Abstract—In this paper, a robust nonlinear control scheme is proposed for a nonlinear multi-input multi-output (MIMO) system subject to bounded time varying uncertainty which satisfies a certain integral quadratic constraint condition. The scheme develops a robust feedback linearization approach which uses standard feedback linearization approach to linearize the nominal nonlinear dynamics of the uncertain nonlinear system and linearizes the nonlinear time varying uncertainties at an arbitrary point using the mean value theorem. This approach transforms uncertain nonlinear MIMO systems into an equivalent MIMO linear uncertain system model with unstructured uncertainty. Finally, a robust minimax linear quadratic Gaussian (LQG) control design is proposed for the linearized model. The scheme guarantees the internal stability of the closed loop system and provides robust performance. In order to illustrate the effectiveness of this approach, the proposed method is applied to a tracking control problem for an air-breathing hypersonic flight vehicle (AHFV).

I. INTRODUCTION

The general problem of feedback linearization for nonlinear systems with uncertainty has been typically approached in the literature by imposing some conditions on the uncertainty description which are known as matching conditions [1] and the strict triangularity condition [2]. Methods considering mismatched uncertainties also exist, in which uncertainties are decomposed into matched and mismatched parts. These methods typically require the mismatched parts not to exceed some maximum allowable bound [3]. In an attempt to solve a related issue arising in feedback linearization, in our previous work [4], [5] we propose a method of robust feedback linearization to feedback linearize a nonlinear system with uncertainties by representing the uncertainties in a realistic way and relaxing the matching condition requirements on the description of the uncertainties. In this approach, we linearized the nominal part of the system using the feedback linearization approach and linearized the remaining nonlinear terms with respect to each uncertainty and state to obtain an acceptable linear form for the uncertainty model at arbitrary operating points. However, in order to express the system in a more convenient set of coordinates, we have defined a diffeomorphism $T$ which depends on the nominal values (without uncertainty) of the uncertain states. This definition of the diffeomorphism requires that the system either satisfies the generalized matching conditions [1] which are relaxed versions of the matching condition or allows for additional uncertainty inputs in the system.

In order to relax the generalized matching condition and the strict triangularity requirement, in this paper, we have introduced a notion of an uncertain diffeomorphism. This definition of uncertain diffeomorphism is similar to the one used in our previous work (see [6]). Furthermore, in order to deal with the nonlinear uncertain terms which are subject to time varying uncertainty, here we use a mean value approach similar to the approach used in [5]. The uncertain diffeomorphism used in this paper is the function of system states and uncertain time varying parameters. Generally, feedback linearization of higher order systems involves higher order derivatives of the state and feedback linearization is proposed in this paper.

In the later part of this paper, we apply the proposed method to design a tracking controller for a nonlinear MIMO system subject to time varying uncertainty using an uncertain diffeomorphism along with a mean value approach. The main contributions of this work as compared to previously published work [4], [5], [6] are as follows:

1. Feedback linearization of uncertain systems subject to time varying uncertainty using an uncertain diffeomorphism.

2. Estimation of uncertain states and design of a tracking controller using the minimax LQG design method for the linearized model.

The paper is organized as follows. Section II describes the class of nonlinear systems and uncertainties considered in the paper. A complete derivation of the robust feedback linearization of the uncertain system is presented in Section III. In Section IV, the minimax LQG control design
method is presented for the feedback linearized system with linearized uncertainty. For the case study using an uncertain nonlinear model of the AHFV, the uncertainty modeling and control design methods with tracking simulation results are presented in Section VI. Conclusions are presented in Section VI.

II. SYSTEM DEFINITION

Here, we consider an uncertain multi-input multi-output nonlinear system having same number of inputs and outputs and which is subject to time varying uncertainty \( p(t) \):

\[
\dot{x}(t) = f(x(t), p(t)) + \sum_{k=1}^{m} g_k(x(t), p(t))u_k(t),
\]

\[
y_i(t) = \nu_i(x(t)), \quad i = 1, 2, \ldots, m
\]

where \( x(t) \in \mathbb{R}^n, u(t) = [u_1, \ldots, u_m]^T \in \mathbb{R}^m \) and \( y(t) = [y_1, \ldots, y_m]^T \in \mathbb{R}^m \). The nonlinear functions \( f(x(t), p(t)) \) and \( g_k(x(t), p(t)) \) and \( \nu_i(x(t), p(t)) \) for \( i = 1, 2, \ldots, m \) are infinitely differentiable (or differentiable to a sufficiently large degree) arguments of their functions. Also, \( p(t) \in \mathbb{R}^q \in \Omega \) is a vector of unknown parameters or disturbances which takes values in the set \( \Omega \subset \mathbb{R}^p \). The subsuperscript indices \( k \) and \( i \) indicate \( k\)th and \( i\)th elements of the corresponding vectors respectively. The full state vector \( x(t) \) is assumed to be available for measurement and the uncertainty in the system satisfies an integral quadratic constraint condition (IQC) (see [7]). It is assumed that the uncertain functions can be written as \( f(x(t), p(t)) = f(x(t), p_0) + \Delta f(x(t), p(t)) \) and \( g(x(t), p(t)) = g(x(t), p_0) + \Delta g(x(t), p(t)) \) where \( p_0 \) is the nominal value of the parameter. In addition, the uncertain functions \( \Delta f(x(t), p(t)) \) and \( \Delta g(x(t), p(t)) \) are smooth and contain all the uncertainties in the system, including disturbances and uncertain nonlinear terms. Furthermore, there exist an isolated equilibrium point which is not affected by the vector \( p(t) \); i.e. \( f(0) = 0 \) and \( \Delta f(0, p(t)) = 0 \) and system has full relative degree with respect to the regulated output.

III. ROBUST FEEDBACK LINEARIZATION

In this section a robust feedback linearized method is used to linearize [1] using a technique developed in our previous work; see [4], [5]. We decompose the system [1] into nominal and uncertain parts as follows:

\[
\begin{align*}
\dot{x}(t) &= f_0(x(t), p_0) + \sum_{k=1}^{m} g_k(x(t), p_0)u_k(t) \\
&= \text{Nominal part} + \Delta f(x(t), p(t)) + \sum_{k=1}^{m} \Delta g_k(x(t), p(t))u_k(t), \\
y_i(t) &= \nu_i(x(t)), \quad i = 1, 2, \ldots, m.
\end{align*}
\]

The nominal nonlinearities in the equation [1] can be canceled using a standard feedback linearization approach [8]. Let us assume that \( r_i \) for \( i = 1, 2, \ldots, m \) is the relative degree of each regulated output, the Lie derivative of each output \( \nu_i, r_i \) number of times, for each subsystem can be written as follows (we drop the argument \( t \) from the functions for the sake of brevity):

\[
y_i^r = L_{\Delta f}^r(\nu_i) + L_{\Delta f}^r(\nu_i), \\
y_i^* = L_{\Delta f}^*(\nu_i) + L_{\Delta f}^*(\nu_i), \\
\vdots
\]

\[
y_i^{r+1} = L_{\Delta f}^{r+1}(\nu_i) + \sum_{k=1}^{m} L_{\Delta f}^{r+1}[L_{\Delta f}(\nu_i)]u_k + \sum_{k=1}^{m} L_{\Delta f}^{r+1}[L_{\Delta f}(\nu_i)]u_k.
\]

In order to write the system in a form suitable for feedback linearization, we write the \( r^\text{th} \) derivative of each output as follows:

\[
\begin{bmatrix}
y_i^{r+1} \\
y_i^{r+2} \\
\vdots \\
y_i^{r+m}
\end{bmatrix} = f_*(x, p_0) + \Delta f_*(x, p_0)u
\]

\[
+ \begin{bmatrix}
L_{\Delta f}^{r+1}(\nu_i) + \sum_{k=1}^{m} L_{\Delta f}^{r+1}[L_{\Delta f}(\nu_i)]u_k \\
L_{\Delta f}^{r+2}(\nu_i) + \sum_{k=1}^{m} L_{\Delta f}^{r+2}[L_{\Delta f}(\nu_i)]u_k \\
\vdots \\
L_{\Delta f}^{r+m}(\nu_i) + \sum_{k=1}^{m} L_{\Delta f}^{r+m}[L_{\Delta f}(\nu_i)]u_k
\end{bmatrix}.
\]

where,

\[
\begin{aligned}
f_*(x, p_0) &= [L_{\Delta f}^{r+1}(\nu_i) \ldots L_{\Delta f}^{r+m}(\nu_i)]^T, \\
g_*(x, p_0) &= \begin{bmatrix}
L_{\Delta x_1}^{r+1}(\nu_i) \\
L_{\Delta x_2}^{r+1}(\nu_i) \\
\vdots \\
L_{\Delta x_m}^{r+1}(\nu_i)
\end{bmatrix},
\end{aligned}
\]

and the Lie derivative of the functions \( \nu_i \) with respect to the vector fields \( f \) and \( g_k \) are given by

\[
L_{f\nu_i} = \frac{\partial \nu_i(x)}{\partial x} f, \quad L_{g_k\nu_i} = \frac{\partial \nu_i(x)}{\partial x} g_k.
\]

The nominal feedback linearizing control law

\[
u = -g_*(x, p_0)^{-1}f_* + \Delta g_*(x, p_0)^{-1}v
\]

partially linearizes the input-output map [4] in the presence of uncertainties as follows:

\[
y_i^{r+1} = \begin{bmatrix}
v_1 \\
\vdots \\
v_m
\end{bmatrix} + \begin{bmatrix}
\Delta W_1^{r+1}(x, u, p_0, \Delta p(t)) \\
\vdots \\
\Delta W_m^{r+1}(x, u, p_0, \Delta p(t))
\end{bmatrix}.
\]

where \( \Delta W_i^{r+1}(x, p_0, \Delta p(t)) = L_{\Delta f}^{r+1}(\nu_i) + \sum_{k=1}^{m} L_{\Delta f}^{r+1}[L_{\Delta f}(\nu_i)]u_k, \quad y_* = \left[y_1^{r+1} \ldots y_m^{r+1}\right]^T, \quad \text{and} \quad v = [v_1, \ldots, v_m]^T \) is the new control input vector. Furthermore, we define an uncertainty vector \( \Delta W_i \), which represents the uncertainty in each derivative of the \( r^\text{th} \) regulated output as

\[
\Delta W_i(x, u, p_0, \Delta p(t)) = \begin{bmatrix}
\Delta W_1^{r+1}(x, u, p_0, \Delta p(t)) \\
\Delta W_2^{r+1}(x, u, p_0, \Delta p(t)) \\
\vdots \\
\Delta W_m^{r+1}(x, u, p_0, \Delta p(t))
\end{bmatrix} = \begin{bmatrix}
L_{\Delta f}(\nu_i) \\
L_{\Delta f}(\nu_i) \\
\vdots \\
L_{\Delta f}(\nu_i) + \sum_{k=1}^{m} L_{\Delta f}^{r+1}[L_{\Delta f}(\nu_i)]u_k
\end{bmatrix}.
\]
and write $y_i$ for $i = 1, 2, \ldots, m$ as given below.

$$
\begin{bmatrix}
  y_i^1 \\
  y_i^2 \\
  \vdots \\
  y_i^m \\
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  v_i \\
\end{bmatrix}
+ \begin{bmatrix}
  \Delta W_i^1(x, u, p, \Delta p(t)) \\
  \Delta W_i^2(x, u, p, \Delta p(t)) \\
  \vdots \\
  \Delta W_i^m(x, u, p, \Delta p(t))
\end{bmatrix}.
$$

(8)

Let us define an uncertain diffeomorphism for each partially linearized system in (9) for $i = 1, \ldots, m$ as given below:

$$
\chi_i = T_i(x, p(t)) = 
\left[
\begin{array}{c}
  y_i - y_{ic} \\
  y_i - y_{ic} \\
  \vdots \\
  y_i^m - y_i^{m-1}
\end{array}
\right]^T.
$$

(9)

Using the diffeomorphism (9) and system (8) we obtain the following:

$$
\hat{\chi} = A\chi + Bu + \Delta W(x, v, p, \Delta p(t)),
$$

(10)

where $\chi(t) = [\chi_1(t), \ldots, \chi_m(t)]^T \in \mathbb{R}^m$, $v(t) = [v_1, v_2, \ldots, v_m]^T \in \mathbb{R}^m$ is the new control input vector, $\Delta W(x, v, p(t)) \in \mathbb{R}^{m \times r}$ is the transformed version of $\Delta W(x, u, p(t))$ and $\Delta W_i(t) = \Delta W_i(0, 0, \Delta p(t))$ for $i = 1, 2, \ldots, m$. Also,

$$
A = \begin{bmatrix}
  A_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & A_m
\end{bmatrix},
B = \begin{bmatrix}
  B_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & B_m
\end{bmatrix}.
$$

In order to obtain a fully linearized form for (10), here, we use a similar approach as used in [5]. In this work, we perform the linearization of $\Delta W(\chi, v, p)$ using the generalized mean value theorem [9], [10] such that no higher order uncertain terms exist after the linearization process. Since in this scheme an uncertain diffeomorphism is used, therefore, this scheme provides a bound which would be less conservative than the bound obtained in [5].

**Theorem 1:** [10] Let $\tilde{w}_i^{(j)}: \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable on $\mathbb{R}^n$ with a Lipschitz continuous gradient $\nabla \tilde{w}_i^{(j)}$. Then for given $\chi$ and $\chi(0)$ in $\mathbb{R}^n$, there is a $c = \chi + \ell(\chi - \chi(0))$ with $\ell \in [0, 1]$, such that

$$
\tilde{w}_i^{(j)}(\chi) - \tilde{w}_i^{(j)}(\chi(0)) = \nabla \tilde{w}_i^{(j)}(c_i)(\chi - \chi(0)).
$$

(11)

In order to extend Theorem 1 to the case of $\hat{w}$, we can write

$$
\hat{w}(\chi) - \hat{w}(\chi(0)) = \hat{w}'(c)(\chi - \chi(0)),
$$

where $\hat{w}'$ is the Jacobian of the function $\hat{w}(\chi)$ and $c$ is a point on the straight line between $\chi$ and $\chi(0)$ which may be different for different rows of $\hat{w}'(c)$ [10]. We may estimate the norm of $\|\hat{w}(\chi) - \hat{w}(\chi(0))\|$ as follows:

$$
\|\hat{w}(\chi) - \hat{w}(\chi(0))\| = \|\hat{w}'(c)(\chi - \chi(0))\|
\leq \|\hat{w}'(c)\| \|\chi - \chi(0)\|.
$$

(12)

The Lipschitz constant $\hat{w}'$ may be estimated by $\max_i \|\tilde{w}'(c_i)\|$ where, $\|\cdot\|$ represents the Euclidean norm.

We can apply the result of Theorem 1 to the nonlinear uncertain part of (10). Let us define a hyper rectangle

$$
\mathcal{B} = \left\{ \begin{bmatrix} \chi \\ v \end{bmatrix} : \chi_i \leq \chi_i \leq \hat{\chi}_i, \quad v_i \leq v_i \leq \hat{v}_i \right\},
$$

(13)

where $\hat{\chi}_i$ and $\hat{v}_i$ denote the lower bounds and $\chi_i$ and $v_i$ denote the upper bounds on the new states and inputs respectively. For this purpose, the Jacobian of $w_i^{(j)}(\cdot)$ is found by differentiating it with respect to $\chi$ and $v$ at an arbitrary operating point $c_i = [\chi \quad v \quad \tilde{p}(t)]$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, r_i$ where, $\chi, \tilde{v} \in \mathcal{B}$, and $\hat{p}(t) \in \Omega$. Since we assume $w_i^{(j)}(0, 0, \Delta p(t)) = 0$, $\chi(0) = 0$, and $v(0) = 0$; $w_i^{(j)}(\cdot)$ can be written as follows:

$$
\hat{w}_i^{(j)}(\chi, v, \Delta p(t)) = w_i^{(j)}(c_i) \cdot [\chi \quad v \quad \Delta p(t)]^T.
$$

(14)

And then $\Delta \hat{W}(\cdot)$ can be written as

$$
\Delta \hat{W}(\cdot) = \Phi \begin{bmatrix} \chi \\ v \\ \Delta p(t) \end{bmatrix},
$$

(15)

where,

$$
\Phi = \begin{bmatrix}
  w_1^{(c_1)}(c_{1r_1}) \\
  \vdots \\
  w_m^{(c_{mr_m})}
\end{bmatrix},
$$

(16)

Also, the bound on $\Delta \hat{W}(\cdot)$ can be obtained as follows:

$$
\tilde{p} = \max_{c_i, r_i} \|\Phi\|,
$$

where, $i = 1, 2, \ldots, m$.

A. Linearized model with an Uncertainty Representation

In (15), $c_i$ is chosen such that it gives the maximum induced matrix norm on $\Phi$. Once these bounds are obtained, we can write (15) in a suitable MIMO stochastic uncertain system form so that the minimax LQG control approach [7] can be utilized to design a tracking controller. We define, $\zeta(t) = \Delta t(\cdot)\tilde{C}_1\chi(t) + D_1\tilde{v}(t) \in \mathbb{R}^m$, and $W(t) = [\zeta(t) \tilde{w}_1^2]^T \in \mathbb{R}^{m+r_1}$, where $\zeta = \zeta(t) + \tilde{w}_1$ and $\tilde{w}_1$ is a disturbance input corresponding to $\Delta p(t)$. Also $\tilde{w}_2$ is a unity covariance noise input. We write linearized model as follows:

$$
\begin{align}
\chi(t) &= A\chi(t) + B_1\tilde{v}(t) + B_2\tilde{W}(t); \\
\tilde{z}(t) &= C_1\chi(t) + D_1\tilde{v}(t); \\
\tilde{y}(t) &= C_2\chi(t) + D_2\tilde{W}(t);
\end{align}
$$

(17)

where, $B_1 = B$, $B_2 = [B_1 E_1 0]$, $C_1 = \begin{bmatrix} 0 & \cdots & \tilde{p} & 0 \\ 0 & \cdots & \tilde{p} & 0 \end{bmatrix}$,

$$
\tilde{D}_1 = \tilde{p}I_2,
D_2 = \begin{bmatrix} 0 & I \end{bmatrix},
E_1 = \begin{bmatrix} \tilde{\Delta} p(t) \Delta p(t) \end{bmatrix}^T,
$$

C_1 = E_1^{-1}C_1, D_1 = E_1^{-1}D_1 and $\|\Delta\| \leq 1$. Note that $\Delta(t)$ is $m \times m$ which satisfies the following stochastic uncertainty constraint condition:

$$
E \int_0^\infty \|\tilde{g}(t)\|^2 \leq E \int_0^\infty \|\tilde{z}(t)\|^2,
$$

(18)

where $\|\cdot\|$ indicates the Euclidean norm.
IV. MINIMAX LQG DESIGN

The model developed in above section uses an uncertain diffeomorphism \(T(x, p(t))\) which is unknown and hence any control system design using this model must contain a robust filter which able to estimates the uncertain states. Therefore, in this section we propose a minimax LQG design approach which uses a robust Kalman filter to estimates the elements of \(T(x, p(t))\) and guarantees the stability and robust performance of the closed loop system. Here we present a summary of the minimax LQG design procedure. Interested readers are referred to [7] for more details on results and related proofs.

The minimax LQG control problem [7] involves finding a controller which minimizes the maximum value of the following cost function:

\[
J = \lim_{\tau \to \infty} \frac{1}{2T} \mathbf{E} \int_0^T (\chi(t)^T R \chi(t) + v(t)^T G v(t))dt,
\]

where \(R > 0\) and \(G > 0\). The maximum value of the cost is taken over all uncertainties satisfying the uncertainty constraint [19]. If we define a variable

\[
\Psi = \begin{bmatrix} R^{1/2} \chi \\ G^{1/2} v \end{bmatrix},
\]

the cost function [19] can be written as follows:

\[
J = \lim_{\tau \to \infty} \frac{1}{2T} \mathbf{E} \int_0^T \|\Psi\|^2 dt.
\]

The minimax optimal controller problem can now be solved by scaling a risk-sensitive control problem [7] which corresponds to a scaled \(H_{\infty}\) control problem; e.g. see [11]. In this control problem the system is described by (17) and (20) by solving a scaled risk-sensitive control problem [7] which is unknown and hence (22) and guarantees the stability and robust performance of the closed loop system. Here we present a summary of the minimax LQG design procedure. Interested readers are referred to [7] for more details on results and related proofs.

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J = \lim_{\tau \to \infty} \frac{1}{2T} \mathbf{E} \int_0^T (\chi(t)^T R \chi(t) + v(t)^T G v(t))dt,
\]

where \(R > 0\) and \(G > 0\). The maximum value of the cost is taken over all uncertainties satisfying the uncertainty constraint [19]. If we define a variable

\[
\Psi = \begin{bmatrix} R^{1/2} \chi \\ G^{1/2} v \end{bmatrix},
\]

the cost function [19] can be written as follows:

\[
J = \lim_{\tau \to \infty} \frac{1}{2T} \mathbf{E} \int_0^T \|\Psi\|^2 dt.
\]

V. AIR-BREATHEING HYPERSONIC FLIGHT VEHICLE EXAMPLE

A. Vehicle Model

In this section, we consider the same example as considered in our previous work [5], [6]. The nonlinear model for the longitudinal dynamics of an AHVF is presented in [12]:

\[
\dot{Y} = \frac{T \cos \alpha - D - g \sin \gamma}{m} \gamma = \frac{L + T \sin \alpha - g \cos \gamma}{V},
\]

\[
\dot{h} = V \sin \gamma, \quad \quad \dot{\alpha} = \dot{Q} - \dot{\gamma}, \quad \dot{\dot{Q}} = M_{yy}/I_{yy},
\]

\[
\dot{n}_i = 2\zeta_s w_m n_i - w_m n_i + N_i, \quad i = 1, 2, 3.
\]

Approximations to the forces and moments occurring in these equations are given as follows:

\[
L \approx q S C_L(\alpha, \delta_e, \delta_r, \Delta \tau_1, \Delta \tau_2),
\]

\[
D \approx q S C_D(\alpha, \delta_e, \delta_r, \Delta \tau_1, \Delta \tau_2),
\]

\[
M_{yy} \approx z \tau T + q S \delta C_M(\alpha, \delta_e, \delta_r, \Delta \tau_1, \Delta \tau_2),
\]

\[
T \approx q \delta C_{\tau_0} (\alpha, \Delta \tau_1, M_\text{s}), + C_T(\alpha, \Delta \tau_1, M_\text{s}, A_d),
\]

\[
N_i \approx q C_{N_i}[\alpha, \delta_e, \delta_r, \Delta \tau_1, \Delta \tau_2], \quad i = 1, 2, 3.
\]

The coefficients obtained by fitting curves corresponding to these quantities are given as follows; here, we remove the function arguments for the sake of brevity:

\[
C_L = C_{L_0} + C_{L_0} \delta_e + C_{L_0} \delta_r + C_{L_0} \Delta \tau_1 + C_{L_0} \Delta \tau_2 + C_{L_0} \gamma,
\]

\[
C_M = C_{M_0} + C_{M_0} \delta_e + C_{M_0} \delta_r + C_{M_0} \Delta \tau_1 + C_{M_0} \Delta \tau_2 + C_{M_0} \gamma,
\]

\[
C_D = C_{D_0} \Delta \tau_1^2 (\alpha + \Delta \tau_1) + C_{D_0} \Delta \tau_2 (\alpha + \Delta \tau_2) + C_{D_0} \Delta \tau_1 \Delta \tau_2 + C_{D_0} \alpha \Delta \tau_1 + C_{D_0} \alpha \Delta \tau_2 + C_{D_0} \alpha \gamma,
\]

\[
C_{T, \tau_0} = C_{T_0, \tau_0} + C_{T_0, \tau_0} \Delta \tau_1 + C_{T_0, \tau_0} \Delta \tau_2 + C_{T_0, \tau_0} \alpha \Delta \tau_1 + C_{T_0, \tau_0} \alpha \Delta \tau_2 + C_{T_0, \tau_0} \alpha \gamma,
\]

\[
C_{N_i} = N_{i,0} \alpha + N_{i,0} \delta_e + N_{i,0} \delta_r + N_{i,0} \Delta \tau_1 + N_{i,0} \Delta \tau_2 + N_{i,0} \gamma,
\]

where \(n = [n_1, n_2, n_3]^T\), and \(E_1 \in \mathbb{R}^{1 \times 3}\) are vectors which describe the linear relationships \(\Delta \tau_j = E_1 n\) for \(j = 1, 2\) [12]. The terms \(M_{\text{ms}}\) and \(\psi\) are defined as follows:

\[
\psi = \frac{1}{M_{\text{ms}}} = \frac{V}{M_0}.
\]

The nonlinear equations of motion in (20) have 11 states (in the vector \(x\)) in which there are five rigid body states; i.e., velocity \(V\), altitude \(h\), angle of attack \(\alpha\), flight path angle \(\gamma\), and pitch rate \(Q\) and there are three vibrational modes which are represented by generalized modal coordinates \(n_1\), \(n_2\) and \(n_3\), where \(n_1\) is a generalized force. There are four inputs (in the vector \(u\)) and they are the diffuser-area-ratio \(A_d\), the throttle setting or fuel equivalence ratio \(\phi\), the elevator deflection \(\delta_e\), and the canard deflection \(\delta_r\). For tracking control purposes we simplify the model is such a way that the scheme developed...
in Section III can be used and the simplified model closely approximates the real model (see also, [13], [14]). Note that the effect of structural flexibility is entering into the system through the forebody turn angle and aftbody flexure angle, $\Delta \gamma_1$ and $\Delta \gamma_2$ of the vehicle respectively. In the process of simplification, firstly, we remove all the flexible states $n_j$ for $j = 1, 2$ from the CMF and consider the effect of flexibility in the model by considering $\Delta \gamma_1(x)$ and $\Delta \gamma_2(x)$ as uncertain parameters. We simplify the forces and moment coefficients as follows:

$$C_{T,\phi} = C_{T,\phi}^0 + C_{T,\phi}^{M,2} \alpha M_\infty^2 + C_{T,\phi}^{M,2} \alpha M_\infty^2 + C_{T,\phi}^{D,2} \phi + \Delta C_{T,\phi}(x, u),$$

$$C_L = C_L^0 + C_L^0 + \Delta C_L(u),$$

$$C_M = C_M^0 + [C_M^0 - C_M^0 \frac{c_0^2}{C_L^0}] \delta_\alpha + C_M^0 + \Delta C_M(x, u),$$

$$C_D = C_D^0 + [C_D^0 - C_D^0 \frac{c_0^2}{C_L^0}] \delta_\phi + C_D^0 + \Delta C_D(x, u),$$

$$C_T = C_T^0 + C_T^0 \alpha + C_T^{M,2} M_\infty^2 + C_T^{D,2} \phi + \Delta C_T(x, u),$$

where $\Delta C_M(\cdot), \Delta C_D(\cdot), \Delta C_T(\cdot), \Delta C_{T,\phi}(\cdot)$ and $\Delta C_M(\cdot)$ represent the uncertainties in their corresponding functions. Furthermore, in order to obtain full relative degree for the purpose of feedback linearization, we dynamically extend the system by introducing second order actuator dynamics (adding two more states $\phi$ and $\dot{\phi}$) into the fuel equivalence ratio input as follows:

$$\dot{\phi} = -2 \xi_0 \omega_n \phi - \omega_n^2 \phi + \omega_n^2 \phi. \quad (35)$$

After this extension, the sum of the elements of vector relative degree will be equal to the order of the system $n$; i.e. $n = 7$ and thus satisfying one of the conditions for exact feedback linearization [8].

**B. Robust Feedback Linearization of the Simplified Model**

The model obtained through the above simplification is still difficult to feedback linearize due to the presence of uncertainties in the system. We approach this problem by using the technique developed in Section III. The outputs to be regulated are selected as the velocity $V$ and the altitude $h$ using two inputs, elevator deflection $\delta_\alpha$ and fuel equivalence ratio $\phi_c$. Since $\delta_\alpha$ is a function of $\delta_\phi$; i.e related to $\delta_\phi$ via an interconnect gain, we do not consider it as a separate input. Furthermore, we fix the diffuser area ratio $A_d$ to be unity. This manipulation results in a 2-input and 2-output square system. The new simplified model consists of seven rigid states and two additional integral states as follows:

$$x = \begin{bmatrix} V_I & V & h_I & h & \gamma & \alpha & \phi & Q \end{bmatrix}^T,$$

where,

$$V_I = \int_0^t (V(\tau) - V_c) d\tau, \quad h_I = \int_0^t (h(\tau) - h_c) d\tau,$$

and $V_c$ and $h_c$ are the desired command values for the velocity and altitude respectively. The uncertain parameter vector $p \in \mathbb{R}^9$ includes the vehicle inertial parameters, coupling terms and the coefficients which appear in the force and moment approximations described previously and is given as follows:

$$p = [C_L^0, C_M^0, C_T^0, \Delta C_M, \Delta C_T, \Delta C_D, \Delta C_{T,\phi}]^T \in \mathbb{R}^9.$$  \quad (37)

The model of the AHFV can be written in the form (3) as follows:

$$\dot{x}(t) = \bar{f}(x, p) + \sum_{k=1}^2 g_k(x, p)u_k + \Delta \bar{f}(x, p)$$

$$+ \sum_{k=1}^2 \Delta g_k(x, p)u_k;$$

$$y_i(t) = \nu_i(x, p), \quad i = 1, 2$$

where, $\Delta \bar{f}(x, p)$ and $\Delta g_k(x, p)$ are the uncertainty terms appearing in the corresponding functions. The control vector $u$ and output vector $y$ are defined as

$$u = [u_1, u_2]^T = [\delta_c, \phi_c]^T, \quad y = [y_1, y_2]^T = [V, h]^T.$$

We assume that $p(t) \in \Omega$, where $\Omega$ is a compact convex set that represents the admissible range of variation of $p(t)$ such that $p_0$ lies in its interior. In this study, a maximum variation of 10% of the nominal values has been considered. Thus, $\Omega = \{p(t) \in \mathbb{R}^9 \mid 0.9p_0 \leq |p_i(t)| \leq 1.1p_0 \mid i = 1, \cdots, 9\}$. It is worth mentioning the fact that there are no uncertainty terms exists in $V$ and $h$, we can write linearized input-output map for the original model (26) using (6) as follows:

$$\begin{bmatrix} \dot{V} \\ \dot{V} \\ \dot{h} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \Delta V \\ 0 & 0 & \Delta h \\ \nu_1 & 0 & \Delta V(x, u, p) \\ \nu_2 & 0 & \Delta h(x, u, p) \end{bmatrix}.$$  \quad (39)

Corresponding uncertain diffeomorphisms for each system as in (\ref{5}), which maps the new vectors $\xi$ and $\eta$ respectively to the original vector $x$ can be written as follows:

$$\xi = T_1(x, p(t), V_c), \quad \eta = T_2(x, p(t), h_c),$$

where,

$$T_1(x, p(t), V_c) = \left[ \int_0^t (V(\tau) - V_c) d\tau \ V - V_c \ V \ \dot{V} \right]^T,$$

$$T_2(x, p(t), h_c) = \left[ \int_0^t (h(\tau) - h_c) d\tau \ h - h_c \ h \ \dot{h} \right]^T,$$

and $V_c$ and $h_c$ are the desired command values for the velocity and altitude respectively. Also,

$$\chi = T(x, p(t), V_c, h_c),$$

where $\chi = [\xi_1, \xi_2, \xi_3, \xi_4, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5]^T$, and $T(x, p(t), V_c, h_c) = [T_1(x, p(t), V_c) \ T_2(x, p(t), h_c)]^T$. And finally we can rewrite (30) using the method given in

$$u = [u_1, u_2]^T = [\delta_c, \phi_c]^T, \quad y = [y_1, y_2]^T = [V, h]^T.$$
Section III as follows:

A robust nonlinear control scheme for an uncertain nonlinear system with time varying uncertainty is presented using robust feedback linearization and minimax linear quadratic Gaussian (LQG) methods. In the proposed method, a linearized uncertainty model is derived for the corresponding uncertain nonlinear system which is followed by a minimax LQG control design method. Simulation results with a large flight envelope simulation is also presented to demonstrate the effectiveness of the scheme. The results show that the proposed method works very well under parameter uncertainties and give satisfactory results. Further, investigation of the results reveals that the minimax LQG based controller works well with parameter variations of up to 10% of their nominal values for which it is designed.

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Fig. 1: Velocity and altitude reference tracking responses using the mean value approach.