Composition operators between two different bilateral grand Lebesgue spaces

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Abstract.

In this paper we consider composition operator generated by nonsingular measurable transformation between two different Grand Lebesgue Spaces (GLS); we investigate the boundedness, compactness and essential norm of composition operators.

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1. INTRODUCTION

Let \((X = \{x\}, A, \mu)\) be measurable space equipped with a non-zero sigma-finite measure \(\mu\). Denote by \(M_0 = M_0(X)\) the linear set of all numerical measurable functions \(f : X \to R\). Let also \(\xi = \xi(x)\) be measurable function from \(X\) to itself: \(\xi : X \to X\).

Definition 1.1.

The linear operator \(U_\xi[f] = U_\xi[f](x)\) defined may be on some Banach subspace \(B_1\) of the space \(M_0(X)\) to another, in general case, space \(B_2\) by the formula

\[ U_\xi[f](x) = f(\xi(x)) \] (1.1)

is said to be composition operator generated by \(\xi(\cdot)\).

Many important properties on these operators acting on different spaces \(B_1\) with values in \(B_2\) : Lebesgue, Lorentz, mainly Orlicz spaces etc., namely: boundedness, compactness, the exact values of norm are investigated, e.g. in [15 - 29].

Applications other than those mentioned appears in ergodic theory, see [34 - 37].

Our purpose in this short article is investigation of these operators in Grand Lebesgue spaces (GLS).
Another operators acting in these spaces: Hardy, Riesz, Fourier, maximal, potential etc. are investigated, e.g. in [9], [12], [13], [30], [31].

2. Grand Lebesgue Spaces (GLS).

We recall first of all here for reader conventions some definitions and facts from the theory of GLS spaces.

Recently, see [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], etc. appear the so-called Grand Lebesgue Spaces GLS

\[ G(\psi) = G = G(\psi; A; B); \; A; B = \text{const}; \; A \geq 1, \; B \leq \infty \]

spaces consisting on all the measurable functions \( f : X \to R \) with finite norms

\[ ||f||_{G(\psi)} \overset{\text{def}}{=} \sup_{p \in (A; B)} \left[ \frac{|f|_p}{\psi(p)} \right], \quad (1.2) \]

where as usually

\[ |f|_p = \left[ \int_X |f(x)|^p \mu(dx) \right]^{1/p}. \]

Here \( \psi = \psi(p), \; p \in (A, B) \) is some continuous positive on the open interval \( (A; B) \) function such that

\[ \inf_{p \in (A; B)} \psi(p) > 0. \quad (1.3) \]

We define formally \( \psi(p) = +\infty, \; p \notin [A, B] \).

We will denote

\[ \text{supp}(\psi) \overset{\text{def}}{=} (A; B). \]

The set of all such a functions with support \( \text{supp}(\psi) = (A, B) \) will be denoted by \( \Psi(A, B) \).

This spaces are rearrangement invariant; and are used, for example, in the theory of Probability, theory of Partial Differential Equations, Functional Analysis, theory of Fourier series, Martingales, Mathematical Statistics, theory of Approximation etc.

Notice that the classical Lebesgue - Riesz spaces \( L_p \) are extremal case of Grand Lebesgue Spaces; the exponential Orlicz spaces are the particular cases of Grand Lebesgue Spaces, see [11], [12].

Let a function \( f : X \to R \) be such that

\[ \exists (A, B) : 1 \leq A < B \leq \infty \Rightarrow \forall p \in (A, B) \; |f|_p < \infty. \]

Then the function \( \psi = \psi(p) \) may be naturally defined by the following way:

\[ \psi_f(p) := |f|_p, \; p \in (A, B). \quad (1.4) \]
3. main result.

0. Recall that the homothety operator \( T_\lambda[g] \) for arbitrary numerical function \( g \) is defined as ordinary:

\[
T_\lambda[g] = \lambda \cdot g, \quad \lambda = \text{const}.
\]

1. We suppose first of all that the source function \( f = f(x) \) belongs to some Grand Lebesgue Space \( G\psi \):

\[
|f|_p \leq ||f||_{G\psi} \cdot \psi(p), \quad p \in (A, B). \tag{3.0}
\]

Evidently, the function \( \psi(\cdot) \) may be natural for \( f(\cdot) \), if it is non-trivial. In this case \( ||f||_{G\psi} = 1 \).

2. Suppose also that the distribution \( \xi \) is absolute continuous relative the measure \( \mu \) with the Radon - Nikodym derivative \( h = h(x) \):

\[
\mu\{y : \xi(y) \in A\} = \int_{A} h(x) \mu(dx). \tag{3.1}
\]

The necessity of this condition is discussed in [18, 19].

Assume further that the function \( h = h(x) \) belongs to some \( G\theta \) space:

\[
|h|_q \leq ||h||_{G\theta} \cdot \theta(q), \quad q \in (a, b). \tag{3.2}
\]

In particular, the function \( \theta(\cdot) \) may be natural for \( h(\cdot) \), if of course it is non-trivial. In this case \( ||h||_{G\theta} = 1 \).

3. Let us introduce a new function \( \nu(p) = \psi \circ \theta(p) = [\psi \circ \theta](p) \) by the following way:

\[
\nu(p) = \psi \circ \theta(p) := \inf_{\alpha > 1} \left\{ \psi(\alpha p) \cdot (T|_{|h|_{G\theta}}) [\theta]^{1/p} \left( \frac{\alpha}{\alpha - 1} \right) \right\}. \tag{3.3}
\]

It will be presumed that it is finite for some non empty interval \( p \in (c, d), \quad 1 \leq c < d \leq \infty \).

**Theorem 3.1.** We propose under all formulated in this section conditions:

\[
||U_\xi[f]||_{G(\psi \circ \theta)} \leq ||f||_{G\psi}. \tag{3.4}
\]

**Proof.** We have using Hölder inequality and denoting for brevity \( \beta = \alpha'/\alpha - 1, \quad \alpha > 1 \):

\[
|U_\xi[f]|_p = \int_{X} |f(\xi(x))|^p \mu(dx) = \int_{X} |f(x)|^p \cdot h(x) \mu(dx) \leq \left[ \int_{X} |f(x)|^{\alpha p} \ d\mu \right]^{1/\alpha} \cdot \left[ \int_{X} |h(x)|^{\beta p} \ d\mu \right]^{1/\beta} = |f|_{\alpha p} \cdot |h|_{\beta}, \tag{3.5}
\]

or equally

\[
|U_\xi[f]|_p \leq |f|_{\alpha p} \cdot |h|_{\beta}^{1/p}. \tag{3.6}
\]
It follows from the direct definition of GLS that

\[ |f|_{\alpha p} \leq ||f||G\psi \cdot \psi(\alpha p); \quad |h|_{\beta} \leq ||h||G\theta \cdot \theta(\beta), \quad (3.7) \]

and we get after substituting into (3.6):

\[ |U_{\xi}[f]|_{p} \leq ||f||G\psi \cdot [\psi(\alpha p) |||h||G\theta \cdot \theta(\alpha/(\alpha - 1))]^{1/p} . \quad (3.8) \]

We deduce after minimization over \( \alpha; \ \alpha > 1 \)

\[ |U_{\xi}[f]|_{p} \leq \nu(p), \]

which is equivalent to the assertion of theorem 3.1.

**Remark 3.1.** We conclude in the case of natural pick of both the functions \( \psi(\cdot) \) and \( \theta(\cdot) \):

\[ \nu(p) = \psi \circ \theta(p) := \inf_{\alpha > 1} \left\{ \psi(\alpha p) \cdot [\theta]^{1/p} \left( \frac{\alpha}{\alpha - 1} \right) \right\} . \quad (3.9) \]

and correspondingly

\[ ||U_{\xi}[f]||G\nu \leq 1. \quad (3.10) \]

4. **Examples.**

**Example 1.** (Exactness.)

Let us show that the (implicit) coefficient ”1” in the right-hand side of inequality (3.4) is in general case non-improvable.

It is sufficient to consider a very simple example when \( \xi(x) = x; \) then \( h(x) = 1. \)

**Example 2.** (Power substitution.)

Let \( X = (0, 1) \) with Lebesgue measure \( d\mu = dx. \) Let now \( f : X \to R \) be any function from the space \( G\psi. \) We choose \( \xi(x) = x^m, \ m = \text{const} > 0, \) (not necessary to be integer), so that

\[ g(x) = g_m[f](x) \overset{\text{def}}{=} f(x^m). \quad (4.1) \]

Define a new function

\[ \psi^{(m)}(p) = \inf_{\alpha > \min(1, m)} \left\{ \left[ m^{-1/\alpha} (\alpha - 1)^{(\alpha - 1)/\alpha} (\alpha - m)^{-1-1/\alpha} \right] \cdot \psi(\alpha p) \right\} , \quad (4.2) \]

and suppose this function is non-trivial: \( \exists (c, d) : 1 \leq c < d \leq \infty \Rightarrow \psi^{(m)}(p) < \infty. \)

It follows from theorem 3.1 that

\[ ||g_m[f]||G\psi^{(m)} \leq 1 \times ||f||G\psi, \quad (4.3) \]

and the constant ”1” in (4.3) the best possible.

**Example 3.** (Counterexample).

Let again \( X = (0, 1) \) and define
\[ f(x) = x^{-1/2}, \quad \xi(x) = x^3. \] (4.4)

Then
\[ |f|_p = \left[ \frac{2}{2 - p} \right]^{1/p} =: \psi(p), \quad 1 \leq p < 2; \] (4.5)

\[ h(z) = 3z^{-2/3}, \quad 0 < z \leq 1; \quad |h|_q = 3^{1/p-1} \cdot (3 - 2p)^{-1/p} =: \theta(p), \quad 1 \leq q < 3/2; \] (4.6)
or equally \( f(\cdot) \in G\psi, \ h \in G\theta, \) but the superposition function \( g(x) = f(\xi(x)) = x^{-3/2} \)
does not belong to any \( L_p(X) \) space with \( p \geq 1 \).

**Example 4.** (Linear substituting).

Let here \( X = \mathbb{R}^d \) with Lebesgue measure and \( f : \mathbb{R}^d \to \mathbb{R} \) be some function belonging to the space \( G\psi \).

Define an operator of a view
\[ V_A[f] = f(Ax). \] (4.7)

Obviously,
\[ |V_A[f]|_p = \int_{\mathbb{R}^d} |f(Ax)|^p \, dx = \int_{\mathbb{R}^d} |\det(A)|^{-1} |f(y)|^p \, dy = \]
\[ |\det(A)|^{-1/2} |f|_p, \]
or equally
\[ |V_A[f]|_p = |\det(A)|^{-1/p} |f|_p \] (4.8)
and following
\[ |V_A[f]|_p \leq |\det(A)|^{-1/p} \cdot ||f||G\psi \cdot \psi(p). \] (4.9)

Let the function \( \psi(\cdot) \) be factorable:
\[ \psi(p) = \frac{\zeta(p)}{\tau(p)}, \quad p \in (A, B), \] (4.10)
where both the functions \( \zeta(\cdot), \ \tau(\cdot) \) are from the set \( G\Psi \), i.e. satisfy all the conditions imposed on the function \( \psi(\cdot) \). We deduce after dividing the inequality (4.9) on the function \( \zeta(p) : \)
\[ \frac{|V_A[f]|_p}{\zeta(p)} \leq ||f||G\psi \cdot \frac{|\det(A)|^{1/p}}{\tau(p)} \] (4.11)

Recall now that the fundamental function \( \phi(G\tau, \ \delta) , \ 0 \leq \delta \leq \mu(X) \) for the Grand Lebesgue Space \( G\tau \) may be calculated by the formula
\[ \phi(G\tau, \ \delta) = \sup_{p \in (A, B)} \left[ \frac{\delta^{1/p}}{\tau(p)} \right]. \]

This notion play a very important role in the theory of operators, Fourier analysis etc., see [1]. The detail investigation of the fundamental function for GLS is done in [9], [11].
Taking the maximum over \( p \in (A, B) \) from both the sides of inequality (4.11), we get to the purpose of this subsection: under condition (4.10)

\[
\|V_A[f]\|_{G\zeta} \leq \|f\|_{G\psi} \cdot \phi(G\tau, |\det(A)^{-1})).
\]  

(4.12)

5. Compactness of the composition operator on GLS.

We suppose in this section that the measure \( \mu \) is diffuse; this imply by known definition that for arbitrary measurable set \( A \) having positive measure: \( \mu(A) > 0 \) there exists a measurable subset \( B : B \subset A \) for which \( \mu(B) = \mu(A)/2 \).

We investigate in this section the compactness of operator \( U_\xi[\cdot] \) as an operator acting from source Grand Lebesgue Space \( G\psi \) into another GLS \( G\gamma \).

For Orlicz’s, Lebesgue-Riesz, Lorentz and other Banach spaces this question is considered in [15], [16], [18], [19], [23], [20], [22], [25] - [28].

Let the function \( \nu = \psi \circ \theta \), \( \text{supp } \nu = (a, b) \) be at the same as defined in (3.3).

Theorem 5.1. Assume that

\[
\sup_{p \in (a, b)} \left[ \frac{\nu(p)}{\gamma(p)} \right] < \infty
\]  

(5.1)

and moreover

\[
\lim_{\gamma(p) \to \infty} \left[ \frac{\nu(p)}{\gamma(p)} \right] = 0.
\]  

(5.2)

(This condition may be omitted if the function \( \gamma = \gamma(p) \) is bounded.)

Then the operator \( U_\xi(\cdot) \) is compact operator from the space \( G\psi \) into the space \( G\gamma \).

Proof. Denote

\[
B = \{ f : \|f\|_{G\psi} \leq 1 \}, \quad U_\xi(B) = \{ U_\xi[f], \|f\|_{G\psi} \leq 1 \}.
\]  

(5.3)

Here the symbol \( B \) stands for unit ball in the space \( G\psi \).

It is sufficient to prove that the set \( U_\xi(B) \) is compact set in the space \( G\gamma \). In turn it is sufficient to prove in accordance with the articles [32], [33] that

\[
\lim_{\gamma(p) \to \infty} \sup_{f \in B} \left[ \frac{\|U_\xi[f]\|_p}{\gamma(p)} \right] = 0.
\]  

(5.4)

It follows from theorem 3.1 that the set \( U_\xi(B) \) is also bounded in the \( G\nu \) norm:

\[
\sup_{f \in B} \|U_\xi(f)\|_{G\nu} \leq 1,
\]

or equally

\[
\sup_{f \in B} |U_\xi(f)|_p \leq \nu(p),
\]

therefore the left-hand side of inequality (5.4) may be estimated as follows:
\[
\lim_{\gamma(p) \to \infty} \sup_{f \in B} \left[ \frac{|U_\xi[f]|_p}{\gamma(p)} \right] \leq \lim_{\gamma(p) \to \infty} \left[ \frac{\nu(p)}{\gamma(p)} \right] = 0,
\]
Q.E.D.

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