Spherical harmonic analysis for multivariate stable distributions\textsuperscript{1}

Zhiyi Chi  
Department of Statistics  
University of Connecticut  
Storrs, CT 06269, USA,  
E-mail: zhiyi.chi@uconn.edu  

October 18, 2021

Abstract

Series representations consisting of spherical harmonics are obtained for characteristic exponents and probability density functions of multivariate stable distributions under various conditions. A result potentially applicable in a practical setting is that for any distribution with stability index not equal to 1 and with a polynomial spectral spherical density, the series representation converges absolutely with all terms being calculable in closed form. Asymptotic expansions consisting of spherical harmonics are also considered for probability density functions.

Keywords and phrases. Spherical harmonics; multivariate stable; special function

2000 Mathematics Subject Classifications: Primary 60G51; Secondary 60E07.

Acknowledgment. The research is partially supported by NSF Grant DMS 1720218.

1 Introduction

The probability density functions (p.d.f.’s) of univariate stable distributions, i.e., stable distributions on \( \mathbb{R} \), have been well known for a long time. In contrast, much less can be said about the p.d.f.’s of multivariate stable distributions except for a few cases, such as spherically symmetric stable distributions, or more generally, subordinated normal distributions [16], and direct products of such distributions and univariate stable distributions. Many efforts have been dedicated to the understanding of multivariate stable distributions; see [9] for an review. One approach is to approximate the distributions by more tractable ones, such as series of simple random variables or stable distributions with discrete spectral spherical measures [4–6]. This approach provides error bounds of approximation but not functional forms of the distributions. On the other hand, [1] gives integral expressions for multivariate stable p.d.f.’s and [8] provides analytic approximations of the p.d.f.’s by solutions of partial differential equations of fractional order. Despite the progress, it has been difficult to extend several important representations for univariate stable distributions ([16], chapter 4) to the multivariate ones. To a large degree, the difficulty is due to a lack of analytic tools. However, in the study on estimation for multivariate stable distributions, spherical harmonic analysis has already been used [11]. Inspired by this, the aim of the

\textsuperscript{1}Research partially supported by NSF Grant DMS 1720218.
paper is to apply spherical harmonic analysis to get more understanding of the p.d.f.’s of multivariate stable distributions.

Let \( \mu \) be an \( \alpha \)-stable probability distribution on \( \mathbb{R}^d \) with no shift, where \( \alpha \in (0, 2) \) is the stability index. Unless \( \mu \) is a unit mass at 0, there is a finite nonzero Borel measure \( \lambda \) on the unit sphere \( S^{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \} \), where \( |x| \) denotes the Euclidean norm, such that the Fourier transform of \( \mu \) is ([14], Theorem 14.10)

\[
\hat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle x, z \rangle} \mu(dx) = \exp\{-\Phi_\mu(z)\}, \quad z \in \mathbb{R}^d,
\]

where

\[
\Phi_\mu(z) = \begin{cases} 
\int_{S^{d-1}} |\langle z, v \rangle|^\alpha \left[ 1 - i \tan \frac{\pi \alpha}{2} \text{sign}(\langle z, v \rangle) \right] \lambda(dv) & \text{if } \alpha \neq 1 \\
\int_{S^{d-1}} \left[ |\langle z, v \rangle| + i \frac{2}{\pi} \langle z, v \rangle \ln |\langle z, v \rangle| \right] \lambda(dv) & \text{if } \alpha = 1
\end{cases}
\tag{1.1}
\]

is known as the characteristic exponent of \( \mu \). The measure \( \lambda \) is unique ([3], Theorem 3.4.2) and will be referred to as the spectral spherical measure of \( \mu \), although it has been called the spectral or Poisson spectral measure elsewhere (cf. [6]). If the support of \( \mu \) is not contained in \( a + E \) for any \( a \in \mathbb{R}^d \) and linear subspace \( E \subset \mathbb{R}^d \) with \( \dim(E) < d \), then \( \mu \) is called nondegenerate and has a bounded continuous p.d.f. with respect to (w.r.t.) the Lebesgue measure ([14], Definition 24.16 and Example 28.2)

\[
g(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle x, z \rangle} \hat{\mu}(z) \, dz.
\tag{1.2}
\]

From (1.1) and (1.2), the calculation of the p.d.f. boils down to two integrals, one for \( \Phi_\mu \), the other for the inverse Fourier transform of \( \exp\{-\Phi_\mu\} \). To tackle the integrals, a significant portion of the paper will consider the case where \( \lambda \) has a density w.r.t. the Haar measure on \( S^{d-1} \), henceforth referred to as the spectral spherical density.

Section 2 sets up notation. It also lists basic facts about spherical harmonics and several other special functions, all selected from the classical books [2] and [15]. Section 3 considers the characteristic exponential \( \Phi_\mu \) of a nondegenerate \( \alpha \)-stable distribution \( \mu \) that has no shift. In general, if \( \alpha \neq 1 \), then \( \Phi_\mu(z) \) can be written as \( |z|^\alpha V(u_z) \), where \( u_z \) is the unit vector with the same direction as \( z \). Under the condition that \( \mu \) has a square-integrable spectral spherical density \( P \), the section shows that \( V \) can be expressed as a series of spherical harmonics, each being an explicit multiple of a spherical harmonic of \( P \). As a result, if \( P \) is a polynomial, then \( \Phi_\mu(z) \) has a closed form. Similar results also hold when \( \alpha = 1 \). Thus, it is possible to get the characteristic exponential in a practical setting, although a direct calculation typically is tedious; see comments in Section 2.2.

Sections 4 and 5 consider the p.d.f. \( g(x) \) for nondegenerate \( \alpha \)-stable distributions with \( \alpha \neq 1 \) and no shift. In both sections, some emphasis is given to series representations that are absolutely convergent (a.c.) and have all terms calculable in closed form. Section 4 deals with the case \( \alpha \in (0, 1) \). It shows that \( g(x) \) can be represented as an infinite series of spherical harmonics, each coming from a positive integral power of the aforementioned
and weighted by a negative fractional power of $|x|$. When $\mu$ has a polynomial spectral spherical density, the series is uniformly a.c. in $\{x : |x| \geq r\}$ for any $r > 0$ and all the spherical harmonics can be written in closed form. On the other hand, for the general case where $\mu$ may not have a spectral spherical density, only convergence in an $L^2$ sense is established, which nevertheless is strong enough to yield the p.d.f. of $|X|$ with $X \sim \mu$. It is of interest to consider cases in between the above two. For example, when $\mu$ has a square-integrable spectral spherical density that is not a polynomial, it would be desirable to have an a.c. series of spherical harmonics for $g(x)$. Results of this sort will require a better understanding of the spherical harmonics of high powers of $V$. The asymptotic expansion of $g(x)$ at 0 is also derived in this section. In contrast to the series representations, the asymptotic expansion consists of spherical harmonics that come from negative powers of $V$ and are weighted by nonnegative integral powers of $|x|$.

Section 5 deals with the case $\alpha \in (1, 2)$, and furnishes two a.c. series representations of $g(x)$ in terms of spherical harmonics. The first one consists of spherical harmonics coming from negative fractional powers of $V$. However, these spherical harmonics do not have easily available closed form even when $V$ is a nonconstant polynomial. The second series representation allows all its spherical harmonic terms to be expressed in closed form when $\mu$ has a polynomial spectral spherical density. However, it requires a somewhat arbitrary parameter. The section also considers the asymptotic expansion of $g(x)$ as $|x| \to \infty$. Compared to the same problem in the univariate case, the analysis is much more subtle. The section obtains the asymptotic expansion of the spherical harmonic of $g$ of every fixed degree. There is still a significant gap between the result and a full asymptotic expansion of $g$, although the former strongly suggests the latter. On the other hand, the result provides useful information such as an asymptotic expansion of the p.d.f. of $|X|$ with $X \sim \mu$.

As applications of the above results, Section 6 illustrates the case where the spectral spherical density is a linear function and Section 7 shows that the series representations can be applied to sample multivariate stable distributions. However, the important issue of efficient sampling, which is available for univariate stable p.d.f.’s (cf. [7], section IV.6), is beyond the scope of the paper. Also, the paper has no result on the p.d.f. for $\alpha = 1$. In view of currently available results in the univariate case ([14], p. 88), it is possible that the multivariate case for $\alpha = 1$ does not admit a simple series representation.

2 Preliminaries

Denote $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$, and $\omega = \omega_{d-1}$ the measure on $\mathbb{S}^{d-1}$ such that for $f \in L^1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} f(x) \, dx = \int_{\mathbb{S}^{d-1}} [\int_0^\infty f(sv)s^{d-1} \, ds] \omega(dv)$. For $0 \neq x \in \mathbb{R}^d$, let $u_x = x/|x|$. Since $u_x$ will be used only when $x \neq 0$ or in functions of the form $|x|^a f(u_x)$ with $a > 0$ and $f$ bounded, $u_0$ need not be specified.

2.1 Spherical harmonics

A polynomial on $\mathbb{R}^d$ of degree $j \in \mathbb{Z}_+$ has the form $f(x) = \sum c_a x^a$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, where the sum is taken over all $a = (a_1, \ldots, a_d) \in \mathbb{Z}_+^d$ with $\sum a_i \leq j$, $x^a$ denotes $\prod x_i^{a_i}$.
and each \( c_a \in \mathbb{C} \) is a constant. If \( c_a = 0 \) unless \( \sum a_i = j \), then \( f \) is said to be homogeneous of degree \( j \) and \( f(x) = |x|^j f(u_x) \) for \( x \neq 0 \). If \( \sum \frac{\partial f}{\partial x_i} = 0 \), then \( f \) is said to be harmonic. Restrictions of homogeneous harmonic polynomials to \( S^{d-1} \) are called spherical harmonics ([2], Section 9.4). Due to \(|x|^2 = 1\) on \( S^{d-1} \), different polynomials when restricted to \( S^{d-1} \) can be equal, e.g., \( f(x) \) and \( |x|^2 f(x) \). However, a spherical harmonic is the restriction of a unique homogeneous harmonic polynomial on \( \mathbb{R}^d \) and the two have the same degree; see [2], Theorem 9.4.1 or [15], Corollary VI.2.4.

Recall that \( L^2(S^{d-1}) \) is equipped with inner product \( \langle f, g \rangle = \int_{S^{d-1}} f g \omega, \) where \( \omega \) stands for complex conjugation. Denote by \( \mathcal{P}_{j,d} \) the set of polynomials of degree \( j \) restricted to \( S^{d-1} \), and \( \mathcal{H}_{j,d} \) that of spherical harmonics of degree \( j \). Both \( \mathcal{P}_{j,d} \) and \( \mathcal{H}_{j,d} \) are finite dimensional subspaces of \( L^2(S^{d-1}) \). From [15], p. 140 or [2], p. 450,

\[
\dim(\mathcal{H}_{j,d}) = c_{j,d} = \begin{cases} 
1, & j = 0 \\
\left( 1 + j + d - 2 \right) \frac{(i + d - 3)}{(j - 1)!}, & j > 0,
\end{cases} \tag{2.1}
\]

\( \mathcal{H}_{j,d} \perp \mathcal{H}_{j',d} \) for \( j \neq j' \), \( \mathcal{P}_{k,d} = \bigoplus_{k=0}^{\infty} \mathcal{H}_{j,d} \) for \( k \geq 0 \), and \( L^2(S^{d-1}) = \bigoplus_{j=0}^{\infty} \mathcal{H}_{j,d} \).

The projection from \( L^2(S^{d-1}) \) to \( \mathcal{H}_{j,d} \) will be denoted by \( \pi_j \). For \( f \in L^2(S^{d-1}) \), \( \pi_j f \) will be referred to as the \( j \)th spherical harmonic of \( f \). If \( d \geq 2 \), then given \( j \geq 0 \) and any real-valued orthonormal basis \( S_{j,i} \), \( i = 1, \cdots, c_{j,d} \), of \( \mathcal{H}_{j,d} \), for \( f \in L^2(S^{d-1}) \),

\[
(\pi_j f)(u) = \frac{c_{j,d}}{A(S^{d-1})} \int_{S^{d-1}} \tilde{C}^{(d-2)/2}(\langle u, v \rangle) f(v) \omega(dv), \quad u \in S^{d-1}, \tag{2.2}
\]

where \( A(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2) \) is the surface area of \( S^{d-1} \) and for \( b > 0 \),

\[
\tilde{C}^b_j(t) = C^b_j(t)/C^b_1(1)
\]

is an ultraspherical polynomial with \( C^b_j(t) \) known as the Gegenbauer polynomial of degree \( j \) ([2], p. 302). The latter can be written as

\[
C^b_j(t) = \sum_{m=0}^{[j/2]} (-1)^m 2^{j-2m} \frac{(b)_j}{m! (j-2m)!} t^{j-2m} \tag{2.3}
\]

([11], p. 233) and for \(|t| \leq 1\), \(|C^b_j(t)| \leq C^b_1(1) = (2b)_j/j!\), where for \( z \in \mathbb{C} \) and \( n \in \mathbb{N} \), \((z)_0 = 1\) and \((z)_n = \prod_{m=0}^{n-1}(z + m)\). For \( d = 2 \), if \( j \geq 1 \), then \( C^0_j(t) \equiv 0 \), so \( \tilde{C}^0_j(t) \) cannot be defined as the ratio of \( C^0_j(t) \) and \( C^0_1(1) \). However, \( \lim_{b \to 0} C^b_j(t)/C^b_1(t) = T_j(t) \) ([2], Eq. (6.4.13’)), where \( T_j(t) \) is the Tchebyshev polynomial of the first kind of degree \( j \) defined by the formula \( T_j(\cos \theta) = \cos(j \theta) \) ([2], p. 101). Then (2.2) still holds with \( \tilde{C}^0_j(t) = T_j(t) \) (cf. [2], Remark 9.6.1). It is useful to note that

\[
|\tilde{C}_j^b(t)| \leq 1, \quad -1 \leq t \leq 1, \quad b \geq 0, \quad j \in \mathbb{Z}_+ \tag{2.4}
\]

From (2.2), \( \pi_j \) is an integral operator with kernel \( K(u,v) = \frac{c_{j,d}}{A(S^{d-1})} C^{(d-2)/2}(\langle u, v \rangle) \).

Given \( u \in S^{d-1} \), the polynomial \( \tilde{C}^b_j(\langle u, \cdot \rangle) \) is known as a zonal harmonic with pole \( u \) and
belongs to $\mathcal{H}_{j,d}$ ([2], p. 455–456). The Funk-Hecke formula ([2], Theorem 9.7.1) states that, for any continuous function $f$ on $[-1,1]$ and any $S \in \mathcal{H}_{j,d}$, $j \in \mathbb{Z}_+$, $d \geq 2$,

$$
\int_{S^{d-1}} f(\langle u, v \rangle) S(v) \omega(dv) = \lambda_{j,d} S(u), \quad u \in S^{d-1}
$$

with $\lambda_{j,d} = \lambda_{j,d}(f) = A(S^{d-2}) \int_{-1}^{1} f(t) \widetilde{C}_j^{(d-2)/2}(t)(1 - t^2)^{(d-3)/2} dt$. \hfill (2.5)

### 2.2 Closed form of spherical harmonics of a polynomial

Let $f$ be a polynomial. Then $\widetilde{C}_j^{(d-2)/2}(\langle u, v \rangle) f(v)$ can be written as a linear combination of a finite number of $\langle u, v \rangle^k v^c$ with explicit coefficients, where $k \in \mathbb{Z}_+$ and $c \in \mathbb{Z}_+^d$. By expanding $\langle u, v \rangle^k v^c$ and integrating term by term, (2.2) can be written as

$$(\pi_j f)(u) = \sum m_{a,b} \int_{S^{d-1}} v^a \omega(dv) \cdot u^b,$$

where the sum only has a finite number of terms, and for each pair $a, b \in \mathbb{Z}_+^d$, $m_{a,b}$ is an explicit number. In polar coordinates, $v_i = \cos \theta_i \prod_{j=1}^{i-1} \sin \theta_j$ for $i < d$ and $v_d = \prod_{j=1}^{d-1} \sin \theta_j$, where $\theta = (\theta_1, \ldots, \theta_d) \in E = [0, \pi]^{d-2} \times [0, 2\pi)$. Then ([2], Eq. (9.6.4))

$$
\omega(dv) = \prod_{j=1}^{d-2} (\sin \theta_j)^{d-1-j} d\theta = (\sin \theta_1)^{d-2} \cdots (\sin \theta_{d-3})^2 \sin \theta_{d-2} \, d\theta_1 \cdots d\theta_{d-2} \, d\theta_{d-1}
$$

and

$$
\int_{S^{d-1}} v^a \omega(dv) = \int_E \prod_{j=1}^{d-1} (\sin \theta_j)^{p_j} (\cos \theta_j)^{q_j} d\theta
$$

with $p_j = p_j(a)$ and $q_j = q_j(a) \in \mathbb{Z}_+$. Thus $\pi_j f$ can be found in closed form by trigonometric integration. Unfortunately, if $f$ has a high degree or $j$ is large, the calculation involves a large number of $\int_{S^{d-1}} v^a \omega(dv)$ and becomes tedious.

For example, let $d = 3$ and $f(u) = f(u_1, u_2, u_3) = u_1^2$. Then $\pi_j f = 0$ for $j > 2$. To find $\pi_2 f$, from (2.1), $c_{2,3} = 5$ and from (2.3), $\widetilde{C}_2^{1/2}(t) = (3t^2 - 1)/2$. By (2.2),

$$(\pi_2 f)(u) = \frac{5}{4\pi} \int_{S^2} \frac{1}{2} [3(u_1 v_1 + u_2 v_2 + u_3 v_3)^2 - 1] v_1^2 \omega(dv).$$

As described above, let $v_1 = \cos \theta$, $v_2 = \sin \theta \cos \phi$, and $v_3 = \sin \theta \sin \phi$, $\theta \in [0, \pi)$, $\phi \in [0, 2\pi]$. Routine trigonometric integration gives $\int_{S^2} v_1^a \omega(dv) = \int_0^{2\pi} \int_0^\pi \cos^a \theta \sin \theta d\theta d\phi = 4\pi / 5$, $\int_{S^2} v_2^a \omega(dv) = 4\pi / 3$, and for $i = 1, 2$, $\int_{S^2} v_1^2 v_i^2 \omega(dv) = 4\pi / 15$ and $\int_{S^2} v_2^2 v_3 \omega(dv) = \int_{S^2} v_1^2 v_1^2 \omega(dv) = 0$. Then $(\pi_2 f)(u) = (3u_1^2 + u_2^2 + u_3^2)^2 - 1/2 - 5/6$. This may seem contrary to $\pi_2 f$ being a harmonic homogeneous polynomial restricted to $S^2$, but is in fact correct as $(\pi_2 f)(u) = (2u_2^2 - u_2^2 - u_3^3)/3$ for $u \in S^2$. More calculation yields $\pi_1 f = 0$ and $\pi_0 f = 1/3$.

For this example, simpler calculation can be made using the Funk-Hecke formula (2.5).

Let $e = (1,0,0)$ and $g(t) = t^2$. Then $(\pi_j f)(u) = (5/4\pi) \int_{S^2} g(\langle e, v \rangle) \widetilde{C}_2^{1/2}(\langle u, v \rangle) \omega(dv)$.

Given $u$, $\widetilde{C}_2^{1/2}(\langle u, \cdot \rangle) \in \mathcal{H}_{2,3}$, so $(\pi_2 f)(u) = (5/4\pi) \lambda_{2,3} \widetilde{C}_2^{1/2}(\langle u, e \rangle) = (3u_1^2 - 1)/3$ with $\lambda_{2,3} = A(S^1) \int_{-1}^{1} t^2[(3t^2 - 1)/2] dt = 8\pi/15$. 
2.3 Other special functions

For \(a_1, \ldots, a_p, b_1, \ldots, b_q \in \mathbb{C}, \ p, q \geq 0, \) with \(b_i \not\in \{0, -1, -2, \ldots\}, \) the series

\[
\sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k z^k}{(b_1)_k \cdots (b_q)_k k!},
\]

where \((a_1)_k \cdots (a_p)_k = 1\) if \(p = 0\) and likewise for \((b_1)_k \cdots (b_q)_k,\) is known as the hypergeometric series and denoted by \(_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\) or \(_pF_q\left(\frac{a_1}{b_1}, \ldots, \frac{a_p}{b_q}; z\right)\).

The series is a.c. for all \(z \in \mathbb{C}\) if \(p \leq q,\) and for \(|z| < 1\) if \(p = q + 1.\) When \(p = 2\) and \(q = 1,\) the analytic continuation of the series \(_2F_1(a, b; c; z)\) as a function of \(z\) is known as the hypergeometric function. Gauss’ formula states that \([2, \text{Theorem 2.2.2}]\)

\[
_2F_1\left(a, b; c; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{if Re}(c-a-b) > 0. \quad (2.6)
\]

Several classical formulas for the Gamma function will be used \([2, \text{Chapter 1}]\):

- **Legendre’s duplication** \(\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) = 2^{1-z}\Gamma(z)\sqrt{\pi}, \) \(2.7\)
- **Euler’s reflection** \(\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \) \(2.8\)
- **Stirling** \(\Gamma(z) \sim \sqrt{2\pi z}(z/e)^z, \) \(\text{Re} z \to \infty. \quad (2.9)\)

From Stirling’s formula, given \(c \in \mathbb{R}, \ \Gamma(z+c)/\Gamma(z) \sim z^c.\) Then from \((2.1)\)

\[
c_{j,d} = O(j^{d-2}), \quad j \to \infty. \quad (2.10)
\]

Since \(\Gamma(z) = \int_0^\infty t^{z-1}e^{-t} \, dt\) if \(\text{Re} z > 0,\) by Hölder’s inequality, the Gamma function is log-convex on \((0, \infty),\) i.e., given \(c > 0, \ \Gamma(z+c)/\Gamma(z)\) is increasing in \(z \in (0, \infty).\) The reciprocal of the Gamma function can be continuously extended into an entire function whose set of zeros is exactly the set of negative integers; this continuation is still denoted by \(1/\Gamma(x)\) \([2, \text{p. 3}]\). Thus, for functions of the form \(f = g/\Gamma,\) if \(g\) has a finite value at \(-x\) with \(x \in \mathbb{N},\) then \(f\) is well defined at \(-x\) with value zero.

The Bessel function of the first kind of order \(a\) is defined by \([2, \text{Eq. (4.5.3)}]\)

\[
J_a(x) = \frac{(x/2)^a}{\Gamma(a+1)} {}_0F_1\left(-\frac{a}{a+1}; -(x/2)^2\right) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+a}}{k!\Gamma(k+a+1)}. \quad (2.11)
\]

It is known that \([10, \text{10.14.4}]\)

\[
|J_a(z)| \leq \frac{|z/2|^a e^{\text{Im}z}}{\Gamma(a+1)}, \quad a \geq -1/2, \ \text{z} \in \mathbb{C}. \quad (2.12)
\]
Finally, let \( d \geq 2 \). For \( r > 0, u, v \in S^{d-1} \), and \( S \in \mathcal{H}_{j,d}, j \geq 0 \), letting \( b = d/2 - 1 \),
\[
\int_{S^{d-1}} e^{-ir(u,v)} S(v) \omega(dv) = (2\pi)^{d/2} (-i)^j S(u) \frac{J_{j+b}(r)}{r^b}.
\]
This is essentially Eq. (9.10.2) in [2], except that the latter incorrectly uses factor \( i^j \) instead of \( (-i)^j \) on the r.h.s. If \( f \in L^2(S^{d-1}) \), then from \( f = \sum_{j=0}^{\infty} \tau_j f \) and the above formula,
\[
\int_{S^{d-1}} e^{-ir(u,v)} f(v) \omega(dv) = \frac{(2\pi)^{d/2}}{r^b} \sum_{j=0}^{\infty} (-i)^j J_{j+b}(r)(\tau_j f)(u).
\] (2.13)
In particular, for any \( w \in S^{d-1} \), since \( \tilde{C}_j^b(\langle w, \cdot \rangle) \in \mathcal{H}_{j,d} \), then (2.2) and (2.13) yield
\[
(\tau_j e^{-ir(u,\cdot)})(w) = \frac{c_{j,d}}{A(S^{d-1})} \int_{S^{d-1}} e^{-ir(u,v)} \tilde{C}_j^{(d-2)/2}(\langle w, v \rangle) \omega(dv)
= \frac{(2\pi)^{d/2}}{A(S^{d-1})} (-i)^j c_{j,d} \tilde{C}_j^b(\langle u, w \rangle) \frac{J_{j+b}(r)}{r^b}.
\] (2.14)

### 3 Calculation of characteristic exponent

Throughout the section, let \( d \geq 2 \). The main result of this section is the following.

**Theorem 1.** Let the characteristic exponent \( \Phi_\mu \) of an \( \alpha \)-stable distribution \( \mu \) be given by (1.1). Suppose \( \lambda(dv) = P(v) \omega(dv) \) with \( P \in L^2(S^{d-1}) \). Let \( P_j = \pi_j P, j \in \mathbb{Z}_+ \). Then
\[
\Phi_\mu(z) = |z|\alpha \left[ \sum_{j \text{ even}} w_j(\alpha) P_j(u_z) - i \tan \frac{\pi \alpha}{2} \sum_{j \text{ odd}} w_j(\alpha) P_j(u_z) \right],
\] (3.1)
if \( \alpha \neq 1 \), and
\[
\Phi_\mu(z) = |z| \left[ \sum_{j \text{ even}} w_j(1) P_j(u_z) + \frac{2}{\pi} \sum_{j \text{ odd}} w_j^*(P_j(u_z)) \right] + i |z| \ln |z| \frac{2\pi^{d/2-1}}{\Gamma(d/2 + 1)} P_1(u_z)
\] (3.2)
if \( \alpha = 1 \), where
\[
w_j(\alpha) = \frac{\pi^{d/2} \Gamma(\alpha + 1)}{2^{\alpha-1} \Gamma((j + \alpha + d)/2) \Gamma((\alpha - j)/2 + 1)}
= \frac{\pi^{d/2-1} \sin((j - \alpha)\pi/2) \Gamma(\alpha + 1) \Gamma((j - \alpha)/2)}{2^{\alpha-1} \Gamma((j + \alpha + d)/2)},
\] (3.3)
and
\[
w_j^* = \begin{cases} 
\frac{\pi^{d/2}}{\Gamma(d/2 + 1)^2} & j = 1 \\
\frac{(-1)^{(j-3)/2} \pi^{d/2} \Gamma((j - 1)/2)}{2 \Gamma((d + 1 + j)/2)} & j > 1 \text{ odd},
\end{cases}
\] (3.4)
where, letting \( s_0 = 0 \) and \( s_n = s_{n-1} + 1/n \) for \( n \geq 1 \),

\[
\beta_d = \begin{cases} 
1 - \ln 2 - s_{d/2}/2 & d \text{ even}, \\
1 - s_{d+1} + s_{(d+1)/2}/2 & \text{else}. 
\end{cases}
\]

The series in (3.1) and (3.2) are a.c.

Remark.

1) If \( \alpha = 1 \) and \( \mu \) is strictly stable, then by [14, Theorem 14.10],

\[
\int_{S^{d-1}} v\lambda(dv) = \int_{S^{d-1}} vP(v)\omega(dv) = 0.
\]

Consequently, by (2.2), \( P_1 \equiv 0 \) in (3.2).

2) The two expressions in (3.3) are equal by Euler’s reflection formula (2.8). Both are presented because either one may be more convenient to use in certain cases.

3) If \( P \) is a polynomial of degree \( q \), then from Section 2, \( P_j = \pi_j P = 0 \) for \( j > q \), and \( P_j \) with \( j \leq q \) can be explicitly calculated in closed form by (2.2). As a result, \( \Phi_\mu(z) \) can be obtained in closed form. See Section 6 for an example.

From (1.1), if \( \mu \) has no shift and \( \alpha \neq 1 \), then

\[
\Phi_\mu(z) = |z|\alpha V(u_z),
\]

with

\[
V(u) = \int_{S^{d-1}} |\langle u, v \rangle|^\alpha \left[ 1 - i\tan(\pi\alpha/2)\text{sign}\langle u, v \rangle \right] \lambda(dv).
\]

(3.5)

The following is a standard result.

**Lemma 1.** If \( \mu \) has characteristic exponent (3.5), then

\[
\sup_{S^{d-1}} |V| \leq \frac{\lambda(S^{d-1})}{|\cos(\pi \alpha/2)|} < \infty.
\]

and the distribution is nondegenerate if and only if \( \inf_{S^{d-1}} \text{Re}(V) > 0 \).

From Theorem 1, if \( \mu \) has a square-integrable spectral spherical density \( P \), then

\[
V = \sum_{j=0}^\infty a_j \pi_j P \quad \text{with} \quad a_j = \begin{cases} w_j(\alpha) & j \text{ even}, \\
-i\tan(\pi\alpha/2)w_j(\alpha) & \text{else}. \end{cases}
\]

(3.6)

Moreover, the following is true.

**Corollary 2.** Let \( |z|\alpha V(u_z) \) be the characteristic exponent of an \( \alpha \)-stable distribution \( \mu \) with \( \alpha \neq 1 \). Then \( \mu \) has a square-integrable spectral spherical density if and only if

\[
\sum j^{2\alpha+d}\|\pi_j V\|_{L^2(S^{d-1})}^2 < \infty.
\]
Let $X \sim \mu$. Given $u \in S^{d-1}$, $Y = \langle u, X \rangle$ is univariate $\alpha$-stable. Since its characteristic exponent $\Phi_X(t)$ is equal to $\Phi_\mu(tu)$, from Theorem 1,

$$ \Phi_Y(t) = \begin{cases} 
|t|^\alpha \left( \sum_{j \text{ even}} w_j(\alpha) P_j(u) - i \tan \frac{\pi \alpha}{2} \sign t \sum_{j \text{ odd}} w_j(\alpha) P_j(u) \right), & \alpha \neq 1 \\
|t| \left( \sum_{j \text{ even}} w_j(1) P_j(u) + i \ln |t| \frac{2 \pi^{d/2-1}}{\Gamma(d/2 + 1)} P_1(u) \right) + \frac{2\alpha}{\pi} \sum_{j \text{ odd}} w_j^* P_j(u), & \alpha = 1 
\end{cases} $$

For different $\alpha$, while $\Phi_Y$ is different, it always satisfies the characterization in [14], Theorem 14.15. It follows that the spherical harmonics of $P$ satisfy

$$ \sum_{j \text{ even}} w_j(\alpha) P_j(u) \geq \sum_{j \text{ odd}} w_j(\alpha) P_j(u) $$

(3.7)

for all $\alpha \in (0, 2)$ and $u \in S^{d-1}$. Equality can hold for some $u \in S^{d-1}$, in which case the Lévy measure of $\langle u, X \rangle$ is concentrated on a half line according to the proof of Theorem 14.15 of [14]. On the other hand, the result below holds. For $z \in \mathbb{C}$, denote by $\arg z$ the principal argument of $z$, i.e., the unique $\theta \in (-\pi, \pi]$ with $z = |z|e^{i\theta}$.

**Corollary 3.** Let $P \neq 0$ be a polynomial.

1) Strict inequality holds in (3.7).

2) If $\alpha \neq 1$, then for $V$ in (3.5), $\sup_{u \in S^{d-1}} |\arg V(u)| < (\pi/2) \min(\alpha, 2 - \alpha)$.

### 3.1 Proof of Theorem 1

Recall that $\lambda = P \omega$ is a finite measure. By assumption, $P \in L^2(S^{d-1})$. Since $P_j = \pi_j P$, then $P = \sum_{j=0}^{\infty} P_j$ in $L^2(S^{d-1})$. Write (1.1) as

$$ \Phi_\mu(z) = \begin{cases} 
\int_{S^{d-1}} |\langle z, v \rangle|^\alpha \lambda(dv) - i \tan \frac{\pi \alpha}{2} \int_{S^{d-1}} |\langle z, v \rangle|^\alpha \sign \langle z, v \rangle \lambda(dv) & \alpha \neq 1, \\
\int_{S^{d-1}} |\langle z, v \rangle| \lambda(dv) + \frac{2}{\pi} \int_{S^{d-1}} \langle z, v \rangle \ln |\langle z, v \rangle| \lambda(dv) & \alpha = 1. 
\end{cases} $$

(3.8)

Since $|\langle z, v \rangle| = |z| \cdot |\langle u_z, v \rangle|$ is bounded and symmetric in $v$, by $P_j(-v) = (-1)^j P_j(v)$,

$$ \int_{S^{d-1}} |\langle z, v \rangle|^\alpha \lambda(dv) = |z|^\alpha \sum_{j \text{ even}} \int_{S^{d-1}} |\langle u_z, v \rangle|^\alpha P_j(v) \omega(dv), $$

$$ \int_{S^{d-1}} |\langle z, v \rangle|^\alpha \sign \langle z, v \rangle \lambda(dv) = |z|^\alpha \sum_{j \text{ odd}} \int_{S^{d-1}} |\langle u_z, v \rangle|^\alpha \sign \langle u_z, v \rangle P_j(v) \omega(dv). $$

For $j \in \mathbb{Z}_+$, let $\epsilon_j = 1\{j \text{ is odd}\}$. The next step is to evaluate

$$ \int_{S^{d-1}} |\langle u, v \rangle|^\alpha (\sign \langle u, v \rangle)^\epsilon_j P_j(v) \omega(dv). $$
The following derivation applies to all $\alpha \in (0, \infty)$. Since the mapping $t \mapsto |t|^\alpha (\mathrm{sign} \, t)^\epsilon$ is continuous, by Funk-Hecke formula (2.5), for all $h \in \mathcal{H}_{j,d}$ and $u \in \mathbb{S}^{d-1}$,

$$
\int_{\mathbb{S}^{d-1}} |\langle u, v \rangle|^\alpha (\mathrm{sign} \, (u, v))^{\epsilon_j} h(v) \omega(dv) = w_j(\alpha) h(u), \quad (3.9)
$$

where

$$
w_j(\alpha) = A(\mathbb{S}^{d-2}) \int_{-1}^1 |t|^\alpha (\mathrm{sign} \, t)^\epsilon \tilde{C}^{(d-2)/2}_j(t) (1 - t^2)^{(d-3)/2} \, dt. \quad (3.10)
$$

Other than the exact values of $w_j(\alpha)$, (3.1) follows immediately from (3.8) and (3.9) if $\alpha \neq 1$. With similar consideration, it can be expected that (3.2) holds as well. Thus, the task is to show that $w_j(\alpha)$ and $w_j^*$ are given by (3.3) and (3.4), respectively.

From the polynomial expression of $\tilde{C}^{(d-2)/2}_j(t)$, it is possible to obtain a closed form of $w_j(\alpha)$ from (3.10) by integration term by term. However, this calculation does not directly lead to the desired form of $w_j(\alpha)$. Instead, by [2], Exercise 6.28, if $b > 0$, then

$$
\tilde{C}^b_j(t) = t^{\epsilon_j} \binom{-[j/2], [j/2] + \epsilon_j + b}{b + 1/2} \binom{1}{1 - t^2} \quad (3.11)
$$

and by continuity, the identity still holds if $b = 0$. Then by (3.10),

$$
w_j(\alpha) = A(\mathbb{S}^{d-2}) L_j(\alpha, (d - 2)/2), \quad (3.11)
$$

where for $a > 0$, $b \geq 0$, and $j \in \mathbb{Z}_+$,

$$
L_j(a, b) = \int_{-1}^1 |t|^{a+\epsilon_j} \binom{-[j/2], [j/2] + \epsilon_j + b}{b + 1/2} (1 - t^2)^{b-1/2} \, dt.
$$

To evaluate $L_j(a, b)$, by change of variable $s = 1 - t^2$,

$$
L_j(a, b) = \int_0^1 s^{b-1/2} (1 - s)^{(a+\epsilon_j-1)/2} \binom{-[j/2], [j/2] + \epsilon_j + b}{b + 1/2} s \, ds.
$$

By [2], Theorem 2.2.4, for $p_i, q_i \in \mathbb{C}$ with $\mathrm{Re} \, q_i > 0$, $i = 1, 2$, and $x \in \mathbb{C} \setminus \{1, \infty\}$,

$$
\int_0^1 s^{q_1-1} (1 - s)^{q_2-1} \binom{p_1, p_2}{q_1, q_2} x s \, ds = B(q_1, q_2) \binom{p_1, p_2}{q_1 + q_2} x,
$$

where $B(q_1, q_2) = \Gamma(q_1)\Gamma(q_2)/\Gamma(q_1 + q_2)$ is the Beta function. If $p_1$ or $p_2$ is a nonpositive integer, then on each side of the display the $\binom{p_1, p_2}{q_1, q_2}$ function is a polynomial of finite degree, and so the identity holds for all $x \in \mathbb{C}$, in particular, for $x = 1$. Then

$$
L_j(a, b) = B(b + 1/2, (a + \epsilon_j + 1)/2) \binom{-[j/2], [j/2] + \epsilon_j + b}{b + (a + \epsilon_j)/2 + 1}.
$$
By Gauss’ formula (2.6),
\[
L_j(a, b) = B(b + 1/2, (a + \epsilon_j + 1)/2) \\
\quad \times \frac{\Gamma(b + (a + \epsilon_j)/2 + 1)\Gamma((a - \epsilon_j)/2 + 1)}{\Gamma(b + (a + \epsilon_j)/2 + 1 + \lfloor j/2 \rfloor)\Gamma((a - \epsilon_j)/2 + 1 - \lfloor j/2 \rfloor)} \\
= \frac{\Gamma(b + (a + j)/2 + 1)\Gamma((a - j)/2 + 1)}{\Gamma(b + (a + j)/2 + 1)\Gamma((a - j)/2 + 1)} \times \Gamma((a + 1)/2)\Gamma(a/2 + 1),
\]
the second equality due to \(\Gamma((a + \epsilon_j + 1)/2)\Gamma((a - \epsilon_j)/2 + 1) = \Gamma((a + 1)/2)\Gamma(a/2 + 1)\) and \(\epsilon_j/2 + \lfloor j/2 \rfloor = j/2\). Then by Legendre’s duplication formula (2.7),
\[
L_j(a, b) = \frac{\sqrt{\pi} \Gamma(b + 1/2)\Gamma(a + 1)}{2^a \Gamma(b + (a + j)/2 + 1)\Gamma((a - j)/2 + 1)}.
\] (3.12)
By (3.11), the first expression in (3.3) follows. Then by Euler’s reflection formula (2.8), the second expression in (3.3) follows. This proves (3.1) for \(\alpha \neq 1\) and also yields the terms \(w_j(1)P_j(u_z)\) in (3.2) with \(j\) being even for \(\alpha = 1\).

For \(\alpha = 1\), it only remains to evaluate the second integral on the r.h.s. of (3.8). First,
\[
\int_{\mathbb{S}^{d-1}} \langle z, v \rangle \ln |\langle z, v \rangle| \lambda(dv) = |z| \int_{\mathbb{S}^{d-1}} \langle u_z, v \rangle (\ln |z| + \ln |\langle u_z, v \rangle|) \lambda(dv) \\
= |z| \ln |z| \int_{\mathbb{S}^{d-1}} \langle u_z, v \rangle \lambda(dv) + |z| \int_{\mathbb{S}^{d-1}} \langle u_z, v \rangle \ln |\langle u_z, v \rangle| \lambda(dv).
\]
Given \(z \neq 0, \langle u_z, \cdot \rangle \in \mathcal{H}_{1,d}\), so the first integral on the r.h.s. equals \(\int_{\mathbb{S}^{d-1}} \langle u_z, v \rangle P_1(v) \omega(dv)\). Then by \(\langle u, v \rangle = |\langle u, v \rangle| \operatorname{sign} \langle u, v \rangle\),\n\[
\int_{\mathbb{S}^{d-1}} \langle z, v \rangle \ln |\langle z, v \rangle| \lambda(dv) = |z| \ln |z| w_1(1)P_1(u_z) + |z| I(u_z),
\]
where
\[
I(u) = \int_{\mathbb{S}^{d-1}} \langle u, v \rangle \ln |\langle u, v \rangle| \lambda(dv) = \sum_{j \text{ odd}} \int_{\mathbb{S}^{d-1}} \langle u, v \rangle \ln |\langle u, v \rangle| P_j(v) \omega(dv).
\]
From the first expression in (3.3),
\[
w_1(1) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}.
\] (3.13)
Following the proof for \(\alpha \neq 1\), for each odd \(j \geq 1\), there is a constant \(w_j^*\) such that
\[
\int_{\mathbb{S}^{d-1}} \langle u, v \rangle \ln |\langle u, v \rangle| P_j(v) \omega(dv) = w_j^* P_j(u),
\]
which together with the last two displays yields
\[
\int_{\mathbb{S}^{d-1}} \langle z, v \rangle \ln |\langle z, v \rangle| \lambda(dv) = |z| \ln |z| \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} P_1(u_z) + |z| \sum_{j \text{ odd}} w_j^* P_j(u_z).\] (3.14)
Then by (3.8), (3.2) holds. The task now is to show (3.4) for odd \( j \geq 1 \). Following the proof of (3.11), \( w_j = A(S^{d-2}) L_j(1, (d - 2)/2) \), where for \( a > 0 \) and \( b \geq 0 \),

\[
\tilde{L}_j(a, b) = \int_{-1}^{1} |t|^{a+1} \ln |t| F_1 \left( - \left\lfloor j/2 \right\rfloor, \left\lfloor j/2 \right\rfloor + 1 + b, 1 - t^2 \right) (1 - t^2)^{b-1/2} \, dt.
\]

By dominated convergence, \( \tilde{L}_j(a, b) = \partial L_j(a, b) / \partial a \). Then by (3.12),

\[
\tilde{L}_j(a, b) = \left[ - \ln 2 + \psi(a + 1) - \psi(b + (a + j)/2 + 1/2) \right] L_j(a, b) + \frac{\sqrt{\pi} \Gamma(b + 1/2) \Gamma(a + 1)}{2^a \Gamma(b + (a + j)/2 + 1)} g'((a - j)/2 + 1),
\]

(3.15)

where \( \psi(x) = \Gamma'(x)/\Gamma(x) \) and \( g(x) = 1/\Gamma(x) \). Let \( a = 1 \) and \( b = (d - 2)/2 \). Multiply both sides of (3.15) by \( A(S^{d-2}) = 2^a \Gamma(d-1)/\Gamma(d-1/2) \). Then

\[
w_j^* = [\ln 2 + \psi(2) - \psi((d + 1 + j)/2)/2] w_j(1) + \frac{\pi^{d/2}}{2 \Gamma((d + 1 + j)/2)} g'((3 - j)/2).
\]

From [2], p. 13, for \( n \in \mathbb{Z}_+ \), \( \psi(n + 1) = - \gamma + s_n \), \( \psi(n + 1/2) = - \gamma - 2 \ln 2 + 2 s_{2n} - s_n \), where \( \gamma \) is Euler’s constant and \( s_n = s_{n-1} + 1/n \) for \( n \geq 1 \) with \( s_0 = 0 \). Then \( \psi(2) = 1 - \gamma \) and \( g'(1) = - \psi(1)/\Gamma(1) = \gamma \), which together with (3.13) yields

\[
w_1^* = \left[ - \ln 2 + 1 - \frac{\gamma + \psi(2 + 1)}{2} \right] \frac{\pi^{d/2}}{\Gamma(d+1/2)}.
\]

The expression of \( \psi(d/2 + 1) \) in terms of \( s_n \)’s then yields \( w_1^* \) in (3.4). For \( j > 1 \) odd, since \((j - 1)/2\) is a positive integer, from the first expression in (3.3), \( w_j(1) = 0 \). This can also be derived using Rodrigues formula ([2], Eq. (6.4.14)): for any \( b \),

\[
C_j^b(t)(1 - t^2)^{b-1/2} = \frac{(-2)^j (b)_j}{j!(j + 2b)_j} [(1 - t^2)^{b+j-1/2}(j)].
\]

Then \( w_j(1) = 0 \) for \( d > 2 \) follows from (3.10) and integration by parts. The case \( d = 2 \) can be shown by continuity argument or using the property of Tchebyshev polynomial \( C_j^0(t) = T_j(t) \) (cf. [2], p. 101).

On the other hand, \( g(x) = \pi^{-1} \sin(\pi x) \Gamma(1 - x) \) by Euler’s reflection formula (2.8). Then \( g'(x) = \cos(\pi x) \Gamma(1 - x) - \pi^{-1} \sin(\pi x) \Gamma'(1 - x) \) and so for \( n \in \mathbb{Z}_+ \), \( g'(-n) = (-1)^n \Gamma(n + 1) \), which together with (3.13) gives \( w_j^* \) for odd \( j > 1 \) in (3.4). This shows (3.2) for \( \alpha = 1 \).

It only remains to show that (3.1) and (3.2) are a.c. It suffices to show

\[
\sum_{j=0}^{\infty} |w_j(\alpha)| |P_j| < \infty, \quad \sum_{j=0}^{\infty} |w_j^*| |P_j| < \infty.
\]

(3.16)

**Lemma 2.** For \( j \geq 1 \) and \( P \in \mathcal{H}_{j,d} \),

\[
\sup_{S^{d-1}} |P|^2 \leq \frac{C_{j,d}}{A(S^{d-1})} \int_{S^{d-1}} |P|^2 \, d\omega,
\]

where \( c_{j,d} = \dim(\mathcal{H}_{j,d}) \) is specified in (2.1).
Proof. Let \( \sigma^2 = \int_{S^d-1} |P|^2 \, d\omega \) and \( \varphi_1 = P/\sigma \). Then \( \varphi_1 \in H_{j,d} \) and there are \( \varphi_2, \ldots, \varphi_{c_{j,d}} \in H_{j,d} \), such that together with \( \varphi_1 \) they form an orthonormal basis of \( H_{j,d} \). Then by [2], Theorem 9.6.3 or [15], Corollary IV.2.9(b),

\[
\sup_{S^d-1} |P|^2 = \sigma^2 \sup_{S^d-1} |\varphi_1|^2 \leq \sigma^2 \sup_{S^d-1} \sum_{i=1}^{c_{j,d}} |\varphi_i|^2 = \sigma^2 \cdot \frac{c_{j,d}}{A(S^{d-1})}. \]

Let \( \sigma_j^2 = \int_{S^d-1} |P_j|^2 \, d\omega \). Then \( \sum_{j=0}^{\infty} \sigma_j^2 = \sigma^2 = \int_{S^d-1} |P|^2 \, d\omega \). By Cauchy–Schwarz inequality and Lemma 2,

\[
\left( \sum_{j=0}^{\infty} |w_j(\alpha)| \sup_{S^d-1} |P_j| \right)^2 \leq \left( \sum_{j=0}^{\infty} \frac{\sup_{S^d-1} |P_j|^2}{\sigma_j^2} \right) \left( \sum_{j=0}^{\infty} \sigma_j^2 \right) \left( \sum_{j=0}^{\infty} c_{j,d} w_j(\alpha)^2 \right) \leq \frac{\sigma^2}{A(S^{d-1})} \left( \sum_{j=0}^{\infty} \sigma_j^2 \right) \sum_{j=0}^{\infty} c_{j,d} w_j(\alpha)^2.
\]

From the second expression in (3.3) and Stirling’s formula (2.9), as \( j \to \infty \), \( w_j(\alpha) = O(j^{-d/2}/\alpha) \). Then by (2.10), \( c_{j,d} w_j(\alpha)^2 = O(j^{-d/2}) \), which implies the first half of (3.16). Next, from (3.4), \( w_j^2 = O(j^{-d/2}) \), so by similar argument, the second half of (3.16) follows.

3.2 Proof of other results

Proof of Corollary 2. Let \( \lambda \) be the spectral spherical measure of \( \mu \). First show that \( \lambda \) has a density in \( L^2(S^{d-1}) \) if and only if \( \Delta = \sum_{j=0}^{\infty} \|a_j\|^2 \|\pi_j V\|^2 < \infty \), where \( \|\cdot\| \) denotes \( \|\cdot\|_{L^2(S^{d-1})} \) for simplicity and \( a_j \) are defined in (3.6). Notice that since \( \alpha \neq 1 \), \( a_j \neq 0 \) for all \( j \in \mathbb{Z}_+ \). Also, since \( V \) is bounded by Lemma 1, it is clearly in \( L^2(S^{d-1}) \).

Suppose \( \lambda = P \omega \) with \( P \in L^2(S^{d-1}) \). Then by (3.6), \( a_j^{-1} \pi_j V = \pi_j P \), so \( \Delta = \sum_{j=0}^{\infty} \|\pi_j P\|^2 = \|P\|^2 < \infty \). Conversely, suppose \( \Delta < \infty \). Then \( \sum_{j=0}^{\infty} a_j^{-1} \pi_j V \) converges in \( L^2(S^{d-1}) \). Let \( Q \) be the limit. We need the following result, which is also useful later.

Lemma 3. \( Q \) is real-valued on \( S^{d-1} \).

Proof. Let \( S_j = (-i)^j \pi_j (V) \), \( j \in \mathbb{Z}_+ \). From \( V(-u) = \overline{V(u)} \), \( S_j \) is real-valued. Indeed, in general, suppose \( f \in L^2(S^{d-1}) \) and define \( f^* \in L^2(S^{d-1}) \) such that \( f^*(u) = f(-u) \). Then from (2.2), \( \pi_j f^* = (-1)^j \pi_j f \) and \( \overline{\pi_j f} = \pi_j \overline{f} \). If \( f^* = \overline{f} \), then \( (-i)^j \pi_j f = i^j \pi_j f^* = -(-i)^j \pi_j f \), so \( (-i)^j \pi_j f \) is real-valued. Now \( Q = \sum (i^j a_j^{-1}) S_j \). From (3.6), \( i^j a_j^{-1} \) is real-valued for all \( j \in \mathbb{Z}_+ \). Then \( Q \) is real-valued.

Continuing the proof of Corollary 2, since by Cauchy–Schwarz inequality \( \int_{S^d-1} |Q|^2 \, d\omega \leq \{A(S^{d-1}) \int_{S^d-1} |P|^2 \, d\omega\}^{1/2} < \infty \), \( \tilde{\lambda} = Q \omega \) is a finite (signed) measure on \( S^{d-1} \). Denote by \( \nu \) the Lévy measure of \( \mu \). From Remark 14.6 and the proof of Theorem 14.10 of [14],

\[
\nu(B) = \frac{1}{C_\alpha} \int_{S^{d-1}} \lambda(d\nu) \int_{r>0} 1\{rv \in B\} r^{-1-\alpha} \, dr, \quad (3.17)
\]
Then by a Cingz
Note that in general the first two equalities only hold for a.e. from (3.17),
Proof of Corollary 3. 1) Fix α ∈ (0, 2). If X ∼ µ has Lévy measure ν, then for u ∈ S^{d-1},
\langle u, X \rangle has Lévy measure ν_1(B) = ν(\{x: \langle u, x \rangle \in B \}). If equality holds for u in (3.7), then
ν_1 is concentrated on a half line, so that, say  ν_1((0, ∞)) = ν(\{x: \langle u, x \rangle > 0 \}) = 0. Then
from (3.17), \Lambda(\{v ∈ S^{d-1} : \langle u, v \rangle > 0 \}) = 0, giving P(v) = 0 for all v ∈ S^{d-1} with \langle u, v \rangle > 0.
Since P is a polynomial, this implies P ≡ 0, a contradiction.

2) From Lemma 1 and 1),
\[ \arg V = \arg[1 - ir(V) \tan(\pi \alpha/2)] = \begin{cases} -\arctan[r(V) \tan(\pi \alpha/2)] & \text{if } \alpha \in (0,1) \\ \arctan[r(V) \tan(\pi/2)] & \text{if } \alpha \in (1,2) \end{cases} \]
where \( r(V) = \text{Im}(V)/\text{Re}(V) \) is continuous on \( S^{d-1} \) with maximum absolute value strictly less than 1. Then the claim follows.

4 Density when \( \alpha \in (0,1) \)

In this section, let \( \mu \) be a nondegenerate \( \alpha \)-stable distribution with \( \alpha \in (0,1) \). Without loss of generality, assume \( \mu \) has no shift, so that its characteristic exponent is \( \Phi_{\mu}(z) = |z|^\alpha V(u_z) \) as in (3.5). From Lemma 1, \( V^p \) is bounded for any \( p \in \mathbb{R} \), where the principal branch of the power function is used when \( p \) is a noninteger. Denote
\[ S_{j,p} = (-i)^{j} \pi_j(V^p), \quad j \in \mathbb{Z}_+. \] (4.1)

Then \( S_{j,p} \in \mathcal{H}_{j,d} \) and
\[ V^p = \sum_{j=0}^{\infty} i^j S_{j,p} \text{ in } L^2(S^{d-1}). \] (4.2)

Since \( [V(-u)]^p = [V(u)]^p \), from the proof of Lemma 3, it is seen that \( S_{j,p} \) is real-valued.

**Theorem 4.** Let \( \mu \) be an \( \alpha \)-stable distribution on \( \mathbb{R}^d \), \( d \geq 2 \), with \( \alpha \in (0,1) \) and characteristic exponent (3.5). If \( \lambda = P\omega \) with \( 0 \neq P \in \mathcal{P}_{q,d} \) for some \( q \in \mathbb{Z}_+ \), then \( \hat{\mu} \in L^1(\mathbb{R}^d) \) and the p.d.f. of \( \mu \) is
\[ g(x) = \sum_{k=1}^{\infty} \frac{(-2\alpha)^k \pi^{-d/2}}{k! |x|^{\alpha+d}} \sum_{j=0}^{kq} \frac{\Gamma((j + k\alpha + d)/2)}{\Gamma((j - k\alpha)/2)} S_{j,k}(u_x), \quad x \neq 0. \] (4.3)

The series is uniformly a.c. in \( \{x: |x| \geq r\} \) for any \( r > 0 \).

**Remark.**

1) By (3.6), if \( P \) is a polynomial of degree \( q \), i.e., \( P \in \mathcal{P}_{q,d} \), then for each \( k \in \mathbb{Z}_+ \), \( V^k \in \mathcal{P}_{kq,d} \), so from Section 2, \( S_{j,k} = 0 \) for \( j > kq \), and each \( S_{j,k} \) with \( j \leq kq \) has a closed form. As a result, all the terms in the series (4.3) have closed form expressions.

2) If \( V \) is a constant \( c > 0 \), then \( S_{j,k} = c^k \) for \( j = 0 \) and 0 for \( j > 0 \). Consequently, (4.3) yields the well known result for the spherically symmetric case ([16], Eq. (7.5.6)).

3) It would be more satisfactory if an a.c. series similar to (4.3) could be obtained for \( P \in L^2(S^{d-1}) \). Since given \( k \), \( \Gamma((j + k\alpha + d)/2)/\Gamma((j - k\alpha)/2) \) grows in the same order as \( j^{k\alpha} \), the desired absolute convergence would require that \( S_{j,k}(u) = (\pi_j V^k)(u) \) vanish rapidly as \( j \to \infty \). However, it is unclear whether this is actually the case.
In general, the tail asymptotic behavior of a multivariate stable distribution is quite complicated, depending on the fractal dimensions on the spectral spherical measure [17]. However, [12], Theorem 4.2 suggests that under the condition of Theorem 4, as \(|x| \to \infty\), \(g(x)\) should behave as \(CP(u_x)/|x|^{a+d}\) for some constant \(C > 0\). Indeed, from (4.3),

\[
g(x) = -\frac{2\alpha\pi^{-d/2}}{|x|^{a+d}} \sum_{j=0}^{\infty} \frac{\Gamma((j + \alpha + d)/2)}{\Gamma((j - \alpha)/2)} S_{j,1}(u_x) + O(|x|^{-2\alpha-d}).
\]

On the other hand, from (3.1) and (3.6),

\[
S_{j,1} = (-i)^j a_j P_j = (-1)^{(j+\epsilon_j)/2}[\tan(\pi\alpha/2)]^{\epsilon_j} w_j(\alpha) P_j,
\]

where \(\epsilon_j = 1\{j \text{ is odd}\}\) and \(P_j = \pi_j P\). Then by the second expression in (3.3),

\[
\frac{\Gamma((j + \alpha + d)/2)}{\Gamma((j - \alpha)/2)} S_{j,1} = 2^{1-\alpha} \pi^{d/2-1}\Gamma(\alpha + 1)(-1)^{(j+\epsilon_j)/2}[\tan(\pi\alpha/2)]^{\epsilon_j} \sin((j - \alpha)\pi/2) P_j
\]

\[
= -2^{1-\alpha} \pi^{d/2-1}\Gamma(\alpha + 1) \sin(\pi\alpha/2) P_j.
\]

It follows that

\[
g(x) = \frac{2\alpha\Gamma(\alpha) \sin(\pi\alpha/2)}{\pi} \frac{P(u_x)}{|x|^{a+d}} + O(|x|^{-2\alpha-d}). \tag{4.4}
\]

For the general case, where \(\mu\) may not have a spectral spherical density, by using some of the arguments in the proof for Theorem 4, the following result can be proved.

**Proposition 5.** Let \(\mu\) be a nondegenerate \(\alpha\)-stable distribution on \(\mathbb{R}^d\), \(d \geq 2\), with \(\alpha \in (0,1)\) and characteristic exponent (3.5). Let the p.d.f. of \(\mu\) be \(g\). For \(x \neq 0\), put

\[
\phi_j(x) = \sum_{k=1}^{\infty} \frac{(-2\alpha)^k \pi^{-d/2}}{k! |x|^{\alpha+d}} \frac{\Gamma((j + k\alpha + d)/2)}{\Gamma((j - k\alpha)/2)} S_{j,k}(u_x), \quad j \in \mathbb{Z}_+.
\]

1) The series in (4.5) is uniformly a.c. in \(\{x: |x| \geq c\}\) for any \(c > 0\).

2) Given \(r > 0\), \(\phi_j(r \cdot) \in \mathcal{H}_{j,d}\) and \(g(r \cdot) = \sum_{j=0}^{\infty} \phi_j(r \cdot)\) in \(L^2(\mathbb{S}^{d-1})\), that is, \(\pi_j g(r \cdot) = \phi_j(r \cdot)\).

3) The p.d.f. of \(|X|\) with \(X \sim \mu\) is equal to

\[
g_{|X|}(r) = \sum_{k=1}^{\infty} \frac{(-2\alpha)^k \pi^{-d/2}}{k! |x|^{\alpha+d+1}} \frac{\Gamma((k\alpha + d)/2)}{\Gamma(-k\alpha/2)} \int_{\mathbb{S}^{d-1}} V^k \, d\omega. \tag{4.6}
\]

Combining the argument that leads to (4.4) and Proposition 5, it can be seen that if \(\mu\) has a square-integrable spectral spherical density \(P\), then for each \(j \in \mathbb{Z}\),

\[
(\pi_j g)(x) = \frac{2\alpha\Gamma(\alpha) \sin(\pi\alpha/2)}{\pi} \frac{(\pi_j P)(u_x)}{|x|^{a+d}} + O(|x|^{-2\alpha-d})
\]

as \(|x| \to \infty\). This suggests the tail asymptotic in (4.4) may hold, which would be a more satisfactory result. Finally, consider the asymptotic expansion of \(g(x)\) as \(x \to 0\). The treatment is simpler than the previous results and follows the one for the univariate case (cf. [16], Section 4.2).
Proposition 6. Let $\mu$ be a nondegenerate $\alpha$-stable distribution on $\mathbb{R}^d$, $d \geq 2$, with $\alpha \in (0,1)$ and characteristic exponent (3.5). Then given $n \in \mathbb{Z}_+$, as $x \to 0$,

$$g(x) = \frac{2^{1-d}}{\pi^{d/2} \alpha} \sum_{k=0}^{n} \Gamma((k + d)/\alpha) \left(\frac{|x|}{2}\right)^{k/2} \sum_{m=0}^{k} \frac{(-1)^m S_{k-2m-\alpha}^{-\alpha}(u_x)}{m! \Gamma(k - m + d/2)} + O(|x|^{n+1}). \quad (4.7)$$

Furthermore,

$$g(0) = \frac{\Gamma(d/\alpha)}{\alpha(2\pi)^d} \int V^{-d/\alpha} \, d\omega.$$

4.1 Proof of Theorem 4

By Lemma 1, $\hat{\mu}(x) = e^{-|x|^\alpha V(u_x)}$ is integrable. Then by dominated convergence, $g(x) = \lim_{x \to 0} g(x)$, where

$$g(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i(x,z)} e^{-\epsilon|x| - |z|^\alpha V(u_x)} \, dz.$$

Since $\alpha < 1$ and $V$ is bounded on $\mathbb{S}^{d-1}$, $e^{-\epsilon|x| - |z|^\alpha V(u_x)}$ is integrable. Then

$$g(x) = (2\pi)^{-d} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{R}^d} e^{-i(x,z)} e^{-\epsilon |z|^{\alpha} [V(u_x)]^k} \, dz.$$

Put $b = (d - 2)/2$. Given $\epsilon > 0$, for each $k \geq 0$,

$$\int_{\mathbb{R}^d} e^{-i(x,z)} e^{-\epsilon |z|^{\alpha} [V(u_x)]^k} \, dz$$

$$= \int_{0}^{\infty} e^{-\epsilon s} s^{k \alpha + d - 1} \left\{ \int_{\mathbb{S}^{d-1}} e^{-i|x| s(u_x,v)} [V(v)]^k \omega(dv) \right\} \, ds$$

$$= \int_{0}^{\infty} e^{-\epsilon s} s^{k \alpha + d} \left\{ \frac{(2\pi)^d/2}{|x|^b} \sum_{j=0}^{k} \langle i \rangle^{j+b} J_{j+b}(|x| s) \pi_j V^k(u_x) \right\} \, ds$$

$$= \int_{0}^{\infty} e^{-\epsilon s} s^{k \alpha + d/2} \left\{ \frac{(2\pi)^d/2}{|x|^b} \sum_{j=0}^{k} J_{j+b}(|x| s) S_{j,k}(u_x) \right\} \, ds,$$

where the second equality follows from (2.13) and the last one from $(-i)^j \pi_j V^k = S_{j,k}$ for $j > kq$. Then

$$\int_{\mathbb{R}^d} e^{-i(x,z)} e^{-\epsilon |z|^{\alpha} [V(u_x)]^k} \, dz = \sum_{j=0}^{kq} F_{j,k}(x) S_{j,k}(u_x),$$

where

$$F_{j,k}(x) = (2\pi)^{d/2} |x|^{-b} \int_{0}^{\infty} e^{-\epsilon s} s^{k \alpha + d/2} J_{j+b}(|x| s) \, ds$$

$$= \frac{(2\pi)^{d/2}}{|x|^{k \alpha + d}} \int_{0}^{\infty} e^{-\epsilon s} s^{k \alpha + d/2} J_{j+b}(s) \, ds.$$
with $\delta = \delta(x) = \epsilon/|x|$. As a result,

$$g_\epsilon(x) = (2\pi)^{-d} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \sum_{j=0}^{kq} F_{j,k,\epsilon}(|x|) S_{j,k}(u_x). \quad (4.8)$$

Given $x \neq 0$, let $\epsilon \downarrow 0$. Then $\delta \downarrow 0$ and by [2], Eq. (9.10.5),

$$F_{j,k,\epsilon}(|x|) \to \frac{(2\pi)^{d/2} 2^{k\alpha+d/2} \Gamma((j + k\alpha + d)/2)}{|x|^{k\alpha+d}} \Gamma((j - k\alpha)/2).$$

Note that when $j = k\alpha$, the r.h.s. is zero. Thus, once it is shown that on the r.h.s. of (4.8), the order of the infinite sum over $k$ and the limit can interchange, then (4.3) follows.

The rest of the proof except for its very end is devoted to the proof of the interchangeability. For $a \geq 0$ and $c > 0$, by [2], Theorem 4.11.3,

$$\int_0^\infty e^{-bs} J_a(s) s^{c-1} \, ds = \frac{\Gamma(a+c)}{2^a \Gamma(a+1)} \frac{\Gamma((a+c)/2, (a+c+1)/2)}{a+1}. \quad (4.9)$$

Then by Pfaff identity ([2], Theorem 2.2.5), the r.h.s. is equal to

$$\frac{\Gamma(a+c)}{2^a \Gamma(a+1)} \left(1 + \frac{1}{\delta^2}\right)^{(a+c)/2} \frac{\Gamma((a+c)/2, (a+c+1)/2)}{a+1} \frac{1}{1+\delta^2}.$$

Then, with $a = j + b$ and $c = k\alpha + d/2 + 1$,

$$F_{j,k,\epsilon}(|x|) = \frac{(2\pi)^{d/2} (1 + \delta^2)^{-\frac{j+k\alpha+d}{2}}}{|x|^{k\alpha+d}} \frac{2^j \Gamma((j + k\alpha + d)/2)}{2j+1+d/2} L_{jk} \left(\frac{1}{1+\delta^2}\right),$$

where

$$L_{jk}(x) = \frac{\Gamma(j + k\alpha + d)}{\Gamma(j + d/2)} \frac{\Gamma((j + k\alpha + d)/2, (j - k\alpha - 1)/2)}{j + d/2} x^m \frac{2^j \Gamma((j + k\alpha + d)/2)}{2j+1+d/2} \frac{\Gamma((j + k\alpha + d)/2, (j - k\alpha - 1)/2)}{j + d/2} x^m.$$

As a result, by (4.8)

$$g_\epsilon(x) = \sum_{k=0}^\infty \frac{(-1)^k (2\pi)^{-d/2}}{|x|^{k\alpha+d}} \sum_{j=0}^{kq} \frac{1 + \delta^2)^{-\frac{j+k\alpha+d}{2}}}{2j+1+d/2} L_{jk} \left(\frac{1}{1+\delta^2}\right) S_{j,k}(u_x). \quad (4.9)$$

To prove the interchangeability of the infinite sum over $k$ and the limit as $\epsilon \to 0$, it suffices to consider the sum over $k \gg 1$. For $j = 0, \ldots, kq$, if $j \geq k\alpha + 1$, then every
coefficient in the series expression of $L_{jk}(x)$ is nonnegative, so for $|x| \leq 1$, $|L_{jk}(x)| \leq L_{jk}(1)$. By Gauss’ formula (2.6) followed by Legendre’s duplication formula (2.7),

$$L_{jk}(1) = \frac{\Gamma(j + k\alpha + d)\Gamma(1/2)}{\Gamma((j - k\alpha)/2)\Gamma((j + k\alpha + 1 + d)/2)} \frac{2^{j+k\alpha+d-1}\Gamma((j + k\alpha + d)/2)}{\Gamma((j - k\alpha)/2)}.$$  \hfill (4.10)

Since $\Gamma(x)$ is log-convex on $x > 0$, and since $j \leq kq$, from the above display,

$$L_{jk}(1) \leq \frac{2^{k(q+\alpha)+d-1}\Gamma((kq + k\alpha + d)/2)}{\Gamma((kq - k\alpha)/2)}.$$  

Since $j \leq kq$, by assuming $j \geq k\alpha + 1$, $q > \alpha$. Apply Stirling’s formula (2.9) to get

$$L_{jk}(1) \leq \text{const}^k \left( \frac{kq + k\alpha + d}{2e} \right)^{(kq+k\alpha+d)/2} \left( \frac{kq - k\alpha}{2e} \right)^{(kq-k\alpha)/2} \leq \text{const}^k \times k^{k\alpha+d/2} \leq \text{const}^k \times \Gamma(k\alpha),$$

where $\text{const}$ denotes a constant independent of $(k,j)$. It follows that

$$|L_{jk}(x)| \leq \text{const}^k \times \Gamma(k\alpha).$$ \hfill (4.11)

On the other hand, if $j < k\alpha + 1$, then for $|x| \leq 1$,

$$|L_{jk}(x)| \leq \frac{\Gamma(j + k\alpha + d)}{\Gamma(j + d/2)} \sum_{m=0}^{\infty} b_m$$  \hfill (4.12)

with

$$b_m = \frac{((j + k\alpha + d)/2)_m((j - k\alpha - 1)/2)_m}{m!(j + d/2)_m}.$$

Let $m_{jk} = \lfloor (k\alpha + 1 - j)/2 \rfloor$. Then for $m \geq m_{jk}$, $((j - k\alpha - 1)/2)_m \geq 0$, so $((j - k\alpha - 1)/2)_m$ has the same sign as $((j - k\alpha - 1)/2)_{m_{jk}}$. Then

$$\sum_{m=0}^{\infty} b_m = \sum_{m < m_{jk}} b_m + \left| \sum_{m \geq m_{jk}} \frac{((j + k\alpha + d)/2)_m((j - k\alpha - 1)/2)_m}{m!(j + d/2)_m} \right| \leq 2 \sum_{m < m_{jk}} b_m + \left| _2F_1 \left( \frac{j + k\alpha + d}{2}, \frac{j - k\alpha - 1}{2}; \frac{j + d}{2} \right) \right|.$$  

By Gauss’ formula (2.6) and Euler’s reflection formula (2.8),

$$_2F_1 \left( \frac{j + k\alpha + d}{2}, \frac{j - k\alpha - 1}{2}; 1 \right) = \frac{\Gamma(j + d/2)\sqrt{\pi}}{\Gamma((j - k\alpha)/2)\Gamma((j + k\alpha + 1 + d)/2)} \frac{\Gamma(j + d/2)\Gamma(1 + (k\alpha - j)/2)\sin((j - k\alpha)\pi/2)}{\sqrt{\pi}\Gamma((j + k\alpha + 1 + d)/2)}.$$  

19
In particular, when \( j = k = 0 \), \( _2F_1(d/2, −1/2; d/2; 1) = 0 \). Then

\[
\sum_{m=0}^{\infty} b_m \leq 2 \sum_{m<m_{jk}} b_m + \frac{\Gamma(j + d/2)\Gamma(1 + (k\alpha - j)/2)}{\Gamma((j + k\alpha + 1 + d/2)}/. \tag{4.13}
\]

For \( 0 \leq m < m_{jk} \), \((j - k\alpha - 1)/2 \leq (j - k\alpha - 1)/2 + m < 0 \), so \(|(j - k\alpha - 1)/2 + m| < (k\alpha + 1 - j)/2\). As a result, \(|((j - k\alpha - 1)/2)_m| \leq |(k\alpha + 1 - j)/2|^m\). On the other hand,

\[
\frac{(j + k\alpha + d)/2}_m \leq \frac{(j + k\alpha + d + 1)/2}_m
\]

and by \( j < k\alpha + 1 \), the r.h.s. is increasing in \( m \). Therefore,

\[
\sum_{m<m_{jk}} b_m \leq \frac{(j + k\alpha + d + 1)/2}_{m_{jk}} \sum_{m<m_{jk}} [(k\alpha + 1 - j)/2]^m
\]

\[
\leq \frac{(j + k\alpha + d + 1)/2}_{m_{jk}} e^{(k\alpha + 1/2)} \leq \text{const}^k \times \frac{(j + k\alpha + d)/2}_{m_{jk}}.
\]

Combine this with (4.12) and (4.13), and then plug in \( m_{jk} = [(k\alpha + 1 - j)/2] \) to get

\[
|L_{jk}(x)| \leq \text{const}^k \times \frac{\Gamma(j + k\alpha + d)}{\Gamma((j + k\alpha + d)/2)} \frac{\Gamma((j + k\alpha + d)/2 + m_{jk})}{\Gamma(j + d/2 + m_{jk})}
\]

\[
+ \frac{\Gamma(j + k\alpha + d)\Gamma(1 + (k\alpha - j)/2)}{\Gamma(j + k\alpha + 1 + d/2)}
\]

\[
\leq \text{const}^k \times \frac{\Gamma(j + k\alpha + d)}{\Gamma((j + k\alpha + d)/2)} \frac{\Gamma(k\alpha + (d + 3)/2)}{\Gamma((j + k\alpha + d + 1)/2)}
\]

\[
+ \frac{\Gamma(j + k\alpha + d)\Gamma(1 + (k\alpha - j)/2)}{\Gamma((j + k\alpha + 1 + d)/2)}
\]

\[
\leq \text{const}^k \times \frac{1}{\Gamma(k\alpha + (d + 3)/2 + \Gamma((j + k\alpha + d)/2)\Gamma(1 + (k\alpha - j)/2)),
\]

where Legendre’s duplication formula (2.7) is used in the last line. Since \( k \gg 1 \), from (2.9),

\[
|L_{jk}(x)| \leq \text{const}^k \times [\Gamma(k\alpha) + \Gamma(1 + (j + k\alpha)/2)\Gamma(1 + (k\alpha - j)/2)].
\]

Recall \( 0 \leq j < k\alpha + 1 \). By the log-convexity of \( \Gamma(x) \) on \( x > 0 \), the r.h.s. is no greater than the value when \( j = k\alpha + 1 \). Then (4.11) again holds.

As a result, for \( k \gg 1 \), the \( k \text{th} \) summand in (4.9) has absolute value no greater than

\[
\text{const}^k \times \frac{\Gamma(k\alpha)}{[x]^{k\alpha + d k!}} \sum_{j=0}^{kq} |S_{j,k}(u_x)|. \tag{4.14}
\]

Since \( S_{j,k} = (−i)^j \pi_j(V^k) \in \mathcal{H}_{j,d} \), by Lemma 2,

\[
\sup_{\mathbb{S}^{d-1}} |S_{j,k}|^2 \leq \frac{c_{j,d}}{A(\mathbb{S}^{d-1})} |S_{j,k}|^2_{L^2(\mathbb{S}^{d-1})} \leq \frac{c_{j,d}}{A(\mathbb{S}^{d-1})} \|V^k\|^2_{L^2(\mathbb{S}^{d-1})} \leq c_{j,d} \sup_{\mathbb{S}^{d-1}} |V|^{2k}.
\]

20
Then by (2.10) and Lemma 1
\[
\sup_{S_{d-1}} |S_{j,k}| \leq \sqrt{c_{j,d}} \sup_{S_{d-1}} |V|^k \leq \text{const} \times (j + 1)^{d/2 - 1} \sup_{S_{d-1}} |V|^k.
\] (4.15)

It follows that (4.14) is bounded by \( D_k = \text{const}^k \times \Gamma(k\alpha)/(|x|^{k\alpha+d}k!) \). Since \( \alpha \in (0,1) \), by Stirling’s formula (2.9), \( \sum_{k=0}^{\infty} D_k < \infty \). Then by dominated convergence, the desired interchangeability follows. Finally, by combining (4.11) and the bound on \( \sup_{S_{d-1}} |S_{j,k}| \), for any constant \( M > 0 \),
\[
\sum_{k=1}^{\infty} \frac{M^k}{k!} \sum_{j=0}^{kq} |L_{jk}(1)| \sup_{S_{d-1}} |S_{j,k}| \leq \sum_{k=1}^{\infty} \frac{M^k}{k!} \sum_{j=0}^{kq} [\text{const}^k \times \Gamma(k\alpha)(j + 1)^{d/2 - 1}] < \infty.
\]

Then from the form of \( L_{jk}(1) \) in (4.10), it easily follows that the series (4.3) is uniformly a.c. in \( \{x: |x| \geq r\} \) for any \( r > 0 \).

**4.2 Proof of Proposition 5**

1) From (4.15), Euler’s reflection formula (2.8), and log-convexity of the Gamma function, each term in (4.5) with \( k \geq j/\alpha \) has absolute value no greater than
\[
\frac{2\pi^k}{k!|x|^{k\alpha+d}} \Gamma((j + k\alpha + d)/2) \Gamma(1 + (k\alpha - j)/2) \times \text{const}^k \times (j + 1)^{d/2 - 1}
\leq \frac{\text{const}^k}{k!|x|^{k\alpha+d}} \Gamma(k\alpha + d/2)(k\alpha + 1)^{d-2}.
\]

Since \( \alpha \in (0,1) \), for any \( c > 0 \), the sum of the above terms over \( k \geq j/\alpha \) is uniformly convergent on \( \{x: |x| \geq c\} \), yielding the proof of 1).

2) Fix \( r > 0 \). From 1), \( \phi_j(r) \) is a continuous function on \( S_{d-1} \). Then by \( S_{j,k} \in \mathcal{H}_{j,d} \), \( \phi_j(r) \in \mathcal{H}_{j,d} \). On the other hand, \( g(r) \) is bounded so it is in \( L^2(S_{d-1}) \). Thus, to prove 2), it suffices to show \( \pi_j g(r) = \phi_j(r) \).

Write \( h = \pi_j g(r) \) and \( b = d/2 - 1 \). Since \( \inf_{S_{d-1}} \text{Re}(V) > 0 \) by Lemma 2, from the integral representation of \( \pi_j \) in (2.2) and Fubini’s theorem, for each \( u \in S_{d-1} \),
\[
h(u) = \pi_j \left[ (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\pi |z|^{(u_z,\cdot)} e^{-|z|^{\alpha}V(u)} \, dz} \right] (u)
= (2\pi)^{-d} \int_{\mathbb{R}^d} (\pi_j e^{-i\pi |z|^{(u_z,\cdot)}})(u) e^{-|z|^{\alpha}V(u)} \, dz.
\]

Change the last integral into polar and coordinates and plug in (2.14). Then
\[
h(u) = \frac{(2\pi)^{-d/2}}{r_b} \int_0^{\infty} s^{d/2} (-i)^j J_{j+b}(rs) \left[ \frac{C_{j,d}}{A(S_{d-1})} \int_{S_{d-1}} \tilde{C}_{j}^b((u,v)) e^{-s^\alpha V(v)} \, \omega(dv) \right] \, ds.
\]

Then by (2.2) again,
\[
h(u) = \frac{(2\pi)^{-d/2} (-i)^j}{r_b} \int_0^{\infty} s^{d/2} J_{j+b}(rs) \pi_j(e^{-s^\alpha V})(u) \, ds.
\] (4.16)
Note that by Fubini’s theorem, each integral in the above three displays is well-defined in the \( L^1 \) sense. Then by dominated convergence, \( h(u) = \lim_{\epsilon \downarrow 0} h_\epsilon(u) \), where

\[
h_\epsilon(u) = \frac{(2\pi)^{-d/2}}{r_b} \int_0^\infty s^{d/2} J_{j+b}(rs) \pi_j(e^{-\epsilon u V})(u)e^{-\epsilon s} \, ds
\]

\[
= \frac{(2\pi)^{-d/2}}{r_b} \int_0^\infty s^{d/2} J_{j+b}(rs) \sum_{k=0}^{\infty} \frac{(-s^\alpha k)(\pi_j V^k)(u)e^{-\epsilon s} \, ds}{k!}
\]

\[
= \frac{(2\pi)^{-d/2}}{r_b^{k\alpha+d}} \sum_{k=0}^{\infty} \frac{(-1)^k S_{j,k}(u)}{k!} \int_0^\infty s^{k\alpha+d/2} J_{j+b}(s)e^{-\delta s} \, ds
\]

with \( \delta = \epsilon/r \). From (4.8),

\[
h_\epsilon(u) = \sum_{k=0}^{\infty} \frac{(-1)^k (2\pi)^{-d/2}}{r^{k\alpha+d} k!} \left( 1 + \frac{\delta^2}{2} \right)^{-(j+k\alpha+d)/2} \frac{L_{jk} \left( \frac{1}{1 + \delta^2} \right) S_{j,k}(u)}{2^{j-1+d/2}}
\]

and from the last paragraph in the proof of Theorem 4, each term on the r.h.s. has absolute value bounded by \( C^k \int_{S^{d-1}} |V|^2k \, d\omega \cdot \Gamma(k\alpha)/(r^{k\alpha+d} k!) \), where \( C \) is a constant. Then by the same dominated convergence argument following (4.8), the proof of 2) follows.

3) The joint p.d.f. of \(|X|\) and \(u_X\) is \( f(r, \theta) = r^{d-1} g(r\theta)\), which is bounded on \( S^{d-1}\) for given \( r > 0 \). Integrate over \( \theta \in S^{d-1}\). From 2) and \( \int_{S^{d-1}} \phi \, d\omega = 0 \) for \( \phi \in \mathcal{H}_{j,d} \) if \( j \geq 1 \), \( g|X|(r) = r^{d-1} \int_{S^{d-1}} \phi(0) \, \omega(\nu) \, d\nu \). By 1), the last integral can be done term by term for the series in (4.5). By (2.2), \( S_{0,k} \) is constant \( \frac{1}{A(S^{d-1})} \int_{S^{d-1}} V^k \, d\omega \). Then \( \int_{S^{d-1}} S_{0,k} \, d\omega = \int_{S^{d-1}} V^k \, d\omega \) and (4.6) follows.

### 4.3 Proof of Proposition 6

Fix \( n \in \mathbb{Z}_+ \). By Taylor’s theorem, for \( t \in \mathbb{R} \),

\[
e^{-it} = \sum_{k=0}^{n} \frac{(-it)^k}{k!} + \frac{1}{n!} \int_0^t s^n(-i)^{n+1} e^{-i(s-t)} \, ds.
\]

Denote the integral on the r.h.s. by \( R_n(t) \). Then \( |R_n(t)| = O(|t|^{n+1}) \). As a result, for \( x, z \in \mathbb{R}^d \), \( |R_n(\langle x, z \rangle)| = |z|^{n+1} O(|x|^{n+1}) \). Since by Lemma 1, \( \inf_{S^{d-1}} \text{Re}(V) > 0 \), then \( \int_{\mathbb{R}^d} |z|^{n+1} e^{-|z|^{\alpha} \text{Re}(V)(u_z)} \, dz < \infty \). Then

\[
g(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \left[ \sum_{k=0}^{n} \frac{(-i\langle x, z \rangle)^k}{k!} + R_n(\langle x, z \rangle) \right] e^{-|z|^{\alpha} V(u_z)} \, dz
\]

\[
= (2\pi)^{-d} \sum_{k=0}^{n} \frac{(-i)^k}{k!} \int_{\mathbb{R}^d} \langle x, z \rangle^k e^{-|z|^{\alpha} V(u_z)} \, dz + O(|x|^{n+1}).
\]

Given \( k = 0, \ldots, n \), in polar coordinates,

\[
\int_{\mathbb{R}^d} \langle x, z \rangle^k e^{-|z|^{\alpha} V(u_z)} \, dz = |x|^k \int_{S^{d-1}} \langle u_x, v \rangle^k \left( \int_0^\infty s^{k+d-1} e^{-V(v)s^n} \, ds \right) \omega(\nu) \, d\nu
\]

\[
= \frac{\Gamma\left( \frac{k+d}{\alpha} \right)}{\alpha} |x|^k \int_{S^{d-1}} \langle u_x, v \rangle^k [V(v)]^{-(k+d)/\alpha} \omega(\nu) \, d\nu.
\]
Since \( \langle u_x, \cdot \rangle \) and \( V^{-(k+d)/\alpha} \) are both in \( L^2(S^{d-1}) \), by (4.2)

\[
\int_{\mathbb{R}^d} \langle x, z \rangle^k e^{-|z|^\alpha V(u_z)} \, dz = \frac{\Gamma(k+d)}{\alpha |x|^k} \sum_{j=0}^{\infty} j! \int_{S^{d-1}} \langle u_x, v \rangle^k S_{j,-(k+d)/\alpha}(v) \, \omega(\mathrm{dv}).
\]  

(4.17)

Put \( P_j = S_{j,-(k+d)/\alpha} \). Because \( P_j \) is a homogeneous polynomial with degree \( j \) and given \( u \in S^{d-1}, \langle u, \cdot \rangle^k \) is a homogeneous polynomial with degree \( k \), if \( k - j \) is odd, then the integral \( \int_{S^{d-1}} \langle u, v \rangle^k P_j(v) \, \omega(\mathrm{dv}) \) is equal to 0, while if \( k - j \) is even, it is equal to

\[
\int_{S^{d-1}} |\langle u, v \rangle|^k (\text{sign} \langle u, v \rangle)^{j} P_j(v) \, \omega(\mathrm{dv}).
\]

From (3.9) and the first expression in (3.3), the integral is equal to \( w_j(k) P_j(u) \) with

\[
w_j(k) = \frac{\pi^{d/2} \Gamma(k+1)}{2^{k-1} \Gamma(\frac{k+j+d}{2}) \Gamma(\frac{k-j}{2} + 1)}.
\]

Write \( j = k - 2m \), where \( m \) is an integer. Then \( i^j = i^k(-1)^m \) and \( w_{k-2m}(k) = 0 \) if \( m < 0 \). Meanwhile, since \( j \geq 0 \), then \( m \leq k/2 \). Combine all the information with the r.h.s. of the above display for \( g(x) \). Then (4.7) follows. Finally, let \( n = 0 \) in (4.7). Notice that for \( u \in S^{d-1}, S_{0,-d/\alpha}(u) \) is the constant \( \frac{1}{A(S^{d-1})} \int V^{-d/\alpha} \, d\omega \). Then by the continuity of \( g \) at 0, the expression of \( g(0) \) is obtained.

5 Density when \( \alpha \in (1, 2) \)

In this section, let \( \mu \) be a nondegenerate \( \alpha \)-stable distribution with \( \alpha \in (1, 2) \). As before, let \( \mu \) have no shift, so that \( \Phi_{\mu}(z) = |z|^\alpha V(u_z) \) as in (3.5).

With \( \alpha > 1 \), the first step to get the p.d.f. of \( \mu \) follows the one for the univariate case; cf. [16], Section 4.2 and also the proof of Proposition 6. That is, in evaluating the inverse transform (1.2), expand \( e^{-i(x,z)} \) into a power series of \( z \) and then integrate term by term. This leads to the result below. Let the spherical harmonics \( S_{i,p} \) be as in (4.1).

**Proposition 7.** Let \( \mu \) be a nondegenerate \( \alpha \)-stable distribution on \( \mathbb{R}^d, d \geq 2 \), with \( \alpha \in (1, 2) \) and characteristic exponent (3.5). Then the p.d.f. of \( \mu \) is

\[
g(x) = \frac{2^{1-d}}{\pi^{d/2} \alpha} \sum_{n=0}^{\infty} \Gamma((n+d)/\alpha) \left( \frac{|x|}{2} \right)^n \frac{|n/2|}{m! \Gamma(n - m + d/2)} (-1)^m S_{n-2m,-(n+d)/\alpha}(u_x), \quad x \in \mathbb{R}^d.
\]

(5.1)

The series is uniformly a.c. in \( \{x: |x| \leq M\} \) for any \( M > 0 \). If \( X \sim \mu \), then \( |X| \) has p.d.f.

\[
g_{|X|}(r) = \frac{1}{\pi^{d/2} \alpha} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma((2n+d)/\alpha)}{n! \Gamma(n + d/2)} \left( \frac{r}{2} \right)^{2n+d-1} \int_{S^{d-1}} V^{(2n+d)/\alpha} \, d\omega.
\]

(5.2)
For spherically symmetric \( \alpha \)-stable distributions, the following construction based on subordination is well known (cf. [14], section 30). Let \( Z = (Z_1, \ldots, Z_d) \), where \( Z_i \sim N(0, 1) \) are independent. Let \( \zeta \) be a positive \((\alpha/2)\)-stable random variable independent of \( Z \) such that for \( t > 0 \), \( E(e^{-t\zeta}) = e^{-t^{\alpha}/2} \). Given constant \( V_0 > 0 \), for \( z \in \mathbb{R}^d \), \( E(e^{i(z,V_0^{\alpha}/\alpha)\sqrt{\mathbb{R}Z}}) \) gives \( E(e^{-V_0^{\alpha}/\alpha|z|^2\zeta}) = e^{-|z|^\alpha V_0} \). As a result, \( V_0^{1/\alpha} \sqrt{2\mathbb{C}Z} \) is spherically symmetric and \( \alpha \)-stable with characteristic exponent \( \Phi(z) = |z|^\alpha V_0 \). For more general \( \alpha \)-stable distributions, Proposition 7 has the useful consequence below.

**Corollary 8.** The p.d.f. in (5.2) can be written as

\[
g_{x}(r) = \frac{r^{d-1}}{2\pi^{d/2}} \int_{\mathbb{S}^{d-1}} \frac{1}{V^{d/\alpha}} \exp \left\{ -\frac{r^2}{4V^{2/\alpha} \zeta} \right\} \, d\omega. \tag{5.3}
\]

From (2.2), the series in (5.1) consists of integrals of the form \( \int \langle u, v \rangle^k [V(u)]^{-p} \omega(du) \) with \( k \in \mathbb{Z}_+ \) and \( p > 0 \). No closed form expressions of the integrals are known except when \( V \) is a constant \( c \), in which case \( S_{n-2m-(n+d)/\alpha} = c^{-(n+d)/\alpha} \) if \( n = 2m \) and 0 otherwise, yielding the p.d.f. of a spherically symmetric stable distribution ([16], p. 212). To get a more explicit representation, one way is to expand \( h(z) = z^{-p} \) into a power series, plug in \( z = V \), and integrate the resulting series term by term. Because \( h(z) \) is singular at 0 and \( V \) only takes values in the right half of the complex plane, one may expand \( h(z) \) at some \( R \in (0, \infty) \). Since the resulting power series is convergent only in the disc \( \{ z : |z-R| < R \} \), it is necessary that \( |V - R| < R \). This is guaranteed by the result below.

**Lemma 4.** Let \( g(z) = |z|^2/(2 \text{Re } z) \). Then

\[
0 < g_0 = \sup_{\mathbb{S}^{d-1}} g(V) < \frac{\lambda(\mathbb{S}^{d-1})}{1 + \cos(\pi \alpha)} < \infty.
\]

Moreover, given any \( R > g_0 \), letting \( V_* = R - V \), \( \sup_{\mathbb{S}^{d-1}} |V_*| < R \).

Fixing \( R \) as in Lemma 4, \( V^{-p} = (R - V_*)^{-p} \) can be expanded into a power series of \( V_* \) that can be integrated term by term. Since \( [V_*(-u)]^k = [V_* u]^k \), similar to (4.2),

\[
S_{j,k}^* = (-i)^j \pi_j(V_*^k) \in H_{j,d} \text{ is real-valued, } \quad j \geq 0,
\]

\[
V_*^k = \sum_{j=0}^{\infty} i^j S_{j,k}^* \text{ in } L^2(\mathbb{S}^{d-1}). \tag{5.4}
\]

**Theorem 9.** Let \( \mu \) be a nondegenerate \( \alpha \)-stable distribution on \( \mathbb{R}^d \), \( d \geq 2 \), with \( \alpha \in (1, 2) \) and characteristic exponent (3.5). Let \( R \) and \( S_{j,k}^* \) be defined as above. Then the p.d.f. of \( \mu \) can be written as

\[
g(x) = \frac{2^{1-d}}{\pi^{d/2} \alpha} \sum_{n=0}^{\infty} \binom{|x|}{2}^{n/2} \frac{(-1)^m}{m! \Gamma(n - m + d/2)} \sum_{k=0}^{\infty} \frac{\Gamma(k + (n + d)/\alpha)}{k! R^{k + (n + d)/\alpha}} S_{n-2m,k}^*(u_x), \quad x \in \mathbb{R}^d. \tag{5.5}
\]
The series is uniformly a.c. in \( \{ x : |x| \leq M \} \) for any \( M > 0 \). If \( X \sim \mu \), then \( |X| \) has p.d.f.

\[
g_{|X|}(r) = \frac{(r/2)^{d-1}}{\pi^{d/2} \alpha} \sum_{n=0}^{\infty} \frac{(-r^2/4)^n}{n! \Gamma(n + d/2)} \sum_{k=0}^{\infty} \frac{\Gamma(k + (2n + d)/\alpha)}{k! R^{k+(2n+d)/\alpha}} \int_{S^{d-1}} V_k^* d\omega. \tag{5.6}
\]

Remark. If \( \mu \) has a polynomial spectral spherical density \( P \) of degree \( q \), i.e., \( P \in \mathcal{P}_{q,d} \), then by (3.6), \( V_* = R - V \in \mathcal{P}_{q,d} \). Consequently, the first remark that follows Theorem 4 applies to \( S^*_n \). Similarly, \( \int_{S^{d-1}} V_k^* d\omega \) can be expressed in closed form. \( \square \)

Finally, consider the asymptotic expansion of \( g(x) \) as \( |x| \to \infty \). It is well known that all the partial derivatives of \( g(x) \) tend to 0 as \( |x| \to \infty \) ([14], Proposition 28.1). However, it is hard to get a grasp on the asymptotic behavior of \( g(x) \) from Proposition 7 or Theorem 9, as their representations of \( g(x) \) are in terms of positive integral powers of \( |x| \). Since given \( |x| \), \( g \) is characterized by its behavior on the sphere \( |x| \cdot S^{d-1} \), one essentially needs to consider the asymptotic expansion of the function \( g(r \cdot) \) as \( r \to \infty \). The following partial result supplies an asymptotic expansion of the spherical harmonic of \( g(r \cdot) \) of a given degree.

**Proposition 10.** Let \( \mu \) be a nondegenerate \( \alpha \)-stable distribution on \( \mathbb{R}^d \), \( d \geq 2 \), with \( \alpha \in (1, 2) \), characteristic exponent (3.5), and p.d.f. \( g \). Fix \( j \in \mathbb{Z}_+ \). Then for each \( n \geq 0 \), as \( r \to \infty \),

\[
\sup_{S^{d-1}} \left| \pi_j g(r \cdot) - \sum_{k=0}^{n} \frac{(-2\alpha)^k\pi^{-d/2} \Gamma((j + k\alpha + d)/2)}{k! \Gamma((j - k\alpha)/2)} S_{j,k} \right| = O(r^{-(n+1)\alpha-d}). \tag{5.7}
\]

Furthermore, if \( g_{|X|}(r) \) denotes the p.d.f. of \( |X| \), then as \( r \to \infty \),

\[
g_{|X|}(r) = \sum_{k=0}^{n} \frac{(-2\alpha)^k\pi^{-d/2} \Gamma((1 + k\alpha + d)/2)}{k! \Gamma((1 - k\alpha)/2)} \int V_k^* d\omega + O(r^{-(n+1)\alpha-1}). \tag{5.8}
\]

It would be more satisfactory to have a full asymptotic expansion of \( g \). Eq. (5.7) indicates that the expansion would have the form as the series representation (4.3), and in view of [12], Theorem 4.2 or (4.4), its first term would be \( 2\pi^{-1}\alpha \Gamma(\alpha) \sin(\pi\alpha/2) r^{-\alpha-d} P \), where \( P \) is the spectral spherical density of \( \mu \). Thus, if \( \mu \) has no spectral spherical density, an asymptotic expansion of \( g \) is unlikely to exist. On the other hand, even if \( \mu \) has a polynomial spectral spherical density \( P \), except for \( P \) being a constant, the method of the paper has yet been able to yield the desired expansion.

### 5.1 Proof of Proposition 7

Given \( x \), since \( \int_{\mathbb{R}^d} e^{i\langle x,z \rangle - |z|^\alpha \text{Re}(V(z))} \, dz < \infty \) due to \( \alpha > 1 \) and \( \inf_{S^{d-1}} \text{Re}(V) > 0 \), by dominated convergence followed by changing to polar coordinates,

\[
g(x) = (2\pi)^{-d} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^d} \langle x,z \rangle^n e^{-|z|^\alpha V(z)} \, dz.
\]
Treat each of the integrals on the r.h.s. the same way as in the proof of Proposition 6; see the argument from (4.17) to the end of that proof. Then (5.1) follows. Given \( M > 0 \), to show the uniform and absolute convergence of (5.1) on \( \{ x : |x| \leq M \} \), for \( 0 \leq m \leq \lfloor n/2 \rfloor \), similar to (4.15),

\[
\sup_{S_{d-1}} |S_{n-2m,-(n+d)/\alpha}| \leq \sqrt{c_{n-2m,d}} \sup_{S_{d-1}} |V^{-(n+d)/\alpha}| \leq \frac{\text{const} \times (n - 2m + 1)^{d/2-1}}{\inf_{S_{d-1}} \text{Re}(V))^{(n+d)/\alpha}}.
\]

Then by Lemma 1, there is a constant \( C > 0 \), such that \( \sup_{S_{d-1}} |S_{n-2m,-(n+d)/\alpha}| \leq C^n \) for all \( n \) and \( 0 \leq m \leq n/2 \). Consequently, for all \( |x| \leq M \) and \( 0 \leq m \leq n/2 \),

\[
\left| \Gamma((n + d)/\alpha) \left( \frac{|x|}{2} \right)^n \frac{(-1)^m S_{n-2m,-(n+d)/\alpha}(u_x)}{m! \Gamma(n - m + d/2)} \right| \leq \frac{\Gamma((n + d)/\alpha)(CM/2)^n}{m! \Gamma(n - m + d/2)}
\]

so the uniform and absolute convergence of the series (5.1) follows from

\[
\sum_{0 \leq m \leq n/2} \frac{\Gamma((n + d)/\alpha)C^n}{m! \Gamma(n - m + d/2)} < \infty \quad \text{for any } C > 0.
\] (5.9)

To see (5.9), first, by log-convexity of the Gamma function,

\[
m! \Gamma(n - m + d/2) \geq [\Gamma((n + 1 + d/2)/2)]^2 \quad \text{for } 0 \leq m \leq n/2.
\]

Since \( \Gamma(x) \) is increasing on \([1, \infty)\) (\([10], \S 5.4(iii)\)) and \( d \geq 2 \), by Legendre’s duplication formula (2.7), for \( n \geq 1 \), \( [\Gamma((n + 1 + d/2)/2)]^2 \geq \Gamma((n + 1)/2)\Gamma((n + 1)/2) = 2^{-n}n!\sqrt{\pi} \). Then (5.9) follows by \( \alpha > 1 \).

From (5.1), the joint p.d.f. of \(|X|\) and \(u_X\) is

\[
r^{d-1}g(r\theta) = \frac{(r/2)^{d-1}}{\pi^{d/2}\alpha} \sum_{0 \leq m \leq n/2} \Gamma((n + d)/\alpha)(r/2)^n \frac{(-1)^m S_{n-2m,-(n+d)/\alpha}(\theta)}{m! \Gamma(n - m + d/2)}.
\]

Given \( r > 0 \), since the series in (5.1) is uniformly a.c. on \( \{ x : |x| = r \} \), its integral over \( \theta \) can be done term-by-term, so that

\[
g_{|X|}(r) = \int_{S_{d-1}} r^{d-1}g(r\theta) \omega(d\theta)
\]

\[
= \frac{(r/2)^{d-1}}{\pi^{d/2}\alpha} \sum_{0 \leq m \leq n/2} \Gamma((n + d)/\alpha)(r/2)^n \int_{S_{d-1}} \frac{(-1)^m S_{n-2m,-(n+d)/\alpha}(\theta)}{m! \Gamma(n - m + d/2)} \omega(d\theta).
\]

Since \( \int_{S_{d-1}} S_{0,-(n+d)/\alpha} \omega = \int_{S_{d-1}} V^{-(n+d)/\alpha} \omega \) and \( \int_{S_{d-1}} S_{n-2m,-(n+d)/\alpha} \omega = 0 \) if \( n > 2m \), the p.d.f. of \(|X|\) in (5.2) is obtained.

### 5.2 Proof of Corollary 8

Since \( \zeta \) has characteristic exponent \(|t|^{\alpha/2}\exp\{-i(\alpha/2)(\pi/2) \text{ sign } t\} \), the Mellin transform of its p.d.f., denoted \( q \), yields that for all \( s \in \mathbb{C} \) with \(-1 < \text{Re } s < \alpha/2\),

\[
E(\zeta^s) = \frac{2\Gamma(-2s/\alpha)}{\alpha \Gamma(-s)}
\]
where all the inequalities are strict. By Cauchy–Schwarz inequality, the first line yields

\[ g_{|\mathcal{X}|}(r) = \frac{1}{2\pi d/2} \int_{S^{d-1}} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n 2\Gamma((2n + d)/\alpha)}{n! \Gamma(n + d/2)} \left( \frac{r}{2} \right)^{2n + d - 1} \frac{1}{V(2n+d)/\alpha} \right] d\omega \]

\[ = \frac{1}{2\pi d/2} \int_{S^{d-1}} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n n! E(\zeta^{n-d/2})}{\alpha} \left( \frac{r}{2} \right)^{2n + d - 1} \frac{1}{V(2n+d)/\alpha} \right] d\omega, \]

yielding (5.3).

5.3 Proof of Lemma 4

By Lemma 1, \( \inf_{S^{d-1}} \Re(V) > 0 \), giving \( \varrho_0 > 0 \). It is easy to check that if \( \Re z > 0 \), then \( |z - \varrho(z)| = \varrho(z) \). Geometrically, \( \varrho(z) \) is the radius of the unique circle that is centered on \( \mathbb{R} \) and passes both 0 and \( z \). Then for any \( r > \varrho(z) \), the circle with center and radius both equal to \( r \) encircles \( z \), i.e., \( |z - r| < r \). Provided that \( \varrho_0 < \infty \), from the geometric interpretation, the last claim of the lemma holds. Thus, it only remains to show the upper bound on \( \varrho_0 \). From (1.1) and \( \alpha \in (1, 2) \),

\[ |V(u)| < \frac{1}{\cos(\pi \alpha/2)} \int_{S^{d-1}} |\langle u, v \rangle|^\alpha \lambda(dv) < \frac{1}{\cos(\pi \alpha/2)} \int_{S^{d-1}} |\langle u, v \rangle| \lambda(dv), \]

\[ \Re(V(u)) = \int_{S^{d-1}} |\langle u, v \rangle|^\alpha \lambda(dv) > \int_{S^{d-1}} \langle u, v \rangle^2 \lambda(dv), \]

where all the inequalities are strict. By Cauchy–Schwarz inequality, the first line yields

\[ |V(u)|^2 < \frac{\lambda(S^{d-1})}{\cos(\pi \alpha/2)^2} \int_{S^{d-1}} \langle u, v \rangle^2 \lambda(dv). \]

Then for each \( u \in S^{d-1} \),

\[ \varrho(V(u)) = \frac{|V(u)|^2}{2 \Re(V(u))} < \frac{\lambda(S^{d-1})}{4 |\cos(\pi \alpha/2)|^2} = \frac{\lambda(S^{d-1})}{1 + \cos(\pi \alpha)}. \]

By the continuity of \( \varrho(V) \) on \( S^{d-1} \), the upper bound on \( \varrho_0 \) follows.

5.4 Proof of Theorem 9

By the definitions of \( S_{n-2m,-(n+d)/\alpha} \) and \( V_* \),

\[ \Gamma((n + d)/\alpha)S_{n-2m,-(n+d)/\alpha} = (-i)^{n-2m} \Gamma((n + d)/\alpha) \pi_{n-2m} \Gamma(V^{-(n+d)/\alpha} \pi_{n-2m} \Gamma((n + d)/\alpha) \pi_{n-2m} (1 - V_*/R)^{-(n+d)/\alpha}). \]
Then by $S_{n-2m,k}^* = (-i)^{n-2m} \pi_{n-2m}(V_*)^k$ and $\sum_{k=0}^\infty \frac{(n+d)/k!}{R^k} (-z)^k = (1+z)^{-c}$ for $c > 0$ and $|z| < 1$, the r.h.s. equals

$$(-i)^{n-2m} R^{-(n+d)/\alpha} \Gamma((n+d)/\alpha) \pi_{n-2m} \left[ \sum_{k=0}^\infty \frac{(n+d)/k!}{R^k} (V_*/R)^k \right]$$

$$= \sum_{k=0}^\infty \frac{\Gamma(k+(n+d)/\alpha)}{k! R^{k+(n+d)/\alpha}} S_{n-2m,k}^*.$$  

This combined with (5.1) then yields (5.5). Put $D = \sup_{S^d-1} |V_*|$. Similar to (4.15), for any $n \geq 0$ and $k \geq 0,$

$$\sup_{S^d-1} |S_{n-2m,k}^*| \leq \sqrt{c_{n-2m,d}} \sup_{S^d-1} |V_*|^k \leq \text{const} \times (n+1)^{d/2} D^k.$$  

Since $D < R$, then for any $M > 0,$

$$\sum_{n=0}^\infty M^n \sum_{m=0}^{[n/2]} \frac{1}{m! \Gamma(n - m + d/2)} \sum_{k=0}^\infty \frac{\Gamma(k+(n+d)/\alpha)}{k! R^{k+(n+d)/\alpha}} \sup_{S^d-1} |S_{n-2m,k}^*|$$

$$\leq \text{const} \times \sum_{n=0}^\infty (n+1)^{d/2} M^n \sum_{m=0}^{[n/2]} \frac{\Gamma((n+d)/\alpha)}{m! \Gamma(n - m + d/2) R^{(n+d)/\alpha}} \sum_{k=0}^\infty \frac{((n+d)/\alpha)_k}{(D/R)^k}$$

$$= \text{const} \times \sum_{n=0}^\infty (n+1)^{d/2} M^n \sum_{m=0}^{[n/2]} \frac{\Gamma((n+d)/\alpha)}{m! \Gamma(n - m + d/2) (R - D)^{-(n+d)/\alpha}},$$

which is finite by similar argument for (5.9). It follows that the series in (5.5) is uniformly a.c. in $\{x: |x| \leq M\}$. Finally, (5.6) follows by similar argument for (5.2).

### 5.5 Proof of Proposition 10

For $r > 0$ and $u \in S^{d-1},$ by change of variable,

$$g(ru) = (2\pi)^{-d} r^{-d} \int_{\mathbb{R}^d} e^{-i(u,z)} e^{-r^{-\alpha} |z|^\alpha V(u_z)} \, dz = (2\pi)^{-d/2} r^{-d} G_{r^{-\alpha}}(u),$$  

(5.10)

where for $t > 0,$

$$G_t(u) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i(u,z)} e^{-t|z|^\alpha V(u_z)} \, dz.$$  

From (4.16), for $j \in \mathbb{Z}_+,$

$$\pi_j G_t(u) = (-i)^j \int_0^\infty s^{d/2} J_{j+b}(s) \pi_j(e^{-ts^\alpha V})(u) \, ds.$$  

(5.11)
Lemma 5. Given \( k \in \mathbb{Z}_+ \), as \( t \to 0^+ \),

\[
\frac{\partial^k}{\partial t^k}(\pi_j G_t)(u) \to \frac{2^{d/2}(-2^\alpha)^k \Gamma((j + k\alpha + d)/2)}{\Gamma((j - k\alpha)/2)} S_{j,k}(u)
\]

uniformly in \( u \in \mathbb{S}^{d-1} \).

Lemma 6. Suppose \( f : (0,1] \to \mathbb{R} \) is \( n \) times differentiable such that for each \( 0 \leq k \leq n \), \( f^{(k)}(t) \) converges as \( t \to 0^+ \). Let the limits be \( a_k \). Then by defining \( f(0) = a_0 \), \( f \) is \( n \) times differentiable at \( 0 \) with (one-sided) derivatives \( f^{(k)}(0) = a_k \), \( 0 \leq k \leq n \).

Proof. For each \( t \in (0,1) \), by the mean value theorem, there is \( s \in (0,t) \) such that \( f'(s) = [f(t) - f(0)]/t \). As \( t \to 0^+ \), \( s \to 0^+ \) and so by assumption, \( [f(t) - f(0)]/t \to a_1 \). This shows \( f'(0) = a_1 \). The proof for higher order derivatives of \( f \) at \( 0 \) follows by induction. \( \square \)

Assuming Lemma 5 is true for now, by Lemma 6 and Taylor’s theorem, given \( n \geq 0 \), for \( u \in \mathbb{S}^{d-1} \) and \( t > 0 \),

\[
(\pi_j G_t)(u) = \sum_{k=0}^{n} \frac{2^{d/2}(-2^\alpha)^k \Gamma((j + k\alpha + d)/2)}{\Gamma((j - k\alpha)/2)} S_{j,k}(u) \frac{t^k}{k!} + \frac{\partial^{k+1}}{\partial t^{k+1}}(\pi_j G_{t^*})(u) \frac{t^{k+1}}{(k+1)!},
\]

where \( t^* = t^*(u) \) is some number in \( (0,t) \). As \( t \to 0^+ \), \( t^* \to 0^+ \), so by Lemma 5, the remainder term is \( O(t^{k+1}) \) with the implicit constant being uniform in \( u \). From the relationship (5.10), (5.7) follows. Since by (2.2),

\[
g_{|X|}(r) = r^{d-1} \int_{\mathbb{S}^{d-1}} g(\rho u) \omega(du) = r^{d-1} A(\mathbb{S}^{d-1}) \pi_0 g(r \cdot),
\]

by (5.7),

\[
g_{|X|}(r) = A(\mathbb{S}^{d-1}) \sum_{k=0}^{n} \frac{(-2^\alpha)^k \pi^{-d/2}}{k! r^{k\alpha + d}} \frac{\Gamma((1 + k\alpha + d)/2)}{\Gamma((1 - k\alpha)/2)} S_{0,k} + O(r^{-(n+1)\alpha-1}).
\]

Since \( S_{0,k} = \frac{1}{A(\mathbb{S}^{d-1})} \int_{\mathbb{S}^{d-1}} V^k \omega \), (5.8) follows, completing the proof of Proposition 10.

To prove Lemma 5, the following result is needed.

Lemma 7. Let \( \alpha \in (1,2) \). Given \( \tau > 0 \), \( p > 0 \), for \( \xi \in \mathbb{C} \) with \( \text{Re} \xi > 0 \) and \( s > 0 \), let

\[
f_s(\xi) = \int_0^\infty J_\tau(r) r^{p-1} e^{-sr^\alpha \xi} dr.
\]

Then, as \( s \to 0^+ \),

\[
f_s(\xi) \to \frac{2^{p-1} \Gamma((\tau + p)/2)}{\Gamma(1 + (\tau - p)/2)}
\]

uniformly in any compact set of \( \xi \in \{ z \in \mathbb{C} : \text{Re} z > 0 \} \).
For $\alpha = 1$, the limit is essential for the proof of Theorem 4; see the passage below (4.8). For $\alpha = 2$, the limit is also known (cf. [2], Theorem 4.11.7 followed by Theorem 4.2.2).

**Proof.** Let $r_0 = \tau \pi / 2 + \pi / 4$. By [2], Eq. (4.8.5), given $N \in \mathbb{N}$, $J_\tau(r) = \text{Re}(k_N(r)) + R_N(r)$, $r > 0$, where

$$k_N(r) = \sqrt{\frac{2}{\pi r}} \sum_{n=0}^{N} \frac{e^{i(r-r_0)}(1/2 - \tau)n(1/2 + \tau)_n}{n!(2ri)^n}$$

and $R_N(r) = O(r^{-N-1})$ as $r \to \infty$. Then

$$f_s(\xi) = \int_1^{\infty} J_\tau(r)r^{p-1}e^{-sr^\alpha\xi} dr + \int_1^{\infty} R_N(r)r^{p-1}e^{-sr^\alpha\xi} dr + \int_1^{\infty} \text{Re}(k_N(r))r^{p-1}e^{-sr^\alpha\xi} dr.$$  
Since $\text{Re} \xi > 0$, $|e^{-sr^\alpha\xi}| \leq 1$. By (2.12), $|J_\tau(r)|r^{p-1} = O(r^{\tau+p-1})$ as $r \to 0+$. Also, clearly $|R_N(r)r^{p-1}| = O(r^{p-2-N})$ as $r \to \infty$. Fix $N > p - 1$. Then by dominated convergence, as $s \to 0+$, the sum of the first two integrals on the r.h.s. tends to $\int_0^{\infty} J_\tau(r)r^{p-1} dr + \int_1^{\infty} R_N(r)r^{p-1} dr$. It is easy to see that the convergence is uniform in any compact set of $\xi \in \{ \xi : \text{Re} \xi > 0 \}$. For the third integral, since it is equal to

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N} \frac{(1/2 - \tau)_n(1/2 + \tau)_n}{n!2^n} \int_1^{\infty} [(-i)^n e^{i(r-r_0)} + i^n e^{-i(r-r_0)}]r^{p-n-3/2}e^{-sr^\alpha\xi} dr,$$

it suffices to consider, for any given $q \in \mathbb{R}$, the convergence of the integrals

$$\int_1^{\infty} e^{\pm ir^q}e^{-sr^\alpha\xi} dr.$$

Put $H_s(z) = e^{iz^q}e^{-sr^\alpha\xi}$. Then $H_s(z)$ is analytic on the right half of $\mathbb{C}$. Given a compact $D \subset \{ z : \text{Re} z > 0 \}$, fix $\theta_* \in (0, \pi/2)$ such that $a\theta_* + |\text{arg} \xi| < \pi/2$ for all $\xi \in D$. Given $R > 1$, consider the contour consisting of $[1, R]$, $\{ re^{i\theta} : r \in [1, R] \}$, $C = \{ e^{i\theta} : \theta \in [0, \theta_*] \}$, and $C_R = \{ Re^{i\theta} : \theta \in [0, \theta_*] \}$. Fixing $s > 0$, for $z \in C_R$, $|H_s(z)| = Re^{-R\sin(\text{arg} z) - sr^\alpha \xi |\cos(a\arg z + \text{arg} \xi)}$. By $|\alpha \arg z + \text{arg} \xi| \leq \alpha \theta_* + |\text{arg} \xi| < \pi/2$, $\inf_{z \in C_R} \cos(\alpha \arg z + \text{arg} \xi) > 0$. Then by $\alpha > 1$, as $R \to \infty$, $\int_{C_R} H_s(z) dz \to 0$, and the integral of $H_s(z)$ along the ray from 1 to $R$ and the integral along the ray from $e^{i\theta_*}$ to $Re^{i\theta_*}$ both converge. Therefore,

$$\int_1^{\infty} H_s(r) dr = \left( \int_C + \int_{e^{i\theta_*}}^{\infty} \right) H_s(z) dz$$

$$= i \int_0^{\theta_*} e^{i\theta + i(q+1)\theta - s \xi e^{i\theta}} d\theta + e^{i(q+1)\theta_*} \int_{\infty}^{e^{i\theta_*}} e^{ir^q} r^q e^{-sr^\alpha} e^{i\alpha \theta} \xi dr,$$

where the integral along $C$ is from 1 to $e^{i\theta_*}$. As $s \to 0+$, the first integral on the r.h.s. converges uniformly in $\xi \in D$. On the other hand,

$$|e^{ir^q + i(q+1)\theta} e^{-sr^\alpha} e^{i\alpha \theta} \xi| = e^{-r \sin \theta_* + r^q e^{-sr^\alpha} |\cos(a\theta_* + \text{arg} \xi)}.$$
By \( \sin \theta_\ast > 0 \) and dominated convergence, the second integral on the r.h.s. converges uniformly in \( \xi \in D \). As a result,

\[
\int_{1}^{\infty} H_s(r) \, dr \to i \int_{0}^{\theta_\ast} e^{i(q+1)\theta} \, d\theta + e^{i(q+1)\theta} \int_{1}^{\infty} e^{i\theta^*} r^{-q} \, dr
\]

uniformly in \( \xi \in D \). Likewise, by considering the contour consisting of \( [1, R], \{r e^{-i\theta}: r \in [1, R]\}, C = \{e^{-i\theta}: \theta \in [0, \theta_\ast]\}, \) and \( C_R = \{Re^{-i\theta}: \theta \in [0, \theta_\ast]\}, \) similar argument leads to

\[
\int_{1}^{\infty} e^{-ir^q e^{-s\alpha} \xi} \, dr \to -i \int_{0}^{\theta_\ast} e^{-ie^{-i(q+1)\theta} + e^{-i(q+1)\theta} \int_{1}^{\infty} e^{-ir^{-i\theta^*} r^{-q} \, dr}
\]

uniformly in \( \xi \in D \).

In each of the above convergence results, the limit is independent of \( \xi \). Thus, as \( s \to 0+, \) \( f_s(\xi) \) converges to the same constant \( \lambda \) uniformly in any compact set of \( \xi \in \{z: \Re z > 0\} \). It only remains to find out \( \lambda \). For this purpose, fix \( \xi = 1 \). For \( t > 0, \) let

\[
f(s, t) = \int_{0}^{\infty} J_r(r) r^{-p-1} e^{-sr^\alpha - tr} \, dr.
\]

Then exactly the same argument as above shows that as \( (s, t) \to (0+, 0+) \), \( f(s, t) \to \lambda \). On the other hand, given \( t > 0, \) by dominated convergence, as \( s \to 0+, \)

\[
f(s, t) \to f(0, t) = \int_{0}^{\infty} J_r(r) r^{-p-1} e^{-r^\alpha} \, dr.
\]

Then \( \lambda = \lim_{t \to 0+} f(0, t) \). From [2], Eq. (9.10.5), the last limit is the r.h.s. of (5.12).

**Proof of Lemma 5.** By (5.11) and dominated convergence, for \( j, k \in \mathbb{Z}_+ \),

\[
\frac{d^k}{d s^k}(\pi_j G_s)(u) = \frac{(-i)^j c_{j,d}}{A(S^{d-1})} \frac{d^k}{d s^k} \int_{S^{d-1}} \tilde{C}_j^b((u, v)) \left[ \int_{0}^{\infty} r^{d/2} f_{j+b}(r) e^{-tr^\alpha V(v)} \, dr \right] \omega(\,dv)
\]

\[
= \frac{(-1)^k(-i)^j c_{j,d}}{A(S^{d-1})} \int_{S^{d-1}} \tilde{C}_j^b((u, v)) V(v)^k \left[ \int_{0}^{\infty} r^{d/2+ka} f_{j+b}(r) e^{-tr^\alpha V(v)} \, dr \right] \omega(\,dv).
\]

By Lemma 7, as \( t \to 0+ \),

\[
\int_{0}^{\infty} r^{d/2+ka} f_{j+b}(r) e^{-tr^\alpha V(v)} \, dr \to \frac{2^{d/2+ka} \Gamma((j + ka + d)/2)}{\Gamma((j - ka)/2)}
\]

uniformly in \( v \in S^{d-1} \). Then

\[
\frac{d^k}{d s^k}(\pi_j \omega_s G_s)(u)
\]

\[
\to \frac{(-1)^k}{A(S^{d-1})} \frac{(-i)^j c_{j,d}}{A(S^{d-1})} \int_{S^{d-1}} \tilde{C}_j^b((u, v)) V(v)^k \frac{2^{d/2+ka} \Gamma((j + ka + d)/2)}{\Gamma((j - ka)/2)} \omega(\,dv)
\]

\[
= \frac{2^{d/2}(-2^\alpha)^k \Gamma((j + ka + d)/2)}{\Gamma((j - ka)/2)} S_{j,k}(u)
\]

uniformly in \( u \in S^{d-1} \).
6 An example with linear spectral spherical density

Fix $c \geq 1$ and $\theta \in S^{d-1}$, where $d \geq 2$. This section applies the results in previous sections to the case where $\mu$ is an $\alpha$-stable distribution on $\mathbb{R}^d$ with spectral spherical density

$$P(u) = c + \langle \theta, u \rangle$$

and no shift. This is the simplest case other than the one that assumes a constant spectral spherical density. Similar treatment also applies if the spectral spherical density is $F(\langle \theta, \cdot \rangle)$. Denote the zonal harmonic $\tilde{C}^{(d-2)/2}(\langle \theta, \cdot \rangle)$ by $Z_j(\cdot)$.

**Proposition 11.** Let

$$\kappa_0 = \frac{c}{\Gamma(\frac{\alpha+d}{2})\Gamma(\frac{d}{2}+1)}, \quad \kappa_1 = -\frac{\tan(\frac{\pi\alpha}{2})}{\Gamma(\frac{\alpha+d+1}{2})\Gamma(\frac{d+1}{2})}, \quad \kappa_2 = \frac{\pi^{d/2}\Gamma(\alpha+1)}{2^{\alpha-1}}. \quad (6.1)$$

1) If $\alpha \neq 1$, then the characteristic exponent of $\mu$ is

$$\Phi_\mu(z) = |z|^{\nu}V(u_\nu) \quad \text{with} \quad V(u) = \kappa_3[\kappa_0 + i\kappa_1(\theta, u)], \quad (6.2)$$

and if $\alpha = 1$, then

$$\Phi_\mu(z) = 2|z|^{\nu}V(u_\nu) \left[ \frac{c}{\Gamma(\frac{1+d}{2})} - \frac{i\beta_d(\theta, u_\nu)}{\Gamma(\frac{d}{2}+1)\sqrt{\pi}} \right] + i|z| \ln |z| \frac{\pi^{d/2}-1}{\Gamma(\frac{d}{2}+1)}(\theta, u_\nu), \quad (6.3)$$

where $\beta_d$ is given in Theorem 1.

2) If $\alpha \in (0, 1)$, then the p.d.f. of $\mu$ at $x \neq 0$ is

$$g(x) = \frac{\Gamma(\frac{d}{2})}{\pi^{d/2}} \sum_{k=1}^{\infty} \frac{[-2\Gamma(\alpha+1)\pi^{d/2}]^k}{k!|x|^{\alpha+d}} \sum_{j=0}^{k} c_{j,d}\gamma_{j,k}Z_j(u_x), \quad (6.4)$$

where

$$\gamma_{j,k} = \frac{\Gamma(\frac{j+k+\alpha+d}{2})}{\Gamma(\frac{j-k+\alpha}{2})} \sum_{m=0}^{\infty} \frac{k!}{(k-j-2m)!\Gamma(j+m+\frac{d}{2})m!} \frac{(-1)^m k^{-j-2m} k_1^{j+2m}}{2^{j+2m}},$$

and if $\alpha \in (1, 2)$, then the p.d.f. at any $x$ is

$$g(x) = \frac{2^{1-d}\Gamma(\frac{d}{2})}{\pi^{d/2}\alpha} \sum_{n=0}^{\infty} \kappa_2^{-(n+d)/\alpha} \left( \frac{|x|}{2} \right)^{n/2} \sum_{m=0}^{\infty} \frac{(-1)^m c_{n-2m,d}}{m!\Gamma(n-m+\frac{d}{2})} \times \sum_{k=n-2m}^{\infty} \frac{\Gamma(k+(n+d)/\alpha)}{(\kappa_0 + \kappa_1^{2}/\kappa_0)^{k+(n+d)/\alpha}} \gamma_{n-2m,k}^{*} Z_{n-2m}(u_x), \quad (6.5)$$

where for $k \geq j$,

$$\gamma_{j,k}^{*} = \sum_{l=0}^{[(k-j)/2]} \frac{1}{(k-j-2l)!\Gamma(j+l+\frac{d}{2})l!} \frac{(-1)^l k_1^{k-j-2l} (\kappa_1/\kappa_0)^{k-j-2l}}{2^{j+2l}}.$$
To prove Proposition 11, the next technical result is needed.

**Lemma 8.** Let $f_k(u) = \langle \theta, u \rangle^k$. Then for $j, k \in \mathbb{Z}_+$ and $u \in \mathbb{S}^{d-1}$,

$$
\pi_j f_k = \begin{cases} 
\frac{c_{j,d}w_j(k)}{A(\mathbb{S}^{d-1})} Z_j, & \text{if } k - j \geq 0 \text{ is even} \\
0 & \text{else},
\end{cases}
$$

where $w_j(k)$ are given in (3.3).

**Proof.** If $j > k$, then $\pi_j f_k = 0$ as $f_k \in \mathcal{P}_{k,d}$. If $j \leq k$, then by (2.2) and (2.5), for $u \in \mathbb{S}^{d-1}$,

$$(\pi_j f_k)(u) = \frac{c_{j,d}}{A(\mathbb{S}^{d-1})} \int_{\mathbb{S}^{d-1}} \widetilde{C}^{(d-2)/2}_j(u,v) \omega(dv) = \frac{c_{j,d}}{A(\mathbb{S}^{d-1})} \lambda_{j,k} \widetilde{C}^{(d-2)/2}_j(\langle \theta, u \rangle),$$

where

$$
\lambda_{j,k} = A(\mathbb{S}^{d-1}) \int_{-1}^{1} t^k \widetilde{C}^{(d-2)/2}_j(t)(1 - t^2)^{(d-3)/2} dt.
$$

Since $\widetilde{C}^{(d-2)/2}_j(-t) = (-1)^j \widetilde{C}^{(d-2)/2}_j(t)$, if $j$ and $k$ have different parities, then $\lambda_{j,k} = 0$. On the other hand, if $j$ and $k$ have the same parity, then by comparing the above display and (3.10), $\lambda_{j,k}$ is exactly $w_j(k)$.

**Proof of Proposition 11.** Denote $f(u) = \langle \theta, u \rangle$.

1) From $P = c + f$ and $f \in \mathcal{H}_{1,d}$, $P_0 = \pi_0 P = c$, $P_1 = \pi_1 P = f$, and $P_j = \pi_j P = 0$ for $j > 1$. Then (6.2) and (6.3) directly follow from Theorem 1.

2) Let $\alpha \in (0,1) \cup (1,2)$. If $k < j$, then by $(\kappa_0 + i\kappa_1 f)^k \subset \mathcal{P}_{k,d}$, $\pi_j[(\kappa_0 + i\kappa_1 f)^k] = 0$. On the other hand, if $k \geq j$, then by Lemma 8,

$$
\pi_j[(\kappa_0 + i\kappa_1 f)^k] = \sum_{l=0}^{k} \binom{k}{l} \kappa_0^{-l} (i\kappa_1)^l \pi_j f_l
$$

$$
= \sum_{j \leq l \leq k, \text{ even}} \binom{k}{l} \kappa_0^{-l} (i\kappa_1)^l \frac{c_{j,d} w_j(l)}{A(\mathbb{S}^{d-1})} Z_j
$$

$$
= \frac{i^j c_{j,d} Z_j}{A(\mathbb{S}^{d-1})} \sum_{m=0}^{\lfloor (k-j)/2 \rfloor} \binom{k}{j+2m} \kappa_0^{-j-2m} (-1)^m \kappa_1^{j+2m} w_j(j+2m).
$$

Apply the first expression in (3.3) to $w_j(j+2m)$ on the r.h.s. It follows that

$$
\pi_j[(\kappa_0 + i\kappa_1 f)^k] = \frac{i^j c_{j,d} \Gamma(d/2) Z_j}{(k-j-2m)! \Gamma(j+m+\frac{d}{2})m!} \frac{(-1)^m \kappa_0^{-j-2m} \kappa_1^{j+2m}}{2^{j+2m}}.
$$

(6.6)
Let $\alpha \in (0, 1)$. Then by Theorem 4 and (6.2), for $x \neq 0$,
\[
g(x) = \sum_{k=1}^{\infty} \frac{[-2\Gamma(\alpha + 1)\pi^{d/2}]^k}{\pi^{d/2}k!} |x|^{\alpha + d} \sum_{j=0}^{k} \frac{\Gamma(\frac{j + \alpha + d}{2})}{\Gamma(\frac{j - \alpha}{2})} (-i)^j \pi_j [(\kappa_0 + i\kappa_1 f)^k](u_x).
\]

Then by (6.6), (6.4) follows.

Let $\alpha \in (1, 2)$. From Lemma 4 and (6.2), $\sup_{z \in \mathbb{C}} |R - V| < R$ for any
\[
R > \varrho_0 = \sup_{z \in \mathbb{C}} \frac{|V|^2}{2 \Re(V)} = \kappa_2 \frac{\kappa_0^2 + \kappa_1^2}{2\kappa_0}.
\]

Choose $R = 2\varrho_0$, so that $V_* = R - V = \kappa_2 [\kappa_1^2 / \kappa_0 - i\kappa_1 f]$. Then by Theorem 9,
\[
g(x) = \frac{2^{1-d}}{\pi^{d/2} \alpha} \sum_{n=0}^{\infty} \kappa_2^{-(n+d)/\alpha} \left(\frac{|x|}{2}\right)^n \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{m! \Gamma(n - m + 1)} \times \sum_{k=n-2m}^{\infty} \frac{\Gamma(k + (n+d)/\alpha)}{k! (\kappa_0^2 / \kappa_0 + \kappa_1^2 / \kappa_0)^{(n+d)/\alpha}} (-i)^{n-2m} \pi_{n-2m} [(\kappa_1^2 / \kappa_0 - i\kappa_1 f)^k](u_x).
\]

Then from (6.6), but with $\kappa_0$ and $\kappa_1$ therein replaced with $\kappa_1^2 / \kappa_0$ and $-\kappa_1$, respectively, (6.5) follows.

## 7 Sampling issues

The results in Section 4 and 5 allow sampling with the series method, which is a version of the rejection method ([7], sections II.2 and IV.5). In general, suppose we wish to sample from a p.d.f. $g$ that is specified as $\text{const} \times g^*$, where $g^* \geq 0$ is known. The multiplicative factor need not be specified and is often intractable. The rejection method exploits a p.d.f. $f$ that is relatively easy to sample. Similar to $g$, $f$ is specified as $\text{const} \times f^*$, where the so-called dominating function $f^*$ satisfies $g^* \leq f^*$. Denote by $U$ a random variable uniformly distributed on $(0, 1)$. Then $f$ can be sampled as follows.

**Rejection method.**

- Keep sampling $X \sim f$ and $U$ independently until $Uf^*(X) \leq g^*(X)$. Then output $X$ and stop.

As a version of the rejection method, the series method deals with $g^*(x)$ and $f^*(x)$ that are specified as convergent series $\sum_{k=1}^{\infty} a_k(x)$ and $\sum_{k=1}^{\infty} b_k(x)$, respectively, where $a_k(x)$ and $b_k(x)$ are real valued. Write $g_k^*(x) = \sum_{n=1}^{k} a_n(x)$ and $f_k^*(x) = \sum_{n=1}^{k} b_n(x)$. Suppose functions $A_{k+1}(x)$ and $B_{k+1}(x)$ are available, such that
\[
|g^*(x) - g_k^*(x)| \leq A_{k+1}(x), \quad |f^*(x) - f_k^*(x)| \leq B_{k+1}(x)
\]
and $A_{k+1}(x) \to 0$ and $B_{k+1}(x) \to 0$ as $k \to \infty$. Then $f$ can be sampled as follows.
Series method.
1. Sample $X \sim f$ and $U$ independently.
2. Find the first $k$ such that $|g_k^*(X) - U f_k^*(X)| > A_{k+1}(X) + UB_{k+1}(X)$.
3. If $g_k^*(X) > U f_k^*(X)$, then output $X$ and stop, otherwise go back to step 1 and repeat. \hfill $\Box$

Note that if $a_1(x) = g^*(x)$, $b_1(x) = f^*(x)$, and $a_k(x) = b_k(x) = A_{k+1}(x) = B_{k+1}(x) \equiv 0$ for $k > 1$, then the series method reduces to the rejection method. To verify that its output follows $f$, with probability one, $|g^*(X) - U f^*(X)| > 0$, so for large $k$, $|g_k^*(X) - U f_k^*(X)| > A_{k+1}(X) + UB_{k+1}(X)$. If $g_k^*(X) > U f_k^*(X)$, then we have $g_k^*(X) - U f_k^*(X) > A_{k+1}(X) + UB_{k+1}(X)$, and so

$$g^*(X) \geq g_k^*(X) - A_{k+1}(X) > U f_k^*(X) + UB_{k+1}(X) \geq U f^*(X).$$

Likewise, if $g_k^*(X) < U f_k^*(X)$, then $g^*(X) < U f^*(X)$. Thus, the above algorithm implements the rejection method.

### 7.1 Case 1: $\alpha \in (0, 1)$

The series method will rely on the following.

**Proposition 12.** Let the conditions in Theorem 4 be satisfied. That is, $\mu$ is an $\alpha$-stable distribution on $\mathbb{R}^d$, $d \geq 2$, with $\alpha \in (0, 1)$ and characteristic exponent (3.5), and $\lambda = P \omega$ with $0 \neq P \in \mathcal{P}_{q,d}$ for some $q \in \mathbb{Z}_+$. Then

$$g(x) \leq C_1 \mathbb{1}\{|x| \leq 1\} + \frac{C_2}{|x|^\alpha+1} \mathbb{1}\{|x| > 1\},$$

where $C_1 = \frac{\Gamma(d/\alpha)}{\alpha(2\pi)^d} \left[\inf_{S^{d-1}} \text{Re}(V)\right]^{-d/\alpha}$ and

$$C_2 = \frac{1}{\pi^{d/2}} \sum_{k=1}^{\infty} \frac{(2^\alpha)^k}{k!} \sup_{S^{d-1}} \left| V \right|^k \sum_{j=0}^{kq} \frac{\sqrt{\Gamma((j + k\alpha + d)/2)}}{\Gamma((j - k\alpha)/2)}.$$

**Proof.** From (1.2), (3.5), and Lemma 1, for any $x \in \mathbb{R}^d$,

$$g(x) \leq (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-|z|^{\alpha} \text{Re}(V(z))} \, dz \leq (2\pi)^{-d} \int_0^{\infty} r^{d-1} e^{-r^{\alpha} \inf_{S^{d-1}} \text{Re}(V)} \, dr = C_1.$$ 

On the other hand, from Theorem 4, for $x \neq 0$,

$$g(x) \leq \sum_{k=1}^{\infty} \frac{(2^\alpha)^k \pi^{-d/2}}{k! |x|^{k\alpha+d}} \sum_{j=0}^{kq} \frac{\Gamma((j + k\alpha + d)/2)}{\Gamma((j - k\alpha)/2)} \sup_{S^{d-1}} |S_{j,k}|.$$

When $|x| > 1$, by (4.15), the r.h.s. is bounded by $C_2/|x|^{\alpha+d}$. Then (7.1) follows. \hfill $\Box$

From Proposition 12, $g$ can be sampled by the series method with the following inputs.
1. \( g^*(x) = g(x) = \sum_{k=1}^{\infty} a_k(x) \), where
\[
a_k(x) = \frac{(-2^\alpha)^k \pi^{-d/2}}{k!|x|^{\alpha+d}} \sum_{j=0}^{kq} \frac{\Gamma((j+k\alpha+d)/2)}{\Gamma((j-k\alpha)/2)} S_{j,k}(u_x),
\]
and the bound \( A_{k+1}(x) \) is any convenient upper bound of
\[
\frac{1}{\pi^{d/2}} \sum_{n=k+1}^{\infty} \frac{(2\alpha)^n}{n!|x|^{\alpha+d}} \sup_{|x| \leq 1} |V|^k \sum_{j=0}^{nq} \sqrt{\Gamma((j+n\alpha+d)/2)} \bigg| \Gamma((j-k\alpha)/2) \bigg|
\]
as long as \( A_{k+1}(x) \to 0 \) as \( k \to \infty \).

2. \( f^*(x) = D_1|x| \{ |x| \leq 1 \} + D_2|x|^{-\alpha-d} \{ |x| > 1 \} \), where \( D_1 \geq C_1 \) and \( D_2 \geq C_2 \) are bounds that are easy to evaluate, and the series \( \sum_{k=1}^{\infty} b_k(x) \) for \( f^* \) and corresponding bounds on remainders are given by \( b_1(x) = f^*(x) \) and \( b_k(x) = B_{k+1}(x) = 0 \) for \( k > 1 \).

With these inputs, in each iteration, the series method needs to sample from the p.d.f. \( f \neq f^* \), which is not hard because \( f \) is a mixture of the uniform p.d.f. in the unit ball and the p.d.f. of \( U^{-1/\alpha} \vartheta \), where \( U \) and \( \vartheta \) are independent random variables, with \( U \) uniformly distributed on \((0,1)\) and \( \vartheta \) uniformly distributed on \( \mathbb{S}^{d-1} \). Furthermore, for \( X \sim f \), with probability 1, \( X \neq 0 \), so all \( a_k(X) \) and \( A_{k+1}(X) \) are well defined.

### 7.2 Case 2: \( \alpha \in (1,2) \)

The sampling when \( \alpha \in (1,2) \) is more complicated because the series in Proposition 7 and Theorem 9 do not indicate a simple integrable function to dominate the p.d.f. Instead, for \( X \sim \mu \), we will first sample \(|X|\), and then sample \( U \) and \( V \) conditioning on \(|X|\).

To sample \(|X|\), the case where \( \mu \) is symmetric turns out to be quite simple. From Lemma 1 and (3.6), \( V \) is real valued and positive in this case. Let \( Z_1, \ldots, Z_d \), and \( \zeta \) be independent random variables with \( Z_i \sim N(0,1) \) and \( \zeta \) positive and \((\alpha/2)-\text{stable} \) such that \( E(e^{-t\zeta}) = e^{-t^{\alpha/2}} \) for \( t \geq 0 \). Put \( Z = (Z_1, \ldots, Z_d) \).

**Proposition 13.** Let \( X \) have the p.d.f. \( g \) in Proposition 7. If \( g \) is symmetric, i.e., \( g(x) = g(-x) \), then \( V^{1/\alpha}(Z/|Z|) \sqrt{2\zeta} |Z| \sim |X| \).

**Proof.** From Corollary 8 as well as the discussion preceding it,
\[
\frac{A(S^{d-1})(r/2)^{d-1}}{2\pi^{d/2}} \mathbb{E} \left[ \zeta^{-d/2} \exp \left\{ -\frac{r^2}{4\zeta} \right\} \right]
\]
is the p.d.f. of \( \sqrt{2\zeta} |Z| \). Then from (5.3),
\[
P\{ |X| \in dr \} = \frac{1}{A(S^{d-1})} \int_{S^{d-1}} P\{ V^{1/\alpha}(u) \sqrt{2\zeta} |Z| \in dr \} \omega(du).
\]

The r.h.s. can be written as \( P\{ V^{1/\alpha}(\vartheta) \sqrt{2\zeta} |Z| \in dr \} \), where \( \vartheta \) is uniformly distributed on \( \mathbb{S}^{d-1} \) and independent of \((|Z|, \zeta)\). Since \( Z/|Z| \) is uniformly distributed on \( \mathbb{S}^{d-1} \) and independent of \((|Z|, \zeta)\), then \( \vartheta \) can be replaced with \( Z/|Z| \) and the claim follows. \( \square \)
For the general case, the series method is based on the following.

**Proposition 14.** Let the conditions in Theorem 9 be satisfied. That is, \( \mu \) is a nondegenerate stable distribution on \( \mathbb{R}^d \), \( d \geq 2 \), with \( \alpha \in (1, 2) \) and characteristic exponent (3.5). Let \( w = \inf_{S^{d-1}} \text{Re}(V^{-2/\alpha}) \).

1) If \( w > 0 \), then letting \( V_0 = w^{-\alpha/2} \) and \( f \) be the p.d.f. of \( V_0^{1/\alpha} \sqrt{2|Z|} \), for \( X \sim g \),

\[
g_{|X|}(r) \leq Cf(r) \quad \text{with} \quad C = \sup_{S^{d-1}} |V_0/V|^{d/\alpha}.
\]

2) Suppose \( \alpha \in [4/3, 2) \) and \( \mu \) has a polynomial spectral spherical density. Then \( w > 0 \).

**Proof.** 1) From (5.3),

\[
g_{|X|}(r) \leq \frac{(r/2)^{d-1}}{2\pi d/2} \int_{S^{d-1}} \frac{1}{|V|^{d/\alpha}} E \left[ \zeta^{-d/2} e^{-\text{Re}(V^{-2/\alpha})r^2/4\zeta} \right] \, d\omega
\]

\[
\leq \sup_{S^{d-1}} |V_0/V|^{d/\alpha} \times \frac{(r/2)^{d-1}}{2\pi d/2} \int_{S^{d-1}} \frac{1}{V_0^{d/\alpha}} E \left[ \zeta^{-d/2} e^{-V_0^{-2/\alpha}r^2/4\zeta} \right] \, d\omega.
\]

Then from Proposition 13 the proof follows.

2) From Corollary 3, the assumption implies \( \sup_{S^{d-1}} |\arg(V^{2/\alpha})| < \pi/2 \) and so the proof follows by noticing \( \text{Re}(V^{-2/\alpha}) = |V|^{-2/\alpha} \cos(\arg(V^{2/\alpha})) \).

From Proposition 14, provided \( w = \inf_{S^{d-1}} \text{Re}(V^{-2/\alpha}) > 0 \), \( g_{|X|} \) can be sampled using the series method with \( f^* = Cf \) as the dominating function. However, while \( f \) is easy to sample, neither \( g_{|X|} \) nor \( f^* \) has a closed form. We will rely on the series representations in Proposition 7 and Theorem 9. The inputs to the series method can be as follows.

1. \( g^*(r) = g_{|X|}(r) = \sum_{k=0}^{\infty} a_k(r) \), where \( a_k(r) \) is any ordering of the terms in the a.c. double series in (5.6). As noted after Theorem 9, when \( \mu \) has a polynomial spectral spherical density, all the terms can be expressed in closed form.

2. \( f^*(r) = Cf(r) = \sum_{k=0}^{\infty} b_k(r) \), where \( C \) and \( f \) are as in Proposition 14, and by (5.2),

\[
b_k(r) = \frac{CA(S^{d-1}) (-1)^k \Gamma((2k+d)/\alpha)}{\pi^{d/2} k! \Gamma(k+d/2)} \left( \frac{r}{2} \right)^{2k+d-1} \frac{1}{V_0^{(2k+d)/\alpha}}.
\]

where \( V_0 = w^{-\alpha/2} \).

Suppose \( |X| \) has been sampled, the next step is to sample \( u_X \) conditional on \( |X| \). Suppose \( \mu \) has a polynomial spectral spherical density \( P \in \mathcal{P}_{q,d} \). Then from Theorem 9, the p.d.f. of \( u_X \) w.r.t. \( \omega(du) \) given \( |X| = r \) is in proportion to

\[
g^*(u) = \sum_{k,m=0}^{\infty} \sum_{n=2m}^{2m+q} \frac{(-1)^m (r/2)^n}{m! \Gamma(n - m + d/2)} \frac{\Gamma(k + (n + d)/\alpha)}{k! R^{k+(n+d)/\alpha}} S_{n-2m,k}^*(u)
\]

\[
\leq C = \sum_{k,m=0}^{\infty} \sum_{n=2m}^{2m+q} \frac{(r/2)^n}{m! \Gamma(n - m + d/2)} \frac{\Gamma(k + (n + d)/\alpha)}{k! R^{k+(n+d)/\alpha}} \sup_{S_{n-2m,k}} |S_{n-2m,k}^*| < \infty.
\]

37
Then the conditional distribution can be sampled using a constant $f^*$ as the dominating function, with the value of the constant being any convenient upper bound of $C$. The corresponding p.d.f. is uniform on $S^{d-1}$. The way to describe the inputs to the series method pretty much follows the one for $|X|$, so for brevity the description is omitted.

**References**

[1] **Abdul-Hamid, H. and Nolan, J. P.** (1998). Multivariate stable densities as functions of one-dimensional projections. *J. Multivariate Anal.* 67, 1, 80–89.

[2] **Andrews, G. E., Askey, R., and Roy, R.** (1999). *Special Functions*. Encyclopedia of Mathematics and its Applications, Vol. 71. Cambridge University Press, Cambridge.

[3] **Bochner, S.** (1955). *Harmonic Analysis and the Theory of Probability*. University of California Press, Berkeley and Los Angeles.

[4] **Byczkowski, T., Nolan, J. P., and Rajput, B.** (1993). Approximation of multidimensional stable densities. *J. Multivariate Anal.* 46, 1, 13–31.

[5] **Davydov, Y. and Nagaev, A. V.** (2002a). Limit theorems and simulation of stable random vectors. In *Limit Theorems in Probability and Statistics, Vol. I* (Balatonlelle, 1999). János Bolyai Math. Soc., Budapest, 495–519.

[6] **Davydov, Y. and Nagaev, A. V.** (2002b). On two approaches to approximation of multidimensional stable laws. *J. Multivariate Anal.* 82, 1, 210–239.

[7] **Devroye, L.** (1986). *Nonuniform Random Variate Generation*. Springer-Verlag, New York.

[8] **Fallahgoul, H., Hashemiparast, S. M., Fabozzi, F. J., and Klebanov, L.** (2014). Analytical-numeric formulas for the probability density function of multivariate stable and geo-stable distributions. *J. Statist. Theory and Practice* 8, 2, 260–282.

[9] **Nolan, J. P.** (1998). Multivariate stable distributions: approximation, estimation, simulation and identification. In *A Practical Guide to Heavy Tails: Statistical Techniques and Applications*, R. J. Adler, R. E. Feldman, and M. S. Taqqu, Eds. Birkhäuser, Boston, 509–525.

[10] **Olver, F. W. J., Lozier, D. W., Boisvert, R. F., and Clark, C. W.,** Eds. (2010). *NIST Handbook of Mathematical Functions*. U.S. Department of Commerce National Institute of Standards and Technology, Washington, DC.

[11] **Pivato, M. and Seco, L.** (2003). Estimating the spectral measure of a multivariate stable distribution via spherical harmonic analysis. *J. Multivariate Anal.* 87, 2, 219–240.

[12] **Rvačeva, E. L.** (1962). On domains of attraction of multi-dimensional distributions. In *Selected Translations in Math. Statist. and Probability, Vol. 2*. American Mathematical Society, Providence, R.I., 183–205.
[13] Samorodnitsky, G. and Taqqu, M. S. (1994). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Stochastic Modeling. Chapman & Hall, New York.

[14] Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Studies in Advanced Mathematics, Vol. 68. Cambridge University Press, Cambridge.

[15] Stein, E. M. and Weiss, G. (1971). *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton Mathematical Series, Vol. 32. Princeton University Press, Princeton, New Jersey.

[16] Uchaikin, V. V. and Zolotarev, V. M. (1999). *Chance and Stability: Stable Distributions and Their Applications*. Modern Probability and Statistics. VSP, Utrecht.

[17] Watanabe, T. (2007). Asymptotic estimates of multi-dimensional stable densities and their applications. *Trans. Amer. Math. Soc.* 359, 6, 2851–2879.