LINEAR COMPLEMENTARY PAIR OF ABELIAN CODES OVER FINITE CHAIN RINGS

CEM GÜNERİ, EDGAR MARTÍNEZ-MORO, SELCEN SAYICI

Abstract. Linear complementary dual (LCD) codes and linear complementary pair (LCP) of codes over finite fields have been intensively studied recently due to their applications in cryptography, in the context of side channel and fault injection attacks. The security parameter for an LCP of codes \((C, D)\) is defined as the minimum of the minimum distances \(d(C)\) and \(d(D^\perp)\). It has been recently shown that if \(C\) and \(D\) are both abelian codes over a finite field \(\mathbb{F}_q\), and the length of the codes is relatively prime to \(q\), then \(C\) and \(D^\perp\) are equivalent. Hence the security parameter for an LCP of abelian codes \((C, D)\) is simply \(d(C)\). In this work, we first extend this result to the non-semisimple case, i.e. the code length is divisible by the characteristic of the field of definition. Then we use the result over the finite fields to prove the same fact for an LCP of abelian codes over any finite chain ring.

Keywords: LCP of codes, abelian code, group algebra, finite chain ring, code equivalence.

1. Introduction

Let \(\mathbb{F}_q\) be a finite field, where \(q\) is a power of some prime number \(p\). A pair of linear codes \((C, D)\) over \(\mathbb{F}_q\) of length \(n\) is called a linear complementary pair (LCP) of codes if \(C \cap D = \{0\}\) and \(C + D = \mathbb{F}_q^n\) (i.e. \(C \oplus D = \mathbb{F}_q^n\)). In the case \(C = D^\perp\) the dual code of \(D\), \(C\) is referred as a linear complementary dual (LCD) code. The duality in this paper will be relative to the Euclidean inner product.

LCD codes were introduced by Massey [13] in 1992 and there is a revived interest in LCD and LCP of codes due to their application in protection against side channel and fault injection attacks ([11]). In this context, the security parameter of an LCP \((C, D)\) is defined to be \(\min\{d(C), d(D^\perp)\}\), where \(d(C)\) stands for the minimum distance of the code \(C\). In the LCD case, this parameter is simply \(d(C)\), since \(D^\perp = C\).

It has been recently shown by Carlet et al. ([22]) that if \(C\) and \(D\) are both cyclic or both 2D cyclic codes, then \(C\) is equivalent to \(D^\perp\) and therefore \(d(C) = d(D^\perp)\) as in the case of LCD codes case. In other words, there is an LCP of cyclic codes which has as good security parameter as the cyclic code with the best minimum distance. The same also holds for 2D cyclic codes.

If \(C_a\) denotes the cyclic group of order \(a\), for any positive integer \(a\), then a length \(n\) cyclic code is an ideal in the group algebra \(\mathbb{F}_q[C_a]\) and a length \(n \times m\) 2D cyclic code is an ideal in \(\mathbb{F}_q[C_n \times C_m]\).

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C. Güneri and S. Sayıcı are with Sabancı University, Faculty of Engineering and Natural Sciences, İstanbul, Turkey. Email: guneri@sabanciuniv.edu, selcensayici@sabanciuniv.edu.
E. Martínez-Moro is with University of Valladolid, Institute of Mathematics, Castilla, Spain. Email: edgar.martinez@uva.es.
If $G$ is an arbitrary finite abelian group, which is a direct product of $n$ cyclic groups, then ideals of $\mathbb{F}_q[G]$ are called $n$D cyclic codes or abelian codes (\cite{10}). In the case $\gcd(q, |G|) = 1$ (i.e. semisimple case), Güneri et. al. extended the result in \cite{7} to abelian codes in $\mathbb{F}_q[G]$. That is, if $(C, D)$ is an LCP of abelian codes in a semisimple group algebra $\mathbb{F}_q[G]$, then $C$ and $D^\perp$ are equivalent, and hence $d(C) = d(D^\perp)$ (see \cite{9}).

Although LCD and LCP of codes have been extensively studied over finite fields, these code classes have not been as well-understood over rings (particularly, chain rings). In \cite{13}, the authors prove some sufficient conditions, particularly in terms of generator matrices, for a code over a chain ring to be LCD. Recently in \cite{14} the authors prove some results on LCD codes over general finite commutative rings. Over a finite chain ring, they extend the Massey’s LCD characterization (\cite{18}, Proposition 1) in terms of code’s generator matrix. Moreover, Yang-Massey characterization of LCD cyclic codes in \cite{22}, in terms of the code’s generator polynomial, is extended to cyclic codes over chain rings (\cite{14}, Theorem 25). However, LCP of codes over chain rings is addressed for the first time in the present work. Our main contribution in this article is the extension of the results in \cite{7, 9} to abelian codes over finite chain rings. Namely, we prove that for an LCP of abelian codes $(C, D)$ in $R[G]$, where $R$ is a finite chain ring and $G$ is any finite abelian group, $C$ and $D^\perp$ are equivalent codes (Theorem 4.10). So we settle the security parameter problem for LCP of abelian codes over chain rings, just as it has been settled for abelian codes over finite fields.

Before addressing the problem over chain rings, we also prove some further results on LCP of abelian codes over finite fields. As mentioned above, Yang and Massey characterized an LCD cyclic code over a finite field in terms of the code’s generator polynomial in \cite{22}. This result was later extended to LCP of cyclic codes by Carlet et al. in \cite{7}. In this work we prove an analogous statement for LCP of abelian codes in a semisimple group algebra $\mathbb{F}_q[G]$ (Proposition 2.2), where one can define a generator polynomial for an abelian code. Equivalence of $C$ and $D^\perp$, for an LCP $(C, D)$ of abelian codes over finite fields, was proved in the semisimple case in \cite{9}. We also extend this result to LCP of abelian codes in $\mathbb{F}_q[G]$, where $p$ divides the order of $G$ (Theorem 3.5). Let us note that Borello et. al. very recently proved in \cite{4} the main theorem for LCP of codes for a pair of group codes in $\mathbb{F}_q[G]$, where $G$ is any finite group (including nonabelian groups and groups of arbitrary order). Our proof for Theorem 3.5 is different, and we also elaborate on the equivalence map which is very explicit when the codes $C$ and $D^\perp$ are viewed “vectorially”. This explicit form of the equivalence is also crucial for the proof of our main result (Theorem 4.10) over chain rings.

2. On Generators of LCP of Abelian Codes over Finite Fields

From now on $R$ will denote a finite commutative ring with identity and $G$ will be a finite abelian group. We denote by $R[G]$ be the group ring of $G$ over $R$ thus the elements of $R[G]$ are of the form $\sum_{g \in G} \alpha_g g$, where $\alpha_g \in R$ and nonzero for finitely many $g \in G$. An abelian code over $R$ is defined to be an ideal in $R[G]$.

Let $\mathbb{F}_q$ denote the finite field with $q$ elements, where $q$ is a power of the prime $p$. We will be interested in abelian codes over $\mathbb{F}_q$ in the group algebra $\mathbb{F}_q[G]$ in this section. Since $G$ is a finite
abelian group, one can write it as
\[ G = C_1 \times C_2 \times \cdots \times C_n, \]
where \( C_i = \langle g_i \rangle \) is a cyclic group of order \( m_i \) for all \( 1 \leq i \leq n \). Let \( \mathbb{F}_q^{n_1 \times \cdots \times n_m} \) denote the \( \mathbb{F}_q \)-space of dimension \( n_1 \cdots n_m \) whose elements can be viewed as \( n_1 \times \cdots \times n_m \) arrays \( (a_{i_1, i_2, \ldots, i_m}) \), where \( a_{i_1, i_2, \ldots, i_m} \in \mathbb{F}_q \) for all \( 0 \leq i_j \leq m_j - 1 \) and \( 1 \leq j \leq n \). We also consider the quotient ring of polynomials in \( n \) variables \( R_n = \mathbb{F}_q[x_1, \ldots, x_n]/(x_1^{m_1} - 1, \ldots, x_n^{m_n} - 1) \). We have the following mappings
\[
\mathbb{F}_q^{n_1 \times \cdots \times n_m} \leftrightarrow R_n \leftrightarrow \mathbb{F}_q[G]
\]
(2.1)
\[
(a_{i_1, i_2, \ldots, i_m}) \leftrightarrow \sum_{j=1}^{n} \sum_{i_j=0}^{m_j-1} a_{i_1, i_2, \ldots, i_m} x_1^{i_1} \cdots x_n^{i_n} \leftrightarrow \sum_{j=1}^{n} \sum_{i_j=0}^{m_j-1} a_{i_1, i_2, \ldots, i_m} (g_1^{i_1}, \ldots, g_n^{i_n})
\]
In fact they are \( \mathbb{F}_q \)-linear isomorphisms and moreover, \( R_n \) and \( \mathbb{F}_q[G] \) are isomorphic as rings. Hence an abelian code \( C \) can be viewed as an ideal in \( R_n \) or in \( \mathbb{F}_q[G] \). When viewed in \( \mathbb{F}_q^{n_1 \times \cdots \times n_m} \), \( C \) is a linear code with symmetries induced from the ideal structure. We note that when \( n = 1 \), \( C \) is a cyclic code of length \( m_1 \). Moreover, \( C \subset R_n \) is also referred to as \( n \)D cyclic code or multidimensional cyclic code (\[8, 9, 10\]).

Yang and Massey characterized LCD cyclic codes in terms of the generator polynomial (\[22\]). This result was extended to LCP of cyclic codes by Carlet et. al. (\[11\] Theorem 2.1). Our goal in this section is to extend the same result to abelian codes.

One has that the abelian group \( G \) can be decomposed as
\[
G = A \oplus P,
\]
where \( |G| = N = mp^t \) with \( |A| = m, |P| = p^t \) and \( \gcd(m, p) = 1 \). In other words, \( P \) is the unique \( p \)-Sylow subgroup of \( G \). It is noted in \[11\] that if \( P \) is a cyclic \( p \)-group, then \( \mathbb{F}_q[G] \) is a principal ideal group algebra (PIGA). Clearly, \( \mathbb{F}_q[G] \) is also a PIGA when \( P \) is trivial (i.e. when \( \mathbb{F}_q[G] \) is semisimple). Hence an abelian code \( C \) in a PIGA \( \mathbb{F}_q[G] \) can be generated by one element, though not uniquely, as in the case of cyclic codes (Note that if \( R \) is a chain ring and \( P \) is a non-trivial \( p \)-Sylow of \( G \), there is not hope for \( R[G] \) to be principal ideal group algebra other than \( R \) being a finite field and \( R[G] \) a PIGA, see \[15\] Corollary 1)). Let \( u, v \in \mathbb{F}_q[G] \) such that
\[
C = \mathbb{F}_q[G]u = \{ x \in \mathbb{F}_q[G] : xv = 0 \} =: \text{Ann}(v) \quad (\text{cf. } \[11\] Proposition 3.1)).
\]
Here, \( \text{Ann}(v) \) is the annihilator of \( v \). Hence, one can define generator and check elements for an abelian code in a PIGA \((u\) and \( v \) in this case). Moreover, for \( v = \sum_{g \in G} v_g g \in \mathbb{F}_q[G] \), if we set
\[
\bar{v} := \sum_{g \in G} v_{-g} g,
\]
then \( C^\perp = \mathbb{F}_q[G][\bar{v}] \), see \[11\] Proposition 3.1. We will also need the following fact.

**Proposition 2.1.** (\[11\] Corollary 5.8) For \( \mathbb{F}_q[G]u = \text{Ann}(v) \) in a semisimple algebra \( \mathbb{F}_q[G] \), we have \( \mathbb{F}_q[G]u \cap \mathbb{F}_q[G]v = \{0\} \).
With generator and check elements defined as above for an abelian code in a PIGA, we can now extend the relation between the generator polynomials of an LCP of cyclic codes (Theorem 2.1) to the abelian codes in a semisimple PIGA.

**Proposition 2.2.** Assume that \( \gcd(q, |G|) = 1 \) and let \( C = F_q[G]u = \text{Ann}(v) \) and \( D = F_q[G]w \) be abelian codes, where \( u, v, w \in F_q[G] \). Then, \((C, D)\) is an LCP of abelian codes if and only if \( D = F_q[G]v \).

**Proof.** Assume that \( F_q[G]u \oplus F_q[G]w = F_q[G] \) (i.e. \((C, D)\) is LCP). Then,

\[
F_q[G]v = (F_q[G]u \oplus F_q[G]w) \cap F_q[G]v = (F_q[G]u \cap F_q[G]v) \oplus (F_q[G]w \cap F_q[G]v) = F_q[G]w \cap F_q[G]v \quad \text{(Proposition 2.1)}.
\]

Hence, \( F_q[G]v \subseteq F_q[G]w \).

Note that \( |F_q[G]u||F_q[G]v| = |F_q[G]| = |F_q[G]u||F_q[G]w| \). The first equality follows since \((C^\perp = F_q[G]v, and the second follows since \((C, D)\) is LCP. Hence, \( |F_q[G]v| = |F_q[G]w| \). It is easy to see that \( |F_q[G]v| = |F_q[G]v| \) (cf. [11 Corollary 3.2]). Therefore \( |F_q[G]v| = |F_q[G]w| \). Thus we obtain \( F_q[G]v = F_q[G]w \).

For the converse statement, let us assume that \( F_q[G]w = F_q[G]v \). Then \( F_q[G]u \cap F_q[G]w = \{0\} \) by Proposition 2.1. The fact that \( |F_q[G]| = |F_q[G]u||F_q[G]w| \) follows using the same argument above. Hence, \( F_q[G] \) is the direct sum of \( F_q[G]u \) and \( F_q[G]w \).

**Remark 2.3.** Theorem 2.1 in [7] states in the semisimple case that a pair of cyclic codes \((C, D)\) of length \( n \) with generator polynomials \( g(x), h(x) \), respectively, is LCP if and only if \( h(x) = (x^n - 1)/g(x) \). Note that these are codes in \( F_q[C_n] \), or in \( F_q[x]/(x^n - 1) \). Hence, \( g(x)h(x) = 0 \) in \( F_q[x]/(x^n - 1) \) and \( C = \text{Ann}(h(x)) \). Hence, Proposition 2.2 indeed extends the result of Carlet et al.

Let us also note that [7, Theorem 2.1] extends the Yang-Massey characterization of cyclic LCD codes (i.e. \((C, C^\perp)\) is LCP), which states that \( C \) is LCD if and only if \( g(x) \) is a self-reciprocal polynomial. In the general semisimple abelian code case, since \( C^\perp = F_q[G]v \), Proposition 2.2 concludes that \( C \) is LCD if and only if \( C^\perp = F_q[G]v = F_q[G]v \). This is analogous to the Yang-Massey result, since \( v \) amounts to “reciprocal” of \( v \). Moreover, \( F_q[G]v = F_q[G]v \) and \( F_q[G]u = F_q[G]u \) are equivalent statements, as shown in [11 Theorems 5.4 and 5.9], for LCD abelian codes in the semisimple case.

### 3. LCP of Abelian Codes over Finite Fields: Non-Semisimple Case

For an LCP \((C, D)\) of cyclic or 2D cyclic codes, it has been shown in [7] that \( D \) is uniquely determined by \( C \), and \( D^\perp \) is equivalent to \( C \). Hence the minimum distances of both codes satisfy \( d(C) = d(D^\perp) \). Later in [9], the same result is extended to an arbitrary \( n \)-D cyclic (abelian) codes in the semisimple case (i.e. \( \gcd(q, m_i) = 1 \) for all \( 1 \leq i \leq n \), using the correspondence between ideals of \( R_n \) (as described in Section 2) and their zero sets. The goal in this section is to extend the same result to all abelian codes by proving it when \( \gcd(q, |G|) \neq 1 \).
For a finite abelian group $G = A \oplus P$ as in (2.2), we have
\[
\mathbb{F}_q[A] \simeq \prod_{i=1}^{a} \mathbb{F}_q \times \prod_{j=1}^{b} \mathbb{K}_j \times \prod_{\ell=1}^{c} (\mathbb{L}_\ell \times \mathbb{L}_\ell),
\]
where $\mathbb{K}_j, \mathbb{L}_\ell$ are finite proper extensions of $\mathbb{F}_q$ for each $1 \leq j \leq b$ and $1 \leq \ell \leq c$, for some nonnegative integers $a, b, c$ (see [11]). Hence, $\mathbb{F}_q[G] = \mathbb{F}_q[A][P]$ can be decomposed as
\[
\mathbb{F}_q[G] = \mathbb{F}_q[A][P] \simeq \prod_{i=1}^{a} \mathbb{F}_q[P] \times \prod_{j=1}^{b} \mathbb{K}_j[P] \times \prod_{\ell=1}^{c} (\mathbb{L}_\ell[P] \times \mathbb{L}_\ell[P]).
\]

Therefore abelian codes $C, D$ in $\mathbb{F}_q[G]$ decompose as
\[
C = \prod_{i=1}^{a} C_{1,i} \times \prod_{j=1}^{b} C_{2,j} \times \prod_{\ell=1}^{c} (C_{3,\ell} \times C'_{3,\ell}),
\]
(3.1)
\[
D = \prod_{i=1}^{a} D_{1,i} \times \prod_{j=1}^{b} D_{2,j} \times \prod_{\ell=1}^{c} (D_{3,\ell} \times D'_{3,\ell}),
\]
where $C_{1,i}, D_{1,i} \subseteq \mathbb{F}_q[P], C_{2,j}, D_{2,j} \subseteq \mathbb{K}_j[P]$ and $C_{3,\ell}, C'_{3,\ell}, D_{3,\ell}, D'_{3,\ell} \subseteq \mathbb{L}_\ell[P]$ are abelian codes in respective group algebras, for all $i, j, \ell$.

The following result is not difficult to prove using the fact that $\mathbb{F}[P]$ is a local group algebra for a finite field $\mathbb{F}$ of characteristic $p$ and any finite abelian $p$-group $P$ (see [19]).

**Proposition 3.1.** ([5] Theorem 2) Let $\mathbb{F}$ be a finite field of characteristic $p$ and $P$ be a finite abelian $p$-group. Then the ideals $\{0\}$ and $\mathbb{F}[P]$ are the only direct summands of the group algebra $\mathbb{F}[P]$.

A straightforward consequence of Proposition 3.1 is the following characterization.

**Proposition 3.2.** For a finite abelian group $G$ as in (2.2), let $C$ and $D$ be abelian codes in $\mathbb{F}_q[G]$ with the decompositions as in (3.1). Then, $(C, D)$ is an LCP of abelian codes if and only if

1. $(C_{1,i}, D_{1,i}) \in \{\{0\}, \mathbb{F}_q[P], \{0\}\}$ for all $i = 1, \ldots, a$,
2. $(C_{2,j}, D_{2,j}) \in \{\{0\}, \mathbb{K}_j[P], \{0\}\}$ for all $j = 1, \ldots, b$,
3. $(C_{3,\ell}, D_{3,\ell}), (C'_{3,\ell}, D'_{3,\ell}) \in \{\{0\}, \mathbb{L}_\ell[P], \{0\}\}$ for all $\ell = 1, \ldots, c$.

Hence, given an abelian code $C$ in $\mathbb{F}_q[G]$, the complementary abelian code $D$ is uniquely determined by $C$.

For each $i, j, \ell$, set
\[
\tilde{C}_{1,i} := \begin{cases} 
\{0\}, & \text{if } C_{1,i} = \{0\} \\
\mathbb{F}_q & \text{if } C_{1,i} = \mathbb{F}_q[P]
\end{cases},
\]
\[
\tilde{C}_{2,j} := \begin{cases} 
\{0\}, & \text{if } C_{2,j} = \{0\} \\
\mathbb{K}_j & \text{if } C_{2,j} = \mathbb{K}_j[P]
\end{cases}.
\]
\[ \tilde{C}_{3,\ell} \ (\tilde{C}'_{3,\ell}) := \begin{cases} \{0\}, & \text{if } C_{3,\ell} = \{0\} \ (\text{if } C'_{3,\ell} = \{0\}) \\ \mathbb{L}_\ell, & \text{if } C_{3,\ell} = \mathbb{L}_\ell[P] \ (\text{if } C'_{3,\ell} = \mathbb{L}_\ell[P]) \end{cases}. \]

Define \( \tilde{D}_{1,i}, \tilde{D}_{2,j}, \tilde{D}_{3,\ell}, \tilde{D}'_{3,\ell} \) analogously. Let

\begin{align*}
\tilde{C} &= \prod_{i=1}^{a} \tilde{C}_{1,i} \times \prod_{j=1}^{b} \tilde{C}_{2,j} \times \prod_{\ell=1}^{c} \left( \tilde{C}_{3,\ell} \times \tilde{C}'_{3,\ell} \right), \\
\tilde{D} &= \prod_{i=1}^{a} \tilde{D}_{1,i} \times \prod_{j=1}^{b} \tilde{D}_{2,j} \times \prod_{\ell=1}^{c} \left( \tilde{D}_{3,\ell} \times \tilde{D}'_{3,\ell} \right).
\end{align*}

Then \((\tilde{C}, \tilde{D})\) is an LCP of abelian codes in \( \mathbb{F}_q[A] \). Moreover, \( C = \tilde{C}[P] \) and \( D = \tilde{D}[P] \) in \( \mathbb{F}_q[A][P] = \mathbb{F}_q[G] \).

**Proposition 3.3.** With the above notation, let \((C, D) = (\tilde{C}[P], \tilde{D}[P])\) be LCP of abelian codes in \( \mathbb{F}_q[G] \). Then \( \tilde{C}[P] \) and \( \tilde{D}^\perp[P] \) are equivalent codes.

**Proof.** We observed that \((\tilde{C}, \tilde{D})\) is an LCP of codes in \( \mathbb{F}_q[A] \). In the semisimple case, it was proved that there is an equivalence \( \sigma \) between \( \tilde{C} \) and \( \tilde{D}^\perp \) (\([9, \text{Theorem 8}]\)). Then the following bijection is the equivalence desired:

\[ \pi : \tilde{C}[P] \rightarrow \tilde{D}^\perp[P], \quad \sum_{h \in P} c_h h \mapsto \sum_{h \in P} \sigma(c_h) h. \]

\[ \square \]

**Remark 3.4.** The equivalence \( \sigma \) between \( \tilde{C} \) and \( \tilde{D}^\perp \) is explicitly given in the proof of Theorem 8 in \([9]\). Since the map \( \pi \) simply applies this permutation on each coefficient \( c_h \in \tilde{C} \), we also have an explicit permutation equivalence established between \( \tilde{C}[P] \) and \( \tilde{D}^\perp[P] \). It is also helpful to visualize elements of the group algebra \( \mathbb{F}_q[A][P] \) as \( |P| = p^t \)-tuple of elements of \( \mathbb{F}_q[A] \) by ordering the elements in \( P \) as \((h_1, \ldots, h_{p^t})\). Then we can view elements of \( C = \tilde{C}[P] \) as

\[ \sum_{i=1}^{p^t} c_i h_i \leftrightarrow (c_1, \ldots, c_{p^t}) \in \mathbb{F}_q[A]^{p^t}, \]

where each \( c_i \) belongs to \( \tilde{C} \).

We are ready to prove the main result of this section, which extends \([9, \text{Theorem 8}]\) from abelian codes in \( \mathbb{F}_q[A] \) to those in \( \mathbb{F}_q[G] \) (i.e. all abelian codes over finite fields).

**Theorem 3.5.** Let \((C, D)\) be an LCP of abelian codes in \( \mathbb{F}_q[G] \). Then \( C \) and \( D^\perp \) are equivalent codes.

**Proof.** We need to show that \( \tilde{D}^\perp[P] \) and \( (\tilde{D}[P])^\perp \) are equal. Note that if \( \dim_{\mathbb{F}_q} \tilde{D} = k \), then \( \dim_{\mathbb{F}_q} \tilde{D}^\perp[P] = \dim_{\mathbb{F}_q}(\tilde{D}[P])^\perp = (m - k)p^t \) (recall that \( m = |A| \) and \( p^t = |P| \)). Hence it is enough to show that one of these codes is contained in the other. By Remark 3.4, an element of \( \tilde{D}^\perp[P] \)
can be viewed as a $p^t$-tuple $(d_1^t, \ldots, d_k^t)$ of elements of $\tilde{D}^\perp$. Same also holds for the elements of $\tilde{D}[P]$ for which the elements can be viewed as $p^t$-tuples of elements of $\tilde{D}$. Since the Euclidean inner product on $\mathbb{F}_q[A]$ is “coordinate-wise”, $(d_1^t, \ldots, d_k^t)$ is orthogonal to all elements in $\tilde{D}[P]$. Hence $\tilde{D}^\perp[P] \subseteq (\tilde{D}[P])^\perp$ and the result follows. □

Remark 3.6. One can also view $C$ as a “matrix-product (MP) code” ([3]). Namely,

$$C = \tilde{C}[P] = [\tilde{C} \cdots \tilde{C}]I_p^t,$$

where $I_p^t$ denotes the identitiy matrix of size $p^t \times p^t$. In other words, $C$ is simply the MP code $[\tilde{C} \cdots \tilde{C}]$.

For a nonsingular matrix $M$, the dual of the MP code $[C_1 \cdots C_s]\,M$ is described as $[C_1^\perp \cdots C_s^\perp](M^{-1})^T$ ([3 Proposition 6.2]). We have the identitiy matrix for $M$ in our case. Hence the dual of $\tilde{D}[P]$ is simply $[\tilde{D}^\perp \cdots \tilde{D}^\perp]$ in MP notation. This code is nothing but $\tilde{D}^\perp[P]$. So, Theorem 3.5 can also be proved via MP codes.

Remark 3.7. We elaborate further on the equivalence between $C$ and $D^\perp$, since this will be useful in the proof of the main result over a chain ring (cf. proof of Theorem 3.10). If $G$ and $G'$ are finite multiplicative groups which are isomorphic via a map $\psi$, and if $R$ is any ring, then it is easy to see that $\psi$ extends to a ring isomorphism

$$\psi : \frac{R[G]}{\sum g \in G r_gg} \longrightarrow \frac{R[G']}{\sum g \in G r_g\psi(g)}$$

Hence such a map takes an ideal in $R[G]$ to an ideal of $R[G']$.

For $R = \mathbb{F}_q$ and $G = C_{m_1} \times \cdots \times C_{m_n}$, where the order $m_i$ of each cyclic component is relatively prime to $q$, recall that the rings $R_n = \mathbb{F}_q[x_1, \ldots, x_n]/(x_1^{m_1}-1, \ldots, x_n^{m_n}-1)$ and $\mathbb{F}_q[G]$ are isomorphic (cf. [2,1]). For an LCP $(C, D)$ of abelian codes in $R_n$, the equivalence between $C$ and $D^\perp$ is induced from the map (cf. proof of Theorem 8 in [9])

$$\psi : R_n \longrightarrow R_n
\frac{f(x_1, \ldots, x_n)}{x_1^{m_1}-1 \cdots x_n^{m_n}-1 f(x_1^{-1}, \ldots, x_n^{-1})}$$

Let us note that in the group ring interpretation, this is equivalent to the map

$$\psi : \sum \alpha_g g \longrightarrow \sum \alpha_g g^{-1},$$

or

$$\psi : \mathbb{F}_q[C_{m_1} \times \cdots \times C_{m_n}] \longrightarrow \mathbb{F}_q[C_{m_1} \times \cdots \times C_{m_n}]
\sum_{j=1}^n \sum_{i_j=0}^{m_j-1} \alpha_{i_1, i_2, \ldots, i_n} (g_1^{i_1}, \ldots, g_n^{i_n}) \longrightarrow \sum_{j=1}^n \sum_{i_j=0}^{m_j-1} \alpha_{i_1, i_2, \ldots, i_n} (g_1^{i_1}, \ldots, g_n^{i_n}),$$

if $g_j$ denotes a generator for $C_{m_j}$ for each $1 \leq j \leq n$ (cf. [2,1]). If $G = A \oplus P$, where $A = C_{m_1} \times \cdots \times C_{m_n}$ and $P = C_{p^\alpha_1} \times \cdots \times C_{p^\alpha_2}$, we have seen in Proposition 3.3 and Theorem 3.5 (see
also Remark 3.4) that the equivalence between \( C \) and \( D \perp \) for an \( LCP \) \((C, D)\) of abelian codes in \( \mathbb{F}_q[G] \) is induced from the following automorphism of \( G \):

\[
(g_1, \ldots, g_n, h_1, \ldots, h_t) \mapsto (g_1^{-1}, \ldots, g_n^{-1}, h_1, \ldots, h_t)
\]

So for any ring \( R \) the automorphism (3.3), or its “semisimple version” (i.e. \( t = 0 \)), induces a ring isomorphism from \( R[G] \) to itself.

4. LCP of Abelian Codes over Chain Rings

Our goal in this section is to generalize Theorem 3.5 to abelian codes over finite chain rings. We start with brief background on chain rings. Let us note that unless otherwise specified, \( R \) will denote a finite chain ring in this section.

A finite commutative ring \( R \) with identity is called a chain ring if its lattice of ideals is a chain. It is then clear that \( R \) is a local ring and it is well-known that \( R \) is a principal ideal ring. Let \( \gamma \) be a generator of the maximal ideal and let the ideals of \( R \) be

\[
R = R\gamma^0 \supset R\gamma \supset \cdots \supset R\gamma^{v-1} \supset R\gamma^v = \{0\}.
\]

The number \( v \) with \( \gamma^v = 0 \) is called the nilpotency index of \( \gamma \). Note that since \( R \) is a commutative ring, \( R\gamma^i = \gamma^i R \) for all \( i \).

As described in Section 2, we consider \( R[G] \) for a finite abelian group \( G \) of order \( N \). Recall that \( R[G] \) can be identified with \( R^N \) via the map that takes \( \sum_{g \in G} r_g g \in R[G] \) to the vector \((r_g)_{g \in G} \in R^N \). Note that \( R[G] \) is also isomorphic to a quotient ring of a polynomial ring in several variables, where the number of variables and the ideal quotienting the polynomial ring are determined by the number and the cardinalities of the cyclic groups in the decomposition of \( G \) (cf. (2.1)).

It is clear that \( R/R\gamma \) is a finite field, which we will denote by \( \mathbb{F}_q \). The natural projection map \( \varphi : R \to \mathbb{F}_q \) takes a ring element to its coset modulo \( R\gamma \). This map is a surjective ring homomorphism and it extends to \( R[G] \) and takes values in \( \mathbb{F}_q[G] \) via

\[
\sum_{g \in G} r_g g \mapsto \sum_{g \in G} \varphi(r_g) g.
\]

We will denote the extended map by \( \varphi \) as well, which is a surjective \( R \)-module homomorphism. In particular, \( \varphi \) maps an \( R \)-submodule of \( R[G] \) to an \( \mathbb{F}_q \)-subspace of \( \mathbb{F}_q[G] \). The kernel of this map is \( R\gamma[G] \) (i.e. those formal sums in \( R[G] \) whose coefficients are multiples of \( \gamma \)). In vectorial view, this is the set of all \( N \) tuples whose coordinates are multiples of \( \gamma \) (i.e. \( \gamma R^N \)).

A (linear) code over a ring \( R \) is defined as an \( R \)-submodule of \( R[G] \). A code \( C \) in \( R[G] \) is called an abelian code if \( C \) is an ideal in \( R[G] \) (i.e. \( R[G] \)-submodule in \( R[G] \)). A pair of codes \((C, D)\) in \( R[G] \) is called LCP if \( C \oplus D = R[G] \). We start with a simple observation on LCP of codes over a chain ring. Note we follow the same reasoning as in [2, Theorem 2] for LCD codes.

**Lemma 4.1.** If \((C, D)\) is LCP of codes over a chain ring, then both \( C \) and \( D \) are free modules (codes).
Proof. Note that by definition (being direct summands of the free module \( R[G] = R^N \)), both \( C \) and \( D \) are projective modules over \( R \). A chain ring is local and by [12] Theorem 2], a projective module over a local ring is free.

**Proposition 4.2.** If \((C, D)\) is an LCP of abelian codes in \( R[G] \), then \((\varphi(C), \varphi(D))\) is an LCP of abelian codes in \( F_q[G] \).

**Proof.** As noted above, the map \( \varphi \) in \([11]\) sends a (linear) code in \( R[G] \) to an \( F_q \)-linear code in \( F_q[G] \). Multiplication of an element \( r(g) = \sum g \in G \) by any \( g' \in G \) amounts to a certain permutation of the coefficients of \( r(g) \). Multiplication of \( r(g) \) by a ring element \( s \in R \) amounts to multiplying each coefficient in \( r(g) \) by \( s \). Therefore, the image under \( \varphi \) of an ideal (abelian code) in \( R[G] \) is an ideal (abelian code) in \( F_q[G] \).

If \( x \in F_q[G] \), there exists \( y = c + d \in R[G] = C + D \) such that \( x = \varphi(y) = \varphi(c) + \varphi(d) \). Hence, \( \varphi(C) + \varphi(D) = F_q[G] \).

If \( x \in \varphi(C) \cap \varphi(D) \), then \( x = \varphi(c) = \varphi(d) \) for some \( c, d \in D \). Then \( \varphi(c - d) = 0 \), which implies that \( (c - d) \in R\gamma[G] \). This yields \( \gamma^{u-1}(c - d) = 0 \) and hence \( z := \gamma^{u-1}c = \gamma^{u-1}d \). Since \( C \) and \( D \) are \( R \)-submodules in \( R[G] \), we have \( z \in C \cap D \). Then \( z = 0 \) since \((C, D)\) is LCP. So, \( \gamma^{u-1}c = 0 \) and hence \( c \in R\gamma[G] \). But then \( x = \varphi(c) = 0 \). Therefore \( \varphi(C) \cap \varphi(D) = \{0\} \). \( \square \)

For an element \( r \in R \) and \( r(g) = \sum g \in G \), we have \( rr(g) := \sum rr \). For any code \( C \) in \( R[G] \), we set \( rC := \{rc : c \in C\} \). We also define

\[ (C : r) := \{x \in R[G] : rx \in C\}. \]

It is clear that

\[ C = (C : \gamma^0) \subseteq (C : \gamma) \subseteq \cdots \subseteq (C : \gamma^{u-1}), \]

which implies that

\[ \varphi(C) = \varphi((C : \gamma^0)) \subseteq \varphi((C : \gamma)) \subseteq \cdots \subseteq \varphi((C : \gamma^{u-1})). \]

We collect some facts which will be needed. Let us note that the dual code of \( C \subseteq R[G] \) (with respect to Euclidean product in \( R[G] \)) is defined as in codes over finite fields, and it is denoted by \( C^\perp \).

**Proposition 4.3.** ([20] Theorem 3.10] Let \( C \) be a code in \( R[G] \). Then,

1. \( |C^\perp| = |R[G]|/|C| \).
2. \( \varphi((C : \gamma^{u-1-i}))^\perp = \varphi((C^\perp : \gamma^i)) \), for all \( i \).

Note that an explicit polynomial description of the dual code of a code \( C \) in \( R[G] \) in the non-repeated root case and the repeated root case can be found in [17] and [16] respectively.

**Proposition 4.4.** ([20] Proposition 3.13, [21] Proposition 3.11 and Corollary 3.12] The following holds for a free code \( C \) in \( R[G] \).

1. \( C^\perp \) is free.
2. \( \varphi(C) = \varphi((C : \gamma)) = \cdots = \varphi((C : \gamma^{u-1})) \).
3. \( C \cap \gamma^i R[G] = \gamma^i C \), for all \( i \).
(iv) For $\tilde{C} := C \setminus \gamma R[G] = C \setminus \gamma C$, we have $C = \tilde{C} \cup \gamma \tilde{C} \cup \cdots \cup \gamma^{v-1}\tilde{C} \cup \{0\}$.

We are ready to proceed with the steps of our proof.

**Proposition 4.5.** If $(C, D)$ is LCP of codes in $R[G]$, then $(C^⊥, D^⊥)$ is also LCP.

**Proof.** Let $x$ be an element of $C^⊥ \cap D^⊥$ and let $u = u_C + u_D$ be an arbitrary element in $R[G]$, where $u_C \in C$ and $u_D \in D$. Then the Euclidean product of $x$ and $u$ is

$$x \cdot (u_C + u_D) = x \cdot u_C + x \cdot u_D = 0,$$

since $x$ is orthogonal to both $C$ and $D$. So, $x = 0$ since its inner product with any element in $R[G]$ is 0. Therefore $C^⊥ \cap D^⊥ = \{0\}$.

For $c, c' \in C^⊥$ and $d, d' \in D^⊥$, if $c + d = c' + d'$ then $c - c' = d' - d \in C^⊥ \cap D^⊥$. But this intersection is shown to be trivial, hence $c = c'$ and $d = d'$. Therefore the number of elements in $C^⊥ + D^⊥ = \{c' + d' : c' \in C^⊥, d' \in D^⊥\}$ is $|C^⊥||D^⊥|$. By Proposition 4.3,

$$|C^⊥||D^⊥| = \frac{|R[G]|^2}{|C||D|} = |R[G]|.$$

Hence, $C^⊥ + D^⊥ = R[G]$. The result follows since the two dual codes intersect only at 0. □

**Proposition 4.6.** (i) For a free code $C \subset R[G]$, we have $\varphi(C)^⊥ = \varphi(C^⊥)$.

(ii) If $(C, D)$ is LCP of abelian codes in $R[G]$, then $\varphi(C)$ and $\varphi(D^⊥)$ are equivalent codes.

**Proof.** (i) We have $\varphi(C)^⊥ = \varphi((C^⊥ : \gamma^{v-1}))$ by part (ii) of Proposition 4.3. By Proposition 4.4 (i) and (ii), $\varphi((C^⊥ : \gamma^{v-1})) = \varphi(C^⊥)$. Hence the result follows.

(ii) By [19, Theorem 8] and Theorem 3.5, $\varphi(C)$ and $\varphi(D^⊥)$ are equivalent abelian codes via a permutation $\tau$ on $|G| = N$ letters acting on elements of $\mathbb{F}_q[G]$ (for any finite abelian group $G$).

Since $D$ is a free code, we have $\varphi(D^⊥) = \varphi(D^⊥)$ by part (i). □

**Remark 4.7.** Note that for an LCP of abelian codes $(C, D)$ in $R[G]$, we have

$$|D^⊥| = \frac{|R[G]|}{|D|} \quad \text{(by Proposition 4.3(i))}$$

$$(4.2) \quad = \frac{|C||D|}{|D|} \quad \text{(since $C \oplus D = R[G]$)}$$

$$= |C|.$$

Our aim is to lift the equivalence $\tau$ between $\varphi(C)$ and $\varphi(D^⊥)$ to an equivalence between $C$ and $D^⊥$, whose cardinalities have been shown to be the same in $(4.2)$.

If we restrict the map $\varphi : R[G] \to \mathbb{F}_q[G]$ to the (free) abelian codes $C$ and $D^⊥$, and use Proposition 4.4(iii), we obtain the isomorphisms

$$(4.3) \quad C/(C \cap \gamma R[G]) = C/\gamma C \simeq \varphi(C) \quad \text{and} \quad D^⊥/(D^⊥ \cap \gamma R[G]) = D^⊥/\gamma D^⊥ \simeq \varphi(D^⊥).$$

Let $t := |\varphi(C)| = |\varphi(D^⊥)|$ and set the elements of the cosets $C/\gamma C$ and $D^⊥/\gamma D^⊥$ as follows:

$$C/\gamma C := \{c_1 + \gamma C, c_2 + \gamma C, \ldots, c_t + \gamma C\},$$

$$D^⊥/\gamma D^⊥ := \{d_1 + \gamma D^⊥, d_2 + \gamma D^⊥, \ldots, d_t + \gamma D^⊥\}.$$
(i.e. \( c_1 = 0 = d_1 \) in \( R[G] \)). Clearly, cosets partition the codes \( C \) and \( D^\perp \):

\[
(4.4) \quad C = \bigcup_{1 \leq i \leq t} (c_i + \gamma C) \quad \text{and} \quad D^\perp = \bigcup_{1 \leq i \leq t} (d_i + \gamma D^\perp)
\]

Note that \( \varphi \) is constant on cosets, since a multiple of \( \gamma \) is mapped to 0. Namely for all \( i = 1, \ldots, t \), we have

\[
\varphi(c_i + \gamma c) = \varphi(c_i) + \gamma \varphi(c) = \varphi(c_i) \quad \text{for all} \quad c \in C,
\]

\[
\varphi(d_i + \gamma d) = \varphi(d_i) + \gamma \varphi(d) = \varphi(d_i) \quad \text{for all} \quad d \in D^\perp.
\]

Moreover \( \varphi(c_i) \neq \varphi(c_j) \) (for \( i \neq j \)), since otherwise \( c_i \) and \( c_j \) would be in the same coset modulo \( \gamma C \). The same holds for representatives of cosets of \( D^\perp \) modulo \( \gamma D^\perp \). Hence, we have

\[
\varphi(C) = \{ \varphi(c_1) = 0, \varphi(c_2), \ldots, \varphi(c_t) \},
\]

\[
\varphi(D^\perp) = \{ \varphi(d_1) = 0, \varphi(d_2), \ldots, \varphi(d_t) \}.
\]

Without loss of generality, we assume that the coset representatives are indexed so that the permutation \( \tau \) between the equivalent codes \( \varphi(C) \) and \( \varphi(D^\perp) \) (cf. Proposition 4.6) satisfies

\[
(4.5) \quad \varphi(\tau(c_i)) = \varphi(d_i), \quad \text{for all} \quad i = 1, \ldots, t.
\]

Note that this implies

\[
(4.6) \quad \tau(c_i) - d_i \in \gamma R[G] \quad \text{for all} \quad i = 1, \ldots, t.
\]

**Remark 4.8.** We saw in Remark 4.7 that \( C \) and \( D^\perp \) have the same cardinalities. At this point, using [21, Corollary 4.3], we can also conclude that they have the same minimum distance since \( \varphi(\tau(C)) = \varphi(D^\perp) \). Namely, since \( C \) and \( D^\perp \) are free codes, [21, Corollary 4.3] implies

\[
d(C) = d(\tau(C)) = d(\varphi(\tau(C))) = d(\varphi(D^\perp)) = d(D^\perp).
\]

Before the proof of the main result, let us state the following which gives a generating set as an \( R \)-module for a free code \( C \) in \( R[G] \).

**Proposition 4.9.** Let \( C \) be a free code in \( R[G] \) with the following representation (cf. (4.4)):

\[
C = \bigcup_{1 \leq i \leq t} (c_i + \gamma C).
\]

Let \( S := \{ c_2, \ldots, c_t \} \). Then any element of \( C \) can be represented as sum of the elements in

\[
S \cup \gamma S \cup \cdots \cup \gamma^{v-1} S.
\]

**Proof.** By Proposition 4.4, we have

\[
C = \tilde{C} \cup \gamma \tilde{C} \cup \cdots \cup \gamma^{v-1} \tilde{C} \cup \{ 0 \},
\]

where \( \tilde{C} = C \setminus \gamma C \). Since cosets modulo \( \gamma C \) partition \( C \), and recalling that \( c_1 = 0 \), we have

\[
\tilde{C} = (c_2 + \gamma C) \cup \cdots \cup (c_t + \gamma C),
\]

\[
\gamma C = \gamma \tilde{C} \cup \cdots \cup \gamma^{v-1} \tilde{C} \cup \{ 0 \}.
\]
Hence,
\[ \tilde{C} = \bigcup_{2 \leq i \leq t} (c_i + \gamma C) = \bigcup_{2 \leq i \leq t} c_i + (\gamma \tilde{C} \cup \cdots \cup \gamma^{v-1} \tilde{C} \cup \{0\}). \]

Since \( \gamma^v = 0 \), we have
\[ \gamma^{v-1} \tilde{C} = \bigcup_{i=2}^{t} \gamma^{v-1} c_i, \]
\[ \gamma^{v-2} \tilde{C} = \bigcup_{i=2}^{t} \left( \gamma^{v-2} c_i + (\gamma^{v-1} \tilde{C}) \right) \]
\[ = \bigcup_{i=2}^{t} \left( \gamma^{v-2} c_i + \left( \bigcup_{j=2}^{t} \gamma^{v-1} c_i \right) \right). \]

Continuing in the same manner until \( \gamma \tilde{C} \), we obtain the desired result. \( \square \)

We are ready to prove the main result for LCP of abelian codes over a chain ring.

**Theorem 4.10.** Let \((C, D)\) be an LCP of abelian codes in \(R[G]\), where \(R\) is a finite chain ring and \(G\) is a finite abelian group. Then \(C\) and \(D^\perp\) are equivalent codes.

**Proof.** By Proposition 4.6, \(\varphi(C)\) and \(\varphi(D^\perp)\) are equivalent codes. Let \(\tau\) be the permutation between them (i.e. \(\varphi(\tau(C)) = \varphi(D^\perp)\)). Note that \((C^\perp, D^\perp)\) is also an LCP of codes in \(R[G]\) by Proposition 4.5 and hence \((\varphi(C^\perp), \varphi(D^\perp))\) is LCP in \(F_q[G]\) (Proposition 4.2). If \(\{c'_1, c'_2, \ldots, c'_s\}\) denotes the coset representatives of \(C^\perp\) modulo \(\gamma C\) and \(\{d_1, d_2, \ldots, d_t\}\), as before, denotes the coset representatives of \(D^\perp\) modulo \(\gamma D\), we have

\[ F_q[G] = \varphi(C^\perp) \oplus \varphi(D^\perp) = \{\varphi(c'_i) + \varphi(d_j) : 1 \leq i \leq s, 1 \leq j \leq t\}. \]

Since \(C\) is free, \(\tau(C)\) is also a free code in \(R[G]\) and partitions as
\[ \tau(C) = \bigcup_{1 \leq i \leq t} (\tau(c_i) + \gamma \tau(C)) \quad (\text{cf. (4.4)}), \]
where \(\{c_1, c_2, \ldots, c_t\}\) is the set of coset representatives of \(C\) modulo \(\gamma C\).

If \(\tau(C) \cap C^\perp\) contains an element \(x\) in a coset \(c'_i + \gamma C^\perp\) for some \(i \in \{2, \ldots, s\}\), then
\[ \varphi(x) = \varphi(c'_i) \notin \varphi(\tau(C)) = \{\varphi(d_1) = 0, \varphi(d_2), \ldots, \varphi(d_t)\} \quad (\text{cf. (4.7)}). \]

Therefore \(\tau(C) \cap C^\perp\) is contained in \(\gamma C^\perp\), hence in \(\gamma \tau(C)\) (cf. Proposition 4.4 (iii)). Let \(x \in \tau(C) \cap C^\perp\) be \(x = \gamma \tau(c(1)) = \gamma c'(1)\), where \(c(1) \in C\) and \(c'(1) \in C^\perp\). Then \(\gamma(\tau(c(1)) - c'(1)) = 0\) and hence the difference \(\tau(c(1)) - c'(1)\) is a multiple of \(\gamma^{v-1}\):
\[ \text{i.e. } \tau(c(1)) = c'(1) + \gamma^{v-1} y_1, \text{ for some } y_1 \in R[G]. \]

If \(c'(1) \in C^\perp \setminus \gamma C^\perp\), then \(\varphi(\tau(c(1))) = \varphi(c'(1)) \notin \varphi(\tau(C))\) again. Hence, \(c'(1) = \gamma c'(2)\) for some \(c'(2) \in C^\perp\) and
\[ x = \gamma^2 c'(2) = \gamma^2 \tau(c(2)), \]
where $c(2) \in C$. This yields $\gamma^2(\tau(c(2)) - c'(2)) = 0$ and hence the difference $\tau(c(2)) - c'(2)$ is a multiple of $\gamma^{v-2}$. In other words, $\tau(c(2)) = c'(2) + \gamma^{v-2}y_2$ for some $y_2 \in R[G]$. By the same reasoning, $c'(2) \in \gamma C^\perp$ and hence

$$x = \gamma^3 \tau(c(3)) = \gamma^3 c'(3)$$

for some $c(3) \in C$ and $c'(3) \in C^\perp$.

If we follow this process, we conclude that the element $x$ in the intersection $\tau(C) \cap C^\perp$ must be 0.

Note that any permutation does not necessarily take an ideal of $R[G]$ to an ideal of $R[G]$. However $\tau$ does, as noted in Remark 3.7, since it is induced from an automorphism of $G$. So, $\tau(C)$ is an ideal of $R[G]$. By (4.6), we have (for all $1 \leq i \leq t$)

$$\tau(c_i) = d_i + \gamma x + \gamma y,$$

for uniquely determined $x \in D^\perp$ and $y \in C^\perp$, since $R[G] = C^\perp \oplus D^\perp$. Let $1 = a + b$ for $a \in C^\perp$, $b \in D^\perp$. Then, $\tau(c_i) = \tau(c_i)a + \tau(c_i)b$. Since $\tau(C)$ is an ideal, $\tau(c_i)a$ belongs to both $\tau(C)$ and $C^\perp$, whose intersection is 0. Hence,

$$\tau(c_i) = (d_i + \gamma x + \gamma y)b = (d_i + \gamma x)b + \gamma y b.$$

Note that $y b = 0$ since it belongs to $C^\perp \cap D^\perp = \{0\}$. Hence, $\tau(c_i) \in D^\perp$ for each $i$. This implies, by Proposition 4.9, that $\tau(C) \subset D^\perp$. Since $\tau(C)$ and $D^\perp$ have the same cardinalities (cf. Remark 4.8), we have $\tau(C) = D^\perp$. This concludes the proof. 

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