Non-Abelian Fractional Supersymmetry in Two Dimensions

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Abstract. Non-Abelian fractional supersymmetry algebra in two dimensions is introduced utilizing $U_q(sl(2,\mathbb{R}))$ at roots of unity. Its representations and the matrix elements are obtained. The dual of it is constructed and the corepresentations are studied. Moreover, a differential realization of the non-Abelian fractional supersymmetry generators is given in the generalized superspace defined by two commuting and two generalized Grassmann variables. An invariant action under the fractional supersymmetry transformations is given.
1. Introduction

Supersymmetry is formulated in terms of $Z_2$ graded algebras whose realizations can be obtained by Grassmann coordinates. A step beyond supersymmetry is to consider $Z_3$ graded algebras or the spaces given by generalized Grassmann variables $\eta^p = 0$, where $p$ is a positive integer and fractional supersymmetry generators, which are defined as the $p^{th}$ root of space–time translation operators. Although, the latter approach can be studied in terms of the deformed algebras and spaces where the deformation parameter $q$ is a primitive root of unity, its group theoretical understanding was lacking. Recently, we proposed to study the two dimensional fractional supersymmetry in terms of a formulation of the quantum Poincarè group at roots of unity. This gave us the possibility of utilizing the well developed representation theory techniques to study the fractional supersymmetry in two dimensions.

In two fractional supersymmetry generators which are mutually commuting were considered. Here we generalize it to the fractional supersymmetry generators which are mutually noncommuting utilizing the formulation given in [11]. Hence, we call it the non-Abelian fractional supersymmetry. Our formulation straightforwardly leads to a differential realization of the fractional supersymmetry generators which would be very difficult, if not impossible, to find in terms of $q$–calculus. The general theory of representations of $U_q(sl(2))$ at roots of unity is well known and the references therein, but for physical applications their explicit forms are needed as it is discussed in the last section.

Another advantage of our formalism shows up when one deals with higher dimensions. Although it is not presented here, higher dimensional fractional supersymmetry can be introduced in terms of $SL_q(n, \mathbb{R})$ at roots of unity and its subgroups by generalizing our procedure of using $SL_q(2, \mathbb{R})$ to obtain the properties of the two dimensional fractional supersymmetry, in a straightforward fashion.

After presenting the two dimensional non-Abelian fractional supersymmetry algebra $U_{FS}$ and its dual $A_{FS}$, we deal with their representations and corepresentations. We study the $*$-representations and in terms of them find differential realizations of the non-Abelian fractional supersymmetry generators in the superspace with two commuting and two generalized Grassmann coordinates. Then, we discuss how one can utilize these representations in possible physical applications.
2. Non-Abelian fractional supersymmetry algebra and its dual

The two dimensional non-Abelian fractional supersymmetry algebra denoted $U_{FS}$, is the $\star$-algebra generated by $P_\pm$, $H$, $E_\pm$, $K$, ($q^p = 1$, $p$ being an odd positive integer)

$$[P_+, P_-] = 0, \quad [P_\pm, H] = \pm iP_\pm,$$

$$KE_\pm K^{-1} = q^{\pm 1}E_\pm, \quad [E_+, E_-] = \frac{K^2 - K^{-2}}{q - q^{-1}},$$

$$[K, H] = 0, \quad [E_\pm, H] = \pm \frac{i}{p}E_\pm,$$

$$P_\star^p = P_\pm, \quad H^* = H, \quad E_\star^\pm = E_\pm, \quad K^* = K,$$

and $E_\pm$ are the $p^{\text{th}}$ root of the space-time translations $P_\pm$:

$$E_\pm^p = P_\pm.$$

Moreover, we put the condition

$$K^p = 1_{U},$$

where $1_U$ indicates the unit element of the algebra.

The basis elements of $U_{FS}$ are

$$\phi^{nmkrsl} = E_+^n E_+^m K^k P_+^r P_-^s H^l,$$

where $n, m, k = 0, 1, \cdots, p - 1$, and $r, s, l$ are positive integers.

We can equip $U_{FS}$ with the Hopf algebra structure

$$\Delta(P_\pm) = P_\pm \otimes 1_U + 1_U \otimes P_\pm, \quad \varepsilon(P_\pm) = 0, \quad S(P_\pm) = -P_\pm,$$

$$\Delta(H) = H \otimes 1_U + 1_U \otimes H, \quad \varepsilon(H) = 0, \quad S(H) = -H,$$

$$\Delta(E_\pm) = E_\pm \otimes K + K^{-1} \otimes E_\pm, \quad \varepsilon(E_\pm) = 0, \quad S(E_\pm) = -q^{\pm 1}E_\pm,$$

$$\Delta(K) = K \otimes K, \quad \varepsilon(K^{\pm 1}) = 1, \quad S(K^{\pm 1}) = K^{\mp 1}.$$

Let us present the dual of the non-Abelian fractional supersymmetry algebra $U_{FS}$: It is the $\star$-algebra $A_{FS}$ with $q^p = 1$ and generated by $z_\pm$, $\lambda$, $\eta_\pm$, $\delta$ satisfying

$$\eta_+ \eta_- = q^2 \eta_- \eta_+, \quad \eta_\pm \delta = q^2 \delta \eta_\pm,$$

$$\eta_\pm^p = 0, \quad \delta^p = 1_A,$$

$$\eta_\pm^* = \eta_\pm, \quad \delta^* = \delta,$$

$$z_\pm^* = z_\pm, \quad \lambda^* = \lambda.$$
where $z_\pm$, $\lambda$ commute with the others and $1_A$ is the unit element of $A_{FS}$. Its Hopf algebra structure is given by the coproducts

$$
\Delta \delta = \delta \otimes \delta + q^{-2}\delta^{-1}\eta_+^2 \otimes \eta_-^2 \delta + (1_A + q^{-2})\eta_+ \otimes \eta_-, \\
\Delta \eta_+ = \eta_+ \otimes 1_A + \delta \otimes \eta_+ + (1_A + q^2)\eta_+ \otimes \eta_+ \eta_- + q^{-2}\delta^{-1}\eta_+^2 \otimes (1_A + q^2\eta_+ \eta_-)\eta_-, \\
\Delta \eta_- = \eta_- \otimes 1_A + \delta^{-1} \otimes \eta_- + \sum_{k=1}^{p-2}(-1)^k q^{-k(k+1)}\delta^{-k-1} \eta_+^k \otimes \eta_-^{k+1}, \\
\Delta \lambda = \lambda \otimes 1_A + 1_A \otimes \lambda, \\
\Delta z_+ = z_+ \otimes 1_A + e^\lambda \otimes z_+ + \sum_{k=1}^{p-1} \frac{q^{k^2}}{[k][p-k]} \eta_+^{p-k} \delta^k e^{n\lambda/p} \otimes (-q^2\eta_+ \eta_-; q^2)_{(p-k)} \eta_+^k, \\
\Delta z_- = z_- \otimes 1_A + e^{-\lambda} \otimes z_- + \sum_{k=1}^{p-1} \frac{q^{-k^2}}{[k][p-k]} \eta_-^{p-k} \delta^k e^{-n\lambda/p} (-\eta_+ \eta_-; q^{-2})_k \otimes \eta_-^k,
$$

(13)

the antipodes

$$
S(\delta) = \delta^{-1}(1_A + q^{-2}\eta_+ \eta_-)(1_A + \eta_+ \eta_-), \quad S(\eta_\pm) = -\delta^{\mp1}\eta_\pm, \\
S(\lambda) = -\lambda, \quad S(z_\pm) = -z_\pm,
$$

and the counits

$$
\epsilon(\delta) = 1, \quad \epsilon(\eta_\pm) = 0, \quad \epsilon(\lambda) = -\lambda, \quad \epsilon(z_\pm) = 0.
$$

We use the notation

$$
(a; q)_k \equiv \prod_{j=1}^{k} (1 - aq^{j-1}),
$$

the symmetric $q$–number

$$
[n] = \frac{q^n - q^{-n}}{q - q^{-1}}
$$

and the $q$–factorial $[n]! = [n][n-1]\cdots[1]$

Any element of $A_{FS}$ can be written as

$$
\sum_{n,m,k=0}^{p-1} f_{nmk}(z_+, z_-, \lambda) \eta_+^n \eta_-^m \lambda^k \in A_{FS},
$$

where $f_{nmk}(z_+, z_-, \lambda)$ are infinitely differentiable functions on $\mathbb{R}^3$. Definition of the Hopf algebra operations on these functions were given in [11]. Hence, a local basis of $A_{FS}$ can be given by

$$
a^{nmktsl} \equiv \eta_+^n \eta_-^m \zeta(k; \delta) z_+^t z_-^s \lambda^l,
$$

(14)
where \( n, m, k = 0, 1, \ldots, p - 1; \) \( t, s, l \) are positive integers and we defined
\[
\zeta(m, \delta) \equiv \frac{1}{p} \sum_{n=0}^{p-1} q^{-nm} \delta^n.
\]
Observe that
\[
\langle \phi^{nmktsl}, a^{n'm'k't's'l'} \rangle = i^{n+m+t+s+l} q^{-m} \delta_{nm} \delta_{t+l} \delta_{s+l} \delta_{l+n+m,k'}.
\]
are the duality relations between \( \mathcal{A}_{FS} \) and \( \mathcal{U}_{FS} \).

3. Pseudo-unitary, irreducible corepresentations of \( \mathcal{A}_{FS} \)

Let \( C_0^\infty(\mathbb{R}) \) be the space of all infinitely differential functions of finite support in \( \mathbb{R} \) and \( P(t) \) denote the algebra of polynomials in \( t \) subject to the conditions \( t^p = 1 \) and \( t^* = t \), i.e. any element of \( P(t) \) can be written as \( a(t) \equiv \sum_{n=0}^{p-1} a_n t^n \).
The irreducible representation of \( \mathcal{U}_{FS} \) in \( C_0^\infty(\mathbb{R}) \times P(t) \) is defined by the linear map
\[
\pi_{\lambda_\pm}(U_{FS}) : C_0^\infty(\mathbb{R}) \times P(t) \to C_0^\infty(\mathbb{R}) \times P(t)
\]
given as
\[
\begin{align*}
\pi_{\lambda_+}(E_+) f(x) a(t) &= \lambda_+^{1/p} e^{x/p} f(x) a(t), \\
\pi_{\lambda_+}(E_-) f(x) a(t) &= e^{-x/p} f(x) \sum_{n=0}^{p-1} M_n a_n t^{n-1}, \\
\pi_{\lambda_+}(P_\pm) f(x) a(t) &= \lambda_\pm e^{\pm x} f(x) a(t), \\
\pi_{\lambda_+}(H) f(x) a(t) &= -i \frac{d}{dx} f(x) a(t), \\
\pi_{\lambda_+}(K) f(x) a(t) &= f(x) a(qt),
\end{align*}
\]
where we used the notation
\[
M_0 = \lambda_+^{-1/p}, \ M_n = \lambda_+^{-1/p} \{ \lambda_+ - [n][n-1] \} \text{ for } n \neq 0,
\]
and
\[
\lambda_- = \prod_{n=0}^{p-1} M_n.
\]
Let us introduce the following hermitian forms for the space $C^\infty_0(\mathbb{R}) \times P(t)$,

$$(f_1, f_2) = \int_{-\infty}^{+\infty} dx f_1(x) \overline{f_2(x)},$$

$$(a_1, a_2) = \Phi(a_1(t)a_2^*(t)),$$

where

$$\Phi(t^s) = \delta_{s,0(\text{mod } p)}.$$  

$C^\infty_0(\mathbb{R})$ endowed with the norm induced by (17) leads to the Hilbert space of the square integrable functions on $\mathbb{R}$. On the other hand $P(t)$ with the norm $||a||^2 \equiv (a, a)$ is the pseudo-Euclidean space with $\frac{p+1}{2}$ positive and $\frac{p-1}{2}$ negative signatures [11]. Now, one can verify that $\pi_{\lambda \pm}$ defines pseudo-unitary, irreducible $\ast$–representation of $U_{FS}$ for real $\lambda \pm$.

The irreducible corepresentation of $A_{FS}$ can be found in terms of the duality relations (15) and the representation (16) of $U_{FS}$ as

$$T_{\lambda \pm}(f(x)a(t)) = \sum_{n,m,k=0}^{p-1} \sum_{t,s,l=0}^{\infty} a^{nmktsl} \pi_{\lambda \pm}(\phi^{nmktsl}) f(x)a(t) \langle \phi^{nmktsl}, a^{nmktsl} \rangle,$$

which is pseudo-unitary for real $\lambda \pm$.

Consider the Fourier transform of $f(x) \in C^\infty_0(\mathbb{R})$

$$F(\nu) = \int_{-\infty}^{+\infty} f(x)e^{\nu x} dx.$$  

This integral converges for any complex $\nu$. $F(\nu)$ is an analytic function and moreover, satisfies

$$|F(\text{Re } \nu + i\text{Im } \nu)| < \omega e^{c|\text{Re } \nu|},$$

for some real constants $\omega$ and $c$. Then we can write the inverse transform as

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(\nu)e^{-\nu x} d\nu.$$  

The Fourier transform of $T_{\lambda \pm}$ (20) yields the pseudo-unitary corepresentation in the space of functions $F(\nu)a(t)$ as

$$Q_{\lambda \pm}(F(\nu)t^k) = \int_{c-i\infty}^{c+i\infty} d\mu \sum_{l=0}^{p-1} Q_{kl}^{\lambda \pm}(\nu, \mu, g) F(\mu)t^l,$$
where \( g \equiv (g_0; \ g_p) \equiv (z_+, z_-, \ \lambda; \ \eta_+, \eta_-, \delta) \). The kernel \( Q^\pm_{kl} \) is

for \( l \geq k \),

\[
Q^\pm_{kl}(\nu, \mu, g) = K^\pm_{l-k}(\nu, \mu, g_0)\delta_l^\pm \Omega_{k,l}(\xi) + K^\pm_{l-k-p}(\nu, \mu, g_0)\eta_-^{p-k-l}\delta_{l}^\pm \tilde{\Omega}_{k,l}(\xi),
\]

(25)

for \( l < k \),

\[
Q^\pm_{kl}(\nu, \mu, g) = K^\pm_{l+k}(\nu, \mu, g_0)\delta_l^\pm \tilde{\Omega}_{k,p+l}(\xi) + K^\pm_{l-k}(\nu, \mu, g_0)\eta_-^{k-l}\delta_l^\pm \tilde{\Omega}_{k,p+l}(\xi),
\]

(26)

where we introduced, in terms of \( \xi = q\eta_+\eta_- \),

\[
\Omega_{k,l}(\xi) = \sum_{m=0}^{p+l-k-1} \frac{i^{k-l}(-1)^{m+k+l}q^{l-k}(l+1/2)-m(k+l)}{[m]![m-k+l]} \left( \prod_{s=0}^{m} M_{s+l} \right) \xi^m,
\]

(27)

\[
\tilde{\Omega}_{k,l}(\xi) = \sum_{m=0}^{k-l-1} \frac{i^{l-k-p}(-1)^{m+1}q^{l-k-l+m(k+l)}M_{m+l}}{[m]![m+p+k-l]} \left( \prod_{s=1}^{p+k-l+m} M_{s-p+l} \right) \xi^m.
\]

(28)

The functions \( K^\pm_{s} \) are

\[
K^\pm_{s}(\nu, \mu, g_0) = \frac{1}{2\pi i} e^{s\lambda_\pm} \int_{-\infty}^{+\infty} e^{ir(z_+e^{-x}+e^{-x}z_-)+x(\nu-\mu/s+p)} dx.
\]

(29)

By utilizing the analog of polar coordinates \( \rho > 0, \ \beta \in \mathbb{R} \), the pseudo-Euclidean plane defined by the axis \( z_- = 0 \) and \( z_+ = 0 \) can be studied in terms of the quadrants

Quad.1 : \( z_+z_- > 0 \), \( z_\pm = \frac{1}{2}\rho e^{\pm \beta} \), \quad Quad.2 : \( z_+z_- < 0 \), \( z_\pm = \pm\frac{1}{2}\rho e^{\pm \beta} \),

Quad.3 : \( z_+z_- > 0 \), \( z_\pm = \frac{1}{2}\rho e^{\pm \beta} \), \quad Quad.4 : \( z_+z_- < 0 \), \( z_\pm = \pm\frac{1}{2}\rho e^{\pm \beta} \).

In these quadrants \((29)\) will lead to the Hankel functions \( H^{(1)}_{\nu} \), \( H^{(2)}_{\nu} \) or cylindrical functions of imaginary argument \( K_{\nu} \):

Quad.1 : \( K^\pm_{s}(\nu, \mu, g_0) = \frac{1}{2} e^{(\nu-\mu-s/p)(\beta+\frac{\pi i}{2})+\mu\lambda_\pm} H^{(1)}_{\nu-\mu-s/p}(r\rho), \)

Quad.2 : \( K^\pm_{s}(\nu, \mu, g_0) = \frac{1}{2} e^{(\nu-\mu-s/p)(\beta-\frac{\pi i}{2})+\mu\lambda_\pm} H^{(2)}_{\nu-\mu-s/p}(r\rho), \)

Quad.3 : \( K^\pm_{s}(\nu, \mu, g_0) = \frac{1}{\pi i} e^{(\nu-\mu-s/p)(\beta+\frac{\pi i}{2})+\mu\lambda_\pm} K_{\nu-\mu-s/p}(r\rho), \)

Quad.4 : \( K^\pm_{s}(\nu, \mu, g_0) = \frac{1}{\pi i} e^{(\nu-\mu-s/p)(\beta-\frac{\pi i}{2})+\mu\lambda_\pm} K_{\nu-\mu-s/p}(r\rho), \)
with the condition \(-1 < \text{Re}(\nu - \mu + s/p) < 1\).

4. Quasi-regular corepresentation of \(A_{FS}\), \(\ast\)-representation of the
non-Abelian fractional supersymmetry algebra and a differential
realization

The comultiplication

\[ \Delta : A \rightarrow A_{FS} \otimes A \]  

(30)

defines the pseudo-unitary left quasi-regular corepresentation of \(A_{FS}\) in its
subspace \(A\) consisting of the finite sums

\[ X = \sum_{s} a_{s}(\eta_{+}, \eta_{-})f_{s}(z_{+}, z_{-}) \]

where \(a_{s}(\eta_{+}, \eta_{-})\) are polynomials in \(\eta_{+}, \eta_{-}\) and \(f_{s}(z_{+}, z_{-}) \in C_{0}^\infty(\mathbb{R}^{2})\). The
space \(A\) can be endowed with the hermitian form

\[ (X, Y)_{E} = I_{E}(XY^{\ast}), \]  

(31)

\(X, Y \in A\) and the linear functional \(I_{E} : A \rightarrow \mathbb{C}\)

\[ I_{E}(X) = \sum_{s} I(a_{s})I_{C}(f_{s}) \]  

(32)

was shown to be the left invariant integral\[\text{[11]}\] in terms of the integrals on
generalized superspace\[\text{[3],[4],[5],[8],[13]-[17]}\]

\[ I(\eta_{+}^{n}\eta_{-}^{m}) = q^{-1}\delta_{n,p-1}\delta_{m,p-1}, \]  

(33)

\[ I_{C}(f_{s}) = \int_{-\infty}^{+\infty} dz_{+}dz_{-}f_{s}(z_{+}, z_{-}). \]  

(34)

The right representation of the non-Abelian fractional supersymmetry
algebra \(U_{FS}\) corresponding to the quasi-regular representation \(\text{[10]}\),

\[ \mathcal{R}(\phi)X = (\phi \otimes \text{id})\Delta(X), \]  

(35)

\(\phi \in U_{FS}\), is a \(\ast\)-representation

\[ (\mathcal{R}(\phi)X, Y)_{E} = (X, \mathcal{R}(\phi^{\ast})Y)_{E}, \]
due to the fact that the hermitian form \([31]\) is defined in terms of the left invariant integral \([32]\).

The right representations on the variables \(\eta_\pm\) and \(f(z_+, z_-)\) can explicitly be written as

\[
\mathcal{R}(E_+)\eta^\pm_\mp = \pm i q^{1/2}[n] \eta^{\pm n-1}_\mp + i q^{1/2-n}[2n] \eta_{\mp} \eta^n_\mp, \quad \mathcal{R}(E_-)\eta^\pm_\mp = -i q^{-1/2}[n] \eta^{n+1}_\mp, \\
\mathcal{R}(E_-)\eta^\pm_\mp = 0, \quad \mathcal{R}(K)\eta^\pm_\mp = q^{\pm n} \eta^n_\mp, \\
\mathcal{R}(P_\pm)\eta^n_\mp = 0, \quad \mathcal{R}(E_+)f(z_+, z_-) = \frac{-i q^{1/2}[p-1]}{p-1} \eta^{p-1}_\pm \frac{df(z_+, z_-)}{dz_\pm}, \\
\mathcal{R}(H)f = iz_+ \frac{df}{dz_+} - iz_- \frac{df}{dz_-}.
\]

The relations satisfied by the right representation \(\mathcal{R}\)

\[
\mathcal{R}(\phi\phi') = \mathcal{R}(\phi')\mathcal{R}(\phi), \quad \mathcal{R}(E_\pm)(XY) = \mathcal{R}(E_\pm)X\mathcal{R}(K)Y + \mathcal{R}(K^{-1})X\mathcal{R}(E_\pm)Y, \\
\mathcal{R}(K)(XY) = \mathcal{R}(K)X\mathcal{R}(K)Y, \quad \mathcal{R}(H)(XY) = \mathcal{R}(H)XY + X\mathcal{R}(H)Y, \\
\mathcal{R}(P_\pm)(XY) = \mathcal{R}(P_\pm)XY + X\mathcal{R}(P_\pm)Y,
\]

permit us to define the action of an arbitrary operator \(\mathcal{R}(\phi)\) on any function in \(\mathcal{A}\).

The quantum algebra which we deal with possesses two Casimir elements

\[
C_1 = E_- E_+ + \frac{(qK - q^{-1}K^{-1})^2}{(q^2 - q^{-2})^2}, \quad C_2 = P_+ P_-.
\]

As the complete set of commuting operators we can choose \(\mathcal{R}(C_1), \mathcal{R}(C_2), \mathcal{R}(H), \mathcal{R}(K)\) and \(\mathcal{L}(H), \mathcal{L}(K)\) where \(\mathcal{L}(\phi)\) is the left representation of the element \(\phi\) defined similar to \([33]\) with the interchange of \(\phi\) with the identity \(id\). Observe that \(\mathcal{L}(H)X = 0\) and \(\mathcal{L}(K)X = X\) for any \(X \in \mathcal{A}\), thus, in the space \(\mathcal{A}\) the matrix elements can be labeled as \(D^{\lambda}_n\). Indeed, in terms of the kernel \(Q^{\lambda}_n\) \([29]\) one observes that

\[
D_{n00} = Q^{\lambda}_0(n, 0, g),
\]

\(n \in [0, p - 1]\), satisfy

\[
\mathcal{R}(K)D_{n00} = q^n D_{n00}
\]
\[ \mathcal{R}(H)D_{n\nu,00} = -i(\nu + n/p)D_{n\nu,00} \]
\[ \mathcal{R}(C)D_{n\nu,00} = \lambda_+ D_{n\nu,00} \]
\[ \mathcal{R}(E_+)D_{n\nu,00} = \lambda_+^{1/p} D_{n+1\nu,00}, \]
\[ \mathcal{R}(E_-)D_{n\nu,00} = M_\nu D_{n-1\nu,00}, \]
\[ \mathcal{R}(P_\pm)D_{k\nu,00} = \lambda_\pm D_{k\nu,00}, \]

where we introduced the notation \( D_{n\nu,00} \equiv D_{0\nu+1,00} \) and \( D_{-1\nu,00} \equiv D_{p-1\nu-1,00}. \)

To write field theory actions in the generalized superspace given by \( \eta_{\pm}, z_{\pm} \), one should be equipped with \((q-)\)differential realizations of the fractional supersymmetry generators, \( E_{\pm} \). When the functions of only \( \eta_+, z_+ \) or \( \eta_-, z_- \) are considered, one can use either the algebraic or the group theoretical properties to find the differential realizations of the generators \([4],[7] - [11]\) in terms of the \( q \)-derivatives \( D_q^\pm \) satisfying

\[ D_q^\pm \eta_{\pm} = \frac{1 - q^n}{1 - q} \eta_{\pm}^{n-1}. \] (39)

When we deal with the generalized superspace \( \eta_+, z_+ \), using the algebraic properties \((q\)-calculus\) to find the differential realizations of the non-Abelian fractional supersymmetry generators is hopeless. However, the right representation obtained in (39) can be used to write the operators corresponding to \( E_{\pm} \) in the generalized superspace given by \( \eta_{\pm}, z_{\pm} \) with the ordering

\[ F(\eta_{\pm}, z_{\pm}) = \sum_{m,n} f_{mn}(z_+, z-) \eta_{\pm}^m \eta_{\pm}^n, \]

in terms of the operators

\[ \hat{\eta}_{\pm} F = F \eta_{\pm}, \]

and the \( q \)-derivatives \( D_q^\pm \) satisfying

\[ D_q^\pm \hat{\eta}_{\pm} = q^2 \hat{\eta}_{\pm} \]
\[ D_q^\pm \hat{\eta}_{\pm} = q^2 \hat{\eta}_{\pm} \] (40)

This can be achieved by using the well known mutually commuting “dilatation” operators \( T_{\pm} \) and their inverses

\[ T_{\pm} = D_q^\pm \eta_{\pm} - \eta_{\pm} D_q^\pm = 1 - (1 - q)\hat{\eta}_{\pm} \]
\[ T_{\pm}^{-1} = 1 - (1 - q^{-1})\hat{\eta}_{\pm}, \] (41)

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satisfying
\[ T_\pm \eta^\pm_n = q^n \eta^\pm_n, \]
\[ T_\pm \eta^\mp_n = \eta^\pm_n. \] (42)

Differential realizations of the fractional supersymmetry generators in the generalized superspace are
\[ \mathcal{R}(E_+) = \frac{iq^{1/2}}{1 + q} T_- (D_q^{q^{-1}} + qD_q) + \frac{iq^{1/2}}{q^2 - q^{-2}} \hat{\eta}^- T_- T_+^3 + \frac{iq^{-3/2}}{q^2 - q^{-2}} \hat{\eta}^- T_- T_+^{-1} T_+^{-1} \]
\[- \frac{iq^{-1/2}}{q - q^{-1}} \hat{\eta}^- T_- T_+^{-1} + \frac{iq^{1/2}}{[p - 1]!} \eta^p_+ \frac{d}{dz^+}, \]
\[ \mathcal{R}(E_-) = \frac{iq^{-1/2}}{1 + q} T_+^{-1} (D_q^{-1} + qD_q^q) + \frac{iq^{-1/2}}{[p - 1]!} \eta^p_- \frac{d}{dz^-}. \] (43) (44)

In terms of the scalar product (32) we can define the involutions on the \( q \)-derivatives as
\[ D_q^{q^\pm} = -q D_q^{q^\pm}, \] (45)
\[ D_q^{q^{-1} \pm} = -q^{-1} D_q^{q^{-1} \pm}, \] (46)

and verify that the realizations (43)-(44) satisfy the involution conditions
\[ \mathcal{R}(E_\pm)^* = \mathcal{R}(E_\pm), \]
as they should. Moreover, \( \mathcal{R}(K) = T_-^{-1} T_+ \) can easily be observed.

5. Discussions

The formalism presented here can directly be used to define actions which are invariant under the transformations of the generalized superspace, (13), as far as we know the Casimir elements of the algebra.

Let \( \mathcal{C} \) be any function of the Casimir elements \( C_1, C_2, (37)-(38) \), and the identity. The action
\[ S[\Phi] = \mathcal{I}_E (\Phi^* \mathcal{R}(\mathcal{C}) \Phi), \] (47)
where \( \Phi(z_\pm, \eta_\pm) \) is any function on the generalized superspace, is invariant under the fractional supersymmetry transformations (13): Being the Casimir element, \( \mathcal{C} \) satisfies
\[ \Delta \mathcal{R}(\mathcal{C}) = (id \otimes \mathcal{R}(\mathcal{C})) \Delta. \]
Since the Casimir operators $\mathcal{R}(\mathcal{C}_{1,2})$ commute with the quasiregular representation $\Delta$ of $\mathcal{A}_{FS}$, (30), we get

\[
S[\Delta \Phi] = (id \otimes I_E) (\Delta \Phi^* (id \otimes \mathcal{R}(\mathcal{C})) \Delta \Phi) = (id \otimes I_E) (\Delta \Phi^* \Delta (\mathcal{R}(\mathcal{C}) \Phi)) \\
= (id \otimes I_E) \Delta (\Phi^* \mathcal{R}(\mathcal{C}) \Phi) = I_E (\Phi^* \mathcal{R}(\mathcal{C}) \Phi) \\
= S[\Phi].
\]

The choice of $\mathcal{C}$ depends on the physical system which we would like to deal with. Equipped with these representations, (33)–(44), we hope that one can find a fractional supersymmetric action which can be useful to understand some physical problems like the quantum Hall effect where the $U_q(sl(2))$ at $q$ roots of unity appears[18] in a natural way.

Fractional supersymmetric quantum mechanics[19] and field theories[4]–[7],[20] were discussed in terms of $q$–calculus. Because of being a group theoretical approach, we hope that our way of treating fractional supersymmetry will shed some light on how to overcome the difficulties which show up in defining fractional supersymmetric models.

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