Non-Malleable Codes with Leakage and Applications to Secure Communication

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Abstract. Non-malleable codes are randomized codes that protect coded messages against modification by functions in a tampering function class. These codes are motivated by providing tamper resilience in applications where a cryptographic secret is stored in a tamperable storage device and the protection goal is to ensure that the adversary cannot benefit from their tamperings with the device. In this paper we consider non-malleable codes for protection of secure communication against active physical layer adversaries. We define a class of functions that closely model tampering of communication by adversaries who can eavesdrop on a constant fraction of the transmitted codeword, and use this information to select a vector of tampering functions that will be applied to a second constant fraction of codeword components (possibly overlapping with the first set). We derive rate bounds for non-malleable codes for this function class and give two modular constructions. The first construction adapts and provides new analysis for an existing construction in the new setting. The second construction uses a new approach that results in an explicit construction of non-malleable codes. We show applications of our results in securing message communication against active physical layer adversaries in two settings: wiretap II with active adversaries and Secure Message Transmission (SMT) in networks. We discuss our results and directions for future work.

1 Introduction

Non-Malleable codes (NM-codes) provide protection against active adversaries who can tamper with coded messages using a function from a family $F$, of tampering functions. NM-codes were motivated by providing tamper resilience in cryptographic applications such as protection of secret keys that are stored in tamperable storage devices (e.g. smart cards) that can be subjected to physical manipulations that would affect the values of the stored secret. NM-codes ensure the basic security requirement that the tampering (using functions from the function class $F$) cannot be used to generate related cryptographic values (e.g. a digital signature for related keys). Roughly speaking, a coding scheme $(Enc, Dec)$ provides non-malleability with respect to the tampering family $F$ if for any $f \in F$, a codeword $c$ that encodes a message $m$, the decoding of $f(c)$ results in either the original message $m$, or a value $\tilde{m}$ that is unrelated to $m$, and the probability of which of the two happens is independent of $m$. This
property in the application scenario above will ensure that the tampering with the device (stored codeword of the key) will result in either an unchanged key, or a key that is unrelated to the original key (and hence an unrelated digital signature). A slightly stronger notion is strong non-malleability that effectively requires that the decoded message $\tilde{m}$ of a modified codeword $f(c) = \tilde{c}$, where $\tilde{c} \neq c$, be independent of $m$. NM-codes have found other applications in computational cryptography, including construction of non-malleable commitment \cite{31,32}, and domain extension for public key encryption systems \cite{28}. NM-codes have been studied in both information-theoretic and computational setting. In this paper we consider information-theoretic setting.

Traditional protection goals against tampering of codewords are error correction and error detection: correction allows the original message to be recovered, and detection allows the decoder to detect that the message has been modified. These protections are achieved for the class of additive functions with a bound on the number of tampered codeword components (a codeword $c$ is tampered to $\tilde{c} = c \oplus e$ and $\text{wt}(e)$ is bounded). NM-codes can provide protection against much more powerful adversaries with access to much larger function families by using randomised coding schemes and weakening the protection goal to only ensuring that the adversary cannot benefit by manipulating a particular message. Storage efficiency of non-malleable codes is measured by the rate of these codes, given by the ratio of the message length to the codeword length. The highest achievable rate of coding schemes for a function family is the capacity of the coding scheme for the family.

The ultimate goal of NM-codes is to construct high rate codes with efficient (computational complexity) encoding and decoding algorithms for large families of functions. In \cite{10} it is proved that if $|\mathcal{F}| \leq 2^{2^n}$, then the capacity is lower-bounded by $1 - \alpha$. The proof uses a probabilistic construction of codes that achieves this rate but the code obtained is inefficient (the construction uses a greedy algorithm). Efficient (i.e. polynomial time) non-malleable codes have been constructed \cite{30} when the size of the family is $|\mathcal{F}| < 2^{O(n)}$ for some polynomial $p$.

A widely studied family of NM-codes is the bit-wise independent NM-codes that is defined with respect to the bit-wise independent tampering family $\mathcal{F}_{\text{BIT}}^{(n)}$: for binary codewords of length $n$, the tampering function is represented by a vector of $n$ independently chosen functions $(f_1, \cdots, f_n)$, where $f_i$ is a binary tampering function belonging to $\mathcal{F}_{\text{BIT}}^{(1)} = \{\text{Set0, Set1, Keep, Flip}\}$, where Set0 and Set1 set the value of the bit to 0 and 1, respectively, and Keep and Flip will keep and flip the bit, respectively. Non-malleable codes for protection against (simultaneous) independent bit-wise tampering and permutation have been proposed in \cite{31,32}. A second widely studied function family is called $C$-split state model where for a constant $C$, the codeword consists of $C$ blocks, and each block is tampered independently. A number of constructions of these codes for $C = 2$ in \cite{33,38} and other values of $C$, for example $C = 10$ in \cite{11}, have been proposed. All these function families are naturally suited to the motivating scenario of protecting a stored secret against tampering of the device, and protection approaches that
are based on splitting the secret and storing each part on a different hardware (with the assumption that they are not all accessible to the adversary).

In this paper we consider non-malleable codes for protection of communication against physical layer adversaries who tamper with transmitted codewords. Physical layer security has been pioneered by Wyner [1] who showed message transmission with perfect information-theoretic secrecy and without a shared secret key is possible if the adversary does not have full view of the codeword. This incomplete view may be due to the random noise in the adversary’s channel, or their limited reception and access to the codeword. In Wyner’s original wiretap model [1] the eavesdropper’s view of the channel is partially obstructed by the noise, and in Ozarow and Wyner’s wiretap II model [24], the eavesdropper can select the codeword components that they want to eavesdrop, subject to an upper bound on the number of such components. In both models the adversary is a passive eavesdropping adversary. Our goal is to provide protection against active physical layer adversaries that are modelled by a function family.

Our work

We study NM-codes for a class of functions that closely reflect capabilities of physical layer adversaries. We consider adversaries who have access to directional antennas and advanced transceivers, and can selectively read (eavesdrop) and tamper with different parts of a codeword. The adversary can choose an index set \( S_r \) of codeword components to eavesdrop, and an index set \( S_w \) of codeword components to tamper with, and the tampering is bitwise (component-wise). The choice of \( S_r \) and \( S_w \) is adaptive and for each component, taking into account all previous choices that had been made and codeword component values accessed, until that point. The sizes of the two sets are bounded by \( |S_r| \leq n\rho_r \) and \( |S_w| \leq n\rho_w \), for two constants \( 0 \leq \rho_r, \rho_w \leq 1 \). We model these adversaries, when the codewords are binary, by a function family denoted by \( F_{\text{BIT}}^{[n],\rho_r,\rho_w} \). The size of this family depends on the actual values of \( \rho_r, \rho_w \) and is at least \( 2^{n(\rho_r + \rho_w)} \), which is exponentially larger than the size of \( F_{\text{BIT}}^{[n]}(4^n) \). (This latter class can be seen as a special case of the former when \( \rho_r = 0 \) and \( \rho_w = 1 \).) This is because the eavesdropping set of the adversary allows them to choose their tampering functions depending on the read components of the codeword. Thus each tampering function \( f_i \) will be a function of \( \alpha = c_{S_r} \), that is, the read value of the codeword \( c \) on the \( S_r \) positions. We obtain rate bounds and give constructions for this class of functions. We also give applications of our results in two types of communication settings: a wiretap II channel with active adversaries and secure message transmission in networks.

Rate bounds. Storage efficiency of NM-codes for a function family is measured using the rate of the codes for the family. We give two sets of results, depending on the non-malleability notion. For strong non-malleability, we prove capacity of non-malleable codes with respect to \( F_{\text{BIT}}^{[n],\rho_r,\rho_w} \) is \( 1 - \rho_r \). The proof is by deriving an upper bound and a lower bound on the rate of these codes. The proof of the upper bound (Lemma 3) is by proving that strong non-malleability with respect
to $F_{\text{BIT}}^{[n],\rho_r,\rho_w}$ implies indistinguishability security of the code against an adversary who can eavesdrop $\rho_r$ fraction of codeword components (wiretap II adversary), and then using rate upper bound of wiretap II codes for this adversary to obtain the upper bound for NM-codes. The lower bound uses [16, Theorem 3.1] and reduces to finding an upper bound on the number of functions in $F_{\text{BIT}}^{[n],\rho_r,\rho_w}$.

For (default) non-malleability, we prove that if $\rho_r \leq \rho_w$, the capacity of the coding scheme is $1 - \rho_r$. The rate lower bound follows from the rate lower bound for strong non-malleability, as it is proved [34, Theorem 3.1] that the latter codes also provide default non-malleability. To prove the upper bound we build on a result from [18, Theorem 5.3] that was proved for the C-split state model. Our proof requires $\rho_r \leq \rho_w$. When $\rho_r > \rho_w$, we show that the rate of NM-codes with respect to $F_{\text{BIT}}^{[n],\rho_r,\rho_w}$ can exceed $1 - \rho_r$. We leave the upper bound (and hence capacity) for this case as an open question.

Constructions. We give two constructions. The first one is based on a modular construction that had been proposed for the function family $F_{\text{BIT}}^{[n],\rho_r,\rho_w}$ [34]. We construct a new proof that shows that with appropriate choice of parameters, one can obtain non-malleability against our new class of tampering functions where the choice of the tampering functions depends on the read values. The second construction uses a novel approach that relies on a new (not used in the context of NM-codes) building block and using the security notion of indistinguishability security.

Construction 1 uses an Algebraic Manipulation Detection (AMD) code [26] and a Linear Error Correcting Secret Sharing (LECSS) [34]: the encoding of a message $m$ is given by LECSS(AMD($m$)). AMD codes protects against additive errors of oblivious adversaries (the codeword is not seen by the adversary). A $(t, d)$-LECSS has $t$-uniformity (every $t$ components is $t$-wise independent, and each bit is uniformly distributed), and minimum (Hamming) distance $d$. To prove non-malleability, for each function $f$ we construct a probability distribution $D_f$ that for all messages $m$, can be used to simulate the decoding of the tampered codeword. The distribution $D_f$ is obtained by averaging a set of distributions, one for each read value of the eavesdropped part of the codeword. We borrow techniques from [34] and extend them to cater for the new much larger function class. Theorem 3 shows that for $F_{\text{BIT}}^{[n],\rho_r,\rho_w}$ function class, to achieve the level of security that is provided by a $(t, d)$-LECSS for the function class $F_{\text{BIT}}^{[n]}$ (same as $\rho_r = 0$ and $\rho_w = 1$), we need a $(t', d')$-LECSS with $t' = t + n\rho_r$ and $d' = (1 - \rho_r)d$. That is we need to increase $t$-uniformity of LECSS to $t' = t + n\rho_r$, but the minimum distance can be reduced. There is no known construction of LECSS that meets the requirements of the construction in [34] or our construction, and so it is unclear if the new set of parameters is harder (or easier) to achieve in concrete constructions.

Construction 2 uses a novel approach to the construction of non-malleable codes in the sense that instead of relying on the $t$-uniformity of LECSS, uses indistinguishability security of wiretap II codes. The construction uses an AMD code and a linear wiretap II code with indistinguishability security WT: the encoding
of a message $m$ is given by $\text{WT}(\text{AMD}(m))$. Wiretap II codes are randomised codes that provide indistinguishability security against an eavesdropping adversary that can adaptively eavesdrop a fraction of codeword components. The indistinguishability security is defined as follows: for $|S| \leq n\rho$, and any two messages $m_0$ and $m_1$, $\text{SD}(\text{Enc}(m_0)_S;\text{Enc}(m_1)_S) \leq \varepsilon$. Theorem 4 shows that using a wiretap II code for $\rho = \frac{1}{1+\rho^2}$ and security parameter $\varepsilon$ and an AMD code with error parameter $\delta$, results in an NM-code with security parameter $\delta + 2\varepsilon$. An important advantage of this construction is that there are explicit constructions for linear wiretap II codes satisfying $\rho = \frac{1}{1+\rho^2}$ that use cosets of linear error correcting codes \cite{19}, and so we obtain an explicit construction of NM-codes with respect to $F^{[n],\rho_r,\rho_w}_{\text{BIT}}$ function class using error correcting codes (and using efficient AMD code construction in \cite{26} that has flexible parameters). A by-product of this construction is an explicit construction of non-malleable codes for $F^{[n]}_{\text{BIT}}$ using error correcting codes. To our knowledge this is the first and the only known direct construction of non-malleable codes for this function family. The code exists for all $n = 2^h - 1$ and $h \geq 5$.

**Applications.** We motivated the function class $F^{[n],\rho_r,\rho_w}_{\text{BIT}}$ by considering physical layer adversaries who can eavesdrop the communication and then choose their tampering functions accordingly. The function class also models adversaries in storage scenarios \cite{20,34,14} where the storage is partially leaked to the adversary. In the following we apply our results to two physical layer communication security scenarios that have been widely studied. Before outlining our results, we discuss applicability of non-malleability as a protection goal in communication security.

In the basic physical layer security setting (e.g. wiretap models), Alice wants to send a message to Bob and protection is against an eavesdropping adversary. Using NM-codes allows protection against active adversaries with access to a family of tampering functions for which traditional error correction and detection is not possible. This protection is desirable in cases such as key agreement protocols where the eavesdropper’s goal is to influence the shared key.

The protection through NM-code for securing message transmission in the above setting however, does not allow Bob to know if the received message is the one sent by Alice, or an unrelated one that is the result of tampering. An interesting application of NM-codes to protection of message transmission is against collusion attacks, where a dishonest protocol participant (Alice or Bob) uses a helper to modify the transmitted message to a desired value. Consider a malicious sender who sends a message $m$, and uses the helper to modify the codeword during transmission so that the decoded message is a desired value $m'$. The sender does not have access to an out-of-band channel to send extra information to the helper and the only help they can receive is defined by the class of tampering functions that are available to the helper. Using non-malleable codes with protection against this function family will guarantee that helper cannot help the sender in anyway. An example of such setting is known as Terrorist Fraud in authentication protocols \cite{36}. 
Protecting wiretap II channel against active adversaries. A \((\rho_r,1)\)-active adversary wiretap II code is a coding scheme that provides (i) indistinguishability security against \(\rho_r\) leakage, and (ii) non-malleability against \(\mathcal{F}_{\text{BIT}}^{[n],\rho_r,-1}\) \(\rho_w = 1\). Our results on strong non-malleability can be used to show that the secrecy capacity\(^1\) for \((\rho_r,1)\)-active adversary wiretap II code, is \(1 - \rho_r\) (Theorem 5). Our (default) non-malleable code constructions give (explicit) constructions for \((\rho_r,1)\)-active adversary wiretap II codes (Theorem 6). The rate of the second construction that uses wiretap II codes (Section 4.2) is \(1 - \rho_r\). If the wiretap II code in the construction was to provide protection only against eavesdropping, then it could achieve the rate \(1 - \rho_r\). Their function family is considerably smaller than the family \(\mathcal{F}_{\text{BIT}}^{[n],\rho_r,-1}\), considered here.

Protecting communication in networks. Secure Message Transmission (SMT) in networks that are partially controlled by a Byzantine adversary has been studied in [5], where the network is modelled as a set of \(n\) node disjoint paths (also called wires) that connect the sender to the receiver. The adversary is active and controls a subset of size \(t\) of the wires. An \((\varepsilon,\delta)\)-SMT protocol guarantees that the information leakage (indistinguishability of adversary’s view for two messages) is bounded by \(\varepsilon\), and reliability guarantee is given by \(\Pr[M_S \neq M_R] \leq \delta\), where \(M_S\) and \(M_R\) are the sent and received messages, respectively. It has been proved that SMT exists only if \(n \geq 2t + 1\) [20]. We define \((\varepsilon,\delta)\)-NM-SMT for a network adversary whose tampering capability is defined by a function class \(\mathcal{F}\), and require indistinguishability privacy and reliability in terms of non-malleability, against this adversary. Our construction in Section 5.2 is for \(\mathcal{F}\) defined as follows. Let \(w_i\) denote the transcript of the \(i\)th wire, and let \(w_i \in \mathcal{W}\) for all \(i\). The adversary adaptively chooses a set \(S \subset [n]\) of \(t\) wires, eavesdrop and arbitrarily tampers with them. The adversary also uses the values of the eavesdropped wires in \(S\) to tamper with the remaining \([n] \setminus S\) wires, each by either replacing \(w_i\) with a chosen constant \(a_i \in \mathcal{W}\), or choosing a constant \(a_i \in \mathcal{W}\) and adding it to \(w_i\). The function family defined by the above adversary on \(\mathcal{W}^n \rightarrow \mathcal{W}^n\) is denoted by \(\mathcal{F}_{\Lambda O}^{[n],\rho_r,-1}\). For \(\mathcal{W} = \mathbb{F}_q\) we show that the construction in Section 4.2 can be extended to \(q\)-ary alphabet, resulting in a \((0,\delta)\)-NM-SMT, where \(\delta\) is the security parameter of AMD code (Theorem 7).

Other related work.

The concept of non-malleability in cryptography was introduced by Dolev et al. [9] and has since become a fundamental notion in cryptographic systems.

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\(^1\) The highest achievable code rate satisfying (i) and (ii).
Dziembowski et al. [34] introduced non-malleable codes in the context of tamper resilience and providing protection for secrets that are stored in tamperable hardware. There is a large body of works on NM-codes including computational NM-codes [13,6,2], and codes with extra properties such as continuously tampering models [29,37,28], locally decodable/updatable [11,22], and block-wise [23] that are not directly related to our work. In particular, leakage resilient NM-codes [13] consider a tampering family for non-malleability and a different leakage family for leakage resilience. In our model of NM-codes, there is only one family of functions $F_{\text{BIT}}^{[n],\rho_r,\rho_w}$ and the goal is non-malleability only. Bound on the rate of non-malleable codes was first studied in [16]. Authors present a general lower bound for any family of tampering functions that only depends on the size of the family, and an upper bound for a family of tampering functions that arbitrarily act on a subset $S \subset [n]$ of codeword components. A tampering class with apparent similarity with our work is [18,10]. This function class Local$^L$ consists of functions $f : \{0,1\}^n \rightarrow \{0,1\}^n$ where each output bit depends on at most $L_o(n)$ input bits. The tampering functions in $F_{\text{BIT}}^{[n],\rho_r,\rho_w}$ are vector of bit tampering functions where each bit function depends on a subset of size $n\rho_c$ of read components of the vector. That is unlike the function class Local$^L$ where each output bit is determined by a subset of input bits, in $F_{\text{BIT}}^{[n],\rho_r,\rho_w}$ a subset of components of the input codeword determines the vector of functions that will then be applied to the whole codeword.

Non-malleable code constructions for $F_{\text{BIT}}^{[n]}$ include, the first construction in [34], the first capacity-achieving construction [17], capacity-achieving and additionally non-malleable against permutation [32], capacity-achieving and linear time encode/decode [27]. Explicit constructions of information-theoretic C-split state include 2-split state with one-bit message [33], multi-bit message [8], constant rate [7] and 10-split state [11]. Non-malleable codes for non-binary alphabets are considered in [27] and constructions of linear-time encode/decode non-malleable codes with respect to a tampering class $\mathcal{F}^+$ that similar to $F_{\text{BIT}}^{[n]}$ consists of a vector of independently chosen tampering functions, where each function can be from $\mathcal{F}$ or an overwrite functions introduced above (referred to as $\mathcal{F}_{\text{const}}$ in [27]), or an identity function $\text{id}$. The code constructions are $\ell$-fold.

2 Preliminaries

Coding schemes define the basic properties for codes that are used in this paper. Let $\bot$ denote a special symbol.

**Definition 1** ([34]). A $(k,n)$-coding scheme consists of two functions: a randomised encoding function $\text{Enc} : \{0,1\}^k \rightarrow \{0,1\}^n$, where the randomness is implicit, and a deterministic decoding function $\text{Dec} : \{0,1\}^n \rightarrow \{0,1\}^k \cup \{\bot\}$ such that, for each $m \in \{0,1\}^k$, $\Pr[\text{Dec}(\text{Enc}(m)) = m] = 1$ (correctness), and the probability is over the randomness of the encoding algorithm.

The rate of a $(k,n)$-coding scheme is the ratio $\frac{k}{n}$. For a family of $(k,n(k))$-coding schemes, the achievable rates of the family is the supremum of the rates
of schemes as $k$ grows to infinity. A tampering function for a $(k, n)$-coding scheme is any function $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$.

**Definition 2** ([34]). Let $\mathcal{F}$ be a family of tampering functions. For each $f \in \mathcal{F}$ and $m \in \{0, 1\}^k$, define the tampering-experiment

$$\text{Tamper}_f^m = \begin{cases} x \leftarrow \text{Enc}(m), \bar{x} = f(x), \bar{m} = \text{Dec}(\bar{x}) \\ \text{Output } \bar{m}, \end{cases}$$

which is a random variable over the randomness of the encoding function $\text{Enc}$. A coding scheme $(\text{Enc}, \text{Dec})$ is non-malleable with respect to $\mathcal{F}$ if for each $f \in \mathcal{F}$, there exists a distribution $\mathcal{D}_f$ over the set $\{0, 1\}^k \cup \{\perp, \text{same}^*\}$, such that, for all $m \in \{0, 1\}^k$, we have:

$$\text{Tamper}_f^m \approx \mathcal{D}_f$$  \hspace{1cm} (1)

and $\mathcal{D}_f$ is efficiently sampleable given oracle access to $f(\cdot)$. Here $\approx$ refers to statistical or computational indistinguishability. In the case of statistical indistinguishability, the scheme has exact-security $\varepsilon$ if the above statistical distance is at most $\varepsilon$.

The right hand side of (1), denoted by $\text{Patch}(\mathcal{D}_f, m)$ in [34], is a random variable defined by the distribution $\mathcal{D}_f$ and the message $m$. Using this notation, (1) can be written as,

$$\text{Tamper}_f^m \approx \text{Patch}(\mathcal{D}_f, m).$$  \hspace{1cm} (2)

A stronger notion of non-malleability is the following.

**Definition 3** ([34]). Let $\mathcal{F}$ be a family of tampering functions. For each $f \in \mathcal{F}$ and $m \in \{0, 1\}^k$, define the tampering-experiment

$$\text{StrongNM}_f^m = \begin{cases} x \leftarrow \text{Enc}(m), \bar{x} = f(x), \bar{m} = \text{Dec}(\bar{x}) \\ \text{Output same}^* \text{ if } \bar{x} = x, \text{ and } \bar{m} \text{ otherwise.} \end{cases}$$

which is a random variable over the randomness of the encoding function $\text{Enc}$. A coding scheme $(\text{Enc}, \text{Dec})$ is strongly non-malleable w.r.t. $\mathcal{F}$ if for any $m_0, m_1 \in \{0, 1\}^k$ and any $f \in \mathcal{F}$, we have:

$$\text{StrongNM}_f^{m_0} \approx \text{StrongNM}_f^{m_1}.$$  \hspace{1cm} (3)

It is proved [34, Theorem 3.1] that strong non-malleability implies (default) non-malleability. The (default) non-malleability however is strictly weaker than strong non-malleability and does not imply strong non-malleability.

We will use the following coding schemes in our constructions in Section 4.

**Definition 4** ([26]). Let $(\text{AMDenc}, \text{AMDdec})$ be a coding scheme with $\text{AMDenc} : \{0, 1\}^k \rightarrow \{0, 1\}^n$. We say that $(\text{AMDenc}, \text{AMDdec})$ is a $\delta$-secure Algebraic Manipulation Detection (AMD) code if for all $m \in \{0, 1\}^k$ and all non-zero $\Delta \in \{0, 1\}^n$, we have $\Pr[\text{AMDdec}(\text{AMDenc}(m) + \Delta) \notin \{m, \perp\}] \leq \delta$, where the probability is over the randomness of the encoding.
Efficient AMD codes can be constructed using polynomials over finite fields.

Lemma 1 ([26]). There exists an AMD code \((\text{AMDenc}, \text{AMDdec})\) with encoder \(\text{AMDenc} : \{0, 1\}^k \rightarrow \{0, 1\}^{k+2u}\) that satisfies \(\Pr[\text{AMDdec}(\text{AMDenc}(m) + \Delta) \neq \bot] \leq (k/u + 1)/2^u\).

Note that the AMD code constructed in Lemma 1 is in fact a tamper detection code [37], which requires that a tampered codeword is always decoded to \(\bot\).

We say an AMD code achieves \(\delta\)-tamper detection security if for all \(\Delta \neq 0^n\), \(\Pr[\text{AMDdec}(\text{AMDenc}(m) + \Delta) \neq \bot] \leq \delta\).

The first construction of NM-codes with respect to \(\mathbb{F}_q[\text{BIT}]\) in [34] uses the following Linear Error-Correcting Secret Sharing (LECSS) scheme.

Definition 5 ([34]). Let \((\text{LECSSenc}, \text{LECSSdec})\) be a coding scheme with messages \(m \in \{0, 1\}^k\) and codewords \(x \in \{0, 1\}^n\). We say that the scheme is a \((d, t)\)-LECSS if the following properties hold:

- Linearity: For all \(x \in \{0, 1\}^n\) such that \(\text{LECSSdec}(x) \neq \bot\), and for all \(x' \in \{0, 1\}^n\), we have \(\text{LECSSdec}(x+x') = \begin{cases} \bot, & \text{if } \text{LECSSdec}(x') = \bot; \\ \text{LECSSdec}(x) + \text{LECSSdec}(x'), & \text{otherwise}. \end{cases}\)

- \(d\)-distance: For all non-zero \(\tilde{x} \in \{0, 1\}^n\) with Hamming weight less than \(d\), we have \(\text{LECSSdec}(\tilde{x}) = \bot\).

- \(t\)-uniform: For any fixed \(m \in \{0, 1\}^k\), we define the random variables \(X = (X_1, \cdots, X_n) = \text{LECSSenc}(m)\), where \(X_i\) denotes the bit of \(X\) in position \(i\) and randomness is from the encoding algorithm. Then the random variables \(\{X_i\}_1 \leq i \leq n\) are individually uniform over \(\{0, 1\}\) and \(t\)-wise independent.

In wiretap II model [24] Alice wants to send messages to Bob over a reliable channel that is eavesdropped by an adversary, Eve, who for a codeword of length \(n\), is allowed to choose any subset of size \(\rho n\) of the codeword components for eavesdropping.

Definition 6. A \((\rho, \varepsilon)\)-Wiretap code, or \((\rho, \varepsilon)\)-WT code for short, is a \((k, n)\)-(\(q\)-ary) coding scheme that satisfies the following privacy property. For any \(m_0, m_1 \in \mathbb{F}_q^k\), any \(S \subset [n]\) of size \(|S| \leq \rho n\),

\[
\text{SD}(\text{Enc}(m_0)_S; \text{Enc}(m_1)_S) \leq \varepsilon.
\]  \hspace{1cm} (4)

A \((\rho, \varepsilon)\)-WT code is called linear if for two vectors \(x_0, x_1 \in \mathbb{F}_q^n\),

\[
\text{Dec}(x_0 + x_1) = \begin{cases} \\
\bot, & \text{either } \text{Dec}(x_0) = \bot \text{ or } \text{Dec}(x_1) = \bot; \\
\text{Dec}(x_0) + \text{Dec}(x_1), & \text{otherwise}. \end{cases}
\]

The above indistinguishability based definition of security is equivalent to semantic security which is the strongest notion of security in cryptography.
Lemma 2 ([24]). The rate of wiretap II code with leakage parameter $\rho$ is upper bounded by $1 - \rho$.

This bound was proved with respect to weak secrecy [24] that assume uniform message distribution and use security measure $H(M|e_S)$, where $H()$ is Shannon entropy and $M$ is the random variable associated with the message. A $(t,d)$-LECSS construction can be used as a linear wiretap II code with $\rho = \frac{t}{d}$ and $\varepsilon = 0$, however the converse is not true in general. This is because privacy requirement of wiretap code is in terms of almost $t$-wise independence instead of $t$-unifformity, and minimum distance of these codes can be 1. Another closely related primitive is linear secret sharing scheme, which is usually studied over large alphabets (share size) and requires reconstruction of message from subset of codeword components of size above the reconstruction threshold.

3 Bit Tampering with Leakage

Our proposed tampering class is defined by two parameters $(\rho_r, \rho_w)$. We first define the function class, and then prove rate bounds for codes that provide non-malleability for this class.

3.1 $(\rho_r, \rho_w)_{\text{BIT}}$-NMC

Let $\mathcal{F}_{\text{BIT}}^{[1]} = \{\text{Set0, Set1, Keep, Flip}\}$ denote the set of functions that tamper with one bit, and $\mathcal{F}_{\text{BIT}}^{[n]}$ denote the set of $n$-bit bit-wise independent tampering functions. Each $f \in \mathcal{F}_{\text{BIT}}^{[n]}$ is specified by a vector $(f_1, f_2, \cdots, f_n)$ where $f_i \in \mathcal{F}_{\text{BIT}}^{[1]}$. For a vector $x = (x_1, x_2, \cdots, x_n) \in \{0,1\}^n$, $f(x)$ is a vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \in \{0,1\}^n$, where $\tilde{x}_i = f_i(x_i), i = 1, \cdots, n$.

Let $[n] = \{1, 2, \cdots, n\}$. We define the set $\mathcal{F}_{\text{BIT}}^{(n),\rho_r,\rho_w}$ as the set of bitwise tampering functions where the adversary (i) adaptively selects a subset $S_r \subset [n]$ of size $n\rho_r$ codeword components for eavesdropping, and (ii) tampers bitwise with a subset $S_w \subset [n]$ of size $n\rho_w$ of codeword components, each using a function from $\mathcal{F}_{\text{BIT}}^{[n]}$. The adversary can choose any pair of subsets $S_r, S_w$, subject to the bound on their sizes. Let $S_r^{[n]}$ denote the set of subsets of size $n\rho_r$ of $[n]$. We use $S_{w}^{[n]}$ and $S_{w}^{[n]}$, with cardinality $(n\rho_r)$ and $(n\rho_w)$, respectively. For a vector $x = (x_1, x_2, \cdots, x_n) \in \{0,1\}^n$ and a set $S = \{i_1, i_2, \cdots, i_{|S|}\} \subset [n]$, let $x_S$ denote the subvector $(x_{i_1}, x_{i_2}, \cdots, x_{i_{|S|}})$.

Definition 7. Let $x$ be a binary vector of length $n$, and $x_{S_r}$ and $x_{S_w}$ denote the subvectors with components in the set $S_r \subset [n]$ and $S_w \subset [n]$, respectively. The function $g : \{0,1\}^{n\rho_r} \rightarrow \mathcal{F}_{\text{BIT}}^{[n\rho_w]}$ defines a vector of bit tampering functions, dependent on a vector of length $n\rho_r$ (read values in $x_{S_r}$). For fixed $S_r, S_w$ and $g$ values, the tampering function $f_{S_r, S_w, g} : \{0,1\}^n \rightarrow \{0,1\}^n$ takes a vector $x$ and results in a vector $\tilde{x} = f_{S_r, S_w, g}(x)$ where:

$$\tilde{x}_{S_r} = (g(x_{S_r}))(x_{S_w}) \text{ and } \tilde{x}_{S_w} = x_{S_w}. \quad (5)$$
In other words, \( f_{S_r, S_w, g} \) modifies components of the input vector that are in \( S_w \), using the tampering functions \( g(x_{S_r}) \) and leaves the other components unchanged. Let \( g^x_{S_r} \) denote a vector of \( n_{ρ_r} \) bit tampering functions. For a fixed \((ρ_r, ρ_w)\) pair, we define the tampering function family,

\[
F_{\text{BIT}}^{[n], ρ_r, ρ_w} \triangleq \left\{ f_{S_r, S_w, g} \mid S_r \in S^{[n]}_{ρ_r}, S_w \in S^{[n]}_{ρ_w}, g : \{0, 1\}^{n_{ρ_r}} \rightarrow F^{[n_{ρ_w}]} \right\}. \tag{6}
\]

**Sizes of \( S_r \) and \( S_w \).** Note that in the above definition, the sets \( S_r \) and \( S_w \) have the exact sizes \( n_{ρ_r} \) and \( n_{ρ_w} \), respectively. The function set however includes all functions with \(|S_r| ≤ n_{ρ_r}\) and \(|S_w| ≤ n_{ρ_w}\). This is because a set \( S_r \) of size \( n_{ρ_r} - \ell \), where \( \ell \) is an integer satisfying \( 1 ≤ \ell < n_{ρ_r} \), is a subset of a set \( S'_r \) of size \( n_{ρ_r} \), where \( \ell \) components have not been used in selecting the tampering functions in \( g \). Similarly, a set \( S_w \) of size \( n_{ρ_w} - \ell \), where \( \ell \) is an integer satisfying \( 1 ≤ \ell < n_{ρ_w} \), is a subset of a set \( S'_w \) of size \( n_{ρ_w} \), where \( \ell \) components are Keep function. Thus although we focus on \( S_r \) and \( S_w \) with exact sizes \( n_{ρ_r} \) and \( n_{ρ_w} \), our results hold if one considers all function vectors that are determined by \( S_r \) and \( S_w \) of any size up to the corresponding upper bounds. This is particularly important as we use this function class for modelling physical layer adversaries, and adversaries can choose set sizes arbitrarily (up to their reading and writing capabilities).

**Special Case Example** \( F_{\text{BIT}}^{[n], 0, 1} = F_{\text{BIT}}^{[n]} \). It is easy to see that in \( F_{\text{BIT}}^{[n], ρ_r, ρ_w} \), when \( ρ_r = 0 \) the adversary does not have any access to the codeword, and \( ρ_w = 1 \) implies that all components of a codeword will be tampered bitwise and independently. The function class is thus the same as \( F_{\text{BIT}}^{[n]} \).

**Subsets of \( F_{\text{BIT}}^{[n], ρ_r, ρ_w} \) for Fixed \((S_r, S_w)\).** For a fixed pair of reading index set \( S_r \) and writing index set \( S_w \), let,

\[
F_{\text{BIT}}^{[n], S_r, S_w} = \left\{ f_{S_r, S_w, g} \mid g : \{0, 1\}^{n_{ρ_r}} \rightarrow F^{[n_{ρ_w}]} \right\}. \tag{7}
\]

According to \[\text{(6)}\], we have:

\[
F_{\text{BIT}}^{[n], ρ_r, ρ_w} = \bigcup_{S_r \in S^{[n]}_{ρ_r}, S_w \in S^{[n]}_{ρ_w}} F_{\text{BIT}}^{[n], S_r, S_w}. \tag{8}
\]

**Definition 8.** A \((k, n)\)-coding scheme is called a \((ρ_r, ρ_w)\)\-Non-Malleable Code \((\rho_r, ρ_w)\)-NMC if it is a non-malleable coding scheme with respect to \( F_{\text{BIT}}^{[n], ρ_r, ρ_w} \).

### 3.2 Rate Bounds for \((ρ_r, ρ_w)\)\-NMC

The highest achievable rate of coding schemes for the function family \( F_{\text{BIT}}^{[n], ρ_r, ρ_w} \) is the capacity of the coding scheme for this family. We provide rate results for the two notions of non-malleability.

**Strong non-malleability.** This stronger notion puts more stringent requirement on the code and allows us to characterise the capacity for the \( F_{\text{BIT}}^{[n], ρ_r, ρ_w} \) function family.
Theorem 1. The capacity of strong \((\rho_r, \rho_w)_{\text{BIT}-\text{NMC}}\) is \(1 - \rho_r\).

The proof of this theorem uses a theorem in [16] and Lemma 3 below, to derive a lower bound and an upper bound on the achievable rates of the coding schemes, respectively. We include the theorem for completeness.

(Theorem 3.1, [16].) Let \(\mathcal{F}\) be any family of tampering functions from \(n\)-bit to \(n\)-bit. There exists a construction parameterized by \(T\) and \(\delta\), such that for any \(\varepsilon, \eta > 0\), with probability at least \(1 - \eta\), the \((k, n)\)-coding scheme obtained is a strong non-malleable code with respect to \(\mathcal{F}\) with exact security \(\varepsilon\) and relative distance \(\delta\), provided that both of the following conditions are satisfied.

1. \(T \geq T_0\), for some
\[
T_0 = O \left( \frac{1}{\varepsilon^6} \left( \log \frac{|\mathcal{F}|2^n}{\eta} \right) \right).
\]

2. \(k \leq k_0\), for some
\[
k_0 \geq n(1 - h_2(\delta)) - \log T - 3\log \left( \frac{1}{\varepsilon} \right) - O(1),
\]
where \(h_2(\cdot)\) denotes the binary entropy function.

Thus by choosing \(T = T_0\) and \(k = k_0\), the construction satisfies
\[
k \geq n(1 - h_2(\delta)) - \log \log \frac{|\mathcal{F}|}{\eta} - \log n - 9\log \left( \frac{1}{\varepsilon} \right) - O(1).
\]

In particular, if \(|\mathcal{F}| \leq 2^{2^{\alpha n}}\) for any constant \(\alpha \in (0, 1)\), the rate of the code can be made arbitrarily close to \(1 - h_2(\delta) - \alpha\) while allowing \(\varepsilon = 2^{-\Omega(n)}\).

The following lemma relates strong non-malleability with respect to \(\mathcal{F}_{\text{BIT}}^{[n], \rho_r, \rho_w}\) to indistinguishability security of \((\rho_r, \varepsilon)\)-wiretap II codes.

Lemma 3. If a coding scheme is strongly non-malleable with respect to \(\mathcal{F}_{\text{BIT}}^{[n], \rho_r, \rho_w}\) with exact security \(\varepsilon\), then it is a \((\rho_r, \varepsilon)\)-WT code.

Proof. Proof is by contradiction: we show that if a strongly non-malleable coding scheme with respect to \(\mathcal{F}_{\text{BIT}}^{[n], \rho_r, \rho_w}\) does not satisfy wiretap II indistinguishability security, then we can construct a tampering function that violates the strong non-malleability property of the coding scheme.

Assume a strongly non-malleable coding scheme with respect to \(\mathcal{F}_{\text{BIT}}^{[n], \rho_r, \rho_w}\) does not satisfy wiretap II indistinguishability security. Then, there exists a reading set \(S_r \subset [n]\) of size \(|S_r| = n\rho_r\), and a pair of messages \(m_0, m_1\) such that
\[
\text{SD}(\text{Enc}(m_0)_{S_r}; \text{Enc}(m_1)_{S_r}) > \varepsilon.
\]

By the definition of statistical distance, there exists a set \(D_\varepsilon \subset \{0, 1\}^{n\rho_r}\) such that
\[
|\text{Pr}[\text{Enc}(m_0)_{S_r} \in D_\varepsilon] - \text{Pr}[\text{Enc}(m_1)_{S_r} \in D_\varepsilon]| > \varepsilon.
\]
Now consider a tampering function \( f_{S_r, \{1\}} \), that reads the codeword components in \( S_r \) positions, and tampers with the first bit of the codeword based on the read value. We define \( g : \{0, 1\}^{w_n} \rightarrow \{\text{Set0, Set1, Keep, Flip}\} \) using the set \( D_\epsilon \) as follows.

\[
g(\alpha) = \begin{cases} 
\text{Keep, } \alpha \in D_\epsilon; \\
\text{Flip, otherwise.} 
\end{cases}
\]

Note that \( f_{S_r, \{1\}} \cdot g \) when applied to a codeword in \( c \), will leave it unchanged if \( c_{S_r} \in D_\epsilon \), and flips its first component otherwise.

According to Definition \( \delta \) we should have

\[
|\Pr[\text{StrongNMC}_{f_{S_r, \{1\}} \cdot g} = \text{same}] - \Pr[\text{StrongNMC}_{f_{S_r, \{1\}} \cdot g} = \text{same}^*]| > \epsilon.
\]

This leads to

\[
\text{SD}(\text{StrongNMC}_{f_{S_r, \{1\}} \cdot g}; \text{StrongNMC}_{f_{S_r, \{1\}} \cdot g}^*) > \epsilon,
\]

which contradicts the strong non-malleability of the coding scheme. \( \square \)

We use the above two results leads to the following proof.

\textbf{Proof (of Theorem 7).} Theorem 3.1 in [16] shows that for any function family \( \mathcal{F} \) of size upper bounded by \( |\mathcal{F}| \leq 2^{2n^\alpha} \), there is a family of coding schemes that can achieve the rate \( 1 - \alpha \) arbitrarily close, by using sufficiently long codes (e.g. let \( \delta = 1/2 \)). To use this theorem to find a lower bound on the achievable rate of \( \mathcal{F}_{\mathcal{BIT}}^{[n],S_r,S_w} \), we need to upper bound the number of functions in the family. We note that the representation in (7) may not be unique for a function in \( \mathcal{F}_{\mathcal{BIT}}^{[n],S_r,S_w} \).

In particular, when \( S_r \cap S_w \neq \emptyset \), it is possible to have \( f_{S_r,S_w,g'} = f_{S_r,S_w,g''} \) for \( g' \neq g'' \). For example, let \( S_r = S_w = \{1\} \). Then two functions \( g' \) and \( g'' \) from the set \( \{g : \{0, 1\} \rightarrow \{\text{Set0, Set1, Flip, Keep}\}\} \), given by \( (g'(x_i) = 0) = \text{Set1}; g'(x_i) = 1) = \text{Set0}, \) and \( (g''(x_i) = 0) = \text{Flip}; g''(x_i) = 1) = \text{Flip}, \) will represent the same function in \( \mathcal{F}_{\mathcal{BIT}}^{[n],S_r,S_w} \). However only requires an upperbound on the number of functions. Using (3) it is easy to see that \( |\mathcal{F}_{\mathcal{BIT}}^{[n],\rho_r,\rho_w}| \leq \binom{n_\rho_r}{n_\rho_w} \cdot 2^{n(n_\rho_w)} \cdot 2^{n^\rho_r}. \)

From the above computation, we have

\[
\log \log |\mathcal{F}_{\mathcal{BIT}}^{[n],\rho_r,\rho_w}| \leq 2(\log n + \log \log n) + n n_\rho_r + \log(n n_\rho_w) + 1 \leq n (\rho_r + \xi),
\]

where \( \xi \) is an arbitrarily small constant and the inequality holds for large enough \( n \). Theorem 3.1 in [16] shows that for any tampering family \( \mathcal{F} \) that satisfies \( \log \log |\mathcal{F}| \leq na \), there is a coding scheme with rate arbitrarily close to \( 1 - \alpha \). Thus the achievable rate of \( (\rho_r, \rho_w)_{\mathcal{BIT}} \)-NMC with strong non-malleability is lower bounded by \( 1 - \rho_r \).

The upper bound on the rate of these codes follows from Lemma 3 that implies that the rate of a coding scheme with strong non-malleability for function family \( \mathcal{F}_{\mathcal{BIT}}^{[n],\rho_r,\rho_w} \), cannot exceed the rate of wiretap II codes of length \( n \) and with leakage parameter \( \rho_r \), and noting that the upper bound on the rate of these latter codes is \( 1 - \rho_r \) (see Lemma 3). \( \square \)
Default non-malleability. For (default) non-malleability, we have a general lower bound. But the upper bound (and so capacity) is only known for $\rho_r \leq \rho_w$.

**Theorem 2.** The capacity of $(\rho_r, \rho_w)$-BIT-NMC for $\rho_r \leq \rho_w$ is $1 - \rho_r$.

The rate lower bound in the case of (default) non-malleability follows from the lower bound on strong non-malleable codes for the same function class, and noting that a coding scheme that provides strong non-malleability also provides default non-malleability ([34, Theorem 3.1]). To prove a rate upper bound for default non-malleability we use the following theorem.

**(Theorem 5.3, [16]).** Let $S \subset [n]$ be of size $\rho n$ and consider the family of tampering functions that only acts on the coordinate positions in $S$. Then, there is a $\xi_0 = O(\frac{\log n}{n})$ such that the following holds. Let $(\text{Enc}, \text{Dec})$ be any $(k,n)$-coding scheme which is non-malleable for the family and achieves rate $1 - \rho + \xi$, for any $\xi \in [\xi_0, \rho]$ and error $\varepsilon$. Then $\varepsilon \geq \frac{1}{16\rho}$. In particular, when $\rho$ and $\xi$ are absolute constants, $\varepsilon = \Omega(1)$.

**Proof (of Theorem 2).** The lower bound $1 - \rho_r$ for strong $(\rho_r, \rho_w)$-BIT-NMC is also a lower bound for $(\rho_r, \rho_w)$-BIT-NMC.

In the rest of the proof we show that $1 - \rho_r$ is also an upper bound when $S_w \subset S_r$. Theorem 5.3 in [16] shows that $1 - \frac{\rho n}{n}$ is a rate upper bound for non-malleable codes with respect to the family of tampering functions that only act on the coordinate positions in $S \subset [n]$. We first show that the set of functions considered in this theorem is the same as the set $\mathcal{F}_{\text{BIT}}^{[n],S,S}$. Towards this goal, we first show that the size of the two sets are the same. The total number of functions that arbitrarily tamper with coordinate positions in $S$ is $(2^{\rho n})2^{2n}$. On the other hand, the set $\mathcal{F}_{\text{BIT}}^{[n],S,S}$ contains the subset of functions

$$| \{ f_{S,S,g} : \{0,1\}^{n\rho} \rightarrow \{\text{Set0, Set1}\}^{n\rho} \}| = (2^{\rho n})2^{2n}.$$ 

Note that each function in the above description is distinct because tampering of each codeword component in $S$ can be done in one of the two ways. Thus, $|\mathcal{F}_{\text{BIT}}^{[n],S,S}| \geq (2^{\rho n})2^{2n}$. Noting that the set $\mathcal{F}_{\text{BIT}}^{[n],S,S}$ is a subset of all functions that tamper with the coordinate positions in $S$, we conclude that the two sets have the same size and contain the same functions. The rate upper bound of $1 - \rho_r$ for $(\rho_r, \rho_w)$-BIT-NMC follows from Theorem 5.3 in [16] because, when $\rho_r \leq \rho_w$, there exists an $S \subset [n]$ of size $n\rho_r$ where $\mathcal{F}_{\text{BIT}}^{[n],S,S} \subset \mathcal{F}_{\text{BIT}}^{[n],\rho_r,\rho_w}$, and for this subset of functions, the upperbound holds. \(\square\)

**Remark 1.** The proof of Theorem 2 requires $\rho_r \leq \rho_w$. For $\rho_r > \rho_w$, the rate lower bound $1 - \rho_r$ remains valid but the upper bound is an open question. It is interesting to note that capacity in this case can be higher than $1 - \rho_r$. This is because for small values of $\rho_w$, error correcting codes with non-zero rate exists and in the case of error correcting codes $\rho_r = 1$ which using $1 - \rho_r = 0$, suggests zero rate for NM-codes. This is however not true because error correcting codes are non-malleable and in this case have non-zero rate.
4 Code Constructions

Using the results in [34], one can construct NM-codes for tampering family (including $F_{\text{BIT}}^{[n], \rho_t, \rho_w}$ family) in the Common Reference String (CRS) model. We construct explicit and efficient $(\rho_t, \rho_w)_{\text{BIT}}$-NMC without any setup conditions.

Our first construction is based on a construction proposed by Dziembowski, Pietrzak and Wichs [34] for the set of Bit-wise Independent Tampering (BIT) functions ($F_{\text{BIT}}^{[n], \rho_t, \rho_w}$ in our notation). This construction has inspired a number of other NM-code constructions [31], [17], [18] and more recently [27], [28]. The construction uses two coding schemes: an AMD (Algebraic Manipulation Detection, see Definition 4) code and a LECSS (Linear Error Correcting Secret Sharing, see Definition 5) with appropriate parameters. Explicit construction of LECSS with the required parameters has been an open question. Our second construction uses two coding schemes: an AMD code (ρ, ρ_t, ρ_w) and an LECSS (Linear Error Correcting Secret Sharing) code.

4.1 Construction 1: LECSS$\circ$AMD

We consider the function class $F_{\text{BIT}}^{[n], \rho_t, \rho_w}$ with size at least $(2^{\rho_w})^{2^{n_{\rho_t}}}$, which is much larger than $F_{\text{BIT}}^{[n]}$ (of size $4^n$) that was considered in [34].

**Theorem 3.** Let $(\text{LECSSenc}, \text{LECSSdec})$ be a $(d', t')$-LECSS with an encoder $\text{LECSSenc} : \{0, 1\}^t \rightarrow \{0, 1\}^{n'}$. Let $(\text{AMDenc}, \text{AMDdec})$ be an AMD code from $\{0, 1\}^{10}$ to $\{0, 1\}^t$ with δ-tamper detection security. Let $(\text{Enc}, \text{Dec})$ be defined as follows.

\[
\begin{align*}
\text{Enc}(m) &= \text{LECSSenc}(\text{AMDenc}(m)) \\
\text{Dec}(x) &= \text{AMDdec}(\text{LECSSdec}(x))
\end{align*}
\]  

Then the $(k, n)$-coding scheme $(\text{Enc}, \text{Dec})$ is a $(\rho_t, 1)_{\text{BIT}}$-NMC with exact security $\max\{\delta, 2^{-t'(t'-n_{\rho_t})}\}$, if $t' > n_{\rho_t}$ and $d' > n_{(1-\rho_t)}$.

Compared to Theorem 4.1 in [34], the result implies that for the same security level, one needs to use LECSS with higher uniformity parameter ($t' = t + n_{\rho_t}$) but the minimum distance of the LECSS can be somewhat relaxed. The intuition of the proof is as follows. We need to show that for an arbitrary function $f_{S_r, [n], g}$ there is a distribution $D_{f_{S_r, [n], g}}$ that satisfies (2) for any message $m$. For a function with read index set $S_r$, the set of codewords corresponding to the message $m$ can be partitioned into subsets $C_\alpha$ consisting of codewords $c$ where $c_{S_r} = \alpha$ and $\alpha \in \{0, 1\}^{n_{\rho_t}}$. For all codewords in $C_\alpha$, the function $g^\alpha \overset{\text{def}}{=} g(\alpha) \in F_{\text{BIT}}^{[n]}$ will be used. By choosing appropriate parameters for LECSS and AMD code, we can construct distribution $D_{f_{S_r, [n], g}}$ which is the “average” of the distributions $D_{\alpha}^{f_{S_r, [n], g}}$ corresponding to $C_\alpha$.

**Proof.** Consider a message $m \in \{0, 1\}^{k}$, and a tampering function $f_{S_r, [n], g} \in F_{\text{BIT}}^{[n], \rho_t, \rho_w}$. We define two (vector) random variables $X, \tilde{X} \in \{0, 1\}^n$ representing the codeword and the tampered codeword, respectively.

\[
m \xrightarrow{\text{Enc}} X \xrightarrow{f_{S_r, [n], g}} \tilde{X} \xrightarrow{\text{Dec}} \text{Tamper}_m f_{S_r, [n], g}.
\]
The randomness of the variables Tamper_{m,r}^{f, \alpha, \gamma} and X are from the randomness of the encoding. Since Tamper_{m,r}^{f, \alpha, \gamma} and X are correlated, we have

\[ \Pr \left[ \text{Tamper}_{m,r}^{f, \alpha, \gamma} = \gamma \right] = \sum_{\alpha \in \{0,1\}^{n\rho_r}} \Pr[X_{S_r} = \alpha] \cdot \Pr[\text{Tamper}_{m,r}^{f, \alpha, \gamma} = \gamma | X_{S_r} = \alpha] \] (10)

\[ = \sum_{\alpha \in \{0,1\}^{n\rho_r}} \frac{1}{2^{n\rho_r}} \Pr[\text{Tamper}_{m,r}^{f, \alpha, \gamma} = \gamma | X_{S_r} = \alpha], \]

where the last equality follows from the \( t' \)-uniform property of the LECSS, and assuming that \( t' > n\rho_r \).

To construct the distribution \( D_{\alpha}^{f, \alpha, \gamma} \) that satisfies (2), we start by constructing a set of distributions \( \{D_{\alpha}^{f, \alpha, \gamma} | \alpha \in \{0,1\}^{n\rho_r} \} \), each over the set \( \{0,1\}^n \cup \{\bot\} \cup \{\text{same}\} \) and satisfying

\[ \text{Tamper}_{m,\alpha}^{f, \alpha, \gamma} \sim \text{Patch}(D_{\alpha}^{f, \alpha, \gamma}, m) \] (11)

for \( \varepsilon = \max\{\delta, 2^{-O(t' - n\rho_r)}\} \). The distribution \( D_{\alpha}^{f, \alpha, \gamma} \) is used to simulate the function \( f_{S_r}[\alpha, \gamma] \) when applied to codewords in \( C_\alpha \), the set of encodings \( c \) of \( m \) that for the chosen index set \( S_r \), have \( c_{S_r} = \alpha \). From (11) we have,

\[ \text{SD} \left( \text{Tamper}_{m,\alpha}^{f, \alpha, \gamma}, D_{\alpha}^{f, \alpha, \gamma} \right) \leq 2^{-n\rho_r} \sum_{\alpha} \text{SD} \left( \text{Tamper}_{m,\alpha}^{f, \alpha, \gamma}, D_{\alpha}^{f, \alpha, \gamma} \right), \]

where \( \text{Tamper}_{m,\alpha}^{f, \alpha, \gamma} \) is the tampering variable defined in (11). We will have

\[ \text{SD} \left( \text{Tamper}_{m,\alpha}^{f, \alpha, \gamma}, D_{\alpha}^{f, \alpha, \gamma} \right) \leq \varepsilon \]

because for all \( \alpha \), we have \( \text{SD} \left( \text{Tamper}_{m,\alpha}^{f, \alpha, \gamma}, D_{\alpha}^{f, \alpha, \gamma} \right) \leq \varepsilon. \)

To construct \( D_{\alpha}^{f, \alpha, \gamma} \), consider \( X \in C_\alpha \), namely, assume \( X_{S_r} = \alpha \). Let \( g(\alpha) = g^{S_r} = (g^{S_r}_1, \cdots, g^{S_r}_n) \), where \( g^{S_r} \in \{\text{Set0}, \text{Set1}, \text{Keep, Flip}\}. \)

Firstly, \( g^{S_r}(X) \) on condition \( X_{S_r} = \alpha \) will be constant and can be computed from \( \{g^{S_r}_i | i \in S_r\} \) and \( \alpha \).

Next consider application of \( \{g^{S_r}_i | i \in S_r\} \) to \( X_S \). The analysis below is all under the condition \( X_{S_r} = \alpha \). For all component functions in \( \{g^{S_r}_i | i \in S_r\} \) that are in \( \{\text{Set0, Set1}\} \) the values of \( g^{S_r}(X) \) at these positions will be constant values 0 and 1, respectively. For all component functions in \( \{g^{S_r}_i | i \in S_r\} \) that are in \( \{\text{Keep, Flip}\} \) the values of \( g^{S_r}(X) \) in these positions will be kept the same and flipped, respectively. In the latter case this means that the statistical properties of columns (\( C_\alpha \) seen as an array of row vectors) will stay the same. Since columns of \( C_\alpha \) in \( S_r \) are \( (t' - n\rho_r) \)-wise independent, we will have, (i) each non-overwritten column of \( g^{S_r}(C_\alpha) \) (also as an array of row vectors) in \( S_r \) is uniformly distributed,
and (ii) non-overwritten columns of $g^\alpha(C_a)$ in $\tilde{S}_r$ are jointly $(t' - n\rho_r)$-wise independent.

Let $n^\text{ow}_{S_r}$ denote the number of overwrite bit functions in $\{g^\alpha_i | i \in S_r\}$ defined as,

$$n^\text{ow}_{S_r} = \{|i \in S_r | g^\alpha_i = \text{Set0 or } g^\alpha_i = \text{Set1}\}.$$  

The above analysis shows that $n\rho_r + n^\text{ow}_{S_r}$ components of $g^\alpha(X)$ will have constant values independent of the initial value of $X$, while the remaining $(|S_r| - n^\text{ow}_{S_r})$ non-overwritten components in $\tilde{S}_r$ are individually uniformly distributed, and are jointly $(t' - n\rho_r)$-wise independent.

For $g^\alpha \in F^{|n|}_{\text{BIT}}$, the difference function $\Delta g^\alpha \in F^{|n|}_{\text{BIT}}$ is defined as:

$$\Delta g^\alpha(x) = g^\alpha(x) \oplus x.$$  

(12)

Using (12) it can be seen that if $g^\alpha_i \in \{\text{Set0, Set1}\}$, then $\Delta g^\alpha_i \in \{\text{Keep, Flip}\}$, and if $g^\alpha_i \in \{\text{Keep, Flip}\}$, then $\Delta g^\alpha_i \in \{\text{Set0, Set1}\}$. Thus applying a non-overwrite bit function in $\{\Delta g^\alpha_i | i \in \tilde{S}_r\}$ on a column of $C_a$, will correspond to applying an overwrite function of $\{g^\alpha_i | i \in \tilde{S}_r\}$ on that column, and vice versa.

The distribution $D^{|\alpha|}_{\tilde{S}_r-[n], \tilde{r}}$ is constructed by considering four cases according to the number of overwrite component functions, denoted by $n^\text{ow}_{S_r}$, in the set $\tilde{S}_r$. In the following analysis, following the approach of [34, Appendix B], we consider four cases.

1. $n^\text{ow}_{S_r} \in [0, t' - n\rho_r)$: rely on linearity, $t'$-uniform of LECSS and AMD;
2. $n^\text{ow}_{S_r} \in (t' - n\rho_r, \frac{|S_r|}{2})$: rely on linearity, $t'$-uniform and $d'$-distance of LECSS;
3. $n^\text{ow}_{S_r} \in (\frac{|S_r|}{2}, n - t')$: rely on $t'$-uniform and $d'$-distance of LECSS;
4. $n^\text{ow}_{S_r} \in [n - t', |S_r|]$: rely on $t'$-uniform of LECSS.

For each case we show how the distribution can be constructed. The complete proof is given in Appendix A.

\[ \square \]

**Lemma 4.** When $\rho_w < \frac{1 - \rho_r}{2}$, the coding scheme $(\text{Enc, Dec})$ in Theorem 3 is a $(\rho_r, \rho_w)_{\text{BIT}}$-NMC with exact security $\max\{\delta, 2^{-\Omega(t' - n\rho_r)}\}$, if the $(d', t')$-LECSS satisfies $t' > n\rho_r$ and $d' > \frac{n\rho_r}{2}$.

*Proof.** Lemma 4 is a special case of Theorem 3 when $\rho_w$ is small and we have $\rho_w < \frac{1 - \rho_r}{2}$. Using the proof steps of this theorem leads to four cases that are distinguished according to $n^\text{ow}_{S_r}$, the number of overwrite component functions of $g^\alpha$ in $\tilde{S}_r$. Note that we always have $n^\text{ow}_{S_r} \leq n\rho_w$ because $n\rho_w$ is the total writing budget. When $\rho_w < \frac{1 - \rho_r}{2}$, we have $n^\text{ow}_{S_r} \leq n\rho_w < \frac{|S_r|}{2}$ and hence Case 3 and Case 4 in the proof above will not occur. If Case 2 occurs (i.e. when $n\rho_w > t' - n\rho_r$), the range of $n^\text{ow}_{S_r}$ is $(t' - n\rho_r, n\rho_w] \subseteq (t' - n\rho_r, \frac{|S_r|}{2})$ and this leads to the relaxation of the parameter $d'$ from $d' > \frac{|S_r|}{2}$ to $d' > \frac{n\rho_w}{2}$. The rest of the argument is as before, and is given in the detailed arguments of Case 2, given in Appendix A. \[ \square \]
Explicit construction of LECSS that satisfies the required minimum distance and uniformity for arbitrary security level is an open question. The probabilistic construction in [34, Lemma C.2] could be used to estimate the achievable rate of LECSS with the required parameters. The estimate for the original parameters of LECSS shows positive achievable rate [34, Theorem 4.2]. The second construction uses building blocks for which explicit constructions do exist. However estimating achievable rate of these codes remain open.

4.2 Construction 2: WTοAMD

This is a modular construction that uses a wiretap II code and an AMD code with appropriate parameters.

**Theorem 4.** Let \((\text{AMDenc}, \text{AMDdec})\) be an AMD code from \(\{0, 1\}^k\) to \(\{0, 1\}^\ell\) with \(\delta\)-tamper detection security. Let \((\text{WTenc}, \text{WTdec})\) be a linear \((\rho, \varepsilon)\)-WT code with encoder \(\text{WTenc} : \{0, 1\}^\ell \rightarrow \{0, 1\}^n\). Let \((\text{Enc}, \text{Dec})\) be defined as follows.

\[
\begin{align*}
\text{Enc}(m) &= \text{WTenc}(\text{AMDenc}(m)); \\
\text{Dec}(x) &= \text{AMDdec}(\text{WTdec}(x)).
\end{align*}
\]  

Then the \((k, n)\)-coding scheme \((\text{Enc}, \text{Dec})\) is a \((\rho, 1)\)-BIT-NMC with exact security \(2\varepsilon + \delta\), if \(\rho \geq 1 + \frac{1}{2\delta^2}\). The rate of the NM-code is upper bounded by \(1 - \frac{\rho}{2}\).

**Proof.** We use the approach of Theorem 3 and express Tamper\(_{f_{Sr}[\alpha]}\)\(_m\)\(_g\) as in expression (10). We thus need to find distribution \(D^{f_{Sr}[\alpha]}_\alpha\) \((\independent \text{message } m)\) that corresponds to each

\[
\text{Tamper}^{f_{Sr}[\alpha]}_m = \left(\text{Tamper}^{f_{Sr}[\alpha]}_m | X_{Sr} = \alpha\right).
\]

The proof however, replaces \(t\)-uniformity in LECSS with \(t\)-privacy in wiretap II code which is expressed in terms of indistinguishability security. We use this property and the linearity of the code, to show that the distributions \(D^{f_{Sr}[\alpha]}_\alpha\) can be found using wiretap II encodings of \(0^\ell\). Define a (vector) random variable \(Y = \text{WTenc}(0^\ell)\) and then construct the distribution \(D^{f_{Sr}[\alpha]}_\alpha\) as follows:

\[
\Pr[D^{f_{Sr}[\alpha]}_\alpha = \gamma] = \sum_{\alpha \in \{0, 1\}^{n_{Sr}}} \Pr[Y_{Sr} = \alpha] \cdot \Pr[D^{f_{Sr}[\alpha]}_\alpha = \gamma].
\]

We use the following notations. Let \(X = \text{Enc}(m)\) and \(X^\alpha = (X|X_{Sr} = \alpha)\). The tampered version is given by \(X^\alpha = f_{Sr[\alpha]}(X^\alpha) = g^\alpha(X^\alpha)\), where \(g^\alpha = g(\alpha) \in \mathcal{F}_{\text{BT}}^{[n]}\) and \(g^\alpha = (g^\alpha_1, \ldots, g^\alpha_n)\). Since \(X^\alpha_{Sr} = \alpha\), then \(X^\alpha_{Sr}\) is a constant. Components of \(X^\alpha_{Sr}\) can be constant or random bits, depending on the corresponding components \(g^\alpha_i\), \(i \in S_r\). We consider two cases that are distinguished by the number \(n^\text{once}_{Sr}\) of overwrite bit functions in \(\{g^\alpha_i| i \in S_r\}\).
1. “At most half of $S_\alpha$ are overwrite functions” ($n_{S_\alpha}^{ow} \leq \frac{|S_\alpha|}{2}$):
   The difference function $\Delta g^\alpha$ (i.e., $g^\alpha(x) = x \oplus \Delta g^\alpha(x)$) has at most half non-overwrite bit functions over $S_\alpha$. Let $S$ be the index set of the non-overwrite components of $\Delta g^\alpha$ in $S_\alpha$. Then $|S| \leq \frac{|S_\alpha|}{2} = \frac{n_{S_\alpha}^{\bot}}{2}$ and hence $\frac{|S_\alpha \cup S|}{n_{S_\alpha}^{\bot}} \leq \frac{1}{2} + \epsilon$. We use the short hand $Y^\alpha = (Y|Y_{S_\alpha} = \alpha)$ (similar to $X^\alpha = (X|X_{S_\alpha} = \alpha)$). Then according to the indistinguishability privacy of the $(\frac{1}{2} + \epsilon)$-WT code, we have
   \[
   SD(X_\alpha^S; Y_\alpha^S) \leq SD(X_{S \cup S'}; Y_{S \cup S'}) \leq \epsilon. \tag{14}
   \]

   Define the following distribution using $f_{S_\alpha,[n_\alpha]}$ and $\alpha$.
   \[
   D^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{\alpha} \defeq \begin{cases} y \leftarrow Y^\alpha & \text{Output same}, \\ \text{if } \Delta g^\alpha(y) = 0^n; \bot, & \text{otherwise.} \end{cases}
   \]

   In order to show that the real tampering experiment
   \[
   Tamper^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{m,\alpha} = AMDdec(AMDenc(m) \oplus WTdec(\Delta g^\alpha(X^\alpha)))
   \]

   is close to its simulation $Patch(D^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{\alpha}, m)$, we use the intermediate variable
   \[
   T' = AMDdec(AMDenc(m) \oplus WTdec(\Delta g^\alpha(Y^\alpha))).
   \]

   Now,
   \[
   SD(Tamper^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{m,\alpha}; Patch(D^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{\alpha}, m))
   \leq SD(Tamper^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{m,\alpha}; T') + SD(T'; Patch(D^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{\alpha}, m))
   \leq SD(X_\alpha^S; Y_\alpha^S) + SD(T'; Patch(D^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{\alpha}, m))
   \leq \epsilon + SD(T'; Patch(D^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{\alpha}, m))
   \leq \epsilon + \delta,
   \]

   where inequality (i) follows from the fact that $Tamper^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{m,\alpha}$ and $T'$ are only different at $X^\alpha$ and $Y^\alpha$, inequality (ii) follows from \[14\] and inequality (iii) follows from the fact that $T'$ and $Patch(D^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{\alpha}, m)$ have different values only when $\Delta g^\alpha(Y^\alpha) \neq 0^n$ and $T' \neq \bot$, which happens with probability at most $\delta$ according to the $\delta$-tamper detection security of the AMD code.

2. “More than half of $S_\alpha$ are overwrite functions” ($n_{S_\alpha}^{ow} > \frac{|S_\alpha|}{2}$):
   In this case, let $S$ be the index set of the non-overwrite components of $g^\alpha$ in $S_\alpha$. From the assumption, we have $|S| < \frac{|S_\alpha|}{2}$ and \[14\] holds. Let $D^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{\alpha}$ be the distribution of the random variable $Dec(g^\alpha(Y^\alpha))$. Note that $Tamper^{f_{S_\alpha,[n_\alpha]} \cdot \alpha}_{m,\alpha} = Dec(g^\alpha(X^\alpha))$. We need to show these two random variables are close.
   \[
   SD(Dec(g^\alpha(X^\alpha)); Dec(g^\alpha(Y^\alpha))) \leq SD(X_\alpha^S; Y_\alpha^S) \leq \epsilon,
   \]

   where the first inequality follows because $(g^\alpha(X^\alpha))_S = (g^\alpha(Y^\alpha))_S$ and the second inequality follows from \[14\].
To bound the exact security of the NM-code, we define an intermediate distribution $D_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma}$ that (unlike $D_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma}$) depends on message $m$.

$$\Pr\left[D_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma} = \gamma\right] = \sum_{\alpha \in \{0,1\}^{nP\rho}} \Pr[X_{\overline{S}_r} = \alpha] \cdot \Pr[D_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma} = \gamma].$$

Let $\tilde{M} = \text{Patch}(D_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma}, m)$ and $\tilde{M} = \text{Patch}(D_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma}, m)$. We compute

$$\text{SD}(\text{Tamper}_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma}; \tilde{M}) \leq \text{SD}(\text{Tamper}_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma}; \tilde{M}) + \text{SD}(\tilde{M}; \tilde{M})$$

where (i) follows because $\text{Tamper}_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma}$ is written as expected value (over $\alpha$) of $\text{Tamper}_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma}$ according to (10) and for each $\alpha$ it is shown above that $\text{SD}(\text{Tamper}_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma}; \text{Patch}(D_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma}, m)) \leq \epsilon + \delta$; (ii) follows because $\tilde{M}$ and $\tilde{M}$ are defined in the same way (see $D_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma}$ and $D_{\alpha}^{f_{\overline{S}_r\cdot|\cdot},\gamma}$) with different distributions $X_{\overline{S}_r}$ and $Y_{\overline{S}_r}$, which are $\epsilon$ close according to privacy of wiretap II.

The rate of the coding scheme is $\frac{k}{n} < \frac{r}{n}$, which according to Lemma 2 is upper bounded by $1 - \rho$. \hfill $\Box$

**Lemma 5.** When $\rho_w < \frac{1-\rho}{2}$, the coding scheme $(\text{Enc}, \text{Dec})$ in Theorem 1 is a $(\rho, \rho_w)$-NMC with exact security $2\epsilon + \delta$, if $\rho \geq \rho_r + \rho_w$.

**Proof.** This is a special case of Theorem 4 when $\rho_w \leq \frac{1-\rho}{2}$. In this case the number of overwrite components can not exceed $\frac{|S_r|}{2}$. Following the proof steps of the theorem, the case $n_{\text{ow}}^{\text{sw}} > \frac{|S_r|}{2}$ in the proof will not occur and one only has to make sure non-malleability is provided for $n_{\text{ow}}^{\text{sw}}$ values in the range $[0, n\rho_w]$. Let $S$ be the index set of the non-overwrite components of $\Delta q^\alpha$ (or equivalently the overwrite components of $q^\alpha$) in $\overline{S}_r$. We have $|S| \leq n\rho_w$ and hence $|S_r \cup S| = |S_r| + |S| \leq n(\rho_r + \rho_w)$. Now (14) will hold as long as the $(\rho, \epsilon)$-WT satisfies $\rho \geq \rho_r + \rho_w$. The rest is identical to the proof above (using only Case 1). \hfill $\Box$

It has been proved that the capacity of binary $(\rho, \epsilon)$-WT codes with indistinguishability security is $1 - \rho$. It is however unknown if linear $(\rho, \epsilon)$-WT codes can achieve this rate. The rate of the resulting NM-code is at most $1 - \rho$, which is less than $1 - \rho_r$ the achievable rate of $(\rho_r, \rho_w)$-NMC. We leave explicit construction of capacity-achieving $(\rho_r, \rho_w)$-NMC as an open question.

**Constructions of linear wiretap II codes.** The construction in Theorem 4 requires a Wiretap II code with leakage parameter $\rho = \frac{1/2 - \rho}{2} \geq \frac{1}{2}$. The following explicit construction (coset coding) gives binary linear $(\rho, 0)$-WT codes.

**Lemma 6 ([24]).** Let $G_{R(k, -k)}$ be a generator matrix of an $[n, n - k, d]$-code $C$ with dual distance $d^\perp$. Append $k$ rows to $G$ such that the obtained matrix $\begin{bmatrix} G & G \end{bmatrix}$
is of full rank. Define the encoder $\text{WTenc} : \mathbb{F}_q^k \to \mathbb{F}_q^n$ as follows.

$$\text{WTenc}(m) = [R \; m] \begin{bmatrix} G \\ \hat{G} \end{bmatrix},$$

where $R \sim \mathbb{F}_q^{n-k}$.

The message set $\mathbb{F}_q^k$ is in one-to-one correspondence with the cosets $\mathbb{F}_q^n/C$ of $C$ in the space $\mathbb{F}_q^n$. The decoder $\text{WTdec}$ uses a parity-check matrix $H$ of the code $C$ to efficiently identify the coset of the received word and output the corresponding message. Then $(\text{WTenc}, \text{WTdec})$ is a linear $(d_{\perp} - 1, n, 0)$-WT code.

Binary linear codes with minimum distance $d > n/2$ exist; see for example [19]. Instantiating $C$ in Lemma 6 with the dual of such codes result in binary linear $(\rho, 0)$-WT codes with $\rho \geq \frac{1}{2}$.

An explicit family of $(0, 1)_{\text{BIT}}$-NMC. For a $[2^h - 1, 2^h - 1 - h, 3]$-Hamming code, the dual code is a $[2^h - 1, h, 2^h - 1]$-Simplex code and has $d_{\perp} = 2^h - 1$. The WtII code can tolerate $\rho = \frac{1}{2}$ with $\varepsilon = 0$. Using the AMD code in Lemma 1 with this code gives a bit-wise independent non-malleable code of length $2^h - 1$. This is an explicit construction of non-malleable codes for the function family $\mathcal{F}^{[n]}_{\text{BIT}}$. This code can be made linear time encoding/decoding if the AMD construction is replaced with the linear time AMD construction in [27].

5 Applications to Communication Security

Our motivation for introducing the function family $\mathcal{F}^{[n]}_{\text{BIT}}$ is to model physical layer adversaries. In the following we give two applications of the NM-codes for $\mathcal{F}^{[n]}_{\text{BIT}}$ family in widely studied communication settings. In both cases we only consider one round protocols.

5.1 Wiretap Channel II with Active Adversary

Wiretap II model with active adversary was first studied in [35], where the eavesdropped and tampered components were restricted to the same set. In the model proposed in [24] the adversary can read a fraction $\rho_r$, and add noise to a fraction $\rho_w$, and the goal is to provide secrecy (indistinguishability) and correct message recovery. It was proved that the rate upper bound for these codes is $1 - \rho_r - \rho_w$, and so when $\rho_r + \rho_w > 1$, one needs to relax privacy or reliability requirements. We consider a wiretap II model where the active adversary can tamper with the codeword using functions in $\mathcal{F}^{[n]}_{\text{BIT}}$.

Definition 9. A $(\rho_r, 1)$-active adversary wiretap II channel is a communication channel between Alice and Bob that is (partially) controlled by an adversary Eve with two capabilities: Read and Write.

- Read: Eve selects a fraction $\rho_r$ of the components of the codeword to read.
Write: Eve uses the read components to add errors to, or write over, possibly all components of the codeword. This is equivalent to applying a function in $F_{\text{BIT}}^{[n],\rho_r,1}$ to the codeword.

Codes for this channel must provide security (indistinguishability) and non-malleability.

**Definition 10.** A $(\rho_r,1)$-active adversary wiretap II code is a coding scheme $(\text{Enc},\text{Dec})$ that guarantees the following two security properties.

- **Secrecy:** For any pair of messages $m_0$ and $m_1$, any reading set $S_r \subset [n]$ of size $|S_r| \leq n\rho_r$, $SD(\text{Enc}(m_0)_{S_r};\text{Enc}(m_1)_{S_r}) \leq \varepsilon$.

- **Non-malleability:** $(\text{Enc},\text{Dec})$ is non-malleable with respect to $F_{\text{BIT}}^{[n],\rho_r,1}$.

The capacity of a $(\rho_r,1)$-active adversary wiretap II channel is the highest achievable rate of coding schemes for this channel. Using the results in Section 3.2 we can prove the following theorem.

**Theorem 5.** The capacity of $(\rho_r,1)$-active adversary wiretap II code is $1 - \rho_r$.

**Proof.** We first show that a strong $(\rho_r,1)_{\text{BIT}}$-NMC is a $(\rho_r,1)$-active adversary wiretap II code. The secrecy property follows from Lemma 3. The non-malleability property is satisfied because a strongly non-malleable coding scheme is non-malleable. The lower bound $1 - \rho_r$ for strong $(\rho_r,1)_{\text{BIT}}$-NMC then gives a lower bound for $(\rho_r,1)$-active adversary wiretap II code.

The upper bound $1 - \rho_r$ follows because a $(\rho_r,1)$-active adversary wiretap II code is a wiretap II code with leakage $\rho_r$ (passive adversary). \qed

We have the following two explicit constructions of efficient $(\rho_r,1)$-active adversary wiretap II codes using our constructions of $(\rho_r,\rho_w)_{\text{BIT}}$-NMC. Constructing capacity-achieving $(\rho_r,1)$-active adversary wiretap II codes is an open question.

**Theorem 6.** The constructions in Theorem 3 with $t' > n\rho_r$ and $d' > n(1 - \rho_r)/4$, and Theorem 4 with $\rho = \frac{1+\rho_r}{2}$, each gives a $(\rho_r,1)$-active adversary wiretap II code.

**Proof.** The non-malleability property follows directly from Theorem 3 and Theorem 4 respectively. The secrecy property follows from the $t'$-uniform property of LECS and the privacy of the $(\rho,\varepsilon)$-WT code due to the parameter choices $t' > n\rho_r$ and $\rho = \frac{1+\rho_r}{2} > \rho_r$, respectively. \qed

### 5.2 Secure Message Transmission in Networks

In the model of Secure Message Transmission (SMT) \cite{5}, Alice is connected to Bob by a set of $n$ node-disjoint network paths (also called wires). The adversary can adaptively choose a subset of paths to eavesdrop and arbitrarily modify.
Although the original model considered adversaries who can select possibly distinct sets of wires for listening, corrupting, and blocking, SMT problem has been mainly studied for \((t, n)\)-threshold adversaries who adaptively select \(t\) out of \(n\) wires and arbitrarily modify them. In the following we only consider this model.

A \(1\)-round \((\varepsilon, \delta)\)-SMT protocol is a coding scheme with a pair of algorithms \((\text{SMTenc}, \text{SMTdec})\): \text{SMTenc} encodes a message \(ms\) in \(M\) to a codeword (also called protocol transcript) \(c = (c_1, \ldots, c_n) \in (\mathbb{F}_q)^n\) where \(c_i\) is sent over wire \(i\) (referred to as wire \(i\) transcript), and \text{SMTdec} decodes the received transcripts to \(mr\) in \(M \cup \{\bot\}\), guaranteeing privacy loss (indistinguishability security) is at most \(\varepsilon\), and probability of error \((\Pr[ms \neq mr])\) is bounded by \(\delta\).

It has been proved \cite{20} that \((\varepsilon, \delta)\)-SMT protocols exist only if \(n \geq 2t + 1\), and this is irrespective of the number of protocol rounds. In the following our goal is to show that one can remove this restriction if the reliability goal is reduced to non-malleability.

We consider adversaries that tamper with the protocol transcript according to functions in a tampering function family defined below. We consider an SMT adversary with the following capabilities: the adversary (i) controls \(t = n\rho_r\) wires and (ii) tampers obliviously with all other wires by either (algebraically) adding an offset, or setting the value to a new value (overwrite). Compared to the traditional threshold \((t, n)\) adversary, this new adversary has the extra capability of tampering with all the wires also. We consider the following set of symbol-wise independent Add and Overwrite (AO) tampering functions.

\[
\mathcal{F}^{[n]}_\text{AO} = (\mathcal{F}^{\text{add}} \cup \mathcal{F}^{\text{ow}})^n,
\]

where \(\mathcal{F}^{\text{add}} = \{f_{\Delta}(x) = x + \Delta | \Delta \in \mathbb{F}_q\}\) denotes the set of additive tampering functions and \(\mathcal{F}^{\text{ow}} = \{f_c(x) = c | c \in \mathbb{F}_q\}\) denotes the set of overwrite tampering functions.

Relation with \(\mathcal{F}^{[n]}_\text{BIT}\). Generalisation of \(\mathcal{F}^{[n]}_\text{BIT}\) from binary to \(q\)-ary alphabet \(\mathbb{F}_q\), is symbol-wise independent tampering family \(\mathcal{F}^{[n]}_\text{SIT} \overset{\text{def}}{=} (\mathcal{F}^{\text{all}})^n\) where \(\mathcal{F}^{\text{all}}\) denotes all functions from \(\mathbb{F}_q\) to \(\mathbb{F}_q\). The class \(\mathcal{F}^{[n]}_\text{AO}\) is a subset of \(\mathcal{F}^{[n]}_\text{SIT}\) and has size \(|\mathcal{F}^{[n]}_\text{AO}\| = (2q)^n\). This is much smaller than \(\mathcal{F}^{[n]}_\text{SIT}\) that is of size \(|\mathcal{F}^{[n]}_\text{SIT}\| = (q^n)^n\). It is only in the special case for \(q = 2\), we have \(\mathcal{F}^{[n]}_\text{AO} = \mathcal{F}^{[n]}_\text{BIT}\).

Definition 11.

\[
\mathcal{F}^{[n],\rho_r,1}_\text{AO} \overset{\text{def}}{=} \left\{ f_{S_r,[n],g} | S_r \in \mathbb{F}^{[n],\rho_r}_q, g : \mathbb{F}^{[n],\rho_r}_q \to \mathcal{F}^{[n]}_\text{AO} \right\},
\]

where the tampering function \(f_{S_r,[n],g} : \mathbb{F}^n_q \to \mathbb{F}^n_q\) is given as follows: for \(x \in \mathbb{F}^n_q\), depending on the value of \(x_{S_r}\), \(g\) selects a symbol-wise tampering function \(g(x_{S_r})\) from \(\mathcal{F}^{[n]}_\text{AO}\) which is applied to \(x\). That is,

\[
f_{S_r,[n],g}(x) = g(x_{S_r})(x).
\]

To simplify notations, we let \(g^{x_{S_r}} \overset{\text{def}}{=} g(x_{S_r})\) and write \(g^{x_{S_r}} = (g_1^{x_{S_r}}, \ldots, g_n^{x_{S_r}})\), where \(g_i^{x_{S_r}}\) is either an additive function or an overwrite function, for \(i = 1, \ldots, n\).
Definition 12. A (1-round) \((n, F, \epsilon, \delta)\)-non-malleable secure message transmission or \((n, F, \epsilon, \delta)\)-NM-SMT is a protocol over \(n\) wires, defined by a pair of algorithms \((\text{SMTenc}, \text{SMTdec})\), that for an adversary with access to corruption strategies in \(F\), the following properties are satisfied:

- **Secrecy:** For any pair of messages \(m_0, m_1 \in \mathcal{M}\) and for any adversary strategy \(A\) embodied by \(F\),
  \[
  \max_{m_0, m_1} SD(\text{View}_A(\text{SMTenc}(m_0)); \text{View}_A(\text{SMTenc}(m_1))) \leq \epsilon,
  \]
  where \(\text{View}_A(\cdot)\) is a random variable representing leakage.

- **Non-malleability:** \((\text{SMTenc}, \text{SMTdec})\) is non-malleable with respect to \(F\) and with exact security \(\delta\).

Theorem 7. The construction in Theorem 4 with a \((1+\rho_r, \epsilon)\)-WT code over the alphabet \(F_q\) and an AMD code with \(\delta\)-tamper detection security gives a \((n, F_{\mathcal{M}_0}^{[n], \rho_r, 1}, \epsilon, 2\epsilon + \delta)\)-NM-SMT.

**Proof.** The construction in Theorem 4 with the above parameter setting gives a \(q\)-ary NM-code with respect to \(F_{\mathcal{M}_0}^{[n], \rho_r, 1}\). The proof relies on the properties of difference function that will hold for functions in \(F_{\mathcal{M}_0}^{[n]}\) only (and not \(F_{\mathcal{SIT}}^{[n]}\)). Secrecy follows from the indistinguishability security of \(q\)-ary wiretap II code. \(\Box\)

Linear \((1+\rho_r, \epsilon)\)-WT codes over large alphabet can be constructed using the coset code construction in Lemma 6 using for example a Maximum Distance Separable (MDS) code. This gives explicit \((1+\rho_r, 0)\)-WT codes with rate \(1 - \frac{1+\rho_r}{2} \).

6 Conclusion

We proposed a family of bitwise tampering functions that were motivated by physical layer adversaries and were specified by a pair of parameters \((\rho_r, \rho_w)\), defining the eavesdropping and tampering capabilities of the adversary. Allowing the adversary to select tampering based on the eavesdropped information models powerful adversaries and results in a class of functions that is much larger (double exponential) than the widely studied independent bit tampering class. We defined non-malleable code with respect to this class and proved a number of rate bounds that fully characterize capacity of codes that provide strong non-malleability, as well as capacity of (default) non-malleable codes when \(\rho_r \leq \rho_w\).

We also gave two modular constructions, with the second one using wiretap II codes that, using coset code construction of these codes, effectively constructs NM-codes from linear error correcting codes and AMD codes.

There are many open questions and directions for future research. We left tight upper bound and capacity of (default) NM-codes when \(\rho_w < \rho_r\), as an open problem. Also none of our construction are capacity achieving, and so construction of capacity achieving codes remains open.
Our main results are for binary codes. However in SMT setting, transcripts of wires are $q$-ary values. Extending the results for $F_{\text{BIT}}^{[n],\rho_r,\rho_w}$ to $q$-ary case, strengthens our current NM-SMT construction for function class $F_{\text{AO}}^{[n],\rho_r,1}$, and allows more powerful adversaries in network setting be tolerated. Our explicit construction for NM-SMT uses $q$-ary wiretap II codes. One can also adapt the construction of $q$-ary codes in [27] to obtain secure NM-SMT. We leave this for future work.

Non-malleability was originally motivated for providing tamper resilience in tamperable storages. Our work is the first to consider application in well motivated communication settings of wiretap II codes with active adversaries, and secure communication in networks. Other applications of non-malleability in communication scenarios, including modelling collusion attacks, are interesting directions for future work. The function class $F_{\text{BIT}}^{[n],\rho_r,\rho_w}$ assumes tampering on the components of $S_w$ are independent of each other, and depends on the read values over $S_r$, only. A more general case is when tampering of bits in $S_w$ are correlated.

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Appendices

A. Appendix to the proof of Theorem 3

1. $n_{\alpha}^{\omega} \in [0, t' - n\rho_{r}]$. We consider the effect of the difference function defined by (12) on codewords in $C_\alpha$. According to above, the number of non-overwrite components of $\Delta g^{\alpha}$ in $\bar{S}_r$ will be at most $t' - n\rho_{r}$, and according to the $t'$-uniform property of LECSS, these components are each uniformly distributed, and are jointly $(t' - n\rho_{r})$-wise independent. This means that the non-overwrite components of $\Delta g^{\alpha}(X)$ in $\bar{S}_r$ are uniformly distributed over $\{0, 1\}^{n_{\alpha}^{\omega}}$. The rest of the components of $\Delta g^{\alpha}(X)$ in $\bar{S}_r$ are overwrite components (correspond to non-overwrite components of $g^{\alpha}$). Thus the distribution $\Delta g^{\alpha}(X)$ is independent of the input (an AMD codeword $\text{AMDenc}(m)$) of LECSS. The third step of tampering experiment is applying the decoding function $\text{Dec}$ on the tampered codeword (of the NM-code). Using $g^{\alpha}(x) = x \oplus \Delta g^{\alpha}(x)$, we have

$$\text{Dec}(g^{\alpha}(X)) = \text{Dec}(X \oplus \Delta g^{\alpha}(X)) = \text{AMDdec}(\text{AMDenc}(m) \oplus \text{LECSSdec}(\Delta g^{\alpha}(X))),$$

where the second equality follows from the linearity of the LECSS.

To find the distribution of the output of the tampering experiment in this case, denoted by $\text{Tamper}_{m}^{f_{S_r,n},g}|X_{S_r}=\alpha$, we note that:

- Conditioned on $\text{LECSSdec}(\Delta g^{\alpha}(X)) = 0^{t'}$, the output of $\text{Dec}$ is going to be $\text{AMDdec}(\text{AMDenc}(m)) = m$.
- Consider $\text{LECSSdec}(\Delta g^{\alpha}(X)) \neq 0^{t'}$. Since the distribution of $\Delta g^{\alpha}(X)$ is completely determined by $f_{S_r,n},g$ and $\alpha$, it is independent of the randomness of the AMD code. According to Lemma 1 the probability that the AMD decoder not outputting $\perp$ is at most $\Pr[\text{LECSSdec}(\Delta g^{\alpha}(X)) \neq 0^{t'}] \cdot \delta \leq \delta$.

Thus we can define the distribution $\mathcal{D}_{\alpha}^{f_{S_r,n},g}$ as follows:

$$\mathcal{D}_{\alpha}^{f_{S_r,n},g} \overset{def}{=} \begin{cases} z \leftarrow (\Delta g^{\alpha}(X)|X_{S_r} = \alpha) \\
\text{Output same*}, \text{ if LECSSdec}(z) = 0^{t'}; \text{Output } \perp, \text{ otherwise.}
\end{cases}$$

This distribution will be different from the tampering experiment when the AMD decoder fails to output $\perp$ for $z \not\in \{0^{t'}, \perp\}$. We then have

$$\left(\text{Tamper}_{m}^{f_{S_r,n},g}|X_{S_r}=\alpha\right) \overset{\delta}{=} \text{Patch}(\mathcal{D}_{\alpha}^{f_{S_r,n},g},m).$$
2. $n^w_{\alpha} \in \left(t' - n\rho_r, \frac{|S_r|}{2}\right]$. The distribution $D^f_{\alpha, \rho_r, d}$ will only output ⊥, using LECSS decoder error detection property. Firstly using the LECSS linearity, the decoder output will depend on the result of $\Delta g^\alpha$ on a codeword (in $C_\alpha$). The given number of $n^w_{\alpha}$ translates into the same number of non-overwrite for $\Delta g^\alpha$ on components in $S_r$, and the rest of components being overwrite function. The codeword components of $C_\alpha$ in $S_r$ are $(t' - n\rho_r)$-uniform, and as said earlier non-overwrite functions do not affect a column probability distribution, which are uniform because $C_\alpha$ in $S_r$ is $(t' - n\rho_r)$-uniform.

If none of the vectors in the list $\text{Array}(\Delta g^\alpha(x)|x \in C_\alpha)$ correspond to a valid codeword of LECSS, the LECSS decoder output will be always ⊥. If there is a vector $\omega$ in $\text{Array}(\Delta g^\alpha(x)|x \in C_\alpha)$ that corresponds to a LECSS codeword, there will be an undetected error. Note that $\omega$ may appear more than once in the list. Next, the distance property of LECSS (together with $(t' - n\rho_r)$-uniform property) is utilised to claim that the density of valid LECSS codewords in $\text{Array}(\Delta g^\alpha(x)|x \in C_\alpha)$ is very small. The argument had been used in [34] Proof of Theorem 4.1, Case 3 for the function class $F^{[n]}_{\text{bit}}$, to quantify the decoder error. Authors showed that for a $(t, d)$-LECSS if the minimum distance is $d > n/4$, the error probability is given by,

$$\Pr[\text{LECSSdec}(\Delta) \neq \perp] \leq \frac{1}{2^t} + \left(\frac{t}{n(d/n - 1/4)^2}\right)^{t/2},$$

where $\Delta$ is a vector random variable of $n$ components, more than half but less than $n - t$ of which are fixed values and the rest of components are $t$-uniform.

We use the same argument and make the following adjustments. Firstly, the tampering functions are applied to $C_\alpha$ and so the tampered words $\text{Array}(\Delta g^\alpha(x)|x \in C_\alpha)$ will have fixed values on index set $S_r$. So the part of components that can be different (between two vectors in the list) are in $S_r$. For our proof we consider $S_r$. Thus we only need $d' > (n - n\rho_r)/4$. Also the non-overwrite components of $\text{Array}(\Delta g^\alpha(x)|x \in C_\alpha)$ in $S_r$ are $(t' - \rho_r,n)$-uniform and so we have,

$$\Pr[\text{LECSSdec}(\Delta g^\alpha(X)) \neq \perp|X_{S_r} = \alpha] \leq \frac{1}{2^{t' - n\rho_r}} + \left(\frac{t' - n\rho_r}{n(d'/n - 1/4)^2}\right)^{t'/2}.$$

3. $n^w_{g_{\alpha}} \in \left(\frac{|S_r|}{2}, n - t\right]$. The distribution $D^f_{\alpha, \rho_r, d}$ will only output ⊥, using LECSS decoder error detection property. We study $\text{Array}(g^\alpha(x)|x \in C_\alpha)$, the list of tampered codewords, and bound the probability of LECSS decoder cannot detect the error. The argument is similar to above. This corresponds to Case 4 in the proof of Theorem 4.1 in [34]. Using the required adjustment as outlined above, we will have
\[
\Pr [ \text{LECSSdec} (g^\alpha(X)) \neq \perp | X_{S_r} = \alpha ] \leq \frac{1}{2^{t'-n\rho_r}} + \left( \frac{t' - n\rho_r}{n \left( \frac{d'}{\rho_r} - \frac{1}{2} \right)^2} \right)^{t'-n\rho_r/2} .
\]

(19)

(It is worth noting that the argument in the above two cases use two different sets \( \Delta g^\alpha(x) | x \in \mathcal{C}_\alpha \) and \( g^\alpha(x) | x \in \mathcal{C}_\alpha \) that have the property that overwrite components in one, corresponds to non-overwrite component in the other. The choice of the list is to allow many overwrite components and minimise the \( d' \)-distance requirement.)

4. \( n_{sw}^{S_r} \in [n - t', |S_r|] \).

This is the case that most of the codeword is overwritten, and non-overwritten part is uniformly distributed. This is because less than \( t' \) components are not overwritten, and the set of vectors \( \tilde{C}_\alpha \) is \( t' - n\rho_r \)-uniform. Thus the distribution is independent of \( m \), and the decoder output distribution will have the same property also. The distribution \( D^f_{S_r,\alpha}[n] \) in this case is defined as follows:

\[
D^f_{S_r,\alpha}[n] \overset{def}{=} \begin{cases} 
  z \leftarrow g^\alpha(Y) \\
  \text{Output Dec}(z)
\end{cases}
\]

Since the simulation and the tampering experiment are identical in this case,

\[
\left( \text{Tamper}^f_{m}[S_r,\alpha] | X_{S_r} = \alpha \right) \equiv \text{Patch}(D^f_{S_r,\alpha}[n], m).
\]