PARAMETRICES FOR THE LIGHT RAY TRANSFORM ON MINKOWSKI SPACETIME

YIRAN WANG
Department of Mathematics, University of Washington
Box 354350
Seattle, WA 98195-4350, USA
(Communicated by Mikko Salo)

Abstract. We consider restricted light ray transforms arising from an inverse problem of finding cosmic strings. We construct a relative left parametrix for the transform on two tensors, which recovers the space-like and some light-like singularities of the two tensor.

1. Introduction. Let \((M, g)\) be a smooth Lorentzian manifold. A smooth curve \(\gamma : \mathbb{R} \to M\) is called a light ray if \(g(\dot{\gamma}(t), \dot{\gamma}(t)) = 0, t \in \mathbb{R}\), where \(\dot{\gamma}\) denotes the covariant derivative along \(\gamma\). Let \(\mathcal{C}\) be the set of light rays on \(M\) and \(\text{Sym}^2\) denote the bundle of symmetric 2-tensors on \(M\). For \(f \in C_0^\infty(M; \text{Sym}^2)\), we consider the light ray transform

\[
L_c f(\gamma) = \int_{\mathbb{R}} f_{lm}(\gamma(t)) \dot{\gamma}^l(t) \dot{\gamma}^m(t) dt, \quad \gamma \in \mathcal{C}.
\]

Hereafter, the Einstein summation convention is used i.e. summation is over repeated indices. On 2 + 1 dimensional Minkowski space-time, this transform was studied by Guillemin [10] and it was envisioned to have applications in “cosmological X-ray tomography”, see the concluding remarks [10, Section 17]. Recently in [14], such transform naturally arises from an inverse problem of detecting singularities of the Lorentzian metric of the Universe using Cosmic Microwave Background (CMB) radiation measurements. In particular, let \((M, g)\) be a Friedmann-Lemaître-Robertson-Walker (FLRW) type model for the Universe. For a small parameter \(\epsilon\), consider a family of metrics on \(M\):

\[
g_\epsilon = g + \epsilon f + O(\epsilon^2)
\]

representing small perturbations of \(g\). In [14], it is demonstrated that one can obtain a restricted light ray transform of \(f\) from the linearization of the CMB measurements. Then it is proved in Theorem 4.4 that one can recover the space-like singularities of \(f\). However, as already noted in [14], light-like singularities are of great interest as they correspond to gravitational waves which may be caused for example by cosmic strings. We address this problem in this note.

For restricted geodesic ray transforms on functions (the light ray transform being an example), there is a microlocal framework developed by Greenleaf-Uhlmann [4, 5, 6, 7, 8] to understand their mapping properties. We combine it with some calculations in [14] to show that the normal operator of the light ray transform is a

2010 Mathematics Subject Classification. Primary: 53C55; Secondary: 35S30.
Key words and phrases. Light ray transforms, parametrix, space-times, Lagrangian distributions, inverse problems.
paired Lagrangian distribution and construct parametrices on the elliptic part. This allows us to obtain a relative left parametrix for the restricted light ray transform, which among other things recovers the space-like and some light-like singularities of the metric perturbations. We remark that time-like singularities are in the kernel of \( L_\mathcal{E} \) and there is a good physical explanation for one not being able to determine them from the light ray transform, see [10, Page 188], [14] and the interesting work of Stefanov [17] on support theorems of the light ray transform.

The paper is organized as follows. In Section 2, we state the main results after setting up the problem. We show in Section 3 that the normal operator is a paired Lagrangian distribution and construct parametrices on the elliptic part. This allows us to obtain a relative left parametrix for the restricted light ray transform, see [10, Page 188], [14] and the interesting work of Stefanov [17] on support theorems of the light ray transform.

2. The main results. It is known that a FLRW type space-time is conformal to the Minkowski space-time. Since conformal diffeomorphisms preserve light-like geodesics, as discussed in [14], it suffices to consider light ray transforms on

\[ M = \mathbb{R}^{3+1}, \quad g = -dt^2 + \sum_{i=1}^3 (dx^i)^2, \]

where \( x = (t, x') = (x^0, x^1, x^2, x^3) \) denotes the coordinates on \( M \). In this case, the light rays are straight lines and we denote by \( \mathcal{C} \) the set of light rays. As demonstrated in [14, Lemma 4.3], the light ray transform \( L_\mathcal{E} \) defined as (1) has a non-trivial null space given by

\[ \mathcal{N} = \{ cg + d^t \omega; c \in \mathcal{E}'(M), w \in \mathcal{E}'(M; \Lambda^1) \}, \]

where \( \mathcal{E}'(M) \) denotes the space of distributions with compact support, \( d^t \) is the symmetric differential given in local coordinates by

\[ (d^t \omega)_{ij} = \frac{1}{2} (\nabla_i \omega)_j + (\nabla_j \omega)_i, \quad i, j = 0, 1, 2, 3, \]

with \( \nabla \) the covariant derivative, and \( \Lambda^1 \) denotes the bundle of one forms. Let \( \mathcal{V} \) be an open set of \( \mathbb{R}^3 \) and define the line complex

\[ \mathcal{C}_0 = \{ \gamma \in \mathcal{C} : \gamma \cap \mathcal{V} \neq \emptyset \} \]

i.e. collection of all light rays intersecting \( \mathcal{V} \), see Figure 1. We denote by \( L_{\mathcal{E}_0} = L_{\mathcal{E}|\mathcal{C}_0} \) the restricted light ray transform on \( \mathcal{C}_0 \). \( L_{\mathcal{E}_0} \) also has a non-trivial null space. We denote \( \mathcal{L}(\mathcal{V}) = \{ p \in M : \text{there exists } q \in \mathcal{V} \text{ and a light ray joining } p \text{ and } q \} \) and

\[ \mathcal{P}(\mathcal{V}) = \{ f \in \mathcal{E}'(M; \text{Sym}^2) \setminus \mathcal{N} : \text{supp } (f) \subset \mathcal{L}(\mathcal{V}) \}. \]

It is easy to see that \( \mathcal{P}(\mathcal{V}) \) is a non-trivial null space of \( L_{\mathcal{E}_0} \). This is because for any \( f \in \mathcal{P}(\mathcal{V}) \), we have \( L_{\mathcal{E}_0} f = L_{\mathcal{E}_0} L_{\mathcal{E}_0} f \) as \( \text{supp } (f) \subset \mathcal{L}(\mathcal{V}) \). Since \( \mathcal{P}(\mathcal{V}) \subset \mathcal{N} \), we can conclude from [14, Lemma 4.3] that \( L_{\mathcal{E}_0} f = 0 \).

The microlocal nature of \( L_{\mathcal{E}_0} \) is well-understood. Let

\[ Z = \{ (\gamma, x) \in \mathcal{C}_0 \times M : x \in \gamma \} \]

be the point-line relation. Then the Schwartz kernel of \( L_{\mathcal{E}_0} \) is \( \delta_Z \) the delta distribution on \( \mathcal{C}_0 \times M \) supported on \( Z \). Hence we know from Hörmander’s theory that \( L_{\mathcal{E}_0} \) is a Fourier integral operator of order \(-3/4\) associated with the canonical relation \( N^* Z' \) (see (3)). Although we do not explore this point here, the operator should fit into the framework in [8], see also [3]. In Section 3, we use a more direct approach to show that the Schwartz kernel of the normal operator \( L_{\mathcal{E}_0}^t \circ L_{\mathcal{E}_0} \) is
a paired Lagrangian distribution and we obtain the Sobolev estimate of $L_{\mathcal{E}_0}$, see Theorem 3.1.

To state the main result, we need to describe the two Lagrangians associated to the normal operator. Let $T^*M$ be the cotangent bundle and $(x, \xi)$ be the coordinate for $T^*M$ where $\xi = (\xi_0, \xi_1, \xi_2, \xi_3)$. Consider $p(x, \xi) = g(\xi, \xi) = -\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2$ the (dual) metric function on $T^*M$. We denote $L^*M = \{(x, \xi) \in T^*M : g(\xi, \xi) > 0\}$ the light-like covectors, $\Omega^+ = \{(x, \xi) \in T^*M : g(\xi, \xi) > 0\}$ the space-like covectors and $\Omega^- = \{(x, \xi) \in T^*M : g(\xi, \xi) < 0\}$ the time-like covectors. Then we can decompose $T^*M = \Omega^+ \cup \Omega^- \cup L^*M$. Let $\omega = d\xi \wedge dx$ be the canonical two form on $T^*M$. The Hamilton vector field of $p$ denoted by $H_p$ is defined through

$$H_p = \sum_{i=0}^{3} \frac{\partial p}{\partial \xi_i} \frac{\partial}{\partial x^i} - \frac{\partial p}{\partial x^i} \frac{\partial}{\partial \xi_i} = 2(-\xi_0 \frac{\partial}{\partial x^0} + \sum_{i=1}^{3} \xi_i \frac{\partial}{\partial x^i}).$$

The integral curves of $H_p$ in $L^*M$ are called null bicharacteristics. It is well known that their projections to $M$ are light-like geodesics. We denote $\Delta = \{(x, \xi, x, -\xi) \in T^*M\{0 \times T^*M\{0\}\}$ and $\Sigma = \{(x, \xi, x, -\xi) \in L^*M\{0 \times L^*M\{0\}\}$ where $0$ stands for the zero section. We let $\Lambda$ be the flow out of $\Sigma$ meaning

$\Lambda = \{(x, \xi, y, -\eta) \in (L^*M\{0 \times (L^*M\{0 : (x, \xi) = \exp tH_p(y, \eta), \text{ for some } t \in \mathbb{R}\}.$

Then $\Delta$ and $\Lambda$ are Lagrangian submanifolds of $T^*(M \times M)$ and they form a pair of cleanly intersecting Lagrangians that is

$T_p\Delta \cap T_p\Lambda = T_p\Sigma, \forall p \in \Sigma.$

Now we briefly recall the notion of Lagrangian and paired Lagrangian distributions. Let $\Lambda$ be a smooth conic Lagrangian submanifold of $T^*M\{0$. We denote by $I^\mu(M; \Lambda)$ the space of Lagrangian distributions of order $\mu$ on $M$ associated with $\Lambda$. For two Lagrangians $\Lambda_0, \Lambda_1 \subset T^*M\{0$ intersecting cleanly at a codimension $k$ submanifold $\Lambda_0 \cap \Lambda_1$, the space of paired Lagrangian distributions associated with $(\Lambda_0, \Lambda_1)$ is denoted by $I^{\mu, k}(M; \Lambda_0, \Lambda_1).$ We use $I^{\mu, k}(\Lambda_0, \Lambda_1)$ when the background manifold is clear. By abuse of notations, we also use $I^{\mu, k}(\Lambda_0, \Lambda_1)$ for section valued distributions in Sym$^2$.

For any subset $A$ of $T^*M$, we let $1_A$ be the microlocal cut-off defined as

$$1_A f(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} e^{i(x-z)\eta} \chi_A(x, \eta) f(z) dz d\eta,$$

where $\chi_A$ is the characteristic function for $A$ and $f \in \mathcal{E}'(M; \text{Sym}^2)$. Our main result is

**Theorem 2.1.** There exists a relative left parametrix $A$ for $L_{\mathcal{E}_0}$ such that

$$A \circ L_{\mathcal{E}_0} = 1_{\Omega^+} + B \mod C^\infty \text{ on } \mathcal{P}(\mathcal{U}),$$

where $A = \tilde{A} \circ L_{\mathcal{E}}^t$, $\tilde{A} \in I^\frac{3}{2} \hat{\cdot} (\Delta, \Lambda)$ and $B \in I^{-\frac{3}{2}} (\Lambda).$

Using this result as a reconstruction formula and wave front analysis, we see that for $f \in \mathcal{P}(\mathcal{U})$, one can recover the singularities in $f$ on space-like directions and on some light-like directions. One may not be able to recover all light-like singularities due to the error term, see a related example in [7, Section 2]. However, $Bf$ contains singularities on the flow out which can be regarded as artifacts in the reconstruction. As we already mentioned, light-like singularities corresponds to gravitational waves and the artifacts may help us to identify these singularities. Furthermore, we notice that $B$ is a Fourier integral operator associated with the canonical relation $\Lambda'$. 


The rank of the projection of $\Lambda'$ to $T^*M$ drops by 1. From Hörmander’s result on $L^2$ boundedness of Fourier integral operators [12, Theorem 4.3.2], we conclude that if $f \in H^s(M; \text{Sym}^2) \cap \mathcal{P}(\mathcal{W})$, then $Bf \in H^s(M)$. So the artifacts have the same order of Sobolev regularity as $f$ does. In a different context [16], the problem of reducing and enhancing the artifacts due to a similar mechanism is studied. The same strategy should work as well, however we do not further explore this point here.

Away from the light-like directions, we state Theorem 2.1 as a corollary in the same spirit as [14, Theorem 4.4].

**Corollary 1.** For $f \in \mathcal{P}(\mathcal{W})$ with $\text{WF}(f) \subset \Omega^+$ and $A$ defined in Theorem 2.1, we have

$$A \circ L_{\mathcal{W}_0} f = f \mod C^\infty$$
onumber

on $\mathcal{P}(\mathcal{W})$.

![Illustration of complex $\mathcal{C}_0$](image)

To conclude this section, we review some facts about the space $I^{p,l}(M \times M; \Delta, \Lambda)$. We follow some presentations in [4, Section 3] and refer the reader to the reference for details. It is convenient to consider the local representation of such distributions on the following model pair (see also [2, Section 5]): On $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, $n = 4$, we let:

$$\hat{\Delta} = \{(x, \xi, y, \eta) \in (T^*\mathbb{R}^n \times T^*\mathbb{R}^n) \setminus 0 : x = y = 0\}$$

and

$$\hat{\Lambda} = \{(x, \xi, y, \eta) \in (T^*\mathbb{R}^n \times T^*\mathbb{R}^n) \setminus 0 : x'' = y'', \xi'' = \eta'' = 0\}$$

where $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}, k = 1$. So $\Delta$ intersects $\Lambda$ at submanifold $\hat{\Sigma} = \{(x, \xi, y, \eta) \in (T^*\mathbb{R}^n \times T^*\mathbb{R}^n) \setminus 0 : x = y = 0, \xi' = \eta' = 0, \xi'' = \eta''\}$. In this case, we can write $u \in I^{p,l}(\hat{\Delta}, \hat{\Lambda})$ as an oscillatory integral

$$u(x, y) = \int e^{i(x' - y' - s)\xi' + (x'' - y'')\xi''} a(x, y, s, \xi, \sigma) d\sigma ds d\xi,$$

with $a \in S^{p-n/2+k/2, l-k/2}(\mathbb{R}^{2n+k}, \mathbb{R}^k)$. By definition, this means that for any compact set $K \subset \mathbb{R}^{2n+k}$ and multi-indices $\alpha, \beta, \gamma \geq 0$, we have

$$|\partial_{\xi}^\alpha \partial_{\eta}^\beta \partial^\gamma_x a(x, y, s, \xi, \sigma)| \leq C_{\alpha, \beta, \gamma} |K|^{p-n/2+k/2} (1+|\sigma|)^{-k},$$

with $C_{\alpha, \beta, \gamma} > 0$. Actually, this is the local definition of distributions in $I^{p,l}(M \times M, \Delta, \Lambda)$. According to [4, Proposition 3.1], we know that $u(x, y)$ is the Schwartz kernel of a pseudo-differential operator of order $p + l$ on $\Delta \setminus \hat{\Sigma}$ and a Fourier integral operator of order $p$ on $\Lambda \setminus \hat{\Sigma}$. In an invariant way, $u \in I^{p+l}(\Delta \setminus \Sigma)$ and $u \in I^p(\Lambda \setminus \Sigma)$.

On $\Delta \setminus \hat{\Sigma}$, we can write

$$u(x, y) = \int e^{i(x - y)\xi} a(x, y, 0, \xi, \xi') d\xi$$

**Figure 1. Illustration of complex $\mathcal{C}_0$**
modulo a pseudo-differential operator of lower orders. We see that the principal symbol \( \sigma_0(u) \) (as a half-density tensored with the Maslov factor) of \( u \) on \( \Delta \setminus \Sigma \) has the conormal type singularity at \( \tilde{\Sigma} \) of order \( l - \frac{k}{2} \) in the sense of [11] i.e. \( \sigma_0(u) \in R^{l - \frac{k}{2}}(\Delta, \Sigma) \). Actually, one can define the symbol of \( u \in \text{IP}^l(M \times M; \Delta, \Lambda) \) invariantly as discussed in [11] or [4], and the symbol space is denoted by \( S^p(M \times M; \Delta, \Sigma) \). Then we have the symbol calculus ([4, Prop. 3.4])

\[
0 \to \text{IP}^{l-1}(M \times M; \Delta, \Lambda) + \text{IP}^{-1}(M \times M; \Delta, \Lambda) \to \text{IP}^l(M \times M; \Delta, \Lambda) \xrightarrow{\sigma_0} S^p(M \times M; \Delta, \Sigma) \to 0.
\]

3. The normal operator. We choose a parametrization of \( \mathcal{C}_0 \) and find the normal operator in the parametrization. Some calculations below are done in [14]. Let \( y \in \mathcal{U} \subset \mathbb{R}^3, v \in \mathbb{S}^2 \triangleq \{ z \in \mathbb{R}^3 : |z| = 1 \} \). We let \( \theta = (1, v) \) so that \( \theta \) is a (future pointing) light like vector, see Fig. 1. For \( \gamma \in \mathcal{C}_0 \) and \( \gamma \cap \mathcal{U} = (0, y) \), we can write

\[
\gamma(s) = (s, y + sv), \quad s \in \mathbb{R}.
\]

Then for \( f \in C^\infty_0(M; \text{Sym}^2) \), we have

\[
L_{\mathcal{C}_0}f(y, v) = \int_\mathbb{R} f_{tm}(s, y + sv)\theta^m ds, \quad y \in \mathcal{U}, v \in \mathbb{S}^2.
\]

The point-line relation is parametrized by

\[
Z = \{(y, v, x) \in \mathcal{U} \times \mathbb{S}^2 \times M : x' = y + x^0v\}.
\]

Therefore, we can find the conormal bundle \( N^*Z \) and the canonical relation \( C = N^*Z' \) as

\[
C = \{(y, v, \eta, w, x', \xi_0, \xi') \in (T^*\mathcal{C}_0 \times T^*M) \setminus 0 : y = x' - x^0v, \quad \eta = \xi',
\]

\[
w = x^0\xi'|_{T_x\mathbb{S}^2}, \quad \xi_0 = -\xi'v, \quad y \in \mathcal{U}, v \in \mathbb{S}^2, \eta \in \mathbb{R}^3, x \in M\}, \tag{3}
\]

see (39) of [14]. Now let’s consider the double fibration picture

\[
\begin{array}{c}
\pi \\
C \\
T^*M \\
\downarrow \phi \\
T^*\mathcal{C}_0
\end{array}
\]

If \( \rho \) is an injective immersion, the double fibration satisfies the Bolker condition. In this case, the composition \( L_{\mathcal{C}_0}^t \circ L_{\mathcal{C}_0} \) belongs to the clean intersection calculus, see Hörmander [13]. However, as demonstrated in [14, Lemma 11.1], \( \rho \) fails to be injective on the set \( \mathcal{L} \cap C \) where

\[
\mathcal{L} = \{(y, v, \eta, w, x, \xi) \in T^*\mathcal{C}_0 \times T^*M : \xi \text{ is light like}\}.
\]

Now let’s consider the wave front set of the normal operator. The canonical relation for \( L_{\mathcal{C}_0}^t \) is \( C' \), so by the calculus of wave front set (see e.g. [13]), we have

\[
\text{WF}(L_{\mathcal{C}_0}^t \circ L_{\mathcal{C}_0}) \subset (C' \setminus \mathcal{L})^t \cup (C \cap \mathcal{L})^t \cup (C \cap \mathcal{L})
\]

\[
\quad = \Delta' \cup (C \cap \mathcal{L})^t \circ (C \cap \mathcal{L}) \subset \Delta' \cup \mathcal{N}'.
\]

Here we observed that

\[
(C \cap \mathcal{L})^t \circ (C \cap \mathcal{L}) = \{(x, \xi, z, \zeta) \in (L^*M \setminus 0) \times (L^*M \setminus 0) : (x, \xi) \text{ and } (z, \zeta) \text{ lie on a null bicharacteristics intersecting } L_{\mathcal{Y}}^tM \equiv \bigcup_{x \in \mathcal{Y}} L_x^tM \} \subset \mathcal{N}'.
\]
To show that $L^1_{\mathcal{E}_0} \circ L^2_{\mathcal{E}_0}$ actually belongs to the paired Lagrangian space $\mathcal{I}^{p,1}(\Delta, \Lambda)$ and determine $p, l$, it suffices to show that the symbol belongs to the class of symbols of product type. For convenience, we shall work with $\chi L^2_{\mathcal{E}_0}$ for $\chi \in C^\infty_0(\mathcal{W})$ and find the symbol of $(\chi L^1_{\mathcal{E}_0})^l \circ (\chi L^2_{\mathcal{E}_0})$. Here we can regard $\chi(y)$ as a function $\chi(y, v)$ defined on $\mathcal{E}_0$. Moreover, the analysis below works for any $\chi \in C^\infty_0(\mathcal{E}_0)$.

For $f, h \in C^\infty_0(M; \text{Sym}^2)$, we compute

$$(\chi L^2_{\mathcal{E}_0} f, \chi L^1_{\mathcal{E}_0} h)_{L^2(\mathbb{R}^3 \times S^2)} = \int_{\mathbb{R}^3} \int_{S^2} \int \chi^2(y) f_{lm}(r, y + rv) h_{jk}(s, y + sv) \theta^j \theta^k \theta^m ds dy dr dv$$

and determine

$$I_{\mathcal{E}_0} = \int_{\mathbb{R}^3} \int_{S^2} \chi^2(x' - x^0 v) f_{lm}(r, x' + (r - x^0) v) h_{jk}(x) \theta^j \theta^k \theta^m dx dr dv,$$

where we made the change of variable $x^0 = s, x' = y + sv$. We obtain that

$$(\chi L^1_{\mathcal{E}_0})^l \circ (\chi L^2_{\mathcal{E}_0} f)_{jk}(x) = \int_{\mathbb{R}^3} \chi^2(x' - x^0 v) f_{lm}(r, x' + (r - x^0) v) \theta^j \theta^k \theta^m dr dv$$

We can write this as an oscillatory integral using

$$f_{lm}(r, x' + (r - x^0) v) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{i \cdot (r, x' + (r - x^0) v) - z \cdot x} f_{lm}(z) dz d\eta$$

Therefore, we have

$$(\chi L^1_{\mathcal{E}_0})^l \circ (\chi L^2_{\mathcal{E}_0} f)_{jk}(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{S^2} \int_{\mathbb{R}^4} \chi^2(x' - x^0 v) e^{i r \cdot (\eta^0 + v \cdot \eta')} e^{i \phi(x, z, v)} \theta^j \theta^k \theta^m f_{lm}(z) dz d\eta$$

and the phase function

$$\phi(x, z, v) = (-z^0, x' - x^0 v - z') \cdot \eta = (x^0 - z^0, x' - z') \cdot \eta,$$

since the integrand is supported on $\eta^0 = -v \cdot \eta'$. Therefore, we can write

$$(\chi L^1_{\mathcal{E}_0})^l \circ (\chi L^2_{\mathcal{E}_0} f)_{jk}(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^4} e^{i (x - z) \cdot \eta} a_{jklm}(x, \eta) f_{lm}(z) dz d\eta,$$

where the symbol is given by

$$a_{jklm}(x, \eta) = \int_{S^2} \delta(\eta^0 + v \cdot \eta') \chi^2(x' - x^0 v) \theta^j \theta^k \theta^m dv.$$

The computation in [14, Lemma 8.1] see also [14, Prop. 11.4] showed that $a_{jklm}$ is a locally integrable function and the integral was explicitly evaluated, which we recall now. Consider the set $S^1_\eta = \{ v \in S^2 : \eta^0 + \eta' v = 0 \}$. If $\eta$ is timelike, $S^1_\eta = \emptyset$. If $\eta$ is spacelike, $S^1_\eta$ is a circle of radius $|\eta|^2 - (\eta^0)^2)^\frac{1}{2}$. Then we have

(4)

$$a_{jklm}(x, \eta) = \begin{cases} \frac{1}{(|\eta|^2 - (\eta^0)^2)^\frac{1}{2}} \int_{S^1_\eta} \chi^2(x' - x^0 v) \theta^j \theta^k \theta^m dv, & \text{if } \eta \text{ is spacelike}, \\ 0, & \text{otherwise}. \end{cases}$$

Now we prove the main result of this section.

**Theorem 3.1.** For any $\chi \in C^\infty_0(\mathcal{W})$ (or $C^\infty_0(\mathcal{E}_0)$), the normal operator $$(\chi L^1_{\mathcal{E}_0})^l \circ (\chi L^2_{\mathcal{E}_0}) \in \mathcal{I}^{\frac{1}{2}, \frac{1}{2}}(M \times M; \Delta, \Lambda).$$ Also for $s \geq -\frac{1}{2}$, $\chi L^2_{\mathcal{E}_0} : H^s_{\text{comp}}(M) \to H^0_{\text{loc}}(\mathcal{E}_0)$ is bounded.
Corollary 4.2] to FIO associated to a canonical relation where one projection is a submersion with conjugate points, the light ray transform on a general Lorentzian manifold is an Greenleaf and Seeger Proof. We see that on $\Delta \setminus \Sigma$, $a_{jklm}(x, \eta)$ is a symbol of order $-1$ so that the normal operator is a pseudo-differential operator of order $-1$ microlocally restricted to $\Delta \setminus \Sigma$. Notice that $\Sigma$ consists of light-like vectors $\eta$ such that $|\eta'|^2 = \eta_0^2$. Then $a_{jklm}(x, \eta)$ with the non-vanishing density factor $|dxd\eta|^\frac{1}{2}$ on $\Delta$ has a conormal type singularity at $\Sigma$. Actually, some expressions of $a_{jklm}$ are calculated explicitly in [14, Section 8] and they do not all vanish at $\Sigma$. According to the discussion at the end of Section 2, the symbol $a(x, \eta)|dxd\eta|^\frac{1}{2}$ belongs to the space $S^{p,l}(M \times M; \Sigma, \Delta)$ with $p + l = -1$ and $l - \frac{1}{2} = 0$. Therefore, we get $p = -\frac{3}{2}, l = \frac{1}{2}$ so $(\chi L_{e_0})^1 \circ (\chi L_{e_0}) \in I^{-\frac{3}{2}, \frac{1}{2}}(\Delta, \Lambda)$.

For the Sobolev estimates, we first apply [6, Theorem 3.3] to obtain that $(\chi L_{e_0})^1 \circ (\chi L_{e_0}) : H^s_{\text{comp}}(M) \to H^{s+1}_{\text{loc}}(M)$ is continuous for $s_0 \leq 1, s \in \mathbb{R}$. By duality argument, $\chi L_{e_0} : H^{-1}_{\text{comp}}(M) \to L^2_{\text{loc}}(e_0)$ is bounded. Next, for any $m$-th order differential operator $P$ on $e_0$, we can use the local expression (2) to write $PL_{e_0}f(y, v) = L_{e_0}(\hat{P}f)(y, v)$ where $\hat{P}$ is a $m$-th order differential operator on $M$. Therefore, we get $\chi L_{e_0} : H^{-m+\frac{3}{2}}_{\text{comp}}(M) \to H^{m+\frac{3}{2}}_{\text{loc}}(e_0)$ for non-negative integers $m$. Finally, by interpolation, we prove the desired Sobolev estimates.

We remark that the Sobolev estimates can be seen from a more general result of Greenleaf and Seeger. In [3, Section 4], the authors demonstrated that in absence of conjugate points, the light ray transform on a general Lorentzian manifold is an FIO associated to a canonical relation where one projection is a submersion with folds, and the mapping properties of such operators are analyzed. We can apply [3, Corollary 4.2] to $L_{e_0}$ to obtain $L_{e_0} : H^s_{\text{comp}}(M) \to H^{s+1}_{\text{loc}}(e_0)$ continuous for any $\epsilon > 0$. Furthermore, one can check the proof of [3, Theorem 1.1] to conclude that the $\epsilon$ loss does not happen for $L_{e_0}$ because the Hessian of the submersion with folds in this case is sign-definite.

4. The parametrix construction. We prove Theorem 2.1. Notice that since we shall consider the operator $L_{e_0}$ acting on distributions in $\mathcal{D}(\mathcal{W})$, we actually have $L_{e_0} = L_{\mathcal{W}}$ so we just need to consider the light ray transform $L_{\mathcal{W}}$. The analysis in Section 3 applies to this case by taking $\mathcal{W} = \mathbb{R}^3$ and $\chi = 1$. Notice that $\Delta \setminus \Sigma$ has disjoint components $\Delta^- = \{(x, \xi, x, -\xi) \in T^*M \setminus 0 \times T^*M \setminus 0 : \xi \text{ is time like}\} = \Omega^- \times \Omega^-$ $\Delta^+ = \{(x, \xi, x, -\xi) \in T^*M \setminus 0 \times T^*M \setminus 0 : \xi \text{ is space like}\} = \Omega^+ \times \Omega^+$ so that $\Delta = \Delta^+ \cup \Delta^- \cup \Sigma$. We consider the set where the symbol $a$ in (4) (when $\chi = 1$) is elliptic. For $(x, \eta) \in \Omega^+$, consider $a(x, \eta) : f_{lm} \to a_{jklm}(x, \eta) f_{lm}$ as a linear map on $\text{Sym}^2_x$. Since $\chi = 1$ does not vanish identically on $S^3_{\eta}$, we know from [14, Lemma 9.1] that the kernel of the map is given by $\mathcal{M}_x = \{cg(x) + \eta \otimes w + w \otimes \eta ; c \in \mathbb{R}, w \in \mathbb{R}^4\}$, for any $x \in M$. Therefore, $a|_{\Omega^+}$ is injective on $C^\infty(M; \text{Sym}^2) \setminus \mathcal{N}$. In particular, one can find $b_{ijkl}(x, \eta)$ such that $b_{ijkl}(x, \eta) a_{jklm}(x, \eta)|_{\Omega^+} = \delta_{ij} \delta_{klm}$ on $C^\infty(M; \text{Sym}^2) \setminus \mathcal{N}$. Since $a_{jklm}(x, \eta)$ is a symbol of order $-1$ on $\Omega^+$, we can find $b_{ijkl}(x, \eta)$ a symbol of order 1 on $\Omega^+$.

---

1The author thanks Prof. Greenleaf for pointing this out.
Now we use the calculus of paired Lagrangian distributions to construct parameters for $L_{\varphi_0}$. The argument is quite standard as for elliptic pseudo-differential operators. We will use the symbol calculus [4, Prop. 3.4] and the composition of $I_{p,l}$ for the flow out model [4, Prop. 3.5]. These results can be found in [1, 2, 11] as well. First, we let $A_0 \in I_{1/2,1}(\Delta, \Lambda)$ be an operator with a symbol $\sigma(A_0)(x, \eta) = b(x, \eta)$ on $\Omega^+$ and otherwise 0. Then we have that acting on $\mathcal{P}(\mathcal{H})$,

$$A_0 \circ L_{\varphi_0} \circ L_{\varphi_0} - H \in I^{-1/2,1}(\Delta, \Lambda) + I^{-1/2,1}(\Delta, \Lambda)$$

where $H \in I^{-1/2,1}(\Delta, \Lambda)$. Actually, the full symbol of $H$ on $\Delta \setminus \Sigma$ is $\delta_{\alpha l}\delta_{\beta m} \chi_{\Omega^+}$ so we have $H - \mathbb{1}_{\Omega^+} \in \bigcap I^{-1/2,1}(\Delta, \Lambda) = I^{-1/2}(\Lambda)$. Next, using the ellipticity of the symbol $a$ (on $\Omega^+$), we can follow the argument in [4, Page 226-227] to get $A \in I_{1/2,1}(\Delta, \Lambda)$ such that

$$A \circ L_{\varphi_0} \circ L_{\varphi_0} = \mathbb{1}_{\Omega^+} + B, \quad B \in I^{-1/2}(\Lambda)$$

modulo a smoothing operator and acting on distributions in $\mathcal{P}(\mathcal{H})$. This completes the proof of Theorem 2.1.

Acknowledgments. The author sincerely thanks Prof. Gunther Uhlmann for suggesting the problem and for many helpful discussions. He is also grateful to Prof. Allan Greenleaf for reference [3] and related comments. The anonymous referee is acknowledged for valuable suggestions. Part of the work was done at IAS, HKUST.

REFERENCES

[1] J. Antoniano and G. Uhlmann, A functional calculus for a class of pseudodifferential operators with singular symbols, Proc. Sympos. Pure Math., 43 (1985), 5–16.

[2] M. de Hoop, G. Uhlmann and A. Vasy, Diffraction from conormal singularities, Annales Scientifiques de l’Ecole Normale Superieure, 4e serie, 48 (2015), 351–408.

[3] A. Greenleaf and A. Seeger, Fourier integral operators with fold singularities, J. reine angew. Math., 455 (1994), 35–56.

[4] A. Greenleaf and G. Uhlmann, Nonlocal inversion formulas for the X-ray transform, Duke Math. J., 58 (1989), 205–240.

[5] A. Greenleaf and G. Uhlmann, Composition of some singular Fourier integral operators and estimates for restricted X-ray transforms, Annales de l’institut Fourier, 40 (1990), 443–466.

[6] A. Greenleaf and G. Uhlmann, Estimates for singular Radon transforms and pseudodifferential operators with singular symbols, Journal of Functional Analysis, 89 (1990), 202–232.

[7] A. Greenleaf and G. Uhlmann, Microlocal techniques in integral geometry, Contemporary Mathematics, 113 (1990), 121–135.

[8] A. Greenleaf and G. Uhlmann, Composition of some singular Fourier integral operators and estimates for restricted X-ray transforms. II, Duke Math. J., 64 (1991), 415–444.

[9] A. Greenleaf and G. Uhlmann, Recovering singularities of a potential from singularities of scattering data, Communications in Mathematical Physics, 157 (1993), 549–572.

[10] V. Guillemin, Cosmology in (2 + 1)-Dimensions, Cyclic Models, and Deformations of $M_{2,1}$, Annals of Mathematics Studies, No. 121, Princeton University Press, 1989.

[11] V. Guillemin and G. Uhlmann, Oscillatory integrals with singular symbols, Duke Math. J., 48 (1981), 251–267.

[12] L. Hörmander, Fourier integral operators. I, Acta Mathematica, 127 (1971), 79–183.

[13] L. Hörmander, The Analysis of Linear Partial Differential Operators IV: Fourier Integral Operators, Springer-Verlag, Berlin, Heidelberg, 2009.

[14] M. Lassas, L. Oksanen, P. Stefanov and G. Uhlmann, On the inverse problem of finding cosmic strings and other topological defects, preprint, arXiv:1505.03123.

[15] R. Melrose and G. Uhlmann, Lagrangian intersection and the Cauchy problem, Communications on Pure and Applied Mathematics, 32 (1979), 483–519.

[16] B. Palacios, G. Uhlmann and Y. Wang, Reducing streaking artifacts in quantitative susceptibility mapping, SIAM Journal of Imaging Sciences, 10 (2017), 1921–1934.
[17] P. Stefanov, Support theorems for the light ray transform on analytic Lorentzian manifolds, Proc. Amer. Math. Soc., 145 (2017), 1259–1274. arXiv:1504.01194.

Received for publication February 2017.

E-mail address: wangy257@math.washington.edu