COFINITE CONNECTEDNESS AND COFINITE GROUP ACTIONS

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ABSTRACT. We have defined and established a theory of cofinite connectedness of a cofinite graph. Many of the properties of connectedness of topological spaces have analogs for cofinite connectedness. We have seen that if \( G \) is a cofinite group and \( \Gamma = \Gamma(G, X) \) is the Cayley graph. Then \( \Gamma \) can be given a suitable cofinite uniform topological structure so that \( X \) generates \( G \), topologically iff \( \Gamma \) is cofinitely connected.

Our immediate next concern is developing group actions on cofinite graphs. Defining the action of an abstract group over a cofinite graph in the most natural way we are able to characterize a unique way of uniformizing an abstract group with a cofinite structure, obtained from the cofinite structure of the graph in the underlying action, so that the afore said action becomes uniformly continuous.

1. Introduction

A cofinite graph \( \Gamma \) is said to be **cofinitely connected** if for each compatible cofinite equivalence relation \( R \) on \( \Gamma \), the quotient graph \( \Gamma/R \) is path connected.

Similar to the standard connectedness arguments for finite graphs or general topological spaces we were able to establish that the following statements are equivalent for any cofinite graph \( \Gamma \):

1. \( \Gamma \) is cofinitely connected;
2. \( \Gamma \) is not the union of two disjoint nonempty subgraphs.

As an immediate consequence we obtained the following generalized characterization of connected Cayley graphs of cofinite groups:

Let \( G \) be a cofinite group and let \( \Gamma = \Gamma(G, X) \) be the Cayley graph. Then \( \Gamma \) can be given a suitable cofinite topological graph structure so that \( X \) generates \( G \) (topologically) iff \( \Gamma \) is cofinitely connected.

Our final section is concerned with cofinite group actions on cofinite graphs.

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A group $G$ is said to act uniformly equicontinuously over a cofinite graph $\Gamma$ if and only if for each entourage $W$ over $\Gamma$ there exists an entourage $V$ over $\Gamma$ such that for all $g$ in $G, (g \times g)[V] \subseteq W$. In this case the group action induces a (Hausdorff) cofinite uniformity over $G$ if and only if the aforesaid action is faithful.

We say that a group $G$ acts on a cofinite graph $\Gamma$ residually freely, if there exists a fundamental system of $G$-invariant compatible cofinite entourages $R$ over $\Gamma$ such that the induced group action of $G/N_R$ over $\Gamma/R$ is a free action, where $N_R$ is the Kernel of the action of $G$ on $\Gamma/R$.

Suppose that $G$ is a group acting faithfully and uniformly equicontinuously on a cofinite graph $\Gamma$, then the action $G \times \Gamma \to \Gamma$ is uniformly continuous. Also in that case $\hat{G}$ acts on $\hat{\Gamma}$ uniformly equicontinuously.

2. Connected Cofinite Graphs

A path in a graph $\Gamma$ is a finite string of edges $p = e_1 \cdots e_n \in E(\Gamma)$ such that $t(e_i) = s(e_{i+1})$ for $1 \leq i \leq n - 1$. The source and target of this path $p$ are the vertices $s(p) = s(e_1)$ and $t(p) = t(e_n)$. We say that $\Gamma$ is path connected if there is a path in $\Gamma$ joining any two vertices.

**Definition 2.1.** A cofinite graph $\Gamma$ is cofinitely connected if for each compatible cofinite equivalence relation $R$ on $\Gamma$, the quotient graph $\Gamma/R$ is path connected.

**Proposition 2.2.** The following statements are equivalent for any cofinite graph $\Gamma$:

1. $\Gamma$ is cofinitely connected;
2. $\Gamma$ is not the uniform sum of two disjoint nonempty subgraphs.
3. the completion $\overline{\Gamma}$ of $\Gamma$ is cofinitely connected.

**Proof.** (1) $\implies$ (2): If possible, let us assume that $\Gamma$ is the uniform sum of two disjoint subgraphs $\Gamma_1$ and $\Gamma_2$. Let $R_{\Gamma_1}$ be a compatible cofinite entourage over $\Gamma_1$ and $S_{\Gamma_2}$ be another compatible cofinite entourage over $\Gamma_2$. Then $W = R_{\Gamma_1} \cup S_{\Gamma_2}$ is a compatible cofinite entourage over $\Gamma$. Moreover $\Gamma/W$ is not path connected, a contradiction.

(2) $\implies$ (3): If possible, let us assume that $\overline{\Gamma}$ is not cofinitely connected. Hence there exists a cofinite entourage $W$ over $\overline{\Gamma}$ such that $\overline{\Gamma}/W$ is not path connected.

Let $\Sigma$ be a path connected component of $\overline{\Gamma}/W$. Hence $\Sigma$ is a subgraph of $\overline{\Gamma}/W$ and thus $(\overline{\Gamma}/W) \setminus \Sigma$ is a subgraph of $\overline{\Gamma}/W$ as well. Let $\Gamma_1 = \phi^{-1}(\Sigma)$ and $\Gamma_2 = \phi^{-1}(\overline{\Gamma} \setminus \Sigma)$, where $\phi: \overline{\Gamma} \to \overline{\Gamma}/W$ is the...
canonical quotient map. Then $\Gamma_1, \Gamma_2$ are closed subgraphs of $\Gamma$ such that $\Gamma$ is equal to the disjoint union of two closed subgraphs of $\Gamma$ and then $\Gamma$ is equal to the uniform sum of two disjoint subgraphs of $\Gamma$, a contradiction.

$(3) \Rightarrow (1)$: If possible assume that $\Gamma$ is not cofinitely connected. Then there exists a cofinite entourage $R$ over $\Gamma$ such that $\Gamma/R$ is not path connected. But, $R$ is a compatible cofinite entourage over $\Gamma$ such that $\Gamma/R$ is graph isomorphic to $\Gamma/R$. Hence $\Gamma/R$ is not path connected as well, a contradiction. □

Many of the properties of connectedness of topological spaces have analogs for cofinite connectedness. Next we list a few of them.

**Proposition 2.3.** Let $\Gamma$ be a cofinite graph and let $\Sigma$ be a uniform subgraph.

1. If $\Sigma$ is path connected, then it is also cofinitely connected.
2. If $\Sigma$ is cofinitely connected, then so is the cofinite subgraph $\Sigma$.
3. If $\Sigma$ is cofinitely connected and $f: \Gamma \to \Delta$ a uniformly continuous map of graphs, then $f(\Sigma)$ is also cofinitely connected (as a cofinite subgraph of $\Delta$).

**Proof.** Note that $\Sigma$ is also a cofinite graph

1. If $\Sigma$ is path connected then any quotient graph of $\Sigma$ is path connected as well and thus our claim follows.
2. We will first see that $\Sigma = V(\Sigma) \cup E(\Sigma) = V(\bar{\Sigma}) \cup E(\bar{\Sigma})$ and that equals $V(\bar{\Sigma}) \cup E(\bar{\Sigma})$ and thus is a cofinite subgraph of $\Gamma$ as well. Now, if possible suppose $\Sigma = \Sigma_1 \coprod \Sigma_2$, where $\Sigma_1, \Sigma_2$ are two disjoint nonempty cofinite subgraphs of $\Sigma$. Then $\Sigma_1 \cap \Sigma, \Sigma_2 \cap \Sigma$ are two disjoint connected cofinite subgraphs of $\Sigma$. Let $R_1, R_2$ be two compatible cofinite entourage over $\Sigma_1 \cap \Sigma, \Sigma_2 \cap \Sigma$ respectively. Then there exist two compatible cofinite entourages $\bar{R}_1, \bar{R}_2$ over $\Sigma_1, \Sigma_2$ respectively such that $R_1 \supseteq \bar{R}_1 \cap (\Sigma \times \Sigma)$ and $R_2$ contains $\bar{R}_2 \cap (\Sigma \times \Sigma)$. But as $\bar{R}_1 \cup \bar{R}_2$ is a compatible cofinite entourage over $\Sigma$, then $(\bar{R}_1 \cup \bar{R}_2) \cap (\Sigma \times \Sigma)$ is equal to $\bar{R}_1 \cap (\Sigma \times \Sigma) \cup \bar{R}_2 \cap (\Sigma \times \Sigma)$ which is a subset of $R_1 \cup R_2$. So $R_1 \cup R_2$ is a compatible entourage over $\Sigma$. Hence $\Sigma = (\Sigma_1 \cap \Sigma) \coprod (\Sigma_2 \cap \Sigma)$. Now suppose $\Sigma_1 \cap \Sigma = \emptyset$. Then $\Sigma \subseteq \Sigma_2$. However $\Sigma_2$ is closed in $\Sigma$ and hence closed in $\Gamma$. Thus $\Sigma \subseteq \Sigma_2$ and therefore $\Sigma_1 = \emptyset$, a contradiction. Thus $\Sigma$ is cofinitely connected.
3. Let $S$ be a compatible cofinite entourage over $f(\Sigma)$. Then as $f|_\Sigma: \Sigma \to f(\Sigma)$ is uniformly continuous there is a compatible
cofinite entourage \( R \) over \( \Sigma \) such that \( R \subseteq (f \times f)^{-1}[S] \). Let us define \( g: \Sigma/R \to f(\Sigma)/S \) via \( g(R[a]) = S[f(a)] \), for all \( a \in \Sigma \). Now if \( R[a] = R[b] \), then \( (a, b) \in R \). Hence \( (f(a), f(b)) \) is in \( S \) which implies that \( S[f(a)] = S[f(b)] \). Therefore \( g \) is well defined and as \( f \) is a map of graphs and both of \( \Sigma/R, f(\Sigma)/S \) are discrete, \( g \) is a surjective uniformly continuous map of graphs. Since \( \Sigma/R \) is path connected then so is \( g(\Sigma/R) = f(\Sigma)/S \).

\[ \square \]

3. Cofinite Groups and their Cayley Graphs

**Definition 3.1.** Let \( G \) be an abstract group and \( X = \{\ast\} \cup E(X) \) be an abstract graph such that there is a map of sets \( \alpha: X \to G \) with \( \alpha(\ast) = 1_G \), \( (\alpha(e))^{-1} = \alpha(\overline{e}) \), for all \( e \in E(X) \). Then the Cayley Graph \( \Gamma(G, X) \) is defined as follows:

1. \( V(\Gamma(G, X)) = G \times \{\ast\}, E(\Gamma(G, X)) = G \times E(X) \).
2. \( s(g, e) = (g, \ast), t(g, e) = (g\alpha(e), \ast), (g, e) = (g\alpha(e), \overline{e}) \).

Thus it follows that

1. \( \Gamma(G, X) = V(\Gamma(G, X)) \cup E(\Gamma(G, X)) \).
2. \( s, t, \overline{\cdot} \) are well defined and \( t((g, e)) = t(g\alpha(e), \overline{e}) = (g\alpha(e)\alpha(e)^{-1}, \ast) = (g, \ast) = s(g, e); s((g, e)) = s(g\alpha(e), \overline{e}) = (g\alpha(e), \ast) = t(g, e) \).
3. If possible, let \( (g, e) = (g, \overline{e}) = ((g\alpha(e), \overline{e}) \) and thus \( e = \overline{e} \), a contradiction. Finally, \( (g, e) = (g\alpha(e), \overline{e}) \) = (\( g\alpha(e)\alpha(e)^{-1}, e \) = (\( g, e \). Hence \( \Gamma(G, X) \) is indeed a graph.

We say that \( \alpha: X \to G \) generates \( G \) algebraically if \( \langle \alpha(X) \rangle = G \). Equivalently, \( \alpha: X \to G \) generates \( G \) algebraically if the unique extension to \( \alpha: E(X)^* \to G \) is onto.

**Lemma 3.2.** The Cayley graph \( \Gamma(G, X) \) is path connected if and only if \( \alpha: X \to G \) generates \( G \) algebraically.

**Definition 3.3.** Let \( G \) be a cofinite group and \( X = \{\ast\} \cup E(X) \) be a cofinite graph such that there is a uniform continuous map of spaces \( \alpha: X \to G \) with \( \alpha(\ast) = 1_G \), \( (\alpha(e))^{-1} = \alpha(\overline{e}) \), for all \( e \in E(X) \). Then the cofinite Cayley Graph \( \Gamma(G, X) \) is defined as follows:

1. \( V(\Gamma(G, X)) = G \times \{\ast\}, E(\Gamma(G, X)) = G \times E(X) \).
2. \( s(g, e) = (g, \ast), t(g, e) = (g\alpha(e), \ast), (g, e) = (g\alpha(e), \overline{e}) \).

\( \Gamma(G, X) \) is endowed with the product uniform topological structure obtained from \( G \times X = G \times V(\Gamma(G, X)) \) \( \cup G \times E(\Gamma(G, X)) \).
We have already seen that $\Gamma(G, X)$ is an abstract graph. Also being the product of Hausdorff, cofinite spaces, $\Gamma(G, X)$ is a Hausdorff, cofinite space as well. So in order to check that $\Gamma(G, X)$ is a cofinite graph it remains to prove that the compatible cofinite entourages over $\Gamma(G, X)$ forms a fundamental system of entourages. So it suffices to show that the family of cofinite entourages of the form $R \times S$, where $R$ is a cofinite congruence over $G$ and $S$ is a compatible cofinite entourage over $X$ such that $(\alpha \times \alpha)[S] \subseteq R$ forms a fundamental system of entourages.

To establish the above claim let us first see that the cofinite entourages of the form $R \times S$ are indeed compatible.

1. Let $((x, y), (p, q)) \in R \times S$. So $(x, p) \in R \subseteq G \times G$ and $(y, q)$ is in $S$. Thus either $(y, q) \in S_E$ or $(y, q) \in S_V$ which implies that $y = * = q$ or $(y, q) \in S_E$. Hence $(x, y), (p, q) \in V(\Gamma(G, X))$ or $(x, y), (p, q) \in E(\Gamma(G, X))$. Hence $R \times S \subseteq (R \times S)_V \cup (R \times S)_E$. The other direction of the inclusion follows more immediately.

2. Let $((g_1, e_1), (g_2, e_2)) \in R \times S$. Then $(g_1, g_2) \in R$ and $(e_1, e_2)$ is in $S$. This implies that $(\alpha \times \alpha)(e_1, e_2) = (\alpha(e_1), \alpha(e_2)) \in R$ and $(g_1, g_2) \in S$. Hence $(g_1 \alpha(e_1), g_2 \alpha(e_2)) \in R$, which implies $((g_1, *), (g_2, *))$ and $((g_1 \alpha(e_1), *), (g_2 \alpha(e_2), *))$ as well as $((g_1 \alpha(e_1), \overline{e_1}), (g_2 \alpha(e_2), \overline{e_2}))$ is in $R \times S$. Hence $(s(g_1, e_1), s(g_2, e_2)), (t(g_1, e_1), t(g_2, e_2))$ and $((g_1, e_1), (g_2, e_2)) \in R \times S$.

3. If possible let $((\overline{g_1}, e_1), (g_1, e_1)) \in R \times S$ so $((g_1 \alpha(e_1), \overline{e_1}), (g_1, e_1))$ is in $R \times S$. Thus $((\overline{e_1}, e_1), (g_1, e_1)) \in R \times S$, a contradiction.

Now let $R \times T$ be any cofinite entourage over $G \times X$. Note that since $\alpha$ is uniformly continuous and $R$ is a cofinite congruence over $G$, $T$ is a cofinite entourage over $X$, $(\alpha \times \alpha)^{-1}[R] \cap T$ is a cofinite entourage over $X$ and $(\alpha \times \alpha)[(\alpha \times \alpha)^{-1}[R] \cap T] \subseteq R$. So in particular one can take $S$ to be a compatible cofinite entourage over $X$ such that $S \subseteq (\alpha \times \alpha)^{-1}[R] \cap T$. Then $(\alpha \times \alpha)[S] \subseteq R$ and $R \times S \subseteq R \times T$. This proves that $\Gamma(G, X)$ is a cofinite graph. We say that $\alpha: X \to G$ generates $G$ topologically if $[\alpha(X)] = G$.

**Theorem 3.4.** Let $\Gamma = \Gamma(G, X)$ be the cofinite Cayley graph. $\alpha$ from $X$ to $G$ generates $G$ topologically iff $\Gamma$ is cofinitely connected.

**Proof.** Let us first assume that $\alpha: X \to G$ topologically generates $G$ and let $T$ be a compatible cofinite entourage over $\Gamma$, say $T$ is equal to $R \times S$ where $R$ is a cofinite congruence over $G$ and $S$ is a compatible cofinite entourage over $X$ where $S \subseteq (\alpha \times \alpha)^{-1}[R]$. Let us define $\alpha_{RS}: X/S \to G/R$ via $\alpha_{RS}(S[x]) = R[\alpha(x)]$. Clearly, $\alpha_{RS}$ is well defined and $\alpha_{RS}(S[*]) = R[1_G]$ and $\alpha_{RS}(S[e]) = R[\alpha(e)] = R[(\alpha(e))^{-1}]$.
and that is equal to $R$ which we know is equal to $T$ if and only if $(h, y, (g, x)) \in T$ for all $x$ in $X$ and all $g$ in $G$. Clearly, it is well defined injection as $T([(h, y)] = T([(g, x)]$ if and only if $(h, y, (g, x)) \in T$ if and only if $(h, g) \in R, (y, x) \in S$ if and only if $R[h] = R[g]$ and $S[x] = S[y]$ if and only if $(R[h], S[y]) = (R[g], S[x])$. Also for all $(R[g], S[x]) \in \Gamma(G/R, X/S)$, there exists $T([(g, x)]) \in \Gamma/T$ such that $\theta(T([(g, x)])) = (R[g], S[x])$. Moreover it can easily be seen that $\theta$ is a map of graphs as $\theta(T([(g, *)])$ which is equal to $(R[g], S[*])$ belongs to $V(\Gamma(G/R, X/S))$ and $\theta(T([(g, e)])$ which equals to $(R[g], S[e])$ belongs to $E(\Gamma(G/R, X/S))$. Further more for all $(T([(g, e)])$ in $E(\Gamma/T)$ we see that $\theta(s(T([(g, e)])) = \theta(T[s(g, e)])$ which also equals to $\theta(T([(g, *)])$ equal to $(R[g], S[*])$ and that equals to $s(R[g], S[e])$. We also notice that $\theta(t(T([(g, e)])) = \theta(T(t(g, e)]) = \theta(T([(g\alpha(e), *)]) = (R[g\alpha(e)], S[*])$ which we know is equal to $(R[g]R[\alpha(e)], S[*]) = (R[g\alpha](S[e]), S[*])$ and that is equal to $\theta(R[g], S[e])$. Finally, $\theta(T([(g\alpha(e), \overline{e})])) = \theta(T([(g, e)])$ and that equals $\theta(T([(g\alpha(e), \overline{e})])) = (R[g\alpha(e)], S[\overline{e}])$ which can be written as $(R[g]R[\alpha(e)], S[\overline{e}]) = (R[g\alpha](S[e]), S[\overline{e}])$ and that equals $(R[g], S[e])$. Since $\Gamma/T, \Gamma(G/R, X/S)$ are discrete cofinite graphs, our claim follows.

Now we wish to prove that $\langle \alpha_{RS}(X/S) \rangle = G/R$. Let $R[g] \in G/R$. Then as $\langle \alpha(X) \rangle = G$, we have $R[g] \cap \langle \alpha(X) \rangle \neq \emptyset$. Let $a \in R[g] \cap \langle \alpha(X) \rangle$. So, $R[g] = R[a]$. Also, since $a \in \langle \alpha(X) \rangle$, $a = \alpha_e \cdots \alpha_{e_n}$, for some $e_1, e_2, \cdots, e_n \in E(X)$. Hence $R[a] = R[\alpha(e_1)] \cdots R[\alpha(e_n)]$, and one can represent this as $\alpha_{RS}(S[e_1]) \cdots \alpha_{RS}(S[e_n])$. Thus $R[g] = R[a] \in \langle \alpha_{RS}(X/S) \rangle$. Therefore $\langle \alpha_{RS}(X/S) \rangle = G/R$ and consequently, $\Gamma/T = \Gamma(G/R, X/S)$ is path connected. Hence $\Gamma$ is cofinitely connected.

Conversely, let us now take $\Gamma$ to be cofinitely connected. We want to show that $\langle \alpha(X) \rangle = G$. So we intend to show that for any $g$ in $G$ and any open set $R[g]$ on $G, R[g] \cap \langle \alpha(X) \rangle \neq \emptyset$. We can form a compatible cofinite entourage $T = R \times S$ where $S$ is a compatible cofinite entourage over $X$ and $S \subseteq (\alpha \times \alpha)^{-1}[R]$. As earlier we can form the Cayley graph $\Gamma/T = \Gamma(G/R, X/S)$ and as $\Gamma$ is cofinitely connected, $\Gamma/T$ and therefore $\Gamma(G/R, X/S)$, is path connected. This implies $\langle \alpha_{RS}(X/S) \rangle = G/R$. So there is $e_1, e_2, \cdots, e_n$ in $E(X)$ such that $\alpha_{RS}(S[e_1]) \alpha_{RS}(S[e_2]) \cdots \alpha_{RS}(S[e_n]) = R[g]$. Thus we can finally say that $\alpha_{e_1} \cdots \alpha_{e_n} \in R[g]$ which means $\langle \alpha(X) \rangle \cap R[g] \neq \emptyset$ and thus $\langle \alpha(X) \rangle = G$. Hence $\alpha: X \to G$ topologically generates $G$. 

\[ \text{\square} \]
4. Groups Acting on Cofinite Graphs

Let $G$ be a group and $\Gamma$ be a cofinite graph. We say that the group $G$ acts over $\Gamma$ if and only if

1. For all $x$ in $\Gamma$, for all $g$ in $G$, $g.x$ is in $\Gamma$
2. For all $x$ in $\Gamma$, for all $g_1, g_2$ in $G$, $g_1(g_2.x) = (g_1g_2).x$
3. For all $x$ in $\Gamma$, $1.x = x$
4. For all $v$ in $V(\Gamma)$, for all $g$ in $G$, $g.v$ is in $V(\Gamma)$ and for all $e$ in $E(\Gamma)$, for all $g$ in $G$, $g.e$ is in $E(\Gamma)$.
5. For all $e$ in $E(\Gamma)$, for all $g$ in $G$, $g.s(e) = s(g.e), g.t(e) = t(ge), g.(\overline{e}) = \overline{g.e}$
6. There exists a $G$-invariant orientation $E^+(\Gamma)$ of $\Gamma$.

Note that the aforesaid group action restricted to a singleton group element $g \in G$ can be treated as a well defined map of graphs, $\Gamma \to \Gamma$ taking $x \mapsto g.x$.

**Definition 4.1.** A group $G$ is said to act uniformly equicontinuously over a cofinite graph $\Gamma$, if and only if for each entourage $W$ over $\Gamma$ there exists an entourage $V$ over $\Gamma$ such that for all $g$ in $G$, $(g \times g)[V]$ is a subset of $W$.

**Lemma 4.2.** If $G$ acts uniformly equicontinuously over a cofinite graph $\Gamma$, then there exists a fundamental system of entourages consisting of $G$-invariant compatible cofinite entourages over $\Gamma$, i.e. for any entourage $U$ over $\Gamma$ there exists a compatible cofinite entourage $R$ over $\Gamma$ such that for all $g \in G, (g \times g)[R] \subseteq R \subseteq U$.

**Proof.** Let $U$ be any cofinite entourage over $\Gamma$. Then as $G$ acts uniformly equicontinuously over $\Gamma$, there exists a compatible cofinite entourage $S$ over $\Gamma$ such that for all $g \in G, (g \times g)[S] \subseteq U$. Choose a $G$-invariant orientation $E^+(\Gamma)$ of $\Gamma$. Without loss of generality, we can assume that our compatible equivalence relation $S$ on $\Gamma$ is orientation preserving i.e. whenever $(e, e') \in R$ and $e \in E^+(\Gamma)$, then also $e' \in E^+(\Gamma)$. Clearly, $S \subseteq \cup_{g \in G}(g \times g)[S] \subseteq U$. Now if $S_0 = \cup_{g \in G}(g \times g)[S]$ and $T = \langle S_0 \rangle$, note that $S \subseteq T \subseteq U$. Since for all $h \in G, (h \times h)[S_0] = S_0$ and $S_0^{-1} = S_0$ it follows that $T$ is in the transitive closure of $S_0$. Let $(x, y) \in T$. Then there exists a finite sequence $x_0, x_1, \ldots, x_n$ such that $(x_i, x_{i+1}) \in S_0$, for all $i = 0, 1, 2, \ldots, n - 1$ and $x = x_0, y = x_n$. Hence $(gx_i, gx_{i+1}) \in S_0$, for all $i = 0, 1, 2, \ldots, n - 1$, for all $g \in G$. Thus $(gx_0, gx_n) = (gx, gy) \in T$, for all $g \in G$. Hence for all $g \in G, (g \times g)[T] \subseteq T$ and our claim that $T$ is a $G$-invariant cofinite entourage, follows. It remains to check that $T$ is compatible. Let $(x, y) \in T$. If $(x, y) \in S_0$, then there is $(t, s) \in S = S_V \cup S_E$ and $g \in G$ such that $(gt, gs) = (x, y)$. Without loss of generality let
$(t, s) \in S_V$. Then $(t, s) \in V(\Gamma) \times V(\Gamma)$ which implies that $(x, y) \in T_V$. Thus $(x, y) \in T \setminus S_0$. Then there exists a finite sequence $x_0, x_1, ..., x_n$ such that $(x_i, x_{i+1}) \in S_0$, for all $i = 0, 1, 2, ..., n-1$ and $x = x_0, y = x_n$. Hence by the previous argument if $(x_0, x_1) \in T_V$ then $(x_i, x_{i+1}) \in T_V$, for all $i = 1, 2, ..., n-1$. Finally, to show that for any $e, p, q \in G$ such that $(gp, gq) = (e_1, e_2)$. Then $(s(p), s(q)) \in S$. So $(s(e_1), s(e_2))$ which equals to $(gs(p), gs(q))$ is in $(g \times g)[S] \subseteq S_0$ so that $(s(e_1), s(e_2)) \in T$. Now let $(e_1, e_2) \in T \setminus S_0$. Then there exists a finite sequence $x_0, x_1, ..., x_n$ such that $(x_i, x_{i+1}) \in S_0, \forall i = 0, 1, 2, ..., n-1$ and $e_1 = x_0, e_2 = x_n$. Hence by the previous argument $(s(x_i), s(x_{i+1})) \in T, \forall i = 0, 1, 2, ..., n-1$ and thus $(s(e_1), s(e_2)) \in T$. Similarly, $(t(e_1), t(e_2)) \in T$ and $(e_1, e_2) \in T$. Finally, to show that for any $e \in E^+(\Gamma)$, $(e \Gamma) \in T$ it suffices to note that $T$ is orientation preserving. Alternatively, if possible let $(e, \overline{e}) \in T$. If $(e, \overline{e}) \in S_0$, then there is $(p, q) \in S$ and $g \in G$ such that $(gp, gq) = (e, \overline{e})$. Then $\overline{e} = \overline{g \overline{p}} = g \overline{p} = gq$ which implies that $\overline{p} = q$, so $(p, \overline{p}) \in S$, a contradiction. Now let $(e, \overline{e}) \in T \setminus S_0$. Then there exists a finite sequence $x_0, x_1, ..., x_n$ such that $(x_i, x_{i+1}) \in S_0$, for all $i = 0, 1, 2, ..., n-1$ and $e = x_0, \overline{e} = x_n$. Now let there is $(p, q) \in S$ and $g \in G$ such that $(gp, gq) = (x_0, x_1)$. Without loss of generality we may assume $(p, q) \in E^+(\Gamma) \times E^+(\Gamma)$. Then $(gp, gq) = (x_0, x_1) \in E^+(\Gamma) \times E^+(\Gamma)$. Hence $(x_i, x_{i+1}) \in E^+(\Gamma) \times E^+(\Gamma),$ for all $i = 1, 2, ..., n-1$ which implies that $(e, \overline{e}) \in E^+(\Gamma) \times E^+(\Gamma)$, a contradiction. Our claim follows. 

**Definition 4.3.** We say a group $G$ acts on a cofinite space $\Gamma$ faithfully, if for all $g \in G \setminus \{1\}$ there exists $x \in \Gamma$ such that $gx$ is not equal to $x$ in $\Gamma$.

**Lemma 4.4.** Let $G$ acts on a cofinite graph $\Gamma$ uniformly equicontinuously. Then $G$ acts on $\Gamma/R$ and $G/N_R$ acts on $\Gamma/R$ as well, where $R$ is a $G$-invariant compatible cofinite entourage over $\Gamma$. If $\{R R \mid R \in I\}$ is a fundamental system of $G$-invariant compatible co-entourages over $\Gamma$, then $\{N_R R \mid R \in I\}$ forms a fundamental system of cofinite congruences for some uniformity over $G$.

**Proof.** Let $R$ be a $G$-invariant compatible cofinite entourage over $\Gamma$. Let us define $G \times \Gamma/R \rightarrow \Gamma/R$ via $g.R[x] = R[g.x]$, for all $g \in G$, for all $x \in \Gamma$. Now let $R[x] = R[y]$ so $(x, y) \in R$ which implies that $(g.x, g.y) \in R$. Then $R[g.x] = R[g.y]$. Hence the induced group action is well defined.

Let us now consider the group action $G/N_R \times \Gamma/R \rightarrow \Gamma/R$, defined via $N_R[g].R[x] = R[g.x]$ for all $x \in \Gamma$, for all $g \in G$. Now let $(N_R[g], R[x]) = (N_R[h], R[y])$ which implies that $(g, h) \in N_R, (x, y)$ is
in $R$. Then $(g.x, h.x) \in R$, as $h^{-1} \in G$, $(h^{-1}g.x, h^{-1}h.x) \in R$. So $(h^{-1}g.x, y) \in R$. Thus $(g.x, h.y) \in R$ which implies that $R[g.x]$ equals to $R[h.g]$. Hence the induced group action is well defined. Let us now show that $N_R$ is an equivalence relation over $G$, for all $G$-invariant compatible cofinite entourage $R$ over $\Gamma$.

(1) for all $g \in G$, for all $x \in \Gamma, (g.x, g.x) \in R$. Hence $(g, g) \in N_R$, for all $g \in G$ which implies that $D(G) \subseteq N_R$.

(2) Now $(h, g) \in N_R^{-1} \iff (g, h) \in N_R \iff (g.x, h.x) \in R$, for all $x \in \Gamma$. Thus $(g.x, h.x) \in R \iff (h.x, g.x) \in R$, for all $x \in \Gamma$. Hence $(h.x, g.x) \in R \iff (h, g) \in N_R$. Thus $N_R^{-1} = N_R$.

(3) Let $(g, h), (h, k) \in N_R$. This implies $(g.x, h.x), (h.x, k.x)$ is in $R, \forall x \in \Gamma$. Hence $(g.x, k.x) \in R$, for all $x \in \Gamma$. So $(g, k) \in N_R$ which implies that $(N_R)^2 \subseteq N_R$.

Also we now check that $N_R$ is a congruence over $G$. For, let us take $(g_1, g_2), (g_3, g_4) \in N_R$. Then for all $x \in \Gamma, (g_1.x, g_2.x), (g_3.x, g_4.x) \in R$; for all $x \in \Gamma, g_3.x \in \Gamma$ and so $(g_1.g_3.x, g_2.g_3.x) \in R$ and $(g_2.g_3.x, g_2.g_4.x)$ is in $R$, since $R$ is $G$-invariant. Thus $(g_1.g_3.x, g_2.g_4.x) \in R$, for all $x \in \Gamma$ so that $(g_1.g_3, g_2.g_4) \in N_R$. Thus our claim follows. Let us now show that $G/N_R$ is finite. Furthermore, define $g: \Gamma/R \to \Gamma/R$ as $g$ maps $(R[x])$ into $R[g.x]$. Now, $R[x] = R[y] \iff (x, y) \in R$ if and only if $(g.x, g.y) \in R \iff R[g.x] = R[g.y]$. Hence the map $g$ is a well defined injection. Now for all $R[x] \in \Gamma/R$ there exists $g^{-1}R[x] \in \Gamma/R$ such that $g(g^{-1}R[x])$ equals to $R[x]$. Hence $g \in Sym(\Gamma/R)$. Now let us define a map $\theta: G/N_R \to Sym(\Gamma/R)$ via $\theta(N_R[g]) = g$. Now $N_R[g_1]$ equals to $N_R[g_2]$ if and only if $(g_1, g_2) \in N_R$ if and only if $(g_1.x, g_2.x) \in R$ for all $x \in \Gamma$. Hence $(g_1.x, g_2.x) \in R$ if and only if $R[g_1.x] = R[g_2.x]$ if and only if $g_1(R[x]) = g_2(R[x])$ at $g_1 = g_2$ in $Sym(\Gamma/R)$. Hence $\theta$ is a well defined injection. Thus $|G/N_R| \leq |Sym(\Gamma/R)| < \infty$ as $|\Gamma/R| < \infty$.

So, next we will like to show that $\{N_R \mid R \in I\}$ forms a fundamental system of cofinite congruences over $G$.

(1) $D(G) \subseteq N_R$, for all $R \in I$, as $N_R$ is reflexive.

(2) Now for some $R, S \in I, (g_1, g_2) \in N_R \cap N_S$ if and only if $(g_1.x, g_2.x) \in R \cap S$, for all $x \in \Gamma$ \iff $(g_1, g_2) \in N_R \cap N_S$. Thus $N_R \cap N_S = N_R \cap N_S$.

(3) For all $N_R, N_R^2 = N_R$, as $N_R$ is transitive.

(4) For all $N_R, N_R^{-1} = N_R$, as $N_R$ is symmetric.

Hence our claim follows. □

**Definition 4.5.** We say that a group $G$ acts on a cofinite graph $\Gamma$ residually freely, if there exists a fundamental system of $G$-invariant
compatible cofinite entourages $R$ over $\Gamma$ such that the induced group action of $G/N_R$ over $\Gamma/R$ is a free action.

**Lemma 4.6.** $N_R[1]$ is a finite index normal subgroup of $G$ and $G/N_R[1]$ is isomorphic with $G/N_R$. More generally, if $N$ is a congruence on $G$, then $N[1]$ is a normal subgroup of $G$ and $G/N[1] \cong G/N$.

**Proof.** Let us first see that $N_R[1] \triangleleft_f G$ for all $G$-invariant compatible cofinite entourage $R$ over $\Gamma$. Let $g, h \in N_R[1]$. This implies $(1, g) \in N_R$ and hence $(g, 1), (1, h) \in N_R$. Thus $(g, h) \in N_R$. This implies $(g, x, h, x)$ is in $R$, for all $x \in \Gamma$ and so $(x, g^{-1}h, x) \in R$, for all $x \in \Gamma$. Hence, $(1, g^{-1}h)$ is in $N_R$ and thus $g^{-1}h \in N_R[1]$. So, $N_R[1] \leq G$. For all $g \in G$, for all $x \in \Gamma, g.x \in \Gamma$. Hence for all $k \in N_R[1], (x, k.x) \in R$, hence $(k.x, x)$ is in $R$. Thus $(kg.x, g.x) \in R$ and $(g^{-1}kg.x, g^{-1}g.x) = (g^{-1}kg.x, x) \in R$. Hence $(g^{-1}kg, 1) \in N_R$. So, $g^{-1}kg \in N_R[1]$ and thus $N_R[1] \triangleleft G$. Now let us define $\eta$ from $G/N_R[1]$ to $G/N_R$ via $\eta(gN_R[1]) = gN_R[1]$. Then, $gN_R[1]$ is equal to $hN_R[1]$ if and only if $h^{-1}g \in N_R[1]$ if and only if $(1, h^{-1}g) \in N_R$ if and only if $(x, h^{-1}g, x) \in R$ if and only if $(h, x, g.x) \in R$ if and only if $(h, g) \in N_R$ if and only if $N_R[h] = N_R[g]$, for all $x \in \Gamma$. Thus $\eta$ is a well defined injection and hence $|G/N_R[1]| \leq |G/N_R| < \infty$. Hence $N_R[1] \triangleleft_f G$. Let us check that $G/N_R$ is a group. For, let $N_R[g_i]$ is in $G/N_R$, $i = 1, 2$. Then $N_R[g_1]N_R[g_2] = N_R[g_1g_2] \in G/N_R$. Let $N_R[g_i]$ in $G/N_R$, for $i = 1, 2, 3$. Then $(N_R[g_1]N_R[g_2])N_R[g_3]$ which is equal to $N_R[g_1g_2]N_R[g_3]$ and that equals to $N_R[g_1g_2g_3] = N_R[g_1]N_R[g_2g_3]$ which is equal to $N_R[g_1](N_R[g_2]N_R[g_3])$. For all $g \in G/N_R$, there exists $N_R[g]$ in $G/N_R$, such that $N_R[1]N_R[g] = N_R[g] = N_R[g]N_R[1]$. For all $g \in G/N_R$, there exists $N_R[g^{-1}]$ in $G/N_R$, such that $N_R[g^{-1}]N_R[g] = N_R[g^{-1}]N_R[g]N_R[1]$ which equals to $N_R[g^{-1}]g = N_R[1] = N_R[g]g^{-1} = N_R[g]N_R[g^{-1}]$. Hence our claim. Now let us define $\zeta: G/N_R[1] \to G/N_R$ via $\zeta(gN_R[1]) = N_R[g]$.

Then for $g_1, g_2$ in $G$, $g_1N_R[1] = g_2N_R[1]$ if and only if $g_2^{-1}g_1 \in N_R[1]$ if and only if $(1, g_2^{-1}g_1) \in N_R$ if and only if $(x, g_2^{-1}g_1, x) \in R$ if and only if $(g_2, g_1) \in N_R$ if and only if $N_R[g_2]$ equals to $N_R[g_1]$. Hence $\zeta$ is a well defined injection. Also for all $g \in G/N_R$, there exists $gN_R[1] \in G/N_R[1]$ such that $\zeta(gN_R[1]) = N_R[g]$. Thus $\zeta$ is surjective as well. Also for $g_1N_R[1], g_2N_R[1] \in G/N_R[1]$, we have $\zeta((g_1N_R[1]g_2N_R[1])) = \zeta(g_1N_R[1], g_2N_R[1])$ and that equals to $N_R[g_1g_2]$ which equals to $N_R[g_1]N_R[g_2] = \zeta(g_1N_R[1]N_R[g_2])$. Hence $\zeta$ is a group homomorphism and thus a group isomorphism. Also, both $G/N_R[1], G/N_R$, are finite discrete topological groups, so $\zeta$ is an isomorphism of uniform cofinite groups as well. 

**Lemma 4.7.** The induced uniform topology over $G$ as in Lemma 4.4 is Hausdorff if and only if $G$ acts faithfully over $\Gamma$. 


Proof. Let us first assume that $G$ acts faithfully over $\Gamma$. Now let $g \neq h$ in $G$. Then $h^{-1}g \neq 1$. So there exists $x \in \Gamma$ such that $h^{-1}g.x \neq x$ implying that $g.x \neq h.x$. Then there exists a $G$-invariant compatible cofinite entourage $R$ over $\Gamma$ such that $(g.x, h.x) \notin R$, as $\Gamma$ is Hausdorff. Hence $(g, h) \notin N_R$. Thus $G$ is Hausdorff.

Conversely, let us assume that $G$ is Hausdorff and let $g \neq 1$ in $G$. Then there exists some $G$-invariant compatible cofinite entourage $R$ over $\Gamma$ such that $(1, g) \notin N_R$. Thus there exists $x \in \Gamma$ such that $(x, g.x) \notin R$. Hence $R[x] \neq R[g.x]$ so that $x \neq g.x$. Our claim follows.

□

Lemma 4.8. Suppose that $G$ is a group acting uniformly equicontinuously on a cofinite graph $\Gamma$ and give $G$ the induced uniformity as in Lemma 4.4. Then the action $G \times \Gamma \to \Gamma$ is uniformly continuous.

Proof. Let $R$ be a $G$-invariant cofinite entourage over $\Gamma$. Now let $((g.x), (h.y)) \in N_R \times R$, i.e. $(g, h) \in N_R, (x, y) \in R$. Now $x$ in $\Gamma$ and $(gx, hx) \in R$ this implies $(h^{-1}gx, x) \in R$. We have $(h^{-1}gx, y) \in R$ and hence $(gx, hy) \in R$. Thus our claim. □

Now if $R \leq S$ in $I$, then $S \subseteq R$. Let $(g_1, g_2) \in N_S$. Then $(g_1x, g_2x) \in S$, for all $x \in \Gamma$ and hence $(g_1x, g_2x) \in R$, for all $x \in \Gamma$ which implies $(g_1, g_2) \in N_R$. Thus $N_S \subseteq N_R$. For all $R \leq S$, in $I$, let us define $\psi_{RS} : G/N_S \to G/N_R$ via $\psi_{RS}(N_S[g]) = N_R[g]$. Then $\psi_{RS}$ is a well defined uniformly continuous group isomorphism, as each of $G/N_R, G/N_S$ are finite discrete groups. If $R = S$, then $\psi_{RR} = id_{G/N_R}$. And if $R \leq S \leq T$, then $\psi_{RS}\psi_{ST} = \psi_{RT}$. Then $\{G/N_R \mid R \in I, \psi_{RS}, R \leq S \in I\}$, forms an inverse system of finite discrete groups. Let $\widehat{\Gamma} = \lim_{\leftarrow \subseteq \Gamma} \Gamma/R$ and $\widehat{G} = \lim_{\leftarrow \subseteq \Gamma} G/N_R$, where $\psi_R : G \to G/N_R$ is the corresponding canonical projection map. Now if $I_1, I_2$ are two fundamental systems of $G$-invariant cofinite entourages over $\Gamma$, clearly $I_1, I_2$ will form fundamental systems of cofinite congruences, for two induced uniformities, over $G$. Now let $N_{R_1}$ be a cofinite congruence over $G$ for some $R_1 \in I_1$. Then there exists a $R_2$, cofinite entourage over $\Gamma$, such that $R_2 \in I_2$ and $R_2 \subseteq R_1$. Hence $N_{R_2} \subseteq N_{R_1}$. Now let $N_{S_2}$ be a cofinite congruence over $G$ for some $S_2 \in I_2$. Then there exists $S_1$, cofinite entourage over $\Gamma$, such that $S_1 \in I_1$ and $S_1 \subseteq S_2$. Hence $N_{S_1} \subseteq N_{S_2}$. Thus any cofinite congruence corresponding to the directed set $I_1$ is a cofinite congruence corresponding to the directed set $I_2$ and vice versa. Thus the two induced uniform structures over $G$ are equivalent and so the completion of $G$ with respect to the induced uniformity, from the cofinite graph $\Gamma$, is unique up to both algebraic and topological isomorphism.
Theorem 4.9. If $G$ acts on $\Gamma$, as in Lemma 4.4, faithfully then $\hat{G}$ acts on $\hat{\Gamma}$ uniformly equicontinuously.

Proof. The group $G$ acts on $\Gamma$ uniformly equicontinuously. We fix a $G$-invariant orientation $E^+(\Gamma)$ of $\Gamma$. By Lemma 4.8, the action is uniformly continuous as well. Let $\chi : G \times \Gamma \to \Gamma$ be this group action. Now since $\Gamma$ is topologically embedded in $\hat{\Gamma}$ by the inclusion map, say, $i$, the map $i \circ \chi : G \times \Gamma \to \hat{\Gamma}$ is a uniformly continuous map. Then there exists a unique uniformly continuous map $\hat{\chi} : \hat{G} \times \hat{\Gamma} \to \hat{\Gamma}$ that extends $\chi$. We claim that $\hat{\chi}$ is the required group action. We can take $\hat{\Gamma} = \lim \Gamma/R$ and $\hat{G} = \lim G/N_R$, where $R$ runs throughout all $G$-invariant compatible cofinite entourages of $\Gamma$ that are orientation preserving. Then $\hat{G} \times \hat{\Gamma} = \lim (G/N_R \times \Gamma/R)$ and $G \times \Gamma$ is defined coordinatewise via $(N_R[g])_R.(R[x])_R = (R[gr,x])_R$. If possible let, $((N_R[g])_R,(R[x])_R) = ((N_R[h])_R,(R[yr])_R)$. So, $N_R[gr]$ equals to $N_R[h]$, and $R[x] = R[yr], \forall R \in I, (g_R, h_R) \in N_R$ and $(x_R, y_R) \in R$. This implies that $(g_{R,x_R}, h_{R,y_R}) \in R$ which further ensures that $(h^{-1}_{R}g_{R,x_R}, x_R) \in R$. Then $(h^{-1}_{R}g_{R,x_R}, y_R) \in R$ and $(g_{R,x_R}, h_{R,y_R}) \in R$. Hence $(R[g_{R,x_R}]_R = (R[h_{R,y_R}]_R)$. So, the action is well defined. Let $g = (N_R[g])_R$ and $h = (N_R[h])_R$ in $\hat{G}$, $x = (R[x])_R \in \hat{\Gamma}$. Now $h.(g.x) = h.(R[g_{R,x_R}]_R) = (R[h_{R,y_R}]_R)$ which then equals to $(N_R[h_{R,y_R}]_R).x = (h_{R})_R.x$. Hence the action is associative. Now $(N_R[1])_R.(R[x])_R = (R[1x])_R = (R[x])_R$. Furthermore for all $v$ equal to $(R[v])_R \in V(\hat{\Gamma})$ and for all $g$ equal to $(N_R[gr])_R \in \hat{G}$ one can say that $g.v = (R[gr,v])_R \in V(\hat{\Gamma})$ as each $g_{R}.v_R \in V(\Gamma)$. Similarly, for all $e$ equal to $(R[e])_R \in E(\hat{\Gamma})$ and for all $g$ equal to $(N_R[gr])_R \in \hat{G}$, we have $s(g,e) = s((R[gr,e])_R)$ and so $(R[gr,s(e)])_R$ equals to $(g.(R[s(e)])_R)$ and that equals to $g.s(e)$. Hence the properties $t(g.e) = g.t(e)$ and $\overline{g.e} = g.\overline{e}$ follow similarly. Finally, let $E^+(\hat{\Gamma})$ consists of all the edges $(R[e])_R$, where $e \in E^+(\Gamma)$. Since each $R$ is orientation preserving, it follows that $E^+(\hat{\Gamma})$ is an orientation of $\hat{\Gamma}$. Since $E^+(\Gamma)$ is $G$-invariant, we see that $E^+(\hat{\Gamma})$ is $\hat{G}$-invariant. Hence this is a well defined group action. Also for all $g \in G$, and $x \in \Gamma$, $(N_R[g])_R.(R[x])_R$ equals to $(R[g.x])_R$ which equals to $g.x$ in $\Gamma$. Thus the restriction of this group action agrees with the group action $\chi$. Now $\{R \mid R \in I, \{N_R \mid R \in I\}$ is a fundamental system of cofinite entourages over $\Gamma$, is a fundamental system of cofinite congruences over $G$. Hence $\{R \mid R \in I\}$ is a fundamental system of cofinite entourages over $\hat{\Gamma}$ and $\{N_R \mid R \in I\}$
is a fundamental system of cofinite congruences over \( \hat{G} \) respectively. Let us now see that the aforesaid group action is uniformly continuous. For let us consider the group action \( G/N_R \times \Gamma/R \to \Gamma/R \) defined via \( N_R[g]R[x] = R[g.x] \), which is uniformly continuous as both \( G/N_R \times \Gamma/R \) and \( \Gamma/R \) are finite discrete uniform topological spaces. Hence the group action, \( \hat{G} \times \hat{\Gamma} \to \hat{\Gamma} \) is uniformly continuous. Thus the aforesaid group action is our choice of \( \hat{\chi} \), by the uniqueness of \( \hat{\chi} \). So the restriction of the aforesaid action \( \{ \hat{g} \} \times \hat{\Gamma} \to \hat{\Gamma} \) is a uniformly continuous map of graphs, for all \( \hat{g} \in \hat{G} \). We check that for all \( (x, y) \in R \) and for all \( \hat{g} \in \hat{G} \) the ordered pair \( (\hat{g}.x, \hat{g}.y) \in \hat{R} \). For, let \( \hat{g} = (N_R[g_R])_R \in \hat{G} \) and for \( x, y \in \Gamma , ((R[x]_R), (R[y]_R)) \in R \). Now \( \hat{R}[(R[\hat{g}.x]_R)] = \hat{R}[\hat{g}.x] = \hat{R}[h.x] = \hat{R}[h(R[x]_R)] \) which implies that \( (g, h) \in N_R \cap G \times G \). Thus, \( N_R \subseteq N_R \cap G \times G \). Again, if \( (g, h) \) belongs to \( N_R \cap G \times G \), then for all \( x \in \Gamma \subseteq \hat{\Gamma} \), and so \( (g.x, h.x) \in \hat{R} \cap \Gamma \times \Gamma = R \) and this implies \( (g, h) \in N_R \). Our claim follows. Then as uniform subgraphs \( (G, \tau_{\phi_1}) \simeq (G, \tau_{\phi_2}) \), both algebraically and topologically, their corresponding completions \( (\hat{G}, \tau_{\phi_1}) \simeq (\hat{G}, \tau_{\phi_2}) \), both algebraically and topologically. Since for all \( S \in I \), \( \psi_S : G \to G/N_S \) is a uniform continuous group homomorphism and \( G/N_S \) is discrete, there exists a unique uniform continuous extension of \( \psi_S \), namely, \( \hat{\psi}_S : \hat{G} \to G/N_S \). Let us define \( \lambda_S : \hat{G} \to G/N_S \) via \( \lambda_S(g) = N_S[g_S] \), where \( g = (N_R[g_R])_R \). Now let \( g = (N_R[g_R])_R, h = (N_R[h_R])_R \in \hat{G} \) be such that \( g = h \) which implies that \( N_S[g_S] = N_S[h_S] \) and hence \( \lambda_S \) is well defined. Now let \( (g, h) \in N_{\hat{R}} \). First of all \( N_{\hat{R}}[g_S] = N_{\hat{R}}[g] = N_{\hat{R}}[h] = N_{\hat{R}}[h_S] \). So, \( (g_S, h_S) \in N_{\hat{R}} \cap \hat{G} = G \times G \). Hence \( N_S[g_S] = N_S[h_S] \), which implies that \( \lambda_S(g) = \lambda_S(h) \), so \( (\lambda_S(g), \lambda_S(h)) \in D(G/N_R) \). Thus \( N_{\hat{R}} \) is a sub set of \( \lambda_S x \lambda_S^{-1} D(G/N_R) \). Hence \( \lambda_S \) is uniformly continuous. Now for
all \( g, h \in \hat{G} \), \( \lambda_S(gh) = N_S[gh] = N_S[gs]N_S[hs] = \lambda_S(g)\lambda_S(h) \) and for all \( g \in G \), \( \lambda_S(g) = \lambda_S((N_R[g])_R) = N_S[g] = \psi_S(g) \). Thus \( \lambda_S \) is an well defined uniformly continuous group homomorphism that extends \( \psi_S \). Then by the uniqueness of the extension, \( \psi_S = \lambda_S \). Now \( N_g \) is a closed subspace of \( \hat{G} \), then \( \overline{N_g} \cap G \times G = N_S \) which implies that \( N_S \) is a sub set of \( \overline{N_g} \) which equals to \( N_{g} \). Let us define \( \theta \) from \( \hat{G}/N_S \) to \( G/N_S \) as \( \theta \) takes \( N_S[g] \) into \( N_S[g_s] \), where \( g = (N_R[g_R])_R \). Now \( N_S[g] = N_S[h] \) in \( \hat{G}/N_S \) will imply \( (g_s, h_s) \) is in \( N_S \) and this implies for all \( x \) in \( X \) the ordered pair \( (g_sx, h_sx) \) in \( \hat{S} \bigcap \Gamma \times \Gamma \) which is eventually equal to \( S \). Thus \( (g_s, h_s) \in N_S \). Then \( \theta(N_S[g]) \) equals to \( N_S[g_s] \) which is equal to \( N_S[h_s] \) and that equals \( \theta(N_S[h]) \). Hence \( \theta \) is well defined. On the other hand let \( N_S[g], N_S[h] \) be such that \( \theta(N_S[g]) = \theta(N_S[h]) \). Thus \( N_S[g] \) equal to \( N_S[h] \) implies that \( (g_s, h_s) \in N_S \subseteq N_{g} \). Hence \( N_S[g] = N_S[g_s] = N_S[h_s] = N_S[h] \). So \( \theta \) is injective as well. Also for all \( N_S[g] \in G/N_S \) there exists \( N_S[g] = \hat{G}/N_S \) such that \( \theta(N_S[g]) = N_S[g] \). So \( \theta \) is surjective. Finally, \( \theta(N_S[g], N_S[h]) \) equals to \( \theta(N_S[h]) \) and that equals to \( N_S[g, h] \) which is \( N_S[gs]N_S[hs] \) and finally that equals to \( \theta(N_S[gs]N_S[hs]) \). So \( \theta \) is an well defined group isomorphism, both algebraically and topologically. Hence \( \hat{G}/N_S \cong G/N_S \cong \hat{G}/N_S \) which implies that \( \hat{G}/N_S[1] \) is equal to \( \hat{G}/N_S[1] \). But since \( N_S \subseteq N_{g} \) one obtains \( N_S[1] \leq N_S[1] \leq \hat{G} \) and thus \( \hat{G}/N_S[1] \rightarrow N_S[1] : N_S[1] \) equals to \( \hat{G}/N_S[1] \). Hence \( N_S[1] : N_S[1] \) equals to \( 1 \) which implies that \( N_S[1] = N_S[1] \) and thus \( N_S = N_S \) as each of them are congruences. Thus our claim. \( \square \)

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