On-brane solutions in a bulk spacetime with torsion

Sumanta Chakraborty † and Soumitra SenGupta ‡
IUCAA, Post Bag 4, Ganeshkhind
Pune University Campus, Pune 411 007, India
and
Department of Theoretical Physics
Indian Association for the Cultivation of Science, Kolkata-700032, India

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Abstract

Effective gravitational field equation on a brane has been derived, when the bulk spacetime is endowed with torsion originating from the field strength of the second rank antisymmetric Kalb-Ramond field. From the effective Einstein equation on the brane, we derive a spherically symmetric solution, which represents a black hole or naked singularity depending on the parameter space of the model. The stability of the model is also discussed. Cosmological solutions have been obtained, where the Kalb-Ramond field is found to behave as normal pressure free matter. For certain specific choices of the parameter space the solution exhibits a transition in the nature of the scale factor. The possibility of accelerated expansion of the universe in this scenario is also discussed.

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*sumantac.physics@gmail.com
†sumanta@iucaa.ernet.in
‡tpssg@iacs.res.in
1 Introduction

Einstein-Cartan theory is a generalization of Einstein’s general relativity where the affine connection has a non-symmetric extension, known as torsion. While matter-energy is known to be the source of space-time curvature, the origin of torsion is believed to have its origin in particle spin [1-4]. Subsequently in the context of string-inspired models it has been argued that the rank-3 antisymmetric field strength of the rank-2 anti-symmetric tensor field (known as Kalb-Ramond field) which appears as closed string excitation, can be identified to space-time torsion in the effective low energy action of a type IIB string theory in higher dimensions [5,6]. Apart from the string theory viewpoint, the torsion plays important role in other places as well, which can be listed as,

- Modified theories of gravity, formulated using twistors, require the inclusion of torsion [7,8].
- Attempts to unify gravity and electromagnetism necessitates the inclusion of torsion in higher-dimensional theories [9,10].
- In supergravity, the curvature tensor, torsion field and matter fields are treated in identical manner, with torsion field having important physical contributions [11,12].
- In the universe undergoing one or several phase transitions, the torsion field could give rise to topological defects leading to intrinsic angular momenta for structures in the universe, e.g. galaxies [13-16].
- In theories of gravitation including torsion, the helicity of fermions are not conserved and the probability of spin flip is related to the torsion field [17].
- Spacetime endowed with torsion becomes optically active exhibiting birefringence [18,19].

In general, from the action of a theory with additional spatial dimensions one needs to compactify to extract an effective 4-dimensional action to describe our universe which is an artefact of the geometry of the compact manifold. However instead of implementing the compactification procedure in the action, an effective gravitational field equation on a lower dimensional hypersurface (often known as a brane [20-23]) embedded in a higher dimensional space-time (known as bulk) can be derived by computing the induced metric from the Gauss-Codazzi equation using the appropriate junction condition [24-26].

For example, in the context of brane-world models with one extra dimension, an effective Einstein’s equation on a surface with 3-spacelike dimensions, called 3-brane, was derived which later was extended for two-brane models as well. These calculations are based on an underlying 5-dimensional Einstein’s gravity in the bulk (for derivation of Gauss Codazzi equation in $f(R)$ gravity see [27-29]).

In this work we start with a bulk spacetime endowed with torsion (such a scenario is phenomenologically quiet important, see [30,31]). We show that the effective gravitational field equation on the brane provides valuable insight to the nature of the gravitational field equations modified by the torsion [32-36]. This idea assumes special significance from the point of view of different extra dimensional models. For example in warped geometry models, like the Randall-Sundrum model only gravity is assumed to propagate in the bulk spacetime as a closed string mode excitation. It is now worthwhile to derive the effective theory on the brane when torsion is also present in the bulk as one of the closed string excitations.

We will follow the setup proposed in Ref. [37] where an antisymmetric tensor field $B_{\mu\nu}$ [called Kalb-Ramond (KR) field] acts as the source of space-time torsion. The $U(1)$ gauge invariance of a background electromagnetic field theory however remains intact due to the Chern-Simons extension for gauge anomaly cancellation which in turn results into a KR-photon coupling term [37].
Having obtained the induced field equations we move forward to determine the corresponding spherically symmetric solutions. The spherically symmetric solution reveals non-local effect originating from the bulk and transmitted by the induced electric part of the Weyl tensor. The role of the bulk torsion field in this solution is discussed. We further show that both these bulk quantities lead to a spherically symmetric solution which exhibits standard black hole characteristic for a certain range of parameter space while one can also obtain naked singularity in another region of the parameter space. Appropriate thermodynamic analysis brings out a negative specific heat for the black hole. We then turn our attention into cosmological solution that results from the induced equation on the brane. The KR field is shown to behave as an additional matter field and thus may be considered as a candidate for dark matter.

The paper is organized as follows: At first we provide a detailed calculation of the Gauss-Codazzi equation in presence of torsion and derive the effective Einstein’s equation with torsion. In the next section, we obtain the equations of motion for the KR field which brings out the dynamics of the torsion. The next section is attributed to the calculation of spherically symmetric solution on the brane and the corresponding black hole solution. The following section explores the nature of the cosmological solution in presence of torsion as a generalization of the FRW metric. We finally conclude with a discussion on our results.

2 Effective Einstein’s Equation on the brane with torsion present in the bulk

We consider a five-dimensional bulk spacetime endowed with a metric $g_{ab}$ and the antisymmetric torsion field $T^{abc}$. All the results presented below are in bulk spacetime unless explicitly mentioned. Also bulk coordinates are denoted by Latin indices, while brane coordinates are denoted by Greek indices. The torsion field is defined through the connections such that, for an arbitrary vector field we have the following relation:

$$\nabla_a A^b = \partial_a A^b + \hat{\Gamma}^b_{ca} A^c$$

(1a)

$$\hat{\Gamma}^b_{bc} = \Gamma^b_{bc} + T^b_{bc}$$

(1b)

Thus the connection breaks up into two parts, one is the usual Christoffel symbol depending only on the metric and symmetric in the lower two indices. The remaining part is the torsion tensor, which in general is antisymmetric in the lower two indices. However in this work we assume it to be more restrictive i.e. completely antisymmetric in all the indices. The justification of this assumption is rooted in a possible origin of torsion via string inspired antisymmetric tensor field [39], namely the KR field and the gauge freedom which we will discuss later. The next important quantity in such modified gravity theory would be the curvature tensor, $\hat{R}^{a}_{bcd}$, with torsion field included. For that the standard treatment can be generalized to obtain $\hat{R}^{a}_{bcd}$ and can be obtained from the result:

$$\nabla_a \nabla_b A_c - \nabla_b \nabla_a A_c = - \left( \partial_a \hat{\Gamma}^p_{bc} - \partial_b \hat{\Gamma}^p_{ac} + \hat{\Gamma}^p_{aq} \hat{\Gamma}^q_{bc} - \hat{\Gamma}^p_{aq} \hat{\Gamma}^q_{ac} \right) A_p - 2T^p_{ab} \nabla_p A_c$$

$$= -\hat{R}^p_{cab} A_p - 2T^p_{ab} \nabla_p A_c$$

(2)

Here the curvature tensor has been generalized as

$$\hat{R}^a_{bcd} = \partial_a \hat{\Gamma}^a_{db} - \partial_d \hat{\Gamma}^a_{cb} + \hat{\Gamma}^a_{cp} \hat{\Gamma}^p_{db} - \hat{\Gamma}^a_{dp} \hat{\Gamma}^p_{cb}$$

$$= R^a_{bcd} + \nabla_c T^a_{db} - \nabla_d T^a_{cb}$$

$$- (T^a_{pb} T^p_{dc} + T^a_{cp} T^p_{db} - T^a_{pb} T^p_{cd} - T^a_{dp} T^p_{cb})$$

(3)
where $R^a_{bcd}$ is the usual Riemann curvature tensor. Note that the generalized curvature tensor $\tilde{R}^a_{bcd}$, defined with torsion has only two properties, antisymmetry in the first two and last two indices. From the above relation between $\tilde{R}^a_{bcd}$ and $R^a_{bcd}$ one obtains the Ricci tensor and Ricci scalar as

$$\tilde{R}^{bd} = R^{bd} + \nabla^a T_{db} + T_{p}^{a} T_{pd} - (2 T_{p}^{a} T_{da} - T_{dp}^{a} T_{ab})$$

$$\tilde{R} = R - T_{abc} T^{abc}$$  (4a)

$$\tilde{R}^{bd} = \nabla^a T_{db} + T_{p}^{a} T_{pd} + T_{a}^{p} T_{pa}$$  (4b)

where Ricci tensor with torsion i.e. $\tilde{R}_{ab}$ is not symmetric in the indices. Also in this bulk spacetime with torsion included, Frobenius identity gets modified to:

$$n^c \nabla_b n_a = 1 \frac{1}{3!} (n_c \nabla_b n_a + n_b \nabla_c n_a + n_a \nabla_c n_b - n_c \nabla_a n_b - n_b \nabla_c n_a - n_a \nabla_b n_c)$$

$$= \frac{1}{3} n_d (n_b T_{ca} + n_a T_{bc} + n_c T_{ab})$$  (5)

Note that the condition $\nabla_c g_{ab} = 0$ only determines $\Gamma_{ab}^c$ to its standard expression while the torsion tensor $T_{bc}$ continues to be an arbitrary field.

Now we construct the effective Einstein’s equation on the brane by using induced metric components on $y = 0$ hypersurface. The normal is spacelike i.e. $n^a n_a = +1$. The induced metric turns out to be, $h_{ab} = g_{ab} - n_a n_b$. Note that $h_a^b h_b^c = (\delta^a_c - n^a n_c)(\delta^b_{c} - n^b n_c) = (\delta^a_c - n^a n_c) = h_a^c$. Two other useful relations are: $h_a^n n_a = 0$ and $n^b \nabla_a n_b = 0$. Also note that for a vector $X_b$ lying on the brane we have $h_b^n X_b = X^n$, since when projected twice we get the same quantity. The next important quantity to define is the extrinsic curvature:

$$\tilde{K}_{ab} = -h_a^m h_b^n \nabla_m n_n$$  (6)

Due to existence of torsion it is no longer symmetric and the anti-symmetric part has the following expression:

$$\tilde{K}_{ab} - \tilde{K}_{ba} = +2 n^c n_d (n_b T_{ca} + n_a T_{bc} + n_c T_{ab}) = 2 T_{ab}^{d} n_{d}$$  (7)

which implies, $T_{ab}^{d} n_{d} = \frac{1}{2} \left( \tilde{K}_{ab} - \tilde{K}_{ba} \right)$. Thus torsion is intimately connected to the extrinsic curvature. With all these results, we now determine the effective Einstein’s equation on the 3-brane. For that we adopt the following steps:

- First we connect the four-dimensional curvature tensor to five-dimensional one.
- Then obtain the relation between four-dimensional Ricci tensor and Ricci scalar to their five-dimensional counterpart.
- Similar relations for Weyl tensor, specially its electric part are obtained.
- Finally, we use the relation between induced torsion tensors and five-dimensional torsion tensors to obtain a relation in which tensors constructed from metric and torsion decouples.

With this scheme (elaborated in App. A.1) and using Eqs. (100) and (101) in Eq. (99), we arrive at the
effective Einstein’s equation on 3-brane as,

\[ R_{ab} - \frac{1}{2} h_{ab}^4 R = \frac{2}{3} h_a^3 h_b^3 \left( R_{qs} - \frac{1}{2} g_{qs} R \right) + \frac{2}{3} h_{ab} n^q n^s \left( R_{qs} - \frac{1}{2} g_{qs} R \right) - E_{ab} + \frac{1}{4} h_{ab} R \\
+ \frac{1}{2} h_{ab} n^q n^s \left( R_{qs} - \frac{1}{2} g_{qs} R \right) - E_{ab} + \frac{1}{4} h_{ab} R
\]

Thus we note that in the effective Einstein equation in four dimension only the terms \( T_{4qr} \) contributes as all the torsion terms have their origin from contraction of torsion tensor with the normal \( n_a \). Only the fourth term in the last line does not have that structure, however as we will observe later that this term has no contribution.

### 3 Identifying torsion with bulk Kalb-Ramond field

A possible physical origin of the torsion tensor \( T_{abc} \) is from antisymmetric spin field density generated from Kalb-Ramond field \( B_{ab} \), such that \([37]\):

\[ T_{abc} = \kappa_5 H_{abc} = \kappa_5 \partial_{[a} B_{bc]} \]  

Then the five dimensional action for the antisymmetric tensor field reduces to,

\[ S = \int d^5 x \sqrt{-g} \left[ \frac{R}{\kappa_5^2} - \frac{1}{2} H_{abc} H^{abc} + \frac{1}{\kappa_5} T_{abc} H^{abc} + L_{\text{matter}} \right] \]  

where \( L_{\text{matter}} \) is the matter Lagrangian.

The variation of the metric and the antisymmetric torsion field leads to two equations of motion:

\[ G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = \frac{\kappa_5}{2} \left( T_{ab}^{\text{matter}} + T_{ab}^{KR} \right) \]

\[ \nabla_a H^{abc} = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} H^{abc}) = 0 \]

where \( T_{ab}^{\text{matter}} \) represents matter energy momentum tensor and the other component \( T_{ab}^{KR} \) has the following expression:

\[ T_{ab}^{KR} = \frac{1}{2} \left[ \frac{3}{2} (g_{ba} H_{pqa} H^{pqr} + g_{ar} H_{pqb} H^{pqr}) - \frac{1}{2} g_{ab} H^{pqr} H_{pqr} \right] \]

Substituting the relation \( \nabla_a H^{abc} = 0 \) in Eq. (8) and Eq. (9) we arrive at:

\[ (4) R_{ab} - \frac{1}{2} h_{ab}^4 R = \frac{2}{3} h_a^3 h_b^3 \left( R_{qs} - \frac{1}{2} g_{qs} R \right) + \frac{2}{3} h_{ab} n^q n^s \left( R_{qs} - \frac{1}{2} g_{qs} R \right) - E_{ab} + \frac{1}{4} h_{ab} R \\
+ \frac{1}{2} h_{ab} n^q n^s \left( R_{qs} - \frac{1}{2} g_{qs} R \right) - E_{ab} + \frac{1}{4} h_{ab} R
\]  

\[ + \frac{1}{2} h_{ab} n^q n^s \left( R_{qs} - \frac{1}{2} g_{qs} R \right) - E_{ab} + \frac{1}{4} h_{ab} R
\]

\[ + \kappa_5 \left( K \right) n_{p} H_{pa} - K_{qa} n_{p} H_{pq} - K_{qa} n_{m} H_{bm} - n^p n^r h_a^3 h_b^3 \nabla_r H_{psq} \]  

(13)
We now explore the gauge invariance associated with the gauge field $B_{ab}$ defined in Eq. (9). The change of the antisymmetric field $B_{ab}$ as $B_{ab} \rightarrow B_{ab} + \partial_a \Lambda_b$, keeps $H_{abc}$ invariant. Then we can eliminate four $B_{ab}$ using four $\Lambda_b$. Using this gauge freedom we eliminate the four components: $B_{a0}, B_{a1}, B_{a2}$ and $B_{a3}$.

The only contribution to the object $n^a H_{abc}$ comes from the term $\partial_a B_{\mu \nu}$. Hence if $B_{\mu \nu}$ is independent of the extra coordinate then the term $n^a H_{abc} = 0$. Thus all the extra terms in Eq. (13) vanishes, and the effective Einstein’s equation does not depend on torsion explicitly. We however should emphasize that this merely shows that in order to determine the five dimensional metric, we require to solve the field equations and this in turn will involve the torsion field, since the five dimensional Einstein equation involves the torsion field as its source.

## 4 Static spherically symmetric Brane

We observe that with proper gauge conditions imposed, and torsion tensor being independent of extra dimensional coordinate, the Gauss-Codazzi equation retains the same form as if torsion is not present. Let us now start by considering bulk Einstein equation, which is presented in Eq. (11a). Substitution of the bulk equation in effective Einstein’s equation leads to:

$$
(4) R_{ab} - \frac{1}{2} h^{4(4)} R = \frac{2}{3} \kappa_5^2 h^a_b h^s T^a_{qs} (T^m_{qs} + T^{KR}) + \frac{2}{3} \kappa_5^2 h_{ab} n^s T_{qs}^m + \frac{1}{6} \kappa_5^2 h_{ab} T^{KR} - E_{ab} - \frac{1}{6} \kappa_5^2 h_{ab} (T^m_{qs} + T^{KR}) + K K_{ba} - g^{pq} K_{pa} K_{qb} - \frac{1}{2} h_{ab} K^2 + \frac{1}{2} h_{ab} K_p K^{qp} \tag{14}
$$

The above equation can be simplified by using Eq. (12) leading to the following identities which will be helpful later.

$$
n^a H_{bca} n_a H^{bca} = \frac{1}{4} H_{abc} H^{abc} = \frac{1}{4} H_{abc} H^{abc} \tag{15a}
$$

$$
n^a n_a T^{KR} = \frac{3}{4} h^a_b n_a H_{bmnq} H^{mnce} + \frac{3}{4} n_a n_c H_{mnbb} H^{mnce} - \frac{1}{4} n_a n_b H_{pqrr} H^{pqrr} = \frac{1}{4} n_a n_b H_{pqrr} H^{pqrr} \tag{15b}
$$

$$
h^a_b h^s T^{KR} = (\delta^s_a - n^s n_a) (\delta^s_b - n^s n_b) T^{KR} = T^{KR}_{ab} - n^q n_a T^{KR}_{pq} - n^q n_b T^{KR}_{qa} + n^q n^s n_a n_b T^{KR}_{qs} = T^{KR}_{ab} + \frac{1}{4} n_a n_b H_{ppqr} H^{pqfr} - \frac{1}{4} h_{ab} H_{pqrr} H^{pqrr} \tag{15c}
$$

Eq. (14) now can be written using the above identities as:

$$
(4) R_{ab} - \frac{1}{2} h^{4(4)} R = \left[ \frac{2}{3} \kappa_5^2 h^a_b h^s T^a_{qs} (T^m_{qs} + T^{KR}) + \frac{2}{3} \kappa_5^2 h_{ab} n^s T^m_{qs} + \frac{1}{6} \kappa_5^2 h_{ab} T^{KR} \right] - E_{ab} + K K_{ba} - g^{pq} K_{pa} K_{qb} - \frac{1}{2} h_{ab} K^2 + \frac{1}{2} h_{ab} K_p K^{qp} \tag{16}
$$

$$
+ \frac{2}{3} \kappa_5^2 h^a_b h^s T^{KR} + \frac{2}{3} \kappa_5^2 h_{ab} n^s T^{KR} - \frac{1}{6} \kappa_5^2 h_{ab} T^{KR} = \frac{2}{3} \kappa_5^2 h^a_b h^s T^a_{qs} + \frac{2}{3} \kappa_5^2 h_{ab} n^s T_{qs}^m - \frac{1}{6} \kappa_5^2 h_{ab} T^{KR} - E_{ab} + K K_{ba} - g^{pq} K_{pa} K_{qb} - \frac{1}{2} h_{ab} K^2 + \frac{1}{2} h_{ab} K_p K^{qp} + \kappa_5^2 \left[ \frac{1}{2} (h_{ba} H_{pqrr} H^{pqfr} + h_{ab} H_{ppqr} H^{pqfr}) - \frac{1}{4} h_{ab} H_{pqrr} H^{pqrr} \right] \tag{16}
$$

6
In order to proceed further we make the standard coordinate choice with vanishing shift function, such that the five-dimensional line element takes the following form:

\[ ds^2 = dy^2 + h_{\mu\nu}dx^\mu dx^\nu \]  

(17)

Here the 4-metric \( h_{\mu\nu} \) is connected to the induced metric \( h_{ab} \) through the relation, \( h_{\mu\nu} = e^a_\mu e^b_\nu h_{ab} \), where \( e^a_\mu = (\partial x^a / \partial y^\mu) \). We will assume that the bulk contributes to the energy momentum tensor only through the cosmological constant term while normal matter contribution comes from the brane. This implies

\[ T_{ab} = -\Lambda g_{ab} + \delta(y) ( -\lambda_T h_{ab} + \tau_{ab}) \]  

(18)

where, \( \lambda_T \) is the brane tension and \( \tau_{ab} \) is the brane energy momentum tensor. Imposing junction conditions and making use of \( Z_2 \) symmetry, Eq. (16) reduces to the following form:

\[ (4) G_{ab} = -\Lambda_4 h_{ab} + 8\pi G\tau_{ab} + \kappa_5^4 \pi_{ab} - E_{ab} \]

\[ + \kappa_5^2 \left[ \frac{1}{2} (h_{bd} H_{pqd} H^{pq} + h_{ad} H_{pqd} H^{pq}) - \frac{1}{6} h_{ab} H^{pq} H_{pq} \right] \]  

(19)

where:

\[ \Lambda_4 = \frac{1}{2} \kappa_5^2 \left( \Lambda + \frac{1}{6} \kappa_5^2 \lambda_T^2 \right) \]  

(20a)

\[ G = \frac{\kappa_4^2 \lambda_T}{48\pi} = \frac{\kappa_4^2}{8\pi} \]  

(20b)

\[ \pi_{ab} = -\frac{1}{4} \tau_{ac} \tau^c_b + \frac{1}{12} \tau_{ab} + \frac{1}{8} h_{ab} \tau_{pq} \tau^{pq} - \frac{1}{24} h_{ab} \tau^2 \]  

(20c)

Using the brane coordinates the final induced equation on the brane located at \( y = 0 \) can be written as:

\[ (4) G_{\mu\nu} = (4) G_{ab} e^a_\mu e^b_\nu \]

\[ = -\Lambda_4 h_{\mu\nu} + 8\pi \tau_{\mu\nu} + \kappa_5^4 \pi_{\mu\nu} - E_{\mu\nu} \]

\[ + \kappa_5^2 \left[ \frac{1}{2} (h_{\nu\gamma} H_{\alpha\beta\mu} H^{\alpha\beta\gamma} + h_{\mu\gamma} H_{\alpha\beta\nu} H^{\alpha\beta\gamma}) - \frac{1}{6} h_{\mu\nu} H^{\alpha\beta\gamma} H_{\alpha\beta\gamma} \right] \]  

(21)

where we have used the following two identities:

\[ e^a_\mu e^b_\nu h_{bd} H_{pqd} H^{pq} = e^a_\mu e^b_\nu h_{bd} e^c_\mu e^\gamma_\nu H_{\alpha\beta\gamma} e^\delta_\xi e^\epsilon_\zeta H^{\delta\xi\epsilon\zeta} \]

\[ = h_{\nu\gamma} H_{\alpha\beta\mu} H^{\alpha\beta\gamma} \]  

(22a)

\[ H_{pqd} H^{pq} = e^a_\mu e^b_\nu H_{pqd} H_{\alpha\beta\gamma} e^\delta_\xi e^\epsilon_\zeta H^{\mu\nu\xi\zeta} \]

\[ = H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} \]  

(22b)

These results are consequences of the fact that only four-dimensional part of the torsion tensor \( H_{pq\gamma} \) contributes. We will henceforth assume that the energy-momentum tensor on the brane vanishes i.e. \( \tau_{\mu\nu} = 0 \), which from Eq. (20c) leads to \( \pi_{\mu\nu} = 0 \). Then the dominant contribution comes from the electric part of the bulk Weyl tensor. Also from the reduced Bianchi identity, \( D_\mu (4) G^{\mu\nu} = 0 \) and the equation of motion \( D_\mu H^{\mu\nu\alpha} = 0 \), we observe that the surface covariant derivative of the electric part of Weyl tensor vanishes.
This bulk contribution can be simplified substantially following the arguments in Ref. [38] where the electric part of the projected Weyl tensor has the expression:

\[ E_{\mu\nu} = -k^4 \left[ U(r) \left( u_\mu u_\nu + \frac{1}{3} \xi_{\mu\nu} \right) + P_{\mu\nu} + 2Q(\mu u_\nu) \right] \]  \hspace{1cm} (23)

In the above expression, \( k \) is a new constant defined as, \( k = k_5/k_4 \) with \( k_4^2 = 8\pi G \), \( u_\mu \) is the normal to \( t = \) constant surface and \( \xi_{\mu\nu} = h_{\mu\nu} + u_\mu u_\nu \) is the induced metric on the three-surface. The term \( U \) is the “dark radiation” term obtained by contracting \( E_{\mu\nu} \) with the velocities and is a scalar quantity. The vector \( Q_\mu \) is obtained by contraction of \( E_{\mu\nu} \) with the four velocity \( u_\mu \) and induced metric \( \xi_{\alpha\beta} \). However for static situation this vector vanishes identically. Finally, \( P_{\mu\nu} \) is a trace free, symmetric three-tensor constructed from \( E_{\mu\nu} \). In the spherically symmetric static metric ansatz we have, \( P_{\mu\nu} = P(r) \left( r_\mu r_\nu - (1/3)\xi_{\mu\nu} \right) \), where \( r_\mu \) is unit radial vector. The spherically symmetric static metric ansatz is taken as:

\[ ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2 \]  \hspace{1cm} (24)

The completely antisymmetric \( H_{\mu\nu\alpha} \) has only four independent elements, \( H_{012} = h_1, H_{013} = h_2, H_{023} = h_3 \) and \( H_{123} = h_4 \). The respective contravariant objects are denoted by \( h^1, h^2, h^3 \) and \( h^4 \) respectively. Then we have the following contractions:

\[ H_{\mu\nu\rho} H^{\mu\nu\rho} = 6 \left( h_1 h^1 + h_2 h^2 + h_3 h^3 + h_4 h^4 \right) \]  \hspace{1cm} (25a)

\[ H_{0\mu\nu} H^{0\mu\nu} = 2 \left( h_1 h^1 + h_2 h^2 + h_3 h^3 \right) \]  \hspace{1cm} (25b)

\[ H_{1\mu\nu} H^{1\mu\nu} = 2 \left( h_1 h^1 + h_2 h^2 + h_4 h^4 \right) \]  \hspace{1cm} (25c)

\[ H_{2\mu\nu} H^{2\mu\nu} = 2 \left( h_1 h^1 + h_3 h^3 + h_4 h^4 \right) \]  \hspace{1cm} (25d)

\[ H_{3\mu\nu} H^{3\mu\nu} = 2 \left( h_4 h^1 + h_2 h^2 + h_3 h^3 \right) \]  \hspace{1cm} (25e)

Using these contractions along with the metric ansatz given in Eq. (24) and the Weyl tensor part from Eq. (23), we obtain the following field equations:

\[ -e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) + \frac{1}{r^2} = \Lambda_4 + \frac{3}{4\pi G \lambda T} U - \kappa_5^2 \left( h_1 h^1 + h_2 h^2 + h_3 h^3 - h_4 h^4 \right) \]  \hspace{1cm} (26a)

\[ e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = -\Lambda_4 + \frac{1}{4\pi G \lambda T} \left( U + 2P \right) + \kappa_5^2 \left( h_1 h^1 + h_2 h^2 - h_3 h^3 + h_4 h^4 \right) \]  \hspace{1cm} (26b)

\[ e^{-\lambda} \left( \frac{\nu'' + \nu^2}{2} - \frac{\nu' \lambda'}{2} + \frac{\nu' - \lambda'}{r} \right) = -2\Lambda_4 + \frac{1}{2\pi G \lambda T} \left( U - P \right) + 2\kappa_5^2 \left( h_1 h^1 - h_2 h^2 + h_3 h^3 + h_4 h^4 \right) \]  \hspace{1cm} (26c)

\[ e^{-\lambda} \left( \frac{\nu'' + \nu^2}{2} - \frac{\nu' \lambda'}{2} + \frac{\nu' - \lambda'}{r} \right) = -2\Lambda_4 + \frac{1}{2\pi G \lambda T} \left( U - P \right) + 2\kappa_5^2 \left( -h_1 h^1 + h_2 h^2 + h_3 h^3 + h_4 h^4 \right) \]  \hspace{1cm} (26d)

\[ h_2 h^4 = h_1 h^1 = h_2 h^2 = h_3 h^3 = h_4 = 0 \]  \hspace{1cm} (26e)

In addition, the conservation of energy momentum tensor leads to the equation:

\[ \nu' = -\frac{U' + 2P'}{2U + P} + \frac{6P}{r (2U + P)} \]  \hspace{1cm} (27)

On being subtracted, Eqs. (26c) and (26d), leads to: \( h_1 h^1 = h_2 h^2 \). However from Eq. (26e) it is evident from the last term that either \( h_3 \) or \( h_4 \) must vanish. Let us work with the choice \( h^4 = 0 \) (from Ref. [39] the other
choice leads to non-physical solutions). Then Eq. \((26c)\) will be satisfied provided both \(h_1\) and \(h_2\) vanishes. Thus ultimately the torsion tensor becomes a function of a single object \(h_3\) which depends only on the radial coordinate. For notational convenience we define the following object:

\[
 h_3^3 = - [h(r)]^2
\]  

(28)

This enables us to express the torsion tensor in terms of a scalar field, namely the axion field. We can write the torsion field in terms of the axion field \(\Phi\) using the definition:

\[
 H_{\alpha\beta\mu} = \epsilon_{\alpha\beta\mu\nu} \partial_{\nu} \Phi
\]

(29)

Then from Eq. \((29)\) we get, \(h_3 = \epsilon_{0231} h_{11} \partial_1 \Phi\), where \(\epsilon_{0231} = \sqrt{-g} [0231] = r^2 \sin \theta e^{(\lambda + \nu)/2}\). Again \(h_3 = \epsilon_{0231} \partial_1 \Phi\), with \(\epsilon_{0231} = - (r^2 \sin \theta)^{-1} e^{-(\nu + \lambda)/2}\). Thus from Eq. \((28)\) we get \([h(r)]^2 = h_1^1 (\partial_1 \Phi)^2\). By simple algebra we arrive at:

\[
 h(r) = e^{-\lambda/2} \partial_r \Phi
\]

(30)

The equation of motion for \(h\) can be obtained by noting that KR field \(H_{\mu\nu\rho}\) is expressible in terms of derivative of an antisymmetric field \(B_{\mu\nu}\). Then we have the following identity:

\[
 \epsilon^{\mu\nu\rho\sigma} \partial_{\sigma} H_{\mu\nu\rho} = \epsilon^{\mu\nu\rho\sigma} \partial_{\sigma} \partial_{\mu} B_{\nu\rho} = 0
\]

(31)

Writing the torsion tensor explicitly in terms of the axion field we get:

\[
 \epsilon^{\mu\nu\rho\sigma} \partial_{\sigma} H_{\mu\nu\rho} = \epsilon^{0231} \partial_1 H_{023} = -(1/\sqrt{-g}) \partial_r (\sqrt{-g} h_{11} \partial_r \Phi).
\]

This leads to the following equation of motion for the axion field \(\Phi\) or equivalently for \(h(r)\) as:

\[
 \partial_r \left( r^2 e^{(\nu - \lambda)/2} \partial_r \Phi \right) = \partial_r \left( r^2 e^{\nu/2} h \right) = 0
\]

(32)

Armed with all these, we can write down the field equations in terms of the only non-zero variable \(h(r)\) (The structure of field equations with axion field in general has been provided in App. \(\text{A.2}\)). The field equations from Eqs. \((26a)\), \((26b)\) and \((26c)\) takes the form (since four-dimensional cosmological constant in the present epoch is very small, it is neglected):

\[
 e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} = -3\alpha U - \kappa_{5}^2 h^2
\]

(33a)

\[
 e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = +\alpha \left( U + 2P \right) + \kappa_{5}^2 h^2
\]

(33b)

\[
 e^{-\lambda} \left( \frac{\nu''}{2} + \frac{\nu' \lambda'}{2} + \frac{\nu' - \lambda'}{r} \right) = 2\alpha \left( U - P \right) - 2\kappa_{5}^2 h^2
\]

(33c)

where, \(\alpha = (1/4\pi G\lambda T)\). This leads to the differential equation satisfied by \(e^{-\lambda}\) as:

\[
 \frac{d}{dr} \left( r e^{-\lambda} \right) = 1 - 3\alpha U r^2 - \kappa_{5}^2 r^2 h^2
\]

(34)

which can be integrated leading to:

\[
 e^{-\lambda} = 1 + \frac{C_1}{r} - \frac{Q(r)}{r} - \frac{\tau(r)}{r}
\]

(35)
where $C_1$ is an arbitrary constant. The other two functions are defined as:

$$Q(r) = 3\alpha \int dr r^2 U(r); \quad \tau(r) = \kappa_5^2 \int dr r^2 h^2(r)$$  \hspace{1cm} (36)

To obtain a solution for $e^\nu$ we add Eqs. (33a) and (33b), which leads to the differential equation satisfied by $\nu$ as

$$(r e^{-\lambda}) \left[ \nu' + \frac{d}{dr} \left\{ \ln (r^2 e^{-\lambda}) \right\} \right] = 2 + 2\alpha r^2 (P - U)$$  \hspace{1cm} (37)

This equation can be integrated leading to a solution for $e^\nu$ which can be presented in the integral form as,

$$e^\nu = \frac{C_2}{r [r + C_1 - Q(r) - \tau(r)]} \exp \left[ \int dr \frac{2 + 2\alpha r^2 (P - U)}{r + C_1 - Q(r) - \tau(r)} \right]$$  \hspace{1cm} (38)

where $C_2$ is an arbitrary constant of integration. We now try to find out the differential equation satisfied by dark pressure and dark radiation term. This can be achieved by substituting $\nu'$ from Eq. (27) into Eq. (33b). After some algebra the final equation turns out to be:

$$\frac{dU}{dr} = \frac{-2 dP}{dr} - \frac{6P}{r} + \frac{(2U + P)}{r e^{-\lambda}} \left[ e^{-\lambda} - \alpha r^2 (U + 2P) - \kappa_5^2 r^2 h^2 \right]$$  \hspace{1cm} (39)

A solution to this problem can be obtained when we put some relations between dark pressure and dark radiation terms as an equation of state. In this case a convenient choice is $2U + P = 0$. Then the above differential equation can be solved leading to the following set of solutions:

$$P(r) = \frac{P_0}{r^4}; \quad U(r) = -\frac{P_0}{2r^4}; \quad Q(r) = Q_0 + \frac{3\alpha P_0}{2r}$$  \hspace{1cm} (40)

With these solutions the metric elements turns out to be,

$$e^{-\lambda} = 1 - \frac{2GM + Q_0}{r} \frac{3\alpha P_0}{2r^2} - \frac{\tau(r)}{r}$$  \hspace{1cm} (41)

$$e^\nu = \frac{2C_2}{[2r^2 - 2(2GM + Q_0) r - 3\alpha P_0 - 2\tau(r)]} \exp \left[ \int dr \frac{4r^2 + 6\alpha}{2r^3 - 2(2GM + Q_0) r^2 - 3\alpha P_0 r - 2\tau(r) r^2} \right]$$  \hspace{1cm} (42)

However we still need to determine the unknown function $\tau(r)$. The differential equation for $\tau(r)$ can be obtained by substituting $\lambda'$ and $\nu'$ from Eqs. (33a) and (33b) into Eq. (33c), which after a little algebra leads to:

$$\tau'' + \frac{\tau'}{r} = \frac{\tau' \left( \tau' - 1 - \frac{3\alpha P_0}{2r^2} \right)}{r - (2GM + Q_0) - \frac{3\alpha P_0}{2r} - \tau(r)}$$  \hspace{1cm} (43)

The above differential equation of $\tau$ is non linear and there exists no exact analytic solution. Under the assumption that $P_0$ is small compared to the mass term, and keeping terms of $O(1/r^5)$, we obtain the following solution for $\tau(r)$,

$$\tau(r) = -(2GM + Q_0) - \frac{b}{r} - \frac{b(a - b)}{3r^3}$$  \hspace{1cm} (44)
where, $b$ is an arbitrary constant. The solution for the metric elements turns out to be:

\[ e^{-\lambda} = 1 - \frac{3\alpha P_0}{2r^2} + \frac{b[(3\alpha P_0/2) - b]}{3r^4} \]  

\[ e^\nu = 1 - \frac{[(3\alpha P_0/2) - b]}{r^2} + \frac{[(3\alpha P_0/2) - b]^2}{r^4} \]  

The horizon can be obtained by solving the equation $e^\nu = 0$, which for the given condition cannot have real solutions. Thus the only singularity is at $r = 0$. However $e^{-\lambda} = 0$ is possible at finite radial distance and the respective solution for $r$ can be given by:

\[ r_h = \frac{1}{\sqrt{2}} \sqrt{[(3\alpha P_0/2) - b] + \sqrt{[(3\alpha P_0/2) - b]^2 - 4b}} \]  

Thus the solution would be real provided $3\alpha P_0 > 10b$, otherwise both the metric elements have singularity only at $r = 0$ and hence the solution would represent naked singularity. For real and finite $r_h$ the spacetime however has a horizon at $r = r_h$, resembling normal black hole solutions. The solution for the Kalb-Ramond field turns out to be:

\[ h(r) = \sqrt{\frac{b}{\kappa \Sigma}} \frac{1}{r^2} \left\{ 1 + \frac{[(3\alpha P_0/2) - b]}{2r^2} - \frac{[(3\alpha P_0/2) - b]^2}{4r^4} \right\} \]  

Note that this contains higher order terms compared to the expression for $h(r)$ in four dimensional theories with torsion [39]. The normal vector to $r = constant$ surface is defined as, $\ell_a = \nabla_a r$. The surface gravity, therefore can be obtained as:

\[ \ell^a \nabla_a \ell_b = \frac{1}{r_h} \ell_b \]  

Using this the surface gravity turns out to be $\kappa = 1/r_h$. Then from the expression of entropy $S = 4\pi r_h^2$ and using Eq. (47), the specific heat turns out to be:

\[ C_v = -8\pi r_h^2 \]  

Thus the specific heat is always negative, and the black hole shows no phase transition.

Stability of black hole solutions under perturbation up to linear order is an important problem in the study of black hole physics. Here we concentrate on gravitational perturbation in the static spherically symmetric spacetime endowed with bulk torsion. The positivity of the Hamiltonian, for certain ranges of the parameter space, guarantee the self-adjoint extension of it. The linearized perturbation considered in this work can be grouped into three classes, namely: scalar, vector and tensor perturbations. Expansion of these perturbations in harmonic function basis lead to a set of equations, which on further reduction reduces to a set of decoupled wave equation with the following form,

\[ \left( \Box - \frac{1}{f(r)} V \right) \Phi = 0 \]  

where, $\Box$ represents the d’Alembertian operator with respect to two dimensional metric. Here $\Phi_S, \Phi_V$ and $\Phi_T$ represent scalar, vector and tensor perturbations respectively. The potential function for these perturbation
modes are given as [40–42]:

\[ V_T = \frac{f(r)}{r^2} \left( r \frac{df(r)}{dr} + \ell(\ell + 1) \right) \]  
\[ V_V = \frac{f(r)}{r^2} \left( 2 f(r) - r \frac{df(r)}{dr} + (\ell - 1)(\ell + 2) \right) \]  
\[ V_S = \frac{f(r) U(r)}{16r^2 (m + 3x)^2} \]

where we have used the following expressions:

\[ U(r) = 144x^3 + 144mx^2 + 48mx + 16m^3 \]  
\[ x \equiv 1 - f(r), \quad m \equiv (\ell - 1)(\ell + 2) \]

It must be emphasized that the total number of independent components of the scalar, vector and tensor modes add up to two, the number of independent degrees of freedom for the graviton on the brane. Since the tensor mode has no degrees of freedom we only need to concentrate on the vector and scalar modes.

Let us now compute the potentials for the solution as presented in Eq. (45). In four dimensions the condition \( b < (3\alpha P_0)/2 \) automatically implies that the coefficient of \( (1/r^4) \) is positive. These two conditions are sufficient to ensure that \( V_V \) and \( V_S \) are positive with self-adjoint extension for all \( r \). However for this choice \( V_T < 0 \). Since we have argued that tensor mode in four dimension does not contain any degrees of freedom, the stability of our solution is assured, not only for black hole cases but for naked singularity solution as well [43, 44]. Hence under above criteria both the black hole and naked singularity solutions are stable under linear perturbations.

For the other choice, \( b > (3\alpha P_0)/2 \), the potential \( V_T \) is positive while the other potentials are negative for certain ranges of \( r \) up to \( r = 0 \). As this situation depicts a naked singularity solution it is unstable under linear perturbations in regions near the singularity.

5 Cosmological Solutions

Let us now turn our attention to cosmological solutions. In the cosmological context we can assume \( E_{\mu\nu} = 0 \), however the four-dimensional cosmological constant cannot be neglected. Thus the effective four-dimensional Einstein’s equation takes the following form:

\[ (4)G_{\mu\nu} = -\Lambda_4 h_{\mu\nu} + 8\pi\tau_{\mu\nu} + \kappa_5^4 \tau_{\mu\nu} \]
\[ + \kappa_5^2 \left[ \frac{1}{2} (h_{\nu\gamma} H_{\alpha\beta\mu} H_{\alpha\beta\nu} H_{\alpha\beta\gamma} + h_{\mu\gamma} H_{\alpha\beta\nu} H_{\alpha\beta\gamma}) - \frac{1}{6} h_{\mu\nu} H_{\alpha\beta\gamma} H_{\alpha\beta\gamma} \right] \]

For the cosmological context we start with the FRW metric ansatz for flat model i.e. with \( k = 1 \). In this case the four dimensional line element is,

\[ ds^2 = -dt^2 + a^2(t) (dr^2 + r^2 d\Omega^2) \]

The components of Einstein tensor in this spacetime have the following expressions:

\[ G_{tt} = 3 \frac{\dot{a}^2}{a^2}; \quad G_{rr} = -a^2 \left[ 2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right]; \quad G_{\theta\theta} = r^2 G_{rr}; \quad G_{\phi\phi} = \sin^2 \theta G_{\theta\theta}; \]

\[ \]
For perfect fluid the energy momentum tensor $\tau_{\mu\nu}$ has the expression:

$$\tau_{\mu\nu} = \text{diag} (-\rho, p, p, p)$$ (60)

Using these, the components of the tensor $\pi_{\mu\nu}$, defined in Eq. (20c), can be obtained as:

$$\pi_{tt} = \frac{\rho^2}{12}; \quad \pi_{rr} = a^2\left(\frac{\rho^2}{12} + \frac{p\rho}{6}\right); \quad \pi_{\theta\theta} = r^2\pi_{rr}; \quad \pi_{\phi\phi} = \sin^2\theta\pi_{\theta\theta}$$ (61)

The field equations now reduce to:

$$H^2 = \frac{\Lambda_4}{3} + \frac{8\pi G \rho}{3} + \frac{\kappa_4^2}{36} + \frac{\kappa_4^2}{3} (h_1 h_1 + h_2 h_2 + h_3 h_3 - h_4 h_4)$$ (62)

$$\frac{\ddot{a}}{a} + H^2 = \frac{\Lambda_4}{3} - 8\pi G p - \frac{\kappa_4^2}{36} - \frac{\kappa_4^2}{3} (h_1 h_1 + h_2 h_2 - h_3 h_3 + h_4 h_4)$$ (63)

$$h_3 h_4 = h_1 h_3 = h_2 h_3 = h_1 h_2 = h_3 h_2 = h_4 h_1 = h_3 h_4 = 0$$ (64)

Here the non-zero component of the Kalb-Ramond field is $h_4$, since $h_3$ is related to radial derivative of the axion field which is zero. Also, we have $h_4 h_4 = \epsilon_{1230} h_0 \Phi \epsilon_{1230} \Phi = (\partial_t \Phi)^2 = \tilde{h}(t)^2$. Using these Eq. (31) yields,

$$\partial_t \left(a^3 \tilde{h}\right) = 0$$ (65)

Hence, we have $\tilde{h} = \text{constant}/a^3$. Thus the Kalb-Ramond field acts as a source for pressure free matter. The modified Friedmann equations now turn out to be:

$$H^2 = \frac{\Lambda_4}{3} + \frac{8\pi G \rho}{3} + \frac{\kappa_4^2}{36} + \frac{1}{3} \kappa_5^2 \tilde{h}^2$$ (66)

$$\frac{\ddot{a}}{a} + H^2 = \frac{\Lambda_4}{3} - 8\pi G p - \frac{\kappa_4^2}{36} - \frac{1}{3} \kappa_5^2 \tilde{h}^2$$ (67)

From Eqs. (66) and (67) we have,

$$\frac{\ddot{a}}{a} = \frac{\Lambda_4}{3} - 4\pi G \left(p + \frac{\rho}{3}\right) - \frac{\kappa_4^2}{2} \left(\frac{\rho^2}{9} + \frac{p\rho}{6}\right) - \frac{2}{3} \kappa_5^2 \tilde{h}^2$$ (68)

Note that when energy conditions are satisfied, we get $\tilde{h}^2 > 0$, and the $KR$ field will act as ordinary matter, while for violation of energy conditions, we have $\tilde{h}^2 < 0$ and then $KR$ field can act as an alternative source for cosmological constant. The differential equation satisfied by $a(t)$ takes the following form:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda_4}{3} + \frac{8\pi G \rho_0}{3} a^3 + \frac{\kappa_4^2 \rho_0^2}{36} a^6 + \frac{\kappa_5^2 \tilde{h}_0^2}{3} a^6$$ (69)

where, $\rho_0$ and $\tilde{h}_0$ are the density of non relativistic matter and the Kalb-Ramond field strength at the present epoch. Multiplying the above equation by $a^6$, introducing a new variable, $x = a^3$ and defining the following constants:

$$C_\Lambda = \frac{\Lambda_4}{3}; \quad C_\rho = \frac{8\pi G \rho_0}{3}; \quad C_{KR} = \frac{\kappa_4^2}{3} \left(\frac{\rho_0^2}{12} + \frac{\tilde{h}_0^2}{\kappa_5^2}\right)$$ (70)
We get the following differential equation:

\[ \dot{x} = 3\sqrt{C_\Lambda x^2 + C_\rho x + C_{KR}} \]  

(71)

For which we have the following solution for the scale factor:

\[
\exp \left[ 3\sqrt{C_\Lambda} (t - t_0) \right] = \frac{2\sqrt{C_\Lambda} \sqrt{C_\Lambda a^6 + C_\rho a^3 + C_{KR}} + 2C_\Lambda a^3 + C_\rho}{2\sqrt{C_\Lambda} \sqrt{C_\rho + C_{KR}} + 2C_\Lambda + C_\rho}; \quad C_\Lambda > 0
\]  

(72)

\[ 3\sqrt{-C_\Lambda} (t_0 - t) = \sin^{-1} \left( \frac{2C_\Lambda a^3 + C_\rho}{\sqrt{C_\rho^2 - 4C_\Lambda C_{KR}}} \right) - \sin^{-1} \left( \frac{2C_\Lambda + C_\rho}{\sqrt{C_\rho^2 - 4C_\Lambda C_{KR}}} \right); \quad C_\Lambda < 0, \quad C_\rho^2 > 4C_\Lambda C_{KR}
\]  

(73)

Though the solutions look complicated, under certain conditions the situation simplifies quite a bit, yielding clearer physical insight. For that consider the situation \( \Lambda_5 = 0 \). This would presumably true for matter dominated era of the universe. This can be achieved if we assume the bulk cosmological constant is negative, such that: \( \Lambda_5 = -{(\kappa_2^2 \lambda_5^2)}/6 \). Under this condition the Hubble parameter from Eq. (66) turns out to be,

\[ H^2 = \frac{8\pi G}{3} \rho \left[ 1 + \frac{\rho}{2\lambda T} \left( 1 + \frac{12}{\kappa_5^2} \frac{\dot{h}_0^2}{\rho_0^2} \right) \right] \]  

(74)

If we assume that \( \rho \) represents energy density of non-relativistic matter we arrive at the following differential equation

\[ H^2 = \frac{8\pi G \rho_0}{3} \left[ 1 + \frac{4\pi G \rho_0^2}{3\lambda T} \left( 1 + \frac{12}{\kappa_5^2} \frac{\dot{h}_0^2}{\rho_0^2} \right) \right] \frac{1}{a^3}
\]  

(75)

which has the following solution:

\[ a^3 = (6\pi G \rho_0) t^2 + \left\{ \sqrt{\frac{12\pi G \rho_0^2}{\lambda T} \left( 1 + \frac{12}{\kappa_5^2} \frac{\dot{h}_0^2}{\rho_0^2} \right)} t \right\}
\]  

(76)

It is clear from the above expression that the universe undergoes a transition in the expansion rate at a timescale such that

\[ t \sim \sqrt{\frac{1}{3\pi G \lambda T}} = \frac{4}{\sqrt{3} \rho_0 \kappa_5 \lambda_5} = -\frac{2}{3} \Lambda_5^{-1}; \quad (\dot{h}_0^2 \ll \rho_0^2)
\]  

(77)

\[ t \sim \sqrt{\frac{4}{\pi G \lambda T - \rho_0 \kappa_5 \lambda_5}} = \frac{\dot{h}_0}{\sqrt{3} \rho_0 \kappa_5 \lambda_5 \Lambda_5^{-1}}; \quad (\dot{h}_0^2 \gg \rho_0^2)
\]  

(78)

Thus at early universe we have a high energy regime, where \( a \sim t^{1/3} \), while at late time low energy regime the scale factor variation with time modifies to \( a \sim t^{2/3} \), which is the standard evolution of the matter field.

Finally, we consider another situation, where it is assumed that \( \Lambda_5 = 0 \). Then the linear term in energy density cancels with the brane tension term coming from \( \Lambda_4 \). In that case the evolution will be governed by the quadratic term. Then for \( \dot{h}_0^2 \ll \rho_0^2 \), we obtain, the Hubble parameter to be: \( H^2 = \kappa_3^2 \rho^2/36 \), leading to solution \( a \sim t^{1/3} \), for matter dominated universe. However this is in contrast to the standard \( t^{2/3} \) behaviour. The same conclusion can be reached in the other situation \( \dot{h}_0^2 \gg \rho_0^2 \) as well. This puts constraint on choosing \( \Lambda_5 = 0 \). From other places, like nucleosynthesis experiments also rules out such possibilities imposing constraints on the models.
6 Concluding Remarks

In this work, we have considered a bulk spacetime manifold, which inherits along with the usual symmetric Christoffel symbol, an antisymmetric counterpart, known as torsion tensor. Due to introduction of such a tensor field the effective Einstein equation on the brane gets modified by additional terms. Note that the derivation of the effective field equation is non-trivial, in the sense that, the extrinsic curvature is not symmetric, the Frobenius identity no longer holds. Even due to the existence of an additional tensor field, the effective equation can be simplified substantially, by matching the torsion field with KR field, a closed string excitation and using gauge freedom. Afterwards the effective Einstein’s equation on the brane is constructed, where the torsion appears as an additional source of energy momentum tensor.

The spherically symmetric solution corresponding to the effective field equation on the brane with bulk spacetime endowed with torsion can be obtained by using symmetry properties of the KR field. These spherically symmetric solution gets influenced from the non local bulk, through Weyl tensor. The spherically symmetric solution shows standard black hole horizon structure as well as existence of naked singularity depending on the parameter space of the model. It turns out that for black hole solution, the specific heat is always negative, a characteristic feature for Schwarzschild-like gravitating system. Also it does not show any kind of phase transition.

In cosmological context presence of the Kalb-Ramond field modifies the Friedmann equations. It turns out that the KR field exhibits a behaviour of normal pressure free matter when energy conditions are satisfied. However for violation of energy conditions the Kalb-Ramond field can also act as a source for an accelerating phase of the universe.

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A Appendix

Below we present some detailed calculations, which we believe will be helpful to the reader. These have not been presented in the main text in order to keep the flow of the work unhindered.

A.1 Derivation of Effective Einstein’s Equation

Here we elaborate our derivation of effective field equations for gravity on the brane. We start with a vector $X^b$ solely on the brane, for which we have the following relation:

$$D_m D_n X_b - D_n D_m X_b = h^t_m h^n_r h^s_b (\nabla_t \nabla_r X_s - \nabla_r \nabla_t X_s) + \left( \tilde{K}_m b \tilde{K}_n s \right) X^s + h^t_m h_r n^r \nabla_r X_s \left( \tilde{K}_m n - \tilde{K}_n m \right)$$

Then from Eq. (2) the above relation can be written as,

$$- (4) \tilde{R}^p_{b m n} X_p - 2 (4) T^p_{m n} D_p X_b = h^t_m h^n_r h^s_b (\nabla_t \nabla_r X_s - \nabla_r \nabla_t X_s) + \left( \tilde{K}_m b \tilde{K}_n s - \tilde{K}_n b \tilde{K}_m s \right) X^s$$

$$+ h^t_m h_r n^r \nabla_r X_s \left( \tilde{K}_m n - \tilde{K}_n m \right)$$
Now for the brane confined vector $X^a$ we have the following relation,

$$D_a X_b = h^p_a h^q_b \nabla_m X_n$$

(81)

This implies that the torsion tensor satisfy the following relation:

$$(4)^T_{ab} X_p = h^m_a h^n_b T^p_{mn} X_p$$

(82)

Since $X_p$ is a vector on the brane we cannot eliminate it from both sides of the above equation, rather we should write $X_p = h^p_s Y_s$, with an arbitrary vector $Y_s$. Thus we get the relation between four-dimensional torsion tensor and five-dimensional torsion tensor as,

$$(4)^T_{ab} = h^m_a h^n_b T^m_{nk}$$

(83)

Then substitution of the above result in Eq. (80) leads to,

$$-(4)^\tilde{R}_{bmn} X_p - 2 (h^t_m h^r_n h^s_p T^p_{tr} \nabla_p X_s - h^t_m h^r_n h^s_p T^p_{tr} n^u n_p \nabla_u X_s)$$

$$= h^t_m h^r_n h^s_p (-\tilde{R}^p_{str} X_p - 2 T^p_{tr} \nabla_p X_s) + \left( \tilde{K}_{mb} \tilde{K}_{ns} - \tilde{K}_{nb} \tilde{K}_{ms} \right) X^s$$

$$+ 2 h^s_p n^r \nabla_r X_s T^m_{mn} X_p$$

(84)

Note that,

$$h^t_m h^r_n T^p_{tr} n_p = n^p (\delta^t_m \delta^r_n - n^t n_m \delta^r_n - n^r n_m \delta^t_n + n^t n_m n^r n_n) T_{ptr}$$

$$= n^p T^p_{mn} - n^p n^t n_m T_{ptr} - n^p n^r n_m T_{ptr} + n^p n^t n_m n^r n_n T_{ptr}$$

$$= n^p T^p_{mn}$$

(85)

due to antisymmetric nature of torsion tensor in all the indices. Finally from Eq. (84) we arrive at,

$$(4)^\tilde{R}_{bmn} X_p = h^t_m h^r_n h^s_p T^p_{tr} X_p - \left( \tilde{K}_{mb} \tilde{K}_{ns} - \tilde{K}_{nb} \tilde{K}_{ms} \right) X^s$$

(86)

In this case also we can introduce $X_p = h^p_s Y_s$ with arbitrary vector $Y_s$, which leads to the Gauss-Coddazzi equation for spacetime with torsion as:

$$(4)^\tilde{R}_{bmn} = h^t_m h^r_n h^s_p h^q_r \tilde{R}^q_{str} - h^p_s \left( \tilde{K}_{mb} \tilde{K}_{ns} - \tilde{K}_{nb} \tilde{K}_{ms} \right)$$

(87)

Since $n^a K_{aq} = 0$, the above equation reduces to:

$$(4)^\tilde{R}_{bmn} = h^t_m h^r_n h^s_p \tilde{R}^q_{str} - g^{pq} \left( \tilde{K}_{mb} \tilde{K}_{aq} - \tilde{K}_{nb} \tilde{K}_{mq} \right)$$

(88)

From the above equation two more relations can be easily constructed, first one by contracting $p$ and $m$ as,

$$(4)^\tilde{R}_{bd} = h^r_q h^q_d \tilde{R}^p_{qrs} + \tilde{K} \tilde{K}_{db} - g^{aq} \tilde{K}_{dq} \tilde{K}_{ab}$$

$$= h^r_q h^q_d \tilde{R}^p_{qrs} - h^r_q h^q_d n^r n_p \tilde{R}^p_{qrs} + \tilde{K} \tilde{K}_{db} - g^{aq} \tilde{K}_{dq} \tilde{K}_{ab}$$

(89)

and the second one by constructing scalar out of the above second rank tensor such as:

$$(4)^\tilde{R} = h^{qd} h^d_q \tilde{R}^q_{qrs} - h^{qd} h^d_q n^r n_p \tilde{R}^p_{qrs} + \tilde{K}^2 - \tilde{K}_{dq} \tilde{K}_{ab} g^{bd} g^{aq}$$

$$= \tilde{R} - n^a n^b \left( \tilde{R}_{ab} + \tilde{R}_{ba} \right) + \tilde{K}^2 - \tilde{K}_{ab} \tilde{K}_{ba}$$

(90)
Note that $h_{sa}^a h_{tb}^b g_{ab} = h_{ab}$. Thus the effective Einstein Equation on the brane has the following expression:

$$(4) \tilde{G}_{ab} = (4) \tilde{R}_{ab} - \frac{1}{2} h_{ab} \tilde{R}$$

$$= h_{ab}^a h_{b}^b \left( \tilde{R}_{qs} - \frac{1}{2} g_{qs} \tilde{R} \right) - h_{ab}^a h_{b}^b n^r n_p \tilde{R}_{pr}^p + \frac{1}{2} h_{ab} n_s n^s \left( \tilde{R}_{qs} + \tilde{R}_{sq} \right)$$

$$+ \tilde{K} \tilde{K}_{ba} - g^{pq} \tilde{K}_{pa} \tilde{K}_{qb} - \frac{1}{2} h_{ab} \tilde{K}^2 + \frac{1}{2} h_{ab} \tilde{K}_{pq} \tilde{K}^{pq}$$

(91)

Now the Weyl tensor has the following expression in terms of curvature tensor and its by product such that,

$$\tilde{C}_{abcd} = \tilde{R}_{abcd} - \frac{1}{3} \left[ (g_{ac} \tilde{R}_{bd} - g_{ad} \tilde{R}_{bc}) - (g_{bc} \tilde{R}_{ad} - g_{bd} \tilde{R}_{ac}) \right]$$

$$+ \frac{1}{12} \tilde{R} (g_{ac} g_{bd} - g_{ad} g_{bc})$$

(92)

Then we have the following result:

$$\tilde{C}_{abcd} h_{dp}^a h_{eq}^b n^c n^e = \tilde{R}_{abcd} h_{dp}^a h_{eq}^b n^c n^e - \frac{1}{3} \left[ h_{dp}^a h_{eq}^b \tilde{R}_{bd} + h_{pq} n^a n^c \tilde{R}_{ac} \right] + \frac{1}{12} \tilde{R} h_{pq}$$

(93)

Thus substitution for $\tilde{R}_{abcd} h_{dp}^a h_{eq}^b n^c n^e$ in Eq. (91) leads to the following expression:

$$(4) \tilde{G}_{ab} = \frac{2}{3} h_{ab} h_{b}^b \tilde{R}_{qs} - \frac{1}{3} h_{ab} n_s n^s \tilde{R}_{qs} + \frac{1}{2} h_{ab} n_s n^s \left( \tilde{R}_{qs} - \frac{1}{2} g_{qs} \tilde{R} \right) - \frac{1}{2} h_{ab} \tilde{R}$$

$$+ \tilde{K} \tilde{K}_{ba} - g^{pq} \tilde{K}_{pa} \tilde{K}_{qb} - \frac{1}{2} h_{ab} \tilde{K}^2 + \frac{1}{2} h_{ab} \tilde{K}_{pq} \tilde{K}^{pq}$$

(94)

Now using Eqs. (3), (4a), (4b) and the following one:

$$\tilde{K}_{ab} = K_{ab} + h_{ap}^m T_{mb} n_p$$

$$= K_{ab} + T_{ab} n_p$$

(95)

with immediate corollary $\tilde{K} = K$, we finally arrive at the following decomposition:

$$(4) \tilde{G}_{ab} = \frac{2}{3} h_{ab} h_{b}^b \left( R_{qs} - \frac{1}{2} g_{qs} R \right) + \frac{1}{2} h_{ab} n_s n^s \left( R_{qs} - \frac{1}{2} g_{qs} R \right) - \tilde{E}_{ab} + \frac{1}{4} h_{ab} R$$

$$+ \tilde{K} \tilde{K}_{ba} - g^{pq} \tilde{K}_{pa} \tilde{K}_{qb} - \frac{1}{2} h_{ab} \tilde{K}^2 + \frac{1}{2} h_{ab} \tilde{K}_{pq} \tilde{K}^{pq}$$

$$+ \left[ \frac{5}{12} h_{ab} T_{pq} T_{pq} + K n_p T_{pa} - K_{ba} n_p T_{pq} - K_{ba} n_m T_{pm} T_{pq} - T_{pq} n_p T_{m} n_m \right]$$

$$+ \frac{1}{2} h_{ab} T_{pa} n_p T_{ab} n_q$$

(96)
Then we can use Eq. (92) to obtain,
\[
\hat{E}_{pq} = n^a n^c h^b h^d \xi_{abcd}
\]
\[
= E_{pq} + \left[ n^a n^c h^b h^d \nabla_c T^a_{db} - n^a n^c h^b h^d \nabla_d T^a_{cb} + 2n^a n^c h^b h^d T^a_{mb} T^m_{cd} + n^a n^c h^b h^d T^a_{dp} T^p_{cb} - n^a n^c h^b h^d T^a_{db} T^p_{cp} - \frac{1}{3} \{ h^b h^d T^m_{nb} T^m_{md} + n^a n^c h_{pq} T^p_{qa} T^q_{pc} \} - \frac{1}{12} h_{pq} T^{abc} T_{abc} \right]
\]
(97)

Substituting the above result in Eq. (96), we readily obtain the following form of the effective equation:
\[
(4)^{th} \hat{G}_{ab} = \frac{2}{3} h^a_\alpha h^b_\beta \left( R_{qs} - \frac{1}{2} g_{qs} R \right) + \frac{2}{3} h_{ab} n^q n^s \left( R_{qs} - \frac{1}{2} g_{qs} R \right) - E_{ab} + \frac{1}{4} h_{ab} R
\]
\[+ K K_{ba} - g^{pq} K_{pa} K_{qb} - \frac{1}{2} h_{ab} K^2 + \frac{1}{2} h_{ab} K p_{pq} + \left[ - \frac{2}{3} h^a_\alpha h^b_\beta \nabla_q T^p_{ps} + h^a_\alpha h^b_\beta T^q_{ps} T^p_{ms} + h_{ab} n^q n^s T^m_{pq} T^p_{ms} + \frac{1}{2} h_{ab} T^{pq} T_{pq} \right] + K n_{pa} T^p_{ba} - K_{aq} n_{pq} T^p_{ba} - K^q n_{pq} T^m_{ba} - T_{uaw} T^{aw} + n w_{nb} T_{uaw} T^{uw} + n k n_a T_{uaw} T^{uw} - n^b n^c h^a_\alpha h^b_\beta \nabla_q T_{pq} - n^b n^c h^a_\alpha h^b_\beta \nabla_q T_{pq} - n^b n^c h^a_\alpha h^b_\beta \nabla_q T_{pq} + 2n^b n^c h^a_\alpha h^b_\beta \nabla_q T_{pq} - \frac{1}{2} h_{ab} T^{pq} T_{pq} - \frac{1}{2} h_{ab} T^{pq} T_{pq} - \frac{1}{2} h_{ab} T^{pq} T_{pq} \right]
\]
(98)
The above equation can easily be simplified leading to the following expression:
\[
(4)^{th} \hat{G}_{ab} = \frac{2}{3} h^a_\alpha h^b_\beta \left( R_{qs} - \frac{1}{2} g_{qs} R \right) + \frac{2}{3} h_{ab} n^q n^s \left( R_{qs} - \frac{1}{2} g_{qs} R \right) - E_{ab} + \frac{1}{4} h_{ab} R
\]
\[+ K K_{ba} - g^{pq} K_{pa} K_{qb} - \frac{1}{2} h_{ab} K^2 + \frac{1}{2} h_{ab} K p_{pq} + \left[ - \frac{2}{3} h^a_\alpha h^b_\beta \nabla_q T^p_{ps} + K n_{pa} T^p_{ba} - K_{aq} n_{pq} T^p_{ba} - K^q n_{pq} T^m_{ba} - T_{uaw} T^{aw} + n w_{nb} T_{uaw} T^{uw} + n k n_a T_{uaw} T^{uw} - n^b n^c h^a_\alpha h^b_\beta \nabla_q T_{pq} + 2n^b n^c h^a_\alpha h^b_\beta \nabla_q T_{pq} \right]
\]
(99)
Note that the induced four dimensional torsion tensor \( (4)^{th} T_{abc} \) has the following expression in terms of the bulk torsion tensor \( T_{abc} \)
\[
(4)^{th} T_{abc} = h^a_\alpha h^b_\beta h^c_\gamma T_{pqrs} h^{a}_{ps} n^q n^s T^{mns}
\]
\[= h^a_\alpha h^b_\beta h^c_\gamma T_{pqrs} T^{mns} - T_{pqrs} T^{mns} - 3n^p n_m T_{pqrs} T^{mns}
\]
(100)
\[
(4)^{th} T_{pq} = -h_m T_{pq} T^{mns}
\]
\[= -h_m T_{pq} T^{mns} - T_{uaw} T^{uw} + n w_{nb} T_{uaw} T^{uw} + n k n_a T_{uaw} T^{uw} - n k n_w n_a n_b T_{uaw} T^{uw} + 2n^b n^c h^a_\alpha h^b_\beta \nabla_q T_{pq} + 2n^b n^c h^a_\alpha h^b_\beta \nabla_q T_{pq} \]
(101)
All these results have been used in order to obtain the effective field equation on the brane presented in Eq. (8).
A.2 Field equations with axion field

Below we provide the gravitational field equations for the axion field using the metric ansatz in Eq. (24). Writing the KR field $H_{\alpha\beta\mu}$ in terms of a single scalar field using the complete antisymmetric tensor $\epsilon^{\alpha\beta\mu\nu}$ we obtain,

$$H_{\alpha\beta\mu} = \epsilon_{\alpha\beta\mu\rho} \partial^\rho \Phi \partial_\mu \Phi$$

$$H_{\alpha\beta\nu} = \epsilon_{\alpha\beta\nu\sigma} \partial^\sigma \Phi \partial_\nu \Phi$$

$$H_{\alpha\beta\mu} H_{\alpha\beta\nu} = (\sqrt{-g} \partial_{\alpha} \Phi \partial_\beta \Phi) (\sqrt{-g} \partial^\nu \Phi \partial^\rho \Phi)$$

$$H_{\alpha\beta\mu} H_{\alpha\beta\nu} = (\sqrt{-g} \partial_{\alpha} \Phi \partial_\beta \Phi) (\sqrt{-g} \partial^\nu \Phi \partial^\rho \Phi)$$

With the help of the above identities the field equations for gravity reduce to the following forms:

$$-e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda^2}{r^2} \right) + \frac{1}{r^2} = \Lambda_4 + \frac{4\pi G}{k^4 \lambda} U$$

$$e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = -\Lambda_4 + \frac{16\pi G}{k^4 \lambda} (U + 2P)$$

$$e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = -\Lambda_4 + \frac{32\pi G}{k^4 \lambda} (U - P)$$

Again from Eqs. (103c) and (103d) we obtain the relation:

$$\left( \partial_\theta \Phi \right) = \sin^2 \theta \left( \partial_\phi \Phi \right)$$

The above differential equations represent the effective Einstein equations on the brane with axion field present in the bulk.

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