Computational Wavelet Method for Multidimensional Integro-Partial Differential Equation of Distributed Order

Yashveer Kumar\textsuperscript{1} \textsuperscript{*}, Somveer Singh\textsuperscript{1} \textsuperscript{†}, Reshma Singh\textsuperscript{2} \textsuperscript{‡}, Vineet Kumar Singh\textsuperscript{1} \textsuperscript{§}

\textsuperscript{1}Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi, India.

\textsuperscript{2}Department of Mathematics, Ram Briksh Benipuri Mahila College, Muzaffarpur Babasaheb Bhimrao Ambedkar Bihar University, Muzaffarpur India.

Abstract

This article provides an effective computational algorithm based on Legendre wavelet (LW) and standard tau approach to approximate the solution of multi-dimensional distributed order time-space fractional weakly singular integro-partial differential equation (DOT-SFWSIPDE). To the best of our understanding, the proposed computational algorithm is new and has not been previously applied for solving DOT-SFWSIPDE. The matrix representation of distributed order fractional derivatives, integer order derivatives and weakly singular kernel associated with the integral based on LW are established to find the numerical solutions of the proposed DOT-SFWSIPDE. Moreover, the association of standard tau rule and Legendre-Gauss quadrature (LGQ) techniques along with constructed matrix representation of differential and integral operators diminish DOT-SFWSIPDE into system of linear algebraic equations. Error bounds, convergence analysis, numerical algorithms and also error estimation of the DOT-SFWSIPDE are rigorously investigated. For the reliability of the proposed computational algorithm, numerous test examples has been incorporated in the manuscript to ensure the robustness and theoretical results of proposed technique.

Keywords: Multi-dimensional distributed order time-space fractional weakly singular integro-differential equation, Fractional order Caputo derivative, Legendre wavelets, Operational matrices, Convergence analysis, Error estimation.

1 Introduction

Fractional differential and integral models have sparked a lot of interest, because of their applications in many fields of science, finance as well as in engineering [1]. There are some fascinating implementations of fractional calculus in viscoelasticity model [2], electromagnetic waves [3], chaotic systems [4], physical systems [5], optimization [6], nonlinear dynamical systems [7], in the modeling of heat transfer [8], and dynamics of interfaces between nano particles and substrates [9]. Furthermore, the use of fractional calculus in viscoelasticity has emerged as a
promising area of research. For example, fractional derivatives without singular kernels, have been proposed as mathematical methods for describing viscoelasticity models. A new fractional-order algorithm to explain the dynamic behaviours of general fractional-order viscoelasticity with memory effect, Maxwell and Voigt models are suggested and used within the framework of general fractional derivatives [10]. A new model is presented in [11] to demonstrate the efficiency of fractional-order operators in the case of line viscoelasticity.

Fractional differential equations (FDEs) and fractional integral equations have attracted the interest of many researchers due to their practical applications in various fields of science and engineering. Although some techniques exist for obtaining analytical solutions to some FDEs, analytical solutions to FDEs remain unknown in the overwhelming majority of cases. As a result, several scholars have devised numerous computational methods for obtaining approximate solutions to fractional order integral and fractional order differential equations. The most commonly used methods are variational iteration method [12], generalized transform method [13,14], adomian decomposition method [15,16] and wavelet method [17,18]. A. Saadatmandi & M. Dehghan debated on the solution of space-fractional diffusion equation with Caputo derivative by the tau approach [19]. In [20], authors discussed semi-discrete scheme for Riesz- FDE. In [21], a finite element/finite difference scheme has been proposed to solve the 2D time and space fractional partial integro differential equation with weakly singular kernel. For more study about the methods to solve FDE readers one can see [22–25].

Now a days, distributed order operator is an attractive tool to explain the physical phenomena of mathematical models governed by the fields of science, finance and engineering. The distributed order fractional (DOF) derivative has a long and illustrious history. In 1969, Caputo was the first to introduce the concept of DOF operator, and in 1995, he was also the first to solve it. The distributed order fractional differential equation (DOFDE) is stated in its generic form as [26]

\[
\sum_{i=1}^{k} a_i \int_{\alpha_1}^{\alpha_2} \rho_i(\alpha) D^{\alpha-i}_\alpha \mathbf{U}(\kappa, \varrho) d\alpha + \sum_{j=0}^{k} b_j \mathbf{U}^{(j)}(\kappa, \varrho) = f(\kappa, \varrho),
\]

where, \( \rho_i(\alpha) \) denotes the weight function of distribution of order \( \alpha \in [\alpha_1, \alpha_2] \) and \( k \in \mathbb{Z}^+ \). Hence, the above equation can be viewed as a generalisation form of

- If \( \rho_i(\alpha) \equiv 0 \), then we get differential equation of integer order.
- If \( \rho_i(\alpha) \) takes any discrete values in \( [\alpha_1, \alpha_2] \), then we get FDE’s.

As a result, differential equation with integer and non-integer order can be considered as special cases of distributed order fractional differential equations (DOFDE). In the fields of engineering, science and financial mathematics, distributed-order differential equations have a wide range of applications. For instance, they are used in the modelling of dielectric induction and diffusion [27]. In 2004, Sokolov and Chechkin [28] debated on the DOF kinetics. Umarov et al. provided random walk models [29] are governed with the help of DOFDE. The financial mathematical model governed with the help of DOF derivative defined in [30] considered time DOF Black-Scholes equation. With the development of DOFDE, various numerical methods were constructed for their solutions. In [31] the authors presented a numerical wavelet scheme for DOFDE. In [32], Riesz-space DOFDE using second-order finite difference scheme has been proposed. For solving the time DOF advection-diffusion equation, the authors of [33] discovered a special point for the linear combination of multi-term fractional derivatives interpolation approximation and obtained a numerical differentiation formula with second-order precision. In [34] authors applied Crank-Nicolson/Galerkin spectral method for solving two-dimensional time-space DOF integro-partial differential equation with weakly singular by using
Riesz derivative in the space direction, whereas in [19] authors used the tau approach for solving space fractional diffusion equation by using Caputo derivative in space. In this work, we consider the following DOT-SFWSIPDE [34–37] in 1D & 2D, using Caputo derivative in both directions define as,

\[
\int_{\alpha_1}^{\alpha_2} \rho(\alpha) \frac{\partial^\alpha U(\kappa, \varrho)}{\partial \kappa^\alpha} d\alpha + U(\kappa, \varrho) = K^* \int_{\beta_1}^{\beta_2} \rho(\beta) \frac{\partial^\beta U(\kappa, \eta, \varrho)}{\partial \eta^\beta} d\beta + \int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(\kappa, \xi, \varrho)}{\partial \kappa^2} \right] d\xi + f(\kappa, \varrho),
\]

(1.1)

where, \( K^* \) is viscosity constant and \((\kappa, \varrho) \in \Omega, \quad \alpha_1 = 0, \alpha_2 = 1, \beta_1 = 1, \beta_2 = 2 \) and \( \Omega = [0, 1] \times [0, T] \).

The above equation 1.1 is with the initial condition

\( U(\kappa, 0) = \nu(\kappa), \quad 0 < \kappa < 1 \), (1.2)

and Dirichlet boundary conditions

\( U(0, \varrho) = p_1(\varrho), \quad 0 < \varrho < T \), (1.3)

\( U(0, \varrho) = p_2(\varrho), \quad 0 < \varrho < T \). (1.4)

The 2D form of the above problem is defined as

\[
\int_{\alpha_1}^{\alpha_2} \rho(\alpha) \frac{\partial^\alpha U(\kappa, \eta, \varrho)}{\partial \kappa^\alpha} d\alpha + U(\kappa, \eta, \varrho) = K^* \int_{\beta_1}^{\beta_2} \rho(\beta) \left[ \frac{\partial^\beta U(\kappa, \eta, \varrho)}{\partial \eta^\beta} + \frac{\partial^\beta U(\kappa, \eta, \varrho)}{\partial \eta^\beta} \right] d\beta + \int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(\kappa, \eta, \xi)}{\partial \kappa^2} + \frac{\partial^2 U(\kappa, \eta, \xi)}{\partial \eta^2} \right] d\xi + f(\kappa, \eta, \varrho),
\]

(1.5)

where, \( K^* \) is viscosity constant and \((\kappa, \eta, \varrho) \in \Omega, \alpha_1 = 0, \alpha_2 = 1, \beta_1 = 1, \beta_2 = 2 \) and \( \Omega = [0, 1] \times [0, 1] \times [0, T] \).

The initial condition for the above equation 1.5 is

\( U(\kappa, \eta, 0) = \nu(\kappa, \eta), \quad 0 < \kappa < 1 \) and \( 0 < \eta < 1 \) (1.6)

and the boundary conditions are

\( U(0, \eta, \varrho) = p_1(\eta, \varrho), \quad 0 < \eta < 1 \) and \( 0 < \varrho < T \), (1.7)

\( U(1, \eta, \varrho) = p_2(\eta, \varrho), \quad 0 < \eta < 1 \) and \( 0 < \varrho < T \). (1.8)

\( U(\kappa, 0, \varrho) = q_1(\kappa, \varrho), \quad 0 < \kappa < 1 \) and \( 0 < \varrho < T \), (1.9)

\( U(\kappa, 1, \varrho) = q_2(\kappa, \varrho), \quad 0 < \kappa < 1 \) and \( 0 < \varrho < T \). (1.10)

Here, \( \rho(\alpha), \rho(\beta) \) are the weight functions that satisfy the following criteria [38]

\[
\rho(\alpha) \geq 0, \quad \int_{\alpha_1}^{\alpha_2} \rho(\alpha) d\alpha = \lambda_1 > 0 \quad \text{and} \quad \rho(\beta) \geq 0, \quad \int_{\beta_1}^{\beta_2} \rho(\beta) d\beta = \lambda_2 > 0.
\]

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The existence and uniqueness of the solution of DOFDE can be seen in [39,40]. The application of considered problem 1.1-1.4 & 1.5-1.10 can be found in the modeling of physical phenomena involving viscoelastic model. Based on the current literature and referring to the data, there is no computational approach available for solving DOT-SFIPDE centred on LW. The proposed technique based on LW operational matrices is employed in this article, to solve this newly created model 1.1-1.4 & 1.5-1.10 in the sense of fractional Caputo derivative.

The operational matrices have been proved to be an effective tool for solving FDEs. Saadatmandi and Dehghan developed the operational matrix of fractional derivative (OMFD) for shifted Legendre polynomials (SLPs) in 2010 [41]. In [42], Zhao et al. constructed the OMFD using Haar wavelet. In [43] authors developed the Bernoulli wavelets OMFD. In [44], Taha et al. invented the Laguerre polynomials OMFD. Pourbabaei and Saadatmandi [45] recently devised a helpful technique based on Legendre polynomials OMFD for finding the approximated solution of DOFDE. Readers can see [46–48], to learn more about operational matrix approaches.

Wavelets are a special type of orthogonal functions that have become very useful and effective tools in computational science. Wavelet methods have recently received increased recognition for numerically solving integral and differential equations; they were first utilized to discuss the solution of differential equations in the early 1990 [49]. Many papers have been recently published that use LWs to include numerical solutions of fractional differential and integro-differential equations (IDEs). The LW operational matrix approach is used to solve the nonlinear Volterra IDEs [50]. To solve the Dirichlet boundary value problem for fractional partial differential equation, LWs were used [51]. In [52] Meng et. al. used LW to evaluate the solution of linear and nonlinear fractional IDEs.

It is observed that majority of papers that use the LWs approach to obtain numerical solution of FDEs use a LWs operational matrix. As a result, we use the LWs operational matrix approach to solve linear time-space DOF integro-differential equations with weakly singular kernels in this article. The operational matrix approach is also proven to be an effective and resilient numerical methodology for solving DOFDE, as shown in [45]. The goal of this article is to construct the DOF derivative matrix based on LWs. The motivation for using a wavelets-based approach is convenient. There are two approaches to improve the accuracy of the solution in such methods: raising the level of resolution of wavelets family and increasing the number of wavelet basis functions. Furthermore, because LWs are made up of orthogonal polynomials, they have indefinitely differentiable functions & small compact support. Moreover, the LW operational matrices are sparse, reducing calculation time. The Legendre-Gauss quadrature (LGQ) rule and the tau technique are used to solve the DOT-SFWSIPDE using such matrices. Know more information about wavelet and DOFDE readers can be see [33,53–56]. The goal of this technique is to have successful experimental tests that are less computationally expensive.

The remainder of the paper is structured as follows: Fractional derivatives, the distributed differential operator, LWs, and their approximation characteristics are all briefly defined in Section 2. This section also includes the Gauss-Legendre quadrature integration formula. The operational matrices are covered in the reference section 3. Operational matrices of derivative are produced in this section for both integer and distributed order LWs. In section 4, the suggested technique is implemented together with a numerical algorithm to solve DOFIDEs. Section 5 discusses the error bound and convergence analysis for the described scheme. The suggested method’s error estimation is described in section 6. Finally, in section 7, numerical tests of a viscoelastic model regulated by DOT-SFWSIPDE are carried out.
2 Some basic definitions

**Definition 2.1. (Fractional Caputo derivative):** The fractional derivative of order $\alpha > 0$ in Caputo sense is devoted as [1]

$$D^\alpha \varrho U(\kappa, \varrho) = \begin{cases} \frac{1}{\Gamma(n - \alpha)} \int_0^\varrho \frac{1}{(\varrho - \tau)^{\alpha + 1 - n}} \frac{\partial^n U(\kappa, \tau)}{\partial \tau^n} d\tau, & n - 1 < \alpha < n, \\ \frac{\partial^n U(\kappa, \varrho)}{\partial \varrho^n}, & \alpha = n \in \mathbb{N}. \end{cases} \quad (2.1)$$

Here, $\Gamma(.)$ represents the Gamma function. This Caputo operator have some basic properties:

- $D^\alpha \varrho J = 0$, here, $J$ denotes arbitrary constant.
- The Caputo derivative of $u(\varrho) = \varrho^n$, $n \in \mathbb{Z}^+$ is given as:

$$D^\alpha \varrho^n = \begin{cases} 0, & n < \lceil \alpha \rceil, \\ \frac{\Gamma(n + 1)}{\Gamma(n + 1 - \alpha)} \varrho^{n - \alpha}, & n \geq \lceil \alpha \rceil, \end{cases}$$

where, $\lceil \cdot \rceil$ represent the ceiling function.

- $D^\alpha \varrho$ fulfills the linearity property, i.e.

$$D^\alpha \left( \sum_{j=1}^s b_j u_j(\varrho) \right) = \sum_{j=1}^s b_j D^\alpha \varrho u_j(\varrho).$$

Here, $b_j$ denotes an arbitrary constants for $j = 1, 2, \ldots, s$, $s \in \mathbb{Z}^+$

**Definition 2.2. (Distributed order fractional derivative):** The $D_{\varrho}^{\rho(\alpha)}$ DOF derivative is defined as [26]

$$D_{\varrho}^{\rho(\alpha)} u(\varrho) = \int_{\alpha_1}^{\alpha_2} \rho(\alpha) D^\alpha \varrho u(\varrho) d\alpha. \quad (2.2)$$

Here, $\rho(\alpha)$ defines the distribution weight function of order $\alpha$ and $\alpha \in [\alpha_1, \alpha_2]$, where $\alpha_1$ and $\alpha_2$ are non-negative real numbers. The DOFD operator has the following properties:

- $D_{\varrho}^{\rho(\alpha)} J = 0$, where, $J$ is any arbitrary constant.
- $D_{\varrho}^{\rho(\alpha)}$ is a linear operator, i.e,

$$D_{\varrho}^{\rho(\alpha)} \left( \sum_{j=1}^s b_j D_{\varrho}^{\rho(\alpha)} u_j(\varrho) \right) = \sum_{j=1}^s b_j D_{\varrho}^{\rho(\alpha)} u_j(\varrho), \quad (2.3)$$

where, $b_j$ are arbitrary constants for $j = 1, 2, \ldots, s, s \in \mathbb{Z}^+$

- If $\rho(\alpha) = \delta(\alpha - \mu)$, where, $\alpha_1 < \mu < \alpha_2$ and $\delta$ is delta Dirac function. Then we have

$$D_{\varrho}^{\rho(\alpha)} u(\varrho) = \int_{\alpha_1}^{\alpha_2} \delta(\alpha - \mu) D^\alpha \varrho u(\varrho) d\alpha = D_{\varrho}^\mu u(\varrho). \quad (2.4)$$

In other words, we obtain a fractional derivative of order $\mu$. 

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Definition 2.3. (Legendre wavelets): Legendre wavelets $\Psi_{b,g}(\varphi) = \Psi(\mathcal{R}, \hat{h}, g, \varphi)$ have four arguments: $\hat{h} = 2h - 1, h = 1, 2, 3, \ldots , 2^{R-1}, \mathcal{R} \in \mathbb{Z}^+, \ g$ is degree of Legendre polynomials and $\varphi$ denotes the norm of normalization. Then the one dimension LWs definition over $[0,1]$ are described as [47]:

$$
\Psi_{b,g}(\varphi) = \begin{cases} 
\sqrt{\frac{\varphi + 1}{2}}2^\varphi \mathcal{P}(2^{\varphi} \varphi - 2h + 1), & \frac{h-1}{2^\varphi-1} \leq \varphi \leq \frac{h}{2^\varphi-1}; \\
0, & \text{elsewhere.} 
\end{cases}
$$

(2.5)

Where, $g = 0, 1, 2, \cdots , \Lambda - 1, h = 1, 2, 3, \cdots , 2^{R-1}$.

Remark 2.1. Two-dimensional LWs are represented as follows: [47]:

$$
\Psi_{b,g,b',g'}(x, \varphi) = \begin{cases} 
\Psi_{b,g}(x)\Psi_{b',g'}(\varphi), & \frac{h-1}{2^\varphi-1} \leq x \leq \frac{h}{2^\varphi-1}, \frac{h'-1}{2^{\varphi'-1}} \leq \varphi \leq \frac{h'}{2^{\varphi'-1}}; \\
0, & \text{elsewhere.} 
\end{cases}
$$

(2.6)

Definition 2.4. (Function approximation):

The function $f(x, \varphi)$ defined over $L^2(\Omega = [0, 1] \times [0, T])$ can be written as the sum of LW infinite series such as

$$
f(x, \varphi) = \sum_{b=1}^{\infty} \sum_{g=0}^{\infty} \sum_{g'}^{\infty} \sum_{g''}^{\infty} \sum_{f_{hgh}^{b'}} \varphi_{hgh}^{b'}(x, \varphi). 
$$

(2.7)

The truncation of the above series leads to

$$
f(x, \varphi) \approx \sum_{b=1}^{2^{R-1} \Lambda-1} \sum_{g=0}^{2^{R'-1} \Lambda'-1} \sum_{g'} \sum_{g''} \sum_{f_{hgh}^{b'}} \varphi_{hgh}^{b'}(x, \varphi) = \Psi^T(\varphi)\mathcal{F}\Psi(x),
$$

(2.8)

where, $f_{hgh}^{b'} = \langle (f, \Psi(\varphi)), \Psi(x) \rangle$ and $\langle , , \rangle$ represents the inner product and $\mathcal{F}$ is $2^{R-1} \Lambda \times 2^{R'-1} \Lambda'$ vector and $\Psi(\varphi, \Psi(x)$ are $2^{R-1} \Lambda \times 1, 2^{R'-1} \Lambda' \times 1$ vectors, respectively.

$$
\Psi(\varphi) = [\varphi_{1,0}, \varphi_{1,1}, \cdots , \varphi_{1,\Lambda-1}, \varphi_{2,0}, \cdots , \varphi_{2,\Lambda'-1}, \cdots , \varphi_{2^{R-1},0}, \cdots , \varphi_{2^{R-1},\Lambda-1}]^T.
$$

(2.9)

Similarly,

$$
f(x, \eta, \varphi) = \sum_{b=1}^{\infty} \sum_{g=0}^{\infty} \sum_{g'} \sum_{g''} \sum_{f_{hgh}^{b'}} \varphi_{hgh}^{b'}(x, \eta, \varphi). 
$$

(2.10)

The truncation of the above series leads to

$$
f(x, \eta, \varphi) \approx \sum_{b=1}^{2^{R-1} \Lambda-1} \sum_{g=0}^{2^{R'-1} \Lambda'-1} \sum_{g'} \sum_{g''} \sum_{f_{hgh}^{b'}} \varphi_{hgh}^{b'}(x, \eta, \varphi) = \Psi^T(\eta)\mathcal{F}\Psi(x, \eta),
$$

(2.11)

where, the function $f(x, \eta, \varphi)$ defined over $L^2(\Omega = [0, 1] \times [0, 1] \times [0, T]), \Psi(x, \eta) = \Psi(x) \otimes \Psi(\eta), \otimes$ denotes the Kronecker product and $f_{hgh}^{b'} = \langle (f, \Psi(\varphi)), \Psi(x), \Psi(\eta) \rangle$. $\mathcal{F}$ is $2^{R-1} \Lambda \times (2^{R'-1} \Lambda') \times (2^{R''-1} \Lambda''$ vector and $\Psi(\varphi, \Psi(x)$ and $\Psi(\eta)$ are $2^{R-1} \Lambda \times 1, 2^{R'-1} \Lambda' \times 1, 2^{R''-1} \Lambda'' \times 1$ vectors, respectively.
Definition 2.5. (Legendre-Gauss quadrature (LGQ) formula for Numerical integration): Let \( \{\tau_s\}_{s=1}^P \) denotes the collection of \( P \) distinct roots of Legendre polynomial of degree \( P \), where, \( P \in \mathbb{Z}^+ \). The \( P \)-point LGQ formula approximates the function integral over the interval \( (\alpha_1, \alpha_2) \) as

\[
\int_{\alpha_1}^{\alpha_2} u(\varrho) d\varrho \approx \sum_{q=1}^P w_q u(\sigma_q),
\]

(2.12)

where,

\[
w_s = \frac{\alpha_2 - \alpha_1}{(1 - \tau_s^2)(L'_P(\tau_s))^2}, \quad \sigma_s = \frac{\alpha_2 - \alpha_1}{2} \tau_s + \frac{\alpha_2 + \alpha_1}{2}, \quad s = 1, 2, \cdots, P.
\]

(2.13)

Here, \( \{w_s\}_{s=1}^P \) and \( \{\sigma_s\}_{s=1}^P \) are LGQ weights and nodes, respectively. The LGQ formula is correct upto for all polynomials, of degree atmost \( 2P - 1 \).

3 Construction of operational matrices

3.1 Derivative operational matrix for integer order

Theorem 3.1. The derivative of \( m \)-degree shifted Legendre polynomial \( p_m(\kappa) \) defined over \([0, 1]\) is given as:

\[
p'_m(\kappa) = 2 \sum_{k=0, k+m \text{ odd}}^{m-1} (2k + 1)p_k(\kappa)
\]

Proof. Given in reference [58].

Theorem 3.2. Suppose \( \Psi(\kappa) \) denotes the LW vector. The derivative of \( \Psi(\kappa) \) can be determined as:

\[
\frac{d\Psi(\kappa)}{d\varrho} = D\Psi(\kappa),
\]

(3.1)

here, \( D \) denotes derivative operational matrix for LW of order \( 2^g-1(g+1) \), described as follows:

\[
D = \begin{bmatrix}
H & 0 & 0 & \cdots & 0 \\
0 & H & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & H
\end{bmatrix}.
\]

\( H \) represents matrix of order \( (g+1) \times (g+1) \) and its \( (p, q) \)th component is described as follows:

\[
H_{p,q} = \begin{cases} 
2^g \sqrt{(2p-1)(2q-1)} & p = 2, 3, \cdots, (g+1), \quad q = 1, 2, \cdots, p - 1 \text{ and } (p + q) \text{ odd}, \\
0, & \text{otherwise.}
\end{cases}
\]

(3.2)

In general,

\[
\frac{d^{n_1}\Psi(\kappa)}{d\varrho^{n_1}} = (D)^{n_1}\Psi(\kappa), \quad n_1 = 1, 2, 3, \cdots
\]

(3.3)

Proof. Given in reference [58]
3.2 Construction of DOF matrix

Let $D_{\rho}^{(\alpha)}$ be the DOF derivative w.r.t time component defined in equation 2.2. Then we have

$$D_{\rho}^{(\alpha)} \varrho^n = \int_{\alpha_1}^{\alpha_2} \rho(\alpha) D_{\rho}^{(\alpha)} \varrho^{n} d\alpha = \int_{\alpha_1}^{\alpha_2} \rho(\alpha) \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} \varrho^{n-\alpha} d\alpha, \quad n \in \mathbb{N}, k \geq \lceil \alpha_2 \rceil. \quad (3.4)$$

The LGQ rule approximates the above equation as follows:

$$D_{\rho}^{(\alpha)} \varrho^n \approx \sum_{s=1}^{P} w_s \rho(\sigma_s)n! \frac{\Gamma(n+1)}{\Gamma(n+1-\sigma_s)}. \quad (3.5)$$

For the general class

$$D_{\rho}^{(\alpha)} \Psi(\varrho) = \int_{\alpha_1}^{\alpha_2} \rho(\alpha) D_{\rho}^{(\alpha)} \Psi(\varrho) d\alpha = \int_{\alpha_1}^{\alpha_2} \rho(\alpha) \begin{bmatrix} D_{\rho}^{(\alpha)} \varphi_{10}(\varrho) \\ D_{\rho}^{(\alpha)} \varphi_{11}(\varrho) \\ \vdots \\ D_{\rho}^{(\alpha)} \varphi_{1g}(\varrho) \\ D_{\rho}^{(\alpha)} \varphi_{20}(\varrho) \\ \vdots \\ D_{\rho}^{(\alpha)} \varphi_{2g}(\varrho) \\ \vdots \\ D_{\rho}^{(\alpha)} \varphi_{2R-10}(\varrho) \\ \vdots \\ D_{\rho}^{(\alpha)} \varphi_{2R-1g}(\varrho) \end{bmatrix} d\gamma$$

$$= \int_{\alpha_1}^{\alpha_2} \rho(\alpha) \begin{bmatrix} b_{10}(\varrho, \alpha) \\ b_{11}(\varrho, \alpha) \\ \vdots \\ b_{1g}(\varrho, \alpha) \\ b_{20}(\varrho, \alpha) \\ \vdots \\ b_{2g}(\varrho, \alpha) \\ \vdots \\ b_{2R-10}(\varrho, \alpha) \\ \vdots \\ b_{2R-1g}(\varrho, \alpha) \end{bmatrix} d\alpha = \begin{bmatrix} \int_{\alpha_1}^{\alpha_2} \mathcal{H}_{10}(\varrho, \alpha) d\alpha \\ \int_{\alpha_1}^{\alpha_2} \mathcal{H}_{11}(\varrho, \alpha) d\alpha \\ \vdots \\ \int_{\alpha_1}^{\alpha_2} \mathcal{H}_{1g}(\varrho, \alpha) d\alpha \\ \int_{\alpha_1}^{\alpha_2} \mathcal{H}_{20}(\varrho, \alpha) d\alpha \\ \vdots \\ \int_{\alpha_1}^{\alpha_2} \mathcal{H}_{2g}(\varrho, \alpha) d\alpha \\ \vdots \\ \int_{\alpha_1}^{\alpha_2} \mathcal{H}_{2R-10}(\varrho, \alpha) d\alpha \\ \vdots \\ \int_{\alpha_1}^{\alpha_2} \mathcal{H}_{2R-1g}(\varrho, \alpha) d\alpha \end{bmatrix}.$$
By using the LGQ rule for numerical integration, one can write

\[
D_\varepsilon^{(\alpha)} \Psi(\varrho) \approx \left[ \sum_{s=1}^{P} w_s \mathcal{H}_{10}(\varrho, \sigma_q) \mathcal{H}_{11}(\varrho, \sigma_q) \right] = \left[ Q_{10}(\varrho) \quad Q_{11}(\varrho) \quad \cdots \quad Q_{1g}(\varrho) \quad Q_{20}(\varrho) \quad \cdots \quad Q_{2g-10}(\varrho) \quad \cdots \quad Q_{2g-1g}(\varrho) \right]
\]

Thus we obtain

\[
D_\varepsilon^{(\alpha)} \Psi(\varrho) \approx \hat{D}^{(\alpha_1, \alpha_2, \varrho(\alpha))} \Psi(\varrho),
\]

where, the matrix \(\hat{D}^{(\alpha_1, \alpha_2, \varrho(\alpha))}\) defined as:

\[
\begin{pmatrix}
   d_{1010} & d_{1011} & \cdots & d_{101g} & d_{1020} & \cdots & d_{102g} & \cdots & d_{102g-10} & \cdots & d_{102g-1g} \\
   d_{1110} & d_{1111} & \cdots & d_{111g} & d_{1120} & \cdots & d_{112g} & \cdots & d_{112g-10} & \cdots & d_{112g-1g} \\
   \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
   d_{1g10} & d_{1g11} & \cdots & d_{1g1g} & d_{1g20} & \cdots & d_{1g2g} & \cdots & d_{1g2g-10} & \cdots & d_{1g2g-1g} \\
   d_{2010} & d_{2011} & \cdots & d_{201g} & d_{2020} & \cdots & d_{202g} & \cdots & d_{202g-10} & \cdots & d_{202g-1g} \\
   \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
   d_{2g10} & d_{2g11} & \cdots & d_{2g1g} & d_{2g20} & \cdots & d_{2g2g} & \cdots & d_{2g2g-10} & \cdots & d_{2g2g-1g} \\
   d_{2g-1010} & d_{2g-1011} & \cdots & d_{2g-101g} & d_{2g-1020} & \cdots & d_{2g-102g} & \cdots & d_{2g-102g-10} & \cdots & d_{2g-102g-1g} \\
   \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
   d_{2g-1g10} & d_{2g-1g11} & \cdots & d_{2g-1g1g} & d_{2g-1g20} & \cdots & d_{2g-1g2g} & \cdots & d_{2g-1g2g-10} & \cdots & d_{2g-1g2g-1g}
\end{pmatrix}
\]

whose entries can be calculated as follows

\[
d_{ijkl} = \langle Q_{ij}(\varrho), \varphi_{kl}(\varrho) \rangle, \quad i, k = 1, 2, \cdots, 2^{g-1}, \quad j, l = 0, 1, 2, \cdots, g \quad \text{and} \quad <, , > \text{ denotes the inner product.}
\]

The above defined matrix \(\hat{D}^{(\alpha_1, \alpha_2, \varrho(\alpha))}\) of order \(2^{g-1}(g + 1 \times g + 1)\) is represents the DOF operational matrix.
Remark 3.1. Similarly, one can construct the DOF operational matrix namely: $\hat{D}_{\alpha}^{(\beta_1,\beta_2,\rho(\beta))_\alpha}$, $\hat{D}_{\beta}^{(\beta_1,\beta_2,\rho(\beta))_\beta}$, for space direction.

4 Numerical method

In this section we discuss numerical procedure to solve one and two dimensional DOT-SFIDEs.

4.1 1-D distributed order time-space fractional weakly singular integro differential equation

We consider the following DOT–SFWSIPDE of the form:

$$
\int_{\alpha_1}^{\alpha_2} \rho(\alpha) \frac{\partial^\alpha U(x, \varrho)}{\partial \varrho^\alpha} d\alpha + U(x, \varrho) = K^* \int_{\beta_1}^{\beta_2} \rho(\beta) \frac{\partial^\beta U(x, \varrho)}{\partial x^\beta} d\beta + \int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(x, \xi)}{\partial x^2} \right] d\xi
+ f(x, \varrho),
$$

(4.1)

where, $K^*$ is viscosity constant and $(x, \varrho) \in \Omega, \alpha_1 = 0, \alpha_2 = 1, \beta_1 = 1, \beta_2 = 2$ and $\Omega = [0, 1] \times [0, T]$.

The above equation 4.1 is endowed with the initial condition (IC)

$$
U(x, 0) = \nu(x), \quad 0 < x < 1,
$$

(4.2)

and Dirichlet boundary conditions (BCs)

$$
U(0, \varrho) = p_1(\varrho), \quad 0 < \varrho < T,
$$

(4.3)

$$
U(0, \varrho) = p_2(\varrho), \quad 0 < \varrho < T.
$$

(4.4)

Consider the approximation of the known and unknown function as

$$
f(x, \varrho) \approx \Psi^T(\varrho) \mathcal{F} \Psi(x),
$$

(4.5)

$$
U(x, \varrho) \approx \Psi^T(\varrho) \mathcal{A} \Psi(x),
$$

(4.6)

where, the matrix $\mathcal{F}$ is known and $\mathcal{A} = [a_{ij}]$ denotes the unknown matrix that must be evaluated.

The left hand side (L.H.S) of 4.1, by using the approximation of $U(x, \varrho)$ can be written as

$$
\int_{\alpha_1}^{\alpha_2} \rho(\alpha) \frac{\partial^\alpha U(x, \varrho)}{\partial \varrho^\alpha} d\alpha \approx \left( \int_{\alpha_1}^{\alpha_2} \rho(\alpha) (D_{e}^\alpha \Psi^T(\varrho)) d\alpha \right) \mathcal{A} \Psi(x)

\approx (D_{e}^{\rho(\alpha)}) \Psi^T(\varrho) \mathcal{A} \Psi(x)

\approx \Psi^T(\varrho) \left( \hat{D}^{(\alpha_1,\alpha_2,\rho(\alpha))}_\alpha \right)^T \mathcal{A} \Psi(x).
$$

(4.7)

Here, $\hat{D}^{(\alpha_1,\alpha_2,\rho(\alpha))}_\alpha$ denotes the time-DOF operational matrix.

Now, the R.H.S of equation 4.1, with the help of the approximation of $U(x, \varrho)$ can be described as

$$
\int_{\beta_1}^{\beta_2} \rho(\beta) \frac{\partial^\beta U(x, \varrho)}{\partial x^\beta} d\beta \approx \Psi^T(\varrho) \mathcal{A} \left( \int_{\beta_1}^{\beta_2} \rho(\beta) (D_{x}^\beta \Psi(x)) d\beta \right)

\approx \Psi^T(\varrho) \mathcal{A} (D_{x}^{\rho(\beta)} \Psi(x))

\approx \Psi^T(\varrho) \mathcal{A} \left( \hat{D}^{(\beta_1,\beta_2,\rho(\beta))}_\beta \Psi(x) \right).
$$

(4.8)
Here, \( \hat{D}_{\alpha_1, \alpha_2, \rho(\beta)} \) denotes the space-DOF operational matrix.

Now, approximation of the second term of R.H.S of equation 4.1 by using the derivative operational matrix of integer order can be written as

\[
\frac{\partial^2 U(\kappa, \varrho)}{\partial \kappa^2} \approx \Psi^T(\varrho)A\left(\frac{d^2}{d\kappa^2} \Psi(\kappa)\right) = \Psi^T(\varrho)AD^{(2)}(\kappa). \tag{4.9}
\]

\[
\int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} \left[\frac{\partial^2 U(\kappa, \xi)}{\partial \kappa^2}\right] d\xi \approx \int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} (\Psi^T(\xi)AD^{(2)}(\kappa)) d\xi
\approx \left[\int_0^\varrho \frac{\Psi^T(\xi)}{(\varrho - \xi)^{\frac{1}{2}}} d\xi\right]AD^{(2)}(\kappa).
\tag{4.10}
\]

By the use of orthogonal property of shifted Legendre polynomials [58] one can write

\[
(\Psi_{0, \Lambda}(\kappa), \Psi_{0, \Lambda'}^T(\kappa)) = I_0^{(\Lambda, \Lambda')} = (h^0_{ij})_{0 \leq i \leq \Lambda, 0 \leq j \leq \Lambda'}. \tag{4.11}
\]

Grouping equations 4.5-4.10, the residual term for the equation 4.1 is

\[
\text{Res}_{p, q}(\kappa, \varrho) \approx \Psi^T(\varrho) \left( (\hat{D}^{(\alpha_1, \alpha_2, \rho(\alpha))})^T A + A - K^* A (\hat{D}^{(\beta_1, \beta_2, \rho(\beta))}) - P^* T AD^{(2)} - \mathcal{F} \right) \Psi(\kappa)
= \Psi^T(\varrho) \mathcal{G} \Psi(\kappa), \tag{4.12}
\]

where,

\[
\mathcal{G} = \left( (\hat{D}^{(\alpha_1, \alpha_2, \rho(\alpha))})^T A + A - K^* A (\hat{D}^{(\beta_1, \beta_2, \rho(\beta))}) - P^* T AD^{(2)} - \mathcal{F} \right).
\]

Now, the standard tau method [59] is used to construct the following \( g(g - 1) \) linear algebraic equations

\[
I_T^{(\Lambda - 1, \Lambda)} \left( (\hat{D}^{(\alpha_1, \alpha_2, \rho(\alpha))})^T A + A - K^* A (\hat{D}^{(\beta_1, \beta_2, \rho(\beta))}) - P^* T AD^{(2)} - \mathcal{F} \right) I_{\Lambda'}^{(\Lambda', \Lambda' - 2)} = 0. \tag{4.13}
\]

The IC 4.2 and BCs 4.3-4.4, with the help of equation 4.6 can be utilised to obtain

\[
\Psi^T(0) A \Psi(\kappa) = \nu(\kappa), \tag{4.14}
\]

\[
\Psi^T(\varrho) A \Psi(0) = p(\varrho), \tag{4.15}
\]

\[
\Psi^T(\varrho) A \Psi(1) = q(\varrho). \tag{4.16}
\]

Equation 4.14, acquired with the help of IC and equations 4.15-4.16 are obtained through BCs. We collocate IC 4.14 at \( g + 1 \) & BCs 4.15-4.16 at \( g \) points. Equations 4.13-4.16 constitute a linear algebraic set of \( (g + 1)^2 \) equations which are solved for the unknowns \( a_{ij}, i, j = 0, \ldots, g \).
Here we chose the roots of shifted Legendre polynomials as a collocation point.

**Algorithm 1:** To evaluate the numerical solution of one dimensional distributed order time-space fractional integro-partial differential equation (4.1)-(4.4)

**Input:** The constant $\mathcal{R} \in \mathbb{N}$ and $g \in \mathbb{N}$ or $N \in \mathbb{N}$, $f(\mathcal{R}, g) : L^2(\Omega) \rightarrow \mathbb{R}$.

**Output:** The approximate solutions $\mathbb{U}(\mathcal{R}, g) \approx \Psi^T(g)A\Psi(\mathcal{R})$ for numerical solution of DOT-SFWSPDE (4.1-4.4) by using of operational matrix method.

**for** Numerical solution of DOT-SFWSPDE 4.1-4.4 by using operational matrix method do

**Step-2.1** Generate the basis function $\varphi_i(\mathcal{R})$, $\varphi_j(g)$; $i, j = 0, \ldots, g$, by using LWs as given in section 2.

**Step-2.2** Approximate the unknown function $U(\mathcal{R}, g)$ as given in equation 4.6 to get unknown vector $\mathcal{A}$.

**Step-2.3** Approximate the term $f(\mathcal{R}, g)$ as $f(\mathcal{R}, g) \approx \Psi^T(g)F\Psi(\mathcal{R})$ and obtain known vector $F$.

**Step-2.4** Approximate the distributed order time-fractional and space-fractional operational matrix using section 3.1 as

$$D_\mathcal{R}^{(\alpha)} Z(\mathcal{R}, g) \approx \Psi^T(g)\left(\hat{D}^{(\alpha_1, \alpha_2, \alpha(\mathcal{R}))}\right)^T A\Psi(\mathcal{R})$$

and

$$D_\mathcal{R}^{(\beta)} Z(\mathcal{R}, g) \approx \Psi(g)^T A\left(\hat{D}^{(\beta_1, \beta_2, \beta(\mathcal{R}))}\right) \Psi(\mathcal{R})$$

**Step-2.5** Approximate the singular integral operational matrix using section 3.1 and equations 4.9-4.12 as $\Psi^T(g)P^sTA^2(\mathcal{R})\Psi(\mathcal{R})$.

**Step-2.6** Compute the residual function $Res_{p,q}(\mathcal{R}, g)$ using equations 4.5-4.10 for equation 4.1 we get equation 4.12 as follows $Res_{p,q}(\mathcal{R}, g) \approx \Psi^T(g)\left((\hat{D}^{(\alpha_1, \alpha_2, \alpha(\mathcal{R}))})^T A + \mathcal{A} - \mathcal{K}^* A(\hat{D}^{(\beta_1, \beta_2, \beta(\mathcal{R}))}) - \mathcal{P}^sTA^{2(\mathcal{R})} - F\right) \Psi(\mathcal{R})$.

**Step-2.7** Apply standard tau method (use equation 4.13) to get unknown vector $f(g - 1)$ system of linear algebraic equations.

**Step-2.8** Use the initial condition (use equation 4.14) to construct $g + 1$ system of linear algebraic equations with the help of collocation points.

**Step-2.9** Use the boundary conditions (equation 4.15 and 4.16) to create $g + 2g$ linear algebraic system of equations with the help of collocation points.

**Step-2.10** For getting unknown vector $\mathcal{A}$, to solve the system of $(g + 1)^2$ algebraic linear equations which is evaluated in step (2.6)-(2.8).

**Step-2.11** Put the value of $\mathcal{A}$ in step (2.2) and we get the estimated solution $\mathbb{U}(\mathcal{R}, g)$.

end

### 4.2 2-D distributed order time-space fractional weakly singular integro differential equation

In order to describe the numerical method for solving two dimensional DOT-SFIPDEs, we consider the following DOT-SFWSIDE of the form:

$$\int_{\alpha_1}^{\alpha_2} \rho(\mathcal{R}) \frac{\partial^n \mathbb{U}(\mathcal{R}, \eta, \xi)}{\partial \eta^\mathcal{R}} d\mathcal{R} + \mathbb{U}(\mathcal{R}, \eta, \xi) = \mathcal{K}^* \int_{\beta_1}^{\beta_2} \rho(\mathcal{R}) \left[ \frac{\partial^3 \mathbb{U}(\mathcal{R}, \eta, \xi)}{\partial \eta^3} + \frac{\partial^3 \mathbb{U}(\mathcal{R}, \eta, \xi)}{\partial \eta^3} \right] d\beta$$

$$+ \int_{0}^{\xi} (\xi - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 \mathbb{U}(\mathcal{R}, \eta, \xi)}{\partial \eta^2} + \frac{\partial^2 \mathbb{U}(\mathcal{R}, \eta, \xi)}{\partial \eta^2} \right] d\xi + f(\mathcal{R}, \eta, \xi), (4.17)$$

Where, $\mathcal{K}^*$ is viscosity constant and $(\mathcal{R}, \eta, \xi) \in \Omega, \alpha_1 = 0, \alpha_2 = 1, \beta_1 = 1, \beta_2 = 2$ and $\Omega = [0, 1] \times [0, 1] \times [0, T]$. 
The above equation 4.17 is acquired with the initial condition (IC)
\[ U(\kappa, \eta, 0) = \nu(\kappa, \eta), \quad 0 < \kappa < 1 \text{ and } 0 < \eta < 1, \] (4.18)
and the boundary conditions (BCs)
\[ U(0, \eta, \rho) = p_1(\eta, \rho), \quad 0 < \eta < 1 \text{ and } 0 < \rho < T, \] (4.19)
\[ U(1, \eta, \rho) = p_2(\eta, \rho), \quad 0 < \eta < 1 \text{ and } 0 < \rho < T, \] (4.20)
\[ U(\kappa, 0, \rho) = q_1(\kappa, \rho), \quad 0 < \kappa < 1 \text{ and } 0 < \rho < T, \] (4.21)
\[ U(\kappa, 1, \rho) = q_2(\kappa, \rho), \quad 0 < \kappa < 1 \text{ and } 0 < \rho < T. \] (4.22)

Before discussing the method, we need to give brief about Kronecker product of two matrices [59].

If
\[
A = \begin{bmatrix}
  a_{00} & a_{01} & \cdots & a_{0g} \\
  a_{10} & a_{11} & \cdots & a_{1g} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{g0} & a_{g1} & \cdots & a_{gg}
\end{bmatrix}_{(g+1) \times (g+1)},
\]
\[
B = \begin{bmatrix}
  b_{00} & b_{01} & \cdots & b_{0g} \\
  b_{10} & b_{11} & \cdots & b_{1g} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{g0} & b_{g1} & \cdots & b_{gg}
\end{bmatrix}_{(g+1) \times (g+1)}.
\]

Then
\[
A \otimes B = \begin{bmatrix}
  a_{00}B & a_{01}B & \cdots & a_{0g}B \\
  a_{10}B & a_{11}B & \cdots & a_{1g}B \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{g0}B & a_{g1}B & \cdots & a_{gg}B
\end{bmatrix}_{(g+1)^2 \times (g+1)^2},
\]

- If A and B are lower(upper) triangular, then \( A \otimes B \) is also lower(upper) triangular.
- If A and B are band matrices, then \( A \otimes B \) is also a band matrix.

Consider the approximation of the known and unknown function as
\[ f(\kappa, \eta, \rho) \approx \Psi^T(\rho) \mathcal{F} \Psi(\kappa, \eta), \] (4.23)
\[ U(\kappa, \eta, \rho) \approx \Psi^T(\rho) \mathcal{F} \Psi(\kappa, \eta), \] (4.24)
where,
\[
\mathcal{F} = \begin{bmatrix}
  f_{00} & f_{01} & f_{02} & \cdots & f_{0g^2} \\
  f_{10} & f_{11} & f_{12} & \cdots & f_{1g^2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  f_{g-10} & f_{g-11} & f_{g-12} & \cdots & f_{g-1g^2} \\
  f_{g0} & f_{g1} & f_{g2} & \cdots & f_{gg^2}
\end{bmatrix}_{(g+1) \times (g+1)^2}.
where, $\Psi(\kappa, \eta) = (\Psi(\kappa) \otimes \Psi(\eta))$, the matrix $F$ is known and $A = [a_{ij}]$ denotes the unknown matrix that must be evaluated. The left hand side (L.H.S) of 4.17, by using the approximation of $U(\kappa, \eta, \varrho)$ can be written as

$$\begin{aligned}
\int_{\alpha_1}^{\alpha_2} \rho(\alpha) \frac{\partial^2 U(\alpha, \eta, \varrho)}{\partial \eta^2} d\alpha & \approx \left( \int_{\alpha_1}^{\alpha_2} \rho(\alpha) (D_\varrho^a \Psi(\varrho)) d\alpha \right) A \Psi(\kappa, \eta) \\
& \approx (D_\varrho^{\rho(\alpha)}) \Psi(\varrho) \right) A \Psi(\kappa, \eta) \\
& \approx \Psi(\varrho) \left( D(\alpha_1, \alpha_2, \rho(\alpha)) \right)^T A \Psi(\kappa, \eta).
\end{aligned} \tag{4.25}$$

Here, $D(\alpha_1, \alpha_2, \rho(\alpha))$ is the time-DOF operational matrix.

Now, the R.H.S of equation 4.17, with the help of the approximation of $U(\kappa, \eta, \varrho)$ can be described as

$$\begin{aligned}
\int_{\beta_1}^{\beta_2} \rho(\beta) \frac{\partial^2 U(\beta, \eta, \varrho)}{\partial \kappa^2} d\beta & \approx \Psi(\varrho) \left( \int_{\beta_1}^{\beta_2} \rho(\beta) (D_\kappa^{\beta} \Psi(\kappa, \eta)) d\beta \right) \\
& \approx \Psi(\varrho) \left( (D_\kappa^{\rho}) \Psi(\kappa) \right) \otimes \Psi(\eta) \\
& \approx \Psi(\varrho) \left( D(\kappa_1, \kappa_2, \rho(\beta)) \Psi(\kappa) \right) \otimes \Psi(\eta), \tag{4.26}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\beta_1}^{\beta_2} \rho(\beta) \frac{\partial^2 U(\beta, \eta, \varrho)}{\partial \eta^2} d\beta & \approx \Psi(\varrho) \left( \int_{\beta_1}^{\beta_2} \rho(\beta) (D_\eta^{\beta} \Psi(\kappa, \eta)) d\beta \right) \\
& \approx \Psi(\varrho) \left( \Psi(\kappa) \otimes (D_\eta^{\rho(\beta)}) \Psi(\eta) \right) \\
& \approx \Psi(\varrho) \left( \Psi(\kappa) \otimes (D(\kappa_1, \kappa_2, \rho(\beta)) \Psi(\eta)) \right). \tag{4.27}
\end{aligned}$$

Here, $D(\kappa_1, \kappa_2, \rho(\beta))$, $D(\kappa_1, \kappa_2, \rho(\beta))$ are the space-DOF operational matrices.

Now, approximation of the second term of R.H.S of equation 4.17 with the help of derivative operational matrix of integer order can be written as

$$\begin{aligned}
\frac{\partial^2 U(\kappa, \eta, \varrho)}{\partial \kappa^2} & \approx \Psi(\varrho) \left( \left( \frac{d^2}{d\kappa^2} \Psi(\kappa) \right) \otimes \Psi(\eta) \right) \approx \Psi(\varrho) \left( (D_\kappa^{2}) \Psi(\kappa) \right) \otimes \Psi(\eta), \tag{4.28}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 U(\kappa, \eta, \varrho)}{\partial \eta^2} & \approx \Psi(\varrho) \left( \Psi(\kappa) \otimes \left( \frac{d^2}{d\kappa^2} \Psi(\eta) \right) \right) \approx \Psi(\varrho) \left( \Psi(\kappa) \otimes (D_\kappa^{2}) \Psi(\eta) \right). \tag{4.29}
\end{aligned}$$

$$\begin{aligned}
\int_{\xi_0}^{\varrho} (\varrho - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(\xi, \eta, \varrho)}{\partial \kappa^2} \right] d\xi & \approx \left[ \int_{\xi_0}^{\varrho} (\varrho - \xi)^{-\frac{1}{2}} \Psi(\xi) \right] \Psi(\varrho) \right) \otimes \Psi(\eta)) \right) d\xi \\
& \approx \Psi(\varrho) \left( (D_\kappa^{2}) \Psi(\kappa) \right) \otimes \Psi(\eta)) \right) \right) d\xi. \tag{4.30}
\end{aligned}$$
\[
\int_0^g (\varrho - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(\varphi, \eta, \xi)}{\partial \eta^2} \right] d\xi \approx \int_0^g (\varrho - \xi)^{-\frac{1}{2}} \left( \Psi^T(\xi) A (\Psi(\varphi) \otimes (D^2(2)\Psi(\eta))) \right) d\xi
\]

\[
\approx \left[ \int_0^g \frac{\Psi^T(\xi)}{\varrho - \xi^\frac{1}{2}} d\xi \right] A (\Psi(\varphi) \otimes (D^2(2)\Psi(\eta)))
\]

\[
\approx \Psi^T(\varrho) P^T A (\Psi(\varphi) \otimes (D^2(2)\Psi(\eta))) \quad \text{.} (4.31)
\]

substituting all these approximations in equation 4.17

\[
\Psi^T(\varrho) \left( \hat{D}^{(\alpha_1,\alpha_2,\rho(\alpha))} \right)^T A (\Psi(\varphi) \otimes \Psi(\eta)) - \mathcal{K}^* \Psi^T(\varrho) A \left( \left( \hat{D}^{(\beta_1,\beta_2,\rho(\beta))} \Psi(\varphi) \right) \otimes \Psi(\eta) \right)
\]

\[
- \mathcal{K}^* \Psi^T(\varrho) A \left( \left( \hat{D}^{(\beta_1,\beta_2,\rho(\beta))} \Psi(\eta) \right) \right) - \Psi^T(\varrho) P^T A \left( (D^2(2)\Psi(\varphi)) \otimes \Psi(\eta) \right)
\]

\[
- \Psi^T(\varrho) P^T A \left( (D^2(2)\Psi(\varphi)) \right) - \Psi^T A (\Psi(\varphi) \otimes \Psi(\eta)) = 0. \quad (4.32)
\]

Using the tau method [59] with LW operational matrix, we generate \((\Lambda - 1) \times (\Lambda' - 2) \times (\Lambda'' - 2)\) linear algebraic equations.

\[
I_T^{(\Lambda-1,\Lambda)}(\varrho) \left( \hat{D}^{(\alpha_1,\alpha_2,\rho(\alpha))} \right)^T A (I_l^{(\Lambda',\Lambda''-1)} \otimes I_h^{(\Lambda-1,\Lambda)}) + I_T^{(\Lambda-1,\Lambda)} A (I_l^{(\Lambda'-1,\Lambda')} \otimes I_h^{(\Lambda''-1,\Lambda'')})
\]

\[
- \mathcal{K}^* I_T^{(\Lambda-1,\Lambda)} A \left( \left( \hat{D}_{\varphi}^{(\beta_1,\beta_2,\rho(\beta))} I_l^{(\Lambda'-1,\Lambda')} \otimes I_h^{(\Lambda''-1,\Lambda'')} \right) \right)
\]

\[
- \mathcal{K}^* I_T^{(\Lambda-1,\Lambda)} A \left( I_l^{(\Lambda'-1,\Lambda')} \otimes \left( \hat{D}_{\eta}^{(\beta_1,\beta_2,\rho(\beta))} I_h^{(\Lambda''-1,\Lambda'')} \right) \right)
\]

\[
- I_T^{(\Lambda-1,\Lambda)} P^T A \left( (D^2(2)) I_l^{(\Lambda'-1,\Lambda')} \otimes I_h^{(\Lambda''-1,\Lambda'')} \right)
\]

\[
- I_T^{(\Lambda-1,\Lambda)} P^T A \left( I_l^{(\Lambda'-1,\Lambda')} \otimes (D^2(2)I_h^{\Lambda''-1,\Lambda'\prime}) \right)
\]

\[
- I_T^{(\Lambda-1,\Lambda)} \mathcal{F} (I_l^{(\Lambda'-1,\Lambda')} \otimes I_h^{(\Lambda''-1,\Lambda'\prime)}) = 0. \quad (4.33)
\]

The IC 4.18 and BCs 4.19-4.22, with the help of equation 4.24 can be utilised to obtain

\[
\Psi^T(0) A (\Psi(\varphi) \otimes \Psi(\eta)) = \nu(\varphi, \eta). \quad (4.34)
\]

\[
\Psi^T(\varrho) A (\Psi(0) \otimes \Psi(\eta)) = p_1(\eta, \varrho). \quad (4.35)
\]

\[
\Psi^T(\varrho) A (\Psi(1) \otimes \Psi(\eta)) = p_2(\eta, \varrho). \quad (4.36)
\]

\[
\Psi^T(\varrho) A (\Psi(\varphi) \otimes \Psi(0)) = q_1(\varphi, \varrho). \quad (4.37)
\]

\[
\Psi^T(\varrho) A (\Psi(\varphi) \otimes \Psi(1)) = q_2(\varphi, \varrho). \quad (4.38)
\]

Equation 4.34, acquired with the help of IC and equations 4.35-4.38 are obtained through BCs. We collocate IC 4.34 at \((g + 1)^2\) & BCs 4.35-4.38 at \(4g^2\) points. Equations 4.33-4.38 constitute linear algebraic set of \((g + 1)^3\) equations which are solved for the unknowns.
$a_{ij}$, $i = 0, 1, \ldots, g$ & $j = 0, 1, \ldots, g^2$. Here we chose the roots of shifted Legendre polynomials as a collocation points.

**Algorithm 2:** To evaluate the numerical solution of two dimensional distributed order time-space fractional integro-partial differential equation 4.17-4.22.

| **Input:** | The constant $K \in \mathbb{N}$ and $g \in \mathbb{N}$ or $N \in \mathbb{N}$, $f(x, \eta, \varrho) : L^2(\Omega = [0, 1] \times [0, 1] \times [0, 1]) \rightarrow \mathbb{R}$. |
| **Output:** | The approximate solutions $w.r.t.$ continuous functions $\varphi_i(x)$, $\varphi_j(\varrho)$; $i, j, k = 0, \ldots, g$ by using LWs as given in section 2. |

for Numerical solution of DOT-SFWSIPDE 4.17-4.22 by using of operational matrix method do

| **Step-2.1** Generate the basis function $\varphi_i(x), \varphi_j(\varrho), \varphi_k(\eta)$; $i, j, k = 0, \ldots, g$ by using LWs as given in section 2. |
| **Step-2.2** Approximate the unknown function $U(x, \eta, \varrho)$ as given in equation 4.24 to get unknown vector $A$. |
| **Step-2.3** Approximate the source term $f(x, \eta, \varrho)$ as $f(x, \eta, \varrho) \approx \Psi^T(\varrho) F \Psi(x, \eta)$ to get the known vector say $F$. |
| **Step-2.4** Approximate the distributed order time-fractional and space-fractional operational matrix using section 3.2 $D^\rho(\alpha)U(x, \eta, \varrho) \approx \Psi^T(\varrho) \left( \hat{D}^{(\alpha_1, \alpha_2, \rho(\alpha))} \right)^T \mathcal{A} \Psi(x) \otimes \Psi(\eta)$, |
| **Step-2.5** Approximate the singular integral operational matrix using section 3.1 and equation 4.30-4.31 as $\Psi^T(\varrho) P_1^T A (D^{(2)}) \Psi(x) \otimes \Psi(\eta)$ and $\Psi^T(\varrho) P_2^T A (\Psi(x) \otimes D(2)) \Psi(\eta)$, respectively. |
| **Step-2.6** Compute the matrix $I_T^{(\Lambda, \Lambda, \Lambda-2)}$, $I_T^{(\Lambda, \Lambda', \Lambda'-2)}$ and $I_h^{(\Lambda, \Lambda, \Lambda-2)}$ with the help of equation 4.11. |
| **Step-2.7** Apply standard tau method (use equation 4.33) to create $g(g - 1)^2$ system of linear algebraic equations. |
| **Step-2.8** Use the initial condition (use equation 4.34) to construct $(g + 1)^2$ system of linear algebraic equations with the help of collocation points. |
| **Step-2.9** Use the boundary conditions (equations 4.35-4.38) to create $4g^2$ linear algebraic system of equations with the help of collocation points. |
| **Step-2.10** For getting unknown vector $A$, to solve the system of $(g + 1)^3$ linear algebraic equations which is evaluated in step (2.7)-(2.9). |
| **Step-2.11** Put the value of $A$ in step (2.2) and we get the estimated solution $U(x, \eta, \varrho)$. |

end

### 5 Error bounds and convergence analysis

**Theorem 5.1.** Assume $\{f(x, \varrho)\}_N$, $\{v(x, \varrho)\}_N$, $\{w(x, \varrho)\}_N$, $\{w_1(x, \varrho)\}_N$, $\{U(x, \varrho)\}_N$ be the approximate solutions w.r.t. continuous functions $f(x, \varrho)$, $v(x, \varrho)$, $w(x, \varrho)$, $w_1(x, \varrho)$, $U(x, \varrho)$, respectively, defined over the domain $\Omega = [0, 1] \times [0, 1]$ with second order bounded mixed derivative, say $\left| \frac{\partial^4 f(x, \varrho)}{\partial x^2 \partial \varrho^2} \right| \leq B_0$, $\left| \frac{\partial^4 v(x, \varrho)}{\partial x^2 \partial \varrho^2} \right| \leq B_1$, $\left| \frac{\partial^4 w(x, \varrho)}{\partial x^2 \partial \varrho^2} \right| \leq B_2$, $\left| \frac{\partial^4 w_1(x, \varrho)}{\partial x^2 \partial \varrho^2} \right| \leq B_3$, $\left| \frac{\partial^4 U(x, \varrho)}{\partial x^2 \partial \varrho^2} \right| \leq B_0$, for some positive constant $B$, $B_1$, $B_2$, $B_3$, $B_0$, where $v = \frac{\partial U(x, \varrho)}{\partial \varrho}$.
where, \( w = \frac{\partial^2 U(x, \varrho)}{\partial x^2} \) and \( w_1 = \frac{\partial U(x, \varrho)}{\partial x} \). Then

(a) \( H(x, \varrho) \) can be expressed in terms of LWs infinite series which converges uniformly to the function \( H(x, \varrho) \), that is

\[
H(x, \varrho) = \sum_{h=1}^{\infty} \sum_{g=1}^{\infty} \sum_{s'g'=1}^{\infty} C_{bgh'g'} \Psi_{bgh'g'}(x, \varrho),
\]

where, \( C_{bgh'g'} = \langle H(x, \varrho), \Psi_{bgh'g'} \rangle \) \( L^2(\Omega) \) and \( H(x, \varrho) \in \left\{ f(x, \varrho), \frac{\partial U}{\partial \varrho}, \frac{\partial^2 U}{\partial x^2}, \frac{\partial U}{\partial x}, U(x, \varrho) \right\} \).

(b) The bound of error is

\[
\|\epsilon_H\|^2_{L^2} \leq 9B_*^2 \sum_{h=2^{N-1}+1}^{\infty} \sum_{g=2^{N'-1}+1}^{\infty} \sum_{h'=2^{N'-1}+1}^{\infty} \frac{1}{256(hh')^5(2g - 3)(2g' - 3)},
\]

where,

\[
\|\epsilon_H\|^2_{L^2} = \int_0^1 \int_0^1 |H(x, \varrho) - \sum_{h=1}^{2^N-1} \sum_{g=0}^{2^{N'-1}-1} \sum_{h'=1}^{2^{N'-1}-1} \sum_{g'=0}^{2^{N'-1}-1} C_{bgh'g'} \Psi_{bgh'g'}(x, \varrho)|^2 \, dx \, d\varrho,
\]

and \( B_* \in \{B, B_1, B_2, B_3, B_0\} \). It should be noted that \( B, B_1, B_2, B_3, B_0 \) correspond to \( f, v, w, w_1, U \), respectively.

**Proof.** The proof of this theorem is similar to the proof of theorem 1 given in reference [47]. Consider, equation (22) and applying the inequality \( 2^{N} \geq 2h \) & \( 2^{N'} \geq 2h' \) for \( R, R' \), \( h, h' \in \mathbb{Z}^+ \), we have,

\[
\frac{1}{2^N} \leq \frac{1}{2h} \text{ and } \frac{1}{2^{N'}} \leq \frac{1}{2h'}.
\]

Then

\[
\|\epsilon_H\|^2_{L^2} \leq 9B_*^2 \sum_{h=2^{N-1}+1}^{\infty} \sum_{g=2^{N'-1}+1}^{\infty} \sum_{h'=2^{N'-1}+1}^{\infty} \frac{1}{256(hh')^5(2g - 3)(2g' - 3)},
\]

where,

\[
\|\epsilon_H\|^2_{L^2} = \int_0^1 \int_0^1 |H(x, \varrho) - \sum_{h=1}^{2^N-1} \sum_{g=0}^{2^{N'-1}-1} \sum_{h'=1}^{2^{N'-1}-1} \sum_{g'=0}^{2^{N'-1}-1} C_{bgh'g'} \Psi_{bgh'g'}(x, \varrho)|^2 \, dx \, d\varrho.
\]

\( H(x, \varrho) \in \left\{ f(x, \varrho), \frac{\partial U}{\partial \varrho}, \frac{\partial^2 U}{\partial x^2}, \frac{\partial U}{\partial x}, U(x, \varrho) \right\} \) and \( B_* \in \{B, B_1, B_2, B_3, B_0\} \). \( Q.E.D. \)

**Theorem 5.2.** Let \( \{J(x, \varrho)\}_N \) represents the approximate solution of continuous function \( J(x, \varrho) \) for \( x, \varrho \in [0, 1] \) such that \( \left| \frac{\partial^6 U(x, \varrho)}{\partial x^2 \partial \xi^2} \right| \leq B_2 \), where \( B_2 \) is a positive constant, then

\[
\|J(x, \varrho) - (J(x, \varrho))_N\|^2_{L^2} \leq 36B_2 \sum_{h=2^{N-1}+1}^{\infty} \sum_{g=2^{N'-1}+1}^{\infty} \sum_{h'=2^{N'-1}+1}^{\infty} \frac{1}{256(hh')^5(2g - 3)(2g' - 3)},
\]

where, \( J(x, \varrho) = \int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(x, \xi)}{\partial x^2} \right] d\xi \).

**Proof.** Consider the term \( \|J(x, \varrho) - \{J(x, \varrho)\}_N\|^2_{L^2} \)

\[
\|J(x, \varrho) - \{J(x, \varrho)\}_N\|^2_{L^2} = \int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} \left( \frac{\partial^2 U(x, \xi)}{\partial x^2} - \left( \frac{\partial^2 U(x, \xi)}{\partial x^2} \right)_N \right) d\xi^2
\]
Now using theorem 5.1, then we have

\[
||J(x, \varrho) - \{J(x, \varrho)\}_N||_2^2 \leq \left[ \int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} d\xi \right]^2 \left\| \partial^2 U(x, \varrho) \right\|_{\partial x^2}^2 - \left( \partial^2 U(x, \varrho) \right)_{L^2}^2 \leq \left[ 9B_2 \sum_{h=2^{N-1}+1}^{\infty} \sum_{g=\Lambda}^{\infty} \sum_{g'=\Lambda}^{\infty} \sum_{g'=2^{N-1}+1}^{\infty} 256(hh')^5(2g - 3)^4(2g' - 3)^4 \right] \times \left[ \int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} d\xi \right]^2 \leq 36B_2 \sum_{h=2^{N-1}+1}^{\infty} \sum_{g=\Lambda}^{\infty} \sum_{g'=\Lambda}^{\infty} \sum_{g'=2^{N-1}+1}^{\infty} 256(hh')^5(2g - 3)^4(2g' - 3)^4.
\]

\[ \text{Theorem 5.3. Let} \ \left( \frac{\partial^\alpha U(x, \varrho)}{\partial \varrho^\alpha} \right) \text{and} \ \left( \frac{\partial^\beta U(x, \varrho)}{\partial x^\beta} \right) \text{be the approximation of continuous function} \ \frac{\partial^\alpha U(x, \varrho)}{\partial \varrho^\alpha} \text{and} \ \frac{\partial^\beta U(x, \varrho)}{\partial x^\beta} \ \alpha \in (0, 1), \ \beta \in (1, 2) \text{ such that} \ \left| \frac{\partial^\alpha U(x, \varrho)}{\partial x^\beta \partial \varrho^\beta} \right| < B_3 \text{, where} \ B_1, B_3 \text{ are positive constants, then} \]

\[
\left\| \frac{\partial^\alpha U(x, \varrho)}{\partial \varrho^\alpha} - \left( \frac{\partial^\alpha U(x, \varrho)}{\partial \varrho^\alpha} \right)_{N} \right\|_{L^2}^2 \leq \frac{9B_2}{\Gamma(2 - \alpha)^2} \sum_{h=2^{N-1}+1}^{\infty} \sum_{g=\Lambda}^{\infty} \sum_{g'=\Lambda}^{\infty} \sum_{g'=2^{N-1}+1}^{\infty} 256(hh')^5(2g - 3)^4(2g' - 3)^4,
\]

\[ \text{and,} \]

\[
\left\| \frac{\partial^\beta U(x, \varrho)}{\partial x^\beta} - \left( \frac{\partial^\beta U(x, \varrho)}{\partial x^\beta} \right)_{N} \right\|_{L^2}^2 \leq \frac{(2!)^2 \times 9B_3}{\Gamma(3 - \beta)^2} \sum_{h=2^{N-1}+1}^{\infty} \sum_{g=\Lambda}^{\infty} \sum_{g'=\Lambda}^{\infty} \sum_{g'=2^{N-1}+1}^{\infty} 256(hh')^5(2g - 3)^4(2g' - 3)^4.
\]

\[ \text{Proof. Consider the term} \ \left\| \frac{\partial^\alpha U(x, \varrho)}{\partial \varrho^\alpha} - \left( \frac{\partial^\alpha U(x, \varrho)}{\partial \varrho^\alpha} \right)_{N} \right\|_{L^2}^2 \text{ and using equation 2.1 & theorem 5.1, we have} \]

\[
\left\| \frac{\partial^\alpha U(x, \varrho)}{\partial \varrho^\alpha} - \left( \frac{\partial^\alpha U(x, \varrho)}{\partial \varrho^\alpha} \right)_{N} \right\|_{L^2}^2 = \left\| \frac{1}{\Gamma(1 - \alpha)} \int_0^\varrho (\varrho - s)^{-\alpha} \left( \frac{\partial U(x, \varrho)}{\partial \varrho} - \left( \frac{\partial U(x, \varrho)}{\partial \varrho} \right)_{N} \right) ds \right\|_{L^2}^2.
\]
Now consider the term \( \left\| \frac{\partial^\beta \mathbf{U}(\mathbf{z}, \varrho)}{\partial \mathbf{z}^\beta} - \left( \frac{\partial^\beta \mathbf{U}(\mathbf{z}, \varrho)}{\partial \mathbf{z}^\beta} \right)_{\mathcal{N}} \right\|_2^2 \) and again applying the definition of Caputo derivative (section 2) for vector \( \mathbf{z} \) along with theorem 5.1, we can write

\[
\left\| \frac{\partial^\beta \mathbf{U}(\mathbf{z}, \varrho)}{\partial \mathbf{z}^\beta} - \left( \frac{\partial^\beta \mathbf{U}(\mathbf{z}, \varrho)}{\partial \mathbf{z}^\beta} \right)_{\mathcal{N}} \right\|_2^2 = \left\| \frac{1}{(2-\beta)} \int_0^\infty (\mathbf{z}-s)^{-1-\beta} \left( \frac{\partial^2 \mathbf{U}(\mathbf{z}, \varrho)}{\partial \mathbf{z}^2} - \left( \frac{\partial^2 \mathbf{U}(\mathbf{z}, \varrho)}{\partial \mathbf{z}^2} \right)_{\mathcal{N}} \right) \, ds \right\|_2^2
\]

Now, define

\[
L_1(\mathbf{U}(\mathbf{z}, \varrho)) = \int_{\alpha_1}^{\alpha_2} \rho(\alpha) \frac{\partial^\alpha \mathbf{U}(\mathbf{z}, \varrho)}{\partial \alpha^\alpha} \, d\alpha + \mathbf{U}(\mathbf{z}, \varrho) - \mathbf{K}^* \int_{\beta_1}^{\beta_2} \rho(\beta) \frac{\partial^\beta \mathbf{U}(\mathbf{z}, \varrho)}{\partial \beta^\beta} \, d\beta - \int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 \mathbf{U}(\mathbf{z}, \xi)}{\partial \mathbf{z}^2} \right] \, d\xi - f(\mathbf{z}, \varrho),
\]

\[
L_2(\mathbf{U}(\mathbf{z}, \varrho)) = \sum_{s=1}^P w_s \rho(\sigma_s) \left( \frac{\partial \sigma_s \mathbf{U}(\mathbf{z}, \varrho)}{\partial \sigma^s} \right) + \mathbf{U}(\mathbf{z}, \varrho) - \mathbf{K}^* \sum_{r=1}^{P^*} w_r \rho(\sigma_r) \left( \frac{\partial \sigma_r \mathbf{U}(\mathbf{z}, \varrho)}{\partial \sigma^{r*}} \right) - \int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 \mathbf{U}(\mathbf{z}, \xi)}{\partial \mathbf{z}^2} \right] \, d\xi - f(\mathbf{z}, \varrho).
\]

Also consider

\[
L_1(\mathbf{U}(\mathbf{z}, \varrho)) - L_2(\mathbf{U}(\mathbf{z}, \varrho)) = P_1(P, \mathbf{z}, \varrho) + P_2(P^*, \mathbf{z}, \varrho).
\]
Let $P_1(P, \varphi, \vartheta)$ and $P_2(P^*, \varphi, \vartheta)$ denote the error for using P-point and $P^*$-point LGQ formula, respectively then

$$P_1(P, \varphi, \vartheta) = \frac{(P!)^4}{(2P + 1)(2P!)^4} \frac{\partial^{2P}}{\partial \alpha^{2P}} H_1(\varphi, \vartheta, \alpha)$$

$$\approx \frac{\pi}{4^P} \frac{\partial^{2P}}{\partial \alpha^{2P}} H_1(\varphi, \vartheta, \alpha), \quad \alpha \in [0, 1].$$

(5.1)

And

$$P_2(P^*, \varphi, \vartheta) = \frac{(P^*)^4}{(2P^* + 1)(2P^*)^4} \frac{\partial^{2P^*}}{\partial \beta^{2P^*}} H_2(\varphi, \vartheta, \beta)$$

$$\approx \frac{\pi}{4^{P^*}} \frac{\partial^{2P^*}}{\partial \beta^{2P^*}} H_2(\varphi, \vartheta, \beta), \quad \beta \in [1, 2].$$

(5.2)

Where,

$$H_1(\varphi, \vartheta, \alpha) = \rho(\alpha) \frac{\partial^\alpha}{\partial \vartheta^\alpha} \U(\varphi, \vartheta) \quad \text{and} \quad H_2(\varphi, \vartheta, \beta) = \rho(\beta) \frac{\partial^\beta}{\partial \vartheta^\beta} \U(\varphi, \vartheta).$$

Now, for $H_1(\varphi, \vartheta, \alpha) \in C^{2P}([0, 1])$, $H_2(\varphi, \vartheta, \beta) \in C^{2P^*}([1, 2])$ and fixed $(\varphi, \vartheta) \in [0, 1]$, we have

$$\|P_1(P, \varphi, \vartheta)\|_2^2 = \int_0^1 \int_0^1 |P_1(P, \varphi, \vartheta)|^2 d\vartheta d\varphi$$

$$= \int_0^1 \int_0^1 \pi^2 \left| \frac{\partial^{2P}}{\partial \alpha^{2P}} H_1(\varphi, \vartheta, \alpha) \right|^2 d\vartheta d\varphi$$

$$\leq \frac{C_1^2 \pi^2}{4^{2P^*}},$$

(5.3)

And

$$\|P_2(P^*, \varphi, \vartheta)\|_2^2 = \int_0^1 \int_0^1 |P_2(P^*, \varphi, \vartheta)|^2 d\vartheta d\varphi$$

$$= \int_0^1 \int_0^1 \pi^2 \left| \frac{\partial^{2P^*}}{\partial \beta^{2P^*}} H_2(\varphi, \vartheta, \beta) \right|^2 d\vartheta d\varphi$$

$$\leq \frac{C_2^2 \pi^2}{4^{2P^*}},$$

(5.4)

where,

$$C_1 = \max \left\{ \left| \frac{\partial^{2P}}{\partial \alpha^{2P}} H_1(\varphi, \vartheta, \alpha) \right| : 0 \leq \varphi, \vartheta, \alpha \leq 1 \right\} \quad \text{and}$$

$$C_2 = \max \left\{ \left| \frac{\partial^{2P^*}}{\partial \beta^{2P^*}} H_2(\varphi, \vartheta, \beta) \right| : 0 \leq \varphi, \alpha \leq 1, \ 1 \leq \beta \leq 2 \right\}.$$

Now, define the residual function as:

$$Res_N(\U(\varphi, \vartheta)) = \sum_{s=1}^P w_s \rho(\sigma_s) \left( \frac{\partial \U(\varphi, \vartheta)}{\partial \vartheta^s} \right)_N + (\U(\varphi, \vartheta))_N - K^* \sum_{r=1}^{P^*} w_r \rho(\sigma_r) \left( \frac{\partial \U(\varphi, \vartheta)}{\partial \vartheta^r} \right)_N$$

$$- \left( \int_0^\varphi (\varphi - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 \U(\varphi, \xi)}{\partial \alpha^2} \right] d\xi \right)_N - (f(\varphi, \vartheta))_N.$$
By theorems 5.1, 5.3, it is evident that

\[
\|L_2(U(\mathbf{x}, \varphi)) - \text{Res}_{\mathcal{N}}(U(\mathbf{x}, \varphi))\|_2 = \left\| \sum_{s=1}^{P} w_s \rho(\sigma_s) \left( \frac{\partial^{\tau_s} U(\mathbf{x}, \varphi)}{\partial \varphi^{\tau_s}} - \left( \frac{\partial^{\tau_s} U(\mathbf{x}, \varphi)}{\partial \varphi^{\tau_s}} \right)_\mathcal{N} \right) \right. \\
+ \left( U(\mathbf{x}, \varphi) - (U(\mathbf{x}, \varphi))_\mathcal{N} \right) \\
- \mathcal{K}^* \sum_{r=1}^{P^*} w_r \rho(\sigma_r) \left( \frac{\partial^{\tau_r} U(\mathbf{x}, \varphi)}{\partial \varphi^{\tau_r}} - \left( \frac{\partial^{\tau_r} U(\mathbf{x}, \varphi)}{\partial \varphi^{\tau_r}} \right)_\mathcal{N} \right) \\
- \left( \int_0^\theta (\varphi - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(\mathbf{x}, \xi)}{\partial \varphi^2} \right] d\xi - \left( \int_0^\theta (\varphi - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(\mathbf{x}, \xi)}{\partial \varphi^2} \right] d\xi \right)_{\mathcal{N}} \right) \\
- (f(\mathbf{x}, \varphi) - (f(\mathbf{x}, \varphi))_\mathcal{N}) \\
\leq \left\| \sum_{s=1}^{P} w_s \rho(\sigma_s) \left( \frac{\partial^{\tau_s} U(\mathbf{x}, \varphi)}{\partial \varphi^{\tau_s}} - \left( \frac{\partial^{\tau_s} U(\mathbf{x}, \varphi)}{\partial \varphi^{\tau_s}} \right)_\mathcal{N} \right) \right. \\
+ \left( U(\mathbf{x}, \varphi) - (U(\mathbf{x}, \varphi))_\mathcal{N} \right) \\
+ \mathcal{K}^* \left\| \sum_{r=1}^{P^*} w_r \rho(\sigma_r) \left( \frac{\partial^{\tau_r} U(\mathbf{x}, \varphi)}{\partial \varphi^{\tau_r}} - \left( \frac{\partial^{\tau_r} U(\mathbf{x}, \varphi)}{\partial \varphi^{\tau_r}} \right)_\mathcal{N} \right) \right. \\
+ \left( \int_0^\theta (\varphi - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(\mathbf{x}, \xi)}{\partial \varphi^2} \right] d\xi - \left( \int_0^\theta (\varphi - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(\mathbf{x}, \xi)}{\partial \varphi^2} \right] d\xi \right)_{\mathcal{N}} \right) \\
+ \left( (f(\mathbf{x}, \varphi) - (f(\mathbf{x}, \varphi))_\mathcal{N}) \right) \\
\leq PM_1M_2 \left[ \frac{9B_1^2}{\Gamma(2 - \alpha)^2} \sum_{b = 2^{K-1}+1}^{\infty} \sum_{g = \Lambda}^{\infty} \sum_{h' = 2^{K'-1}+1}^{g} \sum_{g' = \Lambda'}^{\infty} \frac{1}{256(hh')^5(2g - 3)^4(2g' - 3)^4} \right]^{\frac{1}{2}} \\
+ 9B_0^2 \left[ \frac{(2l)^2 \times 9B_2^2}{\Gamma(3 - \beta)^2} \sum_{b = 2^{K-1}+1}^{\infty} \sum_{g = \Lambda}^{\infty} \sum_{b' = 2^{K'-1}+1}^{g} \sum_{g' = \Lambda'}^{\infty} \frac{1}{256(hh')^5(2g - 3)^4(2g' - 3)^4} \right]^{\frac{1}{2}} \\
+ P^* M_3M_3 \left[ \frac{(2l)^2 \times 9B_2^2}{\Gamma(3 - \beta)^2} \sum_{b = 2^{K-1}+1}^{\infty} \sum_{g = \Lambda}^{\infty} \sum_{b' = 2^{K'-1}+1}^{g} \sum_{g' = \Lambda'}^{\infty} \frac{1}{256(hh')^5(2g - 3)^4(2g' - 3)^4} \right]^{\frac{1}{2}} \\
+ 4 \times 9B_2^2 \left[ \frac{(2l)^2 \times 9B_2^2}{\Gamma(3 - \beta)^2} \sum_{b = 2^{K-1}+1}^{\infty} \sum_{g = \Lambda}^{\infty} \sum_{b' = 2^{K'-1}+1}^{g} \sum_{g' = \Lambda'}^{\infty} \frac{1}{256(hh')^5(2g - 3)^4(2g' - 3)^4} \right]^{\frac{1}{2}} \\
+ 9B_2^2 \left[ \frac{(2l)^2 \times 9B_2^2}{\Gamma(3 - \beta)^2} \sum_{b = 2^{K-1}+1}^{\infty} \sum_{g = \Lambda}^{\infty} \sum_{b' = 2^{K'-1}+1}^{g} \sum_{g' = \Lambda'}^{\infty} \frac{1}{256(hh')^5(2g - 3)^4(2g' - 3)^4} \right]^{\frac{1}{2}}. \tag{5.5} \right.

Where, \( M_1 = \max\{w_s, s = 1, 2, \cdots P\} \) and \( M_2 = \max\{|\rho(\sigma_s)|, s = 1, 2, \cdots P\} \) and \( M_3 = \max\{|w_r|, r = 1, 2, \cdots P^*\} \) and \( M_4 = \max\{|\rho(\sigma_r)|, r = 1, 2, \cdots P^*\} \).
Finally, we have

\[
\|Res_N(U(\alpha, \theta))\|_2 = \|0 - Res_N(U(\alpha, \theta))\|_2 \\
= \|L_1(U(\alpha, \theta)) - Res_N(U(\alpha, \theta))\|_2 \\
\leq \|L_1(U(\alpha, \theta)) - L_2(U(\alpha, \theta))\|_2 + \|L_2(U(\alpha, \theta)) - Res_N(U(\alpha, \theta))\|_2
\]

\[
\leq C_1 \pi + \frac{3B_1}{\Gamma(2 - \alpha)} \left[ \sum_{b=2^{m-1}+1}^{\infty} \sum_{g=\Lambda}^{2^{b'1}+1} \sum_{g'=\Lambda}^{\infty} \sum_{\ell=\Lambda}^{1} \frac{1}{256(hh')^5(2g - 3)^4(2g' - 3)^4} \right]^{\frac{1}{2}} \\
+ 3B_0 \left[ \sum_{b=2^{m-1}+1}^{\infty} \sum_{g=\Lambda}^{2^{b'1}+1} \sum_{g'=\Lambda}^{\infty} \sum_{\ell=\Lambda}^{1} \frac{1}{256(hh')^5(2g - 3)^4(2g' - 3)^4} \right]^{\frac{1}{2}} \\
+ C_2 \pi + 6PM_3M_4B_2|\mathcal{K}^*| \left[ \sum_{b=2^{m-1}+1}^{\infty} \sum_{g=\Lambda}^{2^{b'1}+1} \sum_{g'=\Lambda}^{\infty} \sum_{\ell=\Lambda}^{1} \frac{1}{256(hh')^5(2g - 3)^4(2g' - 3)^4} \right]^{\frac{1}{2}} \\
+ 6B_2 \left[ \sum_{b=2^{m-1}+1}^{\infty} \sum_{g=\Lambda}^{2^{b'1}+1} \sum_{g'=\Lambda}^{\infty} \sum_{\ell=\Lambda}^{1} \frac{1}{256(hh')^5(2g - 3)^4(2g' - 3)^4} \right]^{\frac{1}{2}} \\
+ 3B \left[ \sum_{b=2^{m-1}+1}^{\infty} \sum_{g=\Lambda}^{2^{b'1}+1} \sum_{g'=\Lambda}^{\infty} \sum_{\ell=\Lambda}^{1} \frac{1}{256(hh')^5(2g - 3)^4(2g' - 3)^4} \right]^{\frac{1}{2}}
\]

\[
\leq C_1 \pi + \frac{C_2 \pi}{4\pi} + 3\mathcal{K} \left[ \frac{PM_1M_2}{\Gamma(2 - \alpha)} + 1 + |\mathcal{K}^*| \frac{2P^*M_5M_4}{\Gamma(3 - \beta)} + 2 + 1 \right] \left[ \sum_{b=2^{m-1}+1}^{\infty} \sum_{g=\Lambda}^{2^{b'1}+1} \sum_{g'=\Lambda}^{\infty} \sum_{\ell=\Lambda}^{1} \frac{1}{256(hh')^5(2g - 3)^4(2g' - 3)^4} \right]^{\frac{1}{2}},
\]

where, \(\mathcal{K} = max\{B, B_0, B_1, B_2\}\).

6 Error estimation

In this section, we discuss about an error estimation for the DOT–SFWSIPDE. We achieve this by rewriting the equations.

\[
\int_{\alpha_1}^{\alpha_2} \rho(\alpha) \frac{\partial^3 U(\alpha, \theta)}{\partial \xi^3} d\alpha + U(\alpha, \theta) = \mathcal{K}^* \int_{\beta_1}^{\beta_2} \rho(\beta) \frac{\partial^3 U(\beta, \theta)}{\partial \xi^3} d\beta + \int_{0}^{\theta} (\theta - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(\alpha, \theta)}{\partial \xi^2} \right] d\xi + f(\alpha, \theta),
\]

where, \(\mathcal{K}^*\) is viscosity constant and \((\alpha, \theta) \in [0, 1] \times [0, T]\), \(\alpha_1 = 0, \alpha_2 = 1, \beta_1 = 1, \beta_2 = 2\).

Equation 6.1 is acquired with the IC and BCs

\[
U(\alpha, 0) = \nu(\alpha), \quad 0 < \alpha < 1, \\
U(0, \theta) = p_1(\theta), \quad 0 < \theta < T, \\
U(1, \theta) = p_2(\theta), \quad 0 < \theta < T.
\]
Let $E_N(\kappa, \varrho) = U(\kappa, \varrho) - U_N(\kappa, \varrho)$ denote the error function, where $U(\kappa, \varrho)$, $U_N(\kappa, \varrho)$ denotes the exact & approximate solution, respectively, of equation 6.1. Inserting the approximate solution into the equation 6.1 then we get

$$\int_{\alpha_1}^{\alpha_2} \rho(\alpha) \frac{\partial^n U_N(\kappa, \varrho)}{\partial \alpha^n} d\alpha + U_N(\kappa, \varrho) = K^* \int_{\beta_1}^{\beta_2} \rho(\beta) \frac{\partial^3 U_N(\kappa, \varrho)}{\partial \beta^3} d\beta + \int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U_N(\kappa, \xi)}{\partial \xi^2} \right] d\xi$$

$$+ f(\kappa, \varrho) + R_N(\kappa, \varrho),$$

(6.3)

where, $(\kappa, \varrho) \in L^2(\Omega)$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 2$ and $\Omega = [0, 1] \times [0, T]$,

with

$$(U(\kappa, 0))_N = (\nu(\kappa))_N,$$

$$(U(0, \varrho))_N = (p_1(\varrho))_N,$$

$$(U(1, \varrho))_N = (p_2(\varrho))_N.$$

(6.4)

Now we deduct equations 6.3 and 6.4 from equations 6.1 and 6.2, respectively to get

$$\int_{\alpha_1}^{\alpha_2} \rho(\alpha) \frac{\partial^n E_N(\kappa, \varrho)}{\partial \alpha^n} d\alpha + E_N(\kappa, \varrho) = K^* \int_{\beta_1}^{\beta_2} \rho(\beta) \frac{\partial^3 E_N(\kappa, \varrho)}{\partial \beta^3} d\beta + \int_0^\varrho (\varrho - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 E_N(\kappa, \xi)}{\partial \xi^2} \right] d\xi - R_N(\kappa, \varrho),$$

(6.5)

where, $(\kappa, \varrho) \in L^2(\Omega)$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 2$ and $\Omega = [0, 1] \times [0, T]$,

with

$$(E(\kappa, 0))_N = 0,$$

$$(E(0, \varrho))_N = 0,$$

$$(E(1, \varrho))_N = 0.$$

(6.6)

Where, $R_N(\kappa, \varrho)$ represents the function of perturbation, which depends on $U_N$, $(U_{\infty})_N$ and IC & BCs. The above mentioned equations 6.5 & 6.6 can be solved for $E_N \approx \Psi_T(\varrho)C^* \Psi(\kappa)$ by using the mechanism, depicted in section 4.2 to find the value of vector $C^*$. Therefore, the maximum absolute error can be evaluated approximately by the following formula

$$E_N = max\{|E_N(\kappa, \varrho)|, 0 \leq \kappa, \varrho < 1\}.$$  

(6.7)

7 Numerical examples

In this part two subsections are incorporated. Four test examples are considered, two for one dimensional case study, which are described in first subsection and two for two dimensional case study, which are described in second subsection. To ensure the method’s robustness and utility the numerical results are taken in the form of Figures & Table for various values of $\Lambda$ and $P$ using the presented method. Tables 2–5 offer numerical results for pointwise error, $L_2$-error, $L_\infty$-error and mean error, as well as used CPU time. Figures 1-10 show numerical results of approximate, exact solutions as well as absolute errors. Example 7.1-7.3 are examined at the ultimate time level, $\varrho = 0.5$, whereas examples 7.4 is assessed at $\varrho = 1.0$ time level.
The following formula’s for one & two dimensional will be used

\[
\|\mathbf{U}_{ex} - \mathbf{U}_N\| = \begin{cases} 
\sqrt{\left(\sum_{i=0}^{N_p-1} h \|\mathbf{U}_{ex}(\kappa_i, \varrho) - \mathbf{U}_N(\kappa_i, \varrho)\|\right)}, & \text{ } L_2 - \text{error} \\
\max_{0 \leq i \leq N_p-1} |\mathbf{U}_{ex}(\kappa_i, \varrho) - \mathbf{U}_N(\kappa_i, \varrho)|, & \text{ } L_\infty - \text{error} \\
\frac{1}{N_p} \sum_{i=0}^{N_p-1} |\mathbf{U}_{ex}(\kappa_i, \varrho) - \mathbf{U}_N(\kappa_i, \varrho)|, & \text{mean error} 
\end{cases} 
\]

\[
\|\mathbf{U}_{ex} - \mathbf{U}_N\| = \begin{cases} 
\sqrt{\left(\sum_{i=0}^{N_p-1} \sum_{j=0}^{N_p-1} h_{\kappa} h_{\eta} \|\mathbf{U}_{ex}(\kappa_i, \eta_j, \varrho) - \mathbf{U}_N(\kappa_i, \eta_j, \varrho)\|\right)}, & \text{ } L_2 - \text{error} \\
\max_{0 \leq i \leq N_p-1} \max_{0 \leq j \leq N_p-1} |\mathbf{U}_{ex}(\kappa_i, \eta_j, \varrho) - \mathbf{U}_N(\kappa_i, \eta_j, \varrho)|, & \text{ } L_\infty - \text{error} \\
\frac{1}{N_p} \sum_{i=0}^{N_p-1} \sum_{j=0}^{N_p-1} |\mathbf{U}_{ex}(\kappa_i, \eta_j, \varrho) - \mathbf{U}_N(\kappa_i, \eta_j, \varrho)|, & \text{mean error} 
\end{cases} 
\]

Remark 7.1. In graphs, we denotes \( E_1, E_2, E_3, E_4 \) as errors corresponding to various values of fixed \( \Lambda \) and variable \( P \) or vice-versa. For their identification, they are colored red, blue, green, and black, accordingly.

Table 1: Table of Notation

| General notation | Notation meaning |
|------------------|------------------|
| \( \mathcal{L}W \) | Legendre wavelet (LW) |
| \( \Lambda, \Lambda', \Lambda'' \) | Number of basis element (upto degree \( \Lambda - 1, \Lambda' - 1, \Lambda'' - 1 \)) & \( \Lambda = g + 1 \) |
| \( P, P^*, P^{**} \) | Number of node points in LGQ for \( \varrho, \kappa, \eta \) direction vectors |
| \( \Lambda, \Lambda', \Lambda'' \) | For time vector \( (\varrho) \) and space vectors \( (\kappa, \eta) \) |
| \( h = \left( \frac{1}{N_p - 1} \right) \) | Step size |
7.1 Solving one dimensional time-space DOF integero-differential equation

Example 7.1. We used the proposed approach on the following DOT-SFWSSIPDE in one dimension

\[ \int_0^1 \rho(\alpha) \frac{\partial^\alpha U(\kappa, \varrho)}{\partial \varrho^\alpha} d\alpha + U(\kappa, \varrho) = K^* \int_1^2 \rho(\beta) \frac{\partial^{3\beta} U(\kappa, \varrho)}{\partial \kappa^{3\beta}} d\beta + \int_0^\varrho (\varrho-\xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(\kappa, \xi)}{\partial \kappa^2} \right] d\xi + f(\kappa, \varrho), \]

where, the source term

\[ f = \frac{2 \kappa^2 \varrho (\varrho - 1)}{\log(\varrho)} + \kappa^2 \varrho^2 - K^* \frac{\varrho^2 (\kappa - 1)}{\log(\kappa)} - \frac{32 \varrho^{5/2}}{15}, \]

and \( U_{ex} = \varrho^2 \kappa^2 \) be the exact solution, with the given IC \( U(\kappa, 0) = 0 \) and the BCs \( U(0, \varrho) = 0, \) \( U(1, \varrho) = \varrho^2. \) Here we take the distributed weight function as follows, \( \rho(\alpha) = \frac{\Gamma(3-\alpha)}{2} \)

For various parameters \( P, P^*, \Lambda, \Lambda^*, \mathfrak{R}, \) and \( \mathfrak{R}' \), numerical results are produced and displayed for this example. This example is solved for \( \Lambda = \Lambda^* = 4, P = P^* = 3, 5, 7, 9 \) and \( \mathfrak{R} = \mathfrak{R}' = 1. \)

- Table 2 shows numerical results of pointwise errors, \( L_2 \)-errors , \( L_\infty \)-errors and mean errors for \( \Lambda = \Lambda^* = 4, P = P^* = 3, 5, 7, 9, \) and \( \mathfrak{R} = \mathfrak{R}' = 1. \) Here we take value of viscosity constant \( K^* = 1. \)

- Figure 1 displays the results related to absolute errors for \( \Lambda = \Lambda^* = 4, \mathfrak{R} = \mathfrak{R}' = 1 \) and \( P = P^* = 9 \) of example 7.1.

- Figure 2 displays the results related to absolute errors for \( \Lambda = \Lambda^* = 4, \mathfrak{R} = \mathfrak{R}' = 1 \) and \( P = 3, 5, 7, 9 \) of example 7.1. Here \( E_1, E_2, E_3, E_4 \) denotes the errors at \( \varrho=0.5 \) corresponds for fixed \( \Lambda = \Lambda^* = 4, \mathfrak{R} = \mathfrak{R}' = 1 \) and \( P = P^* = 3, 5, 7, 9, \) respectively.

- We labeled the graphs in Figure 2, to check the nature of produced error. For this purpose, we multiply \( E_1, E_2, E_3, E_4 \) with \( 10^{-9}, 10^{-6}, 10^{-6}, 1, \) respectively.

- Figure 3 refers to exact solution and Figure 4 show the approximate solution for \( \Lambda = \Lambda^* = 4, \mathfrak{R} = \mathfrak{R}' = 1 \) and \( P = P^* = 9 \) of example 7.1.
Table 2: Results of errors and used CPU time of example 7.1 for $K^* = 1$, $h = 0.05$, $\Lambda = \Lambda' = 4$, $R = R' = 1$.

| $(\chi, \eta)$ | $P = P^* = 3$ | $P = P^* = 5$ | $P = P^* = 7$ | $P = P^* = 9$ |
|--------------|--------------|--------------|--------------|--------------|
| $(0.0,0.0)$  | 1.093E-40    | 1.877E-41    | 1.404E-41    | 3.411E-41    |
| $(0.1,0.1)$  | 7.880E-09    | 1.559E-11    | 1.672E-14    | 3.659E-19    |
| $(0.2,0.2)$  | 2.955E-08    | 8.926E-11    | 8.655E-14    | 8.659E-19    |
| $(0.3,0.3)$  | 1.575E-07    | 2.145E-10    | 1.973E-13    | 1.388E-17    |
| $(0.4,0.4)$  | 3.551E-07    | 3.453E-10    | 3.052E-13    | 6.939E-17    |
| $(0.5,0.5)$  | 5.463E-07    | 4.236E-10    | 3.609E-13    | 1.804E-16    |
| $(0.6,0.6)$  | 6.276E-07    | 4.031E-10    | 3.305E-13    | 3.608E-16    |
| $(0.7,0.7)$  | 5.162E-07    | 2.717E-10    | 2.121E-13    | 4.718E-16    |
| $(0.8,0.8)$  | 2.145E-07    | 7.118E-11    | 4.735E-14    | 4.441E-16    |
| $(0.9,0.9)$  | 1.054E-07    | 8.563E-11    | 7.161E-14    | 1.110E-16    |
| $(1.0,1.0)$  | 1.016E-04    | 1.110E-16    | 1.016E-40    | 7.772E-17    |
| $L_2$-error  | 1.762E-07    | 1.465E-10    | 1.289E-12    | 6.910E-17    |
| $L_{\infty}$-error | 6.880E-07 | 5.870E-10    | 5.100E-13    | 2.640E-16    |
| mean error   | 3.471E-07    | 2.789E-10    | 2.404E-13    | 1.384E-16    |
| CPU time(s)  | 30.675       | 30.148       | 32.417       | 35.396       |

Figure 1: Error graph for $\Lambda = \Lambda' = 4$, $P = P^* = 9$, $R = R' = 1$ of example 7.1.
Figure 2: Error graph of for $\Lambda = \Lambda' = 4, P = P^* = 3, 5, 7, 9, R = R' = 1$ and $\rho=0.5$ example 7.1.

Figure 3: Exact solution graph for $\Lambda = \Lambda' = 4, P = P^* = 9, R = R' = 1$ and $\rho=0.5$ of example 7.1.
Example 7.2. Take the following DOT–SFWSIPDE

\[
\int_0^1 \rho(\alpha)D^a_\alpha U(x,\eta)d\alpha + \mathbb{U}(x,\varrho) = \mathcal{K}^{*}\int_1^2 \rho(\beta)D^b_\beta U(x,\eta)d\beta + \int_0^\varrho \frac{\mathbb{U}_{xx}(\varrho - \xi)}{(\varrho - \xi)^{3/2}}d\xi + f(x,\varrho)
\]

with IC \(\mathbb{U}(x,0) = x^b\) and BCs \(\mathbb{U}(0,\varrho) = \varrho^a, \mathbb{U}(1,\varrho) = 1+\varrho^a\), where, \(f = \Gamma(a+1)\frac{\varrho-1}{\log(\varrho)} + (\varrho^a + x^b) - \mathcal{K}^{*}\Gamma(b+1)\frac{(x-1)x^{b-2}}{\log(x)} - 2b(b-1)\sqrt{\varrho}x^{b-2}\). The exact solution for this example is \(U_{ex} = x^b + \varrho^a\) and parameters are \(a=2, b=2\). Distributed weight functions are \(\rho(\alpha) = \Gamma(a+1-\alpha)\) and \(\rho(\beta) = \Gamma(b + 1 - \beta)\).

- Table 3 shown the numerical results of errors and used CPU time for \(P=P^*=9, \Lambda = \Lambda^*=3,5,7,9\) and \(\mathcal{R} = \mathcal{R}^*=1\).

- Figure 5 displays the results of absolute error for fixed \(P = P^*=9, \Lambda = \Lambda^*=9\) and \(\mathcal{R} = \mathcal{R}^*=1\) of example 7.2.

- Figure 6 shows the results of absolute errors at time \(\varrho=0.5\) for example 7.2. In this figure \(E_1, E_2, E_3, E_4\) correspond for fixed \(P=P^*=9, \mathcal{R} = \mathcal{R}^*=1\) and \(\Lambda = \Lambda^*=3,5,7,9\) respectively.

- For labeling the graph in Figures 6 we multiply \(E_1, E_2, E_3, E_4\) with factors \(0.3,2,1,1\), respectively.
Table 3: Results of errors with used CPU time for $K^* = 1$, $h = 0.05$, $P = P^* = 9$, $R = R' = 1$ of example 7.1.

| $(\varepsilon, \vartheta)$ | $\Lambda = \Lambda' = 3$ | $\Lambda = \Lambda' = 5$ | $\Lambda = \Lambda' = 7$ | $\Lambda = \Lambda' = 9$ |
|-----------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| $|U_{ex} - U_X|_{L^2}$      | 2.322E-40                | 5.632E-41                | 1.549E-41                | 8.185E-41                |
| $|U_{ex} - U_X|_{L^\infty}$     | 6.051E-14                | 8.882E-15                | 3.331E-16                | 7.772E-16                |
| $|U_{ex} - U_X|_{L^\infty}$     | 6.772E-14                | 1.110E-15                | 1.110E-16                | 4.441E-16                |
| $L_2$-error                 | 7.327E-14                | 1.776E-14                | 4.441E-16                | 2.220E-16                |
| $L_\infty$-error            | 7.994E-15                | 3.111E-15                | 8.882E-16                | 2.220E-16                |
| mean error                  | 9.104E-14                | 4.441E-15                | 1.110E-15                | 1.663E-17                |
| mean error                  | 1.132E-14                | 2.210E-40                | 1.036E-40                | 3.742E-41                |
| mean error                  | 2.794E-14                | 1.889E-15                | 5.030E-16                | 1.115E-16                |
| mean error                  | 6.050E-14                | 8.880E-15                | 2.227E-15                | 6.110E-16                |
| mean error                  | 4.879E-14                | 4.097E-15                | 3.332E-16                | 1.929E-16                |
| CPU time(s)                 | 18.417                   | 56.719                   | 164.539                  | 267.399                  |

Figure 5: Error graph for $\Lambda = \Lambda' = 9$, $P = P^* = 9$, $R = R' = 1$ of example 7.2.
7.2 Two dimensional time-space DOF integro-differential equation

**Example 7.3.** We consider the following DOT–SFWSIPDE with nonhomogeneous boundary conditions

\[
\int_{a_1}^{a_2} \rho(\alpha) \frac{\partial^n U(x, \eta, \vartheta)}{\partial \alpha^n} d\alpha + U(x, \eta, \vartheta) = K^* \int_{\beta_1}^{\beta_2} \rho(\beta) \left[ \frac{\partial^3 U(x, \eta, \vartheta)}{\partial \beta^3} + \frac{\partial^3 U(x, \eta, \vartheta)}{\partial \eta^3} \right] d\beta \\
+ \int_{0}^{\vartheta} (\vartheta - \xi)^{-\frac{1}{2}} \left[ \frac{\partial^2 U(x, \eta, \vartheta)}{\partial x^2} + \frac{\partial^2 U(x, \eta, \vartheta)}{\partial \eta^2} \right] d\xi + f(x, \eta, \vartheta),
\]

with the IC \( U(x, \eta, 0) = ax^2 + b\eta^2, \) and the BCs

\[
U(0, \eta, \vartheta) = b\eta^2 + c\vartheta^2, \quad U(1, \eta, \vartheta) = a + b\eta^2 + c\vartheta^2, \\
U(x, 0, \vartheta) = ax^2 + c\vartheta^2, \quad U(x, 1, \vartheta) = ax^2 + b + c\vartheta^2.
\]

The source function, \( f = \frac{2c\vartheta(\vartheta - 1)}{\log(c)}(ax^2 + b\eta^2 + c\vartheta^2) - K^* \left( \frac{2a(x - 1)}{\log(x)} + \frac{2b(\eta - 1)}{\log(\eta)} \right) - 4(a + b)\sqrt(\vartheta) \) and exact solution is \( U_{\text{ex}} = ax^2 + b\eta^2 + c\vartheta^2. \) The values of parameters are \( a=1, b=1, c=1 \) and the value of distributed weight functions are \( \rho(\alpha) = \Gamma(3 - \alpha) \) and \( \rho(\beta) = \Gamma(3 - \beta). \)

- **Table 4** shows the results of pointwise errors, \( L_2 \)-errors, \( L_{\infty} \)-errors, mean errors for \( \Lambda = \Lambda' = \Lambda'' = 4, \ \mathcal{R} = \mathcal{R}' = \mathcal{R}'' = 1, \) and \( P=P^* = P^{**} = 5, 7, 9, 11 \) of example 7.3.

- **Figure 7** describes the results of absolute errors for \( \Lambda = \Lambda' = \Lambda'' = 4, \ \mathcal{R} = \mathcal{R}' = \mathcal{R}'' = 1 \) and \( P=P^* = P^{**} = 11. \)

- **Figures 8 and 9** are referred to exact and approximate solution graph, respectively.
Table 4: Results of errors with used CPU time for $K^*=1$, $h_x=0.05$, $h_\eta=0.05$, $\Lambda = \Lambda' = \Lambda'' = 4, \mathcal{R} = \mathcal{R}' = \mathcal{R}'' = 1$ of example 7.3.

| $(\chi, \eta)$ | $P^*=P^*=P^{**}=5$ | $P^*=P^*=P^{**}=7$ | $P^*=P^*=P^{**}=9$ | $P^*=P^*=P^{**}=11$ |
|----------------|---------------------|---------------------|---------------------|---------------------|
| (0.0,0.0) | 5.551E-17 | 5.551E-17 | 7.772E-15 | 2.770E-17 |
| (0.1,0.1) | 3.214E-09 | 2.838E-12 | 1.021E-14 | 1.110E-16 |
| (0.2,0.2) | 6.840E-09 | 6.036E-12 | 1.299E-14 | 2.220E-16 |
| (0.3,0.3) | 7.356E-09 | 6.484E-12 | 1.332E-14 | 2.221E-16 |
| (0.4,0.4) | 5.473E-09 | 4.810E-12 | 1.188E-14 | 1.110E-16 |
| (0.5,0.5) | 3.159E-09 | 2.756E-12 | 1.010E-14 | 1.111E-16 |
| (0.6,0.6) | 1.773E-09 | 1.528E-12 | 9.104E-15 | 4.301E-17 |
| (0.7,0.7) | 1.314E-09 | 1.128E-12 | 8.660E-15 | 3.221E-17 |
| (0.8,0.8) | 7.592E-10 | 6.530E-13 | 8.438E-15 | 7.115E-17 |
| (0.9,0.9) | 4.800E-10 | 4.243E-13 | 7.55E-15 | 2.221E-16 |
| (1.0,1.0) | 1.132E-40 | 3.410E-40 | 7.550E-15 | 3.742E-41 |

$L_2$-error | 3.2961E-09 | 9.516E-12 | 9.948E-15 | 1.626E-16 |

$L_\infty$-error | 1.0747E-08 | 2.906E-12 | 1.598E-14 | 4.440E-16 |

Mean error | 3.2043E-09 | 2.8151E-12 | 1.017E-14 | 2.430E-16 |

CPU time(s) | 142.440 | 148.064 | 148.916 | 152.864 |

Figure 7: Error graph for $\Lambda = \Lambda' = \Lambda'' = 4, P = P^* = P^{**} = 11, \mathcal{R} = \mathcal{R}' = \mathcal{R}'' = 1$ of example 7.3.
Example 7.4. We consider the following DOT-SFWSIPDE in 2D

$$
\int_0^1 \rho(\alpha) D^\alpha_\theta U(\xi, \eta, \varrho) d\alpha + U(\xi, \eta, \varrho) = K^* \int_1^2 \rho(\beta) [D^\beta_\xi U(\xi, \eta, \varrho) + D^\beta_\eta U(\xi, \eta, \varrho)] d\beta
$$

$$
+ \int_0^\varrho U_{\xi\eta}(\xi, \eta, \varrho) + U_{\eta\eta}(\xi, \eta, \varrho) \frac{1}{(\varrho - \xi)^2} d\xi + f(\xi, \eta, \varrho),
$$

Figure 8: Exact solution graph for $\varrho=0.5$ of example 7.3

Figure 9: Approximate solution graph for $\Lambda = \Lambda' = \Lambda'' = 5$, $P = P^* = P^{**} = 4$, $\mathfrak{R} = \mathfrak{R}' = \mathfrak{R}'' = 1$ and $\varrho=0.5$ of example 7.3.
where,
\[
f(\kappa, \eta, \varrho) = a(\kappa + \eta) \frac{(\varrho - 1)}{\log(\varrho)} + (\varrho \eta + \varrho \kappa) - b(\eta + \varrho) \frac{(\kappa - 1)}{\kappa \log(\kappa)} - b(\kappa + \varrho) \frac{(\eta - 1)}{\eta \log(\eta)}.
\]

Furthermore, the equation is facilitated with the IC \( U(\kappa, \eta, 0) = \varrho \eta \) and BCs
\[
U(0, \eta, \varrho) = \eta \varrho, \quad U(1, \eta, \varrho) = \eta + \varrho + \eta \varrho,
\]
\[
U(\kappa, 0, \varrho) = \varrho \kappa, \quad U(\kappa, 1, \varrho) = \kappa + \varrho + \kappa \varrho.
\]
The exact solution is \( U_{ex} = \varrho \eta + \eta \varrho + \varrho \kappa \). Here we take distributed weight functions as \( \rho(\alpha) = a \Gamma(2 - \alpha) \) and \( \rho(\beta) = b \Gamma(2 - \beta) \). Numerical results are obtained and depicted for \( \Lambda = \Lambda' = \Lambda'' = 3, P = P^* = P^{**} = 4, 6, 8, 10, R = R' = R'' = 1. \)

- Table 5 shows the numerical results of errors for \( \Lambda = \Lambda' = \Lambda'' = 3, R = R' = R'' = 1 \) and \( P = P^* = P^{**} = 4, 6, 8, 10 \).

- The graph of absolute errors for \( \Lambda = \Lambda' = \Lambda'' = 3, R = R' = R'' = 1 \) and \( P = P^* = P^{**} = 8 \) is shown in Figure 10.

Table 5: Results of errors with used CPU time for \( \mathcal{K} = 1, h_{\kappa} = 0.05, h_{\eta} = 0.05, \Lambda = \Lambda' = \Lambda'' = 3, R = R' = R'' = 1 \) of example 7.4.

| (\kappa, \eta) | P=4, a=1, b=0.5 | P=6, a=0.5, b=1 | P=8, a=2, b=1.5 | P=10, a=1.5, b=2 |
|---------------|----------------|-----------------|-----------------|-----------------|
| \( 0.0, 0.0 \) | 4.471E-15 | 4.700E-16 | 3.727E-17 | 2.282E-17 |
| \( 0.1, 0.1 \) | 5.551E-15 | 5.154E-16 | 2.776E-17 | 2.7345E-17 |
| \( 0.2, 0.2 \) | 5.251E-15 | 3.245E-16 | 2.540E-17 | 2.568E-18 |
| \( 0.3, 0.3 \) | 1.110E-14 | 1.658E-16 | 2.115E-17 | 5.235E-18 |
| \( 0.4, 0.4 \) | 1.111E-14 | 2.124E-16 | 1.758E-17 | 5.267E-18 |
| \( 0.5, 0.5 \) | 1.234E-15 | 2.554E-16 | 5.667E-17 | 5.246E-18 |
| \( 0.6, 0.6 \) | 1.236E-15 | 2.650E-16 | 9.457E-17 | 4.324E-18 |
| \( 0.7, 0.7 \) | 2.220E-14 | 1.254E-16 | 5.325E-17 | 4.821E-17 |
| \( 0.8, 0.8 \) | 1.348E-15 | 4.441E-16 | 4.257E-17 | 4.857E-16 |
| \( 0.9, 0.9 \) | 1.257E-15 | 8.335E-16 | 3.448E-17 | 2.325E-17 |
| \( 1.0, 1.0 \) | 1.532E-40 | 2.412E-40 | 3.550E-40 | 7.732E-41 |

\( L_2 \)-error | 8.108E-15 | 7.876E-16 | 9.788E-17 | 3.792E-17 |

\( L_{\infty} \)-error | 4.440E-14 | 4.440E-16 | 4.441E-16 | 7.795E-17 |

mean error | 3.700E-15 | 2.114E-16 | 3.700E-17 | 2.231E-17 |

CPU time(s) | 68.884 | 71.312 | 72.934 | 81.361 |
8 Conclusion

In this manuscript, a robust numerical method based on standard tau approach and collocation technique has been constructed for solving multi-dimensional DOT-SFWSIPDE. To this end, the original defined problem is converted into a system of linear algebraic equations using a variety of operational matrices, the standard tau technique, and the LGQ rule. Four test examples are performed for testing of the proposed method. To determined the method’s efficiency and accuracy pointwise errors, $L_2$-errors, $L_\infty$-errors are evaluated and summed up in Tables 2-5 and Figures 1-10. The used CPU time is also evaluated. The produced numerical results conclude that the method has the capability of providing high accuracy with minimal computing efforts. It is also clear from the error Tables 2-5 that error is minimized when we continuously increase basis functions or increase the nodes points in LGQ. Hence it is observed that the proposed method is proved to be an effective numerical procedure in terms of accuracy & computational cost for handling multi-dimensional DOT-SFWSIPDE. The method is also applicable on nonlinear time-space DOFDEs, which is one of our goals for future study. In this article, we have provided the convergence analysis for 1D case only, the analysis for 2D case is a task for future study.

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