Generalized Forchheimer flows in heterogeneous porous media

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Abstract
We study the generalized Forchheimer flows of slightly compressible fluids in heterogeneous porous media. The media’s porosity and coefficients of the Forchheimer equation are functions of the spatial variables. The partial differential equation for the pressure is degenerate in its gradient and can be both singular and degenerate in the spatial variables. Suitable weighted Lebesgue norms for the pressure, its gradient and time derivative are estimated. The continuous dependence on the initial and boundary data is established for the pressure and its gradient with respect to those corresponding norms. Asymptotic estimates are derived even for unbounded boundary data as time tends to infinity.

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1. Introduction and preliminaries

Forchheimer equations [6, 7] are commonly used in place of Darcy’s law to describe the fluid dynamics in porous media when the Reynolds number is large. Their nonlinear structure as opposed to the linear Darcy’s equation requires new mathematical investigations. For more thorough introduction to Forchheimer flows and their generalizations, the reader is referred to [1, 10], see also [2, 17, 18, 23].

In previous articles either for incompressible fluids, e.g. [20, 24], or compressible ones, e.g. [1, 8–10, 13], the porous media considered are always homogeneous. In reality, however, the porous media such as soil, geological media or multi-layer media are not homogeneous. The
current paper is to start our investigation of Forchheimer fluid flows in heterogeneous porous media. We will develop the model and analyze it mathematically. This lays the foundation for our subsequent study including maximum estimates, higher integrability of the gradient, as well as the structural stability. Such analysis will be needed in mathematical theory of homogenization and upscale computation for non-Darcy fluid flows in heterogeneous porous media.

Let a porous medium be modeled as a bounded domain $U$ in space $\mathbb{R}^n$ with $C^1$-boundary $\Gamma = \partial U$. Throughout this paper, $n \geq 2$ even though for physics problems $n = 2$ or 3. Let $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ be the spatial and time variables. The porosity of this heterogeneous medium is denoted by $\phi = \phi(x)$ which depends on the location $x$ and has values in $(0, 1]$.

For a fluid flow in the media, we denote the velocity by $v(x, t) \in \mathbb{R}^n$, pressure by $p(x, t) \in \mathbb{R}$, and density by $\rho(x, t) \in \mathbb{R}^+ = [0, \infty)$.

A generalized Forchheimer equation for heterogeneous porous media is

\[ g(x, v)v = -\nabla p, \]

where $g(x, s) \geq 0$ is a function defined on $\bar{U} \times \mathbb{R}^+$. It is a generalization of Darcy and Forchheimer equations [1, 9, 10].

The dependence of function $g$ in (1.1) on the spatial variable $x$ is used to model the heterogeneous media. For homogeneous media, $g$ is independent of $x$. For instance, when

\[ g(x, s) = \alpha + \beta s, \quad \alpha \geq 0, \quad s \geq 0, \]

where $\alpha, \beta, \gamma, m \in (1, 2]$, $\gamma_m$ are empirical constants, we have Darcy’s law, Forchheimer’s two term, three term and power laws, respectively, for homogeneous media, see e.g. [2, 17]. Moreover, many models of two-term Forchheimer law obtained from experiments, see e.g. [2], have $\alpha$ and $\beta$ in (1.2) depending on the porosity $\phi$. For heterogeneous porous media, $\phi = \phi(x)$, thus, these coefficients become functions of $x$. This motivates the $x$-dependent model (1.1).

In this paper, we study the model when the function $g$ in (1.1) is a generalized polynomial with non-negative coefficients. More precisely, the function $g$ is of the form

\[ g(x, s) = a_0(x)s^{\alpha_0} + a_1(x)s^{\alpha_1} + \cdots + a_N(x)s^{\alpha_N}, \quad s \geq 0, \]

where $N \geq 1, \alpha_0 = 0 < \alpha_1 < \cdots < \alpha_N$ are fixed real numbers, the coefficient functions $a_0(x), a_1(x), ..., a_N(x)$ are non-negative, and $a_0(x), a_N(x) > 0$. The number $\alpha_0$ is the degree of $g$ and is denoted by $\deg(g)$. Such a model (1.3) is sufficiently general to cover most examples in [2].

From (1.1), we have

\[ g(x, |v|)|v| = |\nabla p|. \]

Note that the function $s \in [0, \infty) \mapsto sg(x, s)$ is strictly increasing, mapping $[0, \infty)$ onto $[0, \infty)$. Hence, for each $\xi \in [0, \infty)$, there is a unique non-negative solution $s = s(x, \xi)$ of the equation $sg(x, s) = \xi$. Thus, $|v| = s(x, |\nabla p|)$, and by (1.1) again

\[ v = -\frac{\nabla p}{g(x, |v|))} = -\frac{\nabla p}{g(x, s(x, |\nabla p|))}. \]

We rewrite this key relation between $v$ and $\nabla p$ as

\[ v = -K(x, |\nabla p|)|\nabla p|, \]

where the function $K : \bar{U} \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

\[ K(x, \xi) = \frac{1}{g(x, s(x, \xi))} \quad \text{for } x \in \bar{U}, \xi \geq 0. \]
Equation (1.4) can be seen as a nonlinear generalization of Darcy’s equation. The case when \( a_i(x) \)'s are independent of \( x \) was studied in depth in [1, 9–13].

In addition to (1.1) we have the equation of continuity

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1.6}
\]

and the equation of state which, for (isothermal) slightly compressible fluids, is

\[
\frac{1}{\rho} \frac{d \rho}{dp} = \varpi, \quad \text{where the compressibility } \varpi = \text{const.} > 0. \tag{1.7}
\]

We rewrite (1.6) as

\[
\phi \frac{d \rho}{dp} \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \frac{d \rho}{dp} \nabla \cdot \mathbf{v} = 0,
\]

and by (1.7),

\[
\phi \varpi \frac{d \rho}{dp} = -\rho \nabla \cdot \mathbf{v} - \rho \varpi \nabla \cdot \mathbf{v}.
\]

Dividing both sides by \( \rho \varpi \) and using formula (1.4) for \( \mathbf{v} \), we obtain a scalar partial differential equation (PDE) for the pressure:

\[
\phi(x) \frac{\partial \rho}{\partial t} = \frac{1}{\varpi} \nabla \cdot (K(x, |\nabla \rho|) \nabla \rho) + K(x, |\nabla \rho|) |\nabla \rho|^2. \tag{1.8}
\]

On the right hand side of (1.8) the constant \( \varpi \) is very small for most slightly compressible fluids in porous media, hence we neglect its second term and study the following reduced equation

\[
\phi(x) \frac{\partial \rho}{\partial t} = \frac{1}{\varpi} \nabla \cdot (K(x, |\nabla \rho|) \nabla \rho). \tag{1.9}
\]

(This simplification is used commonly in petroleum engineering. For a full treatment without such a simplification, see [3].)

For our mathematical study of (1.9) below, by scaling the time variable \( t \to \varpi^{-1} t \), we can assume, without loss of generality, that \( \varpi = 1 \).

Throughout the paper, function \( g(x, s) \) in (1.3) is fixed, hence so is \( K(x, \xi) \). The initial boundary value problem (IBVP) of our interest is

\[
\begin{aligned}
\phi(x) \frac{\partial \rho}{\partial t} &= \nabla \cdot (K(x, |\nabla \rho|) \nabla \rho) \quad \text{on } U \times (0, \infty), \\
p &= \psi \quad \text{on } \Gamma \times (0, \infty), \\
p(x, 0) &= p_0(x) \quad \text{on } U,
\end{aligned} \tag{1.10}
\]

where \( p_0(x) \) and \( \psi(x, t) \) are given initial and boundary data.

Although the porosity function \( \phi(x) \) is bounded in applications, in this mathematical treatment we consider any \( \phi(x) > 0 \). Problem (1.10) deals with the Dirichlet boundary data; for other boundary conditions, see remark 5.7 below.

The main goals of this paper are to estimate the solution of (1.10) in different norms and to establish its continuous dependence on the initial and boundary data. Regarding the PDE of (1.10), the fact that \( \phi(x) \) can be close to zero, alone, makes its left-hand side degenerate. In addition, as we will see in lemma 1.1 below and the discussion right after that, the \( K(x, |\nabla \rho|) \)
is degenerate when $|\nabla p|$ is large, and can be either very small or very large at different $x$. Therefore, we have to deal with a parabolic equation with different types of degeneracy and singularity. For the degeneracy/singularity in $x$, we use appropriate weighted Lebesgue and Sobolev norms. To identify the weight functions for these norms, we carefully examine the structure of the PDE in (1.10), particularly, the function $K(x, \xi)$. It turns out that the porosity function $\phi(x)$ and the function $W_j(x)$, which will be computed explicitly in (1.17), are the essential weights. In order to derive differential inequalities for the weighted norms, we make use of a suitable two-weight Poincaré-Sobolev inequality, see (2.2). We then proceed and obtain the estimates for the solution in section 2, for its gradient in section 3, and for its time derivative in section 4. The continuous dependence in corresponding weighted norms for both solution and its gradient are obtained in section 5. The results for large time are particularly emphasized to show the long time dynamics of the problem. Their formulations are made simpler than those in the previous works [10–13].

For the remainder of this section, we present main properties of $K(x, \xi)$. First, we recall some elementary inequalities that will be needed. Let $x, y \geq 0$, then

\[(x + y)^p \leq 2^{p-1}(x^p + y^p) \quad \text{for all } p \geq 1, \tag{1.11}\]
\[x^\beta \leq 1 + x^\gamma \quad \text{for all } \gamma \geq \beta \geq 0. \tag{1.12}\]

The following exponent will be used throughout in our calculations

\[a = \frac{\alpha_N}{\alpha_N + 1} \in (0, 1). \tag{1.13}\]

We have from lemmas III.5 and III.9 in [1] that

\[-aK(x, \xi) \leq \xi \frac{\partial K(x, \xi)}{\partial \xi} \leq 0 \quad \forall \xi \geq 0. \tag{1.14}\]

This implies $K(x, \xi)$ is decreasing in $\xi$, hence

\[K(x, \xi) \leq K(x, 0) = \frac{1}{g(x, 0)} = \frac{1}{a_0(x)}. \tag{1.15}\]

The function $K(x, \xi)$ can also be estimated from above and below in terms of $\xi$ and coefficient functions $a_i(x)$’s as follows. Let us define the main weight functions

\[M(x) = \max\{a_j(x) : j = 0, ..., N\}, \quad m(x) = \min\{a_0(x), a_N(x)\}, \tag{1.16}\]
\[W_j(x) = \frac{\alpha_N(x)^a}{2NM(x)}, \quad W_\xi(x) = \frac{NM(x)}{m(x)a_N(x)^{1-a}}. \tag{1.17}\]

**Lemma 1.1.** For $\xi \geq 0$, one has

\[\frac{2W_j(x)}{\xi^a + a_0(x)^a} \leq K(x, \xi) \leq \frac{W_\xi(x)}{\xi^a} \tag{1.18}\]

and, consequently,

\[W_j(x)\xi^{2-a} - \frac{a_0(x)^a}{2} \leq K(x, \xi)\xi^2 \leq W_\xi(x)\xi^{2-a}. \tag{1.19}\]
**Proof.** Let \( s = s(x, \xi) \) be defined in (1.5). Then
\[
\xi = sg(x, s) = a_0(x)s + a_1(x)s^{\alpha_1 + 1} + \cdots + a_N(x)s^{\alpha_N + 1} \geq a_N(x)s^{\alpha_N + 1},
\]
hence
\[
s \leq \left( \frac{\xi}{a_N(x)} \right)^{\frac{1}{\alpha_N + 1}}. \tag{1.20}
\]

Since the exponents \( \alpha_j \) are increasing in \( j \), then by (1.12), one has \( s^\alpha \leq 1 + s^{\alpha_N} \) for \( j = 1, \ldots, N - 1 \). Thus, we have
\[
g(x, s) \leq M(x)(1 + s + \cdots + s^{\alpha_N}) \leq M(x)N(1 + s^{\alpha_N}). \tag{1.21}
\]
Combining (1.20) and (1.21) yields
\[
g(x, s) \leq M(x)N\left[ 1 + \left( \frac{\xi}{a_N(x)} \right)^{\frac{\alpha}{\alpha_N + 1}} \right] = NM(x)\left[ 1 + \left( \frac{\xi}{a_N(x)} \right)^{\alpha} \right] = \frac{NM(x)(\xi^\alpha + a_0(x)^\alpha)}{a_0(x)^{\alpha}},
\]
Therefore
\[
K(x, \xi) = \frac{1}{g(x, s)} \geq \frac{a_0(x)^\alpha}{NM(x)(\xi^\alpha + a_0(x)^\alpha)} = \frac{2W_2(x)}{\xi^\alpha + a_0(x)^\alpha},
\]
which proves the first inequality in (1.18).

Now, using (1.21)
\[
\xi = sg(x, s) \leq s \cdot M(x)N(1 + s^{\alpha_N}). \tag{1.22}
\]
Note that \( g(x, s) \geq m(x)(1 + s^{\alpha_N}) \), then by (1.22) and (1.20), we have
\[
g(x, s) \geq \frac{m(x)\xi}{NM(x)} \cdot \frac{1}{s} \geq \frac{m(x)\xi}{NM(x)} \cdot \frac{a_0(x)^{\alpha_N + 1}}{1} = \frac{m(x)a_0(x)^{1-a}\xi^\alpha}{NM(x)}.
\]
Therefore,
\[
K(x, \xi) = \frac{1}{g(x, s)} \leq \frac{NM(x)}{m(x)a_0(x)^{1-a}\xi^\alpha} = \frac{W_2(x)}{\xi^\alpha},
\]
hence we obtain the second inequality of (1.18).

Next, multiplying (1.18) by \( \xi^2 \), we have
\[
\frac{2W_2(x)\xi^2}{\xi^\alpha + a_0(x)^{\alpha}} \leq K(x, \xi)\xi^2 \leq W_2(x)\xi^{2-a}. \tag{1.23}
\]
The second inequality of (1.23) is exactly that of (1.19). For the first inequality of (1.19), if \( \xi \geq a_0(x) \) then (1.23) gives
\[
K(x, \xi)\xi^2 \geq \frac{2W(x)\xi^2}{2a^2} = W(x)\xi^{2-a}.
\] (1.24)

Thus, for all \( \xi \geq 0 \)
\[
K(x, \xi)\xi^2 \geq W(x)(\xi^{2-a} - a_0(x)^2^{-a}) = W(x)\xi^{2-a} - W(x)a_0(x)^2^{-a}.
\] (1.25)

Note that
\[
W(x)\xi^{2-a} = \frac{a_0(x)^2}{2NM(x)} \leq \frac{a_0(x)}{2N} \leq \frac{a_0(x)}{2}.
\] (1.26)

Hence (1.23), (1.25) and (1.26) yield the first inequality of (1.19). The proof is complete. \( \square \)

We now discuss some characters of the PDE in (1.10). On the left-hand side, the porosity \( \phi(x) \) can be close to zero, hence giving the degeneracy in variable \( x \). On the right-hand side, the dependence of \( K(x, \xi) \) on \( \xi \) as seen in (1.18) shows that the PDE is degenerate in \( |\nabla p| \) as \( |\nabla p| \to \infty \). Moreover, since the weights \( W_1(x) \) and \( W_2(x) \) can tend to either zero or infinity at different location \( x \), then, thanks to (1.18) again, so can \( K(x, \xi) \). Therefore the PDE can become singular and/or degenerate in \( x \). The above fact about the weights \( W_1(x) \) and \( W_2(x) \) is supported by practical models in [2]. For example, the two-term Forchheimer law (i.e. \( N = 1 \)) has the coefficients \( a_0 \) and \( a_1 \) going to zero as \( \phi \to 1 \), and to infinity as \( \phi \to 0 \). In this case, \( \phi \) is required to be in \( (0, 1) \). For heterogeneous media, constant \( \phi \) becomes function \( \phi(x) = \phi(x_0, 1) \) and it may be close 1 or 0 at different values of \( x \). Therefore, thanks to the mentioned behavior of \( a_0(x) \) and \( a_1(x) \), the weights \( W_1(x) \) and \( W_2(x) \) possess the stated property.

Lastly for this section, we recall an important monotonicity property for the PDE in (1.10).

**Lemma 1.2 (see [1], proposition III.6 and lemma III.9).** For any \( y, y' \in \mathbb{R}^n \), one has
\[
(K(x, |y|)|y - y'|) \cdot (y - y') \geq (1 - a)K(x, \max(|y|, |y'|))|y - y'|^2.
\]

In order to estimate the pressure gradient, similar to [1, 9, 10], we will make use of the function
\[
H(x, \xi) = \int_0^{\xi^2} K(x, \sqrt{s})ds \quad \text{for } x \in U, \; \xi \geq 0.
\] (1.27)

Same as (96) of [1], we have the comparison
\[
K(x, \xi)\xi^2 \leq H(x, \xi) \leq 2K(x, \xi)\xi^2.
\] (1.28)

Combining (1.28) with (1.19) gives
\[
W(x)\xi^{2-a} - \frac{a_0(x)^2}{2NM(x)} \leq H(x, \xi) \leq 2W(x)\xi^{2-a}.
\] (1.29)

### 2. Estimates for the pressure

We start analyzing the IBVP (1.10). To deal with the non-homogeneous boundary condition, let \( \Psi(x, t) \) be an extension (in \( x \)) of \( \psi(x, t) \) from boundary \( \Gamma \) to \( U \). Our results are stated in terms of \( \Psi \), but can be easily converted to \( \psi \); see e.g. [9].
Let \( \bar{\rho} = p - \Psi \), then we have
\[
\phi(x) \frac{\partial \rho}{\partial t} = \nabla \cdot (K(x, |\nabla p|)\nabla p) - \phi(x)\Psi \quad \text{on } U \times (0, \infty),
\]
\[
\bar{\rho} = 0 \quad \text{on } \Gamma \times (0, \infty).
\]
The analysis will make use of the following two-weight Poincaré-Sobolev inequality
\[
\left( \int_U |u|^2 \phi(x)dx \right)^{\frac{1}{2}} \leq c_p \left( \int_U W_1(x)|\nabla u|^{2-a} dx \right)^{\frac{1}{2-a}}
\]
(2.2) for functions \( u \) in certain classes that satisfy \( u = 0 \) on \( \Gamma \).

For some classes of functions \( \phi, W_1, u \) such that the inequality (2.2) is valid, see e.g. [5, 21]. Here we give a simple example that (2.2) holds under the so-called Strict Degree Condition (SDC) in section 5.

Assume further that \( r > q^* > 2 \), where the constant \( c \) depends on \( q, n \) and the domain \( U \), and \( q^* = n q/(n - q) \).

We recall the standard Poincaré-Sobolev’s inequality. Let \( \dot{W}^{1,q}(U) \) be the space of functions in \( W^{1,q}(U) \) with vanishing trace on the boundary. If \( 1 \leq q < n \) then
\[
\|f\|_{\dot{W}^{1,q}(U)} \leq c\|\nabla f\|_{L^q(U)} \quad \text{for all } f \in \dot{W}^{1,q}(U),
\]
(2.3) where the constant \( c \) depends on \( q, n \) and the domain \( U \), and \( q^* = n q/(n - q) \).

Assume (SDC). Let \( q < 2 - a \) such that \( r = q^* > 2 \). Let \( c \) be the positive constant in (2.3). Assume further that
\[
c_p \overset{\text{def}}{=} c \left( \int_U W_1(x) \frac{q}{2-a-q} dx \right)^{\frac{2-a-q}{2-a-q}} \left( \int_U \phi(x)\frac{r}{q-n} dx \right)^{\frac{q-n}{r}} < \infty.
\]
(2.4)

Let \( u \in \dot{W}^{1,q}(U) \). Then by Hölder’s inequality and standard Poincaré-Sobolev inequality (2.3)
\[
\left( \int_U |u|^2 \phi(x)dx \right)^{\frac{1}{2}} \leq \left( \int_U |u|^{q} dx \right)^{\frac{1}{2}} \left( \int_U \phi(x)\frac{r}{q-n} dx \right)^{\frac{q-n}{2-a-q}} \leq c \left( \int_U |\nabla u|^{q} dx \right)^{\frac{1}{2}} \left( \int_U \phi(x)\frac{r}{q-n} dx \right)^{\frac{q-n}{2-a-q}}.
\]
Since \( q < 2 - a \), applying Hölder’s inequality again to the second to last integral, we obtain (2.2) with \( c_p \) defined by (2.4).

The above example shows the validity of (2.2) for reasonable \( \phi, W_1 \) while \( u \) belongs to a standard Sobolev space. Nonetheless, for the purpose of this paper, it suffices to assume, without focusing on technical weighted Sobolev spaces, that the inequality (2.2) always holds true for \( u = \bar{\rho} \), as well as \( u = \bar{P} \) in section 5.

**Notation.** The following notations will be used throughout the paper.

- If \( f(x) > 0 \) is a function on \( U \), then define
  \[
  L^p_f(U) = \left\{ u(x) : |u| \leq f(x) \right\}
  \]
  (5.2)

  Notation \( \| \cdot \|_{L^p_f(U)} \) will be used as a short form of \( \| \cdot \|_{L^p_f(U)} \).
We will use the symbol $C$ to denote a generic positive constant which may change its values from place to place, depends on number $a$ defined by (1.13) and the Sobolev constant $c_P$ in (2.2), but not on individual functions $\phi(x)$ and $a_i(x)$’s, and not on the initial data and boundary data. Constants $C_0, C_1, C_2, \ldots$ have fixed values within a proof, while $d_1, d_2, \ldots$ are fixed positive constants throughout the paper.

• The notation $p_i$ stands for $\frac{\partial p}{\partial x_i}$. Similarly, $p_{1,i} = \frac{\partial p_1}{\partial x_i}$, $p_{2,i} = \frac{\partial p_2}{\partial x_i}$, etc.

• For a function $f(x, t)$, we denote $f(t) = f(\cdot, t)$.

In this section, we derive estimates for $\bar{p}(x, t)$ in $L^2(\Omega)$.

**Lemma 2.1.** If $t > 0$ then

\[
\frac{d}{dt} \int_{\Omega} \bar{p}^2(x, t)\phi(x)dx + \int_{\Omega} K(x, |\nabla p(x, t)|) |\nabla \bar{p}(x, t)|^2 dx \leq CG_0(t),
\]  

where $C > 0$ and

\[
G_0(t) = B_1 + \int_{\Omega} a_0(x^{-1}) |
\begin{array}{l}
\end{array}
\]

\[
B_1 = \int_{\Omega} a_0(x) dx.
\]  

**Proof.** Multiplying equation (2.1) by $\bar{p}(x, t)$, integrating over $\Omega$, and using the integration by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \bar{p}^2 dx - \int_{\Omega} K(x, |\nabla p|) \nabla p \cdot \nabla \bar{p} dx - \int_{\Omega} \bar{p} \Psi \phi dx.
\]

Substituting $\bar{p} = p - \Psi$ into the first integral on the right-hand side, and applying Cauchy–Schwarz inequality give

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \bar{p}^2 dx + \int_{\Omega} K(x, |\nabla p|) |\nabla \bar{p}|^2 dx = \int_{\Omega} K(x, |\nabla p|) \nabla \Psi \cdot \nabla p dx - \int_{\Omega} \bar{p} \Psi \phi dx
\]

\[
\leq \int_{\Omega} K(x, |\nabla p|) |\nabla \Psi| |\nabla p| dx + \int_{\Omega} |\bar{p}| |\Psi| \phi dx \defeq I_1 + I_2.
\]  

For $I_1$ in (2.9), applying Cauchy’s inequality, we have

\[
I_1 = \int_{\Omega} K(x, |\nabla p|) |\nabla \Psi| |\nabla p| dx \leq \frac{1}{4} \int_{\Omega} K(x, |\nabla p|) |\nabla p|^2 dx + \int_{\Omega} K(x, |\nabla p|) |\nabla \Psi|^2 dx.
\]

Estimating the last integral by (1.15), we get

\[
I_1 \leq \frac{1}{4} \int_{\Omega} K(x, |\nabla p|) |\nabla p|^2 dx + \int_{\Omega} \frac{1}{a_0(x)} |\nabla \Psi|^2 dx.
\]
For $I_2$ in (2.9), applying Hölder’s inequality and using the weighted Poincaré-Sobolev inequality (2.2), and applying Young’s inequality with powers $2 - a$ and $\frac{2}{1 - a}$, we have

$$I_2 \leq \left( \int_U |p|^2 \phi \, dx \right)^{\frac{1}{2}} \left( \int_U |\Psi|^2 \phi \, dx \right)^{\frac{1}{2}} \leq c_p \left( \int_U W(x) |\nabla p|^{2 - a} \, dx \right)^{\frac{1}{2 - a}} \left( \int_U |\Psi|^2 \phi \, dx \right)^{\frac{1}{2 - a}}$$

$$\leq 2^{a - 1} \int_U W(x) |\nabla p|^{2 - a} \, dx + C \left( \int_U |\Psi|^2 \phi \, dx \right)^{\frac{2 - a}{2(1 - a)}}.$$  

By triangle inequality, (1.11), and relation (1.19) we estimate

$$\int_U W(x) |\nabla p|^{2 - a} \, dx \leq 2^{1 - a} \int_U (W(x) |\nabla p|^{2 - a} + W(x) |\nabla \Psi|^{2 - a}) \, dx \leq 2^{1 - a} \int_U \left( K(|\nabla p|) |\nabla p|^2 + \frac{\alpha(y(x))}{2} + W(x) |\nabla \Psi|^{2 - a} \right) \, dx$$

$$\leq 2^{1 - a} \int_U K(|\nabla p|) |\nabla p|^2 \, dx + 2^{- a} B_1 + 2^{- a} \int_U W(x) |\nabla \Psi|^{2 - a} \, dx.$$  

Therefore,

$$I_2 \leq \frac{1}{4} \int_U K(x, |\nabla p|) |\nabla p|^2 \, dx + CB_1 + C \int_U W(x) |\nabla \Psi|^{2 - a} \, dx + C \left( \int_U |\Psi|^2 \phi \, dx \right)^{\frac{2 - a}{2(1 - a)}}.$$

Combining (2.9), (2.10), and (2.12), we have

$$\frac{1}{2} \frac{d}{dt} \int_U p^2 \phi \, dx + \frac{1}{2} \int_U K(x, |\nabla p|) |\nabla p|^2 \, dx$$

$$\leq \int_U \frac{1}{\alpha_0(x)} |\nabla \Psi|^2 \, dx + CB_1 + C \int_U W(x) |\nabla \Psi|^{2 - a} \, dx + C \left( \int_U |\Psi|^2 \phi \, dx \right)^{\frac{2 - a}{2(1 - a)}}.$$  

Thus we obtain (2.6). □

For the sake of future estimates’ simplicity, we replace $B_1$ in (2.7) by

$$B_* = \max \{ B_1, 1 \}.$$  

(2.13)

Thus, (2.6) gives

$$\frac{d}{dt} \int_U p^2(x, t) \phi(x) \, dx + \int_U K(x, |\nabla p(x, t)|) |\nabla p(x, t)|^2 \, dx \leq CG(t),$$

(2.14)

where

$$G(t) = G[\Psi](t) \overset{\text{def}}{=} B_* + \int_U \alpha_0(x)^{-1} |\nabla \Psi(x, t)|^2 \, dx + \int_U W(x) |\nabla \Psi(x, t)|^{2 - a} \, dx$$

$$+ \left( \int_U |\Psi(x, t)|^2 \phi(x) \, dx \right)^{\frac{2 - a}{2(1 - a)}}.$$  

(2.15)
With this change, we have
\[ G(t) \geq 1 \quad \forall t \geq 0. \] (2.16)

Using (2.11) and (2.14), we derive
\[
\frac{d}{dt} \int_U \rho^2 \phi \, dx + 2^{a-1} \int_U W_i(x) |\nabla \rho|^2 \, dx
\leq \frac{d}{dt} \int_U \rho^2 \phi \, dx + \int_U K(x, |\nabla p|) |\nabla p|^2 \, dx + CB_u + C \int_U W_i(x) |\nabla \psi|^2 \, dx
\leq CG(t) + CB_u + C \int_U W_i(x) |\nabla \psi|^2 \, dx \leq CG(t),
\]
which, by setting \( d_1 = 2^{a-1} \), proves
\[
\frac{d}{dt} \int_U \rho^2(x) \phi(x) \, dx + d_1 \int_U W_i(x) |\nabla \rho(x,t)|^2 \, dx \leq CG(t). \tag{2.17}
\]

Applying inequality (2.2) to \( u = \bar{\rho} \) and utilizing it in (2.17) give
\[
\frac{d}{dt} \int_U \rho^2(x,t) \phi(x) \, dx \leq - d_2 \left( \int_U \rho^2(x,t) \phi(x) \, dx \right)^{\frac{1}{2-a}} + CG(t), \tag{2.18}
\]
where \( d_2 = d_1 2^{a-2} \).

This nonlinear differential inequality enables us to obtain estimates for \( \bar{\rho} \) in terms of initial and boundary data. They are described by the following function and numbers.

Let \( \mathcal{M}(t) = \mathcal{M}[\psi](t) \) be a continuous function on \([0, \infty)\) that satisfies
\[
\mathcal{M}(t) \text{ is increasing and } \mathcal{M}(t) \geq G(t) \quad \forall t \geq 0.
\] (2.19)

Denote
\[
A = A[\psi] \overset{\text{def}}{=} \limsup_{t \to \infty} G(t) \quad \text{and} \quad B = B[\psi] \overset{\text{def}}{=} \limsup_{t \to \infty} [G'(t)]^+.
\] (2.20)

Note from (2.16) that
\[
\mathcal{M}(t) \geq 1 \quad \forall t \geq 0, \quad \text{and } A \geq 1. \tag{2.21}
\]

Assumptions. Throughout the paper, we assume that each solution \( p(x, t) \) and its corresponding function \( \psi(x, t) \) have enough regularity in spatial and time variables such that all calculations are carried out legitimately. Also, the time dependent quantities such as the above \( G(t) \) and others introduced in subsequent sections are required to belong to \( C([0, \infty)) \), and when needed, \( C^1([0, \infty)) \). The purpose of this requirement is to allow application of Gronwall’s inequality and other types of estimates in appendix.

**Theorem 2.2.**
(i) If \( t > 0 \) then
\[
\int_U \rho^2(x,t) \phi(x) \, dx \leq \int_U \rho^2(x,0) \phi(x) \, dx + \mathcal{M}(t)^\frac{2}{2-a}. \tag{2.22}
\]
(ii) If \( A < \infty \) then
\[
\limsup_{t \to \infty} \int_U \rho^2(x,t) \phi(x) \, dx \leq C A^{\frac{2}{2-a}}. \tag{2.23}
\]
(iii) If $B < \infty$ then there is $T > 0$ such that for all $t > T$
\[
\int_U \beta^2(x,t)\phi(x)dx \leq C(B^{-1/a} + G(t)^{2/a}).
\] (2.24)

Proof.

(i) Define $y(t) = \int_U \beta^2(x,t)\phi(x)dx$. We rewrite (2.18) as
\[
y'(t) \leq -\varphi^{-1}(y(t)) + CG(t),
\] (2.25) where $\varphi(z) = C_0e^{z^{2/a}}$ with $C_0 = d_2^2$. Using nonlinear Gronwall’s inequality in Lemma A.1(i), we have for all $t \geq 0$
\[
y(t) \leq y(0) + \varphi(C,M(t)),$

hence obtaining (2.22).

(ii) Applying Lemma A.1(ii) to the differential inequality (2.25), we get
\[
\limsup_{t \to \infty} y(t) \leq C \limsup_{t \to \infty} G(t)^{2/a} = CA^{2/a},
\] which proves (2.23).

(iii) Finally, note that $\varphi(z) \leq \varphi_0(z) \overset{\text{def}}{=} C_0(z + z^\gamma)$ for $z \geq 0$, where $1 < \gamma = \frac{2}{2 - a} < 2$.

Clearly, $\varphi_0^{-1}(y) \leq \varphi^{-1}(y)$, then we have from (2.25) that
\[
y'(t) \leq -\varphi_0^{-1}(y(t)) + CG(t).
\]
Hence by Lemma A.2, there is $T > 0$ such that for all $t > T$
\[
y(t) \leq C(1 + B^{1/a} + G(t)^{2/a}),
\]
which, together with (2.22), yields (2.24). \hfill \Box

3. Estimates for the pressure’s gradient

In this section we estimate the weighted $L^{2/a}_w$-norm for the gradient of $p$. Due to the structure of equation (2.1), we start with estimates for $H(x,|\nabla p(x,t)|)$ defined by (1.27), and will use relation (1.29) to derive the ones desired. We define
\[
G_i(t) = G_i[\Psi](t) \overset{\text{def}}{=} \int_U a_0(x)^{-1/2} |\nabla \Psi_i(x,t)|^2 dx.
\] (3.1)

Theorem 3.1.

(i) For $t > 0$,
\[
\int_U H(x,|\nabla p(x,t)|)dx \leq e^{-\frac{t}{2}} \int_U H(x,|\nabla p(x,0)|)dx
\]
\[
+ C \left( \int_U \beta^2(x,0)\phi(x)dx + M^{2/a}(t) + \int_0^t e^{-\frac{t}{2}(t-\tau)}G_i(t)\,d\tau \right).
\] (3.2)
(ii) If \( A < \infty \) then

\[
\limsup_{t \to \infty} \int_U H(x, |\nabla p(x,t)|)dx \leq C \left( A \rightarrow + \limsup_{t \to \infty} G_i(t) \right).
\]  

(3.3)

Proof.

(i) Multiplying equation (2.1) by \( \bar{p}_t \), integrating over \( U \), and using integration by parts we have

\[
\int_U \bar{p}_t^2 \phi dx = \int_U \bar{p}_t K(x, |\nabla p|) \nabla p \cdot \nabla \phi dx - \int_U \bar{p}_t \psi \phi dx
\]

\[
= \int_U \bar{p}_t K(x, |\nabla p|) \nabla p \cdot \nabla \phi dx + \int_U K(x, |\nabla p|) \nabla p \cdot \nabla \psi dx - \int_U \bar{p}_t \psi \phi dx
\]

\[
= \frac{1}{2} \frac{d}{dt} \int_U H(x, |\nabla p|)d x + \int_U K(x, |\nabla p|) \nabla p \cdot \nabla \psi dx - \int_U \bar{p}_t \psi \phi dx.
\]

Let \( \varepsilon > 0 \). Applying Cauchy’s inequality, we derive

\[
\int_U \bar{p}_t^2 \phi dx + \frac{1}{2} \frac{d}{dt} \int_U H(x, |\nabla p|)dx \leq \varepsilon \int_U K(x, |\nabla p|) |\nabla p|^2 dx + \frac{1}{4\varepsilon} \int_U K(x, |\nabla p|) |\nabla \psi|^2 dx + \frac{1}{2} \int_U |\psi| \phi dx.
\]

By using (1.28) to estimate the first term on the right-hand side, and using (1.15) to estimate the second term on the right-hand side, we obtain

\[
\frac{1}{2} \int_U \bar{p}_t^2 \phi dx + \frac{1}{2} \frac{d}{dt} \int_U H(x, |\nabla p|)dx
\]

\[
\leq \varepsilon \int_U H(x, |\nabla p|)dx + \frac{1}{4\varepsilon} \int_U a^{-1}(x) |\nabla \psi|^2 dx + \frac{1}{2} \int_U |\psi| \phi dx.
\]

Using inequality (1.12) and the fact \( 2 - a > 2(1-a) \) one has

\[
\int_U |\psi|^2 \phi dx \leq 1 + \left( \int_U |\psi|^2 \phi dx \right)^{2-a\over 2(1-a)} \leq G(t).
\]

Thus,

\[
\int_U \bar{p}_t^2 \phi dx + \frac{d}{dt} \int_U H(x, |\nabla p|)dx \leq 2 \varepsilon \int_U H(x, |\nabla p|)dx + \frac{1}{2\varepsilon} G(t) + G(t).
\]  

(3.4)

From (2.14) and (1.28), we have

\[
\frac{d}{dt} \int_U \bar{p}^2 \phi dx + \frac{1}{2} \int_U H(x, |\nabla p|)dx \leq CG(t).
\]  

(3.5)

Combining (3.4) and (3.5) with \( \varepsilon = 1/8 \), we obtain

\[
\frac{d}{dt} \int_U \bar{p}_t^2 \phi dx + \frac{d}{dt} \int_U H(x, |\nabla p(x,t)|)dx + \int_U \bar{p}_t^2 \phi dx + \frac{1}{4} \int_U H(x, |\nabla p(x,t)|)dx
\]

\[
\leq C(G(t) + G_i(t)).
\]  

(3.6)
We rewrite the first term on the left-hand side and apply Cauchy’s inequality as follows
\[
\frac{d}{dt} \int_U \rho^2 \phi \, dx = 2 \int_U \rho \rho_t \phi \, dx \geq -\frac{1}{2} \int_U \rho_t^2 \phi \, dx - 2 \int_U \rho^2 \phi \, dx,
\]
hence
\[
\frac{d}{dt} \int_U H(x, |\nabla p(x, t)|) \, dx + \frac{1}{4} \int_U \rho^2 \phi \, dx + \frac{1}{4} \int_U H(x, |\nabla p(x, t)|) \, dx \leq 2 \int_U \rho^2 \phi \, dx + C(G(t) + G_1(t)).
\]
(3.7)

Particularly, neglecting the second integral of the left-hand side of (3.7) reduces it to
\[
\frac{d}{dt} \int_U H(x, |\nabla p(x, t)|) \, dx + \frac{1}{4} \int_U H(x, |\nabla p(x, t)|) \, dx \leq 2 \int_U \rho^2 \phi \, dx + C(G(t) + G_1(t)).
\]
(3.8)

Using (2.22) to estimate the integral term on the right-hand side, and then properties (2.19), (2.21), we have
\[
\frac{d}{dt} \int_U H(x, |\nabla p(x, t)|) \, dx + \frac{1}{4} \int_U H(x, |\nabla p(x, t)|) \, dx \leq 2 \int_U \rho^2 \phi \, dx + C M(t)^{2-\sigma} + C(G(t) + G_1(t)) \leq C \int_U \rho^2 \phi \, dx + C M(t)^{2-\sigma} + C G(t).\]

Consequently, by Gronwall’s inequality,
\[
\int_U H(x, |\nabla p(x, t)|) \, dx \leq C \int_U H(x, |\nabla p(x, 0)|) \, dx
\]
\[
+ C \int_0^t e^{-\frac{1}{2}(t-\tau)} \left( \int_U \rho^2 \phi \, dx + C M(\tau)^{2-\sigma} + C G(\tau) \right) \, d\tau.
\]
(3.9)

Since \( M(\tau) \leq M(t) \) for all \( \tau \in [0, t] \), estimate (3.2) follows (3.9).

(ii) Applying Lemma A.1(ii) to differential inequality (3.8), and using limit estimate (2.23), we have
\[
\limsup_{t \to \infty} \int_U H(x, |\nabla p(x, t)|) \, dx \leq C \limsup_{t \to \infty} \int_U \rho^2 \phi \, dx + C \limsup_{t \to \infty} (G(t) + G_1(t))
\]
\[
\leq C (A^{\sigma-\sigma_0} + A + \limsup_{t \to \infty} G_1(t)).
\]

Since \( A \geq 1 \), by (2.21), we obtain (3.3). \( \square \)

For large time, we improve the estimates in theorem 3.1 by establishing inequalities of uniform Gronwall-type \([22, 25]\).

Lemma 3.2. For \( t \geq 1 \),
\[
\int_U H(x, |\nabla p(x, t)|) \, dx + \frac{1}{2} \int_U \rho_t^2(x, \tau) \phi(x) \, dx \, d\tau \leq C \left( \int_U \rho^2 \phi \, dx + \int_{t-1}^t (G(\tau) + G_1(\tau)) \, d\tau \right).
\]
(3.10)
Proof. The proof follows [10] by using basic differential inequalities (3.4) and (3.5).

Integrating (3.5) from \( t - 1 \) to \( t \) yields

\[
\int_U \tilde{p}^2(x, t) \phi \, dx + \frac{1}{2} \int_{t-1}^t \int_U H(x, |\nabla p(x, \tau)|) \, dx \, d\tau \leq \int_U \tilde{p}^2(x, t - 1) \phi \, dx + C \int_{t-1}^t G(\tau) \, d\tau.
\]

Neglecting first term on the left-hand side, we have

\[
\int_{t-1}^t \int_U H(x, |\nabla p(x, \tau)|) \, dx \, d\tau \leq 2 \int_U \tilde{p}^2(x, t - 1) \phi \, dx + C \int_{t-1}^t G(\tau) \, d\tau. \tag{3.11}
\]

Using (3.4) with \( \epsilon = \frac{1}{2} \) gives

\[
\int_U \tilde{p}^2(x, t) \phi \, dx + \frac{d}{dt} \int_U H(x, |\nabla p(x, t)|) \, dx \leq \int_U H(x, |\nabla p(x, t)|) \, dx + C(G(t) + G_i(t)).
\]

Let \( s \in [t - 1, t]. \) Integrating the previous inequality in time from \( s \) to \( t \), we obtain

\[
\int_s^t \int_U \tilde{p}^2(x, \tau) \phi \, dx \, d\tau + \int_U H(x, |\nabla p(x, t)|) \, dx \\
\leq \int_U H(x, |\nabla p(x, s)|) \, dx + \int_s^t \int_U H(x, |\nabla p(x, \tau)|) \, dx \, d\tau + C \int_s^t (G(\tau) + G_i(\tau)) \, d\tau \\
\leq \int_U H(x, |\nabla p(x, s)|) \, dx + \int_s^t \int_U H(x, |\nabla p(x, \tau)|) \, dx \, d\tau + C \int_{t-1}^t (G(\tau) + G_i(\tau)) \, d\tau.
\]

Integrating the last inequality in \( s \) from \( t - 1 \) to \( t \) results in

\[
\int_{t-1}^t \int_s^t \tilde{p}^2(x, \tau) \phi \, dx \, d\tau + \int_U H(x, |\nabla p(x, t)|) \, dx \\
\leq 2 \int_{t-1}^t \int_U H(x, |\nabla p(x, \tau)|) \, dx \, d\tau + C \int_{t-1}^t (G(\tau) + G_i(\tau)) \, d\tau.
\]

Using (3.11) to estimate the first term on the right-hand side, we have

\[
\int_{t-1}^t \int_s^t \tilde{p}^2(x, \tau) \phi \, dx \, d\tau + \int_U H(x, |\nabla p(x, t)|) \, dx \tag{3.12}
\]

\[
\leq 4 \int_U \tilde{p}^2(x, t - 1) \phi \, dx + C \int_{t-1}^t (G(\tau) + G_i(\tau)) \, d\tau.
\]

For the first integral on the left-hand side, we observe that

\[
\int_{t-1}^t \int_s^t \tilde{p}^2(x, \tau) \phi \, dx \, d\tau \geq \int_{t-1}^{t-1/2} \int_s^t \tilde{p}^2(x, \tau) \phi \, dx \, d\tau = \frac{1}{2} \int_{t-1}^t \int_s^t \tilde{p}^2(x, \tau) \phi \, dx \, d\tau.
\]

Utilizing this estimate in (3.12), we obtain inequality (3.10). □

Combining lemma 3.2 with theorem 2.2 results in the following specific estimates.
Theorem 3.3.

(i) If $t \geq 1$ then

$$
\int_\Omega H(x, |\nabla p(x, t)|) dx \leq C \left( \int_\Omega \tilde{\rho}^2(x, 0)\phi(x) dx + \mathcal{M}(t)\frac{1}{t-\alpha} + \int_{t-1}^t G(\tau) d\tau \right). \tag{3.13}
$$

(ii) If $\mathcal{A} < \infty$ then

$$
\lim_{t \to \infty} \sup \int_\Omega H(x, |\nabla p(x, t)|) dx \leq C \left( A\frac{1}{t-\alpha} + \lim_{t \to \infty} \sup \int_{t-1}^t G(\tau) d\tau \right). \tag{3.14}
$$

(iii) If $B < \infty$ then there is $T > 1$ such that for all $t > T$,

$$
\int_\Omega H(x, |\nabla p(x, t)|) dx \leq C \left( B\frac{1}{t-\alpha} + G(t)\frac{2}{t-\alpha} + \int_{t-1}^t G(\tau) d\tau \right). \tag{3.15}
$$

Proof.

(i) Combining (3.10) with estimate (2.22) and property (2.19) yields

$$
\int_\Omega H(x, |\nabla p(x, t)|) dx \leq C \left( \int_\Omega \tilde{\rho}^2(x, 0)\phi(x) dx + \mathcal{M}(t)\frac{1}{t-\alpha} + \int_{t-1}^t (G(\tau) + G_1(\tau)) d\tau \right) \leq C \left( \int_\Omega \tilde{\rho}^2(x, 0)\phi(x) dx + \mathcal{M}(t)\frac{2}{t-\alpha} + \mathcal{M}(t) + \int_{t-1}^t G(\tau) d\tau \right).
$$

Then using the fact $\mathcal{M}(t) \geq 1$ from (2.21), we obtain (3.13).

(ii) Taking limit superior of (3.10), and using limit estimate (2.23), we have

$$
\lim_{t \to \infty} \sup \int_\Omega H(x, |\nabla p(x, t)|) dx \leq C \lim_{t \to \infty} \sup G(t)\frac{2}{t-\alpha} + C \lim_{t \to \infty} \sup \int_{t-1}^t [G(\tau) + G_1(\tau)] d\tau.
$$

Note that

$$
\lim_{t \to \infty} \sup \int_{t-1}^t G(\tau) d\tau \leq \lim_{t \to \infty} \sup G(t).
$$

Then

$$
\lim_{t \to \infty} \int_\Omega H(x, |\nabla p(x, t)|) dx \leq C \left( A\frac{1}{t-\alpha} + A + \lim_{t \to \infty} \sup \int_{t-1}^t G_1(\tau) d\tau \right).
$$

Estimate (3.14) then follows since $A \geq 1$.

(iii) Using (2.24) to estimate the term $\int_\Omega \tilde{\rho}^2(x, t-1)\phi dx$ in (3.10), we obtain

$$
\int_\Omega H(x, |\nabla p(x, t)|) dx \leq C \left( B\frac{1}{t-\alpha} + G(t-1)\frac{2}{t-\alpha} + \int_{t-1}^t (G(\tau) + G_1(\tau)) d\tau \right). \tag{3.17}
$$

Note from Lemma A.4 that

$$
G(\tau) \leq G(t) + B + 1 \quad \forall \tau \in [t-1, t].
$$
Hence (3.17) implies
\[
\int_U H(x, |\nabla p(x, t)|) \, dx \leq C \left( B^{\frac{1}{2\sigma}} + (G(t) + B + 1)^{\frac{2}{2\sigma}} + (G(t) + B + 1) + \int_{t-1}^t G(\tau) \, d\tau \right).
\]
Then inequality (3.15) follows by using (1.11), (1.12) and the fact $G(t) \geq 1$.

\[\square\]

**Remark 3.4.** (a) Compared to (3.2), estimate (3.13) does not require $\nabla p(x, 0)$. Also, (3.14) improves (3.3) slightly, particularly when $G(t)$ fluctuates strongly in time. (b) The estimate (3.15) is simpler than (3.17) which is the form usually presented in previous papers [10–13].

The statements in theorems 3.1 and 3.3 can be rewritten to give estimates for the integral
\[
\int_U W(x)|\nabla p(x, t)|^{2-a} \, dx,
\]
that is, $\|\nabla p(t)\|_{L^a}^{2-a}$.

**Corollary 3.5.** For $t > 0$,
\[
\int_U W(x)|\nabla p(x, t)|^{2-a} \, dx \leq C \left( \int_U \tilde{p}^2(x, 0) \phi(x) \, dx + \mathcal{M} t^{-\frac{2}{2\sigma}} + \int_0^t e^{-\frac{1}{4(t-t')}} G(t) \, d\tau \right).
\]

For $t \geq 1$,
\[
\int_U W(x)|\nabla p(x, t)|^{2-a} \, dx \leq C \left( \int_U \tilde{p}^2(x, 0) \phi(x) \, dx + \mathcal{M} t^{-\frac{2}{2\sigma}} + \int_{t-1}^t G(\tau) \, d\tau \right).
\]

If $\mathcal{A} < \infty$ then
\[
\limsup_{t \to \infty} \int_U W(x)|\nabla p(x, t)|^{2-a} \, dx \leq C \left( \mathcal{A} t^{-\frac{2}{2\sigma}} + \limsup_{t \to \infty} \int_{t-1}^t G(\tau) \, d\tau \right).
\]

If $\mathcal{B} < \infty$ then there is $T > 1$ such that for all $t > T$,
\[
\int_U W(x)|\nabla p(x, t)|^{2-a} \, dx \leq C \left( B^{\frac{1}{2\sigma}} + (G(t) + B)^{\frac{2}{2\sigma}} + \int_{t-1}^t G(\tau) \, d\tau \right).
\]

**Proof.** Using property (1.29), definitions (2.8) and (2.13) we have
\[
\int_U W(x)|\nabla p(x, t)|^{2-a} \, dx \leq \int_U \left[ \frac{d_b(x)}{2} + H(x, |\nabla p(x, t)|) \right] \, dx \leq \frac{B}{2} + \int_U H(x, |\nabla p(x, t)|) \, dx.
\]
Also, from definition (2.15), $G(t) \geq B_\epsilon$. With these relations, the estimates (3.18)–(3.21) immediately follow (3.2), (3.13)–(3.15), respectively.

**4. Estimates for the pressure’s time derivative**

In this section, we estimate the pressure’s time derivative. Let
\[
q(x, t) = p(x, t) \quad \text{and} \quad \bar{q}(x, t) = \bar{p}(x, t) = p(x, t) - \Psi.
\]
Then $\tilde{q}$ solves

$$\phi(x) \frac{\partial \tilde{q}}{\partial t} = \nabla \cdot (K(x, |\nabla p|) \nabla p) - \phi(x) \Psi_t \quad \text{on } U \times (0, \infty),$$

$$\tilde{q} = 0 \quad \text{on } \Gamma \times (0, \infty).$$

In the following estimates, we use

$$G_2(t) = G_2[\Psi(t)] \overset{\text{def}}{=} \int_U |\Psi_0(x, t)|^2 \phi(x) dx.$$  \hfill (4.2)

**Lemma 4.1.** One has for any $t > 0$ and $\varepsilon > 0$ that

$$\frac{d}{dt} \int_U \tilde{q}^2(x, t) \phi(x) dx \leq - (1 - a) \int_U K(x, |\nabla p(x, t)|)|\nabla q(x, t)|^2 dx$$

$$+ \varepsilon \int_U \tilde{q}^2(x, t) \phi(x) dx + CG_1(t) + C\varepsilon^{-1}G_2(t).$$

**Proof.** Multiplying (4.1) by $\tilde{q}$, integrating over $U$, and using integration by parts we have

$$\int_U \frac{\partial}{\partial t} \tilde{q} \phi dx = \int_U \nabla \cdot (K(x, |\nabla p|) \nabla p) \tilde{q} dx - \int_U \Psi_t \tilde{q} dx$$

$$= - \int_U (K(x, |\nabla p|) \nabla p) \cdot \nabla \tilde{q} dx - \int_U \Psi_t \tilde{q} dx.$$

Taking the derivative in $t$ for the first integral on the right-hand side, we derive

$$\frac{1}{2} \frac{d}{dt} \int_U \tilde{q}^2 \phi dx = - \int_U \frac{\partial K(x, |\nabla p|)}{\partial \xi} \frac{(\nabla p \cdot \nabla q)}{|\nabla p|} (\nabla p \cdot \nabla q) dx$$

$$- \int_U K(x, |\nabla p|) \nabla q \cdot \nabla \tilde{q} dx - \int_U \Psi_t \tilde{q} \phi dx.$$

Using the fact that $\tilde{q}(x, t) = q(x, t) - \Psi_t$ for the first two integrals on the right-hand side, we rewrite

$$\frac{1}{2} \frac{d}{dt} \int_U \tilde{q}^2 \phi dx = - \int_U \frac{\partial K(x, |\nabla p|)}{\partial \xi} \frac{|\nabla p \cdot \nabla q|^2}{|\nabla p|} dx + \int_U \frac{\partial K(x, |\nabla p|)}{\partial \xi} \frac{\nabla p \cdot \nabla q}{|\nabla p|} \nabla p \cdot \nabla \Psi_t dx$$

$$- \int_U K(x, |\nabla p|) |\nabla q|^2 dx + \int_U K(x, |\nabla p|) \nabla q \cdot \nabla \Psi_t dx - \int_U \Psi_t \tilde{q} \phi dx.$$

Next, by derivative property (1.14) and Cauchy–Schwarz’s inequality we obtain

$$\frac{1}{2} \frac{d}{dt} \int_U \tilde{q}^2 \phi dx \leq a \int_U K(x, |\nabla p|) |\nabla q|^2 dx + a \int_U K(x, |\nabla p|) |\nabla q||\nabla \Psi_t| dx$$

$$- \int_U K(x, |\nabla p|)|\nabla q|^2 dx + \int_U K(x, |\nabla p|) |\nabla q||\nabla \Psi_t| dx - \int_U \Psi_t \tilde{q} \phi dx$$

$$\leq - (1 - a) \int_U K(x, |\nabla p|) |\nabla q|^2 dx + (a + 1) \int_U K(x, |\nabla p|) |\nabla q||\nabla \Psi_t| dx + \int_U |\Psi_t||\tilde{q}| \phi dx.$$
Let $\varepsilon' > 0$. Applying Cauchy’s inequality to the last two integrals gives
\[
\frac{1}{2} \frac{d}{dt} \int_U \tilde{q}^2 \phi dx \leq (\varepsilon'(a + 1) - (1 - a)) \int_U K(x, |\nabla p|) |\nabla q|^2 dx + \varepsilon \int_U \tilde{q}^2 \phi dx + \frac{1}{4\varepsilon} \int_U |\nabla \varepsilon|^2 \phi dx.
\]
We estimate $K(x, |\nabla p|)$ in the second integral on the right-hand side by (1.15), then it follows
\[
\frac{1}{2} \frac{d}{dt} \int_U \tilde{q}^2 \phi dx \leq (\varepsilon'(a + 1) - (1 - a)) \int_U K(x, |\nabla p|) |\nabla q|^2 dx + \frac{1}{4\varepsilon} \int_U |\nabla \varepsilon|^2 \phi dx + \frac{G_2(t)}{4\varepsilon}.
\]
Selecting $\varepsilon' = (1 - a)/(2(1 + a))$ gives
\[
\frac{1}{2} \frac{d}{dt} \int_U \tilde{q}^2 \phi dx \leq \frac{1 - a}{2} \int_U K(x, |\nabla p|) |\nabla q|^2 dx + CG_1(t) + \varepsilon \int_U \tilde{q}^2 \phi dx + \frac{CG_2(t)}{\varepsilon},
\]
which proves (4.3).

The next theorem contains different estimates of $\|\tilde{p}(t)\|_{L^2}$ for both small and large time in terms of the initial and boundary data. Note that we cannot estimate the norm at $t = 0$.

**Theorem 4.2.**

(i) For $t_0 \in (0, 1]$ and $t \geq t_0$,
\[
\int_U H(x, |\nabla p(x, t)|) dx + \int_U \tilde{p}_1^2(x, t) \phi(x) dx \leq C \left\{ t_0^{-1} \int_U [H(x, |\nabla p(x, 0)|) + \tilde{p}_1^2(x, 0) \phi(x)] dx + t_0 \int_0^t G_1(\tau) d\tau + M(t) \int_0^t \int_0^\tau (G_1(\tau) + G_2(\tau)) d\tau \right\}. \tag{4.4}
\]

(ii) If $t \geq 1$ then
\[
\int_U \tilde{p}_1^2(x, t) \phi(x) dx \leq C \left\{ \int_U \tilde{p}_1^2(x, 0) \phi(x) dx + M(t) \int_0^t \int_0^\tau (G_1(\tau) + G_2(\tau)) d\tau \right\}. \tag{4.5}
\]

(iii) If $A < \infty$ then
\[
\limsup_{t \to \infty} \int_U \tilde{p}_1^2(x, t) \phi(x) dx \leq C \left\{ A \int_0^\infty (G_1(\tau) + G_2(\tau)) d\tau \right\}. \tag{4.6}
\]

Consequently,
\[
\limsup_{t \to \infty} \int_U \tilde{p}_1^2(x, t) \phi(x) dx \leq C \left\{ A + \limsup_{t \to \infty} \int_0^t (G_1(\tau) + G_2(\tau)) d\tau \right\}. \tag{4.7}
\]

(iv) If $B < \infty$ then there is $T > 0$ such that for all $t > T$,
\[
\int_U \tilde{p}_1^2(x, t) \phi(x) dx \leq C \left\{ B \int_0^\infty (G_1(\tau) + G_2(\tau)) d\tau + \int_0^T (G_1(\tau) + G_2(\tau)) d\tau \right\}. \tag{4.8}
\]
Proof. Denote $I(t) = \int_U H(x, |\nabla p(x, t)|)dx + \int_U \tilde{q}^2(x, t)\phi(x)dx$ for $t > 0$.

(i) Adding (3.7) and (4.3) with $\varepsilon = 1/4$ yields

$$\frac{d}{dt}I(t) + \frac{1}{4}I(t) \leq C\int_U \tilde{p}^2\phi dx + CG_3(t), \quad \text{where } G_3 = G + G_1 + G_2. \quad (4.9)$$

Integrating (3.6) in time from 0 to $t$, we have

$$\int_0^t I(\tau)d\tau \leq C_1 J_0 + C_1 \int_0^t (G(\tau) + G_1(\tau))d\tau \quad (4.10)$$

for some $C_1 > 0$, where $J_0 = \int_U [H(x, |\nabla p(x, 0)|) + \tilde{p}^2(x, 0)\phi(x)] dx$. Applying (4.10) to $t = t_0$, then there exists $t_* \in (0, t_0)$ such that

$$I(t_*) \leq \frac{2}{t_0} \int_0^{t_0} I(\tau)d\tau \leq \frac{2C_1}{t_0} J_0 + \frac{2C_1}{t_0} \int_0^{t_0} (G(\tau) + G_1(\tau))d\tau. \quad (4.11)$$

For $t \geq t_0$, applying Gronwall’s inequality to (4.9) on the interval $[t_*, t]$, and then combining with (4.11), we have

$$I(t) \leq e^{-\frac{1}{4}(t-t_0)}I(t_*) + C\int_{t_*}^t e^{-\frac{1}{4}(t-\tau)}\left[\int_U \tilde{p}^2(x, \tau)\phi(x)dx + G_3(\tau)\right]d\tau \leq \frac{C_1 J_0}{t_0} + \frac{C_1}{t_0} \int_0^{t_0} (G(\tau) + G_1(\tau))d\tau + C \int_{t_*}^t e^{-\frac{1}{4}(t-\tau)}\left[\int_U \tilde{p}^2(x, \tau)\phi(x)dx + G_3(\tau)\right]d\tau.$$

Using (2.22) to estimate the integral $\int_U \tilde{p}^2(x, \tau)\phi(x)dx$ yields

$$I(t) \leq \frac{C}{t_0} J_0 + \frac{C}{t_0} \int_0^{t_0} (G(\tau) + G_1(\tau))d\tau + C\int_{t_*}^t e^{-\frac{1}{4}(t-\tau)}\left[\int_U \tilde{p}^2(x, \tau)\phi(x)dx + M(\tau)\tilde{q}^2 + G(\tau) + G_1(\tau) + G_2(\tau)\right]d\tau.$$

Since $G(t) \leq M(t)$ is increasing, see (2.19), it follows that

$$I(t) \leq \frac{C}{t_0} J_0 + \frac{C}{t_0} \int_0^{t_0} G(\tau)d\tau + CM(t_0) + C \int_U \tilde{p}^2(x, 0)\phi(x)dx + M(t_0)^{\frac{1}{2}} + M(t_0) + C \int_{t_*}^t e^{-\frac{1}{4}(t-\tau)}(G_1(\tau) + G_2(\tau))d\tau \quad (4.12)$$

Also, from (2.21), $M(t) \geq 1$, then $M(t_0) \leq M(t) \leq M(t)^{\frac{1}{2}}$. Thus, we obtain estimate (4.4) from (4.12).

(ii) Using (4.3) with $\varepsilon = 1/2$, and dropping the first integral on the right-hand side, we have

$$\frac{1}{2} \frac{d}{dt} \int_U \tilde{q}^2\phi dx \leq \frac{1}{2} \int_U \tilde{q}^2\phi dx + C_2(G_1(t) + G_2(t)) \quad (4.13)$$
for some $C_2 > 0$. For $s \in [t - \frac{1}{2}, t]$, integrating (4.13) in time from $s$ to $t$ gives

$$\frac{1}{2} \int_U \tilde{q}^2(x, t) \phi dx \leq \frac{1}{2} \int_U \tilde{q}^2(x, s) \phi dx + \frac{1}{2} \int_s^t \int_U |\tilde{q}(x, \tau)|^2 \phi d\tau d\tau + C_2 \int_{t-\frac{1}{2}}^t (G_1(\tau) + G_2(\tau)) d\tau \leq \frac{1}{2} \int_U \tilde{q}^2(x, s) \phi dx + \frac{1}{2} \int_{t-\frac{1}{2}}^t \int_U |\tilde{q}(x, \tau)|^2 \phi d\tau d\tau + C_2 \int_{t-\frac{1}{2}}^t (G_1(\tau) + G_2(\tau)) d\tau.$$

Next, integrating in $s$ from $t - \frac{1}{2}$ to $t$, we have

$$\frac{1}{2} \frac{1}{2} \int_U \tilde{q}^2(x, t) \phi dx \leq \frac{1}{2} \int_{t-\frac{1}{2}}^{t} \int_U \tilde{q}^2(x, s) \phi d\tau d\tau + \frac{1}{2} \frac{1}{2} \int_{t-\frac{1}{2}}^{t} \int_U |\tilde{q}(x, \tau)|^2 \phi d\tau d\tau + \frac{3}{4} \int_{t-\frac{1}{2}}^{t} \int_U \tilde{q}^2(x, s) \phi d\tau d\tau + \frac{C_2}{2} \int_{t-\frac{1}{2}}^{t} (G_1(\tau) + G_2(\tau)) d\tau.$$

Hence,

$$\int_U \tilde{q}^2(x, t) \phi dx \leq 3 \int_{t-\frac{1}{2}}^{t} \int_U \tilde{q}^2(x, s) \phi d\tau d\tau + C \int_{t-\frac{1}{2}}^{t} (G_1(\tau) + G_2(\tau)) d\tau.$$

Using (3.10) to bound the first integral on the right-hand side, we obtain

$$\int_U \tilde{p}^2(x, t) \phi dx \leq C \int_U \tilde{p}^2(x, t-1) \phi dx + C \int_{t-\frac{1}{2}}^{t} (G(\tau) + G_1(\tau) + G_2(\tau)) d\tau.$$

Combining (2.22) and (4.14) gives

$$\int_U \tilde{p}^2(x, t) \phi dx \leq C \int_U \tilde{p}^2(x, 0) \phi dx + C \mathcal{M}(t-1) \mathcal{T}^2 + C \int_{t-\frac{1}{2}}^{t} (G(\tau) + G_1(\tau) + G_2(\tau)) d\tau.$$

Again, by properties (2.19) and (2.21), estimate (4.5) follows.

(iii) Taking limit superior of (4.14) and using (2.23), we obtain

$$\limsup_{t \to \infty} \int_U \tilde{p}^2(x, t) \phi dx \leq C A \tilde{T} + C A + C \limsup_{t \to \infty} \int_{t-\frac{1}{2}}^{t} (G_1(\tau) + G_2(\tau)) d\tau,$$

which yields (4.6). The estimate (4.7) follows (4.6) and property (3.16) for functions $G_1$ and $G_2$ in place of $G$.

(iv) For sufficiently large $t$, estimating the first term on the right-hand side of (4.14) by (2.24), and then applying Lemma A.4 to bound $G(t-1)$ and $G(\tau)$ in terms of $B$ and $G(t)$, we obtain

$$\int_U \tilde{p}^2(x, t) \phi dx \leq C(B^{\frac{1}{2-a}} + G(t-1)^{\frac{2}{2-a}}) + C \int_{t-\frac{1}{2}}^{t} (G(\tau) + G_1(\tau) + G_2(\tau)) d\tau \leq C B^{\frac{1}{2-a}} + C(1 + B + G(t)^{\frac{2}{2-a}} + C(1 + B + G(t)) + C \int_{t-\frac{1}{2}}^{t} (G_1(\tau) + G_2(\tau)) d\tau.$$

Then (4.8) follows by simple manipulations using inequalities (1.11), (1.12).

**5. Continuous dependence**

In this section, we establish the continuous dependence of the solution on the initial and boundary data.
Let \( p_1(x, t) \) and \( p_2(x, t) \) be two solutions of (1.9) with boundary data \( \psi_1(x, t) \) and \( \psi_2(x, t) \), respectively. For \( i = 1, 2 \), let \( \Psi_i(x, t) \) be an extension of \( \psi_i(x, t) \), and define \( \bar{p}_1 = p_1 - \Psi_1 \). Denote
\[
P = p_1 - p_2, \quad \Phi = \Psi_1 - \Psi_2 \quad \text{and} \quad \bar{P} = \bar{p}_1 - \bar{p}_2 = P - \Phi.
\]

Then
\[
\phi(x) \frac{\partial \bar{P}}{\partial t} = \nabla \cdot (K(x, |\nabla p_1|)\nabla p_1 - K(x, |\nabla p_2|)\nabla p_2) - \phi(x)\Phi \quad \text{on} \quad U \times (0, \infty),
\]
\[
P = 0 \quad \text{on} \quad \Gamma \times (0, \infty).
\]

The weighted norms of \( P \) and \( \Phi \) are related by the following differential inequalities.

**Lemma 5.1.** For all \( t > 0 \), one has
\[
\frac{d}{dt} \int_U \bar{P}^2(x, t)\phi(x)dx \leq -d_3 h_1(t)^{-\frac{a}{2-a}} \left( \int_U W(x)|\nabla \bar{P}(x, t)|^{2-a}dx \right)^{\frac{2-a}{2}} + CD(t)h_2(t)^{\frac{1}{2}},
\]
where \( d_3, d_4 > 0, \)
\[
D(t) = \int_U a_0(x)^{-1}|\nabla \Phi(x, t)|^2dx + \left( \int_U a_0(x)^{-1}|\nabla \Phi(x, t)|^2dx \right)^{\frac{1}{2}} + \left( \int_U |\Phi(x, t)|^2|\phi(x)dx| \right)^{\frac{1}{2}},
\]
\[
h_1(t) = B_1 + \sum_{i=1}^{2} \int_U H(x, |\nabla p_i(x, t)|)dx,
\]
\[
h_2(t) = 1 + \sum_{i=1}^{2} \int_U \left[ H(x, |\nabla p_i(x, t)|) + p_i^2(x, t)|\phi(x)\right] dx.
\]

**Proof.** We define
\[
D_1(t) = \int_U |\Phi(x, t)|^2|\phi(x)dx|, \quad D_2(t) = \int_U a_0(x)^{-1}|\nabla \Phi(x, t)|^2dx,
\]
\[
h_3(t) = \sum_{i=1}^{2} ||\bar{p}_i(t)||_{L^2_x}^2, \quad h_4(t) = \sum_{i=1}^{2} \int_U H(x, |\nabla p_i(x, t)|)dx.
\]

Multiplying equation (5.1) by \( \bar{P} \) and integrating over \( U \) give
\[
\int_U \bar{P} \cdot \Phi \phi dx = \int_U (\nabla \cdot (K(x, |\nabla p_1|)\nabla p_1 - K(x, |\nabla p_2|)\nabla p_2))\bar{P}dx - \int_U \Phi_1 \bar{P}\phi dx.
\]
Using integration by parts for the first integral on the right-hand side, we have
\[\frac{1}{2} \frac{d}{dt} \int_U \hat{P}^2 \phi \, dx = -\int_U (K(x, |\nabla p_1|)\nabla p_1 - K(x, |\nabla p_2|)\nabla p_2) \cdot \nabla \hat{P} \, dx - \int_U \Phi \hat{P} \phi \, dx\]
\[= -\int_U (K(x, |\nabla p_1|)\nabla p_1 - K(x, |\nabla p_2|)\nabla p_2) \cdot (\nabla p_1 - \nabla p_2) \, dx + \int_U (K(x, |\nabla p_1|)\nabla p_1 - K(x, |\nabla p_2|)\nabla p_2) \cdot \nabla \Phi \, dx - \int_U \Phi \hat{P} \phi \, dx.\]

Applying lemma 1.2 to the first integrand on the right-hand side of the last identity, we obtain
\[\frac{1}{2} \frac{d}{dt} \int_U \rho^2 \phi \, dx \leq -(1 - a) \int_U K(x, |\nabla p_1| \vee |\nabla p_2|) |\nabla p_1 - \nabla p_2|^2 \, dx + \int_U (K(x, |\nabla p_1|) |\nabla p_1| + K(x, |\nabla p_2|) |\nabla p_2|) |\nabla \Phi|^2 \, dx + \int_U |\Phi| |\hat{P}| \phi \, dx.\]

Above, we use the notation \(|\nabla p_1| \vee |\nabla p_2| = \max(|\nabla p_1|, |\nabla p_2|)|.

For the first integral on the right-hand side, we note that
\[|\nabla p_1 - \nabla p_2|^2 = |\nabla \hat{P} + \nabla \Phi|^2 \geq \frac{1}{2} |\nabla \hat{P}|^2 - |\nabla \Phi|^2,\]

hence,
\[\frac{1}{2} \frac{d}{dt} \int_U \rho^2 \phi \, dx \leq \frac{1}{2} \cdot \frac{1 - a}{2} \int_U K(x, |\nabla p_1| \vee |\nabla p_2|) |\nabla \hat{P}|^2 \, dx + C \int_U K(x, |\nabla p_1| \vee |\nabla p_2|) |\nabla \Phi|^2 \, dx + \int_U |\Phi| |\hat{P}| \phi \, dx + \int_U |\Phi|^2 \, dx.\]

(5.7)

\[\cdot \text{Consider } I_1. \text{ Let } K(x, t) = K(x, |\nabla p_1(x, t)| \vee |\nabla p_2(x, t)|). \text{ Then by H"older's inequality,}\]
\[\int_U W_1(x) |\nabla \hat{P}|^{2-a} \, dx \leq \left( \int_U K(x, t) |\nabla \hat{P}|^2 \, dx \right)^\frac{2-a}{2} \cdot J_1^\frac{a}{2}, \text{ where } J_1 = \int_U \frac{W_1(x)^\frac{2}{2-a} \, dx.}\]

(5.8)

Applying (1.18) to bound \(K(x, t)\) from below, and then using (1.11), we estimate \(J_1\) as
\[J_1 \leq \int_U W_1(x)^\frac{2}{2-a} \left( \left( |\nabla p_1| \vee |\nabla p_2| \right)^a + a_0(x)^a \right)^\frac{2-a}{2} \, dx\]
\[\leq C \left( \int_U W_1(x) a_0(x)^{2-a} \, dx + \int_U W_1(x) (|\nabla p_1| \vee |\nabla p_2|)^2 \, dx \right)\]
\[\leq C \left( \int_U W_1(x) a_0(x)^{2-a} \, dx + \int_U W_1(x) (|\nabla p_1|^{2-a} + |\nabla p_2|^{2-a}) \, dx \right).\]

Then by (1.26) and (1.29), we have
\[ J_1 \leq C \left( \int_U a_0(x) \, dx + \int_U [H(x, |\nabla p_1|) + H(x, |\nabla p_2|)] \, dx \right) = C h_1(t). \]

This and (5.8) yield
\[ I_1 = \int_U K(x, t)|\nabla \tilde{P}|^2 \, dx \geq C \left( \int_U W_1(x)|\nabla \tilde{P}|^{2-a} \, dx \right)^{\frac{2}{2-a}} h_1(t)^{-\frac{a}{2-a}}. \tag{5.9} \]

• For \( I_2 \), by using (1.15)
\[ I_2 \leq C \int_U a_0(x)^{-1} |\nabla \Phi|^2 \, dx = CD_2(t). \tag{5.10} \]

• For \( I_3 \), applying Hölder’s inequality gives
\[ I_3 \leq \sum_{i=1,2} \left\{ \left( \int_U K(x, |\nabla p_i|)|\nabla p_i|^2 \, dx \right)^{\frac{1}{2}} \left( \int_U K(x, |\nabla p_i|)|\nabla \Phi|^2 \, dx \right)^{\frac{1}{2}} \right\}. \]

Using (1.28) for the first integral and (1.15) the second integral, and then applying Cauchy–Schwarz inequality, we have
\[ I_3 \leq \sqrt{2} \left( \sum_{i=1,2} \int_U H(x, |\nabla p_i|) \, dx \right)^{\frac{1}{2}} \left( \int_U a_0(x)^{-1} |\nabla \Phi|^2 \, dx \right)^{\frac{1}{2}}. \]

Thus,
\[ I_3 \leq Ch_1(t)^{\frac{1}{2}}D_2(t)^{\frac{1}{2}}. \tag{5.11} \]

• For \( I_4 \), applying Hölder’s inequality gives
\[ I_4 \leq \|\tilde{P}\|_{L^2} \|\Phi\|_{L^2} \leq (\|\tilde{p}_1\|_{L^2} + \|\tilde{p}_2\|_{L^2}) \|\Phi\|_{L^2} \leq \sqrt{2} h_3(t)^{\frac{1}{2}}D_1(t)^{\frac{1}{2}}. \tag{5.12} \]

Then combining (5.7), (5.9)–(5.11) and (5.12) yields
\[ \frac{d}{dt} \int_U \tilde{P}^2(x, t) \phi(x) \, dx \leq -d_2 h_1(t)^{-\frac{a}{2-a}} \left( \int_U W_1(x)|\nabla \tilde{P}|^{2-a} \, dx \right)^{\frac{2}{2-a}} h_1(t)^{-\frac{a}{2-a}} + CD_2(t) + Ch_4(t)^{\frac{1}{2}}D_2(t)^{\frac{1}{2}} + Ch_3(t)^{\frac{1}{2}}D_1(t)^{\frac{1}{2}}. \]

Hence
\[ \frac{d}{dt} \int_U \tilde{P}^2(x, t) \phi(x) \, dx \leq -d_2 h_1(t)^{-\frac{a}{2-a}} \left( \int_U W_1(x)|\nabla \tilde{P}(x, t)|^{2-a} \, dx \right)^{\frac{2}{2-a}} + CD_3(t), \tag{5.13} \]

where
\[ D_3(t) = D_2(t) + h_4(t)^{\frac{1}{2}}D_2(t)^{\frac{1}{2}} + Ch_4(t)^{\frac{1}{2}}D_1(t)^{\frac{1}{2}}. \]

We estimate
\[ D_3(t) \leq C(D_2(t) + D_2(t)^{\frac{1}{2}} + D_1(t)^{\frac{1}{2}})(1 + h_4(t)^{\frac{1}{2}} + h_3(t)^{\frac{1}{2}}) \leq CD(t)h_2(t)^{\frac{1}{2}}. \tag{5.14} \]
Therefore, (5.2) follows (5.13) and (5.14). Finally, using Poincaré-Sobolev’s inequality (2.2) for \( u = \bar{P} \), (5.2) implies (5.3).

To describe more specific estimates, we introduce

\[
\hat{\mathcal{P}}_0 = \sum_{i=1}^{2} \int_{U} \bar{P}_i^2(x, 0) \phi(x) \, dx, \quad \mathcal{H}_0 = \sum_{i=1}^{2} \int_{U} H(x, |\nabla p_i(x, 0)|) \, dx,
\]

and, referring to (2.15), (2.19), (3.1), (4.2), define for \( t \geq 0 \)

\[
\hat{\mathcal{G}}(t) = \sum_{i=1}^{2} G_i[\Psi_i](t), \quad \hat{\mathcal{M}}(t) = \sum_{i=1}^{2} \mathcal{M}[\Psi_i](t),
\]

\[
\hat{\mathcal{G}}_1(t) = \sum_{i=1}^{2} G_i[\Psi_i](t), \quad \hat{\mathcal{G}}_2(t) = \sum_{i=1}^{2} G_i[\Psi_i](t).
\]

In the following, we show that the \( L^2 \)-norm of \( \bar{P}(t) \) for \( t > 0 \) can be bounded by the initial difference \( \bar{P}(0) - \bar{P}_0 \) and the difference between the boundary data expressed by \( D(t) \). It means that the solution of (1.10) depends continuously on the initial and boundary data.

**Theorem 5.2.** For \( t \geq 0 \),

\[
\|\bar{P}(t)\|_{L^2}^2 \leq e^{-\int_0^t \mathcal{M}(\tau) \frac{d}{d\tau} d\tau} \|\bar{P}(0)\|_{L^2}^2 + C \int_0^t e^{-\int_0^{\tau} \mathcal{M}(\tau') \frac{d}{d\tau'} d\tau'} \mathcal{M}_0(s) D(s) \, ds, \tag{5.15}
\]

where \( d_5 > 0 \), and

\[
\mathcal{M}_0(t) = \mathcal{H}_0 + \hat{\mathcal{P}}_0 + \hat{\mathcal{M}}(t)^{\frac{d}{d\tau}} + \sup_{\tau \in [0,t]} \hat{\mathcal{G}}_1(\tau).
\]

In particular, for any \( T > 0 \),

\[
\sup_{t \in [0,T]} \|\bar{P}(t)\|_{L^2}^2 \leq \|\bar{P}(0)\|_{L^2}^2 + C \mathcal{M}_0(T)^{\frac{d}{d\tau}} \int_0^T D(t) \, dt. \tag{5.16}
\]

**Proof.** Define \( y(t) = \int_0^t \bar{P}^2(x, t) \phi(x) \, dx \). We rewrite (5.3) as

\[
y'(t) \leq -d_5 h(t) \frac{d}{dt} y(t) + C_0 D(t) h_2(t)^{\frac{d}{d\tau}}. \tag{5.17}
\]

By Gronwall’s inequality

\[
y(t) \leq y(0) e^{-\int_0^t h(t) \frac{d}{d\tau} d\tau} + C \int_0^t e^{-\int_0^{\tau} h(t) \frac{d}{d\tau} d\tau} h_2(s)^{\frac{d}{d\tau}} D(s) \, ds. \tag{5.18}
\]

Let \( t \geq 0 \). Using definition (5.5) of \( h_1(t) \) and by applying (3.2) to each \( p = p_i \) for \( i = 1, 2 \), we have

\[
h_1(t) \leq B_1 + e^{-\frac{d}{d\tau}} \mathcal{H}_0 + C \hat{\mathcal{P}}_0 + C \hat{\mathcal{M}} \frac{d}{d\tau} \hat{\mathcal{G}}_1(t) + C \int_0^t e^{-\frac{d}{d\tau}(t - \tau)} \hat{\mathcal{G}}_1(\tau) \, d\tau.
\]
Thus,

\[ h_1(t) \leq C(\mathcal{H}_0 + \tilde{P}_0 + \tilde{\mathcal{M}} + \sup_{\tau \in [0,t]} \tilde{G}(\tau)) = C\mathcal{M}_1(t). \]  

(5.19)

Similarly, by (5.6), estimates (3.2) and (2.22), we have

\[ h_2(t) \leq 1 + e^{\frac{1}{a^2}h_0 + \tilde{P}_0 + C\mathcal{M}_1(t)^2} + C \int_0^t e^{-\frac{1}{a^2}(t-\tau)\tilde{G}(\tau)d\tau}, \]

which implies

\[ h_2(t) \leq C\mathcal{M}_1(t). \]  

(5.20)

Therefore, we obtain (5.15) from (5.18), (5.19) and (5.20).

Now, let \( T > 0 \). Neglecting the exponentials in (5.15) and noting that \( \mathcal{M}_1(t) \leq \mathcal{M}_1(T) \) for all \( t \in [0, T] \), we obtain (5.16).

Next, we estimate \( \tilde{P}(t) \) when \( t \to \infty \). The estimate is independent of the initial data and only depends on the asymptotic behavior of \( \Psi(x, t), \Phi(x, t), \) and \( \Phi(x, t) \) as \( t \to \infty \).

If \( \int_0^\infty h_1(t) \tilde{G}(\tau) d\tau = \infty \), then from (5.17) and Lemma A.1(ii) we have

\[ \limsup_{t \to \infty} \|\tilde{P}(t)\|^2_{L_2} \leq C \limsup_{t \to \infty} \frac{D(t) h_2(t)^2}{h_1(t)} = C \limsup_{t \to \infty} R(t), \]  

(5.21)

where

\[ R(t) = h_2(t)h_1(t)^2D(t). \]  

(5.22)

To estimate the last limit in (5.21), we define, referring to (2.20), (3.1), and (4.2), the following numbers

\[ \tilde{A} = \sum_{i=1}^2 \tilde{A}[\Psi_i] = \sum_{i=1}^2 \lim_{t \to \infty} G[\Psi_i](t), \quad \tilde{B} = \sum_{i=1}^2 \tilde{B}[\Psi_i] = \sum_{i=1}^2 \lim_{t \to \infty} \sup[G[\Psi_i](t)]^\prime, \]

\[ \tilde{G}_1 = \sum_{i=1}^2 \lim_{t \to \infty} \int_{t-1}^{t+1} G_i[\Psi_i](\tau)d\tau, \quad \tilde{G}_2 = \sum_{i=1}^2 \lim_{t \to \infty} \int_{t-1}^{t+1} G_2[\Psi_i](\tau)d\tau. \]

The asymptotic behavior of \( \Phi(x, t) \) as \( t \to \infty \) will be characterized by

\[ \mathcal{D} = \limsup_{t \to \infty} D(t). \]

Denote also that

\[ \kappa_0 = \frac{a}{2-a} + \frac{1}{2} = \frac{2 + a}{2(2 - a)}. \]
Theorem 5.3. If $\tilde{A}$ and $G_1$ are finite, then
\[
\limsup_{t \to \infty} \|P(t)\|_{L_0^2}^2 \leq C(\tilde{A}^{\frac{1}{2-a}} + G_1)^{\omega^*}D.
\] (5.23)

Proof. Note from (2.23) and (3.14) that
\[
\limsup_{t \to \infty} h_1(t), \limsup_{t \to \infty} h_2(t) \leq C(\tilde{A}^{\frac{1}{2-a}} + G_1) < \infty.
\] (5.24)

Then $h_1(t)$ and $h_2(t)$ are bounded on $[0, \infty)$. Thus, $\int_0^\infty h_1(t)^{\frac{2}{2-a}} dt = \infty$ and, consequently, estimate (5.21) holds. By (5.22) and (5.24),
\[
\limsup_{t \to \infty} R(t) \leq C(\tilde{A}^{\frac{1}{2-a}} + G_1)^{\omega^*}D = C(\tilde{A}^{\frac{1}{2-a}} + G_1)^{\omega^*}D.
\] (5.25)

Therefore, (5.23) follows this and (5.21). □

Now, we focus on the case when the boundary data is unbounded as $t \to \infty$.

• If $t > 0$, then, by (2.22),
\[
\sum_{i=1}^{2} \int_U \rho_i^2(x,t)\phi(x)dx \leq C(P_0 + \tilde{M}(t))^{\frac{2}{2-a}}.
\] (5.26)

If $t \geq 1$, then, by (3.13),
\[
\sum_{i=1}^{2} \int_U H(x,|\nabla P_i(x,t)|)dx \leq C\left(P_0 + \tilde{M}(t)^{\frac{2}{2-a}} + \int_{t-1}^{t} (\tilde{G}_i(\tau) + \tilde{G}_2(\tau))d\tau\right).
\] (5.27)

and, by (4.5),
\[
\sum_{i=1}^{2} \int_U \rho_{i,t}^2(x,t)\phi(x)dx \leq C\left(P_0 + \tilde{M}(t)^{\frac{2}{2-a}} + \int_{t-1}^{t} (\tilde{G}_i(\tau) + \tilde{G}_2(\tau))d\tau\right).
\] (5.28)

• In case $\tilde{B} < \infty$, then $B[\Psi_i]$ and $B[\Psi_2]$ are finite. Using estimates (2.24), (3.15), (4.8) for $P_i$ with $i = 1, 2$, there is $T_0 > 0$ such that for $t > T_0$, one has
\[
\sum_{i=1}^{2} \int_U \rho_i^2(x,t)\phi(x)dx \leq C\left(\tilde{B}^{\frac{1}{1-a}} + \tilde{G}_i(t)^{\frac{2}{2-a}}\right).
\] (5.29)

\[
\sum_{i=1}^{2} \int_U H(x,|\nabla P_i(x,t)|)dx \leq C\left(\tilde{B}^{\frac{1}{1-a}} + \tilde{G}_i(t)^{\frac{2}{2-a}} + \int_{t-1}^{t} (\tilde{G}_i(\tau) + \tilde{G}_2(\tau))d\tau\right).
\] (5.30)

\[
\sum_{i=1}^{2} \int_U \rho_{i,t}^2(x,t)\phi(x)dx \leq C\left(\tilde{B}^{\frac{1}{1-a}} + \tilde{G}_i(t)^{\frac{2}{2-a}} + \int_{t-1}^{t} (\tilde{G}_i(\tau) + \tilde{G}_2(\tau))d\tau\right).
\] (5.31)
Assume \( \tilde{A} = \infty \). Then \( \tilde{M}(t) = \infty \), and for sufficiently large \( t \), one has \( \tilde{M}(t) \geq \tilde{P}_0 \).

From (5.27), (5.28), and (5.30), (5.31), we have for large \( t \) that
\[
\sum_{i=1}^{2} \int_{U} H(x, |\nabla P_i(x, t)|) dx + \sum_{i=1}^{2} \int_{U} \tilde{P}_{i}^{2}(x, t, \phi(x)) dx \leq CV(t),
\]
where
\[
V(t) = \begin{cases} 
\tilde{M}(t)^{2-a} + \int_{t-1}^{t} (\tilde{G}^1(\tau) + \tilde{G}^2(\tau)) d\tau & \text{in general,} \\
\tilde{B} \frac{1}{1-a} + \tilde{G}(t)^{2-a} + \int_{t-1}^{t} (\tilde{G}^1(\tau) + \tilde{G}^2(\tau)) d\tau & \text{when } \tilde{B} < \infty.
\end{cases}
\]

With the above preparations, we are ready to estimate \( \|\tilde{P}(t)\|_{L^2} \) as \( t \to \infty \) in the case \( \tilde{A} = \infty \).

**Theorem 5.4.** Assume \( \tilde{A} = \infty \). If \( \int_{0}^{\infty} V(t)^{-\alpha} dt = \infty \), then
\[
\limsup_{t \to \infty} \|\tilde{P}(t)\|_{L^2}^2 \leq C \limsup_{t \to \infty} (V(t)^{\alpha} D(t)).
\]

**Proof.** By (5.29) and (5.30), or (5.26) and (5.27), we have for large \( t \)
\[
h_2(t), h_1(t) \leq CV(t).
\]
Combining this with (5.21), we have
\[
\limsup_{t \to \infty} \|\tilde{P}(t)\|_{L^2}^2 \leq C \limsup_{t \to \infty} (h_2(t)^{2}h_1(t)^{2}\tilde{P}(t)^{2}d\tilde{D}(t)) \leq C \limsup_{t \to \infty} (V(t)^{2}V(t)^{2}D(t))
\]
\[
= C \limsup_{t \to \infty} (V(t)^{2}D(t)).
\]
This proves (5.34).

The estimate (5.34) can be interpreted as follows. As \( t \to \infty \), even though \( V(t) \to \infty \), if the boundary data’s difference characterized by \( D(t) \) decays very fast, it can diminish the growth of \( V(t) \) and result in \( \|\tilde{P}(t)\|_{L^2} \) going to zero.

Now, we turn to the continuous dependence for the pressure gradient. What the results obtained below mean for \( \|\nabla \tilde{P}(t)\|_{L^2} \) are the same as theorems 5.2–5.4 for \( \|\tilde{P}(t)\|_{L^2} \).

**Theorem 5.5.** Let \( t_0 \in (0, 1] \). For \( t \geq t_0 > 0 \),
\[
\|\nabla \tilde{P}(t)\|_{L^2}^2 \leq C \mathcal{M}_2(t)^{\gamma} \left( e^{-dt} \int_{t_0}^{t} \tilde{M}(r)^{2-a} \|\tilde{P}(0)\|_{L^2}^2 + D(t)^{2} + \int_{0}^{t} e^{-dr} \int_{r}^{t} \tilde{M}(s)^{2-a} \mathcal{M}_1(s)^{\frac{1}{2}} D(s) ds dr \right)^{\frac{1}{2}},
\]
where
\[
\mathcal{M}_2(t) = t_0^{-1}(\mathcal{H}_0 + \tilde{P}_0) + \tilde{M}(t)^{2-a} + \sup_{\tau \in [0,t]} (\tilde{G}^1(\tau) + \tilde{G}^2(\tau)).
\]
Moreover,

\[
\limsup_{t \to \infty} \|\nabla \tilde{P}(t)\|_{L_{p_{1}}}^{2} \leq C \left[ (\hat{A} \tilde{\tau}^{\alpha} + G_{1} + G_{2})^{3/\sigma} D \right] + C (\hat{A} \tilde{\tau}^{\alpha} + G_{1})^{\nu} D. \tag{5.37}
\]

**Proof.** Multiplying (5.2) by \(d^{1/3} h_{0}(t) \tilde{\tau}^{-\alpha} \), we have

\[
\left( \int_{\Omega} W_{k}(x) \left| \nabla \tilde{P}(x, t) \right|^{2} \, dx \right)^{1/2} \leq - \frac{d^{1/3} h_{0}(t) \tilde{\tau}^{-\alpha}}{d} \int_{\Omega} \tilde{P}^{2}(x, t) \phi(x) \, dx + C h_{0}(t) \tilde{\tau}^{-\alpha} D(t) h_{2}(t) t^{1/2} \leq Ch_{0}(t) \tilde{\tau}^{-\alpha} \int_{\Omega} |\tilde{P}| \, \phi \, dx + CR(t) \leq Ch_{0}(t) \tilde{\tau}^{-\alpha} \|\tilde{P}\|_{L_{2}} + CR(t).
\]

Applying triangle inequality to \(\|\tilde{P}\|_{L_{2}}\) gives

\[
\|\nabla \tilde{P}(t)\|_{L_{p_{1}}}^{2} \leq Ch_{0}(t) \tilde{\tau}^{-\alpha} (\|\tilde{P}_{1}(t)\|_{L_{2}} + \|\tilde{P}_{2}(t)\|_{L_{2}}) \|\tilde{P}\|_{L_{2}} + CR(t) \tag{5.38}
\]

By (5.22), (5.19) and (5.20),

\[
R(t) \leq C M_{0}(t) \tilde{\tau}^{\alpha} D(t) = C M_{0}(t)^{\nu} D(t). \tag{5.39}
\]

By (4.4),

\[
\|\tilde{P}_{1}(t)\|_{L_{2}(D)} + \|\tilde{P}_{2}(t)\|_{L_{2}(D)} \leq C \left( \int_{0}^{t} \frac{1}{|\tilde{\tau}^{-\alpha}(\tau)|} \int_{0}^{t} \tilde{G}(\tau) \, d\tau + t_{0} \left( H_{0} + \tilde{P}_{0} \right) + \tilde{M}_{0}(t) \tilde{\tau}^{-\alpha} + \int_{0}^{t} \frac{1}{|\tilde{\tau}^{-\alpha}(\tau)|} \int_{0}^{t} \tilde{G}(\tau) \, d\tau \right)^{1/2} \leq C M_{2}(t)^{1/2}. \tag{5.40}
\]

Combining estimates (5.19), (5.40), (5.15), (5.39) with (5.38) yields

\[
\|\nabla \tilde{P}(t)\|_{L_{p_{1}}}^{2} \leq C M_{0}(t) \tilde{\tau}^{-\alpha} M_{0}(t)^{1/2} \left\{ e^{-d_{0} \int_{0}^{t} M_{0}(s) \, ds} \right\}^{1/2} \|\tilde{P}(0)\|_{L_{2}}^{2} \leq C M_{1}(t)^{\nu} D(t).
\]

Estimating the first and last \(M_{0}(t)\) terms on the right-hand side by \(M_{0}(t) \leq M_{2}(t)\), we obtain (5.36).

Let \(D_{0} = (\hat{A} \tilde{\tau}^{\alpha} + G_{1})^{\nu} D\). Taking limit superior of (5.38), and using the limit estimates (5.24), (5.23), (5.25) and (4.6), we have

\[
\limsup_{t \to \infty} \|\nabla \tilde{P}(t)\|_{L_{p_{1}}}^{2} \leq C (\hat{A} \tilde{\tau}^{\alpha} + G_{1})^{\nu} (\hat{A} \tilde{\tau}^{\alpha} + G_{1} + G_{2})^{1/2} D_{0} + CD_{0} \leq C (\hat{A} \tilde{\tau}^{\alpha} + G_{1} + G_{2})^{3/\sigma} D_{0} + C (\hat{A} \tilde{\tau}^{\alpha} + G_{1})^{\nu} D,
\]

hence obtaining (5.37). \(\square\)
Finally, we derive the gradient estimates for the case $\tilde{\mathcal{A}} = \infty$.

**Theorem 5.6.** Assume $\tilde{\mathcal{A}} = \infty$. Let $V(t)$ be defined by (5.33). Suppose
\[ \int_1^\infty V^{-\frac{d-2}{2}}(t)\,dt = \infty \quad \text{and} \quad \lim_{t \to \infty} (V^{-\frac{d}{2}}(t))' = 0. \] (5.41)

Then
\[ \limsup_{t \to \infty} \|\nabla \tilde{P}(t)\|_{L^2_{\eta_1}}^2 \leq C \lim sup \|V(t)^{\frac{\beta_0}{2}}D(t)\|^\frac{1}{2} + C \lim sup [V(t)^{\beta_0}D(t)]. \] (5.42)

**Proof.** By (5.38), (5.35), (5.32) and (5.22), we have for large $t$ that
\[ \|\nabla \tilde{P}(t)\|_{L^2_{\eta_1}}^2 \leq CV(t)^{\beta_0}\|\tilde{P}\|_{L^2_{\eta_1}}^2 + CV(t)^{\beta_0}D(t). \]

Taking limit superior of the previous inequality yields
\[ \limsup_{t \to \infty} \|\nabla \tilde{P}(t)\|_{L^2_{\eta_1}}^2 \leq C \lim sup \left( V(t)^{\frac{\beta_0}{2}} \int_U |\tilde{P}|^2 \phi \,dx \right)^\frac{1}{2} + C \lim sup [V(t)^{\beta_0}D(t)]. \] (5.43)

Consider first limit on the right-hand side of the (5.43). Let $y(t) = \int_U |\tilde{P}(x, t)|^2 \phi(x)\,dx$. By (5.17) and (5.35), we have for large $t$ that
\[ y'(t) \leq - CV(t)^{\frac{\beta_0}{2}} y(t) + C_2 D(t) V(t)^\beta. \]

We apply Lemma A.3 to $h(t) = CV(t)^{\frac{\beta_0}{2}}$, $f(t) = C_2 D(t) V(t)^\beta$, and $g(t) = V(t)^{\beta_0}$, noticing that condition (A.2) is met thanks to (5.41). It follows that
\[ \limsup_{t \to \infty} [V(t)^{\beta_0}y(t)] \leq C \limsup [V(t)^{\beta_0}]^{\frac{\beta_0}{2}} V(t)^{\beta}D(t)] = C \limsup [V(t)^{\beta_0}D(t)]. \]

Then inequality (5.42) follows this and (5.43). \qed

**Remark 5.7.** Finally, we remark on some issues related to the main problem (1.10).

(a) In IBVP (1.10), the Dirichlet boundary data is considered. The Neumann boundary condition is studied in [10, 13], and the nonlinear Robin condition is studied in [3] only for the homogeneous media. Currently, there is no analysis for these types of boundary conditions in heterogeneous media. Due to the extra dependence on the spatial variables, it is not clear which space of functions without vanishing trace on the boundary, and which version of Poincaré-Sobolev inequality or trace theorem will be suitable.

(b) There have been more work on the numerical analysis of generalized Forchheimer flows for incompressible fluids [4, 16, 19], and slightly compressible fluids [14, 15]; see the cited papers for additional references. However, all of these papers are only for homogeneous porous media. It will be interesting to see numerical analysis for heterogeneous media as proposed by equation (1.1), and particularly, problem (1.10). We hope that the analysis results obtained in the current paper will give hints to the functional spaces and types of estimates expected for the numerical study.
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Appendix

We collect here some useful lemmas on solutions of differential inequalities.

Lemma A.1 (see [8], lemma A.1). Let $\phi$ be a continuous, strictly increasing function from $[0, \infty)$ onto $[0, \infty)$. Suppose $y(t) \geq 0$ is a continuous function on $[0, \infty)$ such that

$$y'(t) \leq -h(t)\phi^{-1}(y(t)) + f(t) \quad \forall t > 0,$$

where $h(t) > 0$, $f(t) \geq 0$ are continuous functions on $[0, \infty)$.

(i) If $M(t)$ is an increasing, continuous function on $[0, \infty)$ that satisfies $M(t) \geq f(t)/h(t)$ for all $t \geq 0$, then

$$y(t) \leq y(0) + \phi(M(t)) \quad \forall t \geq 0.$$

(ii) If $\int_0^\infty h(\tau)d\tau = \infty$ then

$$\limsup_{t \to \infty} y(t) \leq \phi\left(\limsup_{t \to \infty} \frac{f(t)}{h(t)}\right).$$

Here, we use the notation $\phi(\infty) = \infty$.

Lemma A.2 (see [10], proposition 3.7). Let $\phi(z) = c(z + \gamma z^2)$ for all $z \geq 0$, where $c > 0$ and $1 < \gamma < 2$. Suppose $y(t) \geq 0$ is a continuous function on $[0, \infty)$ such that

$$y'(t) \leq -\phi^{-1}(y(t)) + f(t) \quad \forall t > 0,$$

where $f(t) \geq 0$ is a function in $C([0, \infty)) \cap C^1([0, \infty))$.

Assume $\beta \overset{\text{def}}{=} \limsup_{t \to \infty} |f'(t)|$ is finite. Then there is $T > 0$ such that

$$y(t) \leq C(1 + \beta^{2-\gamma} + f(t)^\gamma) \quad \text{for all } t > T, \quad (A.1)$$

where $C = 3[32(1 + c)]^{\frac{1}{\gamma-1}}$.

Proof. We track and calculate the constant $C$ explicitly. From (3.19) in the proof of proposition 3.7 [10],

$$y(t) \leq \phi(2f(t) + (1 + 8c\beta)\frac{1}{\beta\gamma}) \leq 2c(1 + 2f(t) + (1 + \beta)^\frac{1}{\gamma-1}[16(1 + c)]^\frac{1}{\gamma-1}).$$

Estimating $(1 + \beta)^\frac{1}{\gamma-1}$ by (1.11), we have
\[
y(t) \leq 2\varepsilon(1 + 2f(t) + 2^{\frac{1}{2\gamma - 1}}(1 + \beta^{\frac{1}{2\gamma - 1}}(16(1 + c)^{\frac{1}{2\gamma - 1}})^{\frac{3}{2\gamma - 1}}(f(t) + 1 + \beta^{\frac{1}{2\gamma - 1}})^{\frac{1}{2\gamma - 1}}) - \varepsilon[32(1 + c)^{\frac{2}{2\gamma - 1}}(f(t) + 1 + \beta^{\frac{1}{2\gamma - 1}})^{\frac{2}{2\gamma - 1}} + f(t)^{\frac{2}{2\gamma - 1}}(1 + \beta^{\frac{1}{2\gamma - 1}} + f(t)^{\frac{1}{2\gamma - 1}}),
\]
which proves (A.1).

**Lemma A.3.** Let \( T \in \mathbb{R} \). Suppose the continuous functions \( y(t), f(t) \geq 0 \) and \( h(t), g(t) > 0 \) on \( [T, \infty) \) satisfy

\[
y'(t) \leq -h(t)y(t) + f(t) \quad \forall \; t > T,
\]

\[
\int_T^\infty h(\tau)d\tau = \infty \quad \text{and} \quad \lim_{t \to \infty} \frac{g(t)}{h(t)} = 0,
\]

then

\[
\lim_{t \to \infty} \sup_{\tau \geq t}(g(\tau)y(\tau)) \leq \lim_{t \to \infty} \left( \frac{g(t)f(t)}{h(t)} \right).
\]

**Proof.** Same as lemma A.3 of [13].

**Lemma A.4.** Let \( f(t) \geq 0 \) be a \( C^1 \)-function on \((0, \infty)\). Assume

\[
\beta = \limsup_{t \to \infty} |f'(t)| < \infty.
\]

Then there is \( T > 0 \) such that for any \( t_2 > t_1 > T \),

\[
f(t_1) \leq f(t_2) + (t_2 - t_1)(\beta + 1).
\]

**Proof.** There exists \( T > 0 \) such that for all \( t > T \) one has \(-f'(t) \leq \beta + 1\). Let \( t_2 > t_1 > T \). Then

\[
f(t_1) = f(t_2) - \int_{t_1}^{t_2} f'(\tau)d\tau \leq f(t_2) + \int_{t_1}^{t_2} (\beta + 1)d\tau = f(t_2) + (t_2 - t_1)(\beta + 1),
\]

which proves (A.3).

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