Angular Functions with Complex Angular Momenta

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In the study of the amplitudes for many-particle processes, and also for processes involving particles with spin, the use is made of matrix elements of the rotation group $d^{j}_{\mu \nu}(z)$. In this paper the generalization of the functions $d^{j}_{\mu \nu}(z)$ to arbitrary arguments and indices is studied. At the same time the functions of the second kind, analogous to Legendre functions of the second kind, are investigated. The results obtained play an important part in the introduction of complex angular momenta in many-particle processes. [Added in 2013: The generalized Legendre functions considered here may be applied as well to many other problems.]

The study of the partial amplitudes for elastic scattering of two spinless particles for complex orbital angular momenta requires a knowledge of the properties of the Legendre functions $P_{j}(z)$ and $Q_{j}(z)$ for arbitrary values of the argument and index. In the case of the scattering of particles with spin or of many-particle processes the expansion of the total amplitude is carried out in terms of matrix elements of the rotation group, $d^{j}_{\mu \nu}(z)$, which satisfy the equation

$$\left[(1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} + j(j + 1) - \frac{\mu^2 - 2\mu\nu z + \nu^2}{1 - z^2}\right] y(z) = 0.$$ (1)

Therefore in order to continue the partial amplitudes of such processes to complex values of the angular momentum it is first necessary to generalize the functions $d^{j}_{\mu \nu}(z)$ and the corresponding functions of the second kind to arbitrary values of the indices.

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and the argument. In the treatment of the elastic scattering of particles with spin it is sufficient to have the functions \(d_{j\mu\nu}(z)\) for arbitrary \(j\) and \(z\), but for “physical” values of the helicities \(\mu\) and \(\nu\) (i.e., for \(\mu\) and \(\nu\) either both integers or both half-integers). This case has been investigated, for example, in [3]. For many-particle amplitudes, however, even a preliminary study shows that they must obviously be continued not only with respect to the total angular momentum, but also with respect to the helicities [4]. Owing to this we shall consider in the present paper the continuation of the functions \(d_{j\mu\nu}(z)\) to arbitrary values of \(z, j, \mu, \nu\). Since, however, it turns out that \(d_{j\mu\nu}(z)\) themselves have nonessential cuts in \(j, \mu, \nu\), we study instead of them functions which differ from \(d_{j\mu\nu}\) by a factor which does not depend on \(z\). These functions are a generalization of the associated Legendre functions, so that it is natural to call them generalized Legendre functions.

The present paper contains a description of the fundamental properties of the generalized Legendre functions, and is an aid in the study of many-particle amplitudes for complex angular momenta.

I. DEFINITION OF THE GENERALIZED LEGENDRE FUNCTIONS

We define the Legendre functions of the first kind, \(P^j_{\mu\nu}(z)\), and of the second kind, \(Q^j_{\mu\nu}(z)\), for arbitrary values of \(j, \mu, \nu, \) and \(z\), by the equations

\[
P^j_{\mu\nu}(z) = \frac{1}{\Gamma(\nu - \mu + 1)} \left( \frac{z - 1}{2} \right)^{(\nu - \mu)/2} \left( \frac{z + 1}{2} \right)^{(\nu + \mu)/2}
\times F \left( j + \nu + 1, -j + \nu; \nu - \mu + 1; \frac{1 - z}{2} \right),
\]

\[
Q^j_{\mu\nu}(z) = e^{i\pi(\mu-\nu)} \frac{\Gamma(j + \mu + 1)\Gamma(j - \nu + 1)}{2\Gamma(2j + 2)} \left( \frac{z - 1}{2} \right)^{-(j+1)} \left( \frac{z + 1}{z - 1} \right)^{(\nu + \mu)/2}
\times F \left( j + \nu + 1, j + \mu + 1; 2j + 2; \frac{2}{1 - z} \right).
\]

Here \(F(a, b; c; x)\) is the hypergeometric function, and \(\text{arg}(z - 1) = \text{arg}(z + 1) = 0\) for real \(z > 1\). It is not hard to verify that the functions \(P^j_{\mu\nu}(z)\) and \(Q^j_{\mu\nu}(z)\) defined by (2) and (3) satisfy Eq. (1).
It follows directly from (2) and (3) that for $\nu = 0$ the functions $P^j_{\mu\nu}(z)$ and $Q^j_{\mu\nu}(z)$ go over into the associated Legendre functions:

$$P^j_{\mu0}(z) = P^\mu_j(z), \quad Q^j_{\mu0}(z) = Q^\mu_j(z),$$

where $P^\mu_j(z)$ and $Q^\mu_j(z)$ are the associated Legendre functions \[5, 6\]. In the general case the generalized Legendre functions are connected in a simple way with the Jacobi functions; for example,

$$P^j_{\mu\nu}(z) = \frac{\Gamma(j - \nu + 1)}{\Gamma(j - \mu + 1)} \left(\frac{z - 1}{2}\right)^{(\nu-\mu)/2} \left(\frac{z + 1}{2}\right)^{(\nu+\mu)/2} \frac{P^{\nu-\mu,\nu+\mu}_{j-\nu}(z)}{P^{\nu,\nu}_{j-\nu}},$$

$$Q^j_{\mu\nu}(z) = e^{i\pi(\mu-\nu)} \frac{\Gamma(j - \nu + 1)}{\Gamma(j - \mu + 1)} \left(\frac{z - 1}{2}\right)^{(\nu-\mu)/2} \left(\frac{z + 1}{2}\right)^{(\nu+\mu)/2} \frac{Q^{\nu-\mu,\nu+\mu}_{j-\nu}(z)}{Q^{\nu,\nu}_{j-\nu}},$$

where $P^{\alpha,\beta}_n(z)$ and $Q^{\alpha,\beta}_n(z)$ are the Jacobi functions\[1\] of the first and second kinds \[5\].

We shall list some simple properties of the functions $P^j_{\mu\nu}$ and $Q^j_{\mu\nu}$ which follow directly from the definitions (2) and (3). It is obvious from (2) that

$$P^j_{\mu\nu}(z) = P^{j-1}_{\nu-\mu}(z).$$

It is clear from the definition (3) that $Q^j_{\mu\nu}$ and $Q^{j-1}_{\nu\mu}$ differ only by a factor:

$$Q^j_{\mu\nu}(z) = e^{2i\pi(\mu-\nu)} \frac{\Gamma(j + \mu + 1)\Gamma(j - \nu + 1)}{\Gamma(j - \mu + 1)\Gamma(j + \nu + 1)} Q^{\nu,\nu}_{j\nu}(z).$$

The functions $P^j_{\mu\nu}$ and $P^{j-1}_{\nu\mu}$, on the other hand, are in general linearly independent solutions of Eq. (1), with the Wronskian determinant

$$W(P^j_{\mu\nu}, P^{j-1}_{\nu\mu}) = \frac{2}{\pi} \frac{\sin\pi(\nu - \mu)}{1 - z^2},$$

where $W(f_1, f_2) = f_1 f_2' - f_1' f_2$. The functions $Q^j_{\mu\nu}$ and $Q^{j-1}_{\nu\mu}$ are likewise linearly independent. Their Wronskian determinant is

$$W(Q^j_{\mu\nu}, Q^{j-1}_{\nu\mu}) = \frac{\pi}{2} \frac{\sin 2\pi j}{\sin \pi(j + \mu) \sin \pi(j - \nu)} \frac{1}{1 - z^2}.$$

Using the properties of hypergeometric functions, one can also derive other relations for the generalized Legendre functions. The simplest of these are the equations

$$P^j_{\mu\nu}(z) = P^j_{-\nu, -\mu}(z), \quad Q^j_{\mu\nu}(z) = Q^j_{-\nu, -\mu}(z),$$

1 Note that $P^{\alpha,\beta}_n(z)$ with positive integer $n$ is the Jacobi polynomial of order $n$. 
which follow from Eq. 2.1(23) in \[5\] (9.131.1 in \[6\]). From Eq. 2.1(2.2) in \[5\] one also gets another definition for \(P_{\mu \nu}^j(z)\), which is equivalent to the definition \(2\):

\[
P_{\mu \nu}^j(z) = \frac{1}{\Gamma(\nu - \mu + 1)} \left( \frac{z + 1}{2} \right)^j \left( \frac{z - 1}{z + 1} \right)^{(\nu - \mu)/2} \times F \left( -j + \nu, -j - \mu; \nu - \mu + 1; \frac{z - 1}{z + 1} \right). \tag{2a}
\]

The form \(2a\) for \(P_{\mu \nu}^j\) makes the first of the equations \(8\) obvious.

\[\text{II. RECURRENCE RELATIONS}\]

As is well known, adjacent hypergeometric functions are connected by linear relations. These lead to the following recurrence relations for the generalized Legendre functions:

\[
(2j + 1) \sqrt{\frac{z - 1}{2}} P_{\mu - 1/2, \nu + 1/2}^j(z) = P_{\mu \nu}^{j+1/2}(z) - P_{\mu \nu}^{j-1/2}(z), \tag{9}
\]

\[
(2j + 1) \sqrt{\frac{z - 1}{2}} P_{\mu + 1/2, \nu - 1/2}^j(z) = (j + \nu + 1/2)(j - \mu + 1/2) P_{\mu \nu}^{j+1/2}(z) - (j - \nu + 1/2)(j + \mu + 1/2) P_{\mu \nu}^{j-1/2}(z), \tag{10}
\]

\[
(2j + 1) \sqrt{\frac{z + 1}{2}} P_{\mu + 1/2, \nu + 1/2}^j(z) = (j - \mu + 1/2) P_{\mu \nu}^{j+1/2}(z) + (j + \mu + 1/2) P_{\mu \nu}^{j-1/2}(z), \tag{11}
\]

\[
(2j + 1) \sqrt{\frac{z + 1}{2}} P_{\mu - 1/2, \nu - 1/2}^j(z) = (j + \nu + 1/2) P_{\mu \nu}^{j+1/2}(z) + (j - \nu + 1/2) P_{\mu \nu}^{j-1/2}(z), \tag{12}
\]

Equations \(9\) and \(10\) follow from the respective formulas 9.137.4 and 9.137.5 in \[6\], and \(11\) and \(12\) follow from Eqs. 2.8(37) and 2.8(32) in \[5\].

By iteration of the relations \(9\)–\(12\) we can derive a large number of other recurrence relations. One example of these is

\[
j(j + 1)(2j + 1) z P_{\mu \nu}^j(z) = j(j + \nu + 1)(j - \mu + 1) P_{\mu \nu}^{j+1}(z) + \nu \mu (2j + 1) P_{\mu \nu}^j(z) + (j + 1)(j + \mu)(j - \nu) P_{\mu \nu}^{j-1}(z). \tag{13}
\]
Another type of recurrence relations can be derived by using the differentiation properties of hypergeometric functions [Eqs. 2.8(27) and 2.8(20) in [5]]. We thus get the following equations:

\[
\frac{d^n}{dz^n}[(z - 1)^{(\nu - \mu)/2}(z + 1)^{(\nu + \mu)/2} P_{\mu\nu}^j(z)] = (z - 1)^{(\nu - \mu - n)/2}(z + 1)^{(\nu + \mu - n)/2} P_{\mu,\nu-n}^j(z),
\]

\[
\frac{\Gamma(j + \nu + 1)}{\Gamma(j - \nu + 1)} \frac{d^n}{dz^n}[(z - 1)^{-(\nu - \mu)/2}(z + 1)^{-(\nu + \mu)/2} P_{\mu\nu}^j(z)] = \frac{\Gamma(j + \nu + n + 1)}{\Gamma(j - \nu - n + 1)} (z - 1)^{-(\nu - \mu + n)/2}(z + 1)^{-(\nu + \mu + n)/2} P_{\mu,\nu+n}^j(z). \tag{15}
\]

Two further formulas of this type, giving changes of the index \(\mu\), are obtained if we make the replacement \(\mu \leftrightarrow -\nu\) in (14) and (15) and use the property (8).

All of the formulas (9)–(15) are valid not only for \(P_{\mu\nu}^j\), but also for the functions of the second kind, \(Q_{\mu\nu}^j\).

III. RELATIONS BETWEEN THE FUNCTIONS OF 1ST AND 2ND KINDS

Equation (1) has two linearly independent solutions. Therefore it is clear that there must exist relations between the four different solutions of this equation: \(P_{\mu\nu}^j(z)\), \(P_{\nu\mu}^j(z)\), \(Q_{\mu\nu}^j(z)\), and \(Q_{\nu\mu}^{-j-1}(z)\). We can derive these relations easily by using Eq. 2.10(2) of [3] (or Eq. 9.132.2 of [4]). Applying this equation to (3), we have

\[
\frac{2}{\pi} e^{-i\pi(\mu - \nu)} \sin \pi(\mu - \nu) Q_{\mu\nu}^j(z) = P_{\mu\nu}^j(z) - \frac{\Gamma(j + \mu + 1) \Gamma(j - \nu + 1)}{\Gamma(j - \mu + 1) \Gamma(j + \nu + 1)} P_{\nu\mu}^j(z). \tag{16}
\]

Recalling the property (6), we easily define a further equation

\[
Q_{\mu\nu}^j(z) - Q_{\mu\nu}^{-j-1}(z) = \frac{\pi}{2} e^{i\pi(\mu - \nu)} \frac{\sin 2\pi j}{\sin \pi(j - \mu) \sin \pi(j + \nu)} \frac{\Gamma(j + \mu + 1) \Gamma(j - \nu + 1)}{\Gamma(j - \mu + 1) \Gamma(j + \nu + 1)} P_{\nu\mu}^j(z). \tag{17}
\]

It is furthermore obvious that Eq. (11) remains unchanged if we change the sign of the variable \(z\) and at the same time change the sign of one of the indices, \(\mu\) or \(\nu\). Therefore there are also relations connecting generalized Legendre functions of \(z\) and
of $-z$. The simplest of these is obtained by applying to Eq. (3) the formula 2.10(6) from [5] (or 9.131.1 from [6]) for the hypergeometric function:

$$Q^j_{\mu\nu}(z) = e^{\mp i(j+1)\pi} e^{-2i\pi\nu} \frac{\Gamma(j - \nu + 1)}{\Gamma(j + \nu + 1)} Q^j_{\mu,-\nu}(-z)$$

$$= e^{\mp i(j+1)\pi} e^{-2i\pi\mu} \frac{\Gamma(j + \mu + 1)}{\Gamma(j - \mu + 1)} Q^j_{-\mu,\nu}(-z).$$  \(18\)

The sign $\mp$ corresponds to values $\text{Im}z \geq 0$.

When we now use (17) and (18), it is not hard to derive the following relations:

$$\frac{\Gamma(j + \nu + 1)}{\Gamma(j - \nu + 1)} P^j_{\mu\nu}(z) = e^{\pm i\pi j} P^j_{\mu,-\nu}(-z)$$

$$- \frac{2}{\pi} e^{\pm i\pi\nu} e^{-i\pi(\mu+\nu)} \sin \pi(j + \mu) Q^j_{\mu,-\nu}(-z),$$ \(19\)

$$\frac{\Gamma(j - \mu + 1)}{\Gamma(j + \mu + 1)} P^j_{\mu\nu}(z) = e^{\pm i\pi j} P^j_{-\mu,\nu}(-z)$$

$$- \frac{2}{\pi} e^{\pm i\pi\mu} e^{i\pi(\mu+\nu)} \sin \pi(j - \nu) Q^j_{-\mu,\nu}(-z),$$ \(20\)

$$\frac{1}{\pi} \sin \pi(\mu + \nu) P^j_{\mu\nu}(z) = \frac{e^{\mp i\pi\mu}}{\Gamma(j + \nu + 1) \Gamma(-j + \nu)} P^j_{\mu,-\nu}(-z)$$

$$- \frac{e^{\pm i\pi\nu}}{\Gamma(j - \mu + 1) \Gamma(-j - \mu)} P^j_{-\mu,\nu}(-z).$$ \(21\)

In Eqs. (19)–(21), as in (18), the upper signs correspond to the case $\text{Im}z > 0$, and the lower signs to $\text{Im}z < 0$.

**IV. ANALYTIC PROPERTIES**

The analytic properties of the generalized Legendre functions are clear from their definitions (2), (3). The function of the first kind, $P^j_{\mu\nu}(z)$, is an entire function of each of the three indices $j$, $\mu$, and $\nu$. It has zeros,

$$P^j_{j+n+1,j-m}(z) \equiv 0, \quad P^j_{-j+m,-j-n-1}(z) \equiv 0,$$

\(^2\) Recall that the definitions (2) and (3) unambiguously fix the phases of the Legendre functions at real $z > +1$. Relations (18)–(21) define analytical continuation of these functions to real $z < -1$ through upper or lower complex half-plane.
if $m \geq 0$ and $n \geq 0$ are nonnegative integers.

The function of the second kind, $Q_{j\mu\nu}(z)$, is a meromorphic function of the indices. It has poles at $j + \mu + 1 = -n$ or $j - \nu + 1 = -n$, where $n$ is a nonnegative integer. The residues at the poles of the function $Q_{j\mu\nu}(z)$ can be expressed in terms of $Q_{j\nu\mu}(z)$ or $P_{j\nu\mu}(z)$ by means of Eqs. (7) and (16). Coincidence of two poles of $Q_{j\mu\nu}(z)$ in the $j$ plane in general leads to a pole of second order with respect to $j$. For $\nu = 0$ there is no second order pole.

As functions of the variable $z$ the generalized Legendre functions are analytic in the complex plane with two cuts drawn along the real axis from $-\infty$ to $-1$ and from $-1$ to $+1$. The discontinuities on the cut that goes from $-\infty$ to $-1$ can be calculated easily from (18)–(21):

$$\frac{1}{2i} [Q_{j\mu\nu}(x + i\epsilon) - Q_{j\mu\nu}(x - i\epsilon)] = e^{-2i\nu} \sin \pi j \frac{\Gamma(j - \nu + 1)}{\Gamma(j + \nu + 1)} Q_{j\mu\nu}(-x),$$

$$-\infty < x < -1,$$  \hspace{0.5cm} (22)

$$\frac{1}{2i} [P_{j\mu\nu}(x + i\epsilon) - P_{j\mu\nu}(x - i\epsilon)] = \frac{\Gamma(j - \nu + 1)}{\Gamma(j + \nu + 1)} [\sin \pi j P_{j\mu\nu}(-x)$$

$$- \frac{2}{\pi} e^{-i\pi(\mu+\nu)} \sin \pi \nu \sin \pi (j + \mu) Q_{j\mu\nu}(-x)], \hspace{0.5cm} -\infty < x < -1. \hspace{0.5cm} (23)$$

Other expressions for these discontinuities are obtained by interchanging $\mu \leftrightarrow -\nu$ and using the properties (8).

To study the cut that goes from $-1$ to $+1$ it is convenient to introduce the function $\tilde{P}_{j\mu\nu}(x)$ defined by the equations

$$\tilde{P}_{j\mu\nu}(x) = e^{i\pi(\mu-\nu)/2} P_{j\mu\nu}(x + i\epsilon) = e^{-i\pi(\mu-\nu)/2} P_{j\mu\nu}(x - i\epsilon), \hspace{0.5cm} -1 < x < +1. \hspace{0.5cm} (24)$$

We then have the obvious expression

$$\frac{1}{2i} [P_{j\mu\nu}(x + i\epsilon) - P_{j\mu\nu}(x - i\epsilon)] = \sin \frac{\pi}{2}(\nu - \mu) \tilde{P}_{j\mu\nu}(x), \hspace{0.5cm} -1 < x < +1. \hspace{0.5cm} (25)$$

The discontinuity of $Q_{j\mu\nu}(z)$ can be expressed in terms of $\tilde{P}_{j\mu\nu}$ and $\tilde{P}_{j\nu\mu}$ by means of (16) and (25). A more important formula, however, is

$$\frac{1}{2i} [e^{i\pi(\nu-\mu)/2} Q_{j\mu\nu}(x + i\epsilon) - e^{-i\pi(\nu-\mu)/2} Q_{j\mu\nu}(x - i\epsilon)]$$

$$= -\frac{\pi}{2} e^{-i\pi(\nu-\mu)} \tilde{P}_{j\mu\nu}(x), \hspace{0.5cm} -1 < x < +1. \hspace{0.5cm} (26)$$
V. THE ASYMPTOTIC BEHAVIOR

The definitions (2) and (3) allow us to study with ease the asymptotic behavior of the generalized Legendre functions with respect to the variable $z$. For example, (2) describes the behavior of the function of the first kind, $P_{j}^{\mu\nu}(z)$, for $(z - 1) \to 0$. The behavior of the functions of the second kind is then found from the expression (16). The asymptotic behavior for $z \to \infty$ is given by (3) and (17). The behavior of the functions $P_{j}^{\mu\nu}$ and $Q_{j}^{\mu\nu}$ for $(z + 1) \to 0$ can be found without difficulty by means of (21).

Let us now proceed to consider the asymptotic behavior of the generalized Legendre functions with respect to the indices. For this it is convenient to rewrite (1) in a different form, using the change of variable $z = \cosh \alpha$

\[
\left[ \frac{d^2}{d\alpha^2} + \coth \alpha \frac{d}{d\alpha} - \frac{1}{\sinh^2(\alpha/2)} \left( \frac{\nu - \mu}{2} \right)^2 + \frac{1}{\cosh^2(\alpha/2)} \left( \frac{\nu + \mu}{2} \right)^2 \right] y(\cosh \alpha) = j(j + 1) y(\cosh \alpha). \tag{1a}
\]

With this way of rewriting (1) and the asymptotic form of $P_{\mu\nu}^{j}(z)$ for $z \to 1$, we can get the following value of a limit:

\[
\lim_{t \to \infty} \left[ t^{(\nu-\mu)/2} P_{\mu\nu}^{j} \left( \cosh \frac{y}{\sqrt{t}} \right) \right] = J_{\nu-\mu}(y), \tag{27}
\]

where $J_{\kappa}(y)$ is the Bessel function and $t = (\nu + \mu)^2/4 - j(j + 1)$; it is supposed that $\nu - \mu = \text{const}$ and $(\nu + \mu)/t \to 0$ as $t \to \infty$.

In the more general case we have

\[
\lim_{t \to \infty} \left[ t^{(\nu-\mu)/2} P_{\mu\nu}^{j} \left( 1 + \frac{2x}{t} \right) \right] = \frac{x^{(\nu-\mu)/2}}{\Gamma(\nu - \mu + 1)} e^{ax/2} \Phi \left( \frac{\nu - \mu + 1}{2} + \frac{b}{a}; \nu - \mu + 1; -ax \right), \tag{28}
\]

where

\[
a = \lim_{t \to \infty} (\nu + \mu)/t, \quad b = \lim_{t \to \infty} [(\nu + \mu)^2/4 - j(j + 1)]/t,
\]

and $\Phi$ is the confluent hypergeometric function [5, 6]. It is not hard to verify that

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3 In this limit, the limiting expressions for the generalized Legendre functions of the 1st and 2nd kinds may be expressed also through the Whittaker functions [5, 6], of the 1st and 2nd kinds respectively.
for \( a \to 0, \ b \to 1 \) the relation (28) goes into (27). Equations (27) and (28) can be used as an approximate expression for \( P_{\mu \nu}^j(z) \) when \( z \) is very close to unity.

The asymptotic expansion for \( \mu \to \infty \) and fixed \( z, j, \nu \) is obtained from the definition (2). To study other cases of the asymptotic behavior of generalized Legendre functions with respect to the indices for fixed \( z \) it is convenient to apply to Eq. (1a) a method analogous to the quasiclassical approximation of Wentzel, Kramers, and Brillouin, and use the well known behavior of the solutions with respect to \( z \). In this way it is not hard to find that with \( \mu \) and \( \nu \) constant

\[
Q_{\mu \nu}^j(\cosh \alpha)_{j \to \infty} \sim e^{i\pi(\mu-\nu)} j^{\mu-\nu-1/2} \sqrt{\frac{\pi}{2 \sinh \alpha}} e^{-\alpha(j+1/2)} . \tag{29}
\]

The following asymptotic cases can be treated similarly:

1) for \( j, \mu + \nu = \text{const} \)

\[
P_{\mu \nu}^j(\cosh \alpha)_{\nu - \mu \to \infty} \sim \frac{1}{\Gamma(\nu - \mu + 1)} \frac{[2 \tanh(\alpha/4)]^{\nu - \mu}}{\sqrt{\cosh(\alpha/2)}} , \tag{30}
\]

2) for \( \nu, j - \mu = \text{const}, \text{and } j, \mu \to \infty \)

\[
Q_{\mu \nu}^j(\cosh \alpha) \sim e^{i\pi(\mu-\nu)} \frac{\Gamma(j + \mu + 1)}{\Gamma(j + \nu + 1)} \sqrt{\frac{2\pi}{j + \mu + 1}}
\times [\tanh(\alpha/2)]^\nu (\tanh \alpha)^{-(j+1)} (2 \cosh \alpha)^{-(j+1)} ,
\]

\[
Q_{\mu \nu}^{j-1}(\cosh \alpha) \sim e^{i\pi(\mu-\nu)} \frac{\Gamma(-j + \mu)}{\Gamma(-j + \nu)} \sqrt{\frac{2\pi}{-j - \mu}}
\times [\tanh(\alpha/2)]^{-\nu} (\tanh \alpha)^{\mu} (2 \cosh \alpha)^{j} , \tag{31}
\]

The asymptotic formulas (27)–(31) are written out for the functions for which they have the simplest forms. The asymptotic behaviors of other functions in these cases can be found by using the relations (6), (7), (16), (17).

VI. INTEGRALS OF PRODUCTS OF TWO FUNCTIONS.

ORTHOGONAL SYSTEMS

Equation (11) allows us to calculate easily the indefinite integral of the product of \( f_{j_1} \) and \( f_{j_2} \), where \( f_{j_1} \) and \( f_{j_2} \) are solutions of (11) with the same \( \mu, \nu \) but different values of \( j \)

\[
\int f_{j_1}(z) f_{j_2}(z) \, dz = \frac{1 - z^2}{(j_1 - j_2)(j_1 + j_2 + 1)} (f_{j_2} f'_{j_1} - f'_{j_2} f_{j_1}) . \tag{32}
\]
A simple example is the integral
\[ \int_1^\infty dz \, P_{\mu \nu}^j(z) Q_{\nu \mu}^l(z) = \frac{e^{i\pi(\nu-\mu)}}{(l-j)(l+j+1)}. \] (33)

Convergence of the integral (33) at the upper limit imposes the requirement \( \text{Re} l > \text{Re} j \geq -1/2 \), and convergence at the lower limit requires \( \text{Re}(\nu - \mu + 1) > 0 \).

In precisely the same way we can calculate the integral from \(-1\) to \(+1\) of the product \( \tilde{P}_{\mu \nu}^j(x) \tilde{P}_{\mu \nu}^j(x) \). In general the existence of this integral requires some other conditions besides \( \text{Re}(\nu - \mu + 1) > 0 \). Both the result and the conditions in general form are rather cumbersome, and we shall not write them out here. It is essential, however, to point out that the functions \( \tilde{P}_{\mu \nu}^j(x) \) contain two systems orthogonal on the interval \([-1, 1]\). One of these systems is determined by the requirements
\[ \text{Re}(\nu - \mu + 1) > 0, \quad \text{Re}(\nu + \mu + 1) > 0, \quad j - \nu = n \geq 0, \] (34)
where \( n \) is an integer. The other system is determined by the requirements that are obtained from (34) by the interchange \( \mu \leftrightarrow -\nu \). As the first of the equations (5) shows, both orthogonal systems are connected with the Jacobi polynomials. If the function \( \tilde{P}_{\mu \nu}^j(x) \) belongs to one of the orthogonal systems, its norm is
\[ \int_{-1}^{+1} dx \, [\tilde{P}_{\mu \nu}^j(x)]^2 = \frac{2}{2j+1} \frac{\Gamma(j+\mu+1) \Gamma(j-\nu+1)}{\Gamma(j-\mu+1) \Gamma(j+\nu+1)}. \] (35)

As an example of the expansions that can occur we can present the following series:
\[ \frac{1}{\zeta - z} = e^{i\pi(\mu-\nu)} \left( \frac{z-1}{\zeta-1} \right)^{-(\nu-\mu)/2} \left( \frac{z+1}{\zeta+1} \right)^{-(\nu+\mu)/2} \times \sum_{j-\nu=n=0}^{\infty} (2j+1) P_{\mu \nu}^j(z) Q_{\nu \mu}^j(\zeta). \] (36)

**VII. THE FUNCTIONS** \( d_{\mu \nu}^j(x) \)

Let us now establish the connection between the generalized Legendre functions and the functions \( d_{\mu \nu}^j(x) \). If all of the indices \( j, \mu, \) and \( \nu \) are either integers, or else all half-integers, and if \( -|j + 1/2| < (\mu, \nu) < |j + 1/2| \), then we can define
\[ d_{\mu \nu}^j(x) = \tilde{P}_{\mu \nu}^j(x) \left[ \frac{\Gamma(j-\mu+1) \Gamma(j+\nu+1)}{\Gamma(j+\mu+1) \Gamma(j-\nu+1)} \right]^{1/2}. \] (37)
By the use of (6), (8), (21) and (24) it is easy to show that the following equations hold for $d_{j}^{\mu\nu}(x)$:

$$d_{j}^{\mu\nu}(x) = d_{j-1}^{\mu\nu}(x) = d_{-\nu,-\mu}^{j}(x) = (-1)^{\mu-\nu}d_{\nu\mu}^{j}(x)$$

$$= (-1)^{j-\nu}d_{-\mu,\nu}^{j}(-x) = (-1)^{j+\mu}d_{\mu,-\nu}^{j}(-x). \quad (38)$$

Besides this, the functions $d_{j}^{\mu\nu}(x)$ with different $j$ are orthogonal, and it follows from (35) that

$$\int_{-1}^{+1} d_{j}^{\mu\nu}(x) d_{j}^{\mu\nu}(x) dx = \frac{2}{2j+1} \delta_{ji}. \quad (39)$$

It is not hard to verify that the definition (37) corresponds to the choice of the functions $d_{j}^{\mu\nu}$ that is commonly used (cf., e.g., [1]).

**VIII. ADDITION AND MULTIPLICATION FORMULAS**

Addition theorems play a large part in the theory of Legendre functions, and also in that of the functions $d_{j}^{\mu\nu}$. It turns out that also for the generalized Legendre functions one can establish addition theorems in general form.

We first establish the equation

$$Q_{\mu\lambda}^{j}(z_{1}) Q_{\lambda\mu}^{j}(z_{2}) = \frac{1}{2} \int_{-\infty}^{\infty} d\alpha e^{-\lambda\alpha} e^{\mu\alpha_{2}} Q_{\mu\nu}^{j}(z_{\alpha}) e^{\nu\alpha_{1}}; \quad (40)$$

here $z_{\alpha}$, $\alpha_{1}$, and $\alpha_{2}$ are defined by the relations

$$z_{\alpha} = z_{1} z_{2} + \sqrt{z_{1}^{2} - 1} \sqrt{z_{2}^{2} - 1} \cosh \alpha,$$

$$z_{1} = z_{2} z_{\alpha} - \sqrt{z_{2}^{2} - 1} \sqrt{z_{\alpha}^{2} - 1} \cosh \alpha_{1},$$

$$z_{2} = z_{\alpha} z_{1} - \sqrt{z_{\alpha}^{2} - 1} \sqrt{z_{1}^{2} - 1} \cosh \alpha_{2},$$

$$\sinh \alpha \sqrt{z_{1}^{2} - 1} \sqrt{z_{2}^{2} - 1} = \sinh \alpha_{1} \sqrt{z_{2}^{2} - 1} \sqrt{z_{\alpha}^{2} - 1} = \sinh \alpha_{2} \sqrt{z_{\alpha}^{2} - 1} \sqrt{z_{1}^{2} - 1}$$

$$= \pm \sqrt{z_{\alpha}^{2} + z_{1}^{2} + z_{2}^{2} - 2z_{\alpha} z_{1} z_{2} - 1}. \quad (41)$$

---

4 We assume (at least, initially) that $z_{1}$, $z_{2} > +1$; then the sigh before the last square root coincides with the sign of $\alpha$ (the same is true for signs of $\alpha_{1}$, $\alpha_{2}$).
The requirement that the integral in (40) converges imposes the restrictions

\[ \Re(j - \lambda + 1) > 0, \quad \Re(j + \lambda + 1) > 0. \]

To prove Eq. (40), it suffices to note the following. The right member of (40) satisfies the same equation with respect to \( z_1 \) and \( z_2 \) as the left member. (This can be verified by a direct but rather cumbersome calculation.) When \( z_1 \) or \( z_2 \) increases, there is an equally rapid decrease of the right and left members. For \( z_1 \to \infty \) and \( z_2 \to \infty \) the coefficient in the asymptotic form is the same for the right and left members.

From (40) one easily gets the inverse equation

\[ e^{\mu \alpha_2} Q^j_{\mu \nu}(z_\alpha) e^{\nu \alpha_1} = \frac{1}{\pi i} \int_{-i\infty}^{i\infty} d\lambda e^{\lambda \alpha} Q^j_{\mu \lambda}(z_1) Q^j_{\lambda \nu}(z_2). \]  

(42)

In (42) it is assumed that \( \Re j \) is sufficiently large so that the poles of the integrand do not fall on the contour. \( \alpha \) is originally real, but (42) can be continued analytically with respect to \( \alpha_1, \, \zeta_1, \) and \( z_2 \) as long as the integral does not become divergent at infinity. For further continuation it is necessary to break up the integrand into components and rotate the path of integration. The path is then turned differently for different terms.

For \( \mu = \nu = 0 \) (42) is an integral form of the usual addition theorem for Legendre functions of the second kind [Eq. 3.11(4) in [5] or 8.795.2 in [6]]. In the general case it determines the addition theorem for generalized Legendre functions. Equation (40) is the inverse of (42), and it is natural to call it a multiplication formula.

By means of (40) and (42) we can obtain a large number of other addition and multiplication formulas.

For example, using (26) and (24) we can show that for \( z_1 > z_2 \) and \( \Re(j - \lambda + 1) > 0 \)

\[ e^{i\pi(\lambda - \nu)} Q^j_{\mu \lambda}(z_\lambda) P^j_{\lambda \nu}(z_\nu) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-i\lambda \theta} e^{-i\mu \theta_2} Q^j_{\mu \nu}(z_\theta) e^{-i\nu \theta_2}, \]

and for \( z_1 < z_2 \) and \( \Re(j + \lambda + 1) > 0 \)

\[ e^{i\pi(\mu - \lambda)} P^j_{\mu \lambda}(z_1) Q^j_{\lambda \nu}(z_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-i\lambda \theta} e^{-i\mu \theta_2} Q^j_{\mu \nu}(z_\theta) e^{-i\nu \theta_2}, \]

(43)
The quantities $z_\theta, \theta_1,$ and $\theta_2$ in (43) and (44) are defined by the relations\footnote{Again, we assume $z_1, z_2 > +1$; the sign before the square root (as well as the signs of $\sin \theta_1$ and $\sin \theta_2$) coincides with the sign of $\sin \theta$.}

\[
\begin{align*}
  z_\theta &= z_1 z_2 - \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1} \cos \theta, \\
  z_1 &= z_2 z_\theta - \sqrt{z_2^2 - 1} \sqrt{z_\theta^2 - 1} \cos \theta_1, \\
  z_2 &= z_\theta z_1 - \sqrt{z_\theta^2 - 1} \sqrt{z_1^2 - 1} \cos \theta_2, \\
  \sin \theta \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1} - 1 &= \sin \theta_1 \sqrt{z_2^2 - 1} \sqrt{z_\theta^2 - 1} - 1 = \sin \theta_2 \sqrt{z_\theta^2 - 1} \sqrt{z_1^2 - 1} - 1 = \pm \sqrt{1 + 2z_\theta z_1 z_2 - z_\theta^2 - z_1^2 - z_2^2}. \quad (45)
\end{align*}
\]

We note that in the limit $z_1 \to \infty$ Eqs. (40) and (43) give integral representations for the generalized Legendre functions, for example,

\[
Q^j_{\mu \nu}(z) = \frac{1}{2} e^{i\pi (\mu - \nu)} \frac{\Gamma(j - \nu + 1)}{\Gamma(j - \mu + 1)} \int_{-\infty}^{\infty} d\alpha e^{-\mu \alpha} \times \\
\times (\sqrt{z^2 - 1} + z \cosh \alpha + \sinh \alpha)^{-(j-\nu+1)/2} \times (\sqrt{z^2 - 1} + z \cosh \alpha - \sinh \alpha)^{-(j+\nu+1)/2}. \quad (46)
\]

The relations (43) and (44) allow us to write the addition theorem directly in the form of a series:

\[
\begin{align*}
  e^{-i\mu \theta_1} Q^j_{\mu \nu}(z\theta) e^{-i\nu \theta_1} &= \vartheta(z_1 - z_2) \sum_{\lambda - \nu = \eta} (-1)^{\lambda - \nu} Q^j_{\mu \lambda}(z_1) P^j_{\lambda \nu}(z_2) e^{i\lambda \theta} \\
  + \vartheta(z_2 - z_1) \sum_{\lambda - \mu = \eta} (-1)^{\mu - \lambda} P^j_{\mu \lambda}(z_1) Q^j_{\lambda \nu}(z_2) e^{i\lambda \theta}.
\end{align*} \quad (47)
\]

In (47) $n$ runs through all integer values, and $\vartheta(x)$ is the usual discontinuous function: $\vartheta(x) = 1$ for $x > 0$ and $\vartheta(x) = 0$ for $x < 0$. It is easy to write down one further addition theorem:

\[
\begin{align*}
  e^{-i\mu \theta_2} P^j_{\mu \nu}(z\theta) e^{-i\nu \theta_2} &= \vartheta(z_1 - z_2) \sum_{\lambda - \nu = \eta} (-1)^{\nu} P^j_{\mu \lambda}(z_1) P^j_{\lambda \nu}(z_2) e^{i\lambda \theta} \\
  + \vartheta(z_2 - z_1) \sum_{\lambda - \mu = \eta} (-1)^{\mu} P^j_{\mu \lambda}(z_1) P^j_{\lambda \nu}(z_2) e^{i\lambda \theta}.
\end{align*}
\]
\[ + \delta(z_2 - z_1) \sum_{\lambda-\mu=n=-\infty}^{+\infty} (-1)^n P^j_{\mu\lambda}(z_1) P^j_{\lambda\nu}(z_2) e^{i\lambda\theta}. \quad (48) \]

It is not hard to verify that (47) is equivalent to the analytic continuation of (42) to imaginary values of \( \alpha \). The addition theorem for \( P^j_{\mu\nu} \) can also be written in integral form. It is, however, much more cumbersome than (42) and we shall not write it out here.

In conclusion we note the following. As can be seen from the definitions (2) and (3), the theory of generalized Legendre functions is equivalent to the theory of hypergeometric functions. Therefore for each relation for generalized Legendre functions there are some corresponding relations for hypergeometric functions. In particular, there are some rather complicated nonlinear relations between hypergeometric functions which correspond to the addition and multiplication theorems. The author does not know whether these relations exist in the mathematical literature. In any case they are not in the well-known reference books [5, 6].

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\[ ^6 \] Even easier is to continue Eq. (48) into the region \(-1 < (z_1, z_2) < +1\). Then, if all the indices \( j, \mu, \nu \) are either integer or half-integer with \(-|j+1/2| < (\mu, \nu) < |j+1/2|\), the continued relation appears to be equivalent to the known addition theorem for the functions \( d^j_{\mu\nu} \).
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