Meromorphic Cubic Differentials and Convex Projective Structures

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Abstract. Extending the Labourie-Loftin correspondence, we establish, on any punctured oriented surface of finite type, a one-to-one correspondence between convex projective structures with specific types of ends and punctured Riemann surface structures endowed with meromorphic cubic differentials whose poles are at the punctures. This generalizes previous results of Loftin, Benoist-Hulin and Dumas-Wolf.

1. Introduction

A convex projective structure on a surface \( \Sigma \) is given by a developing pair \((\text{dev}, \text{hol})\), where the holonomy \( \text{hol} \) is a representation of \( \pi_1(\Sigma) \) in the group of projective transformations \( \text{PGL}(3, \mathbb{R}) \cong \text{SL}(3, \mathbb{R}) \), while the developing map \( \text{dev} \) is a \( \text{hol} \)-equivariant diffeomorphism from the universal cover \( \tilde{\Sigma} \) to some properly convex open set \( \Omega \) in \( \mathbb{R}P^2 \), i.e. a bounded open set of some affine chart.

Convex projective structures on manifolds of arbitrary dimension are extensively studied, see the survey [Ben08]. However, a good understanding of the space \( \mathcal{P}(\Sigma) \) of all convex projective structures is available only for surfaces, as we review now.

Assume that \( \Sigma \) is oriented and is not the 2-sphere. Let \( \mathcal{C}(\Sigma) \) be the space of pairs \((J, b)\), where \( J \) is a complex structure on \( \Sigma \) and \( b \) is a holomorphic cubic differential with respect to \( J \). Then results of Cheng-Yau [CY77, CY86] and Wang [Wan91] in affine differential geometry provide a natural map

\[
\mathcal{P}(\Sigma) \to \mathcal{C}(\Sigma).
\]

(1.1)

If \( \Sigma \) is further assumed to be closed, Labourie [Lab07] and Loftin [Lof01] showed independently that the map (1.1) is bijective, establishing a homeomorphism between the quotients \( \mathcal{P}(\Sigma)/\text{Diff}^0(\Sigma) \) and \( \mathcal{C}(\Sigma)/\text{Diff}^0(\Sigma) \), where \( \text{Diff}^0(\Sigma) \) is the identity component of the diffeomorphism group of \( \Sigma \). This recovers a previous result of Choi and Goldman [CG93] that \( \mathcal{P}(\Sigma)/\text{Diff}^0(\Sigma) \) is homeomorphic to \( \mathbb{R}^{16(g-1)} \).

The purpose of the present paper is to extend the Labourie-Loftin bijection to punctured surfaces of finite type, i.e. the case where \( \Sigma \) is obtained from a closed oriented surface \( \Sigma \) by removing a finite set of punctures.

Let \( \mathcal{C}_0(\Sigma) \) denote the space of \((J, b) \in \mathcal{C}(\Sigma)\) satisfying the following requirements.

- The complex structure \( J \) is cuspidal at each puncture.
- The cubic differential \( b \) has removable or meromorphic singularity at each puncture. If \( \Sigma \) is homeomorphic to either \( \mathbb{C} \) or \( \mathbb{C}^* \), we further assume that \( b \) does not vanish identically.
Our main theorem says that the map (1.1) restricts to a bijection between \( C_0(\Sigma) \) and a subset \( P_0(\Sigma) \) of \( P(\Sigma) \) consisting of convex projective structures with specific types of ends. We proceed to give a detailed description of those ends.

1.1. **The main theorem.** Given a convex projective structure on \( \Sigma \) with developing map \( dev \), we put \( \Omega := \text{dev}(\tilde{\Sigma}) \subset \mathbb{RP}^2 \) and assign to each puncture \( p \) some connected closed subsets of \( \partial \Omega \), called developing boundaries at \( p \), as follows.

If \( \Sigma \) has negative Euler characteristic, we fix a complete hyperbolic metric of finite volume on \( \Sigma \) so as to identify \( \tilde{\Sigma} \) with the Poincaré disk \( D \). We choose an ideal point \( \tilde{p} \in \partial D \) corresponding to the puncture \( p \), then take a decreasing sequence of horodisks \( D_1 \supset D_2 \supset \cdots \) based at \( \tilde{p} \) and converging to \( \tilde{p} \). The developing boundary \( \partial_{\tilde{p}} \Omega \) is defined as the set of accumulation points of the sequence of developing images \( \text{dev}(D_1), \text{dev}(D_2), \cdots \). Also let \( \text{hol}_{\tilde{p}} \in \text{SL}(3, \mathbb{R}) \) denote the holonomy of the projective structure around \( p \) which preserves \( \partial_{\tilde{p}} \Omega \).

If \( \Sigma \) is homeomorphic to \( \mathbb{C} \) or \( \mathbb{C}^* \), the definition of developing boundaries at the puncture \( \infty \) or 0 is more straightforward since no choice of \( \tilde{p} \) is involved.

The allowed types of developing boundaries for \( P_0(\Sigma) \) are as follows. Here we employ the standard classification of projective automorphisms of properly convex sets into hyperbolic, quasi-hyperbolic, planar and parabolic ones (c.f. §3.2.1 below).

- **A point.** In this case \( \text{hol}_{\tilde{p}} \) is parabolic and we call \( p \) a cusp of the convex projective structure. Marquis [Mar12] proved that we are in this case if and only if a punctured neighborhood of \( p \) has finite volume with respect to the Hilbert metric.

- **A segment.** In this case \( \text{hol}_{\tilde{p}} \) is either hyperbolic, quasi-hyperbolic or planar (planar holonomy can occur only when \( \Sigma \) is an annulus though). We call \( p \) a geodesic end.

- **Two non-collinear segments sharing an endpoint.** In this case \( \text{hol}_{\tilde{p}} \) is hyperbolic and the three endpoints of the two segments are the fixed points of \( \text{hol}_{\tilde{p}} \). We call such \( p \) a triangular end. Choi [Cho13] studied ends of convex projective manifolds of arbitrary dimensions and referred to this case as a “properly convex radial end”.

- **A twisted \( n \)-gon.** This is the case where \( \partial_{\tilde{p}} \Omega \) consists of consecutive segments \( (Y_k)_{k \in \mathbb{Z}} \) such that \( Y_k \) and \( Y_{k+1} \) are not collinear and \( \text{hol}_{\tilde{p}} \) maps \( Y_k \) to \( Y_{k+n} \) for a fixed \( n \geq 1 \).

\( P_0(\Sigma) \) is defined as the set of convex projective structure whose developing boundary at each puncture is among the above types.

Our main result is the following theorem.

**Theorem 1.1.** Let \( \Sigma \) be a punctured oriented surface. Then

1. The natural map (1.1) restricts to a bijection from \( P_0(\Sigma) \) to \( C_0(\Sigma) \).
2. For each projective structure in \( P_0(\Sigma) \), the type of end at a puncture is determined by the type of singularity of the corresponding cubic differential as in the following table. Here \( R \) denotes the residue of the cubic differential.
Recall that the residue $R \in \mathbb{C}$ of a cubic differential at a pole of order $\leq 3$ is by definition the coefficient of $z^{-3}dz^3$ in the Laurent expansion with respect to a conformal local coordinate $z$ centered at the pole. $R$ does not depend on the choice of coordinate (this is false when the order exceeds 3).

The above theorem generalizes many previous works. Namely,

- Restricting to the space of those $(J, b) \in C_0(\Sigma)$ where $b$ only has poles of order at most 3, Lofin [Lof04] provided substantial informations towards the bijection.
- Benoist and Hulin [BH13] proved the theorem for those $(J, b) \in C_0(\Sigma)$ where $b$ only has removable singularities or poles of order at most 2.
- Dumas and Wolf [DW14] proved the theorem for $\Sigma = \mathbb{R}^2$. In this case, an element in $C_0(\Sigma)$ is a complex structure biholomorphic to $\mathbb{C}$ together with a polynomial cubic differential, whereas an element in $P(\Sigma)$ is a diffeomorphism from $\mathbb{R}^2$ to a properly convex polygon $\Omega \subset \mathbb{R}P^2$.

Lofin [Lof04] also showed that the residue of $b$ at a third order pole determines the eigenvalues of the holonomy of the convex projective structure around that pole. We can state his result as follows.

**Theorem 1.2** (Lofin). Let $(J, b) \in C_0(\Sigma)$. Equip $\Sigma$ with the projective structure corresponding to $(J, b)$ given by Theorem 1.1. Let $p$ be a third order pole of $b$ with residue $R$ and $\text{hol}_p$ be the holonomy of the projective structure around $p$.

Then the eigenvalues of $\text{hol}_p$ are $\exp\left(-\frac{4\pi i \mu_i}{3}\right)$, $i = 1, 2, 3$, where $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ are the imaginary parts of the three cubic roots of $R/2$, respectively.

Theorem 1.2 and many other results in [Lof04] are obtained as byproducts of our proof of Theorem 1.1.

Around poles of order $\geq 4$, it does not seem possible to capture the holonomy from local information of the cubic differential. For example, when the order $r$ is not divisible by 3, there always exists a conformal local coordinate $z$ in which $b = z^{-r}dz^3$. Thus $b$ does not have any local invariant, nevertheless the holonomy can be quite arbitrary.

### 1.2. Sketch of proof.

Theorem 1.1 is proved by assembling and adapting various techniques developed by Lofin [Lof04], Benoist-Hulin [BH13] and Dumas-Wolf [DW14]. In order to give a sketch, we first recall the construction of the natural map (1.1).

Let $\mathcal{W}(\Sigma)$ be the space of pairs $(g, b)$ where $g$ is a complete Riemannian metric on $\Sigma$ and $b$ is a holomorphic cubic differential (with respect to the conformal structure.

| type of end of the projective structure | type of singularity of the cubic differential |
|----------------------------------------|-----------------------------------------------|
| cusp                                   | removable singularity or pole of order at most 2 |
| geodesic end with planar or quasi-hyperbolic holonomy | third order pole with residue $R \in i\mathbb{R}^*$ |
| geodesic end with hyperbolic holonomy   | third order pole with $\text{Re}(R) > 0$ |
| triangular end                          | third order pole with $\text{Re}(R) < 0$ |
| twisted $n$-gon end                     | pole of order $n + 3$ |
of \( g \) satisfying Wang’s equation

\[
\kappa_g = -1 + 2\|b\|^2_g.
\]

Here \( \kappa_g \) is the curvature of \( g \) and \( \|b\|_g \) is the pointwise norm of \( b \) with respect to \( g \).

The map (1.1) is defined as a composition \( P(\Sigma) \xrightarrow{\sim} W(\Sigma) \xrightarrow{} C(\Sigma) \)

- \( W(\Sigma) \to C(\Sigma) \) is the forgetful map, i.e. the map sending \( (g, b) \) to \( (J, b) \) where \( J \) is the conformal structure underlying \( g \).
- The bijection \( P(\Sigma) \xrightarrow{\sim} W(\Sigma) \) is given by assigning to each convex projective structure an affine sphere structure, and then taking the Blaschke metric \( g \) and Pick differential \( b \).
- The inverse \( W(\Sigma) \xrightarrow{\sim} P(\Sigma) \) to the above bijection is provided by integrating a certain flat connection associated to \( (g, b) \), defined on the vector bundle \( E = T\Sigma \oplus \mathbb{R} \) (where \( \mathbb{R} = \Sigma \times \mathbb{R} \) is the trivial line bundle). A so-obtained developing map is call Wang’s developing map in the literature.

Let \( W_0(\Sigma) \) denote the set of \( (g, b) \in W(\Sigma) \) satisfying the following conditions.

- \( (g, b) \) is sent into \( C_0(\Sigma) \) by the forgetful map \( W(\Sigma) \to C(\Sigma) \);
- the ratio between \( g \) and the flat metric \( |b|^2 \) has positive upper and lower bounds around poles of order \( \geq 3 \)
- \( \kappa_g \) has negative upper and lower bounds around removable singularities or poles of order \( \leq 2 \).

We obtain Theorem 1.1 as a combination of the following independent results.

(\textbf{I}) \( P(\Sigma) \xrightarrow{\sim} W(\Sigma) \) maps \( P_0(\Sigma) \) into \( W_0(\Sigma) \). More precisely, we shall prove that if an end \( p \) of a convex projective structure is among the previously described types, then the conformal structure underlying the Blaschke metric \( g \) is cuspidal at \( p \) and the Pick differential \( b \) is meromorphic at \( p \), and furthermore, \( g \) satisfies the conditions in the above definition of \( W_0(\Sigma) \).

For cusps, this is proved in [BH13]. We will review this proof and give a proof for the other types of ends, generalizing a result from [DW14]. The main tool is the Hausdorff continuity property of Blascke metrics and Pick differentials discovered in [BH13].

(\textbf{II}) \( W_0(\Sigma) \to C_0(\Sigma) \) is bijective. In other words, for any \( (J, b) \in C_0(\Sigma) \), there is a unique metric \( g \) in the conformal class of \( J \) satisfying Wang’s equation (1.2) and the condition \( (g, b) \in W_0(\Sigma) \).

Note that this is just the uniformization theorem when \( \Sigma \) has negative Euler characteristic and \( b = 0 \). Thus we suppose in this step that \( b \) does not vanish identically.

We will present a proof following the standard approach as in [Lof04, BH13, DW14], using the method of sup/sub solutions to establish existence and using the Yau-Omori maximum principle to prove uniqueness.

(\textbf{III}) \( W(\Sigma) \xrightarrow{\sim} P(\Sigma) \) maps \( W_0(\Sigma) \) to \( P_0(\Sigma) \) in the way described by the table from Theorem 1.1. More precisely, if we pick \( (g, b) \in W_0(\Sigma) \) and use the corresponding Wang’s developing map to endow \( \Sigma \) with a convex projective structure, then at each puncture \( p \), the type of end of the projective structure is determined by the nature of \( b \) as shown in the table.
If \( p \) is a pole of order \( r \geq 4 \), we show \( \partial \tilde{p} \Omega \) is a twisted polygon using the method of [DW14]. A crucial observation is that the pair \( (2^{1/3} |b|^{1/3}, b) \) satisfies Wang’s equation, hence provides us with an “auxiliary developing map” \( \text{dev}_0 : \Sigma \to \mathbb{RP}^2 \), although in general the metric \( 2^{1/3} |b|^{1/3} \) is singular and \( \text{dev}_0 \) does not define a projective structure. The map \( \text{dev}_0 \) can be fully understood by virtue of its explicit expressions. The twisted polygon is found by comparing the actual developing map \( \text{dev} \) with \( \text{dev}_0 \) using ODE techniques.

We then apply the same method to tackle the case of third order poles. Many results that we obtained in this case have already appeared more or less explicitly in [Lof04]. A small innovation here is a refinement of the ODE technique which enables us to treat the “unstable directions”, i.e. directions in \( T_p \Sigma \) along which \( \text{dev}_0 \) and \( \text{dev} \) are not comparable.

For poles of order \( \leq 2 \), the required result is simpler and is already proved in [Lof04], using ODE techniques as well. There is also a new proof in [BH13].

Once statements (I), (II) and (III) are proved, Theorem 1.1 follows immediately. Indeed, we obtain two bijections \( \mathcal{P}_0(\Sigma) \sim \mathcal{W}_0(\Sigma) \sim \mathcal{C}_0(\Sigma) \), the first one provided by (I) and (III) and the second by (II).

1.3. **Organization of the paper.** In Section 2 we review in detail the construction of the natural map (1.1) mentioned above. In Section 3, ends of convex projective structures are discussed, where we carefully define the developing boundary \( \partial \tilde{p} \Omega \) and related notions, and establish some fundamental properties. Section 5 describes a local model for a cubic differential around a pole of order \( \geq 4 \). In Section 4, 6 and 7 we prove assertions (I), (II) and (III) in the above sketch, respectively. Section 7 also contains a proof of Theorem 1.2.

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2. A recap of the identification $P(\Sigma) \cong W(\Sigma)$

Through this paper, $\Sigma$ is a surface obtained from a closed oriented $C^\infty$ surface $\Sigma$ by removing a finite (possibly empty) set of punctures. Moreover, $\Sigma$ is assumed not to be the 2-sphere.

In this section, we first give a concise review of the bijection between the space $P(\Sigma)$ of convex projective structures and the space $W(\Sigma)$ of pairs $(g, b)$ satisfying Wang’s equation. This lies at the foundation of the proof of Theorem 1.1 as we have seen in the introduction. We then investigate the particular pair $(g, b) = (2|dz|^2, 2|dz|^3)$ for later use.

Besides the seminal works [Lof01, Lab07], the theory relating convex projective structures, affine spheres and cubic differentials is surveyed in many subsequent works, e.g. [Lof04, BH13, DW14]. Here we merely aim at covering the notions and results used later on as quickly as possible. The reader is referred to the above cited papers whenever a detailed explanation or proof is in need.

2.1. From $P(\Sigma)$ to $W(\Sigma)$: Blaschke metrics and Pick differentials.

2.1.1. Support functions. Given a bounded convex open set $\Omega \subset \mathbb{R}^2$, a support function of $\Omega$ is by definition a strictly convex function $u \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ satisfying the non-linear PDE

$$\det(\text{Hess}(u)) = u^{-4}, \quad u|_{\partial \Omega} = 0.$$ 

A theorem of Cheng and Yau [CY77] ensures existence and uniqueness of the support function $u = u_\Omega$ for any $\Omega$.

Significance of support functions lies in the fact that the above PDE is the condition for the hypersurface

$$S = \left\{ -\frac{1}{u(x)}(x, 1) \right\}_{x \in \Omega} \subset \mathbb{R}^3$$

to be an hyperbolic affine sphere, whereas the vanishing boundary condition means that $S$ is asymptotic to the convex cone $\mathbb{R}_+ \cdot \{(x, 1)\}_{x \in \Omega}$. Thus the Cheng-Yau theorem establishes unique-existence of hyperbolic affine sphere asymptotic to a convex cone.

The following Hausdorff continuity result on $u_\Omega$ is due to Benoist and Hulin (see [BH13] Corollary 3.3 and [DW14] Theorem 4.4).
Theorem 2.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded convex open set. Pick any compact set $K \subset \Omega$, non-negative integer $k$ and real number $\varepsilon > 0$. Then for any $k \in \mathbb{N}$ and any convex open set $\Omega' \subset \mathbb{R}^2$ sufficiently close to $\Omega$ in the Hausdorff topology, we have $K \subset \Omega'$ and

$$\|u_{\Omega} - u_{\Omega'}\|_{K,k} < \varepsilon.$$  

Here $\| \cdot \|_{K,k}$ denote the $C^k$-norm on $K$.

2.1.2. Blaschke metrics and Pick differentials. Any properly convex open set $\Omega \subset \mathbb{R}P^2$ carries a canonical complete Riemannian metric $g_{\Omega}$ and holomorphic cubic differential $b_{\Omega}$ (with respect to the conformal structure of $g$), called the Blaschke metric and Pick differential of $\Omega$, respectively. We let $\nu_{\Omega}$ denote the volume form of $g_{\Omega}$.

The precise definitions of $g_{\Omega}$ and $b_{\Omega}$ are irrelevant to our purpose. We only mention here that they are defined as affine invariants of the hyperbolic affine sphere asymptotic to a cone in $\mathbb{R}^3$ projecting to $\Omega$. More important to us are the following properties of them.

- $g_{\Omega}$ and $b_{\Omega}$ satisfy Wang’s equation $\kappa_{g_{\Omega}} = -1 + 2\|b\|_{g_{\Omega}}^2$.
- If we view $\Omega$ as a bounded subset of an affine chart $\mathbb{R}^2 \subset \mathbb{R}P^2$ and let $u = u_{\Omega}$ be the support function, then the coefficients of $g_{\Omega}$ and $b_{\Omega}$ are linear combinations of $u_{\Omega}$ and its derivatives of order up to 3. Indeed, we have

$$g_{\Omega} = -\frac{1}{u} \sum_{i,j=1,2} \partial_i \partial_j u \, dx^i dx^j,$$

while $b_{\Omega}$ is basically the difference between the Levi-Civita connection of $g_{\Omega}$ and the Blaschke connection $\nabla$ associated to the affine sphere, given by

$$\nabla = d - \frac{1}{u} \left( \begin{array}{c} \partial_1 u \, dx^1 \\ \partial_2 u \, dx^2 \end{array} \right).$$

- $g_{\Omega}$ and $b_{\Omega}$ are invariant by projective transformations in the sense that for any $a \in SL(3, \mathbb{R})$ we have $g_{a(\Omega)} = a_* g_{\Omega}$ and $b_{a(\Omega)} = a_* b_{\Omega}$.  
- $\Omega$ is a triangle if and only if

$$\Omega, g_{\Omega}, b_{\Omega} \cong (\mathbb{C}, |dz|^2, 2 \, dz^2),$$

whereas $\Omega$ is an ellipse if and only if

$$\Omega, g_{\Omega}, b_{\Omega} \cong (\mathbb{D}, g_{\text{hyp}}, 0).$$

Here $(\mathbb{D}, g_{\text{hyp}})$ is the Poincaré disk with the hyperbolic metric.

2.1.3. Hilbert metrics. Any properly convex set $\Omega \subset \mathbb{R}P^2$ also carries a natural Finsler metric $g^H_{\Omega}$, called the Hilbert metric. By definition, in an affine chart $\mathbb{R}^2$ containing $\Omega$, we have

$$g^H_{\Omega}(v) = \left( \frac{1}{|x-a|} + \frac{1}{|x-b|} \right) |v|, \quad \forall x \in \Omega, \ v \in T_x \Omega \cong \mathbb{R}^2,$$

where $a, b \in \partial \Omega$ are the intersections of $\partial \Omega$ with the line passing through $x$ along the direction $v$ and $| \cdot |$ is a Euclidean norm on $\mathbb{R}^2$. Let $\nu_{\Omega}^H$ denote the volume form of $g^H_{\Omega}$.

The Hilbert metric is invariant by projective transformations in the same sense as $g_{\Omega}$ and $b_{\Omega}$ are.
2.1.4. Continuity. Let \( \mathcal{C} \) denote the space of all properly convex open sets in \( \mathbb{R}P^2 \), equipped with the Hausdorff topology. Let \( \mathcal{C}_* \) denote the topological subspace of \( \mathcal{C} \times \mathbb{R}P^2 \) consisting of pairs \( (\Omega, x) \) such that \( x \in \Omega \).

The Benzécri Compactness Theorem (see [BH13] Theorem 2.7) says that the quotient of \( \mathcal{C}_* \) by the natural action of \( SL(3, \mathbb{R}) \) is compact. Combining this with Theorem 2.1, we get the following consequences.

**Corollary 2.2 (Hausdorff continuity).**

1. The proportion between the Hilbert volume form and the Blaschke volume form, viewed as a map
   \[
   \mathcal{C}_* \rightarrow \mathbb{R}_+, \quad (\Omega, x) \mapsto \frac{\nu^H}{\nu^\Omega}(x),
   \]
   is continuous and is bounded from above and below by positive constants.

2. On any properly convex open set \( \Omega \) we define the function
   \[
   f_\Omega := 2\|b_\Omega\|_g^2 = \kappa_{g_\Omega} + 1 : \Omega \rightarrow \mathbb{R}_{\geq 0}.
   \]
   Then the map
   \[
   \mathcal{C}_* \rightarrow \mathbb{R}_{\geq 0}, \quad (\Omega, x) \mapsto f_\Omega(x)
   \]
   is continuous and is bounded from above.

**Outline of proof.** We only need to prove continuity for these maps, then the bounds follow from Benzécri Compactness theorem. But \( (\Omega, x) \mapsto \nu^H_{g_\Omega}(x) \) is continuous by definition, while \( (\Omega, x) \mapsto \nu^\Omega_{g_\Omega}, (\Omega, x) \mapsto b_\Omega(x) \) and \( (\Omega, x) \mapsto g_\Omega(x) \) are continuous by Theorem 2.1 and the fact that \( \nu^\Omega_{g_\Omega}, g_\Omega \) and \( b_\Omega \) can be expressed in terms of \( u_\Omega \) and its derivatives. Continuity of the required maps follows. \( \square \)

2.1.5. Convex projective structures. A projective structure on \( \Sigma \) is an equivalence class of developing pairs \( (dev, hol) \), where

- The holonomy \( hol \) is a homomorphism from \( \pi_1(\Sigma) \) to \( PGL(3, \mathbb{R}) \cong SL(3, \mathbb{R}) \).
- The developing map \( dev : \tilde{\Sigma} \rightarrow \mathbb{R}P^2 \) is a \( hol \)-equivariant local diffeomorphism.
- Two pairs \( (dev, hol) \) and \( (dev', hol') \) are equivalent if there is \( g \in SL(3, \mathbb{R}) \) such that
  \[
  dev' = g \circ dev, \quad hol'(\alpha) = g \, hol(\alpha) \, g^{-1}, \quad \forall \alpha \in \pi_1(\Sigma).
  \]

A projective structure is said to be convex if \( dev \) sends \( \tilde{\Sigma} \) bijectively to a properly convex open set \( \Omega \subset \mathbb{R}P^2 \). The space of all convex projective structure on \( \Sigma \) is denoted by \( \mathcal{P}(\Sigma) \).

Given a convex projective structure on \( \Sigma \) with developing map \( dev \) and developing image \( \Omega = dev(\Sigma) \), the previously defined \( g_\Omega, b_\Omega \) and \( \nu^H_{g_\Omega} \) pull back to \( \Sigma \) through \( dev \). These pullbacks are simply called the Blaschke metric, the Pick differential and the Hilbert metric of the convex projective structure, respectively.

Recall from the introduction that we let \( \mathcal{W}(\Sigma) \) denote the space of pairs \( (g, b) \) such that

- \( g \) is a complete Riemannian metric on \( \Sigma \),
- \( b \) is a holomorphic cubic differential on \( \Sigma \) with respect to the conformal structure underlying \( g \).
\[ \kappa_g = -1 + 2\|b\|^2. \]

For later use, we record here the coordinate expression of Eq. (2.1) in a conformal local coordinate \( z \): suppose \( g = h|dz|^2 \) and \( b = b\, dz^3 \), then (2.1) reads

\[-2\, \frac{h}{h} \partial_z \partial_{\bar{z}} \log h = -1 + \frac{2|b|^2}{h^3}. \]

Assigning to each convex projective structure its Blaschke metric and Pick differential gives rise to a canonical map \( \mathcal{P}(\Sigma) \to \mathcal{W}(\Sigma) \) which is known to be bijective.

We give some more details on the bijectivity. Let \( \mathcal{A}(\Sigma) \) be the space of hyperbolic affine structures on \( \Sigma \), i.e., equivalence classes of "developing pairs" \((\text{Dev}, \text{hol})\), where \( \text{Dev} : \Sigma \to \mathbb{R}^3 \) is now a hol-equivariant affine-spherical embedding. Then the map \( \mathcal{P}(\Sigma) \to \mathcal{W}(\Sigma) \) is essentially defined as a composition

\[ \mathcal{P}(\Sigma) \to \mathcal{A}(\Sigma) \to \mathcal{W}(\Sigma). \]

The first map is bijective by virtue of the Cheng-Yau theorem, while bijectivity of the second map follows from the fundamental theorem of affine differential geometry (an analogue of the fundamental theorem of surface theory which determines a hyperspace in Euclidean space from its first and second fundamental form).

The inverse \( \mathcal{A}(\Sigma) \to \mathcal{P}(\Sigma) \) to the first map in (2.2) trivially assigns to each pair \((\text{Dev}, \text{hol})\) the underlying pair \((\text{dev} = \mathcal{P} \circ \text{dev}, \text{hol})\), where \( \mathcal{P} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{RP}^2 \) is the projectivization. In the next section we give a concrete construction of the inverse \( \mathcal{W}(\Sigma) \to \mathcal{A}(\Sigma) \) to the second map in (2.2). This immediately yields the inverse to the canonical map \( \mathcal{P}(\Sigma) \to \mathcal{W}(\Sigma) \).

2.2. From \( \mathcal{W}(\Sigma) \) to \( \mathcal{P}(\Sigma) \): Wang’s developing pair.

2.2.1. Parallel transports. Let \((E, D, \mu)\) be a flat \( \text{SL}(3, \mathbb{R})\)-vector bundle over \( \Sigma \) (i.e. \( E \) is a rank 3 vector bundle, \( D \) is a flat connection and \( \mu \) is a fiberwise volume form preserved by \( D \)). The parallel transport of the connection \( D \) along a \( C^1 \) path \( \gamma : [0, 1] \to \Sigma \) is a specific linear isomorphism between fibers

\[ T(\gamma) : E_{\gamma(0)} \to E_{\gamma(1)} \]

which preserves \( \mu \). A precise definition goes as follows. Let \( e(t) = (e_1(t), e_2(t), e_3(t)) \) be a frame of determinant 1 (with respect to \( \mu \)) of the pullback bundle \( \gamma^* E \to [0, 1] \). The pullback connection \( \gamma^* D \) is expressed under the frame as \( d + A \), where \( A \) is a 3 \times 3 matrix of 1-forms. The initial value problem of linear ODE

\[ \frac{d}{dt} T(t) + A(\partial_t) T(t) = 0, \quad T(0) = \text{id} \]

admits a unique solution \( T : [0, 1] \to \text{SL}(3, \mathbb{R}) \). We then define \( T(\gamma) \) as the linear map \( E_{\gamma(0)} \to E_{\gamma(1)} \) which is represented by the matrix \( T(1) \) under the bases \( e(\gamma(0)) \) and \( e(\gamma(1)) \). In particular, given a base point \( m \in \Sigma \), \( T \) associates to each loop \( \gamma \in \pi_1(\Sigma, m) \) an element \( T(\gamma) \) in \( \text{SL}(E_m, \mu_m) \). The resulting group homomorphism \( \pi_1(\Sigma, m) \to \text{SL}(E_m, \mu_m) \) is called the holonomy of \( D \) at \( m \).

The parallel transport \( T(\gamma) \) remains unchanged when \( \gamma \) undergoes a (endpoints-fixing) homotopy. It also satisfies the composition property

\[ T(\beta) T(\alpha) = T(\beta \cdot \alpha). \]

Here, we warn the reader of the following convention of notations used throughout this paper.
Notation. Let $\alpha$ and $\beta$ be oriented paths on $\Sigma$ such that $\alpha$ terminates at the point where $\beta$ starts. Then the concatenation of $\alpha$ and $\beta$ (running from the starting point of $\alpha$ to the ending point of $\beta$) is denoted by $\beta \cdot \alpha$.

Now let $e = (e_0, e_1, e_2)$ be a frame field of $E$ over a contractible open set $U \subset \Sigma$ such that $\mu(e_1 \wedge e_2 \wedge e_3) = 1$. Suppose $D$ is expressed under $e$ as $D = d + A$. Since paths in $U$ are determined up to homotopy by their endpoints, parallel transports are expressed by a two-pointed parallel transport map

$$T : U \times U \to \text{SL}(3, \mathbb{R}), \quad (y, x) \mapsto T(y, x).$$

For any $x, y \in U$, let $\gamma$ be a path in $U$ going from $x$ to $y$, then $T(y, x)$ is by definition the matrix of $T(\gamma) : E_x \to E_y$ with respect to the bases $e(x)$ and $e(y)$.

The map (2.3) is characterized by the following properties:

1. $T(x, x) = \text{id}$.
2. $T(x, y) = T(y, x)^{-1}$.
3. Let $\partial_u^{(2)} T(y, x)$ (resp. $\partial_u^{(1)} T(y, x)$) denote the derivative of $T(y, x)$ with respect to the variable $x$ (resp. $y$) along the tangent vector $u \in T_x U$ (resp. $v \in T_y U$). Then
   $$\partial_u^{(2)} T(y, x) = T(y, x)A(u), \quad \partial_u^{(1)} T(y, x) = -A(v)T(y, x).$$

2.2.2. Wang’s developing pair. In the rest of this paper, we let $E$ denote the rank 3 vector bundle $\Sigma \otimes \mathbb{R}$ over $\Sigma$ and let $E_m$ be the fiber of $E$ over a point $m \in \Sigma$. Let $1$ be the canonical section of $\mathbb{R} \subset E$ and $1^*$ be the section of $E^*$ such that $\langle 1^*, 1 \rangle = 1$ and $1^*(v) = 0$ for any $v \in T \Sigma$.

We now present the construction mentioned at the end of §2.1.5. Namely, we assign to each $(g, b) \in W(\Sigma)$ a pair $(\text{Dev}, \text{hol})$ which represents the pre-image of $(g, b)$ by the second map in (2.2). The construction depends on a choice of base point $m \in \Sigma$ and the resulting map $\text{Dev}$ takes values in $E_m$.

Given $m \in \Sigma$, let $\tilde{\Sigma}_m$ denote the universal cover of $\Sigma$ viewed as the space of endpoints-fixing-homotopy classes of oriented paths on $\Sigma$ terminating at $m$. The covering map $\pi : \tilde{\Sigma}_m \to \Sigma$ assigns to each path class its starting point, while $\pi_1(\Sigma, m)$ acts on $\tilde{\Sigma}_m$ by concatenation.

We first associate to $(g, b) \in W(\Sigma)$ a flat $\text{SL}(3, \mathbb{R})$-vector bundle $(E, D, \mu)$ over $\Sigma$. Suppose $g = h|dz|^2$ and $b = b dz^3$ in a conformal local coordinate $z$. We define

- $E := T \Sigma \oplus \mathbb{R}$, $\mu := \nu_g \wedge 1^*$, where $\nu_g$ is the volume form of $g$;
- Let $D$ be the connection on $E$ whose extension (by complex linearity) to $E \otimes \mathbb{C} = T \Sigma \otimes \mathbb{C}$ is expressed under the local frame $(\partial_z, \partial_{\bar{z}}, 1)$ as

$$D = d + \begin{pmatrix} \partial \log h & \frac{b}{h} d\bar{z} & d\bar{z} \\ \frac{\bar{b}}{h} dz & \bar{\partial} \log h & d\bar{z} \\ \frac{\bar{b}}{h} d\bar{z} & \frac{b}{h} dz & 0 \end{pmatrix}.$$  

Straightforward computations show $D$ is coordinate-independent, $D\mu = 0$ and $D$ is flat.

**Theorem 2.3 (Wang’s developing pair).** Let

$$\text{hol} : \pi_1(\Sigma, m) \to \text{SL}(E_m) \cong \text{SL}(3, \mathbb{R})$$
be the holonomy of the connection $D$ at $m$ and define

$$\text{Dev} : \tilde{\Sigma}_m \to E_m \cong \mathbb{R}^3, \quad \text{Dev}(\gamma) := T(\gamma) \frac{1}{\pi(\gamma)},$$

where $T(\gamma)$ is the parallel transport of $D$ along $\gamma$. Then $(\text{Dev}, \text{hol})$ represents the pre-image of $(g, b)$ under the second map in (2.2).

As a consequence, $(\text{dev} := \mathbb{P} \circ \text{Dev}, \text{hol})$ is a developing pair of the convex projective structure with Blaschke metric $g$ and Pick differential $b$.

Here, $\text{SL}(E_m)$ denotes the group of $\mu_m$-preserving linear automorphisms of the fiber $E_m$.

The developing pair $(\text{dev}, \text{hol})$ produced by Theorem 2.3 is called Wang’s developing pair associated to $(g, b) \in \mathcal{W}(\Sigma)$ relative to the base point $m$. Note that different choices of $m$ give rise to equivalent developing pairs.

2.2.3. Equilateral frames. It is conceptually clearer to express the connection (2.4) under a frame of $E$, rather than a frame of the complexification $E \otimes \mathbb{C}$. An obvious choice of such a frame is $(\partial_x, \partial_y, 1)$, where $z = x + iy$. Another choice, which is more convenient for our purpose, is

$$\begin{aligned}(e_0, e_1, e_2) &: = \left(\partial_x + \frac{1}{2}, -\frac{1}{2}\partial_x + \frac{\sqrt{3}}{2}\partial_y + 1, -\frac{1}{2}\partial_x - \frac{\sqrt{3}}{2}\partial_y + 1\right) \\
&= (\partial_z, \partial \bar{z}, 1) B,
\end{aligned}$$

where the base change matrix $B$ and its inverse are given by

$$B = \begin{pmatrix} 1 & \omega & \omega^2 \\
1 & \omega & \omega^2 \\
1 & 1 & 1 \end{pmatrix}, \quad B^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\
\omega^2 & \omega & \omega \\
\omega & \omega^2 & 1 \end{pmatrix}, \quad \omega = e^{2\pi i/3}.$$ We refer to the frame (2.5) as the equilateral frame of $E$ associated to the coordinate $z$, because its projection to $T \Sigma$ gives vertices of an equilateral triangle in each $T_m \Sigma$. Note that the canonical section $1$ of $E$ is $e_1 + e_2 + e_3$.

Such a frame is useful when $b$ is a constant multiple of $dz^3$. Throughout this paper, we always use the letter $\zeta$ to denote a local coordinate of $\Sigma$ in which $b = 2d\zeta^3$, and let $u(\zeta)$ be a function such that the Blaschke metric is expressed as $g = 2e^{u}|d\zeta|^2$.

The connection (2.4) is expressed under the equilateral frame associated to $\zeta$ as

$$D = d + B^{-1} \begin{pmatrix} \partial u & e^{-u}d\zeta & d\zeta \\
e^{-u}d\zeta & \partial u & d\zeta \\
e^{u}d\zeta & e^{u}d\zeta & 0 \end{pmatrix} B.$$

The resulting parallel transport and developing map will be studied in detail in the next subsection.

2.3. Wang’s developing map for $g = 2|d\zeta|^2$ and $b = 2d\zeta^3$. 

2.3.1. The Titeica affine sphere. The simplest example of an element \((g, b)\) in \(\mathcal{W}(\Sigma)\) is given by
\[
\Sigma = \mathbb{C}, \quad g_0 = 2|d\zeta|^2, \quad b_0 = 2\, d\zeta^3. \tag{2.8}
\]

Let \(D_0\) denote the connection \((2.4)\) in this case. It turns out that the matrix expression of \(D_0\) under the equilateral frame associated to \(\zeta\) is diagonal:
\[
D_0 = d + B^{-1} \begin{pmatrix} 0 & d\zeta & d\zeta \\ d\zeta & 0 & d\zeta \\ d\zeta & d\zeta & 0 \end{pmatrix} B = \begin{pmatrix} 2\, \text{Re}(d\zeta) & & \\ & 2\, \text{Re}(\omega^2 d\zeta) & \\ & & 2\, \text{Re}(\omega d\zeta) \end{pmatrix}, \tag{2.9}
\]
where \(\omega = e^{2\pi i/3}\).

Taking \(0 \in \mathbb{C}\) to be the base point, we apply Theorem 2.3 and get an affine spherical embedding \(\text{Dev}_0 : \mathbb{C} \to E_0\) (here \(E_0\) is the fiber of \(E = \mathbb{C} \oplus \mathbb{R}\) at \(0\)), known as the Titeica affine sphere. In order to get an explicit expression, we let \(T_0 : \mathbb{C} \times \mathbb{C} \to \text{SL}(3, \mathbb{R})\) denote the two-pointed parallel transport map of \(D_0\) under the equilateral frame \((e_0, e_1, e_2)\) associated to \(\zeta\). Since \(D_0\) is invariant under translations, we have
\[
T_0(\zeta_2, \zeta_1) = T_0(0, \zeta_1 - \zeta_2),
\]
whereas the expression of \(D_0\) and the properties of two-pointed parallel transport maps given at the end of §2.2.1 yield
\[
T_0(0, \zeta) = \begin{pmatrix} e^{2\, \text{Re}(\zeta)} & & \\ & e^{2\, \text{Re}(\omega^2 \zeta)} & \\ & & e^{2\, \text{Re}(\omega \zeta)} \end{pmatrix}. \tag{2.10}
\]

Since \(1 = e_1 + e_2 + e_3\), we get
\[
\text{Dev}_0(\zeta) = T_0(0, \zeta)1_\zeta = e^{2\, \text{Re}(\zeta)} e_1(0) + e^{2\, \text{Re}(\omega^2 \zeta)} e_2(0) + e^{2\, \text{Re}(\omega \zeta)} e_3(0).
\]

The image of \(\text{Dev}_0(\zeta)\) is the hypersurface
\[
\{x\, e_1(0) + y\, e_2(0) + z\, e_3(0) \in E_0 \mid xyz = 1\}
\]
asymptotic to the first octant in \(E_0 \cong \mathbb{R}^3\). The image of the developing map
\[
\text{dev}_0 = \mathbb{P} \circ \text{Dev}_0 : \Sigma_m \to \mathbb{P}(E_0)
\]
is a triangle \(\Delta \subset \mathbb{P}(E_0)\), in accordance with the last property in §2.1.2.

Let \([v] \in \mathbb{P}(E_0)\) denote the projectivization of \(v \in E_0\). We denote the vertices of the triangle \(\Delta\) by \(X_i := [e_i(0)] \) \((i = 1, 2, 3)\) and the edges by
\[
X_{ij} := \left\{ \left. (1 - s)\, e_i(0) + s\, e_j(0) \right\} \right\}_{s \in [0, 1]}
\]
Let \(X_{ij}^\circ\) denote the interior of the edge \(X_{ij}\).

Although the norm of \(\text{Dev}_0(\zeta)\) tends to \(+\infty\) as \(\zeta\) tends to \(\infty\), the projectivization \(\text{dev}_0(\zeta)\) has limit points lying on \(\partial \Delta\), as described by the next proposition. The proof amounts to elementary limit calculations and we omit it.

**Proposition 2.4.** Let \(\alpha : [0, +\infty) \to \mathbb{C}\) be a path in \(\mathbb{C}\) such that \(|\alpha(t)|\) tends to \(+\infty\) and the argument \(\arg(\alpha(t))\) admits a limit \(\theta(\alpha) \in [0, 2\pi)\) as \(t \to +\infty\).
(1) Put
\[ I_1 = (-\frac{\pi}{3}, \frac{\pi}{3}), \quad I_2 = (\frac{\pi}{3}, \pi), \quad I_3 = (\pi, \frac{5\pi}{3}) \]
then \( \lim_{t \to +\infty} \text{dev}_0(\alpha(t)) = X_i \) whenever \( \theta(\alpha) \in I_i \). Furthermore, let \( I_i^- \) (resp. \( I_i^+ \)) denote the first (resp. second) half of \( I_i \), i.e. \( I_1^- = (-\frac{\pi}{3}, 0), I_2^- = (0, \frac{\pi}{3}) \), etc., then the curve \( t \mapsto \text{dev}_0(\alpha(t)) \) is asymptotic to \( X_{i,i\pm 1} \) (here the indices are counted modulo 3) whenever \( \theta(\alpha) \in I_i^\pm \).

(2) For any \( \theta \in \{ \pm \frac{\pi}{3}, \pi \} \) and \( s \in \mathbb{R} \), put
\[ \alpha_{\theta,s}(t) := e^{\theta i}(t + si). \]
Then
\[ \lim_{t \to +\infty} \text{dev}_0(\alpha_{\theta,\pi/3,s}(t)) = [e^{-\sqrt{3}s}e_1(0) + e^{\sqrt{3}s}e_2(0)] \in X_{12}^\circ, \]
\[ \lim_{t \to +\infty} \text{dev}_0(\alpha_{-\pi/3,s}(t)) = [e^{-\sqrt{3}s}e_3(0) + e^{\sqrt{3}s}e_1(0)] \in X_{31}^\circ, \]
\[ \lim_{t \to +\infty} \text{dev}_0(\alpha_{\pi,s}(t)) = [e^{-\sqrt{3}s}e_2(0) + e^{\sqrt{3}s}e_3(0)] \in X_{23}^\circ. \]

As a consequence, each point of \( X_{12}^\circ \) (resp. \( X_{23}^\circ, X_{31}^\circ \)) is the limit of \( \text{dev}_0(\alpha(t)) \) for some \( \alpha \) satisfying \( \theta(\alpha) = \frac{\pi}{3} \) (resp. \( \pi, -\frac{\pi}{3} \)).

In the first part of the proposition, a curve \( t \mapsto x(t) \) in \( \mathbb{R}^2 \) converging to a point \( x_0 \) is said to be asymptotic to a ray or a segment issuing from \( x_0 \) if, for any small neighborhood \( U \) of \( x_0 \) and any sector \( S \) of small angle delimited by \( t \) and another ray issuing from \( x_0 \), \( x(t) \) is contained in \( U \cap S \setminus \{ x_0 \} \) for \( t \) sufficiently large.

2.3.2. Eigenvalues of \( \text{Ad}_{\mathcal{T}_0(0,\zeta)} \). For our purpose in Section 7, we need to know the particulars on eigenvalues of the Lie algebra inner automorphism
\[ \text{Ad}_{\mathcal{T}_0(0,\zeta)} : \mathfrak{sl}_3 \mathbb{R} \to \mathfrak{sl}_3 \mathbb{R} \]
where \( \zeta \in \mathbb{C} \). We record here some calculations used later on.

Let \( E_{ij} \) denote the matrix whose \((i,j)\)-entry is 1 and the other entries vanish. \( \text{Ad}_{\mathcal{T}_0(0,\zeta)} \) acts trivially on the diagonal subalgebra, while acting on \( E_{ij} \) \((i \neq j)\) with eigenvalue
\[ \exp \left( 2 \text{Re}(\omega^{1-i} \zeta \omega^{1-j} \bar{\zeta}) \right) = \exp \left( \varpi_{ij}(\arg(\zeta))|\zeta| \right), \]
where
\[ \varpi_{ij}(\theta) := 2 \text{Re}(\omega^{1-i}e^{\theta i} - \omega^{1-j}e^{\theta i}). \]

One easily checks the following facts.
(1) The spectral radius of \( \text{Ad}_{\mathcal{T}_0(0,\zeta)} \) is
\[ \rho(\text{Ad}_{\mathcal{T}_0(0,\zeta)}) = \exp \left( \varpi(\arg(\zeta))|\zeta| \right), \]
where the function
\[ \varpi(\theta) := \max_{i,j} \varpi_{ij}(\theta) \]
is continuous and reaches the maximum \( 2\sqrt{3} \) if and only if \( \theta \) is an odd multiple of \( \frac{\pi}{6} \).
(2) When \( \theta \) equals each odd multiple of \( \frac{\pi}{6} \), there is exactly one \((i,j)\) satisfying \( \varpi_{ij}(\theta) = \varpi(\theta) = 2\sqrt{3} \), given by the following table.

| \( \theta \) | \( \frac{\pi}{6} \) | \( \frac{\pi}{3} \) | \( \frac{\pi}{2} \) | \( \frac{5\pi}{6} \) | \( \pi \) | \( \frac{7\pi}{6} \) | \( \frac{\pi}{2} \) | \( \frac{5\pi}{6} \) |
| (i,j) | (1,3) | (2,3) | (2,1) | (3,1) | (3,2) | (1,2) |

\( \varpi_{ij}(\theta) = \varpi(\theta) = 2\sqrt{3} \).
2.3.3. Convex projective structures whose developing image are triangles. A convex projective structure whose developing image is a triangle must be a quotient of the one given by the Tiţeica affine sphere. The following proposition enumerates all the possibilities.

**Proposition 2.5.** Let \((\dev, \hol)\) be a developing pair of a convex projective structure \(\Sigma\) with Blaschke metric \(g\) and Pick differential \(b\). Then \(\Omega = \dev(\Sigma)\) is a triangle if and only if \((\Sigma, g, b)\) is equivalent to either of the following examples.

(a) \((\mathbb{C}, g_0, b_0) = (\mathbb{C}, 2|d\zeta|^2, 2d\zeta^3)\);

(b) \((\mathbb{C}^*, 2\frac{1}{\pi}|R|^{\frac{2}{3}}|z|^{-2}|dz|^2, Rz^{-3}dz^3)\), where \(R \in \mathbb{C}^*\);

(c) \((\mathbb{C}/\Lambda, 2|d\zeta|^2, 2d\zeta^3)\), where \(\Lambda \cong \mathbb{Z}^2\) is a lattice in \(\mathbb{C}\).

Furthermore,

1. In case (b), the holonomy of the convex projective structure along a loop going around 0 counter-clockwise is conjugate to

\[
\begin{pmatrix}
e^{-4\pi \mu_1} & e^{-4\pi \mu_2} \\
e^{-4\pi \mu_3}
\end{pmatrix},
\]

where \(\mu_1, \mu_2, \mu_3 \in \mathbb{R}\) are the imaginary parts of the three cubic roots of \(R/2\), respectively.

2. If \(\Sigma\) is a torus then \((\Sigma, g, b)\) is always equivalent to (c). As a result, the developing image of any convex projective structure on a torus is a triangle.

Note that part (1) is the simplest example of Theorem 1.2 from the introduction.

**Proof.** The developing map \(\dev\) induces an isomorphism between \((\Sigma, g, b)\) and the quotient of \((\mathbb{C}, g_0, b_0)\) by a free and properly discontinuous action of \(\pi_1(\Sigma)\) preserving \(b_0\) and hence preserving \(g_0\). But the only non-trivial such actions are the actions of either \(\mathbb{Z}\) or \(\mathbb{Z}^2\) by translations.

In the case of \(\mathbb{Z}\), the action is generated by \(\zeta \mapsto \zeta + 2\pi i\mu\) for some \(\mu \in \mathbb{C}, \mu \neq 0\). The map

\[
(2.11) \quad \mathbb{C} \to \mathbb{C}^*, \quad \zeta \mapsto z = e^{\zeta / \mu},
\]

identifies the quotient \((\mathbb{C}, g_0, b_0)/\mathbb{Z}\) with \((\mathbb{C}^*, 2|\mu|^2|d\zeta|^2, \frac{2\lambda^2}{\zeta^3}d\zeta^3)\) in such a way that a path in \(\mathbb{C}\) going from \(2\pi i\mu\) to 0 corresponds to a loop in \(\mathbb{C}^*\) going around 0 counter-clockwise. Eq. (2.10) gives the holonomy along such a loop as

\[
T_0(0, 2\pi \lambda i) = \begin{pmatrix}
e^{-4\pi \im \lambda} & e^{-4\pi \im (\omega^2 \lambda)} \\
e^{-4\pi \im (\omega \lambda)} & e^{-4\pi \im (\omega^2 \lambda)}
\end{pmatrix}, \quad \omega = e^{2\pi i / 3}.
\]

Statement (1) is proved by setting \(R = 2\mu^3\).

We now assume that \(\Sigma\) is a torus and prove statement (2). Fix \((g, b) \in \mathcal{W}(\Sigma)\). \((\Sigma, g)\) is conformally equivalent to \(\mathbb{C}/\Lambda\) for some lattice \(\Lambda \subset \mathbb{C}\) and we have \(b = b(z)\ dz^3\) for a \(\Lambda\)-periodic entire function \(b(z)\), which must be a constant. We can assume \(b = 2\) by scaling and assume \(g = h|dz|^2\) for some smooth function \(h : \mathbb{C}/\Lambda \to \mathbb{R}_+\). Wang’s equation (2.1) then becomes

\[-\frac{2}{h} \partial_{zz} \log h + 1 - \frac{8}{h^3} = 0.\]
Applying the maximum principle to this equation shows that the maximal and minimal values of \( h \) are both 2, hence the proof is complete. \( \square \)

### 2.3.4. Wang’s developing pair associated to \((2 \frac{1}{2} | b | \frac{3}{2}, b)\).

It is important for our purpose later on to remark that the construction of Wang’s developing pair \((\text{dev}, h\text{ol})\) in Theorem 2.3 makes sense not only for \((g, b) \in \mathcal{W}(\Sigma)\), but also for some more general \((g, b)\).

Indeed, the definition of \(\text{Dev}\) and \(\text{hol}\) in Theorem 2.3 only requires the connection \(D\) defined by (2.4) to be flat. But the flatness is actually equivalent to Wang’s equation (2.1) and does not particularly require \(g\) to be complete as in the definition of \(\mathcal{W}(\Sigma)\). We can even allow \(g\) to have zeros as long as the ratio \(b/h\) in the expression of \(D\) makes sense at the zeros.

Therefore, we can put the singular flat metric \(2 \frac{1}{2} | b | \frac{3}{2}\) in place of \(g\) and carry out the same construction, i.e. integrating the flat connection (2.4), denote by \(D_0\) in this case, to get a pair \((\text{dev}_0, h\text{ol}_0)\), where

\[
\text{dev}_0 : \tilde{\Sigma}_m \to \mathbb{P}(E_m), \quad h\text{ol}_0 : \pi_1(\Sigma, m) \to \text{SL}(E_m, \mu_m).
\]

This is called Wang’s developing pair associated to \((2 \frac{1}{2} | b | \frac{3}{2}, b)\), although in general it is not really a developing pair of any projective structure.

As in §2.2.3, let \(\zeta\) be a conformal local coordinate defined on a contractible open set \(U\) such that \(b = 2 d\zeta^3\). Let \(T_0 : U \times U \to \text{SL}(3, \mathbb{R})\) be the two-pointed parallel transport map of \(D_0\) with respect to the equilateral frame associated to \(\zeta\). Then the local expression of \((2 \frac{1}{2} | b | \frac{3}{2}, b)\) coincide with (2.8), hence \(D_0\) and \(T_0\) also have the same expressions (2.9) and (2.10) as the Ţiţeica affine sphere. This explains our choice of notations. Indeed, the main use of Ţiţeica example in this paper is to provide a local model for Wang’s developing map associated to \((2 \frac{1}{2} | b | \frac{3}{2}, b)\).

### 3. Ends of convex projective surfaces

In this section, we fix an oriented punctured surface of finite type \(\Sigma\), a base point \(m \in \Sigma\) so as to define the universal cover \(\tilde{\Sigma} = \tilde{\Sigma}_m\) as in §2.2.2, and a developing pair \((\text{dev}, h\text{ol})\) of a convex projective structure on \(\Sigma\). Denote \(\Omega := \text{dev}(\tilde{\Sigma}) \subset \mathbb{R}\mathbb{P}^2\).

#### 3.1. Developing boundaries.

##### 3.1.1. The Farey set.

**Proposition/Definition 3.1 (Farey set).** Let \(p\) be a puncture of \(\Sigma\). The following transitive \(\pi_1(\Sigma)\)-sets are canonically isomorphic and are called the Farey set of \(\Sigma\) at \(p\), denoted by \(\text{Farey}(\Sigma, p)\).

(i) The conjugacy class in \(\pi_1(\Sigma)\) carried by a simple loop going around \(p\) clockwise (with respect to the orientation of \(\Sigma\)). \(\pi_1(\Sigma)\) acts by conjugation.

(ii) (If \(\Sigma\) has negative Euler characteristic and we identify \(\tilde{\Sigma}\) with the Poincaré disk \(\mathbb{D}\) by means of a complete hyperbolic metric of finite volume) the set of points on the idea boundary \(\partial \mathbb{D}\) corresponding to \(p\). \(\pi_1(\Sigma)\) acts on this set by Möbius transformations.

(iii) The set of equivalence classes of oriented paths on \(\Sigma\) issuing from \(m\) and converging to \(p\), where two such paths are considered equivalent if they are homotopic through a family of such paths.
Remark 3.2. (1) If $\Sigma$ has non-negative Euler characteristic (i.e. homeomorphic to $\mathbb{C}$ or $\mathbb{C}^*$) then only (i) and (iii) makes sense and Farey($\Sigma, p$) consist of a single point.

(2) The choice of hyperbolic metric in (ii) is not essential because identifications $\tilde{\Sigma} \cong \tilde{\mathbb{D}}$ induced by different hyperbolic metrics are intertwined by $\pi_1(\Sigma)$-equivariant homeomorphisms $\mathbb{D} \to \mathbb{D}$ which extend to the boundary.

(3) We call the above defined object “Farey set” following [FG06]. This name stems from the fact that when $\tilde{\Sigma}$ is the punctured torus, an ideal triangulation of $\Sigma$ with two triangles lifts to the classic Farey triangulation of $\mathbb{D}$, whose vertices form the set (ii).

Isomorphisms between the three $\pi_1(\Sigma)$-sets in Proposition/Definition 3.1 follows from standard geometric arguments. Their constructions are briefly outline below. We assume $\Sigma$ has negative Euler characteristic, otherwise it is trivial.

Outline of proof of Proposition/Definition 3.1. Isomorphism between (i) and (ii): given a complete hyperbolic metric of finite volume on $\Sigma$, an element $\gamma$ in (i) is realized as a parabolic isometry of $\mathbb{D}$. The point in (ii) corresponding to $\gamma$ is the fixed point of the $\gamma$-action on $\mathbb{D}$.

Isomorphism between (ii) and (iii): given a point $\tilde{p}$ in (ii), the corresponding element in (iii) consists of projections of paths in $\tilde{\Sigma} \cong \tilde{\mathbb{D}}$ going from $\tilde{m}$ to $\tilde{p}$ and meeting any open horodisk based at $\tilde{p}$. □

In view of Proposition/Definition 3.1, we admit the following terminologies concerning the Farey set.

Notations. When $\Sigma$ has negative Euler characteristic, each $\tilde{p} \in \text{Farey}(\Sigma, p)$ is understood as a point in $\partial \mathbb{D}$ as in (ii). The corresponding element in (i) is denoted by $\gamma_{\tilde{p}} \in \pi_1(\Sigma)$, while a path belonging to the corresponding equivalence class in (iii) is said to belong to the class $\tilde{p}$. Moreover, for any connected punctured neighborhood $U$ of $p$, $\pi^{-1}(U) \subset \tilde{\Sigma}$ has exactly one connected component $\tilde{U}$ whose closure in $\mathbb{D}$ contains $\tilde{p}$. We call $\tilde{U}$ the lift of $U$ attached at $\tilde{p}$.

3.1.2. Developing boundaries. We introduce a point-set-topological notion: given a sequence of subsets $S_1, S_2, \cdots$ of a topological space $X$, the set of accumulation points of the $S_i$’s, as $i$ tends to infinity, is defined as

$$\text{Accum}(S_i) := \{ x \in X \mid \exists x_i \in S_i \text{ such that } \lim_{i \to +\infty} x_i = x \}. $$

Note that if $(S_i)$ is a decreasing sequence, i.e. $S_1 \supset S_2 \supset \cdots$, then

$$\text{Accum}(S_i) = \bigcap_{i=1}^{+\infty} S_i.$$

Return to the punctured surface $\Sigma$. A nested sequence of punctured neighborhoods $U_1 \supset U_2 \supset \cdots$ of $p$ is said to be exhausting if any punctured neighborhood of $p$ contains some $U_i$.

Definition 3.3 (Developing Boundary and end holonomy). Given $\tilde{p} \in \text{Farey}(\Sigma, p)$, the developing boundary and end holonomy of the convex projective structure at $\tilde{p}$, denoted by $\partial_\text{d} \Omega$ and $\text{hol}_\text{end}$, respectively, are defined as follows.
Let $U_1 \supset U_2 \supset \cdots$ be an exhausting sequence of connected punctured neighborhoods of $p$ and let $\tilde{U}_i$ be the lift of $U_i$ attached at $\tilde{p}$ (see §3.1.1; if $\Sigma$ is homeomorphic to $\mathbb{C}$ or $\mathbb{C}^*$, we just set $\tilde{U}_i = \pi^{-1}(U_i)$). Then we define

$$\partial_{\tilde{p}} \Omega := \text{Accum}(\text{dev}(\tilde{U}_i)) \subset \mathbb{RP}^2, \quad \text{hol}_\gamma := \text{hol}(\gamma_\beta) \in \text{SL}(3, \mathbb{R}).$$

Since $(U_i)$ is exhausting, the definition of $\partial_{\tilde{p}} \Omega$ does not depend on the choice of $(U_i)$. Some fundamental properties of $\partial_{\tilde{p}} \Omega$ are given by the next proposition.

**Proposition 3.4.** The developing boundary $\partial_{\tilde{p}} \Omega$ is a nonempty connected closed subset of $\partial \Omega$ preserved by $\text{hol}_\gamma$. Moreover, we have

$$\partial_{\gamma, \tilde{p}} \Omega = \text{hol}(\gamma) \cdot \partial_{\tilde{p}} \Omega.$$  

**Proof.** $\partial_{\tilde{p}} \Omega$ is preserved by $\text{hol}_\gamma$ because each $\tilde{U}_i$ is preserved by the deck action of $\gamma_\tilde{p}$ and $\text{dev}$ is $\text{hol}$-equivariant.

Since $\Omega$ is compact, the developing boundary $\partial_{\tilde{p}} \Omega = \bigcap_i \text{dev}(\tilde{U}_i)$ is nonempty and is contained in $\Omega$. We proceed to prove that $\partial_{\tilde{p}} \Omega$ is contained in $\partial \Omega$. Otherwise, $\text{dev}(\tilde{U}_1), \text{dev}(\tilde{U}_2), \cdots$ accumulates at some $q \in \Omega$. Projecting $\text{dev}^{-1}(q) \in \tilde{\Sigma}$ to $\Sigma$, we get an accumulation point of the sequence of punctured neighborhoods $(U_i)$, contradicting the exhausting hypothesis.

Finally, since each $\text{dev}(\tilde{U}_i)$ is connected, connectedness of $\partial_{\tilde{p}} \Omega$ follows from the elementary fact that in a compact metrizable topological space $X$, if $(S_i)$ is a sequence of connected subsets, then $\text{Accum}_{i \to +\infty}(S_i)$ is connected as well. \hfill $\square$

### 3.1.3. Developing limits

Fix $\tilde{p} \in \text{Farey}(\Sigma, p)$. Let $\beta$ be an oriented path on $\Sigma$ issuing from $m$, converging to $p$ and belonging to the class $\tilde{p}$. Take a parametrization

$$[0, +\infty) \to \Sigma, \quad t \mapsto \beta(t).$$

Consider the truncated path $\beta_{[0,t]} := \beta([0,t])$. Reversing its orientation, we get a path $\beta_{[0,t]}^{-1}$ going from $\beta(t)$ to $m$, which represents a point in the universal cover $\tilde{\Sigma}_m$. We simply denote the image of this point under the developing map $\text{dev} : \tilde{\Sigma}_m \to \Omega$ by $\text{dev}(\beta_{[0,t]}^{-1})$.

As $t \to +\infty$, the point in $\tilde{\Sigma}_m$ represented by $\beta_{[0,t]}^{-1}$ eventually enters any open set $\tilde{U}_i$ from Definition 3.3. As a consequence, any limit point of the path $t \mapsto \text{dev}(\beta_{[0,t]}^{-1})$ is contained in $\partial_{\tilde{p}} \Omega$. For our purpose later on, we only need to consider the case where there is a single limit point, as in the following definition.

**Definition 3.5 (Developing limits).** If the limit $\lim_{t \to +\infty} \text{dev}(\beta_{[0,t]}^{-1}) \in \partial_{\tilde{p}} \Omega$ exists, we call it the developing limit of $\beta$ and denoted it by $\text{Lim}(\beta)$. Otherwise, we just say “$\text{Lim}(\beta)$ does not exists”.

For example, consider the convex projective structure on $\mathbb{C}$ given by the Ţiţeica affine sphere embedding (c.f. §2.3.1). Proposition 2.4 is a description of $\text{Lim}(\beta)$ when $\beta$ tends to the puncture $\infty$ along specific directions.

### 3.2. Simple and non-simple ends
3.2.1. Automorphisms of properly convex sets. Let \( a \in \text{SL}(3, \mathbb{R}) \) be a projective transformation preserving a properly convex open set \( \Omega \subset \mathbb{RP}^2 \) such that \( a \) does not have any fixed point in \( \Omega \). It is well known that \( a \) belongs to either of the following four classes.

1. \( a \) is said to be **hyperbolic** if it is conjugate to
   \[
   \begin{pmatrix}
   \lambda_+ & 0 \\
   \lambda_0 & \lambda_-
   \end{pmatrix}
   \]
   where \( \lambda_+ > \lambda_0 > \lambda_- > 0 \) and \( \lambda_1 \lambda_0 \lambda_- = 1 \). The fixed point of \( a \) in \( \mathbb{RP}^2 \) corresponding to the eigenvalue \( \lambda_+ \) (resp. \( \lambda_0, \lambda_- \)) is called the attracting (resp. saddle, repelling) fixed point and is denoted by \( x_+ \) (resp. \( x_0, x_- \)).

   Any properly convex open set \( \Omega \) preserved by such \( a \) can be described as follows. There is a segment \( I \subset \mathbb{RP}^2 \) joining \( x_+ \) and \( x_- \), called the principal segment of \( a \), such that \( I \subset \partial \Omega \). Let \( \Delta_1 \) and \( \Delta_2 \) be the two open triangles preserved by \( a \) which are adjacent along \( I \). Then we can write
   \[
   \Omega = \Omega_1 \cup \Omega_2 \cup I^\circ,
   \]
   where each \( \Omega_i \) is either the empty set, or the triangle \( \Delta_i \), or some open subset of \( \Delta_i \) delimited by \( I \) and a convex curve in \( \Delta_i \) joining \( x_+ \) and \( x_- \).

2. \( a \) is said to be **quasi-Hyperbolic** if it is conjugate to
   \[
   \begin{pmatrix}
   \lambda & 1 \\
   \mu & \lambda
   \end{pmatrix}
   \]
   with \( \lambda > 0, \lambda \neq 1 \) and \( \lambda^2 \mu = 1 \). We call the fixed points corresponding to the eigenvalues \( \lambda \) and \( \mu \) the double fixed point and the simple fixed point, respectively.

   For any properly convex open set \( \Omega \) preserved by \( a \), the boundary \( \partial \Omega \) contains a segment \( I \) joining \( x_0 \) as a vertex and contains a segment on \( l \) as an edge. In analogy with the quasi-hyperbolic case, we call the two edges issuing from \( x_0 \) the principal segments of \( a \), and call each vertex different from \( x_0 \) a double fixed point.

3. \( a \) is said to be **planar** if it is conjugate to
   \[
   \begin{pmatrix}
   \lambda & 0 \\
   \mu & \lambda
   \end{pmatrix}
   \]
   with \( \lambda > 0, \lambda \neq 1 \) and \( \lambda^2 \mu = 1 \). \( a \) has an isolated fixed point \( x_0 \) and a pointwise fixed line \( l \), corresponding to the eigenvalues \( \mu \) and \( \lambda \), respectively.

   Any properly convex open set \( \Omega \) preserved by \( a \) is a triangle whose boundary contains \( x_0 \) as a vertex and contains a segment on \( l \) as an edge. In analogy with the quasi-hyperbolic case, we call the two edges issuing from \( x_0 \) the principal segments of \( a \), and call each vertex different from \( x_0 \) a double fixed point.

4. \( a \) is said to be **parabolic** if it is conjugate to
   \[
   \begin{pmatrix}
   1 & 1 \\
   1 & 1
   \end{pmatrix}.
   \]
In this case, \(a\) fixes a single point \(x_0 \in \mathbb{R}^2\) and preserves a line \(l\) passing through \(l\). For any properly convex open set \(\Omega\) preserved by \(a\), the boundary \(\partial \Omega\) contains \(x_0\) and is tangent to \(l\) at \(x_0\).

We refer to [Mar12] Section 2 for a systematic treatise of the above classification. Dynamical properties of each type of automorphisms, such as asymptotic behaviors of any iteration sequence \((a^n(x))\), are well known and are used in proving the above statements about the shape of \(\Omega\) relative to fixed points. We will make use of these properties a few times below without detailed explanation.

3.2.2. Simple ends. In addition to the surface \(\Sigma\) and the convex projective structure \(y\) fixed at the beginning of this section, we choose a punctured \(p\) of \(\Sigma\) and choose \(\tilde{p} \in \text{Farey}(\Sigma, p)\) for the rest of this section so as to define \(\partial \tilde{p} \Omega\) and \(\text{hol} \tilde{p}\) (c.f. §3.1).

**Definition 3.6 (Simple ends).** We call \(p\) a simple end of the convex projective structure if the developing boundary \(\partial \tilde{p} \Omega\) is either

1. a point, or
2. a segment, or
3. the union of two non-collinear segments sharing an endpoint.

In the three cases, we call \(p\) a cusp, a geodesic end and a triangular end, respectively.

By definition, if \(\Sigma \cong \mathbb{C}\) then \(\partial \tilde{p} \Omega\) is just the whole \(\partial \Omega\) and the puncture \(\infty\) is always a non-simple end.

We first establish the following technical result for later use, claiming that planar end holonomies only occur for some particular projective structures on an annulus and only give rise to geodesic ends.

**Lemma 3.7 (Ends with planar holonomy).** A convex projective structure on \(\Sigma\) has planar end holonomy \(\text{hol} \tilde{p}\) if and only if \(\Sigma\) is an annulus and the Blaschke metric \(g\) and Pick differential \(b\) are as in case (b) of Proposition 2.5 with \(R \in i \mathbb{R}^+\). In this case, the developing boundaries of each puncture is a principal segment of \(\text{hol} \tilde{p}\).

**Proof.** If a convex projective structure has a planar holonomy, then the developing image is a triangle (see §3.2.1), hence Proposition 2.5 applies. But among the three cases in Proposition 2.5, the only one with planar holonomy is (b) with \(R \in i \mathbb{R}^+\). This proves the first statement.

Let \(x_1\) and \(x_2\) denote the vertices of \(\Delta\) other than \(x_0\). Let \((y_i)_{i \geq 1}\) be a sequence of points contained in the interior of the edge \(x_1 x_2\) and converging to \(x_1\). Let \(\Delta_i\) be the sub-triangle of \(\Delta\) with vertices \(x_0, x_1\) and \(y_i\), which is invariant by the holonomy \(h\).

The projective structure identifies the annulus \(\Sigma\) with the quotient of \(\Delta\) by \(h\). Hence the quotients the \(\Delta_i\)'s form an exhausting sequence of punctured neighborhoods of a punctured \(p\). By definition, the developing boundary at \(p\) is \(\bigcap_i \overline{\Delta_i} = \overline{x_0 x_2}\). Similarly, the developing boundary of the other puncture is \(\overline{x_0 x_2}\). □

The holonomy of a general simple end is described by the following proposition.

**Proposition 3.8 (Holonomy of simple ends).**

1. If \(p\) is a cusp of the convex projective structure, then \(\text{hol} \tilde{p}\) is parabolic.
2. \(p\) is a triangular end if and only if \(\text{hol} \tilde{p}\) is hyperbolic with saddle fixed point contained in \(\partial \tilde{p} \Omega\). In this case, \(\partial \tilde{p} \Omega\) consists of a segment joining the saddle and attracting fixed points and a segment joining the saddle and repelling fixed points.
3. If \(p\) is a geodesic end then \(\text{hol} \tilde{p}\) is either hyperbolic, quasi-hyperbolic or planar. In the former two cases, \(\partial \tilde{p} \Omega\) is the principal segment of \(\text{hol} \tilde{p}\).
Proof. (1) Suppose by contradiction that \( a := \text{hol}_p \) is not parabolic, so that \( a \) is either hyperbolic, quasi-hyperbolic or planar and \( x_0 := \partial_p \Omega \) is one of the fixed points of \( a \). Looking at the dynamics of those types of projective transformations, one sees that for a generic point \( x \) in a neighborhood of \( x_0 \), the iteration sequence \( (a^n(x)) \) converges to another fixed point \( x_1 \) of \( a \) as \( n \) goes to either \( +\infty \) or \( -\infty \). It follows that the closure of each \( a \)-invariant set \( \text{dev}(U_i) \) in Definition 3.3 contains \( x_1 \), hence so does \( \partial_p \Omega \), a contradiction.

(2) Suppose \( p \) is a triangular end. By the classification in §3.2.1, the projective transformation \( a \) restricts to an orientation-preserving homeomorphism of \( \partial \Omega \), thus the three endpoints of the two segments are all fixed by \( a \), hence \( a \) is either hyperbolic or planar. The planar case is excluded by Lemma 3.7. This proves the “only if” part.

Conversely, suppose that \( a \) is hyperbolic with saddle fixed point contained in \( \partial_p \Omega \). The description of \( \Omega \) in §3.2.1 for hyperbolic \( a \) implies that \( \Omega \) contains some open triangle \( \Delta \) preserved by \( a \), so that the edge \( x_\pm x_0 \) of \( \Delta \) joining \( x_\pm \) and \( x_0 \) is a part of \( \partial \Omega \).

Let \( (U_i) \) be an exhausting sequence of punctured neighborhood of \( p \) whose boundaries are embedded closed curves. Let \( \hat{U}_i \subset \hat{\Sigma} \) be the lift of \( U_i \) attached at \( p \). The developing image \( C_i \) of the boundary curve of \( \hat{U}_i \) is an \( a \)-invariant embedded curve in \( \Omega \). Then dynamical properties of \( a \) imply that \( C_i \) is contained in \( \Delta \), joins \( x_+ \) and \( x_- \) and is asymptotic to \( x_\pm x_0 \) at \( x_\pm \). Since \( \text{dev}(\hat{U}_i) \) is the connected component of \( \Omega \setminus C_i \), whose closure contains \( x_0 \), the properties of \( C_i \) imply that \( \text{dev}(\hat{U}_i) \cap \partial \Omega \) is exactly \( \overline{x_- x_0} \cup \overline{x_+ x_0} \) and does not depend on \( i \). Thus we get \( \partial_p \Omega = \overline{x_- x_0} \cup \overline{x_+ x_0} \).

This proves the “if” part and the second statement.

(3) Both endpoints of the segment \( \partial_p \Omega \) are fixed by \( \text{hol}_p \). But only hyperbolic, quasi-hyperbolic and planar projective transformations have at least two fixed points. This proves the first statement.

The second statement is obvious when \( \text{hol}_p \) is quasi-hyperbolic and follows from part (2) above when \( \text{hol}_p \) is hyperbolic.

\( \square \)

The next lemma gives some information about non-simple ends and will be used in Section 7 below.

**Lemma 3.9 (Rough classification of non-simple ends).** If \( \Sigma \) is not homeomorphic to \( \mathbb{C} \) and \( p \) is not a simple end, then we are in one of the following cases.

(a) \( \text{hol}_p \) is hyperbolic, while \( \partial_p \Omega \) is a convex curve joining the attracting fixed point \( x_+ \) and the repelling fixed point \( x_- \) of \( \text{hol}_p \) such that \( \partial_p \Omega \setminus \{x_+, x_-\} \) is contained in an open triangle \( \Delta \subset \mathbb{R}P^2 \) preserved by \( \text{hol}_p \).

(b) \( \Sigma \) is an annulus, \( \text{hol}_p \) is quasi-hyperbolic and \( \partial_p \Omega = \partial \Omega \setminus I^\circ \). Here \( I^\circ \) is the interior of the principal segment \( I \) of \( \text{hol}_p \). The other end of \( \Sigma \) is geodesic.

(c) \( \Sigma \) is an annulus, \( \text{hol}_p \) is parabolic and \( \partial_p \Omega = \partial \Omega \). The other end of \( \Sigma \) is a cusp.

Proof. Being a connected closed subset of \( \partial \Omega \) not degenerating to a point, \( \partial_p \Omega \) belongs to either of the following two cases.

**Case 1: \( \partial_p \Omega \) is a curve with two endpoints.** Both endpoints are fixed by \( a := \text{hol}_p \), thus \( a \) is either hyperbolic, quasi-hyperbolic (the planar case being excluded by Lemma 3.7). It follows from the descriptions of \( \Omega \) in §3.2.1 and the non-simple
assumption that if \( a \) is hyperbolic then statement (a) holds. Let us show that if \( a \) is quasi-hyperbolic then statement (b) holds.

First, it also follows from the descriptions of \( \Omega \) in §3.2.1 and the non-simple assumption that \( \partial \Omega = \partial \Omega \setminus I^0 \).

We prove next that \( \Sigma \) is an annulus. Suppose by contradiction that \( \Sigma \) has negative Euler characteristic. Then \( \text{Farey}(\Sigma, p) \) has infinitely many elements. Take \( \tilde{p}' \in \text{Farey}(\Sigma, p) \) different from \( \tilde{p} \). We identify \( \tilde{\Sigma} \) with the Poincaré disk \( \mathbb{D} \) via a hyperbolic metric of finite volume and view \( \text{Farey}(\Sigma, p) \) as a subset of \( \partial \mathbb{D} \). Take disjoint horodisks \( \tilde{U} \) and \( \tilde{U}' \) based at \( \tilde{p} \) and \( \tilde{p}' \), respectively. Let \( C \) denote the developing image of the horocycle bounding \( \tilde{U} \), which is an embedded curve in \( \Omega \). Dynamical properties of the quasi-hyperbolic holonomy imply that \( C \) joins the two endpoints of \( I \). Thus \( \Omega \setminus C \) has two connected components: one is just \( \text{dev}(\tilde{U}) \), whose closure contains \( \partial \tilde{\Omega} = \partial \Omega \setminus I^0 \), whereas the other, denoted by \( W \), contains \( I \) in its closure. \( \tilde{U}' \) does not meet \( \tilde{U} \), so we have \( \text{dev}(\tilde{U}') \subset W \), whence \( \partial \tilde{p}' \Omega \subset \text{dev}(\tilde{U}') \subset W \subset I \) by definition of developing boundaries. But this would imply that \( \partial \tilde{p}' \Omega \) is either a segment or a point, which is impossible because \( \partial \tilde{p}' \Omega \) is projectively equivalent to \( \partial \tilde{p} \Omega \) by transitivity of the \( \pi_1(\Sigma) \) action on \( \text{Farey}(\Sigma, p) \) and Proposition 3.4. Therefore \( \Sigma \) is an annulus.

The projective structure identifies \( \Sigma \) with the quotient of \( \Omega \) by \( a \). The dynamics of the \( a \)-action on \( \overline{\Omega} \) is the same as that of a loxodromic isometry of \( \mathbb{D} \) acting on \( \overline{\mathbb{D}} \). Using the definition, one checks that in this case the developing boundaries of the two punctures are exactly the two intervals on \( \partial \Omega \) bounded by fixed points. Thus the developing boundary of the other puncture is \( I \). The proof of statement (b) is now complete.

Case 2: \( \partial \tilde{p} \Omega = \partial \Omega \). We shall show that statement (c) holds in this case. This would complete the proof of the lemma.

Let us first show that \( \text{hol}_p \) is parabolic. Let \( U \) be a punctured neighborhood of \( p \) whose boundary is an embedded closed curve and let \( \tilde{U} \subset \tilde{\Sigma} \) be the lift of \( U \) attached at \( \tilde{p} \). If \( \text{hol}_p \) is not parabolic, then it is hyperbolic or quasi-hyperbolic and has two fixed points on \( \partial \Omega \). Similarly as above, dynamical properties imply that the boundary curve of \( \tilde{U} \) develops into an embedded curve in \( \Omega \) joining the two fixed points, dividing \( \Omega \) into two regions. \( \partial \tilde{p} \Omega \) is contained in the closure of one of the regions and can not be the whole \( \partial \Omega \), a contradiction.

It remains to be shown that \( \Sigma \) is an annulus and the other puncture is a cusp. But this follows from similar arguments in the above quasi-hyperbolic case. We omit the details. \( \square \)

3.2.3. Twisted polygonal ends, the space \( P_0(\Sigma) \). A definition for the main object of study of the present paper, the subset \( P_0(\Sigma) \) of the space of convex projective structures \( P(\Sigma) \), is given below. \( P_0(\Sigma) \) consists of projective structures with simple ends as well as a particular type of non-simple ends which we define first.

**Definition 3.10.** (Twisted polygons) Given a projective transformation \( a \in \text{SL}(3, \mathbb{R}) \), a \( n \)-gon twisted by \( a \) is by definition a continuous curve \( C \subset \mathbb{RP}^2 \) formed by a sequence of segments \( (Y_i)_{i \in \mathbb{Z}} \) where \( Y_i \) and \( Y_{i+1} \) share an endpoint and are non-collinear, such that \( Y_{i+n} = a(Y_i) \). Points in \( \text{Accum}_{i \rightarrow +\infty}(Y_i) \) and \( \text{Accum}_{i \rightarrow -\infty}(Y_i) \) (see §3.1.2 for the notation) are called accumulation points of the twisted polygon.
(2) **(Twisted polygonal ends)** A convex projective structure on a punctured surface $\Sigma$ is said to have **twisted $n$-gon end** (or **twisted polygonal end**, if $n$ is not specified) at a puncture $p$ if $\partial_{\tilde{p}}\Omega$ (where $\tilde{p} \in \text{Farey}(\Sigma, p)$) consists of an $n$-gon twisted by $\text{hol}_{\tilde{p}}$ together with the accumulation points of the twisted polygon.

(3) **(The space $\mathcal{P}_0(\Sigma)$)** We let $\mathcal{P}_0(\Sigma)$ denote the subset of $\mathcal{P}(\Sigma)$ consisting of convex projective structures on $\Sigma$ such that each puncture is either a simple end or a twisted polygonal end.

The most common instances of twisted polygonal ends occur when $\text{hol}_{\tilde{p}}$ is hyperbolic, so that we are in case (a) of Lemma 3.9 and the accumulation points of the twisted polygon are $x_+$ and $x_-$. Twisted polygonal ends with non-hyperbolic holonomy only appears in the following particular situations:

- $\Sigma \cong \mathbb{C}$. In this case, a convex projective structure is given by a single diffeomorphism $\text{dev} : \Sigma \to \Omega \subset \mathbb{RP}^2$. The puncture $\infty$ being a twisted polygonal end just means that $\Omega$ is a convex polygon in the usual sense.
- $\Sigma$ is an annulus. In this case, a convex projective structure can have a twisted polygonal end with quasi-hyperbolic or parabolic holonomy, so that we are in case (b) or (c) of Lemma 3.9.

4. From $\mathcal{P}_0(\Sigma)$ to $\mathcal{W}_0(\Sigma)$

The goal of this section is to prove the following theorem, which contains the statement (I) in the sketch of proof of Theorem 1.1 from the introduction.

**Theorem 4.1 (g and $b$ around simple or twisted polygonal ends).** Let $g$ and $b$ be the Blaschke metric and Pick differential of a convex projective structure, respectively, and let $J$ be the conformal structure underlying $g$.

1. If $p$ is either a geodesic end, a triangular end or a twisted polygonal end of the projective structure, then $J$ is cuspidal at $p$, and $p$ is a pole of order $\geq 3$. The ratio between $g$ and $\frac{2}{3}|b|_f$ at $x \in \Sigma$ tends to 1 as $x \to p$.

2. If $p$ is a cusp of the projective structure, then $J$ is cuspidal at $p$, and $p$ is a removable singularity or a pole of order $\leq 2$. The curvature $\kappa_g(x)$ tends to $-1$ as $x \to p$. Moreover, $g$ is bilipschitz to a cuspidal hyperbolic metric around $p$.

Here, a conformal structure $J$ on $\Sigma$ is said to be **cuspidal** at a puncture $p$ if a punctured neighborhood of $p$ is equivalent to the punctured disk $\{0 < |z| < 1\}$ in such a way that points near $p$ correspond to points in the disk near 0. Two conformally equivalent metrics are said to be **bilipschitz** if their ratio has positive upper and lower bounds. We discuss the notion of “cuspidal hyperbolic metrics” in detail in §4.2. Part (1) and (2) of the theorem are proved in §4.1 and §4.3 below, respectively.

4.1. **Geodesic, triangular and twisted polygonal ends.** Recall from §2.1.2 that, on any properly convex open set $\Omega \subset \mathbb{RP}^2$, we defined a function $f_{\Omega} : \Omega \to \mathbb{R}_{\geq 0}$ by

$$f_{\Omega} := \kappa_{\Omega} + 1 = 2\|b_{\Omega}\|_{g_{\Omega}}^2 = \left(\frac{2^{\frac{1}{2}}|b_{\Omega}|_{\frac{1}{2}}}{g_{\Omega}}\right)^3.$$ 

It captures the ratio between $2^{\frac{1}{2}}|b_{\Omega}|_{\frac{1}{2}}$ and $g_{\Omega}$. The following lemma is close in spirit to Proposition 3.1 of [BH13] and Lemma 7.2 of [DW14].
Lemma 4.2 (Controlling $f_\Omega$). Let $\Omega \subset \mathbb{RP}^2$ be a properly convex open set such that $\partial \Omega$ contains a line segment. Assume that $l$ is a maximal segment (i.e. $l = L \cap \partial \Omega$, where $L \subset \mathbb{R}^2$ is the line containing $l$).

(1) For any $K > 1$ and any segment $l'$ contained in the interior of $l$, there exists an open quadrilateral $V \subset \Omega$ with edge $l'$, as in the following picture, such that the inequality

\[ \frac{1}{K} \leq f_\Omega \leq K \]

holds on $V$.

(2) Further assume that $\partial \Omega$ is not $C^1$ at an endpoint $y_0$ of $l$, i.e. the two tangent directions to $\partial \Omega$ at $y_0$ form an angle $\theta < \pi$. Then for any $K > 1$, any $\theta' < \theta$ and any sub-interval $l' \subseteq l$ containing $y_0$, there is an open quadrilateral $V$ with edge $l'$ and with angle $\theta'$ at $y_0$, as in the following picture, such that (4.1) holds on $V$.

Proof. (1) Fix a distance $d$ on $\mathbb{RP}^2$ compatible with the topology. We define the Hausdorff distance $d_{\text{Hausdorff}}$ between two properly convex open sets $\Omega_1$ and $\Omega_2$ as the usual Hausdorff distance between the closures $\overline{\Omega}_1$ and $\overline{\Omega}_2$ with respect to $d$.

Let $\Delta \subset \Omega$ be the open triangle spanned by $l$ and a point $x_0$ in $\Omega$ (see the first picture in Figure 4.1). Fix $x_1 \in \Delta$. Fix $K$ and $l'$ as in the statement. By Corollary 2.2 (2) and the last property in §2.1.2, there exists a constant $\varepsilon > 0$ such that for any properly convex open set $\Omega'$ satisfying

\[ d_{\text{Hausdorff}}(\Delta, \Omega') < \varepsilon, \]

we have $x_1 \in \Omega'$ and

\[ \frac{1}{K} \leq f_{\Omega'}(x_1) \leq K. \]
We claim that there is a quadrilateral $V$ with edge $l'$ such that, for any $x \in V$, if we let $\Phi_{x_1,x}$ denote the unique projective transformation stabilizing $\triangle$ and mapping $x$ to $x_1$, then
\begin{equation}
\label{eq:4.2}
d_{\text{Hausdorff}}(\triangle, \Phi_{x_1,x}(\Omega)) < \varepsilon.
\end{equation}
Such a $V$ proves the required statement because $f_{\Omega}(x) = f_{\Phi_{x_1,x}(\Omega)}(x_1)$.

To prove the claim, we work with the affine chart $\mathbb{R}^2 \subset \mathbb{R}P^2$ in which the points $x_0$, $x_1$ and the two endpoints of $l$ correspond to $(0,0)$, $(1,1)$ and $(0,\infty)$, $(\infty,0)$, respectively. So $\triangle$ is the first quadrant of $\mathbb{R}^2$, while the segments $l_1$ and $l_2$ joining $x_0$ and the two endpoints of $l'$ are rays in $\mathbb{R}^2$ with slopes $k_1$ and $k_2$, respectively. We can suppose $0 < k_1 < k_2$. See the second picture in Figure 4.1. Put
\[
\triangle' = \{(x^1, x^2) \in \mathbb{R}^2 \mid k_1 \leq x^2/x^1 \leq k_2\}.
\]

Figure 4.1. Proof of Lemma 4.2 (I)

Given $a = (a^1, a^2) \in \mathbb{R}^2$ with $a^1, a^2 < 0$ and $0 \leq \lambda, \mu < 1$, we let $D_{a,\lambda,\mu}$ denote the convex region in $\mathbb{R}^2$ delimited by the two rays issuing from $a$ which slope $-\lambda$ and $-\mu^{-1}$, respectively. For any $x = (x^1, x^2) \in \triangle$, the previously defined projective transformation $\Phi_{x_1,x}$ is the linear map
\[
\Phi_{x_1,x}(y^1, y^2) = \left(\frac{y^1}{x^1}, \frac{y^2}{x^2}\right).
\]
One checks that
\begin{equation}
\label{eq:4.3}
\Phi_{x_1,x}(D_{a,\lambda,\mu}) = D\left(\frac{a^1}{\lambda}, \frac{a^2}{\mu}\right).
\end{equation}

A crucial implication of this expression is that, when $x$ tends to infinity within $\triangle'$, the origin of the image sector $\Phi_{x_1,x}(D_{a,\lambda,\mu})$ tends to $0$, while the slopes of its boundary rays remain bounded.

Now the claim is a consequence of the following assertions.

(a) If $a'$ is close enough to $0$ and $\lambda', \mu'$ are small enough, then
\[
d_{\text{Hausdorff}}(\triangle, D_{a',\lambda',\mu'}) < \varepsilon.
\]
(b) For any $\lambda, \mu > 0$, we can take $a$ far enough from $0$, such that $\Omega \subset D_{a,\lambda,\mu}$.

These assertions can be easily verified by transferring back to the affine chart in the first picture of Figure 4.1. We omit detailed proofs.

To prove the claim, we first fix $\lambda'$ and $\mu'$ as small as assertion (a) requires. Set $\lambda = k_1^{-1} \lambda'$ and $\mu = k_2 \mu'$. Using assertion (b), we find a point $a$ such that $\Omega \subset D_{a,\lambda,\mu}$.

Now the expression (4.3) implies that if we take $V \subset \triangle'$ far away enough from
0, as in Figure 4.1, then for any \( x \in V \), the origin \( a'' \) of the sector \( D_{a''} = \Phi_{x_1, x}(D_{a, \lambda, \mu}) \) is as close to 0 as assertion (a) requires, while the slopes satisfy

\[
\lambda'' = \frac{x_1}{x_2} \lambda \leq k_1^{-1} \lambda', \quad \mu'' = \frac{x_2}{x_1} \mu \leq k_2 \mu = \mu'.
\]

Thus assertion (a) gives

\[
d_{\text{Hausdorff}}(\Delta, \Phi_{x_1, x}(\Omega)) \leq d_{\text{Hausdorff}}(\Delta, \Phi_{x_1, x}(D_{a, \lambda, \mu})) < \varepsilon
\]

as we wished.

(2) In view of part (1), it is sufficient to prove that the inequality (4.1) holds on an open triangle \( V_0 \) such that \( y_0 \) is a vertex of \( V_0 \), some segment on \( l \) is an edge of \( V_0 \), and the angle of \( V_0 \) at \( y_0 \) is \( \theta' \). See the first picture of Figure 4.2.

![Figure 4.2. Proof of Lemma 4.2 (2)](image)

We take an open triangle \( \Delta \) containing \( \Omega \) and tangent to \( \Omega \) at \( y_0 \) as shown in the picture. Fix \( x_1 \in \Delta \). Let \( \Phi_{x_1, x} \) and \( \varepsilon \) be defined as in the proof of part (1). As in the proof of part (1), it is sufficient to show that (4.2) holds for any \( x \in V_0 \).

Let \( \Delta_1 \) and \( \Delta_2 \) be small enough sub-triangles of \( \Delta \) as in the picture, such that for any convex subset \( \Omega' \subset \Delta \), we have \( d_{\text{Hausdorff}}(\Delta, \Omega') < \varepsilon \) whenever \( \Omega' \) meets both \( \Delta_1 \) and \( \Delta_2 \). We only need to find a small enough \( V_0 \) such that for any \( x \in V_0 \), the convex set \( \Phi_{x_1, x}(\Omega) \) meets \( \Delta_1 \) and \( \Delta_2 \), or equivalently, \( \Omega \) meets \( \Phi_{x_1, x}(\Delta_1) \) and \( \Phi_{x_1, x}(\Delta_2) \). But this is easily done by working on an affine chart where \( \Delta \) corresponds to the first quadrant (see the second picture of Figure 4.2), taking account of the fact that the slope of the ray delimiting \( \Phi_{x_1, x}(\Delta_1) \) is bounded because the slope of any \( x \in V_0 \) is bounded. \( \square \)

The above lemma readily implies the last statement in part (1) of Theorem 4.1, as we will see below. We need the following notion in proving the other statements.

**Definition 4.3.** A metric \( g \) on a punctured surface \( \Sigma \) is said to be complete at a puncture \( p \) if for any sequence of points \( (x_n) \) tending to \( p \), the distance \( d(x_n, y) \) tends to \( +\infty \) as \( n \to +\infty \). Here \( y \in \Sigma \) is a point whose choice is not essential.

It is easy to see that \( g \) is complete “globally” (i.e. every bounded closed subset is compact) if and only if \( g \) is complete at every puncture.

**Proof of Theorem 4.1 (1).** Let \( \text{dev}, \text{hol} \) be a developing pair of the convex projective structure and put \( \Omega = \text{dev}(\Sigma) \). Fix \( \bar{p} \in \text{Farey}(\Sigma, p) \).
We first prove the last statement. The developing map \( \text{dev} \) induces a diffeomorphism \( \Sigma \cong \Omega/\text{hol}(\pi_1(\Sigma)) \), whereas the Blaschke metric \( g \) and Pick differential \( b \) are pullbacks of \( g_0 \) and \( b_0 \) by this diffeomorphism. Therefore, fixing \( K > 1 \), we only need to find an open set \( V \subset \Omega \) where the inequality (4.1) holds such that the quotient map \( \Omega \to \Omega/\text{hol}(\pi_1(\Sigma)) \cong \Sigma \) projects \( V \) to punctured neighborhood of \( p \).

Such \( V \) can be constructed for geodesic ends, triangular ends and twisted polygonal ends respectively as follows.

- Assume \( \partial_p \Omega = \{ \} \) is a segment. Let \( l' \subset \text{int}(l) \) be a sub-interval such that
  \[
  \bigcup_{n \in \mathbb{Z}} \text{hol}_{\delta_p}^{n}(\text{int}(l')) = \text{int}(l).
  \]

We can take a quadrilateral given by Lemma 4.2 (1) to be the required \( V \).

- Assume \( \partial_p \Omega = l_1 \cup l_2 \), where \( l_1 \) and \( l_2 \) are non-colicar segments. We can take a quadrilateral given by Lemma 4.2 (2) with big enough \( \theta' \) to be \( V \).

- Assume \( \partial_p \Omega = \bigcup_{b \in \mathbb{Z}} Y_b \) is a twisted \( n \)-gon. \( V \) is found by applying Lemma 4.2 (2) to the \( n + 2 \) edges \( Y_0, Y_1, \ldots, Y_{n+1} \).

This completes the proof of the last assertion in (1). Thus \( |b|^{\frac{2}{3}} \) is complete at \( p \) since \( g \) is. Let us show that the conformal structure \( J \) is cuspidal at \( p \) through a deep result of A. Huber in [Hub57]. If \( J \) is not cuspidal, a punctured neighborhood of \( p \) is conformally equivalent to a annulus \( \{ z \in \mathbb{C} \mid 1 < |z| < r \} \) in such a way that points near \( p \) correspond to points near the inner circle \( \Delta := \{ |z| = 1 \} \). The hypothesis “\( \Phi(\Delta) < +\infty \)” of Theorem 4 in [Hub57] is fulfilled for the metric \( |b|^{\frac{2}{3}} \) by virtue of the discussions at the same page of loc. cit.. That theorem then yields a rectifiable path tending to \( p \) with finite length with respect to \( |b|^{\frac{2}{3}} \), contradicting completeness. The conformal structure \( J \) is therefore cuspidal.

Let \( z \) be a conformal local coordinate such that \( \{ 0 < |z| < 1 \} \) corresponds to a punctured neighborhood of \( p \) and \( z = 0 \) corresponds to \( p \). Assume \( b = b(z)dz^3 \). It remains to be shown that \( z = 0 \) is a pole of \( b(z) \) of order at least 3. To this end, we use another theorem of A. Huber in [Hub67]. Let \( u : \{ 0 < |z| < 1 \} \to \mathbb{R} \) be the function defined by

\[
|b(z)|^{\frac{2}{3}} = e^{u(z)} |z|^{-\frac{1}{3}},
\]

i.e. \( u(z) = \frac{1}{3} \log |z^3 b(z)| \). Since \( |b|^{\frac{2}{3}} \) is complete at \( p \) and \( u \) is harmonic, conditions (a) and (b) in Satz 4 of [Hub67] are fulfilled, so we conclude that \( u \) can be extended to a super-harmonic function \( \{ |z| < 1 \} \to \mathbb{R} \cup \{ +\infty \} \). By lower semi-continuity, \( |z^3 b(z)| \) is bounded from below by a positive constant in vicinity of \( z = 0 \). As a result, \( b(z) \) has a pole of order at least 3 at \( z = 0 \).

4.2. Cuspidal hyperbolic metrics. Lemma 5.2 in [BH13] can be reformulated as the follows.

**Lemma 4.4.** Let \( g_1 \) and \( g_2 \) be \( C^\infty \) Riemannian metrics on \( D^* := \{ x \in \mathbb{R}^2 \mid 0 < |x| < 1 \} \) such that \( g_1 \) and \( g_2 \) are conformally equivalent and both satisfy the following conditions:

(i) complete at 0 (see Definition 4.3);
(ii) the curvature is negatively pinched (i.e. bounded from above and below by negative constants) around 0.

Then \( g_1 \) and \( g_2 \) are bilipschitz near 0.
Although the original statement in [BH13] is global in nature, we can clearly deduce the above local version from it by completing \( g_1 \) and \( g_2 \) to metrics on \( \mathbb{R}^2 \).

As a consequence of Lemma 4.4, any metric \( g \) on \( D^* \) satisfying conditions (i) and (ii) belongs to either of the following bilipschitz classes.

- If the conformal structure underlying \( g \) is cuspidal at 0, then \( g \) is bilipschitz to the hyperbolic metric of finite volume

\[
\frac{4|dz|^2}{|z|^2(\log |z|)^2}.
\]

Here we identify a punctured neighborhood of \( p \) with \( \{ z \in \mathbb{C} \mid 0 < |z| < \varepsilon \} \). The metric (4.4) is what we call a cuspidal hyperbolic metric. The expression is obtained from the Poincaré metric on the upper-half plane \( \mathbb{H} \) by identifying \( \{ \zeta \in \mathbb{H} \mid \text{Im}(\zeta) > -\log \varepsilon \} / \mathbb{Z} \) via \( z = \exp(i\zeta) \). Note that a different choice of the coordinate \( z \) gives rise to a different cuspidal hyperbolic metric, but Lemma 4.4 implies that these metrics are bilipschitz around 0.

- Otherwise, a punctured neighborhood of 0 is conformally equivalent to \( \{ z \in \mathbb{C} \mid 1 < |z| < 1 + \varepsilon \} \). In this case, \( g \) is bilipschitz to a hyperbolic metric of infinite volume.

### 4.3. Cusps

Part (2) of Theorem 4.1 is proved by Benoist and Hulin [BH13]. We recap their proof for convenience of the reader.

**Proof of Theorem 4.1 (2).** Put \( f = \kappa_g + 1 \) as before. We first prove the assertion about \( \kappa_g \) by showing that, for any \( \varepsilon > 0 \), there exists a punctured neighborhood \( U \) of \( p \) on which \( |f| < \varepsilon \).

The developing boundary \( \partial \tilde{p} \Omega \) consists of a single point \( x_0 \) by assumption. Take coordinates of \( \mathbb{RP}^2 \) such that

\[
x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{hol}_p = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

The holonomy \( \text{hol}_p \) preserves a family of conic curves \( (C_\lambda)_{\lambda \in \mathbb{R}} \) given by

\[
C_\lambda = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{RP}^2 \mid y^2 + \lambda z^2 = 2xz \right\}.
\]

Let \( E_\lambda \subset \mathbb{RP}^2 \) denote the properly convex open set bounded by \( C_\lambda \). Using convexity of \( \Omega \) and invariance of \( \Omega \) under \( \text{hol}_p \), one sees that there exists \( \lambda_0 < \lambda_1 \) such that \( E_{\lambda_0} \supset \Omega \supset E_{\lambda_1} \). Up to conjugating the developing pair by \( x_0 \)-fixing projective transformation, we can assume \( \lambda_0 = 0 \). See the following picture.

For any \( \lambda_2 \geq \lambda_1 \), under the covering map \( \Omega \to \Omega/\text{hol}(\pi_1(\Sigma)) \cong \Sigma \), the open set \( E_{\lambda_2} \) projects to a punctured neighborhood of \( p \). Hence, in order to prove the required statement, it is sufficient to show that if \( \lambda_2 \) is big enough then

\[
|f_{\Omega}(x)| < \varepsilon, \quad \forall x \in E_{\lambda_2}.
\]

To this end, let \( G \subset \text{Aut}(E_0) \subset \text{SL}(3, \mathbb{R}) \) denote the group of \( x_0 \)-fixing projective automorphisms of \( E_0 \), which acts freely and transitively on \( E_0 \). Given \( x_1, x \in E_0 \), let \( \Phi_{x_1, x} \) denote the unique element of \( G \) sending \( x \) to \( x_1 \). The required inequality
(4.5) is proved with the same method as part (1): given \( x_1 \in E_0 \), on one hand, there exists \( \delta > 0 \) such that for any properly convex open set \( \Omega' \subset \mathbb{RP}^2 \),
\[
d_{\text{Hausdorff}}(E_0, \Omega') < \delta \implies x_1 \in \Omega' \text{ and } |f_{\Omega'}(x_1)| < \varepsilon;
\]
on the other hand, we can choose \( \lambda_2 > \lambda_1 \) big enough such that
\[
d_{\text{Hausdorff}}(E_0, \Phi_{x_1, x}(\Omega)) < \delta, \quad \forall x \in E_{\lambda_2}.
\]
This implies (4.5) because \( f_{\Omega}(x) = f_{\Phi_{x_1, x}(\Omega)}(x_1) \). Thus we have shown that \( \kappa_g(x) \) tends to \(-1\) as \( x \to p \).

It is proved in [Mar12] that a punctured neighborhood \( U \) of \( p \) has finite volume with respect to the Hilbert metric if and only if it is a cusp. By Corollary 2.2 (1), \( U \) has finite volume with respect to the Blaschke metric \( g \) as well. Since \( g \) has negatively pinched curvature on \( U \), the discussions following Lemma 4.4 imply that the conformal structure of \( g \) is cuspidal at \( p \) and \( g \) is bilipschitz to a cuspidal hyperbolic metric.

Finally, let \( z \) be a conformal local coordinate centered at \( p \) and assume \( b = b(z)dz^3 \) near \( p \). Since \( g \) is bilipschitz to the cuspidal hyperbolic metric (4.4) while the ratio between \( |b|^{\frac{2}{3}} \) and \( g \) is bounded from above by Corollary 2.2 (2), we have
\[
|b(z)|^{\frac{2}{3}} < \frac{C}{|z|^2(\log |z|)^2}
\]
whenever \( |z| \) is small. It follows that \( b \) has a pole of order \( \leq 2 \) at \( p \) and the proof is complete. \( \square \)

5. Local model for \((\Sigma, b)\) around poles of order \( \geq 4 \)

Let \( \Sigma \) be a punctured Riemann surface of finite type, i.e. \( \Sigma \) is obtained from a closed Riemann surface \( \overline{\Sigma} \) by removing finitely many punctures. Let \( b \) be a holomorphic cubic differential not vanishing constantly on \( \Sigma \), such that each puncture is a removable singularity or a pole.

It is well known that for each puncture \( p \), there exists a conformal local coordinate \( z \) of \( \overline{\Sigma} \) centered at \( p \) such that \( b \) has the following normal form.

\[
b(z) = \begin{cases} 
  z^m dz^3 & (m \in \mathbb{Z} \setminus 3\mathbb{Z}_-) \quad \text{if } p \text{ is a removable singularity or a pole of order not divisible by } 3, \\
  \frac{R}{z^2} dz^3 & (R \in \mathbb{C}^*) \quad \text{if } p \text{ is a third order pole}, \\
  \left( \frac{1}{z^l} + A \right)^3 dz^3 & (A \in \mathbb{C}) \quad \text{if } p \text{ is a pole of order } 3l, \text{ where } l \geq 2.
\end{cases}
\]
See [Str84] §6 for a proof for quadratic differentials, which readily adapts to cubic differentials. We always let \( z \) denote such a coordinate and let \( U = \{ 0 < |z| < a \} \) denote a punctured neighborhood of \( p \) where \( z \) is defined.

In this section, we only consider a pole \( p \) of order \( n + 3 \geq 4 \). The goal is to elaborate a construction of a local model of \((\Sigma, b) \) around \( p \), used in Section 7 below.

5.1. Special directions and sectors in \( T_p \Sigma \). Identifying \( T_p \Sigma \) with \( \mathbb{C} \) via the coordinate \( z \) introduced above, we set, for each \( k \in \mathbb{Z}/n\mathbb{Z} \),

\[
C_k := \left\{ v \in T_p \Sigma \cong \mathbb{C} \mid v \neq 0, \quad \frac{2\pi(k-1)}{n} < \arg(v) < \frac{2\pi k}{n} \right\}
\]

and let \( C_{k,k+1} \) denote the ray along which \( C_k \) and \( C_{k,k+1} \) are adjacent, \( i.e. \)

\[
C_{k,k+1} := e^{2\pi ik/n} \mathbb{R}_+.
\]

For each \( k \), consider the three rays which divide \( C_k \) into four equally angled sectors. The two aside rays among the three are called unstable directions and are denoted by \( U_k^- \) and \( U_k^+ \) in the counter-clockwise order. In other words,

\[
U_k^- := e^{(4k-3)\pi i/2n} \mathbb{R}_+, \quad U_k^+ := e^{(4k-1)\pi i/2n} \mathbb{R}_+.
\]

There are \( 2n \) unstable directions in total, dividing \( T_p \Sigma \setminus \{0\} \) into \( 2n \) open sectors, called stable sectors. We let \( S_k \) denote the stable sector contained in \( C_k \) and let \( S_{k,k+1} \) denote the stable sector containing \( C_{k,k+1} \). In other words,

\[
S_k = \left\{ v \in T_p \Sigma \mid \frac{(4k-3)\pi}{2n} < \arg(v) < \frac{(4k-1)\pi}{2n} \right\},
\]

\[
S_{k,k+1} = \left\{ v \in T_p \Sigma \mid \frac{(4k-1)\pi}{2n} < \arg(v) < \frac{(4k+1)\pi}{2n} \right\}.
\]

The following figure illustrate the case \( n = 5 \).

Significance of these special directions and sectors will be clear in Section 7. Here we only mention that the \( C_k \)'s correspond to the vertices of the twisted polygon asserted by Theorem 1.1, while the \( C_{k,k+1} \)'s correspond to the edges.

\[1\] We always use the boldfaced letter \( k \) to denote an element in \( \mathbb{Z}/n\mathbb{Z} \) and let \( k \) denote an integer in the congruence class \( k \). This will be convenient in Section 7 below.
5.2. Natural half-planes. Let $\mathcal{H} := \{ \zeta \mid \Re(\zeta) > 0 \}$ be the right half-plane. Following [DW14], we introduce certain conformal maps $\mathcal{H} \to U$, referred to as “natural half-planes”, which pull $b$ back to the constant cubic differential $2d\zeta^3$.

We first describe how these maps looks like before going into the construction. Actually we will build $n$ maps

$$ \Phi_k : \mathcal{H} \to U = \{ 0 < |z| < a \} \subset \Sigma, \quad k \in \mathbb{Z}/n\mathbb{Z}. $$

Each $\Phi_k$ sends $\mathcal{H}$ to a teardrop-like region whose tangent cone at $p$ is the sector $S_{k-1, k} \cup C_k \cup S_{k, k+1}$, while mapping the imaginary axis $\partial \mathcal{H}$ to a curve asymptotic to the unstable directions $U_{k-1}^{+}$ and $U_{k+1}^{-}$. See the picture below, where the image of $\Phi_1$ is shown in the cases $n = 1, 2$ and $5$ respectively.

![Diagram](image)

The $n = 1$ case is special in that there is a single map $\Phi = \Phi_1$, which is not injective and the image covers a punctured neighborhood of $p$. When $n \geq 2$, each $\Phi_k$ is injective and the union of images covers a punctured neighborhood.

To construct $\Phi_k$, we first consider the case where $n$ is not divisible by 3. By definition of the coordinate $z$ at the beginning of this section, in this case we have $b = \frac{dz^3}{z^5}$. We define

$$ \Phi_k(\zeta) := \left( \frac{2 + n}{3} \right)^{-\frac{3}{n}} \exp \left( -\frac{3}{n} \log(\zeta + B) + \frac{(2k + 1)\pi i}{n} \right), $$

where $B > 0$ is big enough such that $\Phi_k(\mathcal{H}) \subset U$.

A straightforward computation shows that $\Phi_k$ pulls back $b$ to

$$ \Phi_k^*(b) = 2d\zeta^3. $$

This can be done, for example, through the following equalities.

$$ \Phi_k^* \left( \frac{dz}{z} \right) = -\frac{3}{n} \cdot \frac{d\zeta}{(\zeta + B)}, $$

Here and below, we take the branch of $\log(\zeta)$ such that $-\pi < \Im(\log(\zeta)) < \pi$ for any $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.\footnote{Here and below, we take the branch of $\log(\zeta)$ such that $-\pi < \Im(\log(\zeta)) < \pi$ for any $\zeta \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.}
\[ \Phi_k \left( \frac{1}{z^2} \right) = \Phi_k(\zeta)^{-\frac{3}{2}} = \frac{2^\frac{3}{2} n}{3} \exp \left( \log(\zeta) + \frac{2k+1}{3} \pi i \right) \]
\[ = \frac{2^\frac{3}{2} n}{3} e^{-\frac{2k+1}{3} \pi i} (\zeta + B). \]

When \( n \) is a multiple of 3, we can still construct \( \Phi_k \) with similar properties, as statement in the next proposition. We construct the maps on the closed half-plane \( \overline{\mathbb{H}} = \{ \text{Re}(\zeta) \geq 0 \} \) instead of the open one so as to make sense of \( \Phi_k(0) \).

**Proposition 5.1 (Natural half-planes).** There exists \( n \) conformal maps
\[ \Phi_k : \overline{\mathbb{H}} \to U \subset \Sigma, \quad k \in \mathbb{Z}/n\mathbb{Z} \]
such that

(i) \( \Phi_k^*(b) = 2d\zeta^3 \).

(ii) Given a parametrized path
\[ \beta : [0, +\infty) \to \Sigma, \quad \lim_{t \to +\infty} \beta(t) = p, \]
if \( \beta \) is asymptotic to a ray \( \mathbb{R}_{\geq 0} \cdot v \subset T_p\Sigma \) for some \( v \in S_{k-1,k} \cup C_k \cup S_{k,k+1} \), then
\[ \beta([t_0, +\infty)) \text{ is contained in } \Phi_k(\overline{\mathbb{H}}) \text{ for sufficiently large } t_0 \text{ and } \beta|_{[t_0, +\infty)} \text{ pulls back through } \Phi_k \text{ to a path } \tilde{\beta} : [t_0, +\infty) \to \overline{\mathbb{H}} \text{ tending to } \infty \text{ with asymptotic argument} \]
\[ \theta(\tilde{\beta}) := \lim_{t \to +\infty} \arg(\tilde{\beta}(t)) = -\frac{n}{3} \arg(v) + \frac{(2k+1)\pi}{3}. \]

(iii) If \( n \geq 2 \), each \( \Phi_k \) is injective. The union of images \( \bigcup_{k \in \mathbb{Z}/n\mathbb{Z}} \Phi_k(\overline{\mathbb{H}}) \) is a punctured neighborhood of \( p \). Moreover, the pre-image of \( \Phi_k(\overline{\mathbb{H}}) \cap \Phi_{k+1}(\overline{\mathbb{H}}) \) by \( \Phi_k \) and \( \Phi_{k+1} \) are sectors of the form
\[ \left\{ \zeta \in \mathbb{H} \mid -\frac{\pi}{2} \leq \arg(\zeta - a) \leq -\frac{\pi}{2}, \quad \zeta \in \mathbb{H} \mid \frac{\pi}{6} \leq \arg(\zeta - b) \leq \frac{\pi}{3} \right\}, \]
respectively, for some \( a, b \in \partial \mathbb{H} \), whereas the transition map \( \Phi_{k+1} \circ \Phi_k^{-1} \) from the former to the latter is the map
\[ \zeta \mapsto \omega \zeta - \omega a + b, \quad \omega = e^{2\pi i/3}. \]

(iv) If \( n = 1 \), the image of \( \Phi = \Phi_1 \) is a punctured neighborhood of \( p \). Moreover, there are two disjoint sectors of the same form as in (iii), such that \( \Phi(\zeta_1) = \Phi(\zeta_2) \) if and only if \( \zeta_1 \) and \( \zeta_2 \) lie in the two sectors respectively and \( \zeta_1 \) is sent to \( \zeta_2 \) by the map (5.5).

In the last condition, the notion of “asymptotic ray” is similar to the one used in §2.3.1 in a different context and we omit the precise definition. The main consequence that we draw from the expression of \( \theta(\tilde{\beta}) \) is the following table, which gives the range of \( \theta(\tilde{\beta}) \) when \( v \) belongs to each of the special directions/sectors.

| \( v \) | \( C_{k,k+1} \) | \( C_k \) | \( C_{k-1,k} \) | \( S_{k,k+1} \) | \( U_k \) | \( S_k \) | \( U_k \) | \( S_{k-1,k} \) |
|-------|-------------|----------|-------------|-------------|---------|--------|---------|-------------|
| \( \theta(\beta) \) | \( -\frac{\pi}{3} \) | \( \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \) | \( \frac{\pi}{3} \) | \( \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \) | \( -\frac{\pi}{3} \) | \( \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \) | \( \frac{\pi}{3} \) | \( \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \) |

**Proof.** When \( n \) is not divisible by 3, we have already seen that the \( \Phi_k \) defined by (5.2) satisfies properties (i). The other properties are also straightforward.
When $n$ is divisible by 3, we put $n' = n/3$. The expression of $b$ at the beginning of this section can be written as

$$b = \left(\frac{1}{z^{n'+1}} + \frac{A}{z}\right)^3 \, dz^3.$$  

We use a construction from [DW14] Appendix A. For $\epsilon > 0$ small, put

$$H_\epsilon = \{ \xi \in \mathbb{C} \mid \arg(\xi) \in (-\pi/2 - \epsilon, \pi/2 + \epsilon) \}.$$  

The key property of $H_\epsilon$ is that any $\xi_1, \xi_2 \in H_\epsilon$ are joint by a path in $H_\epsilon$ with length inferior to $\lambda|\xi_1 - \xi_2|$, where $\lambda > 1$ is a constant determined by $\epsilon$.

The map $\Phi_k : H \to U$ will be constructed as a composition

$$\Phi_k = \Psi_k \circ \phi : H \xrightarrow{\phi} H_\epsilon \xrightarrow{\Psi_k} U,$$

where $\phi$ and $\Psi_k$ are defined as follows

- $\Psi_k$ is just the previously defined map (5.2) extended to $H_\epsilon$ by the same expression, where we take $B$ big enough to ensure $\Psi_k(H_\epsilon) \subset U$. Using (5.3) and (5.4), one checks that $\Psi_k$ pulls $b$ back to

$$\Psi_k(b) = 2 \left(1 + \frac{C}{\xi + B}\right)^3 \, d\xi^3$$

for a constant $C \in \mathbb{C}$ depending on $A$ but not on $B$. Put $f(\xi) := 1 + \frac{C}{\xi + B}$. We take $B$ further bigger such that $|f(\xi) - 1| = \left|\frac{C}{\xi + B}\right| < \lambda^{-1}$ for any $\xi \in H_\epsilon$.

- Put $F(\xi) := \xi + C \log(\xi + B)$. This is a primitive of $f$. We define $\phi$ as

$$\phi(\xi) = F^{-1}(\xi + D)$$

for a big enough constant $D > 0$ which we determine later.

In order for $\phi$ to make sense, we need to show that $F$ maps $H_\epsilon$ injectively to a domain in $\mathbb{C}$ containing $H + D$.

To check injectivity, assume by contradiction that $F(\xi_1) = F(\xi_2)$ for distinct $\xi_1, \xi_2 \in H_\epsilon$. Let $\gamma$ be a path going from $\xi_1$ to $\xi_2$ with length less than $\lambda|\xi_1 - \xi_2|$ as mentioned above. Then

$$0 = F(\xi_2) - F(\xi_1) = \int_\gamma f(\xi) d\xi = \xi_2 - \xi_1 + \int_\gamma (f(\xi) - 1) d\xi.$$  

But

$$\left|\int_\gamma (f(\xi) - 1) d\xi\right| \leq \sup_{\xi \in H_\epsilon} |f(\xi) - 1| \cdot \text{Length}(\gamma) < \lambda^{-1} \cdot \lambda|\xi_1 - \xi_2| = |\xi_1 - \xi_2|,$$

a contradiction. $F$ is therefore injective on $H_\epsilon$.

Furthermore, $F(H_\epsilon)$ contains $H + D$ for some $D > 0$ because $\lim_{\xi \to \infty} F(\xi) = \infty$ and $\sup_{\xi \in \partial H_\epsilon} \text{Re}(F(\xi)) < +\infty$. This finishes the construction of $\Phi_k$.

We briefly outline the proofs of the properties (i) to (iv) for $\Phi_k$. Property (i) is an immediate consequence of the above expressions of $\Psi_k(b)$ and $\phi$. Property (ii) follows from the explicit expression of $\Psi_k$ and the fact that the map $\phi$ preserves asymptotic arguments. As for (iii), firstly, the injectivity follows from injectivity of both $\phi$ and $\Psi_k$. Secondly, $\phi(H)$ contains a half-plane of the form $H + D'$ for some $D' > 0$, but the union $\bigcup_k \Psi_k(H + D')$ is a punctured neighborhood of $p$. Lastly, to prove the assertion about the pre-image of $\Phi_k(H) \cap \Phi_{k+1}(H)$, it is sufficient to show that the curve $\Phi_k^{-1}(\Phi_k(\partial H))$ (resp. $\Phi_k^{-1}(\Phi_{k+1}(\partial H))$) in $\mathbb{H}$ is a ray with asymptotic
argument $\frac{\pi}{3}$ (resp. $-\frac{\pi}{3}$). This curve is a ray because each $\Phi_k(\partial H)$ is a geodesic with respect to the flat metric $|b|^2$, whereas the asymptotic argument $\pm\frac{\pi}{3}$ can be obtained using property (ii). Property (iv) is proved similarly as (iii).

Figure 5.1. A local model for $(\Sigma, b)$ by gluing $n$ right half-planes: $H_k$ is glued to $H_{k+1}$ through a map of the form $\zeta \mapsto e^{2\pi i/3} \zeta + \zeta_0$ which sends $\xi_k$ to $\eta_{k+1}$. The cubic differential $2d\zeta^3$ is invariant under this map, hence defines a cubic differential on the resulting surface.

5.3. **A local model for $(\Sigma, b)$**. Proposition 5.1 implies that we can build a local model of $(\Sigma, b)$ by patching together $n$ half-planes, each equipped with the cubic differential $2d\zeta^3$. The following corollary gives a formal statement.

**Corollary 5.2 (Local model).** Let $H_k$ ($k \in \mathbb{Z}/n\mathbb{Z}$) be $n$ copies of the right half-plane. Then there exists $\xi_k, \eta_k \in \partial H_k$ with $\text{Im}(\xi_k) < 0 < \text{Im}(\eta_k)$, such that a punctured neighborhood $U'$ of $p$ endowed with the cubic differential $b$ is isomorphic to the Riemann surface $S$ obtained by gluing the half-planes in the way described by Figure 5.1, endowed with the cubic differential $2d\zeta^3$.

Clearly, we can further identify the closure $\overline{U'}$ with the surface with boundary $\mathcal{S}$ obtained by gluing the closed half-planes $H_k$ ($k \in \mathbb{Z}/n\mathbb{Z}$) in the same way. In what follows, we view $\mathcal{S}$ and the $H_k$’s as subsets of $\Sigma$. This point of view is not honest in the $n = 1$ case, where a quotient of $\overline{H}$ rather than $\overline{H}$ itself is subset of $\Sigma$. In fact, some arguments in §7.1 below should indeed be adjusted when applied to this case. However, the adjustment is just a matter of notations and thus we will not single out the $n = 1$ case anymore.

6. **Wang’s equation**

Let the Riemann surface $\Sigma$ and the cubic differential $b$ be as at the beginning of Section 5. In this section, we first prove statement (II) in the sketch of proof of Theorem 1.1 from the introduction. This is given by Theorem 6.1 below. We then establish some asymptotic estimates for $g$ around poles of order $\geq 4$ for later use.
6.1. Existence and uniqueness.

**Theorem 6.1 (Solving Wang's equation).** $\Sigma$ admits a unique complete conformal metric $g$ which satisfies Wang's equation

$$\kappa_g = -1 + 2\|b\|_g^2$$

and the following conditions around each puncture $p$.

(i) If $p$ is a pole of $b$ of order $\geq 3$, then $g$ is bilipschitz to the flat metric $2^{\frac{1}{2}}|b|^{\frac{3}{2}}$ around $p$.

(ii) If $p$ is a removable singularity or a pole of order $\leq 2$, then $\kappa_g$ is negatively pinched around $p$.

Using the notations from the introduction, we can state the theorem as saying that there exists a unique $g$ such that $(g, b) \in \mathcal{W}_0(\Sigma)$. Note that condition (ii) implies $g$ is bilipschitz to a cuspidal hyperbolic metric as shown in §4.2.

**Proof.** For any conformal metric $g$ on $\Sigma$, we put

$$L(g) = \kappa_g + 1 - 2\|b\|_g^2.$$

We need to find a $g$ satisfying $L(g) = 0$. By the standard method of super/sub solutions (see e.g. [Lof04, BH13, DW14]), it is sufficient to find continuous conformal metrics $g_+$ and $g_-$ satisfying $g_- \leq g_+$ and

$$L(g_+) \geq 0, \quad L(g_-) \leq 0$$

in the sense of distributions. The method then produces a smooth conformal metric $g$ satisfying $L(g) = 0$ and $g_- \leq g \leq g_+$.

We first construct $g_-$ and $g_+$ when $\Sigma$ has negative Euler characteristic. In this case $\Sigma$ carries a unique complete conformal hyperbolic metric of finite volume, denote by $g_{hyp}$. The required $g_-$ is defined as

$$g_- = \max(g_{hyp}, 2^{\frac{1}{2}}|b|^{\frac{3}{2}}).$$

Since $L(g_{hyp}) = -2\|b\|_{g_{hyp}}^2 \leq 0$ and $L(2^{\frac{1}{2}}|b|^{\frac{3}{2}}) = 0$ (away from zeros of $b$), the required inequality $L(g_-) \leq 0$ holds on the open set where $g_{hyp} \neq 2^{\frac{1}{2}}|b|^{\frac{3}{2}}$. It also holds in the distribution sense at the points where $g_{hyp} = 2^{\frac{1}{2}}|b|^{\frac{3}{2}}$, because $g_-$ has non-positive curvature in the distribution sense. Indeed, a conformal metric $g = e^u|dz|^2$ has non-positive curvature in the distribution sense if and only if $u$ is a continuous subharmonic function, while the maximum of two subharmonic functions is again subharmonic.

To construct $g_+$, we first let $g_0$ be a smooth conformal metric which equals $2^{\frac{1}{2}}|b|^{\frac{3}{2}}$ around poles of $b$ of order $\geq 3$, and equals $g_{hyp}$ around removable singularities or poles of order $\leq 2$. Using the local expression (4.4) of the metric $g_{hyp}$, we easily verify the following properties:

- $g_0$ coincides with $g_-$ around each puncture.
- The pointwise normal $\|b\|_{g_0}^2$ and the curvature $\kappa_{g_0}$ are bounded.

In order to check the second property, one only need to note that $\|b\|_{g_0}^2$ and $\kappa_{g_0}$ are bounded around each puncture.

By virtue of these properties, we can take

$$g_+ = \lambda g_0$$

with $\lambda > 1$ big enough so as to fulfill the requirements that $g_+ \geq g_-$ and

$$L(g_+) = 1 + \lambda^{-1}\kappa_{g_0} - 2\lambda^{-3}\|b\|_{g_0}^2 \geq 0.$$
This completes the construction of $g_-$ and $g_+$ when $\Sigma$ has negative Euler characteristic.

When $\Sigma$ has non-negative Euler characteristic, i.e. $\Sigma = \mathbb{C}^*$ or $\mathbb{C}$, it does not carry a complete conformal hyperbolic metric. By the Riemann-Roch theorem, the cubic canonical bundle $K^3$ of the Riemann sphere $\mathbb{CP}^1$ has degree $-6$, so we are in one of the following cases.

(a) $\Sigma = \mathbb{C}$ and the puncture $\infty$ is a pole of order $\geq 6$.
(b) $\Sigma = \mathbb{C}^*$ and exactly one of the two punctures $0$ and $\infty$ is a pole of order $\geq 3$.
(c) $\Sigma = \mathbb{C}^*$ and both punctures are poles of order $\geq 3$.

We assume without loss of generality that $\infty$ is such a pole.

Given $r > 1$, we define an open subset $\Sigma_r \subset \Sigma$ in three cases above respectively, by (a) $\Sigma_r = \{ |z| < r \}$, (b) $\Sigma_r = \{ 0 < |z| < r \}$ and (c) $\Sigma_r = \{ r^{-1} < |z| < r \}$. Then $\Sigma_r$ carries a unique complete conformal hyperbolic metric $g_r$ (of infinite volume) expressed by

\[ (a) \frac{4r^2|dz|^2}{(r^2 - |z|^2)^2}, \quad (i) \quad \frac{|dz|^2}{|z|^2 \left( \log \frac{r}{|z|} \right)^2}, \quad (c) \quad \frac{\left( \frac{\pi}{2} \right)^2 |dz|^2}{(\log r)^2 |z|^2 \cos^2 \left( \frac{\pi}{2} \log |z| \right)} \]

Here, the last two expressions are obtained from the hyperbolic metric $\frac{|dz|^2}{\cos^2(\text{Im} \, \zeta)}$ on the upper-half plane $\{ \text{Im} \, \zeta > 0 \}$ and the hyperbolic metric $\frac{|dz|^2}{\cos^2(\text{Im} \, \zeta)}$ on the band $\{ -\frac{\pi}{2} < \text{Im} \, \zeta < \frac{\pi}{2} \}$ by scaling and the exponential map.

Using these expressions, we check that $g_r$ is less than the flat metric $2^{\frac{1}{2}} |b|^{\frac{3}{2}}$ on the boundary of $\Sigma_{\sqrt{T}} \subset \Sigma_r$ when $r$ is big enough. Fixing such a $r$, we can define the required metric $g_-$ by

\[ g_- = \begin{cases} \max(g_r, 2^{\frac{3}{2}} |b|^{\frac{3}{2}}) & \text{ on } \Sigma_{\sqrt{T}}, \\ 2^{\frac{1}{2}} |b|^{\frac{3}{2}} & \text{ outside } \Sigma_{\sqrt{T}}. \end{cases} \]

It satisfies $L(g_-) \leq 0$ in the distribution sense for the same reason as in the negative Euler characteristic case. The metric $g_+$ is also defined similarly: let $g_0$ be a smooth conformal metric which coincides with $g_-$ around punctures and set $g_+ = \lambda g_0$ for $\lambda$ big enough.

This completes the constructions of $g_{\pm}$ and produces a metric $g$ satisfying Wang’s equation such that $g_- \leq g \leq g_+$. By construction, $g_{\pm}$ satisfy the asymptotic properties (i) and (ii), hence $g$ satisfies as well.

Uniqueness of $g$ is established in a similar way as in [Lof04, BH13, DW14]. Let $g' = e^w g$ be another metric satisfying Wang’s equation and the properties (i) and (ii). Wang’s equation for $g$ and $g'$ give rise to

\[ \Delta_g w = 2(e^w - 1) + 4\|b\|^2 g(1 - e^{-2w}) \]

where $\Delta_g$ is the Laplacian associated to $g$.

Note that $g$ and $g'$ are bilipschitz on the whole $\Sigma$ since they are both bilipschitz to $|b|^{\frac{3}{2}}$ around poles of order $\geq 3$ and bilipschitz to a cuspidal hyperbolic metric around other punctures. Hence $w$ is bounded. Therefore, there is a constant $k > 0$ only depending on $M := \sup_{\Sigma} w$ such that, at the points where $w \geq 0$, we have

\[ \Delta_g w \geq 2(e^w - 1) \geq k w. \]

Let us prove $M \leq 0$. Since $g$ is complete and has negatively pinched curvature, we can apply the Omori-Yau maximum principle [?] to $(\Sigma, g)$ and conclude that for
any $\epsilon > 0$ there exists $x_\epsilon \in \Sigma$ such that

$$w(x_\epsilon) \geq M - \epsilon, \quad \Delta_g w(x_\epsilon) \leq \epsilon.$$  

If $M > 0$, we take $\epsilon \in (0, M)$ and get

$$\epsilon \geq \Delta_g w(x_\epsilon) \geq k w(x_\epsilon) \geq k(M - \epsilon),$$

hence $M \leq \epsilon (1 + k^{-1})$ for any $\epsilon \in (0, M)$.

Thus we get $M \leq 0$, or equivalently, $g' \leq g$. Exchanging the roles of $g$ and $g'$, we get $g' \geq g$ as well. This establishes the uniqueness.

The above proof also yields bounds of $g$ in terms of $g_+$ and $g_-$. By construction, we have $g_- \geq 2^{\frac{1}{2}} |b|_2^2$ on the whole $\Sigma$ and $g_+ = \lambda \cdot 2^{\frac{1}{2}} |b|_2^2$ around poles of order $\geq 3$. Therefore we have

**Corollary 6.2 (Coarse estimate).** The metric $g$ from Theorem 6.1 satisfies $g \geq 2^{\frac{1}{2}} |b|_2^2$ on the whole $\Sigma$ and $g \leq \lambda \cdot 2^{\frac{1}{2}} |b|_2^2$ for some constant $\lambda > 1$ around poles of order $\geq 3$.

Since $g$ satisfies Wang’s equation, an equivalent statement of the above corollary is that $g$ has non-positive curvature on the whole $\Sigma$ while $\kappa_g \geq -1 + \epsilon$ for some $\epsilon > 0$ around poles of order $\geq 3$.

6.2. **Strong estimate for $g$ around poles of order $\geq 3$.** Let $p$ be a pole of order $n + 3$, where $n \geq 1$. Let $u$ be the function on the local model $S \subset \Sigma$ (see §5.3) such that $g$ is expression in each natural half-plane $H_k \subset S$ as

$$g = 2e^u |d\zeta|^2,$$

where $\zeta$ is the natural coordinate on $H_k$. Note that $u$ can be understood in a coordinate-free way as the logarithm of the ratio between $g$ and $2^{\frac{1}{2}} |b|_2^2$.

The next theorem, due to Dumas and Wolf, implies that $u$ and its gradient decays exponentially with respect to the $|b|_2^2$-distance from $\zeta$ to a fixed point. This is a far-reaching refinement of Corollary 6.2. Intriguingly, the factor $|\zeta|^{-\frac{1}{2}}$ in the bound plays an essential role in applications (Theorem 7.8 (3)).

**Theorem 6.3 (Strong estimate).** There is a constant $C$ such that for each $k \in \mathbb{Z}/n\mathbb{Z}$ and $\zeta \in H_k$, we have

$$0 \leq u(\zeta), |\partial_\zeta u(\zeta)| \leq C |\zeta|^{-\frac{1}{2}} e^{-2\sqrt{3}|\zeta|}.$$  

The proof presented here slightly simplifies the original one.

**Proof.** Since $b = 2d\zeta^3$ and $g = 2e^u |d\zeta|^2$, we have $2|b|_g^2 = e^{-3u}$. Wang’s equation becomes $\Delta u = 4e^u - 4e^{-2u}$. Corollary 6.2 implies that $u$ is non-negative and bounded on $S$.

When $n \geq 4$, we can identify $\overline{H}_{k-1} \cup \overline{H}_k \cup \overline{H}_{k+1} \subset \overline{S}$ with the surface $\overline{X} = \overline{H} \cup \overline{H} \cup \overline{H}$ considered in Lemma A.2 from the appendix. When $n \leq 3$, we identify it rather with a quotient of $\overline{X}$. In any case, the restriction of $u$ to $\overline{H}_{k-1} \cup \overline{H}_k \cup \overline{H}_{k+1}$ can be viewed as a function on $\overline{X}$ satisfying the hypotheses of Lemma A.2, hence that lemma and Corollary A.4 give us the required estimates.  

$$\square$$
In this section, we still let the Riemann surface $\Sigma$ and the cubic differential $b$ be as in the previous two sections. Moreover, let $g$ be the conformal metric produced by Theorem 6.1, so that $(g, b) \in \mathcal{W}_0$.

The goal of this section is to prove statement (III) in the sketch of proof of Theorem 1.1 from the introduction. Therefore we fix a base point $m \in \Sigma$ in order to define Wang’s developing pair $(\text{dev}, \text{hol})$ associated to $(g, b)$ (c.f. §2.2.2). Also fix a puncture $p$ and a point $\tilde{p}$ in the Farey set $\text{Farey}(\Sigma, p)$ so as to define the developing boundary $\partial \tilde{p} \Omega$ (c.f. §3.1.1).

7. Poles of order $\geq 4$. We first establish statement (III) when $p$ is a pole of order $n+3 \geq 4$. Theorem 7.4 below is a precise statement, where the required twisted polygon is found as developing limits of some particular paths on $\Sigma$, defined as follows.

7.1. Special collections of paths converging to $p$.

**Definition 7.1.** We let $\mathcal{C}$ denote the space of oriented smooth paths $\beta$ on $\Sigma$ issuing from $m$, converging to $p$ and fulfilling the following requirements.

(i) $\beta$ belongs to the class $\tilde{p}$ (see §3.1.1 for the definition).

(ii) $\beta$ is eventually $3$-geodesic with respect to the flat metric $|b|^2$.

(iii) $\beta$ is asymptotic to some ray $r(\beta) = \mathbb{R}_+ \cdot v \subset T_p \Sigma$ (c.f. Proposition 5.1 for the notion of “asymptotic ray”).

Two such paths $\beta, \beta' \in \mathcal{C}$ are said to be equivalent if they satisfy

(a) $\beta$ is eventually the same geodesic as $\beta'$;

(b) $\beta$ can be homotoped to $\beta'$ by a homotopy which fixes the starting point $m$ and a common final portion of $\beta$ and $\beta'$.

Paths $\beta, \beta' \in \mathcal{C}'$ merely satisfying condition (1) are not necessarily equivalent basically because they can wrap around $p$ differently many times before becoming a geodesic. The equivalence class can be pinned down by taking the “winding number” into account.

More precisely, we define the winding number $\vartheta(\beta) \in \mathbb{R}$ of $\beta$ with respect to a reference path $\beta_0 \in \mathcal{C}$. Since both $\beta_0$ and $\beta$ belong to the class $\tilde{p}$, we can deform $\beta_0$ into $\beta$ through a continuous family of paths $\beta_s \in \mathcal{C}$ (where $s \in [0,1]$, $\beta_1 = \beta$). Roughly speaking, $\vartheta(\beta)$ is the angle swept out by the asymptotic ray $r(\beta_s)$ as $s$ goes from 0 to 1. See Figure 7.1 for some examples. In what follows, we always choose $\beta_0$ such that $r(\beta_0) = \mathbb{R}_+ \subset T_p \Sigma \cong \mathbb{C}$ as in the figure.

In this paper, we content ourselves with the above intuitive definition of $\vartheta(\beta)$ and will make use of the following intuitively obvious facts.

1. $\vartheta(\beta) \equiv \text{arg}(r(\beta)) \mod 2\pi$.

2. Two paths $\beta, \beta' \in \mathcal{C}$ are equivalent if and only if $\beta$ eventually coincides with $\beta'$ and $\vartheta(\beta) = \vartheta(\beta')$.

3. $\vartheta(\beta \cdot \gamma_{\tilde{p}}) = \vartheta(\beta) - 2\pi$ (see §3.1.1 for the definition of $\gamma_{\tilde{p}}$).

Some special subsets of $\mathcal{C}$ are defined as follows.

---

3 The path $\beta$ eventually satisfying a property means that a final portion of $\beta$, i.e. the part of $\beta$ in some neighborhood of $p$, satisfies that property.
Figure 7.1. Examples of winding numbers: we have $\vartheta(\alpha) = \theta$, $\vartheta(\beta) = \theta + 2\pi$ and $\vartheta(\gamma) = \theta - 2\pi$ with respect to $\beta_0$.

**Definition 7.2.** For each $k \in \mathbb{Z}$, we define the following subsets of $C$.

$C_k := \{ \beta \in C \mid \frac{2(k-1)}{n} \pi < \vartheta(\beta) < \frac{2k}{n} \pi \}$,

$C_{k,k+1} := \{ \beta \in C \mid \vartheta(\beta) = \frac{2\pi k}{n} \}$,

$U_k^- := \{ \beta \in C \mid \vartheta(\beta) = \frac{(4k-3)}{2n} \pi \}$, $U_k^+ := \{ \beta \in C \mid \vartheta(\beta) = \frac{(4k-1)}{2n} \pi \}$,

$S_k := \{ \beta \in C \mid \frac{(4k-3)}{2n} \pi < \vartheta(\beta) < \frac{(4k-1)}{2n} \pi \}$,

$S_{k,k+1} := \{ \beta \in C \mid \frac{(4k-1)}{2n} \pi < \vartheta(\beta) < \frac{(4k+1)}{2n} \pi \}$.

A path $\beta \in C$ is said to be unstable if it belongs to $U_k^-$ or $U_k^+$ for some $k$, otherwise it is said to be stable.

These definitions clearly stem from the special directions and sectors defined in §5.1. For example, given $k \in \mathbb{Z}/n\mathbb{Z}$, the union $\bigcup_{k \in \mathbb{Z}/n\mathbb{Z}} C_k$ is the set of all $\beta \in C$ such that the asymptotic ray $r(\beta)$ is in $C_k$; an unstable path is a path $\beta$ such that $r(\beta) \in \bigcup_{k \in \mathbb{Z}/n\mathbb{Z}} U_k^-$ or $U_k^+$ for some $k$.

We proceed to single out some particular representatives in each equivalence class of paths $\beta \in C$ in order to apply the local model built in §5.3 to the present study. First let us introduce some more notations for certain particular paths. See the picture below.

- Concatenating the segment of $\partial H_k$ from $0$ to $\xi_k$ and the segment of $\partial H_{k+1}$ from $\eta_{k+1}$ to $0$ (see §5.3 for the notations) yields a path going from $0 \in \overline{H}_k$ to $0 \in \overline{H}_{k+1}$, denoted by $\gamma_{k,k+1}$.

- Let $(\alpha_k)_{k \in \mathbb{Z}}$ be a family of paths such that $\alpha_k$ goes from $m$ to $0 \in \overline{H}_k$ and $\alpha_{k+1}$ is homotopic to $\gamma_{k,k+1} \cdot \alpha_k$ for any $k$.

- Let $\beta_0 \in C$ be the concatenation of $\alpha_0$ with the ray $e^{-\pi i/3}$ in $H_0$. From on now, we take this $\beta_0$ as the reference path when considering winding numbers.

Using property (ii) in Proposition 5.1 and the table following Proposition 5.1, we get the following lemma.

**Lemma 7.3 (Reducing paths to a half-plane).** A path $\beta \in C$ belongs to $\mathcal{S}_{k-1,k} \cup \mathcal{S}_k \cup \mathcal{S}_{k,k+1}$ if and only if $\beta$ is equivalent to $\beta \cdot \alpha_k$ for some path $\beta \in \overline{H}_k$ such that $\beta$ is eventually a ray with asymptotic argument

$\theta(\tilde{\beta}) := \lim_{t \to +\infty} \arg(\tilde{\beta}(t)) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. 
Moreover, $\beta$ belongs to each of the sets in the first row of the following table if and only if $\theta(\tilde{\beta})$ belongs to the interval or takes the specific value indicated by the second row.

| $\beta$ | $C_{k,k+1}$ | $C_k$ | $C_{k-1,k}$ | $U_{k+1}$ | $U_k$ | $U_{k-1}$ |
|---------|-------------|-------|-------------|-----------|-------|-----------|
| $\theta(\beta)$ | $-\pi/3$ | $(-\pi/3, \pi/3)$ | $\pi/3$ | $(-\pi/6, \pi/6)$ | $\pi/6$ | $(\pi/6, \pi/2)$ |

7.1.2. Statement of the main result. Given $\beta \in \mathcal{C}$, we always let $[0, +\infty) \to \Sigma$, $t \mapsto \beta(t)$ denote the parametrization of $\beta$ by arc-length with respect to the metric $|b|^{3/2}$. Recall from §3.1.3 that $\beta_{[0,t]} := \beta([0,t])$ denotes the truncation at length $t$ and $\text{dev}(\beta_{[0,t]}) \in \mathbb{P}(E_m)$ denotes the developing image of the point in $\tilde{\Sigma}_m$ represented by $\beta_{[0,t]}$. The developing limit is defined as $\text{Lim}(\beta) := \lim_{t \to +\infty} \text{dev}(\beta_{[0,t]}) \in \mathbb{P}(E_m)$ if the latter limit exists.

**Theorem 7.4 (Developing boundary of higher order pole).** The developing limit $\text{Lim}(\beta)$ exists for any stable $\beta \in \mathcal{C}$ and admits the following descriptions.

1. There exists $X_k \in \mathbb{P}(E_m)$ such that $\text{Lim}(\beta) = X_k$ for any $\beta \in \mathcal{C}_k$.
2. As $\beta$ runs over $\mathcal{C}_{k,k+1}$, the set of developing limits
   
   $$X_{k,k+1}^\circ := \{\text{Lim}(\beta) \mid \beta \in \mathcal{C}_{k,k+1}\}$$
   
   is the interior of a segment $X_{k,k+1} \subset \mathbb{P}(E_m)$ with endpoints $X_k$ and $X_{k+1}$. The adjacent segments $X_{k-1,k}$ and $X_{k,k+1}$ are not collinear.
3. The holonomy $\text{hol}_\beta$ maps $X_k$ to $X_{k+n}$, thus $\bigcup_{k \in \mathbb{Z}} X_{k,k+1}$ is a $n$-gon twisted by $\text{hol}_\beta$. This twisted $n$-gon and its accumulation points constitute $\partial \mathbb{P}\Omega$.

This theorem is a generalization of Theorem 6.3 in [DW14] by Dumas and Wolf. Our proof is an adaptation of theirs as well.

7.1.3. Auxiliary developing limits. We let $\text{dev}_0 : \tilde{\Sigma}_m \to \mathbb{P}(E_m)$ be Wang’s developing map associated to $(2^3/4[b]^{3/4}, b)$ (see §2.3.4). Let $\text{Lim}_0(\beta)$ denote the corresponding developing limit for $\beta \in \mathcal{C}$, i.e.

$$\text{Lim}_0(\beta) := \lim_{t \to +\infty} \text{dev}_0(\beta_{[0,t]})$$

if the limit exists.
The pair $\langle 2^{1/2}\beta, \beta \rangle$ is expressed as $\langle 2|\zeta|^2, 2|\zeta|^3 \rangle$ on each half-plane $H_k$ in the local model. But Wang’s developing map associated to the latter pair is well known as we have seen in §2.3.1. Therefore, in order to determine all the $L_0(\beta)$'s, one only need to patch together the developing limits given by Proposition 2.4 for each $H_k$. We introduce some notations in order to state the result.

Let $(e_1^k, e_2^k, e_3^k)$ be the equilateral frame (see §2.2.2 for the definition) of $E$ over $\mathbb{H}_k$ associated to the natural coordinate. Let $E_{0,k}$ be the fiber of $E$ at $0 \in \mathbb{H}_k$ and let $\Delta^k \subset \mathbb{P}(E_{0,k})$ be the triangle

$$\Delta^k = \{ [xe^k_1(0) + ye^k_2(0) + ze^k_3(0)] \mid xyz \geq 0 \}.$$  

Here $[v]$ stands for the projectivization of the vector $v$. As in §2.3.1, we put $X_i^k = [e_i^k(0)]$, $X_{ij}^k = \{ [(1-s)e_i^k(0) + se_j^k(0)] \}_{s \in [0,1]}$.

We emphasize that the “$0$” in the notation “$e_i^k(0)$” means the origin of $\mathbb{H}_k$ and represents different points for different $k$.

It turns out that for any $k \in \mathbb{Z}$, the triangle $\Delta^k$ gets mapped by the parallel transport $T_0(\alpha_k^{-1}): \mathbb{P}(E_{0,k}) \to \mathbb{P}(E_m)$ to the same triangle in $\mathbb{P}(E_m)$. However, the map permutes the vertices according to $k$. To describe this precisely, we put

$$\hat{X}_1 := T_0(\alpha_1^{-1})X_1^k, \quad \hat{X}_2 := T_0(\alpha_1^{-1})X_2^k, \quad \hat{X}_3 := T_0(\alpha_1^{-1})X_3^k,$$

$$\hat{X}_{12} := T_0(\alpha_1^{-1})X_{13}^k, \quad \hat{X}_{23} := T_0(\alpha_1^{-1})X_{23}^k, \quad \hat{X}_{31} := T_0(\alpha_1^{-1})X_{31}^k.$$

That is, the $\hat{X}_i$'s and $\hat{X}_{ij}$'s are the vertices and edges of the triangle $T_0(\alpha_k^{-1})\Delta^k$ in $\mathbb{P}(E_m)$. Note that they are labelled differently from those of $\Delta^k$.

**Lemma 7.5.** For any $k \in \mathbb{Z}$, let $[k] \in \{1, 2, 3\}$ denote the mod 3 value of $k$ (not to be confused with the notation for projectivization). Then

$$T_0(\alpha_k^{-1})X_i^k = \hat{X}_{[k]}, \quad T_0(\alpha_k^{-1})X_{ij}^k = \hat{X}_{[k]-1}[i,j].$$

Proof. When $k = 1$, these are exactly the definitions. In order to prove for other $k$’s, we claim that

$$T_0(\alpha_k^{-1})[e_i^k(0)] = T_0(\alpha_{k-1}^{-1})[e_{i-1}^{k-1}(0)]$$

for any $k$ and $i$. Here the indices $i, i-1 \in \{1, 2, 3\}$ are counted mod 3. The required equalities then follow from the claim and the $k = 1$ case by induction.

To prove the claimed, we rewrite it as

$$[e_i^k(0)] = T_0(\gamma_{k-1,k})[e_{i-1}^{k-1}(0)].$$

Here $\gamma_{k-1,k} \equiv \alpha_k \cdot \alpha_{k-1}^{-1}$ is defined in §7.1.1 as the concatenation of a segment $\gamma' \subset \partial \mathbb{H}_{k-1}$ followed by a segment $\gamma'' \subset \partial \mathbb{H}_k$. Now (7.1) is a consequence of the following two facts: first, the transition map from $\mathbb{H}_{k-1}$ to $\mathbb{H}_k$ is $\zeta \mapsto e^{2\pi i/3} \zeta + \zeta_0$, so a straightforward computation with the definition of equilateral frames shows that $e_i^k = e_{i-1}^{k-1}$ on $\mathbb{H}_{k-1} \cap \mathbb{H}_k$; second, as we have seen in §2.3.1, the matrix expressions of $T_0$ are diagonal with respect to the equilateral frame, thus $T_0(\gamma')$ (resp. $T_0(\gamma'')$) preserves the sections $[e_i^{k-1}]$ (resp. $[e_i^k]$) of the projectivized bundle $\mathbb{P}(E)$. \hfill $\Box$

**Proposition 7.6 (Auxiliary developing limits).** The developing limit $\text{Lim}_0(\beta)$ exists for any $\beta \in \mathcal{C}$. Moreover,

(1) $\text{Lim}_0(\beta) = \hat{X}_{[k]}$ for any $\beta \in \mathcal{C}_k$;
(2) \( \{ \lim_0(\beta) \}_{\beta \in \mathcal{C}_{k,k+1}} = \hat{X}_{[k],[k+1]}^\circ \).

**Proof.** By Lemma 7.3, \( \beta \in \mathcal{C}_k \) if and only if \( \beta = \tilde{\beta} \cdot \alpha_k \), where the path \( \tilde{\beta} \subset \overline{H}_k \) is eventually a ray with asymptotic argument \( \theta \in (-\frac{\pi}{3}, \frac{\pi}{3}) \). The developing limit \( \lim_0(\tilde{\beta}) \in \mathbb{P}(E_0,k) \), defined using the developing map with respect to the base point \( 0 \in \mathbb{H}_k \), is related to \( \lim_0(\beta) \) by

\[
\lim_0(\beta) = T_0(\alpha_k^{-1})\lim_0(\tilde{\beta})
\]

Proposition 2.4 gives \( \lim_0(\tilde{\beta}) = X^k_1 \). Using Lemma 7.5, we get

\[
\lim_0(\beta) = T_0(\alpha_k^{-1})X^k_1 = \hat{X}_k
\]

for any \( \beta \in \mathcal{C}_k \), as required. This is proofs statement (1).

Similarly, if \( \beta \in \mathcal{C}_{k,k+1} \) then \( \theta = -\frac{\pi}{3} \) and Proposition 2.4 gives \( \{ \lim_0(\tilde{\beta}) \}_{\beta \in \mathcal{C}_{k,k+1}} = X^k_{\beta_1} \). Again, we use Lemma 7.5 to get statement (2). \( \square \)

7.1.4. **Comparing parallel transports.** Now that the \( \lim_0(\beta) \)'s are known, in order to find \( \lim(\beta) \), we compare the parallel transports \( T_0 \) and \( T \) which generate \( dev_0 \) and \( dev \), respectively.

**Definition 7.7.**

1. Given \( \beta \in \mathcal{C} \), we define the comparison transformation along the truncated path \( \beta_{[0,1]} \) as the projective transformation

\[
P_\beta(t) := T_0(\beta_{[0,1]}^{-1})T_0(\beta_{[0,1]}) \in SL(E_m).
\]

Denote \( P_\beta := \lim_{t \to +\infty} P_\beta(t) \in SL(E_m) \) if the limit exists.

2. For each \( k \), we define one-parameter unipotent subgroups \( G_k^\pm \subset SL(E_m) \) as follows. Endow \( E_m \) with a basis whose projection to \( \mathbb{P}(E_m) \) is the triple of points \( (\hat{X}_k, \hat{X}_k^{-1}, \hat{X}_k) \) (see §7.1.3 for the notation). Writing elements in \( SL(E_m) \) as matrices under this basis, we define

\[
G_k := \begin{cases}
1 & 1 \\
1 & 1
\end{cases}
\]

and

\[
G_k^+ := \begin{cases}
1 & * \\
1 & 1
\end{cases}
\]

Let \( \Pi_k^\pm \) denote the projection \( SL(E_m) \to SL(E_m)/G_k^\pm \).

The following theorem is a reformulation of Lemma 6.4 and Lemma 6.5 in [DW14]. We include the proof here for the sake of completeness.

**Theorem 7.8 (Limit of comparison transformation).**

1. The limit in (7.3) exists if \( \beta \) is stable.
2. Given \( k \), any \( \beta \in \mathcal{J}_k \) (resp. \( \beta \in \mathcal{J}_{k,k+1} \)) gives rise to the same \( P_\beta \), which we denote by \( P_k \) (resp. \( P_{k,k+1} \)).
3. \( P_{k-1,k}P_k \) and \( P_{k,k+1}P_k \) belong to \( G_k^- \) and \( G_k^+ \), respectively.

We first reformulate the problem in terms of the local model from §5.3.

We fix \( k \) henceforth and denote the half-plane \( \mathbb{H}_k \) simply by \( H \). To prove the theorem, we can restrict our attention to paths \( \beta \in \mathcal{J}_k \cup \mathcal{J}_{k,k+1} \cup \mathcal{J}_{k+1} \). By Lemma 7.3, each such \( \beta \) is equivalent to a composition \( \tilde{\beta} \cdot \alpha_k \) for some path \( \tilde{\beta} \in \mathbb{H} \) issuing from \( 0 \). We shall study alternatively the comparison transformation \( P_\beta(t) \in SL(E_0) \) of \( \tilde{\beta} \), defined with respect to the base point \( 0 \in \mathbb{H} \).
Let $T, T_0 : \mathbb{H} \times \mathbb{H} \to \text{SL}(3, \mathbb{R})$ be the two-pointed parallel transport maps of the connections $D = d + A$ and $D_0 = d + A_0$ (see §2.2.2 and §2.3.4 for the notation), respectively, under the equilateral frame $(e_1, e_2, e_3)$ of $E$ over $\mathbb{H}$. The matrix representative of $P_\beta(t)$ under the basis $(e_1(0), e_2(0), e_3(0))$ is

$$P_\beta(t) = T(0, \tilde{\beta}(t))T_0(\tilde{\beta}(t), 0).$$

Here $\tilde{\beta}(t)$ is the parametrization of $\tilde{\beta}$ by arc-length with respect to $|b|^\frac{2}{3}$. Note that the original comparison transformation that we wish to study is given by

$$(7.4) \quad P_\beta(t + t_0) = T(\alpha_k^{-1}) P_\beta(t) T(\alpha_k),$$

where $t_0$ is the $|b|^\frac{2}{3}$-length of $\alpha_k$.

Clearly $P_\beta(t)$ converges if and only if $P_\tilde{\beta}(t)$ does. In view of Lemma 7.3, we can restate part (1), (2) and (3) of the proposition as statements (1'), (2') and (3') below, respectively. In these statements, $\tilde{\beta} : [0, +\infty) \to \mathbb{H}$ is any path satisfying $\tilde{\beta}(0) = 0$, $|\frac{d}{dt} \tilde{\beta}(t)| \equiv 1$ and

$$\tilde{\beta}(t) = e^{\theta t} + \zeta_0, \quad \forall t \geq M$$

for some $\zeta_0 \in \mathbb{H}$, $\theta = \theta(\tilde{\beta}) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $M > 0$. The paths $\tilde{\beta}_0$ and $\tilde{\beta}_1$ are under the same hypotheses.

(1') If $\theta(\tilde{\beta})$ belongs to either of the intervals $(-\frac{\pi}{2}, -\frac{\pi}{6})$, $(-\frac{\pi}{6}, \frac{\pi}{6})$ or $(\frac{\pi}{6}, \frac{\pi}{2})$, then the limit $P_\beta := \lim_{t \to +\infty} P_\beta(t) \in \text{SL}(E_0)$ exists.

(2') Given $\tilde{\beta}_0$ and $\tilde{\beta}_1$, if $\theta(\tilde{\beta}_0)$ and $\theta(\tilde{\beta}_1)$ are both in one of the intervals $(-\frac{\pi}{2}, -\frac{\pi}{6})$, $(-\frac{\pi}{6}, \frac{\pi}{6})$ or $(\frac{\pi}{6}, \frac{\pi}{2})$, then $P_{\tilde{\beta}_0} = P_{\tilde{\beta}_1}$.

(3')

$$P_{\tilde{\beta}_1}^{-1} P_{\tilde{\beta}_0} = \begin{cases} \begin{pmatrix} 1 & * \\ 1 & 1 \end{pmatrix} & \text{if } \theta(\tilde{\beta}_0) \in (-\frac{\pi}{6}, \frac{\pi}{6}) \text{ and } \theta(\tilde{\beta}_1) \in (\frac{\pi}{6}, \frac{\pi}{2}), \\ \begin{pmatrix} 1 & * \\ 1 & 1 \end{pmatrix} & \text{if } \theta(\tilde{\beta}_0) \in (-\frac{\pi}{2}, -\frac{\pi}{6}) \text{ and } \theta(\tilde{\beta}_1) \in (-\frac{\pi}{2}, -\frac{\pi}{6}). \end{cases}$$

Statement (3') is equivalent to (3) because, on one hand, $P_{\tilde{\beta}_1}^{-1} P_{\tilde{\beta}_0}$ is conjugate to $P_{\tilde{\beta}_1}^{-1} P_{\tilde{\beta}_0}$ through $T_0(\alpha_k^{-1})$ by (7.4); on the other hand, by Lemma 7.5, the basis used in the definition of $G_k^\pm$ is the $T_0(\alpha_k^{-1})$-translates of the basis $(e_0^k(0), e_1^k(0), e_2^k(0))$ used in (3').

We proceed to prove the statements (1'), (2') and (3'). The idea is to take derivatives of the SL(3, $\mathbb{R}$)-valued functions in question, resulting in linear ODEs, and then apply certain asymptotic result for such ODEs to get the required limit.

In the following proof, we re-denote $\beta$ by $\tilde{\beta}$ for tidiness. This is not to be confused with the $\beta$ from the original statement of Theorem 7.8.
Proof. (1') We compute the derivative of \( P_\beta(t) \) using the last property of two-pointed parallel transport maps given in §2.2.1, obtaining
\[
\frac{d}{dt} P_\beta(t) = T(0, \beta(t)) A(\hat{\beta}(t)) T_0(\beta(t), 0) - T(0, \beta(t)) A_0(\hat{\beta}(t)) T_0(\beta(t), 0) = P_\beta(t) \text{Ad}_{T_0(0, \beta(t))} (A(\hat{\beta}(t)) - A_0(\hat{\beta}(t))) = P_\beta(t) N(t),
\]
where we put
\[
N(t) := \text{Ad}_{T_0(0, \beta(t))} (A(\hat{\beta}(t)) - A_0(\hat{\beta}(t))).
\]
A result on asymptotics of linear ODEs (Lemma B.1 in [DW14]) then ensures that \( \lim_{t \to +\infty} P_\beta(t) \) exists if
\[
\int_0^{+\infty} \|N(t)\| dt < +\infty.
\]
Here \( \| \cdot \| \) denotes a matrix norm.

The expressions of \( D \) and \( D_0 \) from (2.7) and (2.9) yield
\[
A - A_0 = B^{-1} \begin{pmatrix} \partial u & (e^{-u} - 1)d\zeta & 0 \\ (e^{-u} - 1)d\zeta & \tilde{\partial} u & 0 \\ (e^{u} - 1)d\zeta & (e^{u} - 1)d\zeta & 0 \end{pmatrix} B,
\]
where the function \( u \) is as in §6.2. Therefore, Lemma 6.3 and the assumption that \( |\hat{\beta}(t)| = 1 \) imply
\[
\|A(\hat{\beta}(t)) - A_0(\hat{\beta}(t))\| \leq C |\hat{\beta}(t)|^{-\frac{1}{2}} e^{-2\sqrt{\delta}|\hat{\beta}(t)|}, \quad \forall t \geq 0
\]
(6.6)

On the other hand, by virtue of the discussion on the spectral radius \( \text{Ad}_{T_0(0, \zeta)} \) in §2.3.2, our assumption on \( \theta(\beta) = \lim_{t \to +\infty} \arg(\beta(t)) \) implies that
\[
\rho(\text{Ad}_{T_0(0, \beta(t))}) = e^{\arg(\beta(t))|\beta(t)|} \leq e^{(2\sqrt{3} - \delta)|\beta(t)|}
\]
for some constant \( \delta > 0 \) when \( t \) is sufficiently large.

Combining the estimates (6.6) and (7.7), we get
\[
\|\text{Ad}_{T_0(0, \beta(t))} (A(\hat{\beta}(t)) - A_0(\hat{\beta}(t)))\| \leq C|\beta(t)|^{-\frac{1}{2}} e^{-\delta|\beta(t)|} \leq C' t^{-\frac{1}{2}} e^{-\delta t},
\]
where the second equality is because \( t - \mu \leq |\beta(t)| \leq t \) for some constant \( \mu \), as implied by the hypotheses on \( \beta \). The required condition (7.5) follows.

(2') For \( t \geq 0 \) and \( s \in [0, 1] \), put
\[
\beta_s(t) := (1 - s)\beta_0(t) + s \beta_1(t) \in \mathbb{H}
\]
It is easy to see that there exist \( \lambda > 1 \) and \( \mu > 0 \) such that
\[
\frac{1}{\lambda} t - \mu \leq |\beta_s(t)| \leq t, \quad \left| \frac{d}{dt} \beta_s(t) \right| \leq \lambda t + \mu
\]
for any \( s \) and \( t \). Put
\[
Q(s, t) := T_0(0, \beta_s(t)) T(\beta_s(t), \beta_0(t)) T_0(\beta_0(t), 0) \in \text{SL}(3, \mathbb{R}).
\]
Note that
\[
Q(1, t) = T_0(0, \beta_1(t)) T(\beta_1(t), 0) T(0, \beta_1(t)) T_0(\beta_0(t), 0) = P_{\beta_1}(t)^{-1} P_{\beta_0}(t).
\]
Therefore \( P_{\beta_1}^{-1} P_{\beta_0} \) is the limit of \( Q(1, t) \) as \( t \to +\infty \). By part (1'), this limit exists if neither \( \theta(\beta_0) \) nor \( \theta(\beta_1) \) equals \( \pm \frac{\pi}{2} \).
In order to evaluate the limit of $Q(1, t)$, we compute $\frac{\partial}{\partial s} Q(s, t)$ similarly as above, obtaining
\[
\frac{\partial}{\partial s} Q(s, t) = T_0(0, \beta_s(t)) A_0 \left( \frac{\partial}{\partial s} \beta_s(t) \right) T(\beta_s(t), \beta_0(t)) T_0(\beta_0(t), 0) \\
- T_0(0, \beta_s(t)) A \left( \frac{\partial}{\partial s} \beta_s(t) \right) T(\beta_s(t), \beta_0(t)) T_0(\beta_0(t), 0) \\
= \text{Ad}_{T_0(0, \beta_s(t))} \left( A_0 \left( \frac{\partial}{\partial s} \beta_s(t) \right) - A \left( \frac{\partial}{\partial s} \beta_s(t) \right) \right) Q(s, t).
\]
Let us set
\[
M(s, t) := \text{Ad}_{T_0(0, \beta_s(t))} \left( A_0 \left( \frac{\partial}{\partial s} \beta_s(t) \right) - A \left( \frac{\partial}{\partial s} \beta_s(t) \right) \right),
\]
so that $\frac{\partial}{\partial s} Q(s, t) = M(s, t) Q(s, t)$. This can be viewed as a family of ODEs on the interval $[0, 1]$ depending on the parameter $t \in [0, +\infty)$. The following asymptotic result applies.

**Lemma 7.9.** Let $Q(s, t)$ and $M(s, t)$ be smooth functions on $[0, 1] \times [0, +\infty)$ taking values in $\text{GL}(3, \mathbb{R})$ and $\mathfrak{gl}_3 \mathbb{R}$, respectively, satisfying the following conditions:

(i) $\frac{\partial}{\partial s} Q(s, t) = M(s, t) Q(s, t)$,

(ii) $Q(0, t) = \text{id}$ for any $t$.

(iii) There exist $X \in \mathfrak{gl}_3 \mathbb{R}$ and a real-valued function $f(s, t)$ such that

\[
\begin{align*}
&\text{(a)} \quad \lim_{t \to +\infty} \max_{s \in [0, 1]} \| M(s, t) - f(s, t) X \| = 0, \\
&\text{(b)} \quad \sup_{t \in [0, +\infty]} \int_0^1 |f(s, t)| \, ds < +\infty.
\end{align*}
\]

Then we have
\[
\lim_{t \to +\infty} \left\| Q(1, t) - \exp \left( X \int_0^1 f(s, t) \, ds \right) \right\| = 0.
\]

This lemma is an immediate consequence of Lemma B.2 in [DW14].

We need to prove $\lim_{t \to +\infty} Q(1, t) = \text{id}$. Using the above lemma, it is sufficient to show that
\[
\lim_{t \to +\infty} \max_{s \in [0, 1]} \| M(s, t) \| = 0.
\]
But this is similar to the proof of part (1'): on one hand, Lemma 6.3 yields
\[
\| A_0 \left( \frac{\partial}{\partial s} \beta_0(s) \right) - A \left( \frac{\partial}{\partial s} \beta_0(s) \right) \| \leq C \| \beta_0(s) \|^{-\frac{1}{2}} e^{-2\sqrt{3} |\beta(s)|} \| \frac{\partial}{\partial s} \beta_0(s) \|,
\]
whereas the discussion on $\rho(\text{Ad}_{T_0(0, \xi)})$ in §2.3.2 and the assumption that $\theta(\beta_0)$ and $\theta(\beta_1)$ are both in either $(-\frac{\pi}{2}, -\frac{\pi}{6})$, $(-\frac{\pi}{6}, \frac{\pi}{6})$ or $(\frac{\pi}{6}, \frac{\pi}{2})$ yield
\[
\rho(\text{Ad}_{T_0(0, \beta_3(s))}) \leq e^{(2\sqrt{3} - \delta)} |\beta(s)|.
\]
Eq.(7.11) now follows from the above two inequalities and (7.8).

(3') First assume
\[
\theta(\beta_0) \in (-\frac{\pi}{6}, \frac{\pi}{6}), \quad \theta(\beta_1) \in (\frac{\pi}{6}, \frac{\pi}{2}).
\]
Part (2') implies that $P^{-1}_{\beta_1} P_{\beta_0}$ is the same for any such $\beta_0$ and $\beta_1$, thus we can choose
\[
\beta_0(t) = t, \quad \beta_1(t) = t e^{\pi i}.
\]
For \( t \geq 0 \) and \( s \in [0, 1] \), put
\[
\beta_s(t) := t e^{\frac{s}{3} i}.
\]

Then define \( Q(s, t) \) and \( M(s, t) \) in the same way as above.

We still have \( \frac{\partial}{\partial s} Q(s, t) = M(s, t)Q(s, t) \) and \( P_{\beta_1}^{-1} P_{\beta_0} = \lim_{t \to +\infty} Q(1, t) \). We shall apply Lemma 7.9 again to obtain the latter limit.

The estimate (7.12) still holds and gives
\[
\| A_0 \left( \frac{\partial}{\partial s} \beta_s(t) \right) - A \left( \frac{\partial}{\partial s} \beta_s(t) \right) \| \leq C \sqrt{t} e^{-2\sqrt{3} t}.
\]

One the other hand, with the notations from §2.3.2, the eigenvalue of \( \text{Ad}_{T_0(0, \beta_s(t))} \) on \( E_{ij} \) is
\[
\exp \left( \omega_{ij}(\arg(\beta_s(t))) | \beta_s(t) \right) = \exp \left( \omega_{ij}(\frac{s}{3} t) \right).
\]

Since \( \frac{s}{3} t \) takes values in the interval \( [0, \frac{t}{3}] \) in which the only odd multiple of \( \frac{t}{3} \) is \( \frac{2}{3} \) itself, fact (2) in §2.3.2 implies that, for any \( (i, j) \neq (1, 3) \), the eigenvalue on \( E_{ij} \) is bounded by \( e^{(2\sqrt{3} - \delta) t} \) for some \( \delta > 0 \). In view of (7.13), we conclude that
\[
\max_{s \in [0, 1]} \| M(s, t) - M_{13}(s, t) E_{13} \| \leq C \sqrt{t} e^{-\delta t},
\]
where we let \( M_{13}(s, t) \) denote the \((1, 3)\) entry of \( M(s, t) \). Thus condition (a) in Lemma 7.9 is satisfied for \( X = E_{13} \) and \( f = M_{13} \). Moreover, by definition,
\[
\omega_{13}(\theta) = 2 \text{Re}(e^{\theta i} - e^{2\pi i/3} e^{\theta i}) = 2(\cos(\theta) - \cos(\theta + 2\pi/3)) = 2\sqrt{3} \cos(\theta - \frac{\pi}{6}).
\]

Therefore
\[
|M_{13}(s, t)| \leq C \sqrt{t} e^{-2\sqrt{3} t \left( 1 - \cos(\theta - \frac{\pi}{6}) \right)} = C \sqrt{t} e^{-\frac{4 \sqrt{3} t}{3} \sin^2 \left( \frac{\theta - \pi/6}{2} \right)} \leq C \sqrt{t} e^{-c t (s - \frac{1}{2})^2}
\]
for some constants \( c > 0 \). Up to a constant factor, the last term above, viewed as a function of \( s \) parametrized by \( t \), is a Gaussian distribution with average \( \frac{1}{2} \) and with variance tending to 0 as \( t \to +\infty \). Hence its integral over \([0, 1]\) tends to a constant. Thus condition (b) is satisfied as well.

Applying Lemma 7.9, we conclude that if we let \( (t_n) \) be a sequence in \( \mathbb{R}_{\geq 0} \) tending to \(+\infty\) such that \( \int_0^1 f(s, t_n) ds \) admits a limit \( r \in \mathbb{R} \) as \( n \to +\infty \) (such a sequence exists because \( \int_0^1 |f(s, t_n)| ds \leq \int_0^1 |f(s, t_n)| ds \) is bounded as we have just seen), then \( \lim_{n \to +\infty} Q(1, t_n) = \exp(r E_{13}) \). But we already know that \( Q(1, t) \) converges, so we obtain \( P_{\beta_1}^{-1} P_{\beta_0} = \lim_{t \to +\infty} Q(1, t) = \exp(r E_{13}) \) as required.

In the case \( \theta(\beta_0) \in (-\frac{\pi}{6}, -\frac{\pi}{6}) \), \( (\beta_1) \in (-\frac{\pi}{6}, \frac{\pi}{6}) \), we apply the same argument to
\[
\beta_0(t) = t, \quad \beta_1(t) = t e^{-\frac{\pi}{3} i}, \quad \beta_s(t) = t e^{-\frac{s}{3} i}.
\]

Now \( \arg(\beta_s(t)) = -\frac{s}{3} t \) takes values in \([0, \frac{3}{2}]\), in which the only odd multiple of \( \frac{3}{2} \) is \( \frac{3}{2} \). Fact (2) in §2.3.2 now implies that the only eigenvalue not controlled by \( e^{2\sqrt{3} - \delta} \) is the one on \( E_{12} \). Applying Lemma 7.9 to \( X = E_{12} \) and \( f = M_{12} \) similarly as above, we obtain \( P_{\beta_1}^{-1} P_{\beta_0} = \lim_{t \to +\infty} Q(1, t) = \exp(r E_{12}) \) as required. \( \square \)
7.1.5. Proof of Theorem 7.4. We now deduce Theorem 7.4 from Proposition 7.6 and Theorem 7.8.

By definition of Wang’s developing map (see §2.2.2), if the limits in the definitions of $P_\beta$ and $\lim_0(\beta)$ both exist, then

$$
\lim(\beta) = \lim_{t \to +\infty} T_k(\beta^{-1}_{[0,t]}) \lim(\beta^{-1}_{[0,t]}) = \lim_{t \to +\infty} T_k(\beta^{-1}_{[0,t]}) T_0(\beta^{-1}_{[0,t]}) \lim(\beta^{-1}_{[0,t]}) = \lim_{t \to +\infty} P_\beta(t) \dev_0(\beta^{-1}_{[0,t]}) = P_\beta(\lim(\beta)).
$$

Here, recall that $\pi : \Sigma_m \to \Sigma$ is the projection and $1$ is the canonical section of $E$.

As $\beta$ runs over each of the sets given in the first row of the following table, Proposition 7.6 and Theorem 7.8 guarantee existence of both limits and provide a description of $P_\beta$ and the set formed by the $\lim_0(\beta)$’s, as shown in the second and third row of the table, respectively.

| $\beta$ | $\mathcal{C}_{k-1,k}$ | $\mathcal{C}_k \cap \mathcal{F}_{k-1,k}$ | $\mathcal{F}_k$ | $\mathcal{C}_k \cap \mathcal{F}_{k,k+1}$ | $\mathcal{C}_{k,k+1}$ |
|---------|---------------------|-------------------------------|-------------|---------------------------------|-----------------|
| $P_\beta$ | $P_{k-1,k}$ | $P_{k-1,k}$ | $P_k$ | $P_{k,k+1}$ | $P_{k,k+1}$ |
| $\{\lim_0(\beta)\}$ | $X_{[k-1],[k]}^\circ$ | $X_{[k]}^\circ$ | $X_{[k],[k+1]}^\circ$ |

Stable paths belonging to $\mathcal{C}_k$ correspond to the three columns in the middle where $\lim_0(\beta) = \hat{X}_{[k]}$. Theorem 7.8 (3) implies that both $P^{-1}_{k-1,k} P_k$ and $P^{-1}_{k,k+1} P_k$ fixes $\hat{X}_{[k]}$, thus $\lim(\beta) = P_\beta(\lim_0(\beta))$ is the point

$$
X_k := P_{k-1,k}(\hat{X}_{[k]}) = P_k(\hat{X}_{[k]}) = P_{k,k+1}(\hat{X}_{[k]})
$$

for any stable $\beta \in \mathcal{C}_k$. Part (1) of the theorem is proved.

Part (2) is proved by setting

$$
X_{k-1,k} := P_{k-1,k}(\hat{X}_{[k-1],[k]}) = P_k(\hat{X}_{[k-1],[k]}),
$$

$$
X_{k,k+1} := P_{k,k+1}(\hat{X}_{[k],[k+1]}) = P_k(\hat{X}_{[k],[k+1]}),
$$

where the second equalities in both lines follows from the fact that $P^{-1}_{k-1,k} P_k$ and $P^{-1}_{k,k+1} P_k$ pointwise fix the segments $\hat{X}_{[k-1],[k]}$ and $\hat{X}_{[k],[k+1]}$, respectively, as implied by Theorem 7.8 (3). Thus we get

$$
\{\lim(\beta)\}_{\beta \in \mathcal{C}_{k-1,k}} = \{P_\beta(\lim_0(\beta))\}_{\beta \in \mathcal{C}_{k-1,k}} = P_{k-1,k}(\hat{X}_{[k-1],[k]}^\circ) = X_{k-1,k}^\circ,
$$

as required, and similarly $\{\lim(\beta)\}_{\beta \in \mathcal{C}_{k,k+1}} = X_{k,k+1}^\circ$. Moreover, $X_{k-1,k}$ and $X_{k,k+1}$ are non-collinear segments sharing the endpoint $X_k = P_k(\hat{X}_{[k]})$ because they are $P_k$ translates of the segments $\hat{X}_{[k-1],[k]}$ and $\hat{X}_{[k],[k+1]}$, which are non-collinear and share the endpoint $\hat{X}_{[k]}$ by construction.

To prove the first statement of part (3), we note that, by property (3) of winding numbers in §7.1.1, $\beta$ belongs to $\mathcal{C}_{k+n}$ if and only if $\beta \cdot \gamma_{\beta}$ belongs to $\mathcal{C}_k$. Using hol-equivariance of $\dev$, we get, for any $\beta \in \mathcal{C}_{k+n}$

$$
X_k = \lim(\beta \cdot \gamma_{\beta}) = \lim_{t \to +\infty} \dev(\gamma_{\beta}^{-1} \beta^{-1}_{[0,t]}) = \lim_{t \to +\infty} \hol^{-1}_P(\dev(\beta^{-1}_{[0,t]})) = \hol^{-1}_P(X_{k+n})
$$

as required.

---

4 Here and below, by an abuse of notation, we do not distinguish the point $\hat{X}_{[k]}$ and the set with one element formed by this point.
Finally, looking at each case in Lemma 3.9 respectively, one sees that, in general, if a developing boundary $\partial_\beta \Omega$ contains some polygon $C$ twisted by $\text{hol}_\beta$, then the whole $\partial_\beta \Omega$ is the union of $C$ and the accumulation points of $C$. The last statement in part (3) follows.

7.2. Poles of order 3. Now assume that $p$ is a third order pole. We can suppose either $\Sigma = \mathbb{C}^*$ or $\Sigma$ has negative Euler characteristic, because if $\Sigma = \mathbb{C}$ then the pole $\infty$ has order at least 6. Let $R \in \mathbb{C}^*$ be the residue of $b$ at $p$.

In this case, statement (III) from the introduction is contained in Theorem 7.10 below. As an intermediate step in the proof, we determine the end holonomy $\text{hol}_\beta$ in Theorem 7.17. This covers Theorem 1.2 from the introduction.

7.2.1. Statements of the main results. We define the collection of paths $\mathcal{C}$ using the same Definition 7.1 as in the discussion of higher order poles. The counterpart to Theorem 7.4 in the present case is the following

**Theorem 7.10 (Developing boundary of third order pole).**

1. If $\text{Re}(R) < 0$ then $p$ is a triangular end. $\text{Lim}(\beta)$ is the saddle fixed point of the hyperbolic holonomy $\text{hol}_\beta$ for any $\beta \in \mathcal{C}$.

2. If $\text{Re}(R) > 0$ then $p$ is a geodesic end with hyperbolic holonomy. Let $x_+$ and $x_-$ denote the attracting and repelling fixed points of $\text{hol}_\beta$, respectively.
   - if $\text{Im}(R) < 0$ then $\text{Lim}(\beta) = x_+$ for any $\beta \in \mathcal{C}$.
   - if $\text{Im}(R) > 0$ then $\text{Lim}(\beta) = x_-$ for any $\beta \in \mathcal{C}$.
   - if $R \in \mathbb{R}_+$ then $\text{Lim}(\beta)$ exists for any $\beta \in \mathcal{C}$ and $\{\text{Lim}(\beta)\}_{\beta \in \mathcal{C}}$ is the interior of a segment joining $x_+$ and $x_-$.

3. If $R \in i\mathbb{R}^+$ then $p$ is a geodesic end with quasi-hyperbolic or planar holonomy and $\text{Lim}(\beta)$ is the double fixed point of $\text{hol}_\beta$ for any $\beta \in \mathcal{C}$.

In part (3), although a planar end holonomy has two double fixed points, only one of them is in the developing boundary and $\text{Lim}(\beta)$ is supposed to be this one. Note that planar holonomy only occurs in case (b) of Proposition 2.5 (see Lemma 3.7). Theorem 7.10 can actually be proved by direct calculations in this case.

The statements concerning $\text{Lim}(\beta)$ will be proved using similar ideas as in our treatment of higher order poles: we first investigate the “auxiliary developing limits” given by Wang’s developing map associated to $(2^\frac{1}{3}|b|^\frac{2}{3}, b)$; then we compare the actual developing limits with the auxiliary ones by studying the comparison transformation $P_{\beta}$ defined in §7.1.4.

However, while a serious investigation of unstable directions is avoidable for poles of order $\geq 4$, it is inevitable for third order poles. In fact, when $\text{Re}(R) = 0$, all directions that we are concerned with are “unstable”. A improvement of the ODE techniques used earlier will be developed in order to tackle this situation.

Another complication for third order poles is that the information about $\text{Lim}(\beta)$ does not determine $\partial_\beta \Omega$ immediately. In fact, when $R \notin \mathbb{R}_{\geq 0}$, as stated in Theorem 7.10, the $\text{Lim}(\beta)$’s only detect a single point, while $\partial_\beta \Omega$ is expected to have a continuum of points. Our idea to determine $\partial_\beta \Omega$ is to further study the asymptotic direction along which the path $\text{dev}(\beta_{\Omega}^{-1})$ converges to $\text{Lim}(\beta)$.

7.2.2. Local model for $(\Sigma, b)$. As at the beginning of §5, let $z$ be a coordinate around $p$ such that $b$ has the normal form

$$(7.14) \quad b = Rz^{-3}dz^3.$$
Let $U = \{0 < |z| < a\} \subset \Sigma$ be a punctured neighborhood of $p$ where $z$ is defined. Since a dilation of $z$ does not change the expression (7.14), we can assume $a > 1$.

By virtue of the expression (7.14), one easily constructs a local model of $(\Sigma, b)$ in the spirit of §5.3 using the exponential map: pick a cubic root $(R/2)^{\frac{1}{3}}$ of $R/2$ and consider the rotated half-plane

$$H = -(R/2)^{\frac{1}{3}}H,$$

where $H$ is the right half-plane as before.

The map $H \to U, \zeta \mapsto \exp((R/2)^{-\frac{1}{3}}\zeta)$ is invariant under the translation

$$H \to H, \quad \zeta \mapsto \zeta + 2\pi i(R/2)^{\frac{1}{3}}$$

and pulls $b$ back to $2d\zeta^{3}$. Let $H/\sim$ denote the quotient of $H$ by this translation. We can identify the punctured neighborhood $\{0 < |z| < 1\}$ of $p$, endowed with the cubic differential $b$, with $(H/\sim, 2d\zeta^{3})$. Furthermore, we identify $\{0 < |z| \leq 1\} \subset \Sigma$ with $H/\sim$.

As before, assume that the metric $g$ is expression in $H/\sim$ as

$$g = 2e^{u}|d\zeta|^{2}.$$  

Then $u$ satisfies Wang’s equation $\Delta u = 4e^{u} - 4e^{-2u}$. Corollary 6.2 implies that $u$ is non-negative and bounded.

The asymptotic estimate for $u$ in Theorem 6.3 no longer holds, but we have a slightly weaker estimate. Indeed, since $u$ can be viewed as a function on $H/\sim$, we can apply Lemma A.1 (2) and Corollary A.4 from the appendix and get the following

**Lemma 7.11 (Strong estimate for third order pole).** There is a constant $C$ such that

$$0 \leq u(\zeta), |\partial_{\zeta}u(\zeta)| \leq Ce^{-2\sqrt{3}\text{dist}(\zeta, \partial H)}$$

for any $\zeta \in H$. Here $\text{dist}(\zeta, \partial H)$ denotes the distance from $\zeta$ to $\partial H$ with respect to the metric $|d\zeta|^{2}$.

7.2.3. Notations. We slightly simplify the settings and introduce some notations for the proof of Theorem 7.10.

First of all, recall that the definition of $H$ depends on a choice of the cubic root $(R/2)^{\frac{1}{3}}$. For the sake of determinancy, we now only let $(R/2)^{\frac{1}{3}}$ denote the root satisfying

$$\arg((R/2)^{\frac{1}{3}}) \in (-\frac{\pi}{6}, \frac{\pi}{2}).$$

The choices of the base point $m \in \Sigma$ and the point $\bar{p} \in \text{Farey}(\Sigma, p)$ are not essential for the statement of Theorem 7.10. Thus we work henceforth with the following simplest choices:

- Take the origin $0 \in \mathcal{H}/\sim$ of the local model to be the base point, so that Wang’s developing map $\text{dev}$ takes values in $\mathbb{F}(E_{0})$, where $E_{0}$ denotes the fiber of the vector bundle $E \to \Sigma$ at the base point.
- Let $\bar{p} \in \text{Farey}(\Sigma, p)$ be represented by the ray $\alpha$ in $\mathcal{H}/\sim$ issuing from $0$ and perpendicular to $\partial H$.

With this choice of $\bar{p}$, we take $\gamma_{\bar{p}}$ (see §3.1.1 for the definition) to be the loop whose lift to $\mathcal{H}$ is the segment on $\partial H$ going from $0$ to $-2\pi i(R/2)^{\frac{1}{3}}$.

Any $\beta \in C$ is equivalent (in the sense of Definition 7.1) to a path in $\mathcal{H}/\sim$ which issues from $0$ and eventually becomes a ray. The ray is parallel to $\alpha$, otherwise $\beta$ is
a spiral viewed in the coordinate $z$ and does not fulfill condition (iii) in Definition 7.1. We do not need to distinguish equivalent paths, thus we re-define $C$ as

$$C = \{ \beta : [0, +\infty) \to \overline{H} \mid \beta(0) = 0, |\dot{\beta}(t)| = 1, \beta(t) = e^{\theta t} + \zeta_0 \text{ for } t \text{ big enough} \},$$

where $\theta = \arg(-(R/2)^{\frac{1}{3}})$.

The equilateral frame of $\overline{TH} \oplus \mathbb{R}$ (see §2.2.2 for the definition) associated to the natural coordinate $\zeta$ is invariant under translation, hence defines a frame $(e_1, e_2, e_3)$ of $E$ over $\overline{H}/ \sim$. Let $E_0$ denote the fiber of $E$ at $0 \in \overline{H}$. In what follows, whenever we write an element of $\text{SL}(E_0)$ as a metric, it is with respect to the basis $(e_1(0), e_2(0), e_3(0))$. Finally, as in §2.3.1, put

$$X_i = [e_i(0)], \quad X_{ij} := \{ [(1 - s) e_i(0) + s e_j(0)] \}_{s \in [0,1]}$$

7.2.4. Auxiliary developing limits: the case $\text{Re}(R) \neq 0$. Let $(\text{dev}_0, \text{hol}_0)$ be Wang’s developing pair associated to $(2 \frac{1}{2} |b|^{\frac{1}{3}}, b)$ (c.f. §2.3.4) and let $\text{Lim}_0(\beta)$ be the corresponding developing limit as in §7.1.3.

Within $\overline{H}/ \sim$, the pair $(2 \frac{1}{2} |b|^{\frac{1}{3}}, b)$ is expressed under the natural coordinate $\zeta$ of $\overline{H}$ as $(2d|\zeta|^2, 2d|\zeta|^3)$, thus the connection $D_0$, the parallel transport $T_0$ and the map $\text{dev}_0$ have the same expressions as in the Tițeica example. The investigation in §2.3.1 yields the next proposition.

**Proposition 7.12.** Set

$$\lambda_1 := e^{-4\pi \text{Im}\left((R/2)^{\frac{1}{3}}\right)}, \quad \lambda_2 := e^{-4\pi \text{Im}\left(\omega^2(R/2)^{\frac{1}{3}}\right)}, \quad \lambda_3 := e^{-4\pi \text{Im}\left(\omega(R/2)^{\frac{1}{3}}\right)},$$

where $\omega = e^{2\pi i/3}$. Then

(1) \hspace{1cm} \text{hol}_0(\gamma_{\beta}) = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}.

(2) The developing limit $\text{Lim}_0(\beta) \in \mathbb{P}(E_0)$ exists for any $\beta \in C$ and we have

$$\{\text{Lim}_0(\beta)\}_{\beta \in C} = \begin{cases} \{X_2\} & \text{if } \text{Re}(R) > 0, \text{Im}(R) < 0, \\
X_{23} & \text{if } R \in \mathbb{R}_+, \\
\{X_3\} & \text{otherwise.} \end{cases}$$

**Proof.** (1) The lift of the loop $\gamma_{\beta}$ to $\overline{H}$ goes from $0$ to $-2\pi i(R/2)^{\frac{1}{3}}$, hence, in terms of the two-pointed parallel transport map $T_0(\cdot, \cdot) : \overline{H} \times \overline{H} \to \text{SL}(3, \mathbb{R})$ with respect to the equilateral frame,

$$\text{hol}_0(\gamma_{\beta}) = T_0(-2\pi i(R/2)^{\frac{1}{3}}, 0).$$

The expression of $T_0(\cdot, \cdot)$ given in §2.3.1 then yields the required result.

(2) In the three cases of the required equality, $\theta = \arg(-(R/2)^{\frac{1}{3}}) = \lim \arg(\beta(t))$ takes values in $\left(\frac{5\pi}{6}, \pi\right)$, $\{\pi\}$ and $\left[\pi, \frac{7\pi}{6}\right]$, respectively. In the first and last cases, the required equality follows from Proposition 2.4 (1), whereas in the second case it follows from Proposition 2.4 (2).

The hyperbolic/planar nature of $\text{hol}_0(\gamma_{\beta})$, as well as the attracting/repelling/saddle nature of each fixed point, depends on relative largeness of the eigenvalues, as summarized in the following table.
Each column of the table stands for a case with \( \arg(R) \) contained in the interval
given in the first row; a "+" (resp. "−", "0") sign on the row of \( X_i \) means that \( \lambda_i \) is
the largest (resp. the smallest, intermediate) amongst the three eigenvalues. Thus
two "−" or two "+" on the same column means \( \text{hol}_0(\gamma_p) \) is planar.

From the table we read off the following

Corollary 7.13. If \( \Re(R) \neq 0 \) then \( \text{hol}_0(\gamma_p) \) is hyperbolic. Let \( \hat{X}_+, \hat{X}_- \) and \( \hat{X}_0 \) denote the
attracting, repelling and saddle fixed points of \( \text{hol}_0(\gamma_p) \), respectively. Then the \( \text{Lim}_0(\beta) \)'s
can be described as follows.

- If \( \Re(R) < 0 \), then \( \text{Lim}_0(\beta) = \hat{X}_0 \) for any \( \beta \in \mathcal{C} \).
- If \( R \in \mathbb{R}_+ \), then \( \{ \text{Lim}_0(\beta) \}_{\beta \in \mathcal{C}} \) is the interior of a segment in \( \mathbb{P}(E_m) \) joining \( \hat{X}_+ \)
  and \( \hat{X}_- \).
- If \( \Re(R) > 0 \) and \( \Im(R) < 0 \), then \( \text{Lim}_0(\beta) = \hat{X}_+ \) for any \( \beta \in \mathcal{C} \).
- If \( \Re(R) > 0 \) and \( \Im(R) > 0 \), then \( \text{Lim}_0(\beta) = \hat{X}_- \) for any \( \beta \in \mathcal{C} \).

7.2.5. Comparing parallel transports: the case \( \Re(R) \neq 0 \). As in §7.1.4, using the equi-
lateral frame and two-pointed parallel transport map, the comparison transformation \( P_{\beta}(t) := T(\beta_{0,1})^{-1}T_0(\beta_{0,1}) \in \text{SL}(E_0) \) is expression as

\[
P_{\beta}(t) = T(0, \beta(t)) T_0(\beta(t), 0).
\]

The following proposition is a counterpart to Theorem 7.8.

Proposition 7.14. Suppose \( \Re(R) \neq 0 \). Then the limit \( \hat{P}_\beta := \lim_{t \to +\infty} P_{\beta}(t) \) exists and takes the same value \( P \in \text{SL}(E_0) \) for any \( \beta \in \mathcal{C} \).

Proof. The proof is basically the same as parts (1) and (2) of Theorem 7.8, so we only
give here a sketch.

The derivative of \( P_{\beta}(t) \) is given by \( \frac{\partial}{\partial \beta} P_{\beta}(t) = P_{\beta}(t) N(t) \), where

\[
N(t) = \text{Ad}_{T_0(0, \beta(t))}(A(\hat{\beta}(t)) - A_0(\hat{\beta}(t))).
\]

Note that \( |\beta(t)| \leq t \) because \( \beta(0) = 0 \) and \( |\hat{\beta}(t)| \equiv 1 \). The asymptotic argument \( \theta = \lim_{t \to +\infty} \arg(\beta(t)) = \arg(-R/2)^{\frac{1}{2}} \) is not an odd multiple of \( \frac{\pi}{2} \) because \( \Re(R) \neq 0 \).

By virtue of fact (1) in §2.3.2, the spectral radius of \( \text{Ad}_{T_0(0, \beta(t))} \) is controlled by
\( e^{(2\sqrt{\gamma} - \delta)t} \) for some \( \delta > 0 \), while Lemma 7.11 implies that \( \|A(\hat{\beta}(t)) - A_0(\hat{\beta}(t))\| \) is
controlled by \( e^{-2\sqrt{\gamma}t} \). So \( \|N(t)\| \) decays exponentially, hence \( P_{\beta}(t) \) converge by
Lemma B.1 in [DW14].

In order to prove that \( P_{\beta_0}^{-1} P_{\beta_0} = \text{id} \) for any \( \beta_0, \beta_1 \in \mathcal{C} \), define \( \beta_s(t) \) and \( Q(s, t) \) in
the same way in the proof of Theorem 7.8, so that we have \( P_{\beta_0}^{-1} P_{\beta_0} = \lim_{t \to +\infty} Q(1, t) \) and

\[
\frac{\partial}{\partial \beta} Q(s, t) = M(s, t) Q(s, t),
\]
where
\[ M(s, t) = \text{Ad}_{\mathbf{T}^0(0, \beta_3(t))} \left( A_0 \left( \frac{\partial}{\partial s} \beta_3(t) \right) - A \left( \frac{\partial}{\partial s} \beta_3(t) \right) \right). \]
Noting that \[ \left| \frac{\partial}{\partial s} \beta_3(t) \right| = |\beta_0(t) - \beta_1(t)| \] is bounded, Lemma 7.11 again implies \[ \|A(\beta(t)) - A_0(\beta(t))\| = O(e^{-2\sqrt{3}t}) \], while the spectral radius of \( \text{Ad}_{\mathbf{T}^0(0, \beta_3(t))} \) is \( O(e^{(2\sqrt{3}-\delta)t}) \) for the same reason as above. Therefore \( \max_{s \in [0, 1]} \|M(s, t)\| \) decays exponentially and Lemma 7.9 gives \( \lim_{t \to +\infty} Q(1, t) = 0 \) as we wished.

7.2.6. Comparing parallel transports: the case \( \text{Re}(R) = 0 \). Suppose \( \text{Re}(R) = 0 \), so that \( \text{hol} \circ \gamma \in \text{SL}(E_0) \) is planar.

In this case, every \( \beta \in \mathcal{E} \) is eventually a ray with asymptotic argument \( \frac{2\pi}{3} \) or \( \frac{4\pi}{3} \). Such rays are “unstable” in the same sense as in the case of higher order poles, that is, the proof for existence of \( \lim_{t \to +\infty} P_\beta(t) \) based on ODE asymptotics (Proposition 7.14) fails. However, modifying the method, we can still obtain some asymptotic information on \( P_\beta(t) \), as stated in the next proposition.

Let \( G \subset \text{SL}(E_0) \) denote the subgroup consisting of unipotent elements which commute with \( \text{hol}_0 \circ \gamma \) and fix the point \( \text{Lim}_0(\beta) = X_0 \). In terms of matrices under the basis \( (e_1(0), e_2(0), e_3(0)) \),

\[
G = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ \ast & \ast & 1 \end{pmatrix} : R \in i\mathbb{R}_- \right\} \quad \text{if } R \in i\mathbb{R}_-.
\]

\[
G = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ \ast & \ast & 1 \end{pmatrix} : R \in i\mathbb{R}_+ \right\} \quad \text{if } R \in i\mathbb{R}_+.
\]

Let \( \Pi : \text{SL}(E_0) \to \text{SL}(E_0)/G \) be the projection.

**Proposition 7.15.** There exists a \( G \)-valued function \( \Xi(t) \) on \([0, +\infty)\) with the following properties.

- The only non-vanishing off-diagonal entry of \( \Xi(t) \), denoted by \( F(t) \), is a primitive of a bounded smooth function. In particular, \( |F(t)| = O(t) \).
- For any \( \beta \in \mathcal{E} \), \( P_\beta(t) \Xi(t)^{-1} \) admits a limit in \( \text{SL}(E_0) \), denoted by \( P_\beta' \).

Moreover, the left \( G \)-coset \( \Pi(P_\beta) \in \text{SL}(E_0)/G \) is independent of \( \beta \).

The proof is based on the following generalization of Lemma B.1 in [DW14].

**Lemma 7.16.** Let \( P(t) \) be a \( \text{SL}(3, \mathbb{R}) \)-valued \( C^1 \)-function and \( N(t) \) a \( \text{sl}(3, \mathbb{R}) \)-valued continuous function on \([0, +\infty)\) such that \( \dot{P}(t) = P(t)N(t) \). Denote the \((i, j)\)-entry of \( N(t) \) by \( N_{ij}(t) \). Let \( f(t) := N_{i_0j_0}(t) \) be an off-diagonal entry and \( F(t) \) be a primitive of \( f(t) \). Suppose

\[
\int_{0}^{+\infty} |N_{i_j}(t) \cdot F(t)| \, dt < +\infty,
\]
for any \((i, j) \neq (i_0, j_0)\) and \( l = 0, 1, 2 \). Then \( P(t) \) is asymptotic to

\[
\Xi(t) := \exp \left( F(t)E_{i_0j_0} \right)
\]
in the sense that \( P(t) \Xi(t)^{-1} \) admits a limit in \( \text{SL}(3, \mathbb{R}) \) as \( t \to +\infty \).
Proof. We compute the derivative of $R(t) := P(t) \Xi(t)^{-1}$:

$$
\dot{R} = \dot{P} \Xi^{-1} - P \Xi^{-1} \dot{\Xi} \Xi^{-1} = R(\Xi N \Xi^{-1} - \dot{\Xi} \Xi^{-1}).
$$

A straightforward computation shows that each entry of $\Xi N \Xi^{-1} - \dot{\Xi} \Xi^{-1}$ is a sum of terms of the form $N_{ij}(t) \cdot F(t)$ with $(i, j) \neq (i_0, j_0)$ and $l = 0, 1, 2$. Therefore

$$
\int_0^{+\infty} \| \Xi N \Xi^{-1} - \dot{\Xi} \Xi^{-1} \| dt < +\infty
$$

by assumption. Applying Lemma B.1 from [DW14], we conclude that $R(t)$ admits a limit. \qed

Proof of Proposition 7.15. We will actually prove the following assertions.

1. For each $\beta \in \mathcal{C}$, there exists $\Xi_\beta(t) \in G$ satisfying the two requirements.
2. Fix a $\Xi_\beta(t)$ as above for each $\beta \in \mathcal{C}$ and set $P_\beta' := \lim_{t \to +\infty} P_\beta(t) \Xi_\beta(t)^{-1}$. Then

$$
\Xi_\beta_0(t) P_\beta_1(t)^{-1} P_\beta_0(t) \Xi_\beta_0(t)^{-1}
$$

admits a limit in $G$ for any $\beta_0, \beta_1 \in \mathcal{C}$.

To deduce the proposition from these assertions, note that

$$
\Xi_\beta_1(t) P_\beta_1(t)^{-1} P_\beta_0(t) \Xi_\beta_0(t)^{-1} = \left( \Xi_\beta_1(t) \Xi_\beta_0(t)^{-1} \right) \left( \Xi_\beta_0(t) P_\beta_1(t)^{-1} P_\beta_0(t) \Xi_\beta_0(t)^{-1} \right).
$$

The left-hand side converges to $P_\beta_0' P_\beta_0'$ by (1), while the first factor on the right-hand side is in $G$ and the second factor converges to an element in $G$ by (2). Thus, we deduce two consequences: $\Xi_\beta_1(t) \Xi_\beta_0(t)^{-1}$ converges, and $P_\beta_1' P_\beta_0'$ is contained in $G$. The first consequence implies that we can take the same $G$-valued function $\Xi(t) := \Xi_\beta_0(t)$ as $\Xi_\beta(t)$ for every $\beta \in \mathcal{C}$; the second implies the last statement in the proposition.

In order to prove the assertions, we define $P_\beta(t)$, $N(t)$, $Q(s, t)$ and $M(s, t)$ in the same way as in the proof of Proposition 7.14 and follow the same line of arguments.

First consider the case $R \in i\mathbb{R}_{++}$, so that $\theta = \arg(-R/2) = \frac{\pi}{2}$. Looking at the exponential growth rate of each eigenvalue of $\text{Ad}_{P_\beta}(0, \beta(t))$ as in the proof of Proposition 7.8 (3), we see that the biggest eigenvalue is asymptotic to $e^{2\sqrt{3}t}$ and is associated to the eigenspace $\mathbb{R}E_{31}$, while all the other eigenvalues are controlled by $e^{(2\sqrt{3} - \delta)t}$. Therefore, every entry of $N(t)$ except the $(3, 1)$-one decays to 0 exponentially while $f(t) := N_{31}(t)$ is bounded. We let $F(t)$ be a primitive of $f(t)$ and define

$$
\Xi_\beta(t) := \exp(F(t)E_{31}).
$$

Then $P_\beta(t) \Xi_\beta(t)^{-1}$ converges by Lemma 7.16 as required. Assertion (1) is proved. Put

$$
Q'(s, t) = \Xi_\beta_0(t) Q(s, t) \Xi_\beta_0(t)^{-1}, \quad M'(s, t) = \Xi_\beta_0(t) M(s, t) \Xi_\beta_0(t)^{-1}.
$$

Assertion (2) just says that $Q'(1, t)$ has a limit in $G$ as $t \to +\infty$. But we have

$$
\frac{\partial}{\partial t} Q'(s, t) = M'(s, t) Q'(s, t).
$$

Similarly as in the proof of Proposition 7.8 (3), applying Lemma 7.9 to $X = E_{31}$ and $f = M'_{31}$ we see that assertion (2) holds if $M'(s, t)$ fulfills the following conditions:
(a) If \((i, j) \neq (3, 1)\) then \(|M'_{ij}(s, t)| = O(e^{-\delta t})\) for some \(\delta > 0\);
(b) \(\sup_{t \in [0, +\infty)} \int_0^1 |M'_{31}(s, t)| ds < +\infty\)

Using the expression
\[
\Xi_{\beta_0}(t) = \begin{pmatrix}
1 & 1 \\
O(t) & 1
\end{pmatrix},
\]
we see that \(M'(s, t)\) satisfies the above conditions if and only if \(M(s, t)\) does. It is thus sufficient to verify (a) and (b) for \(M(s, t)\) instead of \(M'(s, t)\).

As before, by Lemma 7.11 we have
\[
(7.16) \quad \|A_0(\frac{\partial}{\partial s}\beta_s(t)) - A(\frac{\partial}{\partial s}\beta_s(t))\| = O(e^{-2\sqrt{3} t}).
\]

To analyze the effect of \(\text{Ad}_{T_0(0, \beta_s(t))}\), we put \(\theta_{s,t} := \text{arg}(\beta_s(t))\). With the notations from §2.3.2, the eigenvalue of \(\text{Ad}_{T_0(0, \beta_s(t))}\) on \(E_{ij}\) is
\[
\lambda_{ij}(s, t) := \exp(\varpi_{ij}(\theta_{s,t})|\beta_s(t)|).
\]
Since \(\beta_0(t)\) and \(\beta_1(t)\) are parallel rays with asymptotic argument \(\frac{7\pi}{6}\) for \(t\) sufficiently large, \(\theta_{s,t}\) converges to \(\frac{7\pi}{6}\) uniformly in \(s\). Noting that \(|\beta_s(t)| \leq t\), fact (2) in §2.3.2 implies that if \((i, j) \neq (3, 1)\) then \(\lambda_{ij}(s, t) \leq e^{(2\sqrt{3} - \delta) t}\). Combining this with (7.16), we get condition (b).

As for \(\lambda_{31}(s, t)\), we have
\[
\varpi_{31}(\theta) = 2 \text{Re}(e^{2\pi i / 3} e^{\theta i} - e^{\theta i}) = 2(\cos(\theta + \frac{2\pi}{3}) - \cos(\theta)) = 2\sqrt{3} \cos\left(\theta - \frac{7\pi}{6}\right).
\]

Let \(\alpha\) be the ray parallel to \(\beta_0\) passing through 0 and let \(y_i\) be the signed distance from \(\alpha\) to \(\beta_i\) \((i = 0, 1)\). We have
\[
|\theta_{s,t} - \frac{7\pi}{6}| \geq c \left| \tan(\theta_{s,t} - \frac{7\pi}{6}) \right| \approx c \left( \frac{(1-s)y_0 + sy_1}{t} \right)
\]
for some constant \(c > 0\) when \(t\) is sufficiently large. Therefore,
\[
-2\sqrt{3} + \varpi_{31}(\theta_{s,t}) = -2\sqrt{3}(1 - \cos(\theta - \frac{7\pi}{6})) = -4\sqrt{3} \sin^2\left( \frac{\theta_{s,t} - \frac{7\pi}{6}}{2} \right) \leq -c' \left( \frac{(1-s)y_0 + sy_1}{t} \right)^2
\]
for some \(c'\). Combining these with (7.16), we get
\[
|M_{31}(s, t)| \leq C e^{\lambda_{31}(s, t)} e^{-2\sqrt{3} t} \leq C e^{(2\sqrt{3} + \varpi_{31}(\theta_{s,t})) t} \leq C \exp\left( -c' \left( \frac{(1-s)y_0 + sy_1}{t} \right)^2 \right).
\]
The last term, viewed as a family of functions of \(s\) parametrized by \(t\), converges uniformly to 1 as \(t \to +\infty\). This implies property (b) and concludes the proof of assertion (2) in the case \(R \in \mathbb{R}_+\).

The proof for the case \(R \in \mathbb{iR}_-\) is almost identical, the only difference being that \(\theta_{s,t}\) now tends to \(\frac{7\pi}{6}\), hence the fastest-growing eigenvalue is \(\lambda_{32}(s, t)\), as the table in §2.3.2 shows.

\(\square\)

7.2.7. Determining the end holonomy. From the above obtained information on comparison transformations, we can derive an expression of \(\text{hol}_P\), as stated in the next theorem. It contains Theorem 1.2 from the introduction.

Here \(P\) and \(P'\) are given by Proposition 7.14 and Proposition 7.15.

Theorem 7.17 (Holonomy of third order pole).
(1) If \( \text{Re}(R) \neq 0 \) then the \( \lambda_i \)'s are distinct and
\[
\text{hol}_\beta = P\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 \end{pmatrix} P^{-1};
\]

(2) If \( R \in i\mathbb{R}_- \) then \( \lambda_1 < \lambda_2 = \lambda_3 \) and there is \( a \in \mathbb{R} \) such that for any \( \beta \in \mathcal{C} \),
\[
\text{hol}_\beta = P_\beta \begin{pmatrix} \lambda_1 & \lambda_2 \\ a & \lambda_2 \end{pmatrix} P_\beta^{-1}.
\]

(3) If \( R \in i\mathbb{R}_+ \) then \( \lambda_1 = \lambda_3 < \lambda_2 \) and there is \( a \in \mathbb{R} \) such that for any \( \beta \in \mathcal{C} \),
\[
\text{hol}_\beta = P_\beta \begin{pmatrix} \lambda_1 & \lambda_2 \\ a & \lambda_1 \end{pmatrix} P_\beta^{-1}.
\]

Remark that if statements (2) and (3) hold for one \( \beta \in \mathcal{C} \) then they hold for any other \( \beta \) as well, because changing \( \beta \) amounts to right-multiplying \( P_\beta \) by an element of \( G \), but any element of \( G \) commutes with
\[
\begin{pmatrix} \lambda_1 & \lambda_2 \\ a & \lambda_2 \end{pmatrix} \quad \text{(resp.} \quad \begin{pmatrix} \lambda_1 & \lambda_2 \\ a & \lambda_1 \end{pmatrix}\).
\]

when \( R \in i\mathbb{R}_- \) (resp. \( R \in i\mathbb{R}_+ \)).

**Proof.** In this proof we work on the punctured neighborhood \( U = \{0 < |z| < a\} \) of \( p \) directly rather than using the local model \( H/\sim \).

Recall that \( t \mapsto \beta(t) \) is the parametrization of \( \beta \) by arc-length with respect to \( |b|^{\frac{2}{3}} \) and we have, by definition,
\[
(7.17) \quad P_\beta(t) = T(\beta^{-1}_{[0,t]} \beta_{[0,t]}).
\]

For each \( t \), let \( \gamma_t \) denote the oriented loop based at \( \beta(t) \) such that \( \gamma_t \) runs over the circle \( \{|z| = |\beta(t)|\} \) clockwise. The loop \( \beta^{-1}_{[0,t]} \cdot \gamma_t \cdot \beta_{[0,t]} \) is homotopic to \( \tilde{\beta}_0 \), thus
\[
(7.18) \quad \text{holo}(\gamma_t) = T_0(\beta^{-1}_{[0,t]} \beta_{[0,t]}).
\]

Put \( \tilde{\beta} := \beta \cdot \gamma_t \). Suppose \( \gamma_t \) has length \( t_0 \) with respect to \( |b|^{\frac{2}{3}} \). Then \( \beta_{[0,t_0+t]} \) is homotopic to \( \beta_{[0,t]} \cdot \gamma_t \) thus
\[
(7.19) \quad P_\beta(t + t_0)^{-1} = T_0(\beta^{-1}_{[0,t+t_0]} \beta_{[0,t+t_0]}) = T_0(\beta^{-1}_{[0,t]} \gamma_t \cdot \beta_{[0,t]}).
\]

Concatenating (7.17), (7.18) and (7.19), we obtain, for any \( t \),
\[
P_\beta(t) \cdot \text{holo}(\gamma_t) \cdot P_\beta(t + t_0)^{-1} = T(\beta^{-1}_{[0,t]} \gamma_t \cdot \beta_{[0,t]}) = \text{hol}_\beta P_\beta(t + t_0)^{-1} \cdot \text{holo}(\gamma_t) \cdot \text{holo}(\gamma_{t_0}) \cdot \text{holo}(\gamma_{t+t_0})^{-1} = \text{hol}_\beta P_\beta(t + t_0)^{-1}.
\]

When \( \text{Re}(R) \neq 0 \), letting \( t \) tend to \( +\infty \) and using Proposition 7.14, we get part (1) of the theorem.

When \( \text{Re}(R) = 0 \), we take the \( G \)-valued function \( \Xi(t) \) provided by Proposition 7.15 into account. Members of \( G \) commute with \( \text{holo}(\gamma_t) \) by definition, hence (7.19) yields
\[
(7.20) \quad P_\beta(t) \Xi(t)^{-1} \cdot \text{holo}(\gamma_t) \Xi(t) \Xi(t + t_0)^{-1} \cdot \Xi(t + t_0) P_\beta(t + t_0)^{-1} = \text{hol}_\beta P_\beta(t + t_0)^{-1}.
\]

The non-vanishing off-diagonal entry of \( \Xi(t) \) is a primitive of some bounded function, so the non-vanishing off-diagonal entry of \( \Xi(t) \Xi(t + t_0)^{-1} \) is bounded. As
a result, $\Xi(t) \Xi(t + t_0)^{-1}$ converges to some $\Xi_0 \in G$ at least along a subsequence $(t_n) \subset [0, +\infty)$. Letting $t$ tend to $+\infty$ along this subsequence in (7.20), we get

$$\text{hol}_\beta = P_\beta' \cdot \text{hol}_0(\gamma_\beta) \Xi_0 \cdot P_\beta'^{-1}.$$  

The last statement in Proposition 7.15 implies $P_\beta' = P_\beta' \Xi_1^{-1}$ for some $\Xi_1 \in G$, so we further get

$$\text{hol}_\beta = P_\beta' \cdot \text{hol}_0(\gamma_\beta) \Xi_0 \Xi_1 \cdot P_\beta'^{-1}.$$  

This establishes the required expressions in part (2) and (3) of the theorem. \hfill $\square$

Remark 7.18. Eq. (7.19) has nothing to do with the order of pole, hence it also provides some information on $\text{hol}_\beta$ for higher order poles. Suppose that $p$ is a pole of order $\geq 4$ and let $\beta \in \mathcal{C}$ be a stable path. Taking the limit $t \to +\infty$ in (7.19), we see that $\text{hol}_\beta$ is conjugate to $\text{hol}_0(\gamma_\beta) \cdot P_\beta^{-1} P_\beta'$.

Assuming $\beta \in \mathcal{J}_{0,1}$ and applying Theorem 7.8 (3) consecutively, we can write $P_\beta^{-1} P_\beta'$ as a product $u_1 u_2 u_3 \cdots u_n u_n^+$, where $u_n^+ \in G_k$.

7.2.8. Proof of Theorem 7.10. If $\text{Re}(R) \neq 0$ then Corollary 7.17 (1) says that $\text{hol}_\beta$ is conjugate to $\text{hol}_0(\gamma_\beta)$ through $P$, whereas $\text{Lim}(\beta)$ is the translate of $\text{Lim}_0(\beta)$ by $P$ because $\text{Lim}(\beta) = P_\beta(\text{Lim}_0(\beta))$ (see §7.1.5). Therefore, the statements about $\text{Lim}(\beta)$ in parts (1) and (2) of Theorem 7.10 are consequences of Corollary 7.13.

If $\text{Re}(R) \in i\mathbb{R}^*$, the above argument fails because $P_\beta = \lim P_\beta(t)$ does not make sense. But we can use the $\Xi(t)$ provided by Proposition 7.15 to remedy it. Using the expressions

$$\text{dev}_{0}(\beta_0^{-1}) = \begin{bmatrix} e^{2 \text{Re}(\beta(t))} \\ e^{2 \text{Re}(\omega^2(t))} \\ e^{2 \text{Re}(\omega(t))} \end{bmatrix}, \quad \Xi(t) = \begin{pmatrix} 1 & 1 \\ O(t) & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 \\ O(t) & 1 \end{pmatrix}$$

and, for $t$ sufficiently large,

$$\beta(t) = \begin{cases} e^{\frac{3\pi i}{2}} t + \zeta_0 & \text{if } R \in i\mathbb{R}_-, \\ e^{\frac{\pi i}{2}} t + \zeta_0 & \text{if } R \in i\mathbb{R}_+, \end{cases}$$

we see that

$$\lim_{t \to +\infty} \Xi(t) \text{dev}_{0}(\beta_0^{-1}) = X_3$$

(a short reasoning: $\text{dev}_{0}(\beta_0^{-1})$ converges exponentially fast to $\text{Lim}_0(\beta) = X_3$, hence operating it by the sub-linearly growing $\Xi(t)$ does not effect the limit). Therefore,

$$\text{Lim}(\beta) = \lim_{t \to +\infty} P_\beta(t) \text{dev}_{0}(\beta_0^{-1}) = \lim_{t \to +\infty} P_\beta(t) \Xi(t)^{-1} \lim_{t \to +\infty} \Xi(t) \text{dev}_{0}(\beta_0^{-1}) = P_\beta'(X_3).$$

But this is exactly the double fixed point of $\text{hol}_\beta$ as Theorem 7.17 implies.

It remains to prove the statements about the type of end.

In the case $R \in \mathbb{R}_+$, $\text{hol}_\beta$ is hyperbolic and the $\text{Lim}(\beta)$’s form the interior of the principal segment $I$, but a case-by-case check for simple ends and non-simple ends (see Proposition 3.8 and Lemma 3.9) shows that the only situation where $\partial_\beta \Omega$ contain a point in the interior of $I$ is when $\partial_\beta \Omega = I$.

In the case $\text{Re}(R) < 0$, where $\text{hol}_\beta$ is hyperbolic and $\text{Lim}(\beta)$ is the saddle fixed point, Proposition 3.8 (2) implies that $p$ is a triangular end.
In the remaining cases, if hol$_p$ is planar, of course the type of end is immediately determined by Lemma 3.7. Otherwise, we study in addition the asymptotic direction of the path $t \mapsto \text{dev}((y_{0,t})^{-1})$, obtaining the following proposition.

**Proposition 7.19.** Suppose $\text{Re}(R) \geq 0$, $\text{Im}(R) \neq 0$ and assume that hol$_p$ is not planar. Let $I$ denote the principal segment of hol$_p$. Then for any $\beta \in \mathcal{C}$, the path $t \mapsto \text{dev}((y_{0,t})^{-1})$ converges to an endpoint of $I$ and is asymptotic to $I$.

**Proof.** The proof is based on the following simple fact: if $t \mapsto \xi(t)$ is a parametrized curve in $\mathbb{RP}^2$ converging to $x_0 \in \mathbb{RP}^2$ and asymptotic to a ray $l$ issuing from $x_0$, and $a(t)$ is a continuous $\text{SL}(3,\mathbb{R})$-valued function converging to $a_0 \in \text{SL}(3,\mathbb{R})$, then the curve $t \mapsto a(t)(\xi(t))$ converges to $a_0(x_0)$ and is asymptotic to $a_0(l)$.

First consider the case $\text{Re}(R) > 0$, $\text{Im}(R) > 0$, so that $\arg(-(R/2)^t) \in (\pi, \pi/6)$. Proposition 2.4 (1) says that $t \mapsto \text{dev}_0(y_{0,t})$ is asymptotic to $X_{23}$ while converging to $X_3$. By the above mentioned fact, $t \mapsto \text{dev}_0((y_{0,t})^{-1}) = P_\beta(t)(\text{dev}_0((y_{0,t})^{-1}))$ converges to $P(X_3)$ and is asymptotic to $P(X_{23})$. But the matrix expression of hol$_p$ in Theorem 7.17 and the table in §7.2.4 shows that $P(X_{23})$ is a segment joining the attracting and repelling fixed points of hol$_p$, hence can only be the principal segment (otherwise there could not be a path in $\Omega$ asymptotic to it).

The proof for the case $\text{Re}(R) > 0$, $\text{Im}(R) < 0$ is similar.

Let us treat the case $R \in \mathfrak{i}\mathbb{R}_{-}$. Using the expressions of $\text{dev}_0((y_{0,t})^{-1})$ and $\Xi(t)$ exhibited earlier, we not only see that the path

$$t \mapsto \xi_\beta(t) := \Xi(t)\text{dev}_0((y_{0,t})^{-1})$$

converges to $X_3$, but also see that it is asymptotic to $X_{31}$. Noting that

$$\text{dev}_0((y_{0,t})^{-1}) = P_\beta(t)\Xi(t)^{-1}(\xi_\beta(t)),$$

we apply the above mentioned fact and conclude that $t \mapsto \text{dev}_0((y_{0,t})^{-1})$ is asymptotic to $P'_\beta(X_{31})$. The latter is a segment joining the two fixed points of hol$_p$ as Theorem 7.17 implies, hence it can only be the principal segment. \hfill $\square$

In view of this proposition, the following lemma implies that $p$ is a geodesic end in the remaining cases, completing the discussion of third order poles.

**Lemma 7.20.** Let $(\text{dev}, \text{hol})$ be a convex projective structure on $\Sigma$ such that hol$_p$ is either hyperbolic or quasi-hyperbolic. Let $I$ be the principal segment of hol$_p$. Suppose there exists a path $t \mapsto \beta(t) \in \Sigma$ issuing from $m$ and converging to $p$, such that $t \mapsto \text{dev}((y_{0,t})^{-1})$ converges to an endpoint of $I$ and is asymptotic to $I$. Then $p$ is a geodesic end.

We remark that, in contrast to the situation described in the lemma, when $p$ is a triangular end or a non-simple end with hyperbolic holonomy (cf. Lemma 3.9), one can still construct $\beta$ such that $x = \lim(\beta)$ is an endpoint of the principal segment – just construct the developed path $t \mapsto \text{dev}((y_{0,t})^{-1})$ first and then take the quotient by $\tau_1(\Sigma)$. However, $\text{dev}((y_{0,t})^{-1})$ should be asymptotic to the segment joining $x$ and the saddle fixed point, otherwise $\beta$ does not converges to $p$.

**Proof.** By definition of $\partial_p \Omega$, it is sufficient to prove that, for any punctured neighborhood $U$ of $p$, the closure of $\text{dev}(\tilde{U})$ contains $I$. 

Let $C$ denote the trajectory of $\text{dev}(\beta^{-1}_{[0,t]})$ in $\text{dev}(\tilde{U})$. The assumption that $C$ is asymptotic to $I$ and dynamical properties of (quasi-)hyperbolic projection transformations imply that either the accumulation set $\text{Accum}(\text{hol}_{\tilde{p}}^k(C))$ for $k \to +\infty$ or $\text{Accum}(\text{hol}_{\tilde{p}}^k(C))$ (see §3.1.2 for the notation) is $I$. Since $\text{dev}(\tilde{U})$ is $\text{hol}_{\tilde{p}}$-invariant, the $\text{hol}_{\tilde{p}}^k(C)$'s are contained in $\text{dev}(\tilde{U})$, hence the accumulation set $I$ is in the closure of $\text{dev}(\tilde{U})$, as required.

7.3. Poles of order $\leq 2$. The following theorem completes the proof of statement (III) from the introduction, hence finishes the proof of Theorem 1.1.

**Theorem 7.21 (Loftin, Benoist-Hulin).** Suppose $p$ is a pole of order $\leq 2$ or a removable singularity of $b$. Then $p$ is a cusp of the convex projective structure.

This theorem was proved by Loftin [Lof04] and Benoist-Hulin [BH13] using different methods. We outline both (short) proofs for convenience of the reader. Note that Loftin only treated the case where $\Sigma$ has negative Euler characteristic, so we add to it a discussion for annuli.

**First proof.** Lofin [Lof04] showed that if $p$ is a pole of order $\leq 2$ then the eigenvalues of $\text{hol}_{\tilde{p}}$ are all 1 (in fact, he proved Theorem 1.2 from the introduction for all poles of order $\leq 3$ in a unified way, the result that we quote here being the $R = 0$ case), so $\text{hol}_{\tilde{p}}$ is parabolic by the classification in §3.2.1.

Let us show that $p$ is a cusp of the convex projective structure. A case-by-case check for simple ends and non-simple ends (see Proposition 3.8 and Lemma 3.9) shows that the only situation where $\text{hol}_{\tilde{p}}$ is parabolic and $p$ is not a cusp is case (c) of Lemma 3.9. Suppose by contradiction that this is the case, so that we can assume that $\Sigma = \mathbb{C}^*$, $p = 0$ and the other puncture $\infty$ is a cusp. But $\infty$ is a pole of order $\geq 4$ because the cubic canonical bundle $K^3$ of $\mathbb{CP}^1$ has degree $-6$. Our earlier result says that $\infty$ is a polygonal end, a contradiction. □

**Second proof.** A straightforward computation shows that a punctured neighborhood $U$ of $p$ has finite volume with respect to the metric $|b|^2$ if and only if $p$ is a removable singularity or a pole of order $\leq 2$. In this case, $U$ has finite volume with respect to the Hilbert metric as well by Corollary 2.2 (1). But ends with finite Hilbert volume are exactly cusps as shown in [Mar12]. □

**Appendix A. Estimates for solutions to the equation** $\Delta u = 4e^u - 4e^{-2u}$

In this appendix, we study bounded non-negative solutions to the PDE

(A.1) $\Delta u = 4e^u - 4e^{-2u}$

first on a half-plane $H \subset \mathbb{C}$, then on a specific surface $X$ obtained by gluing three half-planes by translations. $X$ is a part of the local model for $(\Sigma, b)$ constructed in §5.3. In both settings, $\Delta = 4 \partial_\zeta \partial_{\bar{\zeta}}$ is the usual Laplacian with respect to the natural coordinate $\zeta$ on each half-plane.

The results presented here are mostly adaptations of results from §5.4 of [DW14].

**Lemma A.1.** Let $H \subset \mathbb{C}$ be a half-plane (i.e. a region whose boundary is a straight line). Let $u \in C^2(H) \cap C^0(\bar{H})$ be a bounded non-negative function satisfying Eq.(A.1). Then
(1) There exists a constant $C$ only depending on the upper bound of $u$, such that
\[ u(\zeta) \leq C \operatorname{dist}(\zeta, \partial H)^{\frac{1}{2}} e^{-2\sqrt{3} \operatorname{dist}(\zeta, \partial H)} \]
for any $\zeta \in H$. Here $\operatorname{dist}(\zeta, \partial H)$ is the distance from $\zeta$ to $\partial H$, measured by the metric $|d\zeta|^2$.

(2) If $u$ is invariant under a translation $\zeta \mapsto \zeta + a$ which preserves $H$, then for any $\zeta \in H$ we have
\[ u(\zeta) \leq C' e^{-2\sqrt{3} \operatorname{dist}(\zeta, \partial H)}, \]
where the constant $C'$ only depends on the upper bound of $u$.

(3) If $u|_{\partial H}$ is integrable and $\lim_{|\zeta| \to +\infty} u(\zeta) = 0$, then for any $\zeta \in H$ we have
\[ u(\zeta) \leq C'' \operatorname{dist}(\zeta, \partial H)^{-\frac{1}{2}} e^{-2\sqrt{3} \operatorname{dist}(\zeta, \partial H)}, \]
where the constant $C''$ only depends on the upper bound of $u$ and the integral of $u|_{\partial H}$.

**Proof.** Eq. (A.1) and the three statements are all invariant under translations and rotations, hence we can assume without loss of generality that $H$ is the right half-plane $H = \{ \zeta \in \mathbb{C} \mid \Re(\zeta) > 0 \}$, so that $\operatorname{dist}(\zeta, \partial H) = \Re(\zeta)$.

(1) We shall prove the following result.

For any nonnegative bounded function $u$ satisfying (A.1) on a disk $D \subset \mathbb{C}$ of radius $r$, the value of $u$ on the center of $D$ is bounded by $Cr^\frac{1}{2} e^{-2\sqrt{3} r}$, where $C$ only depends on the upper bound of $u$.

The required estimate follows from this immediately by considering the biggest disk in $H$ containing $\zeta$.

Eq. (A.1) is invariant under translations, so we can assume $D = \{ |\zeta| < r \}$. Let $M$ be a upper bound of $u$ and let $v$ be the solution to the Dirichlet problem
\[ \Delta v = 12v \text{ in } D, \quad v = M \text{ on } \partial D. \]

Using the maximum principle, we see that $u \leq v$ over $D$: otherwise, $u - v$ is non-positive on $\partial D$ but positive somewhere in $D$, hence takes positive maximum at some $\zeta_0 \in D$; on the other hand, since $e^x - e^{-2x} \geq 12x$ whenever $x \geq 0$, we have
\[ \Delta(u - v) = 4e^u - 4e^{-2u} - 12v \geq 12(u - v), \]
thus $\Delta(u - v)(\zeta_0) > 0$, contradicting maximality of $(u - v)(\zeta_0)$.

But $v$ can be written as
\[ v(\zeta) = \frac{M I_0(2\sqrt{3} |\zeta|)}{I_0(2\sqrt{3} r)} \]
where $I_0(x)$ is the modified Bessel function of the first kind and has an asymptotic expansion
\[ I_0(x) = \frac{e^x}{\sqrt{2\pi x}} (1 + O(x^{-1})). \]

This yields the required estimate
\[ u(0) \leq v(0) \leq \frac{M I_0(0)}{I_0(2\sqrt{3} r)} \leq Cr^\frac{1}{2} e^{-2\sqrt{3} r}. \]

(2) Set $v(\zeta) := Me^{-2\sqrt{3} \Re(\zeta)}$, where $M$ is a upper bound of $u$. Then $v$ satisfies $\Delta v = 12v$. Let us prove that $u \leq v$ using the maximum principle in a similar way as above.
Part (1) implies that \( u \) tends to 0 as \( \text{Re}(\zeta) \to +\infty \). The function \( v \), and hence \( u - v \), has this property as well. Moreover, \( u - v \) is invariant under a translation \( \zeta \mapsto \zeta + ia \) \((a \in \mathbb{R}^*)\). Therefore, if we suppose by contradiction that \( u - v \) takes a positive value \( \delta \) somewhere in \( H \), then the quotient of the set \( \{ \zeta \in H \mid (u - v)(\zeta) \geq \delta \} \) by the translation is compact. It follows that \( u - v \) takes positive maximum at some point \( \zeta_0 \in H \). This leads to a contradiction as before.

(3) Let \( v \in C^2(H) \cap C^0(\overline{H}) \) be the solution to the Dirichlet problem

\[ \Delta v = 12v, \quad v|_{\partial H} = u|_{\partial H} \]

such that \( \lim_{|\zeta| \to +\infty} v(\zeta) = 0 \). A similar maximum principle argument as above shows \( u \leq v \).

As shown in the proof of Lemma 5.8 in [DW14], \( v \) can be expressed as

\[ v(\zeta) = \int_{\partial H} \frac{2\sqrt{3} \text{Re}(\zeta)}{\pi|\zeta - \xi|} K_1(2\sqrt{3}|\zeta - \xi|) u(\xi) d\xi, \]

where \( K_1(x) \) is the modified Bessel function of the second kind. Using the asymptotic expansion

\[ K_1(x) = \sqrt{\frac{\pi}{2x}} e^{-x} (1 + O(x^{-1})), \]

we see that \( v \) satisfies the required estimate, hence \( u \) satisfies as well. \( \square \)

Let \( \overline{H} = \{ \zeta \in \mathbb{C} \mid \text{Re}(\zeta) \geq 0 \} \) be the closure of \( H \). Given \( \xi', \xi'' \in \partial H \) with \( \text{Im}(\xi'') < 0 < \text{Im}(\xi') \), we let \( \overline{X} \) denote the surface with boundary obtained by gluing the half-planes \( \overline{H}' = e^{2\pi i/3} \overline{H} \) and \( \overline{H}'' = e^{4\pi i/3} \overline{H} \) to \( \overline{H} \) via translations, as in Figure A.1, such that \( \partial \overline{H}' \) and \( \partial \overline{H}'' \) are glued to \( \partial \overline{H} \) at the points \( \xi' \) and \( \xi'' \), respectively. Let \( X \) denote the interior of \( \overline{X} \).

We view \( H, \overline{H}', \) and \( \overline{H}'' \) as subsets of \( \overline{X} \). There is a natural projection \( \overline{X} \to \mathbb{C} \) mapping \( \overline{H} \subset \overline{X} \) identically to \( \overline{H} \subset \mathbb{C} \). A disk/half-plane in \( \overline{X} \) is by definition a subset of \( \overline{X} \) whose projection to \( \mathbb{C} \) is a disk/half-plane.

![Figure A.1. The translation surface X](image)

**Lemma A.2.** Let \( u \in C^2(X) \cap C^0(\overline{X}) \) be a bounded non-negative function satisfying Eq. (A.1). Then for any \( \zeta \in \overline{H} \subset \overline{X} \) we have

\[ u(\zeta) \leq C|\zeta|^{-\frac{1}{2}} e^{-2\sqrt{3}|\zeta|}, \]

where the constant \( C \) only depends on the upper bound of \( u \).
Proof. Let \( Y \subset \overline{X} \) denote the union of \( \overline{H} \) with the two half-planes in \( \overline{X} \) which project to \( \{ \zeta \in \mathbb{C} \mid \text{Im}(\zeta) \geq \text{Im}(\xi') \} \) and \( \{ \zeta \in \mathbb{C} \mid \text{Im}(\zeta) \leq \text{Im}(\xi'') \} \), respectively. The projection \( \overline{X} \to \mathbb{C} \) is injective on \( Y \), hence we can represent a point in \( Y \) by the coordinate \( \zeta \) of its projection.

We first show that the restriction of \( u \) on \( Y \) decays exponentially in a uniform rate as \( |\zeta| \to +\infty \).

Let \( D_\zeta \) denote the biggest disk in \( \overline{X} \) centered at \( \zeta \in Y \) and let \( r(\zeta) \) be the radius of \( D_\zeta \). In the proof of Lemma A.1 (1), we have shown that

\[
   u(\zeta) \leq C r(\zeta)^{-\frac{1}{2}} e^{-2\sqrt{3} r(\zeta)},
\]

where \( C \) only depends on the upper bound of \( u \). On the other hand, it is easy to see that there is a constant \( c > 0 \) only depending on \( \xi' \) and \( \xi'' \), such that

\[
   r(\zeta) \geq \frac{1}{2} |(\zeta) - c|.
\]

Since the function \( x \mapsto x^{\frac{1}{2}} e^{-2\sqrt{3} x} \) is decreasing whenever \( x \geq x_0 := \frac{1}{4\sqrt{3}} \), for any \( \zeta \in Y \) with \( |\zeta| \geq c + x_0 \), we get

\[
   u(\zeta) \leq C \left( \frac{|\zeta| - c}{2} \right)^{\frac{1}{2}} e^{-\sqrt{3} |(\zeta)| - c}
\]

as required.

To prove the lemma, we consider the biggest half-plane \( H_\theta \subset X \) whose boundary intersects the ray \( e^{\theta i \mathbb{R}_{\geq 0}} \subset \overline{H} \) perpendicularly (where \( \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \)). The estimate (A.2) implies that \( \lim_{|\zeta| \to +\infty} u(\zeta) = 0 \) and, for any \( \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \),

\[
   \int_{\partial H_\theta} u \leq K
\]

for some constant \( K \) only depends on the upper bound of \( u \). Put

\[
   d = \max_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \text{dist}(0, \partial H_\theta).
\]

For any \( \zeta \in \overline{H} \) with \( |\zeta| \geq d \), we have \( \zeta \in \overline{H}_{\arg(\zeta)} \) and

\[
   |\zeta| = \text{dist}(\zeta, \partial H_{\arg(\zeta)}) + \text{dist}(0, \partial H_{\arg(\zeta)}) \leq \text{dist}(\zeta, \partial H_{\arg(\zeta)}) + d.
\]

Applying Lemma A.1 (3) to the half-plane \( \overline{H}_{\arg(\zeta)} \), we get

\[
   u(\zeta) \leq C \text{dist}(\zeta, \partial H_{\arg(\zeta)})^{-\frac{1}{2}} e^{-2\sqrt{3} \text{dist}(\zeta, \partial H_{\arg(\zeta)})}
\]

where \( C \) only depends on \( K \) and the upper bound of \( u \). Combining the last two inequalities, we get the required estimate. \( \square \)

We deduce an estimate for \( |\partial_\zeta u| \) by means of the following well known result.

Lemma A.3 (Gradient estimate for the Poisson equation). For any \( r > 0 \), there exists a constant \( C \) such that any bounded \( C^2 \) function \( u \) defined on a disk \( B_r(z_0) \subset \mathbb{C} \) centered at \( z_0 \) with radius \( r \) satisfies

\[
   |\partial_\zeta u(z_0)| \leq C \left( \sup_{B_r(z_0)} u - \inf_{B_r(z_0)} u + \sup_{B_r(z_0)} |\Delta u| \right)
\]

See e.g. [Jos13] Corollary 1.2.7 for a proof.
Corollary A.4. In each of the estimates in Lemma A.1 and Lemma A.2, one can take the constant to be big enough such that \( |\partial_{\zeta} u| \) satisfies the same estimate.

The corollary is proved by applying Lemma A.3 to a disk of fixed radius centered at each \( \zeta \), taking account of the fact that
\[
\Delta u = 4e^u - 4e^{-2u} \leq ku
\]
for a constant \( k \) only depending on the upper bound of \( u \).

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