SIMPLIFIED AND EQUIVALENT CHARACTERIZATIONS OF 
BANACH LIMIT FUNCTIONAL AND STRONG ALMOST 
CONVERGENCE

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This paper is dedicated to Prof. Lixin Xuan for his great encouragement.

Abstract. In this paper, we give simplified and equivalent characterizations of Banach limit functional, which is the minimum requirement to characterize strong almost convergence. With this machinery, we show that Hajduškuvič’s quasi-almost convergence is equivalent to strong almost convergence.

1. Introduction

Let $l^\infty$ be the Banach space of bounded sequences of real numbers $x := \{x(n)\}_{n=1}^\infty$ with supremum norm $\|x\|_\infty := \sup_n |x(n)|$. As an application of Hahn-Banach theorem, a Banach limit $L$ is a bounded linear functional on $l^\infty$, which satisfies the following properties:

(i) If $x = \{x(n)\}_{n=1}^\infty \in l^\infty$ and $x(n) \geq 0$, then $L(x) \geq 0$;

(ii) If $x = \{x(n)\}_{n=1}^\infty \in l^\infty$ and $Tx = \{x(2), x(3), \ldots\}$, where $T$ is the left-shift operator, then $L(x) = L(Tx)$;

(iii) $\|L\| = 1$;

(iv) If $x = \{x(n)\}_{n=1}^\infty \in c$, where $c$ is the Banach subspace of $l^\infty$ consisting of convergent sequences, then $L(x) = \lim_{n \to \infty} x(n)$.

Since the Hahn-Banach norm-preserving extension is not unique, there must be many Banach limits in the dual space of $l^\infty$, and usually different Banach limits have different values at the same element in $l^\infty$. However, there indeed exist sequences whose values of all Banach limits are the same. Condition (iv) is a trivial example. Besides that, there also exist nonconvergent sequences satisfying this property, for such examples please see [1] and [2]. In [3], G. G. Lorentz called a sequence $x = \{x(n)\}_{n=1}^\infty$ almost convergent, if all Banach limits of $x$, $L(x)$, are the same, and this unique Banach limit is called $F$-limit of $x$. In his paper, Lorentz proved the following criterion for almost convergent sequences:

Theorem 1.1. A sequence $x = \{x(n)\}_{n=1}^\infty \in l^\infty$ is almost convergent with $F$-limit $L(x)$ if and only if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{i+n-1} x(t) = L(x)$$

uniformly in $i$. 

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Similar to this theorem, recently D. Hajduković[4] and S. Shaw et al[5] generalized the concept of almost convergence to bounded sequences in normed vector space and bounded continuous vector-valued functions, respectively.

Suppose \((V, \| \cdot \|_V)\) is a complex normed vector space. Let \(l^\infty(V)\) be the normed vector space of bounded \(V\)-valued sequences \(x := \{x_n\}_{n=1}^\infty\) with supremum norm \(\|x\|_\infty := \sup_n \|x_n\|_V\). In particular, \(c(V)\) is the subspace of \(l^\infty(V)\), which consists of convergent \(V\)-valued sequences. For any \(v \in V\), let \(\bar{v} := \{v, v, \ldots\}\) denote the sequence with constant entry \(v\), clearly \(\bar{v} \in c(V)\).

**Definition 1.2** ([4]). Suppose \(x = \{x_n\}_{n=1}^\infty \in l^\infty(V)\) and \(v \in V\). \(\{x_n\}_{n=1}^\infty\) is called almost convergent to \(v\) if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} = v
\]

uniformly in \(j\).

Let \(C_b((0, \infty), V)\) be the normed vector space of bounded \(V\)-valued continuous functions \(f\) with supremum norm \(\|f\| := \sup_{t \in [0, \infty)} \|f(t)\|_V\).

**Definition 1.3** ([5]). Suppose \(f \in C_b((0, \infty), V)\) and \(v \in V\). \(f(t)\) is called almost convergent to \(v\) when \(t \to \infty\) if

\[
\lim_{t \to \infty} \frac{1}{t} \int_a^{a+t} f(s) ds = v
\]

uniformly in \(a\).

In[4], Hajduković also gave the concept of quasi-almost convergence in terms of some kind of linear functionals, which are similar to Banach limit in the real sequence case. First, Hajduković defined a semi-norm \(q\) on \(l^\infty(V)\) as following:

For \(x = \{x_n\}_{n=1}^\infty \in l^\infty(V)\),

\[
q(x) = \lim_{n \to \infty} \left( \sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+jn} \right\|_V \right). \tag{1.1}
\]

And then, he showed that there exists the family \(\Pi\) of nontrivial linear functionals \(L\) defined on \(l^\infty(V)\) such that for all \(x = \{x_n\}_{n=1}^\infty \in l^\infty(V)\), the following assertions are valid:

(i) \(L(Tx) = L(x)\);
(ii) \(|L(x)| \leq q(x)\);
(iii) \(L(x - \bar{v}) = 0\) if and only if \(q(x - \bar{v}) = 0\).

**Definition 1.4.** A sequence \(x = \{x_n\}_{n=1}^\infty \in l^\infty(V)\) is called quasi-almost convergent to \(v \in V\) if \(\forall L \in \Pi, L(x - \bar{v}) = 0\).

Similar to the definition of almost convergence, Hajduković gave the following equivalent characterization of quasi-almost convergence:

**Theorem 1.5.** Suppose \(x = \{x_n\}_{n=1}^\infty \in l^\infty(V)\) and \(v \in V\). \(\{x_n\}_{n=1}^\infty\) is quasi-almost convergent to \(v\) if and only if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} x_{i+jn} = v
\]

uniformly in \(j\).
From this theorem, it seems that quasi-almost convergence is weaker than almost convergence. However, in this paper, we will show that actually they are equivalent! In section 2 we will define the concept of Banach limit functional, which is a generalization of Banach limit in bounded real sequence case but much simpler, even than Hajduković’s linear functionals \( \Pi \). To show the existence and sufficiency of Banach limit functionals, we provide a natural construction of Banach limit functionals induced from \( B_1(V^*) \). Then we will give an equivalent characterization of Banach limit functional, which shows that some items in traditional or Hajduković’s definition of Banach limit are equivalent or one could imply another, so it is unnecessary to put them together in the definition.

In Section 3, we define the concept of strong almost convergence in terms of Banach limit functionals, and show that it is equivalent to almost convergence in \( [4] \), then it is immediate that Hajduković’s quasi-almost convergence is equivalent to almost convergence too. We also show that our almost convergence is stronger than that of J. Kurtz’s\[6\], so that’s why we call it strong almost convergence. Some basic properties of strong almost convergence are also discussed. In particular, we show that though strong almost convergence is weaker than norm convergence, corresponding completenesses with respect to the two convergences are the same.

In the end, we point out that all definitions and results here could be applied to bounded continuous functions exactly word by word from summation to integration. Thus, to save space, we don’t restate them again.

2. Banach Limit Functional of Bounded Sequences in Normed Vector Space

**Definition 2.1.** A bounded linear functional \( L \) on \( l^\infty(V) \) is called a Banach limit functional if it satisfies the following two conditions:

(i) \( \|L\| \leq 1 \);

(ii) \( \forall x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V) \) and \( Tx = \{x_2, x_3, \ldots\} \), then \( L(Tx) = L(x) \).

To see the existence and sufficiency of Banach limit functionals, let us begin with the following lemma, which is similar to that in Sucheston’s paper\[7\].

**Lemma 2.2.** \( \forall x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V) \),

\[
\lim_{n \to \infty} \left( \sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V \right)
\]

exists.

**Proof.** Set

\[
c_n = \sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V .
\]
We need to show that \( \lim_{n \to \infty} c_n \) exists. For each \( m, n \), one has
\[
\sup_j \left\| \sum_{i=0}^{m+n-1} x_{i+j} \right\|_V \leq \sup_j \left( \left\| \sum_{i=0}^{m-1} x_{i+j} \right\|_V + \left\| \sum_{i=m}^{m+n-1} x_{i+j} \right\|_V \right)
\leq \sup_j \left\| \sum_{i=0}^{m-1} x_{i+j} \right\|_V + \sup_j \left\| \sum_{i=m}^{m+n-1} x_{i+j} \right\|_V
\leq \sup_j \left\| \sum_{i=0}^{m-1} x_{i+j} \right\|_V + \sup_j \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V,
\]
i.e., \((m+n)c_{m+n} \leq mc_m + nc_n\). Thus
\[
(r + km)c_{r+km} \leq rc_r + kmc_km \leq rc_r + kmc_m.
\]
Dividing by \( r + km \) and letting \( k \to \infty \) with \( r, m \) fixed, we obtain
\[
\lim_{k \to \infty} \sup_{c_{r+km}} \leq c_m.
\]
Since this holds for \( r = 0, 1, \ldots, m - 1 \), \( \limsup_{n \to \infty} c_n \leq c_m \) for each \( m \), and hence \( \limsup_{n \to \infty} c_n \leq \liminf_{n \to \infty} c_n \), which implies that \( \lim_{n \to \infty} c_n \) exists. \( \square \)

**Definition 2.3.** For any \( x \in \ell^\infty(V) \), define
\[
(2.1) \quad p(x) = \lim_{n \to \infty} \left( \sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V \right).
\]

From Lemma 2.2, it is easy to see that \( p \) is a well-defined seminorm on \( \ell^\infty(V) \).

**Lemma 2.4.** If \( x = \{x_n\}_{n=1}^\infty \in \ell^\infty(V) \) such that
\[
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V = m
\]
uniformly in \( j \), then
\[
\lim_{n \to \infty} \left( \sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V \right) = \lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V = m.
\]

**Proof.** \( \forall \varepsilon > 0 \), since
\[
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V = m
\]
uniformly in \( j \), there exists \( N \in \mathbb{N} \) such that for any \( j \in \mathbb{N} \) if \( n > N \), then
\[
\left\| \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} \right\|_V - m < \varepsilon,
\]
i.e.,
\[
m - \varepsilon < \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V < m + \varepsilon.
\]
Hence
\[
m - \varepsilon < \sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V \leq m + \varepsilon.
\]
Since $\varepsilon$ is arbitrary, it follows that
\[
\lim_{n \to \infty} \left( \sup_i \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V \right) = m.
\]

Lemma 2.5. If $x = \{x_n\}_{n=1}^\infty \in c(V)$ such that $\lim_{n \to \infty} x_n = 0$, then
\[
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V = 0
\]
uniformly in $j$.

Proof. $\forall \varepsilon > 0$, since $\lim_{n \to \infty} x_n = 0$, there exists $N_1 \in \mathbb{N}$ such that $\|x_n\|_V < \varepsilon/2$ if $n > N_1$. Choose $N_2$ such that \((\|x_1\|_V + \|x_2\|_V + \cdots + \|x_{N_1}\|_V)/N_2 < \varepsilon/2\). Let $N = \max\{N_1, N_2\}$. Let $n > N$, for any $j \in \mathbb{N}$, if $j > N_1$, then
\[
\frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V \leq \sum_{i=0}^{n-1} \frac{\|x_{i+j}\|_V}{n} < \frac{n\varepsilon/2}{n} = \varepsilon/2;
\]
if $j \leq N_1$,
\[
\frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V \leq \sum_{i=0}^{n-1} \frac{\|x_{i+j}\|_V}{n} = \frac{\sum_{i=0}^{N_1-1} \|x_{i+j}\|_V + \sum_{i=N_1-1}^{n-1} \|x_{i+j}\|_V}{n} < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]
Hence
\[
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V = 0
\]
uniformly in $j$. \qed

Corollary 2.6. If $x = \{x_n\}_{n=1}^\infty \in c(V)$ with $\lim_{n \to \infty} x_n = v \in V$, then
\[
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V = \|v\|_V
\]
uniformly in $j$.

Proof. Since $\lim_{n \to \infty} x_n = v$, i.e., $\lim_{n \to \infty} (x_n - v) = 0$, it follows from Lemma 2.5 that
\[
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} - nv \right\|_V = 0
\]
uniformly in $j$. Since
\[
\left\| \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V - \|v\|_V \right\| = \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V - \|nv\|_V \leq \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} - nv \right\|_V,
\]
it follows that
\[
\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V = \|v\|_V
\]
uniformly in $j$. \qed
Definition 2.7. Suppose that $f \in V^*$ and $\|f\| \leq 1$, define the induced bounded linear functional $L_f$ on $c(V)$ as following: for any $x = \{x_n\} \in c(V)$ with $\lim_{n \to \infty} x_n = v \in V$, $L_f(x) = f(v)$.

Proposition 2.8. For any $x = \{x_n\}_{n=1}^\infty \in c(V)$ with $\lim_{n \to \infty} x_n = v \in V$, $|L_f(x)| \leq p(x)$.

Proof. From Lemma 2.3 and Corollary 2.6 we have

$$|L_f(x)| = |f(v)| \leq \|v\|_V = \lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V = \lim_{n \to \infty} \left( \sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V \right) = p(x).$$

Corollary 2.9. $\|L_f\| \leq 1$.

Proof. $\forall x = \{x_n\}_{n=1}^\infty \in c(V)$, from Proposition 2.8 $|L_f(x)| \leq p(x) \leq \|x\|_\infty$. So $\|L_f\| \leq 1$.

From Hahn-Banach Theorem, we know that there must exist a norm-preserving extension $\overline{L}_f$ of $L_f$ on whole $l^\infty(V)$ such that

$$(2.2) \quad |\overline{L}_f(x)| \leq p(x),$$

$\forall x = \{x_n\}_{n=1}^\infty \in l^\infty(V)$. Now we will show that such $\overline{L}_f$ is an example of Banach limit functional as defined in Definition 2.4.

Theorem 2.10. If $L \in l^\infty(V)^*$ and $x = \{x_n\}_{n=1}^\infty \in l^\infty(V)$ such that $|L(x)| \leq p(x)$, then $L(Tx) = L(x)$.

Proof. Define sequence $y := \{y_n\}_{n=1}^\infty$ as $y_n := x_{n+1} - x_n$, i.e., $y = Tx - x$. Since $x$ is bounded, $y$ is also bounded, i.e., $y \in l^\infty(V)$. Then we have

$$p(y) = \lim_{n \to \infty} \left( \sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} (x_{i+j+1} - x_{i+j}) \right\|_V \right)$$

$$= \lim_{n \to \infty} \left( \sup_j \frac{1}{n} \left\| x_{n+j} - x_j \right\|_V \right)$$

$$\leq \lim_{n \to \infty} \frac{2\|x\|_\infty}{n} = 0.$$

Since $|L(y)| \leq p(y) = 0$, i.e., $L(y) = 0$, we have

$$L(y) = L(Tx - x) = L(Tx) - L(x) = 0,$$

i.e., $L(Tx) = L(x)$.

So far, we have shown that $\overline{L}_f$ is indeed a Banach limit functional. Since that $f$ is an arbitrary choice from $B_1(V^*)$ and Hahn-Banach norm-preserving extension is not unique, we can see that $l^\infty(V)$ has sufficiently many Banach limit functionals. Let us denote all the Banach limit functionals of $l^\infty(V)$ by $\mathcal{L}(V)$.

Remark 2.11. Our definition of Banach limit functional here has greatly improved and simplified corresponding definition in D. Hajdučki’s paper [14]. First of all, you will find that we don’t confine $V$ to be only real normed vector space. Actually, since there is no longer positive element in normed vector space, we don’t need real scalars. And we also improve the definition of $p(x)$ from $\lim_{n \to \infty} \left( \sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V \right)$.
Remark 2.14

simplification of Banach limit. Hence, our definition of Banach limit functional is essentially a
the following proposition shows that in some sense item (iv) can be implied from

Since \( \lim_{\mathcal{L} \Rightarrow} \) convergent, so item (iv) is always included in the definitio

\[ \forall \mathcal{L} \in \mathcal{L}(\mathcal{V}) \]

Proof. \( \forall \mathcal{L} \in \mathcal{L}(\mathcal{V}) \) could be implied from item (i)(positivity) could be implied to classical Banach limit of bounded real sequences, item (i)(positivity) could be implied from linear functional \( f(x) = x, \forall x \in \mathbb{R} \). Moreover, the following proposition shows that in some sense item (iv) can be implied from item (ii) and (iii). Hence, our definition of Banach limit functional is essentially a

\( \forall \mathcal{L} \in \mathcal{L}(\mathcal{V}) \), the following two statements are equivalent:

(i) \( \mathcal{L} \) is a Banach limit functional;

(ii) \( |\mathcal{L}(x)| \leq p(x), \forall x = \{x_n\}_{n=1}^{\infty} \in l^\infty(\mathcal{V}). \)

Proof. (ii) \( \implies \) (i) is exactly Theorem 2.10.

For (i) \( \implies \) (ii), let \( c_n = \{c_{n,j}\}_{j=1}^{\infty} \in l^\infty(\mathcal{V}) \), where \( c_{n,j} = \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} \), i.e.,

\( c_n = \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{T}^i x \). Then for any Banach limit functional \( \mathcal{L} \), we have

\( \sup_{n} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\| \mathcal{V} = \left\| c_n \right\|_{\infty} \geq |\mathcal{L}(c_n)| = |\mathcal{L}(\frac{1}{n} \sum_{i=0}^{n-1} \mathcal{T}^i x)| = \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{L}(\mathcal{T}^i x) = |\mathcal{L}(x)|. \)

Hence \( |\mathcal{L}(x)| \leq \lim_{n \to \infty} \left( \sup_{n} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\| \mathcal{V} \right) = p(x). \)

Classical Banach limit on bounded real sequences is a generalization of ordinary convergence, so item (iv) is always included in the definition of Banach limit. From our viewpoint of Banach limit functional here, Banach limit actually is the Banach limit functional induced from linear functional \( f(x) = x, \forall x \in \mathbb{R} \). Moreover, the following proposition shows that in some sense item (iv) can be implied from item (ii) and (iii). Hence, our definition of Banach limit functional is essentially a

**Proposition 2.13.** If \( \mathcal{L} \in \mathcal{L}(\mathcal{V}) \) and \( x = \{x_n\}_{n=1}^{\infty} \in c(\mathcal{V}) \) with \( \lim_{n \to \infty} x_n = v \in \mathcal{V} \), then \( \mathcal{L}(x) = \mathcal{L}(v). \)

Proof. Since \( \lim_{n \to \infty} (x_n - v) = 0 \), it follows from Lemma 2.14 and Lemma 2.15 that \( p(x - \tilde{v}) = 0 \). From Theorem 2.12 \( |\mathcal{L}(x - \tilde{v})| \leq p(x - \tilde{v}) = 0 \), i.e., \( \mathcal{L}(x) = \mathcal{L}(\tilde{v}). \)

**Remark 2.14.** Before finishing this section, we remark that in the definition of classical Banach limit of bounded real sequences, item (i)(positivity) could be implied from item (ii) and (iii), so this item could be excluded. Moreover, due to Proposition 2.13 item (iv) could be replaced by (iv’) \( \mathcal{L}(\tilde{1}) = 1 \). We leave the proofs as easy exercises to interested readers.

3. **Strong Almost Convergence of Bounded Sequences in Normed Vector Space**

**Definition 3.1.** A sequence \( x = \{x_n\}_{n=1}^{\infty} \in l^\infty(\mathcal{V}) \) is called strongly almost convergent to \( v \in \mathcal{V} \) if for any Banach limit functional \( \mathcal{L} \in \mathcal{L}(\mathcal{V}) \), it holds that \( \mathcal{L}(x) = \mathcal{L}(\tilde{v}). \) Let us denote it by \( x_n \overset{s.a.}{\longrightarrow} v \), and \( v \) is called strong almost limit of \( x \).

Next we will give an equivalent characterization of strong almost convergence, and show that our strong almost convergence is equivalent to almost convergence given by Hajdukovitch. Moreover, as an immediate corollary, his quasi-almost convergence is equivalent too.

**Lemma 3.2.** Suppose \( x = \{x_n\}_{n=1}^{\infty} \in l^\infty(\mathcal{V}) \). \( p(x) = 0 \) if and only if \( \mathcal{L}(x) = 0 \), \( \forall \mathcal{L} \in \mathcal{L}(\mathcal{V}). \).
Proposition 3.5. (i) If $L$ is a limit functional, then for any $\lambda, \mu \in \mathbb{C}$, so we can see that quasi-almost convergence given by Hajdуковић is actually equivalent to strong almost convergence.

An immediate corollary of Lemma 3.2 is the following important theorem:

Theorem 3.4. A sequence $x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V)$ is strongly almost convergent to $v \in V$ if and only if $p(x - \overline{v}) = 0$.

Proposition 3.5. (i) If $x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V)$ is strongly almost convergent in $V$, then its strong almost limit is unique.

(ii) Suppose $x = \{x_n\}_{n=1}^{\infty}, y = \{y_n\}_{n=1}^{\infty} \in l^\infty(V)$. If $x_n \overset{s.a.}{\to} u$ and $y_n \overset{s.a.}{\to} v$, then for any $\lambda, \mu \in \mathbb{C}$, $\lambda x_n + \mu y_n \overset{s.a.}{\to} \lambda u + \mu v$.

(iii) If $\{x_n\}_{n=1}^{\infty}$ is a sequence from $V$ such that $\lim_{n \to \infty} x_n = v \in V$, then $x_n \overset{s.a.}{\to} v$.

Proof. (i) If $x_n \overset{s.a.}{\to} v_1$ and $x_n \overset{s.a.}{\to} v_2$ simultaneously, then it follows from Theorem 3.4 that $\|v_1 - v_2\|_V = p(v_1 - v_2) \leq p(\overline{v_1} - x) + p(x - \overline{v_2}) = 0 + 0 = 0$. Hence $v_1 = v_2$.

(ii) $p(\lambda x + \mu y - \lambda \overline{u} - \mu \overline{v}) \leq |\lambda| p(x - \overline{u}) + |\mu| p(y - \overline{v})$.

(iii) From Theorem 2.12.

Remark 3.6. Please notice that if $x_n \overset{s.a.}{\to} v \in V$, it doesn’t mean that each subsequence of $x = \{x_n\}_{n=1}^{\infty}$ is also strongly almost convergent, let alone strongly almost convergent to the same vector. For example, consider bounded real sequence $x = \{1, 0, 1, 0, \ldots\}$. Then $x_n \overset{s.a.}{\to} 1/2$. However, $\lim_{k \to \infty} x_{2k} = 1$, while $\lim_{k \to \infty} x_{2k-1} = 0$.

Lemma 3.7. Suppose $x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V)$ and $p(x) = 0$, then

$$\lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V = 0$$

uniformly in $j$.

Proof. Since $p(x) = 0$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_j \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V < \varepsilon$$

when $n > N$. In other words, for any $j \in \mathbb{N}$, when $n > N$

$$\frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V < \varepsilon.$$
So
\[ \lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} x_{i+j} \right\|_V = 0 \]
uniformly in \( j \).
\[ \square \]

**Theorem 3.8.** Suppose \( x = \{x_n\}_{n=1}^{\infty} \subseteq l^\infty(V) \). \( x_n \xrightarrow{s.a.} v \in V \) if and only if
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} = v \]
uniformly in \( j \).

**Proof.** \( x_n \xrightarrow{s.a.} v \iff p(x - v) = 0 \). From Lemma 3.7
\[ \lim_{n \to \infty} \frac{1}{n} \left\| \sum_{i=0}^{n-1} (x_{i+j} - v) \right\|_V = \lim_{n \to \infty} \left\| \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} - v \right\|_V = 0 \]
uniformly in \( j \), i.e.,
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} = v \]
uniformly in \( j \). \[ \square \]

This theorem shows that strong almost convergence is equivalent to almost convergence in \( H \), and so is quasi-almost convergence.

**Remark 3.9.** In the definition of strong almost convergence, we require \( x = \{x_n\}_{n=1}^{\infty} \) to be bounded. Actually this is not constrained, because from
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} = v \]
uniformly in \( j \), we can easily imply that \( \{x_n\}_{n=1}^{\infty} \) is bounded.

**Corollary 3.10.** Suppose \( x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V) \). If \( x_n \xrightarrow{s.a.} v \in V \), then \( v \in \overline{\sigma} \{x_n : n \in \mathbb{N}\} \).

**Definition 3.11.** \( V \) is a normed vector space and \( A \subseteq V \). \( A \) is called **s.a.-sequentially closed** if \( \forall \{x_n\}_{n=1}^{\infty} \) from \( A \) such that \( x_n \xrightarrow{s.a.} v \in V \), then \( v \in A \).

**Theorem 3.12.** Suppose \( V \) is a normed vector space and \( A \subseteq V \) is convex. \( A \) is **(norm) closed** if and only if \( A \) is s.a.-sequentially closed. In particular, a subspace of \( V \) is (norm) closed if and only if it is s.a.-sequentially closed.

**Proof.** Suppose \( A \) is s.a.-sequentially closed. If \( \{x_n\}_{n=1}^{\infty} \subseteq A \) and \( \lim_{n \to \infty} x_n = v \in V \). From Theorem 3.9 (iii), \( x_n \xrightarrow{s.a.} v \in A \). Hence \( A \) is (norm) closed.

Conversely, suppose \( A \) is (norm) closed. If \( \{x_n\}_{n=1}^{\infty} \subseteq A \) and \( x_n \xrightarrow{s.a.} v \in V \). From Corollary 3.10 \( v \in \overline{\sigma} \{x_n : n \in \mathbb{N}\} \subseteq A = A \). Hence \( A \) is s.a.-sequentially closed. \[ \square \]

**Definition 3.13.** A bounded sequence \( \{x_n\}_{n=1}^{\infty} \) of normed vector space \( V \) is called an **s.a.-Cauchy sequence** if for any \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for any \( j \in \mathbb{N} \) if \( n, m > N \), then
\[ \left\| \frac{1}{n} \sum_{i=0}^{n-1} x_{i+j} - \frac{1}{m} \sum_{i=0}^{m-1} x_{i+j} \right\|_V < \varepsilon \]. \( V \) is called s.a.-complete if every s.a.-Cauchy sequence in \( V \) is strongly almost convergent to a vector in \( V \).
Corollary 3.14. A normed vector space $V$ is (norm) complete if and only if it is s.a.-complete.

Remark 3.15. This shows that though strong almost convergence is weaker than (norm) convergence, considering completion, it doesn’t enlarge the space further.

In the end, we will explain why we use the terminology strong almost convergence.

Definition 3.16 (J. Kurtz[6]). Suppose $x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V)$. We say that $x = \{x_n\}_{n=1}^{\infty}$ is weakly almost convergent to $v \in V$ if for any $f \in V^*$, $\hat{f}(x) := \{f(x_n)\}_{n=1}^{\infty} \in l^\infty(\mathbb{C})$ is almost convergent to $f(v)$. Let us denote it by $x_n \overset{w.a.}{\to} v$.

Remark 3.17. From the definition, it is immediate that any weakly convergent sequence is weakly almost convergent to its weak limit.

Theorem 3.18. Suppose $x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V)$ and $v \in V$. If $x_n \overset{s.a.}{\to} v$, then $x_n \overset{w.a.}{\to} v$.

Proof. From Theorem 3.4 we just need to show that for any $f \in V^*$, $p(\hat{f}(x) - \hat{f}(v)) = 0$. Since $p(x - v) = 0$, we have

\[
p(\hat{f}(x) - \hat{f}(v)) = \lim_{n \to \infty} \left( \sup_j \frac{1}{n} \sum_{i=1}^{n-2} (f(x_{i+j}) - f(v)) \right) = \lim_{n \to \infty} \left( \sup_j \frac{1}{n} \sum_{i=1}^{n-2} x_{i+j} - v \right) \leq \|f\| \|p(x - v)\|_V = 0.
\]

Theorem 3.19 (J. Kurtz[6]). Suppose $x = \{x_n\}_{n=1}^{\infty} \in l^\infty(V)$ and $v \in V$. If \{x_n : n \in \mathbb{N}\} is precompact and $x_n \overset{s.a.}{\to} v$, then $x_n \overset{w.a.}{\to} v$.

Remark 3.20. When $V = \mathbb{C}$, strong almost convergence and weak almost convergence coincide, since each bounded sequence in $\mathbb{C}$ is precompact. So we just say almost convergence there.

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