ISOMETRIC EMBEDDING WITH NONNEGATIVE GAUSS CURVATURE UNDER THE GRAPH SETTING

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Abstract. We study the regularity of the isometric embedding $X : (B(O, r), g) \to (\mathbb{R}^3, g_{can})$ of a 2-ball with nonnegatively curved $C^4$ metric into $\mathbb{R}^3$. Under the assumption that $X$ can be expressed in the graph form, we show $X \in C^{2,1}$ near $P$, which is optimal by Iaia’s example.

1. Introduction

Weyl posted the following problem in 1916 [15]: Consider a positively curved 2-sphere $(S^2, g)$. Does there exist a global $C^2$ isometric embedding $X : (S^2, g) \to (\mathbb{R}^3, g_{can})$, where $g_{can}$ is the standard flat metric on $\mathbb{R}^3$? Weyl himself suggested the continuity method to solve this problem and obtained a priori estimates up to the second derivatives. Lewy [9] solved the problem under the assumption the $g$ is analytic. In 1953, Nirenberg [12] solved the Weyl problem under the mild smoothness assumption that the metric is $C^4$.

P. Guan and Y.Y. Li [3] considered the question that if the Gauss curvature of the metric $g$ is nonnegative, whether does $(S^2, g)$ still have a smooth isometric embedding? They proved in [3] that for any $C^4$ metric $g$ on $S^2$, there is a global $C^{1,1}$ isometric embedding into $\mathbb{R}^3$. Examples in Iaia [7] show that for some analytic metrics with positive Gauss curvature on $S^2$ except at one point, there exists only a $C^{2,1}$ but not a $C^3$ global isometric embedding into $\mathbb{R}^3$.

Then a natural question, posted in [3], is that if a smooth metric $g$ on $S^2$ has nonnegative Gauss curvature, whether does it have a $C^{2,\alpha}$, for some $0 < \alpha < 1$, or even a $C^{2,1}$ global isometric embedding? To study this problem, we can look at the degenerate Monge-Ampère equation

$$(1.1) \quad \det(D^2u) = k$$

where $k(x, y) \geq 0$, in $B_r(O)$ for small $r > 0$. Guan [2] considered the case $k \in C^\infty(B_r(O))$, and

$$(1.2) \quad \frac{1}{A}(x^{2n} + By^{2m}) \leq k(x, y) \leq A(x^{2n} + By^{2m}),$$

for some $A > 0, B \geq 0$ and positive integers $n \leq m$. It’s shown in [2] that a $C^{1,1}$ solution $u$ of (1.1) is smooth near the origin if (1.2) holds, and if, additionally

$$(1.3) \quad u_{xx} \geq C_0 > 0.$$


Guan and Sawyer [4] improved this result by replacing (1.3) by a weaker condition \( \Delta u \geq C_0 > 0 \).

Daskalopoulos and Savin [1] considered (1.1) in the case that \( k \) is radial. It’s shown that in [1] that if \( k(x, y) = (x^2 + y^2)^{\frac{3}{2}} \) for some \( \delta > 0 \), then \( u \in C^{2, \epsilon} \) for a small \( \epsilon \) which depends on \( \delta \).

In this paper, we consider a \( C^{1,1} \) isometric embedding \( X : (B(O, r), g) \to (\mathbb{R}^3, g_{\text{can}}) \), where \( B(O, r) \) is a ball in \( \mathbb{R}^2 \), centred at the origin with radius \( r \), and \( g \) is a \( C^4 \) metric with Gauss curvature \( k \geq 0 \). We regard the image of \( X \), as the graph of a function \( u \). In fact, we can assume the isometric embedding is of form

\[
X : (x, y) \mapsto (\alpha(x, y), \beta(x, y), u(x, y)),
\]

where \( k = 0 \) at \( O \). Under normalization, we may assume

\[
u(0, 0) = 0, u_x(0, 0) = u_y(0, 0) = 0.
\]

Notice (1.5) implies that the \( \alpha \beta \)-plane in \( \mathbb{R}^3 \), is tangent to the image \( X(S^2) \) at \( X(O) \).

An example is that \( \alpha = x, \beta = y, u = r^3 = (x^2 + y^2)^{\frac{3}{2}} \).

Then \( g = dx^2 + dy^2 + du^2 \) is smooth in \( x, y \), \( k(x, y) = 18(x^2 + y^2)^{\frac{3}{2}} > 0 \) except at the origin, but the embedding is only \( C^{2,1} \).

First we have the following theorem when \( \alpha(x, y) \equiv x, \beta(x, y) \equiv y \), and the Gauss curvature only degenerates at a single point.

**Theorem 1.1.** Assume that we have a \( C^4 \) metric \( g \) on a ball \( B(O, r) \subseteq \mathbb{R}^2 \), for some \( r > 0 \), and that the Gauss curvature \( k > 0 \) in \( B(O, r) \setminus \{O\} \). Assume that a \( C^{1,1} \) isometric embedding \( X : (B(O, r), g) \to (\mathbb{R}^3, g_{\text{can}}) \) is of form

\[
X : (x, y) \mapsto (x, y, u(x, y)),
\]

under local coordinates \( x, y \) such that (1.5) holds. Then \( X \in C^{2,1}(B(O, r)) \).

Here we only need the sign of the Gauss curvature \( k \). By the example \( u = r^3 \), we see that the \( C^{2,1} \) smoothness of \( X \) is optimal.

Enlightened by Guan and Sawyer [4], if \( \Delta u \), or the mean curvature \( H \), has a uniform positive lower bound near but not necessarily at the origin, we have

**Corollary 1.2.** Assume the same assumptions as in Theorem 1.1. In addition, we assume that \( g \in C^\infty \), and \( \Delta u = u_{xx} + u_{yy} > C_0 > 0 \) for some constant \( C_0 \), around \( O \) but not necessarily at \( O \), then \( X \in C^\infty(B_g(P, r)) \).

For the Monge-Ampère equation (1.1), in the case that \( u \) is radial, we have the following corollary showing \( u \in C^{2,1} \), which is optimal by the example \( u = r^3 \). This result is expected to be true. We list it here as a quick corollary of lemmas in Section 2.

**Corollary 1.3.** Assume that a \( C^{1,1} \) convex function \( u \) satisfies (1.1) in \( B(O, \rho) \), the ball of radius \( \rho \) centered at the origin, and \( k \geq 0 \) in \( B(O, \rho) \setminus O \). In addition, we assume
that \( u = \Phi(r) \) for some function \( \Phi \), where \( r = \sqrt{x^2 + y^2} \), and \( k \in C^3(B(O, \rho)) \). Then \( u \in C^{2,1}(B(O, \rho)) \).

Here \( k \) could vanish at infinite order at \( r = 0 \). We see that \( \Phi r \) is the square root of a \( C^4 \) function.

Under the general nonnegative Gauss curvature, Pogorelov’s counterexample in [13] shows that a \( C^{2,1} \) metric with nonnegative Gauss curvature may not have a \( C^2 \) isometric embedding. However, given a \( C^4 \) metric, under the graph setting, our result is positive.

**Theorem 1.4.** Assume that we have a \( C^4 \) metric \( g \) on a ball \( B(O, r) \subseteq \mathbb{R}^2 \), for some \( r \geq 0 \), with Gauss curvature \( k \geq 0 \). Assume that a \( C^{1,1} \) isometric embedding \( X : (B(O, r), g) \rightarrow (\mathbb{R}^3, g_{can}) \) is of form

\[
X : (x, y) \mapsto (x, y, u(x, y)),
\]

under local coordinates \( x, y \) such that the normalization (1.5) holds. If in addition, \( u \) is (weakly) convex, then \( X \in C^{2,1}(B(O, r')) \), for any \( r' < r \).

The paper is organized as follows. In Section 2 we will discuss the one dimensional model. In Section 3 we will prove Theorem 1.1. In Section 4 we prove Corollary 1.2. In Section 5 we prove Theorem 1.4.

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### 2. A Model in Dimension One

In this section, we derive \( C^{2,1} \) estimates of \( u \) in an one dimension model.

Assume a nonnegative function \( u = u(x) \in C^1(2I) \), where \( I = [-1, 1] \), and \( u(0) = 0 \). In addition, we assume that \( f = u^2 \) is in \( C^4(2I) \), and \( f'(x)x \geq 0 \). The goal is to show \( u \in C^{2,1}(I) \). The condition that \( f'(x)x \geq 0 \) is necessary, since there is a nonnegative function

\[
f(x) = e^{-\frac{1}{x^2}} \sin^2(\frac{1}{x}) + e^{-\frac{2}{x^2}}
\]

which is smooth, vanishes at infinite order at \( x = 0 \), and \( (\sqrt{f})'' \) blows up when \( x \to 0 \). In fact, \( \sqrt{f} \in C^{1,\alpha} \) for any \( 0 < \alpha < 1 \), and \( (\sqrt{f})'' \leq C \frac{|f|}{|x|} \) for some fixed \( C \). \( x^4 \sqrt{f} \) is a \( C^{1,1} \) function. It shows that \( |u'''| = |(\sqrt{f})''| \) is not bounded when \( x \) tend to zero.

We have the following well known lemma,

**Lemma 2.1.** Assume \( f \in C^2(2I) \), where \( I = [-1, 1] \). \( f \geq 0 \) for \( x \in 2I \). Then for every \( x \in I \),

\[
|f'(x)| \leq \frac{3}{2} \frac{\|f\|_{C^2(2I)}^\frac{1}{2} f(x)^{\frac{1}{2}}}{|x|}.
\]
Proof. Assume first \(||f||_{C^2(2I)} = 1\). We only need to consider at \(x\) where \(f(x) > 0\). If \(f(x) \geq 1\) at some \(x \in I\), then \(|(\sqrt{f})'(x)| = \left|\frac{f'(x)}{2\sqrt{f(x)}}\right| \leq \frac{1}{2}||f||_{C^2(2I)} = \frac{1}{2}\). We assume \(0 < f(x) < 1\) at some \(x \in I\), then by the Taylor expansion, if \(x + t \in I\),

\[
f(x + t) = f(x) + f'(x)t + \frac{f''(\bar{x})}{2}t^2 \geq 0,
\]

for some \(\bar{x}\) between \(x\) and \(x + t\). Then

\[
f'(x)t \geq -f(x) - \frac{f''(\bar{x})}{2}t^2.
\]

When \(f'(x) > 0\), we set \(t = -f(x)^\frac{1}{2}\), and divide \(t\) on both hand sides to derive,

\[
0 < f'(x) \leq \frac{3}{2}f(x)^\frac{1}{2}.
\]

When \(f'(x) < 0\), we set \(t = -f(x)^\frac{1}{2}\), and divide \(t\) on both hand sides to derive,

\[
0 > f'(x) \geq -\frac{3}{2}f(x)^\frac{1}{2}.
\]

Notice the choice of \(t\) is valid, since \(|x + t| \leq |x| + |t| \leq |x| + \sqrt{f} < 2\).

So we derived,

\[
|f'(x)| \leq \frac{3}{2}f(x)^\frac{1}{2}.
\]

If general, when \(||f||_{C^2(2I)} \neq 1\), by a scaling, we see that

\[
|f'(x)| \leq \frac{3}{2}||f||_{C^2(2I)}^\frac{1}{2}f(x)^\frac{1}{2}.
\]

\(\Box\)

The following is a standard interpolation lemma,

**Lemma 2.2.** Assume that \(G(y)\) is a \(C^4\) function defined on \([-1, 0]\) such that \(G(y) \geq 0\), and is non-decreasing. Then there exist universal constants \(A, B\) such that

\[
|G'(0)| + |G''(0)| + |G'''(0)| \leq AG(0) + B \max_{y \in [-1, 0]} |G^{(4)}(y)|.
\]

**Proof.** By the Taylor expansion,

\[
G(-1) = G(0) + G'(0)(-1) + \frac{G''(0)}{2}(-1)^2 + \frac{G'''(0)}{6}(-1)^3 + \frac{G^{(4)}(\xi_1)}{24}(-1)^4,
\]

\[
G\left(-\frac{1}{2}\right) = G(0) + G'(0)(-\frac{1}{2}) + \frac{G''(0)}{2}(-\frac{1}{2})^2 + \frac{G'''(0)}{6}(-\frac{1}{2})^3 + \frac{G^{(4)}(\xi_2)}{24}(-\frac{1}{2})^4,
\]

\[
G\left(-\frac{1}{4}\right) = G(0) + G'(0)(-\frac{1}{4}) + \frac{G''(0)}{2}(-\frac{1}{4})^2 + \frac{G'''(0)}{6}(-\frac{1}{4})^3 + \frac{G^{(4)}(\xi_3)}{24}(-\frac{1}{4})^4,
\]

for some \(-1 < \xi_1 < 0, -\frac{1}{2} < \xi_2 < 0, -\frac{1}{4} < \xi_3 < 0\). Regard these as 3 linear equations in \(G'(0), G''(0), G'''(0)\), and solve them in terms of \(G(-1) - G(0), G\left(-\frac{1}{2}\right) - G(0), G\left(-\frac{1}{4}\right) - G(0)\).
Assume that Theorem 2.3. G(0), G^{(4)}(ξ_1), G^{(4)}(ξ_2), G^{(4)}(ξ_3). Since G(x) is non-decreasing and nonnegative, we have |G(1) - G(0)|, |G(1/2) - G(0)|, |G(-1/2) - G(0)| \leq G(0). Then the lemma follows. □

Next is the key theorem in this section.

Theorem 2.3. Assume that u is a C^4 function such that f = u_x^2 \in C^4(2I). In addition, assume that u_x(0) = 0 and f'(x)x is nonnegative.

Then u is C^5 in I - \{f = 0\}, and for every x \in I - \{f = 0\},

\begin{equation}
|u_{xxx}(x)| \leq C||f||_{C^4(2I)}^{1/2},
\end{equation}

for some universal constant C.

Proof. At x \in I - \{f = 0\}, f(x) > 0. So u_x = \sqrt{f} > 0 in a neighborhood of x, or u_x = -\sqrt{f} < 0 in a neighborhood of x. In both cases, f \in C^1(I) implies u is C^5 near x.

First we assume ||f||_{C^4(2I)} = 1. Our goal is to prove, for x \in I - \{f = 0\},

\begin{equation}
|u_{xxx}| = |(\sqrt{f})''| = \frac{2ff'' - (f')^2}{4f^{3/2}} \leq C,
\end{equation}

for some universal constant C.

Denote g = \frac{f}{x^2}, h = \frac{f}{x}. By Lemma (A.1), g, h \in C^2(2I) and the C^2 norms of g, h are bounded by C||f||_{C^4} = C for some universal constant C. Notice g, h are nonnegative. If we apply Lemma 2.1 to g, then we derive, for x \in I,

\begin{equation}
|x'g'| = \frac{x'g' - 2f}{x^2} \leq Cf^{1/2}.
\end{equation}

And applying Lemma 2.1 to h, we derive, for x \in I,

\begin{equation}
|h'| = \frac{x'g' - f'}{x^2} \leq C\left(\frac{f'}{x}\right)^{1/2} = C(xg' + 2f \frac{f}{x^2})^{1/2} \leq C(f^{1/2} + |x|^{-1} f^{1/2}).
\end{equation}

Case 1: Consider points x \in I - \{f = 0\}, such that f(x) \geq x^4. We see that |x|f^{1/2} \leq f^{1/2}, |x|f^{1/2} \leq f^{1/2}. Then by (2.3), (2.4),

\begin{equation}
|x^2g'| = |f' - 2f | \leq C f^{1/2},
\end{equation}

\begin{equation}
|xh'| = |f'' - f | \leq C f^{1/2},
\end{equation}

and by (2.3), (2.5), (2.6),

\begin{equation}
|fxh'| = f|xh'| \leq C f^{3/2},
\end{equation}

\begin{equation}
x^4(g')^2 = (x^2g')^2 \leq C f^{3/2},
\end{equation}

\begin{equation}
|xf^g| = f|g'| \leq C f^{3/2}.
\end{equation}

which further implies 2ff'' - (f')^2 = 2fxh' - x^4(g')^2 - 2xfg' is bounded by C f^{3/2}. So (2.2) holds.
Case 2: Consider points $x \in I - \{f = 0\}$, such that $f(x) \leq x^4$. We are to prove at such $x$,

\begin{equation}
|f'| \leq C f^\frac{3}{4}, |f''| \leq C f^\frac{1}{4}, |f^{(3)}| \leq C f^\frac{1}{4},
\end{equation}

for some universal constant $C$, which implies (2.2).

If $0 < \epsilon \leq x$, Lemma 2.2 can be applied to the function $G(y) = f(x + \epsilon y)$ for $-1 \leq y \leq 0$. Since

\begin{align*}
G'(0) &= \epsilon f'(x), \\
G''(0) &= \epsilon^2 f''(x), \\
G'''(0) &= \epsilon^3 f'''(x), \quad \max_{y \in [0, 1]} |G^{(4)}(y)| \leq \epsilon^4 \max_{x \in I} |f^{(4)}(x)|,
\end{align*}

we derive

\begin{equation}
\epsilon |f'(x)| + \epsilon^2 |f''(x)| + \epsilon^3 |f'''(x)| \leq Af(x) + \epsilon^4 B \max_{x \in I} |f^{(4)}(x)|.
\end{equation}

By setting $\epsilon = f(x)^{\frac{1}{2}} \leq x$, (2.8) implies (2.7). If for some $x$, $x < 0$ and $f(x) \leq x^4$, we set $G(y) = f(x - \epsilon y)$ for $-1 \leq y \leq 0$, then we derive (2.7) in a similar way.

In sum, (2.2) is verified, and we derive (2.1) by a scaling. 

\begin{theorem}
Assume the same assumption as Theorem 2.3, and in addition, $u_{xx} \geq 0$ for any $x \in 2I - \{f = 0\}$. Then $u \in C^{2,1}(I)$, with (2.1) holds.
\end{theorem}

\begin{proof}
We show $u \in C^{2}(I)$, then the theorem follows from Theorem 2.3

First assume $f = 0$ only at $x = 0$. Taking the Taylor expansion of $f(x)$ at $0$, if $f(x) = M x^2 + R(x)$, for some $M > 0$, and $R(x) \in O(x^3)$, then for $x \neq 0$,

\begin{equation}
u_{xx} = \text{sign}(x) \frac{f_x}{2 \sqrt{f}} = \text{sign}(x) \frac{2 M x + R_x}{2 \sqrt{M x^2 + R}},
\end{equation}

which approaches $\sqrt{M}$ as $x$ tends to 0. Also we check

\begin{equation}u_{xx}(0) = \lim_{x \to 0} \frac{u_{xx}(x)}{x} = \lim_{x \to 0} u_{xx}(x) = \sqrt{M}
\end{equation}

by L’Hospital’s Rule.

If $M = 0$, then $f(x) = O(x^4)$. After a scaling, we assume $f(x) \leq x^4$. Then for $x$ near the origin,

\begin{equation}
u_{xx} = \text{sign}(x) \frac{f_x}{\sqrt{f}} \leq C f_x^\frac{3}{2} = C \sqrt{f},
\end{equation}

which approaches $\sqrt{M}$ as $x$ tends to 0. Also we check

\begin{equation}u_{xx}(0) = \lim_{x \to 0} \frac{u_{xx}(x)}{x} = \lim_{x \to 0} u_{xx}(x) = \sqrt{M}
\end{equation}

by L’Hospital’s Rule.
by (2.7). So \( u_{xx}(0) \) exists and equals 0.

Secondly, if \( f = 0 \) at some \( x_0 \neq 0 \), without loss of generality, assume \( x_0 > 0 \). Since \( f'(x)x \geq 0 \), \( f \) is non-decreasing as \( x > 0 \). Then \( f \equiv 0 \), for \( 0 \leq x \leq x_0 \). Denote \( x_1 = \max\{x \in 2I : f(x) = 0\} \). We only have to consider \( u_{xx} \) at \( x_1 \) if \( x_1 \in I \). We can use a new coordinate that translates \( x_1 \) to the origin, and apply an argument like (2.10) to show \( u_{xx}(x_1) = 0 \). □

Example \( u = |x|^3 \) shows that in general \( u \notin C^3(\frac{1}{2}I) \) under the assumption of Theorem 2.5 even if \( f \in C^\infty(I) \).

For the case \( f > 0 \) in \( 2I \), we see that \( u \) is a \( C^5 \) function on \( 2I \).

**Corollary 2.6.** Assume \( u \) is a \( C^2 \) function such that \( f = u_x^2 \in C^4(2I) \). In addition, \( f > 0, u_{xx} \geq 0 \) in \((-2, 2)\). Then for every \( x \in I \),

\[
|u_{xxx}(x)| \leq C||f||_{C^4(2I)}^{\frac{1}{2}},
\]

for some universal constant \( C \).

**Proof.** \( u_{xx} \geq 0 \) implies \( f \) is non-increasing or non-decreasing on \((-2, 2)\), depending on the sign of \( u_x(0) \). Without loss of generality, we shift the origin to \(-2\), and assume \( f \) is non-decreasing on \((0, 4)\).

Assume \( ||f||_{C^4(2I)} = 1 \) first. Our goal is to prove (2.7) in \([1, 3]\). In fact, for any \( x \in [1, 3] \), we can derive (2.8) for \( \epsilon \in (0, x) \). We select \( \epsilon = \left(\frac{1}{2} f(x)\right)^{\frac{1}{4}} < f(x)^{\frac{1}{4}} \leq 1 \leq x \). The rest follows as Theorem 2.3. □

For the applications in Section 3 and 5, we need a scaling version of Theorem 2.3, 2.5. Assume \( u \) is a \( C^{1,1} \) function such that \( f = u_x^2 \in C^4(2sI) \) for some \( 0 < s < 1 \). In addition, \( u_x(0) = 0 \) and \( f'(x)x \) is nonnegative. In addition \( u_{xx} \geq 0 \) in \( 2sI - \{f = 0\} \). We define

\[
\bar{u}(x) = u(s^{-1}x),
\]

then \( \bar{u} \) satisfies the assumption of Theorem 2.3. We derive \( \bar{u} \in C^2(I) \) and

\[
|\bar{u}_{xxx}| \leq C||\bar{u}_x^2||_{C^4(2I)}^{\frac{1}{2}},
\]

in \( I - \{\bar{u}_x = 0\} \), implying \( u \in C^2(sI) \), and

\[
|u_{xxx}| \leq Cs^{-3}||u_x^2||_{C^4(2sI)}^{\frac{1}{2}},
\]

in \( sI - \{u_x = 0\} \). (2.12) is also right under the assumption of Corollary 2.6 if we shrink the interval by multiplying the factor \( s \).
3. Two Dimensional Case with One Singular Point

In this section, we prove Theorem 1.1. Maybe making $r$ a bit smaller, we assume that the $C^{1,1}$ function $u$ in (1.6) is defined in $rI \times rI$. In addition, we assume that $f = u_x^2 > 0$, $u_x$ exists and be positive, except at the origin. Here $u_{xx}$ exists in $B(O, r) - \{O\}$, since the Gauss curvature $k > 0$ except at the origin. Then by the classic theory of Monge-Ampère equations, $g$ is $C^4$ implies that $u$ is $C^{3,\alpha}$, except at the origin, for any $\alpha \in (0, 1)$. See Section 10.3 of [14].

For $(x, y) \neq 0$, \{u_x = 0\} is locally a curve, since at any point except the origin, $u_{xx}(x, y) > 0$, then we can solve out $x = A(y)$ as a function of $y$ from the equation $u_x(x, y) = 0$. Furthermore, $\frac{d}{dy} A(y) = \frac{u_{xy}(y)}{u_{xx}(A(y), y)}$, which is uniformly bounded when $y \in (\delta, r) \cup (-r, -\delta)$ for any $\delta > 0$. In addition, for each $y$, we can only have at most one $x$, such that $u_x(x, y) = 0$, since $u_x$ is strictly increasing. Though the gradient of $A(y)$ may blow up when $y$ approaches 0, we show that \{u_x = 0\} is a continuous curve.

**Lemma 3.1.** Assume that $u \in C^{1,1}(rI \times rI)$, $u_x(0, 0) = 0$ and $u_x$ is an increasing function in $x$ for any fixed $y$. Then \{u_x = 0\} is a continuous curve near the origin.

**Proof.** We only have to show $A(y)$ is continuous at $y = 0$, i.e. for any $\epsilon \in (0, 1)$, we can find an $\delta > 0$, such that $-\delta < y < \delta$ implies $-\epsilon < A(y) < \epsilon$.

On the segment \{(x, y) : -r \leq x \leq r, y = 0\}, $u_x$ is increasing. Assume $u_x(\epsilon, 0) > \eta$, and $u_x(-\epsilon, 0) < -\eta$, for some $\eta > 0$. Then there is an $\delta > 0$, such that when $|y| < \delta$, $u_x(\epsilon, y) > 0$, $u_x(-\epsilon, y) < 0$.

The choice of $\delta$ depends on $\eta, ||u||_{C^{1,1}}$. Hence, for any fixed $y \in (-\delta, \delta)$, the zero of $u_x$ must be unique and the value of $x$ lies in $(-\epsilon, \epsilon)$, by the assumption that $u_x$ is an increasing function in $x$ for any fixed $y$. So we derive $-\epsilon < A(y) < \epsilon$ as $|y| < \delta$. \hfill $\square$

Now we prove Theorem 1.1.

**Proof.** By the assumption, the metric

$$g = dx^2 + dy^2 + du^2 = (1 + u_x^2)dx^2 + u_x u_y dx dy + u_x u_y dy dx + (1 + u_y^2)dy^2$$

is $C^4$, which implies that $u_x^2, u_y^2, u_x u_y \in C^4$. Hence $u_x^2 \in C^4$ for any $z = lx + my$, where $l, m$ are fixed numbers in $\mathbb{R}$. At points except the origin, $k > 0$. Then the classic theory of Monge-Ampère equation shows that $u \in C^{2,\alpha}$ at any point $(x, y)$ away from the origin. We get $u_{zz} > 0$ except at the origin.

Now fix a small $\epsilon << r$. Assume $u_x(-\epsilon, 0) < -2\eta_1, u_x(\epsilon, 0) > 2\eta_1$ for some $\eta_1 > 0$. Then $u_x(-\epsilon, y) < -\eta_1, u_x(\epsilon, y) > \eta_1$, if $|y| < \delta_1 = \frac{\eta_1}{||u||_{C^{1,1}(rI \times rI)}}$. We set

$$s = \max_{|y| < \delta_1} \{\epsilon - A(y), \epsilon + A(y)\},$$

which has bound $\epsilon \leq s \leq 2\epsilon$. Then the interval $[A(y) - 2s, A(y) + 2s] \subset I$ since $\epsilon$ is small. And for any $y \in (-\delta_1, \delta_1), f(x, y) = u_x^2(x, y) > \eta_1^2$,
if \( x = A(y) - 2s \) or \( A(y) + 2s \). So by (2.12), for any \( y \in (-\delta_1, \delta_1) \), \( u \) has \( C^{2,1} \) estimates for \( x \in [A(y) - s, A(y) + s] \), which depends only on \( s, |g|_{C^4(rI \times rI)} \). For points \((x, y) \in (rI - [A(y) - s, A(y) + s]) \times \delta_1 I\),

\[
 f(x, y) = u_x^2(x, y) > \eta_1^2,
\]

and hence by (2.2),

\[
 |u_{xxx}| = \left| \frac{2ff'' - (f')^2}{4f^2} \right| \leq \eta_1^{-3}||f||_{C^2(rI \times rI)}.
\]

Now \( u \in C^3(rI \times rI - (0, 0)) \), since the metric \( g \in C^4 \). In \( rI \times \delta_1 I \), \( u \) has \( C^{2,1} \) estimates in \( x \), which depends only on \( \eta_1, \epsilon, |g|_{C^4} \), that is,

\[
 ||u_{xxx}(x, y)|| \leq C(\eta_1, \epsilon, |g|_{C^4}),
\]

for \((x, y) \in rI \times \delta_1 I - (0, 0)\).

Similarly, switching \( x \) and \( y \), we can find \( \eta_2 \) such that

\[
 ||u_{yyy}(x, y)|| \leq C(\epsilon, \eta_2, |g|_{C^4}),
\]

for \((x, y) \in \delta_2 I \times rI - (0, 0)\), where \( \delta_2 = \frac{\eta_2}{||u||_{C^4(rI \times rI)}} \).

Apply the same argument to coordinates \( z = \frac{x+y}{2} \) and \( w = \frac{x-y}{2} \), we have

\[
 u_{zzz} = u_{xxx} + 3u_{xxy} + 3u_{xyy} + u_{yyy},
\]

\[
 u_{www} = u_{xxx} - 3u_{xxy} + 3u_{xyy} - u_{yyy},
\]

are uniformly bounded in two rectangular neighborhoods of the origin minus the origin, respectively, where the bounds only depend on \( \epsilon, \eta_1, \eta_2, \eta_3, \eta_4, |g|_{C^4} \).

So we derive the bounds of \( u_{xxy}, u_{xyy} \) in a neighborhood of the origin, since

\[
 u_{xxy} = \frac{u_{zzz} - u_{wwww} - 2u_{yyy}}{6},
\]

\[
 u_{xyy} = \frac{u_{zzz} + u_{wwww} - 2u_{xxx}}{6}.
\]

We derived that \( u \) has uniform \( C^3 \) estimates except at the origin. Then, by a basic argument in calculus, we derive that \( u \in C^{2,1} \) near the origin. \( \square \)

4. PROOF OF COROLLARIES

Proof of Corollary 1.2. By Theorem 1.1 \( u \) is \( C^{2,1} \) in \( x, y \). By the assumption, \( \Delta u > C_0 > 0 \) around the origin but not necessarily at the origin. Without loss of generality, under a rotation of coordinates, we assume for coordinates \( \hat{x}, \hat{y} \),

\[
 u_{\hat{x}\hat{x}}(0, 0) \geq C_0 > 0, \quad u_{\hat{y}\hat{y}}(0, 0) = 0
\]

which should hold for some \( \hat{x}, \hat{y} \), since the Gauss curvature \( k = 0 \) at the origin.
We can rotate $x, y$ a little bit, such that none of the $x, y$ direction confirms with the $\tilde{y}$ direction, and so

$$u_{xx} > C_1 > 0, u_{yy} > C_1 > 0,$$

for some $C_1$ depends only on $C_0$ and the angle between the $x, y$ direction and the $\tilde{y}$ direction. This does not change that fact that $u_{xx}u_{yy} - u_{xy}^2 = 0$ at the origin, so we cannot apply the classic theory for Monge-Ampère equations.

Recall in the proof of Theorem 2.5, for the one dimensional model, if $u_{xx} > C_1 > 0$, we derive $f = u_x^2 = Mx^2 + Rx^3$, where $M > C_1^2$ is independent of $x$, and $R$ is smooth by the assumption that $f = g_{xx} - 1$ is smooth. Then

$$u_x = \text{sign}(x) \cdot \sqrt{Mx^2 + Rx^3} = x\sqrt{M + Rx},$$

which has $C^k$ bounds for any integer $k > 0$, which only depends on $M, k, ||f||_{C^{k+3}}$. In sum, $|D_x^k u| \leq B_k(M, k, ||f||_{C^{k+3}})$ for any $k$, and explicitly we have

$$D_x^2 u(0) = \sqrt{M},$$

$$D_x^3 u(0) = \frac{R(0)}{\sqrt{M}},$$

$$D_x^4 u(0) = \frac{12R_x(0)M - 3R(0)^2}{4M^4},$$

$$\ldots$$

Then we check the two dimensional model, and derive $|D_x^k u| \leq B_k(M, k, ||f||_{C^{k+3}})$ around the origin. The estimates also hold for the two pairs of coordinate systems $z = \frac{x+y}{2}, w = \frac{x-y}{2}, z_1 = \frac{x+2y}{5}, w_1 = \frac{2x-y}{5}$, if none of these four coordinates points to the $\tilde{y}$ or $-\tilde{y}$ direction. We can rotate the $x, y$ coordinates system a little bit if one of them does. Then

$$D_x^4 u = u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}$$

$$D_x^4 u = u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}$$

$$D_x^4 u = u_{xxxx} + 8u_{xxxy} + 24u_{xxyy} + 32u_{xyyy} + 16u_{yyyy}$$

are bounded, implying $u \in C^4$ near the origin.

Inductively, we can prove $u$ is $C^k$ near the origin by introducing more pairs of coordinate systems, and the corollary is verified.

In Theorem 1.1 The forms of $\alpha, \beta$ that are allowed can be slightly generalized.

**Corollary 4.1.** Assume that $(B(O, r), g)$ satisfies the same assumption as in Theorem 1.1. If a $C^{1, 1}$ isometric embedding $X : (B(O, r), g) \rightarrow (\mathbb{R}^3, g_{can})$ which is of form (1.4) under local coordinates near $O$, satisfies the normalization (1.5), and if $\alpha, \beta$ are $C^5$ in $x, y$, then $X \in C^{2, 1}(B_0(O, r))$. 

\[\square\]
Proof. Under the assumption of Corollary 4.1 in any domain which does not include the origin, we have \( u \in C^{2,\mu} \ (0 < \mu < 1) \), and
\[
(4.1) \quad u_{\alpha\alpha} > 0, \quad u_{\alpha\alpha}u_{\beta\beta} - u_{\alpha\beta}u_{\alpha\beta} > 0,
\]
if we regard \( u \) as a function of \( \alpha, \beta \). Notice we does not necessarily have \( u_{xx} > 0 \).

The system of equations
\[
\alpha^2 + \beta^2 + u_x^2 = g_{xx}, \quad \alpha_x\alpha_y + \beta_x\beta_y + u_xu_y = g_{xy}, \quad \alpha^2 + \beta^2 + u_y^2 = g_{yy},
\]
implies
\[
\begin{align*}
 u_{\alpha\alpha}^2 &= g_{xx}x_{\alpha}^2 + 2g_{xy}x_{\alpha}y_{\alpha} + g_{yy}y_{\alpha}^2 - 1 \\
 u_{\beta\beta}^2 &= g_{xx}x_{\beta}^2 + 2g_{xy}x_{\beta}y_{\beta} + g_{yy}y_{\beta}^2 - 1 \\
 u_{\alpha\beta} &= g_{xx}x_{\alpha}y_{\beta} + g_{xy}(x_{\beta}y_{\beta} + x_{\alpha}y_{\alpha}) + g_{yy}y_{\alpha}y_{\beta},
\end{align*}
\]
where the terms on the right hand side are \( C^4 \) in \( \alpha, \beta \) by the assumption of Corollary 4.1. Here \( x, y \) can be regarded as \( C^5 \) functions of \( \alpha, \beta \), since at the origin, we may choose \( x, y \) as normal coordinates, and after a rotation, we assume \( \alpha_y(0,0) = 0, \beta_x(0,0) = 0 \).

Then at the origin
\[
\begin{align*}
 \alpha_x^2 &= g_{xx} = 1, \quad \beta_y^2 = g_{yy} = 1.
\end{align*}
\]

The Jacobian \( \frac{\partial(\alpha, \beta)}{\partial(x, y)} = 1 \) at the origin. By the implicit function theorem, we can solve out \( x, y \) as functions of \( \alpha, \beta \).

We derive that \( u_{\alpha\alpha}, u_{\beta\beta}, u_{\alpha\beta} \in C^4 \) and (4.1) holds. We can apply the same method as in Section 3 to derive \( u \in C^{2,1} \) near the origin. Then Corollary 4.1 follows. \( \square \)

Proof of Corollary 1.3. Without loss of generality, we assume (1.5) holds and \( k = 0 \) at \( O \). Since \( u = \Phi(r) \), we can check \( \Phi \in C^{1,1} \), as \( u \in C^{1,1} \). We compute,
\[
\det(D^2 u) = \frac{\Phi_r \Phi_{rr}}{r} = k.
\]
So \( k = \Psi(r) \), for some \( C^3 \) function \( \Psi \) in \( r \), by the assumption that \( k \) is \( C^3 \) in \( x, y \) and Lemma A.2. For \( r < 0 \), we define \( \Phi(r) = \Phi(-r), k(-r) = k(r) \). Then \( \Phi \in C^{1,1}(\rho I) \), \( \Psi \in C^{2,1}(\rho I) \), since \( \Phi(0) = \Phi_r(0) = k(0) = k_r(0) = 0 \). And when \( r < 0 \), \( \Phi_r, \Phi_{rr} = \Psi r \) still holds. Notice \( \Psi r \) is \( C^3(\rho I) \). Then
\[
\Phi_r^2 = \int_0^r \Psi(s)sds,
\]
and \( \Phi(0) = \Phi_r(0) = 0 \). Notice \( \int_0^r \Psi(s)sds \) is in \( C^4(\rho I) \) and \( \frac{\partial}{\partial r} \int_0^r \Psi(s)sds \cdot r = \Psi(r)r^2 \geq 0 \).

Then we can apply (2.12), and derive
\[
\Phi \in C^{2,1}(\frac{\rho}{2} I),
\]
which further implies $u$ is a $C^{2,1}$ function.

5. Nonnegative Gauss curvature case

Set $\delta = \frac{r}{\eta}$ in this section. Assume we have the Gauss curvature $k \geq 0$ on $(B(P, r), g)$. In addition, we assume $u$ is convex, which implies

\begin{equation}
(5.1) \quad u_{xx}, u_{yy}, u_{zz}, u_{ww} \geq 0,
\end{equation}

(at where they exist) in $B_g(P, r)$. Here $(z, w)$ is the new coordinate system such that $z = \frac{x+y}{2}, w = \frac{x-y}{2}$. Note that convexity of $u$ is necessary, since we have an example

$$u = \text{sign}(x) \cdot (x^2 + |y|^2),$$

where $u_x^2$ and the metric of the graph are smooth, with nonnegative Gauss curvature, but $u$ is not $C^2$ with respect to $x$ on the $y$-axis. The main obstruction is that the graph of $u$ is not convex.

For a point $(x_0, y_0) \in [-\delta, \delta] \times [-\delta, \delta]$, if $u_x(x_0, y_0) = 0$ and $u_x(x, y_0) < 0$ for every $x \in (-\delta, x_0)$, we call $(x_0, y_0)$ a left touch point of $u_x$. Notice the left touch point is unique for any $y \in [-\delta, \delta]$ if it exists, according to (5.1). Similarly, we can define right touch points.

**Lemma 5.1.** Assume $u(x, y) \in C^{1,1}(rI \times rI)$, $u_x^2 \in C^4(rI \times rI)$, $u_{xx}$ exists and $u_{xx} \geq 0$ in $\delta I \times \delta I$. Then in the square $[-\delta, \delta] \times [-\delta, \delta]$, the left and right touch points, form two sets of measure zero in $\mathbb{R}^2$.

**Proof.** Without loss of generality, we only consider set of the left touch points. Define the left touch function $T_L(y)$ on $[-\delta, \delta]$, where $T_L(y_0)$ equals $x_0$ if we can find an $x_0$ such that $(x_0, y_0)$ is the left touch point on the line $y = y_0$, which is unique if any; otherwise, $u_x(x, y_0) \geq 0$ for $x \in (-\delta, \delta)$, for which case we set $T_L(y_0) = -\delta$, or $u_x(x, y_0) < 0$ for $x \in (-\delta, \delta)$ for which case we set $T_L(y_0) = \delta$.

We check $T_L$ is a lower semi-continuous function. If $T_L(y) = -\delta$, then it’s trivial since $T_L(x) \geq -\delta$.

If $T_L(y) = x$ for some $x > -\delta$, then $(x, y)$ is a touch point, or $x = \delta$. In this case, for any $\epsilon \in (0, \delta + x)$, $u_x(x - \epsilon, y) < 0$, so there is a neighborhood of $(x - \epsilon, y)$ in $\mathbb{R}^2$, such that $u_x < 0$ in the neighborhood. So there is a $\eta > 0$, for any $y_1 \in (y - \eta, y + \eta)$, $u_x(x - \epsilon, y_1) < 0$, implying $T_L(y_1) > x - \epsilon$.

So $T_L$ is measurable as a lower semi-continuous function, and by the Fubini Theorem, its graph has measure zero.

Lemma [5.1] shows that $u_{xxx}$ exists and is uniformly bounded in $\delta I \times \delta I$ minus the sets of left and right touch points, i.e. $u_{xxx}$ exists and is uniformly bounded, almost everywhere in $rI \times rI$.

We are ready to prove Theorem [1.3] using mollifiers to help with applying (3.1).
proof of Theorem 1.4. Consider the $x$ direction first. Denote $r_0 = 6\delta, r_1 = \sqrt{r^2 - \delta^2}$. Then for any $x_0 \in (-2\delta, 2\delta)$, we have

$$-r_1 < x_0 - r_0 < x_0 + r_0 < r_1,$$

(5.2)

$$x_0 + \frac{r_0}{2} \geq \delta,$$

$$x_0 - \frac{r_0}{2} \leq -\delta.$$

On each integral curve of $\frac{\partial}{\partial x}$ in $[-2\delta, 2\delta] \times \delta I$, we check whether $f = u_x^2$ has a zero.

If not on the segment $[-2\delta, 2\delta] \times \{y_0\}$, we apply Corollary 2.6 to show $u$ is $C^{2,1}$ in $x$ on $[\delta, \delta] \times \{y_0\}$ and (2.12) holds with $s = \delta$.

If there is any zero on the segment $[-2\delta, 2\delta] \times \{y_0\}$, we apply Corollary 2.6 to show $u$ is $C^{2,1}$ in $x$ on $[\delta, \delta] \times \{y_0\}$ and (2.12) holds with $s = \delta$.

Hence $u_{xx}$ exists everywhere in $\delta I \times \delta I$ and is Lipschitz in $x$ on every integral curve of $\frac{\partial}{\partial x}$. In addition, by Lemma 5.3, $u_{xxx}$ exists almost everywhere in $\delta I \times \delta I$, and has uniform bound (2.12) with $s = \delta$. Consider the regulation of $u$ using the mollifier (A.1),

$$u_\tau(x, y) = \tau^{-2} \rho\left(\frac{x}{\tau}, \frac{y}{\tau}\right) * u(x, y).$$

By Lemma A.3, $|(u_\tau)_{xxx}| \leq C(r, g)$ in $B(P, \frac{\delta}{2})$, for any $\tau < \text{dist} (\partial(\delta I \times \delta I), \partial B(P, \frac{\delta}{2}))$.

We have similar results in the $y, z, w$ direction. In the ball $B(P, \frac{\delta}{2})$, $u_{xz}, u_{xw}, u_{yzz}, (u_\tau)_{zzz}, (u_\tau)_{yyyy}, (u_\tau)_{yyyy}, (u_\tau)_{yyyy}$ have a uniform bound which is independent of $\tau$. Thus, $u_\tau$ has a uniform $C^3$ bound as well. Now that $u_\tau$ has a uniform $C^3$ bound, so we can apply the Arzela-Ascoli Theorem to derive $u \in C^{2,1}(B(P, \frac{\delta}{2}))$.

\[\square\]

**Appendix A. Calculus lemmas**

**Lemma A.1.** Assume that $f$ satisfies conditions of Theorem 2.3. Then for $g = \frac{f'}{x^2}, h = \frac{f''}{x^3}$, we have $g, h \in C^2(I)$, and

$$\|g\|_{C^2(I)} + \|h\|_{C^2(I)} \leq C\|f\|_{C^4(I)},$$

for some universal constant $C$.

**Proof.** We can express

$$f(x) = \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \int_0^x \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} f^{(4)}(s_4) ds_4 ds_3 ds_2 ds_1.$$
Then \( g'' \) includes terms like
\[
\frac{1}{x^2} \int_0^x \int_0^{s_3} f^{(4)}(s_4) ds_4 ds_3 \\
\frac{1}{x^3} \int_0^x \int_0^{s_2} \int_0^{s_3} f^{(4)}(s_4) ds_4 ds_3 ds_2 \\
\frac{1}{x^4} \int_0^x \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} f^{(4)}(s_4) ds_4 ds_3 ds_2 ds_1
\]
which are all bounded by \( ||f||_{C^4(I)} \). In addition, as \( x \to 0 \), all these terms have limits by L'Hospital's rule, which shows \( g'' \) is continuous.

For \( h \), the proof is similar.

\[\Box\]

**Lemma A.2.** If \( k = k(x,y) \) lies in \( C^3(B(O,\rho)) \), and \( k = \Psi(r) \), where \( r = \sqrt{x^2 + y^2} \), then \( \Psi \in C^3((0,\rho)) \).

**Proof.** It follows directly from the fact that \( \Psi(r) = k(r,0) \).

Denote \( T = \frac{\partial}{\partial x_1} \), a tangential vector on \( \mathbb{R}^n \). Then it’s integral curves are lines \((x_2,x_3,\cdots,x_n) = \text{const.}\) Then we have the following lemma,

**Lemma A.3.** Assume \( w \) is a measurable function on \( \Omega \subseteq \mathbb{R}^n \), and absolute continuous on every integral curves of \( T \). In addition, \( Tw \) exists almost everywhere, and it is integrable in \( \Omega \). Then \( Tw \) is a derivative of \( w \) in the weak sense, i.e., for any smooth function \( v \) which has compact support in \( \Omega \),
\[
\int_\Omega Tw \cdot v dx = -\int_\Omega w \cdot T v dx
\]

**Proof.** We need to show the one-dimensional case, then the general case is done by the Fubini’s theorem. Denote \( T = \frac{\partial}{\partial x} \). By the assumption, \( w \) is a continuous function, and \( Tw \) exists almost everywhere.

It’s integration by parts. In fact,
\[
Twv = Tw \cdot v + w \cdot Tv.
\]

\( wv \) is absolute continuous so we can integrate the equation using the fundamental theorem of calculus.

Consider a mollifier, for \( x \in \Omega \subseteq \mathbb{R}^n \),

(A.1)
\[
\rho(x) = c \exp \left( \frac{1}{|x|^2 - 1} \right),
\]
when \( |x| < 1 \), and \( \rho(x) = 0 \) when \( |x| \geq 1 \). Here \( c \) is selected such that \( \int_{\mathbb{R}^n} \rho(x) dx = 1 \). For any \( w \in L^1(\Omega) \) and \( \tau > 0 \), the regulation of \( w \) is defined to be
\[
w_\tau(x) = \tau^{-n} \int_{\Omega} \rho \left( \frac{x-y}{\tau} \right) w(y) dy,
\]
where $\tau < \text{dist}(x, \partial \Omega)$. Then $w_\tau$ is a smooth function in a domain $\Omega'$, if $\overline{\Omega'} \subseteq \Omega$ and $\tau < \text{dist}(\partial \Omega', \partial \Omega)$.

**Lemma A.4.** Assume that $w$ satisfies the assumption of Lemma A.3. In addition, $|T_w| < C$ in $\Omega$. Then $|T_{w_\tau}| < C$ in a domain $\Omega'$, if $\overline{\Omega'} \subseteq \Omega$ and $\tau < \text{dist}(\partial \Omega', \partial \Omega)$.

**Proof.**

$$T_{w_\tau}(x) = \tau^{-n} \lim_{\epsilon \to 0} \int_{\Omega} \rho \left( \frac{x}{\tau} \right) \cdot \frac{w(x - y + \epsilon e_1) - w(x - y)}{\epsilon} dy,$$

where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$. Since $w$ is absolute continuous on the integral curves of $T$,

$$\left| \frac{w(x - y + \epsilon e_1) - w(x - y)}{\epsilon} \right| = \left| \frac{1}{\epsilon} \int_{0}^{\epsilon} T_w(x - y + se_1) ds \right| < C.$$ 

By the dominated convergence theorem,

$$T_{w_\tau}(x) = \tau^{-n} \int_{\Omega} \rho \left( \frac{x}{\tau} \right) \cdot T_w(x - y) dy,$$

and has uniform bound $C$. □

If we reduce the absolute continuity condition to being continuous on integral curves of $T$, then the lemma is not right. Cantor function in the one dimensional case is a counterexample.

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