Renormgroup algorithm for the theory of the relativistic plasma resonance

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Abstract. Renormgroup algorithm is used to reveal the nonlinear space-time stationary structure of the potential electric field in the vicinity of a plasma resonance. A two parameter transformation group with two renormgroup parameters related to non-relativistic and relativistic nonlinear effects is constructed and then used to transform the perturbation theory approximation to the desired electric field structure. Further applications of the obtained solution to problems of laser-plasma physics are discussed.

1. Introduction

The renormalization group (RG) algorithm was introduced in mathematical physics at the end of XX-th century [1–3] with the same aim of finding an improved solution (in comparison with the initial approximate solution) as the classical algorithm of Bogoliubov’s RG method [4–7], but its implementation in mathematical physics is based on techniques of the modern group analysis. Since the appearance and the first successful application of RG algorithm in plasma physics [3] the approach proved its efficiency for many nonlinear problems of mathematical physics [8]. Here this approach is used to find the nonlinear space-time stationary structure of the potential electric field, excited by a powerful laser radiation in the vicinity of a relativistic plasma resonance. Nonlinear effects in the vicinity of the plasma resonance are of considerable importance for analyzing nonlinear absorption and harmonics generation in the laser plasma [9], and the component of the electric field, polarized along a plasma inhomogeneity gradient, plays the dominant role in these processes. The stationary structure of this field was discussed in detail in [10, 11] in the linear, perturbation theory (PT) approximation. Increasing the laser intensity makes it topical to involve nonlinear effects of the laser-plasma interaction into consideration even for non-relativistic electron velocities. At even higher laser intensities relativistic effects for plasma electrons near the plasma resonance becomes essential. Here we present an analytical approach for finding the stationary solution for the equations that describe nonlinear relativistic plasma oscillations in the vicinity of the plasma resonance. The key point in constructing this solution is the use of the RG algorithm.
2. Solution of basic equations via RG algorithm

Basic equations for the potential electric field, excited by the \( p \)-polarized laser wave in the inhomogeneous plasma with a density scale \( L \) in the vicinity of a plasma resonance \( x = 0 \), where the laser frequency \( \omega \) coincides with the Langmuir frequency of plasma electrons \( \omega_L \), are presented by a system of two nonlinear partial differential equations for electron dynamics and electric field structure,

\[
\partial_t v + av \partial_x v = P (1 - \beta v^2)^{3/2}, \quad \partial_t P + av \partial_x P = -\omega_0^2 v .
\]

Here \( a \) is the dimensionless constant proportional to the magnetic field amplitude in the plasma resonance point \( x = 0 \) in the linear approximation, which is defined by the laser wave amplitude; \( \beta = a^2/c^2 \), \( c \) is the speed of light; \( v \) and \( P \) are values of the electron velocity and the potential electric field along the gradient of the plasma inhomogeneity, which are normalized to the parameter \( a \).

Notice that equations (1) contain two parameters, \( a \) and \( \beta \), which make contributions to the nonrelativistic and the relativistic nonlinearities, respectively. Generally speaking, a measure of relativistic nonlinearity is the parameter \( \beta \sim 1/c^2 \) which tends to zero in the nonrelativistic limit. However, the parameter selection in the form \( \beta = a^2/c^2 \) simplifies further calculations. In the limiting case of \( a \to 0, \beta \to 0 \) equations (1) have the well-known solution \cite{10} that describes the plasma resonance in the linear approximation. In non-relativistic case, \( \beta = 0 \), equations (1) by the use of the RG algorithm were solved and analyzed in \cite{3}. We extend now this result to the relativistic case with \( \beta \neq 0 \).

Following the general scheme of the RG algorithm \cite{8} we first define the basic RG-manifold as equations (1) in the extended space of independent variables that now include parameters \( a \) and \( \beta \) and construct the PT solution of (1) in the form of truncated perturbation series in powers of \( a \) and \( \beta \). Next, using the group analysis technique we calculate the symmetry group admitted by the basic equations via RG algorithm \cite{8}. The coordinates of (2) can be defined from the solution of an overdetermined system of differential equations, named as determining equations,

\[
X_{(1)} (\partial_t v + av \partial_x v - (1 - \beta v^2)^{3/2} P)|_{(1)} = 0 ,
\]

\[
X_{(1)} (\partial_t P + av \partial_x P + \omega_0^2 v)|_{(1)} = 0 ,
\]

that follow from the invariance test for the basic RG-manifold (1) with respect to the extended group with the group generator \( X_{(1)} \). Here \( X_{(1)} \) defines the prolongation of the generator \( X \) to the first derivatives and \( |_{(1)} \) means that the result of the action of the operator \( X_{(1)} \) is taken on the manifold defined by the equations (1) and all their differential consequences,

\[
X_{(1)} = X + \xi^1 \partial_v + \xi^2 \partial_x + \xi^3 \partial_\beta + \eta^1 \partial_v + \eta^2 \partial_P ,
\]

\[
\xi^1 = D_1 \eta^1 - \partial_t v D_1 \xi^1 - \partial_x v D_1 \xi^2 - \partial_\beta v D_1 \xi^3 - \partial_\beta v D_1 \xi^4 ,
\]

\[
\xi^2 = D_2 \eta^2 - \partial_t v D_2 \xi^1 - \partial_x v D_2 \xi^2 - \partial_\beta v D_2 \xi^3 - \partial_\beta v D_2 \xi^4 ,
\]

\[
\xi^3 = D_3 \eta^3 - \partial_t v D_3 \xi^1 - \partial_x v D_3 \xi^2 - \partial_\beta v D_3 \xi^3 - \partial_\beta v D_3 \xi^4 ,
\]

\[
\partial_t \equiv \partial_t + \partial_t v \partial_v + \partial_t P \partial_P , \quad D_x \equiv \partial_x + \partial_x v \partial_v + \partial_x P \partial_P .
\]
In view of (1), (2) and (4), the system of determining equations for coordinates $\xi^i$ and $\eta^j$ is reduced to the following form,

$$a\eta^1 + avY \xi^1 - Y \xi^2 + v \xi^3 = 0, \quad Y \eta^1 - z^{3/2} (PY \xi^1 + \eta^2) + \frac{3}{2} (1 - z) Pz^{1/2} \left[ \frac{\xi^4}{\beta} + 2 \eta^1 \right] = 0,$$

$$Y [a\eta^2 + \xi^2 \omega_0^2 + \xi^3 P] = 0, \quad Y \xi^3 = 0, \quad Y \xi^4 = 0,$$

where we introduce the operator

$$Y \equiv \partial_t + av\partial_x + Pz^{3/2} \partial_v - \omega_0^2 v \partial_P, \quad z \equiv 1 - \beta v^2. \quad (6)$$

Omitting the calculation details we present the final expressions for the coordinates $\xi^i$ and $\eta^j$ of the group generator (2),

$$\xi^1 = -\eta^2/\omega_0^2 v + \xi_1(I_1, I_2, I_3) + \frac{1}{\beta} \left[ \frac{6 + \beta I_1}{2\sqrt{4 + \beta I_1}} E(\rho, \sigma) - \frac{1}{\sqrt{4 + \beta I_1}} F(\rho, \sigma) + \sqrt{\beta P/\omega_0} \sqrt{\frac{1/\sqrt{z} - 1}{1 + 1/\sqrt{z}}} \right],$$

$$\xi^2 = -a\eta^2/\omega_0^2 - \xi^3 P/\omega_0^2 + \xi_2(I_1, I_2, I_3), \quad \eta^1 = \frac{z^{3/2}}{v} \left[ \eta_1(I_1, I_2, I_3) - (P/\omega_0^2) \eta^2 - \xi^4 \left( 2 - 3/\sqrt{z} + 1/z^{3/2} \right) \right],$$

$$\eta^2 = \eta^2(t, x, v, P), \quad \xi^{3,4} = \xi^{3,4}(\alpha, \beta, I_1, I_2, I_3). \quad (7)$$

Here $\xi_1, \xi_3, \xi_4$, and $\eta^2$ are arbitrary functions of their arguments; $I_1$, $I_2$, and $I_3$ are invariants of the operator $Y$, and are defined as follows:

$$I_1 = \frac{2}{\beta} (1/\sqrt{z} - 1) + P^2/\omega_0^2, \quad I_2 = x + aP/\omega_0^2, \quad I_3 = t + \sqrt{4 + \beta I_1} E(\rho, \sigma) - \frac{2}{\sqrt{4 + \beta I_1}} F(\rho, \sigma). \quad (8)$$

In (7),(8) $F(\rho, \sigma)$ and $E(\rho, \sigma)$ are incomplete elliptic integrals of the first and second kind, respectively:

$$F(\lambda, k) = \int_0^{\lambda} \frac{d\mu}{\sqrt{(1 - k^2 \mu^2)(1 - \mu^2)}}, \quad E(\lambda, k) = \int_0^{\lambda} \frac{\sqrt{1 - k^2 \mu^2}}{\sqrt{1 - \mu^2}} d\mu. \quad (9)$$

The operator (2) with coordinates (7) determines the most general symmetry group of continuous point transformations admitted by (1). Following the scheme of the RG algorithm now we perform the restriction procedure of this group on a particular PT solution of our problem. This procedure appears as a “combining” of different coordinates of the group generator (2) admitted by the RG-manifold. The vanishing condition for this combination on a particular solution leads to algebraic equalities that couple different coordinates $\xi^i$ and $\eta^j$ and give rise to desired RG
characterized by the dielectric permittivity $\varepsilon$, the electric field $P$, 

Thus, the nonlinear structure of the electric field and the electron velocity depend on the form of the index $\lambda$.

The velocity and the electric field here are defined by the linear theory solution (indicated by $(\lambda, \sigma)$). 

Formulas (12) define nonlinear solution which takes nonrelativistic nonlinearity into account.

Here $\chi = \omega_0 x_{\text{linear}}$, and $\eta = x(0, \eta)$ denotes the value of the coordinate $x = x(a, \eta)$ at $a = 0$. Formulas (12) define nonlinear solution which takes nonrelativistic nonlinearity into account. 

The velocity and the electric field here are defined by the linear theory solution (indicated by the index $\text{linear}$), because they are invariant under transformation with the group parameter $a$.

Thus, the nonlinear structure of the electric field and the electron velocity depend on the form of the electric field $P_{\text{linear}}$, which arises from the solution of the linearized system of equations (1).

For definiteness sake we select for $P_{\text{linear}}$ the linear solution obtained for cold electron plasma, characterized by the dielectric permittivity $\varepsilon = 1 - \omega_0^2/\omega_0^2$, with a linear ion density profile [1]. In this case electric field has a singularity of the order of $1/\varepsilon$ at the critical point $x = 0$, however it is eliminated in view of either finite values of the electron plasma temperature, or a finite electron electron collision frequency.

For thus specified choice of the linear theory field structure, the formulas (12) for the electric field and the electron velocity are rewritten as follows:

$$P = P_{\text{linear}} = -\frac{\omega_0^2 L^2}{\Delta^2 + \eta^2} (\eta \cos \chi + \Delta \sin \chi),$$

$$v_1 = \frac{\omega_0 L^2}{\Delta^2 + \eta^2} (\eta \sin \chi - \Delta \cos \chi),$$

$$x = x_1 = \eta + \frac{a L^2}{\Delta^2 + \eta^2} (\eta \cos \chi + \Delta \sin \chi).$$

Here the plasma resonance width $\Delta$ is determined either by the thermal motion of electrons with the thermal velocity $V_T$, or by a finite value of the collision frequency $\nu$:

$$\Delta = \max\{\nu L/\omega_0 : (3 V_T^2 L/\omega_0^2)^{1/3}\}. $$
Now we consider finite transformations generated by $R_2$. These transformations involve the group parameter $\beta$, and relate non-relativistic nonlinear solutions (13) with $\beta = 0$ to relativistic solutions with $\beta \neq 0$. As coordinate $x$ and the electric field $P$ are invariants of the group transformations with the generator $R_2$ that is $x_{II} = x_I \equiv x$ and $P_{II} = P_I \equiv P$, they are defined by the corresponding expressions (14). The finite transformations for the remaining variables, namely the velocity $v \equiv v_{II}$ and “time” $\tau \equiv \omega_0 t_{II}$ are given by the following formulas:

$$v = \pm \frac{1}{\sqrt{\beta}} \left[ 1 - \frac{1}{(1 + \beta v_I^2/2)^2} \right]^{1/2}, \quad \tau = \chi - \left( \frac{\sqrt{4 + \beta I_1} E(\varphi;k) - 2F(\varphi;k)}{\sqrt{4 + \beta I_1}} - \varphi \right), \quad (15)$$

where

$$\varphi = \arcsin \frac{P}{\omega_0 \sqrt{I_1}}, \quad k = \sqrt{-\frac{\beta I_1}{4 + \beta I_1}}, \quad I_1 = \frac{2}{\beta} \left( 1/\sqrt{z} - 1 \right) + P^2/\omega_0^2. \quad (16)$$

The final formulas (13), (15), generated by RG algorithm, describe the sought-for nonlinear structure of the electric field $P$ and the plasma electron velocity $v$ that takes into account both relativistic, $\beta \neq 0$, and nonrelativistic, $\beta = 0$, plasma nonlinearities. The dependence of $P$ and $v$ upon coordinate $x$ and time $\tau$ is defined implicitly through the parametric variables $\eta$ and $\chi$.

3. Conclusions

Using the RG algorithm we have constructed an analytical solution to the system of equations that describe relativistic plasma oscillations in the vicinity of the plasma critical density. The distinctive feature of the method is that we have employed the two-parameter algebra of RG generators to obtain exact nonlinear solutions in a wide range of nonlinear and relativistic plasma parameters. The two-parameter renormgroup gives the possibility to separate inputs from nonrelativistic and relativistic nonlinearities. This fact is an added reason for utilizing multi-dimensional renormgroups for systems with several parameters [12]. Notice that although to obtain the solution we used the hydrodynamic-type equations for the cold collisionless plasma, the RG algorithm looks attractive for pursuing further research in this area with different plasma models. The detailed analysis of the space-time nonlinear structure of the potential plasma field in the vicinity of a plasma resonance opens a possibility to calculate the power-law spectra of stationary plasma oscillations, to estimate the threshold for plasma wave-breaking, to analyze the nonlinear absorption and harmonics generation in a laser irradiated low-density plasma corona. These results will be published elsewhere.

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