Foundation ranks and supersimplicity

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Abstract

We introduce a new foundation rank based in the relation of dividing between partial types. We call $DU$ to this rank. We also introduce a new way to define the $D$ rank over formulas as a foundation rank. In this way, $SU$, $DU$ and $D$ are foundation ranks based in the relation of dividing. We study the properties and the relations between these ranks.

Next, we discuss the possible definitions of a supersimple type. This is a notion that it is not clear in the previous literature. In this paper we give solid arguments to set up a concrete definition of this notion and its properties. We also see that $DU$ characterizes supersimplicity, while $D$ not.

1 Conventions

We denote by $L$ a language and $T$ a complete theory. We denote by $\mathfrak{C}$ a monster model of $T$, that is a $\kappa$-saturated and strongly $\kappa$-homogeneous model for a cardinal $\kappa$ large enough. Models $M, N, \ldots$ are considered elementary substructures of $\mathfrak{C}$ with cardinal less than $\kappa$ and every set of parameters $A, B, \ldots$ is considered as a subset of $\mathfrak{C}$ with cardinal less than $\kappa$.

We denote by $a, b, \ldots$ tuples of elements of the monster model, possibly infinite (of length less than $\kappa$). We often use these tuples as ordinary sets regardless of their order. We often omit union symbols for sets of parameters, for example we write $ABc$ to mean $A \cup B \cup c$. Given a sequence of sets $(A_i : i \in \alpha)$ we use $A_{<i}$ and $A_{\leq i}$ to denote $\bigcup_{j<i} A_j$ and $\bigcup_{j\leq i} A_j$ respectively. We use $I$ to denote a infinite index set without order and use $O$ for a infinite lineal ordered set. Unless otherwise stated, all the types are finitary. We use $\downarrow^d$ and $\downarrow^f$ to denote the independence relations for non-dividing and non-forking respectively. By $\text{dom}(p)$ we denote the set of all parameters that appear in some formula of $p$. 
2 The DU-rank

We are going to introduce a new rank that we call \( \text{DU} \). \( \text{DU} \) is a foundation rank as it is the known rank \( \text{SU} \) (for their definitions and properties, see for example, Casanovas[4]). We will define the rank \( \text{DU} \) as the foundation rank of the relation of dividing between pairs \((p, A)\) of partial types and set of parameters satisfying \( \text{dom}(p) \subseteq A \). Similarly we will define \( \text{DU}^f \) using the relation of forking, although we will check a little later (Proposition 4.4) that both ranks are the same.

Let us begin by remembering the notion of foundation rank:

**Definition 2.1.** Let \( R \) be a binary relation defined in a set or class of mathematical objects. The foundation rank of \( R \) is the mapping \( r \) assigning to every element \( a \) of the domain of \( R \) an ordinal number or \( \infty \) according to the following rules:

1. \( r(a) \geq 0 \).
2. \( r(a) \geq \alpha + 1 \) if and only if there exists \( b \) such that \( aRb \) and \( r(b) \geq \alpha \).
3. \( r(a) \geq \alpha \) with \( \alpha \) a limit ordinal, if and only if \( r(a) \geq \beta \) for all \( \beta < \alpha \).

One defines \( r(a) \) as the supremum of all \( \alpha \) such that \( r(a) \geq \alpha \). If such supremum does not exist we set \( r(a) = \infty \).

Now, we define \( \text{DU} \) and \( \text{DU}^f \) and we will check that really \( \text{DU} \) does not depend of the set of parameters. We denote provisionally by \( \text{DU}(p, A) \) the \( \text{DU} \) rank of the pair \((p, A)\).

**Definition 2.2.** \( \text{DU} \) and \( \text{DU}^f \) are the foundation ranks of the following relations \( R_d \) and \( R_f \):

- \( (p(x), A)R_d(q(x), B) \) if and only if \( p(x) \subseteq q(x) \) and \( q \) divides over \( A \)
- \( (p(x), A)R_f(q(x), B) \) if and only if \( p(x) \subseteq q(x) \) and \( q \) forks over \( A \)

where \( p \) is a partial type over \( A \) and \( q \) is a partial type over \( B \).

**Remark 2.3.** It is immediate to verify by induction that both ranks are invariant under conjugation (automorphism).

**Lemma 2.4.** Let \( p(x) \) be a partial type dividing over \( A \). Let \( B \supseteq A \). Then, there exists \( f \in \text{Aut}(\mathcal{C}/A) \) such that \( p^f \) divides over \( B \).
Proof. Let \( p(x) = q(x, a) \) for some \( q(x, y) \) without parameters and \( a \subseteq A \). For \( \lambda \) big enough there exist a set \( \{a_i : i \in \lambda\} \) such that \( a_i \equiv_A a \) for any \( i \in \lambda \) and \( \bigcup_{i \in \lambda} q(x, a_i) \) is \( k \)-inconsistent. So, we can choose an infinite subset all having the same type over \( B \), witnessing division over \( B \).

**Proposition 2.5.** The rank \( DU \) does not depend on the set of parameters \( A \). That is, if \( p(x) \) is a partial type with parameters in \( A \cap B \) then \( DU(p, A) = D(p, B) \). So, from now on we will use the notation \( DU(p) \).

**Proof.** It suffices to prove that given \( p \) be a partial type over \( A \) and \( A' \supseteq A \) then \( DU(p, A) = DU(p, A') \). Obviously \( DU(p, A) \geq DU(p, A') \). For the proof of \( DU(p, A) \leq DU(p, A') \), we show, by induction on \( \alpha \), \( DU(p, A) \geq \alpha \) implies \( DU(p, A') \geq \alpha \).

If \( DU(p, A) \geq \alpha + 1 \) then there exists \( q \supseteq p \) over \( B \) such that \( q \) divides over \( A \) and \( DU(q, B) \geq \alpha \). By the previous lemma, there exists an \( A \)-automorphism \( f \) such that \( q^f \) divides over \( A' \). Then, \( DU(q^f, B^f) \geq \alpha \). By the induction hypothesis, \( DU(q^f, A'B^f) \geq \alpha \). As \( p \subseteq q^f \) and \( q^f \) divides over \( A' \), \( DU(p, A') \geq \alpha + 1 \).

3 Properties of the DU-rank

We begin by setting some basic properties of \( DU \). From the first property, it follows that two equivalent partial types have identical \( DU \)-rank. So, the \( DU \)-rank of a type-definable set makes sense.

**Remark 3.1.** Let \( p(x), q(x) \) be partial types.

1. If \( p \vdash q \) then \( DU(p) \leq DU(q) \).
2. \( DU(p \lor q) = \max(DU(p), DU(q)) \).
3. \( DU(p) = 0 \) if and only if \( p \) is algebraic.
4. Two type-definable sets with a definable bijection between them have the same \( DU \)-rank.

**Proof.** We Assume that \( p \) and \( q \) are over the same set of parameters \( A \).

1. We prove \( DU(p) \geq \alpha \) implies \( DU(q) \geq \alpha \) by induction on \( \alpha \). Assume \( DU(p) \geq \alpha + 1 \). Then, there exists \( p_1 \supseteq p \) such that \( p_1 \) divides over \( A \) and \( DU(p_1) \geq \alpha \). Now \( p_1 \cup q \) extends \( q \), divides over \( A \), and by the inductive hypothesis, \( DU(p_1 \cup q) \geq \alpha \). Therefore, \( DU(q) \geq \alpha + 1 \).

2. By the previous point, \( DU(p \lor q) \geq \max(DU(p), DU(q)) \). The other inequality is done by induction on \( \alpha \). Assume \( DU(p \lor q) \geq \alpha + 1 \). There
exists \( r(x) \supseteq p(x) \lor q(x) \) such that \( r \) divides over \( A \) and \( DU(r) \geq \alpha \). By inductive hypothesis, as \( r \equiv (p \cup r) \lor (q \cup r) \), \( DU(p \cup r) \geq \alpha \) or \( DU(q \land r) \geq \alpha \). So, \( DU(p) \geq \alpha + 1 \) or \( DU(q) \geq \alpha + 1 \).

3. \( DU(p) \geq 1 \) iff \( p \) has some extension dividing over \( A \) iff \( p \) is non-algebraic. 4. Let \( p(x), q(y) \) be partial types and let \( f : p(\mathcal{E}) \rightarrow q(\mathcal{E}) \) be a definable bijection. We assume \( p, q \) are over \( A \) and \( f \) is defined over \( A \). We prove by induction that \( DU(p(\mathcal{E})) \geq \alpha \) implies \( DU(q(\mathcal{E})) \geq \alpha \). If \( DU(p(\mathcal{E})) \geq \alpha + 1 \) there is some \( p'(x) \supseteq p(x) \) such that \( p' \) divides over \( A \) and \( DU(p'(\mathcal{E})) \geq \alpha \). Then \( f(p'(\mathcal{E})) \) is type-definable and, by inductive hypothesis, \( DU(f(p'(\mathcal{E}))) \geq \alpha \). It is not difficult to prove that if \( q(y) \) type-defines \( f(p'(\mathcal{E})) \) then \( q(y) \) divides over \( A \).

\[ \square \]

We are going to see some equivalences for \( DU \):

**Proposition 3.2.** Let \( p(x) \) be a partial type over a set of parameters \( A \) and \( \alpha \) an ordinal. Denote \( \mu = (2^{\aleph_1 + |A|})^+ \). The following are equivalent:

1. \( DU(p) \geq \alpha + 1 \).

2. There are \( \psi(x, y) \in L \) and a countable sequence \( (a_i : i < \omega) \) such that
   
   \( (a) \) \( (a_i : i < \omega) \) is \( A \)-indiscernible.
   
   \( (b) \) \{\( \psi(x, a_i) : i < \omega \}\} is inconsistent.
   
   \( (c) \) For every \( i < \omega \), we have \( DU(p(x) \cup \{\psi(x, a_i)\}) \geq \alpha \).

3. There are \( \psi(x, y) \in L \) and a number \( k \geq 2 \) such that for every cardinal \( \lambda \), there is a sequence \( (a_i : i < \lambda) \) such that
   
   \( (a) \) \( \{\psi(x, a_i) : i < \lambda\} \) is \( k \)-inconsistent.
   
   \( (b) \) For every \( i < \lambda \), we have \( DU(p(x) \cup \{\psi(x, a_i)\}) \geq \alpha \).

4. There are \( \psi(x, y) \in L \), a number \( k \geq 2 \) and a sequence \( (a_i : i < \mu) \) such that
   
   \( (a) \) \( \{\psi(x, a_i) : i < \mu\} \) is \( k \)-inconsistent.
   
   \( (b) \) For every \( i < \mu \), we have \( DU(p(x) \cup \{\psi(x, a_i)\}) \geq \alpha \).

5. There are a partial type \( p'(x, y) \) over \( \emptyset \) with \( |y| \leq |A| + |T| \), a number \( k \geq 2 \) and a sequence \( (a_i : i < \mu) \) such that
   
   \( (a) \) The union of any \( k \) types in \( \{p'(x, a_i) : i \in \mu\} \) is inconsistent.
   
   \( (b) \) \( p'(x, a_i) \vdash p(x) \) for each \( i < \mu \).
(c) \( DU(p'(x, a_i)) \geq \alpha \) for each \( i < \mu \).

6. There are a partial type \( p'(x, y) \) over \( \emptyset \) and a sequence \( (a_i : i < \omega) \) such that

(a) \( (a_i : i < \omega) \) is \( A \)-indiscernible.
(b) \( \bigcup_{i \in \omega} p'(x, a_i) \) is inconsistent.
(c) \( p'(x, a_i) \vdash p(x) \) for each \( i < \omega \).
(d) \( DU(p'(x, a_i)) \geq \alpha \) for each \( i < \omega \).

7. There are a partial type \( p'(x, y) \) over the same set of parameters \( A \) and a sequence \( (a_i : i < \omega) \) such that

(a) \( (a_i : i < \omega) \) is \( A \)-indiscernible.
(b) \( \bigcup_{i \in \omega} p'(x, a_i) \) is inconsistent.
(c) \( p'(x, a_i) \vdash p(x) \) for each \( i < \omega \).
(d) \( DU(p'(x, a_i)) \geq \alpha \) for each \( i < \omega \).

Proof.

1 \( \Rightarrow \) 2. If \( DU(p) \geq \alpha + 1 \) there exist \( q(x) \) extending \( p(x) \), dividing over \( A \) with \( DU(q) \geq \alpha \). Let \( \psi(x, a) \in q \) dividing over \( A \). So, there exist a sequence \( (a_i : i < \omega) \) indiscernible over \( A \) with \( a_0 = a \) such that \( \{\psi(x, a_i) : i < \omega\} \) is inconsistent. By point 1 in Remark 3.1, \( DU(p(x) \cup \psi(x, a)) \geq \alpha \). By conjugation, conditions (c) is satisfied.

2 \( \Rightarrow \) 3. We can extend the indiscernible sequence to an indiscernible sequence of length \( \lambda \). This sequence satisfies the required conditions.

3 \( \Rightarrow \) 4. Immediate.

4 \( \Rightarrow \) 5. Let \( p(x) = p(x, a) \), where \( p(x, y) \) is without parameters and \( a \) enumerates \( A \) (we assume the variables \( y \) in \( p(x, y) \) and \( \psi(x, y) \) are the same). Then \( p'(x, y) = p(x, y) \cup \{\psi(x, y)\} \) and \( (b_i = aa_i : i < \mu) \) satisfy 5.

5 \( \Rightarrow \) 6. Choose an infinite subsequence of \( (a_i : i \in \mu) \) with all elements having the same type over \( A \). Then apply the standard lemma (Lemma 7.1.1 in Tent, Ziegler[10]) to obtain a sequence \( (a'_i : i \in \omega) \) indiscernible over \( A \) and satisfying the Ehrenfeucht-Mostowski type of the subsequence. Then \( (a'_i : i \in \omega) \) satisfy the conditions of 6.

6 \( \Rightarrow \) 7. Immediate.

7 \( \Rightarrow \) 1. The closure under conjunction of \( p'(x, a_0) \) divides over \( A \), extends \( p(x) \) and has \( DU \)-rank at least \( \alpha \). Therefore \( DU(p) \geq \alpha + 1. \)

\[ \square \]
Now we want to see that $DU$ may be characterized by the existence of certain trees of formula with certain properties.

**Definition 3.3.** We define recursively a rooted tree $T_{\alpha,\lambda}$ for every ordinal $\alpha$ and cardinal $\lambda$:

1. $T_{0,\lambda}$ is a tree with a unique node.

2. For an ordinal $\alpha + 1$, we take $\lambda$ disjoint copies of $T_{\alpha,\lambda}$ and add a new node related with all nodes, that is, a new root.

3. For a limit ordinal $\alpha$, we take a disjoint union of all trees $\{T_{\beta,\lambda} : \beta \in \alpha\}$ and add a new node related with all nodes, that is, a new root. The node added in this step will be called a limit node of the tree.

**Remark 3.4.** It is immediate that every $T_{\alpha,\lambda}$ is a tree. That is, the binary relation $R$ defined in the tree is a strict partial order (irreflexive and transitive) and for each node $t$, the set $\{s : sRt\}$ is well-ordered.

We use standard tree terminology: we say that a node $s$ is a child of a node $r$ (or $r$ is the parent of $s$) if $rRs$ and there are no nodes $t$ with $sRt$ and $tRs$. The root of the tree will be the minimum. An end-node is a node without children. We will denote by $F_{\alpha,\lambda}$ the set of parent nodes in $T_{\alpha,\lambda}$ which are not limit. $P_{\alpha,\lambda}$ will denote the set of nodes of $T_{\alpha,\lambda}$ which are a child of a non-limit.

Next lemma characterizes the value of $DU$ using the trees defined above. Compare to the definition of the rank $DD$ in Cárdenas, Farré [2].

**Lemma 3.5.** Let $p(x)$ be a partial type over $A$ in $T$, $\alpha$ and ordinal and $\mu = (2^{|T|+|A|})^+$. The following are equivalent:

1. $DU(p) \geq \alpha$.

2. There is a sequence of formulas $(\varphi_s(x, y_n) : s \in F_{\alpha,\mu})$, a sequence of numbers $(k_s : s \in F_{\alpha,\mu})$ and a sequence of parameters $(a_s : s \in P_{\alpha,\mu})$ such that

   (a) For every $s \in F_{\alpha,\mu}$, the set of formulas $\{\varphi_s(x, a_t) : t \text{ is a child of } s\}$ is $k_s$-inconsistent.

   (b) For every end-node $s$, the set of formulas $p(x) \cup \{\varphi_s(x, a_r) : tRs, r \text{ a child of } t\}$ is consistent.

**Proof.** It is easily proved by induction using the equivalence 4 in Proposition 3.2. \qed
Proposition 3.6. Let \( p(x) \) be a partial type over \( A \). Then, there exists a set of parameters \( B \subseteq A \) such that \( |B| \leq |T|^{DU(p)} \) and \( DU(p \mid B) = DU(p) \).

Proof. We may assume \( DU(p) < \infty \) and fix \( \alpha = DU(p) + 1 \). For every partial type \( q(x) \) over \( A \) consider the type \( \Sigma_{q,\overline{\varphi},\overline{k}} \) in the variables \( (y_s : s \in P_{\alpha,\mu}) \). Here \( \overline{\varphi} = (\varphi_s(x, y_n) : s \in F_{\alpha,\mu}) \) and \( \overline{k} = (k_s : s \in F_{\alpha,\mu}) \) denote sequences of formulas and numbers and \( \mu = (2^{|T|} + |A|)^+ \). That is, \( DD(q) < \alpha \) if and only if for every \( \varphi \) and \( k \), \( \Sigma_{q,\overline{\varphi},\overline{k}} \) is inconsistent.

As \( DD(p) < \alpha \), for every \( \varphi \) and \( k \), by compactness, there is some finite \( A_{\overline{\varphi},\overline{k}} \subseteq A \) such that \( \Sigma_{p|A_{\overline{\varphi},\overline{k}}} \) is inconsistent. Taking \( B = \bigcup A_{\overline{\varphi},\overline{k}} \), we get \( \Sigma_{p|B,\overline{\varphi},\overline{k}} \) is inconsistent for every \( \varphi, \overline{k} \). We are using that \( p \subseteq q \) implies \( \Sigma_{\varphi,\overline{k}} \subseteq \Sigma_{q,\overline{\varphi},\overline{k}} \). \( \square \)

Proposition 3.7. Let \( p(x) \) be a partial type over \( A \) such that \( DU(p) = \infty \). Then there exists a partial type \( q(x) \) such that \( p \subseteq q \), \( q \) divides over \( A \) and \( DU(q) = \infty \).

Proof. For each \( \alpha \), there is a \( p_\alpha \) such that \( p_\alpha \vdash \varphi_\alpha \) with \( \varphi_\alpha \) dividing over \( A \), \( p \subseteq p_\alpha \) and \( DU(p_\alpha) \geq \alpha \). We may assume all formulas \( \varphi_\alpha \) are conjugate over \( A \). This is true because there are only boundedly many formulas and boundedly many types over \( A \).

By conjugation over \( A \) we may assume all \( p_\alpha \) contain a formula that divides over \( A \). So, \( q = \bigcap p_\alpha \) is a partial type dividing over \( A \). Then, \( q(x) \) is a dividing extension of \( p(x) \) with \( DU(q) = \infty \). \( \square \)

4 Relation between \( DU \) and other ranks

The \( SU \)-rank has traditionally been defined as the foundation rank of the forking relation. In the same way, we can define the rank \( SU^d \) using dividing instead of forking. Namely, \( SU^d \) will be the foundation rank of the relation of dividing extension between complete types. To avoid confusion we will write \( SU^f \) to refer to the ordinary rank \( SU \) for forking. Obviously \( SU^f(p) \geq SU^d(p) \).

Now, we are going to see that we can define the known \( D \)-rank (for their definitions and properties, see for example, Casanovas[4]) for formulas, as a foundation rank. More precisely, as the foundation rank of the relation of dividing between pairs \( (\varphi, A) \) of formulas and set of parameters satisfying \( dom(\varphi) \subseteq A \). Using Lemma 2.4 one can easily show that \( D \) does not depend on the set of parameters. We can define similarly \( D^f \) using the forking
relation instead of dividing. Later, we will check (Proposition 4.4) that both ranks are the same and therefore \( D_f \) does no depend on the set of parameters.

**Definition 4.1.** \( D, D_f, S^d \) and \( S^f \) are the foundation ranks of the following relations \( R_{dd}, R_{df}, R_{sd} \) and \( R_{sf} \):

- \((\varphi(x), A)R_{dd}(\psi(x), B)\) if and only if \( \models \psi \rightarrow \varphi \) and \( \psi \) divides over \( A \)
- \((\varphi(x), A)R_{df}(\psi(x), B)\) if and only if \( \models \psi \rightarrow \varphi \) and \( \psi \) forks over \( A \)
- \( p(x)R_{sd}q(x) \) if and only if \( q \) is a dividing extension of \( p \)
- \( p(x)R_{sf}q(x) \) if and only if \( q \) is a forking extension of \( p \)

where \( \varphi \) is a formula over \( A \), \( \psi \) is a formula over \( B \) and \( p \) and \( q \) are complete types.

It is not difficult to verify that this definition of \( D \) for formulas coincides with the traditional definition. For indeed, we can proceed as in Proposition 3.2.

Next remark states well known properties of \( S^f \) (and therefore, of \( S \) in the context of simple theories where \( S^d \) and \( S^f \) coincide). We can check that \( S^d \) satisfy them in any theory. The proofs are similar to the proofs for \( DU \) in Remark 3.1.

**Remark 4.2.** Let \( p(x) \in S(A) \) and \( q(x) \in S(B) \). The rank \( S^d \) satisfies:

1. If \( q \subseteq p \) then \( S^d(p) \leq S^d(q) \).
2. For every \( r \) completion of \( p \lor q \), \( S^d(r) \leq \max(S^d(p), S^d(q)) \).
3. \( S^d(p) = 0 \) if and only if \( p \) is algebraic.

It is easy to verify that \( D \) and \( DU \) coincide for formulas:

**Lemma 4.3.** For every formula \( \varphi(x) \), we have \( D(\varphi) = DU(\varphi) \).

**Proof.** We only need to prove \( DU(\varphi) \leq D(\varphi) \). A proof by induction reduces the problem to show \( DU(\varphi) \geq \alpha + 1 \) implies \( D(\varphi) \geq \alpha + 1 \). Assume \( \varphi \) is over \( A \) and \( DU(\varphi) \geq \alpha + 1 \). Then, there exists a partial type \( q \) such that \( \varphi \in q \), \( q \) divides over \( A \) and \( DU(q) \geq \alpha \). Assuming \( q \) closed under conjunction, there exists a formula \( \psi \in q \) such that \( \psi \) divides over \( A \). Obviously \( \varphi \land \psi \) also divides over \( A \) and \( DU(\varphi \land \psi) \geq \alpha \). By the induction hypothesis, \( D(\varphi \land \psi) \geq \alpha \). So, \( D(\varphi) \geq \alpha + 1 \). \( \square \)
A variation of the proof above also shows $D^f = DU^f$ for formulas. Now, we are going to prove that $D^f$ and $DU^f$ are the same as $D$ and $DU$ respectively (and therefore do not depend on the set of parameters). So, from now on, we will use only $D$ and $DU$.

**Proposition 4.4.** Let $p$ a partial type and $\varphi$ a formula both over $A$. Then,

1. $DU(p) = DU^f(p, A)$,
2. $D(\varphi) = D^f(\varphi, A)$.

**Proof.** To prove 1 it suffices to show that $DU(p) \geq DU^f(p)$. A proof by induction reduces to prove the following: $DU^f(p) \geq \alpha + 1$ implies $DU(p) \geq \alpha + 1$, assuming it is true for $\alpha$. If $DU^f(p) \geq \alpha + 1$, there exists $q \supseteq p$ such that $q$ forks over $A$ and $DU^f(q) \geq \alpha$ and by the induction hypothesis, $DU(q) \geq \alpha$. Then, there exists $\{q_i : i \in n\}$ such that $q \equiv \bigvee_i q_i$ with each $q_i$ extending $q$ and dividing over $A$. Then, $DU(q) = \max\{DU(q_i) : i \in n\}$. So, for some $q_i$, $DU(q_i) \geq \alpha$ and therefore, $DU(p) \geq \alpha + 1$.

2 follows from 1, since $D^f = DU^f$ for formulas. $\square$

From that proposition is immediate deduce that $SU^d(p) \leq SU^f(p) \leq DU(p)$ for any complete type $p$.

$D$ is extended in a standard way to partial types $p$ as follows:

$$D(p) = \min\{D(\varphi) : \varphi \text{ is a finite conjunction of formulas in } p\}$$

As $D = DU$ for formulas, it is obvious that $DU(p) \leq D(p)$ for a partial type $p$, but in some cases they are not equal. In the next example we even see how $D$ can be $\infty$ while $DU$ not.

**Example 4.5.** Let the language contain an infinite set of disjoint unary predicates $\{Q_i : i \in \omega\}$ and binary relations $\{\leq_i : i \in \omega\}$. Each $\leq_i$ being a dense linear order without endpoints defined in $Q_i$. Let $p$ denote $\{-Q_i(x) : i \in \omega\}$. Then $DU(p) = 1$ while $D(p) = \infty$.

**Proof.** As $p$ is not algebraic, $DU(p) \geq 1$. Suppose $DU(p) \geq 2$. Then, by the equivalence 4 in Proposition [3.2], there exist $\varphi(x, y)$ and $(a_i : i \in \mu)$ such that for each $i \in \mu$, $DU(p \cup \{\varphi(x, a_i)\}) \geq 1$ and $\{\varphi(x, a_i) : i \in \mu\}$ is $k$-inconsistent for some $k$. Here $\mu = (2^{[T]} + [A])^+$. Any two realizations of $p$ different from $a_i$ have the same type over $a_i$, so any realization of $p$ except maybe $a_i$ satisfy $\varphi(x, a_i)$. This shows that $\{\varphi(x, a_i) : i \in \mu\}$ is realized by every realization of $p$, except maybe $\{a_i : i \in \mu\}$ and therefore $\{\varphi(x, a_i) : i \in \omega\}$ is not $k$-inconsistent. This shows $DU(p) = 1$. 

For each fine subset \( S \subseteq I \), we will check that \( D(\bigwedge_{i \in S} \neg Q_i) = \infty \), so \( D(p) = \infty \). Fix \( j \in \omega - S \) and choose \( \{a_i, b_i : i \in \omega\} \) in \( Q_j \) such that
\[
a_0 < a_1 < \ldots < a_n < \ldots < b_n < \ldots < b_1 < b_0
\]
Then, the formula \( a_n < x < b_n \) divides over \( \{a_0b_0, \ldots, a_{n-1}b_{n-1}\} \), so there is an infinite dividing sequence of formulas and therefore (see 14.3.3 Casanovas\[4\]) \( D(\bigwedge_{i \in S} \neg Q_i) = \infty \). □

In some cases \( DU \) and \( SU^d \) coincide for complete types:

**Remark 4.6.** Assume \( DU \) has extension, i.e. for every partial type \( p(x) \) over \( A \), there exists \( q(x) \in S(A) \) such that \( p \subseteq q \) and \( DU(p) = DU(q) \). Then for every complete type \( p \) \( DU(p) = SU^d(p) \).

**Proof.** We prove that \( SU^d(p) \geq DU(p) \) by induction on \( \alpha \). Let \( p \in S(A) \) such that \( DU(p) \geq \alpha + 1 \). Then, there exists \( q \) over \( B \) such that \( p \subseteq q \), \( q \) divides over \( A \) and \( DU(q) \geq \alpha \). By the extension property, there exists \( q' \in S(B) \) such that \( q \subseteq q' \) and \( DU(q') \geq \alpha \). By the induction hypothesis, \( SU^d(q') \geq \alpha \) and therefore \( SU^d(p) \geq \alpha + 1 \). □

In addition to the other mentioned ranks, we are going also to explore the relations with the \( DD \)-rank defined in Cárdenas, Farré\[2\]. In that paper we can found a definition of \( DD \) from Shelah trees and several equivalences. Here, we define \( DD \) by dividing chains of complete types, which is the equivalence that we are going to use.

**Definition 4.7.** Let \( p \) be a partial type over \( A \). A **dividing chain of partial types of depth** \( \alpha \) **in** \( p \) is a sequence of partial types \( (p_i(x) : i \in \alpha) \) and a sequence of sets of parameters \( \{A_i : i \in \alpha\} \), each \( p_i \) a partial type over \( A_i \), \( p \subseteq p_0 \), \( A \subseteq A_0 \), \( p_0 \) divides over \( A \) and for every \( 0 < i < \alpha \), \( p_{<i} \subseteq p_i \), \( A_{<i} \subseteq A_i \) and \( p_i \) divides over \( A_{<i} \).

If there is not such dividing chain for \( p(x) \) we set \( DD(p) = 0_+ \). Let \( \beta \) the supremum of all possible depths of dividing chains of partial types in \( p \) If this supremum does not exist we write \( DD(p) = \infty \). Otherwise, if \( \beta \) is attained we put \( DD(p) = \beta_+ \) and \( DD(p) = \beta_- \) if it is not attained. We call \( DD(p) \) the **Dividing Depth of** \( p \).

In Cárdenas, Farré\[2\] is shown that \( DD \) does not depend of the set of parameters.

Ranks \( DU \), \( SU^d \) and \( SU^f \) take the value \( \infty \) at the same time:

**Proposition 4.8.** Let \( p \) be a partial type. Then, \( DD(p) \geq \omega_+ \) if and only if \( DU(p) = \infty \). Moreover, if \( p \) is complete, then \( SU^d(p) = \infty \), \( SU^f(p) = \infty \) are also equivalent to \( DU(p) = \infty \).
**Proof.** The first equivalence has a standard proof based in properties of foundations ranks (see for example Remark 13.6 in Casanovas[4]) and Proposition 3.7.

For the second equivalence, assume $DU(p) = \infty$. By Lemma 3.7 we can build a dividing chain of partial types $(p_i : i \in \omega)$ and sets of parameters $(A_i : i \in \omega)$ such that $p = p_0$, $A = A_0$ and for every $i \in \omega$, $p_i \subseteq p_{i+1}$, $A_i \subseteq A_{i+1}$, $p_{i+1}$ divides over $A_i$. Let $a = \bigcup_{i \in \alpha} p_i$. Then $(tp(a/A_i) : i \in \omega)$ is a dividing chain of complete types. It is easy to check by induction over $\alpha$ that for every $i \in \omega$, $SU^d(a/A_i) \geq \alpha$. □

In the following two results, in order to compare $DD$ with other ranks, we will suppress the subscripts in the values of $DD$.

**Proposition 4.9.** Let $p$ a partial type with $DD(p)$ finite, then $DD(p) = DU(p)$. Moreover, if $p$ is complete, then $DD(p) = SU^d(p) = SU^f(p) = DU(p)$.

**Proof.** If $DU(p) \geq n$ we can build a dividing chain of partial types of length $n$, $(p_i : i < n)$ and $(A_i : i < n)$. So, $DD(p) \geq n$. The converse is immediate. If moreover $p$ is complete, let $a = \bigcup_{i < n} p_i$. Then, the sequence $(tp(a/A_i) : i < n)$ forms a dividing chain of complete types witnessing $SU^d(p) \geq n$. □

The inequality $SU^f(p) \leq D(p)$ is well known (see Kim[8]) but a standard proof needs simplicity of the theory and does not work in full generality. Actually, it is true in general:

**Remark 4.10.** Let $p$ be a partial type. Then, $DD(p) \leq DU(p) \leq D(p)$. Moreover if $p$ is a complete type, $DD(p) \leq SU^d(p) \leq SU^f(p) \leq DU(p) \leq D(p)$.

**Proof.** It is immediate by Proposition 4.4 that $SU^d(p) \leq SU^f(p) \leq DU(p) \leq D(p)$. And by Proposition 4.8 and Proposition 4.9 we obtain $DD(p) \leq SU^d(p)$ and $DD(p) \leq DU(p)$. □

We are going to use a result about $DD$ in Cárdenas, Farré[2] to prove that $DU$, $SU^d$ and $SU^f$ have a bounded number of different values. We take next lemma from Proposition 3.10 in Cárdenas, Farré[2]:

**Lemma 4.11.** Let $p(x)$ be a partial type over $A$. Then, there exists a set of parameters $B \subseteq A$ such that $|B| \leq |T|^{DD(p)}$ and $DD(p \upharpoonright B) = DD(p)$.

**Proposition 4.12.** There is some ordinal $\alpha$ such that $DU(p) \geq \alpha$ implies $DU(p) = \infty$. 

Proof. Observe that, as $DU$ takes the same values over conjugate sets of parameters and there are boundedly many non-conjugate sets of parameters of size $\leq |T|^\aleph_0$, the $DU$-values on types over a set of parameters of size $\leq |T|^\aleph_0$ is upper bounded. By Proposition 4.8 and the previous lemma, for any partial type $p$ over $A$ with $DU(p) < \infty$ there is some $B \subseteq A$ with $|B| \leq |T|^\aleph_0$ and $DU(p) \leq DU(p \upharpoonright B) < \infty$. Therefore the set of non-infinite values taken by $DU$ is bounded.

Remark 4.13. The same $\alpha$ as in Proposition 4.12 satisfies that $SU^d(p) \geq \alpha$ implies $SU^d(p) = \infty$ and $SU^f(p) \geq \alpha$ implies $SU^f(p) = \infty$.

We have seen so far that $D = D^f$ and $DU^d = DU^f$, but we do not know if, in general, $SU^d = SU^f$ or $SU^f = DU$. We know that the three ranks are equal for finite values and the value $\infty$, but in all intermediate cases, when $DD(p) = \omega_-$, we do not have the answer. So, we have these two open questions:

Question 4.14. Is there a complete type $p$ such that $SU^d(p) < SU^f(p)$?

Question 4.15. Is there a complete type $p$ such that $SU^f(p) < DU(p)$?

In Cárdenas, Farré [3] we prove that in an $\text{NTP}_2$ theory, for any stable complete type $p$, $SU^d(p) = SU^f(p)$. We also prove that if $SU^d$ has extension then $SU^d = SU^f$.

5 Supersimple types

The following are two equivalent definitions of a simple type (see in Hart, Kim, Pillay [7] and Chernikov [6]).

Definition 5.1. Let $p(x)$ be a partial type over $A$. $p$ is simple if and only if one of the following two equivalent conditions are satisfied:

1. for every $B \supseteq A$ and every realization $a$ of $p(x)$, there is some $B_0 \subseteq B$ with $|B_0| < |T|^+$ such that $a \downarrow^d_{B_0} B$.

2. for every $B \supseteq A$ and every realization $a$ of $p(x)$, there is some $B_0 \subseteq B$ with $|B_0| < |T|^+$ such that $a \downarrow^d_{AB_0} B$.

From this, one might think in defining a supersimple type in two different ways, replacing in both definitions the bound $|T|^+$ by $\aleph_0$. In fact, in Hart, Kim, Pillay [7], they suggest to define a supersimple type through the first alternative, although they do not develop the implications of this possibility.
We will see through the Example 5.10 that these possible definitions are not equivalent and that the first one depends on the set of parameters while by Corollary 5.6 the second not. In addition, in this example we show also a superstable complete type not satisfying the first possible definition of supersimple. All of that indicate us that the correct way of defining a supersimple type will be the second:

**Definition 5.2.** Let $p(x)$ be a partial type over $A$. $p$ is **supersimple** if and only if for every $B \supseteq A$ and every realization $a$ of $p(x)$, there exists a finite set $B_0 \subseteq B$ with $a \downarrow_{AB_0} B$.

**Remark 5.3.** It is obvious that the discarded definition of supersimple type implies our definition of supersimple type.

The notion of supersimple type satisfies the following expected properties:

**Remark 5.4.** The following are satisfied:

1. If $p(x)$ is supersimple and $p(x) \subseteq q(x)$, then $q(x)$ is supersimple.
2. If $p(x, y)$ is supersimple, then the type $\exists y p(x, y)$ is supersimple.
3. $\text{tp}(ab/A)$ is supersimple if and only if $\text{tp}(a/A)$ and $\text{tp}(b/Aa)$ are supersimple. More generally, $\text{tp}(\langle a_i : i \in n \rangle/A)$ is supersimple if and only if $\text{tp}(\langle a_i : i \in n \rangle/A)$ is supersimple.
4. Assume that $x$ and $y$ are disjoint. $p(x)$ and $q(y)$ are supersimple if and only if $p(x) \cup q(y)$ is supersimple.
5. Let $p(x)$ be over $A$. Then $p$ is supersimple if and only if every $q(x) \in S(A)$ extending $p(x)$ is supersimple.
6. $p_1(x), p_2(x)$ are supersimple if and only if $p_1 \lor p_2$ is supersimple.

**Proof.** 1 is obvious assuming $p$ and $q$ are over the same set of parameters.

2. Let $a \models \exists y p(x, y)$, then $ab \models p(x, y)$ for some $b$. Let $B \supseteq A$. As $p(x, y)$ is supersimple, there exists a finite $B_0 \subseteq B$ with $ab \downarrow_{AB_0} B$. So, $a \downarrow_{AB_0} B$.

3$\Rightarrow$). $\text{tp}(a/A) = \exists y \text{tp}(ab/A)$ and $\text{tp}(b/Aa) = \exists x \text{tp}(ab/A)$ are supersimple by 2 and therefore by 1 $\text{tp}(b/Aa)$ is also supersimple.

3$\Leftarrow$). Let $B \supseteq A$ and $a'b' \equiv_A ab$. By the first condition, as $a' \equiv_A a$, there exists $B_1 \subseteq B$ finite such that $a' \downarrow_{AB_1} B$. As $\text{tp}(b'/Aa')$ is supersimple, there exists $B_2 \subseteq B$ finite such that $b' \downarrow_{Aa'B_2} B a'$. Now, taking $B_0 = B_1B_2$ we have $a' \downarrow_{AB_0} B$ and $b' \downarrow_{Aa'B_0} B$. By left transitivity we obtain $a'b' \downarrow_{AB_0} B$. 
4. Assume \( p(x) \) and \( q(y) \) are supersimple over \( A \) and let \( B \supseteq A \) and \( ab \models p(x) \cup q(y) \). Then \( tp(a/A) \) and \( tp(b/Aa) \) are supersimple by 1. By 3, \( tp(ab/A) \) is supersimple. So, there is some finite \( B_0 \subseteq B \) with \( ab \downarrow^d_{AB_0} B \).

5 \( \Rightarrow \) is trivial by 1.

5 \( \Leftarrow \). Let \( B \supseteq A \) and \( a \models p \). As \( tp(a/A) \) is supersimple, there exists \( B_0 \subseteq B \) finite such that \( a \downarrow^d_{AB_0} B \).

6 follows from 5, since any completion of \( p \lor q \) is either a completion of \( p \) or a completion of \( q \). \( \square \)

We remember the following result for \( DD \) from Cárdenas, Farré\[2\]:

Lemma 5.5. Let \( p \) be a partial type over a set of parameters \( A \). Let \( \kappa \) be any regular cardinal number. The following are equivalent:

1. \( DD(p) < \kappa_+ \).
2. For every \( B \supseteq A \) and \( a \models p \), there exists a set \( B_0 \subseteq B \) with \( |B_0| < \kappa \) such that \( a \downarrow^d_{AB_0} B \).

Corollary 5.6. The definition of supersimple does not depend on the set of parameters. Moreover, the following are equivalent for a partial type \( p \) over \( A \): 1. \( p \) is supersimple, 2. \( DD(p) < \omega_+ \), 3. \( DU(p) < \infty \), 4. For every completion \( q \in S(A) \) of \( p \), \( SU^d(q) < \infty \). 5. For every completion \( q \in S(A) \) of \( p \), \( SU^f(q) < \infty \).

Proof. By previous Lemma, the fact that \( DD(p) \) does not depend on the set of parameters, Remark 4.10 and Proposition 4.8. \( \square \)

We can obtain other two equivalences of the notion of supersimple type replacing dividing by forking. We can define a forking chain of partial types in \( p \) in a similar way to dividing chain (see Cárdenas, Farré\[2\]).

Remark 5.7. Let \( p(x) \) be a partial type over \( A \). \( p \) is supersimple if and only it verifies the following equivalent conditions:

1. For every \( B \supseteq A \) and every realization \( a \) of \( p(x) \), there exists a finite set \( B_0 \subseteq B \) with \( a \downarrow^f_{AB_0} B \).
2. There is not a forking chain of partial types in \( p \) of depth \( \omega \).

Proof. Arguing as in Proposition 4.8 it follows that 2 is equivalent to the fact that \( DU^f(p) < \infty \). Therefore, 2 is equivalent to supersimple. The equivalence between 1 and 2 is similar to the proof of lemma 5.5 changing dividing by forking. \( \square \)
We remember the definition of the Lascar rank $U$ and the definition of a superstable type of Poizat:

**Definition 5.8.** The $U$-rank for a complete type $p(x) \in S(A)$ is defined as follows:

1. $U(p) \geq 0$.
2. $U(p) \geq \alpha + 1$ if and only if for each cardinal number $\lambda$ there is a set $B \supseteq A$ and there are at least $\lambda$ many types $q(x) \in S(B)$ extending $p$ and such that $U(q) \geq \alpha$.
3. $U(p) \geq \alpha$ with $\alpha$ a limit ordinal, if and only if $U(p) \geq \beta$ for all $\beta < \alpha$.

$U(p)$ is the supremum of all $\alpha$ such that $U(p) \geq \alpha$. If such supremum does not exist we set $U(p) = \infty$.

**Definition 5.9.** Let $p$ be a complete type. $p$ is **superstable** if and only if $U(p) < \infty$.

In Cárdenas, Farré we prove that a complete type is stable and supersimple if and only if it is superstable.

**Example 5.10.** There is an example of a superstable and supersimple type $p \in S(A)$ not satisfying the discarded definition of supersimple. However, for some $b$, $p$ considered over $A b$ satisfies the alternative definition, so the discarded definition depend on the set of parameters.

**Proof.** Consider the theory of infinitely many refining equivalence relations, which is a stable non supersimple theory with quantifier elimination. The language consists in $\omega$ equivalence relations $\{E_i : i \in \omega\}$, $E_0$ has infinite many classes, $E_{i+1}$ refines $E_i$ and each $E_i$-class is partitioned into infinitely many $E_{i+1}$-classes. Given $d$ and $C$, denote $\rho(d/C) = \infty$ if $d \in C$, $\rho(d/C) = \sup\{n : d \mathbin{E_n} c \text{ for some } c \in C\}$ otherwise. Here we consider $\omega < \infty$. One can verify that $tp(d/BC)$ divides over $C$ if and only if $\rho(d/C) < \rho(d/BC)$. So,

$$D \downarrow B \iff \text{for every } d \in D : \rho(d/C) = \rho(d/BC)$$

Now we choose and fix $a$ and $b$ such that $a E_i b$ for every $i \in \omega$ and take $A = \{a_i : i \in \omega\}$ such that $a E_i a_i$ and $a E_{i+1} a_i$ for every $i \in \omega$. The example is $p = tp(a/A)$.

$p$ is supersimple: Given $B \supseteq A$ and $a' \models p$, $\rho(a'/A) = \omega$ and $\rho(a'/B) \geq \omega$. So, taking $B_0 = \{a'\} \cap B$, we have $a' \downarrow_{A B_0} B$. 
Proof. We will prove that \( p \) is simple, using symmetry (Proposition 5.12). Assume then \( q \equiv p \). We have \( \rho(a/B_0) < \omega \), So, \( a \not\models B_0 \). But given \( B \supseteq A \) and \( a' \models p \), we have \( \rho(a'/b) = \omega \) and \( \rho(a'/B) \geq \omega \). So, taking \( B_0 = \{a'b\} \cap B \), we have \( a' \equiv B_0 \).

Although for a particular type the discarded definition is not equivalent to supersimplicity, for a fixed theory the fact that all types satisfy one of the definitions is equivalent to all types satisfy the other.

Remark 5.11. The following are equivalent:
1: \( T \) is supersimple.
2: \( \{x = x\} \) is supersimple.
3: Every complete type is supersimple.
4: Every partial type is supersimple.
5: For every \( p \in S(A) \), there exists a finite subset \( A_0 \subseteq A \) such that \( p \) does not divide over \( A_0 \).
6: For every \( p \in S(A) \), every \( B \supseteq A \) and every realization \( a \) of \( p \), there exists a finite subset \( B_0 \subseteq B \) such that \( a \equiv B_0 \).

Proof. The equivalence between 3 and 4 follows from remark 5.4. The other are standard, see 13.1, and 13.4 in Casanovas[4].

Now we improve slightly for \( SU^d \) and \( SU^f \) the known fact that in simple theories \( SU \) is preserved by non-forking extensions. We recall that a theory is called Extensible if forking has existence, that is every complete type does not fork over its parameter set. For instance, simple theories are extensible.

Proposition 5.12. In a extensible theory, let \( p(x) \in S(A) \) and \( q(x) \in S(B) \) be such that \( p(x) \subseteq q(x) \) and \( tp(B/A) \) is simple. If \( q \) does not fork over \( A \) then \( SU^d(p) \leq SU^d(q) \) and \( SU^f(q) = SU^f(p) \).

Proof. We will prove that \( SU^d(p) \leq SU^d(q) \). The proof is similar for \( SU^f \). If \( SU^d(q) = \infty \) the result is immediate, so we can assume without loss of generality that \( q \) is supersimple. We use induction on \( \alpha \) to prove that if \( SU^d(p) \geq \alpha \) then \( SU^d(q) \geq \alpha \). This is clear for \( \alpha = 0 \) or limit ordinal.

Assume \( SU^d(p) \geq \alpha + 1 \). Now, by the definition of \( SU^d \), there is a dividing extension \( p_1 \in S(C) \) of \( p \) such that \( SU^d(p_1) \geq \alpha \).

Let \( d \models q \) and \( d' \models p_1 \). As \( d' \equiv_A d \), there exists \( C' \) such that \( d'C \equiv_A d'C' \). Using \( T \) extensible we can choose \( C'' \) such that \( C'' \equiv_{Ad} C' \) and \( C'' \equiv_{Ad} B \).

As \( d \equiv_{Ad} B \) and \( C'' \equiv_{Ad} B \), by left transitivity, \( C''d \equiv_{A} B \). As \( tp(B/A) \) is simple, using symmetry (Proposition 7.3 in Casanovas[5]), \( B \equiv_{A} C''d \) and
therefore $B \downarrow_{C''} d$. Since $tp(d/B)$ and $tp(B/A)$ are simple, $tp(dB/A)$ is simple and therefore $tp(d/C'')$ is simple. Using symmetry again $d \downarrow_{C''} B$. Since $tp(B/A)$ is simple $tp(B/C'')$ is simple and therefore $tp(C''B/C'')$ is also simple. By induction hypothesis, $SU^d(d/C''B) \geq \alpha$. By $B \downarrow_{A}^d C''$ we get $B \downarrow_{A}^d C''$. With $d \downarrow_{A}^d C''$, we obtain $d \downarrow_{B}^d C''$ and therefore $tp(d/C''B)$ divides over $B$. So, finally $SU^d(q) \geq \alpha + 1$.

**Corollary 5.13.** In an extensible theory, let $p$ be a complete type over $A$ and $q$ be a partial type over $B$ such that $q$ is a non forking extension of $p$ and $tp(B/A)$ is simple. If $q$ is supersimple then $p$ is supersimple.

By Proposition 3.9 in Cárdenas, Farré[2], we have a similar corollary using $DD$ with somewhat different hypotheses:

**Corollary 5.14.** Let $p$ be a complete type over $A$ and $q$ be a partial type over $B$ such that $q$ is a non forking extension of $p$ and $tp(B/A)$ is simple and co-simple. If $q$ is supersimple then $p$ is supersimple.

**Proof.** If $q$ is supersimple, any completion $\bar{q}$ of $q$ is supersimple and $DD(\bar{q}) \leq \omega_-$. By Proposition 3.9 in Cárdenas, Farré[2], $DD(p) \leq \omega_-$ and therefore is supersimple.

It is immediate to conclude from Proposition 3.11 in Cárdenas, Farré[2] the following:

**Corollary 5.15.** Let $p(x)$ be a non-supersimple partial type over $A$. Then,

1. There exists $B \supset A$ and a completion $q \in S(B)$ of $p$ dividing over $A$ with $q$ non-supersimple.

2. If $M \supset A$ is an $|A|^+$-saturated model, there exists a completion $q \in S(M)$ of $p$ dividing over $A$ with $q$ non-supersimple.

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