Position of the centroid of a planar convex body

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Abstract. It is well known that any planar convex body \( A \) permits to inscribe an affine-regular hexagon \( H_A \). We prove that the centroid of \( A \) belongs to the homothetic image of \( H_A \) with ratio \( \frac{4}{21} \) and the center in the center of \( H_A \). This ratio cannot be decreased.

Keywords: convex body, centroid, affine-regular hexagon

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1 Introduction

This paper concerns the position of the centroid of a planar convex body, i.e., a closed bounded convex set. Recall that the notion of centroid is discussed by, among others, Bonnesen and Fenchel [2], Grünbaum [3], Hammer [4] and Neumann [5].

As usual, by an affine-regular hexagon we understand a non-degenerated affine image of the regular hexagon. Besicovitch [1] proved that for every planar convex body \( A \) there exists an affine-regular hexagon \( H_A \) inscribed in \( A \). Our aim is to prove that the centroid of \( A \) belongs to the homothetic image \( \frac{4}{21}H_A \) of \( H_A \) with ratio \( \frac{4}{21} \) and the center in the center of \( H_A \). In general, this ratio cannot be lessened, which is explained at the end of the paper.

For a compact set \( C \) of the Euclidean plane \( E^2 \) denote by \( cen_x(C) \) and \( cen_y(C) \) the first and the second coordinates of the centroid of \( C \). Let compact sets \( B_1, \ldots, B_n \subset E^2 \) with non-empty interiors have disjoint interiors and \( B = \bigcup_{j=1}^n B_j \). It is well known that

\[
\text{cen}_x(B) = \frac{\sum_{j=1}^n \text{cen}_x(B_j) \cdot \text{area}(B_j)}{\sum_{j=1}^n \text{area}(B_j)}, \quad \text{cen}_y(B) = \frac{\sum_{j=1}^n \text{cen}_y(B_j) \cdot \text{area}(B_j)}{\sum_{j=1}^n \text{area}(B_j)}.
\]

2 The position of the centroid of a convex body with respect to an inscribed affine-regular hexagon

Let \( D \subset E^2 \) and \( \ell \) be a straight line. Imagine \( D \) as the union of segments (including one-point segments) being intersections of \( D \) by straight lines perpendicular to \( \ell \). Shift every such a segment perpendicularly to \( \ell \) in order to obtain its image centered at \( \ell \). Denote the union of
all these obtained segments by \( \operatorname{sym}_D D \). It is the result of the Steiner symmetrization of \( D \). The proof of the following lemma is given in a number of books. For instance in Section 40 of [2].

**Lemma.** If \( D \subset E^2 \) is convex, then \( \operatorname{sym}_D D \) is convex.

**Theorem.** Let \( A \subset E^2 \) be a convex body and \( H_A \) be an affine-regular hexagon inscribed in \( A \). Then the centroid of \( A \) belongs to the homothetic image of \( H_A \) with ratio \( \frac{4}{21} \) and center in the center of \( H_A \).

**Proof.** For better clarity, we divide the proof into a preliminary text mostly on notations, and then Parts 1–8 with considerations.

We do not lose the generality assuming that the successive vertices \( a_1, \ldots, a_6 \) of \( H_A \) are \((1, 1), (-1, 1), (-2, 0), (-1, -1), (1, -1), (2, 0)\), see Figure. Denote by \( o \) the center \((0, 0)\) and by \( a \) the midpoint of \( a_1 a_2 \). Since we deal with \( \operatorname{cen}_y(A) \), by Lemma we may assume that \( x = 0 \) is an axis of symmetry of \( A \).

In order to prove the assertion, let us show that for any side of \( \frac{4}{21} H_A \) the centroid of \( A \) is on the same side of the straight line containing this side which contains \( o \). Observe that it is enough to show this for one side of the hexagon \( \frac{4}{21} H_A \). Let us provide this task for the side connecting \( a_{\frac{4}{21}} a_1 a_2 \).

Denote by \( \bar{a}_i \) the intersection of the straight lines containing \( a_i a_{i+1} \) and \( a_{i-1} a_{i-2} \) for \( i = 1, \ldots, 6 \) (mod 6), see Figure. We define the star \( S(H_A) \) over \( H_A \) as the union of \( H_A \) and six triangles \( T_i(H_A) = a_{i-1} \bar{a}_i a_i \), where \( i = 1, \ldots, 6 \) and where \( a_0 \) means \( a_6 \). From the convexity of \( A \) we conclude that \( A \subset S(H_A) \).

We do not make our considerations narrower assuming that the centroid of \( A \) is over or on the axis \( y = 0 \). Since our aim is to show that \( \operatorname{cen}_y(A) \leq \frac{4}{21} \) for every convex body \( A \), it is sufficient to consider only such convex bodies \( A \) which are disjoint with the interiors of \( T_4(H_A) \), \( T_5(H_A) \) and \( T_6(H_A) \). Still the closure of \( A \setminus \bigcup_{i=4}^6 T_i(H_A) \) is a convex body with \( H \) inscribed and the centroid at the same or higher level.

Provide any supporting straight line \( L_1 \) of \( A \) at \( a_1 \) and the symmetric (with respect to \( x = 0 \)) supporting line \( L_2 \) of \( A \) at \( a_2 \). Denote by \( u = (0, w) \) the intersection point of \( L_1 \) (and thus of \( L_2 \)) with the axis \( x = 0 \). Since the second coordinates of \( a \) and \( \bar{a}_2 \) are equal to 1 and 2, respectively, we have \( w \in [1, 2] \).

Since \( L_1 \) passes through \( u = (0, w) \) and \( a_1 = (1, 1) \), it has the equation \( y - 1 = (-w + 1)(x - 1) \). Its point of intersection with the segment \( a_6 \bar{a}_1 \) (being a subset of the straight line \( y = x - 2 \)) is \( m_1 = (\frac{2+w}{w}, \frac{2-w}{w}) \). Similarly, we get the symmetric point \( m_2 \) being the intersection of \( L_2 \) with the segment \( a_3 \bar{a}_3 \).
Later we explain the geometric meaning of the following number

\[ w_0 = \frac{1}{3}(\sqrt[3]{44 - 3\sqrt{177}} + \sqrt[3]{44 + 3\sqrt{177}} - 1) = 1.6589670\ldots \]

Parts 3–7 lead to the proof of our theorem for \( w \in [w_0, 2] \) and Part 8 for \( w \in [1, w_0] \).

**Part 1** where we introduce a heptagon and find its cen\( y \).

Let \( z \in [0, 1] \). Since \( a_1 = (1, 1) \) and \( m_1 = (\frac{2+w}{w}, \frac{2-w}{w}) \), every point \( p_1(z) \), or shortly \( p_1 \), of \( a_1m_1 \) has the form \((1-z)a_1 + zm_1\). So \( p_1 = ((1-z) + z\cdot\frac{2+w}{w}, (1-z) + z\cdot\frac{2-w}{w}) = (z\cdot\frac{2}{w} + 1, z\cdot\frac{2-2w}{w} + 1) \).

The symmetric point with respect to \( x = 0 \) is denoted by \( p_2 \). The second coordinates of them are \( z\cdot\frac{2-2w}{w} + 1 \).

Consider the heptagon \( G = up_2a_3a_4a_5a_6p_1 \). The area of each of the two symmetric wings \( W_1 = a_1a_5p_1 \) and \( W_2 = a_2a_3p_2 \) of \( G \) is \( z \cdot \frac{2-w}{w} \) and cen\( y \) of each wing of this heptagon is \( 1 + z\frac{2-2w}{w} + 1 = 2 + z\frac{2-2w}{w} \). The area of the triangle \( a_1ua_2 \) is \( \frac{1}{2} \cdot 2 \cdot (w - 1) = w - 1 \) and its cen\( y \) is \( 2 + w \). Moreover, the area of \( H_A \) is 6 and its cen\( y \) is 0. Taking all this into account and having in mind that \( G = H_A \cup a_1a_6p_1 \cup a_2a_3p_2 \cup a_1ua_2 \), by the right part of (1) we conclude that

\[
\text{cen}_y(G) = \frac{0 + \frac{2}{3}(2 + z\frac{2-2w}{w})\frac{2-w}{w} + \frac{2+w}{3}(w - 1)}{6 + 2z\frac{2-w}{w} + w - 1} \]

which, after a simplification, equals to

\[
\frac{2(2 + z\frac{2-2w}{w})\frac{2-w}{w} + w^2 + w - 2}{6z\frac{2-w}{w} + 3w + 15}. \quad (2)
\]
Part 2 whose aim is to show the following statement

Denote by $\nu$ the numerator and by $\delta$ the denominator of $\text{cen}_y(G)$ as in (2) (so $\text{cen}_y(G) = \frac{\nu}{\delta}$). Consider a truncation of the wings $W_i$ of $G$ to symmetric convex subsets $A_i = W_i \cap A$ for $i = 1, 2$. Put $V_i = W_i \setminus A_i$ for $i = 1, 2$ and $V = V_1 \cup V_2$. We have

$$\frac{\nu - \text{area}(V)\text{cen}_y(V)}{\delta - \text{area}(V)} \leq \frac{\nu}{\delta} \iff \text{cen}_y(V) \geq \frac{\nu}{\delta}. \quad (3)$$

Let us confirm this. We have $\frac{\nu - \text{area}(V)\text{cen}_y(V)}{\delta - \text{area}(V)} \leq \frac{\nu}{\delta} \iff \delta(\nu - \text{area}(V)\text{cen}_y(V)) \leq \nu(\delta - \text{area}(V)) \iff \nu \cdot \text{area}(V) \leq \delta \cdot \text{area}(V)\text{cen}_y(V) \iff \frac{\nu}{\delta} \leq \text{cen}_y(V)$ if $\text{cen}_y(V) \geq \frac{\nu}{\delta}$.

Observe that $\frac{\nu - \text{area}(V)\text{cen}_y(V)}{\delta - \text{area}(V)}$ is nothing else but $\text{cen}_y(A')$, where $A' = G \setminus V$.

Part 3 where we start considerations for $w \in [w_0, 2]$.

For every $w \in [w_0, 2]$ we are looking for the positions of $p_1$ and thus of $p_2$ such that $\text{cen}_y(G)$ is the largest. For this reason let us find the derivative of the function (2) with respect to $z$:

$$\frac{2(w - 2)[4z^2(-w^2 + 3w - 2) + 4z(2w^2 + 4w - 5) + (w^4 - w^3 - 12w^2)]}{3w(2w^2 - 2wz + 5w + 4z)^2}. \quad (4)$$

The discriminant of the quadratic function in the square bracket is $16w^2(2w^4 + 4w^3 - w^2 - 6w + 1)$. Hence (4) equals 0 for $z = \frac{w(w^2 + 4w - 5 + \sqrt{2w^4 + 4w^3 - w^2 + 6w + 1})}{2(w^2 - 3w + 2)}$. Take into account only the root

$$z_w = \frac{w(w^2 + 4w - 5 - \sqrt{2w^4 + 4w^3 - w^2 - 6w + 1})}{2(w^2 - 3w + 2)}. \quad (5)$$

which is positive for every $w \in [w_0, 2]$ (the other one is always negative here). Moreover, put $z_2 = \lim_{w \to 2^-} z_w$. This is $z_2 = \frac{5}{7}$.

We see that for any fixed $w \in [w_0, 2]$ the global maximum of (2) as a function of $z$ from the interval $[0, 1]$ can be only for $z = 0$, $z = z_w$ or $z = 1$. Substituting these three $z$ into (2) we see that the global maximum of (2) in the interval $[0, 1]$ is at $z = z_w$ for every fixed $w \in [w_0, 2]$.

Part 4 where our aim is to show that for each $w \in [w_0, 2]$ the value of (2) for $z = z_w$ is at most $\frac{4}{21}$.

This task with substituting $z = z_w$ into (2) seems to be very complicated to perform. We can get it around by performing the more general task to show that for every $w \in [w_0, 2]$ and $z \in [\frac{5}{7}, 1]$ we have

$$\frac{2(2 + z\frac{2 - w}{w})z \frac{2 - w}{w} + w^2 + w - 2}{6z \frac{2 - w}{w} + 3w + 15} \leq \frac{4}{21}. \quad (6)$$
This task is more general since $z_w$ belongs to $[\frac{5}{7}, 1]$ for every $w \in [w_0, 2]$. Really, the inequality $z_w \leq 1$ is equivalent to $w^4 - 7w^3 + 2w^2 + 16w^2 - 8w - 16 \geq 0$ and thus to $(w-1)(w-2)(w^3 + w^2 - 2w - 4) \geq 0$, which means that it holds true in $[1, 2]$ if and only if $w \in [w_0, 2]$ (still $w_0$ is the only real root of this polynomial). Moreover, the inequality $\frac{5}{7} \leq z_w$ is equivalent to $-7(w-2)(7w^2 + 22w + 20) \geq 0$, which means that it holds true in the whole interval $[1, 2]$, so in particular for every $w \in [w_0, 2]$.

Equivalently to (6), it is sufficient to show that

$$28z^2(w^2 - 3w + 2) + 20zw(2 - w) + w^2(7w^2 + 3w - 34) \quad (7)$$

is at most 0 for every point $(w, z)$ of the rectangle $[w_0, 2] \times \left[\frac{5}{7}, 1\right]$.

In order to simplify evaluations consider this task in the larger rectangle $[1, 2] \times \left[\frac{5}{7}, 1\right]$.

Let us apply the following method of finding the global maximum of a continuous function $f(w, z)$ in a polygon $R \subset E^2$. Namely, first we find the points being the solutions of the system of two equations when partial derivatives of our function $f(w, z)$ are 0 in the interior of $R$. Next we write the equations of the sides in the forms $z = g(w)$ or $w = g(z)$. We find the critical points in the relative interiors of each side, where the derivative of the respective equation is 0. Finally, we check the values of $f(w, z)$ at the vertices of $R$. The largest value at all the found points gives the maximum value of $f(w, z)$ in $R$.

In our particular case our function $f(w, z) = 28z^2(w^2 - 3w + 2) + 20zw(2 - w) + w^2(7w^2 + 3w - 34)$ is given by (7). Moreover, $R = [1, 2] \times \left[\frac{5}{7}, 1\right]$. According to the recalled method we find the partial derivatives $f'_w(w, z) = 28w^3 + 9w^2 + 56wz^2 - 40wz - 68w - 84z^2 + 40z$ and $f'_z(w, z) = 56w^2z - 20w^2 - 168wz + 40w + 112z$. Consider the system of equations when both are 0. Finding $z$ from the second and substituting to the first we get three solutions: $w \approx -1.8, w = 0$ and $w \approx 1.544$. None of them is in the interval $[w_0, 2]$. Hence the system of equations has no solution in our $R$, and thus in its interior.

Let us find the critical points in the relative interiors of the sides. After substituting $z = \frac{5}{7}$ to $f(w, z)$ we get $\frac{1}{7}(49w^4 + 21w^3 - 238w^2 - 100w + 200)$. Its derivative $\frac{1}{7}(196w^3 + 63w^2 - 476w - 100)$ is 0 only at $w_1 = 1.5103\ldots$. Placing $z = 1$ to our $f(w, z)$ we get $7w^4 + 3w^3 - 26w^2 - 44w + 56$. Its derivative $28w^3 + 9w^2 - 52w - 44$ equals 0 in $[1, 2]$ only at $w_2 = 1.5427\ldots$. Substituting $w = 1$ to $f(w, z)$ we get $20z - 24$, which is negative for every $z \in \left[\frac{5}{7}, 1\right]$. Placing $w = 2$ to $f(w, z)$ we get 0 for every $z \in \left[\frac{5}{7}, 1\right]$.

We have $f(w_1, \frac{5}{7}) \approx -23.803$, $f(w_2, 1) \approx -23.094$, $f(1, 1) = -4$, $f(1, \frac{5}{7}) = -9.714$, $f(2, \frac{5}{7}) = 0$, and $f(2, 1) = 0$. Thus the global maximum of $f(w, z)$ in $R$ is 0. Hence (7) is at most 0 and thus (6) holds true in $R$. We conclude that (2) for $z = z_w$ is at most $\frac{34}{21}$ for every $w \in [1, 2]$ and so for every $w \in [w_0, 2]$. 

\textit{Centroid of a planar convex body}
Part 5} where we show that \( \frac{\nu}{3} = \text{cen}_y(G) \). Looking at the second coordinates of \( a_1, a_6 \) and \( p_1 \) we get \( \text{cen}_y(a_1a_6p_1) = (z_w - \frac{2w}{w} + 1) / 2 \). Hence \( \text{cen}_y(V_1) \geq (z_w - \frac{2w}{w} + 1) / 2 \) and so \( \text{cen}_y(V) \geq (z_w - \frac{2w}{w} + 1) / 2 \).

We see that in order to confirm the promise of Part 5 it is sufficient to show that

\[
\frac{z_w - \frac{2w}{w} + 1}{2} \leq \frac{2(2 + z_w - \frac{2w}{w})z_w - \frac{2w}{w} + w^2 + w - 2}{6z_w - \frac{2w}{w} + 3w + 15},
\]

for every \( w \in [w_0, 2] \), where the right side is taken from (2). Instead, let us show the inequality

\[
\frac{z^2 - \frac{2w}{w} + 1}{2} \leq \frac{2(2 + z^2 - \frac{2w}{w})z - \frac{2w}{w} + w^2 + w - 2}{6z - \frac{2w}{w} + 3w + 15},
\]

or equivalently, let us show that

\[
8z^2 - 12z^2w - 22zw^2 + 26zw + 4zw^2 - 6zw^3 - 2w^4 + w^3 + 19w^2
\]

is at most 0 for every point \((w, z)\) of the piece of the curve \( z = z_w \) when \( w \in [w_0, 2] \).

Instead, let us find the global maximum of (9) in a triangle containing it. Namely, in the triangle \( T \) between the straight lines \( w = 2 \), \( z = -\frac{5}{7}w + \frac{15}{7} \), and \( z = 1 \). Its vertices are \((2, 1), \left(\frac{5}{7}, 1\right)\) and \((2, \frac{13}{21})\).

First let us show that the piece of the curve \( z = z_w \) for \( w \in [w_0, 2] \) is a subset of \( T \). The reason is that \(-\frac{5}{7}w + \frac{15}{7} \leq z_w \leq 1\) for every \( w \in [w_0, 2] \). The left inequality is equivalent to the inequality \((829w^4 - 1057w^3 + 18960w^2 - 27840w + 22472)(w - 1)(2 - w) \geq 0\) which holds true for every \( w \in (-\infty, \infty) \). Thus in [1, 2] and so for every \( w \in [\frac{5}{7}, 2] \). The right inequality \( z_w \leq 1 \) is shown just after (6).

Next let us find the global maximum of (9) in \( T \) by the method described in Part 4.

Consider the system of equations \(-8w^3 - 18w^2 + 3w^2 + 8wz^2 - 44wz + 38w - 12z^2 + 26z = 0\) and \(-6w^3 + 8w^2z - 12w^2 - 24wz + 26w + 16z = 0\) (where the left sides are the partial derivatives of (9)). Finding \( z = \frac{3w^3 + 11w^2 - 13w}{4w^2 - 12w + 8} \) from the second and substituting it into the first we get the equation \((68w^6 - 141w^5 - 262w^4 + 359w^3 + 356w^2 - 447w + 68) = 0\) whose solutions are \( w = 0, w \approx 0.183, w \approx 0.951, w \approx 1.037 \) and \( w \approx 2.614 \). All these \( w \) are out of the interval \([\frac{5}{7}, 2]\) which implies that all the obtained points \((w, z)\) are out of \( T \). Thus the system of equations has no solution in the interior of \( T \).

Look for critical points in the relative interiors of the sides. Substituting \( z = -\frac{5}{7}w + \frac{15}{7} \) into (9) we get \( \frac{1}{15}(212w^3 - 511w^2 - 789w + 1830) \). Its derivative \( \frac{1}{15}(848w^3 - 1533w^2 - 1578w + 1830) \) is never 0 in \([\frac{5}{7}, 2]\). Putting \( z = 1 \) into (9) we get \(-2w^4 + 5w^3 + w^2 + 22w \). Its derivative \(-8w^3 + 15w^2 + 2w + 22 \) is never 0 in \([\frac{5}{7}, 2]\). Putting \( w = 2 \) into (9) we get \( 8z^2 - 84z + 52 \). Its derivative \( 16z - 84 \) is never 0 in \([\frac{5}{7}, 1]\).
The value of (9) at (2, 1) is \(-44\), at \((\frac{5}{7}, 1)\) is \(-11.827\ldots\), and at \((2, \frac{13}{27})\) is \(-4.598\ldots\). So the global maximum of the function (9) in \(T\) is \(-11.827\ldots\). Hence (9) is always negative in \(T\).

Consequently, we have shown that (9) is at most 0 in \(T\) and thus that (8) is true for every \(w \in [w_0, 2]\). Therefore \(cen_y(V) \geq \frac{w}{5}\) for \(G\) with \(z_w\) in the part of \(z\).

**Part 6** where we show that \(cen_y(A') \leq \frac{4}{21}\) for \(w \in [w_0, 2]\).

Recall that \(cen_y(G) = \frac{w}{5}\). By Part 5 and by (3) we have \(\frac{\nu - \text{area}(V)cen_y(V)}{\delta - \text{area}(V)} \leq \frac{w}{5}\). The left side is \(cen_y(A')\) and the right one is \(cen_y(G)\) with \(G\) is taken for \(z = z_w\). By (6) it is at most \(\frac{4}{21}\). So \(cen_y(A') \leq \frac{4}{21}\).

**Part 7** on enlarging \(A'\) up to \(A\) which leads to the proof of our theorem for \(w \in [w_0, 2]\).

Put \(A'_1 = A \cap p_1a_6m_1, A'_2 = A \cap p_2a_3m_2\) and \(A'' = A''_1 \cup A''_2\). Clearly \(A = A' \cup A''\). We intend to show that adding \(A''\) to \(A'\) does not increase \(cen_y\), so that \(cen_y(A) \leq cen_y(A')\).

First let us show that if the triangles \(p_1a_6m_1\) and \(p_2a_3m_2\) are added to \(A'\), then \(cen_y\) does not increase. Applying the easy to show implication: “if \(\text{int}(X) \cap \text{int}(Y) = \emptyset\), \(cen_y(X) \leq \mu\) and \(cen_y(Y) \leq \mu\), then \(cen_y(X \cup Y) \leq \mu\) as well” and having in mind that \(cen_y(A') \leq \frac{4}{21}\) (see Part 6), it is sufficient to show that \(cen_y(p_1a_6m_1) \leq \frac{4}{21}\) (then also \(cen_y(p_2a_3m_2) \leq \frac{4}{21}\)).

Let us show this. Since \(cen_y(p_1a_6m_1) = (z_w \cdot \frac{2-4w}{w} + 1 + \frac{2-w}{w})/3\), we have to show that this is at most \(\frac{4}{21}\). This task is equivalent to \(7z_w(2 - 2w) \leq 4w - 14\). After substituting \(z_w\) and providing some simplifications, this inequality is equivalent to \(h(w) \geq 0\), where \(h(w) = 49w^6 - 196w^5 + 105w^4 + 1946w^2 + 1800w - 400\). We have \(h''(w) = 14(105w^4 - 280w^3 + 90w^2 + 278)\).

From the fact that \(h''(w)\) always positive we conclude that \(h'(w)\) is an increasing function. Thus from \(h(1) = 3304\) we see that \(h(w) \geq 0\) for \(w \geq 1\), and thus for every \(w \in [w_0, 2]\). Hence \(cen_y(p_1a_6m_1) \leq \frac{4}{21}\).

Also for adding only \(A''\) to \(A'\), the value of \(cen_y\) does not increase. The reason is that \(cen_y(A'_2) \leq cen_y(p_1a_6m_1)\) and \(cen_y(A''_2) \leq cen_y(p_2a_3m_2)\). The first follows from the convexity of \(A''_2\) and from the observation that every segment jointing \(a_6\) with a point of \(p_1m_1\) has in common with \(A''_2\) only a segment which is lower. Analogously, we confirm the second inequality.

We conclude that \(cen_y(A) \leq cen_y(A')\). This and \(cen_y(A') \leq \frac{4}{21}\) (see Part 6) imply \(cen_y(A) \leq \frac{4}{21}\).

**Part 8** where we prove our theorem for \(w \in [1, w_0]\).

Consider the pentagon \(P = m_1um_2a_4a_5\). It is the special case of \(G\) for \(z = 1\). Thus substituting \(z = 1\) to (2) we see that \(cen_y(P)\) equals to \(\frac{w^4 + w^3 - 2w^2 - 4w + 8}{5w(w^3 + 3w^2 + 4)}\). In order to show that this is at most \(\frac{4}{21}\) for every \(w \in [1, w_0]\) take into account the equivalent inequality \((w - 2)(7w^3 + 17w^2 + 8w - 28) \leq 0\). Its left side equals 0 only for \(w = 2\) and \(w = w_3 \approx 0.934\).
Consequently, this inequality and thus also the preceding one hold true in \([w_3, 2]\). Hence also in \([1, w_0]\). Resuming, \(\text{cen}_y(P) \leq \frac{4}{21}\).

From \(z = 1\) we see that \(V_1 = a_1a_6m_1\) for our \(P = G\). The second coordinates of \(a_1, a_6\) and \(m_1\) give \(\text{cen}_y(a_1a_6m_1) = \frac{4-2w}{3w}\). Hence \(\text{cen}_y(V_1) \leq \frac{4-2w}{3w}\). Thus by \(\text{cen}_y(V) = \text{cen}_y(V_1)\) we get \(\text{cen}_y(V) \leq \frac{4-2w}{3w}\).

In order to show that the right side of (3) is now true we have to show that \(\text{cen}_y(V) \geq \text{cen}_y(P)\), where, as \(P\) takes the role of \(G\), the right side is denoted by \(\frac{w}{3}\) in (3). Hence we have to show that \(\frac{4-2w}{3w} \geq \frac{w^4 + w^3 - 2w^2 - 4w + 8}{3w(w^2 + 3w + 4)}\). This is equivalent to the inequality \(w^4 + 3w^3 - 8w - 8 \leq 0\). A simple evaluation confirms that it is true in \([1, w_0]\) (by the way, we have here the equality just for \(w = w_0\)). Thus \(\text{cen}_y(V) \geq \text{cen}_y(P)\).

The shown inequality means that the right side of (3) is fulfilled. Hence the left side of (3), i.e., \(\text{cen}_y(A') \leq \frac{w}{3}\) holds true. From \(A = A'\) for our \(P = G\) we obtain \(\text{cen}_y(A) \leq \frac{w}{3}\). Consequently, from \(\frac{w}{3} = \text{cen}_y(P) \leq \frac{4}{21}\) we conclude that \(\text{cen}_y(A) \leq \frac{4}{21}\).

Thanks to results of Parts 7 and 8 the thesis of our theorem holds true.

The ratio \(\frac{4}{21}\) in Theorem cannot be lessened as it follows from the example of the pentagon \(\tilde{a}_2a_3a_4a_5a_6\) in the part of \(A\), and the hexagon \(a_1 \ldots a_6\) as \(H_A\). The author expects that there are no more such examples besides the affine images of the above presented one.

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