Comment on “Regional Versus Global Entanglement in Resonating-Valence-Bond States”

In a recent Letter [1], Chandran and coworkers study the entanglement properties of valence bond (VB) states. Their main result is that VB states do not contain (or only an insignificant amount of) two-site entanglement, whereas they possess multi-body entanglement. Two examples (“RVB gas and liquid”) are given to illustrate this claim, which essentially comes from a lower bound derived for spin correlators in VB states. While we do not question that two-site entanglement is generically “small” for isotropic VB states, we show in this Comment that (i) for the “RVB liquid” on the square lattice, the calculations and conclusions of Ref. [1] are incorrect. (ii) A simple analytical calculation gives the exact value of the correlator for the “RVB gas”, showing that the bound found in Ref. [1] is tight. (iii) The lower bound for spin correlators in VB states is equivalent to a celebrated result of Anderson dating from more than 50 years ago.

The SU(2) symmetry of VB states guarantees that any two-spin reduced density matrix is a “Werner state” fully characterized by a parameter $p$. The considered pair of spins is entangled if $p > 1/3$. Chandran et al. used quantum information concepts such as monogamy of entanglement and quantum telecloning to obtain bounds on $p$. The number $p$ is simply related to the correlator $\langle S_i S_j \rangle$ between these two spins $1/2$ ($S = \sigma/2$, with $\sigma$ Pauli matrices). We have $\langle S_i S_j \rangle = -3/4p$ (and not “exactly equal to the parameter $p^2$” as stated in Ref. [1]).

(i) The “RVB liquid” is the equal amplitude superposition of all nearest-neighbour (NN) VB coverings of a bipartite lattice. Exact results can be obtained for small sizes $L$ of the square lattice with periodic BC. For $L = 4$, we do not recover the value $p \approx 0.2004$ of Ref. [1], but find $p = 0.4457579115872$ for periodic boundary conditions (BC) and $p = 0.2281115037$ in the interior of a sample with open BC. However, what really matters is the behavior for large $L$. Exact calculations are difficult in this case, but Monte Carlo calculations are possible [2]. We computed the NN correlator $\langle S_i S_j \rangle$ for large samples (up to $L = 128$) on the square lattice with periodic BC. The data of Fig. 1 shows that $p$ is larger than $1/3$ in the thermodynamic limit (we find $p = 0.3946(3)$, resulting in an entanglement of formation of $\approx 0.0215$). Therefore, the “RVB liquid” on the square lattice does possess two-site (NN) entanglement, contrary to the claim of Ref. [1].

(ii) The “RVB gas” is the equal amplitude superposition of all bipartite VB coverings of a bipartite lattice. This is in fact the projection into the singlet sector of the (magnetically ordered) Néel state on this lattice. This observation can be used to calculate $p$ exactly. The total spins $S_A$ and $S_B$ on sublattices A and B are maximal, couple antiferromagnetically and form a singlet (total spin $S = 0$). For a system of $2N$ spins, $S_A = S_B = N/2$. One then easily obtains that $\langle S_i S_j \rangle = -1/4 - 1/(2N)$ if $i$ and $j$ belong to different sublattices. The equivalent exact result $p = 1/3 + 2/(3N)$ shows that the telecloning bound $p \leq 1/3 + 2/(3N)$ is tight. Two-site entanglement is therefore present in any finite “RVB gas” and vanishes only in the thermodynamic limit.

![FIG. 1: Werner parameter $p$ as a function of inverse linear system size $1/L$ for the square lattice “RVB liquid”.](image)

In conclusion, the bound obtained with quantum information techniques [1] has been familiar in the condensed matter context for a long time. Nevertheless, it is interesting to see that it can be derived in a totally different framework. For the two examples chosen in Ref. [1], typical condensed matter methods allowed us to provide in one case an exact solution, and to show that the results of Ref. [1] are incorrect in the other one.

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