Analytical holographic superconductor with backreaction using $AdS_3/CFT_2$

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Abstract

The holographic model for a two-dimensional superconductor has been investigated by considering the three-dimensional gravity in the bulk. To find the critical temperature, we used the Sturm-Liouville variational method. Whereas the same method is applied for calculating the condensation of the dual operators on the boundary. We included the back reactions on the metric by a combination of the perturbation method of the fields with respect to the small parameter and then applying the variational integrals on the resulting equations of the motion. The critical temperature has been successfully obtained on the backreaction effects, and we showed that it dropped with a rise in the backreaction of the fields, and it makes the condensation harder. We can use our analytical results to support the numerical data which was reported previously.

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I. INTRODUCTION

Maldacenda discovered the relation between a $d+1$ dimensional gravitational system as the weak limit of the string theory, and a quantum theory on the $d$ dimensional boundary, which is illustrated by the conformal field theory \[1\]. It is called anti-de Sitter/conformal field theories (AdS/CFT) communication. In recent years, its applications has been frequently observed to explain the behaviors of the simple strongly correlated systems \[2-4\] in the theoretical condensed matter physics \[5-34\]. A large number of the holographic superconductors in a wide class of the gravitational models, have been investigated from the non relativistic regime in the form of Horava-Lifshitz to the Gauss-Bonnet and Weyl corrections. If we ignore the field’s backreaction effects on gravity sector and fix the metric as static and with a definitive symmetry and a well behavior horizon’s temperature, this approximation is called as the probe approximation in the literatures. The full description of the phase transition in such systems needs to amount the affects of the backreacted fields on the fix geometry. Also in this case and apart from the probe limit, people find similar scalar condensation in the boundary \[35-45\]. All these calculations in the probe limit or with back reactions have been done numerically, based on the shooting method for solving the coupled of the differential equations by appropriate boundary conditions. From the analytical point of view of holographic superconductors, there are three main methods in literatures:

1- The Sturm-Liouville variational method \[19\], which near the critical point, we replace the electromagnetic scalar gauge by an approximated solution which satisfy the boundary conditions and then reduce the form of the scalar field equation to a usual form of a self-adjoint equation in the functional theory of the real valued functions. By minimizing such functional by a suitable trial function one can find the best optimized critical temperature.

2- The small parameter perturbation theory, in which we expand all the functions as a perturbative series with respect to the small parameter $\langle O_+ \rangle$ (it remains small just near the critical point $T = T_c$) and finds the corrections on the background or fields on it \[13, 56\].

3- The Matching method. In this method, the asymptotic solutions of fields near the AdS horizon to the horizon solutions at an arbitrary (completely arbitrary) mid point are matched, and then we find
the expectation value of dual operator $\langle O_+ \rangle$ and the critical temperature $T_c$ analytically [20, 21]. Many models of the holographic superconductors have been investigated in four dimensions or higher-dimensions. However, it is a possibility to consider a toy model for lower dimensional holographic superconductors. Such lower dimensional models have been proposed for simplicity and also to find the nature of the phase transitions in the dilatonic black holes which have a very important role in the string theory. Another reason for consideration of a non-realistic lower dimensional superconductors back to the ability of the AdS/CFT dictionary in the reduction of the dimensions from $d > 4$ to $2 < d < 4$. Here you need to a lower dimension AdS/CFT dictionary. The existence of such a dictionary depends upon the string theory. But in any case, we know that, for example such theory works in $AdS_3/CFT_2$ correspondence [46]. Different aspects of such lower dimensional holographic superconductors have been discussed in the probe limit effectively by the authors [46, 50].

In addition, they showed that explicitly the behaviors of the fields as the dynamical quantities near the phase transition points. In the present work, we will use the variational method to observe the analytical behavior of such a lower dimensional holographic superconductors. Although people investigated the problem before by applying the numerical algorithms, however, here we will derive the critical properties just be applying the analytical method. Firstly, we review the main ideas of the holographic superconductors in such lower dimensional systems. Generally, we know that now, [20, 51], in the quantum theory on the boundary, both dual operators correspond to the conformal dimension $\Delta_{\pm}$, and explicitly they can be written as $m^2$, where $m$ is mass of the scalar field. The work is planned as: in section II, an $U(1)$ gauge field model along with the scalar field is presented, within the background of a Banados-Teitelboim-Zanelli (BTZ) like planar black hole; in section III, the phase transition is analytically investigated; in section IV, we obtain the critical temperature $T_c$ versus the backreaction up to the order $\kappa^2$. Finally we conclude in the last section.

II. 1 + 1 HOLOGRAPHIC SUPERCONDUCTOR

The general lower dimensional, in fact $2 + 1$ gravitational bulk action depicting a charged complex scalar field (we set Stuckelberg field $\theta = 0$, it means we take the scalar field read) with negative cosmological constant of Einstein-Maxwell action reads [46]

$$S = \int d^3x \sqrt{-g} \left[ \frac{1}{2\kappa^2} (R + \frac{2}{l^2}) - \frac{1}{4} F^{ab} F_{ab} - |\nabla \zeta - ieA\zeta|^2 - m^2 |\zeta|^2 \right].$$

(1)
Here, $\kappa$ is the usual three dimensional gravitational constant $\kappa^2 = 8\pi G_3$, $G_3$ is the Newton constant in three dimensions and $g = |g_{\mu\nu}|$. We need an AdS radius $l$ (non effective, because here we used just the Einstein-Hilbert gravitational term), $e$ appears in the covariant derivative exactly as a standard electric charge and $m = m_\phi$. We are interesting in the effects of backreaction on the holographic superconductor. We focus just on the s-wave cases. We must clarify the motivation of the s-wave approximation in holographic models of the type II high temperature superconductors. In the relativistic models of the gravity, we know that s-wave approximation is not a good approximation [47]. s-wave refers to a scalar condensation not more. But you can have the Yang-Mills fields with $SU(2)$ symmetry which potentially they can generate another symmetry breaking of an axial vector type. The last case resembles the p-wave models. So, just for more clarifications, we mention here that the two dimensional model of the superconductors which we proposed is a toy model and it will be more interesting that we can find a direct relation between this toy model and the results of a 4 dimensional model, by a principle like the detailed balance.

Another additional point is, here we study only the case of a single horizon and not multi horizon cases [48]. In the holographic set up for superconductors one must identify a temperature in his gravitational bulk model to the CFT temperature on the boundary because the partition function of bulk (stringy) equals to the partition function of the CFT. If the black hole has only one horizon, in this case, we can use the Hawking-Bekenstein (horizon) temperature as a reasonable candidate. But if our asymptotically AdS bulk has more than one horizon, for example, in the case of the charged BTZ like black holes, then we take the temperature of the real physical horizon (the temperature which is obtained by calculation the surface gravity of the biggest null hypersurface orthogonal surface) as the candidate for temperature of the CFT. In fact the effects of the quantum corrections and charged Maxwell field on the background of the bulk is very interesting problem and can be investigated in more details. Also, may be it become possible to relate the instability of such charged dilaton configurations in the AdS spacetime [49] to the symmetry breaking mechanism of the superconductors. The idea has motivation enough as a new work. Also, the effect of the charge in the dilatonic BTZ like black holes can produce the hyperscaling violations which it can modify the thermodynamics and the critical point of the second order phase transitions.

In this paper, by ignoring the quantum corrections or instabilities for the gravity bulk sector, we use
from an ansatz which is described by the static spherically symmetric spacetime as

$$ ds^2 = -f(\gamma)e^{-\beta(\gamma)}dt^2 + \frac{d\gamma^2}{f(\gamma)} + \frac{\gamma^2}{l^2}dx^2 . $$

(2)

The $U(1)$ usual electromagnetic gauge field, $\gamma$ stands for the radial coordinate, and the electromagnetic gauge is in the following one form (in the language of differential geometry) and the scalar field

$$ A_t = a_0(\gamma)dt, \quad \zeta \equiv \zeta(\gamma). $$

(3)

The scalar field phase is set to be zero, so it’s natural that we treat it as the real function. In the gravity sector of the bulk, we must identify the temperature. The unique well behavior of this static black hole is the Hawking-Unruh killing-Horizon temperature, which it can be construed as the temperature of the dual CFT, This temperature’s reading may be is not a very trivial case, and can be calculated by different methods, for example, by the method of the weak rotation in the action to make it Euclidean or also by expansion of the metric near the horizon and comparison of the resulting metric by the Rindler geometry. By any method, the result for the horizon is the same as

$$ T = \left. \frac{f'(\gamma)e^{-\beta(\gamma)/2}}{4\pi} \right|_{\gamma=\gamma^+} . $$

(4)

To find the equations of motion, we remember that these equations are the Maxwell equation for electromagnetic gauge and the generalized Klein-Gordon equation, which we list them here

$$ \nabla_\mu F^{\mu\nu} = J^\nu, $$

(5)

$$ (D_\mu D^\mu - m^2)\zeta = 0. $$

(6)

Here the current $J^\nu$ can be derived easily by the Noether theorem. The closed forms of the equations presented before as the following.

$$ \zeta''(\gamma) + \zeta'(\gamma) \left[ \frac{1}{\gamma} + \frac{f'(\gamma)}{f(\gamma)} - \frac{\beta'(\gamma)}{2} \right] + \zeta(\gamma) \left[ \frac{e^2\Phi(\gamma)^2e^{\beta(\gamma)}}{f(\gamma)^2} - \frac{m^2}{f(\gamma)} \right] = 0, $$

$$ a''_0(\gamma) + a'_0(\gamma) \left[ \frac{1}{\gamma} + \frac{\beta'(\gamma)}{2} \right] - \frac{2e^2a_0(\gamma)\zeta(\gamma)^2}{f(\gamma)} = 0, $$

$$ f'(\gamma) + 2\kappa^2\gamma \left[ \frac{e^2a_0(\gamma)^2\zeta(\gamma)^2e^{\beta(\gamma)}}{f(\gamma)} + f(\gamma)\zeta'(\gamma)^2 + m^2\zeta(\gamma)^2 + \frac{1}{2}e^{\beta(\gamma)}a_0(\gamma)^2 \right] - \frac{2\gamma}{l^2} = 0, $$

$$ \beta'(\gamma) + 4\kappa^2\gamma \left[ \frac{q^2a_0(\gamma)^2\zeta(\gamma)^2e^{\beta(\gamma)}}{f(\gamma)^2} + \zeta'(\gamma)^2 \right] = 0. $$

(7)
Here, \( f' = \partial_\gamma f \). The full numerical solutions to the given system is reported in [55].

Using the scaling symmetry, we can set the charge parameter equal to \( e = 1 \). However, our approach is the analytical approach, and we will not reproduce the wellknown results of the [55]. For this reason, we will use the Sturm-Liouville (S-L) variational method. We have to think about two different boundaries. When system stays in the normal phase, \( \zeta(\gamma) = 0 \), we find that the lapse function of the metric (redshift function) \( \beta \) is a constant and the analytic solutions to system (7) lead to the well known exact black holes with the metric coefficient and the potential function

\[
f(\gamma) = k + \frac{\gamma^2}{l^2} - \kappa^2 \mu^2 \log \gamma, \tag{8}
\]

\[
a_0(\gamma) = \rho + \mu \log \gamma. \tag{9}
\]

Here \( k = -\frac{\gamma^2}{l^2} + \kappa^2 \mu^2 \log \gamma_+ \), where, \( \mu, \rho \) correspond to the chemical potential charge density in the dual field theory. Also about the (9), we mention here that for example, a Maxwell field in \( AdS_3 \) is also logarithmic, but is physical once appropriate counterterms are added (and describes a vector operator that is not a conserved current). When \( \kappa = 0 \), the metric coefficient \( f \) recovers the case of the Banados, Teitelboim, Zanelli(BTZ) black hole. If we are interested to solve in superconducting stage, where \( \zeta \neq 0 \), we must locate the appropriate boundary conditions. We must clarify these boundary conditions here. At the black hole horizon \( \gamma_+ \), which is the positive real root of \( f(\gamma_+) = 0 \), all fields have regular solutions [55]

\[
a_0(\gamma_+) = 0, \quad \zeta'(\gamma_+) = \frac{m^2}{f'(\gamma_+)} \zeta(\gamma_+), \tag{10}
\]

and the metric ansatz satisfy [55]

\[
f'(\gamma_+) = \frac{2\gamma_+}{l^2} - 2\kappa^2 \gamma_+ \left[ m^2 \zeta(\gamma_+)^2 + \frac{1}{2} e^{\beta(\gamma_+)} a_0'(\gamma_+)^2 \right],
\]

\[
\beta'(\gamma_+) = -4\kappa^2 \gamma_+ \left[ e^{2} a_0'(\gamma_+)^2 \zeta(\gamma_+)^2 e^{\beta(\gamma_+)} + \zeta'(\gamma_+)^2 \right]. \tag{11}
\]

Far from the horizon boundary, at the spatial infinity which it coincides exactly on the AdS boundary, the asymptotic performance of the solutions is

\[
\beta \to 0, \quad f(\gamma) \sim \frac{\gamma^2}{l^2}
\]

\[
a_0(\gamma) \sim \mu \log \gamma, \quad \zeta(\gamma) \sim \frac{\zeta_-}{\gamma^{\Delta_-}} + \frac{\zeta_+}{\gamma^{\Delta_+}}, \tag{12}
\]

where as the usual exponent \( \Delta_{\pm} \) denote the conformal dimension. For \(-1 \leq m^2 < 0 \), both the fields are normalizable, so, the boundary conditions are applied in such a way that one just becomes zero.
After applying these appropriate boundary conditions that either \( \zeta^- \) or \( \zeta^+ \) becomes extinct, a one parameter family of solutions is obtained using the shooting method algorithms \([55]\).

### III. ANALYTICAL INVESTIGATION OF THE HOLOGRAPHIC SUPERCONDUCTORS

The S-L method \([19]\) is used here for the analytical investigation of the properties of holographic superconductor with backreactions. Further, the relation between the critical temperature \( T_c \) and charge density \( \rho \) will be derived near the phase transition point and resolve the backreaction effects. As the usual, it’s adequate to introduce a new variable \( z = \frac{\gamma + \gamma}{\gamma} \), so, the Einstein, Maxwell and the scalar equations can be written as

\[
\zeta''(z) + \frac{\zeta'(z)}{z} \left[ 1 + \frac{zf'(z)}{f(z)} - \frac{z\beta'(z)}{2} \right] + \frac{\gamma^2 \zeta(z)}{z^4} \left[ \frac{e^2 a_0(z)^2 e^{\beta(z)}}{f(z)^2} - \frac{m^2}{f(z)} \right] = 0 \tag{13}
\]

\[
a_0''(z) + \frac{a_0'(z)}{z} \left[ 1 - \frac{z\beta'(z)}{2} \right] - \frac{2e^2 \gamma^2 a_0(z)\zeta(z)^2}{z^4 f(z)} = 0 \tag{14}
\]

\[
f'(z) - 2\kappa^2 \frac{\gamma^2}{z^3} \left[ \frac{e^2 a_0(z)^2 \zeta(z)^2 e^{\beta(z)}}{f(z)} + \frac{f(z)\zeta'(z)^2 z^4}{\gamma^2} + m^2 \zeta(\gamma)^2 + \frac{1}{2} \frac{e^{\beta(z)} a_0(z)^2 z^4}{\gamma^2} - \frac{2\gamma^2}{f^2 z^3} \right] = 0 \tag{15}
\]

\[
\beta'(z) - 4\kappa^2 \frac{\gamma^2}{z^3} \left[ \frac{e^2 a_0(z)^2 \zeta(z)^2 e^{\beta(z)}}{f(z)^2} - \frac{z^4 \zeta'(z)^2}{\gamma^2} \right] = 0 \tag{16}
\]

where, in this new system, \( f' = \frac{df}{dz} \). Following the perturbation scheme in the \([56]\), since the value of the scalar operator \( <O_+> \) (or \( <O_-> \)) is small close to the critical point, it can be introduced as an expansion parameter

\[
\epsilon \equiv \langle O_i \rangle, \quad i = +, - \tag{17}
\]

Note that, in the perturbation method and close to the critical point, our interest is in the solutions for which the scalar field \( \zeta \) is small, therefore from equations \([13-16]\) we must extend the gauge field \( a_0 \), the scalar field \( \zeta \), and the metric functions \( f(z), \beta(z) \) as

\[
\zeta = \sum_{k=1}^{\infty} \epsilon^k \zeta_k, \quad a_0 = \sum_{k=0}^{\infty} \epsilon^{2k} a_0(2k), \tag{18}
\]

\[
f = \sum_{k=0}^{\infty} \epsilon^{2k} f_{2k}, \quad \beta = \sum_{k=1}^{\infty} \epsilon^{2k} \beta_{2k}. \tag{19}
\]

where the metric function \( f(z) \) and \( \beta(z) \) can be expanded around the BTZ spacetime which are the exact solutions in the probe limits. Also for the chemical potential \( \mu \), we will allow it to be expanded
as the following series form

$$\mu = \sum_{k=0}^{\infty} e^{2k}\delta \mu_k.$$  

(20)

where at least $\delta \mu_2 > 0$. Thus, close to the phase transition,

$$\epsilon \approx \left(\frac{\mu - \mu_0}{\delta \mu_2}\right)^{\frac{1}{2}},$$  

(21)

whose critical exponent from the quantum holographic picture $\beta = \frac{1}{2}$ as a result of mean field theory. Obviously, as observed, the order parameter vanishes and phase transition can happen if $\mu \to \mu_0$, which shows $\mu$ is $\mu_c = \mu_0$.

At the zeroth order, we can get the solution $a_0$ from (14), i.e., the electromagnetic field behaves like $a_0(z) = \rho + \mu \log \frac{s}{\gamma^2}$, which by applying the boundary condition $a(\gamma_+) = 0$ gives a relation $\mu_0 = -\frac{\rho}{\log \gamma_+}$. At the critical point $\mu_c$, we can find $\mu_0 = \mu_c = -\frac{\rho}{\log \gamma_c}$, where $\gamma_c$ is the radius of the horizon at the critical point. For employing the analytical S-L method, we will set

$$a_0 = -\lambda \gamma_c \log z, \quad \lambda = -\frac{\rho}{\gamma_c \log \gamma_c}.$$  

(22)

Since $\zeta = 0$, by inserting this solution into (15), we obtain the metric function in the probe limit

$$f_0(z) = \gamma_+^2 g(z) = \gamma_+^2 \left(\kappa^2 \lambda^2 \log z - \frac{1}{l^2 z^2} + \frac{1}{l^2}\right).$$  

(23)

Here, we define a new function $g(z)$ for simplicity in the following calculation, using the boundary conditions on the horizon $f_0(z = 1) = 0$, the constant term $\frac{1}{l^2}$ is obtained. Now in the first order approximation, the asymptotic AdS solution for $\zeta_1$ can be expressed as

$$\zeta_1(z) \sim \frac{\zeta_x}{\gamma_+^2 \log z + \Delta^+}, \quad \zeta_\pm = \langle \Omega_\pm \rangle.$$  

(24)

So, introducing a variational completely trial function $F(z)$ close to $z = 0$

$$\zeta_1(z) \sim \frac{\zeta_x}{\gamma_+^2 \log z + \Delta^+} F(z).$$  

(25)

where, $F(0) = 1$ and $F'(0) = 0$. Substituting equation (25) into equation (13), we get,

$$F'' + \left[\frac{2\Delta_i + 1}{z} + \frac{g'}{g}\right]F' + \left[\frac{\lambda^2 (\log z)^2}{z^4 g^2} - \frac{m^2}{z^4 g} + \frac{\Delta_i + \frac{z g'}{g}}{z^2}\right]F = 0.$$  

(26)
We can convert (26) to be

\[(TF')' + T\left[U + \lambda^2 V\right] F = 0, \tag{27}\]

with

\[T = g z^{2\Delta_i + 1}, \quad U = -\frac{m^2}{z^4 g} + \frac{\Delta_i}{z^2} (\Delta_i + \frac{z g'}{g}), \quad V = \frac{(\log z)^2}{z^4 g^2}. \tag{28}\]

From the SL eigenvalue problem in the real valued functional theory, the expression to minimize eigenvalue of \(\lambda^2\) is

\[\lambda^2 = \frac{\int_0^1 T(F'^2 - UF^2)dz}{\int_0^1 TVF^2dz}. \tag{29}\]

Note that, here we are interesting just to the corrections of the backreaction term, i.e. we want to obtain the \(\lambda^2\) up to the order of \(\kappa^2\). So it is useful to write the functions

\[T = T_0(z) + \kappa^2 \lambda^2 T_1(z), \quad T_1(z) = z^{2\Delta_i + 1} \log z, \quad T_0(z) = \frac{z^{2\Delta_i + 1}}{l^2} (1 - z^{-2}), \tag{30}\]

\[g = g_0(z) + \kappa^2 \lambda^2 g_1(z), \quad g_1(z) = \log z, \quad g_0(z) = \frac{1}{l^2} (1 - z^{-2}), \tag{31}\]

\[U = U_0(z) + \kappa^2 \lambda^2 U_1(z), \quad U_0(z) = \left.\right. -\frac{m^2}{z^4 g_0(z)} + \frac{\Delta_i}{z^2} \left(\frac{z g_0'(z)}{g_0(z)}\right), \quad U_1(z) = \frac{m^2 g_1(z)}{z^4 g_0(z)^2} + \frac{\Delta_i}{z^2} \left(\frac{g_1(z)}{g_0(z)}\right)', \tag{32}\]

\[V = V_0(z) + \kappa^2 \lambda^2 V_1(z), \quad V_0(z) = \frac{(\log z)^2}{z^4 g_0(z)^2}, \quad V_1(z) = \frac{(\log z)^2 g_1(z)}{z^4 g_0(z)^3}. \tag{33}\]

So, the (29) revised as

\[\lambda^2 \approx \lambda^2 + \kappa^2 \lambda_1^2 + O(\kappa^4), \tag{34}\]

\[\lambda^2 = \frac{\alpha_1}{\alpha_2}, \quad \lambda_1^2 = \frac{\alpha_1}{\alpha_2^3} (\alpha_2 \beta_1 - \alpha_1 \beta_2), \tag{35}\]

where

\[\alpha_1 = \int_0^1 T_0(z)(F'^2 - U_0(z)F^2)dz, \tag{36}\]

\[\beta_1 = \int_0^1 \left[T_1(z)(F'^2 - U_0(z)F^2) - F^2 T_0(z)U_1(z)\right]dz, \tag{37}\]

\[\alpha_2 = \int_0^1 T_0(z)V_0(z)F^2dz, \tag{38}\]

\[\beta_2 = \int_0^1 F^2 (T_0(z)V_1(z) + T_1(z)V_0(z))dz. \tag{39}\]

The first term in (35) is the probe limit (\(\kappa^2 = 0\)) value for the eigenvalue while the first order correction of the back reaction \(\kappa^2\), is shown in second term.
In the following section, the analytical results are presented in solving (35) for several values of the backreaction $\kappa^2$ with $m^2 = 0$ to compare with the numerical results in [55]. We will use $F(z) = 1 - az^2$.

**IV. THE CRITICAL TEMPERATURE $T_c$**

When we are choosing the massless scalar field, the conformal dimension $\Delta_i$ takes the form

$$\Delta_i = \Delta_+ = 2. \quad (40)$$

For describing the condensate, we choose only $\zeta_+\text{dual to the scalar operator in the boundary field theory. So in this case the (36-39) read as}$

$$\alpha_1 = -1 + 1.33a - 0.67a^2, \quad (41)$$
$$\beta_1 = 0.25 + 0.28a - 0.22a^2, \quad (42)$$
$$\alpha_2 = -0.0505142 + 0.0385285a - 0.010005a^2, \quad (43)$$
$$\beta_2 = -0.0280578 + 0.0263808a - 0.00758224a^2. \quad (44)$$

So, for (35) we have

$$\lambda^2 = \Sigma_{n=0}^7 c_n a^n \quad (45)$$
$$c_0 = 746496 - 77.5004\kappa^2, \quad (46)$$
$$c_1 = -1.99066 \times 10^6 + 231.354\kappa^2, \quad (47)$$
$$c_2 = 2.32243 \times 10^6 - 328.431\kappa^2, \quad (48)$$
$$c_3 = -1.3271 \times 10^6 + 287.341\kappa^2, \quad (49)$$
$$c_4 = 331776 - 168.438\kappa^2, \quad (50)$$
$$c_5 = 67.323\kappa^2, \quad (51)$$
$$c_6 = -17.8282\kappa^2, \quad (52)$$
$$c_7 = 2.85173\kappa^2. \quad (53)$$

We list the $\lambda^2_{Min}$ with the chosen strength of the backreaction $\kappa$ for the condensates of the scalar operator $< O_+ >$ 3-dimensional case of hairy black hole background.

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1. The case with $m^2 = -1$ does not have convergence.
TABLE I: The dependence of the eigenvalue $\lambda^2_{Min}$ and the optimal value of the parameter $a$ on the backreaction $\kappa^2$ with $m^2 = 0$.

| $\kappa^2$ | 0   | 0.05 | 0.1  | 0.15 | 0.2  |
|------------|-----|------|------|------|------|
| $a$        | 0.759109 | 0.759108 | 0.759107 | 0.759106 | 0.759105 |
| $\lambda^2_{Min}$ | 189.394 | 187.312 | 179.392 | 169.216 | 158.125 |

Now, critical temperature $T_c$ can be taken for different values of the backreaction $\kappa$ and from the following relation

$$T_c = \frac{\gamma^+ c}{4\pi} - \frac{\kappa^2 \mu^2}{4\pi \gamma^+ c}, \quad \mu = \mu_c = -\frac{\rho}{\log \gamma^+ c}.$$  \hspace{1cm} (54)

Finally we have,

$$\frac{T_c}{\mu_c} = \frac{1}{4\pi \lambda_{Min}} (1 - \kappa^2 \lambda^2_{Min}).$$ \hspace{1cm} (55)

So we can put the data in the following table. We observe that the analytic results for the critical temperature are consistent with the numerical result [55].

TABLE II: The dependence of the critical temperature $\frac{T_c}{\mu_c}$ on the backreaction $\kappa^2$. Obviously, growth of back reaction decrease the critical temperature, with $m^2 = 0$.

| $\kappa^2$ | 0    | 0.05 | 0.1  | 0.15 | 0.2  |
|----------|------|------|------|------|------|
| $\frac{T_c}{\mu_c}$ | 0.0578239 | 0.0489749 | 0.0193732 | 0.0158489 | 0.01313246 |

V. CONCLUSIONS AND DISCUSSIONS

We investigated the holographic superconductors in the three dimensional gravitational bulk background with backreactions. We have used the SturmLiouville eigenvalue variational method for the analytical investigation of the holographic super conductors properties with backreactions. We found that in the fully backreacted BTZ spacetime, our analytical results shows a very good coincidence with numerically computed results. According to our analytical results, the backreaction
FIG. 1: (Left) The curve depicts the eigen value $\lambda^2$ as a function of the $\kappa^2$. (Right) Variation of the $\frac{T_c}{\mu_c}$ versus $\kappa^2$. It shows that the backreaction decreases $T_c$.

decrease the critical temperature of the superconductor, which can be used to support the numerical results that the condensation can be hindered by the backreactions.

[1] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998),[Int. J. Theor. Phys. 38, 1113 (1999)].
[2] S.A. Hartnoll, Class. Quant. Grav. 26, 224002 (2009).
[3] C.P. Herzog, J. Phys. A 42, 343001 (2009).
[4] G.T. Horowitz, arXiv: 1002.1722 [hep-th].
[5] G.T. Horowitz and M.M. Roberts, Phys. Rev. D 78, 126008 (2008).
[6] E. Nakano , W.Y. Wen, Phys. Rev. D 78, 046004 (2008).
[7] I. Amado, M. Kaminski, and K. Landsteiner, J. High Energy Phys. 05, 021 (2009).
[8] G. Koutsoumbas, E. Papantonopoulos, and G. Siopsis, J. High Energy Phys. 07, 026 (2009).
[9] O.C. Umeh, J. High Energy Phys. 08, 062 (2009).
[10] J. Sonner, Phys. Rev. D 80, 084031 (2009).
[11] S.S. Gubser, C.P. Herzog, S.S. Pufu, and T. Tesileanu, Phys. Rev. Lett. 103, 141601 (2009).
[12] J.P. Gauntlett, J. Sonner, and T. Wiseman, Phys. Rev. Lett. 103, 151601 (2009).
[13] J.L. Jing and S.B. Chen, Phys. Lett. B 686, 68 (2010).
[14] S. Franco, A.M. Garcia-Garcia, and D. Rodriguez-Gomez, Phys. Rev. D 81, 041901(R) (2010).
[15] C.P. Herzog, Phys. Rev. D 81, 126009 (2010) arXiv:1003.3278.
[16] R.A. Konoplya and A. Zhidenko, Phys. Lett. B 686, 199 (2010).
[17] X. He, B. Wang, R.G. Cai, and C.Y. Lin, Phys. Lett. B 688, 230 (2010) arXiv:1002.2679 [hep-th].
[18] R.G. Cai, Z.Y. Nie, B. Wang, and H.Q. Zhang, arXiv:1005.1233 [gr-qc].
[19] G. Siopsis and J. Therrien, J. High Energy Phys. 05, 013 (2010).
[20] R. Gregory, S. Kanno, and J. Soda, J. High Energy Phys. 10, 010 (2009).
[21] D. Momeni, R. Myrzakulov, L. Sebastiani, M. R. Setare, arXiv:1210.7965.
[22] X.H. Ge, B. Wang, S.F. Wu, and G.H. Yang, J. High Energy Phys. 08, 108 (2010).
[23] Q.Y. Pan and B. Wang, Phys. Lett. B 693, 159 (2010).
[24] R.G. Cai, Z.Y. Nie, and H.Q. Zhang, Phys. Rev. D 82, 066007 (2010).
[25] J.W. Chen, Y.J. Kao, D. Maity, W.Y. Wen, and C.P. Yeh, Phys. Rev. D 81, 106008 (2010).
[26] D.-Z. Ma, Y. Cao, J.-P. Wu, Phys. Lett. B 704, 604 (2011).
[27] J.-P. Wu, Y. Cao, X.-M. Kuang, W.-J. Li, Phys. Lett. B 697, 153 (2011).
[28] D. Momeni, M. R. Setare, N. Majd, J. High Energy Phys. 05, 118 (2011), arXiv:1003.0370.
[29] D. Momeni, M. R. Setare, Mod. Phys. Lett. A, 26, 2889 (2011), arXiv:1106.0431.
[30] M. R. Setare, D. Momeni, EPL, 96, 60006 (2011), arXiv:1106.1025.
[31] D. Momeni, N. Majd, R. Myrzakulov, EPL, 97, 61001 (2012), arXiv:1204.2146.
[32] D. Momeni, E. Nakano, M. R. Setare, W.-Y. Wen, arXiv:1108.4340.
[33] D. Momeni, M. R. Setare, R. Myrzakulov, Int. J. Mod. Phys. A27, 1250128 (2012), arXiv:1209.3104.
[34] M. R. Setare, D. Momeni, R. Myrzakulov, M. Raza, arXiv:1210.1062.
[35] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, J. High Energy Phys. 12, 015 (2008), arXiv:0810.1563.
[36] F. Aprile and J.G. Russo, Phys. Rev. D 81, 026009 (2010).
[37] Y. Liu and Y.W. Sun, J. High Energy Phys. 07, 008 (2010).
[38] G.T. Horowitz and B. Way, J. High Energy Phys. 11, 011 (2010).
[39] S.S. Gubser and A. Nellore, J. High Energy Phys. 04, 008 (2009).
[40] M. Ammon, J. Erdmenger, V. Grass, P. Kerner, and A. O’Bannon, Phys. Lett. B 686, 192 (2010).
[41] Y. Brihaye and B. Hartmann, Phys. Rev. D 81, 126008 (2010).
[42] L. Barclay, R. Gregory, S. Kanno, and P. Sutcliffe, J. High Energy Phys. 12, 029 (2010).
[43] M. Siani, J. High Energy Phys. 12, 035 (2010).
[44] R.G. Cai, Z.Y. Nie, and H.Q. Zhang, arXiv:1012.5559.
[45] Q.Y. Pan and B. Wang, arXiv:1101.0222.
[46] J. Ren, J. High Energy Phys. 11, 055 (2010), arXiv:1008.3904.
[47] S. Nojiri, S. D. Odintsov, Phys. Lett. B 463, 57, (1999), hep-th/9904146.
[48] S. Nojiri, S. D. Odintsov, Mod. Phys. Lett. A13:2695 (1998), arXiv:gr-qc/9806034.
[49] S. Nojiri, S. D. Odintsov, Phys. Rev. D59, 044003(1999), arXiv:hep-th/9806055.
[50] D. Birmingham, I. Sachs, and S. N. Solodukhin, Phys. Rev. Lett. 88, 151301 (2002).
[51] S.A. Hartnoll, C.P. Herzog, and G.T. Horowitz, Phys. Rev. Lett. 101, 031601 (2008).
[52] K. Maeda, M. Natsuume, and T. Okamura, Phys. Rev. D 79, 126004 (2009).
[53] B. Wang, C.Y. Lin, and E. Abdalla, Phys. Lett. B 481, 79 (2000).
[54] Y.Q. Liu, Q.Y. Pan, B. Wang, and R.G. Cai, Phys. Lett. B 693, 343 (2010).
[55] Y. Liu, Q. Pan, B. Wang, Phys. Lett. B 702, 94 (2011), arXiv:1106.4353.
[56] S. Kanno, Class. Quant. Grav. 28:127001(2011), arXiv:1103.5022.