The cohomology of simple Lie groups

Haibao Duan∗ and Xuezhi Zhao
Institute of Mathematics, Chinese Academy of Sciences
dhb@math.ac.cn
Department of Mathematics, Capital Normal University,
zhaoxve@mail.cnu.edu.cn

Abstract
The problem of computing the cohomology ring $H^*(G; F)$ of a Lie group $G$ was initiated by E. Cartan in 1929, and has been one of the main focuses in the 20th century algebraic topology. However, despite jumbled efforts and great achievements by many mathematicians in about one century two important questions remain open:
1) find a single procedure computing the ring $H^*(G; F_p)$ for all $G$ and $p$ (see Kač [24]);
2) determine the integral cohomology ring $H^*(G; \mathbb{Z})$ for the most difficult and subtle cases of $G = E_6, E_7$ and $E_8$.

Let $G$ be a simple Lie group with a maximal torus $T$. Based on recent progress in Schubert calculus on the complete flag manifold $G/T$ [20] we construct the ring $H^*(G; F)$ uniformly for all $G$ and $F = \mathbb{Q}, \mathbb{F}_p, \mathbb{Z}$.

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1 Introduction
Let $G$ be a compact, 1–connected and simple Lie group, namely, $G$ is one of the classical groups $SU(n), Spin(n), Sp(n)$, or one of the exceptional groups $G_2, F_4, E_6, E_7, E_8$. These groups constitute the cornerstones in all compact Lie groups in views of Cartan’s classification Theorem [42 p.674]. Our main concern is the cohomology ring $H^*(G; \mathbb{F})$ with the coefficients $\mathbb{F}$ either the ring $\mathbb{Z}$ of integers, the field $\mathbb{Q}$ of rationals, or one of the finite fields $\mathbb{F}_p$.

The problem of determining the ring $H^*(G; \mathbb{F})$ was initiated by E. Cartan in 1929, and has been one of the main focuses in the 20th century algebraic topology

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for its relevance to the topologies of the loop space $\Omega G$, classifying space $BG$, homogeneous spaces $G/H$, as well as the homotopy theory of $H$–spaces [34] [25]. Just some of the many mathematician involved were Brauer, Pontryagin, Hopf, Samelson, Yan, Leray, Chevalley, Miller, Borel, Araki [15] [38] [24]. However, despite great achievements in about one century two important questions remain open.

Historically the rings $H^*(G;\mathbb{F}_p)$ were obtained by quite different methods, presented by generators with various origins and using case by case computations depending on $G$ and $p$ [5] [6] [7] [8] [9] [10] [11] [12] [8] [27] [30]. The question to find a unified procedure to compute the ring $H^*(G;\mathbb{F}_p)$ was raised and studied by Kać [24], who succeeded in translating the ring structure for $p \neq 2$ and additive structure for $p = 2$ into a purely “Weyl group question” by showing that these structures are entirely determined by the degrees of basic $W$–invariants over $\mathbb{Q}$ and the degrees of the basic “generalized invariants” over $\mathbb{F}_p$. Unfortunately, apart from the fact that the ring $H^*(G;\mathbb{F}_2)$ remains undetermined in this scheme (see Remark 5.1), in order to compute the degrees of those invariants Kać made extensively use of the previous computation of $H^*(G;\mathbb{F}_p)$ “in the inverse order” [24]. Similar question was asked by Lin [31] for the case of $p = 2$ in the context of homotopy theory of finite $H$–spaces.

The traditional method computing the algebra $H^*(G;\mathbb{F})$ with $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ relies largely on the classification of the finite dimensional Hopf algebras due Hopf, Samelson and Borel [35] which does not directly apply to the most important but subtle case for $\mathbb{F} = \mathbb{Z}$. Apart from the facts that the ring $H^*(G;\mathbb{Z})$ has been determined for the classical $G$ by Borel [5] and Pittie [36], and for $G = G_2, F_4$ by Borel [8] [9], the cases of $G = E_6, E_7$ and $E_8$ remain a challenge and beckon us for decades.

This paper is devoted to a unified procedure constructing the cohomology ring $H^*(G;\mathbb{F})$ with $\mathbb{F} = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{F}_p$. The main results are stated in Theorems 1–2 in §5, and Theorems 3–6 in §6. The method of this paper has been extended in [21] to obtain a complete characterization for $H^*(G;\mathbb{F}_p)$ as a module over the Steenrod algebra $\mathbb{A}_p$.

Our approach begins with the Leray–Serre spectral sequence $\{E^r_{p,q}(G;\mathbb{F}), d_r\}$ of the fibration

\[(1.1)\quad T \rightarrow G \xrightarrow{\pi} G/T,\]

where $G$ is a simple Lie group with a fixed maximal torus $T \subset G$. Let $W$ be the Weyl group of $G$ and let $\{\omega_i\}_{1 \leq i \leq n} \subset H^2(G/T;\mathbb{Z})$ be a set of fundamental dominant weights of $G$, $n = \dim T$ [13]. It is well known that (23) [32]

\[(1.2)\quad E^p_{2,q}(G;\mathbb{F}) = H^p(G/T; H^q(T;\mathbb{F}))) = H^p(G/T) \otimes \Lambda^2_{\mathbb{F}}(t_1, \ldots, t_n);\]

\[(1.3)\quad \text{the differential } d_2 : E^p_{2,q}(G;\mathbb{F}) \rightarrow E^{p+2,q-1}_2(G;\mathbb{F}) \text{ is given by } d_2(x \otimes t_k) = x\omega_k \otimes 1,\]

where $t_k \in H^1(T;\mathbb{F})$ the class that is mapped to $\omega_i$ under the transgression $\tau : H^1(T;\mathbb{F}) \rightarrow H^2(G/T;\mathbb{F})$ [4], and where $\Lambda^2_{\mathbb{F}}(t_1, \cdots, t_n)$ is the exterior algebra in $t_1, \cdots, t_n$. 

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An intimate relationship between the two additive groups $H^*(G;\mathbb{F})$ and $E_3^*(G;\mathbb{F})$ has been expected by many authors. Early in 1950 Leray [25] established that $H^*(G;\mathbb{Q}) = E_3^*(G;\mathbb{Q})$. Serre [39] in 1965 showed that $H^*(G;\mathbb{F}_p) = E_3^*(G;\mathbb{F}_p)$. Marlin [32], in 1991 conjectured that $E_3^*(G;\mathbb{Z}) = H^*(G;\mathbb{Z})$. Conceivably, a thorough understanding of $E_3^*(G;\mathbb{Z})$ may bring a way to $H^*(G;\mathbb{Z})$.

In principle, the two ingredients (1.2), (1.3) required by computing $E_3^*(G;\mathbb{Z})$ are known in the context of Schubert’s calculus [23, 32]: the basis theorem of Chevalley [10] asserts that the set of Schubert classes $\{s_w\}_{w \in W}$ on $G/T$ furnishes $H^*(G/T;\mathbb{Z})$ with an additive basis, the Chevalley formula [10, 12, 17] that expands the product of a Schubert class $s_w$ with a weight $\omega_i$ characterizes $d_2$ explicitly. However, direct calculation based on these ideas fails to access $E_3^*(G;\mathbb{Z})$ [23, 32], not to mention that our interest is the ring structure on $H^*(G;\mathbb{Z})$.

We explain the main idea in this paper. The basis theorem merely describes $H^*(G/T;\mathbb{Z})$ additively. Resorting to its ring structure one may attempt a compact presentation of $H^*(G/T;\mathbb{Z})$ such as the quotient of a polynomial ring. To merge the geometry of Schubert classes into calculating $H^*(G;\mathbb{Z})$ and to eliminate the computation cost from the very beginning, one may expect further that the generators for $H^*(G/T;\mathbb{Z})$ are taken in a minimal set of Schubert classes on $G/T$. It is the fulfilment of this task in [20] for all exceptional $G$ that reveals certain general features of the ring $H^*(G/T;\mathbb{Z})$ that are summarized in Lemma 2.1 of §2.

Starting from Lemma 2.1 the ring $E_3^*(G;\mathbb{F})$ (resp. $H^*(G,\mathbb{F})$) can be calculated effectively and uniformly for all $G$ and $\mathbb{F}$. More precisely,

i) a set of explicit generators for $E_3^*(G;\mathbb{F})$ (resp. for $H^*(G,\mathbb{F})$) can be constructed from certain polynomials in the Schubert classes on $G/T$;

ii) in terms of these generators, the ring $H^*(G,\mathbb{F})$ can be explicitly presented (e.g. Theorem 3–6 in §6).

Along the way we confirm the conjecture that additively

$$E_3^*(G;\mathbb{Z}) = E_3^*(G;\mathbb{Z})$$

by Kač [23] in §4, and the stronger one

$$E_3^*(G;\mathbb{Z}) = H^*(G;\mathbb{Z})$$

by Marlin [32] in §5.

Hopefully our approach to $H^*(G,\mathbb{F})$ may admit generalizations. Let $A$ be an extended Cartan matrix with associated group $G$ of the Kač–Moody type, and let $B \subset G$ be a Borel subgroup [26]. One may expect that result analogue to Lemma 2.1 hold for the complete flag variety $G/B$, and that the method in this paper is extendable to formulate $H^*(G;\mathbb{F})$. We emphasize at this point that the problem of understanding $H^*(G;\mathbb{F})$ in the context of Schubert calculus has been studied by Kač [23] twenty years ago, and that, as the partition of $G/B$ by the Schubert cells is completely determined by the Cartan matrix $A$ (as in the classical situation $G/T$), the techniques developed in our previous works [17, 18, 19, 20] may be applicable to extend Lemma 2.1 from $G/T$ to $G/B$.

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2 Construction based on Schubert presentation of the ring $H^*(G/T; \mathbb{Z})$

For a compact Lie group $G$ with a maximal torus $T$, the quotient space $G/T$ is known as the complete flag manifold of $G$. In the modern form taken by the classical topic known as " enumerative geometry" in the 19th century, Schubert calculus amounts to the determination of the cohomology ring $H^*(G/T; \mathbb{Z})$ [41, p 331].

Based on the formula for multiplying Schubert classes in [17], explicit presentation of the ring $H^*(G/T; \mathbb{Z})$ as the quotient of a polynomial ring in certain Schubert classes on $G/T$ was obtained in [20]. In Lemmas 2.1–2.4 we demonstrate concise presentation of $H^*(G/T; \mathbb{F})$ (in accordance to $\mathbb{F} = \mathbb{Z}, \mathbb{Q}$ and $\mathbb{F}_p$) that are barely sufficient for the purpose of this paper. In terms of the defining polynomials for the ideal in those presentation a set of so called primary polynomials are specified in §2.3. They are utilized in §2.4 to construct a set of elements in $E_3^{*,1}(G; \mathbb{F})$, temporarily named as the primary forms in $E_3^{*,1}(G; \mathbb{F})$. The ring $H^*(G; \mathbb{F})$ will be formulated by these elements in §5 and §6.

2.1. Presentation of $H^*(G/T; \mathbb{Z})$. In this paper all elements in a graded vector space (or graded $\mathbb{Z}$-module) are homogeneous. Given a subset $\{f_1, \cdots, f_m\}$ in a ring write $\langle f_1, \cdots, f_m \rangle$ for the ideal generated by $f_1, \cdots, f_m$. For a set $S$ denote by $|S|$ its cardinality.

We begin with the next result shown in [20, Theorem 6], that summarizes certain general properties in the Schubert presentations of the rings $H^*(G/T; \mathbb{Z})$ for all simple Lie group $G$.

Lemma 2.1. For each simple $G$ there exist a set $\{y_1, \cdots, y_m\}$ of Schubert classes $y_1, \cdots, y_m$ on $G/T$ of $\deg y_i > 2$, so that the set $\{\omega_1, \cdots, \omega_n; y_1, \cdots, y_m\}$ is a minimal set of generators for the ring $H^*(G/T; \mathbb{Z})$.

Moreover, with respect to those generators, one has the presentation

\begin{equation}
H^*(G/T; \mathbb{Z}) = \mathbb{Z}[\omega_1, \cdots, \omega_n; y_j]/\langle e_i, f_j, g_j \rangle_{1 \leq i \leq k; 1 \leq j \leq m},
\end{equation}

in which

i) $k = n - |\{\deg g_j \mid 1 \leq j \leq m\}|$;

ii) for each $1 \leq i \leq k$, $e_i \in \langle \omega_1, \cdots, \omega_n \rangle$;

iii) for each $1 \leq j \leq m$, the pair $(f_j, g_j)$ of polynomials is related to the Schubert class $y_j$ in the fashion

$$
f_j = p_j y_j + \alpha_j, \quad g_j = y_j^{k_j} + \beta_j, \quad 1 \leq j \leq m,
$$

with $p_j \in \{2, 3, 5\}$, $\alpha_j, \beta_j \in \langle \omega_1, \cdots, \omega_n \rangle$.

Furthermore, for each simple $G$ the sets of integers

$$
\{k, m\}, \{\deg e_i\}, \{\deg y_j\}, \{p_j\}, \{k_j\}
$$

emerging in (2.1), called the basic data of $G$, are given in Tables 1 and 2 below:
In our unified approach to the Schubert presentation (2.1) describes Schubert classes of the form \( f \in \mathbb{Z}[\omega_1, \ldots, \omega_n; y_1, \ldots, y_m] \). In contrast, in terms of the minimal constraint on \( m \) in Lemma 2.1, the basic data for \( E_8 \) given in the last column of Table 2, there appears the following phenomenon which will cause a few additional concerns for the case of \( G = E_8 \) in our unified approach to \( H^*(G; \mathbb{F}) \) (see formula (5.3) in [20]).

**Lemma 2.2.** For \( G = E_8 \) there exists a polynomial \( f \in \mathbb{Z}[\omega_1, \ldots, \omega_n; y_1, \ldots, y_r] \) of the form \( f = 2y_1^2 - y_2^2 + y_3^2 + \beta \), \( \beta \in \langle \omega_1, \ldots, \omega_n \rangle \), so that

\[
\begin{align*}
g_4 &= -12f + 5y_1^4f_4 - 4y_0^5f_6 + 6y_7f_7; \\
g_5 &= -10f + 4y_1^4f_4 - 3y_0^5f_6 + 5y_7f_7; \\
g_7 &= 15f - 6y_1^4f_4 + 5y_0^5f_6 - 7y_7f_7.
\end{align*}
\]

**Remark 2.1.** With the minimal constraint on \( m \) in Lemma 2.1, the basic data of \( G \) can be shown to be invariants of \( G \).

Since the set \( \{\omega_1, \cdots, \omega_n\} \) consists of all Schubert classes on \( G/T \) with cohomology degree 2 [20], (2.1) describes \( H^*(G/T; \mathbb{Z}) \) by certain Schubert classes on \( G/T \) and therefore, is called a Schubert presentation of the ring \( H^*(G/T; \mathbb{Z}) \). In addition to \( \{\omega_1, \cdots, \omega_n\} \) elements in the set \( \{y_i\}_{1 \leq i \leq m} \) are called the special Schubert classes on \( G/T \) [20]. To be precise for each exceptional \( G \) a set of special Schubert classes on \( G/T \) (satisfying Lemmas 2.1 and 2.2) is given by their Weyl coordinates [20] in the table below.

| \( G/T \) | \( G_2/T \) | \( F_4/T \) | \( E_6/T, n = 6, 7, 8 \) |
|---|---|---|---|
| \( y_1 \) | \( \sigma_{[1,2,1]} \) | \( \sigma_{[3,2,1]} \) | \( \sigma_{[5,4,2]}, n = 6, 7, 8 \) |
| \( y_2 \) | \( \sigma_{[3,2,1]} \) | \( \sigma_{[4,3,2,1]} \) | \( \sigma_{[6,5,4,2]}, n = 6, 7, 8 \) |
| \( y_3 \) | \( \sigma_{[5,4,2]} \) | \( \sigma_{[7,6,5,4,2]}, n = 7, 8 \) |
| \( y_4 \) | \( \sigma_{[6,5,4,2]} \) | \( \sigma_{[1,3,6,5,4,2]}, n = 8 \) |
| \( y_5 \) | \( \sigma_{[7,6,5,4,2]} \) | \( \sigma_{[1,5,4,3,7,6,5,4,2]}, n = 7, 8 \) |
| \( y_6 \) | \( \sigma_{[1,6,5,4,3,7,6,5,4,2]} \) | \( \sigma_{[5,4,2,3,1,4,6,5,4,3,8,7,6,5,4,2]}, n = 8 \) |
| \( y_7 \) | \( \sigma_{[5,4,2,3,1,4,6,5,4,3,8,7,6,5,4,2]} \) | | |

Table 3. The special Schubert classes on \( G/T \) for all exceptional \( G \).
2.2. The algebra \( H^*(G/T; \mathbb{F}) \) with \( \mathbb{F} = \mathbb{Q} \) or \( \mathbb{F}_p \). Since the ring \( H^*(G/T; \mathbb{Z}) \) is torsion free one may deduce presentation of \( H^*(G/T; \mathbb{F}) \) directly from Lemma 2.1 and the isomorphism \( H^*(G/T; \mathbb{F}) = H^*(G/T; \mathbb{Z}) \otimes \mathbb{F} \).

One of the attempts in this work is to describe the ring \( H^*(G; \mathbb{F}) \) by a minimal set of generators. As an initial step we need to characterize \( H^*(G/T; \mathbb{F}) \) by a minimal system of generators and relations. The following notion subsequent to the basic data of \( G \) serves this purpose.

**Definition 2.1.** For each \( G \) and a prime \( p \) we set

\[
G(p) = \{ j \mid 1 \leq j \leq m, p_j = p \} \text{ (see Tables 1 and 2).}
\]

We shall also put for \( G \neq E_8 \) that

\[
\overline{G}(\mathbb{F}) = \begin{cases} 
\{1, \ldots, m\} & \text{if } \mathbb{F} = \mathbb{Z} \text{ or } \mathbb{Q}; \\
\text{the complement of } G(p) \text{ in } \{1, \ldots, m\} & \text{if } \mathbb{F} = \mathbb{F}_p,
\end{cases}
\]
and let

\[
E_8(\mathbb{F}) = \begin{cases} 
\{1, 2, 3, 5, 6\} & \text{if } \mathbb{F} = \mathbb{Z}, \mathbb{Q} \text{ or } \mathbb{F}_p \text{ with } p \neq 2, 3, 5; \\
\{2\} & \text{if } \mathbb{F} = \mathbb{F}_2; \\
\{1, 3, 5\} & \text{if } \mathbb{F} = \mathbb{F}_3; \\
\{1, 2, 3, 5\} & \text{if } \mathbb{F} = \mathbb{F}_5.
\end{cases}
\]

**Lemma 2.3.** Let \( e_i^{(0)}, g_j^{(0)} \in \mathbb{Q}[\omega_1, \ldots, \omega_n] \) be the polynomials obtained from \( e_i, g_j \) in Lemma 2.1 by eliminating the classes \( y_j \) using \( f_j, 1 \leq j \leq m \). Then

\[
H^*(G/T; \mathbb{Q}) = \mathbb{Q}[\omega_1, \ldots, \omega_n] / \langle e_i^{(0)}, g_j^{(0)} \rangle_{1 \leq i \leq k, j \in \overline{G}(\mathbb{Q})}.
\]

**Proof.** Rationally \( y_j = -\frac{1}{p_j} a_j \) by the relation \( f_j \) in Lemma 2.1. It implies that

\[
H^*(G/T; \mathbb{Q}) = \mathbb{Q}[\omega_1, \ldots, \omega_n] / \langle e_i^{(0)}, g_j^{(0)} \rangle_{1 \leq i \leq k, 1 \leq j \leq m},
\]
which verifies Lemma 2.3 for \( G \neq E_8 \). For \( G = E_8 \) we get from (2.2) that \( g_4^{(0)} = \frac{6}{5} g_6^{(0)}, g_7^{(0)} = -\frac{4}{5} g_6^{(0)} \). This completes the proof. \( \square \)

**Lemma 2.4.** For a prime \( p \) let \( e_i^{(p)}, \alpha_i^{(p)}, g_j^{(p)}, \beta_j^{(p)} \) be the polynomials obtained respectively from \( e_i, \alpha_i, \mu_j, \beta_j \) in Lemma 2.1 by eliminating \( y_s, s \notin G(p) \), using \( f_s \). Then

\[
H^*(G/T; \mathbb{F}_p) = \mathbb{F}_p[\omega_1, \ldots, \omega_n, y_t] / \langle e_i^{(p)}, \alpha_i^{(p)}, g_j^{(p)}, \beta_j^{(p)}, g_s^{(p)} \rangle_{1 \leq i \leq k, t \in G(p), s \in \overline{G}(\mathbb{F}_p)},
\]
where

i) \( g_t^{(p)} = y_t^{k_t} + \beta_t^{(p)} \); \( t \in G(p) \);

ii) \( \langle e_i^{(p)}, \alpha_i^{(p)}, \beta_j^{(p)}, g_s^{(p)} \rangle \subset \langle \omega_1, \ldots, \omega_n \rangle_{\mathbb{F}_p} \),

and where \( \langle \omega_1, \ldots, \omega_n \rangle_{\mathbb{F}_p} \) is the ideal in \( \mathbb{F}_p[\omega_1, \ldots, \omega_n, y_t] \) generated by \( \omega_1, \ldots, \omega_n \).

**Proof.** After mod \( p \) the relations \( f_t \) in Lemma 2.1 become
a) \( \alpha_i \equiv 0 \mod p \) for \( t \in G(p) \);
b) \( y_t - q_t \alpha_t \equiv 0 \mod p \) for \( t \notin G(p) \),

where \( q_t > 0 \) is the smallest integer satisfying \( q_t p_t \equiv -1 \mod p \). a) implies that the relations \( f_t \) with \( t \in G(p) \) should be replaced by \( \alpha_t \equiv 0 \). In view of b) we can eliminate all \( y_s \) with \( s \notin G(p) \) from the set of generators and replace it in the remaining relations by \( q_s \alpha_s \) to obtain the presentation

c) \( \HH^*(G/T; \mathbb{F}_p) = \mathbb{F}_p[\omega_1, \cdots, \omega_n, y_t] / \langle e_i^{(p)}, \alpha_t^{(p)}, g_j^{(p)} \rangle_{1 \leq i, k, t \in G(p), 1 \leq j \leq m} \).

For \( G \neq E_8 \) the result is verified by \( \{1, \cdots, m\} = G(p) \cup \overline{G}(\mathbb{F}_p) \). For \( G = E_8 \) reduction mod \( p \) of the system (2.2) yields the next relations

\[
g_4^{(p)} \equiv 0; \; g_6^{(p)} \equiv y_7 \alpha_7^{(p)} \quad \text{for} \; p = 2; \\
g_4^{(p)} \equiv g_7^{(p)} \equiv -y_6 \alpha_6^{(p)} \quad \text{for} \; p = 3; \\
g_6^{(p)} \equiv g_7^{(p)} \equiv -y_4 \alpha_4^{(p)} \quad \text{for} \; p = 5; \\
g_4^{(p)} \equiv s g_6^{(p)}; \; g_7^{(p)} \equiv t g_6^{(p)} \quad \text{if} \; p \neq 2, 3, 5 \quad \text{(for some} \; s, t \in \mathbb{F}_p)\).
\]

Combining these with c) establishes Lemma 2.4 for \( G = E_8 \). □

2.3. Notations The ring \( \HH^*(G; \mathbb{F}) \) may vary considerably with respect to \( G \) and \( \mathbb{F} \). The following notations allow us to carry out construction and calculation uniformly for all \( G \) and \( \mathbb{F} \).

(2.3) \( P_{G, \mathbb{F}} := \) the numerator (ring) in the presentation of \( \HH^*(G/T; \mathbb{F}) \) in Lemmas 2.1, 2.3 and 2.4 (in accordance with \( \mathbb{F} = \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{F}_p \)).

(2.4) \( P_{G, \mathbb{F}}^* := \) the subring of \( P_{G, \mathbb{F}} \) obtained by setting \( \omega_i = 0 \) in \( P_{G, \mathbb{F}} \).

(2.5) \( I_{G, \mathbb{F}} := \) the ideal in \( P_{G, \mathbb{F}} \) appearing as the denominator in the presentation of \( \HH^*(G/T; \mathbb{F}) \) in Lemmas 2.1, 2.3 and 2.4.

(2.6) \( \langle \omega_1, \cdots, \omega_n \rangle_{\mathbb{F}} := \) the ideal in \( P_{G, \mathbb{F}} \) generated by \( \omega_1, \cdots, \omega_n \).

In Lemmas 2.1, 2.3 and 2.4, the polynomials enclosed to specify \( I_{G, \mathbb{F}} \) will be called the defining polynomials of \( I_{G, \mathbb{F}} \). Precisely if we write \( \Sigma_{G, \mathbb{F}} \subset I_{G, \mathbb{F}} \) for the set of these polynomials, then

\[
\Sigma_{G, \mathbb{Z}} = \{ e_i, f_j, g_k \}_{1 \leq i, k, j \leq m}; \\
\Sigma_{G, \mathbb{Q}} = \{ e_i^{(0)}, g_j^{(0)} \}_{1 \leq i, j \in \mathbb{Q}}; \\
\Sigma_{G, \mathbb{F}_p} = \{ e_i^{(p)}, \alpha_t^{(p)}, g_j^{(p)} \}_{1 \leq i, k, j \in \mathbb{Z}(p), s \in \mathbb{F}_p};
\]

With the presence of \( \Sigma_{G, \mathbb{F}} \) we single out a subset of \( I_{G, \mathbb{F}} \) which will give rise to a minimal set of generators for the ring \( \HH^*(G; \mathbb{F}) \).

Definition 2.2. In accordance with \( \mathbb{F} = \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{F}_p \) the set \( \Phi_{G, \mathbb{F}} \) of primary polynomials in \( I_{G, \mathbb{F}} \) consists of
i) \( \Phi_{G,Z} = \{ e_i, h_j \}_{1 \leq i \leq j, j \in \mathbb{Z}(G)} \) where \( h_j = p_j g_j - y_j^{k_j-1} f_j \);

ii) \( \Phi_{G,Q} = \{ e_i^{(0)}, g_j^{(0)} \}_{1 \leq i \leq j, j \in \mathbb{R}(Q)} \);

iii) \( \Phi_{G,F_p} = \{ e_i^{(p)}, \alpha_i^{(p)}, g_j^{(p)}, \omega_j^{(p)} \}_{1 \leq i \leq j, t \in \mathbb{Z}(G(p), s \in \mathbb{F}(p))} \).

Useful properties of the set \( \Phi_{G,F} \) are collected in the next result.

**Lemma 2.5.** One has \( |\Phi_{G,F}| = n \) and

i) \( \Phi_{G,F} \subset \{ \omega_1, \cdots, \omega_n \}_F \cap I_{G,F} \);

ii) \( \dim G = \left\{ \sum_{u \in \Phi_{G,F}} (\deg u - 1) \right\} \) for \( F = \mathbb{Q} \) or \( \mathbb{Z} \);

iii) the mod \( p \) reduction from \( P_{G,Z} \) to \( P_{G,F_p} \) satisfies

\[
\begin{align*}
\text{and if } G &= E_{8}, \text{ from the alternative expression by } (2.2) \\
h_6 &= -30 f + 13 g_4^2 f_4 - 10 y_6^2 f_6 + 15 y_7 f_7. \square
\end{align*}
\]

**Proof.** i) is trivial for \( F = \mathbb{Q} \), and has been shown by i) of Lemma 2.4 for \( F = \mathbb{F}_p \). For the case \( F = \mathbb{Z} \) substituting in the formula of \( h_j \) the expressions of \( f_j \) and \( g_j \) in Lemma 2.1 yields that \( h_j = p_j \beta_j - y_j^{k_j-1} \alpha_j \). i) follows from \( \rho_i, \alpha_i, \beta_i \in \langle \omega_1, \cdots, \omega_n \rangle_Z \) and \( e_i, h_j \in I_{G,Z} \).

With the basic data for \( G \) given in Tables 1 and 2 and taking into account of Definition 2.1, ii) and iii) can be directly verified (when \( F = \mathbb{F}_p \) these may be done in accordance with \( p = 2, 3, 5 \) and \( p \neq 2, 3, 5 \)).

Finally the relations in iii) are clear from the proof of Lemma 2.3, and when \( G = E_8 \), from the alternative expression by \( (2.2) \)

\[
h_6 = -30 f + 13 g_4^2 f_4 - 10 y_6^2 f_6 + 15 y_7 f_7. \square
\]

2.4. **Construction in** \( E_8^{*-1}(G; F) \). The ideal \( \langle \omega_1, \cdots, \omega_n \rangle_F \subset P_{G,F} \) is a module over the ring \( P_{G,F}^* \) with basis \( \{ \omega_1^{b_1} \cdots \omega_n^{b_n} \mid b_i \geq 0, \sum b_i \geq 1 \} \). This simple fact gives rise to the well defined \( P_{G,F}^* \)-linear map

\[
(2.7) \varphi : \langle \omega_1, \cdots, \omega_n \rangle_F \to E_8^{*-1}(G; F) = H^*(G/T; F) \otimes \Lambda^1_F
\]

by \( \varphi(\omega_1^{b_1} \cdots \omega_n^{b_n}) = \omega_1^{b_1} \cdots \omega_k^{b_k-1} \cdots \omega_n^{b_n} \otimes t_k \), where \( k \in \{1, \cdots, n\} \) is the least one with \( b_k \geq 1 \). Since \( \Phi_{G,F} \subset \langle \omega_1, \cdots, \omega_n \rangle_F \) by i) of Lemma 2.5, \( \varphi \) acts on \( \Phi_{G,F} \). Let
\( \iota_F : \langle \omega_1, \cdots, \omega_n \rangle_F \rightarrow H^*(G/T; \mathbb{F}) \)

be the composition of the inclusion \( \langle \omega_1, \cdots, \omega_n \rangle_F \rightarrow P_{G,F} \) followed by the
obvious quotient map \( P_{G,F} \rightarrow H^*(G/T; \mathbb{F}) \). Then, from \( \iota_F = d_2 \circ \varphi \) and
\( E_2^{*,0}(G; \mathbb{F}) = H^*(G/T; \mathbb{F}) = P_{G,F}/I_{G,F} \) we find that

\( \varphi(\Phi_{G,F}) \subset \ker E_2^{*,1}(G; \mathbb{F}) \). \( d_2 \rightarrow E_2^{*,0}(G; \mathbb{F}) \).

**Definition 2.3.** For a \( d_2 \)-cocycle \( h \in E_2^{*,*}(G; \mathbb{F}) \) write \([h] \in E_3^{*,*}(G; \mathbb{F})\) for its
cohomology class. Elements in the subset
\( \mathcal{O}_{G,F} = \{ \langle \varphi(g) \rangle \in E_3^{deg,9-2,1}(G; \mathbb{F}) \mid g \in \Phi_{G,F} \} \)
are called the primary forms in \( E_3^{*,*}(G; \mathbb{F}) \).

For the notational convenience we adopt the abbreviations for all primary forms in accordance with \( F = \mathbb{Z}, \mathbb{Q} \) and \( F_p \):

(2.8) if \( F = \mathbb{Z} \) let \( \xi_i := [\varphi(e_i)], \eta_j := [\varphi(h_j)], 1 \leq i \leq k, j \in \mathcal{O}(\mathbb{Z}). \)

(2.9) if \( F = \mathbb{Q} \) set \( \xi_i^{(0)} := [\varphi(e_i^{(0)})], \eta_j^{(0)} := [\varphi(h_j^{(0)})], 1 \leq i \leq k, j \in \mathcal{O}(\mathbb{Q}). \)

(2.10) if \( F = F_p \) put \( \xi_i^{(p)} := [\varphi(e_i^{(p)})], \theta_t^{(p)} := [\varphi(\alpha_t^{(p)})], \eta_s^{(p)} := [\varphi(g_s^{(p)})], \)
where \( 1 \leq i \leq k, t \in G(p), s \in \mathcal{O}(F_p). \)

**Example 2.1.** For a given pair \((G,F)\) all elements in \( \mathcal{O}_{G,F} \), together with their
degrees, can be enumerated from the basic data of \( G \) in Tables 1 and 2. For an
exceptional Lie group \( G \) see Tables 4-8 in \S 6 for the set \( \mathcal{O}_{G,F} \), as well as the
degrees of its elements, so obtained. \( \square \)

It follows from Lemma 2.5 that

**Lemma 2.6.** We have \( |\mathcal{O}_{G,F}| = n \) and

\[
\text{dim } G = \begin{cases} 
\sum_{u \in \mathcal{O}_{G,F}} \text{deg } u & \text{for } F = \mathbb{Q} \text{ or } \mathbb{Z}; \\
\sum_{u \in \mathcal{O}_{G,F,p}} \text{deg } u + \sum_{t \in G(p)} (k_t - 1) \text{deg } y_t & \text{for } F = F_p. \end{cases} \]

**2.5. Preliminaries in algebra.** We conclude this section with two results in
algebra for further use.

**Definition 2.4.** Let \( \mathcal{R}^r = \oplus_{i \geq 0} \mathcal{R}^i \) be a graded algebra over a field \( \mathbb{F} \) (resp. a
graded ring over \( \mathbb{F} = \mathbb{Z} \)) and let \( u = t_1^{c_1} \cdots t_h^{c_h} \in \mathcal{R}^r \) be a decomposed element
in degree \( r, b_h \geq 1 \).

We call \( \mathcal{R}^r \) monotone in degree \( r \) with respect to \( u \) if \( \mathcal{R}^r = \mathbb{F} \) is generated by
\( u, \) and \( t_1^{c_1} \cdots t_h^{c_h} = 0 \) for all \((c_1, \cdots, c_h) \neq (b_1, \cdots, b_h)\) with \( \text{deg } t_1^{c_1} \cdots t_h^{c_h} = r. \)

**Lemma 2.7.** Let \( \mathcal{R}^r \) be a graded algebra (resp. ring) which is monotone
with respect to \( u = t_1^{c_1} \cdots t_h^{c_h} \in \mathcal{R}^r. \)

Then the set \( \{ t_1^{k_1} \cdots t_n^{k_n} \}_{0 \leq k_i \leq b_i} \) of monomials is linearly independent, and
spans a direct summand of \( \mathcal{R}^r \) (resp. of the free part of \( \mathcal{R}^r \)). \( \square \)

**Lemma 2.8.** Let \( A, B, C \) be three abelian groups, and let \( f : A \oplus B \rightarrow C \) be
an epimorphism. If \( a \in A, b \in B \) are such that
i) the element \( a \) spans a direct summand of \( A \);

ii) \( f(a) = f(b) \),

then \( f \) induces an epimorphism \( \hat{f} : A/\langle a \rangle \oplus B \to C \), where \( \langle a \rangle \subset A \) is the cyclic subgroup spanned by \( a \).

The following standard notations will be adopted in this paper. Given a ring \( A \) and a finite set \( S = \{ u_1, \ldots, u_t \} \) we write

\[
\begin{align*}
(2.11) \quad & A\{S\} = A\{u_i\}_{1 \leq i \leq t} \text{ for the free } A\text{-module with basis } \{ u_1, \ldots, u_t \}; \\
(2.12) \quad & A[S] = A[u_i]_{1 \leq i \leq t} \text{ for the ring of polynomials in } u_1, \ldots, u_t \text{ with coefficients in } A; \\
(2.13) \quad & \Lambda F(u_i)_{1 \leq i \leq t} \text{ for the exterior algebra over } F \text{ generated by } u_1, \ldots, u_t; \\
(2.14) \quad & A \otimes \Delta(S) = A \otimes \Delta(u_i)_{1 \leq i \leq t} \text{ for the } A\text{-module in the simple system of generators } u_1, \ldots, u_t \text{ [3].}
\end{align*}
\]

In addition, if \( A = F \), \( \Delta F(S) \) is used instead of \( F \otimes \Delta(S) \).

\section{Computing with \( E_3^{*,r}(G, F) \), \( r = 0, 1 \)}

From Lemma 2.1 we determine \( E_3^{*,0}(G; F) \) in Lemma 3.1. Using primary forms in \( E_3^{*,1}(G; F) \) introduced in §2.4 we deduce a partial presentation for \( E_3^{*,1}(G; F) \) (with \( F \) a field) in Lemmas 3.2. The relationship between \( E_3^{*,1}(G; F_p) \) and \( E_3^{*,1}(G; \mathbb{Z}) \) with respect to the mod \( p \) reduction and the Bockstein homomorphism is discussed in Lemma 3.3. These results will be summarized in §4 as to give a complete characterization for \( E_3^{*,*}(G; \mathbb{Z}) \).

### 3.1. The Chow rings of reductive algebraic groups.

In term of (1.1) define the subring \( A^*_{G,F} \) of \( H^*(G; F) \) by

\[
A^*_{G,F} := \text{Im}\{ \pi^* : H^*(G/T; F) \to H^*(G; F) \}.
\]

Grothendieck [22] showed that it is the Chow ring (with \( F \) coefficient) of the reductive algebraic group \( G^* \) corresponding to \( G \), and \( \pi^* \) induces an isomorphism

\[
A^*_{G,F} = H^*(G/T; F) / \omega_1 = \cdots = \omega_n = 0.
\]

On the other hand, according to (1.2) and (1.3), \( E_3^{*,0}(G; F) \) is the cokernel of the differential \( d_2 : H^*(G/T; F) \otimes A^1_F \to H^*(G/T; F) \), where \( d_2(a \otimes t_k) = a \omega_k \) implies that \( \text{Im} \ d_2 \) in the ideal in \( H^*(G/T; F) \) generated by \( \omega_1, \ldots, \omega_n \). Therefore, we get directly from Lemma 2.1 that

**Lemma 3.1.** \( E_3^{*,0}(G; F) = A^*_{G,F} = A^*_{G,Z} \otimes F \), where

\[
A^*_{G,Z} = \mathbb{Z}[y_1, \ldots, y_m] / \left\langle p_j y_i, y_j^k \right\rangle.
\]
Example 3.1. To facilitate with calculation in §6 concrete presentations of $A_{G*Z}$ for the exceptional $G$ are needed. These can be obtained by inputting in the formula in Lemma 3.1 the values of $p_j$, $k_j$ given in Table 2. To emphasize the cohomology degrees of the Schubert classes $y_j$ (see Table 3), $x_{deg y_j}$ is used instead of $y_j$.

By irggives rise to the short exact sequence of cochain complexes

$$
\Lambda = \cdots \to 0.
$$

Lemma 3.1 presents [33]. In [24] Kač obtained presentations for the algebras $A_{G*F}$ for all simple $G$ in which the generators are specified only up to their degrees. In comparison Lemma 3.1 presents $A_{G*Z}$ in terms of the special Schubert classes on $G/T$ whose geometric configuration can be read from their Weyl coordinates [20]. □

3.2. $E^{n,1}_3(G;F)$ with $F$ a field. The exact sequence of $F$-modules

$$0 \to I_{G,F} \to P_{G,F} \to H^*(G/T;F) \to 0 \ (see \ (2.3) \ and \ (2.5))$$

gives rise to the short exact sequence of cochain complexes

$$
\cdots \to 0 \to I_{G,F} \otimes \Lambda^* \to P_{G,F} \otimes \Lambda^* \to H^*(G/T;F) \otimes \Lambda^* \to 0,
$$

where $\Lambda^* = \Lambda(t_1, \cdots, t_n), d(a \otimes t_k) = a \omega_k \otimes 1$. Since (as is clear)

$$
H^0(P_{G,F} \otimes \Lambda^*) = P_{G,F}^0; \ H^r(P_{G,F} \otimes \Lambda^*) = 0 \ for \ r \geq 1;
$$

$$
H^r(H^*(G/T;F) \otimes \Lambda^*) = E^{r,r}_3(G;F),
$$

the cohomology exact sequence of (3.1) contains the section

$$
0 \to E^{n,1}_3(G;F) \xrightarrow{\delta} I_{G,F}/J_{G,F} \xrightarrow{\theta} P_{G,F}^0 \xleftarrow{\chi} A_{G,F} = E^{0,0}_3(G;F) \to 0,
$$

where $J_{G,F} = Im[d : I_{G,F} \otimes \Lambda^1 \to I_{G,F}]$. This implies that

$$
E^{n,1}_3(G;F) \ can \ be \ calculated \ as \ ker \ \theta.
$$

In addition, the map $\theta$ in (3.2) can be computed by the simple algorithm: for each $f \in I_{G,F}$ write $[f]$ for its residue class in $I_{G,F}/J_{G,F}$, then

$$
\theta[f] = f \mid_{\omega_1 = \cdots = \omega_n = 0}.
$$

The natural action $E^{n,0}_3(G;F) \otimes E^{r,r}_3(G;F) \to E^{n+r,0}_3(G;F)$ of $E^{0,0}_3(G;F)$ furnishes $E^{r,r}_3$ with the structure of an $A_{G,F}$ module. We apply (3.3) and (3.4) to show

Lemma 3.2. If $F$ is a field, the $A_{G,F}$ module $E^{n,1}_3(G;F)$ is spanned by the set $O_{G,F}$ of primary forms (Definition 2.3).

Proof. The map $\psi : P_{G,F}(\Sigma_{G,F}) \to I_{G,F}$ induced from the inclusion $\Sigma_{G,F} \subset I_{G,F}$ is clearly surjective. Since $\omega_1^{b_1} \cdots \omega_n^{b_n} \cdot \Sigma_{G,F} \subset J_{G,F}$ for any $(b_1, \cdots, b_n)$ with $\Sigma b_i \geq 1$, $\psi$ restricts to a surjective map

11
(3.5) \( \psi_1 : P_{G,F}^\ast \Sigma_{G,F} \rightarrow I_{G,F}/J_{G,F} \rightarrow 0 \)

by Lemma 2.8. On the other hand, as the injection \( \delta \) in (3.2) carries \( \varphi(h) \in \mathcal{O}_{G,F} \) to \([h] \in I_{G,F}/J_{G,F}, h \in \Phi_{G,F} \) (§2.4), Lemma 3.2 is established once we show that

(3.6) \( \psi_1 \) factors through the subquotient \( A_{G,F}^\ast \Sigma_{G,F} \) of \( P_{G,F}^\ast \Sigma_{G,F} \) as to yield a surjection \( \psi_2 : A_{G,F}^\ast \Sigma_{G,F} \rightarrow \ker \theta. \)

Case 1. \( F = \mathbb{Q} \). Since \( P_{G,F}^\ast \Sigma_{G,F} = A_{G,F}^\ast \Sigma_{G,F} = \mathbb{Q} \) by Lemma 3.1 and \( \Sigma_{G,F} = \Phi_{G,F} \) by Definition 2.2, \( \delta : E_{G,F}^{1,1}(G; \mathbb{Q}) \rightarrow I_{G,F}/J_{G,F} \) in (3.2) is an isomorphism. The map \( \psi_2 \) asserted in (3.6) is given by \( \psi_1 \).

Case 2. \( F = F_p \). With \( P_{G,F_p}^\ast \Sigma_{G,F_p} = \mathbb{F}_p[y_t \in G(p)] \) the surjection in (3.5) is

\[ \psi_1 : \mathbb{F}_p[y_t \in G(p)] \Sigma_{G,F_p} \rightarrow I_{G,F_p}/J_{G,F_p} \rightarrow 0. \]

The set \( G(p) \), regarded as a subsequence of \( \{1, \ldots, m\} \), will be denoted by \( G(p) = \{i_1, \ldots, i_r\}, r = |G(p)|. \) For a sequence \( c_1, \ldots, c_r \) of \( r \) non-negative integers and \( s \in G(p) \) set

\[ h_s^{c_1 \ldots c_r} = g_{i_1}^{c_1} \cdots g_{i_r}^{c_r} \in \mathbb{F}_p[y_t \in G(p)] \Sigma_{G,F_p}. \]

Then, with respect to the partition \( \Sigma_{G,F_p} = \Phi_{G,F_p} \cup \{g_t^{(p)} \in G(p)\} \) by iii) of Definition 2.2, we have the splitting of \( \mathbb{F}_p[y_t \in G(p)] \Sigma_{G,F_p} \)

(3.7) \( \mathbb{F}_p[y_t \in G(p)] \Sigma_{G,F_p} = \mathbb{F}_p[y_t \in G(p)] \Phi_{G,F_p} \oplus_{s \in G(p)} \mathbb{F}_p\{h_s^{c_1 \ldots c_r} \}_{c_i \geq 0}. \)

Since \( \Phi_{G,F_p} \) is \( (\omega_1, \ldots, \omega_n)_F \cap I_{G,F_p} \) by i) of Lemma 2.5, and since \( g_s^{(p)} = g_s^{k_s} + \beta_s^{(p)} \) with \( \beta_s^{(p)} \in (\omega_1, \ldots, \omega_n)_F \) for \( s \in G(p) \) by ii) of Lemma 2.4, we get from (3.2) that, for \( h \in \Sigma_{G,F_p}, g \in \Phi_{G,F_p} \) and \( s \in G(p) \)

a) \( h \cdot g \in J_{G,F_p} \); b) \( (y_s^{k_s} - \beta_s^{(p)}) \cdot h(= -\beta_s^{(p)} \cdot h) \in J_{G,F_p}. \)

In particular, for \( s, t \in G(p) \)

c) \( \psi_1(y_s^{k_s} \cdot g) \equiv [\tilde{\mu}_s \cdot g] \equiv 0 \mod J_{G,F_p} \) by b) and a);

d) \( \psi_1(y_s^{k_s} \cdot g_t^{(p)}) \equiv [g_s^{(p)} \cdot g_t^{(p)}] \equiv \psi_1(y_s^{k_s} \cdot g_t^{(p)}) \mod J_{G,F_p} \) by b).

It follows from c), d) and Lemma 2.8 that, with respect to the decomposition (3.7), \( \psi_1 \) induces an epimorphism

\[ \psi'_1 : A_{G,F_p}^\ast \Phi_{G,F_p} \oplus_{s \in G(p)} \mathbb{F}_p\{h_s^{c_1 \ldots c_r} \}_{c_i \leq k_i} \text{ for } s < i_j \rightarrow I_{G,F_p}/J_{G,F_p} \rightarrow 0. \]

Further, from (3.4) we find that

\[ \theta \circ \psi'_1(h) = 0, h \in \Phi_{G,F_p}; \quad \theta \circ \psi'_1(h_s^{c_1 \ldots c_r}) = y_{i_1}^{c_1} \cdots y_{i_r}^{c_r}. \]

These imply respectively that the composition

\[ \theta \circ \psi'_1 : A_{G,F_p}^\ast \Phi_{G,F_p} \oplus_{s \in G(p)} \mathbb{F}_p\{h_s^{c_1 \ldots c_r} \}_{c_i \leq k_i} \text{ for } s < i_j \rightarrow P_{G,F_p}^\ast \equiv \mathbb{F}_p[y_t \in G(p)] \]
is trivial on the first summand and is injective on the second. That is, \( \psi'_1 \) restricts to a surjection \( \psi_2 : \Lambda^G_{F_p, k} \rightarrow \ker \theta \). This shows (3.6), hence completes the proof of Lemma 3.2. □

3.3. Relationship between \( E_3^{r+1}(G; \mathbb{F}_p) \) and \( E_3^{r+1}(G; \mathbb{Z}) \). For a prime \( p \) consider the Bockstein sequence

\[
\cdots \rightarrow E_3^{r+1}(G; \mathbb{Z}) \xrightarrow{\times p} E_3^{r+1}(G; \mathbb{Z}) \xrightarrow{\beta_p} E_3^{r+1}(G; \mathbb{F}_p) \rightarrow \cdots
\]

associated to the exact sequence \( 0 \rightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \rightarrow \mathbb{F}_p \rightarrow 0 \) of coefficients, where \( \beta_p \) is the Bockstein homomorphisms and \( r_p \) is the mod \( p \) reduction.

Lemma 3.3. On \( E_3^{r+1}(G; \mathbb{Z}) \) \( (r = 0, 1) \) the reduction \( r_p \) is given by

(3.8) \( r_p(\xi_i) \equiv \xi_i^{(p)}; \)

\[
r_p(y_j) \equiv \begin{cases} y_j \text{ for } j \in G(p) \\ 0 \text{ for } j \notin G(p) \end{cases}; \quad r_p(\eta_j) \equiv \begin{cases} -y_j \theta_j^{(p)} \text{ for } j \in G(p) \\ p_j \eta_j^{(p)} \text{ for } j \in \overline{G}(\mathbb{F}_p) \end{cases},
\]

where \( 1 \leq i \leq k, j \in \overline{G}(\mathbb{Z}) \). In particular, when \( G = E_8 \),

(3.9) \( r_p(\eta_s) \equiv \begin{cases} y_s \theta_s^{(2)} \text{ if } p = 2; \\ -y_s \theta_s^{(3)} \text{ if } p = 3; \\ 2y_s \theta_s^{(5)} \text{ if } p = 5; \\ 3\eta_s^{(p)} \text{ if } p \neq 2, 3, 5. \end{cases} \)

On \( E_3^{r+1}(G; \mathbb{F}_p) \) the Bockstein \( \beta_p \) satisfies:

(3.10) \( \beta_p(\xi_i^{(p)}) = \beta_p(\eta_i^{(p)}) = 0; \beta_p(\theta_i^{(p)}) = -y_i, 1 \leq i \leq k, s \in \overline{G}(\mathbb{F}_p), t \in G(p). \)

Proof. Reduction mod \( p \) yields the commutative diagram

\[
0 \rightarrow E_3^{r+1}(G; \mathbb{Z}) \xrightarrow{\delta} I_{G, \mathbb{Z}}/J_{G, \mathbb{Z}} \xrightarrow{r_p} E_3^{r+1}(G; \mathbb{F}_p) \xrightarrow{\delta} I_{G, \mathbb{F}_p}/J_{G, \mathbb{F}_p} \rightarrow 0
\]

by which the relations in (3.8) and (3.9) are verified by iii) of Lemma 2.5. Turning to (3.10) we get from \( r_p(\xi_i) \equiv \xi_i^{(p)}; r_p(\eta_s) \equiv p_s \eta_s^{(p)} \) that

\[
\beta_p(\xi_i^{(p)}) = \beta_p(\eta_i^{(p)}) = 0.
\]

Finally, the relation \( \beta_p(\theta_t^{(p)}) = -y_t, t \in G(p) \), comes from the diagram chasing

\[
\varphi(\alpha_t) \rightarrow \theta_t^{(p)} \quad d \downarrow \quad d \downarrow
\]

\[-y_t \xrightarrow{p} \alpha_t \quad \alpha_t^{(p)} = 0
\]

in the short exact sequence of cochain complexes

\[
0 \rightarrow H^*(G/T; \mathbb{Z}) \otimes \Lambda^* \xrightarrow{\rho} H^*(G/T; \mathbb{Z}) \otimes \Lambda^* \xrightarrow{\tau_p} H^*(G/T; \mathbb{F}_p) \otimes \Lambda^* \rightarrow 0,
\]

where \( \varphi \) is the map in (2.7), and where \( \alpha_t \) and \( \alpha_t^{(p)} \) are respectively the polynomials specified in Lemmas 2.1 and 2.4. □
4 The structure of $E^*_{\infty} (G; F)$

In this section, we determine $E^*_{\infty} (G; F)$ and show that $E^*_{3} = E^*_{\infty}$ in Lemma 4.1 for $F$ a field, and in Lemma 4.5 for $F = \mathbb{Z}$, respectively. We begin by mentioning useful properties of $E^*_{\infty} (G; F)$ in (4.1)–(4.4):

(4.1) $E^*_{3} (G; F)$ is a module over the ring $A^*_G \otimes F$;

(4.2) the product in $E^*_{\infty} (G; F)$ satisfies $x^2 = 0$ for $x \in E^*_{\infty} (G; F)$;

(4.3) if $F$ is a field, $E^*_{\infty} (G; F)$ is generated by $E^*_3 (G; F)$ and $E^*_1 (G; F)$.

Finally, letting $n = \dim T$ and $g = \dim G/T$, then

$E^*_{n} (G; F) = E^*_{g,n} (G; F)$ (since $E^*_{g,n} = E^*_{g+2,n-1} = 0$).

4.1. The algebra $E^*_{\infty} (G; F)$ with $F$ a field. Let $F$ be a field and let $O_{G,F}$ be the set of primary forms in $E^*_{\infty} (G; F)$. Combining Lemmas 3.1 and 3.2 with (4.2) and (4.3), we find that

$E^*_{\infty} (G; F)$ is spanned by the single element $\psi = \psi_{O_{G,F}} (O_{G,F}) \in E^*_1 (G; F)$.

Lemma 4.1. The map $\psi_{O_{G,F}}$ is a ring isomorphism. Consequently,

i) $E^*_{\infty} (G; F) = E^*_3 (G; F)$;

ii) $\dim E^*_{\infty} (G; F) = \begin{cases} 2^n \text{ if either } F = \mathbb{Q} \text{ or } F_p \text{ with } G(p) = \emptyset; \\
2^n \prod_{t \in G(p)} k_t \text{ if } F = \mathbb{F}_p \text{ with } G(p) \neq \emptyset. \end{cases}$

Proof. Granted with (4.5) it suffices to show that $\psi_{O_{G,F}}$ is injective.

If $F = \mathbb{Q}$ then $A^*_G \otimes \mathbb{Q}$ by Lemma 3.1. In the top degree the algebra $\Lambda_{\mathbb{Q}} (O_{G,\mathbb{Q}})$ is spanned by the single element $u = \prod_{v \in O_{G,\mathbb{Q}}} v$. Since $\deg u = \dim G(\mathbb{Q}) = g + n$ by Lemma 2.6, $\psi_{\mathbb{Q}} (u) \in E^*_{3,n} (G; \mathbb{Q}) = \mathbb{Q}$ must be a generator by (4.5). The proof for $F = \mathbb{F}_p$ is done by

$2^n = \dim \Lambda_{\mathbb{Q}} \otimes \mathbb{F}_p \geq \dim E^*_3 (G; \mathbb{Q}) \geq 2^n$

in which the first inequality $\geq$ comes from (4.5), and the second is obtained by applying Lemma 2.7 to the class $\psi_{\mathbb{Q}} (u) = \prod_{v \in O_{G,\mathbb{Q}}} \psi_{\mathbb{Q}} (v)$, with respect to which the algebra $E^*_3 (G; \mathbb{Q})$ is monotone in bi-degree $(g, n)$ by (4.2).

The same argument applies equally well to the case $F = \mathbb{F}_p$. It follows from

$A^*_G \otimes \mathbb{F}_p \ni [y_t]_{t \in G(p)} / \begin{pmatrix} l^k \\ y_t^{l_t} \end{pmatrix}$

that, in the top degree, the algebra $A^*_G \otimes \mathbb{F}_p$ is spanned by the single element

$u_p = \prod_{1 \leq i \leq k} \xi^{(p)}_{\eta^k_i} \prod_{t \in G(p)} y_t^{l_t-1} \eta^{(p)} \prod_{v \in \overline{G}(\mathbb{F}_p)} \eta^{(p)}.$

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Since \( \deg u_p = \dim G(= g + n) \) by Lemma 2.6, \( \psi_{F_p}(u_p) \in E^3_n(G; F_p) = F_p \) must be a generator by (4.5). The proof is done by

\[
\dim A^*_{G,F_p} \otimes \Lambda_{F_p}(O_{G,F_p}) = 2^n \prod_{i \in G(p)} k_i \geq \dim E^*_3(G; F_p) \geq 2^n \prod_{i \in G(p)} k_i,
\]

where the second inequality \( \geq \) is obtained by applying Lemma 2.7 to the class \( \psi_{F_p}(u_p) \), with respect to it \( E^*_3(G; F_p) \) is monotone in bi-degree \( (g, n) \) by (4.2). \( \Box \)

For a prime \( p \) the differential of bi-degree \( (2, -1) \)

\[
\partial_p = r_p \circ \beta_p : E^*_3(G; F_p) \xrightarrow{\beta_p} E^*_3(G; Z) \xrightarrow{\tau_p} E^*_3(G; F_p)
\]

clearly satisfies that \( \partial_p^2 = 0 \). In view of the presentation

\[
E^*_3(G; F_p) = A^*_{G,F_p} \otimes \Lambda_{F_p}(O_{G,F_p})
\]

by Lemma 4.1, its action on \( E^*_3(G; F_p) \) has been determined by Lemma 3.3 as

(4.7) \( \partial_p(\theta_i^{(p)}) = -y_i; \partial_p(y_i) = \partial_p(\xi_i^{(p)}) = \partial_p(\eta_i^{(p)}) = 0. \)

In preparation for calculating the torsion ideal in \( E^*_3(G; Z) \) we show that

**Lemma 4.2.** dim\( \_p \) \( H^*(E^*_3(G; F_p); \partial_p) = 2^n. \)

**Proof.** Granted with Lemmas 3.1 and 4.1 we have the ring decomposition

\[
E^*_3(G; F_p) = \otimes_{i \in G(p)} C_i \otimes \Lambda_{F_p}(\xi_i^{(p)}, \eta_i^{(p)}) \] \[
\text{where } C_i = (F_p[y_i]/ \langle \theta_i^{(p)} \rangle) \otimes \Lambda_{F_p}(\theta_i^{(p)}). \text{ Since by (4.7)}
\]

i) each factor \( C_i \) is invariant with respect to \( \partial_p \); and

ii) \( \partial_p \) acts trivially on the factor \( \Lambda_{F_p}(\xi_i^{(p)}, \eta_i^{(p)}) \) \( \leq i \leq k, s \in G(F_p) \),

we get from the Künneth formula and the universal–coefficient theorem that

\[
H^*(E^*_3(G; F_p), \partial_p) = \otimes_{i \in G(p)} H^*(C_i, \partial_p) \otimes \Lambda_{F_p}(\xi_i^{(p)}, \eta_i^{(p)}) \] \[
\text{where } H^*(C_i, \partial_p) = 2^n - \#G(p).
\]

The proof is completed by

\[
\dim_{F_p} H^*(C_i, \partial_p) = 2
\]

since \( H^*(C_i, \partial_p) \) has a basis represented by \( 1, y_i^{k_i-1} \theta_i^{(p)} \), and by

\[
\dim_{F_p} \Lambda_{F_p}(\xi_i^{(p)}, \eta_i^{(p)}) = 2^n - \#G(p)
\]

since \( |G(p)| + |G(F_p)| + k = n \) by Lemma 2.6. \( \Box \)

**Remark 4.1.** From the foregoing it is clear that \( \otimes_{i \in G(p)} C_i \) is a subcomplex of \( \{ E^*_3; \partial_p \} \) whose cohomology is spanned by its subset \( \{ 1, \prod_{i \in I} y_i^{k_i-1} \theta_i^{(p)} \} \) \( i \leq G(p), \).

4.2. **The free part of** \( E^*_3(G; Z) \). In view of (4.2) the inclusion \( O_{G,Z} \subset E^*_3(G; Z) \) extends to a ring map
(4.8) \( \psi : \Lambda_{\mathbb{Z}(O_{G,Z})} \rightarrow E_3^{g,*}(G;\mathbb{Z}) \).

**Lemma 4.3.** The map \( \psi \) in (4.8) is injective and induces a splitting
\[
E_3^{g,*}(G;\mathbb{Z}) = \Lambda_{\mathbb{Z}(O_{G,Z})} \oplus T(G),
\]
where \( T(G) \) is the torsion ideal of \( E_3^{g,*}(G;\mathbb{Z}) \).

**Proof.** According to Lemma 3.3 the reduction \( r_p : E_3^{g,n}(G;\mathbb{Z}) \rightarrow E_3^{g,n}(G;\mathbb{F}_p) \) maps the class \( u = \prod_{v \in O_{G,Z}} v \) to
\[
r_p(u) \equiv \sum_{1 \leq i \leq k} s_{i(p)} \prod_{v \in (G(p))} y_{k_i}^{s_i} \prod_{s \in (\mathbb{F}_p)} p_s n_s^{(p)} \equiv (\sum_{s \in (\mathbb{F}_p)} p_s n_s^{(p)}) u_p,
\]
where, if \( G = E_8 \) the factor \( r_p(\eta_k) \) in \( r_p(u) \) is evaluated as in (3.9), and where
\[
i) a = \begin{cases} (-1)^{|G(p)|} & \text{if either } G \neq E_8 \text{ or } G = E_8, p \neq 2, 5; \\ (-1)^2 & \text{if } G = E_8, p = 2; \\ 2 & \text{if } G = E_8, p = 5. \end{cases}
\]

ii) \( u_p \) is the class given by (4.6).

Since \( u_p \) generates \( E_3^{g,n}(G;\mathbb{F}_p) = \mathbb{F}_p \) by the proof of Lemma 4.1 and since the coefficient \( (\prod_{p} p_s) \) is always co–prime to \( p \), \( r_p(u) \) generates \( E_3^{g,n}(G;\mathbb{F}_p) = \mathbb{F}_p \) for every prime \( p \). It follows now from (4.2) and (4.4) that the ring \( E_3^{g,*}(G;\mathbb{Z}) \) is monotone with respect to \( u \in E_3^{g,n}(G;\mathbb{Z}) = \mathbb{Z} \) and consequently, the set \( \{ \prod_{v \in O_{G,Z}} v^{x_v} \}_{x_v = 0,1} \) is linearly independent, and spans a direct summand of rank \( 2^n \) for the free part of \( E_3^{g,*}(G;\mathbb{Z}) \) by Lemma 2.7.

It remains to show that tensoring \( \mathbb{Q} \) yields an isomorphism
\[
\psi \otimes 1 : \Lambda_{\mathbb{Z}(O_{G,Z})} \otimes \mathbb{Q} \rightarrow E_3^{g,*}(G;\mathbb{Q}).
\]

But this comes directly from \( \dim_{\mathbb{Q}} E_3^{g,*}(G;\mathbb{Q}) = 2^n \) by Lemma 4.1, and the injectivity of \( \psi \otimes 1 \). \( \square \)

**4.3. The ring** \( E_3^{g,*}(G;\mathbb{Z}) \). **For a prime** \( p \) **the** \( p \)-**primary component of the torsion ideal** \( T(G) \) **is the subgroup**
\[
T_p(G) = \{ x \in T(G) \mid p^r \cdot x = 0 \text{ for some } r \geq 1 \}.
\]

Consider the Bockstein sequence
\[
\cdots \rightarrow E_3^{g,*}(G;\mathbb{F}_p) \xrightarrow{\beta_p} E_3^{g,*}(G;\mathbb{Z}) \xrightarrow{p} E_3^{g,*}(G;\mathbb{Z}) \xrightarrow{\beta_p} E_3^{g,*}(G;\mathbb{F}_p) \xrightarrow{p} \cdots.
\]

With the presentation of \( E_3^{g,*}(G;\mathbb{Z}) \) in Lemma 4.3 the universal coefficients theorem yields the exact sequence
\[
(4.9) \quad 0 \rightarrow \Lambda_{\mathbb{F}_p}(O_{G,Z}) \oplus T_p(G) \otimes \mathbb{F}_p \rightarrow E_3^{g,*}(G;\mathbb{F}_p) \rightarrow Tor(T_p(G);\mathbb{F}_p) \rightarrow 0
\]
in which
\[
a) \Lambda_{\mathbb{F}_p}(O_{G,Z}) \oplus T_p(G) \otimes \mathbb{F}_p = \text{Im } r_p \subseteq \ker \beta_p;
\]

b) \( \beta_p \) maps \( Tor(T_p(G);\mathbb{F}_p) \) isomorphically onto the subgroup
\[
t_p(G) = \{ x \in T_p(G) \mid px = 0 \}.
\]
Lemma 4.4. For a prime $p$ we have

i) $\text{Im } \beta_p = T_p(G) \cong \text{Im } \partial_p$ under $r_p$;

ii) $\dim_{F_p} T_p(G) = \begin{cases} 0 & \text{if } G(p) = \emptyset; \\ 2^{n-1} \left(\prod_{t \in G(p)} k_t - 1\right) & \text{if } G(p) \neq \emptyset. \end{cases}$

Proof. For i) it suffices to show that $t_p(G) = T_p(G)$. Assuming on the contrary that there exists $x \in T_p(G)$ with $p^r x = 0$ but $p^{r-1} x \neq 0$, $r \geq 2$, then $r_p(p^{r-1} x) = 0$ and $p^{r-1} x \in \text{Im } \beta_p$ imply that the restriction of $\partial_p = r_p \circ \beta_p$ on $\text{Tor}(T_p(G); F_p)$ has a nontrivial kernel. Since $\partial_p$ maps $\text{Tor}(T_p(G); F_p)$ into the summand $T_p(G) \otimes F_p$ in a) and since $\dim F_p \mathcal{O}_G(\mathbb{Z}) = 2^n$, we have $\dim_F H^*(E_3^{*,*}; (G; F)) > 2^n$. This contradiction to Lemma 4.2 shows that $t_p(G) = T_p(G)$, hence verifies i) of Lemma 4.4.

By i) $\partial_p = r_p \circ \beta_p$ must map $\text{Tor}(T_p(G); F_p)$ isomorphically onto the summand $T_p(G) \otimes F_p$ in a). With $\dim_{F_p} E_3^{*,*}(G; F_p)$ being given in Lemma 4.1 the equalities in ii) are obtained from $T_p(G) \otimes F_p \cong T_p(G)$ and

$$2 \dim_{F_p} T_p(G) \otimes F_p + 2^n = \dim_{F_p} E_3^{*,*}(G; F_p)$$

by (4.9). □

Since $G(p) = \emptyset$ for $p \neq 2, 3, 5$ by Lemma 2.1, Lemmas 4.3 and 4.4 yield the next result, which implies a conjecture by Kac (23).

Lemma 4.5. $E_3^{*,*}(G; \mathbb{Z}) = E_3^{*,*}(G; \mathbb{Z}) = \Lambda Z(\mathcal{O}_G, \mathbb{Z}) \oplus \{(2,3,5,5) \mid \text{Im } \beta_p. □

Remark 4.2. Lemma 4.5 is trivial for $G = SU(n), Sp(n)$, and has been shown for $G = Spin(n)$ by Pittie (36). □

5 Additive presentations of $H^*(G; \mathbb{F})$

We obtain additive presentations for $H^*(G; \mathbb{F})$ in Theorem 1 for $\mathbb{F}$ a field, and in Theorem 2 for $\mathbb{F} = \mathbb{Z}$, that are very close to our eventual characterization for $H^*(G; \mathbb{F})$ as a ring. We begin by singling out certain terms $E_\infty^{*,*}(G; \mathbb{F})$ that are naturally subgroups of $H^*(G; \mathbb{F})$. First of all, combining Lemma 3.1 with Lemmas 4.1 and 4.5 we have

(5.1) $E_\infty^{0*}(G; \mathbb{F}) = \mathcal{A}_{G,\mathbb{F}}$ is the subring $\text{Im } \pi^*$ of $H^*(G; \mathbb{F})$.

Next, let $\mathcal{F}$ be the filtration on $H^*(G; \mathbb{F})$ induced from $\pi$. For all $p + q = g + n$ with $g = \dim G/T$ and $n = \dim T$, we have by Lemmas 4.1, 4.5 and (4.4) that

$$E_\infty^{g,q}(G; \mathbb{F}) = E_3^{g,q}(G; \mathbb{F}) = \begin{cases} \mathbb{F} & \text{if } (p, q) = (g, n); \\ 0 & \text{otherwise}. \end{cases}$$

This implies that the filtration $\mathcal{F}$ on $H^{g+n}(G; \mathbb{F})$ reads

$$H^{g+n}(G; \mathbb{F}) = F^{g}H^{g+n} \supset F^{g+1}H^{g+n} = 0,$$

and therefore,

(5.2) $H^{g+n}(G; \mathbb{F}) = F^{g}H^{g+n}(G; \mathbb{F}) = E_\infty^{g,n}(G; \mathbb{F}) = \mathbb{F}$. 

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Further, as $E_2^{p,q}(G; \mathbb{F}) = 0$ for odd $p$, we have the canonical monomorphism

(5.3) $\kappa : E_\infty^{2k+1}(G; \mathbb{F}) \to \mathcal{F} \to H^{2k+1}(G; \mathbb{F}) \subset H^{2k+1}(G; \mathbb{F})$ (see Pittie [36])

which interprets directly elements (in particular, the primary forms) in $E_3^{\ast,1}(G; \mathbb{F})$ as cohomology classes of $G$.

**Definition 5.1.** Elements in the subset $O_{G, \mathbb{F}}^\ast = \{ (u) \in H^*(G; \mathbb{F}) \mid u \in O_{G, \mathbb{F}} \}$ will be called primary generators of $H^*(G; \mathbb{F})$.

The inclusion $\kappa$ in (5.3) has three useful properties that are explained in (5.4)–(5.6) below. Firstly, since the products in $\mathcal{F}$ is compatible with that in $H^*(G; \mathbb{F})$, one infers from (5.2) and (5.3) that

(5.4) for all $k_1, \ldots, k_n$ with $2(k_1 + \cdots + k_n) = g$, the diagram commutes

$$
\begin{array}{c}
\begin{array}{ccc}
E_\infty^{2k_1+1}(G; \mathbb{F}) \times \cdots \times E_\infty^{2k_n+1}(G; \mathbb{F}) & \longrightarrow & E_\infty^{g+n}(G; \mathbb{F}) \\
\kappa \times \cdots \times \kappa & \downarrow & \\
H^{2k_1+1}(G; \mathbb{F}) \times \cdots \times H^{2k_n+1}(G; \mathbb{F}) & \longrightarrow & H^{g+n}(G; \mathbb{F}) \\
\end{array}
\end{array}
$$

where the horizontal maps are the products in $E_\infty^\ast(G; \mathbb{F})$ and $H^*(G; \mathbb{F})$ respectively. Secondly, for $x \in E_\infty^{2k,1}$ we get from $x^2 = 0$ in

$$
E_\infty^{4k,2} = \mathcal{F} \to H^{4k+2}/\mathcal{F} \to H^{4k+2}
$$

that $\kappa(x)^2 \in \mathcal{F} \to H^{4k+2}$. It follows then from

$$
\mathcal{F} \to H^{4k+2}/\mathcal{F} \to H^{4k+2} = E_\infty^{4k+1,1} = 0 \text{ (since } E_\infty^{p,q} = 0 \text{ for odd } p) \n$$

that $\kappa(x)^2 \in \mathcal{F} \to H^{4k+2} = E_\infty^{4k+2,0}$. This implies by (5.1) that

(5.5) $\kappa(x)^2 \in A_{G, \mathbb{F}}^\ast \subset H^*(G; \mathbb{F})$ for all $x \in E_\infty^{2k,1}(G; \mathbb{F})$.

Finally, $\kappa$ is compatible with the Bockstein $\delta_p$ on $H^*(G; \mathbb{F}(p))$ in the sense that the following diagram is commutative (in which the vertical map on the right is given by the inclusion (5.1))

$$
\begin{array}{ccc}
E_\infty^{2k,1}(G; \mathbb{F}(p)) & \xrightarrow{\delta_p} & A_{G, \mathbb{Z}}^\ast = E_\ast^0(G; \mathbb{Z}) \\
\kappa & \downarrow & \\
H^{2k,1}(G; \mathbb{F}(p)) & \xrightarrow{\delta_p} & H^*(G; \mathbb{Z}).
\end{array}
$$

### 5.1. $H^*(G; \mathbb{F})$ with $\mathbb{F}$ a field.

**Theorem 1.** If $\mathbb{F}$ is a field, the inclusions $A_{G, \mathbb{F}}^\ast, O_{G, \mathbb{F}}^\ast \subset H^*(G; \mathbb{F})$ by (5.1) and (5.3) induces an isomorphism of $A_{G, \mathbb{F}}^\ast$ modules

(5.7) $H^*(G; \mathbb{F}) = A_{G, \mathbb{F}}^\ast \otimes \Delta_p(O_{G, \mathbb{F}}^\ast)$.

Consequently,

$$
\dim_{\mathbb{F}} H^*(G; \mathbb{F}) = \left\{ \begin{array}{ll}
2^n & \text{if either } \mathbb{F} = \mathbb{Q} \text{ or } \mathbb{F}_p \text{ with } G(p) = \emptyset, \\
2^n \prod_{i \in G(p)} k_i & \text{if } \mathbb{F} = \mathbb{F}_p \text{ with } G(p) \neq \emptyset.
\end{array} \right.
$$
Proof. If \( F = \mathbb{Q} \), then \( A^*_{G; F} = \mathbb{Q} \) and \( H^\text{dim}G(G; \mathbb{Q}) = \mathbb{Q} \) is spanned by \( u = \prod_{v \in O_{G; \mathbb{Q}}} \kappa(v) \) by (5.4). Since \( \kappa(v)^2 = 0 \) for \( v \in O_{G; \mathbb{Q}} \), the graded algebra \( H^*(G; \mathbb{Q}) \) is monotone with respect to \( u \) in degree \( \text{dim}G \). By Lemma 2.7 the subset

\[
\{ \prod_{v \in O_{G; \mathbb{Q}}} \kappa(v)^{e_v} \}_{e_v = 0, 1} \subset H^*(G; \mathbb{Q})
\]

of cardinality \( 2^n \) is linearly independent. The proof for \( F = \mathbb{Q} \) is completed by

\[
\text{dim} H^*(G; \mathbb{Q}) = \text{dim} E^*_{\infty}(G; \mathbb{Q}) = 2^n,
\]

where the last equality comes from Lemma 4.1.

Consider next the case of \( F = \mathbb{F}_p \). By (4.6) and (5.4) \( H^{\theta+u}(G; \mathbb{F}_p) = \mathbb{F}_p \) is spanned by

\[
\kappa(u_p) = \prod_{1 \leq i \leq k} \kappa(\xi_i^{(p)}) \prod_{t \in G(p)} y_t^{k_t-1} \kappa(\theta_t^{(p)}) \prod_{s \in \mathcal{G}(\mathbb{F}_p)} \kappa(\eta_s^{(p)}).
\]

It should be noted that the ring \( H^*(G; \mathbb{F}_p) \) in general is not monotone with respect to \( \kappa(u_p) \) when \( p = 2 \). However, we can establish the next assertion without using Lemma 2.7:

(5.8) the set of monomials

\[
\{ \prod_{1 \leq i \leq k} \kappa(\xi_i^{(p)})^{e_i} \prod_{t \in G(p)} y_t^{r_t} \kappa(\theta_t^{(p)})^{e_t} \prod_{s \in \mathcal{G}(\mathbb{F}_p)} \kappa(\eta_s^{(p)})^{e_s} \}_{e_i = 0, 1; 0 \leq r_t \leq k_t - 1}
\]

is linearly independent in \( H^*(G; \mathbb{F}_p) \),

from which (5.7) comes from \( \text{dim} H^*(G; \mathbb{F}_p) = \text{dim} E^*_{\infty}(G; \mathbb{F}_p) \).

Denote by \( B \) the set in (5.8), and let \( \mathcal{V} \) be the graded subspace of \( H^*(G; \mathbb{F}_p) \) spanned by \( B \). Consider the involution \( \tau \) on \( B \) defined by

\[
\tau(\prod \kappa(\xi_i^{(p)})^{e_i} \prod y_t^{r_t} \kappa(\theta_t^{(p)})^{e_t} \prod \kappa(\eta_s^{(p)})^{e_s}) = \prod \kappa(\xi_i^{(p)})^{e'_i} \prod y_t^{r'_t} \kappa(\theta_t^{(p)})^{e'_t} \prod \kappa(\eta_s^{(p)})^{e'_s},
\]

where \( e'_i = 0 \) or 1 in accordance with \( e_i = 1 \) or 0, and where \( r'_t = k_t - 1 - r_t \). It follows from (5.5) that, for any pair \( (x, y) \in B \times B \) with \( \deg x + \deg y = g + n \) (= \( \text{dim} G \)), their product in \( H^*(G; \mathbb{F}_p) \) satisfies

\[
xy = \begin{cases} 
\pm \kappa(u_p) & \text{if } y = \tau(x); \\
0 & \text{if } y \neq \tau(x).
\end{cases}
\]

This implies that \( \text{dim} \mathcal{V} = |B| \), hence verifies (5.8). \( \Box \)

Remark 5.1. In [24] Theorem 6] Kač obtained the presentation

\[
H^*(G; \mathbb{F}_p) = A^*_{G; \mathbb{F}_p} \otimes \Delta_{\bar{\mathbb{F}}_p}(\xi_{2b_1-1}, \cdots, \xi_{2b_n-1}), \deg \xi_r = r,
\]

in which the set \( \{2b_1, \cdots, 2b_n\} \) of integers agree with the set of degrees of a regular sequence of homogeneous generators for the ideal of generalized \( W \)-invariants. It should be noticed that
i) for a given \( G \) and \( p \) the number \( \tau(G; p) \) of different choices of a set \( \xi_{2n-1}, \cdots, \xi_{2k-1} \) of generators subject the same degree constraints can be very large, as shown by the formula
\[
\tau(G; p) = \prod_{1 \leq i \leq k} [p - 2 + p^{\dim H^* (G; \mathbb{F}_p)} - 1]
\]
and that

ii) the presentation of \( H^* (G; \mathbb{F}_2) \) as a ring as well as the action of the Steenrod algebra \( A_p \) on \( H^* (G; \mathbb{F}_p) \) vary considerably with respect to different choices of a set of such generators.

In comparison, with respect to the fixed set \( \mathcal{O}_{G; \mathbb{F}} \) of generators in (5.7) stemming from Lemma 2.1, the ring \( H^* (G; \mathbb{F}_2) \) and the action of \( A_p \) on \( H^* (G; \mathbb{F}_p) \) can be effectively analyzed, see Theorem 5 in §6 and [21].

In view of the presentation (5.7) for \( H^* (G; \mathbb{F}_p) \) we examine the differential of degree 1 on \( H^* (G; \mathbb{F}_p) \)
\[
\delta_p = r_p \circ \beta_p : H^* (G; \mathbb{F}_p) \to H^* (G; \mathbb{Z}) \to H^* (G; \mathbb{F}_p).
\]
Since \( H^* (G; \mathbb{F}_p) \) has a basis consisting of certain monomials in \( \kappa (\theta_1^{(p)}), y_1, \kappa (\xi_i^{(p)}) \) and \( \kappa (\eta_s^{(p)}) \) by Theorem 1, the behavior of \( \delta_p \) is determined by the equations
\[
(5.9) \quad \delta_p (\kappa (\theta_t^{(p)})) = -y_t, \quad \delta_p (y_t) = \delta_1 \kappa (\xi_i^{(p)}) = \delta_p \kappa (\eta_s^{(p)}) = 0, \text{ see (4.7) and (5.6)}.
\]
In particular, invoking to the presentation (5.7) one has
\[
(5.10) \quad H^* (G; \mathbb{F}_p) = \otimes_{t \in G(p)} ((\mathbb{F}_p[y_t]/(y_t^{k_t})) \otimes \Delta (\kappa (\theta_t^{(p)})) \otimes \Delta (\kappa (\xi_i^{(p)}), \kappa (\eta_s^{(p)})).
\]
The same argument as that in the proof of Lemma 4.2 shows that
\[
(5.11) \quad \dim_{\mathbb{F}_2} H^* (G; \mathbb{F}_p) \delta_p = 2^n.
\]

5.2. An additive presentation of \( H^* (G; \mathbb{Z}) \). Let \( \tau(G) \) be the torsion ideal of \( H^* (G; \mathbb{Z}) \), and let \( \tau_p (G) \) be the \( p \)-primary component of \( \tau(G) \). A subset \( I \subset G(p) \) defines in \( H^* (G; \mathbb{F}_p) \) the elements:

\[
(5.12) \quad \theta^{(p)}_t = \prod_{t \in I} \kappa (\theta_t^{(p)}), \quad \xi^{(p)}_t = \delta_p (\theta_t^{(p)}),
\]
\[
D^{(p)} = \prod_{t \in I} y_t^{-1} C^{(p)}_t \quad R^{(p)} = \sum_{t \in I} y_t C^{(p)}_t,
\]
where \( I_t \) is obtained by deleting \( t \in I \) from \( I \). We note that if \( I = \{ t \} \) is a singleton, then \( \theta_t^{(p)} = \kappa (\theta_t^{(p)}), C^{(p)} = y_t \) by (5.9).

If \( V^* = V^0 \oplus V^1 \oplus V^2 \oplus \cdots \) is a graded vector space (resp. a graded ring), define its subspace (resp. subring) \( V^+ \) by \( V^+ = V^1 \oplus V^2 \oplus \cdots \).

Theorem 2. The inclusion \( \mathcal{O}_{G; \mathbb{Z}} \subset H^* (G; \mathbb{Z}) \) by (5.3) induces a splitting
\[
(5.13) \quad H^* (G; \mathbb{Z}) = \Delta_2 (\mathcal{O}_{G; \mathbb{Z}}) \oplus \tau_p (G),
\]
on which the reduction \( r_p, p \in \{ 2, 3, 5 \} \), restricts to an additive isomorphism
\[(5.14) \quad \tau_p(G) \cong \frac{\mathbb{F}_p[u_t](1, \mathcal{C}_t^{(p)\ast})^+}{\langle y_t^{k_t}, D_t^{(p)}(r_k^{(p)}) \rangle} \otimes \Delta_{\mathbb{F}_p}(\kappa(\xi_i^{(p)}), \kappa(\eta_i^{(p)}))_{1 \leq i \leq k, s \in \mathbb{G}(\mathbb{F}_p)}, \quad t \in G(p),\]

where

i) \( I, J, K \subseteq G(p) \) with \(|I|, |J| \geq 2, |K| \geq 3 \), and

ii) \( \langle y_t^{k_t}, D_t^{(p)}(r_k^{(p)}) \rangle \) is the \( \mathbb{F}_p[u_t]_{G(p)} \)-module spanned by \( y_t^{k_t}, D_t^{(p)}(r_k^{(p)}) \).

**Proof.** The identification (5.2) carries the generator \( \prod_{v \in \mathcal{O}_{G,Z}} v \) of \( E_{\mathbb{Z}}^n(G; \mathbb{Z}) = \mathbb{Z} \) to the generator \( u = \prod_{v \in \mathcal{O}_{G,Z}} \kappa(v) \) of \( H^{g+n}(G; \mathbb{Z}) = \mathbb{Z} \) by (5.4). Since \( \kappa(v)^2 \in \tau(G) \) for \( v \in \mathcal{O}_{G,Z} \) by (5.5), the ring \( H^*(G; \mathbb{Z}) \) is monotone with respect to \( u \) in degree \( \dim G = g + n \). By Lemma 2.7 the set of monomials

\[ \{ \prod_{v \in \mathcal{O}_{G,Z}} \kappa(v)^{e_v} \}_{e_v=0,1} \]

is linearly independent, and spans a direct summand of rank \( 2^n \) for the free part of \( H^*(G; \mathbb{Z}) \). From \( \dim_\mathbb{Q} H^*(G; \mathbb{Q}) = 2^n \) by Theorem 1 we get

\[(5.15) \quad H^*(G; \mathbb{Z}) = \Delta_{\mathbb{Z}}(\mathcal{O}_{G,Z}) \oplus \tau(G).\]

Granted with Theorem 1, (5.11) and (5.15), the same argument as that in the proof of Lemma 4.4 shows that for any prime \( p \):

\[(5.16) \quad \text{Im} \beta_p = \tau_p(G) \cong \text{Im} \delta_p \text{ under } r_p,\]

\[(5.17) \quad \dim_{\mathbb{F}_p} \tau_p(G) = \begin{cases} 
0 & \text{if } G(p) = \emptyset; \\
2^{n-1}\prod_{v \in \mathcal{O}_{G,Z}} k_v - 1 & \text{if } G(p) \neq \emptyset.
\end{cases}\]

In particular, we get (5.13) from (5.15), (5.17) and that \( G(p) = \emptyset \) for \( p \neq 2, 3, 5 \) by Lemma 2.1.

In view of (5.16) it remains for us to establish the presentation (5.14) for \( \text{Im} \delta_p \). Consider the decomposition obtained from Theorem 1

\[ H^*(G; \mathbb{F}_p) = \oplus_{0 \leq r \leq |G(p)|} L_r \oplus \Delta_{\mathbb{F}_p}(\kappa(\xi_i^{(p)}), \kappa(\eta_i^{(p)}))_{1 \leq i \leq k, s \in \mathbb{G}(\mathbb{F}_p)}, \]

where \( L_0 = A^*_G; \mathbb{F}_p; L_r = A^*_G; \mathbb{F}_p; \{ \theta_r^{(p)} \}_{1 \leq g \leq |G(p)|, |I| = r} \). Since the \( \delta_p \) action satisfies

\[ \delta_p(L_r \otimes 1) \subseteq L_{r-1} \otimes 1, \quad \delta_p(1 \otimes \Delta_{\mathbb{F}_p}(\kappa(\xi_i^{(p)}), \kappa(\eta_i^{(p)}))) = 0 \]

by (5.9), if we let \( \delta_{p,r} \) be the restriction of \( \delta_p \) on \( L_r = L_r \otimes 1 \), then

\[(5.18) \quad \text{Im} \delta_p = \oplus_{1 \leq r \leq |G(p)|} \text{Im} \delta_{p,r} \oplus \Delta_{\mathbb{F}_p}(\kappa(\xi_i^{(p)}), \kappa(\eta_i^{(p)}))_{1 \leq i \leq k, s \in \mathbb{G}(\mathbb{F}_p)}.\]

If \( r = 1 \) we have \( \text{Im} \delta_{p,1} = A^*_G; \mathbb{F}_p \). Assuming next that \( r \geq 2 \), the isomorphism

\[ f : L_r \to A^*_G; \mathbb{F}_p; \{ \mathcal{C}_r^{(p)} \}_{1 \leq g \leq |G(p)|, |I| = r} \]

of \( A^*_G; \mathbb{F}_p \)-modules defined by \( f(\theta_r^{(p)}) = \mathcal{C}_r^{(p)} \) clearly fits in the commutative diagram
The isomorphism (5.14) has now been established by (5.18) and (5.20).

Remark 5.2. i) From (5.13) and (5.16) we have

$$H^*(G;\mathbb{Z}) = \Delta_2(O_{G,\mathbb{Z}}) \oplus \bigoplus_{p \in \{2,3,5\}} \text{Im} \beta_p.$$ 

Comparing this with Lemma 4.5 and taking into account of (5.7), we obtain an additive isomorphism $E_{\text{add}}^*(G;\mathbb{Z}) = H^*(G;\mathbb{Z})$ for all simple $G$. This was conjectured by Marlin [32] who has also checked this up to $n = 4$.

ii) The additive presentation of $\tau_p(G)$ in (5.14) not only implies the classical result that an $1$–connected compact Lie group has no $p^2$–torsion in its integral cohomology, see Lin [29, 30] or Kane [25], but is also very close to our eventual characterization of $\tau_p(G)$ as a ring, see computations in Lemmas 6.1 and 6.2. □

6 The ring $H^*(G,\mathbb{F})$ for exceptional $G$

Assume in this section that $G$ is exceptional. Based on Theorems 1 and 2 we recover the classical results about algebra $H^*(G;\mathbb{F})$ with $\mathbb{F}$ a field in §6.1–6.3, and determine the ring $H^*(G;\mathbb{Z})$ in §6.4–6.5. To emphasize the degrees of the primary generators the notation $\zeta_{\deg u} \in H^*(G;\mathbb{F})$, for instance, is used in the place of $\kappa(u)$, $u \in O_{G,\mathbb{F}}$ (see (5.3)). In addition $x_{\deg y_j}$ is used instead of $y_j \in A_{G,\mathbb{F}}^*$ (as in Example 3.1).

Historically the cohomologies $H^*(G;\mathbb{F}_p)$ were calculated case by case, presented using generators with different origins and characterized mainly by their degrees. As a result one could hardly analyzing $H^*(G;\mathbb{Z})$ from the existing information on $H^*(G;\mathbb{F}_p)$. In comparison, with our primary generators in various coefficients stemming solely from the system $\{e_i, f_j, g_j\}$ in the Schubert classes on $G/T$, the relationships between $H^*(G;\mathbb{Z})$ and $H^*(G;\mathbb{F}_p)$ are transparent for all prime $p$, compare iii) of Lemma 2.5 with Lemma 3.3. It is for this reason
that, starting from the presentation in Theorem 2 we can proceed to determine the structure of \( H^*(G; \mathbb{Z}) \) as a ring.

6.1. The ring \( H^*(G; \mathbb{Q}) \). For \( u \in O_{G, \mathbb{Q}} \) write \( \vartheta_{\deg u} \in H^*(G; \mathbb{Q}) \) instead of \( \kappa(u) \). The elements in \( O_{G, \mathbb{Q}} \), together with their \( \kappa \)-images, are listed in Table 4 below (see Example 2.1).

| \( u \in O_{G, \mathbb{Q}} \) | \( \xi_1(0) \) | \( \eta_1(0) \) | \( \xi_2(0) \) | \( \eta_2(0) \) | \( \xi_3(0) \) | \( \eta_3(0) \) | \( \xi_4(0) \) | \( \eta_4(0) \) |
| \( u \in O_{F_1, \mathbb{Q}} \) | \( \xi_1(0) \) | \( \eta_1(0) \) | \( \xi_2(0) \) | \( \eta_2(0) \) | \( \xi_3(0) \) | \( \eta_3(0) \) | \( \xi_4(0) \) | \( \eta_4(0) \) |
| \( u \in O_{E_6, \mathbb{Q}} \) | \( \xi_1(0) \) | \( \eta_1(0) \) | \( \xi_2(0) \) | \( \eta_2(0) \) | \( \xi_3(0) \) | \( \eta_3(0) \) | \( \xi_4(0) \) | \( \eta_4(0) \) |
| \( u \in O_{E_7, \mathbb{Q}} \) | \( \xi_1(0) \) | \( \eta_1(0) \) | \( \xi_2(0) \) | \( \eta_2(0) \) | \( \xi_3(0) \) | \( \eta_3(0) \) | \( \xi_4(0) \) | \( \eta_4(0) \) |
| \( \kappa(u) \in O^*_{E_6, \mathbb{Q}} \) | \( \vartheta_1 \) | \( \vartheta_9 \) | \( \vartheta_{11} \) | \( \vartheta_{15} \) | \( \vartheta_{17} \) | \( \vartheta_{19} \) | \( \vartheta_{23} \) | \( \vartheta_{27} \) | \( \vartheta_{35} \) | \( \vartheta_{39} \) | \( \vartheta_{47} \) |

Table 4. The elements in \( O_{G, \mathbb{Q}} \) and their \( \kappa \)-images

Since \( u^2 = 0 \) for \( u \in H^{odd}(G; \mathbb{Q}) \), the factor \( \Delta_Q(O_{G, \mathbb{Q}}) \) in (5.7) can be replaced by \( \Lambda_Q(O_{G, \mathbb{Q}}) \). Therefore, Theorem 1, together with contents in Table 4, yields the next result that implies the classical computation of Yen [40]:

Theorem 3. The inclusion \( O^*_{G, \mathbb{Q}} \subseteq H^*(G; \mathbb{Q}) \) induces the ring isomorphisms

\[
\begin{align*}
H^*(G_2; \mathbb{Q}) &= \Lambda_Q(\vartheta_3, \vartheta_{11}); \\
H^*(F_4; \mathbb{Q}) &= \Lambda_Q(\vartheta_3, \vartheta_{11}, \vartheta_{15}, \vartheta_{23}); \\
H^*(E_6; \mathbb{Q}) &= \Lambda_Q(\vartheta_3, \vartheta_9, \vartheta_{11}, \vartheta_{15}, \vartheta_{17}, \vartheta_{23}); \\
H^*(E_7; \mathbb{Q}) &= \Lambda_Q(\vartheta_3, \vartheta_{11}, \vartheta_{15}, \vartheta_{19}, \vartheta_{21}, \vartheta_{27}, \vartheta_{35}); \\
H^*(E_8; \mathbb{Q}) &= \Lambda_Q(\vartheta_3, \vartheta_9, \vartheta_{15}, \vartheta_{23}, \vartheta_{27}, \vartheta_{35}, \vartheta_{39}, \vartheta_{47}, \vartheta_{59}).
\end{align*}
\]

6.2. The ring \( H^*(G; \mathbb{F}_p) \) for \( p = 3, 5 \). Write \( \vartheta_{\deg u} \in H^*(G; \mathbb{F}_p) \) instead of \( \kappa(u) \), \( u \in O_{G, \mathbb{F}_p} \). The elements in \( O_{G, \mathbb{F}_p} \), together with their \( \kappa \)-images, are given in the tables below (see Example 2.1).

| \( u \in O_{G, \mathbb{F}_3} \) | \( \xi_1(0) \) | \( \eta_1(0) \) | \( \xi_2(0) \) | \( \eta_2(0) \) | \( \xi_3(0) \) | \( \eta_3(0) \) | \( \xi_4(0) \) | \( \eta_4(0) \) |
| \( u \in O_{F_4, \mathbb{F}_3} \) | \( \xi_1(0) \) | \( \eta_1(0) \) | \( \xi_2(0) \) | \( \eta_2(0) \) | \( \xi_3(0) \) | \( \eta_3(0) \) | \( \xi_4(0) \) | \( \eta_4(0) \) |
| \( u \in O_{E_6, \mathbb{F}_3} \) | \( \xi_1(0) \) | \( \eta_1(0) \) | \( \xi_2(0) \) | \( \eta_2(0) \) | \( \xi_3(0) \) | \( \eta_3(0) \) | \( \xi_4(0) \) | \( \eta_4(0) \) |
| \( u \in O_{E_7, \mathbb{F}_3} \) | \( \xi_1(0) \) | \( \eta_1(0) \) | \( \xi_2(0) \) | \( \eta_2(0) \) | \( \xi_3(0) \) | \( \eta_3(0) \) | \( \xi_4(0) \) | \( \eta_4(0) \) |
| \( \kappa(u) \in O^*_{E_6, \mathbb{F}_3} \) | \( \vartheta_1 \) | \( \vartheta_9 \) | \( \vartheta_{11} \) | \( \vartheta_{15} \) | \( \vartheta_{17} \) | \( \vartheta_{19} \) | \( \vartheta_{23} \) | \( \vartheta_{27} \) | \( \vartheta_{35} \) | \( \vartheta_{39} \) | \( \vartheta_{47} \) |

Table 5. The elements in \( O_{G, \mathbb{F}_3} \) and their \( \kappa \)-images

| \( u \in O_{G, \mathbb{F}_5} \) | \( \xi_1(0) \) | \( \eta_1(0) \) | \( \xi_2(0) \) | \( \eta_2(0) \) | \( \xi_3(0) \) | \( \eta_3(0) \) | \( \xi_4(0) \) | \( \eta_4(0) \) |
| \( u \in O_{F_4, \mathbb{F}_5} \) | \( \xi_1(0) \) | \( \eta_1(0) \) | \( \xi_2(0) \) | \( \eta_2(0) \) | \( \xi_3(0) \) | \( \eta_3(0) \) | \( \xi_4(0) \) | \( \eta_4(0) \) |
| \( u \in O_{E_6, \mathbb{F}_5} \) | \( \xi_1(0) \) | \( \eta_1(0) \) | \( \xi_2(0) \) | \( \eta_2(0) \) | \( \xi_3(0) \) | \( \eta_3(0) \) | \( \xi_4(0) \) | \( \eta_4(0) \) |
| \( u \in O_{E_7, \mathbb{F}_5} \) | \( \xi_1(0) \) | \( \eta_1(0) \) | \( \xi_2(0) \) | \( \eta_2(0) \) | \( \xi_3(0) \) | \( \eta_3(0) \) | \( \xi_4(0) \) | \( \eta_4(0) \) |
| \( \kappa(u) \in O^*_{E_6, \mathbb{F}_5} \) | \( \vartheta_1 \) | \( \vartheta_9 \) | \( \vartheta_{11} \) | \( \vartheta_{15} \) | \( \vartheta_{17} \) | \( \vartheta_{19} \) | \( \vartheta_{23} \) | \( \vartheta_{27} \) | \( \vartheta_{35} \) | \( \vartheta_{39} \) | \( \vartheta_{47} \) |

Table 6: The elements in \( O_{G, \mathbb{F}_5} \) and their \( \kappa \)-images
Theorem 4. The inclusions $A_{G,F_2}^*, O_{G,F_2}^* \subset H^*(G; F_2)$ induces the ring isomorphisms

$$H^*(G_2; F_2) = \Lambda_{F_2}(\zeta_3, \zeta_11);$$

$$H^*(F_4; F_2) = F_2[x_8]/(x_8^3) \otimes \Lambda_{F_2}(\zeta_3, \zeta_7, \zeta_11, \zeta_15);$$

$$H^*(E_6; F_2) = F_2[x_8]/(x_8^3) \otimes \Lambda_{F_2}(\zeta_3, \zeta_7, \zeta_9, \zeta_{11}, \zeta_{15}, \zeta_{17});$$

$$H^*(E_7; F_2) = F_2[x_8]/(x_8^3) \otimes \Lambda_{F_2}(\zeta_3, \zeta_7, \zeta_{11}, \zeta_{15}, \zeta_{19}, \zeta_{27}, \zeta_{35});$$

$$H^*(E_8; F_2) = F_2[x_8, x_20]/(x_8^3, x_{20}^3) \otimes \Lambda_{F_2}(\zeta_3, \zeta_7, \zeta_{11}, \zeta_{15}, \zeta_{19}, \zeta_{27}, \zeta_{35}, \zeta_{39}, \zeta_{47});$$

$$H^*(G_2; F_3) = \Lambda_{F_3}(\zeta_3, \zeta_{11});$$

$$H^*(F_4; F_3) = \Lambda_{F_3}(\zeta_3, \zeta_{11}, \zeta_{15}, \zeta_{23});$$

$$H^*(E_6; F_3) = \Lambda_{F_3}(\zeta_3, \zeta_9, \zeta_{11}, \zeta_{15}, \zeta_{17}, \zeta_{23});$$

$$H^*(E_7; F_3) = \Lambda_{F_3}(\zeta_3, \zeta_{11}, \zeta_{15}, \zeta_{19}, \zeta_{23}, \zeta_{27}, \zeta_{35});$$

$$H^*(E_8; F_3) = F_3[x_{12}]/(x_{12}^3) \otimes \Lambda(\zeta_3, \zeta_{11}, \zeta_{15}, \zeta_{23}, \zeta_{27}, \zeta_{35}, \zeta_{39}, \zeta_{47}).$$

6.3. The ring $H^*(G; F_2)$. All elements in $O_{G,F_2}$, and their $\kappa$-images in $H^*(G; F_2)$, are given by the table below (see Example 2.1)

| $u \in O_{G_2,F_2}$ | $\xi^{(2)}_2$ | $\theta^{(2)}_2$ | $\zeta^{(2)}_2$ | $\eta^{(2)}_2$ |
|--------------------|---------------|---------------|---------------|---------------|
| $u \in O_{F_2,F_2}$ | $\xi^{(2)}_1$ | $\theta^{(2)}_1$ | $\zeta^{(2)}_1$ | $\eta^{(2)}_1$ |
| $u \in O_{E_6,F_2}$ | $\xi^{(2)}_1$ | $\theta^{(2)}_2$ | $\zeta^{(2)}_2$ | $\eta^{(2)}_2$ |
| $u \in O_{E_7,F_2}$ | $\xi^{(2)}_1$ | $\theta^{(2)}_2$ | $\zeta^{(2)}_3$ | $\eta^{(2)}_3$ |
| $u \in O_{E_8,F_2}$ | $\xi^{(2)}_1$ | $\theta^{(2)}_2$ | $\zeta^{(2)}_4$ | $\eta^{(2)}_4$ |

Table 7. The elements in $O_{G,F_2}$ and their $\kappa$-images.

where $\zeta_{\text{deg } u} = \kappa(u)$, $u \in O_{G,F_2}$. By Theorem 1 we have

$$H^*(G; F_2) = A_{G,F_2}^* \otimes \Delta_{F_2}(\zeta_{\text{deg } u})_{u \in O_{G,F_2}},$$

in which $A_{G,F_2}^*$ has been decided in Example 3.1. The determination of the ring $H^*(G; F_2)$ now amounts to expressing all the squares $\zeta_2^2$ as elements in $A_{G,F_2}^*$ (see (5.5)). This requires computation with explicit presentations of primary polynomials in $\Phi_{G,F_2}$ (Definition 2.2), and has been done in [21 Corollary 2]:

Theorem 5. The inclusions $A_{G,F_2}^*, O_{G,F_2}^* \subset H^*(G; F_2)$ induces the ring isomorphisms

(6.2) $H^*(G_2; F_2) = F_2[x_6]/(x_6^3) \otimes \Delta_{F_2}(\zeta_3) \otimes \Lambda_{F_2}(\zeta_5);$

(6.3) $H^*(F_4; F_2) = F_2[x_6]/(x_6^3) \otimes \Delta_{F_2}(\zeta_3) \otimes \Lambda_{F_2}(\zeta_5, \zeta_{15}, \zeta_{23});$

(6.4) $H^*(E_6; F_2) = F_2[x_6]/(x_6^3) \otimes \Delta_{F_2}(\zeta_3) \otimes \Lambda_{F_2}(\zeta_5, \zeta_9, \zeta_{15}, \zeta_{17}, \zeta_{23});$

(6.5) $H^*(E_7; F_2) = F_2[x_6, x_{10}]/(x_6^3, x_{10}^3) \otimes \Delta_{F_2}(\zeta_3, \zeta_5, \zeta_9) \otimes \Lambda_{F_2}(\zeta_{15}, \zeta_{17}, \zeta_{23}, \zeta_{27});$
\[(6.6) \quad H^* (E_6; \mathbb{F}_2) = \frac{F_7[x, x_{10}, x_{18}, x_{20}]}{(x_6^2 - x_{10} x_{18})} \otimes \Delta_{F_7} (\zeta_3, \zeta_5, \zeta_9, \zeta_{15}, \zeta_{23}) \otimes \Delta_{F_2} (\zeta_{17}, \zeta_{27}, \zeta_{29}), \]

where

i) \( \xi_3^2 = x_6 \) in \( G_2, F_4, E_6, E_7, E_8 \);

ii) \( \xi_5^2 = x_{10}, \xi_9^2 = x_{18} \) in \( E_7, E_8 \);

iii) \( \xi_{15}^2 = x_{30}, \xi_{23}^2 = x_{6}^9 x_{10} \) in \( E_{8, \mathbb{F}_2} \).

**Remark 6.1.** Historically, the rings \( H^* (G; \mathbb{F}_2) \) for exceptional \( G \) were first obtained by Borel, Araki and Shikata, Kono [8, 9, 2, 1, 27], using generators specified mainly by their degrees. It should be noted that the generators \( \xi_i \)'s that we utilized may not coincide with those in the classical descriptions, compare Corollaries 1 and 2 in [21].

### 6.4. The torsion ideal \( \tau_p (G) \) in \( H^* (G; \mathbb{Z}) \).

The strategy in determining the ring structure on \( \tau_p (G) \), \( p = 2, 3, 5 \), is the following one: the formula (5.14) in Theorem 2 has already characterized \( \tau_p (G) \) as a module over the ring \( A_{G, \mathbb{F}_p}^+ \):

\[(6.7) \quad \tau_p (G) \cong \frac{F_p [\eta_{[\eta]}] (\xi_{2}^{(2)}) \otimes \Delta_{F_p} (\kappa (\xi_{2}, \eta_{2}^{(2)})))_{2 \leq k \leq 6} \text{ Im}(F_p). \]

where \( I \subseteq G(p) \) with \( |I| \geq 2 \). It remains to express

a) all the squares \( \kappa (\xi_{2}, \eta_{2}^{(2)})^2 \) as elements in \( A_{G, \mathbb{F}_p}^+ \) (see (5.5));

b) all the products \( C_i \cdot C_j \) as \( A_{G, \mathbb{F}_p}^+ \)-linear combinations of \( \xi_i \)'s.

By considering \( \tau_p (G) \) as the subring \( \text{ Im}(\delta_p) \subset H^* (G; \mathbb{F}_p) \) via \( r_p \), the tasks in a) and b) can be implemented by computation in the ring \( H^* (G; \mathbb{F}_p) \) whose structures has already been determined in Theorems 4 and 5.

One reads from Tables 5–7 those \( \xi_i \)'s that correspond to \( \kappa (\xi_{2}, \eta_{2}^{(2)}) \) in (6.7).

**Lemma 6.1.** Under the ring isomorphisms \( r_p : \tau_p (G) \cong \text{ Im}(\delta_p) \) in (6.7), all nontrivial \( \tau_p (G) \) with \( p = 3, 5 \) are given by

\[(6.8) \quad \tau_3 (F_4) \cong \mathbb{F}_3 [x_8]^+ / \langle x_8^2 \rangle \otimes A_{F_3} (\zeta_3, \zeta_{11}, \zeta_{15}); \]

\[(6.9) \quad \tau_3 (E_6) \cong \mathbb{F}_3 [x_8]^+ / \langle x_8^2 \rangle \otimes A_{F_3} (\zeta_3, \zeta_9, \zeta_{11}, \zeta_{15}, \zeta_{17}); \]

\[(6.10) \quad \tau_3 (E_7) \cong \mathbb{F}_3 [x_8]^+ / \langle x_8^2 \rangle \otimes A_{F_3} (\zeta_3, \zeta_{11}, \zeta_{15}, \zeta_{19}, \zeta_{27}, \zeta_{35}); \]

\[(6.11) \quad \tau_3 (E_8) \cong \frac{\mathbb{F}_3 [x_{20}, x_{22}, x_{24}, x_{26}, x_{28}, x_{30}, x_{32}, x_{34}, x_{36}, x_{38}, x_{40}]}{(x_4 - x_{12})^2} \otimes A_{F_3} (\zeta_3, \zeta_{15}, \zeta_{27}, \zeta_{35}, \zeta_{39}, \zeta_{47}); \]

\[(6.12) \quad \tau_5 (E_8) \cong \mathbb{F}_5 [x_{12}]^+ / \langle x_{12}^2 \rangle \otimes A_{F_5} (\zeta_3, \zeta_{15}, \zeta_{23}, \zeta_{27}, \zeta_{35}, \zeta_{39}, \zeta_{47}). \]

**Proof.** Since \( u^2 = 0 \) for \( u \in H^{odd} (G; \mathbb{F}_p) \) with \( p \neq 2 \), we have in (6.7) that

\[(6.13) \quad \Delta_{F_p} (\kappa (\xi_{p}^{(p)}), \kappa (\eta_{p}^{(p)})) = A_{F_p} (\kappa (\xi_{p}^{(p)}), \kappa (\eta_{p}^{(p)})), p = 3, 5. \]

For \( p = 3 \) we get from Example 3.1 and Table 2 that
i) \(G_2(3) = \emptyset\);

ii) \(A_{G_2} = \mathbb{P}_3[x_8]/\langle x_8^3 \rangle\), \(G(3) = \{2\}\) for \(G = F_4, E_6, E_7\);

iii) \(A_{E_6} = \mathbb{P}_3[x_8, x_{20}]/\langle x_8^3, x_{20}^3 \rangle\), \(E_6(3) = \{2, 6\}\).

With (6.7) we get \(\tau_2(G_2) = 0\) from i); (6.8)–(6.10) from (6.13) and ii) (where the class of the type \(C_7\) is absent since \(G(3)\) is a singleton). The isomorphism (6.11) comes from (6.13) and iii) by notifying further that \(C_{2,6}\) is the only class of the type \(C_7\) with \(|I| \geq 2\), whose square is trivial for the degree reason.

Similarly, granted with (6.7) and (6.13), we get \(\tau_5(G) = 0\) for \(G \neq E_8\) and (6.12) respectively from

\[G_2(5) = F_4(5) = E_6(5) = E_7(5) = \emptyset\]

and \(E_8(5) = \{4\}\) by Table 2. \(\Box\)

**Lemma 6.2.** With \(\xi^2 = x_6\) for all \(G\) and \(\xi^2 = x_{30}, \xi^2 = x_8^6x_10\) for \(E_8\) being understood, the ring isomorphisms \(r_2 : \tau_2(G) \cong \text{Im} \delta_2\) in (6.7) are given by

\[(6.14) \quad \tau_2(G_2) \cong \mathbb{P}_2[x_6]/\langle x_6^2 \rangle \otimes \Delta F_2(\xi);
\]

\[(6.15) \quad \tau_2(F_4) \cong \mathbb{P}_2[x_6]/\langle x_6^2 \rangle \otimes \Delta F_2(\xi) \otimes \Lambda F_2(\xi_{15}, \xi_{23});
\]

\[(6.16) \quad \tau_2(E_6) \cong \mathbb{P}_2[x_6]/\langle x_6^2 \rangle \otimes \Delta F_2(\xi) \otimes \Lambda F_2(\xi_9, \xi_{15}, \xi_{17}, \xi_{23});
\]

\[(6.17) \quad \tau_2(E_7) \cong \frac{\mathbb{P}_2[x_6, x_{10}, x_{18}, C^{(2)}_{(2)}]}{(x_6^2, x_{10}^2, x_{18}^2, D_{(2)}^{(2)}, R^{(2)}_{K}, S^{(2)}_{I,J})} \otimes \Delta F_2(\xi) \otimes \Lambda F_2(\xi_{15}, \xi_{23}, \xi_{27})\]

with \(I, J \subseteq K = \{1, 3, 4\}, |I|, |J| \geq 2;\)

\[(6.18) \quad \tau_2(E_8) \cong \frac{\mathbb{P}_2[x_6, x_{10}, x_{18}, x_{30}, C^{(2)}_{(2)}]}{(x_6^2, x_{10}^2, x_{18}^2, x_{30}^2, D_{(2)}^{(2)}, R^{(2)}_{K}, S^{(2)}_{I,J})} \otimes \Delta F_2(\xi_{15}, \xi_{23}) \otimes \Lambda F_2(\xi_{27})\]

with \(K, I, J \subseteq \{1, 3, 5, 7\}, |I|, |J| \geq 2, |K| \geq 3;\)

in which the relations \(S_{(2)}^{(2)}_{I,J}\) in (6.17) and (6.18) are

\[(6.19) \quad S_{(2)}^{(2)}_{I,J} = C_{I,J}^{(2)} C_{I,J}^{(2)} + \sum_{t \in I} x_{deg y_t} \prod_{s \in I \cap J} \xi_{deg y_s}^{c_{I,J}^{(2)}} C_{(2)}^{(2)} (I_t, J_t),\]

where \(\langle I, J \rangle = \{t \in I \cup J | t \notin I \cap J\}, \prod_{s \in I \cap J} \xi_{deg y_s}^{c_{I,J}^{(2)}} = 1\) when \(I_t \cap J = \emptyset\), and where the squares \(\xi_{deg y_t}^{c_{I,J}^{(2)}}\) (see in Table 7) are evaluated by

\[(\xi^2, \xi^2, \xi^2, \xi^2) = (x_{10}, x_{18}, 0, 0).\]

**Proof.** The cases \(G \neq E_8\) are fairly simple. We may therefore focus on the relatively nontrivial case \(G = E_8\), for which the formula (6.7) turns to be

\[(6.20) \quad \tau_2(E_8) = \frac{\mathbb{P}_2[x_6, x_{10}, x_{18}, x_{30}]}{(x_6^2, x_{10}^2, x_{18}^2, x_{30}^2, D_{(2)}^{(2)}, R^{(2)}_{K})} \otimes \Delta F_2(\xi_{15}, \xi_{23}, \xi_{27}),\]

where

\[
\begin{align*}
\kappa_{\xi_{(2)}} = \kappa_{\xi_{(2)}}, \quad \kappa_{\xi_{(2)}} &= \kappa_{\xi_{(2)}}, \quad \kappa_{\xi_{(2)}} &= \kappa_{\xi_{(2)}}, \quad \kappa_{\xi_{(2)}} = \kappa_{\xi_{(2)}} \quad (Table 7); \\
A_{E_8}^{(2)} &= \frac{\mathbb{P}_2[x_6, x_{10}, x_{18}, x_{30}]}{(x_8^2, x_{10}^2, x_{18}^2, x_{30}^2)} \quad (Example 3.1);
\end{align*}
\]

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and where \( I, J, K \subseteq E_8(2) = \{1, 3, 5, 7\} \) by Table 2. Since \( \tau_2(E_4) \cong \text{Im} \delta_2 \subset H^*(E_8; \mathbb{F}_2) \) via \( \tau_2 \), the relations

\[
\zeta_4 = x_6, \quad \zeta_{15}^2 = x_{30}, \quad \zeta_{23}^2 = x_6^2 x_{10}, \quad \zeta_{27}^2 = 0
\]

obtained in Theorem 5 implies that

(6.21) the factor \( \Delta_{E_8}^2(\zeta_3, \zeta_{15}, \zeta_{23}, \zeta_{27}) \) in (6.20) is \( \Delta_{E_8}^2(\zeta_3, \zeta_{15}, \zeta_{23}) \otimes \Delta_{E_8}(\zeta_{27}) \).

It remains to decide the multiplicative rule among the classes \( C_I \)'s. For \( I, J \subseteq E_8(2) = \{1, 3, 5, 7\} \) with \( |I|, |J| \geq 2 \) we have, in \( H^*(E_8; \mathbb{F}_2) \), that

i) \( \delta_2(\theta_2^{(2)}I) = \sum_{t \in I} x_{\text{deg} y_t} \theta_2^{(2)}t \) by (5.12) and (5.9);

ii) \( \theta_2^{(2)}J = \prod_{s \in I \cap J} \zeta_2^{\text{deg} \theta_2^{(2)}(j,s)} \) with \( \prod_{s \in I \cap J} \zeta_2^{\text{deg} \theta_2^{(2)}} = 1 \) for \( I \cap J = \emptyset \),

where \( \zeta_{\text{deg} \theta_2^{(2)}} = \kappa(\theta_2^{(2)}) \) (Table 7). It follows from \( C_{\text{deg} \theta_2^{(2)}} = \delta_2(\theta_2^{(2)}J) \) that

\[
C_{\text{deg} \theta_2^{(2)}} = \delta_2(\theta_2^{(2)}) \delta_2(\theta_2^{(2)}) = \delta_2(\theta_2^{(2)}) \delta_2(\theta_2^{(2)}) (\text{since} \ \delta_2 = 0)
\]

\[
= \delta_2(\sum_{t \in I} x_{\text{deg} y_t} \theta_2^{(2)}t) (\text{by i})
\]

\[
= \delta_2(\sum_{t \in I} x_{\text{deg} y_t} \prod_{s \in I \cap J} \zeta_2^{\text{deg} \theta_2^{(2)}(j,s)}) (\text{by ii})
\]

From

\[
(\zeta_2^{\text{deg} \theta_2}, \zeta_2^{\text{deg} \theta_2}, \zeta_2^{\text{deg} \theta_2}, \zeta_2^{\text{deg} \theta_2}) = (x_{10}, x_{18}, 0, 0)
\]

by Theorem 5,

\[
\delta_2(x_{\text{deg} y_t}) = 0 \text{ by (5.9)}
\]

and from \( \delta_2(\theta_2^{(2)}I, J) = C_{(I, J)}^{(2)} \) by (5.11) we get

(6.22) \( C_{(I, J)}^{(2)} = \sum_{t \in I \cap J} x_{\text{deg} y_t} \prod_{s \in I \cap J} \zeta_2^{\text{deg} \theta_2^{(2)}(j,s)} C_{(I, J)}^{(2)} \),

where \( \prod_{s \in I \cap J} \zeta_2^{\text{deg} \theta_2^{(2)}} = 1 \) when \( I \cap J = \emptyset \). By taking (6.22) as the relation (6.19) on \( \tau_2(E_8) \), the first factor in (6.20) can be written as

(6.23) \( \frac{F_3[x_{x_{10}}, x_{x_{18}}, x_{x_{30}}, C_{(I, J)}^{(2)}]}{x_{x_{10}}, x_{x_{18}}, x_{x_{30}}, C_{(I, J)}^{(2)}, C_{(I, J)}^{(2)}, C_{(I, J)}^{(2)}} \).

Combining (6.21) with (6.23) establishes (6.18). \( \Box \)

**Example 6.1.** The formula (6.22) (i.e. the relation (6.19)) is effective in computing with the products \( C_{(I)}^{(2)}C_{(J)}^{(2)} \). Taking \( G = E_8 \) as an example and noting that \( E_8(2) = \{1, 3, 5, 7\} \), we have by (6.22) that

i) \( C_{(1,3)}^{(2)}C_{(1,3)}^{(2)} = x_6 C_{(1,3)}^{2}C_{(1,3)}^{2} + x_{10} C_{(1,3)}^{2}C_{(1,3)}^{2} = x_6^2 x_{18} + x_{10}^3 \)

since \( (\zeta_2^{\text{deg} \theta_2^{(2)}}, \zeta_2^{\text{deg} \theta_2^{(2)}}) = (x_{18}, x_{10}) \) and \( (C_{(1,3)}^{(2)}, C_{(1,3)}^{(2)}) = (x_6, x_{10}) \) in \( H^*(E_8; \mathbb{F}_2) \);

ii) \( C_{(1,3)}^{(2)}C_{(1,5)}^{(2)} = x_6 C_{(1,5)}^{2}C_{(1,5)}^{2} + x_{10} C_{(1,3,5)}^{2} = x_6^2 x_{18} + x_{10}^3 \)
6.4. The ring $H^*(G; \mathbb{Z})$. We specify the generators that will be utilized in describing the ring $H^*(G; \mathbb{Z})$. Firstly, all elements in $O_{G, \mathbb{Z}}$, together their $\kappa$-images, are tabulated in Table 8 below (see Example 2.1):

| $u \in O_{G, \mathbb{Z}}$ | $\xi_1$ | $\eta_1$ | $\xi_2$ | $\eta_2$ | $\xi_3$ | $\eta_3$ | $\xi_4$ | $\eta_4$ |
|-------------------------|---------|---------|---------|---------|---------|---------|---------|---------|
| $u \in O_{F_4, \mathbb{Z}}$ | $\xi_1$ | $\eta_1$ | $\xi_2$ | $\eta_2$ | $\eta_3$ | $\xi_3$ | $\eta_4$ | $\xi_4$ |
| $u \in O_{E_6, \mathbb{Z}}$ | $\xi_1$ | $\xi_2$ | $\eta_1$ | $\eta_2$ | $\eta_3$ | $\xi_3$ | $\eta_4$ | $\xi_4$ |
| $u \in O_{E_7, \mathbb{Z}}$ | $\xi_1$ | $\eta_1$ | $\eta_2$ | $\xi_2$ | $\eta_3$ | $\eta_4$ | $\xi_3$ | $\eta_4$ |
| $u \in O_{E_8, \mathbb{Z}}$ | $\xi_1$ | $\eta_2$ | $\xi_2$ | $\eta_3$ | $\eta_4$ | $\xi_3$ | $\eta_4$ | $\xi_4$ |
| $\kappa(u) \in O_{G, \mathbb{Z}}$ | $\varphi_3$ | $\varphi_9$ | $\varphi_{11}$ | $\varphi_{15}$ | $\varphi_{17}$ | $\varphi_{19}$ | $\varphi_{23}$ | $\varphi_{27}$ | $\varphi_{28}$ | $\varphi_{33}$ | $\varphi_{37}$ | $\varphi_{39}$ | $\varphi_{41}$ | $\varphi_{47}$ | $\varphi_{49}$ |

Table 8. the elements in $O_{G, \mathbb{Z}}$ and their $\kappa$-images

Next, the Bockstein $\beta_p$ carries the classes $\theta^p(I) \subset H^*(G; F_p)$, $I \subseteq G(p)$ (see (5.12)), to the elements in $H^*(G; \mathbb{Z})$

(6.25) $\xi^p(I) := \beta_p(\theta^p(I)) \subset H^*(G; \mathbb{Z})$, $I \subseteq G(p)$, with $\xi^p(I) = -y_t$.

Let $A(G)$ be the subring of $H^*(G; \mathbb{Z})$ generated multiplicatively by $\xi^p(I)$, $I \subseteq G(p)$, and the unit 1 in $H^*(G; \mathbb{Z})$. Precisely, by the proofs of Lemma 6.1 and Lemma 6.2 we have that

(6.26) one has the ring splitting $A(G) = \mathbb{Z} \oplus_{p=2,3,5} A_p(G)$ in which

$$A_2(E_7) = \frac{A_{E_7, \mathbb{Z}}[1, \xi^{(2)}(I)]}{(\beta^p, \beta^p)} I, J \subseteq K = \{1, 3, 4\}, |I|, |J| \geq 2;$$

$$A_2(E_6) = \frac{A_{E_6, \mathbb{Z}}[1, \xi^{(2)}(J)]^+}{(\beta^p, \beta^p)} I, J, K \subseteq \{1, 3, 5, 7\}, |I|, |J|, |K| \geq 2;$$

$$A_3(E_6) = \frac{A_{E_6, \mathbb{Z}}[1, \xi^{(3)}(I)]^+}{(\beta^p, \beta^p)} I, J, K \subseteq \{1, 3, 5, 7\}, |I|, |J|, |K| \geq 2;$$

and $A_p(G) = A_p^+(G)$ in the remaining cases.

where

$$D^p(I) = (\prod_{t \in J} y_t^{k_t-1}) \xi^p(I);$$

$$S^p_{I,J} = \sum_{t \in K} y_t S^p_{I,J};$$

and $x_{\deg y_t} = y_t$ as in Example 3.1.

Theorem 6. The inclusions $O_{G, \mathbb{Z}}, A(G) \subset H^*(G; \mathbb{Z})$ induce a ring isomorphism

(6.27) $H^*(G; \mathbb{Z}) = A(G) \otimes \Delta_{\mathbb{Z}}(O_{G, \mathbb{Z}})/ (F_r, H_{r,t})$,

where $r \in \{ \deg u \}_{u \in O_{G, \mathbb{Z}}}$, $t \in G(p)$, $I \subseteq G(p)$ with $p = 2, 3, 5$, and where

1 We observe from Table 8 that elements in $O_{G, \mathbb{Z}}$ have distinct degrees for every $G$. 28
(6.28) the relations \( F_r \) are given by \( q_{1}^{2} = 0 \) with three exceptions:
\[ q_{1}^{2} = x_{6} \] for all \( G \); \( q_{15}^{2} = x_{30} \); \( q_{23}^{2} = x_{0}^{2} x_{10} \) for \( G = E_{8} \);

(6.29) the relations \( H_{I,J} \) are given by the three possibilities
\[ q_{\text{deg},y_{I}^{(p)}} E_{I}^{(p)} = \begin{cases} 
\xi_{x}^{(p)} & \text{if } t \notin I; \\
0 & \text{if either } t \in I, p \text{ is odd or } I = \{ t \}, p = 2; \\
x_{\text{deg},y_{I}^{(p)}} (\eta_{I}^{(p)} / \xi_{x}^{(p)}) & \text{if } p = 2, t \in I \text{ and } |I| \geq 2
\end{cases} \]
with \((\theta_{1}^{(2)})^{2}\) being evaluated by
\[(\theta_{1}^{(2)})^{2} = (x_{10}, x_{18}, 0) \text{ for } G = E_{7}, \text{ and } \]
\[(\theta_{1}^{(2)})^{2}, (\theta_{1}^{(2)})^{2}, (\theta_{1}^{(2)})^{2}, (\theta_{1}^{(2)})^{2} = (x_{10}, x_{18}, 0, 0) \text{ for } G = E_{8}. \]

In particular, one has

i) \( H^{*}(G_{2}; \mathbb{Z}) = A_{G_{2},\mathbb{Z}}^{*} \otimes \Delta_{\mathbb{Z}}(\eta_{3}) \otimes \Lambda_{\mathbb{Z}}(\eta_{11}) / (q_{1}^{2} - x_{6}; \eta_{11} x_{6}) \);
ii) \( H^{*}(F_{4}; \mathbb{Z}) = A_{F_{4},\mathbb{Z}}^{*} \otimes \Delta_{\mathbb{Z}}(\eta_{3}) \otimes \Lambda_{\mathbb{Z}}(\eta_{11}, \eta_{15}, \eta_{23}) / (q_{1}^{2} - x_{6}; \eta_{11} x_{6}; \eta_{23} x_{6}) \);
iii) \( H^{*}(E_{6}; \mathbb{Z}) = A_{E_{6},\mathbb{Z}}^{*} \otimes \Delta_{\mathbb{Z}}(\eta_{3}) \otimes \Lambda_{\mathbb{Z}}(\eta_{9}, \eta_{11}, \eta_{15}, \eta_{17}, \eta_{23}) / (q_{1}^{2} - x_{6}; \eta_{11} x_{6}; \eta_{23} x_{6}) \);
iv) \( H^{*}(E_{7}; \mathbb{Z}) = A_{E_{7},\mathbb{Z}}^{*} \otimes \Delta_{\mathbb{Z}}(\eta_{3}) \otimes \Lambda_{\mathbb{Z}}(\eta_{11}, \eta_{15}, \eta_{19}, \eta_{23}, \eta_{27}, \eta_{33}) \text{ with } t \in \{ 1, 3, 4 \}, I \subseteq \{ 1, 3, 4 \} \);
v) \( H^{*}(E_{8}; \mathbb{Z}) = A_{E_{8},\mathbb{Z}}^{*} \otimes \Delta_{\mathbb{Z}}(\eta_{3}) \otimes \Lambda_{\mathbb{Z}}(\eta_{9}, \eta_{15}, \eta_{19}, \eta_{23}, \eta_{27}, \eta_{33}) \text{ with } s \in \{ 2, 6 \}, I \subseteq \{ 2, 6 \}, t \in \{ 1, 3, 5, 7 \}, J \subseteq \{ 1, 3, 5, 7 \} \)

**Proof.** Since \( r_{p} : H^{*}(G; \mathbb{Z}) \to H^{*}(G; \mathbb{F}_{p}) \) satisfies (by \( \delta_{p} = r_{p} \beta_{p}, \kappa_{r_{p}} = r_{p} \kappa \) and Lemma 3.3) that
\[ r_{p}(\xi_{x}^{(p)}) = \xi_{x}^{(p)} ; r_{p}(\kappa_{\eta_{x}}) = \kappa_{\eta_{x}} ; r_{p}(\kappa_{\eta_{y}}) = p_{y} \kappa_{y}, s \in \mathfrak{G}(\mathbb{F}_{p}) , \]
we get from Theorem 2 the presentations without resorting to \( r_{p} \).

(6.30) \( H^{*}(G; \mathbb{Z}) = \Delta_{\mathbb{Z}}(\mathcal{O}_{G,\mathbb{Z}}) \oplus \kappa_{r_{p}}(G) \) in which
\[ \kappa_{r_{p}}(G) = \mathbb{F}_{p}[y_{I}] / \{ 1, \xi_{x}^{(p)} + y_{I}^{(p)} ; D_{I}^{(p)}(\eta_{I}^{(p)} / \xi_{x}^{(p)}) \otimes \Delta_{\mathbb{Z}}(\kappa(\eta_{I}^{(p)}), \kappa(\eta_{s}^{(p)})) \} _{1 \leq k, s \leq \mathfrak{T}_{p}} . \]

In view of (6.6) and (6.30) the \( \mathcal{A}(G) \)-module map
\[ \psi : \mathcal{A}(G) \otimes \Delta_{\mathbb{Z}}(\mathcal{O}_{G,\mathbb{Z}}) \to H^{*}(G; \mathbb{Z}) \]
induced from the inclusions \( \mathcal{O}_{G,\mathbb{Z}} \times \mathcal{A}(G) \subset H^{*}(G; \mathbb{Z}) \) is surjective by Lemmas 6.1 and 6.2. Since the square of any odd dimensional integral cohomology class of a space \( X \) lands in the 2-primar component of \( H^{*}(X; \mathbb{Z}) \), and since \( \kappa_{r_{p}}(G) \) is an ideal in \( H^{*}(G; \mathbb{Z}) \), to modify \( \psi \) into a ring isomorphism it suffices to

(6.31) express all the squares \( q_{\text{deg},u}^{2}, u \in \mathcal{O}_{G,\mathbb{Z}}, \) as elements in \( \kappa_{r_{p}}(G) \);
(6.32) determine the actions of \( \varrho_{\deg u} \), \( u \in O_{G, \mathbb{Z}} \), on \( \tau_p(G) \).

The proof of Theorem 6 is done by showing that the relations (6.28) and (6.29) take care of these two concerns respectively.

For (6.31) we note from Lemma 3.3 that

\[
\begin{align*}
r_2(\kappa(\xi_i)^2) &\equiv \kappa(\xi_i^{(2)})^2, \ 1 \leq i \leq k; \\
r_2(\kappa(\eta_s)^2) &\equiv p_s^2 \kappa(\eta_s^{(2)})^2 \text{ for } s \in \mathbb{G}(\mathbb{F}_2); \\
r_2(\kappa(\eta_t)^2) &\equiv \eta_t^{2(k_t-1)} \kappa(\eta_t^{(2)})^2 \equiv 0 \text{ for } t \in G(2) \text{ (since } \eta_t^{k_t} \equiv 0). 
\end{align*}
\]

The relations \( \mathcal{F}_r \) in (6.28) are verified by

i) \( \kappa(\xi_i)^2, \kappa(\eta_s)^2 \in \tau_2(G) \);

ii) \( r_2 \) restricts to an isomorphism \( \tau_2(G) \rightarrow \text{Im } \delta_2 \subset H^*(G; \mathbb{F}_2) \); and

iii) the results on \( \kappa(\xi_i^{(2)})^2, \kappa(\eta_s^{(2)})^2 \) in Theorem 5, \( 1 \leq i \leq k, \ s \in \mathbb{G}(\mathbb{F}_2) \).

For (6.32) it suffices, in view of the presentation for \( \tau_p(G) \) in (6.30), to express every product \( \kappa(\eta_s)\mathcal{C}_i^{(p)} \), \( t \in G(p) \), \( I \subseteq G(p) \), as an element in \( \tau_p(G) \). Since \( r_p \) restricts to an isomorphism \( \tau_p(G) \cong \text{Im } \delta_p \), the relation \( \mathcal{H}_{t, I} \) in (6.29) is obtained from the calculation in \( H^*(G; \mathbb{F}_p) \)

\[
r_p(\kappa(\eta_r)\mathcal{C}_i^{(p)}) \equiv y_t^{k_t-1}\kappa(\theta_i^{(p)})\mathcal{C}_i^{(p)} \quad \text{by Lemma 3.3 and } r_p(\mathcal{C}_i^{(p)}) = \mathcal{C}_i^{(p)} \]
\[
\equiv y_t^{k_t-1}\kappa(\theta_i^{(p)})\delta_p(\theta_i^{(p)}) \quad \text{(by (5.12))} \]
\[
\equiv y_t^{k_t-1}\delta_p(\theta_i^{(p)}) \quad \text{since } \theta_i^{(p)} = \kappa(\theta_i^{(p)}), \ y_t^{k_t-1}\delta_p(\theta_i^{(p)}) \equiv y_t^{k_t} \equiv 0 \]
\[
\equiv \begin{cases} 
  y_t^{k_t-1}\mathcal{C}_i^{(p)} & \text{if } t \notin I; \\
  0 & \text{if } I = \{ t \}; \\
  y_t^{k_t-1}(\theta_i^{(p)})\mathcal{C}_i^{(p)} & \text{if } t \in I, \ |I| \geq 2,
\end{cases}
\]

where, in the third instance, \( (\theta_i^{(p)})^2 = 0 \) for \( t \in \mathbb{G}(\mathbb{F}_2) \) since \( \theta_i^{(p)} \) is of odd dimensional with order \( p_t \neq 2 \), and where by Theorem 5, \( (\theta_i^{(2)})^2 \) with \( t \in G(2) \) should be evaluated as that in c) of (6.29).

Finally, concerning the presentations in i)–v), we remark that each \( \varrho_{\deg u} \) with free square contributes to a generator in the exterior part, and that if \( G(p) \) is a singleton, the relations of the type \( \mathcal{H}_{t, I}, t \in G(p), I \subseteq G(p) \), is unique, and can be concretely given as \( x_{\deg u, \varrho_{\deg \eta}} = 0 \).

**Remark 6.2.** One may compare i), ii) of Theorem 6 with the descriptions for \( H^*(G_2; \mathbb{Z}) \) and \( H^*(F_2; \mathbb{Z}) \) by Borel [8, 9].

Theorem 6 summarizes the rings \( H^*(G; \mathbb{Z}) \) into the compact form (6.27). As for the visibility of their structure in terms of free part and torsion parts, the following alternative presentations based on Theorem 2, together with Lemmas 6.1 and 6.2, may appear more practical. To explain this we note that in the isomorphisms in Lemmas 6.1 and 6.2 we have \( \zeta_j = r_p(g_j) \) by Lemma 3.3. Therefore, taking into account of the relations \( \mathcal{H}_{t, I}, t \in G(p), I \subseteq G(p), \) that specify the actions of the free part of \( H^*(G; \mathbb{Z}) \) on \( \tau_p(G) \) we have, as examples,
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