Abstract

Submodular optimization generalizes many classic problems in combinatorial optimization and has recently found a wide range of applications in machine learning (e.g., feature engineering and active learning). For many large-scale optimization problems, we are often concerned with the adaptivity complexity of an algorithm, which quantifies the number of sequential rounds where polynomially-many independent function evaluations can be executed in parallel. While low adaptivity is ideal, it is not sufficient for a distributed algorithm to be efficient, since in many practical applications of submodular optimization the number of function evaluations becomes prohibitively expensive. Motivated by these applications, we study the adaptivity and query complexity of submodular optimization.

Our main result is a distributed algorithm for maximizing a monotone submodular function with cardinality constraint \( k \) that achieves a \((1 - 1/e - \varepsilon)\)-approximation in expectation. This algorithm runs in \( O(\log(n)) \) adaptive rounds and makes \( O(n) \) calls to the function evaluation oracle in expectation. The approximation guarantee and query complexity are optimal, and the adaptivity is nearly optimal. Moreover, the number of queries is substantially less than in previous works. We also extend our results to the submodular cover problem to demonstrate the generality of our algorithm and techniques.
1 Introduction

Submodular functions have the natural property of diminishing returns, making them prominent in applied fields such as machine learning and data mining. There has been a surge in applying submodular optimization for data summarization [TIWB14, SSS07, SSSJ12], recommendation systems [EAG11], and feature selection for learning models [DK08, KED+17], to name a few applications. There are also numerous recent works that focus on maximizing submodular functions from a theoretical perspective. Depending on the setting where the submodular maximization algorithms are applied, new challenges emerge and hence more practical algorithms have been designed to solve the problem in distributed [MKSK13, MZ15, BENW15], streaming [BMKK14], and robust [M KK17, MBNF+17, KZK18] optimization frameworks. Most of the existing work assumes access to an oracle that evaluates the submodular function. However, function evaluations (oracle queries) can take a long time to process—for example, the value of a set depends on interactions with the entire input like in Exemplar-based Clustering [DF07] or when the function is computationally hard to evaluate like the log-determinant of sub-matrices [KZK18]. Although distributed algorithms for submodular maximization partition the input into smaller pieces to overcome these problems, each distributed machine may run a sequential algorithm that must wait for the answers of its past queries before making its next query. This motivates the study of the adaptivity complexity of submodular maximization, introduced by Balkanski and Singer [BS18] to study the number of rounds needed to interact with the oracle. As long as we can ask polynomially-many queries in parallel, we can ask them altogether in one round of interaction with the oracle.

To further motivate this adaptive optimization framework, note that in a wide range of machine learning optimization problems, the objective function can only be computed with oracle access to the function. In settings where the oracle computation is a time-consuming optimization problem that is treated as a black box (e.g., parameter tuning), it is desirable to optimize a function with minimal number of rounds of interaction with the oracle. For example, consider the feature selection problem [DK08, KED+17], which is a critical step for improving the accuracy of machine learning models. The accuracy of a model trained with a subset of features does not necessarily have a closed-form formula, and in many settings must be evaluated by re-training the model from scratch. The training accuracy of certain models (e.g., generalized linear models) is known to be weakly submodular [DK08, KED+17]. In this case, we only have black-box access to the model accuracy function, and it can be time-consuming to compute. However, the model accuracy of many different feature subsets can be computed independently in parallel. The adaptive optimization framework [BS18] is a realistic model for this type of distributed problem, and the insights from lower bounds and algorithms developed in this framework have a deep impact on distributed computing for machine learning applications in practice. For further motivation on the importance of round complexity in the adaptive optimization framework, we refer the reader to [BS18].

While the number of rounds is an important quantity to optimize, the complexity of answering oracle queries also motivates designing algorithms that are efficient in terms of the total number of oracle queries. Typically, we need to make at least a constant number of queries per element in the ground set to have a constant approximation guarantee. A fundamental question is how many queries per element are needed to achieve optimal approximation guarantees without compromising the minimum number of adaptive rounds. In this paper, we address this issue and design a simple algorithm for monotone submodular maximization subject to a cardinality constraint that achieves optimal guarantees for the approximation factor and query complexity. Our algorithm also achieves nearly-optimal adaptivity complexity using the lower bound in [BS18].
1.1 Results and Techniques

Our main result is a simple distributed algorithm for maximizing a monotone submodular function with cardinality constraint $k$ that achieves an expected $(1 - 1/e - \varepsilon)$-approximation in $O((\log(n)/\varepsilon^2)$ adaptive rounds and makes $O(n \log(1/\varepsilon)/\varepsilon^3)$ queries to the function evaluation oracle in expectation. We emphasize that while our algorithm runs in $O((\log(n)/\varepsilon^2)$ rounds, only a constant number of queries are made per element. We note that, due to known lower bounds [BS18, MBK+15], the query complexity of the algorithm is optimal up to factors of $1/\varepsilon$ and the adaptivity is optimal up to factors of $1/\log \log(n)$ and $1/\varepsilon$. To achieve this result, we develop a number of techniques and subroutines that can be used in a variety of submodular optimization problems.

First, we develop the algorithm \textsc{Threshold-Sampling} in Section 3, which returns a subset of items from the ground set in $O((\log(n)/\varepsilon)$ adaptive rounds such that the expected marginal gain of each item in the solution is at least the input threshold. Furthermore, upon terminating it guarantees that all unselected items have marginal gain to the returned set less than the threshold. This effectively clears out all high-value items. To achieve $O((\log(n)/\varepsilon)$ adaptivity complexity, \textsc{Threshold-Sampling} adds a random subset of candidate items to its current solution in each round in such a way that probabilistically filters out an $\varepsilon$-fraction of the remaining candidates. We then use \textsc{Threshold-Sampling} as a subroutine in a submodular maximization algorithm that constructs a solution by gradually reducing its threshold for acceptance. This algorithm runs \textsc{Threshold-Sampling} in parallel starting from many different initial thresholds, one of which is guaranteed to be sufficiently closed to the optimal starting threshold. Consequently, we do not increase the adaptivity complexity because these processes are independent. One of the challenges that arises when analyzing the approximation factor of this algorithm is that \textsc{Threshold-Sampling} returns a random set of (possibly) variable size. We overcome this by constructing an averaged random process that agrees with the state of the maximization algorithm at the beginning and end, but otherwise acts as an intermediate proxy. In Section 4, we demonstrate how to use \textsc{Threshold-Sampling} as a subroutine in a greedy maximization algorithm to achieve an expected $(1 - 1/e - \varepsilon)$-approximation to OPT.

Our second main technical contribution is the \textsc{Subsample-Preprocessing} algorithm. This algorithm iteratively subsamples the ground set and uses the output guarantees of \textsc{Threshold-Sampling} to reduce the ratio of the interval containing OPT from $k$ to a constant. The adaptivity complexity of this subroutine is $O((\log(n))$ and its query complexity is $O(n)$. In particular, we show how to reduce the ratio of the interval in each step from $R$ to $O(poly(\log(R))$ by subsampling the ground set and using a key lemma that relates OPT to the optimum in the subsampled set. This approximation guarantee (Lemma 5.2) for OPT is a function of the subsampling rate and may be of independent interest. Our ratio-reduction technique and the \textsc{Subsample-Preprocessing} algorithm are presented in Section 5. Finally, in Section 6 we show how to use \textsc{Threshold-Sampling} to solve the submodular cover problem, demonstrating that our techniques are readily applicable to problems beyond submodular maximization subject to a cardinality constraint.

1.2 Related Work

The problem of optimizing query complexity for maximizing a submodular function subject to cardinality constraints has been studied extensively. In fact, a linear-time $(1 - 1/e - \varepsilon)$-approximation algorithm called stochastic greedy was recently developed for this problem in [MBK+15]. We achieve the same optimal query complexity in this paper, combined with nearly optimal $O((\log(n))$
adaptive round complexity. The applications of efficient algorithms for submodular maximization are widespread due to the numerous applications in machine learning and data mining. Submodular maximization has also recently attracted a significant amount of attention in the streaming and distributed settings [LMSV11, KMVV15, MKSK13, BMKK14, MZ15, BENW15, BENW16, CQ19]. We note that the distributed MapReduce model and adaptivity framework of [BS18] are different in that the latter model does not allow for adaptivity within each round. In many previously studied distributed models, such as MapReduce, sequential algorithms on a given machine are allowed to be adaptive within one round for the part of the data that they are processing locally. To highlight the difference between these models, Balkanski and Singer [BS18] showed that no constant-factor approximation is achievable in $O((\log(n) / \log \log(n))$ non-adaptive rounds; however, it is possible to achieve a constant-factor approximation in the MapReduce model in two rounds [MZ15].

Balkanski and Singer [BS18] introduced the adaptive framework model for submodular maximization and showed that a $(1/3)$-approximation is achievable in $O(\log(n))$ rounds. Furthermore, they showed that $\Omega((\log(n) / \log \log(n)))$ rounds are necessary for achieving any constant-factor approximation. Balkanski and Singer [BS18] showed that for any constant-factor approximation the dependence on $1/\varepsilon$ is $O(nk^2)$ queries [BS18]. While writing this paper, another related work (on arXiv) was brought to our attention [EN19]. While [EN19] has a similar goal to ours and aims to minimize the number of adaptivity rounds and oracle queries, their query complexity is $O(n \text{poly}(\log(n)))$, or $O(\text{poly}(\log(n)))$ calls per element. In contrast, we present a simple algorithm that achieves optimal query complexity (i.e., a constant number of oracle queries per element). The query complexity of our algorithm is optimal up to factors of $1/\varepsilon$. While we did not aggressively optimize the dependence on $1/\varepsilon$, the dependence is better than that in the related works [BS18, BRS19, EN19].

2 Preliminaries

For a set function $f : 2^N \rightarrow \mathbb{R}$ and any $S, T \subseteq N$, let $\Delta(T, S) \overset{\text{def}}{=} f(S \cup T) - f(S)$ be the marginal gain of $f$ at $T$ with respect to $S$. We call $N$ the ground set and let $|N| = n$. A function $f : 2^N \rightarrow \mathbb{R}$ is submodular if for every $S \subseteq T \subseteq N$ and $x \in N \setminus T$ we have $\Delta(x, S) \geq \Delta(x, T)$, where we overload the marginal gain notation for singletons. A natural class of submodular functions are those which are monotone, meaning that for every $S \subseteq T \subseteq N$ we have $f(S) \leq f(T)$. In the inputs to our algorithms, we let $f_S(T) \overset{\text{def}}{=} \Delta(T, S)$ denote a new submodular function with respect to $S$. We also assume the ground set is global to all algorithms. Let $S^*$ be a solution set to the maximization problem $\max_{S \subseteq N} f(S)$ subject to the cardinality constraint $|S| \leq k$. Lastly, let $U(A, t)$ denote the uniform distribution over all subsets of $A$ of size $t$.

Our algorithms take as input an evaluation oracle for $f$, which for any query $S \subseteq N$ returns the value of $f(S)$ in $O(1)$ time. Given an evaluation oracle, we define the adaptivity of an algorithm to be the minimum number of rounds such that in each round the algorithm can make polynomially-many independent queries to the evaluation oracle. We measure the complexity of our distributed algorithms in terms of their query and adaptivity complexity. Finally, we note that in our runtime guarantees, we take $1/\delta = \Omega(\text{poly}(n))$ so the claims hold with high probability.
3 Threshold-Sampling Algorithm

We start by giving a high-level description of the Threshold-Sampling algorithm. For an input threshold $\tau$, the algorithm iteratively builds a solution $S$ and maintains a set of unselected candidate elements $A$ over $O(\log(n)/\varepsilon)$ adaptive rounds. Initially, the solution is empty and all elements are candidates. In each round, the algorithm starts by filtering out candidate elements whose current marginal gain is less than the threshold. Then the algorithm efficiently finds the largest set size $t^* = \min\{\lfloor (1 + \hat{\varepsilon})i \rfloor, |A|\}$, where $i \in \mathbb{Z}_{\geq 0}$ and $\hat{\varepsilon}$ is a small fixed parameter defined in the algorithm, such that for $T \sim U(A, t^*)$ uniformly at random we have $\mathbb{E}[\Delta(T, S)/|T|] \geq (1-\varepsilon)\tau$. Next, the algorithm samples $T \sim U(A, t^*)$ and updates the current solution to $S \cup T$. This probabilistic guarantee has two beneficial effects:

1. It ensures the average contribution of each element in the output set $S$ is at least $(1 - \varepsilon)\tau$.

2. It implies that an $\varepsilon$-fraction of candidates are filtered out of $A$ in each round in expectation.

Therefore, the number of remaining elements that the algorithm considers in each round decreases geometrically in expectation. It follows that $O(\log(n)/\varepsilon)$ rounds are sufficient to guarantee with high probability that when the algorithm terminates, we either have $|S| = k$ or the marginal gain of all remaining elements to $S$ is below the threshold.

Before presenting the Threshold-Sampling algorithm, we define the probability distribution $D_t$ from which Threshold-Sampling draws samples when estimating the maximum set size $t^*$ in each round. Sampling from $D_t$ can be implemented with two calls to the evaluation oracle.

**Definition 3.1.** Conditioned on the current state of the algorithm, consider the random process where we sample $T \sim U(A, t)$ and then $x \sim A \setminus T$ uniformly at random. For all $t \in \{0, 1, \ldots, |A| - 1\}$, let $D_t$ denote the Bernoulli distribution for the indicator random variable

$$I_t \overset{\text{def}}{=} 1[\Delta(x, S \cup T) \geq \tau].$$

For completeness, we define $I_{|A|} = 0$.

It is useful to think of $\mathbb{E}[I_t]$ as the probability that the $(t+1)$-st marginal gain is at least threshold $\tau$ if the candidates in $A$ are inserted into $S$ according to a uniformly random permutation.

Now that $D_t$ is defined, we present the Threshold-Sampling algorithm and its guarantees below. Observe that this algorithm calls the Reduced-Mean subroutine, which detects when the mean of $D_t$ falls below $1 - \varepsilon$. We give the exact guarantees of Reduced-Mean in Lemma 3.3. Relating the mean of $D_t$ to threshold values, this means that after sampling $T \sim U(A, t^*)$ and adding the elements of $T$ to $S$, the expected marginal gain of the remaining candidates to $S$ is at most $(1-\varepsilon)\tau$. This is the invariant we want to maintain in each iteration for an $O(\log(n/\delta)/\varepsilon)$ adaptive algorithm. We explain the mechanics of Threshold-Sampling in detail and prove Lemma 3.2 in Section 3.1.
Algorithm 1 Threshold-Sampling

Input: evaluation oracle for $f : 2^N \to \mathbb{R}_{\geq 0}$, constraint $k$, threshold $\tau$, error $\varepsilon$, failure probability $\delta$

1: Set smaller error $\hat{\varepsilon} \leftarrow \varepsilon/3$
2: Set iteration bounds $r \leftarrow \lceil \log(1-\hat{\varepsilon})^{-1}(2n/\delta) \rceil$, $m \leftarrow \lceil \log(1+\hat{\varepsilon})(k) \rceil$
3: Set smaller failure probability $\hat{\delta} \leftarrow \delta/(2r(m+1))$
4: Initialize $S \leftarrow \emptyset$, $A \leftarrow N$
5: for $r$ rounds do
6: Filter $A \leftarrow \{ x \in A : \Delta(x, S) \geq \tau \}$
7: if $|A| = 0$ then
8: break
9: for $i = 0$ to $m$ do
10: Set $t \leftarrow \min\{(1+\hat{\varepsilon})^i, |A|\}$
11: if Reduced-Mean($D_t, \hat{\varepsilon}, \hat{\delta}$) then
12: break
13: Sample $T \sim U(A, \min\{t, k-|S|\})$
14: Update $S \leftarrow S \cup T$
15: if $|S| = k$ then
16: break
17: return $S$

Lemma 3.2. Let $Z$ be the event that all calls to Reduced-Mean give correct outputs (i.e., the reported property in Lemma 3.3 holds). For any monotone, nonnegative submodular function $f$, Threshold-Sampling outputs $S \subseteq N$ with $|S| \leq k$ in $O(\log(n/\delta)/\log(1/(1-\varepsilon)))$ adaptive rounds (for small $\varepsilon$, this becomes $O(\log(n/\delta)/\varepsilon)$) such that the following properties hold conditioned on $Z$:

1. The algorithm makes $O(n/\varepsilon)$ oracle queries in expectation.
2. The average marginal gain satisfies $\mathbb{E}[f(S)/|S|] \geq (1-\varepsilon)\tau$.
3. With probability at least $1-\delta/2$, if $|S| < k$, then $\Delta(x, S) < \tau$ for all $x \in N$.

Further, event $Z$ happens with probability at least $1-\delta/2$.

Algorithm 2 Reduced-Mean

Input: Bernoulli distribution $D$, error $\varepsilon$, failure probability $\delta$

1: Set number of samples $m \leftarrow 16\lceil \log(2/\delta)/\varepsilon^2 \rceil$
2: Sample $X_1, X_2, \ldots, X_m \sim D$
3: Set $\overline{\mu} \leftarrow \frac{1}{m}\sum_{i=1}^{m} X_i$
4: if $\overline{\mu} \leq 1-1.5\varepsilon$ then
5: return true
6: return false

1In the subsequent work [FMZ19], the authors modify Threshold-Sampling to give guarantees for non-monotone submodular functions.
Lemma 3.3. For any Bernoulli distribution $D$, Reduced-Mean uses $O(\log(\delta^{-1})/\varepsilon^2)$ samples to report one of the following properties, which is correct with probability at least $1 - \delta$:

1. If the output is true, then the mean of $D$ is $\mu \leq 1 - \varepsilon$.
2. If the output is false, then the mean of $D$ is $\mu \geq 1 - 2\varepsilon$.

We briefly remark that the Reduced-Mean subroutine is a standard unbiased estimator for the mean of a Bernoulli distribution. Since $D_t$ is a uniform distribution over indicator random variables, it is a Bernoulli distribution. The guarantees of in Lemma 3.3 are consequences of Chernoff bounds and the proof of Lemma 3.3 is given in Appendix A.2.

3.1 Analysis of Threshold-Sampling Algorithm

To prove the guarantees of Threshold-Sampling in Lemma 3.2, we first show that $D_t$ has monotonic behavior at any point in the algorithm. This is a simple consequence of submodularity, and the proof can be found in Appendix A.1.

Lemma 3.4. In each round of Threshold-Sampling, we have $1 = \mathbb{E}[I_0] \geq \mathbb{E}[I_1] \geq \cdots \geq \mathbb{E}[I_{|A|}]$.

Now we show that if we choose the maximum set size $t^*$ in each round such that the average marginal gain of a randomly sampled subset of size $t^*$ is at least $(1 - \varepsilon)\tau$, then we expect to filter an $\varepsilon$-fraction of the remaining candidates in the subsequent round. In Lemma 3.6 we show that our choice of the number of rounds is sufficient to guarantee that all unchosen elements have marginal gain less than $\tau$ with high probability.

Lemma 3.5. Conditioned on event $Z$, in each round of Threshold-Sampling, we expect to filter out an $\hat{\varepsilon}$-fraction of the elements in $A$.

Proof. This is a consequence of our choice of $t^* = \min\{t, k - |S|\}$ when sampling $T$. If $t^* = k - |S|$ or $t^* = |A|$, then the algorithm breaks from the loop and there is no subsequent filtering. Otherwise, for any given round, we condition on the state of the algorithm. Let $A_i$ be the value of $A$ after the filtering step in the $i$-th round, and let $A_{i+1}$ be the random variable for the future value of $A_i$ after being filtered in the next round. The algorithm draws $T \sim \mathcal{U}(A_i, t^*)$ uniformly at random, so by the process in Definition 3.1, filtering has the property that for $x \sim A_i \setminus T$,

$$\mathbb{E}[I_{t^*}] = \Pr(\Delta(x, S \cup T) \geq \tau) = \mathbb{E}\left[\frac{|A_{i+1}|}{|A_i \setminus T|}\right].$$

We have $\mathbb{E}[I_{t^*}] \leq 1 - \hat{\varepsilon}$ by Lemma 3.3. Since $|A_i \setminus T| \leq |A_i|$ for all choices of $T$, it follows that

$$\mathbb{E}[|A_{i+1}|] \leq (1 - \hat{\varepsilon}) \cdot |A_i|.$$

Therefore, an expected $\hat{\varepsilon}$-fraction of elements are filtered out in each round. \qed

Lemma 3.6. Conditioned on event $Z$, if Threshold-Sampling terminates with $|S| < k$, then we have $|A| = 0$ with probability at least $1 - \delta/2$. 

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Proof. Denote by $A_i$ the random variable for the value of $A$ after it is filtered in the $i$-th round of Threshold-Sampling, and recall that $A_0 = N$. Lemma 3.5 gives us $E[|A_{i+1}|] \leq (1 - \hat{\varepsilon}) \cdot E[|A_i|]$, which implies

$$E[|A_r|] \leq (1 - \hat{\varepsilon})^r \cdot E[|A_0|] = (1 - \hat{\varepsilon})^r n.$$ 

Therefore, by Markov’s inequality and our choice of the number of rounds $r$, we have

$$\Pr(|A_r| \geq 1) \leq (1 - \hat{\varepsilon})^{\log(1 - \hat{\varepsilon}) - 1} (2n/\delta) n = \delta/2.$$ 

It follows that $\Pr(|A_r| = 0) = 1 - \Pr(|A_r| \geq 1) \geq 1 - \delta/2$, as desired, conditioned on event $Z$. □

Using the guarantees for Reduced-Mean and the two lemmas above, we can prove Lemma 3.2.

Proof of Lemma 3.2. We start by showing that the adaptivity complexity of Threshold-Sampling is always $O(\log(n/\delta)/\varepsilon)$. By construction, the number of rounds is $O(\log(1 - \hat{\varepsilon})^{-1}(n/\delta))$, and in each round the algorithm makes polynomially-many queries, all of which are independent and only rely on the current state of $S$.

Assume that event $Z$ is true. For Property 1, the total number of oracle queries incurred from calling Reduced-Mean is $O(rm \log(\delta^{-1})/\varepsilon^2) = O(\log(n/\delta) \log(k) \log(\delta^{-1})/\varepsilon^2)$ by Lemma 3.3. Note that we can sample from $D_i$ with two oracle calls. Now we bound the total expected number of queries made while filtering over the course of the algorithm. Let $A_i$ be a random variable for the value of $A$ in the $i$-th round. It follows from the inequality $E[|A_{i+1}|] \leq (1 - \hat{\varepsilon}) \cdot E[|A_i|]$ in the proof of Lemma 3.5 and the linearity of expectation that the expected number of queries is bounded by

$$E \left[ \sum_{i=0}^{r-1} |A_i| \right] = \sum_{i=0}^{r-1} E[|A_i|] \leq n \sum_{i=0}^{r-1} (1 - \hat{\varepsilon})^i \leq n/\hat{\varepsilon}.$$ 

Since we set $\delta^{-1} = O(\text{poly}(n))$, the number of expected queries made when filtering dominates the sum of queries made when calling Reduced-Mean.

For Property 2, it suffices to lower bound the expected marginal gain of every element added to $S$ if we think of adding each set $T$ to the output $S$ one element at a time according to a uniformly random permutation. Let $t^* = \min\{t, k - |S|\}$ be the size of $T$ in an arbitrary round. If $t^* = 1$, then $E[\Delta(T, S)] \geq \tau$ by the definition of $A$. Otherwise, the size $t \leq t^*$ in the previous step, where $t = \lfloor(1 + \hat{\varepsilon})^{t-1} \rfloor \leq \lfloor(1 + \hat{\varepsilon})^{t^*} \rfloor = t^*$, satisfies $E[I_t] \geq 1 - 2\hat{\varepsilon}$ since the algorithm did not break on Line 11. Then, since $T \sim U(A, t^*)$ is sampled uniformly at random, we can lower bound $E[\Delta(T, S)]$ by the contribution of the first $\min\{t + 1, t^*\}$ elements, giving us

\[
E[\Delta(T, S)] \geq (E[I_0] + E[I_1] + \cdots + E[I_{t^* - 1}])\tau \\
\geq (\min\{t + 1, t^*\})(1 - 2\hat{\varepsilon})\tau \\
\geq \frac{t^*}{1 + \hat{\varepsilon}}(1 - 2\hat{\varepsilon})\tau \\
\geq t^*(1 - \varepsilon)\tau.
\]

Inequality (1) uses the definition of $I_t$ and is essentially Markov’s inequality. Note that (1) focuses only on the elements for which we achieve a marginal gain of at least $\tau$. For the rest of the elements in $T$, we use the monotonicity of function $f$ and argue that their marginal gains are nonnegative. Inequality (2) uses Lemma 3.4 and the fact that $E[I_t] \geq 1 - 2\hat{\varepsilon}$. Inequality (3) uses the observation
that \( t^* \leq (1+\varepsilon)(t+1) \). Thus, Property 2 follows since the expected marginal gain of any individual element in \( T \) is at least \((1 - \varepsilon)\tau\).

Property 3 follows from Lemma 3.6, the definition of \( A \), and submodularity.

Finally, it remains to show that event \( Z \) happens with probability at least \( 1 - \delta/2 \). Recall that each call to \textsc{Reduced-Mean} succeeds with probability at least \( 1 - \hat{\delta} \) by Lemma 3.3. Therefore, by a union bound, all \( r(m+1) \) calls succeed with probability at least \( 1 - r(m+1)\hat{\delta} = 1 - \delta/2 \). \( \square \)

### 4 Exhaustive-Maximization Algorithm

In this section we show how \textsc{Threshold-Sampling} fits into a greedy framework for maximizing monotone submodular functions subject to a cardinality constraint. We start by presenting the \textsc{Exhaustive-Maximization} algorithm, and then we prove its guarantees in Section 4.1. The algorithm \textsc{Exhaustive-Maximization} works as follows. Given an initial threshold \( \tau \), \textsc{Exhaustive-Maximization} constructs a solution by repeatedly running \textsc{Threshold-Sampling} at decreasing thresholds \( (1-\varepsilon)^j \tau \) conditioned on the current partial solution. It is greedy in the sense that every time \textsc{Threshold-Sampling} is called, the expected average contribution of the element in the returned set is at least \((1-\varepsilon)\tau\) and the marginal gain of all remaining elements is less than the current threshold. These properties allow us to prove an approximation guarantee with respect to the initial threshold \( \tau \).

To relate the quality of the solution to \( \text{OPT} \), we first let \( \Delta^* = \max \{ f(x) : x \in N \} \) be an upper bound for all marginal contributions by submodularity and observe that \( \Delta^* \leq \text{OPT} \leq k\Delta^* \). The threshold that \textsc{Exhaustive-Maximization} searches for is \( \tau^* = \text{OPT}/k \), so it suffices to run the greedy thresholding algorithm for \( O(\log(k)/\varepsilon) \) initial thresholds \( (1+\varepsilon)^j\Delta^*/k \) in parallel and return the solution with maximum value. Since the algorithm will try some threshold \( \tau \) close enough to \( \tau^* \), specifically \( \tau \leq (1+\varepsilon)\tau^* \), the approximation to \( \text{OPT} \) follows. Note that by trying all thresholds in parallel, the adaptivity complexity of the algorithm does not increase. In Section 5 we present efficient preprocessing methods to reduce the ratio of the interval containing \( \text{OPT} \).

**Algorithm 3** \textsc{Exhaustive-Maximization}

**Input:** evaluation oracle for \( f : 2^N \to \mathbb{R}_{\geq 0} \), constraint \( k \), error \( \varepsilon \), failure probability \( \delta \)

1: Set upper bounds \( \Delta^* \leftarrow \arg \max \{ f(x) : x \in N \} \), \( r \leftarrow \lceil 2 \log(k)/\varepsilon \rceil \), \( m \leftarrow \lceil \log(4)/\varepsilon \rceil \)
2: Set smaller failure probability \( \hat{\delta} \leftarrow \delta/(r(m+1)) \)
3: Initialize \( R \leftarrow \emptyset \)
4: for \( i = 0 \) to \( r \) in parallel do
5: \hspace{1em} Set \( \tau \leftarrow (1+\varepsilon)^i\Delta^*/k \)
6: \hspace{1em} Initialize \( S \leftarrow \emptyset \)
7: \hspace{1em} for \( j = 0 \) to \( m \) do
8: \hspace{2em} Set \( T \leftarrow \textsc{Threshold-Sampling}(f_S, k-|S|, (1-\varepsilon)^j\tau, \varepsilon, \hat{\delta}) \)
9: \hspace{2em} Update \( S \leftarrow S \cup T \)
10: \hspace{2em} if \( |S| = k \) then
11: \hspace{3em} break
12: \hspace{1em} if \( f(S) > f(R) \) then
13: \hspace{2em} Update \( R \leftarrow S \)
14: return \( R \)
**Theorem 4.1.** For a monotone, nonnegative submodular function $f$, Exhaustive-Maximization outputs a set $S \subseteq N$ with $|S| \leq k$ in $O(\log(n/\delta)/\varepsilon^2)$ adaptive rounds and with $O(n \log(k)/\varepsilon^3 + \delta n^2)$ oracle queries in expectation such that $E[f(S)] \geq (1 - 1/e - \varepsilon)(1 - \delta)OPT$. Setting $\delta < 1/n$ yields a good trade-off between the number of adaptive rounds and oracle calls.

### 4.1 Analysis of Exhaustive-Maximization Algorithm

To analyze the expected approximation factor of Exhaustive-Maximization, we first assume that all calls to Threshold-Sampling give correct outputs by our choice of $\delta$ and a union bound (i.e., event $Z$ in Lemma 3.2 always holds). The analysis is for one execution of the block in the for loop of Exhaustive-Maximization (Line 5 to Line 13), and it assumes the initial value of $\tau$ is sufficiently close to $\tau^* = OPT/k$, satisfying $\tau \leq \tau^* \leq (1 + \varepsilon)\tau$. Furthermore, we assume that the final output set is of size $k$. We refer to this modified block of Exhaustive-Maximization as the algorithm.

For a fixed input, as the algorithm runs it produces nonempty sets $T_1, T_2, \ldots, T_m$, inducing a probability distribution over sequences of subsets. Denote their respective sizes by $t_1, t_2, \ldots, t_m$ and the input values $(1 - \varepsilon)/\tau$ for which they were returned by $\tau_1, \tau_2, \ldots, \tau_m$. We view the algorithm as a random process that adds elements to the output set $S$ one at a time instead of set by set. Specifically, for each new $T_i$ the algorithm adds each $x \in T_i$ to $S$ in lexicographic order, producing a sequence of subsets $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k$ with $S_0 = \emptyset$. Note that there is no randomness in adding the elements of $T_i$ once $T_i$ is drawn.

Instead of analyzing the expected value of the partial solution $E[f(S_i)]$ at each step, we consider an averaged version of this random process that is easier to analyze and whose final expected value is equal to $E[f(S_k)]$. In particular, when the process draws a set $T_\ell$, each element $x \in T_\ell$ contributes the same amount $\Delta(T_\ell, T_1 \cup \cdots \cup T_{\ell-1})/|T_\ell|$ to the value of the output set. Formally, we define the averaged version of the random process as

$$\hat{f}(S_i) \overset{\text{def}}{=} f(T_1 \cup \cdots \cup T_{\ell-1}) + \frac{i - (t_1 + \cdots + t_{\ell-1})}{t_\ell} \cdot \Delta(T_\ell, T_1 \cup \cdots \cup T_{\ell-1}),$$

where we use the overloaded notation $S_i = (i, T_1, T_2, \ldots, T_\ell)$ to record the history of the process up to adding the $i$-th element. This means that $t_1 + t_2 + \cdots + t_{\ell-1} < i$ and $t_1 + t_2 + \cdots + t_\ell \geq i$. Note that for a given history $T_1, T_2, \ldots, T_m$ of subsets, both processes agree after adding a complete subset. Analogously, we define the marginal of the $i$-th element $X_i$ of this process to be

$$\hat{\Delta}(X_i, S_{i-1}) \overset{\text{def}}{=} \frac{\Delta(T_\ell, T_1 \cup \cdots \cup T_{\ell-1})}{t_\ell}.$$

Since the original algorithm induces a probability distribution over sequences of returned subsets, this defines a distribution over the values of $\hat{f}(S_i)$ and $\hat{\Delta}(X_i, S_{i-1})$ for all indices $i \in [k]$.

Lastly, it will be useful to define the distribution $\mathcal{H}$ over all possible (random bit) histories $(T_1, T_2, \ldots, T_m)$ at the termination of the algorithm, and also the distributions $\mathcal{H}_i$, for all $i \in [k]$, over the possible histories immediately before adding the $i$-th element. This means that for each $h = (T_1, T_2, \ldots, T_{\ell-1}) \in \text{supp}(\mathcal{H}_i)$ we have $t_1 + t_2 + \cdots + t_{\ell-1} < i$ and there exists a set $T_\ell$ that can be drawn such that $t_1 + t_2 + \cdots + t_\ell \geq i$. Let $H_i(h)$ be the event over $\mathcal{H}_i$ such that the history is $h = (T_1, T_2, \ldots, T_{\ell-1})$ and the next returned subset $T_\ell$ adds the $i$-th element. We condition our statements on $H_i(h)$, as this captures the state of the algorithm just before adding the $i$-th element. To provide intuition for Lemma 4.3, it is worth noting that $\mathcal{H}$ is a refinement of $\mathcal{H}_i$ conditioned
on \( H_i(h) \). This can be seen by recursively joining leaves in the probability tree of \( \mathcal{H} \) until the \( i \)-th element is reached. The result is the probability tree of \( H_i \) conditioned on \( H_i(h) \). In the statements that follow, the probabilities and expectations conditioned on \( H_i(h) \) are over the distribution \( \mathcal{H}_i \) and all other expressions are over the distribution \( \mathcal{H} \) of final outcomes.

**Lemma 4.2.** Conditioned on event \( Z \) (defined in Lemma 3.2), for any \( i \in [k] \), event \( H_i(h) \), and threshold \( \tau \) such that \( \tau \leq \tau^* \leq (1+\varepsilon)\tau \), we have

\[
E \left[ \Delta(X_i, S_{i-1}) \mid H_i(h) \right] \geq \frac{(1-\varepsilon)^2}{k} \cdot E \left[ \text{OPT} - \hat{f}(S_{i-1}) \mid H_i(h) \right].
\]

*Proof.* First we prove the claim for \( i = 1 \) and then we proceed by case analysis. If \( i = 1 \) there is no history, so it suffices to show that \( E[\hat{f}(S_1)] \geq (1-\varepsilon)^2\text{OPT}/k \). The first element belongs to the subset \( T_1 \) returned by \textsc{Threshold-Sampling}, so by Property 2 of Lemma 3.2, it follows that

\[
E \left[ \hat{f}(S_1) \right] = E \left[ \frac{f(T_1)}{|T_1|} \right] \geq (1-\varepsilon)\tau \geq (1-\varepsilon)^2 \cdot \frac{\text{OPT}}{k}.
\]

Assuming that \( i > 1 \), let \( i^* = t_1 + \cdots + t_{\ell-1} + 1 \) be the size of the partial solution after adding the first element in \( T_\ell \). We consider the cases \( i = i^* \) and \( i > i^* \) separately. If \( i = i^* \), observe that for monotone submodular functions we have

\[
f(S^*) \leq f(S^* \cup S_{i-1}) \\
\leq f(S_{i-1}) + \sum_{x \in S^*} \Delta(x, S_{i-1}) \\
\leq f(S_{i-1}) + k \cdot \tau_{\ell-1} \\
= f(S_{i-1}) + k \cdot \tau_{\ell-1} \left( 1 - \varepsilon \right) \\
\leq f(S_{i-1}) + \frac{k}{(1-\varepsilon)^2} \cdot E \left[ \frac{\Delta(T_\ell, S_{i-1})}{|T_\ell|} \mid H_i(h) \right] \quad \text{(Property 3 of Lemma 3.2)} \\
= f(S_{i-1}) + \frac{k}{(1-\varepsilon)^2} \cdot E \left[ \hat{\Delta}(X_i, S_{i-1}) \mid H_i(h) \right].
\]

In the fourth line, we have \( \tau_{\ell-1}/(1-\varepsilon) \) because \textsc{Threshold-Sampling} was run with parameter \( \tau_{\ell-1} \) immediately before running with threshold \( \tau_\ell \), which returned \( T_\ell \). The upper bound for the marginal gain \( \Delta(x, S_{i-1}) \) for \( x \in S^* \) is then a consequence of Property 2 and Property 3 of Lemma 3.2.

The history \( h = (T_1, T_2, \ldots, T_{\ell-1}) \) is known since we are conditioning on \( H_i(h) \), so it follows that

\[
f(S^*) - f(S_{i-1}) = E \left[ \text{OPT} - \hat{f}(S_{i-1}) \mid H_i(h) \right],
\]

because there is no randomness in the expectation. Recall that \( f(S_{i-1}) = E[\hat{f}(S_{i-1}) \mid H_i(h)] \) since the set \( S_{i-1} = T_1 \cup \cdots \cup T_{\ell-1} \) is a union of complete sets, and hence there are no partial, averaged contributions. Rearranging the previous inequalities gives

\[
E \left[ \hat{\Delta}(X_i, S_{i-1}) \mid H_i(h) \right] \geq \frac{(1-\varepsilon)^2}{k} \cdot E \left[ \text{OPT} - \hat{f}(S_{i-1}) \mid H_i(h) \right],
\]

as desired.
Now we consider the case when $i > i^*$. Because we condition on the history $h = (T_1, T_2, \ldots, T_{i-1})$ immediately before drawing a set $T_i$ that necessarily contains the $i$-th element, the averaging property of $\hat{f}$ and the analysis for the previous case give us

$$
\mathbb{E} \left[ \Delta(X_i, S_{i-1}) \mid H_i(h) \right] = \mathbb{E} \left[ \Delta(X_{i^*}, S_{i^*-1}) \mid H_i(h) \right] \\
\geq \frac{(1-\epsilon)^2}{k} \cdot \mathbb{E} \left[ \text{OPT} - \hat{f}(S_{i^*-1}) \mid H_i(h) \right] \\
\geq \frac{(1-\epsilon)^2}{k} \cdot \mathbb{E} \left[ \text{OPT} - \hat{f}(S_{i-1}) \mid H_i(h) \right].
$$

The final inequality makes use of

$$
\mathbb{E} [\hat{f}(S_{i-1}) \mid H_i(h)] \geq \mathbb{E} [\hat{f}(S_{i^*}) \mid H_i(h)],
$$

which is a consequence of monotonicity and the averaging property of $\hat{f}$. This completes the proof for all $i \in [k]$. 

**Lemma 4.3.** Conditioned on event $Z$, if for all $i \in [k]$ and events $H_i(h)$ we have

$$
\mathbb{E} \left[ \Delta(X_i, S_{i-1}) \mid H_i(h) \right] \geq \frac{(1-\epsilon)^2}{k} \cdot \mathbb{E} \left[ \text{OPT} - \hat{f}(S_{i-1}) \mid H_i(h) \right],
$$

then the algorithm returns a set $S_k$ such that $\mathbb{E} [f(S_k)] \geq (1-1/e - \epsilon)\text{OPT}.

**Proof.** Let $\delta_i = \text{OPT} - \hat{f}(S_i)$, and observe that

$$
\mathbb{E} \left[ \Delta(X_i, S_{i-1}) \mid H_i(h) \right] = \mathbb{E}[\delta_{i-1} \mid H_i(h)] - \mathbb{E}[\delta_i \mid H_i(h)]
$$

by the linearity of expectation. It follows from the assumption that

$$
\mathbb{E}[\delta_i \mid H_i(h)] \leq \left(1 - \frac{(1-\epsilon)^2}{k}\right) \cdot \mathbb{E}[\delta_{i-1} \mid H_i(h)].
$$

Since $\mathcal{H}_i$ conditioned on the event $H_i(h)$ is a partition of the final outcome distribution $\mathcal{H}_i$, it follows from the law of total probability that

$$
\mathbb{E}[\delta_i] = \sum_{h \in \text{supp}(\mathcal{H}_i)} \mathbb{E}[\delta_i \mid H_i(h)] \cdot \Pr(H_i(h)) \\
\leq \left(1 - \frac{(1-\epsilon)^2}{k}\right) \sum_{h \in \text{supp}(\mathcal{H}_i)} \mathbb{E}[\delta_{i-1} \mid H_i(h)] \cdot \Pr(H_i(h)) \\
\leq \left(1 - \frac{(1-\epsilon)^2}{k}\right) \cdot \mathbb{E}[\delta_{i-1}].
$$

Iterating this inequality over the sequence of expectations $\mathbb{E}[\delta_i]$ and the using the fact $1 - x \leq e^{-x}$,

$$
\mathbb{E}[\delta_k] \leq \left(1 - \frac{(1-\epsilon)^2}{k}\right)^k \cdot \mathbb{E}[\delta_0] \leq \left(\frac{1}{e} + \epsilon\right) \cdot \mathbb{E}[\delta_0].
$$

We have $\mathbb{E}[\hat{f}(S_0)] = \mathbb{E}[f(S_0)]$ and $\mathbb{E}[\hat{f}(S_k)] = \mathbb{E}[f(S_k)]$ by the construction of $\hat{f}$. Further, since $f$ is nonnegative, we have $\delta_0 = \text{OPT} - \hat{f}(S_0) \leq \text{OPT}$. Therefore, we have $\mathbb{E}[f(S_k)] \geq (1-1/e - \epsilon)\text{OPT}$, which completes the proof. 

\[\square\]
Proof of Theorem 4.1. The cardinality constraint is satisfied by construction, so we start by proving the adaptivity complexity of Exhaustive-Maximization. Lowering bounding OPT by $\Delta^*$ takes one adaptive round, and each execution of the block in the parallelized for loop is independent of all previous iterations. Therefore, it suffices to bound the adaptivity complexity of the for loop block (Lines 5–13). Each invocation of Threshold-Sampling is potentially dependent on the last since $S$ can be updated in each round. Therefore, because there are $m = O(1/\varepsilon)$ iterations in the block, the total adaptivity complexity is $O(m \log(n/\delta)/\varepsilon) = O((\log(n/\delta)/\varepsilon)^2)$ by Lemma 3.2.

Now we analyze the query complexity of the algorithm. Each call to Threshold-Sampling behaves as intended with probability at least $1 - \delta$, so by our choice of $\delta$ and a union bound, all calls to Threshold-Sampling are correct with probability at least $1 - \delta$ (i.e., event $Z$ holds). If this is the case, then the expected query complexity of Threshold-Sampling is $O(n/\varepsilon)$ by Lemma 3.2. Therefore, it follows that the overall expected query complexity of Exhaustive-Maximization is $O(n + nm(n/\varepsilon) + \delta n^2) = O(n \log(k)/\varepsilon^3 + \delta n^2)$.

To prove the approximation guarantee, first observe that $\Delta^* \leq \text{OPT} \leq k\Delta^*$ by submodularity. Therefore, we know that $\Delta^*/k \leq \tau^* \leq \Delta^*$. The values of $\tau$ considered are $(1 + \varepsilon)^i\Delta^*/k$, so by our choice for the number of iterations $r = O(\log(k)/\varepsilon)$, there exists a $\tau$ satisfying $\tau \leq \tau^* \leq (1 + \varepsilon)\tau$. Although we do not know this value of $\tau$, we can use its existence to give a guarantee by taking the maximum over all potential solutions. Therefore, conditioned on $Z$ which happens with probability at least $1 - \delta$, we have

$$E[f(S) \mid Z] \geq (1 - 1/e - \varepsilon)\text{OPT}$$

by Lemma 4.2 and Lemma 4.3, assuming the returned set satisfies $|S| = k$. If instead $|S| < k$, then all unchosen elements $x \in N \setminus S$ have marginals $\Delta(x, S) \leq \tau/4$ by our choice of $m$ and Property 3 of Lemma 3.2. Thus, for any $\tau \leq \tau^*$, monotonicity and submodularity give

$$f(S^*) \leq f(S) + \sum_{x \in S^*} \Delta(x, S) \leq f(S) + k\tau/4,$$

which implies $f(S) \geq (1 - 1/4)\text{OPT} \geq (1 - 1/e)\text{OPT}$. Putting everything together and using the nonnegativity of $f$, we have $E[f(S)] \geq (1 - 1/e - \varepsilon)(1 - \delta)\text{OPT}$, which completes the proof.  

5 Achieving Linear Query Complexity via Preprocessing

In this section we demonstrate different ways of using Threshold-Sampling to preprocess the interval containing OPT and reduce the total query complexity of the algorithm without increasing its adaptivity. In Section 5.1 we show how to reduce the ratio of the interval containing OPT from $O(k)$ to $O(\log(k))$ in $O(\log \log(k))$ iterations of an imprecise binary search, reducing the query complexity from $O(n \log(k))$ to $O(n \log \log(k))$. In Section 5.2 we show how to reduce the ratio of the interval from $R$ to $O(\log^3(R))$ in each step, until the ratio is a constant. This second approach subsamples the ground set and uses the imprecise binary search subroutine. By adaptively setting the parameters at each step according to the current ratio, we reduce the query complexity to $O(n)$ while maintaining $O(\log(n))$ adaptivity.

5.1 Reducing the Query Complexity with an Imprecise Binary Search

To see how we can use a binary search, consider the output of Threshold-Sampling($f, k, \tau, 1-p, \delta$) for an arbitrary value of $\tau$. If $|S| = k$, then by Property 2 of Lemma 3.2 we have $pk\tau \leq E[f(S)] \leq \frac{E[f(S)]}{\varepsilon}$.
OPT. Otherwise, if \( |S| < k \) then by Property 3 of Lemma 3.2 we have \( \Delta(x, S) \leq \tau \). In the second case, it follows for monotone submodular functions that \( f(S) \leq \OPT \leq f(S) + k\tau \). If \( f(S) \leq k\tau \) then \( \OPT \leq 2k\tau \), and if \( k\tau < f(S) \leq \OPT \) then \( p\tau \leq \OPT \). Therefore, after each call to \textsc{Threshold-Sampling} we can determine with probability at least \( 1 - \hat{\delta} \) that one of the following inequalities is true: \( \OPT \leq 2k\tau \) or \( p\tau \leq \OPT \). Note that these decisions may overlap, hence the term \textit{imprecise binary search}. We give the guarantees for \textsc{Binary-Search-Maximization} below and defer the proof of Corollary 5.1 to Appendix B.1.

\begin{algorithm}[h]
\caption{\textsc{Binary-Search-Maximization}}
\label{alg:binary-search-maximization}
\begin{algorithmic}[1]
\Require evaluation oracle for \( f : 2^N \to \mathbb{R}_{\geq 0} \), constraint \( k \), error \( \epsilon \), failure probability \( \delta \)
\State Set maximum marginal \( \Delta^* \leftarrow \max \{ f(x) : x \in N \} \)
\State Set interval bounds \( L \leftarrow \Delta^* \), \( U \leftarrow k\Delta^* \)
\State Set balancing parameter \( p \leftarrow 1/\log(k) \)
\State Set upper bound \( m \leftarrow \lceil \log_2(\log(k)) \rceil \)
\State Set smaller failure probability \( \hat{\delta} \leftarrow \delta/(m + 1) \)
\For {\( i = 1 \) to \( m \) do}
\State Set \( \tau \leftarrow \sqrt{LU/(2p)}/k \)
\State Set \( S \leftarrow \textsc{Threshold-Sampling}(f, k, \tau, 1 - p, \hat{\delta}) \)
\If {\( |S| < k \) and \( f(S) \leq k\tau \)} \Comment{Imprecise binary search decision}
\State Update \( U \leftarrow 2k\tau \)
\Else
\State Update \( L \leftarrow p\tau \)
\EndIf
\EndFor
\State \Return \textsc{Exhaustive-Maximization}(\( f, k, \epsilon, \hat{\delta} \)) modified to search over \( [L/k, U/k] \)
\end{algorithmic}
\end{algorithm}

\textbf{Corollary 5.1.} For any monotone, nonnegative submodular function \( f \), the algorithm \textsc{Binary-Search-Maximization} outputs a set \( S \subseteq N \) with \( |S| \leq k \) in \( O(\log(n/\delta)/\epsilon^2) \) adaptive rounds and with \( O(n \log \log(k)/\epsilon^3 + \delta n^2) \) expected oracle queries such that \( \mathbb{E}[f(S)] \geq (1 - 1/e - \epsilon)(1 - \delta)\OPT \).

\subsection{5.2 Reducing the Query Complexity by Subsampling}

Now we describe how to combine \textsc{Threshold-Sampling} and subsampling to preprocess the interval containing \( \OPT \) in \( O(\log(n)) \) adaptive rounds and with a total of \( O(n) \) queries so that the final interval has a constant ratio. There are three main ideas underlying the algorithm \textsc{Subsample-Preprocessing}:

1. We subsample the ground set \( N \) so that the query complexity of each \textsc{Threshold-Sampling} call is sublinear (Lemma 3.2).

2. We relate the optimal solution in the sampled space to \( \OPT \) via the subsampling probability.

3. We repeatedly subsample the ground set \( N \) with a granularity that depends on the current ratio of the feasible interval \( R \). In each of these iterations, we run \( O(\log(R)) \) imprecise binary search decisions in parallel (by calling \textsc{Threshold-Sampling} with error \( 1 - p \) as described in Section 5.1) to reduce the ratio from \( R \) to \( O(\log^5(R)) \).
The adaptivity of each step is $O(\log(n/\delta)/\log(1/p))$ by Lemma 3.2 because the calls are distributed. There are $O(\log^*(R))$ ratio reduction rounds, but by our choice of parameters $\ell$ and $p$ in each round, the total number of adaptive rounds is $O(\log(n/\delta))$. Therefore, when Subsample-Preprocessing terminates, the interval containing $\OPT$ has a constant ratio. In the last step, we run Exhaustive-Maximization modified to search over this new interval for the final solution.

Now we formally present Subsample-Preprocessing and state the lemmas that are prerequisites for its guarantees. All proofs for this preprocessing step are deferred to Appendix B.2. We first show how the optimal solution in a subsampled ground set is related to $\OPT$ in terms of the subsampling probability.

**Lemma 5.2.** For any monotone, nonnegative submodular function $f$, sample each element in $N$ independently with probability $1/\ell$. Let the subsampled set be $N'$, and denote the optimal solution restricted to $N'$ by $\OPT'$. Let $\Delta^*$ be an upper bound for the max marginal gain in $N$. Then, for any $\delta \in (0, 1]$, with probability at least $1 - \delta$, we have

$$\frac{1}{2}(\Delta^* + \OPT') \leq \OPT \leq \frac{2\ell}{\delta}(\Delta^* + \OPT').$$

Next we show that in each round of Subsample-Preprocessing, the current ratio $R$ becomes polylogarithmically smaller until it drops below a constant lower bound threshold $R^*$. The adaptivity and query complexity of this iteration is sublinear in $R$, so by summing over the $O(\log^*(k))$ rounds of Subsample-Preprocessing, the total number of adaptive rounds and expected number of queries are $O(\log(n/\delta))$ and $O(n)$, respectively.

**Lemma 5.3.** For any monotone, nonnegative submodular function $f$, let $[L, U]$ be an interval containing $\OPT$ with $U/L = R$. For any ratio $R > 0$, with probability at least $1 - \delta$, we can compute a new feasible interval with ratio at most $(64/\delta)^5 R$ such that:

- The number of adaptive rounds is $O\left(\frac{\log(n/\delta)}{\log \log(R)}\right)$.
- The expected number of queries is $O(n/\log(R))$.

**Lemma 5.4.** For any monotone, nonnegative submodular function $f$ and constant $0 < \delta \leq 1$, with probability at least $1 - \delta$, the algorithm Subsample-Preprocessing returns an interval containing $\OPT$ with ratio $O(\alpha^{4\log(\alpha)})$ in $O(\log(n/\delta))$ adaptive rounds and uses $O(n)$ queries in expectation, where $\alpha = 64/\delta$.

Last, we show how to use the reduced interval returned by Subsample-Preprocessing with the Exhaustive-Maximization algorithm to get Subsample-Maximization.

**Theorem 5.5.** For any monotone, nonnegative submodular function $f$ and constant $0 < \epsilon \leq 1$, the algorithm Subsample-Maximization outputs a set $S \subseteq N$ with $|S| \leq k$ in $O(\log(n/\epsilon)/\epsilon^3)$ adaptive rounds and with $O(n \log^2(1/\epsilon)/\epsilon^3)$ expected queries such that $\mathbb{E}[f(S)] \geq (1 - 1/e - \epsilon)\OPT$.

**Proof.** Set a smaller error $\tilde{\epsilon} = \epsilon/4$ and run Subsample-Preprocessing($f, k, \tilde{\epsilon}, \tilde{\epsilon}$) to obtain an interval with ratio $O(\alpha^{4\log(\alpha)})$ that contains $\OPT$ with probability at least $1 - \tilde{\epsilon}$ where $\alpha = 64/\tilde{\epsilon}$. Next, modify and run Exhaustive-Maximization($f, k, \tilde{\epsilon}, \tilde{\epsilon}/n$) so that it searches over the interval with

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2We note that $\log^*(n)$ denotes the *iterated logarithm* defined as $\log^*(n) := 1 + \log^*(\log(n))$ if $n > 1$ and 0 otherwise.
Algorithm 5 Subsample-Preprocessing

Input: evaluation oracle for \( f : 2^N \rightarrow \mathbb{R}_{\geq 0} \), constraint \( k \), error \( \varepsilon \), constant failure probability \( \delta \)

1: Set maximum marginal \( \Delta^* \leftarrow \max\{ f(x) : x \in N \} \)
2: Set interval bounds \( L \leftarrow \Delta^*, U \leftarrow k\Delta^*, R^* \leftarrow 2000 \)
3: while \( U/L \geq R^* \) do
4:  Set \( R \leftarrow U/L \)
5:  Set sampling ratio \( \ell \leftarrow \log^2(R) \)
6:  Set imprecise decision accuracy \( p \leftarrow 1/\log(R) \)
7:  Set upper bound \( m = \lceil \log_2(R) \rceil \)
8:  Set smaller failure probability \( \delta_R \leftarrow \delta/(2(m+1)\log(R)) \)
9:  Set \( N' \leftarrow \text{Subsample}(N, 1/\ell) \)
10: for \( i = 0 \) to \( m \) in parallel do
11:  Set \( \tau_i \leftarrow 2^i(L/k) \)
12:  Set \( S'_i \leftarrow \text{Threshold-Sampling}(f', k, \tau_i, 1-p, \delta_R/n) \)
13:  Decide if \( \text{OPT}' \leq 2k\tau_i \) or \( \text{OPT}' \geq pk\tau_i \) using \( S'_i \) \( \triangleright \) Imprecise binary search decision
14:  if \( \text{OPT}' \leq 2k\tau_0 \) then \( \triangleright \) Update interval
15:     Update \( L \leftarrow (\Delta^* + L)/2 \)
16:     Update \( U \leftarrow (4\ell/\delta_R)(\Delta^* + L) \)
17:  else if \( \text{OPT}' \geq pk\tau_m \) then
18:     Update \( L \leftarrow pU \)
19:     Update \( U \leftarrow U \) \( \triangleright \) \( U \) stays intact
20:  else
21:     Set \( i^* \leftarrow \) first \( i = 0 \) to \( m \) such that \( \text{OPT}' \geq pk\tau_i \) and \( \text{OPT}' \leq 2k\tau_{i+1} \)
22:     Update \( L \leftarrow (p/2)(\Delta^* + 2^i L) \)
23:     Update \( U \leftarrow (8\ell/\delta_R)(\Delta^* + 2^i L) \)
24:  if \( \log(\log(R)) < 2\log(\log(U/L)) \) then \( \triangleright R \) is sufficiently small
25:     break
26: return \([L, U]\)
Both Subsample-Preprocessing and Exhaustive-Maximization succeed with probability at least \( 1 - 2\hat{\epsilon} \) by a union bound. Therefore, conditioning on the success of both events, we have

\[
\mathbb{E}[f(S)] \geq (1 - 1/e - \hat{\epsilon}) \text{OPT} \cdot (1 - 2\hat{\epsilon}) \geq (1 - 1/e - \epsilon) \text{OPT},
\]

as desired.

The adaptivity complexity follows from the guarantees of Lemma 5.4 and Theorem 4.1. For the query complexity, observe that if the preprocessed interval does not have ratio \( O(\alpha^4 \log(\alpha)) \), then we can just output the empty set without calling Exhaustive-Maximization (to avoid a superlinear number of queries), as this happens with probability at most \( \hat{\epsilon} \) by Lemma 5.4. \( \Box \)

### 6 Using the Threshold-Sampling Algorithm for Submodular Cover

In the submodular cover problem, we aim to find a minimum cardinality subset \( S \) such that \( f(S) \) is at least some target goal \( L \). In some sense, this problem can be viewed as the dual of submodular maximization with a cardinality constraint. To formalize the submodular cover problem, we want to solve \( \min_{S \subseteq N} |S| \) subject to the value lower bound \( f(S) \geq L \). We overload the notation \( S^* \) to denote the lexicographically least minimum size set satisfying the value lower bound. Therefore, the value of OPT is the cardinality \( |S^*| \). To overcome granularity issues resulting from arbitrarily small marginal gains, a standard assumption is to work with integer-valued submodular functions.

The greedy algorithm provides the state-of-the-art approximation for submodular cover by outputting a set of size \( O(\log(L)|S^*|) \). There have been recent attempts (e.g., [MZK16]) to develop distributed algorithms based on the greedy approach that achieve similar approximation factors, but these algorithms have suboptimal adaptivity complexity because the summarization algorithm of the centralized machine is sequential. Here, we show how to apply the ideas behind the Threshold-Sampling algorithm to submodular cover so that the algorithm runs in a logarithmic number of adaptive rounds without losing the approximation guarantee.

We start by giving a high-level description of our algorithm. Similar to [MZK16], which attempts to imitate the greedy algorithm, we initialize \( S = \emptyset \) and set the threshold \( \tau \) to the highest marginal value \( \Delta^* = \max_{x \in N} f(x) \). Then the algorithm repeatedly adds sets of items to \( S \) whose average value to \( S \) is at least \((1 - \epsilon)\tau\). When we run out of high value items, we lower the threshold from \( \tau \) to \((1 - \epsilon)\tau\) and repeat the process. Unlike the cardinality constraint setting, the stopping condition of this algorithm is when the value of \( f(S) \) reaches the target value \( L \).

Specifically, for each threshold \( \tau \) we run a variant of Threshold-Sampling called Threshold-Sampling-For-Cover (Algorithm 6) as a subroutine to find a maximal set of valuable items in \( O(\log(n)) \) adaptive rounds. The first difference between this algorithm and Threshold-Sampling is that it takes the value lower bound \( L \) as part of its input instead of a cardinality constraint \( k \). Therefore, we slightly modify the Threshold-Sampling algorithm as follows. Since we do not have an explicit constraint on the number of elements that we can to add, we set \( m \) such that it is possible to add all elements at once. This change is reflected in Lines 2 and 13 of Threshold-Sampling-For-Cover. Next, we use

\[
k = \left\lceil \frac{L - f(S)}{(1 - \epsilon)\tau} \right\rceil
\]
as an upper bound for the number of elements that can be added in each round. This is a consequence of our initial choice of \( \tau = \Delta^* \) and the method for lowering the threshold as the algorithm
progresses. We know that for the current threshold \( \tau \), no element has marginal gain more than \( \tau/(1-\varepsilon) \) to the selected set \( S \) by Property 3 of Lemma 3.2. Furthermore, the average contribution of elements in \( T \) for this stage satisfies \( \mathbb{E}[\Delta(T,S)/|T|] \geq (1-\varepsilon)\tau \) by an analog of Property 2 of Lemma 3.2. This then justifies our choice for the upper bound.

To give intuition for why this leads to an acceptable approximation factor, observe that for the current threshold \( \tau \), the optimum (conditioned on our current choice of \( S \)) must have at least \( (L-f(S))/((1-\varepsilon)\tau) \) elements since the marginal gains are upper bounded. Our cardinality bound \( \lceil(L-f(S))/((1-\varepsilon)\tau) \rceil \) implies that the algorithm does not add too many more elements than the optimum in each round. The final modification is the stopping condition on Line 15, where we check whether or not we have reached the value lower bound \( L \).

Algorithm 6 Threshold-Sampling-For-Cover

Input: evaluation oracle for \( f : 2^N \rightarrow \mathbb{Z}_{\geq 0} \), value goal \( L \), threshold \( \tau \), error \( \varepsilon \), failure probability \( \delta \)

1: Set smaller error \( \hat{\varepsilon} \leftarrow \varepsilon/3 \)
2: Set iteration bounds \( r \leftarrow \lceil \log_{(1-\hat{\varepsilon})^{-1}}(2n/\delta) \rceil \), \( m \leftarrow \lceil \log_{(1+\hat{\varepsilon})}(n) \rceil \)
3: Set smaller failure probability \( \hat{\delta} \leftarrow \delta/(2r(m+1)) \)
4: Initialize \( S \leftarrow \emptyset \), \( A \leftarrow N \)
5: for \( r \) rounds do
6: Filter \( A \leftarrow \{x \in A : \Delta(x,S) \geq \tau\} \)
7: if \( |A| = 0 \) then
8: break
9: for \( i = 0 \) to \( m \) do
10: Set \( t \leftarrow \min\{(1+\hat{\varepsilon})^i, |A|\} \)
11: if Reduced-Mean(\( D_t, \hat{\varepsilon}, \hat{\delta} \)) then
12: break
13: Set \( T \sim U(A, \min\{t, \lceil(L-f(S))/((1-\varepsilon)\tau) \rceil \}) \)
14: Update \( S \leftarrow S \cup T \)
15: if \( f(S) \geq L \) then
16: break
17: return \( S \)

Corollary 6.1. Let \( Z \) be the event that all calls to Reduced-Mean give correct outputs (i.e., the reported property in Lemma 3.3 holds). For any integer-valued, monotone, nonnegative submodular function \( f \), Threshold-Sampling-For-Cover outputs \( S \subseteq N \) in \( O(\log(n/\delta)/\varepsilon) \) adaptive rounds such that the following properties hold conditioned on \( Z \):

1. The average marginal gain satisfies \( \mathbb{E}[f(S)/|S|] \geq (1-\varepsilon)\tau \).
2. With probability at least \( 1-\delta/2 \), if \( f(S) < L \), then \( \Delta(x,S) < \tau \) for all \( x \in N \).

Further, event \( Z \) happens with probability at least \( 1-\delta/2 \).

Proof. The proof is a direct consequence of the proof for Lemma 3.2. \( \square \)

As explained above, we iteratively use Threshold-Sampling-For-Cover as a subroutine starting from the highest threshold \( \tau = \Delta^* \) to ensure that we either reach the value goal \( L \) or that there is no element with marginal value above \( \tau \) to the current set \( S \). If we have not reached the value
lower bound $L$, we reduce the threshold by a factor of $(1-\varepsilon)$ and repeat. This idea is summarized in the Adaptive-Greedy-Cover algorithm. We note that by the integrality assumption on $f$, this algorithm is guaranteed to output a feasible solution in a deterministic amount of time, since the threshold can be lowered enough such that any element with positive marginal gain can eventually be added to the solution.

**Algorithm 7 Adaptive-Greedy-Cover**

**Input:** evaluation oracle for $f : 2^N \to \mathbb{Z}_{\geq 0}$, value goal $L$

1. Set error $\varepsilon \leftarrow 1/2$
2. Set upper bounds $\Delta^* \leftarrow \max\{f(x) : x \in N\}$, $m \leftarrow \lceil \log(\Delta^*)/\varepsilon \rceil + 1$
3. Set failure probability $\delta \leftarrow 1/(n(m+1))$
4. Initialize $S \leftarrow \emptyset$
5. for $i = 0$ to $m$ do $\triangleright$ Until $(1-\varepsilon)^i \Delta^* < 1$
6. Set $\tau \leftarrow (1-\varepsilon)^i \Delta^*$
7. Set $T \leftarrow \text{Threshold-Sampling-For-Cover}(N, f_S, L - f(S), \tau, \varepsilon, \delta)$
8. Update $S \leftarrow S \cup T$
9. if $f(S) \geq L$ then
   10. break
11. return $S$

**Theorem 6.2.** For any integer-valued, nonnegative, monotone submodular function $f$, the algorithm Adaptive-Greedy-Cover outputs a subset $S \subseteq N$ with $f(S) \geq L$ in $O\left(\log^2(n \log(L)) \log(L)\right)$ adaptive rounds such that $E[|S|] = O\left(\log(L)|S^*|\right)$.

We defer the proof of Theorem 6.2 to Appendix C.1 and remark that it follows a similar line of reasoning to the analysis of the approximation factor for Exhaustive-Maximization in Lemma 4.3. The main difference between these proofs is that for Theorem 6.2 we need to show that Adaptive-Greedy-Cover makes geometric progress towards its value constraint $L$ not only in expectation, but also with constant probability. We do this by considering the progress of the algorithm in intervals of $O(|S^*|)$ elements, which ultimately allows us to analyze $E[|S|]$ using properties of the negative binomial distribution. Lastly, we have not optimized the constant in the approximation factor, but one could do this by more carefully considering how the error $\varepsilon$ affects the lower bound for the constant probability term in our analysis.

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A Missing Analysis from Section 3

A.1 Proof of Lemma 3.4

Lemma 3.4. In each round of Threshold-Sampling, we have $1 = E[I_0] \geq E[I_1] \geq \cdots \geq E[I_{|A|}]$.

Proof. We have $E[I_0] = 1$ by the definition of $I_t$ and the fact that the candidates in $A$ are filtered at the beginning of each round.

Now, let $m = |A|$ and $(n)_t = \prod_{i=1}^{t} (n-(i-1))$ denote the falling factorial. For $t \in \{1, \ldots, |A|-1\}$, summing over all $(t+1)$-truncated permutations of the elements $x_1, x_2, \ldots, x_m \in A$ gives

$$E[I_t] = \frac{1}{(m)_{t+1}} \sum_{x_1, \ldots, x_t, x_{t+1}} 1[\Delta(x_{t+1}, S \cup \{x_1, \ldots, x_t\}) \geq \tau]$$

$$\leq \frac{1}{(m)_{t+1}} \sum_{x_1, \ldots, x_t, x_{t+1}} 1[\Delta(x_{t+1}, S \cup \{x_1, \ldots, x_{t-1}\}) \geq \tau]$$  \hspace{1cm} \text{(submodularity)}

$$= \frac{m-t}{(m)_{t+1}} \sum_{x_1, \ldots, x_{t-1}, x_{t+1}} 1[\Delta(x_{t+1}, S \cup \{x_1, \ldots, x_{t-1}\}) \geq \tau]$$

$$= \frac{1}{(m)_t} \sum_{x_1, \ldots, x_{t-1}, x_t} 1[\Delta(x_t, S \cup \{x_1, \ldots, x_{t-1}\}) \geq \tau]$$  \hspace{1cm} \text{(symmetry)}

$$= E[I_{t-1}].$$

The second-to-last equality is a change of variables. Last, the boundary case $E[I_{|A|-1}] \geq E[I_{|A|}] = 0$ holds by the definition of $I_t$.

A.2 Analysis of Reduced-Mean Algorithm

Lemma A.1 (Chernoff bounds, [BS06]). Suppose $X_1, \ldots, X_n$ are binary random variables such that $\Pr(X_i = 1) = p_i$. Let $\mu = \sum_{i=1}^{n} p_i$ and $X = \sum_{i=1}^{n} X_i$. Then for any $a > 0$, we have

$$\Pr(X - \mu \geq a) \leq e^{-a \min\left(\frac{1}{\mu}, \frac{a}{\mu^2}\right)}.$$  \hspace{1cm} \text{Moreover, for any $a > 0$, we have}  

$$\Pr(X - \mu \leq -a) \leq e^{-\frac{a^2}{2\mu}}.$$

Lemma 3.3. For any Bernoulli distribution $\mathcal{D}$, REDUCED-MEAN uses $O(\log(\delta^{-1})/\varepsilon^2)$ samples to report one of the following properties, which is correct with probability at least $1 - \delta$:

1. If the output is true, then the mean of $\mathcal{D}$ is $\mu \leq 1 - \varepsilon$.
2. If the output is false, then the mean of $\mathcal{D}$ is $\mu \geq 1 - 2\varepsilon$.  

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Proof. By construction the number of samples used is \( m = 16\lceil \log(2/\delta)/\varepsilon^2 \rceil \). To show the correctness of Reduced-Mean, it suffices to prove that \( \Pr(|\bar{\mu} - \mu| \geq \varepsilon/2) \leq \delta \). Letting \( X = \sum_{i=1}^{m} X_i \), this is equivalent to

\[
\Pr \left( |X - m\mu| \geq \frac{\varepsilon m}{2} \right) \leq \delta.
\]

Using the Chernoff bounds in Lemma A.1 and a union bound, for any \( a > 0 \) we have

\[
\Pr( |X - m\mu| \geq a) \leq e^{-a^2/(2m\mu)} + e^{-a \min \left( \frac{1}{5}, \frac{a}{4m\mu} \right)}.
\]

Let \( a = \varepsilon m/2 \) and consider the exponents of the two terms separately. Since \( \mu \leq 1 \), we bound the left term by

\[
a^2/(2m\mu) = \frac{\varepsilon^2 m^2}{8m\mu} \geq \frac{\varepsilon^2}{8\mu} \cdot \frac{16 \log(2/\delta)}{\varepsilon^2} \geq \log(2/\delta).
\]

For the second term, first consider the case when \( 1/5 \leq a/(4m\mu) \). For any \( \varepsilon \leq 1 \), it follows that

\[
a \min \left( \frac{1}{5}, \frac{a}{4m\mu} \right) = \frac{1}{5} \geq \frac{\varepsilon}{10} \cdot \frac{16 \log(2/\delta)}{\varepsilon^2} \geq \log(2/\delta).
\]

Otherwise, we have \( a/(4m\mu) \leq 1/5 \), and by previous analysis we have \( a^2/(4m\mu) \geq \log(2\delta) \). Therefore, in all cases we have

\[
\Pr \left( |X - m\mu| \geq \frac{\varepsilon m}{2} \right) \leq 2e^{-\log(2/\delta)} = \delta,
\]

which completes the proof. \( \square \)

B Missing Analysis from Section 5

B.1 Analysis of Binary-Search-Maximization Algorithm

**Corollary 5.1.** For any monotone, nonnegative submodular function \( f \), the algorithm Binary-Search-Maximization outputs a set \( S \subseteq N \) with \( |S| \leq k \) in \( O(\log(n/\delta)/\varepsilon^2) \) adaptive rounds and with \( O(n \log \log(k)/\varepsilon^3 + \delta n^2) \) expected oracle queries such that \( \mathbb{E}[f(S)] \geq (1 - 1/e - \varepsilon)(1 - \delta)OPT \).

**Proof.** At the beginning of the algorithm, the interval \([L, U] = [\Delta^*, k\Delta^*]\) contains OPT by submodularity. In each step of the binary search we can choose \( \tau \in [L, U] \) and use Threshold-Sampling to reduce the interval by some amount such that the updated interval contains OPT. This decision process is described in Section 5.1. Our goal is to run Exhaustive-Maximization on a smaller feasible interval with ratio \( U/L = O(1/p) \) so that we can set \( \tau = O(\log(1/p)/\varepsilon) \) instead of \( O(\log(k)/\varepsilon) \). This objective stems from the fact that Exhaustive-Maximization grows \((1 + \varepsilon)^i\)-sized balls until the interval is covered to approximate \( \tau^* \). Therefore, at each step of the binary search we let

\[
\tau = \arg \min_{\tau' \in [L, U]} \max \left\{ \frac{2k \tau'}{L}, \frac{U}{pk \tau'} \right\},
\]

by considering the worst ratio of both outcomes. Since one function is increasing in \( \tau \) and the other is decreasing, we equate the two expressions to optimize \( \tau \), which gives us \( \tau = \sqrt{UL/(2pk^2)} \). It follows that the ratio of the updated interval is at most \( \sqrt{2U/(pL)} \).
Starting with a ratio \( R = U/L \), it follows for any \( p \in (0, 1] \), the \( i \)-th interval ratio is at most
\[
\left( \frac{2}{p} \right)^{\frac{1}{2^i+1} + \cdots + \frac{1}{2^i}} \cdot R^{\frac{1}{2^i+1}} = \left( \frac{2}{p} \right)^{\frac{1}{2^i+1}} \cdot R^{\frac{1}{2^i}} \leq \frac{2}{p} \cdot R^{\frac{1}{2^i}}.
\]

Initially \( R = k \), so setting \( m = \lceil \log_2(\log(k)) \rceil \) gives
\[
\frac{2}{p} \cdot R^{\frac{1}{2^m}} \leq \frac{2}{p} \cdot R^{\frac{1}{2^{\log_2(\log(k))}}} = \frac{2}{p} \cdot R^{\frac{1}{\log(k)}} = \frac{2}{p} \cdot k^{\frac{1}{\log(k)}} = \frac{2e}{p}.
\]

Setting \( p = 1/\log(k) \), the final ratio is at most \( 2e \log(k) \). Running Exhaustive-Maximization on this preprocessed \([L, U]\) interval, it suffices to set \( r = \lceil 2 \log(2e/p)/\varepsilon \rceil \).

The adaptivity complexity of Threshold-Sampling is \( O(\log(n/\delta)/\log(1/p)) \) by Lemma 3.2, so it follows from the number of iterations \( m \) in the binary search of Binary-Search-Maximization that the adaptivity complexity of the entire preprocessing step is
\[
O\left( m \cdot \frac{\log(n/\delta)}{\log(1/p)} \right) = O\left( \log \log(k) \cdot \frac{\log(n/\delta)}{\log \log(k)} \right) = O(\log(n/\delta)).
\]

Thus, the overall adaptivity of Binary-Search-Maximization is \( O(\log(n/\delta)/\varepsilon^2) \) by Theorem 4.1.

Now we analyze the expected query complexity of Binary-Search-Maximization. By our choice of \( \delta \) and a union bound, assume all subroutines produce their guaranteed outputs (i.e., event \( Z \) holds for all calls to Reduced-Mean). Each call to Threshold-Sampling in the binary search makes \( O(n/(1-p)) \) oracle queries in expectation by Lemma 3.2. Therefore, the total expected query complexity for the binary search preprocessing is
\[
O\left( m \cdot \frac{n}{1-p} \right) = O\left( \log \log(k) \cdot \frac{n}{1-\frac{1}{\log(k)}} \right) = O(n \log \log(k)).
\]

Next, since the ratio of the updated interval \([L, U]\) after the binary search is \( O(\log(k)) \), it follows that by modifying the search for \( \tau^* \) in Exhaustive-Maximization, the expected query complexity in this stage is \( O(n \log \log(k)/\varepsilon^3) \) by Theorem 4.1. This term dominates the query complexity of the binary search, so the result follows. The approximation factor then holds by Theorem 4.1 since the updated interval contains \( \tau^* \).

### B.2 Analysis of Subsample-Maximization Algorithm

**Lemma B.1** (Chebyshev’s inequality). Let \( X_1, X_2, \ldots, X_n \) be independent random variables with \( \mathbb{E}[X_i] = \mu_i \) and \( \text{Var}(X_i) = \sigma_i^2 \). Then, for any \( a > 0 \),
\[
\Pr\left( \left| \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i \right| \geq a \right) \leq \frac{1}{a^2} \sum_{i=1}^{n} \sigma_i^2.
\]
Lemma 5.2. For any monotone, nonnegative submodular function $f$, sample each element in $N$ independently with probability $1/\ell$. Let the subsampled set be $N'$, and denote the optimal solution restricted to $N'$ by $\text{OPT}'$. Let $\Delta^*$ be an upper bound for the max marginal gain in $N$. Then, for any $\delta \in (0, 1]$, with probability at least $1 - \delta$, we have

$$
\frac{1}{2} (\Delta^* + \text{OPT}') \leq \text{OPT} \leq \frac{2\ell}{\delta} (\Delta^* + \text{OPT}').
$$

Proof. Let $x_1, x_2, \ldots, x_k$ be the elements in $S^*$ in lexicographic order. By summing the marginal gain for each element when they are added in lexicographic order, we have

$$
f(S^*) = \sum_{x \in S^*} \Delta(x, \pi_x),
$$

where $\pi_x$ denotes the set of elements before $x$ in the lexicographic order. Subsample the ground set $N$ such that each element is included in the set $N'$ independently with probability $1/\ell$, and let $S'$ be the random set denoting the elements in $S^*$ that remain after subsampling. It follows from submodularity that for any value that $S'$ takes, we have

$$
f(S') \geq \sum_{x \in S'} \Delta(x, \pi_x).
$$

For each $x \in S^*$ define the random variable

$$
Z_x = \begin{cases} 
\Delta(x, \pi_x) & \text{with probability } 1/\ell, \\
0 & \text{otherwise}.
\end{cases}
$$

It follows that

$$
\mathbb{E}[Z_x] = \Delta(x, \pi_x) \cdot \frac{1}{\ell}
$$

$$
\text{Var}(Z_x) = \Delta(x, \pi_x)^2 \cdot \frac{1}{\ell} \left(1 - \frac{1}{\ell}\right).
$$

Let $g(S^*)$ be the random variable

$$
g(S^*) = \sum_{x \in S^*} Z_x,
$$

which is always a lower bound for the optimal solution $\text{OPT}'$ in $N'$. It follows that

$$
\mathbb{E}[g(S^*)] = \frac{1}{\ell} \sum_{x \in S^*} \Delta(x, \pi_x) = \frac{1}{\ell} \cdot f(S^*) = \frac{\text{OPT}}{\ell}.
$$

Ultimately, we want to show that with probability at least $1 - \delta$, we have

$$
\frac{\ell \Delta^*}{\delta} + \ell \cdot \text{OPT}' \geq \text{OPT} \geq \text{OPT}',
$$

as this implies the lower and upper bounds

$$
\frac{\ell}{\delta} \cdot (\Delta^* + \text{OPT}') \geq \text{OPT} \geq \frac{\text{OPT}' + \Delta^*}{2}.
$$
Consider the probability
\[
\Pr\left(\frac{\ell \Delta^*}{\delta} + \frac{\OPT}{2} + \ell g(S^*) \geq \OPT\right) = \Pr\left(\ell g(S^*) - \OPT \geq -\frac{\ell \Delta^*}{\delta} - \frac{\OPT}{2}\right)
\]
\[
= \Pr\left(\OPT - \ell g(S^*) \leq \frac{\ell \Delta^*}{\delta} + \frac{\OPT}{2}\right)
\]
\[
= \Pr\left(\OPT - g(S^*) \leq \frac{\Delta^*}{\delta} + \frac{\OPT}{2}\right).
\]
Using Chebyshev’s inequality (Lemma B.1), the probability of the complementary event is
\[
\Pr\left(\frac{\OPT}{\ell} - g(S^*) > \frac{\Delta^*}{\delta} + \frac{\OPT}{2}\right) \leq \Pr\left(\left|\frac{\OPT}{\ell} - g(S^*)\right| \geq \frac{\Delta^*}{\delta} + \frac{\OPT}{2}\right)
\]
\[
\leq \frac{1}{\left(\frac{\Delta^*}{\delta} + \frac{\OPT}{2\ell}\right)^2} \cdot \frac{1}{\ell} \cdot \left(1 - \frac{1}{\ell}\right) \sum_{x \in S^*} \Delta(x, \pi_x)^2
\]
\[
\leq \frac{1}{\left(\frac{\Delta^*}{\delta} + \frac{\OPT}{2\ell}\right)^2} \cdot \left(\frac{\ell - 1}{\ell^2}\right) \Delta^* \cdot \sum_{x \in S^*} \Delta(x, \pi_x)
\]
\[
\leq \frac{4(\ell - 1)}{4\ell^2(\Delta^*)^2/\delta^2 + 4\ell \Delta^* \OPT + \OPT^2}
\]
\[
\leq \frac{4\ell \Delta^* \OPT}{4\ell \Delta^* \OPT / \delta}
\]
\[
= \delta.
\]
Therefore, it follows that
\[
\Pr\left(\frac{\ell \Delta^*}{\delta} + \frac{\OPT}{2} + \ell g(S^*) \geq \OPT\right) \geq 1 - \delta.
\]
Since for all random outcomes we have \(\OPT' \geq g(S^*)\) and \(\OPT \geq \OPT'\), it follows that
\[
\frac{\ell \Delta^*}{\delta} + \ell g(S^*) \geq \frac{\OPT}{2} \implies \frac{2\ell}{\delta}(\Delta^* + \OPT') \geq \OPT \geq \frac{1}{2}(\Delta^* + \OPT').
\]
Therefore, if we query all marginals to compute \(\Delta^*\) and then subsample by \(1/\ell\), then with probability at least \(1 - \delta\) we know that \(\OPT\) lies within an interval of ratio \(4\ell / \delta\).

\[\square\]

**Lemma 5.3.** For any monotone, nonnegative submodular function \(f\), let \([L,U]\) be an interval containing \(\OPT\) with \(U/L = R\). For any ratio \(R > 0\), with probability at least \(1 - \delta\), we can compute a new feasible interval with ratio at most \((64/\delta)\log^5(R)\) such that:

- The number of adaptive rounds is \(O\left(\frac{\log(n/\delta)}{\log\log(R)}\right)\).
- The expected number of queries is \(O(n/\log(R))\).
Proof. Subsample the ground set $N$ with probability $1/\ell$ to get $N'$. We decide the value of $\ell$ later as a function of $R$. Let $m = \lceil \log(R) \rceil$ and set a smaller failure probability $\delta_R = \delta/(2(m + 1) \log(R))$. For a value $p \in [0, 1)$ that we also set later, run \textsc{Threshold-Sampling}($f, k, \tau, 1 - p, \delta_R/n$) on $N'$ in parallel for the values $\tau_i = 2^i(L/k)$ for $i = 0, 1, \ldots, m$. For each call, we determine if $\text{OPT}' \leq 2k\tau$ or $pk\tau \leq \text{OPT}'$ as explained in Section 5.1. There are three cases to consider:

1. If we have $\text{OPT}' \leq 2k\tau_0$, then $\text{OPT}' \leq 2L$.

2. If we have $\text{OPT}' \geq pk\tau_m$, then $\text{OPT} \in [pU, U]$ since $\text{OPT}' \leq U$.

3. Otherwise, find the least index $i^*$ such that $\text{OPT}' \geq pk(2^i L/k)$ and $\text{OPT}' \leq 2k(2^{i^*+1} L/k)$. This implies that $\text{OPT}' \in [p2^{i^*} L, 2^{i^*+2} L]$.

For the first case, we apply Lemma 5.2 and observe that with probability at least $1 - \delta_R$,

$$\frac{1}{2} (\Delta^* + L) \leq \text{OPT} \leq \frac{2\ell}{\delta_R} (\Delta^* + \text{OPT}') \leq \frac{2\ell}{\delta_R} (2\Delta^* + 2L).$$

Therefore, we have a new interval containing $\text{OPT}$ whose ratio is $4\ell/\delta_R$.

For the second case, we have $\text{OPT} \in [pU, U]$, so the ratio of the new feasible region is $1/p$.

For the third case, it follows from Lemma 5.2 and the case assumption that, with probability at least $1 - \delta_R$,

$$\frac{p}{2} (\Delta^* + 2^{i^*} L) \leq \text{OPT} \leq \frac{8\ell}{\delta_R} (\Delta^* + 2^{i^*} L).$$

It follows that the ratio of the new feasible region is $16\ell/(p\delta_R)$. Thus, in all cases the ratio $R$ maps to a new ratio of size at most $16\ell/(p\delta_R)$ with probability at least $1 - 2(m + 1)\delta_R = 1 - \delta/\log(R)$ by a union bound. Setting $\ell = \log^2(R)$ and $p = 1/\log(R)$, for any $R \geq 100$, the new ratio is at most

$$\frac{16\ell}{p\delta_R} = \frac{32 \log^4(R)(m + 1)}{\delta} \leq \frac{64 \log^5(R)}{\delta}.$$

Now we assume $R \geq 100$ and analyze the adaptivity and query complexity of the ratio reduction procedure. The adaptivity is that of \textsc{Threshold-Sampling}($f, k, \tau, 1 - p, \delta_R/n$) because we try all values of $\tau_i$ in parallel. Thus, by Lemma 3.2 and our choice of $\delta_R$, the adaptivity complexity\footnote{Note that the expected size of the ground after subsampling is $n/\ell$. To make this upper bound deterministic, we can instead randomly permute the ground set and take the first $\lceil n/\ell \rceil$ elements.} is

$$O\left( \frac{\log(n^2/\ell)}{\log(1/p)} \right) = O\left( \frac{\log(n/\ell)}{\log(1/p)} \right) = O\left( \frac{\log(n/\log^2(R))}{\log(1/p)} \right) = O\left( \frac{\log(n/\delta)}{\log(1/p)} \right).$$
Similarly, the expected number of queries is

\[ O \left( m \cdot \frac{n/\ell}{1 - \alpha} \right) = O \left( \log(R) \cdot \frac{n/\ell}{1 - \alpha} \right) = O \left( \log(R) \cdot \frac{n}{\log(R)} \right) = O \left( \frac{n}{\log(R)} \right), \]

which completes the proof. \(\square\)

**Lemma 5.4.** For any monotone, nonnegative submodular function \(f\) and constant \(0 < \delta \leq 1\), with probability at least \(1 - \delta\), the algorithm **Subsample-Preprocessing** returns an interval containing \(\text{OPT}\) with ratio \(O(\alpha^{4 \log(\alpha)})\) in \(O(\log(n/\delta))\) adaptive rounds and uses \(O(n)\) queries in expectation, where \(\alpha = 64/\delta\).

**Proof.** Assume \(\delta > 0\) is a constant failure probability. We start by bounding the number of adaptive rounds. The algorithm **Subsample-Preprocessing** consists of multiple iterations of the while loop. The first iteration starts with \(R = U/L = k\), and we progress to the next iteration with a new value of \(R\). We denote the sequence of these values by \(R_0 = k, R_1, \ldots, R_X\) where \(R_X\) is the final ratio of \(R = U/L\) for which we complete the iteration of the while loop. In other words, we terminate the while loop after the iteration with \(R = R_X\) is complete either on Line 3 or 25. Lemma 5.3 upper bounds the number of adaptive rounds in iteration \(0 \leq j \leq X\) by \(O(\log(n/\delta) / \log \log(R_j))\). Thus, it suffices to upper bound the sum \(\sum_{j=0}^{X} 1 / \log \log(R_j)\). Note that the if condition on Line 24 ensures that

\[ \frac{1}{\log \log(R_j)} \leq 2 \frac{1}{\log \log(R_{j+1})}. \]

The summands form a geometrically increasing series, so the summation is upper bounded by twice its largest term:

\[ \sum_{j=0}^{X} \frac{1}{\log \log(R_j)} \leq \frac{2}{\log \log(R_X)} \leq 1, \quad (5) \]

where the last inequality holds because the while condition on Line 3 tells us \(R_X \geq 2000\).

The expected number of queries can be bounded similarly. Lemma 5.3 bounds the expected number of queries in each iteration by \(O(n/\log(R))\). Therefore, we need to upper bound the sum \(\sum_{j=0}^{X} 1 / \log(R_j)\) by a constant. This is evident using (5) and the fact that \(1/\log(R) \leq 1/\log \log(R)\).

Now let us bound the overall failure probability. In iteration \(0 \leq j \leq X\), we set \(\delta_R = \delta / (2(m + 1) \log(R))\) where \(R = R_j\). We call **Threshold-Sampling** \(m + 1\) times and we may also use the bounds on OPT stated in Lemma 5.2 for any of the \(m + 1\) threshold values \(\{\tau_i\}_{i=1}^{m}\). We apply the union bound on all of these \(2(m + 1)\) potential failure events and conclude that the results of iteration \(j\) is reliable (i.e., the run is successful) with probability at least \(1 - \delta / \log(R_j)\). Using the above arguments and (5), the overall failure probability across all iterations is at most:

\[ \sum_{j=0}^{X} \frac{\delta}{\log(R_j)} \leq \delta. \]
Finally, it remains to prove that after running Subsample-Preprocessing, we bound $\text{OPT}$ in the desired constant-ratio range. Let us define $\alpha = 64/\delta$ and the function $h(R) = \alpha \cdot \log^5(R)$. Using Lemma 5.3, we know that in each iteration we reduce $R$ to some value at most $h(R)$.

If Subsample-Preprocessing terminates after the while condition on Line 3 fails, for the final $L$ and $U$, we have $U/L < R^\ast = 2000$.

Otherwise, we terminate because of the if condition on Line 24. So, for the final $R$, we have:

$$\log \log(R) < 2 \log \log(h(R)) \implies \log(R) < \log^2(h(R)) = (\log(\alpha) + 5 \log \log(R))^2.$$  

Therefore,

$$\log(R) < 4 \max\{\log^2(\alpha), 25(\log \log(R))^2\}.$$  

The max could take each of its two terms values. If the first case occurs, we have $\frac{1}{4} \log(R) < \log^2(\alpha)$. In this case, we have

$$\log(R^{1/4}) < \log^2(\alpha) \implies R^{1/4} < e^{\log^2(\alpha)} = (e^{\log(\alpha)})^{\log(\alpha)} = \alpha^{\log(\alpha)}.$$  

This means $R$ is at most $\alpha^{4\log(\alpha)}$, yielding the upper bound we want.

If the second case occurs, we have $\log(R) < 100(\log \log(R))^2$. The two sides of this inequality do not have the same asymptotic complexity, so the limit of their ratio (RHS divided by LHS) goes to zero as $R$ approaches infinity. This means there exists a constant $C$ such that for $R > C$, the inequality $\log(R) < 100(\log \log(R))^2$ cannot hold. Therefore, $R$ is at most this constant $C$ and consequently $R = O(1)$, which completes the proof.  

\section{Missing Analysis from Section 6}

\subsection{Analysis of the Adaptive-Greedy-Cover Algorithm}

\begin{theorem}
For any integer-valued, nonnegative, monotone submodular function $f$, the algorithm Adaptive-Greedy-Cover outputs a subset $S \subseteq N$ with $f(S) \geq L$ in $O(\log(n \log(L)) \log(L))$ adaptive rounds such that $E[|S|] = O(\log(L)|S^\ast|)$.
\end{theorem}

\begin{proof}
Start by assuming all calls to Reduced-Mean give correct outputs (i.e., event $Z$ occurs). This happens with probability at least $1 - 1/n$ by our choice of $\delta$ and a union bound. Furthermore, assume that $\Delta^\ast < L$, since if $\Delta^\ast \geq L$ then the algorithm can trivially output the singleton with the largest marginal value.

Next, observe that we have $f(S) \geq L$ upon termination since we assumed $f$ is integer-valued and the threshold can eventually reach $\tau < 1$. To bound the adaptivity complexity, observe that Threshold-Sampling-For-Cover runs in $O(\log(n^2(m+1))) = O(\log(n(\log(\Delta^\ast) + 1)))$ adaptive rounds by Corollary 6.1 and our choice of $\varepsilon$ and $\delta$ in Lines 1–3 of Adaptive-Greedy-Cover. Adaptive-Greedy-Cover calls this subroutine $O(m) = O(\log(\Delta^\ast) + 1)$ times. Thus, the adaptivity complexity is $O(\log(n \log(L)) \log(L))$ by our initial assumption $1 \leq \Delta^\ast < L$.

For the approximation factor of Adaptive-Greedy-Cover, we begin by mirroring the analysis of the approximation factor for submodular maximization in Theorem 4.1. Recall the value and marginal gain of the averaged process $\hat{f}(S_i)$ and $\hat{\Delta}(X_i, S_{i-1})$ defined in Section 4, and let $k^\ast = |S^\ast|$ denote the size of the optimal set $S^\ast$. Call the subsets that are added to $S$ during the course of the
algorithm $T_1, T_2, \ldots, T_m$ and let the remaining gap be $\delta_i = f(S^*) - \hat{f}(S_i)$. Following the proofs of Lemmas 4.2 and 4.3, and noticing that $f(S^*)/k^* \leq \Delta^*$ by submodularity, for all $i \geq 1$, we have

$$
\mathbb{E}[\delta_i \mid H_{i-1}(h)] \leq \left( 1 - \frac{(1 - \varepsilon)^2}{k^*} \right) \cdot \mathbb{E}[\delta_{i-1} \mid H_{i-1}(h)].
$$

(6)

While this expected inequality holds when conditioned on histories $h = (T_1, T_2, \ldots, T_{i-1})$, we show how to iterate it so that the gap $L - f(S)$ decreases geometrically with constant probability.

We start by showing that the size of every subset $T_i$ is at most $|T_i| \leq k^*/(1 - \varepsilon)^2 = 4k^*$. Since $f$ is a monotone submodular function and $\tau$ is reduced by a factor of $(1 - \varepsilon)$ each time THRESHOLD-SAMPLING-FOR-COVER is called starting from $\tau = \Delta^*$, it follows from Property 2 of Corollary 6.1 that

$$
f(S^*) \leq f(S) + \sum_{x \in S^*} \Delta(x, S) \leq f(S) + k^* \frac{\tau}{1 - \varepsilon}. $$

It follows from Line 13 in THRESHOLD-SAMPLING-FOR-COVER that an upper bound for $|T_i|$ is

$$
|T_i| \leq \left\lceil \frac{L - f(S)}{(1 - \varepsilon)^2 \tau} \right\rceil \leq \left\lceil \frac{f(S^*) - f(S)}{(1 - \varepsilon)^2 \tau} \right\rceil \leq \left\lceil \frac{k^*}{(1 - \varepsilon)^2} \right\rceil = 4k^*.
$$

Now we consider the progress of reducing the gap $L - f(S)$ after adding blocks of sets $T_i$. Define the first block $B_1 = T_1 \cup T_2 \cup \cdots \cup T_{t - 1}$ such that $t_1 + t_2 + \cdots + t_{\ell} \geq 4k^*$ for the least possible value of $\ell$. Similarly, define the blocks $B_2, B_3, \ldots$ to be the union of the sets $T_i$ after the previous block such that the cardinality first exceeds $4k^*$. Since $|T_i| \leq 4k^*$, we have the upper bound $|B_i| \leq 8k^*$, which we use to ensure that the algorithm processes sufficiently many blocks. Since we analyze the algorithm by blocks, it is convenient to let $S_{B_i} = \bigcup_{j=1}^{i} B_j$ denote the union of the first $i$ blocks. Lastly, observe that

$$
\Delta(B_i, S_{B_{i-1}}) \leq 4(f(S^*) - f(S_{B_{i-1}})),
$$

for all $i \geq 1$, because the addition of each block never exceeds the gap $L - f(S_{B_{i-1}})$ by a factor of more than $1/(1 - \varepsilon)^2$ and $L \leq f(S^*)$.

By analyzing the algorithm with blocks of size $O(k^*)$, we show that the addition of each block independently reduces the current gap $L - f(S_{B_{i-1}})$ by a constant factor with probability $p \geq 0.05$. This allows us to analyze the expected output size $\mathbb{E}[|S|]$ via the negative binomial distribution. Using an analogous block indexing for the gap $\delta_i, \varepsilon = 1/2$ imply that

$$
\mathbb{E}[\delta_{B_i} \mid S_{B_{i-1}}] \leq \left( 1 - \frac{(1 - \varepsilon)^2}{k^*} \right)^{4k^*} \cdot \mathbb{E}[\delta_{B_{i-1}} \mid S_{B_{i-1}}] \leq (1/e) \cdot \mathbb{E}[\delta_{B_{i-1}} \mid S_{B_{i-1}}].
$$

Since blocks are unions of complete sets $T_j$, the averaged process $\hat{f}(S_{B_i})$ and the true value $f(S_{B_i})$ always agree, conditioned on the previous state $S_{B_{i-1}}$. Using the inequality above,

$$
\mathbb{E}[\Delta(B_i, S_{B_{i-1}}) \mid S_{B_{i-1}}] = \mathbb{E}[\delta_{B_{i-1}} \mid S_{B_{i-1}}] - \mathbb{E}[\delta_{B_i} \mid S_{B_{i-1}}] \geq \mathbb{E}[\delta_{B_{i-1}} \mid S_{B_{i-1}}] - (1/e) \cdot \mathbb{E}[\delta_{B_{i-1}} \mid S_{B_{i-1}}] = (1 - 1/e) \cdot \mathbb{E}[\delta_{B_{i-1}} \mid S_{B_{i-1}}].
$$

This means that the addition of each block $B_i$ decreases the current gap $L - f(S_{B_{i-1}})$ by a constant factor in expectation. However, we can make a stronger claim since $\Delta(B_i, S_{B_{i-1}})$ is upper bounded.
Let $X_i$ be the indicator random variable conditioned on $S_{B_{i-1}}$ such that

$$X_i = \begin{cases} 
0 & \text{if } \Delta(B_i, S_{B_{i-1}}) < (1 - 2/e) \cdot \delta_{B_{i-1}}, \\
1 & \text{otherwise.}
\end{cases}$$

We claim that $X_i = 1$ with probability $p \geq 0.05$; otherwise, we would have

$$\mathbb{E}[\Delta(B_i, S_{B_{i-1}}) \mid S_{B_{i-1}}] < (1 - p)(1 - 2/e) \cdot \mathbb{E}[\delta_{B_{i-1}} \mid S_{B_{i-1}}] + 4p \cdot \mathbb{E}[\delta_{B_{i-1}} \mid S_{B_{i-1}}],$$

which is a contradiction. Therefore, for each block $B_i$ we have

$$\Pr(\delta_{B_i} \leq (2/e) \cdot \delta_{B_{i-1}} \mid S_{B_{i-1}}) \geq 0.05.$$

In other words, with probability $p \geq 0.05$, the addition of each block independently decreases the remaining gap to $f(S^*)$ by a constant factor.

Thus, if after the addition of $\ell$ blocks there are $a = \lceil \log(L)/\log(e/2) \rceil$ events such that $X_i = 1$, then the current gap to $f(S^*)$ satisfies

$$\delta_{B_{\ell}} \leq (2/e)^a \cdot \delta_{B_0} \leq \frac{1}{L} \cdot \delta_{B_0}.$$

By the definition of $\delta_{B_i}$ and the assumption that $f$ is nonnegative, this implies that

$$f(S^*) - f(S_{B_{\ell}}) \leq \frac{1}{L} \cdot f(S^*) \implies L \left(1 - \frac{1}{L}\right) = L - 1 \leq f(S_{B_{\ell}}).$$

Since we assumed that $f$ is integer-valued, the algorithm reaches the value lower bound $L$ after the addition of the next item.

It follows that we can upper bound $\mathbb{E}[|S|]$ by the expected number of blocks needed to have $a$ successful events plus one more block to ensure that we exceed the target value $L$ (conditioned on all calls to THRESHOLD-SAMPLING-FOR-COVER succeeding, which by our choice of $\delta$ happens with probability at least $1 - 1/n$). Since each block has at most $8k^*$ elements, noticing that this stopping criterion is given by the negative binomial distribution yields

$$\mathbb{E}[|S|] \leq 8k^* \left(1 + \sum_{\ell=0}^{\infty} (\ell + a - 1) \frac{p^\ell}{\ell!} \right)\left(1 - p \right)^\ell \leq 8k^* \left(1 + a \cdot \frac{1}{p} \right) \leq 8k^* \cdot (20a + 1).$$

Here we use the fact that the expected value of a negative binomial distribution parameterized by $a$ successes and failure probability $1 - p$ is $(1 - p)a/p$. Since $a = O(\log(L))$, it follows that we have the conditional expectation $\mathbb{E}[|S|] = O(k^* \log(L))$ with probability at least $1 - 1/n$. Conditioned on the algorithm failing (which happens with probability at most $1/n$), we have $|S| \leq n$. Therefore, in total we have

$$\mathbb{E}[|S|] = (1 - 1/n) \cdot O(k^* a) + (1/n) \cdot n = O(k^* \log(L)),$$

as desired. This completes the proof of the expected approximation factor.