An algorithm for classifying origamis into components of Teichmüller curves

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Abstract

Nontrivial examples of Teichmüller curves have been studied systematically with notions of combinatorics invariant under affine homeomorphisms. An origami (a square-tiled surface) induces a Teichmüller curve for which the absolute Galois group acts on the embedded curve in the moduli space. In this paper, we study general origamis admitting pure half-translation (i.e. the quadratic differential is not a global square). The canonical double covering of such a flat surface is determined by a pair of an abelian origami and a tuple of signs. We present an algorithm for describing the $\text{PSL}(2,\mathbb{Z})$-action on origamis and calculating their Teichmüller curves by comparing their parallelogram decompositions. We list the concrete result for degree $d \leq 7$ and consider the Galois conjugacy of origamis.

1 Introduction

A holomorphic quadratic differential on a Riemann surface induces a flat structure, on which several notions of affine geometry are well-defined. Sometimes a square root of a quadratic differential defines an abelian differential and a translation structure: we call such a case abelian. Affine deformations of a flat structure induce a geometric holomorphic disk on the Teichmüller space. Its projected image in the moduli space is an orbifold isomorphic to

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the quotient of the unit disk by the Veech group, the group of derivatives of self affine deformations on a flat surface. Veech groups are originally studied by Veech [15] in the context of the billiard flow (a geodesic flow on a surface that represents ‘orbits of billiard balls on a billiard table’). We note that that the first nontrivial examples of Veech group are presented in his paper. Nontrivial examples of Veech groups have been studied systematically with the notion of combinatorics coming from coverings. An abelian origami is a covering of once punctured torus equipped with a natural translation structure. Schmithüsen [13] proved that the universal Veech group \( SL(2, \mathbb{Z}) \) acts on abelian origamis as automorphisms of the free group \( F_2 \) and the Veech group of an abelian origami is the stabilizer of a \( F_2 \)-subgroup corresponding to the fundamental group. Shinomiya [14] considered the Veech groups of coverings of the the regular \( 2n \)-gon translation surface. He proved that their universal Veech group is generated by two matrices that correspond to a Dehn twist and a rotation. The Veech group of such a surface is obtained as a stabilizer of the action of the universal Veech group on the fundamental group. By combining the Reidemeister-Schreier method [10] with their results, we can specify the Veech group for each kind of translation surface. Abelian origamis are also studied in the context of the Galois action on combinatorial objects, as well as \textit{dessins d’enfants}. An origami induces a Teichmüller curve defined over a number field, for which the two Galois actions are compatible: for the belonging curves of the Teichmüller curve and for the embedded Teichmüller curve in the moduli space. A crucial result is given by Möller [11] and an overview of this ground is described in [5] and [9]. The faithfulness of the Galois action on origamis is proved in [11] and [12] using ‘M-origamis’ constructed from dessins where any nontrivial Galois conjugacy is inherited. There are some examples of nontrivial Galois conjugate origamis in [12].

In this paper, we deal with a general \textit{origami} which is a surface obtained from finitely many unit squares equipped with natural flat structure. Such a surface corresponds to an abelian origami with a sign list of squares. We give a condition for isomorphism class and a description of parallelogram decomposition in directions deformed by elements in \( PSL(2, \mathbb{Z}) \) for each origami. By comparing decompositions we obtain a picture of Teichmüller curve. We exhaust the calculation for all origamis of degree \( d \leq 7 \) and classify their Teichmüller curves by some classical Galois invariants. Our result suggests that to consider general origamis gives rich examples in the context of Galois conjugacy.
The structure of this paper is as follows: In section 2 we mention some background concepts and results to explain the main theory. In section 3 we define some notations to deal with origamis and prove key lemmas used for the calculations. In section 4 we state the main algorithms. In section 5 we present the calculation results.

2 Preliminaries

2.1 flat surface

Let \( \mathbb{R} \) be a Riemann surface of type \((g,n)\) with \(2g - 2 + n > 0\).

**Definition 2.1.** A holomorphic quadratic differential \( \phi \) on \( \mathbb{R} \) is a tensor on \( \mathbb{R} \) whose restriction to each chart \((U,z)\) on \( \mathbb{R} \) is of the form \( \phi_U(z)dz^2 \) where \( \phi_U : U \rightarrow \hat{\mathbb{C}} \) is holomorphic. A pair \((\mathbb{R},\phi)\) is called a flat surface. We say that the set \( \text{Sing}(\mathbb{R},\phi) \) of marked points of \( \mathbb{R} \) and zeros and poles of \( \phi \) is the set of the singularities of \((\mathbb{R},\phi)\).

Let \( p_0 \in \mathbb{R}^* = \mathbb{R} \setminus \text{Sing}(\mathbb{R},\phi) \) and \((U,z)\) be a chart around \( p_0 \). Then, \( \phi \) defines a natural coordinate \((\phi\text{-coordinate})\) on \( U \) by

\[
\zeta_{\phi}(p) = \int_{p_0}^{p} \sqrt{\phi_U(z)}dz, \quad p \in U,
\]

for which \( \phi = (d\zeta_{\phi})^2 \) holds. The \( \phi \)-coordinates give an atlas \( A_{\phi} \) on \( \mathbb{R}^* \) any of whose transition map is a half-translation \( \zeta \mapsto \pm \zeta + c \) \((c \in \mathbb{C})\). Such a structure, an atlas any of whose coordinate transformation is a half-translation is called a flat structure. The atlas \( A_{\phi} \) extends to each singularity \( p_0 \in \text{Sing}(\mathbb{R},\phi) \) of order \( m \), with local representation

\[
\zeta_{\phi}(p) = \int_{p_0}^{p} \sqrt{z^m}dz = z(p)^{\frac{m}{2}+1}, \quad p \in U \setminus \{p_0\},
\]

for a suitable chart \((U,z)\) around \( p_0 \).

**Definition 2.2.** Let \((\mathbb{R},\phi)\) and \((S,\psi)\) be flat surfaces of genus \( g \).

1. For \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}) \), we denote by \([A] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) the quotient class of \( A \) in \( PSL(2,\mathbb{R}) \). We define \( f_A : \mathbb{C} \rightarrow \mathbb{C} \) by

\[
f_A(\xi + i\eta) = (a\xi + c\eta) + i(b\xi + d\eta), \quad \xi, \eta \in \mathbb{R}.
\]
(2) We say that a branched covering \( f : (R, \phi) \to (S, \psi) \) is locally-affine if there exist finite subsets \( \text{Sing}(R, \phi) \subset \Sigma_R \subset R \), \( \text{Sing}(S, \psi) \subset \Sigma_S \subset S \) such that \( f \) is restricted to a covering \( f : R \setminus \Sigma_R \to S \setminus \Sigma_S \) that is locally represented by \( z \mapsto f_A(z) + c \) for some \( A \in SL(2, \mathbb{R}) \) and \( c \in \mathbb{C} \) with respect to natural coordinates of \( \phi \) and \( \psi \). A locally-affine biholomorphism is called an isomorphism.

(3) For a locally-affine covering \( f : (R, \phi) \to (S, \psi) \), the local derivative \( A \) on \( R \setminus \Sigma_R \) is constant up to a factor \( \{\pm 1\} \). We call \( D(f) := [A] \in PSL(2, \mathbb{R}) \) the derivative of \( f \).

(4) The group \( \text{Aff}^+(R, \phi) \) of locally-affine self-homeomorphisms of \( (R, \phi) \) is called the affine group of \( (R, \phi) \). The group \( \Gamma(R, \phi) \) of the derivatives of elements in \( \text{Aff}^+(R, \phi) \) is called the Veech group of \( (R, \phi) \).

For a flat surface \( (R, \phi) \), the family of affine homeomorphisms on \( (R, \phi) \) forms a disk \( \Delta(R, \phi) \) isometrically embedded in the Teichmüller space of \( R \) by Teichmüller’s theorem. The stabilizer of \( \Delta(R, \phi) \) in the Teichmüller-modular group is known to be the affine group \( \text{Aff}^+(R, \phi) \). Furthermore, the action \( \text{Aff}^+(R, \phi) \) on \( \Delta(R, \phi) \) is given by the Veech group \( \Gamma(R, \phi) \) acting on the unit disk by Möbius transformations. (See [1] for instance.)

It is first observed by Veech [15] that a Veech group is a discrete group. In particular, the projected image \( C(R, \phi) \) of \( \Delta(R, \phi) \) in the moduli space is an orbifold isomorphic to \( \mathbb{H}/\Gamma(R, \phi) \). If \( \Gamma(R, \phi) \) has finite covolume, \( C(R, \phi) \) is an algebraic curve called the Teichmüller curve.

**Definition 2.3.** We say that a flat surface \( (R, \phi) \) is abelian if \( \phi \) becomes the square of an abelian differential on \( R \) and otherwise non-abelian.

In this paper, we consider integrable flat surfaces, equivalently flat surfaces having singularities of order no less than \(-1\). The space \( \mathcal{Q}_g \) of isomorphism classes of integrable flat surfaces of genus \( g \) is naturally straftified as

\[
\mathcal{Q}_g = \left( \bigsqcup A_g(2m_1, \ldots, 2m_k) \right) \sqcup \left( \bigsqcup Q_g(m'_1, \ldots, m'_{k'}) \right),
\]

(2.4)

where \( A_g(2m_1, \ldots, 2m_k) \) (resp. \( Q_g(m'_1, \ldots, m'_{k'}) \)) is the stratum of abelian (resp. non-abelian) flat surfaces of genus \( g \) with precisely \( k \) (resp. \( k' \)) singularities of orders \( 2m_1 \leq \cdots \leq 2m_k \) (resp. \( m'_1 \leq \cdots \leq m'_{k'} \)). It follows from Riemann-Roch theorem that the orders of singularities of a flat surface of genus \( g \) sum up to \( 4g - 4 \). Thus the indices in the stratification (2.4) run...
through integers such that
\[ m_1, \ldots, m_k \in \mathbb{N}, \quad m'_1, \ldots, m'_k \in \{-1\} \cup \mathbb{N}, \quad \sum_{j=1}^k 2m_j = \sum_{j=1}^{k'} m'_j = 4g - 4. \tag{2.5} \]

Kontsevich-Zorich \[6] \text{ and Lanneau } \[8] \text{ showed that strata } \mathcal{A}_g(2m_1, \ldots, 2m_k), \quad \mathcal{Q}_g(m'_1, \ldots, m'_k) \text{ with } (2.5) \text{ are noempty except for few exceptional cases.}

**Definition 2.4.** The canonical double covering of a flat surface \((R, \phi)\) is the double covering \(\pi_{\phi} : \hat{R} \to R\) obtained from the continuation of branches of locally defined abelian differential \(\sqrt{\phi}\) on \(R\).

By definition, the canonical double covering \(\pi_{\phi} : \hat{R} \to R\) induces a holomorphic abelian differential \(\hat{\phi} = \pi_{\phi}^* \sqrt{\phi}\) on \(\hat{R}\). As a consequence of [2, Theorem 3.6], a canonical double covering satisfies the following property.

**Proposition 2.5.** Let \((R, \phi)\) be a flat surface, \((S, \psi)\) be an abelian flat surface, and \(f : (S, \psi) \to (R, \phi)\) be a locally-affine covering. Then \(f\) is the composition of a locally-affine covering \(\hat{f} : (S, \psi) \to (\hat{R}, \hat{\phi})\) and the canonical double covering \(\pi_{\phi} : \hat{R} \to R\).

**Remark 2.6.** By continuing natural coordinates, every flat surface is represented by finitely many Euclidian polygons \((C_i)_{i \in I}\) with their edges glued by half-translations. On this representation, the canonical double covering is obtained as follows (see Figure 1):

1. For each \(i \in I\), take a copy \(C_i^+\) and a half-rotated copy \(C_i^-\) of the polygon \(C_i\). Denote by \(e_+\) (resp. \(e_-\)) the edge of the polygon \(C_i^+\) (resp. \(C_i^-\)) corresponding to an edge \(e\) of the polygon \(C_i\).

2. If an edge \(e\) of a polygon \(C_i\) is glued with an edge \(e'\) of a polygon \(C_i'\) by transition map of the form \(z \mapsto z + c\) (resp. \(-z + c\)), glue \(e_+\) with \(e_+\) and \(e_-\) with \(e_-\) (resp. \(e_+\) with \(e'_+\) and \(e_-\) with \(e'_-\)) by translation.

**2.2 \(\phi\)-metric**

Let \((R, \phi)\) be a flat surface of genus \(g\). The Euclidian metric lifts via \(\phi\)-coordinates to a flat metric on \(R\), called the \(\phi\)-metric. A geodesic of \(\phi\)-metric is called a \(\phi\)-geodesic. Via the \(\phi\)-coordinates, a \(\phi\)-geodesic is locally a line segment on the plane whose direction is uniquely determined in \(\mathbb{R}/\pi\mathbb{Z}\).
Definition 2.7. Let \((R, \phi)\) be a flat surface of genus \(g\).

(1) The \(\phi\)-cylinder generated by a \(\phi\)-geodesic \(\gamma\) is the union of all \(\phi\)-geodesics parallel (with same direction) and free homotopic to \(\gamma\). We define the direction of a \(\phi\)-geodesic by the one of its generator.

(2) \(\theta \in \mathbb{R}/\pi\mathbb{Z}\) is called Jenkins-Strebel direction of \((R, \phi)\) if almost every point in \(R\) lies on some closed \(\phi\)-geodesic in the direction \(\theta\). Let \(JS(R, \phi)\) denote the set of Jenkins-Strebel directions of \((R, \phi)\).

If a flat surface \((R, \phi)\) admits two Jenkins-Strebel directions \(\theta_1, \theta_2 \in JS(R, \phi)\), then the surface is decomposed into parallelograms which are intersections of cylinders in the directions \(\theta_1, \theta_2\). By continuing local segments in each directions, we may define a combinatorial structure of such a decomposition similar to an origami (see section 2.3). This structure is uniquely determined by the isomorphism class of \((R, \phi)\), and we may determine the existence of an affine map of prescribed derivative by comparing decompositions as follows.

Proposition 2.8 ([7]). Let \((R, \phi)\) be a flat surface with two distinct finite Jenkins-Strebel directions \(\theta_1, \theta_2 \in JS(R, \phi)\). A matrix \(A \in PSL(2, \mathbb{R})\) belongs to \(\Gamma(R, \phi)\) if and only if the following holds.

(1) \(A\theta_1, A\theta_2\) belongs to \(JS(R, \phi)\).

(2) Two origamis with compatible moduli lists given by the decomposition of \((R, \phi)\) in \((\theta_1, \theta_2)\) and \(A(\theta_1, \theta_2) := (A\theta_1, A\theta_2)\) are equivalent.
To calculate the Veech group of a flat surface \((R, \phi)\), it is good to consider the action of \(\text{Stab}_{PSL(2, \mathbb{R})} JS(R, \phi)\) on the set of parallelogram decompositions of \((R, \phi)\) defined by taking the decomposition in \(A^{-1}(\theta_1, \theta_2)\) for each \(\theta_1, \theta_2 \in JS(R, \phi)\) and \(A \in PSL(2, \mathbb{R})\). Then the image under \(A\) determines the isomorphism class of \((R, f_A^*\phi)\), and the stabilizer of a decomposition becomes the Veech group of \((R, \phi)\).

### 2.3 origami

**Definition 2.9.** An origami of degree \(d\) is a flat surface obtained by gluing \(d\) Euclidian unit squares at edges equipped with the flat structure induced from the natural coordinates of squares.

![An origami of degree 4: edges with the same marks are glued.](image)

In this paper, we shall mark all the corner points of an origami by treating nonsingular corner points as singularities of order 0. Figure 2 shows an example of origami of degree 4 that belongs to the stratum \(\mathcal{A}_2(0, 4)\). A cell-to-cell correspondence defines a locally-affine covering \(p : R \to E\) of the unit square torus \((E = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}), dz^2)\) branched at most over one point \(\infty \in E\). It induces the following characterizations of an abelian origami. (See [5] for details.)

**Lemma 2.10.** An origami of degree \(d\) is up to equivalence (in brackets) uniquely determined by each of the following.

(a) A connected, oriented graph \((\mathcal{V}, \mathcal{E})\) with \(|\mathcal{V}| = d\) such that each vertex has precisely one incoming edge and one outgoing edge labelled one with \(x\) and one with \(y\), respectively [up to equivalence of the natural filling graph embedding].

(b) A \(d\)-fold branched covering \(p : R \to E\) of a torus \(E\) branched at most over one point \(\infty \in E\) [up to covering equivalence].
(c) A pair \((x, y)\) of two elements in the symmetric group \(\mathfrak{S}_d\) generating a transitive permutation group \(G\) [up to conjugation in \(\mathfrak{S}_d\)].

(d) A subgroup \(H\) of the free group \(F_2\) of index \(d\) [up to conjugation in \(F_2\)].

The two generators \(x, y \in G\) correspond to the monodromies along the horizontal and vertical cylinder curves on \(E\), respectively. The permutation \(z = xyx^{-1}y^{-1} \in \mathfrak{S}_d\) corresponds to the monodromy around the corner point of square cells. An abelian origami belongs to the stratum \(A_g(2m_1, \ldots, 2m_n)\) if and only if \(z\) consists precisely of \(n\) cycles of lengths \(m_1 + 1, \ldots, m_n + 1\).

**Remark 2.11.**

- For an origami \((R, \phi)\), the set \(JS(R, \phi)\) is the set \(PSL(2, \mathbb{Z}) \cdot (0 + \pi \mathbb{Z}) \subset \mathbb{R}/\pi \mathbb{Z}\). By Proposition 2.8, the Veech group of an origami is a subgroup of \(PSL(2, \mathbb{Z})\).

- Let \(\bar{\Omega}_d\) be the set of all isomorphism classes of origamis of degree \(d \in \mathbb{N}\). \(PSL(2, \mathbb{Z})\) acts on \(\bar{\Omega}_d\) by \(A \in PSL(2, \mathbb{Z}) : \mathcal{O} \mapsto \mathcal{O}_A\) where \(\mathcal{O}_A\) is the origami obtained as the parallelogram decomposition of \(\mathcal{O}\) in \(A^{-1}(0, \frac{\pi}{2})\). In this situation, the Veech group of \(\mathcal{O}\) is the stabilizer of \(\mathcal{O}\).

- Schmithüsen [13] showed that the action of \(PSL(2, \mathbb{Z})\) on the set of abelian origamis is described by the automorphisms \(\gamma_T, \gamma_S\) on \(F_2 = \langle x, y \rangle\) defined by

\[
\gamma_T(x, y) = (x, xy), \quad \gamma_S(x, y) = (y, x^{-1}),
\]

(2.6)

where \([T] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\) and \([S] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\) are matrices generating \(PSL(2, \mathbb{Z})\).

Let \(\mathcal{O}\) be an arbitrary origami. As the Veech group \(\Gamma(\mathcal{O})\) is a finite index \(PSL(2, \mathbb{Z})\)-subgroup, it induces a Teichmüller curve \(C(\mathcal{O})\) as a Belyï covering \(\beta_{C(\mathcal{O})} : \mathbb{H}/\Gamma(\mathcal{O}) \to \mathbb{H}/PSL(2, \mathbb{Z})\) i.e. a covering of the Riemann sphere \(\hat{\mathbb{C}}\) branched at most over the three points \(0, 1, \infty\). The origami \(\mathcal{O}\) also admits a Belyï covering \(\beta_\mathcal{O}\) induced from the covering \(\beta(x, y) = 4x^2\) on \(y = 4x^3 - x\).

Both are algebraic curves defined over \(\bar{\mathbb{Q}}\) by Belyï theorem. Möller [11] proved the compatibility of the action of the absolute Galois group \(G_\mathbb{Q} = Gal(\bar{\mathbb{Q}}/\mathbb{Q})\) for the curve \(C(\mathcal{O})\) embedded in the moduli space. That is, for any \(\sigma \in G_\mathbb{Q}\), the curve \(C(\mathcal{O})^\sigma\) coincides with the Teichmüller curve \(C(\mathcal{O}^\sigma)\) induced from the origami \(\mathcal{O}^\sigma\) and every point \(\mathcal{O}_t \in C(\mathcal{O})\) that corresponds to a curve defined over \(\bar{\mathbb{Q}}\) is conjugated to a point \(\mathcal{O}_t^\sigma \in C(\mathcal{O})^\sigma\).
The valency list of a Belyi covering $\beta : C \to \hat{C}$ is the list $(k_0^0, \ldots, k_0^0 \mid k_1^1, \ldots, k_1^1 \mid k_{\infty}^\infty, \ldots k_{\infty}^\infty)$ of all the ramification indices over the three branched points. The genus of $C$, the degree and the valency list of $\beta$ are classically known as $G_{Q}$-invariants. It is classically known that the action of $G_{Q}$ preserves local behavior of a branched covering. With Möller’s work, we can see that the valency list of $\beta_{C(O)}$, the genus of $C(O)$, and the stratum of $O$ are $G_{Q}$-invariants of an origami $O$.

3 Notations and key lemmas

**Definition 3.1.** Let $I_d = \{1, \ldots, d\}$ be the set of $d$ indices and $\bar{I}_d = \{\pm 1, \ldots, \pm d\}$ be its double. Let $\mathcal{E}_d := \{\varepsilon : \{\pm 1\}^{I_d} \mid \varepsilon(-i) = \varepsilon(i), \; i \in \bar{I}_d\}$ be the set of even functions on $\bar{I}_d$. Let $\mathfrak{S}_d := \text{Sym}(\bar{I}_d)$ naturally embeds the symmetric group $\mathfrak{S}_d$ as the group of even functions. For each $x \in \mathfrak{S}_d$ and $\varepsilon \in \{\pm 1\}^{I_d}$, let $x^\varepsilon$ denote the mapping on $\bar{I}_d$ defined by

$$x^\varepsilon(i) = \begin{cases} 
  x(i) & \text{if } \varepsilon(i) = +1 \\
  x^{-1}(i) & \text{if } \varepsilon(i) = -1 
\end{cases} \text{ for each } i \in \bar{I}_d. \quad (3.1)$$

**Definition 3.2.** Let $\Omega_d := \mathfrak{S}_d \times \mathfrak{S}_d$ be the set of (possibly disconnected) abelian origamis of degree $d$, $\Omega_{2d} := \{O \in \Omega_{2d} \mid$ there exists an origami whose canonical double covering surface is $O\}$, and $\bar{\Omega}_d := \Omega_d \times \mathcal{E}_d$. For each $O = (x, y, \varepsilon) \in \bar{\Omega}_d$ and $i \in \bar{I}_d$, define

$$\begin{cases} 
  x_O(i) = x^{\text{sign}(i)} \\
  y_O(i) = \varepsilon(i) \cdot y^\varepsilon(i) \cdot \varepsilon(y^\varepsilon(i)) 
\end{cases} \quad (3.2)$$

Observe that the $\pi$-symmetry $w(-w(-i)) = i, \; i \in \bar{I}_d$ holds for $w = x_O, y_O$ and thus (3.2) defines two permutations $x_O, y_O \in \mathfrak{S}_d$. We may assign each triple $O = (x, y, \varepsilon) \in \bar{\Omega}_d$ to an origami of degree $d$ whose canonical double covering is the abelian origami $(x_O, y_O) \in \Omega_{2d}$ in the following way:

1. Cut the abelian origami $(x, y)$ at all edges (with the edge-pairings remembered).
2. Apply the vertical reflection to the $i$-th cell if $\varepsilon(i) = -1$ for each $i \in I_d$.
3. Glue all paired edges in such a way that with the natural coordinates, the quadratic differential $(dz)^2$ is globally defined on the resulting surface.
If we take an additional half-rotated copy with cells labelled by \( \{-1, \ldots, -d\} \), we can make similar constructions so that the abelian differential \( dz \) is globally defined on the resulting surface. In the construction, every horizontal sides remain the same, and \( x \) is uniquely extended to \( x_\Omega \) by \( \pi \)-symmetry. Natural coordinates around the upper side of the \( i \)-th cell are multiplied by \( \varepsilon(i) \), paired to the upper or lower sides of the \( \pm y^{\varepsilon(i)}(i) \)-th cells, and continued by translation to the lower side of the \( \varepsilon(i) \cdot y^{\varepsilon(i)}(i) \cdot \varepsilon(y^{\varepsilon(i)}(i)) \)-th cell. In this way, we obtain the abelian origami represented by \((x_\Omega, y_\Omega)\) as to be the canonical double covering. (see Figure 3)

![Figure 3: The origami given by \((x, y, \varepsilon)\) where \(x = (1)(2 3 4), y = (1 2)(3 4), \varepsilon = (+, +, +, -):\) edges with the same marks are glued.](image)

Conversely, an abelian origami \((x, y)\) admits an involutive deck transformation \(\tau\) locally represented by \(z \mapsto -z\). We may label the cells of the origami \((x, y)\) in \(\tilde{I}_d\) so that \(\tau\) acts as \(i \mapsto -i\), and then \(x, y\) have \(\pi\)-symmetry.

**Lemma 3.3.** For a cyclic permutation \(a = (a_1 a_2 \cdots a_m)\), denote \(a' := (-a_1 -a_2 \cdots -a_m)^{-1}\) and \(|a| := (|a_1| |a_2| \cdots |a_m|)\). Let \(y \in \tilde{S}_d\) be a permutation with \(\pi\)-symmetry and fix a cycle decomposition \(y = c_1 c'_1 \cdots c_n c'_n\). For \(j = 1, \ldots, n\), define \(\varepsilon_j \in \mathcal{E}_d\) so that \(\varepsilon_j(i) \cdot i\) belongs to the cycle \(c_j\) for each \(i \in \tilde{I}_d\). Then, \(y = \tilde{y} := |c_1| \cdots |c_n| \in \tilde{S}_d\) and \(\varepsilon = \varepsilon_y := \varepsilon_1 \cdots \varepsilon_n \in \mathcal{E}_d\) satisfy that

\[
y(i) = \varepsilon(i) \cdot y^\varepsilon(i) \cdot \varepsilon(y^\varepsilon(i)) \quad \text{for each} \ i \in \tilde{I}_d.
\]

**Proof.** Suppose \(n = 1\). Denote \(y = \alpha c' = (a_1 a_2 \cdots a_d)(-a_1 -a_2 \cdots -a_d)^{-1}\), \(y = (|a_1| |a_2| \cdots |a_d|)(-|a_1| -|a_2| \cdots -|a_d|)\), and \(a_{d+1} = a_1\). By definition,
we have $\varepsilon(a_i) = 1$ and $y(a_i) = \varepsilon(|a_{i+1}|a_{i+1}|)$, for all $i \in I_d$. For each $i \in I_d$,

$$y(y, \varepsilon)(a_i) = \varepsilon(a_i) \cdot y^\varepsilon(a_i)(a_i) \cdot \varepsilon(y^\varepsilon(a_i)(a_i)) = y(\text{sign}(a_i)|a_{i+1}|) \cdot \varepsilon(y(\text{sign}(a_i)|a_{i+1}|)) = \text{sign}(a_i)|a_{i+1}| \cdot \text{sign}(a_i)\varepsilon(|a_{i+1}|) = \varepsilon(|a_{i+1}|)|a_{i+1}| = y(a_i). \quad (3.4)$$

Thus $y(y, \varepsilon) := \varepsilon \cdot y^\varepsilon \cdot \varepsilon(y^\varepsilon)$ equals to $y$. Applying this to each cycle in $y = c_1c'_1 \cdots c_n c'_n$, we obtain the claim for general $n$. □

Remark that $\bar{y}, \varepsilon_y$ depends on the way choosing cycles $c_1, \ldots, c_n$ in the above proof. For any $x \in \bar{S}_d$ with $\pi$-symmetry, we may take $x = \bar{x}$ in similar way so that $x(i) = x^\text{sign}(i)$ for each $i \in I_d$. In particular, $O \mapsto (x_O, y_O)$ defines a 1-1 correspondence $\bar{\Omega}_d \rightarrow \Omega_{2d}^0$ up to isomorphism.

**Lemma 3.4.** Let $O_j = (x_j, y_j, \varepsilon_j) \in \bar{\Omega}_d$ ($j = 1, 2$) be two origamis. Then $O_1, O_2$ are isomorphic as flat surfaces if and only if there exists $\sigma = \delta \bar{\sigma} \in \bar{S}_d$ ($\delta \in \{\pm 1\}^I d, \bar{\sigma} \in \bar{S}_d$) such that the following holds on $I_d$:

1. $\delta = \delta \circ x_1$,
2. $x_2 = \bar{\sigma}^\#(x_1^\delta)$,
3. $\xi(y_2, \delta \circ \bar{\sigma}^{-1} \circ \varepsilon_1 \circ \bar{\sigma}^{-1} \circ \varepsilon_2) = 1$ where $\xi(\tau, \lambda) := \lambda \cdot \lambda(\tau) \in \mathcal{E}_d$,
4. $y_2 = \bar{\sigma}^\#(y_1^{\delta \varepsilon_1 \varepsilon_2 \varepsilon_2 \varepsilon_2 \varepsilon_2})$.

**Proof.** Assume that there exists an isomorphism between $O_1$ and $O_2$. By Proposition [2.5] it lifts via their canonical double covering and induces a cell-to-cell correspondence $\sigma \in \bar{S}_d$ such that $x_2(i) = \sigma^\#x_1(i)$ and $y_2(i) = \sigma^\#y_1(i)$ for $i \in I_d$. Since $x_1$ and $x_2$ have $\pi$-symmetry, it follows that $x_2(\sigma(-i)) = x_2(-\sigma(i))$ for each $i \in I_d$ and thus $\sigma \in \bar{S}_d$. For $i \in I_d$ and $\varepsilon \in \{\pm 1\}$, we
have the following:

\[
\sigma(x_1)(\varepsilon i) = (\delta \bar{\sigma})(x_1^{\text{sign}(\varepsilon i)}(\varepsilon i)) = \varepsilon \delta(x_i^\varepsilon(i))\bar{\sigma}(x_i^\varepsilon(i)), \quad \cdots (a)
\]

\[
x_2(\sigma(\varepsilon i)) = x_2^{\text{sign}(\sigma(\varepsilon i))}(\varepsilon \delta(i)\bar{\sigma}(i)) = \varepsilon \delta(i)x_2^{\varepsilon\delta(i)}(\bar{\sigma}(i)), \quad \cdots (b)
\]

\[
\sigma(y_1)(\varepsilon i) = (\delta \bar{\sigma})(\xi(y_1,\varepsilon_1)(\varepsilon i) \cdot y_1^{\varepsilon\xi_1(i)}(\varepsilon i)) = \varepsilon \xi(y_1^{\varepsilon\xi_1(i)}(\varepsilon i)) \cdot \delta(y_1^{\varepsilon\xi_1(i)}(\varepsilon i)) \cdot \bar{\sigma}(y_1^{\varepsilon\xi_1(i)}(\varepsilon i)), \quad \cdots (c)
\]

\[
y_2(\sigma(\varepsilon i)) = \xi(y_2^{\varepsilon_2}(\varepsilon i)) \cdot y_2^{\varepsilon_2}(\varepsilon \delta(i)\bar{\sigma}(i)) = \varepsilon \delta(i)\xi(y_2^{\varepsilon_2\delta(i)\varepsilon_2}(\varepsilon i)) \cdot y_2^{\varepsilon_2\delta(i)\varepsilon_2}(\bar{\sigma}(i)) \cdot \bar{\sigma}(i))(\bar{\sigma}(i)). \quad \cdots (d)
\]

By considering the equation \(x_2(\sigma(\varepsilon \bar{\sigma}^{-1}(i))) = \sigma(x_1(\varepsilon \bar{\sigma}^{-1}(i)))\), we obtain (1) and (2). Similarly for \(y_1, y_2\), setting \(\varepsilon = \varepsilon_2(i) \cdot \delta \circ \bar{\sigma}^{-1}(i)\), we obtain (4) and the following:

\[
\delta \circ \bar{\sigma}^{-1}(i) \cdot \xi(y_2^{\varepsilon_2\delta(i)\varepsilon_2}(\varepsilon i)) = \xi(y_1^{\varepsilon\xi_1(i)}(\varepsilon i)) \cdot \delta(y_1^{\varepsilon\xi_1(i)}(\varepsilon i)) \cdot \bar{\sigma}(y_1^{\varepsilon\xi_1(i)}(\varepsilon i)) = \xi(\bar{\sigma}#(y_1^{\varepsilon_1\xi_1(i)}(\varepsilon i)) \cdot \delta \circ \bar{\sigma}^{-1}(\bar{\sigma}#y_1^{\varepsilon_1\xi_1(i)}(\varepsilon i))).
\]

With (4) \(\bar{\sigma}#y_1^{\varepsilon_1\xi_1(i)}(\varepsilon i) = y_2(i)\), we conclude (3).

Suppose (1)-(4) conversely. Then for each \(i \in I_d\), we have (a) = (b) and (c) = (d) for one of \(\varepsilon \in \{\pm 1\}\). We may fill the equations for the other \(\varepsilon \in \{\pm 1\}\) as follows. First, the signs of (a) and (b) coincide by (1). The equality of the other parts of (a) and (b) follows from (2) taking inverse mappings of both sides. We can say the same for the other parts of (c) and (d). Finally, the equality of the signs of (c) and (d) follows from (3) for each \(y_2^{-1}(i) = \bar{\sigma}#(y_1^{-\varepsilon_1\xi_1}(\varepsilon i)) \cdot (\varepsilon i) \in I_d\). This observation completes the proof. \(\square\)

**Lemma 3.5.** Let \(O\) be an origami and \(\pi_\circ : \hat{O} \rightarrow O\) be the canonical double covering. Then, a point \(p \in O\) is a singularity of order \(m\) (\(m \in \{-1, 0\} \cup N\)) if and only if either

1. \(m\) is even and \(\pi_\circ^{-1}(p)\) consists of two points of order \(m\), or
2. \(m\) is odd and \(\pi_\circ^{-1}(p)\) consists of one point of order \(2m\).

In particular, the singularities of \(O\) are of orders \(2m_1, \ldots, 2m_k, m'_1, \ldots, m'_{k'}\) (\(m'_1, \ldots, m'_{k'} : \text{odd}\)) if and only if the permutation \(\pi_\circ = x_0 y_0 x_0^{-1} y_0^{-1}\) has a cycle decomposition \(a_1 b_1 \cdots a_k b_k c_1 \cdots c_{k'}\) such that
(1) $a_j, b_j$ are cycles of length $m_j + 1$ with $(i \mapsto -i)^\#(x_\mathcal{O}y_\mathcal{O})^\#a_j = b_j$, and

(2) $c_j$ is a cycle of length $m'_j + 1$ with $(i \mapsto -i)^\#(x_\mathcal{O}y_\mathcal{O})^\#c_j = c_j$.

**Proof.** The former claim directly follows from Definition 2.4. For the latter claim, recall that a cycle of $z_\mathcal{O}$ of length $m + 1$ uniquely corresponds to a singularity of the abelian origami $\mathcal{O}$ of order $2m$. The same holds for the half-rotated monodromy $(x_\mathcal{O}y_\mathcal{O})^\#z_\mathcal{O} = x_\mathcal{O}^{-1}y_\mathcal{O}^{-1}x_\mathcal{O}y_\mathcal{O}$. The involutive Deck transformation $\tau$ of $\pi_\mathcal{O}$ acts on $\bar{1}_d$ by $i \mapsto -i$ and on the monodromy group of $\mathcal{O}$ by $x_\mathcal{O} \mapsto x_\mathcal{O}^{-1}, y_\mathcal{O} \mapsto y_\mathcal{O}^{-1}$. With the former claim, we know that a singularity $p$ on $\mathcal{O}$ uniquely corresponds to either

(1) two cycles $a, b$ of $z_\mathcal{O}$ of length $m + 1$ with $\tau \cdot a = (i \mapsto -i)^\#b$ if $p$ has even order $2m$, or

(2) one cycle $c$ of $z_\mathcal{O}$ of length $m' + 1$ with $\tau \cdot c = (i \mapsto -i)^\#c$ if $p$ has odd order $m'$.

So the latter claim follows.

If an origami of degree $d$ has $n$ singularities, the Euler characteristic calculation shows that the genus $g$ is given by

$$g = 1 + \frac{d - n}{2}.$$  \hfill (3.5)

## 4 Algorithm

A *partition* of $d$ is a finite sequence of weakly decreasing positive integers that sum to $d$. The *partition number* $p(d)$, which counts the number of partitions of $d$, defines the following rapidly increasing sequence.

$$1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, \ldots$$

(cf. [http://oeis.org/A000041](http://oeis.org/A000041)) The following asymptotic formula [3] is known:

$$p(d) \sim \frac{1}{4d\sqrt{3}} \cdot e^{\sqrt{2d/3}}$$ \hfill (4.1)

The algorithm by Hashiguchi, Niki, and Nakagawa [4] enumerates all the partitions of given integer. We will accept the set $P(d) = \{(j_1, j_2, \ldots, j_n) :$ partition of $d\}$ as a known data.
To describe the isomorphism class of each origami $O = (x, y, \varepsilon) \in \bar{\Omega}_d$, we enumerate all the conjugators $\sigma = \delta \bar{\sigma} \in \bar{S}_d$ ($\delta \in \{\pm 1\}^I_d$, $\bar{\sigma} \in S_d$) satisfying the conditions in Lemma 3.4. By (2) in Lemma 3.4, up to isomorphisms, we only have to think of $x$ with the normalized cycle decompositions

$$x = (1 \cdots j_1)(j_1 + 1 \cdots j_2) \cdots \left(\sum_{k=1}^{n-1} j_k + 1 \cdots d\right)$$

(4.2)

for each partition $(j_1, j_2, \ldots, j_n) \in P(d)$. We will consider the restricted class of an origami, the set of origamis with the same ‘$x$’ and isomorphic to it. By Lemma 3.4, the restricted class is the conjugacy class in $\text{Stab}(x) := \{\sigma = \delta \bar{\sigma} \in \bar{S}_d \mid \delta = \delta \circ x \text{ and } x = \bar{\sigma}^\#(x^\delta) \text{ on } I_d\}$. Remark that for general $y \in \bar{S}_d$ and $\varepsilon \in E_d$, the mapping $y^\varepsilon : \bar{I}_d \to \bar{I}_d$ is not a bijection. It will be checked at (2) in Algorithm 4.1.

First, we present algorithms for enumerating all the $\text{Stab}(x)$-conjugacy classes of origamis of the form $(x, y, \varepsilon) \in \bar{\Omega}_d$ satisfying the conditions in Lemma 3.4 for each $x \in P(d)$.

**Algorithm 4.1.** For each $O = (x, y, \varepsilon) \in \bar{\Omega}_d$, we construct its restricted class $[O] = \{(x, y', \varepsilon') \in \bar{\Omega}_d \mid (x, y', \varepsilon') \sim (x, y, \varepsilon)\}$ in the following steps:

1. Take an element in $\text{Stab}(x)$: $\sigma = \delta \bar{\sigma} \in \bar{S}_d$ such that $\delta = \delta \circ x$ and $x = \bar{\sigma}^\#(x^\delta)$ on $I_d$.
2. For each $\varepsilon' \in E_d$, let $y_{\sigma, \varepsilon'} := \bar{\sigma}^\#(y^{\varepsilon' \circ \bar{\sigma}^{-1}})$. Verify $\varepsilon' \in E_d$ such that $y_{\sigma, \varepsilon'} \in \bar{S}_d$ and $\xi(y_{\sigma, \varepsilon'}, \delta \circ \bar{\sigma}^{-1} \cdot \varepsilon \circ \bar{\sigma}^{-1} \cdot \varepsilon') = 1$ on $I_d$.
3. Let $C_\sigma := \{(x, y_{\sigma, \varepsilon'}, \varepsilon') \mid \varepsilon' \text{ passes the test in (2)}\}$.
4. Go back to (1) for some other leftover $\sigma \in \text{Stab}(x)$. When we have been through all elements in $\text{Stab}(x)$, finish the algorithm and we conclude that $[O] = \bigcup_{\sigma \in \text{Stab}(x)} C_\sigma$.

**Algorithm 4.2.** Let $P(d) = \{(j_1, j_2, \ldots, j_d) : \text{partition of } d\}$. We obtain the set $C\bar{\Omega}_d$ of the restricted classes of all origamis of degree $d$ in the following steps.

1. $C\bar{\Omega}_d := \emptyset$
2. Take $j = (j_1, j_2, \ldots, j_d) \in P(d)$. Define as follows:
   - $d'_j := \max\{k \mid j_k > 0\}$,
   - $x_j := (1 \cdots j_1)(j_1 + 1 \cdots j_1 + j_2) \cdots (\sum_{k=1}^{d'-1} j_k + 1 \cdots d) \in \bar{S}_d$,
   - $R_j := \bar{S}_d \times E_d$. 

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(3) Take \((y, \varepsilon) \in R_j\). Apply Algorithm 4.1 to \((x_j, y, \varepsilon) \in \bar{\Omega}_d\) to get \([x_j, y, \varepsilon]\).

(4) Add \([x_j, y, \varepsilon]\) to \(C\bar{\Omega}_d\). After that, remove \((y(\mathcal{O}), \varepsilon(\mathcal{O}))\) from \(R_j\) for every \(\mathcal{O} = (x_j, y, \varepsilon) \in [(x_j, y, \varepsilon)]\).

(5) Go back to (3) until \(R_j = \emptyset\). If so, go to the next step.

(6) Go back to (2) for other leftover \(j \in P(d)\). When we have been through all elements in \(P(d)\), finish the algorithm.

Next, we calculate the permutations \(\varphi_T, \varphi_S \in \text{Sym}(C\bar{\Omega}_d)\) which correspond to \(T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in PSL(2, \mathbb{Z})\) acting on \(\bar{\Omega}_d\) as decomposing origamis into pairs of directions \(T(0, \pi/2) = (0, \pi/4)\) and \(S(0, \pi/2) = (-\pi/2, 0)\), respectively. Let \(C\bar{\Omega}_d\) be the output in Algorithm 4.2.

1. To obtain the permutation \(\varphi_T\), we consider as follows:

\[
(x, y, \varepsilon) \xrightarrow{\text{Def. 3.2}} (x, y) \xrightarrow{\gamma_T} (x_T, y_T) \xrightarrow{\text{conj.}} (x_T, y_T, \varepsilon_T).
\]

To apply Lemma 3.3, we calculate \(\varepsilon_{y_T}\) and a cycle decomposition of \(y_T\).

Remark that the decomposition into \(T(0, \pi/2) = (0, \pi/4)\) is given by \(\gamma_T\) and the conjugation in \((-i \mapsto ix^{-1}(i) \mid i \in I_d\) as shown in Figure 4.

Figure 4: Decomposition of origami (a) into \(T(0, \pi/2) = (0, \pi/4)\): The desired decomposition (c) is obtained from (b) applying \(\gamma_T\) and the conjugation in \((-i \mapsto -x^{-1}(i) \mid i \in I_d\).
For $\mathcal{O} = (x, y, \varepsilon) \in \bar{\bar{\Omega}}_d$, $a \in I_d$, and $\varepsilon' \in \{\pm 1\}$, we have:

$$
\gamma_T(y_\mathcal{O})(\varepsilon'a) = y_\mathcal{O} \circ x_\mathcal{O}(\varepsilon'a)
= y_\mathcal{O}(\varepsilon'x^{\varepsilon'}(a))
= \varepsilon(\varepsilon'x^{\varepsilon'}(a)) \circ y(\varepsilon'(x^{\varepsilon'}(a))) \circ \varepsilon(\varepsilon'x^{\varepsilon'}(a))
= \varepsilon(\varepsilon'x^{\varepsilon'}(a)) \circ \varepsilon'(x^{\varepsilon'}(a)) \circ \varepsilon(\varepsilon')(x^{\varepsilon'}(a))
= \varepsilon(\varepsilon'(x^{\varepsilon'}(a))) \circ \varepsilon(\varepsilon'(x^{\varepsilon'}(a))) \circ \varepsilon'(x^{\varepsilon'}(a)).
$$

(4.3)

Algorithm 4.3. Let $C = [(x, y, \varepsilon)] \in C\bar{\bar{\Omega}}_d$ be a restricted class. By (4.3), we obtain $\varphi_T(C)$ in the following steps:

1. $I'_d := I_d$, $j := 0$.
2. $a_{0,j} := \min(I'_d)$, $\varepsilon_{0,j} := 1$, $i := 0$.
3. $b_{i,j} := x^{\varepsilon_{i,j}}(a_{i,j})$, $a'_{i+1,j} := y^{\varepsilon_{i+1,j}}(b_{i,j})$, $\varepsilon_{i+1,j} := \varepsilon_{i,j}(b_{i,j}) \varepsilon(a_{i+1,j})$.
4. Remove $a_{i,j}$ from $I'_d$. Define:

$$
a_{i+1,j} := \begin{cases} 
a'_{i+1,j} & \text{if } \varepsilon_{i+1,j} = 1, \\
x^{-1}(a'_{i+1,j}) & \text{otherwise.}
\end{cases}
$$

5. If $a_{i+1,j} = a_{0,j}$, let $c_j := (a_{0,j} a_{1,j} \cdots a_{i,j})$. Otherwise, go back to (3) for the next $i$.
6. If $I'_d \neq \emptyset$, go back to (2) for the next $j$. Otherwise, finish the loop and let $x_T = x$, $y_T := c_1c_2 \cdots c_j$, and $\varepsilon_T := (a_{i,j} \mapsto \varepsilon'_{i,j})$.
7. Search for the isomorphism class $C_T \in C\bar{\bar{\Omega}}_d$ represented by $(x_T, y_T, \varepsilon_T)$ and we conclude that $\varphi_T(C) = C_T$.

2. To obtain the permutation $\varphi_S$, we consider as follows:

$$
(x, y, \varepsilon) \xrightarrow{\text{Def.3.2}} (x, y) \xrightarrow{\gamma_{S, \text{conj.}}} (x_S, y_S) \xrightarrow{\text{Lem.3.3, conj.}} (x_S, y_S, \varepsilon_S).
$$

We use two conjugators in $\bar{\bar{\mathcal{S}}}_d$: the former collects signs of cells in each vertical cylinder to apply Lemma 3.3, and the latter makes $x_S$ to be the normalized form 4.2. The former conjugator is given by $\sigma_\delta := (\pm i \mapsto \pm \delta(i)i \mid i \in I_d) \in \bar{\bar{\mathcal{S}}}_d$ where $\delta \in \mathcal{E}_d$ satisfies that for every cycle $c$ in $x$, $\{\delta(|i|)i \mid i \in c\}$ forms a cycle either $c$ or $c'$.
For \( O = (x, y, \varepsilon) \in \bar{\Omega}_d \), \( a \in I_d \), and \( \delta' \in \{\pm 1\} \), we have:

\[
\gamma_S(x)(\delta' a) = y(\delta' a) \\
= \varepsilon(\delta' a) \cdot y^{\varepsilon(\delta' a)}(\delta' a) \cdot \varepsilon(y^{\varepsilon(\delta' a)}(\delta' a)) \\
= \delta' \varepsilon(a) \cdot \delta' y^{\varepsilon(a)}(a) \cdot \delta' \varepsilon(y^{\varepsilon(a)}(a)) \\
= \delta' \varepsilon(a) \varepsilon(y^{\varepsilon(a)}(a)) \cdot y^{\varepsilon(a)}(a).
\]

(4.4)

Algorithm 4.4. By (4.4), we obtain \( \delta \) in the following steps:

1. \( I'_d := I_d \), \( j := 0 \).
2. \( a_{0,j} := \min(I'_d) \), \( \delta_{0,j} := 1 \), \( i := 0 \).
3. \( a_{i+1,j} := y^{\delta_{i,j}}(a_{i,j}), \delta_{i+1,j} := \delta_{i,j} \varepsilon(a_{i,j}) \varepsilon(a_{i+1,j}) \)
4. Remove \( a_{i,j} \) from \( I'_d \).
5. If \( a_{i+1,j} = a_{0,j} \), let \( c_j := (a_{0,j}, a_{1,j}, \ldots, a_{i,j}) \). Otherwise go back to (3) for the next \( i \).
6. If \( I'_d \neq \emptyset \) then go back to (2) for the next \( j \). Otherwise finish the loop and let \( x'_S := c_1 c_2 \cdots c_j \) and \( \delta := (a_{i,j} \mapsto \delta_{i,j}) \).

To apply Lemma 3.3, we will calculate \( \varepsilon_{\sigma \delta} y_S \) and a cycle decomposition of \( \sigma \delta y_S \). After that, we apply the conjugator which makes \( x_S \) to the normalized form (4.2). So in advance, we will prepare the list \( \{\sigma x_p \mid \sigma \in \mathfrak{S}_d\} \) equipped with information of conjugator for each \( p \in P(d) \). Note that ‘x’s of any isomorphic two origamis share the same partition by Lemma 3.4. So the restricted classes calculated from Algorithm 4.1 with this list exhausts all the patterns of origamis.

For \( (x, y, \varepsilon) \in \bar{\Omega}_d \), \( a \in I_d \) and \( \varepsilon' \in \{\pm 1\} \), we have:

\[
\delta^\#(y_S)(\varepsilon' a) = \delta(x^{-1}(\varepsilon(\varepsilon' a))\varepsilon' a)) \\
= \delta(x^{-1}(\varepsilon' \delta(a) a)) \cdot x^{-1}(\varepsilon' \delta(a) a) \\
= \varepsilon' \delta(a) \cdot \delta(x^{-\varepsilon' \delta(a)}(a)) \cdot x^{-\varepsilon' \delta(a)}(a).
\]

(4.5)

Algorithm 4.5. Let \( C = [(x, y, \varepsilon)] \in C\bar{\Omega}_d \) be a restricted class. By (4.5), we obtain \( \varphi_S(C) \) in the following steps:

(1) \( I'_d := I_d \), \( j := 0 \).
(2) \( a_{0,j} := \min(I'_d) \), \( \varepsilon_{0,j} := 1 \), \( i := 0 \)

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(3) Remove \(a_{i,j}\) from \(I'_d\). Let \(a_{i+1,j} := x^{-\varepsilon'_{i,j}\delta(a_{i,j})}(a_{i,j}), \varepsilon'_{i+1,j} := \varepsilon'_{i,j}\delta(a_{i,j})\delta(a_{i+1,j})\).

(4) If \(a_{i+1,j} = a_{0,j}\), let \(c_j := (a_{0,j} \ a_{1,j} \cdots \ a_{i,j})\). Otherwise go back to (3) for the next \(i\).

(5) If \(I'_d \neq \emptyset\), go back to (2) for the next \(j\). Otherwise finish the loop and let \(x'_S := \delta^\#x_S, y'_S := c_1c_2\cdots c_j\) and \(\varepsilon'_S := (a_{i,j} \mapsto \varepsilon'_{i,j})\).

(6) Search for the conjugator \(\sigma \in \mathfrak{S}_d\) such that \(\sigma^\#x'_S\) is of normalized form. Let \((x_S, y_S, \varepsilon_S) := (\sigma^\#x'_S, \sigma^\#y'_S, \varepsilon'_S \circ \sigma^{-1})\).

(7) Search for the isomorphism class \(C_S \in C\bar{\Omega}_d\) represented by \((x_S, y_S, \varepsilon_S)\) and we conclude that \(\varphi_S(C) = C_S\).

Finally, we present an algorithm for a simultaneous calculation of the Veech groups of origamis in \(\bar{\Omega}_d\) using Proposition 2.8.

**Algorithm 4.6.** Let \(\varphi_T, \varphi_S \in \text{Sym}(C\bar{\Omega}_d)\). We obtain the \(\langle \varphi_T^{-1}, \varphi_S^{-1} \rangle\)-orbit decomposition of \(C\bar{\Omega}_d\) in the following steps.

1. \(I'_N := I_N\).
2. For \(t \in \mathbb{N}\), \(O_t := \emptyset\).
3. Take \(i \in I'_N\) and add \(i\) to \(O_t\).
4. Take \(j \in O_t\) and let \(O(j) := \{\varphi_T^{-k}(j), \varphi_S^{-k}(j) \mid k \in \mathbb{N}\}\).
5. Add all elements in \(O(j)\) to \(O_t\) and remove them from \(I'_N\).
6. Go back to (4) for other leftover \(j \in O_t\). When we have been through all elements in \(O_t\), go to the next step.
7. Go back to (2) for the next \(t\) until \(I'_N = \emptyset\). If so, finish the algorithm.

Note that we may apply the Reidemeister-Schreier method \([10]\) to the result of Algorithm 4.6 to obtain the list of generators and the list of representatives of the Veech group of each origami.

5 Calculation results

In the following, we show some calculation results obtained by the algorithms stated in the previous section powered by Python. For each degree \(d\), we number classes of origamis according to Algorithm 4.2 (i.e. lexicographic order with respect to permutations and signs). We first note that all classes
representing disconnected origamis are removed and abelian origamis are treated as flat surfaces.

Figure 5 and Figure 6 show all classes of origamis of degree 4 and their positions in Teichmüller curves. There are 26 classes of abelian origamis summing up to 5 components and 34 classes of non-abelian origamis summing up to 6 components. In the figures, we denote copies of the standard fundamental domain of the group \( PSL(2, \mathbb{Z}) \) by isosceles triangles where the keen vertices correspond to the cusp. Every two edges with the same symbol are glued so that the cusps match. Every edge with no symbol is glued individually, making a conical point of angle \( \pi/2 \).

Table 1 shows the number of classes of origamis, the number of components of Teichmüller curves, the range of genus of Teichmüller curves, and the number of classes of possible \( G_Q \)-conjugacy for degree \( 1 \leq d \leq 7 \). The possibility of \( G_Q \)-conjugacy is checked by the following \( G_Q \)-invariants: the degree (=the index of \( \Gamma(O) \)), the genus, and the valency list of \( C(O) \). We calculate strata of origamis by Lemma 3.5. The distribution of the number of classes origamis per genus is shown in Table 2.

| \( d \) | abelian \#(O) | abelian \#(C(O)) | abelian \( g(C(O)) \) | possible \( G_Q \)-conjugacy | non-abelian \#(O) | non-abelian \#(C(O)) | non-abelian \( g(C(O)) \) | possible \( G_Q \)-conjugacy |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | none | 0 | 0 | 0 | none |
| 2 | 3 | 1 | 0 | none | 1 | 1 | 0 | none |
| 3 | 7 | 2 | 0 | none | 4 | 1 | 0 | none |
| 4 | 26 | 5 | 0 | none | 34 | 6 | 0 | none |
| 5 | 91 | 8 | 0 | none | 227 | 13 | 0 | none |
| 6 | 490 | 28 | 0 | 1 class | 2316 | 88 | 0 | 13 classes |
| 7 | 2785 | 41 | 0 \( \sim \) 1 class | 26574 | 88 | 0 \( \sim \) 11 classes |

Table 1: Summary of the result for degree \( 1 \leq d \leq 7 \)

| \( d \) \( \backslash \) \( g \) | abelian 0 1 2 3 4 total | non-abelian 0 1 2 3 4 total |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 1 0 0 0 1 | 0 0 0 0 0 1 |
| 2 | 0 3 0 0 0 3 | 1 0 0 0 0 1 |
| 3 | 0 4 0 0 0 4 | 0 4 0 0 0 4 |
| 4 | 0 7 19 0 0 26 | 3 15 16 0 0 34 |
| 5 | 0 6 51 34 0 91 | 0 49 138 40 0 227 |
| 6 | 0 15 142 333 0 490 | 10 181 1085 1040 0 2316 |
| 7 | 0 8 250 1735 792 2785 | 0 534 6449 16000 3591 26574 |

Table 2: The number of classes of origamis of given genus and degree.
Figure 5: (Part 1/2) All classes of origamis of degree 4 and their positions in Teichmüller curves.
Figure 6: (Part 2/2) All classes of origamis of degree 4 and their positions in Teichmüller curves.
Theorem 5.1. All the Teichmüller curves induced from origamis of degree $d \leq 7$ are distinguished by Galois invariants except for the 13 cases in Table 3 and the 9 cases in Table 4. Figure 7 and Figure 8 show origamis that induce Teichmüller curves in each of the exceptional cases.

| No. | stratum | index | valency list of $C(Q)$ | relationship between $C(Q)$ |
|-----|---------|-------|-------------------------|----------------------------|
| 6-1 | $A_0(0, 8)$ | 15 | $(3^6, 2^1, 1, 5, 4, 3)$ | two identical, mirror-closed curves |
| 6-2 | $Q_2(-1^2, 0^1, 2)$ | 12 | $(3^4 | 2^6, 6, 3, 2, 1)$ | two identical, mirror-closed curves |
| 6-3 | $Q_2(-1^2, 0, 6)$ | 12 | $(3^4 | 2^6, 6, 3, 2, 1)$ | two identical, mirror-closed curves |
| 6-4 | $Q_2(0^1, 2^2)$ | 12 | $(3^4 | 2^6, 6, 3, 2, 1)$ | three identical, mirror-closed curves |
| 6-5 | $Q_2(2, 6)$ | 12 | $(3^4 | 2^6, 6, 3, 2, 1)$ | two identical, mirror-closed curves |
| 6-6 | $Q_2(-1^2, 2^2)$ | 15 | $(3^5 | 2^7, 1, 6, 5, 3, 1)$ | two distinct, mirror-closed curves |
| 6-7 | $Q_2(-1^2, 3^2)$ | 15 | $(3^5 | 2^7, 1, 5, 4, 3^2)$ | one pair of mirror-symmetric curves, mirroring each other |
| 6-8 | $Q_2(-1, 1, 9)$ | 22 | $(3^7, 1, 2^{11}, 6, 5, 4, 3^3)$ | one pair of mirror-symmetric curves, mirroring each other |
| 6-9 | $Q_2(-1^2, 0, 6)$ | 24 | $(3^8 | 2^{12}, 6, 5, 4, 3, 2)$ | one mirror-conjugate pair |
| 6-10 | $Q_2(-1^1, 7)$ | 27 | $(3^9 | 2^{13}, 1, 6, 5, 4, 3^3)$ | one mirror-conjugate pair & one mirror-closed curve |
| 6-11 | $Q_2(-1, 0, 1, 4)$ | 36 | $(3^{12} | 2^{18}, 6, 5, 4, 3^2)$ | one mirror-conjugate pair & one mirror-closed curve |
| 6-12 | $Q_2(1, 7)$ | 54 | $(3^{18} | 2^{27}, 6, 5, 4, 3^3)$ | one mirror-conjugate pair & one mirror-closed curve |
| 6-13 | $Q_2(-1, 9)$ | 66 | $(3^{22} | 2^{33}, 6, 5, 4, 3^3)$ | two mirror-conjugate pairs |

Table 3: Classes of possible $G_Q$-conjugacy for degree 6

| No. | stratum | index | valency list of $C(Q)$ | relationship between $C(Q)$ |
|-----|---------|-------|-------------------------|----------------------------|
| 7-1 | $A_0(12)$ | 7 | $(3^6, 1 | 2^5, 1, 4, 3)$ | one pair of mirror-symmetric curves, mirroring each other |
| 7-2 | $A_0(0, 2, 6)$ | 16 | $(3^6 | 1, 2^5 | 7, 4, 3, 2)$ | two distinct, mirror-closed curves |
| 7-3 | $A_0(12)$ | 21 | $(3^7 | 2^{11}, 6, 5, 4, 3^2)$ | two distinct, mirror-closed curves |
| 7-4 | $A_0(12)$ | 42 | $(3^{14} | 2^{22}, 7^2, 5^2, 4^3, 3^2)$ | two distinct, mirror-closed curves |
| 7-5 | $A_0(0, 2, 6)$ | 48 | $(3^{16} | 2^{24}, 7^2, 5^2, 4^3, 3^2)$ | one mirror-conjugate pair |
| 7-6 | $Q_2(-1^1, 2^2)$ | 16 | $(3^5 | 2^7, 1, 6, 2, 1)$ | one mirror-conjugate pair |
| 7-7 | $Q_2(-1, 2)$ | 28 | $(3^9 | 1 | 2^{14}, 7, 6, 3^2, 2)$ | two distinct, mirror-closed curves |
| 7-8 | $Q_2(-1^2, 10)$ | 36 | $(3^{12} | 2^{18}, 7^2, 6, 3^2, 2, 1)$ | two distinct, mirror-closed curves |

Table 4: Classes of possible $G_Q$-conjugacy for degree 7

Remark 5.2. The mirror relation implies a Galois conjugacy which induces complex conjugacy. For example, the situation ‘one pair of mirror-symmetric curves, mirroring each other’ is caused by such a Galois conjugacy modifying the embeddings of Teichmüller curves into the moduli space. If there is no mirror relation, we only know the invariant agreement and do not know if there is a nontrivial $G_Q$-conjugacy.
Figure 7: Origamis that induce Teichmüller curves in Table 3. Unmarked edges are glued with the opposite.
Figure 8: Origamis that induce Teichmüller curves in Table 4. Unmarked edges are glued with the opposite.
Data Availability Statement. The data that support the findings of this study are openly available in GitHub at https://github.com/ShunKumagai/origami.

Acknowledgments. Author thanks to Prof. Toshiyuki Sugawa for his helpful advices and comments. The computation was carried out using the computer resource offered under the category of General Projects by Cyber Science Center, Tohoku University.

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