_K\)-morphism between \( C \) and \( D \) is a family of degree 0 maps \( \imath = \{\imath_i\}_{i \in \mathbb{Z}_{\geq 0}} \) which satisfies the following equality:

\[
\vartheta^{(K)} \circ \vartheta^{(K)} \equiv \imath^{(K)} \circ \imath^{(K)} \mod(u^{K+1})
\]

33
We define a $\Lambda_0$-algebra $\Lambda_0^{(K)}$ as follows:

$$
\Lambda_0^{(K)} = \begin{cases} 
\Lambda_0[[u]]/(u^{K+1}) & K < +\infty \\
\Lambda_0[[u]] & K = +\infty
\end{cases}
$$

For any $\Lambda_0$-module $C$, we define $\Lambda_0^{(K)}$-module $C^{(K)}$ by

$$
C^{(K)} = C \otimes_{\Lambda_0} \Lambda_0^{(K)}.
$$

We define $\deg(u) = 2$, so $C^{(K)}$ is also $\mathbb{Z}_2$-graded. Note that $(C, \partial)$ is an $X_K$-homotopy if and only if $\partial^{(K)} \circ \partial^{(K)} = 0$ holds on $C^{(K)}$. $f : (C, \partial) \to (D, l)$ is an $X_K$-morphism if and only if

$$
f^{(K)} : (C^{(K)}, \partial^{(K)}) \longrightarrow (D^{(K)}, l^{(K)})
$$

is a cochain map.

Let $f_0$ and $f_1$ be $X_K$-morphisms from $(C, \partial)$ to $(D, l)$. We say $f_0$ and $f_1$ are $X_K$-homotopic if there is a family of maps $h = \{h_i : C \to D\}_{i=0}^{K}$ which satisfies the following relations:

$$
f_0^{(K)} - f_1^{(K)} \equiv h^{(K)} \circ \partial^{(K)} + f^{(K)} \circ h^{(K)} \mod(u^{K+1})
$$

A composition of two $X_K$-morphism $\{f_i : C \to D\}$ and $\{g_j : D \to E\}$ is defined by $(f \circ g)_i = \sum_{j+i=1} f_j g_i$. Note that

$$
(f \circ g)^{(K)} \equiv f^{(K)} \circ g^{(K)} \mod(u^{K+1})
$$

holds. We call an $X_K$-morphism $f : (C, \partial) \to (D, l)$ $X_K$-homotopy equivalence if there is a $X_K$-morphism $g : (D, l) \to (C, \partial)$ such that $fg$ and $gf$ are $X_K$-homotopic to the identity.

We also give an equivalent definition of $X_K$-homotopy. For this purpose, we introduce a new $X_K$-module $(C \times [0, 1], \partial = \{\partial_i\})$ for an $X_K$-module $(C, \partial)$. We define $d^{(K)} = \sum_{i=0}^{K} \partial_i u^i$ as follows:

$$
C \times [0, 1] = C \oplus C[-1] \oplus C
$$

$$
\begin{align*}
\bar{d}^{(K)}(x, 0, 0) &= (d^{(K)}(x), (-1)^{\deg x} x, 0) \\
\bar{d}^{(K)}(0, y, 0) &= (0, d^{(K)}(y), 0) \\
\bar{d}^{(K)}(0, 0, z) &= (0, -(-1)^{\deg z} z, d^{(K)}(z))
\end{align*}
$$

Note that $C[-1]$ is a copy of $C$ with degree shift 1.

$$
C[-1]^m = C^{m-1}
$$

It is straightforward to see that $\bar{d}^{(K)} \bar{d}^{(K)} = 0$ holds. So $(C \times [0, 1], \bar{\partial})$ is an $X_K$-module. We also define $X_K$-morphisms Incl = $\{(\text{Incl})_i\}_{i \leq K}$, Eval$_{i=0} = \{(\text{Eval})_{i=0}\}_{i \leq K}$ and Eval$_{i=1} = \{(\text{Eval})_{i=1}\}_{i \leq K}$ between $(C, \partial)$ and $(C \times [0, 1], \bar{\partial})$ as follows:
homotopy between \( f \)

By comparing the middle factors of the right hand sides, we can see that 

\[ \text{Eval}_{s=0} : (C \times [0, 1], \partial) \rightarrow (C, \partial) \]

\[ (\text{Eval}_{s=0})_{i}(x, y, z) = \begin{cases} x & i = 0 \\ 0 & i \geq 1 \end{cases} \]

\[ \text{Eval}_{s=1} : (C \times [0, 1], \partial) \rightarrow (C, \partial) \]

\[ (\text{Eval}_{s=1})_{i}(x, y, z) = \begin{cases} z & i = 0 \\ 0 & i \geq 1 \end{cases} \]

**Definition 6.2** Let \( f_0 \) and \( f_1 \) be \( X_K \)-morphisms between \((C, \partial) \) and \((D, l)\). We say \( f_0 \) and \( f_1 \) are \( X_K \)-homotopic if there is an \( X_K \)-morphism \( h : C \rightarrow D \times [0, 1] \) so that \( \text{Eval}_{s=0} \circ h = f_0 \) and \( \text{Eval}_{s=1} \circ h = f_1 \) hold. \( h \) is called an \( X_K \)-homotopy between \( f_0 \) and \( f_1 \).

The next lemma implies that the above two definitions of \( X_K \)-homotopy are equivalent.

**Lemma 6.1** Let \( f_0 \) and \( f_1 \) be \( X_K \)-morphisms between \((C, \partial) \) and \((D, l)\). A family of degree \(-1\) maps \( h = \{ h_i \}_{i=0}^{K} \subset \text{Hom}(C, D) \) satisfies

\[ f_0^{(K)} - f_1^{(K)} = h^{(K)} \circ \partial^{(K)} + (l^{(K)} \circ h^{(K)}) \mod(u^{K+1}) \]

if and only if a family of maps

\[ \tilde{h} = \{ \tilde{h}_i \}; (C, \partial) \rightarrow (D \times [0, 1], \tilde{l}) \]

\[ \tilde{h}_i(x) = ((f_0)_i(x), (-1)^{\text{deg} h_i}(x), (f_1)_i(x)) \]

is an \( X_K \)-homotopy between \( f_0 \) and \( f_1 \).

**proof:** We calculate \( \tilde{h}^{(K)} \circ \partial^{(K)} \) and \( \tilde{h}^{(K)} \circ \partial^{(K)} \). Note that \( \tilde{h} \) is an \( X_K \)-homotopy between \( f_0 \) and \( f_1 \) if and only if they coincide modulo \( u^{K+1} \).

\[ \tilde{h}^{(K)} \circ \partial^{(K)}(x) = (f_0^{(K)})(x), -(-1)^{\text{deg} f_0^{(K)}}(x) + (-1)^{\text{deg} f_1^{(K)}}(x) - (-1)^{\text{deg} f_0^{(K)}}(x) + f_1^{(K)}(x) \]

\[ \tilde{h}^{(K)} \circ \partial^{(K)}(x) = (f_0^{(K)})(x), (-1)^{\text{deg} f_0^{(K)}}(x) + (-1)^{\text{deg} f_1^{(K)}}(x) + f_1^{(K)}(x) \]

By comparing the middle factors of the right hand sides, we can see that \( \tilde{h} \) is an \( X_K \)-homotopy between \( f_0 \) and \( f_1 \) if and only if

\[ f_0^{(K)} - f_1^{(K)} = h^{(K)} \circ \partial^{(K)} + (l^{(K)} \circ h^{(K)}) \mod(u^{K+1}) \]

holds.
Lemma 6.2

\[(\text{Incl})_0 : (C, d_0) \longrightarrow (C \times [0, 1], \tilde{d}_0)\]
\[(\text{Eval}_{s = s_0})_0 : (C \times [0, 1], \tilde{d}_0) \longrightarrow (C, d_0) \quad (s_0 = 0, 1)\]

are cochain homotopy equivalences.

\textbf{proof}: It suffices to prove that \((\text{Eval}_{s = s_0})_0(\text{Incl})_0\) and \((\text{Incl})_0(\text{Eval}_{s = s_0})_0\) are homotopic to the identity. The former case is trivial because it is equal to the identity.

\[(\text{Eval}_{s = s_0})_0(\text{Incl})_0 = \text{Id}\]

\[\{\text{Id} - (\text{Incl})_0 \circ (\text{Eval}_{s = s_0})_0\}(x, y, z) = (0, y, z - x)\]

\[\{\text{Id} - (\text{Incl})_0 \circ (\text{Eval}_{s = 1})_0\}(x, y, z) = (x - z, y, 0)\]

For the sake of simplicity, we do not use the shifted degree of \(C \times [0, 1]\) here. We can define cochain homotopies \(h_{s_0}\) and \(h_{s_1}\) as follows.

\[h_{s_0} : C \oplus C \oplus C \longrightarrow C \oplus C \oplus C\]

\[(x, y, z) \mapsto (0, 0, -(-1)^{\text{deg}_y} y)\]

\[h_{s_1} : C \oplus C \oplus C \longrightarrow C \oplus C \oplus C\]

\[(x, y, z) \mapsto ((-1)^{\text{deg}_y} y, 0, 0)\]

The following calculations imply that \(h_{s = s_0}\) is a homotopy between \(\text{Id}\) and \((\text{Incl})_0 \circ (\text{Eval}_{s = s_0})_0\).

\[\{h_{s_0} \circ \tilde{d}_0 + \tilde{d}_0 \circ h_{s_0}\}(x, y, z)\]

\[= h_{s_0}(d_0(x), (-1)^{\text{deg}_x} x + d_0(y) - (-1)^{\text{deg}_z} z, d_0(z)) + \tilde{d}_0(0, 0, -(-1)^{\text{deg}_y} y)\]

\[= (0, 0, -x + (-1)^{\text{deg}_y} d_0(y) + z) + (0, y, -(-1)^{\text{deg}_y} d_0(y))\]

\[= (0, y, z - x) = \{\text{Id} - (\text{Incl})_0 \circ (\text{Eval}_{s = s_0})_0\}(x, y, z)\]

\[\{h_{s_1} \circ \tilde{d}_0 + \tilde{d}_0 \circ h_{s_1}\}(x, y, z)\]

\[= h_{s_1}(d_0(x), (-1)^{\text{deg}_x} x + d_0(y) - (-1)^{\text{deg}_z} z, d_0(z)) + \tilde{d}_0((-1)^{\text{deg}_y} y, 0, 0)\]

\[= (x - (-1)^{\text{deg}_y} d_0(y) - z, 0, 0) + ((-1)^{\text{deg}_y} d_0(y), y, 0)\]

\[= (x - z, y, 0) = \{\text{Id} - (\text{Incl})_0 \circ (\text{Eval}_{s = 1})_0\}(x, y, z)\]

\[\square\]

36
We need a cohomology theory on the space of homomorphisms between two cochain complexes $\text{Hom}((A, d_A), (B, d_B))$.

**Definition 6.3** Let $(A, d_A)$ and $(B, d_B)$ be $\mathbb{Z}_2$-graded differential modules. We define a coboundary operator on $\text{Hom}(A, B)$ as follows:

$$\partial_{\text{Hom}((A, d_A), (B, d_B))} : \text{Hom}(A, B) \to \text{Hom}(A, B)$$

$$\phi \mapsto d_B \circ \phi - (-1)^{\text{deg} \phi} \circ d_A$$

We denote the cohomology of this cochain complex by $H\left(\text{Hom}((A, d_A), (B, d_B))\right)$.

**Lemma 6.3** Let $(A, d_A)$, $(A', d_{A'})$, $(B, d_B)$ and $(B', d_{B'})$ be $\mathbb{Z}_2$-graded complexes. Assume that there are cochain maps $f : (A', d_{A'}) \to (A, d_A)$ and $g : (B, d_B) \to (B', d_{B'})$. Then, $(g, f)_* : \text{Hom}(A, B) \to \text{Hom}(A', B')$

$$\phi \mapsto g \circ \phi \circ f$$

is a cochain map. Moreover, if $f'$ is cochain homotopic to $f$ and $g'$ is a cochain homotopic to $g$, $(g', f')_*$ is also cochain homotopic to $(g, f)_*$.

**Proof:**

$$(g, f)_*(\partial_{\text{Hom}(A, B)}(\phi)) = g \circ (d_B \phi - (-1)^{\deg \phi} \circ d_A) \circ f$$

$$= d_{B'} \circ g \circ f - (-1)^{\deg \phi} g \circ f \circ d_{A'} = \partial_{\text{Hom}(A', B')}(g \circ f)$$

$$= \partial_{\text{Hom}(A', B')}((g, f)_*(\phi))$$

So, $(g, f)_*$ is a cochain map. Assume that $f$ is cochain homotopic to $f'$ and $g'$ is cochain homotopic to $g$. It suffices to prove that $(g, \text{Id})_*$ and $(g', \text{Id})_*$ are cochain homotopic and $(\text{Id}, f)_*$ and $(\text{Id}, f')_*$ are cochain homotopic. Let $h_{f, f'}$ be a homotopy between $f$ and $f'$ and let $h_{g, g'}$ be a homotopy between $g$ and $g'$. Then, we define cochain homotopies $H_{(f, f')}$ and $H_{(g, g')}$ as follows:

$$H_{(f, f')} : \text{Hom}(A, B) \to \text{Hom}(A', B)$$

$$\phi \mapsto (-1)^{\deg \phi} \circ h_{(f, f')}(\phi)$$

$$H_{(g, g')} : \text{Hom}(A, B) \to \text{Hom}(A, B')$$

$$\phi \mapsto h_{(g, g')}(\phi)$$
The following calculations imply that $H_{f,f'}$ is a cochain homotopy between $(\text{Id}, f)_*$ and $(\text{Id}, f')_*$, and $H_{g,g'}$ is a cochain homotopy between $(g, \text{Id})_*$ and $(g', \text{Id})_*$. 

$$\left\{ \partial_{\text{Hom}(A',B)} \circ H_{(f,f')} + H_{(f,f')} \circ \partial_{\text{Hom}(A,B)} \right\}(\phi)$$

$$= \partial_{\text{Hom}(A',B)}((-1)^{\deg \phi} \circ h_{(f,f')}) + H_{(f,f')}(d_B \phi - (-1)^{\deg \phi} d_A)$$

$$= (-1)^{\deg \phi} d_B h_{(f,f')} + \phi h_{(f,f')} d_A' + (-1)^{\deg \phi + 1} (d_B \phi h_{(f,f')} - (-1)^{\deg \phi} d_A h_{(f,f')})$$

$$= \phi (h_{(f,f')} d_A' + d_A h_{(f,f')}) = \phi f - \phi f'$$

$$= \{(\text{Id}, f)_* - (\text{Id}, f')_*\}(\phi)$$

Next proposition describes an obstruction for extensions of $X_K$-morphisms.

**Proposition 6.1.** Let $(C, \partial)$ and $(D, \partial)$ be $X_{K+1}$-modules. Assume that $\hat{f} : C \to D$ is an $X_K$-morphism. Then there is degree 1 element

$$\sigma_{K+1}(\hat{f}) \in \text{Hom}(C, D)$$

which satisfies the following properties.

1. $\partial_{\text{Hom}(C, D, \partial)}(\text{df}_{K+1}(\hat{f})) = 0$ holds. So $\sigma_{K+1}(\hat{f})$ is a cocycle.

2. We can extend $\hat{f}$ to an $X_{K+1}$-morphism if and only if $\sigma_{K+1}(\hat{f})$ is a coboundary ($[\sigma_{K+1}(\hat{f})] = 0$).

3. If $\hat{f}$ and $\hat{f}'$ are $X_K$-homotopic,

$$[\sigma_{K+1}(\hat{f})] = [\sigma_{K+1}(\hat{f}')] \in H(\text{Hom}(C, D, \partial))$$

holds.

4. Let $g : C' \to C$ and $g' : D \to D'$ be $X_{K+1}$-morphisms. Then,

$$[\sigma_{K+1}(g' \circ \hat{f} \circ g)] = (g' \circ \text{id})_*[\sigma_{K+1}(\hat{f})]$$

holds.
proof: Let \( f : (C, \partial) \to (D, l) \) be an \( X_K \)-morphism. Then,

\[ f^{(K+1)} \circ f^{(K)} - f^{(K)} \circ g^{(K+1)} \equiv o_{K+1}(f) \otimes u^{K+1} \mod(u^{K+2}) \]

holds for some \( o_{K+1}(f) \in \text{Hom}(C, D) \). This is the definition of \( o_{K+1}(f) \). Note that

\[ f^{(K+1)} \circ f^{(K)} - f^{(K)} \circ g^{(K+1)} = \partial_{\text{Hom}((C(K+1), \partial(K+1)) (D(K+1), l(K+1)))}(f^{(K)}) \]

holds. This implies that

\[ 0 = \left\{ \partial_{\text{Hom}((C(K+1), \partial(K+1)) (D(K+1), l(K+1)))} \right\}^{2}(f^{(K)}) \]

\[ = \partial_{\text{Hom}((C(K+1), \partial(K+1)) (D(K+1), l(K+1)))}(o_{K+1}(f) \otimes u^{K+1}) \]

\[ = \partial_{\text{Hom}((C, d_0), (D, l_0))}(o_{K+1}(f)) \otimes u^{K+1} \]

holds on \( \text{Hom}((C^{(K+1)}, \partial^{(K+1)})(D^{(K+1)}, l^{(K+1)}) \). So

\[ \partial_{\text{Hom}((C, d_0), (D, l_0))}(o_{K+1}(f)) = 0 \]

holds and \( o_{K+1}(f) \in \text{Hom}(C, d_0), (D, l_0) \) is a cocycle. \( f = \{ f_0, \cdots, f_K \} \) and \( f_{K+1} \) determines an \( X_{K+1} \)-morphism if and only if

\[ 0 \equiv f^{(K+1)} \circ (f^{(K)} + f_{K+1} \otimes u^{K+1}) - (f^{(K)} + f_{K+1} \otimes u^{K+1}) \circ g^{(K+1)} \]

\[ \equiv (o_{K+1}(f) + \partial_{\text{Hom}((C, d_0), (D, l_0))}(f_{K+1})) \otimes u^{K+1} \mod(u^{K+2}) \]

holds. So, \( f \) can be extended to an \( X_{K+1} \)-morphism if and only if \( o_{K+1}(f) \) is a coboundary. Next, we prove (iv). Let \( g : C' \to C \) and \( g' : D \to D' \) be \( X_{K+1} \)-morphisms. Then there is \( e : C' \to D' \) so that

\[ (g' \circ f \circ g)^{(K)} \equiv g'^{(K)} \circ f^{(K)} \circ g^{(K)} + e \otimes u^{K+1} \mod(u^{K+2}) \]

holds. Then,

\[ o_{K+1}(g' \circ f \circ g) \otimes u^{K+1} \equiv g'^{(K)} \circ f^{(K)} \circ g^{(K)} - (g' \circ f \circ g)^{(K)} \circ g^{(K+1)} \]

\[ \equiv g'^{(K+1)} \circ (g'^{(K)} \circ f^{(K)} \circ g^{(K)}) - (g'^{(K+1)} \circ f^{(K)} \circ g^{(K)}) \circ g^{(K+1)} \]

\[ + \partial_{\text{Hom}((C', d'_{0}), (D', l'_{0}))}(e) \otimes u^{K+1} \mod(u^{K+2}) \]

holds. \( g'^{(K+1)} \circ (g'^{(K)} \circ f^{(K)} \circ g^{(K)}) - (g'^{(K+1)} \circ f^{(K)} \circ g^{(K)}) \circ g^{(K+1)} \mod(u^{K+2}) \)

is divided into a sum of the following three terms:

(1)

\[ (g'^{(K+1)} \circ g'^{(K)} \circ f^{(K)} \circ g^{(K)}) - (g'^{(K+1)} \circ f^{(K)} \circ g^{(K+1)}) \]

\[ \equiv (o_{K+1}(g') \otimes u^{K+1}) \circ f^{(K)} \circ g^{(K)} \equiv (o_{K+1}(g') f_0 g_0) \otimes u^{K+1} \mod(u^{K+2}) \]

(2)

\[ g'^{(K)} \circ (g'^{(K+1)} \circ f^{(K)} \circ g^{(K)}) - (g'^{(K+1)} \circ f^{(K)} \circ g^{(K)}) \]

\[ \equiv g'^{(K)} \circ (o_{K+1}(f) \otimes u^{K+1}) \circ g^{(K)} \equiv (g'_0 o_{K+1}(f) g_0) \otimes u^{K+1} \mod(u^{K+2}) \]
\[ g'(K) \circ f(K) \circ (g(K+1) \circ g(K) - g(K) \circ g(K+1)) \]
\[ \equiv g'(K) \circ f(K) \circ (\sigma_{K+1}(g) \otimes u_{K+1}) \equiv (g'_0 f_0 \sigma_{K+1}(g)) \otimes u_{K+1} \mod(u_{K+2}) \]

So,
\[ \sigma_{K+1}(g' \circ g) = \sigma_{K+1}(g') f_0 g_0 + g'_0 \sigma_{K+1}(f) g_0 + g'_0 f_0 \sigma_{K+1}(g) + \partial_{\text{Hom}(C', \delta_0)(D', \delta_0)}(e) \]

Note that \([\sigma_{K+1}(g)] = [\sigma_{K+1}(g')] = 0\) because \(g\) and \(g'\) are \(X_{K+1}\)-morphisms. This implies that
\[ [\sigma_{K+1}(g' \circ g)] = [g'_0 \sigma_{K+1}(f) g_0] = (g'_0, g_0) \ast [\sigma_{K+1}(f)] \]
holds. Next, we prove (iii). Let \(h : C \to D \times [0, 1]\) be an \(X_K\)-homotopy between \(f\) and \(f'\). Then,
\[ [\sigma_{K+1}(f)] = [\sigma_{K+1}(\text{Eval}_{s=0} \circ h)] = ((\text{Eval}_{s=0})_0, \text{Id})_{\ast} [\sigma_{K+1}(h)] \]
\[ = ((\text{Eval}_{s=1})_0, \text{Id})_{\ast} [\sigma_{K+1}(h)] = [\sigma_{K+1}(\text{Eval}_{s=1} \circ h)] = [\sigma_{K+1}(f')] \]
holds.

\[ \square \]

We have the following corollary.

**Corollary 6.1** Let \(f : C \to D\) be an \(X_{K+1}\)-morphism. Assume that there are \(X_K\)-morphisms \(g : C \to D\) and \(h : C \to D \times [0, 1]\) such that \(h\) is an \(X_K\)-homotopy between \(f\) and \(g\). Then, we can extend \(g\) and \(h\) to \(X_{K+1}\)-morphisms so that \(h\) is an \(X_{K+1}\)-homotopy between \(f\) and \(g\).

**proof**: We define \(h'_{K+1} : C \to D \times [0, 1]\) by \(h'_{K+1} = (\text{Incl})_0 \circ f_{K+1}\). Then, \(h'_{K+1}\) satisfies the following equalities:
\[ ((\text{Eval}_{s=0})_0, \text{Id})_{\ast} (\sigma_{K+1}(h) + \partial_{\text{Hom}(C, \delta_0)(D \times [0, 1], \delta_0)}(h'_{K+1})) \]
\[ = \sigma_{K+1}(f) + \partial_{\text{Hom}(C, D)}(f_{K+1}) = 0 \]
\[ (\text{Eval}_{s=0})_0 \circ h'_{K+1} = f_{K+1} \]

Let \(N\) be a subcomplex of \((D \times [0, 1], \bar{\delta}_0)\) as follows:
\[ N = \text{Ker}((\text{Eval}_{s=0})_0) = \{(0, y, z) \in D \times [0, 1]\} \]
Then \(\sigma_{K+1}(h) + \partial_{\text{Hom}(C, \delta_0)(D \times [0, 1], \delta_0)}(h'_{K+1})\) is a cocycle in \(\text{Hom}(C, N)\). Note that \(N\) is \(l_0\)-acyclic. Indeed,
\[ H : N = 0 \oplus D \oplus D \longrightarrow 0 \oplus D \oplus D \]
\[ (0, y, z) \mapsto (0, 0, (-1)^{\text{deg}_y}) \]
satisfies \( \text{Id}_N - 0 = \tilde{\ell}_0 H + H \tilde{\ell}_0 \). So,

\[
H(\text{Hom}(C, N), \partial_{\text{Hom}((C,d_0),(N,\tilde{\ell}_0)))} = 0
\]

holds and we can choose \( \Delta h_{K+1} : C \to D \times [0,1] \) so that

\[
\text{Eval}_{s=0} \circ \Delta h_{K+1} = 0
\]

\[
o_{K+1}(h) + \partial_{\text{Hom}(C,D \times [0,1])}(h'_{K+1}) = -\partial_{\text{Hom}(C,D \times [0,1])}(\Delta h_{K+1})
\]

holds. \( h_{K+1} = h'_{K+1} + \Delta h_{K+1} \) satisfies the following equalities:

\[
o_{K+1}(h) + \partial_{\text{Hom}(C,D \times [0,1])}(h_{K+1}) = 0
\]

\[
\text{Eval}_{s=0} \circ h_{K+1} = f_{K+1}
\]

The first equality implies that \( h_0, \cdots, h_K, h_{K+1} \) determines an \( X_{K+1} \)-morphism. So this is an \( X_{K+1} \)-extension of the original \( X_K \)-morphism \( h \). The second equality implies that the extended \( h \) is an \( X_{K+1} \)-homotopy between \( f \) and \( \text{Eval}_{s=1} \circ h \). Note that \( \text{Eval}_{s=1} \circ h \) is an \( X_{K+1} \)-extension of \( g \) by \( \text{Eval}_{s=1} \circ h \).

\[\square\]

Next, we prove a very important proposition.

**Proposition 6.2** Let \( f : (C, \partial) \to (D, l) \) be an \( X_K \)-morphism and assume that

\[
f_0 : (C, d_0) \to (D, l_0)
\]

is a cochain homotopy equivalence. Then, \( f \) is an \( X_K \)-homotopy equivalence.

**proof** : Assume that \( g : D \to C \) is an \( X_L \)-morphism \( (L < K) \) such that there is an \( X_L \)-homotopy \( h : C \to C \times [0,1] \) between \( \text{Id} \) and \( g \circ f \). Our first purpose is to extend \( g \) and \( h \) to \( X_{L+1} \)-morphisms. Corollary 6.1 implies that we can choose \( h'_{L+1} : C \to C \times [0,1] \) so that \( h' = \{h_0, \cdots, h_L, h'_{L+1}\} \) is an \( X_{L+1} \)-morphism which satisfies the following properties.

- \( \text{Eval}_{s=0} \circ h' = \text{Id} \)
- \( \text{Eval}_{s=1} \circ h' \) is an \( X_{L+1} \)-extension of \( g \circ f \).

Note that

\[
0 = [\text{Eval}_{L+1}(g \circ f)] = (\text{Id}, f_0)_* [\text{Eval}_{L+1}(g)]
\]

implies that \( [\text{Eval}_{L+1}(g)] = 0 \) holds because \( f_0 \) is a cochain homotopy equivalence and hence \( (\text{Id}, f_0)_* \) is an isomorphism. So, we can choose \( g'_{L+1} : D \to C \) so that
$g' = \{g_0, \cdots, g_L, g'_L+1\}$ is an $X_{L+1}$-morphism. Then, $g' \circ f$ and $\text{Eval}_{s=1} \circ h'$ are $X_{L+1}$-extensions of $g \circ f$. This implies that the difference of these two extensions

$$\Theta = (g' \circ f)_{L+1} - (\text{Eval}_{s=1})_0 \circ h'_{L+1}$$

is a cocycle in $(\text{Hom}(C, C), \partial_{\text{Hom}(C, d_0), (C, d_0)})$ because

$$\partial((g' \circ f)_{L+1}) = \partial((\text{Eval}_{s=1})_0 \circ h'_{L+1}) = -\sigma_{L+1}(g \circ f)$$

holds. Recall that

$$(\text{Id}, f_0)_* : (\text{Hom}(D, C), \partial_{\text{Hom}(D, C)}) \longrightarrow (\text{Hom}(C, C), \partial_{\text{Hom}(C, C)})$$

is a cochain homotopy equivalence. So we can choose a cocycle $\Delta g'_{L+1} \in \text{Hom}(D, C)$ so that

$$[\Delta g'_{L+1} \circ f_0 + \Theta] = 0$$

holds. $g = \{g_0, \cdots, g_L, g'_L+1 + \Delta g'_{L+1}\}$ is another $X_{L+1}$-extension of $g$ because

$$\partial(g'_{L+1} + \Delta g'_{L+1}) = \partial(g'_{L+1}) = -\sigma_{L+1}(g)$$

holds. We fix $\Delta_1 h_{L+1} \in \text{Hom}(C, C)$ so that

$$\partial(\Delta_1 h_{L+1}) = \Delta g'_{L+1} \circ f_0 + \Theta = (g \circ f - \text{Eval}_{s=1} \circ h')_{L+1}$$

holds. We also fix $\Delta h_{L+1} \in \text{Hom}(C, C \times [0, 1])$ so that

$$(\text{Eval}_{s=0})_0 \circ \Delta h_{L+1} = 0$$

and

$$(\text{Eval}_{s=1})_0 \circ \Delta h_{L+1} = \Delta_1 h_{L+1}$$

holds. Note that $h = \{h_0, \cdots, h_L, h'_L+1 + \partial(\Delta h_{L+1})\}$ is another $X_{L+1}$-extension of $h$ because

$$\partial(h'_{L+1} + \partial(\Delta h_{L+1})) = \partial(h'_{L+1}) = -\sigma_{L+1}(h)$$

holds. The equalities

$$\text{Eval}_{s=0} \circ h = \text{Eval}_{s=0} \circ h' = \text{Id}$$

$$(\text{Eval}_{s=1} \circ h)_{L+1} = (\text{Eval}_{s=1} \circ h')_{L+1} + \partial(\Delta h_{L+1})$$

$$(\text{Eval}_{s=1} \circ h')_{L+1} + (g \circ f - \text{Eval}_{s=1} \circ h')_{L+1} = (g \circ f)_{L+1}$$

imply that $h$ is an $X_{L+1}$-homotopy between $\text{Id}$ and $g \circ f$. Inductively, we can construct an $X_K$-morphism $g : D \rightarrow C$ so that $g \circ f$ is $X_K$-homotopic to the identity. By applying this arguments to this $g$, we can construct an $X_K$-morphism $f' : C \rightarrow D$ so that $f' \circ g$ is $X_K$-homotopic to the identity. $f$ is $X_K$-homotopic to $f'$ because $f$ is homotopic to $f' \circ g \circ f$ and $f' \circ g \circ f$ is homotopic to $f'$. So $f$ is an $X_K$-homotopy equivalence and $g$ is a homotopy inverse of $f$. 

42
Next, we explain how to extend $X_K$-modules to $X_{K+1}$-modules.

**Proposition 6.3** Let $(C, \partial)$ be an $X_K$-module. There is an obstruction class
\[ p_{K+1}(\partial) \in (\text{Hom}(C, C), \partial_{\text{Hom}(C,d_0),(C,d_0)}) \]
such that we can extend $(C, \partial)$ to an $X_{K+1}$-module if and only if $[p_{K+1}(\partial)] = 0$ holds. Moreover, if $f : C \to D$ is an $X_K$-homotopy equivalence, $[p_{K+1}(\partial)] = 0$ holds if and only if $[p_{K+1}(f)] = 0$ holds.

**proof**: We define $p_{K+1}(\partial) \in \text{Hom}(C, C)$ by
\[ \partial(K) \circ \partial(K) \equiv p_{K+1}(\partial) \otimes u^{K+1} \mod(u^{K+2}). \]
\[
\partial_{\text{Hom}(C,C)}(p_{K+1}(\partial)) \otimes u^{K+1} \equiv (d_0 \circ p_{K+1}(\partial) - p_{K+1}(\partial) \circ d_0) \otimes u^{K+1} \\
\equiv (\partial(K) \circ p_{K+1}(\partial) - p_{K+1}(\partial) \circ \partial(K)) u^{K+1} \\
\equiv \partial(K) \circ \partial(K) \circ \partial(K) - \partial(K) \circ \partial(K) \circ \partial(K) \equiv 0 \mod(u^{K+2})
\]
So, $p_{K+1}(\partial)$ is a cocycle. Note that $d_{K+1} \in \text{Hom}(C, C)$ satisfies
\[ \partial_{\text{Hom}(C,C)}(d_{K+1}) = -p_{K+1}(\partial) \]
if and only if $\{d_0, \cdots, d_k, d_{K+1}\}$ determines an $X_{K+1}$-module structure on $C$. So, we can extend $(C, \partial)$ to an $X_{K+1}$-module if and only if $[p_{K+1}(\partial)] = 0$ holds. Assume that $f : C \to D$ and $g : D \to C$ are $X_K$-morphisms so that $f \circ g$ and $g \circ f$ are $X_K$-homotopic to the identity. It suffices to prove that
\[ [p_{K+1}(f)] = (f_0, g_0) \ast [p_{K+1}(\partial)] \]
holds. We define $e : C \to D$ by
\[ f(K) \circ \partial(K) - f(K) \circ f(K) \equiv e \otimes u^{K+1} \mod(u^{K+2}), \]
\[
\partial(K) \circ \partial(K) \circ \partial(K) \circ g(K) \\
\equiv (e u^{K+1} + f(K) \partial(K)) g(K) \\
\equiv e d_0 g_0 \otimes u^{K+1} + f(K) (e u^{K+1} + f(K) \partial(K)) g(K) \\
\equiv (e d_0 g_0 + l_0 e g_0) \otimes u^{K+1} + p_{K+1}(1) f_0 g_0 \otimes u^{K+1} \\
\equiv (\partial_{\text{Hom}(C,D)}(e)) g_0 \otimes u^{K+1} + p_{K+1}(1) f_0 g_0 \otimes u^{K+1} \mod(u^{K+2})
\]
On the other hands,
\[ f(K) \circ \partial(K) \circ \partial(K) \circ g(K) \\
\equiv f(K) \circ (p_{K+1}(\partial) \otimes u^{K+1}) \circ g(K) \equiv (f_0 \circ p_{K+1}(\partial) \circ g_0) \otimes u^{K+1} \mod(u^{K+2})
\]
This implies that
\[ (f_0, g_0) \ast [p_{K+1}(\partial)] = [f_0 \circ p_{K+1}(\partial) \circ g_0] = [p_{K+1}(1) f_0 g_0] = [p_{K+1}(1)] \]
holds.
We apply the above proposition to the next proposition.

**Proposition 6.4** Let \((C, d)\) be an \(X_K\)-module and let \((D, l)\) be an \(X_{K+1}\)-module. Assume that \(f : C \rightarrow D\) is an \(X_K\)-homotopy equivalence. Then we can extend \((C, d)\) to an \(X_{K+1}\)-module and we can also extend \(f\) to an \(X_{K+1}\)-homotopy equivalence.

**proof:** Proposition 6.3 implies that \([p_{K+1}(d)] = 0\) holds. So we can choose \(d'_{K+1} \in \text{Hom}(C, C)\) so that \(\{d_0, \cdots, d_K, d'_{K+1}\}\) determines an \(X_{K+1}\)-module structure on \(C\). Our purpose is to extend \(f\) to an \(X_{K+1}\)-morphism. However, \([o_{K+1}(f)] = 0\) does not hold in general. So, we add a cocycle \(\Delta d_{K+1} \in \text{Hom}(C, C)\) to \(d'_{K+1}\) so that \([o_{K+1}^{\text{new}}(f)] = 0\) holds. Note that \(\{d_0, \cdots, d_k, d'_{K+1} + \Delta d_{K+1}\}\) is a new \(X_{K+1}\)-extension of \(d\) and

\[ o_{K+1}^{\text{new}}(f) = o_{K+1}(f) - f_0 \circ \Delta d_{K+1} \]

holds. We can choose \(\Delta d_{K+1}\) so that \([o_{K+1}^{\text{new}}(f)] = 0\) holds because \(f_0 : (C, d_0) \rightarrow (D, l_0)\) is a cochain homotopy equivalence and hence

\[(f_0, \text{Id})_* : H(\text{Hom}(C, C), \partial_{\text{Hom}(C,C)}) \longrightarrow H(\text{Hom}(C, D), \partial_{\text{Hom}(C,D)}) \]

is an isomorphism. Then, \([o_{K+1}^{\text{new}}(f)] = 0\) implies we can extend \(f\) to an \(X_{K+1}\)-morphisms. Proposition 6.2 implies that this \(f\) is also an \(X_{K+1}\)-homotopy equivalence.

\[ \square \]

We also consider \(X_K\)-modules and \(X_K\)-morphisms over \(\mathbb{F}_p\).

**Definition 6.4 ((local \(X_K\)-modules, local \(X_K\)-morphisms))**

1. Let \((\overline{C}, \overline{d})\) be a \(\mathbb{Z}\)-graded finite dimensional \(\mathbb{F}_p\)-vector space. Let \(\overline{d}_{\text{loc}} = \{d_{i,\text{loc}}\}_{i=0}^{K}\) be a family of maps \(0 \leq K \leq \infty\) so that \(\deg(d_{i,\text{loc}}) = 1 - 2i\) holds. We call \((\overline{C}, \overline{d}_{\text{loc}})\) a local \(X_K\)-module if \(\overline{d}_{\text{loc}}\) satisfies

\[ \overline{d}_{\text{loc}}^{(K)} \circ \overline{d}_{\text{loc}}^{(K)} = 0 \mod(u^{K+1}). \]

If we assume that \(\deg(u) = 2\), \(\overline{d}^{(K)} = \sum_i d_{i,\text{loc}} \otimes u^i\) is a degree 1 map.

2. Let \((\overline{C}, \overline{d}_{\text{loc}})\) and \((\overline{D}, \overline{d}_{\text{loc}})\) be local \(X_K\)-modules. A local \(X_K\)-morphism between \(\overline{C}\) and \(\overline{D}\) is a family of maps \(\overline{f}_{\text{loc}} = \{f_{i,\text{loc}}\}_{i=0}^{K}\) which satisfies the following conditions:

\[ \deg(f_{i,\text{loc}}) = -2i \]

\[ f_{i,\text{loc}}^{(K)} \circ \overline{d}_{\text{loc}}^{(K)} = \overline{f}_{i,\text{loc}}^{(K)} \circ f_{i,\text{loc}}^{(K)} \mod(u^{K+1}) \]

If we assume that \(\deg(u) = 2\), \(f_{i,\text{loc}}^{(K)} = \sum_i f_{i,\text{loc}} \otimes u^i\) is a degree 0 map.
Next we introduce “\(\epsilon\)-gapped condition” to \(X_K\)-modules and \(X_K\)-morphisms. We fix \(\epsilon > 0\).

**Definition 6.5 ((\(\epsilon\)-gapped condition))** (1) Let \(\overline{C}\) be a \(\mathbb{Z}\)-graded finite dimensional \(\mathbb{F}_p\)-vector space and let \(C\) be a \(\Lambda_0\) module defined by \(C = \overline{C} \otimes_{\mathbb{F}_p} \Lambda_0\).

Assume that \(\overline{d}_{i,\text{loc}} = \{d_{i,\text{loc}} : \overline{C} \rightarrow \overline{C}\}_{i \leq K}\) determines a local \(X_K\)-module structure on \(\overline{C}\). Let \(d_{i,\text{loc}} \in \text{Hom}(C, C)\) be the natural extension of \(d_{i,\text{loc}} \in \text{Hom}(\overline{C}, \overline{C})\).

An \(X_K\)-module \((C, \mathcal{D})\) is called \(\epsilon\)-gapped if
\[
d_i = d_{i,\text{loc}} + \epsilon^d_{i,\text{loc}} \quad (d_{i,\text{loc}} \in \text{Hom}(C, C))
\]
holds.

(2) Let \((\overline{C}, \overline{d}_{\text{loc}})\) and \((\overline{D}, \overline{d}_{\text{loc}})\) be local \(X_K\)-modules. Assume that \((C, \mathcal{D})\) and \((D, \mathcal{I})\) are \(\epsilon\)-gapped \(X_K\)-modules. An \(X_K\)-morphisms \(f : (C, \mathcal{D}) \rightarrow (D, \mathcal{I})\) is called \(\epsilon\)-gapped if there is a local \(X_K\)-morphism \(\overline{f}_{\text{loc}} : (\overline{C}, \overline{d}_{\text{loc}}) \rightarrow (\overline{D}, \overline{d}_{\text{loc}})\) so that
\[
f_i = f_{i,\text{loc}} + \epsilon^f_{i,\text{loc}} \quad (f_{i,\text{loc}} \in \text{Hom}(C, D))
\]
holds. Note that \(f_{i,\text{loc}} \in \text{Hom}(C, D)\) is the natural extension of \(\overline{f}_{i,\text{loc}} \in \text{Hom}(\overline{C}, \overline{D})\).

(3) Let \(\mathcal{F}\) and \(\mathcal{G}\) be \(\epsilon\)-gapped \(X_K\)-morphisms between \((C, \mathcal{D})\) and \((D, \mathcal{I})\). We say that \(\mathcal{F}\) and \(\mathcal{G}\) are \(\epsilon\)-gapped \(X_K\)-homotopic if there is an \(\epsilon\)-gapped \(X_K\)-morphism
\[
h : (C, \mathcal{D}) \rightarrow (D \times [0, 1], \overline{I})
\]
so that
\[
\text{Eval}_{s=0} \circ h = \mathcal{F}
\]
\[
\text{Eval}_{s=1} \circ h = \mathcal{G}
\]
holds.

Note that \(f_{\text{loc}} : (C, \mathcal{D}_{\text{loc}}) \rightarrow (D, \mathcal{I}_{\text{loc}})\) is an \(X_K\)-morphism if \(\overline{f}_{\text{loc}}\) is an \(X_K\)-morphism.

**Proposition 6.5** Let \((C, \mathcal{D})\) and \((D, \mathcal{I})\) be \(\epsilon\)-gapped \(X_K\)-modules. Let \(\overline{f}_{\text{loc}} : (\overline{C}, \overline{d}_{\text{loc}}) \rightarrow (\overline{D}, \overline{d}_{\text{loc}})\) be a local \(X_K\)-morphism.

(1) There is an obstruction class
\[
o_{\epsilon}(f_{\text{loc}}) \in \text{Hom}((C^{(K)}, \mathcal{D}^{(K)}), (C^{(K)}, \mathcal{D}^{(K)}))
\]
so that we can extend \(f_{\text{loc}}\) to \(\epsilon\)-gapped \(X_K\)-morphism if and only if \([o_{\epsilon}(f_{\text{loc}})] = 0\) holds.
(2) Assume that $\tilde{f}_{\text{loc}}$ and $\tilde{f}'_{\text{loc}}$ are $X_K$-homotopic. Then,
\[ [\sigma_\epsilon(f_{\text{loc}})] = [\sigma_\epsilon(f'_{\text{loc}})] \]
holds.

(3) Assume that $g' : C' \to C$ and $g : D \to D'$ are $\epsilon$-gapped $X_K$-morphism. Then,
\[ [\sigma_\epsilon(g'_{\text{loc}} \circ f_{\text{loc}} \circ g_{\text{loc}})] = (g'_{\text{loc}}, g_{\text{loc}})_{\epsilon} \cdot [\sigma_\epsilon(f_{\text{loc}})] \]
holds.

**proof**: We define $\sigma_\epsilon(f_{\text{loc}})$ as follows:
\[ f(K) \circ f_{\text{loc}}'(K) - f_{\text{loc}}'(K) \circ d(K) \equiv T^\epsilon \sigma_\epsilon(f_{\text{loc}}) \mod(u^{K+1}) \]

Note that $\sigma_\epsilon(f_{\text{loc}}) \in \text{Hom}(C(K), D(K))$ holds because $f_{\text{loc}} : (C, d_{\text{loc}}) \to (D, i_{\text{loc}})$ is an $X_K$-morphisms.

\[ 0 = \{ \partial_{\text{Hom}}(C(K), d_{\text{loc}}), (D(K), i_{\text{loc}}) \} f_{\text{loc}}'(K) \]
\[ = \{ \partial_{\text{Hom}}(C(K), d_{\text{loc}}), (D(K), i_{\text{loc}}) \} (f(K) \circ f_{\text{loc}}'(K) - f_{\text{loc}}'(K) \circ d(K)) \]
\[ = \{ \partial_{\text{Hom}}(C(K), d_{\text{loc}}), (D(K), i_{\text{loc}}) \} (T^\epsilon \sigma_\epsilon(f_{\text{loc}})) \]
\[ = T^\epsilon \partial_{\text{Hom}}(C(K), d_{\text{loc}}), (D(K), i_{\text{loc}})) (\sigma_\epsilon(f_{\text{loc}})) \]

So, $\sigma_\epsilon(f_{\text{loc}})$ is a cocycle. Let $f_\epsilon = \{ f_{\text{loc}}' \}$ be a family of maps in $\text{Hom}(C, D)$. Then, $f = f_{\text{loc}} + T^\epsilon f_{\text{loc}}$ is an $\epsilon$-gapped $X_K$-morphism if and only if
\[ 0 \equiv f(K) \circ f_{\text{loc}}'(K) - f_{\text{loc}}'(K) \circ d(K) \]
\[ \equiv f(K) \circ f_{\text{loc}}'(K) + T^\epsilon f_{\text{loc}}'(K) - (f_{\text{loc}}'(K) + T^\epsilon f_{\text{loc}}'(K)) \circ d(K) \]
\[ \equiv T^\epsilon \{ \sigma_\epsilon(f_{\text{loc}}) + \partial_{\text{Hom}}(C(K), d_{\text{loc}}), (D(K), i_{\text{loc}})) (f_{\text{loc}}) \} \mod(u^{K+1}) \]
holds. So, we can extend $f_{\text{loc}}$ to an $\epsilon$-gapped $X_K$-morphism if and only if $[\sigma_\epsilon(f_{\text{loc}})] = 0$ holds. Next, we prove (iii). By definition,
\[ T^\epsilon \cdot \sigma_\epsilon(g_{\text{loc}} \circ f_{\text{loc}} \circ g_{\text{loc}}) \equiv (T^\epsilon \sigma_\epsilon(g_{\text{loc}})) (g_{\text{loc}}(K) \circ f'_{\text{loc}}(K) - f'_{\text{loc}}(K) \circ d'(K), g_{\text{loc}}(K)) \mod(u^{K+1}) \]
holds. The right hand side is a sum of the following three terms.

(1)
\[ (f(K) - f(K))_{\text{loc}} \cdot g_{\text{loc}}(K) \]

(2)
\[ g_{\text{loc}}(K) (f(K) - f(K))_{\text{loc}} \cdot d'(K) \]
Next we prove the following proposition.

**Proposition 6.6** Let \( f : (C, 0) \to (D, 1) \) be an \( \epsilon \)-gapped \( X_K \)-morphism. Assume that \( f_0 : (C, d_0) \to (d, l_0) \) is a cochain homotopy equivalence. Then we can construct an \( \epsilon \)-gapped \( X_K \)-morphism \( g : (D, 1) \to (C, 0) \) so that \( f \circ g \) and \( g \circ f \) are \( \epsilon \)-gapped \( X_K \)-homotopic to the identity.

**proof**: \( f_{0, \text{loc}} : (\overline{C}, \overline{d}_{0, \text{loc}}) \to (\overline{D}, \overline{l}_{0, \text{loc}}) \) is also a cochain homotopy equivalence. So Proposition 6.2 implies that we can construct an \( X_K \)-morphism

\[
g_{\text{loc}} : (\overline{D}, \overline{l}_{\text{loc}}) \to (\overline{C}, \overline{d}_{\text{loc}})
\]

and an \( X_K \)-homotopy

\[
\overline{h}_{\text{loc}} : (\overline{C}, \overline{d}_{\text{loc}}) \to (\overline{C} \times [0, 1], \overline{d}_{\text{loc}})
\]

between \( \text{Id} \) and \( g_{\text{loc}} \circ f_{\text{loc}} \). Our purpose is to extend \( g_{\text{loc}} \) and \( h_{\text{loc}} \) to \( \epsilon \)-gapped \( X_K \)-morphisms so that \( h \) becomes an \( \epsilon \)-gapped \( X_K \)-homotopy between \( \text{Id} \) and \( g \circ f \).

\[0 = [\sigma_{\epsilon}(\text{Id})] = (\text{Eval}_{s=0}, \text{Id})_* [\sigma_{\epsilon}(h_{\text{loc}})]\]

implies that \([\sigma_{\epsilon}(h_{\text{loc}})] = 0\) holds. Moreover, note that the image of

\[
\sigma_{\epsilon}(h_{\text{loc}}) : C^{(K)} \to (C \times [0, 1])^{(K)} = C^{(K)} \oplus C[1]^{(K)} \oplus C^{(K)}
\]

is included in \( \text{Ker}(\text{Eval}_{s=0}) = \{(0, y, z) \in (C \times [0, 1])^{(K)}\} \). Because \( \text{Ker}(\text{Eval}_{s=0}) \) is acyclic, we can choose \( h'_{t, \epsilon} = \{h'_{t, \epsilon}\}_{0 \leq t \leq K} \) so that

\[
h'_{t, \epsilon} : C \to C \times [0, 1]
\]

\[
\text{Eval}_{s=0}(h'_{t, \epsilon}) = 0
\]

\[
\partial_{\text{Hom}(C^{(K)}, (C \times [0, 1])^{(K)}), (h'_{t, \epsilon}(K))} = -\sigma_{\epsilon}(h_{\text{loc}})
\]

47
holds. Then, $\mathfrak{h}_{\text{loc}} + T^*\mathfrak{h}'_e$ is an $\epsilon$-gapped $X_K$-morphism which satisfies $\text{Eval}_{s=0} \circ (\mathfrak{h}_{\text{loc}} + T^*\mathfrak{h}'_e) = \text{Id}$. On the other hands, 

$$0 = \{a, (\text{Id}), f_{\text{loc}}^{(K)}\} \ast \{a, (\mathfrak{g}_{\text{loc}})\}$$

implies that $a, (\mathfrak{g}_{\text{loc}}) = 0$ holds. So we can choose $g' = \{g'_{i,\epsilon}\}_{0 \leq i \leq K}$ which satisfies

$$g'_{i,\epsilon} : D \rightarrow C$$

$$\partial_{\text{Hom}(D^{(K)}, C^{(K)})}(g'^{(K)}_{\epsilon}) = -a, (\mathfrak{g}_{\text{loc}}).$$

Then $\mathfrak{g}_{\text{loc}} + T^*g'_e$ is an $\epsilon$-gapped $X_K$-morphism. So $(\mathfrak{g}_{\text{loc}} + T^*g'_e) \circ f$ and $\text{Eval}_{s=1} \circ (\mathfrak{h}_{\text{loc}} + T^*\mathfrak{h}'_e)$ are $\epsilon$-gapped $X_K$-morphisms which extends $g_{\text{loc}} \circ f_{\text{loc}}$. Let $\Theta$ be the difference of these two $X_K$-morphisms.

$$\Theta : C^{(K)} \rightarrow C^{(K)}$$

$$T^*\Theta \overset{\text{def}}{=} \{(\mathfrak{g}_{\text{loc}} + T^*g'_e) \circ f^{(K)} - \text{Eval}_{s=1}^{(K)} \circ (\mathfrak{h}_{\text{loc}} + T^*\mathfrak{h}'_e)^{(K)}\}$$

$$T^* \cdot \{(\mathfrak{g}_{\text{loc}} \circ f_{\text{loc}}^{(K)} + g'^{(K)}_{\epsilon} \circ f_{\text{loc}}^{(K)}) - \text{Eval}_{s=1} \circ h'^{(K)}_{\text{loc}}\}$$

$\Theta$ is a cocycle because

$$\partial_{\text{Hom}(C^{(K)}), C^{(K)}}(\Theta) = \partial \{(\mathfrak{g}_{\text{loc}} \circ f_{\text{loc}}^{(K)} + g'^{(K)}_{\epsilon} \circ f_{\text{loc}}^{(K)}) - \partial (\text{Eval}_{s=1} \circ h'^{(K)}_{\text{loc}})\}$$

$$= a, (\mathfrak{g}_{\text{loc}} \circ f_{\text{loc}}) - a, (\mathfrak{g}_{\text{loc}} \circ f_{\text{loc}}) = 0$$

holds. Proposition 6.2 implies that

$$f^{(K)} : (C^{(K)}, \partial^{(K)}) \rightarrow (D^{(K)}, t^{(K)})$$

is a cochain homotopy equivalence. This implies that

$$(\text{Id}, f^{(K)}) \ast \text{Hom}\{(D^{(K)}, t^{(K)}), (C^{(K)}, \partial^{(K)})\} \rightarrow \text{Hom}\{(C^{(K)}, \partial^{(K)}), (C^{(K)}, \partial^{(K)})\}$$

$$\phi \mapsto \phi \circ f^{(K)}$$

is also a cochain homotopy equivalence. So we can choose $\Delta g'_e = \{\Delta g'_{i,\epsilon}\}_{0 \leq i \leq K}$ so that

$$g'_{i,\epsilon} : D \rightarrow C$$

$$\partial(\Delta g'^{(K)}_{\epsilon}) = 0$$

$$[\Delta g'^{(K)}_{\epsilon} \circ f^{(K)}] = -[\Theta]$$

holds. $g = \mathfrak{g}_{\text{loc}} + T^*(g'_e + \Delta g'_e)$ is an $\epsilon$-gapped $X_K$-morphism which extends $g_{\text{loc}}$. We choose $\Delta h = \{\Delta h_{i,\epsilon}\}_{0 \leq i \leq K}$ so that

$$\Delta h_{i,\epsilon} : C \rightarrow C$$

$$\partial_{\text{Hom}(C^{(K)}, \partial^{(K)}, (C^{(K)}, \partial^{(K)}))}(\Delta h_{i}^{(K)}) = \Theta + \Delta g'^{(K)}_{\epsilon} \circ f^{(K)}$$
holds. Next we choose \( \Delta h_e = \{ \Delta h_{i,e} \}_{0 \leq i \leq K} \) so that
\[
\Delta h_{i,e} : C \to C \times [0, 1]
\]
\[
\text{Eval}_{s=0}(\Delta h_{i,e}) = 0
\]
\[
\text{Eval}_{s=1}(\Delta h_{i,e}) = \Delta_1 h_{i,e}
\]
holds. We define \( h = \{ h_i \}_{0 \leq i \leq K} \) as follows:
\[
h_i : C \to C \times [0, 1]
\]
\[
h^{(K)} = h^{(K)}_{loc} + T^e(\partial h^{(K)} + \partial(\Delta h^{(K)}))
\]
h is an \( \varepsilon \)-gapped \( X_K \)-morphism because
\[
\partial(h^{(K)} + \partial(\Delta h^{(K)})) = \partial(h^{(K)}) = -\alpha(h_{loc})
\]
holds. Moreover, \( g \circ f = \text{Eval}_{s=1} \circ h \) is satisfied because
\[
g^{(K)} \circ f^{(K)} - \text{Eval}_{s=1} \circ h^{(K)} = T^e\{\Theta + \Delta g^{(K)} \circ f^{(K)} - \partial(\Delta_1 h^{(K)})\} \equiv 0 \mod(\nu^{K+1})
\]
holds. So \( h \) is an \( \varepsilon \)-gapped \( X_K \)-homotopy between \( \text{Id} \) and \( g \circ f \). By applying these arguments to \( g \), we can construct an \( \varepsilon \)-gapped \( X_K \)-morphism \( f' : C \to D \) so that \( f' \circ g \) is homotopic to the identity. Then \( f \) is homotopic to \( f' \) because \( f \) is homotopic to \( f' \circ g \circ f \). So we proved the proposition.

Next, we explain how to extend \( \varepsilon \)-gapped \( X_K \)-modules to \( \varepsilon \)-gapped \( X_{K+1} \)-modules.

**Proposition 6.7** (1) Let \((C, \mathfrak{d})\) be an \( \varepsilon \)-gapped \( X_K \)-module. Assume that there is a map \( \mathfrak{d}_{K+1, loc} : C \to C \) so that \( \{ \mathfrak{d}_{0, loc}, \ldots, \mathfrak{d}_{K+1, loc} \} \) determines a local \( X_K \)-module structure on \( C \). There is an obstruction class
\[
p_{K+1, e}(\mathfrak{d}, d_{K+1, loc}) \in \text{Hom}(C, C)
\]
so that there is an \( \varepsilon \)-gapped \( X_K \)-module structure \( \{ d_0, \ldots, d_{K+1, loc} + T^e d_{K+1, e} \} \) on \( C \) if and only if \( [p_{K+1, e}(\mathfrak{d}, d_{K+1, loc})] = 0 \) holds.

(2) Let \( f : (C, \mathfrak{d}) \to (D, l) \) be an \( \varepsilon \)-gapped \( X_K \)-morphism which is also an \( X_K \)-homotopy equivalence. Assume that there are maps
\[
\bar{f}_{K+1, loc} : C \to D
\]
\[
\bar{d}_{K+1, loc} : C \to C
\]
\[
\bar{I}_{K+1, loc} : D \to D
\]
so that \( \{ \bar{d}_{0, loc}, \ldots, \bar{d}_{K+1, loc} \} \) and \( \{ I_{0, loc}, \ldots, I_{K+1, loc} \} \) are \( X_K \)-module structures on \( C \) and \( D \), and \( \{ \bar{f}_{0, loc}, \ldots, \bar{f}_{K+1, loc} \} \) is an \( X_{K+1} \)-homotopy equivalence between them. Then \( [p_{K+1, e}(\mathfrak{d}, d_{K+1, loc})] = 0 \) holds if and only if \( [p_{K+1, e}(l, I_{K+1, loc})] = 0 \) holds.
**proof**: We define $\tilde{d} : C^{(K+1)} \to C^{(K+1)}$ as follows:

$$\tilde{d} : C^{(K+1)} \longrightarrow C^{(K+1)}$$

$$\tilde{d}(x \otimes 1) = d_0(x) \otimes 1 + \cdots + d_K(x) \otimes u^K + d_{K+1,loc}(x) \otimes u^{K+1}$$

Then

$$\tilde{d} \circ \tilde{d} = T^e \cdot p_{K+1,\varepsilon}(\mathfrak{d}, d_{K+1,loc}) \otimes u^{K+1} \mod(u^{K+2})$$

holds because $\mathfrak{d}$ is an $X_K$-module structure and $\overline{\mathfrak{d}}_{loc}$ is an $X_{K+1}$-module structure. $p_{K+1,\varepsilon}(\mathfrak{d}, d_{K+1,loc})$ can be regarded as an element of $\text{Hom}(C, C)$ because it vanishes on the subset $u \cdot C^{(K+1)} \subset C^{(K+1)}$ and its image is included in $C \subset C^{(K+1)}$.

$$T^e \cdot \partial_{\text{Hom}(C, C)} \big( p_{K+1,\varepsilon}(\mathfrak{d}, d_{K+1,loc}) \big) \otimes u^{K+1} \equiv d_0 \circ \tilde{d} \circ \tilde{d} - \tilde{d} \circ \tilde{d} \circ d_0$$

$$\equiv d \circ \tilde{d} \circ \tilde{d} - \tilde{d} \circ \tilde{d} \circ \tilde{d} \equiv 0 \mod(u^{K+2})$$

So $p_{K+1,\varepsilon}(\mathfrak{d}, d_{K+1,loc})$ is a cocycle. $\{d_0, \cdots, d_{K+1,loc} + T^e d_{K+1,\varepsilon}\}$ is an $\varepsilon$-gapped $X_{K+1}$-module structure on $C$ if and only if

$$(\tilde{d} + T^e d_{K+1,\varepsilon} \otimes u^{K+1})^2 \equiv T^e \big( p_{K+1,\varepsilon}(\mathfrak{d}, d_{K+1,loc}) + \partial(d_{K+1,\varepsilon}) \big) \otimes u^{K+1} \equiv 0 \mod(u^{K+2})$$

holds. In other words, $\{d_0, \cdots, d_{K+1,loc} + T^e d_{K+1,\varepsilon}\}$ is an $\varepsilon$-gapped $X_{K+1}$-module structure on $C$ if and only if $[p_{K+1,\varepsilon}(\mathfrak{d}, d_{K+1,loc})] = 0$ holds. So we proved (i). Next we prove (ii). By assumption and Proposition 6.2 and Proposition 6.6, we can choose

$$\overline{g} = \{g_{i,loc}\}_{i \leq K+1} : (\overline{\mathfrak{g}}, \overline{\mathfrak{d}}) \longrightarrow (\overline{\mathfrak{d}}, \mathfrak{d})$$

$$g : (D, \mathfrak{d}) \longrightarrow (C, \mathfrak{d})$$

so that $\overline{g}$ is an $X_{K+1}$-morphism and $g$ is an $\varepsilon$-gapped $X_K$-morphism which extends $\overline{g}$. Moreover, $\overline{g}$ is a homotopy inverse of $\overline{f}$ and $g$ is a homotopy inverse of $f$. We define $\tilde{f}$, $\overline{g}$, $\tilde{d}$ and $\overline{l}$ as follows:

$$\tilde{f} = f_0 \otimes 1 + \cdots + f_K \otimes u^K + f_{K+1,loc} \otimes u^{K+1}$$

$$\overline{g} = g_0 \otimes 1 + \cdots + g_K \otimes u^K + g_{K+1,loc} \otimes u^{K+1}$$

$$\tilde{d} = d_0 \otimes 1 + \cdots + d_K \otimes u^K + d_{K+1,loc} \otimes u^{K+1}$$

$$\overline{l} = l_0 \otimes 1 + \cdots + l_K \otimes u^K + l_{K+1,loc} \otimes u^{K+1}$$

Then

$$\tilde{f} \circ \tilde{d} - \overline{l} \circ \overline{f} \equiv T^e \cdot q \otimes u^{K+1} \mod(u^{K+2})$$

holds for some $q \in \text{Hom}(C, D)$ because $\tilde{f}$ is an $\varepsilon$-gapped $X_K$-morphism and $\overline{f}$ is an $X_{K+1}$-morphism. By the above definition of $p_{K+1,\varepsilon}(\cdot)$ we have the following
Assume that we have maps $X$. Let $\phi: (D, l) \rightarrow (X, g)$ be an $\epsilon$-gapped $X_K$-module and let $(D, l)$ be an $\epsilon$-gapped $X_{K+1}$ module. Let $f: C \rightarrow D$ be an $\epsilon$-gapped $X_K$-homotopy equivalence. Assume that we have maps

\[
\tilde{d}_{K+1, loc}: \overline{C} \rightarrow \overline{C} \\
\tilde{l}_{K+1, loc}: \overline{C} \rightarrow \overline{D}
\]

so that $\{\tilde{d}_0, \tilde{d}_1, \ldots, \tilde{d}_{K+1, loc}\}$ is an $X_K$-module structure on $\overline{C}$ and $\{\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_{K+1, loc}\}$ is an $X_{K+1}$-homotopy equivalence between $\overline{C}$ and $\overline{D}$. Then we can extend $(\tilde{d}, d_{K+1, loc})$ to an $\epsilon$-gapped $X_{K+1}$-module structure on $C$ and we can extend $(\tilde{f}, f_{K+1, loc})$ to an $\epsilon$-gapped $X_{K+1}$-homotopy equivalence between $C$ and $D$.

**proof:** By Proposition 6.7, $[\phi_{K+1, \epsilon}(\tilde{d}, d_{K+1, loc})] = 0$ holds and we can choose $d'_{K+1, \epsilon}: C \rightarrow C$ so that $\{d'_{0}, \ldots, d'_{K+1, loc} + T^*d'_{K+1, \epsilon}\}$ becomes an $\epsilon$-gapped $X_{K+1}$-module structure on $C$. Our next purpose is to extend $(\tilde{f}, f_{K+1, loc})$ to $\epsilon$-gapped $X_{K+1}$-morphism. We define $\tilde{f}$, $\tilde{d}$ and $\tilde{l}$ as follows:

\[
\tilde{f} = f_0 \otimes 1 + \cdots + f_K \otimes u^K + f_{K+1, loc} \otimes u^{K+1} \\
\tilde{d} = d_0 \otimes 1 + \cdots + d_K \otimes u^K + (d_{K+1, loc} + T^*d'_{K+1, \epsilon}) \otimes u^{K+1} \\
\tilde{l} = l_0 \otimes 1 + \cdots + l_{K+1} \otimes u^{K+1}
\]

We define $\phi_{K+1, \epsilon}(\tilde{f}, f_{K+1, loc}): C \rightarrow D$ as follows:

\[
\tilde{l} \circ \tilde{f} \circ \tilde{d} \equiv T^* \phi_{K+1, \epsilon}(\tilde{f}, f_{K+1, loc}) \otimes u^{K+1} \mod(u^{K+2})
\]

So,

\[
f_0 \circ \phi_{K+1, \epsilon}(\tilde{d}, d_{K+1, loc}) \circ g_0 = \partial(g_0) + \phi_{K+1, \epsilon}(l, l_{K+1, loc})f_0g_0
\]

holds. This implies that

\[
(f_0, g_0)\cdot [\phi_{K+1, \epsilon}(\tilde{d}, d_{K+1, loc})] = [\phi_{K+1, \epsilon}(l, l_{K+1, loc})f_0g_0] = [\phi_{K+1, \epsilon}(l, l_{K+1, loc})]
\]

holds and $[\phi_{K+1, \epsilon}(\tilde{d}, d_{K+1, loc})] = 0$ holds if and only if $[\phi_{K+1, \epsilon}(l, l_{K+1, loc})] = 0$ holds.

□

We apply this to the following proposition.

**Proposition 6.8** Let $(C, \tilde{d})$ be an $\epsilon$-gapped $X_K$-module and let $(D, l)$ be an $\epsilon$-gapped $X_{K+1}$ module. Let $f: C \rightarrow D$ be an $\epsilon$-gapped $X_K$-homotopy equivalence. Assume that we have maps

\[
\tilde{d}_{K+1, loc}: \overline{C} \rightarrow \overline{C} \\
\tilde{l}_{K+1, loc}: \overline{C} \rightarrow \overline{D}
\]

so that $\{\tilde{d}_0, \tilde{d}_1, \ldots, \tilde{d}_{K+1, loc}\}$ is an $X_K$-module structure on $\overline{C}$ and $\{\tilde{l}_0, \tilde{l}_1, \ldots, \tilde{l}_{K+1, loc}\}$ is an $X_{K+1}$-homotopy equivalence between $\overline{C}$ and $\overline{D}$. Then we can extend $(\tilde{d}, d_{K+1, loc})$ to an $\epsilon$-gapped $X_{K+1}$-module structure on $C$ and we can extend $(\tilde{f}, f_{K+1, loc})$ to an $\epsilon$-gapped $X_{K+1}$-homotopy equivalence between $C$ and $D$. We define $\tilde{f}$, $\tilde{d}$ and $\tilde{l}$ as follows:

\[
\tilde{f} = f_0 \otimes 1 + \cdots + f_K \otimes u^K + f_{K+1, loc} \otimes u^{K+1} \\
\tilde{d} = d_0 \otimes 1 + \cdots + d_K \otimes u^K + (d_{K+1, loc} + T^*d'_{K+1, \epsilon}) \otimes u^{K+1} \\
\tilde{l} = l_0 \otimes 1 + \cdots + l_{K+1} \otimes u^{K+1}
\]

We define $\phi_{K+1, \epsilon}(\tilde{f}, f_{K+1, loc}): C \rightarrow D$ as follows:

\[
\tilde{l} \circ \tilde{f} \circ \tilde{d} \equiv T^* \phi_{K+1, \epsilon}(\tilde{f}, f_{K+1, loc}) \otimes u^{K+1} \mod(u^{K+2})
\]
Note that
\[ T^r(\partial(\sigma_{K+1,\epsilon}(\tilde{f}, f_{K+1,\text{loc}}))) \equiv \partial_{\text{Hom}((C'(K+1), \mathcal{A}),(D(K+1), \mathcal{B}))}(f \circ \tilde{f} - \tilde{f} \circ \tilde{d}) \]
\[ \equiv (\partial_{\text{Hom}((C'(K+1), \mathcal{A}),(D(K+1), \mathcal{B}))})^2(\tilde{f}) \equiv 0 \pmod{(K+2)} \]
implies \( \sigma_{K+1,\epsilon}(\tilde{f}, f_{K+1,\text{loc}}) \) is a cocycle. \( \{f_0, \ldots, f_K, f_{K+1,\text{loc}} + T^r f_{K+1,\epsilon}\} \) is an \( \epsilon \)-gapped \( X_{K+1} \)-morphism if and only if
\[ 0 \equiv \tilde{f} + (T^r f_{K+1,\epsilon}) - (\tilde{f} + T^r f_{K+1,\epsilon}) \circ \tilde{d} \]
\[ \equiv T^r(\sigma_{K+1,\epsilon}(\tilde{f}, f_{K+1,\text{loc}}) + \partial(f_{K+1,\epsilon})) \pmod{(K+2)} \]
holds. However, \( \sigma_{K+1,\epsilon}(\tilde{f}, f_{K+1,\text{loc}}) = 0 \) does not hold in general. So we add a cocycle \( \Delta d_{K+1,\epsilon} \in \text{Hom}(C, C) \) to \( d_{K+1}' \) so that \( \sigma_{K+1,\epsilon}^\text{new}(\tilde{f}, f_{K+1,\text{loc}}) = 0 \) holds.

Note that
\[ \sigma_{K+1,\epsilon}^\text{new}(\tilde{f}, f_{K+1,\text{loc}}) = \sigma_{K+1,\epsilon}(\tilde{f}, f_{K+1,\text{loc}}) - f_0 \circ \Delta d_{K+1,\epsilon} \]
holds.

\[ (f_0, \text{Id}) : (\text{Hom}(C, C), \partial(\text{Hom}(C, C))) \longrightarrow (\text{Hom}(C, D), \partial(\text{Hom}(C, D))) \]
is a cochain homotopy equivalence and we can choose a cocycle \( \Delta d_{K+1} \) so that \( \sigma_{K+1,\epsilon}^\text{new}(\tilde{f}, f_{K+1,\text{loc}}) = 0 \) holds. Then, \( \{f_0, \ldots, f_{K,\text{loc}} + T^r(d_{K+1}^\epsilon + \Delta d_{K+1,\epsilon})\} \) is an \( \epsilon \)-gapped \( X_K \)-module structure and we can extend \( (\tilde{f}, f_{K+1,\text{loc}}) \) to an \( \epsilon \)-gapped \( X_{K+1} \)-morphism. Proposition 6.6 implies that this is an \( \epsilon \)-gapped \( X_{K+1} \)-homotopy equivalence.

\[ \square \]

The following corollary is the \( \epsilon \)-gapped version of corollary 6.1.

**Corollary 6.2** Let \( (C, \mathcal{A}) \) and \( (D, \mathcal{B}) \) be \( \epsilon \)-gapped \( X_{K+1} \) modules. Assume that \( \tilde{f} : C \to D \) is an \( \epsilon \)-gapped \( X_{K+1} \)-morphism and \( \tilde{g} : C \to D \) is an \( \epsilon \)-gapped \( X_K \)-morphism so that \( \tilde{f} \) and \( \tilde{g} \) are \( \epsilon \)-gapped \( X_K \)-homotopic. Then we can extend \( \tilde{g} \) to an \( \epsilon \)-gapped \( X_{K+1} \)-morphism so that it is \( \epsilon \)-gapped \( X_{K+1} \) homotopic to \( \tilde{f} \).

**proof**: Let \( \tilde{h} : C \to D \times [0, 1] \) be an \( \epsilon \)-gapped \( X_K \)-homotopy between \( \tilde{f} \) and \( \tilde{g} \). Corollary 6.1 implies that we can choose \( \mathcal{T}_{K+1,\text{loc}} : C \to D \times [0, 1] \) and \( \mathfrak{T}_{K+1,\text{loc}} : C \to \mathcal{D} \) so that \( \{T_{i,\text{loc}}\}_{i=0}^{K+1} \) is a local \( X_{K+1} \)-morphism and \( \{T_{0,\text{loc}}\}_{i=0}^{K+1} \) is an \( X_{K+1} \)-homotopy between \( \{T_{1,\text{loc}}\}_{i=0}^{K+1} \) and \( \{T_{1,\text{loc}}\}_{i=0}^{K+1} \). We consider the obstruction \( \sigma_{K+1,\epsilon}^\text{new}(\tilde{g}, (G_{K+1,\text{loc}}) \text{ defined in the proof of Proposition 6.8.} \)

\[ h'_{K+1,\epsilon} = (\text{Incl})_0 \circ f_{K+1,\epsilon} \]
satisfies the following equations:

\[ ((\text{Eval})_{=0})_0(\text{Id})_*(\sigma_{K+1,\epsilon}(\tilde{h}, h_{K+1,\text{loc}}) + \partial_{\text{Hom}(C, D \times [0, 1])}(h'_{K+1,\epsilon})) \]
\[ = \sigma_{K+1,\epsilon}(\tilde{f}, f_{K+1,\text{loc}}) + \partial_{\text{Hom}(C, D)}(f_{K+1,\epsilon}) = 0 \]
\[ ((\text{Eval})_{=0})_0 h'_{K+1,\epsilon} = f_{K+1,\epsilon} \]
\[ 52 \]
The first equation implies that \( \varnothing_{K+1,\epsilon}(h, h_{K+1,loc}) + \partial(h'_{K+1,\epsilon}) \) is a cycle in \( \text{Hom}(C, N) \) where \( N \) is the kernel of \( (\text{Eval}_{s=0})_0 : D \times [0, 1] \to D \). Recall that \( (\text{Hom}(C, N), \partial_{\text{Hom}(C, N)}) \) is acyclic. So we can find \( \Delta h_{K+1,\epsilon} : C \to D \times [0, 1] \) so that

\[
(Eval_{s=0})_0 \circ \Delta h_{K+1,\epsilon} = 0
\]

\[
\varnothing_{K+1,\epsilon}(h, h_{K+1,loc}) + \partial_{\text{Hom}(C, D \times [0, 1])}(h'_{K+1,\epsilon}) = -\partial_{\text{Hom}(C, D \times [0, 1])}((\Delta h_{K+1,\epsilon})
\]

holds. So \( h_{K+1,\epsilon} = h'_{K+1,\epsilon} + \Delta h_{K+1,\epsilon} \) satisfies the following equations:

\[
\varnothing_{K+1,\epsilon}(h, h_{K+1,loc}) + \partial_{\text{Hom}(C, D \times [0, 1])}(h_{K+1,\epsilon}) = 0
\]

\[
(Eval_{s=0})_0 \circ h_{K+1,\epsilon} = f_{K+1,\epsilon}
\]

The first equality implies that \( h = \{h_0, \ldots, h_{K+1,loc} + T' h_{K+1,\epsilon}\} \) is an \( \epsilon \)-gapped \( X_{K+1} \)-morphism. \( Eval_{s=1} \circ h \) is an \( \epsilon \)-gapped \( X_{K+1} \)-morphism which extends \( g \). The second equality implies that \( h \) is an \( \epsilon \)-gapped \( X_{K+1} \)-homotopy between \( f \) and \( g \).

\[\square\]

### 6.3 Geometric constructions

First, we consider a family of \( X_K \)-modules and \( X_K \)-morphisms.

**Definition 6.6** We call \( \mathcal{C} = \{C_{K,i}, \iota_{K,i\to j}, \tau_{K \to K+1,i}\} \) is a directed family of \( X_K \)-modules if it satisfies the following conditions.

1. \( C_{K,i} \) is an \( X_K \)-module (\( K \in \mathbb{Z}_{\geq 0}, i \in \mathbb{N} \)).
2. \( \iota_{K,i\to j} : C_{K,i} \to C_{K,j} \) is an \( X_K \)-homotopy equivalence. Moreover, \( \iota_{K,j\to l} \circ \iota_{K,i\to j} \) is \( X_K \)-homotopic to \( \iota_{K,i\to l} \) for any \( i, j, l \in \mathbb{N} \).
3. For any \( K \in \mathbb{Z}_{\geq 0} \), there are constant \( N(K, K+1) \in \mathbb{N} \) and a family of \( X_K \)-morphisms as follows:
   \[
   \tau_{K \to K+1,i} : C_{K,i} \to C_{K+1,i} \quad (\forall i \geq N(K, K+1))
   \]
4. \( \tau_{K \to K+1,i} \circ \iota_{K,i\to j} \) and \( \iota_{K+1,i\to j} \circ \tau_{K \to K+1,i} \) are \( X_K \)-homotopic for any \( i, j \geq N(K, K+1) \).

**Definition 6.7** Let \( \mathcal{C} = \{C_{K,i}, \iota_{K,i\to j}, \tau_{K \to K+1,i}\} \) and \( \mathcal{D} = \{D_{K,i}, \iota'_{K,i\to j}, \tau'_{K \to K+1,i}\} \) be directed families of \( X_K \)-modules. \( \mathcal{F} = \{f_{K,i}\} \) is a morphism between \( \mathcal{C} \) and \( \mathcal{D} \) if it satisfies the following conditions.

1. There are constant \( M(K) \in \mathbb{N} \) and \( X_K \)-morphisms \( f_{K,i} \) for every \( K \) as follows:
   \[
   f_{K,i} : C_{K,i} \to D_{K,i} \quad (\forall i \geq M(K))
   \]
\( (2) \) \( f_{K,j} \circ i_{K,i \to j} \) is \( X_K \)-homotopic to \( i'_{K,i \to j} \circ f_{K,i} \) for any \( i, j \geq M(K) \).

\( (3) \) \( f_{K+1,i} \circ \tau_{K \to K+1,i} \) is \( X_K \)-homotopic to \( \tau'_{K \to K+1,i} \circ f_{K,i} \) for any \( i \geq \max\{M(K), M(K + 1)\} \).

**Definition 6.8** We call two directed families of \( X_K \)-modules \( C \) and \( D \) are equivalent if there is a morphism \( F : C \to D \) so that \( f_{K,i} : C_{k,i} \to D_{k,i} \) are \( X_K \)-equivalent.

**Definition 6.9** Let \( C \) be an \( X_{\infty} \)-module. Then it defines a natural directed family of \( X_K \)-modules by \( C_{k,i} = C \). We call this directed family of \( X_K \)-modules constant directed family of \( X_K \)-modules.

**Proposition 6.9** (1) Let \( C = \{C_{K,i}, i_{K,i \to j}, \tau_{K \to K+1,i}\} \) be a directed family of \( X_K \)-modules. Then, there is a constant directed family of \( X_K \)-modules \( C \) and a morphism \( I : C \to C \) which gives an equivalence between them. In other words, we can construct an \( X_{\infty} \)-module from \( C \). Moreover, this \( X_{\infty} \)-module \( C \) is unique up to \( X_{\infty} \)-homotopy equivalence.

(2) Let \( C \) and \( D \) be directed families of \( X_K \)-modules and let \( F : C \to D \) be a morphism. We assume that \( C \) and \( D \) be \( X_{\infty} \)-modules constructed from \( C \) and \( D \) and \( I : C \to C \) and \( F : D \to D \) be morphisms constructed in (i) \( (F \) is an inverse of \( D \to D) \). Then we can construct an \( X_{\infty} \)-morphism \( g : C \to D \) so that \( g \) is \( X_K \)-homotopy equivalent to the composition of \( C \to C_{K,i} \) and \( f_{K,i} : C_{k,i} \to D_{k,i} \) and \( D_{k,i} \to D \) for any \( k \in \mathbb{Z}_{\geq 0} \). In particular, \( g : C \to D \) is unique up to \( X_K \)-homotopy equivalence for any \( K < \infty \).

**Proof**: (i) We fix \((K, i)\) and assume that \( C = C_{K,i} \). So \( C \) is an \( X_K \)-module. We extend this to an \( X_{\infty} \)-module. Proposition 6.4 implies that we can extend \( C \) to an \( X_{K+1} \)-modules so that \( C \) is \( X_{K+1} \)-homotopy equivalent to every \( C_{K+1,i} \). Inductively, we can extend \( C \) to an \( X_{\infty} \)-module and we can construct an equivalence between constant directed family of \( X_K \)-modules \( C \) and \( C \). Next we prove the uniqueness. Let \( C' \) be another \( X_{\infty} \)-module which is equivalent to \( C \). Then we have a family of \( X_K \)-equivalence \( f_{K} : C \to C' \) so that \( f_{K} \) and \( f_{K+1} \) are \( X_K \)-homotopic. Corollary 6.1 implies that we can construct \( X_{\infty} \)-morphism \( \bar{f} : C \to C' \) so that \( \bar{f} \) is \( X_K \)-homotopic to \( f_{K} \) for any \( k \in \mathbb{Z}_{\geq 0} \). Proposition 6.2 implies that this \( \bar{f} \) is an \( X_{\infty} \)-homotopy equivalence.

(ii) We have a morphism \( \bar{g} : C \to D \) between constant directed families of \( X_K \)-modules. By choosing \( i(K) \geq M(K) \) for every \( K \), we have a family of \( X_K \)-morphisms \( g_{K} = g_{K,i(K)} \) so that \( g_{K} \) is \( X_K \)-homotopic to the composition of \( C \to C_{K,i} \) and \( f_{K,i} : C_{K,i} \to D_{K,i} \) and \( D_{K,i} \to D \). Corollary 6.1 implies that we can construct an \( X_{\infty} \)-morphism \( \bar{g} : C \to D \) so that \( \bar{g} \) is \( X_K \)-homotopic to the above composition. In particular, \( \bar{g} \) is unique up to \( X_K \)-homotopy for \( K < \infty \).

\[ \square \]

We can also define directed families of \( \varepsilon \)-gapped \( X_K \)-modules, morphisms between directed families of \( \varepsilon \)-gapped \( X_K \)-modules and equivalence by replacing “\( X_K \)” to “\( \varepsilon \)-gapped \( X_K \)” . We also have \( \varepsilon \)-gapped version of Proposition 6.9 as follows.
Proposition 6.10  (1) Let $\mathcal{C} = \{C_{K,i} : \tau_{K,i} \rightarrow K_{i} \rightarrow K_{i+1} \}$ be a directed family of $\epsilon$-gapped $X_K$-modules. Then, there is a constant directed family of $\epsilon$-gapped $X_K$-modules $C$ and a morphism $\mathcal{I} : C \rightarrow C$ which gives an equivalence between them. In other words, we can construct an $\epsilon$-gapped $X_\infty$-modules from $C$. Moreover, this $\epsilon$-gapped $X_\infty$-module $C$ is unique up to $\epsilon$-gapped $X_\infty$-homotopy equivalence.

(2) Let $\mathcal{C}$ and $\mathcal{D}$ be directed families of $\epsilon$-gapped $X_K$-modules and let $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ be a morphism. We assume that $\mathcal{C}$ and $\mathcal{D}$ be $\epsilon$-gapped $X_\infty$-modules constructed from $\mathcal{C}$ and $\mathcal{D}$ and $\mathcal{I} : C \rightarrow C$ and $\mathcal{J} : D \rightarrow D$ be morphisms constructed in (i). Then we can construct an $\epsilon$-gapped $X_\infty$-morphism $g : C \rightarrow D$ so that $g$ is $X_K$-homotopy equivalent to the composition of $C \rightarrow C_{K,i}$ and $f_{K,i} : C_{K,i} \rightarrow D_{K,i}$ and $D_{K,i} \rightarrow D$ for any $K \in \mathbb{Z}_{\geq 0}$. In particular, $g : C \rightarrow D$ is $\epsilon$-gapped $X_K$-homotopic. So we can choose a family of $\epsilon$-gapped $X_\infty$-modules $\mathcal{C}$ and $\mathcal{D}$. Let $C'$ be another $\epsilon$-gapped $X_\infty$-module which is equivalent to $C$. Then we have a family of $\epsilon$-gapped $X_K$-equivalence $f_K : C \rightarrow C'$ so that $f_K$ and $f_{K+1}$ are $\epsilon$-gapped $X_K$-homotopic. So we can construct an $\epsilon$-gapped $X_\infty$-homotopy equivalence between $C$ and $C'$.

Proof: (i) The proof is almost the same as in the proof of Proposition 6.9. We fix $(K, i)$ and assume that $C = C_{K,i}$. So $C$ is an $\epsilon$-gapped $X_K$-module. First, we apply Proposition 6.4 to extend $\mathcal{C}$ to a local $X_{K+1}$-module so that it is $X_{K+1}$-homotopy equivalent to every $\mathcal{C}_{K+1,j}$. Next, we apply Proposition 6.8 to extend $C$ to an $\epsilon$-gapped $X_{K+1}$-module so that it is $\epsilon$-gapped $X_{K+1}$-homotopy equivalent to every $C_{K+1,j}$. Inductively, we can extend $C$ to an $\epsilon$-gapped $X_\infty$-module and construct an $\epsilon$-gapped $X_\infty$-equivalence between constant directed family of $\epsilon$-gapped $X_\infty$-module $C$ and $\mathcal{C}$. Let $C'$ be another $\epsilon$-gapped $X_\infty$-module which is equivalent to $\mathcal{C}$. Then we have a family of $\epsilon$-gapped $X_K$-equivalence $f_K : C \rightarrow C'$ so that $f_K$ and $f_{K+1}$ are $\epsilon$-gapped $X_K$-homotopic. So we can construct an $\epsilon$-gapped $X_\infty$-homotopy equivalence between $C$ and $C'$.

(ii) We have a morphism $g : C \rightarrow D$ between constant directed families of $\epsilon$-gapped $X_K$-modules. So we can choose a family of $\epsilon$-gapped $X_K$-morphisms $\mathcal{g}_K$ so that $\mathcal{g}_K$ is $\epsilon$-gapped $X_K$-homotopic to $\mathcal{g}_{K+1}$. Our purpose is to construct an $\epsilon$-gapped $X_\infty$-morphism $\mathcal{g}$ from this family. Corollary 6.2 implies that we can construct an $\epsilon$-gapped $X_\infty$-morphism $g : C \rightarrow D$ so that $g$ is $\epsilon$-gapped $X_K$-homotopic to $\mathcal{g}_K$. Next we prove the uniqueness of the underlying local $X_\infty$-morphism $\mathcal{g}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{D}$ up to $X_\infty$-homotopy. Let $\mathcal{g}' : C \rightarrow D$ be another $\epsilon$-gapped $X_\infty$-morphism which is $\epsilon$-gapped $X_K$-hotopic to $g$ for any $K < \infty$. We fix a large positive integer $L \in \mathbb{N}$ so that

$$|\deg(x)| < \frac{1}{10} L$$

for any $x \in \overline{C} \cup \overline{D}$. Let $\overline{g}_{\mathcal{C}} = \{\overline{g}_{j,\mathcal{C}}\}_{j=0}^L$ be a family of maps $\overline{g}_{j,\mathcal{C}} : \overline{C} \rightarrow \overline{D}$ so that

$$\overline{g}_{(L)} - \overline{g}'_{(L)} = \overline{g}_{(L)} \circ \overline{u}_{\mathcal{C}} + \overline{u}_{\mathcal{C}} \circ \overline{g}_{(L)} \mod(u^{L+1})$$

holds. This is possible because $\overline{g}_{\mathcal{C}}$ and $\overline{g}'_{\mathcal{C}}$ are $X_L$-homotopic. Then the
condition (*) implies that
\[ g(\infty)_{\text{loc}} = g'(\infty)_{\text{loc}} = \eta_{\text{loc}} \circ h_{\text{loc}} + h_{\text{loc}} \circ \eta_{\text{loc}} \]
holds. In particular, \( \eta_{\text{loc}} \) is \( X_\infty \)-homotopic to \( \eta_{\text{loc}} \).

\( \Box \)

We apply this machinery to the following situation. Let \( \{ H_k \in C^\infty(S^1 \times M) \}_{k=1}^\infty \) be a family of Hamiltonian functions which satisfies the following conditions.

- \( H_k \rightarrow H \) in \( C^\infty \)-topology
- \( (H_k, J) \) is a Floer regular pair

Assume that \( P(H, \gamma) \) is a finite set \( \{ x_1, \ldots, x_l \} \). For fixed \( 1 \leq i \leq l \), \( x_i \) splits into \( \{ x_{i1}, \ldots, x_{il} \} \subset P(H_k, \gamma) \).

As in section 4, we slightly modify the Floer differential operator of \( (H_k, J) \) as follows.

Let \( v_i^j : [0, 1] \times S^1 \rightarrow M \) be a small cylinder connecting \( v_i^j(0, t) = x_i(t) \) and \( v_i^j(1, t) = x_j^i(t) \). Let \( c(x_i, x_j^i) \in \mathbb{R} \) be the action gap
\[ c(x_i, x_j^i) = \int_{[0, 1] \times S^1} (v_i^j)^* \omega + \int_0^1 H(t, x_i(t)) - H_k(t, x_j^i(t)) \, dt. \]

and let \( \tau \) be a correction map defined as follows.
\[ \tau : CH(H_k, \gamma : \Lambda) \rightarrow CH(H_k, \gamma : \Lambda) \]
\[ x_j^i \mapsto T^{c(x_i, x_j^i)} x_j^i \]

The modified differential operator \( \tilde{d}_F \) was defined by
\[ \tilde{d}_F = \tau^{-1} d_F \tau \]

By using the modified Floer differential \( \tilde{d}_F \), we can define a modified differential
\[ d_{Z_p} : CF(H_k, \gamma : \Lambda_0)^{\otimes p} \otimes \Lambda_0[[u]](\theta) \rightarrow CF(H_k, \gamma : \Lambda_0)^{\otimes p} \otimes \Lambda_0[[u]](\theta) \]
as follows:
\[ d_{Z_p}(x \otimes 1) = \tilde{d}_F(x) \otimes 1 + (1 - \tau) \otimes \theta \]
\[ d_{Z_p}(x \otimes \theta) = \tilde{d}_F(x) \otimes \theta + N(x) \otimes u \theta \]

Then \( d_{Z_p} \) determines an \( X_\infty \)-module structure on
\[ C_k = CF(H_k, \gamma : \Lambda_0)^{\otimes p} \otimes \Lambda_0(\theta). \]

We define a degree of \( C_k \) as follows. For any periodic orbit \( x \in P(H_k, \gamma) \), we can define the Conley-Zehnder index \( \mu_{CZ}(x) \in \mathbb{Z}_2 \) (we normalize \( \mu_{CZ} \) so that the Conley-Zehnder index of a local maximum of \( C^2 \)-small Morse function is
equal to $n$). We define the degree of $x$ by $\deg(x) = n - \mu_C(x)$. The degree of $\theta$ is equal to 1. Moreover, we fix a localization of $x^*TM$ so that $\mu_C(x) \in \mathbb{Z}$ and $\deg(x) \in \mathbb{Z}$ are well-defined. Then, $C_k$ becomes an $\epsilon$-gapped $X_\infty$-module.

Note that $(C_k, d_{xP})$ and $(C_{k'}, d_{xP})$ are chain homotopy equivalence (hence $X_\infty$-homotopy equivalence) for any $k$ and $k'$. In particular, $C_{K,i} = C_t$ becomes a directed system of $\epsilon$-gapped $X_{K'}$-modules.

Next we construct a directed family of $\epsilon$-gapped $X_K$-modules for $\mathbb{Z}_\epsilon$-equivariant Floer cohomology. Assume that $P(H^{(\epsilon)}, p\gamma)$ is a finite set. Let $\epsilon_1$ and $\epsilon_2$ be positive constants as follows:

$$\epsilon_1 = \min \left\{ \int_{\mathbb{C}P^1} u^* \omega \neq 0 \mid u : \mathbb{C}P^1 \to M \right\}$$

$$\epsilon_2 = \min \left\{ E(u) \neq 0 \mid u : \mathbb{R} \times S^1 \to M \quad \partial_s u + J(\partial_t u - X_{H^{(\epsilon)}}(u)) = 0 \right\}$$

We fix a positive constant $\epsilon < \min \{\epsilon_1, \epsilon_2\}$. Let $\{G_{(K,i)}\}$ be a family of Hamiltonian functions in $C^\infty(S^1 \times M)$ which satisfies the following conditions:

- $G_{(K,i)} \to H^{(\epsilon)}$ ($i \to \infty$) in $C^\infty$-topology
- $(G_{(K,i)}, J)$ is Floer regular

Let $G_{(K,i)}^{(w,t)}$ be a family of Hamiltonian functions parametrized by $(w, t) \in S^{2K+1} \otimes S^1 \subset S^\infty \otimes S^1$ which satisfies the following conditions:

- (locally constant at critical points) For all $w$ in a small neighborhood of $Z_\mu^m \in S^{2K+1}$,

$$G_{(K,i)}^{(w,t)}(x) = G_{(K,i)}(t - \frac{m}{\lambda}, x)$$

holds.

- (equivariance) $G_{(K,i)}^{(w,t)} = G_{(K,i)}^{(w-t/\lambda)}$

- (invariance under the shift $\tau$) $G_{(K,i)}^{(\tau(w), t)} = G_{(K,i)}^{(w,t)}$ holds.

As in the definition of the $\mathbb{Z}_\epsilon$-equivariant Floer differential operator for toroidally monotone symplectic manifolds, we consider the following equation for $x, y \in P(G_{(K,i)}, p\gamma)$, $m \in \mathbb{Z}_\mu$, $\alpha \geq 0$, $\alpha \in \{0, 1\}$ and $0 \leq l \leq 2K + 1$. Assume that $\bar{x}, \bar{y} \in P(H^{(\epsilon)}, p\gamma)$ are periodic orbits which split into $\{x, \cdots \} \subset P(G_{(K,i)}, p\gamma)$ and $\{y, \cdots \} \subset P(G_{(K,i)}, p\gamma)$.

$$\left( u, v \right) \in C^\infty(\mathbb{R} \times S^1, M) \times C^\infty(\mathbb{R}, S^{2K+1})$$

$$\partial_s u(s, t) + J(u(s, t))(\partial_t u(s, t) - X_{G_{(K,i)}^{(w,t)}}) = 0$$

$$\frac{dv(s)}{ds} = \text{grad}(\bar{F}) = 0$$

$$\lim_{s \to \infty} v(s) = Z_{\alpha}^0, \lim_{s \to -\infty} v(s) = Z_{\alpha}^m, \lim_{s \to \infty} u(s, t) = x(t), \lim_{s \to -\infty} u(s, t) = y(t - \frac{m}{\lambda})$$

$$\int_{\mathbb{R} \times S^1} \bar{u}^* \omega + \int_0^1 H^{(\epsilon)}(t, \bar{x}(t)) - H^{(\epsilon)}(t, \bar{y}(t)) dt = \lambda$$
Moreover, we assume that \( v \) is contained in a small neighborhood of \( \tilde{\partial} \) the boundary:

\[
\text{have the natural compactifications and we have the following decomposition of moduli spaces. As in the case of the Floer cohomology, these moduli spaces (Lemma 6.4)}
\]

\[
\text{Remark 6.1 If } G_{(K,i)} \text{ is sufficiently close to } H^{(\rho)}, \lambda \geq 0 \text{ holds. The proof is the same as for Lemma 4.1. Moreover we assume that } G_{(K,i)} \text{ is sufficiently close to } H^{(\rho)} \text{ so that } \lambda = 0 \text{ or } \lambda \geq \epsilon \text{ holds.}
\]

We denote the space of solutions modulo the natural \( \mathbb{R} \)-action by \( N_{\alpha,l,m}^{\lambda}(K,i)(x,y) \).

We define \( d_{\alpha,l}^{(K,i)} \) as follows.

\[
d_{\alpha,l}^{(K,i)} : CF(G_{(K,i),p\gamma} : \Lambda_0) \to CF(G_{(K,i),p\gamma} : \Lambda_0)
\]

\[
x \mapsto \sum_{m \in \mathbb{Z}, \lambda \geq 0, y \in P(G_{(K,i),p\gamma})} \sum \mathbb{C}^{\lambda} N_{\alpha,l,m}^{\lambda}(K,i)(x,y) \cdot T^\lambda y
\]

We define an \( \epsilon \)-gapped \( X_K \)-module structure on

\[
D_{K,i} = CF(G_{(K,i),p\gamma} : \Lambda_0) \otimes \Lambda_0(\theta).
\]

We determine \( \{\delta_i^{(K,i)} : D_{K,i} \to D_{K,i}\}_{i=0}^k \) as follows.

\[
\delta_i^{(K,i)}(x \otimes 1) = d_{0,2i}^{(K,i)}(x) \otimes 1 + d_{0,2i+1}^{(K,i)}(x) \otimes \theta
\]

\[
\delta_i^{(K,i)}(x \otimes \theta) = d_{1,2i}^{(K,i)}(x) \otimes 1 + d_{1,2i+1}^{(K,i)}(x) \otimes \theta
\]

**Lemma 6.4** \( (D_{K,i}, \{\delta_i^{(K,i)}\}_{i=0}^k) \) is an \( \epsilon \)-gapped \( X_K \)-module.

**proof:** It suffices to prove that

\[
(\delta^{(K,i)})(K) \circ (\delta^{(K,i)})(K) = 0 \mod(u^{K+1})
\]

holds. Let \( N_{\alpha,l,m}^{\lambda}(K,i)(x,y)(\mu) \) be the sum of the dimension \( \mu \) components of the moduli spaces. As in the case of the Floer cohomology, these moduli spaces have the natural compactifications and we have the following decomposition of the boundary:

\[
\partial N_{\alpha,l,m}^{\lambda}(K,i)(x,y)(1) = \bigcup_{\alpha'' = \{\lambda'\}, \mu = \mu', m = m' + m''} N_{\alpha,l',m'}^{\lambda',(K,i)}(x,z)(0) \times N_{\alpha,,l',m'}^{\lambda',\mu'(K,i)}(z,y)(0)
\]

This implies that the above equality holds.
Next, we define $e$-gapped $X_K$-morphisms $\iota_{(K,i \to j)} : D_{K,i} \to D_{K,j}$ for $i, j \in \mathbb{N}$.

We consider a family of Hamiltonian functions which connects $G_{w,t}^{(K,i)}$ and $G_{w,t}^{(K,j)}$.

Let $G_{s,w,t}^{(K,i \to j)}$ be a family of Hamiltonian functions parametrized by $(s, w, t) \in \mathbb{R} \times S^{2K+1} \times S^1$ which satisfies the following conditions:

- $G_{s,w,t}^{(K,i \to j)}(x) = \begin{cases} G_{w,t}^{(K,i)}(x) & s \ll 0 \\ G_{w,t}^{(K,j)}(x) & s \gg 0 \end{cases}$
- (Z-$p$-equivariance) $G_{s,mw,t}^{(K,i \to j)}(x) = G_{s,w,t}^{(K,i \to j)}(x)$ ($\forall m \in \mathbb{Z}_p$)
- (invariance under the shift $\iota$) $G_{s,\iota(w),t}^{(K,i \to j)}(x) = G_{s,w,t}^{(K,i \to j)}(x)$

We consider the following equation for $x \in P(G_{K,i}, p\gamma)$, $y \in P(G_{K,j}, p\gamma)$, $m \in \mathbb{Z}_p$, $\lambda \geq 0$, $\alpha \in \{0, 1\}$ and $0 \leq l \leq 2K + 1$. Assume that $\bar{x}, \bar{y} \in P(H^{(p)}, p\gamma)$ are periodic orbits which split into $\{x, \cdots\} \subset P(G_{K,i}, p\gamma)$ and $\{y, \cdots\} \subset P(G_{K,j}, p\gamma)$.

\[
(u, v) \in C^\infty(\mathbb{R} \times S^1, M) \times C^\infty(\mathbb{R}, S^{2K+1})
\]
\[
\partial_su(s, t) + J(u(s, t))(\partial_ku(s, t) - X_{G_{s,v(x),t}}^{(K,i \to j)}) = 0
\]
\[
\frac{dv(s)}{ds} - \operatorname{grad}(\tilde{F}) = 0
\]
\[
\lim_{s \to -\infty} v(s) = Z_\alpha, \lim_{s \to +\infty} v(s) = Z_l, \lim_{s \to -\infty} u(s, t) = x(t), \lim_{s \to +\infty} u(s, t) = y(t - \frac{m}{p})
\]
\[
\int_{\mathbb{R} \times S^1} \bar{u}^*\omega + \int_0^1 H^{(p)}(t, \bar{x}(t)) - H^{(p)}(t, \bar{y}(t))dt = \lambda
\]

Here, $\bar{u}$ is a cylinder obtained as before (So $\bar{u}$ connects $\bar{x}$ and $\bar{y}$). We denote the space of solutions by $N_{\alpha,l,m}^{\lambda,(K,i \to j)}(x, y)$. We define $\iota_{\alpha,l,(K,i \to j)}$ as follows:

\[
\iota_{\alpha,l,(K,i \to j)}(x) = \sum_{m \in \mathbb{Z}_p, \lambda \geq 0, y \in P(G_{K,j}, p\gamma)} \sharp N_{\alpha,l,m}^{\lambda,(K,i \to j)}(x, y) \cdot T^\lambda y
\]

We define an $e$-gapped $X_K$-morphism $\{\iota_{(K,i \to j), l}\}_{l=0}^K : D_{K,i} \to D_{K,j}$ as follows:

\[
\iota_{l,(K,i \to j), l}(x \otimes 1) = \iota_{0,l,(K,i \to j)}(x) \otimes 1 + \iota_{0,l+1,(K,i \to j)}(x) \otimes \theta
\]
\[
\iota_{l,(K,i \to j), l}(x \otimes \theta) = \iota_{1,l,(K,i \to j)}(x) \otimes 1 + \iota_{1,l+1,(K,i \to j)}(x) \otimes \theta
\]

We assume that $G_{s,w,t}^{(K,i \to j)}(x)$ is sufficiently close to $H^{(p)}(t, x)$ so that $\iota_{(K,i \to j)}$ is an $e$-gapped $X_K$-morphism over $\Lambda_0$ (see also Remark 6).
Lemma 6.5 $\iota(K,i\to j)$ is an $\epsilon$-gapped $X_K$-morphism and $\iota(K,j\to k) \circ \iota(K,i\to j)$ is $\epsilon$-gapped $X_K$-homotopic to $\iota(K,i\to k)$ for any $i, j, k \in \mathbb{N}$. In particular, $\iota(K,j\to i) \circ \iota(K,i\to j)$ is $\epsilon$-gapped $X_K$-homotopic to the identity and $\iota(K,i\to j)$ is an $\epsilon$-gapped $X_K$-homotopy equivalence between $D_{K,i}$ and $D_{K,j}$ for any $i, j \in \mathbb{N}$.

Proof: Let $\mathcal{N}_{\alpha,l,m}^{\lambda,(K,i\to j)}(x,y)^{(\mu)}$ be the sum of dimension $\mu$ components of $\mathcal{N}_{\alpha,l,m}^{\lambda,(K,i\to j)}(x,y)$. Then, the boundary of $\mathcal{N}_{\alpha,l,m}^{\lambda,(K,i\to j)}(x,y)$ is written as follows:

$$
\partial \mathcal{N}_{\alpha,l,m}^{\lambda,(K,i\to j)}(x,y)^{(1)} = \bigcup_{\alpha''} \mathcal{N}_{\alpha'',l',m'}^{\lambda',(K,i\to j)}(x,z)^{(0)} \times \mathcal{N}_{\alpha'',l',m'}^{\lambda'',(K,i\to j)}(z,y)^{(0)}
$$

This implies that

$$(\iota(K,i\to j))^{(K)} \circ (\delta(K,i))^{(K)} = (\delta(K,j))^{(K)} \circ (\iota(K,i\to j))^{(K)} \mod(u^{K+1})$$

holds and $\iota(K,i\to j)$ is an $\epsilon$-gapped $X_K$-morphism.

Next, we fix $R > 0$ so that

$$
\mathcal{G}_{s,w,t}^{(K,i\to j)}(x) = \begin{cases} 
\mathcal{G}_{w,t}^{(K,i)}(x) & s \leq -R \\
\mathcal{G}_{w,t}^{(K,i)}(x) & s \geq R
\end{cases}
$$

and

$$
\mathcal{G}_{s,w,t}^{(K,j\to k)}(x) = \begin{cases} 
\mathcal{G}_{w,t}^{(K,j)}(x) & s \leq -R \\
\mathcal{G}_{w,t}^{(K,k)}(x) & s \geq R
\end{cases}
$$

hold. We define a family of Hamiltonian functions parametrized by

$$(s, \rho, w, t) \in \mathbb{R} \times \mathbb{R}_{\geq 0} \times S^{2K+1} \times S^1$$

as follows:

$$
\mathcal{G}_{(s,\rho,w,t)}(x) = \begin{cases} 
\mathcal{G}_{w,t}^{(K,i)}(x) & s \ll 0 \\
\mathcal{G}_{w,t}^{(K,k)}(x) & s \gg 0
\end{cases}
$$

$$
\mathcal{G}_{(s,0,w,t)}(x) = \mathcal{G}_{s,w,t}^{(K,i\to k)}(x)
$$

$$
\mathcal{G}_{(s,\rho,w,t)}(x) = \begin{cases} 
\mathcal{G}_{s+\rho,w,t}^{(K,i\to j)}(x) & s \leq 0, \ \rho \geq 2R \\
\mathcal{G}_{s,w,t}^{(K,j\to k)}(x) & s \geq 0, \ \rho \geq 2R
\end{cases}
$$

$$(\mathbb{Z}_p\text{-equivariance}) \ \mathcal{G}_{(s,\rho,w,t)}(x) = \mathcal{G}_{(s,\rho,w,t-\frac{2}{p})}(x)
$$
we consider the following equation:

$G_{(s,p,\tau(w),t)}(x) = G_{(s,p,w,t)}(x)$

For $x \in P(G_{(K,i),p\gamma})$, $y \in P(G_{K,j,p\gamma})$, $m \in \mathbb{Z}_p$, $\lambda \geq 0$, $\alpha \in \{0, 1\}$ and $0 \leq l \leq 2K + 1$, we consider the following partial differential equation:

$\frac{d}{ds} f(s) - \text{grad}(F) = 0$

$$\lim_{s \to -\infty} v(s) = Z_{t, s}^0, \quad \lim_{s \to -\infty} v(s) = Z_{t}^{m}, \quad \lim_{s \to +\infty} u(s, t) = x(t), \quad \lim_{s \to +\infty} u(s, t) = y(t - \frac{m}{p}),$$

$$\int_{\mathbb{R} \times S^1} \tilde{u}^* \omega + \int_{0}^{1} H(p)(t, \tilde{x}) - H(p)(t, \tilde{y}) dt = \lambda$$

Here, $\tilde{u}$ is a cylinder obtained from $u$ as before. We denote the space of solutions by $\mathcal{M}_{\alpha, l, m}(x, y)$. We define $h_{\alpha, l}$ as follows:

$$h_{\alpha, l}(x) = \sum_{m \in \mathbb{Z}_p, \lambda \geq 0, y \in P(D_K, \mu)} \mathcal{M}_{\alpha, l, m}(x, y) : T^\lambda y$$

We define a family of maps $\{h_1\}_{l=0}^K$ as follows:

$h_1(x \otimes 1) = h_{0, 2l}(x) \otimes 1 + h_{0, 2l+1}(x) \otimes \theta$

$h_1(x \otimes \theta) = h_{1, 2l}(x) \otimes 1 + h_{1, 2l+1}(x) \otimes \theta$

Let $\mathcal{N}_{\alpha, l, m}^x(K, i \to j)(x, y)^{(\mu)}$, $\mathcal{N}_{\alpha, l, m}^x(K, j \to i)(x, y)^{(\mu)}$ and $\mathcal{M}_{\alpha, l, m}(x, y)^{(\mu)}$ be the sum of the dimension $\mu$ components of the moduli spaces. As in the case of the Floer cohomology, these moduli spaces have the natural compactifications. In particular, we have the following decomposition:

$$\partial \mathcal{M}_{\alpha, l, m}^x(x, y)^{(1)} = \bigsqcup_{\alpha'' = \beta'' \forall \beta'} \mathcal{N}_{\alpha', l, m'}^x(K, i \to j)(x, z)^{(0)} \times \mathcal{N}_{\alpha'', l, m''}^x(K, j \to i)(z, y)^{(0)}$$

The identity is to be understood with orientations (see [37]). This implies that

$l_{(K, j \to k)}^{(K)} \circ l_{(K, i \to j)}^{(K)} + (\delta^{(K, k)})^{(K)} \circ h_{(K)}^{(K)} + h_{(K)}^{(K)} \circ (\delta^{(K, i)})^{(K)} - l_{(K, i \to k)}^{(K)} = 0 \mod(u^{K+1})$

holds. In particular, $l_{(K, j \to k)}^{(K)} \circ l_{(K, i \to j)}^{(K)}$ is $c$-gapped $X_K$-homotopic to $l_{(K, i \to k)}^{(K)}$. 

$\square$
We define $\epsilon$-gapped $X_K$-morphisms

$$\tau_{(K \to K+1, i)} : D_{(K, i)} \to D_{(K+1, i)}$$

for $i \geq N(K)$ in the same way we defined $\iota_{(K, i \to j)}$. We assume that $N(K)$ is sufficiently large so that $\tau_{(K \to K+1, i)}$ is defined over $A_0$.

**Lemma 6.6** $\tau_{(K \to K+1, j)} \circ \iota_{(K, i \to j)}$ is $\epsilon$-gapped $X_K$-homotopic to $\iota_{(K+1, i \to j)} \circ \tau_{(K \to K+1, i)}$ for any $i, j \geq N(K)$.

The proof is essentially the same as the proof of Lemma 6.5.

Next, we define a morphism $\{P_{(K, i)} : C_{K, i} \to D_{K, i}\}$ between two directed families of $\epsilon$-gapped $X_K$-modules $C$ and $D$. By proposition 6.10, this morphism determines an $\epsilon$-gapped $X_\infty$-morphism and this is the $\mathbb{Z}_p$-equivariant pants product. First, we define a $p$-legged pants $\Sigma_p$ as follows:

$$\Sigma_p = \left( \bigsqcup_{0 \leq k \leq p-1} \mathbb{R} \times [k, k+1] \right) / \sim$$

The equivalence relation $\sim$ is defined as follows:

- $(s, k) \in [0, \infty) \times [k-1, k]$ is equivalent to $(s, k) \in [0, \infty) \times [k, k+1]$ for $1 \leq k \leq p-1$.
- $(s, 0) \in [0, \infty) \times [0, 1]$ is equivalent to $(s, p) \in [0, \infty) \times [p-1, p]$.
- $(s, k) \in (\infty, 0] \times [k, k+1]$ is equivalent to $(s, k+1) \in (\infty, 0] \times [k, k+1]$ for $0 \leq k \leq p-1$.

We determine a complex structure near $[(0, 0)] \in \Sigma_p$. We give an explicit local coordinate of a neighborhood of $[(0, 0)] \in \Sigma_p$ as follows:

$$w : \{ z \in \mathbb{C} \mid |z| \leq \frac{1}{2} \} \to \Sigma_p$$

$$w(z) = \begin{cases} 
[z^p] & 0 \leq \text{arg}(z) \leq \frac{\pi}{p} \\
[z^p + k\sqrt{-1}] & \frac{(2k-1)p}{p} \leq \text{arg}(z) \leq \frac{(2k+1)p}{p} \\
[z^p + p\sqrt{-1}] & \frac{(2p-1)p}{p} \leq \text{arg}(z) \leq 2\pi
\end{cases}$$

Here we identify $\mathbb{R}^2$ with $\mathbb{C}$. Without loss of generality, we assume that $H(t, x) = 0$ near $t = 0$. Let $K_{w,z}^{(K, i)}$ be a family of Hamiltonian functions parametrized by $(w, z) \in S^{2K+1} \times \Sigma_p$ as follows:

- $K_{w,z}^{(K, i)}(x) = H_K([t], x)$ \quad $(z = [s, t] \in \Sigma_p, s \ll 0)$
- $K_{w,z}^{(K, i)}(x) = \frac{1}{p} G_{(K, i)}(x, \frac{t}{p})$ \quad $(z = [s, t] \in \Sigma_p, s \gg 0)$
- $K_{w,z}^{(K, i)} \xrightarrow{i \to \infty} H(t, x)$ \quad $z = [s, t]$
we consider the following equation:

\[ K^{(K,i)}_{m,w,z}(x) = K^{(K,i)}_{w,z+m,\sqrt{-1}}(x) \]

((invariance under the shift \( \tau \)) \( K^{(K,i)}_{\tau(w),z}(x) = K^{(K,i)}_{w,z}(x) \))

For \( \{x_k\}_{k=1}^p \subset P(H_K, \gamma), y \in P(D_{K,i}, p) \), \( m \in \mathbb{Z}_p, \lambda \geq 0, \alpha \in \{0, 1\} \), and \( 0 \leq l \leq 2K + 1 \), we consider the following equation:

\[
(u, v) \in C^\infty(\Sigma_p, M) \times C^\infty(\mathbb{R}, S^{2K+1}) \nonumber
\]

\[
\partial_s u([s,t]) + J(u([s,t]))(\partial_t u([s,t]) - X_{K^{(K,i)}_{v(s)},[s,t]})) = 0
\]

\[
\frac{dv}{ds} = -\text{grad}(F) = 0
\]

\[
\lim_{s \to -\infty} v(s) = Z_0^\alpha, \lim_{s \to +\infty} v(s) = Z_\infty^m, \lim_{s \to -\infty} u([s,t]) = x_i([t]) \ (t \in [i-1, i])
\]

\[
\lim_{s \to +\infty} u([s,t]) = y(\frac{t}{p} - m)
\]

\[
\int_{\Sigma_p} \bar{u}^* \omega + \int_0^1 H^{(p)}(t, \bar{y}(t)) dt - \sum_{i=1}^p \int_0^1 H(t, \bar{x}_i(t)) dt = \lambda
\]

Here \( \{\bar{x}_i\} \subset P(H, \gamma) \) and \( \bar{y} \in P(H^{(p)}, p\gamma) \) are periodic orbits as before and \( \bar{u} \) is a map which connects \( \{\bar{x}_i\} \) and \( \bar{y} \). We denote the space of solutions by \( M^{(K,i),\alpha,l,m}(x_1, \cdots, x_p : y) \). We define \( f_{\alpha,l}^{(K,i)} \) as follows:

\[
f_{\alpha,l}^{(K,i)} : C_{K,i} \rightarrow D_{K,i} \nonumber
\]

\[
x_1 \otimes \cdots \otimes x_p \mapsto \sum_{m \in \mathbb{Z}_p, \lambda \geq 0, y \in P(G_{(K,i)}, p\gamma)} \nu M^{(K,i),\alpha,l,m}(x_1, \cdots, x_p : y) T^\lambda y
\]

Then, we define \( \{f^{(K,i)}_l\}_{l=0}^K \) as follows:

\[
f^{(K,i)}_l((x_1 \otimes \cdots \otimes x_p) \otimes 1) = f^{(K,i)}_{0,2l}(x_1 \otimes \cdots \otimes x_p) \otimes 1 + f^{(K,i)}_{0,2l+1}(x_1 \otimes \cdots \otimes x_p) \otimes \theta
\]

\[
f^{(K,i)}_l((x_1 \otimes \cdots \otimes x_p) \otimes \theta) = f^{(K,i)}_{1,2l}(x_1 \otimes \cdots \otimes x_p) \otimes 1 + f^{(K,i)}_{1,2l+1}(x_1 \otimes \cdots \otimes x_p) \otimes \theta
\]

**Lemma 6.7**: \( \{f^{(K,i)}_l\} \) determines a morphism between a directed family of \( \epsilon \)-gapped \( X_K \)-module \( \{C_{K,i}\} \) and a directed family of \( \epsilon \)-gapped \( X_K \)-module \( \{D_{K,i}\} \).

**Proof**: If \( k^{(K,i)}_{w,z}(x) \) is sufficiently close to \( H(t, x) \), \( f^{(K,i)}_l \) becomes a morphism between \( \epsilon \)-gapped \( X_K \)-modules. This follows from the same argument as in the proof of Lemma 4.1. It suffices to prove that \( f^{(K,i)}_l \circ \iota_{(K,i)} \) and \( \iota_{(K,i)} \circ f^{(K,i)}_l \) are \( \epsilon \)-gapped \( X_K \)-homotopic and \( f^{(K,i)}_{l+1} \circ \tau_{(K-K+1,i)} \) and \( \tau_{(K-K+1,i)} \circ f^{(K,i)}_l \) are \( \epsilon \)-gapped \( X_K \)-morophism for sufficiently large \( i \in \mathbb{N} \). However, these facts follow from the same arguments as in the proof of Lemma 6.5.
We fix \((K, i)\) and we denote \(H_K\) by \(\overline{H}\) and \(G_{(K, i)}\) by \(\overline{H}^{(p)}\). Proposition 6.10 implies that we have the following two \(\epsilon\)-gapped \(X_\infty\)-modules

\[
(C, \mathfrak{d}) = ((CF(\overline{H}, \gamma))^\otimes_p \otimes \Lambda_0(\theta), \{d_i\})
\]

\[
(D, l) = (CF(\overline{H}^{(p)}, p\gamma) \otimes \Lambda_0(\theta), \{l_i\})
\]

and an \(\epsilon\)-gapped \(X_\infty\)-morphism

\[
f = \{f_i\} : (C, \mathfrak{d}) \longrightarrow (D, l).
\]

\((C, \mathfrak{d})\) is an extension of \(C_{K,i}\) and \((D, l)\) is an extension of \(D_{K,i}\). The cochain complex \((C \otimes \Lambda_0[[u]], \mathfrak{d}^{(\infty)})\) is the \(\mathbb{Z}_p\)-equivariant cochain complex for \(CF(H, \gamma)^{\otimes p}\) and the cochain complex \((D \otimes \Lambda_0[[u]], l^{(\infty)})\) is the \(\mathbb{Z}_p\)-equivariant Floer cochain complex for \(H^{(p)}\). So

\[
H(Z_p, CF(H, \gamma)^{\otimes p}) = H(C \otimes \Lambda_0[[u]], \mathfrak{d}^{(\infty)})
\]

\[
\hat{H}(Z_p, CF(H, \gamma)^{\otimes p}) = H(C \otimes \Lambda_0[u^{-1}, u], \mathfrak{d}^{(\infty)})
\]

\[
HF_{Z_p}(H^{(p)}, p\gamma) = H(D \otimes \Lambda_0[[u]], l^{(\infty)})
\]

\[
\hat{HF}_{Z_p}(H^{(p)}, p\gamma) = H(D \otimes \Lambda_0[u^{-1}, u], l^{(\infty)})
\]

are well-defined and they are unique. \(f\) determines the \(\mathbb{Z}_p\)-equivariant pair of pants product.

\[
P : H(Z_p, CF(H, \gamma)^{\otimes p}) \longrightarrow HF_{Z_p}(H^{(p)}, p\gamma)
\]

\[
\hat{P} : \hat{H}(Z_p, CF(H, \gamma)^{\otimes p}) \longrightarrow \hat{HF}_{Z_p}(H^{(p)}, p\gamma)
\]

Moreover, the uniqueness of \(\overline{H}_{loc}\) in Proposition 6.10 (ii) implies that

\[
P_{loc} : \bigoplus_{x \in P(H, \gamma)} H(Z_p, CF^{loc}(H, x)^{\otimes p}) \longrightarrow \bigoplus_{x \in P(H, \gamma)} HF_{Z_p}^{loc}(H^{(p)}, x^{(p)})
\]

\[
\hat{P}_{loc} : \bigoplus_{x \in P(H, \gamma)} \hat{H}(Z_p, CF^{loc}(H, x)^{\otimes p}) \longrightarrow \bigoplus_{x \in P(H, \gamma)} \hat{HF}_{Z_p}^{loc}(H^{(p)}, x^{(p)})
\]

are equal to the local \(\mathbb{Z}_p\)-equivariant pair of pants product defined in [37]. In particular, \(\hat{P}_{loc}\) is an isomorphism. By the same arguments as in the toroidally monotone case, we can apply this local isomorphism to prove that \(\hat{P}\) is an isomorphism. The only non-trivial point is the proof of Lemma 5.3 in the weakly monotone case. In the proof of Lemma 5.3, we used a cochain homotopy \(L\) to prove that for any cocycle \(x \in F^{q+r}A\) we can find \(y \in F^{q+1}A\) so that \(d(y) = x\). In order to construct \(L\) for \(A = (D \otimes [u^{-1}, u], l^{(\infty)})\), we need the uniqueness of \(X_\infty\)-morphism up to \(X_\infty\)-homotopy. However, we only proved the uniqueness up to \(X_K\)-homotopy for \(K < \infty\). So we have to give another proof in the weakly monotone case.

64
proof(Lemma 5.3) : It suffices to prove that for any cocycle \( x \in (D \otimes \Lambda_0[u^{-1}, u]) \), we can find \( y \in T^{-p[H]}(D \otimes \Lambda_0[u^{-1}, u]) \) so that \( t^{(\infty)}(y) = x \) holds. Note that \((D \otimes \Lambda, l_0)\) is acyclic and we can construct a cochain homotopy \( H \):

\[
H : D \otimes \Lambda \longrightarrow D \otimes \Lambda
\]

\[
H(D) \subset T^{-p[H]}D
\]

\[
\text{Id}_{D \otimes \Lambda} = l_0H + Hl_0.
\]

Assume that \( x \) is an infinite sum \( x = \sum_{k \geq 0} x_k u^k \) (\( x_k \in D \)). Our purpose is to construct \( y = \sum_{k \geq 0} y_k u^k \) (\( y_k \in T^{-p[H]}D \)) so that \( t^{(\infty)}(y) = x \) holds. We define \( y_0 = H(y_0) \). Then, \( t^{(\infty)}(y_0) = x_0 + \sum_{k \geq 1} x_k u^k \) holds. \( x - t^{(\infty)}(y_0) = \sum_{k \geq 1} (x_k - x_k') u^k \) is a cocyle. Inductively, we can determine \( y_1, y_2, \cdots \) so that \( t^{(\infty)}(y) = x \) and \( y \in T^{-p[H]}D \) hold.

\[\square\]

The rest of the proof of Theorem 1.1 for the weakly monotone case is the same as in the toroidally monotone case.

**Remark 6.2** We can prove the uniqueness of \( f \) up to \( \epsilon \)-gapped \( X_\infty \)-homotopy by applying “homotopy of homotopy” theory. In particular, we can also prove the uniqueness of \( P \) and \( \hat{P} \). However, the uniqueness of \( \hat{P}_{loc} \) is sufficient for our proof of Theorem 1.1. So, Proposition 6.10 is sufficient for our purpose. The proof of the uniqueness of \( f \) up to \( X_\infty \)-homotopy is beyond the scope of this paper.

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65
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