NEW BOUNDS OF WEYL SUMS

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Abstract. We augment the method of Wooley (2015) by some new ideas and among a series of other bounds, we show that for a set of \((u_2, \ldots, u_{d-1}) \in [0, 1)^{d-2}, \ d \geq 3,\) of full Lebesgue measure, one has

\[
\sup_{(u_1, u_d) \in [0, 1)^2} \left| \sum_{n=1}^{N} \exp(2\pi i (u_1 n + \ldots + u_d n^d)) \right| \leq N^{1/2 + \frac{d+2}{2d^2 + 2d + 6} + o(1)}
\]

as \(N \to \infty,\) while the result of Wooley (2015) coupled with the modern advances on the Vinogradov Mean Value Theorem gives the exponent \(1/2 + (2d + 5)/(2d^2 + 2d + 6).\) We also improve the bound of Wooley (2015) on the discrepancy of polynomial sequences. Finally, we extend these results and ideas to very general settings of arbitrary orthogonal projections of the vectors of the coefficients \((u_1, \ldots, u_d)\) onto a lower dimensional subspace, which seems to be a new point of view.

1. Introduction

1.1. Motivation and background. For an integer \(d \geq 2,\) let \(T_d = (\mathbb{R}/\mathbb{Z})^d\) be the \(d\)-dimensional unit torus.

Given a family \(\varphi = (\varphi_1, \ldots, \varphi_d) \in \mathbb{Z}[T]^d\) of \(d\) distinct nonconstant polynomials and a sequence of complex weights \(a = (a_n)_{n=1}^\infty,\) for \(u = (u_1, \ldots, u_d) \in T_d\) we define the trigonometric polynomials

\[
T_\varphi(a, u; N) = \sum_{n=1}^{N} a_n e(u_1 \varphi_1(n) + \ldots + u_d \varphi_d(n)),
\]

where throughout the paper we denote \(e(x) = \exp(2\pi i x).\)

Furthermore, decomposing

\[
T_d = T_k \times T_{d-k}
\]

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with $T_k = [0, 1)^k$ and $T_{d-k} = [0, 1)^{d-k}$, for $k = 1, \ldots, d$, given $x \in T_k$, $y \in T_{d-k}$ we refine the notation (1.1) and write

$$T_\varphi(a, x, y; N) = \sum_{n=1}^{N} a_n e \left( \sum_{j=1}^{k} x_j \varphi_j(n) + \sum_{j=1}^{d-k} y_j \varphi_{k+j}(n) \right).$$

For the classical case $a_n = 1$ for all $n \in \mathbb{N}$ and the polynomials

$$\{\varphi_1(T), \ldots, \varphi_d(T)\} = \{T, \ldots, T^d\}$$

satisfying some natural necessary conditions, the result of Wooley [8, Theorem 1.1] together with the modern knowledge towards the Vinogradov Mean Value Theorem asserts that for almost all $x \in T_k$ in a sense of the $k$-dimensional Lebesgue measure on $T_k$, one has

$$\sup_{y \in T_{d-k}} |T_\varphi(a, x, y; N)| \leq N^{1/2+\Gamma_0+o(1)}, \quad N \to \infty,$$

where

$$\Gamma_0 = \frac{d - k + 2\sigma_k(\varphi) + 1}{2d^2 + 4d - 2k + 2}$$

and

$$\sigma_k(\varphi) = \sum_{j=k+1}^{d} \deg \varphi_j.$$

We remark that the bound (1.3) is presented in a more explicit form than in [8, Theorem 1.1] as we have used the optimal result Wooley [9, Theorem 1.1] for the parameter $u$ of [8, Theorem 1.1]. Furthermore the results in [8, Theorem 1.1] have the restriction that $k < d$, but our methods works for $k = d$ also. Naturally, for the case $k = d$ we consider $x$ only and remove the variable $y$ from each statement for this special case.

Here we use some new ideas to extend the method and results of Wooley [8] in several directions. In particular, we obtain an improvement of (1.3).

We note that it is also interesting to find the upper bound for the almost all points $u \in T_d$ for the following classical Weyl sums,

$$S_d(u; N) = \sum_{n=1}^{N} e(u_1n + \ldots + u_d n^d).$$

In this direction the authors [2, Appendix A] have shown that for almost all $u \in T_d$ one has

$$|S_d(u; N)| \leq N^{1/2+o(1)}, \quad N \to \infty.$$
It is very natural to conjecture that the exponent $1/2$ can not be improved, however there seems to be no any results in this direction.

As in [8] we give applications of our bounds of exponential sums to bounds on the discrepancy (see Section 1.10 for a definition) of the sequence of fractional parts of polynomials. However, we modify the approach of Wooley [8] of passing from exponential sums to the discrepancy and obtain stronger results.

Here we obtain results of three different types:

(i) We study the scenario of Wooley [8] when the vector $u \in T_d$ is split into two parts $x$ and $y$ formed by its components which is related to the coordinate-wise projections of $u \in T_d$.

(ii) We introduce and study an apparently new problem related to arbitrary orthogonal projections of $u \in T_d$.

(iii) As in [8], we study the uniform distribution of polynomials modulo one and obtain bounds for the discrepancy, which improve those of [8, Theorem 1.4].

1.2. Results for coordinate-wise projections of $u$: a traditional point of view. Let $\varphi = (\varphi_1, \ldots, \varphi_d) \in \mathbb{Z}[T]^d$ be $d$ distinct nonconstant polynomials and a sequence of complex weights $a = (a_n)_{n=1}^{\infty}$ with $a_n = n^{o(1)}$.

We start with a very broad generalisation of (1.3).

**Theorem 1.1.** Supposed that $\varphi \in \mathbb{Z}[T]^d$ is such that the Wronskian $W(T; \varphi)$ does not vanish identically and $\sigma_k(\varphi) < d(d+1)/2$, then for almost all $x \in T_k$ one has

$$\sup_{y \in T_{d-k}} |T\varphi(a, x, y; N)| \leq N^{1/2+\Gamma_0+o(1)}, \quad N \to \infty,$$

where $\Gamma_0$ is given by (1.4).

We remark that under the condition $\sigma_k(\varphi) < d(d+1)/2$ we have $\Gamma_0 < 1/2$ which gives a nontrivial upper bound. Furthermore for the classical choice of $\varphi$ as in (1.2) we always have $\sigma_k(\varphi) < d(d+1)/2$. However in this case we have a stronger result which improves the bound (1.3) of Wooley [8, Theorem 1.1].

In fact our method works for the following more general polynomials $\varphi \in \mathbb{Z}[T]^d$ such that for some $\{j_1, \ldots, j_\ell\} \subseteq \{1, \ldots, d\}$ we have

$$\{\varphi_{j_1}(T), \ldots, \varphi_{j_\ell}(T)\} = \{T, \ldots, T^\ell\},$$

which contains (1.2) as a special case.
Theorem 1.2. Suppose that $\varphi \in \mathbb{Z}[T]^d$ is as in (1.7). If $\sigma_k(\varphi) < \ell(\ell + 1)/2$ then for almost all $x \in T_k$ one has
\[
\sup_{y \in T_{d-k}} |T_\varphi(a, x, y; N)| \leq N^{1/2 + \Gamma_1 + o(1)}, \quad N \to \infty,
\]
where
\[
\Gamma_1 = \frac{d - k + 2\sigma_k(\varphi)}{2\ell^2 + 2\ell + 2d - 2k}.
\]

Corollary 1.3. Suppose that $\varphi \in \mathbb{Z}[T]^d$ is as in (1.2). Then for almost all $x \in T_k$ one has
\[
\sup_{y \in T_{d-k}} |T_\varphi(a, x, y; N)| \leq N^{1/2 + \Gamma_2 + o(1)}, \quad N \to \infty,
\]
where
\[
\Gamma_2 = \frac{d - k + 2\sigma_k(\varphi)}{2d^2 + 4d - 2k}.
\]

Note that in the settings of Corollary 1.3 implies $\sigma_k(\varphi) < (d + 1)d/2$ for each $k = 1, \ldots, d$. Elementary calculations show that $\Gamma_2 < \Gamma_0$ for each $k = 1, \ldots, d$, thus Corollary 1.3 gives a direct improvement of the result of (1.3) which is due to Wooley [8, Theorem 1.1].

Note that $\sigma_d(\varphi) = 0$ and the Corollary 1.3 gives the bound (1.6) when $k = d$, and hence we obtain a different proof for the bound (1.6).

1.3. Results for arbitrary orthogonal projections of $u$: a new point of view. We now consider other projections which seems to be a new scenario which has not been studied in the literature prior this work.

We need to introduce some notation first.

Let $G(d, k)$ denote the collections of all the $k$-dimensional linear subspaces of $\mathbb{R}^d$. For $V \in G(d, k)$, let $\pi_V : \mathbb{R}^d \to V$ denote the orthogonal projection onto $V$. For $0 < \alpha < 1$, we consider the set
\[
E_{\varphi, a, \alpha} = \{ u \in T_d : |T_\varphi(a, u; N)| \geq N^\alpha \text{ for infinity many } N \in \mathbb{N} \}.
\]
We also use $\lambda (S)$ to denote the Lebesgue measure of $S \subseteq T^d$ (and also for sets in other spaces).

We are interested in the following apparently new point of view:

Question 1.4. Given $\varphi \in \mathbb{Z}[T]^d$, for what $\alpha$ we have $\lambda(\pi_V(E_{\varphi, a, \alpha})) = 0$ for all $V \in G(d, k)$?

We now see that Corollary 1.3 implies that for $\varphi \in \mathbb{Z}[T]^d$ is as in (1.2) and $a_n = n^{\alpha(1)}$, for any $\alpha > 1/2 + \Gamma_2$ we have
\[
\lambda(\pi_{d,k}(E_{\varphi, a, \alpha})) = 0,
\]
where $\pi_{d,k}$ is the orthogonal projection of $T_d$ onto $T_k$, that is,
\[
\pi_{d,k} : (u_1, \ldots, u_d) \to (u_1, \ldots, u_k).
\]
For the degree sequence \( \deg \varphi_1, \ldots, \deg \varphi_d \) we denote them as
\[
(1.8) \quad r_1 \leq \ldots \leq r_d,
\]
and define
\[
(1.9) \quad \tilde{\sigma}_k(\varphi) = \sum_{i=k+1}^{d} r_i.
\]

We remark that our results in the following are similar to the results of Theorem 1.1, Theorem 1.2 and Corollary 1.3, with the change of \( \tilde{\sigma}_k(\varphi) \) only.

**Theorem 1.5.** Supposed that \( \varphi \in \mathbb{Z}[T]^d \) is such that the Wronskian \( W(T; \varphi) \) does not vanish identically and \( \tilde{\sigma}_k(\varphi) < \frac{d(d+1)}{2} \), then for any \( \mathcal{V} \in \mathcal{G}(d,k) \) one has
\[
\lambda(\pi_\mathcal{V}(\mathcal{E}_{\varphi,a,\alpha})) = 0
\]
provided that \( \alpha > \frac{1}{2} + \tilde{\Gamma}_0 \)
where
\[
\tilde{\Gamma}_0 = \frac{d - k + 2\tilde{\sigma}_k(\varphi) + 1}{2d^2 + 4d - 2k + 2}.
\]

**Theorem 1.6.** Suppose that \( \varphi \in \mathbb{Z}[T]^d \) is as in (1.7). If \( \tilde{\sigma}_k(\varphi) < \ell(\ell+1)/2 \) then for any \( \mathcal{V} \in \mathcal{G}(d,k) \) one has
\[
\lambda(\pi_\mathcal{V}(\mathcal{E}_{\varphi,a,\alpha})) = 0
\]
provided that \( \alpha > \frac{1}{2} + \tilde{\Gamma}_1 \)
where
\[
\tilde{\Gamma}_1 = \frac{d - k + 2\tilde{\sigma}_k(\varphi)}{2\ell^2 + 2\ell + 2d - 2k}.
\]

We now consider Question 1.4 in the classical case (1.2) and the sums (1.5). That is, we study the following set
\[
\mathcal{E}_{d,\alpha} = \{ u \in \mathbb{T}_d : |S_d(u; N)| \geq N^\alpha \text{ for infinity many } N \in \mathbb{N} \},
\]
which we define for \( 0 < \alpha < 1 \) and integer \( d \geq 2 \).

**Corollary 1.7.** For any \( \mathcal{V} \in \mathcal{G}(d,k) \) one has
\[
\lambda(\pi_\mathcal{V}(\mathcal{E}_{d,\alpha})) = 0
\]
provided that \( \alpha > \frac{1}{2} + \tilde{\Gamma}_2 \)
where
\[
\tilde{\Gamma}_2 = \frac{(d - k)(d + k + 2)}{2d^2 + 4d - 2k}.
\]
We remark that the orthogonal projection of sets is a fundamental topic in fractal geometry and geometric measure theory. Recall the classical Marstrand–Mattila projection theorem: Let $A \subseteq \mathbb{R}^d$, $d \geq 2$, be a Borel set with Hausdorff dimension $s$, see [6, Chapter 5] for more details and related definitions. Then we have:

- **Dimension part:** If $s \leq k$, then the orthogonal projection of $A$ onto almost all $k$-dimensional subspaces has Hausdorff dimension $s$.
- **Measure part:** If $s > k$, then the orthogonal of $A$ onto almost all $k$-dimensional subspaces has positive $k$-dimensional Lebesgue measure.

From the Marstrand–Mattila projection theorem and Corollary 1.7 we obtain the following results. For $A \subseteq \mathbb{R}^d$ we use $\dim A$ to denote the Hausdorff dimension of $A$.

**Corollary 1.8.** Let $k, d$ be two integers with $1 \leq k < d$ and $d \geq 2$. Then $\dim E_{d,\alpha} \leq k$ for any

$$\alpha > 1/2 + \frac{(d-k)(d+k+2)}{2d^2 + 4d - 2k}.$$ 

In particular, taking $k = d - 1$ we obtain

**Corollary 1.9.** For any integer $d \geq 2$ one has $\dim E_{d,\alpha} \leq d - 1$ for any

$$\alpha > \frac{1}{2} + \frac{2d + 1}{2d^2 + 2d + 2}.$$ 

We remark that the authors [2] have obtained a lower bound of the Hausdorff dimension of $E_{d,\alpha}$. Among other things, it is shown in [2] that for any $d \geq 2$ and $\alpha \in (0, 1)$ one has

$$\dim E_{d,\alpha} \geq \xi(d, \alpha)$$

with some explicit function $\xi(d, \alpha) > 0$. As a counterpart to Corollary 1.9, we remark that we expect $\dim E_{d,\alpha} = d$ for $\alpha \in (0, 1/2)$, see also [2]. On the other hand, we do no have any plausible conjecture about the exact behaviour of $\dim E_{d,\alpha}$ for $\alpha \in [1/2, 1)$.

1.4. Uniform distribution modulo one. Let $x_n, n \in \mathbb{N}$, be a sequence in $[0, 1)$. The discrepancy of this sequence at length $N$ is defined as

$$D_N = \sup_{0 \leq a < b \leq 1} \left| \# \{1 \leq n \leq N : x_n \in (a, b) \} - (b - a)N \right|.$$ 

We note that sometimes in the literature the scaled quantity $N^{-1}D_N$ is called the discrepancy, since our argument looks cleaner with the definition (1.10), we adopt it here.
For \( x \in T_k, y \in T_{d-k} \) we consider the sequence
\[
\sum_{j=1}^{k} x_j n^j + \sum_{j=k+1}^{d} y_j n^j, \quad n \in \mathbb{N},
\]
and for each \( N \) we denote by \( D_d(x, y; N) \) the corresponding discrepancy.

Wooley [8, Theorem 1.4] has proved that \((d \geq 3)\) for almost all \( x \in T_k \) with \( 1 \leq k \leq d-1 \) one has
\[
\sup_{y \in T_{d-k}} D_d(x, y; N) \leq N^{1/2+\gamma_0+o(1)}, \quad N \to \infty,
\]
where
\[
\gamma_0 = \frac{d - k + 2\sigma_k(\varphi) + 2}{2d^2 + 4d - 2k + 4}.
\]
We improve this bound as follows.

**Theorem 1.10.** Let \( 1 \leq k \leq d \) be an integer then for almost all \( x \in T_k \) one has
\[
\sup_{y \in T_{d-k}} D_d(x, y; N) \leq N^{1/2+\gamma_1+o(1)}, \quad N \to \infty,
\]
where
\[
\gamma_1 = \frac{d - k + 2\sigma_k(\varphi) + 1}{2d^2 + 4d - 2k + 2}.
\]
Since \( \sigma_k(\varphi) < d(d+1)/2 \), we have \( \gamma_1 < \gamma_0 \).

2. Preliminaries

2.1. Notation and conventions. Throughout the paper, the notation \( U = O(V) \), \( U \ll V \) and \( V \gg U \) are equivalent to \( |U| \leq cV \) for some positive constant \( c \), which throughout the paper may depend on the degree \( d \) and occasionally on the small real positive parameter \( \varepsilon \).

For any quantity \( V > 1 \) we write \( U = V^{o(1)} \) (as \( V \to \infty \)) to indicate a function of \( V \) which satisfies \( |U| \leq V^\varepsilon \) for any \( \varepsilon > 0 \), provided \( V \) is large enough.

We use \( \#S \) to denote the cardinality of a finite set \( S \).

We always identify \( T_d \) with half-open unit cube \( [0, 1)^d \), in particular we naturally associate the Euclidean norm \( \|x\| \) with points \( x \in T_d \).

We say that some property holds for almost all \( x \in [0, 1]^k \) if it holds for a set \( T_k \subseteq [0, 1]^k \) of \( k \)-dimensional Lebesgue measure \( \lambda(T_k) = 1 \).

Let
\[
s(q) = q(q+1)/2.
\]
We always assume that \( \varphi \in \mathbb{Z}[T]^d \) consists of polynomials \( \varphi_j \) of degrees
\[
\text{(2.2)} \quad \deg \varphi_j = e_j, \quad j = 1, \ldots, d.
\]

2.2. **Generalised mean value theorems.** For the classical case of the Weyl sums \( S_d(u; N) \) as in (1.5), the Parseval identity gives
\[
\int_{T^d} |S_d(x; N)|^2 dx = N.
\]
Furthermore, we have the following Vinogradov mean value theorem, in the currently known form
\[
\text{(2.3)} \quad \int_{T^d} |S_d(x; N)|^{2s(d)} dx \leq N^{s(d) + o(1)}, \quad N \to \infty,
\]
where \( s(d) = d(d+1)/2 \). This is due to Bourgain, Demeter and Guth [1] for \( d \geq 4 \) and Wooley [7] for \( d = 3 \).

We use the following a general form due to Wooley [9, Theorem 1.1], which extends the bound (2.3) to the sums \( T_{\varphi}(a, u; N) \).

We recall that the Wronskian of the functions \( \psi = (\psi_1, \ldots, \psi_d) \) is defined as
\[
W(T; \psi) = \det \left( \psi_j^{(i)}(T) \right)_{i,j=1}^d
\]
(assuming these derivatives exist).

**Lemma 2.1.** For any a family \( \varphi \in \mathbb{Z}[T]^d \) of \( d \) polynomials such that the Wronskian \( W(T; \varphi) \) does not vanish identically, any sequence of complex weights \( a = (a_n)_{n=1}^{\infty} \), and any integer \( N \geq 1 \), we have the upper bound
\[
\int_{T^d} |T_{\varphi}(a, u; N)|^{2\sigma} d\mathbf{u} \leq N^{o(1)} \left( \sum_{n=1}^{N} |a_n|^2 \right)^{\sigma}
\]
for any real positive \( \sigma \leq s(d) \), where \( s(d) \) is given by (2.1).

Using translation invariance we also have the following result for sums over short intervals. In the case of the polynomials (1.2) the translation invariance immediately leads to the the following bound of short sums.

**Lemma 2.2.** If \( \varphi \in \mathbb{Z}[T]^d \) is as in (1.2) then for any sequence of complex weights \( a = (a_n)_{n=1}^{\infty} \) with \( a_n = n^{o(1)} \) and any integers \( L, N \geq 1 \) we have the upper bound
\[
\int_{T^d} |T_{\varphi}(a, u; N + L) - T_{\varphi}(a, u; N)|^{2s(d)} d\mathbf{x} \leq L^{s(d)} (LN)^{o(1)},
\]
as $LN \to \infty$, where $s(d)$ is given by (2.1).

**Proof.** Using the orthogonality of exponential functions, we immediately see that

$$
\int_{T_d} |T_\varphi(a, x; N + L) - T_\varphi(a, x; N)|^{2s(d)} dx = J
$$

where $J$ is number of solutions to the system of equations

$$
\sum_{j=1}^{s(d)} n_j^i = \sum_{j=s(d)+1}^{2s(d)} n_j^i, \quad i = 1, \ldots, d,
$$

$$
n_j = N + 1, \ldots, N + L, \quad j = 1, \ldots, 2s(d),
$$

counted with the weights

$$
\prod_{j=1}^{s(d)} a_{n_j} \prod_{j=s(d)+1}^{2s(d)} \overline{a_{n_j}} = N^{o(1)}
$$

where $\overline{z}$ means complex conjugation. Thus $J \leq N^{o(1)} J_0$ where $J_0$ is the number of solutions to the above system of equations counted with the weight 1. By the translation invariance $J_0$ does not depend on $N$ and so can be estimated via (2.3) (applied with $L$ instead of $N$). $\square$

We now immediately derive from Lemma 2.2:

**Corollary 2.3.** Suppose that $\varphi \in \mathbb{Z}[T]^d$ is as in (1.7). Then for any sequence of complex weights $a = (a_n)_{n=1}^{\infty}$ with $a_n = n^{o(1)}$ and any integers $L, N \geq 1$ we have the upper bound

$$
\int_{T_d} |T_\varphi(a, u; N + L) - T_\varphi(a, u; N)|^{2s(\ell)} du \leq L^{s(\ell)} (LN)^{o(1)},
$$

as $LN \to \infty$, where $s(\ell)$ is given by (2.1).

**2.3. Continuity of exponential sums.** For $u \in \mathbb{R}^d$ and $\zeta = (\zeta_1, \ldots, \zeta_d)$ with $\zeta_j > 0$, $j = 1, \ldots, d$, we define the $d$-dimensional box centred at $u$ and side lengths $2\zeta$ by

$$
\mathcal{R}(u, \zeta) = [u_1 - \zeta_1, u_1 + \zeta_1] \times \ldots \times [u_d - \zeta_d, u_d + \zeta_d].
$$

**Lemma 2.4.** Suppose that $\varphi \in \mathbb{Z}[T]^d$ satisfies (2.2). Let $0 < \alpha < 1$ and let $\varepsilon > 0$ be sufficiently small. If $|T_\varphi(a, u; N)| \geq N^\alpha$ for some $u \in T_d$, then

$$
|T_\varphi(a, v; N)| \geq N^\alpha / 2
$$

holds for any $v \in \mathcal{R}(u, \zeta)$ provided that $N$ is large enough and

$$
0 < \zeta_j \leq N^{\alpha - \epsilon_j - 1 - \varepsilon}, \quad j = 1, \ldots, d.
$$
Proof. Let $\mathbf{u} = (u_1, \ldots, u_d)$ and $\mathbf{v} = (v_1, \ldots, v_d)$. For $1 \leq n \leq N$ the continuity of function $e(x)$ and the choice of $\zeta$ implies
\[
|a_n e\left(\sum_{j=1}^{d} u_j \varphi_j(n)\right) - a_n e\left(\sum_{j=1}^{d} v_j \varphi_j(n)\right)| \ll |a_n| \sum_{j=1}^{d} \zeta_j |\varphi_j(n)|.
\]
Observe that $\varphi_j(n) \ll n^{\varepsilon_j}$ for each $j = 1, \ldots, d$. It follows that
\[
|T_\varphi(\mathbf{a}, \mathbf{u}; N) - T_\varphi(\mathbf{a}, \mathbf{v}; N)| \ll \sum_{n=1}^{N} |a_n| \sum_{i=1}^{d} \zeta_j |\varphi_j(n)|
\leq \sum_{j=1}^{d} \zeta_j N^{\varepsilon_j+\varepsilon o(1)} \leq N^\alpha/2.
\]
The last estimates hold when $N$ is large enough, which finishes the proof. □

Lemma 2.5. Suppose that $\varphi \in \mathbb{Z}[T]^d$ satisfies (2.2). Let $\varepsilon > 0$ be sufficiently small, $\rho > 1$ and $N_i = i^\rho$, $i \in \mathbb{N}$. Let $N_i \leq K < L \leq N_{i+1}$ for some $i \in \mathbb{N}$. Suppose that
\[
|T_\varphi(\mathbf{a}, \mathbf{u}; L) - T_\varphi(\mathbf{a}, \mathbf{u}; K)| \geq N_i^\alpha
\]
for some $\mathbf{u} \in T_d$, then
\[
|T_\varphi(\mathbf{a}, \mathbf{v}; L) - T_\varphi(\mathbf{a}, \mathbf{v}; K)| \geq N_i^\alpha/2
\]
holds for any $\mathbf{v} \in R(\mathbf{u}, \zeta)$ provided that $i$ is large enough and
\[
0 < \zeta_j \leq N_i^{\alpha - \varepsilon_j - \varepsilon} (L - K)^{-1}, \quad 1 \leq j \leq d.
\]
Proof. The proof is similar to the proof of Lemma 2.4, we show a sketch here. We observe that
\[
T_\varphi(\mathbf{a}, \mathbf{u}; L) - T_\varphi(\mathbf{a}, \mathbf{u}; K) - T_\varphi(\mathbf{a}, \mathbf{v}; L) + T_\varphi(\mathbf{a}, \mathbf{v}; K)
= \sum_{n=K+1}^{L} a_n e\left(u_1 \varphi_1(n) + \ldots + u_d \varphi_d(n)\right)
- \sum_{n=K+1}^{L} a_n e\left(v_1 \varphi_1(n) + \ldots + v_d \varphi_d(n)\right).
\]
By the continuity of function $e(x)$ and the choice of $\zeta$ we obtain
\[
|T_\varphi(\mathbf{a}, \mathbf{u}; L) - T_\varphi(\mathbf{a}, \mathbf{u}; K) - T_\varphi(\mathbf{a}, \mathbf{v}; L) + T_\varphi(\mathbf{a}, \mathbf{v}; K)|
\leq \sum_{n=K+1}^{L} |a_n| \sum_{j=1}^{d} \zeta_j |\varphi_j(n)| \leq \sum_{j=1}^{d} \zeta_j N_i^{\varepsilon_j+\varepsilon o(1)} (L - K) \leq N_i^\alpha/2.
\]
provided that \( i \) is large enough, and by the triangle inequality we obtain the desired statement. \( \square \)

2.4. Distribution of large values of exponential sums. We adapt the arguments of [8, Lemma 2.2] to our setting.

Firstly we show the following useful box counting lemma. We note that any better bound of the exponent of \( N \) immediately yields an improvement of our results.

Suppose that \( \varphi \in \mathbb{Z}[T]^d \) satisfies (2.2).

Let \( 0 < \alpha < 1 \) and let \( \varepsilon \) be sufficiently small. For each \( j = 1, \ldots, d \) let

\[
\zeta_j = \frac{1}{\lceil Ne_j^{1+1+\varepsilon-\alpha} \rceil}.
\]

We divide \( T_d \) into

\[
U = \prod_{j=1}^{d} \zeta_j^{-1}
\]

boxes of the type

\[
[n_1 \zeta_1, (n_1 + 1) \zeta_1) \times \ldots \times [n_d \zeta_d, (n_d + 1) \zeta_d),
\]

where \( n_j = 1, \ldots, 1/\zeta_j \) for each \( j = 1, \ldots, d \). Let \( \mathcal{R} \) be the collection of these boxes, and

\[
\tilde{\mathcal{R}} = \{ \mathcal{R} \in \mathcal{R} : \exists u \in \mathcal{R} \text{ with } |T_\varphi(a, u; N)| \geq N^\alpha \}.
\]

Lemma 2.6. In the above notation, we have

\[
\#\tilde{\mathcal{R}} \leq UN^{s(d)(1-2\alpha)+o(1)}.
\]

Proof. Let \( \mathcal{R} \in \mathcal{R} \). By Lemma 2.4 if \( |T_\varphi(a, u; N)| \geq N^\alpha \) for some \( u \in \mathcal{R} \), then \( |T_\varphi(a, v; N)| \geq N^\alpha/2 \) for any \( v \in \mathcal{R} \). Combining this with Lemma 2.1 we have

\[
(N^\alpha/2)^{2s(d)} \#\tilde{\mathcal{R}} \prod_{j=1}^{d} \zeta_j \leq \int_{T_d} |T_\varphi(a, u; N)|^{2s(d)} \, du \leq N^{s(d)+o(1)},
\]

which yields the desired bound. \( \square \)

Note that the above bound of \( \#\tilde{\mathcal{R}} \) is nontrivial when \( 1/2 < \alpha < 1 \).

Corollary 2.7. Let \( 0 < \alpha < 1 \) and let \( \varepsilon > 0 \) be sufficiently small. Then

\[
\lambda(\{ x \in T_k : \exists y \in T_{d-k} \text{ with } |T_\varphi(a, x, y; N)| \geq N^\alpha \}) \leq N^{s(d-2s(d)+\sigma_k(\varphi)+1-\alpha+\varepsilon)(d-k)+o(1)}.
\]
Proof. Using above notation, and defining the set
\[ \mathcal{U} = \bigcup_{\mathcal{R} \in \mathcal{R}} \mathcal{R}. \]
Observe that
\[ \{ x \in T_k : \exists y \in T_{d-k} \text{ with } |T_\varphi(a, x, y; N)| \geq N^\alpha \} \subseteq \pi_{d,k}(\mathcal{U}). \]
Clearly we have
\[ \lambda(\pi_{d,k}(\mathcal{U})) \leq \#\mathcal{R} \prod_{j=1}^{k} \zeta_j. \]
By Lemma 2.6 and the choice (2.4) of \( \zeta \), we obtain the desired result. \( \square \)

We have the following bound on the amount of certain boxes which admit values for the difference \( |T_\varphi(a, u; L) - T_\varphi(a, u; K)| \).

**Lemma 2.8.** Let \( \varphi \) be as in (1.7) with
\[ \varphi_{jm}(T) = T^{e_{jm}}, \quad m = 1, \ldots, \ell. \]
Let \( \varepsilon > 0 \) be sufficiently small, \( \rho > 1 \) and \( N_i = i^\rho, \ i \in \mathbb{N} \). Let \( N_i \leq K < L \leq N_i + 1 \) for some \( i \in \mathbb{N} \). For each \( j = 1, \ldots, d \) let
\[ \eta_j = 1/\left[ N^{e_j + \varepsilon - \alpha}(L - K) \right]. \]
We divide the \( T_d \) into
\[ W = \prod_{j=1}^{d} \eta_j^{-1} \]
boxes in the same way as in the beginning of this subsection. Denote \( \mathcal{Q} \) the collection of these boxes. Then one has
\[ \#\{ \mathcal{R} \in \mathcal{Q} : \exists u \in \mathcal{R} \text{ with } |T_\varphi(a, u; L) - T_\varphi(a, u; K)| \geq N_i^\alpha \} \]
\[ \leq W(L - K)^{s(\ell)} N_i^{-2as(\ell) + o(1)}, \]
where \( s(\ell) \) is given by (2.1).

Applying Lemma 2.5 and Lemma 2.8, and similar arguments as in the proof of Corollary 2.7, we immediately obtain the following result. We omit the details here.

**Corollary 2.9.** Suppose that \( \varphi \in \mathbb{Z}[T]^d \) is as in (1.7). Let \( 0 < \alpha < 1 \) and let \( \varepsilon > 0 \) be sufficiently small. Let \( N_i \leq K < L \leq N_i + 1 \) for some \( i \in \mathbb{N} \). Using the same notation as in Lemma 2.8, we have
\[ \lambda(\{ x \in T_k : \exists y \in T_{d-k} \text{ with } |T_\varphi(a, x, y; L) - T_\varphi(a, x, y; K)| \geq N_i^\alpha \}) \]
\[ \leq N_i^{\sigma_k(\varphi)-2as(\ell)-(d-k)\alpha+\varepsilon(d-k)+o(1)}(L - K)^{s(\ell)+d-k}, \]
as \( i \to \infty \), where \( s(\ell) \) is given by (2.1).

3. Proofs of exponential sum bounds for coordinate-wise projections

3.1. Proof of Theorem 1.1. We fix some \( \alpha > 1/2 \) and \( \rho > 1 \) and set

\[
N_i = \lfloor i^\rho \rfloor, \quad i = 1, 2, \ldots.
\]

In particular, the Lagrange mean value theorem implies that

\[
N_{i+1} - N_i \leq 2 \rho i^{\rho - 1} \ll i^{\rho - 1}, \quad i = 1, 2, \ldots.
\]

We now consider the set

\[
B_i = \{ x \in T_k : \exists y \in T_{d-k} \text{ with } |T_\varphi(a, x, y; N_i)| \geq N_i^{\alpha} \}.
\]

By Corollary 2.7 we have

\[
\lambda(B_i) \leq N_i^{s(d) - 2\alpha s(d) + \sigma_k(\varphi) + (d-k)(1-\alpha) + \varepsilon(d-k) + o(1)}.
\]

We ask that the fixed \( \alpha \) and \( \rho \) satisfy the following condition

\[
\rho(s(d) - 2\alpha s(d) + \sigma_k(\varphi) + (d-k)(1-\alpha) + \varepsilon(d-k)) < -1.
\]

By choosing small enough \( \varepsilon \) it is sufficient to have the condition

\[
\rho(s(d) - 2\alpha s(d) + \sigma_k(\varphi) + (d-k)(1-\alpha)) < -1.
\]

Combining (3.5) with the Borel–Cantelli lemma, we obtain that

\[
\lambda\left(\bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} B_i\right) = 0.
\]

Since

\[
\{ x \in T_k : \exists y \in T_{d-k} \text{ with } |T_\varphi(a, x, y; N_i)| \geq N_i^{\alpha} \}
\]

for infinite many \( i \in \mathbb{N} \) \( \subseteq \bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} B_i \),

we conclude that for almost all \( x \in T_k \) there exists \( i_x \) such that for any \( i \geq i_x \) one has

\[
\sup_{y \in T_{d-k}} |T_\varphi(a, x, y; N_i)| \leq N_i^{\alpha}.
\]

We fix this \( x \) in the following arguments. For any \( N \geq N_{i_x} \) there exists \( i \) such that

\[
N_i \leq N \leq N_{i+1}.
\]

Clearly we have the following trivial upper bound,

\[
\sup_{y \in T_{d-k}} |T_\varphi(a, x, y; N) - T_\varphi(a, x, y; N_i)| \leq (N_{i+1} - N_i) N_i^{\alpha(1)}.
\]
We choose $\rho$ and $\alpha$ such that

$$\alpha > 1 - 1/\rho,$$

and then by (3.2) we have

$$(N_{i+1} - N_i)N_i^{\alpha(1)} = o(N_i^\alpha).$$

Applying (3.6) and (3.7) for all $y \in T_{d-k}$ we have

$$|T_\varphi(a, x, y; N)| \leq |T_\varphi(a, x, y; N) - T_\varphi(a, x, y; N_i)| + |T_\varphi(a, x, y; N_i)| \leq N_i^{\alpha+o(1)} \leq N^{\alpha+o(1)}.$$

Now we intend to put the conditions (3.5) and (3.8) together to obtain the smallest possible value for $\alpha$, which gives the desired bound. First note that the condition (3.5) can be written as

$$\alpha > \frac{s(d) + \sigma_k(\varphi) + d - k + 1/\rho}{2s(d) + d - k},$$

which is a monotonically increasing decrease function for the variable $\rho$. The condition $\sigma_k(\varphi) < d(d+1)/2$ gives

$$\frac{s(d) + \sigma_k(\varphi) + d - k}{2s(d) + d - k} < 1.$$

Clearly $1 - 1/\rho$ is a monotonically increasing function of the variable $\rho$, and it has the limit 1 when $\rho$ goes to infinity. Thus there exists an unique $\rho_0 > 1$ such that

$$\frac{s(d) + \sigma_k(\varphi) + d - k + 1/\rho_0}{2s(d) + d - k} = 1 - 1/\rho_0.$$

Thus it is sufficient to take any $0 < \alpha < 1$ such that

$$\alpha > 1 - 1/\rho_0.$$

An elementary calculation gives

$$1 - 1/\rho_0 = \frac{1}{2} + \frac{d - k + 2\sigma_k(\varphi) + 1}{2d^2 + 4d - 2k + 2},$$

which finishes the proof.

3.2. **Proof of Theorem 1.2.** We proceed as in the proof of Theorem 1.1 and choose the sequence $N_i$ as in (3.1). We also have the condition (3.5) for the parameters $\alpha$ and $\rho$. For each $i \in \mathbb{N}$ denote

$$L_i = N_{i+1} - N_i.$$ 

Note that we have

$$i^{\rho-1} \ll L_i \ll i^{\rho-1}.$$
Let $M$ be a large integer number to be determined later. We choose $\vartheta_i \in (0, 1)$ as the largest real number with
\begin{equation}
1/\vartheta_i \in \mathbb{N} \quad \text{and} \quad \vartheta_i^M L_i \leq N_i^\alpha.
\end{equation}

Let $\mathcal{I} = [a, b] \subseteq [N_i, N_{i+1}]$ with $a, b \in \mathbb{N}$ and denote $I = b - a$. We divide $\mathcal{I}$ into $1/\vartheta_i$ intervals with the equal length $\vartheta_i I$. For each $j = 1, \ldots, 1/\vartheta_i$ let
\begin{equation}
\mathcal{B}_{i, I, j} = \{x \in \mathbb{T}_k : \exists y \in \mathbb{T}_{d-k} \text{ with } |T_\varphi(a, x, y; a + j\vartheta_i I) - T_\varphi(a, x, y; a)| \geq N_i^\alpha\},
\end{equation}
and
\[ \mathcal{B}_{i, I} = \bigcup_{j=1}^{1/\vartheta_i} \mathcal{B}_{i, I, j}. \]

We remark that $a + j\vartheta_i I$ may not be an integer for some $j = 1, \ldots, 1/\vartheta_i$. For this situation we choose one of the nearby integers. Under this modification we still get $1/\vartheta_i$ intervals and each of them has the length about $\vartheta_i I$. However this does not effect our results. In the following and through out the project we use this modification, furthermore we may assume that $a + j\vartheta_i I$ is integer for each $j = 1, \ldots, 1/\vartheta_i$.

Applying Corollary 2.9 we obtain
\begin{equation}
\lambda(\mathcal{B}_{i, I}) \leq \sum_{j=1}^{1/\vartheta_i} N_i^{\sigma_k(\varphi) - 2s(\ell)a - (d-k)\alpha + \varepsilon(d-k) + o(1)} |j\vartheta_i I|^{s(\ell) + d - k} \leq N_i^{\sigma_k(\varphi) - 2s(\ell)a - (d-k)\alpha + \varepsilon(d-k) + o(1)} f s(\ell) + d - k \vartheta_i^{d-1}.
\end{equation}

Now we divide interval $[N_i, N_{i+1}]$ into $1/\vartheta_i$ intervals with the equal length $\vartheta_i L_i$. Let $\mathcal{D}_1$ be the collection of these $1/\vartheta_i$ intervals. Precisely
\[ \mathcal{D}_1 = \{[N_i + j\vartheta_i L_i, N_i + j\vartheta_i L_i + \vartheta_i L_i] : 0 \leq j < 1/\vartheta_i\}. \]

For each interval of $\mathcal{D}_1$ we repeat this process again, and let $\mathcal{D}_2$ be the collection of all these intervals. Now $\mathcal{D}_2$ has $(1/\vartheta_i)^2$ intervals of equal lengths $\vartheta_i^2 L_i$. We continue this process until $M$ steps.

For $1 \leq m \leq M$, we write
\[ \mathcal{D}_m = \{[N_i + j\vartheta_i^m L_i, N_i + j\vartheta_i^m L_i + \vartheta_i^m L_i] : 0 \leq j < (1/\vartheta_i)^m\}. \]
We also write $\mathcal{D}_0 = \{[N_i, N_{i+1}]\}$ for convenience. Note that each interval of $\mathcal{D}_m$ has length $\vartheta_i^m L_i$.

For each $1 \leq m \leq M - 1$ we define
\[ \mathcal{B}_{i}(m) = \bigcup_{\mathcal{I} \in \mathcal{D}_{m-1}} \mathcal{B}_{i, I}. \]
By (3.11) we have

\[
\lambda(B_i(m)) \leq \#D_{m-1} N_i^{\sigma_k(\varphi) - 2s(\ell)\alpha - (d-k)\alpha + \varepsilon(d-k) + o(1)} \times \left( \vartheta_i^{m-1} L_i \right) s(\ell) + d-k \vartheta_i^{-1}
\]

(3.12)

\[
\leq N_i^{\sigma_k(\varphi) - 2s(\ell)\alpha - (d-k)\alpha + \varepsilon(d-k) + o(1)} \times L_i^{s(\ell) + d-k} \vartheta_i^{s(\ell) + d-k}(m-1) - m
\]

as \(i \to \infty\).

Observe that for each \(m \geq 1\) we have

\[
\vartheta_i^{s(\ell) + d-k}(m-1) - m \leq 1/\vartheta_i.
\]

Thus, if

(3.13) \[\sum_{i=1}^{\infty} N_i^{\sigma_k(\varphi) - 2s(\ell)\alpha - (d-k)\alpha + \varepsilon(d-k) + o(1)} L_i^{s(\ell) + d-k} \vartheta_i^{-1} < \infty,\]

then for each \(1 \leq m \leq M - 1\) we obtain

\[
\sum_{i=1}^{\infty} \lambda(B_i(m)) < \infty,
\]

and hence, by the Borel–Cantelli lemma, one has

\[
\lambda \left( \bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} B_i(m) \right) = 0.
\]

Let \(M = \lfloor \rho/\varepsilon \rfloor\). Then by (3.9) we have the following trivial inequality

(3.14) \[1/\vartheta_i \ll \left( \frac{L_i}{N_i^\alpha} \right)^{1/M} \ll N_i^{1/M} \ll i^\varepsilon.\]

Combining this with (3.12) for \(m = 1\) and taking into account that \(\varepsilon\) can be chosen arbitrary small, we see that for the purpose of convergence (3.13) it is sufficient to take

(3.15) \[\rho(\sigma_k(\varphi) - 2s(\ell)\alpha - (d-k)\alpha) + (\rho - 1)(s(\ell) + d-k) < -1.\]

The above arguments imply that if the parameters \(\alpha \in (0, 1), \rho > 1\) and \(M \in \mathbb{N}\) satisfy the conditions (3.5) and (3.15), then we have

\[
\lambda \left( \bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} B_i \right) = 0,
\]
and for each $m = 1, \ldots, M$

$$
\lambda \left( \bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} B_i(m) \right) = 0.
$$

It follows that for almost all $x \in T_k$ there exists $i_x$ such that for all $i \geq i_x$ and $1 \leq m \leq M$ we have $x \notin B_i$ and $x \notin B_i(m)$. If $x \notin B_i$ then

$$
\sup_{y \in T_{d-k}} |T_{\varphi}(a, x, y; N_i)| \leq N_i^\alpha.
$$

Combining this with $x \notin B_i(1)$, we obtain that for any $j = 1, \ldots, 1/\vartheta_i$ and any $y \in T_{d-k}$

$$
|T_{\varphi}(a, x, y; N_i + j \vartheta_i L_i)| \leq |T_{\varphi}(a, x, y; N_i + j \vartheta_i L_i) - T_{\varphi}(a, x, y; N_i)|
$$

$$
+ |T_{\varphi}(a, x, y; N_i)|
$$

$$
\ll N_i^\alpha.
$$

Note that the implied constant does not depend on $y$. By using this process, finally for all $y \in T_{d-k}$ we obtain

(3.16)

$$
T_{\varphi}(a, x, y; N_i + j \vartheta_i^M L_i) \ll N_i^\alpha.
$$

Now for any $N \geq N_{i_x}$ there exists $i \geq i_x$ and $1 \leq j \leq (1/\vartheta_i)^M$ such that

$$
N_i + j \vartheta_i^M L_i \leq N \leq N_i + (j + 1) \vartheta_i^M L_i.
$$

Combining (3.16) with (3.9) and the trivial bound $|e(x)| \leq 1$ we obtain

$$
|T_{\varphi}(a, x, y; N)| \leq |T_{\varphi}(a, x, y; N) - T_{\varphi}(a, x, y; N_i + j \vartheta_i^M L_i)|
$$

$$
+ |T_{\varphi}(a, x, y; N_i)|
$$

$$
\leq N_i^{\alpha + o(1)}.
$$

The conditions (3.5) and (3.9) can be written as

$$
\alpha > \frac{\sigma_k(\varphi) + s(d) + d - k + 1/\varrho}{2s(d) + d - k}
$$

and

$$
\alpha > \frac{\sigma_k(\varphi) + (1 - 1/\varrho)(s(\ell) + d - k) + 1/\varrho}{2s(\ell) + d - k}.
$$

Observe that by taking $\varrho$ large enough it is sufficient to take any $\alpha$ such that

$$
\alpha > \frac{s(\ell) + \sigma_k(\varphi) + d - k}{2s(\ell) + d - k}.
$$

Elementary calculations give

$$
\frac{s(\ell) + \sigma_k(\varphi) + d - k}{2s(\ell) + d - k} = \frac{1}{2} + \frac{d - k + 2\sigma_k(\varphi)}{2\ell^2 + 2\ell + 2d - 2k},
$$
which finishes the proof.

4. Proofs of exponential sum bounds for arbitrary orthogonal projections

4.1. Orthogonal projections of boxes. We start with the following general result which is perhaps well known.

**Lemma 4.1.** Let $\mathcal{R} \subseteq \mathbb{R}^d$ be a box with side lengths $h_1 \geq \ldots \geq h_d$. Then for all $\mathcal{V} \in \mathcal{G}(d, k)$ we have

$$\lambda(\pi_{\mathcal{V}}(\mathcal{R})) \ll \prod_{i=1}^{k} h_k,$$

where the implied constant depends on $d$ and $k$ only.

**Proof.** The idea is to cover a box by balls, and use that their orthogonal projections do not depend on the choice of $\mathcal{V} \in \mathcal{G}(d, k)$.

More precisely, without lose generality we can assume that $\mathcal{R} = [0, h_1) \times \ldots \times [0, h_d)$.

Let $\mathcal{R}_k = [0, h_1) \times \ldots \times [0, h_k) \times \{0\} \times \ldots \times \{0\}$ be a subset of $\mathcal{R}$. Since for any $x \in \mathcal{R}$ there exists $y \in \mathcal{R}_k$ such that

$$\|x - y\| \leq \left( \sum_{j=k+1}^{d} h_j^2 \right)^{1/2} \leq dh_{k+1},$$

we obtain

(4.1) $\mathcal{R} \subseteq \mathcal{R}_k + \mathcal{B}(0, dh_{k+1}),$

where $\mathcal{B}(0, dh_{k+1})$ is the ball of $\mathbb{R}^d$ with center $0$ and radius $dh_{k+1}$ and for $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$, as usual, we define:

$$\mathcal{A} + \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}.$$

Now we intend to cover $\mathcal{R}_k$ by a family of balls of $\mathbb{R}^d$ such that each of these balls has radius roughly $h_{k+1}$.

For each $1 \leq j \leq k$ we have

$$[0, h_j) \subseteq \bigcup_{q=0}^{Q_j} \mathcal{I}_{j,q},$$

where $\mathcal{I}_{j,q} = [qh_{k+1}, (q + 1)h_{k+1})$ and

(4.2) $Q_j = \lfloor h_j/h_{k+1} \rfloor.$
Then
\[ R_k \subseteq \bigcup_{0 \leq q_1 \leq Q_1, \ldots, 0 \leq q_k \leq Q_k} I_{1,q_1} \times \cdots \times I_{k,q_k} \times \{0\} \times \cdots \times \{0\}. \]

Observe that for each choice on integers \(q_1, \ldots, q_k\) with
\[ 0 \leq q_1 \leq Q_1, \ldots, 0 \leq q_k \leq Q_k \]
there exists a ball \(B_{q_1, \ldots, q_k}\) of \(\mathbb{R}^d\) of radius \(dh_{k+1}\) such that
\[ I_{1,q_1} \times \cdots \times I_{k,q_k} \times \{0\} \times \cdots \times \{0\} \subseteq B_{q_1, \ldots, q_k}. \]

Denote the collection of these balls by
\[ \mathcal{B} = \{B_{q_1, \ldots, q_k} : 0 \leq q_1 \leq Q_1, \ldots, 0 \leq q_k \leq Q_k\}. \]

It follows that
\[
(4.3) \quad R_k \subseteq \bigcup_{B \in \mathcal{B}} B.
\]

Since each ball \(B \in \mathcal{B}\) has radius \(dh_{k+1}\), we have
\[ B + B(0, dh_{k+1}) \subseteq 2B, \]
where \(2B(x, r) = B(x, 2r)\). Together with (4.1) and (4.3) we obtain
\[ R \subseteq R_k + B(0, dh_{k+1}) \subseteq \bigcup_{B \in \mathcal{B}} 2B. \]

It follows that for any \(\mathcal{V} \in \mathcal{G}(d, k)\) we have
\[ \pi_{\mathcal{V}}(R) \subseteq \bigcup_{B \in \mathcal{B}} \pi_{\mathcal{V}}(2B). \]

Since for each ball \(B \in \mathcal{B}\) the projection \(\pi_{\mathcal{V}}(2B)\) is a ball of the \(k\)-dimensional subspace \(\mathcal{V}\) with radius \(2dr_{k+1}\), one has
\[ \lambda(\pi_{\mathcal{V}}(2B)) \ll h_{k+1}^k. \]

Combining this with (4.2) we obtain
\[ \lambda(\pi_{\mathcal{V}}(R)) \ll h_{k+1}^k \prod_{j=1}^k Q_j \ll \prod_{i=1}^k h_i \]
which gives the result. \(\square\)
4.2. Orthogonal projections and large values of exponential sums. We now provide some basic tools for the proofs of Theorems 1.5 and 1.6. We remark that once these tools were established then we obtain the desired results by applying similar arguments as in the proofs of Theorems 1.1 and 1.2. We omit these similar arguments here.

We use the same notation as in Section 2.4, including the choice (2.4). For $\mathcal{R} \in \mathfrak{R}$ with side lengths $\zeta_1, \ldots, \zeta_d$ we denote them as $\tilde{\zeta}_1 \geq \ldots \geq \tilde{\zeta}_d$.

For $j = 1, \ldots, d$ by (1.8) we obtain

\[ \tilde{\zeta}_j = 1 / \left\lceil N r_j + 1 + \varepsilon - \alpha \right\rceil. \]

Applying Lemma 2.6 and Lemma 4.1 we obtain the following result.

**Corollary 4.2.** Let $0 < \alpha < 1$ and let $\varepsilon > 0$ be sufficiently small. For any $\mathcal{V} \in \mathcal{G}(d, k)$ we have

\[ \lambda(\{ x \in \mathcal{V} : \exists u \in \mathbb{T}_d \text{ with } \pi_{\mathcal{V}}(u) = x \& |T_{\varphi}(a, u; N)| \geq N^\alpha \}) \leq N^{s(d) - 2\alpha s(d) + \tilde{\sigma}_k(\varphi) + (d - k)(1 - \alpha) + \varepsilon(d - k) + o(1)}, \]

as $N \to \infty$. Here $\tilde{\sigma}_k(\varphi)$ is given by (1.9).

**Proof.** Using the same notation as in Section 2.4. We also define the set

\[ \mathcal{U} = \bigcup_{\mathcal{R} \in \mathfrak{R}} \mathcal{R}. \]

Observe that

\[ \{ x \in \mathcal{V} : \exists u \in \mathbb{T}_d \text{ with } \pi_{\mathcal{V}}(u) = x \& |T_{\varphi}(a, u; N)| \geq N^\alpha \} \subseteq \pi_{\mathcal{V}}(\mathcal{U}). \]

Combining this with Lemma 2.6, Lemma 4.1 and (4.4) we obtain

\[ \lambda(\pi_{\mathcal{V}}(\mathcal{U})) \leq \#\mathfrak{R} \prod_{i=1}^{k} \tilde{\zeta}_i \leq N^{s(d) - 2\alpha s(d) + o(1)} \prod_{j=k+1}^d \tilde{\zeta}_j^{-1} \]

\[ \leq N^{s(d) - 2\alpha s(d) + o(1)} \prod_{j=k+1}^d N^{r_j + 1 + \varepsilon - \alpha}. \]

By the definition of $\tilde{\sigma}_k(\varphi)$ at (1.9) we obtain the desired bound. \qed

Applying Lemma 2.8 and Lemma 4.1 we have the following result. We use the same notation as in Lemma 2.8.
Corollary 4.3. Let $0 < \alpha < 1$ and $\varepsilon > 0$ be sufficiently small. Let $N_i \leq K < L \leq N_{i+1}$ for some $i \in \mathbb{N}$. If $\varphi$ is as in (1.7), then one has

$$
\lambda\{x \in V : \exists u \in T_d \text{ with } \pi_V(u) = x \&
|T_\varphi(a, u; L) - T_\varphi(a, u; K)| \geq N_i^\alpha\}\leq N_i^\tilde{\alpha}(\varphi) - 2s_\alpha(a - d - k) + o(1)(L - K)^{s(\ell) + d - k}.
$$

4.3. Concluding the proofs. As we have claimed, Theorems 1.5 and 1.6 now follow by applying Corollaries 4.2 and 4.3 and using similar arguments as in the proofs of Theorems 1.1 and 1.2, respectively.

5. Proof of discrepancy bounds

5.1. Preliminaries. We start with recalling the classical Erdős–Turán inequality (see, for instance, [3, Theorem 1.21]).

Lemma 5.1. Let $x_n$, $n \in \mathbb{N}$, be a sequence in $[0, 1)$. Then for any $H \in \mathbb{N}$ for the discrepancy $D_N$ given by (1.10), we have

$$
D_N \leq 3\left(\frac{N}{H + 1} + \sum_{h=1}^{H} \frac{1}{h} \left|\sum_{n=1}^{N} e(x_n h)\right|\right).
$$

We also use the following trivial property of the Lebesgue measure, see [8, Section 3] for a short proof.

Lemma 5.2. Let $A \subseteq T_d$ and $h \in \mathbb{N}$, then

$$
\lambda\{x \in T_d : (hx \pmod{1}) \in A\} = \lambda(A).
$$

5.2. Proof of Theorem 1.10. We note that in what follows, the vector of polynomials $\varphi$ is as in (1.2). We proceed as in the proof of Theorem 1.1 and choose the sequence $N_i$ as in (3.1).

For each $N$ there exits $i \in \mathbb{N}$ such that

$$
N_i \leq N < N_{i+1},
$$

and we denote $H(N) = H_i = \lfloor N_i^{\eta}\rfloor$ for some $\eta > 0$ to be chosen later. For each $h = 1, \ldots, H_i$ let

$$
B_{i,h} = \{x \in T_k : \exists y \in T_{d-k} \text{ with } |T_\varphi(h x, h y; N_i)| \geq N_i^\alpha\},
$$

and

$$
\tilde{B}_i = \bigcup_{h=1}^{H_i} B_{i,h}.
$$

Observe that

$$
B_{i,h} = \{x \in T_k : (h x \pmod{1}) \in B_i\},
$$
where the notation $\mathcal{B}_i$ is given by (3.3). By Lemma (5.2) and (3.4) we conclude that

$$\lambda(\mathcal{B}_i) \leq H_i N_i^{s(d) - 2\alpha s(d) + \sigma_k(\varphi) + (d-k)(1-\alpha) + \varepsilon(d-k) + o(1)}.$$ 

We ask that the fixed $\alpha, \rho$ and $\eta$ satisfy the following condition

$$\rho \eta + \rho (s(d) - 2\alpha s(d) + \sigma_k(\varphi) + (d-k)(1-\alpha)) < -1. \tag{5.1}$$

Combining this with the Borel–Cantelli lemma, and choosing a small enough $\varepsilon$, we obtain that

$$\lambda(\bigcap_{q=1}^{\infty} \bigcup_{i=q}^{\infty} \tilde{B}_i) = 0.$$ 

It follows that for almost all $x \in T_k$ there exists $i_x$ such that for any $i \geq i_x$ and any $h = 1, \ldots, H_i$, one has

$$\sup_{y \in T_{d-k}} \left| T_{\varphi}(h, h x, h y; a_i + j \vartheta_i) - T_{\varphi}(h, h y; a) \right| \leq N_i^\alpha.$$ 

We are now going to adapt the iterated construction in the proof of Theorem 1.2 to contain the new parameter $H_i$. Let $\vartheta_i, L_i$ and $M$ be the same as in the proof of Theorem 1.2.

Let $\mathcal{I} = [a, b] \subseteq [N_i, N_{i+1}]$. For each $j = 1, \ldots, 1/\vartheta_i$ and $h = 1, \ldots, H_i$ let

$$\mathcal{B}_{i, \mathcal{I}, j, h} = \{x \in T_k : \exists y \in T_{d-k} \text{ with} \}

|T_{\varphi}(h x, h y; a + j \vartheta_i) - T_{\varphi}(h x, h y; a)| \geq N_i^\alpha\},$$

and

$$\tilde{B}_{i, \mathcal{I}} = \bigcup_{h=1}^{H_i} \bigcup_{j=1}^{1/\vartheta_i} \mathcal{B}_{i, \mathcal{I}, j, h}.$$ 

Using similar arguments as the above, we obtain

$$\lambda(\tilde{B}_{i, \mathcal{I}, j, h}) = \lambda(\mathcal{B}_{i, \mathcal{I}, j}),$$

where $\mathcal{B}_{i, \mathcal{I}, j}$ is as in (3.10). Combining this with (3.11) for the case $\ell = d$, we have

$$\lambda(\tilde{B}_{i, \mathcal{I}}) \leq H_i N_i^{s(d) - 2s(d)\alpha - (d-k)\alpha + \varepsilon(d-k) - o(1)} I^s(d) + d-k \vartheta_i^{-1} \tag{5.2}.$$ 

For each $m = 1, \ldots, M$ we define

$$\tilde{B}_i(m) = \bigcup_{\mathcal{I} \in \mathcal{D}_m} \tilde{B}_{i, \mathcal{I}}.$$
Note that each \( I \in D_m \) has length \( \vartheta_i^m L_i \). By (5.2) we have
\[
\lambda(\tilde{B}_i(m)) \leq H_i N_i^\sigma_k(\varphi) - 2s(d)\alpha - (d - k)\alpha + \varepsilon(d - k) + o(1)
\times L_i^{s(d) + d - k} \vartheta_i^{s(d) + d - k}(m - 1) - m
\]
as \( i \to \infty \). Applying the same arguments as in the proof of Theorem 1.2 we conclude that the convergence of the series
\[
\sum_{i=1}^{\infty} H_i N_i^\sigma_k(\varphi) - 2s(d)\alpha - (d - k)\alpha + \varepsilon(d - k) + o(1) L_i^{s(d) + d - k} \vartheta_i^{s(d) + d - k}(m - 1) - m < \infty,
\]
implies that for each \( m = 1, \ldots, M - 1 \) one has
\[
\sum_{i=1}^{\infty} \lambda(\tilde{B}_i(m)) < \infty.
\]
Recalling the choice of \( H_i = \lfloor N_i^\eta \rfloor \) and the upper bound of \( \theta_i^{-1} \) which is given by (3.14), for the purpose of convergence (5.3) it is sufficient to take
\[
\rho \eta + \rho(\sigma_k(\varphi)) - 2s(d)\alpha - (d - k)\alpha + (\rho - 1)(s(d) + d - k) < -1.
\]
Thus putting all together we conclude that if \( \alpha, \rho, \eta \) satisfy the conditions (5.1) and (5.4) then for almost all \( x \in T_k \) there exists a \( i_x \) such that for any \( N \geq N_i \), we have
\[
\sup_{h=1, \ldots, H(N)} \sup_{y \in T_{d-k}} |T_{\varphi}(hx, hy; N)| \ll N_i^\alpha.
\]
In the following we fix this \( x \) and \( y \). Applying Lemma 5.1 for \( N_i \leq N < N_{i+1} \) and \( H = H_i \) we conclude that
\[
D_d(x, y; N) \ll N_i^{1-\eta} + N_i^\alpha \log H_i.
\]
Let \( \eta = 1 - \alpha \). The conditions (5.1) and (5.4) can be written as
\[
\alpha > \frac{\sigma_k(\varphi) + s(d) + d - k + 1 + 1/\rho}{2s(d) + d - k + 1}
\]
and
\[
\alpha > \frac{\sigma_k(\varphi) + (1 - 1/\rho)(s(d) + d - k) + 1 + 1/\rho}{2s(d) + d - k + 1}.
\]
Thus by taking \( \rho \) large enough it is sufficient to take any \( \alpha \) such that
\[
\alpha > \frac{\sigma_k(\varphi) + s(d) + d - k + 1}{2s(d) + d - k + 1}.
\]
Elementary calculations give
\[
\frac{\sigma_k(\varphi) + s(d) + d - k + 1}{2s(d) + d - k + 1} = \frac{1}{2} + \frac{d - k + 2\sigma_k(\varphi) + 1}{2d^2 + 4d - 2k + 2},
\]
which finishes the proof.

6. Comments

Similarly to our notation for the Weyl sums \( S_d(u; N) \), for \( u \in T_d \) and the sequence
\[ u_1 n + \ldots + u_d n^d, \quad n \in \mathbb{N}, \]
we denote by \( D_d(u; N) \) the corresponding discrepancy (we always take the sequence modulo one that is, take fractional parts, when we talk about the discrepancy). By taking \( k = d \) in Theorem 1.10 (which is an admissible choice) and using that in this case \( \sigma_d(\varphi) = 0 \), we obtain
that for almost all \( u \in T_d \) one has
\[ D_d(u; N) \leq N^{1/2 + 1/(2d^2 + 2d + 2) + o(1)}, \quad N \to \infty. \]
However, for this special case \( k = d \), combining [5, Theorem 5.13] with some additional arguments, one can show that for almost all \( u \in T_d \) with \( d \geq 2 \) we have the following stronger bound,
\[
D_d(u; N) \leq N^{1/2} (\log N)^{3/2 + o(1)}, \quad N \to \infty.
\]
which we conjecture is the best possible.

This is based on [5, Theorem 5.13] (see below Proposition 6.2) and the following general statement which is perhaps well-known but the authors have not been able to find it in the literature.

**Proposition 6.1.** Let \( \mathcal{U} \subseteq T_d \) be a set of positive Lebesgue measure \( \lambda(\mathcal{U}) > 0 \). Then there is a vector \( \mathbf{v}_0 \in T_d \) and a set real numbers \( \mathcal{W} \subseteq [0, \sqrt{d}] \) of positive Lebesgue measure \( \lambda(\mathcal{W}) > 0 \), such that for every \( w \in \mathcal{W} \) we have \( w\mathbf{v}_0 \in \mathcal{U} \).

**Proof.** Let \( \chi_\mathcal{U} \) be the characteristic function of \( \mathcal{U} \), then clearly we have
\[
\lambda(\mathcal{U}) = \int_{T_d} \chi_\mathcal{U}(u) \, du.
\]
Applying the polar coordinates [4, Theorem 3.12] to the function \( \chi_\mathcal{U} \), we obtain
\[
\int_{T_d} \chi_\mathcal{U}(u) \, du = \int_0^{\sqrt{d}} \left( \int_{\{u : \|u\| = r\}} \chi_\mathcal{U}(u) d\mathcal{H}^{d-1}(u) \right) dr,
\]
where $H^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure which is given by [4, Chapter 2]. By taking $u = rv$ for some $v \in S^{d-1}$ in the second term of (6.3) we obtain
\[
\int_{\{u: \|u\|=r\}} \chi_U(u) dH^{d-1}(u) = \int_{S^{d-1}} \chi_U(rv) r^{d-1} dH^{d-1}(v),
\]
where $S^{d-1} \subseteq \mathbb{R}^d$ is the unit sphere centred at the origin. Combining this with (6.2) and (6.3) and applying Fubini’s theorem, we arrive to
\[
\lambda(U) = \int_{S^{d-1}} \left( \int_0^{\sqrt{d}} \chi_U(rv) r^{d-1} dr \right) dH^{d-1}(v).
\]
Since $\lambda(U) > 0$, we conclude that there exists a vector
\[
v = (v_1, \ldots, v_d) \in S^{d-1}
\]
and a set of $r \in [0, \sqrt{d}]$ of positive Lebesgue measure such that $rv \in U$, which gives the desired result. \qed

We formulate [5, Theorem 5.13] in the following.

**Proposition 6.2.** Let $a_n, n \in \mathbb{N}$, be a sequence increasing sequence of real numbers such that $a_{n+1} - a_n \geq \delta > 0$ and let $\varepsilon > 0$. Then for almost all $w \in \mathbb{R}$ we have
\[
D(wa_n; N) \ll N^{1/2} (\log N)^{3/2+\varepsilon},
\]
where $D(wa_n; N)$ means the discrepancy of the sequence $wa_n \pmod{1}$, $n = 1, \ldots, N$.

Let us now fix some $\varepsilon > 0$ and denote by $U \subseteq T_d$ the set of $u \in T_d$, for which
\[
(6.4) \quad D_d(u; N) \geq N^{1/2} (\log N)^{3/2+\varepsilon},
\]
for infinitely many $N \in \mathbb{N}$. Assume that $\lambda(U) > 0$. By Proposition 6.1 there exists a vector
\[
v = (v_1, \ldots, v_d) \in T_d
\]
and a set of $w \in [0, \sqrt{d}]$ of positive Lebesgue measure such that we have (6.4) with $u = wv$ and for infinitely many $N \in \mathbb{N}$. On the other hand applying Proposition 6.2 to the sequence $v_1n + \ldots + v_dn^d$, $n \in \mathbb{N}$ we obtain that for almost all $w \in \mathbb{R}$ one has
\[
D_d(wv; N) \ll N^{1/2} (\log N)^{3/2+\varepsilon}.
\]
This now gives the contradiction and therefore (6.1) holds.
Finally, we note that in the case $d = 1$ the celebrated result of Khintchine, see [3, Theorem 1.72], implies that for almost all $x \in [0,1)$ one has

$$D_1(x; N) \leq N^{o(1)}, \quad N \to \infty.$$ 

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