EQUIVARIANT ALGEBRAIC KK-THEORY AND ADJOINTESS THEOREMS

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ABSTRACT. We introduce an equivariant algebraic kk-theory for $G$-algebras and $G$-graded algebras. We study some adjointness theorems related with crossed product, trivial action, induction and restriction. In particular we obtain an algebraic version of the Green-Julg Theorem which gives us a computational tool.

1. Introduction

Algebraic $kk$-theory has been introduced by G. Cortiñas and A. Thom in [2]. This is a bivariant $K$-theory on the category of $\ell$-algebras where $\ell$ is a commutative ring with unit. For each pair $(A, B)$ of $\ell$-algebras a group $\text{kk}(A, B)$ is defined. A category $\text{RK}$ is obtained whose objects are $\ell$-algebras and where the morphisms from $A$ to $B$ are the elements of the group $\text{kk}(A, B)$. The category $\text{RK}$ is triangulated and there is a canonical functor $j : \text{Alg}_\ell \to \text{RK}$ with universal properties. These properties are algebraic homotopy invariance, matrix invariance and excision.

The definition of algebraic $kk$-theory was inspired by the work of J. Cuntz [4] and N. Higson [7] on the universal properties of Kasparov $KK$-theory [8]. The $KK$-theory of separable $C^*$-algebras is a common generalization both of topological $K$-homology and topological $K$-theory as an additive bivariant functor. Let $A, B$ be separable $C^*$-algebras. Then

$$KK_*(\mathbb{C}, B) \simeq K_*^{top}(B)$$
$$KK^*(A, \mathbb{C}) = K_{\text{hom}}^*(A)$$

(1.1)

here $K_*^{top}(B)$ denotes the $K$-theory of $B$ and $K_{\text{hom}}^*(A)$ the topological $K$-homology of $A$. J. Cuntz in [3] gave another equivalent definition of the original one given in [8]. This new approach allowed to put bivariant $K$-theory in algebraic context. Higson in [7] stated the universal property of $KK$ whose algebraic analogue is studied in [2], where also an analogue of (1.1) is proved. On the algebraic side, if $A$ is an $\ell$-algebra then

$$\text{kk}(\ell, A) \simeq \text{KH}(A)$$

here KH is Weibel’s homotopy $K$-theory defined in [12]. We can start to build a dictionary between Kasparov’s $KK$-theory and algebraic $kk$-theory in the following way

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In this paper we obtain an equivariant version of the dictionary stated above in the following sense

| Equivariant Kasparov’s $KK$-theory | Equivariant algebraic $kk$-theory |
|-----------------------------------|----------------------------------|
| bivariant $K$-theory on separable $G$-$C^*$-algebras | bivariant $K$-theory on $G$-algebras |
| $k : G$-$C^*$-$Alg \rightarrow KK^G$ | $j^G : G$-$Alg \rightarrow \mathcal{A}^G$ |
| $k$ is stable with respect to $K$($\mathcal{E}^2(G \times N)$) | $j^G$ is $G$-stable |
| $k$ is continuous homotopy invariant | $j^G$ is polynomial homotopy invariant |
| $k$ is split exact | $j^G$ is excisive |
| $k$ is universal for the properties described above | $j^G$ is universal for the properties described above |
| $KK^G_*(\mathbb{C}, A) \simeq K^G_{*+}(A \times G)$ with $G$ compact | $kk^G_*(\ell, A) \simeq KH_*(A \times G)$ with $G$ finite and $1/|G| \in \ell$ |

We also introduce a dual theory $\mathcal{A}^G$ for $G$-graded algebras and establish a duality result similar to that proved by Baaj and Skandalis in [1].

In Section 2 we recall some results from [2]. We take special care in the definition of equivariant matrix invariance. We introduce the concept of $G$-stable functor in Section 3, the rest of the concepts which appears in the equivariant dictionary are straightforward. The definition of $G$-stability was inspired by the definition of equivariant stability for $G$-$C^*$-algebras (see [11]). In sections 2 and 3 $G$ is a group without any other assumption but from Section 4 and for the rest of the paper $G$ is a countable group.

In Section 4 we introduce the appropriate brand of equivariant algebraic $kk$-theory in each case and establish its universal properties. For a countable group $G$, we
define an equivariant algebraic $KK$-theory $KK_G$ for the category of $G$-algebras and $\hat{KK}_G$ for the category of $G$-graded algebras.

We study adjointness theorems in equivariant $KK$-theory. We put in an algebraic context some of the adjointness theorems which appear in Kasparov $KK$-theory. In Section 5, we define the functors of trivial action and crossed product between $KK$ and $\hat{KK}_G$. The first adjointness theorem is Theorem 5.2.1 which is an algebraic version of the Green-Julg Theorem. This result gives us the first computation related with homotopy K-theory. If $G$ is a finite group, $A$ is a $G$-algebra, $B$ is an algebra and $\frac{1}{|G|} \in \ell$ then there is an isomorphism

$$\psi_{GJ} : KK_G(B^\tau, A) \to \hat{KK}(B, A \rtimes G).$$

In particular, if $B = \ell$ then

$$KK_G(\ell, A) \cong KH(A \rtimes G).$$

In Section 6 we consider $H$ a subgroup of $G$. We define induction and restriction functors between $KK_G$ and $\hat{KK}_H$ and study the adjointness between them. If $B$ is an $H$-algebra and $A$ is a $G$-algebra then there is an isomorphism

$$\psi_{IR} : KK_G(\text{Ind}_H^G B, A) \to KK_H(B, \text{Res}_G^H A).$$

This result gives us another computation. Taking $H$ the trivial group and $B = \ell$ we obtain that

$$KK_G(\ell, A) \cong KH(A) \quad \forall A \in G - \text{Alg}.$$

Here $\ell(G) = \bigoplus_{g \in G} \ell$ with the regular action of $G$. More general, if $H$ is a finite subgroup of $G$ and $1/|H| \in \ell$ we combine $\psi_{GJ}$ and $\psi_{IR}$ and obtain

$$KK_G(\ell(G/H), A) \cong KH(A \rtimes H) \quad \forall A \in G - \text{Alg}.$$

In Section 7 we obtain an algebraic version of the Baaj-Skandalis duality theorem. We show that the functors

$$\rtimes G : \hat{KK}_G \to \hat{KK}_G \quad G \rtimes : \hat{KK}_G \to \hat{KK}_G$$

are inverse category equivalences.

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2. Algebraic $KK$-theory

In this section we recall some results from [2] and we adapt them to our setting. Let $\ell$ be a commutative ring with unit. We consider an $\ell$-bimodule $A$ such that $x \cdot a = a \cdot x$, with $x \in \ell$ and $a \in A$. An $\ell$-bimodule $A$ is an $\ell$-algebra if it is an associative and not necessarily unital algebra. Let $G$ be a group. A $G$-algebra $A$ is an $\ell$-algebra with an action of $G$, i.e. with a group homomorphism $\alpha : G \to \text{End}_\ell(A)$, where $\text{End}_\ell(A)$ denote the group of $\ell$-linear endomorphisms of $A$. We shall denote by $g(a)$ or $g \cdot a$ the element $\alpha(g)(a)$. An equivariant morphism $f : A \to B$ between
$G$-algebras is a $G$-equivariant $\ell$-linear map. A $G$-graded algebra $A$ is an $\ell$-algebra with a family of $\ell$-submodules $\{A_s\}_{s \in G}$ such that

$$A = \bigoplus_{s \in G} A_s \quad A_s A_t \subseteq A_{st} \quad s, t \in G.$$  

We write $|a| = s$ if $a \in A_s$. An homogeneous morphism $f : A \to B$ of $G$-graded algebras is an algebra morphism such that $f(A_s) \subseteq B_s$, for all $s \in G$. We consider the category of $G$-algebras with equivariant morphisms and the category of $G$-graded algebras with homogeneous morphisms.

In this section we define an algebraic $kk$-theory for the categories of $G$-algebras and $G$-graded algebras. We write $C$ to refer to either of these categories.

2.1. Homotopy invariance. Let $A$ be an object of $C$. Put $A^\Delta^1 := A[t] = A \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t]$. Consider the trivial action of $G$ (trivial $G$-grading) on $\mathbb{Z}[t]$ and the diagonal structure on $A[t]$. Then $A[t]$ is an object of $C$. Let us write $c_A : A \to A[t]$ for the inclusion of $A$ as constant polynomials in $A[t]$ and $ev_i : A[t] \to A$ for the evaluation of $t$ at $i$ ($i = 0, 1$). Note that these maps are morphisms in $C$ and that $c_A$ is a section of $ev_i$.

Let $f_0, f_1 : A \to B$ be morphisms in $C$. We call $f_0$ and $f_1$ elementarily homotopic, and write $f_0 \sim_e f_1$, if there exists a morphism $H : A \to B[t]$ such that $ev_i H = f_i$, $i = 0, 1$. It is easy to check that elementary homotopy is a reflexive and symmetric relation but in general it is not transitive. Let $f, g : A \to B$ be morphisms in $C$. We call $f$ and $g$ homotopic, and write $f \sim g$, if they can be connected by a finite chain of elementary homotopies,

$$f \sim_e h_0 \sim_e \ldots \sim_e h_n \sim_e g.$$  

We denote the set of homotopy classes by $[A, B]_C$. A morphism $f : A \to B$ is an elementary homotopy equivalence if there exists a morphism $g : B \to A$ such that $f \circ g \sim_e id_B$ and $g \circ f \sim_e id_A$. We say $A$ is elementarily contractible if the null morphism and the identity morphism are elementarily homotopic.

The category of ind-objects of $C$ is the category $\text{ind-}C$ of directed diagrams in $C$. An object in $\text{ind-}C$ is described by a filtering partially ordered set $(I, \leq)$ and a functor $A : I \to C$. The set of homomorphisms in $\text{ind-}C$ is defined by

$$\text{hom}_{\text{ind-}C}((A, I), (B, J)) := \lim_{i \in I} \lim_{j \in J} \text{hom}_{C}(A_i, B_j).$$

Let $A = (A, I)$ and $B = (B, J)$ be objects of $\text{ind-}C$, we have a map

$$(2.1.1) \quad \text{hom}_{\text{ind-}C}(A, B) \ni [A, B]_C = \lim_{i \in I} \lim_{j \in J} [A_i, B_j]_C.$$  

We say two morphisms in $\text{ind-}C$ are homotopic if their images by $(2.1.1)$ are equal. An object of $\text{ind-}C$ is contractible if the null morphism and the identity morphism are homotopic. Sometimes we shall omit the poset in the notation.

Let $D$ be an arbitrary category. A functor $F : C \to D$ is homotopy invariant if it maps the inclusion $c_A : A \to A[t]$ to an isomorphism. It is easy to check that $F : C \to D$ is an homotopy invariant functor if and only if $F(f) = F(g)$ when $f \sim g$ (or equivalently when $f \sim_e g$). If $C$ is the category of $G$-algebras we say that an homotopy invariant functor is equivariantly homotopy invariant and if $C$ is the category of $G$-graded algebras we say that a homotopy invariant functor is graded homotopy invariant.
2.2. **Matrix stability.** Consider $M_n$ the algebra of $n \times n$-matrices with coefficients in $\mathbb{Z}$ with the trivial action (grading) of $G$ and $M_\infty = \cup_{n \geq 0} M_n$. Let $A$ be an object in $\mathcal{C}$. We define

$$M_n A = M_n \otimes \mathbb{Z} A \quad M_\infty A = M_\infty \otimes \mathbb{Z} A$$

which are objects in $\mathcal{C}$ with the diagonal action (grading). Denote by $\iota_n : \mathbb{Z} \to M_n$ and $\iota_\infty : \mathbb{Z} \to M_\infty$ the inclusions at the upper left corner. A functor $F : \mathcal{C} \to \mathcal{D}$ is $M_n$-stable ($M_\infty$-stable) if $F(\iota_n \otimes \text{id}_A)$ ($F(\iota_\infty \otimes \text{id}_A)$) is an isomorphism for all $A \in \mathcal{C}$.

2.3. **Algebra of polynomial functions.** Consider the following simplicial ring

$$(2.3.1) \quad \mathbb{Z}^\Delta : [n] \mapsto \mathbb{Z}^{\Delta^n} \quad \mathbb{Z}^{\Delta^n} := \mathbb{Z}[t_0, \ldots, t_n]/\left< 1 - \sum t_i \right>$$

$$\Theta : [n] \mapsto [m] \quad \Theta^* : \mathbb{Z}^{\Delta^m} \to \mathbb{Z}^{\Delta^n}$$

$$\Theta^*(t_i) = \begin{cases} 0 & \text{si } \Theta^{-1}(i) = \emptyset \\ \sum_{j \in \Theta^{-1}(i)} t_j & \text{si } \Theta^{-1}(i) \neq \emptyset \end{cases}$$

Let $A$ be an object in $\mathcal{C}$. Define

$$A^\Delta : [n] \mapsto A^{\Delta^n} \quad A^{\Delta^n} := A \otimes_\mathbb{Z} \mathbb{Z}^{\Delta^n}$$

Note $A^\Delta$ is a simplicial $\mathcal{C}$-object. Let $X$ be a simplicial set. Define

$$A^K := \text{map}_\mathbb{S}(X, A^\Delta)$$

If $(K, \ast)$ is a pointed simplicial set, put

$$A^{(K, \ast)} := \text{map}_\mathbb{S}_* \left( (K, \ast), A^\Delta \right) = \ker \left( \text{map}_\mathbb{S}(K, A^\Delta) \to \text{map}_\mathbb{S}(\ast, A^\Delta) \right) = \ker (A^K \to A)$$

If $A$ is a $G$-algebra ($G$-graded algebra) and $K$ a finite simplicial set, by [2, Lemma 3.1.3] we can consider $A^K$ and $A^{(K, \ast)}$ as a $G$-algebras ($G$-graded algebras) taking the diagonal structure with the trivial structure in $\mathbb{Z}^K$ and $\mathbb{Z}^{(K, \ast)}$.

We will denote by $\text{sd}^*X$ the following pro-simplicial set

$$\text{sd}^*X : \ldots \to \text{sd}^n X \xrightarrow{h} \text{sd}^{n-1} X \to \ldots \to \text{sd}X \xrightarrow{h} X$$

where $\text{sd}X$ is the subdivision of $X$ and $h$ is the last vertex map, see [5, III.4]. If $A$ is an object in $\mathcal{C}$ we consider the following object in $\text{ind-}\mathcal{C}$

$$A^{\text{sd}^*X} : A^K \to A^{\text{sd}X} \to \ldots \to A^{\text{sd}^{n-1}X} \to A^{\text{sd}^nX} \to \ldots$$

2.4. **Extensions and classifying map.** A sequence of morphisms in $\text{ind-}\mathcal{C}$

$$(2.4.1) \quad A \xrightarrow{f} B \xrightarrow{g} C$$

is called an **extension** if $f$ is a kernel of $g$ and $g$ is a cokernel of $f$. Let $\mathcal{U}(\mathcal{C})$ be the category of modules with linear and equivariant maps (graded maps), i.e. we forget the multiplication in $\mathcal{C}$ keeping the module structure. Let $F : \mathcal{C} \to \mathcal{U}(\mathcal{C})$ be the forgetful functor. This functor can be extended to $F : \text{ind-}\mathcal{C} \to \text{ind-}\mathcal{U}(\mathcal{C})$. We will call an extension $(2.4.1)$ **weakly split** if $F(g)$ has a section in $\text{ind-}\mathcal{U}(\mathcal{C})$.

Let $M$ be an object in $\mathcal{U}(\mathcal{C})$. Consider in

$$\bar{T}(M) = \bigoplus_{n \geq 1} M^{\otimes^n} \quad M^{\otimes^n} = \underbrace{M \otimes \ldots \otimes M}_{n}\text{-times}$$
the usual structure of \( \ell \)-algebra. If \( \mathcal{C} \) is the category of \( G \)-algebras, we consider in \( M^{\otimes n} \) the following action,
\[
g \cdot (m_1 \otimes m_2 \otimes \ldots \otimes m_n) = g \cdot m_1 \otimes g \cdot m_2 \otimes \ldots \otimes g \cdot m_n
\]
which gives to \( \tilde{T}(M) \) a \( G \)-algebra structure. If \( \mathcal{C} \) is the category of \( G \)-graded algebras, take in \( M^{\otimes n} \) the following \( G \)-gradation
\[
|m_1 \otimes m_2 \otimes \ldots \otimes m_n| = |m_1||m_2| \ldots |m_n|.
\]
By this way \( \tilde{T}(M) \) is a \( G \)-graded algebra. Both constructions are functorial hence we consider the functor \( \tilde{T} : \mathcal{U}(\mathcal{C}) \rightarrow \mathcal{C} \).

Put
\[
T := \tilde{T} \circ F : \mathcal{C} \rightarrow \mathcal{C}
\]
If \( A \) is an object in \( \mathcal{C} \) there exists an morphism in \( \mathcal{C} \)
\[
\eta_A : T(A) \rightarrow A \quad \eta_A(a_1 \otimes \ldots \otimes a_n) = a_1 \ldots a_n
\]
and a morphism in \( \mathcal{U}(\mathcal{C}) \) \( \mu_A : A \rightarrow T(A) \) which is the inclusion at the first summand of \( T(A) \). Let \( A, B \) be objects in \( \mathcal{C} \), it is easy to check that
\[
(2.4.2) \quad \text{hom}_\mathcal{C}(T(A), B) \simeq \text{hom}_\mathcal{U}(\mathcal{C})(F(A), F(B)).
\]
Hence if we have a morphism \( A \rightarrow B \) in \( \mathcal{U}(\mathcal{C}) \), we can extend it to a morphism \( T(A) \rightarrow B \) in \( \mathcal{C} \). It shows that \( T \) is the left adjoint of \( F \). The counit of the adjunction is \( \eta_A : T(A) \rightarrow A \) and it is surjective (see [10, IV.3 Thm 1]). We define
\[
J(A) := \ker \eta_A.
\]

The universal extension of \( A \) is
\[
J(A) \xrightarrow{\iota_A} T(A) \xrightarrow{\eta_A} A.
\]
Let \( A \xrightarrow{f} B \xrightarrow{s} C \) be a weakly split extension. Let \( s \) be a section of \( F(g) \) and define \( \hat{\xi} = \eta_B \circ \tilde{T}(s) \), then
\[
\eta_C = \eta_C \circ T(g) \circ \tilde{T}(s) = g \circ \eta_B \circ \tilde{T}(s) = g \circ \hat{\xi}.
\]
Define \( \xi : J(C) \rightarrow A \) as the restriction of \( \hat{\xi} \) to \( J(C) \) We obtain a commutative diagram of extensions
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow{\xi} & & \downarrow{\xi} & & \downarrow{id_C} \\
J(C) & \xrightarrow{\iota_C} & T(C) & \xrightarrow{\eta_C} & C
\end{array}
\]
This construction of \( \xi \) depends on which \( s \) is chosen. If \( \xi_1, \xi_2 : J(C) \rightarrow A \) are morphisms constructed taking different sections of \( F(g) \) we will show \( \xi_1, \xi_2 \) are homotopic. Define a linear equivariant (graded) map
\[
H : C \rightarrow A[t] \quad H(c) = (1 - t)\xi_1(c) + t\xi_2(c).
\]
By (2.4.2) it extends to a homomorphism and there exists a morphism \( H : T(C) \rightarrow A[t] \) in \( \mathcal{C} \) such that
\[
ev_0 \circ H|_{J(C)} = \xi_1 \quad \text{ev}_1 \circ H|_{J(C)} = \xi_2.
\]
Then the map $\xi$ is unique up to elementary homotopy. We call $\xi$ the classifying map of the extension $A \xrightarrow{f} B \xrightarrow{g} C$. This construction is functorial, that means if we have a diagram of weakly split extensions,

\[
\begin{array}{c}
A \xrightarrow{f} B \xrightarrow{h} C \\
A' \xrightarrow{f'} B' \xrightarrow{h'} C'
\end{array}
\]

then there is a diagram

\[
\begin{array}{c}
J(C) \xrightarrow{\xi} A \\
J(g) \xrightarrow{\xi} A' \\
J(C') \xrightarrow{\xi} A'
\end{array}
\]

of classifying maps, which is commutative up to elementary homotopy.

Let $L$ be a ring and $A$ an object in $C$. Then the extension

\[
J(A) \otimes_{\mathbb{Z}} L \rightarrow T(A) \otimes_{\mathbb{Z}} L \rightarrow A \otimes_{\mathbb{Z}} L
\]

is weakly split, and there is a choice for classifying map

\[
\phi_{A,L} : J(A) \otimes_{\mathbb{Z}} L \rightarrow J(A) \otimes_{\mathbb{Z}} L
\]

of (2.4.3), which is natural in both variables. In particular, if $K$ is a finite pointed simplicial set. There is a homotopy class of maps

\[
\phi_{A,K} : J(A^K) \rightarrow J(A)^K
\]

natural with respect to $K$, which is represented by a classifying map of the following extension

\[
J(A^K) \xrightarrow{\xi} T(A^K) \xrightarrow{\eta} A^K
\]

Let $S^1$ be the simplicial circle $\Delta^1/\partial\Delta^1$, we define

\[
\Omega := \mathbb{Z}^{(S^1,\ast)} \quad P := \mathbb{Z}^{(\Delta^1,\ast)}
\]

The path extension of $A$ is the extension induced by the cofibration $\partial\Delta^1 \subset \Delta^1$ (see [2, Lemma 3.1.2]),

\[
\begin{array}{c}
\Omega A \xrightarrow{} PA \xrightarrow{\text{ev}_0,\text{ev}_1} A \oplus A
\end{array}
\]

The extension (2.4.5) is weakly split because we have a linear and equivariant (graded) section of $(\text{ev}_0, \text{ev}_1)$

\[
(a,b) \mapsto (1-t)a + tb.
\]

The loop extension of $A$ is

\[
\begin{array}{c}
\Omega A \rightarrow PA \xrightarrow{\text{ev}_1} A
\end{array}
\]

Note $a \mapsto at$ is a natural section of $F(\text{ev}_1)$. Thus we can pick a natural choice for the classifying map of (2.4.7). We call it

$$\rho_A : J(A) \rightarrow \Omega A$$

Let $P := \mathbb{Z}^{(sd\ast\Delta^1,\ast)}$ we have an extension in $\text{ind-}C$

\[
A^{S^1} := A^{(sd\ast S^1,\ast)} \quad A^{S^1} \rightarrow PA \xrightarrow{\text{ev}_1} A
\]
which is naturally weakly split. The classifying map \( J(A) \to A^{S^1} \) is the following composition

\[
(2.4.8) \quad J(A) \xrightarrow{\rho_A} \Omega A \xrightarrow{h} A^{S^1}
\]

where \( h \) is induced by the last vertex map. We will sometimes abuse notation and write \( \rho_A \) for the map \((2.4.8)\).

Let \( f : A \to B \) be a morphism in \( C \). The mapping path extension of \( f \) is the extension obtained from the loop extension of \( B \) by pulling it back to \( A \)

\[
P_f := PB \times_B A \quad \Omega B \xrightarrow{e} PB \times_B A \xrightarrow{\pi_f} A \quad E'
\]

We call \( P_f \) the path algebra of \( f \). Note the extension \( E' \) is naturally weakly split because \( \tilde{s}(a) = (s \circ f(a), a) \) is a natural section of \( \pi_f \) where \( s \) is the natural section of \( E \). We define

\[
(2.4.9) \quad P_f := PB \times_B A \quad B^{S^1} \to PB \times_B A \xrightarrow{\pi_f} A.
\]

Note \( \rho_f := \rho_B J(f) \) is the classifying map of the extension \((2.4.9)\).

2.5. Excisive homology theories. We consider triangulated categories in terms of a loop functor \( \Omega \). A triangulated category \((\mathcal{T}, \Omega, Q)\) is an additive category \( \mathcal{T} \) with an equivalence \( \Omega : \mathcal{T} \to \mathcal{T} \) and a class \( Q \) of sequences in \( \mathcal{T} \) called distinguished triangles

\[
(\mathcal{T}) \quad \Omega C \to A \to B \to C
\]

satisfying some axioms, see [9], [2]. A triangle functor from \((\mathcal{T}_1, \Omega_1, Q_1)\) to \((\mathcal{T}_2, \Omega_2, Q_2)\) is a pair consisting of an additive functor \( R : \mathcal{T}_1 \to \mathcal{T}_2 \) and a natural transformation \( \alpha : \Omega_2 R \to R \Omega_1 \) such that

\[
\Omega_2 R(C) \xrightarrow{R(f) \circ \alpha_C} R(A) \xrightarrow{R(g)} R(B) \xrightarrow{R(h)} R(C)
\]

is a distinguished triangle in \( \mathcal{T}_2 \) for each triangle \( \Omega_1 C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \).

Let \((\mathcal{T}, \Omega, Q)\) be a triangulated category. An excisive homology theory for \( C \) with values in \( \mathcal{T} \) is an \( \mathcal{E} \)-excisive homology, as defined in [2, Sec 6.6], when \( \mathcal{E} \) is the class of weakly split extensions.

Let \( P \) be a property for functors defined on \( C \) to some triangulated category. A functor \( u : C \to \mathcal{V} \) is a universal functor with \( P \) if it has the property \( P \) and if \( F : \mathcal{C} \to \mathcal{T} \) is another functor with \( P \) there exists a unique triangle functor \( G : \mathcal{V} \to \mathcal{T} \) such that the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{u} & \mathcal{V} \\
\downarrow \quad F & & \downarrow \quad G \\
\mathcal{T} & & \mathcal{V}
\end{array}
\]
2.6. The category \( \mathcal{R}_C \). Let \( A \) and \( B \) be objects in \( C \). Consider \( \mathcal{M}_\infty \) the ind-ring defined in [2, Sec 4.1] and define inductively \( B^{S^n+1} := (B^{S^n})^{S^1} \). Define

\[
E_n(A, B)_C := [J^n(A), \mathcal{M}_\infty B^{S^n}]_C.
\]

Consider the following morphism \( \iota_n : E_n(A, B)_C \to E_{n+1}(A, B)_C \) such that

\[
J^n(A) \xrightarrow{J(f)} J(\mathcal{M}_\infty B^{S^n}) \xrightarrow{\phi_{E_{S^n}B^{S^n}}} \mathcal{M}_\infty J(B^{S^n}) \xrightarrow{\rho_{B^{S^n}}} \mathcal{M}_\infty B^{S^{n+1}}.
\]

Define

\[
kk_C(A, B) = \operatorname{colim}_{n \in \mathbb{N}} E_n(A, B)_C.
\]

Let \( A, B \) and \( C \) be objects in \( C \). There exists an associative product

\[
o : kk_C(B, C) \times kk_C(A, B) \to kk_C(A, C)
\]

which extends the composition of algebra homomorphisms. If \( [\alpha] \in kk_C(B, C) \) is an element represented by \( \alpha : J^n(B) \to C^{S^n} \) and \( [\beta] \in kk_C(A, B) \) is an element represented by \( \beta : J^m(A) \to B^{S^m} \) then \( [\alpha] \circ [\beta] \) is an element \( kk_C(A, C) \) represented by

\[
J^{n+m}(A) \xrightarrow{J^n(\beta)} J^n(B^{S^m}) \xrightarrow{\alpha^{S^m}} C^{S^{n+m}}.
\]

This product allows us to define a composition in the category \( \mathcal{R}_C \) whose objects are the same objects of \( C \) and the morphisms from \( A \) to \( B \) are the elements of \( kk_C(A, B) \). Denote by

\[
j_C : C \to \mathcal{R}_C
\]

the functor which at the level of objects is the identity and at level of morphism sends \( f : A \to B \) to \( [f] \in kk_C(A, B) \).

Remark 2.6.2. A morphism \( f : C \to \mathcal{M}_\infty C \) in \( C \) represents an element \( [f] \) in \( kk_C(C, \mathcal{M}_\infty C) \). But also represents an element in \( kk_C(C, C) \) because

\[
kk_C(C, C) = \operatorname{colim} E_n(C, C)_C \quad \text{and} \quad E_0(C, C)_C = [C, \mathcal{M}_\infty C]_C.
\]

Consider the functor \( \Omega : \mathcal{R}_C \to \mathcal{R}_C \) which sends an object \( A \) of \( C \) to the path object \( \Omega A \). Let \( [\alpha] \) be an element of \( kk_C(A, B) \) represented by \( \alpha : J^n(A) \to B^{S^n} \). The class of \( \Omega [\alpha] \) is represented by

\[
J^n(A^{S^1}) \to J^n(A)^{S^1} \xrightarrow{\alpha^{S^1}} B^{S^{n+1}}.
\]

Note (2.6.3) represents an element of \( kk_C(A^{S^1}, B^{S^1}) \) (see Lemma 6.3.8 [2] to check it is well defined). As \( \iota : \Omega A \to A^{S^1} \) is a \( kk_C \)-equivalence (see corollary of Lemma 6.3.2 [2]), we have the natural isomorphism

\[
kk_C(A^{S^1}, B^{S^1}) \xrightarrow{\sim} kk_C(\Omega A, \Omega B).
\]

A diagram

\[
\Omega C \to A \to B \to C
\]

of morphisms in \( \mathcal{R}_C \) is called a distinguished triangle if it is isomorphic in \( \mathcal{R}_C \) to the path sequence

\[
\Omega B' \xrightarrow{j(f)} P_f \xrightarrow{j(\pi_1)} A' \xrightarrow{j(f)} B'
\]

associated with a homomorphism \( f : A' \to B' \) in \( C \). Denote the class of distinguished triangles by \( Q \). The category \( \mathcal{R}_C \) is triangulated with respect to the
endofunctor $\Omega : \mathcal{R}_C \rightarrow \mathcal{R}_C$ and the class $Q$ of distinguished triangles, see [2, Theorem 6.5.2].

Let $E : A \xrightarrow{f} B \xrightarrow{g} C$ be a weakly split extension and let $c_E \in \text{kk}_C(J(C), A)$ be the classifying map of $E$. As the natural map $\rho_A : J(A) \rightarrow \Omega A$ induces a $\text{kk}_C$-equivalence (see Lemma 6.3.10, [2]) we can consider the following morphisms in $\text{kk}_C(\Omega C, A)$

\[
\partial_E := c_E \circ \rho_C^{-1}.
\]

The functor $j_C : C \rightarrow \mathcal{R}_C$ with the morphisms $\{\partial_E : E \in \mathcal{E}\}$ is an excisive homology theory, homotopy invariant and $M_\infty$-stable.

**Theorem 2.6.5.** The functor $j_C : C \rightarrow \mathcal{R}_C$ is universal with the properties defined above. In other words, if $T$ is a triangulated category and $G : C \rightarrow T$ together a class of morphisms $\{\partial_E : E \in \mathcal{E}\}$ is an excisive, homotopy invariant and $M_\infty$-stable functor, then there exists a unique triangle functor $G : \mathcal{R}_C \rightarrow T$ such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{j_C} & \mathcal{R}_C \\
\downarrow G & & \downarrow j_C \\
T & & T
\end{array}
\]

**Proof.** By definition $j_C$ is homotopy invariant and $M_\infty$-stable. Let us show that $j_C : C \rightarrow \mathcal{R}_C$ is an excisive homology theory with the family $\{\partial_E : E \in \mathcal{E}\}$. Let $E : A \xrightarrow{f} B \xrightarrow{g} C$ be a weakly split extension. Take the path sequence asociated to $g$ and the following diagram in $\mathcal{R}_C$.

\[
\begin{array}{ccc}
\Omega C & \xrightarrow{j(f)} & P_g \\
\downarrow \partial_E & & \downarrow \partial_E \\
A & \xrightarrow{j(f)} & B
\end{array}
\]

The first square commutes because $j(f) \circ c_E$ is elementarily homotopic to $j(f) \circ c_E$. By [2, Lemma 6.3.2], the morphism $j(f)$ is a $\text{kk}_C$-equivalence. Finally $T$ and $T'$ are isomorphic in $\mathcal{R}_C$ and $T'$ is a distinguished triangle. For the rest of the proof see [2, Theorem 6.6.2].

Algebraic $K$-theory is not excisive, nor is it $M_\infty$-stable or homotopy invariant. However the homotopy algebraic $K$-theory, defined in [12] and denoted by $KH$, have all these properties. Consider the funtor $KH = KH_0 : \text{Alg}_\ell \rightarrow \text{Ab}$. Because it satisfies [2, Theorem 6.6.6] we have a natural map

\[
\text{kk}_{\text{Alg}_\ell}(\ell, A) \rightarrow \text{hom}_{\text{Ab}}(KH(\ell), KH(A))
\]

As $\ell$ is unital, there is a map $Z \rightarrow \ell$ which induce a map

\[
\text{hom}_{\text{Ab}}(KH(\ell), KH(A)) \rightarrow \text{hom}_{\text{Ab}}(KH(Z), KH(A)) \simeq KH(A).
\]

Composing both maps we obtain a homomorphism $\text{kk}_{\text{Alg}_\ell}(\ell, A) \rightarrow KH(A)$ and the main theorem from [2] prove that it is an isomorphism.

**Theorem 2.6.6.** Consider $C = \text{Alg}_\ell$ the category of $\ell$-algebras, then

\[
\text{kk}_{\text{Alg}_\ell}(\ell, A) \simeq KH(A).
\]

**Proof.** See [2, Theorem 8.2.1].
3. Equivariant matrix invariance

In the equivariant setting we replace the property of $M_\infty$-stability by a stability condition depending on $C$. We consider the different cases of $C$ separately.

3.1. $G$-equivariant stability. Regard

\[ M_G = \{ f : G \times G \to \ell : \text{supp}(f) < \infty \} \]

as the algebra of matrices with coefficients in $\ell$ indexed by $G \times G$, with translation action of $G$:

\[ g \cdot e_{s,t} = e_{gs,gt}. \]

We are going to identify a $G$-algebra $A$ with the $G$-algebra $M_G \otimes A$ with the diagonal action. Note that the map $a \mapsto a \otimes e_{1_G,1_G}$ is not equivariant then we can not define $G$-stability as in Section 2.2. For this reason we define $G$-stability in terms of $G$-modules.

A pair $(W, B)$ is a $G$-module with basis if $W$ is a $G$-module, free as an $\ell$-module and $B$ is a basis of $W$. A pair $(W', B')$ is a submodule with basis of $(W, B)$ if $W'$ is a submodule of $W$ and $B' \subset B$. Note that if $(W_1, B_1)$ and $(W_2, B_2)$ are $G$-modules with basis then $(W_1 \oplus W_2, B_1 \cup B_2)$ is a $G$-module with basis.

Let $(W, B)$ be a $G$-module with basis $B$. We define

\[ \mathcal{L}(W, B) := \{ \psi : B \times B \to \ell : \{ v : \psi(v, w) \neq 0 \} \text{ is finite for all } w \} \]

Note that $\mathcal{L}(W, B)$ and $\text{End}_\ell(W) = \{ f : W \to W : f \text{ is } \ell\text{-linear} \}$ are isomorphic; indeed we have inverse isomorphisms

\begin{align*}
\text{(3.1.1)} & \quad \text{End}_\ell(W) \to \mathcal{L}(W, B) \quad f \mapsto \psi_f \\
& \quad \psi_f(v, w) = p_v(f(w))
\end{align*}

Here $p_v : W \to \ell$ is the projection to the submodule of $W$ generated by $v$.

\begin{align*}
\text{(3.1.2)} & \quad \mathcal{L}(W, B) \to \text{End}_\ell(W) \\
& \quad \psi \mapsto f_\psi \\
& \quad f_\psi(w) = \sum_{v \in B} \psi(v, w)v.
\end{align*}

Define

\[ C(W, B) := \{ \psi \in \mathcal{L}(W, B) : \{ w : \psi(v, w) \neq 0 \} \text{ is finite for all } v \} \]

\[ F(W, B) := \{ \psi \in \mathcal{L}(W, B) : \{ (v, w) : \psi(v, w) \neq 0 \} \text{ is finite } \} \]

\[ \text{End}^C_\ell(W, B) := \{ f \in \text{End}_\ell(W) : \psi_f \in F(W, B) \} \]

\[ \text{End}^C_\ell(W, B) := \{ f \in \text{End}_\ell(W) : \psi_f \in C(W, B) \} \]

Note that $C(W, B)$ is a ring with the matrix product and $\text{End}^C_\ell(W, B)$ is a ring with the composition. These rings are isomorphic.

Let $(W, B)$ be a $G$-module with basis. Consider the representation

\[ \rho : G \to \text{End}_\ell(W) \]

\[ \rho_g(w) = g \cdot w \]

We say that $(W, B)$ is a $G$-module by finite automorphisms if $\rho(G) \subset \text{End}^C_\ell(W, B)$.

We say that $(W, B)$ is a $G$-module by locally finite automorphisms if $\rho(G) \subset \text{End}^C_\ell(W, B)$.

If $(W, B)$ is a $G$-module by locally finite automorphisms, $\text{End}^C_\ell(W, B)$ and $\text{End}^C_\ell(W, B)$ are $G$-algebras with the following action

\[ g \cdot f = \rho(g)f(\rho(g))^{-1} \]

Note that $\text{End}^C_\ell(W, B)$ is an ideal of $\text{End}^C_\ell(W, B)$. 
Remark 3.1.5. Let $\mathcal{W}$ be a $G$-module. Let $\mathcal{W}^T$ be $\mathcal{W}$ considered as a $G$-module with trivial action. Recall that $\ell G \otimes \mathcal{W} \simeq \ell G \otimes \mathcal{W}^T$. Inverse isomorphisms are given by

$T : \ell G \otimes \mathcal{W}^T \rightarrow \ell G \otimes \mathcal{W} \quad S : \ell G \otimes \mathcal{W} \rightarrow \ell G \otimes \mathcal{W}^T$

and

$T(\delta_g \otimes h) = \delta_g \otimes g(h) \quad S(\delta_g \otimes h) = \delta_g \otimes g^{-1}(h)$

If $(\mathcal{W}, B)$ is a $G$-module with basis we will write $\text{End}_{\ell}^G(\mathcal{W})$ and $\text{End}_{\ell}^F(\mathcal{W})$ omitting the basis when there is no confusion.

Remark 3.1.6. Let $(\mathcal{W}, B)$ be a $G$-module with basis. Let us check that

$\text{End}_{\ell}^F(\ell G \otimes \mathcal{W}) \simeq \text{End}_{\ell}^F(\ell G) \otimes \text{End}_{\ell}^F(\mathcal{W})$.

Define a $G$-algebra homomorphism $T : \text{End}_{\ell}^F(\ell G) \otimes \text{End}_{\ell}^F(\mathcal{W}) \rightarrow \text{End}_{\ell}^F(\ell G \otimes \mathcal{W})$ by

$T(e_{g,h} \otimes e_{v,w}) = e_{g,v,h,w} \quad v, w \in B \quad g, h \in G$

As $T$ is a bijection between the basis, $T$ is an isomorphism.

Let $(\mathcal{W}_1, B_1)$ and $(\mathcal{W}_2, B_2)$ be $G$-modules by locally finite automorphisms such that $\text{card}(B_i) \leq \text{card}(G) \times \text{card} \mathbb{N}, \quad i = 1, 2$. The inclusion $\iota : \mathcal{W}_1 \rightarrow \mathcal{W}_1 \oplus \mathcal{W}_2$ induces a morphism of $G$-algebras

(3.1.7) \quad \iota : \text{End}_{\ell}^F(\mathcal{W}_1) \rightarrow \text{End}_{\ell}^F(\mathcal{W}_1 \oplus \mathcal{W}_2) \quad f \mapsto \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$

Let $A$ be a $G$-algebra and consider

$\iota \otimes 1 : \text{End}_{\ell}^F(\mathcal{W}_1) \otimes A \rightarrow \text{End}_{\ell}^F(\mathcal{W}_1 \oplus \mathcal{W}_2) \otimes A$.

A functor $F : G\text{-Alg} \rightarrow \mathcal{D}$ is $G$-stable if for $(\mathcal{W}_1, B_1)$, $(\mathcal{W}_2, B_2)$ and $A$ as above $F(\iota \otimes 1)$ is an isomorphism in $\mathcal{D}$.

Let us show that if $F : G\text{-Alg} \rightarrow \mathcal{D}$ is a $G$-stable functor then $F$ is $M_\infty$-stable. Consider $(\mathcal{W}_1, B_1) = (\ell, \{1\})$ and $(\mathcal{W}_2, B_2) = (\ell^{(b)}, \{e_i : i \in \mathbb{N}\})$ with $\ell^{(b)} = \bigoplus_{i=1}^\infty \ell$, $\{e_i : i \in \mathbb{N}\}$ is the canonical basis and both modules have the trivial action of $G$. Note

$\text{End}_{\ell}^F(\ell) = \text{End}_{\ell}(\ell) = \ell \quad \text{End}_{\ell}^F(\ell \oplus \ell^{(b)}) = \text{End}_{\ell}^F(\ell^{(b)}) = \ell^{(b)}$

and $\iota_\infty : \ell \rightarrow M_\infty$ is the inclusion at the upper left corner. Then $\iota \otimes 1 = \iota : A \rightarrow M_\infty(A)$ and $F(\iota)$ is an isomorphism. Observe that if $G = \{e\}$, $F$ is $G$-stable if and only if $F$ is $M_\infty$-stable.
Let $A$, $B$ be $G$-algebras and $F : G\text{-Alg} \to \mathcal{D}$ a functor. A zig-zag between $A$ and $B$ by $F$ is a diagram in $G\text{-Alg}$

$$A \xrightarrow{f_1} C_1 \xrightarrow{g_1} \cdots \xrightarrow{f_n} C_n \xrightarrow{g_n} B$$

such that $F(g_i)$, $i = 1, \ldots, n$, are isomorphisms in $\mathcal{D}$.

**Example 3.1.8.** Let $A$ be a $G$-algebra and $F$ a $G$-stable functor. There exists a zig-zag between $A$ and $M_G A$ by $F$. Consider $\mathcal{W}_1 = (\ell G, B)$ as in the example 3.1.3 and consider $\mathcal{W}_2 = (\ell, \{1\})$ with the trivial action of $G$. Put $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ and $C = \text{End}_F(\mathcal{W})$ with the induced action, then

$$\ell : A = A \otimes \ell = A \otimes \text{End}_F(\ell) \to A \otimes C \leftarrow A \otimes M_G : \ell'$$

is a zig-zag between $A$ and $M_G A$ by $F$.

**Proposition 3.1.9.** Suppose $G$ is countable. Let $F : G\text{-Alg} \to \mathcal{D}$ be an $M_\infty$-stable functor. Define

$$\tilde{F} : G\text{-Alg} \to \mathcal{D} \quad A \mapsto F(M_G \otimes A)$$

Then $\tilde{F}$ is $G$-stable.

**Proof.** Let $(\mathcal{W}_1, B_1)$, $(\mathcal{W}_2, B_2)$ be $G$-modules by locally finite automorphisms and let $A$ be a $G$-algebra. Consider

$$\tilde{i} \otimes 1 : \text{End}_F(\mathcal{W}_1) \otimes A \to \text{End}_F(\mathcal{W}_1 \oplus \mathcal{W}_2) \otimes A.$$ 

We have to prove that

$$\tilde{F}(\tilde{i} \otimes 1) : F(M_G \otimes \text{End}_F(\mathcal{W}_1) \otimes A) \to F(M_G \otimes \text{End}_F(\mathcal{W}_1 \oplus \mathcal{W}_2) \otimes A)$$

is an isomorphism. By remarks 3.1.6 and 3.1.5 we know that

$$F(M_G \otimes \text{End}_F(\mathcal{W}_1) \otimes A) \simeq F(\text{End}_F(\mathcal{W}_1^\gamma) \otimes M_G \otimes A)$$

and

$$F(M_G \otimes \text{End}_F(\mathcal{W}_1 \oplus \mathcal{W}_2) \otimes A) \simeq F(\text{End}_F((\mathcal{W}_1 \oplus \mathcal{W}_2)^\gamma) \otimes M_G \otimes A).$$

Note that $\text{End}_F(\mathcal{W}_1^\gamma)$ and $\text{End}_F((\mathcal{W}_1 \oplus \mathcal{W}_2)^\gamma)$ are equivariantly isomorphic to $M_n$ or $M_\infty$. As $F$ is $M_\infty$-stable, (3.1.10) is an isomorphism.

**Remark 3.1.11.** Suppose $G$ is a finite group of order $n$ such that $1/n \in \ell$. Then the element $\xi = (1/n) \sum_{g \in G} \delta_g$ in $\ell G$ is idempotent. The map $s : \ell \to \ell G$, $s(1) = \xi$, is a $G$-equivariant section of the canonical augmentation $\varphi : \ell G \to \ell$. Thus the sequence of $G$-modules

$$0 \longrightarrow I \longrightarrow \ell G \xrightarrow{\varphi} \ell \longrightarrow 0$$

splits. Hence $\ell G = \ell \xi \oplus I$. Notice that $I$ is a $G$-module with basis $\{\delta_e - \delta_g : g \neq e\}$. Define

$$\lambda_g = \begin{cases} \xi & g = e \\ \delta_e - \delta_g & g \neq e \end{cases}$$

The set $\Lambda = \{\lambda_g : g \in G\}$ is a basis of $\ell G$ and the relations with the elements of $B = \{\delta_g : g \in G\}$ are the following

$$\begin{align*}
\lambda_e &= \frac{1}{n} \sum_{g \in G} \delta_g \\
\delta_e &= \lambda_e + \frac{1}{n} \sum_{g \neq e} \lambda_g \\
\lambda_h &= \delta_e - \delta_h \\
\delta_h &= \lambda_e - \lambda_h + \frac{1}{n} \sum_{g \neq e} \lambda_g \quad \text{for } h \neq e
\end{align*}$$
Consider $W_1 = \ell = (\ell \xi, \{\xi\})$ and $W_2 = (I, \{\lambda_g\}_{g \neq e})$, in this case the morphism (3.1.7) is
\[
T : \ell \to M_G \simeq \text{End}(\ell G, A) \quad 1 \mapsto \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]
If we write it in the canonical basis $B$ we have
\[
T : \ell \to M_G = \text{End}(\ell G, B) \quad 1 \mapsto \begin{pmatrix}
1 & \frac{1}{n} & \cdots & \frac{1}{n} \\
\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n}
\end{pmatrix}
\]
If $F : G\text{-Alg} \to \mathcal{D}$ is a $G$-stable functor then $F(T)$ is an isomorphism in $\mathcal{D}$.

3.2. $G$-graded stability. In this section we consider a dual notion of $G$-equivariant stability. We want to identify a $G$-graded algebra $A$ with the $G$-graded matrix algebra $M_G A$. The definition of $G$-graded stability is easier than that of $G$-equivariant stability because the morphism $A \to M_G A, a \mapsto e_{1G, 1G} \otimes a$, is homogeneous.

Write $G_{gr}$-$\text{Alg}$ for the category of $G$-graded algebras. Let $A$ be a $G$-graded algebra. Define the following grading in $M_G A$
\[
(M_G A)_g := \langle e_{s,t} \otimes a : g = s|a|t^{-1} \rangle.
\]
Depending on the context $M_G A$ will be considered as an algebra, a $G$-algebra or a $G$-graded algebra. Let be
\[
\iota_A : A \to M_G A \quad a \mapsto e_{1G, 1G} \otimes a,
\]
note $\iota_A$ is homogeneous. A functor $F : G_{gr}$-$\text{Alg} \to \mathcal{D}$ is $G_{gr}$-stable if $F(\iota_A)$ is an isomorphism in $\mathcal{D}$ for all $A \in G_{gr}$-$\text{Alg}$.

**Proposition 3.2.2.** Suppose $G$ is countable. Let $F : G_{gr}$-$\text{Alg} \to \mathcal{D}$ be an $M_\infty$-stable functor. Define
\[
\hat{F} : G_{gr}$-$\text{Alg} \to \mathcal{D} \quad A \mapsto F(M_G A)
\]
with $M_G A$ as in (3.2.1). Then $\hat{F}$ is $G_{gr}$-stable.

**Proof.** Denote by $M_{[G]}$ to $M_G$ with the trivial grading. Because $G$ is countable we have that $M_\infty M_{[G]}$ and $M_\infty$ are isomorphic in $G_{gr}$-$\text{Alg}$. We also have the following isomorphisms of $G$-graded algebras
\[
\begin{align*}
\eta : M_G M_G &\to M_G M_{[G]} \\
\mu : M_G M_{[G]} &\to M_{[G]} M_G
\end{align*}
\]
Consider the following commutative diagram
\[
\begin{tikzcd}
M_G M_G \arrow{r}{\eta} \arrow{rd}{\mu} & M_G M_{[G]} \arrow{d}{\iota_{M_G M_{[G]}}} \\
M_{[G]} M_G \arrow{ru}{\iota_{M_G M_G}} \arrow{r}{\mu} & M_{[G]} M_{[G]} \arrow{d}{\iota_{M_{[G]} M_{[G]}}}
\end{tikzcd}
\]
It follows that \( \hat{F}(\iota_A) = F(\text{id}_{M_G} \otimes \iota_A) \) is an isomorphism in \( D \) because \( F \) is \( M_\infty \)-stable.

\[ \Box \]

4. **EQUIVARIANT ALGEBRAIC KK-THEORY**

From now to the rest of the paper we suppose \( G \) is a countable group.

4.1. **The category \( \mathcal{R}R^G \)**. In this section we introduce an equivariant algebraic kk-theory for \( G \)-algebras. Let \( A, B \) be \( G \)-algebras, we define

\[
\text{kk}^G(A, B) := \text{kk}_{G, \text{Alg}}(M_G \otimes A, M_G \otimes B).
\]

Consider the category \( \mathcal{R}R^G \) whose objects are the \( G \)-algebras and where the morphisms between \( A \) and \( B \) are the elements of \( \text{kk}^G(A, B) \). Let \( j^G : G\text{-Alg} \to \mathcal{R}R^G \) be the functor defined as the identity on objects and which sends each morphism of \( G \)-algebras \( f : A \to B \) to its class \([\text{id}_{M_G} \otimes f] \in \text{kk}^G(A, B)\). The composition law in \( \mathcal{R}R^G \) is the same that in (2.6.1) taking \( M_G \otimes A \), \( M_G \otimes B \) and \( M_G \otimes C \) instead of \( A \), \( B \) and \( C \). As in Section 2.6 we have an equivalence \( \Omega : \mathcal{R}R^G \to \mathcal{R}R^G \) and distinguished triangles

\[
\Omega C \to A \to B \to C
\]

in \( \mathcal{R}R^G \) which gives to \( \mathcal{R}R^G \) a triangulated category structure.

**Theorem 4.1.1.** The functor \( j^G : G\text{-Alg} \to \mathcal{R}R^G \) is an excisive, equivariantly homotopy invariant, and \( G \)-stable functor. Moreover, it is the universal functor for these properties. In other words, if \( T \) is a triangulated category and \( R : G\text{-Alg} \to T \) together a class of morphisms \( \{ \partial_E : E \in \mathcal{E} \} \) is an excisive, equivariantly homotopy invariant and \( G \)-stable functor, then there exists a unique triangle functor \( \overline{R} : \mathcal{R}R^G \to T \) such that the following diagram commutes

\[
\begin{array}{ccc}
G\text{-Alg} & \xrightarrow{j^G} & \mathcal{R}R^G \\
\downarrow R & & \downarrow \overline{R} \\
\mathcal{T} & & \mathcal{T}
\end{array}
\]

**Proof.** Let \( E \) be a weakly split extension. Define

\[
\partial^G_E \in \text{hom}_{\mathcal{R}R^G}(\Omega C, A) = \text{hom}_{\mathcal{R}R_{G, \text{Alg}}}(\Omega M_G \otimes C, M_G \otimes A)
\]

as the morphism \( \partial^G_E \) defined in (2.6.4) associated to the following weakly split extension

\[
M_G \otimes A \to M_G \otimes B \to M_G \otimes C \quad (E')
\]

By theorem 2.6.5 and proposition 3.1.9 the functor \( j^G : G\text{-Alg} \to \mathcal{R}R^G \) with the family \( \{ \partial^G_E : E \in \mathcal{E} \} \) is an excisive, homotopy invariant and \( G \)-stable functor. Let us check it is universal for these properties. Let \( X : G\text{-Alg} \to T \) be a functor which has the mentioned properties with a family \( \{ \partial^G_E : E \in \mathcal{E} \} \). By theorem 2.6.5 there exists a unique triangle functor \( \overline{X} : \mathcal{R}R_{G, \text{Alg}} \to T \) such that the following diagram
We will define \( X' : \mathcal{R}G \to \mathcal{T} \). We know that \( X' = X \) on objects. As \( X \) is \( G \)-stable the following morphisms are a zig-zag between \( A \) and \( M_G \otimes A \) by \( X \), (see Example 3.1.8)

\[
(4.1.3) \quad A \xrightarrow{\iota_A} M_{G,\{*\}} \otimes A \xrightarrow{\iota'_A} M_G \otimes A
\]

Let \( \alpha \in \text{kk}^G(A, B) \) and define

\[
X'(\alpha) := X(\iota_B)^{-1}X(\iota'_B)X(\alpha)X(\iota'_A)^{-1}X(\iota_A).
\]

Note this definition is the unique possibility to make the diagram (4.1.2) commutative. \( \square \)

4.2. **The category \( \hat{\mathcal{R}}^G \).** In this section we introduce the equivariant algebraic \( \text{kk} \)-theory for \( G \)-graded algebras. Let \( A, B \) be \( G \)-graded algebras. We define

\[
\text{kk}^G(A, B) := \text{kk}_{Ggr,\text{Alg}}(M_G A, M_G B).
\]

Consider the category \( \hat{\mathcal{R}}^G \) whose objects are the \( G \)-graded algebras and the morphisms between \( A \) and \( B \) are the elements of \( \text{kk}^G(A, B) \). Let \( j^{Ggr} : G_{gr,\text{Alg}} \to \hat{\mathcal{R}}^G \) be the functor defined as the identity on objects and which sends each morphism of \( G \)-graded algebras \( f : A \to B \) to its class \( [\text{id}_{M_G f}] \in \text{kk}^G(A, B) \). As in Section 4.1 we can consider a triangulated category structure on \( \hat{\mathcal{R}}^G \).

**Theorem 4.2.1.** The functor \( j^{Ggr} : G_{gr,\text{Alg}} \to \hat{\mathcal{R}}^G \) is an excisive, graded homotopy invariant, and \( G \)-graded stable functor. Moreover, it is the universal functor for these properties. In other words, if \( \mathcal{T} \) is a triangulated category and \( R : G_{gr,\text{Alg}} \to \mathcal{T} \) together a class of morphisms \( \{ \overline{\partial_E} : E \in \mathcal{E} \} \) is an excisive, graded homotopy invariant and \( G \)-graded stable functor, then there exists a unique triangle functor \( \overline{\mathcal{R}} : \hat{\mathcal{R}}^G \to \mathcal{T} \) such that the following diagram commutes

\[
\begin{array}{ccc}
G_{gr,\text{Alg}} & \xrightarrow{j^{Ggr}} & \hat{\mathcal{R}}^G \\
\downarrow R & & \downarrow \overline{\mathcal{R}} \\
\mathcal{T} & \xrightarrow{} & \mathcal{T}
\end{array}
\]

**Proof.** The proof is similar to Theorem 4.1.1. \( \square \)
5. Algebraic Green-Julg theorem

5.1. Crossed product and trivial action. Let $A$ be an $\ell$-algebra. Write $A^\tau$ for $A$ with the trivial action of $G$. This gives us a functor $\tau: \text{Alg} \to G\text{-Alg}$. It is easy to check that $j^G \circ \tau$ satisfies excision, is homotopy invariant and is $M_\infty$-stable. Write $\mathcal{R} = \mathcal{R}_{\text{Alg}}$. By Theorem 2.6.5 there exists a unique functor $\tau: \mathcal{R} \to \mathcal{R}^G$ such that the following diagram is commutative

$$
\begin{array}{ccc}
\text{Alg} & \xrightarrow{\tau} & G\text{-Alg} \\
j & & j^G \\
\mathcal{R} & \xrightarrow{\tau} & \mathcal{R}^G
\end{array}
$$

Let $A$ be a $G$-algebra. The crossed product algebra $A \rtimes G$ is the $\ell$-module $A \otimes \ell G$ with the following multiplication

$$(a \rtimes g)(b \rtimes h) = a(g \cdot b) \rtimes gh \quad a,b \in A, g,h \in G.$$

**Proposition 5.1.1.** Let $A$ be a $G$-algebra and $\mathcal{W}$ a $G$-module by locally finite automorphisms. The following algebras are naturally isomorphic

$$(A \rtimes G) \otimes \text{End}_F^\ell(\mathcal{W}) \simeq (A \otimes \text{End}_F^\ell(\mathcal{W})) \rtimes G.$$

**Proof.** Let $\rho: G \to (\text{End}^\ell_F(\mathcal{W}))^\times$ be the structure map. Note that the homomorphisms

$$\begin{align*}
\phi: (A \rtimes G) \otimes \text{End}_F^\ell(\mathcal{W}) & \to (A \otimes \text{End}_F^\ell(\mathcal{W})) \rtimes G \\
\psi: (A \otimes \text{End}_F^\ell(\mathcal{W})) \rtimes G & \to (A \rtimes G) \otimes \text{End}_F^\ell(\mathcal{W})
\end{align*}$$

are inverse of each other. □

**Proposition 5.1.2.** There exists a unique functor $\rtimes G: \mathcal{R}^G \to \mathcal{R}$ such that the following diagram is commutative

$$
\begin{array}{ccc}
G\text{-Alg} & \xrightarrow{\rtimes G} & \text{Alg} \\
j^G & & j \\
\mathcal{R}^G & \xrightarrow{\tau} & \mathcal{R}
\end{array}
$$

**Proof.** We shall show $j(\times G)$ is excisive, homotopy invariant and $G$-stable. Because $\times G$ maps split exact sequences to split exact sequences and $j$ is excisive, then $j(\times G)$ is excisive. That $j(\times G)$ is homotopy invariant follows from the fact that

$$A[t] \rtimes G = (A \rtimes G)[t].$$

Let $(\mathcal{W}_1, B_1), (\mathcal{W}_2, B_2)$ be $G$-modules by locally finite automorphisms and $A$ a $G$-algebra. Consider the isomorphism $\psi$ defined in Proposition 5.1.1. Note that the following diagram is commutative

$$
\begin{array}{ccc}
(A \otimes \text{End}_F^\ell(\mathcal{W}_1)) \rtimes G & \xrightarrow{1 \otimes \psi} & (A \otimes \text{End}_F^\ell(\mathcal{W}_1 \oplus \mathcal{W}_2)) \rtimes G \\
\psi & & \psi \\
(A \rtimes G) \otimes \text{End}_F^\ell(\mathcal{W}_1) & \xrightarrow{(1 \otimes G) \otimes \psi} & (A \rtimes G) \otimes \text{End}_F^\ell(\mathcal{W}_1 \oplus \mathcal{W}_2)
\end{array}
$$
Because \( j \) is \( M_\infty \)-stable, \( j((1 \times G) \otimes \bar{1}) \) is an isomorphism. Hence \( j(- \times G)(1 \otimes \bar{1}) \) is an isomorphism by the diagram above. 

\[ \square \]

**Remark 5.1.3.** Let \( [\iota_\alpha \alpha] \in \text{kk}^G(A, B) \) be an element represented by \( \alpha : J^n(M_G A) \to (M_G B)^{sd^n} S^n \) which is a morphism in \([J^n(M_G A), M_\infty(M_G B)^{sd^n} S^n]\). Consider the classifying map

\[ J^n(M_G A \times G) \to J^n(M_G A) \times G \]

The element \([\alpha] \times G\) is represented by the following composition

\[ J^n(M_G A \times G) \to J^n(M_G A) \times G \xrightarrow{\alpha \times G} (M_G B)^{sd^n} S^n \times G. \]

**Proposition 5.1.4.** The functor \( \times G : \mathfrak{R}_G \to \mathfrak{R} \) is a triangle functor.

**Proof.** A distinguished triangle in \( \mathfrak{R}_G \) is a diagram isomorphic to

\[ \Omega B \xrightarrow{j^G(\iota_\alpha)} P_f \xrightarrow{j^G(\pi_0)} A \xrightarrow{j^G(\iota_1)} B \]

for some morphism of \( G \)-algebras \( f : A \to B \). That means, if it is isomorphic to

\[ (5.1.5) \quad M_G \Omega B \xrightarrow{j^G(\iota_\alpha)} M_GP_f \xrightarrow{j^G(\pi_0)} M_GA \xrightarrow{j^G(\iota_1)} M_GB \]

in \( \mathfrak{R}_{G, \text{Alg}} \). The functor \( \times G : \mathfrak{R}_G \to \mathfrak{R} \) sends the triangle (5.1.5) to

\[ (5.1.6) \quad (M_G \Omega B) \times G \to (M_G P_f) \times G \to (M_GA) \times G \to (M_GB) \times G \]

which by Proposition 5.1.1 and \( M_\infty \)-stability is isomorphic to

\[ (5.1.7) \quad \Omega(B \times G) \xrightarrow{P_f \times G} A \times G \to B \times G \]

As \( P_f \times G \simeq P_f \times G \), (5.1.7) is a distinguished triangle in \( \mathfrak{R} \). \( \square \)

### 5.2. Green-Julg Theorem for \( \mathfrak{R}_G \)

In this section we shall see an algebraic version of the **Green-Julg Theorem**, see [11] and [6] for versions of this result in Kasparov KK-theory and \( E \)-theory setting.

**Theorem 5.2.1.** Let \( G \) be a finite group of \( n \) elements and \( 1/n \in \ell \). The functors \( \tau : \mathfrak{R} \to \mathfrak{R}_G \) and \( \times G : \mathfrak{R}_G \to \mathfrak{R} \) are adjoint functors. Hence

\[ \text{kk}^G(A^\tau, B) \simeq \text{kk}(A, B \times G) \quad A \in \text{Alg} \quad B \in \text{G-Alg} \]

**Proof.** By [10, Theorem 2, pag 81], it is enough to prove that there exist natural transformations \( \overline{\sigma}_A \in \text{kk}(A, A^\tau \times G) \) and \( \overline{\pi}_B \in \text{kk}^G((B \times G)^\tau, B) \) such that the following compositions

\[ A^\tau \xrightarrow{\overline{\pi}(\sigma_A)} (A^\tau \times G)^\tau \xrightarrow{\overline{\sigma}(\pi_A)} A^\tau \]

are the identities in \( \text{kk}^G(A^\tau, A^\tau) \) and \( \text{kk}(B \times G, B \times G) \) respectively.

Put \( \epsilon = 1/n \sum_{g \in G} g \in \ell G \) and define

\[ (5.2.2) \quad \alpha_A : A \to A^\tau \times G = A \otimes \ell G \quad \alpha(a) = a \otimes \epsilon. \]

Note \( \alpha_A \) is an algebra morphism since \( \epsilon \) is idempotent. Consider the element \( \overline{\sigma}_A \in \text{kk}(A, A^\tau \times G) \) represented by \( \alpha_A \). Let

\[ \beta_B : (B \times G)^\tau \to MB_G \quad \beta_B(b \times g) = \sum_{s \in G} s(b)\epsilon_s, \sigma_g \]

One can check that \( \beta_B \) is an equivariant algebra morphism. Let \( \overline{\pi}_B \in \text{kk}^G((B \times G)^\tau, B) \) be represented by \( \beta_B \). The composite \( \beta_{\tau(\sigma_A)} \tau(\alpha_A) \) is \( \text{id}_{A^\tau} \otimes \tau \) where \( \tau \) is the map defined in 3.1.11. As \( j^G \) is \( G \)-stable, \( j^G(\text{id}_{A^\tau} \otimes \tau) \) is the identity in \( \text{kk}^G(A^\tau, A^\tau) \).
Let \( \psi \) be the morphism defined in the proof of the Proposition 5.1.1. By Remark 5.1.3, \( \beta_B \times G \) is represented by \( \psi \circ (\beta_B \times G) \). We want to prove that \( \psi \circ (\beta_B \times G) \circ \alpha_{B \times G} \) represents the identity in \( \text{kk}(B \times G, B \times G) \). Note that

\[
(\psi \circ (\beta_B \times G) \circ \alpha_{B \times G})(b \times g) = \frac{1}{n} \sum_{h,s \in G} (s(b) \times h)e_{s,h^{-1}sg}.
\]

Put \( t = h^{-1}sg \) and note

\[
(5.2.3) \quad \frac{1}{n} \sum_{h,s \in G} (s(b) \times h)e_{s,h^{-1}sg} = \frac{1}{n} \sum_{t,s \in G} (s(b) \times sgt^{-1})e_{s,t}
\]

and

\[
s(b) \times sgt^{-1} = (1 \times s)(b \times g)(1 \times t^{-1}) \quad \text{in } \tilde{B} \times G
\]

We can write (5.2.3) as \( T A_{b \times g} T^{-1} \), where

\[
A_{b \times g} = \frac{1}{n} \sum_{t,s \in G} (b \times g)e_{s,t} \quad T = \sum_{t \in G} (1 \times t)e_{t,t}
\]

Because \( b \times g \mapsto A_{b \times g} \) represents the identity, the same is true for \( b \times g \mapsto T A_{b \times g} T^{-1} \), see \([2, \text{Proposition 5.1.2}]\). \( \square \)

**Example 5.2.4.** We give an example to show that the adjointness between of \( \tau \) and \( \times G \) of Theorem 5.2.1 fails to hold at the algebra level. Let \( G = \mathbb{Z}_2 = \{1, \sigma\} \), \( A = \ell \) and \( B = (\ell G)^* \) the dual algebra of \( \ell G \) with the regular action. Note \( \text{hom}_{G Alg}(A^\ast, B) \) has two elements only:

\[
\varphi_1 : \ell \to (\ell G)^* \quad \varphi_0(1) = 0 \quad \varphi_1(1) = \chi_1 + \chi_\sigma
\]

One the other hand \( \text{hom}_{\ell Alg}(A, B \times G) = \text{hom}_{\ell Alg}(\ell, (\ell G)^* \times G) \) has at least as many elements as \( \ell \). For each \( \lambda \in \ell \) we can define

\[
\varphi_\lambda : \ell \to (\ell G)^* \times G \quad \varphi_\lambda(1) = \chi_1 \times 1 + \lambda(\chi_1 \times \sigma) \quad \lambda \in \ell
\]

Note \( \varphi_\lambda \) is an algebra morphism because \( \chi_1 \times 1 + \lambda(\chi_1 \times \sigma) \) is an idempotent element.

Write \( \psi_{G,J} \) for the isomorphism of the Theorem 5.2.1

\[
(5.2.5) \quad \psi_{G,J} : \text{kk}^G(B^\tau, A) \to \text{kk}(B, A \times G) \quad \psi_{G,J} = \alpha^* \circ \times G
\]

where \( \alpha \) is the morphism defined in (5.2.2).

**Corollary 5.2.6.** Let \( G \) be a finite group such that \( 1/|G| \in \ell \). Let \( A \) be a \( G \)-algebra, then

\[
\text{kk}^G(\ell, A) \simeq \text{kk}(\ell, A \times G) \simeq \text{KH}(A \times G)
\]

\( \square \)

6. **Induction and Restriction**

In this section we study the adjointness property of the functors of induction and restriction between \( \text{R}_{G}^{H} \) and \( \text{R}_{G}^{G} \) where \( G \) is a group and \( H \) is a subgroup of \( G \).

Let \( A \) be a \( G \)-algebra and \( H \subset G \) a subgroup. If we restrict the action to \( H \) we obtain an \( H \)-algebra \( \text{Res}_{G}^{H}(A) \). It is clear this construction defines a functor
\[ \text{Res}_G^H : G\text{-Alg} \to H\text{-Alg}. \]

It is easily seen that we can extend \( \text{Res}_G^H : G\text{-Alg} \to H\text{-Alg} \)
to a triangle functor \( \text{Res}_G^H : \mathcal{R}_G \to \mathcal{R}_H \) so that the following diagram commutes:

\[
\begin{array}{ccc}
G\text{-Alg} & \xrightarrow{\text{Res}_G^H} & H\text{-Alg} \\
\downarrow{j^G} & & \downarrow{j^H} \\
\mathcal{R}_G & \xrightarrow{\text{Res}_G^H} & \mathcal{R}_H
\end{array}
\]

Let \( \pi : G \to G/H \) the projection and \( A \) an \( H \)-algebra. Consider
\[
A^{(G,H)} := \{ f : G \to A : \# \pi(\text{supp}(f)) < \infty \}
\]

and define
\[
\text{Ind}_G^H(A) = \{ f \in A^{(G,H)} : f(s) = h(f(sh)) \ \forall s \in G, \ h \in H \}.
\]

One checks that \( \text{Ind}_G^H(A) \) is a \( G \)-algebra with pointwise multiplication and the following action of \( G \)
\[
(g \cdot f)(s) = f(g^{-1}s) \quad f \in \text{Ind}_G^H(A) \quad g, s \in G.
\]

Observe this construction is functorial, if \( \varphi : A \to B \) is a morphism of \( H \)-algebras then \( \text{Ind}_G^H(\varphi(f)) = \varphi \circ f \).

If \( g \in G \), write \( \chi_g : G \to \mathbb{Z} \) for the characteristic function. If \( a \in A \) and \( g \in G \), define
\[
\xi_H(g,a) = \sum_{h \in H} \chi_{sh} h^{-1}(a) \quad \xi_H(g,a)(s) = \begin{cases} h^{-1}(a) & s = gh \\ 0 & s \notin gh \end{cases}
\]

It is easy to check that these elements belong to \( \text{Ind}_G^H(A) \) and every element \( \phi \in \text{Ind}_G^H(A) \) can be written as a finite sum
\[
\phi = \sum_{g \in \mathcal{R}} \xi_H(g, \phi(g))
\]

where \( r : G/H \to G \) is a pointed section and \( \mathcal{R} = r(G/H) \). Note that we have the following relations
\[
s \cdot \xi_H(g,a) = \xi_H(sg,a)
\]
\[
\xi_H(g,a) \xi_H(\tilde{g},\tilde{a}) = \begin{cases} \xi_H(\tilde{g}, \tilde{g}^{-1}g(a)\tilde{a}) & \tilde{g}^{-1}g \in H \\ 0 & \tilde{g}^{-1}g \notin H \end{cases}
\]
\[
\xi_H(g,a) = \xi_H(gh, h^{-1} \cdot a) \quad h \in H
\]

**Proposition 6.7.** Let \( A \) be a \( G \)-algebra and \( B \) be an \( H \)-algebra, then
\[
\text{Ind}_G^H(B \otimes \text{Res}_G^H A) \simeq \text{Ind}_G^H(B) \otimes A
\]

**Proof.** The isomorphisms are given by
\[
S : \text{Ind}_G^H(B) \otimes A \to \text{Ind}_G^H(B \otimes \text{Res}_G^H A) \quad \xi_H(g,b) \otimes a \mapsto \xi_H(g,b \otimes g^{-1} \cdot a)
\]
\[
T : \text{Ind}_G^H(B \otimes \text{Res}_G^H A) \to \text{Ind}_G^H(B) \otimes A \quad \xi_H(g,b \otimes a) \mapsto \xi_H(g,b) \otimes g \cdot a
\]
\[
\square
\]
Corollary 6.8. Let $A$ be a $G$-algebra. Then
\[
\Ind_G^H \Res_G^H A \to \ell^{(G/H)} \otimes A \quad \xi_H(s, b) \mapsto \chi_s H \otimes s \cdot b
\]
is an isomorphism of $G$-algebras. \hfill \Box

Proposition 6.9. Let $\Ind : H\text{-Alg} \to G\text{-Alg}$ be the following functor
\[
\Ind(A) = \Ind_G^H(M_H \otimes A).
\]
There exists a functor $\Ind_H^G : \mathfrak{R}^H \to \mathfrak{R}^G$ such that the following diagram is commutative
\[
\begin{array}{ccc}
H\text{-Alg} & \xrightarrow{\Ind} & G\text{-Alg} \\
\mathfrak{R}^H & \downarrow{\scriptstyle \pi^H} & \mathfrak{R}^G \\
\Ind_H^G & \downarrow{\scriptstyle \pi^G} & \Ind_H^G
\end{array}
\]
Proof. Straightforward. \hfill \Box

Proposition 6.11. The functor $\Ind_H^G : \mathfrak{R}^H \to \mathfrak{R}^G$ is a triangle functor.
Proof. Let $f : A \to B$ be a morphism of $H$-algebras. The following is an isomorphism of $G$-algebras
\[
\Theta : \Ind_H^G(P_f) \to P_{\Ind_H^G(f)} \quad \xi_H(g, (tp(t), a)) \mapsto (\xi_H(g, tp(t)), \xi_H(g, a)).
\]
The image of the induction functor applied to the path extension of $f$ is
\[
\begin{array}{c}
\Ind_H^G(\Omega B) \\
\xrightarrow{\Ind_H^G(\pi^H)} \\
\Ind_H^G(\Omega B) \\
\xrightarrow{\Ind_H^G(\pi^H)} \\
\Ind_H^G(\Omega B)
\end{array}
\]
By Proposition 6.7 and (6.12), the extension (6.13) is isomorphic to
\[
\begin{array}{c}
\Ind_H^G(\Omega B) \\
\xrightarrow{\Ind_H^G(\pi^H)} \\
\Ind_H^G(\Omega B) \\
\xrightarrow{\Ind_H^G(\pi^H)} \\
\Ind_H^G(\Omega B)
\end{array}
\]
Then $\Ind_H^G : \mathfrak{R}^H \to \mathfrak{R}^G$ is a triangle functor. \hfill \Box

Theorem 6.14. Let $G$ be a group and $H$ a subgroup of $G$. Then the functors
\[
\Ind_H^G : \mathfrak{R}^H \to \mathfrak{R}^G \quad \Res_G^H : \mathfrak{R}^G \to \mathfrak{R}^H
\]
are adjoint. Hence
\[
\kk^G(\Ind_H^G(B), A) \simeq \kk^H(B, \Res_G^H(A)) \quad \forall B \in H\text{-Alg} \quad A \in G\text{-Alg}
\]
Proof. Let $A \in G$-Alg and $B \in H$-Alg. We need natural transformations
\[
\alpha_A \in \kk^G(\Ind_G^H(\Res_G^H(A), A) \quad \beta_B \in \kk^H(B, \Res_G^H(\Ind_G^H(B))
\]
which verify the unit and counit condition, respectively.
Define $\varphi_A : \Ind_G^H(\Res_G^H(A)) \to M_G/H \otimes A$ such that
\[
\varphi_A(\xi_H(s, b)) = e_{sH,sH} \otimes s \cdot b.
\]
One checks that $\varphi_A$ is a $G$-equivariant algebra morphism. Put
\[
\psi_B : B \to \Res_G^H(\Ind_G^H(B)) \quad \psi_B(b) = \xi_H(e, b)
\]
It is easy to check that $\psi_B$ is well-defined and is a map of $H$-algebras. Let $\alpha_A \in \kk^G(\Ind_G^H(\Res_G^H(A), A)$ the element represented by $\varphi_A$ and $\beta_B \in \kk^H(B, \Res_G^H(\Ind_G^H(B)$ the element represented by $\psi_B$. 

The composite $\text{Res}^H_G(\alpha_A) \circ \beta_{\text{Res}^G_H A}$ is represented by $\text{Res}^H_G(\varphi_A) \circ \psi_{\text{Res}^G_H A}$ which is $\text{kk}^H$-equivalent to the identity in the sense of remark 2.6.2. The element $\alpha_{\text{Ind}^G_H B} \circ \text{Ind}^G_H(\beta_B) \in \text{kk}^G(\text{Ind}^G_H B, \text{Ind}^G_H B)$ is represented by

$$
\gamma : \text{Ind}^G_H B \to M_{G/H} \otimes \text{Ind}^G_H B \quad \xi_H(g, b) \mapsto e_{gH,gH} \otimes \xi_H(g, b)
$$

The following morphism of $H$-algebras

$$
\theta : C \to M_{G/H} \otimes C \quad \theta(c) = e_{H,H} \otimes c
$$

represents to the identity in the sense of remark 2.6.2. Then $\text{Ind}^G_H(\theta) \simeq \text{kk}^G(\ell, A) \simeq \text{KH}(A \rtimes H)$

Corollary 6.17. Let $G$ be a group, $H$ a finite subgroup of $G$ and $A$ a $G$-algebra then

$$
\text{kk}^G(\ell^{(G/H)}, A) \simeq \text{kk}(\ell, A \rtimes H) \simeq \text{KH}(A \rtimes H)
$$

Proof. The isomorphism is the composition of $\psi_{GJ}$ and $\psi_{IR}$ defined in (5.2.5) and in (6.16).

7. Baaj-Skandalis Duality

In this section we define crossed product functors between the categories $G$-$\text{Alg}$ and $G_{gr}$-$\text{Alg}$. We prove that they extend to equivalences between $\mathfrak{R}^G$ and $\mathfrak{R}^G_{gr}$. In this way we obtain an algebraic duality theorem similar to the duality given by Baaj-Skandalis in [1].

Let $A$ be a $G$-algebra. Then

$$
A \rtimes G = \bigoplus_{s \in G} A \rtimes s \quad \text{and} \quad (A \rtimes s)(A \rtimes t) \subset A \rtimes st
$$

thus $A \rtimes G$ is a $G$-graded algebra. If $f : A \to B$ is a homomorphism of $G$-algebras then $f \rtimes G : A \rtimes G \to B \rtimes G$ is a graded homomorphism. Hence we have a functor

$$
\rtimes G : G$-$\text{Alg} \to G_{gr}$-$\text{Alg}
$$

We can also define a functor

$$
G \hat{\rtimes} : G_{gr}$-$\text{Alg} \to G$-$\text{Alg}
$$

as follows. Let $B$ be a $G$-graded algebra. Let $G \hat{\rtimes} B$ be the algebra which as a module is $\ell(G) \otimes B$ and the product is the following

$$
(\chi_g \rtimes a)(\chi_h \rtimes b) := \chi_g \rtimes a_{g^{-1}h}b.
$$

Here $b_g$ is the homogeneous element associated to $g$ in the decomposition

$$
b = \sum_{g \in G} b_g.
$$
One checks that the product (7.1) is associative and the action of $G$, $s \cdot \chi_g \times a = \chi_{sg} \times a$, makes it into a $G$-algebra. The crossed product $G \hat{\times} B \times G$ is a $G$-graded algebra which contains $B$ as a graded subalgebra by

$$b \mapsto \sum_{g \in G} \chi_{1G} \times b \times g.$$  

In (3.2.1) we defined $M_G B$ a $G$-graded matrix algebra associated to $B$. Through the inclusion $b \mapsto e_{1G,1G} \otimes b$ we can see $B$ as a $G$-graded subalgebra of it. In Proposition 7.4 below, we prove that $G \hat{\times} B \times G$ is isomorphic to $M_G B$ as a $G$-graded algebras.

If $f : A \to B$ is a homogeneous homomorphism we define a $G$-algebra homomorphism $(G \hat{\times} f)$ in the obvious way. Thus we have a functor

$$\hat{\times} G : G_{gr}-\text{Alg} \to G-\text{Alg}.$$

**Proposition 7.4.** Let $A$ be a $G$-algebra and let $B$ be a $G$-graded algebra.

a) There are natural isomorphisms of $G$-algebras

$$\hat{\times} G(A \times G) \simeq M_G \otimes A$$

b) There are natural isomorphisms of $G$-graded algebras

$$(G \hat{\times} B) \times G \simeq M_G \otimes B$$

**Proof.**

a) Define $T : G \hat{\times} (A \times G) \to M_G \otimes A$ as

$$T(\chi_g \times a \times s) = g \cdot a \otimes e_{g,gs}.$$

It is easy to check that $T$ is an equivariant algebra isomorphism with inverse given by

$$S(a \otimes e_{r,t}) := \chi_r \times r^{-1} \cdot a \times r^{-1} t.$$

b) Define $T : (G \hat{\times} B) \times G \to M_G \otimes B$ as

$$T(\chi_h \times b \times s) = \sum_{r \in G} e_{h,s^{-1}hr} \otimes b_r.$$

It is easy to check that $T$ is a graded algebra isomorphism with inverse given by

$$S(e_{r,s} \otimes b_q) = \chi_r \times b_q \times rqs^{-1}.$$

$\square$

**Theorem 7.6.** The functors $\times G$ and $G \hat{\times}$ extend to inverse equivalences

$$- \times G : \hat{\text{R}G} \to \hat{\text{R}G} \quad G \hat{\times} : \hat{\text{R}G} \to \hat{\text{R}G}$$

Hence if $A$ and $B$ are $G$-algebras and $C$ and $D$ are $G$-graded algebras then

$$\text{kk}^G(A,B) \simeq \hat{\text{kk}}^G(A \times G, B \times G) \quad \hat{\text{kk}}^G(C,D) \simeq \text{kk}^G(G \hat{\times} C, G \hat{\times} D)$$

**Proof.** As $\times G$ maps split sequences to split sequences, $j^G(- \times G)$ is excisive. By Proposition 5.1.2 $j^G(- \times G)$ is $G$-stable and homotopy invariant, whence it extends to $- \times G : \hat{\text{R}G} \to \hat{\text{R}G}$ by universality. Similarly, as $G \hat{\times}$ maps split exact sequences to split exact sequences then $j^G(G \hat{\times} -)$ is excisive. Because $G \hat{\times}$ maps graded homotopies to equivariant homotopies and $j^G(G \hat{\times} -)$ is $M_{\infty}$-stable, $j^G(G \hat{\times} -)$ extends to $G \hat{\times} : \hat{\text{R}G} \to \hat{\text{R}G}$ by universality. To finish we must show that the maps

$$\text{kk}^G(A,B) \to \hat{\text{kk}}^G(A \times G, B \times G) \quad \text{and} \quad \hat{\text{kk}}^G(C,D) \to \text{kk}^G(G \hat{\times} C, G \hat{\times} D)$$

are isomorphisms. This is true by Proposition 7.4. $\square$
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