A NON-LOCAL POPULATION MODEL OF LOGISTIC TYPE EQUATION

LI MA, LIANG CHENG

Abstract. In this paper, we propose a new non-local population model of logistic type equation on a bounded Lipschitz domain in the whole Euclidean space. This model preserves the $L^2$ norm, which is called mass, of the solution on the domain. We show that this model has the global existence, stability and asymptotic behavior at time infinity.

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1. Introduction

In this work, we propose a new population model with non-local term. This non-local term keeps the mass, a global quantity to be defined below, constant for all time. Let’s first review some previous study of population modelings.

After a critical study of Malthus’s population model, people believe that a good population model should have good behavior. This makes it come to the logistic model, which is a slight modification of Malthus’s model. By definition, the logistic model is a population model such that it describes the changes over time of a population occupying a single small region. For more models and the history of population modeling, we refer to the work [15], and the books [7] and [13]. In mathematical language, the logistic model can be stated as below. Let $P$ be the population quantity. Then the change rate of $P$ is the difference between the birth rate $\frac{dB}{dt}$ and the death rate $\frac{dD}{dt}$, i.e.,

$$\frac{dP}{dt} = \frac{dB}{dt} - \frac{dD}{dt}.$$

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From the experimental observation, we put
\[
\frac{dB}{dt} = aP + bP^2
\]
and
\[
\frac{dD}{dt} = cP + dP^2.
\]
where \(a, b, c, d\) are constants such that \(a > c\) and \(d > b\). Hence we obtain
\[
\frac{dP}{dt} = (a - c)P - (d - b)P^2.
\]
Let
\[
r = a - c
\]
and
\[
K = \frac{a - c}{d - b}.
\]
Then we get the logistic model
\[
\frac{dP}{dt} = rP(1 - \frac{P}{K}).
\]
Here the growth rate \(r\) represents the change at which the population may grow if it were unencumbered by environmental degradation, and the parameter \(K\) represents the carrying capacity of the system considered. By definition, the carrying capacity is the population level at which the birth and death rates of a species exactly match, resulting in a stable population over time. Hence in some modeling, one may assume that \(K = K(t)\) is a periodic function in time variable.

The drawback of the model above is that it ignores the impact of the environmental condition to the population. When the environmental condition on the region \(D\), a bounded Lipschitz domain in \(\mathbb{R}^n\), is considered, one encounters the following diffusion model of logistic type on \(D\)
\[
(1) \quad u_t = \Delta u + ru(1 - \frac{u}{K}),
\]
where \(u = u(t, x)\) is the population quantity such that \(u = u(t, x) > 0\) for \(x \in D\) and \(u(t, x) = 0\) on \(\partial D\), \(t > 0\). (1) will be also called the Logistic equation as considered in [3]. Note that the equation (1) is a local model such that the value \(u(x, t)\) at \((x, t)\) depends only on its immediate surroundings. We refer to [3], [6], and [14] for related models.

We now turn to our main subject. We now turn to our new subject. In this paper, we shall modify (2) to the following non-local logistic
equation

\[
\begin{cases}
\partial_t u = \Delta u + \lambda(t)u + a(x)(u - u^p) & \text{in } D \times \mathbb{R}_+, \\
 u(x, 0) = g(x) & \text{in } D, \\
 u(x, t) = 0 & \text{on } \partial D
\end{cases}
\]

where \( p > 1 \) and \( a(x) > 0 \) is a non-trivial Lipschitz function on the closure of the domain \( D \), which has the positive solution and preserves the \( L^2 \) norm. By definition, we call the integral quantity of \( u(x, t) \),

\[
\int_D u^2 \, dx,
\]

the mass of the population model. Likewise,

\[
\frac{1}{2} \frac{d}{dt} \int_D u^2 \, dx = - \int_D \nabla u \cdot \nabla u \, dx + \lambda(t) \int_D u^2 \, dx + \int_D a(u^2 - u^{p+1}) \, dx.
\]

Thus, one must have

\[
\lambda(t) = \frac{\int_D (|\nabla u|^2 + a(u^{p+1} - u^2)) \, dx}{\int_D g^2 \, dx}
\]

to preserve the \( L^2 \) norm. Without loss of generality we assume \( \int_D g^2 \, dx = 1 \). Then we consider the following problem on the bounded domain \( D \)

\[
\begin{cases}
\partial_t u = \Delta u + \lambda(t)u + a(u - u^p) & \text{in } D \times \mathbb{R}_+ \\
 u(x, 0) = g(x) & \text{in } D \\
 u(x, t) = 0 & \text{on } \partial D
\end{cases}
\]  

(2)

where \( p > 1 \), \( \lambda(t) = \int_D (|\nabla u|^2 + a(u^{p+1} - u^2)) \, dx \), \( g(x) \geq 0 \) in \( D \), \( \int_D g^2 \, dx = 1 \) and \( g \in C^1(D) \).

Similar to the global existence results obtained in C.Caffarelli and F.Lin [2] and our previous work [10] (see also related works [11], [9], and [8]), we have following result.

**Theorem 1.** Problem (2) has a global solution \( u(t) \in L^\infty(\mathbb{R}_+, H^1_0(D)) \cap L^\infty(\mathbb{R}_+, L^{p+1}(D)) \cap L^2_{\text{loc}}(\mathbb{R}_+, H^2(D)) \).

**Remark 2.** We note that solutions of (2) have automatically higher regularity for \( t > 0 \). Indeed, the bound of \( \lambda(t) \) (see (11)) and the standard parabolic estimates imply that solutions are Hölder continuous. Then coming back to \( \lambda(t) \), it would be a Hölder continuous function in time. A bootstrap argument implies that \( u \) is smooth in both spatial and time variables if we assume \( a \) is smooth function.

We also have the stability results for (2).

**Theorem 3.** Let \( u, v \) be the two bounded solutions to problem (2) with initial data \( g_u, g_v \) at \( t = 0 \), where \( g_u, g_v \in H^1(D) \cap L^\infty(D) \). Then

\[
||u - v||_{L^2}^2 \leq ||g_u - g_v||_{L^2}^2 \exp(C_1t)
\]

and

\[
||u - v||_{H^1}^2 \leq ||g_u - g_v||_{H^1}^2 \exp(C_2t),
\]
where $C_1, C_2$ are the constants depending on the upper bound of $\|g_u\|_{H^1(D)}, \|g_v\|_{H^1(D)}$ and $\|g_u\|_{L^\infty}, \|g_v\|_{L^\infty}$. In particular, the solution to problem (2) is unique.

As the simple applications to theorem 1 we can study asymptotic behavior of $u(t)$ of problem (2).

**Corollary 4.** Suppose $u(t)$ is the solution to problem (2). Then one can take $t_i \to \infty$ such that $\lambda(t_i) \to \lambda_\infty$, $u(x, t_i) \to u_\infty(x)$ in $H^1_0(D)$ and $u_\infty$ solves the equation $\Delta u_\infty + \lambda_\infty u_\infty + a(u_\infty - u^p_\infty) = 0$ in $D$ with $\int_D |u_\infty|^2 dx = 1$.

In the next section, we prove the global existence, stability and asymptotic behavior of solutions to the problem (2). In particular, we give the proofs of theorem 1 to theorem 3, and corollary 4. In section ?? we give our conclusion based on our study of problem (2).

## 2. GLOBAL EXISTENCE AND STABILITY PROPERTY

In this section we study the global existence, stability and asymptotic behavior of solutions to the problem (2).

**Proof of theorem 1.** Let us define a series $u^{(k)}$ as

\[\begin{align*}
&\left\{\begin{array}{l}
 u^{(0)} = g, \\
 \lambda^{(k)}(t) = \int_D (|\nabla u^{(k)}|^2 + a(x))(u^{(k)})^{p+1} - (u^{(k)})^2 dx, \\
 \partial_t u^{(k+1)} = \Delta u^{(k+1)} + \lambda^{(k)}(t) u^{(k+1)} + a(x)(u^{(k+1)} - (u^{(k+1)})^p), \\
 u^{(k+1)}(x, 0) = g(x), \\
 u^{(k+1)}(x, t) = 0, \quad \text{on } \partial D
\end{array}\right. \\
&\hspace{1cm}k \geq 0
\end{align*}\]

a series of initial boundary value problems of linear parabolic systems.

To prove the convergence of series $\{u^{(k)}\}$ constructed above, we estimate for $k \geq 0$

\[\frac{1}{2} \frac{d}{dt} \int_D |\nabla u^{(k+1)}|^2 dx + \frac{1}{2} \frac{d}{dt} \int_D |\Delta u^{(k+1)}|^2 dx + p \int_D a(x)(u^{(k+1)})^{p-1} |\nabla u^{(k+1)}|^2 dx = \lambda^{(k)}(t) \int_D |\nabla u^{(k+1)}|^2 dx + \int_D a(x) |\nabla u^{(k+1)}|^2 dx + \int_D (\nabla u^{(k+1)} \cdot \nabla a)(u^{(k+1)} - (u^{(k+1)})^p dx, \]

\[\frac{1}{2} \frac{d}{dt} \int_D |\nabla u^{(k+1)}|^2 dx + \int_D |u^{(k+1)}|^2 dx + \frac{1}{p + 1} \frac{d}{dt} \int_D a(x)(u^{(k+1)})^{p+1} dx = \frac{\lambda^{(k)}(t)}{2} \frac{d}{dt} \int_D |u^{(k+1)}|^2 dx + \frac{1}{2} \frac{d}{dt} \int_D a(x)|u^{(k+1)}|^2 dx. \]

\[\frac{1}{p + 1} \frac{d}{dt} \int_D (u^{(k+1)})^{p+1} dx + \int_D p(u^{(k+1)})^{p-1} |\nabla u^{(k+1)}|^2 dx + \int_D a(x)(u^{(k+1)})^{2p} dx \]
\[ \lambda^{(k)}(t) \int_D (u^{(k+1)})^{p+1} dx + \int_D a(x)(u^{(k+1)})^{p+1} dx. \]

Now we denote \( M = ||a||_{C^1(D)}. \) By \((4)\), we get

\[
\frac{1}{2} \frac{d}{dt} \int_D |\nabla u^{(k+1)}|^2 dx
\leq (\lambda^{(k)}(t) + M) \int_D |\nabla u^{(k+1)}|^2 dx + \int_D (\nabla u^{(k+1)} \cdot \nabla a)(u^{(k+1)} - (u^{(k+1)})^p) dx
\leq (\lambda^{(k)}(t) + M) \int_D |\nabla u^{(k+1)}|^2 dx + \frac{1}{4\epsilon} \int_D |\nabla u^{(k+1)}|^2 |\nabla a|^2 dx
\]

\[
+ \epsilon \int_D (u^{(k+1)} - (u^{(k+1)})^p)^2 dx
\leq (\lambda^{(k)}(t) + M + \frac{M^2}{4\epsilon}) \int_D |\nabla u^{(k+1)}|^2 dx + \epsilon c_1 \int_D (u^{(k+1)})^{2p} dx,
\]

where \( c_1 \) is a constant only depending on \( D \). Moreover, by \((3)\), we have

\[
\frac{1}{p+1} \frac{d}{dt} \int_D (u^{(k+1)})^{p+1} dx + \int_D a(x)(u^{(k+1)})^{2p} dx \leq (\lambda^{(k)}(t) + M) \int_D (u^{(k+1)})^{p+1} dx.
\]

Now, we denote \( \tilde{\lambda}(t) = \int_D |\nabla u|^2 dx + M \int_D u^{p+1} dx \). Note that \( a(x) \) is a positive Lipschitz function on the compact domain \( D \) and we may assume \( a(x) \geq c_0 > 0 \). Hence by \((7)\) and \((8)\), we have

\[
\frac{d}{dt} \tilde{\lambda}^{k+1}(t) + ((p + 1)Mc_0 - 2\epsilon c_1) \int_D (u^{(k+1)})^{2p} dx \leq c_2(\lambda^k(t) + c_2)\tilde{\lambda}^{k+1}(t),
\]

where \( c_2 \) is a constant only depending on \( p, M, \epsilon \). We choose \( \epsilon \) such that \((p + 1)Mc_0 = 3\epsilon c_1 \) and denote \( c_3 = \epsilon c_1 \), combining with the fact \( \lambda^k(t) \leq \tilde{\lambda}^k(t) \), we get

\[
\lambda^{(k+1)}(t) + c_3 \int_D (u^{(k+1)})^{2p} dx \leq c_2(\lambda^k(t) + c_2)\tilde{\lambda}^{k+1}(t).
\]

Hence

\[
\frac{d}{dt} \tilde{\lambda}^{k+1}(t) + c_3 \int_D (u^{(k+1)})^{2p} dx \leq c_2(\lambda^k(t) + c_2)\tilde{\lambda}^{k+1}(t).
\]

By induction, there is \( \delta \) depending only on \( \int_D |\nabla g|^2 dx, M \int_D g^{p+1} dx \) and \( c_2 \) such that

\[
\lambda^{(k+1)}(t) \leq \tilde{\lambda}^{k+1}(t) \leq c_4, \ \text{for} \ t \in [0, \delta], k \geq 1,
\]
where \( c_4 \) is a constant depending on \( \int_D |\nabla g|^2 \, dx \), \( M \int_D g^{p+1} \, dx \) and \( c_2 \).

Hence

\[
\int_D |\nabla u^{(k+1)}|^2 \, dx \leq c_4, \quad M \int_D |u^{(k+1)}|^{p+1} \, dx \leq c_4 \quad \text{for } t \in [0, \delta], k \geq 1.
\]

Integrate (9) with \( t \), we can conclude that

\[
(13) \quad \int_0^\delta \int_D |u^{(k+1)}|^{2p} \, dx \, dt \leq c_5.
\]

Now integrate (11) with \( t \), also by (13), we get

\[
\frac{1}{2} \int_D |\nabla u(\delta)^{(k+1)}|^2 \, dx - \frac{1}{2} \int_0^\delta \int_D |\nabla u(0)^{(k+1)}|^2 \, dx + \int_0^\delta \int_D |\Delta u^{(k+1)}|^2 \, dx \, dt
\]

\[
+ \int_0^\delta \int_D p(u^{(k+1)})^{p-1} |\nabla u^{(k+1)}|^2 \, dx \, dt \leq \int_0^\delta (\lambda^{(k)}(t) + M + \frac{M^2}{4\epsilon}) \int_D |\nabla u^{(k+1)}|^2 \, dx \, dt + c_3 c_5.
\]

Hence

\[
(14) \quad \int_0^\delta \int_D |\Delta u^{(k+1)}|^2 \, dx \, dt \leq c_6,
\]

where \( c_6 \) depending on \( \int_D |g|^2 \, dx \), \( \int_D |\nabla g|^2 \, dx \), \( c_3, c_5 \), \( M \) and \( \delta \). Integrate (5) with \( t \), we get

\[
\frac{1}{2} \int_D |\nabla u(\delta)^{(k+1)}|^2 \, dx - \frac{1}{2} \int_D |\nabla u(0)^{(k+1)}|^2 \, dx + \int_0^\delta \int_D |u^{(k+1)}|^2 \, dx \, dt
\]

\[
+ \frac{1}{p+1} \int_D a(x)(u^{(k+1)}(t))^{p+1} \, dx - \frac{1}{p+1} \int_D a(x)g^{p+1} \, dx
\]

\[
= \frac{\int_0^\delta \lambda^{(k)}(t) \frac{d}{dt} \int_D |u^{(k+1)}|^2 \, dx \, dt}{2} + \frac{1}{2} \int_D a(x)|u^{(k+1)}|^{2} \, dx - \frac{1}{2} \int_D a(x)|u(0)^{(k+1)}|^{2} \, dx.
\]

Hence

\[
(15) \quad \int_0^\delta \int_D |u_t^{(k+1)}|^2 \, dx \, dt \leq c_7,
\]

where \( c_7 \) depending on \( \int_D |g|^2 \, dx \), \( \int_D |\nabla g|^2 \, dx \), \( \int_D g^{p+1} \, dx \), \( c_3, c_5 \), \( M \) and \( \delta \).

By (11), (12), (14) and (15), there is a subsequence of \( \{u^{(k)}\} \) (still denoted by \( \{u^{(k)}\} \)) and a function \( u(t) \in L^\infty([0, \delta], H^1(D)) \cap L^2([0, \delta], L^2(D)) \cap L^\infty([0, \delta], L^{p+1}(D)) \) with \( \partial_t u(t) \in L^2([0, \delta], L^2(D)) \) such that \( u^{(k)} \rightharpoonup u \) weak* in \( L^\infty([0, \delta], H^1(D)) \), weakly in \( L^2([0, \delta], H^2(D)) \) and weakly in \( L^\infty([0, \delta], L^{p+1}(D)) \). Then we have \( u^{(k)} \rightarrow u \) strongly in \( L^2([0, \delta], H^1(D)) \) and \( u(t) \in C([0, \delta], L^2(D)) \). Hence \( \lambda^{(k)}(t) \rightarrow \lambda(t) \) strongly in \( L^2([0, \delta]) \).

Thus, we get a local strong solution to problem (2).
where \( \delta \) we can extend the local solution to \([0, 2\delta]\) in exactly the same way as above. By induction, we have a global solution to problem (2).

The stability result will be proved in the similar manner as in [10].

Proof of theorem 3. By the arguments in theorem 1, we can take a constant \( C \) such that all \( \|u\|_{L^\infty(\mathbb{R}_+, H^1(D))} \), \( \|v\|_{L^\infty(\mathbb{R}_+, H^1(D))}, \|u\|_{L^\infty(\mathbb{R}_+, L^\infty(D))}, \|v\|_{L^\infty(\mathbb{R}_+, L^\infty(D))}, \|\lambda_u(t)\|_{L^\infty(\mathbb{R}_+)} \), and \( \|\lambda_v(t)\|_{L^\infty(\mathbb{R}_+)} \) not less than \( C \), where \( C \) is only depending on upper bound of \( \|g_u\|_{H^1(D)}, \|g_v\|_{H^1(D)}, \|g_u\|_{L^\infty(D)}, \|g_v\|_{L^\infty(D)} \). We still denote \( M = \|a\|_{W^{1,\infty}} \). First we calculate

\[
\frac{1}{2} \frac{d}{dt} \int_D (u - v)^2 \, dx = \int_D (u - v)(u_t - v_t) \, dx
\]

\[
= \int_D (u - v)(\Delta (u - v) + \lambda_u(t)u - \lambda_v(t)v + a(x)(u - v)) \, dx
\]

\[
- a(x)(u^p - v^p) \, dx
\]

\[
\leq - \int_D |\nabla (u - v)|^2 \, dx + \int_D (u - v)(\lambda_u(t)u - \lambda_v(t)v) \, dx
\]

\[
+ \int_D a(x)(u - v)^2 \, dx
\]

Note that

\[
\int_D (u - v)(\lambda_u(t)u - \lambda_v(t)v) \, dx
\]

\[
= (\lambda_u(t) - \lambda_v(t)) \int_D (u - v) \, dx + \lambda_v(t) \int_D (u - v)^2 \, dx
\]

\[
\leq |\lambda_u(t) - \lambda_v(t)|(\int_D (u - v)^2 \, dx)^{\frac{1}{2}}(\int_D u^2 \, dx)^{\frac{1}{2}} + \lambda_v(t) \int_D (u - v)^2 \, dx
\]

\[
\leq C|\lambda_u(t) - \lambda_v(t)|(\int_D (u - v)^2 \, dx)^{\frac{1}{2}} + C \int_D (u - v)^2 \, dx,
\]

and

(16) \[ |\lambda_u(t) - \lambda_v(t)| \]

\[
= |\int_D (|\nabla u|^2 - |\nabla v|^2) + a(u^{p+1} - v^{p+1})) - a(u^2 - v^2) \, dx|
\]

\[
\leq \int_D |\nabla (u - v)||(|\nabla u| + |\nabla v|)\, dx + \int_D a(u - v)(\frac{u^{p+1} - v^{p+1}}{u - v}) \, dx
\]

\[
+ \int_D a(u - v)(u + v) \, dx
\]

\[
\leq C(\int_D |\nabla (u - v)|^2 \, dx)^{\frac{1}{2}} + MC(\int_D (u - v)^2 \, dx)^{\frac{1}{2}}.
\]
We have
\[
\frac{1}{2} \frac{d}{dt} \int_D (u - v)^2 dx \\
\leq - \int_D |\nabla (u - v)|^2 dx + C^2 \left( \int_D |\nabla (u - v)|^2 dx \right)^{\frac{1}{2}} \left( \int_D (u - v)^2 dx \right)^{\frac{1}{2}} \\
+ (MC^2 + M + C) \int_D (u - v)^2 dx \\
\leq -\frac{1}{2} \int_D |\nabla (u - v)|^2 dx + \left( \frac{C^4}{2} + MC^2 + M + C \right) \int_D (u - v)^2 dx.
\]

By the Gronwall inequality \cite{5}, we have
\[
||u - v||_{L^2}^2 \leq ||g_u - g_v||_{L^2}^2 \exp(\left( \frac{C^4}{2} + MC^2 + M + C \right) t).
\]

Further more,
\[
\frac{1}{2} \frac{d}{dt} \int_D |\nabla (u - v)|^2 dx \\
= - \int_D \Delta (u - v) \cdot (u - v) dx \\
= - \int_D \Delta (u - v) \cdot (\Delta (u - v) + \lambda_u(t)u + a(u - v) - a(u^p - v^p)) dx \\
= - \int_D (\Delta (u - v))^2 dx + \int_D \nabla (u - v) \cdot \nabla (\lambda_u(t)u - \lambda_v(t)v) dx \\
+ \int_D \nabla (u - v) \cdot \nabla (a(u - v)) dx + \int_D \Delta (u - v) \cdot a(u^p - v^p) dx.
\]

Note that
\[
\int_D \Delta (u - v) \cdot a(u^p - v^p) dx \\
\leq \frac{1}{2} \int_D (\Delta (u - v))^2 dx + \frac{1}{2} \int_D a^2(u^p - v^p)^2 dx \\
= \frac{1}{2} \int_D (\Delta (u - v))^2 dx + \frac{1}{2} \int_D a^2(u - v)^2 \left( \frac{u^p - v^p}{u - v} \right)^2 dx \\
\leq \frac{1}{2} \int_D (\Delta (u - v))^2 dx + \frac{M^2 C}{2} \int_D (u - v)^2 dx.
\]
Likewise,
\[
\int_D \nabla (u - v) \cdot \nabla (\lambda_u(t) u - \lambda_v(t) v) dx
\]
\[
= (\lambda_u(t) - \lambda_v(t)) \int_D \nabla (u - v) \cdot \nabla u dx + \lambda_v(t) \int_D |\nabla (u - v)|^2 dx
\]
\[
\leq |\lambda_u(t) - \lambda_v(t)| \left( \int_D |\nabla (u - v)|^2 dx \right)^{\frac{1}{2}} \left( \int_D |\nabla u|^2 dx \right)^{\frac{1}{2}} + \lambda_v(t) \int_D |\nabla (u - v)|^2 dx
\]
\[
\leq C|\lambda_u(t) - \lambda_v(t)| \left( \int_D |\nabla (u - v)|^2 dx \right)^{\frac{1}{2}} + C \int_D |\nabla (u - v)|^2 dx
\]
\[
\leq (C^2 + \frac{MC^2}{2} + C) \int_D |\nabla (u - v)|^2 dx + \frac{MC^2}{2} \int_D (u - v)^2 dx,
\]
where the second inequality follows by (16). Furthermore, we calculate
\[
\int_D \nabla (u - v) \cdot \nabla (a(u - v)) dx
\]
\[
= \int_D a|\nabla (u - v)|^2 dx + \int_D (u - v)(\nabla (u - v) \cdot \nabla a) dx
\]
\[
\leq \frac{3M}{2} \int_D |\nabla (u - v)|^2 dx + \frac{M}{2} \int_D (u - v)^2 dx.
\]
Then we have
\[
\frac{1}{2} \frac{d}{dt} \int_D |\nabla (u - v)|^2 dx \leq C_3 \int_D |\nabla (u - v)|^2 dx + C_4 \int_D (u - v)^2 dx,
\]
where $C_3$ and $C_4$ are the constants depending on $C$ and $M$. By the Gronwall inequality [5], we have
\[
||\nabla (u - v)||_{H^1}^2 \leq ||\nabla (g_u - g_v)||_{H^1}^2 \exp(C_2t).
\]

\[
\square
\]

Similar to [10], we have

**Proof of corollary 4**

Since
\[
\frac{1}{2} \frac{d}{dt} \int_D |\nabla u|^2 dx = - \int_D (u_t)^2 dx - \frac{1}{p+1} \frac{d}{dt} \int_D u^{p+1} dx + \frac{1}{2} \frac{d}{dt} \int_D au^{p+1} dx,
\]
we have
\[
(17)
\]
\[
\lambda(t)+2 \int_0^t \int_D |u_t|^2 dx dt = \int_D (|\nabla g|^2 + \frac{2}{p+1} ag^{p+1} - ag^2) dx + \frac{p-1}{p+1} \int_D au^{p+1} dx.
\]

By the arguments in theorem 1, we have $\lambda(t)$ is continuous, uniformly bounded in $t \in [0, \infty)$. Moreover, $u \in L^\infty(\mathbb{R}_+, H^1(D))$ and $u \in L^\infty(\mathbb{R}_+, L^{p+1}(D))$. Then we can take a subsequence \{\ti\} with $\ti \to \infty$
such that \( u_i(x) = u(x, t_i) \), \( \lambda(t_i) \to \lambda_\infty \). By (17) and theorem 1, we have

\[
\begin{cases}
    u_i \rightharpoonup u_\infty \text{ in } L^2(D), \\
    u_i \to u_\infty \text{ in } H^1(D) \text{ and } L^p(D), \\
    \partial_t u_i - (\lambda(t_i) - \lambda_\infty) u_i \to 0 \text{ in } L^2(D).
\end{cases}
\]

Since \( \partial_t u_i - (\lambda(t_i) - \lambda_\infty) u_i = \Delta u_i + \lambda_\infty u_i + a(u - u_p^\infty) \), \( u_i \in H^1 \) solves the equation \( \Delta u_\infty + \lambda_\infty u_\infty + a(u_\infty - u_p^\infty) = 0 \) in \( M \) and \( \int_D |u_\infty|^2 dx = 1 \).

\[ \square \]

3. Conclusion

In general, we may assume that \( a \) is a smooth function both in space variable and time variable and \( a = a(x, t) \) is a periodic function in time variable \( t \). We may also assume the spatial domain \( D \) is a compact manifold (with or without boundary) as in the works [1] and [12]. We leave this subject for future research.

Based on our study of (2) above we would like to point out that the non-local population model of logistic type has the advantage that the parameter \( \lambda(t) \) plays a role like a control term so that the flow exists globally and has nice behavior at time infinity. This research shows that global terms in population modeling should be considered in the future.

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**DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA**

*E-mail address: lma@math.tsinghua.edu.cn*