ON A TRANSMISSION PROBLEM FOR EQUATION AND
DYNAMIC BOUNDARY CONDITION OF CAHN–HILLIARD TYPE
WITH NONSMOOTH POTENTIALS

PIERLUIGI COLLI, TAKESHI FUKAO, AND HAO WU

Abstract. This paper is concerned with well-posedness of the Cahn–Hilliard equation subject to a class of new dynamic boundary conditions. The system was recently derived in Liu–Wu (Arch. Ration. Mech. Anal. 233 (2019), 167–247) via an energetic variational approach and it naturally fulfills three physical constraints such as mass conservation, energy dissipation and force balance. The target problem examined in this paper can be viewed as a transmission problem that consists of Cahn–Hilliard type equations both in the bulk and on the boundary. In our approach, we are able to deal with a general class of potentials with double-well structure, including the physically relevant logarithmic potential and the non-smooth double-obstacle potential. Existence, uniqueness and continuous dependence of global weak solutions are established. The proof is based on a novel time-discretization scheme for the approximation of the continuous problem. Besides, a regularity result is shown with the aim of obtaining a strong solution to the system.

Key words: Cahn–Hilliard system, dynamic boundary condition, transmission problem, non-smooth potentials, well-posedness, regularity.

AMS (MOS) Subject Classification: 35K61, 35K25, 74N20, 80A22

1. Introduction

In this paper, we consider the following initial boundary value problem for a Cahn–Hilliard equation subject to a dynamic boundary condition that is also of Cahn–Hilliard type. Let $0 < T < \infty$ be some fixed time and let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded domain with smooth boundary $\Gamma := \partial \Omega$. We aim to find four unknown functions $\phi, \mu : Q := (0, T) \times \Omega \to \mathbb{R}$ and $\psi, w : \Sigma := (0, T) \times \Gamma$ satisfying

\begin{align*}
\frac{\partial \phi}{\partial t} - \Delta \mu &= 0 \quad \text{in } Q, \\
\mu &= -\Delta \phi + W'(\phi) \quad \text{in } Q, \\
\partial_\nu \mu &= 0 \quad \text{on } \Sigma, \\
\phi_{|\Gamma} &= \psi \quad \text{on } \Sigma, \\
\frac{\partial \psi}{\partial t} - \Delta_{\Gamma} w &= 0 \quad \text{on } \Sigma, \\
w &= \partial_\nu \phi - \Delta_{\Gamma} \psi + W'_{\Gamma}(\psi) \quad \text{on } \Sigma,
\end{align*}

where $\partial_t$ and $\partial_\nu$ denote the partial time derivative and the outward normal derivative on $\Gamma$, respectively; $\Delta$ denotes the Laplacian and $\Delta_{\Gamma}$ denotes the Laplace–Beltrami operator on $\Gamma$ (see, e.g., [22, Chapter 3]); $\phi_{|\Gamma}$ stands for the trace of $\phi$ on the boundary $\Gamma$. In
view of (1.4), system (1.1)–(1.6) is a sort of transmission problem between the Cahn–Hilliard equation in the bulk $\Omega$ and the Cahn–Hilliard equation on the boundary $\Gamma$. The nonlinear functions $W$ and $W_\Gamma$ are usually referred as the double-well potentials, with two minima and a local unstable maximum in between. Typical and physically significant examples of such potentials are the so-called classical potential, the logarithmic potential, and the double obstacle potential, which are given, in this order, by

$$W_{\text{reg}}(r) := \frac{1}{4} (r^2 - 1)^2, \quad r \in \mathbb{R},$$

$$W_{\text{log}}(r) := (1 + r) \ln(1 + r) + (1 - r) \ln(1 - r) - c_1 r^2, \quad r \in (-1, 1),$$

$$W_{2\text{obs}}(r) := c_2 (1 - r^2) \quad \text{if } |r| \leq 1 \quad \text{and} \quad W_{2\text{obs}}(r) := +\infty \quad \text{if } |r| > 1. \quad (1.9)$$

where the constants in (1.8) and (1.9) satisfy $c_1 > 1$ and $c_2 > 0$, so that $W_{\text{log}}$ and $W_{2\text{obs}}$ are nonconvex. The nonlinear terms $W'(\phi)$ in (1.2) and $W'_\Gamma(\psi)$ in (1.6) characterize the dynamics of the Cahn–Hilliard system. In cases like (1.7) and (1.8), $W'$ and $W'_\Gamma$ denote simply the derivatives of the related potentials; while non-smooth potentials like (1.9) are considered, then $W'$ and $W'_\Gamma$ denote the subdifferential of the convex part plus the derivative of the smooth concave contribution, i.e., for (1.9) it is

$$s \in W'_{2\text{obs}}(r) \quad \text{if} \quad r \in [-1, 1], \quad s + 2c_2 r \begin{cases} \in (-\infty, 0] & \text{if } r = -1 \\ = 0 & \text{if } r \in (-1, 1) \\ \in [0, +\infty) & \text{if } r = 1 \end{cases}.$$ 

Of course, in this case one should replace the equalities in (1.2) and (1.6) by inclusions. In this paper, we are able to handle completely general potentials $W$ and $W_\Gamma$ including all the three cases (1.7)–(1.9) mentioned above.

The system (1.1)–(1.6) was first derived by Liu and Wu [29] in a more general form (see also [33]) on the basis of an energetic variational approach. It describes effective short-range interactions between the binary mixture and the solid wall (boundary), furthermore, it has the feature that the related model naturally fulfills important physical constraints such as conservation of mass, dissipation of energy and force balance relations. In its current formulation, we see that equations (1.1) and (1.2) yield a Cahn–Hilliard system subject to a no-flux boundary condition (1.3) together with a non-homogeneous Dirichlet boundary condition (1.4), while the dynamic boundary condition (1.5) and equation (1.6) provide an evolution system of Cahn–Hilliard type on the boundary $\Gamma$. These two Cahn–Hilliard systems in the bulk and on the boundary are coupled through the trace condition (1.4) and the normal derivative term $\partial_\nu \phi$ in (1.6).

The total energy functional for system (1.1)–(1.6) given by

$$E(\phi, \psi) := \int_{\Omega} \left( \frac{1}{2} |\nabla \phi|^2 + W(\phi) \right) \, dx + \int_{\Gamma} \left( \frac{1}{2} |\nabla_\Gamma \psi|^2 + W_\Gamma(\psi) \right) \, d\Gamma \quad (1.10)$$

is decreasing in time (see [29]) and furthermore, system (1.1)–(1.6) can be interpreted as a gradient flow of $E(\phi, \psi)$ in a suitable dual space (see (1.8)). In light of (1.11), (1.5) and (1.6), we easily deduce that the following properties on mass conservation:

$$\int_{\Omega} \phi(t) \, dx = \int_{\Omega} \phi(0) \, dx, \quad \int_{\Gamma} \psi(t) \, d\Gamma = \int_{\Gamma} \psi(0) \, d\Gamma \quad \text{for all } t \in [0, T]. \quad (1.11)$$
In this paper, we study the well-posedness of system (1.1)–(1.6) for a weak solution subject to the following initial data
\[ \phi(0) = \phi_0 \quad \text{in } \Omega, \quad \psi(0) = \psi_0 \quad \text{on } \Gamma. \] (1.12)

Moreover, we also establish a regularity theory in order to obtain a strong solution. In particular, we are able to treat the initial value problem for system (1.1)–(1.6) in a wider class of nonlinearities \( \mathcal{W} \) and \( \mathcal{W}_\Gamma \). Indeed, in the previous contributions, the well-posedness was investigated only in the case of smooth potentials like (1.7) (cf. [29, Remark 3.2] and [18, Remark 2.1]): this is the point of emphasis of our present paper.

We would like to mention some related problems in the literature. In 2011, Goldstein, Miranville and Schimperna [21] studied a different type of transmission problem between the Cahn–Hilliard system in the bulk and on the boundary with non-permeable walls (cf. a previous work Gal [15] for the case with permeable walls). Their system can be derived from the same energy functional (1.10) by a variational method, however, the corresponding boundary conditions turn out to be different from (1.3) and (1.5). This also leads to a different property on the mass conservation comparing with (1.11) such that the total (bulk plus boundary) mass is conserved. We refer to [29] for more detailed information on the comparison between these models. In addition, we mention the contributions [6, 8, 15, 21] related to the well-posedness, [9, 12, 13, 16, 17, 20] for the study of long time behavior and the optimal control problems, [7, 14] for numerical analysis and [24] for the maximal regularity theory. Comparing the large number of known results on the previous model [15, 21], we are only aware of the recent papers [18, 29] that analyze the well-posedness of system (1.1)–(1.6) with (1.12).

Let us now describe the contents of the present paper. In Section 2, we state the main well-posedness result for global weak solutions. We consider the problem within a general framework by setting \( \mathcal{W}' := \beta + \pi \) and \( \mathcal{W}'_\Gamma := \beta_\Gamma + \pi_\Gamma \), where \( \beta \) and \( \beta_\Gamma \) are maximal monotone graphs with \( 0 \in \beta(0) \) and \( 0 \in \beta_\Gamma(0) \), while \( \pi \) and \( \pi_\Gamma \) yield the anti-monotone terms that are Lipschitz continuous functions. The main theorems are concerned with the existence of a global weak solution (Theorem 2.1) and the continuous dependence on the given data (Theorem 2.2), which implies the uniqueness.

In Section 3, we study the time-discrete approximate problem for (1.1)–(1.6) with (1.12). We start from the viscous Cahn–Hilliard system by inserting two additional terms, \( \tau \partial_t \phi \) and \( \sigma \partial_t \psi \) in the right hand sides of (1.2) and (1.6), respectively, with the parameters \( \tau, \sigma > 0 \). Moreover, we take the Yosida approximations \( \beta_\varepsilon \) and \( \beta_{\Gamma, \varepsilon} \) in place of the maximal monotone graphs \( \beta \) and \( \beta_\Gamma \) and in terms of the parameter \( \varepsilon > 0 \). Then we apply a time discretization scheme using the approach in [10, 11]. We can show the existence of a discrete solution taking advantage of the general maximal monotone theory. After that, we proceed to derive a sequence of uniform estimates. For this purpose, we apply the technique of [5] in order to treat different potentials in the bulk and on the boundary. In the subsequent iterations, we prove the existence results by performing the limiting procedures, with respect to the time step first, then as \( \varepsilon \to 0 \), finally taking the limit as either \( \tau \to 0 \) or \( \sigma \to 0 \), or both \( \tau, \sigma \to 0 \), in order to obtain a partially viscous Cahn–Hilliard system or a pure Cahn–Hilliard system in the limit. The continuous dependence result is then proved by using the energy method.
In Section 4, we discuss the regularity for weak solutions. Returning to the time discrete approximation, we gain some necessary higher order estimates at all the different levels up to the final limits. Thus, we are able to obtain enough regularity as to guarantee a strong solution for the pure Cahn–Hilliard system as well (see Theorem 4.1).

Here, for the reader’s convenience, let us include a detailed index of sections and sub-sections.

1. Introduction
2. Main results
3. Well-posedness
   3.1. Time-discrete approximate solution
   3.2. A priori estimates and limiting procedure
   3.3. From viscous to pure Cahn–Hilliard system
4. Existence of strong solution

2. Main results

We now formulate our target problem (1.1)–(1.6) and (1.12) as follows:

\[
\begin{align*}
\partial_t \phi - \Delta \mu &= 0 \quad \text{a.e. in } Q, \\
\mu &= -\Delta \phi + \xi + \pi(\phi) - f, \quad \xi \in \beta(\phi) \quad \text{a.e. in } Q, \\
\partial_\nu \mu &= 0 \quad \text{a.e. on } \Sigma, \\
\phi|_{\Gamma} &= \psi \quad \text{a.e. on } \Sigma, \\
\partial_t \psi - \Delta \Gamma w &= 0 \quad \text{a.e. on } \Sigma, \\
w &= \partial_\nu \phi - \Delta \Gamma \psi + \zeta + \pi_\Gamma(\psi) - g, \quad \zeta \in \beta_\Gamma(\psi) \quad \text{a.e. on } \Sigma, \\
\phi(0) &= \phi_0 \quad \text{a.e. in } \Omega, \\
\psi(0) &= \psi_0 \quad \text{a.e. on } \Gamma.
\end{align*}
\]

where \( f : Q \to \mathbb{R}, g : \Sigma \to \mathbb{R}, \phi_0 : \Omega \to \mathbb{R}, \psi_0 : \Gamma \to \mathbb{R} \) are given functions. Moreover, \( \beta \) stands for the subdifferential of the convex part \( \hat{\beta} \) and \( \pi \) stands for the derivative of the concave perturbation \( \hat{\pi} \) of a double well potential \( W(r) = \hat{\beta}(r) + \hat{\pi}(r) \). The same setting holds for \( \beta_\Gamma \) and \( \pi_\Gamma \). Typical examples of \( \beta, \pi \) are given by (cf. (1.7)–(1.9)):

- \( \beta(r) = r^3, \pi(r) = -r, r \in \mathbb{R} \), for the prototype potential \( W_{\text{reg}}(r) \);

- \( \beta(r) = \ln((1 + r)/(1 - r)), \pi(r) = -2c_1r, \) with \( r \in (-1, 1) \) for the logarithmic potential \( W_{\text{log}}(r) \);

- \( \beta(r) = \partial I_{[-1, 1]}(r), \pi(r) = -2c_2r, \) with \( r \in [-1, 1] \), for the nonsmooth potential \( W_{\text{2obs}}(r) \).

Same considerations apply to \( \beta_\Gamma, \pi_\Gamma \) and \( W_\Gamma \). Since the bulk and boundary potentials are allowed to be different, in order to handle the nontrivial bulk-boundary interaction of the transmission problem, an assumption for the relationship between \( \beta \) and \( \beta_\Gamma \) will be needed. We shall present it later.

Hereafter, we use the spaces

\[
H := L^2(\Omega), \quad H_\Gamma := L^2(\Gamma), \quad V := H^1(\Omega), \quad V_\Gamma := H^1(\Gamma)
\]
with their dual spaces $V^*$ and $V^*_Γ$ of $V$ and $V^*_Γ$, respectively; and

$$ W := \{ z ∈ H^2(Ω) : \partial_ν z = 0 \text{ a.e. on } Γ \} $$

equipped with the usual norms and inner products, denote them by $|·|_H$ and $(·,·)_H$, and so on.

Now, we define the weak solution of problem (2.1)–(2.7):

**Definition 2.1.** The sextuplet $(φ, μ, ξ, ψ, w, ζ)$ is called a weak solution of problem (2.1)–(2.7), if

$$ φ ∈ H^1(0, T; V^∗) ∩ L^∞(0, T; V) ∩ L^2(0, T; H^2(Ω)), $$

$$ μ ∈ L^2(0, T; V), \quad ξ ∈ L^2(0, T; H), $$

$$ ψ ∈ H^1(0, T; V^*_Γ) ∩ L^∞(0, T; V^*_Γ) ∩ L^2(0, T; H^2(Γ)), $$

$$ w ∈ L^2(0, T; V^*_Γ), \quad ζ ∈ L^2(0, T; H^*_Γ) $$

and $φ, μ, ξ, ψ, w, ζ$ satisfy

$$ (\partial_t φ, z)_{V^*, V} + ∫_Ω \nabla μ · \nabla z \, dx = 0 \quad \text{for all } z ∈ V, \quad \text{a.e. in } (0, T), \quad (2.8) $$

$$ μ = −Δ φ + ξ + π(φ) − f, \quad ξ ∈ β(φ) \quad \text{a.e. in } Q, \quad (2.9) $$

$$ φ|_{Γ} = ψ \quad \text{a.e. on } Σ, \quad (2.10) $$

$$ (\partial_t ψ, z)_{V^*_Γ, V^*_Γ} + ∫_Γ \nabla Γ w · \nabla Γ z \, dΓ = 0 \quad \text{for all } z ∈ V^*_Γ, \quad \text{a.e. in } (0, T), \quad (2.11) $$

$$ w = ∂_ν φ − Δ Γ ψ + ζ + π Γ (ψ) − g, \quad ζ ∈ β Γ (ψ), \quad \text{a.e. on } Σ, \quad (2.12) $$

$$ φ(0) = φ_0 \quad \text{a.e. in } Ω, \quad ψ(0) = ψ_0 \quad \text{a.e. on } Γ. \quad (2.13) $$

We note that, due to the lack of the regularities of time derivatives, the equations (2.1) and (2.5) are replaced by the variational formulations (2.8) and (2.11), respectively. Moreover, the boundary condition (2.3) is hidden in the weak form (2.8).

Next, we define the strong solution of problem (2.1)–(2.7):

**Definition 2.2.** The sextuplet $(φ, μ, ξ, ψ, w, ζ)$ is called a strong solution of problem (2.1)–(2.7) if

$$ φ ∈ W^{1,∞}(0, T; V^*) ∩ H^{1}(0, T; V) ∩ L^∞(0, T; H^2(Ω)), $$

$$ μ ∈ L^∞(0, T; V) ∩ L^2(0, T; W ∩ H^3(Ω)), \quad ξ ∈ L^∞(0, T; H), $$

$$ ψ ∈ W^{1,∞}(0, T; V^*_Γ) ∩ H^{1}(0, T; V^*_Γ) ∩ L^∞(0, T; H^2(T)), $$

$$ w ∈ L^∞(0, T; V^*_Γ) ∩ L^2(0, T; H^3(T)), \quad ζ ∈ L^∞(0, T; H^*_Γ) $$

and they satisfy (2.1)–(2.7).

Before we state our main theorems, we recall the structure of mass conservation of problem (2.1)–(2.7). Taking $z = 1$ in (2.8) and integrating from 0 to $t$ with the help of (2.13), we obtain the first equality in (1.11). Analogously, from (2.11) and (2.13) we
obtain the second condition in (1.11). Therefore, it is useful to define the following mean value functions:

$$m_\Omega(z) := \frac{1}{|\Omega|} \int_{\Omega} z \, dx, \quad |\Omega| := \int_{\Omega} 1 \, dx,$$

$$m_\Gamma(z_\Gamma) := \frac{1}{|\Gamma|} \int_{\Gamma} z_\Gamma \, d\Gamma, \quad |\Gamma| := \int_{\Gamma} 1 \, d\Gamma,$$

for any $z \in L^1(\Omega)$ and $z_\Gamma \in L^1(\Gamma)$.

Throughout this paper, we make the following assumptions:

(A1) $\phi_0 \in V, \widehat{\beta}(\phi_0) \in L^1(\Omega), \psi_0 \in V_\Gamma, \widehat{\beta}_\Gamma(\psi_0) \in L^1(\Gamma)$, and $\phi_0 \big|_\Gamma = \psi_0$. Moreover, $m_0 := m_\Omega(\phi_0) \in \text{int}D(\beta), m_{\Gamma 0} := m_\Gamma(\psi_0) \in \text{int}D(\beta_\Gamma)$;

(A2) $f \in L^2(0,T;V), g \in L^2(0,T;V_\Gamma)$;

(A3) $\beta, \beta_\Gamma$ are maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$, that is, they are the subdifferentials

$$\beta = \partial \widehat{\beta}, \quad \beta_\Gamma = \partial \widehat{\beta}_\Gamma$$

of some proper lower semicontinuous and convex functions $\widehat{\beta}$ and $\widehat{\beta}_\Gamma : \mathbb{R} \to [0, \infty]$ satisfying $\widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0$ with the corresponding effective domains denoted by $D(\beta)$ and $D(\beta_\Gamma)$, respectively;

(A4) $\pi, \pi_\Gamma : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions with Lipschitz constants $L$ and $L_\Gamma$, respectively;

(A5) $D(\beta_\Gamma) \subseteq D(\beta)$ and there exist positive constants $\varrho, c_0 > 0$ such that

$$|\beta^0(r)| \leq \varrho |\beta^c_\Gamma(r)| + c_0 \quad \text{for all } r \in D(\beta_\Gamma),$$

where $\beta^0$ and $\beta^c_\Gamma$ denote the minimal sections of $\beta$ and $\beta_\Gamma$.

The assumption (A3) implies that $0 \in \beta(0)$ and $0 \in \beta_\Gamma(0)$. Moreover, the minimal section $\beta^0$ of $\beta$ is defined by $\beta^0(r) := \{r^* \in \beta(r) : |r^*| = \min_{s \in \beta(r)} |s|\}$ and same definition applies to $\beta^c_\Gamma$ in (A5). These assumptions are the same as in [2,8], in particular, the compatibility condition (A5) is essential to treat different potentials $\beta$ in the bulk and $\beta_\Gamma$ on the boundary. Of course, if one chooses $\beta = \beta_\Gamma$ to be the same potential, then (A5) holds automatically.

Our first result is related to the existence of global weak solutions.

**Theorem 2.1.** Under the assumptions (A1)–(A5), there exists a global weak solution of problem (2.1)–(2.7) in the sense of Definition 2.1.

The existence of strong solutions will be discussed in Section 4 (see Theorem 4.1).

Our second result is the continuous dependence on the initial data and external sources, which immediately yields the uniqueness of weak solutions:

**Theorem 2.2.** Assume that (A3) and (A4) hold. Moreover, let $f^{(1)}, f^{(2)} \in L^2(0,T;V^*), g^{(1)}, g^{(2)} \in L^2(0,T;V^*_\Gamma), \phi_0^{(1)}, \phi_0^{(2)} \in V^*, \psi_0^{(1)}, \psi_0^{(2)} \in V^*_\Gamma$ and

$$\langle \phi_0^{(1)}, 1 \rangle_{V^*, V} = \langle \phi_0^{(2)}, 1 \rangle_{V^*, V} = m_0 |\Omega|,$$

$$\langle \psi_0^{(1)}, 1 \rangle_{V^*_\Gamma, V_\Gamma} = \langle \psi_0^{(2)}, 1 \rangle_{V^*_\Gamma, V_\Gamma} = m_{\Gamma 0} |\Gamma|.$$


Let sextuplets of functions \((\phi^{(i)}, \mu^{(i)}, \xi^{(i)}, \psi^{(i)}, w^{(i)}, \zeta^{(i)})\) be weak solutions of problem (2.1)–(2.7) corresponding to the given data \(f^{(i)}, g^{(0)}\) and \(\psi^{(0)}\) for \(i = 1, 2\). Then, there exists a positive constant \(C > 0\), depending on \(L, L_\Gamma\) and \(T\), such that

\[
\left| \phi^{(1)} - \phi^{(2)} \right|_{C((0,T);V^*)} + \left| \psi^{(1)} - \psi^{(2)} \right|_{C((0,T);V^*)} + \left| \psi^{(1)} - \phi^{(2)} \right|_{L^2(0,T;V)} + \left| \psi^{(1)} - \psi^{(2)} \right|_{L^2(0,T;V^*)} \leq C \left( \left| \phi^{(1)} - \phi^{(2)} \right|_{V^*} + \left| \psi^{(1)} - \psi^{(2)} \right|_{V^*} + \left| f^{(1)} - f^{(2)} \right|_{L^2(0,T;V^*)} + \left| g^{(1)} - g^{(2)} \right|_{L^2(0,T;V^*)} \right).
\]

In order to prove Theorem 2.1, we quote the abstract framework as in [25, 26] and we also prepare the following function spaces:

\[
\begin{align*}
V_0 & := \{ z \in V : m_\Omega(z) = 0 \}, \\
V_{0*} & := \{ z^* \in V^* : z^* \cdot 1_{V^*,V} = 0 \}, \\
V_{T,0} & := \{ z_T \in V_T : m_T(z_T) = 0 \}, \\
V_{T,0*} & := \{ z_T^* \in V_T^* : z_T^* \cdot 1_{V_T^*,V_T} = 0 \}.
\end{align*}
\]

From the Poincaré–Wirtinger inequalities (see, e.g., [23]), we see that there exists a positive constant \(C_P\) such that

\[
\begin{align*}
|z|_{V_0}^2 & \leq C_P |z|_{V_0}^2, \\
|z|_{V_{T,0}}^2 & \leq C_P |z|_{V_{T,0}}^2.
\end{align*}
\]

Then, based on the Lax–Milgram theorem, we introduce the operator \(N_\Omega : V_{0*} \to V_0\) by:

\[
u = N_\Omega u \text{ if and only if } m_\Omega(u) = 0 \text{ and } \int_\Omega \nabla u \cdot \nabla z dx = \langle v, z \rangle_{V^*,V} \text{ for all } z \in V;
\]

(2.19)

analogously, we define \(N_T : V_{T,0*} \to V_{T,0}\) by:

\[
u_T = N_T v_T \text{ if and only if } m_T(u_T) = 0 \text{ and } \int_\Gamma \nabla_T u_T \cdot \nabla_T z_T d\Gamma = \langle v_T, z_T \rangle_{V_T^*,V_T} \text{ for all } z_T \in V_T.
\]

(2.20)

By virtue of these definitions, we can also introduce the norms

\[
|z|_{V_{0*}} := \left( \int_\Omega |\nabla N_\Omega z|^2 dx \right)^{1/2} \text{ for all } z \in V_{0*},
\]

equivalent to the usual norm \(|\cdot|_{V^*}\) for the elements of \(V_{0*}\); and

\[
|z_T|_{V_{T,0*}} := \left( \int_\Gamma |\nabla_T N_T z_T|^2 d\Gamma \right)^{1/2} \text{ for all } z_T \in V_{T,0*},
\]

equivalent to the usual norm \(|\cdot|_{V_T^*}\) for the elements of \(V_{T,0*}\), respectively.

3. WELL-POSEDNESS

In this section, we prove the existence of global weak solutions and the continuous dependence with respect to given data. To do so, we introduce an approximate problem for problem (2.1)–(2.7). The idea is based on a time-discretization scheme, the Moreau–Yosida regularization, together with a viscous Cahn–Hilliard approach.

Let \(N \in \mathbb{N}\) and put \(h := T/N\), the time step of discretization. Moreover, \(\tau, \sigma \in (0, 1]\) stand for viscosity coefficients; \(\varepsilon \in (0, 1]\) is used as a parameter of Moreau–Yosida
regularization for maximal monotone graphs. We consider the following equations and conditions for \( n = 0, 1, \ldots, N - 1 \):
\[
\frac{\phi_{n+1} - \phi_n}{h} + \mu_{n+1} - \mu_n - \Delta \mu_{n+1} = 0 \quad \text{a.e. in } \Omega, \quad (3.1)
\]
\[
\mu_{n+1} = \tau \frac{\phi_{n+1} - \phi_n}{h} - \Delta \phi_{n+1} + \beta_\varepsilon(\phi_{n+1}) + \pi(\phi_{n+1}) - f_n \quad \text{a.e. in } \Omega, \quad (3.2)
\]
\[
\partial_v \mu_{n+1} = 0 \quad \text{a.e. on } \Gamma, \quad (3.3)
\]
\[
(\phi_{n+1})_{|_r} = \psi_{n+1} \quad \text{a.e. on } \Gamma, \quad (3.4)
\]
\[
\frac{\psi_{n+1} - \psi_n}{h} + w_{n+1} - w_n - \Delta_r w_{n+1} = 0 \quad \text{a.e. on } \Gamma, \quad (3.5)
\]
\[
w_{n+1} = \partial_v \phi_{n+1} + \sigma \frac{\psi_{n+1} - \psi_n}{h} - \Delta_r \psi_{n+1} + \beta_\varepsilon r(\psi_{n+1}) + \pi_r(\psi_{n+1}) - g_n \quad \text{a.e. on } \Gamma. \quad (3.6)
\]
Note that \( \phi_0 \) and \( \psi_0 \) are known and, in order to solve the system (3.1)–(3.6), we need to prepare initial data \( \mu_0 \) and \( w_0 \), respectively. In the level of time-discrete approximation, we set up as follows:
\[
\mu_0 := 0, \quad w_0 := 0. \quad (3.7)
\]
Indeed, the terms \( \mu_{n+1} - \mu_n \) in the equation (3.1) and \( w_{n+1} - w_n \) in the equation (3.5) play a role of viscosities with the parameter \( h \). In (3.2) and (3.6), \( f_n \) and \( g_n \) are known too, defined by
\[
f_n := \frac{1}{h} \int_{nh}^{(n+1)h} f(s)ds; \quad g_n := \frac{1}{h} \int_{nh}^{(n+1)h} g(s)ds \quad \text{for } n = 0, 1, \ldots, N - 1.
\]
In order to approximate the maximal monotone graphs, we recall the Moreau–Yosida regularization (see, e.g., [1][2]). For each \( \varepsilon \in (0, 1) \), we define \( \beta_\varepsilon, \beta_\varepsilon r : \mathbb{R} \to \mathbb{R} \), along with the associated resolvent operators \( J_\varepsilon, J_\varepsilon r : \mathbb{R} \to \mathbb{R} \) given by
\[
\beta_\varepsilon(r) := \frac{1}{\varepsilon}(r - J_\varepsilon(r)), \quad J_\varepsilon(r) := (I + \varepsilon \beta)^{-1}(r),
\]
\[
\beta_\varepsilon r(r) := \frac{1}{\varepsilon \varrho}(r - J_\varepsilon r(r)), \quad J_\varepsilon r(r) := (I + \varepsilon \varrho \beta_\varepsilon)^{-1}(r),
\]
for all \( r \in \mathbb{R} \), where \( \varrho > 0 \) is same as in the condition (2.16). As a remark, the above two definitions are not symmetric, more precisely, the parameter of approximation is not directly \( \varepsilon \) but \( \varepsilon \varrho \) in the definition of \( \beta_\varepsilon r \) and \( J_\varepsilon r \). This is important in order to apply [3] Lemma 4.4], which ensures that
\[
|\beta_\varepsilon(r)| \leq \varrho|\beta_\varepsilon r(r)| + c_0 \quad \text{for all } r \in \mathbb{R}, \quad (3.8)
\]
for all \( \varepsilon \in (0, 1] \) with the same constants \( \varrho \) and \( c_0 \) as in (2.16). We also have \( \beta_\varepsilon(0) = \beta_\varepsilon r(0) = 0 \). Moreover, the related Moreau–Yosida regularizations \( \beta_\varepsilon, \beta_\varepsilon r \) of \( \beta, \beta_\varepsilon r : \mathbb{R} \to \mathbb{R} \) fulfill
\[
\hat{\beta}_\varepsilon(r) := \inf_{s \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon} |r - s|^2 + \hat{\beta}(s) \right\} = \frac{1}{2\varepsilon} |r - J_\varepsilon(r)|^2 + \beta(J_\varepsilon(r)) = \int_0^r \beta_\varepsilon(s)ds,
\]
\[
\hat{\beta}_\varepsilon r(r) := \inf_{s \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon \varrho} |r - s|^2 + \hat{\beta}_\varepsilon r(s) \right\} = \int_0^r \beta_\varepsilon r(s)ds,
\]
for all \( r \in \mathbb{R} \). Then, we see that \( \beta_\varepsilon \) and \( \beta_{\Gamma,\varepsilon} \) are Lipschitz continuous with constants \( 1/\varepsilon \) and \( 1/(\varepsilon\varrho) \), respectively. Additionally, we also use the following facts:

\[
\begin{align*}
|\beta_\varepsilon(r)| & \leq |\beta^*(r)|, \quad |\beta_{\Gamma,\varepsilon}(r)| \leq |\beta^*_\Gamma(r)|, \\
0 & \leq \hat{\beta}_\varepsilon(r) \leq \beta(r), \quad 0 \leq \hat{\beta}_{\Gamma,\varepsilon}(r) \leq \beta_{\Gamma}(r),
\end{align*}
\]

(3.9)

for all \( r \in \mathbb{R} \).

### 3.1. Time-discrete approximate solution.

In this subsection, firstly we discuss the existence of solutions to the time-discrete approximate problem (3.1)–(3.6) for all \( n = 0, 1, \ldots, N - 1 \), for arbitrary but fixed parameters \( \tau, \sigma \in (0, 1] \). Secondly, by introducing the piecewise linear and constant interpolants, we construct the approximate problem of a viscous Cahn–Hilliard system.

**Proposition 3.1.** There is a value \( h^* \in (0, 1] \), depending on \( \tau \) and \( \sigma \), such that for every \( h \in (0, h^*) \), there exists a unique quadruplet \( (\phi_{n+1}, \mu_{n+1}, \psi_{n+1}, w_{n+1}) \) with \( \phi_{n+1} \in H^2(\Omega) \), \( \mu_{n+1} \in W \), \( \psi_{n+1}, w_{n+1} \in H^2(\Gamma) \) such that (3.1)–(3.6) holds for all \( n = 0, 1, 2, \ldots, N - 1 \).

**Proof.** Define \( \Delta_N : W \to H \) be the Laplace operator, subject to the homogeneous Neumann boundary condition. From (3.1) and (3.3), we infer that

\[
\mu_{n+1} = (I - \Delta_N)^{-1} \left( \mu_n - \frac{\phi_{n+1} - \phi_n}{h} \right) \quad \text{in } H,
\]

(3.10)

where \( I - \Delta_N \) is a linear operator from its domain \( W \subset H \) to \( H \). At the same time, from (3.5) we obtain

\[
w_{n+1} = (I - \Delta_\Gamma)^{-1} \left( w_n - \frac{\psi_{n+1} - \psi_n}{h} \right) \quad \text{in } H_\Gamma,
\]

(3.11)

where \( I - \Delta_\Gamma \) is a linear operator from \( H^2(\Gamma) \subset H_\Gamma \) to \( H_\Gamma \). As a consequence, equation (3.2) can be rewritten as

\[
(I - \Delta_N)^{-1} \phi_{n+1} + \tau \phi_{n+1} - h\Delta \phi_{n+1} + h\beta_\varepsilon(\phi_{n+1}) + h\pi(\phi_{n+1}) = (I - \Delta_N)^{-1} \phi_n + h(I - \Delta_N)^{-1} \mu_n + \tau \phi_n + hf_n \quad \text{in } H
\]

(3.12)

and the condition (3.6) becomes

\[
h\partial_\nu \phi_{n+1} + h(I - \Delta_\Gamma)^{-1} \psi_{n+1} + \sigma \psi_{n+1} - h\Delta_\Gamma \psi_{n+1} + h\beta_{\Gamma,\varepsilon}(\psi_{n+1}) + h\pi_\Gamma(\psi_{n+1}) = (I - \Delta_\Gamma)^{-1} \psi_n + h(I - \Delta_\Gamma)^{-1} w_n + \sigma \psi_n + hg_n \quad \text{in } H_\Gamma.
\]

(3.13)

Now, the map

\[
(z, z_\Gamma) \mapsto (-h\Delta z, h\partial_\nu z - h\Delta_\Gamma z_\Gamma)
\]

gives a maximal monotone operator \( A \) from its domain \( D(A) := \{(z, z_\Gamma) \in H^2(\Omega) \times H^2(\Gamma) : z_\Gamma = z_\Gamma \text{ a.e. on } \Gamma \} \) to \( H := H \times H_\Gamma \), with reference to [11] p. 47, Theorem 2.8. Indeed, it coincides with the subdifferential of the proper, lower semicontinuous and convex functional \( J : H \to [0, \infty) \) defined by

\[
J(z, z_\Gamma) := \begin{cases} 
\frac{h}{2} \int_\Omega |\nabla z|^2 dx + \frac{h}{2} \int_\Gamma |\nabla z_\Gamma|^2 d\Gamma & \text{if } (z, z_\Gamma) \in V, \\
+\infty & \text{otherwise,}
\end{cases}
\]
where \( V := \{(z, z_\Gamma) \in V \times V_\Gamma : z_{1\Gamma} = z_\Gamma \text{ a.e. on } \Gamma\} \). This also implies that the subdifferential of \( J \) in \( H \) at \((z, z_\Gamma)\) coincides with \( A(z, z_\Gamma) = (-h\Delta z, h\partial_\nu z - h\Delta_\Gamma z_\Gamma) \). Next, we define another operator \( B : H \to H \) by
\[
B(z, z_\Gamma) := \left( (I - \Delta_N)^{-1}z + \tau z + h\beta_\varphi(z), \right. \\
\left. (I - \Delta_\Gamma)^{-1}z_\Gamma + \sigma z_\Gamma + h\beta_{\Gamma,e}(z_\Gamma) + \pi \right)
\]
with its domain \( D(B) = H \). Then, we see that \( B \) is Lipschitz continuous and monotone provided that \( h \) is sufficiently small compared to \( \tau \) and \( \sigma \). Thus, we can complete the proof of Proposition 3.1 by iterating from the strict coerciveness of \( B \) for some suitable function space (3.16) is obtained from the direct calculation as follows:

\[
\phi = (z, z_\Gamma) \quad \text{is a consequence of the strict coerciveness of} \quad B
\]

Next, we can recover \( \phi_{n+1}, \psi_{n+1} \) in \( V \) solving (3.12) and (3.13), where the uniqueness is a consequence of the strict coerciveness of \( B \). Next, we can recover \( \mu_{n+1} \in W \) and \( w_{n+1} \in H_2(\Gamma) \) from (3.11) and (3.1) respectively. By comparison in the equations (3.2) and (3.6), we also deduce that \( \phi_{n+1} \in H^2(\Omega) \) and \( \psi_{n+1} \in H^2(\Gamma) \), using the elliptic regularity theory (see, e.g., [31] Lemma A.1). Thus, we can complete the proof of Proposition 3.1 by iterating from \( n = 0 \) to \( n = N - 1 \).

According to the standard manner, we now define the following piecewise linear functions and steps functions:

\[
\hat{\phi}_h(t) := \phi_n + \frac{\phi_{n+1} - \phi_n}{h}(t - nh) \quad \text{for } t \in [nh, (n + 1)h], \quad n = 0, 1, \ldots, N - 1,
\]
\[
\hat{\psi}_h(t) := \phi_{n+1} \quad \text{for } t \in (nh, (n + 1)h], \quad n = 0, 1, \ldots, N - 1,
\]
\[
f_h(t) := f_n \quad \text{for } t \in (nh, (n + 1)h], \quad n = 0, 1, \ldots, N - 1,
\]

and analogously for \( \hat{\mu}_h, \bar{\mu}_n, \bar{\psi}_h, \bar{\psi}_h, \bar{w}_h, \bar{w}_h, \bar{g}_h \). Then, we have the following useful properties:

\[
|\hat{\phi}_h|_{L^2(0,T;X)} \leq \frac{h}{2} |\phi_0|_{X} + |\hat{\phi}_h|_{L^2(0,T;X)}, \quad (3.14)
\]
\[
|\hat{\phi}_h|_{L^\infty(0,T;X)} = \max \{|\phi_0|_X, |\hat{\phi}_h|_{L^\infty(0,T;X)}\} \quad (3.15),
\]
\[
|\hat{\phi}_h - \hat{\phi}_h|_{L^2(0,T;X)} = \frac{h^2}{3} |\hat{\phi}_h|_{L^2(0,T;X)}, \quad (3.16)
\]

for some suitable function space \( X \). Indeed, (3.15) is clear from the definition, the equality (3.16) is obtained from the direct calculation as follows:

\[
|\hat{\phi}_h - \hat{\phi}_h|_{L^2(0,T;X)} = \sum_{n=0}^{N-1} \frac{|\phi_{n+1} - \phi_n}{h} \int_{nh}^{(n+1)h} (t - h(n + 1))^2 dt
\]
Under these settings, we see from (3.1)–(3.6) that the functions $h$ for every inequality we infer that constructed above solve the following polygonal approximate problem of the viscous Cahn–Hilliard system:

$$
\frac{h}{3} \sum_{n=0}^{N-1} \left| \frac{\phi_{n+1} - \phi_n}{h} \right|^2 = \frac{h^2}{3} \left| \partial_t \phi_h \right|^2_{L^2(0,T;X)}.
$$

Concerning the inequality (3.14), invoking the convexity and Jensen’s inequality we obtain that

$$
\left| \dot{\phi}_h \right|^2_{L^2(0,T;X)} = \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} \left\{ \left( \frac{t - nh}{h} \right) \phi_{n+1} + \left[ 1 - \left( \frac{t - nh}{h} \right) \right] \phi_n \right\} dt
$$

$$
\leq \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} \left\{ \left( \frac{t - nh}{h} \right) |\phi_{n+1}|^2_X + \left[ 1 - \left( \frac{t - nh}{h} \right) \right] |\phi_n|^2_X \right\} dt
$$

$$
= \sum_{n=0}^{N-1} \left( \frac{h}{2} |\phi_{n+1}|^2_X + \frac{h}{2} |\phi_n|^2_X \right)
$$

$$
\leq \frac{h}{2} |\phi_0|^2_X + |\phi_h|^2_{L^2(0,T;X)}.
$$

Under these settings, we see from (3.1)–(3.6) that the functions $\dot{\phi}_h, \ddot{\phi}_h, \dddot{\phi}_h, \dddot{\bar{\psi}}_h, \ddot{\bar{w}}_h, \bar{w}_h$ constructed above solve the following polygonal approximate problem of the viscous Cahn–Hilliard system:

$$
\partial_t \dddot{\phi}_h + h \partial_t \dddot{\bar{w}}_h - \Delta \dddot{\bar{w}}_h = 0 \quad \text{a.e. in } Q, \quad (3.17)
$$

$$
\dddot{\bar{w}}_h = \tau \partial_t \dddot{\bar{\psi}}_h - \Delta \dddot{\bar{\psi}}_h + \beta_{e}(\dddot{\bar{\psi}}_h) + \pi(\dddot{\bar{\psi}}_h) - \dddot{f}_h \quad \text{a.e. in } Q, \quad (3.18)
$$

$$
\dddot{\bar{\psi}}_h = 0 \quad \text{a.e. on } \Sigma, \quad (3.19)
$$

$$
\dddot{\phi}_h|_\Gamma = \dddot{\bar{\psi}}_h \quad \text{a.e. on } \Sigma, \quad (3.20)
$$

$$
\partial_t \dddot{\bar{\psi}}_h + h \partial_t \dddot{\bar{w}}_h - \Delta \dddot{\bar{w}}_h = 0 \quad \text{a.e. on } \Sigma, \quad (3.21)
$$

$$
\dddot{\bar{w}}_h = \partial_{\nu} \dddot{\phi}_h + \sigma \partial_t \dddot{\psi}_h - \Delta \dddot{\psi}_h + \beta_{\Gamma,e}(\dddot{\psi}_h) + \pi_{\Gamma}(\dddot{\psi}_h) - \dddot{g}_h \quad \text{a.e. on } \Sigma, \quad (3.22)
$$

$$
\dddot{\phi}_h(0) = \dddot{\bar{\psi}}_0, \quad \dddot{\bar{\psi}}_h(0) = \dot{\psi}_0, \quad \dddot{\bar{w}}_h(0) = 0 \quad \text{a.e. in } \Omega, \quad \dddot{\bar{w}}_h(0) = 0 \quad \text{a.e. on } \Gamma, \quad (3.23)
$$

for every $h \in (0, h^*)$. By virtue of the definitions of $f_n$ and $g_n$ we see that $\{f_h\}_{h>0}$ and $\{g_h\}_{h>0}$ are bounded in $L^2(0, T; V)$ and $L^2(0, T; V_T)$, respectively. Indeed, from the Hölder inequality we infer that

$$
\int_0^T \left| \frac{f(t)}{h} \right|^2_{V} dt = \sum_{n=0}^{N-1} h |f_n|^2_V
$$

$$
= \sum_{n=0}^{N-1} h \left| \frac{1}{h} \int_{nh}^{(n+1)h} f(s) ds \right|^2_{V}
$$

$$
\leq \frac{1}{h} \sum_{n=0}^{N-1} \left( \int_{nh}^{(n+1)h} |f(s)|^2_V ds \right) \left( \int_{nh}^{(n+1)h} 1 ds \right)
$$
Lemma 3.1. There exists a positive constant $M_1$, independent of $h \in (0, h^{**}]$, $\tau, \sigma, \varepsilon \in (0, 1]$, such that

\[
\begin{align*}
&\|\tilde{\phi}_h\|_{L^\infty(0,T;V)} + \|\tilde{\psi}_h\|_{L^\infty(0,T;\overline{V})} + \|\partial_t \tilde{\phi}_h + h\partial_t \tilde{\mu}_h\|_{L^2(0,T;V^*)} + \|\partial_t \tilde{\psi}_h + h\partial_t \tilde{w}_h\|_{L^2(0,T;V^*)} \\
&+ h\|\tilde{\mu}_h\|_{L^\infty(0,T;\overline{H})} + h^2\|\partial_t \tilde{\mu}_h\|_{L^2(0,T;\overline{H})} + h\|\tilde{w}_h\|_{L^\infty(0,T;\overline{H})} + h^2\|\partial_t \tilde{w}_h\|_{L^2(0,T;\overline{H})} \\
&+ \tau\|\tilde{\phi}_h\|_{L^2(0,T;\overline{H})} + \sigma\|\partial_t \tilde{\psi}_h\|_{L^2(0,T;\overline{H})} + h\|\partial_t \tilde{w}_h\|_{L^2(0,T;\overline{H})} + h\|\partial_t \tilde{\psi}_h\|_{L^2(0,T;\overline{V})} \\
&+ \|\tilde{\beta}_\varepsilon(\tilde{\phi}_h)\|_{L^\infty(0,T;L^1(\Gamma))} + \|\tilde{\beta}_\varepsilon(\tilde{\psi}_h)\|_{L^\infty(0,T;L^1(\Gamma))} \leq M_1,
\end{align*}
\]  
for all $h \in (0, h^{**}]$, where $h^{**} \in (0, h^*)$ is a threshold value for the step size depending on $\tau, \sigma \in (0, 1]$.  

Proof. By integrating (3.1) over $\Omega$, with the help of (3.3) we deduce the following relation for the mean values defined in (2.14):

\[
m_{\Omega}(\phi_{n+1} + h\mu_{n+1}) = m_{\Omega}(\phi_n + h\mu_n) = m_0,
\]  
for all $n = 0, 1, \ldots, N - 1$, where (cf. (A1) and (3.7))

\[
m_0 := m_{\Omega}(\phi_0) = \frac{1}{|\Omega|} \int_{\Omega} \phi_0 dx = \frac{1}{|\Omega|} \int_{\Omega} (\phi_0 + h\mu_0) dx.
\]

In a similar manner, from (3.5) and (2.14) it follows that

\[
m_{\Gamma}(\psi_{n+1} + hw_{n+1}) = m_{\Gamma}(\psi_n + hw_n) = m_{\Gamma_0},
\]  
for all $n = 0, 1, \ldots, N - 1$, where, thanks to (A1) and (3.7),

\[
m_{\Gamma_0} := m_{\Gamma}(\psi_0) = \frac{1}{|\Gamma|} \int_{\Gamma} \psi_0 d\Gamma = \frac{1}{|\Gamma|} \int_{\Gamma} (\psi_0 + hw_0) d\Gamma.
\]

Then, as (3.26) entails that

\[
\langle \phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n, 1 \rangle_{V^*, V} = \int_{\Omega} (\phi_{n+1} + h\mu_{n+1}) dx - \int_{\Omega} (\phi_n + h\mu_n) dx = 0,
\]

that is, $\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n \in V_0^*$, we can test (3.1) by

\[N_{\Omega}(\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n)
\]

and, using (3.3) and (2.19), we obtain

\[
\begin{align*}
&h \left| \frac{\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n}{h} \right|_{V_0^*}^2 + \int_{\Omega} \mu_{n+1}(\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n) dx = 0,
\end{align*}
\]  
for all $n = 0, 1, 2, \ldots, N - 1$. Next, from (3.27) we see that

\[
\langle \psi_{n+1} + hw_{n+1} - \psi_n - hw_n, 1 \rangle_{\overline{V}'_T, \overline{V}_T} = \int_{\Gamma} (\psi_{n+1} + hw_{n+1}) d\Gamma - \int_{\Gamma} (\psi_n + hw_n) d\Gamma = 0,
\]
that is, \( \psi_{n+1} + hw_{n+1} - \psi_n - hw_n \in V_{\Gamma,\sigma} \); therefore, testing (3.5) by
\[
\mathcal{N}_\Gamma(\psi_{n+1} + hw_{n+1} - \psi_n - hw_n),
\]
and using (2.20) we obtain
\[
\int \left| \frac{\psi_{n+1} + hw_{n+1} - \psi_n - hw_n}{h} \right|^2 + \int w_{n+1}(\psi_{n+1} + hw_{n+1} - \psi_n - hw_n) dx = 0, \tag{3.29}
\]
for all \( n = 0, 1, \ldots, N - 1 \). Next, we add \( \phi_{n+1} \) to both sides of (3.24), multiply the resultant by \( \phi_{n+1} - \phi_n \) and use the condition (3.4) and the equation (3.6), to find out that
\[
\int \mu_{n+1}(\phi_{n+1} - \phi_n) dx + \int \psi_{n+1} - \psi_n) d\Gamma \geq \tau h \left( \phi_{n+1} - \phi_n \right)^2 + \sigma h \left( \psi_{n+1} - \psi_n \right)^2 + \frac{1}{2} \left| \phi_{n+1} \right|^2 + \frac{1}{2} \left| \phi_n \right|^2
\]
\[
+ \frac{1}{2} \left| \psi_{n+1} \right|^2 + \frac{1}{2} \left| \psi_n \right|^2 - \frac{1}{2} |\psi_{n+1}|^2 - \frac{1}{2} |\psi_n|^2
\]
\[
+ \int \tilde{\beta}_\varepsilon(\phi_{n+1}) dx - \int \tilde{\beta}_\varepsilon(\phi_n) dx
\]
\[
+ \int (\pi(\phi_{n+1}) - f_n - \phi_{n+1})(\phi_{n+1} - \phi_n) dx
\]
\[
+ \int (\pi(\psi_{n+1}) - g_n - \psi_{n+1})(\psi_{n+1} - \psi_n) d\Gamma \tag{3.30}
\]
for all \( n = 0, 1, \ldots, N - 1 \), where we used the elementary inequality \( r(r - s) = (r^2 + (r - s)^2 - s^2)/2 \) for \( r, s \in \mathbb{R} \).

Now, we collect (3.28)–(3.30), sum up for \( n = 0, 1, \ldots, m - 1 \) and apply (3.7) and (3.9), obtaining
\[
\sum_{n=0}^{m-1} h \left| \phi_{n+1} - \phi_n \right|^2 + \mu_{n+1} - \mu_n \right|_{V_{\Gamma,\sigma}}^2 + \sum_{n=0}^{m-1} h \left| \psi_{n+1} - \psi_n \right|_{H_H}^2 + w_{n+1} - w_n \right|_{V_{\Gamma,\sigma}}^2
\]
\[
+ \frac{1}{2} \left| \phi_{n+1} \right|^2 + \frac{1}{2} \left| \phi_n \right|^2 + \int \tilde{\beta}_\varepsilon(\phi_m) dx + \int \tilde{\beta}_\varepsilon(\psi_m) d\Gamma
\]
\[
\leq \frac{1}{2} |\phi_0|^2 + \frac{1}{2} |\psi_0|^2 + \int \tilde{\beta}(\phi_0) dx + \int \tilde{\beta}(\psi_0) d\Gamma
\]
\[
- \sum_{n=0}^{m-1} \int (\pi(\phi_{n+1}) - f_n - \phi_{n+1})(\phi_{n+1} - \phi_n) dx
\]
\[
- \sum_{n=0}^{m-1} \int (\pi(\psi_{n+1}) - g_n - \psi_{n+1})(\psi_{n+1} - \psi_n) d\Gamma \tag{3.31}
\]
for all \( m = 1, 2, \ldots, N \). We know that there exists a positive constant \( C_1 \) such that 
\[
|z|_{V_{0*}}^2 \leq C_1|z|_V^2
\]
for all \( z \in V_{0*} \), as well as 
\[
|z|_{H}^2 \leq C_1|z|_V^2
\]
for all \( z \in H \). Therefore, in order to estimate the terms on the right hand side of (3.31) we can argue with the help of assumptions (A2) and (A4). First, we have that
\[
\left| \sum_{n=0}^{m-1} \int_{\Omega} \left( \pi(\phi_{n+1}) - f_n - \phi_{n+1} \right)(\phi_{n+1} - \phi_n) dx \right| 
\leq \delta \sum_{n=0}^{m-1} \left| \frac{\phi_{n+1} - \phi_n}{h} \right|^2_{V^*} + C_\delta \sum_{n=0}^{m-1} h \left( 1 + |\phi_{n+1}|_{V^*}^2 + |f_n|_V^2 \right)
\leq 2\delta C_1 \sum_{n=0}^{m-1} h \left| \frac{\phi_{n+1} - \phi_n}{h} \right|^2_{V^*} + \mu_{n+1} - \mu_n \left| \phi_{n+1} \right|^2_{V_{0*}} + 2\delta C_1 \sum_{n=0}^{m-1} h \left| \mu_{n+1} - \mu_n \right|^2_H
+ C_\delta \sum_{n=0}^{m-1} h \left( 1 + |\phi_{n+1}|_V^2 + |f_n|_V^2 \right),
\]
(3.32)
for all \( \delta > 0 \), where we also use Young’s inequality with \( \delta > 0 \); \( C_\delta \) is a positive constant such that \( C_\delta \to \infty \) as \( \delta \to 0 \). Indeed, taking care of (3.26), we have 
\[
\left( \phi_{n+1} - \phi_n \right) / h + \mu_{n+1} - \mu_n \in V_{0*} \quad \text{for all} \quad n = 0, 1, \ldots, N - 1.
\]
From (3.27), a very similar procedure can be used to estimate the other contribution
\[
\left| \sum_{n=0}^{m-1} \int_{\Gamma} \left( \pi(\psi_{n+1}) - g_n - \psi_{n+1} \right)(\psi_{n+1} - \psi_n) d\Gamma \right| 
\leq 2\delta C_2 \sum_{n=0}^{m-1} h \left| \frac{\psi_{n+1} - \psi_n}{h} \right|^2_{V_{0*}} + w_{n+1} - w_n \left| \psi_{n+1} \right|^2_{V_{0*}} + 2\delta C_2 \sum_{n=0}^{m-1} h \left| w_{n+1} - w_n \right|^2_{H_{\Gamma}}
+ C_\delta \sum_{n=0}^{m-1} h \left( 1 + |\psi_{n+1}|_{V_{0*}}^2 + |g_n|_{V_{0*}}^2 \right),
\]
(3.33)
where \( C_2 \) is a positive constant such that 
\[
|z|_{V_{0*}}^2 \leq C_2|z|_{V_{0*}}^2, \quad \text{for all} \quad z \in V_{0*}
\]
and 
\[
|z|_{H_{\Gamma}}^2 \leq C_2|z|_{H_{\Gamma}}^2, \quad \text{for all} \quad z \in H_{\Gamma}.
\]
Then, we can choose \( \delta > 0 \) in order that
\[
\delta \leq \min \{1/(8C_1), 1/(8C_2)\}
\]
and consequently we fix the constant \( C_\delta \) in the above estimates. Next, we choose a threshold value for the step size \( h^{**} \in (0, h^*) \) with the requirement that
\[
C_\delta h^{**} \leq \frac{1}{4}.
\]
Then, collecting (3.31)–(3.33) and recalling (2.17)–(2.18), it is not difficult to obtain
\[
\frac{1}{2} |\phi_m|_{V}^2 + \frac{1}{2} |\psi_{m-1/2}|_{V_{0}}^2 
\leq \frac{1}{2} |\phi_0|_{V}^2 + \frac{1}{2} |\psi_0|_{V_{0}}^2 + \int_{\Omega} \hat{\beta}(\phi_0) dx + \int_{\Gamma} \hat{\beta}_\Gamma(\psi_0) d\Gamma
+ C_\delta \sum_{n=0}^{m-1} h \left( 1 + |f_n|_V^2 \right) + C_\delta \sum_{n=1}^{m-1} h |\phi_n|_V^2 + C_\delta h |\phi_m|_V^2
\]
and consequently

\[ \frac{1}{2} |\phi_m|^2_V + \frac{1}{2} |\psi_m|^2_{Vr} \leq \frac{1}{2} |\phi_0|^2_V + \frac{1}{2} |\psi_0|^2_{Vr} + \int \Omega \hat{\beta}(\phi_0) dx + \int \Gamma \hat{\beta}_T(\psi_0) d\Gamma \]

\[ + C_\delta \left( T + |f|^2_{L^2(0,T;V)} \right) + C_\delta \sum_{n=0}^{m-1} h |\phi_n|^2_V + \frac{1}{4} |\phi_m|^2_V \]

\[ + C_\delta \left( T + |g|^2_{L^2(0,T;V)} \right) + C_\delta \sum_{n=0}^{m-1} h |\psi_n|^2_{Vr} + \frac{1}{4} |\psi_m|^2_{Vr}, \]

for all \( m = 1, 2, \ldots, N \). Therefore, applying the discrete Gronwall lemma with assumptions (A1) and (A2), we conclude that there exists a positive constant \( M_1 \), independent of \( h \in (0, h^*) \), \( \tau, \sigma, \varepsilon \in (0, 1) \), such that

\[ |\phi_m|^2_V + |\psi_m|^2_{Vr} \leq M_1 \]

for all \( m = 1, 2, \ldots, N \). Moreover, going back to (3.34)–(3.36), we plainly deduce (3.25) for some positive constant \( M_1 \geq M_1 \) independent of \( h \in (0, h^*) \) and \( \tau, \sigma, \varepsilon \in (0, 1) \).

**Lemma 3.2.** There exist two functions \( \Lambda_1, \Lambda_2 \in L^2(0,T) \) and a positive constant \( M_2 \), independent of \( h \in (0, h^*) \), \( \tau, \sigma, \varepsilon \in (0, 1) \), such that

\[ |\bar{\mu}_h(t) - m_\Omega(\bar{\mu}_h(t))|_{Vr} + |\bar{w}_h(t) - m_T(\bar{w}_h(t))|_{Vr} \leq \Lambda_1(t), \quad (3.34) \]

\[ |\beta_\varepsilon(\bar{\phi}_h(t))|_{L^1(\Omega)} + |\beta_T(\bar{\psi}_h(t))|_{L^1(\Gamma)} \leq M_2 \left( \Lambda_2(t) + |\partial_\nu \bar{\phi}_h(t)|_{H^1} \right), \quad (3.35) \]

for a.a. \( t \in (0,T) \).

**Proof.** Firstly, multiplying (3.17) by \( \bar{\mu}_h - m_\Omega(\bar{\mu}_h) \), integrating the resultant over \( \Omega \), using the boundary condition (3.19), and applying the Young and Poincaré–Wirtinger inequalities, we obtain

\[ \int_\Omega |\nabla (\bar{\mu}_h - m_\Omega(\bar{\mu}_h))|^2 dx \leq |\partial_t \bar{\phi}_h + h \partial_t \bar{\mu}_h|_{Vr}, |\bar{\mu}_h - m_\Omega(\bar{\mu}_h)|_{V} \]

\[ \leq \frac{1}{2\delta} |\partial_t \bar{\phi}_h + h \partial_t \bar{\mu}_h|^2_{Vr}, + \frac{\delta}{2} |\bar{\mu}_h - m_\Omega(\bar{\mu}_h)|^2_V \]

\[ \leq \frac{1}{2\delta} |\partial_t \bar{\phi}_h + h \partial_t \bar{\mu}_h|^2_{Vr} + \frac{C_\delta}{2} |\bar{\mu}_h - m_\Omega(\bar{\mu}_h)|^2_{V_0} \]

a.e. in \((0,T)\), for all \( \delta > 0 \). Similarly, multiplying (3.21) by \( \bar{w}_h - m_T(\bar{w}_h) \) and integrating the resultant over \( \Gamma \), we obtain

\[ \int_\Gamma |\nabla_T (\bar{w}_h - m_T(\bar{w}_h))|^2 d\Gamma \leq |\partial_t \bar{\psi}_h + h \partial_t \bar{w}_h|_{Vr}, |\bar{w}_h - m_T(\bar{w}_h)|_{Vr} \]

\[ \leq \frac{1}{2\delta} |\partial_t \bar{\psi}_h + h \partial_t \bar{w}_h|^2_{Vr} + \frac{C_\delta}{2} |\bar{w}_h - m_T(\bar{w}_h)|^2_{V_{r,0}} \]
a.e. in \((0, T)\). Letting \(\delta := 1/C_P\) and applying the Poincaré–Wirtinger inequality again, we deduce that
\[
|\tilde{\mu}_h - m_\Omega(\bar{\mu}_h)|^2 \leq C_P|\tilde{\mu}_h - m_\Omega(\bar{\mu}_h)|^2 \leq C_P^2|\partial_t \tilde{\phi}_h + h\partial_t \bar{\mu}_h|^2_{V^*},
\]
\[
|\tilde{w}_h - m_\Gamma(\bar{w}_h)|^2 \leq C_P^2|\partial_t \tilde{\psi}_h + h\partial_t \bar{w}_h|^2_{V^*_\Gamma},
\]
a.e. in \((0, T)\). Taking \((3.25)\) into account, we conclude the estimate \((3.34)\) with
\[
\Lambda_1(t) := \sqrt{2}C_P \left( |\partial_t \tilde{\phi}_h(t) + h\partial_t \tilde{\psi}_h(t)|_{V^*} + |\partial_t \tilde{\psi}_h(t) + h\partial_t \bar{w}_h(t)|_{V^*_\Gamma} \right),
\]
for a.a. \(t \in (0, T)\).

Secondly, recalling \((3.26)\) and \((3.27)\), we have
\[
m_\Omega(\tilde{\phi}_h + h\bar{\mu}_h) = m_0, \quad m_\Gamma(\tilde{\psi}_h + h\bar{w}_h) = m_{\Gamma_0} \quad \text{a.e. on} \ (0, T).
\]
Then, we multiply \((3.15)\) by \(\tilde{\phi}_h - m_0\) and integrate the resultant over \(\Omega\). Also, we use \((3.20)\) and exploit the argument devised in \([30, \text{Appendix, Prop. A.1}]\) (see also \([19]\) for a complete proof) along with \((A1)\) to infer that there exist two positive constants \(C_3, C_4 > 0\), independent of \(h \in (0, h^{**}]\), \(\tau, \sigma, \varepsilon \in (0, 1]\), such that
\[
C_3 \int_\Omega |\beta_x(\tilde{\phi}_h)| \, dx - C_4
\leq \int_\Omega \beta_x(\tilde{\phi}_h)(\tilde{\phi}_h - m_0) \, dx
\]
\[
= -\int_\Omega (\tau \partial_t \tilde{\phi}_h + \pi(\tilde{\phi}_h) - f_h)(\tilde{\phi}_h - m_0) \, dx - \int_\Omega |\nabla \tilde{\phi}_h|^2 \, dx
\]
\[
+ \int_\Omega \tilde{\mu}_h(\tilde{\phi}_h + h\bar{\mu}_h - m_0) \, dx - h \int_\Omega |\tilde{\mu}_h|^2 \, dx + \int_\Gamma \partial_\nu \tilde{\phi}_h(\tilde{\psi}_h - m_{\Gamma_0}) \, d\Gamma
\]
\[
+ \int_\Gamma \partial_\nu \tilde{\phi}_h(m_{\Gamma_0} - m_0) \, d\Gamma,
\]
a.e. in \((0, T)\). Similarly, we multiply \((3.22)\) by \(\tilde{\psi}_h - m_{0\Gamma}\) and integrate the resultant over \(\Gamma\) to infer that there exist two positive constants \(C_5, C_6 > 0\), independent of \(h \in (0, h^{**}]\), \(\tau, \sigma, \varepsilon \in (0, 1]\), such that
\[
C_5 \int_\Gamma |\beta_{\Gamma, x}(\tilde{\psi}_h)| \, d\Gamma - C_6
\leq \int_\Gamma \beta_{\Gamma, x}(\tilde{\psi}_h)(\tilde{\psi}_h - m_{\Gamma_0}) \, d\Gamma
\]
\[
= -\int_\Gamma \partial_\nu \tilde{\phi}_h(\tilde{\psi}_h - m_{\Gamma_0}) \, d\Gamma - \int_\Gamma (\sigma \partial_t \tilde{\psi}_h + \pi_\Gamma(\tilde{\psi}_h) - g_h)(\tilde{\psi}_h - m_{\Gamma_0}) \, d\Gamma
\]
\[
- \int_\Gamma |\nabla_\Gamma \tilde{\psi}_h|^2 \, d\Gamma + \int_\Gamma \tilde{w}_h(\tilde{\psi}_h + h\bar{w}_h - m_{\Gamma_0}) \, d\Gamma - h \int_\Gamma |\tilde{w}_h|^2 \, d\Gamma,
\]
a.e. in \((0, T)\). Adding \((3.37)\) and \((3.38)\), we obtain that
\[
C_3|\beta_x(\tilde{\phi}_h)|_{L^1(\Omega)} + C_5|\beta_{\Gamma, x}(\tilde{\psi}_h)|_{L^1(\Gamma)}
\leq C_4 + C_6 + |\tau \partial_t \tilde{\phi}_h + \pi(\tilde{\phi}_h) - f_h|_H|\tilde{\phi}_h - m_0|_H + |\bar{\mu}_h - m_\Omega(\bar{\mu}_h)|_H|\tilde{\phi}_h + h\bar{\mu}_h - m_0|_H
\]
Lemma 3.4. There exist a function $M_{\text{ht}}$ of resultant over $\Gamma$, we have
\[ a.e. \text{ in } (0, T). \] Hence, from Lemma 3.1 and (3.31) it follows that there exist a function $\Lambda_2 \in L^2(0, T)$ and a positive constant $M_2$, independent of $h \in (0, h^*)$, $\tau, \sigma, \varepsilon \in (0, 1)$, such that (3.35) holds.

**Lemma 3.3.** There exist functions $\Lambda_3, \Lambda_4 \in L^2(0, T)$ and positive constants $M_3, M_4$, independent of $h \in (0, h^*)$, $\tau, \sigma, \varepsilon \in (0, 1)$, such that
\[
\left| m_\Omega(\mu_h(t)) \right| + \left| m_\Gamma(\tilde{\omega}_h(t)) \right| \leq M_3 \left( \Lambda_3(t) + \left| \partial_\nu \tilde{\phi}_h(t) \right|_{H^r} \right),
\]
\[
\left| \tilde{\mu}_h(t) \right|_{V} + \left| \tilde{\omega}_h(t) \right|_{V_t} \leq M_4 \left( \Lambda_4(t) + \left| \partial_\nu \tilde{\phi}_h(t) \right|_{H^r} \right),
\]
for a.a. $t \in (0, T)$.

**Proof.** Multiplying the equation (3.18) by $1/|\Omega|$ and the equation (3.22) by $1/|\Gamma|$, using integration by parts, and adding the resultants together, we obtain
\[
\left| m_\Omega(\mu_h) \right| + \left| m_\Gamma(\tilde{\omega}_h) \right| \leq \frac{1}{|\Omega|} \int_\Omega |\beta_\varepsilon(\tilde{\phi}_h)| \, dx + \frac{1}{|\Omega|} \int_\Omega |\tau \partial_\nu \tilde{\phi}_h + \pi(\tilde{\phi}_h) - f_h| \, dx + \frac{1}{|\Omega|} \int_\Gamma |\partial_\nu \tilde{\phi}_h| \, d\Gamma
\]
\[
+ \frac{1}{|\Gamma|} \int_\Gamma |\beta_{\Gamma, \varepsilon}(\tilde{\psi}_h)| \, d\Gamma + \frac{1}{|\Gamma|} \int_\Gamma |\partial_\nu \tilde{\phi}_h + \sigma \partial_\nu \tilde{\psi}_h + \pi(\tilde{\psi}_h) - g_h| \, d\Gamma,
\]
a.e. in $(0, T)$. Therefore, using Hölder’s inequality and recalling (3.25) and (3.35), we see that there exist a function $\Lambda_3 \in L^2(0, T)$ depending on $M_2, A_2$ and a positive constant $M_3$ depending on $M_2, L, L_\Gamma, |f|_{L^2(\Omega, T; H)}, |g|_{L^2(\Gamma, T; H)}, |\Omega|$ and $|\Gamma|$, independent of $h \in (0, h^*)$, $\tau, \sigma, \varepsilon \in (0, 1)$, such that the estimate (3.40) holds. Additionally, from (3.34) and the above estimate on mean values of $\tilde{\mu}_h, \tilde{\omega}_h$, we infer (3.41), where the function $\Lambda_4 \in L^2(0, T)$ and the positive constant $M_4$ are independent of $h \in (0, h^*)$, $\tau, \sigma, \varepsilon \in (0, 1)$.

**Lemma 3.4.** There exist a function $\Lambda_5 \in L^2(0, T)$ and a positive constant $M_5$, independent of $h \in (0, h^*)$, $\tau, \sigma, \varepsilon \in (0, 1)$, such that
\[
\left| \beta_\varepsilon(\tilde{\phi}_h(t)) \right|_{H^r} + \left| \beta_{\Gamma, \varepsilon}(\tilde{\psi}_h(t)) \right|_{H^r} \leq M_5 \left( \Lambda_5(t) + \left| \partial_\nu \tilde{\phi}_h(t) \right|_{H^r} \right)
\]
for a.a. $t \in (0, T)$.

**Proof.** Multiplying the equation (3.18) by $\beta_\varepsilon(\tilde{\phi}_h)$, integrating over $\Omega$, and using (3.20), we have
\[
\int_\Omega \beta_\varepsilon(\tilde{\phi}_h) |\nabla \tilde{\phi}_h|^2 \, dx + \int_\Omega |\beta_\varepsilon(\tilde{\phi}_h)|^2 \, dx
\]
\[
\leq \frac{1}{2} \left| \tilde{\mu}_h - \tau \partial_\nu \tilde{\phi}_h - \pi(\tilde{\phi}_h) + f_h \right|_{H^r}^2 + \frac{1}{2} |\beta_\varepsilon(\tilde{\phi}_h)|^2_{H}^2 + \int_\Gamma |\partial_\nu \tilde{\phi}_h \beta_\varepsilon(\tilde{\psi}_h) \, d\Gamma,
\]
a.e. in $(0, T)$. Next, multiplying the the equation (3.22) by $\beta_{\Gamma, \varepsilon}(\tilde{\psi}_h)$ and integrating the resultant over $\Gamma$, we have
\[
\int_\Gamma \beta_{\Gamma, \varepsilon}(\tilde{\psi}_h) |\nabla_{\Gamma} \tilde{\psi}_h|^2 \, d\Gamma + \int_\Gamma |\beta_{\Gamma, \varepsilon}(\tilde{\psi}_h)|^2 \, d\Gamma
\]
Accounting for (3.25), (3.41) and (3.42), we see that there exist a function \( \tilde{\Lambda} \) and a positive constant \( \tilde{\mu} \) as (3.47) and (3.48), we infer

\[
\begin{aligned}
\int_{\Gamma} \partial_{\nu} \tilde{\phi}_h \beta_{t,\varepsilon}(\tilde{\psi}_h) d\Gamma - \int_{\Gamma} \partial_{\nu} \tilde{\phi}_h \beta_{t,\varepsilon}(\tilde{\psi}_h) d\Gamma \\
\leq |\partial_{\nu} \tilde{\phi}_h|_{H^r} \left( |\beta_{t,\varepsilon}(\tilde{\psi}_h)|_{H^r} + |\Gamma|^{1/2} c_0 + |\beta_{t,\varepsilon}(\tilde{\psi}_h)|_{H^r} \right) \\
\leq \frac{1}{4} |\beta_{t,\varepsilon}(\tilde{\psi}_h)|_{H^r}^2 + \tilde{M}_5 \left( 1 + |\partial_{\nu} \tilde{\phi}_h|_{H^r}^2 \right),
\end{aligned}
\]

by Young’s inequality. Finally, adding (3.43), (3.44), recalling the monotonicity of \( C_{\beta} \), such that (3.42) holds a.e. in (0, T).

Proof. There exists a function \( \tilde{\Lambda}_6 \in L^2(0, T) \), independent of \( h \in (0, h^*], \tau, \sigma, \varepsilon \in (0, 1] \), such that (3.45)

\[
|\tilde{\phi}_h(t)|_{H^2(\Omega)} + |\tilde{\psi}_h(t)|_{H^2(\Gamma)} + |\partial_{\nu} \tilde{\phi}_h(t)|_{H^r} \leq \tilde{M}_6(t)
\]

for a.a. \( t \in (0, T) \).

Lemma 3.5. There exists a function \( \Lambda_6 \in L^2(0, T) \), independent of \( h \in (0, h^*], \tau, \sigma, \varepsilon \in (0, 1] \), such that

\[
|\tilde{\phi}_h(t)|_{H^2(\Omega)} + |\tilde{\psi}_h(t)|_{H^2(\Gamma)} + |\partial_{\nu} \tilde{\phi}_h(t)|_{H^r} \leq \Lambda_6(t)
\]

for a.a. \( t \in (0, T) \).

Proof. We rewrite (3.13), (3.21), (3.22) in the following elliptic problem for \( \tilde{\phi}_h \) and \( \tilde{\psi}_h \)

\[
\begin{aligned}
\begin{cases}
-\Delta \tilde{\phi}_h = \tilde{\mu}_h - \tau \partial_{t,\varepsilon} \tilde{\phi}_h - \beta_{t,\varepsilon}(\tilde{\phi}_h) + \pi(\tilde{\phi}_h) + f_h & \text{a.e. in } \Omega, \\
(\tilde{\phi}_h)_{\nu} = \tilde{\psi}_h & \text{a.e. on } \Gamma, \\
-\Delta \tilde{\psi}_h + \tilde{\psi}_h + \partial_{\nu} \tilde{\phi}_h = \tilde{w}_h - \sigma \partial_{t,\varepsilon} \tilde{\psi}_h - \beta_{t,\varepsilon}(\tilde{\psi}_h) - \pi(\tilde{\psi}_h) + g_h + \tilde{\psi}_h & \text{a.e. on } \Gamma,
\end{cases}
\end{aligned}
\]

a.e. in (0, T). Then it follows from [31] Lemma A.1] that the following estimate holds

\[
|\tilde{\phi}_h|_{H^2(\Omega)} + |\tilde{\psi}_h|_{H^2(\Gamma)} \leq C_{MZ} \left( |\tilde{\mu}_h - \tau \partial_{t,\varepsilon} \tilde{\phi}_h - \beta_{t,\varepsilon}(\tilde{\phi}_h) - \pi(\tilde{\phi}_h) + f_h|_{H^r} \\
+ |\tilde{w}_h - \sigma \partial_{t,\varepsilon} \tilde{\psi}_h - \beta_{t,\varepsilon}(\tilde{\psi}_h) - \pi(\tilde{\psi}_h) + g_h + \tilde{\psi}_h|_{H^r} \right)
\]

a.e. in (0, T), where \( C_{MZ} \) is a positive constant independent of \( h \in (0, h^*], \tau, \sigma, \varepsilon \in (0, 1] \).
for a.a. \( t \in (0, T) \), whence, also by (3.25), we conclude that (3.15) holds for some function \( \Lambda_6 \in L^2(0, T) \), independent of \( h \in (0, h^*\]) \( \), \( \tau, \sigma, \varepsilon \in (0, 1] \). \( \square \)

Now, using these uniform estimates, we can discuss the existence of weak solutions to the viscous problem for the original system (2.1), (2.7), by taking \( h \to 0 \) and \( \varepsilon \to 0 \). The subscripts of \( \tau \) and \( \sigma \) for functions mean the dependence on parameters \( \tau, \sigma \in (0, 1] \), however in the next proposition we omit them for simplicity.

**Proposition 3.2.** Assume the \((A1)–(A5)\) are satisfied. For each \( \tau, \sigma \in (0, 1] \), there exists a sextuplet \((\phi, \mu, \xi, \psi, w, \zeta)\) of functions
\[
\phi := \phi_{\tau, \sigma} \in H^1(0, T; V^{\ast}) \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)),
\]
\[
\partial_t \phi \in L^2(0, T; H^1), \quad \tau \phi \in H^1(0, T; H^1),
\]
\[
\mu := \mu_{\tau, \sigma} \in L^2(0, T; V), \quad \tau \mu \in L^2(0, T; W), \quad \xi := \xi_{\tau, \sigma} \in L^2(0, T; H),
\]
\[
\psi := \psi_{\tau, \sigma} \in H^1(0, T; V^{\ast}) \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Gamma)), \quad \sigma \psi \in H^1(0, T; H^1),
\]
\[
w := w_{\tau, \sigma} \in L^2(0, T; V), \quad \sigma w \in L^2(0, T; H^2(\Gamma)),
\]
such that they satisfy the following viscous Cahn–Hilliard system:
\[
\begin{align*}
\partial_t \phi - \Delta \mu &= 0 & \text{a.e. in } Q, \quad (3.49) \\
\mu &= \tau \partial_t \phi - \Delta \phi + \xi + \pi(\phi) - f, \quad \xi \in \beta(\phi) & \text{a.e. in } Q, \quad (3.50) \\
\partial_t \mu &= 0 & \text{a.e. on } \Sigma, \quad (3.51) \\
\phi|_{\Sigma} &= \psi & \text{a.e. on } \Sigma, \quad (3.52) \\
\partial_t \psi - \Delta \tau w &= 0 & \text{a.e. on } \Sigma, \quad (3.53) \\
w &= \sigma \partial_t \psi + \partial_t \phi - \Delta \tau \psi + \zeta + \pi_t(\psi) - g, \quad \zeta \in \beta_t(\psi) & \text{a.e. on } \Sigma, \quad (3.54) \\
\phi(0) &= \phi_0 & \text{a.e. in } \Omega, \quad \psi(0) = \psi_0 & \text{a.e. on } \Gamma. \quad (3.55)
\end{align*}
\]

**Proof.** Let \( \tau, \sigma \in (0, 1] \). Recalling (3.25), (3.41), (3.42) and (3.45), we deduce that there exist positive constants \( M_6, M_7 \), independent of \( h \in (0, h^*\]) \( \), \( \tau, \sigma, \varepsilon \in (0, 1] \), such that
\[
|\check{\mu}_h|_{L^2(0,T;V)} + |\check{w}_h|_{L^2(0,T;V^*)} + |\check{\xi}_h|_{L^2(0,T;H^1)} + |\check{\psi}_h|_{L^2(0,T;H^2)} + |\check{\phi}_h|_{L^2(0,T;H^1)} \\ + |\check{\phi}_h|_{L^2(0,T;H^2(\Omega))} + |\check{\psi}_h|_{L^2(0,T;H^2(\Gamma))} + |\partial_t \check{\phi}_h|_{L^2(0,T;H^1)} \leq M_6,
\]
\[
\sqrt{\tau} |\check{\mu}_h|_{L^2(0,T;W)} + \sqrt{\sigma} |\check{w}_h|_{L^2(0,T;H^2(\Gamma))} \leq M_7,
\]
for all \( h \in (0, h^*\]) \( \) and \( \varepsilon \in (0, 1] \), where also a comparison in (3.17), (3.19) and (3.21) has been used for (3.57). Next, using (3.14), (3.16), we observe that
\[
|\check{\phi}_h|_{L^\infty(0,T;V)} \leq \max\{ |\phi_0|_V, |\check{\phi}_h|_{L^\infty(0,T;V)} \} \leq \sqrt{M_1} + |\phi_0|_V,
\]
\[
\tau |\check{\phi}_h - \phi_0|_{L^2(0,T;H)}^2 = \frac{h^2}{3} \tau |\partial_t \check{\phi}_h|_{L^2(0,T;H)}^2 \leq \frac{h^2}{3} M_1 \leq \frac{1}{3} M_1,
\]
\[
|\partial_t \check{\phi}_h|_{L^2(0,T;V^*)} \leq 3 \sqrt{M_1},
\]
have that

\[ |\hat{\mu}_h|_{L^2(0,T;V)}^2 \leq \frac{h}{2} |\mu_0|^2 + |\hat{\mu}_h|_{L^2(0,T;V)}^2 \leq M_0^2, \]

\[ |\hat{\nu}_h|_{L^\infty(0,T;V_T^\ast)} \leq \max\{ |\psi_0|_{V_T^\ast}, |\psi_0|_{L^\infty(0,T;V_T^\ast)} \} \leq \sqrt{M_1} + |\psi_0|_{V_T}, \]

\[ \sigma |\hat{\psi}_h - \bar{\psi}_h|_{L^2(0,T;H_T)}^2 = 2 |\sigma_\psi\hat{\psi}_h|_{L^2(0,T;H_H)}^2 \leq \frac{h^2}{3} M_1 \leq \frac{1}{3} M_1, \]

\[ |h\partial_\nu \hat{\omega}_h|_{L^2(0,T;H_T)} + |\partial_\nu \hat{h}_\psi|_{L^2(0,T;V_T^\ast)} \leq 3 \sqrt{M_1}, \]

\[ |\bar{\omega}_h|_{L^2(0,T;V_T^\ast)} \leq \frac{h}{2} |\omega_0|^2 + |\bar{\omega}_h|_{L^2(0,T;V_T^\ast)} \leq M_0^2, \]

for all \( h \in (0, h^\ast] \) and \( \varepsilon \in (0, 1] \). Then, there exist functions \( \phi_\varepsilon, \mu_\varepsilon, \psi_\varepsilon, w_\varepsilon \) and a subsequence \( \{h_k\}_{k \in \mathbb{N}} \) of \( h \to 0 \) such that

\[ \hat{\phi}_{h_k} \to \phi_\varepsilon \text{ weakly star in } H^1(0, T; V^*) \cap L^\infty(0, T; V), \]

and strongly in \( C([0, T]; H) \),

\[ \tau \hat{\phi}_{h_k} \to \tau \phi_\varepsilon \text{ weakly in } H^1(0, T; H), \]

\[ \bar{\phi}_{h_k} \to \phi_\varepsilon \text{ weakly star in } L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)), \]

and strongly in \( L^2(0, T; H) \),

\[ \bar{\mu}_{h_k} \to \mu_\varepsilon \text{ weakly in } L^2(0, T; W), \]

\[ h_k \bar{\mu}_{h_k} \to 0 \text{ weakly in } H^1(0, T; H) \text{ and strongly in } L^2(0, T; V), \]

\[ \hat{\psi}_{h_k} \to \psi_\varepsilon \text{ weakly star in } H^1(0, T; V_T^* \cap L^\infty(0, T; V_T) \cap L^2(0, T; H^2(\Gamma)) \text{ and strongly in } C([0, T]; H_T), \]

\[ \sigma \hat{\psi}_{h_k} \to \sigma \psi_\varepsilon \text{ weakly in } H^1(0, T; H_T), \]

\[ \bar{\psi}_{h_k} \to \psi_\varepsilon \text{ weakly star in } L^\infty(0, T; V_T^* \cap L^2(0, T; H^2(\Gamma)) \text{ and strongly in } L^2(0, T; H_T), \]

\[ \bar{w}_{h_k} \to w_\varepsilon \text{ weakly in } L^2(0, T; H^2(\Gamma)), \]

\[ h_k \bar{w}_{h_k} \to 0 \text{ weakly in } H^1(0, T; H_T) \text{ and strongly in } L^2(0, T; V_T), \]

\[ \partial_\nu \bar{\phi}_{h_k} \to \partial_\nu \phi_\varepsilon \text{ weakly in } L^2(0, T; H_T) \]

as \( k \to +\infty \), where we applied the compactness results 32 Section 8, Corollary 4] and 35.10] to obtain the strong convergences. We recall that \( \tau \) and \( \sigma \) are positive and fixed in this limit procedure. Moreover, due to the Lipschitz continuity of \( \beta_\varepsilon, \pi, \beta_{T, \varepsilon} \) and \( \pi_T \), we have that

\[ \beta_\varepsilon(\bar{\phi}_{h_k}) \to \beta_\varepsilon(\phi_\varepsilon), \quad \pi(\bar{\phi}_{h_k}) \to \pi(\phi_\varepsilon) \quad \text{strongly in } L^2(0, T; H), \]

\[ \beta_{T, \varepsilon}(\bar{\psi}_{h_k}) \to \beta_{T, \varepsilon}(\psi_\varepsilon), \quad \pi_T(\bar{\psi}_{h_k}) \to \pi_T(\psi_\varepsilon) \quad \text{strongly in } L^2(0, T; H_T) \]

as \( k \to +\infty \). Besides, it is not difficult to check that

\[ f_h \to f \quad \text{strongly in } L^2(0, T; V), \]

\[ g_h \to g \quad \text{strongly in } L^2(0, T; V_T) \]
as $h \to 0$ (in Appendix) the argument is fully detailed. Based on all these convergence results, we can pass to the limit as $h_k \to 0$ in problem (3.17)--(3.23) and find that the quadruplet $(\phi_\varepsilon, \mu_\varepsilon, \psi_\varepsilon, w_\varepsilon)$ solves

$$
\partial_t \phi_\varepsilon - \Delta \mu_\varepsilon = 0 \quad \text{a.e. in } Q, \quad (3.58)
$$

$$
\mu_\varepsilon = \tau \partial_t \phi_\varepsilon - \Delta \phi_\varepsilon + \beta_\varepsilon(\phi_\varepsilon) + \pi(\phi_\varepsilon) - f \quad \text{a.e. in } Q, \quad (3.59)
$$

$$
\partial_\nu \mu_\varepsilon = 0 \quad \text{a.e. on } \Sigma, \quad (3.60)
$$

$$
(\phi_\varepsilon)|_\Gamma = \psi_\varepsilon \quad \text{a.e. on } \Sigma, \quad (3.61)
$$

$$
\partial_t \psi_\varepsilon - \Delta_\Gamma w_\varepsilon = 0 \quad \text{a.e. on } \Sigma, \quad (3.62)
$$

$$
w_\varepsilon = \partial_\nu \phi_\varepsilon + \sigma \partial_t \psi_\varepsilon - \Delta_\Gamma \psi_\varepsilon + \beta_{\Gamma,\varepsilon}(\psi_\varepsilon) + \pi_\Gamma(\psi_\varepsilon) - g \quad \text{a.e. on } \Sigma, \quad (3.63)
$$

$$
\phi_\varepsilon(0) = \phi_0 \quad \text{a.e. in } \Omega, \quad \psi_\varepsilon(0) = \psi_0 \quad \text{a.e. on } \Gamma. \quad (3.64)
$$

Moreover, by weak or weak star lower semicontinuity of norms, we see that the following estimates hold (see (3.29), (3.30), (3.31))

$$
|\partial_t \phi_\varepsilon|_{L^2(0,T;V^*)} \leq \liminf_{k \to \infty} |\partial_t \hat{\phi}_{h_k}|_{L^2(0,T;V^*)} \leq 3 \sqrt{M_1}, \quad (3.65)
$$

$$
|\phi_\varepsilon|_{L^\infty(0,T;V)} \leq \liminf_{k \to \infty} |\hat{\phi}_{h_k}|_{L^\infty(0,T;V)} \leq \sqrt{M_1} + |\phi_0|_V, \quad (3.66)
$$

$$
|\phi_\varepsilon|_{L^2(0,T;H^2(\Omega))} \leq \liminf_{k \to \infty} |\hat{\phi}_{h_k}|_{L^2(0,T;H^2(\Omega))} \leq M_6, \quad (3.67)
$$

$$
\sqrt{\tau} |\partial_t \phi_\varepsilon|_{L^2(0,T;H)} \leq \liminf_{k \to \infty} \sqrt{\tau} |\partial_t \hat{\phi}_{h_k}|_{L^2(0,T;H)} \leq \sqrt{M_1}, \quad (3.68)
$$

$$
|\mu_\varepsilon|_{L^2(0,T;V)} \leq \liminf_{k \to \infty} |\hat{\mu}_{h_k}|_{L^2(0,T;V)} \leq M_6, \quad (3.69)
$$

$$
\sqrt{\tau} |\mu_\varepsilon|_{L^2(0,T;W)} \leq \liminf_{k \to \infty} \sqrt{\tau} |\hat{\mu}_{h_k}|_{L^2(0,T;W)} \leq M_7, \quad (3.70)
$$

$$
|\partial_t \psi_\varepsilon|_{L^2(0,T;V^*)} \leq \liminf_{k \to \infty} |\partial_t \hat{\psi}_{h_k}|_{L^2(0,T;V^*)} \leq 3 \sqrt{M_1}, \quad (3.71)
$$

$$
|\psi_\varepsilon|_{L^\infty(0,T;V^*)} \leq \liminf_{k \to \infty} |\hat{\psi}_{h_k}|_{L^\infty(0,T;V^*)} \leq \sqrt{M_1} + |\psi_0|_{V^*}, \quad (3.72)
$$

$$
|\psi_\varepsilon|_{L^2(0,T;H^2(\Gamma))} \leq \liminf_{k \to \infty} |\hat{\psi}_{h_k}|_{L^2(0,T;H^2(\Gamma))} \leq M_6, \quad (3.73)
$$

$$
\sqrt{\sigma} |\partial_t \psi_\varepsilon|_{L^2(0,T;H^*)} \leq \liminf_{k \to \infty} \sqrt{\sigma} |\partial_t \hat{\psi}_{h_k}|_{L^2(0,T;H^*)} \leq \sqrt{M_1}, \quad (3.74)
$$

$$
|w_\varepsilon|_{L^2(0,T;V)} \leq \liminf_{k \to \infty} |\hat{w}_{h_k}|_{L^2(0,T;V)} \leq M_6, \quad (3.75)
$$

$$
\sqrt{\sigma} |w_\varepsilon|_{L^2(0,T;H^2(\Gamma))} \leq \liminf_{k \to \infty} \sqrt{\sigma} |\hat{w}_{h_k}|_{L^2(0,T;H^2(\Gamma))} \leq M_7, \quad (3.76)
$$

$$
|\partial_\nu \phi_\varepsilon|_{L^2(0,T;H^*)} \leq \liminf_{k \to \infty} |\partial_\nu \hat{\phi}_{h_k}|_{L^2(0,T;H^*)} \leq M_6, \quad (3.77)
$$

$$
|\beta_\varepsilon(\phi_\varepsilon)|_{L^2(0,T;H)} = \lim_{k \to \infty} |\beta_\varepsilon(\hat{\phi}_{h_k})|_{L^2(0,T;H)} \leq M_6, \quad (3.78)
$$

$$
|\beta_{\Gamma,\varepsilon}(\psi_\varepsilon)|_{L^2(0,T;H^*)} = \lim_{k \to \infty} |\beta_{\Gamma,\varepsilon}(\hat{\psi}_{h_k})|_{L^2(0,T;H^*)} \leq M_6. \quad (3.79)
$$

Due to the uniform estimates (3.65)–(3.79), we are able to pass to the limit along a subsequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ of $\varepsilon$, in the problem (3.58)--(3.64) by finding elements

$$
\phi := \phi_{r,\sigma}, \quad \mu := \mu_{r,\sigma}, \quad \xi := \xi_{r,\sigma}, \quad \psi := \psi_{r,\sigma}, \quad w := w_{r,\sigma}, \quad \zeta := \zeta_{r,\sigma}
$$
such that
\[
\phi_{\varepsilon_k} \to \phi \quad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \\
\text{and strongly in } C([0,T];H) \cap L^2(0,T;V),
\]
\[
\tau \phi_{\varepsilon_k} \to \tau \phi \quad \text{weakly in } H^1(0,T;H),
\]
\[
\mu_{\varepsilon_k} \to \mu \quad \text{weakly in } L^2(0,T;W),
\]
\[
\beta_{\varepsilon_k}(\phi_{\varepsilon_k}) \to \xi \quad \text{weakly in } L^2(0,T;H),
\]
\[
\psi_{\varepsilon_k} \to \psi \quad \text{weakly star in } H^1(0,T;V_\Gamma^*) \cap L^\infty(0,T;V_\Gamma) \cap L^2(0,T;H^2(\Gamma)) \\
\text{and strongly in } C([0,T];H_\Gamma) \cap L^2(0,T;V_\Gamma),
\]
\[
\sigma \psi_{\varepsilon_k} \to \sigma \psi \quad \text{weakly in } H^1(0,T;H_\Gamma),
\]
\[
w_{\varepsilon_k} \to w \quad \text{weakly in } L^2(0,T;H^2(\Gamma)),
\]
\[
\beta_{\Gamma,\varepsilon_k}(\psi_{\varepsilon_k}) \to \zeta \quad \text{weakly in } L^2(0,T;H_\Gamma),
\]
\[
\partial_\nu \phi_{\varepsilon_k} \to \partial_\nu \phi \quad \text{weakly in } L^2(0,T;H_\Gamma)
\]
as \(k \to +\infty\), due to the compactness theorems again. Now, we observe that \(\xi \in \beta(\phi)\) a.e. in \(Q\) and \(\zeta \in \beta(\psi)\) a.e. on \(\Sigma\), due to the maximal monotonicity of \(\beta\) and \(\beta_\Gamma\), and the weak-strong convergence for \(\beta_{\varepsilon_k}(\phi_{\varepsilon_k})\) and \(\phi_{\varepsilon_k}\) in \(L^2(0,T;H)\), and for \(\beta_{\Gamma,\varepsilon_k}(\psi_{\varepsilon_k})\) and \(\psi_{\varepsilon_k}\) in \(L^2(0,T;H_\Gamma)\), respectively. Finally, we observe that
\[
\pi(\phi_{\varepsilon_k}) \to \pi(\phi) \quad \text{strongly in } C([0,T];H),
\]
\[
\pi_\Gamma(\psi_{\varepsilon_k}) \to \pi_\Gamma(\psi) \quad \text{strongly in } C([0,T];H_\Gamma).
\]
Thus, we can pass to the limit as \(k \to \infty\) in the regularized problem \(3.58-3.64\) to obtain the viscous Cahn–Hilliard system \(3.49-3.55\). \(\square\)

3.3. From viscous to pure Cahn–Hilliard system. As a summary of the previous subsection, we can find a sextuplet \((\phi_{\tau,\sigma}, \mu_{\tau,\sigma}, \xi_{\tau,\sigma}, \psi_{\tau,\sigma}, w_{\tau,\sigma}, \zeta_{\tau,\sigma})\) of functions, depending on \(\tau, \sigma \in (0,1]\), such that it satisfies the viscous Cahn–Hilliard system \(3.49-3.55\). Moreover, in \(3.65-3.79\), from weak or weak star lower semicontinuity of norms, we also know that there exists a positive constant \(M_8\), independent of \(\tau, \sigma \in (0,1]\), such that
\[
|\phi_{\tau,\sigma}|_{H^1(0,T;V^*)} + |\phi_{\tau,\sigma}|_{L^\infty(0,T;V)} + |\phi_{\tau,\sigma}|_{L^2(0,T;H^2(\Omega))} + \sqrt{\tau} |\phi_{\tau,\sigma}|_{H^1(0,T;H)} \\
+ |\mu_{\tau,\sigma}|_{L^2(0,T;V)} + \sqrt{\tau} |\mu_{\tau,\sigma}|_{L^2(0,T;W)} + |\psi_{\tau,\sigma}|_{H^1(0,T;V_\Gamma^*)} + |\psi_{\tau,\sigma}|_{L^\infty(0,T;V_\Gamma)} \\
+ |\psi_{\tau,\sigma}|_{L^2(0,T;V_\Gamma^*)} + \sqrt{\sigma} |\psi_{\tau,\sigma}|_{H^1(0,T;H_\Gamma)} + |w_{\tau,\sigma}|_{L^2(0,T;V_\Gamma)} + \sqrt{\sigma} |w_{\tau,\sigma}|_{L^2(0,T;H^2(\Gamma))} \\
+ |\partial_\nu \phi_{\tau,\sigma}|_{L^2(0,T;H_\Gamma)} + |\xi_{\tau,\sigma}|_{L^2(0,T;H)} + |\zeta_{\tau,\sigma}|_{L^2(0,T;H_\Gamma)} \leq M_8.
\] (3.80)

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. We obtain from \(3.49, 3.51\) and \(3.55\), the following variational formulations:
\[
\langle \partial_\varepsilon \phi_{\tau,\sigma}, z \rangle_{V^*,V} + \int_\Omega \nabla \mu_{\tau,\sigma} \cdot \nabla z \, dx = 0 \quad \text{for all } z \in V,
\] (3.81)
\[
\langle \partial_\varepsilon \psi_{\tau,\sigma}, z_\Gamma \rangle_{V^*_\Gamma,V} + \int_\Gamma \nabla \Gamma w_{\tau,\sigma} \cdot \nabla \Gamma z_\Gamma \, d\Gamma = 0 \quad \text{for all } z_\Gamma \in V_\Gamma,
\] (3.82)
a.e. in \((0,T)\). At this point, we can pass to the limit as either \(\tau \to 0\) or \(\sigma \to 0\), or both \(\tau, \sigma \to 0\) in order to obtain a partially viscous Cahn–Hilliard system or a pure Cahn–Hilliard system at the limit. Let us detail only the last case with \((\tau, \sigma) \to (0, 0)\) along a joint subsequence \((\tau_k, \sigma_k)\). We see that there exists a sextuplet \((\phi, \mu, \xi, \psi, w, \zeta)\) such that

\[
\begin{align*}
\phi_{\tau_k, \sigma_k} &\to \phi \quad \text{weakly star in } H^1(0, T; \mathcal{V}^*) \cap L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; H^2(\Omega)) \\
&\quad \text{and strongly in } C([0, T]; H) \cap L^2(0, T; \mathcal{V}), \\
\tau_k \phi_{\tau_k, \sigma_k} &\to 0 \quad \text{strongly in } H^1(0, T; \mathcal{H}), \\
\mu_{\tau_k, \sigma_k} &\to \mu \quad \text{weakly in } L^2(0, T; \mathcal{V}), \\
\xi_{\tau_k, \sigma_k} &\to \xi \quad \text{weakly in } L^2(0, T; \mathcal{H}), \\
\psi_{\tau_k, \sigma_k} &\to \psi \quad \text{weakly star in } H^1(0, T; \mathcal{V}^*_\Gamma) \cap L^\infty(0, T; \mathcal{V}_\Gamma) \cap L^2(0, T; H^2(\Gamma)) \\
&\quad \text{and strongly in } C([0, T]; \mathcal{H}_\Gamma) \cap L^2(0, T; \mathcal{V}_\Gamma), \\
\sigma_k \psi_{\tau_k, \sigma_k} &\to 0 \quad \text{strongly in } H^1(0, T; \mathcal{H}_\Gamma), \\
w_{\tau_k, \sigma_k} &\to w \quad \text{weakly in } L^2(0, T; \mathcal{V}_\Gamma), \\
\zeta_{\tau_k, \sigma_k} &\to \zeta \quad \text{weakly in } L^2(0, T; \mathcal{H}_\Gamma), \\
\partial_\nu \phi_{\tau_k, \sigma_k} &\to \partial_\nu \phi \quad \text{weakly in } L^2(0, T; \mathcal{H}_\Gamma)
\end{align*}
\]

as \(k \to +\infty\). Different from the previous subsection, we can pass to the limit in

\[
\zeta_{\tau_k, \sigma_k} \in \beta(\phi_{\tau_k, \sigma_k}) \quad \text{a.e. in } Q, \quad \zeta_{\tau_k, \sigma_k} \in \beta_\Gamma(\psi_{\tau_k, \sigma_k}) \quad \text{a.e. in } \Sigma
\]

just using the demi-closedness of \(\beta\) and \(\beta_\Gamma\), respectively, to obtain the same inclusions at the limit. To complete this limiting procedure, we pass to the limit in \(\eqref{3.50}, \eqref{3.52}, \eqref{3.54}, \eqref{3.55}, \eqref{3.81}, \eqref{3.82}\) to obtain \(\eqref{2.8} - \eqref{2.13}\). Hence, we arrive at the conclusion. \(\square\)

Let us remark that, if we let only one of the parameters \(\tau, \sigma\) go to \(0\), then we also have the convergence

\[
\mu_{\tau, \sigma_k} \to \mu_\tau \quad \text{weakly in } L^2(0, T; \mathcal{W}) \quad \text{if only } \sigma_k \to 0,
\]

or the convergence

\[
w_{\tau_k, \sigma} \to w_\sigma \quad \text{weakly in } L^2(0, T; H^2(\Gamma)) \quad \text{if only } \tau_k \to 0.
\]

In these cases, we can keep the smoothness of the time derivative, more precisely, \(\phi_\tau \in H^1(0, T; \mathcal{H})\) if \(\tau > 0\), or \(\psi_\sigma \in H^1(0, T; \mathcal{H}_\Gamma)\) if \(\sigma > 0\), respectively.

**Proof of Theorem 2.2.** We now prove a continuous dependence estimates with two weak solutions

\[
(\phi^{(i)}, \mu^{(i)}, \xi^{(i)}, \psi^{(i)}, w^{(i)}, \zeta^{(i)}) \quad \text{for } i = 1, 2,
\]

corresponding to the initial data

\[
(\phi_0^{(i)}, \psi_0^{(i)}) \in \mathcal{V}^* \times \mathcal{V}^*_\Gamma \quad \text{for } i = 1, 2,
\]

satisfying \(\eqref{2.17}, \eqref{2.13}\), and the sources

\[
(f^{(i)}, g^{(i)}) \in L^2(0, T; \mathcal{V}^*) \times L^2(0, T; \mathcal{V}^*_\Gamma) \quad \text{for } i = 1, 2.
\]
We take the difference of (2.8) and choose \( z := 1 \) to obtain that
\[
\langle \partial_t (\phi^{(1)} - \phi^{(2)}), 1 \rangle_{V^*, V} = 0
\]
a.e. in \((0, T)\), whence
\[
\langle \phi^{(1)}(t) - \phi^{(2)}(t), 1 \rangle_{V^*, V} = \langle \phi^{(1)}_0 - \phi^{(2)}_0, 1 \rangle_{V^*, V} = 0
\]
for all \( t \in [0, T] \). Then, we can take \( z := \mathcal{N}_T(\phi^{(1)} - \phi^{(2)}) \) as a test function in the difference of (2.8) and obtain
\[
\frac{1}{2} \langle \phi^{(1)}(t) - \phi^{(2)}(t), \frac{1}{2} \rangle_{V^*, V} + \int_0^t \int_{\Omega} (\mu^{(1)} - \mu^{(2)}) (\phi^{(1)} - \phi^{(2)}) \, dx \, ds = \frac{1}{2} \langle \phi^{(1)}_0 - \phi^{(2)}_0, \frac{1}{2} \rangle_{V^*, V} \tag{3.83}
\]
for all \( t \in [0, T] \). By operating on the difference of (2.11) in the same way, that is, \( z_T := 1 \) first and \( z_T := \mathcal{N}_T(\psi^{(1)} - \psi^{(2)}) \) second, we obtain the similar formula
\[
\frac{1}{2} \langle \psi^{(1)}(t) - \psi^{(2)}(t), \frac{1}{2} \rangle_{V^*, V} + \int_0^t \int_{\Gamma} (\gamma^{(1)} - \gamma^{(2)}) (\psi^{(1)} - \psi^{(2)}) \, d\Gamma \, ds = \frac{1}{2} \langle \psi^{(1)}_0 - \psi^{(2)}_0, \frac{1}{2} \rangle_{V^*, V} \tag{3.84}
\]
for all \( t \in [0, T] \). Next, we multiply the difference of the equalities in (2.9) by \( \phi^{(1)} - \phi^{(2)} \) and integrate the resultant with respect to space and time. Using (2.10) and (2.12), we infer that
\[
\int_0^t \int_{\Omega} (\mu^{(1)} - \mu^{(2)}) (\phi^{(1)} - \phi^{(2)}) \, dx \, ds + \int_0^t \int_{\Omega} (w^{(1)} - w^{(2)}) (\psi^{(1)} - \psi^{(2)}) \, d\Gamma \, ds
\]
\[
= \int_0^t \int_{\Omega} |\nabla (\phi^{(1)} - \phi^{(2)})|^2 \, dx \, ds + \int_0^t \int_{\Gamma} |\nabla (\psi^{(1)} - \psi^{(2)})|^2 \, d\Gamma \, ds
\]
\[
+ \int_0^t \int_{\Omega} (\zeta^{(1)} - \zeta^{(2)}) (\phi^{(1)} - \phi^{(2)}) \, dx \, ds + \int_0^t \int_{\Gamma} (\zeta^{(1)} - \zeta^{(2)}) (\psi^{(1)} - \psi^{(2)}) \, d\Gamma \, ds
\]
\[
+ \int_0^t \int_{\Omega} (\pi (\phi^{(1)} - \pi (\phi^{(2)})) (\phi^{(1)} - \phi^{(2)}) \, dx \, ds - \int_0^t \langle f^{(1)} - f^{(2)}, \phi^{(1)} - \phi^{(2)} \rangle_{V^*, V} \, ds
\]
\[
+ \int_0^t \int_{\Gamma} (\pi_T (\psi^{(1)} - \pi_T (\psi^{(2)})) (\psi^{(1)} - \psi^{(2)}) \, d\Gamma \, ds - \int_0^t \langle g^{(1)} - g^{(2)}, \psi^{(1)} - \psi^{(2)} \rangle_{V^*, V} \, ds,
\]
for all \( t \in [0, T] \). Then, we take the sum of (3.83), (3.84) and combine with the above equality. Thanks to the monotonicity of \( \beta, \beta_T \), the Lipschitz continuity of \( \pi, \pi_T \), and the Poincaré–Wirtinger inequality, we deduce that
\[
|\phi^{(1)}(t) - \phi^{(2)}(t)|_{V^*, V}^2 + |\psi^{(1)}(t) - \psi^{(2)}(t)|_{V^*, V}^2
\]
\[
\leq |\phi^{(1)}_0 - \phi^{(2)}_0|_{V^*, V}^2 + |\psi^{(1)}_0 - \psi^{(2)}_0|_{V^*, V}^2 + 2L \int_0^t \int_{\Omega} |\phi^{(1)} - \phi^{(2)}|^2 \, dx \, ds
\]
\[
+ 2L \int_0^t \int_{\Gamma} |\psi^{(1)} - \psi^{(2)}|^2 \, d\Gamma \, ds + \frac{1}{2} \int_0^t \int_{\Omega} |\phi^{(1)} - \phi^{(2)}|^2 \, dx \, ds + \frac{C_P}{2} |f^{(1)} - f^{(2)}|_{L^2(0, T; V^*)}^2
\]
\[
+ \frac{1}{2} \int_0^t \int_{\Gamma} |\psi^{(1)} - \psi^{(2)}|^2 \, d\Gamma \, ds + \frac{C_P}{2} |g^{(1)} - g^{(2)}|_{L^2(0, T; V^*)}^2,
\]
for all \( t \in [0, T] \). Here, we observe that

\[
2L \int_0^t \int_\Omega |\phi^{(1)} - \phi^{(2)}|^2 \, dx \, ds
\]

\[
= 2L \int_0^t \int_\Omega \nabla N_\Omega (\phi^{(1)} - \phi^{(2)}) \cdot \nabla (\phi^{(1)} - \phi^{(2)}) \, dx \, ds
\]

\[
\leq \frac{1}{2} \int_0^t \int_\Omega |\nabla (\phi^{(1)} - \phi^{(2)})|^2 \, dx \, ds + 2L^2 \int_0^t \int_\Omega |\nabla N_\Omega (\phi^{(1)} - \phi^{(2)})|^2 \, dx \, ds
\]

\[
= \frac{1}{2} \int_0^t |\phi^{(1)} - \phi^{(2)}|^2 \, d\Gamma ds + 2L^2 \int_0^t |\phi^{(1)} - \phi^{(2)}|^2 \, ds,
\]

and similarly,

\[
2L_T \int_0^t \int_\Gamma |\psi^{(1)} - \psi^{(2)}|^2 \, d\Gamma ds \leq \frac{1}{2} \int_0^t \int_\Gamma |\psi^{(1)} - \psi^{(2)}|^2 \, ds + 2L_T^2 \int_0^t |\psi^{(1)} - \psi^{(2)}|^2 \, ds,
\]

for all \( t \in [0, T] \). Therefore, applying the Gronwall lemma and invoking the equivalences of norms, we conclude the proof of Theorem 2.2. As an immediate sequence, the continuous dependence implies the uniqueness of the weak solution obtained in Theorem 2.1. \( \square \)

The continuous dependence estimate can be extended to the viscous or partially viscous cases, with the following additional assumptions: there exists a constant \( C \) such that the inequality

\[
|\phi^{(1)} - \phi^{(2)}|_{C([0,T];V^*)} + |\psi^{(1)} - \psi^{(2)}|_{C([0,T];V^*)} + |\phi^{(1)} - \phi^{(2)}|_{L^2(0,T;V)} + |\psi^{(1)} - \psi^{(2)}|_{L^2(0,T;V_T)}
\]

\[
+ \sqrt{\tau} |\phi^{(1)} - \phi^{(2)}|_{C([0,T];H)} + \sqrt{\sigma} |\psi^{(1)} - \psi^{(2)}|_{C([0,T];H_T)}
\]

\[
\leq C \left\{ |\phi^{(1)} - \phi^{(2)}|_{V^*} + |\psi^{(1)} - \psi^{(2)}|_{V^*_T} + |f^{(1)} - f^{(2)}|_{L^2(0,T;V^*)} + |g^{(1)} - g^{(2)}|_{L^2(0,T;V^*_T)}
\right.
\]

\[
+ \sqrt{\tau} |\phi^{(1)} - \phi^{(2)}|_H + \sqrt{\sigma} |\psi^{(1)} - \psi^{(2)}|_{H_T}\right\}
\]

holds for \( \tau \geq 0 \) and \( \sigma \geq 0 \).

### 4. Existence of Strong Solution

In this section, we establish a regularity result, which leads to the existence of a strong solution in the case of the pure Cahn–Hilliard system (2.1)–(2.7).

Now, we point out the additional assumptions we need on the given data:

(A6) \( f \in H^1(0, T; H) \) and \( g \in H^1(0, T; H_T) \);

(A7) \( -\Delta \phi_0 + \beta_\epsilon (\phi_0) + \pi(\phi_0) - f(0) \) remains bounded in \( V \) as \( \epsilon \to 0 \), and \( \partial_N \phi_0 - \Delta \psi_0 + \beta_{\Gamma,\epsilon}(\psi_0) + \pi(\psi_0) - g(0) \) remains bounded in \( V_\Gamma \) as \( \epsilon \to 0 \).

Our third result of this paper is related to the existence of strong solutions:

**Theorem 4.1.** Under assumptions (A1)–(A7), the unique weak solution of problem (2.1)–(2.7) obtained in Theorem 2.1 is a strong one in the sense of Definition 2.2.
Proof. Let us take the difference of equations \((3.1)\) written for \(n\) and \(n - 1\). Then multiplying the resultant by
\[
z := \mathcal{N}_\Omega(\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n)
\]
and integrating the resultant over \(\Omega\), we obtain
\[
\frac{1}{h} \int_{\Omega} (\phi_{n+1} + h\mu_{n+1} - 2\phi_n - 2h\mu_n + \phi_{n-1} + h\mu_{n-1}) \, z \, dx
\]
\[
+ \int_{\Omega} \nabla(\mu_{n+1} - \mu_n) \cdot \nabla z \, dx = 0,
\]
whence
\[
\frac{1}{2h} |\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n|^2_{V_0} - \frac{1}{2h} |\phi_n + h\mu_n - \phi_{n-1} - h\mu_{n-1}|^2_{V_0}
\]
\[
+ \frac{1}{2h} |\phi_{n+1} + h\mu_{n+1} - 2\phi_n - 2h\mu_n + \phi_{n-1} + h\mu_{n-1}|^2_{V_0}
\]
\[
+ \int_{\Omega} (\mu_{n+1} - \mu_n)(\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n) \, dx = 0,
\]
for all \(n = 1, \ldots, N - 1\). Next, we use \((3.2)\) to derive that
\[
\int_{\Omega} (\mu_{n+1} - \mu_n)(\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n) \, dx
\]
\[
= \frac{\tau}{2h} |\phi_{n+1} - \phi_n|^2_H - \frac{\tau}{2h} |\phi_n - \phi_{n-1}|^2_H + \frac{\tau}{2h} |\phi_{n+1} - 2\phi_n + \phi_{n-1}|^2_H
\]
\[
+ \int_{\Omega} |\nabla(\phi_{n+1} - \phi_n)|^2 \, dx - \int_{\Gamma} \partial_\nu(\phi_{n+1} - \phi_n)(\psi_{n+1} - \psi_n) \, d\Gamma
\]
\[
+ \int_{\Omega} (\beta_e(\phi_{n+1}) - \beta_e(\phi_n))(\phi_{n+1} - \phi_n) \, dx + \int_{\Omega} (\pi(\phi_{n+1}) - \pi(\phi_n))(\phi_{n+1} - \phi_n) \, dx
\]
\[
- \int_{\Omega} (f_n - f_{n-1})(\phi_{n+1} - \phi_n) \, dx + h \int_{\Omega} |\mu_{n+1} - \mu_n|^2 \, dx,
\]
for all \(n = 0, 1, \ldots, N - 1\). Next, in order to treat the fifth term on the right hand side of \((4.2)\), we recall equation \((3.6)\) and infer that
\[
- \int_{\Gamma} \partial_\nu(\phi_{n+1} - \phi_n)(\psi_{n+1} - \psi_n) \, d\Gamma
\]
\[
= - \int_{\Gamma} (w_{n+1} - w_n)(\psi_{n+1} - \psi_n) \, d\Gamma + \frac{\sigma}{2h} |\psi_{n+1} - \psi_n|^2_{H_T} - \frac{\sigma}{2h} |\psi_n - \psi_{n-1}|^2_{H_T}
\]
\[
+ \frac{\sigma}{2h} |\psi_{n+1} - 2\psi_n + \psi_{n-1}|^2_{H_T} + \int_{\Gamma} |\nabla_T(\psi_{n+1} - \psi_n)|^2 \, d\Gamma
\]
\[
+ \int_{\Gamma} (\beta_{T,e}(\psi_{n+1}) - \beta_{T,e}(\psi_n))(\psi_{n+1} - \psi_n) \, d\Gamma
\]
\[
+ \int_{\Gamma} (\pi_T(\psi_{n+1}) - \pi_T(\psi_n))(\psi_{n+1} - \psi_n) \, d\Gamma
\]
\[
- \int_{\Gamma} (g_n - g_{n-1})(\psi_{n+1} - \psi_n) \, d\Gamma,
\]
for all $n = 0, 1, \ldots, N - 1$. We now exploit (3.5) to discuss the first term on the right hand side of (4.3). Taking the test function

$$z_{\Gamma} := N_{\Gamma} (\psi_{n+1} + hw_{n+1} - \psi_n - hw_n),$$

we have that

$$\frac{1}{h} \int_{\Gamma} (\psi_{n+1} + hw_{n+1} - \psi_n - hw_n - (\psi_n + hw_n - \psi_{n-1} - hw_{n-1})) z_{\Gamma} d\Gamma$$

$$= (\Delta_{\Gamma}(w_{n+1} - w_n), z_{\Gamma})_{H_{\Gamma}}$$

$$= - \int_{\Gamma} \nabla_{\Gamma}(w_{n+1} - w_n) \cdot \nabla z_{\Gamma} d\Gamma$$

$$= - \int_{\Gamma} (w_{n+1} - w_n)(\psi_{n+1} - \psi_n) d\Gamma - h \int_{\Gamma} |w_{n+1} - w_n|^2 d\Gamma,$$  \(4.4\)

for all $n = 1, \ldots, N - 1$.

Now, by collecting the identities (4.1)–(4.4) and using (A4), we deduce from Young’s inequality that

$$\frac{1}{2h} |\phi_{n+1} + h\mu_{n+1} - \phi_n - h\mu_n|_{V_{0,0}}^2 + \frac{1}{2h} |\psi_{n+1} + hw_{n+1} - \psi_n - hw_n|_{V_{0,0}}^2$$

$$- \frac{1}{2h} |\phi_n + h\mu_n - \phi_{n-1} - h\mu_{n-1}|_{V_{0,0}}^2 - \frac{1}{2h} |\psi_n + hw_n - \psi_{n-1} - hw_{n-1}|_{V_{0,0}}^2$$

$$+ \frac{1}{2h} |\phi_{n+1} + h\mu_{n+1} - 2\phi_n - 2h\mu_n + \phi_{n-1} + h\mu_{n-1}|_{V_{0,0}}^2$$

$$+ \frac{1}{2h} |\psi_{n+1} + hw_{n+1} - 2\psi_n - 2hw_n + \psi_{n-1} + hw_{n-1}|_{V_{0,0}}^2$$

$$+ \frac{\tau}{2h} |\phi_{n+1} - \phi_n|^2_H + \frac{\sigma}{2h} |\psi_{n+1} - \psi_n|^2_H - \frac{\tau}{2h} |\phi_n - \phi_{n-1}|^2_H - \frac{\sigma}{2h} |\psi_n - \psi_{n-1}|^2_H$$

$$+ \frac{\tau}{2h} |\phi_{n+1} - 2\phi_n + \phi_{n-1}|^2_H + \frac{\sigma}{2h} |\psi_{n+1} - 2\psi_n + \psi_{n-1}|^2_H + \int_{\Omega} |\nabla (\phi_{n+1} - \phi_n)|^2 dx$$

$$+ \int_{\Gamma} |\nabla_{\Gamma}(\psi_{n+1} - \psi_n)|^2 d\Gamma + h \int_{\Omega} |\mu_{n+1} - \mu_n|^2 dx + h \int_{\Gamma} |w_{n+1} - w_n|^2 d\Gamma$$

$$\leq (L + 1) \int_{\Omega} |\phi_{n+1} - \phi_n|^2 dx + \frac{1}{4} \int_{\Omega} |f_n - f_{n-1}|^2 dx + (L + 1) \int_{\Gamma} |\psi_{n+1} - \psi_n|^2 d\Gamma$$

$$+ \frac{1}{4} \int_{\Gamma} |g_n - g_{n-1}|^2 d\Gamma$$

for $n = 1, 2, \ldots, N - 1$. We divide the above inequality by $h$ and sum up for $n = 1$ to $n = m$, by finding that

$$\frac{1}{2} \left| \frac{\phi_{m+1} - \phi_m}{h} + \mu_{m+1} - \mu_m \right|_{V_{0,0}}^2 + \frac{1}{2} \left| \frac{\psi_{m+1} - \psi_m}{h} + w_{m+1} - w_m \right|_{V_{0,0}}^2$$

$$+ \frac{\tau}{2} \left| \frac{\phi_{m+1} - \phi_m}{h} \right|_H^2 + \frac{\sigma}{2} \left| \frac{\psi_{m+1} - \psi_m}{h} \right|_H^2 + \sum_{n=1}^{m} h \int_{\Omega} \left| \nabla \left( \frac{\phi_{n+1} - \phi_n}{h} \right) \right|^2 dx$$

$$+ \sum_{n=1}^{m} h \int_{\Gamma} \left| \nabla_{\Gamma} \left( \frac{\psi_{n+1} - \psi_n}{h} \right) \right|^2 d\Gamma + \sum_{n=1}^{m} |\mu_{n+1} - \mu_n|_{H}^2 + \sum_{n=1}^{m} |w_{n+1} - w_n|_{H_{\Gamma}}^2$$
\[ \leq \frac{1}{2} \left| \frac{\phi_1 - \phi_0}{h} + \mu_1 - \mu_0 \right|_{V_{0,\sigma}}^2 + \frac{1}{2} \left| \frac{\psi_1 - \psi_0}{h} + w_1 - w_0 \right|_{V_{\Gamma,0,\sigma}}^2 + \frac{\tau}{2} \left| \frac{\phi_1 - \phi_0}{h} \right|_H^2 \\
+ \frac{\sigma}{2} \left| \frac{\psi_1 - \psi_0}{h} \right|_{H_{\Gamma}}^2 + (L + 1) \sum_{n=1}^{m} \left| \frac{\phi_{n+1} - \phi_n}{h} \right|_{H_{\Gamma}}^2 + \frac{1}{4} \sum_{n=1}^{m} \left| \frac{f_n - f_{n-1}}{h} \right|_{H_{\Gamma}}^2 \]
\]
for all \( m = 1, 2, \ldots, N - 1 \). In order to estimate the first four terms on the right hand side of (4.5), we multiply (3.1) at \( n = 0 \) by
\[ \frac{1}{h} N_{\Omega}(\phi_1 + h\mu_1 - \phi_0 - h\mu_0) \]
and obtain
\[ \left| \frac{\phi_1 - \phi_0}{h} + \mu_1 - \mu_0 \right|_{V_{0,\sigma}}^2 + \int_{\Omega} \mu_1 \left( \frac{\phi_1 - \phi_0}{h} + \mu_1 - \mu_0 \right) \, dx = 0. \] (4.6)
From (3.2) and (3.7), it follows that
\[ \int_{\Omega} \mu_1 \left( \frac{\phi_1 - \phi_0}{h} + \mu_1 - \mu_0 \right) \, dx \\
= \tau \int_{\Omega} \left| \frac{\phi_1 - \phi_0}{h} \right|^2 \, dx + \int_{\Omega} \nabla \phi_1 \cdot \nabla \left( \frac{\phi_1 - \phi_0}{h} \right) \, dx - \int_{\Gamma} \partial_{\nu} \phi_1 \left( \frac{\psi_1 - \psi_0}{h} \right) \, d\Gamma \\
+ \int_{\Omega} (\beta_{\epsilon}(\phi_1) - \beta_{\epsilon}(\phi_0)) \left( \frac{\phi_1 - \phi_0}{h} \right) \, dx + \int_{\Omega} \beta_{\epsilon}(\phi_0) \left( \frac{\phi_1 - \phi_0}{h} \right) \, dx \\
+ \int_{\Omega} (\pi(\phi_1) - \pi(\phi_0)) \left( \frac{\phi_1 - \phi_0}{h} \right) \, dx + \int_{\Omega} \pi(\phi_0) \left( \frac{\phi_1 - \phi_0}{h} \right) \, dx \\
- \int_{\Omega} f_0 \left( \frac{\phi_1 - \phi_0}{h} \right) \, dx + \int_{\Omega} |\mu_1 - \mu_0|^2 \, dx. \] (4.7)
Next, we multiply (3.6) at \( n = 0 \) by \( (\psi_1 - \psi_0)/h \) to obtain
\[ - \int_{\Gamma} \partial_{\nu} \phi_1 \left( \frac{\psi_1 - \psi_0}{h} \right) \, d\Gamma \\
= - \int_{\Gamma} w_1 \left( \frac{\psi_1 - \psi_0}{h} \right) \, d\Gamma + \sigma \int_{\Gamma} \left| \frac{\psi_1 - \psi_0}{h} \right|^2 \, d\Gamma + \int_{\Gamma} \nabla \psi_1 \cdot \nabla \left( \frac{\psi_1 - \psi_0}{h} \right) \, d\Gamma \\
+ \int_{\Gamma} (\beta_{\Gamma,e}(\psi_1) - \beta_{\Gamma,e}(\psi_0)) \left( \frac{\psi_1 - \psi_0}{h} \right) \, d\Gamma + \int_{\Gamma} \beta_{\Gamma,e}(\psi_0) \left( \frac{\psi_1 - \psi_0}{h} \right) \, d\Gamma \\
+ \int_{\Gamma} (\pi_{\Gamma}(\psi_1) - \pi_{\Gamma}(\psi_0)) \left( \frac{\psi_1 - \psi_0}{h} \right) \, d\Gamma + \int_{\Gamma} \pi_{\Gamma}(\psi_0) \left( \frac{\psi_1 - \psi_0}{h} \right) \, d\Gamma \\
- \int_{\Gamma} g_0 \left( \frac{\psi_1 - \psi_0}{h} \right) \, d\Gamma. \] (4.8)
Besides, we go back to (3.5) at \( n = 0 \), multiply it by
\[ \frac{1}{h} N_{\Gamma}(\psi_1 + hw_1 - \psi_0 - hw_0), \]
and integrate the resultant over \( \Gamma \), to obtain
\[
\left\| \frac{\psi_1 - \psi_0}{h} + w_1 - w_0 \right\|_{V_{\Gamma,0}}^2 + \int_\Gamma w_1 \left( \frac{\psi_1 - \psi_0}{h} + w_1 - w_0 \right) \, d\Gamma = 0. \tag{4.9}
\]

Collecting (4.6)–(4.9) and applying the assumption (A7) along with the monotonicity of \( \beta_\varepsilon, \beta_{\Gamma,\varepsilon} \), the Lipschitz continuity of \( \pi, \pi_\Gamma \) and Young’s inequality, we deduce that there exists a positive constant \( \tilde{M}_0 \), independent of \( h \in (0, h^{**}] \), \( \tau, \sigma, \varepsilon \in (0, 1] \), such that
\[
\left\| \frac{\phi_1 - \phi_0}{h} + \mu_1 - \mu_0 \right\|_{V_0}^2 + \left\| \frac{\psi_1 - \psi_0}{h} + w_1 - w_0 \right\|_{V_{\Gamma,0}}^2 + \tau \int_\Omega \left\| \frac{\phi_1 - \phi_0}{h} \right\|^2 \, dx \\
+ \sigma \int_\Gamma \left\| \frac{\psi_1 - \psi_0}{h} \right\|^2 \, d\Gamma + h \int_\Omega \left\| \nabla \left( \frac{\phi_1 - \phi_0}{h} \right) \right\|^2 \, dx + h \int_\Gamma \left\| \nabla_\Gamma \left( \frac{\psi_1 - \psi_0}{h} \right) \right\|^2 \, d\Gamma \\
+ \int_\Omega |\mu_1 - \mu_0|^2 \, dx + \int_\Gamma |w_1 - w_0|^2 \, d\Gamma \\
\leq - \int_\Omega (-\Delta \phi_0 + \beta_\varepsilon(\phi_0) + \pi(\psi_0) - f_0) \left( \frac{\phi_1 - \phi_0}{h} \right) \, dx + Lh \left\| \frac{\phi_1 - \phi_0}{h} \right\|^2_H \\
- \int_\Gamma (\partial_\varepsilon \phi_0 - \Delta_\Gamma \psi_0 + \beta_{\Gamma,\varepsilon}(\psi_0) + \pi_\Gamma(\psi_0) - g_0) \left( \frac{\psi_1 - \psi_0}{h} \right) \, d\Gamma + L_{\Gamma} h \left\| \frac{\psi_1 - \psi_0}{h} \right\|^2_{H_\Gamma} \\
\leq \tilde{M}_0 + \left\| \frac{\phi_1 - \phi_0}{h} + \mu_1 - \mu_0 \right\|_{V_0}^2 + \frac{1}{2} |\mu_1 - \mu_0|^2_H + Lh \left\| \frac{\phi_1 - \phi_0}{h} \right\|^2_H \\
+ \frac{1}{2} \left\| \frac{\psi_1 - \psi_0}{h} + w_1 - w_0 \right\|_{V_{\Gamma,0}}^2 + \frac{1}{2} |w_1 - w_0|^2_{H_\Gamma} + L_{\Gamma} h \left\| \frac{\psi_1 - \psi_0}{h} \right\|^2_{H_\Gamma}. \tag{4.10}
\]

Then, we can add (4.5) and (4.10) to obtain that
\[
\left\| \frac{\phi_{m+1} - \phi_m}{h} + \mu_{m+1} - \mu_m \right\|_{V_0}^2 + \frac{1}{2} \left\| \frac{\psi_{m+1} - \psi_m}{h} + w_{m+1} - w_m \right\|_{V_{\Gamma,0}}^2 \\
+ \frac{\tau}{2} \left\| \frac{\phi_{m+1} - \phi_m}{h} \right\|^2_H + \frac{\sigma}{2} \left\| \frac{\psi_{m+1} - \psi_m}{h} \right\|^2_{H_\Gamma} + \frac{m}{2} \int_\Gamma \left\| \nabla_\Gamma \left( \frac{\psi_{n+1} - \psi_n}{h} \right) \right\|^2 \, d\Gamma \\
+ \sum_{n=0}^m h \int_\Gamma \left\| \nabla_\Gamma \left( \frac{\psi_{n+1} - \psi_n}{h} \right) \right\|^2 \, d\Gamma + \frac{1}{2} \sum_{n=0}^m |\mu_{n+1} - \mu_n|^2_H + \frac{1}{2} \sum_{n=0}^m |w_{n+1} - w_n|^2_{H_\Gamma} \\
\leq M_0 + (L + 1) \sum_{n=0}^m \left\| \frac{\phi_{n+1} - \phi_n}{h} \right\|^2_H + (L_{\Gamma} + 1) \sum_{n=0}^m \left\| \frac{\psi_{n+1} - \psi_n}{h} \right\|^2_{H_\Gamma}.
\]

for all \( m = 1, 2, \ldots, N - 1 \), where \( M_0 \) is a positive constant depending on \( |\partial_\varepsilon f|_{L^2(0,T;H)} \), \( |\partial_\varepsilon g|_{L^2(0,T;H_\Gamma)} \) and \( M_0 \). Finally, we apply the compactness inequality on the right hand side of that above inequality as follows:
\[
(L + 1) \sum_{n=0}^m \left\| \frac{\phi_{n+1} - \phi_n}{h} \right\|^2_H + (L_{\Gamma} + 1) \sum_{n=0}^m \left\| \frac{\psi_{n+1} - \psi_n}{h} \right\|^2_{H_\Gamma}.
for all $\delta > 0$. Observe now that by taking $\delta := 1/2$, the last two terms are already bounded due to (3.25). Thus, we conclude that there exists a positive constant $M_{10}$, independent of $h \in (0, h^*)$, $\tau, \sigma, \varepsilon \in (0, 1)$, such that

$$
|\partial_t \tilde{\phi}_h + h \partial_t \tilde{\mu}_h|_{L^\infty(0,T;V^*)} + |\partial_t \tilde{\psi}_h + h \partial_t \tilde{w}_h|_{L^\infty(0,T;V)} + \tau |\partial_t \tilde{\phi}_h|_{L^\infty(0,T;H)}
$$

$$
+ \sigma |\partial_t \tilde{\psi}_h|_{L^\infty(0,T;H^2)} + |\partial_t \tilde{\phi}_h|_{L^2(0,T;V)} + |\partial_t \tilde{\psi}_h|_{L^2(0,T;V^*)} + h |\partial_t \tilde{\mu}_h|_{L^2(0,T;H)}
$$

$$
+ h |\partial_t \tilde{w}_h|_{L^2(0,T;H)} + |h \partial_t \tilde{w}_h|_{L^\infty(0,T;H^2)} \leq M_{10}, \quad (4.11)
$$

for all $h \in (0, h^*)$. The subsequent estimates repeat the previous ones, that is, from Lemmas 3.2 to 3.6 as follows:

$$
\text{D} \quad \text{From (3.34), using (4.11) we infer that}
$$

$$
|\tilde{\mu}_h - m_{1\varepsilon}(\tilde{\mu}_h)|_{L^\infty(0,T;V)} + |\tilde{w}_h - m_{1\varepsilon}(\tilde{w}_h)|_{L^\infty(0,T;V^*)} \leq \Lambda_1(t),
$$

for all $h \in (0, h^*)$ and $\varepsilon \in (0, 1)$, with $\Lambda_1$ being defined by (3.36). Now $\Lambda_1$ is bounded in $L^\infty(0,T)$;

$$
\text{D} \quad \text{arguing as in (3.37)–(3.39) and checking the right hand side of (3.39), using (3.25) and (4.11) we arrive at (3.35) with $\Lambda_2 \in L^\infty(0,T)$;}
$$

$$
\text{D} \quad \text{we can repeat the estimates in Lemmas 3.3 and 3.4, so that we arrive at (3.40)–(3.42) with $\Lambda_3, \Lambda_4, \Lambda_5 \in L^\infty(0,T)$;}
$$

$$
\text{D} \quad \text{now, we consider the same elliptic system (3.40) and observe that we can derive (3.45) with $\Lambda_6 \in L^\infty(0,T)$;}
$$

$$
\text{D} \quad \text{instead of (3.56) and (3.57), here we derive from the above modifications the final estimates}
$$

$$
|\tilde{\mu}_h|_{L^\infty(0,T;V)} + |\tilde{w}_h|_{L^\infty(0,T;V^*)} + |\beta_\varepsilon(\tilde{\phi}_h)|_{L^\infty(0,T;H)} + |\beta_{\Gamma,\varepsilon}(\tilde{\psi}_h)|_{L^\infty(0,T;H^2)}
$$

$$
+ |\tilde{\phi}_h|_{L^\infty(0,T;H^2)} + |\tilde{\psi}_h|_{L^\infty(0,T;H^2)} + |\partial_{\nu}\tilde{\phi}_h|_{L^\infty(0,T;H^2)} \leq M_{11},
$$

$$
\sqrt{\tau} |\tilde{\mu}_h|_{L^\infty(0,T;W)} + \sqrt{\sigma} |\tilde{w}_h|_{L^\infty(0,T;H^2)} \leq M_{11},
$$

for all $h \in (0, h^*)$ and $\varepsilon \in (0, 1)$, where $M_{11}$ is a positive constant independent of $h \in (0, h^*)$, $\tau, \sigma, \varepsilon \in (0, 1)$.

Thus, we have obtained sufficient additional estimates that can be extended to the limit functions as $h_k \to 0$, by weak or weak star lower semicontinuity of norms. In particular, the approximate solution $(\phi_\varepsilon, \mu_\varepsilon, \psi_\varepsilon, w_\varepsilon)$ to problem (3.58)–(3.64), which is unique due to the continuous dependence estimate stated in Theorem 2.2, additionally satisfies (cf. (3.65)–(3.79))

$$
|\phi_\varepsilon|_{W^{1,\infty}(0,T;V^*)} + |\phi_\varepsilon|_{H^1(0,T;V)} + |\phi_\varepsilon|_{L^\infty(0,T;H^2)} + \sqrt{\tau} |\phi_\varepsilon|_{W^{1,\infty}(0,T;H)}
$$

$$
+ |\mu_\varepsilon|_{L^2(0,T;W)} + |\mu_\varepsilon|_{H^1(0,T;H^2)} + \sqrt{\tau} |\mu_\varepsilon|_{L^\infty(0,T;W)}
$$

$$
+ |\psi_\varepsilon|_{W^{1,\infty}(0,T;V^*)} + |\psi_\varepsilon|_{H^1(0,T;V)} + |\psi_\varepsilon|_{L^\infty(0,T;H^2)} + \sqrt{\sigma} |\psi_\varepsilon|_{W^{1,\infty}(0,T;H^2)}
$$
TRANSMISSION PROBLEM OF CAHN–HILLIARD TYPE

\[ + |w_\varepsilon|_{L^\infty(0,T;V^*)} + |w_\varepsilon|_{L^2(0,T;H^3(\Gamma))} + \sqrt{\sigma}|w_\varepsilon|_{L^\infty(0,T;H^2(\Gamma))} + |\partial_\nu \phi_\varepsilon|_{L^\infty(0,T;H')} + |\beta_\varepsilon(\phi_\varepsilon)|_{L^\infty(0,T;H^2)} + |\beta_\Gamma,\varepsilon(\psi_\varepsilon)|_{L^\infty(0,T;H')} \leq M_{12}, \]

for all \( \varepsilon \in (0,1] \), where \( M_{12} \) is a positive constant independent of \( \tau, \sigma, \varepsilon \in (0,1] \). Let us point out that, in order to obtain the above estimate, we have to make comparison of terms in (3.58) with (3.60) and (3.62), as well as to apply the elliptic regularity results. In this framework, the final regularity of the sextuplet \((\phi, \mu, \xi, \psi, w, \zeta)\) solving problem (2.1)–(2.7) is

\[ \phi \in W^{1,\infty}(0, T; V^*) \cap H^1(0, T; V) \cap L^\infty(0, T; H^2(\Omega)), \]
\[ \mu \in L^\infty(0, T; V) \cap L^2(0, T; W \cap H^3(\Omega)), \quad \xi \in L^\infty(0, T; H), \]
\[ \psi \in W^{1,\infty}(0, T; V^*_\Gamma) \cap H^1(0, T; V_\Gamma) \cap L^\infty(0, T; H^2(\Gamma)), \]
\[ w \in L^\infty(0, T; V_\Gamma) \cap L^2(0, T; H^3(\Gamma)), \quad \zeta \in L^\infty(0, T; H_\Gamma) \]

and under these regularities, (2.8) and (2.10) can be replaced by

\[ \partial_t \phi - \Delta \mu = 0 \quad \text{a.e. in } Q, \]
\[ \partial_\nu \mu = 0 \quad \text{a.e. on } \Sigma, \]
\[ \partial_t \psi - \Delta_\Gamma w = 0 \quad \text{a.e. on } \Sigma. \]

Moreover, in order to obtain the regularities for \( \mu \) and \( w \) stated above, we simply use the weak star lower semicontinuity property. Thus, we complete the proof of Theorem 4.1. \( \square \)

ACKNOWLEDGMENTS

This work was started during a visit of P. Colli and T. Fukao at the School of Mathematical Sciences of Fudan University: the contributed support and warm hospitality of the School are very gratefully acknowledged. P. Colli also acknowledges other support from the Italian Ministry of Education, University and Research (MIUR): Dipartimenti di Eccellenza Program (2018–2022) – Dept. of Mathematics “F. Casorati”, University of Pavia, and the GNAMPA (Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica). T. Fukao acknowledges the support from the JSPS KAKENHI Grant-in-Aid for Scientific Research(C), Japan Grant Number 17K05321. H. Wu acknowledges the support from the NNSFC Grant No. 11631011 and the Shanghai Center for Mathematical Sciences at Fudan University.

REFERENCES

[1] V. Barbu, Nonlinear Differential Equations of Monotone Types in Banach Spaces, Springer, London 2010.
[2] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland, Amsterdam, 1973.
[3] F. Brezzi and G. Gilardi, Chapters 1–3 in Finite Element Handbook, H. Kardestuncer and D. H. Norrie (Eds.), McGraw–Hill Book Co., New York, 1987.
[4] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system I. Interfacial free energy, J. Chem. Phys., 2 (1958), 258–267.
[5] L. Calatroni and P. Colli, Global solution to the Allen–Cahn equation with singular potentials and dynamic boundary conditions, Nonlinear Anal., 79 (2013), 12–27.
[6] L. Cherfils, S. Gatti, and A. Miranville, A variational approach to a Cahn–Hilliard model in a domain with nonpermeable walls, J. Math. Sci. (N.Y.), 189 (2013), 604–636.
[7] L. Cherfils and M. Petcu, A numerical analysis of the Cahn–Hilliard equation with non-permeable walls, Numer. Math., 128 (2014), 518–549.
[8] P. Colli and T. Fukao, Equation and dynamic boundary condition of Cahn–Hilliard type with singular potentials, Nonlinear Anal., 127 (2015), 413–433.
[9] P. Colli, G. Gilardi, and J. Sprekels, On the longtime behavior of a viscous Cahn–Hilliard system with convection and dynamic boundary conditions, J. Elliptic Parabol. Equ., 4 (2018), 327–347.
[10] P. Colli, G. Gilardi, and J. Sprekels, Well-posedness and regularity for a generalized fractional Cahn–Hilliard system, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., to appear (see also Preprint arXiv:1804.11290 [math.AP] (2018), 1–36).
[11] P. Colli and S. Kurima, Global existence for a phase separation system deduced from the entropy balance, Preprint arXiv:1901.10158 [math.AP] (2019), 1–40.
[12] T. Fukao and H. Wu, Separation property and convergence to equilibrium for the equation and dynamic boundary condition of Cahn–Hilliard type with singular potential, Preprint (2019).
[13] T. Fukao and N. Yamazaki, A boundary control problem for the equation and dynamic boundary condition of Cahn–Hilliard type, pp. 255–280 in “Solvability, Regularity, Optimal Control of Boundary Value Problems for PDEs”, Springer INdAM Series, Vol.22, Springer, 2017.
[14] T. Fukao, S. Yoshikawa, and S. Wada, Structure-preserving finite difference schemes for the Cahn–Hilliard equation with dynamic boundary conditions in the one-dimensional case, Commun. Pure Appl. Anal., 16 (2017), 1915–1938.
[15] C. Gal, A Cahn–Hilliard model in bounded domains with permeable walls, Math. Methods Appl. Sci., 29 (2006), 2009–2036.
[16] C. Gal, Robust family of exponential attractors for a conserved Cahn–Hilliard model with singularly perturbed boundary conditions, Commun. Pure Appl. Anal., 7 (2008), 819–836.
[17] C. Gal and H. Wu, Asymptotic behavior of a Cahn–Hilliard equation with Wentzell boundary conditions and mass conservation, Discrete Contin. Dyn. Syst., 22 (2008), 1041–1063.
[18] H. Garcke and P. Knopf, Weak solutions of the Cahn–Hilliard system with dynamic boundary conditions: a gradient flow approach, Preprint arXiv:1810.09817 [math.AP] (2018), 1–27.
[19] G. Gilardi, A. Miranville, and G. Schimperna, On the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions, Commun. Pure Appl. Anal., 8 (2009), 881–912.
[20] G. Gilardi and J. Sprekels, Asymptotic limits and optimal control for the Cahn–Hilliard system with convection and dynamic boundary conditions, Nonlinear Anal., 178 (2019), 1–31.
[21] G. R. Goldstein, A. Miranville, and G. Schimperna, A Cahn–Hilliard model in a domain with non-permeable walls, Phys. D, 240 (2011), 754–766.
[22] A. Grigor’yan, Heat Kernel and Analysis on Manifolds, American Mathematical Society, International Press, Boston, 2009.
[23] E. Hebey, Sobolev Spaces on Riemannian Manifolds, Lecture Notes in Math., 1635, Springer, Berlin, 1996.
[24] N. Kajiwara, Global well-posedness for a Cahn–Hilliard equation on bounded domains with permeable and non-permeable walls in maximal regularity spaces, Adv. Math. Sci. Appl., 27 (2018), 277–298.
[25] N. Kenmochi, M. Niezgódka, and I. Pawłow, Subdifferential operator approach to the Cahn–Hilliard equation with constraint, J. Differential Equations, 117 (1995), 320–354.
[26] M. Kubo, The Cahn–Hilliard equation with time-dependent constraint, Nonlinear Anal., 75 (2012), 5672–5685.
[27] J. L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod Gauthier-Villas, Paris, 1968.
[28] J. L. Lions and E. Magenes, Non-homogeneous Boundary Value Problems and Applications I, Springer-Verlag, New York-Heidelberg, 1972.
[29] C. Liu and H. Wu, An energetic variational approach for the Cahn–Hilliard equation with dynamic boundary conditions: model derivation and mathematical analysis, Arch. Ration. Mech. Anal., 233 (2019), 167–247.

[30] A. Miranville and S. Zelik, Robust exponential attractors for Cahn–Hilliard type equations with singular potentials, Math. Methods Appl. Sci., 27 (2004), 545–582.

[31] A. Miranville and S. Zelik, Exponential attractors for the Cahn–Hilliard equation with dynamical boundary conditions, Math. Meth. Appl. Sci., 28 (2005), 709–735.

[32] J. Simon, Compact sets in the spaces $L^p(0,T;B)$, Ann. Mat. Pura. Appl. (4), 146 (1987), 65–96.

[33] H. Wu, The Cahn–Hilliard equation with a new class of dynamic boundary conditions, pp. 117–131 in “Theory of Evolution Equation and Mathematical Analysis of Nonlinear Phenomena”, RIMS Kôkyûroku, 2090, Kyoto University, 2018.

Pierluigi Colli: Dipartimento di Matematica, Università degli Studi di Pavia, and Research Associate at the IMATI – C.N.R. Pavia, Via Ferrata 5, 27100 Pavia, Italy
E-mail address: pierluigi.colli@unipv.it

Takeshi Fukao: Department of Mathematics, Faculty of Education, Kyoto University of Education, 1 Fujinomori, Fukakusa, Fushimi-ku, Kyoto 612-8522 Japan
E-mail address: fukao@kyoyo-u.ac.jp

Hao Wu: School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Han Dan Road 220, Shanghai 200433, China; Key Laboratory of Mathematics for Nonlinear Science (Fudan University), Ministry of Education, Han Dan Road 220, Shanghai 200433, China
E-mail address: haowufd@fudan.edu.cn