Mathematical and computer modeling of component surface shaping

A Lyashkov
Department of Engineering geometry and CAD, Omsk State Technical University, 11, Mira ave., Omsk, 644050, Russia
E-mail: 3dogibmod@mail.ru

Abstract. The process of shaping technical surfaces is an interaction of a tool (a shape element) and a component (a formable element or a workpiece) in their relative movements. It was established that the main objects of formation are: 1) a discriminant of a surfaces family, formed by the movement of the shape element relatively the workpiece; 2) an enveloping model of the real component surface obtained after machining, including transition curves and undercut lines; 3) The model of cut-off layers obtained in the process of shaping. When modeling shaping objects there are a lot of insufficiently solved or unsolved issues that make up a single scientific problem – a problem of qualitative shaping of the surface of the tool and then the component surface produced by this tool. The improvement of known metal-cutting tools, intensive development of systems of their computer-aided design requires further improvement of the methods of shaping the mating surfaces. In this regard, an important role is played by the study of the processes of shaping of technical surfaces with the use of the positive aspects of analytical and numerical mathematical methods and techniques associated with the use of mathematical and computer modeling. The author of the paper has posed and has solved the problem of development of mathematical, geometric and algorithmic support of computer-aided design of cutting tools based on computer simulation of the shaping process of surfaces.

1. Introduction
The manufacture of products in several branches of mechanical engineering industry is connected with technological processes of shaping of geometrically complex component surfaces. These components include rotors of screw pumps, impellers turbines, compressors and pumps, impeller fans, compressors and other products.

An important role at this is given to the issues of shaping modeling during the process of the cutting tool design. One of the elements of this process is designing its shaping surface. The solution to this problem is considered in many papers. Many of them state that in order to perform the calculation it is necessary to derive the appropriate dependencies as applied to the different initial data. In this case the classical method of differential geometry or the kinematic method is basically used [1]. One of the new approaches was proposed in [2] and to some extent it was used in papers [3, 4]. However the computational dependencies have the form of transcendental equations. Numerical methods are used to solve them. All this complicates the process of the cutting tool profiling.

The definition of the envelope of the family of surfaces is based on the development of mathematical models of the shaping process. This approach does not use differential methods.
However, the effective solution of the considered problem of complex surfaces shaping can be implemented using the methods of geometrical simulation by means of computer graphics and CAD systems [5, 6]. Modern computer technologies, providing the possibility of modeling the shaping process, allow us to determine the influence of various parameters of the tool, as well as its location, on the shape of the component surface. In addition, they provide the opportunity to develop software modeling the movement of the tool and the workpiece in the automated mode. In this case the problem of shaping is performed with necessary accuracy [4, 7].

Such approach provides a solution to a number of problems of technological support of dimensional processing of different surfaces: (1) profiling a shaping element of the cutting tool; (2) assigning the conditions for the relative motion of the tool and the workpiece; (3) forecasting of the possible errors in shaping of the component surface and other tasks.

This paper concerns the study of the shaping processes of technical surfaces by means of using analytical and numerical mathematical methods and techniques associated with the use of geometric and computer modeling. The study, combining the above-mentioned methods, deals with the mappings of the orthogonal projection of two-dimensional surfaces and three-dimensional hypersurfaces, specified by the equation in an implicit form and parametric equations, on the coordinate plane and the hyperplane.

2. Mathematic modeling

Let the original surface be defined by the equation in an implicit form:

\[ F(x, y, z) = 0 \tag{1} \]

At points of the criminant of this surface relatively coordinate plane XY the following condition is met:

\[ F_x(x, y, z) = 0 \tag{2} \]

Considering equation (2) as an equation of one more surface and having determined its intersection with original surface (1), we obtain surface criminant (1) with respect to coordinate plane XY. The projection of this criminant on coordinate plane XY is surface discriminant (1) and, correspondingly, the envelope of the family of plane congruent curves. Surface (1) is obtained by mapping of these curves in space \( \mathbb{R}^3 \).

It was found that equation (2), reflecting the condition of tangency of a ‘vertical’ plane with a specified surface can also be considered as a necessary condition of existence of conditional extrema of some functions. The graphs of these functions are curves obtained at the intersection of surfaces (1) by planes parallel to coordinate planes XZ or YZ (Figure 1).

In order to determine the sufficient conditions for the existence of extrema on the curves under consideration, the second differential of the Lagrangian function is defined:

\[ d^2 L = -\frac{F_{zz}}{F_y} dz^2 \tag{3} \]

Then from (3) it follows that if \( \frac{F_{zz}}{F_y} > 0 \), the point of the studied intersection of the surface by the plane parallel to coordinate plane XZ or YZ is the point of conditional maximum; if \( \frac{F_{zz}}{F_y} < 0 \), the corresponding point is the point of a conditional minimum.

Condition \( F_{zz} = 0 \) requires additional studies.

As a result criminant \( D \) of the surface is the combination of a set of extreme points, namely

\[ D = \sum_{i=1}^{n} \min f(x, z) \vee \max f(x, z) \bigg|_{z=a_i}. \tag{4} \]

The variable is coordinate \( z \) in its domain of definition.
The conducted analysis of the total curvature of the surface allowed establishing the following:

- if \( F_{zz} = 0 \) and \( F_y \cdot F_{xx} - F_x \cdot F_{xy} \neq 0 \) then the Gaussian curvature at points of criminant line of the surface is negative. At the same time the average curvature of the surface in it is directly proportional to the value of the curve curvature obtained at the intersection of the surface by the plane parallel to XY;

- if \( F_{zz} \neq 0, |F_x| + |F_y| \neq 0 \), the corresponding point on the surface is hyperbolic. The curve on the surface in a section plane perpendicular to axis \( z \), has a point of inflection;

- if \( F_{xx} = 0 \) and \( F_y \cdot F_{xx} - F_x \cdot F_{xy} = 0 \), the corresponding point of the surface is parabolic;

- if \( F_{yy} \cdot F_{xx} - 2F_x \cdot F_y \cdot F_{xy} + F_x^2 \cdot F_{yy} = 0 \) and \( F_y \cdot F_{xx} - F_x \cdot F_{xy} = 0 \) the corresponding point of the surface is parabolic;

- if \( F_x = F_y = 0 \), the point is specific on both the original curve and the surface as well as on its discriminant;

- if \( 2F_x \cdot F_y \cdot F_{xy} - F_{xx} \cdot F_{yy} - F_{xy} \cdot F_{yy} \neq 0 \), \( F_{xx} \neq 0 \), \( F_y \cdot F_{xx} - F_x \cdot F_{xy} \neq 0 \), the corresponding point of the surface may be either elliptical or hyperbolic.

By analogy with the previous goal we have conducted a study of mapping of the orthogonal projection of the surface specified by the following parametric equations:

\[
\begin{align*}
x &= f_1(u,v), \\
y &= f_2(u,v), \\
z &= f_3(u,v).
\end{align*}
\]  

(5)

In this case criminant \( D \) of the surface is the combination of a set of extreme points, namely:

\[
D = \min_{i=1}^{6} f_i(u,v) \vee \max_{f_i (u,v) = a_i} f_i(u,v) = f_3(u,v).
\]  

(6)

A variable parameter is one of the parameters of the surface in its definition domain.

From (6) it follows that:

\[
F(u,v) = f_{1u} \cdot f_{2v} - f_{2u} \cdot f_{1v} = 0.
\]  

(7)

Equation (7) establishes the relationship between parameters \( u, v \). Along with equations (5) they determine the criminant of the surface.

**Figure 1.** The model of the body and lines of its intersection with the planes parallel to coordinate planes XZ (line \( m \)) and YZ (line \( n \)); \( t \) - tangent to curve \( m \) and \( n \).
The obtained results are extended to the definition of the envelope of one-parameter family of surfaces. First, let us investigate the mapping of the three-dimensional hypersurface specified in four-dimensional space by the equation in an implicit form:

\[ F(x, y, z, t) = 0. \] (8)

For its discriminant \( D \) is a combination of the following set of extreme points:

\[ D = \sum_{j=1}^{n} \left( \sum_{i=1}^{g} \min f(x, y, t) \bigg|_{x=x_i} \cdot \max f(x, y, t) \bigg|_{y=y_j} \right), \] (9)

where the variable is coordinate \( t \).

Let us now consider the three-dimensional hypersurface in four-dimensional space specified by the parametric equations:

\[ x = f_1(u, v, \varphi), \quad y = f_2(u, v, \varphi), \quad z = f_3(u, v, \varphi), \quad t = f_4(u, v, \varphi). \] (10)

Then criminant \( D \) of the surface is a combination of a set of extreme points, namely

\[ D = \sum_{j=1}^{n} \left( \sum_{i=1}^{g} \min f_1(u, v, \varphi) \bigg|_{y=y_j} \cdot \max f_1(u, v, \varphi) \bigg|_{z=z_j} \right). \] (11)

A variable parameter is one of the parameters of the hypersurface in its domain of definition.

Using the method of Lagrange multipliers from equation (11) we will obtain the connection of parameters \( u, v \) and \( \varphi \)

\[ F(u, v, \varphi) = f_{1u}(f_{2v} \cdot f_{3p} - f_{2p} \cdot f_{3v}) - f_{1v}(f_{2u} \cdot f_{3p} - f_{2p} \cdot f_{3u}) + f_{1p}(f_{2u} \cdot f_{3v} - f_{2v} \cdot f_{3u}) = 0 \] (12)

This equation along with equations (10) determines the discriminant of the hypersurface. Equation (12) can be regarded as the equation of the two-dimensional surface in cartesian coordinates \( U, V, \varphi \). Mapping \( \Omega : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) of this surface distinguishes its criminant (a two-dimensional surface) on a specified hypersurface relatively hyperplane XYZ.

Thus, the obtained results allow us to determine the discriminant of the two-dimensional surface and the three-dimensional hypersurface, and, consequently, the envelope of the family of plane curves as well as surfaces from the unified positions. From the analytical position one can realise it by means of conditional extrema of the corresponding functions, based on the method of Lagrange multipliers, and by means of computational methods, that is, without the use of differential surface parameters.

3. Solid-state modeling of components

The next step in the application of computer technology in the problems of surfaces shaping is to develop solid-state models of the components having a geometric shape different from 3D primitives used in the known CAD.

For these purposes one may use in particular the polygonal model of surfaces.

Thus, Figure 2 shows a model of a component with a helical groove. It is derived by cutting off a part of the body with a helical surface from the workpiece. The construction of the solid-state model of the impeller and of the models of removed allowances (Figure 3) is based on the same technology. The very solid-state model of the impeller billet was obtained using standard means of most of CAD systems. For the development of the most effective schemes of allowances removal of the interblade volume it is necessary to analyse the geometric model of the body of this allowance – \( P \). The methods of geometric synthesis and capabilities of modern hardware and software to shape three-dimensional bodies lie in the heart of the proposed approach to the selection of alternatives of removal of the most part of the allowance.
The sequence of selection of rational schemes of removal of the part of interblade volume allowance involves the following tasks:

- analysis of the surface shape bounding the volume under investigation;
- removal of the part of the interblade volume allowance using the operations of geometric modeling;
- selection of a rational technological operation or their combinations of allowance part removal proceeding from the analysis of productivity of each of them;
- selection of the technological process, equipment and tool-set for dimensional preliminary processing of the impeller.

**Figure 2.** Shaping of the surface of the component model – cutting a segment from the workpiece

**Figure 3.** The solid-state model of the impeller; $P$ – removable interblade volume

4. **Computer Simulation of Shaping**

The modeling results obtained at the previous stage are used to develop algorithms and programmes in order to appoint necessary technological conditions of the component shaping using the most rational methods of the dimensional processing. For that one uses the known solid-state modeling operations. So the kinematic operation is used to specify a multitude of tool positions in the process of the component shaping. The choice of setting and the orientation of the tool can be automated using programming languages. This operation allows modeling more complex schemes of formation of surfaces that have a complex profile. Then, using Boolean operations, the required surface is formed.

In the tasks of surfaces shaping of the components processed by the method of centroid or noncentroid rounding one uses effectively the capabilities of modern CAD-systems and the corresponding programming languages. The solution of direct and inverse problems of modeling of surface shaping processed by the method of noncentroid rounding, such as a helicoid surface - rotational surface – helicoid surface, can be one of the examples.
When solving the direct problem, the initial data are a body model with a spiral groove and a workpiece model as a section of the cylinder for the rotary body. The cylinder dimensions and its location are determined by the parameters of setting of the shaping element relatively the workpiece.

Modeling of shaping of the rotary surface on the basis of the previously created model of the product and the installation parameters is performed in accordance with the developed algorithms and programmes (Figure 4).

**Figure 4.** Shaping of the workpiece of a disc tool: an intermediate position of the models

Due to the fact that the rotary body surface is uniquely determined by its axial cross-section, only a part of the workpiece is subjected to shaping simulation. The resulting axial section may be edited both for the purpose of replacing its profile with technological curves and for some other reasons.

This section is used to obtain a solid-state model of the rotary body, on the basis of which the model of a disk cutter and its drawing is created.

The solid-state model of the tool, created on the basis of the axial section, is used to solve the inverse problem of shaping. In this case, the simulation result of shaping is a body with a spiral groove (Figure 5). The same figure shows a model of its end section, which is used for comparative analysis with the original face profile. If the resulting deflection is within the permissible range, the modeling process is completed. Otherwise, it is possible to change the parameters of mutual arrangement of the models of the tool and the product. Along with solution of direct and inverse problems of shaping the developed algorithms and programmes allow obtaining solid-state models of the cut layers in order to select the optimal technological parameters of cutting. Modeling of shaping of the component and cut layers is shown in Figure 6.

**Figure 5.** Shaping of the model with a screw surface: a disc tool in the initial and final states and a cross-section of the model
5. Conclusions
On the basis of the above stated information the use of geometric modeling and computer technologies allows performing the following tasks:

- to develop (on the basis of the established laws of geometry) mathematical models of surfaces shaping of the components processed by the method of centroid and decentroid rounding implemented by numerical methods; at this there is no need to obtain dependencies, which establish the connection of the surface parameters and parameters of their family;
- to obtain solid-state models of the component in order to choose the method of dimensional processing corresponding to the removal of the greatest volume of the allowance at the stage of the preliminary treatment,
- to provide the possibility of modeling of the shaping process in an automated mode;
- to assign the conditions of installation, securing and relative movement of the shaping tool and the processed surfaces;
- to predict the conditions preventing technological losses of products during the production process, associated with the failure of the relative position and the relative movement of the workpiece and the tool.

![Figure 6. Computer modeling of the component and the cut layers](image_url)

References
[1] Litvin F L 2004 Cambridge University Press 816
[2] Thom R 1962 J. de math. pur et apple 41(2) 177-192
[3] Bruce J W and Giblin P G 1988 Publisher ‘World’ 263
[4] Lyashkov A A 2012 Russian Engineering Research, Allerton Press, Inc. 32(4) 404-406
[5] Nikolaos T 2012 Journal of Manufacturing Science and Engineering 134
[6] Pottmann H and Peternell M 2000 Comenius University, Bratislava 3-23
[7] Lyashkov A A and Panchuk K L 2015 Procedia Engineering 113 174-180