Dualised $\sigma$-models at the two-loop order

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Abstract

We address ourselves the question of the quantum equivalence of non abelian dualised $\sigma$-models on the simple example of the T-dualised $SU(2)$ $\sigma$-model. This theory is classically canonically equivalent to the standard chiral $SU(2)$ $\sigma$-model. It is known that the equivalence also holds at the first order in perturbations with the same $\beta$ functions. However, this model has been claimed to be non-renormalisable at the two-loop order. The aim of the present work is the proof that it is - at least up to this order - still possible to define a correct quantum theory. Its target space metric being only modified in a finite manner, all divergences are reabsorbed into coupling and fields (infinite) renormalisations.

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1 Introduction

The subject of classical versus quantum equivalence of T-dualised \( \sigma \)-models has been strongly studied in recent years, and extensive reviews covering abelian, non-abelian dualities and their applications to string theory and statistical physics are available [1, 2, 3]. More recent developments on the geometrical aspects of duality can be found in [4].

The interpretation of T-duality as a canonical transformation, for constant backgrounds, was first given by [5, 6]. Its more general formulation [7] was applied to the non-abelian case in [8, 9].

After the settling of the classical equivalence, the most interesting problem was its study at the quantum level. This was done mostly for dualisations of Lie groups, with emphasis put on \( SU(2) \). For this model the one-loop equivalence was established in [10, 11]. This one-loop quantum equivalence was recently settled for the general class of models built on \( G_L \times G_R/G_D \), with an arbitrary breaking of \( G_R \) [12]. An interesting intermediary result is an expression for the Ricci tensor of the dualised geometry (with torsion) exhibiting its dependence with respect to the geometrical quantities of the original model. In the same work, the two-loop renormalisability problem was tackled and the need for extra (non-minimal) one-loop order finite counter-terms was emphasized. Some years ago, it was noted that in the minimal dimensional scheme, two-loop renormalisability does not hold for the \( SU(2) \) T-dualised model [13, 14].

The aim of the present work is a more precise analysis of this two-loop (in)equivalence for the non-abelian T-dualised \( \sigma \)-models, still on the simple example of the original \( SU(2) \) T-dualised model.

The main remark is that, part of the isometries being somehow lost, the T-dualised models are not - as they should be if one wants to give an all-order analysis - defined by a sufficient system of Ward identities. For example, in our simple case there is, \textit{a priori} only a linear \( SU(2) \) [or \( O(3) \)] invariance, and any \( O(3) \) invariant action is allowed (let us remind the reader that in higher-loop corrections to a classical action, all the terms which are not prohibited by some reason such as power counting, isometries or conservation laws..., would appear). To our present knowledge, the extra constraints coming from the origin of the model (dualisation of an \( SU(2)_L \times SU(2)_R/SU(2)_D \) chiral model) are not understood. As it is highly probable that they are linked with the space-time dimension, it is not surprising that a minimal dimensional renormalisation scheme fails : as is well known, when the regularization method does not respect all the properties that define the theory, extra finite counter-terms are needed [18].

The content of this article is the following : in Section 2 we recall the expression of the classical action of the dualised theory and set the notations. In Section 3, we start from the corresponding \textit{a priori} quantum bare action and obtain through \( \hbar \) expansion the possible counter-terms that may be added to the classical action in order to reabsorb the divergences. Then in Section 4 we give the 2-loop divergences and in Section 5 we discuss how they match with the candidates in Section 3. Our result is that coupling constant and field renormalisations (infinite and finite ones) are not sufficient to ensure the two-loop existence of the T-dualised theory but the metric itself has to be deformed (in a \textit{finite way}). Some concluding remarks are offered in Section 6.

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1For abelian T-duality, similar works were achieved in [15, 16].
2In [17] the quantisation of a \( U(1) \)-invariant non-linear \( \sigma \) model, the so-called Complex SineGordon model, was performed by imposing as extra constraints its classical property of factorisation and non-production ; there it was shown that \textit{definite extra finite} one-loop counter-terms are needed to enforce this property to one-loop order and then they also restore the two-loop renormalisability.
2 The classical action

At the classical level and in light-cone co-ordinates, the dual action can be written \([10, 12]\): \[ S = \frac{1}{\lambda} \int G_{ij} \partial_+ \phi^i \partial_- \phi^j , \]

where \(g_{ij} = G_{(ij)}\) is the target space metric and \(h_{ij} = G_{[ij]}\) is the torsion potential. The torsion \(T_{ijk}\) is defined by \(T_{ijk} = \frac{3}{2} \partial_i h_{jk}\). The connections with torsion \(\Gamma^i_{jk}\) and without torsion \(\gamma^i_{jk}\) respectively write:

\[ \Gamma^i_{jk} = \frac{1}{2} g^{is} (\partial_j G_{ks} + \partial_k G_{sj} - \partial_s G_{kj}) = T^i_{jk}, \quad \gamma^i_{jk} = \frac{1}{2} g^{is} (\partial_j g_{sk} + \partial_k g_{sj} - \partial_s g_{kj}) , \]

and the corresponding covariant derivatives are:

\[ D_i k_j = \partial_i k_j - \Gamma^s_{ij} k_s = \nabla_i k_j - T^s_{ij} v_s , \quad D_i k^j = \partial_i k^j + \Gamma^j_{is} k^s = \nabla_i k^j + T^j_{is} v^s . \]

The Riemann tensor without torsion will be noted \(R_{ij,kl}\) whereas we will denote the one with torsion as \(\bar{R}_{ij,kl}\).

The expression of the dualised target space metric \(G_{ij}\) as a function of the original one is well known and in \([12]\) the various geometrical quantities (Ricci tensor,..) were also related. In the special case considered here, where the original model is the \(SU(2)\times SU(2)\) non-linear \(\sigma\) model, the metric writes:

\[ G_{ij}[\bar{\phi}] = \frac{1}{1 + \bar{\phi}^2} [\delta_{ij} + \phi^i \phi^j + \epsilon_{ijk} \phi^k] , \quad (1) \]

where \(\bar{\phi}\) is a \(SU(2)\) (real) vector representation and the \(\phi^i\), \(i = 1, 2, 3\), are the co-ordinates on the dualised manifold. Then \(\bar{\phi}^2\) is a \(SO(3)\) invariant and the symmetry is linearly realised. Torsion breaks parity, but the model is invariant under the simultaneous change \(\phi \to -\phi\) and \(\epsilon_{ijk} \to -\epsilon_{ijk}\). Let us emphasize that no other local symmetry exists for that model.

3 The two-loop order bare action

In order to analyse the two-loop renormalisability of the dualised \(SU2\) \(\sigma\)-model, we first examine all the possible ways to reabsorb the divergences through local counter-terms. As usual, we allow for finite and infinite renormalisations of both fields and coupling. But, as we shall see later on, this appears as insufficient to reabsorb the various divergences. Thus, we also allow for a finite deformation of the classical metric and torsion potential \(g_{ij} + h_{ij} = G_{ij}\) to describe its quantum extension: of course, this à la Friedan \([19]\) extension of the notion of renormalisability involves \(a\) \(priori\) an infinite number of new parameters. Let us emphasize that we shall consider only finite deformations.

Even if by doing so we obviously introduce too many parameters, we first let them all independent in order to show the announced need for such intrinsic metric deformation.

Let us first write the bare action:

\[ S^o = \frac{1}{\lambda^o} \int G^o_{ij} \partial_+ \phi^{oi} \partial_- \phi^{oj} \quad (2) \]
where:

\[
\begin{align*}
\frac{1}{\lambda^0} &= \frac{1}{\lambda} \left[ 1 + \frac{\hbar \lambda}{2\pi} \left( \frac{A_1}{\varepsilon} + b \right) + \left( \frac{\hbar \lambda}{2\pi} \right)^2 \left( \frac{c}{\varepsilon^2} + \frac{A_2}{\varepsilon} + d \right) + \cdots \right],
\end{align*}
\]

\[
\begin{align*}
\tilde{\phi}^0 &= \tilde{\phi} + \frac{\hbar \lambda}{2\pi} \left( \tilde{v}_1(\tilde{\phi}) + \tilde{w}_1(\tilde{\phi}) \right) + \left( \frac{\hbar \lambda}{2\pi} \right)^2 \left( \tilde{v}_2(\tilde{\phi}) + \tilde{w}_2(\tilde{\phi}) \right) + \tilde{x}(\tilde{\phi}) + \cdots,
\end{align*}
\]

\[
\begin{align*}
G^0_{ij} &= G_{ij} + \frac{\hbar \lambda}{2\pi} \tilde{G}_{ij} + \left( \frac{\hbar \lambda}{2\pi} \right)^2 \hat{G}_{ij} + \cdots
\end{align*}
\]

(3)

To express (2) we shall need the Lie derivative \( \mathcal{L}_k \) and a “second order” Lie derivative \( \mathcal{L}^{(2)}_k \).

Indeed, for any tensor \( S_{ij} \) defined on a manifold with co-ordinates \( \phi^j \), in a change of co-ordinates:

\[
S^0_{ij}(\tilde{\phi}) \partial_+ \phi^m \partial_- \phi^n = S_{ij}(\tilde{\phi}) \partial_+ \phi^i \partial_- \phi^j,
\]

and if \( \tilde{\phi}^0 = \tilde{\phi} + \eta \tilde{k} \) (note that \( \tilde{k} \) is not a vector field on the manifold):

\[
S_{ij}(\tilde{\phi}) = S^0_{ij}(\tilde{\phi}) - \eta \mathcal{L}^0_{k}(S^0_{ij}(\tilde{\phi})) + \frac{1}{2} \eta^2 \mathcal{L}^{(2)}_{k}(S^0_{ij}(\tilde{\phi})) + \mathcal{O}(\eta^3).
\]

(4)

We remind the reader that

\[
\mathcal{L}_k^0(S_{ij}) = k^s \nabla_s S_{ij} + S_{sj} \nabla_i k^s + S_{is} \nabla_j k^s.
\]

(5)

One can show that:

\[
\mathcal{L}_k^{(2)}(S_{ij}) = \left( \mathcal{L}_k(S_{ij}) \right) - \mathcal{L}_{k+s}^0 k^s (S_{ij})
\]

(6)

With \( \nabla_i g_{jk} = 0 \), we rewrite the equations (5,6) for \( S_{ij} \equiv G_{ij} \) as:

\[
\begin{align*}
\mathcal{L}_k^0(G_{ij}) &= 2D_j k_i + \partial [\xi_j], \\
\mathcal{L}_k^{(2)}(G_{ij}) &= 2k^s k^u (k_{sj} + 2D_j k^s D_j k_s - 4T_{ius} k^u D_j k_s + \mathcal{L}_{(k^a \xi_{js})} (G_{ij}) + \partial [\xi_j]).
\end{align*}
\]

(7)

\( \xi \) is some quantity whose computation is useless as, in the same manner as \( \xi \), it gives a vanishing contribution to the action or, the torsion potential being always defined up to a gauge transformation, such term can always be put into \( h_{ij} \) (moreover, in our particular situation, the \( O(3) \) symmetry implies that such \( \partial [\xi_j] \) terms vanish). Then, we shall not write them anymore.

Then, expanding (2) with the help of (3,4), one gets the possible counter-terms at lowest orders:

- **0 order in \( \frac{\hbar \lambda}{2\pi} \):**

\[
\frac{1}{\lambda} G_{ij} \partial_+ \phi^i \partial_- \phi^j
\]
at the two-loop order

- first order in $\frac{\hbar\lambda}{2\pi}$:

$$\frac{1}{\lambda} \left[ \left( \frac{\Lambda_1}{\varepsilon} + b \right) G_{ij} + \mathcal{L}_{\vec{w}_1\vec{w}_1} (G_{ij}) + \bar{G}_{ij} \right] \partial_+ \phi^i \partial_- \phi^j \quad (8)$$

- at second order in $\frac{\hbar\lambda}{2\pi}$:

$$\frac{1}{\lambda} \left[ \frac{1}{\varepsilon^2} (\cdots) + \left( \frac{\Lambda_1}{\varepsilon} \bar{G}_{ij} + \mathcal{L}_{\vec{w}_1} (\bar{G}_{ij}) + b \mathcal{L}_{\vec{w}_1} (G_{ij}) + \frac{\Lambda_2}{\varepsilon} G_{ij} + \mathcal{L}_{\vec{w}_1} (G_{ij}) + \frac{1}{2} \mathcal{L}_{\vec{w}_1}^2 (G_{ij}) \right) \right] \partial_+ \phi^i \partial_- \phi^j \quad (9)$$

where $Q \Big|_{\vec{w}_1}$ means that we only take the term in $\frac{1}{\varepsilon}$ in the expression $Q$.

As we don’t consider the 3-loop order, in expression (9) we only need the coefficient of $\frac{1}{\varepsilon}$ (the double poles $\frac{b^2}{\varepsilon^2}$ are not new quantities as they are directly related to first order simple poles and it has already been proved that the dualised $SU2\sigma$-model is one-loop renormalisable $[10]$).

Using the following identity between Lie derivatives:

$$\mathcal{L}_{\vec{w}}} \mathcal{L}_{\vec{w}_1} = \mathcal{L}_{\vec{w}} \mathcal{L}_{\vec{w}_1} + \mathcal{L}_{\vec{w}_1} \mathcal{L}_{\vec{w}} - 2 \mathcal{L}_{\vec{w}_1 \vec{w}_1}$$

with $Z_i^j = X^j \partial_i Y^i - Y^j \partial_i X^i$,

the term with the “second order” Lie derivative may be re-expressed:

$$\varepsilon \left[ \mathcal{L}_{\vec{w}_1}^2 (G_{ij}) \right] = \mathcal{L}_{\vec{w}_1} \left( \mathcal{L}_{\vec{w}_1} (G_{ij}) \right) + \mathcal{L}_{\vec{w}_1} \left( \mathcal{L}_{\vec{w}_1} (G_{ij}) \right) - \mathcal{L}_{\vec{w}_1 \vec{w}_1} \left( \mathcal{L}_{\vec{w}_1} (G_{ij}) \right) = 2 \left[ \mathcal{L}_{\vec{w}_1} \left( \mathcal{L}_{\vec{w}_1} (G_{ij}) \right) - \mathcal{L}_{\vec{w}_1 \vec{w}_1} \left( G_{ij} \right) \right].$$

So, the $\mathcal{O}(h)$ term $[8]$ may be rewritten as:

$$\frac{1}{\lambda} \left[ \frac{1}{\varepsilon} \left( A_1 G_{ij} + \mathcal{L}_{\vec{v}_1} (G_{ij}) \right) + \left( \mathcal{L}_{\vec{w}_1} (G_{ij}) + b G_{ij} + \bar{G}_{ij} \right) \right] \partial_+ \phi^i \partial_- \phi^j,$$

and the $\mathcal{O}(h)^2$ term $[3]$ as:

$$\frac{1}{\lambda \varepsilon} \left[ A_1 \left( \mathcal{L}_{\vec{w}_1} (G_{ij}) + b G_{ij} + \bar{G}_{ij} \right) + (A_2 - b A_1) G_{ij} + \mathcal{L}_{\vec{v}_1} \left( \mathcal{L}_{\vec{w}_1} (G_{ij}) + b G_{ij} + \bar{G}_{ij} \right) \right].$$

As a consequence, as expected, any term $\mathcal{L}_{\vec{w}_1} (G_{ij}) + b G_{ij}$ may be reabsorbed into the finite deformation $\bar{G}_{ij}$ (and vice-versa) to the expense of a change in the $\mathcal{O}(h)^2$ parameters:

$$\bar{G}_{ij} + \mathcal{L}_{\vec{w}_1} (G_{ij}) + b G_{ij} \rightarrow \bar{G}_{ij} \rightarrow \{ A_2 \rightarrow A_2 - b A_1, \quad \bar{w}_2 \rightarrow \bar{w}_2 - v_1^k \partial_k \bar{w}_1 \}.$$  \quad (10)
Finally, for the term in $\frac{1}{\varepsilon} \left( \frac{\hbar A}{2\pi} \right)^2$ in the bare action, one has the following expression:

$$\frac{1}{\lambda} \left( \Lambda_1 \tilde{G}_{ij} + \mathcal{L}(\tilde{G}_{ij}) + \Lambda_2 G_{ij} + \mathcal{L}(G_{ij}) + \mathcal{H}_{ij}(v_1, w_1) \right),$$

where

$$\begin{align*}
\tilde{W}_2 &= \tilde{w}_2 + b \tilde{v}_1 + \Lambda_1 \tilde{w}_1 + v_1^s w_1^u \tilde{g}_{su} \\
\mathcal{H}_{ij}(v_1, w_1) &= v_1^s w_1^u \tilde{R}_{isuj} + D_i v_1^s D_j w_1 s - 2T_{ius} v_1^s D_j w_1^u + (\tilde{v}_1 \leftrightarrow \tilde{w}_1).
\end{align*}$$

### 4 The two-loop order divergences

We use the expression of the covariant divergences given by Hull and Townsend, in the background field method and in the minimal dimensional scheme, up to the two-loop order:

$$\begin{align*}
\text{Div}^1_{ij} &= -\frac{\hbar}{2\pi \varepsilon} \tilde{R} \text{ic}_{ij} \\
\text{Div}^2_{ij} &= -\frac{\hbar^2 \lambda}{8\pi^2 \varepsilon} \left( \tilde{R}^{klm} (\tilde{R}_{klnj} - \frac{1}{2} \tilde{R}_{lnkj}) + 2T_{mn} T^{lmn} \tilde{R}_{kijl} \right)
\end{align*}$$

In order to ensure the renormalizability of the theory, these divergences should match with the candidate counter-terms given by (8) and (11):

$$\begin{align*}
\text{CT}^1_{ij} &= \frac{\hbar}{2\pi \varepsilon} \left( \Lambda_1 G_{ij} + \mathcal{L}(G_{ij}) \right) \\
\text{CT}^2_{ij} &= \frac{\hbar^2 \lambda}{4\pi^2 \varepsilon} \left( \Lambda_1 \tilde{G}_{ij} + \mathcal{L}(\tilde{G}_{ij}) + \Lambda_2 G_{ij} + \mathcal{L}(G_{ij}) + \mathcal{H}_{ij}(v_1, w_1) \right)
\end{align*}$$

It has been previously proven that the dualised metric is quasi-Einstein as soon as the original metric is Einstein. In our special case, we get:

$$\tilde{R} \text{ic}_{ij} = \Lambda G_{ij} + 2D_j v_i \quad , \quad \Lambda = \Lambda_1 = \frac{1}{2} \quad , \quad \tilde{v} = \tilde{v}_1 = \frac{1}{2} \left( \frac{1 - \phi^2}{1 + \phi^2} \right) \tilde{\phi}$$

The addition to the effective action of a $\hbar$ finite deformation of the metric and of some finite renormalisations for the coupling and fields (non-minimal scheme) modifies the $\hbar^2$ divergences. The addition term is easily obtained as

$$-\frac{\hbar}{2\pi \varepsilon} \left( \tilde{R} \text{ic}_{ij} \left( G_{kl} + \frac{\hbar \lambda}{2\pi \tilde{w}_1} \left( \mathcal{L}(G_{kl}) + b G_{kl} + \tilde{G}_{kl} \right) \right) - \tilde{R} \text{ic}_{ij}(G_{kl}) \right) \equiv -\frac{\hbar^2 \lambda}{4\pi^2 \varepsilon} \Delta_{ij} + \mathcal{O}(\hbar^3).$$

Here also, only the combination $\tilde{G}_{ij} + \mathcal{L}(G_{ij}) + b G_{ij}$ appears. Then, we could decide to reabsorb $b G_{ij}$ and $\mathcal{L}(G_{ij})$ into $\tilde{G}_{ij}$, but, as announced at the beginning of Section 3, in order to see if they would be sufficient by themselves, we keep them apart in a first step.

Finally, the dualised $SU(2)$ $\sigma$-model will be renormalisable at two loops if and only if we can find $\left\{ \tilde{G}_{ij}[\tilde{\phi}], b, \tilde{w}_1[\tilde{\phi}] ; \Lambda_2, \tilde{W}_2[\tilde{\phi}] \right\}$ such that:

\footnote{We checked for our example that the two other calculations in $\cite{21,22}$ give the same result.}
\[ \text{Div}_{ij}^2 - \frac{\hbar^2 \lambda}{4 \pi^2 \varepsilon} \Delta_{ij} + \frac{\hbar^2 \lambda}{4 \pi^2 \varepsilon} \left( \Lambda_1 \tilde{G}_{ij} + \mathcal{L}(\tilde{G}_{ij}) + \Lambda_2 G_{ij} + \mathcal{L}(G_{ij}) + \mathcal{H}_{i}(v_1, w_1) \right) = 0 \] (16)

5 Results

According to the linearly realised symmetry of the T-dualised SU(2)σ-model, the finite deformation of the metric \( \tilde{G}_{ij} \) and the vectors \( \tilde{w}_1(\phi) \) and \( \tilde{W}_2(\phi) \) respectively write:

\[
\tilde{G}_{ij} = \alpha(\tau) \delta_{ij} + \beta(\tau) \phi^i \phi^j + \epsilon_{ijk} \gamma(\tau) \phi^k, \quad \tilde{w}_1 = w_1(\tau) \tilde{\phi}, \quad \tilde{W}_2 = W_2(\tau) \tilde{\phi}
\]

where \( \tau = \phi^2 \). Moreover, the symmetry also implies that terms of the form \( \partial_k k_{ij} \) or of the form \( k^u K^u \gamma^l_{su} \) are equal to zero. It is then possible to re-express (16) as a set of three linear differential equations:

\[
W_2(\tau) = \frac{(1 + \tau) \Lambda_2}{2} + \frac{45 + 68 \tau - 18 \tau^2 - 12 \tau^3 - 3 \tau^4}{16(1 + \tau)^3} - \frac{1 - \tau}{(1 + \tau)^2} w_1(\tau)
\]

\[
= \frac{3 + 10 \tau + 5 \tau^2 + 2 \tau^3}{4(1 + \tau)} \alpha(\tau) + \frac{4 + 5 \tau + 6 \tau^2 + \tau^3}{4(1 + \tau)} \beta(\tau) - \frac{3(1 + \tau)(3 + \tau)}{2} \gamma(\tau)
\]

\[
= \frac{4 + 11 \tau + 5 \tau^2 - \tau^3}{2} \alpha'(\tau) + \frac{\tau}{2} \beta'(\tau) - (1 + \tau)(3 + \tau) \gamma'(\tau) - \tau(1 + \tau)^2 \alpha''(\tau)
\]

\[
3 \Lambda_2 - \frac{3(-5 + 60 \tau + 10 \tau^2 + 12 \tau^3 + 3 \tau^4)}{8(1 + \tau)^4} = \frac{(7 + 10 \tau)}{2} \alpha(\tau) + \frac{(12 + 5 \tau)}{2} \beta(\tau) - 3(11 + 5 \tau) \gamma(\tau)
\]

\[
+ (-17 - 22 \tau + 9 \tau^2) \alpha'(\tau) + (5 + 4 \tau + \tau^2) \beta'(\tau) - 2(5 + 2 \tau)(3 + 5 \tau) \gamma'(\tau)
\]

\[
+ 2(-5 - 19 \tau - 12 \tau^2 + \tau^3) \alpha''(\tau) + 2 \tau \beta''(\tau) - 4(1 + \tau)(3 + \tau) \gamma''(\tau) - 4 \tau(1 + \tau)^2 \alpha'''(\tau)
\]

The need for a true deformation \( \tilde{G}_{ij} \) immediately appears: setting both \( \alpha(\tau) \), \( \beta(\tau) \) and \( \gamma(\tau) \) to zero, equations (18) and (19) cannot be satisfied, even if we allowed for some finite renormalisations of the coupling \( b \) and field \( \tilde{w}_1(\tilde{\phi}) \), both hidden into the vector \( \tilde{W}_2(\tilde{\phi}) \) (see equation (12)). Then, as first proven in [13, 14], we have checked that:

In a purely dimensional scheme (even with non minimal subtractions), the dualised SU(2)σ model is not renormalisable at the two-loop order.

\footnote{One notices also that the parameters \( b \) and \( \tilde{w}_1 \) do not appear in (18) and (19). So, the existence of some solution to this set of differential equations is independent of the finite renormalisations of both coupling and fields, as is usual in perturbation theory. This freedom corresponds to a change of renormalisation scheme. This absence is only true if we take the very vector \( \tilde{v}_1 \) that reabsors the divergences at the one-loop order: otherwise, \( \tilde{w}_1(\tilde{\phi}) \) would appear in (18) and (19). This is a check of a correct renormalisation at the one-loop order.}
So, from the discussion in the previous Sections, and without restricting the generality of our analysis, one can take $b$ and $\bar{w}_1(\phi)$ as vanishing quantities.

Remarks:

- As $\Lambda_2$ is not a function, but a constant, differentiating equations (18) and (19) will relate $\alpha(\tau), \beta(\tau),$ and $\gamma(\tau)$. Then, as soon as $\tilde{G}_{ij}$, the finite one loop renormalisation, has been definitely set, equation (17) will give the infinite two-loop renormalisations $\tilde{W}_2(\phi)$ and $\Lambda_2$.

- From the previous discussions, we know that $\tilde{G}_{ij}$ will be fixed up to some $\tilde{b}G_{ij} + \mathcal{L}(\tilde{G}_{ij})$; it is then natural to use this freedom, for example to reabsorb $\alpha(\tau)$, and to redefine $\tilde{G}_{ij}$ such that:

\[
\tilde{G}_{ij} = \tilde{b}G_{ij} + \mathcal{L}(\tilde{\phi})(G_{ij}) + \tilde{G}_{ij}, \quad \tilde{W}(\tilde{\phi}) = \tilde{W}(\tau)\tilde{\phi}
\]

with $\tilde{W}(\tau) = \frac{(1 + \tau)^2}{2}\alpha(\tau) - \frac{\tilde{b}(1 + \tau)}{2} \Rightarrow \tilde{G}_{ij} = \tilde{\beta}(\tau)\phi^i\phi^j + \epsilon_{ijk}\tilde{\gamma}(\tau)\phi^k$

with $\tilde{\beta} = \beta - \frac{\tilde{b}}{1 + \tau} - \frac{2(2 + \tau)\tilde{W}}{(1 + \tau)^2} - 4\tilde{W}'$ and $\tilde{\gamma} = \gamma - \frac{\tilde{b}}{1 + \tau} - \frac{(3 + \tau)\tilde{W}}{(1 + \tau)^2}$. (20)

We know that, when expressed as functions of $\tilde{\beta}(\tau)$ and $\tilde{\gamma}(\tau)$, equations (17), (18), (19) remain unchanged, up to the substitutions discussed in Section 3 [equation (10)]:

\[
\Lambda_2 \rightarrow \tilde{\Lambda}_2 = \Lambda_2 + \frac{\tilde{b}}{2},
\]

\[
W_2(\phi) \rightarrow \tilde{W}_2(\tau) = W_2(\tau) + \tilde{b} \frac{1 - \tau}{2(1 + \tau)} + \frac{1}{2}\tilde{W}(\tau) + \frac{1 - \tau}{2(1 + \tau)}[\tilde{W}(\tau) + 2\tau\tilde{W}'(\tau)]. (21)
\]

Equations (18) and (19) give $\tilde{\beta}(\tau)$ as a function of $\tilde{\gamma}(\tau)$ which itself satisfies a non-homogeneous linear fourth order differential equation:

\[
\begin{align*}
(a) \quad \tilde{\beta}(\tau) & = \frac{-\tilde{b} + 2\Lambda_2}{6 + \tau} = \frac{3(1 - \tau)(3 + 6\tau + \tau^2)}{4(6 + \tau)(1 + \tau)^3} + \frac{2(17 + 3\tau)}{6 + \tau}\tilde{\gamma}(\tau) - \frac{4(5 - 6\tau - \tau^2)}{6 + \tau}\tilde{\gamma}'(\tau) \\
& - \frac{8\tau}{6 + \tau}\tilde{\gamma}''(\tau)
\end{align*}
\]

\[
\begin{align*}
(b) \quad \tilde{\gamma}^{(4)}(\tau) & + \frac{6 - \tau}{\tau(6 + \tau)}\tilde{\gamma}^{(3)}(\tau) + \frac{1260 - 276\tau - 91\tau^2 + 3\tau^3 + \tau^4}{4\tau^2(6 + \tau)^2}\tilde{\gamma}''(\tau) + \\
& + \frac{-120 + 254\tau + 57\tau^2 + 3\tau^3}{8\tau^2(6 + \tau)^2}\tilde{\gamma}'(\tau) - \frac{138 + 25\tau + \tau^2}{8\tau^2(6 + \tau)^2}[\tilde{\gamma}(\tau) - \frac{\tilde{b} + 2\Lambda_2}{2}] = \\
& = -\frac{3(6402 - 8681\tau - 5856\tau^2 - 22\tau^3 + 390\tau^4 + 39\tau^5)}{64\tau^2(1 + \tau)^5(6 + \tau)^2}.
\end{align*}
\]

Note that under the change

\[
\Gamma(\tau) = [\tilde{\gamma}(\tau) - \frac{\tilde{b} + 2\Lambda_2}{2} ], \quad B(\tau) = [\tilde{\beta}(\tau) - 3(\tilde{b} + 2\Lambda_2)],
\]

(23)
the parameter \( \tilde{b} \) and the constant \( \Lambda_2 \) disappear from the set \( \{22\} \). Then, \( \Lambda_2 \) being an unknown constant, the general solution of the differential equation \( \{22\}-b \) will be

\[
\hat{\gamma}(\tau) = \Gamma(\tau) + c, \text{ where } c \text{ is an arbitrary constant},
\]

and the two-loop coupling constant renormalisation \( \Lambda_2 \) will be:

\[
\Lambda_2 = c - \frac{\tilde{b}}{2}.
\]

The model will be renormalisable up to two loops iff equation \( \{22\}-b \) , where \( \hat{\gamma}(\tau) \) has been replaced by \( \Gamma(\tau) \) according to \( \{23\} \), has a solution which is analytic near \( \tau = 0 \). In order to reach such a conclusion, we use the method of Frobenius for linear differential equations around the singular point 0 has four different solutions : \( \nu = -\frac{3}{2}, \ -\frac{1}{2}, \ 0, \ 1 \). For each one, we can find convergent series \( \tau^\nu \sum_{n=0}^{\infty} c_n \tau^n \) that are independent solutions of the homogeneous equation associated to \( \{22\}-b \). We give here the first terms of such series (it happens that for \( \nu = -\frac{3}{2} \) we have an exact solution ) :

\[
\begin{align*}
\hat{\gamma}_{-\frac{3}{2}}(\tau) &= \frac{1}{\tau^\frac{3}{2}} + \frac{1}{20\sqrt{\tau}} - \frac{\sqrt{\tau}}{20}, \quad \hat{\gamma}_{-\frac{1}{2}}(\tau) = \frac{1}{\sqrt{\tau}} \left( 1 - \frac{11}{6} \tau + \frac{35}{108} \tau^2 + \cdots \right) \text{ (24)} \\
\hat{\gamma}_0(\tau) &= 1 + \frac{23}{840} \tau^2 + \cdots, \quad \hat{\gamma}_1(\tau) = \tau \left( 1 + \frac{1}{42} \tau - \frac{1}{324} \tau^2 + \cdots \right)
\end{align*}
\]

Then, we use the method of variation of parameters to find \( \lambda_{-\frac{3}{2}}(\tau) \), \( \lambda_{-\frac{1}{2}}(\tau) \), \( \lambda_0(\tau) \) and \( \lambda_1(\tau) \) such that

\[
\Gamma(\tau) = \lambda_{-\frac{3}{2}}(\tau) \hat{\gamma}_{-\frac{3}{2}}(\tau) + \lambda_{-\frac{1}{2}}(\tau) \hat{\gamma}_{-\frac{1}{2}}(\tau) + \lambda_0(\tau) \hat{\gamma}_0(\tau) + \lambda_1(\tau) \hat{\gamma}_1(\tau)
\]

is the general solution of the inhomogeneous equation \( \{22\}-b \) where \( \hat{\gamma}(\tau) \) has been replaced by \( \Gamma(\tau) \) according to \( \{23\} \).

The first terms in the expansion of these functions are :

\[
\begin{align*}
\lambda_{-\frac{3}{2}}(\tau) &= \lambda_{-\frac{3}{2}} + \tau^\frac{3}{2} \left( \frac{1067}{1680} - \frac{13691}{3780} \tau + \cdots \right), \quad \lambda_{-\frac{1}{2}}(\tau) = \lambda_{-\frac{1}{2}} + \tau^\frac{1}{2} \left( -\frac{1067}{240} + \frac{2543509}{100800} \tau + \cdots \right) \\
\lambda_0(\tau) &= \lambda_0 + \frac{1067}{192} \tau^2 - \frac{9805}{288} \tau^3 + \cdots, \quad \lambda_1(\tau) = \lambda_1 - \frac{1067}{480} \tau + \frac{27887}{5760} \tau^2 + \cdots
\end{align*}
\]

The analyticity requirement near \( \tau = 0 \) enforces the choice \( \lambda_{-\frac{3}{2},\tilde{b}} = \lambda_{-\frac{1}{2}} = 0 \); \( \hat{\gamma}(\tau) \) is then expressed as a convergent series in \( \tau \), and the same will be true for \( \hat{\beta}(\tau) \). The final expression for the deformation \( \tilde{G}_{ij} \) depends on 3 constants \( \{c, \lambda_0^0, \lambda_1^0\} \) and an arbitrary function \( \{\tilde{W}(\tau)\} \) and is given by the three functions :

\[
\begin{align*}
\alpha(\tau) &= \frac{\tilde{b}}{1 + \tau} + \frac{2\tilde{W}}{(1 + \tau)^2}, \\
\beta(\tau) &= 6c + \frac{\tilde{b}}{1 + \tau} + 2(2 + \tau)\tilde{W} \left( 1 + \tau \right)^2 + 4\tilde{W}' + \frac{3(1 - \tau)(13 + 6\tau + \tau^2)}{4(6 + \tau)(1 + \tau)^3} + \frac{2(17 + 3\tau)}{6 + \tau} \Gamma(\tau) \\
\gamma(\tau) &= c + \frac{\tilde{b}}{1 + \tau} + \frac{(3 + \tau)\tilde{W}}{(1 + \tau)^2} + \Gamma(\tau).
\end{align*}
\]
We now use the up to now free parameter \( \tilde{b} \) to reabsorb the parameter \( c \). Let us define

\[
\tilde{b} = \bar{b} - 2c, \quad \tilde{W}(\tau) = \bar{W}(\tau) + c(1 + \tau),
\]

we get

\[
\tilde{G}_{ij} = \bar{G}_{ij} + \tilde{b}G_{ij} + \mathcal{L}G_{ij}
\]

with \( \bar{W} = \bar{W}(\tau)\bar{\phi} \) and

\[
G_{ij} = \tilde{G}_{ij} \big|_{\text{equ.}\,(26)} \text{ for } c = b = W(\tau) \equiv 0.
\]

The dualised SU(2) \( \sigma \)-model is therefore renormalisable at the two-loop order if and only if we add a finite \( \hbar \) deformation of the classical metric, depending on two new parameters \( \lambda_0 \) and \( \lambda_1 \).

### 6 Concluding remarks

We have been able to exhibit some set of counter-terms that ensures the two-loop renormalisability of the T-dualised chiral non-linear \( \sigma \) model. The one-loop effective metric is defined up to two constants (\( \lambda_0^0 \) and \( \lambda_1^0 \)), and some finite arbitrary field and coupling renormalisations. As is well known (e.g. in [24]), the two-loop Callan-Symanzik \( \beta \) function (related to \( \Lambda_2 \)) depends on these finite counterterms.

We emphasize that, contrarily to D. Friedan’s approach to \( \sigma \) models quantisation, where the classical metric receives infinite perturbative deformations, our candidate for the deformation of the classical metric is a finite one, depending on only two parameters (plus the usual infinite, and finite, renormalisations of the fields and of the coupling constant) : our ansatz is that a proper understanding of the dualisation process will precisely offer the extra constraints that uniquely define the quantum extension of the classical theory, order by order in perturbation theory, in the same spirit as Ward identities determine what otherwise would appear as new parameters (see also footnote 2).

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\[ ^5 \text{The two loops quantities } \Lambda_2 \text{ and } \bar{W}_2 \text{ are fixed as :} \]

\[
\Lambda_2 = \frac{\bar{b}}{2}, \quad \bar{W}_2 \text{ obtained through (21)}.
\]

Notice that the normalisation condition \( \bar{b} = 0 \) (no \( \hbar \) extra finite coupling constant renormalisation) enforces \( \Lambda_2 = 0 \).
... at the two-loop order

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