ABSTRACT. Grundy functions have found many applications in a wide variety of games, in solving relevant problems in Game Theory. Many authors have been working on this topic for over many years, for example: C. Berge, P. Erdős, M. P. Schützenberger, R. P. Sprague. Since the existence of a Grundy function on a digraph implies that it must have a kernel, the problem of deciding if a digraph has a Grundy function is NP-complete, and how to calculate one is not clearly answered.

In this paper, we introduce the concept: Semi-Grundy function, which arises naturally from the connection between kernel and semi-kernel and the connection between kernel and Grundy function. We explore the relationship of this concept with the Grundy function, proving that for digraphs with a defining hereditary property is sufficient to get a semi-grundy function to obtain a Grundy function. Then we prove sufficient and necessary conditions for some products of digraphs to have a semi-Grundy function. Also, it is shown a relationship between the size of the semi-Grundy function obtained for the Cartesian Product and the size of the semi-Grundy functions of the factors. This size is an upper bound of the chromatic number.

We present a family of digraphs with the following property: for each natural number \( n \geq 2 \), there is a digraph \( R_n \) that has two Grundy functions such that the difference between their maximum values is equal to \( n \). Then it is important to have bounds for the Grundy or semi-Grundy functions.

1. Introduction.

The concept of kernel was introduced by Von Neumann and Morgenstern [15] in the context of Game Theory. The problem of the existence of a kernel in a given digraph has been studied by several authors, see for example ([3, 4, 5, 16]). An important concept to study kernels of a digraph, is the concept of semi-kernel, introduced in [9]. The following result is an important example of this relation:

**Theorem 1.1.** [14] If every induced digraph of \( D \) has a semi-kernel then \( D \) has a kernel.

Another concept closely related to kernels of a digraph is Grundy functions, introduced by P. M. Grundy in [12]. This concept have found many applications in Game Theory see for example ([1, 7, 8, 13]) and in graph theory: [6]. The relation of these concepts is shown in the following results:

**Theorem 1.2.** [1] If \( D \) has a Grundy function \( g \), then the set \( N = \{ x \in V(D) | g(x) = 0 \} \) is a kernel of \( D \).

**Theorem 1.3.** [1] If \( D \) is a kernel-perfect digraph, then \( D \) possesses a Grundy function.

The problem of deciding if a digraph has kernel is NP-complete. The research of sufficient conditions for a digraph to have kernel has been addressed by many authors along many years, for a comprehensive survey see [2]. For digraphs with a defining hereditary property is sufficient to get:
Semigrundy function

- a semi-kernel to obtain a kernel,
- a kernel to obtain a Grundy function,

We introduce a new concept, namely semi-Grundy function. This is a non-negative integer function defined on the set of vertices of a digraph. This concept generalizes that of Grundy function and allow us to obtain Grundy functions in an easier way, in those digraphs defined by an hereditary property.

In [11] we prove sufficient and necessary conditions for the Cartesian Product $\sigma(D, \alpha)$ to have a Grundy function in terms of the existence of Grundy function or kernel in $D$ and in each $\alpha_i$. Also, it is shown a relationship between the size of the Grundy function obtained for $\sigma(D, \alpha)$ and the size of the Grundy functions of the factors $\alpha_i$. The most significative results on Grundy Functions are proved for semi-Grundy function, so we obtain a wide generalization of this results.

2. Preliminaries.

For general concepts we refer the reader to [1]. Let $D$ a digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of $D$ respectively. Let $X_1, X_2$ be a subset of $V(D)$: the arc $(u_1, u_2)$ will be called an $X_1X_2$-arc whenever $u_1 \in X_1$ and $u_2 \in X_2$. And $D[X_1]$ will denote the subdigraph of $D$ induced by $X_1$.

A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A semi-kernel $S$ of $D$ is an independent set such that if there exist a $Sz$-arc then must exist a $zS$-arc in $D$. A kernel $N$ of $D$ is an independent set of vertices such that for each $z \in V(D) - N$ there exists a $zN$-arc. A digraph $D$ is a kernel-perfect digraph whenever each one of its induced subdigraphs has a kernel.

A non-negative function $g$ is called a Grundy function on $D$ if for every vertex, $x$, $g(x)$ is the smallest non-negative integer which does not belong to the set $\{g(y) | y \in \Gamma^+(x)\}$.

The concepts of semi-kernel, kernel and Grundy functions of a digraph are nearly related as we can see in Theorem [1.1], Theorem [1.2] and Theorem [1.3].

In [1] C. Berge defined the cartesian sum of $n$ digraphs $D_1, D_2, \ldots, D_n$, denoted by $D_1 + D_2 + \ldots + D_n$, as follows:

\begin{enumerate}
  \item $V(D_1 + D_2 + \ldots + D_n) = \bigoplus_{i=1}^{n} V(D_i)$,
  \item $\Gamma(x_1, x_2, \ldots, x_n) = \bigcup_{i=1}^{n} \left( \{x_1\} \times \ldots \times \{x_{i-1}\} \times \Gamma(x_i) \times \ldots \times \{x_n\} \right)$
\end{enumerate}

This operation comes naturally from the theory of Games. In [1], C. Berge proved:

**Theorem 2.1.** The cartesian sum $D_1 + D_2 + \ldots + D_n$ of digraphs having Grundy function, also has a Grundy function.

Let $D$ a digraph, $\alpha = (\alpha_v)_{v \in V(D)}$ a family where the $\alpha_v$ are mutually disjoint digraphs. The Cartesian product of $\alpha$ over $D$, denoted by $\sigma(D, \alpha)$ is defined as follows:

\begin{enumerate}
  \item $V(\sigma(D, \alpha)) = \bigcup_{v \in V(D)} V(\alpha_v)$,
  \item $A(\sigma(D, \alpha)) = (\bigcup_{v \in V(D)} A(\alpha_v)) \cup \{ (x,y) | x \in V(\alpha_u), y \in V(\alpha_v) \text{ and } (u,v) \in A(D) \}$
\end{enumerate}

3. Semi-Grundy functions.

**Definition 3.1.** Let $D$ be a digraph. A function $s : V(D) \to \mathbb{N}$ is a semi-Grundy function if satisfies:
Semigrundy function

(1) \( s(x) = k \) implies that each \( y \in \Gamma^+(x) \) satisfies \( s(y) \neq k \);

(2) \( s(x) = k \) and \( y \in \Gamma^+(x) \) with \( s(y) > k \) implies that there exists \( z \in \Gamma^+(y) \) such that \( s(z) = k \).

**Remark 3.1.** Every Grundy function is in particular a semi-Grundy function. This tells us that the following families of digraphs has semi-Grundy functions: acyclic, transitive, kernel-perfect and digraphs without odd-cycles.

**Lemma 3.2.** If \( D \) has a semi-Grundy function then \( D \) has a semi-kernel.

**Proof.** Let \( m_0 := \min\{s(x) | x \in V(D)\} \) and we define \( S := s^{-1}(m_0) \). We will prove that \( S \) is a semi-kernel of \( D \).

Let \( x, y \in S \), this implies that \( s(x) = m_0 = s(y) \). From (1) of the definition of semi-Grundy function this implies that \( x \notin \Gamma^+(y) \) and \( y \notin \Gamma^+(x) \). Thus \( S \) is an independent set.

Now suppose that \( x \in S \) and \( y \in \Gamma^+(x) \). Thus \( s(x_0) = m_0 \) and from the definition of \( m_0 \) it follows that \( s(y) > m_0 \). From (2) of the definition of semi-Grundy function, there exists \( z \in \Gamma^+(y) \) such that \( s(z) = m_0 \). Then \( S \) is a semi-kernel of \( D \). \( \Box \)

**Remark 3.2.** This concept differs from the concept of pseudo-Grundy function presented in [1]: a pseudo-Grundy function determines a kernel in a digraph.

Now we point out some interesting facts between the concepts of semi-kernel, kernel, Grundy function and semi-Grundy function.

- A digraph can have more than one semi-Grundy function (see Figure 1).
- A digraph can have a Grundy function \( f \) and semi-Grundy function \( s \), where the maximum value of \( f \) is greater than the maximum value of \( s \) (see Figure 2).
- A digraph can have a Grundy function \( f \) and semi-Grundy function \( s \), where the maximum value of \( f \) is less than the maximum value of \( s \) (see Figure 3).
- If a digraph have semi-kernel, this not implies that it has semi-Grundy function (see Figure 4).
- There are digraphs with semi-Grundy function and no Grundy function (see Figure 5).
- If a digraph has a semi-Grundy function this not implies that it has a kernel (see Figure 6).
- Having semi-Grundy function is not an hereditary property on induced subdigraphs (see Figure 7).
Figure 1 – A digraph with two different Grundy functions: one of them is given by the numbers $\bar{x}$, the other by the subscripts.

Figure 2 – A digraph with a Grundy functions and a semi-Grundy function
A natural generalization of Theorem 3 is the following result:

**Proposition 3.3.** If every induced subdigraph of $D$ has a semi-kernel then $D$ has a semi-Grundy function.

**Proof.** Let $D$ be such a digraph and let $s_0$ a semi-kernel of $D$. Let $S_1$ a semi-kernel of $D_1 := D[V(D) \setminus S_0]$. Let $S_2$ be a semi-kernel of $D_2 := D[V(D) \setminus (S_0 \cup S_1)]$, etc. The sets $S_i$ form a partition of $V(D)$. We define an integer function $s(x)$ on $V(D)$ by:

$$s(x) = k \text{ if and only if } x \in S_k$$

Now we show that $s$ is a semi-Grundy function:
Let $x \in V(D)$ with $s(x) = k$ and $y \in \Gamma^+(x)$. We must show that $s(y) \neq k$. From the definition of $s$ we know that $x \in S_k$ and that $S_k$ is a semi-kernel of $D_k$. Since $(x, y) \in A(D)$ then $y \notin S_k$ and $s(y) \neq k$.

(2) Suppose that $s(x) = k, y \in \Gamma^+(x)$ and $s(y) > k$. We must show that there exists $z \in \Gamma^+(y)$ such that $s(z) = k$.

Since $s(x) = k$ we know that $x \in S_k$ with $S_k$ a semi-kernel of $D_k$. By hypothesis we have that $s(y) > k$, this tells us that $y \in V(D_k)$. Since $(x, y) \in A(D_k) \text{ and } S_k$ is a semi-kernel of $D_k$ it follows that exists $z \in \Gamma^+(y)$ with $z \in S_k$. Then, by the definition of $s, s(z) = k$.

Then $s$ is a semi-Grundy function on $D$. □

As a generalization of Theorem 2.1, we can prove that the cartesian sum of digraphs having semi-Grundy function, also has semi-Grundy function. There is an important difference in the proof of this theorem; in [1] for the proof of Theorem 2.1 C. Berge use the digital sum. In our proof, we do not use the digital sum, just the ordinary sum of the integers.

**Proposition 3.4.** Let $D_i$ be digraphs, for $i = 1, \ldots, n$. If $D_i$ has a semi-Grundy function $s_i$ for $i = 1, \ldots, n$ then the function

$$S(x_1, \ldots, x_n) := \sum_{i=1}^{n} s_i(x_i)$$

is a semi-Grundy function of $D := D_1 + D_2 + \ldots + D_n$. The maximum value of $S$ is $\sum_{i=1}^{n} m_i$, where $m_i$ is the maximum value of $s_i$.

**Proof.** Let $D_i$ and $s_i$ be as in the hypothesis. We will prove that $S$ is a semi-Grundy function of $D := D_1 + D_2 + \ldots + D_n$.

(1) Let $x = (x_1, \ldots, x_j, \ldots, x_n) \in D$ such that $S(x) = s_1(x_1) + \ldots + s_n(x_n) = k$ and $y \in \Gamma^+(x)$. We must show that $S(y) \neq k$.

Since $y \in \Gamma^+(x)$, from the definition of $D$, we have that $y = (x_1, \ldots, x'_j, \ldots, x_n)$ for some $j \in \{1, \ldots, n\}$, where $(x_j, x'_j) \in A(D_j)$. Since $s_j$ is a semi-Grundy function of $D_j$, $s_j(x_j) \neq s_j(x'_j)$ and we have:

$$S(y) = S(x_1, \ldots, x'_j, \ldots, x_n) = s_1(x_1) + \ldots + s_j(x'_j) + \ldots + s_n(x_n) \neq k.$$
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(2) Suppose that $S(x) = k$, $y \in \Gamma^+(x)$ and $S(y) > k$. We must show that there exists $z \in \Gamma^+(y)$ such that $S(z) = k$.

Let $x = (x_1, \ldots, x_j, \ldots, x_n) \in D$ and $y \in \Gamma^+(x)$, then from the definition of $D$ we know

that $y = (x_1, \ldots, x_j', \ldots, x_n)$. From the definition of $S$, we have:

\[
S(y) = S(x_1, \ldots, x_j', \ldots, x_n) = s_1(x_1) + \ldots + s_j(x_j') + \ldots + s_n(x_n) > k \tag{1 - hypothesis}
\]

This implies that $s_j(x_j') < s_j(x_j)$ and since $s_j$ is a semi-Grundy function of $D_i$, there exists $x_j'' \in \Gamma^+_D(x_j')$ such that $s_j(x_j'') = s_j(x_j)$. Then the vertex $z = (x_1, \ldots, x_j'', \ldots, x_n)$ is in $y \in \Gamma^+(x)$ and

\[
S(z) = S(x_1, \ldots, x_j', \ldots, x_n) = s_1(x_1) + \ldots + s_j(x_j'') + \ldots + s_n(x_n) = s_1(x_1) + \ldots + s_j(x_j) + \ldots + s_n(x_n) = k
\]

Then $S$ is a semi-Grundy function of $D$. \hfill \Box

Remark 3.3. Now consider that $D$ is a digraph and has a semi-Grundy function $S'$. From the definition of semi-Grundy function its minimum value could be any integer. It would be easy to work with, if we know which is its minimum value, as happen with the Grundy functions, and which is next value, and so on. With this purpose, we define a new semi-Grundy function, $S$, in the following way: Suppose that the image of $S'$ is $\{m_0, m_1, \ldots, m_r\}$ where $m_i < m_j$ if and only if $i < j$. Then:

$\begin{align*}
S : V(D) &\to \mathbb{N}, \\
S(x) &= k \text{ if and only if } S'(x) = m_k.
\end{align*}$

It is easy to see that this a semi-Grundy function of $D$ and induced the same partition in independent subsets of $V(D)$ that $S'$. This semi-Grundy function starts in 0 and take consecutive positive integer values. So, from now on, we will assume that every semi-Grundy function take consecutive non-negative integer values starting from 0.

4. semi-Grundy function on the cartesian product.

The cartesian product is an operation on digraphs that comes naturally from the Theory of Games. In \cite{11}, we studied the behavior of the Grundy function and the cartesian product, and in some cases we found some bounds of Grundy functions. In this section we generalizes those results to the concept of semi-Grundy function.

Theorem 4.1. Let $D$ be a digraph and $\alpha = (\alpha_v)_{v \in V(D)}$ a family of mutually disjoint digraphs. If $D$ is a kernel-perfect digraph and each $\alpha_v$ has a semi-Grundy function, then $\sigma(D, \alpha)$ possesses a semi-Grundy function.

Proof. Let $D$ and $\alpha = (\alpha_v)_{v \in V(D)}$ as in the hypothesis. We consider for each $v \in V(D)$ any fixed semi-Grundy function $f_v$ of $\alpha_v$ and $S_0$ a kernel of $D$. Now we define the following sets:

$\begin{align*}
N_0 &:= \{x \in V(\sigma(D, \alpha)) | f_v(x) = 0 \text{ for some } y \in S_0\} , \\
M_0 &:= \{y \in V(D) | V(\alpha_y) \subseteq N_0\}.
\end{align*}$

Let $S_1$ a kernel of $D_1 := D[V(D) \setminus M_0]$ (the subdigraph of $D$ induced by $V(D) \setminus M_0$). For $y \in S_1$ we denote:

$\begin{align*}
m(1, y) &:= \min \{f_v(x) | x \in V(\alpha_y) \setminus N_0\} , \\
N_1 &:= \{x \in (V(\sigma(D, \alpha)) \setminus N_0) | f_y(x) = m(1, y) \text{ for some } y \in S_1\} , \\
M_1 &:= \{y \in V(D) | V(\alpha_y) \subseteq (N_0 \cup N_1)\}.
\end{align*}$

(see figure \cite{7})
Now we define the function \( \sigma \) semigrundy function of each vertex and the arcs between the ovals means that there must be an arc from any vertex of the source to any vertex in the target: \( S_0 = \{a,d\}, N_0 = \{a_1, d_1\}, M_0 = \emptyset; S_1 = \{a, d\}, m(1, a) = 1, m(1, d) = 1, N_1 = \{a_2, d_2\}, M_1 = \{a\}; S_2 = \{d\}, m(2, d) = 2, N_2 = \{d_2\} \ldots \\

Continuing this way we define a sequence of subsets of vertices of \( N \) and \( M \) as follows: if \( D_i, N_i \) and \( M_i \) are defined and \( S_i \) is a kernel of \( D_i \) then we defined \( D_{i+1}, S_{i+1}, N_{i+1} \) and \( M_{i+1} \) as follows:

\[
D_{i+1} := D[V(D) \setminus (M_0 \cup \ldots \cup M_i)]; S_{i+1} \text{ is a kernel of } D_{i+1}; \\
m(i + 1, y) := \min \{f_y(x) \mid x \in V(\sigma_y \setminus (N_0 \cup \ldots \cup N_i))\} \text{ for } y \in S_{i+1}; \\
N_{i+1} := \{x \in V(\sigma(D, \alpha)) \setminus (N_0 \cup \ldots \cup N_i) \mid f_y(x) = m(i + 1, y) \text{ for } y \in S_{i+1}\}; M_{i+1} := \{y \in V(D) \setminus V(\sigma) \subseteq (N_0 \cup \ldots \cup N_{i+1})\}.
\]

Clearly \( N_i \cap N_j = \emptyset \) for any \( i, j \) with \( i \neq j \). This procedure finishes when we get the first natural number \( r \) such that \( V(D_r) = \emptyset \). Notice that this natural number \( r \) exists as \( N_i \cap N_j = \emptyset \) whenever \( i \neq j \) and \( N_i \neq \emptyset \) for each \( 0 \leq i \leq r - 1 \). 

Now we define the function \( S : V(\sigma(D, \alpha)) \rightarrow \mathbb{N} \) as follows:

\[
S(x) = k \text{ if and only if } x \in N_k.
\]

\( S \) is well defined as \( N_i \cap N_j = \emptyset \) for any \( i, j \) with \( i \neq j \) and \( V(D_r) = \emptyset \). We will prove that \( S \) is a semi-Grundy function on \( \sigma(D, \alpha) \).

1. Let \( x \in V(\sigma(D, \alpha)) \) such that \( S(x) = k \) and \( y \in \Gamma^+(x) \). We must show that \( S(y) \neq k \).

By definition of \( \sigma(D, \alpha) \) there exists vertices \( u, v \in V(D) \) such that \( x \in V(\alpha_u) \) and \( y \in V(\alpha_v) \). Suppose that \( S(x) = S(y) = k \). If \( u = v \) this implies that \( f_u(x) = f_u(y) \) and then \( y \notin \Gamma^+(x) \) (because \( f_u^{-1}(k) \) is an independent set). This contradicts the assumption \( y \in \Gamma^+(x) \) then, in this case, \( S(y) \neq k \).

Now suppose that \( S(x) = S(y) = k \) and \( u \neq v \). From the definition of the sets \( S_k \), we know \( S_k \) is a semi-kernel of \( D_k \) and that \( u, v \in S_k \). Since \( S_k \) is an independent subset of \( V(D) \) it follows that there is no arc between \( u \) and \( v \) in \( D \), and then by definition of \( \sigma(D, \alpha) \) there is no arc between any vertex from \( \alpha_u \) to any vertex of \( \alpha_v \) or viceversa. In particular,
there is no arc between $x$ and $y$. This contradicts the hypothesis that $y \in \Gamma^+(x)$. Thus in this case also it satisfies $S(y) \neq k$.

2) Suppose that $S(x) = k$ and $y \in \Gamma^+(x)$ with $S(y) > k$. We must show that there exists $z \in \Gamma^+(y)$ such that $S(z) = k$.

By definition of $\sigma(D, \alpha)$ there exists $u, v \in V(D)$ such that $x \in V(\alpha_u)$ and $y \in V(\alpha_v)$. First we prove the case when $u = v$. In this case, $S(x) = k < S(y)$ implies that $f_u(x) < f_u(y)$ (because $f_u$ is a semi-Grundy function) and then, must exists $z \in \Gamma^+_{\alpha_u}(y)$ such that $f_u(z) = f_u(x)$. From the definition of $\sigma(D, \alpha)$ and $S$, we have $S(z) = k = S(x)$ and $z \in \Gamma^+(y)$.

When $u \neq v$, by definition of $\sigma(D, \alpha)$ and the fact there is $xy$ arc in $\sigma(D, \alpha)$, there is a $uv$ arc in $D$. Since $S(x) = k < S(y)$, we know that $u$ is an element of $S_k$ that is a kernel of $D_k$ and $v \in V(D_k)$. Then must exists $w \in S_k$ such that $(v, w) \in A(D_k)$. From this and the definition of $S$, must exists $z \in V(\alpha_w)$ such that $S(z) = k$ and $(y, z) \in A(\sigma(D, \alpha))$.

Then $S$ is a semi-Grundy function of $\sigma(D, \alpha)$.

\[\square\]

**Corollary 4.2.** Let $D$ be a digraph and $\alpha = (\alpha_v)_{v \in V(D)}$ a family of mutually disjoint digraphs. If $D$ is a kernel-perfect digraph and each $f_v$ has a semi-Grundy function, then $\sigma(D, \alpha)$ possesses a semi-Grundy function $S$ such that $\max\{S(x) | x \in V(\sigma(D, \alpha))\} \leq \sum_{v \in V(D)} m_u + |V(D)| - 1$, where $m_u = \max\{f_u(x) | x \in V(\alpha_u)\}$.

**Proof.** Since $D$ is a kernel-perfect digraph then $D$ has a semi-Grundy function $f$ (indeed is a Grundy function). Let $S$ be the semi-Grundy function of $\sigma(D, \alpha)$ defined in the proof of Theorem 4.1. Since $S^{-1}(i) = \cup f^{-1}(j)$ for some $j \in \mathbb{N}$ and $v \in V(D)$ we have that the bound reaches whenever $S^{-1}(i) = f^{-1}(j)$ for some $j \in \mathbb{N}$ and $v \in V(D)$.

**Theorem 4.3.** Let $D$ be a digraph and $\alpha = (\alpha_v)_{v \in V(D)}$ a family of mutually disjoint digraphs. If $\sigma(D, \alpha)$ has a semi-Grundy function then $D$ has a semi-kernel and $\alpha_v$ has a semi-Grundy function for each $v \in V(D)$.

**Proof.** First, we prove that $D$ has a semi-kernel. Let $S$ be a semi-Grundy function of $\sigma(D, \alpha)$; $N := \{x \in V(\sigma(D, \alpha)) | S(x) = 0\}$ and $A := \{u \in V(D) | V(\alpha_u) \cap N \neq \emptyset\}$.

$A$ is a semi-kernel of $D$.

Let $u, v \in A$, with $u \neq v$. Then there exists $x \in V(\alpha_u)$ and $w \in V(\alpha_v)$ such that $S(x) = 0 = S(w)$. Since $S$ is a semi-Grundy function in $\sigma(D, \alpha)$ it follows that there is no arc in $\sigma(D, \alpha)$ between $x$ and $w$. Hence from the definition of $\sigma(D, \alpha)$, there is no arc between $u$ and $v$ in $D$.

Then $A$ is an independent set of $D$.

Let $u \in A$ and $w \in (V(D) \setminus A)$ such that $(u, w) \in A(D)$. Let $x \in V(\alpha_u)$ such that $S(x) = 0$ and $y \in V(\alpha_w)$. From the definition of $\sigma(D, \alpha)$ and since $S$ is a semi-Grundy function it follows that $S(y) > 0$. Then, there exists $z \in S^+(y)$ such that $S(z) = 0$. From the definition of $\sigma(D, \alpha)$, there exists $v \in V(D)$ such that $z \in V(\alpha_v)$; then $v \in A$, and $(w, v) \in A(D)$. Thus $A$ is a semi-kernel of $D$.

Now let $u \in V(D)$ be. We will prove that $\alpha_u$ has a semi-Grundy function. Since $S$ is a Grundy function of $\sigma(D, \alpha)$, there exists a subset of integers $\{i_0, i_1, \ldots, i_r\}$ such that $S^{-1}(i_j) \cap V(\alpha_u) \neq \emptyset$, with $i_j < i_{j+1}$, for every $j = 0, 1, \ldots, r - 1$ and such that $V(\alpha_u) \subseteq \bigcup_{j \in \{i_0, i_1, \ldots, i_r\}} S^{-1}(j)$. Consider the function $s_u : V(\alpha_u) \rightarrow \mathbb{N}$ given by $s_u(x) = j$ if and only if $x \in S^{-1}(j)$. By definition of $s_u$ and since $S$ is a semi-Grundy function, follows that $s_u$ is well defined. Now we need to prove that is a semi-Grundy function of $\alpha_u$.

1) Let $x \in V(\alpha_u)$ be such that $s_u(x) = j$ and $y \in \Gamma^+_{\alpha_u}(x)$. Since $(x, y) \in A(\sigma(D, \alpha))$ then $S(x) \neq S(y)$, i.e, $s_u(x) = S(x) \neq S(y) = s_u(y)$.
2) Suppose that $s_u(x) = k$ and $y \in \Gamma^+_\alpha(x)$ with $s_u(y) > k$. We must show that there exists $z \in \Gamma^+_\alpha(y)$ such that $s_u(z) = k$. Since $S$ is a semi-Grundy function of $\sigma(D, \alpha)$, there exists $z \in \Gamma^+_\alpha(y)$ such that $S(z) = k$. Now, $z \in V(\alpha_u)$, otherwise $(y, z) \in A(\sigma(D, \alpha))$ would implied that $(x, z) \in A(\sigma(D, \alpha))$ and $S(x) = S(z)$, and so, $S$ would not be a semi-Grundy function of $\sigma(D, \alpha)$. Thus, $z \in V(\alpha_u)$ and $s_u(z) = S(z) = k$.

Then, $s_u$ is a semi-Grundy function of $\alpha_u$. \(\square\)

**Theorem 4.4.** Let $D$ be a digraph, $\alpha = (\alpha_u)_{u \in V(D)}$ a family of mutually disjoint digraphs. If $D$ has a semi-Grundy function $f$ and each $\alpha_u$ has a semi-Grundy function $s_u$ ($u \in V(D)$) such that $\max\{s_u(x) : x \in V(\alpha_u)\} = \max\{s_v(x) : x \in V(\alpha_v)\}$ whenever $\{u, v\} \subseteq f^{-1}(i)$ for some $i \in \mathbb{N}$. Then $\sigma(D, \alpha)$ possesses a semi-Grundy function $S$ which satisfies: $\max\{S(x) : x \in V(\sigma(D, \alpha))\} = n + \sum_{i=0}^{\alpha} m_i$, where $n = \max\{f(x) : x \in V(D)\}$ and $m_i = \max\{s_u(x) : x \in V(\alpha_u), \text{ and } v \not\in f^{-1}(i)\}$, for each $0 \leq i \leq n$.

**Proof.** Define $S : V(\sigma(D, \alpha)) \rightarrow \mathbb{N}$ as follows:

Let $x \in V(\sigma(D, \alpha))$ be; we have that there exist a unique $u \in V(D)$ such that $x \in V(\alpha_u)$ and a unique $i \in \{0, 1, \ldots, n\}$ such that $u \in f^{-1}(i)$. Then we define:

$$S(x) = \sum_{j=0}^{i-1} m_j + f(u) + s_u(x);$$

whenever $1 \leq i \leq n$ and $S(x) = s_u(x)$ whenever $i = 0$.

We will prove that $S$ is semi-Grundy function of $\sigma(D, \alpha)$.

1) Let $x \in V(\sigma(D, \alpha))$ such that $S(x) = k$ and $y \in \Gamma^+(x)$. We must show that $S(y) \neq k$.

By definition of $\sigma(D, \alpha)$ exist vertices $u, v \in V(D)$ such that $x \in V(\alpha_u)$ and $y \in V(\alpha_v)$. We will consider two cases: $u = v$ and $u \neq v$. If $u = v$ then we have that $x, y \in V(\alpha_u)$. Since $y \in \Gamma^+(x)$ and the definition of $\sigma(D, \alpha)$ follows that $y \in \Gamma^+_\alpha(x)$. Then, $s_u(x) \neq s_u(y)$ because $s_u$ is a semi-Grundy function of $\alpha_u$. Thus:

$$S(x) = \sum_{j=0}^{i-1} m_j + f(u) + s_u(x) \neq \sum_{j=0}^{i-1} m_j + f(u) + s_u(y) = S(y);$$

(assuming that $f(u) = i$).

Now suppose that $u \neq v$. In this case, from $y \in \Gamma^+(x)$ and the definition of $\sigma(D, \alpha)$ we conclude that $(u, v) \in A(D)$, and since $f$ is a semi-Grundy function $f(u) \neq f(v)$. If $f(u) = i < k = f(v)$ we have:

$$S(x) = \sum_{j=0}^{i-1} m_j + f(u) + s_u(x) \leq \sum_{j=0}^{i-1} m_j + m_i + f(u)$$

$$\leq \sum_{j=0}^{i-1} m_j + m_i + f(v) \leq \sum_{j=0}^{k-1} m_j + f(v)$$

$$\leq \sum_{j=0}^{k-1} m_j + f(v) + s_u(y) = S(y)$$

Then $S(x) < S(y)$. If $f(u) > f(v)$ the argument is similar. Then in either case, $S(x) \neq S(y)$.

2) Suppose that $S(x) = k$ and $y \in \Gamma^+(x)$ with $S(y) > k$. We must show that there exist $z \in \Gamma^+(y)$ with $S(z) = k$.

Let $u, v \in V(D)$ such that $x \in V(\alpha_u)$ and $y \in V(\alpha_v)$. If $u = v$ then from the definition of $S$, follows that $s_u(x) < s_u(y)$ and since $s_u$ is a semi-Grundy function of $\alpha_u$, must
Semigrundy function

exist $z \in \Gamma^+_\alpha(y)$ (and then $z \in \Gamma^+(y)$) such that $s_u(z) = s_u(x)$. Then we have that:

$S(x) = \sum_{j=0}^{n-1} m_j + f(u) + s_u(x) = \sum_{j=0}^{n-1} m_j + f(u) + s_u(z) = S(z)$, thus $S(z) = k$.

If $u \neq v$ then $f(u) \neq f(v)$. From the definition of $S$ and the fact that $S(x) < S(y)$ follows $f(u) < f(v)$. Since $f$ is a semi-Grundy function there exists $w \in V(D)$ with $w \in \Gamma^+_\beta(v)$ and $f(w) = f(u)$. By the hypothesis, we know that $\max\{s_u(x)|x \in V(\alpha_w)\} = \max\{s_w(x)|x \in V(\alpha_w)\}$, and then there exist $z \in V(\alpha_w)$ with $s_w(z) = s_u(x)$. Then we have:

$S(x) = \sum_{j=0}^{n-1} m_j + f(u) + s_u(x) = \sum_{j=0}^{n-1} m_j + f(u) + s_u(z) = S(z)$, thus $S(z) = k$.

Thus $S$ is a semi-Grundy function of $\sigma(D, \alpha)$. 

5. Digraphs $R_n$

Now we construct a digraph $R_n$, for all $n \geq 2$, with the property that has 2 different Grundy functions: $g_1^{(n)}$ and $g_2^{(n)}$. These functions satisfied: $\max\{g_1^{(n)}(x)|x \in V(R_n)\} = 1$ and $\max\{g_2^{(n)}(x)|x \in V(R_n)\} = n$. These digraphs are constructed by recursion.

Let $R_2$ be the digraph in Figure 1. We define two functions on $V(D)$:

$g_1^{(2)}(\bar{x_p}) = \begin{cases} 
0, & \text{if } x + p \equiv 0 \mod 2 \\
1, & \text{if } x + p \equiv 1 \mod 2 
\end{cases}$

and $g_2^{(2)}(\bar{x_p}) = p$. It is easy to see that this functions are Grundy function of $R_2$. Now we construct $R_3$, $R_4$, and so on, by recursion.

Suppose that the digraph $R_n$ is defined and has two Grundy functions $g_1^{(n)}$, $g_2^{(n)}$ defined by:

$g_1^{(n)}(\bar{x_p}) = \begin{cases} 
0, & \text{if } x + p \equiv 0 \mod 2 \\
1, & \text{if } x + p \equiv 1 \mod 2 
\end{cases}$

and $g_2^{(n)}(\bar{x_p}) = p$.

Now, we define the digraph $R_{n+1}$ as follows (see Figure 3):

- $V(R_{n+1}) = V(R_n) \cup \{\bar{0}_{n+1}, \bar{1}_{n+1}, \bar{2}_{n+1}, \bar{3}_{n+1}\}$;
- $A(R_{n+1}) = A(R_n) \cup \bigcup_{x \in \{0,1,2,3\}} A_x$;

where $A_x = \{(\bar{x_{n+1}} , \bar{x_n - 2i})|0 \leq n - 2i \leq n, i \in \mathbb{N}\} \cup \{(\bar{x_{n+1}} , (x + 1)_{n - (2i + 1)})|0 \leq n - (2i + 1) \leq n - 1, i \in \mathbb{N}\}$.

Note that the sum in $(x + 1)$ is the sum of $\mathbb{Z}_4$. 

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Now we define two functions on $g_1^{(n+1)}, g_2^{(n+1)} : V(R_{n+1}) \to \mathbb{N}$ as follows:

$$g_1^{(n+1)}(x_p) = \begin{cases} 
0, & \text{if } x + p \equiv 0 \mod 2 \\
1, & \text{if } x + p \equiv 1 \mod 2 
\end{cases}$$

and $g_2^{(n+1)}(x_p) = p$.

**Remark 5.1.** It's important to note that the Grundy function $g_1^{(n)}$ only takes two values: 0 and 1, and then, $R_n$ is a bipartite graph.

From the definition of both functions is easy to see that $g_1^{(n)}(\bar{x}_p) = g_1^{(n+1)}(\bar{x}_p)$ if $p < n+1$, $i = 1, 2$. So, we just have to prove that $g_1^{(n+1)}$ extends $g_1^{(n)}$.

Note that $\{0_{n+1}, \bar{1}_{n+1}, 2_{n+1}, 3_{n+1}\}$ is an independent set of $R_{n+1}$. In $R_{n+1}$, $\bar{x}_{n+1}$ only is adjacent to $\bar{x}_{n-2i}$ or $\bar{x}+\bar{1}_{n-(2i+1)}$. Since $x + (n + 1) + x + n - 2i = 2x + 2n - 2i + 1$, it follows that $x + (n + 1) \not\equiv x + n - 2i \mod 2$, so $g_1^{(n+1)}(x_{n+1}) \neq g_1^{(n+1)}(x_{n-2i})$.

Since $x + (n + 1) + x + n - (2i + 1) = 2x + 2n + 2 - 2i - 1$, it follows that $x + (n + 1) \not\equiv x + n - (2i + 1) \mod 2$ and then $g_1^{(n+1)}(x_{n+1}) \neq g_1^{(n+1)}(x + \bar{1}_{n-(2i+1)})$.

From this we have: $g_1^{(n+1)}$ is a Grundy function of $R_{n+1}$, and it is a bipartite digraph.

Now we will prove that $g_2^{(n+1)}$ is a Grundy function of $R_{n+1}$. Since there are no arcs in $R_{n+1}$ that ends in $\bar{x}_{n+1}$, with $x = 0, 1, 2, 3$, and the set $\{0_{n+1}, \bar{1}_{n+1}, 2_{n+1}, 3_{n+1}\}$ is an independent set of $R_{n+1}$, we just have to prove that for every $j \in \{0, 1, \ldots, n\}$ there exists $y \in \Gamma_{R_{n+1}}(x_{n+1})$ such that $g_2^{(n+1)}(y) = j$. But, by definition of $R_{n+1}$, there is an arc from $\bar{x}_{n+1}$ to $\bar{x}_n, \bar{x} + \bar{1}_{n-1}, \bar{x}_{n-2}, \bar{x} + \bar{1}_{n-3}, \ldots$.

So, $g_2^{(n+1)}$ is a Grundy function of $R_{n+1}$ and $\max\{g_2^{(n+1)}(u) | u \in V(R_{n+1})\} = n + 1$.

Then with this digraphs, we have prove the following theorem:

**Theorem 5.1.** For any natural number $n$, there is a digraph $D$ with two Grundy functions $f$ and $g$ such that $\max\{|f(x)|x \in V(D)\} - \max\{|g(x)|x \in V(D)\} = n$.

Note that Theorem 5.1 tells us that is important to have bounds for the Grundy function or semi-Grundy function.
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