A Kobayashi-Hitchin correspondence between
Dirac-type singular mini-holomorphic bundles and
HE-monopoles

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Abstract
We prove an analogue of the Kobayashi-Hitchin correspondence on
compact connected 3-folds that is fibered on orbifold Riemann surfaces
and satisfy an integrability condition, which contains compact con-
nected Sasakian 3-folds. We define mini-holomorphic bundles on such
3-folds and the algebraic Dirac-type singularities on mini-holomorphic
bundles, and prove that there exists a special Hermitian metric (admis-
sible BHE-metric) on a Dirac-type singular mini-holomorphic bundle
if the bundle satisfies a slope stability.

1 Introduction
On a connected compact Kähler manifold \((M, g)\) with the Kähler form \(\omega\), a
holomorphic vector bundle \(V\) has a Hermite-Einstein metric if and only if
\(V\) is polystable, which is called the Kobayashi-Hitchin correspondence and
proved by Uhlenbeck and Yau [10]. In this paper, we prove an analog of the
Kobayashi-Hitchin correspondence on compact mini-holomorphic 3-folds.

We describe mini-holomorphic 3-folds and mini-holomorphic vector bun-
dles on them. In our context, they are the counterpart of complex manifolds
and holomorphic bundles. Let \(X\) be a compact oriented 3-fold. Let \(\partial_t\) be
a nowhere vanishing vector field on \(X\) and \(\alpha\) a \(\partial_t\)-invariant 1-form on \(X\)
with a condition \(\langle \partial_t, \alpha \rangle = 1\). Let \((\Sigma, g_\Sigma)\) be an orbifold Riemann surface
and \(\pi : M \to \Sigma\) a \(\partial_t\)-invariant submersion. Set a metric \(g = \alpha^2 + \pi^* g_\Sigma\). As-
sume that \(\alpha \wedge \pi^* \text{vol}_\Sigma\) is positive-oriented. We set \(\Omega_{X,1}^{0,1} := \mathcal{C}_{\alpha} \oplus \pi^* \Omega_{\Sigma}^{0,1}\)
and \(\Omega_{X,2}^{0,2} := \bigwedge^2 \Omega_{X}^{0,1}\), where \(\mathcal{C}_{\alpha}\) is the subbundle spanned by \(\alpha\). Then
the tuple \((\partial_t, \alpha, \Sigma, \pi)\) is called a mini-holomorphic structure on \(X\). We
set a differential operator $\overline{\partial}_X(f) := \partial_t(f)\alpha + \overline{\partial}_z f \pi^*(d\bar{z})$, where $z$ is a holomorphic local chart of $\Sigma$ and $\overline{\partial}_z$ is the lift of $\partial_z$ by the isomorphism $\text{Ker}(\alpha) \simeq \pi^*T\Sigma$. If a vector bundle $V$ on an open subset $U \subset X$ and a differential operator $\overline{\partial}_V : \Omega^{0,1}_X(V) \to \Omega^{0,0}_X(V)$ satisfies the Leibniz rule $\overline{\partial}_V(f \pi^*\omega) = \overline{\partial}_V f \pi^*\omega + f \pi^*\overline{\partial}_V \omega$, for any proper mini-holomorphic subbundle $(E, h, A, \Phi)$ of $(V, h, A, \Phi)$ on $X \setminus Z$, $(V, \overline{\partial}_V)$ is a Dirac-type singular mini-holomorphic bundle on $(X, Z)$ if it satisfies a certain condition (See Definition 2.13). We describe HE-monopoles on mini-holomorphic 3-folds. Let $(V, h, A)$ be a Hermitian vector bundle on an open subset $U \subset X$ with a connection. Let $\Phi$ be a skew-Hermitian endomorphism of $V$. The tuple $(V, h, A, \Phi)$ is HE-monopole of factor $c \in \mathbb{R}$ if it satisfies the Bogomolny-Hermite-Einstein equation $F(A) = \ast \nabla_A(\Phi) + \sqrt{-1}c \cdot (\pi^*\omega_{\Sigma})\text{Id}_V$, where $\omega_{\Sigma}$ is the Kähler form of $\Sigma$. If the constant $c = 0$, the Bogomolny-Hermite-Einstein equation agrees with the ordinary Bogomolny equation. A HE-monopole $(V, h, A, \Phi)$ on $X \setminus Z$ is a Dirac-type singular monopole if it satisfies a certain condition (See Definition 2.25). A HE-monopole $(V, h, A, \Phi)$ has a natural mini-holomorphic structure $\overline{\partial}_V := \nabla^{0,1}_A - \sqrt{-1}\Phi \cdot \alpha$, and underlying mini-holomorphic bundles of Dirac-type singular HE-monopoles are Dirac-type singular. Conversely, for a mini-holomorphic bundle $(E, \overline{\partial}_E)$ and a Hermitian metric $h$ on $E$, there uniquely exist a connection $A_h$ and a skew-Hermitian endomorphism $\Phi_h$ on $E$ such that $\overline{\partial}_E = \nabla^{0,1}_{A_h} - \sqrt{-1}\Phi_h \cdot \alpha$.

For a Dirac-type singular mini-holomorphic bundle $(E, \overline{\partial}_E)$, if the tuple $(E, h, A_h, \Phi_h)$ has the lift by the Hopf-fibration, then $h$ is called an admissible metric (See Definition 3.3). If the tuple $(E, h, A_h, \Phi_h)$ is a HE-monopole, we call $h$ a Bogomolny-Hermite-Einstein metric (or shortly BHE-metric). Moreover, the tuple $(E, h, A_h, \Phi_h)$ is a Dirac-type singular HE-monopole if and only if $h$ is an admissible BHE-metric (See Proposition 3.5). We introduce the stability of Dirac-type singular mini-holomorphic bundles. We set the degree of $(E, \overline{\partial}_E)$ to be $\text{deg}(E, \overline{\partial}_E) := \int_{X \setminus Z} \alpha \wedge c_1(A_{h_0})$, where $h_0$ is an admissible metric. By Proposition 3.6, $\text{deg}(E, \overline{\partial}_E)$ is independent of the choice of $h_0$. The Dirac-type singular mini-holomorphic bundle $(E, \overline{\partial}_E)$ is stable if the inequality $\text{deg}(E, \overline{\partial}_E)/\text{rank}(E) > \text{deg}(F, \overline{\partial}_F)/\text{rank}(F)$ holds for any proper mini-holomorphic subbundle $(F, \overline{\partial}_F)$ of $(E, \overline{\partial}_E)$. Our main result is the following:

**Theorem 1.1 (Theorem 3.7).** If $(E, \overline{\partial}_E)$ is stable, then there exists an admissible BHE-metric $h$ on $E$.

In Section 2, we recall the notations necessary for Section 3. Moreover,
in Proposition [2.30] we give a slight generalization of Theorem 4.5 in [7]. In Section 3, we prove our main result.

Comparison with previous studies

In [4], Charbonneau and Hurtubise introduced the notion of HE-monopoles and mini-holomorphic bundles on a product of $S^1$ and a Riemann surface $\Sigma$. They also proved the Kobayashi-Hitchin correspondence on $S^1 \times \Sigma$.

In [2], Biswas and Hurtubise considered the Kobayashi-Hitchin correspondence on compact Sasakian 3-folds, and proved that two Dirac-type singular monopole on a compact Sasakian 3-folds are isomorphic as monopoles if their underlying mini-holomorphic structures are isomorphic.

In [1], Baraglia and Hekmati constructed the Kobayashi-Hitchin correspondence for compact oriented taut Riemannian foliated manifolds with transverse Hermitian structure. This result seems to be considered as a higher-dimensional generalization of our result under the non-singular condition.

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2 Preliminaries

2.1 Kähler orbifolds

We recall the notion of orbifold by following [8]. For a Lie groupoid $\mathcal{C}$, we denote by $\mathcal{C}_0$ and $\mathcal{C}_1$ the object space and the morphism space of $\mathcal{C}$. Let $s, t, m$ and $u$ be the source map, the target map, the composition map and the unit map. We set $\mathcal{C}(x, y) := (s, t)^{-1}(x, y)$. We also set $\mathcal{C}_2 := \mathcal{C}_1 \times_t \mathcal{C}_1$, and denote by $p_i : \mathcal{C}_2 \to \mathcal{C}_1$ the projection to the $i$-th component.

Definition 2.1. Let $\mathcal{C}$ be a Lie groupoid unless otherwise denoted.

(i) The groupoid $\mathcal{C}$ is called an orbifold if the following conditions are satisfied.

- The maps $s$ and $t$ are local diffeomorphisms.
- The map $(s, t) : \mathcal{C}_1 \to \mathcal{C}_0 \times \mathcal{C}_0$ is proper.
For an orbifold $C$, we define the dimension of $C$ as the one of $C_0$.

(ii) We denote by $|C|$ the underlying topological space $C_0/\sim$, where $x \sim y$ holds if $C(x, y) \neq \emptyset$.

(iii) A vector bundle on $C$ is a vector bundle $V$ on $C_0$ equipped with an isomorphism $\Phi : t^*V \to s^*V$ that satisfies the following commutative diagram:

\[
\begin{array}{ccc}
p_2^*t^*V & \xrightarrow{p_2^*\Phi} & p_2^*s^*V \\
\downarrow & & \downarrow p_1^*\Phi \\
m^*t^*V & \xrightarrow{m^*\Phi} & m^*s^*V
\end{array}
\]

The tangent and cotangent bundles on $C_0$ naturally satisfy the above condition. In particular, a Riemannian metrics on $C$ is a $C^1$-invariant Riemannian metric on $C_0$.

(iv) An orbifold $C$ is a complex orbifold if both $C_0$ and $C_1$ have complex structures such that $s$, $t$, $m$ and $u$ are holomorphic. Moreover, a complex orbifold $C$ is called Kähler orbifold if $C$ equips a Riemannian metric $g$ such that $(C_0, g)$ is Kähler.

(v) Let $M$ be a manifold and $C$ an orbifold. A smooth map $\varphi$ from $M$ to $C$ is a collection of an open covering $\{U_i\}_{i \in I}$ of $M$ and smooth maps $\varphi_i : U_i \to C_0$ and $\varphi_{ij} : U_i \cap U_j \to C_1$ that satisfies $(s, t) \circ \varphi_{ij} = (\varphi_j, \varphi_i)$ and $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ for any $i, j, k \in I$. Moreover, a smooth map $\varphi : M \to C$ is said to be a submersion or an immersion if each $\varphi_i$ is a submersion or an immersion.

For a preparation of following subsections, we show the following lemma.

**Lemma 2.2.** Let $f : M \to C$ be a smooth map from a manifold $M$ to an orbifold $C$, and $\{U_i\}_{i \in I}$ the associated open covering. Let $V$ be a vector bundle on $C$. Then we have the isomorphisms $f_j^*V|_{U_i \cap U_j} \simeq f_i^*V|_{U_i \cap U_j}$ that satisfies the cocycle condition.

**Proof.** Obvious from the definition of vector bundles on a Lie groupoid. □

**Definition 2.3.** Let $f : M \to C$ and $V$ be as in the above lemma. We define the pullback $f^*V$ as the gluing of each pullback $f_i^*V$.

**Remark 2.4.** The pullback of a Riemannian metric is defined in a similar way.
2.2 The boundary value problem of HE-metrics on trivial holomorphic bundles

Let \((M, g)\) be a compact connected Hermitian manifold of dimension \(n\) with boundary \(\partial M\) and \(\omega\) the fundamental form. Let \(\Lambda_\omega : \Omega^1_M \to \Omega^{n-2}_M\) be the contraction by \(\omega\). We assume that \(g\) is of \(L^p\)-class for some \(p \gg 2n\). We denote by \(\partial\) and \(\overline{\partial}\) the \((1, 0)\) and \((0, 1)\)-part of the exterior derivative.

**Definition 2.5.** Let \((V, \overline{\partial}_V)\) be a holomorphic bundle on \(M\). A Hermitian metric \(h\) on \((V, \overline{\partial}_V)\) is Hermite-Einstein of factor \(f \in L^1(M)\) if the Chern connection \(A\) of \((V, \overline{\partial}_V, h)\) satisfies the Hermite-Einstein equation \(\Lambda_\omega F(A) = \sqrt{-1}f \text{Id}_V\).

For a holomorphic line bundle \((L, \overline{\partial}_L)\) on \(M\), a Hermitian metric \(h\) on \(L\) is Hermite-Einstein of factor \(f \in L^1(M)\) if and only if we have \(\tilde{\Delta}(\log(h)) = -f\), where \(\tilde{\Delta} := \sqrt{-1}\Lambda_\omega \overline{\partial}\partial\) is the complex Laplacian.

**Lemma 2.6.** The operator \(S : L^p(M) \ni f \mapsto (\Delta f, f|_{\partial M}) \in L^p(M) \oplus L^p_{2-n/p}(\partial M)\) is an isomorphism.

**Proof.** Set the operator \(\hat{\Delta} := \partial^*\partial\), where \(\partial^*\partial\) is the adjoint of \(\partial\) with respect to \(g\). Then the operator \(\hat{S} : L^p(M) \ni f \mapsto (\hat{\Delta}(f), f|_{\partial M}) \in L^p(M) \oplus L^p_{2-n/p}(\partial M)\) is an isomorphism by the Lax-Milgram argument. The difference \(S - \hat{S}\) is a first-order differential operator with \(C^0\)-coefficient, and hence \(S\) is a Fredholm operator of index 0. Here we have \(\text{Ker}(S) = \{0\}\) by the maximum principle. Therefore \(S\) is an isomorphism.

Let \(V \simeq \mathbb{C}^r \times M\) be a trivial holomorphic bundle on \(M\). By following [5], we solve the Dirichlet problem of HE-metrics on \(V\).

**Proposition 2.7.** For any smooth Hermitian metric \(\tilde{h}\) on \(V|_{\partial M}\) and a real-valued function \(f \in L^p\), there exists a unique Hermitian metric \(h\) on \(V\) such that \(h|_{\partial M} = \tilde{h}\) is a Hermite-Einstein metric of factor \(f \in L^p(M)\) and satisfies \(h|_{\partial M} = \tilde{h}\).

**Proof.** By Lemma 2.6, we may assume \(f = 0\). We first show the uniqueness. Let \(h_1, h_2\) be Hermite-Einstein metrics on \(V\) of factor 0 such that \(h_1|_{\partial M} = h_2|_{\partial M} = \tilde{h}\). We denote by \(A_i\) the Chern connections of \((V, h_i)\) respectively. Set an endomorphism \(\eta\) of \(V\) to be \(\eta := (h_1)^{-1}h_2\). Then Hermite-Einstein condition induces \(\overline{\partial}(\eta) \wedge \eta^{-1}\partial A_1(\eta) = \overline{\partial}\partial A_1(\eta)\). Hence by taking the trace and contraction by \(\omega\), we obtain \(\Delta(\text{Tr}(\eta)) = -|\overline{\partial}_V \eta \cdot \eta^{-1/2}|^2 h_1 \leq 0\) because \(\eta\) is a Hermitian endomorphism with respect to \(h_1\). The same argument...
applies to $\eta^{-1}$ and we also obtain $\tilde{\Delta}(\text{Tr}(\eta^{-1})) \leq 0$. Therefore $\tilde{\Delta}(\text{Tr}(\eta) + \text{Tr}(\eta^{-1})) \leq 0$, and then maximum principle shows $\max_M(\text{Tr}(\eta) + \text{Tr}(\eta^{-1})) \leq \max_{\partial M}(\text{Tr}(\eta) + \text{Tr}(\eta^{-1})) = 2r$. Since we have $(\text{Tr}(\eta) + \text{Tr}(\eta^{-1})) \geq 2r$ by some calculation, the equality $\text{Tr}(\eta) + \text{Tr}(\eta^{-1}) = 2r$ holds identically. Therefore we obtain $\eta = \text{Id}_V$, and which proves the uniqueness.

We prove the existence by the method of continuity. Since the space of smooth Hermitian metric on $V|_{\partial M}$ is contractible, there exists a smooth family $\{\hat{h}_t\}_{t \in [0,1]}$ of smooth Hermitian metrics on $V|_{\partial M}$ such that $\hat{h}_0$ is the trivial metric and $\hat{h}_1$ is the vacuum metric and particularly $\hat{h}_1$ exists in the case $\hat{h} = \hat{h}_1$. Obviously $0 \in I$ and particularly $I \neq \emptyset$. We prove that $I$ is open. We fix an arbitrary $s \in I$. Let $X_t$ be the space of $L^p$-valued Hermitian endomorphism on $V$ with respect to $h_s$. Let $Y$ be the space of $L^p_{2-4/p}$-valued Hermitian endomorphism on $V|_{\partial M}$ with respect to $h_s|_{\partial M}$. Let $O \subset X_0$ be a neighborhood of $\text{Id}_V$ such that $h_s$ is a Hermitian metric for any $e \in O$. We set an operator $\Xi : O \to X_0 \times Y$ to be $\Xi(e) := (\sqrt{-\Lambda}F(A_{h_s,e}), e|_{\partial M})$. Then we have $(d\Xi)|_{\text{Id}_V}(v) = (\tilde{\Delta} h_s(v), v|_{\partial M})$, where $\tilde{\Delta} h_s := \sqrt{-\Lambda} \partial \bar{\partial} A_{h_s}$. By the same argument in Lemma 2.6 $(d\Xi)|_{\text{Id}_V}$ is an isomorphism. Therefore the implicit function theorem shows that $s$ is an interior point of $I$. We prove that $I$ is closed. Let $\{h_i\}_{i \in \mathbb{N}}$ be a sequence of Hermitian-Einstein metrics on $V$ such that $\{h_i|_{\partial M}\}$ converges to a smooth Hermitian metric $h_\infty$ in the sense of $L^p_{2-4/p}$-norm. We introduce the Donaldson metric on Hermitian metric space. For Hermitian metrics $H_1, H_2$ on $\mathbb{C}^n$, we set $\sigma(H_1, H_2) := \text{Tr}(H_1^{-1} H_2) + \text{Tr}(H_2^{-1} H_1) - 2r$. Then $\sigma$ is a complete metric on the space of Hermitian metrics and its topology coincides with the induced one from the linear space of skew-Hermitian forms on $\mathbb{C}^n$. We consider functions $d_{ij} := \sigma(h_i, h_j)$ on $\mathcal{M}$. By the same argument in the uniqueness part, we have $\tilde{\Delta}(d_{ij}) \leq 0$, and hence $\max_{\partial M}(d_{ij}) \leq \max_{\partial M}(d_{ij})$. Since $\{h_i|_{\partial M}\}$ converges to $h_\infty$ in $C^0$-norm, there exists a $C^0$-limit $h_\infty$ of the sequence $\{h_i\}$. By the elliptic regularity and the (vacuum) Hermitian-Einstein equation $\Lambda \partial \bar{\partial}(h^{-1}\partial h) = 0$, it is sufficient to prove the regularity of $h_\infty$ that the $C^1$-norms of $h_\infty$ are bounded. We assume that the norms $||h_i||_{C^1}$ are unbounded. Without loss of generality, we may assume $||h_i||_{C^1} \to \infty$. For $i \in \mathbb{N}$, we take $p_i \in M$ to satisfy $m_i := |d h_i|(p_i) \to \infty$. First we consider the case $\delta = \lim_{\epsilon} \text{dist}(p_i, \partial M) \cdot m_i > 0$. For $i \in \mathbb{N}$, we set $K_i$ to be the rescaling of $h_i|_{B(p_i, \delta/m_i)}$ to $B(0, \delta) \subset \mathbb{C}^n$, where $B(p, \epsilon) := \{z \in \mathbb{C}^n \mid |z - p| \leq \epsilon\}$. Since $\{h_i\}$ converge to $h_\infty$ in $C^0$-sense, $\{K_i\}$ converges to a constant metric $K_\infty$ in $C^0$-sense. We have $||d K_i||_{C^0} = 1$ and each $K_i$ is a Hermitian-Einstein metric on $B(0, \delta)$ with respect to each rescaled Hermitian metric. Hence $\{K_i\}$ converges to the constant metric

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\[ K_{\infty} \] in \( C^1 \)-sense. However, this contradict to the assumption \(|dK_i|(0) = |dh_i|(p)/m_i = 1\). Next we consider the case \( \lim_i \text{dist}(p_i, \partial M) \cdot m_i = 0 \). We use closed half balls instead of closed balls, and then a similar argument induces a contradiction. Therefore the \( C^1 \)-norms of \( h_i \) are bounded and this completes the proof. \[ \square \]

2.3 3-dimensional Sasakian manifolds

By following [3], we recall the notion of Sasakian manifolds.

Definition 2.8. Let \((X, g)\) be a \( 2n + 1 \)-dimensional Riemannian manifold. If there exists a compatible \( \mathbb{R}_+ \)-invariant complex structure \( J \) on \((\mathbb{R}_+ \times X, dr^2 + r^2 g)\) such that \((\mathbb{R}_+ \times X, J, dr^2 + r^2 g)\) is Kähler, then the tuple \((X, g, J)\) is called a Sasakian manifold. For a Sasakian manifold \((X, g, J)\), the Killing vector field \( \xi := -J(\partial_r)|_{\{1\} \times X} \) on \( X \) is called the Reeb vector field on \((X, g, J)\).

Let \((X, g, J)\) be a Sasakian manifold. If any orbits of the Reeb vector field of \((X, g, J)\) is compact, then \((X, g, J)\) is called a quasi-regular Sasakian manifold. For a quasi-regular Sasakian manifold \((X, g, J)\), the Reeb vector field determines an almost free \( S^1 \)-action on \( X \). In particular, \( X/S^1 = (\mathbb{R}_+ \times X)/\mathbb{C}^* \) is Morita-equivalent to a complex orbifold. For a compact 3-dimensional Sasakian manifold, we have the following result in [3].

Theorem 2.9. Any compact 3-dimensional Sasakian manifolds are quasi-regular.

2.4 The Mini-holomorphic Structure

Let \( X \) be a \( 2n + 1 \) dimensional manifold.

Definition 2.10. Let \( \partial_t \) be a nowhere vanishing vector field on \( X \). Let \( \alpha \in \Omega_X^1 \) be a \( \partial_t \)-invariant 1-form such that \( \langle \alpha, \partial_t \rangle = 1 \). Let \( \pi : X \to Y \) be a \( \partial_t \)-invariant submersion to an \( n \)-dimensional complex orbifold \( Y \). For a local vector field \( \nu \) on \( Y_0 \), we will denote by \( \tilde{\nu} \) the lift of \( \nu \) by the isomorphism \( \text{Ker}(\alpha) \cong \pi^*TY \).

- The tuple \((\partial_t, \alpha, Y, \pi)\) is called an almost mini-holomorphic structure on \( X \).

- We define \( \Omega_X^{0,1} \) as a subbundle of \( \Omega_X^1 \) spanned by \( \alpha \).

We define \( \Omega_Y^{0,1} := \pi^* \Omega_Y^{0,1} \oplus \varepsilon \alpha \) and \( \Omega_X^{0,i} := \bigwedge^i \Omega_X^{0,1} \), where \( \varepsilon \alpha \) means a subbundle of \( \Omega_X^{1,\mathbb{C}} \) spanned by \( \alpha \).
An almost mini-holomorphic structure \((\partial_t, \alpha, Y, \pi)\) is called a mini-holomorphic structure on \(X\) if we have \((d\alpha)^{0,2} = 0\), where \((d\alpha)^{0,2}\) is the \(\Omega^0_{X,C} = \Omega^0_X \oplus (\Omega^0_X \otimes \pi^*\Omega^1_Y) \oplus \pi^*\Omega^2_Y\) component of \(d\alpha\) with respect to the decomposition \(\Omega^2_X, C = \Omega^0_X \oplus (\Omega^0_X \otimes \pi^*\Omega^1_Y) \oplus \pi^*\Omega^2_Y\).

We define a differential operator \(\partial_X : \Omega^0_{X,0} \to \Omega^0_{X,1}\) to be \(\partial_X(f) := \partial_t(f)\alpha + \sum_i \overline{\partial} z_i(f)\pi^*(d\overline{z}_i)\), where \((z^i)\) is a holomorphic local chart on \(Y_0\). Moreover, we extend \(\partial_X\) to be an operator from \(\Omega^0_{X,i}\) to \(\Omega^0_{X,i+1}\) by a usual way.

If a function \(f\) locally defined on \(X\) satisfies \(\partial_X(f) = 0\), then \(f\) is said to be a mini-holomorphic function.

**Remark 2.11.**

If \(Y\) is a complex manifold and \(X = \mathbb{R}_t \times Y\) or \(S^1_t \times Y\), then there exists the trivial mini-holomorphic structure \((\partial_t, dt, Y, \pi)\). On the trivial mini-holomorphic structure, the notion of mini-holomorphic bundle as above defined agree with the one in [7].

For a mini-holomorphic structure \((\partial_t, \alpha, Y, \pi)\), we have \(\partial_X \circ \partial_X = 0\).

For a nonempty open subset \(U \subset X\), the tuple \(\left(\partial|_U, \alpha|_U, \pi(U), \pi|_U\right)\) is a mini-holomorphic structure on \(U\).

For example, Any principal \(S^1\)-bundle on a Riemann surface with a connection has a unique mini-holomorphic structure.

Let \((M, g_M)\) be a compact Sasakian 3-fold and \(\partial_t\) the Reeb vector field on \(M\). Let \(\Sigma\) be the orbifold Riemann surface obtained as the quotient of \(M\) by the \(S^1\)-action induced by \(\partial_t\), and \(\pi : M \to \Sigma\) the quotient map. Let \(\alpha \in \Gamma(M, (\pi^*\Omega^1_{\Sigma})^\perp)\) be the unique section that satisfies \(\langle \alpha, \partial_t \rangle = 1\), where \((\pi^*\Omega^1_{\Sigma})^\perp\) is the orthogonal complement bundle of \(\pi^*\Omega^1_{\Sigma}\) in \(\Omega^1_M\).

**Proposition 2.12.** The tuple \((\partial_t, \alpha, \Sigma, \pi)\) is a mini-holomorphic structure on \(M\).

**Proof.** Any conditions other than the \(\partial_t\)-invariance of \(\alpha\) is trivial, and \(\alpha\) is \(\partial_t\)-invariant because \(\text{rank}((\pi^*\Omega^1_{\Sigma})^\perp) = 1\).

**Definition 2.13.** Let \((\partial_t, \alpha, Y, \pi)\) be a mini-holomorphic structure on \(X\).

(i) Let \(V\) be a complex vector bundle on \(X\). A differential operator \(\overline{\partial}_V : \Omega^0_{X,i}(V) \to \Omega^0_{X,i+1}(V)\) is said to be a mini-holomorphic structure on \(V\) if the following conditions are satisfied:
• The Leibniz rule $\overline{\partial}_V(fs) = \overline{\partial}_X(f) \wedge s + f\overline{\partial}_V(s)$ holds for any $f \in C^\infty(X)$ and $s \in \Omega^0_X(V)$.

• The integrability condition $\overline{\partial}_V \circ \overline{\partial}_V = 0$ holds.

We call the pair $(V, \overline{\partial}_V)$ a mini-holomorphic bundle on $X$.

(ii) If a local section $v$ on $V$ satisfies $\overline{\partial}_V(v) = 0$, then we call $v$ a mini-holomorphic section.

(iii) Let $(V_i, \overline{\partial}_{V_i})$ be a mini-holomorphic bundle on $X$ for $i = 1, 2$. A homomorphism $\phi : V_1 \to V_2$ is mini-holomorphic if we have $\overline{\partial}_{V_2} \circ \phi = \phi \circ \overline{\partial}_{V_1}$. Moreover, if $\phi$ is an injective homomorphism of vector bundles, then $V_1$ is called a mini-holomorphic subbundle of $V_2$.

Let $(\partial_t, \alpha, Y, \pi)$ be a mini-holomorphic structure on $X$ and $(V, \overline{\partial}_V)$ a mini-holomorphic bundle on $X$. Let $U \subset X$ be a sufficiently small open subset such that $\pi$ can be lifted to $\pi_U : U \to Y_0$. Let $W \subset \pi_U(U)$ be an open subset. A smooth map $s : W \to X$ is called a section on $W$ i.e., $s$ satisfies $\pi_U \circ s = \text{Id}_W$. For a section $s : W \to X$, the pullback $s^*V$ has a natural holomorphic structure $\overline{\partial}_{s^*V} : \Omega^0_0(s^*V) \to \Omega^0_1(s^*V)$ defined as follows:

$$\overline{\partial}_{s^*V}(v) := \overline{\partial}_V(\tilde{v})|_{s(W)}, \quad (v \in \Omega^0_0(s^*V)),$$

where $\tilde{v}$ is the local section of $V$ on a neighborhood of $s(W)$ that is obtained by the parallel transport of $v$ with the differential equation $\langle \partial_t, \overline{\partial}_V(\tilde{v}) \rangle = 0$ along each integral curve of $\partial_t$.

We define the scattering map of a mini-holomorphic bundle in our context. Since the following arguments are local with respect to $X$ and $Y$, we assume that $Y$ is a domain of $\mathbb{C}^n$ and $X = (-\varepsilon, \varepsilon)_t \times Y$. Let $s_1, s_2$ be a section on $Y$. We set the scattering map $\Psi_{s_1, s_2} : (s_1)^*V \simeq (s_2)^*V$ to be $\Psi_{s_1, s_2}(v) := \tilde{v}|_{s_2(Y)}$, where $\tilde{v}$ is the parallel transport of $v$. The scattering map $\Psi_{s_1, s_2}$ is obviously an isomorphism of differentiable vector bundle.

**Proposition 2.14** ([4]). The scattering map $\Psi_{s_1, s_2}$ is a holomorphic isomorphism.

**Proof.** Let $v$ be a local holomorphic section of $(s_1)^*V$. By the integrability condition $\overline{\partial}_V \circ \overline{\partial}_V = 0$, the parallel transport $\tilde{v}$ satisfies $\overline{\partial}_V(\tilde{v}) = 0$. Therefore $\Psi_{s_1, s_2}(v) = (s_2)^*(\tilde{v})$ is a holomorphic section of $(s_2)^*V$.  

\qed
In the following part of this subsection, we assume \( \dim(X) = 3 \). We define the notion of algebraic Dirac-type singularities of mini-holomorphic bundle on \( X \). Similar in the above argument, we assume that \( Y \) is a neighborhood of \( p \in \mathbb{C} \) and \( X = (−\varepsilon, \varepsilon) \times Y \). Let \( (V, \overline{\nabla}_V) \) be a mini-holomorphic bundle on \( X \setminus \{(0,p)\} \). Set sections \( s_1, s_2 : Y \to X \) as \( s_i(z) := ((−1)^i\varepsilon/2), z) \).

**Definition 2.15.**
- The point \((0,p)\) is an algebraic Dirac-type singularity of \((V, \overline{\nabla}_V)\) if the scattering map \( \Psi_{s_1, s_2} \) can be prolonged to the meromorphic isomorphism \( (s_1)^*V \ast \pi(p) \simeq (s_2)^*V \ast \pi(p) \). Moreover, The algebraic Dirac-type singularity \((0,p)\) is of weight \( \vec{k} = (k_i) \in \mathbb{Z}^r \) if there exist holomorphic frames \( e_i = (e_{i,j}) \) of \((s_i)^*V \) \( i = 1, 2 \) such that we have \( \Psi_{s_1, s_2}(e_{1,j}) = z^{k_j}e_{2,j} \), where \( z \) is a holomorphic local chart on \( Y \) such that \( z(p) = 0 \).
- Let \( Z \subset X \) be a discrete subset. If each point \( p \in Z \) is an algebraic Dirac-type singularity of a mini-holomorphic bundle \((E, \overline{\nabla}_E)\) on \( X \setminus Z \), then we call \((E, \overline{\nabla}_E)\) a Dirac-type singular mini-holomorphic bundle on \((X, Z)\).

**Remark 2.16.** Since \( \mathbb{C}\{z\} \) is a PID, the weight of an algebraic Dirac-type singularity is unique up to permutations.

### 2.5 The Hopf fibration and Dirac-type HE-monopoles
#### 2.5.1 The Hopf fibration
Let \( U \subset \mathbb{R}^3 \) be a neighborhood of \( 0 \in \mathbb{R}^3 \) and \( g \) be a Riemannian metric on \( U \). We assume that the canonical coordinate of \( \mathbb{R}^3 \) is the normal coordinate of \( g \) at \( 0 \in \mathbb{R}^3 \). Set the Hopf fibration \( p : \mathbb{R}^4 = \mathbb{C}^2 \to \mathbb{R}^3 = \mathbb{R} \times \mathbb{C} \) to be \( p(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1z_2) \), where we set \( z_i = x_i + \sqrt{-1}y_i \). We also set the \( S^1(= \mathbb{R}/2\pi\mathbb{Z}) \)-action on \( \mathbb{C}^2 \) to be \( \theta \cdot (z_1, z_2) := (e^{\sqrt{-1}\theta}z_1, e^{-\sqrt{-1}\theta}z_2) \). Then the restriction \( p : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\} \) forms a principal \( S^1 \)-bundle.

**Lemma 2.17.** There exist a harmonic function \( f : U \setminus \{0\} \to \mathbb{R} \) with respect to the metric \( g \) and a 1-form \( \xi \) on \( p^{-1}(U) \) such that the following hold.
- The 1-form \( \omega := p^*f \cdot \xi \) is a connection of \( p : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\} \), i.e. \( \omega \) is \( S^1 \)-invariant, and we have \( \omega(\partial_\theta) = 1 \). Here \( \partial_\theta \) is the generating vector field of the \( S^1 \)-action on \( \mathbb{R}^4 \setminus \{0\} \).
- We have \( d\omega = p^*(-\ast df) \).
We have the following estimates:
\[
\begin{align*}
 f &= 1/2r^3 + o(1) \\
 \xi &= 2(-y_1 dx_1 + x_1 dy_1 + y_2 dx_2 - x_2 dy_2) + O(r^2).
\end{align*}
\]

- The symmetric tensor \( g_4 = p^* f(p^* g + \xi^2) \) is a Riemannian metric of \( L^2_{5,\text{loc}} \)-class on \( p^{-1}(U) \), and we have an estimate \( |g_4 - 2g_4|_{Euc} = O(r^4) \). Here a function on \( p^{-1}(U) \) is of \( L^2_k,\text{loc} \)-class if every derivative of \( f \) up to order \( k \) has a finite \( L^2 \)-norm on any compact subset of \( p^{-1}(U) \).

**Remark 2.18.**
- If \( g = g_{\mathbb{R}^3} \), we can choose \( f = 1/2r^3 \) and \( \xi = 2(-y_1 dx_1 + x_1 dy_1 + y_2 dx_2 - x_2 dy_2) \). Then we have \( g_4 = 2g_{\mathbb{R}^4} \).
- By the Sobolev embedding theorem, the connection matrix of \( A_4 \) is of \( C^3 \)-class.

### 2.5.2 The Hopf fibration of mini-holomorphic manifolds

Let \( I = (-\varepsilon, \varepsilon) \subset \mathbb{R}_t \) and \( W \subset \mathbb{C}_z \) be neighborhoods of origins of \( \mathbb{R} \) and \( \mathbb{C} \) respectively. Set \( U := I \times W \), and take the projection \( \pi : U \to W \). Let \( g_W = \lambda dzd\bar{z} \) be the Kähler metric on \( W \) that satisfies \( \lambda(0) = 1 \). Take an \( \mathbb{R}_t \)-invariant 1-form \( \alpha = dt + \alpha_x dx + \alpha_y dy \in \Omega^1_U \) satisfying \( \alpha_x(0) = \alpha_y(0) = 0 \). Set \( g_U := \alpha^2 + \pi^* g_W \). The tuple \( (\partial_t, \alpha, W, \pi) \) forms a mini-holomorphic structure on \( U \).

Take a normal coordinate \( (x^i) \) at \( (0, 0) \) on \( U \) satisfying \( dt|_0 = dx^1|_0 \) and \( dz|_0 = (dx^2 + \sqrt{-1} dx^3)|_0 \). Let \( U_4 \subset \mathbb{R}^4 \) be a sufficiently small neighborhood of \( 0 \in \mathbb{R}^4 \) and \( p : U_4 \to U \) the Hopf fibration with respect to the coordinate \( (x^i) \). We take a harmonic function \( f \) on \( U \setminus \{0\} \) and a 1-form \( \xi \) on \( U_4 \) as in Lemma 2.17. Set an \( L^2_{5,\text{loc}} \)-metric \( g_4 := p^* (p^* g_U + \xi^2) \) on \( U_4 \). Set an almost complex structure \( J \in \Gamma(U_4, \text{End}(TU_4)) \simeq \Gamma(U_4 \setminus \{0\}, \text{End}(\Omega^1_{U_4})) \) to be
\[
\begin{align*}
 J(-\xi) &:= -p^* \alpha \\
 J(p^* dx) &:= -p^* dy.
\end{align*}
\]

**Lemma 2.19.** The almost complex structure \( J \) is integrable and of \( L^p_{2,\text{loc}} \)-class on \( U_4 \) for any \( p \in [1, \infty) \).

**Proof.** We can check vanishing of the Nijenhuis tensor of \( J \) by an easy calculation, and hence \( J \) is integrable. Take a 1-form \( \xi_0 = 2(-y_1 dx_1 + x_1 dy_1 + y_2 dx_2 - x_2 dy_2) \) and
\begin{align*}
y_2dx_2 - x_2dy_2\end{align*}
and set an almost complex structure \( J_0 \) on \( U_4 \) to be
\[
\begin{cases}
J(-\xi_0) := -p^*dx^1 \\
J(p^*dx^2) := -p^*dx^3.
\end{cases}
\]

Then \( J_0 \) agrees with the canonical complex structure on \( \mathbb{R}^4 \). Hence \( J_0 \) is of \( C^\infty \)-class on \( U_4 \). Then by some calculation the difference \( J_0 - J \) is of \( L^p_{2,\text{loc}} \)-class for any \( p \in [1, \infty) \), which completes the proof. \( \square \)

By the result in [6], we obtain the following corollary.

**Corollary 2.20.** There exists a holomorphic coordinate \((w_1, w_2)\) on \((U_4, J)\) such that \( \theta \cdot w_1 = e^{\sqrt{-1}\theta}w_1 \) and \( \theta \cdot w_2 = e^{-\sqrt{-1}\theta}w_2 \) for any \( \theta \in S^1 \), and \( w_1 \) and \( w_2 \) are of \( L^p_{3,\text{loc}} \)-class for any \( p \in [1, \infty) \). Moreover, we have \( p^*z = w_1w_2 \).

**Proof.** Since the \( S^1 \)-weight of \( \Omega^1_{C^2} \) is \((1,1)\). Hence we can take a holomorphic local chart \( w_1, w_2 \) that \( \theta \cdot w_1 = e^{\sqrt{-1}\theta}w_1 \) and \( \theta \cdot w_2 = e^{-\sqrt{-1}\theta}w_2 \) for any \( \theta \in S^1 \). Since \( p^*z \) is a holomorphic function and \( \eta^1 \)-invariant and of order 2 at origin, we may assume \( p^*z = w_1 \cdot w_2 \). \( \square \)

Set \( U_+ := U \setminus ((-\varepsilon, 0) \times \{0\}) \) and \( U_- := U \setminus ([0, \varepsilon) \times \{0\}) \). For Proposition 2.23, we prepare the following lemma.

**Lemma 2.21.** We have \( \text{Zero}(w_1) = p^*((-\varepsilon, 0) \times \{0\}) \) and \( \text{Zero}(w_2) = p^*([0, \varepsilon) \times \{0\}) \).

**Proof.** We have \( p^*((-\varepsilon, \varepsilon) \times \{0\}) = \text{Zero}(p^*z) = \text{Zero}(w_1) \cup \text{Zero}(w_2) \) because \( p^*z = w_1w_2 \). Hence \( p^*((-\varepsilon, 0) \times \{0\} \cup [0, \varepsilon) \times \{0\}) = (\text{Zero}(w_1) \setminus \{(0,0)\}) \cup (\text{Zero}(w_2) \setminus \{(0,0)\}) \). Since the image of a connected space by a continuous map is connected, we obtain \( \text{Zero}(w_1) = p^*((-\varepsilon, 0) \times \{0\}) \) and \( \text{Zero}(w_2) = p^*([0, \varepsilon) \times \{0\}) \), or \( \text{Zero}(w_1) = p^*((0, \varepsilon) \times \{0\}) \) and \( \text{Zero}(w_2) = p^*((-\varepsilon, 0) \times \{0\}) \). We assume the latter one. Let \( L \) be an \( S^1 \)-equivariant line bundle on \( U_4 \) such that the weight of \( L|_{0} \) is 1. Let \( b \) be a frame of \( L \) such that \( \theta \cdot b = e^{\sqrt{-1}\theta}b \). We denote by \( \tilde{L} \) the quotient of \( L|_{U_4\setminus\{0\}} \). On one hand, for the canonical coordinate \( z_1, z_2 \) on \( \mathbb{C}^2 \), the descent of the sections \( (z_1)^{-1} \cdot b \) on \( U_4 \setminus \text{Zero}(z_1) \) and \( z_2 \cdot b \) on \( U_4 \setminus \text{Zero}(z_2) \) give frames of \( \tilde{L} \) on \( U_+ \) and \( U_- \) respectively. By using this frame, the degree of \( \tilde{L} \) is calculated as 1. On the other hand, the descent of the frame \((w_1)^{-1} \cdot b \) on \( U_4 \setminus \text{Zero}(w_1) \) and \( w_2 \cdot b \) on \( U_4 \setminus \text{Zero}(w_2) \) also gives frames of \( \tilde{L} \) on \( U_- \) and \( U_+ \) respectively. Then the degree of \( \tilde{L} \) is calculated as \(-1 \), which is a contradiction. Therefore we obtain \( \text{Zero}(w_1) = p^*((-\varepsilon, 0) \times \{0\}) \), \( \text{Zero}(w_2) = p^*([0, \varepsilon) \times \{0\}) \). \( \square \)

**Remark 2.22.**
• If $p > 4$, we have $L_i^p \subset C^{i-1,(p-4)/p}$.

• The tuple $(U_4, J, g_4)$ is not a Kähler manifold in general.

We mention the lift of Dirac-type singular mini-holomorphic bundles by the Hopf fibration. Let $(E, \overline{\partial} E)$ be a Dirac-type singular mini-holomorphic bundle on $(U, \{(0,0)\})$ of weight $\vec{k} = (k_i) \in \mathbb{Z}^r$. We set a holomorphic structure $\overline{\partial} E_4 : \Omega^{0,0}(E_4) \to \Omega^{0,1}(E_4)$ on $E_4 := p^*E$ as follows:

$$\overline{\partial} E_4(p^*v) := (p^*\overline{\partial} E_4(v))^{0,1},$$

where $v$ is a local section of $E$. Take mini-holomorphic frames $e_{\pm} = (e_{\pm,i})$ of $E$ on $U_\pm$ such that we have $e_{+,i} = z^{k_i} \cdot e_{-,i}$.

By Lemma 2.21 we get a frame $e_4 = (e_{4,i})$ of $E_4$ on $U_4 \setminus \{(0,0)\}$ given as follows:

$$e_{4,i} = \begin{cases} w_1^{k_i} \cdot p^*(e_{+,i}) & (w_1 \neq 0) \\
 w_2^{-k_i} \cdot p^*(e_{-,i}) & (w_2 \neq 0). \end{cases}$$

We extend $E_4$ over $U_4$ by this frame. Then the weight of $E_4|_0$ is $\vec{k} = (k_i) \in \mathbb{Z}^r$ because $\theta \cdot e_{4,i} = e^{\sqrt{-1}k_i} \cdot e_{4,i}$. Summarizing the above argument, we obtain the following proposition.

**Proposition 2.23.** For the Dirac-type singular mini-holomorphic bundle $(E, \overline{\partial} E)$ of weight $\vec{k} \in \mathbb{Z}^r$ and the mini-holomorphic frames $e_{\pm}$ on $U_\pm$, the lift $E_4 := p^*E$ has the natural $S^1$-equivariant prolongation over $U_4$ and the $S^1$-weight of $E_4|_0$ is $\vec{k}$.

For the proof of Theorem 3.7 we prove the following lemma.

**Lemma 2.24.** Let $(E, \overline{\partial} E)$ be a Dirac-type singular mini-holomorphic bundle on $(U, \{(0,0)\})$ of rank $r$ and $(F, \overline{\partial} F)$ be a mini-holomorphic subbundle of $(E, \overline{\partial} E)$ of rank $r'$. Then the lift $F_4$ of $F$ in Proposition 2.23 is a holomorphic subbundle of the lift $E_4$ of $E$.

**Proof.** Let $e = (e_{\pm,i})$ and $f_{\pm} = (f_{\pm,j})$ be the mini-holomorphic frames of $E$ and $F$ on $U_\pm$ respectively such that there exist $\vec{k} = (k_i) \in \mathbb{Z}^r$ and $\vec{k}' = (k'_j) \in \mathbb{Z}^{r'}$ such that $e_{+,i} = \sum_i z^{k_i} \cdot e_{-,i}$ and $f_{+,j} = z^{k'_j} \cdot f_{-,j}$ for any $i, j$. By shrinking $U$ if necessary, we may assume that for any $j \in \{1, \ldots, r'\}$ there exist mini-holomorphic functions $(a^j_i)_{i=1}^{\vec{k}}$ on $U_+$ such that $f_{j,+} = a^j_i \cdot e_{i,+}$. By the definition of mini-holomorphic functions, $a^j_i$ can be prolonged to whole $U$ uniquely. Since we have $f_{j,-} = \sum_i a^j_i z^{k_i-k'_j} e_{-,i}$,
we obtain \( a_j^i = 0 \) unless \( k_i = k_j^i \). Therefore we obtain \( f_{j,\pm} = \sum_i a_j^i e_{\pm, i} \).

Let \( e_4 = (e_4, i) \) and \( f_4 = (f_4, j) \) be the holomorphic flames of \( E_4 \) and \( F_4 \) that is used in construction of \( E_4 \) and \( F_4 \) respectively. Then we have \( f_{4, j} = \sum_i p^*(a_j^i) \cdot e_{4, i} \).

Let \( e_4 \) and \( f_4 \) be the holomorphic flames of \( E_4 \) and \( F_4 \) respectively. Then we have \( f_{4, j} = \sum_i p^*(a_j^i) \cdot e_{4, i} \).

Since \( F_4|_{U_4}\{ (0, 0) \} \) is a subbundle of \( E_4|_{U_4}\{ (0, 0) \} \), it suffices to show \( \text{rank}(a_j^i(0, 0)) = r' \).

If we have \( \text{rank}(a_j^i(0, 0)) < r' \), then we have \( \text{rank}(a_j^i(-\varepsilon, 0)) = \text{rank}(a_j^i(0, 0)) < r' \) by the definition of mini-holomorphic functions. However, it contradicts to the assumption that \( f_- \) is a frame of \( F|_{U_-} \).

### 2.5.3 HE-monopoles and underlying mini-holomorphic structures

Let \( X \) be an oriented connected 3-fold. Let \((\partial_t, \alpha, \Sigma, \pi)\) be a mini-holomorphic structure on \( X \). Let \( g_\Sigma \) be a Kähler metric on \( \Sigma \) and \( \omega_\Sigma \) the Kähler form of \((\Sigma, g_\Sigma)\). Set \( g_X := \alpha^2 + \pi^* g_\Sigma \). We assume that \( \alpha \wedge \pi^* \omega_\Sigma \) is positive-oriented.

**Definition 2.25.**

(i) Let \((V, h)\) be a Hermitian vector bundle with a unitary connection \( A \) on \( X \). Let \( \Phi \) be a skew-Hermitian section of \( \text{End}(V) \). The tuple \((V, h, A, \Phi)\) is said to be a HE-monopole of degree \( c \in \mathbb{R} \) on \( X \) if it satisfies the Hermite-Einstein-Bogomolny equation \( F(A) = \pm \nabla_A(\Phi) + \sqrt{-1} c \cdot (\pi^* \omega_\Sigma) \text{Id}_V \).

(ii) Let \( Z \subset X \) be a discrete subset. Let \((V, h, A, \Phi)\) be a HE-monopole of rank \( r \in \mathbb{N} \) on \( X \setminus Z \). A point \( p \in Z \) is called a Dirac-type singularity of the monopole \((V, h, A, \Phi)\) with weight \( \vec{k}_p = (k_{p, i}) \in \mathbb{Z}^r \) if the following holds.

- There exists a small neighborhood \( B \) of \( p \) such that \((V, h)|_{B \setminus \{p\}}\) is decomposed into a sum of Hermitian line bundles \( \bigoplus_{i=1}^r F_{p, i} \) with \( \deg(F_{p, i}) = \int_{\partial B} c_1(F_{p, i}) = k_{p, i} \).
- In the above decomposition, we have the following estimates,
  \[
  \Phi = \frac{\sqrt{-1}}{2R_p} \sum_{i=1}^r k_{p, i} \cdot Id_{F_{p, i}} + O(1) \\
  \nabla_A(R_p \Phi) = O(1),
  \]

where \( R_p \) is the distance from \( p \).

For a HE-monopole \((V, h, A, \Phi)\) on \( X \setminus Z \), if each point \( p \in Z \) is a Dirac-type singularity, then we call \((V, h, A, \Phi)\) a Dirac-type singular monopole on \((X, Z)\).
In [9], Pauly proved a characterization of Dirac-type singular monopoles using the Hopf fibration, and it remains valid for HE-monopoles.

**Theorem 2.26.** Let \( U \subset \mathbb{R}_t \times \mathbb{C}_z \) be a neighborhood of \((0, 0)\) and \((\partial_t, \alpha, W, \pi : U \to W)\) a mini-holomorphic structure on \( U \). Let \((V, h, A, \Phi)\) be a HE-monopole on \( U \setminus \{(0, 0)\}\) of degree \( c \in \mathbb{R} \).

- The tuple \((V_4, h_4, A_4) := (p^* V, p^* h, p^* A - \xi \otimes p^* \Phi)\) is a Hermitian holomorphic bundle that satisfies the Hermite-Einstein condition of factor \( c/p^* f \).

- The point \((0, 0)\) is a Dirac-type singularity of the HE-monopole \((V, h, A, \Phi)\) if and only if the tuple \((V_4, h_4, A_4)\) can be prolonged as \(S^1\)-invariant Hermitian holomorphic bundle over \( U_4 \). Moreover, the weight of \((V, h, A, \Phi)\) at \((0, 0)\) agrees with the \(S^1\)-weight of \(V_4 |_{0} \).

**Remark 2.27.** We have \((p^* f)^{-1} \in L^p_{1,\text{loc}}(U_4)\) for any \( p \in [1, \infty) \).

**Proposition 2.28.** A HE-monopole \((V, h, A, \Phi)\) on \(X\) has a natural mini-holomorphic structure \(\partial V(v) := \nabla^0_A(v) - \sqrt{-1} \Phi(v) \alpha\).  

**Proof.** By a direct calculation. \[ \square \]

Let \((E, \overline{\partial}_E)\) be a mini-holomorphic bundle on \(X\), and \(h\) a Hermitian metric on \(E\). As an analogue of the Chern connection, there uniquely exist a connection \(A_h\) and a skew-Hermitian endomorphism \(\Phi_h\) on \(E\). We call \(A_h\) and \(\Phi_h\) the Charbonneau-Hurtubise (or shortly CH) connection and endomorphism. If the tuple \((V, h, A_h \Phi_h)\) is a monopole on \(X\), we call \(h\) a Bogomolny-Hermite-Einstein (or shortly BHE) metric on \((V, \overline{\partial}_V)\).

By following [4], we mention the relation between Dirac-type singular monopole and mini-holomorphic bundles by following [4]. We assume that \(Y\) is a neighborhood of \(0 \in \mathbb{C}\) and \(X = [-\varepsilon, \varepsilon]_t \times Y\). Let \((V, h, A, \Phi)\) be a HE-monopole on \(X \setminus \{(0, 0)\}\) and \((V, \overline{\partial}_V)\) the underlying mini-holomorphic bundle. We denote by \(\tilde{k} \in \mathbb{Z}^r\) the weight of \((V, h, A, \Phi)\) at \((0, 0)\). Take sections \(s_1, s_2\) on \(Y\) to be \(s_i(z) := ((-1)^i \varepsilon, z)\). Let \(\Psi_{s_1, s_2} : (s_1)^* V|_{Y \setminus \{0\}} \to (s_2)^* V|_{Y \setminus \{0\}}\) be the scattering map.

**Proposition 2.29.** The scattering map \(\Psi_{s_1, s_2}\) induces a meromorphic isomorphism \((s_1)^* V(\ast 0) \simeq (s_2)^* V(\ast 0)\). In particular, \((0, 0)\) is an algebraic Dirac-type singularity of \((V, \overline{\partial}_V)\). Moreover, the weight of the algebraic Dirac-type singularity of \((V, \overline{\partial}_V)\) at \((0, 0)\) is \(\tilde{k}\).
Proof. Let \((V_2, \overline{\partial}_2)\) be a Dirac-type singular mini-holomorphic bundle on \((U, \{(0, 0)\})\) such that the weight at \((0, 0)\) is \(\vec{k}\). Take a small neighborhood \(U_4 \subset \mathbb{C}^2\) of \((0, 0) \in \mathbb{C}^2\). Let \(V_{4,1}\) be the holomorphic bundle on \(U_4\) obtained by applying Theorem 2.26 to \((V, h, A, \Phi)\). Let \(V_{4,2}\) be the holomorphic bundle on \(U_4\) obtained by applying Proposition 2.23 to \((V_2, \overline{\partial}_2)\). Since the \(S^1\)-weights of \(V_{4,1}\) and \(V_{4,2}\) agree with each other, there exists an \(S^1\)-equivariant holomorphic isomorphism \(K_4 : V_{4,1} \to V_{4,2}\). Then the descent \(K : V \to V_2\) is a mini-holomorphic isomorphism. Therefore the weights of algebraic Dirac-type singularity of \((V, \overline{\partial}_V)\) and \((V_2, \overline{\partial}_2)\) agrees with each other, which is the assertion of the Proposition. \(\square\)

2.6 A generalization of the Characterization of Dirac-type singularities in \([7]\)

Let \(U = I \times W \subset \mathbb{R} \times \mathbb{C}_z, \alpha \in \Omega^1(U), g_2, g_U\) and \(J\) be as in subsection 2.5.2. We denote by \(r_1 : \mathbb{R} \to \mathbb{R}\) the distance function from the origin. Let \((V, h, A, \Phi)\) be a HE-monopole on \(U \setminus \{0\}\) of rank \(r > 0\) of factor \(c \in \mathbb{R}\). The following proposition is a slight generalization of Theorem 4.5 in \([7]\).

Proposition 2.30. If the estimate \(|\Phi| = O(r_3^{-1})\) is satisfied, then \((V, h, A, \Phi)\) is a Dirac-type monopole on \((U, \{(0, 0)\})\).

Proof. Take the Hopf-fibration \(p : U_4 \to U_4, f : U_4 \setminus \{0\} \to \mathbb{R}, \xi \in \Omega^1(U_4)\), and the holomorphic coordinate \(w_1, w_2\) on \((U_4, J)\) as in 2.5.2. We set \((V_4, h_4, A_4) := (p^*V, p^*h, p^*A + \xi \otimes p^*\Phi)\). Set \(U_+ := U_4 \setminus ((-\varepsilon, 0] \times \{0\})\) and \(U_- := U_4 \setminus ([0, \varepsilon] \times \{0\})\). Let \(\overline{\partial}_V\) be the mini-holomorphic structure of \((V, h, A, \Phi)\). By the assumption, there exist \(\vec{k} = (k_i)\) and mini-holomorphic frames \(e_{\pm}^{1}\) of \((V, \overline{\partial}_V)\) on \(U_\pm\) such that we have \(e_{\pm,i} = z^{k_i}e_{-\pm,i}\) and the estimate \(|e_{\pm,i}|_{h_4} = O(r_4^{-N})\) around the origin for some \(N > 1\). We take the frame \(e_4 = (e_{4,i})\) of \(V_4 := p^*V\) on \(U_4 \setminus \{0\}\) and prolong \(V_4\) over \(U_4\) as in Proposition 2.23. Then we obtain the estimate \(|e_{4,i}|_{h_4} = O(r_4^{-N'})\) for some \(N' > 0\). We prepare the following lemma.

Lemma 2.31. Let \(D \subset \mathbb{C}^2\) be a relatively compact neighborhood of \(0 \in \mathbb{C}^2\) with a smooth boundary \(\partial D\). Let \((E, h_0)\) be a Hermitian holomorphic vector bundle on \(D = D \cup \partial D\) of rank \(r\). Let \(h\) be a HE-metric of factor \(f \in L^p(U)\) on \(E|_{\{D\setminus\{0\}}\) for \(p > 8\). If the estimate \(|e|_h = O(r_4^{-N})\) holds for some positive number \(N > 0\) and for any local smooth section \(e\) of \(E\), then \(h\) can be prolonged as a HE-metric of \(E\) over whole \(D\).

By the above Lemma 2.31, \(h_4\) is at least of \(C^1\)-class. By Theorem 2.26 (ii), \((V, h, A, \Phi)\) is a Dirac-type singular monopole on \((U, \{0\})\). \(\square\)
(The proof of Lemma 2.31). We may assume \( f = 0 \). Since the statement is local, we also may assume that \( E \) is a trivial bundle. Moreover, we may assume that \( h_0 \) is a HE-metric and \( h_0|_{\partial D} = h|_{\partial D} \) by Proposition 2.7. We set the endomorphism \( k := h^{-1}_0 h \). Then log(Tr(\( k \))) satisfies log(Tr(\( k \))|_{\partial \Omega U'}) = log(\( r \)) and \( \tilde{\Delta}(\log \text{Tr}(k)) \leq 0 \) on \( D \setminus \{0\} \) by \[?, \text{Lemma 3.1}\]. We have an estimate |log Tr(\( k \))| = \( O(\log(\( r \)^4)) \), and hence we obtain \( \tilde{\Delta}(\log \text{Tr}(k)) \leq 0 \) on \( D \) as a distribution. Therefore log(Tr(k)) = log(\( r \)) by the maximum principle. Thus we obtain \( k = \text{Id}_E \), and particularly \( h \) is smooth over \( \overline{D} \).

3 The K-H correspondence of Dirac-type singular mini-hol. bundles on compact mini-hol. 3-folds.

3.1 The flat lift of mini-hol. 3-folds

Let \( X \) be a 3-fold with a mini-holomorphic structure \((\partial_t, \alpha, \Sigma, \pi)\). Let \( g_\Sigma \) be a Kähler metric on \( \Sigma \) and set \( g_X := \alpha^2 + \pi^* g_\Sigma \). We set \( M := S^1_\theta \times X \) and \( g_M := d\theta^2 + p^* g_X \), where \( p : M \to X \) is the projection. Let \( J \) be an almost holomorphic structure on \( M \) such that

\[
\begin{cases}
J(d\theta) = -\alpha \\
J(\pi^* a) = \pi^*(J_\Sigma(a)) (a \in \Omega^1_\Sigma),
\end{cases}
\]

where \( J_\Sigma \) is the complex structure on \( \Sigma \).

**Proposition 3.1.** The almost complex structure \( J \) is integrable and the tuple \((M, J, g_M)\) is a Gauduchon manifold.

**Proof.** The integrability is trivial from an easy calculation. For a local holomorphic coordinate \( z = x + \sqrt{-1} y \) on an open subset \( W \subset \Sigma_0 \), there exists a positive function \( \lambda \) on \( W \) such that \( g_\Sigma = \lambda(dx^2 + dy^2) \). Then the fundamental form \( \omega_M \) of \((M, J, g_M)\) can be written as \( \omega = d\theta \wedge \alpha + \lambda dx \wedge dy \). Hence we have \( \bar{\partial} \partial \omega_M = 0 \), and hence \((M, J, g_M)\) is a Gauduchon manifold.

Let \((V, \bar{\partial}_V)\) be a mini-holomorphic bundle on \( X \). The pullback \( \tilde{V} := p^* V \) has a natural holomorphic structure \( \tilde{\partial}_V \) determined as follows:

\[
\tilde{\partial}_V(p^* s) = (p^* \bar{\partial}_V(s))^{0,1},
\]

where \( s \) is a local section of \( V \). We call \((\tilde{V}, \tilde{\partial}_V)\) the flat lift of the mini-holomorphic bundle \((V, \bar{\partial}_V)\). For a Hermitian metric \( h \) on \( V \), the upstairs
connection \( p^*A_h + d\theta \otimes p^*\Phi_h \) is the Chern connection \( A_{p^*h} \). Moreover, \( h \) is a BHE-metric if and only if \( p^*h \) is a HE-metric.

Let \( Z \subset X \) be a finite subset. As a preparation to prove Theorem 3.7, we prove that \( M' = M \setminus (S^1 \times Z) \) satisfies the following assumptions in \([11]\).

(I) The volume of \( M' \) is finite.

(II) There exists an exhaustion function \( f \) of \( M' \) such that \( |\hat{\Delta}(f)| < M \) for some \( M > 0 \).

(III) There exist \( C > 0 \) and an increasing function \( a : [0, \infty) \to [0, \infty) \) with \( a(0) = 0 \) and \( a(x) = x \) for \( x > 1 \), such that if for a bounded positive function \( f \) on \( X \) satisfies \( \hat{\Delta}(f) \leq B \) then \( \sup f \leq C(B) a(\|f\|_{L^1}) \).

Furthermore, if \( \hat{\Delta}(f) \leq 0 \), then \( \hat{\Delta}(f) = 0 \).

Proposition 3.2. If \( X \) is compact, then the assumptions 1-3 in \([11]\) holds for \( M' := M \setminus (S^1 \times Z) \).

Proof. Obviously \( M' \) has finite volume, and hence the assumption (I) holds for \( M' \). We prove the assumption (II). By a direct calculation, there exists a vector field \( \beta \) on \( X \) such that we have \( \hat{\Delta}(p^*f) = 2p^*(\Delta(f) + \beta f) \) for any \( f \in \mathcal{C}^\infty(X) \). For each \( p \in Z \), there exists a smooth function \( \rho_p : B_2(p) \setminus \{p\} \to \mathbb{R} \) that satisfies

\[
\begin{align*}
(\Delta + \beta)(\rho_p) &= 0 \\
\rho_p(x) &= d(x, p)^{-1} + O(1).
\end{align*}
\]

Thus, by using a partition of unity, we obtain a non-negative smooth function \( \rho : X \setminus \{p\} \to \mathbb{R} \) satisfying

\[
\begin{align*}
|\hat{\Delta}(\rho)| &< R \\
\rho(x) &= d(x, p)^{-1} + O(1) \quad (x \in B_\varepsilon(p)).
\end{align*}
\]

Therefore the pullback \( \hat{\rho} := p^*\rho \) is an exhaustion function of \( M' \) and \( |\hat{\Delta}(\hat{\rho})| < R \), and this is the assertion of the assumption (II). We prove the assumption 3. Let \( O(Z) \) be the orbit of \( Z \) with respect to \( \partial_t \)-action. Since \( M \) is compact, \( S^1 \times O(Z) \) is a smooth hypersurface of \( M \). The assumption (III) holds for \( M \setminus (S^1 \times O(Z)) \) by the same argument in \([?\], Proposition 2.2\). Therefore \( M' \) also satisfies the assumption (III) because of the inclusions \( M \setminus (S^1 \times O(Z)) \subset M' \subset M \).
3.1.1 The stability condition for mini-hol. bundles on mini-hol. manifolds

Let $X$ be a compact connected 3-fold with a mini-holomorphic structure $(\partial_t, \alpha, \Sigma, \pi)$ and $Z \subset X$ a finite subset. Let $g_\Sigma$ be a Kähler metric on $\Sigma$ and set $g_X := \alpha^2 + \pi^* g_\Sigma$. We set the orientation of $X$ as $\text{vol}_X = \alpha \wedge \pi^* \text{vol}_\Sigma$.

Let $U_q \subset X$ be a sufficiently small neighborhood of $q \in Z$. Let $U_{q,4} \subset \mathbb{C}^2$ be a neighborhood of $(0,0) \in \mathbb{C}^2$ and $p_q : U_{q,4} \to U_q$ the Hopf-fibration by identifying $U_q$ with a neighborhood of $0 \in \mathbb{R}^3$. We set the holomorphic structure on $U_{q,4}$ by Corollary 2.20. Let $(V, \overline{\partial}_V)$ be a Dirac-type singular mini-holomorphic bundle of rank $r > 0$ on $(X, Z)$ such that each $q \in Z$ is of weight $\vec{k}_q = (k_{q,i}) \in \mathbb{Z}^r$. Let $V_{q,4}$ be the holomorphic bundle on $U_{q,4}$ obtained by applying Proposition 2.23 to $(V, \overline{\partial}_V)|_{U_q}$.

Definition 3.3. (i) A smooth Hermitian metric $h$ on $(V, \overline{\partial}_V)$ is admissible if for any $q \in Z$ the pullback metric $p_q^* h$ can be prolonged to a Hermitian metric of $V_{q,4}$ of $C^1$-class.

(ii) We define the degree $\deg(V, \overline{\partial}_V)$ to be

$$\deg(V, \overline{\partial}_V) := \int_{X \setminus Z} \alpha \wedge c_1(A_h),$$

where $h$ is an admissible Hermitian metric on $(V, \overline{\partial}_V)$.

(iii) We define the slope of $(V, \overline{\partial}_V)$ to be $\mu(V, \overline{\partial}_V) := \deg(V, \overline{\partial}_V)/\text{rank}(V)$.

(iv) A mini-holomorphic bundle $(V, \overline{\partial}_V)$ is said to be stable if $\mu(F, \overline{\partial}_F) < \mu(V, \overline{\partial}_V)$ holds for any proper mini-holomorphic subbundle $(F, \overline{\partial}_F)$ of $(V, \overline{\partial}_V)$. Semistability and polystability of mini-holomorphic bundles are also defined as in the usual case.

Remark 3.4. By Lemma 2.24, the restriction of an admissible metric $h$ to a mini-holomorphic subbundle $(F, \overline{\partial}_F)$ of $(V, \overline{\partial}_V)$ is also admissible.

We show some properties of admissible Hermitian metrics and well-definedness of the degree $\deg(V, \overline{\partial}_V)$.

Proposition 3.5. Any admissible Hermitian metrics on $(V, \overline{\partial}_V)$ are mutually bounded. Conversely, a Hermitian metric $\tilde{h}$ is admissible if the following conditions are satisfied:

- The metric $\tilde{h}$ and an admissible metric $h_0$ are mutually bounded.
• For any \( q \in \mathbb{Z} \) there exists a neighborhood \( U \subset X \) of \( q \) such that the tuple \( (V, h, A_h, \Phi_h)|_{U \setminus \{q\}} \) is a monopole on \( U \setminus \{q\} \).

Proof. By the definition of the admissible metrics, the former claim is trivial. We prove the Converse. Let \( p : U_4 \rightarrow U \) be the Hopf-fibration by identifying \( U \) as a neighborhood of \( 0 \in \mathbb{R}^3 \). The pullback \( p^* \tilde{h} \) is a Hermite-Einstein metric on \( (V_4, \partial V_4)|_{U \setminus \{q\}} \) and \( p^* \tilde{h} \) and \( p^* h_0 \) are mutually bounded. Therefore \( p^* \tilde{h} \) can be prolonged over \( U_4 \) by Lemma \ref{lem:2.31} and hence \( (V_4, \tilde{h}, A_{\tilde{h}}, \Phi_{\tilde{h}})|_{U \setminus \{q\}} \) is a Dirac-type singular monopole on \( (U, \{q\}) \) by Theorem \ref{thm:2.26}.

Proposition 3.6. The degree \( \deg(V, \overline{\partial} V) \) is independent of the choice of admissible connections.

Proof. Let \( h_1 \) and \( h_2 \) be admissible Hermitian metrics on \( (V, \overline{\partial} V) \) and \( (A_i, \Phi_i) \) the CH connections and endomorphisms for \( i = 1, 2 \). Fix \( q \in \mathbb{Z} \) and take a neighborhood \( U \subset X \) of \( q \). As the proof of the last proposition, we take the Hopf fibration \( p : U_4 \rightarrow U \) and the holomorphic bundle \( (V_4, \overline{\partial} V_4) \) on \( U_4 \). Then \( p^* h_i \) are the Hermitian metrics on \( V_4 \) and the upstairs connections \( A_{4,i} = p^* A_i + \xi \otimes p^* \Phi_i \) are the Chern connections of \( p^* h_i \) respectively. Since \( A_{4,i} \) are at least \( C^0 \)-connections on \( V_4 \), we have \( A_{4,1} - A_{4,2} = O(1) \), and hence we obtain an estimate

\[
|A_1 - A_2|, |\Phi_1 - \Phi_2| = O(\text{dist}(\cdot, q)^{-1}).
\]  

Set \( M := S^1_\theta \times X \). Let \( p_M : M \to X \) be the projection and \( \omega_M \) the fundamental form of \( M \). Set \( B_\varepsilon(Z) := \bigsqcup_{p \in Z} B_\varepsilon(p) \) for \( \varepsilon > 0 \). Then by using the flat lift \( (\tilde{V}, \overline{\partial} \tilde{V}) \) of \( (V, \overline{\partial} V) \) we can write

\[
\int_{X \setminus Z} \alpha \wedge (\text{Tr}(F(A_h_1)) - \text{Tr}(F(A_h_2)))
\]

\[
= (2\pi)^{-1} \int_{M \setminus (S^1 \times Z)} \omega_M \wedge \text{Tr} \left( F(A_{p_M}^* h_1) - F(A_{p_M}^* h_2) \right)
\]

\[
= (2\pi)^{-1} \lim_{\varepsilon \to +0} \int_{M \setminus (S^1 \times B_\varepsilon(Z))} \omega_M \wedge \text{Tr} \left( F(A_{p_M}^* h_1) - F(A_{p_M}^* h_2) \right).
\]
Set $\eta := \det(h_1 h_2^{-1})$. Then we have

$$(2\pi)^{-1} \int_{M \setminus (S^1 \times B(\varepsilon)(Z))} \omega_M \wedge \text{Tr} \left( F(A_{p_M}^* h_1) - F(A_{p_M}^* h_2) \right)$$

$$= (2\pi)^{-1} \int_{M \setminus (S^1 \times B(\varepsilon)(Z))} \omega_M \wedge \partial \partial p_M^* \eta$$

$$= (2\pi)^{-1} \int_{M \setminus (S^1 \times B(\varepsilon)(Z))} d\{ \partial (p_M^* \eta \cdot \omega_M) - p_M^* \eta (\partial \omega_M - \partial \omega_M) \}$$

$$= (2\pi)^{-1} \int_{S^1 \times \partial B(\varepsilon)(Z)} \partial (p_M^* \eta \cdot \omega_M) - p_M^* \eta (\partial \omega_M - \partial \omega_M)$$

$$= (2\pi)^{-1} \int_{S^1 \times \partial B(\varepsilon)(Z)} \{ \text{Tr}(A_{p_M} h_1 - A_{p_M} h_2) \wedge \omega_M + p_M^* \eta \cdot \partial \omega_M \} + O(\varepsilon^2)$$

$$= O(\varepsilon) \quad (\because (1)).$$

Taking the limit $\varepsilon \to +0$ we obtain

$$\int_{X \setminus Z} \alpha \wedge \text{Tr}(F(h_1)) = \int_{X \setminus Z} \alpha \wedge \text{Tr}(F(h_2)),$$

which proves the uniqueness.

Polystability of the underlying mini-holomorphic bundle of a Dirac-type HE-monopole easily follows from the Gauss-Codazzi formula as in the ordinary Kobayashi-Hitchin correspondence. We prove the converse.

**Theorem 3.7.** If $(V, \overline{\partial} V)$ is stable, then there exists an admissible BHE-metric $h$ on $(V, \overline{\partial} V)$.

**Proof.** We take an admissible metric $h_0$ on $(V, \overline{\partial} V)$. For a Dirac type singular mini-holomorphic bundle $(E, \overline{\partial} E)$ on $(X, Z)$ and an admissible Hermitian metric $h_E$ on $E$, we have

$$\deg(E, \overline{\partial} E) = (2\pi)^{-1} \int_{M \setminus (S^1 \times Z)} \omega_M \wedge c_1(A_{p}^* h_E) = (2\pi)^{-1} \deg(\tilde{E}, p^* h_E),$$

where $(\tilde{E}, \overline{\partial} \tilde{E})$ is the flat lift of $(E, \overline{\partial} E)$. Therefore the slope inequality $\mu(\tilde{V}, p^* h_0) > \mu(\tilde{F}, p^* (h_0|_F))$ holds for any proper mini-holomorphic subbundle $F$ of $V$. Since an $S^1$-invariant saturated subsheaf $\mathcal{F}$ of $\tilde{V}$ is an $S^1$-invariant holomorphic subbundle, there exists a mini-holomorphic subbundle $(F, \overline{\partial} F)$ such that it satisfies $\mathcal{F} = \tilde{F}$, where $(\tilde{F}, \overline{\partial} \tilde{F})$ is the flat lift of $(F, \overline{\partial} F)$. Hence the slope inequality $\mu(\tilde{V}, p^* h_0) > \mu(\mathcal{F}, p^* h_0|_F)$ also holds.

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for any proper saturated $S^1$-invariant subsheaf $F$. By [11, Theorem 1.1] and Proposition 3.2 there exists a HE-metric $\hat{h}$ of $\hat{V}$ such that $\hat{h}$ and $p^*h_0$ is mutually bounded. Let $h$ be the descent of $\hat{h}$. Then $h$ is BHE-metric and mutually bounded to $h_0$. Therefore $h$ is an admissible BHE-metric by Proposition 3.5 which proves the theorem.

**Remark 3.8.** Indeed group actions are not considered in [11, Theorem 1.1], however the proof of Theorem 1.1 remains valid for the case that $V$ has an action by a group $G$ and satisfies the slope inequality for any $G$-invariant saturated subsheaves.

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