Approximation of signals belonging to generalized Lipschitz class using \((N, p_n, q_n)(E, s)\)-summability mean of Fourier series

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Abstract: Degree of approximation of functions of different classes has been studied by several researchers by different summability methods. In the proposed paper, we have established a new theorem for the approximation of a signal (function) belonging to the \(W(L^r, \xi(t))\)-class by \((N, p_n, q_n)(E, s)\)-product summability means of a Fourier series. The result obtained here, generalizes several known theorems.

Subjects: Science; Mathematics & Statistics; Advanced Mathematics; Analysis - Mathematics; Fourier Analysis

Keywords: Degree of approximation; Fourier series; weighted \(W(L^r, \xi(t))\)-class; \((N, p_n, q_n)(E, s)\)-mean; \((N, p_n, q_n)(E, s)\)-mean; Lebesgue integral

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1. Introduction

The theory of summability arose from the process of summation of series and the significance of the concept of summability has been rightly demonstrated in varying contexts, e.g. in Fourier analysis, approximation theory and fixed point theory and many other fields. The theory of approximation of functions has been originated from a well-known theorem of Weierstrass, it has become an exciting field of research with various applications.

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Public Interest Statement

The theory of summability is a wide field of mathematics as regards to the study of Analysis and Functional Analysis. It has many applications, for instance, in numerical analysis (to speed of the rate of convergence), complex analysis (for analytic continuums), operator theory, the theory of orthogonal series, and approximation theory, etc.; while the classical summability theory deals with the generalization of the convergence of sequences or series of real or complex numbers. Further, in classical summability theory, the use of infinite matrices has been a significant research area in mathematical analysis as regards to the study of summability of divergent sequences and series. Recently, various summability methods have interesting applications in approximation theory. The approximation of functions by positive linear operators is a significant research area in mathematical analysis with key relevance to studies of computer-aided geometric design and solution of differential equations.
interdisciplinary field of study for last 130 years. The approximation of functions by generalized Fourier series, based on trigonometric polynomial is a closely related topic in the recent development of engineering and mathematics. The almost summability method and statistical summability method are now an active area of research in summability theory. The error approximation of periodic functions belonging to various Lipschitz classes through summability method is also an active area of research in the last decades. The engineers and scientist use the properties of approximation of functions for designing digital filters. Psarakis and Moustakides (1997) presented a new $L^2$-based method for designing Finite Impulse Response digital filters for getting optimum approximation. In similar manner, $L^p$-space, $L^2$-space, and $L^\infty$-space play an important role for designing digital filters.

The approximation of functions belonging to various Lipschitz classes, through trigonometric Fourier approximation using different summability means has been proved by various investigators, like Nigam (2011), Lal (2000), Paikray, Jati, Misra, and Sahoo (2012) and many others. Recently, by generalized Hölder’s inequality and Minkowski’s inequality, Mishra, Sonavane, and Mishra (2013) have proved $L^r$ approximation of signals belonging to $W(L^r,\xi(t))$-class by $C^1$.N$_p$-summability means of conjugate series of Fourier series. Mishra and Sonavane (2015) has proved approximation of functions belonging to the Lipschitz class by product mean $(N, p_n, q_n)$ of Fourier series. In an attempt to make an advance study in this direction, in this paper, we obtain a theorem on the approximation of functions belonging to $W(L^r,\xi(t))$ by $(N, p_n, q_n)$-summability means of Fourier series which generalizes several known and unknown results.

2. Definition and notations

Let $\sum u_n$ be an infinite series with the sequence of partial sum $\{s_n\}$. Let $\{p_n\}$ and $\{q_n\}$ be sequences of positive real number such that,

$$p_n = \sum_{k=0}^{n} p_k \quad \text{and} \quad q_n = \sum_{k=0}^{n} q_k$$

and let $R_n = p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0$ (≠0), $p_{-1} = q_{-1} = R_{-1} = 0$.

The sequence to sequence transformation (Mishra, Palo, Padhy, Samanta, & Misra, 2014),

$$t^n_n = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k s_k$$

defines the sequence $\{t^n_n\}$ of the $(N, p_n, q_n)$ mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $p_n$ and $q_n$.

Similarly, we define the extended Riesz mean,

$$\tilde{t}^n_n = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k s_k.$$  \hspace{1cm} (2.1)

where $R_n = p_0 q_0 + p_1 q_1 + \cdots + p_n q_n$ (≠0), $p_{-1} = q_{-1} = R_{-1} = 0$.

If $\sum u_n$, then the series $\sum u_n$ is $(\tilde{N}, p_n, q_n)$ summable to $s$.

Analogous to regularity conditions of Riesz summability (Hardy, 1949), we have

(i) $\frac{p_{n+k}}{R_n} \to 0$, for each integer $k \geq 0$ as $n \to \infty$ and

(ii) \[ \sum_{k=0}^{n} p_k q_k \] < $C |R_n|$, where, $C$ is any positive integer independent of $n$. 
The sequence to sequence transformation (Hardy, 1949),

\[ E_n^s = \frac{1}{(1 + s)^n} \sum_{v=0}^{n} \binom{n}{v} s^{n-v} s_v^r \]  

defines the sequence \( \{E_n^s\} \) of \((E, s)\) means of the sequence \( \{s_n\} \).

If \( E_n^s \rightarrow s \) as \( n \rightarrow \infty \), then \( \sum u_n \) summable to \( s \) with respect to \((E, s)\) summability and \((E, s)\) method is regular (Hardy, 1949).

Now we define, a new composite transformation \((\tilde{N}, p_n, q_n)\) over \((E, s)\) of \( \{s_n\} \) as

\[ T_n^{\tilde{E}} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k (E_k^s) = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k \left\{ \frac{1}{(1 + s)^k} \sum_{v=0}^{k} \binom{k}{v} s^{k-v} s_v^r \right\} \]  

(2.3)

If \( T_n^{\tilde{E}} \rightarrow s \) as \( n \rightarrow \infty \), then \( \sum u_n \) is summable to \( s \) by \((\tilde{N}, p_n, q_n)\)\((E, s)\) means.

Further as \((\tilde{N}, p_n, q_n)\) and \((E, s)\) means are regular, so \((\tilde{N}, p_n, q_n)\)\((E, s)\) mean is also regular.

Remark 1 If we put \( q_n = 1 \) in Equation (2.1) then \((\tilde{N}, p_n, q_n)\)-summability method reduces to \((\tilde{N}, p_n)\)-summability and for \( p_n = 1 \) it reduces to \((\tilde{N}, q_n)\)-summability.

Let \( f \) is a \( 2\pi \) periodic function belonging to \( L^r[0, 2\pi] \), \( r \geq 1 \), with the partial sum \( s_n(f) \), then

\[ s_n(f) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx). \]  

(2.4)

Here, as regards to Lipschitz classes we may recall that, a signal (function) \( f \in \text{Lip}(\alpha) \), if

\[ |f(x + t) - f(x)| = O(|t|^\alpha) \quad \text{for } 0 < \alpha \leq 1, \ t > 0, \]  

and \( f \in \text{Lip}(\alpha, r) \), for \( 0 \leq x \leq 2\pi \), if

\[ \left( \int_{[0,2\pi]} |f(x + t) - f(x)|^r \ dx \right)^{\frac{1}{r}} = O(|t|^\alpha) \quad \text{for } 0 < \alpha \leq 1, \ t > 0, \ r \geq 1. \]

Again, \( f \in \text{Lip}(\zeta(t), r) \), if

\[ \|f(x + t) - f(x)\|_r = \left( \int_{[0,2\pi]} |f(x + t) - f(x)|^r \ dx \right)^{\frac{1}{r}} = O(\zeta(t)), \ r \geq 1, \ t > 0, \]

where \( \zeta(t) \) is a positive increasing function.

Similarly, \( f \in W(L, \zeta(t)) \), if

\[ \|f(x + t) - f(x)\|_\beta = \left( \int_{[0,2\pi]} |f(x + t) - f(x)|^\beta \ dx \right)^{\frac{1}{\beta}} = O(\zeta(t)), \ \beta \geq 0. \]

Further as regards to the norm in \( L_\infty \) and \( L_r \)-spaces, we may recall that \( L_\infty \)-norm of a function \( f:R \rightarrow R \) is defined by

\[ \|f\|_\infty = \sup \{|f(x)|: x \in R\} \]

and \( L_r \)-norm of a function \( f:R \rightarrow R \) is defined by
Next, the degree of approximation of a function \( f: \mathbb{R} \rightarrow \mathbb{R} \) by a trigonometric polynomial \( t_n \) of order \( n \) under \( \| \cdot \|_\infty \) is defined by
\[
\| f - t_n \|_\infty = \sup \{ |f(x) - f(x)| : x \in \mathbb{R} \},
\]
and the degree of approximation of \( E_n(f) \) of a function \( f \in L_1 \) is given by
\[
E_n(f) = \min_{t_n} \| t_n - f \|_r.
\]

We use the following notations throughout this paper:

\[
\phi(t) = f(x + t) + f(x - t) - 2f(x)
\]

and
\[
K_n(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{k} p_k q_k \left\{ \frac{1}{(1 + s)^t} \sum_{v=0}^{k} \left( \frac{k}{v} \right)^s \sin(v + \frac{1}{2}) t \right\}.
\]

**Remark 2** If we take \( \beta = 0 \), then \( W(L_n, \xi(t)) \)-class coincides with the class \( \text{Lip}(\xi(t), r) \); if \( \xi(t) = t^\alpha \) then the class \( \text{Lip}(\xi(t), r) \) coincides with \( \text{Lip}(\alpha, r) \)-class and if \( r \rightarrow \infty \) then \( \text{Lip}(\alpha, r) \)-class reduces to the \( \text{Lip}(\alpha) \).

### 3. Known theorems

Dealing with the product \((C, 1)[E, q]\) mean, in Nigam (2011) proved the following theorem.

**Theorem 1** If \( f \) is a \( 2\pi \)-periodic function, Lebesgue integrable on \([0, 2\pi]\) and belongs to \( W(L_n, \xi(t)) \) class, then its degree of approximation is given by
\[
\| C_n^1 E_n^1 f \|_r - f = O \left( (n + 1)^{\delta' + \frac{1}{2}} \xi \left( \frac{1}{n + 1} \right) \right),
\]
provided \( \xi(t) \) satisfies the following conditions:

\[
\left\{ \frac{\xi(t)}{t} \right\} \text{ be a decreasing sequence},
\]

\[
\int_0^t \left( \frac{t|\phi(t)|}{\xi(t)} \right) \sin^\alpha t \, dt = O \left( \frac{1}{n + 1} \right)
\]

and

\[
\int_0^t \left( \frac{t^{-\delta} |\phi(t)|}{\xi(t)} \right) \, dt = O \left( (n + 1)^{\delta} \right),
\]

where \( \delta \) is any arbitrary number such that \( \delta(1 - \delta) - 1 > 0, \frac{1}{7} + \frac{1}{2} = 1 \), conditions (3.3) and (3.4) hold uniformly in \( x \) and \( C_n^1 E_n^1 \) is \((C, 1)[E, q]\) means of the Fourier series (2.4).

Next, dealing with degree of approximation, in Mishra et. al. (2014) proved the following theorem.

**Theorem 2** For a positive increasing function \( \xi(t) \) and an integer \( l > 1 \), if \( f \) is a \( 2\pi \)-periodic function on the class \( \text{Lip}(\xi(t), l) \), then the degree of approximation by product \((E, s)(N, p_n, q_n)\)-summability mean of Fourier series (2.4) is given by
\[ \| \tau_n - f \|_m = O \left( \frac{1}{(n + 1)^{\alpha - l}} \right), \quad 0 < \alpha < 1, \quad l \geq 1, \]

where \( \tau_n \) is \((E, s)(N, p_n, q_n)\)-summability mean.

4. Main theorem

Here, just by replacing Nörlund summability by extended Riesz summability and taking the reverse composition, we have proved the following theorem.

**Theorem 3** Let \( f \) be a \( 2\pi \) periodic function which is integrable in Lebesgue sense in \([0, 2\pi]\). If \( f \in W(L_r, \zeta(t)) \) class, then its degree of approximation is given by

\[ \| T^R_n - f \|_r = O \left( \frac{1}{(n + 1)} \right) \]

where \( T^R_n \) is the \((N, p_n, q_n)(E, s)\) transform of \( s_n \), provided \( \zeta(t) \) satisfies the following conditions;

\[ \left\{ \frac{\zeta(t)}{t} \right\} \text{ be a decreasing sequence,} \]

(4.2)

\[ \int_{0}^{\frac{1}{n+1}} \left( \frac{t^{|\zeta(t)|}}{\zeta(t)} \right)^r \sin^m t \, dt = O \left( \frac{1}{n + 1} \right) \]

(4.3)

and

\[ \int_{\frac{1}{n+1}}^{1} \left( \frac{t^{|\zeta(t)|}}{\zeta(t)} \right)^r \, dt = O \left( (n + 1)^r \right). \]

(4.4)

To prove the theorem, we need the following lemmas.

**Lemma 1** \( |K_n(t)| = O(n) \), for \( 0 \leq t \leq \frac{1}{n+1} \).

**Proof** For \( 0 \leq t \leq \frac{1}{n+1} \), as \( \sin nt \leq n \sin t \), so we have

\[ |K_n(t)| = \frac{1}{2\pi R_n} \left| \sum_{k=0}^{n} p_k q_k \left\{ \frac{1}{(1 + s)^k} \sum_{v=0}^{k} \binom{k}{v} s^{v-\frac{1}{2}} \frac{\sin \left( v + \frac{1}{2} \right) t}{\sin \left( \frac{t}{2} \right)} \right\} \right| \]

\[ \leq \frac{2(n + 1)}{2\pi R_n} \sum_{k=0}^{n} p_k q_k \frac{1}{(1 + s)^k} \left| \sum_{v=0}^{k} \binom{k}{v} s^{v-\frac{1}{2}} \frac{v + \frac{1}{2}}{\sin \left( \frac{t}{2} \right)} \right| \]

\[ = \frac{2(n + 1)}{2\pi R_n} \sum_{k=0}^{n} p_k q_k \frac{1}{(1 + s)^k} \left| 1 + s \right| \]

\[ = O(n). \]

**Lemma 2** \( |K_n(t)| = O\left( \frac{1}{t} \right) \) for \( \frac{1}{n+1} < t \leq \pi \).

**Proof** For \( \frac{1}{n+1} < t \leq \pi \), as \( \sin \frac{1}{2} \geq \frac{1}{x} \) (Jordan’s lemma) and \( \sin nt \leq 1 \), so
Further, \( I_n = \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_k q_k \left\{ \sum_{v=0}^{k} \left( \begin{array}{c} k \\ v \end{array} \right) s^{k-v} \sin \left( \frac{v + \frac{1}{2}}{2} \right) t \right\} \), and

\[
\int_{[0, \frac{1}{2}]} \int_{[0, \frac{1}{2}]} s_n(f) - f(x) = \frac{1}{2\pi} \int_{[0, \frac{1}{2}]} \phi(t) \frac{\sin(n + \frac{1}{2}) t}{\sin \frac{t}{2}} dt.
\]

As \( |\phi(t, x) - \phi(x)| \leq |f(u + x + t) - f(u + x)| + |f(u - x - t) - f(u - x)| \),

so, by using Minkowski's inequality,

\[
\left\{ \int_{[0, \frac{1}{2}]} \left( \phi(t, x) - \phi(x) \right) \sin^d x \right\}^2 \leq \left\{ \int_{[0, \frac{1}{2}]} \left( f(u + x + t) - f(u + x) \right) \sin^d x \right\}^2 dx + \left\{ \int_{[0, \frac{1}{2}]} \left( f(u - x - t) - f(u - x) \right) \sin^d x \right\}^2 dx
\]

\[= O(\xi(t)).\]

Further \( f \in W(L, \xi(t)) \) implies \( \phi \in W(L, \xi(t)) \), thus

\[|I_1| \leq \int_{[0, \frac{1}{2}]} \left| \frac{t \phi(t) \sin^d t}{\xi(t)} \cdot \xi(t)K_{m1} \right| dt.
\]

Now by Hölder's inequality and Lemma 1, we have
\[ |I_1| \leq \left( \int_{(0, \frac{1}{n+1})} \left| \frac{t^d(t) \sin^\theta t}{\xi(t)} \right|^r \, dt \right)^{\frac{1}{r}} \times \left( \lim_{\varepsilon \to 0} \int_{(\varepsilon, \frac{1}{n+1})} \left| \frac{\xi(t)K_n(t)}{t \sin^\theta t} \right|^s \, dt \right)^{\frac{1}{s}} \text{ for some } 0 < \varepsilon < \frac{1}{n+1} \]

\[ = O\left( \frac{1}{n+1} \right) \left( \lim_{\varepsilon \to 0} \int_{(\varepsilon, 1 \sin^\theta)} \left( \frac{\xi(n(t))}{t \sin^\theta t} \right)^s \, dt \right)^{\frac{1}{s}} \text{ by (4.3)} \]

\[ = O\left( \frac{n}{n+1} \right) \left( \lim_{\varepsilon \to 0} \int_{(\varepsilon, 1 \sin^\theta)} \left( \frac{\xi(t)}{t \sin^\theta t} \right)^s \, dt \right)^{\frac{1}{s}}. \]

Also, by 2nd mean value theorem, we have

\[ |I_1| = O(1)\xi\left( \frac{1}{n+1} \right) \left( \int_{(1/n^\theta, 1]} \left( \frac{1}{t^{1+s_\theta}} \right)^s \, dt \right)^{\frac{1}{s}} \]

\[ = O\left( \xi\left( \frac{1}{n+1} \right) \left( t^{-s_\theta 1+1} \right) \right) \]

\[ = O\left( \xi\left( \frac{1}{n+1} \right) (n+1)^{s_\theta 1} \right) \text{ since } \frac{1}{r} + \frac{1}{s} = 1. \]

Next, \[ |I_2| \leq \int_{\frac{1}{n+1}}^{\frac{1}{n^\theta}} t^{-d} \phi(t) \sin^\theta t \frac{\xi(t)K_n(t)}{\xi(t)} \frac{\xi(t)K_n(t)}{t^{-d} \sin^\theta t} \, dt. \]

Now by Hölder inequality and Lemma 2, we have

\[ |I_2| \leq \left( \int_{(1/n^\theta, 1]} \left| \frac{t^{-d} \phi(t) \sin^\theta t}{\xi(t)} \right|^r \, dt \right)^{\frac{1}{r}} \left( \int_{(1/n^\theta, 1]} \left| \frac{\xi(t)K_n(t)}{t^{-d} \sin^\theta t} \right|^s \, dt \right)^{\frac{1}{s}} \]

\[ = O\left( (n+1)^d \right) \left( \int_{(1/n^\theta, 1]} \left( \frac{\xi(t)}{t^{1-s_\theta 0}} \right)^s \, dt \right)^{\frac{1}{s}} \text{ by (4.4)} \]

\[ = \left( (n+1)^d \right) \left( \int_{(1/n^\theta, 1]} \left( \frac{\xi(t)}{y^{1-s_\theta 0}} \right)^s \, dy \right)^{\frac{1}{s}} \text{ by (4.2).} \]

Again by using 2nd mean value theorem
\[ |I_2| = O\left\{ (n+1)^{\beta} \xi \left( \frac{1}{n+1} \right) \right\} \left\{ \int_{1\over n+1}^{1\over n} \frac{dy}{y^{t(1-\beta)\delta+2}} \right\} ^{\frac{1}{\delta}} \]

\[ = O\left\{ (n+1)^{\beta} \xi \left( \frac{1}{n+1} \right) \right\} \left( \frac{(n+1)^{(s+1)(\beta-\delta)-1}}{s(1+\beta-\delta)-1} \right) ^{\frac{1}{\delta}} \]  

(5.3)

\[ \Rightarrow \|T_n - f\|_r = O\left\{ (n+1)^{\beta+1} \xi \left( \frac{1}{n+1} \right) \right\} \left[ \int_{0}^{2\pi} O\left\{ (n+1)^{\beta+1} \xi \left( \frac{1}{n+1} \right) \right\} \right] ^{\frac{1}{\delta}} \]

\[ = O\left\{ (n+1)^{\beta+1} \xi \left( \frac{1}{n+1} \right) \right\} \left[ \int_{0}^{2\pi} dx \right] ^{\frac{1}{\delta}} \]

\[ = O\left\{ (n+1)^{\beta+1} \xi \left( \frac{1}{n+1} \right) \right\} . \]

Next, by using (5.2) and (5.3), in (5.1) we have

\[ \|T_n - f\|_r = O\left\{ (n+1)^{\beta+1} \xi \left( \frac{1}{n+1} \right) \right\} . \]

Which completes the proof of theorem.

**Corollary 1** If we put \( \beta = 0 \) in Theorem 3, then the generalized Lipschitz \( W(\cdot, \xi(t)) \)-class reduces to \( \operatorname{Lip}(\xi(t), r) \), where \( \xi(t) \) is any positive increasing function and \( 1 > 1 \). If \( f \) is \( 2\pi \)-periodic and belonging to class \( \operatorname{Lip}(\xi(t), r) \), then the degree of approximation by \( \{\hat{N}, p_n, q_n\}(E, s) \)-summability mean of Fourier series is

\[ \|T_n - f\|_r = O\left\{ (n+1)^{\beta+1} \xi \left( \frac{1}{n+1} \right) \right\} r \geq 1. \]

(5.4)

**Corollary 2** If we put \( \beta = 0 \) and \( \xi(t) = t^\alpha \), \( 0 < t \leq 1 \), in Theorem 3, the generalized Lipschitz \( W(\cdot, \xi(t)) \)-class reduces to \( \operatorname{Lip}(\alpha, r) \), then the degree of approximation of \( 2\pi \) periodic function \( f \) belonging to class \( \operatorname{Lip}(\alpha, r) \) by \( \{\hat{N}, p_n, q_n\}(E, s) \)-summability mean of Fourier series is

\[ \|T_n - f\|_r = O\left\{ \frac{1}{(n+1)^{\alpha+1}} \right\}, \quad 0 < \alpha < 1, \quad l \geq 1. \]

(5.5)

**Corollary 3** If we put \( \beta = 0, \xi(t) = t^\alpha \), \( 0 < \alpha < 1 \) and \( r \to \infty \) then the generalized Lipschitz \( W(\cdot, \xi(t)) \)-class reduces to \( \operatorname{Lip}(\alpha) \), then the degree of approximation of \( 2\pi \) periodic function by \( \{\hat{N}, p_n, q_n\}(E, s) \)-summability mean of Fourier series \( s_n(f) \) is

\[ \|T_n - f\|_{\infty} = O\left\{ (n+1)^{-\alpha} \right\}, \quad \text{where} \ 0 < \alpha < 1. \]

(5.6)
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References
Hardy, G. H. (1949). Divergent Series (1st ed.). Oxford: Oxford University Press.
Lal, S. (2000). On degree of approximation of conjugate of a function belonging to weighted \(L_p, \xi(t)\) class by matrix \((N, p_n) (E, s)\)-summability means of conjugate series of Fourier series. Tamkang Journal of Mathematics, 31, 279–288.
Mishra, N., Polo, P., Padhy, B. P., Samanta, P., & Misra, U. K. (2016). Approximation of Fourier series of a function of Lipschitz class by product means. Journal of Advances in Mathematics, 9, 2475–2486.
Mishra, V. N., & Sonavane, V. (2015). Approximation of functions of Lipschitz class by \((N, p_n) (E, s)\)-summability means of conjugate series of Fourier series. Journal of Classical Analysis, 6, 137–151.
Mishra, V. N., Sonavane, V., & Mishra, L. N. (2013). \(L\)-Approximation of signals (functions) belonging to weighted \(W(L_1, \xi(t))\) class by \((C, 1) (E, q)\) means. Tamkang Journal of Mathematics, 42, 31–37.
Paikray, S. K., Jati, R. K., Misra, U. K., & Sahoo, N. C. (2012). On degree of approximation of Fourier series by product means. General Math. Notes, 13, 22–30.
Psarakis, E. Z., & Moustakides, G. V. (1997). An \(L^2\)-based method for the design of 1-D zero phase FIR digital filters. IEEE Transactions on Circuits and Systems, 44, 551–601.