Supersymmetrization of horizontality condition: nilpotent symmetries for a free spinning relativistic particle

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Abstract: We clearly and consistently supersymmetrize the celebrated horizontality condition to derive the off-shell nilpotent and absolutely anticommuting Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetry transformations for the supersymmetric system of a free spinning relativistic particle within the framework of superfield approach to BRST formalism. For the precise determination of the proper (anti-)BRST symmetry transformations for all the bosonic and fermionic dynamical variables of our system, we consider the present theory on a (1, 2)-dimensional supermanifold parameterized by an even (bosonic) variable ($\tau$) and a pair of odd (fermionic) variables $\theta$ and $\bar{\theta}$ (with $\theta^2 = \bar{\theta}^2 = 0$, $\theta\bar{\theta} + \bar{\theta}\theta = 0$) of the Grassmann algebra. One of the most important and novel features of our present investigation is the derivation of (anti-)BRST invariant Curci-Ferrari type restriction which turns out to be responsible for the absolute anticommutativity of the (anti-)BRST transformations and existence of the coupled (but equivalent) Lagrangians for the present theory of a supersymmetric system. These observations are completely new results for this model.

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1 Introduction

One of the most intuitive geometrical approaches [1-3] to Becchi-Rouet-Stora-Tyutin (BRST) formalism is the superfield formulation (see, e.g. [4-10]) where the abstract mathematical properties, associated with the (anti-)BRST symmetries $s_{(a)b}$ of any arbitrary physical system, find their geometrical origin and interpretation in the language of specific entities of the superfield formalism. Within the framework of the latter formalism (i.e. superfield formalism), a given $D$-dimensional physical theory (e.g. gauge theory, reparametrization invariant theory, etc.) is considered on a $(D,2)$-dimensional supermanifold parametrized by the superspace variables $Z^M = (x^\mu, \theta, \bar{\theta})$ where $x^\mu$ (with $\mu = 0, 1, 2, ..., D-1$) are the ordinary bosonic coordinates of the ordinary flat Minkowskian spacetime and $(\theta, \bar{\theta})$ are a pair of Grassmannian variables (with $\theta^2 = \bar{\theta}^2 = 0$, $\theta \bar{\theta} + \bar{\theta} \theta = 0$).

It has been well-established that the proper (i.e. nilpotent and absolutely anticommuting) (anti-)BRST symmetry transformations of the $D$-dimensional ordinary theory are connected with the translational generators along the Grassmannian directions of the $(D,2)$-dimensional supermanifold. These connections have been established by exploiting the theoretical potential and power of the so-called horizontality condition (HC).

In the context of $D$-dimensional $p$-form ($p = 1, 2, 3, ...$) (non-)Abelian gauge theories, the super curvature ($p + 1$)-form [defined on the $(D,2)$-dimensional supermanifold] is covariantly reduced to its counterpart—the ordinary curvature ($p + 1$)-form [defined on the ordinary $D$-dimensional Minkowskian spacetime]. Physically, this reduction amounts to the fact that the gauge covariant quantity (e.g. curvature term) remains independent of the Grassmannian variables of the $(D,2)$-dimensional supermanifold, on which, the ordinary $D$-dimensional (non-)Abelian $p$-form gauge theory is considered. In the process of the above covariant reduction, the proper (i.e. off-shell nilpotent and absolutely anticommuting) (anti-)BRST symmetries of the above $D$-dimensional theories emerge and they are geometrically identified with the translational generators ($\partial_\theta, \partial_{\bar{\theta}}$) along the Grassmannian directions of the $(D,2)$-dimensional supermanifold [4,5]. Exactly similar is the situation with the reparametrization invariant theories where the analogue of the curvature term is found and the above covariant reduction procedure generates the proper (anti-)BRST symmetry transformations for the reparametrization invariant theory as well [9,10].

So far, the above superfield formalism has not been applied to the supersymmetric systems of some physical interest. As a consequence, to the best of our knowledge, one is not clear about the systematic generalization of the HC for the description of a supersymmetric physical system within the framework of superfield formalism. One of the main motivations of our present endeavor is to theoretically state the supersymmetric version of HC and apply it to the derivation of proper (anti-)BRST symmetry transformations for the supersymmetric system of a one $(0 + 1)$-dimensional (1D) model of a free relativistic spinning particle which represents an interesting toy model for the supergravity theories. In our present investigation, we derive the proper (anti-)BRST symmetry transformations $s_{(a)b}$ for the above supersymmetric system within the framework of augmented version of

*To be precise, the geometrical approaches adopted in [1-3] do not utilize superfields. Rather, these exploit the ordinary fields which depend on the vertical directions of the fiber bundle. At their very best, these endeavors can be treated as a set of precursors to the superfield formalism proposed in [4-7].
Bonora-Tonin (BT) superfield formalism where the HC, (super)gauge invariant restrictions [(S)GIR] and supersymmetric version of HC (SUSY-HC) are all exploited together. In fact, we are theoretically compelled to exploit all these restrictions for the derivation of the full set of proper and precise (anti-)BRST symmetry transformations.

The off-shell nilpotency \( s_{(a)b}^2 = 0 \) and absolute anticommutativity property \( (s_b s_{(a)b} + s_{(a)b} s_b = 0) \) of the (anti-)BRST transformations \( s_{(a)b} \) are very sacrosanct because they physically imply the fermionic nature of \( s_{(a)b} \) and the linear independence of \( s_b \) and \( s_{(a)b} \), respectively. The judicious combination of the (S)GIR and (SUSY-)HC leads to the derivation of such type of proper (anti-)BRST symmetry transformations for the supersymmetric system of spinning relativistic particle. We have shown that the SUSY-HC [cf. (22)] resembles very much like the Maurer-Cartan equation (which defines the curvature tensor for the non-Abelian gauge theory). The other novel feature of our present investigation is the derivation of (anti-)BRST invariant Curci-Ferrari (CF)-type restriction [cf. (30)] from the SUSY-HC of our present theory. It will be noted that the idea of (super)gauge invariant restrictions [(S)GIR] also plays a pivotal role in the derivation of all the (anti-)BRST transformations for all the dynamical variables of our present supersymmetric system.

In our present endeavor, we have captured all the key properties of (anti-)BRST symmetry transformations in the language of specific objects of our augmented version of BT superfield formalism. For instance, the nilpotency and absolute anticommutativity properties of the (anti-)BRST symmetry transformations as well as the (anti-)BRST invariance of the coupled (but equivalent) Lagrangians have been incorporated within the framework of our superfield formalism. Finally, we have shown the (anti-)BRST invariance of the CF-type restriction (cf. Sec. 5) which plays a decisive role in our present investigation. In all these superfield descriptions, the mappings between the (anti-)BRST symmetry transformations \( s_{(a)b} \) and the translation generators \( (\partial_{\theta}, \partial_{\bar{\theta}}) \) along the Grassmannian directions of the supermanifold [cf. (43) below] have played a role of paramount importance. In fact, the nilpotency and absolute anticommutativity properties of the BRST symmetries are automatically captured within our superfield formalism because \( \partial_{\theta}^2 = \partial_{\bar{\theta}}^2 = 0, \partial_{\theta} \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_{\theta} = 0 \).

The main motivating factors behind our present investigation are as follows. First, it is, for the first-time, that the augmented version of BT superfield formalism [8-10] is being applied to a supersymmetric model of a relativistic particle. Second, the generalization of HC to the supersymmetric gauge theory (christened as SUSY-HC in our present work) was a challenging problem which we have resolved in our present endeavor. In fact, we have obtained a neat expression for the SUSY-HC which resembles very much like the Maurer-Cartan equation [cf. (22)]. Third, a completely new result is the derivation of CF-type restriction for the present supersymmetric system of spinning relativistic particle. Finally, the derivation of the proper (i.e. off-shell nilpotent and absolutely anticommuting) anti-BRST symmetry transformations and coupled (but equivalent) Lagrangians are novel observations for the supersymmetric model of a free spinning relativistic particle.

Our present paper is organized as follows. In Sec. 2, we set up the notations and conventions for our paper by discussing the bare essentials of (super)gauge transformations

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†The BT superfield formalism, applied to a given (non-)Abelian gauge theory, utilizes only the HC. In a set of papers (see, e.g. [8-10]), we have generalized the HC in a consistent fashion by incorporating some other appropriate restrictions for the derivation of the (anti-)BRST symmetry transformations for the gauge as well as matter fields of a given gauge/reparametrization invariant theory.
for the supersymmetric system of massless spinning relativistic particle. Our Sec. 3 is devoted to the derivation of (anti-)BRST transformations of super-gauge (fermionic) variable \( \chi(\tau) \) and its associated bosonic (anti-)ghost variables \( (\bar{\beta})\beta \). An off-shoot of this exercise is the derivation of (anti-)BRST transformations for the fermionic Lorentz vector variable \( \psi_\mu(\tau) \). Section 4 deals with the discussion of supersymmetric version of HC which enables us to derive the (anti-)BRST symmetry transformations for the bosonic gauge variable \( e(\tau) \) and its associated fermionic (anti-)ghost variables \( (\bar{c})c \). A by-product of this exercise is the derivation of (anti-)BRST symmetry transformations for the bosonic target space variable \( x_\mu(\tau) \). We discuss the (anti-)BRST invariance of the CF-type restriction, within the framework of superfield formalism, in Sec. 5. Our Sec. 6 is devoted to the derivation of coupled Lagrangians and the proof of their (anti-)BRST invariance within the framework of superfield formalism. Finally, we make some concluding remarks in Sec. 7.

In our Appendices A and B, we discuss a set of interesting Lagrangians but show that they are not appropriate in one way or the other ways. Our Appendix C is devoted to a brief discussion of a free massive spinning relativistic particle within the framework of superfield approach to BRST formalism where we derive some specific (anti-)BRST symmetries.

2 Preliminaries: (super)gauge transformations

Let us begin with the following first-order Lagrangian for the one \((0+1)\)-dimensional \((1D)\) supersymmetric model of a massless spinning relativistic particle \([11]\)

\[
L_0 = p_\mu \dot{x}^\mu - \frac{c}{2} p^2 + i \psi_\mu \dot{\psi}^\mu + i \chi (p_\mu \psi^\mu),
\]

where the constraints \( p^2 \approx 0 \) and \( p_\mu \psi^\mu \approx 0 \) are first-class in nature (according to Dirac’s prescription for the classification scheme \([12,13]\)) and they have been incorporated in the above Lagrangian \( L_0 \) through the Lagrange multiplier variables \( e(\tau) \) and \( \chi(\tau) \) which are the analogs of the vierbein and Rarita-Schwinger (gravitino) fields of the 4D supergravity theories. In our present theory, these variables are also the analogs of the gauge fields of the usual 4D supersymmetric gauge theories. Here the super world-line, traced out by the motion of the massless relativistic particle, is parameterized by the monotonically increasing parameter \( \tau \). This world-line is embedded in a \( D \)-dimensional Minkowskian flat target supermanifold characterized by the target even (bosonic) position variable \( x_\mu(\tau) \) \( (\mu = 0, 1, 2, \ldots, D - 1) \) and odd (fermionic) spin variable \( \psi_\mu(\tau) \) \( (\mu = 0, 1, 2, \ldots, D - 1) \) which are superpartners of each-other. The fermionic variables of the theory anticommute with one-other (i.e. \( \psi_\mu^2 = 0, \chi^2 = 0, \psi_\mu \psi_\nu + \psi_\nu \psi_\mu = 0, \psi_\mu \chi + \chi \psi_\mu = 0, \) etc.). The conjugate momenta of the target space variables \( x_\mu \) are \( p^\mu = (\partial L_0 / \partial \dot{x}_\mu) \) and \( \dot{x}^\mu = (dx^\mu / d\tau) = e p^\mu - i \chi \psi^\mu, \) \( \psi^\mu = (d\psi^\mu / d\tau) = \chi p^\mu \) are the generalized “velocities” for the massless relativistic spinning particle corresponding to the above mentioned superpartners \( x_\mu \) and \( \psi_\mu \).

The above starting Lagrangian respects the local, continuous and infinitesimal (super)gauge transformations \( (\delta_{(s)g}) \) as follows (see, e.g. \([11]\))

\[
\delta_{sg} x_\mu = \kappa \psi_\mu, \quad \delta_{sg} p_\mu = 0, \quad \delta_{sg} \psi_\mu = i \kappa \psi_\mu, \quad \delta_{sg} \chi = i \kappa, \quad \delta_{sg} e = 2 \kappa \chi, \\
\delta_{s} x_\mu = \xi \mu, \quad \delta_{s} p_\mu = 0, \quad \delta_{s} \psi_\mu = 0, \quad \delta_{s} \chi = 0, \quad \delta_{s} e = \xi. \quad (2)
\]
where \((\kappa)\xi\) are the (fermionic) bosonic (super)gauge parameters. One can check explicitly that the Lagrangian \(L_0\) transforms to a total derivative under \(\delta_{(s)g}\). It will be noted that the (super)gauge transformations are generated by the primary and secondary first-class constraints of the theory. Furthermore, the infinitesimal transformations \(\delta_{sg}\) are nothing but the normal supersymmetric transformations for our present supersymmetric system.

The above transformations can be combined together (i.e. \(\delta = \delta_g + \delta_{sg}\)). The resulting transformations (\(\delta\)) for the dynamical variables of our present theory are

\[
\delta x_\mu = \xi p_\mu + \kappa \psi_\mu, \quad \delta p_\mu = 0, \quad \delta \psi_\mu = i \kappa p_\mu, \quad \delta \chi = i \kappa, \quad \delta e = \dot{\xi} + 2 \kappa \chi.
\] (3)

Under the above local, continuous and infinitesimal transformations, the Lagrangian \(L_0\) transforms to a total derivative as follows

\[
\delta L_0 = \frac{d}{d\tau} \left[ \frac{\xi}{2} p^2 + \frac{\kappa}{2} (p \cdot \psi) \right],
\] (4)

where, for the sake of brevity, we have chosen \(p_\mu \psi^\mu = p \cdot \psi\). Henceforth, we shall follow this notation in the whole body of our present text. It is clear that the action integral \(S = \int d\tau L_0\) remains invariant under the transformations (4) for the physically well-defined dynamical variables of the theory which vanish off at infinity. The limiting cases of (3) and (4) produce results for the transformations (2), separately and independently.

We close this section with the following remarks. First, the BRST transformations corresponding to the gauge transformations (\(\delta_g\)) have been written in [11] which are nilpotent of order two. Second, it has been pointed out in [11] that the BRST-type transformations exist corresponding to the supergauge transformations (\(\delta_{sg}\)) but they are not nilpotent of order two. Third, the BT superfield formalism has been applied to derive the proper nilpotent (anti-)BRST symmetry transformations corresponding to (\(\delta_{(s)g}\)) separately and independently [14]. However, the (anti-)BRST transformations, corresponding to \(\delta\), have not yet been derived within the framework of superfield formalism. Lastly, the BRST transformations, corresponding to the transformations \(\delta = \delta_g + \delta_{sg}\), have been mentioned in [11,14] which are found to be nilpotent of order two. The proper (i.e. nilpotent and absolutely anticommuting) anti-BRST transformations for the above transformations have, however, not been quoted in [11,14]. In our forthcoming sections, we attempt to resolve these contagious issues in a systematic manner by applying the key concepts of superfield formalism proposed in [4,5] and modify it consistently for the derivation of the correct nilpotent and absolutely anticommuting (anti-)BRST transformations (in the case of our supersymmetric system where the bosonic as well as fermionic variables co-exist together).

3 Horizontality condition: (anti-)BRST transformations for variables \(\chi(\tau), \beta(\tau), \bar{\beta}(\tau), \psi_\mu(\tau)\)

It is elementary to note that the supergauge variable \(\chi(\tau)\) is a fermionic auxiliary variable in the theory (described by the Lagrangian \(L_0\)) as there is no kinetic term for this variable (because \(\chi^2 = 0\)). The common folklore in the gauge theory states that the kinetic term of a gauge variable is hidden in the curvature term which owes its origin to the exterior
derivative $d$ (with $d^2 = 0$). For our present 1D theory, the curvature 2-form, in the language of the exterior derivative $d = d\tau \partial_\tau$ and 1-form fermionic connection $f^{(1)} = d\tau \chi(\tau)$ for the variable $\chi(\tau)$, is $F^{(2)} = d f^{(1)} = (d\tau \wedge d\tau) \partial_\tau \chi(\tau) \equiv (d\tau \wedge d\tau) \chi(\tau) = 0$ (due to the basic property of wedge product $d\tau \wedge d\tau = 0$).

To derive the (anti-)BRST transformations for the supergauge variable $\chi(\tau)$ and its associated (anti-)ghost fields $(\bar{\beta})\beta$, we consider the supergauge theory on a $(1, 2)$-dimensional supermanifold where the exterior derivative $d = d\tau \partial_\tau$ and 1-form fermionic connection $f^{(1)} = d\tau \chi(\tau)$ are generalized onto the $(1, 2)$-dimensional supermanifold as [4,5,14]

\[
\begin{align*}
d \to \tilde{d} &= dZ^M \partial_M = d\tau \partial_\tau + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}, \\
f^{(1)} \to \tilde{F}^{(1)} &= dZ^M \tilde{A}_M(\tau, \theta, \bar{\theta}) = d\tau K(\tau, \theta, \bar{\theta})_\tau + d\theta B(\tau, \theta, \bar{\theta}) + i d\bar{\theta} \bar{B}(\tau, \theta, \bar{\theta}),
\end{align*}
\]

where $Z^M = (\tau, \theta, \bar{\theta})$ and $\partial_M = (\partial_\tau, \partial_\theta, \partial_{\bar{\theta}})$ are the superspace coordinates and superspace derivatives that characterize the $(1, 2)$-dimensional supermanifold and supermultiplet vector superfield $\tilde{A}_M(\tau, \theta, \bar{\theta}) \equiv [K(\tau, \theta, \bar{\theta}), B(\tau, \theta, \bar{\theta}), \bar{B}(\tau, \theta, \bar{\theta})]$. Here $\tau$ is an even (bosonic) element and $(\theta, \bar{\theta})$ are the odd (fermionic) elements of the Grassmann algebra and superfield $K$ is fermionic and $(B, \bar{B})$ are bosonic in nature. The superfields $K(\tau, \theta, \bar{\theta})$, $B(\tau, \theta, \bar{\theta})$ and $\bar{B}(\tau, \theta, \bar{\theta})$ can be expanded along the Grassmannian directions $(\theta, \bar{\theta})$ of the $(1, 2)$-dimensional supermanifold in the following manner (see, e.g. [4,5,14])

\[
\begin{align*}
K(\tau, \theta, \bar{\theta}) &= \chi(\tau) + \theta b_1(\tau) + \bar{\theta} b_1(\tau) + \theta \bar{\theta} f_1(\tau), \\
B(\tau, \theta, \bar{\theta}) &= \bar{\beta}(\tau) + \theta \bar{f}_2(\tau) + \bar{\theta} f_2(\tau) + i \theta \bar{\theta} b_2(\tau), \\
\bar{B}(\tau, \theta, \bar{\theta}) &= \bar{\beta}(\tau) + \theta \bar{f}_3(\tau) + \bar{\theta} f_3(\tau) + i \theta \bar{\theta} b_3(\tau).
\end{align*}
\]

It is evident that, in the limiting case $(\theta, \bar{\theta}) \to 0$, we retrieve the basic variables $\chi(\tau), \beta(\tau)$ and $\bar{\beta}(\tau)$, respectively, of the 1D (anti-)BRST invariant theory of a free spinning particle. In the above, the sets $(\chi, f_1, \bar{f}_2, f_2, \bar{f}_3, f_3)$ and $(b_1, b_1, \beta, \bar{\beta}, b_2, b_3)$ are fermionic and bosonic in nature, respectively, and their equality (in numbers) ensures the existence of supersymmetry (SUSY) in the theory. The variables $(b_1, b_1, \beta, \bar{\beta}, b_2, b_3)$ are secondary and they have to be determined in terms of the basic and auxiliary dynamical variables of the (anti-)BRST invariant 1D theory of spinning particle by the application of the usual HC.

We apply now the standard technique of HC which physically requires that the gauge (and/or BRST) invariant curvature 2-form $F^{(2)} = d f^{(1)}$ has to be independent of the Grassmannian (odd) variables $\theta, \bar{\theta}$ of the superspace coordinate $Z^M$ (and the corresponding differentials) in the following relationship

\[
\tilde{d} \tilde{F}^{(1)} = d f^{(1)} = 0,
\]

where the explicit expression for the l.h.s., in terms of the multiplet superfields, is

\[
\tilde{d} \tilde{F}^{(1)} = (d\tau \wedge d\theta) (i \partial_\tau \bar{B} - \partial_\theta K) - i (d\theta \wedge d\theta) (\partial_\theta \bar{B}) + (d\tau \wedge d\bar{\theta}) (i \partial_\tau B - \partial_{\bar{\theta}} K)
\]

\[
- i (d\bar{\theta} \wedge d\bar{\theta}) (\partial_{\bar{\theta}} B + \partial_\theta \bar{B}) - i (d\theta \wedge d\bar{\theta}) (\partial_\theta B).
\]

The above equality of HC, in (7), yields:

\[
\partial_\theta \bar{B} = 0, \quad \partial_{\bar{\theta}} B = 0, \quad \partial_\theta \bar{B} + \partial_{\bar{\theta}} B = 0, \quad \partial_\theta K = i \partial_\tau \bar{B}, \quad \partial_{\bar{\theta}} K = i \partial_\tau B.
\]
The substitution of the expansions (6), leads to the following relations

\[ b_2 = b_3 = 0, \quad f_2 = \bar{f}_3 = 0, \quad f_3 + \bar{f}_2 = 0, \quad (10) \]

from the first three conditions of (9). If we choose \( f_3 = \gamma \), it implies that \( \bar{f}_2 = -\gamma \). Thus, after the application of HC, we obtain the following expansions [4,5,14]

\[ B^{(h)}(\tau, \theta, \bar{\theta}) = \beta(\tau) + \theta \left( -i \gamma(\tau) \right) + \bar{\theta} \left( 0 \right) + \theta \bar{\theta} \left( 0 \right) \]

\[ \equiv \beta(\tau) + \theta \left( s_{ab} \beta(\tau) \right) + \bar{\theta} \left( s_b \beta(\tau) \right) + \theta \bar{\theta} \left( s_b s_{ab} \beta(\tau) \right), \]

\[ \bar{B}^{(h)}(\tau, \theta, \bar{\theta}) = \bar{\beta}(\tau) + \theta \left( 0 \right) + \bar{\theta} \left( i \gamma(\tau) \right) + \theta \bar{\theta} \left( 0 \right) \]

\[ \equiv \bar{\beta}(\tau) + \theta \left( s_{ab} \bar{\beta}(\tau) \right) + \bar{\theta} \left( s_b \bar{\beta}(\tau) \right) + \theta \bar{\theta} \left( s_b s_{ab} \bar{\beta}(\tau) \right), \quad (11) \]

where the superscript \((h)\) on the superfields denotes the expansion of these superfields after the application of HC. The above expansions would turn out to be very useful later on.

Now, the stage is set to derive the nilpotent (anti-)BRST symmetry transformations for the supergauge variable \( \chi(\tau) \). In this connection, we can exploit the expansions (11) in the relationship (9) to obtain the secondary variables of \( K(\tau, \theta, \bar{\theta}) \) as

\[ b_1(\tau) = i \dot{\beta}(\tau), \quad \bar{b}_1(\tau) = i \dot{\bar{\beta}}(\tau), \quad f_1(\tau) = -\dot{\gamma}(\tau). \quad (12) \]

Thus, the expansion of \( K(\tau, \theta, \bar{\theta}) \), after the application of HC, is

\[ K^{(h)}(\tau, \theta, \bar{\theta}) = \chi(\tau) + \theta \left( i \dot{\beta}(\tau) \right) + \bar{\theta} \left( i \dot{\bar{\beta}}(\tau) \right) + \theta \bar{\theta} \left( -\dot{\gamma}(\tau) \right) \]

\[ \equiv \chi(\tau) + \theta \left( s_{ab} \chi(\tau) \right) + \bar{\theta} \left( s_b \chi(\tau) \right) + \theta \bar{\theta} \left( s_b s_{ab} \chi(\tau) \right). \quad (13) \]

The relation \( \psi_\mu = \chi p_\mu \) (that emerges as the equation of motion from \( L_0 \)) is a supergauge invariant quantity [cf. (2)] on the on-shell where \( p_\mu = 0 \). Within the framework of augmented BT superfield formalism [8-10,14], such relations should remain invariant on the supersymmetric (1, 2)-dimensional supermanifold. Thus, we have the following supergauge invariant restriction (SGIR) on the superfields of the theory:

\[ \dot{\Psi}_\mu(\tau, \theta, \bar{\theta}) - K^{(h)}(\tau, \theta, \bar{\theta}) P_\mu(\tau, \theta, \bar{\theta}) = \psi_\mu(\tau) - \chi(\tau) p_\mu(\tau) = 0. \quad (14) \]

However, as we have seen, the momentum operator \( p_\mu \) is a gauge invariant quantity [cf. (2)]. Thus, we have the gauge invariant restriction (GIR) \( P_\mu(\tau, \theta, \bar{\theta}) = p_\mu(\tau) \) where

\[ P_\mu(\tau, \theta, \bar{\theta}) = p_\mu(\tau) + \theta F^{(1)}_\mu(\tau) + \bar{\theta} \bar{F}^{(1)}_\mu(\tau) + \theta \bar{\theta} B^{(1)}_\mu(\tau). \quad (15) \]

The equality \( P_\mu(\tau, \theta, \bar{\theta}) = p_\mu(\tau) \), however, implies that we have \( \bar{F}^{(1)}_\mu = F^{(1)}_\mu = B^{(1)}_\mu = 0 \). As a consequence, the explicit expansion for the gauge invariant quantity \( P^{(g)}_\mu(\tau, \theta, \bar{\theta}) \) is [4,5]

\[ P^{(g)}_\mu(\tau, \theta, \bar{\theta}) = p_\mu(\tau) + \theta \left( 0 \right) + \bar{\theta} \left( 0 \right) + \theta \bar{\theta} \left( 0 \right) \]

\[ \equiv p_\mu(\tau) + \theta \left( s_{ab} p_\mu + \bar{\theta} \left( s_b \bar{\theta} \right) + \theta \bar{\theta} \left( s_b s_{ab} p_\mu \right), \quad (16) \]
which shows that \( s_{(a)b} p_\mu = 0 \). It is evident that the target space momenta \( p_\mu \) are trivially (anti-)BRST invariant quantities (as there are no transformations for them). Thus, the above equation (14) can be re-expressed as:

\[
\ddot{\Psi}_\mu(\tau, \theta, \bar{\theta}) = K^{(h)}(\tau, \theta, \bar{\theta}) \, p_\mu(\tau).
\]  

(17)

Exploiting the expansion (13) for \( K^{(h)}(\tau, \theta, \bar{\theta}) \) and taking the super-expansion of \( \Psi_\mu(\tau, \theta, \bar{\theta}) \), along the Grassmannian directions of the \((1, 2)\)-dimensional supermanifold, as

\[
\Psi_\mu(\tau, \theta, \bar{\theta}) = \psi_\mu(\tau) + \theta \, \dot{b}_\mu(\tau) + \bar{\theta} \, b_\mu(\tau) + \theta \, \bar{\theta} \, f_\mu(\tau),
\]

we obtain the secondary variables of \( \Psi_\mu(\tau, \theta, \bar{\theta}) \). As a consequence, we have the following expansion for \( \Psi_\mu(\tau, \theta, \bar{\theta}) \), along the Grassmannian directions of the \((1, 2)\)-dimensional supermanifold

\[
\Psi_\mu^{(sg)}(\tau, \theta, \bar{\theta}) = \psi_\mu(\tau) + \theta \left( i \, \bar{\beta} \, p_\mu \right) + \bar{\theta} \left( i \, \beta \, p_\mu \right) + \theta \, \bar{\theta} \, \left( -\gamma \, p_\mu \right)
\]

\[
\equiv \psi_\mu(\tau) + \theta \left( s_{ab} \, \psi_\mu \right) + \bar{\theta} \left( s_b \, \psi_\mu \right) + \theta \, \bar{\theta} \left( s_b \, s_{ab} \, \psi_\mu \right),
\]

(19)

where the superscript \((sg)\) denotes the expansion after the application of SGIR \([\text{cf. } (14)].\)

Taking the help of expansions in (11), (13), (16) and (19), we obtain the following off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations:

\[
\begin{align*}
\beta & = \gamma, & \gamma & = 0, & \bar{\beta} & = 0, & \chi & = i \, \bar{\beta}, \\
\psi_\mu & = i \, \bar{\beta} \, p_\mu, & s_{ab} \, p_\mu & = 0, & s_b \, \chi & = i \, \beta, & \beta & = 0, \\
\bar{\beta} & = i \, \gamma, & s_b \, \psi_\mu & = i \, \beta \, p_\mu, & \gamma & = 0, & s_b \, p_\mu & = 0.
\end{align*}
\]

(20)

We can verify easily that \( s_{(a)b}^2 = 0 \) and \( s_b \, s_{ab} + s_{ab} \, s_b = 0 \). In other words, it is clear that the two consecutive applications of \( s_{(a)b} \) results in zero and the operator form of anticommutator (i.e. \( \{s_b, s_{ab}\} \)), acting on any variable, produces zero result.

We close this section with the remarks that the idea of HC leads to the derivation of off-shell nilpotent (anti-)BRST symmetry transformations only for the gauge and its associated (anti-)ghost variables of the theory. However, these inputs help in the determination of (anti-)BRST symmetries for the other variables of the theory when we demand the additional restriction, within the framework of augmented superfield formalism [8-10,14], where the (super)gauge invariant quantities are also required to be independent of these Grassmannian variables. For instance, we have been able to obtain the (anti-)BRST symmetry transformations for the \( \psi_\mu(\tau) \) variable by demanding the additional restriction \([\text{cf. } (14)]\) on the \((1, 2)\)-dimensional supermanifold because \( \bar{\psi}_\mu = \chi \, p_\mu \) is a supergauge invariant quantity. Thus, the importance of (S)GIR in our work is very decisive and crucial.

4 Supersymmetrization of HC: (anti-)BRST symmetry transformations for \( e(\tau), \, c(\tau), \, \bar{c}(\tau), \, x_\mu(\tau) \)

The HC plays a central role in the BT superfield approach to BRST formalism when it is applied to the BRST analysis of gauge theories [4,5]. However, our present model is a
supersymmetric model where, for the first-time, the ideas of BT superfield formalism is being applied. The bosonic gauge dynamical variable of our present theory is $e(\tau)$. Thus, the 1-form $(A^{(1)})$, is defined in terms of it as $A^{(1)} = d\tau \ e(\tau)$. The curvature tensor turns out to be zero for this gauge dynamical variable because $dA^{(1)} = 0$ (due to $d\tau \wedge d\tau = 0$). This is, however, only the bosonic part of the supersymmetric theory. The total super 2-form $\chi$ is, the one where $\chi(\tau)$ should be present in a consistent and clear fashion, namely:

$$\ dA^{(1)} + i \ (f^{(1)} \wedge f^{(1)}) = 0, \quad (21)$$

where the fermionic 1-form $f^{(1)} = d\tau \ \chi(\tau)$ has already been discussed earlier.

We have to generalize the relation (21) onto the (1, 2)-dimensional supermanifold and demand its equality to itself as given below

$$\ dA^{(1)} + i \ (\tilde{F}^{(1)} \wedge \tilde{F}^{(1)}) = dA^{(1)} + i \ (f^{(1)} \wedge f^{(1)}) = 0. \quad (22)$$

The above restriction is the generalization of the ordinary HC to a supersymmetric HC (i.e. SUSY-HC). The individual symbols on the l.h.s. of (22) are

$$\ \tilde{A}^{(1)} = d\tau \ \partial_\tau + d\theta \ \partial_\theta + d\bar{\theta} \ \partial_{\bar{\theta}}, \quad \tilde{F}^{(1)} = d\tau \ K^{(h)}(\tau, \theta, \bar{\theta}) + i \ d\theta \ \tilde{B}^{(h)}(\tau, \theta, \bar{\theta}) + i \ d\bar{\theta} \ B^{(h)}(\tau, \theta, \bar{\theta}), \quad (23)$$

where $K^{(h)}(\tau, \theta, \bar{\theta}), \tilde{B}^{(h)}(\tau, \theta, \bar{\theta}), B^{(h)}(\tau, \theta, \bar{\theta})$ are the superfields obtained after the application of HC (cf. Sec. 3). It will be noted that the superfield $E(\tau, \theta, \bar{\theta})$ is bosonic and the pair $[\tilde{F}(\tau, \theta, \bar{\theta}), \ F(\tau, \theta, \bar{\theta})]$ is fermionic in $\tilde{A}^{(1)}$. As has been already mentioned, the r.h.s. of (22) is zero on its own. The l.h.s. of (22) is explicitly expressed as follows:

$$\ dA^{(1)} + i \ (\tilde{F}^{(1)} \wedge \tilde{F}^{(1)}) = (d\tau \wedge d\theta) \ (\partial_\tau \tilde{F} - \partial_\theta E + 2 \tilde{B}^{(h)} K^{(h)}) + (d\tau \wedge d\bar{\theta}) \ (\partial_\tau \tilde{F} - \partial_{\bar{\theta}} E + 2 \tilde{B}^{(h)} K^{(h)}) - (d\theta \wedge d\bar{\theta}) \ (\partial_\theta \tilde{F} + \partial_{\bar{\theta}} \tilde{F} + 2i \ B^{(h)} \ B^{(h)}) - (d\theta \wedge d\bar{\theta}) \ (\partial_{\bar{\theta}} \tilde{F} + i \ \bar{\tilde{B}}^{(h)} B^{(h)}) - (d\theta \wedge d\bar{\theta}) \ (\partial_{\bar{\theta}} \tilde{F} + i \ \bar{B}^{(h)} \ B^{(h)}) \quad (24)$$

The restriction in (22) implies that the coefficients of $(d\tau \wedge d\theta), (d\tau \wedge d\bar{\theta}), (d\theta \wedge d\bar{\theta}), (d\theta \wedge d\theta), (d\bar{\theta} \wedge d\bar{\theta})$ have to be set equal to zero. These requirements lead to the following

$$\ \partial_\tau \tilde{F} - \partial_\theta E + 2 \tilde{B}^{(h)} K^{(h)} = 0, \quad \partial_\tau \tilde{F} - \partial_{\bar{\theta}} E + 2 \tilde{B}^{(h)} K^{(h)} = 0, \quad \partial_\theta \tilde{F} + \partial_{\bar{\theta}} \tilde{F} + 2i \ B^{(h)} \ B^{(h)} = 0, \quad \partial_{\bar{\theta}} \tilde{F} + i \ \bar{\tilde{B}}^{(h)} B^{(h)} = 0, \quad \partial_{\bar{\theta}} \tilde{F} + i \ \bar{B}^{(h)} B^{(h)} = 0. \quad (25)$$

The above relationships allow the derivation of secondary variables in terms of the auxiliary and dynamical variables of the BRST-invariant 1D theory of spinning relativistic particle when the proper expansions of superfields are plugged in the above equation (25).

---

3 We note that the above equation looks exactly like the well-known Maurer-Cartan equation which defines the curvature tensor for the $SU(N)$ non-Abelian gauge theories. However, in such usual non-Abelian theories, there are no fermionic 1-form connections as we have in our present analysis.
To achieve the above theoretically important goals, first of all, we have to expand the superfields $E(\tau, \theta, \bar{\theta})$, $F(\tau, \theta, \bar{\theta})$ and $\bar{F}(\tau, \theta, \bar{\theta})$ along the Grassmannian directions as:

$$
E(\tau, \theta, \bar{\theta}) = e(\tau) + \theta \bar{f}(\tau) + \bar{\theta} f(\tau) + i \theta \bar{\theta} B(\tau),
$$

$$
F(\tau, \theta, \bar{\theta}) = c(\tau) + i \theta \bar{b}_1(\tau) + i \bar{\theta} b_1(\tau) + i \theta \bar{\theta} s_1(\tau),
$$

$$
\bar{F}(\tau, \theta, \bar{\theta}) = \bar{c}(\tau) + i \theta \bar{\bar{b}}_2(\tau) + i \bar{\theta} b_2(\tau) + i \theta \bar{\theta} \bar{s}_1(\tau),
$$

(26)

where the basic tenet of SUSY is satisfied accurately because the number of bosonic variables ($e, B, \bar{b}_1, b_1, b_2$) and fermionic variables ($\bar{f}, f, c, \bar{c}, s_1, \bar{s}_1$) match. We have to determine all the secondary variables ($f, \bar{f}, B, b_1, \bar{b}_1, b_2, \bar{s}_1, s_1$) in terms of the auxiliary and basic variables of the BRST invariant 1D model of massless spinning relativistic particle.

The last three relationships of (25) yield the following:

$$
b_1 = -\beta^2, \quad s_1 = -2 i \beta \gamma, \quad \bar{b}_2 = -\bar{\beta}^2, \quad \bar{s}_1 = -2 i \bar{\beta} \gamma, \quad \bar{b}_1 + b_2 = -2 \beta \bar{\beta}. \quad (27)
$$

As a consequence, we have the following expansions for the fermionic superfields [4,5]

$$
F^{(sh)}(\tau, \theta, \bar{\theta}) = c + \theta (i \bar{b}_1) + \bar{\theta} (-i \beta^2) + \theta \bar{\theta} (2 \beta \gamma)
$$

$$
\equiv c + \theta (s_{ab} c) + \bar{\theta} (s_{b} c) + \theta \bar{\theta} (s_{b} s_{ab} c),
$$

$$
\bar{F}^{(sh)}(\tau, \theta, \bar{\theta}) = \bar{c} + \theta (-i \bar{\beta}^2) + \bar{\theta} (i b_2) + \theta \bar{\theta} (2 \bar{\beta} \gamma)
$$

$$
\equiv \bar{c} + \theta (s_{ab} \bar{c}) + \bar{\theta} (s_{b} \bar{c}) + \theta \bar{\theta} (s_{b} s_{ab} \bar{c}),
$$

(28)

where the superscript $(sh)$ on the superfields denotes the expansions of the superfields after the application of SUSY-HC. Identifying $\bar{b}_1 = \bar{b}, \ b_2 = b$, we obtain the following (anti-)BRST transformations

$$
sb c = -i \beta^2, \quad sb \bar{c} = i b, \quad s_{ab} \bar{c} = -i \bar{\beta}^2, \quad s_{ab} c = i \bar{b},
$$

(29)

which are off-shell nilpotent of order two (i.e. $s^2_{ab} = 0$) and absolutely anticommuting $s_b s_{ab} + s_{ab} s_b = 0$ in nature because the following is sacrosanct, namely;

$$
sb \beta = 0, \quad sb b = 0, \quad s_{ab} \bar{\beta} = 0, \quad s_{ab} \bar{b} = 0, \quad b + \bar{b} = -2 \beta \bar{\beta}. \quad (30)
$$

The last entry in the above is nothing but the CF-type restriction which is very crucial for our further discussions and it emerges from, setting equal to zero, the coefficient of $(d\theta \wedge d\bar{\theta})$ in the restriction (22). The first two relations of (25), with inputs from (28), (29) and $(B^{(h)}, B^{(b)}, K^{(h)})$, lead to the determination of secondary variables of $E(\tau, \theta, \bar{\theta})$ as:

$$
f = \dot{c} + 2 \beta \chi, \quad \dot{f} = \dot{\bar{c}} + 2 \bar{\beta} \chi, \quad B = (\dot{b} + 2 \bar{\beta} \dot{\beta} - 2 \chi \gamma) \equiv -(\dot{b} + 2 \bar{\beta} \dot{\beta} - 2 \chi \gamma). \quad (31)
$$

As a consequence, we have the following expansion for this bosonic gauge superfield [4,5]

$$
E^{(sh)}(\tau, \theta, \bar{\theta}) = e + \theta (\dot{c} + 2 \bar{\beta} \chi) + \bar{\theta} (\dot{\bar{c}} + 2 \beta \chi) + \theta \bar{\theta} (i \dot{b} + 2 i \bar{\beta} \dot{\beta} - 2 i \chi \gamma)
$$

$$
\equiv e + \theta (s_{ab} e) + \bar{\theta} (s_{b} e) + \theta \bar{\theta} (s_{b} s_{ab} e),
$$

(32)
which shows that \( s_b e = \dot{c} + 2 \beta \chi \), \( s_{ab} e = \dot{c} + 2 \beta \chi \), \( s_b s_{ab} e = i \left( \dot{b} + 2 \beta \dot{\beta} + 2 \gamma \chi \right) \).

Now, we are in a position to derive the nilpotent (anti-)BRST symmetry transformations for the target space position variable \( x_\mu(\tau) \). It can be seen that \( \dot{x}_\mu = e p_\mu - i \chi \psi_\mu \) is a supergauge invariant quantity on the on-shell where \( \dot{p}_\mu = 0 \). As a consequence, we demand the invariance of this relationship on the (1, 2)-dimensional supermanifold, namely;

\[
\dot{X}_\mu(\tau, \theta, \bar{\theta}) = E^{(sh)}(\tau, \theta, \bar{\theta}) p_\mu(\tau) - i K^{(h)}(\tau, \theta, \bar{\theta}) \Psi^{(sg)}_\mu(\tau, \theta, \bar{\theta}),
\]

where we have taken \( P^{(g)}_\mu(\tau, \theta, \bar{\theta}) = p_\mu(\tau) \) for the obvious reasons. Taking the following general super-expansion for the target space superfield \( X_\mu(\tau, \theta, \bar{\theta}) \) along the Grassmannian directions of (1, 2)-dimensional supermanifold, namely;

\[
X_\mu(\tau, \theta, \bar{\theta}) = x_\mu(\tau) + \theta \bar{R}_\mu(\tau) + \bar{\theta} R_\mu(\tau) + i \theta \bar{\theta} S_\mu(\tau),
\]

we obtain the following explicit expansion for the target space superfield

\[
\dot{X}_\mu(\tau, \theta, \bar{\theta}) = E^{(sh)}(\tau, \theta, \bar{\theta}) p_\mu(\tau) - i K^{(h)}(\tau, \theta, \bar{\theta}) \Psi^{(sg)}_\mu(\tau, \theta, \bar{\theta}),
\]

where \( s_b e = \dot{c} + 2 \beta \chi \), \( s_{ab} e = \dot{c} + 2 \beta \chi \), \( s_b s_{ab} e = i \left( \dot{b} + 2 \beta \dot{\beta} + 2 \gamma \chi \right) \).

The nilpotency can be checked explicitly by taking the (anti-)BRST symmetry transformations for \( c, p_\mu, \beta, \psi_\mu, \dot{c}, \beta \) from our earlier equations

\[
X^{(sg)}_\mu(\tau, \theta, \bar{\theta}) = x_\mu(\tau) + \theta \left( \dot{c} p_\mu + \beta \psi_\mu \right) + \bar{\theta} \left( c p_\mu + \beta \psi_\mu \right)
\]

\[
\equiv x_\mu(\tau) + \theta \left( s_b x_\mu \right) + \bar{\theta} \left( s_b x_\mu \right) + \theta \bar{\theta} \left( s_b s_{ab} x_\mu \right).
\]

\[
\dot{X}_\mu(\tau, \theta, \bar{\theta}) = E^{(sh)}(\tau, \theta, \bar{\theta}) p_\mu(\tau) - i K^{(h)}(\tau, \theta, \bar{\theta}) \Psi^{(sg)}_\mu(\tau, \theta, \bar{\theta}),
\]

where \( s_b x_\mu = c p_\mu + \beta \psi_\mu \), \( s_{ab} x_\mu = \dot{c} p_\mu + \beta \psi_\mu \), \( s_b s_{ab} x_\mu = (i b + i \beta \beta) p_\mu + i \gamma \psi_\mu \).

We wrap up this section with the statement that it is the requirement of nilpotency property that we obtain the (anti-)BRST transformations like \( s_b \beta = 0, s_b \bar{\beta} = 0, s_{ab} \bar{\beta} = 0, \) etc. The sanctity of the absolute anticommutativity property also helps us in the determination of the (anti-)BRST symmetries for the Nakanoishi-Lautrup type auxiliary variables like \( s_b \bar{b} = -2 i \beta \gamma \) and \( s_{ab} \bar{b} = +2 i \beta \gamma \), etc.

\[\text{It should be noted that the comparison in (35), after the use of equations of motion } \dot{p}_\mu = 0, \psi_\mu = \chi p_\mu, \text{ might differ by a constant factor w.r.t. the evolution parameter } \tau. \text{ However, for the sake of simplicity, we have taken that constant equal to zero because physics would not be affected by this choice.}\]
5 CF-type restriction: (anti-)BRST invariance

We have been able to obtain all the expansions of the superfields in terms of the basic and auxiliary variables of the (anti-)BRST invariant Lagrangian for our present supersymmetric model in one (0 + 1)-dimension (1D) of spacetime. These expansions, after the application of HC, GIR, SGIR and SUSY-HC, can be written together in an explicit form as:

\[
X_\mu^{(sg)}(\tau, \theta, \bar{\theta}) = x_\mu(\tau) + \theta \left( \bar{\epsilon} p_\mu + \bar{\beta} \psi_\mu \right) + \bar{\theta} \left( c p_\mu + \beta \psi_\mu \right) + \theta \bar{\theta} \left( i b + i \bar{\beta} \beta \right) p_\mu + i \gamma \psi_\mu,
\]

\[
E^{(sh)}(\tau, \theta, \bar{\theta}) = e(\tau) + \theta \left( \bar{\epsilon} + 2 \bar{\beta} \chi \right) + \bar{\theta} \left( \bar{\epsilon} + 2 \beta \chi \right) + \theta \bar{\theta} \left( i \bar{b} + 2 \bar{i} \bar{\beta} \beta - 2 i \chi \gamma \right),
\]

\[
\Psi_\mu^{(sg)}(\tau, \theta, \bar{\theta}) = \psi_\mu(\tau) + \theta \left( i \bar{\beta} p_\mu \right) + \bar{\theta} \left( i \beta p_\mu \right) + \theta \bar{\theta} \left( - \gamma p_\mu \right),
\]

\[
K^{(h)}(\tau, \theta, \bar{\theta}) = \chi(\tau) + \theta \left( i \bar{\beta} \right) + \bar{\theta} \left( i \beta \right) + \theta \bar{\theta} \left( - \gamma \right),
\]

\[
F^{(sh)}(\tau, \theta, \bar{\theta}) = c(\tau) + \theta \left( i \bar{b} \right) + \bar{\theta} \left( - i \beta^2 \right) + \theta \bar{\theta} \left( 2 \beta \gamma \right),
\]

\[
\bar{F}^{(sh)}(\tau, \theta, \bar{\theta}) = \bar{c}(\tau) + \theta \left( - i \bar{\beta}^2 \right) + \bar{\theta} \left( i \bar{b} \right) + \theta \bar{\theta} \left( 2 \bar{\beta} \gamma \right),
\]

\[
B^{(h)}(\tau, \theta, \bar{\theta}) = \beta(\tau) + \theta \left( - i \gamma \right) + \bar{\theta} \left( 0 \right) + \theta \bar{\theta} \left( 0 \right),
\]

\[
\bar{B}^{(h)}(\tau, \theta, \bar{\theta}) = \bar{\beta}(\tau) + \theta \left( 0 \right) + \bar{\theta} \left( i \gamma \right) + \theta \bar{\theta} \left( 0 \right),
\]

\[
P_\mu^{(g)}(\tau, \theta, \bar{\theta}) = p_\mu(\tau) + \theta \left( 0 \right) + \bar{\theta} \left( 0 \right) + \theta \bar{\theta} \left( 0 \right).
\]

(38)

A close look at the above expansions leads to the derivation of (anti-)BRST symmetry transformations \( s_{(a)b} \), in their full blaze of glory, as listed below:

\[
s_{ab} x_\mu = \bar{c} p_\mu + \bar{\beta} \psi_\mu, \quad s_{ab} e = \bar{c} + 2 \bar{\beta} \chi, \quad s_{ab} \psi_\mu = i \bar{\beta} p_\mu,
\]

\[
s_{ab} \bar{c} = -i \bar{\beta}^2, \quad s_{ab} c = i \bar{b}, \quad s_{ab} \bar{\beta} = 0, \quad s_{ab} \beta = -i \gamma, \quad s_{ab} p_\mu = 0,
\]

\[
s_{ab} \gamma = 0, \quad s_{ab} \bar{b} = 0, \quad s_{ab} \chi = i \bar{\beta}, \quad s_{ab} b = 2 i \bar{\beta} \gamma,
\]

(39)

\[
s_b x_\mu = c p_\mu + \beta \psi_\mu, \quad s_b e = \bar{c} + 2 \beta \chi, \quad s_b \psi_\mu = i \beta p_\mu,
\]

\[
s_b \bar{c} = -i \beta^2, \quad s_b c = i b, \quad s_b \bar{\beta} = 0, \quad s_b \beta = i \gamma, \quad s_b p_\mu = 0,
\]

\[
s_b \gamma = 0, \quad s_b \bar{b} = 0, \quad s_b \chi = i \beta, \quad s_b b = -2 i \beta \gamma.
\]

(40)

It is elementary to check that the transformations in (39) and (40) are nilpotent of order two (i.e. \( s_{(a)b}^2 = 0 \)) which establishes the fermionic nature of (anti-)BRST transformations.

Now we discuss a bit about the CF-type condition \((b + \bar{b} + 2 \beta \bar{\beta} = 0)\) of our present theory, its importance and its (anti-)BRST invariance within the framework of superfield formalism. We re-emphasize that, from the SUSY-HC (22), we obtain the CF-type restriction \((b + \bar{b} + 2 \beta \bar{\beta} = 0)\) when we set equal to zero the coefficient of \((d\theta \wedge d\bar{\theta})\). It can be checked explicitly
that the operator form of \( \{s_b, s_{ab}\} = s_b s_{ab} + s_{ab} s_b \), acting on all the variables of the present theory, is trivially zero except for the following:

\[
\{s_b, s_{ab}\} e(\tau) \neq 0, \quad \{s_b, s_{ab}\} x_\mu(\tau) \neq 0.
\] (41)

Thus, it appears that one of the sacrosanct properties of BRST formalism is lost. However, at this stage, the CF-type restriction comes to our rescue. One can verify that there is an absolute anticommutativity in the theory because the r.h.s. of (41) is also zero on a constrained super world-line, defined by the CF-type equation \( (b + \bar{b} + 2 \beta \bar{\beta} = 0) \). We conclude that the absolute anticommutativity of the off-shell nilpotent (anti-)BRST symmetries is respected in the theory because of the presence of CF-type restriction which emerges from the augmented version of BT superfield formalism [8-10,14]. One of the key features of CF-type restriction is that it is an (anti-)BRST invariant quantity because

\[
s_{(a)b} [b + \bar{b} + 2 \beta \bar{\beta}] = 0,
\] (42)

where we have used explicitly the transformations listed in (39) and (40). The above observation establishes the fact that CF-type restriction is a physical restriction (in some sense) because it is an (anti-)BRST invariant quantity.

The (anti-)BRST invariance of CF-type restriction [cf. (42)] can be captured within the framework of augmented BT superfield formalism as well. Towards this goal in mind, it can be seen [from our expansions (38)] that we have the following generic relationship between the Grassmannian derivatives \( (\partial_\theta, \partial_{\bar{\theta}}) \) of the \((1, 2)\)-dimensional supermanifold and the nilpotent (anti-)BRST transformations \( s_{(a)b} \) on the 1D dynamical variables, namely;

\[
\begin{align*}
\lim_{\theta \to 0} \frac{\partial}{\partial \theta} \Omega^{(h,sh,g,sg)}(\tau, \theta, \bar{\theta}) &= s_b \, w(\tau), \\
\lim_{\theta \to 0} \frac{\partial}{\partial \theta} \Omega^{(h,sh,g,sg)}(\tau, \theta, \bar{\theta}) &= s_{ab} \, w(\tau), \\
\frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} \Omega^{(h,sh,g,sg)}(\tau, \theta, \bar{\theta}) &= s_b \, s_{ab} \, w(\tau),
\end{align*}
\] (43)

where \( \Omega^{(h,sh,g,sg)}(\tau, \theta, \bar{\theta}) \) are the generic expansions of superfields [cf. (38)], derived after the application of (SUSY-)HC as well as (S)GIR and \( \omega(\tau) \) is the generic dynamical variable of the (anti-)BRST invariant 1D theory [cf. (51), (52) below] for the description of a free spinning relativistic particle [e.g. \( \omega(\tau) = e(\tau), \chi(\tau), x_\mu(\tau), \psi_\mu(\tau), c(\tau), \bar{c}(\tau), \) etc.].

In view of the mapping in (43), it can be seen that the coefficient of \( (d\theta \wedge d\bar{\theta}) \), which yields CF condition [cf. (25), (30)], is independent of \( \theta \) and \( \bar{\theta} \). To see it clearly, first of all, we note the following [which have been already quoted in (25) and (30)], namely;

\[
\partial_{\bar{\theta}} F^{(sh)} + \partial_{\bar{\theta}} \bar{F}^{(sh)} + 2i \, \bar{B}^{(h)} \, \mathcal{B}^{(h)} = 0 \quad \implies \quad b + \bar{b} + 2 \beta \bar{\beta} = 0.
\] (44)

Next step is to prove the BRST invariance of \( (b + \bar{b} + 2 \beta \bar{\beta} = 0) \), in the language of superfield formalism. In this context, we observe very explicitly [in view of (43)] that

\[
\lim_{\theta \to 0} \partial_{\bar{\theta}} \left[ \partial_{\bar{\theta}} F^{(sh)} + \partial_{\bar{\theta}} \bar{F}^{(sh)} + 2i \, \bar{B}^{(h)} \, \mathcal{B}^{(h)} \right] = 2 \beta \gamma - 2 \beta \gamma = 0.
\] (45)
Similarly, for the anti-BRST invariance of \((b + \bar{b} + 2 \beta \bar{\beta} = 0)\), we can clearly check that
\[
\lim_{\bar{\theta} \to 0} \partial_{\bar{\theta}} \left[ \partial_{\bar{\theta}} F^{(sh)} + \partial_{\theta} \bar{F}^{(sh)} + 2 i \bar{B}^{(h)} B^{(h)} \right] = 2 \beta \gamma - 2 \bar{\beta} \gamma = 0. \tag{46}
\]
It is transparent that the equations (45) and (46) do capture (42). Thus, from the above equations, it is clear that we have captured the (anti-)BRST invariance of CF-type restriction within the framework of our augmented version of BT superfield formalism.

One of the most important features of CF-type restriction is its role in the derivation of coupled (but equivalent) Lagrangians for a given theory. The beauty of these Lagrangians is the fact that they are (anti-)BRST invariant on the constraint surface (in our 1D case a super world-line) which is defined by the CF-type equation \((b + \bar{b} + 2 \beta \bar{\beta} = 0)\). Furthermore, such coupled Lagrangians yield the CF-type restriction as an off-shoot of their Euler-Lagrange equations of motion. In our next section, we are going to discuss this aspect of the (anti-)BRST invariant CF-type restriction in the full blaze of its glory.

6 Coupled Lagrangians: (anti-)BRST invariance

We derive here the most appropriate coupled (but equivalent) Lagrangians for the spinning relativistic particle which respect the (anti-)BRST symmetry transformations (39) and (40) on the constrained super world-line defined by the CF-type restriction \((b + \bar{b} + 2 \beta \bar{\beta} = 0)\). Using the standard techniques of BRST-formalism, one can write the following

\[
L_{b}^{(0)} = L_{0} + s_{b} \left[ \frac{i}{2} e^{2} + c \bar{c} \right],
\tag{47}
\]

\[
L_{\bar{b}}^{(0)} = L_{0} - s_{ab} s_{b} \left[ \frac{i}{2} e^{2} + c \bar{c} \right],
\tag{48}
\]

where \(L_{0}\) is the starting Lagrangian (1) and \(s_{(a)b}\) are the nilpotent and absolutely anticommuting (anti-)BRST transformations (39) and (40). It is straightforward to check that

\[
s_{b} s_{ab} \left[ \frac{i}{2} e^{2} + c \bar{c} \right] = -i \hat{c} (\hat{c} + 2 \beta \chi) + 2 i \bar{\beta} \hat{c} \chi - \hat{b} e - 2 e (\gamma \chi + \bar{\beta} \hat{\beta}) + 2 \beta \gamma \bar{c} - b \bar{b} + \beta^{2} \beta^{2} + 2 \bar{\beta} c \gamma, \tag{49}
\]

\[
-s_{ab} s_{b} \left[ \frac{i}{2} e^{2} + c \bar{c} \right] = -i \hat{c} (\hat{c} + 2 \beta \chi) + 2 i \bar{\beta} \hat{c} \chi + \hat{b} e - 2 e (\gamma \chi - \beta \hat{\beta}) + 2 \beta \gamma \bar{c} - b \bar{b} + \beta^{2} \beta^{2} + 2 \bar{\beta} c \gamma, \tag{50}
\]

which are nothing but the gauge-fixing and Faddeev-Popov ghost terms for the present theory (being described within the framework of BRST formalism).

Throwing away the total derivative terms and using the CF-restriction \((b + \bar{b} + 2 \beta \bar{\beta} = 0)\), we obtain the following form of the coupled (but equivalent) Lagrangians

\[
L_{b}^{(0)} = L_{0} + b \hat{e} + b (b + 2 \beta \bar{\beta}) - i \hat{c} (\hat{c} + 2 \beta \chi) + 2 i \bar{\beta} \hat{c} \chi - 2 e (\gamma \chi + \bar{\beta} \hat{\beta}) + 2 \beta \gamma \bar{c} + \beta^{2} \beta^{2} + 2 \bar{\beta} c \gamma, \tag{51}
\]
\[
L_b^{(0)} = L_0 - \bar{b} \dot{\bar{b}} + b (\bar{b} + 2 \bar{\beta} \beta) - i \dot{\bar{c}} (\dot{\bar{c}} + 2 \beta \chi) + 2 i \bar{\beta} \dot{\bar{c}} \chi \\
- 2 e (\gamma \chi - \beta \dot{\bar{\beta}}) + 2 \beta \gamma \bar{c} + \bar{\beta}^2 \beta^2 + 2 \bar{\beta} c \gamma.
\]

(52)

We note that we have expressed \((b \bar{b})\), present in equations (49) and (50), in two different ways by using the CF-type restriction \((b + \bar{b} + 2 \beta \bar{\beta} = 0)\). We can check explicitly that, the following perfect symmetry transformations emerge, namely;

\[
s_b L_b^{(0)} = \frac{d}{d\tau} \left[ \frac{1}{2} c p^2 + \frac{1}{2} \beta (p \cdot \psi) + b (\dot{\bar{c}} + 2 \beta \chi) \right],
\]

(53)

\[
s_{ab} L_b^{(0)} = \frac{d}{d\tau} \left[ \frac{1}{2} \bar{c} p^2 + \frac{1}{2} \bar{\beta} (p \cdot \psi) - \bar{b} (\dot{\bar{c}} + 2 \beta \chi) \right].
\]

(54)

As a consequence, the action integrals \((S_1 = \int d\tau L_b^{(0)}, S_2 = \int d\tau L_{\bar{b}}^{(0)})\) remain invariant under the (anti-)BRST transformations (39) and (40). There are other ways of expressing (47) and (48) as we have done in our Appendices A and B. However, we choose the forms (51) and (52) because, at least, these respect perfect symmetries like (53) and (54).

The Lagrangians (51) and (52) are equivalent on the constrained super world-line, defined by the CF-type condition (30). This can be corroborated by the following observations

\[
s_{ab} L_b^{(0)} = \frac{d}{d\tau} \left[ \frac{1}{2} \bar{c} p^2 + \frac{1}{2} \bar{\beta} (p \cdot \psi) + 2 i e \bar{\beta} \gamma + b (\dot{\bar{c}} + 2 \beta \chi) \right] \\
- (\dot{\bar{c}} + 2 \beta \chi) \frac{d}{d\tau} \left[ b + \bar{b} + 2 \beta \bar{\beta} \right] + (2 i \bar{\beta} \gamma) (b + \bar{b} + 2 \beta \bar{\beta}),
\]

(55)

\[
s_b L_{\bar{b}}^{(0)} = \frac{d}{d\tau} \left[ \frac{1}{2} c p^2 + \frac{1}{2} \beta (p \cdot \psi) + 2 i e \beta \gamma - \bar{b} (\dot{\bar{c}} + 2 \beta \chi) \right] \\
+ (\dot{\bar{c}} + 2 \beta \chi) \frac{d}{d\tau} \left[ b + \bar{b} + 2 \beta \bar{\beta} \right] - (2 i \beta \gamma) (b + \bar{b} + 2 \beta \bar{\beta}),
\]

(56)

which demonstrate the equivalence of \(L_b^{(0)}\) and \(L_{\bar{b}}^{(0)}\) [as far as the (anti-)BRST symmetry transformations (39) and (40) are concerned] on the constrained super world-line defined by the CF-type restriction (30). In other words, both \(L_b^{(0)}\) and \(L_{\bar{b}}^{(0)}\) respect the off-shell nilpotent (anti-)BRST symmetry transformations (39) and (40) if we confine ourselves to the constrained super world-line [defined by the CF-condition (30)] embedded in the D-dimensional target Minkowaskian flat spacetime manifold. We obtain, from the above Lagrangians (51) and (52), the following very useful relationships, namely;

\[
b = -\frac{1}{2} \dot{\bar{c}} - \beta \bar{\beta}, \quad \bar{b} = \frac{1}{2} \dot{\bar{c}} - \beta \bar{\beta},
\]

(57)

as the Euler-Lagrange equations of motion which, ultimately, lead to the derivation of the CF-type condition \((b + \bar{b} + 2 \beta \bar{\beta} = 0)\) in a straightforward manner.

The (anti-)BRST invariance (54) and (53) of \(L_b^{(0)}\) and \(L_{\bar{b}}^{(0)}\) can be captured within the framework of superfield formalism. Towards this goal in mind, first of all, we check that
the super Lagrangian $\tilde{L}_0$, corresponding to the starting Lagrangian $L_0$, can be written in the following form in terms of the appropriate superfields:

$$\tilde{L}_0 = P(g) \cdot \dot{X}^{(sg)} - \frac{1}{2} E^{(sh)} (P(g))^2 + \frac{i}{2} \Psi^{(sg)} \cdot \dot{\Psi}^{(sg)} + i K^{(h)} (P(g) \cdot \Psi^{(sg)}) \equiv L_0,$$

(58)

where the super-expansions of the relevant superfields, after the application of (S)GIR and (SUSY-)HC, are quoted in (38). It is obvious from (58) that the l.h.s. is independent of $\theta$ and $\bar{\theta}$ variables (as $\tilde{L}_0 = L_0$). As a consequence, we have the following relationship

$$\lim_{\theta \to 0} \frac{\partial}{\partial \theta} \tilde{L}_0 = 0, \quad \lim_{\bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}} \tilde{L}_0 = 0,$$

(59)

which, in the ordinary 1D spacetime, imply that the starting Lagrangian $L_0$ remains invariant under the (anti-)BRST transformations (39) and (40).

In view of the above observation, it is straightforward to express the Lagrangians (47) and (48) in terms of the superfields, obtained after the application of (anti-)BRST invariant (SUSY-)HC and (S)GIR, as follows

$$\tilde{L}^{(0)}_b = \tilde{L}_0 + \frac{\partial}{\partial \theta} \left[ \frac{i}{2} E^{(sh)} E^{(sh)} + F^{(sh)} \bar{F}^{(sh)} \right],$$

$$\tilde{L}^{(0)}_{\bar{b}} = \tilde{L}_0 - \frac{\partial}{\partial \bar{\theta}} \left[ \frac{i}{2} E^{(sh)} E^{(sh)} + F^{(sh)} \bar{F}^{(sh)} \right],$$

(60)

where the expansions for the superfields are quoted in (38). The (anti-)BRST invariance and equivalence of the Lagrangians $L^{(0)}_b$ and $L^{(0)}_{\bar{b}}$ can be captured, in a very simple manner, within the framework of superfield formalism as illustrated below

$$\lim_{\theta \to 0} \frac{\partial}{\partial \theta} \tilde{L}_{(b,\bar{b})} = 0, \quad \lim_{\bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}} \tilde{L}_{(b,\bar{b})} = 0,$$

(61)

where the nilpotency property ($\partial^2_\theta = 0$, $\partial^2_{\bar{\theta}} = 0$) of the Grassmannian derivative ($\partial_\theta, \partial_{\bar{\theta}}$) and the their absolute anticommutativity ($\partial_\theta \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_\theta = 0$) play a very decisive role. Furthermore, the (anti-)BRST invariance of the starting Lagrangian $L_0$, in the language of the superfield formalism, has also been taken into consideration in the proof of (61).

We close this section with a few remarks. First and foremost, the Lagrangian, proposed in [11] for the spinning relativistic particle, does not respect the (anti-)BRST symmetries together. Second, the coupled (but equivalent) Lagrangians of (47) and (48) are the correct Lagrangians for the supersymmetric system of a spinning relativistic particle which respect both the above nilpotent symmetries on a constrained super world-line [where the CF-type restriction $b + \bar{b} + 2 \beta \bar{\beta} = 0$ is satisfied]. Finally, it is the augmented version of BT superfield formalism [8-10] that plays a key role in the derivation of the proper (anti-)BRST symmetries and the (anti-)BRST invariant CF-type restriction. The latter, we claim, is the hallmark of a supersymmetric gauge theory within the framework of BRST formalism.

7 Conclusions

One of the key results of our present investigation is the derivation of supersymmetric version of HC (SUSY-HC) in equation (22) which enables us to derive the off-shell nilpotent and
absolutely anticommuting (anti-)BRST transformations for the variables \(e(\tau), c(\tau), \bar{c}(\tau)\) and \(x_\mu(\tau)\) within the framework of supersymmetric version of the augmented BT superfield formalism. The beauty of this SUSY-HC (22) lies in the fact that it utilizes the results of HC [cf. (7), (23)] in a meaningful manner and produces [cf. (30)] the (anti-)BRST invariant CF-type restriction\(^\dagger\). As a consequence, the (anti-)BRST symmetry transformations of \(\chi(\tau), \beta(\tau), \bar{\beta}(\tau)\) and \(\psi_\mu(\tau)\) (derived from the application of HC) turn out to be consistent and complementary to such kind of transformations for \(e(\tau), c(\tau), \bar{c}(\tau)\) and \(x_\mu(\tau)\) (derived from the power and beauty of SUSY-HC). The rest of the (anti-)BRST symmetry transformations (e.g. for the auxiliary variables) are obtained from the requirements of off-shell nilpotency and absolute anticommutativity of the (anti-)BRST transformations.

The supersymmetric system of a massless spinning relativistic particle is a physically very interesting system where we have applied the augmented version of BT-superfield formalism for the first-time and derived the proper (i.e. off-shell nilpotent and absolutely anticommuting) anti-BRST symmetry transformations. Besides HC and SUSY-HC, in our investigation, we have shown that the (super)gauge invariant quantities also play very important roles. This is why, there are different varieties of superscripts in expansions (38). The central observation of our present work is the fact that all the proper nilpotent (anti-) BRST transformations, derived from HC, SUSY-HC, GIR and SGIR, are consistent with one-another. We would like to mention that we have never used so many conditions on the superfields in our earlier study of non-supersymmetric \(p\)-form gauge theories.

It is very interesting to point out that the HC yields the proper (anti-)BRST symmetry transformations for the super (fermionic) gauge variable \(\chi(\tau)\) and its associated bosonic (anti-)ghost variables \((\bar{\beta})\beta\). To obtain the (anti-)BRST symmetry transformations for the fermionic variable \(\psi_\mu(\tau)\), however, we have been theoretically compelled to use the equation of motion \(\dot{\psi}_\mu = \chi p_\mu\) which is a supergauge invariant quantity. In exactly similar fashion, the SUSY-HC [cf. (22)] yields the proper (anti-)BRST symmetry transformations for the bosonic gauge variable \(e(\tau)\) and its associated fermionic (anti-)ghost variables \((\bar{c})c\). However, to obtain the proper (anti-)BRST symmetry transformations for the target space (bosonic) variable \(x_\mu(\tau)\), we have invoked another equation of motion \(\dot{x}_\mu = e p_\mu - i \chi \psi_\mu\) which is also a supergauge invariant quantity. It is precisely, because of these observations, that we have christened our superfield formalism as the supersymmetric version of the augmented BT superfield formalism where (super)gauge invariance plays a decisive role.

We would like to point out that, in [11], only the nilpotent BRST symmetries for the free spinning relativistic particle have been discussed. However, the corresponding proper anti-BRST symmetries have been left untouched. In our recent couple of papers [16,17], we have established that the existence of anti-BRST symmetry transformations is sacrosanct in the context of BRST description of any arbitrary \(p\)-form \((p = 1, 2, 3, \ldots)\) gauge theories as it is crucially connected with the existence of CF-type of restrictions which owe their origin to the geometrical object called gerbes. In fact, we have claimed that, given a local gauge symmetry transformation, the holy grails of the BRST formalism allow us to have both BRST as well as anti-BRST symmetry transformations together in a gauge theory. The decisive feature of a gauge theory, within the framework of BRST formalism, is the

\(^\dagger\) It is to be noted that, for the first-time, the celebrated Curci-Ferrari condition [15] appeared in the description of the 4D non-Abelian 1-form gauge theory within the framework of BRST formalism.
existence of the CF type restriction (which allows totally independent existence of BRST as well as anti-BRST symmetry transformations). For the simple case of Abelian 1-form gauge theory, the CF type restriction is trivial. However, for the rest of the $p$-form gauge theories, the CF type restriction is always non-trivial (see, e.g. [16,17]).

In our present investigation, we have applied our superfield formalism to a simple supersymmetric system of a free (massless as well as massive) spinning relativistic particle. In the future, we plan to extend this work to the description of some other physically interesting supersymmetric models and derive various conserved charges (corresponding to various continuous symmetries) of the theory which would include the conserved (super)charges, (anti-)BRST charges, ghost charges, etc., and we envisage to obtain the underlying algebra and look into its relevance to the algebra of some (super)Lie groups. Furthermore, we hope to apply the supersymmetric version of the augmented BT superfield formalism to more physically realistic SUSY models of phenomenological interest as far as the $p$-form gauge theories are concerned. These are some of the issues that are presently under investigation and our results would be reported in our future publications [18].

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Appendix A

We derive here two Lagrangians, by exploiting (39) and (40), which respect anti-BRST and BRST symmetry transformations but they are not equivalent even on the hyper super world-line defined by the CF-type restriction. As a consequence, they are not as interesting as (51) and (52). Using the standard techniques of BRST formalism, we can obtain the Lagrangian that respects only the BRST transformations (40). Such a Lagrangian is:

\[ L_b^{(1)} = L_0 + s_b \left[ -i \bar{c} \left( \dot{c} + \frac{b}{2} \right) - i \bar{\beta} \dot{\chi} \right], \tag{A.1} \]

where $L_0$ is our starting Lagrangian (1). We have to exploit the BRST transformations (40) to obtain the following explicit form of the BRST invariant Lagrangian

\[ L_b^{(1)} = L_0 + b \dot{\chi} + \frac{1}{2} b^2 - i \dot{\bar{c}} (\dot{c} + 2 \beta \chi) + \gamma \dot{\chi} - \dot{\bar{\beta}} \dot{\beta}, \tag{A.2} \]

\[ \text{We have applied the theoretical arsenal of superfield approach to BRST formalism in the case of massless spinning relativistic particle in great detail. However, we have concisely mentioned the application of superfield formalism to the derivation of (anti-)BRST symmetries for the massive spinning relativistic particle in our Appendix C. The details of the superfield technique can be worked out for the latter system, too, on exactly same lines as that of the massless spinning particle.} \]
which remains quasi-invariant under $s_b$ because

$$s_b L_b^{(1)} = \frac{d}{d\tau} \left[ \frac{1}{2} c p^2 + \frac{1}{2} \beta (p \cdot \psi) + b (\dot{c} + 2 \beta \chi) - i \gamma \dot{\beta} \right]. \quad (A.3)$$

As a consequence, the action integral $S = \int d\tau L_b^{(1)}$ remains invariant for the physically well-defined variables that vanish off at infinity due to Gauss’s divergence theorem.

In exactly similar fashion, we can derive the anti-BRST invariant Lagrangian by exploiting the anti-BRST transformations (39). This Lagrangian ($L_b^{(1)}$) can be written as

$$L_b^{(1)} = L_0 + s_{ab} \left[ i c \left( \dot{e} + \frac{\bar{b}}{2} \right) + i \beta \chi \right], \quad (A.4)$$

where $L_0$ is our starting Lagrangian (1). Exploiting the anti-BRST symmetry transformations (39), we obtain the following explicit form of the anti-BRST invariant Lagrangian

$$L_b^{(1)} = L_0 - \bar{b} \dot{e} - \frac{1}{2} \bar{b}^2 + i \dot{c} (\bar{c} + 2 \beta \chi) + \gamma \dot{\chi} + \dot{\beta} \dot{\beta}. \quad (A.5)$$

The anti-BRST invariance of $L_b^{(1)}$ can be checked by using the transformations (39) as we note that $L_b^{(1)}$ transforms to a total derivative as given below

$$s_{ab} L_b^{(1)} = \frac{d}{d\tau} \left[ \frac{1}{2} \bar{c} p^2 + \frac{1}{2} \beta (p \cdot \psi) - \bar{b} (\dot{c} + 2 \beta \chi) - i \gamma \dot{\beta} \right]. \quad (A.6)$$

As a consequence, the action integral $S = \int d\tau L_b^{(1)}$ remains invariant for the physically well-defined variables of the theory which fall-off rapidly at infinity.

It is evident from (A.2) and (A.5) that both the Lagrangians are coupled because we have already derived the CF-type restriction $b + \bar{b} + 2 \beta \bar{\beta} = 0$ which relates the Nakanishi-Lautrup type of variables $b$ and $\bar{b}$ through the bosonic (anti-)ghost variables ($\beta \bar{\beta}$). We have already captured the CF-type restriction within the framework of superfield formalism. Now we demonstrate that the (anti-)BRST invariance [cf. (A.3), (A.6)] of Lagrangians (A.2) and (A.5) (and the Lagrangians themselves) can also be incorporated in the language of the superfield formalism. We note that the Lagrangians (A.2) and (A.5) can be expressed in the language of superfields [obtained after (SUSY-)HC and (S)GI R], as

$$\tilde{L}^{(1)}_b = \tilde{L}_0 + \lim_{\theta \to 0} \frac{\partial}{\partial \theta} \left[ -i \tilde{F}^{(sh)} \left( \tilde{E}^{(sh)} + \frac{b(\tau)}{2} \right) - i \tilde{B}^{(h)} \tilde{K}^{(h)} \right],$$

$$\tilde{L}^{(1)}_b = \tilde{L}_0 + \lim_{\bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}} \left[ i \tilde{F}^{(sh)} \left( \tilde{E}^{(sh)} + \frac{\bar{b}(\tau)}{2} \right) + i \tilde{B}^{(h)} \tilde{K}^{(h)} \right], \quad (A.7)$$

where the super-expansions of all the superfields (with various superscripts) are given in (38). In view of our observation in (59), it is elementary to show that

$$\lim_{\theta \to 0} \frac{\partial}{\partial \theta} \tilde{L}^{(1)}_b = 0, \quad \lim_{\bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}} \tilde{L}^{(1)}_b = 0, \quad (A.8)$$

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which, because of the mappings (43), establish the (anti-)BRST invariance of \((L_b)L_b\) in the ordinary 1D spacetime [cf. (A.3),(A.6)]. We mention here that it is the (anti-)BRST invariance of \(L_0\) [cf. (59)] and the nilpotency \((s^2)_{ab} = 0\) of (anti-)BRST symmetry transformations (i.e. \(\partial^2_b = \partial^2_{\bar{b}} = 0\)) that have played the key roles in the proof of (A.8).

Before we wrap up this Appendix, we note that, even though, the Lagrangians (A.2) and (A.5) respect the (anti-)BRST symmetries [cf. (A.6), (A.3)], they are not equivalent even on the super world-line defined by CF-type condition \((b + \bar{b} + 2 \beta \bar{\beta} = 0)\). This can be checked in the following manner by taking the help of symmetry properties, namely;

\[
s_b L_b^{(1)} = \frac{d}{d\tau} \left[ \frac{1}{2} c p^2 + \frac{1}{2} \beta (p \cdot \psi) - \bar{b} (\dot{c} + 2 \beta \chi) - i \gamma \dot{\beta} \right] + \left( \dot{b} + \dot{\bar{b}} + 2 \beta \dot{\bar{\beta}} \right) \dot{c}
\]

\[
+ 2 i \beta \gamma (\bar{b} + \dot{c}) + 2 (i \gamma + \beta \dot{c}) \dot{\beta} + 2 \left[ \dot{\bar{b}} \beta + 2 \beta \dot{\bar{\beta}} + \dot{\bar{\beta}} \gamma \right] \chi. \tag{A.9}
\]

\[
s_{ab} L_b^{(1)} = \frac{d}{d\tau} \left[ \frac{1}{2} \bar{c} p^2 + \frac{1}{2} \bar{\beta} (p \cdot \psi) + b (\dot{c} + 2 \beta \chi) - i \gamma \dot{\beta} \right] - \dot{c} \left( \dot{b} + \dot{\bar{b}} + 2 \beta \dot{\bar{\beta}} \right)
\]

\[
+ 2 i \beta \gamma (b + \dot{c}) + 2 (i \gamma - \beta \dot{c}) \dot{\beta} + 2 \left( \dot{\bar{\beta}} \gamma - \dot{\bar{\beta}} \bar{\beta} - 2 \beta \dot{\bar{\beta}} \right) \chi. \tag{A.10}
\]

Thus, we lay emphasis on the fact that the Lagrangians \(L_b^{(1)}\) and \(L_b^{(1)}\) (even though endowed with some interesting properties) are not equivalent because even if we use the equations of motion and/or the celebrated CF-type restriction \((b + \bar{b} + 2 \beta \bar{\beta} = 0)\), we do not find the precise (anti-)BRST invariance of \((L_b^{(1)})L_b^{(1)}\) [as is evident from equations (A.9) and (A.10)]. Furthermore, the other drawback of the Lagrangians (A.3) and (A.5) is the fact that we are unable to obtain the CF-type restriction as an off-shoot from the Euler-Lagrange equations of motion (derived from the Lagrangians \(L_b^{(1)}\) and \(L_b^{(1)}\)).

**Appendix B**

We show here that the Lagrangians (47) and (48) (cf. Sec. 6) can be expressed in a coupled form that are different from the ones given in (51) and (52). As we have seen, the latter forms have their own merits. However, the former ones are more symmetrical (in some sense) because the Nakanishi-Lautrup type of auxiliary variables \(b\) and \(\bar{b}\) appear in these Lagrangians in a symmetrical manner. Such coupled Lagrangians, which are more symmetrical in form, are as follows

\[
L_b^{(2)} = L_0 + b \dot{c} + \frac{1}{2} (b^2 + \bar{b}^2) - i \dot{c} (\dot{c} + 2 \beta \chi) + 2 i \beta \dot{c} \chi - 2 e (\gamma \chi + \bar{\beta} \bar{\dot{\beta}} - 2 \beta \gamma - \bar{\beta}^2 \beta^2 + 2 \beta \bar{\beta} c \gamma), \tag{B.1}
\]

\[
L_b^{(2)} = L_0 - \bar{b} \dot{c} + \frac{1}{2} (\bar{b}^2 + b^2) - i \dot{c} (\dot{c} + 2 \beta \chi) + 2 i \beta \dot{c} \chi - 2 e (\gamma \chi - \beta \bar{\dot{\beta}} - 2 \bar{\beta} \gamma - \beta^2 \bar{\beta} + 2 \beta \bar{\beta} c \gamma), \tag{B.2}
\]
where we have used the following standard relationship

\[ b \bar{b} = \frac{1}{4} (b + \bar{b})^2 - \frac{1}{4} (b - \bar{b})^2 \equiv 2 \beta^2 \beta^2 - \frac{1}{2} (b^2 + \bar{b}^2), \]  

(B.3)
due to the CF-type restriction \( b + \bar{b} = -2 \beta \bar{\beta} \). The coupled Lagrangians (B.1) and (B.2) are equal on the constraint world-line defined by \( b + \bar{b} + 2 \beta \bar{\beta} = 0 \). In other words, the terms, that differ between (B.1) and (B.2), are primarily equal due to the CF-type restriction. We note that the CF-type equation can be derived from the following equations of motion

\[ \dot{b} = -\frac{1}{2} p^2 - 2 \gamma \chi - 2 \bar{\beta} \\bar{\beta}, \]
\[ \dot{\bar{b}} = +\frac{1}{2} p^2 + 2 \gamma \chi - 2 \beta \beta, \]  

(B.4)
that emerge from the Lagrangian \( L^{(2)}_b \) and \( L^{(2)}_{\bar{b}} \).

Now we discuss the symmetry properties of (B.1) and (B.2) under the nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations (39) and (40). It can be easily checked that, under (40), the Lagrangian \( L^{(2)}_b \) transforms as follows

\[ s_b L^{(2)}_b = \frac{d}{d\tau} \left[ \frac{1}{2} c \ p^2 + \frac{1}{2} \beta (p \cdot \psi) + b (\dot{\beta} + 2 \beta \chi) \right] - 2 i \beta \gamma (b + \bar{b} + 2 \beta \bar{\beta}). \]  

(B.5)

Similarly, under the off-shell nilpotent anti-BRST transformations (39), the other Lagrangian \( L^{(2)}_{\bar{b}} \) (of the present physical system) transforms to a total derivative as

\[ s_{ab} L^{(2)}_{\bar{b}} = \frac{d}{d\tau} \left[ \frac{1}{2} \bar{c} \bar{p}^2 + \frac{1}{2} \bar{\beta} (\bar{p} \cdot \bar{\psi}) - \bar{b} (\bar{\dot{\beta}} + 2 \bar{\beta} \bar{\chi}) \right] + 2 i \bar{\beta} \gamma (b + \bar{b} + 2 \beta \bar{\beta}). \]  

(B.6)
Thus, we note that \( (L^{(2)}_{b}) L^{(2)}_{\bar{b}} \) respect (anti-)BRST symmetry transformations \( s_{(a)b} \) only when the CF-type condition \( (b + \bar{b} + 2 \beta \bar{\beta} = 0) \) is satisfied. In other words, the off-shell nilpotent (anti-)BRST transformations (39) and (40) are symmetry transformations for the Lagrangians \( L^{(2)}_b \) and \( L^{(2)}_{\bar{b}} \) only on a constrained super world-line, defined by the CF-type equation \( (b + \bar{b} + 2 \beta \bar{\beta} = 0) \), which is embedded in the D-dimensional target spacetime Minkowskian flat manifold. It can be easily noted that the symmetry transformations in (B.5) and (B.6) are not like the transformations in (53) and (54). Hence, the former are not perfect symmetry transformations as are the latter set [cf. (53),(54)].

We close this Appendix with the following remarks. First, the (anti-)BRST transformations (39) and (40) are not perfect symmetry transformations for the Lagrangians \( L^{(2)}_b \) and \( L^{(2)}_{\bar{b}} \). Rather, these are the symmetry transformations only under the validity of the CF-type restriction. Second, the Lagrangians \( L^{(2)}_b \) and \( L^{(2)}_{\bar{b}} \), it can be checked explicitly, transform exactly same as the Lagrangians \( L^{(0)}_b \) and \( L^{(0)}_{\bar{b}} \) of Sec. 6 under the transformations \( s_{ab} \) and \( s_b \), respectively [cf. (55), (56)]. Third, the (anti-)BRST invariance of Lagrangians \( L^{(2)}_b \) and \( L^{(2)}_{\bar{b}} \) can be captured within the framework of superfield formalism, in exactly same manner, as we have accomplished this goal in Sec. 6. Fourth, as far as the symmetrical form is concerned, the coupled Lagrangians (B.1) and (B.2) are more beautiful than (51)
and (52). However, from the point of view of perfect symmetry, the Lagrangians (51) and (52) are more appropriate [cf. (53),(54)].

Appendix C

For the present paper to be complete and self-contained, we discuss here briefly the superfield approach to BRST analysis of the massive spinning relativistic particle. The analogue of the Lagrangian (1), for the free massive relativistic spinning relativistic particle, is [11]

\[ L_1 = p_\mu \dot{x}^\mu - \frac{e}{2} (p^2 + m^2) + \frac{i}{2} (\psi_\mu \dot{\psi}^\mu - \psi_5 \dot{\psi}_5) + i \chi (p_\mu \psi^\mu - \psi_5 m), \quad (C.1) \]

where the \( \tau \)-independent mass parameter \( m \) (which happens to be the analogue of the cosmological constant term) has been introduced by invoking a Lorentz scalar fermionic (i.e. \( \chi \psi_5 + \psi_5 \chi = 0, \psi_5 \psi_\mu + \psi_\mu \psi_5, \) etc.) variable \( \psi_5 \) (with \( \psi_5^2 = -1 \)).

We note that the analogue of the combined (super)gauge transformations (3), in our present case of a free massive spinning particle, are

\[ \delta x_\mu = \xi p_\mu + \kappa \psi_\mu, \quad \delta p_\mu = 0, \quad \delta \psi_\mu = i \kappa p_\mu, \]
\[ \delta \chi = i \dot{\kappa}, \quad \delta \psi_5 = i \kappa m, \quad \delta e = \dot{\xi} + 2 \kappa \chi. \quad (C.2) \]

Under the above transformations, the Lagrangian \( L_1 \) transforms as

\[ \delta L_1 = \frac{d}{d\tau} \left[ \frac{\xi}{2} (p^2 + m^2) + \frac{\kappa}{2} (p \cdot \psi + m \psi_5) \right]. \quad (C.3) \]

As a consequence, the transformations (C.2) are the symmetry transformations for the action integral \( S = \int d\tau L_1 \) which remains invariant for the physically well-defined dynamical variables of the theory that vanish off at infinity.

The proper (i.e. off-shell nilpotent and absolutely anticommuting) (anti-)BRST transformations, corresponding to the (super)gauge symmetry transformations (C.2), can be obtained by our geometrical superfield formalism. In particular, we can derive the nilpotent (anti-)BRST symmetry transformations of \( \psi_5 \) by using the Euler-Lagrange equation of motion corresponding to the dynamical variable \( \psi_5 \) (which is \( \dot{\psi}_5 = \chi m \)). Furthermore, it can be checked that \( \psi_5 = \chi m \) is a super-gauge invariant quantity. Therefore, the above equation (according to the basic tenets of the augmented version of BT superfield formalism [8-10,14]) can be written in terms of the superfields as

\[ \dot{\Psi}_5(\tau, \theta, \bar{\theta}) = K^{(h)}(\tau, \theta, \bar{\theta}) m \Rightarrow \dot{\Psi}_5(\tau, \theta, \bar{\theta}) - K^{(h)}(\tau, \theta, \bar{\theta}) m = 0, \quad (C.4) \]

where the expansion for the superfield \( \Psi_5(x, \theta, \bar{\theta}) \) is taken to be

\[ \Psi_5(\tau, \theta, \bar{\theta}) = \psi_5(\tau) + \theta \bar{B}_5(\tau) + \bar{\theta} B_5(\tau) + \theta \bar{\theta} f_5(\tau). \quad (C.5) \]

In the above super expansion, it is elementary to note that \( \psi_5(\tau) \) and \( f_5(\tau) \) are fermionic in nature as against the bosonic nature of \( B_5(\tau) \) and \( \bar{B}_5(\tau) \). The secondary variables
$[B_5(\tau), \bar{B}_5(\tau), f_5(\tau)]$ are to be determined in terms of the basic and auxiliary variables of the 1D (anti-)BRST invariant theory by exploiting the restriction (C.4). Furthermore, to obtain the correct results, we have to use the expansions for $K^{(h)}(\tau, \theta, \bar{\theta})$ from equation (38). To be precise, we equate the coefficients of $\theta, \bar{\theta}$ and $\theta \bar{\theta}$ to zero that emerge from the super-gauge invariant restriction (C.4). This requirement, in fact, leads to
\[
\dot{B}_5 = i \beta m \quad \Rightarrow \quad B_5 = i \beta m,
\]
\[
\dot{\bar{B}}_5 = i \bar{\beta} m \quad \Rightarrow \quad \bar{B}_5 = i \bar{\beta} m,
\]
\[
\dot{f}_5 = -\gamma m \quad \Rightarrow \quad f_5 = -\gamma m. \quad (C.6)
\]
As a consequence, the superfield expansion of $\Psi_5(\tau, \theta, \bar{\theta})$ turns out to be
\[
\Psi_5^{(sg)}(\tau, \theta, \bar{\theta}) = \psi_5 + \theta (i \beta m) + \bar{\theta} (i \beta m) + \theta \bar{\theta} (-\gamma m). \quad (C.7)
\]
In the language of the (anti-)BRST symmetry transformations (39) and (40), the above expansion can be re-expressed in the following manner, namely;
\[
\Psi_5^{(sg)}(\tau, \theta, \bar{\theta}) \equiv \psi_5 + \theta (s_a b \psi_5) + \bar{\theta} (s_b \psi_5) + \theta \bar{\theta} (s_b s_a \psi_5). \quad (C.8)
\]
From the above, it is clear that we obtain the proper (anti-)BRST symmetry transformations for the dynamical variable $\psi_5(\tau)$ as follows
\[
s_b \psi_5 = i \beta m, \quad s_a b \psi_5 = i \bar{\beta} m, \quad s_b s_a \psi_5 = -\gamma m. \quad (C.9)
\]
which are found to be nilpotent of order two (i.e. $s_{(a)b}^2 = 0$) and they are absolutely anticommuting ($s_b s_a + s_a s_b = 0$) in nature as can be checked from the (anti-)BRST symmetry transformations listed in (39) and (40).

We state, in passing, that the coupled (but equivalent) Lagrangians, their symmetries and their (anti-)BRST invariance as well as the CF-type restrictions and their (anti-)BRST invariance, etc., for the massive spinning relativistic particle, can also be captured within the framework of superfield formalism. This can be accomplished, in exactly same manner, as we have done for the massless spinning relativistic particle in our present investigation.

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