Effects of the Running Gravitational Constant on the Amount of Dark Matter

A. Bottino*, C. W. Kim†, and J. Song‡

*Department of Fisica Teorica, Università di Torino and I.N.F.N., Sezione de Torino, via P. Giuria 1, 10125 Torino, Italy
and
†Centre de Physique Théorique
Ecole Polytechnique, 91128 Palaiseau Cedex, France
and
‡Department of Physics and Astronomy
The Johns Hopkins University
Baltimore, MD 21218, U.S.A.

Abstract

The amount of dark matter in the Milky Way and beyond is examined by taking into account the possible running of the gravitational constant $G$ as a function of distance scale. If the running of $G$, as suggested by the Asymptotically-Free Higher-Derivative quantum gravity, is incorporated into the calculation of the total dark matter in the galactic halo, the amount of dark matter that is necessary to explain the rotation curve is shown to be reduced by one third compared with the standard calculations. However, this running of $G$ alone cannot reproduce the observed flat behavior of the rotation curve. It is also shown that the running of $G$ cannot explain away the presence of most of the dark matter beyond the scale of $\sim 10$ Mpc in the Universe. We also present a pedagogical explanation for the running of $G(r)$ in the region of large scales which is clearly a classical domain.

*Email Address, BOTTINO@TO.INFN.IT
†Email Address, CWKIM@JHUVMS.HCF.JHU.EDU
‡Email Address, JHSONG@ROWLAND.PHA.JHU.EDU
I. INTRODUCTION

The nature (and the amount) of dark matter in our Milky Way and the Universe is one of the most important issues that we are facing in particle physics and cosmology. In spite of recent advances in establishing many limits on the abundance of various dark matter candidates and in estimating the distribution of dark matter, we are still far from understanding the nature and the amount of dark matter. The recent observation of the MACHOs (Massive Compact Halo Objects)[1] may shed some light on the nature of dark matter in our Milky Way, when statistics of the observations further improves in the coming years. Information about non-baryonic dark matter candidates will also be provided in the future by the new low-background detectors for direct detection and by other indirect means [2].

Recently, it was shown that in the Asymptotically-Free Higher-Derivative (AFHD) quantum gravity [3], the gravitational constant, $G$, may be asymptotically free. Its consequences on the content of dark matter have been briefly discussed in [4,5]. In this article we present more detailed discussions on the amount of dark matter in our Milky Way and in the Universe. Our discussions are based on the recent work on the phenomenological consequences of the running of $G(r)$ as a function of the scale $r$ for the cosmology described in Ref.[6,7]. It is shown that if the $G(r)$ is assumed to be running as indicated by the AFHD quantum gravity, the amount of dark matter in our Milky Way is reduced by one third and the running of $G(r)$ alone cannot explain the rotation curve. The behavior of $G(r)$ is such that it can never mimic all the dark matter distribution necessary to explain the rotation curve in the Milky Way.

We also show that the increase of $G(r)$ as a function of $r$ is not fast enough to explain the observed variation of $\Omega_0(r)$ as a function of distance scale. In particular, $\Omega_0(r)$ due to the running of $G(r)$ becomes much slower than the observed behavior beyond the distance of $\sim 10$ Mpc. Unless the running of $G(r)$ is modified drastically, it can never mimic most of the dark matter in the Universe.

Finally we attempt to give a pedagogical explanation why the effects of the running of $G(r)$ is sizeable only in a purely classical domain where large distances are involved. The running of the coupling constant in gauge theories is attributed to the quantum effects. Therefore, it is expected that the effects are prominent only in the quantum domain. In the case of the gravitational constant, one anticipates the quantum domain to be in the region of the Planck mass, $M_P$. The corresponding distance is the Planck length, $10^{-33}$ cm. The answer to this puzzle lies in the nature of asymptotic freedom of the coupling constant, which is inevitably subject to infrared slavery (i.e., confinement).

II. DARK MATTER IN THE MILKY WAY

We begin with the well-known gravitational potential due to a spherical distribution of matter within radius $R$ with density distribution $\rho(r)$ given by [8]

$$\Phi(r) = -4\pi G_N \left[ \frac{1}{r} \int_0^r dx x^2 f(x) + \int_r^R dx x f(x) \right],$$

(1)
where we have used the definitions

\[ G(r) \equiv G_N g(r) \quad ; \quad G(0) = G_N \quad , \]

and

\[ f(r) \equiv g(r) \rho(r) \quad . \]

In Eqs. (1) and (2), \( G_N \) is the Newton’s gravitational constant. It is to be noted that in Eq. (1), the standard \( \rho(r) \) was replaced by \( f(r) = \rho(r) g(r) \) in order to take into account effects of the running of \( G(r) \), expressed in terms of \( g(r) \). From Eq. (1), the force is given by

\[ |F(r)| = 4\pi G_N \frac{1}{r^2} \int_0^r dx x^2 f(x) \quad . \]

When Eq. (4) is substituted into the equation of motion,

\[ m|F(r)| = m \frac{v^2(r)}{r} \quad , \]

one finds

\[ v^2(r) = 4\pi G_N \frac{1}{r} \int_0^r dx x^2 f(x) \quad . \]

Taking derivative of Eq. (6) with respect to \( r \) yields

\[ f(r) = \frac{1}{4\pi G_N} \frac{v^2 + 2rvv'}{r^2} \quad . \]

It is customary to fit the observed rotation curve by the following two-parameter expression

\[ f(r) = \frac{f(0)}{1 + \left(\frac{r}{r_c}\right)^2} \quad , \]

where \( f(0) = \rho(0) \) and \( r_c \) are, respectively, the galactic core density and the size of the core. Comparison of Eqs. (7) and (8) at large values of \( r \) (with \( v' = 0 \)) gives the well-known expression

\[ \frac{v^2(\infty)}{4\pi G_N} = \rho(0)r_c^2 \quad . \]

Hence, for a given \( r_c \), \( \rho(0) \) is determined by the observed value \( v(\infty) \approx 220 \text{ km/sec} \). However, this would overestimate \( \rho(0) \) because the presence of the spheroid is neglected in the above discussion. The two standard derivations of \( f(0) \) and \( r_c \) are due to Bahcall, Schmidt and Soneira [9], and Caldwell and Ostriker [10] (see also [11,12]). Here, we take, for definiteness, \( r_c = 3 \text{ Kpc} \) and \( \rho(0) = 0.09 M_\odot \text{ pc}^{-3} \) which includes the contribution of the spheroid.

Now, if we adopt the \( g(r) \) given in [6], with \( r \) expressed in units of Kpc,

\[ g(r) \equiv 1 + \delta g(r) = 1 + 0.3 r^{0.15} \quad , \]
we obtain

\[ \rho(r) = \frac{\rho(0)}{1 + \left[ \frac{r}{r_c} \right]^2} \frac{1}{1 + \delta G(r)} = \frac{\rho_S(r)}{1 + \delta G(r)} , \]  

(11)

where \( \rho_S(r) \) denotes the standard density distribution without the running \( G(r) \). The behaviors of \( 4\pi r^2 \rho_S(r) \) and \( 4\pi r^2 \rho(r) \) are shown in Fig.(1), where the area under each curve represents the amount of dark matter in the halo for the corresponding case. The total amounts of dark matter in the halo of a radius 100 Kpc for the standard (without \( G(r) \)) and modified (with \( G(r) \)) cases are given, respectively, by

\[ M_{H}^{(s)} = 4\pi \int_{0}^{100 \text{Kpc}} \rho_S(x)x^2 dx = 9.7 \times 10^{11} M_\odot , \]  

(12)

and

\[ M_{H} = 4\pi \int_{0}^{100 \text{Kpc}} \rho(x)x^2 dx = 6.4 \times 10^{11} M_\odot . \]  

(13)

Therefore, the running \( G(r) \) reduces the dark matter content by one third. The amount of reduction is insensitive to the the size of the Milky Way Galaxy. When integrated up to 150 Kpc in Eqs.(12) and (13), we have 36 % reduction of the total dark matter content.

It is important to note that what is needed to explain the rotation curve is the behavior of \( f(r) \), which is a product of \( \rho(r) \) and \( g(r) \), as given in Eq.(8). Suppose we try to explain the dark matter content in the halo with the spheroid and the running \( G(r) \), then we would need \( G(r) \) which is increasing linearly in \( r \) because the density of the spheroid goes as \( r^{-3} \sim r^{-3.5} \) for large distance. Since it is unlikely that \( G(r) \) increases linearly in \( r \), it cannot mimic the dark matter in the halo. Therefore we conclude that, in spite of the running of \( G(r) \), dark matter is necessary in the halo, although the total amount can be reduced by one third. Another related consequence is that the one third reduction of dark matter in the halo would reduce the microlensing event rates by roughly the same amount, ameliorating the apparent discrepancy between the standard calculations of the event rates and the observed rates [13,14].

III. DARK MATTER IN THE UNIVERSE

We now consider the amount of dark matter beyond the Milky Way. In the earlier works [4,5] attempts were made to explain the dark matter in large scale structures by the increase of the running \( G(r) \). The \( \Omega_0(r) \) was simply written as

\[ \Omega_0(r) = \frac{8\pi}{3} \frac{G(r) \rho_0}{H_0^2} \equiv \Omega_0[1 + \delta G(r)] , \]  

(14)

where \( \rho_0 \) and \( H_0^2 \) were assumed to be constant, i.e. independent of distance scale, and the local value \( \Omega_0(r \simeq 0) \equiv \Omega_0 \) was taken to be 0.2, which is a very generous upper bound of \( \Omega_0 \) obtained from nucleosynthesis arguments. Based on Eq.(14), the inferred dark matter content up to the scales of clusters ( up to the scale of the Virgo Cluster) was drastically
reduced since $\Omega_0$ given in Eq.(14) can mimic most of the dark matter except at very large scales beyond the Virgo cluster.

It was recently shown in [6,7], however, that in a general scale-dependent cosmology, the cosmological quantities such as the Hubble constant and the age of the Universe as well as the gravitational constant become all scale-dependent, including $\Omega_0(r)$ which is given by

$$\Omega_0^N(r) = \frac{\Omega_0[1 + \delta(r)]}{1 + \Omega_0 \delta(r)}. \tag{15}$$

The difference between Eq.(14) and (15) is due to the fact that in the standard Friedman cosmology, $\rho_{c,0}$ is taken to be a constant, whereas it is $r$-dependent in this new scale-dependent cosmology. Therefore, the behavior of $\Omega_0^N(r)$ in Eq.(15) should not be compared with the often quoted plot of the standard $\Omega_0(r)$ given in [15] which is essentially a quantity proportional to the density itself. That is, in the new cosmology, $\Omega_0[1 + \delta(r)]$, which is proportional to the density, should be compared with the standard $\Omega_0(r)$ in Eq.(14). Another important difference between them is that $\Omega_0(r)$ in Eq.(14) continues to increase as $\delta_G(r)$ keeps increasing, whereas $\Omega_0^N(r)$ in Eq.(15) would reach unity asymptotically and never exceeds unity. Thus, the $\Omega_0^N(r)$ in Eq.(15) is a less rapidly growing function of $r$ than Eq.(14) for the same $\delta(r)$ because of the denominator in Eq.(15). In the following, we take the local value of $\Omega_0$ to be 0.1 instead of 0.2, which was used in [4], since the local value determined in and around our Milky Way never exceeds $\Omega_0 \approx 0.1$.

Now it is immediately clear from Fig.(2), where we have plotted $\Omega_0[1 + \delta_G(r)]$ with Eq.(10), that, although the running of $G$ can more or less mimic the observed matter density within large horizontal and vertical error bars up to the scale of $\sim 10$ Mpc, it cannot explain the dark matter beyond that scale. In order to illustrate this more specifically, we also plotted in Fig.(2) our best fit of the data points. (Admittedly, the best fit does not mean much because of huge error bars but it was meant to guide the eye.) This best fit is given by

$$\delta(r) = 0.0028 \times r^{0.69}, \tag{16}$$

where $r$ is given in units of Kpc. Note that below $\sim 10$ Mpc, the contribution from the running of $G$ is larger than that of the best fit. This clearly cannot be the real situation since the observed fit should include all the effects from both the running of $G$ and the dark matter. Note that $\Omega_0[1 + \delta_G(r)]$ is slightly above the experimental error bars, indicating a possibility that the running of $G$ in this region may be overestimated. The important feature, however, is that beyond $\sim 10$ Mpc, $\Omega_0[1 + \delta(r)]$ completely takes over $\Omega_0[1 + \delta_G(r)]$ and becomes dominant. Although Fig.(2) shows the crossover point at $\sim 10$ Mpc, it should serve as a rough estimate. It is not feasible, at present, to pin down the exact location of this crossover point because of the poor data quality. The crossover point depends very sensitively on how one fits $\delta(r)$ with the very poorly known data. In Fig.(2), the difference between the two curves beyond $\sim 10$ Mpc represents the true content of dark matter under the assumption that the prediction on the behavior of $G$ given by Eq.(10) is correct. In order to contrast the difference in another perspective, we compare, in Fig.(3), the behavior of the two $\Omega_0^N(r)$’s, one given by Eq.(15) with $\delta_G(r)$ in Eq.(10) and the other with $\delta(r)$ in Eq.(16) which is nothing but the best fit-curve of the data. Note that these curves are not
to be confused with the usual $\Omega_0(r)$ with a constant $\rho_c$. Also, in this figure the difference between the two curves represents the portion of dark matter which cannot be explained by the running of $G$. (Because of the definition of $\Omega_0(r)$, the areas under the curves do not represent the actual amount of dark matter.)

So far we have discussed the difference between the previous treatment of the running of $G$ as discussed in [4,5] and the one based on the modified scale-dependent cosmology [6]. In the latter, $\delta(r)$ was arbitrary to be determined phenomenologically. In another version of scale-dependent cosmology [7] a new metric and a generalized Einstein equation were introduced to explain the same phenomenon. In this version, the new metric dictates the form of $r$-dependence in $\delta(r)$ and the resulting crossover point appears at much larger scales. The gap between the two $\Omega_0[1 + \delta(r)]$’s in Fig. (2) widens very rapidly beyond the crossover point. Future observations will decide which version is valid, if any. It goes without saying that if and when the behavior of $G(r)$ is modified from that of Eq.(10) due to future advances in better understanding of quantum gravity, then the amount of dark matter as discussed above has to be modified accordingly.

**IV. DISCUSSIONS**

If we assume that the gravitational constant $G(r)$ runs as suggested by the AFHD quantum gravity, it is shown that the amount of dark matter in the Milky Way necessary to explain the observed rotation curve is reduced by one third compared with the amount in the standard calculations with the constant $G_N$. It is also argued that it is very unlikely that the running of $G(r)$ can mimic the total dark matter in the Milky Way.

The running of the newly defined $\Omega_0(r)$ in the previous works [4,5], in which the running of $G(r)$ was added in the standard Friedman cosmology in a straightforward manner, is essentially proportional to the matter density. It is different from the one defined in a new cosmology [6,7] with running cosmological quantities, the difference being due to the fact that in this new cosmology the critical density also increases as scale increases. It was pointed out that the correct local value of $\Omega_0$ should be 0.1 instead of 0.2 which further reduces the previous [4,5] estimates of the contribution from the running $G(r)$. We have demonstrated that the increase of the observed $\Omega_0$ as a function of $r$ is much more rapid than that with the running of $G(r)$ alone, leading to the necessity of a large amount of dark matter, in particular beyond the scale of $\sim 10$ Mpc.

Finally we comment on the running of $G(r)$. Some physics rationale was given in [4] to justify the running of $G(r)$ in the classical domain in spite of the common understanding that the running is due to quantum effects. Here, we add another rationale which is purely pedagogical. The asymptotically-free running of $G(r)$ used in this article was derived by using an effective AFHD quantum gravity, which was motivated by supergravity, at one loop level. The behavior of $G(r)$ is such that it stays as the Newton’s gravitational constant for scales up to around 1Kpc and then it starts rising. The obvious question is then “Why are the quantum effects for the gravitational constant so prominent in a purely classical region where large distances are involved?” After all, the running of a coupling constant according to the Renormalization Group Equation (RGE) is the quantum effect in a gauge theory. Here, we do not of course intend to reproduce the result of [4,5] but instead we
will illustrate why the running can be prominent in the classical region. The answer to this question can be found in the behavior of the well-known example of the running of the strong coupling constant, $\alpha_s(Q^2)$, which is also asymptotically free. The one-loop expression for $\alpha_s(Q^2)$ is

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + b\alpha_s(\mu^2) \ln \frac{Q^2}{\mu^2}},$$

(17)

where $\mu^2$ is some fixed point and $b$ is given by

$$b = \frac{(11 - \frac{2}{3}N_f)}{4\pi},$$

(18)

with the number of quark flavors, $N_f$.

Although Eq.(17) is the result of one-loop perturbative calculations, which is valid beyond a certain value of $Q^2$, we can conjecture, because of infrared slavery or confinement, that $\alpha_s(Q^2)$ becomes very large as $Q^2$ becomes smaller and smaller. This behavior is likely to persist even to extremely small $Q^2$, which, when converted into distance scale, corresponds to classical scales. Often, in order to get an idea of the scale associated with the rising of $\alpha_s(Q^2)$, one defines the value of $Q^2$ at which Eq.(17) diverges. The $\alpha_s(Q^2)$ diverges when $Q^2$ takes the value given by

$$Q^2 = \mu^2 \exp \left[-\frac{1}{b\alpha_s(\mu^2)}\right] \equiv \Lambda_{QCD}^2.$$

(19)

Substituting, for example, $\alpha_s(\mu^2) = 0.3$ for $\mu^2 = 1\text{GeV}^2$ into Eq.(17), we find

$$\Lambda_{QCD} \simeq 110\text{ MeV}.$$

(20)

This $\Lambda_{QCD}$ is the characteristic mass scale of the QCD. The above property is inherent in the asymptotically-free coupling constants in non-Abelian gauge theories. (The fine structure constant, $\alpha$, does not have this property, the QED being an Abelian gauge theory.)

We conjecture that similar quantum effects are in operation for $G(r)$ at very large scales. In this case, the distances involved are larger than 1 Kpc. Hence, the corresponding $\Lambda_G$ must be extremely small, say, $10^{-35} \sim 10^{-33}\text{eV}$, corresponding to distances of $10^6$ and $10^4\text{Mpc}$, respectively. If $\Lambda_{QCD} \sim 110\text{ MeV}$ were to be interpreted as representing an effective gluon mass, then $\Lambda_G \sim 10^{-35}\text{eV}$ may be interpreted as an effective graviton mass. To be more specific, let us parametrize $G(Q^2)$ as, with the Plank mass $M_P$,

$$G(Q^2) \equiv \frac{\alpha_G(Q^2)}{M_P^2},$$

(21)

where

$$\alpha_G(Q^2) = \frac{1}{1 + b \ln \left[\frac{Q^2}{M_P^2}\right]}.$$

(22)

We caution the reader that the parametrization in Eq.(21) is, admittedly, not a physically consistent one because it gives the impression that the gravity is generated by the exchange
of a particle with mass, \( M_P \), as in the case of the effective weak-interaction coupling constant where the force is generated by the exchange of the weak gauge bosons, \( W \) and \( Z \). We do not, of course, mean that. Rather, we use the parametrization in Eq.(21), based on dimensional arguments, to facilitate our pedagogical discussion.

The \( G(Q^2) \) defined above is asymptotically free for \( b > 0 \) and becomes identical to \( G_N \) at \( Q^2 = M_P^2 \). Suppose that \( b \) is very small (for which we have no rigorous explanation, but it is possible to have very small \( b \) if most of the contributions to \( b \) from gauge bosons, fermions and bosons, and their superpartners cancel with each other), then \( G(Q^2) \) remains the same as \( G_N \) for the usual particle-physics range of \( Q^2 \). According to Eq.(22), \( \alpha_G(Q^2) \) becomes infinite when \( Q^2 \) takes the value given by

\[
Q^2 = M_P^2 e^{-\frac{b}{2}} \equiv \Lambda_G^2 .
\]  

It is now easy to see that if \( b \) is very small, \( \Lambda_G \) becomes very small although the only mass scale involved is \( M_P \). For \( b = \frac{1}{290} \), for example, we find \( \Lambda_G = 10^{-35} \text{eV} \). When \( Q \) is converted into distance scale \( r \), Eq.(21) with Eq.(22) and \( b = 1/290 \) yields qualitatively the same behavior of \( G(r) \) obtained in the AFHD quantum gravity. This is our pedagogical explanation for the behavior of the \( G(r) \) at very large distance scales which are clearly a classical domain.

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Figure Captions

Fig. 1 Profiles of $4\pi r^{2}\rho(r)$ as a function of radius $r$ for the standard calculation without the running $G(r)$ (dashed curve) and the one with $G(r)$ (solid curve).

Fig. 2 Plots for $\Omega_0[1 + \delta G(r)]$ (dashed curve) and $\Omega_0[1 + \delta(r)]$ which is a fit to data points (solid curve) as functions of $r$. Data points represent $\Omega_0$ in the standard definition with constant $\rho_c$. Note the crossover point at $\sim 10$ Mpc, beyond which $\Omega_0[1 + \delta(r)]$ becomes dominant.

Fig. 3 Plots for the fitted $\Omega_0^N(r)$ and $\Omega_0^{N,G}(r)$ based on Eq.(15). Note that this $\Omega_0^N(r)$ is different from $\Omega_0(r)$ in Fig.2. The shaded area indicates the portion of dark matter which cannot be explained by the running of $G(r)$. 
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