Cutoff Dependence and Complexity of the CFT$_2$ Ground State

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We present the vacuum of a two-dimensional conformal field theory (CFT$_2$) as a network of Wilson lines in $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ Chern-Simons theory, which is conventionally used to study gravity in 3-dimensional anti-de Sitter space (AdS$_3$). The position and shape of the network encode the cutoff scale at which the ground state density operator is defined. A general argument suggests identifying the ‘density of complexity’ of this network with the extrinsic curvature of the cutoff surface in AdS$_3$, which by the Gauss-Bonnet theorem agrees with the holographic Complexity = Volume proposal.

The cutoff.— Consider a semi-classical, static, asymptotically AdS$_3$ geometry dual to some state of a holographic CFT$_2$. We stipulate the following about how an IR cutoff in the former relates to a UV cutoff in the latter:

1. In longitudinally homogeneous geometries radial curves manifest constant CFT cutoffs.

Examples include $u = $ const. lines in the Poincaré patch

$$ds^2 = (dx^+ dx^- + du^2)/u^2$$

and $\chi = $ const. curves in the BTZ / AdS$_3$-Rindler metric:

$$ds^2 = -\sinh^2 \chi dt^2 + d\chi^2 + \cosh^2 \chi dz^2.$$  

2. Coarsening the CFT UV cutoff pushes the bulk cutoff away from the boundary (deeper in the interior).

3. Maximally coarsening the CFT cutoff (so it encompasses the entire Cauchy slice) reduces the theory to s-waves (constant modes). This places the cutoff surface on the horizon if one exists.

This follows as a limit of point 2 and from subregion duality [10]. If s-waves defined a bulk cutoff shy of the horizon, how would we interpret the region between the horizon and the cutoff? Yet the cutoff cannot reach beyond the horizon lest it violate subregion duality.

4. In locally AdS$_3$ geometries, if the bulk cutoff follows a geodesic $\gamma$ then the corresponding CFT cutoff scheme truncates the modes localized in the interval subtended by $\gamma$ to its s-wave sector.

The last point follows from remark 3 after recognizing the geodesic $\gamma$ as the horizon of an AdS$_3$-Rindler geometry. The appropriate way of smearing local operators to produce s-waves on CFT intervals was discussed in [11].

The state.— A density operator is a map, which sends operators to real numbers:

$$\rho : \mathcal{O} \rightarrow \text{tr}(\rho \mathcal{O})$$ (3)
An emphasis on viewing states as functionals on operators, rather than simply members of the algebra of observables, is central to our thinking about complexity.

In field theory, the state’s dependence on the cutoff restricts the domain of map [3]. Specializing to the holographic context, suppose the cutoff is presented as a smooth and convex curve in the bulk, parameterized by proper length $\lambda$. We do not assume that the cutoff is homogeneous, so the profile of the cutoff curve generically depends on space. We require the map $\rho$—the CFT ground state at the given cutoff scale—to satisfy:

(i) $\rho$ eats up multi-local operators $O_1(\lambda_1) \ldots O_n(\lambda_n)$ and returns $(0|O_1(\lambda_1) \ldots O_n(\lambda_n)|0)$.

(ii) The arguments $O_i(\lambda_i)$ of $\rho$ are labeled by their locations $\lambda_i$ on the bulk cutoff curve.

(iii) If the curve follows a geodesic over some range of $\lambda$’s, $\rho$ takes at most one input $O(\lambda)$ from that range.

(iv) Shifts in the bulk curve and the attendant transformations of $\rho$ enact local changes of scale.

Requirement (iii) follows from point 4 in the cutoff discussion: more than one independent input from a geodesic segment would be outside the $s$-wave sector of the AdS$_3$-Rindler space populated by $\bar{\text{SL}}(2,\mathbb{R})$-valued connection one-forms in $(x^+, x^-, u)$-space, i.e. gauge fields whose components are linear combinations of $L_{-1}, L_0, L_1$. The latter generate the left- and right-moving global conformal symmetries and satisfy:

$$[L_n, L_m] = (n - m)L_{n + m} \quad (4)$$

We choose a flat configuration of $A$ and $\bar{A}$, which satisfies boundary conditions:

$$\lim_{u \to 0} u A_+ = L_{-1} \text{ and } \lim_{u \to 0} u A_- = 0 \quad (5)$$

$$\lim_{u \to 0} u \bar{A}_+ = L_1 \text{ and } \lim_{u \to 0} u \bar{A}_- = 0 \quad (6)$$

With this choice, Wilson lines (and their networks) compute CFT$_2$ ground state correlation functions [13, 14].

The CFT meaning of Wilson lines.—In gauge theories, gauge-invariant quantities are supplied by path-ordered exponentials along paths $(x^+(\sigma), x^-(\sigma), u(\sigma))$:

$$P \exp \int_{\sigma_b}^{\sigma_a} ds \left( \partial_\sigma x^+ A_+ + \partial_\sigma x^- A_- + \partial_\sigma u A_u \right) \times (A \leftrightarrow \bar{A})$$

Ordinarily the path must be closed and this is called a Wilson loop. In the present case the flatness of $A$ and $\bar{A}$ makes all Wilson loops trivial. When the path is open there is a residual gauge dependence at the endpoints, so arbitrary Wilson lines are not gauge-invariant. The exception is when the endpoints are taken to the boundary where the gauge dependence is frozen by boundary conditions. Consequently, we focus below on Wilson lines with endpoints at the asymptotic boundary $u \to 0$.

Consider a path from $(x^+_0, x^-_0, u_0)$ to $(x^+_0, x^-_0, u_0)$. Because $A$ and $\bar{A}$ are flat, the exact course of the path is immaterial so we can pull it onto the $u = u_0$ plane. The Wilson line of $A$ evaluates to $\exp \left( L_{-1}(x^+_0 - x^+_0)/u_0 \right)$, a finite translation by $(x^+_0 - x^+_0)/u_0$. The $\bar{A}$-Wilson line performs a similar translation, only in $1/x^-$. Indeed, any Wilson line is a finite $\text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$ transformation because it is a path-ordered exponential of an $\text{sl}(2,\mathbb{R}) \times \text{sl}(2,\mathbb{R})$ one-form. Due to the boundary conditions [4, 5], Wilson lines that begin and end on the boundary are always finite translations in $x^+$ and in $1/x^-$.

We would like to compute expectation values of such Wilson lines in various $\text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$ representations. Representations of $\text{SL}(2,\mathbb{R})$ are classified by conformal dimensions $h$. Their multiplets are built up by acting with $L_{-1}$ on the highest weight state $|h\rangle \equiv O_h(0)|0\rangle$. One example of an expectation value of our Wilson line is therefore:

$$\langle h|e^{-L_{-1}x^+/u_0}|h\rangle = (0|O_h(0)O_h((x^+_0 - x^+_0)/u_0)|0) \quad (7)$$

We recognize that all CFT 2-point functions are expectation values of translations, taken over $\text{SL}(2,\mathbb{R})$ multiplets in various representations. Once again, equation [5] ensures that boundary-pegged $A$-Wilson lines evaluate to the correct translations up to a rescaling by $u_0$, which we comment on below. For the $\bar{A}$-components of our Wilson lines, the conclusion is the same except the $\text{SL}(2,\mathbb{R})$ multiplets must be built in the $1/x^-$ conformal frame—by acting with $L_{+1}$ from the right on $|h\rangle \equiv |h\rangle$!

The CFT meaning of networks.—For higher-point correlators, we combine Wilson lines to form networks—collections of lines joined by junctions. Such junctions were previously described in [14]. In the discussion above, the flatness of $(A, \bar{A})$ allowed us to pull arbitrary Wilson lines onto $u = \text{const}$, slices. We need to make sure that networks with junctions are similarly deformable.

A junction merges two incoming Wilson lines into one outgoing line. Recall that an individual Wilson line carries an $\text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$ representation—a vector space populated by $|h, \bar{h}\rangle \equiv O_{h, \bar{h}}(0)|0\rangle$ and descendants. Two incoming Wilson lines carry two independent such multiplets, which a junction then sends to one multplet:

$$O_{h_1, \bar{h}_1}(v, \bar{v}) |0\rangle_v \otimes O_{h_2, \bar{h}_2}(w, \bar{w}) |0\rangle_w \xrightarrow{\text{junction}} (8)$$

$$\int dz d\bar{z} \sum_{h, \bar{h}} c_{h_1, \bar{h}_1; h_2, \bar{h}_2}^{h, \bar{h}} (z, \bar{z}, v, \bar{v}, w, \bar{w}) O_{h, \bar{h}}(z, \bar{z}) |0\rangle_z$$

The notation $|0\rangle_v$ distinguishes the ‘ground state’ on $v$-space from the ‘ground states’ on $w$- and $\bar{w}$-spaces—local
copies of the CFT on which the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ multiplets live. Defining a junction boils down to choosing appropriate kernels $c_{h_1,h_2}(z,\bar{z},v,\bar{v},w,\bar{w})$ in (9).

With an arbitrarily chosen kernel, networks of Wilson lines will not be deformable. A condition that guarantees deformability is this: if an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ transformation is simultaneously applied to the $v$- and $w$-space inputs in eq. (5), its $z$-space output must come out transformed in the same way (4). An identical condition governs the Operator Product Expansion (OPE) of local operators in the CFT. Therefore, in a deformable junction, the dependence of $c_{h_1,h_2}(z,\bar{z},v,\bar{v},w,\bar{w})$ on its coordinate arguments is fixed by conformal kinematics up to overall constants $c_{h_1,h_2}$. We recognize that defining a deformable set of junctions leaves out the same flexibility as defining a CFT does: the undetermined data are OPE coefficients.

Naturally, we choose Wilson line junctions to match the fusion algebra of the CFT. Thus, a network shown in the top left panel of the Figure represents a repeated application of the OPE expansion. Once again, this network can be pulled onto the boundary, in which case it is the OPE expansion of the CFT. When the network is contracted at all endpoints with member states of various $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ multiplets, it will by construction return the relevant multi-point correlation function.

The Chern-Simons field encodes the $AdS_3$ geometry.— The $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ connection we exploit is most famous for a reformulation of pure gravity in asymptotically $AdS_3$ spacetimes (15). When a flat $(A,\bar{A})$ which satisfies boundary conditions (5-6) is substituted into

$$g_{\mu\nu} = (\frac{1}{2}) \text{tr}_{\text{fundamental}}(A-\bar{A})_{\mu}(A-\bar{A})_{\nu},$$

it returns the metric of the Poincaré patch of $AdS_3$. The non-vanishing traces entering (9) are $\text{tr} L_0^2 = 1/2$ and $\text{tr} L_{-1}L_1 = -1$. Other locally $AdS_3$ geometries, including global $AdS_3$, arise from (9) with modified boundary conditions (13) (16). (The assertion that networks of Wilson lines compute CFT correlators still applies, with expectation values now taken in the vacuum on a cylinder or in Virasoro descendants.) As an example, the choice

$$A = (L_{-1}/u)dx^+ + L_0d(\log u)$$
$$\bar{A} = (L_1/u)dx^- - L_0d(\log u)$$

produces metric (1) while other, gauge-equivalent choices produce the same geometry in different coordinates. The flatness of $(A,\bar{A})$, which expresses vacuum Einstein’s equations with $\Lambda = -1$, follows from the classical equations of motion of Chern-Simons theory.

Our discussion of CFT correlation functions made no reference to the quantum Chern-Simons theory and $(A,\bar{A})$ were only used as auxiliary technical tools. But because the $AdS_3$ spacetime dual to the ground state of any holographic CFT$_2$ is described by metric (9), we may explain the meaning of bulk cutoff curves with reference to classical solutions such as (10) (11).

The CFT vacuum at a holographically specified scale.— Consider the network of Wilson lines pictured in the top right panel of the Figure. Assume the upper, black part of the network follows the bulk cutoff curve. Now sever all the red tentacles of this network to produce the amputated network. (Amputation creates stumps that transform in dual representations—objects that eat up incoming representations and return numbers.) We claim that this amputated network is the ground state at the scale specified by the cutoff surface. More examples, which represent the vacuum on a cylinder, are shown in the bottom panels of the Figure. (Our amputated network shares certain features with the tensor network of (17).

The amputated network manifestly satisfies conditions (i-iii). It eats up $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ representations (delivered by incoming Wilson lines) to return appropriate multi-point correlators. Incoming lines are sprinkled over the cutoff curve except—as marked in the Figure—on geodesic segments. To verify (iv), inspect one of the amputated tentacles, say in the $u$-direction in gauge (10):

$$\exp \int_{u_0}^{u_\ast} (du/u)L_0 = (u_\ast/u_0)^{L_0}. \quad (12)$$

We recognize that the job of the tentacles is to bring

![FIG. 1. All pictures are equal time snapshots of $(x^+,x^-,u)$-space. Top left: A network of Wilson lines, which computes CFT 6-point functions. Top right: A network which computes multi-point functions. After the red tentacles are amputated, it becomes the ground state at the scale defined by the black line. The density of inputs on the network is $dC \propto Kd\lambda$ so the network accepts no inputs on geodesic segments. Bottom: Two amputated networks in global $AdS_3$, which differ only by crossing symmetry, are shown in black. Crossing symmetry is implemented by shifting the feature $-00--$, which implements the projector $|0\rangle\langle0|$ in computing correlation functions. The blue features—Wilson lines and projections—convert the black network into one at a coarser, uniform cutoff. Projections are necessary to accord with $dC \propto Kd\lambda$ because coarse-graining eliminates degrees of freedom. The coarse-graining scheme is highly non-unique; it varies with gauge choice for $(A,\bar{A})$ and with crossing symmetry.](image-url)
**Networks in the bottom panels of the Figure.** We illustrate this with the two black scales $u_0$ to the scale of the cutoff curve, $u_*$. If we shifted the cutoff curve from $u_*$ to a new scale $u'_*$, individual stumps would get rescaled by $(u_*/u'_*)^{d\lambda_0}$ to absorb the changed scales of their inputs; this is just what requirement (iv) stipulates. (Under coarse-graining, some stumps are also projected out; see the bottom right panel of the Figure and text below.) Such projections are routine in tensor networks that model the RG behavior of CFT states \([18]\). Equation \([12]\) explains why the endpoints of the Wilson line in \([7]\) were put at some arbitrary $u_0$. Sending them to the true asymptotic boundary ($u_0 \to 0$) would have dilated the computation by an overall infinite factor familiar from AdS/CFT.

The amputated network is not gauge-invariant. Its gauge dependence, dual to the gauge dependence of ten-tacles like \([12]\), is an inalienable trait of renormalization. There is no preordained way of renormalizing CFT fields: doing so with $(u_*/u_0)^{d\lambda_0}$ is an artifact of gauge \([10, 11]\) and, by extension, of metric \([1]\). This ambiguity of holographic RG manifests the technicality we mentioned—that CFT operators are sections of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ bundles—and reflects the rule of thumb that boundary global symmetries are bulk gauge symmetries.

Finally, we comment on the loose ends of the amputated network. They carry trivial representations, which are local invariants on smooth curves, but become ill-defined if we discretize the curve into a sequence of geodesic segments.) The density of complexity on the cutoff curve must therefore be a linear combination of the two: $d\mathcal{C}/d\lambda = \#_0 + \#_1K$. As our network takes no inputs on geodesic segments where $K = 0$, $\#_0$ must vanish so $d\mathcal{C} \propto Kd\lambda$. For a sanity check, note that all equal time $u = \text{const.}$ lines in metric \([1]\) have $K = 1$ and $d\lambda = dz/\alpha$, so as expected $d\mathcal{C}/dz \propto 1/\alpha$ in that case.

**Comparison with other proposals.**—Specializing again to cutoff curves confined to the static slice, the Gauss-Bonnet theorem relates the complexity of the amputated network to the volume enclosed by the cutoff surface:

$$\mathcal{C} \propto \int Kd\lambda = \int (-R)dV + 2\pi = \int dV + 2\pi \quad (13)$$

In the convention of eq. \([9]\), the Ricci scalar on the static slice is $R = -1$. We will not attempt to explain the additive Euler term, which does not scale with the cutoff.

Our argument seems to support the Complexity = Volume proposal \([1]\), but it is not compelling evidence. When we take the cutoff curve off the static slice, symmetry allows a new contribution $d\mathcal{C}/d\lambda \propto \#_1K + \#_2\tau$, where $\tau$ is the Lorentzian ‘torsion’ of the curve. Whatever combination of extrinsic curvature and torsion might quantify complexity, it will not match the maximal volume inside the curve, which is a more erratic quantity.

Instead, we emphasize conceptualizing complexity as counting steps of an algorithm which, as in equation \([3]\), sends operators to numbers. This assumption tacitly undergirds equating complexity with counting nodes in tensor networks. Yet the circuit model makes an extra assumption: that all intermediate steps of the algorithm must also be interpretable as states of the theory—all the way back to a reference state at which the algorithm is initialized. We think this is too restrictive. Our amputated network provides an example: it has no reference state and, if you interrupt it at an intermediate stage without inserting the trivial representation, it computes an iterated OPE expansion instead of a density matrix.

**Toward other states.**—Can the amputated network be adapted to excited states? One possibility is an algorithm which transports and fuses modular frequency modes of cutoff-sized intervals instead of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ representations. (Transport of modular modes was sketched in \([19]\).) Because fusing local operators automatically produces vacuum modular modes of intervals \([17]\), this idea is consistent with the model in this paper. Meanwhile, in holography modular modes localize in the bulk \([11, 20]\) on geodesics and other special loci, so the guess maintains contact with holographic proposals. We plan to investigate this idea in CFT$_2$ states produced by heavy operators, which are dual to conical defects and microstates of BTZ black holes.

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