UNIMODULAR ROWS OVER MONOID RINGS

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ABSTRACT. For a commutative Noetherian ring \( R \) of dimension \( d \) and a commutative cancellative monoid \( M \), the elementary action on unimodular \( n \)-rows over the monoid ring \( R[M] \) is transitive for \( n \geq \max(d+2, 3) \). The starting point is the case of polynomial rings, considered by A. Suslin in the 1970s. The main result completes a project, initiated in the early 1990s, and suggests a new direction in the study of \( K \)-theory of monoid rings.

1. Introduction

An \( n \)-row \( \mathbf{a} = (a_1, \ldots, a_n) \) with entries in a commutative ring \( R \) is called unimodular if \( Ra_1 + \cdots + Ra_n = R \). If \( R \) is Noetherian then, using prime avoidance, one can pass from \( \mathbf{a} \) by elementary transformations to a unimodular row \( \mathbf{b} = (b_1, \ldots, b_n) \), such that the height of the ideal \( Rb_1 + \cdots + Rb_i \subset R \) is at least \( i \) for \( i = 1, \ldots, n \); see Section 2.1. Here an ‘elementary transformation’ means adding a multiple of a component to another component. In particular, if the (Krull) dimension of \( R \) is \( d \) and \( n \geq d + 2 \) then every unimodular \( n \)-row over \( R \) can be reduced, by elementary transformations, to \((1, 0, \ldots, 0)\). This is the basis of the classical Serre Splitting and Bass Cancellation Theorems [1, Ch. 4]. Our main result is

**Theorem 1.1.** Let \( R \) be a commutative Noetherian ring of dimension \( d \) and \( M \) be a commutative cancellative (not necessarily torsion free) monoid. Then the elementary action on unimodular \( n \)-rows over \( R[M] \) is transitive for \( n \geq \max(d+2, 3) \).

The starting point is Suslin’s result in [19] that, for \((R, d)\) as above and arbitrary \( r \in \mathbb{N} \), the elementary action on the set of unimodular \( n \)-rows over the polynomial ring \( R[t_1, \ldots, t_r] \) is transitive whenever \( n \geq \max(d+2, 3) \). The polynomial ring \( R[t_1, \ldots, t_r] \) is the monoid ring, corresponding to a free commutative monoid of rank \( r \). Theorem 1.1 completes the project, initiated in [7, 8], where the transitivity was shown for a restricted class of monoids. Theorem 1.1 extends the class of monoids in [7, 8] in the same way as the general convex polytopes extend the class of stacked polytopes [3] (called the ‘polytopes of simplicial growth’ in [8]); see Section 2.5 for the relationship between monoids and polytopes. The class of stacked polytopes is negligibly small within the general convex polytopes.

We now describe consequences and research directions, suggested by Theorem 1.1. Theorem 1.1 is new already when \( R \) is a field, in which case it implies that \( K_1(R[M]) \) is described by Mennnicke symbols [1, Ch. 6]. One would like to know

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what properties of \( K_1(R[M]) \) can be inferred from this fact – as the works [9, 14] show, \( K_1(R[M]) \) exhibits many interesting phenomena.

Using the Quillen induction for projective modules [15, Ch. V.3], one easily deduces from Theorem 1.1 that, for \( R \) and \( M \) as in the statement and \( M \) torsion free with non trivial units, every finitely generated projective \( R[M] \)-module of rank greater than \( d \), which is stably extended from \( R \), is in fact extended from \( R \). For regular \( R \) and seminormal \( M \) this is shown in [21, Corollary 1.4], and for a Dedekind ring \( R \) and a torsion free monoid \( M \) this follows from [21, Theorem 1.5]. It is very likely that the techniques of [2, 18] can be combined with the proof of Theorem 1.1 to yield the full blown \( K_0 \)-part of [10, Conjecture 2.4], claiming that finitely generated projective \( R[M] \)-modules of rank \( \geq \max(d + 1, 2) \) are cancellative and split off free summands. This would extend [2, 18] in the same way as Theorem 1.1 extends Suslin’s mentioned result. It is also worth mentioning that a new proof of Anderson’s conjecture [6] can be derived from Theorem 1.1 and [13], following the outline in [4, Exercises 8.7, 8.8]. Ideologically, this is the same approach, though.

Theorem 1.1 is proved by developing a unimodular row version of the induction techniques, which we call the pyramidal descent. Three versions of this techniques for the corresponding \( K \)-theoretical objects, defined over monoid rings, were developed in [6, 12, 13]. A restricted version of the pyramidal descent for unimodular rows was developed in [7, 8]. The main obstruction for proving the general transitivity of the elementary action was the non-existence of special endomorphisms in monoid rings. Here we find a new approach, allowing to circumvent this difficulty by suitably lifting critical steps to covering polynomial rings, where there is a ubiquity of endomorphisms, and even drop the torsion freeness assumption for monoids. The unimodular row version is different from the previous pyramidal descents in that no lifting to polynomial rings is used in [6, 13, 12]. It also suggests a possible approach to [10, Conjecture 2.4] on stabilizations of all higher \( K \)-groups of \( R[M] \): one could try to show, based on [20, 23], that the mentioned stabilizations are no worse than those for \( R[t_1, \ldots, t_r] \), \( r \in \mathbb{N} \). This would reduce the general case to the free commutative monoids. At present, the appropriate techniques for polynomial rings only exists for \( K_0 \) [2, 18], \( K_1 \) [19], and \( K_2 \) [22]. Of course, Theorem 1.1 already implies the conjectured surjective \( K_1 \)-stabilization.

By a combination of the techniques in Section 6.1 and [21, §15], one easily extends Theorem 1.1 to rings of the form \( R[M]/I \), where \( I \) is generated by a subset of \( M \).

Finally, when \( R \) is a field, the proof of Theorem 1.1 can be converted, along the lines in [16], into an algorithm that, for a given unimodular row over \( R[M] \), finds elementary transformations, leading to \((1, 0, \ldots, 0)\).

A word on the organization of the paper: in Sections 2–6 we overview previous results and make series of reductions in the general case; Section 7 allows involvement of the general coefficient rings, without which Section 8 would lead to the special case of Theorem 1.1 when \( R \) is field; in Section 9 we explain how to involve the monoids with torsion.

Notation: \( \mathbb{N} = \{1, 2, \ldots\} \) and \( \gg \) is for ‘sufficiently larger than’.
2. Unimodular rows, monoids, polytopes

2.1. Unimodular rows. All information we need on unimodular rows (except the case \( l > 0 \) of Theorem 2.2 below), including a detailed exposition of [19], is in [15].

All our rings are commutative and with unit. The elementary subgroup \( E_n(A) \subset \text{GL}_n(A) \) of the general linear group over \( A \) is generated by the elementary \( n \times n \)-matrices, i.e., the matrices which differ from the identity matrix in at most one nonzero off-diagonal entry. The following two statements are, respectively, Lemma 2.1 and Theorem 2.2:

**Lemma 2.1.** For any ring \( A \) and \( n \geq 3 \), \( E_n(A) \) is normal in \( \text{GL}_n(A) \).

**Theorem 2.2.** Assume \( R \) is a Noetherian ring of dimension \( d \) and \( k, l, n \in \mathbb{N} \) with \( n \geq \max(d + 2, 3) \). Then the elementary action on unimodular \( n \)-rows over the Laurent polynomial ring \( R[t_1, \ldots, t_k, s_1^{\pm 1}, \ldots, s_l^{\pm 1}] \) is transitive.

For a ring \( A \), the set of unimodular rows of length \( n \) will be denoted by \( \text{Um}_n(A) \). The group \( E_n(A) \) acts on \( \text{Um}_n(A) \) by multiplication on the right. More precisely, the right multiplication of an elementary matrix \( I_n + aE_{ij} \) corresponds to adding the \( a \)-multiple of the \( i \)-th component to the \( j \)-th component.

For two elements \( f, g \in \text{Um}_n(A) \) and a subring \( S \subset A \), we write \( f \sim_S g \) if \( f \) and \( g \) are in a same orbit of the \( E_n(S) \)-action. The standard row \((1, 0, \ldots, 0)\) will be denoted by \( e \). In particular, \( f \sim_S e \) means that there a matrix \( \varepsilon \in E_n(S) \), whose first row equals \( f \).

In order to avoid confusion between rows and ideals, the ideal in \( A \), generated by elements \( a_1, \ldots, a_n \), will be denoted by \( Aa_1 + \cdots + Aa_n \).

For an ideal \( I \) in a ring \( A \), its height \( \text{ht } I = \text{ht}_A I \) is the maximal height of a minimal prime ideal over \( I \). The height of the unit ideal is set to be \( \infty \). For an ideal \( J \subset A[t] \), the ideal of leading coefficients of elements in \( J \) will be denoted by \( L(I) \).

**Lemma 2.3.** Let \( A \) be a Noetherian ring.

(a) For any element \( a \in \text{Um}_n(A) \), there exists \( b = (b_1, \ldots, b_n) \in \text{Um}_n(A) \), such that \( a \sim_A b \) and \( \text{ht}_A (Ab_1 + \cdots + Ab_n) \geq i \) for all \( i \).

(b) For an ideal \( J \subset A[t] \), we have \( \text{ht}_A L(J) \geq \text{ht}_{A[t]} J \).

These are, respectively, Lemmas 3.4 and 3.2 in [15, Ch. III].

2.2. Cones and polytopes. For generalities on polytopes and cones, we refer the reader to [4, Ch. 1]. Here we only recall a few basic facts and conventions.

For a subset \( X \subset \mathbb{R}^r \), its convex hull will be denoted by \( \text{conv}(X) \) and the affine hull will be denoted by \( \text{Aff}(X) \).

All our polytopes are assumed to be convex. Let \( \mathbb{R}_+ \) denote the non-negative reals. By convention, a cone will refer to a rational, finite, pointed cone in \( \mathbb{R}^r \) for some \( r \), i.e., a subset of the form \( C = \mathbb{R}_+ z_1 + \cdots + \mathbb{R}_+ z_n \subset \mathbb{R}^r \) for some \( z_1, \ldots, z_n \in \mathbb{Z}^r \), which contains no nontrivial subspace.

For a nonzero cone \( C \subset \mathbb{R}^r \), there is a rational affine hyperplane \( \mathcal{H} \subset \mathbb{R}^r \setminus \{0\} \), such that \( C = \mathbb{R}_+(C \cap \mathcal{H}) \) [4, Proposition 1.21]. In this case \( C \cap \mathcal{H} \) is a rational polytope of dimension \( \dim C - 1 \); i.e., the vertices of \( C \cap \mathcal{H} \) belong to \( \mathbb{Q}^r \).
A pyramid with apex $v$ and base $P$ means $\text{conv}(v, P)$, where $v \notin \text{Aff}(P)$.

The relative interior of a polytope $P$ will be denoted by $\text{int}(P)$. By convention, $\text{int}(P) = P$ when $P$ is a point.

For two polytopes $P \subset Q$, sharing a vertex $v$, we say that $Q$ is tangent to $P$ at $v$ if $\dim P = \dim Q$ and the corner cones, spanned by $P$ and $Q$ at $v$, coincide.

2.3. **Affine monoids.** A detailed information on the monoids of interest can be found in [4, Ch. 2]. Below we give a quick review.

All our monoids are commutative, cancellative, and with unit.

The maximal subgroup of a monoid $M$ will be denoted by $U(M)$.

The group of differences of a monoid $M$, also known as the Grothendieck group of $M$, will be denoted by $\text{gp}(M)$. Thus $M$ embeds into $\text{gp}(M)$ and $\text{gp}(-)$ is a left adjoint of the embedding of the category of monoids into that of abelian groups.

The additive monoid of nonnegative integers will be denoted by $\mathbb{Z}_+$. Unless specified otherwise, (i) we use additive notation for the operation in a monoid $M$ but switch to the multiplicative notation when $M$ is considered inside the monoid ring $R[M]$, and (ii) we will make the natural identifications $R[\mathbb{Z}_+] = R[t]$ and $R[\mathbb{Z}_+^r] = R[t_1, \ldots, t_r]$.

The submonoid of a monoid $M$, generated by elements $m_1, \ldots, m_n$, will be denoted by $\mathbb{Z}_+ m_1 + \cdots + \mathbb{Z}_+ m_n$.

A monoid $M$ is torsion free if $\text{gp}(M)$ has no nonzero torsion. An affine monoid is a finitely generated torsion free monoid. Every affine monoid $M$ isomorphically embeds into $\mathbb{Z}_+^r$, where $r = \text{rank}(\text{gp}(M))$. We put $\text{rank}(M) = \text{rank}(\text{gp}(M))$.

If $M$ is affine and $U(M) = 0$, then $M$ is said to be positive.

**Lemma 2.4.** Assume $M$ is a monoid, $\text{gp}(M) = \mathbb{Z}_+^r$, and $U(M) = 0$.

(a) (Gordan Lemma) $M$ is affine if and only if $\mathbb{R}_+ M$ is a cone in $\mathbb{R}^r$.

(b) If $M$ is affine then, for any ring $R$, the monoid ring $R[M]$ admits a grading $R[M] = R \oplus R_1 \oplus \cdots$ where the elements of $M$ are homogeneous.

These are proved in Proposition 2.17(f) and Corollary 2.10(a) in [4].

An affine positive monoid has the smallest generating set – the set of indecomposable elements in $M$. It is called the Hilbert basis of $M$ and denoted by $\text{Hilb}(M)$.

2.4. **Normal and seminormal monoids.** A torsion free monoid $M$ is normal if $x \in \text{gp}(M)$ and $nx \in M$ for some $n \in \mathbb{N}$ imply $x \in M$. The normalization of a torsion free monoid $M$ is the smallest normal submonoid $n(M) \subset \text{gp}(M)$, containing $M$, i.e., $n(M) = \{x \in \text{gp}(M) \mid nx \in M \text{ for some } n \in \mathbb{N}\}$. If $M$ is affine then $n(M)$ is also affine.

**Lemma 2.5.** Let $M$ be an affine positive monoid and $\text{gp}(M) = \mathbb{Z}_+^r$.

(a) There exists $m \in M$ such that $m + n(M) \subset M$.

(b) $M$ is normal if and only if $M = (\mathbb{R}_+ M) \cap \mathbb{Z}_+^r$.

(c) There exists a basis $n_1, \ldots, n_r \in \mathbb{Z}_+^r$, such that $M \subset \mathbb{Z}_+ n_1 + \cdots + \mathbb{Z}_+ n_r$.

(d) The relative interior $\text{int}(\mathbb{R}_+ M)$ contains a basis $\{m_1, \ldots, m_r\} \subset \mathbb{Z}_+^r$.

These are the statements 2.33, 2.24, 2.17(e), and 2.74 in [4], respectively.
A torsion free monoid $M$ is seminormal if $x \in \text{gp}(M)$ and $2x, 3x \in M$ imply $x \in M$ or, equivalently, $x \in \text{gp}(M)$ and $nx \in M$ for all $n \gg 0$ imply $x \in M$. The seminormalization $\text{sn}(M)$ is the smallest seminormal monoid in $\text{gp}(M)$, containing $M$. Explicitly, $\text{sn}(M) = \{x \in \text{gp}(M) \mid nx \in M \text{ for all } n \gg 0\}$. For $M$ affine, $\text{sn}(M)$ is also affine.

2.5. $\Phi$-correspondence. To describe relations between submonoids of an affine positive monoid $M$ we will follow the following conventions:

- The group $\text{gp}(M)$ will be thought of as $\mathbb{Z}^r$, where $r = \text{rank}(M)$;
- In $\mathbb{R}^r$ there will be implicitly (sometimes explicitly) chosen a rational affine hyperplane $\mathcal{H} \subset \mathbb{R}^r \setminus \{0\}$, such that $\mathbb{R}_+ M = \mathbb{R}_+ (\mathbb{R}_+ M \cap \mathcal{H})$;
- $\Phi(L) := (\mathbb{R}_+ L) \cap \mathcal{H}$ for a submonoid $L \subset M$ and $\Phi(m) := (\mathbb{R}_+ m) \cap \mathcal{H}$ for a nonzero element $m \in M$;
- For a convex subset $P \subset \Phi(M)$ we introduce the submonoid
  $$M(P) = \{m \in M \setminus \{0\} \mid \Phi(m) \in P \} \cup \{0\} \subset M;$$

- If $M$ is normal and $P \subset \mathcal{H}$ is a convex subset then we introduce the submonoid
  $$M(P) = \{z \in \mathbb{Z}^r \setminus \{0\} \mid (\mathbb{R}_+ z) \cap \mathcal{H} \in P \} \cup \{0\}.$$  

(By Lemma 2.5(b), the last two notations are compatible when $P \subset \Phi(M)$.)

Gordan Lemma can be rephrased as follows: a nonzero submonoid $L$ of an affine positive monoid $M$ is affine if and only if $\Phi(L)$ is a polytope.

For an affine positive normal monoid $M$, an element $m \in M$ is an extremal generator if $\Phi(m)$ is a vertex of $\Phi(M)$ and $m$ is the generator of $(\mathbb{R}_+ m) \cap M \cong \mathbb{Z}_+$. Thus, $M$ has as many extremal generators as there are the vertices of $\Phi(M)$.

For an affine positive nonzero monoid $M$, we define the interior submonoid and interior ideal of $M$ by $M_* = M(\text{int}(\Phi(M)))$ and $\text{int}(M) = M_* \setminus \{0\}$, respectively. Because of the convention $\text{int}(P) = P$ for $P$ a point (Section 2.2), we have $M_* = M$ when $\text{rank}(M) = 1$.

**Lemma 2.6.** Let $M$ be an affine positive monoid of rank $r$.

(a) If $M$ is normal and $m \in M$ is an extremal generator then $\mathbb{Z}m + M = \mathbb{Z}m + M_0 \cong \mathbb{Z}m \times M_0$ for an affine positive normal submonoid $M_0 \subset \mathbb{Z}^r$ of rank $r - 1$.

(b) $M$ is seminormal if and only if $M(F)_* = n(M(F))_*$ for every face $F \subset \Phi(M)$ (including $\Phi(M)$ itself).

These are, respectively, Propositions 2.32 and 2.40 in [4].

3. Pyramids and monoids

3.1. Pyramidal decomposition. Pyramidal descent is based on the following

**Definition 3.1.** Let $M$ be an affine positive normal monoid and $m \in M$ be an extremal generator. A representation $M = M(\Delta) \cup M(\Gamma)$ is called a pyramidal decomposition with vertex $m$ if $\Delta, \Gamma \subset \Phi(M)$ are rational polytopes, such that:

(a) $\Phi(M) = \Delta \cup \Gamma$,

(b) $\Delta$ is a pyramid with base $\Delta \cap \Gamma$,  

Lemma 3.2. Let $M$ be an affine positive normal monoid, $M = M(\Delta) \cup M(\Gamma)$ be a pyramidal decomposition, and $\deg : M \to \mathbb{Z}_+$ be an associated pyramidal degree. Consider a system of linear combinations $l_i = \sum_{j=1}^r d_{ij} n_j$, $i = 1, \ldots, s$, where

- $n_j \in M \setminus \{0\}$ and $d_{ij} \in \mathbb{Z}_+$ for all $i, j$,
- $n_1$ is the vertex of the decomposition,
- $(d_{11}, \ldots, d_{1r}) > (d_{21}, \ldots, d_{2r}) > \ldots$ in the lexicographical order,

Assume $d_{ij} = a_{ij} + b_{ij}$, where $a_{ij}, b_{ij} \in \mathbb{Z}_+$, $i = 2, \ldots, s$ and $j = 1, \ldots, r$. Then

$$\deg \left( \sum_{j=1}^r d_{1j} c_j n_1 \right) > \deg \left( \sum_{j=1}^r (a_{ij} n_j + b_{ij} c_j n_1) \right), \quad i = 2, \ldots, s,$$

for $c_1 \gg c_2 \gg \ldots \gg c_r \gg 0$.

Proof. We can assume $c_j > \deg(n_j)$ for $j = 2, \ldots, s$. Then it is enough to achieve

$$\deg \left( \sum_{j=1}^r d_{1j} c_j n_1 \right) > \deg \left( \sum_{j=1}^r d_{ij} c_j n_1 \right), \quad i = 2, \ldots, s,$$

which is equivalent to the inequalities

$$\sum_{j=1}^r d_{1j} c_j > \sum_{j=1}^r d_{ij} c_j, \quad i = 2, \ldots, s,$$

known to be satisfied for $c_1 \gg c_2 \gg \ldots \gg c_r \gg 0$. \qed

3.2. Admissible configurations.

Definition 3.3. For an affine positive normal monoid $M$ of rank $r$ and an extremal generator $m \in M$, a triple $(\mathcal{H}, \Delta_1, \Delta_2)$ is called an admissible configuration if $\mathcal{H} \subset \mathbb{R}^r \setminus \{0\}$ is a rational affine hyperplane, such that $\mathbb{R}_+ M = \mathbb{R}_+ ((\mathbb{R}_+ M) \cap \mathcal{H})$, and $\Delta_1, \Delta_2 \subset \mathcal{H}$ are rational pyramids with apex $\Phi(m)$, satisfying the conditions:

- (a) $\Phi(M) \subset \Delta_1 \subset \Delta_2$,
- (b) $\Delta_1$ and $\Delta_2$ are tangent to $\Phi(M)$ at $\Phi(m)$,
- (c) $M(\Delta_1) = \mathbb{Z}_+ m + M(F_1)$, where $F_1 \subset \Delta_1$ is the facet, opposite to $\Phi(m)$,
- (d) $M(\Delta_2) = \mathbb{Z}_+ m + M(F_2)$, where $F_2 \subset \Delta_2$ is the facet, opposite to $\Phi(m)$,
- (e) $F_2 \cap \Phi(M) = \emptyset$.

In the notation above, $M(F_1)$ and $M(F_2)$ are isomorphic monoids and, consequently, so are the monoids $M(\Delta_1)$ and $M(\Delta_2)$. 

(c) $\Phi(M)$ is tangent to $\Delta$ at $\Phi(m)$ (Section 2.2).

The pyramidal decomposition is non-degenerate if $\dim \Gamma = \dim \Delta$.

For a pyramidal decomposition $M = M(\Delta) \cup M(\Gamma)$, there exists a monoid homomorphism $\deg : M \to \mathbb{Z}_+$, such that $0 \neq \deg(M(\Delta)) \subset \mathbb{Z}_+$ and $\deg(M(\Gamma)) \subset \mathbb{Z}_-$, where $\mathbb{Z}_-$ is the set of nonnegative integers. Such a map will be called a pyramidal degree, associated to the given decomposition. The pyramidal decomposition is non-degenerate if an only if $0 \neq \deg(M(\Gamma))$. 

(c) $\Phi(M)$ is tangent to $\Delta$ at $\Phi(m)$ (Section 2.2).
Lemma 3.4. An admissible configurations exists for any affine positive normal monoid $M$ with $\text{rank}(M) \geq 2$ and any extremal generator $m \in M$.

Proof. By Lemma 2.6(a), $\mathbb{Z}m + M = \mathbb{Z}m + M_0 \cong \mathbb{Z}m \times M_0$ for an affine positive normal submonoid $M_0 \subset \text{gp}(M)$. Let $n_1, \ldots, n_{r-1}$ be a basis of $\text{gp}(M_0) \cong \mathbb{Z}^{r-1}$, such that $M_0 \subset \mathbb{Z}+n_1 + \cdots + \mathbb{Z}+n_{r-1}$ (Lemma 2.5(c)). Then, for $k \gg 0$, the following triple is an admissible configuration:

$$
\mathcal{H} = \text{Aff}(m, n_1 - (k+1)m, \ldots, n_{r-1} - (k+1)m) \subset \mathbb{R}^r,
$$

$$
\Delta_1 = (\mathbb{R}_+m + \mathbb{R}_+(n_1 - km) + \cdots + \mathbb{R}_+(n_{r-1} - km)) \cap \mathcal{H},
$$

$$
\Delta_2 = \text{conv}(m, n_1 - (k+1)m, \ldots, n_{r-1} - (k+1)m).
$$

4. Patching unimodular rows

We say that a commutative square of ring homomorphisms

$$
\begin{array}{ccc}
A & \rightarrow & A_1 \\
\downarrow & & \downarrow \\
A_2 & \rightarrow & A'
\end{array}
$$

has the Milnor patching property for unimodular rows if for every natural number $n$ and every element $f \in \text{Um}_n(A_1)$, whose image $f' \in \text{Um}_n(A')$ satisfies $f' \sim e$, there exists $g \in \text{Um}_n(A)$ with the image $g' \in \text{Um}_n(A_1)$, satisfying $g' \sim f$.

A Karoubi square is a commutative square of rings of the following type

$$
\begin{array}{ccc}
A & \rho & \rightarrow & B \\
\downarrow & & \downarrow & \downarrow \\
S^{-1}A & \rightarrow & \rho(S)^{-1}A'
\end{array}
$$

where $S \subset A$ is a multiplicative subset, $S$ acts regularly on $A$, $\rho(S)$ acts regularly on $B$, and the homomorphism $A/sA \rightarrow B/\rho(s)B$ is an isomorphism for every $s \in S$. A Karoubi square is always a pull-back diagram.

Lemma 4.1. (Milnor Patching) The following two types of commutative squares of ring homomorphisms have the Milnor patching property:

(a) $\begin{array}{ccc}
A & \rightarrow & A_1 \\
\downarrow & & \downarrow \\
A_2 & \rightarrow & A'
\end{array}$ a pull-back diagram, where either $\pi$ or $\sigma$ is surjective,
(b) \[ \begin{array}{c}
A \xrightarrow{\rho} B \\
S^{-1}A \xrightarrow{S^{-1}\rho} \rho(S)^{-1}B
\end{array} \] a Karoubi square.

This is proved in [8, Proposition 9.1(a)], with the details for one sketched step included in [11, Lemma 8]. The basis is the equality \( E_n(A') = E_n(A_1) E_n(A_2) \), obvious for the case (a) and proved in [24, Lemma 2.4] for the case (b). (Vorst actually shows \( E_n(A') = E_n(A_2) E_n(A_1) \), but the two equalities are equivalent.)

Let \( A = A_0 \oplus A_1 \oplus \cdots \) be a graded ring and \( f \in U_m(A) \). Let \( f(0) \) denote the image of \( f \) in \( U_m(A_0) \) under the augmentation \( A \to A_0 \). In this notation we have

**Proposition 4.2.** (Quillen Patching) Assume \( f(0) = e \). Then \( f \sim_A e \) if and only if \( f_\mu \sim_{A_\mu} e \) for every maximal ideal \( \mu \subseteq A \).

This is [7, Corollary 7.4]. It is based on Suslin’s \( K_1 \)-analogue [19, Theorem 3.1] of Quillen’s well-known local-global patching for projective modules [17, Theorem 1].

5. **Integral and subintegral extensions**

Let \( M = M(\Delta) \cup M(\Gamma) \) be a pyramidal decomposition of an affine positive normal monoid \( M \) with vertex \( m \) and \( \deg : M \to \mathbb{Z} \) be an associated pyramidal degree. Call an element \( f = \sum_{j=1}^k r_j m_j \in R[M] \) monic (with respect to \( \deg \)) if \( m_1 = um^c \) for some \( u \in U(R) \), \( c \in \mathbb{N} \), and \( \deg(m_1) > \deg(m_j) \) for \( j = 2, \ldots, k \).

Call a ring homomorphism \( A \to B \) integral if \( B \) is integral over \( \text{Im}(A) \).

**Lemma 5.1.** Let \( M = M(\Delta) \cup M(\Gamma) \) be a pyramidal decomposition, \( \deg : M \to \mathbb{Z}_+ \) be an associated pyramidal degree, and \( f \in R[M] \) be a monic element. Then the ring homomorphism \( R[M(\Gamma)] \to R[M]/fR[M] \) is integral.

**Proof.** Since \( M \) is normal, every element \( g \in R[M] \) admits a representation \( g = fg_1 + f_1 \), where \( g_1, f_1 \in R[M] \) with \( \deg(f_1) < \deg(f) \). In particular, as an \( R[M(\Gamma)] \)-module, \( R[M]/fR[M] \) is generated by the image of the subset \( M_{<\deg(f)} := \{ m \in M \mid \deg(m) < \deg f \} \) and, therefore, by the image of \( M_{\leq\deg(f)} := \{ m \in M \mid \deg(m) \leq \deg f \} \). But, according to [4, Theorem 2.12], \( RM_{\leq\deg(f)} \) is a finitely generated \( R[M(\Gamma)] \)-module. \( \square \)

5.1. **Subintegral extensions.** Call an extension of rings \( A \subseteq B \) elementary subintegral if \( B = A[b] \) for some \( b \in B \) with \( b^2, b^3 \in A \). A ring extension \( A \subseteq B \) is subintegral if it is a filtered union of elementary subintegral extensions.

**Theorem 5.2.** For a subintegral extension of rings \( A \subseteq B \) and an element \( f \in U_m(A) \), where \( n \geq 3 \), we have \( f \sim_A e \) if and only if \( f \sim_B e \).
This is the main result of [11]. It represents the ‘unimodular row’ counterpart of [21, Theorem 14.1]. Although the statement is about arbitrary rings, the proof in [11] uses monoid rings. More precisely, it uses the main result of [7]. A stronger result in the context of the Euler class groups under subintegral extensions was later derived in [5].

6. Reduction to the interior of normal monoids

Lemma 6.1. In order to prove Theorem 1.1 for torsion free monoids of rank $r$, it is sufficient to prove it in the special case when $R$ is a local Noetherian ring and $M = N_*$ for an affine positive normal monoid $N$ with rank$(N) \leq r$.

Proof. Let $L$ be a torsion free monoid. Since $L$ is the inductive limit of its affine submonoids, we can assume that $L$ is itself affine. Consider the pullback diagram

$$
\begin{array}{ccc}
R[(L \setminus U(L)) \cup \{1\}] & \longrightarrow & R[L] \\
\downarrow & & \downarrow \pi \\
R & \longrightarrow & R[U(L)] \\
\end{array}
\quad \text{ker } \pi = R(L \setminus U(L)).
$$

Since $U(L)$ is a free abelian group, Theorem 2.2 yields the transitivity of the elementary actions on $\text{Um}_n(R[U(L)])$. Applying Lemma 4.1(a) and using that $(L \setminus U(L)) \cup \{1\}$ as the union of its affine submonoids, we can also assume that $L$ is an affine positive monoid. Since $\text{sn}(L)$ is a filtered union of subintegral extensions of monoids, by Theorem 5.2 we can further assume $L$ is seminormal.

Since the elementary action on $\text{Um}_n(R)$ is transitive for $n \geq d + 2$, Lemma 2.4(a) and Proposition 4.2 make it possible to reduce the general case to $R$ local.

Let $F_1, \ldots, F_k$ be the set of non-empty faces of the polytope $\Phi(L)$, including $\Phi(L)$ itself, indexed in such a way that $i \leq j$ implies $\text{dim } F_i \leq \text{dim } F_j$. In particular, $F_k = \Phi(L)$. We have the pull-back diagrams of $R$-algebras with the natural horizontal injective maps:

$$
\begin{array}{ccc}
R[L(F_i)_*] & \longrightarrow & R[L]/(\bigcup_{j>i}(R\text{int}(L(F_j)))) \\
\downarrow & & \downarrow \pi_i \\
(\mathbb{D}_i) & \longrightarrow & R[L]/(\bigcup_{j \geq i}(R\text{int}(L(F_j)))) \\
\end{array}
\quad \text{ker } \pi_i = R(L \cap \text{int}(F_i)).
$$

(We assume $\bigcup_{j>k}(R\text{int}(L(F_j))) = 0$.)

By Lemma 2.6(b), for very $i$, the ring at the upper-left corner of $(\mathbb{D}_i)$ is of the type $R[N_*]$ for $N$ affine, positive, and normal. Moreover, the ring at the upper-right
corner of \((\mathbb{D}_k)\) is \(R[L]\) and that at the lower-right corner of \((\mathbb{D}_1)\) is \(R\). Consequently, Lemma 4.1(a) allows induction on \(i\).

7. **Quasi-Monic Elements**

For a commutative ring \(B\), two subsets \(\alpha, \beta \subset B\), and elements \(x, y \in B\) we write

\[ x + \alpha y^\infty \subset \beta \]

if for every element \(a \in \alpha\) and all \(c > 0\), depending on \(a\), we have \(x + ay^c \in \beta\).

**Lemma 7.1.** Assume \(x, y \in B\) satisfy \(Bx + By = B, \alpha \subset B\) is an ideal, and \(\nu_1, \ldots, \nu_p \subset B\) are prime ideals. If \(x + \alpha y^\infty \subset \bigcup_{i=1}^p \nu_i\) then \(Bx + \alpha \subset \nu_i\) for some \(i\).

In the special case when \(y \in U(B)\), this is [15, Lemma 7.4]:

\[ x + \alpha \subset \bigcup_{i=1}^p \nu_i \implies x + \alpha \in \nu_i \text{ for some } i, \]

which is used in the proof of Lemma 2.3(a). In the general case one inducts on the smallest \(p\) for which Lemma 7.1 is false and the same argument as in [15, Lemma 7.4], with obvious adjustments, goes through.

**Corollary 7.2.** Let \(B\) be a Noetherian ring of dimension \(d\), \(a = (b_1, \ldots, b_n) \in \text{Um}_n(B)\) for some \(n \geq 2\), and \(Bb_1 + By = B\). Then for any finite family of prime ideals \(\nu_1, \ldots, \nu_p \subset B\) there exists an elements \(b \in Bb_2 + \cdots + Bb_n\) and an infinite sequence of natural numbers \(c_1 < c_2 < \ldots\), such that

\[ b_1 + by^{c_i} \notin \nu_i, \quad i = 1, \ldots, p, \quad j = 1, 2, \ldots \]

**Definition 7.3.** Let \(R\) be a rings. An \(R\)-subalgebra (not necessarily Noetherian or monomial) \(A \subset R[\mathbb{Z}_+^r]\) is called \(t_1\)-tilted if there is a neighborhood \(\Phi(t_1) \subset U \subset \Phi(\mathbb{Z}_+^r)\) such that \(m \in \mathbb{Z}_+^r \setminus \{0\}\) and \(\varphi(m) \in U\) imply \(m \in A\).

For an element \(g \in \text{Um}_n(R[t_1, \ldots, t_r])\) we denote by \(g|_{t_1=0}\) the image of \(g\) in \(\text{Um}_n(R[t_2, \ldots, t_r])\) after substituting 0 for \(t_1\).

**Lemma 7.4.** Let \(R\) be a local Noetherian ring of dimension \(d\). Assume \(A \subset R[\mathbb{Z}_+^r]\) is a \(t_1\)-tilted subalgebra and \(g = (g_1, \ldots, g_n) \in \text{Um}_n(A)\) for some \(n \geq 2\), such that \(g|_{t_1=0} \in \text{Um}_n(R)\). Then there exists \(h = (h_1, \ldots, h_n) \in \text{Um}_n(A)\), for which \(g \sim_A h\) and \(\text{ht}_{R[\mathbb{Z}_+^r]}(R[\mathbb{Z}_+^r]h_1 + \cdots + R[\mathbb{Z}_+^r]h_i) \geq i\) for all \(i\).

**Proof.** For elements \(l = (l_1, \ldots, l_n) \in \text{Um}_n(A)\) we introduce the condition

\[ R[\mathbb{Z}_+^r]l_1 + R[\mathbb{Z}_+^r]l_2 = \cdots = R[\mathbb{Z}_+^r]l_{n-1} + R[\mathbb{Z}_+^r]l_1 = R[\mathbb{Z}_+^r]. \]

Since \(R\) is local and \(g|_{t_1=0} \in \text{Um}_n(R)\), by a suitable elementary transformation over \(R\) we can achieve \(g_1|_{t_1=0} = \cdots = g_{n-1}|_{t_1=0} = 1\). Consequently, without loss of generality, we can assume that \(g\) satisfies (1).

Let \(0 \leq k < n\). Assume there exists \(h' = (h'_1, \ldots, h'_n) \in \text{Um}_n(A)\), for which

\[ g \sim_A h', \]

. the condition (1) is satisfied.
the components satisfy the inequalities in Lemma 7.4 for $i < k$ (when $k > 1$).

We will induct on $k$ to achieve the height inequalities for all $i \leq n - 1$; there is nothing to prove for $i = n$.

Without loss of generality, $R[Z^+_1]h'_1 + \cdots + R[Z^+_k]h'_{k-1} \neq R[Z^+_k]$. Put

$$\{\nu_1, \ldots, \nu_p\} = \begin{cases} \text{the minimal primes over } R[Z^+_n]h'_1 + \cdots + R[Z^+_k]h'_{k-1}, \text{ if } k > 1, \\ \text{the minimal primes in } R[Z^+_k], \text{ if } k = 1. \end{cases}$$

Since $h'$ satisfies (1), Corollary 7.2 implies the existence of an element

$$\tilde{h} = f_1h'_1 + \cdots + f_{k-1}h'_{k-1} + f_{k+1}h'_{k+1} + \cdots + f_nh'_n,$$

and an infinite sequence of natural numbers $c_1 < c_2 < \cdots$ such that

$$h'_k + \tilde{h}t^{c_j}_1 \notin \nu_s, \quad s = 1, \ldots, p, \quad j = 1, 2, \ldots$$

Since $h'_1, \ldots, h'_{k-1} \in \nu_s$ for every $s$, the element $h' = f_{k+1}h'_{k+1} + \cdots + f_nh'_n$ satisfies (2)

$$h'_k + h't^{c_j}_1 \notin \nu_s, \quad s = 1, \ldots, p, \quad j = 1, 2, \ldots$$

Since $A$ is $t_1$-tilted, for $c \gg 0$ we also have

$$h't^{c_j}_1 \in Ah'_{k+1} + \cdots + Ah'_n.$$

The last inclusion implies that, for $j \gg 0$, we have

$$h' \sim_A (h'_1, \ldots, h'_{k-1}, h'_k + h't^{c_j}_1, h'_{k+1}, \ldots, h'_n).$$

In view of (2), the minimal primes over the ideal

$$R[Z^+_n]h'_1 + \cdots + R[Z^+_k]h'_{k-1} + R[Z^+_k]\left(h'_k + h't^{c_j}_1\right) \subset R[Z^+_k]$$

are not among the $\nu_s$. In particular, the unimodular row on the right of (3) satisfies (1) and the height inequalities for $i = 1, \ldots, k$. \hfill \Box

For an element $f \in R[Z^+_k]$, its leading term in the lexicographical order with respect to $t_1 > \ldots > t_r$ will be denoted by $L(f)$. Call an element $f \in R[Z^+_k]$ quasi-monic is $L(f) = um$ for some $u \in U(R)$ and $m \in Z^+_k$.

**Corollary 7.5.** Let $R$ be a local Noetherian ring of dimension $d$. Assume $A \subset R[Z^+_k]$ is a $t_1$-tilted $R$-subalgebra and $g = (g_1, \ldots, g_n) \in \text{Um}_n(A)$ for some $n \geq \max(d + 2, 2)$. Assume $g|_{t_1=0} \in \text{Um}_n(R)$. Then there exists $h = (h_1, \ldots, h_n)$, such that $h_n$ is quasi-monic and $g \sim_A h$.

**Proof.** By Lemma 7.4, without loss of generality we can assume $\text{ht}(I) \geq d + 1$ for $I = R[Z^+_k]g_1 + \cdots + R[Z^+_k]g_{d+1}$. Then Lemma 2.3(b), applied to the decreasing sequence of rings $R[t_1, t_2, \ldots, t_r] \supset R[t_2, \ldots, t_r] \supset \cdots \supset R[t_r]$, implies that $I$ contains a quasi-monic element $g$. Assume $L(g) = um$ for some $u \in U(R)$ and $m \in Z^+_k$. Then, using that $A$ is $t_1$-tilted, we can take $h = (g_1, \ldots, g_{n-1}, g_n + t^c_1g)$ with $c \gg 0$. In fact, for $c \gg 0$ we have $L(g_n + t^c_1g) = t^c_1L(g)$ and $t^c_1g \in Ag_1 + \cdots + Ag_{d+1}$. \hfill \Box
8. Pyramidal descent

8.1. Local pyramidal descent. For the rest of Section 8.1, we assume that:

- \((R, \mu)\) is a local Noetherian ring of dimension \(d\),
- \(n \geq \max(d + 2, 3)\),
- \(M\) is a positive, affine, normal monoid with \(\text{rank}(M) \geq 2\),
- \(M = M(\Delta) \cup M(\Gamma)\) is a non-degenerate pyramidal decomposition with vertex \(m\),
- \((\mathcal{H}, \Delta_1, \Delta_2)\) is an admissible configuration for the pair \((M, m)\),
- \(m \subset R[M(\Gamma)]\) is the maximal ideal generated by \((M(\Gamma) \setminus \{1\}) \cup \mu\).

Consider the following commutative diagram of rings:

\[
\begin{array}{cccc}
A_2 & & R[M] \\
\downarrow & & \downarrow \\
A_1 & \to & R[M] & \to & R[M(\Delta_1)] \\
R[Z^*_+] & \xrightarrow{\pi_1} & R[M(\Delta_1)] & \xrightarrow{\pi_2} & R[M(\Delta_2)] \\
R[Z^*_+] & \downarrow & & \downarrow \\
\end{array}
\]

where:

- \(r = \# \text{Hilb}(M(\Delta_1)) = \# \text{Hilb}(M(\Delta_2))\) (see Section 2.3 for Hilbert bases),
- the \(R\)-algebra homomorphisms \(\pi_i\) \((i = 1, 2)\) are induced by surjective monoid homomorphisms \(h_i : Z^*_+ \to M(\Delta_i)\), satisfying the conditions: \(h_1(t_1) = h_2(t_1) = m\) and the following triples of points are collinear
  \[
  \Phi(m), \Phi(h_1(t_j)), \Phi(h_2(t_j)) \in \mathcal{H}, \quad j = 2, \ldots, r,
  \]
- the inner and outer squares are pull-back diagrams,
- the slanted arrows are the induced rings embeddings.

We will keep the identification \(R[Z^*_+] = R[t_1, \ldots, t_r]\) at the lower-left corner of the outer square, and think of \(Z^*_+\) at the lower-left corner of the inner square as the multiplicative monoid, generated by \(t_1, t_1 t_2^{k_2}, \ldots, t_1 t_r^{k_r}\) for appropriate \(k_2, \ldots, k_r \in \mathbb{N}\) so that the slanted arrows in (4) become the identity embeddings.

Denote \(Z^*_+(M) = h_2^{-1}(M)\). Then \(A_2 = R \left[Z^*_+(M) + \ker \pi_2 \right] \subset R[Z^*_+]\), the \(R\)-subalgebra generated by \(Z^*_+(M) \cup \ker \pi_2\).

Observe that the subalgebra \(A_2 \subset R[Z^*_+]\) is \(t_1\)-tilted. It is neither a finitely generated nor a monomial \(R\)-algebra as soon as \(\ker \pi_2 \neq 0\), i.e., when \(M(\Delta_2)\) or, equivalently, \(M(\Delta_1)\) is not a free monoid.

**Theorem 8.1.** Assume the elementary action on \(U_{m_n}(R[\Delta_1])\) is transitive. Then \(f_m e \sim e \) for every \(f \in U_{m_n}(R[M])\).
Proof. Let \( f = (f_1, \ldots, f_n) \in \text{Um}_n(R[M]). \) Since \( f \stackrel{R[M(\Delta_1)]}{\sim} e, \) Lemma 4.1 implies the existence of \( g = (g_1, \ldots, g_n) \in \text{Um}_n(A_1) \) such that \( \pi_1(g) \stackrel{R[M]}{\sim} f. \) We can assume \( \pi_1(g) = f. \)

Considering \( g \) as an element of \( \text{Um}_n(A_2), \) we have \( g|_{c_1 = 0} \in \text{Um}_n(R). \) (This is where we use admissible configurations: the corresponding condition may not be satisfied over \( A_1. \)) By Corollary 7.5, without loss of generality we can assume that

\[
g_n \text{ is quasi-monic, not in } R.
\]

Fix a rational subsimplex \( \Delta_0 \subset \Phi(Z^d_+(M)) \) such that \( \Phi(Z^d_+(M)) \) is tangent to \( \Delta_0 \) at \( \Phi(t_1) \) and \( Z^r_+(\Delta_0) \cong Z^r_+. \) This can be done by choosing

\[
\Delta_0 = \text{conv}\left( \Phi(t_1), \Phi(t_1 t_2^k), \ldots, \Phi(t_1 t_d^k) \right).
\]

for some \( k \gg 0. \) We have \( Z^r_+(\Delta_0) \subset Z^r_+(M). \)

By Theorem 2.2, there exists \( \varepsilon \in E_n(R[Z^r_+(M)]), \) whose first row is \( g. \)

For an \( r \)-tuple of natural numbers \( (c_1, \ldots, c_r), \) consider the \( R \)-algebra endomorphism

\[
\tau = \tau(c_1, \ldots, c_r) : R[Z^r_+] \rightarrow R[Z^r_+], \quad \tau(t_j) = t_j + t_j^{c_j}, \quad j = 1, \ldots, r.
\]

The crucial observation is that the non-unit monomials in the reduced forms of the entries of the matrix \( \varepsilon^{-1} \tau(\varepsilon) \) align in the direction of \( t_1 \in Z^d_+ \subset R^d_+ \) as \( c_1, \ldots, c_d \rightarrow \infty. \) This implies

\[
\alpha := \varepsilon^{-1} \tau(\varepsilon) \in \text{GL}_n(R[Z^d_+(\Delta_0)]) \subset \text{GL}_n(A_2) \text{ for } c_1, \ldots, c_r \gg 0.
\]

By Lemma 2.1 and Theorem 2.2, there exist \( \varepsilon_0 \in E_n(R[Z^d_+(\Delta_0)]) \) and \( \alpha_0 \in \text{GL}_{n-1}(R[Z^d_+(\Delta_0)]), \) such that

\[
\alpha = \begin{pmatrix} \alpha_0 & 0 \\ 0 & 1 \end{pmatrix} \varepsilon_0.
\]

Thus, for \( c_1, \ldots, c_r \gg 0, \) we have

\[
\tau(g) = g \cdot \alpha \sim g \cdot \begin{pmatrix} \alpha_0 & 0 \\ 0 & 1 \end{pmatrix} := (h_1, \ldots, h_{n-1}, \tau(g_n)) \in \text{Um}_n(A_2).
\]

Consequently, for every \( c_1, \ldots, c_r \gg 0, \) there exist \( f'_1, \ldots, f'_{n-1} \in R[M] \) such that

\[
f \stackrel{R[M]}{\sim} (f'_1, \ldots, f'_{n-1}, \pi(\tau(g_n)) \in \text{Um}_n(R[M]).
\]

Pick a pyramidal degree \( \deg : M \rightarrow \mathbb{Z}_+, \) associated to the decomposition \( M = M(\Delta) \cup M(\Gamma). \) Because of (5), Lemma 3.2 implies that \( \pi(\tau(g_n)) \) is monic for \( c_1 \gg \ldots \gg c_1 \gg 0. \) By Lemma 5.1, the endomorphism \( \tau = \tau(c_1, \ldots, c_r) : R[Z^r_+] \rightarrow R[Z^r_+] \) gives rise to the integral ring homomorphism \( R[M(\Gamma)]_m \rightarrow R[M]_m/(\pi(\tau(g_n))). \) Since \( R[M] \) is finitely generated over \( R \) and \( R[M(\Gamma)]_m \) is local, the quotient ring \( R[M]_m/(\pi(\tau(g_n))) \) is semi-local. Thus \( f_m \sim e \) over the ring \( R[M]_m/(\pi(\tau(g_n))), \) where the ‘bar’ refers to mod \( \pi(\tau(g_n)) \) [15, Corollary 7.5]. By lifting the involved elementary transformation to \( R[M]_m, \) one derives the desired equivalence \( f_m \stackrel{R[M]_m}{\sim} e; \) see Proposition 5.6 [15, Ch. 1].  \[\square\]
8.2. Pyramidal descent and Theorem 1.1 for torsion free monoids. The following claim is derived in the proof of [21, Proposition 10.3]:

**Lemma 8.2.** Let $M$ be an affine positive normal monoid and $M = M(\Delta) \cup M(\Gamma)$ be a non-degenerate pyramidal decomposition. Then for a local ring $(R, \mu)$ and the maximal ideal $m = R \int(M(\Gamma)) + \mu \subset R[M(\Gamma)_s]$, the following diagram is a Karoubi square:

$$
\begin{array}{c}
R[M(\Gamma)_s] \rightarrow R[M_s] \\
| \\
R[M(\Gamma)_s]_m \rightarrow R[M_s]_m
\end{array}
$$

A sequence of rational polytopes $\{P_i\}_{i=1}^\infty$ is called *admissible* if, for every $i$, either $P_{i+1} = \Gamma_i$, where $P_i = \Delta_i \cup \Gamma_i$ as in the $\Phi$-images of non-degenerate pyramidal decompositions, or $P_i \subset P_{i+1} \subset P_1$. The following is [6, Lemma 2.8]:

**Lemma 8.3.** For a rational polytope $P_1$ and a neighborhood $U \subset P_1$, there exists an admissible sequence of polytopes $\{P_i\}_{i=1}^\infty$ with $P_i \subset U$ for $i \gg 0$.

For an affine positive normal monoid $M$ of rank $r$ we define its *complexity* as the smallest $k$ for which $M \cong \mathbb{Z}_+^{r-k} \times M_0$. Observe that, (i) the monoid $M_0$ is also affine normal positive, and (ii) the monoids of 0 complexity are the free monoids. The proof of the following lemma is straightforward

**Lemma 8.4.** Assume $M = \mathbb{Z}_+^{r-k} \times M_0$ is an affine positive normal monoid of complexity $k$ and $m \in M_0$ is an extremal generator. If $(\mathcal{H}, \Delta_1, \Delta_2)$ is an admissible configuration, corresponding to the extremal generator $(0, m) \in M$, then the complexity of $M(\Delta_1)$ is strictly less than $k$.

Notice, however, that the basis elements of $\mathbb{Z}_+^{r-k} \times \{0\} \subset M$ may not be in $\text{Hilb}(M(\Delta_2))$.

**Proof of Theorem 1.1 for torsion free monoids.** By Lemma 6.1, it is enough to consider a local coefficient ring $(R, \mu)$ and a monoid of type $M_s$, where $M$ is an affine positive normal monoid. We will induct on the pairs

$$(\text{rank}(M), \text{complexity of } M) \in \mathbb{N} \times \mathbb{Z}_+,$$

ordered lexicographically.

When $\text{rank}(M) = 1$ then $M = M_s \cong \mathbb{Z}_+$ and Theorem 2.2 applies. For complexity 0 monoids of arbitrary rank the we are done by the same theorem.

Assume the claim has been shown for all monoid of rank $< r$, where $r \geq 2$, and for all rank $r$ monoids of complexity $< k$.

Assume $\text{rank}(M) = r$ and the complexity of $M$ is $k > 0$. Without loss of generality, $M = \mathbb{Z}_+^{r-k} \times M_0$.

Let $m \in M_0$ be an extremal generator and $M = M(\Delta) \cup M(\Gamma)$ be a pyramidal decomposition with vertex $(0, m)$, and $f \in \text{Um}_a(R[M_s])$. By appropriate polytopal
approximations of open convex sets from within, there exists an affine positive normal submonoid \( N \subset M_\ast \) and a pyramidal decomposition \( N = N(\bar{\Delta}) \cup N(\bar{\Gamma}) \), such that \( f \in U_{n_n}(R[N]) \) and \( \Gamma \subset \Gamma \). By Lemma 8.4 and the induction hypothesis, Theorem 8.1 implies \( f_n \sim e \), where \( n = R(N \setminus \{1\}) + \mu \). Since \( R[N]_n \subset R[M_\ast]_m \), we have \( f_m \sim e \).

By Lemmas 4.1(b) and 8.2, \( f \sim g \) for some \( g \in U_{n_n}(R[M(\Gamma)_\ast]) \). In other words, elements of \( U_{n_n}(R[M_\ast]) \) satisfy the pyramidal descent.

By Lemma 2.5(d), there exists a free submonoid \( M_0 \subset M_\ast \) with \( \text{gp}(M_0) = \text{gp}(M) \). By Lemma 8.3, there exists an admissible sequence of rational polytopes \( \{P_i\}_{i=1}^\infty \), such that \( P_i = \Phi(M) \) and \( P_i \subset \text{int}(\Phi(M_0)) \) for \( i \gg 0 \). Then the pyramidal descent for unimodular rows implies the existence of elements \( h_i \in U_{n_n}(R[M(P_i)_\ast]) \) with \( h_i \sim f \). Since \( R[M(P_i)_\ast] \subset R[M_0] \subset R[M_\ast] \), there exists \( g \in U_{n_n}(R[M_0]) \) with \( g \sim f \) for \( i \gg 0 \), and we are done by Theorem 2.2.

9. Monoids with torsion

Unless specified otherwise, a monoid in this section means a commutative and cancellative monoid, possibly with torsion, i.e., no longer is the group \( \text{gp}(M) \) assumed to be torsion free.

For a monoid \( L \), let \( t(L) \) denote the the torsion subgroup of \( \text{gp}(L) \). For a monoid \( L \), denote \( L = \text{Im}(L \to \text{gp}(L)/t(L)) \). The correspondence \( L \mapsto L \) is a left adjoint functor for the embedding functor of the category of monoids without torsion into that of monoids.

The normalization \( n(L) \) and seminormalization \( sn(L) \) of a monoid \( L \) is defined in the same way as for the class of torsion free monoids in Section 2.4.

For a finitely generated monoid \( L \), we will identify \( \text{gp}(L) \) with \( \text{gp}(\bar{L}) \times t(L) \). As before, we think of \( \text{gp}(\bar{L}) \) as \( \mathbb{Z}^r \), where \( r = \text{rank}(\text{gp}(L)) \).

We need several facts, starting with an extention of Lemma 2.6(b) to monoids with torsion.

**Lemma 9.1.** Let \( L \) be a finitely generated monoid with trivial \( U(L) \) and \( \mathbb{F}(L) \) be the set of faces of the cone \( \mathbb{R}_+ \bar{L} \), including 0 and \( \mathbb{R}_+ \bar{L} \). For every \( F \in \mathbb{F}(L) \), there is a subgroup \( T_F \subset t(L) \), such that:

1. \( sn(L) = \bigcup_{\bar{F}(L)} (\text{int}(n(\bar{L} \cap F)) \times T_F) \);
2. \( T_{\mathbb{R}_+ \bar{L}} = t(L) \);
3. \( T_{F_1} \subset T_{F_2} \) whenever \( F_1 \subset F_2 \).

**Proof.** The extension of monoids \( L \subset n(\bar{L}) \times t(L) \) is integral, i.e., every element of \( n(\bar{L}) \times t(L) \) has a positive multiple in \( L \). In particular, \( n(\bar{L}) = n(\bar{L}) \times t(L) \). In fact, for any element \( (l,t) \in n(\bar{L}) \times t(L) \), there exists \( c_1 \in \mathbb{N} \) such that \( c_1 l \in L \). Assume \((c_1 l, s) \in L \) for some \( s \in t(L) \). Then \( c_1(c_2) \cdot (l, t) = (c_1 c_2 l, 0) \in L \) for any multiple \( c_2 \) of the orders of the elements \( c_1 t, s \in t(L) \).

Since \( n(L) \) is a finitely generated monoid, \( \mathbb{Q}[n(L)] \) is a finitely generated \( \mathbb{Q}[L] \)-algebra. (Despite the use of monoid rings here, we continue using additive notation
for the monoid operation.) But $\mathbb{Q}[L] \subset \mathbb{Q}[n(L)]$ is also an integral extension. Therefore, $\mathbb{Q}[n(L)]$ is module-finite over $\mathbb{Q}[L]$. There is a finite generating subset of this module

\[
\{(l_{i1}, t_{i1}) - (l_{i2}, t_{i2}) \mid (l_{ij}, t_{ij}) \in L, \; i = 1, \ldots, k, \; j = 1, 2\} \subset n(L).
\]

We have $(\lambda, \tau) + n(L) \subset L$, where $(\lambda, \tau) = \sum_{i=1}^{k} (l_{i2}, t_{i2}) \in L$. (In the torsion-free case, we have just recovered Lemma 2.5(a).) Consequently, all sufficiently high multiples of every element $(t, l) \in \text{int}(n(\bar{L})) \times t(L)$ are in $(\lambda, \tau) + n(L)$. Hence, \(\text{int}(n(\bar{L})) \times t(L) \subset \text{sn}(L)\). The same argument, applied to any face $F \subset \mathbb{R}_+ \bar{L}$, yields

\[
\text{int}(n(\bar{L} \cap F)) \times T_F = \{(l, t) \in \text{sn}(L) \mid l \in \bar{L} \cap F\}
\]

for the subgroup

\[
T_F = \{t \in t(L) \mid (l, t) \in \text{sn}(L) \text{ for some } l \in \bar{L} \cap F\} \subset t(L).
\]

This implies (a,b), and the part (c) follows from the inclusion $(l + l', t) \in \text{int}(n(\bar{L} \cap F_2)) \times T_{F_2}$ for any two elements $(l, t) \in \text{int}(n(\bar{L} \cap F_1)) \times T_{F_1}$ and $l' \in \text{int}(n(\bar{L} \cap F_2))$. \(\square\)

**Lemma 9.2.** In order to prove Theorem 1.1, it is enough to prove it when $U(M)$ is trivial.

**Proof.** We can assume $M$ is finitely generated. Let $U(M) = \mathbb{Z}^s \times H$ for a finite group $H$. Since $\dim R = \dim R[H]$, the elementary action on the unimodular $n$-rows over $R[U(M)] = R[H][\mathbb{Z}^s]$ is transitive by Theorem 2.2. The pullback diagram, similar to the first square in the proof of Lemma 6.1, reduces the transitivity question to the submonoid $(M \setminus U(M)) \cup \{1\} \subset M$. \(\square\)

**Lemma 9.3.** Assume $R$ is a local Noetherian ring of dimension $d$, $n \geq \max(d+2, 3)$, and $B \subset R[t]$ is an $R$-subalgebra, containing $t^m$ for some $m$. Then the elementary action on $\text{Um}_n(B)$ is transitive.

**Proof.** For an integral extension of Noetherian rings $R_1 \subset R_2$ and an ideal $I \subset R_2$, a standard commutative algebra argument yields $\text{ht}_{R_1}(I \cap R_1) \geq \text{ht}_{R_2} I$. (See, for instance, Step 2 in the proof of [7, Lemma 6.5].)

Let $f = (f_1, \ldots, f_n) \in \text{Um}_n(B)$. By Lemma 2.3(a), we can assume $\text{ht}(Bf_1 + \cdots + Bf_{d+1}) \geq d + 1$. Because of the integral extension $R[t^m] \subset B$, the height inequality above and Lemma 2.3(b) imply $L((Bf_1 + \cdots + Bf_{d+1}) \cap R[t^m]) = R$. In particular, $(Bf_1 + \cdots + Bf_{d+1}) \cap R[t^m]$ contains a monic polynomial $g$. Without loss of generality, $\deg(g) > 0$. Since $f \sim_B (f_1, \ldots, f_{n-1}, f_n + g^c)$ and $f_n + g^c$ is monic for $c \gg 0$, we can further assume that $f_n$ is monic.

Since $R$ is Noetherian and $R[t]$ is module-finite over $R[t^m]$, the algebra $B$ is also module-finite over $R[t^m]$. In particular, $B$ is a finitely generated $R$-algebra and, in particular, Noetherian. The extension $B \subset R[t]$ is integral. So, every maximal ideal in $B$ lifts to $R[t]$. The extension $R \to R[t]/(f_n)$ is integral and $R$ is local, implying $R[t]/(f_n)$ is semilocal. Since there are only finitely many maximal ideals in $R[t]$, containing $f_n$, the number of maximal ideals in $B$, containing $f_n$, is also finite, i.e., $B/(f_n)$ is semi-local. By Proposition 5.6 and Corollary 7.5 in [15, Ch. I], $f \sim_B e$. \(\square\)
For two monoids $L_1, L_2$ with $U(L_1) = \{0\}$, we introduce the following submonoid

$$L_1 \Join L_2 = L_1 \times L_2 \setminus \{(0, x) \mid x \in L_2, x \neq 0\} \subset L_1 \times L_2.$$ 

**Proof of Theorem 1.1.** By Lemma 9.2, we can assume that $U(M)$ is trivial.

Using Lemma 9.1 and the obvious adjustment of the argument in the proof of Lemma 6.1, the problem reduces to the transitivity of the elementary action over monoid rings of the form

$$R \left[ M \Join (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_p}) \right], \quad n_1, \ldots, n_p \in \mathbb{N},$$

where $M$ is torsion free (not necessarily affine) monoid with trivial $U(M)$. We will induct on $p$, where the base case $p = 0$ means the torsion free case, already considered in Section 8.2.

Assume the transitivity holds for $p - 1$. At this points we can also assume $M$ is an affine positive monoid. In this case, pulling back a grading $R[M] = R \oplus R_1 \oplus \cdots$ as in Lemma 2.4(b) along the projection $M \Join (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_p}) \to M$, we obtain a grading $R \left[ M \Join (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_p}) \right] = R \oplus S_1 \oplus \cdots$ and so, by Proposition 4.2, $R$ can be assumed to be local.

Consider the following pull-back diagram with the vertical identity embeddings:

\[
\begin{array}{ccc}
\Lambda & \longrightarrow & R \left[ M \Join (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_p}) \right] \\
\downarrow & & \downarrow \\
R[\mathbb{Z}_+] \left[ M \Join (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{p-1}}) \right] & \xrightarrow{\pi} & R[\mathbb{Z}_{n_p}] \left[ M \Join (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{p-1}}) \right]
\end{array}
\]

where:

- $\Lambda = A + B$,
- $A = R \left[ (M \Join \mathbb{Z}_+) \Join (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_{p-1}}) \right]$,
- $B$ is the subalgebra $R[t^{np}, t^{np+1} - t, \ldots, t^{2np-1} - t^{np-1}] \subset R[\mathbb{Z}_+]$,
- $\pi$ is induced by $t \mapsto x$ for some generator $x \in \mathbb{Z}_{n_p}$.

Since $\dim R[\mathbb{Z}_{n_p}] = d$, the induction hypothesis implies that the elementary action on the unimodular $n$-rows over the ring at the lower-right corner of the diagram above is transitive. Therefore, by Lemma 4.1(a), it is enough to show that the elementary action on $\text{Um}_n(\Lambda)$ is also transitive.

The very last pull-back diagram to be used by us is the following:

\[
\begin{array}{ccc}
A & \longrightarrow & \Lambda \\
\downarrow & & \downarrow \\
R & \longrightarrow & B
\end{array}
\]
with the vertical surjective $R$-homomorphisms, induced by $M \times \mathbb{Z}_+ \to 0 \in R$, and the natural injective horizontal maps. Since $R$ local, the elementary action on $\text{Um}_m(B)$ is transitive by Lemma 9.3. But the elementary action on $\text{Um}_n(A)$ is also transitive by the induction hypothesis because $m \times \mathbb{Z}_+$ is torsion free. Thus, Lemma 4.1(a) completes the argument.

Question. Does Theorem 1.1 extend to all commutative monoids?

References

[1] Hyman Bass. *Algebraic K*-theory, W. A. Benjamin, Inc., New York-Amsterdam, 1968.
[2] S. M. Bhatwadekar and A. Roy. Some theorems about projective modules over polynomial rings. *J. Algebra*, 86(1):150–158, 1984.
[3] Arne Borel. *An introduction to convex polytopes*, volume 90 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983.
[4] Winfried Bruns and Joseph Gubeladze. *Polytopes, rings, and K-theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2009.
[5] Mrinal Kanti Das and Md. Ali Zinna. On invariance of the Euler class groups under a subintegral base change. *J. Algebra*, 398:131–155, 2014.
[6] J. Gubeladze. The Anderson conjecture and a maximal class of monoids over which projective modules are free. *Mat. Sb. (N.S.)*, 135(177)(2):169–185, 271, 1988.
[7] J. Gubeladze. The elementary action on unimodular rows over a monoid ring. *J. Algebra*, 148(1):135–161, 1992.
[8] J. Gubeladze. The elementary action on unimodular rows over a monoid ring. II. *J. Algebra*, 155(1):171–194, 1993.
[9] J. Gubeladze. Nontriviality of $\text{SK}_1(R[M])$. *J. Pure Appl. Algebra*, 104(2):169–190, 1995.
[10] J. Gubeladze. $K$-theory of affine toric varieties. *Homology Homotopy Appl.*, 1:135–145, 1999.
[11] J. Gubeladze. Subintegral extensions and unimodular rows. In *Geometric and combinatorial aspects of commutative algebra* (Messina, 1999), volume 217 of *Lecture Notes in Pure and Appl. Math.*, pages 221–225. Dekker, New York, 2001.
[12] J. Gubeladze. The nilpotence conjecture in $K$-theory of toric varieties. *Invent. Math.*, 160(1):173–216, 2005.
[13] J. Gubeladze. The Steinberg group of a monoid ring, nilpotence, and algorithms. *J. Algebra*, 307(1):461–496, 2007.
[14] A. Krishna and H. P. Sarwar. $K$-theory of monoid algebras and a question of Gubeladze. Preprint, 2016. [https://arxiv.org/abs/1610.01825](https://arxiv.org/abs/1610.01825).
[15] T. Y. Lam. *Serre’s problem on projective modules*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006.
[16] Reinhard C. Laubenbacher and Cynthia J. Woodburn. An algorithm for the Quillen-Suslin theorem for monoid rings. *J. Pure Appl. Algebra*, 117/118:395–429, 1997. Algorithms for algebra (Eindhoven, 1996).
[17] D. Quillen. Projective modules over polynomial rings. *Invent. Math.*, 36:167–171, 1976.
[18] Ravi A. Rao. A question of H. Bass on the cancellative nature of large projective modules over polynomial rings. *Amer. J. Math.*, 110(4):641–657, 1988.
[19] A. A. Suslin. The structure of the special linear group over rings of polynomials. *Izv. Akad. Nauk SSSR Ser. Mat.*, 41(2):235–252, 477, 1977.
[20] A. A. Suslin. Stability in algebraic $K$-theory. In *Algebraic $K$-theory, Part I (Oberwolfach, 1980)*, volume 966 of *Lecture Notes in Math.*, pages 304–333. Springer, Berlin, 1982.
[21] Richard G. Swan. Gubeladze’s proof of Anderson’s conjecture. In *Azumaya algebras, actions, and modules* (Bloomington, IN, 1990), volume 124 of *Contemp. Math.*, pages 215–250. Amer. Math. Soc., Providence, RI, 1992.
[22] M. S. Tulenbaev. The Steinberg group of a polynomial ring. *Mat. Sb. (N.S.*), 117(159)(1):131–144, 1982.

[23] Wilberd van der Kallen. Homology stability for linear groups. *Invent. Math.*, 60(3):269–295, 1980.

[24] T. Vorst. The general linear group of polynomial rings over regular rings. *Comm. Algebra*, 9(5):499–509, 1981.

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