Faster and More Accurate Trace-based Policy Evaluation via Overall Target Error Meta-Optimization

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Abstract

To improve the speed and accuracy of the trace based policy evaluation method TD(\(\lambda\)), under appropriate assumptions, we derive and propose an off-policy compatible method of meta-learning state-based \(\lambda\)'s online with efficient incremental updates. Furthermore, we prove the derived bias-variance tradeoff minimization method, with slight adjustments, is equivalent to minimizing the overall target error in terms of state based \(\lambda\)'s. In experiments, the method shows significantly better performance when compared to the existing method and the baselines.

1 Introduction

TD(\(\lambda\)), which uses a geometric sequence as the weights of the \(n\)-step returns, stands out from the sea of compound update methods for its good empirical performance, efficiency and interesting mathematical properties.

As we will show, adaptive \(\lambda\)'s can outperform any static \(\lambda\) value. Research has gone into meta-learning the learning rate hyperparameter in Temporal Difference (TD) algorithms \cite{1}, while investigations into meta-learning the trace parameter are fairly limited. The majority of existing methods do not work in the incremental backward view due to design or computational complexity \cite{2, 3}. Recently, there has been success in meta-learning a state-based \(\lambda\) using extra estimates of statistics of returns \cite{11}, but there is still much room for enhancement.

Our goal is to derive a method that meta-learns \(\lambda\) to achieve better learning speed and accuracy. We derive a new meta-objective for optimizing the bias-variance tradeoff for one state. Then, we propose a trust-region style method to tackle the difficulties of the optimization of the proposed objective and prove its capability of minimizing the overall target error. In experiments, we validate its effectiveness by comparing it to the existing method \cite{11} and baselines.

2 Preliminaries

Classical TD(\(\lambda\)) \cite{5} uses a weighted combination of the multi-step targets as the update target, called the \(\lambda\)-return. The weights constitute a geometric sequence controlled by the hyperparameter \(\lambda \in [0, 1]\) and are applied from the 1-step return to the \(\infty\)-step return (MC return). TD(\(\lambda\)) is also interpreted using the “backward view”: the updates towards the
\( \lambda \)-return can be approximated with incremental updates of space and time complexity \( O(n^1) \) using buffer vectors called the “eligibility traces”. These traces are attractive for both their efficiency and interesting mathematical properties.

Recently, there are several important contributions regarding TD(\( \lambda \)). Particularly relevant are the true online algorithms which use auxiliary vectors to make the updates exactly equivalent to updating towards true \( \lambda \)-return; Additionally, gradient TD methods achieve survival in the deadly triad with linear function approximators. These ideas give rise to the possibilities of meta-adapting \( \lambda \) during the learning processes, with compatibility of off-policy learning and convergence guarantees.

2.1 The Trace Adaptation Problem

We aim to adjust \( \lambda \) to achieve faster learning speed and better accuracy in terms of mean squared error of the value function for policy evaluation of given policies in unknown environments. Since time-dependent adaptions are considered as not equipped with a well-defined fixed point \([11]\), we seek only to find a congenial (i.e. incremental and online), state-dependent adaptation for \( \lambda \).

2.2 Background Knowledge

Before providing some background knowledge for understanding the contexts of this paper, we present all the notations to be used in Table 1.

| Notation | Meaning |
|---------|---------|
| \( x_t \) | Feature vector for the state \( S_t \) met at time-step \( t \). |
| \( V(G_t) \) | Estimated expectation of \( G_t \) for \( S_t \), also recognized as the value estimate \( V_\pi(S_t) \) or \( V(S_t) \). |
| \( v(G_t) \) | True expectation of \( G_t \) for \( S_t \), also recognized as the true value \( v_\pi(S_t) \), \( v(S_t) \) or \( E[G_t] \). |
| \( \lambda(t) \) | State dependent \( \lambda \) value for the state \( S_t \) or state feature \( x_t \). |
| \( \Lambda \) | Enumeration vector of all the state dependent \( \lambda \)'s for all states. |
| \( G^\Lambda_t \) | Sampled generalized \( \lambda \)-return of the state \( S_t \) met at time-step \( t \). |
| \( V(x_t, w) \) | Value estimate for the state with feature \( x_t \), using the weights \( w \). |
| \( \rho_t \) | Importance sampling ratio for the action taken at time-step \( t \). |
| \( \gamma_t \) | Discount factor for rewards after meeting the state \( S_t \) at time-step \( t \). The state-based discounting has been introduced in \([8, 11]\). |
| \( d_\pi(s) \) | The frequency of experiencing state \( s \) among all states, rolling out policy \( \pi \) infinitely in the environment. Depends on the starting state distribution \( d(s_0) \), which is policy independent. |

2.2.1 TD(\( \lambda \))

**Fact 2.1.** TD updates using a convex combination of n-step returns as the target are called compound updates. In the tabular case, compound updates converge to the true values with appropriate assumptions.

TD(\( \lambda \)) does compound updates: it interpolates between TD(0) and Monte-Carlo, achieving generally better learning speed and accuracy than the individual multi-step methods. To have convergence guarantees with linear function approximators \([4]\), we use the TD(\( \lambda \)) variant Gradient TD(\( \lambda \)) (GTD(\( \lambda \))) \([7]\).

\(^1n \equiv |X|, \text{ the dimension of the feature space } X \text{ for states.}\)

\(^2\text{It describes the fact that algorithms hardly have convergence guarantees if function approximation, bootstrapping and off-policy learning are all used.}\)

\(^3\text{GTD methods are for linear approximators only. For more complicated function approximators, we do not have well-defined fixed points for the Bellman operators and thus cannot have the guarantees.}\)
Definition 2.1. The (generalized) $\Lambda$-return $G^\Lambda_t$, where $\Lambda \equiv \{\lambda_1 \equiv \lambda(s_1), \lambda_2 \equiv \lambda(s_2), \ldots, \lambda_t \equiv \lambda(s_t), \ldots\}$, from one state $s_t$ in a trajectory $\tau$ is defined as

$$G^\Lambda_t \equiv G^\Lambda(s_t) = R_{t+1} + \gamma_{t+1}[(1 - \lambda^{(t+1)})V(s_{t+1}) + \lambda^{(t+1)}G^\Lambda_{t+1}]$$

The generalized definition of $\Lambda$-return with state dependent $\lambda$’s gives rise to the generalized algorithm TD($\Lambda$).

Fact 2.2 (7). With finite-length trajectories, the updates of GTD using the $\Lambda$-return as targets can be exactly achieved using eligibility traces.

The possibility for accumulating traces ensures the viability of state based $\lambda$’s. Then we have a set $\Lambda$ of all $\lambda$’s and TD($\Lambda$). This is actually the contribution of true online GTD($\lambda$) [9]. The “true online” algorithms [10] use slightly more complicated incremental updates to achieve true equivalence to the $\Lambda$-return with linear function approximators. The true equivalence gives us full control of the $\Lambda$-return and the possibility of provably achieving an adjusted fixed point for better performance. In this paper, we develop our ideas upon the true online GTD method[7], which combines the merits of being truly online and gradient based [9], to achieve full control of the bias-variance tradeoff and the survival of the deadly triad. With its guarantees, we can focus on the problem of meta-adaptation.

2.2.2 Mean Squared Errors

The quality of an estimate for the value function [6], which we aim to enhance, has great connections to the quality of targets.

Definition 2.2. Given the true value function $v$ and its estimate $V$ of target policy $\pi$, the (scaled) mean squared error for $V$ or value error is defined as:

$$f(V) \equiv 1/2 \cdot \|D \cdot (V - v)\|_2^2$$

where $D \equiv \text{diag}(d_v(s_1), d_v(s_2), \ldots, d_v(s_{|S|}))$.

The Mean Squared Error (MSE or error) for a value function is the weighted $L_2$-distance between $v$ and $V$. The weights favor the states that the agent will meet with higher frequency under the target policy $\pi$.

Definition 2.3. Given $v$ and the collection of the update targets $G$ for all states, the (scaled) mean squared error or target error for $G$ is defined as:

$$f(G) \equiv 1/2 \cdot \|D \cdot (E[G] - v)\|_2^2$$

Proposition 2.1. Given an infinite number of updates and learning rates in $(0, 1)$, value estimates using targets with lower target MSE achieve lower value MSE.

Though easy to prove, this idea is very powerful: since higher learning speed represents achieving higher learning accuracy with some number of updates, optimizing learning rate and accuracy can be achieved at the same time by minimizing the target error. This is the basis for the $\lambda$-greedy algorithm which we are about to discuss as well as our proposed method.

2.2.3 $\lambda$-Greedy & Greedy State Meta-Objective

The $\lambda$-greedy method is the first meta-learning method that can achieve incremental updating, compatibility with function approximation and stability during both on and off-policy learning simultaneously. The idea is to minimize the “greedy objective”, i.e. the MSE between a pseudo target $\hat{G}_t$ and the true value, where the pseudo target is defined as:

$$\hat{G}_t \equiv \rho_t(R_{t+1} + \gamma_{t+1}[(1 - \lambda^{(t+1)})V(\pi_{t+1}) + \lambda^{(t+1)}G_{t+1}])$$

With this we can find that $\hat{J}(s_t) \equiv E[(\hat{G}_t - E[\hat{G}_t])^2]$ is a function of $\lambda^{(t+1)}$. The greedy objective corresponds to minimizing the error of the pseudo target $\hat{G}_t$ with $\lambda^{(t+1)} \in [0, 1]$ and $\lambda_{(k)} = 1, \forall k \geq t + 2$:

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4We use $\Lambda$ here to differentiate with fixed $\lambda$’s.

5The pseudocode for True Online GTD is provided in the appendix.
Proposition 2.2. Given the update target $\hat{G}_t$ of state $s_t$, the minimizer $\lambda^*_t$ of the mean squared target error of the state $J(s_t) = \mathbb{E}[(\hat{G}_t - \mathbb{E}[G_t])^2]$ is:

$$
\lambda^*_t = \frac{(\mathbb{E}[G_{t+1}] - V(x_{t+1}, w^t))^2}{\text{Var}[G_{t+1}] + (\mathbb{E}[G_{t+1}] - V(x_{t+1}, w^t))^2}
$$

The proof is in the appendix. The $\lambda$-greedy algorithm shows empirically strong performance; however, the pseudo target $\hat{G}_t$ used for optimization is not actually the target used in TD($\lambda$) algorithms: we will present our interpretation of it being a compromise for estimation stability in the following section. Also, we will see that this compromise eliminates the possibility of jointly optimizing the overall target error.

$\lambda$-greedy decides how $\lambda^{(t+1)}$ should be set on this step to locally optimize the target error for this state, however with no intention to optimize the overall target error since the pseudo target for each state does not care about the $\lambda$'s for other states.

2.2.4 VTD & DVTD

$\lambda$-greedy needs an effective method to estimate the variance of MC return alongside the first moment. The key to solving this problem is to find the correct “reward” for the variance and estimate it using the trace updates. [11] proposed a generalized method to estimate the second moment of $\lambda$-return called VTD. The variance of MC-return is then approximated via assembling the first and second moments of $\lambda = 1$ returns together. More recently, [4] introduced a similar DVTD method for directly estimating the variance of the $\lambda$-return, with empirically more robust performance [4]. DVTD can be written an trace-based style:

$$
\bar{R}_{t+1} \leftarrow \delta^2_t \equiv (R_{t+1} + \gamma_{t+1} V(x_{t+1}, w^t) - V(x_t, w^t))^2
$$

$$
\bar{\gamma}_{t+1} \leftarrow \gamma^2_{t+1} \lambda^2_{t+1}
$$

$$
\bar{\delta}_t \leftarrow \bar{R}_{t+1} + \gamma_{t+1} \bar{V}(x_{t+1}, \bar{w}_t) - \bar{V}(x_t, \bar{w}_t)
$$

$$
\bar{c}_t \leftarrow \rho_t (\bar{\gamma} \bar{c}_{t-1} + \nabla \bar{V}(x_t, \bar{w}_t))
$$

$$
\bar{w}_{t+1} \leftarrow \bar{w}_t + \bar{\delta}_t \bar{c}_t
$$

where the quantities with overlines represent the corresponding necessities for the estimation of the variance of the $\lambda$-return. This is equivalent to estimating an altered “return” with $\delta^2_t$ as rewards, $\bar{\gamma}_{t+1}$ as the discount factor, $\rho_t$ as the importance sampling ratio and $\lambda = 1$ as the trace parameters, which means it can be dealt with true online GTD routine as $\text{logtd}(R_{t+1} = (R_{t+1} + \gamma_{t+1} (x_{t+1} - x_t)^2 w^t)^2, \gamma_{t+1} = \gamma^2_{t+1} \lambda^2_{t+1}, \gamma_t = \gamma^2_t \lambda^2_t, \lambda_t = 1, \lambda_{t+1} = 1, \rho_t = \rho_t)$.

3 MTA: Meta Trace Adaptation with True Objectives

The pseudo target used in $\lambda$-greedy state objectives is disconnected with the overall target error in a way that their summation is not the overall target error. In this section, we propose a true objective that achieves the connection and derive an off-policy compatible meta-learning approach that uses purely incremental updates to achieve overall target error optimization.

3.1 True State Meta-Objective

The greedy objective optimizes the error for pseudo target $\hat{G}_t$. Here we optimize the error for the true target, i.e. $\mathbb{E}[(G^t - \mathbb{E}[G_t])^2]$. We first derive the true state meta-objective for every state encountered in a trajectory as a function of $\lambda^{(t+1)}$. Then, we discuss the characteristics of the true objective and compare it with the greedy one. After that, we will present our method of optimizing the true objective. Excitedly, we will show that our method of optimization, with appropriate correction, is equivalent to minimizing the overall target error, even if off-policy.
Proposition 3.1. Given the update target $G^\lambda_t$ of state $s$, the gradient of the mean squared target error of the state $J(s_t) \equiv \mathbb{E}[(G^\lambda_t - \mathbb{E}_t)^2]$ w.r.t. $\lambda^{(t+1)}$ is:

$$\nabla J(s_t) = \gamma_t^{2} \left[ \lambda^{(t+1)} \left( (V(s_{t+1}) - \mathbb{E}[G^\lambda_{t+1}])^2 + \mathbb{V}[G^\lambda_{t+1}] \right) + (\mathbb{E}[G^\lambda_{t+1}] - V(s_{t+1}))(\mathbb{E}_t[G_{t+1}] - V(s_{t+1})) \right]$$

And its minimizer is:

$$\arg\min_{\lambda^{(t+1)}} J(\lambda^{(t+1)}) = \frac{(V(s_{t+1}) - \mathbb{E}[G^\lambda_{t+1}])[V(s_{t+1}) - \mathbb{E}[G_{t+1}])}{(V(s_{t+1}) - \mathbb{E}[G^\lambda_{t+1}])^2 + \mathbb{V}[G^\lambda_{t+1}]}$$

The proof is in the appendix. The minimizer of the true objective is a generalization of $[1]$, the minimizer of the greedy objective proposed in $[1]$, which can be seen by setting $G^\lambda_{t+1} = G_{t+1}$. However this is more complicated than the minimizer of the greedy objective: we need the estimate of $\mathbb{E}[G_{t+1}]$, $\mathbb{E}[G^\lambda_{t+1}]$ and $\mathbb{V}[G^\lambda_{t+1}]$, while for $[1]$ we only need the estimation of $\mathbb{E}[G_{t+1}]$ and $\mathbb{V}[G_{t+1}]$. Furthermore, if some $\lambda$ for another state is changed, $\mathbb{E}[G^\lambda_{t+1}]$ and $\mathbb{V}[G^\lambda_{t+1}]$ for all states will be destroyed and will have to be re-estimated from scratch, since they all depend on the whole $\lambda$. While the estimates for the greedy objective are invariant to the change of $\lambda$. This is our interpretation of why the greedy objective is crafted this way: though highly biased towards the MC return, it is with more stable estimates to calculate the optimal $\lambda$’s.

Though it is possible to estimate $\mathbb{E}[G_{t+1}]$, $\mathbb{E}[G^\lambda_{t+1}]$ and $\mathbb{V}[G^\lambda_{t+1}]$ $[1]$, the minimizer of the true objective can be very unstable if we change any $\lambda^{(t+1)} \in \Lambda$. Despite such difficulty, there are potential benefits of tackling such optimization.

Theorem 3.1. Let $s$’s be the states that the agent experiences when rolling out behavior policy $b$ and $\rho_{\text{acc}}$ be the cumulative product of importance sampling ratios from the beginning of an episode until state $s$. Gradient descent on the true state objectives corrected by $\rho_{\text{acc}}$ is equivalent to stochastic gradient descent on the overall target error for target policy $\pi$. More specifically:

$$\nabla_{\lambda} J(G(\lambda)) \approx \sum_{s \sim b} \rho_{\text{acc}} \cdot \nabla_{\lambda^{(t+1)}} J(s)$$

The proof is in the appendix. The theorem shows that if correct gradient descent on the true objectives can be done, the overall target error can be optimized. But the correctness requires trustworthy estimates, which cannot be obtained if we adapt $\lambda$’s to the minimizers. We notice that the true objective is continuous w.r.t. $\lambda$. Thus, a small gradient descent step on $\lambda^{(t+1)}$ only yields bounded shifts of the estimates. If we use small enough steps to change $\lambda$, we can actually stabilize the estimates since they will not deviate far and will be corrected by the TD updates quickly. The small step of gradient descent can be interpreted as a trust region method which is similar to the optimization of neural networks. Though we do not know the convexity of $J(G(\lambda))$, we at least have the guarantee to converge to a stationary point of overall target error, with appropriate assumptions. Also, it is likely that the noises of the estimates may let SGD escape saddle points s.t. it may converge to optima.

We now have an off-policy compatible method that is purely incremental with the computational complexity $O(n)$ and costs multiplied by 4 (3 more TD’s to estimate the statistics, i.e. $\mathbb{E}[G_t]$, $\mathbb{E}[G^\lambda_t]$ and $\mathbb{V}[G^\lambda_t]$). We combine the proposed method, which we name MTA, with true online GTD, and obtain the pseudocode in Algorithm $[1]$. 


Algorithm 1: Policy Evaluation: True Online GTD(\(\Lambda\)) with MTA

initialize weights for \(\mathbb{E}[G_i], \mathbb{E}[G^\Lambda_i]\) and \(\text{Var}[G^\Lambda_i]\) to be 0;

for episodes do

Initialize traces for \(\mathbb{E}[G_i], \mathbb{E}[G^\Lambda_i]\) and \(\text{Var}[G^\Lambda_i]\) to be 0 and \(\rho_{\text{acc}} = 1\);

for non-terminal states \(s_0, s_1, \ldots\) do

// rollout behavior policy & get importance sampling ratio
\(a_t \sim b(x_t), p_t = \pi(a_t, x_t)/b(a_t, x_t), \rho_{\text{acc}} = \rho_{\text{acc}} \cdot p_t, x_{t+1} \sim \text{MDP}(s(x_t), a_t)\);

// update \(\mathbb{E}[G_i]\), using true online GTD(1)
\(\text{to gt}(R_{t+1} = R_{t+1}, \gamma_{t+1} = \gamma_{t+1}, \gamma_t = \gamma_t, \lambda_t = 1, \lambda_{t+1} = 1, p_t = p_t)\);

// update \(\mathbb{E}[G^\Lambda_i]\), using true online GTD(\(\Lambda\))
\(\text{to gt}(R_{t+1} = R_{t+1}, \gamma_{t+1} = \gamma_{t+1}, \gamma_t = \gamma_t, \lambda_t = \lambda_t, \lambda_{t+1} = \lambda_t, p_t = p_t)\);

// update \(\text{Var}[G^\Lambda_i]\), using true online GTD(\(\Lambda\)) and DVT
\(\text{to gt}(\hat{R}_{t+1} = \hat{R}_{t+1} + \gamma_{t+1}(x_{t+1} - x_t)^T \omega)^2, \gamma_{t+1} = \gamma_{t+1}^2 \lambda_{t+1}^2, \gamma_t = \gamma_t^2 \lambda_t^2, \lambda_t = 1, \lambda_{t+1} = 1, p_t = p_t)\);

// gradient descent on true objective
\(\lambda^{(t+1)} = \lambda^{(t+1)} - \kappa \rho_{\text{acc}} \lambda^{(t+1)} ((\mathbb{E}[G^\Lambda_i] - \mathbb{E}[G^\Lambda_i])^2 + \text{Var}[G^\Lambda_i] + \mathbb{V}(s_{t+1} - \mathbb{E}[G^\Lambda_i] + \mathbb{E}[G_i] - \mathbb{E}[G^\Lambda_i] \mathbb{E}[G_{t+1}])\);

// update \(\mathbb{V}(s_i)\), using true online GTD(\(\Lambda\))
\(\mathbb{V}(s_i) = \text{to gt}(R_{t+1} = R_{t+1}, \gamma_{t+1} = \gamma_{t+1}, \gamma_t = \gamma_t, \lambda_t = \lambda_t, \lambda_{t+1} = \lambda_t, p_t = p_t)\);

4 Experiments

4.1 Ringworld: Tests with Low Variance

The first set of experiments focus on a lower-variance environment “ringworld”, adopted in [11]. We test on and off-policy tests on 11-state ringworld with 6 pairs of behavior-target policies and compare the algorithms’ performance using value error. For fair comparison, we use a universal setting of \(\alpha = \beta = 0.05\) for all the true online GTD updates, where \(\alpha\) is the learning rate for weight vectors and \(\beta\) for auxiliary vectors. For MTA, we set the hyperparameter \(\kappa = 0.01\), which is to be investigated afterwards. Also, to ensure \(\mathbb{E}[G^\Lambda_i] \neq \mathbb{V}(s_i)\) in MTA, we set the learning rates of \(\hat{\mathbb{E}}[G_i]\) to be 1.1\(\alpha\) and 1.1\(\beta\).

The error curves are presented in Figure [2]. We observe that MTA generally converges the fastest and with the highest accuracy at the end, better than \(\lambda\)-greedy which already generally achieves faster and more accurate convergence than the baselines with fixed \(\lambda\’s\).

We also care about MTA’s sensitivity to the hyperparameter \(\kappa\). If we consider the true online GTD used in \(\lambda\)-greedy and MTA as a blackbox, then \(\lambda\)-greedy is parameter free: it adapts \(\lambda\) to the minimizer, whereas MTA does gradient descent controlled by the step-size \(\kappa\). In theory, we want the \(\kappa\) to be small, in which case the estimates will be stable and good convergence is expected. However with limited episodes, we also want \(\kappa\) to be not too small s.t. \(\lambda\) can be updated quickly in-time. We can expect a problem-dependent and number-of-episode-dependent range of suitable \(\kappa\’s\) that yield good performance. We present the error curves with different \(\kappa\’s\) in Figure [2(a)] and the correspondence of final results to \(\kappa\’s\) in Figure [2(b)].

We can see that MTA is relatively robust w.r.t. \(\kappa\). When the number of episodes is large, we can safely set \(\kappa\) to be a small enough value to achieve good convergence.

4.2 Frozen Lake: Tests with High Variance

We now turn to a higher-variance environment “frozenlake”. We craft a uniform policy that takes the actions with equal probabilities and a heuristic policy that has 0.3 probability for

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6 More details of experiments are presented in the appendix. The source code is to be released in [GitHub](https://github.com/PowerHarry/MTA). It is also interesting to note that the \(\lambda\) changes in different patterns for MTA and \(\lambda\)-greedy, we provide these curves in the appendix.
Figure 1: Ringworld tests with $\alpha = \beta = 0.05$ and $\kappa = 0.01$. The x-axes are the number of episodes and the y-axes are errors. We run 100000 episodes for 160 independent runs. Baselines of true online GTD($\lambda$) with fixed $\lambda$ values are also provided.

Figure 2: Sensitivity for $\kappa$ in ringworld. In (b), the x-axis is the values of $\kappa$. 
5 Conclusion

To achieve faster and more accurate policy evaluation using TD($\lambda$), we have derived an algorithm for optimizing the bias-variance tradeoff of the overall target error, which has once been considered as infeasible, via meta-learning state dependent $\lambda$’s for TD($\lambda$), with the help of DVTD, the $\Lambda$-return variance estimation method. The proposed off-policy compatible method MTA, which uses only incremental updates, demonstrates superior empirical performance, compared to the existing greedy method and the fixed $\lambda$ baselines. The future work is to extend the idea of overall target error optimization on to the control case and onto fancier function approximators.
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Appendix

5.1 Proof of Proposition 5.2

**Proposition.** Given the update target $\tilde{G}_t$ of state $s_t$, the minimizer $\lambda^*_t$ of the mean squared target error of the state $\bar{J}(s_t) = \mathbb{E}[(\tilde{G}_t - \mathbb{E}[G_t])^2]$ is:

$$\lambda^*_t = \frac{\text{Var}[G_{t+1}] + (\mathbb{E}[G_{t+1}] - V(x_{t+1}, w^{(t)}))^2}{\text{Var}[G_{t+1}] + (\mathbb{E}[G_{t+1}] - V(x_{t+1}, w^{(t)}))^2}$$  \hspace{1cm} (2)

**Proof.**

$$\hat{J}(\lambda^{(t+1)}) = \mathbb{E}[(\tilde{G}_t - \mathbb{E}[G_t])^2]$$

$$= \mathbb{E}^2[\tilde{G}_t - \mathbb{E}[G_t]] + \text{Var}[\tilde{G}_t - \mathbb{E}[G_t]] \quad (\text{both } \mathbb{E} \text{ are w.r.t. policy})$$

$$\equiv \text{Bias}^2(\lambda^{(t+1)}) + \text{Variance}(\lambda^{(t+1)})$$

Now we decomposed the objective to the squared bias and the variance. Let us begin by rewriting the bias. Since we have $\lambda_t$, $x_{t+1}$, $\gamma_t$, and $\gamma_{t+1}$,

$$\mathbb{E}[\tilde{G}_t] = \rho_t \mathbb{E}[R_{t+1} + \gamma_{t+1}(1 - \lambda^{(t+1)})V(x_{t+1}, w^{(t)}) + \lambda^{(t+1)}G_{t+1}]$$

$$\hspace{1cm} = \rho_t \mathbb{E}[R_{t+1}] + \rho_t \gamma_{t+1} (1 - \lambda^{(t+1)})V(x_{t+1}, w^{(t)}) + \lambda^{(t+1)} \mathbb{E}[G_{t+1}]$$  \hspace{1cm} (4)

For convenience, define $\text{err}(w, x_{t+1}) \equiv \mathbb{E}[G_{t+1}] - V(x_{t+1}, w^{(t)})$ as the difference between the $\lambda = 1$ return and the current approximate value from $x_{t+1}$ using weights $w^{(t)}$. Using the definition, we can rewrite

$$\mathbb{E}[\tilde{G}_t] = \mathbb{E}[G_t] - \rho_t \gamma_{t+1}(1 - \lambda^{(t+1)})\text{err}(w^{(t)}, x_{t+1})$$  \hspace{1cm} (5)

Thus

$$\text{Bias}^2(\lambda^{(t+1)}) = \mathbb{E}^2[\tilde{G}_t - G_t] = \rho_t^2 \gamma_{t+1}^2 (1 - \lambda^{(t+1)})^2 \text{err}^2(w^{(t)}, x_{t+1})$$  \hspace{1cm} (6)

Assuming the noise in the reward $R_{t+1}$ given $x_t$ and $x_{t+1}$ is independent of other dynamics

$$\text{Variance}(\lambda^{(t+1)}) = \text{Var}[\rho_t (R_{t+1} + \gamma_{t+1}(1 - \lambda^{(t+1)})V(x_{t+1}, w^{(t+1)}) + \lambda^{(t+1)}G_{t+1})]$$

$$\hspace{1cm} = \rho_t^2 \text{Var}[R_{t+1}] + \rho_t^2 \gamma_{t+1}^2 \lambda^{(t+1)} \text{Var}[G_{t+1}]$$  \hspace{1cm} (7)

Now we can use the derivations to simplify the objective:

$$\lambda^*_t = \arg\min_{\lambda_t \in [0,1]} \text{Bias}^2(\lambda^{(t+1)}) + \text{Variance}(\lambda^{(t+1)})$$

$$\hspace{1cm} = \arg\min_{\lambda_t \in [0,1]} (1 - \lambda^{(t+1)})^2 \text{err}^2(w^{(t)}, x_{t+1}) + \lambda_t^2 \text{Var}[G_{t+1}] = \frac{\text{err}^2(w^{(t)}, x_{t+1})}{\text{Var}[G_{t+1}] + \text{err}^2(w^{(t)}, x_{t+1})}$$  \hspace{1cm} (8)

\[\square\]

5.2 Proof of Proposition 5.1

**Proposition.** Given the update target $G^\lambda_t$ of state $s$, the gradient of the mean squared target error of the state $J \equiv \mathbb{E}[(G^\lambda_t - \mathbb{E}[G_t])^2]$ w.r.t. $\lambda^{(t+1)}$ is:

$$\nabla_{\lambda^{(t+1)}} J(s_t) =$$

$$\gamma_{t+1}^2 \left[ (V(s_{t+1}) - \mathbb{E}[G^\lambda_{t+1}])^2 + \text{Var}[G^\lambda_{t+1}] \right] + (\mathbb{E}[G^\lambda_{t+1}] - V(s_{t+1}))(\mathbb{E}[G_{t+1}] - V(s_{t+1}))$$
And its minimizer is:

\[
\arg\min_{\lambda^{(t+1)}} J(\lambda^{(t+1)}) = \frac{(V(s_{t+1}) - \mathbb{E}[G^\lambda_{t+1}])(V(s_{t+1}) - \mathbb{E}[G_{t+1}])}{(V(s_{t+1}) - \mathbb{E}[G^\lambda_{t+1}])^2 + \text{Var}[G^\lambda_{t+1}]}.
\]

Proof:

\[
J(\lambda^{(t+1)}) = \mathbb{E}[G^\lambda_t - G_t]^2 = \mathbb{E}^2[G^\lambda_t - G_t] + \text{Var}[G^\lambda_t] \equiv \text{Bias}^2(\lambda^{(t+1)}) + \text{Variance}(\lambda^{(t+1)})
\]

Variance(\lambda^{(t+1)})

\[= \text{Var}[R_{t+1} + \gamma_{t+1}((1 - \lambda^{(t+1)})V(s_{t+1}) + \lambda^{(t+1)}G^\lambda_{t+1})] \quad \text{(recursive form)} \]

\[= \text{Var}[R_{t+1} + \gamma_{t+1} \text{Var}((1 - \lambda^{(t+1)})V(s_{t+1}) + \lambda^{(t+1)}G^\lambda_{t+1})] \quad (R_{t+1} \& G^\lambda_{t+1} \text{ are assumed to be uncorrelated}) \]

\[= \text{Var}[R_{t+1} + \gamma_{t+1} \lambda^{(t+1)} \text{Var}[G^\lambda_{t+1}]] \quad ((1 - \lambda^{(t+1)})V(s_{t+1}) \text{ not random}) \]

Bias(\lambda^{(t+1)}) \equiv \mathbb{E}[G^\lambda_t - G_t]

\[= \mathbb{E}[R_{t+1} + \gamma_{t+1}((1 - \lambda^{(t+1)})V(s_{t+1}) + \lambda^{(t+1)}G^\lambda_{t+1}) - (R_{t+1} + \gamma_{t+1}G_{t+1})]
\]

\[= \gamma_{t+1}((1 - \lambda^{(t+1)})V(s_{t+1}) + \gamma_{t+1}\lambda^{(t+1)}\mathbb{E}[G^\lambda_{t+1}] - \gamma_{t+1}\mathbb{E}[G_{t+1}])
\]

\[= \gamma_{t+1}[(1 - \lambda^{(t+1)})V(s_{t+1}) + \lambda^{(t+1)}\mathbb{E}[G^\lambda_{t+1}] - \mathbb{E}[G_{t+1}] - V^2(s_{t+1}) - \mathbb{E}[G^\lambda_{t+1}]\mathbb{E}[G_{t+1}]]
\]

The minimizer is achieved by setting the gradient to 0:

\[
\arg\min_{\lambda^{(t+1)}} J(\lambda^{(t+1)}) = \frac{V^2(s_{t+1}) + \mathbb{E}[G^\lambda_{t+1}][\mathbb{E}[G_{t+1}] - V(s_{t+1})(\mathbb{E}[G^\lambda_{t+1}] + \mathbb{E}[G_{t+1}])}{(V(s_{t+1}) - \mathbb{E}[G^\lambda_{t+1}])^2 + \text{Var}[G^\lambda_{t+1}]} \quad (9)
\]

\[
\blacksquare
\]

5.3 Proof of Theorem 3.1

**Theorem 5.1.** Let s’s be the states that the agent experiences when rolling out behavior policy b and \(\rho_{\text{acc}}\) be the cumulative product of importance sampling ratios from the beginning of an episode until state s. Gradient descent on the true state objectives corrected by \(\rho_{\text{acc}}\) is equivalent to stochastic gradient descent on the overall target error for target policy \(\pi\). More specifically:

\[
\nabla_{\lambda} J(G(\Lambda)) \approx \sum_{s \sim b} \rho_{\text{acc}} \cdot \nabla_{\lambda^{(t+1)}} J(s)
\]

Proof:

\[
\frac{1}{2} J(\lambda) = \sum_{s \in \mathcal{S}} d_n(s) \cdot \frac{1}{2} \mathbb{E}[G^\lambda(s) - v(s)]^2 \equiv \sum_{s \in \mathcal{S}} d_n(s) \cdot \frac{1}{2} J(\lambda(s))
\]

If we take the gradient w.r.t. \(\lambda^{(t+1)}\) we can see that:
\[ \nabla_{\lambda} J(\Lambda) = \nabla_{\lambda} \sum_{s \in S} d_{\pi}(s) \cdot J(\lambda^{(t+1)}) \]

\[ = \nabla_{\lambda} \sum_{s \in S} \sum_{k=0}^{\infty} \mathbb{P}[s_0 \rightarrow s, k, \pi, s_0 \sim d(s_0)] J(\lambda(s)) \]

\[ \mathbb{P}[\cdots] \text{ is the prob. of } s_0 \rightarrow s \text{ in exactly } k \text{ steps, } s_0 \text{ is sampled from the starting distribution } d(s_0). \]

\[ = \nabla_{\lambda} \sum_{s \in S} \sum_{k=0}^{\infty} \sum_{\tau} \mathbb{P}[s_0 \rightarrow s, k, \pi, s_0 \sim d(s_0)] J(\lambda(s)) \]

\[ \tau \text{ is a possible trajectory sampled from } s_0 \text{ and transitioning to } s \text{ with exactly } k \text{ steps} \]

\[ = \nabla_{\lambda} \sum_{s \in S} \sum_{k=0}^{\infty} \sum_{\tau} p(\tau_0, a_0, \tau_1) \pi(a_0|\tau_0) \cdots p(\tau_{k-1}, a_0, s) \pi(a_{k-1}|\tau_{k-1}) J(\lambda(s)) \]

\[ \tau_i \text{ is the } i + 1 \text{-th state of the trajectory } \tau \text{ and } p(s, a, s') \text{ is the probability of } s \rightarrow s' \text{ in the MDP} \]

\[ = \nabla_{\lambda} \sum_{s \in S} \sum_{k=0}^{\infty} \sum_{\tau} p(\tau_0, a_0, \tau_1) \pi(a_0|\tau_0) b(a_0|\tau_0) \cdots p(\tau_{k-1}, a_0, s) \pi(a_{k-1}|\tau_{k-1}) \frac{b(a_{k-1}|\tau_{k-1})}{b(a_{k-1}|\tau_{k-1})} J(\lambda(s)) \]

inject importance sampling ratios

\[ = \nabla_{\lambda} \sum_{s \in S} \sum_{k=0}^{\infty} \sum_{\tau} p(\tau_0, a_0, \tau_1) \pi(a_0|\tau_0) b(a_0|\tau_0) \cdots p(\tau_{k-1}, a_0, s) \pi(a_{k-1}|\tau_{k-1}) J(\lambda(s)) \]

\[ \rho_i \equiv \frac{\pi(a_i|\tau_i)}{b(a_i|\tau_i)} \text{ is the importance sampling ratio} \]

\[ = \nabla_{\lambda} \sum_{s \in S} \sum_{k=0}^{\infty} \sum_{\tau} \sum_{i=0}^{\infty} \rho_0 \cdot b(a_0|\tau_0) \cdots p(\tau_{k-1}, a_0, s) \pi(a_{k-1}|\tau_{k-1}) J(\lambda(s)) \]

\[ \rho_{0;\tau} \equiv \prod_{i=0}^{\tau} \rho_i \text{ is the importance sampling ratio of the trajectory } \tau \text{ from } \tau_0 \text{ until } \tau_i \]

\[ = \sum_{s \in S} \sum_{k=0}^{\infty} \rho_{0;\tau} \cdot b(a_0|\tau_0) \cdots p(\tau_{k-1}, a_0, s) b(a_{k-1}|\tau_{k-1}) J(\lambda(s)) \]

\[ \frac{1}{2} \nabla_{\lambda} J(\lambda(s)) \]

the gradient inside and \( \nabla_{\lambda} \) becomes \( \nabla_{\lambda(s)} \)

\[ \approx \sum_{s \in S} \rho_{0;\tau} \cdot \nabla_{\lambda(s)} J(\lambda(s)) \]

\[ \square \]

5.4 Experiment Details

5.4.1 Feature Engineering
We used binary features.

5.4.2 Ground Truths
We use dynamic programming to solve the ground truth values for computing the error for the ringworld tests. Frozenlake cannot be solved, since it has an episode length limit that destroys its Markov property. We do 30 billion Monte Carlo simulations to get the ground truth.
5.4.3 Reproduction of $\lambda$-greedy

Due to the instability experienced while trying to reproduce the Whites’ experiments, we have replaced GTD used in the greedy algorithm with the more stable and robust true online GTD. Also, due to unknown reasons which caused VTD performed extremely unstably, we have replaced the VTD using the direct variance estimation Bellman operator, since they are mathematically equivalent but the latter is more stable and robust empirically. These modifications to the Whites’ algorithms are expected only to improve its stability and robustness, without touching the core mechanisms of their algorithm.

5.4.4 Implementation of MTA

Due to the instability of the estimates, there could be gradient updates that causes a $\lambda$ value outside the range of $[0, 1]$. In the implementation, whenever we detect such kind of gradient descent step, we simply cancel that operation. The code is in python. The ringworld environment is reproduced and the frozenlake environment is taken directly from OpenAI Gym.

5.4.5 Experiment Platform

All the experiments are conducted on the Cedar clusters of Compute-Canada. This is made possible by the support of Mila.

5.4.6 Figure Details

We used exponentially uniform sample points from the episodes: 200 points for each run of the algorithm and then average.

5.5 Pseudocode of True Online GTD($\Lambda$)

**Algorithm 2:** True Online GTD($\Lambda$) \[9]:

\[ w(t+1) = \text{tograd}(R_{t+1}, y_{t+1}, \Lambda_t, \Lambda_{t+1}, \rho_t) \]

\[
\delta_t = R_{t+1} + \gamma y_{t+1} x_{t+1}^T w(t) - x_t^T w(t), \\
z_t = \rho_t \left[ y_t \Lambda_t z_{t-1} + \alpha_t (1 - \rho_t y_t \Lambda_t) x_t \right], \\
z_t^Y = \rho_t \left[ y_t \Lambda_t z_{t-1}^Y + x_t \right], \\
z_t^h = \rho_t \gamma y_t \Lambda_t z_{t-1}^h + \beta_t (1 - \rho_t y_t \Lambda_t) x_t^T z_{t-1}^h x_t^T, \\
w(t+1) = w(t) + \delta_t z_t + (w(t) - w(t-1))^T x_t (z_t - \alpha_t \rho_t x_t) - \alpha_t y_t (1 - \lambda(t+1)) h_t^T z_t^Y x_{t+1}, \\
h_{t+1} = h_t + \rho_t \delta_t z_t^h - \beta_t x_t^T h_t x_t; \\
\]

5.6 $\lambda$ Curves for Tests

We provide the $\lambda$ values of the starting state of the ringworld here. MTA compared to the greedy algorithm: sometimes $\lambda$ converges to zero quickly; Sometimes it bounces and stay relatively high and sometimes it chatters. We think this is a good sign since we are optimizing $\Lambda$ slowly and one $\lambda$ changes along with the changes of the others.
Figure 4: $\lambda(s_0)$ on $\gamma = 0.95$ ringworlds.