Abelian Landau-Pomeranchuk-Migdal Effects
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Abstract

It is shown that the high-energy expansion of the scattering amplitude calculated from Feynman diagrams factorizes in such a way that it can be reduced to the eikonalized form up to the terms of inverse power in energy in accordance with results obtained by solving the Klein-Gordon equation. Therefore the two approaches when applied to the suppression of the emission of soft photons by fast charged particles in dense matter should give rise to the same results. A particular limit of thin targets is briefly discussed.

1 Introduction

In 1953, Ter-Mikaelian [1] noticed that at high energies the longitudinal momentum transferred at each scattering to an electron traversing a medium becomes very small. It enlarges the effective formation length for emitted photons and leads to specific effects in crystals. Soon, Landau and Pomeranchuk [2] remarked that it would result in the suppression of radiation in amorphous media if the formation length becomes large relative to the scattering mean free path of the electron. Nowadays, these effects are confirmed in experiment both for crystals and for amorphous media and are widely discussed.

Several theoretical approaches extending the classical treatments of Refs [1, 2] have been proposed. Migdal [3] has used the Focker-Planck equation and quantified the results for amorphous media so that the corresponding effect is now known as Landau-Pomeranchuk-Migdal (LPM) effect. The polarization of the medium suppresses the soft photon emission as well (Ter-Mikaelian effect, see [4]). The Kharkov group [5] has applied the path-integral technique to treat both effects simultaneously. In connection with recent SLAC experiments [6] on LPM effect, Blankenbecler and Drell [7] have proposed to use at high energies the solution of the Klein-Gordon equation with
account of higher order corrections in expansion of the phase of the wave function in inverse powers of the initial momentum. The purely diagrammatic approach has been advocated in the series of papers done in Orsay [8] which extended the treatment of LPM effect to the non-Abelian case as well, in search for analogous effects in QCD started by Gyulassy and Wang [9]. Somewhat special treatment using the Schrödinger equation and considering compact quark-antiquark-gluon systems has been proposed by B. Zakharov [10] who criticized some approximations used in papers [8]. Soon it was shown [11] that with account of some additional terms omitted in [8] the two approaches are equivalent. Recently, Novosibirsk group [12] carefully considered different limiting cases of LPM effect in QED confronting them to experiment [6]. Even though being common in spirit and close in final results, these approaches use different technique and different models of a medium so that sometimes it is hard to judge the correspondence between them. At the same time, some limiting cases are preferable to treat by either one or another method.

Here, we would like to fill in one link in this chain of proposals and to show that for the Abelian case the solutions of the Klein-Gordon equation obtained in Ref. [7] directly correspond to results of summing up the series of Feynman graphs considered in Ref. [8] in the high energy ($p \rightarrow \infty$) limit i.e. up to the terms of the order of $O(p^{-1})$. In doing this, we show that the problem can be stated in terms of the post-eikonal approximation because of factorization of sums of Feynman diagrams in this limit. To simplify the formulas, we consider charged scalar particles since spin effects can be incorporated in a straightforward manner (see [7]).

2 High-Energy Wave Function in the Post-Eikonal Approximation

Let $p^\mu = (p^0, \vec{p}) = (\sqrt{p^2 + m^2}, 0, p)$ be the incoming four-momentum of a charged scalar particle, and

$$\phi(x) = e^{ip \cdot x} \int d^3q e^{i\vec{q} \cdot \vec{x}} \tilde{\phi}(\vec{p}, \vec{q})$$

(1)
its energy eigenfunction with energy $p^0$ in the presence of a static source

$$A^0(x) = V(x) = \int d^3k e^{ik \cdot x} v(k),$$  \hspace{1cm} (2)

which transfers a total amount of momentum $\vec{q}$ to the particle. We assume the momentum $\vec{k} = (k, k_z)$ provided by the source at each interaction to be much less than the incoming momentum $p$ (more exactly, $\vec{k}^2 \ll 2p \cdot \vec{k}$), so that the post-eikonal approximation

$$\frac{1}{(p + k)^2 - m^2 + i\epsilon} = \frac{1}{2p \cdot \vec{k} + \vec{k}^2 - i\epsilon} \simeq -\frac{1}{2p} \left( \frac{1}{k_z - i\epsilon} - \frac{\vec{k}^2}{2p (k_z - i\epsilon)^2} \right)$$  \hspace{1cm} (3)

can be applied to the particle propagators of the Feynman diagrams. The first term in the expansion constitutes the familiar eikonal approximation [13], but the second term is also needed to describe the Landau-Pomeranchuk-Migdal (LPM) effect. Our aim is to calculate $\phi(x)$ in perturbation theory to an accuracy of order of $1/p$ in all orders.

The triple interaction vertex is simply $2pv(k)$ in momentum space, and the seagull vertex $\frac{1}{2}v(k_1)v(k_2)$ is down by a factor of $p$ compared to the triple interaction. The amplitude in the tree approximation (see Fig. 1) is given by

$$\tilde{\phi}(p; \vec{q}) = \sum_{n=0}^{\infty} \tilde{\phi}_n(p; \vec{q}),$$

$$\tilde{\phi}_n(p; \vec{q}) = \int \left( \prod_{i=1}^{n} d^3k_i \ v(k_i) \right) P_n(p; k_1, \ldots, k_n) \delta \left( \sum_{i=1}^{n} k_i - \vec{q} \right),$$

$$P_n(p; k_1, \ldots, k_n) = \prod_{i=1}^{n} \frac{2p}{(p + K_i)^2 - m^2 + i\epsilon} + \cdots,$$

$$K_i \equiv \sum_{j=1}^{i} k_j,$$

(4)
where the ellipses represent seagull contributions, and \( \tilde{\phi}_0(\vec{p}; \vec{q}) \equiv 1 \). Note that the repeated appearance of \( v(\vec{k})'s \) in (4) is a result of multiple interactions with the same source \( v \); it does not necessarily imply the presence of \( n \) distinct scatterers, though that can be accommodated. Note also that \( P(p; k_1, \cdots, k_n) \) is not symmetric in its variables \( k_i \).

The perturbative expression for the wave function (1) is

\[
\phi(x) = \sum_{n=0}^{\infty} e^{ip \cdot x} \phi_n(x), \\
\phi_n(x) = \int \prod_{i=1}^{n} \left( d^3 k_i \ e^{i\vec{k}_i \cdot \vec{x}} v(\vec{k}_i) \right) P_n(p; k_1, \cdots, k_n). \tag{5}
\]

In the eikonal approximation where the \( O(\vec{k}^2/2p) \) terms in (3) are neglected, \( P_n \) becomes

\[
P_n^{(0)}(p; k_1, \cdots, k_n) = (-)^n \prod_{i=1}^{n} \frac{1}{K_{iz} - i\epsilon}. \tag{6}
\]

If the correction terms in (5) are included to compute \( P_n = P_n^{(0)} + P_n^{(1)}/p + O(1/p^2) \), then the first post-eikonal contribution is

\[
P_n^{(1)}(p; k_1, \cdots, k_n) = -P_n^{(0)}(p; k_1, \cdots, k_n) \frac{1}{2} \sum_{i=1}^{n} \frac{K_i^2}{K_{iz} - i\epsilon} + \cdots. \tag{7}
\]

The corresponding contributions to \( \phi_n \) will be denoted by \( \phi_n^{(0)} \) and \( \phi_n^{(1)} \) respectively.

We shall use \( \langle f(k_1, k_2, \cdots, k_n) \rangle \) to represent the permutation average of any function \( f \). Thus, for example, \( \langle f(k_1, k_2, k_3) \rangle = (f(k_1, k_2, k_3) + f(k_1, k_3, k_2) + f(k_2, k_1, k_3) + f(k_2, k_3, k_1) + f(k_3, k_1, k_2) + f(k_3, k_2, k_1))/3! \). We may and shall replace \( P_n(k_1, \cdots, k_n) \) in (5) by its symmetric form \( \langle P_n(k_1, \cdots, k_n) \rangle \), because this allows us to use the eikonal factorization formula [13] (see also Appendix A)

\[
\langle \prod_{i=1}^{n} \frac{1}{K_{iz} - i\epsilon} \rangle = \frac{1}{n!} \prod_{i=1}^{n} \frac{1}{k_{iz} - i\epsilon} \tag{8}
\]

to compute the eikonal wave function:

\[
\phi_n^{(0)}(x) = \frac{1}{n!} (-i\chi_0(x))^n,
\]
\[ \chi_0(\vec{x}) = \chi_0(z, b) = -i \int d^3k e^{i\vec{k} \cdot \vec{x}} v(k) \frac{1}{k_z - i\epsilon} = \int_{-\infty}^{z} V(z', b) dz', \]
\[ \phi^{(0)}(x) = \exp \left[ -i p^0 x^0 + ipz - i\chi_0(\vec{x}) \right]. \] (9)

When the post-eikonal terms in (3) are included, it is no longer clear how factorization and summation can be carried out. In order to get an idea how the post-eikonal contributions can be organized to yield factorization, let us look at the second-order contribution to \( \phi^{(1)}(\vec{x}) \):

\[ \phi^{(1)}_2(\vec{x}) = \int d^3k_1 d^3k_2 e^{i(k_1 + \vec{k}_2) \cdot \vec{x}} v(k_1) v(k_2) \langle P_2^{(1)}(p; k_1, k_2) \rangle, \]
\[ \langle P_2^{(1)}(p; k_1, k_2) \rangle = -\frac{1}{4} \left( \frac{k_1^2}{(k_1z - i\epsilon)^2} \frac{1}{k_1z + k_2z - i\epsilon} + \frac{1}{k_1z - i\epsilon (k_1z + k_2z - i\epsilon)^2} \right) \]
\[ + \frac{k_2^2}{(k_2z - i\epsilon)^2} \frac{1}{k_1z + k_2z - i\epsilon} + \frac{1}{k_2z - i\epsilon (k_1z + k_2z - i\epsilon)^2} \]
\[ + \frac{1}{2} \frac{1}{k_1z + k_2z - i\epsilon}. \] (10)

The last term comes from the seagull diagram and the first four terms come from the post-eikonal contribution of the \( n! = 2! \) diagrams of the type shown in Fig. 1. The coefficient for \( \vec{k}_1^2 \) is

\[ -\frac{1}{4} \frac{1}{k_1z + k_2z - i\epsilon} \left( \frac{1}{(k_1z - i\epsilon)^2} \frac{1}{k_1z - i\epsilon + \frac{1}{k_2z - i\epsilon} \frac{1}{k_1z + k_2z - i\epsilon}} \right) \]
\[ = -\frac{1}{4} \frac{1}{(k_1z - i\epsilon)^2} \frac{1}{k_2z - i\epsilon}. \] (11)

Similarly, the \( \vec{k}_2^2 \) coefficient is \(- (k_2z - i\epsilon)^{-2} (k_1z - i\epsilon)^{-1}/4 \), and the \( \vec{k}_1 \cdot \vec{k}_2 \) coefficient is \(- [(k_1z - i\epsilon)(k_2z - i\epsilon)(k_1z + k_2z - i\epsilon)]^{-1}/2 \). Combining this last coefficient with the seagull term in (10) reduces \( \vec{k}_1 \cdot \vec{k}_2 \) to the transverse dot product \( k_1 \cdot k_2 \). In short, the rather complicated expression in (10) can be reduced to the vastly simpler form

\[ \langle P_2^{(1)}(p; k_1, k_2) \rangle = -\frac{1}{2!} \frac{1}{k_1z - i\epsilon} \frac{1}{k_2z - i\epsilon} \left( \frac{\vec{k}_1^2}{k_1z + k_2z - i\epsilon} + \frac{\vec{k}_2^2}{k_1z + k_2z - i\epsilon} \right) \]
\[ + 2 \frac{k_1 \cdot k_2}{k_1z + k_2z - i\epsilon}. \] (12)
Moreover, it is shown in Appendix A that this simplification occurs at all $n$, so that

$$
\langle P_n^{(1)}(p; k_1, \ldots, k_n) \rangle = (-1)^{n-1} \frac{1}{n!} \frac{1}{2} \prod_{i=1}^{n} \frac{1}{k_i - i\epsilon} \left( \sum_{i=1}^{n} \frac{\vec{k}_i^2}{k_i} + 2 \sum_{i>j} \frac{k_i \cdot k_j}{k_i + k_j - i\epsilon} \right).
$$

(13)

With this formula, the post-eikonal component of the wave function can be factorized into

$$
\phi_n^{(1)}(\vec{x}) = \phi_n^{(1)}(z, b) = \frac{1}{(n-1)!}(-i\chi_0)^{n-1}\chi_2 + \frac{1}{(n-2)!}(-i\chi_0)^{n-2}(-i\chi_1),
$$

$$
\chi_2(z, b) = \frac{1}{2} \int d^3 k e^{i\vec{k} \cdot \vec{x}} v(\vec{k}) \frac{\vec{k}^2}{(k_z - i\epsilon)^2} = \frac{1}{2} \int_{-\infty}^{z} dz' \nabla^2 \chi_0(z', b),
$$

$$
\chi_1(z, b) = -\frac{i}{2} \int d^3 k d^3 k' e^{i(\vec{k} + \vec{k}') \cdot \vec{x}} v(\vec{k}) v(\vec{k}') \frac{\mathbf{k} \cdot \mathbf{k}'}{(k_z - i\epsilon)(k'_z - i\epsilon)(k_z + k'_z - i\epsilon)}
$$

$$
= \frac{1}{2} \int_{-\infty}^{z} dz' \left( \nabla_{\perp} \chi_0(z', b) \right)^2.
$$

(14)

This factorization allows the sum over $n$ to be carried out to yield, to accuracy $O(p^{-1})$,

$$
\phi(\vec{x}) = \exp \left[ -ip^0 x^0 + ipz - i\chi_0(\vec{x}) - i\frac{1}{p} (\chi_1(\vec{x}) + i\chi_2(\vec{x})) \right].
$$

(15)

This expression agrees with the wave function obtained by Blankenbecler and Drell [7] by solving the Klein-Gordon equation

$$
[(E - V)^2 + \nabla^2 - m^2] \phi(\vec{x}) = 0.
$$

(16)

They looked for a solution of the form $\phi(\vec{x}) = \exp[i\Phi(\vec{x})]$ accurate to order $1/p$ in $\Phi(\vec{x})$, and found (15) to be the solution. As shown in Ref. [7], the extension to spinor particles is straightforward.

In summary, we have demonstrated that factorization of sums of Feynman diagrams does occur even in the post-eikonal approximation, in such a way to enable the perturbation series for the wave function to sum up to an exponential form, a form that agrees with the one obtained directly by solving the Klein-Gordon equation to accuracy $O(p^{-1})$ in the phase of the wave function.
3 Outgoing Wave Function

In the previous section we have computed the energy eigenfunction $\phi(x)$ with incoming momentum $p^\mu$. In a similar way we can compute the energy eigenfunction $\phi'(x)$ with outgoing momentum $p'^\mu$:

$$
\phi'^* (x) = e^{-ip' \cdot x} \int d^3 q' e^{i\vec{q}' \cdot \vec{x}} \tilde{\phi}'^* (\vec{p}'; \vec{q}'),
$$

$$
\tilde{\phi}'^* (\vec{p}'; \vec{q}') = \sum_{n=0}^\infty \tilde{\phi}'_n^* (\vec{p}'; \vec{q}'),
$$

$$
\tilde{\phi}'_n^* (\vec{p}'; \vec{q}') = \int \left( \prod_{i=1}^n d^3 k'_i v(\vec{k}'_i) \right) P'_n (p'; k'_1, \ldots, k'_n) \delta \left( \sum_{i=1}^n \vec{k}'_i - \vec{q}' \right),
$$

$$
P'_n (p'; k'_1, \ldots, k'_n) = \prod_{i=1}^n \frac{2p'}{(p' - K'_i)^2 - m^2 + i\epsilon + \cdots},
$$

$$
K'_i \equiv \sum_{j=1}^i k'_j. \quad (17)
$$

Since the transverse component $p'_t$ of $\vec{p}'$ is not necessarily zero, the post-eikonal expansion (3) must be modified to read

$$
\frac{1}{(p' - k')^2 - m^2 + i\epsilon} \approx \frac{1}{2p'} \left( \frac{1}{k'_z + i\epsilon} + \frac{\vec{k}'^2 - 2p' \cdot k'}{2p' (k'_z + i\epsilon)^2} \right). \quad (18)
$$

Following very similar arguments as before, we finally obtain

$$
\phi'^* (x) = \exp \left[ -ip' \cdot x - i\chi'_0 (\vec{x}) - i\frac{1}{p'} (\chi'_1 (\vec{x}) + i\chi'_2 (\vec{x})) \right],
$$

$$
\chi'_0 (\vec{x}) = i \int d^3 k e^{i\vec{k} \cdot \vec{x}} v(\vec{k}) \frac{1}{k'_z + i\epsilon} = \int_{\vec{z}} V(\vec{z}', \vec{b}) d\vec{z}',
$$

$$
\chi'_2 (z, \vec{b}) = \frac{1}{2} \int d^3 k e^{i\vec{k} \cdot \vec{x}} v(\vec{k}) \frac{k'^2}{(k'_z + i\epsilon)^2} = \frac{1}{2} \int_{\vec{z}} d\vec{z}' \nabla^2 \chi'_0 (\vec{z}', \vec{b}),
$$

$$
\chi'_1 (z, \vec{b}) = \frac{i}{2} \int d^3 k d^3 k' e^{i(\vec{k} + \vec{k}') \cdot \vec{x}} v(\vec{k}) v(\vec{k}') \frac{\vec{k} \cdot \vec{k}'}{(k'_z + i\epsilon)(k'_z + i\epsilon)(k'_z + k'_z + i\epsilon)}. \quad (19)
$$
\[ 2 \int \frac{d^3 k e^{i \vec{k} \cdot \vec{x}} v(\vec{k})}{(k_z + i\epsilon)^2} \]

\[ \frac{1}{2} \int_{\infty}^{-z'} \left[ (\nabla \chi'_0(z', b))^2 + 2 \vec{p}' \cdot \nabla \chi'_0(z', b) \right]. \] \quad (19)

Let us note that the term linear in \( v \) appears in \( \chi'_1 \) because the transverse momentum \( \vec{p}' \) differs from zero while \( \chi_1 \) contains quadratic in \( v \) terms only.

### 4 Multiple Scattering and Bremsstrahlung

The on-shell scattering matrix element, with incoming momentum \( p^\mu \) and outgoing momentum \( p'^\mu \), is given by

\[
m_{fi} = \int d^4 x \phi'^* (x) V(\vec{x}) e^{ip \cdot x} = \int d^4 x e^{-ip' \cdot x} V(\vec{x}) \phi(\vec{x})
\]

\[
= \sum_{n=1}^{\infty} \int \left( \prod_{i=1}^{n} d^3 k_i v(\vec{k}_i) \right) P_n(p'; k_1, \ldots, k_n)(2\pi)^4 \delta^4 \left( p' - p + \sum_{i=1}^{n} k_i \right)
\]

\[
= \sum_{n=1}^{\infty} \int \left( \prod_{i=1}^{n} d^3 k_i v(\vec{k}_i) \right) P_n(p; k_1, \ldots, k_n)(2\pi)^4 \delta^4 \left( p' - p + \sum_{i=1}^{n} k_i \right),
\] \quad (20)

with \( k_0^i = 0 \) because the source is static.

If, as a result of the scattering, a photon of four-momentum \( r^\mu \) and polarization vector \( \varepsilon^*(r) \) is emitted from the scalar particle, then the matrix element is

\[
M_{fi}(r) = -ie \int d^4 x \phi'^* (x) \varepsilon^*(r) \cdot \bar{\Phi} \phi(x)
\]

\[
= \sum_{m,n=0}^{\infty} \tilde{\phi}'_m(\vec{p}'; \vec{q}) \tilde{\varepsilon}^*_n(\vec{r}) (2p + 2 \sum_{i=1}^{n} k_i - r) \tilde{\phi}_n(\vec{p}; \vec{q}) (2\pi)^4 \delta^4 \left( p' + r - p - \sum_{i=1}^{m+n} k_i \right).
\] \quad (21)

The results obtained in the previous sections for the wave functions enable us to claim that up to accuracy \( O(p^{-1}) \), the treatment of Abelian LPM effects by Blankenbecler and Drell [7] who consider the Klein-Gordon equation directly, corresponds to that of R. Baier et al [8] who use the diagrammatic
approach. At high energies and for finite targets the first approach can become preferable because of the usage of simplified expanded propagators and direct treatment of spatial evolution.

It could become especially simple in case of thin targets, where emission at single scattering with some corrections due to double scattering (see Appendix B) dominates, and formulas for $P^{(1)}_1$ and $P^{(1)}_2$ as given by (13) are exploited. The slightly corrected Bethe-Heitler regime is at work.

For finite targets, there are three lengths important in the problem. Those are the target thickness $l$, the mean free path $l_m$ and the formation length $l_f$. For soft photons, $l_f \approx 2\gamma^2/\omega$. Here $\gamma = p^0/m$, $\omega \equiv r^0$ is the photon energy. For screened Coulomb fields in QED, $l^{-1}_m = 4nr_e^2Z^2\ln(183/Z^{1/3})$, where $n$ is the density of the scattering centers, $r_e = \alpha/m \approx 2.8 fm$. Let us consider the case of very thin targets and high energy electrons when $l \ll l_m \ll l_f$. We show that the decline from the Bethe-Heitler formula is determined by the target thickness in units of the mean free path.

Following Ref. [7], one can express the total intensity of the radiation $I_{(tot)}$ as the Bethe-Heitler intensity $I_{(BH)} \approx 4T$ (where $T = \pi l/3l_m$) suppressed by a form factor $F$:

$$I_{(tot)} = I_{(BH)}F.$$  \hspace{1cm} (22)

From formulas (9.16) of Ref. [4], it is easy to find out that for thin targets in the soft photon limit the form factor is given by

$$F \approx 1 - 1.5T.$$  \hspace{1cm} (23)

The correction is small for $T$ small enough and vanishes linearly with target thickness.

At first sight, it seems that the small sizes of hadronic targets favor this limiting case for non-Abelian effects. However, this statement requires further study since the nuclear target thickness, in units of the mean free path $T_{nucl}$ (which is analogous to $T$), could be rather large. Then, inspired by eq. (23) of QED, one would guess that the QCD suppression can become strong, i.e., nuclear LPM effect essential even though the target thickness in units of the formation length is very small.
5 Conclusions

Thus we have shown the equivalence of two approaches to the problems of scattering and radiation of high-energy electrons traversing the dense medium. One of them deals with the solution of the Klein-Gordon equation, and another one with sums of Feynman graphs. Even though the initial formulas have quite different forms, they lead to essentially the same expressions for the solutions of the above problems. This is due to the fact that the sum of Feynman diagrams calculated at \( O(p^{-1}) \) accuracy factorizes still in such a way that it can be represented by the corresponding post-eikonal expansion of the phases of the wave functions.

Besides, we have argued that the nuclear LPM effect can be essential even though the size of nuclear targets in metric units is very small.

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A Factorization in the Post-Eikonal Approximation

The aim of this appendix is to prove eq. (23). It would be useful first to review how (8) is arrived at.

The integral

\[
i^n \int_{R(12\cdots n)} d^n t \exp \left( i \sum_{j=1}^{n} k_{jz} t_j \right) = \frac{1}{\prod_{i=1}^{n} \sum_{j=1}^{n} k_{jz} - i \epsilon}
\]

(24)

gives the left-hand side of (8) before taking the permutation average, where the hyper-triangular integration region \( R(12\cdots n) \) is defined to be \( \{0 \geq t_1 \geq t_2 \geq \cdots \geq t_n > -\infty \} \). The permutation sum on the left of (8) can be
obtained by summing the integral over all permuted regions, whose union
is the hyper-rectangular region \( \{0 \geq t_i > -\infty\} \). Upon integration over the
hyper-rectangle and a division by \( n! \), one obtains the right-hand side of (8)
and hence the eikonal formula.

The \( \vec{k}_m \) coefficient of \( P_n^{(1)}(k_1, \ldots, k_n) \) can be seen from (6) and (7) to be
\[
\frac{1}{2} \frac{\partial}{\partial k_m} P_n^{(0)}(k_1, \ldots, k_n).
\] (25)

Moreover, this relation persists upon permutation averaging. Thus the \( \vec{k}_m \) coefficient of \( \langle P_n^{(1)} \rangle \) can be obtained by differentiating both sides of (8) with
respect to \( k_{mz} \). The result is the one given in (13).

It also follows from (6) and (7) that the \( 2 \vec{k}_\ell \cdot \vec{k}_m \) coefficient of \( P_n^{(1)}(k_1, \ldots, k_n) \)
can be obtained by differentiating \( P_n^{(0)} \) with respect to \( k_{pz} \), where \( p \) is the
larger of the two numbers \( \ell \) and \( m \), viz., the one that stands to the right
of the identity permutation \( (12 \cdots n) \). Upon permutation, this relation still holds provided \( p \) is taken to be the number \( \ell \) or \( m \) that stands to the right
of the other. The permutation sum will now be divided into two sums, one
with \( \ell \) standing to the right of \( m \), and the other with \( m \) standing to the
right of \( \ell \). In the first case, the total integration region is a product of the
triangular region \( \{0 \geq t_m \geq t_\ell > -\infty\} \) with the hyper-rectangular regions
\( \{0 \geq t_i > -\infty\} \) of the remaining \( n - 2 \) variables. The result is
\[
\prod_{i \neq \ell, m}^n \frac{1}{k_{iz} - i\epsilon} \frac{1}{k_{mz} - i\epsilon} \frac{1}{k_{mz} + k_{\ell z} - i\epsilon}.
\] (26)

Upon differentiation with respect to \( k_{pz} = k_{\ell z} \) it produces an additional factor of \( -(k_{\ell z} + k_{mz} - i\epsilon)^{-1} \).

In the second case, \( m \) and \( \ell \) are interchanged, but the differentiation with
respect to \( k_{pz} = k_{mz} \) still produces the same additional factor as before.
Adding up these two cases, we obtain (6) multiplied the additional factor
\( -(k_{\ell z} + k_{mz} - i\epsilon)^{-1} \), which is then equal to the coefficient of \( 2\vec{k}_\ell \cdot \vec{k}_m \) in (13).

As before, when combined with the seagull contributions, \( 2\vec{k}_\ell \cdot \vec{k}_m \) is reduced
to \( 2k_\ell \cdot k_m \) as shown in (13).
B Thin Targets

For thin targets, the photon emission at single scattering (Bethe-Heitler regime) becomes prevailing, though there are corrections due to double-scattering processes. We write down some formulas for this case.

According to eqs (14), (15), (19), (20), and (21), the matrix element of soft photon emission by a scalar electron, scattered once with longitudinal momentum transfer much less than $p$, is given by

$$ M_1 \propto -i \int d^3x \varepsilon^*(r) \cdot p [\chi_0^t + \frac{1}{p}(\chi_1^t + i\chi_2^t)], $$

(27)

where $\chi_j^t = \chi_j + \chi'_j$, and only terms linear in $v$ are kept in $\chi_1$ and $\chi'_1$. It is easy to get

$$ \chi_0^t = 2\pi \int d^2k v(k,0)e^{ik \cdot b} $$

(28)

$$ \chi_1^t + i\chi_2^t = \frac{i}{2} \int d^3k v(k) e^{ik \cdot x} \left[ \frac{\tilde{k}^2}{(k_z - i\epsilon)^2} + \frac{\tilde{k}'^2}{(k_z + i\epsilon)^2} + \frac{\mathbf{p}' \cdot \mathbf{k}}{(k_z + i\epsilon)^2} \right]. $$

(29)

These expressions follow directly both from Feynman diagrams and from the solution of the Klein-Gordon equation obtained in Ref. [7].

The matrix element of photon emission at double scattering is given by eq. (21) for $m + n = 2$. The two terms with $m = 0$ and $n = 0$ describe emission before or after the scattering, while the term with $m = n = 1$ corresponds to photons emitted between two scatterings. It is easy to check that $1/p$-contribution contains the following factor in the integrand:

$$ I^{(2)} \propto k \cdot k' \left[ -\frac{1}{(k_z - i\epsilon)(k'_z - i\epsilon)(k_z + k'_z - i\epsilon)} + \frac{1}{(k_z + i\epsilon)(k'_z + i\epsilon)(k_z + k'_z + i\epsilon)} \right] $$

$$ - \pi i\delta(k_z)\tilde{k}^2 \left[ \frac{1}{(k'_z - i\epsilon)^2} + \frac{1}{(k'_z + i\epsilon)^2} \right] $$

$$ - \pi i\delta(k'_z)\tilde{k}'^2 \left[ \frac{1}{(k_z - i\epsilon)^2} - \frac{1}{(k_z + i\epsilon)^2} \right] $$

$$ - \pi i\delta(k_z)\frac{\mathbf{p}' \cdot \mathbf{k}'}{(k'_z + i\epsilon)^2} - \pi i\delta(k'_z)\frac{\mathbf{p}' \cdot \mathbf{k}}{(k_z + i\epsilon)^2}. $$

(30)
This expression can be also obtained both directly from propagators in Feynman graphs and from phases of the wave functions. Its structure is clear from above formulas.

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