Duality and helicity: the photon wave function approach

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Abstract

The photon wave equation proposed in terms of the Riemann-Silberstein vector is derived from a first-order Dirac/Weyl-type action principle. It is symmetric w.r.t. duality transformations, but the associated Noether quantity vanishes. Replacing the fields by potentials and using instead a quadratic Klein-Gordon-type Lagrangian allows us to recover the double-Chern-Simons expression of conserved helicity and is shown to be equivalent to recently proposed alternative frameworks. Applied to the potential-modified theory the Dirac/Weyl-type approach yields again zero conserved charge, whereas the Klein-Gordon-type approach applied to the original setting yields Lipkin’s “zilch”.

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1. INTRODUCTION

The century-old problem of symmetry of the vacuum Maxwell equations under duality transformations \[ E \rightarrow \cos \theta E + \sin \theta B, \quad B \rightarrow -\sin \theta E + \cos \theta B, \] (1.1)

has attracted considerable recent attention [2-9]. In particular, the associated Noether quantity is the (optical) helicity [10], expressed as the integral of two Chern-Simons terms

\[ \chi = \frac{1}{2} \int_{\mathbb{R}^3} (A \cdot B - C \cdot E) d^3r, \] (1.2)

for the electromagnetic potential \( A \) and its dual \( C \) [2, 11].

While it is possible to obtain (1.2) within standard Maxwell theory using Noether’s theorem, such a derivation is somewhat complicated, since the Maxwell Lagrangian is not invariant under (1.1) [10, 11]. Dual-symmetric Lagrangians, have been proposed in [3, 5]. The aim of our “Variation on the Duality/Helicity Theme” here is to shed some new light on this old subject. We use suitably modified versions of the photon wave function — a concept which has, admittedly, a long history. The Dirac-type transcription of Maxwell’s
electromagnetism has been considered by E. Majorana as early as in 1928 [12]; see [4, 5, 13] for further details.

We start with Dirac/Weyl-type first-order equations advocated by Iwo and Zofia Bialynicki-Birula [14], eqn. (2.2) below. They are duality-symmetric however the associated helicity vanishes. Our main result here, presented in sec. 3, is to show that the non-trivial expression, (1.2) above, can be recovered, though, when the fields are replaced by potentials and a Klein-Gordon-type quadratic action is used.

We note also that while using the Dirac/Weyl-type first-order Lagrangian in the potential-setting would yield again zero charge, our quadratic Klein-Gordon-type approach applied to the original framework yields, instead, Lipkin’s “zilches” [15–17], — which is thus another manifestation of duality symmetry.

Our results provide us with a nice illustration to the theorem of Weinberg and Witten [18] on spin and helicity of massless particles.

2. THE PHOTON WAVE FUNCTION

Following Bialynicki-Birula [14] we rewrite the vacuum Maxwell equations as a wave equation reminiscent of Dirac and/or Weyl. Their starting point is that requiring that the Riemann-Silberstein vector

\[ F = \frac{1}{\sqrt{2}}(E + iB) \]  

satisfies the coupled system

\[ i \partial_t F = \nabla \times F, \quad \nabla \cdot F = 0 \]  

is equivalent to the vacuum Maxwell equations with \( \varepsilon_0 = \mu_0 = 1 \). In terms of the 3 \( \times 3 \) rotation matrices in the spin 1 representation, \((S_j)_{ab} = -i \varepsilon_{j,ab}, j = 1, 2, 3\), the first eqn in (2.2) and its complex conjugate can also be presented as

\[ \partial_t F = -(S \cdot \nabla)F \quad \text{and} \quad \partial_t F^* = (S \cdot \nabla)F^*. \]  

These two equations are plainly equivalent; note here the opposite signs. Then, imitating the Dirac procedure (understood intuitively as “taking the square root of the Klein-Gordon equation” [19, 20] ), they note that the spin-1 rotation matrices satisfy

\[ (S \cdot \nabla)(S \cdot \nabla) = \nabla^2 \]  

(2.4)
provided that the divergence condition $\nabla \cdot \mathbf{F} = 0$ holds also. Iterating (2.3) shows that each component of the electromagnetic field satisfies the wave equation,

$$[\partial_t^2 - \nabla^2] F_i = 0.$$  \hfill (2.5)

Conversely, (2.4) allows us to take the “square root” of the D’Alembert operator and to posit the two, equivalent equations in (2.3), supplemented by $\nabla \cdot \mathbf{F} = 0$.

The next step is to introduce a 6-component vector and the $6 \times 6$ matrices

$$\mathcal{F} = \begin{pmatrix} \mathbf{F}_+ \\ \mathbf{F}_- \end{pmatrix} \quad \rho_1 = \begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix} \quad \rho_3 = \begin{pmatrix} 1_3 & 0 \\ 0 & -1_3 \end{pmatrix} \quad \Sigma^\mu = \begin{pmatrix} 0 & \overline{S}^\mu \\ S^\mu & 0 \end{pmatrix},$$  \hfill (2.6)

$\mu = 0, \ldots, 3$, where $S^\mu = (1, \mathbf{S})$ and $\overline{S}^\mu = (1, -\mathbf{S})$ and to note that putting

$$F_- = \mathbf{F}^*, \quad F_+ = \mathbf{F} = \frac{1}{\sqrt{2}}(\mathbf{E} + i\mathbf{B})$$  \hfill (2.7)

unifies the two eqns (2.3) into a 6-component first-order Dirac-type equation supplemented with the divergence constraint,

$$\Sigma^\mu \partial_\mu \mathcal{F} = 0$$ \hfill (2.8a)

$$\nabla \cdot \mathcal{F} = 0.$$ \hfill (2.8b)

We stress that the conjugacy condition (2.7) is necessary for recovering the Maxwell theory from the extended one here.\footnote{1 Our strategy is analogous to what is usually done for the Schrödinger Lagrangian, where $\psi$ and $\psi^*$ are first viewed as independent; then one identifies the latter with the complex conjugate of the former after deriving the variational equation. A rigorous mathematical treatment would require using a Lagrange multiplier.}

The matrix $\rho_3$ acts diagonally but changes the sign of the lower component, allowing us to identify left and right helicity states as eigenvectors of $\rho_3$ with eigenvalues $\pm 1$.

The two massless 3-component equations with fixed helicities satisfied by $\mathbf{F}_\pm$ are uncoupled; they are the spin-1 counterparts of the Weyl equations, which describe neutrinos and antineutrinos with spin 1/2. $\rho_3$ is the analog of the chirality operator $\gamma^5$; $\rho_1$ intertwines the helicity components, $\rho_1 \mathcal{F}_\mp = \mathcal{F}_\pm$.

Now, inspired by the analogy with the Dirac/Weyl system, we propose an action principle for the Dirac-type equation (2.8a),

$$\mathcal{L}_F = \overline{\mathcal{F}} (\Sigma^\mu \partial_\mu) \mathcal{F} = \left( \mathbf{F}_-^\dagger \overline{\mathbf{S}}^\mu \partial_\mu \mathbf{F}_- + \mathbf{F}_+^\dagger \mathbf{S}^\mu \partial_\mu \mathbf{F}_+ \right), \quad \overline{\mathcal{F}} = \mathcal{F}^\dagger \Sigma^0.$$  \hfill (2.9)
Our Lagrangian is reminiscent of but still different from the one proposed by Drummond [21]. Treating \( \mathcal{F} \) and \( \mathcal{F}^\dagger \) as independent fields, the Euler-Lagrange equations reproduce eqn (2.8a) and its conjugate when \( \mathcal{F} = \mathcal{F}^\dagger \Sigma^0 \) is used. Expressing in electric and magnetic terms,

\[
\mathcal{L}_F = E \cdot (\partial_t E - \nabla \times B) + B \cdot (\partial_t B + \nabla \times E) \tag{2.10a}
\]

\[
= \partial_\mu \left( \frac{1}{2}(E^2 + B^2) \right) + \nabla \cdot (E \times B), \tag{2.10b}
\]

shows that \( \mathcal{L}_F \) is different from the usual e.m. Lagrange density \( \frac{1}{2}(E^2 - B^2) \); it is indeed the divergence of the current \( \mathcal{T}^\mu = (T^{00}, T^{i0}) \) associated with the usual electromagnetic energy-momentum tensor. \( \mathcal{L}_F = \partial_\mu \mathcal{T}^\mu \) vanishes therefore when the Maxwell equations, (2.2), are satisfied.

The theory given by (2.10) is duality invariant: the transformation (1.1), written as

\[
\mathcal{F} \rightarrow e^{-i\rho_3} \mathcal{F}, \tag{2.11}
\]

plainly leaves the Lagrange density (2.9) invariant because \( \rho_3 \) and \( \Sigma_\mu \) anticommute, \( \{\rho_3, \Sigma_\mu\} = 0 \), in analogy with what happens for Dirac/Weyl for spin \( 1/2 \). Then the Noether theorem provides us with the conserved current,

\[
k^\mu = \mathcal{F} \Sigma^\mu \rho_3 \mathcal{F} = F_+^\dagger S^\mu F_+ - F_-^\dagger S^\mu F_-, \quad \partial_\mu k^\mu = 0, \tag{2.12}
\]

which is reminiscent of the chiral current of a massless particle with spin \( 1/2 \). However, this current is identically zero when the conjugacy condition (2.7) is used,

\[
k^\mu \equiv 0 \quad \Rightarrow \quad \chi_{\mathcal{F}} = \int d^3r \, k^0 = \int d^3r \left( F_+^\dagger F_+ - F_-^\dagger F_- \right) = 0. \tag{2.13}
\]

We conclude that the theory in (2.8) is unsuitable to derive helicity, (1.2).

We also mention that this theory has further unusual aspects: for example, unlike the wave function in quantum mechanics, \( \mathcal{F} \) in (2.6) has no gauge degrees of freedom: the strict gauge invariance of the fields implies strict invariance for \( \mathcal{F} \). Note also that neither the action (2.9) or (2.10) nor do the further conserved quantities have correct physical dimension.

3. A WAVE FUNCTION COMPOSED OF POTENTIALS

Now we put forward our theory, – the main result of this paper. Our clue for obtaining nontrivial dual-symmetry is the observation that the wave equation (2.5) is satisfied also by
the electromagnetic potentials when the Lorentz gauge is chosen. We define therefore the new Riemann-Silberstein-type vectors $V_\pm$ by replacing fields by potentials in the definitions (2.1) and (2.2),

$$V_\pm = \frac{1}{\sqrt{2}}(A \pm iC),$$

(3.1a)

$$\nabla \times A = -\partial_t C (= B), \quad \nabla \times C = \partial_t A (= -E), \quad A^0 = C^0 = 0.$$ (3.1b)

Then the conjugacy condition (2.7) — but now for the potentials:

$$V_- = V_+^*,$$ (3.2)

is, once again, built into the theory. We have also incorporated a double gauge freedom, $V_\pm \to V_\pm + \nabla f \pm i\nabla g$, which is not that of a usual wave function but nevertheless legitimates the choice (3.1b), which imply $\nabla \cdot A = \nabla \cdot C = 0$. I.e., we choose the transverse Coulomb gauge, cf. [2, 3, 5]. Then both of our potentials verify the Lorenz gauge condition $\partial_\mu A^\mu = \partial_0 A^0 + \nabla \cdot A = 0$. When the Maxwell equations hold, each component of $V_\pm$ satisfies, once again, the free wave equation, (2.5), allowing us to postulate, conversely, new field equations for the new wave functions $V_\pm$, i.e., to require that

$$\partial_t V_\pm = \mp (S \cdot \nabla)V_\pm,$$ (3.3a)

$$\nabla \cdot V_\pm = 0$$ (3.3b)

hold.

Eqns (3.3) are of the first order in the potentials. Then taking divergences and curls allows us to deduce that they imply the vacuum Maxwell equations. From (3.3) we infer also that $\partial_t^2 V_\pm - (S \cdot \nabla)(S \cdot \nabla)V_\pm = 0$ which implies, using (2.4), two (equivalent) massless Klein-Gordon equations (D’Alembert equations),

$$\partial_\mu \partial^\mu V_\pm \equiv \left[ \partial_t^2 - \nabla^2 \right] V_\pm = 0.$$ (3.4)

(3.3) is therefore a square root of the K-G type wave equations satisfied by $V_\pm$. Remembering Klein-Gordon, we note that (3.4) derive, after putting $V_+ = V$ and $V_- = V^*$, from the manifestly dual-symmetric Lagrangian

$$L_V = \frac{1}{2}(\partial_\mu V_-) \cdot (\partial^\mu V_+).$$ (3.5)

\footnote{For the relativistic invariance of electromagnetic theory in Coulomb gauge, we refer to [22].}
Further insight is gained by noting that (3.5) is in fact equivalent to the one considered in [3, 5],

\[ \frac{1}{2} \left( \partial_{\mu} V_- \right) \cdot \left( \partial^{\mu} V_+ \right) = \frac{1}{8} \left[ F_{\mu\nu} F^{\mu\nu} + * F_{\mu\nu} * F^{\mu\nu} \right] - \frac{1}{4} \partial_i \left( A_j \partial_j A_i + C_j \partial_j C_i \right). \]  

(3.6)

Direct use of (3.1b) in (3.5) yields a vanishing Lagrangian [3, 5]. Therefore, in analogy with what is done for the Schrödinger Lagrangian (see our footnote # 1 above) and also for the complex, scalar K-G theory, we first consider \( V_+ \) and \( V_- \) as independent and derive the equations of motion (3.4) by treating \( V_\pm \) separately, before inserting the constraint (3.2).

The definitions (3.1b) will be used in eqn. (3.8) – (3.11b).

Turning now to duality, it is readily seen that (1.1), implemented on the potentials as \( A \to A \cos \theta + C \sin \theta, \ C \to C \cos \theta - A \sin \theta \), i.e.,

\[ V_\pm \to V_\pm e^{\mp i \theta}, \]  

(3.7)

leaves (3.3) invariant, establishing the duality symmetry of the system we propose. In fact, the action \( S = \int d^4 x \, L_V \) is manifestly invariant under (3.7). The infinitesimal version of the latter, \( \delta V = -i \theta V, \ \delta V^* = i \theta V^* \), allows us to infer the Noether current

\[ j^\mu = \frac{1}{2} \left( \partial^\mu V \cdot \delta V^* + \partial^\mu V^* \cdot \delta V \right) = \frac{1}{2} \left( (\partial^\mu A) \cdot C - (\partial^\mu C) \cdot A \right), \]  

(3.8)

whose conservation, \( \partial_\mu j^\mu = 0 \), can also be checked directly using (3.4). The associated conserved charge is the space integral of the zeroth component,

\[ \chi = \int d^3 r \, \frac{1}{2} \left( \partial_t A \cdot C - \partial_t C \cdot A \right) = \int d^3 r \, \frac{1}{2} \left( -E \cdot C + B \cdot A \right), \]  

(3.9)

by (3.1) — where we recognize (1.2), the “double Chern-Simons” expression of helicity [2, 3, 5, 8, 10, 11]. Conversely, the charge (3.9) [i.e. 1.1] generates the duality action (3.7).

We note also that the space part of (3.8) is [up to a surface term] the spin density [3],

\[ j = S = \frac{1}{2} (E \times A + B \times C). \]  

(3.10)

The constraint (3.2) does not now imply the vanishing of the helicity in (3.9) : one of the factors has been changed into a field strength, cf. (3.1). The integral (3.9) i.e. (1.2) can indeed be evaluated using Fourier transformation to momentum space [2], showing that
the helicity is proportional to the difference of the number of left- and right-handed photons, \( \chi = n_L - n_R \). Similar formulae hold for the current (3.10), [4].

The energy-momentum tensor of our theory, \( T^{\mu\nu} \), is symmetric. Spelled out in terms of fields and potentials it is,

\[
T^{00} = \frac{1}{4} \left( \partial_\mu A \cdot \partial_\nu A + \partial_\mu C \cdot \partial_\nu C + \partial_\mu A \cdot \partial_\nu C + \partial_\mu C \cdot \partial_\nu A \right),
\]

\[
T^{0i} = \frac{1}{2} \left( \partial_\mu A \cdot \partial^i A + \partial_\mu C \cdot \partial^i C \right),
\]

(3.11a, 3.11b)

Its conservation, \( \partial_\nu T^{\mu\nu} = 0 \), can be checked also directly. \( T^{00} \) and \( T^{0i} \) are, up to surface terms, the usual expressions of the energy density and of the Poynting vector, respectively.

4. TWO MORE “VARIATIONS”

(i) Now we shortly discuss our other approaches. The \( \mathbf{V}_\pm \) could again be unified into a 6-component system by putting \( \mathbf{V} = \begin{pmatrix} \mathbf{V}_+ \\ \mathbf{V}_- \end{pmatrix} \). The two helicity components are interchanged by \( \rho_1 \). The two upper equations in (3.3) are also unified and are supplemented with the divergence constraint,

\[
\Sigma^\mu \partial_\mu \mathbf{V} = 0,
\]

\[
\nabla \cdot \mathbf{V} = 0,
\]

(4.1a, 4.1b)

as in (2.8) : once again, we get Dirac / Weyl type analogs. The Lagrangian (3.5) can also be written as in (2.9) but with \( \mathbf{V}_\pm \) replacing \( \mathbf{F}_\pm \),

\[
\mathcal{L}_\mathbf{V} = \nabla (\Sigma^\mu \partial_\mu \mathbf{V}) = \left( \mathbf{V}_+^\dagger S^\mu \partial_\mu \mathbf{V}_- + \mathbf{V}_+^\dagger S^\mu \partial_\mu \mathbf{V}_+ \right), \quad \nabla \mathbf{V} = \mathbf{V}^\dagger \Sigma^0,
\]

(4.2)

which is again 4-divergence,

\[
\mathcal{L}_\mathbf{V} = \partial_i \left( \frac{1}{2} (A^2 + C^2) \right) + \nabla \cdot (A \times C)
\]

(4.3)

which vanishes when the field equations are satisfied, cf. (2.10).

The Lagrangian (4.2) is invariant w.r.t. duality, (3.7), and yields a Noether current similar to (2.12),

\[
\ell^\mu = \nabla \Sigma^\mu \rho_3 \mathbf{V} = \mathbf{V}_+^\dagger S^\mu \mathbf{V}_+ - \mathbf{V}_-^\dagger \bar{S}^\mu \mathbf{V}_-.
\]

(4.4)
However the current and the to-be helicity vanish again due to $V^* = V$,
\[ \ell^\mu \equiv 0 \Rightarrow \chi_V = \int \ell^0 d^3r = \int (V^*_+ + V^*_-) d^3r = 0. \tag{4.5} \]

We conclude that the Dirac-type approach yields, once again, trivial current and charge.

(ii) What would our Klein-Gordon trick yield for the original setting of Section 2? All components of the RS vector $F$ satisfy the wave equation (2.5) which can in turn be derived from the Klein-Gordon-type Lagrangian analogous to (3.5),
\[ L_F = \frac{1}{2} (\partial_\mu E^* \cdot \partial^\mu F) = \frac{1}{4} \left( \partial_\mu E \cdot \partial^\mu E + \partial_\mu B \cdot \partial^\mu B \right). \tag{4.6} \]

This Lagrangian is plainly symmetric under duality (2.11) with associated Noether current
\[ z_\mu = \frac{1}{2} \left( (\partial_\mu E) \cdot B - (\partial_\mu B) \cdot E \right), \tag{4.7} \]
whose time component is a conserved charge,
\[ Z = \int d^3r \frac{1}{2} \left( (\partial_t E) \cdot B - (\partial_t B) \cdot E \right) = \int d^3r \frac{1}{2} \left( B \cdot \nabla \times B + E \cdot \nabla \times E \right), \tag{4.8} \]
upon using the Maxwell equations (4.8). This expression is reminiscent of (3.9) but with field strengths instead of potentials (consistently with (2.1) vs (3.1)). It is in fact Lipkin’s “Z^000-zilch” [15]. Its space part,
\[ z = \frac{1}{2} \int d^3r \left( E \times (\nabla \times B) - B \times (\nabla \times E) \right), \tag{4.9} \]
is in turn Lipkin’s $Z^{0i0} = Z^{00i}$, identified as the optical chirality flow, cf. eqn. # (8.1) in [4].

5. CONSISTENCY WITH THE WEINBERG-WITTEN THEOREM

We established the duality symmetry of four different frameworks, all related to Maxwell’s electromagnetism, — and got different conserved quantities: two of them are identically zero, the two others are non-trivial. How could this come about? The answer is provided by Weinberg and Witten [18]:

Theorem 1. A theory that allows the construction of a Lorentz-covariant conserved four-current $J^\mu$ cannot contain massless particles of spin $J > 1/2$ with non-vanishing values of the conserved charge $\int J^0 d^3r$. 

9
The Lorentz-covariant currents (2.12) and (4.4), derived from a first-order Dirac/Weyl-type Lagrangian for spin-1 duly vanish. In the quadratic Klein-Gordon-type cases the non-trivial currents (3.8) and (4.7) are not Lorentz covariant, though. Consider a Lorentz boost along the \( z \) axis with parameter \( v \). For the zilch we find, for example,

\[
(z^x)' = z^x + \gamma v \left\{ (\partial_x E_x) E'_y - (\partial_x B_y) B'_x + (\partial_x B_x) B'_y - (\partial_x E_y) E'_x \right\}
\]

where \( \gamma = (\sqrt{1 - v^2})^{-1/2} \), instead of \( (z^x)' = z^x \), as would be required for a Lorentz vector. The “helicity-generating current” \( j^\mu \) in (3.8) behaves similarly. Therefore, the Weinberg-Witten theorem does not apply to these cases [18] allowing for non-zero charges — namely optical helicity (1.2), or the “zilch”, (4.8).

6. CONCLUSION

The concept of a “photon wave function” has long been debated; here we merely used it as a trick to rewrite electromagnetism in a Dirac/Weyl resp. Klein-Gordon-type form, allowing us to use field theoretical tools. Our trick of replacing the e.m. fields by the respective potentials works because all components satisfy, in the Lorentz gauge, the wave equation (2.5), allowing for the “square root trick”. Then the transcription (3.1) allows us to derive the duality/helicity correspondence mimicking the procedure used for spin 1/2.

From our four “variations”, the Dirac-types have vanishing helicity. Our preference goes therefore to the Klein-Gordon type theory with potentials, discussed in section 3 and listed in the third row of Table I. It is equivalent to the double-CS-type theories advocated in [3–5]. When applied to the original framework of Section 2, it yields instead Lipkin’s “zilch”, which are hence also associated with duality symmetry.

Our findings fit perfectly into the hierarchy pattern [4, 16, 17] : The original theory presented in Sect. 2 is in fact obtained by replacing the potentials by their curls (i.e. the e.m. fields themselves) in the theory we propose in Sect. 3. The duality action (3.7) goes over into that on the e.m. fields, eqn. (1.1), whereas the “true” helicity, (1.2), goes over into the “zilch”, (4.8) [15–17].
**TABLE I:** Duality-invariant wave transcriptions of electromagnetism. $F$ is the Riemann-Silberstein vector and $V$ is obtained when fields are replaced by potentials. The first-order “Dirac-type” transcriptions are covariant and have zero conserved current and charge. The quadratic “Klein-Gordon-type” theories are not Lorentz-covariant and have non-trivial charges, namely helicity for $V$, and “zilch” for $F$.

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[1] O. Heaviside, Phil. Trans. R. Soc. A 183 423 (1892); J. Larmor, Phil. Trans. R. Soc. A 190 205 (1897)
[2] G. N. Afanasiev, Yu. P. Stepanovsky, “The helicity of the free electromagnetic field and its physical meaning,” Il Nuovo Cimento. 109, 271 (1996).
[3] K. Y. Bliokh, A.Y. Beksaev and F. Nori, “Dual electromagnetism: helicity, spin, momentum and angular momentum,” New Journal of Physics 15 (2013) 033026.
[4] R. P. Cameron, S. M. Barnett and A. M. Yao, “Optical helicity, optical spin and related quantities in electromagnetic theory,” New J. Phys. 14 053050 (2012);
[5] R. P. Cameron and S. M. Barnett, “Electric-magnetic symmetry and Noether’s theorem,” New J. Phys. 14 123019 (2012).
[6] I. Fernandez-Corbaton, et al. “Electromagnetic Duality Symmetry and Helicity Conservation for the Macroscopic Maxwell’s Equations,” Phys. Rev. Lett. 111 (2013) 6, 060401.

[7] C. Manuel and J. M. Torres-Rincon, “Dynamical evolution of the chiral magnetic effect: Applications to the quark-gluon plasma,” Phys. Rev. D 92 (2015) 074018.

[8] M. Elbistan, C. Duval, P. A. Horvathy and P.-M. Zhang, “Duality and helicity: a symplectic viewpoint,” Phys. Lett. B761 265 (2016).

[9] I. Agullo, A. del Rio and J. Navarro-Salas, “Electromagnetic duality anomaly in curved spacetimes,” Phys. Rev. Lett. 118 (2017), 111301.

[10] M. G. Calkin, “An invariance property of the free electromagnetic field,” Am. J. Phys. 33, 958 (1965).

[11] S. Deser and C. Teitelboim, “Duality Transformations of Abelian and Nonabelian Gauge Fields,” Phys. Rev. D 13 (1976) 1592.

[12] R. Mignani, E. Recami, M. Baldo, “About a Dirac-Like Equation for the Photon according to Ettore Majorana,” Lett. Nuovo Cimento, 11, 568 (1974)

[13] J. Dressel, K. Y. Bliokh and F. Nori, “Spacetime algebra as a powerful tool for electromagnetism,” Phys. Rept. 589 (2015) 1.

[14] I. Bialynicki-Birula, “On the wave function of the photon,” Acta Phys. Pol. 86, 97 (1994); “Photon wave function,” Progress in Optics, Vol. 36, E. Wolf, Editor, Elsevier, Amsterdam, (1996) I. Bialynicki-Birula and Z. I. Bialynicki-Birula, “The role of the Riemann-Silberstein vector in classical and quantum theories of electromagnetism,” J. Phys. A 46, 053001 (2013).

[15] D. M. Lipkin, “Existence of a new conservation law in electromagnetic theory,” J. Math. Phys. 5 696 (1964)

[16] D. J. Candlin “Analysis of the new conservation law in electromagnetic theory,” Il Nuovo Cimento 37 1390 (1965)

[17] T.W.B. Kibble, “Conservation laws for free fields,” J. Math. Phys. 6 1022 (1964)

[18] S. Weinberg and E. Witten, “Limits on massless particles,” Phys. Lett. B 96 59 (1980).

[19] P. A. M. Dirac, Principles of Quantum Mechanics, (International Series of Monographs on Physics) 4th Edition. Oxford University Press (1958)

[20] J. D. Björken and S. D. Drell, Relativistic Quantum Mechanics, N.Y. McGraw-Hill (1964)

[21] P. D. Drummond, “Dual symmetric Lagrangians and conservation laws,” Phys. Rev. A 60, R3331(R) (1999); “Dual-symmetric Lagrangians in quantum electrodynamics: I. Conservation
laws and multi-polar coupling," J. Phys. B: At. Mol. Opt. Phys. 39 (2006) S573.

[22] C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, *Photons and Atoms: Introduction to Quantum Electrodynamics*, New York: Wiley (1989).