Lowest-energy representations of non-centrally extended diffeomorphism algebras

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Abstract

We describe a class of non-central extensions of the diffeomorphism algebra in \(N\)-dimensional spacetime, and construct lowest-energy modules thereof, thus generalizing work of Eswara-Rao and Moody. There is one representation for each representation of \(Vir \ltimes \hat{gl}(N)\) (an extension of the semi-direct product). Similar modules are constructed for gauge algebras.

1 Introduction

Let \(\text{diff}(N)\) denote the diffeomorphism algebra in \(N\)-dimensional spacetime. In a significant paper, Eswara-Rao and Moody constructed the first interesting lowest-energy representations of a non-central extension thereof \(\hat{N}\), but they failed to explicitly describe the extension (except for the “spatial” subalgebra generated by time-independent vector fields). In the present paper, a four-parameter non-central extension \(\hat{\text{diff}}(N; c_1, c_2, c_3, c_4)\) is described explicitly, and a realization of this algebra is constructed for each
representation of $Vir\tilde{\otimes}h_{k_0}gl(N)$ (an extension of $Vir \times \hat{gl}(N)$). The representations in [3] are recovered (in a Fourier basis) by picking a particular vertex operator module for $Vir$ and the trivial module for $\hat{gl}(N)$. Thus, my results are related to theirs in about the same way as tensor densities are related to functions. Similar results also hold for the algebra of gauge transformations on spacetime; special cases were previously found by [4, 6, 10]. A supersymmetric generalization of the present work can be found on the web [9].

After this work was completed I became aware of related references [1, 2].

2 Extension

Let $\xi = \xi^\mu \partial_\mu$ be a vector field and $L_\xi$ the Lie derivative. Greek indices $\mu, \nu = 0, 1, \ldots, N - 1$ label spacetime coordinates and the summation convention is used. The diffeomorphism algebra (algebra of vector fields, Witt algebra) $diff(N)$ is generated by Lie derivatives satisfying $[L_\xi, L_\eta] = L_{[\xi,\eta]}$. Define two families of operators $S^{\nu_1,\ldots,\nu_n}$ and $R^{\rho}_{\nu_1,\ldots,\nu_n}$, where $g^{\nu_1,\ldots,\nu_n}$ and $h^{\rho}_{\nu_1,\ldots,\nu_n}$ are arbitrary functions on spacetime. The operators are linear in the arguments and totally symmetric in the indices $\nu_1,\ldots,\nu_n$. The following relations (2.1–2.3) define a Lie algebra extension of $\hat{diff}(N)$, denoted by $\hat{diff}(N; c_1, c_2, c_3, c_4)$.

$$[L_\xi, S^{\nu_1,\ldots,\nu_n}](g^{\nu_1,\ldots,\nu_n}) = S^{\nu_1,\ldots,\nu_n}(\xi^\mu \partial_\mu g^{\nu_1,\ldots,\nu_n} + \sum_{j=1}^{n} \partial_{\nu_j} \xi^\mu g^{\nu_1,\ldots,\mu,\nu_n})$$

$$-(n - 1)S^{\mu_1,\ldots,\nu_n}_{n+1}(\partial_\mu \xi^0 g^{\nu_1,\ldots,\nu_n})$$

$$S^0(\partial_\nu f) \equiv 0,$$

$$S^0_{n+1}(g^{\nu_1,\ldots,\nu_n}) = S^{\nu_1,\ldots,\nu_n}(g^{\nu_1,\ldots,\nu_n}),$$

$$S_0(f) = \frac{1}{2\pi i} \int dt f(t), \quad \text{if } f(t) \text{ depends on time only},$$

$$[L_\xi, R^{\rho}_{\nu_1,\ldots,\nu_n}](h^{\rho}_{\nu_1,\ldots,\nu_n}) = R^{\rho}_{\nu_1,\ldots,\nu_n}(\xi^\mu \partial_\mu h^{\rho}_{\nu_1,\ldots,\nu_n} + \partial_\rho \xi^0 h^{\mu}_{\nu_1,\ldots,\nu_n}$$

$$+ \sum_{j=1}^{n} \partial_{\nu_j} \xi^\mu h^{\rho}_{\nu_1,\ldots,\mu,\nu_n})$$

$$-(n + 1)R^{\rho}_{n+1,\ldots,\nu_n}(\partial_\mu \xi^0 h^{\rho}_{\nu_1,\ldots,\nu_n}) - R^{\rho}_{n+1,\ldots,\nu_n}(\partial_\rho \xi^0 h^{\mu}_{\nu_1,\ldots,\nu_n})$$

$$+ S^{\rho \mu_1,\ldots,\nu_n}_{n+2}(\partial_\rho \partial_\mu \xi^0 h^{\mu}_{\nu_1,\ldots,\nu_n}) - S^{\rho \mu_1,\ldots,\nu_n}_{n+3}(\partial_\rho \partial_\sigma \xi^0 h^{\mu}_{\nu_1,\ldots,\nu_n}),$$

(2.2)
Verification of all Jacobi identities is straightforward; that these equations define a Lie algebra also follows from the explicit realization in theorem 3.1 below. Indeed, the extensions were discovered by working out which algebra is generated by (3.6). The extensions $a_1 - a_3$ (cocycles are labelled by the factors multiplying them) are cohomologically trivial and may be eliminated by the redefinition

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi, \eta]} + c_1 S_1^\rho (\partial_\rho \partial_\sigma \xi^\mu \partial_\mu \eta^\nu) + c_2 S_1^\rho (\partial_\rho \partial_\sigma \xi^0 \partial_\mu \eta^0) + c_3 (R_1^\rho \mu (\partial_\rho \partial_\sigma \xi^0 \partial_\mu \eta^0) + S_3^\rho (\partial_\rho \partial_\sigma \xi^0 \partial_\mu \eta^0)) + \frac{c_4}{2} S_2^\rho^\sigma (\partial_\rho \eta^0 \partial_\sigma \xi^\mu - \partial_\rho \xi^0 \partial_\sigma \eta^\nu) + a_1 (S_1^\rho (\partial_\rho \partial_\sigma \xi^0 \partial_\mu \eta^0) - \partial_\rho \partial_\sigma \eta^0 \partial_\mu \xi^0) - S_3^\rho (\partial_\rho \partial_\sigma \xi^0 \partial_\mu \eta^0) - \partial_\rho \partial_\sigma \eta^0 \partial_\mu \xi^0) - a_2 S_1^\rho (\partial_\rho \xi^0 \eta^0) + a_3 S_1^\rho (\partial_\rho \eta^0 \partial_\mu \xi^0 - \partial_\rho \xi^0 \partial_\mu \eta^0),$$

$$[S_m^{\mu_1 ... \mu_m} (g_{\mu_1 ... \mu_m}), S_n^{\nu_1 ... \nu_n} (h_{\nu_1 ... \nu_n})] = [S_m^{\mu_1 ... \mu_m} (g_{\mu_1 ... \mu_m}), R_n^{\rho \nu_1 ... \nu_n} (h_{\rho \nu_1 ... \nu_n})] = [R_n^{\rho \nu_1 ... \nu_n} (h_{\rho \nu_1 ... \nu_n}), R_m^{\rho \mu_1 ... \mu_m} (h_{\rho \mu_1 ... \mu_m})] = 0.$$
\[ C^{\mu}(j_{\mu
u}) = -C^{\mu
u}(j_{\mu
u}) \]  \hspace{1cm} (2.5) \\
\[ [\mathcal{L}_\xi, C^{\nu\rho}(j_{\mu\nu})] = C^{\nu\rho}(\xi^\mu \partial_\mu j_{\rho\nu} + \partial_\nu \xi^\mu j_{\mu\rho} + \partial_\rho \xi^\mu j_{\nu\mu}). \]

Dzhumadil’daev \[3\] has given a list of \textit{diff}(N) extensions by irreducible modules, but it seems that the extension \( c_1 \) is missing; however, it is essentially \( \psi_1^W - \psi_3^W + \psi_4^W \) in his notation. Moreover, \( c_2 \) is \( \psi_1^W \) and \( c_1 \) and \( c_2 \) become \( \psi_4^W \) and \( \psi_3^W \) upon the substitution (2.5), respectively. The remaining two cocycles are not covered by his theorem, however, because they are extensions by reducible but indecomposable modules.

### 3 Realization

Consider the Heisenberg algebra generated by operators \( q^i(s), p_j(t) \), \( s, t \in S^1 \), where latin indices \( i, j = 1, \ldots, N - 1 \) run over spatial coordinates only.

\[
[p_j(s), q^i(t)] = \delta_j^i \delta(s - t), \\
[p_i(s), p_j(t)] = [q^i(s), q^j(t)] = 0. \hspace{1cm} (3.1)
\]

These operators can be expanded in a Fourier series; e.g.,

\[
p_j(t) = \sum_{n=-\infty}^{\infty} \hat{p}_j(n)e^{int}. \hspace{1cm} (3.2)
\]

This algebra has a Fock module \( \mathcal{F} \) (\( \mathbb{Z} \)-graded by the frequency \( n \)) generated by finite strings in the non-negative Fourier modes of \( q^i(t) \) and the positive modes of \( p_j(t) \). Define time components by \( q^0(t) = t \) and \( p_0(t) = -\dot{q}^i(t)p_i(t) \); in an obvious notation, \( q^\mu(t) = (t, q^i(t)) \), etc. The following relations hold.

\[
[q^\mu(s), q^\nu(t)] = 0, \\
[p_\mu(s), q^\nu(t)] = (\delta_\mu^\nu - \dot{q}^\nu(s)\delta_\nu^0)\delta(s - t), \\
[p_\mu(s), p_\nu(t)] = (\delta_\mu^0 p_\nu(s) + \delta_\nu^0 p_\mu(t))\delta(s - t). \hspace{1cm} (3.3)
\]

Normal ordering is necessary to remove infinities and to obtain a well defined action of diffeomorphisms on \( \mathcal{F} \). For any function of \( q(t) \) and its derivatives, let

\[
:f(q(t), \dot{q}(t))p_j(t): \equiv f(q(t), \dot{q}(t))p_j^<(t) + p_j^>(t)f(q(t), \dot{q}(t)), \hspace{1cm} (3.4)
\]
where \( p_>(^\tilde{\gamma}(t) \ (p_<(^\tilde{\gamma}(t)) \) is the sum over positive (non-positive) Fourier modes only. Let \( L(s) \) and \( T_\mu^\nu(t) \) generate the following algebra \( Vir_c\tilde{\kappa}_k \overline{gl(N)}_{k_1,k_2} \):

\[
[L(s),L(t)] = (L(s)+L(t))\delta(s-t)+\frac{c}{24\pi i}(\dot{\delta}(s-t)+\delta(s-t)),
\]

\[
[L(s),T_\mu^\nu(t)] = T_\mu^\nu(s)\delta(s-t)+\frac{k_0}{4\pi i}\delta_\mu^\nu\delta(s-t),
\]

\[
[T_\mu^\nu(s),T_\sigma^\tau(t)] = (\delta_\nu^\sigma T_\mu^\tau(s)-\delta_\mu^\tau T_\nu^\sigma(s))\delta(s-t)
\]

\[
-\frac{1}{2\pi i}(k_1\delta_\nu^\sigma\delta_\mu^\tau+k_2\delta_\mu^\nu\delta_\tau^\sigma)\delta(s-t)
\]

**Theorem 3.1** Under the conditions above, the following expressions

\[
\mathcal{L}_\xi = \int dt :\xi^\mu(q(t))p_\mu(t):+\xi^0(q(t))L(t)+\partial_\nu\xi^\mu(q(t))T_\mu^\nu(t)
\]

\[
=\int dt :\xi^i(q(t))p_i(t):-:\xi^0(q(t))\dot{q}^i(t)p_i(t):
\]

\[
+\xi^0(q(t))L(t)+\partial_\nu\xi^\mu(q(t))T_\mu^\nu(t),
\]

\[
S_\mu^\nu_1...^\nu_n(g_{\nu_1...\nu_n}) = \frac{1}{2\pi i} \int dt \ \dot{q}^\nu_1(t)...\dot{q}^\nu_n(t)g_{\nu_1...\nu_n}(q(t)),
\]

\[
R_\mu^\nu_1...^\nu_n(h_{\rho|\nu_1...\nu_n}) = \frac{1}{2\pi i} \int dt \ \dot{q}^\rho(t)\dot{q}^\nu_1(t)...\dot{q}^\nu_n(t)h_{\rho|\nu_1...\nu_n}(q(t)),
\]

realize the Lie algebra \( \overline{\text{diff}}(N;1+k_1,k_2,-2+(c+2N-2)/12,1+k_0) \), while the cohomologically trivial parameters are \( a_1 = -1, a_2 = (c+2N-2)/12, a_3 = i/2 \).

The proof is deferred to the appendix. Consequently, this algebra acts on \( \mathcal{F} \otimes \mathcal{M} \) for every \( Vir_c\tilde{\kappa}_k \overline{gl(N)}_{k_1,k_2} \) module \( \mathcal{M} \). It should be stressed that this action is manifestly well defined, at least for the subalgebra of vector fields that are polynomial in the spatial coordinates and a Fourier polynomial in \( x^0 \), because finiteness is preserved when all operators in \( \{3.4\} \) act on finite strings in non-negative Fourier modes in that case. The Hamiltonian

\[
\mathcal{L}_{-i\partial_0} = -i \int dt (-:\dot{q}^i(t)p_i(t):+L(t))
\]
is the operator responsible for computing the $\mathbb{Z}$-grading.

In the absence of normal ordering and central charges in (3.5), (3.6) yields a proper realization of $diff(N)$. The higher-dimensional analogue of a primary field depends on five parameters $\lambda$, $w$ (defined up to an integer), $\kappa$, $p$, and $q$:

$$
[L_\xi, \phi^{\sigma_1, \sigma_p}(t)] = -\xi^0(q(t))\phi^{\sigma_1, \sigma_p}(t) - \lambda\xi^0(q(t))\phi^{\sigma_1, \sigma_p}(t)
$$

$$
+ i\omega\xi^0(q(t))\phi^{\sigma_1, \sigma_p}(t) - \kappa \partial_\mu \xi^\mu(q(t))\phi^{\sigma_1, \sigma_p}(t)
$$

$$
+ \sum_{i=1}^p \partial_\mu \xi^{\sigma_i}(t)\phi^{\sigma_1, \mu, \sigma_p}(t) - \sum_{j=1}^q \partial_{\tau_j} \xi^\mu(q(t))\phi^{\sigma_1, \sigma_p}(t),
$$

where $[L_\xi, q^\mu(t)] = \lambda^\mu(q(t)) - \dot{q}^\mu(t)\xi^0(q(t))$.

The result of Eswara-Rao and Moody 3 is recovered as follows: they work in a Fourier basis on the torus, and denote $q^i(t) = \delta_i(z)$ and $p_j(t) = d_j(z)$, where $z = \exp(it)$. A standard vertex operator realization for the Virasoro generator $L(t)$ was given, based on the remaining roots $\alpha_p$, but they missed the appearance of $gl(N)$. Consequently, $T_\mu^\rho(t) = 0$ and $k_0 = k_1 = k_2 = 0$ in their work.

4 Gauge algebras

Consider the gauge algebra $map(N, \mathfrak{g})$, i.e. maps from $N$-dimensional space-time to a finite-dimensional Lie algebra $\mathfrak{g}$, where $\mathfrak{g}$ has basis $J^a$, structure constants $f^{ab}_{\ c}$, and Killing metric $\delta^{ab}$. Define constants $g^a$ and $g^a$ satisfying $f^{ab}_{\ cd} = f^{ab}_{\ cd}$. Clearly, $g^a = g^a = 0$ if $J^a \in [\mathfrak{g}, \mathfrak{g}]$, but they may be non-zero on abelian factors. Let $X = X_a(x)J^a$, $x \in \mathbb{R}^N$ be a $\mathfrak{g}$-valued function and define $[X, Y]_c = if^{ab}_{\ c}X_aY_b$. $diff(N) \ltimes map(N, \mathfrak{g})$ has the non-central extension $\tilde{diff}(N; c_1, c_2, c_3, c_4) \ltimes g^a \tilde{map}(N; g^a)\tilde{map}(N; g^a \tilde{map}(N; g^a)$, with brackets

$$
[J_X, J_Y] = J_{[X,Y]} - k\delta^{ab}S^a_1(\partial_\rho X_a Y_b),
$$

$$
[L_\xi, J_X] = L_{\xi^0}X - g^a S^a_2(\partial_\rho \xi^0 \partial_\rho X_a) - g^a S^a_1(\partial_\rho \partial_\mu \xi^\mu X_a),
$$

$$
[J_X, S^a_{n_{1\ldots n}}(\phi_{\nu_{1\ldots \nu}})] = [J_X, R_{\nu_1\ldots \nu}^a(h_{\rho|\nu_{1\ldots \nu}})] = 0,
$$

in addition to (2.3). Let $J^a(t)$, $t \in S^1$, generate the Kac-Moody algebra $\tilde{\mathfrak{g}}$. Consider the algebra $Vir_{r \kappa, \tilde{\mathfrak{g}}}(\text{gl}(N))_{k_1, k_2} \oplus \tilde{\mathfrak{g}}$, with brackets (3.5) and

$$
[J^a(s), J^b(t)] = if^{ab}_{\ c}J^c(s)\delta(s - t) + \frac{k}{2\pi i} \delta^{ab} \delta(s - t),
$$

$$
[L_\xi, J^a(t)] = \lambda^a(t)\dot{J}^a(t) - \dot{\lambda} J^a(t) - \kappa \partial_\mu \lambda_\mu(t)J^a(t) + \sum_{i=1}^p \partial_\mu \lambda^{\sigma_i}(t)J^{\mu, \sigma_p}(t) - \sum_{j=1}^q \partial_{\tau_j} \lambda^\mu(t)J^{\sigma_1, \sigma_p}(t),
$$

where $[L_\xi, J^\mu(t)] = \lambda^\mu(t) - \dot{\lambda}^\mu(t)$. Consequently, $T_\mu^\rho(t) = 0$ and $k_0 = k_1 = k_2 = 0$ in their work.
\[ [T^\mu_\nu(s), J^a(t)] = \frac{g^a}{2\pi i} \delta^\mu_\nu \delta(s - t), \quad (4.2) \]
\[ [L(s), J^a(t)] = J^a(s) \delta(s - t) + \frac{g^a}{2\pi i} \delta(s - t). \]

Then
\[ J_X = \int dt \ X_a(q(t)) J^a(t) \quad (4.3) \]
yields a realization of \( \tilde{\text{map}}(N, g; k) \), with the intertwining action of \( \tilde{\text{diff}}(N; c_1, c_2, c_3, c_4) \) described above, and the parameters \( k, g^a \) and \( g'^a \) in (4.1) and (4.2) agree.

\section*{A \ Proof of theorem 3.1}

We first prove that in absence of normal ordering, (3.6) defines a proper realization of \( \text{diff}(N) \). The operators \( \tilde{p}_\nu(t) = p_\nu(t) + \delta^0_\nu L(t) \) satisfy relations (3.3) and also
\[ [\tilde{p}_\mu(s), T^\nu_\sigma(t)] = \delta^\nu_\mu \delta^0_\sigma(s) \delta(s - t). \quad (A.1) \]

Introduce the abbreviated notation \( \xi^\mu(t) \equiv \xi^\mu(q(t)) \). Now,
\[ [\mathcal{L}_\xi, \mathcal{L}_\eta] = \iint dsdt \ [\xi^\mu(s) \tilde{p}_\mu(s) + \partial_\sigma \xi^\mu(s) T^\sigma_\nu(s), \eta^\nu(t) \tilde{p}_\nu(t) + \partial_\tau \eta^\nu(t) T^\nu_\sigma(t) ] \]
\[ = \iint dsdt \ \xi^\mu(s) \left\{ \partial_\rho \eta^\nu(t) (\delta^\rho_\mu - \delta^0_\mu \tilde{q}^\rho(s)) \delta(s - t) \tilde{p}_\nu(t) 
+ \eta^\nu(t) \delta^0_\rho \tilde{p}_\nu(s) \delta(s - t) 
\right\} \quad (A.2) \]
\[ + \xi^\mu(s) \left\{ \partial_\rho \partial_\tau \eta^\nu(t) (\delta^\rho_\mu - \delta^0_\mu \tilde{q}^\rho(s)) \delta(s - t) T^\nu_\sigma(t) 
+ \partial_\tau \eta^\nu(t) \delta^0_\rho T^\nu_\sigma(s) \delta(s - t) - \xi \leftrightarrow \eta, \right. \]
where \( \xi \leftrightarrow \eta \) stands for the same expression with \( \xi \) and \( \eta \) interchanged everywhere. Rewrite the terms proportional to the derivative of the delta function by noting that
\[ \iint dsdt \ f(s) g(t) \delta(s - t) = \int f \dot{g} = - \int \dot{f} g. \quad (A.3) \]
The function arguments were suppressed in the single integrals, because no confusion is possible. This leaves us with

\[
\int \xi^\mu (\partial_\mu \eta^\nu - \delta_\mu^0 \delta^\nu_0) \tilde{p}_\nu + \xi^\mu \eta^\nu \delta_\mu^0 \tilde{p}_\nu \\
+ \xi^\mu \partial_\upsilon \eta^\nu (\delta_\mu^0 - \delta_\mu^0 \delta^\upsilon_0) T^\upsilon_\nu + \xi^\mu \partial_\tau \eta^\nu \delta_\mu^0 T^\sigma_\nu + \partial_\upsilon \xi^\mu \partial_\mu \eta^\nu T^\upsilon_\nu - \xi \leftrightarrow \eta
\]

\[
= \int \xi^\mu \partial_\nu \eta^\nu \tilde{p}_\upsilon + \xi^\mu \partial_\upsilon \eta^\nu T^\upsilon_\nu + \partial_\upsilon \xi^\mu \partial_\mu \eta^\nu T^\upsilon_\nu - \xi \leftrightarrow \eta \tag{A.4}
\]

where we used that \( \partial_\nu \eta^\nu = \dot{q}^\rho \partial_\rho \eta^\nu \). Hence \([\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi, \eta]}\), and it is clear that normal ordering must result in some abelian extension of \( diff(N) \). We now proceed to calculate it.

Split the delta function into positive and negative energy parts.

\[
\delta^>(t) = \frac{1}{2\pi} \sum_{m>0} e^{-imt}, \quad \delta^<(t) = \frac{1}{2\pi} \sum_{m\leq 0} e^{-imt}. \tag{A.5}
\]

Lemma A.1

i. \( \delta^>(t) \delta^<(t) - \delta^>(-t) \delta^<(t) = -\frac{1}{2\pi i} \dot{\delta}(t) \)

ii. \( \delta^>(t) \delta^<(t) - \delta^>(-t) \delta^<(t) = \frac{1}{4\pi i} (\dot{\delta}(t) + i\dot{\delta}(t)) \)

iii. \( \dot{\delta}^>(t) \delta^<(t) - \dot{\delta}^>(-t) \delta^<(t) = \frac{1}{12\pi i} (\ddot{\delta}(t) + \dot{\delta}(t)) \)

Proof:

i. \( 4\pi^2 \cdot LHS = \sum_{m>0} \sum_{n\leq 0} (e^{-i(m-n)t} - e^{i(m-n)t}) = \sum_{k>0} \sum_{m=1}^k (e^{-ikt} - e^{ikt}) \)

\[
= \sum_{k>0} k(e^{-ikt} - e^{ikt}) = \sum_{k} ke^{-ikt} = 2\pi i \dot{\delta}(t) \quad \text{where} \quad k = m - n.
\]

ii. \( 4\pi^2 i \cdot LHS = \sum_{m>0} \sum_{n\leq 0} (me^{-i(m-n)t} - me^{i(m-n)t}) \)

\[
= \sum_{k>0} \sum_{m=1}^k (m-k)e^{-ikt} - me^{ikt} = \sum_{k>0} \frac{k(k-1)}{2} e^{-ikt} - \frac{k(k+1)}{2} e^{ikt}
\]
whereas (A.9) becomes

We use that

**Proof:**

Define

\[ \tilde{\xi}^i(t) \equiv \dot{\xi}^i(q(t), \dot{q}(t)) = \xi^i(q(t)) - \xi^0(q(t))\dot{q}^i(t), \]

\[ \chi^{ij}_{\xi j}(t, s) \equiv [p^j_\pi(t), \tilde{\xi}^i(s)] = \partial_i \tilde{\xi}^j(s)\delta^>(t - s) + \delta^j \xi^0(s)\delta^>(t - s), \]

and \( \chi^i_{\xi j}(t, s) \) analogously. Moreover, set \( \chi^{ij}_{\xi j}(t, s) = \chi^{zi}_{\xi j}(t, s) + \chi^{zi}_{\xi j}(t, s). \)

**Lemma A.2** The expressions defined in (A.6) satisfy the following relations.

\[
\begin{align*}
\partial_i \tilde{\xi}^i &= \partial_\mu \xi^\mu - \xi^0 \\
\partial_j \tilde{\xi}^i \partial_i \xi^j &= \partial_\rho \dot{\xi}^\mu \partial_\mu \eta^\nu + \partial_\nu \xi^0 \eta^\nu \partial_\rho \dot{\xi}^\mu \partial_\mu \eta^0 - \xi^0 \dot{\eta}^0 \\
&- \xi^0 \dot{\eta}^0 \partial_\rho \eta^0 + \dot{\eta}^0 \partial_\rho \xi^0 \eta^0 + \frac{d}{dt}(\dot{\xi}^0 \eta^0 - \partial_\rho \xi^0 \dot{\eta}^0). 
\end{align*}
\]

**Proof:** We use that \( \tilde{\xi}^0 = 0 \). Eq. (A.8) thus equals

\[
\partial_\mu \tilde{\xi}^\mu = \partial_\mu \xi^\mu - \partial_\mu \xi^0 \dot{q}^\mu, \quad (A.10)
\]

whereas (A.9) becomes

\[
\begin{align*}
\partial_\mu \tilde{\xi}^\mu \partial_\mu \eta^\nu &= (\partial_\nu \tilde{\xi}^\mu - \partial_\nu \xi^0 \dot{q}^\mu - \partial_\nu \xi^0 \dot{q}^\mu)(\partial_\mu \eta^\nu - \partial_\mu \eta^0 \dot{q}^\nu) \\
&= \partial_\mu \tilde{\xi}^\mu \partial_\mu \eta^\nu - \partial_\nu \xi^0 (\dot{q}^\mu \partial_\mu \eta^\nu - \dot{q}^\mu \partial_\mu \eta^0) - \partial_\nu \xi^0 \dot{q}^\nu \\
&- \dot{q}^\mu \partial_\mu \eta^0 + \dot{\xi}^0 (\dot{\eta}^0 - \dot{q}^0 \partial_\rho \eta^0) + \dot{q}^\mu \partial_\mu \xi^0 \eta^0. \quad \Box \quad (A.11)
\end{align*}
\]
Consider
\[ \mathcal{L}_\xi^0 = \int dt : \xi^\mu(q(t))p_\mu(t) : \equiv \int dt \ (\tilde{\xi}(t)p_\xi^\leq(t) + p_\xi^\geq(t)\tilde{\xi}(t)). \] (A.12)

\[ [\mathcal{L}_\xi^0, \mathcal{L}_\eta^0] = \int ds dt \ [\tilde{\xi}(s)p_\xi^\leq(t) + p_\xi^\geq(t)\tilde{\eta}(t)] \]
\[ = \int ds dt \left\{ \tilde{\xi}(s)\chi_{\eta i}^{\leq j}(s, t)p_\xi^\leq(t) - \tilde{\eta}(t)\chi_{\xi j}^{\leq i}(t, s)p_\eta^\leq(s) + \tilde{\xi}(s)p_\xi^\geq(t)\chi_{\eta j}^{\leq i}(s, t) - \tilde{\eta}(t)\chi_{\xi i}^{\leq j}(t, s) \right\}. \] (A.13)

Of these eight terms, the third can be rewritten as
\[ p_\xi^\geq(t)\tilde{\xi}(s)\chi_{\eta i}^{\leq j}(s, t) - \chi_{\xi j}^{\geq i}(t, s)\chi_{\eta i}^{\leq j}(s, t) \] (A.14)
and the fifth as
\[ \chi_{\eta i}^{\geq j}(s, t)\tilde{\eta}(t) + \chi_{\eta i}^{\leq j}(s, t)\chi_{\xi j}^{\leq i}(t, s). \] (A.15)

Hence
\[ [\mathcal{L}_\xi^0, \mathcal{L}_\eta^0] = \int ds dt \left\{ \tilde{\xi}(s)\chi_{\eta i}^{\leq j}(s, t)p_\xi^\leq(t) - \tilde{\eta}(t)\chi_{\xi j}^{\leq i}(t, s)p_\eta^\leq(s) \right. \]
\[ + p_\xi^\geq(t)\tilde{\xi}(s)\chi_{\eta i}^{\leq j}(s, t) - \tilde{\eta}(t)\chi_{\xi j}^{\leq i}(t, s)p_\eta^\leq(s) \]
\[ + \tilde{\xi}(s)\chi_{\eta i}^{\geq j}(s, t)p_\xi^\geq(t) - p_\xi^\leq(t)\tilde{\eta}(t)\chi_{\xi j}^{\leq i}(t, s) \]
\[ - p_\xi^\leq(s)\tilde{\eta}(t)\chi_{\xi j}^{\leq i}(t, s) + p_\xi^\geq(t)\tilde{\xi}(s)\chi_{\eta i}^{\leq j}(s, t) \]
\[ - \chi_{\xi j}^{\geq i}(t, s)\chi_{\eta i}^{\leq j}(s, t) + \chi_{\eta i}^{\geq j}(s, t)\chi_{\xi j}^{\leq i}(t, s) \}. \] (A.16)

The regular piece is
\[ \int ds dt \tilde{\xi}(s)\chi_{\eta i}^{\leq i}(s, t)p_\xi^\leq(t) + p_\xi^\geq(t)\tilde{\xi}(s)\chi_{\eta i}^{\geq i}(s, t) - \xi \leftrightarrow \eta. \] (A.17)
We focus on the first term.

\[
\int dsdt \, \tilde{\xi}^i(s) \chi_{\eta j}^{\tilde{j}}(s,t) p_j^\xi(t) - \xi \leftrightarrow \eta
\]

\[
= \int dsdt \, \tilde{\xi}^\mu(s) (\partial_\mu \tilde{\eta}^j(t) \delta(s-t) + \eta^0(t) \delta^i_\mu \delta(s-t)) p_j^\xi(t) - \xi \leftrightarrow \eta
\]

\[
= \int \left\{ (\xi^\mu \partial_\mu \eta^j - \xi^0 (\eta^j - \eta^0 \dot{\eta}^j)) - \tilde{\dot{\xi}}^j \right\} p_j^\xi - \xi \leftrightarrow \eta,
\]

which equals \( L_{[\xi,\eta]}^0 \). We again suppress the integration variable in single integrals, and write \( \xi^\mu(s) = \xi^\mu(q(s)) \), etc. The extension \( \text{ext}_0(\xi,\eta) \) becomes

\[
\int dsdt \left\{ -\chi_{\xi j}^{\tilde{j}}(t,s) \chi_{\eta i}^{\tilde{i}}(s,t) + \chi_{\eta i}^{\tilde{i}}(s,t) \chi_{\xi j}^{\tilde{j}}(t,s) \right\}
\]

\[
= -\int dsdt \left\{ (\partial_\xi \tilde{\xi}^i(s) \delta^\gamma(t-s) + \delta_i^j \xi^\gamma(s) \delta^\gamma(t-s)) \times \right.
\]

\[
\times (\partial_\eta \tilde{\eta}^j(t) \delta^\gamma(s-t) + \delta_i^j \eta^\gamma(t) \delta^\gamma(s-t)) \left\} - \xi \leftrightarrow \eta \right.
\]

\[
= -\int dsdt \left\{ \partial_\xi \tilde{\xi}^i(s) \partial_\eta \tilde{\eta}^j(t) \delta^\gamma(t-s) \delta^\gamma(s-t)
\]

\[
+ \xi^0(s) \partial_\xi \tilde{\eta}^j(t) \delta^\gamma(t-s) \delta^\gamma(s-t)
\]

\[
+ \partial_\xi \tilde{\xi}^i(s) \eta^0(t) \delta^\gamma(s-t) \delta^\gamma(s-t)
\]

\[
+ \delta^i_\xi \xi^0(s) \eta^0(t) \delta^\gamma(t-s) \delta^\gamma(s-t) \left\} - \xi \leftrightarrow \eta \right.
\]

\[
= \frac{1}{2\pi i} \int dsdt \left\{ \partial_\xi \tilde{\xi}^i(s) \partial_\eta \tilde{\eta}^j(t) \delta(t-s)
\]

\[
+ \frac{1}{2} \xi^0(s) \partial_\xi \tilde{\eta}^j(t) (\delta(t-s) - i \delta(t-s))
\]

\[
- \frac{1}{2} \partial_\xi \tilde{\xi}^i(s) \eta^0(t) (\delta(t-s) + i \delta(t-s))
\]

\[
- \frac{N}{6} \xi^0(s) \eta^0(t) (\delta(t-s) + \delta(t-s)) \right\}
\]

\[
= \frac{1}{2\pi i} \int \left\{ \partial_\xi \tilde{\xi}^i \partial_\eta \tilde{\eta}^j - \frac{1}{2} \xi^0 \partial_\xi \tilde{\eta}^j + \frac{1}{2} \partial_\xi \tilde{\xi}^i \eta^0
\]

\[
- \frac{N}{6} (-\xi^0 \eta^0 + \xi^0 \eta^0) + \frac{i}{2} (-\xi^0 \partial_\xi \tilde{\eta}^j + \partial_\xi \tilde{\xi}^i \eta^0) \right\},
\]

where we used Lemma [A.1] and the fact that \( \delta^i_\xi = N - 1 \). Now consider the full algebra. The regular piece follows from the following calculation.

\[
\int (\xi^\mu \partial_\mu \eta^\nu - \xi^0 \eta^\nu)p_\nu + \int dsdt \, \xi^\mu(s) \eta^\nu(t) \delta^0_\mu p_\nu(s) \delta(s-t)
\]

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\[ + \int (\xi^\mu \partial_\mu \eta^0 - \xi^0 \eta^0) L + \int ds dt \xi^0(s) \eta^0(t) L(s) \delta(s - t) \]
\[ + \int \{ \xi^\mu \partial_\mu \partial_\sigma \eta^\tau - \xi^0 \partial_{\sigma \tau} \eta^0 \} T^\sigma_\tau + \partial_\mu \xi^\nu \partial_\sigma \eta^\nu \delta^\sigma_\mu T^\nu_\tau \}
\[ + \int ds dt \xi^0(s) \eta^\tau(t) T^\sigma_\tau (s) \delta(s - t) - \xi \leftrightarrow \eta \]
\[ = \int [\xi, \eta]^{\nu} p_{\nu} + [\xi, \eta]^{0} L + \partial_\mu [\xi, \eta]^{\nu} T^\mu_\nu, \]

and the full extension is
\[ \text{ext}(\xi, \eta) = \text{ext}_0(\xi, \eta) + \int ds dt \left\{ \frac{c}{24 \pi i} \xi^0(s) \eta^0(t) (\delta(s - t) + \delta(s - t)) \right. \]
\[ + \frac{k_0}{4 \pi i} (\xi^0(s) \partial_\mu \eta^\nu(t) - \partial_\mu \xi^\nu(s) \eta^0(t)) \delta(s - t) \]
\[ - \partial_\sigma \xi^\mu(s) \partial_\nu \eta^\nu(t) \left( \frac{k_1}{2 \pi i} \delta^\sigma_\nu \delta^\mu_\nu + \frac{k_2}{2 \pi i} \delta^\sigma_\mu \delta^\nu_\nu \right) \delta(s - t) \left\} \]
\[ = \text{ext}_0(\xi, \eta) + \frac{1}{2 \pi i} \int \left\{ \frac{c}{12} (\xi^0 \eta^0 - \xi^0 \eta^0) \right. \]
\[ + \frac{k_0}{2} (-\xi^0 \partial_\mu \eta^\nu + \eta^0 \partial_\mu \xi^\nu) + k_1 \partial_\mu \xi^\mu \partial_\nu \eta^\nu + k_2 \partial_\mu \xi^\mu \partial_\nu \eta^\nu \left\}. \]

The result now follows by means of lemma \[A.2\].

\[ \text{ext}(\xi, \eta) = \frac{1}{2 \pi i} \int dt \left\{ (1 + k_1) \partial_\mu \xi^\mu \partial_\nu \eta^\nu + k_2 \partial_\mu \xi^\mu \partial_\nu \eta^\nu \right. \]
\[ + \partial_\nu \xi^0 \partial_\mu \eta^\nu - \partial_\mu \xi^0 \partial_\nu \eta^\nu - \xi^0 \partial_\mu \xi^\nu \eta^0 - \xi^0 \partial_\nu \xi^\nu \eta^0 \right. \]
\[ + \frac{c}{2} (\partial_\mu \xi^\mu \eta^0 - \xi^0 \partial_\mu \eta^\nu) - (2 - \frac{c + 2(N - 1)}{12}) \xi^0 \eta^0 \]
\[ - \frac{c + 2(N - 1)}{12} \xi^0 \eta^0 + \frac{i}{2} (\partial_\mu \xi^\mu \eta^0 - \xi^0 \partial_\mu \eta^\nu) \left\}, \]

where \( \dot{f} = \dot{q}^\mu \partial_\mu f \). As a consistency check we note that the extension satisfies \( \text{ext}(\eta, \xi) = -\text{ext}(\xi, \eta) \).

To calculate the remaining brackets is a straightforward task. Note that normal ordering is irrelevant here, because \( S_n^{\nu_1 \cdots \nu_n} \) and \( R_n^{\mu_1 \cdots \nu_n} \) depend on \( q^\mu \) only whereas \( L_\xi \) depends only linearly on \( p_\nu \). \( \square \)

**Note added.** A. Dzhumadil’daev has explained his results [3], which I had slightly misunderstood. The Rao-Moody cocycle \( c_1 \) is included in his list; it is equivalent to his cocycle \( \psi^W_4 \), with coefficients in \( \Omega^1_{\text{DeRham}} / B^1_{\text{DeRham}} = B^2_{\text{DeRham}} \oplus H^1_{\text{DeRham}} \). Similarly, \( c_2 \) is his \( \psi^W_3 \). \( H^1_{\text{DeRham}} \) is an \( N \)-dimensional
trivial $diff(N)$ module; setting it to zero gives the substitution (2.3). The closedness condition $S_{r}^{f} \partial_{r}f = 0$ can be lifted for the cocycle $c_{2}$, (but not for $c_{1}$). One then obtains $\psi_{1}^{W}$, first discovered in [7]. Dzhumadildaev considered extensions by modules of tensor fields, not necessarily irreducible.

$c_{3}$ and $c_{4}$ are not included in his list, because they are extensions by other types of modules.

References

[1] Berman, S. and Y. Billig, *Irreducible representations for toroidal Lie algebras*, preprint (1998).

[2] Billig, Y., *Principal vertex operator representations for toroidal Lie algebras*, J. Math. Phys. 7, 3844–3864 (1998).

[3] Dzhumadildaev A., *Virasoro type Lie algebras and deformations*, Z. Phys. C 72, 509–517 (1996).

[4] Eswara Rao, S., R.V. Moody and T. Yokonuma, *Lie algebras and Weyl groups arising from vertex operator representations*, Nova J. of Algebra and Geometry 1, 15–57 (1992).

[5] Eswara Rao, S. and R.V. Moody, *Vertex representations for N-toroidal Lie algebras and a generalization of the Virasoro algebra*, Commun. Math. Phys. 159, 239–264 (1994).

[6] Fabbri, M. and R.V. Moody, *Irreducible representations of Virasosotoroidal Lie algebras*, Commun. Math. Phys. 159, 1–13 (1994).

[7] Larsson, T.A., *Multi-dimensional Virasoro algebra*, Phys. Lett. A 231, 94–96 (1989).

[8] Larsson, T.A., *Central and non-central extensions of multi-graded Lie algebras*, J. Phys. A. 25, 1177–1184 (1992).

[9] Larsson, T.A., *Fock representations of non-centrally extended super-diffeomorphism algebras*, physics/9710022 (1997).

[10] Moody, R.V., S. Eswara Rao and T. Yokonoma, *Toroidal Lie algebras and vertex representations*, Geom. Ded. 35, 283–307 (1990).