Light-Front Approach for Heavy Pentaquark Transitions

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Abstract

Assuming the two diquark structure for the pentaquark state as advocated in the Jaffe-Wilczek model, there exist exotic parity-even anti-sextet and parity-odd triplet heavy pentaquark baryons. The theoretical estimate of charmed and bottom pentaquark masses is quite controversial and it is not clear whether the ground-state heavy pentaquark lies above or below the strong-decay threshold. We study the weak transitions of heavy pentaquark states using the light-front quark model. In the heavy quark limit, heavy-to-heavy pentaquark transition form factors can be expressed in terms of three Isgur-Wise functions: two of them are found to be normalized to unity at zero recoil, while the third one is equal to 1/2 at the maximum momentum transfer, in accordance with the prediction of the large- \( N_c \) approach or the quark model. Therefore, the light-front model calculations are consistent with the requirement of heavy quark symmetry. Numerical results for form factors and Isgur-Wise functions are presented. Decay rates of the weak decays \( \Theta_b^+ \rightarrow \Theta^0 \pi^+(\rho^+) \), \( \Theta_c^0 \rightarrow \Theta^+ \pi^-(\rho^-) \), \( \Sigma^\prime_{5b} \rightarrow \Sigma^0_{5c} \pi^+(\rho^+) \) and \( \Sigma^0_{5c} \rightarrow N^+_{5b} \pi^-(\rho^-) \) with \( \Theta_Q, \Sigma^\prime_{5Q} \) and \( N_8 \) being the heavy anti-sextet, heavy triplet and light octet pentaquarks, respectively, are obtained. For weakly decaying \( \Theta_b^+ \) and \( \Theta_c^0 \), the branching ratios of \( \Theta_b^+ \rightarrow \Theta_c^0 \pi^+ \), \( \Theta_c^0 \rightarrow \Theta^+ \pi^- \) are estimated to be at the level of \( 10^{-3} \) and a few percents, respectively.

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I. INTRODUCTION

The discovery of an exotic $\Theta^+$ pentaquark by LEPS at SPring-8 \[1\], subsequently confirmed by many other groups [2, 3, 6, 7, 8, 9, 10], marked a new era for testing our understanding of the hadron spectroscopy and promoted a re-examination of the QCD implications for exotic hadrons. The mass of the $\Theta^+$ is of order 1535 MeV and its width is less than 10 MeV from direct observations and can be as narrow as 1 MeV from the analysis of $K$-deuteron scattering data [11]. Another exotic pentaquark $\Xi_{3/2}^-$ with a mass of 1862 $\pm$ 2 MeV and a width smaller than 18 MeV was observed by NA49 [12]. In spite of the confirmation of the $\Theta^+$ from several experiments, the pentaquark candidate signals must be established beyond any doubt as all current experimental signals are weak and the significance is only of 4–6 standard deviations. Indeed, there are several null results of pentaquark search from e.g., BES [13] and HERA-B [14]. For example, the reaction $pA \rightarrow \Xi^\pm \pi^\mp$ at $\sqrt{s} = 41.7$ GeV has been studied by HERA-B and no narrow signal in the $\Xi^\pm \pi^\mp$ invariant mass distribution is seen. The $\Theta^+$ pentaquark is also not observed by NA49, BES and HERA-B.

In the conventional uncorrelated quark model, the $\Theta^+$ mass is expected to be of order 1900 MeV for a $S$-wave ground state with odd parity and 2200 MeV for a $P$-wave state with even parity. The width is at least of order 100 MeV as the strong decay $\Theta^+ \rightarrow KN$ is OZI super-allowed. Therefore, within the naive uncorrelated quark model one cannot understand why $\Theta^+$ is anomalously light and why its width is so narrow. This hints a possible correlation among various quarks; two or three quarks could form a cluster. Several quark cluster models have been proposed in the past. Jaffe and Wilczek [15] advocated a two diquark picture in which the $\Theta^+$ is a bound state of an $\bar{s}$ quark with two $(ud)$ diquarks. The diquark is a highly correlated spin-zero object and is in a flavor anti-triplet and color anti-triplet state. The parity of $\Theta^+$ is flipped from the negative, as expected in the naive quark model, to the positive owing to the diquark correlation. The even parity of the $\Theta^+$ is in agreement with the prediction of the chiral soliton model [16]. In the model of Karliner and Lipkin [17], $\Theta^+$ is composed of the diquark $(ud)$ and the triquark $(uds)$. Under the assumption that the diquark and triquark are in a relative $P$-wave, the resultant $\Theta^+$ also has a positive parity. Note that two of previous lattice calculations imply a negative parity for the $\Theta^+$ [18, 19]. However, based on the Jaffe-Wilczek picture to construct the interpolating operators, a recent quenched lattice QCD calculation with exact chiral symmetry yields a positive parity for the pentaquark states $\Theta^+$ and $\Xi_{3/2}^-$ [20].

A major distinction between the Jaffe-Wilczek model and the chiral soliton model is that the pentaquark baryons including $\Theta^+$ in the latter model are in the pure $U_f$ representation of the flavor SU(3), while the former has not only anti-decuplet but also octet even-parity pentaquark baryons. The nearly ideal mixing of $U_f$ and $S_f$ in the Jaffe-Wilczek model leads to the prediction that the pentaquark $\Xi_{3/2}^-$ is lighter than $\Sigma_{U_f}$, opposite to the chiral soliton model where $\Xi_{3/2}^-$ is the heaviest one among the anti-decuplet pentaquark baryons. Hence, a measurement of the $\Sigma_{U_f}$ mass can be used to discriminate between the chiral soliton model and any quark model with ideal mixed pentaquark states.

Although the Jaffe-Wilczek model is simple, powerful and leads to many concrete predictions,
it has its own difficulties: (i) Assuming that the octet pentaquark state with nucleon quantum numbers can be identified with the Roper resonance $N(1440)$, the predicted $\Xi_{\frac{3}{2}}$ mass of 1750 MeV is smaller than the NA49 measurement by an amount of 110 MeV. (ii) The predicted mass of the anti-decuplet $N_{\overline{10}}$ with nucleon quantum numbers is around 1700 MeV. It is tempted to identify this with $N(1710)P_{11}$ as suggested by Jaffe and Wilczek. However, as pointed out in [21], the identification of the two nucleon pentaquark states as the Roper resonance and the $N(1710)$ is inconsistent with the phenomenology of the widths given the assumptions underlying the model and the narrow $\Theta^+$ width: The broad Roper resonance will imply a large partial width of $N(1710)$. This means that the model requires the existence of at least one presently unknown pentaquark state with nucleon quantum numbers and a mass in the neighborhood of 1700 MeV. (iii) The model also predicts a negative-parity nonet of pentaquark states which are supposed to be lighter than the even-parity octet pentaquark baryons owing to the lack of the $P$-wave excitation energy. However, except for the SU(3)-singlet $\Lambda(1405)$ which may be a pentaquark state as indicted by recent lattice calculations [22], the observed spectroscopy of the $\frac{1}{2}^-$ states $N(1535), \Sigma(1620), \Lambda(1670)$ is known to be well described as the orbital excited states of three-quark baryons. The question is then why the new $\frac{1}{2}^-$ octet pentaquark states have not been found in the energy regime which has been well explored. (iv) Nussinov has pointed out that a strict point-like diquark picture of the $\Theta^+$ conflicts with QCD inequalities [23]. This suggests that the picture of a point-like diquark scalar may miss significant hyperfine interactions between the $\bar{s}$ and each of the four quarks in the pentaquark. This will be elaborated more in Sec. II.B.

Given the existence of the $\Theta^+$ pentaquark, it is natural to consider its heavy flavor analogs $\Theta^0_c$ and $\Theta^+_b$ by replacing the $\bar{s}$ quark in $\Theta^+$ by the heavy antiquark $\bar{c}$ and $\bar{b}$, respectively. Whether the masses of the heavy pentaquark states are above or below the strong-decay threshold has been quite controversial. Very recently, a narrow resonance in $D^{*-}p$ and $D^{*+}\bar{p}$ invariant mass distributions was reported by the H1 Collaboration [24]. It has a mass of $3099\pm3\pm5$ MeV and a Gaussian width of $12\pm3$ MeV and can be identified with the spin 1/2 or 3/2 charmed pentaquark baryon. Although this state is about 300 MeV higher than the $DN$ threshold, it is possible that the observed H1 pentaquark is a chiral partner of the yet undiscovered ground state $\Theta^0_c$ with opposite parity and a mass of order 2700 MeV as implied by several model estimates [25] (see Sec. II.B for more details). The latter pentaquark can be discovered only through its weak decay.

In the Jaffe-Wilczek model, there exist parity-even antisextet and parity-odd triplet heavy pentaquarks containing a single heavy antiquark $\bar{c}$ or $\bar{b}$ and they are all truly exotic. The heavy pentaquark baryons in the $3_f$ representation are lighter than the $6_f$ ones due to the lack of orbital excitation and therefore may be stable against strong decays [26]. Consequently, it becomes important to study the weak and electromagnetic decays of heavy pentaquarks.

It is well known that the study of nonleptonic weak decays of baryons is much more complicated than the meson case for several reasons. First, the baryon is made of three quarks and hence there exist many more quark diagrams responsible for the weak decays of baryons. Second, the factorizable approximation that the hadronic matrix element is factorized into the product of two matrix elements of single currents and that the nonfactorizable term such as the $W$-exchange contribution is negligible relative to the factorizable one works empirically reasonably well for
describing the hadronic weak decays of heavy mesons. However, this approximation is \textit{a priori} not directly applicable to heavy baryon decays. For example, the $W$-exchange diagram is no longer subject to helicity and color suppression. Indeed, it is known that $W$ exchange plays an essential role in describing the data of charmed baryon decays.

At first sight, it is natural to expect a much more complicated dynamics for the nonleptonic weak decays of pentaquark baryons. Nevertheless, the underlying mechanism is greatly simplified if the pentaquark can be approximately described by the diquark picture of Jaffe and Wilczek. In this scenario it is equivalent to working on the effective three-body problem. Moreover, in the heavy quark limit, pentaquark transitions can be described in terms of several universal Isgur-Wise functions. The point is that in the infinite quark mass limit, the heavy quark spin $S_Q$ decouples from the other degrees of freedom of the hadron, so that $S_Q$ and the total angular momentum $j$ of the light quarks are separately good quantum numbers. Whether the total spin of the light degrees of freedom comes from the quark spin plus orbital angular momentum or just from the quark spin is irrelevant. As a result, the previous studies of heavy-to-heavy baryon transitions using heavy quark effective theory in nineties can be easily generalized to the heavy-to-heavy pentaquark transitions. The heavy quark limit results are model independent and hence must be respected by any explicit model calculations.

In the present work we shall study the pentaquark transitions using the relativistic light-front quark model and assuming the two diquark structure for the pentaquarks as described by the Jaffe-Wilczek model. The light-front model allows us to study the transition form factors and their momentum dependence. Furthermore, large relativistic effects which may manifest near the maximum large recoil, i.e. $q^2 = 0$, are properly taken into account in the light-front framework. In the literature the weak decays of heavy pentaquark states have been studied in [26, 27, 28].

The layout of the present paper is organized as follows. In Sec. II we discuss the mass spectrum of heavy pentaquark baryons in the diquark picture. In Sec. III we present a detailed study of the pentaquark weak transitions within the light-front quark model and derive the analytic expressions for form factors. The heavy quark limit behavior of the form factors is studied in Sec. IV and the universal Isgur-Wise functions are obtained. Numerical results for form factors, Isgur-Wise functions and examples of weak decays of heavy pentaquark baryons are worked out in Sec. V. Conclusion is given in Sec. VI.

II. DIQUARK MODEL FOR HEAVY PENTAQUARKS

A. Jaffe-Wilczek model

In the pentaquark model of Jaffe and Wilczek [15], the heavy flavor pentaquark consists of a heavy antiquark $\bar{Q}$ and two diquark pairs $[q_1q_2]$ and $[q_3q_4]$. The diquark $[q_1q_2]$ is a highly correlated spin-zero object and is in a flavor anti-triplet $\bar{3}_f$ and color anti-triplet $\bar{3}_c$ representation. The two diquarks form a flavor anti-sextet $\bar{6}_f$ or triplet $3_f$ configuration. The diquark pair must be in color $3_c$ state in order to form a color-singlet pentaquark. Bose statistics of the scalar diquarks requires that the diquark pairs in the flavor symmetric $\bar{6}_f$ (antisymmetric $3_f$) state be in an orbital
Therefore, there is a non-exotic nonet of pentaquark baryon with negative parity and flavor content $P$ (S) wave or a spatially antisymmetric (symmetric) state. Since the heavy quark is an SU(3) flavor-singlet, it is clear that heavy pentaquarks form an SU(3)-flavor antiseptet with even parity and triplet with odd parity. The members of $\bar{6}_f$ are the quark $3_f$ are $\Sigma_{5c}^{0,0}$ and $\Xi_{5c}^{0,0,0}$, while members of $3_f$ are $\Sigma_{5c}^{0,0}$ and $\Xi_{5c}^{0,0}$ (we follow [29] for the notation). The corresponding flavor wave functions are summarized in Table I. Note that the resulting $J^P$ of $\bar{6}_f$ pentamers is either $1/2^+$ or $3/2^+$, while $J^P$ is equal to $1/2^-$ for $3_f$ heavy pentamers. However, the $3_f$ pentaquark $P_Q(3)$ is expected to be lighter $P_Q(6)$ owing to the absence of the $P$-wave orbital excitation. However, this may be compensated by the Pauli blocking repulsion as we shall discuss below in more detail.

In the Karliner-Lipkin model [17, 30], the pentaquark is composed of a triquark $q_1q_2Q$ and a diquark $[q_3q_4]$. The two quarks $q_1q_2$ in the triquark are in $6_c$ and $\bar{3}_f$ representations, while the spin zero diquark $[q_3q_4]$ is in $\bar{3}_c$ and $\bar{3}_f$ configuration. Consequently, we have $\bar{6}_f$ and $3_f$ heavy pentaquarks. However, unlike the Jaffe-Wilczek model, Karliner and Lipkin assumed a $P$-wave orbital angular momentum between the two clusters. As a consequence, the resulting $3_f$ heavy pentaquark has $J^P = 1/2^+$ and is degenerate with the $\bar{6}_f$ one. Therefore, a study of the parity of triplet heavy pentaquarks will enable us to discriminate between Jaffe-Wilczek and Karliner-Lipkin models.

**TABLE I:** Flavor wave functions of heavy pentaquarks in the Jaffe-Wilczek model, where $Q = \bar{c}$ or $b$ and $[q_1q_2][q_3q_4] = \sqrt{2}(1)$. | State | Flavor wave function | State | Flavor wave function |
|---|---|---|---|
| $\bar{3}_c$, $\bar{3}_b$ | $|ud|^2 \bar{Q}$ | $\Sigma_{5c}^{0,0}$, $\Sigma_{5b}^{0}$ | $|ud|[us]_+ \bar{Q}$ |
| $\Sigma_{5c}^{0,0}$ | $|ud|[us]_+ \bar{Q}$ | $\Sigma_{5c}^{0,0}$, $\Sigma_{5b}^{0,0}$ | $|us|\bar{Q}$ |
| $\Sigma_{5c}^{0}$ | $|ud|[ds]_+ \bar{Q}$ | $\Sigma_{5c}^{0}$, $\Sigma_{5b}^{0}$ | $|us|\bar{Q}$ |
| $\Xi_{5c}^{0}$ | $|us|\bar{Q}$ | $\Xi_{5c}^{0,0}$, $\Xi_{5b}^{0}$ | $|us|\bar{Q}$ |
| $\Xi_{5c}^{0}$, $\Xi_{5b}^{0}$ | $[ds]^2 \bar{Q}$ |

For light pentaquarks in the Jaffe-Wilczek model, the diquark pair $[q_1q_2][q_3q_4]$ is combined with a light antiquark to form a pentaquark baryon. When the diquark pair is in the flavor-symmetric $\bar{6}_f$ representation, the flavor content of the resulting even-parity pentaquark states is $8_f \oplus \bar{10}_f$. When $q_1 \neq q_2$ and $q_3 \neq q_4$, the three diquark pairs $[ud][us]$, $[ds][su]$ and $[su][ud]$ can be antisymmetrized in flavor and hence they are allowed to have symmetric, positive parity, spatial wavefunctions. Therefore, there is a non-exotic nonet of pentaquark baryons with negative parity and flavor content $3_f \otimes 3_f = 1_f \oplus 8_f$. It is naively expected that the $1_f$ nonet is lighter than the $3_f$ octet pentaquarks owing to the lack of orbital excitation. However, since the two diquarks in the former are in a relative

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1 The notation is opposite to the case of charmed baryons where the unprimed states $\Sigma_c$ and $\Xi_c$ are for $6_f$ and the primed states $\Sigma'_c$ and $\Xi'_c$ are for $3_f$. In [29], the charm and bottom pentaquarks are denoted by $T_a$ and $R_a$, respectively. More precisely, $T_s^{0,0}$, $T_{ss}^{0,0}$, $R_s^{+,0}$, $R_{ss}^{0,0}$ correspond to our $\Sigma_{5c}^{0,0}$, $\Xi_{5c}^{0,0}$, $\Sigma_{5b}^{0,0}$, $\Xi_{5b}^{0,0}$. 

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S wave, they have substantial overlap at short distances. Consequently, the identical particles in different diquarks (e.g. the u quark in \([us][ud]\)) will experience a repulsive interaction due to Pauli blocking \[15\]. Jaffe and Wilczek conjectured that this effect is strong enough to elevate the masses of the resultant negative parity pentaquarks to the range of 1.5–2.0 GeV. \(^2\) These baryons are expected to be very board. The negative-parity baryon resonances that have been observed are the nonet states \(N(1535), \Sigma(1620), \Lambda(1670), \Lambda(1405)\) and the octet states \(N(1650), \Sigma(1750), \Lambda(1800)\) \[32\]. As stated in the Introduction, \(N(1535), \Sigma(1620), \Lambda(1670)\) are known to be well described as the orbital excited states of three-quark baryons, while recent lattice calculations indicate that the SU(3)-singlet \(\Lambda(1405)\) could be a pentaquark state \[22\]. Hence, it is not clear if the second octet states \(N(1650), \Sigma(1750), \Lambda(1800)\) can be identified as \(^{1,2}\) pentaquarks. At any rate, it seems plausible to argue that the Pauli blocking effect is at least comparable to the \(P\)-wave excitation energy.

**B. Masses of heavy pentaquark states**

We first consider the masses of \(\Theta^+_{\bar{c}}\) and \(\Theta^0_c\). Arguing that the \([ud]\) diquark in the \(\Lambda_{\bar{c}}\) and \(\Lambda\) experiences nearly the same environment as in \(\Theta^0_c\) and \(\Theta^+\), respectively, Jaffe and Wilczek estimated that \[13\]

\[
m(\Theta^0_c) = m(\Theta) + m(\Lambda_{\bar{c}}) - m(\Lambda) = 2710 \text{ MeV},
\]

\[
m(\Theta^+_c) = m(\Theta) + m(\Lambda_{\bar{b}}) - m(\Lambda) = 6050 \pm 10 \text{ MeV}. \quad (2.1)
\]

Assuming the color-spin interaction as the dominant mechanism for hyperfine splitting, Cheung obtained \[33\]

\[
m(\Theta^0_c) = (2938 - 2997) \text{ MeV}, \quad m(\Theta^+_c) = (6370 - 6422) \text{ MeV} \quad (2.2)
\]

in the Jaffe-Wilczek model, while Karliner and Lipkin found \[30\]

\[
m(\Theta^0_c) = 2985 \pm 50 \text{ MeV}, \quad m(\Theta^+_c) = 6389 \pm 50 \text{ MeV} \quad (2.3)
\]

in their diquark-triquark model. It appears that the latter two calculations yield similar results. Since the threshold for strong decays into \(DN\) and \(BN\) is 2805 and 6217 MeV, respectively, it is clear that the strong decays \(\Theta^0_c \to D^-p\) and \(\Theta^+_c \to B^0p\) will be kinematically allowed if the \(\Theta^0_c\) and \(\Theta^+_c\) masses are those given by \(2.2\) or \(2.3\). However, it should be stressed that the color-spin hyperfine interaction formula employed by Cheung or by Karliner and Lipkin is applicable in principle only to \(S\)-wave hadronic systems \[33\]. Moreover, it is not clear if the color-spin interaction is the dominant hyperfine splitting effect.

It is worth mentioning the estimate of the \(\Theta^0_c\) mass in other calculations. In the chiral soliton model, the heavy pentaquark has been described as a bound state of the SU(2) chiral soliton and a heavy meson in terms of the bound state approach of \[34\]. The predicted \(\Theta^0_c\) mass lies below the

\(^2\) By neglecting the Pauli blocking effect, the masses of the \(^{1,2}\) pentaquark nonet are estimated in \[31\] to lie in the range of 1360 to 1540 MeV.
$$m(\Theta_c) = \frac{m(\Theta_c) + 2m(\Theta^*_c)}{3}$$

More recently, the average $\Theta_c$ mass, defined by

$$m(\Theta_c) = \frac{m(\Theta_c) + 2m(\Theta^*_c)}{3}$$

is consistent with the estimate by Jaffe and Wilczek. It yields 2704 MeV and is again very close to the estimate of (2.21). An early calculation in a Goldstone boson exchange model also obtained a $\Theta_c$ mass below the strong decay threshold (2.37). In contrast, a lattice calculation by Sasaki (19) leads to $m(\Theta^0_c) = 3445$ MeV, which is 640 MeV above the $DN$ threshold. Another quenched lattice QCD calculation with exact chiral symmetry predicts a mass of $2977 \pm 109$ MeV for the $\Theta^0_c$ (28).

At first sight, it appears that the aforementioned controversy about the charmed pentaquark mass will be settled down by the recent H1 observation of a narrow resonance in $D^*-p$ and $D^{*+}\bar{p}$ invariant mass distributions which can be identified with the spin $1/2 \Theta^0_c$ and/or spin $3/2 \Theta^*_0_c$ (24). However, as pointed out in (25), it is possible that the H1 state $\Theta^0_c(3099)$ is a chiral partner of another yet undiscovered ground state pentaquark $\Theta^0_c(2700)$ with opposite parity. In this case, $\Theta^0_c(2700)$ can be discovered only by studying its weak decays. Therefore, it is important to measure the parity of the H1 state $\Theta^0_c(3099)$. If it has an odd parity, this may imply the existence of a parity-even charmed pentaquark baryon with a mass below the $DN$ threshold. In this case, one can follow the argument of Jaffe and Wilczek to estimate the masses of other heavy pentaquark baryons to be

$$m(\Sigma^0_{bc}) = m(N^{++}_{10}) + \frac{1}{2} \left[ m(\Lambda_c) - m(\Lambda) + m(\Xi_c) - m(\Xi) \right] = 2860 \text{ MeV},$$

$$m(\Sigma^+_{bb}) = m(N^{++}_{10}) + \frac{1}{2} \left[ m(\Lambda_b) - m(\Lambda) + m(\Xi_b) - m(\Xi) \right] = 6199 \text{ MeV},$$

$$m(\Xi^0_{bc}) = m(\Xi^{3/2}) + m(\Xi_c) - m(\Xi) = 3014 \text{ MeV},$$

$$m(\Xi^0_{bb}) = m(\Xi^{3/2}) + m(\Xi_b) - m(\Xi) = 6351 \text{ MeV},$$

where use of $m(N^{++}_{10}) = 1700$ MeV (15), $m(\Xi^{3/2}) = 1862$ MeV (12) and $m(\Xi_b) = 5804$ MeV (39) has been made. Note that our numerical results are slightly different than that in (29).

If the H1 state turns out to have a positive parity and is a truly ground-state charmed pentaquark, it will have an important implication for the Jaffe-Wilczek model, namely, the diquark should not be treated as an idealized point-like scalar and there are significant hyperfine attractive interactions between the light antiquark and other four quarks of the pentaquark state (23). The argument goes as follows. The observed $\Theta^0_c$ mass can be accounted for in the uncorrelated quark model in which the constituent quark masses of the charmed quark and the light $u$ or $d$ quark are of order 1700 and 350 MeV, respectively. However, the narrow width of $12 \pm 3$ MeV cannot be explained by this naive model as the pentaquark is in the $S$-wave state. The $\Theta^0_c$ width will be suppressed if its parity is even: The spatial separation between the quarks due to the centrifugal barrier arising from the orbital angular momentum will help reduce the decay width. (Of course, small widths are not necessary the consequence of the centrifugal barrier.) Since the $P$-wave excitation energy is of order 300 MeV (see below), this means that the effective mass of the diquark [$ud$] in the Jaffe-Wilczek model is about 550 MeV. Presumably, the hyperfine interaction of each quark of the diquark pair with the $\bar{c}$ is small. However, in order to account for the $\Theta^+$ mass, the effective mass of the diquark should be only of order 400 MeV. This implies that for the light pentaquark $\Theta^+$, each quark of the diquark pair has significant hyperfine attractive interactions with the $\bar{s}$. In
the constituent quark picture, this is equivalent to stating that the correlation for the $ud$ pair is stronger in $\Theta^+$ than in $\Theta^0_c$. It is worth remarking that the hyperfine splitting arising from the interaction of the diquark pair with the antiquark is found to be proportional to $-m_u/m_s$ and $-m_u/m_c$ for $\Theta^+$ and $\Theta^0_c$, respectively, in [33].

We next turn to the triplet penatquark $P_Q(3)$ and the spin 3/2 antisextet pentaquark $P_Q^*(6)$. In heavy quark effective theory (HQET), the mass of a baryon containing a single heavy quark has the $1/m_Q$ expansion [40]

$$m(H_Q) = m_Q + \bar{\Lambda}_{H_Q} - \frac{\lambda_1}{2m_Q} - d_H \frac{\lambda_2}{2m_Q} + O\left(\frac{1}{m_Q^2}\right),$$

(2.5)

where

$$\lambda_1 = \langle H_Q(v)|\bar{Q}_v(iD)^2Q_v|H_Q(v)\rangle,$$

$$d_H\lambda_2 = \langle H_Q(v)|\bar{Q}_v g\sigma_{\mu\nu}G^{\mu\nu}Q_v|H_Q(v)\rangle.$$  

(2.6)

The three nonperturbative HQET parameters $\bar{\Lambda}_{H_Q}$, $\lambda_1$, $\lambda_2$ in above equations are independent of the heavy quark mass and $\bar{\Lambda}_{H_Q}$, the mass of the light degrees of freedom of the hadron, in general varies for different heavy hadrons. Since $\sigma \cdot G \sim \vec{S}_Q \cdot \vec{B}$ and since the chromomagnetic field is produced by the light cloud inside the hadron, it is clear that $\sigma \cdot G$ is proportional to $\vec{S}_Q \cdot \vec{S}_\ell$. Hence, the Clebsch factor $d_H$ has the expression

$$d_H = -\langle H_Q|4\vec{S}_Q \cdot \vec{S}_\ell|H_Q\rangle$$

$$= -2[S_{tot}(S_{tot} + 1) - S_Q(S_Q + 1) - S_\ell(S_\ell + 1)],$$

(2.7)

where $\vec{S}_Q$ ($\vec{S}_\ell$) is the spin operator of the heavy quark (light cloud). Since $S_\ell = 1$ (0) for the $P_Q^*(6)$ ($P_Q(3)$) states, it follows that $d_H = 0$ for triplet pentaquark states, $d_H = 4$ for spin-$\frac{3}{2}$ antisextet pentaquarks and $d_H = -2$ for the spin-$\frac{3}{2}$ antisextet pentaquark baryons. For a study of $\bar{\Lambda}_{H_Q}$ in terms of $1/N_c$ expansion, see [39].

We first discuss the mass of the spin 3/2 pentaquark $P_Q^*(6)$. It has the expression

$$m(P_Q^*(6)) = m(P_Q(6)) + \frac{6\lambda_2^{penta}}{2m_Q}.$$  

(2.8)

Although we do not know the magnitude of $\lambda_2^{penta}$ for the pentaquark system, the normal heavy baryons can provide a useful guideline. The parameter $\lambda_2^{baryon}$ can be extracted from the mass splitting between $\frac{3}{2}^+$ and $\frac{1}{2}^+$ sextet heavy baryons, for example [32]

$$m[\Sigma_c(3/2^+)] - m[\Sigma_c(1/2^+)] \approx m[\Xi_c(3/2^+)] - m[\Xi_c(1/2^+)] \approx 65\,\text{MeV}.$$  

(2.9)

The result is [41]

$$\lambda_2^{baryon} = \begin{cases} 
0.055\,\text{GeV}^2 & \text{for charmed baryons;} \\
0.041\,\text{GeV}^2 & \text{for } \Xi_b; \\
0.040\,\text{GeV}^2 & \text{for } \Xi_b'; \\
0.039\,\text{GeV}^2 & \text{for } \Omega_b.
\end{cases}$$

(2.10)
We see that the heavy quark symmetry requirement that \( \lambda_2 \) be independent of the heavy quark mass is fairly respected. From Eqs. (2.8) and (2.10) we find

\[
m(\mathcal{P}_c^*(\bar{6})) = m(\mathcal{P}_c(\bar{6})) + 65 \text{ MeV}, \quad m(\mathcal{P}_b^*(\bar{6})) = m(\mathcal{P}_b(\bar{6})) + 20 \text{ MeV},
\]

(2.11)

when \( \lambda_2^{\text{penta}} \) is set to \( \lambda_2^{\text{baryon}} \).

As for the triplet pentaquark state \( \mathcal{P}_Q(3) \), it is naively expected to be lighter than \( \mathcal{P}_Q(\bar{6}) \) by the amount of the orbital excitation energy, which can be roughly estimated from the mass difference between various \( \frac{1}{2}^- \) and \( \frac{1}{2}^+ \) baryons to be [32]:

\[
m[\Lambda(1/2^-)] - m[\Lambda(1/2^+)] = 290 \text{ MeV},
\]

\[
m[\Lambda_c(1/2^-)] - m[\Lambda_c(1/2^+)] = 308.9 \pm 0.6 \text{ MeV},
\]

\[
m[\Xi_c^+(1/2^-)] - m[\Xi_c^+(1/2^+)] = 318.2 \pm 3.2 \text{ MeV},
\]

\[
m[\Xi_c^+(1/2^-)] - m[\Xi_c^+(1/2^+)] = 324.0 \pm 3.3 \text{ MeV}.
\]

(2.12)

However, one should also take into account the Pauli blocking effect and the intrinsic mass difference between \( 3_f \) and \( \bar{6}_f \). Denoting \( \delta P \) and \( \delta B \) as the P-wave excitation energy and Pauli blocking repulsion, respectively, the masses of the triplet heavy pentaquarks have the expressions:

\[
m(\mathcal{P}_Q') = m(\mathcal{P}_Q) + \delta M_Q - \delta P_Q + \delta B_Q
\]

(2.13)

where \( \delta P_Q \approx 310 \text{ MeV} \) is suggested by the charmed baryon data, and the quantity

\[
\delta M_Q = \bar{\Lambda}_3^{\text{penta}} - \bar{\Lambda}_6^{\text{penta}} + \frac{4\lambda_2^{\text{penta}}}{2m_Q}
\]

(2.14)

is basically the mass difference between spin 1/2 triplet and antisextet pentaquarks with the same parity. It is shown in [39] that an 1/3\( N_c \) expansion of \( \Lambda \) yields \( \Lambda = c_0 N_c + c_2 \frac{S^2}{N_c} \) and hence \( \bar{\Lambda}_6 > \bar{\Lambda}_3 \). For normal charmed baryons, \( \bar{\Lambda}_6^{\text{baryon}} - \bar{\Lambda}_3^{\text{baryon}} = 144 \text{ MeV} \) inferred from the measured mass difference of \( \Xi_c' \) and \( \Xi_c^* \): \( \delta M_c^{\text{baryon}} = m(\Xi_c) - m(\Xi_c') = -107 \text{ MeV} \) [32]. This then leads to \( \delta M_b^{\text{baryon}} \approx -131 \text{ MeV} \). Presumably, \( \delta M_Q^{\text{penta}} \) is not far from \( \delta M_b^{\text{baryon}} \) for heavy pentaquark baryons.

By neglecting the Pauli blocking effect and setting \( \delta M_Q^{\text{penta}} = 0 \), it was argued in [26] that \( \mathcal{P}_Q(3) \) is stable against strong decays. We see that whether the triplet pentaquark is stable against strong decays depends on the Pauli blocking repulsion, \( \delta M_Q \) and the mass of \( \mathcal{P}_Q(\bar{6}) \). Nevertheless, it is most likely that the triplet pentaquark is lighter than the antisextet one unless the Pauli blocking effect is unusually large.

C. Decays of heavy pentaquark states

The heavy pentaquarks can be studied via their strong, electromagnetic and weak decays as discussed below.

(i) strong decays:

\[
(a) \quad \mathcal{P}_Q^{(*)}(\bar{6}) \to \mathcal{B}(8) + M_Q,
\]

\[
(b) \quad \mathcal{P}_Q^{(*)}(\bar{6}) \to \mathcal{P}_Q(3) + M,
\]

\[
(c) \quad \mathcal{P}_Q(3) \to \mathcal{B}(8) + M_Q,
\]

(2.15)
where $B$ stands for the normal baryon made of three quarks. Examples in this category are $\Xi_{5c}^0 \rightarrow \Sigma_{5c}^0 \pi^0$, $D_s^- \Sigma^+$, $\Sigma_{5c}^0 \rightarrow \Sigma_{5c}^0 \pi^+$ and $\Sigma_{5c}^0 \rightarrow T^0 \Sigma^0, D^- \Sigma^+, D_s^- p$. Of course, whether the above-mentioned strong decays are kinematically allowed or not depends on the heavy pentaquark masses.

(ii) electromagnetic decays:

$$(d) \quad P_Q^*(\bar{6}) \rightarrow P_Q(\bar{6}) + \gamma,$$

$$(e) \quad P_Q(\bar{6}) \rightarrow P_Q(3) + \gamma.$$  \hspace{1cm} (2.16)

The decays $\Sigma_{5c}^* \rightarrow \Sigma_{5c}^- \gamma$, $\Sigma_{5c} \rightarrow \Sigma_{5c}^0 \gamma$ and $\Xi_{5c} \rightarrow \Xi_{5c}^\prime \gamma$ are examples. These radiative decays are kinematically allowed.

(iii) weak decays:

$$(f) \quad P_6(\bar{6}) \rightarrow P_6(\bar{6}) + M,$$

$$(g) \quad P_6(\bar{6}) \rightarrow P_6(3) + M,$$

$$(h) \quad P_Q(\bar{6}) \rightarrow B(8) + M,$$  \hspace{1cm} (2.17)

$$(i) \quad P_e(\bar{6}) \rightarrow P_e(10) + M,$$

$$(j) \quad P_6(3) \rightarrow P_6(3) + M,$$

$$(k) \quad P_Q(3) \rightarrow B(8) + M.$$  

Examples are $\Theta_{6}^{+} \rightarrow \Theta_{6}^{0} \pi^{+}$, $\Theta_{6}^{0} \rightarrow \Theta_{6}^{+} \pi^{-}$, $\Sigma_{5b}^{+} \rightarrow \Sigma_{5c}^{0} \pi^{+}$, $\Sigma_{5b}^{0} \rightarrow N_{10}^{+} \pi^{-}$, $\Xi_{5c}^{*+} \rightarrow \Lambda \pi^{-}$. In addition to nonleptonic decays, there are also semileptonic weak decays.

Heavy pentaquark states can be produced in weak decays of heavy hadrons, in $e^+ e^-$ annihilation and at hadronic colliders [42, 43, 44]. If the $P_Q$ lies above the strong-decay threshold, it will appear as a narrow $M_Q\bar{B}$ resonance. If pentaquarks lie below the threshold, then $P_Q(3), \Theta_Q, \Xi_{5c}^{0}, \Xi_{5b}^{0}$ will have only weak decays, while $\Sigma_Q, \Xi_{5c}^{-}, \Xi_{5b}^{-}$ will be dominated by the electromagnetic interactions. In the above listed decays, the strong decay (b) can be studied using the soft pion theorem. The strong coupling constant involved in this process can be related through PCAC to the matrix element of the axial-vector current between the initial and final heavy pentaquark states with the same heavy flavor. The weak decay modes (f), (g), (i) and (j) receive factorizable contributions which can be expressed as the product of the meson decay constant and the matrix element between the initial and final heavy pentaquark states with different heavy flavors. The process (g) will be vanished if the meson is emitted from the $b \rightarrow c$ transition. The diquarks of the pentaquark behave as spectators in this category of weak decays. The decays of the pentaquark to ordinary octet baryons (h) and (k), e.g. $\Sigma_{5c}^{0} \rightarrow \Lambda K^{0}$ and $\Xi_{5c}^{0} \rightarrow \Xi^{-} K^{0}$, proceed via nonfactorizable $W$ exchange and hence the diquark correlation is broken. Other hadronic decay channels of $P_Q(3)$ can be found in [20]. It is interesting to notice that the E791 Collaboration has searched for the triplet pentaquark $\Sigma_{5c}^{0}$ via the decays $\Sigma_{5c}^{0} \rightarrow \phi \pi p$ and $K^{*0} K^{-} p$ with null result [45].

### III. FORMALISM OF A LIGHT-FRONT MODEL FOR PENTAQUARKS

As stressed in Sec. II, the theoretical estimate of charmed and bottom pentaquark masses and the issue of whether the ground-state heavy pentaquark lies above or below the strong-decay
threshold are quite controversial. Even if the H1 state $\Theta^0(3099)$ is confirmed, the existence of a ground-state charmed pentaquark with opposite parity and smaller mass is not ruled out. In this section we shall focus on the hadronic weak decays of heavy pentaquarks and study the pentaquark weak transitions within the light-front approach and the Jaffe-Wilczek model. As mentioned above, the hadronic two-body weak decays of a heavy pentaquark receive factorizable contributions if the final-state baryon is a pentaquark. This is not the case for a pentaquark decaying to an ordinary baryon. Therefore, we will concentrate on $P_Q \to P_{Q'}$ weak transitions where the diquark structure is maintained. The corresponding figure is shown in Fig. 1.

![Feynman diagram](image)

FIG. 1: Feynman diagram for a typical $P_Q \to P_{Q'}$ transition, where the spin-zero diquarks ($[qq] = [ud],[us],[ds]$) are denoted by dashed lines and the corresponding $V - A$ current vertex by $X$.

A. Vertex functions in the light-front approach

We adopt the Jaffe-Wilczek picture \[15\] for the heavy pentaquark $P_Q$ which has the quark flavor content $\bar{Q}[q_1q_2][q_3q_4]$. For the purpose of the calculational convenience, we shall treat the antiquark $\bar{Q}$ as a particle $Q^c$ instead of an antiparticle, i.e. we use the charge conjugated field for the heavy flavor. The reason of this seemingly odd choice will become clear in later calculations. Note that although we consider $P_Q$, the formalism developed in this section can be easily generalized to light pentaquark states, especially, for $\Theta^+_s \sim \bar{s}[ud][ud]$.

In the light-front approach, the pentaquark bound state with the total momentum $P$, spin $J = 1/2$ and the orbital angular momentum of the diquark pair $L = 0, 1$ can be written as (see, for example \[16, 47\])

$$|P_Q(P, L, S_z)\rangle = \int \{d^3p_1\} \{d^3p_2\} \{d^3p_3\} \frac{2(2\pi)^3}{\sqrt{P^+}} \delta^3(\tilde{P} - \tilde{p}_1 - \tilde{p}_2 - \tilde{p}_3) \sum_{\lambda_1, m, \alpha - \epsilon, b - e} \Psi_L^S(x_1, x_2, x_3, k_{1\perp}, k_{2\perp}, k_{3\perp}, \lambda_1) C_{\beta\gamma, \delta\epsilon}^{bc, de} F_L^{bc, de} \times \left| Q^c_\alpha(p_1, \lambda_1)[q^\beta_1 q^\gamma_2](p_2)[q^\delta_3 q^\epsilon_4](p_3) \right|, \quad (3.1)$$

where $\alpha, \cdots, \epsilon$ and $b, \cdots, e$ are color and flavor indices, respectively, $\lambda$ denotes helicity, $p_1, p_2$ and $p_3$ are the on-mass-shell light-front momenta,

$$\tilde{p} = (p^+, p_\perp), \quad p_\perp = (p^1, p^2), \quad p^- = \frac{m^2 + p^2_\perp}{p^+}, \quad (3.2)$$
\[ \begin{align*}
\{d^3p\} & \equiv \frac{dp^+ dp^2 dp_\perp}{(2\pi)^3}, \quad \delta^3(\vec{p}) = \delta(p^+) \delta^2(p_\perp), \\
|Q_p(p_1, \lambda_1)[q_a q_c](p_2)[q_d q_e](p_3)\rangle & = d^3\chi_1(p_1) a^\dagger(p_2) a^\dagger(p_3) |0\rangle, \\
[a(p'), a^\dagger(p)] & = 2(2\pi)^3 \delta^3(\vec{p}' - \vec{p}), \quad \{d_\chi(p'), d_\chi(p)\} = 2(2\pi)^3 \delta^3(\vec{p}' - \vec{p}) \delta_{\chi \chi}.
\end{align*} \]

The coefficient \( C_{\beta \gamma, bc}^\alpha \) is a normalized color factor and \( F_{L}^{bc, de} \) is a normalized flavor coefficient, obeying the relation
\[ C_{\beta' \gamma', \delta' e'}^{\alpha'} F_{L}^{\beta' c', \delta' e'} C_{\beta \gamma, bc}^\alpha F_{L}^{bc, de} \bigg( \langle Q_{\alpha'}^c(p_1', \lambda_1)[q_{b'} q_{c'}](p_2')[q_{d'} q_{e'}](p_3') | Q_{\alpha}^c(p_1, \lambda_1)[q_a q_b](p_2)[q_c q_d](p_3) \rangle \bigg) = 2^3(2\pi)^9 \delta^3(\vec{p}_1' - \vec{p}_1) \frac{1}{2} \delta^3(\vec{p}_2' - \vec{p}_2) \delta^3(\vec{p}_3' - \vec{p}_3) + (-)^L \delta^3(\vec{p}_2' - \vec{p}_3) \delta^3(\vec{p}_3' - \vec{p}_2) \delta_{\chi \chi_1}. \tag{3.4} \]

Note that \( C_{\beta \gamma, bc}^\alpha F_{L}^{bc, de} \) is (anti-)symmetric under \((\beta \gamma, bc) \leftrightarrow (\delta e, de)\) for \( L = 1 \) (0) as discussed in Sec. II.A. As we shall see below, the factor of \((-)^L\) will be compensated by the corresponding wave function under the \( p_2 \leftrightarrow p_3 \) interchange.

In terms of the light-front relative momentum variables \((x_i, k_{i\perp})\) for \( i = 1, 2, 3 \) defined by
\[ p_i^+ = x_i P^+, \quad \sum_{i=1}^3 x_i = 1, \]
\[ p_{i\perp} = x_i P_\perp + k_{i\perp}, \quad \sum_{i=1}^3 k_{i\perp} = 0, \tag{3.5} \]
the momentum-space wave-function \( \Psi_{L}^{S_1} \) can be expressed as
\[ \Psi_{L}^{S_1}(x_i, k_{i\perp}, \lambda_1, m) = \langle \lambda_1 | \mathcal{R}_M^\dagger(x_1, k_{1\perp}, m_1)|s_1\rangle \langle L_{1/2}; m s_1 | L_{1/2}; 1/2 S_z \rangle \phi_{Lm}(x_1, x_2, x_3, k_{1\perp}, k_{2\perp}, k_{3\perp}), \tag{3.6} \]
where \( \phi_{Lm}(x_1, x_2, x_3, k_{1\perp}, k_{2\perp}, k_{3\perp}) \) describes the momentum distribution of the constituents in the bound state with the subsystem consisting of the particles 2 and 3 in the orbital angular momentum \( L, L_z = m \) state, \( \langle L_{1/2}; m s_1 | 1/2, 1/2 S_z \rangle \) is the corresponding Clebsch-Gordan coefficient and \( \langle \lambda_1 | \mathcal{R}_M^\dagger(x_1, k_{1\perp}, m_1)|s_1\rangle \) is the well normalized Melosh transform matrix element. Explicitly \[48, 49,\]
\[ \langle \lambda_1 | \mathcal{R}_M^\dagger(x_1, k_{1\perp}, m_1)|s_1\rangle = \frac{\bar{u}(k_{1\perp}, \lambda_1) u_D(k_{1\perp}, s_1)}{2m_1} = \frac{(m_1 + x_1 M_0) \delta_{\lambda s_1} - i \vec{\sigma}_{\lambda s_1} \cdot \vec{k}_{1\perp} \times \vec{n}}{\sqrt{(m_1 + x_1 M_0)^2 + k_{1\perp}^2}}, \tag{3.7} \]
with \( u_D(k, s) = \frac{k + m}{\sqrt{k^0 + m}} \binom{\chi_s}{0}, \quad u(k, \lambda) = \frac{k + m}{\sqrt{2k^0}} \gamma^0 \binom{\chi_\lambda}{0}, \tag{3.8} \)
in the Dirac representation, \( \vec{n} = (0, 0, 1) \), a unit vector in the \( z \)-direction, and
\[ M_0^2 = \sum_{i=1}^3 \frac{m_i^2 + k_{i\perp}^2}{x_i}, \quad k_i = \frac{m_i^2 + k_{i\perp}^2}{x_i M_0}, \quad (e_i - k_{i\perp}, e_i + k_{i\perp}, k_{i\perp}), \]
\[ M_0 = e_1 + e_2 + e_3, \quad e_i = \sqrt{m_i^2 + k_{i\perp}^2 + k_{iz}^2} = \frac{x_i M_0}{2} + \frac{m_i^2 + k_{i\perp}^2}{2x_i M_0}, \quad k_{iz} = \frac{x_i M_0}{2} - \frac{m_i^2 + k_{i\perp}^2}{2x_i M_0}. \tag{3.9} \]
Note that \( u_D(k, s) = u(k, \lambda) \langle \lambda | \mathcal{R}_M^\dagger | s \rangle \) and, consequently, the state \( | Q^\nu(k, \lambda) \rangle \langle \lambda | \mathcal{R}_M^\dagger | s \rangle \) transforms like \( | Q^\nu(k, s) \rangle \) under rotation, i.e. its transformation does not depend on its momentum. A crucial feature of the light-front formulation of a bound state, such as the one shown in Eq. (3.1), is the frame-independence of the light-front wave function \([48, 50]\). Namely, the hadron can be boosted to any (physical) \((P^+, P_L)\) without affecting the internal variables \((x_i, k_i)\) of the wave function, which is certainly not the case in the instant-form formulation.

In practice it is more convenient to use the covariant form for the Melosh transform matrix element

\[
\langle \lambda_1 | \mathcal{R}_M^\dagger (x_1, k_{1\perp}, m_1) | s_1 \rangle \langle L \frac{1}{2} m s_1 | L \frac{1}{2} \frac{1}{2} S_z \rangle = \frac{1}{\sqrt{2(p_1 \cdot \vec{P} + m_1 M_0)}} \bar{u}(p_1, \lambda_1) \Gamma_{L m} u(\vec{P}, S_z), \tag{3.10}
\]

with

\[
\Gamma_{00} = 1, \quad \Gamma_{1m} = -\frac{1}{\sqrt{3}} \gamma_5 \not{\epsilon}(\vec{P}, m),
\]

\[
\bar{P} \equiv p_1 + p_2 + p_3,
\]

\[
\epsilon^\mu(\vec{P}, \pm 1) = \left[ \frac{2}{P^+} \vec{\epsilon}(\pm 1) \cdot \vec{P} \times 0, 0, \vec{\epsilon}(\pm 1) \right], \quad \vec{\epsilon}(\pm 1) = \mp(1, \pm i)/\sqrt{2},
\]

\[
\epsilon^\mu(\vec{P}, 0) = \frac{1}{M_0} \left( -M_0^2 + P_+^2, P^+, P_L \right), \tag{3.11}
\]

for pentaquark states with \( L = 0 \) or \( L = 1 \) diquark pairs. To derive the above expressions we have used the relations

\[
\bar{u}(k_1, \lambda_1) = \bar{u}(k_1, \lambda_1) \frac{u_D(k_1, s_1) \bar{u}_D(k_1, s_1)}{2m_1}
\]

\[
= \langle \lambda_1 | \mathcal{R}_M^\dagger (x_1, k_{1\perp}, m_1) | s_1 \rangle \bar{u}_D(k_1, s_1),
\]

\[
\langle \frac{1}{2} s_1 | \frac{1}{2} S_z \rangle = \frac{1}{\sqrt{2}} \chi_{s_1} \cdot \chi_{s_2}
\]

\[
= \frac{1}{\sqrt{2} M_0 (e_1 + m_1)} \bar{u}_D(k_1, s_1) u(k_1 + k_2 + k_3, S_z),
\]

\[
\langle \frac{1}{2} m s_1 | \frac{1}{2} S_z \rangle = -\frac{1}{\sqrt{3}} \chi_{s_1}^\dagger \vec{\sigma} \cdot \vec{\epsilon}(k_1 + k_2 + k_3, m) \chi_{s_2}
\]

\[
= -\frac{1}{\sqrt{6}} M_0 (e_1 + m_1) \bar{u}_D(k_1, s_1) \gamma_5 \not{\epsilon}(k_1 + k_2 + k_3, m) u(k_1 + k_2 + k_3, S_z),
\]

where \( \chi_s \) is the usual Pauli spinor. The above relations can be easily proved by using the explicit expression of the Dirac spinors shown in Eq. (3.8) and noting that \( k_1 + k_2 + k_3 = (M_0, M_0, 0, \perp) \). Putting these together and boosting \( k_i \rightarrow p_i \) we obtain Eq. (3.11). It should be remarked that in the conventional LF approach \( \vec{P} = p_1 + p_2 + p_3 \) is not equal to the baryon’s four-momentum as all constituents are on-shell and consequently \( \bar{u}(\vec{P}, S_z) \) is not equal to \( u(\vec{P}, S_z) \); they satisfy different equations of motions \((\vec{P} - M_0) u(\vec{P}, S_z) = 0 \) and \((\vec{P} - M) u(\vec{P}, S_z) = 0 \). This is similar to the case of a vector meson bound state where the polarization vectors \( \epsilon(\vec{P}, S_z) \) and \( \epsilon(\vec{P}, S_z) \) are different and satisfy different equations \( \epsilon(\vec{P}, S_z) \cdot \vec{P} = 0 \) and \( \epsilon(\vec{P}, S_z) \cdot \vec{P} = 0 \) [51]. Although \( u(\vec{P}, S_z) \) is different than \( u(\vec{P}, S_z) \), they satisfy the relation

\[
\gamma^+ u(\vec{P}, S_z) = \gamma^+ u(\vec{P}, S_z), \tag{3.13}
\]
followed from $\gamma^+ \gamma^+ = 0$, $\bar{P}^+ = P^+$, $\bar{P}_\perp = P_\perp$. This is again in analogy with the case of $\varepsilon(\bar{P}, \pm 1) = \varepsilon(P, \pm 1)$.

The pentaquark baryon state is normalized as
$$\langle P(P', S'_z)|\mathcal{P}(P, S_z)\rangle = 2(2\pi)^3 P^+ \delta^3(\bar{P}' - \bar{P})\delta_{L'L}\delta_{S'_z S_z},$$
so that [cf. Eqs. (3.1), (3.4) and (3.16)]
$$\int \left(\frac{\prod_{i=1}^3 dx_i d^2k_{i\perp}}{2(2\pi)^3}\right) 2(2\pi)^3 \delta(1 - \sum x_i) \delta^2(\sum k_{i\perp}) \phi_{Lm}(\{x\}, \{k_{i\perp}\}) \phi_{Lm}(\{x\}, \{k_{i\perp}\}) = \delta_{L'L}\delta_{m'm}.$$  

(3.15)

Under the constraint of $1 - \sum_{i=1}^3 x_i = \sum_{i=1}^3 (k_i x_{i,y,z} = 0$, we have the expressions
$$\phi_{Lm}(\{x\}, \{k_{i\perp}\}) = \sqrt{\frac{\partial(k_{2z}, k_{3z})}{\partial(x_2, x_3)}} \varphi_{00}(\bar{k}_1, \beta_1) \varphi_{Lm} \left(\frac{k_2 - k_3}{2}, \beta_{32}\right),$$
$$\frac{\partial(k_{2z}, k_{3z})}{\partial(x_2, x_3)} = \frac{e_1 e_2 e_3}{x_1 x_2 x_3 M_0}, \quad \varphi_{00}(\bar{k}, \beta) = \varphi(\bar{k}, \beta), \quad \varphi_{1m}(\bar{k}, \beta) = k_m \varphi_p(\bar{k}, \beta),$$
where $k_m = \bar{e}(m) \cdot \bar{k}$, or explicitly $k_m = \mp (k_{\perp x} \pm i k_{\perp y})/\sqrt{2}$, $k_m = 0 = k_z$, are proportional to the spherical harmonics $Y_{lm}$ in the momentum space, and $\varphi$, $\varphi_p$ are the distribution amplitudes of s-wave and p-wave states, respectively. For a Gaussian-like wave function, one has
$$\varphi(\bar{k}, \beta) = 4 \left(\frac{\pi}{\beta^2}\right)^\frac{3}{4} \exp\left(-\frac{k^2 + k^2}{2\beta^2}\right), \quad \varphi_p(\bar{k}, \beta) = \sqrt{\frac{2}{\beta^2}} \varphi(\bar{k}, \beta).$$

(3.17)

In order to see that the above wave functions do satisfy the normalization condition, we note that (under the above-mentioned constraints)
$$dx_2 d^2k_{2\perp} dx_3 d^2k_{3\perp} \frac{\partial(k_{2z}, k_{3z})}{\partial(x_2, x_3)} = d^3k_2 d^3k_3 = d^3(k_2 + k_3) d^3\left(\frac{k_2 - k_3}{2}\right),$$
with $d^3(k_2 + k_3) = d^3k_1$ and the Gaussian-like wave functions can be integrated readily. It is easily seen that $\phi_{Lm}$ picks up a factor of $(-)^L$ under the interchange of $k_2 \leftrightarrow k_3$. This will compensate the $(-)^L$ factor appearing in Eq. (3.4).

By using Eq. (3.10) it is straightforward to obtain
$$\langle \lambda_1 | \mathcal{R}_M^3(x_1, k_{1\perp}, m_1)|s_1\rangle \langle \frac{1}{2}; m_1|\frac{1}{2}; 0; S_z\rangle \frac{(k_2 - k_3)_m}{2} = \frac{1}{2\sqrt{6}(p_1 \cdot P + m_1 M_0)} \bar{u}(p_1, \lambda_1) \gamma_5 \left(\begin{array}{c} p_2 - p_3 - \frac{\bar{P} \cdot (p_2 - p_3)}{M_0} \\ 0 \end{array}\right) u(P, S_z).$$

(3.19)

where the factor of $(k_2 - k_3)_m = \varepsilon(\bar{P}, m) \cdot (p_2 - p_3)$ comes from the wave function Eq. (3.16) for the $L = 1$ case. The state $|\mathcal{P}(P, L, S_z)\rangle$ for $\mathcal{P}_Q$ in the light-front model can now be obtained by using Eqs. (3.1)–(3.19).

**B. $\mathcal{P}_Q \rightarrow \mathcal{P}_Q'$ weak transitions**

In this work we consider $\mathcal{P}_b \rightarrow \mathcal{P}_c$ and $\mathcal{P}_c \rightarrow \mathcal{P}_s$ (especially $\Theta_c \rightarrow \Theta_s$) weak transitions as depicted in Fig. 11. The matrix element for the $\mathcal{P}_Q' \rightarrow \mathcal{P}_Q$ weak transition is
$$\langle \mathcal{P}_Q'(P', S'_z)|\bar{Q} \gamma_\mu(1 - \gamma_5) Q|\mathcal{P}_Q(P, S_z)\rangle = \langle \mathcal{P}_Q'(P', S'_z)| - Q^c \gamma_\mu(1 + \gamma_5) Q^c|\mathcal{P}_Q(P, S_z)\rangle,$$

(3.20)
where $Q^c$ and $Q^*$ are the charge conjugated fields of $Q'$ and $Q$, respectively. The reason we use the charge conjugated field for $Q$ in the above equation and in the previous subsection is to have both $Q^c$ and $Q^*$ fermion lines flow in the same direction as the flow of the pentaquark fermion line such that we do not need a further transpose of Dirac matrices in the ensuing calculation. If we treat $Q$ as an anti-fermion we will have to use $\bar{Q}$ charge conjugated field for $\bar{Q}$ with $\tilde{q}$ where the diquark pairs act as spectators, $\bar{\Gamma}$. We are ready to calculate the weak transition matrix element of heavy pentaquarks. For $Q_\mu^c Q^c \gamma^5$ transitions, we have the general expressions (after using Eqs. (3.11), (3.20) and integrating out the heavy quark momentum $p_1$)

$$
\langle P \bar{Q}(P', S_z') | \bar{Q} \gamma^\mu (1 - \gamma_5) Q^c | P, S_z \rangle = \int \{ d^3 p_2 \} \{ d^3 p_3 \} \frac{\phi_{LM}^*(\{x\}, \{k_+\}) \phi_{LM}(\{x\}, \{k_-\})}{2\sqrt{p_1^+ p_{1I}^+ p_{I}^+ + (p_1 \cdot p + m_1 M_0)(p_{1I} \cdot p_{I}^* + m_{1I} M_0^*)}} \times \bar{u}(P', S_z') \bar{\Gamma} \phi_{LM}(p_1^+ + m_1) \gamma^\mu (1 - \gamma_5)(\bar{\phi}_1 + m_1) \Gamma_{LM} u(\bar{\bar{P}}, S_z),
$$

(3.22)

where the diquark pairs act as spectators, $\bar{\Gamma}_L = \gamma_0 \Gamma_{LM} \gamma_0$ and

$$
p_{I}^{(+)} = x_{I}^{(+)} P_{I}^{(+)} , \quad p_{I}^{(0)} = x_{I}^{(0)} P_{I}^{(0)} + k_{I}^{(0)}, \quad 1 - \sum_{I=1}^{3} x_{I}^{(0)} = \sum_{I=1}^{3} k_{I}^{(0)} = 0,
$$

$$
\bar{p}_1 - \bar{p}_I = \bar{q}, \quad \bar{p}_2 = \bar{p}_2^I, \quad \bar{p}_3 = \bar{p}_3^I,
$$

(3.23)

with $\bar{p} = (p^+, p_{I}^0)$ and $\Gamma_{LM}$ given in Eq. (3.11). Explicit expressions for $P_Q(3) \rightarrow P_Q'(3)$ and $P_Q(6) \rightarrow P_Q(\bar{6})$ transition matrix elements will be given later.

We shall follow [52] to project out various form factors from the transition matrix elements. As in [46, 52], we consider the $q^+ = 0, q_{\perp} \neq 0$ case. To proceed, we apply the relations

$$
\bar{u}(P', S_z') \gamma^\mu u(P, S_z) = \delta_{S_z' S_z} \frac{2\sqrt{P + P'}}{P + P'} \bar{u}(P', S_z') \sigma^{+\mu} q_{\mu} u(P, S_z) = (\bar{\sigma} \cdot q_{\perp} \sigma^3)_{S_z' S_z},
$$

$$
\bar{u}(P', S_z') \gamma^5 \gamma^\mu u(P, S_z) = \frac{2\sqrt{P + P'}}{P + P'} \bar{u}(P', S_z') \sigma^{+\mu} q_{\mu} \gamma^5 u(P, S_z) = (\bar{\sigma} \cdot q_{\perp} \sigma^3)_{S_z' S_z},
$$

(3.24)

extended from the first identity (as shown in [52]) by applying Eq. (3.8), and obtain [52]

$$
f_1(q^2) = -\frac{\langle P_Q'(P', \uparrow) | V^+ | P_Q(P, \downarrow) \rangle}{2\sqrt{P + P'}} = -\frac{\langle P_Q'(P', \downarrow) | V^+ | P_Q(P, \uparrow) \rangle}{2\sqrt{P + P'}}.
$$

$$
f_2(q^2) = \frac{\langle P_Q'(P', \uparrow) | V^+ | P_Q(P, \downarrow) \rangle}{2q_{\perp} \sqrt{P + P'}} = -\frac{\langle P_Q'(P', \downarrow) | V^+ | P_Q(P, \uparrow) \rangle}{2q_{\perp} \sqrt{P + P'}}.
$$
where \( q_{\perp L,R} = q_{\perp}^1 \mp iq_{\perp}^2 \), or equivalently, we have

\[
\begin{align*}
\langle P', S'_z'|V^+|P(Q(P, S_z)\rangle &= -2\sqrt{P^+ P'^+} \left[ f_1(q^2) \delta_{S_z S'_z} + \frac{f_2(q^2)}{M + M'} \left( \hat{\sigma} \cdot q \right) \delta_{S_z S'_z} \right], \\
\langle P', S'_z|A^+|P(Q(P, S_z)\rangle &= 2\sqrt{P^+ P'^+} \left[ g_1(q^2) \left( A^3 \right) S'_z S_z + \frac{g_2(q^2)}{M + M'} \left( \hat{\sigma} \cdot q \right) S'_z S_z \right].
\end{align*}
\]

Various form factors can be projected out by applying the orthogonality of the corresponding matrices, \( \delta_{S_z S'_z}, (\sigma^3)_{S_z S'_z}, \) and \( (\sigma^i)_{S_z S'_z} \), under the trace operation. Note that due to the condition \( q^+ = 0 \) we have imposed in passing, the form factors \( f_3(q^2) \) and \( g_3(q^2) \) cannot be extracted in this manner. To extract \( f_{1,2}(q^2) \) and \( g_{1,2}(q^2) \) from the right hand side of Eq. (3.22), the following identities are proved to be useful:

\[
\begin{align*}
\frac{1}{2} \sum_{S_z, S'_z} u(\bar{P}, S_z) \delta_{S_z S'_z} \bar{u}(\bar{P}', S'_z) &= \frac{1}{4\sqrt{P^+ P'^+}} (\bar{P} + M_0) \gamma^+ (\bar{P}' + M'_0), \\
\frac{1}{2} \sum_{S_z, S'_z} u(\bar{P}, S_z) (\sigma^3)_{S_z S'_z} \bar{u}(\bar{P}', S'_z) &= -\frac{i}{4\sqrt{P^+ P'^+}} (\bar{P} + M_0) \gamma^+ (\bar{P}' + M'_0), \\
\frac{1}{2} \sum_{S_z, S'_z} u(\bar{P}, S_z) (\sigma^i)_{S_z S'_z} \bar{u}(\bar{P}', S'_z) &= \frac{1}{4\sqrt{P^+ P'^+}} (\bar{P} + M_0) \gamma^+ \gamma^5 (\bar{P}' + M'_0), \\
\frac{1}{2} \sum_{S_z, S'_z} u(\bar{P}, S_z) (\sigma^i)_{S_z S'_z} \bar{u}(\bar{P}', S'_z) &= -\frac{i}{4\sqrt{P^+ P'^+}} (\bar{P} + M_0) \gamma^+ \gamma^5 (\bar{P}' + M'_0). \tag{3.27}
\end{align*}
\]

With the above generic discussions on \( \mathcal{P}_Q \to \mathcal{P}_Q' \) transition, we are ready to extract \( \mathcal{P}_Q(3) \to \mathcal{P}_Q'(3) \) and \( \mathcal{P}_Q(6) \to \mathcal{P}_Q'(6) \) transition form factors.

1. **Form factors for the \( \mathcal{P}_Q(3) \to \mathcal{P}_Q'(3) \) transition**

Following from Eq. (3.22), we have explicitly

\[
\begin{align*}
\langle \mathcal{P}_Q(3)|Q\gamma^\mu(1 - \gamma_5)Q'|\mathcal{P}_Q(3)\rangle &= \frac{dx dx^2 dx^3 k_{2,1}}{2(2\pi)^3} \frac{dx dx^2 dx^3 k_{3,1}}{2(2\pi)^3} \frac{dx^4 x_5}{2\sqrt{x_1 x_5} (p_1 \cdot \bar{P} + m_1 M_0)} \phi_0^\mu(x', \{k'_1\}) \phi_0^\mu(x, \{k\}) \\
&\times \bar{u}(\bar{P}', S'_z)(\bar{p} + m'_1) \gamma^\mu (-1 - \gamma_5)(\bar{p} + m_1) \bar{u}(\bar{P}, S_z), \tag{3.28}
\end{align*}
\]

for the \( \mathcal{P}_Q(3) \to \mathcal{P}_Q'(3) \) transition. By using Eqs. (3.25), (3.26) and (3.27) we obtain the \( \mathcal{P}_Q(3) \to \mathcal{P}_Q'(3) \) transition form factors

\[
\begin{align*}
f_1(q^2) &= \frac{1}{8P^+ P'^+} \int dx dx^2 dx^3 dx^4 x_5 \frac{dx dx^2 dx^3 k_{2,1}}{2(2\pi)^3} \frac{dx dx^2 dx^3 k_{3,1}}{2(2\pi)^3} \frac{dx^4 x_5}{2\sqrt{x_1 x_5} (p_1 \cdot \bar{P} + m_1 M_0)} \phi_0^\mu(x', \{k'_1\}) \phi_0^\mu(x, \{k\}) \\
&\times \text{Tr}[(\bar{P} + M_0) \gamma^+ (\bar{P}' + M'_0)(\bar{p} + m'_1) \gamma^+ (\bar{p} + m_1)],
\end{align*}
\]
\[
\frac{f_2(q^2)}{M + M'} = -\frac{i}{8P + P^+q_\perp} \int \frac{dx_2^2k_{2\perp}}{2(2\pi)^3} \frac{dx_3^2k_{3\perp}}{2(2\pi)^3} \frac{\phi_0^0(\{x'\}, \{k'_+\}) \phi_0(\{x\}, \{k_+\})}{2\sqrt{x_1x'_1(p_1 \cdot \bar{P} + m_1M_0)(p'_1 \cdot P' + m'_1M'_0)}} \\
\times \text{Tr}[(\bar{P} + M_0)\sigma^+ (P' + M'_0)(p'_1 + m'_1)\gamma^+(\bar{P} + M_0)]
\]
\[
g_1(q^2) = \frac{1}{8P + P^+q_\perp} \int \frac{dx_2^2k_{2\perp}}{2(2\pi)^3} \frac{dx_3^2k_{3\perp}}{2(2\pi)^3} \frac{\phi_0^0(\{x'\}, \{k'_-\}) \phi_0(\{x\}, \{k_-\})}{2\sqrt{x_1x'_1(p_1 \cdot \bar{P} + m_1M_0)(p'_1 \cdot P' + m'_1M'_0)}} \\
\times \text{Tr}[(\bar{P} + M_0)\gamma^+ \gamma_5 (P' + M'_0)(p'_1 + m'_1)\gamma^+ \gamma_5 (\bar{P} + M_0)]
\]
\[
g_2(q^2) = \frac{1}{8P + P^+q_\perp} \int \frac{dx_2^2k_{2\perp}}{2(2\pi)^3} \frac{dx_3^2k_{3\perp}}{2(2\pi)^3} \frac{\phi_0^0(\{x'\}, \{k'_-\}) \phi_0(\{x\}, \{k_-\})}{2\sqrt{x_1x'_1(p_1 \cdot \bar{P} + m_1M_0)(p'_1 \cdot P' + m'_1M'_0)}} \\
\times \text{Tr}[(\bar{P} + M_0)\sigma^+ \gamma_5 (P' + M'_0)(p'_1 + m'_1)\gamma^+ \gamma_5 (\bar{P} + M_0)]
\]
(3.29)

with \(q_\perp^1 = q_\perp^2\) (no sum over \(i\)).

It is straightforward to work out the traces in \(f_{1,2}(q^2)\) as shown in Eq. (3.29) and obtain

\[
\frac{1}{8P + P^+} \text{Tr}[(\bar{P} + M_0)\gamma^+ (P' + M'_0)(p'_1 + m'_1)\gamma^+(\bar{P} + M_0)]
\]
\[
tr = -(p_1 - x_1\bar{P}) \cdot (p'_1 - x'_1P') + (x_1M_0 + m_1)(x'_1M'_0 + m'_1),
\]
\[
\frac{i}{8P + P^+} \text{Tr}[(\bar{P} + M_0)\sigma^+ (P' + M'_0)(p'_1 + m'_1)\gamma^+(\bar{P} + M_0)]
\]
\[
= (m'_1 + x'_1M'_0)(p'_1 - x_1\bar{P}) - (m_1 + x_1M_0)(p'_1 - x'_1P'),
\]
(3.30)

for \(i = 1, 2\), where uses of \(\bar{P}^+(\bar{P}) = P^{(+)in}, P_1^{(o)} = P_1^{(o)in}, P_1^{(i)} = P_1^{(o)in} + k_1^{(+)}\)
have been made. Note that the traces in \(g_1(q^2)\) and \(g_2(q^2)\) under the replacement \(m'_1 \rightarrow -m'_1, M'_0 \rightarrow -M'_0\) are the same as that in \(f_1(q^2)\) and \(f_2(q^2)\), respectively, except for an additional overall minus sign. Then the traces and the common dominator factors in the above form factors can be expressed in terms of the internal variables via

\[
p_1 \cdot \bar{P} = e_1M_0 = \frac{m_1^2 + x_1^2M_0^2 + k_1^2}{2x_1}, \quad p'_1 \cdot \bar{P}' = e'_1M'_0 = \frac{m'_1^2 + x'_1M'_0^2 + k'_1^2}{2x_1},
\]
\[
(p_1 - x_1\bar{P}) \cdot (p'_1 - x'_1P') = -k_1^2 \cdot k'_1^2, \quad p_1^{(i)} - x_1P_1^{(i)} = k_1^{(o)i},
\]
(3.31)

where \(k_1 \cdot k'_1\) is a scalar product in two-dimensional space. Using Eqs. (3.29), (3.30) and (3.31) we obtain the explicit forms of the \(\mathcal{P}_Q(3) \rightarrow \mathcal{P}_Q'(3)\) transition form factors

\[
f_1(q^2) = \int \frac{dx_2^2k_{2\perp}}{2(2\pi)^3} \frac{dx_3^2k_{3\perp}}{2(2\pi)^3} \frac{\phi_0^0(\{x'\}, \{k'_+\}) \phi_0(\{x\}, \{k_+\})}{\sqrt{[(m_1 + x_1M_0)^2 + k_1^2][(m'_1 + x'_1M'_0)^2 + k'_1^2]}}
\]
\[
\times [k_1 \cdot k'_1 + (m_1 + x_1M_0)(m'_1 + x'_1M'_0)],
\]
\[
f_2(q^2) = \frac{1}{q_\perp^1} \int \frac{dx_2^2k_{2\perp}}{2(2\pi)^3} \frac{dx_3^2k_{3\perp}}{2(2\pi)^3} \frac{\phi_0^0(\{x'\}, \{k'_-\}) \phi_0(\{x\}, \{k_-\})}{\sqrt{[(m_1 + x_1M_0)^2 + k_1^2][(m'_1 + x'_1M'_0)^2 + k'_1^2]}}
\]
\[
\times [(m_1 + x_1M_0) k_1^i - (m'_1 + x'_1M'_0) k_1^i],
\]
\[
g_1(q^2) = \int \frac{dx_2^2k_{2\perp}}{2(2\pi)^3} \frac{dx_3^2k_{3\perp}}{2(2\pi)^3} \frac{\phi_0^0(\{x'\}, \{k'_-\}) \phi_0(\{x\}, \{k_-\})}{\sqrt{[(m_1 + x_1M_0)^2 + k_1^2][(m'_1 + x'_1M'_0)^2 + k'_1^2]}}
\]
\[
\times [-k_1 \cdot k'_1 + (m_1 + x_1M_0)(m'_1 + x'_1M'_0)],
\]
\[
g_2(q^2) = \frac{1}{q_\perp^1} \int \frac{dx_2^2k_{2\perp}}{2(2\pi)^3} \frac{dx_3^2k_{3\perp}}{2(2\pi)^3} \frac{\phi_0^0(\{x'\}, \{k'_-\}) \phi_0(\{x\}, \{k_-\})}{\sqrt{[(m_1 + x_1M_0)^2 + k_1^2][(m'_1 + x'_1M'_0)^2 + k'_1^2]}}
\]
\[
\times [(m'_1 + x'_1M'_0) k_1^i + (m_1 + x_1M_0) k_1^i],
\]
(3.32)
with \( q^i_1 = q^i_1 \) or \( q^i_2 \) (no sum over \( i \)).

2. Form factors for the \( \mathcal{P}_Q(6) \rightarrow \mathcal{P}_Q'(6) \) transition

For the \( \mathcal{P}_Q(6) \rightarrow \mathcal{P}_Q'(6) \) transition, Eq. (3.32) leads to

\[
\langle \mathcal{P}_Q'(P', S'_z) | Q \gamma^\mu (1 - \gamma_5) Q | \mathcal{P}_Q(P, S_z) \rangle = \int \frac{d^2 k_2 \cdot d^2 k_{3\perp}}{(2\pi)^3} \frac{d^2 k_{3\perp}}{(2\pi)^3} \frac{\phi'_0(x', \{ k'_i \}) \phi_0(x, \{ k_i \})}{12\beta_2 \beta_3 (p' - p_3)_\rho (p_2 - p_3)_\sigma} \times \bar{u}(P', S'_z) \gamma_5 \left( \gamma^\mu + \frac{p' \cdot \sigma}{M_0} \right) (p'_1 + m'_1) \gamma^\mu (-1 - \gamma_5) (p_1 + m_1) \gamma_5 \left( \frac{\gamma^\rho - \frac{p' \cdot \sigma}{M_0}}{M_0} \right) u(P, S_z). \tag{3.33}
\]

By repeating the same procedure of the form factor extraction as before, we obtain the \( \mathcal{P}_Q(6) \rightarrow \mathcal{P}_Q'(6) \) transition form factors

\[
f_1(q^2) = \frac{1}{8P^+ P'^+} \int \frac{d^2 k_2 \cdot d^2 k_{3\perp}}{(2\pi)^3} \frac{d^2 k_{3\perp}}{(2\pi)^3} \frac{\phi'_0(x', \{ k'_i \}) \phi_0(x, \{ k_i \})}{12\beta_2 \beta_3 x_1 x'_1 (p_1 \cdot P + m_1 M_0)(p'_1 \cdot P' + m'_1 M'_0)} \times \text{Tr} \left[ \left( \frac{\gamma^\rho}{M_0} \right) (p'_1 - m'_1) \gamma^\mu (p_1 - m_1) \right],
\]

\[
f_2(q^2) = -\frac{i}{8P^+ P'^+ q^i_1} \int \frac{d^2 k_2 \cdot d^2 k_{3\perp}}{(2\pi)^3} \frac{d^2 k_{3\perp}}{(2\pi)^3} \frac{\phi'_0(x', \{ k'_i \}) \phi_0(x, \{ k_i \})}{12\beta_2 \beta_3 x_1 x'_1 (p_1 \cdot P + m_1 M_0)(p'_1 \cdot P' + m'_1 M'_0)} \times \text{Tr} \left[ \left( \frac{\gamma^\rho}{M_0} \right) (p'_1 - m'_1) \gamma^\mu \right],
\]

\[
g_1(q^2) = \frac{1}{8P^+ P'^+} \int \frac{d^2 k_2 \cdot d^2 k_{3\perp}}{(2\pi)^3} \frac{d^2 k_{3\perp}}{(2\pi)^3} \frac{\phi'_0(x', \{ k'_i \}) \phi_0(x, \{ k_i \})}{12\beta_2 \beta_3 x_1 x'_1 (p_1 \cdot P + m_1 M_0)(p'_1 \cdot P' + m'_1 M'_0)} \times \text{Tr} \left[ \left( \frac{\gamma^\rho}{M_0} \right) (p'_1 - m'_1) \gamma^\mu \gamma_5 \gamma_5 (p_1 - m_1) \right],
\]

\[
g_2(q^2) = -\frac{i}{8P^+ P'^+ q^i_1} \int \frac{d^2 k_2 \cdot d^2 k_{3\perp}}{(2\pi)^3} \frac{d^2 k_{3\perp}}{(2\pi)^3} \frac{\phi'_0(x', \{ k'_i \}) \phi_0(x, \{ k_i \})}{12\beta_2 \beta_3 x_1 x'_1 (p_1 \cdot P + m_1 M_0)(p'_1 \cdot P' + m'_1 M'_0)} \times \text{Tr} \left[ \left( \frac{\gamma^\rho}{M_0} \right) (p'_1 - m'_1) \gamma^\mu \gamma_5 \gamma_5 (p_1 - m_1) \right], \tag{3.34}
\]

with \( q^i_1 = q^i_1 \) or \( q^i_2 \) (no sum over \( i \)).

For the traces in \( f_{1,2}(q^2) \) in Eq. (3.33), we have

\[
\frac{1}{8P^+ P'^+} \text{Tr} \left[ \left( \frac{\gamma^\rho}{M_0} \right) (p'_1 - m'_1) \gamma^\mu \gamma_5 (p_1 - m_1) \gamma^\rho (p_1 - m_1) \right]
\]

\[
= p_2 \gamma_5 (p_1 - x_1 P') \cdot (p'_1 - x_1 P') + (m_1 + x_1 M_0)(m'_1 + x_1 M'_0)
\]

\[
+ p_2 \gamma_5 (p_1 - p_1 \cdot \gamma_5 + m_1 M_0)(p'_1 - m'_1 M'_0) + (p_1 \cdot p'_1 - m_1 m'_1)(P \cdot P' - M_0 M'_0)
\]

\[
- (p_1 \cdot P' + m_1 M_0)(p'_1 \cdot P + m'_1 M'_0) + \frac{p_1 \cdot P_2}{M_0} \gamma_5 (p_1 - x_1 P_1 \cdot P_2 + m_1 + M_0)(m'_1 + x_1 M'_0)
\]

\[
+ p_2 \gamma_5 (p_1 - p_1 \cdot \gamma_5 + m_1 M_0)(m'_1 + x_1 M'_0) + (p'_1 \cdot P + m'_1 M'_0)(M_0 - x_1 M'_0)
\]

\[
- (p'_1 \cdot P' + m'_1 M'_0)(m_1 + M_0) + (p_1 \cdot p'_1 - m_1 m'_1)(M_0 - M'_0) - x_1 (P \cdot P' - M_0 M'_0)(m'_1 + M_0))
\]

\[
= 18
\]
\[ p_{1} \cdot p_{2} = \frac{1}{2}(m_{2}^{2} + m_{1}^{2} - q^{2}) \]

\[ p_{1} \cdot \bar{P} = \frac{1}{2}(M_{0}^{2} + M_{0}^{2} - q^{2}) \]

\[ p_{1} \cdot \bar{P} = \frac{1}{2}[(x_{1} P' - p_{1} \cdot \bar{q}) \cdot (p_{1} \cdot \bar{q}) - p_{1} \cdot p_{1}] = \frac{m_{1}^{2} + x_{1}^{2}M_{0}^{2} + (k_{1}^{2} + q_{1}^{2})^{2}}{2x_{1}} \]

\[ (3.36) \]

Traces in Eq. (3.36) can be expressed in terms of the internal variables \( i_{1} \), \( i_{2} \), \( M_{0} \). In doing so, we need identities beyond Eq. (3.31). It is useful to note that \( \bar{P} - \bar{P}' = p_{1} - p_{1}' = \bar{q} \), where \( \bar{q} = q^{+} = 0, q_{1} = q_{1} \), and \( q^{2} = q_{2}^{2} \). Then the corresponding traces of \( g_{1,2}(q^{2}) \) in a similar manner.
\[ p_1' \cdot \bar{P} = \frac{1}{x_1} [(x_1 \bar{P} - p_1) \cdot (p_1 - \bar{q}) + p_1' \cdot p_1] = \frac{m_1'^2 + x_1'^2 M_0^2 + (k_{1\perp} - q_{\perp})^2}{2x_1}, \]

\[ p_2'^2 = k_{23}^2 = (k_2^+ - k_3^+)(k_2^- - k_3^-) - k_{23\perp}^2, \quad p_1^{(\ell)} \cdot p_2^{(\ell)} = \bar{P}^{(\ell)} \cdot \bar{p}_{23}^{(\ell)} = M_0^{(\ell)} e_{23}^{(\ell)}, \quad \text{(3.37)} \]

where \( p_{23}^2 = p_{23} \) and \( m_2 = m_3 \) have been used in the last identity.

Putting all the pieces together, we obtain the explicit expressions of the \( \mathcal{P}_Q(\bar{6}) \to \mathcal{P}'Q(\bar{6}) \) transition form factors:

\[ f_1(q^2) = \int \frac{dx_2d^2k_{2\perp} dx_3d^2k_{3\perp}}{2(2\pi)^3} \frac{\phi_{\alpha0}'\{x', \{k_{\perp}'\}\} \phi_{\alpha0}(\{x, \{k_{\perp}\}\})}{6\beta_{23}^2 \beta_{23}' \sqrt{[(m_1 + x_1 M_0)^2 + k_{1\perp}^2][(m_1' + x_1 M_0')^2 + k_{1\perp}'^2]}} \times \left\{ k_{23}^2 [ - k_{1\perp} \cdot k_{1\perp}' - (m_1 + x_1 M_0)(m_1' + x_1 M_0')] \right. \\
\quad + \frac{x_{23}^2}{4x_1^2} \left[ [(m_1 + x_1 M_0)^2 + k_{1\perp}^2][(m_1' + x_1 M_0')^2 + k_{1\perp}'^2] \right. \\
\quad + x_1^2 [(m_1 - m_1')^2 - q_{\perp}^2][(M_0 - M_0')^2 - q_{\perp}^2] \\
\quad - [(m_1 + x_1 M_0)^2 + (k_{1\perp}' + q_{\perp})^2][(m_1' + x_1 M_0')^2 + (k_{1\perp} - q_{\perp})^2] \\
\quad + e_{23} \left\{ - (1 - x_1) M_0 e_{23}(m_1' + x_1 M_0') \right. \\
\quad + \frac{x_{23}^2}{2x_1} \left[ [(m_1 + x_1 M_0')^2 + (k_{1\perp}' + q_{\perp})^2](m_1' + x_1 M_0) \right. \\
\quad + [(m_1' + x_1 M_0')^2 + (k_{1\perp} - q_{\perp})^2](M_0 - x_1 M_0') \right. \\
\quad - [(m_1' + x_1 M_0')^2 + k_{1\perp}'^2](m_1 + M_0) \right. \\
\quad + x_1 [(m_1' - M_0')^2 - q_{\perp}^2](M_0 - M_0') - x^2_1 [(M_0 - M_0')^2 - q_{\perp}^2](m_1' + M_0) \right. \\
\quad \left. + e_{23} e_{23}' [k_{1\perp} \cdot k_{1\perp}' + (m_1 + M_0)(m_1' + M_0') - M_0 M_0'(1 - x_1)^2] \right\}. \]

\[ \frac{f_2(q^2)}{M + M'} = - \frac{1}{q_{\perp}^2} \int \frac{dx_2d^2k_{2\perp} dx_3d^2k_{3\perp}}{2(2\pi)^3} \frac{\phi_{\alpha0}'\{x', \{k_{\perp}'\}\} \phi_{\alpha0}(\{x, \{k_{\perp}\}\})}{6\beta_{23}^2 \beta_{23}' \sqrt{[(m_1 + x_1 M_0)^2 + k_{1\perp}^2][(m_1' + x_1 M_0')^2 + k_{1\perp}'^2]}} \times \left\{ k_{23}^2 [(m_1 + x_1 M_0)k_{1\perp}' - (m_1' + x_1 M_0')k_{1\perp}] \right. \\
\quad + \frac{1}{2x_1} \left[ [(m_1' - x_1 M_0)^2 + (k_{1\perp}' + q_{\perp})^2] x_23 k_{23\perp}'(m_1 + x_1 M_0) + x_23^2 (m_1 q_{1\perp}' - M_0' k_{1\perp}') \right. \\
\quad + x_1 (k_{23\perp}' + x_23 q_{1\perp}') e_{23} - (x_1 k_{23\perp}' - x_23 k_{1\perp}') e_{23}' \right. \\
\quad + [(m_1' - x_1 M_0')^2 + k_{1\perp}'^2] - x_23 k_{23\perp}'(m_1 + x_1 M_0) + x_23^2 M_0 k_{1\perp}' + x_23 k_{1\perp}' e_{23} \right. \\
\quad + [(m_1' + x_1 M_0')^2 + k_{1\perp}'^2] 2x_23 k_{23\perp}'(m_1 + x_1 M_0) - x_23^2 M_0 k_{1\perp}' \right. \\
\quad + [(m_1 + x_1 M_0)^2 + k_{1\perp}^2] - x_23 (k_{23\perp}' + x_23 q_{1\perp}')(m_1' + x_1 M_0') + x_23^2 M_0' k_{1\perp}' - x_23 k_{1\perp}' e_{23}' \right. \
\]
\[+[(m_1 + x_1 M_0)^2 + (k_{1\perp} + q_{\perp})^2] \left[-x_{23} k_{23\perp} (m_1 + x_1 M_0) - x_{23} M_0 (x_1 q_{\perp} - k_{1\perp}^2)\right]
+ (x_1 k_{23\perp} + x_1 x_3 q_{\perp} - x_3 k_{3\perp}^i) e_{23} - x_1 k_{23\perp}^i e_{23}^i
+ x_1 [(m_1 + m_i^2 - q^2) [x_{23} k_{23\perp} (M_0 - M_0^i) + x_{23} M_0 q_{\perp} - (k_{23\perp} + x_{23} q_{\perp}) e_{23} + k_{23\perp}^i e_{23}^i]
+ x_1 [(M_0 - M_0^i)^2 - q^2] [x_{23} k_{23\perp} (m_1 - m_i^2 - 1) + x_{23}^2 (m_1 k_{1\perp} - m_i k_{1\perp} - x_1 m_i q_{\perp})
- x_1 (x_1 k_{23\perp} + x_1 x_3 q_{\perp} - x_3 k_{3\perp}^i) e_{23} + x_1 (x_1 k_{23\perp} - x_1 k_{3\perp}^i) e_{23}^i]
+ k_{1\perp}^i \{x_{23} e_{23} [m_i (m_0 + M_0^i)] + 2 k_{23\perp}^i \{x_{23} e_{23} [(m_1 + M_0) M_0^i + 2 m_1 M_0] - (1 - x_1) M_0^i e_{23}^2\}
+ k_{1\perp}^i \{x_{23} e_{23} [(m_1 + M_0 + (1 - x_1) M_0^i) M_0^i - x_{23} e_{23} (m_1 + x_1 M_0) M_0^i + (1 - x_1) M_0^i e_{23}^2\}
+ k_{23\perp}^i \{x_{23} e_{23} [(m_1 + M_0) M_0^i + 2 (m_1 + x_1 M_0) M_0^i] - e_{23} [(m_1 + x_1 M_0) [m_i^2 + (1 - x_1) M_0^i]]\}
+ e_{23} e_{23} [(m_1 + M_0) k_{1\perp}^i - (m_1 + M_0) k_{1\perp}^i]\}
\]

\[
g_1(q^2) = \int \frac{dx_2^2 k_{2\perp}^i dx_3^2 k_{3\perp}^i}{2(2\pi)^3} \int \frac{dx_3^2 k_{3\perp}^i}{2(2\pi)^3} \frac{\phi_{\eta_0}^i (x', \{k_{1\perp}\}) \phi_{00} (\{x\}, \{k_{1\perp}\})}{6\hat{M}_{23}^{(23)} \beta_{23}^{(23)} \sqrt{[(m_1 + x_1 M_0)^2 + k_{1\perp}^2][(m_1 + x_1 M_0^i)^2 + k_{1\perp}^2]}}
\]

\[
g_2(q^2) = \frac{1}{(M + M')} \int \frac{dx_2^2 k_{2\perp}^i dx_3^2 k_{3\perp}^i}{2(2\pi)^3} \int \frac{dx_3^2 k_{3\perp}^i}{2(2\pi)^3} \frac{\phi_{\eta_0}^i (x', \{k_{1\perp}\}) \phi_{00} (\{x\}, \{k_{1\perp}\})}{6\hat{M}_{23}^{(23)} \beta_{23}^{(23)} \sqrt{[(m_1 + x_1 M_0)^2 + k_{1\perp}^2][(m_1 + x_1 M_0^i)^2 + k_{1\perp}^2]}}\]

\[
\times \left\{ \text{the integrand of } \frac{f_2(q^2)}{M + M'} \right\} \mid m_1 \to -m_i, M_0 \to -M_0, e_{23} \to -e_{23}\}
\]

where \(i = 1 \text{ or } 2\). It is useful to recall that we have \(0 \leq x_{1,2,3} = x_{1,2,3} \leq 1\), \(x_{23} = x_2 - x_3\), \(e_{23} = e_2^2 - e_3^2\), \(k_{23} = k_2 - k_3\), \(k_{1\perp} = k_{1\perp} - (1 - x_1) q_{\perp}\) and \(k_{2,3\perp} = k_{2,3\perp} + x_{2,3} q_{\perp}\).

**IV. FORM FACTORS IN THE HEAVY QUARK LIMIT**

In the heavy quark limit, heavy quark symmetry (HQS) [54] provides model-independent constraints on the form factors. More precisely, all the heavy-to-heavy baryonic decay form factors are reduced to some universal Isgur-Wise functions. Therefore, it is important to study the heavy quark limit behavior of these physical quantities to check the consistency of calculations.

In the limit of HQS, the weak transitions between antitriplet and antitriplet or sextet and sextet heavy baryons (i.e. charmed or bottom baryons) have been studied within the framework of HQET. For heavy baryons, the two light quarks form either a flavor symmetric sextet and spin 1 state, or a flavor antisymmetric antitriplet and spin 0 configuration. As noted in passing, the four light quarks in the Jaffe-Wilczek model form a flavor antisextet and \(L = 1\), \(S = 0\) state, or a flavor triplet and \(L = 0\), \(S = 0\) state. In the infinite quark mass limit, the heavy quark spin \(S_Q\) decouples from the other degrees of freedom of the hadron, so that \(S_Q\) and the total angular momentum \(j\) of the light quarks (the so-called “brown muck”) are separately good quantum numbers. Whether the total spin of the light degrees of freedom comes from the quark spin plus orbital angular momentum or
just from the quark spin is irrelevant. Therefore in the heavy quark limit, the $\bar{6}_f$ ($3_f$) pentaquark is the analog of the $6_f$ ($3_f$) heavy baryon. Consequently, the previous studies of heavy-to-heavy baryon transitions in nineties using HQET can be generalized to the heavy-to-heavy pentaquark transitions.

In the heavy quark limit, the $3_f - 3_f$ ($\bar{6}_f - \bar{6}_f$) pentaquark transition is similar to that of the $3_f - 3_f$ ($6_f - 6_f$) ordinary baryon transition. Hence, we can write

$$\langle P_{Q'}(3; v', S'_z)|\bar{Q}_v\gamma_\mu(1 - \gamma_5)Q'_v|P_{Q}(3; v, S_z)\rangle = -\zeta(\omega)\bar{u}(v', S'_z)\gamma_\mu(1 + \gamma_5)u(v, S_z)$$

(4.1)

and

$$\langle P_{Q'}(\bar{6}; v', S'_z)|\bar{Q}_v\Gamma Q'_v|P_{Q}(\bar{6}; v, S_z)\rangle = \frac{1}{3} [g^{\mu\nu}\xi_1(\omega) - v^{\mu}\gamma^{\nu}\xi_2(\omega)]\bar{u}(v', S'_z)\gamma_\nu(\gamma_\mu + v'_\mu)$$

$$\times (C^{-1}C)^T(\gamma_\mu + v_\mu)\gamma_5 u(v, S_z),$$

(4.2)

or explicitly,

$$\langle P_{Q'}(\bar{6}; v', S'_z)|\bar{Q}_v\gamma_\mu Q'_v|P_{Q}(\bar{6}; v, S_z)\rangle = \frac{1}{3} \bar{u}(v', S'_z)\{[\omega_\gamma_\mu - 2(v + v'_\mu)]\xi_1(\omega)$$

$$+ [(1 - \omega^2)\gamma_\mu - 2(1 - \omega)(v + v'_\mu)]\xi_2(\omega)\}u(v, S_z),$$

(4.3)

$$\langle P_{Q'}(\bar{6}; v', S'_z)|\bar{Q}_v\gamma_\mu\gamma_5 Q'_v|P_{Q}(\bar{6}; v, S_z)\rangle = -\frac{1}{3} \bar{u}(v', S'_z)\{[\omega_\gamma_\mu - 2(v - v'_\mu)]\xi_1(\omega)$$

$$+ [(1 - \omega^2)\gamma_\mu + 2(1 + \omega)(v - v'_\mu)]\xi_2(\omega)\}\gamma_5 u(v, S_z),$$

where $h_{Q'}$ are the dimensionless heavy quark fields, $\omega \equiv v \cdot v'$, $\zeta, \xi_1$ and $\xi_2$ are three universal baryonic Isgur-Wise functions with the normalization $\zeta(1) = \xi_1(1) = 1$, and $C$ is the charge conjugation matrix with $[C^{-1}\gamma_\mu(1 - \gamma_5)C]^T = -\gamma_\mu(1 + \gamma_5)$. The normalization of $\xi_2(\omega)$ at zero recoil is unknown, but the large-$N_c$ approach and the quark model predict that $\xi_2(1) = 1/2$.

From Eqs. (4.1) and (4.3) we obtain

$$f_1 = g_1 = \zeta(\omega), \quad f_2 = f_3 = g_2 = g_3 = 0$$

(4.4)

for $P_{Q}(3) \rightarrow P_{Q'}(3)$ transitions and

$$f_1 = F_1 + \frac{1}{2}(M + M') \left( \frac{F_2}{M} + \frac{F_3}{M'} \right), \quad g_1 = G_1 - \frac{1}{2}(M - M') \left( \frac{G_2}{M} + \frac{G_3}{M'} \right),$$

$$f_2 = F_2 + \frac{1}{2}(M + M') \left( \frac{F_2}{M} + \frac{F_3}{M'} \right), \quad g_2 = \frac{1}{2}(M + M') \left( \frac{G_2}{M} + \frac{G_3}{M'} \right),$$

$$f_3 = F_3 + \frac{1}{2}(M + M') \left( \frac{F_2}{M} - \frac{F_3}{M'} \right), \quad g_3 = \frac{1}{2}(M + M') \left( \frac{G_2}{M} - \frac{G_3}{M'} \right),$$

(4.5)

with

$$F_1 = G_1 = -\frac{1}{3} [\omega \xi_1 + (1 - \omega^2)\xi_2],$$

$$F_2 = F_3 = \frac{2}{3} [\omega \xi_1 + (1 - \omega)\xi_2],$$

$$G_2 = -G_3 = \frac{2}{3} [\omega \xi_1 - (1 + \omega)\xi_2]$$

(4.6)

for $P_{Q}(\bar{6}) \rightarrow P_{Q'}(\bar{6})$ transitions. It is straightforward to show that at zero recoil $q^2 = q_m^2 \equiv (M - M')^2$

$$f_1(q_m^2) = g_1(q_m^2) = 1, \quad f_2,3(q_m^2) = g_2,3(q_m^2) = 0$$

(4.7)
for $\mathcal{P}_Q(3) \to \mathcal{P}_{Q'}(3)$, and

$$
\begin{align*}
f_1(q_m^2) &= -\frac{1}{3} \left[ 1 - (M + M') \left( \frac{1}{M} + \frac{1}{M'} \right) \right], \\
f_2(q_m^2) &= \frac{1}{3} (M + M') \left( \frac{1}{M} + \frac{1}{M'} \right), \\
f_3(q_m^2) &= \frac{1}{3} (M + M') \left( \frac{1}{M} - \frac{1}{M'} \right), \\
g_1(q_m^2) &= -\frac{1}{3}, \\
g_2(q_m^2) &= g_3(q_m^2) = 0
\end{align*}
$$

(4.8)

for $\mathcal{P}_Q(\bar{6}) \to \mathcal{P}_{Q'}(\bar{6})$. The form factors (4.7) and (4.8) are model independent and should be respected by any model calculations.

We now consider the heavy quark limit of transition form factors obtained in the previous section and begin with the $\mathcal{P}_Q(3) \to \mathcal{P}_{Q'}(3)$ transition. Under the HQ limit, we have

$$
Q|\mathcal{P}_Q(P, S_z) \rangle \to \sqrt{m_Q} Q_v |\mathcal{P}_Q(v, S_z) \rangle, \quad u(\bar{P}, S_z) \to \sqrt{m_Q} u(v, S_z), \quad p_1 + m_1 \to m_Q(\not{p} + 1),
$$

$$
\sqrt{\frac{E_1 E_2 E_3}{x_1 x_2 x_3 M_0}} \to m_Q \sqrt{\frac{v \cdot p_2 v \cdot p_3}{X_2 X_3}}, \quad \Phi_{00} \to \frac{m_Q}{\sqrt{X_2 X_3}} \Phi_{00},
$$

(4.9)

with $X_{2,3} \equiv m_Q x_{2,3}$ and

$$
\begin{align*}
\Phi_{00} &= 16 \left( \frac{\pi^2}{\beta_1^2 \beta_2^2} \right)^{\frac{4}{3}} \sqrt{v \cdot p_2 v \cdot p_3} \exp \left\{ - \left( \frac{1}{2 \beta_1^2} - \frac{1}{8 \beta_2^2} \right) \right\} \Phi_{00}(\vec{k}_2, \vec{k}_3) \\
&\quad - \left( \frac{1}{2 \beta_1^2} + \frac{1}{8 \beta_2^2} \right) \left[ (v \cdot p_2)^2 - m_2^2 + (v \cdot p_3)^2 - m_3^2 \right]
\end{align*}
$$

(4.10)

and similar relations for $u(\bar{P}', S_z')$, etc. Since $x_2$ and $x_3$ are of order $\Lambda_{QCD}/m_Q$ in the $m_Q \to \infty$ limit, it is clear that $X_2$ and $X_3$ are of order $\Lambda_{QCD}$. For on-shell $p_i (i = 2, 3)$ we have $p_i^\perp = (p_i^2 + m_i^2)/p_i^+$ and

$$
v \cdot p_i = \frac{1}{2 X_i} \left( p_i^2 + m_i^2 + X_i^2 \right).
$$

(4.11)

Applying Eqs. (4.9) and (3.28), we obtain

$$
\langle \mathcal{P}_{Q'}(3; v'; S_z') | \bar{Q} v^\gamma(1 - \gamma_5) Q' | \mathcal{P}_Q(3; v, S_z) \rangle = \int \frac{dX_2 d^2k_{2\perp}}{2(2\pi)^3 X_2} \frac{dX_3 d^2k_{3\perp}}{2(2\pi)^3 X_3} \Phi_{00}^{\tau}(zX_2, p_{2\perp}; zX_3, p_{3\perp}) \Phi_{00}(X_2, p_{2\perp}; X_3, p_{3\perp})
$$

\times \bar{u}(v', S_z') \gamma^\mu (1 - \gamma_5) u(v, S_z),
$$

(4.12)

where use of $z \equiv X_2'/X_2 = X_3'/X_3$ has been made. By comparing the above equation with Eq. (4.1), we find

$$
\zeta(\omega) \equiv \int \frac{dX_2 d^2k_{2\perp}}{2(2\pi)^3 X_2} \frac{dX_3 d^2k_{3\perp}}{2(2\pi)^3 X_3} \Phi_{00}^{\tau}(zX_2, p_{2\perp}; zX_3, p_{3\perp}) \Phi_{00}(X_2, p_{2\perp}; X_3, p_{3\perp}).
$$

(4.13)

It can be easily seen that $\zeta(1) = 1$ from Eqs. (3.15) and (4.10). Note that $z$ is related to $\omega$ via

$$
z \to z_\pm = \omega \pm \sqrt{\omega^2 - 1}, \quad z_+ = \frac{1}{z_-},
$$

(4.14)

with the $+ (-)$ sign corresponding to $v^3$ greater (less) than $v'^3$. Note that $v^3$ greater (less) than $v'^3$ corresponds the daughter meson recoiling in the negative (positive) $z$ direction in the rest frame of the parent meson. One can check that $\zeta(\omega)$ remains the same under the replacement of $z \to 1/z$. 23
This indicates that the Isgur-Wise function thus obtained is independent of the recoiling direction, namely, it is truly Lorentz invariant.

As to the $P_Q(\vec{6}) \rightarrow P_{Q'}(\vec{6})$ case, the heavy quark limit of Eq. (4.13) leads to

$$
\langle P_{Q'}(\vec{6}; v', S'_z)|Q_v\gamma^\mu(1 - \gamma_5)Q_{v'}|P_Q(\vec{6}; v, S_z)\rangle
$$

$$
= - \int \frac{dX_2 d^2 k_{2,\perp} dX_3 d^2 k_{3,\perp}}{2(2\pi)^3 X_2 2(2\pi)^3 X_3} \Phi_{00}' \Phi_{00} (p_2 - p_3)_{\rho}(p_2 - p_3)_{\sigma}
\times \bar{u}(v', S'_z) \gamma_5 (\gamma^\rho + v'^\rho) (\gamma^\sigma + v'^\sigma) \gamma_5 u(v, S_z)
$$

$$
= - \int \frac{dX_2 d^2 k_{2,\perp} dX_3 d^2 k_{3,\perp}}{2(2\pi)^3 X_2 2(2\pi)^3 X_3} \Phi_{00}' \Phi_{00} (p_2 - p_3)_{\rho}(p_2 - p_3)_{\sigma}
\times \bar{u}(v', S'_z) \gamma_5 (\gamma^\rho + v'^\rho) \gamma_5 (\gamma^\sigma + v'^\sigma) \gamma_5 u(v, S_z),
$$

(4.13)

where uses of Eqs. (4.9, 14.10), $\bar{u}'_5 (\gamma^\rho + 1)(\gamma^\rho + v'^\rho)$ and $(\gamma^\sigma + 1)(\gamma^\sigma + v'^\sigma) \gamma_5 u = 2(\gamma^\sigma + v'^\sigma) \gamma_5 u$ have been made. To proceed we need to express $(p_2' - p_3')_{\rho}(p_2 - p_3)_{\sigma}$ in terms of $g_{\rho\sigma}$, $v_{\rho}v'_{\sigma}$, $v_{\rho}v_{\sigma}$, $v'_{\rho}v_{\sigma}$, and $v'_{\rho}v'_{\sigma}$ via the general expression

$$
\int \frac{dX_2 d^2 k_{2,\perp} dX_3 d^2 k_{3,\perp}}{2(2\pi)^3 X_2 2(2\pi)^3 X_3} \Phi_{00}' \Phi_{00} (p_2' - p_3')_{\rho}(p_2 - p_3)_{\sigma}
$$

$$
= \int d^3 p_2 d^3 p_3 f(v \cdot p_2) f(v \cdot p_3) f(v' \cdot p_2) f(v' \cdot p_3) (p_2' - p_3')_{\rho}(p_2 - p_3)_{\sigma}
$$

$$
= a_1 g_{\rho\sigma} + a_2 v_{\rho}v_{\sigma} + a_3 v_{\rho}v_{\sigma} + a_4 v_{\rho}v_{\sigma} + a_5 v'_{\rho}v'_{\sigma}.
$$

(4.16)

Applying this to Eq. (4.13), it is easily seen that the $a_3$, $a_4$ and $a_5$ terms do not make contributions to the $P_Q(\vec{6}) \rightarrow P_{Q'}(\vec{6})$ transition owing to the relation $\bar{u'}_5 (\gamma^\rho + 1) = (\gamma^\rho + 1) \gamma_5 u = 0$. As we shall see shortly, the only relevant coefficients $a_1$ and $a_2$ are related to the Isgur-Wise functions $\xi_1$ and $\xi_2$, respectively. The coefficients $a_1$ can be solved by contracting the left and right hand sides of Eq. (4.16) with $g_{\rho\sigma}$, $v_{\rho}v_{\sigma}$, $v'_{\rho}v_{\sigma}$, $v_{\rho}v'_{\sigma}$, $v'_{\rho}v'_{\sigma}$. Noting that $p_{23} \equiv p_2 - p_3$ with $p_{2,3} = p'_{2,3}$ in our case, we obtain

$$
a_1 = \int d^3 p_2 d^3 p_3 f(v \cdot p_2) f(v \cdot p_3) f(v' \cdot p_2) f(v' \cdot p_3)
\times \frac{1}{2(1 - \omega^2)} \{(1 - \omega^2) p_{23}^2 - (v \cdot p_{23})^2 + 2\omega v \cdot p_{23} v' \cdot p_{23} - (v' \cdot p_{23})^2\},
$$

$$
a_2 = \int d^3 p_2 d^3 p_3 f(v \cdot p_2) f(v \cdot p_3) f(v' \cdot p_2) f(v' \cdot p_3)
\times \frac{1}{2(1 - \omega^2)^2} \{(1 - \omega^2) p_{23}^2 + 2(1 + \omega^2)v \cdot p_{23} v' \cdot p_{23} - 3\omega[(v \cdot p_{23})^2 + (v' \cdot p_{23})^2]\}.\ 
$$

(4.17)

Comparing Eq. (4.15) with Eq. (4.12), the HQ limit expression of the $P_Q(\vec{6}) \rightarrow P_{Q'}(\vec{6})$ transition matrix element, we obtain

$$
\xi_1(\omega) = - \int \frac{dX_2 d^2 k_{2,\perp} dX_3 d^2 k_{3,\perp}}{2(2\pi)^3 X_2 2(2\pi)^3 X_3} \Phi_{00}' \Phi_{00} \frac{1}{4\beta_{23}^2 \beta_{23}^2 (1 - \omega^2)}
\times \left[(1 - \omega^2) p_{23}^2 - (v \cdot p_{23})^2 + 2\omega v \cdot p_{23} v' \cdot p_{23} - (v' \cdot p_{23})^2\right]
$$

(4.18)

and

$$
\xi_2(\omega) = \int \frac{dX_2 d^2 k_{2,\perp} dX_3 d^2 k_{3,\perp}}{2(2\pi)^3 X_2 2(2\pi)^3 X_3} \Phi_{00}' \Phi_{00} \frac{1}{4\beta_{23}^2 \beta_{23}^2 (1 - \omega^2)^2}
\times \left[(\omega - \omega^3) p_{23}^2 + 2(1 + \omega^2)v \cdot p_{23} v' \cdot p_{23} - 3\omega[(v \cdot p_{23})^2 + (v' \cdot p_{23})^2]\right].
$$

(4.19)
At first sight, it appears that the second IW function $\xi_2$ is divergent at zero recoil. We shall see below that it is not the case.

We proceed to calculate the normalization of $\xi_{1,2}(\omega)$ at $\omega = 1$. Since $p'_{23,\rho}p_{23,\sigma}$ in Eq. (4.15) can be replaced by $p'_{23,\rho}p_{23,\sigma}$ with $p'_{23} = p_{23} - (v' \cdot p_{23}) v'$, the IW functions $\xi_{1,2}$ can be recast to

$$\xi_1(\omega) = - \int \frac{dX_2 d^2 k_{2\perp} dX_3 d^2 k_{3\perp}}{2(2\pi)^3 X_2 2(2\pi)^3 X_3} \Phi_{00}^* \Phi_{00} \Phi_{00} \Phi_{00} \frac{1}{4\beta_{23,\beta_{23}'}(1 - \omega^2)^2} \left[ (1 - \omega^2) p'_{23} \cdot p_{23} + \omega v \cdot p_{23} v' \cdot p_{23} \right], \quad (4.20)$$

$$\xi_2(\omega) = \int \frac{dX_2 d^2 k_{2\perp} dX_3 d^2 k_{3\perp}}{2(2\pi)^3 X_2 2(2\pi)^3 X_3} \Phi_{00}^* \Phi_{00} \Phi_{00} \Phi_{00} \frac{1}{4\beta_{23,\beta_{23}'}(1 - \omega^2)^2} \left[ (1 - \omega^3) p'_{23} \cdot p_{23} + (2 + \omega^2) v \cdot p_{23} v' \cdot p_{23} \right].$$

Since $p'_{23} = (0, k_{23}')$ in the $P_{Q'}$ rest frame and $v' \cdot p'_{23} = 0$, it follows that $v \cdot p'_{23} v' \cdot p_{23} = (1 - \omega^2) k_{23}' \cdot \hat{v} k_{23} \cdot \hat{v}$, where $\hat{v} = \hat{v}/|\hat{v}|$ with $\hat{v}$ in the $P_{Q'}$ rest frame.\(^3\) We then have

$$\xi_1(\omega) = - \int \frac{dX_2 d^2 k_{2\perp} dX_3 d^2 k_{3\perp}}{2(2\pi)^3 X_2 2(2\pi)^3 X_3} \Phi_{00}^* \Phi_{00} \Phi_{00} \Phi_{00} \frac{1}{4\beta_{23,\beta_{23}'}(1 - \omega^2)^2} \left[ (1 - \omega^2) k_{23}' \cdot \hat{v} k_{23} \cdot \hat{v} \right], \quad (4.21)$$

with $k_{23}' = k_{2}' - k_3'$, $k_{23,\perp} = k_{2,\perp} + x_{2,3} q_L = \tilde{k}_{2,\perp}$ \(^4\) and

$$k_{2,3} = \frac{X_{2,3}}{2} - \frac{n_{2,3}^2 + k_{2,3,\perp}^2}{2X_{2,3}}. \quad (4.22)$$

For a further simplification of $\xi_1$, we notice that $p'_{23} = (0, \bar{k}_{23})$, $p_{23} = (\sqrt{\omega^2 - 1} \hat{v} \cdot k_{23}, \bar{k}_{23} + (\omega - 1) \hat{v} \cdot k_{23})$ in the $P_{Q'}$ rest frame. The above equation for $\xi_1$ is then reduced to

$$\xi_1(\omega) = \int \frac{dX_2 d^2 k_{2\perp} dX_3 d^2 k_{3\perp}}{2(2\pi)^3 X_2 2(2\pi)^3 X_3} \Phi_{00}^* \Phi_{00} \Phi_{00} \Phi_{00} \frac{1}{4\beta_{23,\beta_{23}'}(1 - \omega^2)^2} \left[ \frac{k_{23}'}{2} \cdot \hat{v} k_{23} \cdot \hat{v} \right], \quad (4.23)$$

At zero recoil,

$$\xi_1(1) = \int \frac{dX_2 d^2 k_{2\perp} dX_3 d^2 k_{3\perp}}{2(2\pi)^3 X_2 2(2\pi)^3 X_3} \Phi_{00}^* \Phi_{00} \Phi_{00} \Phi_{00} \frac{(k_{23})^2}{4\beta_{23,\beta_{23}'}(1 - \omega^2)^2} = 1, \quad (4.24)$$

where we have made use of the rotational invariance of $k_{23}' \cdot \hat{v} k_{23} \cdot \hat{v} \rightarrow k_{23}' \cdot \bar{k}_{23}$ and Eq. (3.15). Likewise, for $\xi_2(\omega)$ we have

$$\frac{1}{(1 - \omega^2)^2} \left[ (1 - \omega^3) p'_{23} \cdot p_{23} + (2 + \omega^2) v \cdot p_{23} v' \cdot p_{23} \right],$$

$$\frac{1}{(1 - \omega^2)^2} \left[ \omega p_{23} \cdot p_{23} + (2 + \omega^2) k_{23}' \cdot \hat{v} k_{23} \cdot \hat{v} \right]. \quad (4.25)$$

Applying the relation $p'_{23} \cdot p_{23} = -k_{23}' \cdot \bar{k}_{23} + (k_{23} \cdot \hat{v})(\bar{k}_{23} \cdot \hat{v})$ to the right hand side of the above equation leads to

$$\frac{1}{(1 - \omega^2)^2} \left[ -\omega k_{23}' \cdot \bar{k}_{23} - \omega (\omega - 1) k_{23}' \cdot \hat{v} k_{23} \cdot \hat{v} + (2 + \omega^2) k_{23}' \cdot \hat{v} k_{23} \cdot \hat{v} \right], \quad (4.26)$$

\(^3\) Since $v' = (1, \bar{0})$, $v = (w, \hat{v} \sqrt{\omega^2 - 1})$ and $\bar{p}_{23}' = (0, \bar{k}_{23})$ in the $P_{Q'}$ rest frame, the corresponding quantities in the $P_Q$ rest frame are $v = (1, \bar{0})$, $v' = (w, -\hat{v} \sqrt{\omega^2 - 1})$ and $\bar{p}_{23} = (0, \bar{k}_{23})$.

\(^4\) In the infinite quark mass limit, the IW functions can be evaluated directly in the time-like region by choosing the frame where $q_L = 0$.\(^46\).
which reduces to $2\vec{k}_{23} \cdot \vec{r}_{23}/[3(1 + \omega)]$ in the $\omega \to 1$ limit by using the argument of rotational invariance. Therefore, we have $\xi_2(1) = 1/2$, in agreement with the the large-$N_c$ approach and the quark model predictions.

In short, the heavy-to-heavy pentaquark transition form factors in the heavy quark limit can be expressed in terms of three universal Isgur-Wise functions, as shown in Eqs. (4.13) and (4.19). The fact that these IW functions have the correct normalization at zero recoil indicates the consistency of our light-front model calculations.

V. WEAK DECAYS OF HEAVY PENTAQUARK BARYONS

In this section we will perform the numerical calculations of various pentaquark transition form factors shown in Sec. III and the Isgur-Wise functions in Sec. IV. We then proceed to estimate the decay rates of the some weak decays of heavy pentaquarks such as $\Theta^+_b \to \Theta^+_c\pi^+$, $\Theta^+_b\rho^+$, $\Theta^+_c \to \Theta^+\pi^-$, $\Theta^+\rho^-$, $\Sigma^{f+}_{5b} \to \Sigma^{0}_{5c}\pi^+$, $\Sigma^{0}_{5c}\rho^+$ and $\Sigma^{0}_{5c} \to N^+_8\pi^+$, $N^+_8\rho^+$ with $N^+_8$ being an odd-parity octet pentaquark.

The input parameters $m_{[qq']}$, $m_q$, $\beta_1$ (for the heavy quark) and $\beta_{23}$ (for the diquark pair) [see Eq. (3.16)] are summarized in Table II. The quark masses are taken from $[46, 60]$. For our convenience, the pentaquark masses are fixed to be 6 and 3 GeV for $P_b$ and $P_c$, respectively. For light-anti-decuplet and octet pentaquark masses, we take $m_\beta = 1.54$ GeV and $m_N = 1.46$ GeV. Note that the $\beta$ parameters are of order $\Lambda_{QCD}$ and the diquark carries the same color charge as the anti-quark. Since the diquark pair acts like 3, the $Q^\pm([ud][ud])$ system can be considered as the analog of the heavy meson $Q^\pm q$. Therefore, it is plausible to assume that $\beta_{1b} : \beta_{1c} : \beta_{1s} \sim \beta_B : \beta_D : \beta_K$. The $\beta_{23}[ud]$ parameters for the diquark pair are roughly estimated as $\alpha \beta_{3,5}$ for $P_Q(3)$ and $\alpha \beta_{3}$ for $P_Q(6)$. The numerical values of $\beta_{B,D,K,\pi,\rho,a_1}$ are taken from $[46, 60]$ as we shall see shortly, by using these input parameters, the obtained $\Sigma^{0}_{5b} \to \Sigma^{0}_{5c}$ transition form factors $f_1(0)$, $g_1(0)$ are close to their counterparts (in the sense of SU(3) representation) in the $\Lambda_b \to \Lambda_c$ transition.

To perform numerical calculations of the IW functions we choose $\beta_{1Q}^c = 0.65$ GeV and $\beta_{1Q}^s = 0.38$ GeV as those in Table II. The IW functions (4.13), (4.18) and (4.19) can be fitted nicely to the form

$$f(\omega) = f(1) \left[ 1 - \rho^2(\omega - 1) + \frac{\sigma^2}{2}(\omega - 1)^2 \right], \tag{5.1}$$

and it is found that (see Fig. 2)

$$\zeta(\omega) = 1 - 2.22(\omega - 1) + 1.82(\omega - 1)^2,$$

TABLE II: The pentaquark masses and input parameters $m_{[qq']}$, $m_q$ and $\beta$’s (in units of GeV) appearing in the Gaussian-type wave function $[3, 17]$.

| $m_{P_b}$ | $m_{P_c}$ | $m_{[ud]}$ | $m_{[us]}$ | $m_s$ | $m_b$ | $m_c$ | $\beta_{1b}$ | $\beta_{1c}$ | $\beta_{1s}$ | $\beta_{23}[ud]$ | $\beta_{23}[ud]$ |
|-----------|-----------|-------------|-------------|-------|-------|-------|------------|------------|------------|---------------|---------------|
| 6         | 3         | 0.40        | 0.56        | 0.45  | 4.4   | 1.3   | 0.65       | 0.58       | 0.48       | 0.38          | 0.38          |
We next turn to the form factors. Recall that $\Sigma_5'_{b \to c}$ is a $P_{b}(3) \to P_{c}(3)$ transition and $\Theta_{b \to c}$ is a $P_{b}(\bar{6}) \to P_{c}(\bar{6})$ one. It is straightforward to obtain their transition form factors from Eqs. (3.32) and (3.38). Since the flavor wave functions of $N^{8}_{c}((\bar{s}ud)_{-})$ and $\Theta((\bar{s}ud)_{+})$ are similar to $\Sigma_{5c}'((\bar{s}ud)_{-})$ and $\Theta_{c}(\bar{s}ud)_{+}$, respectively, with $\bar{c}$ replaced by $\bar{s}$, the $\Sigma_{5c}' \to N^{8}_{c}$ and $\Theta_{b \to c}$ transition form factor formula are similar to Eqs. (3.32) and (3.38), respectively.

As our calculation of form factors is done in the $q^+ = 0$ frame where $q^2 \leq 0$, we shall follow [62] to analytically continue the form factors to the timelike region. To proceed, we find that the momentum dependence of the form factors in the spacelike region can be well parameterized and reproduced in the three-parameter form:

$$F(q^2) = \frac{F(0)}{(1 - q^2/M^2)[1 - a(q^2/M^2) + b(q^2/M^2)^2]}$$

for $P_{Q} \to P_{Q'}$ transitions. The parameters $a$, $b$ and $F(0)$ are first determined in the spacelike region. We then employ this parametrization to determine the physical form factors at $q^2 \geq 0$. The parameters $a$, $b$ are expected to be of order $O(1)$.

The $P_{Q} \to P_{Q'}$ transition form factors $f_{1,2}(q^2)$ and $g_{1,2}(q^2)$ are given in Table III and shown in Fig. 3. It can be easily seen that in most cases these form factors at the zero recoil point $(q^2 = q^{2}_{\text{max}})$ agree with the HQ expectation Eqs. (4.7) and (4.8). However, the form factor $|f_{2}(q^{2}_{\text{max}})|$ for the $\Sigma_{5b}' \to \Sigma_{5c}'$ transition is larger than the HQ expectation and this may indicate the importance of $1/m_c$ corrections in this case. Note that the aforementioned heavy quark relations are not

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5 For $f_{2}(q^2)$ and $g_{2}(q^2)$, we take $q_{\perp} = (\sqrt{-q^2}, 0)$ in Eqs. (3.32) and (3.38).

6 Since the heavy quark-pair creation is forbidden in the $m_Q \to \infty$ limit, the so-called $Z$-graph which must be incorporated in the form-factor calculations in order to maintain covariance [61] is no longer a problem in the reference frame where $q^+ \geq 0$. This allows us to compute the Isgur-Wise functions directly in the timelike region.
applicable to \( \Theta_c \to \Theta \) and \( \Sigma'_c \to N'^+_8 \) transitions as the SU(3) quantum numbers of the final states are different from that of the corresponding initial states. The parameters \( a \) and \( b \) are of order unity for most of the entries in Table III. Note that if \( F(q^2) \) is parametrized without including the
factor of $(1 - q^2/M^2)$ in the denominator of Eq. (5.3), $a$ and $b$ will become larger. It is worthy remarking that although there is not much information on the parameter $\beta_{23}$ in the diquark system, the dependence of the Gaussian wave function width in transition form factors is mild as diquarks behave as a spectator. For example, when $\beta_{23}$ is increased from 0.38 GeV to 0.55 GeV, all the form factors at $q^2 = 0$ are reduced at most by 15% except for $g_{1\Theta b\Theta c}^\Theta\Theta$ and $g_{2\Theta b\Theta c}^\Theta\Theta$ that get changed by $-40\%$ and $+50\%$, respectively. In all cases, the modification of form factors is smaller at zero recoil than that at maximum recoil. However, $g_{1\Theta b\Theta c}^\Theta\Theta(0)$ and $g_{2\Theta b\Theta c}^\Theta\Theta(0)$ are of order $10^{-2}$ and $10^{-3}$, respectively, and hence their contributions to decay rates are small. Therefore, a variation of $\beta_{23}$ by 50% will modify the transition rates at most by 30%.

Under the factorization approximation, the decay amplitudes for color-allowed $\mathcal{P}_Q \rightarrow \mathcal{P}_{Q'} \pi$, $\mathcal{P}_{Q'} \rho$ decays in $b \rightarrow \bar{c}$ transitions are given by [41]

$$A(\mathcal{P}_Q \rightarrow \mathcal{P}_{Q'} \pi) = i\bar{u}' f_{\pi}(A + B \gamma_5) u,$$

$$A(\mathcal{P}_Q \rightarrow \mathcal{P}_{Q'} \rho) = i\bar{u}' \pi^\mu f_{\rho} (A_1 \gamma_{\mu} \gamma_5 + A_2 P^\mu_\rho \gamma_5 + B_1 \gamma_{\mu} + B_2 P'_\mu) u, \quad (5.4)$$

where

$$A = \frac{G_f}{\sqrt{2}} V_{cb} V_{ud} a_1 f_{\pi}(M' - M) f_1(m^2_{\rho}),$$

$$B = \frac{G_f}{\sqrt{2}} V_{cb} V_{ud} a_1 f_{\pi}(M + M') g_1(m^2_{\rho}),$$

$$A_1 = -\frac{G_f}{\sqrt{2}} V_{cb} V_{ud} a_1 f_{\rho} m_{\rho} \left[ g_1(m^2_{\rho}) + g_2(m^2_{\rho}) \frac{M - M'}{M + M'} \right], \quad (5.5)$$

$$A_2 = -\frac{2G_f}{\sqrt{2}} V_{cb} V_{ud} a_1 f_{\rho} m_{\rho} g_2(m^2_{\rho}) \frac{M^2}{M + M'},$$

$$B_1 = -\frac{G_f}{\sqrt{2}} V_{cb} V_{ud} a_1 f_{\rho} m_{\rho} \left[ f_1(m^2_{\rho}) - f_2(m^2_{\rho}) \right],$$

$$B_2 = -\frac{2G_f}{\sqrt{2}} V_{cb} V_{ud} a_1 f_{\rho} m_{\rho} f_2(m^2_{\rho}) \frac{M^2}{M + M'}.$$
with $V_{ij}$ the CKM matrix element, $f_\pi (\rho) = 131 \ (216)$ MeV the pion (rho) decay constant and $a_1 \sim 1$ the effective color-allowed Wilson coefficient. Likewise, the decay amplitudes for $\bar{c} \to \bar{s}$ transitions have similar expressions with $V_{cb}V_{ud}$ replaced by $V_{ud}V_{cs}$.

The decay rates read \[ \Gamma(P_Q \to P_{Q'} \pi) = \frac{p_c}{8\pi} \left[ \frac{(M + M')^2 - m_\pi^2}{M^2} |A|^2 + \frac{(M - M')^2 - m_\pi^2}{M^2} |B|^2 \right], \]
\[ \Gamma(P_Q \to P_{Q'} \rho) = \frac{p_c}{8\pi} \frac{E' + M'}{M} \left[ 2(|S|^2 + |P_2|^2) + \frac{E^2}{m^2_\rho} (|S + D|^2 + |P_1|^2) \right], \] (5.6)
with \[ S = -A_1, \quad P_1 = -\frac{p_c}{E_\rho} \left( M + M' B_1 + M B_2 \right), \]
\[ P_2 = \frac{p_c}{E' + M'} B_1, \quad D = -\frac{p_c^2}{E_\rho(E' + M')} (A_1 - M A_2), \] (5.7)
where $p_c$ is the c.m. momentum. The decay rates for the weak decays $\Theta_b^+ \to \Theta^0_{c+} \pi^+, \Theta^0_{c+} \rho^+$, $\Theta_c^0 \to \Theta^+ \pi^-, \Theta^+ \rho^-$, $\Sigma_{5b}^+ \to \Sigma_{5c}^{0+} \pi^+$, $\Sigma_{5c}^{0+} \to N_8^{+} \pi^-, N_8^{+} \rho^-$ are summarized in Table IV.

Assuming $\tau(\Theta_b^+) \sim \tau(\Lambda_b) \sim 1.2 \times 10^{-12}$ s and $\tau(\Theta_c^0) \sim \tau(\Lambda_c) \sim 2 \times 10^{-13}$ s for the weakly decaying $\Theta_b^+$ and $\Theta_c^0$, we find $B(\Theta_b^+ \to \Theta^0_{c+} \pi^+) \sim 1 \times 10^{-3}$ and $B(\Theta_c^0 \to \Theta^+ \pi^-) \sim 4\%$, which are consistent with the intuitive estimate made in \[. \]

Finally, it is worth commenting that $\Theta_c^0$ can be produced in $B$ decays via the dominant modes $B^+ \to \Theta^0_c \bar{D}^+$ and $B^0 \to \Theta^0_{c\rho} \pi^+$ \[. \] Theoretically, it is difficult to estimate their branching ratios. Nevertheless, the measured branching ratios by Belle for charmpful baryonic $B$ decays, $B(B^0 \to \Lambda^+_c \bar{p}) = (2.2^{+0.8}_{-0.3} \pm 0.3 \pm 0.6) \times 10^{-5}$ and $B(B^- \to \Lambda^+_c \bar{p}^\pi^-) = (1.87^{+0.43}_{-0.38} \pm 0.28 \pm 0.49) \times 10^{-4}$, provide some useful cue. Since a production of the pentaquark needs one more pair of $q\bar{q}$ compared to the normal baryon, it is plausible to expect that the branching ratios of $B^+ \to \Theta^0_c \bar{D}^+$ and $B^0 \to \Theta^0_{c\rho} \pi^+$ are at most of order $10^{-6}$ and $10^{-5}$, respectively. Hence, they may be barely reachable at $B$ factories. As for the production of the light pentaquark $\Theta^+$ in the decay $B^0 \to \Theta^+ \bar{p}$, for example, it will be greatly suppressed for two reasons. First, a typical two-body baryonic $B$ decay, e.g. $B^0 \to p\bar{p}$, occurs only at the level of $10^{-7}$ \[. \] Indeed, the upper limit of $B^0 \to p\bar{p}$ has been recently pushed to the level of $2.7 \times 10^{-7}$ by BaBar \[. \] Second, $B^0 \to \Theta^+ \bar{p}$ is Cabibbo suppressed relative to the $p\bar{p}$ mode. Therefore, it will be hopeless to detect a light pentaquark production in two-body or three-body baryonic $B$ decays.

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**Table IV:** The decay rates (in units of $10^{10}$ s$^{-1}$) of $P_Q \to P_{Q'} \pi$, $P_{Q'} \rho$ for $a_1 = 1$.

| Mode                  | $\Sigma_{5b}^+ \to \Sigma_{5c}^{0+} \pi^+$ | $\Theta_b^+ \to \Theta^0_{c+} \pi^+$ | $\Sigma_{5c}^{0+} \to N_8^{+} \pi^-$ | $\Theta_c^0 \to \Theta^+ \pi^-$ | $\Gamma(10^{10}$ s$^{-1}$) |
|-----------------------|------------------------------------------|-------------------------------------|-------------------------------------|---------------------------------|-----------------------------|
| $\Gamma(10^{10}$ s$^{-1}$) | 0.23                                      | 0.12                                | 42.62                               | 21.61                           |                             |

| Mode | $\Sigma_{5b}^+ \to \Sigma_{5c}^{0+} \rho^+$ | $\Theta_b^+ \to \Theta^0_{c+} \rho^+$ | $\Sigma_{5c}^{0+} \to N_8^{+} \rho^-$ | $\Theta_c^0 \to \Theta^+ \rho^-$ | $\Gamma(10^{10}$ s$^{-1}$) |
|---|------------------------------------------|-------------------------------------|-------------------------------------|---------------------------------|-----------------------------|
| $\Gamma(10^{10}$ s$^{-1}$) | 0.34                                      | 0.16                                | 82.41                               | 27.74                           |                             |
VI. CONCLUSIONS

Assuming the two diquark structure for the pentaquark as proposed in the Jaffe-Wilczek model, we study the weak transitions of heavy pentaquark states using the light-front approach. The main conclusions are as follows.

1. In the Jaffe-Wilczek model, there exist parity-even antisextet and parity-odd triplet heavy pentaquark baryons and they are all truly exotic. This differs than the Karliner-Lipkin model where the triplet pentaquark states are parity-even. It is very likely that the heavy pentaquarks in the $\bar{3}_f$ representation are lighter than the $\bar{6}_f$ ones owing to the lack of orbital excitation in the latter.

2. The theoretical estimate of charmed and bottom pentaquark masses is rather controversial. It is not clear if the ground-state heavy pentaquark lies above or below the strong-decay threshold. If the narrow state observed by the H1 experiment can be identified as an even-parity charmed pentaquark with a mass of 3099 MeV, then the diquark in the Jaffe-Wilczek picture should not be treated as a point-like object and it can have sizable hyperfine interactions with the antiquark of the pentaquark baryon. Consequently, the effective mass of the diquark $[ud]$ will be smaller in $\Theta^+$ than in $\Theta^0_c$ or $\Theta^+_b$. If the H1 state is a chiral partner of the yet undiscovered ground-state charmed pentaquark with opposite parity, the latter may be below the $DN$ threshold and can only be discovered by studying its weak decays. In the case, the point-like diquark picture could be valid. At any rate, it is important to check and confirm the H1 state from other experiments.

3. The antisextet-antisextet and triplet-triplet heavy pentaquark weak transition form factors are calculated using the light-front quark model. The momentum dependence of the physical form factors is determined by first fitting the form factors obtained in the spacelike region to a 3-parameter function in $q^2$ and then analytically continuing them to the timelike region.

4. In the heavy quark limit, it is found that heavy-to-heavy pentaquark transitions can be expressed in terms of three Isgur-Wise functions $\zeta$, $\xi_1$ and $\xi_2$: The first two are normalized to unity at zero recoil as required by heavy quark symmetry, while $\xi_2(1)$ is found to be equal to $1/2$ at maximum momentum transfer, in accordance with the prediction of the large-$N_c$ approach or the quark model. Therefore, the light-front model calculations are consistent with the requirement of heavy quark symmetry.

5. Numerical results for form factors and Isgur-Wise functions are presented. Decay rates of the weak decays $\Theta^+_b \to \Theta^0_c \pi^+(\rho^+)$, $\Theta^0_c \to \Theta^+ \pi^-(\rho^-)$, $\Sigma^+_b \to \Sigma^0_c \pi^+(\rho^+)$ and $\Sigma^0_c \to N^+_8 \pi^-(\rho^-)$ with $\Theta_Q$, $\Sigma'_Q$ and $N_8$ being the heavy anti-sixtet, heavy triplet and light octet pentaquarks, respectively, are obtained. For weakly decaying $\Theta^+_b$ and $\Theta^0_c$, the branching ratios of $\Theta^+_b \to \Theta^0_c \pi^+$, $\Theta^0_c \to \Theta^+ \pi^-$ are estimated to be of order $10^{-3}$ and $10^{-2}$, respectively.
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