Calculation of $\gamma\pi \rightarrow \pi\pi$

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Abstract

The problem of $\gamma\pi \rightarrow \pi\pi$ is studied using the axial anomaly, elastic unitarity, analyticity and crossing symmetry. The solution of the integral equation for this amplitude is given by an iteration procedure. The final solution disagrees with vector meson dominance models.

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One of the fundamental calculations in particle theory is the $\pi^0 \rightarrow \gamma \gamma$ decay rate. It is a combination of Partial Conserved Axial Current (PCAC) and of the short distance behavior of Quantum Chromodynamics (QCD) \[1\]. The result of this calculation supports the concept of the color in QCD. This calculation is valid in the chiral world where the $\pi^0$ is massless. Some correction has to be made, in principle, to take into account of the physical pion mass. It turns out that this axial anomaly formula is in very good agreement with the pion life time data, implying that the correction to the physical pion mass is very small. This result is expected because the pion mass is small.

Another axial anomaly result is the process $\gamma \pi \rightarrow \pi \pi$ or its analytical continuation $\gamma \rightarrow 3\pi$ \[2\]. This last process, even at moderate energy has a complex singularity and requires extra work and is postponed to a forthcoming study.

The calculation of the process $\gamma \pi \rightarrow \pi \pi$ is in itself interesting because not only it will be checked by future experiments which are being proposed at various accelerator facilities, but it will also be important for the calculation $\pi^0 \rightarrow \gamma \gamma^*$ \[3\].

Omitting the kinematical factor, the $\gamma \rightarrow 3\pi$ and $\gamma \pi \rightarrow \pi \pi$ amplitudes are given as:

$$F_{3\pi}(0) = \frac{1}{4\pi^2 f_\pi^2} = \lambda$$

where $f_\pi = 93$ MeV and the zero in the argument of $F_{3\pi}$ refers to the massless pions, and to the amplitude evaluated at the symmetry point $s = t = u = 0$.

Using the KSRF relation \[4\] $g_{\rho\pi\pi} = m_\rho/\sqrt{2}f_\pi$, it is straightforward to calculate the process $\gamma \pi \rightarrow \pi \pi$ \[3, 5\] using the vector meson dominance model (VMD)\[7\]:

$$F_{3\pi}^{VMD}(s, t, u) = \frac{\lambda}{2} \left( \frac{m_\rho^2}{m_\rho^2 - s} + \frac{m_\rho^2}{m_\rho^2 - t} + \frac{m_\rho^2}{m_\rho^2 - u} \right)$$

which in the chiral limit, $s = t = u = 0$, yields $F_{3\pi}^{VMD}(0, 0, 0) = \frac{3}{2}\lambda$ and is a factor 3/2 larger than the anomaly Eq. (1). It is clear that the VMD model does not always work. In order to rectify this problem, a contact term is introduced to reduce the VMD contribution \[5, 9\]. The VMD equation can now be written as:

$$F_{3\pi}^{CVMD}(s, t, u) = \frac{\lambda}{2} \left( \frac{m_\rho^2 - m_\pi^2}{m_\rho^2 - s} + \frac{m_\rho^2 - m_\pi^2}{m_\rho^2 - t} + \frac{m_\rho^2 - m_\pi^2}{m_\rho^2 - u} - 1 \right)$$
and is seen to satisfy the anomaly Eq. (1) in the chiral limit. In Eq. (3) the smoothness assumption was made by setting $F_{3\pi}(s = t = u = m_{\pi}^2) = \lambda$. This assumption will be assumed to be correct in the remaining of this letter.

Yet another solution was discussed [6]:

$$F_{3\pi}^{MVMD}(s, t, u) = \lambda \left( \frac{m_{\rho}^2 - m_{\pi}^2}{m_{\rho}^2 - s} + \frac{m_{\rho}^2 - m_{\pi}^2}{m_{\rho}^2 - t} + \frac{m_{\rho}^2 - m_{\pi}^2}{m_{\rho}^2 - u} \right)$$

(4)

and is seen to satisfy the anomaly Eq. (1) but differs from the Vector Meson Dominance result by a factor of 3/2.

In this article a different result on the $F_{3\pi}$ amplitude based on the solution of an integral equation is given. The strong P-wave $\pi\pi$ scattering phase shifts is supposed to be known as given by the experimental data up to 1 GeV or higher, i.e. the existence of the $\rho$ resonance at 0.77 GeV with a width of 0.151 GeV. We shall not make an assumption on the Vector Meson Dominance, but try to find a solution which is consistent with the constraint of the elastic unitarity, crossing symmetry and also of the low energy theorem Eq. (1). Our solution does not agree either with Eq. (3) or with Eq. (4).

The process $\gamma\pi \rightarrow \pi\pi$ is a completely symmetric reaction, that is the same amplitude describes not only the reaction $\gamma\pi^0 \rightarrow \pi^+\pi^-$, but also two other amplitudes involving the permutations of the pions. One assumes that the scattering amplitude can be represented by a single spectral function dispersion relation:

$$F(s, t, u) = \lambda + \left( (s - m_{\pi}^2) \int_{4m_{\pi}^2}^{\infty} \frac{\sigma(z) dz}{(z - m_{\pi}^2)(z - s - i\epsilon)} \right) + [s \leftrightarrow t] + [s \leftrightarrow u]$$

(5)

The spectral function can be evaluated, using the unitarity relation: $\pi\sigma(s, t, u) = \langle \pi^0 | j_3(0) | n \rangle \langle n | \pi^+\pi^- \rangle$ where the $j_3$ is the isovector electromagnetic current operator and $| n \rangle$ is a complete set of the physical states; the summation over $| n \rangle$ is understood. Because of the G-parity, only the even number of pions intermediate states contribute. As one is interested in the low energy region (but including possible resonant effect) of the $\gamma\pi^0 \rightarrow \pi^+\pi^-$ amplitudes where the inelastic effects are small, the summation over the intermediate states is truncated to the two pion (P-wave) states. The amplitudes
\[ \langle \pi^0 | j_3(0) | \pi^+\pi^- \rangle \] and \[ \langle \pi^+\pi^- | \pi^+\pi^- \rangle \] are expanded into partial waves, and keeping only the lowest partial wave contributions (physically the P-state), one has:

\[ \pi\sigma(s) = F(s)e^{-i\delta(s)}\sin\delta(s) \]  

where \( F(s) \) denotes the S-wave projection of \( F(s, t, u) \) (and physically corresponds to the P-wave when combining with the kinematical factor); \( \delta \) is the experimental P-wave \( \pi\pi \) phase shifts which can be obtained from the experimental data which shows that the phase shifts must pass through 90° at the \( \rho \) mass and its width is 151 MeV.

Substituting Eq. (6) in the partial wave projection of Eq. (5) and keeping only the lowest partial wave, one has:

\[ F(s) = \lambda + \frac{(s - m^2_\pi)}{\pi} \int_{4m^2_\pi}^{\infty} F(z)e^{-i\delta(z)}\sin\delta(z) \frac{dz}{(z - m^2_\pi)(z - s - i\epsilon)} \]

\[ + \frac{2}{\pi} \int_{4m^2_\pi}^{\infty} F(z)e^{-i\delta(z)}\sin\delta(z) \left\{ \frac{1}{(s - 4m^2_\pi)} \ln[1 + \frac{(s - 4m^2_\pi)}{z}] - \frac{1}{(z - m^2_\pi)} \right\} dz \]  

The unitarity of the S-matrix requires that \( F(s) \) has the phase \( \delta \) and this should be exhibited in the solution for \( F(s) \). Because of the neglect of the contribution of higher partial waves, \( F(s) \) is not exactly equal to \( \lambda \) at \( s = m^2_\pi \). Eq. (7) is a complicated integral equation; it is similar to, but more complicated than the Muskelishvili-Omnes (MO) type, because the \( t \) and \( u \) channel contributions are also expressed in terms of the unknown function \( F(s) \). It should be noticed that the first term has a cut from \( 4m^2_\pi \) to \( \infty \) and the second one has a cut from 0 to \( -\infty \). This remark enables one to solve the integral equation by an iteration scheme as given below which converges very fast.

The iterative and final solutions can be expressed in terms of the function \( D(s, 0) \), normalized to unity at \( s = 0 \) and defined in terms of the phase shift \( \delta \) as:

\[ \frac{1}{D(s, 0)} = \exp\left[ \frac{s}{\pi} \int_{4m^2_\pi}^{\infty} \frac{\delta(z)dz}{z(z - s - i\epsilon)} \right] \]  

Other functions \( D \) normalized to unity at \( s = s_0 \) can be expressed in terms of the function \( D(s, 0) \) by the simple relation \( D(s, s_0) = D(s, 0)/D(s_0, 0) \).
As remarked above, the Integral Equation (9) has both right and left cuts. This allows us to define an iteration procedure defined as follows:

\[ F^{(i)}(s) = \frac{\lambda}{3} + T^{(i-1)}_B(s) + \frac{s - m^2_\pi}{\pi} \int_{4m^2_\pi}^{\infty} \frac{F^{(i)}(z)e^{-i\delta(z)}\sin\delta(z)}{(z - m^2_\pi)(z - s - i\epsilon)} dz \]  

(9)

where \( F^i \) is the value of the function \( F(s) \) calculated at the \( i^{th} \) step in the iteration procedure; the Born term \( T^{(i-1)}_B(s) \) calculated at the \( i^{th} - 1 \) step is defined as

\[ T^{(i-1)}_B(s) = \frac{2\lambda}{3} + \frac{2}{\pi} \int_{4m^2_\pi}^{\infty} F^{(i-1)}(z)e^{-i\delta(z)}\sin\delta(z) \left\{ \frac{1}{(s - 4m^2_\pi)} \ln[1 + (s - 4m^2_\pi)/z] - \frac{1}{(s - m^2_\pi)} \right\} dz \]

(10)

where \( i \geq 1 \). The Born term is real for \( s \geq 0 \).

The solution of the integral equation Eq. (9) is of the MO type:

\[ F^{(i)}(s) = \frac{\lambda}{3D(s, m^2_\pi)} + T^{(i-1)}_B(s) + \frac{s - m^2_\pi}{\pi} \int_{4m^2_\pi}^{\infty} \frac{D(z, m^2_\pi)e^{i\delta(z)}\sin\delta(z)T^{(i-1)}_B(z)dz}{(z - m^2_\pi)(z - s - i\epsilon)} \]

(11)

where it is assumed that the well-known polynomial ambiguity inherited in the MO integral equation is absent. (The polynomial ambiguity could represent some incalculable inelastic effect occurring above the inelastic threshold). It is now straightforward to take out the imaginary part of the integral in Eq. (11) and using \( e^{i\delta} \sin\delta = \rho(s)N(s)/D(s, 0) \) with \( ImD(s, 0) = -\rho(s)N(s) \) to combine with the \( T^{(i-1)}_B \) Born term to show that \( F^{(i)}(s) \) has indeed the phase \( \delta \) as required by the the unitarity of the S-matrix.

One arbitrarily defines the convergence of the iteration scheme at the \( i^{th} \) iteration step when \( |F^{(i)}|/|F^{(i-1)}| \) differs from 1 by less than 1\% or so for the energy range from the \( 2\pi \) threshold to \( 1 GeV \). (Alternatively one can also consider the ratio \( |T^{(i)}_B|/|T^{(i-1)}_B| \). Then combining the \( T^{(i-1)}_B \) Born term in Eq. (11) with higher uncorrected partial waves (for rescattering) from the \( t \) and \( u \) channels, one arrives at the final solution:

\[ F^{(i)}(s, t, u) = \frac{\lambda}{3D(s, m^2_\pi)} \left\{ \frac{1}{(1 + 3I^{(i-1)}(s))} \right\} + \{(s \leftrightarrow t)\} + \{(s \leftrightarrow u)\} \]

(12)

where the function \( I^{(i-1)} \) is the rescattering correction:

\[ I^{(i-1)}(s) = \frac{s - m^2_\pi}{\pi} \int_{4m^2_\pi}^{\infty} \frac{D(z, m^2_\pi)e^{i\delta(z)}\sin\delta(z)T^{(i-1)}_B(z)dz}{(z - m^2_\pi)(z - s - i\epsilon)} \]

(13)
By projecting out the lowest partial wave from Eq. (12) it can be shown that \( F(i)(s) \) has the phase \( \delta \).

In order to carry out the iteration procedure to find the solution of the integral equation one has to parameterize the function \( D(s,0) \) in terms of the experimental P-wave phase shift. Noticing that the phase of \( D(s,0) \) is \(-\delta\), one has [11, 12]:

\[
\frac{1}{D(s,0)} = \frac{1}{1 - s/s_R - \frac{1}{96\pi^2f^2} \left\{ (s - 4m^2)H_{\pi\pi}(s) + 2s/3 \right\}}
\]

where \( H_{\pi\pi}(s) = \left( 2 - 2\sqrt{s - 4m^2} \ln \frac{\sqrt{s} + \sqrt{s - 4m^2}}{2m_\pi} \right) + i\pi \sqrt{s - 4m^2} / s \) for \( s > 4m^2_\pi \); for other values of \( s \), \( H_{\pi\pi}(s) \) can be obtained by analytic continuation. The partial wave amplitude is written as \( e^{i\delta} \sin \delta / \rho(s) = N(s) / D(s,0) \), where \( N(s) = (s - 4m^2_\pi) / (96\pi f^2_\pi) \).

The \( \rho \) mass is defined as the vanishing of the real part of the denominator and is equal to \( s_R \) in the narrow width approximation. Using \( \sqrt{s}_\rho = 0.770 GeV \) in Eq. (14), we have \( \Gamma_\rho = 155 MeV \) which is very close to the experimental value of \( 151.5 \pm 1.5 MeV \).

To make the iteration scheme coherent, the phase theorem for \( F(i)(s) \) has to be satisfied at every step of the iteration. One first guesses a solution for \( F(0) \) which must satisfy this theorem; the following \( F(i)(s) \) satisfies the theorem by construction. One first sets:

\[
F(0)(s) = \frac{\lambda}{3} D^{-1}(s, m^2_\pi); \text{ we then put this solution into Eq. (10) to evaluate the 0th order Born term:}
\]

\[
T_B^0(s) = \frac{2\lambda}{3} \left( \frac{s_\rho - m^2_\pi}{s - 4m^2_\pi} \right) \ln \left| 1 + \frac{s - 4m^2_\pi}{s_\rho} \right|
\]

To arrive at this relation one takes a \( \delta \)-function approximation for the product \( e^{-i\delta} \sin \delta(s) / D(s,0) \) which is sufficiently accurate at this point. Eq. (13) can also be obtained from projecting out the lowest partial wave from the \( t \) and \( u \) channels contribution of Eq. (4).

Using this Born term, \( F(1) \) is calculated, using Eq. (14) with \( i = 1 \).

One next calculates the next Born \( T_B^1(s) \) term by using the solution \( F(1)(s) \) and then proceeds to calculate \( F(2)(s) \) etc. It is found that \( |F(1)(s)| \) differs by less than 2.5% from \( |F(2)(s)| \) and \( |F(3)(s)| \) differs by less than 0.4% from \( |F(3)(s)| \) in the energy range of \( 2m_\pi \) to \( 1 GeV \) which shows indeed the very fast convergence of our iteration scheme.
If we were happy with an accuracy of about 1.5% in the amplitude, then the final solution given by Eq. (12) could be simply expressed in terms of \( I^{(0)}(s) \). The exact expression for \( I^{(0)}(s) \) is a complicated linear combination of the Spence functions, but an approximate expression can simply be given as follows:

\[
I^{(0)}(s) \simeq 1.13 \frac{s_\rho}{144 \pi^2 f_\pi^2} \{ Sp(1 + \frac{s - 4m_\rho^2}{s_\rho}) - Sp(1 - \frac{3m_\rho^2}{s_\rho}) \}
\]

where \( Sp(s) \) is the Spence function. With this approximation, the expression for \( F^{(1)}(s,t,u) \) is accurate to within 2% compared to the final solution constructed from \( F^{(3)}(s) \) in the energy range of 0.1 to 1 GeV.

In order to compare these results with the previous pole models, one has to make a finite width correction for Eqs. (3,4). This can be done by using the self-energy correction for the \( \rho \) propagator using the \( \rho \pi \pi \) coupling given by the KSRF relation. One then has to make the substitution \((s_\rho - m_\rho^2)/s_\rho \) by \( 1/D(s, m_\pi^2) \) and similarly for expressions with \( t \) and \( u \) variables. The lowest S-wave projection yields:

\[
F_{3\pi}^{CVMD}(s) = \lambda_2 [ \frac{1}{2D(s, m_\pi^2)} + T_B^0(s) - 1 ]
\] (17)

and

\[
F_{3\pi}^{MVMD}(s) = \lambda_3 [ \frac{1}{3D(s, m_\pi^2)} + T_B^0(s) ]
\] (18)

where it is sufficiently accurate to use the pole terms for the contribution of the \( t \) and \( u \) channels. The phase theorem is violated in both Eqs. (17,18).

Fig. 1 shows that the final solution lies between the two VMD models, the (CMVD) given by Eq. (17), and the (MVMD) model given by Eq. (18). It is useful to subject our result with future experimental test in the resonant region where the difference with other models are largest.

Fig. 2 shows a similar plot but with only a low energy scale \( s < 0.3 GeV^2 \). The present experimental data at low energy \([13]\) confirms roughly the anomaly but is not sufficiently accurate to distinguish various models. With more accurate data, below \( s < 0.3 GeV^2 \), it will be experimentally difficult to distinguish our result with that of CVMD but it might be possible to distinguish our result with that of MVMD. For a more complete review of
the experimental situation as well as other theoretical models, two recent articles are to be consulted [14, 15].

It will be best to analyze future experimental data using Eq. (12), i.e. in terms of the energy and scattering angle instead of its S-wave projection as given by Figs. 1 and 2.

In conclusion, the process $\gamma\pi \rightarrow \pi\pi$ is calculated using the low energy theorem, analyticity, elastic unitarity and crossing symmetry which are fundamental conditions for a theory involving strong interaction. The final result shows that one effectively takes into account of the (unstable) $\rho$ model in the s, t and u channels and their rescattering effect treated in a self-consistent way. It is important to put these results to experimental tests.

Part of this work was done while the author was a visitor at the KEK National Laboratory and at the Hue University. The author would like to thank these two institutions for their hospitality and in particular, Prof. Y. Shimizu and his collaborators at KEK and Dr Nguyen trung Dan at the Hue University.
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Figure Captions

Fig. 1: $F^{(i)}(s)^2 / \lambda^2$ as a function of the energy squared $s(GeV^2)$. The solid curve is the solution of the integral equation after 3 iterations; the dotted curve is the solution after one iteration. The long dashed curve is from the modified VMD model (MVMD) Eq. (18); the short dashed curve is from the VMD model with a contact term (CVMD) Eq. (17).

Fig. 2: $|F^{(i)}(s)|^2 / \lambda^2$ as given by the Fig. 1 but with a smaller energy scale.
Figure 1:
Figure 2: