Generic vanishing index and the birationality of the bicanonical map of irregular varieties

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Abstract We prove that any smooth complex projective variety with generic vanishing index bigger or equal than 2 has birational bicanonical map. Therefore, if $X$ is a smooth complex projective variety $\varphi$ with maximal Albanese dimension and non-birational bicanonical map, then the Albanese image of $X$ is fibred by subvarieties of codimension at most 1 of an abelian subvariety of $\text{Alb}_X$.

1 Introduction

In the study of smooth complex algebraic varieties, the natural maps provided by the holomorphic forms defined in the variety, have a special importance. For example, the invertible sheaf $\omega_X$ of differential $n$-forms (where $n$ is the dimension of $X$) produces a map to a projective space, known as the canonical map. The multiples of this canonical sheaf $\omega_X^{\otimes m}$ produce in this way the pluricanonical maps $\varphi$

$$\varphi_m : X \rightarrow \mathbb{P}^N = \mathbb{P}(H^0(X, \omega_X^{\otimes m})^\vee).$$

When $\varphi_m$ gives a birational equivalence between $X$ and its image, we will simply say that $\varphi_m$ is birational. We say that $X$ is of general type if for some $m > 0$ the rational map $\varphi_m$ is birational.

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For example, the curves of general type are those of genus \( g \geq 2 \). The tricanonical map \( \varphi_3 \) is always birational for such curves and the bicanonical \( \varphi_2 \) is also birational once that \( g \geq 3 \). Moreover, the canonical map is birational as soon as the curve is non-hyperelliptic.

For surfaces, Bombieri [4] has given sharp numerical conditions for the birationality of \( \varphi_m \) for \( m \geq 3 \). The bicanonical map has revealed to be more complicated and has been studied by many algebraic geometers. In fact, the surfaces with irregularity \( q(S) \leq 1 \) and \( \chi(S, \omega_S) = 1 \) are not completely understood and there is no classification about which ones have birational \( \varphi_2 \). For a modern review of the state of the art in the surface case, we refer to [2, Thm. 8].

For higher dimensions not many results are known in general. Nevertheless, the example of the bicanonical map on surfaces shows that for small irregularity \( q(X) = h^0(X, \Omega^1_X) \) the classification becomes more difficult. For complex varieties, recall that the differential 1-forms give rise to the Albanese map

\[
alb: X \to \text{Alb} X = H^0(X, \Omega^1_X) / H_1(X, \mathbb{Z}).
\]

from \( X \) to an abelian variety of dimension \( q(X) = h^0(X, \Omega^1_X) \). We say that \( X \) is irregular if, and only if, \( \text{Alb} X \) is not trivial, i.e. \( q(X) > 0 \). And we say that \( X \) is of maximal Albanese dimension (m.A.d) if, and only if, the Albanese map \( \text{alb}: X \to \text{Alb} X \) is generically finite onto its image.

It turns out that some properties of m.A.d varieties seem to behave independently of the dimension and, indeed, Chen-Hacon showed that this is the case for their pluricanonical maps.

**Theorem** (Chen-Hacon. [6])

(a) \( X \) m.A.d and \( \chi(\omega_X) > 0 \) \( \Rightarrow \) \( X \) is of general type, furthermore, \( \varphi_3 \) is birational.

(b) \( X \) m.A.d \( \Rightarrow \) \( \varphi_6 \) is the stable pluricanonical map.

For \( \varphi_2 \), we cannot expect to use \( \chi(\omega_X) \) to control directly its birationality. For example, if \( C \) is a curve of genus 2, then the bicanonical map of the product \( C \times Y \) is never birational. In fact, it is clear that any variety that admits a fibration whose general fibre has non-birational \( \varphi_2 \) will have a non-birational bicanonical map. This should be considered, at least at first glance, as the standard case for higher dimensional varieties.

The following theorem provides geometric constraints for the non-birationality of the bicanonical map (see Theorem 5.2).

**Theorem A** Let \( X \) be a smooth projective complex variety of maximal Albanese dimension such that the bicanonical map is not birational. Then, the Albanese image of \( X \) is fibred by subvarieties of codimension at most 1 of an abelian subvariety of \( \text{Alb} X \). The base of the fibration is also of maximal Albanese dimension.

That is, \( X \) admits a fibration onto a normal projective variety \( Y \) with \( 0 \leq \dim Y < \dim X \), such that any smooth model \( \tilde{Y} \) of \( Y \) is of maximal Albanese dimension and

\[
q(X) - \dim X \leq q(\tilde{Y}) - \dim Y + 1.
\]

Hence, when \( q(X) > \dim X + 1 \) this implies the existence of an actual fibration, i.e. \( \dim Y > 0 \), whose general fibre is mapped generically finite through the Albanese map of \( X \) either onto a fixed abelian subvariety of \( \text{Alb} X \), or onto a divisor of this fixed abelian subvariety. When \( \dim Y = 0 \) the theorem simply says that the image of \( X \) in \( \text{Alb} X \) has codimension at most 1.

In particular, when \( X \) does not admit any fibration and \( q(X) > \dim X \), there is only one possible case, i.e. \( X \) is birationally equivalent to a theta-divisor of an indecomposable principally polarized abelian variety (see [1, Thm. A]). When \( X \) does not admit any fibration and
\( q(X) = \dim X \), there is only one known case of variety of general type and non-birational bicanonical map: a double cover of a principally polarized abelian variety \((A, \Theta)\) branched along a reduced divisor \(B \in |2\Theta|\). Is this the only case? The answer is affirmative in the case of surfaces due to Ciliberto-Mendes Lopes [7, Thm. 1.1].

To deduce Theorem A it is useful to consider the generic vanishing index introduced by Pareschi–Popa in [17, Def. 3.1]

\[
\text{gv}(\omega_X) = \min_{i>0} \left\{ \text{codim}_{\text{Pic}^0 X} V^i(\omega_X) - i \right\},
\]

where \(V^i(\omega_X) = \{ \alpha \in \text{Pic}^0 X \mid h^i(X, \omega_X \otimes \alpha) > 0 \}\). As a consequence of Generic Vanishing Theorem of Green–Lazarsfeld [9, Thm. 1], we have that for any irregular variety \(1 - \dim X \leq \text{gv}(\omega_X) \leq q(X) - \dim X\).

Moreover, the negative values of \(\text{gv}(\omega_X)\) can be interpreted in terms of the dimension of the generic fibre of the Albanese map (see Theorem 3.7) and \(X\) is a m.A.d variety if, and only if, \(\text{gv}(\omega_X) \geq 0\). Due to the work of Pareschi–Popa [17] we can interpret the positive values of \(\text{gv}(\omega_X)\) in terms of the local properties of the Fourier-Mukai transform of the structural sheaf (see Theorem 3.3). They have also proved that the positive values of \(\text{gv}(\omega_X)\) give a lower bound for the Euler characteristic \(\chi(\omega_X)\) (see Theorem 3.4).

Using the generic vanishing index we have the following more synthetic result.

**Theorem B** Let \(X\) be a smooth projective complex variety such that \(\text{gv}(\omega_X) \geq 2\). Then, the rational map associated to \(\omega_X^2 \otimes \alpha\) is birational onto its image for every \(\alpha \in \text{Pic}^0 X\).

Theorem A is deduced from this result by an argument of Pareschi-Popa. On the other hand, this result (see Theorem 5.1) is proved using a birationality criterion (see Lemma 4.2) that is a slight modification of [1, Thm. 4.13].

For curves, \(\text{gv}(\omega_C) \geq 2\) is equivalent to \(g(C) \geq 3\). For surfaces, \(\text{gv}(\omega_S) \geq 2\) is equivalent to suppose that \(q(S) \geq 4\) and does not admit an irregular fibration to a curve of genus \(\leq q(S) - 3\) (see Example 5.3).

**2 Generalized Fourier-Mukai transform**

\(X\) will be a smooth projective variety over an algebraically closed field \(k\) (from Sect. 3.3 on, we will restrict to \(k = \mathbb{C}\)). It will be equipped with a morphism \(a : X \to A\) to a non-trivial abelian variety \(A\), in particular, \(X\) will be irregular. Let \(\mathcal{P}\) be a Poincaré line bundle on \(A \times \text{Pic}^0 A\). We will denote

\[
P_a = (a \times \text{id}_{\text{Pic}^0 X})^* \mathcal{P},
\]

the induced Poincaré line bundle in \(X \times \text{Pic}^0 A\). When \(a = \text{alb}\), the Albanese map of \(X\), then the map \(\text{alb}^*\) identifies \(\text{Pic}^0(\text{Alb} X)\) to \(\text{Pic}^0 X\) and the line bundle \(P_{\text{alb}}\) will be simply denoted by \(P\).

Letting \(p\) and \(q\) the two projections of \(X \times \text{Pic}^0 A\), we consider the left exact functor \(\Phi_{p_a} \mathcal{F} = q_*(p^* \mathcal{F} \otimes P_a)\), and its right derived functors

\[
R^i \Phi_{p_a} \mathcal{F} = R^i q_*(p^* \mathcal{F} \otimes P_a).
\]

Sometimes we will have to consider the analogous derived functor \(R^i \Phi_{p_a^{-1}} \mathcal{F}\) as well. By the Seesaw Theorem [14, Cor. 6, p. 54], \(\mathcal{P}^{-1} = (1_A \times (-1)_{\text{Pic}^0 A})^* \mathcal{P}\), so

\[
R^i \Phi_{p_a^{-1}} \mathcal{F} = (-1_{\text{Pic}^0 A})^* R^i \Phi_{p_a} \mathcal{F}
\]

for any \(i\).
Given a coherent sheaf $\mathcal{F}$ on $X$, its $i$-th cohomological support locus with respect to $a$ is

$$V^i_a(\mathcal{F}) = \left\{ \alpha \in \text{Pic}^0 A \left| h^i (\mathcal{F} \otimes a^* \alpha) > 0 \right. \right\}$$

Again, when $a$ is the Albanese map of $X$, we will omit the subscript, simply writing $V^i_a(\mathcal{F})$.

A way to measure the size of all the $V^i_a(\mathcal{F})$’s is provided by the following invariant introduced by Pareschi–Popa.

**Definition 2.1 ([17, Def. 3.1])** Given a coherent sheaf $\mathcal{F}$ on $X$, the **generic vanishing index** of $\mathcal{F}$ (with respect to $a$) is

$$g_{v,a}(\mathcal{F}) := \min_{i>0} \left\{ \text{codim}_{\text{Pic}^0 A} V^i_a(\mathcal{F}) - i \right\}.$$ 

By convention we define $g_{v,a}(\mathcal{F}) = \infty$, when $V^i_a(\mathcal{F}) = \emptyset$ for every $i > 0$. When $a$ is the Albanese map of $X$, we will omit the subscript, simply writing $g_v(\mathcal{F})$.

By base change (see [17, Lem. 2.1]) it is easy to see that $g_{v,a}(\mathcal{F})$ can be also defined as the

$$\min_{i>0} \left\{ \text{codim}_{\text{Pic}^0 A} \text{supp} R^i \Phi_p \mathcal{F} - i \right\}.$$ 

### 3 Generic vanishing index of the canonical sheaf

#### 3.1 Relations between $g_v(\omega_X)$ and the Fourier-Mukai transform of $\mathcal{O}_X$

Here we specialize some general results of Pareschi–Popa [17,18] to the canonical sheaf of a smooth projective variety of dimension $d$. Some of these results were previously obtained by Hacon (see [11]).

The negative values of the $g_v$-index are related with the vanishing of the lowest cohomologies of the Fourier-Mukai transform of its Grothendieck dual. In the case of $\omega_X$ this can be stressed simply as:

**Theorem 3.1** ([17, Thm. 2.2]) **The following are equivalent,**

(a) $g_{v,a}(\omega_X) \geq -e$ for $e \geq 0$;

(b) $R^i \Phi_p \mathcal{O}_X = 0$ for all $i \neq d - e, \ldots, d$.

Hence, when $g_{v,a}(\omega_X) \geq 0$, $R^i \Phi_p \mathcal{O}_X = 0$ for all $i \neq d$, and we usually denote

$$\overline{\mathcal{O}}_X = R^d \Phi_p \mathcal{O}_X.$$ 

Note that, in this case, $H^i(X, \omega_X \otimes a^* \alpha) = 0$ for all $i > 0$ and general $\alpha \in \text{Pic}^0 A$. Therefore, by deformation-invariance of $\chi$, the generic value of $h^0(X, \omega_X \otimes a^* \alpha)$ equals $\chi(\omega_X)$, in particular $\chi(\omega_X) \geq 0$. Since, by base-change, the fibre of $\overline{\mathcal{O}}_X$ at a general point $\alpha \in \text{Pic}^0 A$ is isomorphic to $H^d(X, a^* \alpha) \cong H^0(X, \omega_X \otimes a^* \alpha^{-1})^*$, the (generic) rank of $\overline{\mathcal{O}}_X$ is $\text{rk} \overline{\mathcal{O}}_X = \chi(\omega_X)$.

From Grothendieck-Verdier duality [8, Thm. 4.3.1] and Theorem 3.1 it follows that,

**Corollary 3.2** ([18, Rem. 3.13]) **If** $g_{v,a}(\omega_X) \geq 0$ **then** $\text{Ext}^i_{\text{Pic}^0 A}((-1_{\text{Pic}^0 A})^* \overline{\mathcal{O}}_X, \mathcal{O}_{\text{Pic}^0 A}) \cong R^i \Phi_p \omega_X.$

The following result of Pareschi–Popa gives a dictionary between the positive values of $g_{v,a}(\omega_X)$ and the local properties of the Fourier-Mukai transform of $\overline{\mathcal{O}}_X$. 

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Theorem 3.3 ([17, Cor. 3.2]) Assume that $g_{v_a}(\omega_X) \geq 0$. Then,

$$g_{v_a}(\omega_X) \geq m \text{ if, and only if, } \widehat{\mathcal{O}}_X \text{ is a } m\text{-syzygy sheaf.}$$

(4)

In particular, $g_{v_a}(\omega_X) \geq 1$ is equivalent to $\widehat{\mathcal{O}}_X$ being torsion-free and $g_{v_a}(\omega_X) \geq 2$ to $\widehat{\mathcal{O}}_X$ being reflexive.

3.2 Top Fourier-Mukai transform of the canonical sheaf

In the case of abelian varieties (or complex torus) the following result is well-known and crucial in the proof of the Mukai Equivalence Theorem [13, Thm. 2.2]. We will need it in the proof of Theorem 5.1.

Proposition 3.6 ([1, Prop. 6.1]) If $a^* : \text{Pic}^0 A \to \text{Pic}^0 X$ is an embedding, then

$$R^d \Phi_{P_a} \omega_X \cong k(\hat{0}).$$

3.3 Generic vanishing theorem of Green–Lazarsfeld

The name of the $gv$-index comes from the well-known Generic Vanishing Theorem of Green–Lazarsfeld. As other general vanishing theorems, it requires $\text{char } k = 0$ so from now on we will restrict ourselves to the case $k = \mathbb{C}$. Basically, the following theorem is [9, Thm. 1]. The converse implication was proven independently in [12, Thm. B] and [1, Prop. 2.9].

Theorem 3.7 For any $e > 0$, the following are equivalent:

(a) the generic fibre of $a : X \to A$ has dimension $e$,
(b) $g_{v_a}(\omega_X) = -e$.

Moreover $g_{v_a}(\omega_X) \geq 0$ if, and only if, $a : X \to A$ is generically finite onto its image.

In particular, observe that for any irregular variety $1 - \dim X \leq g_{v}(\omega_X) \leq q(X) - \dim X$.

Remark 3.8 If $g_{v_a}(\omega_X) \geq 0$ and $\chi(\omega_X) > 0$, then $X$ is a variety of general type. Indeed, by the previous result $a : X \to A$ is generically finite and since $\chi(\omega_X) > 0$, we have that $V_{a^*}^0(\omega_X) = \text{Pic}^0 A$, so by [5, Cor.2.4], $\kappa(X) = \dim X$. In particular, if $g_{v_a}(\omega_X) \geq 1$, then $X$ is of general type.

3.4 Subtorus theorem of Green–Lazarsfeld and Simpson

The following theorem is due to Green and Lazarsfeld [10, Thm. 0.1] with an important addition due to Simpson [19, § 4,6,7].
Theorem 3.9 Let $W$ an irreducible component of $V^i(\omega_X)$ for some $i$. Then,

(a) There exists a torsion point $\beta \in \text{Pic}^0 X$ and a subtorus $B$ of $\text{Pic}^0 X$ such that $W = \beta + B$.
(b) There exists a normal variety $Y$ of dimension $\leq d - i$, such that any smooth model of $Y$ has maximal Albanese dimension and a morphism with connected fibres $f: X \to Y$ such that $B$ is contained in $f^* \text{Pic}^0 Y$.

Remark 3.10 It is useful to recall that the morphism $f: X \to Y$ in the second part of the previous theorem, arises as the Stein factorization of the morphism $\pi \circ \text{alb}: X \to \text{Pic}^0 W$, where $\pi: \text{Alb} X \to \text{Pic}^0 W$ is the dual map of the inclusion $W \subseteq \text{Pic}^0 X$. Hence, the key point of the second part of the theorem is the dimensional bound for $Y$.

4 Birationality criterion for maximal Albanese dimension varieties

In this section, we will assume that $a: X \to A$ is a generically finite morphism onto its image, where $A$ is an abelian variety. We introduce another piece of notation.

Notation 4.1 Let $F$ be a subsheaf of a line bundle and suppose that $g^0_a(F) \geq 1$.

(a) We denote $U_F$, the open subset where $h^0(F \otimes a^*\alpha)$ has the minimal value, i.e. $\chi(F)$.
(b) Let $Z$ be the exceptional locus of $a: X \to A$, that is $Z = a^{-1}(T)$, where $T$ is the locus of points in $A$ over which the fibre of $a$ has positive dimension.
(c) We define $B^F_a(x) = \{ \alpha \in U_F \mid x \text{ is a base point of } |F \otimes a^*\alpha| \}$.

By Remark 3.5, $\chi(F) \geq 1$. So, by semicontinuity, it makes sense to speak of the base locus of $F \otimes a^*\alpha$ for all $\alpha \in \text{Pic}^0 A$.

The following lemma is a slight modification of [1, Thm. 4.13] and it is based on [15, Prop. 2.12 and 2.13].

Lemma 4.2 Suppose that $a: X \to A$ is a generically finite morphism onto its image and let $F$ be a subsheaf of a line bundle such that $g^0_a(F) \geq 1$ and $R^ia_\ast F = 0$ for all $i > 0$. Suppose that for a general $x \in X$,

$$\text{codim}_{U_F} B^F_a(x) \geq 2.$$ 

Then, the rational map associated to the linear system $|F \otimes L|$ is birational for every line bundle $L$ such that $g^0_a(L) \geq 1$.

Proof We first compare the Fourier-Mukai transform of $F \otimes I_x$ and $F$.

Claim Let $x \in X$ be a closed point out of $Z$. Then $R^ia_\ast(F \otimes I_x \otimes a^*\alpha) = 0$ for $i > 0$. This follows immediately from the exact sequence

$$0 \to F \otimes I_x \to F \to k(x) \to 0$$

and the hypotheses that $R^ia_\ast F = 0$, $a$ is generically finite and $x \notin Z$. Hence, the degeneration of the Leray spectral sequence yields to

$$V^i_a(F \otimes I_x) = V^i(a_\ast(F \otimes I_x)).$$

By sequence (5), tensored by $a^*\alpha$, it follows that

$$V^i_a(F \otimes I_x) = V^i_a(F) \quad \text{for all } i \geq 2.$$
For $i = 1$ we have the surjection $H^1(\mathcal{F} \otimes \mathcal{I}_x \otimes a^*\alpha) \to H^1(\mathcal{F} \otimes a^*\alpha)$, that is an isomorphism if, and only if, $x$ is not a base point of $|\mathcal{F} \otimes a^*\alpha|$. In other words $V^1_\alpha(\mathcal{F} \otimes \mathcal{I}_x) \subseteq B^\mathcal{F}_\alpha(x) \cup V^1_\alpha(\mathcal{F})$. Since $\text{gv}_\alpha(\mathcal{F}) \geq 1$, the hypothesis on $B^\mathcal{F}_\alpha(x)$ guarantees that

$$\text{codim} V^1_\alpha(\mathcal{F} \otimes \mathcal{I}_x) \geq 2,$$

for a general $x \in X \setminus Z$. Hence by (6), (7) and (8), $\text{gv}(\alpha(\mathcal{F} \otimes \mathcal{I}_x)) \geq 1$. By [15, Prop. 2.13], $\alpha(\mathcal{F} \otimes \mathcal{I}_x)$ is continuously globally generated (CGG, see [15]). Therefore $\mathcal{F} \otimes \mathcal{I}_x$ itself is CGG outside $Z$ (with respect to $a$). Since the same is true for $L$, it follows from [15, Prop 2.12] that for all $\alpha \in \text{Pic}^0 A$, $\mathcal{F} \otimes L \otimes \mathcal{I}_x$ is globally generated outside $Z$. So the rational map associated to $|\mathcal{F} \otimes L|$ is birational.

**Remark 4.3** From the proof we see that if $\text{codim}_{\mathcal{F} \otimes \mathcal{I}_x} B^\mathcal{F}_\alpha(x) \geq 2$ for every $x \in X \setminus Z$, then $\mathcal{F} \otimes L$ is very ample out of $Z$, the exceptional locus of $a$.

### 4.1 Adjoint line bundles

When $\mathcal{F} = \omega_X$ we will call $U_\mathcal{F}$ simply $U_0$ and $B^\omega_\alpha(x)$ simply by

$$B_\alpha(x) = \{ \alpha \in U_0 \mid x \text{ is a base point of } \omega_X \otimes a^*\alpha \}.$$  

Throughout Sects. 4.1 and 4.2, we will assume that $\text{gv}_\alpha(\omega_X) \geq 1$.

**Proposition-Definition 4.4** Let $X$ be a variety such that $\text{gv}_\alpha(\omega_X) \geq 1$ and let $L$ be any line bundle on $X$ such that $\text{gv}_\alpha(L) \geq 1$. Suppose that there exists $\alpha \in \text{Pic}^0 A$ such that $\omega_X \otimes L \otimes a^*\alpha$ is not birational. Then,

$$\text{codim}_{X \times U_0} (\{ (x, \alpha) \in X \times U_0 \mid x \text{ is a base point of } \omega_X \otimes a^*\alpha \} = 1,$$

and its divisorial part is dominant on $X$ and surjects on $U_0$ via the projections $p$ and $q$.

We endow this set with the natural subscheme structure given by the image of the relative evaluation map $q^*(q_x^*L) \otimes L^{-1} \to \mathcal{O}_{X \times U_0}$, where $L = (p^*\omega_X) \otimes P_a)|_{X \times U_0}$ and we call $\mathcal{V}$ the union of its divisorial components that dominate $U_0$. Let $\overline{\mathcal{V}}$ be its closure in $X \times \text{Pic}^0 X$.

Then

(a) $X$ is covered by the scheme-theoretic fibres of the projection $\overline{\mathcal{V}} \to U_0$, that we will call $F_\alpha$, for $\alpha$ varying in $U_0$. By definition, at a general point $\alpha \in U_0$, $F_\alpha$ is the fixed divisor of $\omega_X \otimes a^*\alpha$.

(b) For a general $x \in X$, the fibre of the projection $\overline{\mathcal{V}} \to X$ is a divisor, that we will call $D_x$.

By definition, $D_x$ is the closure of the union of the divisorial components of the locus of $\alpha \in U_0$ such that $x \in \text{Bs}(\omega_X \otimes a^*\alpha)$.

**Proof** Everything follows from taking $\mathcal{F} = \omega_X$ in Lemma 4.2. The surjectivity of the projection to $U_0$ is consequence of the Castelnuovo-de Franchis inequality 3.4, i.e. $\chi(\omega_X) \geq \text{gv}_\alpha(\omega_X) \geq 1$.

### 4.2 Decomposition

In the sequel we will need $a^* : \text{Pic}^0 A \to \text{Pic}^0 X$ to be an embedding. However, for simplicity we will go one step further and we will simply suppose that $A = \text{Alb} X$. Suppose that we are under the hypotheses of the previous Proposition-Definition and consider a fixed point $\alpha_0 \in U_0$, and the map

$$f_{\alpha_0} : U_0 \to \text{Pic}^0 X \quad \alpha \mapsto \mathcal{O}_X(F_\alpha - F_{\alpha_0}),$$

\[ \square \]
where $F_\alpha$ is the divisor defined in Proposition-Definition 4.4(a). For $\alpha \in U_0$, all the $F_\alpha$ are algebraically equivalent since they are the fibres of $\overline{\mathcal{Y}} \to U_0$, so the map is well-defined.

The following lemma shows that this map induces a decomposition of $\text{Pic}^0 X$ and that the divisors $F_\alpha$ move algebraically along a non-trivial factor of $\text{Pic}^0 X$. Although the proof is basically the same as [1, Lem. 5.1], we do not require $V^1(\omega_X)$ to be a finite set, but only a proper subvariety.

**Lemma 4.5** The map defined in (10), induces an homomorphism $f : \text{Pic}^0 X \to \text{Pic}^0 X$ such that,

(a) $f^2 = f$ and $\text{Pic}^0 X$ decomposes as $\text{Pic}^0 X \cong \ker f \times \ker (\text{id} - f)$. Moreover $\dim \ker (\text{id} - f) > 0$.

(b) Fix $\tilde{\beta} \in \ker f$ such that $U_0 \cap (\{\tilde{\beta}\} \times \ker (\text{id} - f))$ is non-empty. Then, for $\gamma \in U_0 \cap \ker (\text{id} - f)$ the line bundle $O_X(F_{\tilde{\beta} \otimes \gamma}) \otimes \gamma^{-1}$ does not depend on $\gamma$. Since it is effective by semicontinuity, we call it $O_X(F)$.

(c) For all $(\beta, \gamma) \in \ker f \times \ker (\text{id} - f)$, $\text{Pic}^0 X$ such that $\beta \otimes \gamma \in U_0$, $|O_X(F) \otimes \gamma|$ is contained in the fixed divisor of $\omega_X \otimes \beta \otimes \gamma$.

**Proof** Let $O_X(M_\alpha) = \omega_X \otimes a^* \alpha \otimes O_X(-F_\alpha)$. Then, the proof of (a) is the same as [1, Lem. 5.1](a). Item (b) follows directly from the definition of $f$. To prove (c), let $(\beta, \gamma) \in \ker f \times \ker(\text{id} - f)$ such that $\beta \otimes \gamma \in U_0$ and $E \in |O_X(F) \otimes \gamma|$. Then $O_X(F_{\tilde{\beta} \otimes \gamma} - E) \cong O_X(F_{\tilde{\beta} \otimes \gamma} - F_{\tilde{\beta} \otimes \gamma}) = f(\beta \otimes \tilde{\beta}^{-1}) \otimes \gamma$. Since $F_{\tilde{\beta} \otimes \gamma}$ is a fixed divisor of $|\omega_X \otimes \beta \otimes \gamma|$, also $E = F_{\tilde{\beta} \otimes \gamma}$ is a fixed divisor in $|\omega_X \otimes \beta \otimes \gamma|$.

Using the decomposition given by the previous Lemma we give an explicit description of the the “half” Poincaré line bundle.

**Lemma 4.6** ([1, Lem. 5.1, 5.3]) We call $B = \text{Pic}^0(\ker f)$ and $C = \text{Pic}^0(\ker(\text{id} - f))$ so that

$$\text{Alb} X \cong B \times C \quad \text{and} \quad \text{Pic}^0 X \cong \text{Pic}^0 B \times \text{Pic}^0 C,$$

with $\dim C > 0$. Then we have the following description of the “half” Poincaré line bundle.

$$(\text{alb} \times \text{id}_{\text{Pic}^0 X})^*(O_B \times \text{Pic}^0_B \boxtimes \mathcal{P}_C) \cong O_{X \times \text{Pic}^0 X}(\overline{\mathcal{Y}}) \otimes p^* O_X(-F) \otimes q^* O_{\text{Pic}^0 X}(-D_{\tilde{x}}),$$

where $\tilde{x}$ is such that $\text{alb}(\tilde{x}) = 0$ in $\text{Alb} X$ and $\mathcal{P}_C$ is the Poincaré line bundle in $C \times \text{Pic}^0 C$.

**Proof** The decomposition of $\text{Pic}^0 X$ comes directly from Lemma 4.5(a). By the definition of $\overline{\mathcal{Y}}$ (see Proposition-Definition 4.4) and the definition of $F$ (see Lemma 4.5(b)) we have that the line bundle

$$O_{X \times \text{Pic}^0 X}(\overline{\mathcal{Y}}) \otimes p^* O_X(-F) \otimes q^* O_{\text{Pic}^0 X}(-D_{\tilde{\beta}}),$$

– restricted to $X \times \{\beta \otimes \gamma\}$ is isomorphic to $O_X(F_{\tilde{\beta} \otimes \gamma} - F) = \gamma$, for all $(\beta, \gamma) \in U_0 \subseteq \ker f \times \ker(\text{id} - f)$;

– restricted to $\{\tilde{x}\} \times \text{Pic}^0 X$ is isomorphic to $O_{\text{Pic}^0 X}(D_{\tilde{x}}) \otimes O_{\text{Pic}^0 X}(-D_{\tilde{x}})$, i.e. trivial.

On the other hand, $(\text{alb} \times \text{id}_{\text{Pic}^0 X})^*(O_B \times \text{Pic}^0 B \boxtimes \mathcal{P}_C)$,

– restricted to $X \times \{\beta \otimes \gamma\}$ is isomorphic to $\gamma$, for all $(\beta, \gamma) \in \ker f \times \ker(\text{id} - f)$;

– restricted to $\{\tilde{x}\} \times \text{Pic}^0 X$ is isomorphic to $O_{\text{Pic}^0 X}$, i.e. trivial.

Then, the Lemma follows from the seesaw principle. \qed
5 The bicanonical map of irregular varieties

The next theorem gives a sufficient numerical condition for the birationality of the bicanonical map, analogous to Pareschi–Popa Theorem [16, Thm. 6.1] for the tricanonical map.

**Theorem 5.1** Let $X$ be a smooth projective complex variety such that $\text{gv}(\omega_X) \geq 2$. Then, the rational map associated to $\omega_X^2 \otimes \alpha$ is birational onto its image for every $\alpha \in \text{Pic}^0 X$.

As a first corollary we have the following geometric interpretation.

**Theorem 5.2** Let $X$ be a smooth projective complex variety of maximal Albanese dimension such that the bicanonical map is not birational. Then, $0 \leq \text{gv}(\omega_X) \leq 1$. Moreover, it admits a fibration onto a normal projective variety $Y$ with $0 \leq \dim Y < \dim X$, any smooth model $\tilde{Y}$ of $Y$ is of maximal Albanese dimension and

$$q(X) - \dim X \leq q(\tilde{Y}) - \dim Y + \text{gv}(\omega_X).$$

**Proof** By Theorems 3.7 and 5.1, it is clear that $0 \leq \text{gv}(\omega_X) \leq 1$. Now, the proof is the same as the proof of [17, Thm. B]. \qed

**Example 5.3** We would like to show examples of varieties with $\text{gv}(\omega_X) \geq 2$. For curves $C$, this is equivalent to $g(C) \geq 3$. For surfaces $S$, it is equivalent to suppose that $q(S) \geq 4$ and $S$ does not admit an irregular fibration to a curve of genus $\leq q(S) - 3$ (see [3, Cor. 2.3]).

On the other hand, if $A$ is a simple abelian variety, then every subvariety $X$ of codimension $\geq 2$ has $\text{gv}(\omega_X) \geq 2$. Moreover, the property of having $\text{gv}(\omega_X) \geq 2$ is closed under taking products and cyclic coverings induced by a torsion point $\alpha \in \text{Pic}^0 X - V^1(\omega_X)$.

The rest of the paper is devoted to the proof of Theorem 5.1.

**Proof** Assume that $\text{gv}(\omega_X) \geq 1$ and there exists $\alpha \in \text{Pic}^0 X$ such that $\omega_X^2 \otimes \alpha$ is non-birational. Then, we want to see that $\text{gv}(\omega_X) = 1$. Under these hypotheses we can apply Proposition-Definition 4.4 and Lemma 4.6, so $\text{Alb} X \cong B \times C$, where $B = \text{Pic}^0(\text{ker}(\text{id} - f))$ and $C = \text{Pic}^0(\text{ker } f)$. We have the following commutative diagram

$$\begin{array}{ccc}
\text{Pic}^0 X & \xleftarrow{q} & X \times \text{Pic}^0 X \\
\downarrow p_b & & \downarrow \text{id} \times p_b \\
\text{Pic}^0 B & \xleftarrow{q} & X \times \text{Pic}^0 B \\
\downarrow b \times \text{id} & & \downarrow p_b \times p_b \\
& & \text{Alb} X \times \text{Pic}^0 X
\end{array} \quad (11)$$

where

- $p_b : \text{Alb} X \to B$ and $p_b : \text{Pic}^0 X \to \text{Pic}^0 B$ are the corresponding projections,
- $b$ is the composition by $b : X \xrightarrow{\text{alb}} \text{Alb} X \xrightarrow{p_b} B$, and
- abusing notation we also call $q$ either the projection $X \times \text{Pic}^0 X \to \text{Pic}^0 X$ or $X \times \text{Pic}^0 B \to \text{Pic}^0 B$ and $p$ the projections $X \times \text{Pic}^0 X \to X$ or $X \times \text{Pic}^0 B \to X$.

The effectiveness of $\overline{\nabla}$ give us the following short exact sequence on $X \times \text{Pic}^0 X$

$$0 \to (\text{alb} \times \text{id})^*(\mathcal{O}_{B \times \text{Pic}^0 B} \boxtimes \mathcal{P}_C)^{-1} \overline{\nabla} p^*\mathcal{O}_X(F) \otimes q^*\mathcal{O}(\mathcal{D}_{\tilde{\chi}}) \to (p^*\mathcal{O}_X(F) \otimes q^*\mathcal{O}(\mathcal{D}_{\tilde{\chi}}))|_{\overline{\nabla}} \to 0.$$
Recall that $P = (\text{alb} \times \text{id}_{\text{Pic}^0 X})^* (\mathcal{P}_B \boxtimes \mathcal{P}_C)$ since the Poincaré line bundle $\mathcal{P}$ in $\text{Alb} X \times \text{Pic}^0 X$ is isomorphic to $\mathcal{P}_B \boxtimes \mathcal{P}_C$. We apply the functor $R^d q_* (\cdot \otimes (\text{alb} \times \text{id}_{\text{Pic}^0 X})^* (\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_C \times \text{Pic}^0 C))$, that is, we tensor by the other “half” Poincaré line bundle and we consider the top direct image. We get

$$\cdots \rightarrow R^d \Phi_{p^{-1}}(\mathcal{O}_X) \rightarrow \quad$$

$$R^d q_* (p^* \mathcal{O}_X (F) \otimes (\text{alb} \times \text{id}_{\text{Pic}^0 X})^* (\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_C \times \text{Pic}^0 C)) \otimes \mathcal{O}_{\text{Pic}^0 X}(D) \rightarrow$$

$$R^d q_* ((p^* \mathcal{O}_X (F) \otimes q^* \mathcal{O}_{\text{Pic}^0 X}(D))|_{V} \otimes (\text{alb} \times \text{id}_{\text{Pic}^0 X})^* (\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_C \times \text{Pic}^0 C)) \rightarrow 0 \quad$$

Using that $R^i \Phi_{p^{-1}} \cong (-1)^*_{\text{Pic}^0 X} R^i \Phi_P$ (see (3)), we have the following short exact sequence,

$$0 \rightarrow (-1)^*_{\text{Pic}^0 X} \mathcal{O}_{\text{Pic}^0 X} \xrightarrow{\mu} \mathcal{E}(D) \rightarrow \mathcal{T} \rightarrow 0 \quad (12)$$

where:

(a) By base change, $\mathcal{E} = R^d q_* (p^* \mathcal{O}_X (F) \otimes (\text{alb} \times \text{id}_{\text{Pic}^0 X})^* (\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_C \times \text{Pic}^0 C))$ is a coherent sheaf of rank $h^d(\mathcal{O}_X (F) \otimes \mathcal{P}^{-1})$ by a general $\beta \in \text{ker} f$, i.e. $h^0(\mathcal{O}_X (F) \otimes \mathcal{P}) = \chi(\mathcal{O}_X)$ by Lemma 4.5.(c). Then,

$$\mathcal{E} = R^d q_* (p^* \mathcal{O}_X (F) \otimes (\text{alb} \times \text{id})^* (\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_C \times \text{Pic}^0 C))$$

$$= R^d q_* (p^* \mathcal{O}_X (F) \otimes (\text{alb} \times \text{id})^* (\mathcal{P}_B^{-1})) \quad \text{right square of (11)}$$

$$= R^d q_* (p^* \mathcal{O}_X (F) \otimes (\text{alb} \times \text{id})^* (\mathcal{P}_B^{-1})) \quad \text{abuse of notation on p}$$

$$= p_b^* R^d q_* (p^* \mathcal{O}_X (F) \otimes (\text{alb} \times \text{id})^* (\mathcal{P}_B^{-1})) \quad \text{flat base change}$$

$$= p_b^* R^d \Phi_{p^{-1}} (\mathcal{O}_X (F)),$$

following the notation of (1) and (2).

(b) $\mathcal{T} = R^d q_* ((p^* \mathcal{O}_X (F) \otimes q^* \mathcal{O}_{\text{Pic}^0 X}(D))|_{V} \otimes (\text{alb} \times \text{id}_{\text{Pic}^0 X})^* (\mathcal{P}_B^{-1} \boxtimes \mathcal{O}_C \times \text{Pic}^0 C))$ is supported at the locus of $\alpha \in \text{Pic}^0 X$ such that the fibre of the projection $q : \mathcal{V} \rightarrow \text{Pic}^0 X$ has dimension $d$, i.e. it coincides with $X$. Such locus is contained in $V(\mathcal{O}_X)$, therefore, since $\text{gv}(\mathcal{O}_X) \geq 1$, codim supp $\mathcal{T} \geq 2$.

(c) The map $\mu$ is injective since it is a generically surjective map of sheaves of the same rank (recall that $\text{rk} \mathcal{O}_X = \chi(\mathcal{O}_X)$), and, as $\text{gv}(\mathcal{O}_X) \geq 1$, the source $\mathcal{O}_X$ is torsion-free (Thm. 3.3).

(d) $\mu$ is $R^d q_* (m_s)$, where $m_s$ is the multiplication by the section defining $\mathcal{V}$. By base change [14, Cor. 3, p. 53], $R^d q_* (m_s) \otimes (\mathcal{O} (\alpha)) = H^d (m_s|_{q^{-1} (\alpha)})$ where $q$ is the projection $q : \mathcal{V} \rightarrow \text{Pic}^0 X$. When $q^{-1} (\alpha) = X$, $m_s|_{q^{-1} (\alpha)} = 0$, so in these points $R^d q_* (m_s) \otimes (\mathcal{O} (\alpha)) = 0$.

**Claim 5.4** $\mathcal{T} \neq 0$.

**Proof of the Claim.** Suppose that $\mathcal{T} = 0$, so $\mu$ is an isomorphism. Taking $\mathcal{E}x^d (\cdot, \mathcal{O}_{\text{Pic}^0 X})$ we get

$$k(\hat{0}) = R^d \Phi_P \omega_X \quad \text{Prop. 3.6}$$

$$= \mathcal{E}x^d (\mathcal{E}, \mathcal{O}_{\text{Pic}^0 X}) \otimes \mathcal{O} (-D) \quad \mathcal{E}x^d (\mu, \mathcal{O}_{\text{Pic}^0 X}) \text{ and Cor. 3.2}$$

$$= p_b^* \mathcal{E}x^d (R^d \Phi_{Pb} (\mathcal{O}_X (F)), \mathcal{O}_{\text{Pic}^0 B}) \otimes \mathcal{O} (-D) \quad \text{see item (a),}$$

which implies that codim$_{\text{Alb} X} B = \dim \ker (\text{id} - f) = 0$ contradicting Lemma 4.6. □
Let $\tau(\mathcal{E}(D_\chi))$ be the torsion part of $\mathcal{E}(D_\chi)$ and $\mathcal{E}(D_\chi)$ the quotient of $\mathcal{E}(D_\chi)$ by its torsion part. Hence $\mathcal{E}(D_\chi)$ is torsion-free. Now consider the following composition

\[
\begin{CD}
(-1)^*_{\text{Pic}^0 X} \mathcal{O}_X @>{\mu}>> \mathcal{E}(D_\chi) @>{\tilde{\mu}}>> \mathcal{E}(D_\chi).
\end{CD}
\]

Since $\tilde{\mu}$ is generically surjective and $(-1)^*_{\text{Pic}^0 X} \mathcal{O}_X$ is torsion-free (recall that $\text{gv}(\omega_X) \geq 1$), we have that $\tilde{\mu}$ is injective. Completing the diagram we get,

\[
\begin{CD}
0 @>>> 0 @>>> \tau(\mathcal{E}(D_\chi)) @>>> \tau(\mathcal{E}(D_\chi)) @>>> 0
\end{CD}
\]

\[
\begin{CD}
0 @>>> (-1)^*_{\text{Pic}^0 X} \mathcal{O}_X @>{\mu}>> \mathcal{E}(D_\chi) @>>> \mathcal{T} @>>> 0
\end{CD}
\]

\[
\begin{CD}
0 @>>> (-1)^*_{\text{Pic}^0 X} \mathcal{O}_X @>{\tilde{\mu}}>> \mathcal{E}(D_\chi) @>>> \mathcal{T} @>>> 0
\end{CD}
\]

If $\mathcal{T} = 0$, then the middle horizontal short exact sequence splits. But, for $\alpha$ a closed point in the support of $\mathcal{T}$ (by the previous claim we know that $\mathcal{T} \neq 0$), $\mu \otimes \mathbb{C}(\alpha) = 0$ by item (d), so $\mu$ cannot split. Therefore $\mathcal{T} \neq 0$.

Let $e = \text{codim}_{\text{Pic}^0 X} \text{supp} \mathcal{T} \geq 2$ (see item (c)). Then $\text{codim}_{\text{Pic}^0 X} \text{supp} \mathcal{E} \mathcal{T}^e(\mathcal{T}, \mathcal{O}_{\text{Pic}^0 X}) = e$. Now, we apply the functor $\mathcal{E} \mathcal{T}^e(\cdot, \mathcal{O}_{\text{Pic}^0 X})$ to the bottom row of (13) using Corollary 3.2

\[
\ldots \rightarrow R^{e-1} \Phi_p \omega_X \rightarrow \mathcal{E} \mathcal{T}^e(\mathcal{T}, \mathcal{O}_{\text{Pic}^0 X}) \rightarrow \mathcal{E} \mathcal{T}^e(\mathcal{E}(D_\chi), \mathcal{O}_{\text{Pic}^0 X}) \rightarrow \ldots
\]

Since $\mathcal{E}(D_\chi)$ is torsion-free, $\text{codim}_{\text{Pic}^0 X} \text{supp} \mathcal{E} \mathcal{T}^e(\mathcal{E}(D_\chi), \mathcal{O}_{\text{Pic}^0 X}) > e$. Therefore, we must have $\text{codim}_{\text{Pic}^0 X} \text{supp} R^{e-1} \Phi_p \omega_X = e$ and $\text{gv}(\omega_X) \leq 1$.

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