The monodromy property for $K3$ surfaces allowing a triple-point-free model

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Abstract

The aim of this thesis is to study under which conditions $K3$ surfaces allowing a triple-point-free model satisfy the monodromy property. This property is a quantitative relation between the geometry of the degeneration of a Calabi-Yau variety $X$ and the monodromy action on the cohomology of $X$: a Calabi-Yau variety $X$ satisfies the monodromy property if poles of the motivic zeta function $Z_{X,\omega}(T)$ induce monodromy eigenvalues on the cohomology of $X$.

Let $k$ be an algebraically closed field of characteristic 0, and set $K = k((t))$. In this thesis, we focus on $K3$ surfaces over $K$ allowing a triple-point-free model, i.e., $K3$ surfaces allowing a strict normal crossings model such that three irreducible components of the special fiber never meet simultaneously. Crauder and Morrison classified these models into two main classes: so-called flowerpot degenerations and chain degenerations. This classification is very precise, which allows to use a combination of geometrical and combinatorial techniques to check the monodromy property in practice.

The first main result is an explicit computation of the poles of $Z_{X,\omega}(T)$ for a $K3$ surface $X$ allowing a triple-point-free model and a volume form $\omega$ on $X$. We show that $Z_{X,\omega}(T)$ can have more than one pole. This is in contrast with previous results: so far, all Calabi-Yau varieties known to satisfy the monodromy property have a unique pole.

We prove that $K3$ surfaces allowing a flowerpot degeneration satisfy the monodromy property. We also show that the monodromy property holds for $K3$ surfaces with a certain chain degeneration. We don’t know whether all $K3$ surfaces with a chain degeneration satisfy the monodromy property, and we investigate what characteristics a $K3$ surface $X$ not satisfying the monodromy property should have. We prove that there are 63 possibilities for the special fiber of the Crauder-Morrison model of a $K3$ surface $X$ allowing a triple-point-free model that does not satisfy the monodromy property.
Samenvatting

In deze thesis bestuderen we onder welke voorwaarden $K3$-oppervlakken voldoen aan de monodromie-eigenschap. Deze eigenschap is een kwantitatieve relatie tussen de degeneratie van een Calabi-Yau-variëteit $X$ en de monodromieactie op de cohomologie van $X$: een Calabi-Yau-variëteit voldoet aan de monodromie-eigenschap als elke pool van de motivische zetafunctie $Z_{X,\omega}(T)$ overeenkomt met een monodromie-eigenwaarde op de cohomologie van $X$.

Zij $k$ een algebraïsch gesloten veld van karakteristiek 0 en neem $K = k((t))$. In deze thesis concentreren we ons op $K3$-oppervlakken over $K$ met een strikt-normale-kruisingenmodel zonder drievoudige punten. Dit zijn $K3$-oppervlakken met een strikt-normale-kruisingenmodel zodat drie irreducibele componenten nooit een gemeenschappelijke intersectie hebben. Crauder en Morrison classificeerden deze modellen in twee klassen: de zogenaamde bloempotdegeneraties en ketendegeneraties. Deze klassificatie is heel precies zodat we een combinatie van meetkundige en combinatorische technieken kunnen gebruiken om de monodromie-eigenschap na te gaan.

Het eerste grote resultaat in deze thesis is een expliciete berekening van de polen van $Z_{X,\omega}(T)$ voor een $K3$-oppervlak $X$ met een strikt-normale-kruisingenmodel zonder drievoudige punten en een volumevorm $\omega$ op $X$. We tonen aan dat $Z_{X,\omega}(T)$ meer dan één pool kan hebben. Dit is in contrast met eerdere resultaten: alle Calabi-Yau-variëteiten waarover we tot nu toe wisten dat ze aan de monodromie-eigenschap voldeden, hadden een unieke pool.

We bewijzen verder dat $K3$-oppervlakken met een bloempotdegeneratie aan de monodromie-eigenschap voldoen. We tonen ook aan dat de monodromie-eigenschap geldt voor sommige $K3$-oppervlakken met een ketendegeneratie. We weten echter niet of alle $K3$-oppervlakken met een ketendegeneratie deze eigenschap hebben, en we bestuderen $K3$-oppervlakken die er misschien niet aan voldoen. We geven een exhaustieve lijst van 63 mogelijkheden voor de speciale vezel van het Crauder-Morrisonmodel van een $K3$-oppervlak dat niet aan de mononodromie-eigenschap voldoet.
# Contents

Dankwoord i
Abstract v
Samenvatting vii
Contents ix
Introduction 1

1 Preliminaries 9
   1.1 Notation and conventions 9
   1.2 Calabi-Yau varieties and $K3$ surfaces 10
   1.3 Models and the Kulikov classification 11
   1.4 The Grothendieck ring of varieties 12
      1.4.1 The Grothendieck ring of varieties 12
      1.4.2 The equivariant Grothendieck ring of varieties 16
   1.5 Motivic integration on Calabi-Yau varieties 18
   1.6 Algebraic spaces 19

2 The monodromy property for Calabi-Yau varieties 21
   2.1 History: the $p$-adic and motivic monodromy conjecture 21
      2.1.1 The $p$-adic monodromy conjecture 22
      2.1.2 The motivic monodromy conjecture 26
   2.2 The motivic zeta function for Calabi-Yau varieties 29
   2.3 Monodromy eigenvalues 31
   2.4 The monodromy property for Calabi-Yau varieties 34

3 The Crauder-Morrison classification 37
   3.1 Notation and basic facts 38
   3.2 The Crauder-Morrison classification 39
| 3.3 | Flowers                                                                 | 41 |
| 3.3.1 | Terminology                                                             | 41 |
| 3.3.2 | Classification of the flowers                                          | 42 |
| 3.3.3 | Self-intersection of a double curve in the flower                      | 43 |
| 3.3.4 | Combinatorial relations on the numerical data                          | 45 |
| 3.4  | Flowerpots                                                             | 48 |
| 3.4.1 | Geometry of the pot                                                     | 48 |
| 3.4.2 | Combinations of flowers in a flowerpot degeneration                    | 49 |
| 3.5  | Cycles                                                                 | 50 |
| 3.6  | Chains                                                                 | 51 |
| 3.6.1 | Combinations of flowers in a chain degeneration                       | 51 |
| 3.6.2 | Structure of the chain                                                  | 51 |
| 3.6.3 | Geometry of the components in the chain                                | 52 |
| 3.7  | Euler characteristics                                                  | 59 |
| 4    | Poles of the motivic zeta function                                     | 63 |
| 4.1  | Computation of $[E^\circ]$ in $\mathcal{M}^\hat{\mu}_k$              | 64 |
| 4.1.1 | Local computations                                                     | 65 |
| 4.1.2 | Structure of $\mathcal{Y}_k$                                           | 69 |
| 4.1.3 | Top of a flower                                                        | 73 |
| 4.1.4 | Middle component of a flower                                           | 76 |
| 4.2  | Contribution of a flower to the motivic zeta function                  | 79 |
| 4.3  | Poles of the motivic zeta function                                     | 83 |
| 4.3.1 | Definition of a pole of a rational function over $\mathcal{M}^\hat{\mu}_k$ | 83 |
| 4.3.2 | Computation of the poles of $Z_{X,\omega}(T)$                          | 87 |
| 4.4  | Example of a $K3$ surface where $Z_{X,\omega}(T)$ has two poles        | 90 |
| 5    | $K3$ surfaces satisfying the monodromy property                        | 93 |
| 5.1  | Strategy and first results                                              | 94 |
| 5.1.1 | Strategy                                                               | 94 |
| 5.1.2 | Minimal weight                                                         | 95 |
| 5.1.3 | Conic-flowers                                                         | 96 |
| 5.2  | Flowerpot degenerations                                                | 97 |
| 5.2.1 | Flowers of type $4C'$                                                  | 97 |
| 5.2.2 | Easy cases                                                             | 98 |
| 5.2.3 | Rational, non-minimal ruled pot                                        | 99 |
| 5.3  | Chain degenerations with an extra condition                            | 108 |
| 6    | Future research: a proof or a counterexample?                          | 113 |
| 6.1  | Some more results                                                      | 114 |
| 6.1.1 | Index of a variety                                                     | 114 |
| 6.1.2 | Number of $(-2)$-curves on a rational, ruled surface                   | 116 |
| 6.1.3 | Number of blow-ups in the chain                                        | 123 |
A. Formulas for the contribution of a flower and Python code
   A.1 Formulas for the contribution of a flower
   A.2 Python code
      A.2.1 Implementation of relevant functions
      A.2.2 Computation of the contribution of a flower, that is not a conic-flower
      A.2.3 Computation of the contribution of a conic-flower

B. List of combinatorial countercandidates and Python code
   B.1 $\beta = k + 1$, and $V_{k+1}$ is a rational, ruled surface
   B.2 $\beta = k + 1$, and $V_{k+1}$ is an elliptic, ruled surface
   B.3 $\beta = k + 1$, and $V_{k+1} \simeq \mathbb{P}^2$
   B.4 $V_{k+1} \simeq \mathbb{P}^2$, and $N(V_{k+1}) = 3N$
   B.5 $V_{k+1} \simeq \mathbb{P}^2$, and $N(V_{k+1}) = \frac{3}{2}N$
   B.6 $V_{k+1} \simeq \Sigma_2$, and $N(V_{k+1}) = 4N$
   B.7 $V_{k+1} \simeq \Sigma_2$, $\beta = k - 1$, and $N(V_{k+1}) = 2N$
   B.8 $V_{k+1} \simeq \Sigma_2$, and $N(V_{k+1}) = \frac{4}{3}N$
   B.9 $V_{k+1}$ is a rational, ruled surface, $\beta = k$, and $N(V_{k+1}) = 2N$
   B.10 $V_{k+1}$ is an elliptic, ruled surface, and $N(V_{k+1}) = 2N

Bibliography
Introduction

In this thesis, we study the monodromy property for a specific type of Calabi-Yau variety: for $K3$ surfaces allowing a triple-point-free model. The monodromy property is a precise relation between the geometry of snc-models of a Calabi-Yau variety $X$ and the cohomology of $X$. Inspired by the $p$-adic and motivic monodromy conjecture, Halle and Nicaise formulated this property in [HN11].

There is a good reason why we study the specific class of $K3$ surfaces allowing a triple-point-free model: Crauder and Morrison classified triple-point-free snc-models of $K3$ surfaces, and such a classification is an extremely useful tool to verify the monodromy property in practice. To our knowledge, this is the only systematic classification of snc-models of Calabi-Yau varieties without a semi-stability condition.

Fix an algebraically closed field $k$ of characteristic 0, and set $K = k((t))$ and $R = k[[t]]$. Let us now explore the results obtained in this thesis in more detail.

The first ingredient: the monodromy property for Calabi-Yau varieties

$p$-adic zeta functions and the $p$-adic monodromy conjecture

For a very long time, mathematicians have been interested in solutions $x \in \mathbb{Z}^m$ of congruences of the form

$$f(x) \equiv 0 \mod n,$$

for $f$ a polynomial in $\mathbb{Z}[x_1, \ldots, x_m]$ and $n$ an integer. For example Euler, Legendre and Gauss made significant progress for quadratic polynomials $f$ in one variable. In 1796, Gauss proved the law of quadratic reciprocity, which
makes it possible to determine for any odd prime $p$, and for any quadratic equation of the form $x^2 \equiv a \mod p$, whether it has a solution or not.

Another natural question mathematicians wonder about is: how many solutions does the equation $f(x) \equiv 0 \mod n$ have? This problem can be simplified, thanks to the Chinese remainder theorem: we actually only need to study the case where $n$ is a power of a prime. Let’s denote by $N_d$ the number of solutions of $f(x) \equiv 0 \mod p^{d+1}$. One way to study the number $N_d$ for $d$ big, is to study the Poincaré series $P_f(T)$, which is defined as the generating power series

$$P_f(T) = \sum_{d \in \mathbb{Z}_{\geq 0}} N_d T^d.$$ 

Another, closely related function is the $p$-adic zeta function $Z_f(s)$, which also encodes the information of all $N_d$ for $d \in \mathbb{Z}_{\geq 0}$.

In [BS66], Borevich and Shafarevich conjectured that $P_f(T)$ is a rational function. In 1974, Igusa gave a proof of this fact in [Igu74] and [Igu75], by proving that $Z_f(s)$ is a rational function in $p^{-s}$, which implies that $P_f(T)$ is rational too. The rationality of these functions implies that there is a meromorphic continuation to $\mathbb{C}$, and therefore we can talk about poles of $P_f(T)$ and $Z_f(s)$. The poles of $Z_f(s)$ give valuable information on the asymptotic behaviour of $N_d$ when $d$ tends to infinity, see for example [Seg11] for a nice explanation.

With the $p$-adic monodromy conjecture, Igusa formulated a fascinating conjecture, linking the arithmetical concept of the $p$-adic zeta function to something of more differential-topological nature: the local monodromy eigenvalues of the analytic function $f: \mathbb{C} \to \mathbb{C}$. Monodromy eigenvalues give valuable information on the singularities of the hypersurface in $\mathbb{C}^n$ defined by $f = 0$. The $p$-adic monodromy conjecture states that every pole of the $p$-adic zeta function corresponds to a local monodromy eigenvalues.

The $p$-adic monodromy conjecture has been proven in some significant cases, for example for polynomials in two variables [Loe88], and for homogeneous polynomials in three variables [RV01] [ACLM02], but the general case remains wide open.

### Motivic zeta functions and the motivic monodromy conjecture for hypersurface singularities

Inspired by Kontsevich’s theory of motivic integration, Denef and Loeser introduced in [DL98] the motivic zeta function $Z_f^{\text{mot}}(T)$ associated with a polynomial $f \in k[x_1, \ldots, x_m]$. This function is some kind of ‘super $p$-adic zeta function’, because it captures the $p$-adic zeta functions $Z_f(s)$, for almost all
primes $p$. It is a more geometrical invariant of the polynomial $f$ than the $p$-adic zeta function.

The motivic zeta function $Z_f^{\text{mot}}(T)$ is a formal power series with coefficients in a certain Grothendieck ring of varieties. The Denef-Loeser formula, proved in [DL01], shows that the motivic zeta function is in fact a rational function. This formula makes it even possible to explicitly compute the motivic zeta function from the data of an embedded resolution of singularities of the hypersurface in $\mathbb{A}^m_k$ defined by $f = 0$.

The $p$-adic monodromy conjecture got a motivic upgrade. This conjecture is called Denef and Loeser’s motivic monodromy conjecture for hypersurface singularities, and states that every pole of the motivic zeta function corresponds to a local monodromy eigenvalue.

A proof of the motivic monodromy conjecture would immediately imply a proof of the $p$-adic monodromy conjecture. And so far, proofs of specific cases of the $p$-adic monodromy conjecture seem to generalize to the motivic monodromy conjecture without major adaptations.

For more information on the $p$-adic and motivic monodromy conjecture, we refer to Section 2.1.

**Motivic zeta functions for Calabi-Yau varieties**

For every pair $(X, \omega)$, where $X$ is a Calabi-Yau variety over $K$ and $\omega$ a volume form on $X$, Halle and Nicaise formulated in [HN11] the invariant $Z_{X, \omega}(T)$, called the motivic zeta function associated with $(X, \omega)$. It is defined as a formal power series

$$Z_{X, \omega}(T) = \sum_{d \in \mathbb{Z}_{>0}} N_d T^d,$$

where $N_d$ is an element in a certain localized Grothendieck ring of varieties. When $K = \mathbb{C}((t))$, the coefficient $N_d$ measures the space of $\mathbb{C}((\sqrt[2]{t}))$-rational points on $X$. The motivic zeta function $Z_{X, \omega}(T)$ is a natural analog of Denef and Loeser’s motivic zeta function for hypersurface singularities.

The Denef-Loeser formula, which allows to compute $Z_f^{\text{mot}}$ in terms of the data of an embedded resolution of singularities of $f$, has an analog in the context of Calabi-Yau varieties. Nicaise and Sebag in [NS07], and Bultot and Nicaise in [BN16], proved a formula for $Z_{X, \omega}(T)$ in terms of the degeneration of $X$ at $t = 0$, or more precisely, in terms of an snc-model of $X$. An snc-model of $X$ is a regular, proper $R$-scheme $\mathcal{X}$ such that $\mathcal{X} \times_R K \simeq X$ and such that the special fiber $\mathcal{X}_k = \mathcal{X} \times_R k$ is a strict normal crossings divisor on $\mathcal{X}$. This formula immediately implies that $Z_{X, \omega}(T)$ is a rational function. Moreover, from the
data of an \textit{snc}-model of $X$, it is possible to explicitly compute the motivic zeta function $Z_{X,\omega}(T)$ with this formula.

As in the context of hypersurface singularities, we are interested in the poles of $Z_{X,\omega}(T)$. So far, we have results for abelian varieties \cite{HN11}, and for Calabi-Yau varieties with a so-called equivariant Kulikov model \cite{HN17}. In both cases, the motivic zeta function $Z_{X,\omega}(T)$ has a unique pole. In this thesis, we will see that the motivic zeta function of a $K3$ surface with a triple-point-free model can have more than one pole.

\textbf{The monodromy property for Calabi-Yau varieties}

Of course, we wonder whether there is an analog of the motivic monodromy conjecture for hypersurface singularities in the context of Calabi-Yau varieties as well. A Calabi-Yau variety is said to satisfy the \textit{monodromy property}, if every pole of the motivic zeta function $Z_{X,\omega}(T)$ corresponds to a monodromy eigenvalue. Halle and Nicaise proved that abelian varieties, and Calabi-Yau varieties with an equivariant Kulikov model, satisfy the monodromy property, in \cite{HN11} and \cite{HN17} respectively. In this thesis, we will see more examples of Calabi-Yau varieties that satisfy the monodromy property.

For more information on the monodromy property, we refer to Chapter 2.

\textbf{The second ingredient: the Crauder-Morrison classification}

In \cite{CM83}, Crauder and Morrison study triple-point-free \textit{snc}-models of proper, smooth surfaces with trivial pluricanonical bundle. These models are strict normal crossings models such that three irreducible components of the special fiber never meet simultaneously. They show that such a model can always be birationally modified to a so-called Crauder-Morrison model, which is classified in one of the following three classes:

- flowerpot degenerations,
- cycle degenerations, and
- chain degenerations.

The Crauder-Morrison classification specifies ‘building blocks’ of which the special fiber of a Crauder-Morrison model is made: flowers, flowerpots, cycles and chains. They also classify the flowers into 21 classes. Below, we find a picture of the dual graph of a possible chain degeneration with three flowers.
The monodromy property for $K3$ surfaces allowing a triple-point-free model

The Denef-Loeser formula for Calabi-Yau varieties makes it possible to explicitly compute the motivic zeta function $Z_{X,\omega}(T)$ in terms of the data of an $snc$-model of the Calabi-Yau variety $X$. This makes a classification of $snc$-models like the one of Crauder and Morrison so useful: it gives us plenty of information that can be used to verify the monodromy property in practice. Therefore, we are interested in Calabi-Yau varieties to which we can apply the Crauder-Morrison classification. These are exactly the abelian surfaces allowing a triple-point-free model, and $K3$ surfaces allowing a triple-point-free model.

We already know that abelian surfaces satisfy the monodromy property by [HN11], so we will focus on $K3$ surfaces allowing a triple-point-free model.

Since the Crauder-Morrison classification is formulated for more general surfaces, we investigate whether the classification can be refined for $K3$ surfaces. Our most important result is the following theorem.
Theorem 3.5.2. Let $X$ be a K3 surface over $K$ and let $\mathcal{X}$ be a Crauder-Morrison model of $X$. Then $\mathcal{X}$ is either a flowerpot degeneration or a chain degeneration, but not a cycle degeneration.

The Crauder-Morrison classification and the description of the building blocks are very precise, which allows us to use a combination of geometrical and combinatorial arguments to verify the monodromy property. The first major result is the computation of the poles of the motivic zeta function $Z_{X,\omega}(T)$ for a K3 surface $X$ allowing a triple-point-free model and a volume form $\omega$ on $X$.

Theorem 4.3.8. Let $X$ be a K3 surface over $K$. Let $\mathcal{X}$ be a Crauder-Morrison model of $X$ over $R$ with special fiber $\mathcal{X}_k = \sum_{i \in I} N_i E_i$. Let $\omega$ be a volume form on $X$ and let $(N_i, \nu_i)$ be the numerical data of $E_i$, for every $i \in I$.

The rational number $q \in \mathbb{Q}$ is a pole of $Z_{X,\omega}(T)$ if and only if there exists an element $i \in I$ with $q = -\nu_i/N_i$ and such that

(i) either $\rho_i$ is minimal,

(ii) or $E_i$ is the top of a conic-flower.

Moreover, in case (i), $q$ is a pole of order 1 if $\mathcal{X}$ is a flowerpot degeneration, and of order 2 if $\mathcal{X}$ is a chain degeneration. In case (ii), $q$ is a pole of order 1.

For the definition of $\nu_i, \rho_i$, and a conic-flower, we refer to the main text of this thesis.

In particular, we see that if the Crauder-Morrison model $\mathcal{X}$ has a conic-flower, then $Z_{X,\omega}(T)$ has more than one pole. This is in contrast to abelian varieties and Calabi-Yau varieties with an equivariant Kulikov-model, because these kinds of Calabi-Yau varieties have a motivic zeta function with a unique pole.

To verify the monodromy property for K3 surfaces allowing a triple-point-free model, we need to check whether every pole of the motivic zeta function $Z_{X,\omega}(T)$ induces a monodromy eigenvalue. The Crauder-Morrison classification also serves as a tool to compute monodromy eigenvalues, and we find the following theorem.

Theorem 5.2.1. Let $X$ be a K3 surface over $K$ with Crauder-Morrison model $\mathcal{X}$. If $\mathcal{X}$ is a flowerpot degeneration, then $X$ satisfies the monodromy property.

For chain degenerations, the situation is more complex. However, we have a partial result:

Theorem 5.3.1. Let $X$ be a K3 surface over $K$ with Crauder-Morrison model $\mathcal{X}$. If $\mathcal{X}$ is a chain degeneration satisfying some extra conditions, the K3 surface $X$ satisfies the monodromy property.
Our results are summarized below.

We don’t know whether all $K_3$ surfaces with a chain degeneration satisfy the monodromy property. The difficulty is that we don’t know which special fibers can occur as a chain degeneration of a $K_3$ surface. In the last chapter, we will produce a list of 63 ‘combinatorial countercandidates’. These are combinatorial descriptions of special fibers such that if there exists a $K_3$ surface $X$ over $K$ with an snc-model $\mathcal{X}$ such that the special fiber satisfies the properties of one of these 63 combinatorial countercandidates, then $X$ does not satisfy the monodromy property. Conversely, if there exists a $K_3$ surface $X$ over $K$ allowing a triple-point-free model that does not satisfy the monodromy property, then the special fiber $\mathcal{X}_k$ of its Crauder-Morrison model $\mathcal{X}$ satisfies the properties of one of these 63 combinatorial countercandidates.

Outline of the thesis

Chapter 1. Preliminaries

We start by fixing notation and terminology, and we explain the minimal background needed for this thesis.

Chapter 2. The monodromy property for Calabi-Yau varieties

In the second chapter, we give an introduction to the monodromy property for Calabi-Yau varieties. First, we will motivate the subject by giving some historical context: we will discuss the $p$-adic and motivic monodromy conjecture for hypersurface singularities. Then we will introduce the two main ingredients of the monodromy property, namely the motivic zeta function, and monodromy eigenvalues. We will formulate the monodromy property and give the results known so far.
Chapter 3. *The Crauder-Morrison classification*

In the third chapter, we will give the Crauder-Morrison classification of triple-point-free \textit{snc}-models of smooth, proper surfaces with trivial pluricanonical bundle. This classification is one of the major tools in this thesis. We will also refine the classification specifically for $K3$ surfaces.

Chapter 4. *Poles of the motivic zeta function*

This chapter is devoted to the first main theorem in this thesis: for a $K3$ surface $X$ allowing a triple-point-free model and a volume form $\omega$ on $X$, we find a set $S^\dagger \subset \mathbb{Z} \times \mathbb{Z}_{>0}$ such that the motivic zeta function $Z_{X,\omega}(T)$ is an element of $\mathcal{M}_\mu^\hat{\hat{}}_k \left[ T, \frac{1}{1-L^aT^b} \right]_{(a,b) \in S^\dagger}$. Moreover, we prove that if $(a, b) \in S^\dagger$, then $a/b$ is a pole of the motivic zeta function, and therefore no element of $S^\dagger$ can be omitted. The main tools in the proof are the Denef-Loeser formula and the Crauder-Morrison classification. We explicitly compute the contribution of a flower to the motivic zeta function by writing Python code, which can be found in Appendix [A].

Chapter 5. *$K3$ surfaces satisfying the monodromy property*

The main result in this chapter is ‘$K3$ surfaces with a Crauder-Morrison model that is a flowerpot degeneration satisfy the monodromy property’. For most flowerpots, this is not very difficult to prove, but flowerpots that are rational, non-minimal ruled surfaces require some more work. We also prove that $K3$ surfaces with a Crauder-Morrison model that is a chain degeneration satisfy the monodromy property, under certain extra conditions.

Chapter 6. *Future research: a proof or a counterexample?*

The last chapter is a guideline for further research and we present a variety of possibly useful techniques. We introduce the concept of combinatorial countercandidate: a combinatorial description of a special fiber that could lead to a counterexample of the monodromy property. We generate a list of 63 combinatorial countercandidates by writing Python code, which can be found in Appendix [B].
Chapter 1

Preliminaries

The aim of this chapter is twofold: we will fix notation used in this thesis, and we will briefly introduce the minimal background needed. For the reader who would like to get a more in-depth introduction, we will give some references.

1.1 Notation and conventions

For any field $k$, a $k$-variety is a reduced, separated $k$-scheme of finite type. A surface over $k$ is a projective, geometrically connected $k$-scheme of dimension 2.

Unless stated otherwise, $k$ will be an algebraically closed field of characteristic 0 and we set $K = k((t))$ and $R = k[t]$. We fix an algebraic closure $K^{alg}$ of $K$.

For any integer $d \geq 1$, the notation $\mu_d$ is used for the group of $d$-th roots of unity in $k$. The profinite group of roots of unity in $k$ will be denoted by

$$\hat{\mu} = \varprojlim \mu_d.$$ 

For $D$ and $D'$ two divisors on an integral, regular, projective scheme, we write $D \sim D'$ if $D$ and $D'$ are linearly equivalent and $D \equiv D'$ if $D$ and $D'$ are numerically equivalent. For a regular, projective scheme $X$ over $R$, we write $\omega_{X/R}$ for the relative canonical sheaf of $X$ over $R$ and $K_{X/R}$ for a divisor associated with $\omega_{X/R}$. Note that $K_{X/R}$ is only defined up to linear equivalence.

A ruled surface is a surface $X$, together with a surjective morphism $\pi: X \rightarrow C$ to a non-singular curve $C$, such that all fibers are connected and that for all but finitely many points $y \in C$, the fiber $X_y$ is isomorphic to $\mathbb{P}^1$. We say $X$
is rational or elliptic ruled if $C$ is a rational or elliptic curve respectively. A ruled surface is said to be a minimal ruled surface, if all fibers are isomorphic to $\mathbb{P}^1$. Notice that what is called a ruled surface in [Har77, Section V.2] and [BHPV95, Section V.4], is called a minimal ruled surface here. It can be proven that a smooth, projective surface $X$ is a ruled surface if and only if there exists a composition of blow-ups $X \to X'$, where $X'$ is a minimal ruled surface. As a consequence, if $X$ is a ruled surface with reducible fiber $\mathcal{F}$, then any irreducible component of $\mathcal{F}$ is rational with negative self-intersection.

### 1.2 Calabi-Yau varieties and $K3$ surfaces

**Definition 1.2.1.** A Calabi-Yau variety is a smooth, proper, geometrically connected variety with trivial canonical sheaf. Contrary to what is often done, we do not require that $h^{i,0}(X)$ vanishes for $0 < i < \dim X$ in the definition of a Calabi-Yau variety $X$.

A well-known class of examples of Calabi-Yau varieties are the abelian varieties. In this thesis, we will focus on $K3$ surfaces.

**Definition 1.2.2.** A $K3$ surface $X$ is a 2-dimensional Calabi-Yau variety with $H^1(X, \mathcal{O}_X) = 0$.

The theory on $K3$ surfaces is vast and we will just state some very basic properties.

**Proposition 1.2.3.** Let $X$ be a $K3$ surface, the dimension of the $\ell$-adic cohomology is

$$\dim H^m(X, \mathbb{Q}_\ell) = \begin{cases} 1 & \text{for } m = 0, 4, \\ 22 & \text{for } m = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For a proof of this proposition, we refer to [Huy16, Section 1.3.3].

**Corollary 1.2.4.** Let $X$ be a $K3$ surface. Then $\chi(X) = 24$, where $\chi$ denotes the topological Euler characteristic.

**Example 1.2.5.** Let $f \in k[x, y, z, w]$ be a homogeneous polynomial of degree 4. If the surface $X$ defined by $f = 0$ in $\mathbb{P}^3_k$ is smooth, then $X$ is a $K3$ surface.

We refer to [Huy16] for more information on $K3$ surfaces.
1.3 Models and the Kulikov classification

For a smooth, proper $K$-scheme $X$, a model of $X$ is a flat $R$-scheme $\mathcal{X}$ endowed with an isomorphism $\mathcal{X} \times_R K \simeq X$. The base change $\mathcal{X}_k = \mathcal{X} \times_R k$ is called the special fiber of the model $\mathcal{X}$.

A model $\mathcal{X}$ is called an snc-model, if it is regular and proper and if $\mathcal{X}_k = \sum_{i \in I} N_i E_i$ is a strict normal crossings divisor. Because we assumed that $k$ has characteristic 0, such a model always exists by Nagata’s compactification theorem [Nag62, Nag63] and Hironaka’s resolution of singularities [Hir64].

An snc-model $\mathcal{X}$ with a reduced special fiber is said to be semi-stable. By Mumford’s semi-stable reduction theorem [KKMS73, Section 4.3], we have that for every proper $K$-scheme $X$, there exists a finite extension $L$ of $K$ such that $X \times_K L$ has a semi-stable model.

An snc-model $\mathcal{X}$ of $X$ is said to be triple-point-free if no three distinct irreducible components in the special fiber intersect. For such a model, we can define the dual graph $\Gamma$ in the following way. Write $\mathcal{X}_k = \sum_{i \in I} N_i E_i$. The vertices of the graph $\Gamma$ are $\{v_i\}_{i \in I}$, and for every irreducible component of $E_i \cap E_j$, where $i \neq j$, there is an edge between $v_i$ and $v_j$. Note that for snc-models that are not triple-point-free, it is possible to define the dual complex of $\mathcal{X}$, see for example [KX16, Definition 12]. When $\mathcal{X}$ is triple-point-free, the dual complex and the dual graph coincide.

The Kulikov classification

Semi-stable snc-models of K3 surfaces are completely classified by Kulikov in [Kul77] and Persson and Pinkham [PP81].

Theorem 1.3.1. Let $X$ be a K3 surface over $K$ and let $\mathcal{X}$ be a semi-stable snc-model of $X$ over $R$. There exists a birational modification $\tilde{\mathcal{X}}$ of $\mathcal{X}$ such that $\tilde{\mathcal{X}}$ is a semi-stable snc-model of $X$ with $K_{\tilde{\mathcal{X}}/R}$ trivial. Moreover, one of the following properties hold.

(i) The special fiber $\tilde{\mathcal{X}}_k$ is smooth.

(ii) The special fiber $\tilde{\mathcal{X}}_k$ is a chain of surfaces $V_0 + V_1 + \cdots + V_k + V_{k+1}$. The surfaces $V_1, \ldots, V_k$ are elliptic ruled, and $V_0$ and $V_{k+1}$ are rational surfaces. The intersection curves are elliptic.

(iii) The special fiber $\tilde{\mathcal{X}}_k$ is a union of rational surfaces, and its dual complex is a triangulated 2-sphere.
The semi-stable $snc$-model $\tilde{X}$ of $X$ from the previous theorem, is called the Kulikov model of $X$.

To construct $\tilde{X}$ from $X$, one has to run the Minimal Model Program on $X$ and then resolve the singularities by means of small resolutions. These resolutions force us to allow $\tilde{X}$ to be an algebraic space.

Note that for every $K3$ surface $X$, there exists a finite extension $L$ of $K$ such that $X \times_K L$ has a Kulikov model, because of the semi-stable reduction theorem.

1.4 The Grothendieck ring of varieties

The Grothendieck ring of varieties is one of the most important concepts in this thesis. The motivic zeta function, one of the major ingredients in the monodromy property, is defined over a localization of this ring. But it is also an interesting topic in itself and a lot of research is still going on to investigate this mysterious object the Grothendieck ring still is. For a nice survey on this subject and some open questions, we refer to [NS11] and [CNS17, Chapter 1]

1.4.1 The Grothendieck ring of varieties

Let $k$ be any field.

**Definition 1.4.1.** The Grothendieck ring of $k$-varieties $K_0(Var_k)$ is the abelian group generated by the isomorphism classes $[X]$ of separated $k$-schemes $X$ of finite type, modulo the scissor relations

$$[X] = [Y] + [X \setminus Y],$$

for any closed subscheme $Y$ of $X$. The group $K_0(Var_k)$ is endowed with a ring structure by considering the unique multiplication such that

$$[X] \cdot [X'] = [X \times_k X']$$

for every pair $(X, X')$ of separated $k$-schemes of finite type.

We immediately see that $[\emptyset]$ is the zero element of $K_0(Var_k)$ and $[\text{Spec}(k)]$ is the unit element. We introduce the symbol $\mathbb{L}$, which stands for the class of $\mathbb{A}_k^1$. We denote by $\mathcal{M}_k$ the localized Grothendieck ring $K_0(Var_k)[[\mathbb{L}^{-1}]]$.

**Remark 1.4.2.** The scissor relations tell us that $[X] = [X_{\text{red}}]$ for every separated $k$-scheme $X$ of finite type, where $X_{\text{red}}$ is the maximal reduced closed subscheme of $X$. This means that if we replace ‘separated $k$-scheme of finite type’ by ‘$k$-variety’ in the definition of Grothendieck ring, we actually define
the same ring. But in that case, one should be careful with the definition of
the ring multiplication: when $k$ is imperfect and $X$ and $X'$ are $k$-varieties, the
product $X \times_k X'$ is not necessarily reduced.

The relation $[X] = [Y] + [X \setminus Y]$ allows to cut and paste varieties. If we cut a
variety $X$ into two pieces $Y$ and $X \setminus Y$, adding (pasting) the classes of these
two pieces in the Grothendieck ring, gives the class of the original variety $X$
again. This is why we call this relation the scissor relation.

The scissor relation can be very helpful to compute the class of a variety in
practice. For instance, the projective line $\mathbb{P}^1_k$ has a subvariety isomorphic to $\mathbb{A}^1_k$
and the complement of this subvariety is a point. This means that

$$[\mathbb{P}^1_k] = \mathbb{L} + 1.$$ 

Another example: the projective plane $\mathbb{P}^2_k$ has a subvariety isomorphic to the
affine plane $\mathbb{A}^2_k$ and its complement is a projective line. Since $\mathbb{A}^2_k \cong \mathbb{A}^1_k \times_k \mathbb{A}^1_k$, we have

$$[\mathbb{P}^2_k] = \mathbb{L}^2 + \mathbb{L} + 1.$$ 

By induction, one finds that

$$[\mathbb{P}^n_k] = \mathbb{L}^n + \cdots + \mathbb{L} + 1.$$ 

**Remark 1.4.3.** A subset $U$ of a $k$-variety $X$ is called *locally closed* if it is
open in its closure in $X$. A locally closed subset $U$ has a unique structure of
subvariety of $X$. We call a subset $C$ of $X$ a *constructible* subset, when it is
the finite union of locally closed subsets of $X$. Although a constructible subset
doesn’t necessarily have a unique structure of subvariety of $X$, its class in the
Grothendieck ring of varieties is well defined. Indeed, if we write $C$ as a finite
disjoint union of locally closed subsets $U_1, \ldots, U_n$, then the class

$$[C] = \sum_{i=1}^n [U_i]$$

does not depend on the choice of partition into locally closed subsets.

Because of the scissor relations, taking the class $[X]$ of a variety $X$ in the
Grothendieck ring satisfies all the requirements a reasonable *measure* should
have: if we cut a variety into two subvarieties, their sizes should add up to the
original size. Moreover, the size of a product variety should be the product
of the sizes of its factors. So every reasonable way to measure the size of a
variety $X$ factors through its class $[X]$ in the Grothendieck ring of varieties.
Or said even differently, the measure $X \rightarrow [X]$ specializes to any other (more
intuitive) measure. That’s why the Grothendieck ring of $k$-varieties should be
thought of as the *universal measure*.

Let’s look at some examples of such specialization morphisms.
Example 1.4.4.

(i) Fix a finite field $k = \mathbb{F}_q$ and consider the invariant $\sharp$ that counts the number of $\mathbb{F}_q$-rational points on a variety over $\mathbb{F}_q$. Since counting points is clearly additive and multiplicative, we have a well-defined morphism

$$\sharp : K_0(Var_{\mathbb{F}_q}) \to \mathbb{Z} : [X] \mapsto \#X(\mathbb{F}_q).$$

This means in particular that two varieties having the same class in $K_0(Var_{\mathbb{F}_q})$ have the same number of rational points. This morphism localizes to a ring morphism $\sharp : M_{\mathbb{F}_q} \to \mathbb{Z}[q^{-1}]$, because

$$\sharp L = \sharp A^1_{\mathbb{F}_q}(\mathbb{F}_q) = q.$$

(ii) For $k = \mathbb{C}$, an important additive, multiplicative invariant of a complex variety $X$ is the compactly supported Euler characteristic $\chi_c$ of $X(\mathbb{C})$ with respect to the complex topology. This means that there exists a well-defined morphism

$$\chi_c : K_0(Var_k) \to \mathbb{Z} : [X] \mapsto \chi_c(X).$$

For complex varieties, we have that $\chi_c(X) = \chi(X)$, where $\chi(X)$ is the topological Euler characteristic of $X(\mathbb{C})$ with respect to the complex topology. This was proven by Laumon in [Lau81]. Therefore, there also exists a well-defined morphism

$$\chi : K_0(Var_k) \to \mathbb{Z} : [X] \mapsto \chi(X).$$

This last morphism, for example, allows to easily compute the Euler characteristic of projective space:

$$\chi(\mathbb{P}_C^n) = \chi(\mathbb{L}^n + \cdots + \mathbb{L} + 1) = \chi(\mathbb{L})^n + \cdots + \chi(\mathbb{L}) + 1 = n + 1.$$

The morphism $\chi$ localizes to a ring morphism $\chi : M_{\mathbb{C}} \to \mathbb{Z}$, since $\chi(\mathbb{L}) = 1$.

For an arbitrary field $k$, one can consider the $\ell$-adic Euler characteristic $\chi$, with $\ell$ an invertible prime in $k$. One can show that $\chi$ does not depend on the choice of $\ell$, see for example [Kat94, point (2a) p. 28].

(iii) For $k$ a field of characteristic 0, there exists a unique ring morphism $P : K_0(Var_k) \to \mathbb{Z}[v]$ such that, when $X$ is a smooth and proper $k$-variety, the class $[X]$ gets mapped to the Poincaré polynomial

$$P(X; v) = \sum_{i \geq 0} (-1)^i b_i(X) v^i,$$
where \( b_i(X) = \dim H^i(X, \mathbb{Q}_\ell) \) is the \( i \)-th \( \ell \)-adic Betti number of \( X \).

In particular, if a class in the Grothendieck ring has a non-zero Poincaré polynomial, then the class is non-zero itself. But we also have some converse result. In [Nic11 Proposition 8.7], it is proven that for any \( k \)-variety \( X \), the degree of \( P(X; v) \) is \( 2 \dim(X) \) and its leading coefficient equals the number of irreducible components of \( X \times k^{alg} \) of dimension \( \dim(X) \). Therefore, a (non-empty) variety has a non-zero Poincaré polynomial.

Notice that \( P(X; 1) = \chi(X) \). Furthermore, we have \( P(\mathbb{L}) = v^2 \) and hence, there is a well-defined localized ring morphism

\[
P : \mathcal{M}_k \to \mathbb{Z}[v, v^{-1}].
\]

The Poincaré polynomial can also be defined over a field \( k \) of positive characteristic \( p \). For any smooth, proper \( k \)-variety \( X \), it still holds that

\[
P(X; v) = \sum_{i \geq 0} (-1)^i b_i(X) v^i,
\]

where \( b_i(X) = \dim H^i(X, \mathbb{Q}_\ell) \) is the \( i \)-th \( \ell \)-adic Betti number of \( X \) with \( \ell \) a prime invertible in \( k \). However, we don’t know whether \( P : K_0(\text{Var}_k) \to \mathbb{Z} \) is uniquely defined by this property, as is the case for characteristic 0. For more information on the Poincaré polynomial and how to define it for fields of positive characteristic, we refer to [Nic11 Section 8].

(iv) For \( k \) a field of characteristic zero, there exists a unique ring morphism \( HD : K_0(\text{Var}_k) \to \mathbb{Z}[u, v] \) such that for a smooth and proper \( k \)-variety \( X \), the class \([X]\) gets mapped to the Hodge-Deligne polynomial

\[
HD(X; u, v) = \sum_{p, q \geq 0} (-1)^p h^{p,q}(X) u^p v^q,
\]

where \( h^{p,q}(X) = \dim_k H^p(X, \Omega^q_X) \) is the \((p, q)\)-th Hodge number of \( X \). Notice that \( HD(X; v, v) = P(X; v) \). Since \( HD(\mathbb{L}; u, v) = uv \), we can localize this morphism to \( HD : \mathcal{M}_k \to \mathbb{Z}[u, u^{-1}, v, v^{-1}] \).

The Hodge-Deligne polynomial of a complex variety \( X \) equals

\[
HD(X; u, v) = \sum_{p, q \geq 0} \sum_{i \geq 0} (-1)^i h^{p,q}(H^i_c(X(\mathbb{C}), \mathbb{C})) u^p v^q,
\]

where \( h^{p,q}(H^i_c(X(\mathbb{C}), \mathbb{C})) \) is the dimension of the \((p, q)\)-component of Deligne’s mixed Hodge structure on \( H^i_c(X(\mathbb{C}), \mathbb{C}) \).
The Grothendieck ring $K_0(Var_k)$ is still a mysterious object and a lot of researchers are investigating the properties of this ring. In 2002, Poonen proved in [Poo02] that $K_0(Var_k)$ is not a domain, if $k$ has characteristic 0. We even know that the class of $L$ is a zero-divisor in $K_0(Var_C)$, as shown by Borisov in [Bor17]. Moreover, when $k$ has characteristic zero, $M_k$ is not a domain either. This can be proven with the same argument as Ekedahl uses in [Eke09]. Cauwbergs writes this down in more detail in [Cau16].

1.4.2 The equivariant Grothendieck ring of varieties

The equivariant Grothendieck ring of varieties $K^G_0(Var_k)$ is a variant of the Grothendieck ring of varieties, where we take into account a group action of a fixed finite group $G$ on the varieties. This subsection is based on [Har16, Sections 3 and 4], and we refer to this paper for more details.

Notation

Let $k$ be a field and fix a finite group $G$. Let $X$ be a separated scheme of finite type over $k$ endowed with a group action of $G$. Unless explicitly stated otherwise, we assume that groups act on schemes from the left. We say the group action is good, if every orbit of this action is contained in an affine open subscheme of $X$. We denote by $(Sch_k, G)$ the category whose objects are separated schemes of finite type over $k$ with a good $G$-action, and whose morphisms are $G$-equivariant morphisms of $k$-schemes. One can check that the fiber product exists in this category, see for example [Har16, Section 2].

Affine bundles

Definition 1.4.5. Let $S$ be a $k$-scheme. An affine bundle over $S$ of rank $d$ is an $S$-scheme $A$ with a vector bundle $V \to S$ of rank $d$ and a morphism of $S$-schemes $\phi: V \times_S A \to A$ such that $\phi \times p_A: V \times_S A \to A \times_S A$ is an isomorphism of $S$-schemes, where $p_A$ is the projection to $A$.

Remark 1.4.6. The inspiration behind the definition of affine bundle is that it should be a generalization of the following fact in linear algebra: assume $A$ is an affine space of dimension $n$ over $k$ and $V = k^n$. We think of $A$ as $k^n$ where we ‘forget’ the origin. Then we can look at the map $V \times A \to A: (v, x) \to x + v,$
which is just adding a vector to a point as in high-school geometry. This map induces a bijection

\[ V \times A \to A \times A: (v, x) \to (x + v, x), \]

which represents the easy fact that for every two points \( x \) and \( x' \) in \( A \), the vector \( x - x' \) exists in \( V \). We see that \( A \), together with \( V \), satisfies the definition of an affine bundle.

**Remark 1.4.7.** It can be shown that every affine bundle of rank \( d \) over \( S \) is a locally trivial fibration over \( S \) with fiber \( \mathbb{A}^d_\mathbb{k} \).

If there is an action on an affine bundle \( A \), ideally, the action on \( A \) should be compatible with the action on \( V \).

**Definition 1.4.8.** The affine bundle \( A \) over \( S \) is said to be \( G \)-equivariant, if \( A \) and \( S \) are in \( (\text{Sch}_\mathbb{k}, G) \) and \( A \to S \) is \( G \)-equivariant. We call the \( G \)-action on \( A \to S \) affine if there is a \( G \)-action on \( V \), linear over the action on \( S \), such that \( \phi: V \times_S A \to A \) is \( G \)-equivariant.

**Remark 1.4.9.** In the special case we discussed in Remark 1.4.6, a \( G \)-action on \( A \) is affine if \( g \cdot (x + v) = g \cdot x + g \cdot v \) for every \( g \in G \), \( x \in A \) and \( v \in V \).

The equivariant Grothendieck ring \( K^G_0(\text{Var}_\mathbb{k}) \)

**Definition 1.4.10.** The equivariant Grothendieck ring of \( k \)-varieties \( K^G_0(\text{Var}_\mathbb{k}) \) is the abelian group generated by the isomorphism classes \([X]\) of objects \( X \in (\text{Sch}_\mathbb{k}, G) \), modulo the following relations:

1. \([X] = [Y] + [X \setminus Y]\), whenever \( Y \) is a closed \( G \)-equivariant subscheme of \( X \) (scissor relations), and

2. \([A] = [S \times_\mathbb{k} \mathbb{A}^d_\mathbb{k}]\), whenever \( A \to S \) is a \( G \)-equivariant affine bundle of rank \( d \) over \( S \in (\text{Sch}_\mathbb{k}, G) \) with affine \( G \)-action and with trivial \( G \)-action on \( \mathbb{A}^d_\mathbb{k} \).

We endow \( K^G_0(\text{Var}_\mathbb{k}) \) with a ring structure by putting the unique multiplication such that

\[ [X] \cdot [X'] = [X \times_\mathbb{k} X'] \]

for all \( X, X' \in (\text{Sch}_\mathbb{k}, G) \) and where the fiber product is taken in the category \( (\text{Sch}_\mathbb{k}, G) \).

Let \( L \in K^G_0(\text{Var}_\mathbb{k}) \) be the class of the affine line with trivial action. In particular, we see that, if \( A \to S \) is a \( G \)-equivariant affine bundle of rank \( d \) over \( S \), then \([A] = [S]L^d\). We denote by \( M^G_\mathbb{k} \) the localization \( K^G_0(\text{Var}_\mathbb{k})[L^{-1}] \).
Remark 1.4.11. When $G = \{e\}$ is the trivial group, we have that the equivariant Grothendieck ring $K_0^G(Var_k)$ equals the usual Grothendieck ring $K_0(Var_k)$. Moreover, $\mathcal{M}_k^G$ equals $\mathcal{M}_k$ in this case.

Remark 1.4.12. A morphism of finite groups $G' \rightarrow G$ induces ring morphisms $K_0^G(Var_k) \rightarrow K_0^{G'}(Var_k)$ and $\mathcal{M}_k^G \rightarrow \mathcal{M}_k^{G'}$.

Remark 1.4.13. In the literature, other definitions of the equivariant Grothendieck ring are often used. We refer to [Har16, Remark 4.5, Remark 4.6] for a discussion.

In this thesis, we will mostly be interested in actions of the profinite group $\hat{\mu} = \lim\leftarrow \mu_d$ of roots of unity of the field $k$. The ring $K_0^\hat{\mu}(Var_k)$ can be defined as follows.

Definition 1.4.14. Let $\prod_{i \in I} G_i$ be a profinite group, where all $G_i$ are finite groups. Then we define

$$K_0^{\prod_{i \in I} G_i}(Var_k) = \lim_{\longrightarrow I} K_0^{G_i}(Var_k) \quad \text{and} \quad \mathcal{M}_k^{\prod_{i \in I} G_i} = \lim_{\longrightarrow I} \mathcal{M}_k^{G_i}.$$  

1.5 Motivic integration on Calabi-Yau varieties

Let $X$ be a Calabi-Yau variety over $K$, and let $\omega$ be a volume form on $X$, i.e., a nowhere-vanishing differential form of maximal rank. For every integer $d \geq 1$, set $R(d) = R[\pi]/(\pi^d - t)$ and $K(d)$ as the fraction field of $R(d)$. The field $K(d)$ is the unique totally ramified extension of $K$ in $K^{alg}$ of degree $d$. We define

$$X(d) = X \times_K K(d),$$

and $\omega(d)$ as the pullback of $\omega$ to $X(d)$. There is a left group-action of $\mu_d$ on $X(d)$. Let $\mathcal{Y}$ be an equivariant, weak Néron model of $X(d)$. This means that $\mathcal{Y}$ is a separated, smooth $R(d)$-scheme, endowed with a good $\mu_d$-action and a $\mu_d$-equivariant isomorphism $\mathcal{Y} \times_{R(d)} K(d) \simeq X(d)$, such that the natural map $\mathcal{Y}(R(d)) \rightarrow X(K(d))$ is a bijection. It can be proven that such an equivariant, weak Néron model always exists.

For every connected component $C$ of $\mathcal{Y}_k$, we denote by $\text{ord}_C \omega(d)$ the unique integer $a$ such that $t^{-a/d}\omega(d)$ is a generator of $\omega_{\mathcal{Y}/R(d)}$, locally at the generic point of $C$.

Definition 1.5.1. Let $X$ be a Calabi-Yau variety over $K$, and let $\omega$ be a volume form on $X$. Fix an integer $d \geq 1$ and let $\mathcal{Y}$ be an equivariant, weak Néron
model of $X(d)$. The motivic integral of $\omega(d)$ on $X(d)$ is defined as

$$\int_{X(d)} |\omega(d)| = \sum_{C \in \pi_0(Y_k)} [C] L^{-\text{ord}_C \omega(d)} \in \mathcal{M}_k^\hat{\mu},$$

where $\pi_0(Y_k)$ is the set of connected components of $Y_k$.

It is non-trivial to prove that this definition does not depend on the choice of equivariant, weak Néron model $Y$. Without the $\hat{\mu}$-action, this is done by Loeser and Sebag in [LS03, Proposition 4.3.1]. The generalization to the equivariant version follows from the change of variables formula for equivariant motivic integrals, proven by Hartmann in [Har15].

### 1.6 Algebraic spaces

In Chapter 3, we allow models $X$ of smooth, proper $K$-varieties to be algebraic spaces. An algebraic space is a generalization of the notion of a scheme. For the reader who is not familiar with the concept of algebraic space, we refer to [Knu71] and [Stacks, Tag 025R, Tag 03BO and Tag 03H8].

The reason why we need to work with algebraic spaces is that in the category of algebraic spaces, more contraction results hold. In particular, we refer to the work of Artin in [Art70, Theorem 6.2].

Let $X$ be a smooth, proper surface over $K$ with $\omega_{X/K}^\otimes m \simeq \mathcal{O}_K$ for some integer $m \geq 1$. In Chapter 3, we define a Crauder-Morrison model $X$ of $X$ as an algebraic space over $K$ such that $X$ is a triple-point-free snc-model of $X$, where no component in the special fiber $X_k$ can be contracted such that the result is again a triple-point-free snc-model. It is essential that these contractions are considered in the category of algebraic spaces. If we consider only contractions in the category of schemes, then the classification of Crauder-Morrison models in Theorem 3.2.1 does not necessarily hold.

An algebraic space of finite type over a field $k$ has a well-defined class in the Grothendieck ring of $k$-varieties $K_0(Var_k)$, see for example [HN17 (7.1.3)]. The theory of motivic integration can also be extended to algebraic spaces, as done by Halle and Nicaise in [HN17 Section 7].
Chapter 2

The monodromy property for Calabi-Yau varieties

In this introductory chapter, we will explain the monodromy property for Calabi-Yau varieties. In the first section, we will motivate the subject by giving the historical context: we will discuss the \( p \)-adic and motivic monodromy conjecture for hypersurface singularities. In the second and third section, we will define the two main ingredients in the monodromy property, namely the motivic zeta function and monodromy eigenvalues. In the last section, we will be ready to define the monodromy property. We will also discuss what is known about varieties satisfying the monodromy property.

2.1 History: the \( p \)-adic and motivic monodromy conjecture

Before we start defining the main concepts of this thesis, let us first look at some history. The monodromy property for Calabi-Yau varieties is a variant of the motivic monodromy conjecture for hypersurface singularities, which, in its turn, is a generalization of the \( p \)-adic monodromy conjecture. To motivate and better understand the monodromy property for Calabi-Yau varieties, it is useful to first explain the \( p \)-adic and motivic monodromy conjecture for hypersurface singularities. For a nice introduction and more details regarding these conjectures, we refer to [Nic10] and [Bor13, Chapter 0]. It is important to keep in mind that there are no direct implications between the motivic monodromy conjecture for hypersurface singularities and the monodromy
property for Calabi-Yau varieties: a proof for the motivic monodromy conjecture for hypersurface singularities would not imply that the monodromy property holds for certain types of Calabi-Yau varieties and vice versa.

2.1.1 The $p$-adic monodromy conjecture

The $p$-adic zeta function and the Poincaré series

Let $f$ be a non-constant polynomial in $\mathbb{Z}[x_1, \ldots, x_m]$ for some integer $m$, and fix a prime number $p$. It is a very natural question to ask how many solutions the congruence $f \equiv 0 \mod p^{d+1}$ has for $d \geq 0$. Let’s denote by $N_d$ the number of such solutions, so

$$N_d = \# \{ a \in (\mathbb{Z}/p^{d+1})^m \mid f(a) \equiv 0 \mod p^{d+1} \}.$$

If we use these numbers as coefficients in a generating series, we get the Poincaré series of $f$.

**Definition 2.1.1.** Let $p$ be a prime number and let $f \in \mathbb{Z}[x_1, \ldots, x_m] \setminus \mathbb{Z}$ be a non-constant polynomial, for some $m \in \mathbb{Z}_{>0}$. Let $N_d$ be the number of solutions $x \in (\mathbb{Z}/p^{d+1})^m$ of $f(x) \equiv 0 \mod p^{d+1}$, for any integer $d \geq 0$. The Poincaré series of $f$ is the generating series

$$P_f(T) = \sum_{d \geq 0} N_d T^d \in \mathbb{Z}[T].$$

**Remark 2.1.2.** This power series has a radius of convergence $R \geq 1/p^m$, since $N_d \leq p^{(d+1)m}$.

The Poincaré series of a polynomial $f$ is closely related to the $p$-adic zeta function of $f$.

**Definition 2.1.3.** Let $| \cdot |_p$ denote the $p$-adic absolute value on $\mathbb{Q}_p$. The $p$-adic zeta function of $f$ is defined by the $p$-adic integral

$$Z_f(s) = \int_{\mathbb{Z}_p^n} |f(x)|_p^s \, d\mu,$$

for every $s \in \mathbb{C}$ with $\text{Re}(s) > 0$ and where $\mu$ is the usual Haar measure on $\mathbb{Z}_p^n$.

By definition of $p$-adic integration, the function $Z_f(s)$ can be written as a power series in $p^{-s}$:

$$Z_f(s) = \sum_{d \geq 0} \mu \left( \{ a \in \mathbb{Z}_p^m \mid v_p(f(a)) = d \} \right) p^{-ds}; \quad (2.1)$$
with $v_p$ the $p$-adic valuation. Readers not familiar with $p$-adic integration, can take (2.1) as the definition of the $p$-adic zeta function.

**Remark 2.1.4.** The Poincaré series $P_f(T)$ and the $p$-adic zeta function $Z_f(s)$ are closely related and their relation

$$P_f(p^{-n}T) = p^n \frac{1 - Z_f(s)}{1 - T},$$

(2.2)
can be shown by direct computation, where $T = p^{-s}$. From this, we see that $Z_f$ encodes the number of solutions of $f$ modulo powers of $p$ as well.

In problem 9 of Section 1.5 of [BS66], Borevich and Shafarevich asked whether the Poincaré series is a rational function. This question got an affirmative answer in 1974, when Igusa proved that the $p$-adic zeta function of $f$ is a rational function in $p^{-s}$ by using Hironaka’s resolution of singularities. The original proof can be found in the papers [Igu74] and [Igu75], and a simplified and more readable proof can be found in the appendix of [Igu77]. The rationality of $Z_f$ immediately implies the rationality of $P_f$ by formula (2.2). In [Den84, Theorem 3.2], Denef gave a completely different proof of the rationality of the $p$-adic zeta function using cell decomposition.

The rationality of $Z_f(s)$ and $P_f(T)$ implies that they admit meromorphic continuations to $\mathbb{C}$. This means that we can talk about poles of $Z_f(s)$ and $P_f(T)$. From formula (2.2), we immediately see that the poles of $P_f$ can be computed from $Z_f$ and vice versa. This means that knowing the poles of $Z_f$ gives information on the asymptotic behaviour of $N_d$ when $d$ tends to infinity. We refer to [Seg11] for a nice explanation of this fact.

Igusa’s proof of the rationality of $Z_f(s)$ even gives a finite set containing the real parts of all the poles of $Z_f(s)$. Let $g : Y \to \mathbb{A}^m_{\mathbb{Q}_p}$ be an embedded resolution of singularities of the zero locus $f^{-1}(0) \subset \mathbb{A}^m_{\mathbb{Q}_p}$. If $\alpha$ is a pole of $Z_f(s)$, then $Re(\alpha)$ is of the form $-\nu_i/N_i$, where $N_i$ and $\nu_i - 1$ are the multiplicities of $f \circ g$ and $g^*dx$ along an irreducible component $E_i$ of $g^{-1}(f^{-1}(0))$. It is remarkable that a lot of irreducible components of $g^{-1}(f^{-1}(0))$ do not contribute to the poles of $Z_f(s)$, a phenomenon that is intimately connected with the $p$-adic monodromy conjecture.

**Monodromy eigenvalues**

Let $f : \mathbb{C}^m \to \mathbb{C}$ be a non-constant analytic map. We denote by $X_0$ the analytic space defined by $f = 0$ and let $x$ be any point of $X_0$. We would like to study the topology of $X_0$ in a neighbourhood of $x$, especially when $X_0$ is not smooth at $x$. The idea is the following: locally around $x$, we will study the analytic
space $X_\delta$ defined by $f = \delta$ instead of $f = 0$, for complex numbers $\delta$ close enough to 0. For such ‘small’ $\delta$, the analytic space $X_\delta$ is smooth and all these spaces $X_\delta$ are diffeomorphic for every choice of $\delta$. When we let the space $X_\delta$ turn once around the space $X_0$, we get an automorphism of $X_\delta$, which is called the monodromy transformation. This automorphism gives valuable information about the singularity $x$ of $X_0$. Let us now make this idea more concrete.

Let $B = B(x, \varepsilon)$ be an open ball around $x$ in $\mathbb{C}^m$ with radius $\varepsilon$ and let $D = D(0, \eta)$ be an open disc around 0 in $\mathbb{C}$ with radius $\eta$. By $D^*$, we denote the punctured disc $D \setminus \{0\}$. For a suitable choice of $0 < \eta \ll \varepsilon \ll 1$, we have that the map

$$f_x: f^{-1}(D^*) \cap B \to D^*$$

is a locally trivial fibration. This fibration is called the Milnor fibration of $f$ at $x$. The fact that the Milnor fibration is locally trivial implies in particular that all fibers of $f_x$ are diffeomorphic to one another. The fiber of $f_x$ over any point of $D^*$ is called the Milnor fiber $F_x$ of $f$ at $x$, which is well defined up to diffeomorphism.

To study the topological structure of $X_0$ around $x$, we let the Milnor fiber $F_x$ turn once around $X_0$ and we look at the transformation $F_x$ underwent. More formally, we choose a loop $\gamma$ in $D^*$ around the origin. Over each point of this loop, we have a fiber that is diffeomorphic to the Milnor fiber. Since $f_x$ is locally trivial, we can move the Milnor fiber in a continuous way through open neighbourhoods where $f_x$ is trivial. This results in an automorphism of $F_x$, which we call the monodromy transformation $M_x$. It can be shown that, up to homotopy, the monodromy transformation is well defined, i.e., it does not depend on the chosen loop $\gamma$ and the chosen trivialization of $f_x$.

We can describe the monodromy transformation even more formally, which will turn out to be useful when we will give a more algebraic definition of monodromy in Section 2.3. First, notice that the Milnor fiber is homotopy-equivalent to

$$(f^{-1}(D^*) \cap B) \times_{D^*} \tilde{D}^*,$$

where $\tilde{D}^*$ is the universal covering space of $D^*$. This is true because $\tilde{D}^*$ is contractible. The group $\pi_1(D^*) \simeq \mathbb{Z}$ of covering transformations of $\tilde{D}^*$ over $D^*$, acts naturally on $\tilde{D}^*$ and this action can be lifted canonically to $(f^{-1}(D^*) \cap B) \times_{D^*} \tilde{D}^*$. The action of the canonical generator of $\pi_1(D^*)$, namely a positively oriented loop turning once around the origin, is homotopic to the monodromy transformation $M_x$.

The monodromy transformation $M_x$ induces linear maps

$$M_{x,i}: H^i(F_x, \mathbb{C}) \to H^i(F_x, \mathbb{C})$$
on the singular cohomology spaces $H^i(F_x, \mathbb{C})$ for all $i \geq 0$. These are uniquely determined, since $M_x$ is well defined up to homotopy.

**Definition 2.1.5.** Let $f: \mathbb{C}^m \to \mathbb{C}$ be a non-constant analytic map. We denote by $X_0$ the analytic space defined by $f = 0$ and let $x$ be any point of $X_0$. Let $F_x = (f^{-1}(D^*) \cap B) \times D^* \to B$ be the Milnor fiber of $f$ at $x$ and let $M_{x,i}$ be the action of the canonical generator of $\pi_1(D^*, \eta)$ on $H^i(F_x, \mathbb{C})$. We say that a complex number $\alpha \in \mathbb{C}$ is a monodromy eigenvalue of $f$ at $x$, if $\alpha$ is an eigenvalue of $M_{x,i}$ for some $i \geq 0$.

All monodromy eigenvalues are roots of unity [SGA7, Théorème de monodromie 2.1]. An important tool to compute monodromy eigenvalues in practice is the monodromy zeta function of $f$ at $x$.

**Definition 2.1.6.** The monodromy zeta function of $f$ at $x$ is the alternating product of the characteristic polynomials of the monodromy transformations $M_{x,i}$ on the cohomology spaces $H^i(F_x, \mathbb{C})$:

$$\zeta_{f,x}(T) = \prod_{i \geq 0} \det (Id - T \cdot M_{x,i} \mid H^i(F_x, \mathbb{C})) (-1)^{i+1}.$$ 

**Remark 2.1.7.** Zeroes and poles of $\zeta_{f,x}$ are monodromy eigenvalues of $f$ at $x$. A priori, not every monodromy eigenvalue of $f$ at $x$ is necessarily a zero or pole of $\zeta_{f,x}$, since cancellations may occur in the definition of $\zeta_{f,x}$. Denef proved in [Den93, Lemma 4.6] that every monodromy eigenvalue of $f$ at $x$ is a zero or pole of $\zeta_{f,y}$ for some, possibly different, point $y \in X_0$.

As already mentioned before, the monodromy zeta function can be useful to compute monodromy eigenvalues in practice. A’Campo gave a particularly nice formula for the monodromy zeta function in terms of an embedded resolution of singularities $g$ of $f$.

**Theorem 2.1.8 (A’C75 Théorème 3).** Let $f: \mathbb{C}^m \to \mathbb{C}$ be a non-constant analytic map. We denote by $X_0$ the analytic space defined by $f = 0$ and let $x$ be any point of $X_0$. Let $g: Y \to \mathbb{A}^m$ be an embedded resolution of singularities of $f$ and denote by $\{E_i\}_{i \in I}$ the irreducible components of $g^{-1}(X_0)$. Set $N_i$ as the multiplicity of $E_i$ and $E_i^\circ = E_i \setminus \bigcup_{j \in I \setminus \{i\}} E_j$. The monodromy zeta function can be expressed as

$$\zeta_{f,x}(T) = \prod_{i \in I} (1 - T^{N_i})^{-\chi(E_i^\circ \cap g^{-1}(x))},$$

where $\chi(\cdot)$ is the topological Euler characteristic.
The $p$-adic monodromy conjecture

The monodromy conjecture is a fascinating conjecture relating the poles of the $p$-adic zeta function of a polynomial $f$ - an arithmetical concept - to the monodromy eigenvalues of $f$ considered as an analytic function - a concept that is differential-topological in nature.

**Conjecture 2.1.9** (The $p$-adic monodromy conjecture). Let $f$ be a non-constant polynomial in $\mathbb{Z}[x_1, \ldots, x_m]$ for some integer $m$. For almost all primes $p$, we have the following: if $\alpha$ is a pole of the $p$-adic zeta function $Z_f(s)$, then $\exp(2\pi i \text{Re}(\alpha))$ is a monodromy eigenvalue of $f$ at some point $x \in X_0$.

The $p$-adic monodromy conjecture has been proven, for example, for polynomials in two variables in [Loe88] and homogeneous polynomials in three variables in [RV01] and [ACLM02]. However, the general case remains wide open.

2.1.2 The motivic monodromy conjecture

In 1995, Kontsevich launched the idea of motivic integration, which can be thought of as a geometrized version of $p$-adic integration. The key idea is to replace the finite field $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{C}$ and the $p$-adic integers $\mathbb{Z}_p$ by the field of formal complex power series $\mathbb{C}[[t]]$. Instead of using the Haar measure as in $p$-adic integration, for motivic integration, the size of a variety is measured by its class in the Grothendieck ring. Kontsevich’s goal was to prove that two birationally equivalent complex Calabi-Yau varieties have the same Hodge numbers.

Kontsevich never published his ideas, but the theory of motivic integration was further developed by other mathematicians, such as Cluckers, Denef, Nicaise, Loeser and Sebag. In [DL98], Denef and Loeser introduce the motivic zeta function associated with a polynomial $f \in k[x_1, \ldots, x_m]$ and the motivic monodromy conjecture. A more modern version of the motivic zeta function appears in [DL01], and it is this version that we will use in this thesis. The motivic zeta function can be thought of as the universal zeta function, since it specializes to the $p$-adic zeta function for almost all primes $p$. This means that if a statement is proven for the motivic zeta function, we immediately have a proof for the $p$-adic equivalent as well.

The motivic zeta function

Let $k$ by any field. We will now formulate a motivic counterpart of the $p$-adic zeta function $Z_f(s) = \int_{\mathbb{Z}_p^n} |f(x)|_p^s \, d\mu$. 
Definition 2.1.10. Let $f \in k[x_1, \ldots, x_m]$ be a polynomial. The motivic zeta function of $f$ is

$$Z_{f}^{\text{mot}}(T) = \sum_{d \geq 0} [\mathcal{X}_{d,1}] \mathbb{L}^{-md} T^d \in \mathcal{M}_k [T],$$

where $\mathcal{X}_{d,1} = \{ \varphi \in (k[t]/(t^{d+1}))^m \mid f(\varphi) \equiv t^d \mod (t^{d+1}) \}$ for every $d \in \mathbb{Z}_{\geq 0}$.

Let us now have a look at how the $p$-adic and motivic zeta functions are related. The $p$-adic zeta function compares best with the naive motivic zeta function, which is a slightly different version than the definition we gave before. Remember that using the definition of $p$-adic integration, we could write the $p$-adic zeta function as

$$Z_f(s) = \sum_{d \geq 0} \mu \left( \{ a \in \mathbb{Z}_p^m \mid v_p(f(a)) = d \} \right) p^{-ds}.$$

The key-idea of motivic integration is to replace $\mathbb{Z}_p$ by $k[t]$, so the set $\{ a \in \mathbb{Z}_p^m \mid v_p(f(a)) = d \}$ changes to

$$\{ \varphi \in k[t]^m \mid \text{ord}_t(f(\varphi)) = d \},$$

where $\text{ord}_t(\sum_{i \geq 0} a_i t^i) = \min \{ i \geq 0 \mid a_i \neq 0 \}$. Compare this to the $p$-adic valuation, which is defined as $v_p(\sum_{i \geq 0} a_i p^i) = \min \{ i \geq 0 \mid a_i \neq 0 \}$. Note that $\text{ord}_t(f(\varphi)) = d$ if and only if $f(\varphi) \equiv a t^d \mod (t^{d+1})$ for some $a \in k \setminus \{ 0 \}$.

In the motivic setting, the role of $p$ will be played by the class of the affine line $\mathbb{L}$ and $p^{-s}$ is replaced by $T = \mathbb{L}^{-s}$. Putting all these ideas together, the naive motivic zeta function associated with $f$ should be

$$Z_{f}^{\text{mot,naive}}(T) = \sum_{d \geq 0} \mu \left( \{ \varphi \in k[t]^m \mid \text{ord}_t(f(\varphi)) = d \} \right) T^d \in \mathcal{M}_k [T],$$

where $\mu$ is the motivic measure. To define this motivic measure, we need, for every integer $d \geq 0$, the truncation map $\pi_d$:

$$\pi_d : (k[t])^m \to (k[t]/(t^{d+1}))^m : \left( \sum_{i \geq 0} a_{ij} t^j \right) \mapsto \left( \sum_{j=0}^d a_{ij} t^j \right).$$

A subset $A \subset (k[t])^m$ is called cylindric, if there exists a constructible subset $C \subset (k[t]/(t^{d+1}))^m$, when $(k[t]/(t^{d+1}))^m$ is viewed as $k^{(d+1)m}$, such that $A = \pi_d^{-1}(C)$, for some integer $d$. The motivic measure $\mu$ assigns a ‘size’ to such cylindric subsets:

$$\mu(A) := [C] \mathbb{L}^{-m(d+1)} \in \mathcal{M}_k.$$
Theorem 2.1.12. Assume $k$ is a field of characteristic 0 and let $f \in k[x_1, \ldots, x_m]$ be a polynomial. Let $g: Y \to \mathbb{A}^m_k$ be an embedded resolution of singularities for $f$ and let $\{E_i \mid i \in I\}$ be the irreducible components of the divisor $g^{-1}(f^{-1}(0))$. Define $N_i$ and $\nu_i - 1$ to be the multiplicities of $f \circ g$ and $g^*dx$ along the component $E_i$. For any non-empty subset $J \subseteq I$, set $E_J = \cap_{j \in J} E_j$ and $E_J^o = E_J \setminus (\cup_{i \in I \setminus J} E_i)$. The motivic zeta function can be written as

$$Z_f^{\text{mot}}(T) = \sum_{J \subseteq I} (\mathbb{L} - 1)^{|J|} |E_J^o|^{-1} \prod_{j \in J} \frac{\mathbb{L}^{-\nu_j} T^{N_j}}{1 - \mathbb{L}^{-\nu_j} T^{N_j}},$$

where $[E_J^o]$ is a certain Galois cover of $E_J^o$ of degree $\gcd_{j \in J} N_j$, and is defined in [DL01, Section 3.3].

In particular, this formula implies that $Z_f^{\text{mot}}$ is rational over $\mathcal{M}_k$. More precisely,

$$Z_f^{\text{mot}}(T) \in \mathcal{M}_k \left[ T, \frac{1}{1 - \mathbb{L}^{-\nu_i} T^{N_i}} \right]_{i \in I} \subset \mathcal{M}_k \left[ T \right].$$
The motivic monodromy conjecture

Since the motivic zeta function specializes to the $p$-adic zeta function for almost all $p$, one hopes that a motivic upgrade of the $p$-adic monodromy conjecture exists as well. One has to be careful though when formulating the motivic monodromy conjecture since $\mathcal{M}_k$ is not a domain and hence the concept of pole is a priori not defined. We will discuss this issue more in-depth in Section 4.3.1. Denef and Loeser formulated the motivic monodromy conjecture as follows:

**Conjecture 2.1.13** (The motivic monodromy conjecture for hypersurface singularities). Let $k$ be a subfield of $\mathbb{C}$ and let $f \in k[x_1, \ldots, x_m]$ be a non-constant polynomial. Set $X_0$ to be the complex analytic space defined by the equation $f = 0$ in $\mathbb{C}^m$. There exists a finite set $S \subset \mathbb{Z}_{<0} \times \mathbb{Z}_{>0}$ such that $Z_{mot}^f(T) \in \mathcal{M}_k \left[T, \frac{1}{1 - L^a T^b}\right]_{(a,b) \in S}$, and such that for each $(a, b) \in S$, the complex number $\exp(2\pi i a/b)$ is a monodromy eigenvalue of $f$ at some point of $X_0$.

So far, it seems that almost all known partial proofs of the $p$-adic monodromy conjecture could be naturally adapted to the motivic setting. Since the motivic zeta function specializes to the $p$-adic zeta function for almost all primes $p$, researchers try to solve this more general conjecture instead of the $p$-adic one.

### 2.2 The motivic zeta function for Calabi-Yau varieties

Fix an algebraically closed field $k$ of characteristic zero. Define $R = k[[t]]$ and $K = k((t))$. Fix an algebraic closure $K^{alg}$ of $K$.

Let $X$ be a Calabi-Yau variety over $K$ of dimension $m$. Let $\omega$ be a volume form on $X$, i.e., a nowhere-vanishing differential form of maximal rank. For every integer $d \geq 1$, define $K(d)$ as the unique totally ramified extension of $K$ in $K^{alg}$ of degree $d$. For example, when $K = \mathbb{C}((t))$, we have $K(d) = \mathbb{C}((\sqrt[d]{t}))$. We define $X(d) = X \times_K K(d)$, and $\omega(d)$ as the pullback of $\omega$ to $X(d)$.

We start by defining the motivic zeta function associated with $X$ and $\omega$. In [HN12, Definition 6.1.4], the motivic zeta function is defined for a specific choice of $\omega$, but this condition on $\omega$ can be omitted.
Definition 2.2.1 ([HN12, Definition 6.1.4]). The motivic zeta function $Z_{X,\omega}(T)$ of $X$ is defined as

$$Z_{X,\omega}(T) = \sum_{d \in \mathbb{Z}_{>0}} \left( \int_{X(d)} |\omega(d)| \right) T^d \in \mathcal{M}_k^\mu [T].$$

The integral $\int_{X(d)} |\omega(d)|$ is a motivic integral measuring the space of $K(d)$-rational points on $X$.

To understand the inspiration for the definition of the motivic zeta function for Calabi-Yau varieties, we should have a look at an alternative interpretation of $Z_{mot}^{f,x}(T)$ that Nicaise and Sebag give in [NS07] and [Nic09, Corollory 9.6] in terms of non-archimedean geometry. The function $Z_{mot}^{f,x}(T)$ is the local motivic zeta function and is defined as follows: let $f \in k[x_1, \ldots, x_m]$ be a polynomial and let $x \in \mathbb{A}_k^m$ be a point such that $f$ is smooth at $x$. We define

$$Z_{mot}^{f,x}(T) = \sum_{d \geq 0} [X_{d,1,x}] \mathbb{L}^{-md} T^d \in \mathcal{M}_k[T],$$

where

$$X_{d,1,x} = \{ \varphi \in X_{d,1} \mid \varphi(0) = x \}.$$ 

Nicaise and Sebag gave an alternative formula for $Z_{mot}^{f,x}(T)$ in terms of the analytic Milnor fiber $\mathcal{F}_x$ of $f$ at $x$ and the Gelfand-Leray form $\omega/df$ on $\mathcal{F}_x$ associated with a volume form $\omega$ on $\mathbb{A}_k^m$:

$$Z_{mot}^{f,x}(\mathbb{L}T) = \sum_{d \in \mathbb{Z}_{>0}} \left( \int_{\mathcal{F}_x(d)} \left| \frac{\omega}{df}(d) \right| \right) T^d.$$ 

This interpretation gave the inspiration to formulate the version of the motivic zeta function discussed for Calabi-Yau varieties.

**Denef-Loeser formula**

The formula in Theorem 2.1.12 can be extremely useful to compute the motivic zeta function $Z_{mot}^{f}(T)$ in practice, when an embedded resolution of $f$ is known. Luckily, we have an analogue for zeta functions of Calabi-Yau varieties, proven by Bultot and Nicaise in [BN16], that requires the data of an snc-model $X$ of $X$. We also refer to [NS07, Corollary 7.7], where Nicaise and Sebag prove a similar result for formal schemes.

Let $X$ be a Calabi-Yau variety over $K$ with a volume form $\omega$ and snc-model $X$. Denote by $X_k = \sum_{i \in I} N_i E_i$ the special fiber of the model $X$. For every $i \in I$,
we define $\nu_i$ as the order of $\text{div}(\omega)$ along $E_i$, when $\omega$ is viewed as a rational section of the relative log-canonical bundle $\omega_{X/R}(\mathcal{X}_{k,\text{red}} - \mathcal{X}_k)$. The couple $(N_i, \nu_i)$ is called the numerical data of $E_i$.

**Theorem 2.2.2** ([BN16, Corollary 4.3.2]). Let $X$ be a Calabi-Yau variety over $K$ with a volume form $\omega$ and snc-model $\mathcal{X}$. Denote by $\mathcal{X}_k = \sum_{i \in I} N_i E_i$ the special fiber of the model $\mathcal{X}$ and let $(N_i, \nu_i)$ be the numerical data of $E_i$.

For every non-empty subset $J \subseteq I$, we define $E_J = \cap_{j \in J} E_j$ and

$$E_J^\circ = E_J \setminus \left( \cup_{i \in I \setminus J} E_i \right).$$

Let $E_J^\circ$ be the finite étale cover $E_J^\circ \times X \mathcal{Y}$ of $E_J^\circ$, where $\mathcal{Y}$ is the normalization of $\mathcal{X} \times_R R[x]/(x^{N_J} - t)$ with $N_J = \gcd\{N_j \mid j \in J\}$. The motivic zeta function of $X$ can be expressed as

$$Z_{X,\omega}(T) = \sum_{\emptyset \neq J \subseteq I} (\mathbb{L} - 1)^{|J| - 1} [\overline{E_J^\circ}] \prod_{j \in J} \frac{\mathbb{L}^{-\nu_j T^{N_J}}}{1 - \mathbb{L}^{-\nu_j T^{N_J}}}$$

in $\hat{\mathcal{M}}^\mu_k[T]$.

**Remark 2.2.3.** In Chapter 4, we need a stronger version of Theorem 2.2.2, where we weaken the assumption of $\mathcal{X}$ being a scheme to $\mathcal{X}$ being an algebraic space. Theorem 2.2.2 still holds for algebraic spaces because of [HN17, Proposition 7.2.2] and the fact that normalized base change commutes with formal completion.

**Remark 2.2.4.** Theorem 2.2.2 immediately implies that $Z_{X,\omega}(T)$ is a rational function over $\hat{\mathcal{M}}^\mu_k$. Although the definition of a pole will not be given until Section 4.3.1, we can intuitively see that all poles of $Z_{X,\omega}(T)$ are of the form $-\nu_i/N_i$ for some $i \in I$. Since a normal crossings model is not unique, one cannot expect all ‘candidate poles’ $-\nu_i/N_i$ to be actual poles of $Z_{X,\omega}(T)$. But even candidate poles that appear in every model, will not always be actual poles. This phenomenon is intimately related with the monodromy property for Calabi-Yau varieties.

### 2.3 Monodromy eigenvalues

In this subsection, we will give a more algebraic definition of monodromy eigenvalues. In Definition 2.1.5, we defined the monodromy transformation $M_{x,i}$ of a hypersurface singularity to be the action of the canonical generator of $\pi_1(D^*)$ on $H^i(F_x, \mathbb{C})$, the singular cohomology of the Milnor fiber $F_x$. Let us have a look at how we can mimic this construction for a smooth and proper $K$-variety $X$. 
In Definition 2.1.5 we considered a small open disc $D$ around the origin and the punctured disc $D^*$. One can think of $\text{Spec}(R)$ as a small open disc around the origin, therefore, it plays the role of $D$. Furthermore, $\text{Spec}(K)$ can be thought of as the punctured disc $D^*$.

The algebraic analogue of $\tilde{D}^*$ is $\text{Spec}(K_{\text{alg}})$. This can be understood as follows: the universal covering space $\tilde{D}^*$ is the inverse limit of the projective system of finite covers $\{D^* \to D^*: t \mapsto t^d\}_{d \in \mathbb{Z}_{\geq 0}}$ ordered by divisibility. The right algebraic analogue of a cover is a finite étale morphism. So we would like to look at the inverse limit of the projective system of finite étale morphisms $\{\text{Spec}(K(d)) \to \text{Spec}(K): t' \to t\}_{d \in \mathbb{Z}_{\geq 0}}$, where $K(d)$ is the uniquely totally ramified extension of $K$ of degree $d$ in $K_{\text{alg}}$ with uniformizer $t'$. This inverse limit is exactly $\text{Spec}(K_{\text{alg}})$, which means that it is indeed the appropriate analogue of $\tilde{D}^*$.

The topological monodromy group $\pi_1(D^*)$ is equal to the group of deck transformations $\text{Aut}(\tilde{D}^*/D^*)$. So the algebraic counterpart of $\pi_1(D^*)$ is $\text{Aut}(\text{Spec}(K_{\text{alg}})/\text{Spec}(K))$, which is exactly $\text{Gal}(K_{\text{alg}}/K)$. The Galois group $\text{Gal}(K_{\text{alg}}/K)$ is isomorphic to the profinite group of roots of unity $\hat{\mu} = \varprojlim \mu_d$.

In the topological setting, we noticed that the Milnor fiber $F_x$ is homotopy equivalent to $(g^{-1}(D^*) \cap B) \times_{D^*} \tilde{D}^*$. So the algebraic match of $F_x$ is the base change $X \times_K K_{\text{alg}}$.

We put all the algebraic counterparts in the following table:

| Complex topological | Algebraic counterpart |
|---------------------|-----------------------|
| $D$                 | $\text{Spec}(R)$     |
| $D^*$               | $\text{Spec}(K)$     |
| $\tilde{D}^*$       | $\text{Spec}(K_{\text{alg}})$ |
| $\pi_1(D^*)$ (=$\text{Aut}(\tilde{D}^*/D^*)$) | $\text{Gal}(K_{\text{alg}}/K)$ (=$\text{Aut}(\text{Spec}(K_{\text{alg}})/\text{Spec}(K))$) |
| $F_x$ (=$g^{-1}(D^*) \cap B) \times_{D^*} \tilde{D}^*$) | $X \times_K K_{\text{alg}}$ |

When we interpret $X$ as a family of varieties over the punctured disc, the monodromy action can be thought of as the automorphism on a variety in the family, when travelling once around the origin. Let’s define this more formally.

**Definition 2.3.1.** Let $X$ be a smooth, proper variety over $K$ and let $\sigma$ be a topological generator of $\text{Gal}(K_{\text{alg}}/K)$. For every $i \geq 0$, the monodromy transformation $M_{X,i}$ is the action of $\sigma$ on the $\ell$-adic cohomology space $H^i(X \times_K K_{\text{alg}}, \mathbb{Q}_\ell)$. A monodromy eigenvalue is an eigenvalue of $M_{X,i}$ for some $i \geq 0$. 
Monodromy zeta function

Definition 2.3.2. The monodromy zeta function is the alternating product of the characteristic polynomials of the monodromy transformations $M_{X,i}$ on the $\ell$-adic cohomology spaces $H^m(X \times K \text{alg}, Q_\ell)$:

$$\zeta_X(T) = \prod_{m \geq 0} \left( \det (T \cdot \text{Id} - \sigma | H^m(X \times K \text{alg}, Q_\ell)) \right)^{(-1)^{m+1}} \in \mathbb{Q}_\ell(T).$$

Remember that in Section 2.1 we found that both the motivic zeta function and the monodromy zeta function of a hypersurface singularity can be computed from the data of an embedded resolution of singularities. In Section 2.2 we found that the motivic zeta function of a Calabi-Yau variety can be computed when the data of an $\text{snc}$-model of the Calabi-Yau variety is given. It turns out that the monodromy zeta function from Definition 2.3.2 can be computed from the data of an $\text{snc}$-model as well.

Nicaise proved in [Nic13, Theorem 2.6.2] the following simple expression for the monodromy zeta function.

Proposition 2.3.3. Let $X$ be a smooth, proper $K$-variety with $\text{snc}$-model $X$. Denote by $X_k = \sum_{i \in I} N_i E_i$ the special fiber of the model $X$.

The monodromy zeta function of $X$ can be written as

$$\zeta_X(T) = \prod_{i \in I} (T^{N_i} - 1)^{-\chi(E_i^\circ)},$$

where $\chi(E_i^\circ)$ is the topological Euler characteristic of $E_i^\circ = E_i \setminus (\cup_{j \in I \setminus \{i\}} E_j)$.

This is a variant of A’Campo’s formula for the local monodromy zeta function of a hypersurface singularity.

By the formula of Proposition 2.3.3 we can find some of the monodromy eigenvalues from the data of an $\text{snc}$-model of $X$, but in general we won’t find all of them, since cancellations may occur: a monodromy eigenvalue is not necessarily a zero or pole of $\zeta_X(T)$. Note that for a $K3$ surface, all monodromy eigenvalues appear as poles of the monodromy zeta function, since the cohomology spaces of a $K3$ surface are trivial in odd degree.
2.4 The monodromy property for Calabi-Yau varieties

Informally, the monodromy property for Calabi-Yau varieties expresses that poles of the motivic zeta function correspond to monodromy eigenvalues. By Theorem 2.2.2, one can interpret the monodromy property for a Calabi-Yau variety $X$ as a precise relation between its cohomology and the geometry of its snc-models. The following formulation of the monodromy property appeared in [HN12, Definition 6.4.1] and a formulation of the equivariant monodromy property can be found in [HN17, Definition 2.3.5]

**Definition 2.4.1.** Let $X$ be a Calabi-Yau variety over $K$ with volume form $\omega$, and let $\sigma$ be a topological generator of the monodromy group $\text{Gal}(K^{alg}/K)$. We say that $X$ satisfies the monodromy property if there exists a finite subset $S$ of $\mathbb{Z} \times \mathbb{Z}_{>0}$ such that

$$Z_{X,\omega}(T) \in \mathcal{M}_k^\mu \left[ T, \frac{1}{1 - L^a T^b} \right]_{(a,b) \in S},$$

and such that for each $(a,b) \in S$, we have that $\exp(2\pi ia/b)$ is an eigenvalue of $\sigma$ on $H^m(X \times_K K^{alg}, \mathbb{Q}_\ell)$, for some $m \geq 0$, and for every embedding of $\mathbb{Q}_\ell$ into $\mathbb{C}$.

**Remark 2.4.2.** Whether a Calabi-Yau variety $X$ satisfies the monodromy property does not depend on the choice of volume form $\omega$. Indeed, let $\omega'$ be another volume form on $X$. Since $X$ has trivial canonical bundle, there exists a unit $u \in K^\times$ such that $\omega' = u \cdot \omega$. It follows immediately from the definition of the motivic zeta function that

$$Z_{X,\omega'}(T) = Z_{X,\omega}(L^{-\text{ord}_\ell(u)} T). \quad (2.3)$$

Let $S$ and $S'$ be as in Definition 2.4.1 for $Z_{X,\omega}(T)$ and $Z_{X,\omega'}(T)$ respectively. Equation (2.3) implies that if $(a,b) \in S$, then $(a - \text{ord}_\ell(u) \cdot b, b) \in S'$. But if $\exp(2\pi ia/b)$ is a monodromy eigenvalue, then so is $\exp(2\pi i(a/b + \text{ord}_\ell u))$. Therefore, the monodromy property does not depend on the chosen volume form.

In [HN11, Theorem 8.5], Halle and Nicaise prove that abelian varieties satisfy the monodromy property, when the $\hat{\mu}$-action is ignored. The proof uses in an essential way properties of abelian varieties and their Néron models. In [HN17, Theorem 4.2.2], they upgrade the result so that the $\hat{\mu}$-action is taken into account.

**Theorem 2.4.3.** Let $X$ be an abelian variety over $K$ and let $\omega$ be a volume form on $X$. The motivic zeta function $Z_{X,\omega}(T)$ has a unique pole and the monodromy property holds for $X$. 
Remark 2.4.4. For a suitable choice of \( \omega \), the unique pole of \( Z_{X,\omega}(T) \) coincides with Chai’s base change conductor.

Halle and Nicaise also proved the following result in [HN17, Corollary 5.3.3]:

**Theorem 2.4.5.** Let \( X \) be a Calabi-Yau variety over \( K = \mathbb{C}((t)) \) with volume form \( \omega \), and assume that \( X \) has an equivariant Kulikov model over \( R(d) = \mathbb{C}[[\sqrt[4]{t}] \) for some \( d > 0 \). The motivic zeta function \( Z_{X,\omega}(T) \) has a unique pole and the monodromy property holds for \( X \).

So far, all Calabi-Yau varieties known to satisfy the monodromy property, have a motivic zeta function with a unique pole. We will give the first class of Calabi-Yau varieties satisfying the monodromy property with a motivic zeta function with more than one pole.

In this thesis, we will focus on Calabi-Yau varieties of dimension 2. To investigate the monodromy property in dimension 2, the only remaining case is that of \( K3 \) surfaces, i.e., 2-dimensional Calabi–Yau varieties \( X \) with \( H^1(X,\mathcal{O}_X) = 0 \). Indeed, if \( X \) is a 2-dimensional Calabi–Yau variety with \( H^1(X,\mathcal{O}_X) \neq 0 \), then \( X \) is an abelian surface and hence the monodromy property holds for \( X \) by Theorem 2.4.3.

Using Kulikov’s classification, Stewart and Vologodsky give in [SV11] an explicit formula for \( Z_{X,\omega}(T) \) when \( X \) is a \( K3 \) surface allowing a semi-stable model. From their formula, it can be deduced that the motivic zeta function \( Z_{X,\omega}(T) \) has only one pole. From Proposition 2.2.2, we immediately see that this pole is an integer and therefore, \( X \) trivially satisfies the monodromy property.

In this thesis, we will therefore focus on \( K3 \) surfaces without a semi-stable model. In Theorem 2.2.2 and Theorem 2.3.3, we explained why information about an snc-model \( X \) is extremely useful to verify the monodromy property for \( X \) in practice. To our knowledge, the only classification of snc-models of \( K3 \) surfaces without the condition of semi-stability, is the classification of Crauder and Morrison [CMS3] where they classify triple-point-free snc-models of surfaces with a numerically trivial canonical bundle. We will study and refine this classification in Chapter 3.
Chapter 3

The Crauder-Morrison classification

In this chapter, we will discuss and extend the results of [CMS83]. In this paper, Crauder and Morrison classify triple-point-free snc-models of smooth, proper surfaces with trivial pluricanonical bundle. The tools to prove this classification where developed by Crauder in [Cra83], and errata were published in [CM94].

In Chapter 2, in particular in Theorem 2.2.2 and Theorem 2.3.3 we explained why information about an snc-model $X$ of a Calabi-Yau variety $X$ is extremely useful to verify the monodromy property for $X$ in practice. This is the reason why the Crauder-Morrison classification is one of the major tools in this thesis: it gives concrete information about snc-models of $K3$ surfaces allowing a triple-point-free model.

In Section 3.1 we fix notation for this chapter and discuss some basic facts. The main theorem of [CMS83] is given in Section 3.2. The statement there is a rather rough classification and a lot more information can be given. The details that are relevant for this thesis are formulated in sections 3.3 to 3.7. Most of these statements appeared in [CMS83], but some are original. In particular, Theorem 3.5.2 has been announced in [Jas17].
3.1 Notation and basic facts

Notation

We fix an algebraically closed field $k$ of characteristic zero. Define $R = k[[t]]$ and $K = k((t))$, the fraction field of $R$. Let $X$ be a smooth, proper surface over $K$ with $\omega_{X/K} \otimes m \cong \mathcal{O}_X$ for some $m \geq 1$. Let $\mathcal{X}$ be an snc-model of $X$ with special fiber $\mathcal{X}_k = \sum_{i \in I} N_i E_i$, where we allow $X$ to be an algebraic space. Suppose moreover that $\mathcal{X}$ is triple-point-free, i.e., $E_i \cap E_j \cap E_k = \emptyset$, whenever $i$, $j$ and $k$ are pairwise distinct. Assume furthermore that $\mathcal{X}$ is relatively minimal for this property, i.e., it is not possible to contract components in the special fiber (in the category of algebraic spaces) such that the result is still a triple-point-free snc-model. A relatively minimal, triple-point-free snc-model is called a Crauder-Morrison model.

Let $\omega$ be a nowhere-vanishing section of $\omega_{X/K} \otimes m$. We can view $\omega$ as a rational section of $\omega_{\mathcal{X}/R}(\mathcal{X}_{k,\text{red}} - \mathcal{X}_k) \otimes m$, which is a multiple of the relative log-canonical bundle. Define $\nu_i$ such that $m \nu_i$ is the multiplicity of $\text{div}(\omega)$ along $E_i$. The couple $(N_i, \nu_i)$ is called the numerical data of $E_i$.

We define the weight $\rho_i$ of the component $E_i$ to be

$$\rho_i = \frac{\nu_i}{N_i} + 1. \quad (3.1)$$

Notice that the definition of $\nu_i$ and $\rho_i$ depends on the choice of $\omega$. If $\omega'$ is another nowhere-vanishing section of $\omega_{X/K} \otimes m$, inducing $\nu'_i$ and $\rho'_i$, then $\nu_i = \nu'_i + cN_i$ and $\rho_i = \rho'_i + c$ for a fixed constant $c \in \mathbb{Z}$.

We define $\Gamma$ to be the dual graph of $\mathcal{X}_k$. Notice that the dual complex of $\mathcal{X}_k$ is indeed a graph, because $\mathcal{X}_k$ is triple-point-free. Denote by $\Gamma_{\text{min}}$ the subgraph of $\Gamma$ spanned by the vertices corresponding to components $E_i$ with minimal weight $\rho_i$.

Basic facts

Let $K_{\mathcal{X}/R}$ be the relative canonical divisor of $\mathcal{X}$ over $R$. By definition of $\nu_i$, we have

$$K_{\mathcal{X}/R} \equiv \sum_{i \in I} (\nu_i + N_i - 1)E_i. \quad (3.2)$$
Since the special fiber $X_k$ is a numerically trivial divisor on $X$, we have the relation
\[ E_j \equiv -\frac{1}{N_j} \sum_{i \neq j} N_i E_i. \]
Together with the adjunction formula [KM98, Proposition 5.73], this relation yields
\[ K_{E_j/k} \equiv \left( (K_{X/R} + E_j) \cdot E_j \right)_{E_j} \]
\[ \equiv \sum_{i \neq j} \left( \nu_i - \frac{N_i}{N_j} \nu_j - 1 \right) (E_i \cdot E_j)_{E_j} \]
\[ = \sum_{i \neq j} (N_i(\rho_i - \rho_j) - 1) (E_i \cdot E_j)_{E_j}. \] (3.3)

For every $j \neq i$, we define
\[ a_{ji} = \begin{cases} \end{cases} N_i(\rho_i - \rho_j) - 1 & \text{if } E_i \cap E_j \neq \emptyset, \\ 0 & \text{otherwise}. \]

Then
\[ K_{E_j/k} \equiv \sum_{i \neq j} a_{ji}(E_i \cdot E_j)_{E_j}, \] (3.4)
and we have
\begin{align*}
a_{ji} &< -1 \quad \text{iff} \quad \rho_j > \rho_i, \\ a_{ji} &= -1 \quad \text{iff} \quad \rho_j = \rho_i, \\ a_{ji} &> -1 \quad \text{iff} \quad \rho_j < \rho_i. \end{align*} (3.5)

### 3.2 The Crauder-Morrison classification

Crauder and Morrison classified the special fiber $X_k$ of relatively minimal, triple-point-free snc-models $X$ of a smooth, proper surface $X$ over $K$ with trivial pluricanonical bundle.

**Theorem 3.2.1** (Crauder-Morrison Classification [CMS83]). Let $X$ be a smooth, proper surface over $K$ with $\omega_{X/k}^n \simeq O_X$ for some $m \geq 1$, and let $X$ be a Crauder-Morrison model of $X$. Then $X$ has the following properties:
(i) $\Gamma_{\text{min}}$ is a connected subgraph of $\Gamma$. It is either a single vertex, a cycle or a chain. We call $X$ a flowerpot degeneration, a cycle degeneration or a chain degeneration respectively.

(ii) Each connected component of $\Gamma \setminus \Gamma_{\text{min}}$ is a chain (called a flower) $F_0 - F_1 - \cdots - F_{\ell}$, where only $F_{\ell}$ meets $\Gamma_{\text{min}}$, and $F_0$ meets a unique vertex of $\Gamma_{\text{min}}$. The weights strictly decrease: $\rho_0 > \rho_1 > \cdots > \rho_\ell$. The surface $F_0$ is either minimal ruled or it is isomorphic to $\mathbb{P}^2$. If $F_0 \simeq \mathbb{P}^2$, then $F_0 \cap F_1$ is either a line or a conic on $F_0$. The surface $F_i$ is minimal ruled with sections $F_{i-1} \cap F_i$ and $F_i \cap F_{i+1}$ for $1 \leq i \leq \ell$.

(iii) Suppose $\Gamma_{\text{min}}$ is a single vertex $P$ (called a flowerpot). The surface $P$ is isomorphic to $\mathbb{P}^2$, or it is a ruled surface, or $K_{P/k} \equiv 0$.

(iv) Suppose $\Gamma_{\text{min}}$ is a cycle $V_1 - V_2 - \cdots - V_k$. Then there are no flowers and all components have the same multiplicity, i.e., there exists an integer $N \geq 1$ such that $X_k = N \left( \sum_{i=1}^{k} V_i \right)$. Furthermore, for every $i = 1, \ldots, k$, the component $V_i$ is an elliptic, minimal ruled surface with sections $V_{i-1} \cap V_i$ and $V_i \cap V_{i+1}$, where we identify $V_0 = V_k$ and $V_{k+1} = V_1$.

(v) Suppose $\Gamma_{\text{min}}$ is a chain $V_0 - V_1 - \cdots - V_k - V_{k+1}$. If $i = 1, \ldots, k$, the surface $V_i$ is an elliptic, ruled surface with sections $V_{i-1} \cap V_i$ and $V_i \cap V_{i+1}$. If $i = 0$ or $k + 1$, the surface $V_i$ is either isomorphic to $\mathbb{P}^2$, or it is a rational or elliptic, ruled surface.

**Remark 3.2.2.** Note that in Theorem 3.2.1 we do not assume the special fiber to be reduced. So in general, the model $X$ does not need to be semi-stable. A classification of semi-stable models of $K3$ surfaces has been given by Kulikov in [Kul77] with corrections by Persson and Pinkham in [PP81]. Crauder and Morrison weaken the hypothesis of semi-stability, but impose the extra condition of the model being triple-point-free.

In the pictures below, you can find illustrations of possible dual graphs of models of surfaces considered in Theorem 3.2.1. We label the vertices of the graphs with $(N_i, \rho_i)$.

![Figure 3.1: A flowerpot degeneration](image)

In Figure 3.1 you can see an example of a flowerpot degeneration with two flowers. Notice that although the dual graph is in fact a chain, there is only one surface with minimal weight $4/3$, and therefore it is a flowerpot degeneration. The weights of the components of the flowers are strictly decreasing towards the flowerpot.
In Figure 3.2 you can see a cycle degeneration. All components have weight 2, and hence there are no flowers. In fact, a cycle degeneration never has flowers. Moreover, all components have the same multiplicity.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{cycle_degeneration.png}
\caption{A cycle degeneration}
\end{figure}

In Figure 3.3 you can see an example of a possible chain degeneration. There are two components in the chain, both with weight $3/2$. Moreover, there are three flowers, all of length 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{chain_degeneration.png}
\caption{A chain degeneration}
\end{figure}

We also refer to Example 4.4.1 for an example of a $K3$ surface allowing a flowerpot degeneration.

## 3.3 Flowers

### 3.3.1 Terminology

**Definition 3.3.1.** Let $X$ be a smooth, proper surface over $K$ with $\omega_{X/K}^\otimes m \simeq \mathcal{O}_X$ for some $m \geq 1$, and let $\mathcal{X}$ be a Crauder-Morrison model of $X$. Suppose that the special fiber $\mathcal{X}_k$ has a flower $N_0F_0 + N_1F_1 + \cdots + N_\ell F_\ell$, where $F_i \cap F_j = \emptyset$ if and only if $j \not\in \{i-1, i, i+1\}$. The top of the flower is $F_0$, and $F_\ell$ meets $\Gamma_{\min}$ in $F_{\ell+1}$.

The double curve $F_\ell \cap F_{\ell+1}$ is called a flowercurve. The genus of the flower is the genus of the flowercurve $F_\ell \cap F_{\ell+1}$. We call the flower rational if $F_0$ is a rational surface.

**Remark 3.3.2.** For $1 \leq i \leq \ell$, the double curves $F_{i-1} \cap F_i$ and $F_i \cap F_{i+1}$ are sections of the minimal ruled surface $F_i$ and therefore, they have the same genus.
So if a flower has genus $g$, all double curves $F_i \cap F_{i+1}$ are curves of genus $g$ for $0 \leq i \leq \ell$.

**Remark 3.3.3.** A flower is rational if and only if it has genus zero. Indeed, if the flower is rational, then either $F_0$ is a rational, ruled surface, or $F_0 \simeq \mathbb{P}^2$. If $F_0$ is a rational, ruled surface, then $F_0 \cap F_1$ is a rational curve, since it is a section on $F_0$. If $F_0 \simeq \mathbb{P}^2$, then the curve $F_0 \cap F_1$ is either a line or a conic, and therefore, it is a rational curve. Remark 3.3.2 explains why the flower has genus 0. Conversely, if the flower is not rational, then $F_0$ is ruled over a curve of genus $g \geq 1$. Since $F_0 \setminus F_1$ is a section of the ruling on $F_0$, it has genus $g$, and hence the flower has genus $g \geq 1$, by Remark 3.3.2.

**Definition 3.3.4.** If $F_0 \simeq \mathbb{P}^2$, and if $F_0 \setminus F_1$ is a conic, then the flower is called a conic-flower.

**Proposition – Definition 3.3.5** ([CM83, Theorem 3.2 and Theorem 3.5]). Let $X$ be a smooth, proper surface over $K$ with $\omega_{X/K}^\otimes m \simeq \mathcal{O}_X$ for some $m \geq 1$, and let $X$ be a Crauder-Morrison model of $X$. Suppose that the special fiber $X_k$ has a flower $N_0 F_0 + N_1 F_1 + \cdots + N_\ell F_\ell$, where $F_i \cap F_j = \emptyset$ if and only if $j \notin \{i-1, i, i+1\}$. The top of the flower is $F_0$, and $F_\ell$ meets $\Gamma_{\min}$ in $F_{\ell+1}$. There is an integer $M \in \{2, 3, 4, 6, 8, 12\}$, called the type of the flower, such that

$$K_{F_{\ell+1}/k} \equiv (-1 + \frac{2}{M})(F_\ell \cdot F_{\ell+1})_{F_{\ell+1}} - D,$$

where $D$ is an effective divisor on $F_{\ell+1}$ disjoint from the flowercurve $F_\ell \cap F_{\ell+1}$. We also call $M$ the type of the flowercurve $F_\ell \cap F_{\ell+1}$.

**Remark 3.3.6.** In the notation of equation (3.4) with $E_i = F_\ell$ and $E_j = F_{\ell+1}$, we have $a_{ji} = -1 + \frac{2}{M}$. So in particular $-1 < a_{ji} \leq 0$ and $a_{ji} = 0$ if and only if $M = 2$.

### 3.3.2 Classification of the flowers

Crauder and Morrison classified the flowers into 21 combinatorial classes.

**Theorem 3.3.7** ([CM83, Theorem 3.9] and [CM94, Appendix 2]). Let $X$ be a smooth, proper surface over $K$ with $\omega_{X/K}^\otimes m \simeq \mathcal{O}_X$ for some $m \geq 1$, and let $X$ be a Crauder-Morrison model of $X$. Suppose that the special fiber $X_k$ has a flower $N_0 F_0 + N_1 F_1 + \cdots + N_\ell F_\ell$, where $F_i \cap F_j = \emptyset$ if and only if $j \notin \{i-1, i, i+1\}$. The top of the flower is $F_0$, and $F_\ell$ meets $\Gamma_{\min}$ in $F_{\ell+1}$. The multiplicity of $F_{\ell+1}$ is $N_{\ell+1}$.

Then there exists an integer $N \geq 1$, such that the flower satisfies one of the 21 possibilities described in Tables 3.1, 3.2 and 3.3.
### Table 3.1: Flowers with $F_0 \simeq \mathbb{P}^2$ and $F_0 \cap F_1$ is a line

| Name | Type | Composition         | $\ell$ | $N_{F_{\ell+1}}$ |
|------|------|---------------------|--------|------------------|
| 2A   | 2    | $NF_0$              | 0      | $2N$             |
| 3A   | 3    | $NF_0 + 2NF_1$      | 1      | $3N$             |
| 3B   | 3    | $NF_0$              | 0      | $3N$             |
| 4A   | 4    | $NF_0 + 2NF_1 + 3NF_2$ | 2   | $4N$             |
| 4B   | 4    | $NF_0$              | 0      | $4N$             |
| 6A   | 6    | $NF_0 + 2NF_1 + 3NF_2 + 4NF_3 + 5NF_4$ | 4   | $6N$             |
| 6B   | 6    | $NF_0$              | 0      | $6N$             |

### Table 3.2: Flowers with $F_0 \simeq \mathbb{P}^2$ and $F_0 \cap F_1$ is a conic

| Name | Type | Composition         | $\ell$ | $N_{F_{\ell+1}}$ |
|------|------|---------------------|--------|------------------|
| 2B   | 2    | $2NF_0$             | 0      | $N$              |
| 2C   | 2    | $2NF_0 + NF_1 + \cdots + NF_\ell$ | $\ell$ | $N$              |
| 4C   | 4    | $2NF_0 + NF_1 + \cdots + NF_\ell$ | $\ell$ | $2N$             |
| 4D   | 4    | $NF_0 + NF_1 + \cdots + NF_\ell$ | $\ell$ | $N$              |
| 6C   | 6    | $2NF_0 + NF_1 + \cdots + NF_{\ell-1} + 2NF_\ell$ | $\ell$ | $3N$             |
| 6D   | 6    | $2NF_0 + NF_1 + \cdots + NF_\ell$ | $\ell$ | $3N$             |
| 6E   | 6    | $2NF_0$             | 0      | $3N$             |

### Table 3.3: Flowers with $F_0$ minimal ruled

| Name | Type | Composition         | $\ell$ | $N_{F_{\ell+1}}$ |
|------|------|---------------------|--------|------------------|
| 4\alpha | 4    | $NF_0$              | 0      | $2N$             |
| 6\alpha | 6    | $NF_0 + 2NF_1$      | 1      | $3N$             |
| 6\beta  | 6    | $NF_0$              | 0      | $3N$             |
| 8\alpha | 8    | $NF_0 + 2NF_1 + 3NF_2$ | 2   | $4N$             |
| 8\beta  | 8    | $NF_0$              | 0      | $4N$             |
| 12\alpha | 12   | $NF_0 + 2NF_1 + 3NF_2 + 4NF_3 + 5NF_4$ | 4   | $6N$             |
| 12\beta | 12   | $NF_0$              | 0      | $6N$             |

#### 3.3.3 Self-intersection of a double curve in the flower

We will now prove some results concerning the self-intersection of the double curves $F_\ell \cap F_{\ell+1}$ for $i = 0, \ldots, \ell$.

**Lemma 3.3.8.** Let $X$ be a smooth, proper surface over $K$ with $\omega_{X/K}^{\otimes m} \simeq O_X$ for some $m \geq 1$, and let $\mathcal{X}$ be a Crauder-Morrison model of $X$. Suppose that
the special fiber \(X_k\) has a flower \(N_0 F_0 + N_1 F_1 + \cdots + N_\ell F_\ell\), where \(F_i \cap F_j = \emptyset\) if and only if \(j \not\in \{i - 1, i, i + 1\}\). The top of the flower is \(F_0\), and \(F_\ell\) meets \(\Gamma_{\min}\) in \(F_{\ell + 1}\).

Denote by \(C\) the flowercurve \(F_\ell \cap F_{\ell + 1}\). Suppose the flower has genus \(g\) and type \(M\), then

\[
(C^2)_{F_{\ell + 1}} = M(g - 1).
\]

So in particular, if the flower is rational, then \((C^2)_{F_{\ell + 1}} = -M\).

Proof. By Proposition-Definition 3.3.5, we can write

\[
K_{F_{\ell + 1}/k} \equiv \left(-1 + \frac{2}{M}\right)C - D,
\]

where \(D\) is an effective divisor on \(F_{\ell + 1}\) disjoint from the flowercurve \(C\). The adjunction formula gives

\[
2g - 2 = K_{F_{\ell + 1}/k} \cdot C + C^2 = \frac{2}{M} C^2.
\]

It follows that \(C^2 = M(g - 1)\).

We also state and prove the following well-known relation:

**Lemma 3.3.9.** Let \(X\) be a smooth, proper surface over \(K\) with \(\omega_X^{\otimes m} \simeq \mathcal{O}_X\) for some \(m \geq 1\), and let \(X\) be a Crauder-Morrison model of \(X\). Suppose that the special fiber \(X_k\) has a flower \(N_0 F_0 + N_1 F_1 + \cdots + N_\ell F_\ell\), where \(F_i \cap F_j = \emptyset\) if and only if \(j \not\in \{i - 1, i, i + 1\}\).

Let \(C_i = F_{i-1} \cap F_i\). For \(i = 1, \ldots, \ell\), we have

\[
(C_{i+1}^2)_{F_i} = -(C_i^2)_{F_i}.
\]

Proof. Since \(F_i\) is minimal ruled, \(\text{Num}(F_i)\) is generated by a section \(C\) and a fiber \(\mathcal{F}\), and moreover, \(\mathcal{F}^2 = 0\) and \(C \cdot \mathcal{F} = 1\) by [Har77, Proposition V.2.3]. As \(C_i\) and \(C_{i+1}\) are sections, we can write

\[
C_i \equiv C + a\mathcal{F} \quad \text{and} \quad C_{i+1} \equiv C + b\mathcal{F},
\]

for some integers \(a, b \in \mathbb{Z}\). On the one hand,

\[
C_{i+1}^2 + C_i^2 = (C + a\mathcal{F})^2 + (C + b\mathcal{F})^2 = C^2 + 2a(C \cdot \mathcal{F}) + \mathcal{F}^2 + C^2 + 2b(C \cdot \mathcal{F}) + \mathcal{F}^2 = 2(C^2 + a + b).
\]
On the other hand,
\[ 0 = C_i \cdot C_{i+1} = (C + aF) \cdot (C + bF) = C^2 + a + b, \]
since there are no triple points. This implies \((C_{i+1}^2)_{F_i} = -(C_i^2)_{F_i}\). \(\square\)

### 3.3.4 Combinatorial relations on the numerical data

For \(i = 0, \ldots, \ell + 1\), let \((N_i, \nu_i)\) be the numerical data of the component \(F_i\). We will derive combinatorial relations between the numerical data of consecutive components in a flower. Corollary 3.3.11 is an important consequence.

**Lemma 3.3.10.** Let \(X\) be a smooth, proper surface over \(K\) with \(\omega_{X/K}^{\otimes m} \cong \mathcal{O}_X\) for some \(m \geq 1\), and let \(\mathcal{X}\) be a Crauder-Morrison model of \(X\). Suppose that the special fiber \(\mathcal{X}_k\) has a flower \(N_0F_0 + N_1F_1 + \cdots + N_\ell F_\ell\), where \(F_i \cap F_j = \emptyset\) if and only if \(j \not\in \{i - 1, i, i + 1\}\). The top of the flower is \(F_0\), and \(F_\ell\) meets \(\Gamma_{\min}\) in \(F_{\ell+1}\).

(i) Suppose \(F_0 \cong \mathbb{P}^2\) and \(F_0 \cap F_1\) is a line on \(F_0\). Then we have
\[ \nu_1 = \frac{N_1}{N_0} \nu_0 - 2. \]

(ii) Suppose \(F_0 \cong \mathbb{P}^2\) and \(F_0 \cap F_1\) is a conic on \(F_0\). Then we have
\[ \nu_1 = \frac{N_1}{N_0} \nu_0 - 1/2. \]

(iii) Suppose \(F_0\) is a minimal ruled surface. Then we have
\[ \nu_1 = \frac{N_1}{N_0} \nu_0 - 1. \]

(iv) For \(j = 1, \ldots, \ell\), we have
\[ \nu_{j+1} = \frac{N_{j-1} + N_{j+1}}{N_j} \nu_j - \nu_{j-1}. \]

**Proof.** (i) Suppose \(C_1 = F_0 \cap F_1\) is a line on \(F_0 \cong \mathbb{P}^2\). From (3.3), it follows that
\[ K_{F_0/k} \equiv \left( \nu_1 - \frac{N_1}{N_0} \nu_0 - 1 \right) C_1. \]
Adjunction on $F_0$ gives
\[-2 = (K_{F_0/k} + C_1) \cdot C_1 = \left( \nu_1 - \frac{N_1}{N_0} \nu_0 - 1 \right) C_1^2 + C_1^2 = \left( \nu_1 - \frac{N_1}{N_0} \nu_0 \right) C_1^2.\]

Since $C_1$ is a line, $C_1^2 = 1$ and therefore $\nu_1 - \frac{N_1}{N_0} \nu_0 = -2$.

(ii) Suppose $C_1 = F_0 \cap F_1$ is a conic on $F_0 \simeq \mathbb{P}^2$. From (3.3), it follows that

\[K_{F_0/k} \equiv \left( \nu_1 - \frac{N_1}{N_0} \nu_0 - 1 \right) C_1.\]

Adjunction on $F_0$ gives
\[-2 = (K_{F_0/k} + C_1) \cdot C_1 = \left( \nu_1 - \frac{N_1}{N_0} \nu_0 - 1 \right) C_1^2 + C_1^2 = \left( \nu_1 - \frac{N_1}{N_0} \nu_0 \right) C_1^2.\]

Since $C_1$ is a conic, $C_1^2 = 4$, and therefore $\nu_1 - \frac{N_1}{N_0} \nu_0 = -1/2$.

(iii) Suppose $C_1 = F_0 \cap F_1$ is a section on the minimal ruled surface $F_0$. From (3.3), it follows that

\[K_{F_0/k} \equiv \left( \nu_1 - \frac{N_1}{N_0} \nu_0 - 1 \right) C_1.\]

Let $F$ be a fiber of the ruling on $F_0$. The adjunction formula gives
\[-2 = 2g(F) - 2 = K_{F_0/k} \cdot F + F^2 = K_{F_0/k} \cdot F.\]

Hence, we get that
\[-2 = \left( \nu_1 - \frac{N_1}{N_0} \nu_0 - 1 \right) C_1 \cdot F = \nu_1 - \frac{N_1}{N_0} \nu_0 - 1.\]

Therefore, $\nu_1 - \frac{N_1}{N_0} \nu_0 = -1$.

(iv) Let $C_j = F_{j-1} \cap F_j$ and $C_{j+1} = F_j \cap F_{j+1}$. Both $C_j$ and $C_{j+1}$ are sections on the minimal ruled surface $F_j$. From (3.3), it follows that

\[K_{F_j/k} \equiv \left( \nu_{j-1} - \frac{N_{j-1}}{N_j} \nu_j - 1 \right) C_j + \left( \nu_{j+1} - \frac{N_{j+1}}{N_j} \nu_j - 1 \right) C_{j+1}.\]

Let $F$ be a fiber of the minimal ruled surface $F_j$. On the one hand, the adjunction formula applied to $F$ gives
\[K_{F_j/k} \cdot F = (K_{F_j/k} + F) \cdot F = -2.\]
On the other hand, the expression for $K_{F_j/k}$ gives that
\[
K_{F_j/k} \cdot \mathcal{F} = \left( \nu_{j-1} - \frac{N_{j-1}}{N_j} \nu_j - 1 \right) C_j \cdot \mathcal{F} + \left( \nu_{j+1} - \frac{N_{j+1}}{N_j} \nu_j - 1 \right) C_{j+1} \cdot \mathcal{F}
\]
\[
= \left( \nu_{j-1} - \frac{N_{j-1}}{N_j} \nu_j \right) + \left( \nu_{j+1} - \frac{N_{j+1}}{N_j} \nu_j \right) - 2
\]
Hence, \( \left( \nu_{j-1} - \frac{N_{j-1}}{N_j} \nu_j \right) + \left( \nu_{j+1} - \frac{N_{j+1}}{N_j} \nu_j \right) = 0 \).

\[\square\]

**Corollary 3.3.11.** Suppose $X$ is a $K3$ surface or an abelian variety over $K$, with Crauder-Morrison model $X$. The special fiber $X_k$ does not have flowers of type $4D$.

**Proof.** Write $X_k = \sum_{i \in I} N_i E_i$. Let $(N_i, \nu_i)$ be the numerical data of $E_i$. Since $X$ is a $K3$ surface or an abelian variety, we have that $\omega_{X/k} \simeq \mathcal{O}_X$, which means that $m = 1$. It follows that $\nu_i \in \mathbb{Z}_{>0}$, for all $i \in I$. For a flower of type $4D$, relation (ii) of Lemma 3.3.10 gives
\[
\nu_1 = \nu_0 - \frac{1}{2},
\]
which is a contradiction. \[\square\]

**Corollary 3.3.12.** For every class in Theorem 3.3.7, there is a fixed relation between $\nu_{\ell+1}$ and $\nu_0$, which can be found in Table 3.4.

| Type | $\nu_{\ell+1}$ | Type | $\nu_{\ell+1}$ | Type | $\nu_{\ell+1}$ |
|------|----------------|------|----------------|------|----------------|
| 2A   | $2\nu_0 - 2$  | 2B   | $(\nu_0 - 1)/2$ | 4A   | $2\nu_0 - 1$  |
| 3A   | $3\nu_0 - 4$  | 2C   | $(\nu_0 - 2\ell - 1)/2$ | 6A   | $3\nu_0 - 2$  |
| 3B   | $3\nu_0 - 2$  | 4C   | $\nu_0 - 2\ell$ | 6B   | $3\nu_0 - 1$  |
| 4A   | $4\nu_0 - 6$  | 4D   | $(2\nu_0 - \ell - 1)/2$ | 8A   | $4\nu_0 - 3$  |
| 4B   | $4\nu_0 - 2$  | 6C   | $(3\nu_0 - 6\ell + 5)/2$ | 8B   | $4\nu_0 - 1$  |
| 6A   | $6\nu_0 - 10$ | 6D   | $(3\nu_0 - 6\ell + 1)/2$ | 12A  | $6\nu_0 - 5$  |
| 6B   | $6\nu_0 - 2$  | 6E   | $(3\nu_0 - 1)/2$ | 12B  | $6\nu_0 - 1$  |

Table 3.4: Relation between $\nu_{\ell+1}$ and $\nu_0$

**Proof.** By induction and Lemma 3.3.10 we can write $\nu_{\ell+1}$ in terms of $\nu_0$. The computations are straightforward and we will illustrate this fact by computing $\nu_{\ell+1}$ for two classes of flowers.
A flower of type $4A$ is given combinatorially by $NF_0 + 2NF_1 + 3NF_2$ for some integer $N \geq 1$, and $N_3 = 4N$. By numerical relation (iv), we get $\nu_2 = 2\nu_1 - \nu_0$. Using the numerical relation (i), we find that $\nu_1 = 2\nu_0 - 2$, and hence $\nu_2 = 3\nu_0 - 4$. By relation (iv), we have $\nu_3 = 2\nu_2 - \nu_1$. Substituting the expressions for $\nu_1$ and $\nu_2$ gives

$$\nu_3 = 4\nu_0 - 6.$$ 

A flower of type $2C$ is given combinatorially by $2NF_0 + NF_1 + \ldots + NF_\ell$, for some integer $N \geq 1$, and $N_{\ell+1} = N$. Relation (ii) gives $\nu_1 = (\nu_0 - 1)/2$. Relation (iv) gives $\nu_2 = 3\nu_1 - \nu_0$, and $\nu_{j+1} = 2\nu_j - \nu_{j-1}$, for $2 \leq j \leq \ell$. An easy induction argument shows that $\nu_j = \frac{\nu_0 - 2j + 1}{2}$, for $1 \leq j \leq \ell + 1$. So in particular, we get

$$\nu_{\ell+1} = \frac{\nu_0 - 2\ell - 1}{2}.$$ 

\[\square\]

### 3.4 Flowerpots

In this section, we will give more details about the structure of flowerpots and the possible combinations of flowers that a flowerpot degeneration can contain.

#### 3.4.1 Geometry of the pot

**Proposition 3.4.1.** Assume that $X$ is a smooth, proper surface over $K$ with $\omega_{X/K}^m \cong O_X$ for some $m \geq 1$, such that it has a Crauder-Morrison model $X$ where the subgraph $\Gamma_{\text{min}}$ of the dual graph $\Gamma$ has a unique vertex $P$.

The geometry of the flowerpot $P$ is one of the following:

- (i) Either $K_{P/k} \equiv 0$,
- (ii) or $P \cong \mathbb{P}^2$,
- (iii) or $P$ is minimal ruled,
- (iv) or $P$ is rational, non-minimal ruled.

**Proof.** Combine the remark directly before [Cra83, Proposition 2.17] with [CMS83, Theorem 3.2]. In [CMS83, Lemma 4.1], it is stated that a non-minimal ruled pot is rational. \[\square\]
3.4.2 Combinations of flowers in a flowerpot degeneration

**Lemma 3.4.2.** Assume that $X$ is a smooth, proper surface over $K$ with $\omega_{X/K}^{\otimes m} \simeq \mathcal{O}_X$ for some $m \geq 1$, such that it has a Crauder-Morrison model $\mathcal{X}$ where the subgraph $\Gamma_{\min}$ of the dual graph $\Gamma$ has a unique vertex $P$.

Assume there are $\lambda$ flowers in $X_k$, for some integer $\lambda \geq 1$. Denote by $C_1, \ldots, C_\lambda$ the flowercurves on $P$. Then one of the following holds:

(i) There exists a $j \in \{1, \ldots, \lambda\}$, such that the flowercurve $C_j$ has genus $g(C_j) > 1$, and for all $i \in \{1, \ldots, \lambda\} \setminus \{j\}$, the flowercurve $C_i$ is rational.

(ii) For all $i \in \{1, \ldots, \lambda\}$, the flowercurve $C_i$ has genus $g(C_i) \leq 1$.

**Proof.** Define $a_i = 1 - \frac{2}{M_i}$ for all $i = 1, \ldots, \lambda$, where $M_i$ is the type of the flowercurve $C_i$. We have

$$K_{P/k} \equiv -\sum_{i=1}^{\lambda} a_i C_i,$$

by Remark 3.3.6. By adjunction and the fact that there are no triple points, we have for any $i = 1, \ldots, \lambda$:

$$2g(C_i) - 2 = (K_{P/k} + C_i) \cdot C_i = (1 - a_i)C_i^2.$$

Suppose (ii) does not hold. Hence there exists a $j \in \{1, \ldots, \lambda\}$ such that the flowercurve $C_j$ has genus $g(C_j) > 1$. We will show that for $i \in \{1, \ldots, \lambda\} \setminus \{j\}$, the flowercurve $C_i$ is rational. By Theorem 3.3.7, the type of $C_j$ is $M_j \geq 4$. This implies that $C_j^2 > 0$, by Lemma 3.3.8. Moreover, for any $i \neq j$, we know that $C_i \cdot C_j = 0$, since there are no triple points. Hence the Hodge Index Theorem [Ful84, Example 19.3.1] implies that $C_i^2 < 0$. Therefore, $g(C_i) = 0$, by Lemma 3.3.8.

We will now discuss some restrictions on the combination of flowers in a flowerpot degeneration, depending on the geometry of the flowerpot.

**Proposition 3.4.3.** Assume that $X$ is a smooth, proper surface over $K$ with $\omega_{X/K}^{\otimes m} \simeq \mathcal{O}_X$ for some $m \geq 1$, such that it has a Crauder-Morrison model $\mathcal{X}$ where the subgraph $\Gamma_{\min}$ of the dual graph $\Gamma$ has a unique vertex $P$.

If $K_{P/k} \equiv 0$, then any flower in $X_k$ has either type 2 or genus 1.

**Proof.** Denote by $C_1, \ldots, C_\lambda$ the flowercurves on $P$ with types $M_1, \ldots, M_\lambda$ respectively. We will show that for every $i = 1, \ldots, \lambda$, we have either $g(C_i) = 1$
or $M_i = 2$. By Remark 3.3.6, we can write
\[ K_{P/k} \equiv \sum_{i=1}^{\lambda} \left( -1 + \frac{2}{M_i} \right) C_i. \]
Since $K_{P/k} \equiv 0$, it follows that $K_{P/k} \cdot C_i = 0$, and therefore, $C_i^2 = 0$ or $M_i = 2$. Adjunction yields that if $C_i^2 = 0$, then $g(C_i) = 1$. \qed

**Proposition 3.4.4** ([CM83, Lemma 3.6]). Assume that $X$ is a smooth, proper surface over $K$ with $\omega_{X/K} \otimes m \simeq \mathcal{O}_X$ for some $m \geq 1$, such that it has a Crauder-Morrison model $X$ where the subgraph $\Gamma_{\min}$ of the dual graph $\Gamma$ has a unique vertex $P$.

If $P \simeq \mathbb{P}^2$, then there exists exactly one flower, and it has type 4 and genus 10, or type 8 and genus 3.

**Proposition 3.4.5** ([CM83, Lemma 3.7]). Assume that $X$ is a smooth, proper surface over $K$ with $\omega_{X/K} \otimes m \simeq \mathcal{O}_X$ for some $m \geq 1$, such that it has a Crauder-Morrison model $X$ where the subgraph $\Gamma_{\min}$ of the dual graph $\Gamma$ has a unique vertex $P$.

If $P$ is a minimal ruled surface, and if there is a rational flower, then there is exactly one other flower, and it has genus $g \geq 2$.

### 3.5 Cycles

In this section, we prove that cycle degenerations do not occur for $K3$ surfaces. This result has been announced in [Jas17].

**Lemma 3.5.1.** Let $X$ be a smooth, proper surface over $K$ with $\omega_{X/K} \otimes m \simeq \mathcal{O}_X$ for some $m \geq 1$, and let $X$ be a Crauder-Morrison model of $X$. Assume that the subgraph $\Gamma_{\min}$ of the dual graph $\Gamma$ is a cycle. Then $X$ has trivial $\ell$-adic Euler characteristic, i.e. $\chi(X) = 0$.

**Proof.** By Theorem 3.2.1 (iv), we can find an integer $N \geq 1$ such that $X_k = N \sum_{i=1}^{k} V_i$, with $V_i$ an elliptic, minimal ruled surface with sections $V_{i-1} \cap V_i$ and $V_i \cap V_{i+1}$, for all $i = 1, \ldots, k$. Here we identify $V_0 = V_k$ and $V_{k+1} = V_1$. So for every $i = 1, \ldots, k$, we have
\[ \chi(V_i) = 0 \text{ and } \chi(V_{i-1} \cap V_i) = \chi(V_i \cap V_{i+1}) = 0. \]
Define $V_i^\circ = V_i \setminus \left( \bigcup_{j \neq i} V_j \right)$, then
\[ \chi(V_i^\circ) = \chi(V_i) - \chi(V_{i-1} \cap V_i) - \chi(V_i \cap V_{i+1}) = 0. \]
By taking the valuation at infinity of both sides of the A’Campo formula in Proposition \ref{ACampo}, we get

\[ \chi(X) = N \sum_{i=1}^{k} \chi(V_i^\circ) = 0. \]

\[ \square \]

**Theorem 3.5.2.** Let \( X \) be a K3 surface over \( K \) and let \( \mathcal{X} \) be a Crauder-Morrison model of \( X \). Then \( \mathcal{X} \) is either a flowerpot degeneration or a chain degeneration, but not a cycle degeneration.

**Proof.** This follows from Lemma \ref{lemma:1} and the fact that the \( \ell \)-adic Euler characteristic of a K3 surface is 24. \( \square \)

### 3.6 Chains

#### 3.6.1 Combinations of flowers in a chain degeneration

**Proposition 3.6.1** (\cite{CM83} Corollary 6.2). Let \( X \) be a smooth, proper surface over \( K \) with \( \omega_{X/K}^\otimes m \simeq 0_X \) for some \( m \geq 1 \), and let \( \mathcal{X} \) be a Crauder-Morrison model of \( X \). Suppose that the subgraph \( \Gamma_{\min} \) of the dual graph \( \Gamma \) is a chain \( V_0 - V_1 - \cdots - V_k - V_{k+1} \). The following properties hold.

(i) If \( j \in \{1, \ldots, k\} \), then all flowers meeting \( V_j \) are rational of type 2.

(ii) If \( V_0 \simeq \mathbb{P}^2 \), then \( V_0 \) does not meet any flowers.

(iii) If \( V_0 \) is a rational, ruled surface, then all flowers meeting \( V_0 \) are rational of type 2.

(iv) If \( V_0 \) is an elliptic, ruled surface, then flowers meeting \( V_0 \) are either rational of type 2 or non-rational of type \( 4\alpha \). There is either a unique non-rational flower of type \( 4\alpha \), or there are two non-rational flowers of type \( 4\alpha \). In the latter case, both these flowers are elliptic.

The analogous result of (ii), (iii), and (iv) holds for \( V_{k+1} \) as well.

#### 3.6.2 Structure of the chain

**Proposition 3.6.2** (\cite{CM83} Theorem 6.7 and Proposition 6.9). Let \( X \) be a smooth, proper surface over \( K \) with \( \omega_{X/K}^\otimes m \simeq 0_X \) for some \( m \geq 1 \), and let \( \mathcal{X} \) be a Crauder-Morrison model of \( X \). Suppose that the subgraph \( \Gamma_{\min} \) of the
dual graph $\Gamma$ is a chain. There exists a birational modification $\tilde{X}$ of $X$ with the following properties:

(i) $\tilde{X}$ is a Crauder-Morrison model of $X$.

(ii) There exist integers $\alpha, \beta$ and $k$ with $0 \leq \alpha \leq \beta \leq k + 1$, such that $\tilde{\Gamma}_{\text{min}}$ is $N_0V_0 + N_1V_1 + \cdots + N_kV_k + N_{k+1}V_{k+1}$ and

$$N_0 > N_1 > \cdots > N_\alpha = \cdots = N_\beta < \cdots < N_k < N_{k+1}.$$

(iii) The surfaces $V_1, \ldots, V_{\alpha-1}, V_{\beta+1}, \ldots, V_k$ are elliptic, minimal ruled surfaces, and don’t meet any flowers.

(iv) If $\alpha \geq 1$, then there exists an integer $N \geq 1$ such that the chain $V_0 - \cdots - V_{\alpha-1}$ satisfies one of the possibilities described in Table 3.5. A similar result holds for the chain $V_{\beta+1} - \cdots - V_{k+1}$, if $\beta \leq k$.

| $V_0$ | $N_0V_0 + \cdots + N_{\alpha-1}V_{\alpha-1}$ | $\alpha$ | $N_\alpha$ | Remark |
|-------|---------------------------------------------|---------|-----------|--------|
| $V_0 \simeq \mathbb{P}^2$ | $3NV_0$ | 1 | $N$ | No flowers on $V_0$ |
| | $3NV_0 + 2NV_1$ | 2 | $N$ | |
| | $3NV_0$ | 1 | $2N$ | |
| $V_0 \simeq \Sigma_2$ | $4NV_0$ | 1 | $N$ | Unique flower on $V_0$ of type $2A$ |
| | $4NV_0 + 3NV_1 + 2NV_2$ | 3 | $N$ | |
| | $4NV_0 + 3NV_1$ | 2 | $2N$ | |
| | $4NV_0$ | 1 | $3N$ | |
| | $2NV_0$ | 1 | $N$ | |

Table 3.5: Classification of the chains $V_0 - \cdots - V_{\alpha-1}$

From now on, we assume that a Crauder-Morrison model with a chain degeneration always satisfies the properties in Proposition 3.6.2.

### 3.6.3 Geometry of the components in the chain

**Proposition 3.6.3.** Let $X$ be a smooth, proper surface over $K$ with $\omega_{X/K}^{\otimes m} \simeq \mathcal{O}_X$ for some $m \geq 1$, and let $\mathcal{X}$ be a Crauder-Morrison model of $X$. Suppose that the subgraph $\Gamma_{\text{min}}$ of the dual graph $\Gamma$ is a chain with components $V_0, V_1, \ldots, V_k, V_{k+1}$, where $V_i \cap V_j \neq \emptyset$ if and only if $j \in \{i-1, i, i+1\}$. 
(i) For $i = 1, \ldots, k$, the double curves $C_i = V_{i-1} \cap V_i$ and $C_{i+1} = V_i \cap V_{i+1}$ are sections on the elliptic, ruled surface $V_i$, and any reducible fiber of the ruling on $V_i$ is a chain

$$F_1 + \ldots + F_s,$$

where $F_1^2 = F_s^2 = -1$ and $F_j^2 = -2$ for $j = 2, \ldots, s - 1$. Moreover, $F_1$ meets $C_i$ transversally in a unique point, and $F_s$ meets $C_{i+1}$ transversally in a unique point as well. For $j = 2, \ldots, s - 1$, the curve $F_j$ does not intersect $C_i$, nor $C_{i+1}$.

(ii) Suppose $V_0$ is ruled over an elliptic curve $D$, and let $C_1 = V_0 \cap V_1$. One of the following holds:

(a) The morphism $C_1 \to D$ is an isomorphism. Any reducible fiber of the ruling on $V_0$ is a chain

$$F_1 + \ldots + F_s,$$

where $F_1^2 = F_s^2 = -1$ and $F_j^2 = -2$ for $j = 2, \ldots, s - 1$. Moreover, $F_1$ meets $C_1$ transversally in a unique point. For $j = 2, \ldots, s$, the curve $F_j$ does not intersect $C_1$.

(b) The morphism $C_1 \to D$ is an étale morphism of degree 2. Any reducible fiber of the ruling on $V_0$ is a chain

$$F_1 + \ldots + F_s,$$

where $F_1^2 = F_s^2 = -1$ and $F_j^2 = -2$ for $j = 2, \ldots, s - 1$. Moreover, $F_1$ and $F_s$ meet $C_1$ transversally in a unique point each. For $j = 2, \ldots, s - 1$, the curve $F_j$ does not intersect $C_1$.

A similar result holds for $V_{k+1}$.

(iii) Suppose $V_0$ is ruled over a rational curve $D$. The double curve $C_1 = V_0 \cap V_1$ is an elliptic curve and the morphism $C_1 \to D$ is a morphism of degree 2 which is branched above four points $\{d_1, \ldots, d_4\} \subset D$.

Moreover, any reducible fiber of the ruling on $V_0$ above a point $P \notin \{d_1, \ldots, d_4\}$, is a chain

$$F_1 + \ldots + F_s,$$

where $F_1^2 = F_s^2 = -1$ and $F_j^2 = -2$ for $j = 2, \ldots, s - 1$. Moreover, $F_1$ and $F_s$ meet $C_1$ transversally in a unique point each. For $j = 2, \ldots, s - 1$, the curve $F_j$ does not intersect $C_1$.

A reducible fiber of the ruling on $V_0$ above a point $P \in \{d_1, \ldots, d_4\}$, has one of the following forms:
(a) $2F_1 + \ldots + 2F_{s-2} + F_{s-1} + F_s$, where $F_1, \ldots, F_{s-1}$ is a chain. The component $F_s$ intersects $F_{s-2}$ transversally in a unique point and is disjoint from $F_j$ for $j = 1, \ldots, s - 3, s - 1$. Moreover, $F_1^2 = -1$ and $F_j^2 = -2$ for $j = 2, \ldots, s$. The component $F_1$ meets $C_1$ in a unique point and $F_j$ is disjoint from $C_1$ for $j = 2, \ldots, s$.

(b) $F_1 + F_2$, where $F_1^2 = F_2^2 = -1$. The curve $C_1$ intersects $F_1$ and $F_2$ both transversally in $F_1 \cap F_2$.

A similar result holds for $V_{k+1}$.

To prove this proposition, we first need several lemmas.

**Lemma 3.6.4.** Let $V$ be a surface over $k$ such that $K_{V/k} \equiv \sum_{j=1}^{n} a_j C_j$ for rational numbers $-1 \leq a_j < 0$, with $a_1 = -1$ and where all $C_j$ are smooth, pairwise disjoint curves. Then the curve $C_1$ has genus $g(C_1) = 1$.

Suppose moreover that $E \subset V$ is a rational curve with $E^2 = -1$ meeting $C_1$ non-trivially. Then $E$ does not meet $C_j$ for $2 \leq j \leq n$, and $E$ intersects $C_1$ transversally in a unique point.

**Proof.** We have that

$$K_{V/k} \equiv -C_1 + \sum_{j=2}^{n} a_j C_j,$$

so the adjunction formula implies $g(C_1) = 1$.

Under the assumptions in the statement, we have that $E \neq C_j$ for all $1 \leq j \leq n$, as all $C_j$ are pairwise disjoint. So $C_1 \cdot E > 0$, and $C_j \cdot E \geq 0$ for $j = 2, \ldots, n$. The adjunction formula gives that $-K_{V/k} \cdot E = 1$. Therefore we find

$$1 = -K_{V/k} \cdot E = C_1 \cdot E - \sum_{j=2}^{n} a_j C_j \cdot E,$$

which implies that $C_1 \cdot E = 1$ and $\sum_{j=2}^{n} a_j C_j \cdot E = 0$. We can conclude that $C_j \cdot E = 0$ for all $2 \leq j \leq n$, and that $C_1$ meets $E$ transversally in a unique point. \qed

**Lemma 3.6.5.** Let $V$ be a surface over $k$ such that $K_{V/k} \equiv \sum_{j=1}^{n} a_j C_j$ for rational numbers $-1 \leq a_j < 0$, with $a_1 = -1$ and where all $C_j$ are smooth, pairwise disjoint curves. Assume that there is a rational curve $E \subset V$, with $E^2 = -1$, meeting $C_1$ transversally in a unique point, and that $E$ meets no other $C_j$ for $2 \leq j \leq n$. Let $f : V \to \overline{V}$ be the contraction of $E$. 
The curves $\overline{C}_j = f(C_j)$ are smooth and pairwise disjoint, and

$$K_{\mathcal{V}/k} \equiv \sum_{j=1}^{n} a_j \overline{C}_j.$$ 

Proof. Since $f$ is the contraction of $E$, it is an isomorphism on $V \setminus E$. Because $E$ is disjoint from $C_j$ for $2 \leq j \leq n$, the curves $\overline{C}_j$ are also smooth, pairwise disjoint curves. Moreover, since $C_1$ meets $E$ transversally in a unique point, it follows from [Har77, Proposition V.3.6 and Proposition V.3.2] that $\overline{C}_1$ is smooth.

To prove the last claim, we have that $f^*\overline{C}_j = C_j$, for $2 \leq j \leq n$, and that $f^*\overline{C}_1 = C_1 + E$, by [Har77, Proposition V.3.6]. Therefore,

$$f^* \left( \sum_{j=1}^{n} a_j \overline{C}_j \right) = \sum_{j=1}^{n} a_j C_j + a_1 E = K_{\mathcal{V}/k} - E,$$

since $a_1 = -1$.

By [Har77 Proposition V.3.3], it holds that $K_{\mathcal{V}/k} - E = f^*K_{\mathcal{V}/k}$, so

$$f^*K_{\mathcal{V}/k} = f^* \left( \sum_{j=1}^{n} a_j \overline{C}_j \right).$$

The map $f^*$ is injective, since $f^*: \text{Pic}(\mathcal{V}) \rightarrow \text{Pic}(V) \simeq \text{Pic}(\mathcal{V}) \oplus \mathbb{Z}$ is an inclusion [Har77, Proposition V.3.2]. Hence

$$K_{\mathcal{V}/k} \equiv \sum_{j=1}^{n} a_j \overline{C}_j.$$

Lemma 3.6.6. Let $V$ be a ruled surface over $k$ such that $K_{\mathcal{V}/k} \equiv \sum_{j=1}^{n} a_j C_j$ for rational numbers $-1 \leq a_j < 0$, with $a_1 = -1$, and where all $C_j$ are smooth, pairwise disjoint curves. Assume that $C_1$ does not meet any rational curve $E$ with $E^2 = -1$. Then the curve $C_1$ does not meet any smooth, rational curve with negative self-intersection. Moreover, $V$ is a minimal ruled surface.

Proof. Assume there exists a smooth, rational curve $E$ on $V$ with $E^2 \leq -2$ and $C_1 \cdot E > 0$. Adjunction implies that

$$C_1 \cdot E - \sum_{j=2}^{n} a_j C_j \cdot E = -K_{\mathcal{V}/k} \cdot E \leq 0.$$
The Crauder-Morrison Classification

Hence \( \sum_{j=2}^{n} a_j C_j \cdot E > 0 \). However, the assumption \( C_1 \cdot E > 0 \) implies that \( E \neq C_j \), for \( 2 \leq j \leq n \), since the curve \( C_1 \) is disjoint from \( C_j \), for every \( 2 \leq j \leq n \). So \( C_j \cdot E \geq 0 \) for every \( 2 \leq j \leq n \). But this is a contradiction, because \( a_j < 0 \), for every \( 2 \leq j \leq n \).

To prove the last claim, notice that if \( V \) is a ruled surface that is not minimal ruled, then there exists a reducible fiber \( b_1 F_1 + b_2 F_2 + \cdots + b_s F_s \) where each \( F_j \) is a smooth, rational curve with negative self-intersection. Since \( g(C_1) = 1 \), by Lemma 3.6.4, the curve \( C_1 \) must be a horizontal component, and hence meets every fiber non-trivially. But we have shown that \( C_1 \) does not meet any smooth, rational curve with negative self-intersection, hence \( V \) must be minimal ruled.

Proof of Proposition 3.6.3. Let \( V = V_i \) be one of the surfaces in the chain for \( 0 \leq j \leq k + 1 \). Then \( K_{V/k} = \sum_{j=1}^{n} a_j C_j \) for rational numbers \(-1 \leq a_j < 0\) by equations (3.4) and (3.5). After renumbering, we can assume \( C_1 \) is the double curve on \( V \), where \( V = V_i \) meets \( V_i-1 \) or \( V_i+1 \). Because \( V_i \) and \( V_i\pm 1 \) have the same weight, \( a_1 = -1 \). Moreover, all \( C_j \) are smooth, pairwise disjoint curves.

If \( V \) is not minimal ruled, then by Lemma 3.6.6, there is a rational curve \( E \) with \( E^2 = -1 \) meeting \( C_1 \). By Lemma 3.6.4, we know that \( E \cdot C_j = 0 \), for \( 2 \leq j \leq n \). Let \( V \to V^{(1)} \) be the contraction of \( E \). By Lemma 3.6.5

\[
K_{V^{(1)}/k} = \sum_{j=1}^{n} a_j C_j^{(1)},
\]

where \( C_j^{(1)} \) is the image of \( C_j \) under the contraction. Furthermore, all \( C_j^{(1)} \) are smooth and pairwise disjoint, by Lemma 3.6.5. If \( V^{(1)} \) is not minimal ruled, then there is a rational curve \( E \) with \( E^2 = -1 \) meeting \( C_1^{(1)} \). We can apply the same reasoning as before, to get a contraction \( V^{(1)} \to V^{(2)} \).

We can repeat this process and obtain a series of morphisms

\[
V = V^{(0)} \to V^{(1)} \to \cdots \to V^{(t)} = V
\]

by successively contracting rational curves with self-intersection \(-1\) meeting \( C_1^{(1)} \).

At any stage of the process, Lemma 3.6.5 guarantees that \( K_{V^{(t)}/k} = \sum_{j=1}^{n} a_j C_j^{(t)} \), where \( C_j^{(t)} \) denotes the image of \( C_j \) in \( V^{(t)} \). The curves \( C_j^{(t)} \) are smooth and pairwise disjoint and \( C_1^{(t)} \) is an elliptic curve on the ruled surface \( V^{(t)} \).

By Lemma 3.6.6, this process stops as soon as there are no rational curves with self-intersection \(-1\) meeting the image of \( C_1 \) and the surface \( V = V^{(t)} \) is minimal ruled.
Going backwards in this process, we find that

\[ V^{(l-1)} \to V^{(l)} \]

is precisely blowing up a smooth point on \( C_1^{(l)} \) for every \( l = 1, \ldots, t \).

(i) Assume \( i = 1, \ldots, k \), then \( V = V_i \) is elliptic ruled, \( n \geq 2 \), and after renumbering, \( a_1 = a_2 = -1 \). The adjunction formula applied to a general fiber of the ruling on \( V \), gives that \( C_1 \) and \( C_2 \) are both sections. Repeatedly blowing up smooth points on \( C_1^{(l)} \), gives the result. We illustrate this in Figure 3.4. You see a series of blow-ups where the centers are the marked points. The horizontal, dashed lines are \( C_1^{(l)} \) and \( C_2^{(l)} \) and the vertical lines are the components of a fiber of the ruling. The labels \(-2, -1, 0\) indicate the self-intersections of the components of the fibers.

(ii) Assume \( i = 0 \) or \( k + 1 \), and that \( V = V_i \) is a ruled surface over an elliptic curve \( D \). First, suppose \( n > 1 \), so there is at least one flower meeting \( V \). The adjunction formula applied to a general fiber of the ruling on \( V \), gives that \( C_1 \) is a section. Repeatedly blowing up smooth points on \( C_1^{(l)} \), gives the result. We illustrate this in Figure 3.5.

Suppose now that \( n = 1 \), then the adjunction formula applied to a general fiber, gives that \( C_1 \) is a bisection, i.e., \( C_1 \cdot F = 2 \) for any fiber \( F \) of the ruling on \( V \). By Lemma 3.6.4, we also know that \( C_1 \) is elliptic. Hurwitz’ formula [Har77, Corollary IV.2.4] asserts that \( C_1 \to D \) is étale. Repeatedly
blowing up smooth points on $C^{(i)}_1$, gives the result. We illustrate this in Figure 3.6.

(iii) Assume $i = 0$ or $k + 1$ and that $V = V_i$ is a ruled surface over a rational curve $D$. By Lemma 3.6.4, the curve $C_1$ is elliptic. The adjunction formula applied to a general fiber $F$ of the ruling on $V$, gives that $C_1 \cdot F$ is either 1 or 2. But since $C_1$ is elliptic and $D$ is rational, $C_1 \cdot F = 2$. The Hurwitz formula gives

$$2g(C_1) - 2 = 2 \cdot (2g(D) - 2) + \sum_{P \in C_1} (e_P - 1) = -4 + \sum_{P \in C_1} (e_P - 1),$$

where $e_P$ denotes the ramification index of a point $P \in C_1$. Since $C_1$ is elliptic, we find

$$\sum_{P \in C_1} (e_P - 1) = 4.$$

The result of [Liu02, Formula (4.8) p. 290] asserts that $e_P \leq 2$, for all $P \in C_1$, and hence $C_1$ is ramified above exactly four points $\{d_1, d_2, d_3, d_4\} \subset D$.

Repeatedly blowing up smooth points on $C^{(i)}_1$ gives the result. For an illustration of the process above a point that is not a branch point, we refer to Figure 3.6. The evolution of the fiber above a branch point is illustrated in Figure 3.7.
Corollary 3.6.7. Let \( X \) be a smooth, proper surface over \( K \) with \( \omega_X^\otimes m \simeq \mathcal{O}_X \) for some \( m \geq 1 \), and let \( \mathcal{X} \) be a Crauder-Morrison model of \( X \). Suppose that the subgraph \( \Gamma_{\text{min}} \) of the dual graph \( \Gamma \) is a chain. Let \( V_i \) be a component in the chain that is an elliptic, ruled surface. Let \( L \) be the number of blow-ups in the contraction \( V_i \to \overline{V}_i \) to the minimal ruled surface \( \overline{V}_i \) and let \( \lambda \) be the number of flowercurves on \( V_i \) of type 2. Then

\[
L \geq 2\lambda.
\]

Proof. Let \( C_1, \ldots, C_\lambda \) be the flowercurves on \( V_i \) of type 2. For \( j = 1, \ldots, \lambda \), the curve \( C_j \) is a rational curve, and therefore it is contained in a reducible fiber on \( V_i \). Moreover, all \( C_j \) are disjoint, because there are no triple points. Lemma 3.3.8 implies that \( C_j^2 = -2 \) for \( j = 1, \ldots, \lambda \). On the other hand, Proposition 3.6.3 describes the structure of the reducible fibers. It follows that \( L \geq 2\lambda \). \( \square \)

3.7 Euler characteristics

In this section, we will compute \( \chi(E_i^\circ) \), for certain components \( E_i \) of the special fiber of a Crauder-Morrison model. From Proposition 2.3.3, it is clear that this is useful for computing monodromy eigenvalues.

Lemma 3.7.1. Let \( X \) be a smooth, proper surface over \( K \) with \( \omega_X^\otimes m \simeq \mathcal{O}_X \) for some \( m \geq 1 \), and let \( \mathcal{X} \) be a Crauder-Morrison model of \( X \). If \( \mathcal{X} \) is a chain degeneration, we may assume that \( \mathcal{X} \) satisfies the properties of Proposition 3.6.2. Write the special fiber as \( \mathcal{X}_k = \sum_{i \in I} N_i E_i \). Denote by \( E_i^\circ = E_i \setminus \left( \bigcup_{j \neq i} E_j \right) \). The following statements hold.

(i) If \( E_i \simeq \mathbb{P}^2 \) is a top of a flower, then \( \chi(E_i^\circ) = 1 \).

(ii) If \( E_i \) is a top of a flower of genus \( g \geq 0 \), and if \( E_i \) is a ruled surface, then \( \chi(E_i^\circ) = 2 - 2g \).

(iii) If \( E_i \) is a component of a flower, but not the top component, then \( \chi(E_i^\circ) = 0 \).

(iv) If \( \mathcal{X} \) is a chain degeneration, and if \( E_i \) is a component of the chain, but not an end component of the chain, then \( \chi(E_i^\circ) \geq 0 \).

(v) If \( \mathcal{X} \) is a chain degeneration, and if \( E_i \) is a component of the chain, but not an end component of the chain, and if the multiplicity \( N_i \) is not minimal among the components in the chain, then \( \chi(E_i^\circ) = 0 \).

(vi) If \( \mathcal{X} \) is a chain degeneration, and if \( E_i \simeq \mathbb{P}^2 \) is an end component of the chain, then \( \chi(E_i^\circ) = 3 \).
(vii) If $X$ is a chain degeneration, and if $E_i$ is an elliptic, ruled surface and an end component of the chain, then $\chi(E_i^0) \geq 0$.

(viii) If $X$ is a chain degeneration, and if $E_i$ is a rational, minimal ruled surface and an end component of the chain, then $\chi(E_i^0) \geq 0$.

Proof. (i) Let $E_i \simeq \mathbb{P}^2$ be a top of a flower. We have $\chi(E_i) = \chi(\mathbb{P}^2) = 3$. Let $C$ be the double curve on $E_i$, which is either a line or a conic by Theorem 3.2.1. In either case, $\chi(C) = 2$. So
$$\chi(E_i^0) = \chi(E_i) - \chi(C) = 1.$$

(ii) Let $E_i$ be a top of a flower of genus $g \geq 0$, such that $E_i$ is a ruled surface. Denote by $C$ the double curve on $E_i$. By Remark 3.3.2, the curve $C$ has genus $g$, so $\chi(C) = 2 - 2g$. Moreover, $C$ is a section of $E_i$, and therefore $\chi(E_i) = \chi(\mathbb{P}^1) \cdot \chi(C) = 2 \cdot (2 - 2g)$. We conclude
$$\chi(E_i^0) = \chi(E_i) - \chi(C) = 2 - 2g.$$

(iii) Let $E_i$ be a component of a flower, but not the top component. Let $C_i$ and $C_{i+1}$ be the two double curves on $E_i$. By Theorem 3.2.1 we know that $E_i$ is a minimal ruled surface with sections $C_i$ and $C_{i+1}$. So $\chi(C_i) = \chi(C_{i+1})$ and $\chi(E_i) = \chi(\mathbb{P}^1) \cdot \chi(C_i) = 2 \cdot \chi(C_i)$. We conclude
$$\chi(E_i^0) = \chi(E_i) - \chi(C_i) - \chi(C_{i+1}) = 0.$$

(iv) Suppose $X$ is a chain degeneration, and let $E_i$ be a component of the chain, but not an end component. Let $L$ be the number of blow-ups in the contraction $E_i \to \overline{E}_i$ to the minimal ruled surface $\overline{E}_i$. We have $\chi(\overline{E}_i) = 0$, because $\overline{E}_i$ is an elliptic, minimal ruled surface. Since blowing up increases the topological Euler characteristic by 1, we have $\chi(E_i) = L$.

Let $C_1$ and $C_2$ be the double curves on $E_i$, where $E_i$ meets another component in the chain. By Theorem 3.2.1 the double curves $C_1$ and $C_2$ are sections of $E_i$, and hence elliptic. So $\chi(C_1) = \chi(C_2) = 0$.

Let $C_3, \ldots, C_{\lambda+2}$ be the flowercurves on $E_i$, where $\lambda$ is the number of flowers meeting $E_i$. By Proposition 3.6.1, all flowers meeting $E_i$ are rational of type 2. This means, in particular, that $C_j$ is rational and hence, $\chi(C_j) = 2$ for $j = 3, \ldots, \lambda + 2$. Therefore
$$\chi(E_i^0) = \chi(E_i) - \sum_{j=1}^{\lambda+2} \chi(C_j) = L - 2\lambda \geq 0,$$

by Corollary 3.6.7.
(v) Suppose $X$ is a chain degeneration, and let $E_i$ be a component of the chain, but not an end component and such that the multiplicity $N_i$ is not minimal among the components in the chain. By Proposition 3.6.2, the component $E_i$ is an elliptic, minimal ruled surface, and doesn’t meet any flowers. Therefore, there are exactly two double curves $C_1$ and $C_2$ and both $C_1$ and $C_2$ are sections on $E_i$. Since $E_i$ is an elliptic, minimal ruled surface, we have that $\chi(E_i) = 0$. Moreover, $C_1$ and $C_2$ are elliptic curves, and hence $\chi(C_1) = \chi(C_2) = 0$. We conclude
$$\chi(E_i^\circ) = \chi(E_i) - \chi(C_1) - \chi(C_2) = 0.$$ 

(vi) Suppose $X$ is a chain degeneration, and let $E_i \simeq \mathbb{P}^2$ be an end component of the chain. So $\chi(E_i) = 3$. By Proposition 3.6.1, the component $E_i$ does not meet any flowers and hence, there is exactly one double curve $C$. By Theorem 3.2.1, the curve $C$ is elliptic. So $\chi(C) = 0$. Therefore
$$\chi(E_i^\circ) = \chi(E_i) - \chi(C) = 3.$$ 

(vii) Suppose $X$ is a chain degeneration, and let $E_i$ be an elliptic, ruled surface and an end component of the chain. Let $L$ be the number of blow-ups in the contraction $E_i \to E_i^\circ$ to the minimal ruled surface $E_i^\circ$. We have $\chi(E_i) = 0$, because $E_i$ is an elliptic, minimal ruled surface. Since blowing up increases the topological Euler characteristic by 1, we have $\chi(E_i) = L$. Let $C_1$ be the intersection curve of $E_i$ with a component in the chain. By Theorem 3.2.1, $C_1$ is a section of $E_i$ and hence elliptic, so $\chi(C_1) = 0$.

By Proposition 3.6.1, all flowers meeting $E_i$ are rational of type 2 or non-rational of type 4. Denote by $\lambda$ the number of flowers of type 2 and by $\lambda'$ the number of flowers of type 4. Let $C_2, \ldots, C_{\lambda+1}$ be rational flowercurves on $E_i$ of type 2. We have, in particular, that $\chi(C_j) = 2$ for $j = 2, \ldots, \lambda + 1$. Let $C_{\lambda+2}, \ldots, C_{\lambda+\lambda'+1}$ be non-rational flowercurves on $E_i$ of type 4. So $\chi(C_j) \leq 0$ for $j = \lambda + 2, \ldots, \lambda + \lambda' + 1$. Therefore
$$\chi(E_i^\circ) = \chi(E_i) - \sum_{j=1}^{\lambda+\lambda'+1} \chi(C_j) \geq L - 2\lambda \geq 0,$$

by Corollary 3.6.7.

(viii) Suppose $X$ is a chain degeneration, and let $E_i$ be a rational, minimal ruled surface and an end component of the chain. We have $\chi(E_i) = 4$. Let $C_1$ be the intersection of $E_i$ with a component in the chain. By Theorem 3.2.1, the curve $C_1$ is elliptic. So $\chi(C_1) = 0$. 

In [BHPV95, Section V.4], it is explained that, on a rational, minimal ruled surface, there is at most one smooth curve with strictly negative self-intersection. By Proposition 3.6.1, all flowers meeting $E_i$ are rational of type 2 and therefore, Lemma 3.3.8 implies that they have strictly negative self-intersection. Therefore, there is at most one flowercurve on $E_i$.

If there is no flowercurve on $E_i$, then
\[ \chi(E_i^\circ) = \chi(E_i) - \chi(C_1) = 4. \]

If there is a flowercurve $C_2$ on $E_i$, then it is rational, and hence $\chi(C_2) = 2$. Therefore,
\[ \chi(E_i^\circ) = \chi(E_i) - \chi(C_1) - \chi(C_2) = 2. \]
Chapter 4

Poles of the motivic zeta function

In this chapter, we will prove the following theorem, which has been announced in [Jas17].

**Theorem.** Let $X$ be a $K3$ surface over $K$ with Crauder-Morrison model $\mathcal{X}$ of $X$. Write the special fiber as $X_k = \sum_{i \in I} N_i E_i$. Let $\omega$ be a volume form on $X$ and let $(N_i, \nu_i)$ be the numerical data of $E_i$. Let $\rho_i = \nu_i / N_i + 1$ be the weight of $E_i$ for every $i \in I$.

Define $I^\dagger \subset I$ to be the set of indices $i \in I$, where either

(i) $\rho_i$ is minimal, or

(ii) $E_i$ is the top of a conic-flower.

Define $S^\dagger = \{(-\nu_i, N_i) \in \mathbb{Z} \times \mathbb{Z}_{>0} \mid i \in I^\dagger\}$. We have

$$ Z_{X,\omega}(T) \in \mathcal{M}_k^\dagger \left[ T, \frac{1}{1 - R_a T^b} \right]_{(a,b) \in S^\dagger}. $$

Moreover, the poles of $Z_{X,\omega}(T)$ are precisely $\{ -\nu_i / N_i \mid i \in I^\dagger \}$.

The main tool in the proof of this theorem is the Denef-Loeser formula (Theorem 2.2.2):

$$ Z_{X,\omega}(T) = \sum_{\emptyset \neq J \subseteq I} (\mathbb{L} - 1)^{|J|-1} [E_J^\sim] \prod_{j \in J} \frac{\mathbb{L}^{-\nu_j} T^{N_j}}{1 - \mathbb{L}^{-\nu_j} T^{N_j}}. $$
So if we define \( S = \{-\nu_i, N_i \in \mathbb{Z} \times \mathbb{Z}_{>0} \mid i \in I\} \), then it holds that

\[
Z_{X,\omega}(T) \in \mathcal{M}_k^\mu \left[ T, \frac{1}{1 - L^a T^b} \right]_{(a,b) \in S}.
\]

In this chapter, we will find all elements that can be omitted from \( S \).

The factor \([\bar{E}_J^\circ]\) that appears in the Denef-Loeser formula needs to be computed for certain components \( E_i \) in the special fiber \( X_k \). We will do so in Section 4.1. In Section 4.2, we will define the *contribution of flowers to the motivic zeta function* and explain how we compute these contributions by writing Python code. As a consequence, we get the first part of the theorem:

\[
Z_{X,\omega}(T) \in \mathcal{M}_k^\mu \left[ T, \frac{1}{1 - L^a T^b} \right]_{(a,b) \in S^\dagger}.
\] (4.1)

In the final section, we will first define a *pole* of a rational function over \( \mathcal{M}_k^\mu \), and then we will compute the poles of the motivic zeta function \( Z_{X,\omega}(T) \). As a result, we get that \( S^\dagger \) is the ‘smallest’ subset of \( S \) such that (4.1) holds, i.e., no elements of \( S^\dagger \) can be omitted such that (4.1) still holds.

**Notation**

In this chapter, we fix an algebraically closed field \( k \) of characteristic zero. Put \( R = k[t] \) and \( K = k(t) \). Let \( X \) be a \( K \)3 surface over \( K \) and let \( \mathcal{X} \) be a Crauder-Morrison model of \( X \) with special fiber \( \mathcal{X}_k = \sum_{i \in I} N_i E_i \).

For any local ring \( \mathcal{O} \) with maximal ideal \( \mathfrak{m} \), we denote by \( \hat{\mathcal{O}} \) the completion of \( \mathcal{O} \) with respect to \( \mathfrak{m} \). For any reduced ring \( A \), we denote by \( \mathbb{Q}(A) \) the total ring of fractions of \( A \), and by \( A' \) the integral closure of \( A \) in \( \mathbb{Q}(A) \).

**4.1 Computation of \([\bar{E}_J^\circ]\) in \( \mathcal{M}_k^\mu \)**

Recall that for a non-empty subset \( J \subseteq I \), we defined

\[
E_J = \bigcap_{j \in J} E_j \quad \text{and} \quad E_J^\circ = E_J \setminus \left( \bigcup_{i \in I \setminus J} E_i \right).
\]

Notice that if \( |J| \geq 3 \), then \( E_J = \emptyset \), because \( \mathcal{X} \) is triple-point-free. Furthermore, when \( |J| = 2 \), we have \( E_J^\circ = E_J \).
We defined $\tilde{E}_J^0$ as the finite étale cover $E_J^0 \times_Y \mathcal{Y}_J$ of $E_J^0$, where $\mathcal{Y}_J$ is the normalization of $Y \times_R R[\pi]/(\pi^{N_J} - t)$, with $N_J = \text{lcm}_{j \in J} N_j$. The group $\mu_{N_j}$ acts on $\tilde{E}_J^0$ via its action on $R[\pi]/(\pi^{N_j} - t)$. This $\mu_{N_J}$-action induces a $\hat{\mu}$-action on $\tilde{E}_J^0$. In [BN16] Lemma 3.2.2, an alternative description of $\tilde{E}_J^0$ is given:

**Lemma 4.1.1 ([BN16] Lemma 3.2.2).**

$$\tilde{E}_J^0 \simeq E_J^0 \times_Y \mathcal{Y},$$

where $\mathcal{Y}$ is the normalization of $X \times_R R[\pi]/(\pi^n - t)$, for any multiple $n$ of $N_J$.

Moreover, the $\hat{\mu}$-action on $\tilde{E}_J^0$ induced by the $\mu_n$-action on $R[\pi]/(\pi^n - t)$ coincides with the $\hat{\mu}$-action induced by the $\mu_{N_j}$-action on $R[\pi]/(\pi^{N_j} - t)$. In practice, we will often choose $n = \text{lcm}_{i \in I} N_i$.

The aim of this section is to compute the class $[\tilde{E}_J^0] \in M_k^\hat{\mu}$, for certain $E_J$. To be precise, we will prove the following proposition.

**Proposition** (Proposition 4.1.9 and Proposition 4.1.12). Let $X$ be a $K3$ surface over $K$ and let $X$ be a Crauder-Morrison model of $X$ with special fiber $X_k = \sum_{i \in I} N_i E_i$. Let $N_0 F_0 + N_1 F_1 + \cdots + N_\ell F_\ell$ be a flower in $X_k$, where $F_i \cap F_j = \emptyset$ if and only if $j \notin \{i-1, i, i+1\}$. The component $F_\ell$ meets $\Gamma_{\text{min}}$ in $F_{\ell+1}$. Denote by $C_j$ the intersection $F_{j-1} \cap F_j$ for $j = 1, \ldots, \ell + 1$. The following relations hold in $M_k^\hat{\mu}$.

(i) If $F_0 \simeq \mathbb{P}^2$ and $C_1$ is a line, then there exists a $k$-variety $\mathcal{P}$ with a good $\hat{\mu}$-action, such that $[\tilde{F}_0^\mathcal{P}] = [\mathcal{P}]\mathbb{L}^2$ and $[\tilde{C}_1] = [\mathcal{P}]\mathbb{L}$,

(ii) If $F_0$ is a minimal ruled surface, then $[\tilde{F}_0^\mathcal{P}] = [\tilde{C}_1]\mathbb{L}$,

(iii) For any $j = 1, \ldots, \ell$, we have $[\tilde{F}_j^\mathcal{P}] = [\tilde{C}_j](\mathbb{L} - 1)$ and $[\tilde{C}_j] = [\tilde{C}_{j+1}]$

In Subsection 4.1.1, we give some local computations and in Subsection 4.1.2, we describe the geometrical structure of $\mathcal{Y}$ and the morphism $\mathcal{Y} \to X$. Finally, we will prove the proposition in Subsections 4.1.3 and 4.1.4.

### 4.1.1 Local computations

Let $n = \text{lcm}_{i \in I} N_i$ and let $R_n$ be the unique totally ramified extension $R[\pi]/(\pi^n - t)$ of $R$ of degree $n$. We then have $R_n \simeq k[\pi]$. Denote by $\mathcal{Y}$ the normalization of $X \times_R R_n$ and let $f : \mathcal{Y} \to \mathcal{X}$ be the induced morphism, which is finite. Let $p'$ be a closed point in the special fiber $\mathcal{Y}_k = \mathcal{Y} \times_{R_n} k$ and let $p = f(p')$.

In this subsection, we will prove the following lemma.
Lemma (Lemma 4.1.4 and Lemma 4.1.5).

(i) Suppose $p \in E_i^o$. Define $n' \in \mathbb{Z}$ such that $n = n' N_i$. Then the morphism $\hat{O}_{X,p} \to \hat{O}_{Y,p'}$ is described by

$$R[x,y,z]/(t - x^{N_i}) \to R_n[x,y,z]/(\pi^{n'} - \xi x)$$

$$g(x,y,z,t) \mapsto g(x,y,z,\pi^n),$$

for some $N_i$-th root of unity $\xi$.

(ii) Suppose $p \in E_i \cap E_j$ and define $c = \gcd(N_i, N_j)$ and $a, b, n' \in \mathbb{Z}$ such that $N_i = ca$, $N_j = cb$ and $n = cn'$ respectively. Let $d \in \mathbb{Z}$ be the integer such that $n' = abd$. Then the morphism $\hat{O}_{X,p} \to \hat{O}_{Y,p'}$ is described by

$$R[x,y,z]/(t - x^{N_i}y^{N_j}) \to R_n[x_0,y_0,z_0]/(\pi^d - \xi x_0y_0)$$

$$g(x,y,z,t) \mapsto g(x_0^a,y_0^b,z_0,\pi^n),$$

for some $c$-th root of unity $\xi$.

To prove this lemma, we first need some results from commutative algebra.

Some commutative algebra

Lemma 4.1.2.

$$\mathcal{Y} \times_X \text{Spec}(\hat{O}_{X,p}) = \coprod_{f(q) = p} \text{Spec}(\hat{O}_{Y,q}).$$

Proof. This is an immediate consequence of [Eis95, Corollary 7.6].

Lemma 4.1.3.

$$\mathcal{Y} \times_X \text{Spec}(\hat{O}_{X,p}) = \text{Spec}(R_n \otimes_R \hat{O}_{X,p})'.$$

Proof. This follows from the fact that the completion of a Noetherian, integrally closed ring is again integrally closed.
Point on a single component

Let $E_i$ be an irreducible component of the special fiber $X_k$, with multiplicity $N_i$. Let $p'$ be a closed point of $Y_k$ such that $p = f(p') \in E_i^\circ$, i.e., $p \in E_i$ is not contained in any other irreducible component $E_j$ with $j \neq i$.

**Lemma 4.1.4.** The morphism $\mathcal{O}_{X,p} \to \mathcal{O}_{Y,p'}$ is described by

$$R[x, y, z]/(t - x^{N_i}) \to R_n[x, y, z]/(\pi^{n'} - \xi x):$$

$$g(x, y, z, t) \mapsto g(x, y, z, \pi^n),$$

for some $N_i$-th root of unity $\xi$ and where $n' \in \mathbb{Z}$ is defined such that $n = n'N_i$.

**Proof.** Because $p$ is contained in $E_i$, but not in any other component of the special fiber, we have

$$\mathcal{O}_{X,p} = R[x, y, z]/(t - x^{N_i}).$$

Recall that $n' \in \mathbb{Z}$ is defined such that $n = n'N_i$. By Lemma 4.1.2 and Lemma 4.1.3, it is clear that we should compute $(R_n \otimes_R \mathcal{O}_{X,p})'$. Consider the ring

$$A = (R_n \otimes_R \mathcal{O}_{X,p}) \simeq R_n[x, y, z]/(\pi^n - x^{N_i}).$$

Define $B$ as

$$B = \prod_{\xi \in \mu_{N_i}} R_n[x, y, z]/(\pi^{n'} - \xi x),$$

where $\mu_{N_i}$ is the group of $N_i$-th roots of unity in $k$. There is an inclusion $A \hookrightarrow B$, because $\pi^n - x^{N_i} = \prod_{\xi \in \mu_{N_i}} (\pi^{n'} - \xi x)$. Since $B$ is integral over $A$, there is also an inclusion $B \hookrightarrow Q(A)$. Therefore, $A' = B'$.

For every $N_i$-th root of unity, the ring $R_n[x, y, z]/(\pi^{n'} - \xi x)$ is regular, because it is isomorphic to $R_n[y, z]$. Hence $R_n[x, y, z]/(\pi^{n'} - \xi x)$ is normal and therefore, $B$ is normal too. So

$$(R_n \otimes_R \mathcal{O}_{X,p})' \simeq \prod_{\xi \in \mu_{N_i}} R_n[x, y, z]/(\pi^{n'} - \xi x).$$

Lemma 4.1.2 implies that

$$\mathcal{O}_{Y,p'} = R_n[x, y, z]/(\pi^{n'} - \xi x),$$

for some $N_i$-th root of unity $\xi$ and the result follows. \qed
Point on the intersection of two components

Let \( E_i \) and \( E_j \) be irreducible components of \( \mathcal{X}_k \), with respective multiplicities \( N_i \) and \( N_j \). Let \( p' \) be a closed point of \( \mathcal{Y}_k \) such that \( p = f(p') \in E_i \cap E_j \). Define \( c = \gcd(N_i, N_j) \) and \( a, b, n' \in \mathbb{Z} \) such that \( N_i = ca, N_j = cb \) and \( n = cn' \).

**Lemma 4.1.5.** The morphism \( \mathcal{O}_{X,p} \to \mathcal{O}_{Y,p'} \) is described by

\[
R[x, y, z]/(t - x^{N_i} y^{N_j}) \to R_n[x_0, y_0, z_0]/(\pi^d - \xi x_0 y_0):
\]

\[
g(x, y, z, t) \mapsto g(x_0^b, y_0^a, z_0, \pi^n),
\]

for some \( c \)-th root of unity \( \xi \) and where \( d \in \mathbb{Z} \) is defined such that \( n' = abd \).

**Proof.** Because \( p \in E_i \cap E_j \), we have \( \mathcal{O}_{X,p} = R[x, y, z]/(t - x^{N_i} y^{N_j}) \). By Lemma 4.1.2 and Lemma 4.1.3, it is clear that we should compute \( (R_n \otimes_R \mathcal{O}_{X,p})' \).

Consider the ring

\[
A = R_n \otimes_R \mathcal{O}_{X,p} \cong R_n[x, y, z]/(\pi^{n} - x^{N_i} y^{N_j}).
\]

Define \( B \) as

\[
B = \prod_{\xi \in \mu_c} R_n[x, y, z]/(\pi^{n'} - \xi x^a y^b),
\]

where \( \mu_c \) is the group of \( c \)-th roots of unity in \( k \). There is an inclusion \( A \hookrightarrow B \), because \( \pi^{n} - x^{N_i} y^{N_j} = \prod_{\xi \in \mu_c} (\pi^{n'} - \xi x^a y^b) \). Since \( B \) is integral over \( A \), there is also an inclusion \( B \hookrightarrow \mathbb{Q}(A) \). Therefore, \( A' = B' \).

Since \( \gcd(a, b) = 1 \), there exists an integer \( d \in \mathbb{Z} \) with \( n' = abd \). Consider the \( R_n \)-algebra homomorphism

\[
R_n[x, y, z]/(\pi^{n'} - \xi x^a y^b) \to R_n[x_0, y_0, z_0]/(\pi^d - \xi x_0 y_0):
\]

\[
g(x, y, z, \pi) \mapsto g(x_0^b, y_0^a, z_0, \pi).
\]

This morphism is injective and integral, and it induces an isomorphism of fraction fields. Because the ring \( R_n[x_0, y_0, z_0]/(\pi^d - \xi x_0 y_0) \) describes a toric singularity, it is normal. Therefore, \( R_n[x_0, y_0, z_0]/(\pi^d - \xi x_0 y_0) \) is the normalization of \( R_n[x, y, z]/(\pi^{n'} - \xi x^a y^b) \). So

\[
(R_n \otimes_R \mathcal{O}_{X,p})' = \prod_{\xi \in \mu_c} R_n[x_0, y_0, z_0]/(\pi^d - \xi x_0 y_0).
\]

Lemma 4.1.3 implies that

\[
\mathcal{O}_{Y,p'} = R_n[x_0, y_0, z_0]/(\pi^d - \xi x_0 y_0),
\]

for some \( c \)-th root of unity \( \xi \) and the result follows. \( \square \)
4.1.2 Structure of $\mathcal{Y}_k$

Lemma 4.1.6. Let $f : \mathcal{Y} \to \mathcal{X}$ be the finite morphism constructed before. Let $p' \in \mathcal{Y}_k$ be a closed point, and let $p = f(p')$. Then the following properties hold.

(i) If $p$ belongs to a unique irreducible component of $\mathcal{X}_k$, or if $p$ belongs to two irreducible components of $\mathcal{X}_k$ with multiplicities $N_i$ and $N_j$ such that $n = \text{lcm}(N_i, N_j)$, then $\mathcal{Y}$ is regular at $p'$. The converse also holds.

(ii) If $p$ belongs to a unique irreducible component of $\mathcal{X}_k$, then $p'$ belongs to a unique irreducible component of $\mathcal{Y}_k$.

(iii) If $p$ belongs to two distinct irreducible components of $\mathcal{X}_k$, then $p'$ belongs to two distinct irreducible components of $\mathcal{Y}_k$.

(iv) The irreducible components of $\mathcal{Y}_k$ are smooth.

Proof. (i) By [Liu02, Lemma 2.26], we have that $\mathcal{Y}$ is regular at $p'$ if and only if $\mathcal{O}_{\mathcal{Y}, p'}$ is regular. Since $\mathcal{X}$ is triple-point-free, $p$ is either contained in a unique irreducible component $E_i$ of the special fiber $\mathcal{X}_k$, or $p$ is contained in $E_i \setminus E_j$ for some $i, j \in I$.

Suppose first that $p$ is contained in a unique irreducible component $E_i$ of the special fiber $\mathcal{X}_k$. Let $N_i$ be the multiplicity of $E_i$ in $\mathcal{X}_k$. By Lemma 4.1.4, we have that $\mathcal{O}_{\mathcal{Y}, p'} \simeq R_n[x, y, z]/(\pi^{n'} - \xi x)$ for some $N_i$-th root of unity $\xi$ and where $n'$ is defined as $n = n'N_i$. This is a regular ring, so $\mathcal{Y}$ is regular at $p'$.

Suppose now that $p \in E_i \cap E_j$ for some $i, j \in I$. Let $N_i$ and $N_j$ be the respective multiplicities of $E_i$ and $E_j$ in $\mathcal{X}_k$. Define $c = \gcd(N_i, N_j)$ and $d \in \mathbb{Z}$ such that $nc = N_iN_jd$. By Lemma 4.1.5, we have $\mathcal{O}_{\mathcal{Y}, p'} \simeq \mathcal{O}_{\mathcal{X}, p'} / (\pi^{d} - \xi x_0y_0)$ for some $c$-th root of unity $\xi$. This ring is regular if and only if $d = 1$, which is equivalent to $n = \text{lcm}(N_i, N_j)$.

(ii) Suppose $p$ belongs to a unique irreducible component of $\mathcal{X}_k$ of multiplicity $N_i$. Because of Lemma 4.1.4, we know that $\mathcal{O}_{\mathcal{Y}, p'} \simeq R_n[x, y, z]/(\pi^{n'} - \xi x)$, for some $N_i$-th root of unity $\xi$ and where $n' \in \mathbb{Z}$ is defined such that $n = n'N_i$. Since $\mathcal{Y}_k \simeq \mathcal{Y} \otimes R_n$, we have

$$\mathcal{O}_{\mathcal{Y}_k, p'} \simeq \mathcal{O}_{\mathcal{Y}, p'} \otimes R_n / (\pi) \simeq k[x, y, z] / (\xi x) \simeq k[y, z],$$

which is a domain. Therefore, $\mathcal{O}_{\mathcal{Y}_k, p'}$ is a domain as well, which implies there is a unique irreducible component of $\mathcal{Y}_k$ passing through $p'$, by [Liu02, Proposition 2.4.12].
(iii) Suppose $p$ belongs to two irreducible components of $X_k$. Because of Lemma 4.1.5, we know that $\hat{O}_{Y,p'} \simeq R_n[x_0, y_0, z_0]/(\pi d - x_0 y_0)$, for some root of unity $\xi$ and where $d \in \mathbb{Z}$ is defined such that $n \gcd(N_i, N_j) = N_i N_j d$. Since $Y_k \simeq Y \otimes_{R_n} k \simeq Y \otimes_{R_n} R_n/(\pi)$, we have

$$\hat{O}_{Y_k,p'} \simeq \hat{O}_{Y,p'} \otimes_{R_n} R_n/(\pi) \simeq k[x_0, y_0, z_0]/(\xi x_0 y_0).$$

This means that there are at most two irreducible components of $Y_k$ passing through $p'$.

On the other hand, the going-down theorem [Liu02, Lemma 10.4.34] says that the irreducible components passing through $p$ can be lifted, via the finite morphism $f$, to irreducible components passing through $p'$. Therefore, we conclude that there are exactly two irreducible components of $Y_k$ passing through $p'$.

(iv) Let $E$ be an irreducible component of $Y_k$, and take $q' \in E$. If $E$ is the only irreducible component of $Y_k$ passing through $q'$, then $O_{E,q'} = O_{Y_k,q'}$. So in that case, we have

$$\hat{O}_{E,q'} \simeq k[y, z],$$

which implies $E$ is smooth at $q'$. If there is another irreducible component of $Y_k$ passing through $q'$, then

$$\hat{O}_{Y,q'} \simeq k[x_0, y_0, z_0]/(x_0 y_0).$$

As there are exactly two components passing through $q'$ by (iii), both components must be smooth at $q'$.

Lemma 4.1.7. Let $f : Y \rightarrow X$ be the finite morphism constructed before. Write $X_k = \sum_{i \in I} N_i E_i$. Let

$$f_{E_i} : f^{-1}(E_i)_{\text{red}} \rightarrow E_i,$$

be the morphism induced by $f$. The following properties hold.

(i) The morphism $f_{E_i}$ is étale of degree $N_i$ above $E_i$.

(ii) Suppose $E_i$ and $E_j$ intersect non-trivially and let $C$ be an irreducible component of $E_i \cap E_j$. Let $C'$ be an irreducible component of $f^{-1}(C)_{\text{red}}$. Then $C'$ is smooth, and the ramification index of $f_{E_i}$ at $C'$ is $N_i / \gcd(N_i, N_j)$. 

\[\square\]
Proof.  (i) Let \( p' \in \mathcal{Y}_k \) be a closed point such that \( p = f(p') \in E_i^\circ \). The map \( \hat{\mathcal{O}}_{X,p} \to \hat{\mathcal{O}}_{Y,p'} \) is described by
\[
R[x,y,z]/(t-x^{N_i}) \to R_n[x,y,z]/(\pi^n - \xi x):
\]
\[
g(x,y,z,t) \mapsto g(x,y,z,\pi^n),
\]
where \( n' = \frac{\text{lcm}_{i \in I} N_i}{N_i} \). Therefore the map \( \hat{\mathcal{O}}_{E_{i,p}} \to \hat{\mathcal{O}}_{f^{-1}(E_i),\text{red},p'} \) is described by
\[
k[x,y,z]/(x) \to k[x,y,z]/(\xi x):
\]
\[
g(x,y,z) \mapsto g(x,y,z).
\]
We conclude that \( f_{E_i} \) is étale at \( p' \), by [Liu02, Proposition 4.3.26]. Moreover, \( f_{E_i} \) has degree \( N_i \), since the fiber of \( f_{E_i} \) above \( p \) consists of \( N_i \) points, by Lemma 4.1.2 and Lemma 4.1.4.

(ii) Let \( p' \in \mathcal{Y}_k \) be a closed point such that \( p = f(p') \in E_i \cap E_j \). The morphism \( \hat{\mathcal{O}}_{X,p} \to \hat{\mathcal{O}}_{Y,p'} \) is described by
\[
R[x,y,z]/(t-x^{N_i}y^{N_j}) \to R_n[x_0,y_0,z_0]/(\pi^d - \xi x_0y_0):
\]
\[
g(x,y,z,t) \mapsto g(x_0^{N_j/\gcd(N_i,N_j)},y_0^{N_i/\gcd(N_i,N_j)},z_0,\pi^n),
\]
where \( d \in \mathbb{Z} \) is defined such that \( (\text{lcm}_{i \in I} N_i)(\gcd(N_i,N_j)) = N_iN_jd \). Suppose that the ideal \( (x) \) describes \( E_i \) and \( (y) \) describes \( E_j \). Then
\[
\hat{\mathcal{O}}_{E_{i,p}} \simeq \hat{\mathcal{O}}_{X,p}/(x) \simeq R[x,y,z]/(x,t-x^{N_i}y^{N_j}) \simeq k[y,z].
\]
Similarly
\[
\hat{\mathcal{O}}_{f^{-1}(E_i),\text{red},p'} \simeq (R_n[x_0,y_0,z_0]/(x_0,\pi^d - \xi x_0y_0))_{\text{red}} \simeq k[y_0,\pi^n].
\]
Therefore, the morphism \( \hat{\mathcal{O}}_{E_{i,p}} \to \hat{\mathcal{O}}_{f^{-1}(E_i),\text{red},p'} \) is described by
\[
k[y,z] \to k[y_0,\pi^n]:
\]
\[
g(y,z) \mapsto g(y_0^{N_i/\gcd(N_i,N_j)},\pi^n).
\]
We see that the ramification index of \( f_{E_i} \) at \( p' \) is \( N_i/\gcd(N_i,N_j) \).

To prove that \( C' \) is smooth at \( p' \), we compute
\[
\hat{\mathcal{O}}_{C',p'} \simeq \left( \hat{\mathcal{O}}_{f^{-1}(E_i),\text{red},p'}/\left( y_0^{N_i/\gcd(N_i,N_j)} \right) \right)_{\text{red}} \simeq k[z_0],
\]
and the result follows.
The following lemma will be used in the proof of Proposition 4.1.9 (ii) and Proposition 4.1.12.

**Lemma 4.1.8.** Let $f : Y \rightarrow X$ be the finite morphism constructed before. Write $X_k = \sum_{i \in I} N_i E_i$. Suppose that $E_j$ is a minimal ruled surface and that any double curve $C$ on $E_j$ is a section of the ruling. Assume moreover that for any fiber $F$ of the ruling on $E_j$, we have that $f^{-1}(F)_{\text{red}}$ is a disjoint union of smooth, rational curves. Then $f^{-1}(E_j)_{\text{red}}$ is a disjoint union of minimal ruled surfaces.

Moreover, let $E'$ be an irreducible component of $f^{-1}(E_j)_{\text{red}}$, and let $C$ be a double curve on $E_j$. Exactly one irreducible component of $f^{-1}(C)_{\text{red}}$ is contained in $E'$, and it is a section of the ruling on $E'$.

**Proof.** It follows from Lemma 4.1.6 that $f^{-1}(E_j)_{\text{red}}$ is a disjoint union of smooth surfaces. Let $E'$ be an irreducible component of $f^{-1}(E_j)_{\text{red}}$. We will show that $E'$ is a minimal ruled surface. Consider the finite morphism $\phi = (f_{E_j})_{|E'} : E' \rightarrow E_j$. Let $F$ be a fiber of the ruling on $E_j$, and let $F'$ be a component of $\phi^{-1}(F)_{\text{red}}$. Since $F$ is transversal to the double curves, the generic point $\xi$ of $F$ is contained in $E''_j$. Because $\phi$ is étale above $E''_j$ by Lemma 4.1.7, we can write $\phi^*F = F'_1 + \cdots + F'_r$, where $F'_i$ is a smooth, rational curve for $i = 1, \ldots, r$ and all $F'_i$ are disjoint. We can assume $F' = F'_1$. On the other hand, we have $\phi^*F' = mF$, where $m$ is the degree of the map $F' \rightarrow F$. By the projection formula [Liu02 Theorem 9.2.12], we get

$$F' \cdot F' = F' \cdot (F'_1 + \cdots + F'_r) = F' \cdot \phi^*F = \phi_*F' \cdot F = mF \cdot F = 0.$$  

Since $E'$ contains a smooth, rational curve with self-intersection 0, it is a ruled surface, by [BHPV95 Proposition V.4.3].

Since $F' \cdot F' = 0$, [BHPV95 Proposition V.4.3] implies that $F'$ is an irreducible fiber of the ruling on $E'$. As $F$ was chosen arbitrarily, we conclude that any component of the inverse image of a fiber on $E_j$ is an irreducible fiber of the ruling on $E'$.

We will now show that $E'$ is minimal ruled. Let $G$ be a component of a fiber of the ruling on $E'$. By the projection formula

$$\phi_*G \cdot F = G \cdot \phi^*F = 0.$$
This implies that $\phi(\mathcal{G})$ is a fiber of the ruling on $E_j$. Since any component of the inverse image of a fiber on $E_j$ is an irreducible fiber of the ruling on $E'$, we know that $\mathcal{G}$ is an irreducible fiber, and we conclude that $E'$ is minimal ruled.

To prove the second claim in the statement, let $C$ be a double curve on $E_j$, and let $\mathcal{G}$ be a fiber of the ruling on $E'$. As proven before, $\phi_*\mathcal{G}$ is a fiber of the ruling on $E_j$, and hence the projection formula yields

$$\phi^* C \cdot \mathcal{G} = C \cdot \phi_* \mathcal{G} = 1.$$ 

This implies that $\phi^{-1}(C)_{\text{red}}$ is irreducible, and that it is a section.

4.1.3 Top of a flower

Proposition 4.1.9. Let $X$ be a K3 surface over $K$ and let $\mathcal{X}$ be a Crauder-Morrison model of $X$ with special fiber $\mathcal{X}_k = \sum_{i \in I} N_i E_i$. Let $N_0 F_0 + N_1 F_1 + \cdots + N_\ell F_\ell$ be a flower in $\mathcal{X}_k$, where $F_i \cap F_j = \emptyset$ if and only if $j \notin \{i - 1, i, i + 1\}$. Denote by $C_1$ the intersection $F_0 \setminus F_1$. The following relations hold in $\mathcal{M}_k^\mu$.

(i) If $F_0 \simeq \mathbb{P}^2$ and $C_1$ is a line, then there exists a $k$-variety $\mathcal{P}$ with a good $\hat{\mu}$-action, such that $[\tilde{F}_0] = [\mathcal{P}]\mathbb{L}^2$ and $[\tilde{C}_1] = [\mathcal{P}]([\mathbb{L}] + 1)$,

(ii) If $F_0$ is a minimal ruled surface, then $[\tilde{F}_0] = [\tilde{C}_1]\mathbb{L}$,

Proof. Let $f : Y \to \mathcal{X}$ be the finite morphism constructed before.

(i) Assume that $F_0 \simeq \mathbb{P}^2$ and that $C_1$ is a line. From Table 3.1 it follows that $\gcd(N_0, N_1) = N_0$.

Since $N_0 / \gcd(N_0, N_1) = 1$, Lemma 4.1.7 gives that $f$ is étale of degree $N_0$ above $F_0$. Since $F_0 \simeq \mathbb{P}^2$ is simply connected, we have

$$f^{-1}(F_0)_{\text{red}} = \bigsqcup_{i=1}^{N_0} G_i,$$

where $G_i \simeq \mathbb{P}^2$.

Let $g : f^{-1}(F_0)_{\text{red}} \to f^{-1}(F_0)_{\text{red}}$ be the automorphism of $f^{-1}(F_0)_{\text{red}}$ induced by a generator of $\text{Gal}(K(n)/K)$. Since the image of an irreducible component is irreducible, there exists an $\alpha_j \in \{1, \ldots, N_0\}$ for every $j \in \{1, \ldots, N_0\}$, such that $g(G_j) = G_{\alpha_j}$.
Set $\mathcal{P} = \bigsqcup_{i=1}^{N_0} P_i$, where $P_i = \text{Spec}(k)$ is a point. Define a $\mu_n$-action on $\mathcal{P}$ by $g \cdot P_j = P_{\alpha_j}$. Since every preimage of a point under $f$ consists of exactly $N_0$ points, there are $\mu_n$-equivariant isomorphisms

$$f^{-1}(F_0)_\text{red} \cong \mathcal{P} \times_k \mathbb{P}^2,$$

with trivial $\mu_n$-action on $\mathbb{P}^2$, and

$$f^{-1}(C_1)_\text{red} \cong \mathcal{P} \times_k \mathbb{P}^1,$$

with trivial $\mu_n$-action on $\mathbb{P}^1$.

We conclude that

$$[\tilde{F}_0] = [\mathcal{P}]L^2,$$

and

$$[\tilde{C}_1] = [\mathcal{P}](L + 1).$$

(ii) Assume that $F_0$ is a minimal ruled surface. In that case, $C_1$ is a section of the ruling on $F_0$. From Table 3.3, it follows that $\gcd(N_0, N_1) = N_0$. Lemma 4.1.7 implies that $f_{F_0}$ is étale of degree $N_0$ above $F_0$.

Let $\mathcal{F}$ be a fiber of the ruling on $F_0$. Since $\mathcal{F} \cong \mathbb{P}^1$ is simply connected, $f^{-1}(\mathcal{F})_\text{red}$ is a disjoint union of smooth, rational curves. Lemma 4.1.8 guarantees that $f^{-1}(F_0)_\text{red}$ is a disjoint union of minimal ruled surfaces and that every component of $f^{-1}(F_0)_\text{red}$ contains exactly one component of $f^{-1}(C_1)_\text{red}$, which is a section.

Therefore, there is a map

$$\psi: f^{-1}(F_0)_\text{red} \to f^{-1}(C_1)_\text{red},$$

consistent with the rulings on the components of $f^{-1}(F_0)_\text{red}$. So, all fibers of this map are isomorphic to $\mathbb{P}^1$. Furthermore, we have a commutative diagram

$$\begin{array}{ccc}
  f^{-1}(F_0)_\text{red} & \xrightarrow{\psi} & F_0 \\
  \downarrow & & \downarrow \\
  f^{-1}(C_1)_\text{red} & \xrightarrow{} & C_1
\end{array}$$

where the map $F_0 \to C_1$ defines the ruling on $F_0$. Notice that for any closed point $x \in f^{-1}(C_1)_\text{red} \subset f^{-1}(F_0)_\text{red}$, we have $\psi(x) = x$.

We will now show that $\psi$ is $\mu_n$-equivariant. Let $x$ be a closed point of $f^{-1}(F_0)_\text{red}$ and fix $g \in \mu_n$. Set $y = g \cdot x$. Notice that $f(x) = f(y)$. Let
Let $E_x$ and $E_y$ be the irreducible components of $f^{-1}(F_0)_{\text{red}}$ containing $x$ and $y$ respectively. Let $F_x$ be the fiber of the ruling on $E_x$, containing $x$ and similarly, let $F_y$ be the fiber of the ruling on $E_y$ containing $y$. From the proof of Lemma 4.1.8, it follows that $f(F_x)$ and $f(F_y)$ are fibers of the ruling on $F_0$. Since $f(x) = f(y)$, we must have $f(F_x) = f(F_y)$. Therefore, for any point $x' \in F_x$, we have that $g \cdot x' \in F_y$.

Let $x'$ be the unique point in the intersection of $F_x$ with the component of $f^{-1}(C_1)_{\text{red}}$ contained in $E_x$. Since $\psi$ is consistent with the ruling, we have that $\psi(x) = \psi(x')$. Moreover, since $x' \in f^{-1}(C_1)_{\text{red}}$, we have $\psi(x') = x'$. So

$$g \cdot \psi(x) = g \cdot x'.$$

On the other hand, we know that $g \cdot x' \in F_y$, and hence $\psi(g \cdot x') = \psi(y)$. Because $g \cdot x' \in f^{-1}(C_1)_{\text{red}}$, we know that $\psi(g \cdot x') = g \cdot x'$. Therefore

$$\psi(y) = g \cdot x'.$$

We conclude that $\psi(g \cdot x) = \psi(y) = g \cdot \psi(x)$. This implies that $\psi$ is $\mu_n$-equivariant.

From the fact that $\psi$ is $\mu_n$-equivariant, it follows that $\widetilde{F}_0^\circ = f^{-1}(F_0) \setminus f^{-1}(C_1)$ is an affine bundle over $f^{-1}(C_1)_{\text{red}}$ of rank 1. So $[\widetilde{F}_0^\circ] = [\widetilde{C}_1]\mathbb{L}$.

Ignoring the $\mu$-action, we also have a relation for the top of a conic-flower.

**Proposition 4.1.10.** Let $X$ be a K3 surface over $\mathbb{K}$ and let $\mathcal{X}$ be a Crauder-Morrison model of $X$ with special fiber $\mathcal{X}_k = \sum_{i \in I} N_i E_i$. Let $N_0 F_0 + N_1 F_1 + \cdots + N_{i_0} F_{i_0}$ be a flower in $\mathcal{X}_k$, where $F_i \cap F_j = \emptyset$ if and only if $j \notin \{i-1, i, i+1\}$. Denote by $C_1$ the intersection $F_0 \cap F_1$. We have

$$[\widetilde{F}_0^\circ] = \frac{N_0}{2}(\mathbb{L} + 1)\mathbb{L} \quad \text{and} \quad [\widetilde{C}_1] = \frac{N_0}{2}(\mathbb{L} + 1),$$

in $\mathcal{M}_k$.

**Proof.** From Table 3.2 it follows that $N_0$ is even and that $\gcd(N_0, N_1) = N_0/2$. Let $f : Y \to \mathcal{X}$ be the finite morphism constructed before. By Lemma 4.1.17 we know that $f_{F_0}$ is étale of degree $N_0$ above $\widetilde{F}_0^\circ = F_0 \setminus C_1$, and ramified of index 2 at $f^{-1}(C_1)$. Therefore, $f^{-1}(C_1)_{\text{red}} \to C_1$ is étale of degree $N_0/2$. Since $C_1 \simeq \mathbb{P}^1$ is simply connected, $\widetilde{C}_1$ has $N_0/2$ irreducible components, all isomorphic to $\mathbb{P}^1$. In particular, we have

$$[\widetilde{C}_1] = \frac{N_0}{2}(\mathbb{L} + 1).$$
Let $E$ be a component of $f^{-1}(F_0)_{\text{red}}$, and let $C$ be a component of $f^{-1}(C_1)$ contained in $E$. The map $E \to F_0$ is a cyclic covering, branched along the conic $C_1$. Let $d$ be the degree of this covering. By [BHPV95, Section 1.17], there exists a unique line bundle $\mathcal{L}$ on $F_0 \simeq \mathbb{P}^2$ with $O_{\mathbb{P}^2}(C_1) = \mathcal{L}^\otimes d$. As $C_1$ is a conic, we get $\mathcal{L}^\otimes d = O_{\mathbb{P}^2}(2)$, and hence $d = 2$. This means that $E$ contains exactly one component of $f^{-1}(C_1)$ and therefore, $f^{-1}(F_0)_{\text{red}}$ has exactly $N_0/2$ irreducible components.

We will now prove that $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$. As mentioned before, $E$ is the double cyclic cover of $\mathbb{P}^2$, branched along a conic. Therefore, it is isomorphic to the quadric surface in $\mathbb{P}^3 = \text{Proj} k[x : y : z : w]$ defined by the equation $w^2 = f(x : y : z)$, where $f$ is a homogeneous polynomial of degree 2, defining the conic $C_1$ in $\mathbb{P}^2$. Because all quadric surfaces in $\mathbb{P}^3$ are isomorphic, $E$ is isomorphic to the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. We conclude that $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Therefore

$$[\widetilde{F_0}] = \frac{N_0}{2} ([\mathbb{P}^1 \times \mathbb{P}^1] - [\mathbb{P}^1]) = \frac{N_0}{2} L(L + 1).$$

\[
\square
\]

**Remark 4.1.11.** We don’t know whether $[\widetilde{F_0}] = [\widetilde{C_1}]L$ also holds in $\mathcal{M}_k^{\mu}$. Let $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible component of $f^{-1}(F_0)_{\text{red}}$, and let $C$ be the component of $f^{-1}(C_1)_{\text{red}}$ contained in $E$, then it does not seem likely that $E \setminus C$ is an equivariant affine bundle over $C$, because the two fibrations on $E$ are interchanged under the $\mu$-action.

### 4.1.4 Middle component of a flower

**Proposition 4.1.12.** Let $X$ be a $K3$ surface over $K$ and let $\mathcal{X}$ be a Crauder-Morrison model of $X$ with special fiber $\mathcal{X}_k = \bigoplus_{i \in I} N_i E_i$. Let $N_0F_0 + N_1F_1 + \cdots + N_\ell F_\ell$ be a flower in $\mathcal{X}_k$, where $F_i \cap F_j = \emptyset$ if and only if $j \notin \{i-1, i, i+1\}$. The component $F_\ell$ meets $\Gamma_{\text{min}}$ in $F_{\ell+1}$. Denote by $C_j$ the intersection $F_{\ell-1} \cap F_\ell$ for $j = 1, \ldots, \ell + 1$. For any $j = 1, \ldots, \ell$, we have

$$[\widetilde{F_j}] = [\widetilde{C_j}](L - 1) \quad \text{and} \quad [\widetilde{C_j}] = [\widetilde{C_{j+1}}],$$

in $\mathcal{M}_k^{\mu}$.

We first need a lemma.

**Lemma 4.1.13.** Fix $j \in \{1, \ldots, \ell\}$. Let $\mathcal{F}$ be a fiber of the minimal ruled surface $F_j$, and let $f : Y \to X$ be the finite morphism constructed before. Then we have that $f^{-1}(\mathcal{F})_{\text{red}}$ is a disjoint union of smooth, rational curves.
Proof. Let $p \in \mathcal{F}$ be a closed point and let $E$ be a component of $f^{-1}(F_j)_{\text{red}}$. Let $p' \in E$ be a point such that $f(p') = p$, and let $\mathcal{F}'$ be an irreducible component of $f^{-1}(\mathcal{F})_{\text{red}}$ passing through $p'$. Suppose first that $p \in F_j^o$. From Lemma 4.1.4 it follows that the map $\widehat{O}_{F_j,p} \to \widehat{O}_{E,p'}$ is given by

$$k[y,z] \to k[y_0,z_0]: g(y,z) \mapsto g(y_0,z_0).$$

Because $\mathcal{F}$ is smooth, [Liu02 Corollary 4.2.12] gives that the ideal defining $\mathcal{F}$ at $p$ is generated by a non-zero element $h \in \mathfrak{m} \setminus \mathfrak{m}^2$, where $\mathfrak{m} = (y,z)$ is the maximal ideal of $k[y,z]$. Since $\mathcal{F}$ and $C_i$ meet transversally at $p$, the maximal ideal $(y,z)$ must be generated by $h$ and $y$. Hence $h \equiv z + ay \mod \mathfrak{m}^2$ for some $a \in k$. Thus $\mathcal{F}'$ is generated by $h' \in k[y_0,z_0]$ with $h' \equiv z_0 + ay_0^{N_j}/\gcd(N_i,N_j) \mod (y_0,z_0)^2$ and therefore $\mathcal{F}'$ is smooth at $p'$. Furthermore, $f^{-1}(\mathcal{F})_{\text{red}} \to \mathcal{F}$ is étale at $p'$, by [Liu02 Proposition 4.3.26].

Let $i = j$ or $j+1$. Assume now that $p \in C_i$. From Lemma 4.1.4 it follows that the map $\widehat{O}_{F_j,p} \to \widehat{O}_{E,p'}$ is given by

$$k[y,z] \to k[y_0,z_0]: g(y,z) \mapsto g\left(y_0^{N_j}/\gcd(N_i,N_j), z_0\right).$$

Let $(y)$ be the ideal defining $C_i \subset F_j$ at $p$. Let $C'$ be an irreducible component of $f^{-1}(C_i)_{\text{red}}$ passing through $p'$.

Because $\mathcal{F}$ is smooth, [Liu02 Corollary 4.2.12] gives that the ideal defining $\mathcal{F}$ at $p$ is generated by a non-zero element $h \in \mathfrak{m} \setminus \mathfrak{m}^2$, where $\mathfrak{m} = (y,z)$ is the maximal ideal of $k[y,z]$. Since $\mathcal{F}$ and $C_i$ meet transversally at $p$, the maximal ideal $(y,z)$ must be generated by $h$ and $y$. Hence $h \equiv z + ay \mod \mathfrak{m}^2$ for some $a \in k$. Thus $\mathcal{F}'$ is generated by $h' \in k[y_0,z_0]$ with $h' \equiv z_0 + ay_0^{N_j}/\gcd(N_i,N_j) \mod (y_0,z_0)^2$ and therefore $\mathcal{F}'$ is smooth at $p'$. As a consequence, $f^{-1}(\mathcal{F})_{\text{red}}$ is a disjoint union of smooth curves. Moreover, $\mathcal{F}'$ and $C'$ meet transversally at $p'$, because $C'$ is defined by the ideal $(y_0)$.

We now show that $\mathcal{F}'$ is rational. Consider the morphism $\phi = f|_{\mathcal{F}'}: \mathcal{F}' \to \mathcal{F}$. By Lemma 4.1.7 we know that $\phi$ is only ramified above the points $p_j = \mathcal{F} \cap C_j$ and $p_{j+1} = \mathcal{F} \cap C_{j+1}$. Hurwitz’ formula [Liu02 Theorem 7.4.16] gives that

$$2g(\mathcal{F}') - 2 = n(2g(\mathcal{F}) - 2) + (e_{p_j} - 1) + (e_{p_{j+1}} - 1),$$

where $n$ is the degree of $\phi$, and $e_{p_j}$ and $e_{p_{j+1}}$ are the ramification indices of $p_j$ and $p_{j+1}$ respectively. Since $\mathcal{F}$ is a fiber of $F_j$, it is rational. Furthermore, [Liu02 Equation (4.8) p. 290] gives that $e_{p_j} = e_{p_{j+1}} = n$. We conclude that $g(\mathcal{F}') = 0$ and hence $f^{-1}(\mathcal{F})_{\text{red}}$ is a disjoint union of smooth, rational curves.

Proof of Proposition 4.1.12. Let $f: \mathcal{Y} \to \mathcal{X}$ be the finite morphism constructed before. By Lemma 4.1.8 and Lemma 4.1.13 we know that $f^{-1}(F_j)_{\text{red}}$ is a
disjoint union of minimal ruled surfaces. Moreover, any component of \( f^{-1}(F_j)_{\text{red}} \) contains exactly one component of \( f^{-1}(C'_j)_{\text{red}} \) and \( f^{-1}(C_{j+1})_{\text{red}} \) and they are sections of the ruling.

Therefore, there is a map
\[
\psi: f^{-1}(F_j)_{\text{red}} \to f^{-1}(C'_j)_{\text{red}},
\]
consistent with the rulings on the components of \( f^{-1}(F_j)_{\text{red}} \). So, all fibers of this map are isomorphic to \( \mathbb{P}^1 \). Furthermore, we have a commutative diagram
\[
\begin{array}{ccc}
 f^{-1}(F_j)_{\text{red}} & \longrightarrow & F_j \\
 \psi & \downarrow & \downarrow \\
 f^{-1}(C'_j)_{\text{red}} & \longrightarrow & C_j
\end{array}
\]
where the map \( F_j \to C_j \) defines the ruling on \( F_j \). Notice that for any closed point \( x \in f^{-1}(C'_j)_{\text{red}} \subset f^{-1}(F_j)_{\text{red}} \), we have \( \psi(x) = x \).

We will now show that \( \psi \) is \( \mu_n \)-equivariant. Let \( x \) be a closed point of \( f^{-1}(F_j)_{\text{red}} \) and denote \( y = g \cdot x \), for some \( g \in \mu_n \). Notice that \( f(x) = f(y) \). Let \( E_x \) and \( E_y \) be the irreducible components of \( f^{-1}(F_j)_{\text{red}} \) containing \( x \) and \( y \) respectively. Let \( \mathcal{F}_x \) be the fiber of the ruling on \( E_x \) containing \( x \) and similarly, let \( \mathcal{F}_y \) be the fiber of the ruling on \( E_y \) containing \( y \). From the proof of Lemma 4.1.8 it follows that \( f(\mathcal{F}_x) \) and \( f(\mathcal{F}_y) \) are fibers of the ruling on \( F_j \). Since \( f(x) = f(y) \), we must have \( f(\mathcal{F}_x) = f(\mathcal{F}_y) \). Therefore, for any point \( x' \in \mathcal{F}_x \), we have that \( g \cdot x' \in \mathcal{F}_y \).

Let \( x' \) be the unique point in the intersection of \( \mathcal{F}_x \) with the component of \( f^{-1}(C'_j)_{\text{red}} \) contained in \( E_x \). Since \( \psi \) is consistent with the ruling, we have that \( \psi(x) = \psi(x') \). Moreover, since \( x' \in f^{-1}(C'_j)_{\text{red}} \), we have \( \psi(x') = x' \). So
\[
g \cdot \psi(x) = g \cdot x'.
\]
On the other hand, we know that \( g \cdot x' \in \mathcal{F}_y \), and hence \( \psi(g \cdot x') = \psi(y) \). Because \( g \cdot x' \in f^{-1}(C'_j)_{\text{red}} \), we know that \( \psi(g \cdot x') = g \cdot x' \). Therefore
\[
\psi(y) = g \cdot x'.
\]
We conclude that \( \psi(g \cdot x) = \psi(y) = g \cdot \psi(x) \). This implies that \( \psi \) is \( \mu_n \)-equivariant.

In particular, when \( x \in f^{-1}(C_{j+1})_{\text{red}} \), the previous argument implies that
\[
[C_j] = [C_{j+1}].
\]
From the fact that $\psi$ is $\mu_n$-equivariant, it follows that $f^{-1}(F_j)_{\text{red}} \setminus f^{-1}(C_j)_{\text{red}}$ is an affine bundle over $f^{-1}(C_j)_{\text{red}}$ of rank 1. So

$$[f^{-1}(F_j)] - [\widetilde{C}_j] = [\widetilde{C}_j] \mathbb{L}.$$ 

This means that

$$[\widetilde{F}_j] = [f^{-1}(F_j)] - [\widetilde{C}_j] - [\widetilde{C}_{j+1}] = [\widetilde{C}_j](\mathbb{L} - 1).$$

\[\square\]

### 4.2 Contribution of a flower to the motivic zeta function

**Definition 4.2.1.** Let $X$ be a K3 surface over $K$ with Crauder-Morrison model $\mathcal{X}$. Let $\omega$ be a volume form on $X$. Let $N_0F_0 + N_1F_1 + \cdots + N_\ell F_\ell$ be a flower in $\mathcal{X}_k$, where $F_i \cap F_j = \emptyset$ if and only if $j \notin \{i - 1, i, i + 1\}$. The component $F_\ell$ meets $\Gamma_{\text{min}}$ in $F_{\ell+1}$. Denote by $C_j$ the intersection $F_j \setminus F_{j+1}$, for $j = 0, \ldots, \ell$. Let $(N_j, \nu_j)$ be the numerical data of $F_j$, for every $j = 0, \ldots, \ell + 1$.

We define the contribution $Z_F(T) \in \hat{M}^\mathbb{A}_{k}[T]$ of the flower $F$ to the motivic zeta function $Z_{X, \omega}(T)$ as

$$Z_F(T) = \sum_{j=0}^{\ell} \left( [\widetilde{F}_j] \frac{\mathbb{L}^{-\nu_j} T^{N_j}}{1 - \mathbb{L}^{-\nu_j} T^{N_j}} \right) + (\mathbb{L} - 1)[\widetilde{C}_j] \frac{\mathbb{L}^{-\nu_j} T^{N_j}}{1 - \mathbb{L}^{-\nu_j} T^{N_j}} \frac{\mathbb{L}^{-\nu_{j+1}} T^{N_{j+1}}}{1 - \mathbb{L}^{-\nu_{j+1}} T^{N_{j+1}}}.$$ 

**Lemma 4.2.2.** Let $X$ be a K3 surface over $K$ allowing a Crauder-Morrison model $\mathcal{X}$ with special fiber $\mathcal{X}_k = \sum_{i \in I} N_i E_i$. Let $\omega$ be a volume form on $X$ and let $(N_i, \nu_i)$ be the numerical data of $E_i$, for every $i \in I$. Let $\rho_i = \nu_i/N_i + 1$ be the weight of $E_i$, for every $i \in I$.

Define $I_{\text{min}} = \{i \in I \mid \rho_i \text{ is minimal}\}$ and

$$Z_{\text{min}}(T) = \sum_{\emptyset \neq J \subseteq I_{\text{min}}} (\mathbb{L} - 1)^{|J|-1} [\widetilde{E}_J] \prod_{j \in J} \frac{\mathbb{L}^{-\nu_j} T^{N_j}}{1 - \mathbb{L}^{-\nu_j} T^{N_j}}.$$ 

We have

$$Z_{X, \omega}(T) = Z_{\text{min}}(T) + \sum Z_F(T),$$

where the sum runs over all flowers in $\mathcal{X}_k$. 
Proof. This immediately follows from Theorem 2.2.2 and Theorem 3.2.1.

Theorem 4.2.3. Let $X$ be a $K3$ surface over $K$ allowing a Crauder-Morrison model $X'$. Let $\omega$ be a volume form on $X$. Let $N_0 F_0 + N_1 F_1 + \cdots + N_\ell F_\ell$ be a flower in $X_k$, where $F_i \cap F_j = \emptyset$ if and only if $j \notin \{i-1, i, i+1\}$. The component $F_\ell$ meets $\Gamma_{\min}$ in $F_{\ell + 1}$. Denote by $C_{j+1}$ the intersection $F_j \cap F_{j+1}$ for $j = 0, \ldots, \ell$. Let $(N_j, \nu_j)$ be the numerical data of $F_j$, for every $j = 0, \ldots, \ell + 1$.

If $F_0 \cong \mathbb{P}^2$ with $F_0 \cap F_1$ a line, or if $F_0$ is a minimal ruled surface, then

$$Z_F(T) \in \mathcal{M}_k^\mu \left[ T, \frac{1}{1 - L - \nu_{\ell+1} T^{N_{\ell+1}}} \right].$$

If $F_0 \cong \mathbb{P}^2$ with $F_0 \cap F_1$ a conic, then

$$Z_F(T) \in \mathcal{M}_k^\mu \left[ T, \frac{1}{1 - L - \nu_0 T^{N_0}}, \frac{1}{1 - L - \nu_{\ell+1} T^{N_{\ell+1}}} \right].$$

Proof. By Theorem 3.3.7 and Lemma 3.3.10, we can compute $(N_j, \nu_j)$ in function of $(N_0, \nu_0)$, for every $j = 1, \ldots, \ell + 1$. When $F_0 \cong \mathbb{P}^2$ with $F_0 \cap F_1$ a conic, we will express $(N_j, \nu_j)$ in function of $(N_1, \nu_1)$ instead of $(N_0, \nu_0)$ to avoid fractions.

By the results in Section 4.1, we know that

(i) if $F_0 \cong \mathbb{P}^2$ and $F_0 \cap F_1$ is a line, then there exists a $k$-variety $\mathcal{P}$ with good $\hat{\mu}$-action, such that $[\widetilde{F}_0] = [\mathcal{P}] L^2$ and $[\widetilde{C}_1] = [\mathcal{P}] (L + 1)$,

(ii) if $F_0$ is a minimal ruled surface, then $[\widetilde{F}_0] = [\widetilde{C}_1](L + 1)$,

(iii) $[\widetilde{F}_j] = [\widetilde{C}_1](L - 1)$ and $[\widetilde{C}_{j+1}] = [\widetilde{C}_1]$, for every $j = 1, \ldots, \ell$.

Therefore, we can explicitely compute $Z_F(T)$. We will illustrate this for a flower of type 3A and for a flower of type 4C.

Suppose $F$ is a flower of type 3A, then $\ell = 1$ and Theorem 3.3.7 and Lemma 3.3.10 imply that the numerical data are given by

$$\begin{align*}
(N_1, \nu_1) &= (2N_0, 2\nu_0 - 1), \\
(N_2, \nu_2) &= (3N_0, 3\nu_0 - 2).
\end{align*}$$

Let $\mathcal{P}$ be the $k$-variety with good $\hat{\mu}$-action, such that $[\widetilde{F}_0] = [\mathcal{P}] L^2$ and $[\widetilde{C}_1] = [\mathcal{P}] (L + 1)$. We then have $[\widetilde{F}_1] = [\mathcal{P}] (L + 1)(L - 1)$ and $[\widetilde{C}_2] = [\mathcal{P}] (L + 1)$. 

Therefore, we compute
\[
Z_F(T) = \left[ \tilde{F}_0 \right] \frac{L^{-\nu_0} T^{N_0}}{1 - L^{-\nu_0} T^{N_0}} + \left[ \tilde{F}_1 \right] \frac{L^{-\nu_1} T^{N_1}}{1 - L^{-\nu_1} T^{N_1}} \\
+ \left[ \tilde{C}_1 \right] \frac{(L - 1)L^{-\nu_0 - \nu_1} T^{N_0 + N_1}}{(1 - L^{-\nu_0} T^{N_0})(1 - L^{-\nu_1} T^{N_1})} + \left[ \tilde{C}_2 \right] \frac{(L - 1)L^{-\nu_1 - \nu_2} T^{N_1 + N_2}}{(1 - L^{-\nu_1} T^{N_1})(1 - L^{-\nu_2} T^{N_2})}
\]
\[
= \left[ \mathcal{P} \right] \frac{L^2 L^{-\nu_0} T^{N_0}}{1 - L^{-\nu_0} T^{N_0}} + \left[ \mathcal{P} \right] \frac{(L + 1)(L - 1)L^{-2\nu_0 + 1} T^{2N_0}}{1 - L^{-2\nu_0 + 1} T^{2N_0}} \\
+ \left[ \mathcal{P} \right] \frac{(L + 1)(L - 1)L^{-3\nu_0 + 1} T^{3N_0}}{(1 - L^{-\nu_0} T^{N_0})(1 - L^{-2\nu_0 + 1} T^{2N_0})} \\
+ \left[ \mathcal{P} \right] \frac{(L + 1)(L - 1)L^{-5\nu_0 + 3} T^{5N_0}}{(1 - L^{-2\nu_0 + 1} T^{2N_0})(1 - L^{-3\nu_0 + 2} T^{3N_0})} \\
= \left[ \mathcal{P} \right] \frac{L^{-\nu_0 + 2} T^{N_0} (L^{-2\nu_0 + 2} T^{2N_0} + L^{-\nu_0 + 2} T^{N_0} + 1)}{1 - L^{-\nu_2} T^{N_2}},
\]
where we computed the last equality with SymPy, a library of the programming language Python for symbolic mathematics.

We conclude that
\[
Z_F(T) \in \mathcal{M}_k\left[ T,\frac{1}{1 - L^{-\nu_{\ell+1}} T^{N_{\ell+1}}} \right].
\]

Assume now that \( F \) is a flower of type \( 4C \) of length \( \ell \). Theorem 3.3.7 and Lemma 3.3.10 imply that the numerical data are given by
\[
(N_0, \nu_0) = (2N_1, 2\nu_1 + 1),
\]
\[
(N_i, \nu_i) = (N_1, \nu_1 - i + 1) \quad \text{for } i = 1, \ldots, \ell,
\]
\[
(N_{\ell+1}, \nu_{\ell+1}) = (2N_1, 2\nu_1 - 2\ell + 1).
\]

We have \( \tilde{F}_j = [C_1](L - 1) \), and \( \tilde{C}_j = [C_1] \) for \( j = 1, \ldots, \ell \).
Therefore, we compute

\[
Z_F(T) = \frac{[F_0^\circ]L^{-\nu_0}T^{N_0}}{1 - L^{-\nu_0}T^{N_0}} + \frac{[\tilde{C}_1](L - 1)L^{-\nu_0 - \nu_1}T^{N_0 + N_1}}{(1 - L^{-\nu_0}T^{N_0})(1 - L^{-\nu_1}T^{N_1})}
\]

\[+ \sum_{j=1}^{\ell-1} \left( \frac{[\tilde{F}_j^\circ]L^{-\nu_j}T^{N_j}}{1 - L^{-\nu_j}T^{N_j}} + \frac{[\tilde{C}_{j+1}](L - 1)L^{-\nu_j - \nu_{j+1}}T^{N_j + N_{j+1}}}{(1 - L^{-\nu_j}T^{N_j})(1 - L^{-\nu_{j+1}}T^{N_{j+1}})} \right)
\]

\[+ \frac{[\tilde{F}_\ell^\circ]L^{-\nu_\ell}T^{N_\ell}}{1 - L^{-\nu_\ell}T^{N_\ell}} + \frac{[\tilde{C}_{\ell+1}](L - 1)L^{-\nu_\ell - \nu_{\ell+1}}T^{N_\ell + N_{\ell+1}}}{(1 - L^{-\nu_\ell}T^{N_\ell})(1 - L^{-\nu_{\ell+1}}T^{N_{\ell+1}})}
\]

\[= \frac{[F_0^\circ]L^{-\nu_0}T^{N_0}}{1 - L^{-\nu_0}T^{N_0}} + \frac{[\tilde{C}_1](L - 1)L^{-3\nu_1 - 1}T^{3N_1}}{(1 - L^{-2\nu_1 - 1}T^{2N_1})(1 - L^{-\nu_1}T^{N_1})}
\]

\[+ \sum_{j=1}^{\ell-1} \left( \frac{[\tilde{C}_1](L - 1)L^{-\nu_1 + j - 1}T^{N_1}}{1 - L^{-\nu_1 + j - 1}T^{N_1}} \right)
\]

\[+ \frac{[\tilde{C}_1](L - 1)L^{-2\nu_1 + 2j - 1}T^{2N_1}}{(1 - L^{-\nu_1 + j - 1}T^{N_1})(1 - L^{-\nu_1 + j}T^{N_1})}
\]

\[+ \frac{[\tilde{C}_1](L - 1)L^{-\nu_1 + \ell - 1}T^{N_1}}{1 - L^{-\nu_1 + \ell - 1}T^{N_1}}
\]

\[+ \frac{[\tilde{C}_1](L - 1)L^{-3\nu_1 + 3\ell - 2}T^{3N_1}}{(1 - L^{-\nu_1 + \ell - 1}T^{N_1})(1 - L^{-2\nu_1 + 2\ell - 1}T^{2N_1})}.
\]

By induction on \(m\), one can compute that

\[\sum_{j=1}^{m} \left( \frac{[\tilde{C}_1](L - 1)L^{-\nu_1 + j - 1}T^{N_1}}{1 - L^{-\nu_1 + j - 1}T^{N_1}} \right)
\]

\[= \frac{[\tilde{C}_1](L^m - 1)L^{-\nu_1}T^{N_1}}{(1 - L^{-\nu_1}T^{N_1})(1 - L^{-\nu_1 + m}T^{N_1})}
\]

Therefore, direct computation (using code in SymPy) shows that

\[
Z_F(T) = \frac{[F_0^\circ]L^{-\nu_0}T^{N_0}}{1 - L^{-\nu_0}T^{N_0}} + \frac{[\tilde{C}_1]L^{-\nu_1}T^{N_1} \Theta(T)}{(1 - L^{-\nu_0}T^{N_0})(1 - L^{-\nu_{\ell+1}}T^{N_{\ell+1}})},
\]
where
\[ \Theta(T) = (\mathbb{L}^\ell - 1) + (\mathbb{L}^{2\ell - 1} - 1)\mathbb{L}^{-\nu_1 T^N_1} \]
\[ + (\mathbb{L}^\ell - 1)\mathbb{L}^{-2\nu_1 + l - 1} T^{2N_1} + (\mathbb{L} - 1)\mathbb{L}^{-3\nu_1 + 2\ell - 2} T^{3N_1}. \]

We conclude that
\[ Z_F(T) \in \mathcal{M}_{\hat{k}}^\mu \left[ T, \frac{1}{1 - \mathbb{L}^{-\nu_0 T^{N_0}}}, \frac{1}{1 - \mathbb{L}^{-\nu_{l+1} T^{N_{l+1}}}} \right]. \]

All computations have been implemented in Python, as well as the computations for the other flowers. The code can be found in Appendix A.2 and can be downloaded from www.github.com/AnneliesJaspers/flowers_contribution. Explicit formulas for the contribution of flowers can be found in Appendix A.1.

**Corollary 4.2.4.** Let \( X \) be a K3 surface over \( K \) allowing a Crauder-Morrison model \( \mathcal{X} \) with special fiber \( \mathcal{X}_k = \sum_{i \in I} N_i E_i \). Let \( \omega \) be a volume form on \( X \) and let \( (N_i, \nu_i) \) be the numerical data of \( E_i \), for every \( i \in I \). Let \( \rho_i = \nu_i / N_i + 1 \) be the weight of \( E_i \), for every \( i \in I \).

Define \( I^\dagger \subset I \) to be the set of indices \( i \in I \) with either

(i) \( \rho_i \) is minimal, or

(ii) \( E_i \) is the top of a conic-flower.

Let \( S^\dagger = \{(\nu_i, N_i) \in \mathbb{Z} \times \mathbb{Z}_{>0} \mid i \in I^\dagger \} \). We have
\[ Z_{\mathcal{X}, \omega}(T) \in \mathcal{M}_{\hat{k}}^\mu \left[ T, \frac{1}{1 - \mathbb{L}^{-\nu_a T^b}} \right]_{(a,b) \in S^\dagger}. \]

**Proof.** This follows immediately from Lemma 4.2.2 and Theorem 4.2.3.

### 4.3 Poles of the motivic zeta function

#### 4.3.1 Definition of a pole of a rational function over \( \mathcal{M}_{\hat{k}}^\mu \)

The motivic zeta function \( Z_{\mathcal{X}, \omega}(T) \) is a rational function over \( \mathcal{M}_{\hat{k}}^\mu \), and we are interested in its poles. Since \( \mathcal{M}_{\hat{k}}^\mu \) is not a domain, it is, a priori, not clear what the definition of a pole of a rational function over \( \mathcal{M}_{\hat{k}}^\mu \) should be. We use the definition of pole as defined by Nicaise and Xu in [NX16, Remark 3.7].
In this thesis, we will only be interested in poles of rational functions over $\mathcal{M}_k^\mu$ that are elements of $\mathcal{M}_k^\mu \left[ T, \frac{1}{1-\mathbb{L}^aT^b} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}}$. This ring is the localization of $\mathcal{M}_k^\mu[T]$ with respect to the multiplicatively closed set $\{1 - \mathbb{L}^aT^b \mid (a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}\}$. Note that this set does not contain zero-divisors and hence

$$\frac{F_1}{G_1} = \frac{F_2}{G_2} \in \mathcal{M}_k^\mu \left[ T, \frac{1}{1-\mathbb{L}^aT^b} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}} \iff F_1 \cdot G_2 = F_2 \cdot G_1 \in \mathcal{M}_k^\mu[T].$$

**Definition 4.3.1.** Let $Z(T) \in \mathcal{M}_k^\mu \left[ T, \frac{1}{1-\mathbb{L}^aT^b} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}}$ be a rational function over $\mathcal{M}_k^\mu$ and let $q \in \mathbb{Q}$ be a rational number. We say that $Z(T)$ has a pole of order at most $m$ at $q$, if we find a set $A$ consisting of multisets in $\mathbb{Z} \times \mathbb{Z}_{>0}$ such that

(i) each multiset $A \in A$ contains at most $m$ elements $(a,b)$ with $\frac{a}{b} = q$, and

(ii) $Z(T)$ belongs to the sub-$\mathcal{M}_k^\mu[T]$-module of $\mathcal{M}_k^\mu[T]$ generated by

$$\left\{ \frac{1}{\prod_{(a,b) \in A} (1-\mathbb{L}^aT^b)} \right\}.$$  

We say that $Z(T)$ has a pole of order $m$ at $q$, if it has a pole of order at most $m$, but not of order at most $m - 1$ (the latter condition is void for $m = 0$). We say that $Z(T)$ has a pole at $q$, if there exists an integer $m \geq 1$ such that $Z(T)$ has a pole or order $m$ at $q$.

To explain the intuition behind this definition, we notice that one often formally substitutes $T = \mathbb{L}^{-s}$ and considers $Z(T)$ as a function in the variable $s$. If $\frac{a}{b}$ is a pole of $Z(\mathbb{L}^{-s})$, then we can think of the factor $1 - \mathbb{L}^aT^b = 1 - \mathbb{L}^a\mathbb{L}^{-bs}$ in the denominator as having a zero in $s = \frac{a}{b}$.

**Lemma 4.3.2.** Let $Z(T) \in \mathcal{M}_k^\mu \left[ T, \frac{1}{1-\mathbb{L}^aT^b} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}}$ be a rational function over $\mathcal{M}_k^\mu$, and let $q \in \mathbb{Q}$ be a rational number. The following are equivalent:

(i) $q$ is a pole of $Z(T)$,

(ii) for every subset $S \subset \mathbb{Z} \times \mathbb{Z}_{>0}$ with $Z(T) \in \mathcal{M}_k^\mu \left[ T, \frac{1}{1-\mathbb{L}^aT^b} \right]_{(a,b) \in S}$, there exists an element $(a,b) \in S$ with $q = \frac{a}{b}$.

**Proof.** Suppose first that there exists a subset $S \subset \mathbb{Z} \times \mathbb{Z}_{>0}$ with $Z(T) \in \mathcal{M}_k^\mu \left[ T, \frac{1}{1-\mathbb{L}^aT^b} \right]_{(a,b) \in S}$ with the property that $q \neq \frac{a}{b}$ for every $(a,b) \in S$. Then $Z(T)$ does not have a pole at $q$. Therefore, if $Z(T)$ has a pole at $q$, then for every subset $S \subset \mathbb{Z} \times \mathbb{Z}_{>0}$, there exists an element $(a,b) \in S$ with $q = \frac{a}{b}$. This implies the equivalence of the two conditions.

We denote the set of poles of $Z(T)$ at $q$ by $\text{poles}(Z(T), q)$. It is clear that $\text{poles}(Z(T), q)$ is a subset of $\mathbb{Q}$. We will use this notation throughout the thesis.
This means that we can write
\[ Z(T) = \frac{F(T)}{\prod_{(a,b) \in S}(1 - \mathbb{L}^a T^b)^{m_{(a,b)}},} \]
for some \( F(T) \in \mathcal{M}_k^\mu[T] \) and \( m_{(a,b)} \in \mathbb{Z}_{\geq 0} \). Let \( A \) be the multiset that contains every \((a, b) \in S\) with multiplicity \( m_{(a,b)} \) and define \( \mathcal{A} = \{ A \} \). It is clear that 
\[ Z(T) \] belongs to the sub-\( \mathcal{M}_k^\mu[T]\)-module of \( \mathcal{M}_k^\mu[J^T K^T] \) generated by
\[ \left\{ \frac{1}{\prod_{(a,b) \in A}(1 - \mathbb{L}^a T^b)} \bigg| A \in \mathcal{A} \right\}. \]
Moreover, \( A \) does not contain an element \((a, b)\) with \( q = \frac{a}{b} \). Hence \( q \) is a pole of order at most 0, which means that it is \textit{not} a pole of \( Z(T) \).

Conversely, suppose that \( q \) is \textit{not} a pole of \( Z(T) \). So there exists a set \( \mathcal{A} \) consisting of multisets in \( \mathbb{Z} \times \mathbb{Z}_{>0} \) such that 
\[ Z(T) \] belongs to the sub-\( \mathcal{M}_k^\mu[T]\)-module of \( \mathcal{M}_k^\mu[J^T K^T] \) generated by
\[ \left\{ \frac{1}{\prod_{(a,b) \in A}(1 - \mathbb{L}^a T^b)} \bigg| A \in \mathcal{A} \right\}, \]
and such that no multiset \( A \in \mathcal{A} \) contains a couple \((a, b)\) with \( q = \frac{a}{b} \). Define \( S = \bigcup_{A \in \mathcal{A}} A \). It is clear that 
\[ Z(T) \in \mathcal{M}_k^\mu[T, \frac{1}{1 - \mathbb{L}^a T^b}] \] \((a,b) \in S\), and that there is no couple \((a, b) \in S\) with \( q = \frac{a}{b} \).

Let \( P: \mathcal{M}_k^\mu[T] \to \mathbb{Z}[v, v^{-1}] \) be the morphism from Example 1.4.4 (iii) that maps the class \([X]\) of a smooth, proper \( k \)-variety \( X \) to its Poincaré polynomial 
\[ \sum_{i \geq 0} (-1)^i b_i(X) v^i. \] This morphism can be extended to
\[ P: \mathcal{M}_k^\mu[T] \to \mathbb{Z}[v, v^{-1}][T]. \]

We will use the morphism \( P \) to compute poles in practice. More specifically, the following lemma and corollary will turn out to be useful.

**Lemma 4.3.3.** Let \( Z(T) \in \mathcal{M}_k^\mu[T, \frac{1}{1 - \mathbb{L}^a T^b}] \) \((a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}\) be a rational function over \( \mathcal{M}_k^\mu \) and let \( q \in \mathbb{Q} \) be a rational number. Write \( Z(T) \) as
\[ Z(T) = \frac{F(T)}{\prod_{(a,b) \in S}(1 - \mathbb{L}^a T^b)^{m_{(a,b)}},} \]
for some finite subset \( S \subset \mathbb{Z} \times \mathbb{Z}_{>0} \), and \( m_{(a,b)} > 0 \) for every \((a, b) \in S\), and where \( F(T) \in \mathcal{M}_k^\mu[T] \). Suppose \( q = \frac{a_0}{b_0} \) for some \((a_0, b_0) \in S\) . Suppose
moreover that $P(F(T))$ does not have a zero in $T = v^{-2a_0/b_0}$, when we consider $P(F(T))$ as an element of $\mathbb{Z}[v^{1/b_0}, v^{-1/b_0}][T]$ by the inclusion $\mathbb{Z}[v, v^{-1}][T] \hookrightarrow \mathbb{Z}[v^{1/b_0}, v^{-1/b_0}][T]$. Then $Z(T)$ has a pole at $q$.

Proof. We will use Lemma 4.3.2. So suppose $Z(T) \in \mathcal{M}_{k}^\mu [T, \frac{1}{1-\alpha T}]_{(a', b') \in S'}$ with $S' \subset \mathbb{Z} \times \mathbb{Z}_{>0}$. We have to show that there is a couple $(a'_0, b'_0) \in S'$ with $q = \frac{a'_0}{b'_0}$. Write $Z(T)$ as

$$Z(T) = \frac{F'(T)}{\prod_{(a', b') \in S'} (1 - \mathbb{L}a'T^{b'})^{m(a', b')}},$$

This means that

$$\frac{F'(T)}{\prod_{(a', b') \in S'} (1 - \mathbb{L}a'T^{b'})^{m(a', b')}} = \frac{F(T)}{\prod_{(a, b) \in S} (1 - \mathbb{L}aT^{b})^{m(a, b)}}$$

in $\mathcal{M}_{k}^\mu [T, \frac{1}{1-\alpha T}]_{(a, b) \in \mathbb{Z} \times \mathbb{Z}_{>0}}$. Therefore

$$F'(T) \prod_{(a, b) \in S} (1 - \mathbb{L}aT^{b})^{m(a, b)} = F(T) \prod_{(a', b') \in S'} (1 - \mathbb{L}a'T^{b'})^{m(a', b')}.$$

If we apply the morphism $P: \mathcal{M}_{k}^\mu [T] \rightarrow \mathbb{Z}[v, v^{-1}][T]$ to both sides, we get

$$P(F'(T)) \prod_{(a, b) \in S} (1 - v^{2a}T^{b})^{m(a, b)} = P(F(T)) \prod_{(a', b') \in S'} (1 - v^{2a'}T^{b'})^{m(a', b')}.$$

Let us now work in $\mathbb{Z}[v^{1/b_0}, v^{-1/b_0}][T]$ by the inclusion $\mathbb{Z}[v, v^{-1}][T] \hookrightarrow \mathbb{Z}[v^{1/b_0}, v^{-1/b_0}][T]$. Since $(a_0, b_0) \in S$, the right-hand side of this equation has a zero in $T = v^{-2a_0/b_0}$ and therefore

$$P(F(v^{-2a_0/b_0})) \prod_{(a', b') \in S'} (1 - v^{2a'}T^{b'})^{m(a', b')} = 0.$$

Since $P(F(v^{-2a_0/b_0})) \neq 0$ by assumption, and $\mathbb{Z}[v^{1/b_0}, v^{-1/b_0}]$ is a domain, there must be a couple $(a'_0, b'_0) \in S'$ with $2a'_0 - 2a_0b'_0 = 0$. This is equivalent to $\frac{a'_0}{b'_0} = \frac{a_0}{b_0} = q$. □

Remark 4.3.4. By taking $v = 1$ in Lemma 4.3.3, we find that, if $\chi(F(T))$ does not have a zero in $T = \exp(-4\pi i a_0/b_0)$, then $Z(T)$ has a pole at $q$. Here, $\chi$ stands for the topological Euler characteristic.
Corollary 4.3.5. Let $Z(T) \in \mathcal{M}_k^{\hat{\mu}} \left[ T, \frac{1}{1-LaT^b} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}}$ be a rational function over $\mathcal{M}_k^{\hat{\mu}}$, and let $q \in \mathbb{Q}$ be a rational number. Suppose we can write

$$Z(T) = Z_1(T) + Z_2(T),$$

with $Z_1(T), Z_2(T) \in \mathcal{M}_k^{\hat{\mu}} \left[ T, \frac{1}{1-LaT^b} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}}$, and such that $Z_1(T)$ satisfies the conditions of Lemma 4.3.3 and $Z_2(T)$ does not have a pole at $q$. Then $Z(T)$ has a pole at $q$.

### 4.3.2 Computation of the poles of $Z_{X,\omega}(T)$

In this section, we will compute the poles of the motivic zeta function.

The following result is proven by Halle and Nicaise in [HN17, Theorem 3.2.3] for general Calabi-Yau varieties.

**Theorem 4.3.6.** Let $X$ be a Calabi-Yau variety over $K$. Let $X$ be an snc-model of $X$ with special fiber $X_k = \sum_{i \in I} N_i E_i$. Let $\omega$ be a volume form on $X$ and let $(N_i, \nu_i)$ be the numerical data of $E_i$.

Define

$$\text{lct}(X) = \min \{\nu_i/N_i \mid i \in I\},$$

$$\delta(X) = \max \{|J| \mid \emptyset \neq J \subseteq I, E_J \neq \emptyset, \nu_j/N_j = \text{lct}(X) \text{ for all } j \in J\} - 1.$$

The motivic zeta function $Z_{X,\omega}(T)$ has a pole at $-\text{lct}(X)$ of order $\delta(X) + 1$, and this is the largest pole of $Z_{X,\omega}(T)$.

As a consequence, we get

**Corollary 4.3.7.** Let $X$ be a K3 surface over $K$. Let $X$ be a Crauder-Morrison model of $X$ with special fiber $X_k = \sum_{i \in I} N_i E_i$. Let $\omega$ be a volume form on $X$ and let $(N_i, \nu_i)$ be the numerical data of $E_i$ for every $i \in I$. If $\rho_i$ is minimal, then $-\nu_i/N_i$ is a pole of $Z_{X,\omega}(T)$, and it is the largest pole of $Z_{X,\omega}(T)$. Moreover, it is a pole of order 1 if $X$ is a flowerpot degeneration, and of order 2 if $X$ is a chain degeneration.

We will now compute the other poles. The following result has been announced in [Jas17].

**Theorem 4.3.8.** Let $X$ be a K3 surface over $K$. Let $X$ be a Crauder-Morrison model of $X$ with special fiber $X_k = \sum_{i \in I} N_i E_i$. Let $\omega$ be a volume form on $X$ and let $(N_i, \nu_i)$ be the numerical data of $E_i$ for every $i \in I$. 
The rational number $q \in \mathbb{Q}$ is a pole of $Z_{X,\omega}(T)$ if and only if there exists an element $i \in I$ with $q = -\nu_i/N_i$ and such that

(i) either $\rho_i$ is minimal,

(ii) or $E_i$ is the top of a conic-flower.

Moreover, in case (i), $q$ is a pole of order 1 if $X$ is a flowerpot degeneration, and of order 2 if $X$ is a chain degeneration. In case (ii), $q$ is a pole of order 1.

Proof. For every $q \in \mathbb{Q}$, we define $I_q = \{i \in I \mid -\nu_i/N_i = q\}$. Remark that, because of equation (3.1), we have $\rho_i = \rho_j$ for every $i, j \in I_q$.

Suppose first that there does not exist an $i \in I$ with $q = -\nu_i/N_i$ and such that

(i) either $\rho_i$ is minimal,

(ii) or $E_i$ is the top of a conic-flower.

If $I_q = \emptyset$, then $Z_{X,\omega}(T)$ does not have a pole at $q$ by the Denef-Loeser formula 2.2.2. So from now on we may assume $I_q \neq \emptyset$. Let $j \in I_q$. Since $\rho_j$ is not minimal, $E_j$ is a component of a flower. Moreover, if $k \in I_q$ and $k \neq j$, then we know that $E_k$ is a component of a different flower than the flower in which $E_j$ is contained. Indeed, the weights $\rho_i = \nu_i/N_i + 1$ strictly decrease within a flower, so no two components in one flower can realize the same $-\nu_i/N_i$. Denote by $F^{(i)}$ the flower containing $E_i$ for every $i \in I_q$. By Lemma 4.2.2, we can write

$$Z_{X,\omega}(T) = \sum_{i \in I_q} Z_{F^{(i)}}(T) + G(T),$$

where $Z_{F^{(i)}}(T)$ is the contribution of $F^{(i)}$ to the motivic zeta function and $G(T) \in M_k^{\hat{\mu}} [T, \frac{1}{1-L(T)}]_{(a,b) \in \mathbb{Z}_{X,\omega > 0}}$. By Lemma 4.3.2, we know that $G(T)$ does not have a pole at $q$. When $i \in I_q$, Theorem 4.2.3 implies that $Z_{F^{(i)}}(T)$ doesn’t have a pole at $q$ either. We conclude that $q$ is not a pole of $Z_{X,\omega}(T)$.

Conversely, suppose that $\rho_i$ is minimal for some $i \in I_q$, and hence for all $i \in I_q$. Corollary 4.3.7 gives that $-\nu_i/N_i$ is a pole of $Z_{X,\omega}(T)$.

Finally, suppose there is a $j \in I_q$ such that $E_j$ is the top of a conic-flower. For every $i \in I_q$, we have that $E_i$ is a component of a flower $F^{(i)}$, since $\rho_i$ is not minimal. Furthermore, for every $i, k \in I_q$ with $i \neq k$, the flower $F^{(i)}$ and $F^{(k)}$ are distinct. Since the weight strictly decreases within a flower, $q$ is a pole of order at most 1 by the Denef-Loeser formula 2.2.2.

Define

$$I'_q = \{i \in I_q \mid E_i \text{ is the top of a conic-flower}\}.$$
By assumption $I'_q \neq \emptyset$. By Lemma 4.2.2 we can write

$$Z_{X,\omega}(T) = \sum_{i \in I'_q} Z_{F(i)}(T) + G(T).$$

where $Z_{F(i)}(T)$ is the contribution of $F^{(i)}$ to the motivic zeta function and $G(T) \in \mathcal{M}_k^\hat{\mu} \left[ T, \frac{1}{1-L^{-\nu}T^\nu} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}}$. By Lemma 4.3.2 and Theorem 4.2.3, $G(T)$ does not have a pole at $q$.

Choose $i \in I'_q$. Theorem 3.3.7 and Lemma 3.3.10 explain why \((\nu_i - 1)/2, N_i/2)\) is the numerical data of the component meeting $E_i$ in a conic. Define

$$Z_{E_i}(T) = \left[ \widetilde{E}_i \circ i \right] \frac{L^{-\nu_i}T^{N_i}}{1 - L^{-\nu_i}T^{N_i}} + (L - 1)\left[ \widetilde{C}_i \right] \frac{L^{-(\nu_i - 1)/2}T^{N_i}/2}{1 - L^{-(\nu_i - 1)/2}T^{N_i}/2},$$

where $C_i$ is the double curve on $E_i$. We see

$$Z_{F(i)}(T) = Z_{E_i}(T) + G_i(T),$$

with $G_i(T) \in \mathcal{M}_k^\hat{\mu} \left[ T, \frac{1}{1-L^{-\nu}T^\nu} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}}$ and $G_i(T)$ does not have a pole at $q$.

Therefore we can write

$$Z_{X,\omega}(T) = \sum_{i \in I'_q} Z_{E_i}(T) + G'(T),$$

where $G'(T) \in \mathcal{M}_k^\hat{\mu} \left[ T, \frac{1}{1-L^{-\nu}T^\nu} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{>0}}$ does not have a pole at $q$.

Define

$$\Theta_i(T) = \left[ \widetilde{E}_i \circ i \right] L^{-\nu_i}T^{N_i} \left( 1 - L^{-(\nu_i - 1)/2}T^{N_i}/2 \right) + (L - 1)\left[ \widetilde{C}_i \right] L^{-(3\nu_i + 1)/2}T^{3N_i}/2,$$

in $\mathcal{M}_k^\hat{\mu}[T]$. Then

$$Z_{E_i}(T) = \frac{\Theta_i(T)}{(1 - L^{-\nu_i}T^{N_i})(1 - L^{-(\nu_i - 1)/2}T^{N_i}/2)}.$$

Let $N = \text{lcm} \{ N_i \mid i \in I'_q \}$ and $\nu = \text{lcm} \{ \nu_i \mid i \in I'_q \}$. Since $-\nu_i/N_i$ is constant, we have $-\nu/N = -\nu_i/N_i$ for every $i \in I'_q$. Hence

$$(1 - L^{-\nu}T^N) = (1 - L^{-\nu_i}T^{N_i}) \cdot \sum_{k=0}^{N/N_i - 1} L^{-k\nu_i}T^{kN_i}.$$
By bringing all terms in $\sum_{i \in I_q'} E_i(T)$ to a common denominator, we get

$$\sum_{i \in I_q'} Z_{E_i}(T) = \frac{\Theta(T)}{(1 - L^{-\nu_1 T^{N_i}}) \prod_{i \in I_q'} (1 - L^{-(\nu_i - 1)/2 T^{N_i}/2})},$$

where

$$\Theta(T) = \sum_{i \in I_q'} \left( \Theta_i(T) \cdot \sum_{k=0}^{N_i/2} (L^{-k\nu_i T^{N_i}}) \cdot \prod_{j \in I_q' \setminus \{i\}} (1 - L^{-(\nu_j - 1)/2 T^{N_j}/2}) \right).$$

Let $P: M_k^\mu \to Z[v, v^{-1}]$ be the morphism from Example 1.4.4 (iii) that maps the class $[X]$ of a smooth, proper $k$-variety $X$ to its Poincaré polynomial $\sum_{i \geq 0} (-1)^i b_i(X) v^i$. This morphism can be extended to $P: M_k^\mu[T] \to Z[v, v^{-1}][T]$.

Lemma 4.3.5 says that $q$ is a pole of $Z_{X,\omega}(T)$, if $P(\Theta_q(T))$ does not have a zero in $T = v^{-2\nu/N}$, when we consider $P(\Theta_q(T))$ as an element of $Z[v^{1/N}, v^{-1/N}][T]$ by the inclusion $Z[v, v^{-1}][T] \hookrightarrow Z[v^{1/N}, v^{-1/N}][T]$.

Because $P([\widetilde{E_i}]) = \frac{N_i}{2} (v^2 + 1) v^2$ and $P([\widetilde{C_i}]) = \frac{N_i}{2} (v^2 + 1)$ by Proposition 4.1.10, direct computation shows that

$$P(\Theta(v^{-2\nu/N})) = -\frac{\sum_{i \in I_q'} (N_i + N)}{2} v(v^2 + 1)(1 - v)|I_q'|,$$

which is clearly non-zero. This concludes the proof. □

### 4.4 Example of a $K3$ surface where $Z_{X,\omega}(T)$ has two poles

Theorem 4.3.8 suggests that the motivic zeta function of a $K3$ surface can have more than one pole, in contrast with abelian varieties [HNT11, Theorem 8.5] and Calabi-Yau varieties with an equivariant Kulikov model [HNT17, Corollary 5.3.3]. We will illustrate this fact with an example. This example first appeared in [HNT17, Example 5.3.4], and it has also been published in [Jas17, Example 2].

**Example 4.4.1.** Let $k = \mathbb{C}$, $R = \mathbb{C}[t]$ and $K = \mathbb{C}(t)$. Let $X$ be the $K3$ surface defined by the homogeneous equation

$$x^2 w^2 + y^2 w^2 + z^2 w^2 + x^4 + y^4 + z^4 + tw^4 = 0 \quad (4.2)$$
in Proj $K[x, y, z, w]$. Let $Y$ be the closed subscheme of Proj $R[x, y, z, w]$ defined by the homogeneous equation (4.2). It is straightforward to check that $Y$ is regular. Moreover, $Y_k$ is a singular irreducible surface with a unique singularity at $P = (0 : 0 : 0 : 1)$, and the singularity is of type $A_1$. We construct an snc-model $X$ of $Y$ by blowing up $Y$ at $P$. The special fiber of $X$ is of the form $X_k = D + 2E$, where $D$ is the strict transform of $Y_k$, and $E \simeq \mathbb{P}^2$ is the exceptional divisor. The strict transform $D$ is a smooth $K3$ surface and intersects $E$ transversally along a smooth conic $C$. For a suitable choice of volume form $\omega$, one has $\nu_D = 0$ and $\nu_E = 1$. Therefore, $D$ has weight $\rho_D = 1$ and $E$ has weight $\rho_E = 3/2$. This means that $X$ is a flowerpot degeneration, where $D$ is the flowerpot, and $E$ is a conic-flower of type 2B.

Using the Denef-Loeser formula (Theorem 2.2.2), the motivic zeta function can be computed as

$$Z_{X, \omega}(T) = \left[ \widetilde{D} \right] \frac{T}{1 - T} + \left[ \widetilde{E} \right] \frac{\mathbb{L}^{-1}T^2}{1 - \mathbb{L}^{-1}T^2} + \left[ C \right] \frac{\mathbb{L}^{-1}T^3}{(1 - T)(1 - \mathbb{L}^{-1}T^2)}$$

$$= \frac{[\widetilde{D}]T(1 - \mathbb{L}^{-1}T^2) + [\widetilde{E}]\mathbb{L}^{-1}T^2(1 - T) + [C]\mathbb{L}^{-1}T^3}{(1 - T)(1 - \mathbb{L}^{-1}T^2)}.$$

Define $F(T) = \left[ \widetilde{D} \right]T(1 - \mathbb{L}^{-1}T^2) + [\widetilde{E}]\mathbb{L}^{-1}T^2(1 - T) + [C]\mathbb{L}^{-1}T^3$, then

$$\chi(F(T)) = \chi(\widetilde{D})T(1 - T^2) + \chi(\widetilde{E})T^2(1 - T) + \chi(C)T^3.$$

Since $\chi(F(T))$ does not have a zero in $T = 1$, Remark 4.3.4 implies that 0 and $-1/2$ are poles of $Z_{X, \omega}(T)$, which is in agreement with Theorem 4.3.8.
Chapter 5

\(K3\) surfaces satisfying the monodromy property

In this chapter, we will prove that certain types of \(K3\) surfaces satisfy the monodromy property. To be precise, we will prove the following theorem, which is slightly more general than the result announced in [Jas17].

**Theorem.** Let \(X\) be a \(K3\) surface over \(K\) with Crauder-Morrison model \(\mathcal{X}\) and with special fiber \(\mathcal{X}_k = \sum_{i \in I} N_i E_i\). Suppose \(\mathcal{X}\) satisfies one of the following:

(i) \(\mathcal{X}\) is a flowerpot degeneration, or

(ii) \(\mathcal{X}\) is a chain degeneration with chain \(V_0 - V_1 - \cdots - V_k - V_{k+1}\). Set \(N = \min\{N(V_i) \mid i = 0, \ldots, k + 1\}\). Assume moreover that one of the following conditions hold:

(a) the components \(V_0, V_1, \ldots, V_{k+1}\) all have multiplicity \(N\), or

(b) neither \(V_0\) nor \(V_{k+1}\) is a rational, non-minimal ruled surface, or

(c) exactly one end component of the chain is a rational, non-minimal ruled surface and it has multiplicity \(N\), or

(d) if \(V_j\) meets a conic-flower, then \(N(V_j) > N\).

Then \(X\) satisfies the monodromy property.

In the first section, we will explain the strategy to prove this theorem. We will also give some first results about monodromy eigenvalues. In Section 5.2, we will prove part (i) of the theorem, i.e., that \(K3\) surfaces allowing a flowerpot degeneration satisfy the monodromy property. Finally, in Section 5.3, we show...
that the monodromy property holds for $K3$ surfaces with a chain degeneration satisfying one of the extra conditions.

**Notation**

In this chapter, we fix an algebraically closed field $k$ of characteristic zero. Put $R = k[t]$ and $K = k((t))$. Fix an algebraic closure $K^{alg}$ of $K$. Let $\sigma$ be a topological generator of the monodromy group $\text{Gal}(K^{alg}/K)$.

Let $X$ be a $K3$ surface over $K$ and let $\mathcal{X}$ be a Crauder-Morrison model of $X$ with special fiber $\mathcal{X}_k = \sum_{i \in I} N_i E_i$. Let $\omega$ be a volume form on $X$ and let $(N_i, \nu_i)$ be the numerical data of $E_i$, for every $i \in I$. To avoid confusion, we will sometimes use the notation $(N(E_i), \nu(E_i))$ instead of $(N_i, \nu_i)$ for the numerical data of a component $E_i$. Let $\rho_i = \nu_i/N_i + 1$ be the weight of $E_i$, for every $i \in I$.

We define

$$\xi(E_i) = \exp\left(-2\pi i \frac{\nu(E_i)}{N(E_i)}\right),$$

the ‘candidate’ monodromy eigenvalue associated with $E_i$.

For any flower $F$ in $\mathcal{X}_k$, we denote by $F_0$ the top component of the flower, i.e., the component in the flower with maximal weight $\rho(F_0)$. By $F_{\ell+1}$, we mean the component in $\Gamma_{\text{min}}$ that intersects with the flower $F$.

### 5.1 Strategy and first results

#### 5.1.1 Strategy

In Corollary 4.2.4, we found that

$$Z_{X, \omega}(T) \in \mathcal{M}_{k}^R\left[T, \frac{1}{1 - \prod_{a \in S^\dagger} T^a}\right]_{(a,b) \in S^\dagger},$$

with $S^\dagger = \{(-\nu_i, N_i) \in \mathbb{Z} \times \mathbb{Z}_{>0} \mid i \in I^\dagger\}$ and $I^\dagger \subset I$ the set of indices $i \in I$ with either

(i) $\rho_i$ is minimal, or

(ii) $E_i$ is the top of a conic-flower.

In order to prove that $X$ satisfies the monodromy property, we need to verify that, for every $i \in I^\dagger$, the complex number $\exp(-2\pi i \nu_i/N_i)$ is an eigenvalue
of $\sigma$ on $H^m(X \times K K^{alg}, \mathbb{Q}_\ell)$, for some $m \geq 0$ and for every embedding of $\mathbb{Q}_\ell$ into $\mathbb{C}$.

Remember that we defined the monodromy zeta function in Definition 2.3.2 as

$$\zeta_X(T) = \prod_{m \geq 0} \left( \det \left( T \cdot \text{Id} - \sigma \mid H^m(X \times K K^{alg}, \mathbb{Q}_\ell) \right) \right)^{(-1)^{m+1}} \in \mathbb{Q}_\ell(T).$$

This rational function has the property that all zeroes and poles are monodromy eigenvalues, but in general there may be more monodromy eigenvalues than can be seen from the monodromy zeta function. The reason is that cancellations may occur in the alternating product. However, for $K3$ surfaces we have the following result.

**Proposition 5.1.1.** Let $X$ be a $K3$ surface. The monodromy zeta function can be written as

$$\zeta_X(T) = \frac{1}{Q(T)},$$

for some $Q \in \mathbb{Q}_\ell[T]$ of degree 24. Moreover, the poles of $\zeta_X(T)$ are exactly the monodromy eigenvalues of $X$.

**Proof.** Because $X$ is a $K3$ surface, the cohomology spaces of $X$ are trivial in odd degree and $\chi(X) = 24$, by Proposition 1.2.3. Therefore, no cancellations can occur, and the poles of the monodromy zeta function are exactly the monodromy eigenvalues. $\Box$

In this chapter, we will compute the poles of $\zeta_X(T)$. A helpful tool will be the A’Campo formula, discussed in Proposition 2.3.3

$$\zeta_X(T) = \prod_{i \in I} \left( T^{N_i} - 1 \right)^{-\chi(E_i^0)},$$

where $\chi(E_i^0)$ is the topological Euler characteristic of $E_i^0 = E_i \setminus \left( \cup_{j \in I \setminus \{i\}} E_j \right)$. In Lemma 3.7.1 we computed Euler characteristics $\chi(E_i^0)$ for some relevant components $E_i$ of the special fiber $X_k = \sum_{i \in I} N_i E_i$.

### 5.1.2 Minimal weight

Let $E_i$ be a component of $X_k$ with numerical data $(N_i, \nu_i)$ and such that the weight $\rho_i$ is minimal. Halle and Nicaise proved that $\xi(E_i) = \exp(-2\pi i \nu_i/N_i)$ is a monodromy eigenvalue of $X$. 
Theorem 5.1.2. Let $X$ be a K3 surface over $K$ with Crauder-Morrison model $X$. Let $\omega$ be a volume form on $X$. Let $E_i$ be a component of $X_k$ with numerical data $(N_i, \nu_i)$ and such that the weight $\rho_i$ is minimal. Then $\xi(E_i) = \exp(-2\pi i \nu_i/N_i)$ is a monodromy eigenvalue of $X$.

Proof. This is a special case of [HN17, Theorem 3.3.3]. □

5.1.3 Conic-flowers

Let $F$ be a conic-flower. The conic-flowers are classified in types $2B$, $2C$, $4C$, $6C$, $6D$ and $6E$. Notice that we proved in Corollary 3.3.11 that flowers of type $4D$ cannot occur. The following lemma gives some information about the ‘candidate monodromy eigenvalue’ induced by the top of a flower of type $2$ or $6$.

Lemma 5.1.3. Let $X$ be a K3 surface over $K$ with Crauder-Morrison model $X$. If $F$ is a conic-flower in $X_k$ of type $2$ or $6$, i.e., of type $2B$, $2C$, $6C$, $6D$ or $6E$, then $\xi(F_0)$ is not an $N(F_{\ell+1})$-th root of unity.

Proof. Let $\xi$ be a primitive $N(F_0)$-th root of unity, such that $\xi(F_0) = \xi^{\nu(F_0)}$.

(i) If $F$ is a conic-flower of type $2$, then $N(F_0) = 2N(F_{\ell+1})$, by Table 3.2. Suppose first that $F$ is a flower of type $2B$. In Corollary 3.3.12 we computed that $\nu(F_0) = 2\nu(F_{\ell+1}) + 1$. Therefore $\xi(F_0)^{N(F_{\ell+1})} = \xi^{\nu(F_0)N(F_{\ell+1})} = \xi^{\nu(F_{\ell+1})N(F_0)}\xi^{N(F_{\ell+1})} = \xi^{N(F_0)/2} \neq 1$, since $\xi$ is a primitive $N(F_0)$-th root of unity.

Analogously, when $F$ is a flower of type $2C$, we have $\nu(F_0) = 2\nu(F_{\ell+1}) + 2\ell - 1$, and it follows that $\xi(F_0)^{N(F_{\ell+1})} \neq 1$.

(ii) If $F$ is a conic-flower of type $6$, then $3N(F_0) = 2N(F_{\ell+1})$, by Table 3.2. Suppose first that $F$ is a flower of type $6C$. In Corollary 3.3.12 we computed that $3\nu(F_0) = 2(\nu(F_{\ell+1}) + 3\ell - 2) - 1$. Therefore, $\xi(F_0)^{N(F_{\ell+1})} = \xi^{\nu(F_0)N(F_{\ell+1})} = \xi^{(\nu(F_{\ell+1}) + 3\ell - 2)N(F_0)}\xi^{-N(F_{\ell+1})/3}$

$= \xi^{-N(F_0)/2} \neq 1$, since $\xi$ is a primitive $N(F_0)$-th root of unity.

Analogously, when $F$ is a flower of type $6D$, we have $3\nu(F_0) = 2\nu(F_{\ell+1}) + 6\ell - 1$, and when $F$ is a flower of type $6E$, we have $3\nu(F_0) = 2\nu(F_{\ell+1}) + 1$. In both cases, it follows immediately that $\xi(F_0)^{N(F_{\ell+1})} \neq 1$. □
Lemma 5.1.4. Let $X$ be a $K3$ surface over $K$ with Crauder-Morrison model $\mathcal{X}$. If $F$ is a conic-flower in $X_k$ of type $4C$, then $N(F_{\ell+1})$ is even and $\xi(F_0)$ is not an $N(F_{\ell+1})/2$-th root of unity.

Proof. In Table 3.2 we can see that $N(F_{\ell+1}) = N(F_0)$, and that $N(F_{\ell+1})$ is even. Let $\xi$ be a primitive $N(F_0)$-th root of unity, such that

$$\xi(F_0) = \xi^{\nu(F_0)}.$$ 

In Lemma 3.3.10 we computed that $\nu(F_0) = 2\nu(F_1) + 1$. Therefore

$$\xi(F_0)^{N(F_{\ell+1})/2} = \xi^{\nu(F_0)N(F_0)/2} = \xi^{\nu(F_1)N(F_0)} \xi^{N(F_0)/2} = \xi^{N(F_0)/2} \neq 1,$$

since $\xi$ is a primitive $N(F_0)$-th root of unity.

5.2 Flowerpot degenerations

The aim of this section is to prove the following theorem.

Theorem 5.2.1. Let $X$ be a $K3$ surface over $K$ with Crauder-Morrison model $\mathcal{X}$. If $X$ is a flowerpot degeneration, then $X$ satisfies the monodromy property.

This is a consequence of Theorems 5.2.3 and 5.2.14 combined with Proposition 3.4.1.

Throughout this section, we will assume that $\mathcal{X}$ is a flowerpot degeneration, i.e., there is a unique component $P$ of $\mathcal{X}_k$ with minimal weight $\rho(P)$.

In Subsection 5.2.1 we discuss why the existence of flowers of type $4C$ causes an extra difficulty. In Subsection 5.2.2 we show that $X$ satisfies the monodromy property if $P \simeq \mathbb{P}^2$, if $P$ is minimal ruled, or if $K_P \equiv 0$. Finally in Subsection 5.2.3 we show that $X$ also satisfies the monodromy property in the only remaining case: when $P$ is a rational, non-minimal ruled surface.

5.2.1 Flowers of type $4C$

Proposition 5.2.2. Let $X$ be a $K3$ surface over $K$ with Crauder-Morrison model $\mathcal{X}$. Suppose that $\mathcal{X}$ is a flowerpot degeneration. If $F$ is a conic-flower of type 2 or 6, then $\xi(F_0)$ is a monodromy eigenvalue for $X$. 
Proof. We have to prove that $\xi(F_0)$ is a pole of $\zeta_X(T) = \prod_{i \in I} (T^{N_i} - 1)^{-\chi(E_i^o)}$. From Lemma 3.7.1, we see that the only factors in this formula with a positive exponent are either

$$\left(T^{N(P)} - 1\right)^{-\chi(P^o)},$$

when $\chi(P^o) < 0$, or

$$\left(T^{N(F_0')} - 1\right)^{2g(F')-2},$$

where $F'$ is a flower of genus $g(F') > 1$. Notice that for any non-rational flower $F'$, the multiplicity $N(F_0')$ is a divisor of $N(P)$, by Table 3.3. Therefore, $T^{N(F_0')} - 1$ is a divisor of $T^{N(P)} - 1$, for any non-rational flower $F'$.

Since $F$ is a flower of type 2 or 6, Corollary 5.1.3 implies that $\xi(F_0)$ is not a zero of $T^{N(P)} - 1$ and $T^{N(F_0')} - 1$. For this reason, $\xi(F_0)$ is a pole of $\zeta_X(T)$.

If a conic-flower in a flowerpot degeneration of a $K3$ surface is not of type 2 or 6, then it is a flower of type 4C. The proof of the previous proposition cannot be adapted to prove that $\xi(F_0)$ is a monodromy eigenvalue for flowers $F$ of type 4C. The reason is that from Table 3.2, we see that $N(F_0) = N(P)$ and therefore we need to argue why there is no cancellation with the term $(T^{N(P)} - 1)^{-\chi(P^o)}$.

### 5.2.2 Easy cases

In Proposition 3.4.1 we proved that the pot $P$ satisfies one of the following:

(i) $P \simeq \mathbb{P}^2$, or

(ii) $K_P \equiv 0$, or

(iii) $P$ is a minimal ruled surface, or

(iv) $P$ is a rational, non-minimal ruled surface.

In the first three cases, it is not difficult to show that $X$ satisfies the monodromy property.

**Theorem 5.2.3.** Let $X$ be a $K3$ surface over $K$ with Crauder-Morrison model $\mathcal{X}$. Suppose $\mathcal{X}$ is a flowerpot degeneration with pot $P$. If $P$ satisfies one of the following:

(i) $P \simeq \mathbb{P}^2$, or

(ii) $K_P \equiv 0$, or

(iii) $P$ is a minimal ruled surface,

then $X$ satisfies the monodromy property.
Proof. Define $I^\dagger \subset I$ to be the set of indices $i \in I$ with either $E_i = P$, or $E_i$ is the top of a conic-flower. Let $S^\dagger = \{ (-\nu_i, N_i) \in \mathbb{Z} \times \mathbb{Z}_{>0} \mid i \in I^\dagger \}$. Corollary 4.2.4 gives that

$$Z_{X,\omega}(T) \in \mathcal{M}_k^{\hat{\mu}} \left[ T, \frac{1}{1 - L^a T^b} \right]_{(a,b) \in S^\dagger}.$$

We still have to prove that $\xi(F_0)$ is a monodromy eigenvalue for any conic-flower $F$, since it has already been proven in Theorem 5.1.2 that $\xi(P)$ is a monodromy eigenvalue.

Suppose first $P \simeq \mathbb{P}^2$. By Proposition 3.4.4 there are no conic-flowers. Therefore $X$ certainly satisfies the monodromy property.

Suppose now that $K_P \equiv 0$. By Proposition 3.4.3 any conic-flower is of type 2. Proposition 5.2.2 shows that $X$ satisfies the monodromy property.

Suppose now that $P$ is a minimal ruled surface. Let $F$ be a conic-flower. By Proposition 3.4.5 there is exactly one other flower $F'$, and it has genus $g \geq 2$. Let $C$ and $C'$ be the flowercurves on $P$ of $F$ and $F'$ respectively. Then

$$\chi(P^\circ) = \chi(P) - \chi(C) - \chi(C') = 4 - 2 - (2 - 2g) = 2g.$$

Lemma 3.7.1 gives that $\chi((F'_0)^\circ) = 2 - 2g$ and $\chi(F'_0^\circ) = 1$. Moreover, $\chi(E_i^\circ) = 0$ for $E_i \neq P, F_0$ or $F'_0$. Therefore

$$\zeta_X(T) = \frac{(T^{N(F'_0^\circ)} - 1)^{2g-2}}{(T^{N(P)} - 1)^{2g}(T^{N(F_0^\circ)} - 1)}.$$

Because $N(F'_0)$ is a divisor of $N(P)$ by Table 3.3 it is clear that $\xi(F_0)$ is a pole of $\zeta_X(T)$, and hence it is a monodromy eigenvalue. This concludes that $X$ satisfies the monodromy property. \qed

5.2.3 Rational, non-minimal ruled pot

From Theorem 5.2.3 and Proposition 3.4.1 it is clear that the only case we still need to prove in this section, is that of $K3$ surfaces allowing a flowerpot degeneration where the pot $P$ is a rational, non-minimal ruled surface. This is not trivial and will require some work. So assume $P$ is a rational, non-minimal ruled surface. One of the main tools will be the fact that $\xi(P)$ is certainly a monodromy eigenvalue by Theorem 5.1.2.

To start, we will derive some information on possible flowers on a rational, non-minimal ruled pot. Then, we will prove some lemmas about $\xi(P)$ and finally, we will prove that the monodromy property holds for a $K3$ surface allowing a flowerpot degeneration with a rational, non-minimal ruled pot.
Flowers on $P$

We will show the following:

- If there is a flower of type $4C$, then there is no flower of type $2A$, $4A$, $4B$, $6A$ or $6B$. (Lemma 5.2.4)
- There are no flowers of type $8\alpha$, $8\beta$, $12\alpha$ or $12\beta$. (Corollary 5.2.8)
- If there is a flower of type $4C$, then any flower of genus $g > 1$ is of type $4\alpha$. (Proposition 5.2.11)

**Lemma 5.2.4.** Let $X$ be a $K3$ surface over $K$ with a Crauder-Morrison model $\mathcal{X}$. Suppose that $\mathcal{X}$ is a flowerpot degeneration with pot $P$. If there exists a flower of type $4C$, then there are no flowers of type $2A$, $4A$, $4B$, $6A$ and $6B$.

**Proof.** Let $F$ be a flower of type $4C$. Lemma 3.3.10 implies that $\nu(F_0) = 2\nu(F_1) + 1$, and hence $\nu(F_0)$ is odd. This yields that $\nu(P)$ is odd as well, by Corollary 3.3.12. On the other hand, if there exists a flower of type $2A$, $4A$, $4B$, $6A$ or $6B$, then Corollary 3.3.12 implies that $\nu(P)$ is even, which is a contradiction.

**Lemma 5.2.5.** Let $X$ be a $K3$ surface over $K$ with a Crauder-Morrison model $\mathcal{X}$. Suppose that $\mathcal{X}$ is a flowerpot degeneration with pot $P$. For any smooth, rational curve $C$ on $P$, the following properties hold.

(i) If $C^2 \leq -3$, then $C$ is a flowercurve.

(ii) If $C^2 = -2$, then $C$ does not meet any flowercurves of type $\geq 3$.

**Proof.** We know that a flowercurve on $P$ does not meet any other flowercurves, because there are no triple points. So let $C$ be a smooth, rational curve on $P$ with $C^2 \leq -2$, and suppose it is not a flowercurve. We will derive a contradiction for $C^2 \leq -3$. Moreover, for $C^2 = -2$, we will find that $C$ does not meet any flowercurves of type $\geq 3$. Let $C_1, \ldots, C_n$ be the flowercurves on $P$ of type $\geq 3$. By (3.4) and Remark 3.3.6 we can write

$$K_{P/k} \equiv \sum_{i=1}^{n} a_i C_i,$$

with $-1 < a_i < 0$. The adjunction formula gives

$$-2 = (K_P + C) \cdot C = C^2 + \sum_{i=1}^{n} a_i (C_i \cdot C).$$
By assumption $C^2 \leq -2$ and therefore
\[ \sum_{i=1}^{n} a_i(C_i \cdot C) \geq 0, \]
where the inequality is strict if and only if $C^2 \leq -3$.

On the other hand, since $C$ is not a flowercurve, we have $C_i \cdot C \geq 0$ for $i = 1, \ldots, n$. Because $a_i < 0$, we find
\[ \sum_{i=1}^{n} a_i(C_i \cdot C) \leq 0. \]
This is a contradiction for $C^2 \leq -3$. For $C^2 = -2$, this is only possible if $C_i \cdot C = 0$ for all $i = 1, \ldots, n$.

**Lemma 5.2.6.** Let $X$ be a $K3$ surface over $K$ with a Crauder-Morrison model $X$. Suppose that $X$ is a flowerpot degeneration with a rational, non-minimal ruled pot $P$. Let $C$ be a flowercurve of type $M \geq 3$. Then $C$ meets a smooth, rational curve $E \subset P$ with $E^2 = -1$.

**Proof.** Let $C$ be a flowercurve of type $M \geq 3$. The flowercurve $C$ meets some component $E$ of a reducible fiber of the ruling on $P$. Indeed, if $g(C) \geq 1$, then $C$ is a horizontal curve, and if $g(C) = 0$, then $C^2 = -M \leq -3$ by Lemma 3.3.8 and hence $C$ is either a horizontal curve or a component of a reducible fiber of the ruling on $P$. Since $E$ is a component of a reducible fiber, it is a smooth, rational curve with $E^2 < 0$. Lemma 5.2.5 implies that $E^2 = -1$.

**Lemma 5.2.7.** Let $X$ be a $K3$ surface over $K$ with a Crauder-Morrison model $X$. Suppose that $X$ is a flowerpot degeneration with pot $P$. Let $E$ be a smooth, rational curve on $P$ with $E^2 = -1$. There are three possibilities:

(i) $E$ meets only flowercurves of type 2 and 3,

(ii) $E$ meets only flowercurves of type 2 and 4,

(iii) $E$ meets exactly one flowercurve of type 3 and exactly one of type 6. All other flowercurves meeting $E$ are of type 2.

**Proof.** Let $E$ be a smooth, rational curve on $P$ with $E^2 = -1$. Let $C_1, \ldots, C_n$ be the flowercurves of type $M_i \geq 3$ on $P$. By adjunction, we must have
\[ \sum_{i=1}^{n} a_i(C_i \cdot E) = 1, \]
where \( a_i = 1 - \frac{2}{M_i} \). Since \( M_i \geq 3 \), we have \( 1/3 \leq a_i < 1 \), and therefore \( 1 < \sum_{i=1}^{n} (C_i \cdot E) \leq 3 \). Moreover, Lemma 3.3.8 implies that \( E \neq C_i \) and therefore \( C_i \cdot E \geq 0 \).

If \( \sum_{i=1}^{n} (C_i \cdot E) = 3 \), then we must have \( M_i = 3 \) for all \( 1 \leq i \leq n \). This is case (i).

If \( \sum_{i=1}^{n} (C_i \cdot E) = 2 \), then we have

\[
\frac{2}{M_1} \cdot 2 = 1 \quad \text{when } n = 1,
\]

\[
\frac{2}{M_1} + \frac{2}{M_2} = 1 \quad \text{when } n = 2.
\]

For \( n = 1 \), it is clear that \( M_1 = 4 \), and hence \( E \) meets only flowercurves of type 2 and 4. For \( n = 2 \), it is straightforward to see that the only possibilities are \( M_1 = M_2 = 4 \), or \( M_1 = 3 \) and \( M_2 = 6 \).

**Corollary 5.2.8.** Let \( P \) be a flowerpot that is ruled, but not minimal ruled. There are no flowers of type \( 8\alpha, 8\beta, 12\alpha, 12\beta \).

**Proof.** This follows from Lemma 5.2.6 and Lemma 5.2.7.

**Lemma 5.2.9.** Let \( X \) be a \( K_3 \) surface over \( K \) with a Crauder-Morrison model \( \mathcal{X} \). Suppose that \( \mathcal{X} \) is a flowerpot degeneration with a rational, non-minimal ruled pot \( P \). Let \( \mathcal{F} \) be a general fiber of the ruling on \( P \). If there is a flowercurve \( C \) of type 6 with \( C \cdot \mathcal{F} \geq 2 \), then there is no horizontal flowercurve of type 4.

**Proof.** Let \( C_1, \ldots, C_n \) be the flowercurves of flowers of type \( M_i \geq 3 \). After renumbering, we can assume \( C_1 = C \). Since \( \mathcal{F} \) is a general fiber on \( P \), we have \( \mathcal{F}^2 = 0 \) and \( g(\mathcal{F}) = 0 \) and hence, the adjunction formula gives

\[-2 = K_{P/k} \cdot \mathcal{F} + \mathcal{F}^2 = -\sum_{i=1}^{n} a_i C_i \cdot \mathcal{F},\]

where \( a_i = 1 - \frac{2}{M_i} \). By Lemma 3.3.8, either \( g(C_i) \geq 1 \) or \( C_i^2 < 0 \) for every \( 1 \leq i \leq n \). Therefore, \( C_i \neq \mathcal{F} \) and hence \( C_i \cdot \mathcal{F} \geq 0 \). In particular, \( C_1 \cdot \mathcal{F} > 0 \) if and only if \( C_i \) is horizontal.

Since \( C_1 \) is a flowercurve of type 6, we have \( a_1 = 2/3 \). Suppose \( C_2 \) is a horizontal flowercurve of type 4, then \( a_2 = 1/2 \). Thus we get

\[
\sum_{i=1}^{n} a_i C_i \cdot \mathcal{F} \geq \frac{2}{3} \cdot 2 + \frac{1}{2} > 2,
\]
where the first inequality holds because $C_1 \cdot F \geq 2$ and $a_i C_i \cdot F \geq 0$ for all $i = 3, \ldots, n$. This is a contradiction.

**Lemma 5.2.10.** Let $V$ be a rational, non-minimal ruled surface. Let $F$ be a reducible fiber of the ruling on $V$ with at least $r + 1$ components, for some integer $r \geq 2$. Let $F_1, \ldots, F_r$ be some components of $F$ such that $F_1^2 = -1$ and $F_i^2 = -2$ for $i = 2, \ldots, r - 1$. Assume moreover that $F_i \cdot F_{i+1} \geq 1$ for $i = 1, \ldots, r - 1$, and otherwise $F_i \cdot F_j = 0$. Then $F_r^2 \leq -2$.

**Proof.** We prove this by induction on $r$. Let $F_1, F_2$ be two components of a reducible fiber with at least 3 components and suppose

$$F_1^2 = -1 \quad \text{and} \quad F_1 \cdot F_2 \geq 1.$$ 

Let $V \to \overline{V}$ be the contraction of $F_1$, and let $\overline{F}$ and $\overline{F}_2$ be the images of $F$ and $F_2$ respectively. Then $\overline{V}$ is a rational, ruled surface, and $\overline{F}$ is a reducible fiber of the ruling on $\overline{V}$. Since $\overline{F}_2$ is a component of $\overline{F}$, we must have

$$\overline{F}_2^2 \leq -1.$$ 

On the other hand, [Har77, Proposition 3.2] and [Har77, Proposition 3.6] give that

$$\overline{F}_2^2 < \overline{F}_2^2.$$ 

This implies $F_2^2 \leq -2$.

Suppose now that there exists a reducible fiber $F$ with at least $r + 1$ components, and let $F_1, \ldots, F_r$ be some components of $F$. Assume $F_1^2 = -1$ and $F_i^2 = -2$ for $i = 2, \ldots, r - 1$. Assume moreover that $F_i \cdot F_{i+1} \geq 1$ for $i = 1, \ldots, r - 1$, and otherwise $F_i \cdot F_j = 0$. Suppose that

$$F_r^2 > -2.$$ 

Since $F_r$ is a component of a reducible fiber, its self-intersection must be strictly negative, and therefore $F_r^2 = -1$. Let $V \to \overline{V}$ be the contraction of $F_r$ and let $\overline{F}$ and $\overline{F}_i$ be the images of $F$ and $F_i$ respectively, for $i = 1, \ldots, r - 1$. Then $\overline{V}$ is a rational, ruled surface, and $\overline{F}$ is a reducible fiber of the ruling on $\overline{V}$ with at least $r$ components. Moreover, $\overline{F}_1, \ldots, \overline{F}_{r-1}$ are components of $\overline{F}$. By [Har77, Proposition 3.2] and [Har77, Proposition 3.6], we know that $\overline{F}_j^2 = \overline{F}_j^2$ for $j = 1, \ldots, r - 2$ and

$$\overline{F}_{r-1}^2 > \overline{F}_{r-1}^2 = -2.$$ 

On the other hand, the surface $\overline{V}$ with the curves $\overline{F}_1, \ldots, \overline{F}_{r-1}$ satisfy the induction hypothesis, and therefore $\overline{F}_{r-1}^2 \leq -2$. We derived a contradiction, so

$$F_r^2 < -1.$$ 

$\square$
**Proposition 5.2.11.** Let $X$ be a K3 surface over $K$ with a Crauder-Morrison model $\mathcal{X}$. Suppose that $\mathcal{X}$ is a flowerpot degeneration with a rational, non-minimal ruled pot $P$. If there exists a flower of type $4C$, then any flower of genus $g > 1$ is of type $4\alpha$.

**Proof.** Let $F$ be a flower of type $4C$ with flowercurve $C$. Let $F'$ be a flower with flowercurve $C'$ such that the genus $g(C') > 1$. Suppose $F'$ is not of type $4\alpha$. We will find a contradiction.

**Step 1:** $C$ is a component of a reducible fiber $F^\dagger$ and $C'$ is horizontal.

Since $P$ is a rational, ruled surface and $g(C') > 1$, we must have $C' \cdot F \geq 2$ for any fiber $F$ of the ruling on $P$, so $C'$ is horizontal. Since $F'$ is not of type $4\alpha$, Table 3.3 and Corollary 5.2.8 give that $F'$ is of type 6. By Lemma 5.2.10, the curve $C$ is vertical. Since $C^2 = -4$ by Lemma 3.3.8, the curve $C$ is a component of a reducible fiber $F^\dagger$.

**Step 2:** For any component $F_1$ of $F^\dagger$, we have

$$F_1^2 \in \{-1, -2, -4\}.$$  

Let $F_1$ be a component of $F^\dagger$. Since any component of a reducible fiber has strict negative self-intersection, we have $F_1^2 < -1$. Suppose $F_1^2 \leq -3$, and we will prove that $F_1^2 = -4$. By Lemma 5.2.5, the curve $F_1$ is a flowercurve. Because $F^\dagger$ is connected, it is path-connected, and hence there are irreducible components $E_0, E_1, \ldots, E_m$ of $F^\dagger$ such that

$$E_i \cdot E_{i+1} > 0,$$

for $i = 0, \ldots, m - 1$ and where $E_0 = C$ and $E_m = F_1$. By removing any cycles, we can assume that $E_i \cdot E_j = 0$ for all $i, j$ with $j \notin \{i - 1, i, i + 1\}$.

We show, by induction, that $E_i^2 \in \{-1, -4\}$ for $i = 0, \ldots, m$. It is given that $E_0^2 = -4$. Furthermore, by Lemma 5.2.5 we also have $E_1^2 = -1$. So suppose $E_i^2 \in \{-1, -4\}$ for $i \leq k$, we will show that $E_{k+1}^2 \in \{-1, -4\}$.

If $E_k^2 = -4$, then $E_k$ is a flowercurve and hence, $E_{k+1}^2 = -1$, by Lemma 5.2.5. If $E_k^2 = -1$, then $E_{k-1}^2 \neq -1$ by Lemma 5.2.10, so $E_{k-1}^2 = -4$. Since Lemma 5.2.10 also implies that $E_{k+1}^2 \neq -1$, Lemma 5.2.7 gives that $E_{k+1}^2 = -2$ or $-4$. We will exclude $E_{k+1}^2 = -2$ by contradiction. So suppose $E_{k+1}^2 = -2$. Lemma 5.2.10 implies that $E_{k+2}^2 \leq -2$ and therefore by Lemma 5.2.5, we find $E_{k+2}^2 = -2$ as well. Applying again Lemma 5.2.5 together with Lemma 5.2.10 we find that for all $j \geq k + 1$, we have $E_j^2 = -2$. This is in contradiction with the fact that $E_m^2 = F_1^2 \leq -3$. We conclude that $E_{k+1}^2 = -4$ which ends the induction argument. In particular, we find that $F_1^2 = -4$.

**Step 3:** For any component $F_1$ of $F^\dagger$ with $F_1^2 = -1$, there exists a component $F_2$ of $F^\dagger$ with $F_2^2 = -4$ and $F_1 \cdot F_2 > 0$. 

Let $\mathcal{F}_1$ be a component of $\mathcal{F}^\dagger$ with $\mathcal{F}_1^2 = -1$. Because $\mathcal{F}^\dagger$ is connected, it is path-connected, and hence there are irreducible components $E_0, E_1, \ldots, E_m$ of $\mathcal{F}^\dagger$ such that

$$E_i \cdot E_{i+1} > 0,$$

for $i = 0, \ldots, m - 1$ and where $E_0 = C$ and $E_m = E_1$. By removing any cycles, we can assume that $E_i \cdot E_j = 0$ for all $i, j$ with $j \notin \{i - 1, i, i + 1\}$. The same argument as in step 2, shows that $E_i^2 \in \{-1, -4\}$. Since no two components with self-intersection $-1$ can meet each other by Lemma 5.2.10, we conclude $E_m^2 = -4$. Therefore $E_m$ satisfies the conditions of the statement.

**Step 4:** Contradiction.

Let $\mathcal{F}_1$ be a component of $\mathcal{F}^\dagger$ such that $C' \cdot \mathcal{F}_1 > 0$. Such a component certainly exists, since $C'$ is horizontal by Step 1. In Step 2, we showed that $\mathcal{F}_1^2 \in \{-1, -2, -4\}$. By Lemma 5.2.5 we have that $\mathcal{F}_1^2 = -1$, since $\mathcal{F}_1$ meets the flowercurve $C'$.

By Step 3, there exists a component $\mathcal{F}_2$ of $\mathcal{F}^\dagger$ with $\mathcal{F}_2^2 = -4$ and $\mathcal{F}_1 \cdot \mathcal{F}_2 > 0$. Lemma 5.2.5 gives that $\mathcal{F}_2$ is a flowercurve. This is a contradiction with Lemma 5.2.7.

**Remark 5.2.12.** If we additionally assume that $X$ has a rational point, then Proposition 5.2.11 is more easily proven. Indeed, let $F$ be a flower of type $4C$ and let $F'$ be a flower of genus $g > 1$, which is not of type $4\alpha$. Lemma 5.2.8 implies that $F'$ is of type $6\alpha$ or $6\beta$. Let $N_P$ be the multiplicity of the pot $P$. Because $F$ is a flower of type $4\alpha$, Table 3.2 gives that $N_P$ is divisible by 2. Moreover, Table 3.3 shows that $N_P$ is divisible by 3, because $F'$ is a flower of type $6\alpha$ or $6\beta$. Therefore, $N_P$ is divisible by 6. Since there are no flowers of type $6A$, $6B$, $12\alpha$ and $12\beta$ by Lemma 5.2.4 and Corollary 5.2.8 Tables 3.1 and 3.2 and 3.3 give that there is no component of $X_k$ with multiplicity 1. Therefore $X$ does not have a rational point.

**The monodromy eigenvalue $\xi(P)$**

**Lemma 5.2.13.** Let $X$ be a K3 surface over $K$ with a Crauder-Morrison model $\mathcal{X}$. Suppose that $\mathcal{X}$ is a flowerpot degeneration with pot $P$. The following statements hold.

(i) If there exists a flower $F$ of type $4C$, then $N(P)$ is even and $\xi(P)$ is not an $N(P)/2$-th root of unity.

(ii) If there exists a flower $F$ of type $3A$, $3B$, $6\alpha$ or $6\beta$, then $N(P)$ is divisible by 3 and $\xi(P)$ is not an $N(P)/3$-th root of unity.
(iii) If there exists a flower $F$ of type $6C$, $6D$ or $6E$, then $N(P)$ is divisible by 3 and $\xi(P)$ is not an $2N(P)/3$-th root of unity.

Proof. Let $\xi$ be a primitive $N(P)$-th root of unity such that $\xi(P) = \xi^{\nu(P)}$.

(i) Let $F$ be a flower of type $4C$. From Table 3.2 we can see that $N(P) = N(F_0)$, and moreover, $N(P)$ is even. From Lemma 3.3.10 it follows that $\nu(F_0)$ is odd, and hence by Corollary 3.3.12 $\nu(P)$ is odd as well. Therefore

$$\xi(P)^{N(P)/2} = \xi^{\nu(P)N(P)/2} = \xi^{N(P)/2} \neq 1,$$

since $\xi$ is a primitive $N(P)$-th root of unity.

(ii) Let $F$ be a flower of type $3A$, $3B$, $6\alpha$ or $6\beta$. From Tables 3.1 and 3.3 we see that $N(P)$ is divisible by 3. If $F$ is a flower of type $3A$, then $\nu(P) = 3\nu(F_0) - 4$, by Corollary 3.3.12. Therefore

$$\xi(P)^{N(P)/3} = \xi^{\nu(P)N(P)/3} = \xi^{(\nu(F_0)-1)N(P)}\xi^{-N(P)/3} = \xi^{-N(P)/3} \neq 1,$$

since $\xi$ is a primitive $N(P)$-th root of unity. Completely analogously, we compute $\xi(P)^{N(P)/3} \neq 1$, when $F$ is a flower of type $3B$, $6\alpha$ or $6\beta$.

(iii) Let $F$ be a flower of type $6C$, $6D$ or $6E$. From Table 3.2 we see that $N(P)$ is divisible by 3. If $F$ is a flower of type $6C$, then $\nu(P) = (3\nu(F_0) - 6\ell + 5)/2$, by Corollary 3.3.12. Therefore

$$\xi(P)^{2N(P)/3} = \xi^{2\nu(P)N(P)/3} = \xi^{(\nu(F_0)-2\ell+2)N(P)}\xi^{-N(P)/3}$$

$$= \xi^{-N(P)/3} \neq 1,$$

since $\xi$ is a primitive $N(P)$-the root of unity. Completely analogously, we compute $\xi(P)^{2N(P)/3} \neq 1$, when $F$ is a flower of type $6D$ or $6E$.

$\square$

The monodromy property holds

Theorem 5.2.14. Let $X$ be a K3 surface over $K$ with Crauder-Morrison model $\mathcal{X}$. Suppose $\mathcal{X}$ is a flowerpot degeneration with pot $P$. If $P$ is a rational, non-minimal ruled surface, then $X$ satisfies the monodromy property.
**Proof.** Define $I^\dagger \subset I$ to be the set of indices $i \in I$ with either $E_i = P$, or $E_i$ is the top of a conic-flower. Let $S^\dagger = \{(-\nu_i, N_i) \in \mathbb{Z} \times \mathbb{Z}_{>0} \mid i \in I^\dagger\}$. Corollary 4.2.4 says that

$$Z_{X,\omega}(T) \in \mathcal{M}^b_k \left[T, \frac{1}{1-\log T^6}\right]^{(a,b) \in S^\dagger}.$$  

Let $F$ be a conic-flower. We need to prove that $\xi(F_0)$ is a monodromy eigenvalue, since it has already been proven in Theorem 5.1.2 that $\xi(P)$ is a monodromy eigenvalue.

To simplify notation, let $N = N(P)$ be the multiplicity of the pot $P$. If $F$ is not a flower of type $4C$, then Proposition 5.2.2 concludes the proof. So suppose that $F$ is a flower of type $4C$. Since $N(F_0) = N$, we have that $\xi(F_0)$ is an $N$-th root of unity.

Denote by $B_*$ the number of flowers meeting $P$ in a rational curve of type $*$. For instance, $B_{6\alpha\beta}$ denotes the number of flowers of type $6\alpha$ and $6\beta$ meeting $P$ in a rational curve.

By Lemma 3.4.2, one of the following two possibilities hold.

(i) There is a unique flower meeting $P$ in a curve of genus $g > 1$. All other flowercurves on $P$ are rational.

(ii) All flowercurves on $P$ have genus 0 or 1.

Suppose first that all flowercurves on $P$ have genus 0 or 1. Define

$$Q_1(T) = (T^{N/3} - 1)^{B_{3AB} + 2B_{6\alpha\beta}} (T^{2N/3} - 1)^{B_{6CDE}}.$$  

The monodromy zeta function can be written as

$$\zeta_X(T) = \frac{(T^N - 1)^{-\chi(P^\circ)}}{(T^{N/2} - 1)^B_{4C}(T^{2N/2} - 1)^B_{2BC}Q_1(T)},$$  

which can be rewritten to

$$\zeta_X(T) = \frac{1}{(T^N - 1)^B_{4C} + B_{2BC} + \chi(P^\circ)(T^N + 1)^B_{2BC}(T^{N/2} - 1)^2B_{4\alpha}Q_1(T)}.$$  

Since $\xi(P)$ is a monodromy eigenvalue, it is a pole of $\zeta_X(T)$ by Theorem 5.1.1. Theorem 5.2.13 (i) gives that $\xi(P)$ is not an $N/2$-th root of unity. Moreover, $\xi(P)$ is not a zero of $Q_1(T)$, by Theorem 5.2.13 (ii) and (iii). Therefore

$$B_{4C} + B_{2BC} + \chi(P^\circ) > 0,$$

which implies that $\xi(F_0)$ is a pole of the monodromy zeta function, and hence it is a monodromy eigenvalue.
Suppose now that there is a (necessarily unique) flower with genus \( g > 1 \). By Proposition 5.2.11, this flower is of type \( 4\alpha \). The monodromy zeta function can be written as

\[
\zeta_X(T) = \frac{(T^N - 1)^{-\chi(P)}(T^{N/2} - 1)^{2g-2}}{(T^N - 1)^{B_{4C}}(T^{2N} - 1)^{B_{2BC}}(T^{N/2} - 1)^{2B_{4\alpha}}Q_1(T)},
\]

where \( Q_1(T) \) is defined as before. This can be rewritten to

\[
\zeta_X(T) = \frac{1}{(T^N - 1)^{B_{4C} + B_{2BC} + \chi(P)}(T^{N} + 1)^{B_{2BC}}(T^{N/2} - 1)^{2B_{4\alpha}} - 2g + 2Q_1(T)}.
\]

Since \( \xi(P) \) is a monodromy eigenvalue, it is a pole of \( \zeta_X(T) \) by Theorem 5.1.1. Theorem 5.2.13 (i) gives that \( \xi(P) \) is not an \( N/2 \)-th root of unity. Moreover, \( \xi(P) \) is not a zero of \( Q_1(T) \), by Theorem 5.2.13 (ii) and (iii). Therefore

\[
B_{4C} + B_{2BC} + \chi(P) > 0,
\]

which implies that \( \xi(F_0) \) is a pole of the monodromy zeta function, and hence it is a monodromy eigenvalue.

\[\square\]

### 5.3 Chain degenerations with an extra condition

In this section, we will prove the following theorem.

**Theorem 5.3.1.** Let \( X \) be a K3 surface over \( K \) with Crauder-Morrison model \( \mathcal{X} \) with special fiber \( \mathcal{X}_k = \sum_{i \in I} N_i E_i \). Suppose \( \mathcal{X} \) is a chain degeneration with chain \( V_0 - V_1 - \cdots - V_k - V_{k+1} \). Set \( N = \min \{ N(V_i) \mid i = 0, \ldots, k + 1 \} \).

Assume moreover that one of the following conditions hold:

(i) the components \( V_0, V_1, \ldots, V_{k+1} \) all have multiplicity \( N \), or

(ii) neither \( V_0 \) nor \( V_{k+1} \) is a rational, non-minimal ruled surface, or

(iii) exactly one end component of the chain is a rational, non-minimal ruled surface, and it has multiplicity \( N \), or

(iv) if \( V_j \) meets a conic-flower, then \( N(V_j) > N \).

Then \( X \) satisfies the monodromy property.

**Proof.** Define \( I^\dagger \subset I \) to be the set of indices \( i \in I \) where either \( \rho_i \) is minimal, or \( E_i \) is the top of a conic-flower. Let \( S^\dagger = \{ (-\nu_i, N_i) \in \mathbb{Z} \times \mathbb{Z}_{>0} \mid i \in I^\dagger \} \).

Corollary 4.2.4 says that

\[
Z_{X,\omega}(T) \in \mathcal{M}_k^\dagger \left[ T, \frac{1}{1 - \frac{1}{L^\dagger}T^\dagger} \right]_{(a,b) \in S^\dagger}.
\]
Let $F$ be a conic-flower. Denote by $F_0$ the top of the flower $F$ and by $F_{k+1}$ the component of the chain meeting the flower $F$. By Proposition 3.6.1, $F$ is a flower of type $2B$ or $2C$. Moreover, Theorem 3.3.7 and Proposition 3.6.2 imply that $N(F_0) = 2N$ or $4N$. We need to prove that $\xi(F_0)$ is a monodromy eigenvalue, since it has already been proven in Theorem 5.1.2 that $\xi(E_i)$ is a monodromy eigenvalue if $\rho_i$ is minimal.

(i) Suppose first that $V_0, V_1, \ldots, V_{k+1}$ all have multiplicity $N$. Table 3.2 gives that $N(F_0) = 2N$. By Proposition 3.6.1, all flowers are of type 2A, 2B, 2C or $4\alpha$, and moreover flowers of type $4\alpha$ can only meet a component of the chain in $V_0$ or $V_{k+1}$.

To show that $\xi(F_0)$ is a monodromy eigenvalue, we will prove that it is a pole of the monodromy zeta function $\zeta_X(T)$. Proposition 2.3.3 gives that

$$\zeta_X(T) = \prod_{i \in I} (T^{N_i} - 1)^{-\chi(E_i^0)},$$

where $\chi(E_i^0)$ is the topological Euler characteristic of $E_i^0$. From Lemma 3.7.1, we see that the only factor in this formula with a positive exponent is

$$(T^N - 1)^{- \sum_{i=0}^{k+1} \chi(V_i^0)},$$

when $\sum_{i=0}^{k+1} \chi(V_i^0) < 0$, or

$$(T^{N(F_0')} - 1)^{2g(F') - 2},$$

if there exists a flower $F'$ of type $4\alpha$ and of genus $g(F') > 1$. Notice that $N(F_0') = N/2$, by Table 3.3. Therefore, $T^{N(F_0')} - 1$ is a divisor of $T^N - 1$. Since $F$ is a flower of type 2, Corollary 5.1.3 implies that $\xi(F_0)$ is not a zero of $T^N - 1$ and $T^{N(F_0')} - 1$. For this reason, $\xi(F_0)$ is a pole of $\zeta_X(T)$.

(ii) Assume now that neither $V_0$ nor $V_{k+1}$ is a rational, non-minimal ruled surface. By Proposition 3.6.1, all flowers are of type 2A, 2B, 2C or $4\alpha$, and moreover flowers of type $4\alpha$ can only meet a component of the chain in $V_0$ or $V_{k+1}$.

To show that $\xi(F_0)$ is a monodromy eigenvalue, we will prove that it is a pole of the monodromy zeta function $\zeta_X(T)$. Proposition 2.3.3 gives that

$$\zeta_X(T) = \prod_{i \in I} (T^{N_i} - 1)^{-\chi(E_i^0)},$$

where $\chi(E_i^0)$ is the topological Euler characteristic of $E_i^0$. From Lemma 3.7.1, we see that the only factor in this formula with a positive exponent is

$$(T^{N(F_0')} - 1)^{2g(F') - 2},$$
if there exists a flower $F'$ of type $4\alpha$ and of genus $g(F') > 1$. Hence, if there is no such flower, then we are done. So suppose there is a flower $F'$ of type $4\alpha$ and of genus $g(F') > 1$. Since only $V_0$ or $V_{k+1}$ can meet flowers of type $4\alpha$, we can assume, without loss of generality, that $V_0$ meets $F'$. From Proposition 3.6.1 we see that $V_0$ is an elliptic, ruled surface and hence $N(V_0) = N$ or $2N$, by Proposition 3.6.2(iv). Therefore, Theorem 3.3.7 gives that $N(F'_0) = N/2$ or $N$. Since $F$ is a flower of type 2 and $N(F_0) = 2N$ or $4N$, Corollary 5.1.3 implies that $\xi(F_0)$ is not a zero of $T^{N(F'_0)} - 1$. For this reason, $\xi(F_0)$ is a pole of $\zeta_X(T)$.

(iii) Assume that there is exactly one end component of the chain that is a rational, non-minimal ruled surface. Without loss of generality, we can assume that this component is $V_0$. Assume that $N(V_0) = N$.

To show that $\xi(F_0)$ is a monodromy eigenvalue, we will prove that it is a pole of the monodromy zeta function $\zeta_X(T)$. Proposition 2.3.3 gives that

$$\zeta_X(T) = \prod_{i \in I} (T^{N_i} - 1)^{-\chi(E_i^\circ)},$$

where $\chi(E_i^\circ)$ is the topological Euler characteristic of $E_i^\circ$. From Lemma 3.7.1 we see that the only factor in this formula with a positive exponent are either

$$(T^N - 1)^{-\sum_{i=0}^\beta \chi(V_i^\circ)},$$

when $\sum_{i=0}^\beta \chi(V_i^\circ) < 0$ and where $\beta$ is defined as in Proposition 3.6.2 or

$$(T^N(V'_0) - 1)^{2g(F') - 2},$$

if there exists a flower $F'$ of type $4\alpha$ and of genus $g(F') > 1$.

Suppose there is a flower $F'$ of type $4\alpha$ and of genus $g(F') > 1$. From Proposition 3.6.1 it follows that only end components that are elliptic ruled can meet flowers of type $4\alpha$, and therefore we know that $V_{k+1}$ meets $F'$, and that $V_{k+1}$ is an elliptic, ruled surface. Hence $N(V_{k+1}) = N$ or $2N$, by Proposition 3.6.2(iv). Therefore, Theorem 3.3.7 gives that $N(F'_0) = N/2$ or $N$.

Since $F$ is a flower of type 2 and $N(F_0) = 2N$ or $4N$, Corollary 5.1.3 implies that $\xi(F_0)$ is not a zero of $T^N - 1$ and $T^{N(F'_0)} - 1$. For this reason, $\xi(F_0)$ is a pole of $\zeta_X(T)$.

(iv) Finally, assume that every component $V_j$ meeting a conic-flower has multiplicity $N(V_j) > N$. Therefore, $N(F_{k+1}) > N$, since $F$ is a conic-flower. Statements (ii) and (iii) of Proposition 3.6.2 imply that we can assume, without loss of generality, that $F_{k+1} = V_0$. Moreover, from
Proposition 3.6.2 (iv), it follows that $V_0$ is a ruled surface and that $N(V_0) = 2N$. Therefore, $N(F_0) = 4N$, by Theorem 3.3.7.

To show that $\xi(F_0)$ is a monodromy eigenvalue, we will prove that it is a pole of the monodromy zeta function $\zeta_X(T)$. Proposition 2.3.3 gives that

$$\zeta_X(T) = \prod_{i \in I} (T^{N_i} - 1)^{-\chi(E_i^0)},$$

where $\chi(E_i^0)$ is the topological Euler characteristic of $E_i^0$.

Suppose first that $N(V_{k+1}) = 2N$. From Lemma 3.7.1, we see that the only factors in this formula with a positive exponent are either

$$(T^{2N} - 1)^{-\chi(V_0^0) - \chi(V_{k+1}^0)},$$

when $-\chi(V_0^0) - \chi(V_{k+1}^0) < 0$, or

$$(T^{N(F_0')} - 1)^{2g(F') - 2},$$

if there exists a flower $F'$ of type $4\alpha$ and of genus $g(F') > 1$. If there is a flower $F'$ of type $4\alpha$ and of genus $g(F') > 1$, then $N(F_0') = N$, by Theorem 3.3.7 and Proposition 3.6.1. Since $F$ is a flower of type $2$ and $N(F_0) = 4N$, Corollary 5.1.3 implies that $\xi(F_0)$ is not a zero of $T^N - 1$ and $T^{N(F_0')} - 1$. For this reason, $\xi(F_0)$ is a pole of $\zeta_X(T)$.

Suppose now that $N(V_{k+1}) \neq 2N$. From Lemma 3.7.1, we see that the only factors in the formula for $\zeta_X(T)$ with a positive exponent are either

$$(T^{2N} - 1)^{-\chi(V_0^0)},$$

when $-\chi(V_0^0) < 0$, or

$$(T^{N(F_0')} - 1)^{2g(F') - 2},$$

if there exists a flower $F'$ of type $4\alpha$ and of genus $g(F') > 1$. If there is a flower $F'$ of type $4\alpha$ and of genus $g(F') > 1$, then $N(F_0') = N/2$ or $N$, by Theorem 3.3.7 and Proposition 3.6.1. Since $F$ is a flower of type $2$ and $N(F_0) = 4N$, Corollary 5.1.3 implies that $\xi(F_0)$ is not a zero of $T^N - 1$ and $T^{N(F_0')} - 1$. For this reason, $\xi(F_0)$ is a pole of $\zeta_X(T)$.
Chapter 6

Future research: a proof or a counterexample?

Let $X$ be a $K3$ surface over $K$ admitting a triple-point-free model, and let $\mathcal{X}$ be a Crauder-Morrison model of $X$. Instead of proving directly that $X$ satisfies the monodromy property, we can ask ourselves the following question:

If $X$ does not satisfy the monodromy property, then what should $X_k$ look like?

Descriptions of such special fibers are called combinatorial countercandidates. A combinatorial countercandidate is a combinatorial characterization of a special fiber, such that, if it occurs as the special fiber of a $K3$ surface $X$, then $X$ does not satisfy the monodromy property. In other words, if there exists a $K3$ surface $X$ admitting an snc-model with a combinatorial countercandidate as special fiber, then $X$ is a counterexample of the monodromy property.

The interest of combinatorial countercandidates is twofold: on the one hand, with a concrete special fiber in mind that would lead to a counterexample of the monodromy property, one can try to prove that there indeed exists a $K3$ surface with a model with this particular special fiber. So if we can smoothen a combinatorial countercandidate to a $K3$ surface, then we will have constructed a $K3$ surface not satisfying the monodromy property. On the other hand, it is clearly beneficial to have deep knowledge about the properties of possible $K3$ surfaces not satisfying the monodromy property. If these properties are contradictory, no such $K3$ surface exists. Therefore, a strategy to prove that $K3$ surfaces admitting a triple-point-free model satisfy the monodromy property, could be to exclude all combinatorial countercandidates, i.e., find reasons why combinatorial countercandidates do not occur as special fibers of $K3$ surfaces.
In the first section, we will give some more useful results that will be used throughout this chapter. In Section 6.2 we will explain a strategy to produce all combinatorial countercandidates. One application of combinatorial countercandidates would be to smoothen one to a $K3$ surface. A strategy to do so, is described in Section 6.3. Another application would be to rule out all combinatorial countercandidates, and we illustrate this strategy in Section 6.4 where we will indeed exclude some combinatorial countercandidates. The final section of this chapter is devoted to discussing some questions for further research.

**Notation**

In this chapter, we fix an algebraically closed field $k$ of characteristic zero. Put $R = k[[t]]$ and $K = k((t))$. Fix an algebraic closure $K^{alg}$ of $K$.

For any ruled surface $V$ over $k$, we define $L$ to be the number of blow-ups in the contraction $V \to \overline{V}$ to the minimal ruled surface $\overline{V}$. In particular, when $V$ is minimal ruled, we have $L = 0$.

If $X$ is a $K3$ surface over $K$ with $\mathcal{X}$ a Crauder-Morrison model of $X$ over $R$ and if $\omega$ is a volume form on $X$, then we write the special fiber as $X_k = \sum_{i \in I} N_i E_i$. We denote by $(N_i, \nu_i)$ the numerical data of $E_i$, for every $i \in I$. To avoid confusion, we will sometimes use the notation $(N(E_i), \nu(E_i))$ instead of $(N_i, \nu_i)$. Set $\rho_i = \nu_i/N_i+1$ to be the weight of $E_i$, for every $i \in I$. For every component $E_i$ of $\mathcal{X}_k$, we define

$$\xi(E_i) = \exp \left( -2\pi i \frac{\nu(E_i)}{N(E_i)} \right),$$

the ‘candidate’ monodromy eigenvalue associated with $E_i$.

### 6.1 Some more results

#### 6.1.1 Index of a variety

**Definition 6.1.1.** Let $X$ be a variety over $K$. The index $\iota(X)$ of $X$ is defined as the greatest common divisor of the degrees $[L : K]$ of all finite extensions $L/K$ such that $X(L) \neq \emptyset$.

In particular, if $X$ has a $K$-rational point, then $\iota(X) = 1$. The converse is not true.

The following characterization of the index is well known.
Proposition 6.1.2. Let $X$ be a variety over $K$. Let $X$ be an snc-model of $X$ with $X_k = \sum_{i \in I} N_i E_i$. Then
\[ \iota(X) = \gcd_{i \in I} N_i. \]

The following statement is a special case of a theorem proved by Esnault, Levine and Wittenberg.

Theorem 6.1.3. Let $X$ be a $K3$ surface over $K$. The index $\iota(X)$ is either 1 or 2.

Proof. This follows from [ELW15, Theorem 2.1], where it is shown that $\iota(X)$ is a divisor of the holomorphic Euler characteristic of $X$, and the fact that the holomorphic Euler characteristic of a $K3$ surface is 2. \qed

Lemma 6.1.4. Let $X$ be a $K3$ surface over $K$ of index $\iota(X) = 2$. Denote by $K(2)$ the unique totally ramified extension of degree 2 of $K$ in $K^{\text{alg}}$, and define $X(2) = X \times_K K(2)$. Then $X(2)$ is a $K3$ surface with index $\iota(X(2)) = 1$.

Proof. The surface $X(2)$ is proper, geometrically connected and smooth, as these properties are stable under base change. Moreover, [Har77, Proposition II.8.10] guarantees that $X(2)$ has trivial canonical bundle, and [Stacks, Tag 02KH] that $H^1(X(2), \mathcal{O}_{X(2)}) = 0$. Therefore, $X(2)$ is a $K3$ surface.

Every finite extension $L$ of $K$ in $K^{\text{alg}}$ of even degree contains $K(2)$, and therefore we have that $X$ has an $L$-rational point if and only if $X(2)$ has an $L$-rational point. Since $[L : K] = 2 \cdot [L : K(2)]$, we conclude that $\iota(X(2)) = 1$. \qed

Lemma 6.1.5. Let $X$ be a $K3$ surface over $K$ of index $\iota(X) = 2$. Denote by $K(2)$ the unique totally ramified extension of degree 2 of $K$ in $K^{\text{alg}}$, and define $X(2) = X \times_K K(2)$. If $X(2)$ satisfies the monodromy property, then $X$ satisfies the monodromy property as well.

Proof. Let $\omega$ be a volume form on $X$, and let $\omega(2)$ be the pullback of $\omega$ to $X(2)$. Let $\sigma$ be a topological generator of $\text{Gal}(K^{\text{alg}}/K)$. Then $\sigma^2$ is a topological generator of $\text{Gal}(K^{\text{alg}}/K(2))$. Note that obviously, we have $X \times_K K^{\text{alg}} = X(2) \times_K K(2)^{\text{alg}}$.

For every integer $d \geq 1$, denote by $K(d)$ the unique totally ramified extension of degree $d$ of $K$ in $K^{\text{alg}}$, and define $X(d) = X \times_K K(d)$. Let $\omega(d)$ be the pullback of $\omega$ to $X(d)$. Because the index $\iota(X) = 2$, we have $X(K(d)) = \emptyset$, whenever $d$ is odd. Therefore, when $d$ is odd, $X(d)$ itself is a weak Néron model of $X(d)$, and hence $\int_{X(d)} |\omega(d)| = 0$. From Definition 2.2.1, we see that
\[ Z_{X,\omega}(T) = Z_{X(2),\omega(2)}(T^2). \]
Assume that $X(2)$ satisfies the monodromy property. Define $S^{(2)} \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ such that

$$Z_{X(2),\omega(2)}(T) \in \mathcal{M}_k^\mu \left[ T, \frac{1}{1 - \frac{a}{b} T^b} \right]_{(a,b) \in S^{(2)}},$$

and such that for each $(a, b) \in S^{(2)}$, we have that $\exp(2\pi ia/b)$ is an eigenvalue of $\sigma$ on $H^m(X \times_K K^{alg}, \mathbb{Q}_\ell)$, for some $m \geq 0$ and for every embedding of $\mathbb{Q}_\ell$ into $\mathbb{C}$. From (6.1), it follows that

$$Z_{X,\omega}(T) \in \mathcal{M}_k^\mu \left[ T, \frac{1}{1 - \frac{a}{b} T^b} \right]_{(a,b) \in S},$$

where $S = \{(a, 2b) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} | (a, b) \in S^{(2)}\}$.

Take $(a, 2b) \in S$, we have to prove that $\alpha = \exp(2\pi ia/(2b))$ is a monodromy eigenvalue. By definition of $S$, we know that $(a, b) \in S^{(2)}$, and therefore $\alpha^2 = \exp(2\pi ia/b)$ is an eigenvalue of $\sigma^2$ on $H^m(X \times_K K^{alg}, \mathbb{Q}_\ell)$, for some $m \geq 0$. This implies that $\alpha$ or $-\alpha$ is an eigenvalue of $\sigma$ on $H^m(X \times_K K^{alg}, \mathbb{Q}_\ell)$.

Suppose $-\alpha$ is a monodromy eigenvalue. By Lemma 5.1.1, we know that $-\alpha$ is a pole of the monodromy zeta function $\zeta_X(T)$. Let $X$ be an $snc$-model of $X$, and write $X_k = \sum_{i \in I} N_i E_i$. Because the index $\iota(X) = 2$, we have that $N_i$ is even, for every $i \in I$. As a consequence, A’Campo’s formula 2.3.3 gives that $\zeta_X(T)$ is a rational function in $T^2$. Therefore, $\alpha$ is a pole of $\zeta_X(T)$ as well, and hence it is a monodromy eigenvalue. We conclude that $X$ satisfies the monodromy property.

6.1.2 Number of $(-2)$-curves on a rational, ruled surface

In Chapter 5 we proved that $K3$ surfaces allowing a triple-point-free model satisfy the monodromy property, except in the following case: if the Crauder-Morrison model is a chain degeneration where the chain has at least one rational, ruled end component $V_0$. The main difficulty to prove the monodromy property in this case, is that the Euler characteristic $\chi(V_0)$ may be negative. Therefore, the factor $T^{N(V_0) - 1}$ may cause cancellations in the A’Campo formula of Theorem 2.3.3. Negative Euler characteristics do not occur for other surfaces in the chain, as can be seen from Lemma 3.7.1.

Let us explore this problem of negative Euler characteristics in more detail. Let $X$ be a $K3$ surface over $K$ with a Crauder-Morrison model $X$ such that $X_k$ is a chain degeneration. Let $V_0$ be an end component of the chain that is a rational, ruled surface. Let $D$ be the elliptic double curve on $V_0$, where $V_0$ meets the next component in the chain.
Let \( C_1, \ldots, C_\lambda \) be the flowercurves on \( V_0 \). By Lemma \[3.6.1\] and Lemma \[3.3.8\], we know that \( C_i \) is a rational curve with \( C_i^2 = -2 \), for every \( i = 1, \ldots, \lambda \). Moreover, they are all disjoint, and don’t meet \( D \).

Computing the Euler characteristic \( \chi(V_0) \) gives

\[
\chi(V_0) = \chi(V_0) - \chi(D) - \sum_{i=1}^{\lambda} \chi(C_i) = \chi(V_0) - 2\lambda,
\]

since \( D \) is elliptic, and all \( C_i \) are rational.

Denote by \( L \) the number of blow-ups in the contraction \( \phi: V_0 \to \overline{V}_0 \) to the minimal ruled surface \( \overline{V}_0 \). Then we have that \( \chi(V_0) = 4 + L \), and hence

\[
\chi(V_0) = 4 + L - 2\lambda.
\]

Moreover, from the proof of Theorem \[3.6.3\], we know that \( \phi \) consists of contractions of rational curves with self-intersection \(-1\) meeting \( D \) (or the image of \( D \)).

Therefore, the following question arises: Let \( \overline{V}_0 \) be a rational, minimal ruled surface with elliptic curve \( \overline{D} \equiv -K_{\overline{V}_0} \), and let \( V_0 \) be the surface obtained by blowing up \( \overline{V}_0 \) in \( L \) points on \( \overline{D} \) or the intermediate strict transforms of \( \overline{D} \). Let \( D \subset V_0 \) be the strict transform of \( \overline{D} \). What is the maximal number \( \lambda \) of pairwise disjoint, rational curves on \( V_0 \) with self-intersection \(-2\) that are disjoint from \( D \)?

If \( L \geq 2\lambda - 4 \), then \( \chi(V_0) \geq 0 \), but we cannot hope for this inequality in general, as Example \[6.1.7\] shows. This example has \( L = \lambda = 6 \), and hence \( \chi(V_0) = -2 \). However, the following lemma gives a bound on \( \lambda \) anyway.

**Lemma 6.1.6.** Let \( V \) be a rational, ruled surface such that there is an elliptic curve \( D \) with \( D \equiv -K_V \). Let \( L \) be the number of blow-ups in the contraction \( \phi: V \to \overline{V} \) to the minimal ruled surface \( \overline{V} \). Assume that \( \phi \) is a composition of blow-ups with center on \( D \), or strict transforms of \( D \). Suppose there exist pairwise disjoint, rational curves \( C_1, \ldots, C_\lambda \) with \( C_i^2 = -2 \) and \( C_i \cdot D = 0 \), for every \( i = 1, \ldots, \lambda \). Then

\[
\lambda \leq L + 1.
\]

**Proof.** We know by [Har77, Proposition V.2.3 and Exercise II.8.5] that

\[
\text{Pic}(V) \simeq \mathbb{Z}^{L+2}.
\]

We will show that \([C_1], \ldots, [C_\lambda] \) and \([D]\) are linearly independent classes in \( \text{Pic}(V) \otimes \mathbb{Q} \), where \([\cdot]\) denotes the class in \( \text{Pic}(V) \otimes \mathbb{Q} \). From this, it follows that \( \lambda \leq L + 1 \).
We start by proving that $[C_1], \ldots, [C_\lambda]$ are linearly independent in the vector space $\text{Pic}(V) \otimes \mathbb{Z} \mathbb{Q}$. Let $a_i \in \mathbb{Q}$ such that $\sum_{i=1}^\lambda a_i[C_i] = 0$. Then for every $j = 1, \ldots, \lambda$, we have
\[-2a_j = \sum_{i=1}^\lambda a_i C_i \cdot C_j = 0,\]
because $C_j^2 = -2$ and $C_i \cdot C_j = 0$ for every $i \neq j$. Hence $a_j = 0$ for every $j$ and therefore, we have that $[C_1], \ldots, [C_\lambda]$ are linearly independent classes in $\text{Pic}(V) \otimes \mathbb{Z} \mathbb{Q}$.

We now prove that $[D]$ is linearly independent from $[C_1], \ldots, [C_\lambda]$ in $\text{Pic}(V) \otimes \mathbb{Z} \mathbb{Q}$. Suppose the contrary, then we can write
\[[D] = \sum_{i=1}^\lambda a_i[C_i],\]
for some $a_i \in \mathbb{Q}$. As linear equivalence implies numerical equivalence, we must have
\[D \cdot C_j = \sum_{i=1}^\lambda a_i C_i \cdot C_j,\]
for every $j = 1, \ldots, \lambda$. Because $D \cdot C_j = 0$ and $\sum_{i=1}^\lambda a_i C_i \cdot C_j = -2a_j$, we have $a_j = 0$ for all $j = 1, \ldots, \lambda$. Therefore $[D] = 0$, which is a contradiction, since $D \equiv -K_V$. \hfill \Box

We don’t know whether the inequality in Lemma 6.1.6 is sharp. In any case, it cannot be improved a lot because of the following example, which was shown to us by W. Veys.

**Example 6.1.7** (due to W. Veys). We will construct a rational, ruled surface $V$ with the following properties: there is an elliptic curve $D$ with $D \equiv -K_V$, and there are six pairwise disjoint, smooth, rational curves $C_1, \ldots, C_6$ with $C_i^2 = -2$ and $C_i \cdot D = 0$, for every $i = 1, \ldots, 6$. The contraction $\phi: V \to \overline{V}$ to the minimal ruled surface $\overline{V}$ is a composition of six contractions of rational $(-1)$-curves meeting $D$.

We will start from $\overline{V} \simeq \Sigma_1$, the first Hirzebruch surface. Then we will define six blow-ups
\[V = V^{(6)} \to V^{(5)} \to V^{(4)} \to V^{(3)} \to V^{(2)} \to V^{(1)} \to V^{(0)} = \overline{V}.\]

The strict transform of any curve $C^{(i)} \subset V^{(i)}$ will be denoted by $C^{(i+1)} \subset V^{(i+1)}$. For any point $P^{(i)} \in V^{(i)}$ that is not the center of the blow-up $V^{(i+1)} \to V^{(i)}$, we denote its inverse image by $P^{(i+1)} \in V^{(i+1)}$. 

Since $\overline{V} = \Sigma_1$ is the blow-up of $\mathbb{P}^2$ in one point, we will define some relevant curves on $\overline{V}$ by defining them on $\mathbb{P}^2$. So let $\overline{V}^{(-1)} \simeq \mathbb{P}^2$ and consider three lines $\ell, C_2^{(-1)}, C_3^{(-1)}$ in $\overline{V}^{(-1)}$, such that $\ell \cap C_2^{(-1)} \cap C_3^{(-1)} = \{P_1^{(-1)}\}$ for some point $P_1^{(-1)} \in \overline{V}^{(-1)}$. Let $D^{(-1)}$ be a cubic through $P_1^{(-1)}$, tangent to the lines $\ell, C_2^{(-1)}, C_3^{(-1)}$, in the points $P_\ell, P_2^{(-1)}, P_3^{(-1)}$ respectively. None of the points $P_\ell, P_2^{(-1)}, P_3^{(-1)}$ equals $P_1^{(-1)}$. It is not difficult to check that such a cubic indeed exists. A sketch of this configuration can be found in Figure 6.1, where the dotted line denotes $D^{(-1)}$.

![Figure 6.1: Sketch of $\overline{V}^{(-1)}$](image)

Let $\overline{V} = V^{(0)}$ be the blow-up of $V^{(-1)}$ in $P_\ell$. So we have $\overline{V} \simeq \Sigma_1$, the first Hirzebruch surface. Let $C_1^{(0)}$ be the exceptional divisor. Define the curves $D^{(0)}, F_1^{(0)}, C_2^{(0)}$ and $C_3^{(0)}$ as the strict transforms of $D^{(-1)}, \ell, C_2^{(-1)}$ and $C_3^{(-1)}$ respectively. Define the points $P_1^{(0)}, P_2^{(0)}$ and $P_3^{(0)}$ as the inverse images of $P_1^{(-1)}, P_2^{(-1)}$ and $P_3^{(-1)}$ respectively under the morphism $V^{(0)} \to V^{(-1)}$. Define $P_4^{(0)} = C_1^{(0)} \cap F_1^{(0)}$. The curve $D^{(-1)}$ meets $F_1^{(0)}$ transversally in $P_1^{(0)}$ and $P_4^{(0)}$, and it is tangent to $C_2^{(0)}$ in $P_2^{(0)}$, and to $C_3^{(0)}$ in $P_3^{(0)}$.

Some important lines and points on $V^{(0)}$ are sketched in Figure 6.2, where the dashed lines are fibers of the ruling on $V_0$, and the solid lines are horizontal curves.
Because $C_1^{(0)}$ is the exceptional curve, and because $\left(C_2^{(-1)}\right)^2 = \left(C_3^{(-1)}\right)^2 = 1$, we have that

$$\left(C_1^{(0)}\right)^2 = -1 \text{ and } \left(C_2^{(0)}\right)^2 = \left(C_3^{(0)}\right)^2 = 1.$$ 

Let $V^{(2)} \rightarrow V^{(0)}$ be the birational morphism obtained by blowing up $V^{(0)}$ in the points $P_1^{(0)}$ and $P_4^{(0)}$. Some important lines and points on $V^{(2)}$ are sketched in Figure 6.3.

The curves $C_2^{(2)}$ and $C_3^{(2)}$ are the strict transforms of $C_2^{(0)}$ and $C_3^{(0)}$ respectively. We have that

$$\left(C_1^{(2)}\right)^2 = -2 \text{ and } \left(C_2^{(2)}\right)^2 = \left(C_3^{(2)}\right)^2 = 0.$$ 

Furthermore, $C_1^{(2)}$, $C_2^{(2)}$ and $C_3^{(2)}$ are disjoint. Moreover, the curve $D^{(2)}$ is tangent to $C_2^{(2)}$ in $P_2^{(2)}$, and to $C_3^{(2)}$ in $P_3^{(2)}$. 

Figure 6.2: Sketch of $V^{(0)}$

Figure 6.3: Sketch of $V^{(2)}$
There is one reducible fiber, with three irreducible components as seen in Figure 6.3. The middle component has self-intersection $-2$ and the other two components have self-intersection $-1$. Furthermore, the component with self-intersection $-2$ is disjoint from $C_1^{(2)}, C_2^{(2)}$ and $C_3^{(2)}$.

Because we blew up smooth points on $D^{(0)}$, we have that $D^{(2)} \equiv -K_{V^{(2)}}$ by [Har77, Proposition V.3.3 and Proposition V.3.6].

Let $V^{(3)} \rightarrow V^{(2)}$ be the birational morphism obtained by blowing up $V^{(2)}$ in the point $P_3^{(2)}$. Some important lines and points on $V^{(3)}$ are sketched in Figure 6.4.

![Figure 6.4: Sketch of $V^{(3)}$](image)

We have that $D^{(3)}$ intersects $C_3^{(3)}$ transversally, and we denote this point by $P_3^{(3)}$.

Let $V^{(4)} \rightarrow V^{(3)}$ be the birational morphism obtained by blowing up $V^{(3)}$ in the point $P_3^{(3)}$. Some important lines and points on $V^{(4)}$ are sketched in Figure 6.5.

![Figure 6.5: Sketch of $V^{(4)}$](image)

The curves $C_2^{(4)}$ and $C_3^{(4)}$ are the strict transforms of $C_2^{(2)}$ and $C_3^{(2)}$ respectively. We have that

$$
\left(C_1^{(4)}\right)^2 = \left(C_3^{(4)}\right)^2 = -2 \quad \text{and} \quad \left(C_2^{(4)}\right)^2 = 0.
$$
Furthermore, $C_1^{(4)}, C_2^{(4)}$ and $C_3^{(4)}$ are disjoint. Moreover, the curve $D^{(4)}$ is tangent to $C_2^{(4)}$ in $P_2^{(4)}$.

There are two reducible fibers, each with three irreducible components as seen in Figure 6.5. The middle component of each of these fibers has self-intersection $-2$, and the other two components have self-intersection $-1$. Furthermore, the two components with self-intersection $-2$ are disjoint from $C_1^{(4)}, C_2^{(4)}$ and $C_3^{(4)}$.

Because we blew up smooth points on $D^{(2)}$ and $D^{(3)}$, we have that $D^{(4)} \equiv -K_{V^{(4)}}$ by [Har77, Proposition V.3.3 and Proposition V.3.6].

Let $V^{(5)} \to V^{(4)}$ be the birational morphism obtained by blowing up $V^{(4)}$ in the point $P_2^{(4)}$. We have that $D^{(5)}$ intersects $C_2^{(5)}$ transversally, and we denote this point by $P_2^{(5)}$. Let $V = V^{(6)} \to V^{(5)}$ be the birational morphism obtained by blowing up $V^{(5)}$ in the point $P_2^{(5)}$.

![Figure 6.6: Sketch of V](image)

The curves $C_2$ and $C_3$ are the strict transforms of $C_2^{(4)}$ and $C_3^{(4)}$ respectively. We have that

$$C_1^2 = C_2^2 = C_3^2 = -2.$$  

Furthermore, $C_1, C_2$ and $C_3$ are disjoint.

There are three reducible fibers, each with three irreducible components. In each reducible fiber, the middle component has self-intersection $-2$, and the other two components have self-intersection $-1$. Furthermore, these vertical curves with self-intersection $-2$ are pairwise disjoint, and disjoint from $C_1, C_2$ and $C_3$.

Because we blew up smooth points on $D^{(4)}$ and $D^{(5)}$, we have that $D \equiv -K_V$ by [Har77, Proposition V.3.3 and Proposition V.3.6].

To conclude, we have constructed a rational, ruled surface $V$ such that there is an elliptic curve $D$ with $D \equiv -K_V$. There are six pairwise disjoint, smooth, rational curves $C_1, \ldots, C_6$ with $C_i^2 = -2$ and $C_i \cdot D = 0$, for every $i = 1, \ldots, 6$. 

The contraction $\phi : V \to V$ to the minimal ruled surface $V$ is a composition of six blow-ups with center on $D$.

### 6.1.3 Number of blow-ups in the chain

Let $X$ be a $K3$ surface over $K$ with Crauder-Morrison model $X$. Suppose $X$ is a chain degeneration, and let $V_0, V_1, \ldots, V_k, V_{k+1}$ be the components in the chain, such that $V_i \cap V_j = \emptyset$ if and only if $j \not\in\{i-1, i, i+1\}$. For every $i = 0, \ldots, k+1$, if $V_i$ is a ruled surface, then we define $L_i$ to be the number of blow-ups in the contraction $V_i \to \overline{V}_i$ to the minimal ruled surface $\overline{V}_i$. If $V_i \cong \mathbb{P}^2$, then we set $L_i = 0$ and $\overline{V}_i = V_i$. The following formula restricts the possibilities of the values $L_i$.

**Lemma 6.1.8.** Let $X$ be a $K3$ surface over $K$ with Crauder-Morrison model $X$ and with special fiber $X_k = \sum_{i \in I} N_i E_i$. Suppose $X$ is a chain degeneration, and let $V_0, V_1, \ldots, V_k, V_{k+1}$ be the components in the chain, such that $V_i \cap V_j = \emptyset$ if and only if $j \not\in\{i-1, i, i+1\}$. We have

$$
\sum_{i=0}^{k+1} \frac{L_i}{N(V_i)} = \frac{K_{V_0}^2}{N(V_0)} + \frac{K_{V_{k+1}}^2}{N(V_{k+1})}.
$$

**Proof.** For $i = 0, \ldots, k$, set $C_i = V_i \cap V_{i+1}$. We have that

$$K_{V_i} \equiv -C_{i-1} - C_i,$$

for every $i = 1, \ldots, k$, because of (3.4) and the fact that $V_i$ only meets flowers of type 2 by Proposition 3.6.1. On the other hand, since $V_i$ is elliptic ruled for $i = 1, \ldots, k$, we know that

$$K_{V_i}^2 = -L_i,$$

by [Har77, Corollary V.2.11 and Proposition V.3.3]. As there are no triple points, this gives

$$
(C_{i-1})_{V_i}^2 + (C_i)_{V_i}^2 = -L_i.
$$

Using techniques similar as in the proof of the triple point formula [Per77, Corollary 2.4.2], one can prove that

$$
(C_i)_{V_i}^2 = -\frac{N(V_i)}{N(V_{i+1})}(C_i)_{V_{i+1}}^2,
$$

for $i = 0, \ldots, k$. 


Combining formulas (6.2) and (6.3), we obtain

\[(C_0)^2 V_0 = - \frac{N(V_0)}{N(V_1)} (C_0)^2 V_1\]

\[= \frac{N(V_0)}{N(V_1)} ((C_1)^2 V_1 + L_1)\]

\[= - \frac{N(V_0)}{N(V_2)} (C_1)^2 V_2 + \frac{N(V_0)}{N(V_1)} L_1\]

\[= \frac{N(V_0)}{N(V_2)} (C_2)^2 V_2 + \sum_{i=1}^{2} \frac{N(V_0)}{N(V_i)} L_i\]

\[= \ldots\]

\[= - \frac{N(V_0)}{N(V_{k+1})} (C_k)^2 V_{k+1} + \sum_{i=1}^{k} \frac{N(V_0)}{N(V_i)} L_i.\]

On the other hand

\[(C_0)^2 V_0 = K_0^2 V_0 = K_0^2 V_0 - L_0,\]

and

\[(C_k)^2 V_{k+1} = K_{k+1}^2 V_{k+1} = K_{k+1}^2 V_{k+1} - L_{k+1},\]

by [Har77] Proposition V.3.3].

Therefore,

\[K_0^2 V_0 - L_0 = - \frac{N(V_0)}{N(V_{k+1})} \left( K_{k+1}^2 V_{k+1} - L_{k+1} \right) + \sum_{i=1}^{k} \frac{N(V_0)}{N(V_i)} L_i,\]

which is equivalent to

\[\sum_{i=0}^{k+1} \frac{L_i}{N(V_i)} = \frac{K_0^2 V_0}{N(V_0)} + \frac{K_{k+1}^2 V_{k+1}}{N(V_{k+1})}.\]

\[\square\]

**Remark 6.1.9.** The equation in the previous lemma can be simplified if one assumes that \(X\) satisfies the properties of Proposition 3.6.2. If we define \(\alpha\) and \(\beta\) as in that proposition, and if we denote \(N = N(V_{\alpha})\), then we have the equality

\[\frac{L_0}{N(V_0)} + \sum_{i=0}^{\beta} \frac{L_i}{N} + \frac{L_{k+1}}{N(V_{k+1})} = \frac{K_0^2 V_0}{N(V_0)} + \frac{K_{k+1}^2 V_{k+1}}{N(V_{k+1})}.\]
6.2 Combinatorial countercandidates

In this section, we will discuss combinatorial countercandidates and a strategy to produce them all. We will start by explaining the concept of combinatorial countercandidates. In Subsection 6.2.2 we produce all combinatorial countercandidates for a specific kind of chain degenerations. Then we will explain how to generalize this strategy to all chain degenerations in Subsection 6.2.3. In the final subsection, we will use this strategy to prove that certain \(K3\) surfaces with a chain degeneration satisfy the monodromy property.

6.2.1 Combinatorial countercandidates

Although the concept of a combinatorial countercandidate is not mathematically precise, we will explain this notion in more detail. To do so, we first need the definition of a strict normal crossings surface.

**Definition 6.2.1.** A \(k\)-surface \(D\) is called a strict normal crossings surface, if for every closed point \(p \in D\), we have \(\hat{O}_{D,p} \simeq k[[x, y, z]]/(x^ay^bz^c)\) for some non-negative integers \(a, b, c \in \mathbb{Z}_{\geq 0}\) and \(a \neq 0\), and if moreover for every irreducible component \(E\) of \(D\), the reduction \(E_{\text{red}}\) is smooth.

If for every closed point \(p \in D\), we have \(\hat{O}_{D,p} \simeq k[[x, y, z]]/(x^ay^b)\) for some integers \(a, b \in \mathbb{Z}_{\geq 0}\) with \(a \neq 0\), then we say \(D\) is triple-point-free.

We write \(D = \sum_{i \in I} N_i E_i\), where \(E_i\) are the reduced irreducible components of \(D\) and where \(N_i\) is the multiplicity of \(E_i\).

We define the *index* of a strict normal crossings surface \(D\) as \(\iota(D) = \gcd_{i \in I} N_i\).

Note that, when \(X\) is a \(K\)-variety with an \(snc\)-model \(X\), then \(\iota(X) = \iota(X_k)\), by Proposition 6.1.2. We also define the dual graph of a triple-point-free, strict normal crossings surface in the obvious way.

A *combinatorial countercandidate* is a combinatorial description of a triple-point-free, strict normal crossings surface (for example with a given dual graph, multiplicities of the components, some geometrical properties of the components, . . .) such that

- if there exists a strict normal crossings surface \(D\) with the described properties, and if there exists a \(K3\) surface \(X\) admitting an \(snc\)-model \(X\) such that \(X_k \simeq D\) (i.e., if \(D\) can be smoothened to a \(K3\) surface \(X\)), then \(X\) does not satisfy the monodromy property,
• we don’t know whether there exists a $K3$ surface $X$ over $K$ with an $snc$-model $\mathcal{X}$ such that the special fiber $\mathcal{X}_k$ satisfies the combinatorial characterization, i.e., there is no reason (yet) to rule it out.

We think of a combinatorial countercandidate as a description of a candidate to produce a $K3$ surface not satisfying the monodromy property. As soon as we find a reason why this candidate will not produce a counterexample of the monodromy property, it is obviously not a candidate anymore. This explains the non-precise nature of the second requirement.

If there exists a $K3$ surface over $K$ allowing a triple-point-free model that does not satisfy the monodromy property, then it has a Crauder-Morrison model. Therefore, we are mainly interested in combinatorial countercandidates that satisfy the properties of the Crauder-Morrison classification. For this reason, we can use the terminology from this classification in the setting of combinatorial countercandidates as well. We will for example talk about chains, flowerpots and flowers in this context. Moreover, from Theorem 5.2.1 it is clear that combinatorial countercandidates are chain degenerations.

Note that by Lemma 6.1.5 we are mainly interested in combinatorial countercandidates $D$ of index 1. Indeed, if we can exclude all combinatorial countercandidates of index 1, then we have proven that the monodromy property holds for all $K3$ surface with a triple-point-free model, also for those of index 2.

From the results in this section and in Appendix B we will see that we have a list of 63 combinatorial countercandidates. If there exists a $K3$ surface $X$ allowing a triple-point-free model, not satisfying the monodromy property, then the special fiber of its Crauder-Morrison model has the properties of one of these 63 combinatorial countercandidates.

6.2.2 Constructing combinatorial countercandidates: example

Suppose there exists a $K3$ surface $X$ over $K$ of index $\iota(X) = 1$ allowing a triple-point-free model that does not satisfy the monodromy property. Let $\mathcal{X}$ be a Crauder-Morrison model of $X$. By Theorem 5.2.1 we have that $\mathcal{X}$ is a chain degeneration. Let $V_0, V_1, \ldots, V_k, V_{k+1}$ be the components in the chain, such that $V_i \cap V_j = \emptyset$ if and only if $j \not\in \{i - 1, i, i + 1\}$. Because of Theorem 5.3.1(ii), we can assume that $V_0$ is a rational, ruled surface.

In this subsection, we will construct combinatorial countercandidates where the chain has the following properties: $V_{k+1}$ is a rational, ruled surface and there
exists an integer $N \geq 1$ such that

$$N(V_i) = \begin{cases} 
2N & \text{if } i = 0, \\
N & \text{otherwise}.
\end{cases}$$

Let $\phi$ be the total number of flowers of type $2B$ and $2C$ meeting $V_1, \ldots, V_{k+1}$, and let $\gamma$ be the total number of flowers of type $2A$ meeting $V_1, \ldots, V_{k+1}$. We have that $\phi \geq 1$, by Theorem 5.3.1 (iv). Moreover, let $\phi'$ be the total number of flowers of type $2B$ and $2C$ meeting $V_0$, and let $\gamma'$ be the total number of flowers of type $2A$ meeting $V_0$.

Define $I^\dagger \subset I$ to be the set of indices $i \in I$, where either $\rho_i$ is minimal, or $E_i$ is the top of a conic-flower. Let $S^\dagger = \{(-\nu_i, N_i) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \mid i \in I^\dagger\}$. Corollary 4.2.4 gives that

$$Z_{X, \omega}(T) \in \mathcal{M}_k^\dagger \left[T, \frac{1}{1 - L a T b}\right] \mid_{(a,b) \in S^\dagger}.$$ 

By Theorem 5.1.2, we know that $\xi(E_i)$ is a monodromy eigenvalue, if $\rho_i$ is minimal. Therefore, because $X$ does not satisfy the monodromy property, there must be a conic-flower $F$ with top $F_0$, such that $\xi(F_0)$ is not a monodromy eigenvalue. By Proposition 3.6.1, the flower $F$ is of type $2B$ or $2C$.

Because of Lemma 3.7.1, the monodromy zeta function is given by

$$\zeta_X(T) = \frac{(T^{2N} - 1)^{-\chi(V_0^c)}}{(T^{4N} - 1)^{\phi'}(T^{2N} - 1)^{\phi}(T^N - 1)^{\gamma'} + \sum_{i=1}^{k+1} \chi(V_i^c) (T^{N/2} - 1)^{\gamma'}}. \tag{6.4}$$

Suppose $F$ meets $V_0$. Then $N(F_0) = 4N$, and by Lemma 5.1.3, we have that $\xi(F_0)$ is not a $2N$-th root of unity. We see from (6.4) that $\xi(F_0)$ is a pole of $\zeta_X(T)$, and hence a monodromy eigenvalue, contrary to our assumption.

This means that $F$ meets one of the components $V_1, \ldots, V_{k+1}$ and $N(F_0) = 2N$. Since $\xi(F_0)$ is not a monodromy eigenvalue, it is neither a zero nor a pole of $\zeta_X(T)$. By Lemma 5.1.3, we know that $\xi(F_0)$ is a $2N$-th root of unity, but not an $N$-th root of unity, and hence the factor $(T^{2N} - 1)$ must cancel completely in $\zeta_X(T)$. Therefore we must have $-\chi(V_0^c) = \phi + \phi'$. Since $\chi(V_0^c) = 4 + L_0 - 2\phi' - 2\gamma'$, we get

$$\phi' + 2\gamma' - \phi - L_0 = 4. \tag{6.5}$$

Therefore, equation (6.4) can be rewritten as

$$\zeta_X(T) = \frac{1}{(T^{2N} + 1)^{\phi'}(T^N - 1)^{\gamma'} + \sum_{i=1}^{k+1} \chi(V_i^c) (T^{N/2} - 1)^{\gamma'}}.$$
Because $X$ has index $\iota(X) = 1$, Proposition 6.1.2 gives that $\gcd_{i \in I} N_i = 1$. This means that

$$N = \begin{cases} 1 & \text{if } \gamma = 0, \\ 2 & \text{if } \gamma > 0. \end{cases}$$

Proposition 5.1.1 gives that

$$\sum_{i=1}^{k+1} \chi(V_i) + 2\phi' + \gamma' + \frac{\gamma}{2} = \frac{24}{N}.$$  

In particular, $\gamma$ is even.

Since $\sum_{i=1}^{k+1} \chi(V_i) = 4 + \sum_{i=1}^{k+1} L_i - 2\phi - 2\gamma$, we get

$$\sum_{i=1}^{k+1} L_i + 2\phi' + \gamma' - 2\phi - \frac{3\gamma}{2} = \frac{24}{N} - 4.$$  

(6.6)

Lemma 6.1.8 implies that

$$L_0 + 2 \sum_{i=1}^{k+1} L_i = 24.$$  

(6.7)

Note that, since $V_0$ is a non-minimal ruled surface by Theorem 5.3.1, we have that $L_0 \geq 1$.

The variables $\phi, \gamma, \phi', \gamma', N, L_0$ and $\sum_{i=1}^{k+1} L_i$ are non-negative integers, and therefore, we have obvious lower bounds for all of these variables. We will now also derive upper bounds. Equation (6.7) immediately gives upper bounds for $L_0$ and $\sum_{i=1}^{k+1} L_i$.

From Lemma 6.1.6 it follows that

$$\phi' + \gamma' \leq L_0 + 1.$$  

Moreover, Lemma 3.6.7 combined with Lemma 6.1.6 gives that

$$\phi + \gamma \leq \left( \sum_{i=1}^{k} L_i \right)/2 + (L_{k+1} + 1),$$  

and therefore

$$\phi + \gamma \leq \sum_{i=1}^{k+1} L_i + 1.$$
As a consequence, finding combinatorial countercandidates with this particular chain, comes down to finding non-negative integer solutions of

\[
\begin{align*}
\phi' + 2\gamma' - \phi - L_0 &= 4, \\
\sum_{i=1}^{k+1} L_i + 2\phi' + \gamma' - 2\phi - \frac{3\gamma}{2} &= \frac{24}{N} - 4, \\
L_0 + 2\sum_{i=1}^{k+1} L_i &= 24, \\
\phi' + \gamma' &\leq L_0 + 1, \\
\phi + \gamma &\leq \sum_{i=1}^{k+1} L_i + 1, \\
\phi &\geq 1, \\
L_0 &\geq 1, \\
N &= \begin{cases} 
1 & \text{if } \gamma = 0, \\
2 & \text{if } \gamma > 0.
\end{cases}
\end{align*}
\]

Since all variables $N, \phi, \gamma, \phi', \gamma', L_0, \sum_{i=1}^{k+1} L_i$ are bounded, it is easy to let a computer check every possibility whether it satisfies these relations. We have implemented this in Python and the code and output can be found in section B.1 of the appendix. The code can also be downloaded from www.github.com/AnneliesJaspers/combinatorial-counterexamples.

In this way, we find 68 combinatorial countercandidates (modulo the length $k + 1$ of the chain), of which we will now give two as an illustration. In Section 6.4, we will exclude 26 of these combinatorial countercandidates, including Combinatorial Countercandidate 6.2.2.

**Combinatorial Countercandidate 6.2.2.** Let $D$ be a strict normal crossings surface with the following dual graph, where the labels denote the multiplicity of the components:

![Image of graph]

The components $V_0$ and $V_1$ are rational, ruled surfaces. Moreover, if we define $L_i$ as the number of blow-ups in the birational morphism $V_i \to \overline{V}_i$ to the minimal ruled surface $\overline{V}_i$ for $i = 0, 1$, then $L_0 = 6$ and $L_1 = 9$. All other ten components are isomorphic to $\mathbb{P}^2$.

The six flowers meeting $V_0$ are all of type 2A. There are two flowers of type 2A meeting $V_1$ and two flowers of type 2B. Note that the surface $V_0$ exists as Example 6.1.7 shows.
Suppose there exists a $K3$ surface $X$ over $K$ with an $snc$-model $\mathcal{X}$ such that $\mathcal{X}_k \simeq D$. Let $\omega$ be a volume form on $V_0$. Then the monodromy property does not hold for $X$. Indeed, Theorem 4.3.8 guarantees that $-\nu_0/4$ is a pole of the motivic zeta function of $X$, where $(4, \nu_0)$ is the numerical data of $F_0$, a conic-flower meeting $V_1$ with $N(F_0) = 4$.

On the other hand, the monodromy zeta function is

$$
\zeta_X(T) = \frac{(T^4 - 1)^2}{(T^2 - 1)^5(T^2 - 1)^6(T^4 - 1)^2(T-1)^2}
$$

because

$$
\chi(V_0^\circ) = 4 + L_0 - 2\lambda_0 = -2,
\chi(V_1^\circ) = 4 + L_1 - 2\lambda_1 = 5,
$$

where $\lambda_i$ is the number of flowers on $V_i$ for $i = 0, 1$. Since $\xi(F_0) = \exp(\pi i \nu_0/2)$ is not a 2-th root of unity by Lemma 5.1.3, we conclude that $\xi(F_0)$ is not a monodromy eigenvalue. Therefore, $X$ does not satisfy the monodromy property.

It is important to note that, a priori, we don’t know whether such a strict normal crossings surface $D$ exists. And if such a strict normal crossings surface exist, we also don’t know if $D$ can be smoothened to a $K3$ surface $X$. However, we will show in Section 6.4, that there does not exist such a $K3$ surface $X$.

**Combinatorial Countercandidate 6.2.3.** Let $D$ be a strict normal crossings surface with the following dual graph, where the labels denote the multiplicity of the components:

[Diagram of a dual graph with vertices and edges labeled with multiplicities, showing components $V_0$, $V_1$, and $F_0$.]

The components $V_0$ and $V_1$ are rational, ruled surfaces. Moreover, if we define $L_i$ as the number of blow-ups in the birational morphism $V_i \rightarrow \overline{V}_i$ to the minimal ruled surface $\overline{V}_i$ for $i = 0, 1$, then $L_0 = 10$ and $L_1 = 7$. All other eleven components are isomorphic to $\mathbb{P}^2$.

There are five flowers of type $2A$ meeting $V_0$, and five of type $2B$. The flower meeting $V_1$ is of type $2B$, and we call the unique component of this flower $F_0$. 
Suppose there exists a $K3$ surface $X$ over $K$ with an snc-model $\mathcal{X}$ such that $\mathcal{X}_k \simeq D$. Let $\omega$ be a volume form on $X$. Then the monodromy property does not hold for $X$. Indeed, Theorem 4.3.8 guarantees that $-\nu_0/2$ is a pole of the motivic zeta function of $X$, where $(2, \nu_0)$ is the numerical data of $F_0$.

On the other hand, the monodromy zeta function is

$$\zeta_X(T) = \frac{(T^2 - 1)^6}{(T - 1)^9(T^4 - 1)^5(T - 1)^5(T^2 - 1)} = \frac{1}{(T^2 + 1)^5(T - 1)^{14}},$$

because

$$\chi(V_0^\circ) = 4 + L_0 - 2\lambda_0 = -6,$$

$$\chi(V_1^\circ) = 4 + L_1 - 2\lambda_1 = 9,$$

where $\lambda_i$ is the number of flowers on $V_i$ for $i = 0, 1$. Since $\xi(F_0)$ is a primitive 2-th root of unity by Lemma 5.1.3, we conclude that $\xi(F_0)$ is not a monodromy eigenvalue. Therefore, $X$ does not satisfy the monodromy property.

It is important to note that, a priori, we don’t know whether such a strict normal crossings surface $D$ exists. Actually, A. Höring suggested a strategy which seems to rule out the existence of $V_0$, and therefore excludes this particular combinatorial countercandidate. Even if such a strict normal crossings surface $D$ existed, we would still need to prove that it can be smoothened to a $K3$ surface $X$. In Section 6.3, we will give a strategy of how one might try to smoothen such a strict normal crossings surface to a $K3$ surface.

### 6.2.3 Constructing combinatorial countercandidates: strategy

Suppose $X$ is a $K3$ surface over $K$ of index $\iota(X) = 1$ with a Crauder-Morrison model $\mathcal{X}$, not satisfying the monodromy property. Theorem 5.2.1 implies that $\mathcal{X}$ is a chain degeneration. Let $V_0, \ldots, V_{\alpha-1}, V_\alpha, \ldots, V_\beta, V_{\beta+1}, \ldots, V_{k+1}$ be the components in the chain, where $\alpha$ and $\beta$ are defined as in Proposition 3.6.2 and $V_i \cap V_j = \emptyset$, except when $j \in \{i - 1, i, i + 1\}$. Set $N = N(V_\alpha)$.

As $X$ does not satisfy the monodromy property, Theorem 5.3.1 and Proposition 3.6.2 (iv) imply that (possibly after renumbering) the component $V_0$ is a rational, non-minimal ruled surface, that $N(V_0) = 2N$, and that $\alpha = 1$. Moreover, Theorem 5.3.1 (iv) also implies that there exists a conic-flower $F$
meeting one of the components $V_1, \ldots, V_\beta$. Notice that this flower must be of type $2B$ or $2C$, because of Theorem 3.6.1

From Table 3.5 in Proposition 3.6.2, we can deduce that one of the following ten cases hold:

1. $\beta = k + 1$, and $V_{k+1}$ is a rational, ruled surface,
2. $\beta = k + 1$, and $V_{k+1}$ is an elliptic, ruled surface,
3. $\beta = k + 1$, and $V_{k+1} \simeq \mathbb{P}^2$,
4. $V_{k+1} \simeq \mathbb{P}^2$, and $N(V_{k+1}) = 3N$,
5. $V_{k+1} \simeq \mathbb{P}^2$, and $N(V_{k+1}) = \frac{3}{2}N$,
6. $V_{k+1} \simeq \Sigma_2$, and $N(V_{k+1}) = 4N$,
7. $V_{k+1} \simeq \Sigma_2$, $\beta = k - 1$, and $N(V_{k+1}) = 2N$,
8. $V_{k+1} \simeq \Sigma_2$, and $N(V_{k+1}) = \frac{4}{3}N$,
9. $V_{k+1}$ is a rational, ruled surface, $\beta = k$, and $N(V_{k+1}) = 2N$,
10. $V_{k+1}$ is an elliptic, ruled surface, and $N(V_{k+1}) = 2N$.

Define $\phi$ to be the number of conic-flowers meeting $V_1, \ldots, V_\beta$, and define $\gamma$ to be the number of flowers of type $2A$ meeting $V_1, \ldots, V_\beta$. We have that $\phi \geq 1$, because of the conic-flower $F$ meeting one of the components $V_1, \ldots, V_\beta$.

Define $\phi'$ and $\gamma'$ as follows:

- in cases 9 and 10, $\phi'$ is defined as the number of conic-flowers meeting $V_0$ and $V_{k+1}$, and $\gamma'$ is defined as the number of flowers of type $2A$ meeting $V_0$ and $V_{k+1}$.
- otherwise, we define $\phi'$ as the number of conic-flowers meeting $V_0$, and $\gamma'$ as the number of flowers of type $2A$ meeting $V_0$.

If $V_{k+1}$ is an elliptic, ruled surface (cases 2 and 10), then either there is exactly one non-rational flower of type $4\alpha$, or there are two non-rational flowers of type $4\alpha$, as seen in Proposition 3.6.1. In the latter case, both flowers are elliptic. If there is exactly one non-rational flower of type $4\alpha$, we denote by $g$ its genus. If there are two elliptic flowers of type $4\alpha$, we set $g = 1$.

In cases 1 to 8, we will consider the variables $L_0$ and $\sum_{i=1}^\beta L_i$. Note that in cases 3 to 8, we have $L_{k+1} = 0$. In cases 9 and 10, we will consider the variables $L_0 + L_{k+1}$ and $\sum_{i=1}^\beta L_i$.

This means that we will work with the variables $\phi, \gamma, \phi', \gamma', N, \sum_{i=1}^\beta L_i$, and $L_0$ or $L_0 + L_{k+1}$. In cases 2 and 10, we also consider the variable $g$. All these
variables are non-negative integers. We will describe how to find three equalities, similar to equations (6.5), (6.6) and (6.7), that need to be satisfied in order to find a combinatorial countercandidate. We will also describe how to deduce upper bounds for all of these variables.

Define $I^\dagger \subset I$ to be the set of indices $i \in I$, where either $\rho_i$ is minimal, or $E_i$ is the top of a conic-flower. Set $S^\dagger = \{(-\nu_i, N_i) \in \mathbb{Z} \times \mathbb{Z}_{>0} \mid i \in I^\dagger\}$. Corollary 4.2.4 says that $Z_{X, \omega}(T) \in M_k^\dagger \left[T, \frac{1}{1 - L_0 T^0}\right]_{(a,b) \in S^\dagger}$.

By Theorem 5.1.2, we know that $\xi(E_i)$ is a monodromy eigenvalue, if $\rho_i$ is minimal. Therefore, since $X$ does not satisfy the monodromy property, there must be a conic-flower $F$ such that $\xi(F_0)$ is not a monodromy eigenvalue. By Proposition 3.6.1, the flower $F$ is of type $2B$ or $2C$. An argument similar as in the proof of Theorem 5.3.1 (iii), gives that $F$ meets the chain in one of the components $V_1, \ldots, V_\beta$, and therefore $N(F_0) = 2N$.

Recall that we have the following formula for the monodromy zeta-function:

$$
\zeta_X(T) = \prod_{i \in I} (T^{N_i} - 1)^{-\chi(E_i^\circ)}.
$$

Using Lemma 3.7.1, we can compute all relevant $\chi(E_i^\circ)$ in terms of the variables $\phi, \gamma, \phi', \gamma', N, \sum_{i=1}^\beta L_i, g$, and $L_0$ or $L_0 + L_{k+1}$. For example, in all cases, except for case 2, we have

$$
\sum_{i=1}^\beta \chi(V_i^\circ) = \sum_{i=1}^\beta L_i - 2\phi - 2\gamma.
$$

Since $\xi(F_0)$ is not a monodromy eigenvalue, it is not a zero nor a pole of $\zeta_X(T)$. Lemma 5.1.3 gives that $\xi(F_0)$ is not an $N$-th root of unity, and it is not difficult to see that the factor $(T^{2N} - 1)$ must cancel completely, by writing down the monodromy zeta function explicitly in all ten cases. This induces the first equality. Moreover, the monodromy zeta function must be of the form $\frac{1}{Q(T)}$, with $Q(T)$ a polynomial of degree 24. Computing the degree of the monodromy zeta function gives the second equality. The third equality is given by Lemma 6.1.8.

Since all variables $\phi, \gamma, \phi', \gamma', N, g, \sum_{i=1}^\beta L_i$, and $L_0$ or $L_0 + L_{k+1}$ are non-negative integers, we have obvious lower bounds. We can deduce upper bounds for all variables as well. Lemma 6.1.8 gives an upper bound for $\sum_{i=1}^\beta L_i$, and $L_0$ or $L_0 + L_{k+1}$. Lemma 3.6.7 and Lemma 6.1.6 give upper bounds for $\phi, \gamma, \phi', \gamma'$. Since we require the index $\iota(X)$ to be 1, the gcd of the multiplicities
of the components in the special fiber must be 1, which gives information on the variable $N$. Finally, also the variable $g$ in cases 2 and 10, is bounded, because for $g$ too big, it is not possible for the monodromy zeta function to have degree 24.

Because of the upper and lower bounds on integer variables, one can verify for every possibility whether the equalities described above are satisfied. This results in a finite list of combinatorial countercandidates. All 63 combinatorial countercandidates are listed in Appendix B, where we omitted those that will be excluded in Section 6.4.

6.2.4 Partial results

The strategy described in Subsection 6.2.3 can also be used to prove the monodromy property for certain chain degenerations. We will see that there are no combinatorial countercandidates in cases 6 to 10, which implies that the monodromy property holds for $K3$ surfaces of index 1 with such chain degenerations.

**Proposition 6.2.4.** Let $X$ be a $K3$ surface over $K$ of index $\iota(X) = 1$, with Crauder-Morrison model $X$ and with special fiber $X_k = \sum_{i \in I} N_i E_i$. Suppose $X$ is a chain degeneration satisfying Proposition 3.6.2. Let $V_0, V_1, \ldots, V_k, V_{k+1}$ be the components in the chain, where $V_i \cap V_j = \emptyset$, except when $j \in \{i-1, i, i+1\}$. Suppose there exists an integer $N \geq 1$, such that

$$N(V_i) = \begin{cases} 
2N & \text{if } i = 0 \text{ or } k + 1, \\
N & \text{otherwise}.
\end{cases}$$

Then $X$ satisfies the monodromy property.

**Proof.** Let $X$ be a $K3$ surface over $K$ of index $\iota(X) = 1$. Suppose $X$ admits a Crauder-Morrison model $X$ with special fiber $X_k = \sum_{i \in I} N_i E_i$ as described in the statement of the proposition. Suppose $X$ does not satisfy the monodromy property.

From Theorem 5.3.1 (ii), it follows that $V_0$ and/or $V_{k+1}$ is a rational, non-minimal ruled surface. Without loss of generality, we can assume $V_0$ is rational, non-minimal ruled. Because $N(V_{k+1}) = 2N$ and $\beta = k$, Proposition 3.6.2 gives that $V_{k+1}$ is a rational or elliptic, ruled surface.

Let $\phi$ be the total number of conic-flowers meeting $V_1, \ldots, V_k$, and let $\gamma$ be the total number of flowers of type $2A$ meeting $V_1, \ldots, V_k$. We have that $\phi \geq 1$, by Theorem 5.3.1 (iv). Moreover, let $\phi'$ be the total number of conic-flowers meeting $V_0$ and $V_{k+1}$, and let $\gamma'$ be the total number of flowers of type $2A$ meeting $V_0$ and $V_{k+1}$.
Defne \( I^\dagger \subset I \) to be the set of indices \( i \in I \), where either \( \rho_i \) is minimal, or \( E_i \) is the top of a conic-flower. Set \( S^\dagger = \{ (-\nu_i, N_i) \in \mathbb{Z} \times \mathbb{Z}_{>0} \mid i \in I^\dagger \} \). Corollary 4.2.4 says that
\[
Z_{X, \omega}(T) \in \mathcal{M}_k^\dagger \left[ T, \frac{1}{1 - T^a T^b} \right]_{(a, b) \in S^\dagger}.
\]

By Theorem 5.1.2, we know that \( \xi(E_i) \) is a monodromy eigenvalue, when \( \rho_i \) is minimal. Therefore, there must be a conic-flower \( F \) with top \( F_0 \), such that \( \xi(F_0) \) is not a monodromy eigenvalue, since \( X \) does not satisfy the monodromy property. By Proposition 3.6.1, the flower \( F \) is of type 2B or 2C.

Suppose first that \( V_{k+1} \) is an elliptic, ruled surface. Proposition 3.6.1 implies that either there is exactly one non-rational flower, which is of type 4\( \alpha \), or there are exactly two non-rational flowers, both elliptic and of type 4\( \alpha \). In the first case, set \( g \) to be the genus of this unique non-rational flower, and in the latter case, set \( g = 1 \). We then have
\[
\chi(V_{k+1}^\circ) = L_{k+1} - (2 - 2g) - 2\lambda,
\]
where \( \lambda \) is the number of rational flowers meeting \( V_{k+1} \).

Because of Lemma 3.7.1, the monodromy zeta function is given by
\[
\zeta_X(T) = \frac{(T^{2N} - 1)^{-\chi(V_0^\circ) - \chi(V_{k+1}^\circ)}(T^N - 1)^{2g - 2}}{(T^{4N} - 1)\phi'(T^{2N} - 1)\phi(T^N - 1)\sum_{i=1}^k \chi(V_i^\circ) + \gamma'(T^{N/2} - 1)^\gamma}. \tag{6.8}
\]

Suppose \( F \) meets \( V_0 \) or \( V_{k+1} \). Then \( N(F_0) = 4N \), and by Lemma 5.1.3, we have that \( \xi(F_0) \) is not a \( 2N \)-th root of unity. Therefore, \( \xi(F_0) \) is a pole of \( \zeta_X(T) \), and hence a monodromy eigenvalue, contrary to our assumption.

This means that \( F \) meets one of the components \( V_1, \ldots, V_k \), and \( N(F_0) = 2N \). Since \( \xi(F_0) \) is not a monodromy eigenvalue, it is not a pole nor a zero of \( \zeta_X(T) \). By Lemma 5.1.3, we have that \( \xi(F_0) \) is not an \( N \)-th root of unity, and therefore the factor \( (T^{2N} - 1) \) must cancel completely in \( \zeta_X(T) \). This means that
\[
-\chi(V_0^\circ) - \chi(V_{k+1}^\circ) = \phi + \phi'.
\]
Since \( \chi(V_0^\circ) + \chi(V_{k+1}^\circ) = 2 + L_0 + L_{k+1} - 2\phi' - 2\gamma' + 2g \), we get
\[
\phi' + 2\gamma' - \phi - 2g - L_0 - L_{k+1} = 2. \tag{6.9}
\]
Moreover, equation (6.8) can be written as
\[
\zeta_X(T) = \frac{1}{(T^{2N} + 1)^\phi'(T^N - 1)^\gamma'\sum_{i=1}^k \chi(V_i^\circ) + \gamma' - 2g + 2(T^{N/2} - 1)^\gamma}.
\]
Proposition 5.1.1 gives that
\[ 2\phi' + \sum_{i=1}^{k} \chi(V_i^\circ) + \gamma' - 2g + 2 + \frac{\gamma}{2} = \frac{24}{N}. \]

Since \( \sum_{i=1}^{k} \chi(V_i^\circ) = \sum_{i=1}^{k} L_i - 2\phi - 2\gamma \), we get
\[ \sum_{i=1}^{k} L_i + 2\phi' + \gamma' - 2g - 2\phi - \frac{3\gamma}{2} = \frac{24}{N} - 2. \]  \hspace{1cm} (6.10)

Furthermore, Lemma 6.1.8 implies that
\[ L_0 + 2 \sum_{i=1}^{k} L_i + L_{k+1} = 8. \]  \hspace{1cm} (6.11)

We will now derive upper bounds for the variables \( \phi, \gamma, \phi', \gamma', g, N, L_0 + L_{k+1} \) and \( \sum_{i=1}^{k} L_i \). First of all, (6.11) gives obvious upper bounds for \( L_0 + L_{k+1} \) and \( \sum_{i=1}^{k} L_i \).

Combining Lemma 6.1.6 and Corollary 3.6.7 we get that
\[ \phi' + \gamma' \leq L_0 + L_{k+1} + 1. \]  \hspace{1cm} (6.12)

Finally, Corollary 3.6.7 implies that
\[ 2\phi + 2\gamma \leq \sum_{i=1}^{k} L_i. \]  \hspace{1cm} (6.13)

We also have a bound on \( g \), because by Lemma 5.1.1, we know that
\[ \sum_{i=1}^{k} \chi(V_i^\circ) + \gamma' - 2g + 2 \geq 0, \]
and therefore
\[ g \leq \left( \sum_{i=1}^{k} L_i - 2\phi - 2\gamma + \gamma' + 2 \right)/2. \]  \hspace{1cm} (6.14)

Because \( X \) has index \( \iota(X) = 1 \), Proposition 6.1.2 gives that \( \gcd_{i \in I} N_i = 1 \). It is not difficult to see that
\[ \gcd_{i \in I} N_i = \begin{cases} N & \text{if } \gamma = 0, \\ N/2 & \text{if } \gamma > 0, \end{cases} \]
and hence
\[ N = \begin{cases} 
1 & \text{if } \gamma = 0, \\
2 & \text{if } \gamma > 0.
\end{cases} \tag{6.15} \]

One can verify that there are no integer solutions of equations (6.9), (6.10) and (6.11) bounded by (6.12) to (6.15). This is a contradiction, and we conclude that \( X \) satisfies the monodromy property. We have implemented this in Python, and the code can be found in Section B.10 of the appendix. The code can also be downloaded from www.github.com/AnneliesJaspers/combinatorial-counterexamples.

Suppose now that \( V_{k+1} \) is a rational, ruled surface. A similar strategy gives the following equations:

\[
\begin{align*}
\phi' + 2\gamma' - \phi - (L_0 + L_{k+1}) &= 8, \\
\sum_{i=1}^{k} L_i + 2\phi' + \gamma' - 2\phi - \frac{3\gamma}{2} &= \frac{24}{N}, \\
L_0 + 2\sum_{i=1}^{k} L_i + L_{k+1} &= 16, \\
\phi' + \gamma' &\leq L_0 + L_{k+1} + 2, \\
2\phi + 2\gamma &\leq \sum_{i=1}^{k} L_i, \\
N &= \begin{cases} 
1 & \text{if } \gamma = 0, \\
2 & \text{if } \gamma > 0.
\end{cases}
\end{align*}
\]

One can verify that there are no non-negative integer solutions satisfying these relations. This is a contradiction, and we conclude that \( X \) satisfies the monodromy property. We have implemented this in Python, and the code can be found in Section B.9 of the appendix. The code can also be downloaded from www.github.com/AnneliesJaspers/combinatorial-counterexamples.

**Proposition 6.2.5.** Let \( X \) be a \( K3 \) surface over \( K \) of index \( \iota(X) = 1 \), with Crauder-Morrison model \( X \) and with special fiber \( X_k = \sum_{i \in I} N_i E_i \). Suppose \( X \) is a chain degeneration. Let \( V_0, V_1, \ldots, V_k, V_{k+1} \) be the components in the chain, where \( V_i \cap V_j = \emptyset \), except when \( j \in \{i - 1, i, i + 1\} \). Suppose there exists an integer \( N \geq 1 \), such that

\[ N(V_i) = \begin{cases} 
2N & \text{if } i = 0, \\
N & \text{otherwise}.
\end{cases} \]

Assume moreover that \( V_0 \) is a rational, ruled surface, and that \( V_{k+1} \) is an elliptic, ruled surface. If \( X_k \not\cong D \), where \( D \) is the strict normal crossings surface described in Combinatorial Countercandidate 6.2.6, then \( X \) satisfies the monodromy property.
Combinatorial Countercandidate 6.2.6. Let $D$ be a strict normal crossings surface as follows: it is a chain degeneration with components $V_0, V_1, \ldots, V_k, V_{k+1}$, where $V_i \cap V_j = \emptyset$, except when $j \in \{i-1, i, i+1\}$. We have $N(V_0) = 4$, and $N(V_i) = 2$, for $i = 1, \ldots, k+1$.

There is a unique flower of type $2B$ or $2C$ meeting $V_0$, and four flowers of type $2A$ are meeting $V_0$. Moreover, there is a unique flower of type $2B$ or $2C$ meeting one of the components $V_1, \ldots, V_{k+1}$, and denote the top of this flower by $F_0$. Finally, there is a unique flower of type $4\alpha$ meeting $V_{k+1}$, and it has genus 7.

Furthermore, we have $L_0 = 4$ and $\sum_{i=1}^{k+1} L_i = 2$.

For $k = 0$ and when there are no flowers of type $2C$, the dual graph of $D$ is given in Figure 6.7.

![Figure 6.7](image-url)

Suppose there exists a $K3$ surface $X$ over $K$ with an $snc$-model $X$ such that $\mathcal{X}_k \simeq D$. Let $\omega$ be a volume form on $X$. Then $X$ does not satisfy the monodromy property. Indeed, Theorem 4.3.8 guarantees that $-\nu_0/4$ is a pole of the motivic zeta function of $X$, where $(4, \nu_0)$ is the numerical data of $F_0$.

Since $\chi(V'^{\circ}_0) = 4 + 4 - 2 \cdot (1 + 4) = -2$, and $\sum_{i=1}^{k+1} \chi(V_i^{\circ}) = 2 - 2 + (2 \cdot 7 - 2) = 12$, Lemma 3.7.1 gives that

$$\zeta_X(T) = \frac{(T^4 - 1)^2(T - 1)^{12}}{(T^2 - 1)^{12}(T^8 - 1)(T^2 - 1)^4(T^4 - 1)} = \frac{1}{(T^4 + 1)(T^2 - 1)^4(T + 1)^{12}}.$$

Since $\xi(F_0)$ is a primitive 4-th root of unity by Lemma 5.1.3, we conclude that $\xi(F_0)$ is not a monodromy eigenvalue. Therefore, $X$ does not satisfy the monodromy property.

It is important to note that, a priori, we don’t know whether such a strict normal crossings surface $D$ exists. And even if such a strict normal crossings surface $D$ exists, we don’t know whether it can be smoothened to a $K3$ surface $X$. 


Proof of Proposition 6.2.5. Let $X$ be a $K3$ surface over $K$ of index $i(X) = 1$. Suppose $X$ admits a Crauder-Morrison model $\mathcal{X}$ with special fiber $\mathcal{X}_k = \sum_{i \in I} N_i E_i$ as described in the statement of the proposition. Suppose $X$ does not satisfy the monodromy property.

Let $\phi$ be the total number of conic-flowers meeting $V_1, \ldots, V_{k+1}$, and let $\gamma$ be the total number of flowers of type 2A meeting $V_1, \ldots, V_{k+1}$. Moreover, let $\phi'$ be the total number of flowers of type 2B and 2C meeting $V_0$, and let $\gamma'$ be the total number of flowers of type 2A meeting $V_0$.

Following the strategy described in Subsection 6.2.3, we find the following equalities

\[
\begin{aligned}
\phi' + 2\gamma' - \phi - L_0 &= 4, \\
4\phi' + 2\gamma' - 4\phi - 3\gamma + 2g + 2 \sum_{i=1}^{k+1} L_i &= 26, \\
L_0 + 2 \sum_{i=1}^{k+1} L_i &= 8, \\
2\phi + 2\gamma &\leq \sum_{i=1}^{k+1} L_i, \\
\phi' + \gamma' &\leq L_0 + 1.
\end{aligned}
\]

One can verify that there is a unique integer solution, which is $L_0 = 4, \sum_{i=1}^{k+1} L_i = 2, \phi = 1, \gamma = 0, \phi' = 1, \gamma' = 4, g = 7$. We have implemented this in Python, and the code can be found in Section B.2 of the appendix. The code can also be downloaded from www.github.com/AnneliesJaspers/combinatorial-counterexamples.

The following proposition shows that a $K3$ surface of index 1 with a chain described by the cases 6 to 8, satisfies the monodromy property.

**Proposition 6.2.7.** Let $X$ be a $K3$ surface over $K$ of index $i(X) = 1$, with Crauder-Morrison model $\mathcal{X}$ and with special fiber $\mathcal{X}_k = \sum_{i \in I} N_i E_i$. Suppose $\mathcal{X}$ is a chain degeneration. Let $V_0, V_1, \ldots, V_k, V_{k+1}$ be the components in the chain, where $V_i \cap V_j = \emptyset$, except when $j \in \{i-1, i, i+1\}$. Set $N = \min \{ N(V_i) \mid i = 0, \ldots, k+1 \}$. Suppose that $V_{k+1}$ is a rational, minimal ruled surface, and that $N(V_{k+1}) > N$. Then $X$ satisfies the monodromy property.

**Proof.** From Theorem 5.3.1 (ii) and (iii), we know that $V_0$ is a rational, non-minimal ruled surface, and $N(V_0) > N$. From Proposition 3.6.2 it follows that one of the following cases hold:

- $V_{k+1} \cong \Sigma_2$, and $N(V_{k+1}) = 4N$,
- $V_{k+1} \cong \Sigma_2$, $\beta = k - 1$, and $N(V_{k+1}) = 2N$,
- $V_{k+1} \cong \Sigma_2$, and $N(V_{k+1}) = \frac{4}{3} N$,
- $V_{k+1}$ is a rational, ruled surface, $\beta = k$, and $N(V_{k+1}) = 2N$. 



The result now follows from the Python code in Sections B.6, B.7, B.8, and B.9, where we have proven that there are no combinatorial countercandidates.

### 6.3 Existence of a counterexample: a strategy

In this section, we will describe a possible strategy to produce a \( K3 \) surface \( X \) that does not satisfy the monodromy property. The idea is to smoothen a combinatorial countercandidate to a \( K3 \) surface, which then will not satisfy the monodromy property. In the literature, there are several smoothening results for semi-stable normal crossings varieties to \( K3 \) surfaces, for example [Fri83, Theorem 5.10] and [Han15, Corollary 5.4.15]. However, we would like to smoothen a strict normal crossings variety that is not semi-stable. In this section, we will describe how it may be possible to use the results for semi-stable normal crossings varieties anyway.

**Step 1: Choose a combinatorial countercandidate.**

The aim is to prove that there exists a \( K3 \) surface admitting a model with the chosen combinatorial countercandidate as special fiber. So we keep one of the 63 combinatorial countercandidates in mind.

**Step 2: Check that there exists a strict normal crossings surface \( D_X \) as described by the combinatorial countercandidate.**

A priori, it is not clear whether there exists a strict normal crossings surface \( D_X \) as described by the combinatorial countercandidate. We need to verify the existence of the irreducible components, and whether they can be glued in the described way. Part of the difficulty is to check that on every irreducible component, the desired double curves exist.

For example, to check the existence of the surface \( V_0 \) in Combinatorial Countercandidate 6.2.2, we need to verify that there exists a rational, ruled surface, which is constructed by blowing up six points on a minimal ruled surface, and that allows seven disjoint, irreducible curves, where one curve is elliptic and numerically equivalent to the anticanonical bundle, and the other six curves are rational of self-intersection \(-2\). This surface exists by Example 6.1.7.

Proving the existence of the components is not at all trivial. For example, to prove that the component \( V_0 \) in Combinatorial Countercandidate 6.2.3 exists, one needs to verify the existence of a rational, ruled surface, which is constructed by blowing up ten points on a minimal ruled surface, with ten disjoint, rational curves of self-intersection \(-2\). A. Höring suggested a strategy to prove that such a surface does not exist, which would exclude this combinatorial countercandidate as a candidate to produce a counterexample of the monodromy property.
If the desired strict normal crossings surface $D_X$ exists, we write $D_X = \sum_{i \in I} N_i E_i$.

**Step 3: Compute the semi-stable strict normal crossings surface $D_Y$ obtained by normalizing the semi-stable reduction.**

Suppose there exists a $K3$ surface $X$ over $K$ with an snc-model $X$ such that the special fiber $X_k \simeq D_X$. Set $n = \text{lcm}_{i \in I} N_i$ and $R_n = R[\pi]/(\pi^n - t)$. Let $\mathcal{Y}$ be the normalization of $X \times_R R_n$. It is possible to compute some properties of the special fiber $Y_k$, using the results in Section 4.1.2 and general results of cyclic coverings, for example found in [BHPV95, Sections I.16 and I.17].

Note that we don’t know whether $Y$ actually exists, but it should be possible to compute some properties of the special fiber $Y_k$, if $Y$ exists. We can then ignore $Y$ and define $D_Y$ as a strict normal crossings surface that satisfies these properties. Similar as in Step 2, we should verify that such a strict normal crossings surface indeed exists.

**Step 4: Check the existence of the $\hat{\mu}$-action on $D_Y$.**

If $Y$ exists, there is a $\hat{\mu}$-action on $Y$, and therefore there should be a $\hat{\mu}$-action on $Y_k$ satisfying the properties found in the proofs of the propositions in Subsections 4.1.3 and 4.1.4.

Therefore, we need to verify that $D_Y$ allows a $\hat{\mu}$-action with the desired properties.

**Step 5: Compute the semi-stable strict normal crossings variety $D_Z$ obtained by $\hat{\mu}$-equivariantly resolving singularities.**

If $Y$ exists, Theorem 4.1.6 (i) gives the singular locus of $Y$. Let $Z$ be the resolution of singularities of $Y$, where the blow-ups are done in a $\hat{\mu}$-equivariant way. Note that we don’t know whether $Z$ actually exists, but it should be possible to compute some properties of the special fiber $Z_k$, if $Z$ exists. We can then ignore $Z$ and define $D_Z$ as a strict normal crossings surface equipped with a $\hat{\mu}$-action, that satisfies these properties.

Similar as in Step 2, we should verify that such a strict normal crossings surface $D_Z$ indeed exists.

**Step 6: Use results on smoothening semi-stable normal crossings varieties to get an algebraic space $\widetilde{X}$ over $R_n$.** This needs to be done $\hat{\mu}$-equivariantly.

We would like to prove that there exists a smooth algebraic space $\widetilde{X}$ over $R_n$ with a $\hat{\mu}$-action, such that there exists a $\hat{\mu}$-equivariant isomorphism $\widetilde{X}_k \simeq D_Z$. Possibly useful results are [Fri83, Theorem 5.10] and [Han15, Corollary 5.4.15].
Step 7: Define $\mathcal{X}$ to be the quotient space $\tilde{\mathcal{X}}/\hat{\mu}$. Prove that $X \cong \mathcal{X} \times_R K$ is a $K3$ surface that does not satisfy the monodromy property.

Define $\mathcal{X}$ to be the quotient space $\tilde{\mathcal{X}}/\hat{\mu}$. This is an algebraic space over $R$, and $X \cong \mathcal{X} \times_R K$ is a $K3$ surface over $K$. In general, we won’t have $\mathcal{X}_k \cong D_X$, but we expect that it should be possible to contract components in $\mathcal{X}_k$ such that we obtain $D_X$ as a special fiber. This means that $X$ admits a model with $D_X$ as a special fiber, and hence is does not satisfy the monodromy property.

6.4 Excluding some combinatorial countercandidates

By executing the steps described in Section 6.3, we may come across an ad hoc reason to exclude a combinatorial countercandidate. For example, for Combinatorial Countercandidate 6.2.3 we are stuck at Step 2. We are not able to prove the existence of the surface $V_0$, and we ask ourselves whether such a surface exists. If there is indeed no such surface $V_0$, then there does not exist a strict normal crossings surface as described in Combinatorial Countercandidate 6.2.3 which would rule out this combinatorial countercandidate. A strategy shown to us by A. Höring, indeed seems to prove that the surface $V_0$ does not exist, and therefore it excludes Combinatorial Countercandidate 6.2.3.

We also tried to execute the strategy for Combinatorial Countercandidate 6.2.2 but failed to complete Step 3. This results in proving that there does not exist an algebraic space $Y$ over $R_n$, as defined in Step 3. Therefore, Combinatorial Countercandidate 6.2.2 cannot be smoothened to a $K3$ surface. This section is devoted to explaining this reasoning. We will prove that there does not exist a $K3$ surface $X$ over $K$ with an snc-model $\mathcal{X}$ such that $\mathcal{X}_k \cong D$, where $D$ is a strict normal crossings surface with the properties described in Combinatorial Countercandidate 6.2.2. Therefore, we can rule out 6.2.2 as a combinatorial countercandidate.

Proposition 6.4.1. There does not exist a $K3$ surface $X$ over $K$ with Crauder-Morrison model $\mathcal{X}$ such that the special fiber $\mathcal{X}_k$ has the following properties: the dual graph is described in Figure 6.8, where the labels denote the multiplicities of the components.
Denote by $V_0$ and $V_1$ the two components in the chain, where $V_0$ has multiplicity 4 and $V_1$ has multiplicity 2. The surfaces $V_0$ and $V_1$ are rational, ruled surfaces, and if we define $L_i$ as the number of blow-ups in the birational morphism $V_i \to \tilde{V}_i$ to the minimal ruled surface $\tilde{V}_i$ for $i = 0, 1$, then $L_0 = 6$ and $L_1 = 9$. The six flowers meeting $V_0$ are all of type $2A$. There are two flowers of type $2A$ meeting $V_1$, and two flowers of type $2B$ meeting $V_1$.

Proof. Suppose there exists a $K3$ surface $X$ with the properties described in the statement. Set $C_1 = V_0 \cap V_1$ and, denote by $C_2, \ldots, C_7$ the flowercurves on $V_0$. We know that $C_1$ is an elliptic curve and $C_2, \ldots, C_7$ are rational curves. Moreover, they are pairwise disjoint as there are no triple points.

Let $\mathcal{Y}$ be the normalization of $\mathcal{X} \times_R R[\pi]/(\pi^4 - t)$, and let $f: \mathcal{Y} \to \mathcal{X}$ be the induced morphism. Denote by $W$ an irreducible component of $f^{-1}(V_0)_{\text{red}}$. Proposition 4.1.7 gives that $f_{V_0}: f^{-1}(V_0)_{\text{red}} \to V_0$ is étale above $V_0^{\circ}$ of degree 4, and that it is ramified of index 2 above $C_1, \ldots, C_7$. Therefore, $W \to V_0$ is an $m$-cyclic cover, with $m = 2$ or 4, ramified above $C_1 + C_2 + \cdots + C_7$.

In [BHPV95, Section I.17], it is explained that the existence of this $m$-cyclic cover implies the existence of a line bundle $\mathcal{L}$ on $V_0$ with

$$\mathcal{L}^{\otimes m} = \mathcal{O}_{V_0}(C_1 + C_2 + \cdots + C_7).$$

Let $D$ be the divisor on $V_0$ associated with the line bundle $\mathcal{L}$. We have $mD \equiv C_1 + C_2 + \cdots + C_7$, and therefore, we find

$$m^2 D^2 = (C_1 + C_2 + \cdots + C_7)^2$$

$$= (C_1^2 + C_2^2 + \cdots + C_7^2),$$

where the second equality is explained by the fact that the curves $C_1, \ldots, C_7$ are pairwise disjoint. From Lemma 3.3.8, we know that $C_2^2 = \cdots = C_7^2 = -2$. Moreover, from equation (3.4) and Remark 3.3.6 it follows that

$$K_{V_0} \equiv -C_1.$$
Because $L_0 = 6$, we have $C_1^2 = 8 - 6 = 2$, by [Har77, Corollary V.2.11 and Proposition V.3.3]

It follows that $m^2D^2 = 10$, which implies that $D^2$ is not an integer, which is a contradiction. As a consequence, an $m$-cyclic cover of $V_0$ ramified along $C_1 + \cdots + C_7$ does not exist. We conclude that there does not exist a $K3$ surface $X$ with the desired properties.

We can repeat this argument in more generality. Let $X$ be a $K3$ surface over $K$ of index $i(X) = 1$ with a Crauder-Morrison model $\mathcal{X}$ that does not satisfy the monodromy property. Write $\mathcal{X}_k = \sum_{i \in I} N_i E_i$ and set $n = \text{lcm}_{i \in I} N_i$. Since $X$ does not satisfy the monodromy property, $\mathcal{X}$ is a chain degeneration. Let $V_0, \ldots, V_{k+1}$ be the surfaces in the chain, where $V_i \cap V_j = \emptyset$, except when $j \in \{i-1, i, i+1\}$. Assume that $V_0$ is a rational, ruled surface, and suppose moreover that $N(V_0) = 2N(V_1)$, and that $N(V_1) = \min\{N(V_i) \mid i = 0, \ldots, k+1\}$.

Since $X$ has index $i(X) = 1$, it is possible to conclude that $N(V_1)$ is either 1 or 2.

Set $C_1 = V_0 \cap V_1$, and let $C_2, \ldots, C_{\gamma'+1}$ be the flowercurves of flowers of type $2A$ on $V_0$. Let $\mathcal{Y}$ be the normalization of $\mathcal{X} \times_R R[\pi]/(\pi^n - t)$, and let $f: \mathcal{Y} \to \mathcal{X}$ be the induced morphism. Denote by $W$ an irreducible component of $f^{-1}(V_0)_{\text{red}}$. The morphism $W \to V_0$ is an $m$-cyclic cover, with $m = 2$ or 4, ramified along $C_1 + C_2 + \cdots + C_{\gamma'+1}$. Therefore, there exists a divisor $D$ on $V_0$ with the property that

$$m^2D^2 = (C_1 + C_2 + \cdots + C_{\gamma'+1})^2$$

$$= 8 - L_0 - 2\gamma'.$$

Since $m$ is 2 or 4, we conclude that $L_0 + 2\gamma'$ is divisible by 4. Therefore, we can exclude all combinatorial countercandidates with $L_0 + 2\gamma' \not\equiv 0 \mod 4$.

By adding this requirement to our Python-code, we are able to exclude 26 of the 68 combinatorial countercandidates where the chain has components $V_0, \ldots, V_{k+1}$ and

$$N(V_i) = \begin{cases} 2N(V_1) & \text{if } i = 0, \\ N(V_1) & \text{otherwise.} \end{cases}$$

The method described above does not rule out Combinatorial Countercandidate 6.2.3 but A. Höring suggested a strategy to also exclude this combinatorial countercandidate. We believe this method could maybe be generalized to rule out even more combinatorial countercandidates.
6.5 Some questions for further research

As is probably clear by now, we don’t know yet whether all $K3$ surfaces allowing a triple-point-free model satisfy the monodromy property. We produced a finite list of combinatorial countercandidates, but we don’t know if one or more of them can be smoothened to a $K3$ surface. This is the first obvious question we suggest for further research:

- Is it possible to smoothen a combinatorial countercandidate to a $K3$ surface? If so, then there exists a $K3$ surface not satisfying the monodromy property.
- Is it possible to exclude all combinatorial countercandidates? If so, then all $K3$ surfaces allowing a triple-point-free model satisfy the monodromy property. A possible strategy would be to generalize a method suggested by A. Höring to exclude Combinatorial Countercandidate 6.2.3.

If it is possible to construct a $K3$ surface not satisfying the monodromy property, then it would be very interesting to investigate the poles of the motivic zeta function that do not give rise to monodromy eigenvalues. Some possible questions are:

- If there is a $K3$ surface $X$ not satisfying the monodromy property, can we classify the extra poles of $Z_{X,\omega}(T)$ that do not induce monodromy eigenvalues? Do these poles have special characteristics?
- Is there a property $P$ such that all $K3$ surfaces admitting a triple-point-free model satisfy the following: all poles of the motivic zeta function with property $P$, induce monodromy eigenvalues?

Although there are several differences between the monodromy conjecture for hypersurface singularities and the context of Calabi-Yau varieties, there are also many mysterious similarities. Therefore, it would be quite interesting if we could find close links between the two settings. Some questions, we ask ourselves are:

- If there is a $K3$ surface $X$ not satisfying the monodromy property, what is the influence of the existence of $X$ on the monodromy conjecture for hypersurface singularities? Could $X$ serve as an inspiration to construct a counterexample of this famous conjecture?
- Are there techniques used in the setting of Calabi-Yau varieties that could turn out to be fruitful in the context of hypersurface singularities as well?

If all combinatorial countercandidates can be excluded, then we know that every $K3$ surface allowing a triple-point-free model satisfies the monodromy property. So far, we distinguished many cases for which we proved the monodromy
property separately, and there were a lot of ad hoc arguments. It would be nice to see a more general principle that summarizes all these individual cases. Furthermore, we would like to know whether all $K3$ surfaces satisfy the monodromy property, also those without a triple-point-free model. And what about general Calabi-Yau varieties? To summarize our questions:

- Is there a general argument that shows that $K3$ surfaces allowing a triple-point-free model satisfy the monodromy property?
- Do all $K3$ surfaces satisfy the monodromy property?
- Do all Calabi-Yau varieties satisfy the monodromy property?
Appendix A

Formulas for the contribution of a flower and Python code

In Section 4.2, we defined the contribution $Z_F(T)$ of a flower $F$ to the motivic zeta function. In the proof of Theorem 4.2.3, we explained how to compute the contribution for every type of flower. In Section A.1, we will list the explicit formulas for the contribution of flowers. In Section A.2, we give the Python code that was used to compute the contributions. This implementation is written in Python 3.6 and SymPy 1.0. The code can be downloaded from www.github.com/AnneliesJaspers/flowers_contribution.

A.1 Formulas for the contribution of a flower

In this section, we list the explicit formulas for the contribution of a flower to the motivic zeta function. For flowers where the top is isomorphic to $\mathbb{P}^2$ and where the double curve is a line, the contribution can be found in Table A.1. In Table A.2, we give the contribution of a conic-flower. Finally, the contribution of a flower with a ruled top is given in Table A.3.
Table A.1: Contribution to the motivic zeta function of a flower with $P_0 \simeq P_2$ and $C_1$ a line.

| $L$ | Contribution |
|-----|--------------|
| $A_6$ | $\frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})} \frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})}$ |
| $V_6$ | $\frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})} \frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})}$ |
| $B_4$ | $\frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})} \frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})}$ |
| $A_4$ | $\frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})} \frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})}$ |
| $B_3$ | $\frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})} \frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})}$ |
| $V_3$ | $\frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})} \frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})}$ |
| $V_2$ | $\frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})} \frac{1+\frac{2}{L+T}}{(1+\frac{2}{L+T})(1+\frac{2}{L+T})}$ |

$(L)^Z$
| Contribution $Z_F(T)$ |
|-------------------------|
| 2B $\frac{[F_0^C]L^{-\nu_0} T^{N_0}}{1-L^{-\nu_0} T^{N_0}} + (L-1)[C_1] \frac{L^{-\nu_1} T^{N_1}}{1-L^{-\nu_0} T^{N_0}} \frac{L^{-\nu_1+1} T^{N_1+1}}{1-L^{-\nu_1+1} T^{N_1+1}}$ |
| 2C $\frac{[F_0^C]L^{-\nu_0} T^{N_0}}{1-L^{-\nu_0} T^{N_0}} + \frac{[C_1]L^{-\nu_1-1} T^{N_1}}{(1-L^{-\nu_0} T^{N_0})(1-L^{-\nu_1+1} T^{N_1+1})}$ |
| 4C $\frac{[F_0^C]L^{-\nu_0} T^{N_0}}{1-L^{-\nu_0} T^{N_0}} + \frac{[C_1]L^{-\nu_1-1} T^{N_1}}{(1-L^{-\nu_0} T^{N_0})(1-L^{-\nu_1+1} T^{N_1+1})}$ |
| $\times (L-1)L^{-4\nu_1+3\ell-4} T^{4N_1} + (L^{-1}-1)L^{-3\nu_1+2\ell-2} T^{3N_1} + (L^\ell-1)L^{-2\nu_1+\ell-1} T^{2N_1} + (L^{2\ell-2}-1)L^{-\nu_1+1} T^{N_1} + (L^{-1}-1)L)$ |
| 6D $\frac{[F_0^C]L^{-\nu_0} T^{N_0}}{1-L^{-\nu_0} T^{N_0}} + \frac{[C_1]L^{-\nu_1-1} T^{N_1}}{(1-L^{-\nu_0} T^{N_0})(1-L^{-\nu_1+1} T^{N_1+1})}$ |
| $\times (L-1)L^{-4\nu_1+3\ell-2} T^{4N_1} + (L^{-1}-1)L^{-3\nu_1+2\ell-1} T^{3N_1} + (L^\ell-1)L^{-2\nu_1+\ell-1} T^{2N_1} + (L^{2\ell-2}-1)L^{-\nu_1+1} T^{N_1} + (L^{-1}-1)L)$ |
| 6E $\frac{[F_0^C]L^{-\nu_0} T^{N_0}}{1-L^{-\nu_0} T^{N_0}} + (L-1)[C_1] \frac{L^{-\nu_1} T^{N_1}}{1-L^{-\nu_0} T^{N_0}} \frac{L^{-\nu_1+1} T^{N_1+1}}{1-L^{-\nu_1+1} T^{N_1+1}}$ |

Table A.2: Contribution to the motivic zeta function of a flower with $F_0 \simeq \mathbb{P}^2$ meeting $F_1$ in a conic.
Table A.3: Contribution to the motivic zeta function of a flower with $F_0$ a minimal ruled surface.

| Expression                                                                 | Contribution |
|---------------------------------------------------------------------------|--------------|
| $(1 + o_{N_L} L_{o_{T}} - T + o_{N_E} L_{o_{T}} - T)(1 + o_{N_L} L_{o_{T}} - T - o_{N_E} L_{o_{T}} - T)(1 + o_{N_L} L_{o_{T}} - T) o_{N_L} L_{1 + o_{T}} - T^1 | 12 |
| $(1 + o_{N_L} L_{1 + o_{T}} - T^1 + o_{N_E} L_{1 + o_{T}} - T + o_{N_E} L_{1 + o_{T}} - T + o_{N_E} L_{1 + o_{T}} - T + o_{N_E} L_{1 + o_{T}} - T) o_{N_L} L_{1 + o_{T}} - T^1 | 12 |
| $(1 + o_{N_L} L_{1 + 2 o_{T}} - T^1)(1 + o_{N_L} L_{o_{T}} - T)(1 + o_{N_L} L_{o_{T}} - T) o_{N_L} L_{1 + o_{T}} - T^1 | 8 |
| $(1 + o_{N_L} L_{1 + 3 o_{T}} - T^1)(1 + o_{N_L} L_{1 + o_{T}} - T)(1 + o_{N_L} L_{1 + o_{T}} - T) o_{N_L} L_{1 + o_{T}} - T^1 | 8 |
| $(1 + o_{N_L} L_{1 + 2 o_{T}} - T^1)(1 + o_{N_L} L_{1 + o_{T}} - T)(1 + o_{N_L} L_{1 + o_{T}} - T) o_{N_L} L_{1 + o_{T}} - T^1 | 6 |
| $(1 + o_{N_L} L_{1 + 3 o_{T}} - T^1)(1 + o_{N_L} L_{1 + o_{T}} - T)(1 + o_{N_L} L_{1 + o_{T}} - T) o_{N_L} L_{1 + o_{T}} - T^1 | 6 |
| $(1 + o_{N_L} L_{1 + 4 o_{T}} - T^1)(1 + o_{N_L} L_{1 + o_{T}} - T)(1 + o_{N_L} L_{1 + o_{T}} - T) o_{N_L} L_{1 + o_{T}} - T^1 | 6 |
| $(1 + o_{N_L} L_{1 + 5 o_{T}} - T^1)(1 + o_{N_L} L_{1 + o_{T}} - T)(1 + o_{N_L} L_{1 + o_{T}} - T) o_{N_L} L_{1 + o_{T}} - T^1 | 6 |
| $(1 + o_{N_L} L_{1 + 2 o_{T}} - T^1)(1 + o_{N_L} L_{1 + o_{T}} - T)(1 + o_{N_L} L_{1 + o_{T}} - T) o_{N_L} L_{1 + o_{T}} - T^1 | 4 |
| $(1 + o_{N_L} L_{1 + 3 o_{T}} - T^1)(1 + o_{N_L} L_{1 + o_{T}} - T)(1 + o_{N_L} L_{1 + o_{T}} - T) o_{N_L} L_{1 + o_{T}} - T^1 | 4 |

$(L)^F Z$
A.2 Python code

In this section, we explain and give the Python code written to compute the contribution of a flower to the motivic zeta function. This implementation is written in Python 3.6 and SymPy 1.0. Note that the code can be downloaded from [www.github.com/AnneliesJaspers/flowers_contribution](http://www.github.com/AnneliesJaspers/flowers_contribution).

A.2.1 Implementation of relevant functions

The following code defines some preliminary functions used to compute the contributions of flowers to the motivic zeta function.

```python
from sympy import *
from enum import Enum

L, T, P, c, F = symbols('L T P c F', positive=True)
N, nu = symbols('N nu', real=True)

# L is class of affine line
# P is class of variety such that F_0 circ = P L^2

class Group(Enum):
    LINE = 1
    CONIC = 2
    RULED = 3

def compute_nu_vector(N_vector, group):
    """Compute the multiplicities nu_i of div (omega) along E_i."

    Tool is Lemma 3.3.10 in thesis 'Monodromy property for K3 surfaces allowing a triple-point-free model'.
    Define nu such that
    nu_0=nu (if F_0 ruled or F_0 isomorphic to P^2 with C_1 a line), or
    nu_0=2nu+1 (if F_0 isomorphic to P^2 with C_1e a conic).

    Arguments:
    N_vector is a vector with the multiplicities of a flower,
    from N_0 to N_{ell+1}
    group is Group.LINE, Group.CONIC, or Group.RULED,
    depending on the type of the flower
    """

    if group == Group.LINE:
        # Define nu_0
        nu_vector = [nu]
        # Define nu_1
        nu_vector.append(expand(nu_vector[0]*N_vector[1]/N_vector[0]-2))
    elif group == Group.CONIC:
        # Define nu_0 (always odd, when C_1 a conic)
        nu_vector = [2*nu+1]
        # Define nu_1
        nu_vector.append(expand(nu_vector[0]*N_vector[1]/N_vector[0]-1/2))
```

elif group == Group.RULED:
    # Define nu_0
    nu_vector = [nu]
    # Define nu_1
    nu_vector.append(expand(nu_vector[0]*N_vector[1]/N_vector[0]-1))

for i in range(2,len(N_vector)):
    # Define nu_i
    nu_i = nu_vector[-1]*(N_vector[i-2]+N_vector[i])/N_vector[i-1]-nu_vector[-2]
    nu_vector.append(expand(nu_i))

assert len(N_vector) == len(nu_vector)
return nu_vector

def compute_fraction(N_1, nu_1):
    return L**(-nu_1) * T**(N_1)/(1 - L**(-nu_1) * T**(N_1))

def compute_contribution(N_vector, group):
    """Computes contribution of the flower to the motivic zeta function.

    For Definition contribution see thesis, Definition 4.2.1.
    ""
    expr = 0
    if group == Group.LINE:
        C = P * (L+1)  # when F_0=P^2 , then [C]= P * (L+1)
        F0 = P * L **2 # when F_0=P_2 with C a line , then [F_0 circ]= P * L^2
    elif group == Group.CONIC:
        C = c
        F0 = F
    else:
        C = c  # when F_0 is ruled , we don't know what [C] is , so variable c
        F0 = c * L  # when F_0 is ruled , C is section , so [F_0 circ] = c * L
    nu_vector = compute_nu_vector(N_vector, group)
    for j in range(0,len(N_vector)-1):
        term = compute_fraction(N_vector[j],nu_vector[j])
        if j==0:
            term = F0 * term
        else:
            term = (L-1) * C * term  # for all i , we have [C_i]=[C_{i+1}]
        expr += term

    # Now, we add the terms of the sum with |J| = 2.
for j in range(0, len(N_vector) - 1):
    # for all i, we have \([C_i] = [C_{i+1}]\)
    term = (L - 1) * C * compute_fraction(N_vector[j], nu_vector[j])
    * compute_fraction(N_vector[j+1], nu_vector[j+1])
    expr += term

return expr

def compute_contribution_middle_conic(N_vector, m):
    """ Compute terms of the contribution of conic-flowers 
    to the motivic zeta function, corresponding to components \(F_1, \ldots, F_m\).
    
    Arguments:
    \(N\) vector is a vector with the multiplicities of a flower, 
    from \(N_0\) to \(N_{(\ell+1)}\)
    \(m\) for how many terms we add: 
    those corresponding to components \(F_1, \ldots, F_m\).
    """
    expr = 0

    C = c # when \(F_0 = P^2\), then \([C]\) = permutation * (L+1)
    nu_vector = compute_nu_vector(N_vector, Group.CONIC)

    # First we add all the terms of the sum with \(|J| = 1\)
    for j in range(1, m+1):
        term = compute_fraction(N_vector[j], nu_vector[j])
        term = (L - 1) * C * term # for all i, we have \([C_i] = [C_{i+1}]\)
        expr += term

    # Now, we add the terms of the sum with \(|J| = 2\).
    for j in range(1, m+1):
        term = (L - 1) * C * compute_fraction(N_vector[j], nu_vector[j])
        * compute_fraction(N_vector[j+1], nu_vector[j+1])
        expr += term

    return expr

A.2.2 Computation of the contribution of a flower, that is not a conic-flower

Let \(F\) be a flower that is not a conic-flower. From Table 3.1 and 3.3, we see that, if we know the type of the flower, then we know its length \(\ell\). For example, when \(F\) is of type 4A, we have \(\ell = 2\). Therefore, it is straightforward to compute the contribution of \(F\) to the motivic zeta function. The following Python code computes these contributions.

```python
from sympy import *
from functions import *

# Flowers of with top isomorphic to P^2 with C_1 a line
N_2A = [N, 2*N]
```
zeta_2A = factor(cancel(compute_contribution(N_2A, Group.LINE)))
N_3A = [N, 2*N, 3*N]
zeta_3A = factor(cancel(compute_contribution(N_3A, Group.LINE)))
N_3B = [N, 3*N]
zeta_3B = factor(cancel(compute_contribution(N_3B, Group.LINE)))
N_4A = [N, 2*N, 3*N, 4*N]
zeta_4A = factor(cancel(compute_contribution(N_4A, Group.LINE)))
N_4B = [N, 4*N]
zeta_4B = factor(cancel(compute_contribution(N_4B, Group.LINE)))
N_4alpha = [N, 2*N]
zeta_4alpha = factor(cancel(compute_contribution(N_4alpha, Group.RULED)))
N_6alpha = [N, 2*N, 3*N]
zeta_6alpha = factor(cancel(compute_contribution(N_6alpha, Group.RULED)))
N_6beta = [N, 3*N]
zeta_6beta = factor(cancel(compute_contribution(N_6beta, Group.RULED)))
N_8alpha = [N, 2*N, 3*N, 4*N]
zeta_8alpha = factor(cancel(compute_contribution(N_8alpha, Group.RULED)))
N_8beta = [N, 4*N]
zeta_8beta = factor(cancel(compute_contribution(N_8beta, Group.RULED)))
N_12alpha = [N, 2*N, 3*N, 4*N, 5*N, 6*N]
zeta_12alpha = factor(cancel(compute_contribution(N_12alpha, Group.RULED)))
N_12beta = [N, 6*N]
zeta_12beta = factor(cancel(compute_contribution(N_12beta, Group.RULED)))

print("The following are the contributions to the motivic zeta function of flowers with top isomorphic to P^2 and C_1 a line: \
")
print("Contribution of flower 2A: ", zeta_2A, "\n")
print("Contribution of flower 3A: ", zeta_3A, "\n")
print("Contribution of flower 3B: ", zeta_3B, "\n")
print("Contribution of flower 4A: ", zeta_4A, "\n")
print("Contribution of flower 4B: ", zeta_4B, "\n")
print("Contribution of flower 6A: ", zeta_6A, "\n")
print("Contribution of flower 6B: ", zeta_6B, "\n")

print("The following are the contributions to the motivic zeta function of flowers with top a ruled surface: \
")
print("Contribution of flower 4 alpha: ", zeta_4alpha, "\n")
print("Contribution of flower 6 alpha: ", zeta_6alpha, "\n")
print("Contribution of flower 6 beta: ", zeta_6beta, "\n")
print("Contribution of flower 8 alpha: ", zeta_8alpha, "\n")
print("Contribution of flower 8 beta: ", zeta_8beta, "\n")
print("Contribution of flower 12 alpha: ", zeta_12alpha, "\n")
print("Contribution of flower 12 beta: ", zeta_12beta, "\n")
A.2.3 Computation of the contribution of a conic-flower

Let $F$ be a conic-flower. If $F$ is a flower of type $2B$ or $6E$, then $\ell = 0$, otherwise, $\ell$ is chosen arbitrarily. In the first case, it is immediately clear that

$$Z_F(T) \in \mathcal{M}^T_{\ell} \left[ T, \frac{1}{1 - L^{-\nu_0}T^{N_0}}, \frac{1}{1 - L^{-\nu_{\ell+1}}T^{N_{\ell+1}}} \right].$$

Therefore, we may assume $F$ has type $2C, 4C, 6C$ or $6D$. We will need an induction argument on $\ell$ to obtain a closed formula for $Z_F(T)$.

Note that for these flowers, we can find an integer $m$ with $1 \leq m \leq \ell$ such that $N(F_i) = N(F_1)$ for all $1 \leq i \leq m + 1$. Note that $N(F_0) = 2N(F_1)$. Define

$$Z_{F,m}(T) = \sum_{j=1}^{m} \left( [\widetilde{F}^0_j] \frac{L^{-\nu_j}T^{N_j}}{1 - L^{-\nu_j}T^{N_j}} + (L - 1)[\widetilde{C}_{j+1}] \frac{L^{-\nu_j}T^{N_j}}{1 - L^{-\nu_j}T^{N_j}} \right).$$

By induction on $m$, one can compute that

$$Z_{F,m}(T) = \frac{[\widetilde{C}_1](L^m - 1)L^{-\nu_1}T^{N_1}}{(1 - L^{-\nu_1}T^{N_1})(1 - L^{-\nu_1+m}T^{N_1})}.$$

Let $m_0$ be the maximal $m$ with $1 \leq m \leq \ell$ such that $N(F_i) = N(F_1)$ for all $1 \leq i \leq m + 1$. Then

$$Z_F(T) = \left[ \frac{\widetilde{F}_0^0}{1 - L^{-\nu_0}T^{N_0}} \right] L^{-\nu_0}T^{N_0} + (L - 1)[\widetilde{C}_1] \frac{L^{-\nu_0}T^{N_0}}{1 - L^{-\nu_0}T^{N_0}} \frac{L^{-\nu_1}T^{N_1}}{1 - L^{-\nu_1}T^{N_1}}$$

$$+ Z_{F,m_0}(T)$$

$$+ \sum_{j=m_0+1}^{\ell} \left( \left[ \frac{\widetilde{F}_j^0}{1 - L^{-\nu_j}T^{N_j}} \right] L^{-\nu_j}T^{N_j} + (L - 1)[\widetilde{C}_{j+1}] \frac{L^{-\nu_j}T^{N_j}}{1 - L^{-\nu_j}T^{N_j}} \frac{L^{-\nu_{j+1}}T^{N_{j+1}}}{1 - L^{-\nu_{j+1}}T^{N_{j+1}}} \right).$$
The following code first computes $Z_F(T)$ for flowers of type $2B$ and for flowers of type $6C$. Then it computes $Z_{F,m}(T)$ for $m = 2, 3$ and $4$. Finally, we compute

$$
(\mathbb{L} - 1)[\widetilde{C}_1] \frac{\mathbb{L}^{-\nu_0}T^{N_0}}{1 - \mathbb{L}^{-\nu_0}T^{N_0}} \frac{\mathbb{L}^{-\nu_1}T^{N_1}}{1 - \mathbb{L}^{-\nu_1}T^{N_1}} + Z_{F,m_0}(T)
$$

$$
+ \sum_{j=m_0+1}^{\ell} \left( [\widetilde{F}_j] \frac{\mathbb{L}^{-\nu_j}T^{N_j}}{1 - \mathbb{L}^{-\nu_j}T^{N_j}} \right)
$$

$$
+ (\mathbb{L} - 1)[\widetilde{C}_{j+1}] \frac{\mathbb{L}^{-\nu_j}T^{N_j}}{1 - \mathbb{L}^{-\nu_j}T^{N_j}} \frac{\mathbb{L}^{-\nu_{j+1}}T^{N_{j+1}}}{1 - \mathbb{L}^{-\nu_{j+1}}T^{N_{j+1}}},
$$

for flowers of type $2C, 4C, 6C$ and $6D$. To get the contribution of the flower, we need to add $[\widetilde{F}_0]\frac{\mathbb{L}^{-\nu_0}T^{N_0}}{1 - \mathbb{L}^{-\nu_0}T^{N_0}}$ to the result.

```python
1 from sympy import *
2 from functions import *
3 4 l = symbols('l', real=True)
5 6 # Computation of Z_{F,m} ####
7 8 print("We first compute Z_{F,m}(T) for some small m.\n")
9 N_v = [2*N, N, N, N, N, N, N, N, N, N]
10 11 #m=2
12 Z_2 = compute_contribution_middle_conic(N_v, 2)
13 print('For m=2: ', factor(cancel(Z_2)), "\n")
14 15 #m=3
16 Z_3 = compute_contribution_middle_conic(N_v, 3)
17 print('For m=3: ', factor(cancel(Z_3)), "\n")
18 19 #m=4
20 Z_4 = compute_contribution_middle_conic(N_v, 4)
21 print('For m=4: ', factor(cancel(Z_4)), "\n")
22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49
```
```python
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Z_4C = Z_F_m + c*(L-1)*compute_fraction(2*N,2*nu+1)*compute_fraction(N,nu) \
    + c*(L-1)*compute_fraction(N,nu-l+1) \
    *compute_fraction(2*N,2*nu-2*l+1)
print('Contribution of flower 4C:', factor(cancel(Z_4C)), '
')

#### Flower of type 6C ####
# Declare Z_{F, m_0}.
# m_0 = ell - 2
Z_F_m = c * (L**(l-2)-1) * L**(-nu) * T**N \
    / ((1-L**(-nu) * T**N)*(1-L**(-nu+l-2) * T**N))
Z_6C = Z_F_m + c*(L-1)*compute_fraction(2*N,2*nu+1)*compute_fraction(N,nu) \
    + c*(L-1)*compute_fraction(N,nu-l+2) \
    + c*(L-1)*compute_fraction(2*N,2*nu-2*l+3) \
    + c*(L-1)*compute_fraction(2*N,2*nu-2*l+3) \
    *compute_fraction(3*N, 3*nu-3*l+4)
print('Contribution of flower 6C:', factor(cancel(Z_6C)), '
')

#### Flower of type 6D ####
# Declare Z_{F, m_0}.
# m_0 = ell - 1
Z_F_m = c * (L**(l-1)-1) * L**(-nu) * T**N \
    / ((1-L**(-nu) * T**N)*(1-L**(-nu+l-1) * T**N))
Z_6D = Z_F_m + c*(L-1)*compute_fraction(2*N,2*nu+1)*compute_fraction(N,nu) \
    + c*(L-1)*compute_fraction(N,nu-l+1) \
    + c*(L-1)*compute_fraction(N,nu-l+1) \
    *compute_fraction(3*N, 3*nu-3*l+2)
print('Contribution of flower 6D:', factor(cancel(Z_6D)), '
')
Appendix B

List of combinatorial countercandidates and Python code

In this appendix, we will give the Python code to compute all combinatorial countercandidates. We will see that we have a total of 63 combinatorial countercandidates, and we will list them all.

Suppose $X$ is a $K3$ surface of index $\iota(X) = 1$ with a Crauder-Morrison model $\mathcal{X}$, not satisfying the monodromy property. Theorem 5.2.1 implies that $X$ is a chain degeneration. Let $V_0, \ldots, V_{\alpha-1}, V_{\alpha}, \ldots, V_\beta, V_{\beta+1}, \ldots, V_{k+1}$ be the components in the chain, where $\alpha$ and $\beta$ are defined as in Proposition 3.6.2, and $V_i \cap V_j = \emptyset$, except when $j \in \{i - 1, i, i + 1\}$. Set $N = N(V_\alpha)$.

As explained in Section 6.2.3, one of the following ten cases hold:

1. $\beta = k + 1$, and $V_{k+1}$ is a rational, ruled surface,
2. $\beta = k + 1$, and $V_{k+1}$ is an elliptic, ruled surface,
3. $\beta = k + 1$, and $V_{k+1} \simeq \mathbb{P}^2$,
4. $V_{k+1} \simeq \mathbb{P}^2$, and $N(V_{k+1}) = 3N$,
5. $V_{k+1} \simeq \mathbb{P}^2$, and $N(V_{k+1}) = \frac{3}{2}N$,
6. $V_{k+1} \simeq \Sigma_2$, and $N(V_{k+1}) = 4N$,
7. $V_{k+1} \simeq \Sigma_2$, $\beta = k - 1$, and $N(V_{k+1}) = 2N$,
\( \beta = k + 1, \text{ AND } V_{k+1} \text{ IS A RATIONAL, RULED SURFACE} \)

8. \( V_{k+1} \simeq \Sigma_2, \text{ and } N(V_{k+1}) = \frac{4}{3} N, \)

9. \( V_{k+1} \text{ is a rational, ruled surface, } \beta = k, \text{ and } N(V_{k+1}) = 2N, \)

10. \( V_{k+1} \text{ is an elliptic, ruled surface, and } N(V_{k+1}) = 2N. \)

For every case, we will compute the combinatorial countercandidates and give the Python code.

This code has been written in Python 3.6. It can be downloaded from [www.github.com/AnneliesJaspers/combinatorial-counterexamples](http://www.github.com/AnneliesJaspers/combinatorial-counterexamples).

**B.1 \( \beta = k + 1, \text{ and } V_{k+1} \text{ is a rational, ruled surface} \)**

Suppose \( \beta = k + 1, \text{ and } V_{k+1} \text{ is a rational, ruled surface}. \) The strategy described in Section 6.2.3 explains that we have to find integer solutions satisfying the following conditions:

\[
\begin{align*}
\phi' + 2\gamma' - \phi - L_0 &= 4, \\
\sum_{i=1}^{k+1} L_i + 2\phi' + \gamma' - 2\phi - \frac{3\gamma}{2} &= \frac{24}{N} - 4, \\
L_0 + 2\sum_{i=1}^{k+1} L_i &= 24, \\
\phi' + \gamma' &\leq L_0 + 1, \\
\phi + \gamma &\leq \sum_{i=1}^{k+1} L_i + 1, \\
N &= \begin{cases} 
1 & \text{if } \gamma = 0, \\
2 & \text{if } \gamma > 0,
\end{cases} \\
\sum_{i=1}^{k+1} L_i, \gamma, \phi' \text{ and } \gamma' &\geq 0, \\
L_0 \text{ and } \phi &\geq 1.
\end{align*}
\]

Moreover, in Section 6.4, we found the extra equation \( L_0 + 2\gamma' \not\equiv 0 \mod 4. \)

One can verify that there are 42 integer solutions satisfying these relations, for example by using the Python code below.

```python
count=0
for L1 in range(0, (24//2)+1): # L0 +2L1 = 24
    L0 = 24 - 2*L1
    if L0 == 0:
        #V_0 is non-minimal ruled, so L_0 \neq 0.
        continue
    for phi in range(1, (L1+1)+1): # phi+gamma \leq L1+1 and phi\geq 1
        for gamma in range(0, (L1-phi+1)+1, 2): # gamma is even because of degree zeta function
            count+=1
```

159
# index = 1
if gamma == 0:
    N=1
else:
    N=2

for phi_ in range(0,(L0 +1)+1):  # phi '+ gamma ' \leq L0 +1
    for gamma_ in range(0,(L0 - phi_ +1) +1):

        # checking nothing went wrong
        assert phi_ >=0 and gamma_ >=0 and phi >=1 and gamma >=0 and L0 >=1 and
        assert gamma % 2 ==0
        assert (L0 +2* L1) == 24
        assert ( phi_ + gamma_ ) <= (L0 +1)
        assert ( phi + gamma ) <= (L1 +1)

        # checking the two conditions
        if (phi_+2*gamma_ -phi -L0) != 4:
            # making sure the monodromy property does not hold
            continue
        if (L1 +2*phi_+gamma_ -2*phi -3*gamma//2) != ((24//N)-4):
            # making sure the degree zeta function is 24
            continue
        if (L0 +2*gamma_)%4 != 0:
            # asserting the existence of line bundle such that cyclic cover
            exists
            continue

        # if we arrived here, we have found a combinatorial counterexample
        count += 1
        print (N, L0, L1, phi, gamma, phi', gamma', count)

print('We have produced', count, 'combinatorial counterexamples.

The output of this code is the following.

N = 1 L0 = 24 sum L_i = 0 phi = 1 gamma = 0 phi' = 5 gamma' = 12
N = 1 L0 = 22 sum L_i = 1 phi = 1 gamma = 0 phi' = 5 gamma' = 11
N = 1 L0 = 22 sum L_i = 1 phi = 2 gamma = 0 phi' = 6 gamma' = 11
N = 1 L0 = 20 sum L_i = 2 phi = 1 gamma = 0 phi' = 5 gamma' = 10
N = 1 L0 = 20 sum L_i = 2 phi = 2 gamma = 0 phi' = 6 gamma' = 10
N = 1 L0 = 20 sum L_i = 2 phi = 3 gamma = 0 phi' = 7 gamma' = 10
N = 1 L0 = 18 sum L_i = 3 phi = 1 gamma = 0 phi' = 5 gamma' = 9
N = 1 L0 = 18 sum L_i = 3 phi = 2 gamma = 0 phi' = 6 gamma' = 9
N = 1 L0 = 18 sum L_i = 3 phi = 3 gamma = 0 phi' = 7 gamma' = 9
N = 1 L0 = 18 sum L_i = 3 phi = 4 gamma = 0 phi' = 8 gamma' = 9
N = 1 L0 = 16 sum L_i = 4 phi = 1 gamma = 0 phi' = 5 gamma' = 8
N = 2 L0 = 16 sum L_i = 4 phi = 1 gamma = 4 phi' = 1 gamma' = 10
N = 1 L0 = 16 sum L_i = 4 phi = 2 gamma = 0 phi' = 6 gamma' = 8
N = 1 L0 = 16 sum L_i = 4 phi = 3 gamma = 0 phi' = 7 gamma' = 8
N = 1 L0 = 16 sum L_i = 4 phi = 4 gamma = 0 phi' = 8 gamma' = 8
N = 1 L0 = 16 sum L_i = 4 phi = 5 gamma = 0 phi' = 9 gamma' = 8
N = 1 L0 = 14 sum L_i = 5 phi = 1 gamma = 0 phi' = 5 gamma' = 7
N = 2 L0 = 14 sum L_i = 5 phi = 1 gamma = 4 phi' = 1 gamma' = 9
N = 1 L0 = 14 sum L_i = 5 phi = 2 gamma = 0 phi' = 6 gamma' = 7
N = 2 L0 = 14 sum L_i = 5 phi = 2 gamma = 4 phi' = 2 gamma' = 9
\[ \beta = k + 1, \text{ AND } V_{k+1} \text{ IS AN ELLIPTIC, RULED SURFACE} \]

We have produced 42 combinatorial counterexamples.

B.2 \( \beta = k + 1, \text{ and } V_{k+1} \text{ is an elliptic, ruled surface} \)

Suppose \( \beta = k + 1, \text{ and } V_{k+1} \text{ is an elliptic, ruled surface} \). The strategy described in Section 6.2.3 explains that we have to find integer solutions satisfying the following conditions:

\[
\begin{align*}
\phi' + 2\gamma' - \phi - L_0 &= 4, \\
4\phi' + 2\gamma' - 4\phi - 3\gamma + 2g + 2 \sum_{i=1}^{k+1} L_i &= 26, \\
L_0 + 2 \sum_{i=1}^{k+1} L_i &= 8, \\
2\phi + 2\gamma &\leq \sum_{i=1}^{k+1} L_i, \\
\phi' + \gamma' &\leq L_0 + 1, \\
\sum_{i=1}^{k+1} L_i, \gamma, \phi' \text{ and } \gamma' &\geq 0, \\
L_0 \text{ and } \phi &\geq 1.
\end{align*}
\]

One can verify that there is exactly 1 integer solution satisfying these relations, for example by using the Python code below.

```python
count=0
# L1 for \sum_{i=1}^{k+1} L_i and L0 for L_0
for L1 in range(0, (8//2)+1): # L0 +2L1 = 8
```
\begin{verbatim}
L0 = 8 - 2*L1
if L0 == 0:
  # V_0 is non-minimal ruled, so L_0 \neq 0.
  continue
for phi in range(1, (L1//2)+1):  # 2*phi+2*gamma \leq \sum L_i and phi \geq 1
  for gamma in range(0, ((L1 - 2*phi) //2) +1):
    for phi_ in range(0, (L0 +1) +1):  # phi' + gamma' \leq L0 +1
      for gamma_ in range(0, (L0 - phi_ +1) +1):
        # checking nothing went wrong
        assert phi_ >=0 and gamma_ >=0 and phi >=1 and gamma >=0 and L0 >=1 and L1 >=0
        assert (L0 +2* L1) == 8
        assert (phi_ + gamma_) <= (L0 +1)
        assert (2* phi +2* gamma) <= L1
        # checking the two conditions
        if (phi_ +2* gamma_ - phi - L0) != 4:
          # making sure the mondromy property does not hold
          continue
        if (26 -4* phi_ -2* gamma_ +4* phi -3* gamma -2* L1)%2 != 0:
          # making sure g is an integer
          continue
        g = (26 -4* phi_ -2* gamma_ +4* phi -3* gamma -2* L1) //2
        # degree zeta function is 24
        if g<1:
          # g \geq 1
          continue
        # if we arrived here, we have found a combinatorial counterexample
        count += 1
        print ('\nWe have produced ', count, ' combinatorial counterexamples.'))

The output of this code is the following.
N = 1, L0 = 4, sum L_i = 2, phi = 1, gamma = 0, phi' = 1, gamma' = 4, g = 7

We have produced 1 combinatorial counterexamples.

B.3 \( \beta = k + 1, \text{ and } V_{k+1} \simeq \mathbb{P}^2 \)

Suppose \( \beta = k + 1, \text{ and } V_{k+1} \simeq \mathbb{P}^2 \). The strategy described in Section 6.2.3 explains that we have to find integer solutions satisfying the following conditions:
\[ \beta = k + 1, \text{ AND } V_{k+1} \simeq \mathbb{S}^2 \]

\[
\begin{align*}
\phi' + 2\gamma' - \phi - L_0 &= 4, \\
\sum_{i=1}^{k+1} L_i + \gamma' + 2\phi' - 2\phi - \frac{3\gamma}{2} &= \frac{24}{N} - 3, \\
L_0 + 2 \sum_{i=1}^{k+1} L_i &= 26, \\
2\phi + 2\gamma &\leq \sum_{i=1}^{k+1} L_i, \\
\phi' + \gamma' &\leq L_0 + 1, \\
N &= \begin{cases} 
1 & \text{if } \gamma = 0, \\
2 & \text{if } \gamma > 0, 
\end{cases} \\
\sum_{i=1}^{k+1} L_i, \gamma, \phi' \text{ and } \gamma' &\geq 0, \\
L_0 \text{ and } \phi &\geq 1.
\end{align*}
\]

Moreover, in Section 6.4, we found the extra equation \( L_0 + 2\gamma' \not\equiv 0 \pmod{4} \).

One can verify that there are 17 integer solutions satisfying these relations, for example by using the Python code below.

```python
count = 0

# L1 for \( \sum_{i=1}^{(k+1)} L_i \) and L0 for L_0
for L1 in range(0, (26//2) +1): # L0+2* L1 =26
    L0 = 26-2* L1
    if L0 == 0:
        # V_0 is non-minimal ruled, so L_0 \neq 0
        continue

    for phi in range(1,( L1 //2) +1): # 2* phi +2* gamma \leq sum L_i and phi \geq 1
        for gamma in range(0, ((L1 -2* phi ) //2) +1 , 2): # gamma even because of
            # degree zeta function
                # checking nothing went wrong
                assert phi_>=0 and gamma_>=0 and phi>=1 and gamma>=0 and L0>=1 and
                L1>0 and (N==1 or N==2)
                assert gamma%2 == 0
                assert (L0+2* L1) == 26
                assert (phi_+gamma_) <= L0+1
                assert (2*phi+2*gamma) <= L1

                # checking the two conditions
                if (phi_+2*gamma_-phi-L0) != 4:
                    # making sure the mondromy property does not hold
                    continue
                if (L1+gamma_+2*phi_-2*phi-3*gamma//2) != (24//N-3):
                    # making sure degree of zeta function is 24
                    continue

                if (L0+2*gamma_)%4 != 0:
                    continue

                count += 1

print(count)
```

17 integer solutions.
# asserting the existence of line bundle such that cyclic cover exists
continue

# if we arrived here, we have found a combinatorial counterexample
count += 1

print ('\n\nWe have produced ' + str(count) + ' combinatorial counterexamples.\n')

The output of this code is the following.

N = 1 L0 = 22 sum L_i = 2 phi = 1 gamma = 0 phi' = 5 gamma' = 11
N = 1 L0 = 20 sum L_i = 3 phi = 1 gamma = 0 phi' = 5 gamma' = 10
N = 1 L0 = 18 sum L_i = 4 phi = 1 gamma = 0 phi' = 5 gamma' = 9
N = 1 L0 = 18 sum L_i = 4 phi = 2 gamma = 0 phi' = 6 gamma' = 9
N = 1 L0 = 16 sum L_i = 5 phi = 1 gamma = 0 phi' = 5 gamma' = 8
N = 1 L0 = 16 sum L_i = 5 phi = 2 gamma = 0 phi' = 6 gamma' = 8
N = 1 L0 = 14 sum L_i = 6 phi = 1 gamma = 0 phi' = 5 gamma' = 7
N = 1 L0 = 14 sum L_i = 6 phi = 2 gamma = 0 phi' = 6 gamma' = 7
N = 1 L0 = 14 sum L_i = 6 phi = 3 gamma = 0 phi' = 7 gamma' = 7
N = 1 L0 = 12 sum L_i = 7 phi = 1 gamma = 0 phi' = 5 gamma' = 6
N = 1 L0 = 12 sum L_i = 7 phi = 2 gamma = 0 phi' = 6 gamma' = 6
N = 1 L0 = 12 sum L_i = 7 phi = 3 gamma = 0 phi' = 7 gamma' = 6
N = 1 L0 = 10 sum L_i = 8 phi = 1 gamma = 0 phi' = 5 gamma' = 5
N = 1 L0 = 10 sum L_i = 8 phi = 2 gamma = 0 phi' = 6 gamma' = 5
N = 1 L0 = 8 sum L_i = 9 phi = 1 gamma = 0 phi' = 5 gamma' = 4
N = 2 L0 = 6 sum L_i = 10 phi = 1 gamma = 4 phi' = 1 gamma' = 5
N = 2 L0 = 4 sum L_i = 11 phi = 1 gamma = 4 phi' = 1 gamma' = 4

We have produced 17 combinatorial counterexamples.

B.4 \( V_{k+1} \simeq \mathbb{P}^2 \), and \( N(V_{k+1}) = 3N \)

Suppose \( V_{k+1} \simeq \mathbb{P}^2 \), and \( N(V_{k+1}) = 3N \). The strategy described in Section 6.2.3 explains that we have to find integer solutions satisfying the following conditions:

\[
\begin{align*}
\phi' + 2\gamma' - \phi - L_0 &= 4, \\
\sum_{i=1}^{k+1} L_i + \gamma' + 2\phi' - 2\phi - \frac{3\gamma}{2} &= \frac{24}{N} - 9, \\
L_0 + 2\sum_{i=1}^{k+1} L_i &= 14, \\
2\phi + 2\gamma &\leq \sum_{i=1}^{k+1} L_i, \\
\phi' + \gamma' &\leq L_0 + 1, \\
N &= \begin{cases} 
1 & \text{if } \gamma = 0, \\
2 & \text{if } \gamma > 0,
\end{cases} \\
\sum_{i=1}^{k+1} L_i, \gamma, \phi' \text{ and } \gamma' &\geq 0, \\
L_0 \text{ and } \phi &\geq 1.
\end{align*}
\]
One can verify that there are 2 integer solutions satisfying these relations, for example by using the Python code below.

```python
count=0
for L1 in range(0, (14//2) +1) : # L0 +2* sum L_i =14
    L0 = 14 - 2*L1
    if L0 == 0:
        # V_0 is non-minimal ruled, so L_0 \neq 0
        continue
    for phi in range(1,(L1 //2) +1) : # 2* phi+2* gamma \leq sum L_i and phi \geq 1
        for gamma in range(0, ((L1 -2* phi ) //2) +1) : # gamma even because of
degree zeta function
            # index = 1
            if gamma == 0:
                N=1
            else:
                N=2
            for phi_ in range (0,(L0 +1) +1) : # phi '+ gamma ' \leq L0 +1
                for gamma_ in range (0,(L0 - phi_ +1) +1):
                    # checking nothing went wrong
                    assert phi_>=0 and gamma_>=0 and phi>=1 and gamma>=0 and L0>=1 and
                        L1>=0 and (N==1 or N==2)
                    assert gamma%2 == 0
                    assert (L0+2* L1) == 14
                    assert (phi_+gamma_) <= L0+1
                    assert (2* phi+2* gamma ) <= L1
                    # checking the two conditions
                    if (L1+gamma_+2*phi_+3* gamma//2) != ((24//N)-9):
                        # making sure degree of zeta function is 24
                        continue
                    # if we arrived here, we have found a combinatorial counterexample
                    count += 1
                    print("N =", N, ', L0 =',L0, ', sum L_i =',L1, ', phi =',phi, ', gamma =', gamma, ', phi' =', phi_, ', gamma' =', gamma_,")
print("We have produced", count, 'combinatorial counterexamples.")
```

The output of this code is the following.

```
N = 1 L0 = 10 sum L_i = 2 phi = 1 gamma = 0 phi' = 5 gamma' = 5
N = 1 L0 = 8 sum L_i = 3 phi = 1 gamma = 0 phi' = 5 gamma' = 4
```

We have produced 2 combinatorial counterexamples.
B.5 \( V_{k+1} \cong \mathbb{P}^2 \), and \( N(V_{k+1}) = \frac{3}{2} N \)

Suppose \( V_{k+1} \cong \mathbb{P}^2 \), and \( N(V_{k+1}) = \frac{3}{2} N \). The strategy described in Section 6.2.3 explains that we have to find integer solutions satisfying the following conditions:

\[
\begin{align*}
\phi' + 2\gamma' - \phi - L_0 &= 4, \\
2 \sum_{i=1}^{k+1} L_i + 2\gamma' + 4\phi' - 4\phi - 3\gamma &= 15, \\
L_0 + 2 \sum_{i=1}^{k+1} L_i &= 20, \\
2\phi + 2\gamma &\leq \sum_{i=1}^{k+1} L_i, \\
\phi' + \gamma' &\leq L_0 + 1, \\
\sum_{i=1}^{k+1} L_i, \gamma, \phi' \text{ and } \gamma' &\geq 0, \\
L_0 \text{ and } \phi &\geq 1.
\end{align*}
\]

Moreover, in Section 6.4 we found the extra equation \( L_0 + 2\gamma' \not\equiv 0 \mod 4 \).

One can verify that there is exactly 1 integer solution satisfying these relations, for example by using the Python code below.

```python
count=0
#L1 for sum_{i=1}beta L_i and L0 for L_0
for L1 in range (0, (20//2)+1):  # L0 +2* sum L_i =20
    L0 = 20 - 2*L1
    if L0 == 0:
        # V_0 is non-minimal ruled, so L_0 \neq 0
        continue

    for phi in range (1, (L1//2)+1):  # 2* phi +2* gamma \leq sum L_i and phi >= 1
        for gamma in range (0, ((L1 -2* phi )//2)+1):
            for phi_ in range (0, (L0 +1)+1):  # phi'+ gamma' \leq L0 +1
                for gamma_ in range (0, (L0 - phi_ +1)+1):
                    # checking nothing went wrong
                    assert phi_ >=0 and gamma_ >=0 and phi >=1 and gamma >=0 and L0 >=1 and L1 >=0
                    assert (L0 +2* L1) == 20
                    assert ( phi_ + gamma_ ) <= (L0 +1)
                    assert (2*phi+2*gamma) <= L1

                    # checking the two conditions
                    if (phi_+2*gamma_-phi-L0) != 4:
                        # making sure the monodromy property does not hold
                        continue
                    if (2*L1+2*gamma_-phi+4*phi_-4*phi-3*gamma) != 15:
                        # making sure degree of zeta function is 24
                        continue
                    if (L0+2*gamma_)%4 != 0:
                        # asserting the existence of line bundle such that cyclic cover exists
                        continue
```

166 LIST OF COMBINATORIAL COUNTERCANDIDATES AND PYTHON CODE
Suppose \( V_{k+1} \cong \Sigma_2 \), and \( N(V_{k+1}) = 4N \). The strategy described in Section 6.2.3 explains that we have to find integer solutions satisfying the following conditions:

\[
\begin{align*}
\phi' + 2\gamma' - \phi - L_0 &= 7, \\
\sum_{i=1}^{k+1} L_i + \gamma' + 2\phi' - 2\phi - \frac{3\gamma}{2} &= \frac{24}{N} - 4, \\
L_0 + 2\sum_{i=1}^{k+1} L_i &= 12, \\
2\phi + 2\gamma &\leq \sum_{i=1}^{k+1} L_i, \\
\phi' + \gamma' &\leq L_0 + 1, \\
N &= \begin{cases} 
1 & \text{if } \gamma = 0, \\
2 & \text{if } \gamma > 0, \\
\sum_{i=1}^{k+1} L_i, \gamma, \phi' \text{ and } \gamma' \geq 0, \\
L_0 \text{ and } \phi \geq 1.
\end{cases}
\end{align*}
\]

One can verify that there are no integer solutions satisfying these relations, for example by using the Python code below.

```python
count = 0
# L1 for \( \sum_{i=1}^{k+1} L_i \) and L0 for L0
for L1 in range(0, (12//2) +1):  # L0 + 2* sum L_i = 12
    L0 = 12 - 2* L1
    if L0 == 0:  # \( V_0 \) is non-minimal ruled, so L_0 != 0
        continue
    for phi in range(1, (L1//2) +1):  # 2*phi + 2*gamma \leq sum L_i and phi >= 1
        for gamma in range(0, ((L1 - 2* phi)//2) +1, 2):  # gamma even because of
            degree zeta function
                if gamma == 0:
                    N = 1
                else:
            # index = 1
```

The output of this code is the following.

\[ N = 1 \quad L_0 = 4 \quad \text{sum } L_i = 8 \quad \phi = 1 \quad \gamma = 3 \quad \phi' = 1 \quad \gamma' = 4 \]

We have produced 1 combinatorial counterexamples.
Suppose $V_{k+1} \simeq \Sigma_2$, $\beta = k - 1$, and $N(V_{k+1}) = 2N$. The strategy described in Section 6.2.3 explains that we have to find integer solutions satisfying the following conditions:

\[
\begin{cases}
\phi' + 2\gamma' - \phi - L_0 = 6, \\
2\sum_{i=1}^{k+1} L_i + 2\gamma' + 4\phi' - 4\phi - 3\gamma = 22, \\
L_0 + 2\sum_{i=1}^{k+1} L_i = 16, \\
2\phi + 2\gamma \leq \sum_{i=1}^{k+1} L_i, \\
\phi' + \gamma' \leq L_0 + 1, \\
\sum_{i=1}^{k+1} L_i, \gamma, \phi' \text{ and } \gamma' \geq 0, \\
L_0 \text{ and } \phi \geq 1.
\end{cases}
\]

Moreover, in Section 6.4, we found the extra equation $L_0 + 2\gamma' \not\equiv 0 \mod 4$.

One can verify that there are no integer solutions satisfying these relations, for example by using the Python code below.
\[ V_{k+1} \cong \Sigma_2, \text{ AND } N(V_{k+1}) = \frac{4}{3} N \]

Suppose \( V_{k+1} \cong \Sigma_2, \) and \( N(V_{k+1}) = \frac{4}{3} N. \) The strategy described in Section 6.2.3 explains that we have to find integer solutions satisfying the following conditions:
\[
\begin{align*}
\phi' + 2\gamma' - \phi - L_0 &= 4, \\
3 \sum_{i=1}^{k+1} L_i + 3\gamma' + 6\phi' - 6\phi - \frac{9\gamma}{2} &= \frac{24}{N} - 10, \\
L_0 + 2 \sum_{i=1}^{k+1} L_i &= 20, \\
2\phi + 2\gamma &\leq \sum_{i=1}^{k+1} L_i, \\
\phi' + \gamma' &\leq L_0 + 1, \\
N &= \begin{cases} 
1 & \text{if } \gamma = 0, \\
2 & \text{if } \gamma > 0,
\end{cases} \\
\sum_{i=1}^{k+1} L_i, \gamma, \phi' \text{ and } \gamma' &\geq 0, \\
L_0 \text{ and } \phi &\geq 1.
\end{align*}
\]

One can verify that there are no integer solutions satisfying these relations, for example by using the Python code below.

```python
count = 0
for L1 in range(0, (20//2)+1):  # L0+2* sum L_i=20
    L0 = 20 - 2*L1
    if L0 == 0:
        # V_0 is non-minimal ruled, so L_0 != 0
        continue
    for phi in range(1, (L1//2)+1):  # 2* phi +2* gamma \leq sum L_i and phi \geq 1
        for gamma in range(0, ((L1 -2* phi )//2) +1 , 2):  # gamma even because of
            # index = 1
            if gamma == 0:
                N=1
            else :
                N=2
            for phi_ in range(0, (L0+1)+1):  # phi'+ gamma' \leq L_0 +1
                for gamma_ in range(0, (L0 - phi_ +1)+1):
                    # checking nothing went wrong
                    assert phi_>=0 and gamma_>=0 and phi>=1 and gamma>=0 and L0>=0 and (N==1 or N==2)
                    assert (L0+2*L1) == 20
                    assert (phi_+ gamma_) <= (L0 + 1)
                    assert (2*phi+2*gamma) <= L1
                    # checking the two conditions
                    if (phi_+2*gamma_-phi-L0) != 4:
                        # making sure the monodromy property does not hold
                        continue
                    if (3*L1+6*phi_+3*gamma_-6*phi-9*gamma//2) != ((24//N) -10):
                        # making sure degree of zeta function is 24
                        continue
                    # if we arrived here, we have found a combinatorial counterexample
                    count += 1
                    print('N =', N, ' L0 =',L0 , ' sum L_i =',L1 , ' phi =',phi , ' gamma =',gamma , ' phi\' =', phi_ , ' gamma\' =', gamma_)
print('\nWe have produced', count, 'combinatorial counterexamples.')
```
**B.9** \( V_{k+1} \) **is a rational, ruled surface,** \( \beta = k, \) **and** \( N(V_{k+1}) = 2N \)**

Suppose \( V_{k+1} \) is a rational, ruled surface, \( \beta = k, \) and \( N(V_{k+1}) = 2N. \) The strategy described in Section 6.2.3 explains that we have to find integer solutions satisfying the following conditions:

\[
\begin{align*}
\phi' + 2\gamma' - \phi - (L_0 + L_{k+1}) &= 8, \\
\sum_{i=1}^{k} L_i + 2\phi' + \gamma' - 2\phi - \frac{3\gamma}{2} &= \frac{24}{N}, \\
L_0 + 2\sum_{i=1}^{k} L_i + L_{k+1} &= 16, \\
\phi' + \gamma' &\leq L_0 + L_{k+1} + 2, \\
2\phi + 2\gamma &\leq \sum_{i=1}^{k} L_i, \\
N &= \begin{cases} 
1 & \text{if } \gamma = 0, \\
2 & \text{if } \gamma > 0,
\end{cases}
\sum_{i=1}^{k} L_i, \gamma, \phi' \text{ and } \gamma' \geq 0, \\
L_0 + L_{k+1} \text{ and } \phi \geq 1.
\end{align*}
\]

One can verify that there are no integer solutions satisfying these relations, for example by using the Python code below.

```python
count=0
# L1 for \sum_{i=1}^{k} L_i and L0 for L_0+L_{k+1}.
for L1 in range(0, (16//2)+1): # L0=2*L1
    L0 = 16 - 2*L1
    if L0 == 0: # V_0 is non-minimal ruled, so L_0 \neq 0.
        continue
    for phi in range(1, (L1//2)+1): # 2*phi+2*gamma \leq \sum L_i and phi \geq 1
        for gamma in range(0, ((L1-2*phi)//2)+1):
            # index = 1
            if gamma == 0:
                N=1
            else:
                N=2
            for phi_ in range(0, (L0+2)+1): # phi'+gamma' \leq L_0+L_{k+1}+2
                for gamma_ in range(0, (L0-phi_+2)+1):
                    # checking nothing went wrong
                    assert phi_>0 and gamma_>0 and phi>1 and gamma>0 and L0>1 and L1>0 and (N==1 or N==2)
                    assert (L0+2*L1) == 16
                    assert (phi_*gamma_) <= (L0+2)
                    assert (2*phi+2*gamma_) <= L1
            # checking the two conditions
            if (phi_*2*gamma_-phi-L0) != 8:
```
LIST OF COMBINATORIAL COUNTERCANDIDATES AND PYTHON CODE

```python
# making sure the mondromy property does not hold
continue
if (L1 + 2*phi_ + gamma_ - 2*phi - 3*gamma // 2) != (24 // N):
    # making sure the degree zeta function is 24
continue

# if we arrived here, we have found a combinatorial counterexample
count += 1
print('N =', N, ' L0 =', L0, ' sum L_i =', L1, ' phi =', phi, ' gamma =', gamma, ' phi \prime =', phi_, ' gamma \prime =', gamma_)
print('nWe have produced ', count, ' combinatorial counterexamples."

B.10 $V_{k+1}$ is an elliptic, ruled surface, and $N(V_{k+1}) = 2N$

Suppose $V_{k+1}$ is an elliptic, ruled surface, and $N(V_{k+1}) = 2N$. The strategy described in Section 6.2.3 explains that we have to find integer solutions satisfying the following conditions:

\[
\begin{align*}
\phi' + 2\gamma' - \phi - 2g - (L_0 + L_{k+1}) &= 2, \\
\sum_{i=1}^{k} L_i + 2\phi' + \gamma' - 2g - 2\phi - \frac{3\gamma}{2} &= \frac{24}{N} - 2, \\
L_0 + 2\sum_{i=1}^{k} L_i + L_{k+1} &= 8, \\
\phi' + \gamma' &\leq L_0 + L_{k+1} + 1, \\
2\phi + 2\gamma &\leq \sum_{i=1}^{k} L_i, \\
g &\leq (\sum_{i=1}^{k} L_i - 2\phi - 2\gamma + \gamma' + 2)/2 \\
N &= \begin{cases} 
1 & \text{if } \gamma = 0, \\
2 & \text{if } \gamma > 0, 
\end{cases} \\
\sum_{i=1}^{k} L_i, \gamma, \phi' \text{ and } \gamma' &\geq 0, \\
L_0 + L_{k+1}, g \text{ and } \phi &\geq 1.
\end{align*}
\]

One can verify that there are no integer solutions satisfying these relations, for example by using the Python code below.

```
V_{k+1} IS AN ELLIPTIC, RULED SURFACE, AND \( N(V_{k+1}) = 2N \)

```python
" index = 1
if gamma == 0:
    N=1
else:
    N=2

for phi_ in range(0,(L0+1)+1): # phi'+gamma' \leq L0+1
    for gamma_ in range(0, (L0-phi_+1)+1):
        for g in range(1, ((L1-2*phi-2*gamma+gamma_+2)//2) +1):
            # checking nothing went wrong
            assert phi_>=0 and gamma_>=0 and phi>=1 and gamma>=0 and (N==1 or N==2)
            assert g>=1 and L0>=1 and L1>=0
            assert (L0+2*L1) == 8
            assert (phi_*gamma_) <= (L0+1)
            assert (2*phi+2*gamma) <= L1
            assert g <= (L1-2*phi*gamma+gamma_+2) //2

            # checking the two conditions
            if (phi_+2*gamma_)!=(2+L0+2*phi):
                # making sure the mondromy property does not hold
                continue
            if (L1+2*phi_+gamma_-2*phi-3*gamma//2)!=(24//N-2):
                # making sure the degree zeta function is 24
                continue

            # if we arrived here , we have found a combinatorial
counterexample
            count += 1
            print('N =', N, ' L0 =',L0 , ' sum L_i =',L1 , ' L_{k+1} =',L2 , '
            phi =',phi , ' gamma =', gamma,\'
            phi' =', phi_ , ' gamma =', gamma_ , ' g =', g)

            print('
We have produced ', count, ' combinatorial counterexamples.')
```
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