Diluted magnetic Dirac-Weyl materials: Susceptibility and ferromagnetism in three-dimensional chiral gapless semimetals

Sanghyun Park\textsuperscript{1}, Hongki Min\textsuperscript{1,}\ E. H. Hwang\textsuperscript{2,}\ and S. Das Sarma\textsuperscript{3}

\textsuperscript{1} Department of Physics and Astronomy, Seoul National University, Seoul 08826, Korea
\textsuperscript{2} SKKU Advanced Institute of Nanotechnology and Department of Nano Engineering, Sungkyunkwan University, Suwon, 16419, Korea
\textsuperscript{3} Condensed Matter Theory Center and Joint Quantum Institute, Department of Physics, University of Maryland, College Park, Maryland 20742-4111

(Dated: May 1, 2018)

We theoretically investigate the temperature-dependent static susceptibility and long-range magnetic coupling of three-dimensional (3D) chiral gapless electron-hole systems (semimetals) with arbitrary band dispersion \[\varepsilon(k) \sim k^N\], where \(k\) is the wave vector and \(N\) is a positive integer. We study the magnetic properties of these systems in the presence of dilute random magnetic impurities. Assuming carrier-mediated Ruderman-Kittel-Kasuya-Yosida indirect exchange interaction, we find that the magnetic ordering of intrinsic 3D chiral semimetals in the presence of dilute magnetic impurities is ferromagnetic for all values of \(N\). Using finite-temperature self-consistent field approximation, we calculate the ferromagnetic transition temperature \((T_c)\). We find that \(T_c\) increases with increasing \(N\) due to the enhanced density of states, and the calculated \(T_c\) is experimentally accessible assuming reasonable coupling between the magnetic impurities and itinerant carriers.

I. INTRODUCTION

In recent years, there has been substantial interest in three-dimensional (3D) Weyl/Dirac semimetals, which have relativistic linear energy dispersion \([1,2]\). These systems are effectively 3D versions of graphene which is the quintessential 2D Dirac system. The magnetic properties of Weyl and Dirac semimetals have been studied theoretically, demonstrating the possibility of magnetic ordering of the dopant magnetic impurities at zero temperature \([3,10]\). Given the significant interest in 3D gapless semimetal systems at present, it is also interesting to find the effects of arbitrary band dispersion and finite temperature on the magnetic properties of 3D gapless systems in the presence of random magnetic impurities. The indirect exchange interactions between magnetic impurities through carriers of a host material (i.e., Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction \([11,12]\) in semimetal systems has become an interesting issue. In addition, the nonlinear energy dispersions of itinerant carriers result in an interesting behavior of the RKKY interaction between magnetic spins, which provides a more complete picture of the qualitative nature of magnetic properties in gapless semimetals with arbitrary band dispersion. In particular, it is useful to know whether 3D Dirac-Weyl gapless semimetals could magnetically order at finite temperatures through the RKKY coupling, and how the resultant magnetic transition temperature depends on the band carrier energy dispersion.

In this paper, we study the magnetic properties of 3D gapless electron-hole systems at finite temperatures with arbitrary band dispersion, focusing on the possibility of long-range ordering in the magnetic moments that are embedded in the system. In order to study the carrier-mediated indirect RKKY exchange interaction among the random magnetic impurities with the itinerant carriers mediating the magnetic interaction between the impurities, we calculate the temperature-dependent response functions and the corresponding long-range magnetic coupling between dilute random magnetic impurities. The effects of finite temperature, disorder, and carrier mean-free path on the RKKY interaction are also considered systematically. Inclusion of the mean-free path in the theory allows us to make a specific prediction about the dependence of the magnetic behavior of the system on the carrier transport properties \([13]\). A smooth interpolation between long-range and short-range magnetic interactions is possible by varying the cutoff parameter \(R\) in the range of the RKKY interaction, which is related to the localization length of the carriers in semimetals \([16,18]\).

By considering all these effects together within one comprehensive mean-field theory, we calculate the ferromagnetic transition temperature in the framework of a finite-temperature self-consistent field approximation \([14]\) for the ferromagnetism in 3D gapless semimetals. We find that in 3D gapless semimetals, the ferromagnetic ordering between magnetic impurities induced by the RKKY exchange interaction is favored with enhanced magnetic coupling, as the energy dispersion has a higher power-law. Our results indicate that within the experimentally accessible range of parameters, ferromagnetic ordering between magnetic impurities is possible in gapless 3D semimetals. Ferromagnetism in 3D semimetals, as predicted in our theory, can be utilized in spintronics applications if our predictions are validated experimentally. We predict that it should be possible to experimentally induce long-range finite-temperature ferromag-
netic ordering in 3D Dirac-Weyl materials by magnetically doping the system.

This paper is organized as follows. In Sec. II we describe our model and calculate the finite-temperature static susceptibilities. In Sec. III we provide the calculated results of the effective magnetic coupling through RKKY interaction in 3D chiral gapless semimetals. The conclusions are provided in Sec. IV with a discussion on the momentum-cutoff effect on long-range oscillations.

II. MODEL

To describe the 3D chiral gapless semimetals including Weyl/Dirac semimetals, we introduce the following Hamiltonian with an isotropic energy dispersion characterized by a positive integer $N$ [19]:

$$H = \varepsilon_0 \left( \frac{|k|}{k_0} \right)^N \hat{k} \cdot \sigma,$$

where $\sigma$ represents the Pauli matrices acting in the space of the two bands near the band touching point, and $\varepsilon_0$ and $k_0$ are materials dependent constants with dimensions of energy and wave vector, respectively. The energy dispersion of the Hamiltonian is given by $\varepsilon_{\lambda,k} = \lambda \varepsilon_0 \left( \frac{|k|}{k_0} \right)^N$, where $\lambda = \pm 1$ is the index for the conduction (valence) band. Note that the Hamiltonian with $N = 1$ corresponds to Weyl semimetals with linear energy dispersion. We assume that the system is intrinsic with the Fermi energy at the band touching point, which we take to be the zero of energy. We are thus considering undoped intrinsic Dirac-Weyl systems with the chemical potential pinned at the Dirac-Weyl point.

Carrier-mediated RKKY indirect exchange interaction between local moments is proportional to the static carrier susceptibility. At finite temperatures, the static susceptibility is given by

$$\chi(q,T) = -g \sum_{\lambda,\lambda'} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\varepsilon_{\lambda,k} - \varepsilon_{\lambda',k'}} F_{\lambda,\lambda'}(k,k')$$

where $g$ is the total (e.g., spin, valley, etc.) degeneracy factor, $\lambda = \pm 1$ is the wavefunction overlap between $\sigma$ and $\sigma'$, and $k' = k + q$. For the 3D chiral gapless system described by Eq. (1), $F_{\lambda,\lambda'}(k,k') = \frac{1}{2} (1 + \lambda \lambda' \cos \theta_{k,k'})$ for all $N$, where $\theta_{k,k'}$ is the angle between $k$ and $k'$. Note that $F$ arises entirely from the chirality of the system.

Dividing the sum in Eq. (2) into interband ($\lambda \neq \lambda'$) and intraband ($\lambda = \lambda'$) contributions, the static susceptibility can be decomposed into $\chi(q,T) = \chi^+(q,T) + \chi^-(q,T)$, where $\chi^\pm$ denote the interband (+) and intraband (−) contributions, respectively. With the density of states (DOS) at $T = 0, D_N(q) = \frac{q^2}{2 \pi^2 N k_0^3}$, the normalized susceptibility $\chi^\pm$ can be rewritten as

$$\frac{\chi^\pm(q,T)}{D_N(q)} = \frac{N}{4} \int_0^\infty x^2 dx \int_0^\pi \sin \theta d\theta \left( \frac{1 + \cos \psi}{x^N} \right)^N \times \left( \frac{\theta}{2T/T_0} \pm \frac{\tan \psi}{2T/T_0} \right), (3)$$

where $\theta$ is the angle between $k$ and $q$, and $\psi$ is the angle between $k$ and $k' = k + q$. In Eq. (3), $x = k/q, x' = k'/q = \sqrt{1 + 2x \cos \theta + x^2}, \cos \psi = (x + \cos \theta)/x'$, and $T_0 = \varepsilon_0/k_B$. Note that for $N = 1$, a finite (ultraviolet) momentum cutoff is required for the convergence of the integral. For the calculation, we set $g = 4$ and $k_0 = a^{-1}$, where $a$ is the lattice constant of the system.

At zero temperature ($T = 0$), due to the phase space restriction, the intraband part $\chi^-$ vanishes and only the interband part $\chi^+$ contributes to the total susceptibility. In the long wavelength limit ($q \to 0$), the susceptibility approaches the DOS and $\chi^+(q,T = 0) \propto q^{1-N}$. Specifically, we find that $\chi^-(q \to 0, T) \propto T^{2-N}$, which shows the same power-law dependence as the DOS, $D_N(\varepsilon) \propto \varepsilon^{1-N}$ with energy $\varepsilon$ replaced by $T$. Therefore, as temperature increases, the total susceptibility at $q = 0$ increases for $N = 1, 2$, remains constant for $N = 3$, and decreases for $N \geq 4$. These analytical findings are helpful in understanding our detailed numerical results presented in the rest of this paper.

Figure II shows the calculated static susceptibility as a function of the wave vector for several temperatures. For $N = 1, 2$, the susceptibility increases with temperature, whereas for $N \geq 4$, it decreases with temperature, as expected. Interestingly, for $N \geq 3$, the finite-temperature result in the $T \to 0$ limit is different from the zero-temperature value, i.e., $\chi(0,T = 0) \neq \chi(0,T \to 0)$. Note that for $N = 3$, the DOS $D_N(q)$ becomes constant and $\chi(0,T \to 0)$ approaches the constant DOS, whereas $\chi(0,T = 0)$ can be obtained from Eq. (3). For $N = 3$, we find that $\chi(0,T = 0)/\chi(0,T \to 0) \approx 0.8229$. This $T = 0$ non-analyticity in the $N = 3$ susceptibility follows from the fact that the DOS has a 'kink' structure at $N = 3$ with $D_N(\varepsilon)$ increasing (decreasing) as a function of increasing energy for $N < 3$ ($N > 3$).

III. RKKY INTERACTION AND EFFECTIVE MAGNETIC COUPLING

To study the effective magnetic coupling between random magnetic impurities (which are treated as quenched
classical magnetic moments), we consider the carrier-mediated RKKY indirect interaction exchange. The direct exchange interaction between magnetic impurities can be accounted for by the interaction between a localized (classical) spin $S_i$ of a magnetic impurity located at $r_i$ and an itinerant electron spin $s$ located at $r$. It is given by $V(r) = J_{\text{ex}} S_i \cdot s(\mathbf{r}_i - \mathbf{r})$, where $J_{\text{ex}}$ is the local exchange coupling between the quenched impurity and the itinerant carriers. ($J_{\text{ex}}$, which depends on the nature of the magnetic impurities, is an unknown parameter in our theory providing the overall magnitude of the magnetic coupling in the system.) Then, the effective Hamiltonian that describes the magnetic interactions between the classical Heisenberg spins $S_i$ and $S_j$ located at $r_i$ and $r_j$, respectively, is given by

$$H = -\sum_{i,j} J_{\text{RKKY}}(r_i - r_j) S_i \cdot S_j,$$

where

$$J_{\text{RKKY}}(r, T) = \frac{J_{\text{ex}} a^3}{4} \chi(r, T).$$

The RKKY range function $\chi(r, T)$ is defined by the Fourier transform of the static susceptibility $\chi(q, T)$. For an isotropic system in 3D, it is given by

$$\chi(r, T) = \frac{1}{2\pi^2} \int_0^\infty q^2 dq j_0(qr) \chi(q, T),$$

where $j_0(x)$ is the spherical Bessel function of the first kind. Since the large momentum cutoff $q_c = a^{-1}$ is natural for a continuum theory, we set $q_c = a^{-1}$ in the numerical calculation of the range function in Eq. (6).

Figure 2 shows the range functions for $N = 1, 2, 3, 4$ and for different temperatures. For $N = 1, 2$, the magnitude of the oscillating range functions increases with temperature, whereas for $N \geq 4$, it decreases with temperature. For $N = 3$, the range function is almost independent of temperature. These behaviors for different $N$ follow from the temperature dependence of the susceptibility, which is shown in Fig. 1. At large distances ($r/a \gg 1$), we find that the range function decays as $\cos(q_c r)/r^2$ for $N \leq 3$, producing long-range oscillations with a periodicity of $2\pi/q_c$ in the spin density. (See App. A for the detailed derivations.) We will discuss the implications of the cutoff $q_c$ in the Discussion and Conclusion section (see Sec. [V]).

The temperature-dependent effective coupling is given by the spatial average of the RKKY interaction $J_{\text{RKKY}}$, [Eq. (7)]

$$J_{\text{eff}}(T) = \frac{1}{\Omega_{\text{unit}}} \int \frac{d^3r}{r} J_{\text{RKKY}}(r, T),$$

where $\Omega_{\text{unit}}$ is the volume of a unit cell. In the dimensionless form, [Eq. (8)] can be rewritten as

$$\frac{J_{\text{eff}}(T)}{r_{\text{eff}}^4} = \frac{1}{D_0(a^{-1})} \int r^2 dr \chi(r, T),$$

FIG. 1: The calculated finite-temperature static susceptibility $\chi(q, T)$ as a function of wave vector for various temperatures $T = 0, 0.02, 0.04, 0.06, 0.08, and 0.1 T_0$, and for different values of $N$ (a) $N = 1$, (b) $N = 2$, (c) $N = 3$, and (d) $N = 4$. Here, $T_0 = \varepsilon_0/k_B$, $D_1(a^{-1}) = \frac{\Delta_0}{\pi^2 a^2}$, and $a = 0.343$ nm (lattice constant of TaAs). For $N = 1$, the finite momentum cutoff $a^{-1}$ is used for the convergence of the integral.
where $J_{\text{eff}}^{(0)} = 4\pi|J_{\text{ex}}a^3|^2D_0(a^{-1})/4\Omega_{\text{unit}}$ and $D_0 = D_1(a^{-1})/a^3$. Note that the normalization factors $J_{\text{eff}}^{(0)}$ and $D_0$ are defined to be independent of both index $N$ and temperature $T$.

In the presence of non-magnetic impurity scattering arising from unintentional background disorder causing momentum relaxation, the RKKY interaction should be cut off at distances larger than a characteristic disorder length scale (i.e., the transport mean-free path), which is determined by the impurity scattering. We include the disorder effect phenomenologically by including an exponential damping at distances larger than the cutoff $R$ in the range of the RKKY interaction. Then, the effective coupling is modified as

$$J_{\text{eff}}(T) = \begin{cases} \frac{1}{\Omega_{\text{unit}}} \int d^3r J_{\text{RKKY}}(r) \left( r < R \right), \\ \frac{1}{\Omega_{\text{unit}}} \int d^3r J_{\text{RKKY}}(r)e^{-\frac{r}{R}} \left( r > R \right). \end{cases}$$

(See Appendix B for the detailed expression of the effective RKKY coupling with exponential cutoff.) In this calculation, we use $R = 100a$, and our calculated results do not depend on the choice of $R$ qualitatively. One should think of $R$ as a disorder-induced phenomenological effective carrier mean-free path parameter, which provides a cutoff for the RKKY interaction range. $R$ should in general be smaller (larger) depending on the system being more (less) disordered. As a matter of principle, $R$ cannot really be very large since the magnetic ordering phenomenon being studied here necessitates the presence of magnetic impurities, which, in addition to providing the quenched magnetic moments for ordering, also serve as momentum scatterers.

Figure 3 shows the calculated effective coupling as a function of temperature for different values of $N = 1, 2, 3, 4$. In this calculation, the ultraviolet cutoff $q_c = a^{-1}$ and exponential cutoff $R = 100a$ are used. Here, the normalization factor $J_{\text{eff}}^{(0)} = 4\pi|J_{\text{ex}}a^3|^2D_1(a^{-1})/4\Omega_{\text{unit}}$ is independent of $N$ and temperature $T$. The dashed line represents $3k_B T/[S(S+1)x]$, and the intersections with $J_{\text{eff}}(T)$ indicate the transition temperatures solved self-consistently. Here, $J_{\text{ex}} = 0.1$ eV, $x = 0.05$ and $S = 5/2$.

function of temperature for different values of $N$. The effective coupling decreases monotonically with increasing temperature and increases with increasing $N$ at a fixed temperature. Since the effective coupling $J_{\text{eff}}(T)$ is positive, the magnetic moments are expected to be ferromagnetically aligned.

From the temperature dependent effective coupling in Eq. (9), we calculate the magnetic transition temperature of the intrinsic chiral 3D semimetals. For the Heisenberg classical spins, the mean-field transition temperature $T_c$ is given by [20,21]

$$k_B T_c = \frac{S(S+1)}{3} x J_{\text{eff}},$$

(10)

where $S$ and $x = n_{\text{imp}}a^3$ are the spin and concentration of the local magnetic moments, respectively, and $n_{\text{imp}}$ is the effective magnetic impurity density. Since the calculated $J_{\text{eff}}$ is a function of temperature, we calculate $T_c$ self-consistently from Eq. (10). In Fig. 3, the intersections between the dashed line [i.e., $3k_B T/[S(S+1)x]$] and solid lines [i.e., $J_{\text{eff}}(T)$] determine the transition temperature for each $N$.

Figure 4 shows the self-consistently calculated transition temperature for different values of $N$ as a function of the exchange coupling $J_{\text{ex}}$, the magnetic impurity concentration $x$, and the degeneracy factor $g$. The ferromagnetic transition temperature increases with $J_{\text{ex}}$, $x$, and $g$ for all $N$. In particular, for $N = 1$, $T_c$ increases linearly with $J_{\text{ex}}^2$, $x$, and $g$, as expected. In general, for $N > 1$, the calculated $T_c$ shows non-trivial dependence on the parameters, which cannot be intuitively guessed without our calculations.

IV. DISCUSSION AND CONCLUSION

We studied theoretically the effective magnetic coupling between magnetic impurities and the consequent ferromagnetic transition temperature in 3D chiral gapless semimetals with arbitrary energy dispersion. In order to
calculate the RKKY magnetic coupling range function, we introduced the momentum cutoff $q_c$, which is natural for the effective continuum model used in this work. We use the inverse lattice constant as the natural ultraviolet momentum cutoff in the theory. As shown in Fig. 2, we find that for $N \leq 3$, the envelope of the oscillatory RKKY range function decays as $r^{-2}$ and the period of the oscillation is $2\pi/q_c$. This decaying pattern arises from the finite ultraviolet cutoff $q_c$ used in the range function. If we set the cutoff to be infinite, then the range function loses its oscillatory characteristics and monotonically decays as $r^{-6+N}$, which gives the typical $r^{-5}$ decay for $N = 1$ \cite{8, 9}. However, in the presence of a finite $q_c$, the overall behavior of the range function is determined by the competition between the oscillatory $r^{-2}$ term and the non-oscillatory $r^{-6+N}$ term. We find that for $N \leq 3$, the oscillatory $r^{-2}$ decay dominates over the non-oscillatory $r^{-6+N}$ term, but for $N > 3$ it is vice versa (see Appendix A for the detailed derivations). We note that the cutoff dependence of the effective coupling and the corresponding transition temperature is insensitive to the precise quantitative choice of $q_c$.

In summary, we investigate the temperature dependent susceptibility, RKKY interaction, and effective magnetic ordering for 3D chiral gapless semimetals with arbitrary energy dispersion in the presence of dilute random magnetic impurities. We find that in 3D chiral gapless semimetals, the ferromagnetic ordering between magnetic impurities is favored with enhanced magnetic coupling as the energy dispersion has a higher power-law. Our results indicate that ferromagnetic ordering between magnetic impurities is possible in 3D gapless semimetals, arising entirely from the carrier-mediated indirect RKKY interaction in the dilute impurity limit. This predicted ferromagnetic ordering between magnetic impurities should be experimentally accessible with suitable magnetic doping. Our theory is valid when quantum fluctuations and direct exchange coupling between the impurity moments are negligible, which should be justified for large impurity spins and dilute impurity concentrations. In this work, we consider only the case of zero Fermi energy, and the effect of finite Fermi energy would be an interesting future research direction. Our finding that even the intrinsic undoped semimetallic system could be converted to a ferromagnet by dilute magnetic doping has obvious experimental implications, which should be explored in the laboratory.

**Acknowledgments**

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2018R1A2B6007837) and Creative-Pioneering Researchers Program through Seoul National University (S.P. and H.M.), NRF-2017R1A2A2A05001403 (EHH), and the Laboratory for Physical Sciences (SDS).

**Appendix A: Cutoff dependence of the range function**

To derive the asymptotic behavior of Eq. (6) at zero temperature, we use the fact that the zero-temperature intrinsic polarization function $\chi(q, T = 0)$ is proportional to the DOS, which is given by

$$\chi(q, T = 0) \propto q^{d-N},$$

(A1)

where $d$ is the dimension of the system. Then, the range function $\chi(r) \equiv \chi(r, T = 0)$ becomes

$$\chi(r) = C \int_0^{q_c} dqq^{2d-N-1}f_d(qr),$$

(A2)

where $C$ is a momentum- and position-independent constant. Here, $f_d(qr)$ is defined by

$$f_d(qr) = \begin{cases} J_0(qr) & (d = 2), \\ J_0(qr) & (d = 3), \end{cases}$$

(A3)

where $J_0(x)$ and $J_0(x) = \sin(x)/x$ are the Bessel and spherical Bessel functions of the first kind, respectively.

First, consider the 3D case. Using the following integral (see Eq. 3.761 in Ref. 23)

$$\int_0^1 dx x^a-1 \sin(ax) = -\frac{i}{2\mu} [M(\mu, \mu + 1, ia) - c.c.]$$

(A4)

where $a > 0$, $\text{Re}(\mu) > -1$, $\mu \neq 0$, and $M(a, b, z) = \frac{1}{2\pi} F_1(a; b; z)$ is the Kummer’s confluent hypergeometric function, we obtain $\chi(r)$ as

$$\chi(r) = \frac{C}{r^{a-N}} \frac{-i(qcr)^{5-N}}{2(5-N)} [M(5-N, 6-N, iqcr) - c.c.]$$

(A5)

The asymptotic behavior of $M(a, b, z)$ at a large $z$ is given by (see p.508 in Ref. 24)

$$M(a, b, z) \approx \Gamma(b) \left[ \frac{e^{z} a^{-b}}{\Gamma(a)} + \frac{(z)^{-a}}{\Gamma(b-a)} \right].$$

(A6)

Therefore, at large distances ($q_c r \gg 1$), $\chi(r)$ in 3D can be expressed as follows:

$$\chi(r) \approx A \cos(qcr) \frac{r}{r^2} + B \frac{1}{r^{6-N}},$$

(A7)

where $A$ and $B$ are constants. When $6-N \leq 2$, i.e., $N \geq 4$, the second term in Eq. (A7) dominates over the first term. Since the magnitude of the oscillating term is smaller than that of the monotonic decaying term, the oscillation of the range function mostly occurs at positive values. In contrast, for $N \leq 3$, the oscillating first term dominates. Therefore, the range function oscillates with a period $2\pi/q_c$, and its amplitude decays as $1/r^2$.

Similarly, for 2D, we find that when $4-N \leq \frac{3}{2}$, i.e., $N = 3, 4, \ldots$, the range function decays as $1/r^{4-N}$, while for $N = 1, 2$, it oscillates with a period $2\pi/q_c$, and its amplitude decays as $1/r^2$. This result is consistent with Min et al. \cite{18} for 2D gapless semimetals.
Appendix B: Effective RKKY coupling with the exponential disorder cutoff

From Eq. (9) in the main text, the normalized effective RKKY coupling with exponential damping is given by

\[
\frac{J_{\text{eff}}(T)}{J_{\text{eff}}(0)} = \int_0^{R/a} \hat{r}^2 d\hat{r} \tilde{\chi}(\hat{r}, T) + \left( \int_0^\infty - \int_0^{R/a} \right) \hat{r}^2 d\hat{r} \tilde{\chi}(\hat{r}, T) e^{-\frac{\hat{r}}{R}} \quad (B1)
\]

\[
F(T) = \int_0^\infty \hat{r}^2 d\hat{r} \tilde{\chi}(\hat{r}, T) e^{-\frac{\hat{r}}{R}} \quad (B2)
\]

where \( \hat{r} = r/a, \tilde{\chi}(\hat{r}, T) = \chi(r, T)/D_0 \), and

which can be rewritten as (see Eq. 6.623 in Ref. [23])

\[
F(T) = \int_0^\infty \hat{r}^2 d\hat{r} \tilde{\chi}(\hat{r}, T) e^{-\frac{\hat{r}}{R}} \quad (B3)
\]

\[
= \int_0^{q_0} \frac{q^2 dq}{2\sqrt{2q^3}} \frac{2e}{R\pi^3} \left( \frac{1}{1 + q^2 a^2} \right) \chi(q, T)
\]

\[
= \int_0^{q_0} \frac{q^2 dq}{\pi^2} \left( 1 + q^2 \right)^{\frac{1}{2}} \chi(q, T)
\]

where \( q = qa, \hat{R} = R/a, \chi(q, T) = \chi(q, T)/D_1(a^{-1}) \), and \( \Gamma \) is the gamma function.

[1] N. P. Armitage, E. J. Mele, and Ashvin Vishwanath, Weyl and Dirac semimetals in three-dimensional solids, Rev. Mod. Phys. 90, 015001 (2018).
[2] A. Burkov, Chiral anomaly and transport in Weyl metals, J. Phys. Condens. Matter 27, 113201 (2015).
[3] A. Bansil, H. Lin, and T. Das, Colloquium: Topological band theory, Rev. Mod. Phys. 88, 021004 (2016).
[4] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Topological semimetal and Fermi-arc surface states in the electronic structure of pyrochlore iridates, Phys. Rev. B 83, 205101 (2011).
[5] A. A. Burkov and M. D. Hook, and L. Balents, Topological nodal semimetals, Phys. Rev. B 84, 235126 (2011).
[6] A. A. Burkov and L. Balents, Weyl Semimetal in a Topological Insulator Multilayer, Phys. Rev. Lett. 107, 127205 (2011).
[7] P. Hosur, S. A. Parameswaran, and A. Vishwanath, Charge Transport in Weyl Semimetals, Phys. Rev. Lett. 108, 046602 (2012).
[8] Hao-Ran Chang, Jianhui Zhou, Shi-Xiong Wang, Wen-Yu Shan, and Di Xiao, RKKY interaction of magnetic impurities in Dirac and Weyl semimetals, Phys. Rev. B 92, 241103(R) (2015).
[9] Mir Vahid Hosseini and Mehdi Askari, Ruderman-Kittel-Kasuya-Yosida interaction in Weyl semimetals, Phys. Rev. B 92, 224435 (2015).
[10] Yong Sun and Anmin Wang, RKKY interaction of magnetic impurities in multi-Weyl semimetals, J. Phys.: Condens. Matter 29, 435306 (2017).
[11] M. A. Ruderman and C. Kittel, Indirect Exchange Coupling of Nuclear Magnetic Moments by Conduction Electrons, Phys. Rev. 96, 99 (1954).
[12] T. Kasuya, A Theory of Metallic Ferro- and Antiferromagnetism on Zener’s Model, Prog. Theor. Phys. 16, 45 (1956).
[13] K. Yosida, Magnetic Properties of Cu-Mn Alloys, Phys. Rev. 106, 893 (1957).
[14] C. Kittel, Indirect Exchange Interactions in Metals, in Solid State Physics, edited by F. Seitz, D. Turnbull, and H. Ehrenreich (Academic, New York, 1969), Vol. 22.
[15] S. Das Sarma and E. H. Hwang, Charge transport in gapless electron-hole systems with arbitrary band dispersion, Phys. Rev. B 91, 195104 (2015).
[16] D. J. Priour, Jr., E. H. Hwang, and S. Das Sarma, Disordered RKKY Lattice Mean Field Theory for Ferromagnetism in Diluted Magnetic Semiconductors, Phys. Rev. Lett. 92, 117201 (2004).
[17] D. J. Priour, Jr., E. H. Hwang, and S. Das Sarma, Quasi-Two-Dimensional Diluted Magnetic Semiconductor Systems, Phys. Rev. Lett. 95, 037201 (2005).
[18] Hongki Min, E. H. Hwang, and S. Das Sarma, Ferromagnetism in chiral multilayer two-dimensional semimetals, Phys. Rev. B 95, 1155414 (2017).
[19] Seongjin Ahn, E. H. Hwang, and Hongki Min, Collective modes in multi-Weyl semimetals, Scientific Reports 6, 34023 (2016).
[20] C. Kittel, Introduction to Solid State Physics (Wiley, New York, 2005).
[21] S. Das Sarma, E. H. Hwang, and A. Kaminski, Temperature-dependent magnetization in diluted magnetic semiconductors, Phys. Rev. B 67, 155201 (2003).
[22] Y. Liu, Z. Li, L. Guo, X. Chen, Y. Yuan, C. Xu, R. Hübner, S. Akhadmaliev, A. V. Krasheninnikov, A. T. N’Diaye, E. Arenholz, M. Helm, and S. Zhou, Towards diluted magnetism in TaAs, Phys. Rev. Materials 1, 044203 (2017).
[23] I. S. Gradshteyn and I. M. Ryzhik, Tables Of Integrals, Series And Products, 8th ed. (Academic, New York, 2015).
[24] M. Abramowitz and I. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (United States Department of Commerce, National Bureau of Standards, 1964).