ON EDGE-PRIMITIVE AND 2-ARC-TRANSITIVE GRAPHS

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Abstract. A graph is edge-primitive if its automorphism group acts primitively on the edge set. In this short paper, we prove that a finite 2-arc-transitive edge-primitive graph has almost simple automorphism group if it is neither a cycle nor a complete bipartite graph. We also present two examples of such graphs, which are 3-arc-transitive and have faithful vertex-stabilizers.

Keywords. Primitive group, almost simple group, edge-primitive graph, 2-arc-transitive graph.

1. Introduction

All graphs and groups considered in this paper are assumed to be finite.

A graph in this paper is a pair $\Gamma = (V, E)$ of a nonempty set $V$ and a set $E$ of 2-subsets of $V$. The elements in $V$ and $E$ are called the vertices and edges of $\Gamma$, respectively. The number $|V|$ of vertices is called the order of $\Gamma$. For $v \in V$, the set $\Gamma(v) = \{ u \in V \mid \{u, v\} \in E \}$ is called the neighborhood of $v$ in $\Gamma$, while $|\Gamma(v)|$ is the valency of $v$. We say that $\Gamma$ has valency $d$ or $\Gamma$ is $d$-regular if its vertices all have equal valency $d$. For an integer $s \geq 1$, an $s$-arc in $\Gamma$ is an $(s + 1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $\{v_i, v_{i+1}\} \in E$ and $v_i \neq v_{i+2}$ for all possible $i$. A 1-arc is also called an arc.

Let $\Gamma = (V, E)$ be a graph. A permutation $g$ on $V$ is called an automorphism of $\Gamma$ if $\{u^g, v^g\} \in E$ for all $\{u, v\} \in E$. Let $\text{Aut}\Gamma$ denote the set of all automorphisms of $\Gamma$. Then $\text{Aut}\Gamma$ is a subgroup of the symmetric group $\text{Sym}(V)$, and called the automorphism group of $\Gamma$. Note that the group $\text{Aut}\Gamma$ has a natural action on the edge set $E$ (and also on the set of $s$-arcs). The graph $\Gamma$ is called edge-transitive if $E \neq \emptyset$ and for each pair of edges there exists some $g \in \text{Aut}\Gamma$ mapping one of these two edges to the other one. (Similarly, we may define vertex-transitive, arc-transitive or $s$-arc-transitive graphs.) An edge-transitive graph is called edge-primitive if some (and hence every) edge-stabilizer, the subgroup of its automorphism group which fixes a given edge, is a maximal subgroup of the automorphism group.

It is well-known that edge-transitive graphs and hence edge-primitive graphs are either bipartite or vertex-transitive. As a subclass of the edge-transitive graphs, edge-primitive graphs posses more restrictions on their symmetries and automorphism groups. For example, a connected edge-primitive graph is necessarily arc-transitive provided that it is not a star graph. In [9], appealing to the O’Nan-Scott Theorem for (quasi)primitive...
groups [22], Giudici and Li investigated the structural properties of edge-primitive graphs, particularly, on their automorphism groups. Let $\Gamma = (V, E)$ be an arc-transitive and edge-primitive graph which is neither a cycle nor a complete bipartite graph. If $\Gamma$ is bipartite then let $\text{Aut}^+ \Gamma$ be the subgroup of $\text{Aut} \Gamma$ preserving the bipartition. By [9], as a primitive group on $E$, only 4 of the eight O’Nan-Scott types for (quasi)primitive groups may occur for $\text{Aut}\Gamma$, say SD, CD, PA and AS. For the first two types, $\Gamma$ is bipartite and $\text{Aut}^+ \Gamma$ is quasiprimitive of type CD on each bipartite half. For the last two types, with one exception case, $\text{Aut}\Gamma$ or $\text{Aut}^+ \Gamma$ is quasiprimitive on $V$ or on each bipartite half respectively of the same type for $\text{Aut}\Gamma$ on $E$. In this paper, we will work on the types of $\text{Aut}\Gamma$ on $E$ and on $V$ under the further assumption that $\Gamma$ is 2-arc-transitive.

The interests for edge-primitive graphs arises partially from the fact that many (almost) simple groups may be represented as the automorphism groups of edge-primitive graphs. Consulting the Atlas [3], one may get first-hand such examples. For example, the sporadic Higman-Sims group HS is the automorphism group of a rank 3 graph with order 100 and valency 22, which is in fact a 2-arc-transitive and edge-primitive graph; the sporadic Rudvalis group Ru is the automorphism group of a rank 3 graph with order 4060 and valency 2304, which is edge-primitive but not 2-arc-transitive. Besides, the almost groups $\text{PSU}(3,5):2$, $M_{22}:2$, $J_2:2$ and McL:2 all have representations on edge-primitive graphs. The reader may refer to [11, 12, 18, 21, 26] for more examples of edge-primitive graphs which have almost simple automorphism groups. Of course, using the constructions given in [9], one can easily construct examples of edge-primitive graphs with automorphism groups not almost simple.

We have a strong impression from the known examples for edge-primitive graphs in the literature that a 2-arc-transitive and edge-primitive graph has almost simple automorphism group unless it is a cycle or a complete bipartite graph. Yet could it be so? Yes, it is true! We shall prove the following result in Section 3.

**Theorem 1.1.** Let $\Gamma = (V, E)$ be an edge-primitive $d$-regular graph for some $d \geq 3$. If $\Gamma$ is 2-arc-transitive, then either $\Gamma$ is a complete bipartite graph, or $\Gamma$ has almost simple automorphism group.

**Remarks on Theorem 1.1.**

1. Li and Zhang [18] proved that 4-arc-transitive and edge-primitive graphs have almost automorphism groups. Further, as a sequence of their classification on almost simple primitive groups with soluble point-stabilizers, they give a complete list for 4-arc-transitive and edge-primitive graphs.

2. By Theorem 1.1, appealing to the classification of almost simple groups with soluble maximal subgroups, it might be feasible to classify 2-arc-transitive and edge-primitive graphs with soluble edge-stabilizers.

2. **Preliminaries**

For the subgroups of (almost) simple groups, we sometimes follow the notation used in the Atlas [3], while we also use $\mathbb{Z}_l$ and $\mathbb{Z}_p^k$ to denote respectively the cyclic group of order $l$ and the elementary abelian group of order $p^k$. 
2.1. Primitive groups. In this subsection, $\Omega$ is nonempty finite set, and $G$ is a transitive subgroup of the symmetric group $\text{Sym}(\Omega)$. Let $\text{soc}(G)$ be the socle of $G$, that is, $\text{soc}(G)$ is generated by all minimal normal subgroups of $G$.

Consider the point-stabilizer $G_\alpha := \{g \in G \mid \alpha^g = \alpha\}$, where $\alpha \in \Omega$. Then

1. $G$ is primitive if $G_\alpha$ is a maximal subgroup of $G$;
2. $G$ is $\frac{3}{2}$-transitive if $G_\alpha$ is $\frac{1}{2}$-transitive on $\Omega \setminus \{\alpha\}$, that is, all $G_\alpha$-orbits on $\Omega \setminus \{\alpha\}$ have equal length > 1;
3. $G$ is a Frobenius group if $G_\alpha$ is semiregular on $\Omega \setminus \{\alpha\}$;
4. $G$ is 2-transitive if $G_\alpha$ is transitive on $\Omega \setminus \{\alpha\}$.

Note that (4) implies (1) and (2), and (2) implies (1) or (3) (refer to [29, Theorem 10.4]).

Let $1 \neq N \leq G$, a normal subgroup of $G$. Then $N$ is $\frac{1}{2}$-transitive, and $N_\alpha = N \cap G_\alpha \leq G_\alpha$, and so $G_\alpha$ is contained in the normalizer $N_G(N_\alpha)$ of $N_\alpha$ in $G$. Thus, if $G_\alpha$ is maximal then either $N_\alpha \leq G$ or $N_G(N_\alpha) = G_\alpha$. The former case yields $N_\alpha = 1$, while the latter case gives

$$N_N(N_\alpha) = N \cap N_G(N_\alpha) = N \cap G_\alpha = N_\alpha.$$ 

Then we have following simple fact for primitive groups.

**Lemma 2.1.** If $G$ is primitive and $N \neq 1$ then either $N$ is regular on $\Omega$ or $N_\alpha$ is self-normalized; if $G$ is 2-transitive and $N \neq 1$ then $N$ is either regular or $\frac{3}{2}$-transitive on $\Omega$.

For an almost simple 2-transitive group $G$, each non-trivial normal subgroup $N$ of $G$ is primitive, and in fact 2-transitive except for the case where $N = \text{soc}(G) = \text{PSL}(2,8)$ acting on 28 pionts, refer to [1, page 197, Table 7.4]. Next we consider the normal subgroups of affine 2-transitive groups. Refer to [1, page 195, Table 7.3] for a complete list of affine 2-transitive groups. We consider the affine 2-transitive groups in their natural actions.

**Lemma 2.2.** Let $G$ be an affine 2-transitive group and $1 \neq N \leq G$. If $N$ is imprimitive on $\Omega$, then $N$ is a soluble Frobenius group, $N_0$ is cyclic, and either $G_0 \leq \Gamma L(1,q)$ or $N_0 \leq \mathbb{Z}(G_0)$, where $q$ is not a prime.

**Proof.** Assume that $N$ is imprimitive. Then $N \neq G$, and so $N_0 \neq G_0$. Further, by Lemma 2.1 and [29, Theorem 10.4], $N$ is a Frobenius group. Let $|\Omega| = p^k$ for a prime $p$. We may write $G_0 \leq \Gamma L(k,p)$, $G = \mathbb{Z}_p^k:G_0$ and $N = \mathbb{Z}_p^k:N_0$. Since $N$ is imprimitive, $N_0$ is not maximal in $N$, and thus $N_0$ is a normal reducible subgroup of $G_0$. Then, by [13, Lemma 5.1], $N_0$ is cyclic and $|N_0|$ is a divisor of $p^l - 1$, where $l < k$ and $l \mid k$. Finally, the lemma follows from checking all affine 2-transitive groups one by one. $\square$

If every minimal normal subgroup of $G$ is transitive on $\Omega$, then $G$ called a quasiprimitive group. Praeger [22, 24] generalized the O’Nan-Scott Theorem for primitive groups to quasiprimitive groups, which says that a quasiprimitive group has one of the following eight types: HA, HS, HC, TW, AS, SD, CD and PA. In particular, if $G$ is quasiprimitive then $G$ has at most two minimal normal subgroups, and if two (for HS and HC) then they are isomorphic and regular.

Suppose that $G$ has a transitive insoluble minimal normal subgroup $N$. Then $G = NG_\alpha$ for $\alpha \in \Omega$. Write $N = T_1 \times \cdots \times T_k$ for isomorphic nonabelian simple groups $T_i$.
and integer $k \geq 1$. Then $G_\alpha$ acts transitively on $\{T_i \mid 1 \leq i \leq k\}$ by conjugation. Note that, for $g \in G_\alpha$ and $1 \leq i \leq k$,

$$(T_i)_\alpha^g = (T_i \cap G_\alpha)^g = T_i^g \cap G_\alpha^g = (T_j)_\alpha^g$$ for some $j$.

Then $G_\alpha$ acts transitively on $\{(T_i)_\alpha \mid 1 \leq i \leq k\}$ by conjugation. Clearly, $(T_1)_\alpha \times \cdots \times (T_k)_\alpha \leq N_\alpha$; however, the equality is not necessarily holds even if $G$ is quasiprimitive. A sufficient condition for this equality is that $G$ is primitive and of type AS or PA, refer to [4, Theorem 4.6] and its proof. In survey, we have the simple fact as follows.

**Lemma 2.3.** Assume that $G$ has a transitive minimal normal subgroup $N = T_1 \times \cdots \times T_k$, where $T_i$ are isomorphic nonabelian simple groups. Let $\alpha \in \Omega$. Then $G_\alpha$ acts transitively on $\{(T_i)_\alpha \mid 1 \leq i \leq k\}$ by conjugation. If further $G$ is primitive and of type AS or PA, then $N_\alpha = (T_1)_\alpha \times \cdots \times (T_k)_\alpha$.

### 2.2. Locally-primitive graphs.

In this subsection, $\Gamma$ is a connected $d$-regular graph for some $d \geq 3$, and $G \leq \text{Aut}\Gamma$. Assume further that the graph $\Gamma$ is $G$-locally primitive, that is, $G_\alpha$ acts primitively on $\Gamma(v)$ for all $v \in V$.

Fix an edge $\{u, v\} \in E$. Note that $G_v$ induces a primitive permutation group $G_v^\Gamma(v)$ (on $\Gamma(v)$). Let $G_v^{[1]}$ be the kernel of $G_v$ acting on $\Gamma(v)$. Then $G_v^\Gamma(v) \cong G_v/G_v^{[1]}$. Set $G_v^{[1]} = G_v^{[1]} \cap G_v^{[2]}$. Then $G_v^{[1]}$ induces a normal subgroup of $(G_u^\Gamma(u))_v$ with the kernel $G_v^{[1]}$. Writing $G_v^{[1]}$ and $G_v$ in group extensions,

$$G_v^{[1]} = (G_v^{[1]}(G_v^{[1]}))^{\Gamma(u)}, \quad G_v = (G_v^{[1]}(G_v^{[1]}))^{\Gamma(u)}G_v^{\Gamma(v)}, \quad G_{uv} = G_v^{[1]}(G_v^{\Gamma(v)})_u.$$  

Assume that $G$ is transitive on $V$. Then $G_v^{[1]}$ is a $p$-group for some prime $p$, refer to [6]. Note that $G$ is transitive on the arcs of $\Gamma$. There is some element in $G$ interchanging $u$ and $v$. This implies that $(G_v^{[1]}(G_v^{[1]}))^{\Gamma(u)} \leq (G_v^{\Gamma(v)})_u \cong (G_v^{\Gamma(v)})_v$. Thus we have the following lemma.

**Lemma 2.4.** Assume that $G$ is transitive on $V$, and $\{u, v\} \in E$. Then $G_{uv}^{[1]}$ is a $p$-group, and $(G_v^{[1]}(G_v^{[1]}))^{\Gamma(u)}$ is isomorphic to a normal subgroup of a point-stabilizer in $G_v^{\Gamma(v)}$. In particular, $G_v$ is soluble if and only if $G_v^{\Gamma(v)}$ is soluble.

The graph $\Gamma = (V, E)$ is said to be $(G, s)$-arc-transitive if $\Gamma$ has an $s$-arc and $G$ acts transitively on the set of $s$-arcs of $\Gamma$, where $s \geq 1$. Note that $\Gamma$ is $(G, 2)$-arc-transitive if and only if $G$ is transitive on $V$, and $G_v^{\Gamma(v)}$ is a 2-transitive group for some (and hence every) $v \in V$. By [7, 27, 28], we have the following result.

**Theorem 2.5.** Assume that $\Gamma = (V, E)$ is $(G, 2)$-arc-transitive. Then $\Gamma$ is not $(G, 8)$-arc-transitive. Further,

1. if $G_{uv}^{[1]} = 1$ then $\Gamma$ is not $(G, 4)$-arc-transitive.
2. if $G_{uv}^{[1]} \neq 1$ then $G_{uv}^{[1]}$ is a nontrivial $p$-group, $O_p(G_v^{\Gamma(v)}) \neq 1$, $\text{PSL}(n, q) \leq G_v^{\Gamma(v)}$, and $|\Gamma(v)| = \frac{q^n - 1}{q - 1}$, where $n \geq 2$ and $q$ is a power of $p$; in this case, $\Gamma$ is $(G, 4)$-arc-transitive if and only if $n = 2$. 


3. The proof of Theorem 1.1

In this section, we let \( \Gamma = (V,E) \) be a connected graph of valency \( d \geq 3 \), and \( G \leq \text{Aut}\Gamma \). Assume that \( \Gamma \) is \( G \)-edge-primitive, that is, \( G \) act primitively on \( E \). Then, by [9, Lemma 3.4], \( G \) acts transitively on the arc set of \( \Gamma \). Thus, for an edge \( \{u,v\} \in E \), \( d = |G_v : G_{uv}| \) and \( |G_{\{u,v\}} : G_{uv}| = 2 \).

Let \( 1 \neq N \triangleleft G \). Then \( N \) is transitive on \( E \), and so either \( N \) is transitive on \( V \) or \( N \) has two orbits on \( V \); for the latter case, \( N_\nu \) is transitive on \( \Gamma(\nu) \). This implies that either \( G = NG_v \), or \( |G : (NG_v)| = 2 \) and \( N_{uv} = N_{\{u,v\}} \). Note that \( G = NG_{\{u,v\}} \) by the maximality of \( G_{\{u,v\}} \) or the transitivity of \( N \) on \( E \). We have

\[
|G| = \frac{|N||G_{\{u,v\}}|}{|NG_v|} = \frac{|N||G_{\{u,v\}}|}{|N_{\{u,v\}}|} = \frac{2|N||G_{uv}|}{2|N_u^2|} = \frac{2(N|G_v|)}{|dN_{\{u,v\}}|}.
\]

Then the next lemma follows.

Lemma 3.1. Let \( 1 \neq N \triangleleft G \). If \( N \) is transitive on \( V \) then \( 2|N_v| = d|N_{\{u,v\}}| \); if \( N \) is intransitive on \( V \) then \( |N_v| = d|N_{\{u,v\}}| = d|N_{uv}| \). In particular, \( N_v \neq 1 \) and \( N_v \neq N_{\{u,v\}} \).

Let \( K_{d,d} \) and \( K_{d+1} \) be the complete bipartite graph and complete graph of valency \( d \), respectively.

**Corollary 3.2.** Let \( 1 \neq N \triangleleft G \). Then either \( \Gamma \cong K_{d,d} \), or \( N_{uv} \neq 1 \) and \( N_{\{u,v\}} \) is self-normalized in \( N \), where \( \{u,v\} \in E \).

**Proof.** Assume that \( \Gamma \not\cong K_{d,d} \). Then, by the O'Nan-Scott Theorem and [9, Lemmas 6.1, 6.2 and Propersition 6.13], \( G \) has no normal subgroup acting regularly on \( E \). Thus \( N_{\{u,v\}} \neq 1 \), and so \( N_{\{u,v\}} \) is self-normalized in \( N \) by Lemma 2.1.

Suppose that \( N_{uv} = 1 \). Then \( N_{\{u,v\}} \) has order 2, and so \( N_{\{u,v\}} \leq C_N(N_{\{u,v\}}) \leq N_N(N_{\{u,v\}}) = N_{\{u,v\}} \). This implies that \( C_N(N_{\{u,v\}}) = N_N(N_{\{u,v\}}) \), and then \( N_{\{u,v\}} \) is a Sylow 2-subgroup of \( N \). By Burnside’s transfer theorem (refer [14, IV.2.6]), \( N \) has normal 2'-Hall subgroup, say \( M \). Then this \( M \) is normal in \( G \) and regular on \( E \), a contradiction.

By [9], if \( \Gamma \not\cong K_{d,d} \) then \( G \) has type SD, CD, AS or PA on \( E \); in particular, \( G \) has a unique (of course, insoluble) minimal normal subgroup. Thus, if \( \Gamma \not\cong K_{d,d} \) then \( G \) is insoluble, and so \( G_{\{u,v\}} \) is not abelian by [14, IV.7.4]. If \( G_{uv} \) is abelian the following result says that \( \Gamma \cong K_{d,d} \) or \( K_{d+1} \).

**Theorem 3.3.** Assume that \( \Gamma \not\cong K_{d,d} \). Let \( 1 \neq N \triangleleft G \).

1. If \( N_{\{u,v\}} \) has a normal Sylow subgroup \( P \neq 1 \) then \( P \) is also a Sylow subgroup of \( N \); in particular, \( N_{\{u,v\}} \) is not abelian.
2. If \( N_{uv} \) is abelian then \( N \) is transitive on the arc set of \( \Gamma \).
3. If \( N_{uv} \) is an abelian 2-group then \( \soc(G) = \PSL(2,q) \) and \( \Gamma \cong \operatorname{K}_{q+1} \), where \( q \) is a power of some prime with \( q - 1 \) a power of 2 greater than 8.
4. If \( G_{uv} \) is an abelian group then \( d = q \) and either \( \soc(G) \cong \PSL(2,q) \) and \( \Gamma \cong \operatorname{K}_{q+1} \), or \( \soc(G) = \operatorname{Sz}(q) \), \( \text{Aut}\Gamma = \text{Aut}(\operatorname{Sz}(q)) \) and \( \Gamma \) is \( (\operatorname{Sz}(q), 2) \)-arc-transitive, where \( q \) is a power of some prime.

**Proof.** (1) Assume that \( P \neq 1 \) is a normal Sylow \( p \)-subgroup of \( N_{\{u,v\}} \). Then \( P \) is a characteristic subgroup of \( N_{\{u,v\}} \), and so \( P \trianglelefteq G_{\{u,v\}} \) as \( N_{\{u,v\}} \trianglelefteq G_{\{u,v\}} \). Thus \( N_G(P) \geq \)}
are isomorphic nonabelian simple groups. This gives \( N_G(P) = N \cap N_G(P) = N \cap G_{\{u,v\}} = N_{\{u,v\}} \). Choose a Sylow \( p \)-subgroup \( Q \) of \( N \) with \( P \leq Q \). Then \( N_Q(P) \leq Q \cap N_G(P) = Q \cap N_{\{u,v\}} = P \). This yields \( P = Q \), so \( P \) is a Sylow \( p \)-subgroup of \( N \).

Suppose that \( N_{\{u,v\}} \) is abelian. Then \( N_{\{u,v\}} \leq C_N(P) \leq N_G(P) = N_{\{u,v\}} \), yielding \( C_N(P) = N_G(P) \). By Burnside’s transfer theorem, \( P \) has a normal complement \( H \) in \( N \), that is \( N = PH \) with \( P \cap H = 1 \) and \( H \leq N \). Note that \( H \) is a Hall subgroup of \( N \). It follows that \( H \) is characteristic in \( N \), and hence \( H \leq G \). Let \( P \) runs over the Sylow subgroup of \( N_{\{u,v\}} \). Then the resulting normal complements intersect at a normal complement of \( N_{\{u,v\}} \) in \( N \), which is normal in \( G \) and regular on \( E \). This contradicts Corollary 3.2. Therefore, \( N_{\{u,v\}} \) is nonabelian, and \( (1) \) of this theorem follows.

(2) Assume that \( N_{uv} \) is abelian. Then \( N_{uv} \neq N_{\{u,v\}} \) by \( (1) \), and thus \( (u, v) = (v, u)^x \) for some \( X \in N_{\{u,v\}} \). Since \( \Gamma \) is \( N \)-edge-transitive, \( \Gamma \) is \( N \)-arc-transitive.

(3) Assume that \( N_{uv} \) is an abelian 2-group. Recall that \( G \) has a unique minimal normal subgroup, say \( M \). Then \( M \leq N \), and \( (1) \) and \( (2) \) hold for \( M \). Then, since \( M_{uv} \) is an abelian 2-group, \( M_{\{u,v\}} \) is a Sylow 2-subgroup of \( M \), and \( M_{\{u,v\}} \) is not abelian.

Write \( M = T_1 \times \cdots \times T_k \), where \( T_i \) are isomorphic nonabelian simple groups. Recall that \( M_{\{u,v\}} \) is a Sylow \( q \)-subgroup of \( M \). For each \( i \), choose a Sylow \( 2 \)-subgroup of \( T_i \) with \( Q_i \leq M_{\{u,v\}} \). Then \( M_{\{u,v\}} = Q_1 \times \cdots \times Q_k \). Noting that \( Q_i \) are all isomorphic, every \( Q_i \) is nonabelian; otherwise, \( M_{\{u,v\}} \) is abelian, a contradiction. In particular, \( Q_1 \not\leq M_{uv} \).

Then \( M_{\{u,v\}} = M_{uv}Q_1 \), and so

\[
Q_2 \times \cdots \times Q_k \cong M_{\{u,v\}}/Q_1 = M_{uv}Q_1/Q_1 \cong M_{uv}/(M_{uv} \cap Q_1).
\]

Since \( M_{uv} \) is abelian, the only possibility is \( k = 1 \). Thus \( M = \text{soc}(G) \) is simple.

By [10, Corollary 5], \( M_{\{u,v\}} \) has cyclic commutator subgroup. Since \( M_{\{u,v\}} \) is nonabelian, by [2], \( M \) is isomorphic to one of the Mathieu group \( M_{11} \), \( \text{PSL}(2, q) \) (with \( q^2 - 1 \) divisible by 16), \( \text{PSL}(3, q) \) (with \( q \) odd) and \( \text{PSU}(3, q) \) (with \( q \) odd). If \( M \cong M_{11} \), then \( G = M \), and so \( M_{\{u,v\}} \) is maximal in \( M \); however, by the Atlas [3], a Sylow \( 2 \)-subgroup of \( M_{11} \) is not a maximal subgroup, a contradiction. Thus we next let \( M \cong \text{PSL}(2, q) \), \( \text{PSL}(3, q) \) or \( \text{PSU}(3, q) \).

Since \( M \) is transitive on \( E \), we know that \( |E| = |M : M_{\{u,v\}}| \) is odd. Thus \( G \) is an almost simple primitive group (on \( E \)) of odd degree. Noting that \( M_{\{u,v\}} = M \cap G_{\{u,v\}} \), by [20], \( M_{\{u,v\}} \) is known. Notice that the isomorphisms among simple groups (refer to [15, Proposition 2.9.1 and Theorem 5.1.1]). Since \( M_{\{u,v\}} \) is a Sylow \( 2 \)-subgroup of \( M \), the only possibility is that \( M \cong \text{PSL}(2, q) \), and \( M_{\{u,v\}} \) is the stabilizer of some orthogonal decomposition of a natural projective module associated with \( M \) into 1-dimensional subspaces. It follows that \( M_{\{u,v\}} \cong D_{q-1} \) or \( D_{q+1} \), and so \( M_{uv} \cong \mathbb{Z}_{q-1} \) or \( \mathbb{Z}_{q+1} \), respectively. Since \( M \) is transitive on the arcs of \( \Gamma \), we have \( |M_v : M_{uv}| = d \geq 3 \).

Checking the subgroups of \( \text{PSL}(2, q) \) (refer to [14, II.8.27]), we conclude that \( M_{uv} \cong \mathbb{Z}_{q-1} \), \( d = q \), \( V = |M : M_v| = q + 1 \) and \( M \) is 2-transitive on \( V \). Thus \( \Gamma \cong K_{q+1} \).

(4) Assume that \( G_{uv} \) is abelian. Let \( M \) be the unique minimal normal subgroup of \( G \). If \( M_{uv} \) is a 2-group, then \( (4) \) of this theorem follows from \( (3) \).

We next assume that \( |M_{uv}| \) has an odd prime divisor \( p \). By \( (1) \), the unique Sylow \( p \)-subgroup of \( M_{uv} \) is also a Sylow \( p \)-subgroup of \( M \). Write \( M = T_1 \times \cdots \times T_k \), where \( T_i \) are isomorphic nonabelian simple groups. By \( (1) \) of this theorem, \( M_{\{u,v\}} \) is not abelian,
so $M_{\{u,v\}} \not< G_{uv}$, and then $G_{\{u,v\}} = M_{\{u,v\}}G_{uv}$. Thus $G = MG_{uv}$, and hence $G_{uv}$ acts transitively on $\{T_1, \ldots , T_k\}$ by conjugation. Choose, for each $i$, a Sylow $p$-subgroup $P_i$ of $T_i$ such that $P_1 \times \cdots \times P_k$ is the unique Sylow subgroup $M_{uv}$. Since $G_{uv}$ is abelian, we have $P_i = P_i^x \leq T_i^x$ for $x \in G_{uv}$. It follows that $P_i \leq T_i$ for all $i$. The only possibility is that $k = 1$, and so $M$ is simple.

Note that $G$ is an almost simple group with a soluble maximal subgroup $G_{\{u,v\}}$. Then, by [18], both $M = soc(G)$ and $M_{\{u,v\}} = M \cap G_{\{u,v\}}$ are known. Since $M_{\{u,v\}}$ has an abelian subgroup of index 2, it follows that either $M \cong PSL(2,q)$ and $M_{\{u,v\}} \cong D_{2(q+1)}$, or $M = Sz(q)$ and $M_{\{u,v\}} \cong D_{2(q-1)}$. Check the subgroups of $M$, refer to [25] for $Sz(q)$. The former case yields that $M_v \cong [q]:\mathbb{Z}_{q-1}$ and $\Gamma \cong K_{q+1}$. Assume that $M = Sz(q)$ and $M_{\{u,v\}} \cong D_{2(q-1)}$. Then $M_v \cong [q]:\mathbb{Z}_{q-1}$ and $d = q$; in this case, $\Gamma$ is $(M,2)$-arc-transitive. By [5], we have that $\text{Aut}\Gamma = \text{Aut}(Sz(q))$ and $\Gamma$ is unique up to isomorphism. Thus (4) of this theorem follows. \hfill \Box

**Lemma 3.4.** Assume that $G$ has type $PA$ on $E$. Let $soc(G) = T_1 \times \cdots \times T_k$. Then $(T_i)_{uv} \neq 1$ for each $i$ and $(u, v) \in E$; in particular, every $T_i$ is not semiregular.

**Proof.** Let $M = soc(G)$. By Lemma 2.3, $M_{\{u,v\}} = (T_1)_{\{u,v\}} \times \cdots \times (T_k)_{\{u,v\}}$, and $(T_i)_{\{u,v\}}$ all have equal order. By Theorem 3.3, $M_{\{u,v\}}$ is nonabelian. Thus $(T_i)_{\{u,v\}}$ is nonabelian for all $i$. Then the lemma follows. \hfill \Box

For the case where $\Gamma$ is a bipartite graph, we let $G^+$ be the subgroup of $G$ preserving the bipartition of $\Gamma$. Then $|G : G^+| = 2$, and each bipartite half of $\Gamma$ is a $G^+$-orbit on $V$.

**Lemma 3.5.** Assume that the graph $\Gamma = (V,E)$ is $(G,2)$-arc-transitive, and $G$ has type $PA$ on $E$. Then either $\Gamma \cong K_{d,d}$, or one of the following holds:

(1) $G$ is quasiprimitive on $V$;

(2) $\Gamma$ is bipartite, and $G^+$ is faithful and quasiprimitive on each bipartite half of $\Gamma$.

**Proof.** Since $G$ is primitive on $E$, every minimal normal subgroup of $G$ is transitive on $E$, and so has at most two orbits on $V$. If $\Gamma$ is not bipartite then quasiprimitive on $V$.

Now let $\Gamma$ be bipartite with bipartition say $V = V_1 \cup V_2$. Note that $G_v \leq G^+$ for each $v \in V$. Then $G^+$ is locally-primitive on $\Gamma$. Suppose that $\Gamma \not\cong K_{d,d}$. Then, by [23], $G^+$ is faithful on both $V_1$ and $V_2$, and either (2) of this lemma holds, or the unique minimal normal subgroup of $G$ is a direct product $M_1 \times M_2$, where $M_1$ and $M_2$ are normal in $G^+$ and conjugate in $G$, and $M_i$ is intransitive on $V_i$ for $i = 1, 2$. For the latter case, if $M_1$ is intransitive on $V_2$ then $M_1$ is semiregular on $V$ by [8, Lemma 5.1]; if $M_1$ is transitive on $V_2$ then $M_2$ is semiregular on $V_2$. These two cases all contradict Lemma 3.4. Thus $G^+$ is quasiprimitive on both $V_1$ and $V_2$. \hfill \Box

As permutation groups on $V$ and on $E$, the types of $G$ (and $G^+$) have been determined in [9]. Then by Lemma 3.5 and combining with the reduction theorems for 2-arc-transitive graphs given by Preager [22, 23], we get the following result.

**Lemma 3.6.** Assume that the graph $\Gamma = (V,E)$ is $(G,2)$-arc-transitive. Suppose that $\Gamma \not\cong K_{d,d}$. If $G$ is not almost simple, then $G$ has type $PA$ on $E$ and either

(1) $G$ is quasiprimitive and of type $PA$ on $V$; or

(2) $\Gamma$ is bipartite, $G^+$ is faithful and quasiprimitive on each bipartite half of $\Gamma$ with type $PA$. 
Now we are ready to give a proof of Theorem 1.1.

**Theorem 3.7.** Let $\Gamma = (V, E)$ be a connected $d$-regular graph for some $d \geq 3$, and let $G \leq \text{Aut}\Gamma$. Assume that $\Gamma$ is both $G$-edge-primitive and $(G, 2)$-arc-transitive. Then either $\Gamma \cong K_{d,d}$, or $G$ is almost simple.

**Proof.** Assume that $\Gamma \not\cong K_{d,d}$, and let $\{u, v\} \in E$. By the 2-arc-transitivity of $G$ on $\Gamma$, we know that $G^\Gamma_v$ is a 2-transitive permutation group of degree $d$.

Let $M = \text{soc}(G) = T_1 \times \cdots \times T_k$, where $T_i$ are isomorphic nonabelian simple groups. Then $M_s \leq G_v$, and $M_v \neq 1$ by Lemma 3.1 or 3.4. Thus $G^\Gamma_v$ is a transitive normal subgroup of $G^\Gamma_v$.

Assume that $M^\Gamma_v$ is primitive on $\Gamma(v)$. Noting that $G$ is transitive on $V$, we conclude that $M^\Gamma_w$ is primitive for every $w \in V$. Thus $\Gamma$ is $M$-locally primitive. Then, byLemma 3.4 and [8, Lemma 5.1], we conclude that $k = 1$, and so $G$ is almost simple.

Next assume that $M^\Gamma_v$ is imprimitive on $\Gamma(v)$.

Note that every non-trivial normal subgroup of an almost simple 2-transitive group is primitive. Then $G^\Gamma_v$ is an affine 2-transitive group, and by Lemma 2.2, $M^\Gamma_v$ is a soluble Frobenius group and $(M^\Gamma_v)_u$ is cyclic. Set $(M^\Gamma_v)_u \cong \mathbb{Z}_d$ and $\text{soc}(G^\Gamma_v) \cong \mathbb{Z}_r^l$ for a prime $r$ and integer $l \geq 1$ with $d = r^l$. Then $e$ is a divisor of $r^l - 1$, and $e < r^l - 1$.

Assume that $e = 1$. Then $M^\Gamma_v = \text{soc}(G^\Gamma_v) \cong \mathbb{Z}_r^l$, and so $M^\Gamma_v$ is regular on $\Gamma(v)$. By [17, Lemma 2.3], $M_e$ is faithful and hence regular on $\Gamma(v)$, and thus $M^\Gamma_v = 1$, which contradicts Corollary 3.2. Thus $e \neq 1$.

Note that $e$ is a proper divisor of $d - 1 = r^l - 1$. Neither $d$ nor $d - 1$ is a prime, in particular, $l > 1$ and $d = r^l \geq 9$. Thus $G^\Gamma_v$ has no normal subgroup isomorphic to a projective special linear group of dimension $\geq 2$. By Theorem 2.5, $G^\Gamma_v = 1$, and so $M^\Gamma_v = 1$.

Let $x \in G_{\{u,v\}} \setminus G_{uv}$. Then $(u, v)^x = (v, u)$, this implies that $M^\Gamma_v$ and $M^\Gamma_u$ are permutation isomorphic. In particular, $(M^\Gamma_u)_v \cong (M^\Gamma_v)_u \cong \mathbb{Z}_e$. Since $M^\Gamma_v \cap M^\Gamma_u = M^\Gamma_{uv} = 1$, we know that $M_{uv}$ is isomorphic to a subgroup of $(M^\Gamma_v/M^\Gamma_u) \times (M^\Gamma_u/M^\Gamma_v)$. Note that $M_{uv}/M^\Gamma_v \cong (M^\Gamma_u)_v$ and $M_{uv}/M^\Gamma_u \cong (M^\Gamma_v)_v$. Then $M_{uv}$ is isomorphic to a subgroup of $\mathbb{Z}_e \times \mathbb{Z}_e$. In particular, $M_{uv}$ is abelian. Then, by Theorem 3.3, $M$ is transitive on the arcs of $\Gamma$, and so $M_{\{u,v\}} = M_{uv} \cdot 2$.

If $e$ is a power of 2 then, by Theorem 3.3, $M \cong \text{PSL}(2, r^l)$, $\Gamma \cong K_{r^l + 1}$; however, in this case, $M$ is locally primitive on $\Gamma$, a contradiction. Thus $e$ has odd prime divisors. Let $s$ be an odd prime divisor of $e$, and $S$ be a Sylow $s$-subgroup of $M_{uv}$. Then, noting that $M_{\{u,v\}} = M_{uv} \cdot 2$, we know that $S$ is also a Sylow $s$-subgroup of $M$ by Theorem 3.3. Thus $S = S_1 \times \cdots \times S_k$, where $S_i$ is a Sylow $s$-subgroup of $T_i$ for $1 \leq i \leq k$. Since $M_{uv}$ is isomorphic to a subgroup of $\mathbb{Z}_e \times \mathbb{Z}_e$, we know that $M_{uv}$ has no subgroup isomorphic to $\mathbb{Z}_e^3$. It follows that $k \leq 2$.

Now we deduce a contradiction by supposing that $k = 2$.

Let $k = 2$. Since $G \leq (\text{Aut}(T_1) \times \text{Aut}(T_1)) \cdot 2$, we have

$$G_{\{u,v\}}/M_{\{u,v\}} = G_{\{u,v\}}/(M \cap G_{\{u,v\}}) \cong MG_{\{u,v\}}/M = G/M \leq (\text{Out}(T_1) \times \text{Out}(T_1)) \cdot 2.$$
It follows that $G_{\{u,v\}}/M_{\{u,v\}}$ is soluble, and so $G_{\{u,v\}}$ is soluble as $M_{\{u,v\}}$ is soluble. Thus $(G_v^{\Gamma(v)})_u$ is soluble, and $G_v^{\Gamma(v)} = \text{soc}(G_v^{\Gamma(v)}):(G_v^{\Gamma(v)})_u$ is also soluble. Checking the soluble affine 2-transitive groups, by Lemma 2.2, $(G_v^{\Gamma(v)})_u \leq \Gamma L(1, r^i)$ or $Z_u \cong (M_v^{\Gamma(v)})_u \leq Z((G_v^{\Gamma(v)})_u \cong Z_2$. Note that $(M_v^{\Gamma(v)})_u$ is a reducible subgroup of $(G_v^{\Gamma(v)})_u$. Recalling that $e$ is not a power of 2, the latter case does not occur.

Since $|M_{\{u,v\}}| = 2$, we have $M_{\{u,v\}} \not\cong G_{uv}$, and so $G_{uv} \neq M_{\{u,v\}}G_{uv} \leq G_{\{u,v\}}$. Then $M_{\{u,v\}}G_{uv} = G_{\{u,v\}}$, and $G = MG_{\{u,v\}} = MG_{uv}$. Recalling that $M = T_1 \times T_2$, it follows that $G_{uv}$ acts transitively on $\{T_1, T_2\}$ by conjugation. Let $H$ be the kernel of this action. Then $|G_{uv} : H| = 2$, and each $T_i$ is normalized by $H$. For $h \in H$, $((T_i)_v)^h = (T_i \cap G_v)^h = T_i^h \cap (G_v)^h = T_i \cap G_v = (T_i)_v$, $i = 1, 2$.

This implies that $H$ normalizes each $(T_i)_v$. Then $(T_i)^{\Gamma(v)}_v$ is normalized by $H^{\Gamma(v)}$. Let $(T_i)^{\Gamma(v)}_v$ be a connected irreducible on $(G_v^{\Gamma(v)})_u$ and $e = |(M_v^{\Gamma(v)})_u|$ is a proper divisor of $r^i - 1$. Let $K_i$ be the Sylow $r$-subgroup of $(T_i)^{\Gamma(v)}_v$. Then $K_i$ is normalized by $H^{\Gamma(v)}$, of course, $K_i \leq \text{soc}(G_v^{\Gamma(v)})$ and $K_1 \cap K_2 = 1$.

Recalling that $|G_{uv} : H| = 2$, we have $|(G_v^{\Gamma(v)})_u : H^{\Gamma(v)}| \leq 2$. Since $G_v^{\Gamma(v)}$ is 2-transitive, $|(G_v^{\Gamma(v)})_u|$ is divisible by $r^i - 1$, and so $|H^{\Gamma(v)}|$ is divisible by $\frac{r^i - 1}{2}$. Note that $\frac{r^i - 1}{2} > \frac{r^i}{2} - 1 > r^{i-1} - 1$. Then $|H^{\Gamma(v)}|$ is not a divisor of $r^b - 1$ for all $1 \leq b < l$. Then, by [13, Lemma 5.1], $H^{\Gamma(v)}$ is irreducible on $\text{soc}(G_v^{\Gamma(v)})$. It implies that $K_1 = K_2 = 1$, and thus $(T_i)^{\Gamma(v)}_v \leq (M_v^{\Gamma(v)})_u$ for $i = 1, 2$. Let $u$ run over $\Gamma(v)$. It follows that $(T_i)^{\Gamma(v)}_v = 1$, and hence $(T_i)_v \leq M_v^{[1]}$, $i = 1, 2$. Since $M$ is transitive on $V$, by [17, Lemma 2.3], we have $(T_1)_v = (T_2)_v = 1$, which contradicts Lemma 3.4. This completes the proof.

As a consequence of Theorems 3.3 and 3.7, an edge-primitive graph of prime valency is 2-arc-transitive, and then it has almost simple automorphism group if it is not a complete bipartite graph. See also [21].

**Corollary 3.8.** Assume that $d$ is a prime and $\Gamma \not\cong K_{d,d}$. Then $G$ is almost simple, and either $G = \text{PSL}(2, d)$ with $d > 11$ and $\Gamma \cong K_{d+1}$ or $G$ is transitive on the 2-arcs of $\Gamma$.

**Proof.** Note that $G$ is transitive on the arc set of $\Gamma$. Let $\{u, v\} \in E$. By Theorem 3.7, it suffices to deal with the case where $G_v^{\Gamma(v)}$ is not 2-transitive.

Suppose that $G_v^{\Gamma(v)}$ is not 2-transitive. Then $G_v^{\Gamma(v)} \cong \mathbb{Z}_d: \mathbb{Z}_l$ with $l < d - 1$ and $l$ a divisor of $d - 1$. If $l = 1$ then $G_v \cong \mathbb{Z}_d$ by [17, Lemma 2.3], and so $G_{uv} = 1$, which contradicts Corollary 3.2. Then $l > 1$, and so $d \geq 5$. By Theorem 2.5, $G_v^{[1]} = 1$. Then $G_{uv}$ is isomorphic to a subgroup of $((G_v^{\Gamma(v)})_u \times (G_v^{\Gamma(v)})_u \cong \mathbb{Z}_l \times \mathbb{Z}_l$. Thus $G_{uv}$ is abelian. By Theorem 3.3, $\Gamma \cong K_{d+1}$, $\text{soc}(G) \cong \text{PSL}(2, d)$, $\text{soc}(G)_v \cong \mathbb{Z}_d: \mathbb{Z}_{d-1}$ and $\text{soc}(G)_{\{u,v\}} \cong D_{d-1}$. If $G \cong \text{PGL}(2, d)$ then $G$ is transitive on the 2-arcs of $\Gamma$, which is not the case. Thus $G \cong \text{PSL}(2, d)$, and so $d > 11$ by the maximality of $G_{\{u,v\}}$. \qed

### 4. Examples

Let $\Gamma = (V, E)$ be a connected $d$-regular graph, where $d \geq 3$. Let $v \in V$ and $G \leq \text{Aut}\Gamma$. Assume that $\Gamma$ is $(G, 2)$-arc-transitive. Choose an integer $s \geq 2$ such that $\Gamma$
Lemma 4.3. Let $\Gamma = (G, s)$-arc-transitive but not $(G, s+1)$-arc-transitive; in this case, we call $\Gamma$ a $(G, s)$-transitive graph. Then $s \leq 7$ by [28]. If $G_v$ is faithful on $\Gamma(v)$ then $s \leq 3$ by Theorem 2.5, and $s = 3$ yields that $d = 7$ and $G_v \cong A_7$ or $S_7$, see [16, Proposition 2.6]. This leads to the following interesting problem: Do there exist 3-arc-transitive graphs with faithful stabilizers? We next answer this problem by giving several examples of edge-primitive graphs which are 3-transitive and have faithful stabilizers.

The first example is the Hoffman-Singleton graph, which has valency 7, order 50 and automorphism group $G = T.2$, where $T = \text{PSU}(3, 5)$. Let $X = T$ or $G$. For an edge $\{u, v\}$ of this graph, $X_v \cong A_7$ or $S_7$ and $X^{(u,v)} \cong M_{10}$ or $\text{PGL}(2,9)$, which are maximal subgroups of $X$. Thus the Hoffman-Singleton graph is both $X$-edge-primitive and $(X, 2)$-arc-transitive. To see the 3-arc-transitivity, we fix an edge $\{u, v\}$ and consider the action of the arc-stabilizer $X_{uv} \cong A_6$ or $S_6$ on $\Gamma(u) \cup \Gamma(v)$. By the 2-arc-transitivity of $X$, we have two faithful transitive actions of $X_{uv}$ on $\Gamma(u)$ and $\Gamma(v)$, respectively. Let $v_1 \in \Gamma(v) \setminus \{u\}$ and $x \in X^{(u,v)} \setminus X_{uv}$. Then $u_1 := v_1^x \in \Gamma(u) \setminus \{v\}$, and

\[(X_{uv})_{u_1} = (X^{(u,v)})_{u_1} = (X^{(u,v)})_{v_1} = ((X^{(u,v)})_{v_1})^x = ((X_{uv})_{v_1})^x.\]

By the choice of $x$, we know that $(X_{uv})_{v_1}$ and $((X_{uv})_{v_1})^x$ are not conjugate in $X_{uv}$, and so do for $(X_{uv})_{u_1}$ and $(X_{uv})_{v_1}$. This implies that the actions of $X_{uv}$ on $\Gamma(u)$ and $\Gamma(v)$ are not equivalent. Thus $(X_{uv})_{v_1}$ acts on $\Gamma(u) \setminus \{v\}$ without fixed-points, this yields that $(X_{uv})_{v_1}$ is transitive on $\Gamma(u) \setminus \{v\}$. It follows that the Hoffman-Singleton graph is $(X, 3)$-arc-transitive.

In general, combining with [16, Proposition 2.6], a similar argument as above yields the following result.

Lemma 4.1. Let $\Gamma = (V, E)$ be a connected $d$-regular graph for $d \geq 3$, $\{u, v\} \in E$ and $G \leq \text{Aut}\Gamma$. If $\Gamma$ is $(G, 2)$-arc-transitive and $G_v$ is faithful on $\Gamma(v)$, then $\Gamma$ is $(G, 3)$-arc-transitive if and only if $d = 7$, $\text{soc}(G_v) \cong A_7$ and $G^{(u,v)} \not\cong S_6$, i.e. $G^{(u,v)} \cong \text{PGL}(2,9)$, $M_{10}$ or $\text{Aut}(A_6)$.

We next give another example.

Example 4.2. By the information given in the Atlas [3] for the O’Nan simple group O’N, there exactly two conjugacy classes $C_1$ and $C_2$ of (maximal) subgroups isomorphic to $A_7$, which are merged into one class in O’N.2. Further, there are $H \in C_1$ and involutions $x_1, x_2 \in \text{O’N.2} \setminus \text{O’N}$ such that $(H \cap H^{x_1}) \langle x_1 \rangle$ are maximal subgroups of O’N.2 with $(H \cap H^{x_2}) \langle x_2 \rangle \cong \text{PGL}(2,9)$ and $(H \cap H^{x_2}) \langle x_2 \rangle \cong \text{PSL}(2,7)$. Define two bipartite graphs $\Gamma_1 = (V, E_1)$ and $\Gamma_2 = (V, E_2)$ with vertex set $V = C_1 \cup C_2$ and edge sets

\[E_1 = \{\{H_1, H_2\} : H_1 \in C_1, H_2 \in C_2, H_1 \cap H_2 \cong A_6\};\]
\[E_2 = \{\{H_1, H_2\} : H_1 \in C_1, H_2 \in C_2, H_1 \cap H_2 \cong \text{PSL}(2,7)\}.\]

Then $\Gamma_1$ and $\Gamma_2$ are both O’N.2-edge-primitive and (O’N.2, 2)-arc-transitive, which have valency 7 and 15 respectively. By Lemma 4.1, only $\Gamma_1$ is (O’N.2, 3)-arc-transitive. □

Lemma 4.3. Let $\Gamma_1$ be as in Example 4.2. Then $\text{Aut}\Gamma_1 = \text{O’N.2}$.

Proof. Let $G = \text{Aut}\Gamma_1$. Then $G \geq \text{O’N.2}$. By Theorem 1.1, $G$ is almost simple, and so $\text{O’N} \leq \text{soc}(G)$. Let $\{u, v\}$ be an edge of $\Gamma_1$. Then $G^{(u,v)} \cong A_7$ or $S_7$, and $G^{(u,v)} = 1$ by Theorem 2.5. Thus, by the group extensions $\langle \ast \rangle$ in Section 2, we conclude that $|G_v|$ has no prime divisor other than 2, 3, 5 and 7. Since O’N.2 is transitive on the vertices
of $I_1$, we have $G = (O'N.2)G_v$. It follows that $|O'N|$ and $|\text{soc}(G)|$ have the same prime divisors. Using [19, Corollary 5], we get $\text{soc}(G) = O'N$, and so $G = O'N.2$. □

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