External Optimal Control of Nonlocal PDEs

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Abstract. Very recently Warma [42] has shown that for nonlocal PDEs associated with the fractional Laplacian, the classical notion of controllability from the boundary does not make sense and therefore it must be replaced by a control that is localized outside the open set where the PDE is solved. Having learned from the above mentioned result, in this paper we introduce a new class of source identification and optimal control problems where the source/control is located outside the observation domain where the PDE is satisfied. The classical diffusion models lack this flexibility as they assume that the source/control is located either inside or on the boundary. This is essentially due to the locality property of the underlying operators. We use the nonlocality of the fractional operator to create a framework that now allows placing a source/control outside the observation domain. We consider the Dirichlet, Robin and Neumann source identification or optimal control problems. These problems require dealing with the nonlocal normal derivative (that we shall call interaction operator). We create a functional analytic framework and show well-posedness and derive the first order optimality conditions for these problems. We introduce a new approach to approximate, with convergence rate, the Dirichlet problem with nonzero exterior condition. The numerical examples confirm our theoretical findings and illustrate the practicality of our approach.

1. Introduction and Motivation

In many real life applications a source or a control is placed outside (disjoint from) the observation domain $\Omega$ where PDE is satisfied. Some examples of inverse and optimal control problems where this situation can arise are: (i) Acoustic testing, when the loudspeakers are placed far from the aerospace structures [28]; (ii) Magnetotellurics (MT), which is a technique to infer earth's subsurface electrical conductivity from surface measurements [37, 44]; (iii) Magnetic drug targeting (MDT), where drugs with ferromagnetic particles in suspension are injected into the body and the external magnetic field is then used to steer the drug to relevant areas, for example, solid tumors [31, 7, 8]; (iv) Electroencephalography (EEG) is used to record electrical activity in brain [45, 32], in case one accounts for the neurons disjoint from the brain, we will obtain an external source problem.

This is different from the traditional approaches where the source/control is placed either inside the domain $\Omega$ or on the boundary $\partial\Omega$ of $\Omega$. This is not surprising since in many cases we do not have a direct access to $\partial\Omega$. See for instance, the setup in Figure 1. In such applications the existing models can be ineffective due to their strict requirements. Indeed think of the source identification problem for the most basic Poisson equation:

$$-\Delta u = f \quad \text{in} \; \Omega, \quad u = z \quad \text{on} \; \partial\Omega,$$

where the source is either $f$ (force or load) or $z$ (boundary control) see [6, 29, 36]. In (1.1) there is no provision to place the source in $\hat{\Omega}$ (cf. Figure 1). The issue is that the operator $\Delta$ has “lesser
Let a diffusion process occurs inside a domain Ω which is the sphere in the left panel and the letter M in the right panel. We are interested in the source identification or controlling this diffusion process by placing the source/control in a set \( \hat{\Omega} \) which is disjoint from Ω. In the above figure \( \hat{\Omega} \) is the triangular pipe in the left panel and the structure on top of the letter M in the right panel.

Recently, nonlocal diffusion operators such as the fractional Laplacian \( (-\Delta)^s \) have emerged as an excellent alternative to model diffusion. Under a probabilistic framework this operator can be derived as a limit of the so-called long jump random walk \([38]\). Recall that \( \Delta \) is the limit of the classical random walk or the Brownian motion. More applications of these models appear in (but not limited to) image denoising and phase field modeling \([4]\); image denosing where \( s \) is allowed to be spatially dependent \([10]\); fractional diffusion maps (data analysis) \([5]\); magnetotellurics (geophysics) \([44]\).

Coming back to the question of source/control placement, we next state the exterior value problem corresponding to \( (-\Delta)^s \). Find \( u \) in an appropriate function space satisfying

\[
(-\Delta)^s u = f \quad \text{in } \Omega, \quad u = z \quad \text{on } \mathbb{R}^N \setminus \Omega.
\]  

As in the case of \((1.1)\), besides \( f \) being the source/control in \( \Omega \) we can also place the source/control \( z \) in the exterior domain \( \mathbb{R}^N \setminus \Omega \). However, the action of \( z \) in \((1.2)\) is significantly different from \((1.1)\). Indeed, the source/control in \((1.1)\) is placed on the boundary \( \partial \Omega \), but the source/control \( z \) in \((1.2)\) is placed outside in \( \mathbb{R}^N \setminus \Omega \) which is what we wanted to achieve in Figure 1. For completeness, we refer to \([11]\) for the optimal control problem, with \( f \) being the source/control and \([12, 13]\) for another inverse problem to identify the coefficients in the fractional \( p \)-Laplacian.

The purpose of this paper is to introduce and study a new class of the Dirichlet, Robin, and Neumann source identification problems or the optimal control problems. We shall use these terms interchangeably but we will make a distinction in our numerical experiments. We emphasize that yet another class of identification where the unknown is the fractional exponent \( s \) for the spectral fractional Laplacian (which is different from the operator under consideration) was recently considered in \([35]\). We shall describe our problems next.

Let \( \Omega \subset \mathbb{R}^N, \ N \geq 1, \) be a bounded open set with boundary \( \partial \Omega \). Let \((Z_D, U_D)\) and \((Z_R, U_R)\), where subscripts \( D \) and \( R \) indicate Dirichlet and Robin, be Banach spaces. The goal of this paper is to consider the following two external control or source identification problems. The source/control in our case is denoted by \( z \).
• **Fractional Dirichlet exterior control problem:** Given \( \xi \geq 0 \) a constant penalty parameter we consider the minimization problem:

\[
\min_{(u,z)\in(U_D,Z_D)} J(u) + \frac{\xi}{2} \|z\|_{Z_D}^2,
\]

subject to the fractional Dirichlet exterior value problem: Find \( u \in U_D \) solving

\[
\begin{aligned}
(-\Delta)^s u &= 0 \quad \text{in } \Omega, \\
u &= z \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]

and the control constraints

\[
z \in Z_{ad,D},
\]

with \( Z_{ad,D} \subset Z_D \) being a closed and convex subset.

• **Fractional Robin exterior control problem:** Given \( \xi \geq 0 \) a constant penalty parameter we consider the minimization problem

\[
\min_{(u,z)\in(U_R,Z_R)} J(u) + \frac{\xi}{2} \|z\|_{Z_R}^2,
\]

subject to the fractional Robin exterior value problem: Find \( u \in U_R \) solving

\[
\begin{aligned}
(-\Delta)^s u &= 0 \quad \text{in } \Omega, \\
\mathcal{N}_s u + \kappa u &= \kappa z \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]

and the control constraints

\[
z \in Z_{ad,R},
\]

with \( Z_{ad,R} \subset Z_R \) being a closed and convex subset. In (1.4b), \( \mathcal{N}_s u \) is the nonlocal normal derivative of \( u \) given in (2.4) below, \( \kappa \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega) \) and is non-negative. We notice that the latter assumption is not a restriction since otherwise we can replace \( \kappa \) throughout by \( |\kappa| \).

The precise conditions on \( \Omega, J \) and the Banach spaces involved will be given in the subsequent sections. Notice that both the exterior value problems (1.3b) and (1.4b) are ill-posed if the conditions are enforced on \( \partial \Omega \). The main difficulties in (1.3) and (1.4) stem from the following facts.

• **Nonlocal operator.** The fractional Laplacian \((-\Delta)^s\) is a nonlocal operator. This can be easily seen from the definition of \((-\Delta)^s\) in (2.3).

• **Double nonlocality.** The first order optimality conditions for (1.3) and the Robin exterior value problem (1.4b) require to study \( \mathcal{N}_s u \) which is the so-called nonlocal-normal derivative of \( u \) (see (2.4)). Thus we not only have the nonlocal operator \((-\Delta)^s\) on the domain but also on the exterior \( \mathbb{R}^N \setminus \Omega \), i.e., a double nonlocality.

• **Exterior conditions in \( \mathbb{R}^N \setminus \Omega \) and not boundary conditions on \( \partial \Omega \).** The conditions in (1.3b) and (1.4b) need to be specified in \( \mathbb{R}^N \setminus \Omega \) instead on \( \partial \Omega \) as otherwise the problems (1.3) and (1.4) are ill-posed as we have already mentioned above.

• **Very-weak solutions of nonlocal exterior value problems.** A typical choice for \( Z \) is \( L^2(\mathbb{R}^N \setminus \Omega) \). As a result, the Dirichlet exterior value problem (1.3b) can only have very-weak solutions (cf. [14, 15, 17] for the case \( s = 1 \)). To the best of our knowledge this is the first work that considers the notion of very-weak solutions for nonlocal (fractional) exterior value problems associated with the fractional Laplace operator.

• **Regularity of optimization variables.** The standard shift-theorem which holds for local operators such as \( \Delta \) does not hold always hold for nonlocal operators such as \((-\Delta)^s\) (see for example [26]).
In view of all these aforementioned challenges it is clear that the standard techniques which are now well established for local problems do not directly extend to the nonlocal problems investigated in the present paper.

The purpose of this paper is to discuss our approach to deal with these nontrivial issues. We emphasize that to the best of our knowledge this is the first work that considers the optimal control of problems (source identification problems) \((1.3b)\) and \((1.4b)\) where the control/source is applied from the outside. Let us also mention that this notion of controllability of PDEs from the exterior has been introduced by M. Warma in \([42]\) for the nonlocal heat equation associated with the fractional Laplacian and in \([30]\) for the wave type equation with the fractional Laplace operator to study their controllability properties. The case of the strong damping wave equation is included in \([43]\) where some controllability results have been obtained. In case of problems with the spectral fractional Laplacian the boundary control has been established in \([9]\).

We mention that we can also deal with the fractional Neumann exterior control problem. That is, given \(\xi \geq 0\) a constant penalty parameter,

\[
\min_{(u,z) \in (U_N,Z_N)} J(u) + \frac{\xi}{2} \|z\|_{Z_N}^2,
\]

subject to the fractional Neumann exterior value problem: Find \(u \in U_N\) solving

\[
\begin{cases}
(-\Delta)^s u + u = 0 & \text{in } \Omega, \\
N_s u = z & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

and the control constraints

\[
z \in Z_{ad,N}.
\]

The term \(u\) is added in \((1.5)\) just to ensure the uniqueness of solutions. The proofs follow similarly as the two cases we consider in the present paper with very minor changes. Since the paper is already long, we shall not give any details on this case.

Below we mention further the key-novelties of the present paper:

(i) For the first time, we introduce and study the notion of very-weak solutions to the Dirichlet exterior value problem \((1.3b)\) which is suitable for optimal control problems. We also study weak solutions of the Robin exterior value problem \((1.4b)\).

(ii) We approximate the weak solutions of nonhomogeneous Dirichlet exterior value problem by using a suitable Robin exterior value problem. This allows us to circumvent approximating the nonlocal normal derivative. This is a new approach to impose non-zero exterior conditions for the fractional Dirichlet exterior value problem. We refer to an alternative approach \([3]\) where the authors use the Lagrange multipliers to impose nonzero Dirichlet exterior conditions.

(iii) We study both Dirichlet and Robin exterior control problems.

(iv) We approximate (with rate) the Dirichlet exterior control problem by a suitable Robin exterior control problem.

The rest of the paper is organized as follows. We begin with Section 2 where we introduce the relevant notations and function spaces. The material in this section is well-known. Our main work starts from Section 3 where at first we study the weak and very-weak solutions for the Dirichlet exterior value problem in Subsection 3.1. This is followed by the well-posedness of the Robin exterior value problem in Subsection 3.2. The Dirichlet exterior control problem is considered in Section 4 and Robin in Section 5. We show how to approximate the weak solutions to Dirichlet problem and the solutions to Dirichlet exterior control problem in Section 6. Subsection 7.1 is devoted to the experimental rate of convergence to approximate the Dirichlet exterior value problem using the Robin problem. In Subsection 7.2 we consider a source identification problem in the classical
sense, however our source is located outside the observation domain where the PDE is satisfied. Subsection 7.3 is devoted to two optimal control problems.

**Remark 1.1 (Practical aspects).** From a practical point of view, having the source/control over the entire \( \mathbb{R}^N \setminus \Omega \) can be very expensive. But this can be easily fixed by appropriately describing \( Z_{ad} \). Indeed in case of Figure 1 we can set the support of functions in \( Z_{ad} \) to be in \( \hat{\Omega} \setminus \Omega \).

### 2. Notation and Preliminaries

Unless otherwise stated, \( \Omega \subset \mathbb{R}^N \) (\( N \geq 1 \)) is a bounded open set and \( 0 < s < 1 \). We let

\[
W^{s, 2}(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy < \infty \right\},
\]

and we endow it with the norm defined by

\[
\|u\|_{W^{s, 2}(\Omega)} := \left( \int_\Omega |u|^2 \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]

In order to study (1.3b) we also need to define

\[
W^{s, 2}_0(\Omega) := \left\{ u \in W^{s, 2}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.
\]

Then

\[
\|u\|_{W^{s, 2}_0(\Omega)} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}
\]

defines an equivalent norm on \( W^{s, 2}_0(\Omega) \).

We shall use \( W^{-s, 2}(\mathbb{R}^N) \) and \( W^{-s, 2}(\Omega) \) to denote the dual spaces of \( W^{s, 2}(\mathbb{R}^N) \) and \( W^{s, 2}_0(\Omega) \), respectively, and \( \langle \cdot, \cdot \rangle \), to denote their duality pairing whenever it is clear from the context.

We also define the local fractional order Sobolev space

\[
W^{s, 2}_{loc}(\mathbb{R}^N \setminus \Omega) := \left\{ u \in L^2(\mathbb{R}^N \setminus \Omega) : u\varphi \in W^{s, 2}(\mathbb{R}^N \setminus \Omega), \ \forall \varphi \in D(\mathbb{R}^N \setminus \Omega) \right\}.
\] (2.1)

To introduce the fractional Laplace operator, we let \( 0 < s < 1 \), and we set

\[
L^1_s(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable}, \ \int_{\mathbb{R}^N} \frac{|u(x)|}{(1 + |x|)^{N+2s}} \, dx < \infty \right\}.
\]

For \( u \in L^1_s(\mathbb{R}^N) \) and \( \varepsilon > 0 \), we let

\[
(-\Delta)^{s}_\varepsilon u(x) = C_{N,s} \int_{\{y \in \mathbb{R}^N : |y-x| > \varepsilon\}} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy, \ \ \ \ x \in \mathbb{R}^N,
\]

where the normalized constant \( C_{N,s} \) is given by

\[
C_{N,s} := \frac{s^{2s}\Gamma\left(\frac{2s+N}{2}\right)}{\pi^N \Gamma(1-s)},
\] (2.2)

and \( \Gamma \) is the usual Euler Gamma function (see, e.g. [18, 20, 21, 22, 23, 40, 41]). The fractional Laplacian \((-\Delta)^s\) is defined for \( u \in L^1_s(\mathbb{R}^N) \) by the formula

\[
(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy = \lim_{\varepsilon \downarrow 0} (-\Delta)^{s}_\varepsilon u(x), \ \ \ x \in \mathbb{R}^N,
\] (2.3)
provided that the limit exists. It has been shown in [19] Proposition 2.2 that for \( u \in \mathcal{D}(\Omega) \), we have that
\[
\lim_{s \to 1} \int_{\mathbb{R}^N} u(-\Delta)^s u \, dx = \int_{\Omega} |\nabla u|^2 \, dx = -\int_{\mathbb{R}^N} u \Delta u \, dx = -\int_{\Omega} u \Delta u \, dx,
\]
that is where the constant \( C_{N,s} \) plays a crucial role.

Next, for \( u \in W^{s,2}(\mathbb{R}^N) \) we define the nonlocal normal derivative \( N_s \) as:
\[
N_s u(x) := C_{N,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}. \tag{2.4}
\]
We shall call \( N_s \) the interaction operator. Clearly \( N_s \) is a nonlocal operator and it is well defined on \( W^{s,2}(\mathbb{R}^N) \) as we discuss next.

**Lemma 2.1.** The interaction operator \( N_s \) maps continuously \( W^{s,2}(\mathbb{R}^N) \) into \( W^{s,2}_{\text{loc}}(\mathbb{R}^N \setminus \Omega) \). As a result, if \( u \in W^{s,2}(\mathbb{R}^N) \), then \( N_s u \in L^2(\mathbb{R}^N \setminus \Omega) \).

**Proof.** We refer to [25] Lemma 3.2] for the proof of the first part. The second part is a direct consequence of (2.1). □

Despite the fact that \( N_s \) is defined on \( \mathbb{R}^N \setminus \Omega \), it is still known as the “normal” derivative. This is due to its similarity with the classical normal derivative as we shall discuss next.

**Proposition 2.2.** The following assertions hold.

(a) **The divergence theorem for \((-\Delta)^s\).** Let \( u \in C^2_0(\mathbb{R}^N) \), i.e., \( C^2 \) functions on \( \mathbb{R}^N \) that vanishes at \( \pm \infty \). Then
\[
\int_{\Omega} (-\Delta)^s u \, dx = -\int_{\mathbb{R}^N \setminus \Omega} N_s u \, dx.
\]

(b) **The integration by parts formula for \((-\Delta)^s\).** Let \( u \in W^{s,2}(\mathbb{R}^N) \) be such that \((-\Delta)^s u \in L^2(\Omega) \). Then for every \( v \in W^{s,2}(\mathbb{R}^N) \) we have that
\[
\frac{C_{N,s}}{2} \int_{\mathbb{R}^2 \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} v(-\Delta)^s u \, dx + \int_{\mathbb{R}^N \setminus \Omega} v N_s u \, dx, \tag{2.5}
\]
where \( \mathbb{R}^2 \setminus (\mathbb{R}^N \setminus \Omega)^2 := (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega) \).

(c) **The limit as \( s \to 1 \).** Let \( u, v \in C^2_0(\mathbb{R}^N) \). Then
\[
\lim_{s \to 1} \int_{\mathbb{R}^N \setminus \Omega} v N_s u \, dx = \int_{\partial \Omega} \frac{\partial u}{\partial \nu} v \, d\sigma.
\]

**Remark 2.3.** Comparing (a)-(c) in Proposition 2.2 with the classical properties of the standard Laplacian \( \Delta \) we can immediately infer that \( N_s \) plays the same role for \((-\Delta)^s\) that the classical normal derivative does for \( \Delta \). For this reason, we call \( N_s \) the nonlocal normal derivative.

**Proof of Proposition 2.2** The proofs of Parts (a) and (c) are contained in [24] Lemma 3.3] and [24] Proposition 5.1], respectively. The proof of Part (b) for smooth functions can be found in [24] Lemma 3.3]. The version given here is obtained by using a density argument (cf. [42] Proposition 3.7)]. □
3. The state equations

Before analyzing the optimal control problems \((1.3)\) and \((1.4)\), for a given function \(z\) we shall focus on the Dirichlet \((1.3b)\) and Robin \((1.4b)\) exterior value problems. We shall assume that \(\Omega\) is a bounded domain with Lipschitz continuous boundary.

3.1. The Dirichlet problem for the fractional Laplacian. We begin by rewriting the system \((1.3b)\) in a more general form

\[
\begin{cases}
(-\Delta)^s u = f & \text{in } \Omega, \\
u = z & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

(3.1)

Here is our notion of weak solutions.

**Definition 3.1 (Weak solution to the Dirichlet problem).** Let \(f \in W^{-s,2}(\Omega)\), \(z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)\) and \(Z \in W^{s,2}(\mathbb{R}^N)\) be such that \(Z|_{\mathbb{R}^N \setminus \Omega} = z\). A \(u \in W^{s,2}(\mathbb{R}^N)\) is said to be a weak solution to \((3.1)\) if \(u - Z \in W^{s,2}_0(\Omega)\) and

\[
\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx dy = \langle f, v \rangle,
\]

for every \(v \in W^{s,2}_0(\Omega)\).

Firstly, we notice that since \(\Omega\) is assumed to have a Lipschitz continuous boundary, we have that, for \(z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)\), there exists \(Z \in W^{s,2}(\mathbb{R}^N)\) such that \(Z|_{\mathbb{R}^N \setminus \Omega} = z\). Secondly, the existence and uniqueness of a weak solution \(u\) to \((3.1)\) and the continuous dependence of \(u\) on the data \(f\) and \(z\) have been considered in [27], see also [25, 39]. More precisely we have the following result.

**Proposition 3.2.** Let \(f \in W^{-s,2}(\Omega)\) and \(z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)\). Then there exists a unique weak solution \(u\) to \((3.1)\) in the sense of Definition 3.1. In addition there is a constant \(C > 0\) such that

\[
\|u\|_{W^{s,2}(\mathbb{R}^N)} \leq C \left(\|f\|_{W^{-s,2}(\Omega)} + \|z\|_{W^{s,2}(\mathbb{R}^N \setminus \Omega)}\right).
\]

(3.2)

Even though such a result is typically sufficient in most situations, nevertheless it is not directly useful in the current context of optimal control problem \((1.3)\) since we are interested in taking the space \(Z_D = L^2(\mathbb{R}^N \setminus \Omega)\). Thus we need existence of solution (in some sense) to the fractional Dirichlet problem \((3.1)\) when the datum \(z \in L^2(\mathbb{R}^N \setminus \Omega)\). In order to tackle this situation we introduce the notion of very-weak solutions for \((3.1)\).

**Definition 3.3 (Very-weak solution to the Dirichlet problem).** Let \(z \in L^2(\mathbb{R}^N \setminus \Omega)\) and \(f \in W^{-s,2}(\Omega)\). A \(u \in L^2(\mathbb{R}^N)\) is said to be a very-weak solution to \((3.1)\) if the identity

\[
\int_{\Omega} u(-\Delta)^s v \, dx = \langle f, v \rangle - \int_{\mathbb{R}^N \setminus \Omega} z N_s v \, dx,
\]

(3.3)

holds for every \(v \in V := \{v \in W^{s,2}_0(\Omega) : (-\Delta)^s v \in L^2(\Omega)\}\).

Next we prove the existence and uniqueness of a very-weak solution to \((3.1)\) in the sense of Definition 3.3.

**Theorem 3.4.** Let \(f \in W^{-s,2}(\Omega)\) and \(z \in L^2(\mathbb{R}^N \setminus \Omega)\). Then there exists a unique very-weak solution \(u\) to \((3.1)\) according to Definition 3.3 that fulfills

\[
\|u\|_{L^2(\Omega)} \leq C \left(\|f\|_{W^{-s,2}(\Omega)} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)}\right),
\]

(3.4)

for a constant \(C > 0\). In addition, if \(z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)\), then the following assertions hold.
(a) Every weak solution of (3.1) is also a very-weak solution. 

(b) Every very-weak solution of (3.1) that belongs to $W^{s,2}(\mathbb{R}^N)$ is also a weak solution.

**Proof.** In order to show the existence of a very-weak solution we shall apply the Babuška-Lax-Milgram theorem.

Firstly, let $(-\Delta)_{D}^s$ be the realization of $(-\Delta)^s$ in $L^2(\Omega)$ with the zero Dirichlet exterior condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$. More precisely, 

$$D((-\Delta)_{D}^s) = V \text{ and } (-\Delta)_{D}^s u = (-\Delta)^s u.$$ 

Then a norm on $V$ is given by $\|v\|_V = \|(-\Delta)_{D}^s v\|_{L^2(\Omega)}$ which follows from the fact that the operator $(-\Delta)_{D}^s$ is invertible (since by [34] $(-\Delta)_{D}^s$ has a compact resolvent and its first eigenvalue is strictly positive). Secondly, let $F$ be the bilinear form defined on $L^2(\Omega) \times V$ by 

$$F(u, v) := \int_{\Omega} u(-\Delta)^s v \, dx.$$ 

Then $F$ is clearly bounded on $L^2(\Omega) \times V$. More precisely there is a constant $C > 0$ such that 

$$|F(u, v)| \leq \|u\|_{L^2(\Omega)} \|(-\Delta)^s v\|_{L^2(\Omega)} \leq C \|u\|_{L^2(\Omega)} \|v\|_V.$$ 

Thirdly, we show the inf-sup conditions. From the definition of $V$, it immediately follows that 

$$v \in W_0^{s,2}(\overline{\Omega}) \text{ and } (-\Delta)^s v \in L^2(\Omega) \iff v \in V.$$ 

By setting $u := \frac{(-\Delta)_{D}^s v}{\|(-\Delta)_{D}^s v\|_{L^2(\Omega)}} \in L^2(\Omega)$, we obtain that 

$$\sup_{u \in L^2(\Omega), \|u\|_{L^2(\Omega)} = 1} |(u, (-\Delta)_{D}^s v)_{L^2(\Omega)}| \geq \frac{|((-\Delta)_{D}^s v, (-\Delta)_{D}^s v)_{L^2(\Omega)}|}{\|(-\Delta)_{D}^s v\|_{L^2(\Omega)}} = \|(-\Delta)_{D}^s v\|_{L^2(\Omega)} = \|v\|_V.$$ 

Next we choose $v \in V$ as the unique weak solution of $(-\Delta)_{D}^s v = u/\|u\|_{L^2(\Omega)}$ for some $0 \neq u \in L^2(\Omega)$. Then we readily obtain that 

$$\sup_{v \in V, \|v\|_V = 1} |(u, (-\Delta)^s v)_{L^2(\Omega)}| \geq \frac{|(u, u)_{L^2(\Omega)}|}{\|u\|_{L^2(\Omega)}} = \|u\|_{L^2(\Omega)} > 0,$$ 

for all $0 \neq u \in L^2(\Omega)$. Finally, we have to show that the right-hand-side in (3.3) defines a linear continuous functional on $V$. Indeed, applying the Hölder inequality in conjunction with Lemma 2.1 we obtain that there is a constant $C > 0$ such that 

$$\int_{\mathbb{R}^N \setminus \Omega} z N_s v \, dx \leq \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|N_s v\|_{L^2(\mathbb{R}^N \setminus \Omega)} \leq C \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|v\|_{W_0^{s,2}(\overline{\Omega})},$$

(3.5)

where in the last step we have used the fact that $\|v\|_{W_0^{s,2}(\overline{\Omega})} = \|v\|_{W^{s,2}(\mathbb{R}^N)}$ for $v \in W_0^{s,2}(\overline{\Omega})$. Moreover 

$$|\langle f, v \rangle| \leq \|f\|_{W^{-s,2}(\overline{\Omega})} \|v\|_{W^{s,2}(\overline{\Omega})}.$$ 

In view of the last two estimates, the right-hand-side in (3.3) defines a linear continuous functional on $V$. Therefore all the requirements of the Babuška-Lax-Milgram theorem holds. Thus, there exists a unique $u \in L^2(\Omega)$ satisfying (3.3). Let $u = z$ in $\mathbb{R}^N \setminus \Omega$, then $u \in L^2(\mathbb{R}^N)$ and satisfies (3.3). We have shown the existence of the uniqueness of a very-weak solution.
Next we show the estimate \((3.4)\). Let \(u \in L^2(\mathbb{R}^N)\) be a very-weak solution. Let \(v \in V\) be a solution of \((-\Delta)_D^s v = u\). Taking this \(v\) as a test function in \((3.3)\) and using \((3.5)\), we get that there is a constant \(C > 0\) such that

\[
\|u\|_{L^2(\Omega)}^2 \leq \|f\|_{W^{-s,2}(\Omega)} \|v\|_{W^{s,2}(\Omega)} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|N_s v\|_{L^2(\mathbb{R}^N \setminus \Omega)}
\]

\[
\leq C \left( \|f\|_{W^{-s,2}(\Omega)} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \right) \|v\|_{W^{s,2}(\Omega)}
\]

\[
\leq C \left( \|f\|_{W^{-s,2}(\Omega)} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \right) \|(-\Delta)^s_D v\|_{L^2(\Omega)}
\]

\[
\leq C \left( \|f\|_{W^{-s,2}(\Omega)} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \right) \|u\|_{L^2(\Omega)},
\]

and we have shown \((3.4)\). This completes the proof of the first part.

Next we prove the last two assertions of the theorem. Assume that \(z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)\).

(a) Let \(u \in W^{s,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)\) be a weak solution of \((3.1)\). It follows from the definition that \(u = z\) in \(\mathbb{R}^N \setminus \Omega\) and

\[
\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy = \langle f, v \rangle,
\]

for every \(v \in V\). Since \(v = 0\) in \(\mathbb{R}^N \setminus \Omega\), we have that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy = \int \int_{\mathbb{R}^2 \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy.
\]

Using \((3.6)\), \((3.7)\), the integration by parts formula \((2.5)\) together with the fact that \(u = z\) in \(\mathbb{R}^N \setminus \Omega\), we get that

\[
\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy = \langle f, v \rangle
\]

\[
= \int_\Omega u (-\Delta)^s v \, dx + \int_{\mathbb{R}^N \setminus \Omega} u N_s v \, dx
\]

\[
= \int_\Omega u (-\Delta)^s v \, dx + \int_{\mathbb{R}^N \setminus \Omega} z N_s v \, dx.
\]

Thus \(u\) is a very-weak solution of \((3.1)\).

(b) Finally let \(u\) be a very-weak solution of \((3.1)\) and assume that \(u \in W^{s,2}(\mathbb{R}^N)\). Since \(u = z\) in \(\mathbb{R}^N \setminus \Omega\), we have that \(z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)\) and if \(Z \in W^{s,2}(\mathbb{R}^N)\) satisfies \(Z|_{\mathbb{R}^N \setminus \Omega} = z\), then clearly \((u - Z) \in W^{s,2}_0(\Omega)\). Since \(u\) is a very-weak solution of \((3.1)\), then by definition, for every \(v \in V = D((-\Delta)^s_D)\), we have that

\[
\int_\Omega u (-\Delta)^s v \, dx = \langle f, v \rangle - \int_{\mathbb{R}^N \setminus \Omega} z N_s v \, dx.
\]
Since \( u \in W^{s,2}(\mathbb{R}^N) \) and \( v = 0 \) in \( \mathbb{R}^N \setminus \Omega \), then using \((2.5)\) again we get that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \, dx \, dy
= \int_{\mathbb{R}^N} \int_{(\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \, dx \, dy
= \int_{\Omega} u(-\Delta)^s v \, dx + \int_{\mathbb{R}^N \setminus \Omega} u \mathcal{N}_s v \, dx
= \int_{\Omega} u(-\Delta)^s v \, dx + \int_{\mathbb{R}^N \setminus \Omega} z \mathcal{N}_s v \, dx.
\tag{3.9}
\]
It follows from \((3.8)\) and \((3.9)\) that for every \( v \in V \), we have that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \, dx \, dy = \langle f, v \rangle.
\tag{3.10}
\]
Since \( V \) is dense in \( W^{s,2}_0(\Omega) \), we have that \((3.10)\) remains true for every \( v \in W^{s,2}_0(\Omega) \). We have shown that \( u \) is a weak solution of \((3.1)\) and the proof is finished. \( \square \)

3.2. The Robin problem for the fractional Laplacian. In order to study the Robin problem \((1.4b)\) we consider the Sobolev space introduced in \([24]\). For \( g \in L^1(\mathbb{R}^N \setminus \Omega) \) fixed, we let
\[
W^{s,2}_{\Omega,g} := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable}, \| u \|_{W^{s,2}_{\Omega,g}} < \infty \right\},
\]
where
\[
\| u \|_{W^{s,2}_{\Omega,g}} := \left( \| u \|^2_{L^2(\Omega)} + \| g \|^{\frac{1}{2}} \| u \|^2_{L^2(\mathbb{R}^N \setminus \Omega)} + \int_{\mathbb{R}^N \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy \right)^{\frac{1}{2}}. \tag{3.11}
\]
Let \( \mu \) be the measure on \( \mathbb{R}^N \setminus \Omega \) given by \( d\mu = |g| \, dx \). With this setting, the norm in \((3.11)\) can be rewritten as
\[
\| u \|_{W^{s,2}_{\Omega,g}} := \left( \| u \|^2_{L^2(\Omega)} + \| u \|^2_{L^2(\mathbb{R}^N \setminus \Omega, \mu)} + \int_{\mathbb{R}^N \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy \right)^{\frac{1}{2}}. \tag{3.12}
\]
If \( g = 0 \), we shall let \( W^{s,2}_{\Omega,0} = W^{s,2}_\Omega \). The following result has been proved in \([24]\) Proposition 3.1].

**Proposition 3.5.** Let \( g \in L^1(\mathbb{R}^N \setminus \Omega) \). Then \( W^{s,2}_{\Omega,g} \) is a Hilbert space.

Throughout the remainder of the article, for \( g \in L^1(\mathbb{R}^N \setminus \Omega) \), we shall denote by \((W^{s,2}_{\Omega,g})^*\) the dual of \( W^{s,2}_{\Omega,g} \).

**Remark 3.6.** We mention the following facts.
(a) Recall that
\[
\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2 = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega),
\]
so that
\[
\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy
+ \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy + \int_{\mathbb{R}^N \setminus \Omega} \int_{\mathbb{R}^N \setminus \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy. \tag{3.13}
\]
Proposition 3.8. Let $\kappa \equiv 0$, Proposition 3.8 only guarantees uniqueness of solutions to (1.4b) up to a constant. In case we assume $\kappa$ to be non-negative. Here is our notion of weak solutions.

Definition 3.7. Let $z \in L^2(\mathbb{R}^N \setminus \Omega, \mu)$ and $f \in (W^{s,2}_{\Omega,\kappa})^*$. A $u \in W^{s,2}_{\Omega,\kappa}$ is said to be a weak solution of (1.4b) if the identity

$$
\int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N \setminus \Omega} \kappa uv \, dx = \langle f, v \rangle_{(W^{s,2}_{\Omega,\kappa})^*, W^{s,2}_{\Omega,\kappa}} + \int_{\mathbb{R}^N \setminus \Omega} \kappa zv \, dx,
$$

holds for every $v \in W^{s,2}_{\Omega,\kappa}$.

We have the following existence result.

Proposition 3.8. Let $\kappa \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega)$, and $f \in (W^{s,2}_{\Omega,\kappa})^*$, there exists a weak solution $u \in W^{s,2}_{\Omega,\kappa}$ of (1.4b).

Proof. Let $\mathcal{E}$ with domain $D(\mathcal{E}) = W^{s,2}_{\Omega,\kappa}$ be given by

$$
\mathcal{E}(u, v) := \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N \setminus \Omega} \kappa uv \, dx.
$$

Then $\mathcal{E}$ is a bilinear, symmetric, continuous and closed form on $L^2(\Omega)$. Hence, for every $z \in L^2(\mathbb{R}^N \setminus \Omega, \mu)$ and $f \in (W^{s,2}_{\Omega,\kappa})^*$, there is a function $u \in W^{s,2}_{\Omega,\kappa}$ such that

$$
\mathcal{E}(u, v) = \langle f, v \rangle_{(W^{s,2}_{\Omega,\kappa})^*, W^{s,2}_{\Omega,\kappa}} + \int_{\mathbb{R}^N \setminus \Omega} \kappa zv \, dx = \langle f, v \rangle_{(W^{s,2}_{\Omega,\kappa})^*, W^{s,2}_{\Omega,\kappa}} + \int_{\mathbb{R}^N \setminus \Omega} \kappa zv \, dx,
$$

for every $v \in W^{s,2}_{\Omega,\kappa}$. That is, $u$ satisfies (3.16). Thus $u$ is a weak solution of (1.4b). The proof is finished. 

Remark 3.9. Notice that similarly to the classical Neumann problem when $\kappa \equiv 0$, Proposition 3.8 only guarantees uniqueness of solutions to (1.4b) up to a constant. In case we assume $\kappa$ to be non-negative.
strictly positive, uniqueness can be guaranteed under Assumption 6.1 below. In that case we can also show that there is a constant $C > 0$ such that
\[ \|u\|_{W^{s,2}_{\Omega,n}} \leq C \left( \|f\|_{W^{s,2}_{\Omega,n}} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega,\rho)} \right). \] (3.18)

4. Fractional Dirichlet boundary control problem

We begin by introducing the appropriate function spaces needed to study (1.3). We let
\[ Z_D := L^2(\mathbb{R}^N \setminus \Omega), \quad U_D := L^2(\Omega). \]

In view of Theorem 3.4 the following (solution-map) control-to-state map
\[ S : Z_D \to U_D, \quad z \mapsto Sz = u, \]
is well-defined, linear and continuous. We also notice that for $z \in Z_D$, we have that $u := Sz \in L^2(\mathbb{R}^N)$. As a result we can write the reduced fractional Dirichlet exterior control problem as follows:
\[ \min_{z \in Z_{ad,D}} J(z) := J(Sz) + \frac{\xi}{2} \|z\|_{Z_D}^2. \] (4.1)

We then have the following well-posedness result for (4.1) and equivalently (1.3).

**Theorem 4.1.** Let $Z_{ad,D}$ be a closed and convex subset of $Z_D$. Let either $\xi > 0$ or $Z_{ad,D}$ be bounded and let $J : U_D \to \mathbb{R}$ be weakly lower-semicontinuous. Then there exists a solution $\bar{z}$ to (4.1) and equivalently to (1.3). If either $J$ is convex and $\xi > 0$ or $J$ is strictly convex and $\xi \geq 0$, then $\bar{z}$ is unique.

**Proof.** The proof uses the so-called direct-method or the Weierstrass theorem [16, Theorem 3.2.1]. We notice for $J : Z_{ad,D} \to \mathbb{R}$, it is always possible to construct a minimizing sequence $\{z_n\}_{n \in \mathbb{N}}$ (cf. [16, Theorem 3.2.1] for a construction) such that
\[ \inf_{z \in Z_{ad,D}} J(z) = \lim_{n \to \infty} J(z_n). \]

If $\xi > 0$ or $Z_{ad,D} \subset Z_D$ is bounded then $\{z_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $Z_D$ which is a Hilbert space. Due to the reflexivity of $Z_D$ we have that (up to a subsequence if necessary) $z_n \rightharpoonup \bar{z}$ (weak convergence) in $Z_D$ as $n \to \infty$. Since $Z_{ad,D}$ is closed and convex, hence is weakly closed, we have that $\bar{z} \in Z_{ad,D}$.

Since $S : Z_{ad,D} \to U_D$ is linear and continuous, we have that it is weakly continuous. This implies that $Sz_n \rightharpoonup S\bar{z}$ in $U_D$ as $n \to \infty$. We have to show that $(S\bar{z}, \bar{z})$ fulfills the state equation according to Definition 3.3. In particular we need to study the identity
\[ \int_{\Omega} u_n(-\Delta)^s v \, dx = -\int_{\mathbb{R}^N \setminus \Omega} z_n \mathcal{N}_s v \, dx, \quad \forall v \in V, \] (4.2)
as $n \to \infty$, where $u_n := Sz_n$. Since $u_n \rightharpoonup S\bar{z} =: \bar{u}$ in $U_D$ as $n \to \infty$ and $z_n \rightharpoonup \bar{z}$ in $Z_D$ as $n \to \infty$, we can immediately take the limit in (4.2) and obtain that $(\bar{u}, \bar{z}) \in U_D \times Z_{ad,D}$ fulfills the state equation in the sense of Definition 3.3.

It then remains to show that $\bar{z}$ is the minimizer of (4.1). This is a consequence of the fact that $J$ is weakly lower semicontinuous. Indeed, $J$ is the sum of two weakly lower semicontinuous functions (\[ \|\cdot\|_{Z_D}^2 \] is continuous and convex therefore weakly lower semicontinuous).

Finally, if $\xi > 0$ and $J$ is convex then $J$ is strictly convex (sum of a strictly convex and convex functions). On the other hand, if $J$ is strictly convex then $J$ is strictly convex. In either case we have that $J$ is strictly convex and thus the uniqueness of $\bar{z}$ follows. \[ \square \]
We will next derive the first order necessary optimality conditions for (4.1). We begin by identifying the structure of the adjoint operator $S^*$.

**Lemma 4.2.** For the state equation (1.3b) the adjoint operator $S^* : U_D \to Z_D$ is given by

$$S^* w = -N_s p \in Z_D,$$

where $w \in U_D$ and $p \in W^{s,2}_0(\Omega)$ is the weak solution to the problem

$$\begin{cases}
(\Delta)^s p = w & \text{in } \Omega \\
p = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}$$

**Proof.** According to the definition of $S^*$ we have that for every $w \in U_D$ and $z \in Z_D$,

$$(w, Sz)_{L^2(\Omega)} = (S^* w, z)_{L^2(\mathbb{R}^N \setminus \Omega)}.$$

Next, testing the adjoint equation (4.3) with $Sz$ and using the fact that $Sz$ is a very-weak solution of (3.1) with $f = 0$, we arrive at

$$(w, Sz)_{L^2(\Omega)} = (Sz, (\Delta)^s p)_{L^2(\Omega)} = -(z, N_s p)_{L^2(\mathbb{R}^N \setminus \Omega)} = (z, S^* w)_{L^2(\mathbb{R}^N \setminus \Omega)}.$$

This yields the asserted result. □

For the remainder of this section we will assume that $\xi > 0$.

**Theorem 4.3.** Let the assumptions of Theorem 4.1 hold. Let $Z$ be an open set in $Z_D$ such that $Z_{ad,D} \subset Z$. Let $u \mapsto J(u) : U_D \to \mathbb{R}$ be continuously Fréchet differentiable with $J'(u) \in U_D$. If $\bar{z}$ is a minimizer of (4.1) over $Z_{ad,D}$, then the first order necessary optimality conditions are given by

$$(-N_s \bar{p} + \xi \bar{z}, z - \bar{z})_{L^2(\mathbb{R}^N \setminus \Omega)} \geq 0, \quad z \in Z_{ad,D}$$

(4.4)

where $\bar{p} \in W^{s,2}_0(\Omega)$ solves the adjoint equation

$$\begin{cases}
(\Delta)^s \bar{p} = J'(\bar{u}) & \text{in } \Omega \\
\bar{p} = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}$$

(4.5)

Equivalently we can write (4.4) as

$$\bar{z} = \mathcal{P}_{Z_{ad}} \left( \frac{1}{\xi} N_s \bar{p} \right),$$

(4.6)

where $\mathcal{P}_{Z_{ad}}$ is the projection onto the set $Z_{ad}$. If $J$ is convex then (4.4) is a sufficient condition.

**Proof.** The proof is a straightforward application of differentiability properties of $J$ and chain rule in conjunction with Lemma 4.2. Indeed for a given direction $h \in Z_{ad,D}$ we have that the directional derivative of $J$ is given by

$$J'(\bar{z}) h = (J'(S\bar{z}), Sh)_{L^2(\Omega)} + \xi (\bar{z}, h)_{L^2(\mathbb{R}^N \setminus \Omega)}$$

$$= (S^* J'(S\bar{z}), h)_{L^2(\Omega)} + \xi (\bar{z}, h)_{L^2(\mathbb{R}^N \setminus \Omega)},$$

where in the first step we have used that $J'(S\bar{z}) \in \mathcal{L}(L^2(\Omega), \mathbb{R}) = L^2(\Omega)$ and in the second step we have used that $S$ is linear and bounded therefore $S^*$ is well-defined. Then using Lemma 4.2 we arrive at the asserted result.

From Lemma 2.1 we recall that $N_s \bar{p} \in L^2(\mathbb{R}^N \setminus \Omega)$. Therefore the equivalence between (4.4) and (4.6) follows by using [16, Theorem 3.3.5]. □
Remark 4.4 (Regularity for optimization variables). We recall a rather surprising result for the adjoint equation (4.13). The standard shift argument that is known to hold for the classical Laplacian on smooth open sets does not hold in the case of the fractional Laplacian i.e., $p$ does not always belong to $W^{2s,2}(\Omega)$. More precisely assume that $\Omega$ is a smooth bounded open set. If $0 < s < \frac{1}{2}$, then by [26, Formula (7.4)] we have that $D((-\Delta)^s) = W^{2s,2}_0(\Omega)$ and hence, $p \in W^{2s,2}(\Omega)$ in that case. But if $\frac{1}{2} \leq s < 1$, an example has been given in [33, Remark 7.2] where $D((-\Delta)^s) \not\subset W^{2s,2}(\Omega)$, thus in that case $p$ does not always belong to $W^{2s,2}(\Omega)$. As a result, the best known result for $N_s\bar{p}$ is as given in Lemma 2.1. Since $P_{zad}$ is a contraction (Lipschitz) we can conclude that $\bar{z}$ has the same regularity as $N_s\bar{p}$, i.e., they are in $L^2(\mathbb{R}^N \setminus \Omega)$ globally and in $W^{s,2}_{loc}(\mathbb{R}^N \setminus \Omega)$ locally. As it is well-known, in case of the classical Laplacian, one can use a boot-strap argument to improve the regularity of $S\bar{z} = \bar{u}$. However this is not the case for fractional exterior value problems.

5. Fractional Robin exterior control problem

In this section we shall study the fractional Robin exterior control problem (1.4b). We begin by setting the functional analytic framework. We let

$$Z_R := L^2(\mathbb{R}^N \setminus \Omega, \mu), \quad U_R := W^{s,2}_{\Omega,\kappa}.$$ 

Notice that $d\mu = \kappa dx$. In addition we shall assume that $\kappa \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega)$ and $\kappa > 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. In view of Proposition 3.8 the following (solution-map) control-to-state map

$$S : Z_R \rightarrow U_R, \quad z \mapsto u,$$

is well-defined. Moreover $S$ is linear and continuous (by (3.18)). Since $U_R \hookrightarrow L^2(\Omega)$ with the embedding being continuous we can instead define

$$S : Z_R \rightarrow L^2(\Omega).$$

We can then write the so-called reduced fractional Robin exterior control problem

$$\min_{z \in Z_{ad,R}} J(z) := J(Sz) + \frac{\xi}{2} \|z\|_{L^2(\mathbb{R}^N \setminus \Omega, \mu)}^2. \quad (5.1)$$

We have the following well-posedness result.

Theorem 5.1. Let $Z_{ad,R}$ be a closed and convex subset of $Z_R$. Let either $\xi > 0$ or $Z_{ad,R} \subset Z_R$ be bounded. Moreover, let $J : L^2(\Omega) \rightarrow \mathbb{R}$ be weakly lower-semicontinuous. Then there exists a solution $\bar{z}$ to (5.1) and equivalently to (1.4). If either $J$ is convex and $\xi > 0$ or $J$ is strictly convex and $\xi \geq 0$ then $\bar{z}$ is unique.

Proof. We proceed as the proof of Theorem 4.1. Let $\{z_n\} \subset Z_{ad}$ be a minimizing sequence such that

$$\inf_{z \in Z_{ad,R}} J(z) = \lim_{n \rightarrow \infty} J(z_n).$$

If $\xi > 0$ or $Z_{ad,R} \subset Z_R$ is bounded then after a subsequence, if necessary, we have $z_n \rightharpoonup \bar{z}$ in $L^2(\mathbb{R}^N \setminus \Omega, \mu)$ as $n \rightarrow \infty$. Now since $Z_{ad,R}$ is a convex and closed subset of $Z_R$, it follows that $\bar{z} \in Z_{ad,R}$.

Next we show that the pair $(S\bar{z}, \bar{z})$ satisfies the state equation. Notice that $u_n := Sz_n$ is the weak solution of (1.4b) with boundary value $z_n$. Thus, by definition, $u_n \in W^{s,2}_{\Omega,\kappa}$ and the identity

$$\mathcal{E}(u_n, v) = \int_{\mathbb{R}^N \setminus \Omega} z_n v \, d\mu, \quad (5.2)$$
holds for every \( v \in W^{s,2}_{\Omega,\kappa} \) and where we recall that \( \mathcal{E} \) is given in (3.17). We also notice that the mapping \( S \) is also bounded from \( Z_R \) into \( W^{s,2}_{\Omega,\kappa} \) (by (3.18)). This shows that the sequence \( \{ u_n \}_{n \in \mathbb{N}} \) is bounded in \( W^{s,2}_{\Omega,\kappa} \). Thus, after a subsequence, if necessary, we have that \( S z_n = u_n \rightharpoonup S \bar{z} = \bar{u} \) in \( W^{s,2}_{\Omega,\kappa} \) as \( n \to \infty \). This implies that

\[
\lim_{n \to \infty} \left( \int_{\mathbb{R}^N \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N \setminus \Omega} u_n \, v \, d\mu \right) = \]

for every \( v \in W^{s,2}_{\Omega,\kappa} \). Since \( z_n \rightharpoonup \bar{z} \) in \( L^2(\mathbb{R}^N \setminus \Omega, \mu) \) as \( n \to \infty \), it follows that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N \setminus \Omega} z_n \, v \, d\mu = \int_{\mathbb{R}^N \setminus \Omega} \bar{z} \, v \, d\mu,
\]

for every \( v \in W^{s,2}_{\Omega,\kappa} \). Therefore we can pass to the limit in (5.2) as \( n \to \infty \) to obtain that \( (S \bar{z}, \bar{z}) = (\bar{u}, \bar{z}) \) satisfies the state equation (1.4b). The rest of the steps are similar to the proof of Theorem 4.1 and we omit them for brevity.

As in the case of the fractional Dirichlet exterior control problem (4.1) we will next identify the adjoint of the control-to-state map \( S \).

**Lemma 5.2.** For the state equation (1.4b) the adjoint operator \( S^* : L^2(\Omega) \to Z_R \) is given by

\[
(S^* w, z)_{Z_R} = \int_{\mathbb{R}^N \setminus \Omega} p z \, d\mu \quad \forall z \in Z_R,
\]

where \( w \in L^2(\Omega) \) and \( p \in W^{s,2}_{\Omega,\kappa} \) is the weak solution to

\[
\begin{cases}
(-\Delta)^s p = w & \text{in } \Omega \\
\mathcal{N}_\kappa p + \kappa p = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

**Proof.** Let \( w \in L^2(\Omega) \) and \( z \in Z_R \). Then \( Sz \in W^{s,2}_{\Omega,\kappa} \to L^2(\Omega) \) with the embedding being continuous. Then we can write

\[
(w, Sz)_{L^2(\Omega)} = (S^* w, z)_{Z_R}.
\]

Next we test (5.3) with \( Sz \) to arrive at

\[
(w, Sz)_{L^2(\Omega)} = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(p(x) - p(y))}{|x-y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N \setminus \Omega} u p \, d\mu = \int_{\mathbb{R}^N \setminus \Omega} z p \, d\mu = (S^* w, z)_{Z_R},
\]

where we have used the fact that \( u \) solves the state equation according to Definition 3.7. This yields the asserted result. 

For the remainder of this section we will assume that \( \xi > 0 \). The proof of next result is similar to Theorem 4.3 and is omitted for brevity.
Theorem 5.3. Let the assumptions of Theorem 5.1 hold. Let \( Z \) be an open set in \( Z_R \) such that \( Z_{ad,R} \subset \Omega \). Let \( u \mapsto J(u) : L^2(\Omega) \to \mathbb{R} \) be continuously Fréchet differentiable with \( J'(u) \in L^2(\Omega) \). If \( \tilde{z} \) is a minimizer of (5.1) over \( Z_{ad,R} \) then the first necessary optimality conditions are given by

\[
\int_{\mathbb{R}^N \setminus \Omega} (\tilde{p} + \xi \tilde{z})(z - \tilde{z}) \, d\mu \geq 0, \quad z \in Z_{ad,R}
\]

where \( \tilde{p} \in W^{s,2}_{\Omega,k} \) solves the adjoint equation

\[
\begin{cases}
(-\Delta)^s p &= J'(\tilde{u}) \quad \text{in } \Omega \\
N_s p + \kappa \tilde{p} &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Equivalently we can write (5.4) as

\[
\tilde{z} = P_{Z_{ad,R}} \left( \frac{-\tilde{p}}{\xi} \right),
\]

where \( P_{Z_{ad,R}} \) is the projection onto the set \( Z_{ad,R} \). If \( J \) is convex then (5.4) is a sufficient condition.

Remark 5.4 (Regularity of optimization variables). As pointed out in Remark 4.4 (Dirichlet case) the regularity for the integral fractional Laplacian is a delicate issue. In fact for the Robin problem, in \( \mathbb{R}^N \setminus \Omega \) we can only guarantee that \( \tilde{p} \) is in \( L^2(\mathbb{R}^N \setminus \Omega, \mu) \). Therefore we cannot use the classical boot-strap argument to further improve the regularity of the control \( \tilde{z} \).

6. Approximation of Dirichlet exterior value and control problems

We recall that the Dirichlet exterior value problem (1.2) in our case is only understood in the very-weak sense (cf. Theorem 3.4). Moreover a numerical approximation of solutions to this problem will require a direct approximation of the interaction operator \( N_s \) which is challenging.

The purpose of this section is to not only introduce a new approach to approximate weak and very-weak solutions to the nonhomogeneous Dirichlet exterior value problem (recall that if \( z \) is regular enough then a very-weak solution is a weak solution, and every weak solution is a very-weak solution cf. Theorem 3.4) but also to consider a regularized fractional Dirichlet exterior control problem. We begin by stating the regularized Dirichlet exterior value problem: Let \( n \in \mathbb{N} \). Find \( u_n \in W^{s,2}_{\Omega,k} \) solving

\[
\begin{cases}
(-\Delta)^s u_n &= 0 \quad \text{in } \Omega \\
N_s u_n + n\kappa u_n &= n\kappa z \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Notice that the fractional regularized Dirichlet exterior problem (6.1) is nothing but the fractional Robin exterior value problem (1.4b). We proceed by showing that as \( n \to \infty \) the solution \( u_n \) to (6.1) converges to \( u \) solving the state equation (1.2) in the weak sense (3.3). This is our new method to solve the non-homogeneous Dirichlet exterior value problem. Recall that the weak formulation of (6.1) does not require access to \( N_s \) (cf. Definition (3.7)) and it is straightforward to implement.

In this section we are interested in solutions \( u_n \) to the system (6.1) that belong to the space \( W^{s,2}_{\Omega,k} \cap L^2(\mathbb{R}^N \setminus \Omega) \) which is endowed with the norm

\[
\| u \|_{W^{s,2}_{\Omega,k} \cap L^2(\mathbb{R}^N \setminus \Omega)} := \left( \| u \|_{W^{s,2}_{\Omega,k}}^2 + \| u \|_{L^2(\mathbb{R}^N \setminus \Omega)}^2 \right)^{\frac{1}{2}}, \quad u \in W^{s,2}_{\Omega,k} \cap L^2(\mathbb{R}^N \setminus \Omega).
\]

In addition, in our application we shall take \( \kappa \) such that its support \( \text{supp}[\kappa] \) is a compact set in \( \mathbb{R}^N \setminus \Omega \). For this reason we shall assume the following.

Assumption 6.1. We assume that \( \kappa \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega) \) and satisfies \( \kappa > 0 \) almost everywhere in \( K := \text{supp}[\kappa] \subset \mathbb{R}^N \setminus \Omega \), where \( K \) is a compact set.
It follows from Assumption 6.1 that \( \int_{\mathbb{R}^N \setminus \Omega} \kappa \, dx > 0 \).

To show the existence of weak solutions to the system in (6.1) that belong to \( W_{\Omega, \kappa}^{s, 2} \cap L^2(\mathbb{R}^N \setminus \Omega) \), we need some preparation.

**Lemma 6.2.** Assume that Assumption 6.1 holds. Then

\[
\|u\|_{W} := \left( \int_{\mathbb{R}^N \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy + \int_{\mathbb{R}^N \setminus \Omega} |u|^2 \, dx \right)^{\frac{1}{2}},
\]

defines an equivalent norm on \( W_{\Omega, \kappa}^{s, 2} \cap L^2(\mathbb{R}^N \setminus \Omega) \).

**Proof.** Firstly, it is readily seen that there is a constant \( C > 0 \) such that

\[
\|u\|_W \leq C \|u\|_{W_{\Omega, \kappa}^{s, 2} \cap L^2(\mathbb{R}^N \setminus \Omega)} \quad \text{for all } u \in W_{\Omega, \kappa}^{s, 2} \cap L^2(\mathbb{R}^N \setminus \Omega).
\]

Secondly we claim that there is a constant \( C > 0 \) such that

\[
\|u\|_{W_{\Omega, \kappa}^{s, 2} \cap L^2(\mathbb{R}^N \setminus \Omega)} \leq C \|u\|_W \quad \text{for all } u \in W_{\Omega, \kappa}^{s, 2} \cap L^2(\mathbb{R}^N \setminus \Omega).
\]

It is clear that

\[
\int_{\mathbb{R}^N \setminus \Omega} |u|^2 \, d\mu \leq \|\kappa\|_{L^\infty(\mathbb{R}^N \setminus \Omega)} \int_{\mathbb{R}^N \setminus \Omega} |u|^2 \, dx.
\]

It suffices to show that there is a constant \( C > 0 \) such that for every \( u \in W_{\Omega, \kappa}^{s, 2} \cap L^2(\mathbb{R}^N \setminus \Omega) \),

\[
\int_{\Omega} |u|^2 \, dx \leq C \left( \int_{\mathbb{R}^N \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy + \int_{\mathbb{R}^N \setminus \Omega} |u|^2 \, dx \right).
\]

We prove (6.7) by contradiction. Assume to the contrary that for every \( n \in \mathbb{N} \), there exists \( (u_n)_{n \in \mathbb{N}} \subset W_{\Omega, \kappa}^{s, 2} \cap L^2(\mathbb{R}^N \setminus \Omega) \) such that

\[
\int_{\Omega} |u_n|^2 \, dx > n \left( \int_{\mathbb{R}^N \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} \, dxdy + \int_{\mathbb{R}^N \setminus \Omega} |u_n|^2 \, dx \right).
\]

By possibly dividing (6.8) by \( \|u_n\|_{L^2(\Omega)}^2 \) we may assume that \( \|u_n\|_{L^2(\Omega)}^2 = 1 \) for every \( n \in \mathbb{N} \). Hence, by (6.8), there is a constant \( C > 0 \) (independent of \( n \)) such that for every \( n \in \mathbb{N} \),

\[
\int_{\mathbb{R}^N \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} \, dxdy + \int_{\mathbb{R}^N \setminus \Omega} |u_n|^2 \, dx \leq C.
\]

Since \( \kappa \in L^\infty(\mathbb{R}^N \setminus \Omega) \), (6.9) and (6.6) imply that for every \( n \in \mathbb{N} \),

\[
\int_{\mathbb{R}^N \setminus \Omega} |u_n|^2 \, d\mu \leq C.
\]

Now (6.9), (6.10) together with \( \|u_n\|_{L^2(\Omega)}^2 = 1 \) implies that \( (u_n)_{n \in \mathbb{N}} \) is a bounded sequence in \( W_{\Omega, \kappa}^{s, 2} \cap L^2(\mathbb{R}^N \setminus \Omega) \). Therefore, after passing to a subsequence, if necessary, we may assume that \( u_n \) converges weakly to some \( u \in W_{\Omega, \kappa}^{s, 2} \cap L^2(\mathbb{R}^N \setminus \Omega) \) and strongly to \( u \) in \( L^2(\Omega) \), as \( n \to \infty \) (as the...
embedding $W^{s,2}_{\Omega,n} \hookrightarrow L^2(\Omega)$ is compact by Remark 3.6(c). It follows from (6.8) and the fact that $\|u_n\|_{L^2(\Omega)}^2 = 1$ that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_{\mathbb{R}^N \setminus \Omega} |u_n|^2 \, dx = 0.$$  

This implies that $u_n|_{\mathbb{R}^N \setminus \Omega}$ converges strongly to zero in $L^2(\mathbb{R}^N \setminus \Omega)$ as $n \to \infty$, and after passing to a subsequence, if necessary, we have that

$$\lim_{n \to \infty} |u_n(x) - u_n(y)| = 0 \quad \text{for a.e.} \quad (x, y) \in \mathbb{R}^N \setminus (\mathbb{R}^N \setminus \Omega)^2,$$  

and

$$u_n \to 0 \quad \text{a.e. in} \quad \mathbb{R}^N \setminus \Omega \quad \text{as} \quad n \to \infty. \quad (6.12)$$  

More precisely, (6.11) implies that

$$\begin{cases} 
\lim_{n \to \infty} |u_n(x) - u_n(y)| = 0 & \text{for a.e.} \quad (x, y) \in \Omega \times \Omega, \\
\lim_{n \to \infty} |u_n(x) - u_n(y)| = 0 & \text{for a.e.} \quad (x, y) \in \Omega \times (\mathbb{R}^N \setminus \Omega), \\
\lim_{n \to \infty} |u_n(x) - u_n(y)| = 0 & \text{for a.e.} \quad (x, y) \in (\mathbb{R}^N \setminus \Omega) \times \Omega.
\end{cases} \quad (6.13)$$

Using (6.13), we get that $u_n$ converges a.e. to some constant function $c$ in $\mathbb{R}^N$ as $n \to \infty$. From (6.12) and the uniqueness of the limit, we have that $c = 0$ a.e. in $\mathbb{R}^N$. Since (after passing to a subsequence, if necessary) $u_n$ converges a.e. to $u$ in $\Omega$ as $n \to \infty$, the uniqueness of the limit also implies that $c = u = 0$ a.e. on $\Omega$. On the other hand, we have $\|u_n\|_{L^2(\Omega)}^2 = \lim_{n \to \infty} \|u_n\|_{L^2(\Omega)}^2 = 1$, and this is a contradiction. Hence, (6.8) is not possible and we have shown (6.7). Finally the lemma follows from (6.4) and (6.5). The proof is finished. \quad \square

The following theorem is the main result of this section.

**Theorem 6.3 (Approximation of weak solutions to Dirichlet problem).** Assume that Assumption 6.1 holds. Then the following assertions hold.

(a) Let $z \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$ and $u_n \in W^{s,2}_{\Omega,n} \cap L^2(\mathbb{R}^N \setminus \Omega)$ be the weak solution of (6.1). Let $u \in W^{s,2}(\mathbb{R}^N)$ be the weak solution to the state equation (1.3b). Then there is a constant $C > 0$ (independent of $n$) such that

$$\|u - u_n\|_{L^2(\mathbb{R}^N)} \leq C \frac{1}{n} \|u\|_{W^{s,2}(\mathbb{R}^N)}. \quad (6.14)$$

In particular $u_n$ converges strongly to $u$ in $L^2(\Omega)$ as $n \to \infty$.

(b) Let $z \in L^2(\mathbb{R}^N \setminus \Omega)$ and $u_n \in W^{s,2}_{\Omega,n} \cap L^2(\mathbb{R}^N \setminus \Omega)$ be the weak solution of (6.1). Then there is a subsequence that we still denote by $\{u_n\}_{n \in \mathbb{N}}$ and a $\tilde{u} \in L^2(\mathbb{R}^N)$ such that $u_n \to \tilde{u}$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$, and $\tilde{u}$ satisfies

$$\int_{\Omega} \tilde{u}(-\Delta)^s v \, dx = - \int_{\mathbb{R}^N \setminus \Omega} \tilde{u}N_s v \, dx, \quad (6.15)$$

for all $v \in V$.

**Remark 6.4 (Convergence to very-weak solution).** Notice that Part (a) of Theorem 6.3 implies strong convergence to a weak solution (with rate). On the other hand, Part (b) “almost” implies weak convergence to a very-weak solution (we still do not know if $\tilde{u}|_{\mathbb{R}^N \setminus \Omega} = z$). We emphasize that such an approximation of very-weak solutions using Robin problem, to the best of
We have shown that there is a constant $C > 0$ such that $v$.

Taking integration by parts formula (2.5) we get that

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy + n \int_{\mathbb{R}^N \setminus \Omega} u_n v \, d\mu = n \int_{\mathbb{R}^N \setminus \Omega} z v \, d\mu,$$

(6.16)

holds for every $v \in W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)$. Proceeding as the proof of Proposition 3.8 we can easily deduce that for every $n \in \mathbb{N}$, there is a unique $u_n \in W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)$ satisfying (6.16).

For $v, w \in W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)$, we shall let

$$\mathcal{E}_n(v, w) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{N+2s}} \, dx \, dy + n \int_{\mathbb{R}^N \setminus \Omega} v w \, d\mu.$$

We notice that proceeding as the proof of Lemma 6.2 we can deduce that there is a constant $C > 0$ such that

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} \, dx \, dy + n \int_{\mathbb{R}^N \setminus \Omega} |u_n|^2 \, dx \leq C\mathcal{E}_n(u_n, u_n).$$

(6.17)

Next, let $u \in W^{s,2}(\mathbb{R}^N)$ be the weak solution of (3.1) and $v \in W^{s,2}_{\Omega,\kappa} \cap L^2(\mathbb{R}^N \setminus \Omega)$. Using the integration by parts formula (2.5) we get that

$$\mathcal{E}_n(u - u_n, v) = \int_{\Omega} (-\Delta)^s(u - u_n)v \, dx + \int_{\mathbb{R}^N \setminus \Omega} N_s(u - u_n)v \, dx$$

$$+ n \int_{\mathbb{R}^N \setminus \Omega} (u - u_n) v \, d\mu$$

$$= \int_{\Omega} (-\Delta)^s(u - u_n)v \, dx + \int_{\mathbb{R}^N \setminus \Omega} v N_s u \, dx$$

$$- \int_{\mathbb{R}^N \setminus \Omega} (N_s u_n + n\kappa(u_n - z)) \, v \, dx$$

$$= \int_{\mathbb{R}^N \setminus \Omega} v N_s u \, dx.$$  

(6.18)

Taking $v = u - u_n$ in (6.18) and using (6.17), we get that there is a constant $C > 0$ (independent of $n$) such that

$$n\|u - u_n\|^2_{L^2(\mathbb{R}^N \setminus \Omega)} \leq \mathcal{E}_n(u - u_n, u - u_n) = \int_{\mathbb{R}^N \setminus \Omega} (u - u_n) N_s u \, dx$$

$$\leq \|u - u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|N_s u\|_{L^2(\mathbb{R}^N \setminus \Omega)}$$

$$\leq C\|u - u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|u\|_{W^{s,2}(\mathbb{R}^N)}.$$  

We have shown that there is a constant $C > 0$ (independent of $n$) such that

$$\|u - u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)} \leq \frac{C}{n} \|u\|_{W^{s,2}(\mathbb{R}^N)},$$

(6.19)
Next, observe that
\[ \|u - u_n\|_{L^2(\Omega)} = \sup_{\eta \in L^2(\Omega)} \frac{\int_{\Omega} (u - u_n)\eta \, dx}{\|\eta\|_{L^2(\Omega)}}. \]  
(6.20)

For any \( \eta \in L^2(\Omega) \), let \( w \in W^{s,2}_0(\Omega) \) be the weak solution of the Dirichlet problem
\[ (-\Delta)^s w = \eta \quad \text{in } \Omega, \quad w = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \]  
(6.21)

It follows from Proposition 3.2 that there is a constant \( C > 0 \) such that
\[ \|w\|_{W^{s,2}(\mathbb{R}^N)} \leq C\|\eta\|_{L^2(\Omega)}. \]  
(6.22)

Since \( w \in W^{s,2}_0(\Omega) \), then using (6.18) we have that
\[ \int_{\Omega} (u - u_n)(-\Delta)^s w \, dx = \int_{\mathbb{R}^N \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u - u_n)(x) - (u - u_n)(y)(w(x) - w(y))}{|x - y|^{N+2s}} \, dx dy \]
\[ - \int_{\mathbb{R}^N \setminus \Omega} (u - u_n)N_s w \, dx \]
\[ = \mathcal{E}_n(u - u_n, w) - \int_{\mathbb{R}^N \setminus \Omega} (u - u_n)N_s w \, dx \]
\[ = \int_{\mathbb{R}^N \setminus \Omega} w N_s u \, dx - \int_{\mathbb{R}^N \setminus \Omega} (u - u_n)N_s w \, dx \]
\[ = - \int_{\mathbb{R}^N \setminus \Omega} (u - u_n)N_s w \, dx. \]

It follows from the preceding identity, (6.19) and (6.22) that
\[ \left| \int_{\Omega} (u - u_n)(-\Delta)^s w \, dx \right| = \left| \int_{\mathbb{R}^N \setminus \Omega} (u - u_n)N_s w \, dx \right| \]
\[ \leq \|u - u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|N_s w\|_{L^2(\mathbb{R}^N \setminus \Omega)} \]
\[ \leq C \|u\|_{W^{s,2}(\mathbb{R}^N)} \|w\|_{W^{s,2}(\mathbb{R}^N)} \]
\[ \leq C \|u\|_{W^{s,2}(\mathbb{R}^N)} \|\eta\|_{L^2(\Omega)}. \]  
(6.23)

Using (6.20) and (6.23) we get that
\[ \|u - u_n\|_{L^2(\Omega)} \leq C \|u\|_{W^{s,2}(\mathbb{R}^N)}. \]  
(6.24)

Now the estimate (6.14) follows from (6.19) and (6.24). Observe that it follows from (6.14) that \( u_n \to u \) in \( L^2(\mathbb{R}^N) \) as \( n \to \infty \) and this completes the proof of Part (a).

(b) Now let \( z \in L^2(\mathbb{R}^N \setminus \Omega) \hookrightarrow L^2(\mathbb{R}^N \setminus \Omega, \mu) \). Notice that \( \{u_n\}_{n \in \mathbb{N}} \) satisfies (6.16). Proceeding as the proof of Lemma 6.2 we can deduce that there is a constant \( C > 0 \) (independent of \( n \)) such that
\[ n\|u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2 \leq C \mathcal{E}_n(u_n, u_n) \leq nC\|\kappa\|_{L^\infty(\mathbb{R}^N \setminus \Omega)} \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \|u_n\|_{L^2(\mathbb{R}^N \setminus \Omega)}, \]
Hence, the sequence
\[ u_n \in L^2(\Omega) \]
and this implies that
\[ \|u_n\|_{L^2(\Omega)} \leq C\|z\|_{L^2(\Omega)}. \]  
(6.25)

Now we proceed as the proof of (6.24). As in (6.20) we have that
\[ \|u_n\|_{L^2(\Omega)} = \sup_{\eta \in L^2(\Omega)} \left| \int_\Omega u_n \eta \, dx \right|. \]  
(6.26)

Let \( \eta \in L^2(\Omega) \) and \( w \in W^{s,2}_0(\Omega) \) the weak solution of (6.21). Since \( w \in W^{s,2}_0(\Omega) \), then using (6.18) we have that
\[ \int_\Omega u_n(-\Delta)^s w \, dx = \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}\setminus(\Omega\times\Omega)} \frac{(u_n(x) - u_n(y))(w(x) - w(y))}{|x-y|^{N+2s}} \, dxdy - \int_{\Omega} u_n N_s w \, dx \]
\[ = - \int_{\Omega} u_n N_s w \, dx. \]

It follows from the preceding identity, (6.25) and (6.22) that
\[ \left| \int_\Omega u_n(-\Delta)^s w \, dx \right| = \left| \int_{\mathbb{R}^{2N}\setminus(\Omega\times\Omega)} u_n N_s w \, dx \right| \leq \|u_n\|_{L^2(\Omega)} \|N_s w\|_{L^2(\Omega)} \]
\[ \leq C \|z\|_{L^2(\Omega)} \|w\|_{W^{s,2}(\Omega)}. \]  
(6.27)

Using (6.25), (6.27) and (6.22) we get that there is a constant \( C > 0 \) (independent of \( n \)) such that
\[ \|u_n\|_{L^2(\Omega)} \leq C \|z\|_{L^2(\Omega)}. \]  
(6.28)

Combing (6.25) and (6.28) we get that
\[ \|u_n\|_{L^2(\Omega)} \leq C \|z\|_{L^2(\Omega)}. \]  
(6.29)

Hence, the sequence \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( L^2(\mathbb{R}^N) \). Thus, after a subsequence, if necessary, we have that \( u_n \) converges weakly to some \( \bar{u} \) in \( L^2(\mathbb{R}^N) \) as \( n \to \infty \).

Using (6.16) we get that for every \( v \in V := \{v \in W^{s,2}_0(\Omega) : (-\Delta)^s v \in L^2(\Omega)\} \),
\[ \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}\setminus(\Omega\times\Omega)} \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy = 0. \]  
(6.30)

Using the integration by part formula (2.5) we can deduce that
\[ \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}\setminus(\Omega\times\Omega)} \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy \]
\[ = \int_\Omega u_n(-\Delta)^s v \, dx + \int_{\Omega} u_n N_s v \, dx, \]  
(6.31)

for every \( v \in V \). Combing (6.30) and (6.31) we get that the identity
\[ \int_\Omega u_n(-\Delta)^s v \, dx + \int_{\Omega} u_n N_s v \, dx = 0, \]  
(6.32)

holds for every \( v \in V \). Passing to the limit in (6.32) as \( n \to \infty \), we obtain that
\[ \int_\Omega \tilde{u}(-\Delta)^s v \, dx + \int_{\Omega} \tilde{u} N_s v \, dx = 0, \]
for every \( v \in V \). We have shown (6.15) and the proof is finished. \( \Box \)

Toward this end we introduce the regularized fractional Dirichlet control problem

\[
\min_{u \in U_R, z \in Z_R} J(u) + \frac{\xi}{2} \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2,
\]

subject to the regularized boundary value problem (Robin problem): Find \( u_n \in U_R \) solving

\[
\begin{aligned}
(-\Delta)^s u &= 0 \quad \text{in } \Omega \\
N_s u + n\kappa u &= n\kappa z \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]

and the control constraints

\[
z \in Z_{ad,R}.
\]

Here \( Z_R := L^2(\mathbb{R}^N \setminus \Omega) \), \( Z_{ad,R} \) is a closed and convex subset and \( U_R := W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega) \). We again remark that (6.33) is nothing but the fractional Robin exterior control problem.

**Theorem 6.5 (Approximation of the Dirichlet control problem).** The regularized control problem (6.33) admits a minimizer \((z_n, u_n(z_n)) \in Z_{ad,R} \times (W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega))\). Let \( Z_R = W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega) \) and \( Z_{ad,R} \subset Z_R \) be bounded. Then for any sequence \( \{n_\ell\}_{\ell=1}^\infty \) with \( n_\ell \to \infty \), there exists a subsequence still denoted by \( \{n_\ell\}_{\ell=1}^\infty \) such that \( z_{n_\ell} \rightharpoonup \bar{z} \) in \( W_{\Omega,\kappa}^{s,2}(\mathbb{R}^N \setminus \Omega) \) and \( u(z_{n_\ell}) \to u(\bar{z}) \) in \( L^2(\mathbb{R}^N) \) as \( n \to \infty \) with \((\bar{z}, u(\bar{z}))\) solving the Dirichlet control problem (1.3) with \( Z_{ad,D} \) replaced by \( Z_{ad,R} \).

**Proof.** Since the regularized control problem (6.33) is nothing but the Robin control problem therefore existence of minimizers follows by directly using Theorem 5.1. Following the proof of Theorem 5.1 and using the fact that \( Z_{ad,R} \) is a bounded subset of the reflexive Banach space \( W_{\Omega,\kappa}^{s,2}(\mathbb{R}^N \setminus \Omega) \), after a subsequence, if necessary, we have that \( z_{n_\ell} \rightharpoonup \bar{z} \) in \( W_{\Omega,\kappa}^{s,2}(\mathbb{R}^N \setminus \Omega) \) as \( n_\ell \to \infty \).

Now since \( Z_{ad,R} \) is closed and convex, then it is weakly closed. Thus \( \bar{z} \in Z_{ad,R} \).

Following the proof of Theorem 6.3 (a) there exists a subsequence \( \{u_{n_\ell}\} \) such that \( u_{n_\ell} \to \bar{u} \) in \( L^2(\mathbb{R}^N) \) as \( n_\ell \to \infty \). Combining this convergence with the aforementioned convergence of \( z_{n_\ell} \) we conclude that \((\bar{z}, \bar{u}) \in Z_{ad,R} \times W_{\Omega,\kappa}^{s,2}(\mathbb{R}^N) \) solves the Dirichlet exterior value problem (1.3)

It then remains to show that \((\bar{z}, \bar{u})\) is a minimizer of (1.3). Let \((z', u')\) be any minimizer of (1.3). Let us consider the regularized state equation (6.33b) but with boundary datum \( z' \). We denote the solution of the resulting state equation by \( u'_{n_\ell} \). By using the same limiting argument as above we can select a subsequence such that \( u'_{n_\ell} \to u' \) in \( L^2(\mathbb{R}^N) \) as \( n \to \infty \). Letting \( j(z', u) := J(u) + \frac{\xi}{2} \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)}^2 \), it then follows that

\[
j(z', u') \leq j(\bar{z}, \bar{u}) \leq \liminf_{n \to \infty} j(z_{n_\ell}, u_{n_\ell}) \leq \liminf_{n \to \infty} j(z', u'_{n_\ell}) = j(z', u')
\]

where the second inequality is due to the weak-lower semicontinuity of \( J \). The third inequality is due to the fact that \( \{(z_{n_\ell}, u_{n_\ell})\} \) is a sequence of minimizers for (6.33). This is what we needed to show. \( \Box \)

We conclude this section by writing the stationarity system corresponding to (6.33): Find \((z, u, p) \in Z_{ad,R} \times (W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)) \times (W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega))\) such that

\[
\begin{aligned}
\mathcal{E}(u, v) &= \int_{\mathbb{R}^N \setminus \Omega} n\kappa z v \, dx, \\
\mathcal{E}(w, p) &= \int_{\Omega} J'(u) w \, dx, \\
\int_{\mathbb{R}^N \setminus \Omega} (n\kappa p + \xi z)(\bar{z} - z) \, dx &\geq 0,
\end{aligned}
\]

for all \((\bar{z}, v, w) \in Z_{ad} \times (W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega)) \times (W_{\Omega,\kappa}^{s,2} \cap L^2(\mathbb{R}^N \setminus \Omega))\).
7. Numerics

The purpose of this section is to introduce numerical approximation of the problems we have considered so far. In Subsection 7.1, we begin with a finite element approximation of the Robin problem (6.1) which is the same as the regularized Dirichlet problem. We approximate the Dirichlet problem using the Robin problem. Next in Subsection 7.2, we introduce an external source identification problem where we clearly see the difference between the nonlocal case and the classical case ($s \sim 1$). Finally, Subsection 7.3 is devoted to optimal control problems.

7.1. Approximation of nonhomogeneous Dirichlet problem via Robin problem.

In view of Theorem 6.3, we can approximate the Dirichlet problem with the help of the Robin (regularized Dirichlet) problem (6.1). Therefore we begin by introducing a discrete scheme for the Robin problem. Let $\tilde{\Omega}$ be a large enough open bounded set containing $\Omega$. We consider a conforming simplicial triangulation of $\Omega$ and $\tilde{\Omega}\setminus\Omega$ such that the resulting partition remains admissible. We shall assume that the support of the datum $z$ and $\kappa$ is contained in $\tilde{\Omega}\setminus\Omega$. We let our finite element space $V_h$ (on $\tilde{\Omega}$) to be a set of standard continuous piecewise linear functions. Then the discrete (weak) version of (6.33b) with nonzero right-hand-side is given by: Find $u_h \in V_h$ such that

$$
\int_{\tilde{\Omega}\setminus\Omega} \frac{(u_h(x) - u_h(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx 
+ \int_{\tilde{\Omega}\setminus\Omega} n\kappa u_h \cdot \nu \, dx
= \langle f, v \rangle_{W^{s,2}(\mathbb{R}^N\setminus\Omega), W^{s,2}(\mathbb{R}^N\setminus\Omega)} + \int_{\tilde{\Omega}\setminus\Omega} n\kappa z \cdot \nu \, dx \quad \forall v \in V_h.
$$

We approximate the double integral over $\mathbb{R}^N\setminus(\mathbb{R}^N\setminus\Omega)^2$ by using the approach from [2, 1]. The remaining integrals are computed using quadrature which is accurate for polynomials of degree less than and equal to 4.

We next consider an example that has been taken from [3]. Let $\Omega = B_{0}(1/2) \subset \mathbb{R}^2$ then our goal is to find $u$ solving

$$
(-\Delta)^s u = 2 \quad \text{in } \Omega
$$

$$
u(\cdot) = \frac{2^{-2s}}{\Gamma(1 + s)^2} \left(1 - |\cdot|^2\right)^{s} \quad \text{in } \mathbb{R}^N \setminus \Omega.
$$

The exact solution in this case is given by

$$
u(x) = u_1(x) + u_2(x) = \frac{2^{-2s}}{\Gamma(1 + s)^2} \left(\left(1 - |x|^2\right)^{s} + \left(\frac{1}{4} - |x|^2\right)^{s}\right),$$

where $u_1$ and $u_2$ solve

$$
\begin{cases}
(-\Delta)^s u_1 = 1 & \text{in } \Omega \\
\frac{2^{-2s}}{\Gamma(1 + s)^2} \left(1 - |\cdot|^2\right)^{s} & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
$$

$$
\begin{cases}
(-\Delta)^s u_2 = 1 & \text{in } \Omega \\
0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
$$

We let $\tilde{\Omega} = B_{0}(1.5)$. We next approximate (7.2) using (7.1) and we set $\kappa = 1$. Notice that we use a quasiuniform mesh. At first we fix $s = 0.5$ and Degrees of Freedom (DoFs) to be DoFs = 2920. For this configuration, we study the $L^2(\Omega)$ error $\|u - u_h\|_{L^2(\Omega)}$ with respect to $n$ in Figure 2 (left). As expected from Theorem 6.3 (a) we observe an approximation rate of $1/n$.

Next for a fixed $s = 0.5$, we check the stability of our scheme with respect to $n$ as we refine the mesh. We have plotted the $L^2$-error as we refine the mesh (equivalently increase DoFs) for
We notice that the error remains stable with respect to $n$ and we observe the expected rate of convergence with respect to DoFs $\|u - u_h\|_{L^2(\Omega)} \approx (\text{DoFs})^{-\frac{1}{2}}$.

In all cases we obtain the expected rate of convergence $\text{(DoFs)}^{-\frac{1}{2}}$.

Figure 2. Left panel: Let $s = 0.5$ and DoFs = 2920 be fixed. We let $\kappa = 1$ and consider $L^2$-error between actual solution $u$ to the Dirichlet problem and its approximation $u_h$ which solves the Robin problem. We have plotted the error with respect to $n$. We observe a rate of $1/n$ which confirms our theoretical result (6.14).

Middle panel: Let $s = 0.5$ be fixed. For each $n = 1e2, 1e3, 1e4, 1e5$ we have plotted the $L^2$-error with respect to degrees of freedom (DOFs) as we refine the mesh. We notice the error is stable with respect to $n$. In addition, the rate of convergence is $(\text{DoFs})^{-\frac{1}{2}}$ (as expected) and is independent of $n$.

Right panel: Let $n = 1e5$ be fixed. We again plot the $L^2$-error with respect to DOFs for various values of $s$. The effective convergence rate is again $(\text{DoFs})^{-\frac{1}{2}}$ and is independent of $s$.

7.2. External source identification problem. We next consider an inverse problem to identify a source that is located outside the observation domain $\Omega$. The optimality system is as given in (6.34) where we have approximated the Dirichlet problem by the Robin problem. We use the continuous piecewise linear finite element discretization for all the optimization variables: state ($u$), control ($z$), and adjoint ($p$). We choose our objective function as

$$J(u, z) = \frac{1}{2} \|u - u_d\|_{L^2(\Omega)}^2 + \frac{\xi}{2} \|z\|_{L^2(\Omega)}^2,$$

and we let $Z_{ad,R} := \{z \in Z_R : z \geq 0, \text{ a.e. in } \hat{\Omega}\}$ where $\hat{\Omega}$ is the support set of control $z$ and $\kappa$ that is contained in $\hat{\Omega} \setminus \Omega$. Moreover $u_d : L^2(\Omega) \rightarrow \mathbb{R}$ is the given data (observations). All the optimization problems below are solved using projected-BFGS method with Armijo line search.

Our computational setup is shown in Figure 3. The centered square region is $\Omega = [-0.4, 0.4]^2$ and the region inside the outermost ring is $\hat{\Omega} = B_0(1.5)$. The smaller square inside $\hat{\Omega} \setminus \Omega$ is $\hat{\Omega}$ which is the support of source/control. The right panel in Figure 2 shows a finite element mesh with DoFs = 6103.

We define $u_d$ as follows. For $z = 1$, we first solve the state equation for $\tilde{u}$ (first equation in (6.34)). We then add a normally distributed random noise with mean zero and standard deviation 0.02 to $\tilde{u}$. We call the resulting expression as $u_d$. Furthermore, we set $\kappa = 1$, and $n = 1e5$.

Our goal is then to identify the source $\bar{z}_h$. In Figure 4, we first show the behavior of optimal $\bar{z}_h$ for different values of the regularization parameter $\xi = 1e-1, 1e-2, 1e-4, 1e-8, 1e-10$. As
Figure 3. Left: computational domain where the inner square is $\Omega$, the region inside the outer circle is $\tilde{\Omega}$ and the outer square inside $\tilde{\Omega} \setminus \Omega$ is $\hat{\Omega}$ which is the region where source/control is supported. Right: A finite element mesh.

expected the larger the value of $\xi$, the smaller the magnitude of $\bar{z}_h$, and this behavior saturates at $\xi = 1e - 8$.

Figure 4. External source identificaiton problem. The panels show the behavior of $\bar{z}_h$ with respect to the regularization parameter: top row from left to right $\xi = 1e - 1, 1e - 2, 1e - 4$; bottom row from left to right: $\xi = 1e - 8, 1e - 10$. As expected the larger $\xi$, the smaller the magnitude of $\bar{z}_h$, but it saturates at $\xi = 1e - 8$.

Next, for a fixed $\xi = 1e - 8$, Figure 5 shows the optimal $\bar{z}_h$ for $s = 0.1, 0.6, 0.7, 0.8, 0.9$. We notice that for large $s$, $\bar{z}_h \equiv 0$. This is expected as larger the $s$ is, the more close we are to the classical Poisson case and we know that we cannot impose external condition in that case.

7.3. Dirichlet control problem. We next consider two Dirichlet control problems. The setup is similar to Subsection 7.2 except now we set $u_d \equiv 1$.

Example 7.1. The computational setup for the first example is shown in Figure 6. Let $\Omega = B_0(1/2)$ (the region insider the innermost ring) and the region inside the outermost ring is $\hat{\Omega} = B_0(1.5)$. 

Figure 5. The panels show the behavior of $\bar{z}_h$ as we vary the exponent $s$. Top row from left to right: $s = 0.1, 0.6, 0.7$. Bottom row from left to right: $s = 0.8, 0.9$. For smaller values of $s$, the recovery of $\bar{z}_h$ is quite remarkable. However, for larger values of $s$, $\bar{z}_h \equiv 0$ as expected – the behavior of $\bar{u}_h$ for large $s$ is close to the classical Poisson problem which does not allow external sources.

The annulus inside $\tilde{\Omega} \setminus \Omega$ is $\hat{\Omega}$ which is the support of control. The right panel in Figure 6 shows a finite element mesh with DoFs = 6069.

In Figures 7 and 8 we have shown the optimization results for $s = 0.2$ and $s = 0.8$, respectively. The top row shows the desired state $u_d$ (left) and the optimal state $\bar{u}_h$ (right). The bottom row shows the optimal control $\bar{z}_h$ (left) and the optimal adjoint variable $\bar{p}_h$ (right). We notice that in both cases we can approximate the desired state to a high accuracy but the approximation is slightly better for smaller $s$, especially close to the boundary. This is to be expected as for large values of $s$ the regularity of the adjoint variable deteriorates significantly (cf. Remark 4.4).

Example 7.2. The computational setup for our final example is shown in Figure 9. The M-shape region is $\Omega$ and the region inside the outermost ring is $\tilde{\Omega} = B_0(0.6)$. The smaller region inside $\Omega \setminus \hat{\Omega}$

Figure 6. Left: computational domain where the inner circle is $\Omega$, the region inside the outer circle is $\tilde{\Omega}$, and the annulus inside $\tilde{\Omega} \setminus \Omega$ is $\hat{\Omega}$ which is the region where the control is supported. Right: A finite element mesh.
Figure 7. Example 1, $s = 0.2$: Top row: Left - Desired state $u_d$; Right - Optimal state $\bar{u}_h$. Bottom row: Left - Optimal control $\bar{z}_h$, Right - Optimal adjoint $\bar{p}_h$.

Figure 8. Example 1, $s = 0.8$: Top row: Left - Desired state $u_d$; Right - Optimal state $\bar{u}_h$. Bottom row: Left - Optimal control $\bar{z}_h$, Right - Optimal adjoint $\bar{p}_h$.

is $\hat{\Omega}$ which is the support of control. The right panel in Figure 6 shows a finite element mesh with DoFs = 4462.
Figure 9. Left: computational domain where the M-shaped region is $\Omega$, the region inside the outer circle is $\tilde{\Omega}$ and the region inside $\tilde{\Omega} \setminus \Omega$ is $\hat{\Omega}$ which is the region where control is supported. Right: A finite element mesh.

In Figure 10 we have shown the optimization results for $s = 0.8$. The top row shows desired state $u_d$ (left) and optimal state $\bar{u}_h$ (right). The bottom row shows the optimal control $\bar{z}_h$ (left) and the optimal adjoint variable $\bar{p}_h$ (right). Even though the control is applied in an extremely small region we can still match the desired state in certain parts of $\Omega$.

Figure 10. Example 3, $s = 0.8$: Top row: Left - Desired state $u_d$; Right - Optimal state $\bar{u}_h$. Bottom row: Left - Optimal control $\bar{z}_h$, Right - Optimal adjoint $\bar{p}_h$.

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