A Curious Use of Extra Dimension in Classical Mechanics

Trung V. Phan\textsuperscript{1} and Anh Doan\textsuperscript{2}

\textsuperscript{1}Department of Physics, Princeton University, Princeton, NJ 08544, USA
\textsuperscript{2}Department of Astronomy and Astrophysics, Pennsylvania State University, University Park, PA 16802, USA

(Dated: May 14, 2021)

Abstract

Extra dimensions can be utilized to simplify problems in classical mechanics, offering new insights. Here we show a simple example of how the motion of a test particle under the influence of an 1D inverse-quadratic potential is equivalent to that of another test particle moving freely in 2D Euclidean space and 3D Minkowskian space.

I. INTRODUCTION

Broadly speaking, physics questions are often defined by or complicated by their dimensions. On the surface, there seems to be a trend that physical questions become more difficult as the dimensionality increases. For example, collisions in two dimensions are harder to deal with than in one dimension, because the velocity becomes a two-dimensional vector not a scalar.\textsuperscript{10} Rigid body rotation in three dimensions is much more complicated than in two dimensions, since the angular velocity becomes a three-dimensional vector not a scalar.\textsuperscript{11} In quantum mechanics, any potential-well in one dimension and two dimensions has at least one bound state, but that claim is no longer correct in three dimensions.\textsuperscript{3,12}

However, many findings in modern theoretical physics indicate that it is also possible to simplify problems by adding more dimensions. In the theory of general relativity, electromagnetism and gravity can be unified by adding an extra compact dimension.\textsuperscript{5,13} In condensed matter physics, quasicrystals can be treated as projections of a higher-dimensional lattice.\textsuperscript{7,14} Finally, in string theory, a strongly interacting system can be more easily understood by considering a gravitational system in one more dimension\textsuperscript{10,11} via gauge/string duality.\textsuperscript{15}

Though typically used in more advanced topics, the addition of extra dimensions can help to simplify problems in classical physics. For example, the electrostatic problem of finding the charge distribution on a thin conducting circular disk, can be easily solved by an orthogonal projection of the charge distribution of a conducting sphere onto an equatorial plane.\textsuperscript{16} But, those are rare. In this note, we will concretely show how adding extra dimensions simplifies the classical mechanics problem of motion under the influence of an inverse-quadratic potential. To the best of our knowledge, this curious example has not been demonstrated elsewhere.

II. FROM 1D TO 2D EUCLIDEAN SPACE

Consider a point particle of mass $m$ in one-dimensional space moving under the influence of an inverse-quadratic potential $V_1(x) = \alpha/x^2$. This potential appears in experimental atomic physics\textsuperscript{13,15} and also is of many theoretical interests because it gives a scale-invariant Schrodinger’s equation.\textsuperscript{16,17} For now, let us focus on a repulsive potential with $\alpha > 0$. Initially, when $t = 0$, the particle is at position $x_0$ with no velocity. The motion of the particle can be described by applying conservation of energy to arrive at the following integral:

$$m\left(\frac{dx}{dt}\right)^2/2 + V_1(x) = V_1(x_0)$$
$$\Rightarrow dx/dt = \left(2(V_1(x_0) - V_1(x))/m\right)^{1/2}$$
$$\Rightarrow \int_{x_0}^{x(t)} dx (2\alpha(x_0^{-2} - x^{-2})/m)^{-1/2} = t$$

However, doing this integration is non-trivial. The solution requires changing variables to $y = (x_0^2 - x^2)^{1/2}$, at which point the integral becomes $\int dy y^{-1/2}$ up to a multiplication factor. We can get the equation of motion:

$$\left(x_0^2(x^2 - t^2)/(2\alpha)\right)^{1/2} = t$$
$$\Rightarrow x(t) = (x_0^2 + 2\alpha t^2/x_0^2)^{1/2}.$$  \hspace{1cm} (2)

While this solution is tractable, there exists another way to describe the motion of the particle without the need of calculus. A “magic” from an extra dimension.
Consider a general central potential $V_2$ in a two-dimensional space. In the polar coordinates $\vec{r} = (r, \theta)$ where the origin is the center of the potential, we have rotational symmetry. The kinetic energy $K_\theta$ stored in the compact angular dimension depends on the angular momentum $p_\theta$ and the moment of inertia $m_\theta = mr^2$:

$$K_\theta = \frac{p_\theta^2}{2m_\theta} = \frac{p_\theta^2}{2mr^2} = K_\theta(r).$$

The effective potential in the radial dimension $\frac{\partial V}{\partial x}$ is just the sum of the central potential $V_2(r)$ and the kinetic energy $K_\theta(r)$. It should be noted that $K_\theta(r)$ and $V_1(x)$ are both inverse-quadratic functions.

We now how an exact correspondence between this 2D scenario and the 1D problem considered above. The motion of the point particle in one-dimensional space under $V_1$ potential is dual to the radial motion of its counterpart moving freely in two-dimensional space (no potential $V_2 = 0$), given that the angular momentum is exactly $p_\theta = (2ma)^{1/2}$:

$$x(t) \bigg|_{V_1=\alpha/x^2} \leftrightarrow r(t) \bigg|_{V_2=0,p_\theta=(2ma)^{1/2}}.$$  \hspace{1cm} (4)

In other words, with an extra angular dimension we can eliminate the potential.

Using $x(0) = x_0$ and $dx/dt(0) = 0$, we have the corresponding radial position $r(0) = x_0$ and radial velocity $dr/dt(0) = 0$. The tangent velocity is given by:

$$v_\theta \equiv rd\theta/dt(0) = p_\theta/mr(0) = (2\alpha/mx_0^2)^{1/2}. \hspace{1cm} (5)$$

See Fig. 1 for the detail of this 1D/2D duality. We can arrive at the same answer \cite{Kaluza} with the Pythagorean theorem:

$$r(t) = (r^2(0) + (v_\theta t)^2)^{1/2} = (x_0^2 + 2\alpha t^2/mx_0^2)^{1/2}. \hspace{1cm} (6)$$

While we arrived at the same answer, the solution this time is purely geometric and does not involve any calculus.

III. FROM 1D TO 3D MINKOWSKIAN SPACE

The tools developed here can also be used for an arbitrary inverse-quadratic potential. However, it is more complicated and requires generalization to three dimensions. For an attractive potential, we can directly use $\alpha = -|\alpha| < 0$ to get the equation of motion $x(t)$ and also the lifetime $\tau$ until the point particle meets the singularity at position $x = 0$:

$$x(t) = \left(x_0^2 - 2|\alpha|t^2/mx_0^2\right)^{1/2},$$

$$x(\tau) = 0 \Rightarrow \tau = \left(mx_0^2/2|\alpha|\right)^{1/2}. \hspace{1cm} (7)$$

However, as we re-examine the problem from a two-dimensional perspective as explained above, this indicates an imaginary value of angular momentum $p_\theta = i(2ma^2)^{1/2}$. To generalize this extra-dimensional trick for all real values of $\alpha$, we need to complexify the angular dimension $\theta = \theta_R + i\theta_I$ (with $\theta_R$ and $\theta_I$ are real). Thus the corresponding space will be three-dimensional with $(+,+,−)$ metric signature.\cite{Levitov} Note that there are now two extra dimensions instead of one: while $\theta_R$ is a compact dimension, $\theta_I$ is an open one. The particle moves in the $(+,+)$ Euclidean plane when $\alpha > 0$, and in the $(+,-)$ Minkowskian plane when $\alpha < 0$. See Fig. 2 for the detail of this 1D/3D duality.

ACKNOWLEDGMENTS

This note comes from our final projects for MIT’s course 8.223 in 2012 and 2013. We thank Duy V. Nguyen and the xPhIo club for their support to share this finding to a wider audience. We thank Qiantan Hong and Leonid Levitov for examples in physics using extra dimensions. We also thank Oak Nelson and Todd Springer for many useful comments to make this note more approachable for general readers.

\footnotesize{1 typhan@princeton.edu
2. L. D. Landau and E. M. Lifshitz, Mechanics, Elsevier (2013).
3. D. Morin, Introduction to classical mechanics: with problems and solutions, Cambridge University Press (2008).
4. K. Yang and M. de Llano, “Simple variational proof that any two-dimensional potential well supports at least one bound state,” American Journal of Physics 57(1): 85-86 (1989).
5. K. Chadan, N.N. Khuri, A. Martin and T.T. Wu, “Bound states in one and two spatial dimensions,” Journal of Mathematical Physics 44(2): 406-422 (2003).
6. T. Kaluza, “Zum unitätsproblem der physik,” Sitzungsber. Preuss. Akad. Wiss. Berlin.(Math. Phys.) 1921.966972: 45-46 (1921).
7. O. Klein, “Quantentheorie und fünfdimensionale Relativitätstheorie,” Zeitschrift für Physik 37:12: 895-906 (1926).
8. P. Kramer and R. Neri, “On periodic and non-periodic space fillings of $E^n$ obtained by projection,” Acta Crystallographica Section A: Foundations of Crystallography 40(5): 580-587 (1984).}
FIG. 2. The 1D/3D duality. An inverse-quadratic potential in an 1D space is dual to no potential in a 3D space with $(+,+,−)$ metric signature. Different signs of the potential correspond to different planes of motion: $(+,+)$ plane when the potential is repulsive and $(+,$ $−)$ plane when the potential is attractive.

8 D. Levine, and P. J. Steinhardt, “Quasicrystals: a new class of ordered structures,” Physical review letters 53(26): 2477 (1984).
9 P. A. Kalugin, A. Y. Kitaev and L. S. Levitov, “6-dimensional properties of Al0. 86Mn0. 14 alloy,” Journal de Physique Lettres 46(13): 601-607 (1985).
10 D. T. Son and A. O. Starinets, “Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications,” Journal of High Energy Physics 2002(09): 042 (2002).
11 S. A. Hartnoll, P. K. Kovtun, M. Müller and S. Sachdev, “Theory of the Nernst effect near quantum phase transitions in condensed matter and in dyonic black holes,” Physical Review B 76(14): 144502 (2007).
12 J. Maldacena, “The large-N limit of superconformal field theories and supergravity,” International journal of theoretical physics 38(4): 1113-1133 (1999).
13 W. Thomson, Reprint of papers on electrostatics and magnetism, Cambridge University Press (2011).
14 J. M. Levy-Leblond, “Electron capture by polar molecules”, Physical Review, 153(1): 1 (1967).
15 C. Desfrancois, H. Abdoul-Carine, N. Khelifa and J. P. Schermann, “From 1/r to 1/r^2 Potentials: Electron Exchange between Rydberg Atoms and Polar Molecules,” Physical Review Letters 73(18): 2436 (1994).
16 A. F. Nicholson, “Bound States and Scattering in an r^−2 Potential,” Australian Journal of Physics 15(2): 174-179 (1962).
17 S. A. Coon and B. R. Holstein, “Anomalies in quantum mechanics: The 1/r^2 potential,” American Journal of Physics 70(5): 513–519 (2002).
18 A. M. Essin and D. J. Griffiths, “Quantum mechanics of the 1/x^2 potential,” American Journal of Physics 74(2): 109-117 (2006).
19 T. X. Nguyen and F. Marsiglio, “Numerical and analytical study of the bound states of the −α/x^2 potential,” American Journal of Physics 88(9): 746-752 (2020).
20 L. D. Landau and E. M. Lifshitz, The classical theory of fields, Elsevier (2013).