Equivariant Morita-Takeuchi Theory

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Abstract

We introduce the notion of $H$-equivariant Morita-Takeuchi theory for coalgebras with symmetries given by a Hopf algebra $H$. A cohomology theory is introduced which classifies the possible lifts of coactions on coalgebras to corresponding comodules. An equivariant Picard groupoid is defined and its connection to the developed cohomology theory investigated.

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1 Introduction

In physical theories one is interested in the classification of observable algebras only up to measuring equivalence. Measuring equivalence in a strict sense is implemented by various types of Morita equivalence, like the classical theory of Morita \[12\] where one realizes the measuring equivalence as equivalence of module categories, the more realistic approach of Rieffel \[13\] for \(C^\ast\)-algebras, or the algebraic essence of Rieffel’s theory for \(*\)-algebras by Bursztyn and Waldmann \[5, 6\].

One is interested in taking care of symmetries, as well. Probably one of the best ways to construct symmetries is by Hopf algebra actions. They incorporate symmetries modeled by groups and Lie algebras and additionally are the natural habitat of quantum groups. One then needs to lift the action of the Hopf algebra on the observable algebra to the module categories used in the various Morita theories. For the algebraic \(*\)-Morita theory this was done in \[8\] by Jansen and Waldmann in the setting of equivariant Morita theory.

Another interesting thing to study are coalgebras, the dual of algebras in the sense of inverting arrows. In the case of coalgebras one is interested in co-measuring equivalence in the sense of equivalence of comodule categories, as well. The first approaches to study this Co-Morita theory were developed by Lin \[9\] and Takeuchi \[16\]. The theory of Takeuchi turned out to be more potent than the theory of Lin. For coalgebras over rings the most reliable theory was developed by Al-Takhman \[1, 2\], which is the approach on which we rely on in this article.

To implement symmetries in the setting of coalgebras it is plausible to rely on coactions of Hopf algebras instead of actions. In Section \[2\] we develop the theory of Hopf algebra coactions in the specific way we need for equivariant Morita-Takeuchi theory. We define a cohomology theory based on the introduced schism complex which allows for the classification of possible lifts of coactions on coalgebras to coactions on corresponding comodules. Special cases of the appearing cohomology groups are the cocharacter group and the coinvariants. A notion of \(H\)-equivariant coderivations is investigated, which injects into the cocharacters by a convolution exponential. The developed cohomology theory is somewhat dual to the Sweedler cohomology. Hence it can be interpreted as the cohomology theory exponential to an analogon of the Hochschild cohomology in the coalgebra setting.

In Section \[3\] we proceed by equipping Morita-Takeuchi theory with coactions in an equivariant setting. We show that the cotensor product is compatible with \(H\)-coactions. We classify all possible Hopf algebra coactions, which give rise to an equivariant Morita-Takeuchi bicomodule, by the first cohomology group of the schism complex.

All appearing coalgebras, Hopf algebras etc. are over the unital ring \(R\), which is assumed to be commutative, of characteristic zero, a principal ideal domain, local and noetherian.

2 Coactions of Hopf algebras and their cohomology

We are interested in symmetries of rather different types, e.g. Lie algebra actions or group actions. Hence we would like to consider symmetries of high generality. To achieve this, we implement the notion of symmetry as Hopf algebras.

Recall that a Hopf algebra is an algebra which has a compatible coalgebra structure, where the unit has an inverse with respect to the convolution product, the antipode. We will denote the algebra multiplication by \(\mu\), the unit by \(\text{id}\), the comultiplication by \(\Delta\), the counit by \(\varepsilon\) and the antipode by \(S\). We will use Sweedler notation

\[\Delta(h) = h_{(1)} \otimes h_{(2)}\] (2.1)

to denote comultiplication.

In this section we investigate how Hopf algebras can be interpreted as symmetries. The classical way of thinking about symmetries is that of an action. By an action we take a point of e.g. an algebra and move it under the influence of an element of the Hopf algebra. But we could consider the dual
situation, i.e. that of a coaction of a Hopf algebra, as well. Here one is interested in how a point of a coalgebra can be reached by elements of a Hopf algebra or more figurative, how an element of a coalgebra decays.

Unlike in many parts of the literature, where one considers only module or comodule structures of a Hopf algebra as action or coaction, respectively, we want to keep track of the additional structures appearing in a Hopf algebra. That is instead of only a module we would consider additionally some flatness condition on it, which ensures compatibility with the comultiplication. And instead of only a comodule we consider some coflatness condition, which ensures compatibility with the product of the Hopf algebra. The terms flat and coflat are not to be considered in the categorical sense, but instead should be thought of as coming from differential geometry, where one considers flat connections, i.e. those giving rise to flat bundles. Thanks to the Serre-Swan theorem, flat bundles correspond to flat modules. Hence the flatness conditions ensures that we can lift the action to the Morita modules. Analogously the coflatness conditions ensures that we have coflat modules, i.e. we can lift the coaction to Morita-Takeuchi comodules.

The structure of this section is as follows. First we define the notion of a Hopf coaction, give some examples and clarify the notion of coinvariants. Then we will study the corresponding cocharacter groups, which are the analogue to character groups of actions. We show how one can associate a Lie algebra of coderivations to the cocharacter groups. Finally we include the cocharacter groups in the bigger context of a cohomology theory for Hopf algebras.

2.1 Coactions of Hopf algebras

In this subsection we investigate the notion dual to that of an action, i.e. the notion of coactions. These coactions describe how an element of a coalgebra decays. We first state the definition, which at first glance might look a bit messy, but afterwards the definition is clarified by an extended version of Sweedler notation. The axioms are just the categorical duals of that of an action. We use the flip morphism \( \tau \) with permutation in the superscript, indicating the elements which are switched.

**Definition 2.1 (Hopf coaction)** Let \( C \) be a coalgebra and \( H \) a Hopf algebra. A (right) coaction of \( H \) on \( C \) is a linear map \( \delta : C \to C \otimes H \) which satisfies the following axioms:

i.) \( C \) is a \( H \)-comodule via \( \delta \), i.e. we have

\[
(\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \Delta_H) \circ \delta
\]

and

\[
c \otimes 1 = (\text{id} \otimes \varepsilon)(\delta c)
\]

for all \( c \in C \).

ii.) The coaction \( \delta \) is coflat, i.e. we have

\[
(\text{id} \otimes \mu) \circ \tau^{(23)} \circ (\delta \otimes \delta) \circ \Delta_C = (\Delta_C \otimes \text{id}) \circ \delta
\]

and

\[
(\varepsilon_C \otimes \varepsilon_H)(\delta(c)) = \varepsilon_C(c) \otimes 1
\]

for \( c \in C \).

The definition of a left coaction is completely analogously.

**Remark 2.2** The equations stated in Definition 2.1 are very abstract. Hence we would like to concretize them using Sweedler notation. However we are dealing with different types of comultiplications and corepresentations and later we would like to have a good notation for coactions on comodules, as well. A way to solve this issue is the following. We extend the sumless Sweedler notation as follows:
for \( c \in C \) and a coaction \( \delta \) one denotes the elements coming from the coaction by upper indices, i.e. one writes
\[
\delta c = c^{(0)} \otimes c^{(1)},
\]
the zero being reserved for elements of the coalgebra. Then the axioms of Definition 2.1 can be stated as follows. For \( c \in C \) we have for part i.)
\[
c^{(0)} \otimes c^{(1)} \otimes c^{(2)} = c^{(0)} \otimes (c^{(1)})^{(1)} \otimes (c^{(1)})^{(2)},
\]
which allows us to always write the comultiplication of \( H \) in the upper indices, and
\[
c \otimes 1 = \varepsilon(c^{(0)}) \otimes \varepsilon(c^{(1)}),
\]
and for the second part one has
\[
(c^{(1)})^{(0)} \otimes (c^{(1)})^{(1)} = c^{(0)} \otimes (c^{(1)})^{(1)} \otimes c^{(1)}
\]
and
\[
\varepsilon(c^{(0)}) \otimes \varepsilon(c^{(1)}) = \varepsilon(c) \otimes 1.
\]
The axioms ensure the notation is valid, as the ambiguity of which comultiplication is used in (2.9) disappears thanks to (2.7).

We give some examples.

**Example 2.3**  

i.) The trivial coaction is given by simply tensorizing with 1, i.e. if \( C \) is a coalgebra and \( H \) any Hopf algebra then the trivial coaction of \( H \) on \( C \) is given by
\[
\delta c = c \otimes 1
\]
for all \( c \in C \).

ii.) Let \( H = RG \) be a group Hopf algebra and let \( C = \bigoplus_{g \in G} C_g \) be a \( G \)-graded coalgebra. Then the grading coaction of \( H \) on \( C \) is given by
\[
\delta c = c \otimes g
\]
for \( c \in C_g \).

iii.) Let \( H \) be a Hopf algebra. Considering the underlying coalgebra we can investigate the adjoint coaction of \( H \) on itself. The adjoint coaction is given by
\[
\delta h = h^{(2)} \otimes S(h^{(1)})h^{(3)}
\]
for all \( h \in H \). For \( H \) being a group Hopf algebra this is again the trivial coaction, since then \( \delta(h) = h \otimes h^{-1}h = h \otimes 1 \). This holds for every cocommutative Hopf algebra aswell, since one has \( \delta(h) = h^{(2)} \otimes S(h^{(1)})h^{(3)} = h^{(1)} \otimes \varepsilon(h^{(2)}) = h \otimes 1 \). For Sweedler’s Hopf algebra \( H_4 = R\{1, g, x, gx | g^2 = 1, x^2 = 0, xg = -gx\} \) one gets \( \delta g = g \otimes 1, \delta x = 1 \otimes xg + x \otimes g + g \otimes gx \), and \( \delta xg = g \otimes gx + gx \otimes g + 1 \otimes xg \).

iv.) One can consider an inner coaction, i.e. if we have an Hopf algebra \( H \), a coalgebra \( C \) and a coalgebra homomorphism \( J: C \to H \) then the inner coaction is given by
\[
\delta c = c^{(2)} \otimes S(J(c_1))J(c^{(3)})
\]
for all \( c \in C \).
As always we need a corresponding notions of morphisms, so we define equivariant coalgebra morphisms:

**Definition 2.4 (H-equivariant coalgebra homomorphism)** Let $H$ be a Hopf algebra coacting on the coalgebras $C$ and $D$. A coalgebra homomorphism $\Phi : C \to D$ is called $H$-equivariant if

$$\phi(c^{(0)}) \otimes c^{(1)} = \phi(c^{(0)}) \otimes \phi(c^{(1)})$$

(2.15)

for all $c \in C$.

Analogously to the invariants of an action one defines the coinvariants of a coaction as those elements which are affected by the coaction like under the trivial coaction.

**Definition 2.5 (Coaction coinvariants)** Let $C$ be a coalgebra with coaction $\delta$ of $H$. The set of coinvariants is denoted by

$$C^{c_{o}H} = \{c \in C \mid \delta(c) = c \otimes 1\}.$$  

(2.16)

From the definition we can deduce a formula for the coaction interacting with the antipode of the Hopf algebra.

**Lemma 2.6** Let $C$ be a coalgebra carrying a coaction of a Hopf algebra $H$. Then

$$c^{(1)} \otimes c^{(0)} \otimes S(c^{(1)}) = c^{(0)} \otimes c^{(0)} \otimes S(c^{(1)})$$

(2.17)

and

$$c^{(0)} \otimes c^{(0)} \otimes c^{(1)} = c^{(1)} \otimes c^{(0)} \otimes c^{(1)} S(c^{(2)})$$

(2.18)

hold for all $c \in C$.

**Proof:** This follows by the following calculation

$$c^{(0)} \otimes c^{(0)} \otimes S(c^{(1)}) \equiv c^{(0)} \otimes c^{(0)} \otimes S(c^{(1)}) c^{(1)} c^{(2)} \equiv c^{(0)} \otimes c^{(0)} \otimes c^{(1)} c^{(1)} c^{(2)} \equiv c^{(0)} \otimes c^{(0)} \otimes c^{(1)} c^{(1)} c^{(2)}$$

where we used counitarity, the properties of the antipode and in ($\ast$) the compatibility with the Hopf product (2.9). The second equation follows completely analogously.

### 2.2 Cocharacter groups

In the representation theory of groups the *characters* play a fundamental role. For Lie groups also the notion of an infinitesimal character (or derivations) is useful. We are interested in a theory of *cocharacters*, i.e. the theory dual to that of characters. This theory will play an important role in characterizing the kernel of the forgetful morphism from the equivariant Picard group to the standard Picard group. In this section we make use of the characteristic zero condition on $R$, as we need to be able to divide through integers and hence $\mathbb{Q} \subseteq R$ is crucial.

A nice investigation of character groups of Hopf algebras with trivial action is given in [4], the equivariant version, even for a noncommutative target algebra, was introduced in [8]. Basically, one considers the algebra $\text{Hom}(H, \mathfrak{A})$ of linear morphisms from the Hopf algebra $H$ to the algebra $\mathfrak{A}$ with product being the convolution product and finds suitable subgroups of this algebra as character groups.
We now go the other way round and pair a coalgebra \( C \) with a Hopf algebra \( H \), that is we consider the set \( \text{Hom}(C, H) \) with convolution product. Recall that for the set of linear maps from a coalgebra \( C \) to an algebra \( \mathcal{A} \) one has the convolution product
\[
f \ast g = \mu \circ (f \otimes g) \circ \Delta
\]
for \( f, g \in \text{Hom}(C, \mathcal{A}) \). This convolution product endows \( \text{Hom}(C, \mathcal{A}) \) with the structure of an algebra. For a Hopf algebra \( H \) one can apply this on both sides, since \( H \) has a coalgebra and an algebra structure.

The first part is remodelling the group of which the character groups are subgroups. We have to replace the unit by the counit. We call a morphism \( \phi \in \text{Hom}(C, H) \) unitary if it satisfies \( \varepsilon_H \circ \phi = \varepsilon_C \). We set
\[
\Gamma_0 = \{ \phi \in \text{Hom}(C, H) \mid \phi \text{ is counitary and convolution invertible} \}
\]
This is completely dual to the situation in the definition of unitary characters as the unitarity condition \( \phi(1) = 1 \) is actually \( \phi \circ \text{unit}_H = \text{unit}_{\mathcal{A}} \) due to linearity. The corresponding Lie algebra is
\[
c_0 = \{ \phi \in \text{Hom}(C, H) \mid \varepsilon_H \circ \phi = 0 \}.
\]
Obviously \( \Gamma_0 \) forms a group with respect to the convolution product \( \ast \) and \( c_0 \) a Lie algebra with Lie bracket being the commutator to the convolution product. The \( c_0 \) injects into \( \Gamma_0 \) via the convolution exponential \( \exp_x(\phi) = \sum_{k \geq 0} \frac{\phi^k}{k!} \) if \( H \) is a filtered Hopf algebra. Recall that a filtration of a Hopf algebra \( H \) is a filtration of the underlying \( R \)-module, i.e. one has submodules \( H^0 \subseteq H^1 \subseteq \cdots \subseteq H^n \subseteq \cdots \) with \( \bigcup_{n \geq 0} H^n = H \), such that one has \( \mu(H^p \otimes H^q) \subseteq H^{p+q} \), \( \Delta(H^n) \subseteq \sum_{p+q=n} H^p \otimes H^q \), and \( S(H^n) \subseteq H^n \). As example of a filtered Hopf algebra consider the following construction of the coradical filtration. Consider \( H \) as \( (H, H) \)-bicomodule and let \( H^0 = \text{soc} \; H \) be the coradical, i.e. the sum of all simple subcoalgebras of \( H \). If the coradical \( H^0 \) is a Hopf subalgebra then one can recursively define a filtration of \( H \) via
\[
H^n = \Delta^{-1}(H \otimes H^0 + H^{n-1} \otimes H)
\]
which is a Hopf algebra filtration, see [10] Thm. II.2.2, Rem. 1]. The condition that \( H^0 \) is a Hopf subalgebra is for example fulfilled for every pointed Hopf algebra, i.e. Hopf algebras whose simple Hopf subalgebras are all one-dimensional. These include all cocommutative Hopf algebras, e.g. the group algebras and universal enveloping algebras of Lie algebras.

Let us denote by \( \delta \) the coaction of \( H \) on the coalgebra \( C \). We adapt the extended Sweedler notation from Remark 2.2.

**Definition 2.7 (\( H \)-equivariant cocharacters)** An element \( \phi \in \text{Hom}(C, H) \) is called \( H \)-equivariant cocharacter if it is convolution invertible and satisfies the following axioms:

i.) \( \varepsilon_H \circ \phi = \varepsilon_C \),

ii.) \( \phi(c^{(1)}) \otimes \phi(c^{(2)}) = \phi(c^{(0)}) \otimes \phi(c^{(1)}) \),

iii.) \( c^{(0)}(1) \otimes c^{(1)} \circ \phi(c^{(2)}) = c^{(0)}(1) \otimes \phi(c^{(1)})c^{(1)} \),

for all \( c \in C \). The set of all \( H \)-equivariant cocharacters is denoted by
\[
\Gamma(C \otimes H) = \{ \phi \in \text{Hom}(C, H) \mid \phi \text{ is } H \text{-equivariant cocharacter} \}.
\]

Note that the conditions can be phrased completely in the terms of morphisms as follows: for \( \mathcal{A} \) one has
\[
\Delta_H \circ \phi = (\text{id} \otimes \mu) \circ (\phi \otimes \text{id} \otimes \phi) \circ (\delta \otimes \text{id}) \circ \Delta_C
\]
and for \((\text{iii})\) one has

\[
(id \otimes \mu) \circ (\delta \otimes \phi) \circ \Delta_C = (id \otimes \mu) \circ \tau^{(12)} \circ (\phi \otimes \delta) \circ \Delta_C.
\] (2.25)

These conditions arise by dualizing the conditions of [8] Definition A.1. We call condition [\(\text{iv}\)] counit condition, condition [\(\text{ii}\)] the coaction condition and condition [\(\text{iii}\)] the comodule condition.

**Proposition 2.8** The set \(\Gamma(C \odot H)\) forms a group. The inverse of \(\phi \in \Gamma(C \odot H)\) is given by

\[
\phi^{*-1}(c) = S(\phi(c(0)))c(1).
\] (2.26)

**PROOF:** Let \(\phi, \psi \in \Gamma(C \odot H)\). We have to check that \(\phi \ast \psi \in \Gamma(C \odot H)\). The counit condition is clear. For the coaction condition we have for \(c \in C\)

\[
(\phi \ast \psi)(c)(1) \otimes (\phi \ast \psi)(c)(2) = \phi(c(1))(\psi(c(2))(1) \otimes \phi(c(1))(2) \otimes \psi(c(2))(1) \otimes \psi(c(2))(2))
\]

\[
= (\mu \otimes \mu) \circ \tau^{(23)}(\phi(c(1))(1) \otimes \phi(c(1))(2) \otimes \psi(c(2))(1) \otimes \psi(c(2))(2))
\]

\[
= \phi(c(0))(\psi(c(0))(1) \otimes c(1)(\phi(c(2)) \otimes \psi(c(3))(1) \otimes c(1)(\psi(c(4))))
\]

\[
= \phi(c(0))(\psi(c(0))(1) \otimes c(1)(\phi(c(2)) \otimes \psi(c(3))(1) \otimes c(1)(\psi(c(4))))
\]

\[
= (\phi \ast \psi)(c(1))(1) \otimes (\phi \ast \psi)(c(1))(2)
\]

where in \((a)\) we used the coaction condition \([\text{iv}]\) for \(\phi\) and \(\psi\), in \((b)\) the comodule condition \([\text{iii}]\), and in \((c)\) we used \([\text{ii}]\). Checking the comodule condition is comparatively easy:

\[
c(0)(1) \otimes c(1)(\phi \ast \psi)(c)(2) = c(0)(1) \otimes c(1)(\phi(c(2)) \otimes \psi(c(3)))
\]

\[
\overset{\text{iii}}{\rightarrow} c(0)(2) \otimes \phi(c(1))c(1)(\psi(c(3)))
\]

\[
\overset{\text{iii}}{\rightarrow} c(0)(3) \otimes \phi(c(1))\psi(c(2))c(1)(3)
\]

\[
= c(0)(2) \otimes (\phi \ast \psi)(c(1))(2)
\]

for \(c \in C\).

The candidate for the inverse satisfies

\[
\phi^{*-1} \ast \phi(c) = S(\phi(c(0)))c(1)(\phi(c(2)))
\]

\[
\overset{\text{a}}{\rightarrow} S(\phi(c(1))c(2))
\]

\[
\overset{\text{b}}{\rightarrow} c(\phi(c(1)))
\]

\[
\overset{\text{c}}{\rightarrow} c(c),
\]

where in \((a)\) we used the coaction condition of \(\phi\), in \((b)\) the fact that \(S\) is the convolution inverse to the identity, applied via \(S \ast \text{id}(\phi(c))\), and in \((c)\) we used the counitarity of \(\phi\). Now we have to check that \(\phi^{*-1} \in \Gamma(C \odot H)\). The counitarity of \(\phi^{*-1}\) is clear by definition. For the comodule condition, we calculate

\[
c(0)(1) \otimes c(1)(\phi^{*-1}(c)(2)) = c(0)(1) \otimes c(1)(S(\phi(c(2)))c(1)(2))
\]

\[
\overset{\text{a}}{\rightarrow} c(0)(1) \otimes c(1)(S(\phi(c(2)))c(1)(2))
\]

\[
= c(0)(1) \otimes c(1)(S(\phi(c(2)))S(c(1))(c(1)))
\]

\[
\overset{\text{a}}{\rightarrow} c(0)(1) \otimes c(1)(S(\phi(c(2)))S(c(1))(c(1)))
\]

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\[\begin{align*}
&\left\{ c^{(1)}(0) \circ \phi(c^{(1)}(0)) \right\}_c(1) + S(c^{(1)}(0) + c^{(1)}(0))_c(2) + \phi(c^{(1)}(0)) + c^{(1)}(1)_c(2)
&= c^{(1)}(0) \circ \phi(c^{(1)}(0)) + c^{(1)}(0) + c^{(1)}(0))_c(2) + \phi(c^{(1)}(0)) + c^{(1)}(1)_c(2)
&= c^{(1)}(0) \circ \phi(c^{(1)}(0)) + c^{(1)}(0) + c^{(1)}(0))_c(2) + \phi(c^{(1)}(0)) + c^{(1)}(1)_c(2)
&= c^{(1)}(0) \circ \phi(c^{(1)}(0)) + c^{(1)}(0) + c^{(1)}(0))_c(2) + \phi(c^{(1)}(0)) + c^{(1)}(1)_c(2)
&= c^{(1)}(0) \circ \phi(c^{(1)}(0)) + c^{(1)}(0) + c^{(1)}(0))_c(2) + \phi(c^{(1)}(0)) + c^{(1)}(1)_c(2)
&= c^{(1)}(0) \circ \phi(c^{(1)}(0)) + c^{(1)}(0) + c^{(1)}(0))_c(2) + \phi(c^{(1)}(0)) + c^{(1)}(1)_c(2)
&= c^{(1)}(0) \circ \phi(c^{(1)}(0)) + c^{(1)}(0) + c^{(1)}(0))_c(2) + \phi(c^{(1)}(0)) + c^{(1)}(1)_c(2)
&= c^{(1)}(0) \circ \phi(c^{(1)}(0)) + c^{(1)}(0) + c^{(1)}(0))_c(2) + \phi(c^{(1)}(0)) + c^{(1)}(1)_c(2)
&= c^{(1)}(0) \circ \phi(c^{(1)}(0)) + c^{(1)}(0) + c^{(1)}(0))_c(2) + \phi(c^{(1)}(0)) + c^{(1)}(1)_c(2)
&= c^{(1)}(0) \circ \phi(c^{(1)}(0)) + c^{(1)}(0) + c^{(1)}(0))_c(2) + \phi(c^{(1)}(0)) + c^{(1)}(1)_c(2)
&= c^{(1)}(0) \circ \phi(c^{(1)}(0)) + c^{(1)}(0) + c^{(1)}(0))_c(2) + \phi(c^{(1)}(0)) + c^{(1)}(1)_c(2)
\end{align*}\]

where in (a) we used the identity \(2.17\), in (b) we used the properties of the antipode, in (c) the comodule condition for \(\phi\) and in (d) the compatibility of the coaction with product of the Hopf algebra.

Finally we have to check the coaction condition. Here we have

\[\phi^*\left( c^{(1)}(0) \circ \phi(c^{(1)}(0)) \right) = S(c^{(1)}(0)) c^{(1)}(1) + S(c^{(1)}(0)) c^{(1)}(2) + \phi(c^{(1)}(0)) + c^{(1)}(1)_c(2)\]

where we used in (a) the fact that \(\text{id} \ast S = e\), whence \(\varepsilon \circ \Delta = (\text{id} \otimes S) \circ \Delta\), in (b) the coaction condition for \(\phi\) and that \(S\) is a coalgebra homomorphism, and the properties of antipode and counit. \(\Box\)

**Example 2.9** Consider the a group-like coalgebra \(R[S]\) for some set \(S\) and a Hopf algebra \(H\). Assume the trivial coaction \(\delta: R[S] \to R[S] \otimes H\) given by \(\delta s = s \otimes 1\). Then the cocharacter group \(\Gamma(C \otimes H)\) is given by those morphisms \(\phi \in \text{Hom}(C, H)\) which send group-like elements to group-like elements, as the coaction condition in this case reads

\[\phi(s)^{(1)} \otimes \phi(s)^{(2)} = \phi(s) \otimes \phi(s)\] (2.27)

for all \(s \in S\).

There is a corresponding infinitesimal notion of coderivations:

**Definition 2.10 (\(H\)-equivariant coderivations)** An element \(\phi \in \text{Hom}(C, H)\) is called \(H\)-equivariant coderivation if it is convolution invertible and satisfies the following:

\[\begin{align*}
i.) & \quad \varepsilon_H \circ \phi = 0 \\
ii.) & \quad \phi(c)^{(1)} \otimes \phi(c)^{(2)} = c^{(1)}(0) \otimes c^{(1)}(1) \phi(c^{(2)}) + \phi(c^{(0)}) \otimes c^{(1)}(1) c^{(2)} \\
iii.) & \quad c^{(0)}(0) \otimes c^{(1)}(1) \phi(c^{(2)}) = c^{(0)}(2) \otimes \phi(c^{(1)}) c^{(2)} \\
& \quad \text{for all } c \in C. \text{ The set of all } H\text{-equivariant coderivations is denoted by } \mathfrak{c}(C \otimes H).\]

**Proposition 2.11** If \(H\) is filtered, the set \(\mathfrak{c}(C \otimes H)\) injects into \(\Gamma(C \otimes H)\) via \(\exp(\phi) = \sum_{k=0}^{\infty} \frac{\phi^k}{k!}\).
**Proof:** The filtration of $H$ is needed to ensure convergence of the exponential map.

We first have to show that

$$
\phi^{*n}(c)^{(1)} \otimes \phi^{*n}(c)^{(2)} = \sum_{k=0}^{n} \binom{n}{k} \phi^{*k}(c^{(0)}_{(1)}) \otimes c^{(1)}_{(1)} \phi^{*(n-k)}(c^{(2)})
$$

holds for all $\phi \in \mathfrak{c}(C \otimes H)$ and all $c \in C$. This is done by induction, for $n = 1$ it is just the infinitesimal coaction condition, and for $n \to n + 1$ we have

$$
\Delta_H \circ \phi^{*n+1}(c) = \mu(\Delta_H(\phi(c^{(1)}))) \otimes \Delta_H(\phi^{*n}(c^{(2)})))
$$

$$
= \sum_{k=0}^{n} \binom{n}{k} (\phi^{*k}(c^{(0)}_{(1)}) \otimes c^{(1)}_{(1)} \phi^{*(n+1-k)}(c^{(2)})) + \phi^{*n+1}(c^{(0)}_{(1)}) \otimes c^{(1)}_{(1)} \phi^{*(n-k)}(c^{(2)})
$$

$$
= \sum_{k=0}^{n} \binom{n}{k} \phi^{*k}(c^{(0)}_{(1)}) \otimes c^{(1)}_{(1)} \phi^{*(n+1-k)}(c^{(2)}) + \sum_{k=0}^{n+1} \binom{n}{k} \phi^{*k}(c^{(0)}_{(1)}) \otimes c^{(1)}_{(1)} \phi^{*(n-k)}(c^{(2)})
$$

$$
= \phi^{*n+1}(c^{(2)}) + \sum_{k=0}^{n+1} \binom{n+1}{k} \phi^{*k}(c^{(0)}_{(1)}) \otimes c^{(1)}_{(1)} \phi^{*(n+1-k)}(c^{(2)}) + \phi^{*n+1}(c^{(0)}_{(1)})
$$

$$
= \sum_{k=0}^{n+1} \binom{n+1}{k} \phi^{*k}(c^{(0)}_{(1)}) \otimes c^{(1)}_{(1)} \phi^{*(n+1-k)}(c^{(2)}).
$$

This now allows to show that the exponential of the infinitesimal coaction condition is indeed the coaction condition:

$$
\exp(\phi)(c)^{(1)} \otimes \exp(\phi)(c)^{(2)} = \sum_{n \geq 0} \frac{1}{n!} \phi^{*n}(c)^{(1)} \otimes \phi^{*n}(c)^{(2)}
$$

$$
= \sum_{n \geq 0} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \phi^{*k}(c^{(0)}_{(1)}) \otimes c^{(1)}_{(1)} \phi^{*(n-k)}(c^{(2)})
$$

$$
= \sum_{k=0}^{\infty} \frac{1}{k!(n-k)!} \phi^{*k}(c^{(0)}_{(1)}) \otimes c^{(1)}_{(1)} \phi^{*(n-k)}(c^{(2)})
$$

$$
= \exp(\phi)(c^{(0)}_{(1)}) \otimes \exp(\phi)(c^{(2)}).
$$

The counit and comodule condition follow immediately, hence the injection is clear.

\[ \square \]

### 2.3 The schism complex

Recall that the first Hochschild cohomology group classifies the derivations up to inner derivations. Hence one might wonder, if one can define a cohomology theory for Hopf algebras, which has the character groups as 1-cocycles and is sort of exponential to the Hochschild cohomology as we can exponentiate the derivations to the characters via the convolution exponential. This is done for cocommutative Hopf algebras with coefficients in a commutative algebra via *Sweedler cohomology* see [13]. In the noncommutative setting there are attempts to generalize this cohomology theory. The definition of the first noncommutative cohomology group is straight-forward but the second
the zeroth homoschism group will be defined to coincide with the coinvariants of the Hopf coaction on the coalgebra.

We first define the general zeroth and first homoschism group, and then embedd them in the commutative, cocommutative case into a bigger setting somewhat dual to that of Sweedler cohomology for Hopf algebras [15]. The definition of the noncommutative homoschism groups is motivated by the definition of the zeroth and first noncommutative homology groups of Hopf algebras.

Recall that a coalgebra morphism $f: C \to D$ is cocentral if it cocommutates with the identity, that is $f(c_{(1)} \otimes c_{(2)}) = f(c_{(2)}) \otimes c_{(1)}$ for all $c \in C$. The cocenter of a coalgebra is the subcoalgebra missing the elements which spoil cocommutativity, i.e. $\mathcal{Z}(C) = C/ \ker f$ see [17] for a more concrete construction.

**Definition 2.12 (Noncommutative homoschism groups)** Let $H$ be a Hopf algebra and let $C$ be a coalgebra with coaction of $H$. The zeroth homoschism group of $H$ with covalues in $C$ is the group of coinvariants

$$S^0(H, C) = \mathcal{Z}(C)^{\text{co}H}.$$ (2.28)

The first homoschism group of $H$ with covalues in $C$ is the group of cocharacters modulo the inner ones

$$S^1(H, C) = \Gamma(C \odot H)/\mathcal{Z}(C),$$ (2.29)

where $\hat{\cdot}: \mathcal{Z}(C) \to \Gamma(C \odot H)$ is given by

$$\hat{\cdot} = (\delta_c \otimes \text{id}) \circ \delta \ast (\delta_c \otimes 1)^{-1}.$$ (2.30)

for all $c \in \mathcal{Z}(C)$ and $\delta_c \in \mathcal{Z}(C)^*$. is the corresponding delta functional.

One can mod out the image of the cocenter, since by dualizing one gets a central subgroup. As we will see later in Section 3 the first Homoschism group allows to classify coaction lifts to comodules.

We are now interested embedding these two groups, at least in the commutative, cocommutative setting, into a larger theory. For the Sweedler cohomology one defines various groups of linear maps going from tensor powers of the Hopf algebra to the algebra. So the idea is to define groups of linear maps going from the coalgebra to tensor powers of the Hopf algebra. These tensor powers of the Hopf algebra carry an algebra structure.

Recall that the differential for the Sweedler cohomology is defined by successively combining elements of the Hopf algebra in various manners and then sending it to the algebra. As the name schism complex suggests, one would suspect to have the differential between the groups defined by successively splitting the coalgebra elements and then sending them to the Hopf algebra. For the splitting process we can either use the coaction or the comultiplication of the Hopf algebra. Hence it is natural to consider all possible combinations of these.

We justified the following definition.

**Definition 2.13 (Schism complex)** Let $H$ be a commutative Hopf algebra and $C$ be a cocommutative coalgebra. The schism complex consists of the sequence $\Gamma^*_{0}(C, H)$ of groups defined by $\Gamma^*_{0}(C, H) = \Gamma_{0}(C, H^{\otimes q})$ and the differential, which is defined for $d^{q-1}: \Gamma^{q-1}_{0}(C, H) \to \Gamma^{q}_{0}(C, H)$ by

$$d^{q-1}(f) = ((f \otimes \text{id}) \circ \delta) \ast ((\Delta_H \otimes \text{id} \otimes \ldots \otimes \text{id}) \circ f)^{-1} \ast ((\text{id} \otimes \Delta_H \otimes \text{id} \otimes \ldots \otimes \text{id}) \circ f)\ast$$
We want to show, that we can apply this to each term separately, i.e. one has $d(f \ast g) = d(f) \ast d(g)$ for all $f, g \in \Gamma^q_0(C \circ H)$, $q \geq 1$. Since $\Gamma^0_0(C, H)$ is abelian, the only problematic part is the one concerning the coaction. So let $f, g \in \Gamma^q_0(C, H)$ and $c \in C$ then

$$((f * g) \otimes \text{id})(c^{(0)} \otimes c^{(1)}) = f(c^{(0)}_{(1)})g(c^{(0)}_{(2)}) \otimes c^{(1)}_{(1)} c^{(1)}_{(2)}$$

where in $(*)$ we used the coflatness of the coaction.

Now we can check $d^q \circ d^{q-1} = e$. So let $f \in \Gamma^q_0(C, H)$ then one has

$$d^{q-1}(f) = (((f \otimes \text{id}) \circ \delta) \ast ((\Delta_H \otimes \text{id} \otimes \ldots \otimes \text{id}) \circ f)^{\ast -1} \ast ((\text{id} \otimes \Delta_H \otimes \text{id} \otimes \ldots \otimes \text{id}) \circ f) \ast \ldots \ast ((\text{id} \otimes \ldots \otimes \text{id} \otimes \Delta_H) \circ f)^{\ast \pm 1} \ast (f \otimes 1)^{\ast \mp 1}.$$ 

Thanks to $d(f \ast g) = d(f) \ast d(g)$, we can now apply $d^q$ to each term separately and get for the first term, using the notation $\Delta_H^i = (\text{id} \otimes \ldots \otimes \Delta_H \otimes \ldots \otimes \text{id})$ with $\Delta_H$ at the $i$-th position,

$$d^q((f \otimes \text{id}) \circ \delta) = (((f \otimes \text{id}) \circ \delta) \otimes (\Delta_H^i \circ ((f \otimes \text{id}) \circ \delta))^{\ast -1} \ast \ldots \ast (\Delta_H^i \circ ((f \otimes \text{id}) \circ \delta))^{\ast \mp 1} \ast (((f \otimes \text{id}) \circ \delta) \otimes 1)^{\ast \mp 1}$$

for the $i$-th middle term

$$d^q((\Delta_H^i \circ f)^{\ast \pm 1}) = (((\Delta_H^i \circ f)^{\ast \pm 1} \otimes \text{id}) \circ \delta) \ast ((\Delta_H^i \circ f)^{\ast \pm 1})^{\ast -1} \ast \ldots \ast ((\Delta_H^i \circ f)^{\ast \pm 1} \otimes 1)^{\ast \pm 1}$$

and for the end term

$$d^q((f \otimes 1)^{\ast \mp 1}) = (((f \otimes 1)^{\ast \pm 1} \otimes \text{id}) \circ \delta) \ast ((\Delta_H^i \circ f \otimes 1)^{\ast \pm 1})^{\ast -1} \ast \ldots \ast ((\Delta_H^i \circ f \otimes 1)^{\ast \pm 1} \otimes 1)^{\ast \pm 1}$$

Now with shrewd eyes one sees that for each term there is one with the opposite sign, hence we get $d \circ d = e$. \qed

Next we investigate the cohomology groups of the schism complex.

**Definition 2.15 (Homoschism groups)** The homoschism groups for a commutative Hopf algebra $H$ with covalues in a cocommutative coalgebra $C$ is defined by

$$S^0(C, H) = \ker d^0$$

and

$$S^q(C, H) = \ker d^q / \text{im } d^{q-1}$$

for $q \geq 1$. 

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We have to check that in the commutative case both definitions of $S^0(H,C)$ and $S^1(H,C)$ coincide.

**Proposition 2.16** The definitions of the homoschism groups given in Definition 2.12 and Definition 2.15 coincide in the commutative, cocommutative case.

**Proof:** The kernel of $d^0$ is given by the coinvariants as $d^0(f)(c) = (f \otimes \text{id})\delta(c) = f(c) \otimes 1$ says $f(c^{(0)}) \otimes c^{(1)} = f(c) \otimes 1$ and hence $\delta(c) = c \otimes 1$ follows, which is the definition of the coinvariants.

For the first group, we use commutativity of $H$ and cocommutativity of $C$, to see that $\ker d^1$ is given by elements of the form

$$(\text{id} \otimes \mu) \circ (f \otimes \text{id} \otimes f) \circ (\delta \otimes \text{id}) \circ \Delta_C = \Delta_H \circ f,$$

where we also used the unitarity of $1$. So it coincides with the cocharacter group, as the comodule condition in this case is for free. The image of $d^0$ are the inner cocharacters, i.e. those given by delta functionals. \(\square\)

### 3 Equivariant Morita-Takeuchi theory

In noncommutative algebra it is a classical observation that not the isomorphisms of algebras give rise to the most interesting category but it is more convenient to extend the notion of morphism and consider equivalence of module categories. Morita [12] showed that one can describe these equivalences via certain bimodules, now termed Morita bimodules. For coalgebras attempts to consider analogous notions where first done by Lin [9] and more fruitfully by Takeuchi [16]. The later approach, now termed Morita-Takeuchi theory, turned out to be more convenient one and was extended by Altakhman to coalgebras over rings [1].

Having a general notion of cosymmetry via the coaction of an Hopf algebra it is natural to wonder how Morita-Takeuchi equivalence changes in the presence of cosymmetries. Therefore we consider in this section the notion of equivariant Morita-Takeuchi equivalence. This section gives a justification for introducing the cocharacter groups and the Schism complex, aswell, since these cohomology groups model obstructions of lifting coactions on coalgebras to the corresponding comodules. This shows that equivariant Morita-Takeuchi theory is equivalently well-behaved as equivariant Morita theory, cf. [8].

We recall the notion of an equivariant comodule.

**Definition 3.1 (H-equivariant comodule)** Let $H$ be a Hopf algebra and $C$ be a coalgebra with coaction $\delta$ of $H$. A (right) $C$-comodule $M$ is called $H$-equivariant comodule, if $\delta$ lifts to $M$, i.e. there is a linear map $\delta: M \to M \otimes H$, which turns $M$ into a $H$-comodule and additionally satisfies

$$m^{(0)}_{(0)} \otimes m^{(0)}_{(1)} \otimes m^{(1)} = m^{(0)}_{(0)} \otimes m^{(0)}_{(1)} \otimes m^{(1)}_{(0)} m^{(1)}_{(1)}$$

(3.1)

for all $m \in M$.

The definition of a left equivariant comodule is analogously.

We need to check that equivariance is compatible with cotensorizing comodules. For this recall that an $H$-module $M$ is $H$-pure in an $H$-module $N$ if the natural map $M \otimes X \to N \otimes X$ is injective for any $H$-module $X$. We remind the reader that the cotensor product of two comodules is given by the equalizer of the coactions, i.e. if $M, N$ are right and left $C$-comodules, respectively, then the cotensor product $M \boxtimes_C N$ is given by

$$M \boxtimes_C N \mathrel{\mathop{\longrightarrow}^{\rho_M \otimes \text{id}}_{\text{id} \otimes \rho_N}} M \otimes_R N \mathrel{\longrightarrow} M \otimes_R C \otimes_R N,$$

(3.2)

where we denote by $\rho_M, \rho_N$ the corresponding coaction.
Then the cotensor product 

\[ (m \square n)^{(0)} \otimes (m \square n)^{(1)} = m^{(0)} \square n^{(0)} \otimes m^{(1)} n^{(1)} \]  

for \( m \in M, n \in N \).

**Proof:** Note that \( H \)-purity is needed to make the definition well-defined as this gives the associativity up to isomorphism of the combination of the cotensor with the tensor product, see [1, Lemma 2.3]. Let \( m \in M, n \in N \), then

\[
(m \square n)^{(0)} \otimes (m \square n)^{(0)} \otimes (m \square n)^{(1)} = (m \square n)^{(0)} \otimes n^{(0)} \otimes m^{(1)} n^{(1)}
\]

\[
= m^{(0)} \otimes n^{(0)} \otimes n^{(0)} \otimes m^{(1)} n^{(1)}
\]

\[
= (\text{id}^{(3)} \otimes \mu) \circ \tau^{(23)(45)} (m^{(0)} \otimes n^{(0)} \otimes n^{(0)} \otimes n^{(1)} \otimes m^{(1)})
\]

\[
= (\text{id}^{(3)} \otimes \mu) \circ \tau^{(23)(45)} (m^{(0)} \otimes n^{(0)} \otimes n^{(0)} \otimes n^{(1)} n^{(1)} \otimes m^{(1)})
\]

\[
= m^{(0)} \otimes n^{(1)} \otimes n^{(0)} \otimes m^{(1)} n^{(1)} n^{(1)}
\]

\[
= m^{(0)} \otimes n^{(1)} \otimes n^{(0)} \otimes m^{(1)} n^{(1)} n^{(1)}
\]

\[
= (\text{id}^{(3)} \otimes \mu) \circ \tau^{(34)} (m^{(0)} \otimes m^{(1)} \otimes m^{(1)} \otimes n^{(0)} \otimes n^{(0)} n^{(1)} n^{(1)})
\]

\[
= (\text{id}^{(3)} \otimes \mu) \circ \tau^{(34)} (m^{(0)} \otimes m^{(1)} \otimes m^{(1)} n^{(1)} \otimes n^{(0)} \otimes n^{(0)} n^{(1)} n^{(1)})
\]

\[
= (m \square n)^{(0)} \otimes (m \square n)^{(0)} \otimes (m \square n)^{(1)} (m \square n)^{(1)}
\]

and we have

\[
(m \square n)^{(0)} \otimes (m \square n)^{(1)} \otimes (m \square n)^{(2)} = m^{(0)} \square n^{(0)} \otimes m^{(1)} n^{(1)} \otimes m^{(2)} n^{(2)}
\]

\[
= m^{(0)} \otimes n^{(0)} \otimes (m^{(1)} n^{(1)}) \otimes (m^{(1)} n^{(1)})
\]

\[
= (m \square n)^{(0)} \otimes ((m \square n)^{(1)}) \otimes ((m \square n)^{(1)})
\]

hence the lemma is proved since \( M \square N \) is a bicomodule as cotensor product of bicomodules.

We also need a corresponding notion of morphisms and isomorphisms.

**Definition 3.3 (H-equivariant comodule morphism)** Let \( C \) be a coalgebra with \( H \)-coaction and let \( M, N \) be equivariant comodules over \( C \). A comodule morphism \( \phi: M \to N \) is called \( H \)-equivariant if

\[
\phi(m)^{(0)} \otimes \phi(m)^{(1)} = \phi(m^{(0)}) \otimes m^{(1)}
\]  

for all \( m \in M \). Two \( H \)-equivariant comodules \((M, \delta), (N, \tilde{\delta})\) are isomorphic, if there exists a \( \phi \in \text{Iso}(M, N) \) such that \( \tilde{\delta} = \delta_{\phi} \) with

\[
\delta_{\phi}(n) = \phi(\phi^{-1}(n)^{(0)}) \otimes \phi^{-1}(n)^{(1)}
\]  

for \( n \in N \).

It is obvious that the defined coaction is actually a coaction.

As we are interested in an equivariant Morita-Takeuchi theory, we define now equivariant Morita-Takeuchi bicomodules. Recall from [1] that for coalgebras over rings the equivalence of their comodule
categories is given by Morita-Takeuchi bicomodules. For the definition of a Morita-Takeuchi bicomodule recall that a \((C, D)\)-bicomodule \(M\) is called quasi-finite, if the functor \(-\Box\) : \(\text{CoMod}-C \to \text{CoMod}-D\) between the comodule categories of \(C\) and \(D\) has a left adjoint, which will be denoted by \(\text{CoHom}(M, -)\), see [2] Theorem 3.6 for the justification of this unfamiliar definition of quasi-finiteness. If the cohom functor \(\text{CoHom}(M, -)\) is exact, one calls \(M\) an injector. The coendomorphism coalgebra is \(\text{CoEnd}(M) = \text{CoHom}(M, M)\). A right \(C\)-comodule is called faithfully coflat if the functor \(\text{CoMod}-C \to R\text{-Mod}\) is exact and faithful.

For two coalgebras \(C\) and \(D\) a Morita-Takeuchi bicomodule is a \((C, D)\)-bicomodules \(M\), which is quasi-finite, faithfully coflat, an injector in \(\text{CoHom} \) and is compatible with the coaction on the coalgebra \(M\), \(\delta\). An equivariant \(\text{Morita-Takeuchi bicomodule}\) is called equivariant Morita-Takeuchi bicomodule if it is quasi-finite, faithfully coflat and an injector in \(\text{CoHom}\). In the sequel we denote by the subscript \(-_H\) the corresponding notion in the category of \(H\)-equivariant bicomodules.

**Definition 3.4 \((H\text{-equivariant Morita-Takeuchi bicomodule})\)** Let \(C\) and \(D\) be coalgebras with \(H\)-coaction. An equivariant \((C, D)\)-bicomodule is called equivariant Morita-Takeuchi bicomodule if it is quasi-finite, faithfully coflat, an injector in \(\text{CoMod}_H\) and \(\text{CoEnd}_H(M) \cong C\) naturally as coalgebras. In the sequel we denote by the subscript \(-_H\) the corresponding notion in the category of \(H\)-equivariant bicomodules.

Next we investigate the notion of a twirled coaction.

**Theorem 3.5** Let \((M, \delta)\) be a \(H\)-equivariant comodule and let \(\phi \in \text{Hom}(C, H)\). The twirled coaction \(\delta\phi\) defined by

\[\delta\phi = (\text{id} \otimes \mu) \circ (\delta \otimes \phi) \circ \rho\]  

(3.6)
i.e.

\[\delta\phi(m) = m^{\phi(0)} \otimes m^{\phi(1)} = m^{(0)}_{(0)} \otimes m^{(1)}_{(0)} \phi(m_{(1)})\]  

(3.7)

for \(m \in M\) defines another \(H\)-equivariant comodule \((M, \delta\phi)\) if \(\phi \in \Gamma(C \Box H)\). We get “iff” if \(M\) is an equivariant Morita-Takeuchi bicomodule.

**Proof:** If \(\phi \in \Gamma(C \Box H)\), we have to check that this really defines a coaction, that is we have to show it is a \(H\)-comodule, i.e. it fulfills

\[m^{\phi(0)} \otimes m^{\phi(1)} \otimes m^{\phi(2)} = m^{\phi(0)} \otimes (m^{\phi(1)})^{(1)} \otimes (m^{\phi(1)})^{(2)}\]

and

\[m \otimes 1 = m^{\phi(0)} \otimes \varepsilon(m^{\phi(1)})\]

and is compatible with the coaction on the coalgebra

\[m^{\phi(0)} \otimes m^{\phi(0)} \otimes m^{\phi(1)} = m^{\phi(0)} \otimes m^{(0)} \otimes m^{\phi(1)} m_{(1)}^{(1)}\]

For the comodule condition we calculate

\[m^{\phi(0)} \otimes m^{(0)} \otimes m^{\phi(1)} m_{(1)}^{(1)} = m^{(0)}_{(0)} \otimes m^{(2)}_{(0)} \otimes m^{\phi(1)} m_{(1)}^{(1)} \]

\[= m^{(0)}_{(0)} \otimes m^{(0)}_{(1)} \otimes m^{(1)} \phi(m_{(2)}) = m^{\phi(0)} \otimes m^{\phi(1)} \otimes m^{(1)} \phi(m_{(2)})\]

where we used in \((a)\) the comodule condition of \(\phi\), in \((b)\) we used that \(\delta\) is compatible coaction on the comodule \(M\), where we have to keep in mind that we used the \(C\)-comodule coaction two times which we do not see in the Sweedler notation.
The counitarity of $\delta_\phi$ is clear as everything involved is counitary. For the $H$-coaction condition we calculate
\[
m^{\phi(0)} \otimes (m^{\phi(1)})^{(1)} \otimes (m^{\phi(1)})^{(2)} = m^{(0)}_{(0)} \otimes m^{(1)}_{(0)} \phi(m_{(1)})^{(1)} \otimes m^{(2)}_{(0)} \phi(m_{(1)})^{(2)}
\]
\[
= m^{(0)}_{(0)} \otimes m^{(1)}_{(0)} \phi(m_{(1)})^{(1)} \otimes m^{(2)}_{(0)} \phi(m_{(1)})^{(2)}
\]
\[
= m^{(0)}_{(0)} \otimes m^{(1)}_{(0)} \phi(m_{(1)})^{(1)} \otimes m^{(2)}_{(0)} \varepsilon(m_{(1)}) \phi(m_{(2)})
\]
\[
= m^{(0)}_{(0)} \otimes m^{(1)}_{(0)} \phi(m_{(1)})^{(1)} \otimes m^{(2)}_{(0)} \varepsilon(m_{(1)}) \phi(m_{(2)})
\]
\[
= m^{(0)}_{(0)} \otimes m^{(1)}_{(0)} \phi(m_{(1)})^{(1)} \otimes m^{(2)}_{(0)} \phi(m_{(1)})^{(2)}
\]
\[
= m^{\phi(0)} \otimes m^{\phi(1)} \otimes m^{\phi(2)},
\]
where in (a) we used the coaction condition for $\phi$, in (b) we used that $\delta$ is a compatible comodule coaction, in (c) the counitarity of $\phi$, and of course many times the properties of the counit.

Now let $\delta_\phi$ be a compatible coaction on $M$. Counitarity is again clear. We calculate
\[
m^{(0)}_{(0)} \otimes m^{(1)}_{(0)} \phi(m_{(1)})^{(0)} \otimes m^{(2)}_{(0)} m^{(1)}_{(1)} \phi(m_{(2)}) = m^{(0)}_{(0)} \otimes m^{(1)}_{(0)} \phi(m_{(1)})^{(1)} \otimes m^{(2)}_{(0)} \phi(m_{(1)})^{(2)}
\]
\[
= m^{\phi(0)} \otimes m^{\phi(1)} \otimes m^{\phi(2)}
\]
\[
= m^{\phi(0)} \otimes (m^{\phi(1)})^{(1)} \otimes (m^{\phi(1)})^{(2)}
\]
\[
= m^{(0)}_{(0)} \otimes m^{(1)}_{(0)} \phi(m_{(1)})^{(1)} \otimes m^{(2)}_{(0)} \phi(m_{(1)})^{(2)}
\]
and by $M$ being a Morita-Takeuchi bicomodule the coaction condition for $\phi$ follows.

For the comodule condition we have
\[
m^{(0)}_{(0)} \otimes m^{(0)}_{(1)} \otimes m^{(1)}_{(1)} \phi(m_{(2)}) = m^{(0)}_{(0)} \otimes m^{(0)}_{(1)} \otimes m^{(1)}_{(1)} \phi(m_{(2)})
\]
\[
= m^{\phi(0)} \otimes m^{\phi(0)} \otimes m^{\phi(1)}
\]
\[
= m^{\phi(0)} \otimes m^{\phi(0)} \otimes m^{\phi(1)} m_{(1)}
\]
\[
= m^{(0)}_{(0)} \otimes m^{(0)}_{(1)} \otimes m^{(1)}_{(1)} \phi(m_{(1)}) m_{(2)}
\]
were the comodule condition for $\phi$ again follows by $M$ being a Morita-Takeuchi bicomodule.

We are interested in those cocharacters which give isomorphic equivariant Morita-Takeuchi bicomodules.

**Lemma 3.6** Let $\phi \in \Gamma(C \odot H)$. Then $(M, \delta)$ and $(M, \delta_\phi)$ are isomorphic iff $\phi = \hat{c}$ for $c \in \mathcal{Z}(C)$, where $\hat{c} : \mathcal{Z}(C) \to \Gamma(C \odot H)$ is again given by
\[
\hat{c} = ((\delta_c \otimes \text{id}) \circ \delta) \star (\delta_c \otimes 1)^{r-1}.
\]
(3.8)

Also $\delta_\hat{c} = \delta$ iff $c \in \mathcal{Z}(C)^{\text{co}H}$.

**Proof:** In this situation we have for $\delta_\hat{c}$ the following
\[
m^{\hat{c}(0)} \otimes m^{\hat{c}(1)} = m^{(0)}_{(0)} \otimes m^{(1)}_{(0)} \delta_\hat{c}(m^{(0)}_{(1)}) \delta_\hat{c}^{-1}(m^{(2)}_{(1)}) m^{(1)}_{(1)}
\]

\[15\]
\[= m_{(0)} \otimes m_{(1)} \psi_c(m_{(1)})\]
\[= \psi_c(m_{(0)}) m_{(1)} \otimes m_{(1)}\]

with \(\psi_c(m_{(1)}) = \delta_c(m_{(1)}) \delta_c^{1-1}(m_{(2)}) \in R\) using counitarity. Now this is of the desired form, as every comodule morphism is by linearity of such a form for \(c \in \mathcal{Z}(C)\). Also one has \(\psi(m_{(1)}) = 1\) iff \(c(0) \otimes c(1) = c \otimes 1\), which is the definition of the coinvariants. \(\square\)

Hence we have the justification of the term homoschism, as every element of the first homoschism group yields a new equivariant comodule. Furthermore all of them arise by such elements.

**Proposition 3.7** Let \((M, \delta)\) be a \(H\)-equivariant Morita-Takeuchi bicomodule. Then the group \(S^1(H, C)\) acts free and transitive on the set of all \(H\)-coactions, which split \(M\) into a \(H\)-equivariant comodule. The group action is given by
\[(\phi, \delta) \mapsto \delta_{\phi}\] (3.9)

for \(\phi \in S^1(H, C)\).

**Proof:** We only have to show, that this is really a group action, i.e. \(\delta_{\phi \ast \psi} = (\delta_{\phi})_{\psi}\). This is seen by the following calculation
\[m_{\phi \ast \psi(0)} \otimes m_{\phi \ast \psi(1)} = m_{(0)}(0) \otimes m_{(0)}(1)(\phi \ast \psi)(m_{(1)})\]
\[= m_{(0)}(0) \otimes m_{(0)}(1) \phi(m_{(1)}) \psi(m_{(2)})\]
\[= m_{\phi(0)}(0) \otimes m_{\phi(1)}(0) \psi(m_{(1)})\]
\[= m_{\psi(\phi(0))} \otimes m_{\psi(\phi(1))}\]

Hence the proposition follows. \(\square\)

**Remark 3.8** Note that all arguments work equivalently if one considers the left twirled coaction \(\delta^\phi\) given by
\[\delta^\phi(m) = m_{(0)}(0) \otimes \phi(m_{(1)}) m_{(1)}(1)\] (3.10)

Hence the classification for left twirled coactions is the same as for right twirled coactions.

We define the equivariant Picard groupoid as equivalence classes of equivariant Morita bimodules and denote it by
\[\text{Pic}_H = \{[M] \mid M \text{ is } H\text{-equivariant Morita-Takeuchi bicomodule}\}\] (3.11)

We get an exact sequence connecting \(\text{Pic}_H\) and \(\text{Aut}_H\), cf. Definition 2.4:
\[1 \longrightarrow \text{InnAut}_H(C) \longrightarrow \text{Aut}_H(C) \overset{\omega_H}{\longrightarrow} \text{Pic}_H(C)\] (3.12)

We can describe the connection between \(\text{Pic}_H \to \text{Pic}\) with the group \(S^1(H, C)\):

**Proposition 3.9** Let \(C, D\) be two Morita-Takeuchi equivalent coalgebras. Then either
\[S^1(H, C) \cong S^1(H, D)\] (3.13)

or
\[\text{Pic}_H(C, D) = \emptyset\] (3.14)
Proof: If \( \text{Pic}_H(C,D) \neq \emptyset \) then \( S^1(H,C) \) and \( S^1(H,D) \) are both acting free and transitive on \( \text{Pic}_H(C,D) \) and are commuting and hence must coincide.

Example 3.10 Consider a coalgebra \( C \) with coaction of a Hopf algebra \( H \). Then we can define the coactions on \( M_n^r(C) \), the matrix coalgebra, and \( C^n \) pointwise and get an equivariant Morita-Takeuchi equivalence between \( M_n^r(C) \) and \( C \).

Remark 3.11 Having in mind that every coalgebra dualizes to an algebra, one can study the morphism \( \text{Pic}(C) \rightarrow \text{Pic}(C^*) \) and try to describe, which parts of the \( \text{Pic}(C^*) \) are missing. Some results in this direction can be found in [7, Thm. 2.9]. In the equivariant setting one would like to consider something like the morphism \( \text{Pic}_H(C) \rightarrow \text{Pic}_H(C^*) \), where now one has to be aware that \( H^* \) might not be a well-defined Hopf algebra. Some steps in this direction can be found in [11, Chap. 9], where the restricted dual for Hopf algebras is investigated. Using these restricted duals one could also wonder, what one can see by the morphism \( \text{Pic}_H(\mathcal{A}^0) \rightarrow \text{Pic}_H(\mathcal{A}) \). We postpone these questions to later projects.

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