REMARK ON EQUICHARACTERISTIC ANALOGUE OF
HESSELHOLT’S CONJECTURE ON COHOMOLOGY OF WITT
VECTORS

AMIT HOGADI AND SUPRIYA PISOLKAR

ABSTRACT. Let $L/K$ be a finite Galois extension of complete discrete valued fields of characteristic $p > 0$. Assume that the induced residue field extension $k_L/k_K$ is separable. For an integer $n \geq 0$, let $W_n(O_L)$ denote the ring of Witt vectors of length $n$ with coefficients in $O_L$. We show that the pro-abelian group $\{H^1(G, W_n(O_L))\}_{n \in \mathbb{N}}$ is zero. This is an equicharacteristic analogue of Hesselholt’s conjecture (see [4]) which was proved in [5] when the discrete valued fields are of mixed characteristic.

1. Introduction

Let $K$ be a complete discrete valued field with residue field of characteristic $p > 0$. $L/K$ be a finite Galois extension with Galois group $G$. Suppose that $k_L/k_K$ is separable. When $K$ is of characteristic zero, Hesselholt conjectured in [4] that the pro-abelian group $\{H^1(G, W_n(O_L))\}_{n \in \mathbb{N}}$ vanishes, where $W_n(O_L)$ is the ring of Witt vectors of length $n$ with coefficients in $O_L$ (w.r.t. to the prime $p$). As explained in [4], this can be viewed as an analogue of Hilbert theorem 90 for the Witt ring $W(O_L)$. This conjecture was proved in some cases in [4] and in general in [5]. In this paper we show that a similar vanishing holds when $K$ is of characteristic $p$. The main result of this paper is as follows.

Theorem 1.1. Let $L/K$ be a finite Galois extension of complete discrete valued equi-characteristic fields with Galois group $G$. Assume that the induced residue field extension is separable. Then the pro-abelian group $\{H^1(G, W_n(O_L))\}$ is zero.

In order to prove this conjecture one easily reduces to the case where $L/K$ is a totally ramified Galois extension of degree $p$ (see [5, Lemma 3.1]). We make the argument in [5] work in the equicharacteristic case using an explicit description of the Galois cohomology of $O_L$ when $L/K$ is an Artin-Schreir extension (see Proposition 2.4).

We remind the reader that a pro-abelian group indexed by $\mathbb{N}$ is an inverse system of abelian groups $\{A_n\}_{n \in \mathbb{N}}$ whose vanishing means that for every $n \in \mathbb{N}$, there exists an integer $m > n$ such that the map

$A_m \to A_n$

2000 Mathematics Subject Classification. 11S25.

Key words and phrases. Galois Cohomology, Witt vectors.
is zero (see [6, Section 1]). This is clearly implies the vanishing of \( \lim \limits_{n \to \infty} H^1(G, W_n(\mathcal{O}_L)) \).

It also implies the vanishing of \( H^1(G, W(\mathcal{O}_L)) \) (with \( W(\mathcal{O}_L) \) being considered as discrete \( G \)-module) by [5, Corollary 1.2].

**Remark 1.2.** One may also consider an analogue of Theorem 1.1 when \( K \) is of equi-characteristic zero. However, in this case, all extensions \( L/K \) are tamely ramified and the vanishing

\[ H^1(G(L/K), W_n(\mathcal{O}_L)) = 0 \quad \forall \ n \geq 0. \]

can be easily deduced from the fact that \( \mathcal{O}_L \) is a projective \( \mathcal{O}_K[G] \) module (see [2, I. Theorem 3]).

**Acknowledgement:** We thank the referee for several useful comments and suggestions.

2. **Cohomology of integers in Artin-Schreier extensions**

Let \( K \) be a complete discrete valued field of characteristic \( p \) as before. Let \( \mathcal{O}_K \) and \( k \) denote the discrete valuation ring and residue field of \( K \) respectively. Let \( L/K \) be a Galois extension of degree \( p \). Recall that the ramification break (or lower ramification jump) of this extension, to be denoted by \( s = s(L/K) \), is the smallest non-negative integer such that the induced action of \( \text{Gal}(L/K) \) on \( \mathcal{O}_L/m_L^{s+1} \) is faithful, where \( m_L \) is the maximal ideal of \( \mathcal{O}_L \) ([1, II, 4.5]). Thus unramified extensions are precisely the extensions with ramification break equal to zero. We recall the following well known result.

**Proposition 2.1.** (see [3] or [7, Proposition 2.1]) Let \( L/K \) be a Galois extension of degree \( p \) of complete discrete valued fields of characteristic \( p \). There exists an element \( f \in K \) such that \( L \) is obtained by joining a root of the polynomial

\[ X^p - X - f = 0 \]

Further one can choose \( f \) such that \( v_K(f) \) is coprime to \( p \). In this case

\[ v_K(f) = -s \]

where \( s \) is the ramification break of \( \text{Gal}(L/K) \).

We now fix an \( f \in K \) given by the above proposition. Clearly, if \( v_K(f) > 0 \) then by Hensel’s lemma \( X^p - X - f \) already has a root in \( K \). If \( v_K(f) = 0 \) then the extension given by adjoining the root of this polynomial is an unramified extension.

**Proposition 2.2.** Let \( L/K, f \in K \) be as above. Assume \( L/K \) is totally ramified. Let \( \lambda \) be a root of \( X^p - X - f \) in \( L \). Let \( s \) be the ramification break of \( \text{Gal}(L/K) \).

Then the discrete valuation ring \( \mathcal{O}_L \), is the subset of \( L \) is given by

\[ \mathcal{O}_L = \left\{ \sum_{i=0}^{p-1} a_i \lambda^i \mid a_i \in \mathcal{O}_K \text{ with } v_K(a_i) \geq \frac{is}{p} \right\}. \]
Proof. Clearly the set \( \{1, \lambda, \cdots, \lambda^{p-1}\} \) is a \( K \)-basis of \( L \). Thus any element \( x \in L \) can be written uniquely in the form
\[
x = \sum_{i=0}^{p-1} a_i \lambda^i.
\]
Note that \( v_L(\lambda) = v_K(f) = -s \) is coprime to \( p \) by the choice of \( f \). Since \( L/K \) is ramified, \( s \) is nonzero. Moreover \( v_L(a_i) = pv_K(a_i) \) is divisible by \( p \). We thus conclude that for each \( 0 \leq i \leq p-1 \), the values of \( v_L(a_i \lambda^i) \) are all distinct modulo \( p \), and hence, distinct.

Thus
\[
v_L\left( \sum_{i=0}^{p-1} a_i \lambda^i \right) \geq 0 \text{ if and only if } v_L(a_i \lambda^i) \geq 0 \text{ for all } 0 \leq i < p.
\]

But \( v_L(a_i \lambda^i) = pv_K(a_i) - is \). This proves the claim. \( \square \)

Lemma 2.3. Let \( p \) be a prime number as before. Let
\[
S_k = \sum_{n=0}^{p-1} n^k.
\]
Then
1. \( S_k \equiv 0 \mod p \) if \( 0 \leq k \leq p-2 \)
2. \( S_{p-1} \equiv -1 \mod p \)

Proof. The first congruence follows from the recursive formula (see \[8, (4)\])
\[
S_k = \frac{1}{k+1} \left( p^{k+1} - p^k - \sum_{j=0}^{k-2} \binom{k}{j} S_{j+1} \right)
\]
and using the fact that when \( k \leq p-2 \), \( k+1 \) is invertible modulo \( p \). (2) follows from Fermat’s little theorem. \( \square \)

We now state an explicit description of \( H^1(G, O_L) \).

Proposition 2.4. Notation as in \[(2.1)\]. Let \( \sigma \) be a generator of \( \text{Gal}(L/K) \). Let \( O^\text{tr}_{L=0} \) denote the set of all trace zero elements in \( O_L \) and
\[
(\sigma - 1)O_L = \{ \sigma(x) - x \mid x \in O_L \}.
\]
Then,
1. \( O^\text{tr}_{L=0} = \{ \sum_{i=0}^{p-2} a_i \lambda^i \mid v_K(a_i) \geq \frac{is}{p} \} \)
2. \( (\sigma - 1)O_L = \{ \sum_{i=0}^{p-2} a_i \lambda^i \mid v_K(a_i) \geq \frac{(i+1)s}{p} \} \)
Proof. Since the sets $\mathcal{O}_L^{\sigma_1 = 0}$ and $(\sigma - 1)\mathcal{O}_L$ are independent of the choice of $\sigma$, we may assume, without loss of generality, that $\sigma$ is the generator satisfying $\sigma(\lambda) = \lambda + 1$.

(1) Let $x = \sum_{i=1}^{p-1} a_i \lambda^i$. Let $S_k$ be as in Lemma 2.3. Then

$$\text{tr}(x) = \sum_{j=0}^{p-1} \sigma^j(x) = \sum_{j=0}^{p-1} \sum_{i=0}^{p-1} a_i (\lambda + j)^i = \sum_{i=0}^{p-1} a_i \left( \sum_{j=0}^{p-1} (\lambda + j)^i \right)$$

By binomially expanding and collecting coefficients of $\lambda^i$, we get

$$\text{tr}(x) = \sum_{i=0}^{p-1} a_i \left( p\lambda^i + \sum_{j=1}^{i} \binom{i}{j} S_j \lambda^{i-j} \right) = -a_{p-1} \quad \text{(by Lemma 2.3)}.$$ 

This together with Proposition 2.2 proves (1).

(2) Suppose $x = \sum_{i=1}^{p-1} a_i \lambda^i \in (\sigma - 1)\mathcal{O}_L$. Then

$$\sum_{i=1}^{p-1} a_i \lambda^i = (\sigma - 1) \sum_{i=1}^{p-1} b_i \lambda^i,$$

where $v_K(b_i) \geq \frac{i+1}{p}$ by (2.2). This gives us the following system of $p$ equations

\[\begin{align*}
a_0 &= b_1 + \cdots + b_{p-1} \\
a_1 &= \binom{1}{1} b_2 + \binom{3}{2} b_3 + \cdots + \binom{p-1}{p-2} b_{p-1} \\
\vdots \\
a_i &= \binom{i+1}{i} b_{i+1} + \cdots + \binom{p-1}{p-(i+1)} b_{p-1} \\
a_{p-2} &= (p - 1) b_{p-1} \\
a_{p-1} &= 0
\end{align*}\]

Since $v_K(b_{i+1}) \geq \frac{(i+1)s}{p}$, we get $v_K(a_i) \geq \frac{(i+1)s}{p}$. Thus

$$(\sigma - 1)\mathcal{O}_L \subset \left\{ \sum_{i=0}^{p-1} a_i \lambda^i \mid v_K(a_i) \geq \frac{(i+1)s}{p} \right\}.$$ 

Conversely assume

$$\sum_{i=1}^{p-1} a_i \lambda^i \in \left\{ \sum_{i=0}^{p-1} a_i \lambda^i \mid v_K(a_i) \geq \frac{(i+1)s}{p} \right\}.$$ 

Since $H^1(G, L) = 0$, we know there exists $\sum b_i \lambda^i \in L$ such that

$$\sum_{i=1}^{p-1} a_i \lambda^i = (\sigma - 1) \sum_{i=1}^{p-1} b_i \lambda^i.$$
These $b'_i$s satisfy the above system of $p$ equations. Using that $v_K(a_i) \geq \frac{(i+1)s}{p}$, it is straightforward to observe by induction that $v_K(b'_i) \geq \frac{i}{p}$. Hence $\sum_{i=1}^{p-1} b'_i \lambda^i \in \mathcal{O}_L$. □

The following corollary is the equi-characteristic analogue of [4, Lemma 2.4].

**Corollary 2.5.** Let $L/K$ be as in Proposition [2.1]. Let $x \in \mathcal{O}_{L}^{tr=0}$ be an element that defines a non zero class in $H^1(G, \mathcal{O}_L)$. Then $v_L(x) \leq s - 1$.

**Proof.** We will show that for any $x \in \mathcal{O}_{L}^{tr=0}$, if $v_L(x) \geq s$, then the class of $x$ in $H^1(G, \mathcal{O}_L)$ is zero. By (2.4), we may write

$$x = \sum_{i=1}^{p-2} a_i \lambda^i \text{ with } v_L(a_i) \geq is.$$ 

Since for all $i$, $v_L(a_i \lambda^i)$ are distinct (see proof of (2.2)), we have

$$v_L(x) = \inf\{v_L(a_i \lambda^i)\}.$$ 

Thus $v_L(x) \geq s$ implies

$$v_L(a_i \lambda^i) = v_L(a_i) - is \geq s \quad \forall i.$$ 

This shows that $v_L(a_i) \geq (i+1)s$ which by Proposition [2.4] implies $x \in (\sigma - 1)\mathcal{O}_L$, and hence defines a trivial class in $H^1(G, \mathcal{O}_L)$. This proves the claim. □

### 3. Proof of the main theorem

Following [5, Lemma 3.1], we reduce the proof of the main theorem to the case when $L/K$ is totally ramified Galois extension of degree $p$. Thus throughout this section we fix an extension $L/K$ which is of this type. We also fix a generator $\sigma \in \text{Gal}(L/K)$. We first define a polynomial $G \in \mathbb{Z}[X_1, ..., X_p]$ in $p$ variables by

$$G(X_1, ..., X_p) = \frac{1}{p} \left( (\sum_{i=1}^{p} X_i)^p - (\sum_{i=1}^{p} X_i^p) \right).$$

Note that despite the occurrence of $\frac{1}{p}$, $G$ is a polynomial with integral coefficients.

Now for an element $x \in L$ define

$$F(x) = G(x, \sigma(x), ..., \sigma^i(x), ..., \sigma^{p-1}(x)).$$

The expression $F(x)$ is formally equal to the expression $\frac{tr(x)^p - tr(x^p)}{p}$ and makes sense in characteristic $p$ since $G$ has integral coefficients. Moreover, since for any $x \in L$, $F(x)$ is invariant under the action of $\text{Gal}(L/K)$, $F(x) \in K$. We now observe that [4, Lemma 2.2] holds in characteristic $p$ in the following form

**Lemma 3.1.** [4 Lemma 2.2] For all $x \in \mathcal{O}_L$, $v_K(F(x)) = v_L(x)$. 

Proof of 1.1. The proof follows [5, Proof of 1.4] verbatim, with (2.5) and (3.1) replacing [5, Lemma 3.2] and [5, Lemma 3.4] respectively. We briefly recall the idea of the proof for the convenience of the reader. By [4, Lemma 1.1], it is enough to show that for large $n$, the map

$$H^1(G, W_n(O_L)) \to H^1(G, O_L)$$

is zero. By (2.5), it is enough to show that for large $n$

$$(x_0, ..., x_{n-1}) \in W_n(O_L)^{tr=0} \implies v_L(x_0) \geq s.$$  

The condition $(x_0, ..., x_{n-1}) \in W_n(O_L)^{tr=0}$ can be rewritten as

$$\sum_{i=0}^{p-1} (\sigma^i(x_0), ..., \sigma^i(x_{n-1})) = 0.$$  

Using the formula for addition of Witt vectors, one analyses the above equation and obtains (see [5, Lemma 3.5])

$$\text{(1)} \quad \text{tr}(x_\ell) = F(x_{\ell-1}) - C \cdot \text{tr}(x_{\ell-1})^p + h_{\ell-2}, \quad 1 \leq \ell \leq n-1$$

where $C$ is a fixed integer and $h_{\ell-2}$ is a polynomial in $x_0, ..., x_{\ell-2}$ and its conjugates such that each monomial appearing in $h_{\ell-2}$ is of degree $\geq p^2$. Using the above equation, Lemma 3.1 and [4, Lemma 2.1] one proves the theorem in the following three steps, for details of which we refer the reader to [5, Proof of 1.4].

Step 1. We claim that for $0 \leq \ell \leq n-2$

$$v_L(x_\ell) \geq \frac{s(p-1)}{p}.$$  

Recall equation (1) for $1 \leq \ell \leq n-1$

$$\text{tr}(x_\ell) = F(x_{\ell-1}) - C \cdot \text{tr}(x_{\ell-1})^p + h_{\ell-2}. $$

One now proves the above claim by induction on $\ell$. Using the fact that $h_{-1} = \text{tr}(x_0) = 0$, the above equation gives

$$-\text{tr}(x_1) = F(x_0).$$

This, together with [4, Lemma 2.1] proves the claim for $\ell = 0$. The rest of the induction argument is straightforward. This claim, together with equation (1), is then used to observe $v_K(h_\ell) \geq s(p-1)$ for all $\ell$.

Step 2. In this step we show that for $2 \leq i \leq n-1$

$$v_L(x_{n-i}) \geq \frac{s(p-1)}{p} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{i-2}}\right).$$
This is proved by induction on $i$, using equation (II) and the estimates
\[ v_L(x_\ell) \geq \frac{s(p - 1)}{p}, \quad v_L(h_\ell) \geq s(p - 1) \]
obtained in Step 1.
Step 3. Since values taken by $v_L$ is a discrete valuation, for an integer $M$ such that
\[ \frac{s(p - 1)}{p} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^{M-2}} \right) > s - 1, \]
we have $v_L(x_0) \geq s$.

\[ \square \]

References

[1] Fesenko, I. B.; Vostokov, S. V.; Local fields and their extensions. Second edition. Translations of Mathematical Monographs, 121. American Mathematical Society.
[2] Fröhlich, Albrecht; Galois module structure of algebraic integers. 3. Springer-Verlag, Berlin, 1983.
[3] H. Hasse; Theorie der relativ-zyklischen algebraischen Funktionenkörper, insbesondere bei endlichem Konstantenkörper. J. Reine Angew. Math. 172 (1934), 37?54.
[4] Lars Hesselholt; Galois cohomology of Witt vectors of algebraic integers. Math. Proc. Cambridge Philos. Soc. 137 (2004), no. 3, 551–557.
[5] Hogadi, Amit; Pisolkar, Supriya. On the cohomology of Witt vectors of p-adic integers and a conjecture of Hesselholt. J. Number Theory 131 (2011), no. 10, 1797–1807.
[6] Jannsen, Uwe; Continuous étale cohomology. Math. Ann. 280 (1988), no. 2, 207–245
[7] Thomas, Lara; Ramification groups in Artin-Schreier-Witt extensions. J. Théor. Nombres Bordeaux 17 (2005), no. 2, 689–720.
[8] Turner, Barbara; Sums of powers of integers via the binomial theorem. Math. Mag. 53 (1980), no. 2, 92–96.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400005, India
E-mail address: amit@math.tifr.res.in

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400005, India
E-mail address: supriya@math.tifr.res.in