Explicit formula for Schrödinger wave operators on the half-line for potentials up to optimal decay

H. Inoue

Graduate school of mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan
E-mail: m16007v@math.nagoya-u.ac.jp

Abstract

We give an explicit formula for the wave operators for Schrödinger operators on the half-line with a potential decaying strictly faster than the polynomial of degree minus two. The formula consists of the main term given by the scattering operator and a function of the generator of the dilation group, and a Hilbert-Schmidt remainder term. Our method is based on the elementary construction of the generalized Fourier transforms in terms of the solutions of the Volterra integral equations. As a corollary, a topological interpretation of Levinson’s theorem is established via an index theorem approach.

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1 Introduction

Common objects in mathematical scattering theory are the (Møller) wave operators. In the time-dependent framework, they are defined by the strong limits

$$W_{\pm} \equiv W_{\pm}(H, H_0) := \text{slim}_{t \to \pm \infty} e^{itH} e^{-itH_0} 1_{ac}(H_0)$$

for a pair of self-adjoint operators $(H, H_0)$ on a Hilbert space $H$ whenever the above limits exist. Here, $1_{ac}(H_0)$ denotes the spectral projection associated with the absolutely continuous subspace $H_{ac}(H_0)$ of $H_0$. $W_{\pm}$ are, whenever they exist, partial isometries with ranges contained in the absolutely continuous subspace $H_{ac}(H)$ of $H$, and therefore it is natural to say $W_{\pm}$ are complete if their ranges coincide with $H_{ac}(H)$. Important consequences of the existence and completeness are that they give rise to unitary equivalences between $H_{ac}(H_0)$ and $H_{ac}(H)$, and that the scattering operator $S := W_+^* W_-$ is a unitary operator on $H_{ac}(H_0)$.

After they were introduced by physicists Møller and Heisenberg, numerous works have been devoted to the problem of the existence and completeness. As a leading example, mathematical scattering theory is developed for and efficiently applied to the study of Schrödinger operators. We can touch only a few aspects of this theory in this paper and we refer to [BW, Yaf2] for more information.

A next step in the mathematical scattering theory is to study some structural formulas or further mapping properties of $W_{\pm}$. It is usually based on the stationary approach. We mention that the $L^p$-boundedness of $W_{\pm}$ for a Schrödinger operator on $\mathbb{R}^d$ initiated by Yajima [Yaj]
Theorem 1.1. Under the assumption (1.2) with \( \rho > 2 \), the equality

\[
W_- = 1 + \phi(A)(S - 1) + K
\]  

holds, where \( \phi(A) := (ie^{\pi A} + 1)^{-1} \) and \( K \) is a Hilbert-Schmidt operator on \( L^2(\mathbb{R}_+) \).

Note that a quite similar formula for one-dimensional Schrödinger operators has already appeared in [KR] but they could not reach the borderline case. Indeed, the potential \( v : \mathbb{R} \to \mathbb{R} \) was assumed to satisfy

\[
\int_{\mathbb{R}} (1 + |x|)^\rho |v(x)| \mathrm{d}x < \infty
\]  

This assumption is strong enough to ensure the existence and completeness of bound states of a quantum system. It has been originally established for a Schrödinger operator \(-\Delta + V(X)\) with a spherically symmetric potential \( V \) i.e. \( V(\cdot) = v(|\cdot|) \) for some \( v : \mathbb{R}_+ \to \mathbb{R} \), by N. Levinson. For the orbital quantum number \( \ell = 0 \), the radial part of the operator reduces to a Schrödinger operator of the form \(-d^2/dx^2 + v(x)\) on the half-line \( \mathbb{R}_+ \) and this relation is often formulated with the phase shift \( \eta(k) = \eta_{\ell=0}(k) \), i.e. if we set \( N \) to be the number of bound states of the system, then the equality

\[
\eta(\infty) - \eta(0) = \pi (N + \delta)
\]  

holds, where the correction term \( \delta = 1/2 \) if there exists a zero-energy resonance (see, Section 2) and \( \delta = 0 \) otherwise. The relation (1.1) is usually proved based on complex analysis (see [RS Thm. XI. 59] or [Ya3 Thm. 4.6.1]), and has been generalized to several quantum systems by many researchers on purely analytical basis.

In contrast to the usual approach, the topological approach [KR] is based on some algebraic methods including \( K \)-theory for \( C^* \)-algebras. Explicit formulas for \( W_- \) have been used to prove its affiliation to a \( C^* \)-subalgebra \( \mathcal{E} \subset B(\mathcal{H}) \) with the quotient algebra \( \mathcal{E}/K(\mathcal{H}) \) isomorphic to \( C(\mathbb{S}^1) \). A concise introduction to this approach can be found in the review paper [R]. More recently in [IR NPR], this approach is generalized to specific potentials of the form \( x^{-2} \) on the half-line \( \mathbb{R}_+ \), with possibly complex coefficients, which create infinitely many negative eigenvalues or finitely many complex eigenvalues.

The aim of the present paper is to provide an explicit formula for \( W_- \) and give a topological interpretation to the relation (1.1) for the Schrödinger operator \( H = H_0 + v(X) \) with \( H_0 = -d^2/dx^2 \) being the Dirichlet Laplacian on \( \mathbb{R}_+ \). In this paper, the real-valued potential \( v \in L^\infty(\mathbb{R}_+) \) is assumed to decay strictly faster than \(-2\), i.e. there exist \( C > 0 \) and \( \rho > 2 \) such that

\[
|v(x)| \leq C(1 + x)^{-\rho}, \quad \forall x \geq 0.
\]  

This assumption is strong enough to ensure the existence and completeness of \( W_+ \) for the pair \((H, H_0)\). It is also well known that \( H \) has no singular continuous spectrum for such potential \( v \) and has finitely many simple negative eigenvalues. Indeed, \( \rho = 2 \) is the borderline case for the finiteness of the number of bound states.

We now set \( A \) to be the generator of the dilation group defined by \([e^{-itA}f](x) := e^{t^2}f(e^tx)\) for \( f \in L^2(\mathbb{R}_+) \) and \( x \in \mathbb{R}_+ \), and \( S \) to be the scattering operator for \((H, H_0)\), in order to state the following main result:

**Theorem 1.1.** Under the assumption (1.2) with \( \rho > 2 \), the equality

\[
W_- = 1 + \phi(A)(S - 1) + K
\]  

holds, where \( \phi(A) := (ie^{\pi A} + 1)^{-1} \) and \( K \) is a Hilbert-Schmidt operator on \( L^2(\mathbb{R}_+) \).
for some $\alpha > 5/2$, and this condition means that $v(x) = o(|x|^{-1-\alpha})$ as $|x| \to \infty$ in the power scale.

Let us mention that the main difficulty in such a work usually consists in proving that the remainder term is compact. Indeed, the relatively new thing developed in this paper is a new scheme to prove the compactness discussed in Section 5. For our model, the remainder term $K$ is at first defined as a kind of oscillatory integral operator with the kernel in terms of the Jost solution, which is a unique solution of an integral equation (see eq.(2.5)). By iterating that integral equation finitely many times, we reduce this operator to a sum of a good part and bad parts. Then, we obtain a rapid decay of the kernel of the good part due to the iterating process. We can decompose each bad part into a product of a Hilbert-Schmidt operator and a bounded operator by using a little algebraic trick of integration by parts.

Let us now be more precise on the contents of this paper. In Section 2, we review basic spectral results on the Schrödinger operator $H$ on the half-line. In particular, we study the Schrödinger equation $-u'' + vu = k^2 u$ in terms of the Volterra integral equations. Basic properties of two distinguished solutions, called the regular solution $\varphi(\cdot,k)$ and the Jost solution $\theta(\cdot,k)$, and their Wronskian $w(k)$, called the Jost function, are recalled in detail. In order to make our exposition self-contained, we repeat the relevant materials from [Yaf3, Chapter 4] without proofs.

We will also borrow the stationary expression of the wave operators in terms of generalized Fourier transforms (see Theorem 3.1) from that reference. In Section 3, we deduce the above explicit formula for $W^-$ from that expression. Let us mention that the formula implicitly appeared in the proof of the equality between the time-dependent definition and the stationary expression. Indeed, the decomposition of the kernel of the generalized Fourier transform, which is defined by $kw(k)^{-1}\varphi(x,k)$, is the main step. Our formula is obtained by rewriting this decomposition as an operator identity.

In Section 4, we briefly recall the $C^*$-algebraic framework and formulate the topological version of Levinson’s theorem for $H$. Since both analytic and topological indices can be computed explicitly, we will not touch $K$-theory behind the index theorem but more information in this direction can be found in [R]. The interest is pushed forward to the computation of the topological index, which is defined as a winding number of a continuous invertible function on the edge of a square and therefore comes with four contributions, one from each edge. This computation enables us to compare the resulting index theorem with the usual Levinson’s theorem (1.1).

In Section 5, the proof of the compactness of the remainder term is provided, and as we have already mentioned, this part is the main contribution of this paper. Based on the classical techniques of integral equations as in [AK, DE, GNP], we study the integral kernels algebraically by playing with trigonometric functions and integration by parts. For the convenience of the reader, we perform some computations explicitly.

As a final remark, let us mention that in the paper [DF], the $L^p$-boundedness of the wave operators for one-dimensional Schrödinger operators for $1 < p < \infty$ is proved under the assumption (1.4) with $\alpha = 1$. We will not develop this point in this paper, but it is natural to wonder if we can deduce such property of the wave operators in terms of our formula.

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2 Preliminaries

Based on [Yaf3, Chapter 4], let us recall some well-known facts about the regular solution $\varphi(x, \zeta)$ and the Jost solution $\theta(x, \zeta)$ of the Schrödinger equation

$$-u''(x) + v(x)u(x) = zu(x), \quad x > 0$$

(2.1)

for $z = \zeta^2$ with $\zeta \in \mathbb{C} \setminus \{0\}$. Here, the real-valued potential $v$ satisfies (1.2) with $\rho > 2$. The direct approach to scattering theory reviewed in this section can be applied to a more general class of potentials, for example one can admit slower decays at infinity or some singularities at $x = 0$. However, we assume (1.2) throughout the present paper to simplify our presentation. Note that one easily infers that the operator $H := H_0 + v(X)$ with $H_0 = -d^2/dx^2$ being the Dirichlet Laplacian on $\mathbb{R}_+$, is self-adjoint in $L^2(\mathbb{R}_+)$ with the same domain as $H_0$.

Recall that the regular solution $\varphi(\cdot, \zeta)$ is the solution of (2.1) satisfying the following boundary condition at $x = 0$:

$$\varphi(0, \zeta) = 0 \quad \text{and} \quad \varphi'(0, \zeta) = 1,$$

(2.2)

where $'$ denotes the derivative with respect to the variable $x$. In the free case, when the potential $v$ is identically zero, the regular solution is given by $\varphi_0(x, \zeta) = \zeta^{-1} \sin(\zeta x)$. For a function $\varphi(\cdot, \zeta)$ satisfying (2.2), the Schrödinger equation is equivalent to the following Volterra integral equation:

$$\varphi(x, \zeta) = \varphi_0(x, \zeta) + \frac{1}{\zeta} \int_0^x \sin(\zeta(x - y))v(y)\varphi(y, \zeta)dy, \quad x \geq 0.$$  

(2.3)

By applying the method of successive approximations, one can prove that there exists a unique solution $\varphi(\cdot, \zeta)$ of (2.3) for any $\zeta \in \mathbb{C}$, including the point $\zeta = 0$. Note also that for each fixed $x \geq 0$ this solution satisfies $\varphi(x, \zeta) = \varphi(x, -\zeta)$, and that the function $\varphi(x, \zeta)$ is an entire function of the variable $z = \zeta^2$ [Yaf3, Lem. 4.1.2].

For $\Im(\zeta)$ with $\zeta \neq 0$, the Jost solution $\theta(\cdot, \zeta)$ is the solution of (2.1) satisfying the following boundary condition at infinity:

$$\theta(x, \zeta) = e^{i\zeta x}(1 + o(1)) \quad \text{and} \quad \theta'(x, \zeta) = i\zeta e^{i\zeta x}(1 + o(1)) \quad \text{as} \ x \rightarrow \infty.$$  

(2.4)

Note that $\theta_0(x, \zeta) := e^{i\zeta x}$ is the Jost solution in the free case, and the corresponding integral equation is

$$\theta(x, \zeta) = \theta_0(x, \zeta) + \frac{1}{\zeta} \int_x^\infty \sin(\zeta(x - y))v(y)\theta(y, \zeta)dy, \quad x \geq 0.$$  

(2.5)

Similarly, one can prove that there exists a unique solution $\theta(\cdot, \zeta)$ of (2.5) for any $\zeta \neq 0$ in the closure of the upper half-plane $\mathbb{C}_+ := \{\zeta \in \mathbb{C} \mid \Im(\zeta) > 0\}$, and also that for each fixed $x \geq 0$, $\theta(x, \zeta)$ is an analytic function of $\zeta$ in the upper-half plane, and continuous up to the real axis except $\zeta = 0$ [Yaf3, Lem. 4.1.4]. The following estimate is also proved in the same lemma: for any $k_0 > 0$ there is $C > 0$ such that for any $|\zeta| > k_0$ and $x \geq 0$

$$|p(x, \zeta)| \leq Ce^{-\Im(\zeta)x} \frac{1}{|\zeta|} \int_x^\infty |v(y)|dy,$$

(2.6)

where $p(x, \zeta) := \theta(x, \zeta) - \theta_0(x, \zeta)$.

Since the function $x \mapsto xv(x)$ still belongs to $L^1(\mathbb{R}_+)$, one gets the following estimate for the low-energy [Yaf3, Lem. 4.3.1]: for fixed $x \geq 0$ the Jost solution $\theta(x, \zeta)$ is continuous as $\zeta \rightarrow 0$, $\Im(\zeta) \geq 0$, and there exists $C > 0$ such that for any $x \geq 0$ and $\Im(\zeta) \geq 0$

$$|p(x, \zeta)| \leq e^{-\Im(\zeta)x} \left\{ \exp \left( C \int_x^\infty y|v(y)|dy \right) - 1 \right\}.$$  

4
One can easily infer from the Taylor expansion of the function $t \mapsto e^t$ that there exists $C' > 0$ such that
\[ |p(x, \zeta)| \leq C' \int_x^\infty y|v(y)|\,dy. \tag{2.7} \]
Moreover, $\theta(x) := \theta(x, 0)$ is a solution of the Schrödinger equation
\[-\frac{d^2}{dx^2}u(x) + v(x)u(x) = 0, \quad x > 0. \tag{2.8} \]

In the following, we assume $\Im(\zeta) \geq 0$, and we write $k$ instead of $\zeta$ if $\Im(\zeta) = 0$ and $\zeta > 0$. The Wronskian $w(\zeta) := \varphi'(x, \zeta)\theta(x, \zeta) - \varphi(x, \zeta)\theta'(x, \zeta)$ is called the Jost function. Since it is independent of the variable $x$, it follows from the boundary condition (2.2) that $w(\zeta) = \theta(0, \zeta)$.

By using the uniqueness of the solutions $\varphi(\cdot, k)$ and $\theta(\cdot, k)$, one has
\[ \theta(x, -k) = \overline{\theta(x, k)} \quad \text{whence} \quad w(-k) = \overline{w(k)}, \tag{2.9} \]

Therefore, $w(k) \neq 0$ for $k > 0$ since otherwise it would follow from (2.9) that $\varphi(\cdot, k)$ is identically $0$. Hence, by continuity of the function $w$, there exists a unique continuous function $\eta : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\lim \limits_{k \to +\infty} \eta(k) = 0$ such that for any $k > 0$
\[ w(k) = A(k)e^{i\eta(k)} \quad \text{with} \quad A(k) := |w(k)|. \tag{2.10} \]
The coefficient $A(k)$ is called the limit amplitude and $\eta(k)$ is called the phase shift associated with $v$. Moreover, the scattering matrix is now defined by
\[ s(k) := \frac{w(-k)}{w(k)} = e^{-2i\eta(k)}, \quad k > 0. \tag{2.10} \]

It is known [Yaf3, Lem. 4.2.2] that the Jost function $w : \mathbb{C}_+ \setminus \{0\} \to \mathbb{C}$ has complex zeros only on $i\mathbb{R}_+$, and they are simple. Moreover, $w(\zeta) = 0$ if and only if $\lambda = \zeta^2$ is an eigenvalue of $H$. Indeed, if $w(\zeta) = 0$ then the Jost solution $\theta(\cdot, \zeta)$ of $H$ is the eigenfunction with eigenvalue $\lambda = \zeta^2$. It is also well known that $H$ has only finitely many negative eigenvalues (see, [Yaf3, Rem. 3.8, 182p]).

As a consequence, $k = 0$ is the only possible real zero of the Jost function $w$. Since $\theta(\cdot, 0)$ is not a $L^2$-function but satisfies the Schrödinger equation with energy $\lambda = 0$, we say that $H$ has a zero-energy resonance if $w(0) = 0$. It is also called a half-bound state since it contributes to the classical Levinson’s theorem (1.1) as a correction term $\delta$ by $1/2$. Note that the zero-energy resonance also affects on the low-energy asymptotic of the scattering matrix $s(k)$. More precisely, as $k \downarrow 0$
\[ s(k) = \begin{cases} 1 + o(1) & \text{if } w(0) \neq 0, \\ -1 + o(1) & \text{if } w(0) = 0, \end{cases} \]
see [Yaf3, Prop. 4.3.4 and Prop. 4.3.9].
3 Formula for the wave operator $W_-$

We turn now to the scattering theory for the Schrödinger operator $H$. In our situation, $W_-$ and $W_+$ resp. exists, and as we will see below, maps $\psi_0(x,k) = \sin(kx)$ to the wave function

$$\psi^-(x,k) := kw(k)^{-1}\varphi(x,k) \quad \text{(and } \psi^+(x,k) := \psi^-(x,k) \text{ resp.)}.$$ 

More precisely, $W_\pm$ map a wave packet of the form

$$[\mathcal{F}_sg](x) := \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(kx)g(k)dk, \quad \forall x \geq 0$$

(3.1)

to the wave packets

$$[\mathcal{F}^\pm g](x) := \sqrt{\frac{2}{\pi}} \int_0^\infty \psi^\pm(x,k)g(k)dk, \quad \forall k \geq 0$$

(3.2)

for $g \in C_c^\infty(\mathbb{R}_+)$.

The equation (3.1) defines the Fourier sine transform $\mathcal{F}_s$, which continuously extends to a self-adjoint unitary operator on $L^2(\mathbb{R}_+)$. It also satisfies the intertwining relation $\mathcal{F}_sH_0 = L\mathcal{F}_s$, where $L$ is the multiplication operator by the function $k \mapsto k^2$. On the other hand, the generalized Fourier transforms $\mathcal{F}^\pm$ associated with $H$ defined by

$$[\mathcal{F}^\pm f](k) := \sqrt{\frac{2}{\pi}} \int_0^\infty \psi^\pm(x,k)f(x)dx, \quad \forall f \in C_c^\infty(\mathbb{R}_+), \ k \geq 0$$

also continuously extend to bounded linear operators on $L^2(\mathbb{R}_+)$, and (3.2) actually correspond to the adjoints of $\mathcal{F}^\pm$. They satisfy $\mathcal{F}^\pm\mathcal{F}^\pm = 1$, $\mathcal{F}^\pm\mathcal{F}^\pm = 1_{ac}(H)$ and the intertwining relations $\mathcal{F}^\pm H = L\mathcal{F}^\pm$, where $1_{ac}(H) = 1_{\mathbb{R}_+}(H)$ is the projection onto the absolutely continuous subspace $\mathcal{H}_{ac}(H)$ of $H$.

**Theorem 3.1.** Under (1.2), the wave operators $W_\pm$ for the pair $(H,H_0)$ exist, satisfy $W_\pm = \mathcal{F}^\pm\mathcal{F}_s$ and are complete.

**Remark 3.2.** The conclusion of Theorem 3.1 is true for more general class, e.g. $v \in L^1(\mathbb{R}_+)$, see [Yaf3] Thm. 4.2.8 and Rem. 2.9, p.176.

We recall the main idea of the proof for $W_-$ but the one for $W_+$ is almost the same. For $f = \mathcal{F}_sg$ with $g \in C_c^\infty(\mathbb{R}_+)$, by virtue of intertwining property one has

$$\|e^{itH}e^{-itH_0} f - \mathcal{F}^{-}g\|_{L^2(\mathbb{R}_+)} = \|\mathcal{F}^{-} - \mathcal{F}_s\|_{L^2(\mathbb{R}_+)}.$$ 

Hence, it suffices to prove that the $L^2$-norm of the function

$$[(\mathcal{F}^{-} - \mathcal{F}_s)e^{-itL}g](x) = \sqrt{\frac{2}{\pi}} \int_0^\infty (\psi^-(x,k) - \sin(kx))e^{-itk^2}g(k)dk$$

(3.3)

tends to 0 as $t \to -\infty$. By using (2.9) and (2.10), one has

$$\sqrt{\frac{2}{\pi}} (\psi^-(x,k) - \sin(kx)) = \sqrt{\frac{2}{\pi}} \left(\frac{e^{ikx}}{2i} (s(k) - 1)\right) + \sqrt{\frac{2}{\pi}} \frac{1}{2i} \left(p(x,k)s(k) - \overline{p(x,k)}\right)$$

$$=: F_1(x,k) + F_2(x,k).$$

(3.4)
One can prove that the norm of the term in (3.3) defined by the kernel $F_1$ tends to 0 as $t \to -\infty$ by the standard argument [Ya93 Lem. 0.4.9] already used in the proof of the invariance principle. By using the estimate (2.6), there exists $C_g > 0$ such that for $k \in \text{supp} g$ and $x \in \mathbb{R}_+$,

$$|F_2(x, k)| \leq C_g \int_x^\infty |v(y)| \, dy.$$ 

Since $x \mapsto \int_x^\infty |v(y)| \, dy$ belongs to $L^2(\mathbb{R}_+)$, the norm of the term defined by $F_2$ in (3.3) tends to 0 as $|t| \to \infty$ by the Riemann-Lebesgue Lemma and the Lebesgue dominated convergence theorem.

It follows from Theorem 3.1 that the scattering operator $S = W_+^* W_-$ is a unitary operator on $L^2(\mathbb{R}_+)$, and since $s\psi^- = \psi^+$, the equality

$$S = \mathcal{F}_s \mathcal{F}$$

holds, where $s$ denotes the multiplication operator by the scattering matrix $k \mapsto s(k)$.

The decomposition (3.4) is our starting point for an explicit formula of the wave operator. Indeed, if we set $F_j$ for the integral operator with the kernel $F_j(x, k), j = 1, 2$, then

$$W_- = \mathcal{F}_s^{-1} \mathcal{F}_s = 1 + F_1 \mathcal{F}_s + F_2 \mathcal{F}_s.$$  

By letting $\mathcal{F}_c$ denote the Fourier cosine transform, one obtains $F_1 \mathcal{F}_s = \frac{1}{2i}(\mathcal{F}_c \mathcal{F}_s + i\mathbf{1})(S - \mathbf{1})$. Moreover, one can deduce from [DR eq. (4.29)] and from some trigonometric identities,

$$\frac{1}{2i}(\mathcal{F}_c \mathcal{F}_s + i\mathbf{1}) = \frac{1}{ie^{\pi A} + 1} =: \phi(A),$$

where $A$ is the generator of the dilation group on $L^2(\mathbb{R}_+)$, that is, $[e^{-itA} f](x) := e^{t/2} f(e^t x)$ for $f \in L^2(\mathbb{R}_+)$ and $x \in \mathbb{R}_+$. Here, we have changed the sign of the generator of dilation group from that in [DR p.875]. Finally, by combining (3.5) with (3.7), we obtain the expression (1.1) from (3.6) with $K := F_2 \mathcal{F}_s$.

**Remark 3.3.** When we consider $W_+$ in place of $W_-$, the adjoint $S^*$ naturally appears. Similarly, one obtains the expression

$$W_+ = \mathbf{1} + \psi(A)(S^* - \mathbf{1}) + F_2' \mathcal{F}_s$$

where $\psi(A) = (1 - ie^{-\pi A})^{-1}$. The kernel of $F_2'$ is given by $F_2(x, k)s(k)$ and is therefore a Hilbert-Schmidt operator. We can also see this from $W_+ = W_- S^*$.

We provide a detailed analysis of the remainder term $K$ in Section 5. Note that in the case $\rho > 5/2$, the proof of the compactness can be very simplified. Indeed, it follows from (1.2), (2.6) and (2.7) that there exists $C > 0$ such that for any $x \geq 0$

$$|F_2(x, k)| \leq \sqrt{\frac{2}{\pi}} |p(x, k)| \leq C \begin{cases} k^{-1}(1 + x)^{-\rho-1} & \text{if } k > 1, \\ (1 + x)^{-(\rho-2)} & \text{if } 0 \leq k \leq 1. \end{cases}$$

Here, we used the triangle inequality and that $|s(k)| = 1$. Hence, $F_2$ belongs to $L^p(\mathbb{R}_+ \times \mathbb{R}_+)$ for $p > 1$ if $\rho > 2 + 1/p$. In particular, it follows that $F_2$ is a Hilbert-Schmidt operator provided $\rho > 5/2$, and so is $K$. 


4 Topological version of Levinson’s theorem

In this section, we provide a $C^*$-algebraic framework, which has already appeared in [KR], and give a topological interpretation of the usual Levinson’s theorem [14]. In the following, $C(R)$ and $C_0(R)$ denote the $C^*$-algebras of continuous functions on $R$ having limits at $±∞$ and of continuous functions vanishing at $±∞$, respectively.

Note first that our assumption on the decay of the potential is sufficient for the absence of singular continuous spectrum for the operator $H$. Together with the finiteness of the number of eigenvalues of $H$, this implies that $W_-$ is a Fredholm operator.

We define the $C^*$-algebra $E_{□}$ on $L^2(\mathbb{R}_+)$ by

$$ E_{□} := C^*\left( a(A)b(B) \mid a, b \in C(\mathbb{R}) \right), $$

where $B := \frac{1}{2}\ln(H_0)$ is a rescaled energy operator. Here, the right hand side of (4.1) stands for the $C^*$-algebra generated by operators of the form $a(A)b(B)$ with $a, b \in C(\mathbb{R})$. Since the Weyl canonical commutation relation $e^{itB}e^{isA} = e^{-ist}e^{isA}e^{itB}$ holds and the multiplicities of the spectra $σ(B)$ and $σ(A)$ are one, the pair $(B, A)$ is unitarily equivalent to the pair $(X, D)$ on $L^2(\mathbb{R})$, where $X$ and $D = -i\frac{d}{dx}$ are the position and momentum operators on $\mathbb{R}$, respectively.

As a consequence, the set $K = K(L^2(\mathbb{R}_+))$ of compact operators on $L^2(\mathbb{R}_+)$ is isomorphic to

$$ C^*\left( a(A)b(B) \mid a, b \in C_0(\mathbb{R}) \right). $$

One easily observes that $K$ is an ideal of $E_{□}$ and the quotient morphism $π_{□} : E_{□} \to E_{□}/K \cong C(□)$ is given by

$$ π_{□}(a(A)b(B)) = \left( a(-\infty)b, b(-\infty)a, a(\infty)b, b(\infty)a \right), $$

where the topological space $□$ is defined to be the boundary of the compact space $\mathbb{R} \times \mathbb{R}$.

We define the rescaled scattering matrix $S(β) := s(e^β)$ for $β \in \mathbb{R}$. Then, we can write

$$ W_- = 1 + φ(A)(S(B) - 1) + K, $$

therefore $W_- \in E_{□}$, and $π_{□}(W_-) = (Γ_1, Γ_2, Γ_3, Γ_4)$ is given by

$$ Γ_1(β) := S(β), $$

$$ Γ_2(α) := \begin{cases} 1 & \text{if } w(0) \neq 0, \\ \tanh(πα) + i \cosh(πα)^{-1} & \text{if } w(0) = 0, \end{cases} $$

$$ Γ_3(β) := 1 \quad \text{and} \quad Γ_4(α) := 1 $$

for $α, β \in \mathbb{R}$. Here, we have used the low-energy asymptotic of $s(k)$ provided in Section 2.

One can easily observe that $□$ is homeomorphic to the unit circle $S^1$. Hence, for a given non-vanishing function $Γ \in C(□)$, the winding number $\text{Wind}_{□}(Γ)$ $∈ \mathbb{Z}$ of the closed curve $Γ(□) ⊂ C \setminus \{0\}$ is well-defined. Here, we turn around $□$ clockwise as a convention (see Figure 11). Then, we obtain the following index formula by applying the Gohberg-Krein index theorem:

**Theorem 4.1** (topological version of Levinson’s theorem). Under the assumption (12) with $ρ > 2$,

$$ \text{Wind}_{□}\left( π_{□}(W_-) \right) = - \text{Index}(W_-), $$

where $\text{Index}(\cdot)$ denotes the Fredholm index.
Theorem 4.1 indeed gives a reformulation of the usual Levinson’s theorem (1.1). We first observe that

\[
\text{Index}(W_{-}) = \dim \ker (W_{-}) - \dim \text{coker}(W_{-}) = 0 - \dim \mathcal{H}_{p}(H) = -N,
\]

by the asymptotic completeness of \(W_{-}\), and therefore the right hand side in the equality (4.3) is the number of eigenvalues of \(H\). On the other hand, by taking \(S(\beta) = \exp(-2i\eta(e^{\beta}))\) into account, one computes the contributions \(w_{n}(\Gamma_{j})\) of \(\Gamma_{j}\) for \(j = 1, 2, 3, 4\) to the winding number as follows:

\[
w_{n}(\Gamma_{1}) = \frac{-2\eta(\infty) - (-2\eta(0))}{2\pi} = \frac{-\eta(\infty) - \eta(0)}{\pi},
\]

\[
w_{n}(\Gamma_{2}) := \begin{cases} 0 & \text{if } w(0) \neq 0, \\ -\frac{1}{2} & \text{if } w(0) = 0, \end{cases}
\]

\[
w_{n}(\Gamma_{j}) = w_{n}(1) = 0 \text{ for } j = 3, 4.
\]

Here, in the case \(w(0) = 0\), \(w_{n}(\Gamma_{2})\) is computed by the formula \(\frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma_{2}^{*}(\alpha)\Gamma_{2}'(\alpha) d\alpha\) since \(\Gamma_{2}\) is continuously differentiable. According to the orientation of \(\Box\), the winding number is given by

\[
\text{Wind}_{\Box}(\pi_{\Box}(W_{-})) = -w_{n}(\Gamma_{1}) + w_{n}(\Gamma_{2}) + w_{n}(\Gamma_{3}) - w_{n}(\Gamma_{4}) = \begin{cases} \frac{\eta(\infty) - \eta(0)}{\pi} & \text{if } w(0) \neq 0, \\ \frac{\eta(\infty) - \eta(0)}{\pi} - \frac{1}{2} & \text{if } w(0) = 0. \end{cases}
\]

Hence, the relation (1.1) can be obtained by rearranging the equality (4.3).

![Figure 1: The boundary \(\Box\) of \(\mathbb{R} \times \mathbb{R}\) and its orientation.](image)

**Remark 4.2.** One can also directly verify the unitary equivalence between \((B, A)\) and \((X, D)\). We set \(\mathcal{R} := \mathcal{R}_{0}\mathcal{F}_{s} : L^{2}(\mathbb{R}_{+}) \rightarrow L^{2}(\mathbb{R})\), where \(\mathcal{R}_{0} : L^{2}(\mathbb{R}_{+}) \rightarrow L^{2}(\mathbb{R})\) is defined by \(\mathcal{R}_{0}f(\beta) := e^{\beta/2}f(e^{\beta})\) for \(f \in L^{2}(\mathbb{R}_{+})\) and \(\beta \in \mathbb{R}\). Then, one can easily see that \(\mathcal{R}\) is a unitary operator and satisfies \(\mathcal{R}e^{-itA} = e^{-itX}\mathcal{R}\) and \(\mathcal{R}e^{-itB} = e^{-itD}\mathcal{R}\) for any \(t \in \mathbb{R}\) by using the relation \(\mathcal{F}_{s}A = -A\mathcal{F}_{s}\).
Remark 4.3. Our investigation on the present model is motivated from the well-known fact that our assumption on the potential prevents any singular continuous spectrum and an infinite number of bound states for $H$. However, we can also observe this fact from formula (1.3). The Atkinson’s characterization states that a bounded operator $a \in B(H)$ on a Hilbert space $H$ is Fredholm if and only if $\pi(a) \in B(H)/K(H)$ is invertible, where $\pi : B(H) \to B(H)/K(H)$ is the quotient map. Since $\pi(W_-) = \pi[\pi(W_-)]$ is invertible in $C(\square) \cong K \subset B(L^2(\mathbb{R}_+)) / K$, $W_-$ is a Fredholm operator on $L^2(\mathbb{R}_+)$. Hence, $W_-$ can never have infinite dimensional kernel or cokernel. $H$ has therefore no singular continuous spectrum nor infinite number of bound states.

Remark 4.4. Note that in [Yaof], the low-energy asymptotic of the scattering matrix $s(k)$ for a potential decreasing slowly, i.e. $v(x) \sim v_0 x^{-\rho}$, $\rho \in (0, 2)$ as $x \to \infty$, is established. Since the situation is rather complicated for $\rho \in (0, 1)$ due to the long-rangeness, let us recall the following result for $\rho \in (1, 2)$: the phase shift $\eta$ associated with $v$ satisfies

$$\eta(k) = \eta_0 k^{1-2/\rho} + O(1) \quad \text{as } k \downarrow 0, \quad (4.4)$$

where $\pm \eta_0 > 0$ if $\pm v_0 > 0$. Hence, the scattering matrix $s(k)$ is oscillating near $k = 0$.

In [IRNPR], it has been proved that the wave operators associated with a specific potential of the form $x^{-2}$ belong to the $C^*$-algebra $E_A$ generated by $a(A)b(B)$ with $a \in C(\mathbb{R})$ and $b$ belongs to a $C^*$-subalgebra $A \subset C_b(\mathbb{R})$. A canonically contains the scattering matrix in the rescaled energy. In the case, when the potential creates the infinitely many bound states, the role of the ideal $K$ is played by the ideal $\mathcal{J}_A$ generated by $a(A)b(B)$ with $a \in C_0(\mathbb{R})$ and $b \in A$. It is then natural to wonder if we can prove an explicit formula for $W_-$ similar to (1.1) for slowly decreasing potentials with the remainder term belonging to $\mathcal{J}_A$ for some suitable $A \subset C_b(\mathbb{R})$. This question is at present far from being solved, but the affirmative answer would allow one to obtain a K-theoretic index formula by using the boundary map in K-theory, as explained in [IR] Sect. 5 (see also [R, Sect. 4.2 and 4.3]). In view of (1.4), such an algebra $A$ should contain $C(\mathbb{R})$, and any function $f \in C_b(\mathbb{R})$ satisfying that there is a continuous periodic function $f_- \sim f_-(e^{\beta(1-2/\rho)})$ as $\beta \to -\infty$, in a suitable sense.

5 Analysis of the remainder term

In this last section, we prove the compactness of the remainder term $K = F_2 \mathcal{F}_s$ in (1.3). In the following, we will denote integral operators by the same symbols as their kernels.

By virtue of the integral equation (2.5), we first observe that

$$F_2(x, k) = \sqrt{\frac{2}{\pi}} \frac{1}{2i} \left( p(x, k) s(k) - p(x, k) \right)$$

$$= \sqrt{\frac{2}{\pi}} \Im \left( p(x, k) e^{-i\eta(k)} \right) e^{-i\eta(k)}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin(k(x_1 - x))}{k} v(x_1) \Im \left( \theta(x_1, k) e^{-i\eta(k)} \right) dx_1 \times e^{-i\eta(k)}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin(k(x_1 - x))}{k} v(x_1) \Im \left( \frac{p(x_1, k) e^{-i\eta(k)}}{e^{-i\eta(k)}} \right) dx_1 \times e^{-i\eta(k)}$$

$$+ \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin(k(x_1 - x))}{k} v(x_1) \sin(k x_1 - \eta(k)) dx_1 \times e^{-i\eta(k)}. $$

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Therefore, by iterating this procedure one obtains the following decomposition of the kernel $F_2$ for each $N \in \mathbb{N}$:

$$F_2(x_0, k) = \{p_N(x_0, k) + R_N(x_0, k)\} e^{-i\eta(k)}, \quad \forall x_0, k \in \mathbb{R}_+$$

(5.1)

where

$$p_N(x_0, k) = \sqrt{\frac{2}{\pi}} \int_{x_0}^\infty dx_1 \cdots \int_{x_{N-1}}^\infty dx_N \left( \prod_{j=1}^N \sin \left( k(x_j - x_{j-1}) \right) \right) v(x_j) \sin \left( k(x_N - \eta(k)) \right)$$

and

$$R_N(x_0, k) = \sum_{n=1}^N r_n(x_0, k)$$

with

$$r_n(x_0, k) = \sqrt{\frac{2}{\pi}} \int_{x_0}^\infty dx_1 \cdots \int_{x_{n-1}}^\infty dx_n \left( \prod_{j=1}^n \sin \left( k(x_j - x_{j-1}) \right) \right) v(x_j) \sin \left( kx_n - \eta(k) \right).$$

Hence, $K = \{p_N \mathcal{F}_s + R_N \mathcal{F}_s\} e^{-i\eta(\sqrt{\eta_0})}$ by taking into account the intertwining relation between $H_0$ and $\mathcal{F}_s$.

**Lemma 5.1.** For any $\rho > 2$ there exists $N \in \mathbb{N}$ such that $p_N \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$, and accordingly the first term $p_N \mathcal{F}_s e^{-i\eta(\sqrt{\eta_0})}$ in the remainder term $K$ is Hilbert-Schmidt on $L^2(\mathbb{R}_+)$.\footnote{For any $x \geq 0$, $p_N$ is a good part in the remainder term while we need a special treatment for $R_N$. Indeed, since we need to deal with the kernel $r_n$ for small $n$, the above argument does not work for the second term $R_N \mathcal{F}_s e^{-i\eta(\sqrt{\eta_0})}$, and therefore the operator $r_n$ should be understood as a kind of oscillatory integral operator.}

**Proof.** By the assumption (1.2), one can prove that there exists $C_0 > 0$ such that for $x_N \geq \ldots \geq x_1 \geq x_0 \geq 0$

$$\prod_{j=1}^N \left| \sin \left( k(x_j - x_{j-1}) \right) \right| v(x_j) \leq C_0 \begin{cases} k^{-N} \prod_{j=1}^N (1 + x_j)^{-\rho} & \text{if } k > 1, \\ \prod_{j=1}^N (1 + x_j)^{-(\rho-1)} & \text{if } 0 \leq k \leq 1. \end{cases}$$

(5.2)

By the estimates (3.8) and (5.2), one has

$$|p_N(x_0, k)| \leq \int_{x_0}^\infty dx_1 \cdots \int_{x_{N-1}}^\infty dx_N \left| \prod_{j=1}^N \sin \left( k(x_j - x_{j-1}) \right) \right| v(x_j) \times \frac{\sqrt{2}}{\pi} |p(x_N, k)|$$

$$\leq C \int_{x_0}^\infty dx_1 \cdots \int_{x_{N-1}}^\infty dx_N \left\{ k^{-N} \prod_{j=1}^N (1 + x_j)^{-\rho} \times k^{-1}(1 + x_N)^{-(\rho-1)} \right. \left. \prod_{j=1}^N (1 + x_j)^{-(\rho-1)} \times (1 + x_N)^{-(\rho-2)} \right\}$$

if $k > 1$, \hspace{1cm} if $0 \leq k \leq 1$.\footnote{for any $x \geq 0$. Hence, if we take $N \in \mathbb{N}$ so that $2(N+1)(\rho-2) > 1$, then $p_N \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$.}

Finally, by direct computations one obtains

$$|p_N(x, k)| \leq C \begin{cases} k^{-(N+1)}(1 + x)^{-(N+1)(\rho-1)} & \text{if } k > 1, \\ (1 + x)^{-(N+1)(\rho-2)} & \text{if } 0 \leq k \leq 1, \end{cases}$$

(5.3)

for any $x \geq 0$. Hence, if we take $N \in \mathbb{N}$ so that $2(N+1)(\rho-2) > 1$, then $p_N \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$.\footnote{for any $x \geq 0$. Hence, if we take $N \in \mathbb{N}$ so that $2(N+1)(\rho-2) > 1$, then $p_N \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$.}
Our approach is based on the following observation: if we set \( V_v(x) := \int_x^\infty v(y)dy \), then \( V_v \) is differentiable and has the derivative equal to \(-v\) almost everywhere, and by an integration by parts and a change of variables one obtains
\[
 r_1(x, k) = \sqrt{\frac{2}{\pi}} \int_x^\infty V_v(y) \sin(k(2y - x) - \eta(k))dy = \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_x^\infty V_v \left(\frac{x+y}{2}\right) \sin(ky - \eta(k))dy. 
\]

Moreover, by using Fubini theorem one can see that \( r_1 \) equals the product of an integral operator with the kernel \( \frac{1}{2} V_v \left(\frac{2y-x}{y}\right) Y(y-x) \), which belongs to \( L^2(\mathbb{R}_+ \times \mathbb{R}_+) \), and a bounded integral operator with the kernel \( \sin(ky - \eta(k)) \). Here, \( Y \) denotes the Heaviside function on \( \mathbb{R} \).

The singular part \( k^{-1} \) in the kernel \( r_n(x, k) \) might be avoided by iterating similar computation with several integration by parts, and one might obtain the compactness of \( r_n \). However, one gets a much more complicated dependence on the potential \( v \) in the formula of \( r_n \) for \( n = 2, 3, \ldots \), after such computations (see Remark [5.4] at the end of this section).

In order to deal with this, we introduce an algebraic framework for integral operators of the type \( r_n \). For \( p = 1, 2 \), let us consider the complex vector space
\[
 \mathcal{V}_p := \bigcup_{\epsilon > 0} L^\infty_{p+\epsilon}(\mathbb{R}_+) \quad \text{with} \quad L^\infty_{p+\epsilon}(\mathbb{R}_+) := \left\{ V \in L^\infty(\mathbb{R}_+) \mid \| (1+\cdot)^{p+\epsilon} V(\cdot) \|_{L^\infty(\mathbb{R}_+)} < \infty \right\}.
\]

For \( V_j \in \mathcal{V}_1, j = 1, 2 \) we set
\[
 V_1 \star V_2(x) := \int_x^\infty V_1(y) V_2(y)dy, \quad x \in \mathbb{R}_+.
\]

One can easily check that \( V_1 \star V_2 \in \mathcal{V}_1 \). As a consequence, \( \mathcal{V}_1 \) becomes a non-associative commutative \( \mathbb{C} \)-algebra under \( \star \). Furthermore, motivated from the computations for \( r_1 \), we define \( V_u(x) := \int_x^\infty u(y)dy \) for any \( u \in \mathcal{V}_2 \). Then, \( V_u \in \mathcal{V}_1 \) and it has the derivative equal to \(-u\) almost everywhere in \( \mathbb{R}_+ \).

**Lemma 5.2.** Let \( n \in \mathbb{N} \). For \( V_1, \ldots, V_n \in \mathcal{V}_1 \), let \( W[V_1, \ldots, V_n] \) be the integral operator with the kernel
\[
 W[V_1, \ldots, V_n](x_0, k) := \sqrt{\frac{2}{\pi}} \int_{x_0}^\infty dx_1 \cdots \int_{x_{n-1}}^\infty dx_n \left( \prod_{j=1}^n V_j(x_j) \right) \times \sin \left( k \left( 2 \sum_{\ell=0}^{n-1} (-1)^{\ell} x_{n-\ell} + (-1)^n x_0 \right) - \eta(k) \right).
\]

Then, the operator \( W[V_1, \ldots, V_n] \) extends continuously to a Hilbert-Schmidt operator on \( L^2(\mathbb{R}_+) \).

**Proof.** Note first that by a change of variables, one can prove that for any \( x_0, k \in \mathbb{R}_+ \)
\[
 2^n \sqrt{\frac{\pi}{2}} \times W[V_1, \ldots, V_n](x_0, k)
\]
\[
 = \begin{cases} 
 \int_{x_0}^\infty V_1 \left( \frac{x_1 + x_0}{2} \right) \sin(kx_1 - \eta(k))dx_1 & \text{if } n = 1, \\
 \int_{x_0}^\infty V_1 \left( \frac{x_1 + x_0}{2} \right) \int_{x_0}^\infty V_2 \left( \frac{x_2 + x_1}{2} \right) \sin(kx_2 - \eta(k))dx_2dx_1 & \text{if } n = 2, \\
 \int_{x_0}^\infty dx_1 \int_{x_0}^\infty dx_2 \int_{x_1}^\infty dx_3 \cdots \int_{x_{n-2}}^\infty dx_n \left( \prod_{j=1}^n V_j \left( \frac{x_j + x_{j-1}}{2} \right) \right) \sin(kx_n - \eta(k)) & \text{if } n > 2.
\end{cases}
\]
Let us prove this in the case $n = 3$ for the convenience of the reader. We do changes of the variables first as $y_1 = 2x_1 - x_0$, second $y_2 = 2x_2 - y_1$ and finally $y_3 = 2x_3 - y_2$:

\[ W[V_1, V_2, V_3](x_0, k) \]

\[ = \sqrt{\frac{2}{\pi}} \int_{x_0}^{\infty} dx_1 \int_{x_1}^{\infty} dx_2 \int_{x_2}^{\infty} dx_3 \left( \prod_{j=1}^{3} V_j(x_j) \right) \sin \left( k (2x_3 - 2x_2 + 2x_1 - x_0) - \eta(k) \right) \]

\[ = \sqrt{\frac{2}{\pi}} \int_{x_0}^{\infty} dy_1 \int_{y_1}^{\infty} dy_2 \int_{y_2}^{\infty} dy_3 V_1 \left( \frac{y_1 + x_1}{2} \right) V_2 \left( \frac{y_2 + x_2}{2} \right) V_3 \left( \frac{y_3 + x_3}{2} \right) \int_{x_0}^{\infty} dx_1 \int_{x_1}^{y_1} dx_2 \int_{x_2}^{y_2} dx_3 \left( \prod_{j=1}^{3} V_j(x_j) \right) \sin \left( k (2x_3 - 2x_2 + y_1) - \eta(k) \right) \]

\[ = \sqrt{\frac{2}{\pi}} \int_{x_0}^{\infty} dy_1 \int_{y_1}^{\infty} dy_2 \int_{y_2}^{\infty} dy_3 V_1 \left( \frac{y_1 + x_1}{2} \right) V_2 \left( \frac{y_2 + y_1}{2} \right) V_3(x_3) \sin \left( k (2x_3 - y_2) - \eta(k) \right) \]

We then define the kernel $(x, y) \mapsto U[V_1, \ldots, V_n](x, y)$ by

\[ 2^n \times U[V_1, \ldots, V_n](x_0, x_n) \]

\[ = \begin{cases} 
V_1 \left( \frac{x_1 + x_0}{2} \right) Y(x_1 - x_0) & \text{if } n = 1, \\
\int_{x_0}^{\infty} V_1 \left( \frac{x_1 + x_0}{2} \right) V_2 \left( \frac{x_2 + x_1}{2} \right) dy_1 Y(x_2 - x_0) & \text{if } n = 2, \\
\int_{x_0}^{\infty} dx_1 \int_{x_1}^{\infty} dx_2 \int_{x_2}^{\infty} dx_3 \cdots \int_{x_{n-3}}^{\infty} dx_{n-1} \left( \prod_{j=1}^{n} V_j \left( \frac{x_j + x_{j+1}}{2} \right) \right) Y(x_n - x_{n-2}) & \text{if } n > 2.
\end{cases} \]

We also set $\Phi := \cos \left( \sqrt{\eta(0)} \right) - F_{c,F_s} \sin \left( \eta(\sqrt{\eta(0)}) \right)$. Then, by using the addition formula for sine and the intertwining relation between $F_s$ and $H_0$, one obtains

\[ W[V_1, \ldots, V_n] = U[V_1, \ldots, V_n] \Phi F_s. \] (5.5)

To obtain an estimate for $U[V_1, \ldots, V_n]$, we take $\varepsilon > 0$ sufficiently small such that $V_j \in L^\infty_{c+}(\mathbb{R}_+)$ for all $j = 1, \ldots, n$. Then one has for instance

\[ |U[V_1, \ldots, V_n](x_0, x_n)| \leq C(1 + x_0)^{-1/2 - (n-1)\varepsilon} (1 + x_n)^{-1/2 - \varepsilon} Y(x_n - x_0). \] (5.6)

Here, we applied the AM-GM inequality to the function $1 + \frac{x_1 + x_2 + \cdots + x_n}{n}$, and then replaced the lower ends of the integrals and $x_{n-2}$ in $Y(x_n - x_{n-2})$ by $x_0$. Hence, $U[V_1, \ldots, V_n] \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$. Since $\Phi$ is a bounded operator on $L^2(\mathbb{R}_+)$, $W[V_1, \ldots, V_n]$ is a Hilbert-Schmidt operator. \(\square\)

We define $W$ and $U$ to be the (algebraic) linear span of the kernels $W[V_1, \ldots, V_n]$ and $U[V_1, \ldots, V_n]$ for any $V_1, \ldots, V_n \in \mathcal{V}_1$ and any $n \in \mathbb{N}$, respectively. Then, according to Lemma 5.2 both $W$ and $U$ are vector subspaces of $L^2(\mathbb{R}_+ \times \mathbb{R}_+)$. Note that by identifying the set of kernels and the set of integral operators on $L^2(\mathbb{R}_+)$ the equality (5.5) can be rewritten as

\[ W = U \Phi F_s. \] (5.7)

It follows from the estimate (5.6) that for any element $U \in U$ there exists some $\varepsilon > 0$ such that the estimate

\[ |U(x, y)| \leq C(1 + x)^{-1/4} (1 + y)^{-1/4} Y(y - x) \] (5.8)

holds.
Lemma 5.3. For \( W \in \mathcal{W} \) and \( u \in \mathcal{V}_2 \), set
\[
[S_uW](x, k) := \int_x^\infty \frac{\sin(k(y-x))}{k} u(y)W(y,k)dy, \quad x, k \in \mathbb{R}_+.
\] (5.9)

Then, \( S_u \) is a well-defined linear map on \( W \) for any \( u \in \mathcal{V}_2 \).

Proof. The linearity is obvious. Hence, it suffices to prove the well-definedness for generators of the form \( W[V_1, \ldots, V_n] \), and therefore we prove it by induction for \( n \). Let \( V \in \mathcal{V}_1 \). Then, by an integration by parts with respect to \( x_1 \) one has for any \( x_0, k \in \mathbb{R}_+ \),
\[
[S_uW[V]](x_0, k) = \int_{x_0}^\infty \frac{\sin(k(x_1-x_0))}{k} u(x_1)W[V](x_1,k)dx_1
\]
\[=
\left[ \frac{2}{\pi} \int_{x_0}^\infty \sin(k(x_1-x_0)) u(x_1) \int_{x_1}^\infty V(x_2) \sin(2kx_2 - x_1 - \eta(k)) dx_2 dx_1
\]
\[=
\left[ \frac{2}{\pi} \int_{x_0}^\infty \cos(k(x_1-x_0)) V_u(x_1) \int_{x_1}^\infty V(x_2) \sin(2kx_2 - x_1 - \eta(k)) dx_2 dx_1
\]
\[= \frac{2}{\pi} \int_{x_0}^\infty V_u(x_1) \int_{x_1}^\infty V(x_2) \sin(2kx_2 - x_1 + \eta(k)) dx_2 dx_1
\]
\[= \frac{2}{\pi} \int_{x_0}^\infty \sin(k(x_1-x_0)) u(x_1) \int_{x_1}^\infty V(x_2) \sin(x_1 - \eta(k)) dx_2 dx_1
\]
\[= \sqrt{2 \pi} \int_{x_0}^\infty \frac{\sin(k(x_1-x_0))}{k} u(x_1)W[V](x_1,k)dx_1 = W[V_u \ast V](x_0, k).
\]

The first term is equal to \( W[V_u, V](x_0, k) \), and again by one integration by parts
\[
\sqrt{2 \pi} \int_{x_0}^\infty \frac{\sin(k(x_1-x_0))}{k} u(x_1)W[V](x_1,k)dx_1 = W[V_u \ast V](x_0, k).
\]

Therefore, one has
\[
[S_uW[V]] = W[V_u, V] - W[V_u \ast V] \in \mathcal{W}.
\] (5.10)

Now, we assume that the statement is true for \( n \) and for any \( u \in \mathcal{V}_2 \). Let \( V_1, \ldots, V_{n+1} \in \mathcal{V}_1 \). We shall prove the equality
\[
[S_uW[V_1, \ldots, V_{n+1}]] = W[V_u, V_1, \ldots, V_{n+1}] - [S_{V_uV_1}W[V_2, \ldots, V_{n+1}]].
\] (5.11)

Then, \( S_uW[V_1, \ldots, V_{n+1}] \in \mathcal{W} \) since \( V_uV_1 \in \mathcal{V}_2 \).

Let us prove (5.11). By an integration by parts, one has
\[
\left[ S_uW[V_1, \ldots, V_{n+1}] \right](x_0, k) = \int_{x_0}^\infty \frac{\sin(k(x_1-x_0))}{k} u(x_1)W[V_1, \ldots, V_{n+1}](x_1,k)dx_1
\]
\[=
\int_{x_0}^\infty \cos(k(x_1-x_0)) V_u(x_1)W[V_1, \ldots, V_{n+1}](x_1,k)dx_1
\]
\[+ \int_{x_0}^\infty \frac{\sin(k(x_1-x_0))}{k} V_u(x_1) \left( \partial_{x_1} W[V_1, \ldots, V_{n+1}] \right)(x_1,k)dx_1
\]
\[= Z_1(x_0, k) + Z_2(x_0, k).
\]
The derivative \( (\partial x_1W[V_1,\ldots,V_{n+1}])(x_1,k) \) in the integrand of \( Z_2(x_0,k) \) is equal to the sum of \(-V_1(x_1)W[V_2,\ldots,V_{n+1}](x_1,k)\), which gives the second term in (5.11), and

\[
w(x_1,k) := k\sqrt{\frac{2}{\pi}} \int_{x_1}^{\infty} dx_2 \cdots \int_{x_{n+1}}^{\infty} dx_{n+2} \left( \prod_{j=1}^{n+1} V_j(x_{j+1}) \right) \times (-1)^{n+1} \cos \left( k \left( 2 \sum_{\ell=0}^{n} (-1)^{\ell} x_{n+2-\ell} + (-1)^{n+1} x_1 \right) - \eta(k) \right).
\]

Now, let us have a look at the trigonometric functions in the integrand of \( Z_1(x_0,k) \) and the expression \( k^{-1} \sin(k(x_1-x_0))w(x_1,k) \). We have

\[
\cos(k(x_1-x_0)) \sin \left( k \left( 2 \sum_{\ell=0}^{n} (-1)^{\ell} x_{n+2-\ell} + (-1)^n x_1 \right) - \eta(k) \right) + (-1)^{n+1} \sin(k(x_1-x_0)) \cos \left( k \left( 2 \sum_{\ell=0}^{n} (-1)^{\ell} x_{n+2-\ell} + (-1)^n x_1 \right) - \eta(k) \right)
= \sin \left( k \left( 2 \sum_{\ell=0}^{n} (-1)^{\ell} x_{n+2-\ell} + (-1)^n x_1 \right) - \eta(k) + (-1)^{n+1}k(x_1-x_0) \right)
= \sin \left( k \left( 2 \sum_{\ell=0}^{n+1} (-1)^{\ell} x_{n+2-\ell} + (-1)^{n+2} x_0 \right) - \eta(k) \right).
\]

Hence, we have

\[
Z_1(x_0,k) + \int_{x_0}^{\infty} \frac{\sin(k(x_1-x_0))}{k} V_u(x_1)w(x_1,k)dx_1 = W[V_u,V_1,\ldots,V_{n+1}](x_0,k).
\]

This finishes the proof of (5.11) \( \square \)

**Proof of Theorem 1.1.** It suffices to prove that the remainder term \( K = F_2\mathcal{F}_s \) is Hilbert-Schmidt. By (5.4) one has \( r_1 = W[V_u] \in \mathcal{W} \). One can also see from the definition that \( r_{n+1} = \mathcal{S}_n r_n \). Therefore, it follows from Lemma 5.3 that \( r_n \) belongs to \( \mathcal{W} \) for any \( n \in \mathbb{N} \). By Lemma 5.2 \( R_N = \sum_{n=1}^{N} r_n \) is a Hilbert-Schmidt operator. Hence, together with Lemma 5.1 we finish the proof. \( \square \)

**Remark 5.4.** One can easily see from Lemma 5.3 that \( \mathcal{S} \) defines a bilinear map \( \mathcal{S} : \mathcal{V}_2 \times \mathcal{W} \rightarrow \mathcal{W} \). Moreover, by combining (5.10) and (5.11) and by induction, one can also prove the equality

\[
\mathcal{S}_u W[V_1,\ldots,V_n] = W[W[V_u,V_1,\ldots,V_n] + (-1)^1 W[V_u \ast V_1, V_2,\ldots,V_n] + \cdots
+ (-1)^{n-1} W \left[ \cdots \left( (V_u \ast V_1) \ast V_2 \cdots \right) \ast V_{n-1}, V_n \right]
+ (-1)^n W \left[ \cdots \left( (V_u \ast V_1) \ast V_2 \cdots \right) \ast V_n \right]
\]

(5.12)

for any \( n \). Note that parentheses in the right hand side of (5.12) are indispensable since the product \( \ast \) is not associative. This property of \( \mathcal{S} \) is the source of the complicated dependence of \( r_n = \mathcal{S}_v^{n-1} W[V_v] \) on the potential \( v \).
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