On Brownian Exit Times
from Some Non-Convex Domains

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July 25, 2018

Abstract

We consider the tail probabilities for Brownian exit time from a
class of perturbed multi-strips in Euclidean plane. Under some assump-
tions we prove that the long stays in a perturbed multi-strip are more
likely than those in a strip of the same width. This effect is related to
the existence of the trapped modes in waveguides.

Keywords: multivariate Brownian motion; exit time; small deviations
heat equation; boundary problems; discrete spectrum.

AMS classification codes: Primary: 60J65. Secondary: 35K05,
35J25.

1 Introduction and results

Let \( W(t) = (W_j(t))_{j=1}^d, t \geq 0, \) be a standard \( d \)-dimensional Brownian
motion with independent coordinates, \( d \geq 1, \) and let \( W_x(t) = x + W(t) \)
denote a Brownian motion starting from a point \( x \in \mathbb{R}^d. \)

We begin from two natural small deviation problems: find the asymptotic
behavior of

\[
\begin{align*}
\mathbb{P}( \| \min_{1 \leq j \leq d} |W_j| \| \leq r), & \quad r \to 0, \\
\mathbb{P}( \| \min_{1 \leq j \leq d} W_j \| \leq r), & \quad r \to 0.
\end{align*}
\]

Here \( \| \cdot \| \) stands for sup-norm, i.e. \( \|X\| = \max_{0 \leq t \leq 1} |X(t)|. \)

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This problem immediately reduces to the so-called exit time asymptotics. Denote by $\tau_{x,\Theta}$ the exit time for $W^x$ from a domain $\Theta \subset \mathbb{R}^d$ containing a point $x$, and put $\tau_\Theta = \tau_{0,\Theta}$. Then, by self-similarity of $W$, we have

$$\mathbb{P}(\| \min_{1 \leq j \leq d} W_j \| \leq r) = \mathbb{P}\{\tau_{\Theta_1} \geq r^{-2}\},$$

$$\mathbb{P}(\| \min_{1 \leq j \leq d} W_j \| \leq r) = \mathbb{P}\{\tau_{\Theta_2} \geq r^{-2}\},$$

where

$$\Theta_1 := \{x \in \mathbb{R}^d : \min_{1 \leq j \leq d} |x_j| < 1\}$$

is a cross, and

$$\Theta_2 := \{x \in \mathbb{R}^d : |\min_{1 \leq j \leq d} x_j| < 1\}$$

is a corner, see Fig. 1 in dimension 2.

![Figure 1: Two-dimensional cross and two-dimensional corner](image)

It is well known that the function

$$u(x,t) := \mathbb{P}\{W^x(s) \in \Theta, 0 \leq s < t\} = \mathbb{P}\{\tau_{x,\Theta} \geq t\}$$

solves the initial-boundary value problem

$$\mathcal{L}u \equiv \partial_t u - \frac{1}{2} \Delta u = 0 \quad \text{in} \quad \Theta \times \mathbb{R}^+; \quad u|_{x \in \partial \Theta} = 0; \quad u|_{t=0} = 1. \tag{2}$$

The solution of (2) for $d = 1$, $\Theta = (-1,1)$,

$$u(x,t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos(\pi(k+1/2)x) \exp(-(k+1/2)^2 \pi^2 t/2), \tag{3}$$

known long ago, was used in the one-dimensional exit probability problem by Chung, cf. [5] or [8, v.2, Ch. X]. Formula (3) yields the asymptotics

$$\mathbb{P}\{\tau_{x,\Theta} \geq t\} = \frac{4}{\pi} \cos(\pi x/2) \exp(-\pi^2 t/8) + O(\exp(-9\pi^2 t/8)), \quad \text{as} \ t \to \infty.$$
Since any one-dimensional projection of a multivariate Brownian motion is a one-dimensional Brownian motion, we obtain the same results when $d > 1$ and
\[ \Theta = \Pi := \{ x \in \mathbb{R}^d : |x| < 1 \} \]
is a layer (strip for $d = 2$). For the exit times from the semi-strip in $\mathbb{R}^2$
\[ \mathcal{P} := \{ x \in \mathbb{R}^2 : x_1 > 0, |x_2| < 1 \} \]
we have essentially the same tail asymptotics because for $x := (x_1, x_2) \in \mathcal{P}$ one has
\[
P(\tau_{x,\mathcal{P}} \geq t) = P(\tau_{x_1,\mathbb{R}_+} \geq t) P(\tau_{x_2,(-1,1)} \geq t) \\
\sim \sqrt{\frac{2}{\pi t}} x_1 \cdot \frac{4}{\pi} \cos(\pi x_2/2) \exp(-\pi^2 t/8), \text{ as } t \to \infty.
\]
One may also cite the works \cite{2, 3, 10, 11} investigating the tails of $\tau_\Theta$ for conic, parabolic, and some other unbounded convex domains of this kind.

Notice that domains $\Theta_1$ and $\Theta_2$ for $d = 2$ look like a bunch of semi-strips when one observes them far from the origin. More formally, we say that a domain $\Theta$ in $\mathbb{R}^2$ is a perturbed multi-strip if outside of some compact $\Theta$ coincides with a finite union of non-intersecting domains $\mathcal{P}_j$ congruent to $\mathcal{P}$.

From the first glance, it might seem that the decay rate of the tail probability of the exit time from a perturbed multi-strip should be the same as for $\mathcal{P}$. Somewhat surprisingly it is not the case in general. For many domains of this type (in particular, for two-dimensional cross and corner) these tail probabilities decay more slowly.

The reason of this phenomenon is as follows. It is well known that the spectrum of the Dirichlet Laplacian in the strip is absolutely continuous:
\[
\sigma(-\Delta, \Pi) = \sigma_{ac}(-\Delta, \Pi) = [\Lambda, \infty),
\]
where $\Lambda = \pi^2/4$ is the first eigenvalue of one-dimensional Dirichlet Laplacian in the cross-section $(-1, 1)$.

The same is true in the semi-strip $\mathcal{P}$. This gives us the essential spectrum of the Dirichlet Laplacian in any perturbed multi-strip $\Theta$:
\[
\sigma_{ess}(-\Delta, \Theta) = \sigma_{ac}(-\Delta, \Theta) = [\Lambda, \infty),
\]
see, e.g., \cite{6, Proposition 1.1.1} for a more general case.

However, this does not exclude the existence of discrete spectrum of the Dirichlet Laplacian in $\Theta$ below the threshold $\Lambda$. This is indeed the case for two-dimensional cross and corner, see \cite{12, 6, Proposition 1.5.2} and \cite{9, 6, Proposition 1.2.2}, respectively. We show below that this discrete spectrum is responsible for the slower decay of the tail probability of the exit time.
Notice that the phenomenon of emerging additional discrete spectrum in perturbed strips is deeply studied in theory of waveguides where it generates so-called trapped modes, see, e.g., [6] and references therein.

Thus, in what follows we assume that the perturbed multi-strip $\Theta$ satisfies

**Assumption A.** The spectrum of the Dirichlet Laplacian in $\Theta$ consists of the ray $[\Lambda, \infty)$ and a non-empty discrete spectrum in the interval $(0, \Lambda)$.

We denote by $\lambda_0$ the smallest eigenvalue and by $\lambda_1$ the infimum of other spectrum (note that $\lambda_1$ either coincides with $\Lambda$ or is another eigenvalue).

**Proposition 1** Let Assumption A be satisfied for a perturbed multi-strip $\Theta$. Then the eigenvalue $\lambda_0$ is simple, and corresponding eigenfunction $v_0$ does not vanish on $\Theta$ and decays exponentially as $x \to \infty$. Namely, if we direct the axis $y_1$ along the axis of semi-strip $P_j$, then for $y \in P_j$

$$v_0(y) = C_j \exp(-\sqrt{\Lambda - \lambda_0} y_1) \cos(\sqrt{\Lambda} y_2) \cdot (1 + o(1)), \quad |y| \to \infty. \quad (4)$$

This fact is well known. We give a sketch of the proof for the reader’s convenience.

The smallest eigenvalue of the Dirichlet Laplacian can be found as minimum of the Rayleigh quotient

$$Q(v) = \frac{\int_\Theta |\nabla v|^2}{\int_\Theta |v|^2}$$

over the Sobolev space $W^{1,2}_0(\Theta)$, and corresponding eigenfunction minimizes this Rayleigh quotient. Since $Q(|v|) = Q(v)$, for arbitrary eigenfunction $\hat{v}$ corresponding to $\lambda_0$, $|\hat{v}|$ is also an eigenfunction. Therefore, $|\hat{v}|$ is superharmonic, and by the maximum principle it cannot vanish inside $\Theta$.

Next, if $\lambda_0$ were a multiple eigenvalue, we could choose a pair of eigenfunctions orthogonal in $L^2(\Theta)$. Then at least one of them should change sign in $\Theta$ that is impossible by the previous argument.

Finally, the relation (4) can be easily obtained for every semi-strip $P_j$ using the separation of variables in the equation for $v_0$. □

Without loss of generality we assume $v_0$ normalized in $L^2(\Theta)$.

Our main result is as follows.

**Theorem 2** Let a perturbed multi-strip $\Theta$ satisfy Assumption A. Then the tail probability $u(x, t)$ admits a representation

$$u(x, t) = A v_0(x) \exp(-\frac{\lambda_0}{2} t) + \hat{u}(x, t), \quad |\hat{u}(x, t)| \leq C(t + 1) \exp(-\frac{\lambda_1}{2} t),$$

where $A = \int_\Theta v_0$ while $\lambda_0 < \lambda_1 \leq \Lambda$ are introduced after Assumption A.
Remark 3 We conjecture that in fact the remainder term admits a better estimate $|\hat{u}(x, t)| \leq C \exp(-\frac{\lambda_1}{2} t)$.

We give the proof of Theorem 2 in Section 2. In Section 3 we discuss this result and give some generalizations.

2 Proof of Theorem 2

In what follows we use a standard notation $(v, w)_{\Theta} := \int_{\Theta} vw$. Various constants depending only on $\Theta$ are denoted by $C$.

A typical example of a perturbed multi-strip $\Theta$ is a connected union of the strip $\Pi \subset \mathbb{R}^2$ and a bounded domain $O \subset \{x \in \mathbb{R}^2 : |x_1| < R\}$. If $\Theta \neq \Pi$ then Assumption A is satisfied for such $\Theta$, see e.g. [6, Theorem 1.4]. We prove the statement for this $\Theta$, the proof for a general case follows the same line.

Let $U(x, t)$ be the solution of the corresponding problem in the strip $\Pi$:

$LU = 0 \quad \text{in} \quad \Pi \times \mathbb{R}_+; \quad U|_{x \in \partial \Pi} = 0; \quad U|_{t=0} \equiv 1.$

Obviously, $U$ does not depend on $x_1$, cf. (3), and $0 < U(\cdot, t) < C \exp(-\frac{\Lambda}{2} t)$.

Now we split $u$ into two parts: $u = u_1 + u_2$, where $u_1(x, t) = U(x_2, t)\chi(x_1)$ and $\chi$ is a smooth cutoff function:

$\chi(x_1) = 0$ for $|x_1| < R; \quad \chi(x_1) = 1$ for $|x_1| > R + 1$.

Notice that $\varphi = u_2(\cdot, 0)$ has compact support, and the support of the function $f := Lu_2 = -Lu_1 = \frac{1}{2} UD^2_{x_1} \chi$

is compact w.r.t. $x$. Therefore, we can apply the spectral decomposition for the operator exponent:

$\begin{align*}
  u_2(\cdot, t) &= \exp\left(\frac{1}{2} \Delta t\right) \varphi(\cdot) + \int_0^t \exp\left(\frac{1}{2} \Delta (t - s)\right) f(\cdot, s) \, ds \\
  &= \int_{[\lambda_0, \infty)} \exp\left(-\frac{\lambda}{2} t\right) dE(\lambda) \varphi(\cdot) + \int_0^t \int_{[\lambda_0, \infty)} \exp\left(-\frac{\lambda}{2} (t - s)\right) dE(\lambda) f(\cdot, s) \, ds, \tag{5}
\end{align*}$

where $E(\lambda)$ is the projector-valued spectral measure generated by the (self-adjoint) Dirichlet-Laplacian operator in $L_2(\Theta)$, see [4, Ch. 6].

Since $\lambda_0$ is a simple eigenvalue, we can rewrite (5) as $u_2 = u_{2,0} + u_{2,1}$, where

$u_{2,0}(\cdot, t) = \left(\exp\left(-\frac{\lambda_0}{2} t\right)(\varphi, v_0)_{\Theta} + \int_0^t \exp\left(-\frac{\lambda_0}{2} (t - s)\right)(f(\cdot, s), v_0)_{\Theta} \, ds\right) v_0(\cdot);$
\[ u_{2,1}(\cdot, t) = u_{2,11}(\cdot, t) + u_{2,12}(\cdot, t) := \int_{[\lambda_1, \infty)} \exp\left(-\frac{\lambda}{2} t\right) dE(\lambda) \varphi(\cdot) \]

\[ + \int_0^t \int_{[\lambda_1, \infty)} \exp\left(-\frac{\lambda}{2} (t - s)\right) dE(\lambda) f(\cdot, s) \, ds. \]

First, we estimate \( L_2 \)-norm of \( u_{2,1} \). Estimate of the first term is evident:

\[ \|u_{2,11}(\cdot, t)\|_{L_2(\Theta)}^2 \leq \exp(-\lambda_1 t) \|\varphi\|_{L_2(\Theta)}^2. \]

Since \( |f(\cdot, s)| < C \exp(-\frac{\lambda}{2} s) \) and \( f(\cdot, s) \) is compactly supported uniformly w.r.t. \( s \), the estimate for \( u_{2,12} \) follows from \( \lambda_1 \leq \Lambda \):

\[ \|u_{2,12}(\cdot, t)\|_{L_2(\Theta)}^2 \leq C \exp(-\lambda_1 t) \left( \int_0^t \exp\left(\frac{\lambda_1 - \Lambda}{2} s\right) ds \right)^2 \leq Ct^2 \exp(-\lambda_1 t). \]

Thus, we have \( \|u_{2,1}(\cdot, t)\|_{L_2(\Theta)} \leq C(t + 1) \exp(-\frac{\lambda_1}{2} t) \). Therefore, standard parabolic estimates give \( |u_{2,1}(\cdot, t)| \leq C(t + 1) \exp(-\frac{\lambda_1}{2} t) \).

Next, we transform the second term in \( u_{2,0} \) using integration by parts:

\[ (f(\cdot, s), v_0)_\Theta = \frac{1}{2} (U(\cdot, s)D^2_{x_1} \chi, v_0)_\Theta = \frac{1}{2} (U(\cdot, s) \chi, D^2_{x_1} v_0)_\Theta \]

\[ = -\frac{\lambda_0}{2} (U(\cdot, s) \chi, v_0)_\Theta - \frac{1}{2} (U(\cdot, s) \chi, D^2_{x_1} v_0)_\Theta \]

\[ = -\frac{\lambda_0}{2} (U(\cdot, s) \chi, v_0)_\Theta - \frac{1}{2} (D^2_{x_1} U(\cdot, s) \chi, v_0)_\Theta \]

\[ = -\frac{\lambda_0}{2} (U(\cdot, s) \chi, v_0)_\Theta - (\partial_x U(\cdot, s) \chi, v_0)_\Theta \]

(\text{notice that all integrals converge due to the exponential decay of } v_0);\]

\[ \int_0^t \exp\left(-\frac{\lambda_0}{2} (t - s)\right) f(\cdot, s), v_0)_\Theta \, ds \]

\[ = -\exp\left(-\frac{\lambda_0}{2} t\right) \int_0^t \exp\left(\frac{\lambda_0}{2} s\right) \left(\frac{\lambda_0}{2} U(\cdot, s) \chi + \partial_x U(\cdot, s) \chi, v_0\right)_\Theta \, ds \]

\[ = -\exp\left(-\frac{\lambda_0}{2} t\right) \cdot \left( \exp\left(\frac{\lambda_0}{2} s\right) U(\cdot, s) \chi, v_0\right)_\Theta \bigg|_0^t \]

\[ = \exp\left(-\frac{\lambda_0}{2} t\right) \cdot (U(\cdot, 0) \chi, v_0)_\Theta - (U(\cdot, 0) \chi, v_0)_\Theta. \]

We sum up all terms and take into account that \( \varphi + U(\cdot, 0) \chi \equiv 1 \). This gives

\[ u(x, t) = (1, v_0)_\Theta v_0(x) \exp\left(-\frac{\lambda_0}{2} t\right) + O\left((t + 1) \exp\left(-\frac{\lambda_1}{2} t\right)\right), \]

as required. \qed
3 Discussion and generalizations

We begin with some comments.

1. The relation \( \lambda_0 < \Lambda \) shows that the long stays in a perturbed multi-strip under Assumption A are more likely than those in a strip of the same width.

2. By Proposition [1], the main term of the tail probability \( 1 \) depends on the initial point \( x \) exponentially. This is a new phenomenon that just does not exist in dimension one. If we start from a remote \( x \), the optimal strategy to stay in \( \Theta \) for a longer time is to run towards the origin and then stay somewhere near it. The relation between the running time and staying time is a subject of optimization, which results in (4). Notice that there is no “running” phase for long stays in a strip.

3. For the case of cross we can give a simple geometric explanation of the inequality \( \lambda_0 < \Lambda \). Consider the square \( \tilde{\Theta}_1 := \{ x \in \mathbb{R}^2 : |x_1| + |x_2| < 2 \} \). Using the version of Chung’s formula \( 3 \) for squares, we easily obtain for \( \tilde{\Theta}_1 \) the same rate of decay as for the strip. On the other hand, since \( \tilde{\Theta}_1 \subset \Theta_1 \), the inequality between the main exponents of the two sets is strict. For the case of corner we do not know an elementary proof.

Numerical calculations give for the cross in \( \mathbb{R}^2 \) \( \lambda_0 \approx 0.66\Lambda \), see, e.g., \([6, \text{Proposition 1.5.2}]\), and for the corner in \( \mathbb{R}^2 \) \( \lambda_0 \approx 0.929\Lambda \), see, e.g., \([7, 6, \text{Proposition 1.2.3}]\). For both domains it is also known that the eigenvalue under threshold is unique.

Next, we notice that the proof of Theorem 2 runs with minor changes for two classes of domains in \( \mathbb{R}^d \), \( d \geq 3 \):

1. A connected union of the layer \( \Pi \subset \mathbb{R}^d \) and a bounded domain \( \mathcal{O} \subset \{ x \in \mathbb{R}^d : |x| < R \} \). In this case \( \Lambda = \pi^2/4 \), and Assumption A for \( \Theta \neq \Pi \) follows from \([6, \text{Theorem 4.5}]\).

2. A \textit{perturbed multi-tube} that is a domain in \( \mathbb{R}^d \) such that outside of a compact it coincides with a finite union of non-intersecting domains congruent to a semitube

\[
\hat{\mathcal{P}} := \{ x \in \mathbb{R}^d : x_1 > 0, \ (x_2, \ldots, x_d) \in \mathcal{M} \},
\]

where \( \mathcal{M} \) is a bounded domain in \( \mathbb{R}^{d-1} \) with “not very bad” (say, Lipschitz) boundary. In this case \( \Lambda \) is the first eigenvalue of \( (d-1) \)-dimensional Dirichlet Laplacian in the cross-section \( \mathcal{M} \).
The Assumption A should be verified separately. For instance, in the paper [1] the existence and uniqueness of the eigenvalue below the threshold was established for the union of two circular unit cylinders whose axes intersect at the right angle.

As for original small deviation problems in dimensions \( d \geq 3 \), they require more thorough study of the corresponding spectral structure and will be investigated in a forthcoming paper.

**Acknowledgement.** This research was supported by Russian Foundation of Basic Research grant 16-01-00258 and by the co-ordinated grants of DFG (GO420/6-1) and St. Petersburg State University (6.65.37.2017).

We are grateful to E. Hashorva for pointing us this problem and to F. Bakharev for valuable discussion and references.

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