Q-balls, -shells of a nonlinear sigma model with finite cosmological constants

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Abstract. We discuss existence of compact gravitating Q-ball and Q-shell solutions of a nonlinear sigma model with the target space $\mathbb{C}P^N$. The models with odd $N$ can be parametrized by $N$-complex scalar field solutions. We obtain solutions the Q-balls and Q-shells with the finite cosmological constant, in the de Sitter and anti-de Sitter spacetimes.

1. Introduction

Q-balls are non-topological solitons which appear in large family of field theories with global U(1) (or more) symmetry. Compactons are extended object of scalar field which has finite radius $r_{\text{out}}$; for $r > r_{\text{out}}$ the scalar field vanishes. The signum-Gordon model with $V-$shaped potential gives rise to such solutions [1, 2]. Interestingly, when the field is coupled with the electromagnetism, the inner radius emerges, i.e., the scalar field vanishes for $r < r_{\text{in}}$. The matter exists in region $r_{\text{in}} \leq r \leq r_{\text{out}}$ which is called Q-shells.

When a complex scalar field is coupled to gravity, boson stars and boson shells arise. Many research results of compact boson stars and shells are already presented [3, 4, 5, 6]. For the boson shell configurations, one possibility is the case that the gravitating boson shells surround a flat Minkowski-like interior region $r < r_{\text{in}}$ while the exterior region $r > r_{\text{out}}$ is the exterior of a Reissner-Nordström solution.

Recently, a new model realizing both Q-balls and Q-shells has been proposed [7]. The authors have proposed the following lagrangian in 3+1 dimensions

$$L = -\frac{M^2}{2} \text{Tr}(X^{-1} \partial^\mu X)^2 - \mu^2 V(X)$$

where the coupling constants $M, \mu$ have the dimension (length)$^{-1}$, (length)$^{-2}$, respectively. The potential $V$ is chosen in the way that the supports the compact solutions; it is called the V-shaped potential. It can be seen as direct extension of a $\mathbb{C}P^1$ nonlinear sigma model on target space $\mathbb{C}P^N$. The field $X$ is called the principal variable and it successfully parameterizes the coset space $SU(N + 1)/U(N) \sim \mathbb{C}P^N$. There exist the compact solutions for all the odd number of $N$; i.e., $N := 2n + 1, n = 0, 1, 2, \cdots$. The salient feature of the model from other models containing Q-balls, is that it supports both Q-ball and Q-shell solutions and the existence of Q-shells does not require the electromagnetic field.
For $n = 0, 1$ the solutions form $Q$-ball while $n \geq 2$ they always are $Q$-shell like. Again there are no upper bound of $|Q|$ and also no limit of $\omega$. The $Q$-ball solutions in the complex signum-Gordon model with local symmetry exist due to presence of the gauge field whereas in the case of the $CP^N$ model they appear as the result of a self-interactions between the scalar fields.

In this paper, we consider model of these scalar fields coupled with gravity and obtain the compact $Q$-ball and $Q$-shell solutions in the de Sitter and anti-de Sitter spacetimes. The resulting self-gravitating regular solutions form boson stars.

2. The model
We consider the action of a self-gravitating complex variable $X$ coupled to Einstein gravity

$$S = \int \sqrt{-g} d^4x \left[ R - \frac{2\Lambda}{16\pi G} - \frac{M^2}{2} \text{Tr}(X^{-1}\partial_\mu X)^2 - \mu^2 V(X) \right]$$

(2)

with curvature scalar $R$, cosmological constant $\Lambda$, and Newton’s constant $G$. The $CP^N$ space has a nice parametrization in terms of the principal variable $X$ [8] (and for the explicit construction of the solutions, see [9, 7]), defined as

$$X(g) = g \sigma(g)^{-1}, \quad g \in SU(N + 1).$$

(3)

It parametrizes the coset space $CP^N = SU(N + 1)/SU(N) \otimes U(1)$ with the subgroup $SU(N) \otimes U(1)$ being invariant under the involutive automorphism ($\sigma^2 = 1$). It follows that $X(gk) = X(g)$ for $\sigma(k) = k, \quad k \in SU(N) \otimes U(1)$. The $CP^N$ model possesses some symmetries. The existence of compact solutions require special class of potentials. An example of such a potential which we shall adopt in this paper has the form

$$V(X) = \frac{1}{2} [\text{Tr}(I - X)]^{1/2}.$$ 

(4)

As we shall see later, the asymptotic behavior of the field implies $X \rightarrow I$.

We assume the $(N + 1)$-dimensional representation in which the $SU(N + 1)$ valued group element $g$ is parametrized by the set of complex fields $u_i$:

$$g = \frac{1}{\vartheta^2} \left( \begin{array}{cc} \Delta & iu \\ iu^\dagger & 1 \end{array} \right), \quad \Delta_{ij} \equiv \vartheta \delta_{ij} - \frac{u_i u_j}{1 + \vartheta}$$

(5)

which fixes the symmetry to $SU(N) \otimes U(1)$. The principal variable is now

$$X(g) = \left( \begin{array}{cc} I_{N \times N} & 0 \\ 0 & -1 \end{array} \right) + \frac{2}{\vartheta^2} \left( \begin{array}{cc} -u \otimes u^\dagger & iu \\ iu^\dagger & 1 \end{array} \right)$$

(6)

where $\vartheta := \sqrt{1 + u^\dagger \cdot u}$. Thus the $CP^N$ model of the model (2) takes the form

$$\mathcal{L}_{CP^N} = -M^2 g^{\mu\nu} \tau_{\mu\nu} - \mu^2 V$$

(7)

with

$$\tau_{\mu\nu} = -\frac{4}{\vartheta^2} \partial_\mu u^\dagger \cdot \Delta^2 \cdot \partial_\nu u, \quad \Delta^2_{ij} := \vartheta^2 \delta_{ij} - u_i u_j^*.$$ 

(8)

Variation of the action with respect to the metric leads to the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - g_{\mu\nu} \Lambda$$

(9)
with the stress-energy tensor

\[ T_{\mu\nu} = 2 \frac{\partial \mathcal{L}_{\mathcal{C}P^N}}{\partial g^\mu\nu} - g_{\mu\nu} \mathcal{L}_{\mathcal{C}P^N} = -2M^2\tau_{\nu\mu} + M^2g_{\mu\nu}\gamma_{\alpha}^\beta \gamma_{\beta\alpha} + g_{\mu\nu}\mu^2V . \] (10)

Varying the action with respect to the field \( u_i, u_i^* \) one obtain equations for the \( \mathcal{C}P^N \) field

\[ \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g}\partial^\mu u_i) + \frac{2}{\mu^2}(\partial_\mu u_\mu \ast u) \partial^\mu u_i + \frac{\mu^2}{4M^2} g^{\mu\nu} \sum_{k=1}^N \left[ (\delta_{ik} + u_i u_k^*) \frac{\partial V}{\partial u_k^*} \right] = 0 . \] (11)

The obtained system of coupled equations is quite complex so we shall adopt numerical techniques to solve it.

It is useful to introduce the the dimensionless variables

\[ x_\mu \rightarrow \frac{\mu}{M} x_\mu . \] (12)

For the solutions with spherical symmetry, we employ the Schwarzschild-like coordinates

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = A^2(r)C(r)dt^2 - \frac{1}{C(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) . \] (13)

We restrict \( N \) to be odd number, i.e., \( N := 2n + 1 \). It allows for reduction of the equations with the help of ansatz

\[ u_m(t, r, \theta, \varphi) = \sqrt{\frac{4\pi}{2n + 1}} f(r)Y_{nm}(\theta, \varphi)e^{i\omega t} . \] (14)

where \( Y_{nm}, -n \leq m \leq n \) are the standard spherical harmonics and \( f(r) \) is called profile function. The \( 2n + 1 \) fields \( u = (u_m) = (u_n, u_{-n+1}, \ldots, u_{n-1}, u_n) \) are realized by the components of the spherical harmonics for given \( n \). The relation \( \sum_{m=-n}^n Y_{nm}^*(\theta, \varphi)Y_{nm}(\theta, \varphi) = \frac{2n+1}{4\pi} \) is very helpful in finding explicit form of many inner products. Substituting (13),(14) into the Einstein equations (9), we obtain the equations for the \( A, C \) as

\[ A' = \alpha rA \left( \frac{4\omega^2f^2}{A^2C(1+f^2)^2} + \frac{4f^2}{(1+f^2)^2} \right) , \] (15)

\[ C' = \frac{1 - C}{r} + r\Lambda - \alpha r \left( \frac{4\omega^2f^2}{A^2C(1+f^2)^2} + \frac{4Cf^2}{(1+f^2)^2} + \frac{4n(n+1)f^2}{r^2(1+f^2)^2} + \frac{f}{\sqrt{1+f^2}} \right) . \] (16)

where \( \alpha \) is a dimensionless coupling constant of the equation defined as \( \alpha := 8\pi G\mu^2 \). In our numerical analysis, we solve the equations for the model parameters, the coupling constant \( \alpha \) and the frequency \( \omega \). Change of \( \alpha \) thus means the variation of the model parameter \( \mu \) (or dimensionful case, \( M \)). In this sense the parameter \( \mu \) works as a kind of “susceptibility” of gravity to matters.

For the equation of the \( \mathcal{C}P^N \) (11) we have

\[ Cf'' + \frac{(1+C)f'}{r} - \frac{n(n+1)f}{r^2} + \frac{\omega^2f(1-f^2)}{A^2C(1+f^2)} - \frac{2Cf f'^2}{1+f^2} - \frac{rf'}{r} \left( \Lambda + \alpha \left( \frac{4n(n+1)f^2}{r^2(1+f^2)^2} + \frac{f}{\sqrt{1+f^2}} \right) \right) - \frac{1}{8} \text{sgn}(f) \sqrt{1+f^2} = 0 . \] (17)

Here the prime ’ means differentiation with respect to \( r \).
From the dimensionless lagrangian
\[ \tilde{\mathcal{L}}_{\mathbb{C}P^N} = -g^{\mu\nu} \tau_{\mu\nu} - V, \]
we obtain the dimensionless hamiltonian of the \( \mathbb{C}P^N \) model
\[ \mathcal{H}_{\mathbb{C}P^N} = -g^{00} \tau_{00} + g^{ii} \tau_{ii} + V = \frac{4}{(1 + f^2)^2} \left( C f^2 + \frac{\omega^2 f^2}{A^2 C} + \frac{n(n + 1)(1 + f^2) f^2}{r^2} \right) + \frac{f}{\sqrt{1 + f^2}}. \] (19)

In terms of the parametrization (5), the symmetry \( SU(N+1) \) reduces to \( SU(N) \otimes U(1) \) and the lagrangian (18) possesses the symmetry. It contains subgroup \( U(1)^N \) given by the transformation
\[ u_i \rightarrow e^{i\beta_i} u_i, \quad i = 1, \ldots, N \] (20)
where \( \beta_i \) are some global parameters. This global symmetry and the Noether charge successfully has a role for stabilizing the solutions. The Noether currents corresponding to (20) are
\[ J^{(i)}_\mu = -\frac{4i}{\partial^i} \sum_{j=1}^N [u^*_i \Delta^2_{ij} \partial_{\mu} u_j - \partial_{\mu} u^*_i \Delta^2_{ji} u_i]. \] (21)

Now, in terms of (14) the currents are estimated via
\[ J^{(m)}_0 = 8\omega \frac{(n-m)!}{(n+m)!} \frac{f^2}{(1 + f^2)^2} (P^m_n(\cos \theta))^2, \] (22)
\[ J^{(m)}_\varphi = 8n \frac{(n-m)!}{(n+m)!} \frac{f^2}{1 + f^2} (P^m_n(\cos \theta))^2 \] (23)
and \( J^{(m)}_\rho = 0 \), for \( m = -n, -n + 1, \ldots, n-1, n \). We directly see that the current is conserved because the components (22),(23) depend only on \( r, \theta \) and
\[ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} J^{(m)}_\nu) = \frac{1}{A^2 C} \partial_0 J^{(m)}_0 + \frac{1}{r^2 \sin^2 \theta} \partial_\varphi J^{(m)}_\varphi = 0. \] (24)

Integration of the conservation law of the current (24)
\[ 0 = \int dt \int d^3 x \sqrt{-g} \left( \frac{1}{A^2 C} \partial_0 J^{(m)}_0 + \frac{1}{r^2 \sin^2 \theta} \partial_\varphi J^{(m)}_\varphi \right). \] (25)

For the condition that the spatial component \( J^{(m)}_\varphi \) quickly decreases at the spatial boundary, we obtain the conserved Noether charge of the form
\[ Q^{(m)} = \frac{1}{2} \int d^3 x \sqrt{-g} \frac{1}{A^2 C} J^{(m)}_0(x) = \frac{16\pi \omega}{2n + 1} \int_0^\infty r^2 dr \frac{f^2}{AC(1 + f^2)^2}. \] (26)
Figure 1. The $Q$-ball snd $Q$-shell solutions of the $\mathbb{C}P^1$ (top), $\mathbb{C}P^3$ (middle) and $\mathbb{C}P^5$ bottom for coupling constant $\alpha = 0$. The left figures are $\Lambda < 0$ and the right figures are $\Lambda > 0$.

3. Boundary conditions

To obtain compact solutions in curved spacetimes, we specify the boundary conditions for the metric and the profile function. For the metric function $A(r)$ we adopt

$$A(R_{\text{out}}) = 1$$

where $R_{\text{out}}$ is the outer compacton radius. For the metric function $C(r)$ we require

$$C(0) = 1$$

for ball-like solutions and for shell-like solutions

$$C(R_{\text{in}}) = 1 + \frac{\Lambda}{3} R_{\text{in}}^2$$

(29)
Figure 2. The gravitating solutions of the $CP^1, CP^5, CP^{11}$ for $\alpha = 0.010, \Lambda = -0.005$. The dotted line indicates solutions of the vacuum Einstein equations.

where $R_{\text{in}}$ is the inner compacton radius.

For the profile function we choose

$$f(R_{\text{out}}) = f'(R_{\text{out}}) = 0$$  \hspace{1cm} (30)

for ball-like solutions and for shell-like solutions

$$f(R_{\text{in}}) = f'(R_{\text{in}}) = 0.$$  \hspace{1cm} (31)
Figure 3. The relation of the $E^{-1/5}Q^{-1/6}$ of the gravitating solutions. The parameters are $\alpha = 0.010$ and $\Lambda = 0.001$ (upper), $\Lambda = -0.001$ (lower).

The solutions in the inner region $r < R_{\text{in}}$ and the exterior region $r > R_{\text{out}}$ are given by the Schwarzschild-de Sitter and Schwarzschild-anti-de Sitter solutions

$$f(r) = 0, \quad C(r) = 1 - \frac{2m}{r} + \frac{\Lambda}{3}r^2, \quad m = \text{const.}$$

Note that the metric function $A(r)$ satisfy $A(r) = 1$ ($r > R_{\text{out}}$) and $A(r) = \text{const.} = A_i < 1$ ($r < R_{\text{in}}$).

We examine the boundary behavior of the solutions. First we consider the expansion at the
origin. For this reason we take series

\[ f(r) = \sum_{k=0}^{\infty} f_k r^k, \quad A(r) = \sum_{k=0}^{\infty} A_k r^k, \quad C(r) = \sum_{k=0}^{\infty} C_k r^k \]

and substitute them into the equations (15),(16),(17). The result for \( n = 0 \) is

\[ f(r) = f_0 + \frac{1}{48} \left( \sqrt{1 + f_0^2} - \frac{8f_0(1-f_0^2)\omega^2}{A_0^2(1+f_0^2)} \right) r^2 + O(r^4), \]
\[ A(r) = A_0 + \frac{2\alpha f_0^2 \omega^2}{A_0(1+f_0^2)^2} r^2 + O(r^4), \]
\[ C(r) = 1 + \frac{1}{3} \left( \Lambda - \alpha \left( \frac{4f_0^2\omega^2}{A_0^2(1+f_0^2)^2} + \frac{f_0}{\sqrt{1+f_0^2}} \right) \right) r^2 + O(r^4) \]

with the free parameters \( f_0, A_0 \). For \( n = 1 \) we obtain

\[ f(r) = f_1 r + \frac{1}{32} r^2 + \frac{1}{10} \left( 2f_1^3(1+6\alpha) - f_1 \left( \frac{\omega^2}{A_0^2} + \frac{4\Lambda}{3} \right) \right) r^3 + O(r^4), \]
\[ A(r) = A_0 + 2\alpha f_1^2 r^3 + \frac{\alpha f_1}{6} r^3 + O(r^4), \]
\[ C(r) = 1 + \left( \frac{\Lambda}{3} - 4\alpha f_1^2 \right) r^2 - \frac{\alpha f_1}{2} r^3 + O(r^4) \]

with the free parameters \( f_1, A_0 \). The solutions are \( Q \)-balls. For \( n \geq 2 \) we have no nontrivial solutions at the vicinity of the origin \( r = 0 \). It means that solution must be trivial in vicinity of zero. Thus we have to solve the equations at some finite radius \( R_{in} \), which give rise to the \( Q \)-shell solutions.

4. The numerical results

In this section we present our numerical solutions. The differential equations (15),(16),(17) of each sector \( n \) with the definite parameters \( \omega, \alpha \) can be easily solved using the standard shooting algorithm. According to (34),(35), the solutions with \( n = 0, 1 \) are regular at the origin thus we have the \( Q \)-ball type solutions, which are designated as boson stars. First we show results of the solutions for \( \alpha = 0 \) with finite cosmological constants (in Fig.1). As discussed in the last section, for \( n \geq 2 \), the solutions are not regular at the origin so exhibit the \( Q \)-shell structure. As a result, the matter field can exist in the range of \( r \in (R_{in}, R_{out}) \).

Next we consider the case of the gravitating solutions where the interior of the \( Q \)-shell is regular. We present some typical results in Fig.2: the case of \( \mathbb{C}P^1, \mathbb{C}P^5 \) and \( \mathbb{C}P^{11} \). Inside the shells the metric functions are the vacuum solutions. On can easily see that shells corresponding with higher \( n \) solutions become larger and thinner.

For the stability of the \( Q \)-ball type solutions, it is mandatory to examine the energy - the charge scaling. In Fig.3, we plot the relation \( E^{-1/5}Q^{-1/6} \) with the solutions of \( \mathbb{C}P^1, \mathbb{C}P^5 \) and \( \mathbb{C}P^{11} \) for both \( \Lambda > 0 \) and \( \Lambda < 0 \). The results strongly indicate that the energy scales as \( E \sim Q^{5/6} \) which means our \( Q \)-balls and \( Q \)-shells are classically stable against decaying into a collection of the fraction with unit charge.

5. Summary

In this paper, we considered \( \mathbb{C}P^N \) nonlinear sigma model with a compact support coupled with gravity. We numerically obtained the compact \( Q \)-ball and \( Q \)-shell solutions with a finite
cosmological constant, in a de Sitter and anti-de Sitter spacetimes. The resulting self-gravitating regular solutions form boson stars.

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