Method for Solving the Problem of Heat Transfer in a Flat Channel

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Abstract. In power engineering, studies related to the distribution of temperatures and velocities in fluids that move, for example, in pipelines or channels, are of theoretical and practical importance. The presented work displays the results of the development of an approximate analytical method for mathematical modeling of the process of heat transfer in laminar flows. By the example of solving the problem of heat transfer in a flat channel with a Couette flow, the main provisions of the method are considered. The combined use of the integral heat balance method and the collocation method made it possible to obtain an analytical solution that is simple in form. The obtained accuracy of solutions depends on the number N of points of the spatial variable at which the original differential equation is exactly satisfied.

1. Introduction

Determination of heat losses in heating and ventilation systems, modeling the modes of thermomechanical and thermal processing of materials, increasing the efficiency of heat exchange equipment at the design stage, etc. are inextricably linked with the need to determine the velocity and temperature fields in the flows of liquids and gases.

In the mathematical modeling of heat and mass transfer processes in moving media, the classical laws of continuum mechanics are used: the Navier - Stokes equations [1 – 3], the continuity equation [4 – 5], the energy equation [6 – 7]. When these equations are supplemented by the dependences of the physical properties of the liquid on pressure and temperature, they form a closed system of equations describing the dynamics of the liquid and convective heat transfer. In engineering practice, only in some of the simplest cases is it possible to apply exact analytical methods for solving the indicated problems. For example, for linear boundary value problems for a plate and other plane bodies, the Fourier method or the method of separation of variables are used [8]; transformations of Hankel, Laplace, Legendre, etc.) [9 – 10]. Such solutions are mainly expressed in complex analytical dependencies with functions, which complicates their practical use.

In the modern world, numerical methods for studying heat and mass transfer processes are widespread. Software products allow you to automatically build design flows, solve systems of complex linear equations, offer a wide range of tools for analyzing the results obtained. At the same time, analytical solutions have some significant advantages over numerical ones.
2. Mathematical statement of the problem

Let us consider heat transfer in a flat channel in the Couette flow. According to the conditions, the temperature is kept constant on both surfaces. A stationary wall can be understood as an unperturbed flow with zero velocity.

The heat transfer scheme is shown in Figure 1.

![Couette flow diagram](image)

**Figure 1.** Couette flow diagram.

The mathematical formulation of the problem under asymmetric boundary conditions of the first kind has the form

\[
\omega(y) \frac{\partial T(x,y)}{\partial x} = a \frac{\partial^2 T(x,y)}{\partial y^2} + \frac{\mu (\partial \omega)}{\rho (\partial y)} \quad (0 < y < h; \quad 0 < x < \infty). \tag{1}
\]

The boundary conditions for equation (1) are formulated as

\[
T(0,y) = T_0; \tag{2}
\]
\[
T(x,0) = T_1; \tag{3}
\]
\[
T(x,h) = T_2, \tag{4}
\]

where \(\lambda\) – coefficient of thermal conductivity; \(x, y\) – longitudinal and transverse coordinates; \(\omega(y) = \omega_0 y / h\) – profile of the speed of the plane-parallel flow; \(\mu\) – dynamic viscosity; \(h\) – channel width; \(\rho\) – density; \(a\) – thermal diffusivity; \(T\) – temperature; \(T_0\) – fluid temperature at the channel inlet \((x = 0)\); \(T_1, T_2\) – temperature of a stationary and moving wall.

Consider the case \(T_i = T_c = T_{cr}\). Problem (1) - (4) can be represented in dimensionless form. Let’s introduce the following parameters:

\[
\Theta = \frac{T - T_{cr}}{T_0 - T_{cr}}; \quad \xi = \frac{y}{h}; \quad \eta = \frac{ax}{\omega_0 h^2}; \quad R = \frac{\mu \omega_0^2}{\lambda}, \tag{5}
\]

where \(\Theta\) – dimensionless temperature; \(\xi, \eta\) – dimensionless transverse and longitudinal coordinates, respectively; \(R\) – internal heat source. Problem (1) - (4) taking into account (5) will be

\[
\xi \frac{\partial \Theta(\eta,\xi)}{\partial \eta} = \frac{\partial^2 \Theta(\eta,\xi)}{\partial \xi^2} + R \quad (0 < \eta < \infty; \quad 0 < \xi < 1); \tag{6}
\]
\[
\Theta(0,\xi) = 1; \tag{7}
\]
\[ \Theta(\eta,0) = 0; \]  
\[ \Theta(\eta,1) = 0. \] \(8\) \(9\)

### 3. Method description

The method presented in this paper for obtaining approximate analytical solutions to nonstationary heat conduction problems consists in satisfying the differential equation of the Sturm-Liouville boundary value problem at a given number of points. Obtaining a solution is limited only by the possibility of separating variables in the original differential equation.

The solution to problem (6) - (9), following the method of separation of variables, is sought in the form

\[ \Theta(\eta, \xi) = \varphi(\eta) \psi(\xi), \] \(10\)

Substituting (10) into (6) we find

\[ \frac{\partial \varphi(\eta)}{\partial \eta} + v\varphi(\eta) = 0; \] \(11\)

\[ \frac{\partial^2 \psi(\xi)}{\partial \xi^2} + \xi \psi(\xi) = 0, \] \(12\)

where \(v\) – some constant.

The solution to equation (11) is known and has the form

\[ \varphi(\eta) = A \exp(-v\eta), \] \(13\)

where \(A\) – unknown coefficient.

Substituting (10) into (8), (9) we obtain

\[ \frac{\partial \psi(\xi)}{\partial \xi} = 0; \] \(14\)

\[ \psi(0) = 0. \] \(15\)

The solution to the Sturm-Liouville problem (12), (14), (15) is taken in the form

\[ \psi(\xi) = \sum_{i=0}^{n+2} b_i \xi^i, \] \(16\)

where \(b_i (i = 0, n+2)\) – unknown coefficients.

Expression (14) allows us to introduce one more boundary condition

\[ \psi(1) = \text{const} = 1 \] \(17\)

Substituting (16) into (17), we find \(b_0 = 0\).

We require that relation (16) satisfy the boundary condition (14) and equation (15) at five points (with a step \(\Delta \xi = 1/5\), in \(\xi = 0\)). We divide the resulting segments into the following parts:

\[ K_1 = 5; K_2 = 5; K_3 = 1; K_4 = 1; K_5 = 5; \]

We get thirteen points (see figure 2) \(\xi = 0; \frac{1}{25}, \frac{2}{25}, \frac{3}{25}, \frac{4}{25}, \frac{1}{5}, \frac{6}{25}, \frac{7}{25}, \frac{8}{25}, \frac{9}{5}, \frac{3}{5}, \frac{4}{5}. \)
Thus, we divide the segments along the coordinate $\xi$ axis into points in order to obtain the most accurate solution in the range of variation of the dimensionless temperature graph.

Substituting (16), limiting ourselves to sixteen terms of the series, into relations (14), (15) and equation (12), as applied to the points regarding the coefficients we obtain the system of equations.

From the solution we find $b_i(i=0,16)$.

Let us find the integral of the equation (12)

$$
\int \left[ \frac{\partial^2}{\partial \xi^2} \sum_{i=0}^{n/2} b_i \xi^i + \xi \sum_{i=0}^{n/2} b_i \xi^i \right] d\xi = 0.
$$

Calculating the integrals in (18), taking into account the found values $b_i(i=0,16)$ we get:

$$\begin{align*}
\psi_1 &= 4553.8; \\
\psi_2 &= 189.2; \\
\psi_3 &= 341.5; \\
\psi_4 &= 81.9; \\
\psi_5 &= 758.4; \\
\psi_6 &= 530.1; \\
\psi_7 &= 18.9.
\end{align*}$$

Substituting (13), (16) into (10), for each eigenvalue we will have particular solutions of the form

$$
\Theta_k(\eta, \xi) = A_k \exp(-\psi_k \sum_{i=0}^{n/2} b_i(\psi_k) \xi^i).
$$

To fulfill the initial condition, it is compiled

$$
\int \left[ \sum_{k=0}^{n/2} A_k \sum_{i=0}^{n/2} b_i(\psi_k) \xi^i \right] \psi_j(\psi_j, \xi) d\xi = 0. (j=1,2; n=2)
$$

Calculating the integrals in (20), to find $A_k(k=0,7)$ obtain a system of equations. Decision

$$A_1 = -108.8; \quad A_2 = -4.5; \quad A_3 = -1.01; \quad A_4 = -1.2; \quad A_5 = 5.3; \quad A_6 = -4.8; \quad A_7 = -4.8.$$

Figure 2. Point values.

Figure 3. Change of dimensionless temperature along the longitudinal coordinate ($R = 0.5$).
4. Discussion of the results
The temperature distribution graph in comparison with the numerical solution is shown in Fig. 3. The same figure shows a numerical solution based on the finite difference method. It can be seen from the analysis of the figure that temperature stabilization occurs in the range of the longitudinal variable. Moreover, the value depends on the distance from the channel walls and the value also has a significant effect on the length of the stabilization interval.

5. References
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