A Rank 18 Waring Decomposition of $sM_{(3)}$ with 432 Symmetries

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ABSTRACT

The recent discovery that the exponent of matrix multiplication is determined by the rank of the symmetrized matrix multiplication tensor has invigorated interest in better understanding symmetrized matrix multiplication. Author present an explicit rank 18 Waring decomposition of $sM_{(3)}$ and describe its symmetry group.

Determining the complexity of matrix multiplication has been a central problem ever since Strassen showed, in 1969, that one can multiply a pair of $n \times n$ matrices using only $O(n^{2.81})$ arithmetic operations [Strassen 69]. Strassen defined the exponent of matrix multiplication $\omega = \inf \{ \tau | \text{matrix multiplication requires } O(n^{\tau}) \text{ arithmetic operations} \}$, and over the following decades a sequence of results has shown $\omega < 2.3729$ [Bini et al. 79, Schönhage 81, Strassen 87, Coppersmith and Winograd 90, Stothers 10, Williams 11, Le Gall 14]. In 2014, however, it was demonstrated that the only method proving new bounds since 1989, Strassen's laser method applied to the Coppersmith–Winograd tensor, cannot prove an upper bound on $\omega$ better than 2.3 [Ambainis et al. 15].

It is necessary then to pursue other methods to make further progress. Strassen showed that $\omega = \inf \{ \tau | R(M(n)) = O(n^{\tau}) \}$, where $M(n) \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2}$ denotes the structure tensor of the $n \times n$ matrix algebra and $R(M(n))$ its tensor rank [Strassen 69]. One new idea then is to exploit the recent result that this latter quantity is furthermore equal to $\inf \{ \tau | R_s(sM(n)) = O(n^{\tau}) \}$, where $sM(n)$ denotes the result of symmetrizing $M(n)$ and $R_s(sM(n))$ its Waring rank, the smallest $r$ such that $sM(n)$ may be written as the sum of $r$ cubes [Chiantini et al. 17]. I present a Waring decomposition of $sM_{(3)}$ and describe its particularly large symmetry group. I hope the large amount of symmetry of this example will lead to strategies to reduce the search space for larger $n$.

Write $V = \mathbb{C}^n$, and define $sM_{(n)} \in S^3(V^* \otimes V)$ as the tensor corresponding to the symmetric multilinear map $(A, B, C) \mapsto \frac{1}{2} (\text{tr}(ABC) + \text{tr}(ACB))$. Consider the 18 matrices $m_1, \ldots, m_{18}$ below, where $\zeta = e^{2\pi i/3}$ and $a = -2^{1/3}$.

\[
\begin{pmatrix}
1 & -1 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & -\zeta^2 & 1 \\
-\zeta^2 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & -\zeta \\
0 & 0 & 0 \\
-\zeta & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & -1 \\
-1 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & -\zeta^2 \\
0 & 0 & 0 \\
-\zeta^2 & 1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & -\zeta & 1 \\
-1 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & -\zeta^2 \\
1 & -\zeta^2 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & -\zeta \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -\zeta \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -\zeta^2 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Theorem 1. \(sM_{(3)} = \frac{1}{6} \sum_{i=1}^{18} m_i^{(3)}\). That is, the $m_i$ form a rank 18 Waring decomposition of $6sM_{(3)}$.

The group $\Gamma = \text{GL}(V^* \otimes V)$ naturally acts on $S^3(V^* \otimes V)$, and the stabilizer of $sM_{(n)}$ is $\Gamma_{sM_{(n)}} = \text{PGL}(V) \rtimes \mathbb{Z}_2$ [Gesmundo et al. 17]. Here the action of PGL($V$) is induced by its natural action on

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Theorem 2. The symmetry group $\Gamma_S \cong (\mathbb{Z}_2^3 \rtimes \text{SL}(2, \mathbb{F}_3)) \rtimes \mathbb{Z}_2$, which has order 432.

The expression in parentheses is the PGL($V$) action, and the $\mathbb{Z}_2$ is generated by matrix transposition with respect to the basis of the decomposition. To describe the PGL($V$) part of the action, we label each $3 \times 3$ block of matrices with elements of the vector space $\mathbb{F}_3^2$ as follows:

$$
(0,0) \quad (0,1) \quad (0,2) \quad (1,0) \quad (1,1) \quad (1,2) \quad (2,0) \quad (2,1) \quad (2,2).
$$

Then $Z_2^3 \rtimes \text{SL}(2, \mathbb{F}_3)$ acts on the first $3 \times 3$ block as affine transformations of $\mathbb{F}_3^2$ according to this labeling: $Z_3^1$ acts by translation and $\text{SL}(2, \mathbb{F}_3)$ acts by linear transformation. On the second $3 \times 3$ block, $\mathbb{Z}_2^3$ acts trivially and $\text{SL}(2, \mathbb{F}_3)$ again acts as linear transformations. One can alternatively view the action of $Z_2^3 \rtimes \text{SL}(2, \mathbb{F}_3)$ on the second $3 \times 3$ block as equivalent to the action on its normal subgroup $\mathbb{Z}_3^3$ by conjugation.

The decomposition is also closed under complex conjugation, which acts by transposing each $3 \times 3$ block. A Galois-type symmetry like this is not in general in $\Gamma$ and represents another kind of symmetry of decompositions of tensors defined over $\mathbb{Q}$. There are no other nontrivial Galois symmetries for this decomposition, for any such symmetry must be an automorphism of $\mathbb{Q}[[\zeta]]$ fixing $\mathbb{Q}$. Including complex conjugation as a symmetry of the decomposition yields a group of order 864.

**Proof of Theorem 2.** We first describe the representation $\rho : Z_3^3 \rtimes \text{SL}(2, \mathbb{F}_3) \to \text{PGL}(V)$ explicitly by giving the images of a generating set. These elements of PGL($V$) can then be observed to act as claimed on the $3 \times 3$ blocks. Let $e_r$ and $e_d$ denote the generators of $\mathbb{Z}_3^3$ corresponding to translation right and down, respectively, and denote elements of $\text{SL}(2, \mathbb{F}_3)$ by their matrices with respect to the standard basis of $\mathbb{F}_3^2$. Then

$$
\rho(e_r) = \begin{pmatrix} 0 & 0 & 1 \\ \zeta^2 & 0 & 0 \\ 0 & \zeta & 0 \end{pmatrix}, \quad \rho(e_d) = \begin{pmatrix} 0 & 0 & 1 \\ \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \end{pmatrix},
$$

$$
\rho\left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} -\zeta + 1 & \zeta^2 - 1 & 2\zeta + 1 \\ \zeta^2 - 1 & -\zeta + 1 & 2\zeta^2 + 1 \\ -\zeta + 1 & \zeta + 1 & -\zeta - 1 \end{pmatrix},
$$

$$
\rho\left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \begin{pmatrix} -\zeta^2 + 1 & \zeta - 1 & -\zeta^2 + 1 \\ \zeta - 1 & -\zeta^2 + 1 & -\zeta^2 + 1 \\ \zeta - 1 & \zeta - 1 & 2\zeta - 1 \end{pmatrix},
$$

$$
\rho\left( \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}.
$$

It remains to show there are no symmetries of the decomposition other than those claimed. Name the entries of a $3 \times 3$ block by the numbers $1, ..., 9$, like on a telephone. Since all symmetries of $\Gamma_{M_{[1]}}$ preserve matrix rank, we first observe that any symmetry of the decomposition must preserve in particular the first $3 \times 3$ block. This, combined with the fact that there is evidently a matrix transposition in $\Gamma_S$, shows it is sufficient to check the set of PGL($V$) symmetries of the first $3 \times 3$ block as is claimed. Call this group $G$. We wish to show $G = Z_3^3 \rtimes \text{SL}(2, \mathbb{F}_3)$. The first block consists of only rank 1 matrices, so they uniquely determine column vectors up to multiplication by scalars. Let $H$ denote the symmetry group of the corresponding projective configuration of points in $\mathbb{P}^2$. The vectors corresponding to matrices $(1, 3, 4, 6)$ are in general linear position, so each element of $H$ determines at most one element of PGL($V$) which induces it. Hence, the natural homomorphism $G \to H$ is injective, so it suffices to show $H \leq Z_3^3 \rtimes \text{SL}(2, \mathbb{F}_3)$.

First, we show $Z_3^3 \rtimes \text{GL}(2, \mathbb{F}_3)$ is the symmetry group of the combinatorial affine plane consisting of

![Figure 1. The configuration determined by column vectors of the rank one block. This is classically known as the Hesse configuration [Hesse 44].](image-url)
9 points and 12 lines determined by the points and collinearity relations of the configuration (Figure 1). Clearly \( \mathbb{Z}_3^2 \) are symmetries of this configuration. To see that \( \text{GL}(2,\mathbb{F}_3) \) are also symmetries, notice that we may identify points of the configuration with \( \mathbb{Z}_3^2/3\mathbb{Z}_3^2 \), and any line through points in the lattice \( \mathbb{Z}_3^2 \) projects down to one of our 12 lines when modding out by \( 3\mathbb{Z}_3^2 \). Then since \( \text{GL}(2,\mathbb{Z}) \) preserves the lines of \( \mathbb{Z}_3^2 \), it must be that \( \text{GL}(2,\mathbb{F}_3) \) preserves the lines of our configuration \( \mathbb{Z}_3^2/3\mathbb{Z}_3^2 \), as desired. Observe that any symmetry is determined by the image of 3 noncollinear points. For instance, fixing the image of 1, 2, and 5 determines by collinearity the image of 3, 8, and 9, which in turn determines the image of 1, 2, and 5 determines by collinearity the image of 3 noncollinear points.

Now we show that the elements of \( \mathbb{Z}_3^2 \rtimes \text{GL}(2,\mathbb{F}_3) \) where the second factor has determinant \(-1\) do not induce symmetries of the projective configuration of points. Because \( \mathbb{Z}_3^2 \rtimes \text{SL}(2,\mathbb{F}_3) \) does induce such symmetries, it suffices to show the failure for only one element. A convenient choice is the map \( \mathbb{F}_3^2 \rightarrow \mathbb{F}_3^2 \) which interchanges coordinates. The unique matrix taking the general frame \((1, 2, 7, 8)\) to \((1, 4, 3, 6)\) is

\[
\begin{pmatrix}
0 & -\zeta^2 & -1 \\
-\zeta & 0 & -\zeta \\
0 & 0 & \zeta^2
\end{pmatrix},
\]

and one readily checks this matrix does not send, e.g. 3 to any of the other points. Hence \( \mathbb{Z}_3^2 \rtimes \text{SL}(2,\mathbb{F}_3) \leq G \leq H \leq \mathbb{Z}_3^2 \rtimes \text{SL}(2,\mathbb{F}_3) \), and the full symmetry group is \( (\mathbb{Z}_3^2 \rtimes \text{SL}(2,\mathbb{F}_3)) \rtimes \mathbb{Z}_2 \), as claimed.

The Waring decomposition presented here was derived from a numerical decomposition given in [Chiantini et al. 17]. I would like to thank Grey Ballard for his work transforming that numerical decomposition into a sparse numerical one.

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