Constructing Calabi–Yau metrics from hyperkähler spaces

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Abstract

Recently, a metric construction for Calabi–Yau threefolds from a four-dimensional hyperkähler space by adding a complex line bundle was proposed. We extend the construction by adding a $U(1)$ factor to the holomorphic $(3, 0)$-form and obtain the explicit formalism for a generic hyperkähler base. We find that a discrete choice arises: the $U(1)$ factor can either depend solely on the fiber coordinates or vanish. In each case, the metric is determined by a differential equation for the modified Kähler potential. As explicit examples, we obtain the generalized resolutions (up to orbifold singularity) of the cone of the Einstein–Sasaki spaces $Y_{p,q}$. We also obtain a large class of new singular CY3 metrics with $SU(2) \times U(1)$ or $SU(2) \times U(1)^2$ isometries.

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1. Introduction

Six-dimensional Calabi–Yau (CY3) manifolds [1–3] play an important role in string theory, since they provide natural internal compactifying spaces, giving rise to four-dimensional theories that preserve one quarter of the ten-dimensional supersymmetry. String compactification on the CY3 has mainly been studied based on their topological properties. This is because although it was demonstrated that Ricci-flat and complete metrics exist on the compact CY3 manifolds [3], one does not expect to see an explicit one, aside from the flat metric. This can be seen from the Killing vector analysis. A Killing vector $K_j$ satisfies the equation

$$-\Box K_j - R_{ij} K^j = 0.$$  \hspace{1cm} (1)
Multiply by \( K' \) and integrate over the manifold. For a compact manifold, integration by parts on the first term gives no boundary contribution, and hence one concludes
\[
\int_\mathcal{M} ((\nabla_i K_j)^2 - R_{ij} K' K^j) = 0. \tag{2}
\]
For the Ricci-flat CY3 metrics, it must be that \( \nabla_i K_j = 0 \) pointwise everywhere in the manifold. Leaving aside the trivial possibility that there are flat \( S^1 \) factors, such a covariantly constant vector will not exist. Therefore, there can be no Killing vectors in a non-flat Ricci-flat compact manifold. Without Killing vectors, there are no continuous symmetries. And without the simplifications that result from supposing that a metric has continuous symmetries, it is essentially impossible to solve the Einstein equations.

Thus, explicit metrics on the CY3 with Killing vectors are necessarily non-compact and/or singular. In fact, in string theory compactification, the CY3 spaces can develop singularities at the limiting values of their modulus parameters, where additional massless four-dimensional states will emerge. The simplest example of such a singular metric is the conifold, which is the Ricci-flat metric on the cone over the homogeneous Einstein–Sasaki space \( T^{1,1} = (S^3 \times S^3)/U(1)_{\text{diag}} \). The metric is singular at the vertex of the cone, and as the moduli are moved slightly away from the singular limit, the metric near to the previous conifold point is then smoothed out. It was shown [4] that there are two possible ways of smoothing out the conifold: in one of those the vertex is blown up to an \( S^2 \); in the other, it is blown up to an \( S^3 \).

In the recent past, it was demonstrated by the explicit construction that there exist an infinite number of Einstein–Sasaki metrics with the toric \( U(1)^3 \) isometry, called \( Y_{p,q} \) [5] and \( L_{p,q,r} \) [6, 7]. These provide an infinite number of generalized (toric) conifolds, some of which can be smoothed out completely [8–10] or up to an orbifold singularity [9, 10]. The construction can be generalized to arbitrary \( 2n \) dimensions, since it turns out that the local metrics can be directly obtained from the BPS limits [6, 11, 12] of the Euclidean version of the general Kerr–AdS black hole [13–15] and Kerr–AdS–NUT solutions [12] in arbitrary dimensions: the odd-dimensional ones give rise to Einstein–Sasaki spaces whilst the even-dimensional ones give rise to Calabi–Yau metrics.

The same Killing vector analysis applies to the higher dimensional manifolds with reduced holonomy and one does not expect to see an explicit metric on compact spaces. Examples of complete metrics on non-compact manifolds in higher dimensions with \( G_2 \) and \( \text{spin}(7) \) holonomy were constructed in the later 1980s [16–19]. Inspired by the AdS/CFT correspondence and applications in M-theory compactification, large classes of explicit metrics on \( G_2 \) and \( \text{spin}(7) \) holonomy spaces have been constructed and their applications in string and M-theory have been discussed [20–39].

Although a large number of Calabi–Yau metrics have been constructed, an organizing principle is still lacking, since many of these metrics are discovered serendipitously, or constructed indirectly through the BPS limit of the known Kerr–AdS–NUT solutions. Recently, a new technique was developed in [40] for constructing CY3 metrics, generalizing the construction of the D6-brane wrapping on a two-cycle of a four-dimensional hyperkähler space [41]. The essence of the construction is to build a CY3 metric from a hyperkähler one by adding a complex line bundle. This follows the same line of constructing \((2n+2)\)-dimensional Einstein–Kähler spaces from \((2n)\)-dimensional ones [17]. Differently, in the new construction, the Kähler potential in four dimensions is allowed to be modified by an arbitrary function \( G \). However, the proof of the existence of such CY3 metrics was presented for the \( \mathbb{R}^4 \) base only. What is curious is that the metrics with the asymptotic structure of cones over Einstein–Sasaki spaces, such as \( \mathbb{R}^6 \) or conifolds, are absent from the construction presented in [40].

In section 2, we extend this construction by considering a generic hyperkähler base. Furthermore, we find that the ansatz for the holomorphic \((3, 0)\)-form presented in [40] can
be supplemented with a $U(1)$ factor. This allows us to construct a much wider class of solutions including the ones that have asymptotic cones over Einstein–Sasaki spaces. There are two discrete possibilities for the $U(1)$ factor. One is that it is dependent solely on the fiber coordinates. In this case the equations for the CY3 are reduced to one differential equation for the modified Kähler potential $G$. The other possibility is that the $U(1)$ factor vanishes, for which the metric is determined by a differential equation which is the singular limiting case of the previous one. In section 3, we consider the simplest $\mathbb{R}^4$ base and obtain the CY3 metrics that describe resolutions of the cone over the $Y_{p,q}$ spaces, when the $U(1)$ factor in the $(3,0)$-form is non-vanishing. For the case with the vanishing $U(1)$ factor, we obtained a large class of new singular metrics. In section 4, we consider triaxial basis for the hyperkähler spaces that preserve the $SU(2)$ isometry. Again solutions with or without the $U(1)$ factors were obtained. We conclude our paper in section 5. In appendices A and B, we present some complicated formulae and detailed derivations.

2. The construction

2.1. The ansatz

In this section, we review and then extend significantly the metric construction for the CY3 from $D=4$ hyperkähler spaces, proposed in [40]. Let us consider a generic hyperkähler space in four dimensions with the complex coordinates $z_i, \bar{z}_i$ ($i = 1, 2$) and the Kähler potential $K_0(z_i, \bar{z}_i)$. The metric is given by

$$ds^2 = \tilde{g}_{ij} dz^i d\bar{z}^j,$$

$$\tilde{g}_{ij} = \frac{1}{2} \partial_i \partial_j K_0 \equiv \tilde{g}_{ji}. \quad (3)$$

Since it is Ricci flat, the Ricci form $R^{(1,1)}$ vanishes, i.e.

$$R^{(1,1)} = i \tilde{\partial} \tilde{\partial} \log \sqrt{V} = 0. \quad (4)$$

Here, $V \equiv \det(\tilde{g}_{ij})^2$ is the volume factor, and $\tilde{\partial}$ and $\bar{\tilde{\partial}}$ are the Dolbeault 1-form differential operators defined by

$$\tilde{\partial} \equiv dz^i \partial_{z^i}, \quad \bar{\tilde{\partial}} \equiv d\bar{z}_j \partial_{\bar{z}_j}. \quad (5)$$

Equation (4) implies that log $V$ is the real part of a holomorphic function, or equivalently, $V$ can be the norm of a holomorphic function. The choice for the complex coordinates is not unique since we can always make a holomorphic coordinate transformation $z_i \to z'_i = f_i(z_j)$. Under such a transformation, the volume factor transforms as

$$V \to |T|^{-4} V, \quad (6)$$

where

$$T = \det \left[ \frac{\partial(f^1, f^2)}{\partial(z'^1, z'^2)} \right]. \quad (7)$$

It is easy to see that $T$ can be any holomorphic function. Thus, we can always set $V = 1$ by choosing appropriate complex coordinates. We shall do this for later convenience.

We now consider a complex vielbein basis of the hyperkähler space, which is given by $\tilde{e}^a, \tilde{\epsilon}^a$ ($a = 1, 2$); then the corresponding metric, Kähler form and holomorphic $(2,0)$-form are given by

$$ds^2 = \tilde{e}^1 \tilde{\epsilon}^1 + \tilde{e}^2 \tilde{\epsilon}^2,$$

$$\tilde{J}^{(1,1)} = \frac{1}{2} (\tilde{e}^1 \wedge \tilde{\epsilon}^1 + \tilde{e}^2 \wedge \tilde{\epsilon}^2),$$

$$\tilde{\Omega}^{(2,0)} = \tilde{e}^1 \wedge \tilde{\epsilon}^2. \quad (8)$$

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According to the properties of hyperkähler spaces, both \( J \) and \( \tilde{\Omega} \) are closed, i.e., \( d_J J = d_J \tilde{\Omega} = 0 \). One can now use this structure to construct a CY3. The metric ansatz is given by \[ \frac{d \tilde{s}^2}{\Omega_4} = \frac{dx^2 + h^2 dy^2 + h^{-2} (dx + A)^2}{\tilde{\Omega}_4} = (\delta_{ab} + G_{ab})e^{\mu} \bar{e}^{\bar{b}} + h^2 dy^2 + h^{-2} (dx + A)^2. \] (9)

The metric components are the functions of the \( y, z_i \), \( K \) gradients. According to the properties of hyperkähler spaces, both \( \tilde{\Omega} \) can be diagonalized by a local \( \text{SU}(2) \) transformation \( U^b_\alpha(z_i, \bar{z}_j) \), namely

\[
\left( \begin{array}{cc}
1 + G_{11} & G_{12} \\
G_{21} & 1 + G_{22}
\end{array} \right)
\]

\( U \) for the CY3. This implies that the complex vielbein of the CY3 is given by

\[
e_1 = e^{\lambda_1} \sqrt{\lambda_1} e^{\mu}(U^1_\alpha), \quad e_2 = e^{\lambda_2} \sqrt{\lambda_2} e^{\mu}(U^2_\alpha), \quad e_3 = h dy + ih^{-1} (dx + A),
\]

(12) where \( \kappa = \kappa(\alpha, y, z_i, \bar{z}_j) \) is a real function. Correspondingly, the Kähler form and the \( (3,0) \)-form for the CY3 are given by

\[
J^{(1,1)} = \frac{1}{2} \left( e_1 \wedge e_1 + e_2 \wedge e_2 + e_3 \wedge e_3 \right) = \frac{1}{2} (\delta_{ab} + G_{ab}) \bar{e}^a \wedge \bar{e}^b + dy \wedge (dx + A),
\]

(13)

\[
\Omega^{(3,0)} = e_1 \wedge e_2 \wedge e_3 = f e^{\mu} \bar{e}^1 \wedge \bar{e}^2 \wedge (h dy + ih^{-1} (dx + A)),
\]

(14) where

\[
f = \sqrt{\lambda_1 \lambda_2} = \sqrt{\det(\delta_{ab} + G_{ab})} = \sqrt{1 + G_{11} + G_{22} + G_{11} G_{22} - G_{12} G_{21}}.
\]

(15)

Note that comparing with the ansatz in [40], we have introduced a factor \( e^{\mu} \) for the holomorphic \( (3,0) \)-form. This factor turns out to be crucial for constructing metrics with asymptotic cones over Einstein–Sasaki spaces. Although the metric is independent of the coordinate \( \alpha \), this \( U(1) \) factor can be.

The requirement that the metric (9) be Calabi–Yau becomes the requirement that the above Kähler form and \( (3,0) \)-form are both closed. Note that

\[
d J = \frac{1}{2} \partial_a G_{ab} dy \wedge \bar{e}^a \wedge \bar{e}^b - dy \wedge d_A A = \frac{1}{2} d y \wedge \bar{d} \partial \partial y G - dy \wedge d_A A
\]

(16)

\[
= - \frac{1}{4} dy \wedge (\bar{d} + \partial)(\partial - \bar{d}) \partial \partial y G - dy \wedge d_A A = - \frac{1}{4} d y \wedge d_A (\partial - \bar{d}) \partial \partial y G - dy \wedge d_A A;
\]

then \( d J = 0 \) implies that

\[
A = -\frac{1}{4} (\bar{d} - \partial) \partial \partial y G + \lambda(y, z_i, \bar{z}_j) dy,
\]

(17)

up to some pure gauge terms. Note that \( d_A \) denotes an exterior derivative with respect to \( z_i \) and \( \bar{z}_j \) only. The exterior derivative for the \( (3,0) \)-form is given by

\[
d \tilde{\Omega} = \bar{\partial}(f e^{\mu} h) \wedge \bar{e}^1 \wedge \bar{e}^2 \wedge (ih^{-1} dy + if e^{\mu} \partial \partial y G - \partial \partial y G)
\]

(18)
The vanishing of the terms containing $\mathrm{d}\alpha \wedge \mathrm{d}y$ implies that
\begin{align}
\partial_y (f h^{-1}) - f h \partial_y \kappa &= 0, \quad (19) \\
\partial_y \kappa - \lambda \partial_y \kappa &= 0. \quad (20)
\end{align}

Since by construction only $\kappa$ can depend on $\alpha$, it follows from (19) that we have
\begin{equation}
\kappa = \alpha \kappa_1 (y, z^i, \bar{z}^i) + \kappa_0 (y, z^i, \bar{z}^i).
\end{equation}

Substituting it back to (20), we find
\begin{align}
\partial_y \kappa &= \kappa_1 g^{-1} f^2, \quad (23)
\end{align}

The vanishing of the other terms containing $\mathrm{d}\alpha$ implies that
\begin{equation}
4 \bar{\epsilon}_a \mu \partial_a (g \epsilon^\mu) + g \epsilon^\mu \epsilon^a_a \partial_a \partial_y G = 0,
\end{equation}

from which, we find
\begin{align}
\bar{\epsilon}_a \mu \partial_a \kappa_1 &= 0, \quad (26) \\
\bar{\epsilon}_a \mu \partial_a \partial_y G &= 0. \quad (27)
\end{align}

Since $\kappa_1$ is real, (26) implies that $\kappa_1$ is a constant. It can be set to either 0, or 1, by rescaling the $\alpha$ coordinate. The vanishing of the rest terms implies that
\begin{equation}
4 \bar{\epsilon}_a \mu \partial_a (g \epsilon^\mu) + g \epsilon^\mu \epsilon^a_a \partial_a \partial_y G = 0,
\end{equation}

where we have used equations (22), (23), (25) and (27).

2.2. Case I: $\kappa_1 = 1$

For $\kappa_1 = 1$, equation (28) can be deduced from (23) and (27). Then the CY3 is determined by the following equations:
\begin{align}
g \partial_y g &= 1 + G_{11} + G_{22} + G_{11} G_{22} - G_{12} G_{21}, \quad (29) \\
\bar{\epsilon}_a \mu \partial_a (\partial_y G + 4 \log g + i 4 \kappa_0) &= 0. \quad (30)
\end{align}

and the other quantities are then given by
\begin{equation}
\lambda = \partial_y \kappa_0, \quad \kappa = \alpha + \kappa_0 (y, z^i, \bar{z}^i). \quad (31)
\end{equation}

Note that a partial derivative of (30) with respect to $y$ gives rise to (28). Equation (30) implies that we have $\partial_y G + 4 \log g + i 4 \kappa_0 = H(y, z^i)$, where
\begin{equation}
H \equiv U + i V
\end{equation}
is any holomorphic function on the hyperkähler base. The real part $U = \partial_y G + 4 \log g$ satisfies
\begin{equation}
\bar{\epsilon}_a \mu \bar{\epsilon}_a \left( \partial_y \partial_a - \Gamma^a_{\mu \nu} \partial_a \right) U = 0.
\end{equation}
The imaginary part is then given by
\[ 4\kappa_0 = V = \int (\bar{\partial} + \partial) V = -i \int (\bar{\partial} - \partial) U. \] (34)

However, note that \( \partial_y G \to \partial_y G - U \) and \( \alpha \to \alpha + \kappa_0 \) are the gauge transformations in our setup. Therefore, we can always set \( H = 0 \). Then we have \( \kappa_0 = \lambda = 0 \), and that
\[ g = \exp \left( -\frac{1}{4} \partial_y G \right). \] (35)

It follows from (29) that the system will be determined solely by the following basic equation for \( G \):
\[ \partial_y \left[ \exp \left( -\frac{1}{4} \partial_y G \right) \right] = 2(1 + G_{11} + G_{22} + G_{11}G_{22} - G_{12}G_{21}). \] (36)

The \( U(1) \) factor depends on the fibre coordinate \( \alpha \) only.

2.3. Case II: \( \kappa_1 = 0 \)

When \( \kappa_1 = 0 \), the CY3 is determined by the following equations:
\[ \bar{\partial}^\mu \partial_\mu \left( \log g + i\kappa_0 \right) = 0, \] (37)
\[ \bar{\partial}^\mu \partial_\mu \left[ \partial_y^2 G + 4g^{-2}(1 + G_{11} + G_{22} + G_{11}G_{22} - G_{12}G_{21}) + 4i\lambda \right] = 0, \] (38)

where \( \kappa_0 = \kappa_0(z^\lambda, \bar{z}^\lambda) \) and \( g = g(z^\lambda, \bar{z}^\lambda) \). It follows that both
\[ g e^{4\kappa_1} = H_1(z^\lambda) \] (39)

and
\[ \bar{\partial}_y^2 G + 4g^{-2}(1 + G_{11} + G_{22} + G_{11}G_{22} - G_{12}G_{21}) + 4i\lambda = H_2(y, z^\lambda) = U + iV \] (40)
are the holomorphic functions on the hyperkähler base. Again, the gauge transformations \( \partial_y^2 G \to \partial_y^2 G - U \) and \( \alpha \to \alpha + \Lambda \) imply that we can always set \( H_2 = 0 \). Then we have \( \lambda = 0 \) and
\[ \partial_y^2 G + 4g^{-2}(1 + G_{11} + G_{22} + G_{11}G_{22} - G_{12}G_{21}) = 0. \] (41)

Furthermore, let us consider a holomorphic coordinate transformation \( z^\lambda \to \omega^\lambda (z^\lambda) \). It induces the following transformation on the four-dimensional Kähler potential:
\[ G \to \tilde{G}(y, z^\lambda, \bar{z}^\lambda) = K_0(\omega^\lambda, \bar{\omega}^\lambda) - K_0(z^\lambda, \bar{z}^\lambda) + G(y, \omega^\lambda, \bar{\omega}^\lambda), \] (42)

where \( K_0 \) is the Kähler potential for the hyperkähler base. If the holomorphic functions \( \omega^\lambda (z^\lambda) \) satisfy
\[ \det \left[ \frac{\partial (z^\lambda, \bar{z}^\lambda)}{\partial (\omega^\lambda, \bar{\omega}^\lambda)} \right] = H_1(\omega^\lambda), \] (43)

it follows from (15) that equation (41) becomes
\[ \partial_y^2 \tilde{G} + 4(1 + \tilde{G}_{11} + \tilde{G}_{22} + \tilde{G}_{11}\tilde{G}_{22} - \tilde{G}_{12}\tilde{G}_{21}) = 0. \] (44)

(Note that here we used the fact that we had chosen the complex coordinates for the initial hyperkähler base such that the volume factor is unit, i.e. \( V = 1 \).)

It is clear that there always exist such \( \omega^\lambda \) and \( \bar{\omega}^\lambda \) that satisfy (43). For example, we can take \( z^\lambda = \int H_1(\omega^\lambda) \, d\omega^\lambda \) and \( \bar{z}^\lambda = \bar{\omega}^\lambda \). Also note that
\[ A = -\frac{i}{4} (\bar{\partial} - \partial) \partial_y G(y, \omega^\lambda, \bar{\omega}^\lambda) = -\frac{i}{4} (\bar{\partial} - \partial) \partial_y (G(y, \omega^\lambda, \bar{\omega}^\lambda) + K_0(\omega^\lambda, \bar{\omega}^\lambda)) \]
\[ = -\frac{i}{4} (\bar{\partial} - \partial) \partial_y (\tilde{G}(y, z^\lambda, \bar{z}^\lambda) + K_0(z^\lambda, \bar{z}^\lambda)) = -\frac{i}{4} (\bar{\partial} - \partial) \partial_y \tilde{G}(y, z^\lambda, \bar{z}^\lambda). \] (45)
Thus, the above transformation is a gauge transformation that preserves our initial ansatz. Hence we can set \( g = 1 \) by this gauge transformation. It follows from (37) that \( \kappa_0 = 0 \). Now, we have demonstrated that the system with \( \kappa_1 = 0 \) is determined solely by the following basic equation for \( G \):

\[
\partial^2 y G + 4(1 + G_{11} + G_{22} + G_{11}G_{22} - G_{12}G_{21}) = 0.
\] (46)

It can be regarded as the \( \kappa_1 \to 0 \) limit of (36) if we recover the \( \kappa_1 \) therein. (In the special case when the base space is the flat \( \mathbb{R}^4 \), equation (46) was also obtained in [40], but with a numerical error. The factor ‘8’ in equation (2.51) of [40] should be ‘16’ instead.)

To summarize, we find that the Calabi–Yau metrics depend on a discrete choice of the \( \kappa \) function. One is that \( \kappa = \alpha \), in which case the solution is completely determined by one basic equation for \( G \), given by (36). The other is that \( \kappa = 0 \), in which case the solution is completely determined by the basic equation (46).

### 3. The \( \mathbb{R}^4 \) base

Having obtained the general formalism for constructing the CY3 metrics from any hyperkähler metric in four dimensions, we consider explicit examples in this and the next sections. Note that all the hyperkähler bases are related by a modification of the Kähler potential. Therefore, they are equivalent to each other in our previously general construction. However, since the general basic equation is impossible to solve fully, different choices of hyperkähler base will give different results when we construct the explicit metric in certain simplified ansatz. The simplest hyperkähler space is the Euclidean space \( \mathbb{R}^4 \). An obvious choice is to use the complex coordinates \((z_1, z_2)\) directly, and the corresponding complex vielbein is given by

\[
\bar{\epsilon}^1 = dz_1, \quad \bar{\epsilon}^2 = dz_2.
\] (47)

Alternatively, one can write \( \mathbb{R}^4 \) in terms of the spherical-polar coordinates, i.e.

\[
d s^2 = dr^2 + \frac{1}{4}r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2),
\] (48)

with the following choice of the complex vielbein

\[
\bar{\epsilon}^1 = \bar{\epsilon}^1 + i\bar{\epsilon}^2 = dr + \frac{i}{2}r\sigma_3, \quad \bar{\epsilon}^2 = \frac{1}{2}r(\sigma_1 + i\sigma_2).
\] (49)

Here, we define the \( SU(2) \) Maurer–Cartan forms by

\[
\sigma_1 = \sin \psi d\theta - \sin \theta \cos \psi d\phi,
\]
\[
\sigma_2 = -\cos \psi d\theta - \sin \theta \sin \psi d\phi,
\]
\[
\sigma_3 = d\psi + \cos \theta d\phi.
\] (50)

From the relation

\[
z_1 = r \cos \frac{\theta}{2} \exp \left( \frac{i(\psi + \phi)}{2} \right), \quad z_2 = r \sin \frac{\theta}{2} \exp \left( \frac{i(\psi - \phi)}{2} \right).
\] (51)

one can show that

\[
\bar{\epsilon}_\mu \partial_\mu z_i = 0.
\] (52)

Therefore, the two choices of the complex vielbein (47) and (51) are compatible.
For our purpose, we find that the vielbein (51) is more useful for simplifying equations under the isometry group of the $S^3$ level surfaces. With this choice, the general ansatz is

$$\text{d}s^2 = (1 + G_{11}) \left( \text{d}r^2 + \frac{1}{4} r^2 \sigma_2^2 \right) + \frac{1}{4} (1 + G_{22}) r^2 (\sigma_1^2 + \sigma_2^2) + (G_{12} + G_{21}) \left( \frac{1}{2} r \text{d}r \sigma_1 + \frac{1}{4} r^2 \sigma_3 \sigma_2 \right) - i (G_{12} - G_{21}) \left( \frac{1}{2} r \text{d}r \sigma_2 - \frac{1}{4} r^2 \sigma_3 \sigma_1 \right) + h^2 \text{d}y^2 + \frac{1}{h^2} (\text{d}x + A)^2.$$ 

(53)

The inverse complex vielbein is given by

$$\tilde{e}_1 = \frac{1}{2} (\tilde{E}_1 - i \tilde{E}_2), \quad \tilde{e}_2 = \frac{1}{2} (\tilde{E}_3 - i \tilde{E}_4),$$

(54)

where

$$\tilde{E}_1 = E_r, \quad \tilde{E}_2 = \frac{1}{r} E_\psi, \quad \tilde{E}_3 = \frac{1}{r} \left( \sin \psi \ E_\theta - \frac{\cos \psi}{\sin \theta} E_\phi - \frac{\cos \theta \cos \psi}{\sin \theta} E_\phi \right), \quad \tilde{E}_4 = \frac{1}{r} \left( -\cos \psi \ E_\theta - \frac{\sin \psi}{\sin \theta} E_\phi - \frac{\cos \theta \sin \psi}{\sin \theta} E_\phi \right).$$

(55)

Note that $\Gamma^k_{ij} = 0$ on the Kähler manifold; thus we have

$$G_{ab} = \tilde{e}^a_{\mu}(\partial_\mu \tilde{e}^b) G = \tilde{e}^a_{\mu}(\partial_\mu \tilde{e}^b) \tilde{e} = \tilde{e}^a_{\mu}(\partial_\mu G) = \tilde{e}^a_{\mu}(\partial_\mu G - \Gamma^b_{\mu \nu} \partial_\nu \tilde{e}).$$

(56)

The explicit form of the $G_{ab}$ is presented in (A.1). The 1-form connection $A$ is given by

$$A = -\frac{1}{4} (\partial - \bar{\partial}) G - \frac{1}{4} \left( \tilde{e}^a_{\mu} \tilde{e}^b_{\nu} - \tilde{e}^a_{\nu} \tilde{e}^b_{\mu} \right) \partial_\mu \partial_\nu G = \frac{1}{4} \left( \partial_\mu G - \partial_\nu G \right) \text{d}r + \frac{1}{2} \partial_\phi \text{d}r G \cos \theta \text{d}\theta.$$ 

(57)

Since the general equations are rather complicated, we shall further suppose that the functions $G$ and $h$ depend on the coordinates $(r, y)$ only as in [40]. In such a radial ansatz, the resulting metric has the $SU(2) \times U(1)^2$ isometry. Note that we have $G_{12} = 0$ when $G = G(r, y)$. Thus, the metric ansatz is reduced to the following form:

$$\text{d}s^2 = f_1 \left( \text{d}r^2 + \frac{1}{4} r^2 \sigma_2^2 \right) + \frac{1}{4} f_2 r^2 (\sigma_1^2 + \sigma_2^2) + \frac{f_1 f_2}{g^2} \text{d}y^2 + \frac{g^2}{f_1 f_2} (\text{d}x + f_3 \sigma_3)^2.$$

(58)

where

$$f_2 \equiv 1 + \frac{1}{2r} \partial_r G, \quad g \equiv f h^{-1} = \sqrt{f_1 f_2} h^{-1},$$

$$f_1 \equiv 1 + \frac{1}{4} \partial_r G + \frac{1}{4r} \partial_r G = \frac{1}{2r} \partial_r (r^2 f_2),$$

$$f_3 \equiv \frac{r}{8} \partial_r G = \frac{1}{4} \partial_r (r^2 f_2).$$

(59)
3.1. Case I: $\kappa_1 = 1$

In this case, we have

$$g = \exp \left(-\frac{1}{2} \partial_y G \right),$$

and the system is determined solely by the basic equation

$$\partial_y \left[ \exp \left(-\frac{1}{2} \partial_y G \right) \right] = \frac{1}{2r^3} \partial_y \left[ r^4 \left( 1 + \frac{1}{2r} \partial_y G \right)^2 \right].$$

It is likely difficult to solve this equation fully. We obtain two special solutions: one is just a direct product of the Eguchi–Hanson instanton and $\mathbb{R}^2$; the other is given by

$$\begin{align*}
\text{d}s^2 &= \frac{\text{d}y^2}{W} + \frac{1}{4} W y^2 (\text{d}a - r^2 \sigma_3)^2 + y^2 \left( \frac{\text{d}r^2}{V} + \frac{1}{4} V r^2 \sigma_3^2 + \frac{1}{4} r^2 (\sigma_1^2 + \sigma_2^2) \right),
\end{align*}$$

where

$$W = 1 - \frac{a}{\sqrt{r}}, \quad V = 1 - r^2 - \frac{b}{r^4}. \quad (63)$$

The detailed derivation can be found in appendix B.1. The metric (62) with $b = 0$ was known in [17], describing a higher dimensional generalization of the Eguchi–Hanson instanton, with the $\mathbb{R}^2 \times \mathbb{C}P^2$ topology and an asymptotic $\mathbb{R}^5 / \mathbb{Z}_3$. For $a = 0$, the metric is a cone of $Y^{p,q}$. The general solution describes a resolution of the $Y^{p,q}$ cone, and the detailed global analysis can be found in [8–10].

It should be emphasized that we have obtained only a special solution to the basic equation (61). It would be interesting to find the general solutions and examine the corresponding metrics.

3.2. Case II: $\kappa_1 = 0$

In this case, we may use $f_2$ instead of $G$ as a basic function. Then the basic equation (46) becomes

$$\partial^2 f_2 + \frac{1}{2r} \partial_y \left( \frac{1}{r^3} \partial_y \left( r^4 f_2^2 \right) \right) = 0.$$  

(This case was discussed in [40]. However, there is an error in equation (2.52) for $G$. The constant factor ‘8’ should be ‘16’ instead. This error propagates to the later metric results.)

3.2.1. Some special solutions.

One way to solve (64) is to consider the following ansatz:

$$f_2 = u(y) r^2 + \xi (y);$$

then we find

$$f_1 = 2u \ r^2 + \xi, \quad f_3 = \frac{1}{4} (u_y \ r^2 + \xi_y) \ r^2.$$  

The functions $u$ and $\xi$ satisfy

$$u_{yy} = -16 u^2, \quad \xi_{yy} = -12 u \ \xi.$$  

An immediate solution is the degenerate case

$$u(y) = 0, \quad \xi(y) = c_1 y + c_2.$$  

Besides this case, note that

$$u_{yy} = -\frac{32}{3} (u^3 - c^3),$$  

$$
\begin{align*}
\end{align*}$$

9
\[
d s_6^2 = \frac{3du^2}{32(c^3 - u^3)},
\]

where \(c\) is an integration constant. It would be better if we use \(u\) instead of \(y\) as the coordinate. This implies that \(\xi\) satisfies

\[
8(c^3 - u^3)\xi_{uu} - 12u^2\xi_u + 9u\xi = 0.
\]

The exact solution for \(\xi(u)\) is

\[
\xi(u) = 2F_1\left(\frac{1}{12} (1 - \sqrt{19}), \frac{1}{12} (1 + \sqrt{19}); \frac{2}{3}; \frac{u^3}{c^3}\right) C_1
\]

\[
\quad + 2F_1\left(\frac{1}{12} (5 - \sqrt{19}), \frac{1}{12} (5 + \sqrt{19}); \frac{4}{3}; \frac{u^3}{c^3}\right) u C_2,
\]

where \(2F_1\) is the hypergeometric function. When \(c = 0\), the solution takes the simple form

\[
\xi(u) = \left|u\right|^{-\frac{1}{2}} \frac{\pi}{\sqrt{3}} C_1 + \left|u\right|^{-\frac{1}{2}} \frac{\pi}{\sqrt{3}} C_2.
\]

Now the metric becomes

\[
d s_6^2 = (2ur^2 + \xi) \left(\frac{d\sigma_3}{u^3} + \frac{3(2ur^2 + \xi)(ur^2 + \xi) + 3(2ur^2 + \xi)(ur^2 + \xi)}{32(c^3 - u^3)} du^2\right)
\]

\[
\quad + \frac{1}{(2ur^2 + \xi)(ur^2 + \xi)} \left(\frac{32(c^3 - u^3)}{3} + \frac{1}{4} (r^2 + \xi^2)\right) d\alpha^2.
\]

We must have \(u < c\) to keep the metric real. If \(u > 0\), the range of \(r\) is \((0, \infty)\). If \(u < 0\), then we must have \(\xi > 0\) and the range of \(r\) is constrained. Especially, when \(c \leq 0\), the \(r = \infty\) region is not reachable. The metrics have no asymptotic cone over Einstein–Sasaki spaces and they develop a power-law curvature singularity when \(f_1 f_2 = 0\). This is rather different from the case of \(\kappa_1 = 1\), where the non-vanishing \(g\) in the metric (58) allows non-singular collapsing of the cycles.

### 3.2.2. Separation of variables.
We can also solve equation (64) by separation of variables, namely \(f_2 = u(y)\xi(r)\). Substituting this ansatz to (64), we have

\[
u_{yy} + \frac{3}{2} ku^2 = 0,
\]

\[
\frac{1}{r} \partial_r \left( r^3 \partial_r (r^3 \xi^2) \right) + 3k\xi = 0,
\]

where \(k\) is an arbitrary constant. The first equation implies that

\[
dy^2 = \frac{du^2}{k(u^3 - c^3)}.
\]

The solution for the second equation clearly exists although there appears to have no explicit analytical form, except for the case with \(k = 0\), for which the solution for \(f_2\) is given by (82).

### 3.2.3. The formal general solution.
Expanding \(f_2\) by the Taylor series of \(y\)

\[
f_2(r, y) = \sum_{n=0}^{\infty} u_n(r) y^n,
\]

Substituting this into (64), we find the recursion relation

\[
\frac{1}{2r} \partial_r \left( \frac{1}{r^3} \partial_r \left( \sum_{q=0}^{n} u_q u_{n-q} r^4 \right) \right) + (n + 2)(n + 1) u_{n+2} = 0.
\]
Given the two arbitrarily functions $u_0(r)$ and $u_1(r)$, we can determine all the $u_n$ for $n \geq 2$. The general solution of our system can thus be written formally by the Taylor series of $y$. If we restrict that both $u_0$ and $u_1$ are in the form $u r^2 + \xi_0$, we find that all the $u_n$’s are in the form $u r^2 + \xi$. It is consistent with the solution we obtained previously.

If we require that there be a maximum $\eta_{\text{max}} = N$ for no vanishing $u_n$, we shall obtain polynomial solutions on $y$. There will be $2N + 1$ constraint equations for an $N$th-order polynomial solution in general except for $N = 0$.

$N = 0$: the only equation in this case is

$$\frac{1}{2r} \partial_r \left( \frac{1}{r^3} \partial_r (u_0^2 r^4) \right) = 0.$$  

We find that the solution is

$$f_2 = \sqrt{1 - \frac{a^2}{r^2}} b_0.$$  

$N = 1$: the equations in this case are

$$\frac{1}{2r} \partial_r \left( \frac{1}{r^3} \partial_r (u_0^2 r^4) \right) = 0, \quad \frac{1}{r} \partial_r \left( \frac{1}{r^3} \partial_r (u_0 u_1 r^4) \right) = 0, \quad \frac{1}{2r} \partial_r \left( \frac{1}{r^3} \partial_r (u_1^2 r^4) \right) = 0.$$  

The solution is

$$f_2 = \sqrt{1 - \frac{a^2}{r^2}} (b_0 + b_1 y).$$  

$N = 2$: the equations in this case are

$$\frac{1}{2r} \partial_r \left( \frac{1}{r^3} \partial_r (u_0^2 r^4) \right) + 2u_2 = 0, \quad \frac{1}{r} \partial_r \left( \frac{1}{r^3} \partial_r (u_0 u_1 r^4) \right) = 0, \quad \frac{1}{2r} \partial_r \left( \frac{1}{r^3} \partial_r (u_1^2 r^4) \right) = 0,$$

$$\frac{1}{2r} \partial_r \left( \frac{1}{r^3} \partial_r (2u_0 u_2 + u_1^2 r^4) \right) = 0, \quad \frac{1}{r} \partial_r \left( \frac{1}{r^3} \partial_r (u_1 u_2 r^4) \right) = 0,$$

$$\frac{1}{2r} \partial_r \left( \frac{1}{r^3} \partial_r (u_2^2 r^4) \right) = 0.$$  

We find that there is no consistent solution since the solution for the last three equations is in contradiction with the first equation. Similarly, it can be shown that there is no consistent solution for all finite $N \geq 2$ since the solution for the last $N + 1$ equations are in contradiction with the $(N - 1)$st equation.

4. The triaxial base

In this section, we consider the triaxial hyperkähler metric with the $SU(2)$ isometry. The metric is of the form

$$ds^2 = d\rho^2 + a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + a_3^2 \sigma_3^2,$$

where $a_i$’s are functions of $\rho$ only, and they satisfy the first-order equations

$$\dot{a}_1 = \frac{a_2^2 + a_3^2 - a_1^2}{2a_2 a_3}, \quad \dot{a}_2 = \frac{a_1^2 + a_3^2 - a_2^2}{2a_1 a_3}, \quad \dot{a}_3 = \frac{a_1^2 + a_2^2 - a_3^2}{2a_1 a_2}.$$  

where a dot denotes a derivative with respect to $\rho$. We find that the most general solution can be written as follows:

$$
\frac{d s^2}{4} = \sqrt{W \tilde{W}} \frac{d r^2}{4 r^2} + \sqrt{W \tilde{W} \sigma_1^2} + \sqrt{W \sigma_2^2} + \sqrt{W \sigma_3^2},
$$

(86)

where

$$
W = 1 - \frac{a^4}{r^4}, \quad \tilde{W} = 1 - \frac{b^4}{r^4}.
$$

(87)

It is of interest to note that one can also introduce a cosmological constant and construct the triaxial Einstein–Kähler spaces. The three first-order equations were obtained in [42]. Only two exact solutions were known: one describes a $\mathbb{CP}^2$ [43] and the other a direct product $S^2 \times S^2$ [34, 44]. The metrics are

$$
d s^2_{\mathbb{CP}^2} = d \rho^2 + \sin^2 \rho \sigma_1^2 + \cos^2 \rho \sigma_2^2 + \cos^2 2 \rho \sigma_3^2,
$$

$$
d s^2_{S^2 \times S^2} = d \rho^2 + \sin^2 \rho \sigma_1^2 + \sigma_2^2 + \cos^2 \rho \sigma_3^2.
$$

(88)

For the generic choice of the constant parameters $a$ and $b$, the metric (86) contains a naked power-law singularity at $r = \max(a, b)$. When $b = a$, the metric reduces to the Eguchi–Hanson instanton, given by

$$
d s^2 = \frac{dr^2}{W} + \frac{1}{4} W r^2 \sigma_1^2 + \frac{1}{4} \frac{r^2}{(\sigma_1^2 + \sigma_2^2)}.
$$

(89)

We shall first consider this biaxial hyperkähler base. One way to choose the complex vielbein is

$$
\tilde{\epsilon}_1 = W^{-\frac{1}{2}} d \rho + \frac{1}{2} W \frac{d r}{r} \sigma_3, \quad \tilde{\epsilon}_2 = \frac{1}{2} r (\sigma_1 + i \sigma_2).
$$

(90)

It is easy to see that such a choice will lead to the same radial ansatz as in the $\mathbb{R}^4$ case. Fortunately, there are three possible Kähler structures for a hyperkähler space. For the flat base, all the three choices give same ansatz. For the Eguchi–Hanson base, the remaining two choices are equivalent to each other but inequivalent to (90). The corresponding complex vielbein is given by

$$
\tilde{\epsilon}_1 = \tilde{\epsilon}^1 + i \tilde{\epsilon}^2 = W^{-\frac{1}{2}} d \rho + \frac{1}{2} W \frac{d r}{r} \sigma_3, \quad \tilde{\epsilon}_2 = \tilde{\epsilon}^3 + i \tilde{\epsilon}^4 = \frac{1}{2} r (W \frac{1}{2} \sigma_1 + i \sigma_2),
$$

(91)

where we have permuted the $\sigma_i$’s for convenience. Correspondingly, the inverse complex vielbein is given by

$$
\tilde{\epsilon}_1 = \frac{1}{2} (\tilde{E}_1 - i \tilde{E}_2), \quad \tilde{\epsilon}_2 = \frac{1}{2} (\tilde{E}_3 - i \tilde{E}_4).
$$

(92)

where

$$
\tilde{E}_1 = W \frac{1}{2} E_r, \\
\tilde{E}_2 = \frac{2}{r} E_\psi, \\
\tilde{E}_3 = \frac{2}{r W^2} \left( \sin \psi E_\theta - \frac{\cos \psi}{\sin \theta} E_\phi - \frac{\cos \theta \cos \psi}{\sin \theta} E_\phi \right), \\
\tilde{E}_4 = \frac{2}{r} \left( - \cos \psi E_\theta - \frac{\sin \psi}{\sin \theta} E_\phi + \frac{\cos \theta \sin \psi}{\sin \theta} E_\phi \right).
$$

(93)
It follows from relation (56) that $G_{ab}$ can be obtained. The result is presented in (A.2). The 1-form $A$ is given by

$$A = -\frac{i}{4} (\bar{\partial} - \partial) \partial_y G = -\frac{i}{4} (\bar{\varepsilon}_a^\nu \varepsilon_\mu^a - \bar{\varepsilon}_\mu^a \varepsilon_a^\nu) \partial_y \partial_y G$$

$$= \frac{\partial_y \partial_y G}{8} r W^2 \sigma_3 + \frac{\partial_y \partial_y G}{4} \left( W^2 \cos \psi \sigma_1 + W^{-\frac{1}{2}} \sin \psi \sigma_2 \right)$$

$$+ \frac{\partial_y \partial_y G}{4 \sin \theta} \left( W^2 \sin \psi \sigma_1 - W^{-\frac{1}{2}} \cos \psi \sigma_2 \right)$$

$$- \frac{\partial_y \partial_y G}{4} \left( \frac{2}{r W^2} dr + W^2 \cot \theta \sin \psi \sigma_1 - W^{-\frac{1}{2}} \cot \theta \cos \psi \sigma_2 \right).$$

The metric in the radial ansatz is then given by the following form:

$$ds^2 = f_1 \left( \frac{dr^2}{W} + \frac{1}{4} r^2 \sigma_3^2 \right) + \frac{1}{4} f_2 r^2 (W \sigma_1^2 + \sigma_2^2) + \frac{g^2}{f_1 f_2} (\partial_y + f_3 \sigma_3)^2,$$

where

$$f_2 = 1 + \frac{1}{2r} \partial_y G,$$

$$g = f h^{-1} = \sqrt{f_1 f_2} h^{-1},$$

$$f_1 = 1 + \frac{1}{4} W \partial_y G + \frac{1}{4r} \left( 1 + \frac{a^4}{r^4} \right) \partial_y G = \frac{W^2}{2r} \partial_y \left( r^2 W^2 f_2 \right).$$

$$f_3 = \frac{r W^2}{8} \partial_y \partial_y G = \frac{1}{4} r^2 W^2 \partial_y f_2.$$

The metric has the isometry of $SU(2) \times U(1)$.

### 4.1. Case I: $\kappa_1 = 1$

In this case, we have

$$g = \exp \left( -\frac{1}{4} \partial_y G \right).$$

and the system is determined solely by the basic equation

$$\partial_y \left[ \exp \left( -\frac{1}{2} \partial_y G \right) \right] = \frac{1}{2r^3} \partial_y \left[ (r^4 - a^4) \left( 1 + \frac{1}{2r} \partial_y G \right)^2 \right].$$

We obtain some special solutions. One describes an $\mathbb{R}^2 \times \mathbb{C}P^2$ instanton that is asymptotic to $\mathbb{R}^6 / \mathbb{Z}_3$. Another describes an $\mathbb{R}^2 \times S^2 \times S^2$ instanton that is asymptotic to the cone over $T^{1,1} / \mathbb{Z}_2$. The details are presented in appendix B.2.

### 4.2. Case II: $\kappa_1 = 0$

In this case, we may use $f_2$ instead of $G$ as the basic function. Then the basic equation (46) becomes

$$\partial_y^2 f_2 + \frac{1}{2r} \partial_y \left( \frac{1}{r^3} \partial_y (r^4 - a^4) f_2^2 \right) = 0.$$
4.2.1. Some special solution. If \( f_2 \) depends only on \( y \), the solution is simply
\[ f_2 = c_1 y + c_2 \]
and the metric is
\[
\begin{align*}
\text{d}s^2 &= \left( c_1 y + c_2 \right) \left( \frac{\text{d}r^2}{W} + \frac{1}{4} r^2 \sigma_3^2 \right) + \frac{1}{4} \left( c_1 y + c_2 \right) r^2 \left( W \sigma_1^2 + \sigma_2^2 \right) \nonumber \\
&
+ \left( c_1 y + c_2 \right)^2 \text{d}y^2 + \frac{1}{\left( c_1 y + c_2 \right)^2} \left( \text{d}\alpha + \frac{c_1}{4} r^2 W^2 \sigma_3 \right)^2.
\end{align*}
\] (100)

If \( f_2 = u(y) r^2 \), the basic function is simplified to
\[
u_{yy} = -16 u^2.
\] (101)
It implies
\[
\text{d}y^2 = \frac{3 \text{d}u^2}{32(c^3 - u^3)}.
\] (102)
Then we can use \( u \) instead of \( y \) as the coordinate, and the metric is given by
\[
\begin{align*}
\text{d}s_6^2 &= \left( 1 + W \right) u r^2 \left( \frac{\text{d}r^2}{W} + \frac{1}{4} r^2 \sigma_3^2 \right) + \frac{1}{4} u r^4 \left( W \sigma_1^2 + \sigma_2^2 \right) \nonumber \\
&
+ \frac{3 \left( 1 + W \right) u r^4}{32(c^3 - u^3)} \text{d}u^2 + \frac{1}{\left( 1 + W \right) u r^4} \left( \text{d}\alpha + \sqrt{\frac{2}{3}} (c^3 - u^3) r^4 W^2 \sigma_3 \right)^2.
\end{align*}
\] (103)

4.2.2. The formal general solution. The general solution can formally be expressed by the Taylor series of \( y \) as previously. Expanding \( f_2 \) as
\[
f_2(r, y) = \sum_{n=0}^{\infty} u_n(r) y^n.
\] (104)
we find the recursion relations
\[
\frac{1}{2r} \frac{\partial}{\partial r} \left( \frac{1}{r^3} \frac{\partial}{\partial r} \left( \sum_{q=0}^{n} u_q u_{n-q} (r^4 - a^4) \right) \right) + (n + 2)(n + 1) u_{n+2} = 0.
\] (105)
Given the two arbitrary functions \( u_0(r) \) and \( u_1(r) \), we can determine all \( u_n(n > 1) \)’s by the above recursion relations. Then the general solution of our system can formally be written by the Taylor series of \( y \). If we restrict that both \( u_0 \) and \( u_1 \) are proportion to \( r^2 \), we find that all \( u_n \)’s are proportional to \( r^2 \) by the recursion relations. It is consistent with the previous result. If we require that there is a maximum \( n_{\text{max}} = N \) for no vanishing \( u_n \), we shall obtain polynomial solutions on \( y \). There are \( 2N + 1 \) constraint equations for an \( N \)th-order polynomial solution.
We now examine these equations as follows.
\[ N = 0: \] the only equation in this case is
\[
\frac{1}{2r} \frac{\partial}{\partial r} \left( \frac{1}{r^3} \frac{\partial}{\partial r} \left( u_0^2 (r^4 - a^4) \right) \right) = 0.
\] (106)
We find that the solution is
\[
f_2 = \lambda \sqrt{\frac{r^4 - b^4}{r^4 - a^4}}.
\] (107)
It gives rise to a direct product of the triaxial metric (86) and \( \mathbb{R}^2 \).
$N = 1$: the equations in this case are
\[
\frac{1}{2r} \partial_r \left( \frac{1}{r^3} \partial_r \left( u_0^2 (r^4 - a^4) \right) \right) = 0, \quad \frac{1}{r} \partial_r \left( \frac{1}{r^3} \partial_r \left( u_0 u_1 (r^4 - a^4) \right) \right) = 0, \\
\frac{1}{2r} \partial_r \left( \frac{1}{r^3} \partial_r \left( u_1^2 (r^4 - a^4) \right) \right) = 0.
\]

(108)

The solution is
\[
f_2 = \sqrt{\frac{r^4 - b^4}{r^4 - a^4} (\lambda_0 + \lambda_1 y)}.
\]

(109)

As in the flat base case, we again find that there are no finite polynomial solutions of the $y$ coordinate, for any order $N \geq 2$.

5. Conclusions

In this paper, we have examined the metric construction for Calabi–Yau threefolds proposed in [40]. The essence of the construction is to add a complex line bundle over a four-dimensional hyperkähler structure, with a simple deformation where the four-dimensional Kähler potential is modified. The resulting metric ansatz has at least one Killing direction. It was demonstrated for the $\mathbb{R}^4$ base that the condition for the CY3 metrics could indeed be satisfied.

We extend the construction and obtain the general formalism for a generic hyperkähler base. Furthermore, we find that the ansatz for the holomorphic $(3, 0)$-form should be generalized to have a $U(1)$ factor. This allows us to construct the CY3 metrics that are asymptotic to the cones of Einstein–Sasaki spaces. There can be a discrete choice for the $U(1)$ factor. One is that it depends on the fiber $U(1)$ coordinate only, and consequently the equations of motion are reduced to one differential equation on the modified Kähler potential. The other is that the $U(1)$ factor vanishes. In this case, the metrics are determined by a differential equation that is the singular limit of the previous one.

We then construct explicit metrics with two examples of the hyperkähler bases. One is $\mathbb{R}^4$, and the other is the triaxial metric with $SU(2)$ isometries. In both cases, we obtain explicit cohomogeneity-2 metrics. With the $U(1)$ factor, we obtain a general class of solutions that describe a resolution of the cone over $Y^{p,q}$ spaces. For the case with the vanishing $U(1)$ factor, we obtain singular metrics with no asymptotically conical region. The solutions are governed by two arbitrary functions of the radial variable of the hyperkähler spaces.

The general construction that we have obtained allows one to construct a wide class of CY3 metrics with at least one Killing direction. It is of great interest to investigate whether new complete metrics on the non-compact manifolds can arise.

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Appendix A. Explicit $G_{ab}$

In this appendix we give the explicit expressions for $G_{ab}$ defined by (56). For the $\mathbb{R}^4$ base discussed in section 3, we find that they are given by

$$G_{11} = \frac{1}{4} \partial_r^2 G + \frac{1}{r^2} \partial_\psi^2 G + \frac{1}{4r} \partial_r G,$$

$$G_{22} = \frac{1}{r^2} \partial_\psi^2 G + \frac{1}{r^2 \sin^2 \theta} \partial_\phi^2 G - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \partial_\phi \partial_\psi G + \frac{\cos^2 \theta}{r^2 \sin^2 \theta} \partial_\psi^2 G + \frac{1}{2r} \partial_\rho G + \frac{\cos \theta}{r^2 \sin \theta} \partial_\rho G,$$

$$G_{12} = e^{i\psi} \left[ -\frac{1}{2r \sin \theta} \partial_r \partial_\phi G + \frac{\cos \theta}{2r \sin \theta} \partial_\phi \partial_\psi G - \frac{1}{r^2} \partial_\phi \partial_\psi G - \frac{\cos \theta}{r^2 \sin \theta} \partial_\psi^2 G \right]$$

$$= (G_{21})^*.$$

For the triaxial base discussed in section 4, they are more complicated, given by

$$G_{11} = \frac{1}{4} \left( 1 - \frac{a^4}{r^4} \right) \partial_r^2 G + \frac{1}{r^2} \partial_\psi^2 G + \frac{1}{4r} \left( 1 + \frac{a^4}{r^4} \right) \partial_r G,$$

$$G_{22} = \frac{r^4 - a^4 \cos^2 \psi}{r^2 (r^4 - a^4)} \partial_\psi^2 G - \frac{a^4 \sin 2\psi}{r^2 (r^4 - a^4) \sin \theta} \partial_\phi \partial_\psi G + \frac{a^4 \cos \theta \sin 2\psi}{r^2 (r^4 - a^4) \sin \theta} \partial_\phi \partial_\psi G$$

$$+ \frac{r^4 - a^4 \sin^2 \psi}{r^2 (r^4 - a^4) \sin^2 \theta} \partial_\phi^2 G - \frac{2(r^4 - a^4 \sin^2 \psi) \cos \theta}{r^2 (r^4 - a^4) \sin^2 \theta} \partial_\phi \partial_\psi G$$

$$+ \frac{(r^4 - a^4 \cos^2 \psi) \cos \theta \sin \psi}{r^2 (r^4 - a^4) \sin^2 \theta} \partial_\phi \partial_\psi G + \frac{1}{2r} \partial_\rho G + \frac{(r^4 - a^4 \sin^2 \psi) \cos \theta}{r^2 (r^4 - a^4) \sin \theta} \partial_\rho G$$

$$+ \frac{a^4 \cos \theta \sin \psi}{r^2 (r^4 - a^4) \sin \theta} \partial_\rho \partial_\psi G = a^4 \left( 1 + \cos^2 \theta \right) \sin \psi,$$

$$G_{12} = \frac{\sin \psi}{2r} \partial_r \partial_\phi G - \frac{\cos \psi}{2r \sin \theta} \partial_\phi \partial_\psi G + \frac{\cos \theta \cos \psi}{2r \sin \theta} \partial_\phi G - \frac{\cos \psi}{r^2} \partial_\phi \partial_\psi G$$

$$= \frac{\sin \psi}{r^2 \sin \theta} \partial_\phi \partial_\psi G + \frac{\cos \psi}{r^2 \sin \theta} \partial_\phi \partial_\psi G - \frac{a^4 \sin \psi}{r^2 (r^4 - a^4)} \partial_\rho G$$

$$+ \frac{a^4 \cos \theta \cos \psi}{r^2 (r^4 - a^4) \sin \theta} \partial_\phi G$$

$$+ \frac{i}{2r} \left[ -W \cos \psi \partial_r \partial_\phi G - W \sin \psi \partial_\phi \partial_\psi G + \frac{W \cos \theta \sin \psi \partial_\phi \partial_\psi G}{2r \sin \theta} \right]$$

$$- \frac{\sin \psi}{r^2 W^2 \sin \theta} \partial_\phi \partial_\psi G - \frac{\cos \psi}{r^2 W^2 \sin \theta} \partial_\phi \partial_\psi G = \frac{a^4 \cos \psi}{r^6 W^2} \partial_\phi G$$

$$= \frac{a^4 \sin \psi}{r^6 W^2 \sin \theta} \partial_\phi G + \frac{a^4 \cos \theta \sin \psi}{r^6 W^2 \sin \theta} \partial_\phi G = (G_{21})^*.$$

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Appendix B. Detailed derivation for the $\kappa = 1$ solutions

B.1. The $\mathbb{R}^4$ base

For the $\kappa = 1$ case, the system is reduced to one basic equation, given by (61). In this appendix, we obtain some special solutions by considering the following ansatz:

$$g^2 = u_1(r)(a_3 y^3 + a_2 y^2 + 3a_1^2 y + \tilde{a}_0).$$  \hspace{1cm} (B.1)

Correspondingly, we have

$$\frac{1}{2} G = (2 \log g - 3) y - \sum_{i=1}^{3} y_i \log(y - y_i) + u_2(r)$$

$$= y \log u_1 + y \log (a_3 y^3 + a_2 y^2 + 3a_1^2 y + \tilde{a}_0) - 3 y - \sum_{i=1}^{3} y_i \log(y - y_i) + u_2(r),$$  \hspace{1cm} (B.2)

where $y_i$'s are the roots of the equation $a_3 y^3 + a_2 y^2 + 3a_1^2 y + \tilde{a}_0 = 0$. Substituting this into the basic equation (61), we have

$$\frac{1}{2r^3} \partial_r \left[ r^2 \left( \frac{u'_1}{u_1} \right)^2 \right] = 3a_3 u_1,$$

$$\frac{1}{2r^3} \partial_r \left[ r^2 (u'_2 - r) \frac{u'_1}{u_1} \right] = a_2 u_1,$$

$$\frac{1}{2r^3} \partial_r \left[ r^2 (u'_2 - r)^2 \right] = 3a_1^2 u_1.$$

(B.3)

If $a_3 = 0$, the consistency also requires $a_2 = 0$. Then we get

$$u_1 = \text{constant}, \quad f_2 = 1 - \frac{u'_2}{r} = a_1 \sqrt{\frac{3u_1}{2}} \left( 1 - \frac{a}{r^4} \right).$$ \hspace{1cm} (B.4)

After absorbing the redundant parameter by rescaling, the metric is given by

$$ds^2 = \frac{1}{\sqrt{1 + \frac{a^4}{r^8}}} \left( dr^2 + \frac{1}{4} r^2 \sigma_3^2 \right) + \frac{r^2}{4} \sqrt{1 + \frac{a^4}{r^8}} (\sigma_1^2 + \sigma_2^2) + dy^2 + d\alpha^2.$$ \hspace{1cm} (B.5)

Further making the coordinate transformation

$$(r^4 + a^4) \frac{1}{2} \rightarrow r,$$ \hspace{1cm} (B.6)

we get

$$ds^2 = \frac{dr^2}{1 - \frac{a^4}{r^8} + \frac{r^2}{4} \left( 1 - \frac{a^4}{r^4} \right) \sigma_3^2 + \frac{1}{4} r^2 (\sigma_1^2 + \sigma_2^2)} + dy^2 + d\alpha^2.$$ \hspace{1cm} (B.7)

This is nothing but a direct product of the Eughchi–Hanson instanton and $\mathbb{R}^2$. If $a_3 \neq 0$, we can always set $a_3 = 1$ by the redefinition of $u$. Then the consistency of the equations implies

$$u_2 = \frac{1}{2} r^2 + a_1 \log u_1, \quad a_2 = 3a_1.$$ \hspace{1cm} (B.8)
Thus,

\[ g^2 = u_1(r)(y^3 + 3a_1y^2 + 3a_1^2y + \tilde{a}_0) = u_1(r)(y + a_1)^3 + a_0. \]

\[ r^2 f_2 = r^2 \left( 1 + \frac{1}{2r} \partial_r G \right) = -(y + a_1)r \partial_r \log u_1, \]

\[ r^2 f_1 = \frac{r}{2} \partial_r (r^2 f_2) = -(y + a_1) \frac{r}{2} \partial_r (\partial_r \log u_1), \]  
\[ f_3 = \frac{1}{4} \partial_y (r^2 f_2) = -\frac{r}{4} \partial_y \log u_1, \]

\[ h^2 = \frac{f_1 f_2}{g^2} = (y + a_1)^2 \partial_r (r \partial_r \log u_1) \partial_r \log u_1 = \frac{3(y + a_1)^2}{2[(y + a_1)^3 + a_0]}. \]

Obviously, we can set \( a_1 = 0 \) by a coordinate transformation. Then the metric is given by

\[ ds^2 = -\frac{y}{2} \partial_r \rho \left( \frac{d\rho^2}{r} + \frac{1}{4} r \sigma^2 \right) - \frac{y}{4} \rho (\sigma_1^2 + \sigma_2^2)
+ \frac{3y^2}{2(y^3 + a_0)} dy^2 + \frac{2(y^3 + a_0)}{3y^2} \left( d\alpha - \frac{1}{4} \rho \sigma_3 \right)^2, \]  
\[(B.10)\]

where

\[ \rho = r \partial_r \log u_1. \]  
\[(B.11)\]

Taking \( \rho \) instead of \( r \) as the radial coordinate and supposing

\[ r \partial_r = V(\rho) \partial_\rho, \]  
\[(B.12)\]

the metric becomes

\[ ds^2 = -\frac{y}{2} \left( \frac{d\rho^2}{V} + \frac{V}{4} \sigma^2 \right) - \frac{y}{4} \rho (\sigma_1^2 + \sigma_2^2)
+ \frac{3y^2}{2(y^3 + a_0)} dy^2 + \frac{2(y^3 + a_0)}{3y^2} \left( d\alpha - \frac{1}{4} \rho \sigma_3 \right)^2. \]  
\[(B.13)\]

The first equation in (B.3) becomes

\[ 3u_1 = \frac{V}{2r^2} \partial_\rho (\rho^2) = \frac{\rho V}{r^2}. \]  
\[(B.14)\]

Thus,

\[ \rho = V \partial_\rho \log u_1 = \frac{1}{\rho} \partial_\rho (\rho V) = 4. \]  
\[(B.15)\]

The solution is

\[ V = \left( \frac{1}{4} \rho^3 + 2\rho^2 + b_0 \right) \rho^{-1}. \]  
\[(B.16)\]

Let \( \rho \to -\rho \) for convenience, then the CY3 metric is given by

\[ ds^2 = \frac{y}{2} \left( \frac{d\rho^2}{2\rho - \frac{1}{4} \rho^2 + \frac{b_0}{4}} + \frac{2\rho - \frac{1}{4} \rho^2 + \frac{b_0}{2}}{4} \sigma^2 \right) + \frac{y}{4} \rho (\sigma_1^2 + \sigma_2^2)
+ \frac{3y^2}{2(y^3 + a_0)} dy^2 + \frac{2(y^3 + a_0)}{3y^2} \left( d\alpha + \frac{1}{4} \rho \sigma_3 \right)^2. \]  
\[(B.17)\]

After making some appropriate coordinate transformations, rescaling of the metric and renaming the constants, the metric can be cast into (62).
B.2. The triaxial base

The basic equation for the $\kappa = 1$ solutions is given by (98). Since the structure is quite similar with that of the flat $\mathbb{R}^3$ base, we take the same ansatz as in that case, namely

$$g^2 = u_1(r)(a_3 y^3 + a_2 y^2 + 3a_1^2 y + \tilde{a}_0).$$  \hfill (B.18)

Thus, we have

$$-\frac{1}{2}G = (2 \log g - 3) y - \sum_{i=1}^{3} y_i \log (y - y_i) + u_2(r)$$

$$= y \log u_1 + y \log (a_3 y^3 + a_2 y^2 + 3a_1^2 y + \tilde{a}_0) - 3y - \sum_{i=1}^{3} y_i \log (y - y_i) + u_2(r),$$  \hfill (B.19)

where $y_i$’s are the roots of the equation $a_3 y^3 + a_2 y^2 + 3a_1^2 y + \tilde{a}_0 = 0$. Then, the basic equation implies

$$\frac{1}{2r^3} \partial_r \left[ r^2 W \left( \frac{u'_{11}}{u_{11}} \right)^2 \right] = 3a_1 u_1,$$

$$\frac{1}{2r^3} \partial_r \left[ r^2 W (u'_{2} - r) \frac{u'_{1}}{u_{1}} \right] = a_2 u_1,$$

$$\frac{1}{2r^3} \partial_r \left[ r^2 W (u'_{2} - r)^2 \right] = 3a_1^2 u_1.$$  \hfill (B.20)

If $a_3 = 0$, the consistency also requires $a_2 = 0$. The nontrivial solution is given by

$$u_1 = \text{constant}, \quad f_2 = 1 - \frac{u'_{2}}{r} = a_1 \sqrt{\frac{3a_1}{2}} \sqrt{\frac{4 - b^2}{4 - a^2}}.$$  \hfill (B.21)

After absorbing the redundant parameter by rescaling, the corresponding metric is

$$ds^2 = \frac{1}{W W} dr^2 + \frac{1}{4} r^2 \left( \sqrt{W W} \sigma_i^2 + \sqrt{W W} \sigma_j^2 + \sqrt{W W} \sigma_k^2 \right) + dy^2 + d\alpha^2.$$  \hfill (B.22)

This is just a direct product of the triaxial metric (86) and $\mathbb{R}^2$. It implies that the radial ansatz for the CY3 metric based on the triaxial hyperkähler base will be equivalent to (95). Therefore, up to coordinate transformations which permutate the three $\sigma_i$’s, there will be no further radial ansatz coming from the triaxial base.

For non-vanishing $a_3$, we can set it to unity by the redefinition of $u$. Then the consistency of the equations implies

$$u_2 = \frac{1}{2} r^2 + a_1 \log u_1, \quad a_2 = 3a_1.$$  \hfill (B.23)

Thus, 

$$g^2 = u_1(r)(y^3 + a_1 y^2 + 3a_1^2 y + \tilde{a}_0) = u_1(r)(y + a_1)^3 + a_0),$$

$$r^2 f_2 = r^2 \left( 1 + \frac{1}{2r} \partial_r G \right) = -(y + a_1) r \partial_r \log u_1,$$

$$r^2 f_1 = \frac{1}{2} r W \partial_r (r^2 W f_2) = -\frac{1}{2} (y + a_1) r W \partial_r (r W \partial_r \log u_1),$$

$$f_3 = \frac{1}{4} r^2 W \partial_r f_2 = -\frac{1}{4} r W \partial_r \log u_1,$$

$$h^2 = \frac{f_1 f_2}{g^2} = \frac{(y + a_1)^2 W \partial_r (r W \partial_r u_1) \partial_r \log u_1}{2[(y + a_1)^3 + a_0] r^2 u_1} = \frac{3(y + a_1)^2}{2[(y + a_1)^3 + a_0]}.$$  \hfill (B.24)
Obviously, we can set \( a_1 = 0 \) by coordinate transformation. Then the metric is given by
\[
ds^2 = -\frac{y}{2} \partial_\rho \left( \frac{dr^2}{r W^{\frac{1}{2}}} + \frac{1}{4} W^{\frac{1}{2}} \sigma_{\frac{1}{2}}^2 \right) - \frac{y}{4} \theta \left( W^{\frac{1}{2}} \sigma_{\frac{1}{2}}^2 + W^{-\frac{1}{2}} \sigma_{\frac{1}{2}}^2 \right) + \frac{3y^2}{2(y^3 + a_0)} dy^2 + \frac{2(y^3 + a_0)}{3y^2} \left( \frac{\partial x}{4} - \frac{1}{4} \rho \sigma_{\frac{3}{2}} \right)^2 \tag{B.25}
\]
where
\[
\rho = r W^{\frac{1}{2}} \partial_r \log u_1.
\tag{B.26}
\]
Supposing
\[
r W^{\frac{1}{2}} \partial_r = \xi \partial_\xi = V(\rho) \partial_\rho,
\tag{B.27}
\]
we find
\[
\partial_\rho \log \xi = \frac{1}{V}, \quad r^2 = \frac{4\xi^4 + a^4}{4\xi_2^2}, \quad W^{\frac{1}{2}} = \frac{4\xi^4 - a^4}{4\xi_2^4 + a^2}.
\tag{B.28}
\]
Taking \( \rho \) instead of \( r \) as the radial coordinate, the metric becomes
\[
ds^2 = -\frac{y}{2} \left( \frac{d\rho^2}{V} + \frac{V}{4} \sigma_{\frac{1}{2}}^2 \right) - \frac{y}{4} \theta \left( W^{\frac{1}{2}} \sigma_{\frac{1}{2}}^2 + W^{-\frac{1}{2}} \sigma_{\frac{1}{2}}^2 \right) + \frac{3y^2}{2(y^3 + a_0)} dy^2 + \frac{2(y^3 + a_0)}{3y^2} \left( \frac{\partial x}{4} - \frac{1}{4} \rho \sigma_{\frac{3}{2}} \right)^2 \tag{B.29}
\]
The first equation in (B.20) becomes
\[
3u_1 = \frac{V}{2r^4 W^{\frac{1}{2}}} \partial_\rho (\rho^3) = \frac{16\xi^4 \rho V}{16\xi^8 - a^8}.
\tag{B.30}
\]
Thus,
\[
\rho = V \partial_\rho \log u_1 = \frac{1}{\rho} \partial_\rho (\rho V) - 4 \frac{16\xi^4 + a^4}{16\xi^8 - a^8}.
\tag{B.31}
\]
Then we have
\[
\frac{16\xi^8}{a^8} = \frac{y}{4} \frac{\partial_\rho (\rho V) - \rho + 4}{\partial_\rho (\rho V) - \rho - 4} \Rightarrow \frac{8}{V} = \partial_\rho \log \left( \frac{y}{4} \frac{\partial_\rho (\rho V) - \rho + 4}{\partial_\rho (\rho V) - \rho - 4} \right) = -8 \partial_\rho \left( \frac{y}{4} \frac{\partial_\rho (\rho V)}{\partial_\rho (\rho V) - \rho} \right)^2 - 16
\tag{B.32}
\]
Let
\[
\check{V} = \rho V - \frac{1}{4} \rho^3, \quad \check{\rho} = \rho^2;
\tag{B.33}
\]
then the above equation can be rewritten as
\[
(\check{V} + \frac{1}{4} \check{\rho}^2) \partial_\check{\rho}^2 \check{V} + (\partial_\check{\rho} \check{V})^2 - 4 = 0.
\tag{B.34}
\]
Supposing
\[
\check{V} = b_1 \check{\rho}^2 + b_2 \check{\rho} + b_3 \check{\rho}^{\frac{1}{2}} + b_0,
\tag{B.35}
\]
the corresponding solutions are given by
\[
(1) \check{V} = \pm 2 \check{\rho} + b_0, \quad (2) \check{V} = -48 \check{\rho}^2, \quad (3) \check{V} = -\frac{1}{12} \check{\rho}^2 - 16 \check{\rho}^{\frac{1}{2}}.
\tag{B.36}
\]
The first solution corresponds to the flat base case discussed in the previous section. The second solution gives rise to the metric

\[ ds^2 = \frac{y}{2} \left( \frac{3d\rho^2}{12 - \rho^2} + \frac{12^2 - \rho^2}{12} \sigma^2 \right) + \frac{y}{4} \left[ (12 - \sqrt{12^2 - \rho^2})\sigma_1^2 + (12 + \sqrt{12^2 - \rho^2})\sigma_2^2 \right] \]

\[ + \frac{3\rho^2}{2(y^3 + a_0)} dy^2 + \frac{2(y^3 + a_0)}{3y^2} \left( d\sigma - \frac{1}{4} \rho \sigma \right)^2, \]

(B.37)

With certain coordinate transformation and rescaling of the metric, it can be expressed as

\[ ds^2 = \frac{dy^2}{W} + \frac{1}{4} W y^2 (d\sigma - 2 \sin(2\rho)\sigma)^2 + y^2 ds^2_{\mathbb{CP}^2}, \]

(B.38)

where \( W \) is given by (63) and \( ds^2_{\mathbb{CP}^2} \) is the triaxial \( \mathbb{CP}^2 \) metric given by (88). Thus, the metric describes an \( \mathbb{R}^2 \times \mathbb{CP}^2 \) instanton that is asymptotic to \( \mathbb{R}^6/Z_3 \).

The third solution gives the metric

\[ ds^2 = \frac{y}{2} \left( \frac{4d\rho^2}{8^3 - \rho^2} + \frac{8^2 - \rho^2}{16} \sigma^2 \right) + \frac{y}{4} \left( \frac{\rho^2}{8} \sigma_1^2 + 8\sigma_2^2 \right) \]

\[ + \frac{3\rho^2}{2(y^3 + a_0)} dy^2 + \frac{2(y^3 + a_0)}{3y^2} \left( d\sigma - \frac{1}{4} \rho \sigma \right)^2, \]

(B.39)

With certain coordinate transformation and rescaling of the metric, it can be expressed as

\[ ds^2 = \frac{dy^2}{W} + \frac{1}{4} W y^2 \left( d\sigma - \frac{4}{3} \sin \rho \sigma \right)^2 + \frac{y^2 ds^2_{S^2 \times S^2}}, \]

(B.40)

where \( W \) is given by (63) and \( ds^2_{S^2 \times S^2} \) is the triaxial \( S^2 \times S^2 \) metric given by (88). Thus, the metric describes an \( \mathbb{R}^2 \times S^2 \times S^2 \) instanton that is asymptotic to the cone over \( T^{1,1}/Z_2 \).

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