Community Detection with Dependent Connectivity

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Abstract

In network analysis, within-community members are more likely to be connected than between-community members, which is reflected in that the edges within a community are intercorrelated. However, existing probabilistic models for community detection such as the stochastic block model (SBM) are not designed to capture the dependence among edges. In this paper, we propose a new community detection approach to incorporate within-community dependence of connectivities through the Bahadur representation. The proposed method does not require specifying the likelihood function, which could be intractable for correlated binary connectivities. In addition, the proposed method allows for heterogeneity among edges between different communities. In theory, we show that incorporating correlation information can lower estimation bias and accelerate algorithm convergence. Our simulation studies show that the proposed algorithm outperforms the popular variational EM algorithm assuming conditional independence among edges. We also demonstrate the application of the proposed method to agricultural product trading networks from different countries.

Key words: Bahadur Representation; high-order approximation; product trading network; stochastic block model; variational EM.

1 Introduction

Network data has arisen as one of the most common forms in information collection. This is due to the fact that the scope of study not only focuses on subjects alone, but also on the relationships

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among subjects. Networks consist of two components: (1) nodes or vertices corresponding to basic units of a system, and (2) edges representing connections between nodes. These two main components can have various interpretations under different context of applications. For example, nodes might be humans in social networks; molecules, genes, or neurons in biology networks or web pages in information networks. Edges could be friendships, alliances, URLs, or citations. The combination of the nodes and the edges defines a network, which can be represented by an adjacency matrix to reflect direct connectivities among nodes.

In this paper, we are interested in identifying community structures, such as community detection of cluster nodes which have more concentrated connectivities in a subnetwork. Identifying communities is essential to provide deep understanding of relationships among nodes within a community and between communities to address scientific, social and political problems [54, 7, 21, 53, 41, 33, 36]. In terms of other applications, community detection plays an important role in decomposing original large-scale network structures [55, 51, 44] into several subnetworks with more simplified structures [12], and facilitates scalable computation for further analyses.

The major community detection methods can be summarized in the following three categories. One approach is to search a partition of nodes which optimizes a global criterion over all possible partitions. The corresponding criterion function measures the goodness of fit of a partition on the observed networks [49, 34, 2], such as the modularity [38] and profile likelihood [9] to capture densely connected communities. However, obtaining a global optimum based on this type of criterion is computationally infeasible. In addition, modularity also suffers from the resolution limit [20], which intrinsically ignores small communities. The second approach is the spectral method [46, 19, 4], which recovers dense connectivities through the eigenvectors of the adjacent matrix of the network. One critical drawback of the spectral method is that it lacks robustness in estimation, especially when networks are sparse or consist of high-degree nodes such as hubs [30]. The third approach is the maximum likelihood method for cluster networks. This includes the popular stochastic block models (SBM) [26] and its extensions to incorporate the heterogeneity of nodes’ degrees [28, 58], and latent distance modeling [23, 24] to handle overlapping communities [1, 6].

The SBM assumes that a membership assignment for each node follows a multinomial distribution. Given the community memberships, edges in the same community are randomly generated from a specified distribution. The common key assumption for SBM algorithms is that connec-
tivities are conditional independent given the membership of nodes. This assumption simplifies
the complexity of the model, and the likelihood function can be explicitly formulated. However,
the network data is likely to show dependency among connectivities, which is also considered in
several random network modelings [25, 32, 29, 14]. For community detection, the conditional
independency assumption typically does not hold in practice and therefore could lead to a mis-
specified model [45, 3, 52]. For example, friendships within a social community or functional
connectivities in brain networks tend to be highly correlated. In addition, under conditional in-
dependence, the community structure can only be recovered based on the marginal discrepancy
of connectivities, i.e., the different connection rates for nodes between within-communities and
across-communities. However, the correlations among connectivities could be non-negligible and
highly informative in identifying communities’ structures.

More recently, the SBM has been extended to address the heterogeneity feature within-community
for multiple network samples. For example, [50, 42] apply a fixed-effect model through an inde-
pendent intercept without incorporating information from other networks. Alternatively, a random-
effects model is proposed to incorporate heterogeneity [43, 57], which borrows information from
multiple networks. However, both of these approaches requires the specification of a distribution
for the random effects. In addition, an EM-type algorithm is implemented to integrate out the
random-effects, [43, 57] which could be computationally expensive when the size of the commu-
nity or the network size is large.

In this paper, we propose a novel community detection method to jointly model commu-
nity structures among multiple networks. The proposed method can simultaneously incorpo-
rate the marginal and correlation information to differentiate within-community and between-
community connectivities. The main idea is to approximate the unknown joint distribution of
within-community connectivities through a pseudolikelihood function using a truncated Bahadur
representation [5], where the correlations among connectivities are incorporated. We identify com-
munities via maximizing the pseudolikelihood function, which also serves as a discriminative func-
tion for membership assignments of nodes. Specifically, within-community correlations serve as
an additional community-concordance measurement to capture high-order discrepancy among net-
works within-community and between-community, and therefore increase discriminative power to
identify communities.

The main advantages and contributions of the proposed method can be summarized as fol-
The proposed method incorporates the information from dependency among connectivities to achieve more accurate community detection than the variational EM method using marginal information only. The improvement of the proposed community detection method is especially powerful when the marginal information is relatively weak in practice. In addition, compared to the existing random-effects model, the proposed method is more flexible in modeling the heterogeneity of communities among different networks, and does not require the specification of a distribution of within-community connectivities.

In addition, we show that our method achieves a lower estimation bias and a faster convergence rate compared to the variational EM algorithm, since additional correlation information is utilized to approximate the likelihood function. The theoretical development here is nontrivial, since we relax our model assumptions to accommodate more realistic settings in that we do not require the mean parameters from different communities to be the same as in [56]. Furthermore, we show that the convergence of the variational EM algorithm [35] is a special case of our method under the conditional independent SBM.

Computationally, we develop a two-step iterative algorithm which is not sensitive to initial values as in the standard variational EM algorithm. In addition, compared to the existing fixed-effects SBM with independent intercepts and the random-effects SBM, the proposed method has lower computational complexity, as it does not involve integrating out the random effects as in [43], or estimating the fixed effects for each network as in [42]. Simulation studies and real data application also confirm that the proposed method outperforms the existing variational EM significantly, with more accurate community detection and parameter estimations, especially when the marginal information of observed networks is weak.

This paper is organized as follows: Section 2 introduces the background of the proposed method. Section 3 introduces the proposed method to incorporate correlation information for community detection. Section 4 provides an algorithm and implementation strategies. Section 5 illustrates the theoretical properties of the proposed method. Section 6 demonstrates simulation studies, and Section 7 illustrates an application to world agricultural products trading data. The last section provides conclusions and some further discussion.
2 Background and Notation

In this section, we provide background and notation of the proposed community detection. The stochastic block model \([26]\) is a form of hierarchical modeling which captures the community structure for networks. Consider \(M\) symmetric and unweighted sample networks \(Y = \{Y^m\}_{m=1}^M = \{(Y^m_{ij})_{N \times N}\}_{m=1}^M\) with \(N\) nodes for \(K\) communities. Let \(\{z_i\}_{i=1}^N\) be the membership for each node and \(z_i \in \{1, 2, \cdots, K\}\), and denote the membership assignment matrix \(Z = \{(Z_{iq})_{n \times K}\} \in \{0, 1\}^{N \times K}\), where \(Z_{iq} = 1\{z_i = q\}\). Here \(Z\) has exactly one 1 in each row and at least one 1 in each column for no-null communities. The unknown membership \(z_i \in \{1, 2, \cdots, K\}\) can be modeled as a latent variable from a multinomial distribution:

\[
z_i \sim \text{Multinominal}(1, \alpha_i),
\]

where \(i = 1, \cdots, N\), \(\alpha_i = \{\alpha_{i1}, \cdots, \alpha_{iK}\}\) and \(\sum_{k=1}^K \alpha_{ik} = 1\). Given the membership of nodes, the observed edges between two nodes \(\{(Y^m_{ij})_{n \times n}\}_{m=1}^M\) typically follow a Bernoulli distribution:

\[
f_{ql}(Y^m_{ij}) := P(Y^m_{ij} | z_i = q, z_j = l) \sim \text{Bern}(\mu_{ql}), \quad \text{for } i, j \in \{1, \cdots, N\}, \ q, l = 1, \cdots, K, \quad (1)
\]

where \(\mu_{ql}\) is the probability of nodes \(i\) and \(j\) being connected.

For the heterogeneous stochastic blocks model, the marginal mean \(\mu_{ql}\) for each block in the \(m\)th network can be modeled as a logistic model to incorporate heterogeneity among edges:

\[
\mu_{ql}^m = \exp(\beta_{ql}x_{ij}^m) / \{1 + \exp(\beta_{ql}x_{ij}^m)\}, \quad (2)
\]

where \(\{(x_{ij}^m)_{n \times n}\}_{m=1}^M\) are edge-wise covariates, and edges within the same community preserve homogeneity by sharing a block-wise parameter \(\beta_{ql}\). The joint likelihood function can be decomposed into a summation of edge-wise terms following the conditional independence assumption:

\[
\log P(Y; Z) = \sum_{m=1}^M \sum_{q=1}^K \sum_{i=1}^N Z_{iq} \log \alpha_q + \sum_{m=1}^M \sum_{q,l=1}^K \sum_{i<j} Z_{iq} Z_{jl} f_{ql}(Y^m_{ij}; \beta_{ql}). \quad (3)
\]

The latent membership \(Z\) is estimated by \(E(Z|Y)\) through the maximum likelihood estimator of model parameters \(\Theta = \{\beta_{ql}; q, l = 1, \cdots, K; \ \alpha_q; q = 1, \cdots, K\}\) in \((3)\). However, the
classical EM algorithm is not applicable here, because the conditional distribution \( P(Z|Y) = \sum_Z P(Y; Z) \) becomes intractable in the expectation step.

The variational EM algorithm \([35, 27]\) is one of the most popular inference methods, and can be applied to approximate the likelihood \( P(Z|Y) \) by a complete factorized distribution \( R(Z, \tau) = \prod_{i=1}^{N} h(Z_i; \tau_i) \), where \( h(\cdot) \) denotes a multinomial distribution, \( \tau = (\tau_1, \cdots, \tau_N) \) and \( \tau_i = (\tau_{i1}, \cdots, \tau_{iK}) \) is a probability vector such that \( \sum_{q=1}^{K} \tau_{iq} = 1 \). In the expectation step, the joint likelihood \( \log P(Y; Z) \) is averaged over \( R(Z) \) such that for any \( \tau \),

\[
E_{R(Z, \tau)} \{ \log P(Y; Z) \} \leq E_{P(Z|Y)} \{ \log P(Y; Z) \}
\]

where,

\[
E_{R(Z, \tau)} \{ \log P(Y; Z) \} = - \sum_{m=1}^{M} \sum_{k=1}^{K} \tau_{ik} \log \mu_{ql} + \sum_{m=1}^{M} \sum_{q,l=1}^{K} \sum_{i<j}^{N} \tau_{iq} \tau_{jl} f_{ql}(Y_{ij}^{m}).
\]

Instead of directly maximizing \( E_{P(Z|Y)} \{ \log P(Y; Z) \} \), the variational EM approach alternatively maximizes its lower bound \( E_{R(Z, \tau)} \{ \log P(Y; Z) \} \) over model parameters \( \Theta \) and variational parameters \( \tau \), and cluster nodes by \( \tau \) through \( \hat{z}_i = \arg \max_k \{ \hat{\tau}_{ik}, k = 1, \cdots, K \} \).

Throughout this paper, we consider the conditional version of SBM (CSBM) \([9, 46, 15]\), where the true membership \( Z^* \) is fixed. Under the conditional stochastic block model framework, it assumes conditional independence among edges, i.e., \( Y_{ij}^{m} \) and \( Y_{ij}^{m} \) are independent given nodes’ membership \( z_{i1}, z_{i2}, z_{j1}, z_{j2} \), and the corresponding log-likelihood of observed sample networks is:

\[
\log L_{\text{ind}}(Y|Z) = \frac{1}{M} \sum_{m=1}^{M} \sum_{q,l=1}^{K} \sum_{i<j}^{N} Z_{iq} Z_{jl} \left\{ y_{ij}^{m} \log \mu_{ql} + (1 - y_{ij}^{m}) \log (1 - \mu_{ql}) \right\}.
\]

(4)

The above log-likelihood can serve as a discriminant function in clustering membership \( Z \) in that if \( \log L_{\text{ind}}(Y|Z_1) > \log L_{\text{ind}}(Y|Z_2) \) given two membership assignments \( Z_1 \) and \( Z_2 \), then \( Z_1 \) is preferred over \( Z_2 \), since the likelihood for the observed sample networks is higher. Naturally, \( Z^* \) can be estimated by

\[
\hat{Z} = \arg \max_Z \log P_{\text{ind}}(Y|Z).
\]

The SBM in \([4]\) allows one to differentiate within-community and between-community nodes via utilizing only the marginal information, in that the average connectivity rates within-communities are higher than those between-communities. However, the underlying conditional independence
assumption among edges is too restrictive and practically infeasible. In most community detection problems it is common that edges within communities are more correlated. For example, social connections among friends are highly correlated in social networks. However, the dependency among edges is not captured by the traditional SBM, which could lead to significant information loss of the community structure.

3 Methodology

3.1 Community detection utilizing dependent connectivity

In this paper, we incorporate within-community correlation to improve the accuracy and efficiency in identifying communities, in addition to utilizing the edges’ marginal mean information, since within-community dependency contains additional information regarding the membership of nodes. This is especially effective when the marginal mean is not informative in differentiating between and within communities’ connectivity.

In this section, we propose a pseudolikelihood to approximate the dependency among within-community edges. Specifically, we propose a new approach to incorporate correlation among edges $Y_{i_1j_1}^m, Y_{i_2j_2}^m$ within a community: $corr(Y_{i_1j_1}^m, Y_{i_2j_2}^m) = \rho_q(i_1, i_2, j_1, j_2)$ given nodes $z_{i_1}, z_{i_2}, z_{j_1}$ and $z_{j_2}$ are in the same community $q$, where $1 \leq i_1 < j_1 \leq N, 1 \leq i_2 < j_2 \leq N, (i_1, j_1) \neq (i_2, j_2)$ and $q = 1, \cdots, K$. Although correlations among pairs of edges could be different, in practice, we can impose a simplified structure $\rho_q(i_1, i_2, j_1, j_2)$ while allowing flexibility in modeling the correlation structure. For example, in spatial data applications, we can obtain spatial relationships among nodes, i.e., the distance $d(i_1, i_2)$ between nodes $i_1$ and $i_2$ and $d(j_1, j_2)$ between nodes $j_1$ and $j_2$, respectively. Therefore, we can model $\rho_q(i_1, i_2, j_1, j_2) = f_q\{d(i_1, i_2), d(j_1, j_2)\} \in (-1, 1)$, where $f_q(\cdot)$ can be a decreasing function of a distance between two edges, with a stronger correlation for closer edges. In the following, we assume a working structure such as a positive exchangeable correlation within-community, i.e.,

$$\rho_q(i_1, i_2, j_1, j_2) = \rho_q \in (0, 1),$$

where $(i_1, i_2, j_1, j_2) \in \{i : 1 \leq i \leq K|z_{i_1} = z_{i_2} = z_{j_1} = z_{j_2} = q\}$. The positive correlation
encourages concordance of connectivities among nodes within the same community.

3.2 The Bahadur Representation

In this section, we propose an informative approximation of the true log-likelihood to cluster $Z$ via incorporating interactions among edges within a community in addition to marginal mean information. Specifically, we facilitate the Bahadur representation [5] to incorporate the dependence information among edges within-communities, since an exact joint likelihood function for correlated binary outcomes is intractable in optimization.

Consider $T$ dependent binary random variables, then the joint likelihood can be represented through the Bahadur representation:

$$P(Y_1 = y_1, \cdots, Y_T = y_T) = \prod_{j=1}^{T} \mu_j^{y_j}(1 - \mu_j)^{1-y_j} \left[ 1 + \sum_{1 \leq j_1 < j_2 \leq T} \rho_{j_1,j_2} \hat{y}_{j_1} \hat{y}_{j_2} + \sum_{1 \leq j_1 < j_2 < j_3 \leq T} \rho_{j_1,j_2,j_3} \hat{y}_{j_1} \hat{y}_{j_2} \hat{y}_{j_3} + \cdots + \rho_{12\cdots T} \hat{y}_1 \hat{y}_2 \cdots \hat{y}_T \right], \quad (6)$$

where

$$\mu_j = E(Y_j), \quad \hat{y}_j = \frac{y_j - E(y_j)}{\sqrt{E(y_j)(1 - E(y_j))}}, \quad (7)$$

and

$$\rho_{j_1,j_2} = E(\hat{y}_{j_1} \hat{y}_{j_2}), \quad \rho_{j_1,j_2,j_3} = E(\hat{y}_{j_1} \hat{y}_{j_2} \hat{y}_{j_3}), \cdots, \quad \rho_{12\cdots T} = E(\hat{y}_1 \hat{y}_2 \cdots \hat{y}_T).$$

The idea of Bahadur representation is to approximate the joint distribution of dependent binary random variables as a function of moments with a sequential order. For the community detection problem, the binary random variables represent within-community edges, and the corresponding joint distribution can be explicitly decomposed into a marginal part and a correlation part. The marginal part consists of all the marginal mean $\mu_{ij}$ for each edge, which can be directly modeled through the dependency of the mean on covariates in (2). The correlation part consists of interactions among all possible pairwise-associations of normalized edges, which add correlation information beyond a conditional independence likelihood model. Note that the conditional independence model is a special case of the proposed model when the correlation is zero, and the
corresponding Bahadur representation collapses to a marginal part only, which is equivalent to the \( \log L_{\text{ind}}(Y|Z) \) in (4).

There are two major challenges in applying the Bahadur representation to model the interactions among within-community edges. First, the dimension of correlation parameters could be high if all the high-order interactions in (6) are incorporated, and this could lead to an increasing computational demand as the size of community grows. To solve this problem, we retain all the second-order interactions, but remove interactions for higher orders beyond the second order, since the pairwise interactions among edges could be most important. Consequently, we can reduce the dimensionality via imposing a simplified correlation structure, such as an exchangeable correlation within-community as in (5).

The second challenge is that the range of the correlation coefficient could be constrained by the marginal means [18]. Consequently, the correlation parameter space is more restrictive if the variability of marginal means among edges is large. Nevertheless, our primary goal is to construct an objective function which can incorporate information from the marginal mean and correlations of edges within-community, and the objective function does not need to be the true likelihood function. In the proposed method, we construct instead a pseudolikelihood which is not confined by this restriction, to incorporate highly dependent communities while still achieving computational efficiency.

Specifically, we construct a pseudolikelihood \( \tilde{L}(Y|Z) \) incorporating correlated within-community edges as follows:

\[
\log \tilde{L}(Y|Z) = \frac{1}{M} \left\{ \sum_{m=1}^{M} \sum_{q,l=1}^{K} \sum_{i<j}^{N} Z_{iq} Z_{jl} \left\{ \gamma_{ij} \log \mu_{ql} + (1 - \gamma_{ij}) \log (1 - \mu_{ql}) \right\} \right. \\
\left. + \sum_{m=1}^{M} \log \left\{ 1 + \sum_{k=1}^{K} \frac{\rho_k}{2} \sum_{i<j; u<v}^{N} Z_{ik} Z_{jk} Z_{uk} Z_{vk} \tilde{y}_{ij} \tilde{y}_{uv} \right\} \right\},
\]

where \( \mu_{ql} \) and \( \gamma_{ij} \) are formulated in (2) and (7). Notice the first term in (8) is the same as the marginal mean model, and the second term in (8) measures the concordance among edges within communities clustering \( Z \).
We denote the second term of (8) as

$$\log L_{\text{cor}}(Y|Z) = \frac{1}{M} \left\{ \sum_{m=1}^{M} \log \left\{ 1 + \frac{1}{2K} \sum_{k=1}^{K} \sum_{i<j;u<v}^{N} Z_{ik} Z_{jk} Z_{uk} Z_{vk} \hat{y}_{ij}^{m} \hat{y}_{uv}^{m} \right\} \right\}. \quad (9)$$

Compared with $\log L_{\text{ind}}(Y|Z)$ in (4), the proposed $\log \tilde{L}(Y|Z)$ has more discriminative power over $Z$, since it utilizes more information of the observed dependency within communities corresponding to clustering $Z$. Therefore, we can achieve higher classification accuracy and estimation efficiency through maximizing (8).

The key part of the proposed method is to predict memberships of nodes through the Bayes factor constructed by the proposed $\log \tilde{L}(Y|Z)$. Suppose the memberships of other nodes $Z_{-i}$ are known, then we classify node $i$ based on the following Bayes factor:

$$\frac{\tilde{L}(Y|Z_{-i}, Z_{iq} = 1)}{\tilde{L}(Y|Z_{-i}, Z_{ik} = 1)} = \exp \left\{ \log \tilde{L}(Y|Z_{-i}, Z_{iq} = 1) - \log \tilde{L}(Y|Z_{-i}, Z_{ik} = 1) \right\}.$$ 

If the above Bayes factor $> 1$, then the probability of node $i$ in community $q$ is larger than that of community $k$. The Bayes factor can be further decomposed as:

$$\frac{\tilde{L}(Y|Z_{-i}, Z_{iq} = 1)}{\tilde{L}(Y|Z_{-i}, Z_{ik} = 1)} = \frac{L_{\text{ind}}(Y|Z_{-i}, Z_{iq} = 1)}{L_{\text{ind}}(Y|Z_{-i}, Z_{ik} = 1)} \frac{L_{\text{cor}}(Y|Z_{-i}, Z_{iq} = 1)}{L_{\text{cor}}(Y|Z_{-i}, Z_{ik} = 1)}, \quad (10)$$

which contains both the marginal ratio and correlation ratio. It is clear that when the marginal information is weak in differentiating two communities, the marginal ratio is close to 1, and if the correlation ratio is informative, it can enhance the Bayes factor to improve community detection. In addition, the correlation ratio also serves as a correction to lower the estimation bias.

We illustrate the advantage of the proposed pseudolikelihood in (8) over the conditional independent likelihood (4). We generate multiple networks based on the SBM with 30 nodes evenly split between two communities. The marginal means of within-community and between-community edges are the same as 0.5, implying that the marginal mean is not informative. We assume a true exchangeable correlation $\rho = 0.6$ for within-community edges. Figure [1] illustrates that the likelihood function changes as memberships of nodes changes with some misclassified nodes. The left graph is based on the conditional independent SBM utilizing only marginal information, which does not differentiate two communities at all due to weak marginal information. However, the pro-
posed pseudolikelihood in the right graph has high differentiation power for nodes’ memberships, and reaches the maximum when the true memberships are selected.

4 Algorithm and Implementation

In this section, we propose a two-step algorithm to maximize the proposed pseudolikelihood function. In addition, we provide implementation strategies to improve the stability and efficiency of the algorithm.

4.1 Algorithm

To estimate the true membership \( Z^* \) of nodes, ideally we can search through all the possible \( Z \) and choose the one with the largest \( \log \tilde{L}(Y|Z) \). However, this becomes infeasible when the number of nodes \( N \) and the number of communities \( K \) increases. In the following, we propose an iterative two-step algorithm to maximize \( \log \tilde{L}(Y|Z) \) in (8). In the expectation step, we alternatively update membership of each node while fixing other nodes, where \( \tilde{L}(Y|\alpha_{-i}; Z_{ik}) \) has the same formulation as \( \tilde{L}(Y|Z) \) in (8) with \( \{ Z_{iq} \}_{N \times K} \) replaced by its expectation \( \{ \alpha_{iq} \}_{N \times K} \), except \( Z_{ik} \). Furthermore, the memberships are updated through the Bayes factor in (10) with the proposed pseudolikelihood \( \tilde{L}(Y|Z) \). In the maximization step, we estimate the community-wise parameters \( \beta_{ql} \) without considering the correlation information, because our empirical studies show that cor-
Algorithm 1

Step 1: Perform a spectral clustering on a sample network and obtain intial \( \{ z_i \}_{i=1}^N \),
set \( \alpha_i^{(0)} = \mathbb{1} \{ z_i = q \} \), \( 1 \leq i \leq N \), \( 1 \leq q \leq K \),

Step 2: At the \( s \)th iteration, given \( \{ \beta_{ql}^{(s-1)} \}_{q,l=1}^K \) and \( \{ \alpha_i^{(s-1)} \}_{i=1}^N \) from the \((s-1)\)th iteration:

(i) Maximization: block-wise update \( \beta_{ql}^{(s)} \) and \( \rho_q^{(s-1)} \), \( q,l = 1, \cdots, K \);

(a) Obtain \( \beta_{ql}^{(s)} = \arg \max_{\beta_{ql}} \sum_{m=1}^M \sum_{q,l=1}^K \sum_{i<j} \alpha_i^{(s-1)} \alpha_j^{(s-1)} \left\{ y_{ij}^m \log \mu_{ql} + (1 - y_{ij}^m) \log (1 - \mu_{ql}) \right\} \).

(b) Update \( \rho_q^{(s)} \) through the method of moments estimator.

(ii) Expectation: given \( \{ \beta_{ql}^{(s)} \}, \rho_q^{(s)} \), update \( \{ \alpha_i^{(s)} \}_{i=1}^N \):

\[
\alpha_i^{(s)} = \frac{\alpha_i^{(s-1)} L(Y|\alpha_i^{(s-1)},Z_{i1}=1)}{\sum_{k=1}^K \alpha_k^{(s-1)} L(Y|\alpha_k^{(s-1)},Z_{ik}=1)}, \quad i = 1, \cdots, N, \quad q = 1, \cdots, K.
\]

Step 3: Iterate until \( \max_{1 \leq i \leq N} \left| \alpha_i^{(s)} - \alpha_i^{(s-1)} \right| < \epsilon \).

Step 4: Obtain the membership \( z_i \) of clusters by

\[
\{ \alpha_i^{(s)} \}_{i=1}^N, \quad z_i = \max_k \{ \alpha_i^{(s)} \}, \quad i = 1, \cdots, N.
\]

relation information plays a more important role in classification than estimation. Note that the variational EM is a special case of the proposed algorithm if the correlation information is ignored and the conditional independent model in (4) is used.

4.2 Computation and Implementation:

To guarantee stability and improve efficiency, we add two modifications related to \( L_{cor}(Y|Z) \) in (9) at the expectation step. First we replace \( L_{cor}(Y|Z) \) by

\[
\frac{1}{M} \sum_{m=1}^M \log \left\{ 1 + \sum_{k=1}^K \rho_k \max_{i<j; u<v \atop (i,j) \neq (u,v)} Z_{ik} Z_{jk} Z_{uk} Z_{vk} y_{ij}^m y_{uv}^m, 0 \right\}
\]

to ensure its nonnegativity of correlation among edges. Therefore \( \log \tilde{L}(Y|Z) \geq \log L_{ind}(Y|Z) \) is guaranteed, which implies that adding additional correlation information among edges can increase that the likelihood function given within-community correlation exists. The reason we only consider positive correlations among within-community edges is that they enhance the concordance
among within-community edges. Equivalently, the edges in community $k$ show concordance only when

$$\sum_{i<j<u<v} N_{iv} Z_{ik} Z_{jk} Z_{uk} Z_{vk} \hat{y}_{ij} \hat{y}_{uv} \geq 0.$$  

Positive correlation among edges has been considered in the existing literature for community detection on multiple networks. For example, [42, 43] utilize random effects to model the heterogeneity of the connectivity rate for an individual network, which is equivalent to imposing a positive correlation among the edges within the same community. In practice, assuming positive correlations among edges is quite sensible. For example, the probability of two people establishing friendship is positively affected by the number of their common friends. In addition, allowing positive correlation also increases the stability of the algorithm as shown in the numerical studies.

Secondly, we can achieve a better approximation to the true likelihood if higher-order moments are incorporated in the Bahadur representation in (6), which also increases its discrimination power. However, higher-order correlation could also increase the computational cost. Alternatively, we can recover partial higher-order interactions (e.g., the fourth order) derived from lower order interactions (e.g., the second order). For example, consider four normalized edges $\hat{Y}_{m}^{i_1 j_1}$, $\hat{Y}_{m}^{i_2 j_2}$, $\hat{Y}_{m}^{i_3 j_3}$, and $\hat{Y}_{m}^{i_4 j_4}$ within the same community $k$ under an exchangeable correlation structure, we have

$$E(\hat{Y}_{m}^{i_1 j_1} \hat{Y}_{m}^{i_2 j_2} \hat{Y}_{m}^{i_3 j_3} \hat{Y}_{m}^{i_4 j_4}) \geq E(\hat{Y}_{m}^{i_1 j_1} \hat{Y}_{m}^{i_2 j_2}) E(\hat{Y}_{m}^{i_3 j_3} \hat{Y}_{m}^{i_4 j_4}) \geq \rho_{k}^{2}.$$  

To simplify notation, denote $(Z_{1k} Z_{2k} \hat{Y}_{m}^{12}, Z_{1k} Z_{3k} \hat{Y}_{m}^{13}, \cdots, Z_{2k} Z_{3k} \hat{Y}_{m}^{23}, \cdots, Z_{(N-1)k} Z_{Nk} \hat{Y}_{m}^{(N-1)N})$ as $(\gamma_{1}^{m}, \gamma_{2}^{m}, \cdots, \gamma_{N_{0}}^{m})$, where $N_{0} = \frac{N^{2} - N}{2}$. Then the second-order interaction term for the community $k$ in $L_{\text{cor}}(Y|Z)$ is

$$\frac{\rho_{k}}{2} \sum_{i<j, u<v} N_{iv} Z_{ik} Z_{jk} Z_{uk} Z_{vk} \hat{y}_{ij} \hat{y}_{uv} = \rho_{k} \sum_{s<t} \gamma_{s}^{m} \gamma_{t}^{m}.$$  

Based on (12) and given $Z$, we can approximate the fourth-order interaction for community $k$ by
its lower bound:

\[
\sum_{s_1 < t_1, s_2 < t_2} \frac{N_0}{2} \sum_{s_1 < t_1, s_2 < t_2} \frac{E(\gamma_{s_1 t_1}^{m_1 m_2} \gamma_{s_2 t_2}^{m_1 m_2})}{\gamma_{s_1 t_1}^{m_1 m_2} \gamma_{s_2 t_2}^{m_1 m_2}} \geq \sum_{s_1 < t_1, s_2 < t_2} \frac{\rho_k^2}{2} \sum_{s_1 < t_1, s_2 < t_2} \gamma_{s_1 t_1}^{m_1 m_2} \gamma_{s_2 t_2}^{m_1 m_2} = \left( \rho_k \sum_{s < t} \gamma_s^m \gamma_t^m \right)^2 - \rho_k^2 \sum_{s < t} \left( \gamma_s^m \gamma_t^m \right)^2.
\]

Note that the above lower bound of the fourth-order interaction can be calculated by the second-order interaction term in \( L_{cor}(Y|Z) \). Therefore, we can still incorporate higher-order terms in \( \log \tilde{L}(Y|Z) \) without additional computational cost. For other types of non-exchangeable correlation structures, we can incorporate partial higher-order correlation similarly as above. The main difference is that each pair of edges is associated with a specific correlation given a dependency structure. Therefore, the simplified lower bound for higher-order correlations such as (13) does not hold in general, and could have a more complex form depending on the specific correlation structure.

In the following, we also provide some guidelines for determining the number of communities \( K \). For a single network, the criterion-based methods choose \( K \) to maximize a certain probabilistic criterion such as the integrated likelihood \([22, 16, 31]\), composite likelihood BIC \([48]\) or modularity criterion \([10]\). In addition, spectral methods estimate \( K \) through the spectral property of the transformed adjacent matrix, such as a Laplacian matrix \([37]\), non-backtracking matrix \([11]\) or Bethe Hessian matrix \([47]\). In the hierarchical Bayesian framework, the number of communities is treated as a model parameter given a certain prior distribution and is jointly estimated with nodes’ memberships using the MCMC \([22, 39, 40]\). For multiple networks, we can extend the above techniques to estimate a consensus number of communities combining observed realizations of the SBM from each individual network.

In the context of the proposed within-community dependent modeling, we can first perform the modularity-maximizing method or spectral clustering on each individual network to obtain \( K \), then the average of these individual estimated \( K \) can be treated as the consensus number of communities. This procedure is sensible under two considerations. First, each sample network is a realization of the SBM so that the individual estimation of \( K \) is randomly distributed around the true underlying \( K \). Thus the average of individual estimations provides an estimation of \( K \) with low-bias and low-variance. Secondly, spectral clustering or modularity methods are more
favorable than other methods, due to its relatively low computational cost in estimating \( K \). This is especially effective when the sample size of networks is large.

5 Theoretical Results

In this section, we establish the convergence property of the proposed algorithm in Section 4 when the correlation information is incorporated. The consistency of the maximum likelihood estimator or the variational estimator for the SBM has been established under the assumption of conditional independent edges \([8, 13, 15]\). In terms of the algorithm, \([56]\) provides the convergence of the variational inference algorithm under the assumptions of marginal parameters. However, the existing theory is not applicable when the within-community edges are correlated.

In this section, we provide a theoretical foundation for the proposed method in terms of the algorithm. Specifically, we investigate the estimation bias and the convergence rate for Algorithm 1 using the proposed pseudolikelihood and the naive conditional independent likelihood. In the following theory development, we denote the true memberships of nodes as a vector \( Z^* \) and the estimated memberships of nodes at the \( s \)th iteration as \( \alpha^{(s)} = (\alpha_1^{(s)}, \cdots, \alpha_N^{(s)}) \) from Algorithm 1.

The main difference between the proposed method and the variational EM lies at the Bayes factor of (10) in the expectation step from Algorithm 1. If we replace the \( \tilde{L}(Y|Z) \) by the conditional independent likelihood \( L_{ind}(Y|Z) \) in (4) in the expectation step, the standard variational EM becomes a special case of Algorithm 1. Theorem 5.1 provides the convergence properties of the proposed algorithm under the conditional independent likelihood. \([56]\) also establish the minimax rate of misclassification under the assumption that the within-community marginal means are all the same. However, this assumption is too restrictive in practice. The following Theorem 5.1 relaxes this assumption and allows that within-community and between-community marginal means could be different.

We first provide the regularity conditions for the following theorems:

(C1). Suppose the distance between initial membership \( \alpha^{(0)} \) and true membership \( Z^* \) is bounded: \( \|\alpha^{(0)} - Z^*\|_1 = c\sqrt{N} \) where \( c = o(1) \).

(C2). The estimated marginal mean \( \hat{\mu}_{ql} \) has a bounded bias from the truth, i.e., \( \|\hat{\mu}_{ql} - \mu_{ql}\|_1 = o(1) \), \( q, l = 1 \cdots, K \).

(C3). Define the within-community marginal mean as \( \mu_{ll}, \ l = 1, \cdots, K \), the between-
community marginal mean as \(\mu_{ql}, 1 \leq l \neq q \leq K\), \(\lambda_1 = \max_{1 \leq l \neq q \leq K} (\mu_{lq})\) and \(\lambda_2 = \min_{1 \leq l \neq q \leq K} (\mu_{lq})\).

There exist upper and lower bounds for both within-community and between-community marginal means. In addition, we assume \(\lambda_1 < \mu_{ll}, l = 1, \cdots, K\). Therefore, \(0 < \lambda_2 \leq \lambda_1 < \mu_{ll}, l = 1, \cdots, K\).

(C4). All community sizes are bounded above and below by \(\kappa_1 N \leq |\{i \in \{1, 2, \cdots, N\} : Z_{iq}^* = 1\}| \leq \kappa_2 N, q = 1, \cdots, K\), where \(\kappa_1\) and \(\kappa_2\) are constants such that \(0 < \kappa_1 < \kappa_2 < 1\).

(C5). The correlation structure for within-community edges is exchangeable with correlation \(\rho\). All the between-community edges and edges from different communities are independent.

**Theorem 5.1.** Under the regularity conditions (C1)-(C5) and given \(N\) is sufficiently large, we establish the theoretical guarantee for the convergence of Algorithm 1 using marginal information \(L_{ind}(Y|Z)\). Specifically, with probability approaching 1 as \(M\) increases, there exist constants \(\alpha < 0, c_1(\rho) > 0, c_2(\rho) > 0\) and \(\eta = o(1)\) such that

\[
\|\alpha^{(s+1)} - Z^*\|_1 \leq NK \exp(\alpha N) + \frac{c_1(\rho)N^2K}{M^{1-\eta}} + \frac{c_2(\rho)K}{M^{1-\eta}} \|\alpha^{(s)} - Z^*\|_1. \tag{14}
\]

In contrast to the existing theories which require the number of nodes \(N\) to increase, the asymptotic property in Theorem 5.1 requires that the sample size \(M\) increases while the number of nodes \(N\) is fixed. This is because the community structure information is accumulated through independent multiple networks, as the within-community edges are no longer independent under our framework, and thus reducing the effective sample size of the edges within each network.

In Theorem 5.1, the first two terms on the right side of the inequality represent the irreducible estimation bias which measures the discrepancy between the community structure and its realization. The second term is also dominant as the first term has an exponential decay with respect to \(N\). The third term provides a decreasing rate of misclassification for iterations. Theorem 5.1 indicates that the estimated memberships are closer to the true memberships compared to the previous iteration step at a rate of \(\frac{c_2(\rho)K}{M^{1-\eta}}\). In addition, a larger sample size \(M\) will lead to a faster convergence and a lower estimation bias. In general, Theorem 5.1 guarantees the convergence of the iterative algorithm even with marginal information.

In the following, we establish theoretical properties when incorporating additional within-community correlation information. Specifically, we are able to improve the accuracy of membership classification of nodes when the proposed pseudolikelihood \(\tilde{L}(Y|Z)\) in Algorithm 1 is
maximized. With additional correlation information, the likelihood function increases and we can show a faster convergence speed and a lower estimation bias compared to Theorem 5.1.

**Theorem 5.2.** Under the regularity conditions (C1)-(C5) and given $N$ is sufficiently large, we establish the convergence property of Algorithm 1 via incorporating correlation information. That is, with probability approaching 1 as $M$ increases, there exist constants $\alpha < 0$, $c_1'(\rho) > 0$, $c_2'(\rho) > 0$ and $\eta = o(1)$ such that

$$\|\alpha^{(s+1)} - Z^*\|_1 \leq NK\exp(\alpha N) + \frac{c_1'(\rho)NK}{M^{1-\eta}} + \frac{c_2'(\rho)K}{M^{2-\eta}}\|\alpha^{(s)} - Z^*\|_1.$$  \hspace{1cm} (15)

In contrast to the convergence rates in Theorem 5.1, incorporating the correlation information enables us to reduce the estimation bias and accelerate the convergence rate. Specifically, it reduces the order of the second dominant term in the estimation bias from $O\left(\frac{NK}{M^{1-\eta}}\right)$ in (14) to $O\left(\frac{NK}{M^{2-\eta}}\right)$ in (15), and the order of the convergence rate from $O\left(\frac{K}{M^{1-\eta}}\right)$ to $O\left(\frac{K}{M^{2-\eta}}\right)$ in the third terms of (14) and (15). In addition, if there exists within-community correlation, Theorem 5.2 implies that misclassification rates for communities from larger networks’ communities decrease faster than those from small networks, and the corresponding convergence speed to the true membership is faster as the sample sizes of networks increases. Intuitively, the above theory indicates that using additional correlation information increases the probability of correctly identifying the nodes’ memberships. Furthermore, although Theorem 5.2 is established under the exchangeable correlation structure for within-community edges, the theory can be extended to other types of correlation structures using the same idea.

### 6 Numerical Studies

In this section, we conduct simulation studies to illustrate the performance of the proposed method on community detection in networks for dependent edges within-community. In particular, we compare our method to the existing variational EM method which assumes conditional independence among edges.
6.1 Study 1: Networks with dependent within-community connectivity

In the first simulation study, we consider networks where edges within the same community are correlated and compare the performance of variance methods under different network sample sizes with various magnitudes of marginal means for within-community and between-community.

Suppose the memberships of nodes $Z^* = \{Z_1, \cdots, Z_n\}$ in the networks are given with $K$ communities, where $Z_i$ is a binary indicator vector corresponding to the membership of nodes $i$. Conditional on $Z^*$, edges in each sample network are generated following the Bernoulli marginal distribution as in (1), where within-community edges follow an exchangeable correlation structure as in (5). Here we assume that edges between-community edges are independent from each other. The block-wise marginal means $\mu_{ql}$ ($q, l = 1, \cdots, K$) are associated with edgewise covariates through (2). In addition, the edgewise covariates follow uniform distribution, where within-communities covariates

$$x_{ij}^m \sim Unif(a_1, a_2) \text{ if } Z_{iq} = Z_{jq} = 1,$$

(16)

and between-community covariates

$$x_{ij}^m \sim Unif(b_1, b_2) \text{ if } Z_{iq} \neq Z_{jq}, q = 1, \cdots, K.$$

(17)

Although the probability of each edge is different, the edges within the same community share the same coefficient $\beta_{ql}$ in (2). In the following simulation studies, we generate correlated unweighted edges through the R package "MultiOrd."

Specifically, the sample networks consist of 40 nodes split into two communities. In a balanced community network, each community has 20 nodes. In an unbalanced case, two communities are comprised of 10 and 30 nodes, respectively. We compare the performance under different sample sizes of networks with $M = 20, 40, 60$, and different intensities of within-community dependency with correlation coefficient $\rho = 0, 0.3$ and $0.6$.

To simulate a weak marginal signal case, we let the block-wise parameters be $\beta_{11} = 1$, $\beta_{22} = 1.5$ and $\beta_{12} = \beta_{21} = 0$. The means of within-community and between-community covariates are $0$ with $a_1 = b_1 = -0.2$ and $a_2 = b_2 = 0.2$ in (16) and (17). Here, although the marginal mean of within-community edges is slightly larger than that of between-community edges on average due
to the convexity of the logistic link function in (2), the marginal means of within-community edges and between-community edges are very close.

For a strong marginal signal case, the block-wise parameters are $\beta_{11} = 0.3$, $\beta_{22} = 0.6$ and $\beta_{12} = \beta_{21} = 0.2$. The within-community covariates are generated via (16) with $a_1 = 0.9$ and $a_2 = 1.1$, and between-community covariates are generated from (17) with $b_1 = -0.8$ and $b_2 = -0.6$. Note that there is a distinct gap between within-community and between-community marginal means, thus the marginal signal is more dominant for nodes within communities.

We use the Adjusted Rand Index (ARI) to measure the performance of clustering. The ARI takes a value between $-1$ and $1$, where 1 represents a perfect matching of true memberships and predicted memberships of clustering, 0 indicates a random clustering and a negative value indicates that the agreement is less than the expectation from a random result. In the following simulations, we choose five fixed initial memberships of nodes in both balanced and unbalanced communities. These initials can be obtained from spectral clustering on sample networks. The Adjusted Rand Indices based on these chosen initials range between 0.30 to 0.34 under the unbalanced community case and between 0.25 to 0.29 under the balanced community case, which are far from the true memberships.

We compare the performance of clustering and parameter estimation for the proposed method applying the second-order (Bahadur\textsubscript{2nd}) and the fourth-order (Bahadur\textsubscript{4th}) Bahadur approximation, and the variational EM (VEM) approach with only marginal information.

In Table 1 and Table 2, the proposed method with the second-order and fourth-order approximations outperform the variational EM in clustering. Specifically, under the weak marginal signal case in Table 1, the Adjusted Rand Index of the variational EM are 0.34 under different network sizes and correlation strengths, which are similar to the ones calculated by fixed initials. In addition, since the distributions of marginal means from within-community and between-community are similar, the variational EM marginal approach barely improves over the initial memberships as it only utilizes the marginal information. However, the proposed method with the second-order or fourth-order Bahadur representation improves on the ARI by about 280%, compared to the VEM when $\rho = 0.3$ and $\rho = 0.6$. In addition, the performance of the proposed method improves by $1 \sim 5\%$ as the number of sample networks increases from 20 to 60. Furthermore, incorporating the fourth-order interaction can slightly improve the accuracy of clustering.

We notice that when the correlation is as moderate as 0.3, the proposed method still achieves
significant improvement over the variational EM and almost fully recovers the true memberships of clustering. We consider this as an intrinsic advantage of the proposed method in capturing the relatively weak dependency among edges to improve the clustering. This is because the proposed method not only captures pairwise dependency but also reflects connectivities among nodes within a community. That is, even a weak dependency among pairwise connectivities can lead to an accumulative information recovery of clustering.

Table 2 illustrates the clustering performance when the marginal signal is strong. In contrast to Table 1, the variational EM significantly improves on clustering because of the large discrepancy between the within-community marginal mean and the between-community marginal mean. Nevertheless, incorporating the correlation among within-community edges still improves the clustering accuracy by 20% to 26% under various sample sizes of networks and intensities of correlation. The clustering accuracy of the proposed method improves when either the sample size or the correlation increases. In general, stronger correlation and a larger sample size lead to better performance when the marginal signal itself is strong.

In additional to clustering, we also provide estimation of the marginal parameters. Tables 3, 4 and 5 compare parameter estimation between the proposed method and the variational EM when the marginal signal is weak. For within-community parameters $\beta_{11}$ and $\beta_{22}$, the estimations of the proposed method consistently reduces bias $30 \sim 99\%$ more than the variational method, except when $M = 20$ and $\rho = 0.6$. This is because that the sample size $M = 20$ is not sufficiently large to offset the high variance among highly-correlated within-community edges. For the between-community parameter $\beta_{12}$, the estimation bias of the proposed method consistently decreases more than $80\%$ compared to the VEM under all settings. Additionally, the standard errors of the proposed estimator decrease faster than the variational method as the size of networks increases.

### 6.2 Study 2: Networks with additional dependence between different communities

In Study 2, we also investigate whether the proposed method holds for a more general dependency structure among edges from different communities. For example, correlation among edges between
different communities

\[ \text{corr}(Y_{i_1 j_1}^m, Y_{i_2 j_2}^m) = \tilde{\rho}, \text{ given } z_{i_1} = z_{j_1} = q, z_{i_2} = z_{j_2} = l, q \neq l, \]  

where \( \tilde{\rho} \leq \rho_q \) in (5) in general. While (5) characterizes the concordance of edges within a community, (18) also captures the heterogeneity of sample networks. The heterogeneity of multi-layer networks is common in community detection.

In this simulation, we demonstrate that the proposed method is still robust when there is heterogeneity of connectivities among sample networks. To simulate the dependency among inter-community connectivity, we split \( M \) sample networks into 10 groups. Within each group, we add the random effects \( \gamma_k \) to the within-community marginal means:

\[ \mu_{qq}^m = \frac{\exp(\beta_q x_{ij}^m)}{1 + \exp(\beta_q x_{ij}^m)} + \gamma_k, \quad M \frac{k - 1}{10} \leq m \leq M \frac{k}{10}, \]

where \( \gamma_k \sim N(0, \sigma^2), k = 1, \cdots, 10, m = 1, \cdots, M, \) and \( q = 1, \cdots, K \). The variance \( \sigma \) of the random effect \( \gamma_k \) captures the intensity of dependency among inter-community connectivities, which increases as \( \sigma \) increases. We set \( \sigma = 0.5 \) to represent a weak inter-community dependency and \( \sigma = 1.5 \) for a strong inter-community dependency while the other settings remain the same as in simulation Study 1. Our primary interest is to compare clustering performance between the proposed method and the variational method under the weak marginal signal case.

Tables 6 and 7 illustrate the clustering performance between the variational method and the proposed method under balanced and unbalanced community sizes respectively. When the within-community correlation is moderate at 0.3, the proposed method improves the clustering accuracy by 170\% to 257\% for various network sizes and \( \sigma \). For strong correlation \( \rho = 0.6 \), the improvement is between 210\% to 257\%. In particular, the proposed method has better performance when the networks have strong intra-community correlation and large sample sizes under both weak and strong inter-community correlation cases. In addition, using the fourth-order Bahadur representation improves the accuracy by 6\% and 14\% when \( \sigma = 0.5 \) and \( \sigma = 1.5 \) compared to the second-order Bahadur representation, indicating that the higher-order method still enhances the clustering outcome under the misspecified model. It is interesting to notice that the performance of the proposed method decreases by 5\% to 15\% when the inter-community correlation is strong
and the number of networks is small, compared to the same setting with weak inter-community correlation. However, the performances under both weak or strong inter-community correlation are similar when the sample size of networks increases. In conclusion, the proposed method is robust against misspecified dependency structure when the sample size increases.

7 Real Data Example

In this section, we apply the proposed method to the 2010 Worldwide Food Import/Export Network dataset from the Food and Agriculture Organization of the United Nations (http://www.fao.org). We create 364 networks among 214 countries with a total of 318,346 edges, where each network captures the trading connections of a specific food product among countries.

The primary goal of the study is to identify the food and agricultural product trading network communities among different countries. One significant feature of these networks is that the average empirical correlation of the pairwise connection among trading countries is 0.29. Therefore, the SBM based on the conditional independent assumption among edges could possibly lead to a biased network clustering of countries.

We first preprocess the data to select nodes corresponding to the trading countries which are most relevant, the number of communities and initial memberships of countries. Note that several major countries dominate the world economy and lead a high number of trading connectivities, while the other countries with limited agricultural product categories have fewer trading connections with other countries for specific product networks. In this paper, we focus on the partial trading networks consisting of major countries whose corresponding degrees of nodes are larger than 9, which results in 51 countries with major economic impact in the world, such as the United States, mainland China, Japan and some European countries. The average empirical correlation of the trading connections among these countries is 0.22, indicating that the connectivity dependency should be considered in clustering these countries’ trading networks.

In general, there are two major procedures to select the number of communities. First, we can perform the Louvain method for community detection on each individual trading network to obtain the number of communities which maximizes the modularity and the size of the largest community. Next we take the average of the number of communities on networks whose number of communities is smaller than 10 and whose largest community size is larger than 14. This procedure
eliminates 18% of the product trading networks whose countries are commercially isolated from other countries, as our goal is to detect the commercial communities among the countries which are more connected with other countries. After preprocessing, the average number of communities is 4.9 and we set it to be 4, and there are 296 sample networks remaining in the following analysis.

Table 8 and Figure 2 provide the estimated agricultural products trading communities among 51 countries based on the variational EM and the proposed method. For the proposed method, we implement the fourth-order Bahadur approximation since it can better capture high-order within-community connectivity dependency. Table 8 presents the clustering outcome among countries according to the variational method and the proposed method. The countries in the same community under the variational method are marked with the same color, while the newly formed communities based on the proposed method are illustrated on the right sides of Table 8 and Figure 2. In general, the Adjusted Rand Index for clustering between the variational method and the proposed method is 0.43, indicating that the communities detected by the two methods are quite different. The clustering results from the proposed method incorporating within-community dependency are more interpretable compared to the variational EM using only marginal information.

In particular, the proposed method identifies communities 1 and 2 (red and cyan color communities on the right panel of Figure 2) which are highly associated with their geographical and climate environments. However, these features are not detected by the variational method. For example, community 1 with the cyan color on the left of Figure 2 based on the variational method mainly consists of two types of countries: one group comprises Nordic and Eastern European countries, and the other group consists of countries in Latin American and Africa. In contrast, the proposed method clusters countries from geographically neighboring countries in east Europe, including Austria, Poland and Romania which are clustered with other communities by the variational method. Community 2 with blue color on the left of Figure 2 based on the variational method contains northern countries such as Canada as well as tropical countries. However, the proposed method identifies community 2 with tropical coastal countries and Arabian Peninsula countries, which provides more meaningful community clusters compared to the variational EM method.

The variational method and proposed method detect the same third community with orange color in Figure 2 which contains 7 major countries from the European Union: Belgium, France, Germany, Italy, Netherlands, Spain and the UK.

The fourth community from the variational method colored with red on the left of Figure 2 con-
sists of 11 Eastern European countries, and all are categorized in community 1 from the proposed method. Community 4 with blue color on the right of Figure 2 in the proposed method includes countries with large populations or more developed agricultural product trading, such as mainland China, U.S.A, India and Japan.

In terms of parameter estimation, the average probability of having trading connections for communities 1 and 2 based on the variational method are 0.21 and 0.52, respectively. For the proposed method, the estimated correlations of connectivities within communities 1 and 2 are both 0.22, and the corresponding average within-communities connection rates are 0.28 and 0.22, respectively. The relatively low connection rates and correlations may be related to the low diversity and high overlaps of product categories due to more restrictive geographical and climate environments.

For community 3, the corresponding estimated marginal parameter $\beta_{33}$ from the proposed method and the variational method are 2.58 and 2.00 respectively, both of which indicate that the trading connection rate within European Union communities is greater than 88% on average. This strong marginal signal of within-community connection explains that the additional correlation information is less influential in clustering. Additionally, the estimated correlation within the third community is 0.58, implying a high connection rate within-community. For community 4, the corresponding average connection rate is 0.49 based on the variational method, and the estimated within-community average connection rate and the correlation are 0.61 and 0.27, respectively. This is because community 4 involves large population countries with more frequent trading on product categories due to their higher food diversity than other countries.

8 Discussion

In this paper, we propose a new community detection method for networks incorporating the underlying dependency structure among connectivities. To model the correlation without specifying a joint likelihood for correlated edges, we construct a pseudolikelihood based on the Bahadur representation which decomposes a joint distribution into a marginal term and high-order interaction terms. The proposed method provides flexible modeling on the correlation structure which can be specified through the interaction term in the pseudolikelihood.

In theory, we show that the proposed iterative algorithm possesses desirable convergence prop-
erties. In particular, we show that the proposed pseudolikelihood approach can achieve a faster convergence and a lower clustering bias compared to the variational EM algorithm. In addition, we show that the variational EM algorithm is a special case of our algorithm under the conditional independent model, which confirms that incorporating correlation information improves the accuracy for community detection.

Our numeric studies indicate that incorporating the within-community correlation among edges can improve the clustering performance compared to the marginal model, even under moderately misspecified model on inter-community dependency. The improvement of community detection is more significant when the marginal signal is weak, which is less informative for distinguishing between within-community and between-community networks. In addition, the proposed method enables us to achieve more accurate parameter estimation.

In this paper, we only consider incorporating the within-community dependency. It would be worthy of further research to investigate more generalized dependency structure to include between-community dependency as well. In addition, in this paper we mainly focus on the exchangeable structure to simplify the model, although the proposed method is also applicable for more complex correlation structures.

Supplementary Materials

The Web Appendix provides proofs of the Theorem 5.1 and Theorem 5.2.

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Table 1: Adjusted Rand Index between estimated membership and true membership for networks with two communities and weak marginal signal averaging on 50 replicates.

|                  | Unbalanced community | Balanced community |
|------------------|----------------------|--------------------|
|                  | $M = 20$  | $M = 40$  | $M = 60$  | $M = 20$  | $M = 40$  | $M = 60$  |
| $\rho = 0$       |            |            |            |            |            |            |
| VEM              | 0.38       | 0.41       | 0.48       | 0.31       | 0.28       | 0.28       |
| Bahadur$_{2nd}$  | 0.36       | 0.41       | 0.47       | 0.32       | 0.29       | 0.29       |
| Bahadur$_{4th}$  | 0.35       | 0.37       | 0.47       | 0.30       | 0.29       | 0.30       |
| $\rho = 0.3$     |            |            |            |            |            |            |
| VEM              | 0.34       | 0.34       | 0.34       | 0.28       | 0.28       | 0.28       |
| Bahadur$_{2nd}$  | 0.94       | 0.98       | 0.99       | 0.96       | 0.99       | 1.00       |
| Bahadur$_{4th}$  | 0.96       | 0.99       | 1.00       | 0.99       | 0.99       | 1.00       |
| $\rho = 0.6$     |            |            |            |            |            |            |
| VEM              | 0.34       | 0.34       | 0.34       | 0.29       | 0.28       | 0.28       |
| Bahadur$_{2nd}$  | 0.96       | 0.99       | 0.99       | 0.97       | 1.00       | 1.00       |
| Bahadur$_{4th}$  | 0.99       | 1.00       | 1.00       | 0.99       | 1.00       | 1.00       |

Table 2: Adjusted Rand Index between estimated membership and true membership for networks with two communities and strong marginal signal averaging on 50 replicates.

|                  | Unbalanced community | Balanced community |
|------------------|----------------------|--------------------|
|                  | $M = 20$  | $M = 40$  | $M = 60$  | $M = 20$  | $M = 40$  | $M = 60$  |
| $\rho = 0$       |            |            |            |            |            |            |
| VEM              | 0.78       | 0.92       | 0.98       | 0.76       | 0.90       | 0.97       |
| Bahadur$_{2nd}$  | 0.73       | 0.91       | 0.97       | 0.77       | 0.92       | 0.98       |
| Bahadur$_{4th}$  | 0.69       | 0.86       | 0.95       | 0.72       | 0.92       | 0.98       |
| $\rho = 0.3$     |            |            |            |            |            |            |
| VEM              | 0.78       | 0.81       | 0.83       | 0.68       | 0.79       | 0.84       |
| Bahadur$_{2nd}$  | 0.99       | 0.99       | 1.00       | 0.98       | 1.00       | 1.00       |
| Bahadur$_{4th}$  | 0.99       | 0.99       | 1.00       | 0.99       | 1.00       | 1.00       |
| $\rho = 0.6$     |            |            |            |            |            |            |
| VEM              | 0.78       | 0.89       | 0.83       | 0.84       | 0.92       | 0.88       |
| Bahadur$_{2nd}$  | 0.99       | 1.00       | 1.00       | 0.99       | 1.00       | 1.00       |
| Bahadur$_{4th}$  | 0.99       | 1.00       | 1.00       | 0.99       | 1.00       | 1.00       |
Table 3: Estimation of within-community parameter $\beta_{11} = 1$ for networks with two communities and weak marginal signal.

|          | Unbalanced community | Balanced community |
|----------|----------------------|--------------------|
|          | $M = 20$  | $M = 40$  | $M = 60$  | $M = 20$  | $M = 40$  | $M = 60$  |
| $\rho = 0$ |          |          |          |          |          |          |
| VEM      | 0.56_{0.42} | 0.59_{0.29} | 0.58_{0.20} | 0.64_{0.32} | 0.57_{0.16} | 0.64_{0.18} |
| Bahadur$_{2nd}$ | 0.57_{0.42} | 0.58_{0.30} | 0.57_{0.21} | 0.61_{0.28} | 0.57_{0.16} | 0.66_{0.20} |
| Bahadur$_{4th}$ | 0.52_{0.42} | 0.55_{0.28} | 0.57_{0.19} | 0.58_{0.27} | 0.58_{0.18} | 0.65_{0.19} |
| $\rho = 0.3$ |          |          |          |          |          |          |
| VEM      | 0.49_{0.30} | 0.50_{0.17} | 0.52_{0.14} | 0.58_{0.24} | 0.58_{0.18} | 0.59_{0.12} |
| Bahadur$_{2nd}$ | 0.81_{0.48} | 0.84_{0.32} | 0.89_{0.27} | 0.95_{0.24} | 0.93_{0.16} | 0.92_{0.14} |
| Bahadur$_{4th}$ | 0.85_{0.47} | 0.83_{0.31} | 0.89_{0.27} | 0.96_{0.24} | 0.93_{0.16} | 0.93_{0.14} |
| $\rho = 0.6$ |          |          |          |          |          |          |
| VEM      | 0.56_{0.22} | 0.54_{0.20} | 0.52_{0.15} | 0.61_{0.27} | 0.61_{0.16} | 0.60_{0.14} |
| Bahadur$_{2nd}$ | 1.01_{0.42} | 1.04_{0.35} | 1.00_{0.29} | 0.95_{0.31} | 1.00_{0.19} | 0.96_{0.15} |
| Bahadur$_{4th}$ | 0.99_{0.25} | 1.05_{0.15} | 1.01_{0.13} | 0.97_{0.31} | 1.01_{0.19} | 0.97_{0.16} |

Table 4: Estimation of within-community parameter $\beta_{22} = 1.5$ for networks with two communities and weak marginal signal.

|          | Unbalanced community | Balanced community |
|----------|----------------------|--------------------|
|          | $M = 20$  | $M = 40$  | $M = 60$  | $M = 20$  | $M = 40$  | $M = 60$  |
| $\rho = 0$ |          |          |          |          |          |          |
| VEM      | 1.43_{0.43} | 1.42_{0.34} | 1.45_{0.26} | 1.18_{0.40} | 0.94_{0.16} | 0.94_{0.15} |
| Bahadur$_{2nd}$ | 1.50_{0.39} | 1.49_{0.31} | 1.45_{0.25} | 1.21_{0.42} | 0.93_{0.21} | 0.97_{0.22} |
| Bahadur$_{4th}$ | 1.56_{0.37} | 1.49_{0.30} | 1.46_{0.23} | 1.19_{0.47} | 0.94_{0.24} | 0.96_{0.22} |
| $\rho = 0.3$ |          |          |          |          |          |          |
| VEM      | 1.31_{0.23} | 1.40_{0.11} | 1.37_{0.11} | 1.05_{0.21} | 0.92_{0.16} | 0.92_{0.16} |
| Bahadur$_{2nd}$ | 1.56_{0.19} | 1.50_{0.10} | 1.49_{0.09} | 1.48_{0.22} | 1.45_{0.19} | 1.44_{0.14} |
| Bahadur$_{4th}$ | 1.55_{0.19} | 1.50_{0.09} | 1.49_{0.09} | 1.48_{0.22} | 1.45_{0.19} | 1.45_{0.14} |
| $\rho = 0.6$ |          |          |          |          |          |          |
| VEM      | 1.46_{0.16} | 1.43_{0.16} | 1.38_{0.13} | 1.16_{0.21} | 1.09_{0.21} | 1.06_{0.22} |
| Bahadur$_{2nd}$ | 1.73_{0.29} | 1.60_{0.15} | 1.52_{0.12} | 1.73_{0.28} | 1.60_{0.29} | 1.64_{0.15} |
| Bahadur$_{4th}$ | 1.69_{0.25} | 1.60_{0.15} | 1.52_{0.13} | 1.73_{0.26} | 1.61_{0.29} | 1.64_{0.15} |
Table 5: Estimation of within-community parameter $\beta_{12} = 0$ for networks with two communities and weak marginal signal.

|        | Unbalanced community | Balanced community |
|--------|----------------------|---------------------|
|        | $M = 20$  | $M = 40$  | $M = 60$  | $M = 20$  | $M = 40$  | $M = 60$  |
| $\rho = 0$ | VEM  | 0.52_{0.35}  | 0.57_{0.24}  | 0.47_{0.22}  | 0.22_{0.31}  | 0.39_{0.14}  | 0.41_{0.11}  |
|        | Bahadur$_{2nd}$  | 0.51_{0.32}  | 0.58_{0.23}  | 0.48_{0.21}  | 0.23_{0.30}  | 0.41_{0.16}  | 0.39_{0.15}  |
|        | Bahadur$_{4th}$  | 0.51_{0.29}  | 0.63_{0.22}  | 0.48_{0.20}  | 0.25_{0.28}  | 0.40_{0.17}  | 0.41_{0.13}  |
| $\rho = 0.3$ | VEM  | 0.68_{0.24}  | 0.68_{0.13}  | 0.69_{0.10}  | 0.42_{0.14}  | 0.35_{0.12}  | 0.40_{0.10}  |
|        | Bahadur$_{2nd}$  | -0.02_{0.25}  | 0.00_{0.15}  | 0.00_{0.11}  | 0.03_{0.20}  | -0.05_{0.16}  | -0.02_{0.12}  |
|        | Bahadur$_{4th}$  | -0.02_{0.24}  | 0.00_{0.14}  | 0.00_{0.11}  | 0.03_{0.18}  | -0.06_{0.16}  | 0.03_{0.12}  |
| $\rho = 0.6$ | VEM  | 0.72_{0.17}  | 0.71_{0.11}  | 0.70_{0.09}  | 0.41_{0.18}  | 0.45_{0.11}  | 0.48_{0.11}  |
|        | Bahadur$_{2nd}$  | -0.05_{0.17}  | -0.03_{0.13}  | 0.02_{0.11}  | 0.00_{0.19}  | 0.01_{0.12}  | 0.03_{0.12}  |
|        | Bahadur$_{4th}$  | -0.04_{0.17}  | -0.03_{0.13}  | -0.02_{0.11}  | -0.02_{0.18}  | 0.00_{0.12}  | 0.03_{0.11}  |

Table 6: Performance comparison given misspecified inter-community correlation with balanced community and weak marginal signal averaging on 50 replicates.

|        | $\sigma = 0.5$ | $\sigma = 1.5$ |
|--------|----------------|----------------|
|        | $M = 20$  | $M = 40$  | $M = 60$  | $M = 20$  | $M = 40$  | $M = 60$  |
| $\rho = 0.3$ | VEM  | 0.28  | 0.28  | 0.29  | 0.28  | 0.28  | 0.29  |
|        | Bahadur$_{2nd}$  | 0.90  | 0.99  | 1.00  | 0.76  | 0.99  | 0.99  |
|        | Bahadur$_{4th}$  | 0.96  | 1.00  | 1.00  | 0.87  | 0.98  | 1.00  |
| $\rho = 0.6$ | VEM  | 0.28  | 0.28  | 0.29  | 0.28  | 0.28  | 0.29  |
|        | Bahadur$_{2nd}$  | 0.94  | 0.99  | 1.00  | 0.87  | 0.99  | 1.00  |
|        | Bahadur$_{4th}$  | 0.99  | 1.00  | 1.00  | 0.94  | 0.99  | 1.00  |

Table 7: Performance comparison given misspecified inter-community correlation with unbalanced community and weak marginal signal averaging on 50 replicates.

|        | $\sigma = 0.5$ | $\sigma = 1.5$ |
|--------|----------------|----------------|
|        | $M = 20$  | $M = 40$  | $M = 60$  | $M = 20$  | $M = 40$  | $M = 60$  |
| $\rho = 0.3$ | VEM  | 0.32  | 0.33  | 0.33  | 0.33  | 0.33  | 0.33  |
|        | Bahadur$_{2nd}$  | 0.89  | 0.98  | 0.99  | 0.89  | 0.95  | 0.97  |
|        | Bahadur$_{4th}$  | 0.95  | 0.99  | 0.99  | 0.93  | 0.94  | 0.94  |
| $\rho = 0.6$ | VEM  | 0.34  | 0.33  | 0.34  | 0.33  | 0.33  | 0.33  |
|        | Bahadur$_{2nd}$  | 0.91  | 0.96  | 0.98  | 0.91  | 0.95  | 0.94  |
|        | Bahadur$_{4th}$  | 0.95  | 0.96  | 0.97  | 0.92  | 0.93  | 0.92  |
Table 8: Clustering of nations in the agricultural products trading networks given 4 communities

| Community 1                  | VEM                          | Bahadur_{4}th                 |
|------------------------------|------------------------------|--------------------------------|
|                              | Brazil, Denmark, Finland, Ireland | Austria, Denmark, Finland, Ireland, Poland |
|                              | Lebanon, Russia, Sweden, Switzerland | Russia, Sweden, Switzerland, Turkey |
|                              | Turkey, Ukraine, Argentina, Israel | Bulgaria, Croatia, Czech, Greece, Hungary |
|                              | Mexico, Norway, Portugal, Chile | Israel, Lithuania, Norway, Portugal |
|                              | South Africa, Qatar           | Romania, Slovakia, Slovenia, Ukraine |

| Community 2                  | Australia, Canada, Hong Kong, Mainland | Brazil, Hong Kong, Taiwan, Indonesia |
|                              | Taiwan, India, Indonesia, Malaysia     | Lebanon, Philippines, Korea, Argentina |
|                              | Japan, Philippines, Korea, Singapore   | Mexico, Chile, New Zealand |
|                              | Thailand, U.S.A, New Zealand          | South Africa, Qatar |

| Community 3                  | Belgium, France, Germany, Italy     | Belgium, France, Germany, Italy |
|                              | Netherlands, Spain, United Kingdom  | Netherlands, Spain, United Kingdom |

| Community 4                  | Austria, Poland, Bulgaria, Croatia  | Australia, Canada, Mainland, India |
|                              | Czech, Greece, Hungary, Lithuania   | Japan, Malaysia, Singapore |
|                              | Romania, Slovakia, Slovenia         | Thailand, U.S.A |

Figure 2: Communities detected on the trading networks. The countries in the same color are in the same community. Left: standard SBM solved by variational EM. Right: proposed method.