Solitons of the Einstein-Yang-Mills Theory

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Abstract

Subject of this talk is an overview of results on self-gravitating solitons of the classical Yang-Mills-Higgs theory. One finds essentially two classes of solitons, one of them corresponding to the magnetic monopoles the other one to the sphalerons of flat space. The coupling to the gravitational field leads to new features absent in flat space. These are the gravitational instability of these solitons at the Planck scale and the existence of black holes with ‘non-abelin hair” in addition to the regular solutions.

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1 Introduction

My talk is an overview of results on self-gravitating solitons of the classical Yang-Mills-Higgs (YMH) theory. It is based on analytical and numerical results obtained in collaboration with P. Breitenlohner and P. Forgács [1]. Many other people, who have contributed in establishing our present understanding of this subject will be mentioned in due course.

Let me start by specifying, what precisely I understand under solitons, since this concept is used with various different meanings in the literature. I will adhere to a rather liberal use of this concept, denoting by it any particle-like solution of a non-linear field theory. Particle-like solutions are localized, time-independent solutions of finite total energy (mass) with some stability against perturbations. A typical, maybe the best known example of relativistic solitons are the non-abelian 't Hooft-Polyakov monopoles. As a genuine non-linear structure they play an important role in the non-perturbative aspects of the YMH theory. For a long time it was believed, that also Einstein's theory of General Relativity had such smooth solitonic solutions - called geons. However, as was shown by Lichnerowicz [2], neither Einstein’s theory in vacuum nor the Einstein-Maxwell theory give rise to regular particle-like solutions. Nevertheless there are alternative candidates - the Schwarzschild resp. Reissner-Nordstrøm (RN) black holes. Although they suffer from a physical singularity at their center, this singularity is hidden from the observer behind an event horizon. Black holes have finite mass and behave in many ways like genuine particles. In fact, they may well be considered as “renormalized” point particles, dressed with their gravitational self-field [3].

Regular particle-like solutions were found for models involving gravitating complex scalar fields (“Boson stars”), but they have rather the properties of exotic cosmic objects than those of particles [4]. Later Lichnerowicz’s No-Go-Theorem could be generalized from the Einstein-Maxwell theory to Kaluza-Klein models and supergravities [5], leading to the belief that no smooth solitons can be found for self-gravitating gauge theories. It is clear that this result cannot apply to self-gravitating versions of flat-space solitons like ’t Hooft-Polyakov monopoles, first studied by van Nieuwenhuizen, Wilkinson and Perry [6]. However for the YM theory without a Higgs field, which has no solitons in flat space, it came as a surprise to many, when Bartnik and McKinnon (BM) [7] discovered a family of regular, localized (finite mass) solutions of the gravitating theory. Although their discovery was based only on “numerical evidence” a rigorous existence proof was found subsequently (with due delay!) [8]. When it was found that they are unstable these “particles” lost some of their glamour. It was realized that they had much in common with the “sphalerons” of the YMH theory [9]. The latter are solutions with a Higgs doublet (for the gauge group SU(2)) in contrast to the monopole obtained with a triplet. Varying the strength of the gravitational coupling of the gravitating version of the flat-space sphaleron one obtains a one-parameter family of solutions interpolating between the flat solution and the (first) BM solution [10]. This strongly suggests to interprete the BM solutions as gravitationally bound counter-parts of the Higgs-bound flat sphaleron. Indeed, still other sphalerons can be obtained replacing the gravitational field by a dilaton [11].

Besides the gravitating sphalerons I will also discuss the effects of the gravitational self-interaction on the non-abelian monopoles. As to be expected they develop a gravitational instability for sufficiently strong gravitational self-force. Contrary to naive expectation (and to claims in the literature) the static monopoles do however not simply turn into black holes as the strength of the gravitational coupling is increased to its critical value.
Figure 1: a) PS-monopole, b) DHN-sphaleron, both for vanishing (solid) and infinite (dashed) Higgs mass

For values close to the critical strength of the gravitational coupling the space-like hypersurfaces of these solutions develop two distinct regions separated by a long throat. The inner part tends to a kind of ‘cosmological” solution representing a closed asymptotically Robinson-Bertotti universe, whereas the outer one becomes the exterior part of the extremal RN black hole.

As already mentioned before there is a second type of soliton in General Relativity - the black hole. Taking the YM resp. YMH model as the matter part one finds a rich spectrum of static black hole solutions. This is to be contrasted with Einstein’s theory in vacuum resp. with the Einstein-Maxwell theory, where according to a theorem of Israel [12] the Schwarzschild resp. RN solution are the only static black holes. In the EYM theory one finds not only the embedding of the abelian RN black hole, but in addition there are genuinely non-abelian (“coloured”) black holes. Their co-existence gives rise to an interesting violation of the “No-Hair-Conjecture”, since they carry the same (magnetic) charge [13].

2 Yang-Mills-Higgs in Flat Space

Before I come to the effects of gravity I would like to give a short reminder of “particle like” solutions of the YMH system in flat space. For simplicity I restrict myself to the gauge group $SU(2)$ from now on. There are two different cases to be considered, leading to rather different types of solutions. The Higgs field can be either in a triplet or in a doublet representation. In either case the action is

$$S = -\frac{1}{4\pi} \int d^4x \left[ \frac{1}{4g^2} \text{Tr} F^2 + \frac{1}{2} |D\phi|^2 + \frac{\lambda}{8} (|\phi|^2 - v^2)^2 \right]. \quad (1)$$

It is important to notice that the expression for the action contains two mass scales, the mass $M_W = g v$ of the YM field and the mass $M_H = \sqrt{\lambda} v$ of the Higgs field. From these we may form the dimensionless ratio $\beta = M_H/M_W$.

The particle like (static, spherically symmetric) solutions in the case of a Higgs triplet are the ’t Hooft-Polyakov magnetic monopoles. They are obtained with the ansatz

$$W_0^a = 0 \quad W_i^a = \epsilon_{ikl} x^l (W(r) - 1) \quad \phi^a = \frac{x^a}{r} H(r). \quad (2)$$
Inserting this ansatz in the action (1) one gets
\[ S = -\int dr \left[ \frac{1}{g^2}(W'^2 + \frac{(1 - W^2)^2}{2r^2}) + \frac{r^2}{2} H'^2 + \frac{\lambda r^2}{8}(H^2 - v^2)^2 + W^2 H^2 \right]. \] (3)

In order to obtain finite total energy the Higgs field has to tend to its vacuum value \( v \) for \( r \to \infty \), forcing in turn \( W \to 0 \) (l.h.s. of Fig. 1).

For large values of \( M_H \) and hence of \( \beta \) the function \( H(r) \) rises quickly to its asymptotic value \( v \). In the limit \( \beta \to \infty \) the Higgs field may be replaced by \( v \) for all \( r > 0 \) and its only role is to give a mass to the YM field. The total energy of the solution stays finite in this limit. In fact, it only varies by a factor \( \approx 1.8 \) as \( \beta \) varies from 0 to \( \infty \).

There is a second possibility to let \( \beta \) go to infinity, holding \( M_H \) fixed, but letting \( M_W \to 0 \) (and hence \( W \equiv 1 \) at the relevant length scale \( 1/M_H \)). This way one obtains the “global monopole” playing the role of a texture in cosmological considerations [14].

Due to the topological character of the magnetic charge, related to the asymptotic vacuum structure of configurations with finite energy, the monopole is a stable solution.

The second possibility is a Higgs field in the doublet representation. The relevant ansatz of the Higgs field is \( \Phi^\alpha = H(r)\xi^\alpha \) with some constant spinor \( \xi \). Although this ansatz is not itself spherically symmetric it leads to a consistent reduction. The corresponding reduced action is
\[ S = -\int dr \left[ \frac{1}{g^2}(W'^2 + \frac{(1 - W^2)^2}{2r^2}) + \frac{r^2}{2} H'^2 + \frac{\lambda r^2}{8}(H^2 - v^2)^2 + \frac{1}{4}(W + 1)^2 H^2 \right]. \] (4)

The only essential difference of this action to the one for the triplet is the form of the mass term. It destroys the symmetry \( W \to -W \) and enforces \( W \) to turn to \( W = -1 \) for \( r \to \infty \) in order to have finite total energy (Fig. 1). This asymptotic behaviour implies that the solution has no magnetic charge in contrast to the previous case with \( W \to 0 \).

In contrast to the stable monopole the sphaleron, i.e. the solution minimizing the energy \( H = -S \), is unstable. In order to understand this instability it is important to consider the most general spherically symmetric ansatz for the YM field.

The ansatz used above for the monopole and the sphaleron corresponds to a consistent reduction putting \( A_0 = A_1 = W_2 = 0 \) and \( W_1 = W \). The sphaleron turns out to be stable under variations staying within the minimal reduction, but not if \( \delta W_2 \neq 0 \) and \( \delta A_1 \neq 0 \).

As was discussed by Manton [15] this instability is due to the non-trivial topology of the configuration space of the spherically symmetric YM potential, again related to the asymptotic vacuum structure of configurations with finite energy.

### 3 Gravitating Monopoles

A spherically symmetric gravitational field is described by a space-time metric of the form
\[ ds^2 = e^{2\nu} dt^2 - e^{2\lambda} dR^2 - r^2 d\Omega^2 , \] (6)

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) and \( \nu, \lambda \) and \( r^2 \) are functions of the coordinates \( t \) and \( R \).
Figure 2: $W$ and $H$ for ($\beta = 0$) a) the gravitating monopole solutions for $\alpha = 0.05$, $\alpha_{\text{max}} = 1.403$ and $\alpha_c = 1.386$; b) first radial excitation for $\alpha = 0.01, 0.2, 0.5$ and 0.86.

The quantity $r^2$ is proportional to the surface area of the invariant 2-spheres and hence has a geometrical significance. Furthermore for static space-times the function $e^\nu$ measures the invariant length of the time-translation Killing vector, thus only the function $e^\lambda$ is gauge dependent, i.e., depends on the choice of the radial coordinate $R$.

A simple gauge choice is $R = r$ (Schwarzschild coordinates), which is however well defined only as long as $dr/dR \neq 0$. Another convenient choice is obtained putting $e^\lambda = r$ (isotropic coordinates), leading to an autonomous form of the field equations.

With gravity a new scale comes in through Newton’s constant $G$, which allows us to define the Planck mass $M_{\text{Pl}} = 1/\sqrt{G}$. The existence of two different scales, the YM scale given by $M_W$ and the Planck scale $M_{\text{Pl}}$ have a very important impact on the structure of the solutions. In particular, studying the limiting case $M_{\text{Pl}} >> M_W$ gives important insights for their interpretation, as we will see in the following.

Together with $M_W$ we can form the dimensionless ratio $\alpha = M_W \sqrt{G}/g = M_W/gM_{\text{Pl}}$. As just mentioned, a special role is played by the limiting case $\alpha \to 0$, which can however be achieved in two different ways:

i) $G \to 0$, $M_W$ fixed, in which the gravitational field decouples (flat space);

ii) $v = M_W/g \to 0$, $G$ fixed, in which the Higgs field becomes trivial and can be ignored.

The reduced EYMH action can be expressed as

$$S = -\int dRe^{(\nu+\lambda)}\left[\frac{1}{2}(1 + e^{-2\lambda}((r')^2 + \nu'(r)^2)) - e^{-2\lambda}r^2V_1 - V_2 - V_3\right] ,$$

with

$$V_1 = \frac{(W')^2}{r^2} + \frac{1}{2}(H')^2 ,$$

$$V_2 = \frac{(1 - W^2)^2}{2r^2} + \frac{\beta^2r^2}{8}(H^2 - \alpha^2)^2$$

and

$$V_3 = W^2H^2 \quad \text{resp.} \quad V_3 = \frac{1}{4}(W + 1)^2H^2$$
for the triplet resp. doublet Higgs. Through a suitable rescaling we have achieved that the action depends only on the dimensionless parameters $\alpha$ and $\beta$.

Well-known exact solutions of the coupled Einstein-YM equations, obtained from the action (7), are the Schwarzschild solution with trivial YM and Higgs fields $W \equiv 1, H \equiv v$ and the abelian (magnetically charged) Reissner-Nordstrøm (RN) solution with $W \equiv 0, H \equiv v$. Both describe static black holes with a curvature singularity at the origin. As was already mentioned, there exist no regular solitons of the Einstein-Maxwell theory.

Besides these trivial abelian solutions there is a rich spectrum of non-abelian solutions found by numerical integration of the field equations with suitable boundary conditions [1, 16].

To begin with there are the self-gravitating versions of the flat-space non-abelian monopoles, which are recovered in the limit $\alpha \to 0$ ($G \to 0$) (l.h.s. of Fig. [4]). These are globally regular solutions with a regular center of symmetry (origin) and finite mass. They exist only up to some maximal value $\alpha_{\text{max}}$ (depending on $\beta$) of the mass ratio $\alpha = M_W / g M_{\text{Pl}}$. Such a maximal value of $\alpha$ is to be anticipated. We expect the monopole to become gravitationally unstable, when its size $R_{\text{mon}} \approx 1 / M_W$ becomes comparable to its Schwarzschild radius $R_{SS} = G M_{\text{mon}} \approx G M_W / g^2$, i.e. for $G M_W^2 / g^2 = \alpha^2 \approx 1$. As $\alpha$ increases the solutions develop a typical limiting behaviour indicating this instability, which may be characterized as "gravitational confinement" of the monopole. The spatial hyper-surface $t = \text{const}$. develops an infinite throat separating an interior region with a smooth origin and non-trivial YM field from an exterior extremal RN solution with $W \equiv 0$. This throat is characterized by a finite limiting value $r_l = \alpha$ of the metric function $r$. Geometrically this means that neighbouring radial light-rays become non-divergent. All this is much like the $t = \text{const}$. surfaces of the extremal RN solution, with the only difference, that the interior part of the throat is not the analytic continuation of the exterior one. In fact the combination of the metric functions $\nu + \lambda$ blows up along the throat coming from the interior, whereas $\nu + \lambda \equiv 0$ for the RN solution.

Amazingly, for small values ($< \sim 0.7$) of the parameter $\beta = M_H / M_W$ this kind of singular limiting behaviour does not occur at the maximal value of $\alpha$ but at some critical value $\alpha_c < \alpha_{\text{max}}$. Running along the 1-paramter family of solutions starting at $\alpha = 0$ and increasing $\alpha$ one runs through a maximum of $\alpha$ before the slightly smaller critical value $\alpha_c$ is reached, i.e. there exist two monopole solutions to the same value of $\alpha$ in the interval $\alpha_c < \alpha < \alpha_{\text{max}}$. This double-valuedness can be avoided using a different parameter for
Figure 4: $W$ for the first two Bartnik-McKinnon solutions

this family of solutions. For solutions with a regular origin the YM potential $W$ has the
behaviour $W = 1 - br^2 + O(r^4)$ for $r \to 0$. Given $\alpha$ the parameter $b(\alpha)$ is fixed by the
requirement of asymptotic flatness. It turns out that the parametrization with $b$ instead
of $\alpha$ is one-to-one (compare Fig. 3).

This figure contains also the first two members of a sequence of excited families of
monopoles showing a different behaviour of $b(\alpha)$ for $\alpha \to 0$. The corresponding values of
$b$ tend to finite values $b_n$ ($n = 1, 2, \ldots$) as $\alpha$ tends to 0 ($v \to 0$) related to the solutions
found by Bartnik and McKinnon (BM). The latter are globally regular solutions of the
EYM equations without a Higgs field. They are labelled by the number $n$ of zeros of
the YM potential $W$. Their mass $M$ is of order one in units of $M_{Pl}/g$ — the only
mass scale in this case — tending rapidly to $M = 1$ for growing $n$. The parameters $b_n$
converge to $b_\infty \approx 0.7064$. In contrast to the monopole solutions they carry zero magnetic
charge, related to their different asymptotic behaviour for $r \to \infty$, where $W \to \pm 1$.
(Compare Fig. 4). The shape of the $n = 1$ solution reminds very much of the flat space
sphaleron. In fact, it is also unstable and may be understood as a gravitationally bound
sphaleron, the gravitational field replacing the Higgs field of the flat sphaleron [9]. That
this interpretation makes good sense is underlined by the fact that similar solutions are
obtained with a scalar dilaton replacing the gravitational field [11].

For finite, but small $\alpha$ the excited monopoles consist essentially of a very small (Planck
size) BM-solution sitting inside a large (size $1/M_W$) flat monopole (r.h.s. of Fig. 4). Let
me recall that for all the monopole solutions $W \to 0$ for $r \to \infty$, even though this is not
clearly visible for all the curves of the plot.

All these families of excited monopole solutions have a common value of $\alpha_c = \sqrt{3}/2$,
which is also the maximal one in this case. For $\alpha \to \alpha_c$ we observe the same limiting
behaviour (infinite throat) as for the fundamental monopole. Thus it seems that the
latter describes a rather universal phenomenon indicating gravitational instability of static
equilibrium configurations and hence is to be expected to occur also for other gravitating
matter systems.

The l.h.s. of Fig. 4 shows the masses of the various monopole solutions as a function of
$\alpha$ ($\beta = 0$). We find that for $\alpha = \alpha_c$ the mass of the solutions becomes $M = M_c \equiv M_{Pl}/g$,
the mass of the extremal RN solution. This is easily understood from the merging of the
exterior throat part of the monopoles with the latter solution as $\alpha \to \alpha_c$. As indicated
in the r.h.s. of Fig 5 the mass at $\alpha_{max}$ is slightly bigger than $M_c$. While $\beta$ increases $\alpha_c$
decreases to the limiting value $\alpha_c = \sqrt{2}/2$ for $\beta \to \infty$. Similar to their flat counterparts
Figure 5: a) Masses (in units of $M_{\text{Pl}}/g$) of fundamental monopole solutions and first and second radial excitations versus $\alpha$ (for $\beta = 0$); b) the critical region for the fundamental solutions in detail.

Figure 6: Domains of existence for non-abelian black holes: a) for $\beta = 0, 1, 2, 3$, and $4$; b) for $\beta = 6$ and $\infty$

solutions for $\beta = \infty$ (i.e. $H \equiv v$) may be considered as cosmological textures.

4 Non-abelian Black Holes

Apart from the solutions with a regular origin there are non-abelian, “coloured” black holes, parametrized by their radius $r_h$ (in geometrical units, i.e. the value of $r$ at the event horizon) in addition to $\alpha$ and $\beta$ [1, 17]. For $r_h << 1/M_W$ these non-abelian black holes may be interpreted as a tiny Schwarzschild black hole sitting inside a monopole. On the other hand, when $r_h$ becomes bigger than $\approx 1/M_W$ this type of solution disappears and only the abelian RN black holes exist. For $r_h \to 0$ the matter fields tend uniformly to those of the globally regular solutions, whereas for the metrical functions this limit is clearly more delicate.

Detailed numerical analysis reveals that non-abelian black holes exist only in a limited domain of the $\alpha$-$r_h$-plane, whose shape undergoes some characteristic changes as $\beta$ varies from 0 to $\infty$. Fig. 6 shows these domains. Observe that we use $\alpha r_h$ instead of $r_h$ as the abscissa - equivalent to expressing $r_h$ in units of $1/M_W$ - in order to obtain domains remaining bounded for $\alpha \to 0$. In the following I shall discuss in some more detail the
structure of these “Phase Diagrams” and the phenomena happening at their boundaries.

Let me start with the case $\beta = 0$.

It is appropriate to subdivide the relevant sector $\alpha \geq 0$, $r_h \geq 0$ into the four subregions I-IV (compare Fig. 7).

In regions I and II we find coloured black holes. Above the diagonal, i.e. in regions II and III we have the abelian RN solutions, the extremal RN black holes sitting on the diagonal. Below the diagonal the RN solution has a naked singularity and does not represent a black hole. No black holes neither abelian nor non-abelian could be found in region IV. Region I may be subdivided in region I$_a$, where only the b.h. version of the fundamental monopole resides and region I$_b$, where in addition their radial excitations are found. Thus region I$_a$ contains essentially one b.h. solution for given values of $\alpha$ and $r_h$ - apart from a small interval $\alpha_c(r_h) < \alpha < \alpha_{\text{max}}(r_h)$, where two solutions exist - whereas in region I$_b$ countably many solutions exist (for given $\alpha$ and $r_h$).

In region II abelian and non-abelian black holes coexist. This establishes an obvious violation of the so-called “No-Hair Conjecture”. According to the latter black holes should (apart from mass and angular-momentum) be uniquely determined through their “gauge charges” - their magnetic charge in the present case. However, abelian and non-abelian black holes carry the same magnetic charge and can also be made degenerate in mass resp. the value of $r_h$. Because in general only one type of the “degenerate” black holes is stable (compare below), a weakened form of the “No-Hair Conjecture”, including the requirement of stability, could be maintained.

As $\beta$ increases from 0 to $\beta = 4$ the structure of the “Phase Diagram” remains essentially the same, the right boundary curve moving in to the left. However, for $\beta > 4$ this boundary curve develops a second, concave branch (compare Fig. 8) determined by another mechanism, the formation of a degenerate (inner or outer) horizon.

The boundary curve above the diagonal is essentially characterized by the bifurcation of the non-abelian with the abelian RN solution. For a given value of $\alpha$ this happens at some value $r_{h,c}(\alpha)$. Approaching this value from below the value $W_h$ of $W$ at the horizon tends to zero.

But again there is a slight complication for small values of $\beta$ ($\beta \lesssim 1.1$); similar to the existence of $\alpha_{\text{max}} > \alpha_c$ there is a $r_{h,\text{max}} > r_{h,c}$-phenomenon (compare Fig. 8).
5 Stability of monopoles and coloured black holes

I shall discuss here only stability against infinitesimal, spherically symmetric perturbations. In view of the time-independence of the solutions this amounts to analyzing the spectrum of perturbations with harmonic time-dependence obeying suitable boundary conditions. Imaginary frequencies correspond to unstable modes of the solution.

As to be expected all the excited regular monopoles turn out to be unstable. The branch of gravitating monopoles connected to the flat space solution is stable up to $\alpha_{\text{max}}$, whereas the corresponding upper branch - existing for $\alpha < \alpha_c$, $\alpha_{\max}$ - is unstable [18]. This change of stability at the bifurcation point at $\alpha_{\text{max}}$ of the mass function is in agreement with general results on 1-parameter families of solutions and well-known from stellar models [19].

Analogous results hold for the non-abelian, magnetically charged black holes. It is, however, interesting to observe that the abelian RN black hole is unstable in the framework of the non-abelian theory for $\alpha$ smaller than some value $\alpha < \sqrt{3}/2$ [20, 1]. In particular, the extremal RN solution is unstable for $\alpha < \sqrt{3}/2$ and stable above this value. At the limiting value $\alpha = \sqrt{3}/2$ the extremal RN solution bifurcates with infinitely many non-abelian solutions and in fact develops infinitely many unstable modes.

6 Gravitating Sphalerons and Sphaleron Black Holes

In complete analogy to gravitating monopoles one may also consider self-gravitating sphalerons, i.e. gravitating versions of the flat-space sphaleron [10]. Although many of the phenomena discussed above for monopoles repeat itself in this case, there are some important differences as far as the domains of existence for sphaleron black holes are concerned. Also the stability properties are clearly different. Again there is a maximal value of the parameter $\alpha$ for which static gravitating sphalerons exist. However, it is a rather different mechanism that is responsible for its existence. For very small values of $\alpha$ the coupling to gravity yields only a small perturbation of the flat-space sphaleron, whose size is $\approx 1/M_W$. As $\alpha$ increases the gravitational attraction leads to shrinking of the sphaleron, until it eventually becomes of size $\approx 1/M_{\text{Pl}}$ (l.h.s. of Fig. 9).

Following the 1-parameter family of solutions obtained by varying $\alpha$ one finds a similar
phenomenon as observed for the gravitating monopoles: $\alpha$ runs through a maximum. In contrast to the situation for the monopole there is however no critical value for $\alpha$ and the family may be continued all the way back to $\alpha = 0$. In fact, there is no analogue of the abelian RN solution in this case, since finite mass requires the asymptotic condition $W \to -1$ for $r \to \infty$. The parameter $b(\alpha)$ (compare l.h.s. of Fig. 7) instead increases monotonously and reaches the value $b_1$ of the BM solution from below as $\alpha$ comes back to 0. Thus as $\alpha$ increases the gravitational force becomes stronger and eventually replaces the effect of the Higgs field and we end up with a gravitationally bound sphaleron - the BM solution. The limit $\alpha \to 0$ corresponds to case ii) discussed above. However, Fig. 10 shows that there is another branch of solutions starting at $b_1$ with $b(\alpha) \gtrsim b_1$ for small values of $\alpha$. Looking at the solution for $b \gtrsim b_1$ we see, that it consists of a tiny BM solution sitting inside an essentially flat-space sphaleron of size $1/M_W$. The solution starts with $W = -1$ at $r = 0$, reaches almost $W = 1$ within distance $1/M_{Pl}$ and then decreases slowly to $W = -1$ (compare Fig. 9 r.h.s.). For all these sphaleron solutions $W$ tends to $-1$ for $r \to \infty$ although this is not obvious from the plots, because of the different length scales involved. As we increase $\alpha$ it again runs through a maximum and eventually comes back to 0 at $b = b_2$. This whole story repeats itself, a new branch of solutions starting at each BM solution. The values $b_n$ may be interpreted as points where the $n^{\text{th}}$ BM solution bifurcates with the same BM solution having a flat-space sphaleron attached to it at large $r$.

Again there are solutions for $\beta = \infty$ (i.e. $H \equiv v$), which one might call "global sphalerons" in analogy to the global monopoles of the triplet model.

Besides the globally regular solutions there are also in this case non-abelian black holes. Like their regular counter-parts they carry no charge. As already mentioned above the abelian RN solution is not allowed in this case. For $\alpha \to 0$ these sphaleronic black holes tend to the corresponding BM-type black holes. Again they exist only in a bounded domain of the $\alpha-r_h$-plane ($r_h$ measured in units of $1/M_W$). The domains for the various radial excitations of the fundamental solution (with $n \geq 2$ zeros of $W$) are nested and shrink very quickly with $n$ (compare r.h.s. of Fig. 10).
7 Stability of sphalerons and sphaleronic black holes

Like their flat counter-parts the gravitating sphalerons are unstable with respect to certain variations involving the component $W_2$ of the YM potential \[21\].

In the gravitating case we observe however an additional new type of instability involving variations of $W$, i.e. within the minimal ansatz for the YM field. In contrast to the first mentioned instability already present in flat space and which we may call “topological”, the second one may be considered a “gravitational” instability \[22\]. It sets in at the right turning point of the lowest branch of $b(\alpha)$, i.e. at the maximal value of $\alpha$ of the gravitating version of the flat-space sphaleron. According to a general result of stability theory the whole upper branch between $\alpha_{\text{max}}$ and $\alpha = 0$ has one unstable mode. At each turning point of the subsequent branches of $b(\alpha)$ another unstable mode appears. This explains the observation that the $n$th BM solution, obtained when $\alpha \to 0$ on the upper part of the $n$th branch, has $n$ unstable modes within the minimal ansatz in addition to the (also $n$) topological unstable modes involving $W_2$ \[23\].

Analogous results are found for the sphaleronic black holes.

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