STABILITY AND CONVERGENCE OF DIFFERENCE SCHEMES APPROXIMATING A TWO-PARAMETER NONLOCAL BOUNDARY VALUE PROBLEM FOR TIME-FRACTIONAL DIFFUSION EQUATION

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Difference schemes for the time-fractional diffusion equation with variable coefficients and nonlocal boundary conditions containing real parameters \( \alpha \) and \( \beta \) are considered. By the method of energy inequalities, for the solution of the difference problem, we obtain a priori estimates, which imply the stability and convergence of these difference schemes.

Keywords: fractional-order diffusion equation, nonlocal boundary value condition, a priori estimate, difference scheme, stability and convergence.

1. Introduction

Consider the nonlocal boundary value problem

\[ \partial^\gamma_0 u = \frac{\partial}{\partial x} \left( k(x) \frac{\partial u}{\partial x} \right) + f(x,t), \quad 0 < x < 1, \quad 0 < t \leq T, \] (1)

\[ u(0,t) = \alpha u(1,t), \quad k(1)u_x(1,t) = \beta k(0)u_x(0,t) + \mu(t), \quad 0 \leq t \leq T, \] (2)

\[ u(x,0) = u_0(x), \quad 0 \leq x \leq 1, \] (3)

where \( 0 < c_1 \leq k(x) \leq c_2 \) for all \( x \in [0,1] \), \( \alpha \) and \( \beta \) are real numbers such that \( \alpha \beta > 0 \), and \( \mu(t) \in C[0, T] \).

\[ \partial^\gamma_0 u(x,t) = \int_0^t u_\tau(x,\tau)(t-\tau)^{-\gamma}d\tau/\Gamma(1-\gamma) \]

is a Caputo fractional derivative of order \( \gamma \), \( 0 < \gamma < 1 \) [1, 2].

The first nonlocal condition in (2) can be replaced by the inhomogeneous condition \( u(0,t) = \alpha u(1,t) + \mu_1(t) \); however, if \( \mu_1(t) \in C^1[0, T] \), then the simple change of variables \( u(x,t) = v(x,t) + (1-x)\mu_1(t) \) reduces this problem to the considered one.

The existence of the solution for the initial boundary value problem of a number of fractional-order differential equations has been proved in [3, 4].

We introduce the space grid \( \omega_h = \{x_i = ih\}_{i=0}^N \) and the time grid \( \omega_\tau = \{t_n = n\tau\}_{n=0}^{N_\tau} \) with increments \( h = 1/N \) and \( \tau = T/N_\tau \). Set

\[ a_i = k(x_i - 0.5h), \quad \varphi^n_i = f(x_i, t_n + \sigma \tau), \quad y_i^n = y(x_i, t_n), \quad y^n_{\bar{x},i} = (y^n_i - y^n_{i-1})/h, \]

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is a difference analogue of Caputo fractional derivative of order $\gamma$, $0 < \gamma < 1$ [5].

It has been shown that if the function $v(t) \in C^2[0, T]$, then $\partial_{0t_n+1}^\alpha v = \Delta_{0t_n+1}^\alpha v + O(\tau)$ [5]. This result can be improved if $v(t) \in C^3[0, T]$.

**Lemma 1.** For any function $v(t) \in C^3[0, T]$ the following equality takes place:

$$\partial_{0t_n+1}^\alpha v = \Delta_{0t_n+1}^\alpha v + O(\tau^{2-\alpha}), \quad 0 < \alpha < 1.$$  \hspace{1cm} (4)

**Proof.** For $t = t_{n+1}$, $n = 0, 1, \ldots, N_T - 1$, one has

$$\partial_{0t_n+1}^\alpha v = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{n+1}} \frac{v'(\eta)d\eta}{(t_{n+1}-\eta)^\alpha} = \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n} \int_{t_s}^{t_{s+1}} \frac{v'(\eta)d\eta}{(t_{n+1}-\eta)^\alpha}$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n} \left( v'(t_{s+1/2}) + v''(t_{s+1/2})(\eta-t_{s+1/2}) + O((\eta-t_{s+1/2})^2) \right) \frac{d\eta}{(t_{n+1}-\eta)^\alpha}$$

$$= \Delta_{0t_n+1}^\alpha v + \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n} v''(t_{s+1/2}) \int_{t_s}^{t_{n+1}} \frac{\eta-t_{s+1/2}}{(t_{n+1}-\eta)^\alpha} d\eta + O(\tau^2).$$

Let us estimate the value

$$\left| \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n} v''(t_{s+1/2}) \int_{t_s}^{t_{n+1}} \frac{\eta-t_{s+1/2}}{(t_{n+1}-\eta)^\alpha} d\eta \right|$$

$$\leq \frac{M}{\Gamma(1-\alpha)} \sum_{s=0}^{n} \left| \int_{t_s}^{t_{n+1}} \frac{\eta-t_{s+1/2}}{(t_{n+1}-\eta)^\alpha} d\eta \right|$$

$$= \frac{M}{\Gamma(1-\alpha)} \sum_{s=0}^{n} \left( \int_{t_s}^{t_{n+1}} \frac{\eta-t_{s+1/2}}{(t_{n+1}-\eta)^\alpha} d\eta - \int_{t_s}^{t_{n+1}} \frac{\eta-t_{s+1/2}}{(t_{n+1}-\eta)^\alpha} d\eta \right)$$

$$= 2^\alpha M \tau^{2-\alpha} \sum_{s=0}^{n} \left( \frac{1}{4\Gamma(1-\alpha)} \int_{0}^{z} \frac{zdz}{(2(n-s)+1-z)^\alpha} - \frac{1}{4\Gamma(1-\alpha)} \int_{0}^{z} \frac{zdz}{(2(n-s)+1+z)^\alpha} \right)$$
the fractional derivative and nonlocal boundary conditions have been proven \cite{22–24}. The principle implies the stability and convergence of these difference schemes. A priori estimates for the difference problems analyzed in \cite{5, 20, 21} by using the maximum energy inequality method, a priori estimates for the solution of the Dirichlet and Robin boundary value problems for the fractional, variable, and distributed order diffusion equation with Caputo fractional derivative have been obtained \cite{17–19}. A priori estimates for the difference problems analyzed in \cite{5, 20, 21} by using the maximum principle imply the stability and convergence of these difference schemes.

The existence and uniqueness of solutions for fractional ordinary differential equations with various kinds of the fractional derivative and nonlocal boundary conditions have been proven \cite{22–24}.

\[ M = \max_{0 \leq t \leq T} |v''(t)|. \] The proof of Lemma 1 is complete.

Consider the weighted scheme

\[ \Delta_{0t_n+1}^\gamma y_i - (ay_x^{(\sigma)})_{x,i} = \varphi_i^n, \quad i = 1, 2, ..., N - 1, \] (5)

\[ \begin{cases} y_0^{n+1} - \alpha y_N^{n+1} = 0, \\
\beta \Delta_{0t_n+1}^\gamma y_0 + \Delta_{0t_n+1}^\gamma y_N + \frac{2}{h} \left( a_N y_{x,N}^{(\sigma)} - \beta a_1 y_{x,0}^{(\sigma)} \right) = \frac{2}{h} \mu(t_{n+\sigma}) + \varphi_N + \beta \varphi_0, \\
y_0^0 = u_0(x_i). \] (6)

The difference scheme \( (5)–(7) \) has approximation order \( O(\tau^{m_\sigma} + h^2) \), where \( m_\sigma = 1 \) if \( 0 \leq \sigma < 1 \) and \( m_\sigma = 2 - \alpha \) if \( \sigma = 1 \) \cite{6}.

The nonlocal boundary value problem with the boundary conditions \( u(b, t) = pu(a, t), \ u_x(b, t) = \sigma u_x(a, t) + \tau u(a, t) \) for the simplest equations of mathematical physics, referred to as conditions of the second class, was studied in the monograph \cite{7}. Results in the case in which \( \rho \sigma - 1 = 0 \) and \( \rho \tau \leq 0 \) were obtained there. Difference schemes for problem \( (1)–(3) \) with \( \alpha = \beta \) and \( \gamma = 1 \) (the classical diffusion equation) were studied in \cite{8}. In this case, the operator occurring in the elliptic part is self-adjoint. Self-adjointness permits one to use general theorems on the stability of two-layer difference schemes in energy spaces and consider difference schemes for equations with variable coefficients. Stability criteria for difference schemes for the heat equation with nonlocal boundary conditions were studied in \cite{9–13}. The difference schemes considered in these papers have the specific feature that the corresponding difference operators are not self-adjoint. The method of energy inequalities was developed in \cite{14–16} for the derivation of a priori estimates for solutions of difference schemes for the classical diffusion equation with variable coefficients in the case of nonlocal boundary conditions. With the use of the energy inequality method, a priori estimates for the solution of the Dirichlet and Robin boundary value problems for the fractional, variable, and distributed order diffusion equation with Caputo fractional derivative have been obtained \cite{17–19}. A priori estimates for the difference problems analyzed in \cite{5, 20, 21} by using the maximum principle imply the stability and convergence of these difference schemes.

\[ \Delta_{0t_n+1}^\gamma y_i - (ay_x^{(\sigma)})_{x,i} = \varphi_i^n, \quad i = 1, 2, ..., N - 1, \] (5)

\[ \begin{cases} y_0^{n+1} - \alpha y_N^{n+1} = 0, \\
\beta \Delta_{0t_n+1}^\gamma y_0 + \Delta_{0t_n+1}^\gamma y_N + \frac{2}{h} \left( a_N y_{x,N}^{(\sigma)} - \beta a_1 y_{x,0}^{(\sigma)} \right) = \frac{2}{h} \mu(t_{n+\sigma}) + \varphi_N + \beta \varphi_0, \\
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The existence and uniqueness of solutions for fractional ordinary differential equations with various kinds of the fractional derivative and nonlocal boundary conditions have been proven \cite{22–24}.
Numerical methods for solving fractional diffusion equations with classical boundary value problems and various kinds of the fractional order derivative have been proposed [25–29].

2. A Priori Estimate for the Differential Problem

Lemma 2. [18] For any function $v(t)$ absolutely continuous on $[0, T]$, the following equality takes place:

$$v(t) \frac{\partial^\nu}{\partial t^\nu} v(t) = \frac{1}{2} \frac{\partial^\nu}{\partial t^\nu} v^2(t) + \frac{\nu}{2\Gamma(1 - \nu)} \int_0^t \frac{d\xi}{(t - \xi)^{1 - \nu}} \left( \int_0^\xi \frac{v'(\eta)d\eta}{(t - \eta)^\nu} \right)^2,$$

(8)

where $0 < \nu < 1$.

Theorem 1. The solution of the nonlocal boundary value problem (1)–(3) satisfies the identity

$$\frac{1}{2} \frac{\partial^\gamma}{\partial t^\gamma} \int_0^1 (1 + \delta p(x))u^2(x, t)dx + \frac{\gamma}{2\Gamma(1 - \gamma)} \int_0^1 (1 + \delta p(x)) dx \int_0^t \left( \int_0^\xi \frac{\partial u}{\partial \eta}(x, \eta)d\eta \right)^2 \frac{d\xi}{(t - \xi)^{1 - \gamma}}$$

$$+ \int_0^1 (1 + \delta p(x))k(x)u_x^2(x, t)dx + \frac{\delta}{2}(\alpha^2 - 1)u^2(1, t) = \int_0^1 (1 + \delta p(x))u(x, t)f(x, t)dx + u(1, t)\mu(t),$$

(9)

where $p(x) = \int_x^1 k^{-1}(s)ds$, $\delta = (\beta\alpha^{-1} - 1)/p(0)$.

Proof. Let us multiply (1) by $u(x, t)$ and integrate the resulting relation over $x$ from 0 to 1:

$$\int_0^1 u(x, t) \frac{\partial^\gamma}{\partial t^\gamma} u(x, t) dx - \int_0^1 (k(x, t)u_x(x, t))xu(x, t) dx = \int_0^1 u(x, t)f(x, t)dx.$$

(10)

This, together with the nonlocal boundary conditions (2) and equality (8), implies the relation

$$\frac{1}{2} \frac{\partial^\gamma}{\partial t^\gamma} \int_0^1 u^2(x, t)dx + \frac{\gamma}{2\Gamma(1 - \gamma)} \int_0^1 dx \int_0^t \frac{d\xi}{(t - \xi)^{1 - \gamma}} \left( \int_0^\xi \frac{\partial u}{\partial \eta}(x, \eta)d\eta \right)^2$$

$$+ \int_0^1 k(x)u_x^2(x, t)dx = \int_0^1 u(x, t)f(x, t)dx + \left( \frac{\beta}{\alpha} - 1 \right) k(0)u_x(0, t)u(0, t) + \frac{1}{\alpha} u(0, t)\mu(t).$$

(11)

We multiply (1) by $u(x, t)$ and integrate with respect to $s$ from 0 to $x$,

$$\int_0^x u(s, t) \frac{\partial^\gamma}{\partial t^\gamma} u(s, t) ds - \int_0^x (k(s)u_x(s, t))u(s, t) ds = \int_0^x u(s, t)f(s, t)ds.$$

(12)
Hence we obtain the identity
\[
\frac{1}{2} \partial^\gamma_0 \int_0^x u^2(s, t) ds + \frac{\gamma}{2\Gamma(1 - \gamma)} \int_0^x ds \int_0^t \frac{d\xi}{(t - \xi)^1-\gamma} \left( \int_0^{\xi} \frac{\partial u(s, \eta) d\eta}{(t - \eta)^\gamma} \right)^2 \\
+ \int_0^x k(s) u_x^2(s, t) ds = \int_0^x u(s, t) f(s, t) ds + k(x) u_x(x, t) u(x, t) - k(0) u_x(0, t) u(0, t).
\] (13)

We divide (13) by \( k(x) \) and integrate with respect to \( x \) from 0 to 1,
\[
\frac{1}{2} \partial^\gamma_0 \int_0^1 p(x) u^2(x, t) dx + \frac{\gamma}{2\Gamma(1 - \gamma)} \int_0^1 p(x) dx \int_0^t \frac{d\xi}{(t - \xi)^1-\gamma} \left( \int_0^{\xi} \frac{\partial u(x, \eta) d\eta}{(t - \eta)^\gamma} \right)^2 \\
+ \int_0^1 p(x) k(x) u_x^2(x, t) dx = \int_0^1 p(x) u(x, t) f(x, t) dx + \frac{1}{2} (1 - \alpha^2) u^2(1, t) - p(0) k(0) u_x(0, t) u(0, t).
\] (14)

By multiplying identity (14) by \( \delta = (\beta \alpha^{-1} - 1)/p(0) \) and by adding relation (11), we obtain identity (9). The proof of Theorem 1 is complete.

**Theorem 2.** If the condition \((\beta \alpha^{-1} - 1)(\alpha^2 - 1) \geq 0\) is satisfied, then the solution of problem (1)–(3) with \( f(x, t) \equiv 0 \) and \( \mu(t) \equiv 0 \) satisfies the estimate
\[
\|u^2(x, t)\|^2_0 + 2c_1 D^\gamma_0 \|u_x(x, s)\|^2_0 \leq \max \left\{ \frac{\alpha}{\beta}, \frac{\beta}{\alpha} \right\} \|u_0(x)\|^2_0,
\] (15)

where \( \|u(x, t)\|^2_0 = \int_0^1 u^2(x, t) dx \), and \( D^\gamma_0 u(x, t) = \int_0^t (t - s)^{\gamma-1} u(x, s) ds / \Gamma(\gamma) \) is the fractional Riemann–Liouville integral of order \( \nu > 0 \).

**Proof.** Applying the fractional integration operator \( D^\gamma_0 \) to both sides of equality (9) and using the inequalities \( \min\{1, \beta \alpha^{-1}\} \leq 1 + \delta p(x) \leq \max\{1, \beta \alpha^{-1}\} \), we obtain the a priori estimate (15). The proof of Theorem 2 is complete.

The inequality \((\beta \alpha^{-1} - 1)(\alpha^2 - 1) \geq 0\) is equivalent to the system of inequalities \(|\beta| \geq |\alpha| \geq 1\) or \(|\beta| \leq |\alpha| \leq 1\). Note that one can obtain a priori estimates for \(|\alpha| \geq |\beta| \geq 1\) or \(|\alpha| \leq |\beta| \leq 1\) by multiplying Eq. (1) by \((k(x) u_x)_x\). To avoid similar calculations, we assume sufficient smoothness of the solution and the input data and show that this holds for the differential problem (1)–(3). Set \( w(x, t) = k(x) u_x(x, t) \). Then the function \( w(x, t) \) satisfies the problem
\[
\partial^\gamma_0 w = k(x) w_{xx} + k(x) f_x(x, t),
\] (16)
\[
w(0, t) = \frac{1}{\beta} w(1, t) - \frac{1}{\beta} \mu(t), \quad w_x(1, t) = \frac{1}{\alpha} w_x(0, t) + \frac{1}{\alpha} f(0, t) - f(1, t),
\] (17)
\[
w(x, 0) = k(0) u_0^0(x).
\] (18)
Obviously, for problem (16)–(18) with \( f(x, t) \equiv 0 \) and \( \mu(t) \equiv 0 \), one can readily obtain an a priori estimate of the form (15) for \( |\alpha| \geq |\beta| \geq 1 \) or \( |\alpha| \leq |\beta| \leq 1 \).

We have the inequalities

\[
\begin{align*}
  u(1, t) \mu(t) &\leq \frac{\varepsilon}{2} u^2(1, t) + \frac{1}{2\varepsilon} \mu^2(t), \\
  \int_0^1 (1 + \delta p(x)) u(x, t) f(x, t) dx &\leq \frac{\varepsilon}{2} u^2(1, t) + \frac{\varepsilon_1}{2} \int_0^1 (1 + \delta p(x)) k(x) u_x^2(x, t) dx + \left( \frac{\gamma_1}{2\varepsilon} + \frac{\gamma_1 \gamma_2}{2\varepsilon_1} \right) \int_0^1 (1 + \delta p(x)) f^2(x, t) dx,
\end{align*}
\]

where \( \varepsilon, \varepsilon_1 > 0, \gamma_1 = \int_0^1 (1 + \delta p(x)) dx, \gamma_2 = \int_0^1 (1 + \delta p(x))^{-1} k^{-1}(x) dx \).

Inequality (19) is obvious. Let us prove inequality (20). By virtue of the relation

\[
u(x, t) = u(1, t) - \int_x^1 u_s(s, t) ds,
\]

we have

\[
\begin{align*}
  \int_0^1 (1 + \delta p(x)) u(x, t) f(x, t) dx &= \int_0^1 (1 + \delta p(x)) f(x, t) \left( u(1, t) - \int_x^1 u_s(s, t) ds \right) dx \\
  &= u(1, t) \int_0^1 (1 + \delta p(x)) f(x, t) dx - \int_0^1 u_x(x, t) dx \int_0^x (1 + \delta p(s)) f(s, t) ds \\
  &\leq \frac{\varepsilon}{2} u^2(1, t) + \frac{\gamma_1}{2\varepsilon} \int_0^1 (1 + \delta p(x)) f^2(x, t) dx \left( u(1, t) - \int_x^1 u_s(s, t) ds \right) dx \\
  &\quad + \int_0^1 \sqrt{(1 + \delta p(x)) k(x)} |u_x(x, t)| dx \int_0^x \frac{(1 + \delta p(s))}{\sqrt{(1 + \delta p(x)) k(x)}} |f(s, t)| ds \\
  &\leq \frac{\varepsilon}{2} u^2(1, t) + \frac{\gamma_1}{2\varepsilon} \int_0^1 (1 + \delta p(x)) f^2(x, t) dx \\
  &\quad + \frac{\gamma_1 \gamma_2}{2\varepsilon_1} \int_0^1 (1 + \delta p(x)) f^2(x, t) dx.
\end{align*}
\]

The proof of inequality (20) is complete.
Theorem 3. If the condition \((\beta \alpha^{-1} - 1)(\alpha^2 - 1) > 0\) is satisfied, then the solution of problem (1)-(3) satisfies the a priori estimates

\[
\|u^2(x,t)\|_0^2 + D_{0t}^{-\gamma}\|u_x(x,t)\|_0^2 \leq M \left( D_{0t}^{-\gamma}\|f(x,t)\|_0^2 + D_{0t}^{-\gamma} \mu^2(t) + \|u_0(x)\|_0^2 \right),
\]  

where \(M = M(\alpha, \beta, c_1) > 0\) is a known constant independent of \(t\).

Proof. Identity (9), together with inequalities (19) and (20) for \(\epsilon = (\beta \alpha^{-1} - 1)(\alpha^2 - 1)\) and \(\epsilon_1 = 1\), implies the inequality

\[
\frac{1}{2} D_{0t}^{-\gamma} \int_0^1 (1 + \delta p(x))u^2(x,t)dx + \frac{\gamma}{\Gamma(1 - \gamma)} \int_0^1 (1 + \delta p(x))dx \int_0^t \left( \int_0^\xi \frac{x_n(x,\eta)d\eta}{(t - \eta)^{1+\gamma}} \right)^2 d\xi
\]

\[
+ \frac{c_1}{2} \int_0^1 (1 + \delta p(x))u^2_x(x,t)dx \leq M_1 \left( \int_0^1 (1 + \delta p(x))f^2(x,t)dx + \mu^2(t) \right),
\]

where \(M_1 = M_1(\alpha, \beta, c_1) > 0\) is a known constant independent of \(t\).

By applying the fractional integration operator \(D_{0t}^{-\gamma}\) to both sides of inequality (22) and by taking into account the inequalities \(\min\{1, \beta \alpha^{-1}\} \leq 1 + \delta p(x) \leq \max\{1, \beta \alpha^{-1}\}\), we obtain the a priori estimate (21). The proof of Theorem 3 is complete.

2.1. A Priori Estimates for the Difference Problem

Lemma 3. [18] For any function \(y(t)\) defined on the grid \(\bar{\omega}_\tau\) one has the equalities

\[
y^{n+1} \Delta_{0t}^\nu y = \frac{1}{2} \Delta_{0t}^\nu (y^2) + \frac{\nu \Gamma(2 - \nu)}{2} (\Delta_{0t}^\nu y)^2 + J_1(y),
\]

\[
y^n \Delta_{0t}^\nu y = \frac{1}{2} \Delta_{0t}^\nu (y^2) - \frac{\nu \Gamma(2 - \nu)}{2(2 - 2^{1-\nu})} (\Delta_{0t}^\nu y)^2 + J_2(y),
\]

where

\[
J_1(y) = \frac{1}{2 \Gamma(2 - \nu)} \sum_{k=0}^{n-1} \tau \left( (t^{1-\nu}_{n-k+1} - t^{1-\nu}_{n-k})^{-1} - (t^{1-\nu}_{n-k} - t^{1-\nu}_{n-k-1})^{-1} \right) \left( \zeta^{k+1} \right)^2,
\]

\[
J_2(y) = \frac{\nu \Gamma(2 - \nu) - 1}{2 \Gamma(2 - \nu)(2 - 2^{1-\nu})} \left( \zeta^{n+1} - \frac{2 - 2^{1-\nu}}{2^{1-\nu} - 1} \zeta^n \right)^2
\]

\[
+ \frac{1}{2 \Gamma(2 - \nu)} \sum_{k=0}^{n-2} \tau \left( (t^{1-\nu}_{n-k+1} - t^{1-\nu}_{n-k})^{-1} - (t^{1-\nu}_{n-k} - t^{1-\nu}_{n-k-1})^{-1} \right) \left( \zeta^{k+1} \right)^2,
\]

\[
\zeta^{k+1} = \sum_{s=0}^k (t^{1-\nu}_{n-k+1} - t^{1-\nu}_{n-k}) y^n_s, \quad J_1(y) \geq 0, \quad J_2(y) \geq 0, \quad 0 < \nu < 1. \quad \text{Here we consider the sums to be equal to zero if the upper summation limit is less than the lower one.}
Lemma 4. For any nonnegative function \( v(x) \) defined on the grid \( \omega_h \) and any solution \( y(x,t) \) of Eq. (5) with zero right–hand side \( \varphi \equiv 0 \), one has the inequality

\[
\| \sqrt{v} \sqrt{a y_x^{(\sigma)}} \|_0^2 \geq \frac{h^2}{4c_2} \| \sqrt{v} \Delta_{0t}^\gamma y \|_0^2 + \frac{h}{2c_2} \sum_{i=1}^{N} v_{x,i}(a_{i} y_{x,i}^{(\sigma)})^2 h + \frac{h}{2c_2} \left( v_N (a_{N} y_{x,N}^{(\sigma)})^2 + v_0 (a_{1} y_{x,0}^{(\sigma)})^2 \right),
\]

where \( \| y \|_0^2 = \sum_{i=1}^{N-1} y_i^2 h, \| y \|_0^2 = \sum_{i=1}^{N} y_i^2 h. \)

**Proof.** Since \( y(x,t) \) is a solution of equation (5) with zero right–hand side, it follows that, for each nonnegative function \( v(x) \) one has the relation \( \sqrt{v} \Delta_{0t}^\gamma y_i = \sqrt{v_i} (a y_x^{(\sigma)})_{x,i} \) for all \( i = 1, 2, \ldots, N-1 \); consequently,

\[
\| \sqrt{v} \Delta_{0t}^\gamma y \|_0^2 = \| \sqrt{v} (a y_x^{(\sigma)})_{x} \|_0^2
\]

\[
= \frac{1}{h^2} \sum_{i=1}^{N-1} v_i \left( (a_{i+1} y_{x,i+1}^{(\sigma)}) - (a_{i} y_{x,i}^{(\sigma)}) \right)^2 h \leq \frac{1}{h^2} \sum_{i=1}^{N-1} v_i \left( (a_{i+1} y_{x,i+1}^{(\sigma)}) + (a_{i} y_{x,i}^{(\sigma)}) \right)^2 h
\]

\[
= \frac{2}{h^2} \sum_{i=1}^{N-1} \left( v_{i+1} (a_{i+1} y_{x,i+1}^{(\sigma)})^2 + v_i (a_{i} y_{x,i}^{(\sigma)})^2 \right) h - \frac{2}{h^2} \sum_{i=1}^{N-1} (v_{i+1} - v_i) \left( a_{i+1} y_{x,i+1}^{(\sigma)} \right)^2 h
\]

\[
= \frac{4}{h^2} \sum_{i=1}^{N} v_i \left( a_{i} y_{x,i}^{(\sigma)} \right)^2 h - \frac{2}{h} \sum_{i=2}^{N} v_{x,i} \left( a_{i} y_{x,i}^{(\sigma)} \right)^2 h - \frac{2}{h} \left( v_N (a_{N} y_{x,N}^{(\sigma)})^2 + v_0 (a_{1} y_{x,0}^{(\sigma)})^2 \right)
\]

\[
\leq \frac{4c_2}{h^2} \| \sqrt{v} \sqrt{a y_x^{(\sigma)}} \|_0^2 - \frac{2}{h} \sum_{i=1}^{N} v_{x,i} \left( a_{i} y_{x,i}^{(\sigma)} \right)^2 h - \frac{2}{h} \left( v_N (a_{N} y_{x,N}^{(\sigma)})^2 + v_0 (a_{1} y_{x,0}^{(\sigma)})^2 \right).
\]

This implies the desired inequality (25).

Lemma 5. The inequality

\[
\| \sqrt{v} \sqrt{a y_x^{(\sigma)}} \|_0^2 \geq \frac{h^2}{4c_2(1+\varepsilon)} \| \sqrt{v} \Delta_{0t}^\gamma y \|_0^2 + \frac{h}{2c_2} \sum_{i=1}^{N} v_{x,i}(a_{i} y_{x,i}^{(\sigma)})^2 h
\]

\[
+ \frac{h}{2c_2} \left( v_N (a_{N} y_{x,N}^{(\sigma)})^2 + v_0 (a_{1} y_{x,0}^{(\sigma)})^2 \right) - \frac{h^2}{4c_2(1+\varepsilon)} \| \sqrt{v} \varphi \|_0^2, \quad \varepsilon > 0
\]

holds for any nonnegative function \( v(x) \) defined on the grid \( \omega_h \) and any solution \( y(x,t) \) of Eq. (5).

**Proof.** Since \( y(x,t) \) is a solution of Eq. (5), it follows that \( \sqrt{v} \Delta_{0t}^\gamma y_i = \sqrt{v_i} (a y_x^{(\sigma)})_{x,i} + \sqrt{v_i} \varphi_i \) for all \( i = 1, 2, \ldots, N-1 \) for any nonnegative function \( v(x) \); consequently,

\[
\| \sqrt{v} \Delta_{0t}^\gamma y \|_0^2 = \| \sqrt{v} (a y_x^{(\sigma)})_{x} \|_0^2
\]

\[
\leq \frac{1+\varepsilon}{h^2} \sum_{i=1}^{N-1} v_i \left( (a_{i+1} y_{x,i+1}^{(\sigma)}) - (a_{i} y_{x,i}^{(\sigma)}) \right)^2 h + \left( 1 + \frac{1}{\varepsilon} \right) \sum_{i=1}^{N-1} v_i \varphi_i^2 h
\]
if \( \beta \), the relation

\[
2(1 + \varepsilon) h^2 \sum_{i=1}^{N-1} v_i \left( (a_i y_{x,i}^{(\sigma)})^2 + (a_i y_{x,i})^2 \right) h + \left( 1 + \frac{1}{\varepsilon} \right) \| \sqrt{v_i} \varphi \|_0^2
\]

= \frac{4c_2 (1 + \varepsilon)}{h^2} \| \sqrt{v_i} \sqrt{a y_{x,i}^{(\sigma)}} \|_0^2 - \frac{2(1 + \varepsilon)}{h} \sum_{i=1}^{N} v_{x,i} \left( a_i y_{x,i}^{(\sigma)} \right)^2 h

- \frac{2(1 + \varepsilon)}{h} \left( v_N (a_N y_{x,N}^{(\sigma)})^2 + v_0 (a_1 y_{x,0}^{(\sigma)})^2 \right) + \frac{1 + \varepsilon}{\varepsilon} \| \sqrt{v_i} \varphi \|_0^2.

Hence we derive the desired inequality (26).

**Theorem 4.** If \((\beta \alpha^{-1} - 1)(\alpha^2 - 1) \geq 0\), then the condition

\[
\sigma \geq \frac{1}{3 - 2^{1-\gamma}} - \frac{h^2(2 - 2^{1-\gamma})}{2c_2 \Gamma(3 - 2^{1-\gamma}) \Gamma(2 - \gamma)}
\]

is sufficient for the stability of the difference scheme (5)–(7) for \( \varphi(x, t) \equiv 0 \) and \( \mu(t) \equiv 0 \), and its solution satisfies the estimate

\[
\| y^{n+1} \|_1 \leq \| y^0 \|_1
\]

here

\[
\| y \|_1^2 = \| y \|_0^2 + \delta_1(\alpha, \beta) \| p_1 y \|_0^2 + \gamma_1(\alpha, \beta) y_0^2 h,
\]

where

\[
p_1^2(x) = \sum_{s=1}^{N-1} h/a_{s+1}, \quad \delta_1(\alpha, \beta) = (\beta \alpha^{-1} - 1)/p_1^2(0), \quad \gamma_1(\alpha, \beta) = (\alpha \beta + 1)/(2\alpha^2),
\]

if \( \beta \alpha^{-1} - 1 \geq 0, \alpha^2 - 1 \geq 0; \)

\[
\| y \|_1^2 = \| y \|_0^2 + \delta_1(\alpha^{-1}, \beta^{-1}) \| p_1 (1 - x) y \|_0^2 + \gamma_1(\alpha^{-1}, \beta^{-1}) \alpha^{-2} y_0^2 h,
\]

if \( \beta \alpha^{-1} - 1 \leq 0, \alpha^2 - 1 \leq 0. \)

The norm \( \| y \|_1^2 \) is equivalent to the norm \( \| y \|_0^2 = 0.5h^2 y_0^2 + 0.5h^2 y_N^2 + \| y \|_0^2. \)

**Proof.** By multiplying equation (5) by \( y^{(\sigma)} h \) and by summing with respect to \( i \) from 1 to \( N - 1 \), we obtain the relation

\[
(\Delta^\gamma_{0\tau} y, y^{(\sigma)}) - ((ay_{x}^{(\sigma)})_{x}, y^{(\sigma)}) = (\varphi, y^{(\sigma)}),
\]

where \((y, v) = \sum_{i=1}^{N-1} y_i v_i h. \)

Let us transform the terms in relation (29),

\[
(\Delta^\gamma_{0\tau} y, y^{(\sigma)}) = \frac{1}{2} \Delta^\gamma_{\tau} (\| y \|_0^2) + \frac{\tau^\gamma \Gamma(2 - \gamma)}{2(2 - 2^{1-\gamma})} ((3 - 2^{1-\gamma})\sigma - 1) \| \Delta_{0\tau}^\gamma y \|_0^2 + \| \sqrt{J}(\sigma)(y) \|_0^2, -((ay_{x}^{(\sigma)})_{x}, y^{(\sigma)})
\]
where \( J^{(\sigma)}(y) = \sigma J_1(y) + (1 - \sigma)J_2(y) \).

Substituting expression (30) into relation (29), we obtain

\[
\frac{1}{2} \Delta_0^\gamma (\|y\|^2_2) + \frac{\tau^\gamma \Gamma(2 - \gamma)}{2(2 - 2^{1 - \gamma})} ((3 - 2^{1 - \gamma})\sigma - 1) (\Delta_0^\gamma y_0)^2 + \|\sqrt{a} y^{(\sigma)}_0\|^2_2 + \|J^{(\sigma)}(y)\|^2_2
\]

\[
= \left(\frac{\beta}{\alpha} - 1\right) a_{1} y_{x,0} y_{0}^{(\sigma)} + y_{N}^{(\sigma)} \mu(t_{n+\sigma}) + \frac{h}{2} (\varphi_N + \beta \varphi_0) y_N^{(\sigma)} + (\varphi, y^{(\sigma)}),
\]

where \( \|y\|^2_2 = \|y\|^2_0 + (\alpha \beta + 1)/(2\alpha^2) y_0^2 h \).

Multiplying Eq. (5) by \( y_i^{(\sigma)} h \) and by summing with respect to \( s \) from 1 to \( i \), we obtain

\[
\sum_{s=1}^{i} y_s^{(\sigma)} \Delta_0^\gamma y_s h - \sum_{s=1}^{i} (a y_x^{(\sigma)})_{x,s} y_s^{(\sigma)} h = \sum_{s=1}^{i} \varphi_s y_s^{(\sigma)} h, \quad i = 0, 1, ..., N - 1.
\]

Here we adopt the convention that the sum is zero if the upper limit of the summation is less than the lower one.

Let us transform the terms in relation (32),

\[
\sum_{s=1}^{i} y_s^{(\sigma)} \Delta_0^\gamma y_s h = \frac{1}{2} \Delta_0^\gamma \left( \sum_{s=1}^{i} y_s^2 h \right) + \frac{\tau^\gamma \Gamma(2 - \gamma)}{2(2 - 2^{1 - \gamma})} ((3 - 2^{1 - \gamma})\sigma - 1) \sum_{s=1}^{i} (\Delta_0^\gamma y_s)^2 h
\]

\[
+ \sum_{s=1}^{i} J^{(\sigma)}(y)_i h - \sum_{s=1}^{i} (a y_x^{(\sigma)})_{x,s} y_s^{(\sigma)} h
\]

\[
= \sum_{s=1}^{i+1} a_s (y_{x,s}^{(\sigma)})^2 h - a_{i+1} y_{x,i+1}^{(\sigma)} y_{x,i+1}^{(\sigma)} + a_{1} y_{0x,0} y_{0}
\]

\[
= \sum_{s=1}^{i} a_s (y_{x,s}^{(\sigma)})^2 h + \frac{h}{2} a_{i+1} (y_{x,i+1}^{(\sigma)})^2 - \frac{1}{2} a_{i+1} \left( y_{x,i}^{(\sigma)} \right)_{x,i+1}^2 + a_{1} y_{0x,0} y_{0}
\]

Substituting expressions (33) into relation (32), we get

\[
\frac{1}{2} \Delta_0^\gamma \left( \sum_{s=1}^{i} y_s^2 h \right) + \frac{\tau^\gamma \Gamma(2 - \gamma)}{2(2 - 2^{1 - \gamma})} ((3 - 2^{1 - \gamma})\sigma - 1) \sum_{s=1}^{i} (\Delta_0^\gamma y_s)^2 h + \sum_{s=1}^{i} a_s (y_{x,s}^{(\sigma)})^2 h
\]

\[
+ \frac{h}{2} a_{i+1} (y_{x,i+1}^{(\sigma)})^2 + \sum_{s=1}^{i} J^{(\sigma)}(y)_i h = \frac{1}{2} a_{i+1} \left( y_{x,i}^{(\sigma)} \right)_{x,i+1}^2 + a_{1} y_{0x,0} y_{0} + \sum_{s=1}^{i} \varphi_s y_s^{(\sigma)} h.
\]
Consider two cases:

1) \( \beta \alpha - 1 - 1 \geq 0 \) and \( \alpha^2 - 1 \geq 0 \);

2) \( \beta \alpha - 1 - 1 \leq 0 \) and \( \alpha^2 - 1 \leq 0 \).

1) Let \( \varphi \) and \( \mu \equiv 0 \). For \( v(x) = 1, \) \( x \in \bar{w}, \) and \( v(x) = p_i^2(x), i = 0, 1, \ldots, N, \) from Lemma 3 we obtain the inequalities

\[
\| \sqrt{ay_x^{(\sigma)}} \|_0^2 \geq \frac{h^2}{4c^2} \| \Delta_{0t} y \|_0^2 + \frac{h}{2c} \left( (a_N y_{x,N}^{(\sigma)})^2 + (a_1 y_{x,0}^{(\sigma)})^2 \right),
\]

\[
\| p_1 \sqrt{ay_x^{(\sigma)}} \|_0^2 \geq \frac{h^2}{4c^2} \| p_1 \Delta_{0t} y \|_0^2 - \frac{h}{2c} \sum_{i=1}^{N} a_i (y_x^{(\sigma)})^2 h + \frac{h}{2c} p_i^2(0) (a_1 y_{x,0}^{(\sigma)})^2.
\]  

Relation (36), together with (37) and (38), implies the inequality

\[
\frac{1}{2} \Delta_{0t} \left( \| y \|_0^2 + \delta_1 \| p_1 y \|_0^2 \right) + \left( \frac{\tau \Gamma(2 - \gamma)}{2(2 - 2^{1-\gamma})} \left( (3 - 2^{1-\gamma}) \sigma - 1 \right) + \frac{h^2}{4c^2} \right) \| \Delta_{0t} y \|_0^2 \\
+ \delta_1 \left( \frac{\tau \Gamma(2 - \gamma)}{2(2 - 2^{1-\gamma})} \left( (3 - 2^{1-\gamma}) \sigma - 1 \right) + \frac{h^2}{4c^2} \right) \| p_1 y \|_0^2 + \frac{\delta_1 h}{2} \left( \| y_x^{(\sigma)} \|_0^2 - \frac{1}{2c} \sum_{i=1}^{N} a_i (y_x^{(\sigma)})^2 h \right) \\
+ \frac{\tau \Gamma(2 - \gamma)}{2(2 - 2^{1-\gamma})} \left( (3 - 2^{1-\gamma}) \sigma - 1 \right) \frac{\alpha \beta + 1}{2 \alpha^2} (\Delta_{0t} y_0)^2 h + \frac{h}{2c} \left( (a_N y_{x,N}^{(\sigma)})^2 + (a_1 y_{x,0}^{(\sigma)})^2 \right) \\
+ \frac{h}{2c^2} \| p_1^2(0) (a_1 y_{x,0}^{(\sigma)})^2 + \frac{\delta_1}{2} (\alpha^2 - 1)(y_N^{(\sigma)})^2 + \| \sqrt{J^{(\sigma)}(y)} \|_0^2 + \delta_1 \| p_1 \sqrt{J^{(\sigma)}(y)} \|_0^2 \leq 0.
\]
By virtue of the condition (27), in this case, we have
\[\sigma \geq \frac{1}{3 - 2^{1-\gamma}} - \frac{h^2(2 - 2^{1-\gamma})}{2c_2\tau\gamma(3 - 2^{1-\gamma})\Gamma(2 - \gamma)},\]
\[\delta_1 \left(\|y_{x,i+1}\|_0^2 - \frac{1}{c_2} \sum_{i=1}^{N} a_i (y_{x,i})^2 h\right) \geq 0,\]
\[\delta_1 \geq 0, \quad \frac{\delta_1}{2}(\alpha^2 - 1) \geq 0, \quad \|\sqrt{J(\sigma)(y)}\|_2^2 + \delta_1 p_1 \sqrt{J(\sigma)(y)}\|_0^2 \geq 0,\]
it follows from (39) that
\[\Delta_0^\gamma (\|y\|_2^2 + \delta_1 \|p_1 y\|_0^2)\]
\[+ \frac{\tau\gamma\Gamma(2 - \gamma)}{2(2 - 2^{1-\gamma})} ((3 - 2^{1-\gamma})\sigma - 1) \frac{\alpha\beta + 1}{\alpha^2} (\Delta_0^\gamma y_0)^2 h + \frac{h}{c_2} \left((a_N y_{x,N})^2 + \frac{\beta}{\alpha} (a_1 y_{x,0})^2\right) \leq 0. \tag{40}\]
If \(\sigma \geq 1/(3 - 2^{1-\gamma}), \varphi \equiv 0\) and \(\mu \equiv 0\), then from (40) follows the inequality
\[\Delta_0^\gamma \|y\|_1^2 \leq 0. \tag{41}\]
where \(\|y\|_1 = (\|y\|_2^2 + \delta_1 \|p_1 y\|_0^2)^{1/2}\).
Consider the case in which \(0 \leq \sigma < 1/(3 - 2^{1-\gamma})\). We introduce the notation
\[\xi = \frac{\tau\gamma\Gamma(2 - \gamma)}{2(2 - 2^{1-\gamma})} ((3 - 2^{1-\gamma})\sigma - 1).\]
By virtue of the condition (27), in this case, we have \(-h^2/(4c_2) \leq \xi < 0\.
It follows from the boundary conditions (6) with \(\varphi\) and \(\mu \equiv 0\) that
\[\frac{\alpha\beta + 1}{\alpha} \Delta_0^\gamma y_0 = -\frac{2}{\xi} \left(a_N y_{x,N}^{(\sigma)} - \beta a_1 y_{x,0}^{(\sigma)}\right).\]
We square both sides of the last relation and divide the resulting relation by \((\alpha\beta + 1)\),
\[\frac{\alpha\beta + 1}{\alpha^2} (\Delta_0^\gamma y_0)^2 = \frac{4}{h^2(\alpha\beta + 1)} (a_N y_{x,N}^{(\sigma)})^2 \]
\[+ \frac{8\beta}{h^2(\alpha\beta + 1)} (a_N y_{x,N}^{(\sigma)})(a_1 y_{x,0}^{(\sigma)}) + \frac{4\beta^2}{h^2(\alpha\beta + 1)} (a_1 y_{x,0}^{(\sigma)})^2. \tag{42}\]
Inequality (40), together with (42), implies that
\[\Delta_0^\gamma (\|y\|_2^2 + \delta_1 \|p_1 y\|_0^2) + \left(\frac{4\xi}{(\alpha\beta + 1)h} + \frac{h}{c_2}\right) (a_N y_{x,N}^{(\sigma)})^2 \]
\[- \frac{8\beta\xi}{(\alpha\beta + 1)h} (a_N y_{x,N}^{(\sigma)})(a_1 y_{x,0}^{(\sigma)}) + \left(\frac{4\beta^2\xi}{(\alpha\beta + 1)h} + \frac{h\beta}{\alpha c_2}\right) (a_1 y_{x,0}^{(\sigma)})^2 \leq 0. \tag{43}\]
Note that
\[
\frac{4\xi}{(\alpha\beta+1)h} + \frac{h}{c_2} \geq \frac{h}{c_2} - \frac{h}{(\alpha\beta+1)c_2} = \frac{h(\alpha\beta)}{(\alpha\beta+1)c_2} > 0,
\]

The quadratic form
\[
\left( \frac{4\xi}{(\alpha\beta+1)h} + \frac{h}{c_2} \right) (a_N y^{(\sigma)}_{x,N})^2 - \frac{8\beta\xi}{(\alpha\beta+1)h} (a_N y^{(\sigma)}_{x,N})(a_1 y^{(\sigma)}_{x,0}) + \left( \frac{4\beta^2\xi}{(\alpha\beta+1)h} + \frac{h\beta}{\alpha c_2} \right) (a_1 y^{(\sigma)}_{x,0})^2
\]
is nonnegative for all values of \(a_N y^{(\sigma)}_{x,N}\) and \(a_0 y^{(\sigma)}_{x,0}\) if and only if
\[
\frac{16\beta^2\xi^2}{(\alpha\beta+1)^2h^2} - \left( \frac{4\xi}{(\alpha\beta+1)h} + \frac{h}{c_2} \right) \left( \frac{4\beta^2\xi}{(\alpha\beta+1)h} + \frac{h\beta}{\alpha c_2} \right) \leq 0,
\]
which, after simple transformations, acquires the form
\[
- \frac{4\beta}{\alpha c_2} \left( \frac{\tau^\gamma \Gamma(2-\gamma)}{2(2-2^{1-\gamma})} ((3-2^{1-\gamma})\sigma - 1) + \frac{h^2}{4c_2} \right) \leq 0.
\]

Consequently, the estimate (41) follows from inequality (43).

Let us rewrite (41) as
\[
\frac{1}{\Gamma(2-\gamma)} \sum_{s=0}^{n} (t_{n-s}^{1-\alpha} - t_{n-s}^{1-\alpha}) \frac{\|y^{s+1}\|^2 - \|y^s\|^2}{\tau} \leq 0. \tag{44}
\]

It is obvious that at \(n = 0\) the a priori estimate (28) follows from (44). Let us prove that (28) holds for \(n = 1, 2, \ldots\) by using the mathematical induction method. For this purpose, let us assume that the a priori estimate (28) takes place for all \(n = 0, 1, \ldots, k-1, k = 1, 2, \ldots\). From (44) at \(n = k\) one has
\[
\tau^{1-\gamma} \|y^{k+1}\|^2_1 \leq \sum_{s=1}^{k} \left( -t_{s+2}^{1-\gamma} + 2t_{s+1}^{1-\gamma} - t_{s}^{1-\gamma} \right) \|y^{s}\|^2_1 + \left( t_{s+1}^{1-\gamma} - t_{s}^{1-\gamma} \right) \|y^{0}\|^2_1. \tag{45}
\]

Since \(-t_{s+2}^{1-\gamma} + 2t_{s+1}^{1-\gamma} - t_{s}^{1-\gamma} > 0\) for all \(k = 1, 2, \ldots [5]\), and by the assumption of the mathematical induction \(\|y^{s}\|^2_0 \leq \|y^{0}\|^2_0\) at \(s = 1, 2, \ldots, k\), from (45) one obtains the following inequality:
\[
\tau^{1-\gamma} \|y^{k+1}\|^2_1 \leq \left( t_{k+1}^{1-\gamma} - t_{k}^{1-\gamma} \right) + \sum_{s=1}^{k} \left( -t_{s+2}^{1-\gamma} + 2t_{s+1}^{1-\gamma} - t_{s}^{1-\gamma} \right) \|y^{0}\|^2_1
\]
\[
= \left( t_{k+1}^{1-\gamma} - t_{k}^{1-\gamma} - t_{k+1}^{1-\gamma} + t_{k}^{1-\gamma} - t_{0}^{1-\gamma} \right) \|y^{0}\|^2_1 = \tau^{1-\gamma} \|y^{0}\|^2_1. \tag{46}
\]

2) The a priori estimate (28) for the second case, in which \(\beta\alpha^{-1} - 1 \leq 0, \alpha^2 - 1 \leq 0\), follows directly from the first case. Indeed, if we set \(y(x, t) = v(1-x, t)\), then the function \(v(x, t)\) satisfies the problem
\[
\Delta^{\gamma}_{0\alpha, n} v_i - (\bar{a} v^{(\sigma)}_{x})_{x, i} = \varphi^\gamma_{n, i}, \quad i = 1, 2, \ldots, N - 1, \tag{47}
\]
The estimate (28) holds for the solution \( v \) due to the relations

\[
\begin{align*}
\frac{1}{\beta} \Delta_{0,t}^\nu v_0 + \Delta_{0,t}^\nu v_N + \frac{2}{h} \left( \tilde{a}_N v_N^{(\sigma)} - \frac{1}{\beta} \tilde{a}_1 v_{x,0}^{(\sigma)} \right) &= \frac{2}{\beta h} \mu(t_{n+1/2}) + \varphi_N + \frac{1}{\beta} \bar{\varphi}_0, \\
v_i^0 &= u_0(1 - x_i),
\end{align*}
\]

where \( \tilde{a}_i = a_{N-i+1}, \bar{\varphi}_i^n = \varphi_{N-i}^n \).

By virtue of the conditions

\[
0 < c_1 \leq \bar{a} \leq c_2, \quad \alpha \beta^{-1} - 1 \geq 0, \quad \alpha^{-2} - 1 \geq 0, \quad \sigma \geq \frac{1}{3 - 2^{1-\gamma}} - \frac{h^2(2 - 2^{1-\gamma})}{2c_2 \tau \gamma (3 - 2^{1-\gamma}) \Gamma(2 - \gamma)},
\]

the estimate (28) holds for the solution \( v(x,t) \) of problem (47)–(49) for \( \varphi \equiv 0 \) and \( \mu \equiv 0 \). Consequently, by virtue of the relations \( \|v\|_0^2 = \|y\|_0^2, \|p_1 v\|_0^2 = \|p_1(1-x)y\|_0^2 \) and \( v_0^2 = \alpha^{-2} y_0^2 \), the solution of problem (5)–(7) for \( \varphi \equiv 0 \) and \( \mu \equiv 0 \) satisfies the a priori estimate (28). The proof of Theorem 4 is complete.

**Theorem 5.** If \( (\beta \alpha^{-1} - 1)(\alpha^2 - 1) > 0 \), then the condition

\[
\sigma \geq \frac{1}{3 - 2^{1-\gamma}} - \frac{h^2(2 - 2^{1-\alpha})(1 - \varepsilon)}{2c_2 \tau \alpha(3 - 2^{1-\alpha}) \Gamma(2 - \alpha)}, \quad 0 < \varepsilon < 1
\]

is sufficient for the stability of the difference scheme (5)–(7), and its solution satisfies the estimate

\[
\frac{1}{\Gamma(2 - \gamma)} \sum_{s=0}^{n} \left( t_{n-s+1}^{1-\gamma} - t_{n-s}^{1-\gamma} \right) \| y^{s+1} \|_0^2 \leq \frac{t_{n+1}^{1-\gamma}}{\Gamma(2 - \gamma)} \| y_0 \|_0^2 + M \left( \sum_{s=0}^{n} \left( \| \varphi_s \|_0^2 + \mu^2(t_{s+\sigma}) \right) \right),
\]

where \( M > 0 \) is a known number independent of \( h, \tau \) and \( t_n \).

**Proof.** The solution of the difference scheme (5)–(7) satisfies identity (36).

Consider two cases:

1) \( \beta \alpha^{-1} - 1 > 0, \alpha^2 - 1 > 0 \)

2) \( \beta \alpha^{-1} - 1 < 0, \alpha^2 - 1 < 0 \).

1) If \( v(x) = 1 \) and \( v(x) = p_1^2(x), i = 0, 1, \ldots, N \), then Lemma 4 readily implies the inequalities

\[
\begin{align*}
\| \sqrt{a} y^{(\sigma)}_x \|_0^2 &\geq \frac{h^2}{4c_2(1 + \varepsilon_1)} \| \Delta_{0,t}^\nu y \|_0^2 + \frac{h}{2c_2} \left( (a_N y^{(\sigma)}_{x,N})^2 + (a_1 y^{(\sigma)}_{x,0})^2 \right) - \frac{h^2}{4c_2 \varepsilon_1} \| \varphi \|_0^2, \\
\| p_1 \sqrt{a} y^{(\sigma)}_x \|_0^2 &\geq \frac{h^2}{4c_2(1 + \varepsilon_1)} \| \Delta_{0,t}^\nu y \|_0^2 - \frac{h}{2c_2} \sum_{i=1}^{N} a_i(y^{(\sigma)}_{x,i})^2 - \frac{h}{2c_2} \| a_1 y^{(\sigma)}_{x,0} \|_0^2 - \frac{h^2}{4c_2 \varepsilon_1} \| p_1 \varphi \|_0^2.
\end{align*}
\]

Let us estimate the right-hand side of identity (36). By virtue of the relation

\[
y^{(\sigma)}_i = y^{(\sigma)}_N - \sum_{s=i+1}^{N} y^{(\sigma)}_{x,s} h, \quad i = 0, 1, \ldots, N - 1,
\]
we have

\[
((1 + \delta_1 p^2_1) \varphi, y^{(\sigma)}) = y_N^{(\sigma)} \sum_{i=1}^{N-1} (1 + \delta_1 p^2_i(x_i)) \varphi_i h - \sum_{i=1}^{N-1} (1 + \delta_1 p^2_i(x_i)) \varphi_i h \sum_{s=i+1}^{N} y^{(\sigma)}_{x,s} h
\]

\[
\leq \frac{\varepsilon_2}{2} (y_N^{(\sigma)})^2 + \frac{\bar{\gamma}_1}{2\varepsilon_2} \sum_{i=1}^{N-1} (1 + \delta_1 p^2_i(x_i)) \varphi_i^2 h
\]

\[- \sum_{i=1}^{N-1} \sqrt{(1 + \delta_1 p_1(x_{i+1})) a_{i+1} y^{(\sigma)}_{x,i} h} \sum_{s=i+1}^{N} \frac{(1 + \delta_1 p^2_s(x_s))}{\sqrt{(1 + \delta_1 p_1(x_{i+1})) a_{i+1}}} \varphi_i h
\]

\[
\leq \frac{\varepsilon_2}{2} (y_N^{(\sigma)})^2 + \frac{\varepsilon_1}{2} \sum_{i=2}^{N} (1 + \delta_1 p_1(x_i)) a_i (y^{(\sigma)}_{x,i})^2 h + \left( \frac{\bar{\gamma}_1}{2\varepsilon_2} + \frac{\bar{\gamma}_1 \bar{\gamma}_2}{2\varepsilon_1} \right) \sum_{i=1}^{N-1} (1 + \delta_1 p^2_i(x_i)) \varphi_i^2 h,
\]

(54)

where \( \varepsilon_1, \varepsilon_2 > 0, \bar{\gamma}_1 = \sum_{i=1}^{N-1} (1 + \delta_1 p_1(x_i)) h, \bar{\gamma}_2 = \sum_{i=1}^{N-1} (1 + \delta_1 p^2_i(x_i))^{-1} a_i^{-1} h, \) and

\[
\mu(t_{n+\sigma}) = \mu(t_{n+\sigma}) + 0.5 h(\varphi_N + \bar{\varphi}_0).
\]

Relation (36), together with inequalities (54) and (55), implies the inequality

\[
\frac{1}{2} \Delta^2_{\mu_t} (\|y\|^2 + \delta_1 \|p_1 y\|^2) + \frac{\tau \Gamma(2 - \gamma)}{2(2 - 2^{1-\gamma})} ((3 - 2^{1-\gamma})\sigma - 1) (\|\Delta^2_{\mu_t} y\|^2 + \delta_1 \|p_1 \Delta^2_{\mu_t} y\|^2)
\]

\[
+ (1 - \varepsilon_1) \|\sqrt{a} y^{(\sigma)}_{x}\|^2 \delta_1 + \delta_1 (1 - \varepsilon_1) \|p_1 \sqrt{a} y^{(\sigma)}_{x}\|^2 \delta_1 \|y^{(\sigma)}_{x}\|^2 + \frac{\delta_1 h}{2} \|y^{(\sigma)}_{x}\|^2
\]

\[
+ \|J^{(\sigma)}(y)\|^2 + \delta_1 \|p_1 \sqrt{J^{(\sigma)}(y)}\|^2 \leq M_2(\varepsilon_1, \varepsilon_2) (\|\varphi\|^2 + \mu^2(t_{n+\sigma})).
\]

(56)

Taking into account inequalities (52) and (53), from inequality (56) with \( \varepsilon_2 = \delta_1 (\alpha^2 - 1)/2, \) we obtain the inequality

\[
\frac{1}{2} \Delta^2_{\mu_t} (\|y\|^2 + \delta_1 \|p_1 y\|^2) + \left( \frac{\tau \Gamma(2 - \gamma)}{2(2 - 2^{1-\gamma})} ((3 - 2^{1-\gamma})\sigma - 1) + \frac{h^2(1 - \varepsilon_1)}{4c_2 (1 + \varepsilon_1)} \right) \|\Delta^2_{\mu_t} y\|^2
\]

\[
+ \delta_1 \left( \frac{\tau \Gamma(2 - \gamma)}{2(2 - 2^{1-\gamma})} ((3 - 2^{1-\gamma})\sigma - 1) + \frac{h^2(1 - \varepsilon_1)}{4c_2 (1 + \varepsilon_1)} \right) \|p_1 y\|^2
\]

\[
+ \frac{\delta_1 h}{2} \left( \|y^{(\sigma)}_{x}\|^2 - \frac{1}{c_2} \sum_{i=1}^{N} a_i (y^{(\sigma)}_{x,i})^2 h \right) + \frac{\tau \Gamma(2 - \gamma)}{2(2 - 2^{1-\gamma})} ((3 - 2^{1-\gamma})\sigma - 1) \frac{\alpha \beta + 1}{2\alpha^2} (\Delta^2_{\mu_t} y_0)^2 h,
\]
+ \frac{h}{2c_2} (a_N y^{(σ)}_{x,N})^2 + (a_1 y^{(σ)}_{x,0})^2 \right) + \frac{h}{2c_2} \delta_1 p_1^2(0)(a_1 y^{(σ)}_{x,0})^2 \leq M_3(ε_1) \left(|φ|_0^2 + μ^2(t_{n+σ})\right), \quad (57)

where \(0 < ε_1 < 1\).

Then we have \(ε = 2ε_1/(1 + ε_1), 0 < ε < 1\).

Inequality (57), together with the assumptions of Theorem 5, implies the inequality

\[ (||y||_2^2 + δ_1 ||p_1 y||_0^2) + \tau \left(α - \frac{1}{2}\right) \frac{αβ + 1}{α^2} y^2 τ h \left(\frac{α}{c_2} \left(\frac{α}{c_2} \left(\frac{a_N y^{(σ)}_{x,N}}{2} + β a_1 y^{(σ)}_{x,0}\right)\right) \right) \leq M_4(ε) \left(|φ|_0^2 + μ^2(t_{n+σ})\right). \quad (58)\]

If \(σ ≥ 1/(3 - 2^{-1-γ})\), then from (58) we have

\[ Δ_0^γ ||y||_2^2 \leq M \left(|φ|_0^2 + μ^2(t_{n+σ})\right). \quad (59)\]

where \(||y||_1 = (||y||_2^2 + δ_1 ||p_1 y||_0^2)^{1/2}\) and \(M = M_4(ε)\).

Consider the case in which \(0 ≤ σ < 1/(3 - 2^{-1-γ})\). We introduce the notation

\[ ξ = \frac{γεΓ(2 − γ)}{2(2 − 2^{-1-γ})} (3 − 2^{-1-γ})σ - 1). \]

In this case, by virtue of the condition (50), we have \(-h^2(1 − ε)/(4c_2) ≤ ξ < 0\).

It follows from the boundary conditions (6) that

\[ \frac{αβ + 1}{α} Δ_0^γ y_0 = -\frac{2}{h} \left(\frac{a_N y^{(σ)}_{x,N}}{2} - \beta a_1 y^{(σ)}_{x,0}\right) + \frac{2}{h} μ(t_{n+σ}) + φ_N + β φ_0, \]

whence we obtain the inequality

\[ \frac{αβ + 1}{α^2} Δ_0^γ y_0^2 \leq (1 + ε_3) \frac{4}{(αβ + 1) h^2} \left(\frac{a_N y^{(σ)}_{x,N}}{2} - \beta a_1 y^{(σ)}_{x,0}\right)^2 + (1 + \frac{1}{ε_3}) \frac{4}{h^2} μ^2(t_{n+σ}), \quad ε_3 > 0, \]

or (after multiplication of the last inequality by \(ξ < 0\))

\[ \frac{αβ + 1}{α^2} ξ Δ_0^γ y_0^2 \geq \frac{4ξ}{1 - ε}(αβ + 1) h^2 \left(\frac{a_N y^{(σ)}_{x,N}}{2} - \beta a_1 y^{(σ)}_{x,0}\right)^2 - \frac{1 - ε}{c_2 ε} μ^2(t_{n+σ}), \quad (60)\]

where \(ε = ε_3/(1 + ε_3), 0 < ε < 1\) for \(ε_3 > 0\).

From inequalities (58) and (60), we have

\[ Δ_0^γ (||y||_2^2 + δ_1 ||p_1 y||_0^2) + \left(\frac{4ξ}{1 - ε}(αβ + 1) h^2 \right) \left(\frac{h}{c_2} \left(\frac{α}{c_2} \left(\frac{a_N y^{(σ)}_{x,N}}{2} - \beta a_1 y^{(σ)}_{x,0}\right)\right) \right) \leq M_5(ε) \left(|φ|_0^2 + μ^2(t_{n+1/2})\right). \quad (61)\]
The quadratic form
\[
\left( \frac{4\xi}{(1-\varepsilon)(\alpha\beta+1)h} + \frac{h}{c_2} \right) \left( a_Ny_x,N \right)^2
- \frac{8\beta\xi}{(1-\varepsilon)(\alpha\beta+1)h} \left( a_Ny_x,N \right) \left( a_1y_{x,0}^{(\sigma)} \right) + \left( \frac{4\beta^2\xi}{(1-\varepsilon)(\alpha\beta+1)h} + \frac{h\beta}{\alpha c_2} \right) \left( a_1y_{x,0}^{(\sigma)} \right)^2
\]
is nonnegative because
\[
\frac{4\xi}{(1-\varepsilon)(\alpha\beta+1)h} + \frac{h}{c_2} \geq \frac{h}{c_2} - \frac{h}{(\alpha\beta+1)c_2} = \frac{\alpha\beta h}{\alpha\beta+1} > 0,
\]
\[
\frac{16\beta^2\xi^2}{(1-\varepsilon)^2(\alpha\beta+1)^2h^2} \geq \left( \frac{4\xi}{(1-\varepsilon)(\alpha\beta+1)h} + \frac{h}{c_2} \right) \left( \frac{4\beta^2\xi}{(1-\varepsilon)(\alpha\beta+1)h} + \frac{\beta h}{\alpha c_2} \right) \leq 0.
\]
The last inequality is equivalent to the relation
\[
-\frac{4\beta}{\alpha(1-\varepsilon)c_2} \left( \frac{\tau^\gamma\Gamma(2-\gamma)}{2(2-1^\gamma)} \left( (3-2^1\gamma)\sigma - 1 \right) + \frac{h^2(1-\varepsilon)}{4c_2} \right) \leq 0.
\]
Consequently, from (61), we have (59) with \( M = M_5(\varepsilon) \).

By multiplying inequality (59) by \( \tau \) and by summing with respect to \( s \) from 0 to \( n \), we obtain the a priori estimate (51).

2) If \( \beta\alpha^{-1} - 1 < 0 \) and \( \alpha^2 - 1 < 0 \), then the second case follows from the first one. Indeed, if we introduce the notation \( y(x, t) = v(1-x, t) \), then the function \( v(x, t) \) satisfies problem (47)–(49).

The solution \( v(x, t) \) of problem (47)–(49) satisfies the estimate (51). Consequently, by virtue of the relations \( \|v\|_0^2 = \|y\|_0^2 \), \( \|pv\|_0^2 = \|p(1-x)y\|^2_0 \) and \( v_0^2 = \alpha^{-2}y_0^2 \), the a priori estimate (28) holds for the solution of problem (5)–(7). The proof of Theorem 5 is complete.

The resulting a priori estimates imply the convergence of the solution of the difference scheme (5)–(7) to the solution of the differential problem (1)–(3).

The a priori estimates (21) and (51) can be obtained for the case of \( \beta = \alpha \neq 1 \) as well. This follows from the inequalities
\[
u^2(1, t) = \left( \frac{1}{1-\alpha} \int_0^1 u_x(x, t) \, dx \right)^2 \leq \frac{1}{(1-\alpha)^2} \|u_x\|_0^2,
\]
\[
(y_N^{(\sigma)})^2 = \left( \frac{1}{1-\alpha} \sum_{i=1}^N y_{x,i}^{(\sigma)} h \right)^2 \leq \frac{1}{c_1(1-\alpha)^2} \|\sqrt{a}y_{x}^{(\sigma)}\|_0^2.
\]

3. Numerical Results

Numerical calculations are performed for a test problem when the function
\[
u(x, t) = \left( (1-3\alpha)x^3 + \alpha x^2 + \alpha x + \alpha \right) \left( t^3 - t^2 + t + 1 \right)
\]
is the exact solution of the problem (1)–(3) with the coefficient \( k(x) = x^2 \).
### Table 1. \( \gamma = 0.5, \alpha = 3, \beta = 2, T = 1, \sigma = 1, h^2 = \tau^{2-\gamma} \)

| \( h \) | \( \max_{0 \leq n \leq N_T} |z^n|_0 \) | CO in \( |\cdot|_0 \) | \( \|z\|_{C(\omega_{hr})} \) | CO in \( \| \cdot \|_{C(\omega_{hr})} \) |
|---|---|---|---|---|
| 1/20 | 3.03169 \cdot 10^{-2} | | 5.50676 \cdot 10^{-2} | |
| 1/40 | 7.61510 \cdot 10^{-3} | 1.993 | 1.38318 \cdot 10^{-2} | 1.993 |
| 1/80 | 1.90780 \cdot 10^{-3} | 1.997 | 3.46463 \cdot 10^{-3} | 1.997 |

### Table 2. \( \gamma = 0.5, \alpha = 2, \beta = 5, T = 1, \sigma = 1, h^2 = \tau^{2-\gamma} \)

| \( h \) | \( \max_{0 \leq n \leq N_T} |z^n|_0 \) | CO in \( |\cdot|_0 \) | \( \|z\|_{C(\omega_{hr})} \) | CO in \( \| \cdot \|_{C(\omega_{hr})} \) |
|---|---|---|---|---|
| 1/20 | 6.35368 \cdot 10^{-3} | | 7.31523 \cdot 10^{-3} | |
| 1/40 | 1.56940 \cdot 10^{-3} | 2.017 | 1.80908 \cdot 10^{-3} | 2.016 |
| 1/80 | 3.90276 \cdot 10^{-4} | 2.008 | 4.49971 \cdot 10^{-4} | 2.007 |

### Table 3. \( \gamma = 0.5, \alpha = 0.7, \beta = 0.1, T = 1, \sigma = 1, h^2 = \tau^{2-\gamma} \)

| \( h \) | \( \max_{0 \leq n \leq N_T} |z^n|_0 \) | CO in \( |\cdot|_0 \) | \( \|z\|_{C(\omega_{hr})} \) | CO in \( \| \cdot \|_{C(\omega_{hr})} \) |
|---|---|---|---|---|
| 1/20 | 2.19544 \cdot 10^{-2} | | 2.67764 \cdot 10^{-2} | |
| 1/40 | 5.50422 \cdot 10^{-3} | 1.996 | 6.71201 \cdot 10^{-3} | 1.996 |
| 1/80 | 1.37776 \cdot 10^{-3} | 1.998 | 1.67992 \cdot 10^{-3} | 1.998 |

### Table 4. \( \gamma = 0.2, \alpha = 1.1, \beta = 1.1, T = 1, \sigma = 1, h^2 = \tau^{2-\gamma} \)

| \( h \) | \( \max_{0 \leq n \leq N_T} |z^n|_0 \) | CO in \( |\cdot|_0 \) | \( \|z\|_{C(\omega_{hr})} \) | CO in \( \| \cdot \|_{C(\omega_{hr})} \) |
|---|---|---|---|---|
| 1/20 | 3.85126 \cdot 10^{-2} | | 4.38852 \cdot 10^{-2} | |
| 1/40 | 9.65615 \cdot 10^{-3} | 1.995 | 1.10031 \cdot 10^{-2} | 1.996 |
| 1/80 | 2.42041 \cdot 10^{-3} | 1.996 | 2.75763 \cdot 10^{-3} | 1.996 |

The errors \( z = y - u \) and convergence order (CO) in the norms \( |\cdot|_0 \) and \( \| \cdot \|_{C(\omega_{hr})} \) at \( \sigma = 1 \) are given in Tables 1–7.

Each of Tables 1–7 shows that when we take \( h^2 = \tau^{2-\gamma} \), as the number of spatial subintervals/time steps is decreased, a reduction in the maximum error takes place, as expected and the convergence order of the approximate scheme is \( O(h^2) \), where the convergence order is given by the formula \( \text{CO} = \log_{2} \frac{\|z_1\|}{\|z_2\|} \).
Table 5. $\gamma = 0.2, \alpha = 0.9, \beta = 0.9, T = 1, \sigma = 1, h^2 = \tau^{2-\gamma}$

| $h$  | $\max_{0 \leq n \leq N_T} \|z^n\|_0$ | CO in $\|\cdot\|_0$ | $\|z\|_{C(\omega_{h\tau})}$ | CO $\|\cdot\|_{C(\omega_{h\tau})}$ |
|------|-------------------------------------|-----------------|-----------------|-----------------|
| 1/20 | 3.26779 · 10^{-2}                   |                 | 3.66507 · 10^{-2} |                 |
| 1/40 | 8.19304 · 10^{-3}                   | 1.996           | 9.18862 · 10^{-2} | 1.996           |
| 1/80 | 2.05366 · 10^{-3}                   | 1.996           | 2.30287 · 10^{-3} | 1.996           |

Table 6. $\gamma = 0.8, \alpha = 200, \beta = 100, T = 1, \sigma = 1, h^2 = \tau^{2-\gamma}$

| $h$  | $\max_{0 \leq n \leq N_T} \|z^n\|_0$ | CO in $\|\cdot\|_0$ | $\|z\|_{C(\omega_{h\tau})}$ | CO $\|\cdot\|_{C(\omega_{h\tau})}$ |
|------|-------------------------------------|-----------------|-----------------|-----------------|
| 1/20 | 1.27484 · 10^{0}                   |                 | 2.14188 · 10^{0} |                 |
| 1/40 | 3.18346 · 10^{-1}                   | 2.002           | 5.35201 · 10^{-1} | 2.001           |
| 1/80 | 7.95685 · 10^{-2}                   | 2.000           | 1.33790 · 10^{-1} | 2.000           |

Table 7. $\gamma = 0.8, \alpha = 100, \beta = 200, T = 1, \sigma = 1, h^2 = \tau^{2-\gamma}$

| $h$  | $\max_{0 \leq n \leq N_T} \|z^n\|_0$ | CO in $\|\cdot\|_0$ | $\|z\|_{C(\omega_{h\tau})}$ | CO $\|\cdot\|_{C(\omega_{h\tau})}$ |
|------|-------------------------------------|-----------------|-----------------|-----------------|
| 1/20 | 6.49129 · 10^{-1}                   |                 | 1.09160 · 10^{0} |                 |
| 1/40 | 1.62100 · 10^{-1}                   | 2.002           | 2.72769 · 10^{-1} | 2.001           |
| 1/80 | 4.05159 · 10^{-2}                   | 2.000           | 6.81875 · 10^{-2} | 2.000           |

Table 8. $\gamma = 0.4, \alpha = 0.1, \beta = 10, T = 1, \sigma = 1, h^2 = \tau^{2-\gamma}$

| $h$  | $\max_{0 \leq n \leq N_T} \|z^n\|_0$ | $\|z\|_{C(\omega_{h\tau})}$ |
|------|-------------------------------------|-----------------|
| 1/20 | 2.41006 · 10^{-3}                   | 4.48421 · 10^{-3} |
| 1/40 | 5.36386 · 10^{-3}                   | 1.02862 · 10^{-3} |
| 1/80 | 5.21782 · 10^{-1}                   | 1.0008 · 10^{-1} |

Table 8 shows that if $\alpha$ and $\beta$ do not satisfy the conditions $|\alpha|, |\beta| \leq 1$ or $|\alpha|, |\beta| \geq 1$, then the difference scheme (5)–(7) may be unstable.

4. Conclusion

The results obtained in the present paper allow us to apply the method of energy inequalities to finding a priori estimate for nonlocal boundary value problems for the time-fractional diffusion equation in differential and
difference settings. It is interesting to note that the condition
\[
\sigma \geq \frac{1}{3 - 2^{1-\gamma}} - \frac{h^2(2 - 2^{1-\gamma})}{2c_2 \tau \gamma(3 - 2^{1-\gamma}) \Gamma(2 - \gamma)}
\]
at \( \gamma = 1 \) turns into the well-known condition
\[
\sigma \geq \frac{1}{2} - \frac{h^2}{4c_2 \tau}
\]
of stability of difference schemes for the classical diffusion equation.

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