The role of the $\rho(1450)$ in low energy observables from a global analysis, in the meson dominance approach.

Gustavo Ávalos, Antonio Rojas, Marxil Sánchez, and Genaro Toledo

Instituto de Física, Universidad Nacional Autónoma de México,

AP 20-364, México D.F. 01000, México

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Abstract

The $\rho(1450)$ vector meson ($\rho'$) is becoming increasingly important to properly describe precision observables. We determine the corresponding couplings, as described in the context of the vector meson dominance model, by performing a global analysis of a set of decay modes and cross sections. In a first step, we determine the parameters of the model involving the light mesons, from 10 decay modes which are insensitive to the $\rho'$. Then, we consider the $\omega \to 3\pi$ decay, and exhibit the need of extending the description by incorporating the $\rho'$ and a contact term as prescribed by the WZW anomaly. In a second step, we incorporate the data from the $e^+e^- \to 3\pi$ cross section (as measured by SND, CMD2, BABAR and BES III), and then the $e^+e^- \to \pi^0\pi^0\gamma$ data (as measured by SND and CDM2) to further restrict the $\rho'$ parameters validity region. As an application of the results, we compute the $e^+e^- \to 4\pi$ cross section for the so-called omega channel, measured by BABAR and find a good description of the data considering the parameters found. As a by product, the coupling $g_{\rho\omega\pi} = 11.314 \pm 0.383$ GeV$^{-1}$ is found to be consistent with all the relevant observables.

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I. INTRODUCTION

The low energy measurements involving hadrons are reaching a high accuracy. In general, the low mass hadron spectra contributing to the processes can be identified and the corresponding parameters obtained. Excited states manifest themselves in low energy observables as modifications to the values of the parameters and as part of the scattering processes for energies reaching the threshold for their nominal masses. The $\rho(1450)$ vector meson (denoted by $\rho'$ wherever possible) is one example of such states. It can be identified as contributing to the $\omega \to 3\pi$ decay width, by noticing that the effective strong coupling associated to such transition deviates from what is observed in other processes insensitive to the $\rho'$ [1]. The spectra obtained in hadronic $\tau$ decays [2, 3] and hadrons production from $e^+e^-$ annihilation [4] exhibit clear indications of its presence and are used to determine its mass and total decay width [5]. This important information needs to be complemented with the partial width of the different decay modes, which have then implications on the parameters for the models attempting to describe them. This information has not been settled, although evidence can be extracted from particular observables [5]. Decay modes such as $\rho' \to \omega \pi$ and $\rho' \to \pi \pi$ are of particular interest to disentangle the contribution of the $\rho'$ and $\rho$ mesons in low energy observables sensitive to both mesons. They are involved in the $e^+e^- \to \pi^0\pi^+\pi^-$ and $e^+e^- \to \pi^0\pi^0\gamma$ processes [11, 12], and in $e^+e^- \to \pi^0\pi^0\pi^+\pi^-$ process driven by the $\omega$ meson as intermediate state, where available data for this particular channel offers an opportunity to test these contributions [13, 14].

In this work, we determine the hadronic couplings of the low energy mesons and the $\rho'$, as described in the context of the vector meson dominance model, by performing a global fit of a set of decay modes and cross sections. We made use of MINUIT package for minimization and Vegas [15] subroutine for the phase space integration to obtain the cross section whenever needed. In a first step, we determine the parameters of the model involving the light mesons, from 10 decay modes which are practically insensitive to the $\rho'$, namely: $\rho \to \pi\pi$ neutral and charged modes, $\rho^0 \to e^+e^-, \mu^+\mu^-$, $\omega \to e^+e^-, \mu^+\mu^-$, $\omega \to \pi^0\gamma$, $\rho \to \pi\gamma$ neutral and charged modes and $\pi^0 \to \gamma\gamma$. Then, we include the $\omega \to 3\pi$ decay, driven by the $\rho$ meson intermediate state, to exhibit the modification of the parameters previously obtained, signaling the inconsistency and therefore the need of extending the description by incorporating the $\rho'$ and a contact term as prescribed by the Wess-Zumino-Witten anomaly.
In a second step, we incorporate the data from the $e^+ e^- \rightarrow 3\pi$ cross section as measured by SND, CMD2, BABAR and BES III \cite{6,9} and then $e^+ e^- \rightarrow \pi^0 \pi^0 \gamma$ data as measured by SND and CDM2 \cite{4,10,12} to further restrict the $\rho'$ parameters validity region. As an application of the results, we compute the $e^+ e^- \rightarrow 4\pi$ cross section for the so-called omega channel, and compare with the data measured by BABAR \cite{13} considering the parameters found. As a by product, we keep track of the behaviour of the coupling of the $\rho - \omega - \pi$ mesons and determine its stability upon the inclusion of the $\rho'$ and contact term in the description of the processes under consideration.

II. THEORETICAL FRAMEWORK

The vector meson dominance model (VMD) is able to account for the low energy manifestation of the strong interaction by considering the hadrons as the relevant degrees of freedom \cite{18}. Incorporation of symmetries such as Isospin and $SU(3)$ flavour symmetry allow to both classify the hadrons and relate their properties. Further considerations associated to the vector mesons manifestation as gauge bosons and incorporation of higher symmetries have been also considered as extensions of the VMD \cite{19,21}. Here, since the hadrons involved are the lightest ones, we restrict ourselves to the part that is common to all the VMD based models. The VMD Lagrangian including the light mesons $\rho$, $\pi$ and $\omega$, and $\rho'$ can be set as:

$$\mathcal{L} = \sum_{V=\rho,\rho'} g_{V\pi\pi} \epsilon_{abc} V^a \mu_b \pi^c + \sum_{V=\rho,\rho'} g_{\omega V\pi} \delta_{ab} \epsilon_{\mu\nu\lambda\sigma} \partial_\mu \omega_\nu \partial_\lambda V^a_\sigma \pi^b + g_{3\pi} \epsilon_{abc} \epsilon^{\mu\nu\lambda\sigma} \omega_\mu \partial_\nu \pi^a \partial_\lambda \pi^b \partial_\sigma \pi^c + \sum_{V=\rho,\rho',\omega} \frac{e m_V^2}{g_V} V^a \mu A^\mu.$$  \hspace{1cm} (1)

We have labelled the couplings with the corresponding interacting fields and, in general, $V$ refers to a vector mesons and $A^\mu$ refers to the photon field. The couplings are free parameters to be determined from experiment. Although, as we mention before, relations between them and even from other descriptions can be drawn \cite{22,27}.

The strong interaction between the $\omega$, $\rho$ and $\pi$ mesons, encoded in the $g_{\omega\rho\pi}$ parameter necessarily involves at least one of the particles off-shell due to phase space restrictions. Thus, the determination of its values might depend on the particular kinematical conditions of the considered observable. For example, these mesons are produced in experiments devoted
to the hadronic production from electron-positron annihilation as mentioned above and tau
decays \[5, 28–32\]. Here we consider only the data from the former. The \(g_{\omega\rho\pi}\) coupling, the\(\rho'\) parameters and the \(\omega \to 3\pi\) contact term \((g_{3\pi})\) usually appear together when describ-
ing experimental data, exhibiting a strong correlation \[1\]. Therefore an analysis involving
data from different sources should help to disentangle their individual contributions. This
information is relevant in the understanding of other scenarios where there is not enough in-
formation to draw an independent analysis and therefore require to rely in a well supported
determination of such parameters to draw conclusions.

In the following we describe the generic processes and the way they are incorporated to the
analysis. We will extend our discussion on each contribution and the works related to them
along the work.

III. \(V \to P_1 P_2\) DECAY AND THE \(g_{VP_1P_2}\) COUPLING

The coupling of a vector meson \((V)\) and two pseudo-scalar mesons \((P)\), denoted in general
by \(g_{VP_1P_2}\), can be extracted from the measurement of the \(V \to P_1 P_2\) decay width. The
amplitude of this process, depicted in Fig. \[1\]a) can be written as:

\[
\mathcal{M} = i g_{VP_1P_2} (p_1 - p_2)^\mu \eta_\mu(q) \tag{2}
\]

where \(q, p_1\) and \(p_2\) are the momenta of the initial vector meson \(V\) and the pseudo-scalar
pair in the final state, respectively. \(\eta_\mu\) is the polarization tensor of the vector particle. The
decay width is given in terms of the coupling and the masses of the particles involved as:

\[
\Gamma_{VP_1P_2} = \frac{g_{VP_1P_2}^2 \lambda^{3/2}(m_V^2, m_{P_1}^2, m_{P_2}^2)}{48 \pi m_V^4} \tag{3}
\]

where \(m_V, m_{P_1}\) and \(m_{P_2}\) are the corresponding masses and \(\lambda(x, y, z) = x^2 + y^2 + z^2 -
2xy - 2xz - 2yz\) is the Källen function. As we can see in the Eq. \[3\], the \(g_{VP_1P_2}\) coupling is
dimensionless. This result can be applied for example to obtain both \(g_{\rho\pi\pi}\) and \(g_{\rho'\pi\pi}\) provided
the data for the partial decay width is available. This is the case for the \(\rho \to \pi\pi\) decay
\[5\]. In Table \[1\] we show the values of \(g_{\rho\pi\pi}\) from two different process: \(\rho^0(770) \to \pi^+ \pi^-\) and
\(\rho^+(770) \to \pi^+ \pi^0\) and their weighted average. The weighted average is defined in general as:

\[
\bar{x} \equiv \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}, \tag{4}
\]
where $x_i$ is the $i$ measurement and $w_i$ is the $i$ weight associate with this measurement. In our specific case $x_i$ is the coupling constant and $w_i$ is the error fraction of the coupling constant. For the $\rho'$, the partial decay width is not settled, thus, we can use Eq. (3) and its experimental total width of 400 MeV to set an upper bound on the $g_{\rho'\pi\pi} = 6.64$ considering this to be the only decay mode. We will enforce this restriction to set the region to search for this parameter, as we discuss later. The $\rho' \to \pi \pi$ decay can be also addressed in an indirect way, by considering it as part of a decay chain. For example, in the $D_s \to \rho(1450) \pi$ decay, where the $\rho(1450)$ is reconstructed using the two pions decay mode, albeit of requiring additional information on other couplings [33].

\[
\begin{array}{|c|c|}
\hline
\text{Process} & g_{\rho\pi\pi} \text{ coupling} \\
\hline
\rho^0(770) \to \pi^+\pi^- & 5.944 \pm 0.018 \\
\rho^+(770) \to \pi^+\pi^0 & 5.978 \pm 0.048 \\
\text{Weighted Average} & 5.953 \pm 0.017 \\
\hline
\end{array}
\]

TABLE I: $g_{\rho\pi\pi}$ coupling from the neutral and charged processes and the weighted average, $\bar{g}_{\rho\pi\pi}$.

IV. $V \to ll$ DECAY AND THE $g_V$ COUPLING

The vector-photon transition depends on the $g_V$ coupling, as given in Eq. (1). It can be extracted from the measurement of the $V \to \ell^+\ell^-$ decay width, with $\ell$ either electrons or muons. The amplitude of this process, depicted in Fig. 1(b), can be written as:

\[
\mathcal{M} = -i \frac{e^2}{g_V} \bar{u}(l_1) \gamma^\nu v(l_2) \eta_\nu(q)
\]

where $q$, $l_1$ and $l_2$ are the momenta of the initial vector meson and the lepton pair in the final state, respectively. $\eta_\nu$ is the polarization tensor of $V$ and $\bar{u}(l_1)$ and $v(l_2)$ are the corresponding spinors of the leptons. Then, the decay width $\Gamma_{V\ell\ell}$ is given in terms of the coupling, the mass of the vector meson $m_V$ and the masses of the leptons $m_\ell$ as:

\[
\Gamma_{V\ell\ell} = \frac{4 \pi \alpha^2 (2m_{l_1}^2 + m_{l_2}^2)(m_V^2 - 4m_{l_1}^2)^{1/2}}{3 m_V^2 g_V^2}.
\]

$g_V$ is dimensionless, as we can see in the equation above. In Table II we show the values of the $g_V$ couplings for a set of vector mesons obtained from decays to muon and/or electron
pairs. Note that we have included the value for $g_\rho(1450)$ obtained from the information in [5] but quote only a central value, as the experimental information provides only an estimate of the decay width. Improvements on this measurement would be very useful. Still, it will help us to guide the analysis on this parameter when considering scattering processes. The weighted average $\bar{g}_V$ from the $V \rightarrow \mu^+ \mu^-$ and $V \rightarrow e^+ e^-$ decays is shown in Table III for $\rho(770)$, $\omega(782)$ and $\phi(1020)$ mesons.

| Process $\quad V$, Coupling $\quad g_V$, Value $\quad$ |
|-----------------|-----------------|-----------------|
| $\rho^0(770) \rightarrow e^+ e^-$ | $g_\rho$ | $4.956 \pm 0.021$ |
| $\rho^0(770) \rightarrow \mu^+ \mu^-$ | $g_\rho$ | $5.037 \pm 0.021$ |
| $\omega(782) \rightarrow e^+ e^-$ | $g_\omega$ | $17.058 \pm 0.292$ |
| $\omega(782) \rightarrow \mu^+ \mu^-$ | $g_\omega$ | $16.470 \pm 2.469$ |
| $\phi(1020) \rightarrow e^+ e^-$ | $g_\phi$ | $13.381 \pm 0.216$ |
| $\phi(1020) \rightarrow \mu^+ \mu^-$ | $g_\phi$ | $13.674 \pm 0.479$ |
| $\rho^0(1450) \rightarrow e^+ e^-$ | $g_\rho(1450)$ | $13.528$ |

TABLE II: $g_V$ ($V = \rho(770), \omega(782), \phi(1020), \rho(1450)$) couplings from decays to muon and/or electron pairs.

| Coupling $\quad V$, Value $\quad$ |
|-----------------|-----------------|
| $\bar{g}_\rho$ | $4.966 \pm 0.021$ |
| $\bar{g}_\omega$ | $16.972 \pm 0.287$ |
| $\bar{g}_\phi$ | $13.528 \pm 0.339$ |

TABLE III: Weighted average couplings $\bar{g}_V$ ($V = \rho, \omega, \phi$).

V. $V_1 \rightarrow P \gamma$ DECAY AND THE $g_{V_1P\gamma}$ AND $g_{V_1V_2P}$ COUPLINGS

The $g_{V_1P\gamma}$ coupling can be extracted from the $V_1 \rightarrow P \gamma$ decay width, where $V_1$ is a vector meson, $P$ is a pseudo-scalar meson and $\gamma$ is the photon. The amplitude of this process, depicted in Fig. 2(a), can be written as:

$$M = i g_{V_1P\gamma} \epsilon^{\beta\nu\alpha\mu} k_{\beta} q_{\alpha} \eta_{\mu} \epsilon_{\nu}^*$$ (7)
FIG. 1: Decay of vector mesons of the form (a) $V \rightarrow PP$ and (b) $V \rightarrow ll$

FIG. 2: Decay of vector mesons of the form (a) $V \rightarrow P\gamma$ and (b) $P \rightarrow \gamma\gamma$

where $k$ ($\eta$) and $q$ ($\epsilon^*$) are the momenta (polarization tensor) of the $V_1$ and $\gamma$, respectively. The decay width $\Gamma_{V_1P\gamma}$ is given in terms of the $g_{P\gamma}$ coupling, the masses of the vector meson $m_V$ and pseudo-scalar meson $m_P$ as:

$$\Gamma_{V_1P\gamma} = g_{V_1P\gamma}^2 \left[ \frac{(m_{V_1}^2 - m_P^2)^3}{96 \pi m_P^3} \right]. \quad (8)$$

As we can notice from Eq. (8), $g_{V_1P\gamma}$ have energy $^{-1}$ units. Another related coupling is the one where two vector mesons interact with a pseudo-scalar meson, denoted by $g_{V_1V_2P}$. It can be obtained from the previous vector meson radiative decay ($V_1 \rightarrow P\gamma$) considering that the photon emission is mediated by a neutral vector meson [34] (See Fig. 2(a)). Then the amplitude and decay width of this process are similar to the previous Eq. (7) and (8), with the replacement $g_{V_1P\gamma} \rightarrow g_{V_1V_2P} (e/g_{V_2})$. It follows that $g_{V_1V_2P}$ also have energy $^{-1}$ units, we use this parameter hereafter. For the analysis we consider the following decays, with their respective charge combinations: The $\omega \rightarrow \pi\gamma$ decay, driven by the $\omega \rightarrow \pi\rho \rightarrow \pi\gamma$ process; The $\rho \rightarrow \pi\gamma$ decay, driven by the $\rho \rightarrow \pi\omega (\phi) \rightarrow \pi\gamma$ processes. The contribution from the $\phi$ meson channel is relatively small and neglected at this stage [35, 36] (the expected ratios for $g_{\rho\omega}\pi/g_\omega \approx 0.7$ can be compared with $g_{\phi\rho\pi}/g_\phi \approx 0.06$, taking $g_\phi$ weighted average and,
as an approach, $|g_{\rho\pi}| = 0.86 \pm 0.01$ GeV$^{-1}$ obtained by considering the decay width of the $\phi \to 3\pi$ to be fully accounted by the $\rho\pi$ channel) contributions from other channels are relatively smaller \[29\]. Note that the $\phi$ meson parameters are not considered as part of the global analysis.

VI. $\pi^0 \to \gamma\gamma$ DECAY AND THE $g_{P\gamma\gamma}$ AND $g_{V_1V_2P}$ COUPLINGS

The $g_{P\gamma\gamma}$ coupling can be extracted from the measurement of the $P \to \gamma\gamma$ decay width. The amplitude of this process, depicted in Fig. 2(b), can be written as:

$$M = ig_{P\gamma\gamma} \epsilon^\alpha_{\mu \beta \gamma} q^\beta_1 q^\alpha_2 \epsilon^\gamma_1 \epsilon^\nu_2,$$  \hspace{1cm} (9)

where $q_1$ ($\eta^1$) and $q_2$ ($\eta^2$) are the momenta (polarization tensors) of the final photons respectively. We can write the decay width $\Gamma_{P\gamma\gamma}$ in terms of the $g_{P\gamma\gamma}$ coupling and the mass of the pseudo-scalar meson $m_P$ as:

$$\Gamma_{P\gamma\gamma} = \left[ \frac{g^2_{P\gamma\gamma} m^3_P}{32 \pi} \right].$$  \hspace{1cm} (10)

As we can notice from the above equation, $g_{P\gamma\gamma}$ have energy$^{-1}$ units. The $g_{V_1V_2P}$ coupling can be related to this decay considering that the photons emission is mediated by two neutral vector mesons, $\pi^0 \to \rho\omega (\phi) \to \gamma\gamma$ \[31\]. Then the amplitude and decay width of this process are similar to Eq. (9) and (10) by replacing $g_{P\gamma\gamma} \to g_{V_1V_2P} \frac{4\pi \alpha}{g_{V_1} g_{V_2}}$. In Table IV we show the values of the $g_{\rho\omega\pi}$ coupling from four different decays: $\omega(782) \to \pi^0\gamma$, $\rho^0(770) \to \pi^0\gamma$, $\rho^+(770) \to \pi^+\gamma$ and $\pi^0 \to \gamma\gamma$. We have used the values of the $g_{V}$ couplings as listed in Table III and neglected the $\rho - \phi$ channel in the $\pi^0 \to \gamma\gamma$ decay. Mixing effects from $\pi^0 - \eta - \eta'$ states are not considered, although they may become relevant in precision observables analysis \[37\]-\[41\].

VII. $\omega \to 3\pi$ DECAY

Let us consider the decay process $\omega(\eta, q) \to \pi^+(p_1) \pi^-(p_2) \pi^0(p_3)$, where $p_i$ refers to the momentum of the pions, $q$ and $\eta$ are the momentum and polarization tensor of the $\omega$ meson respectively. This process can receive contributions from the $\rho$, $\rho'$ and contact channels as shown in Figure 3. The decay amplitude can be set as follow:

$$M_{\omega \to 3\pi} = i \epsilon^\alpha_{\mu \beta \gamma} \eta^\mu p_1^\alpha p_2^\beta p_3^\gamma A(m^2_\omega),$$  \hspace{1cm} (11)
\begin{center}
| Process | $g_{\rho\omega\pi}$ (GeV$^{-1}$) |
|---------|-------------------------------|
| $\omega(782) \rightarrow \pi^0\gamma$ | 11.489±0.039 |
| $\rho^0(770) \rightarrow \pi^0\gamma$ | 14.224±2.227 |
| $\rho^+(770) \rightarrow \pi^+\gamma$ | 12.358±1.806 |
| $\pi^0 \rightarrow \gamma\gamma$ | 15.631±1.6121 |
\end{center}

**TABLE IV:** $g_{\rho\omega\pi}$ coupling obtained from four different processes.

where $A(m_{\omega}^2)$ is given by:

$$A(m_{\omega}^2) = 6 g_{3\pi} + 2 g_{\omega\rho\pi} g_{\rho\pi\pi} \left( D_{\rho^0}[s_{12}] + D_{\rho^+}[s_{13}] + D_{\rho^-}[s_{23}] \right) + 2 g_{\omega\rho^0\pi} g_{\rho^0\pi\pi} \left( D_{\rho^0}[s_{12}] + D_{\rho^0}[s_{13}] + D_{\rho^0}[s_{23}] \right), \quad (12)$$

and $s_{ij} = p_i + p_j$, $D_V[p] = 1/(p^2 - m_V^2 + i m_V \Gamma_V)$. The factors of 6 and 2 in $A$ come from the cyclic permutations and momentum conservation used to bring the amplitude into the current form. The notation is explicit for the $\rho$ and $\rho'$ contributions. The decay width is obtained upon integration over the full 3-body phase space [5]. While for a single decay this procedure faces not major problem, the inclusion in a numerical analysis involving more processes would require a practical approach to speed up. Since we are interested in the couplings (masses and widths are taken at their nominal values) the decay width can be decomposed as a polynomial on the coupling constants as follow:

$$\Gamma_{\omega3\pi} = A_1 g_{3\pi}^2 + A_2 g_{\omega\rho\pi}^2 g_{\rho\pi\pi}^2 + A_3 g_{3\pi} g_{\omega\rho\pi} g_{\rho\pi\pi} + A_4 g_{\omega\rho^0\pi}^2 g_{\rho^0\pi\pi}^2 + A_5 g_{\omega\rho^0\pi} g_{3\pi} g_{\rho^0\pi\pi} + A_6 g_{\omega\rho^0\pi} g_{\rho\pi\pi} g_{\rho\pi\pi}, \quad (13)$$

where the $A_i$ coefficients can be identified with the corresponding part of the decay width for the couplings involved, and are computed only once, following the decay width definition as given in the PDG [5]. We will show that the approach where only the $\rho$ channel is considered requires a large value for the $g_{\omega\rho\pi}$ coupling compared with the previous estimates considering radiative decays. This result motivates the inclusion of the $\rho(1450)$ and the contact term.

The couplings involved in the right hand side are not settled, neither in the theoretical side nor experimentally. Studies on the value of $|g_{\omega\rho'\pi}|$ have found it to lay in the interval 10 - 18 GeV$^{-1}$ [1, 42]. The magnitude for the contact coupling computed in the literature from different approaches is also in a wide range around 29 - 123 GeV$^{-3}$ [1, 22, 23, 25, 43, 44].
\[ \omega(\eta,q) \pi^- - (p_2) \rho_+, \rho'_+ \pi^+ (p_1) \]

\[ \pi^0 (p_3) \]

\[ (a) \]

\[ \omega(\eta,q) \pi^+ (p_1) \]

\[ \pi^- (p_2) \rho_-, \rho'_- \pi^0 (p_3) \]

\[ (b) \]

\[ \omega(\eta,q) \pi^+ (p_1) \]

\[ \pi^- (p_2) \rho_-, \rho'_- \pi^0 (p_3) \]

\[ (c) \]

\[ \omega(\eta,q) \pi^+ (p_1) \]

\[ \pi^- (p_2) \rho_-, \rho'_- \pi^0 (p_3) \]

\[ (d) \]

FIG. 3: Feynman diagrams to \( \omega \to 3\pi \) process. The contribution from the \( \rho, \rho' \ (a,b,c) \) and the contact \( (d) \) channels.

VIII. THE \( e^+e^- \to \omega \to 3\pi \) CROSS SECTION

Now, we proceed to describe the \( e^+(k_+) e^-(k_-) \to \omega(q) \to \pi^+(p_1) \pi^-(p_2) \pi^0(p_3) \) cross section following the same approach as in the previous section. Following the same notation for the decay process, but now at an energy \( q^2 = (k_+ + k_-)^2 \) instead of \( m_\omega^2 \), we can write the amplitude for the \( \omega \) channel as follows:

\[ M_{e^+e^-\to3\pi} = \frac{e}{q^2} \frac{m_\omega^2}{g_\omega} D_\omega(q) A(q^2) \epsilon_{\mu\alpha\beta\gamma} p_1^\alpha p_2^\beta p_3^\gamma l^\mu \]  

where \( e \) is the positron electric charge, \( l^\mu = -ie \bar{v}(k_+) \gamma^\mu u(k_-) \), is the leptonic current, \( D_\omega(q) = 1/(q^2 - m_\omega^2 + i m_\omega \Gamma_\omega) \), with \( m_\omega \) and \( \Gamma_\omega \) the mass and total width of the \( \omega \) meson. \( A(q^2) \) has been defined in Eq. (12), but now taken at \( q^2 \). Following the same approach as for the \( \omega \to 3\pi \) decay, we expand the cross section in terms of the coupling constants and coefficients that are evaluated at the corresponding energies reported by the experiments.

\[ \sigma(e^+e^- \to \omega \to 3\pi) = \frac{1}{g_\omega^2} \left( B_1 g_\omega^2 + B_2 g_\omega^2 g_{\rho\pi}^2 + B_3 g_\omega^2 g_{\rho\pi} g_{\rho\pi} + B_4 g_\omega^2 g_{\rho\rho'}^2 g_{\rho\pi}^2 g_{\rho\pi} + B_5 g_\omega^2 g_{\rho\rho'}^2 g_{\rho\pi} g_{\rho\pi} + B_6 g_\omega^2 g_{\rho\rho'}^2 g_{\rho\pi} g_{\rho\pi} g_{\rho\pi} \right), \]

The \( B_i \) are computed only once at each energy the experimental data exist, using the kinematical description as given in Ref. [45], and implemented in a Fortran program with the Vegas [15] integration subroutine. We have considered the data from SND [6], which uses
data from DM2 to extend its range up to 2 GeV. They find evidence of the $\rho \rightarrow 3\pi$ decay mode, with a branching ratio of the order of $10^{-4}$, thus its smallness justify our decision of not considering this mode. We consider the energy range up to around 0.82 GeV, to avoid the $\phi$ contribution. The CMD2 data [7] updated the previous measurement [46] to include missing contributions in the energy range of 0.76 to 0.821 GeV, in that case they did not perform a spectral analysis. For the BABAR [8] data, we consider energies below 0.9 GeV to avoid the $\phi$ contribution. They also find that the $\rho \rightarrow 3\pi$ decay mode, branching ratio is of the order of $10^{-4}$. Preliminary data from BESIII [9] is available in the energy range form 0.7 - 3 GeV, for consistency with the approach we restrict our consideration for this data to energies below 0.8 GeV. We have verified that the particular upper energy value considered in this region makes no effect on the results.

$$\sigma(e^+ e^- \rightarrow 3\pi) \text{ (nb)}$$

![Graph](image)

**FIG. 4:** $e^+ e^- \rightarrow \omega \rightarrow 3\pi$ cross section data from SND [6], CMD2 [7], BABAR [8] and BES3 [9], and the corresponding results using the parameters from the global analysis, Table VII.
IX. THE $e^+e^- \rightarrow \omega\pi^0 \rightarrow \pi^0\pi^0\gamma$ CROSS SECTION

The notation of momenta for the process is: $e^+(k_+) e^-(k_-) \rightarrow \pi^0(p_1) \pi^0(p_2) \gamma(\eta^*, p_3)$, where $\eta^*$ represents the polarization tensor of the photon. The process is depicted by the diagrams in Fig. 5, where both the $\rho$ and $\rho'$ intermediate states are considered. Further contributions such as the $\phi$ meson or scalars are not considered at this stage, although they may be relevant for precision observable estimates such as the Muon magnetic dipole moment [41]. The amplitude for the diagram of Fig. 5(a) can be written as:

$$\mathcal{M}_{(a)} = \frac{e^2}{q^2} \left( C_{\rho^0} + e^{i\theta} C_{\rho'} \right) D_\omega(q - p_1) \epsilon_{\mu\sigma\epsilon\lambda} q^\sigma (q - p_1)^\epsilon \epsilon_{\alpha\lambda\beta\nu} (q - p_1)^\alpha p_3^\beta \eta^{*\nu} l^\mu, \quad (16)$$

where the global factors are defined by:

$$C_{\rho^0} = \left( \frac{g_{\omega\rho\pi}}{g_\rho} \right)^2 m_{\rho^0}^2 D_{\rho^0}(q), \quad C_{\rho'} = \frac{g_{\omega\rho'\pi} g_{\omega\rho\pi}}{g_\rho g_{\rho'}} m_{\rho'}^2 D_{\rho'}(q), \quad (17)$$

with a relative phase $e^{i\theta}$ between both channels. Note that the amplitude for Fig. 5(b), which considers the exchange of neutral pions, respect to Fig. 5(a), is exactly the same amplitude $\mathcal{M}_{(a)}$ by interchanging $p_1 \leftrightarrow p_2$ momenta.

![FIG. 5: The $e^+ e^- \rightarrow \omega \pi \rightarrow \pi \pi \gamma$ decay.](image)

The cross section is set, in terms of the couplings involved, as:

$$\sigma(e^+e^- \rightarrow 2\pi^0\gamma) = \left( \frac{g_{\omega\rho\pi}}{g_\rho} \right)^4 C_1 + \left( \frac{g_{\omega\rho\pi} g_{\omega\rho'\pi}}{g_\rho g_{\rho'}} \right)^2 C_2 + \left( \frac{g_{\omega\rho\pi} g_{\omega\rho'\pi}}{g_\rho g_{\rho'}} \right) \left( \cos(\theta) C_3 - \sin(\theta) C_4 \right). \quad (18)$$

We have considered the data from three SND Coll. measurements [4, 10, 11], although the later [4] updated the previous ones, they will be useful to illustrate the behavior of the couplings even in such cases were some corrections are missing. Data from CMD2 Coll. [12] is also available and used in this analysis. We can profit from the corresponding analysis that
the experiments carried out, by identifying the parameters region favored from their own fit. In particular we can identify that the relative phase is expected to be large ($\theta = 122 \pm 8^0$) is obtained in Ref. [4]), and the parameter $A_1 \equiv (g_{\omega\rho\pi}/g_{\omega\rho\pi}) (g_{\rho}/g_{\rho})$ is introduced to describe the process, instead of the individual parameters (do not get confused with $A_1$ coefficient in Eq. (13)). We will consider this last as a constrain for the individual couplings and search for the most favored value as well, using the experimental analyses as guidance, we expect the value to be around $A_1 \approx 0.2$.

\[ \sigma(e^+ e^- \rightarrow 2 \pi \gamma) \ (nb) \]

FIG. 6: $e^+ e^- \rightarrow \pi^0 \pi^0 \gamma$ cross section data from SND (SND00 [10] SND13 [11] and SND16 [4] and CMD2 [12], and the corresponding results using the parameters from the global analysis, Table VI.
X. $\chi^2$ ANALYSIS. RESULTS

In order to determine the hadronic couplings of the low energy mesons and the $\rho'$, from the processes described above, we perform a global fit using the MINUIT package. The $\chi^2$ function to minimize is defined by:

$$\chi^2(\theta) = \sum_{i=1}^{N} \frac{(y_i - \mu(x_i; \theta))^2}{E_i^2},$$

where $\theta = (\theta_1, ..., \theta_N)$ are the parameters to determine; $y_i$ and $E_i$ are the experimental data and their corresponding uncertainty. $\mu(x_i; \theta)$ are the theoretical estimate for the corresponding parameters. In a first step we determine the parameters of the model involving the light mesons, from 10 decay modes which are insensitive to the $\rho'(1450)$, namely: $\rho \to \pi \pi$ neutral and charged modes, $\rho^0 \to e^+ e^-$, $\mu^+ \mu^-$, $\omega \to e^+ e^-$, $\mu^+ \mu^-$, $\omega \to \pi^0 \gamma$, $\rho \to \pi \gamma$ neutral and charged modes and $\pi^0 \to \gamma \gamma$, using the experimental information as listed in the PDG [5]. These involve four parameters: $g_\rho$, $g_{\rho \pi \pi}$, $g_\omega$ and $g_{\omega \rho \pi}$. In Table V, we show the result of the fit. The value of the minimization function per degree of freedom ($dof$) is $\chi^2/dof = 0.32$. The correlation between parameters is shown in Fig. 7 as a heat map.

| Parameter       | Central value | Error   |
|-----------------|---------------|---------|
| $g_{\rho \pi \pi}$ | 5.9485        | 0.0536  |
| $g_\rho$        | 4.9619        | 0.0661  |
| $g_\omega$      | 17.038        | 0.603   |
| $g_{\omega \rho \pi}$ (GeV$^{-1}$) | 11.575        | 0.438   |

TABLE V: Fit to 10 decay modes as described in the text.

Then, we include the $\omega \to 3 \pi$ decay mode to exhibit the strong modification of the $g_{\omega \rho \pi}$ parameter previously obtained, which becomes $g_{\omega \rho \pi} = 14.572 \pm 0.22$ and a $\chi^2/dof >> 1$, signaling the inconsistency and therefore the need of extending the description by incorporating the $\rho(1450)$ and a contact term as prescribed by the WZW anomaly. Upon the inclusion of these contributions we obtain $g_{\omega \rho \pi} = 11.576 \pm 0.463$, in accordance with previous results. Hereafter this is the way to describe the $\omega$ decay, and denote this set of data as the 11 decay modes. In a second step, we incorporate the data from the $e^+ e^- \to 3 \pi$ cross section (as measured by SND [6], CMD2 [7], BABAR [8] and BES III [9]) and the
FIG. 7: Correlation matrix for $g_{\rho\pi\pi}$, $g_\rho$, $g_\omega$ and $g_{\omega\rho\pi}$ parameters from 10 decay modes, see text for details.

$e^+ e^- \rightarrow \pi^0 \pi^0 \gamma$ (as measured by SND [4, 10, 11] and CDM2 [12]) to further restrict the $\rho(1450)$ parameters validity region. Global restrictions from other measurements, as the mentioned $A_1$ and upper bound for the $g_{\rho'\pi\pi}$ parameter, are incorporated by setting a consistent region for the search of the parameters in the minimization process. In particular, we obtain $A_1 = 0.125 \pm 0.05$. Table VI shows the parameters values when considering the 11 decay modes plus the experimental data for $e^+ e^- \rightarrow \pi^0 \pi^0 \gamma$ cross section, and Table VII correspond to the results when adding $e^+ e^- \rightarrow 3 \pi$ cross section data. The corresponding correlation matrix are shown as heat maps in Figs. 8 and 9, respectively.

To summarize the results, in Figs. 10 and 11 we have plotted the values of the individual parameters as a function of the data considered for the minimization. For the sake of clarity, in Fig. 12 we show the description of them, corresponding to the $x$ axis labeling of the previous figures. Missing parameter data in any of this $x$ values means that this last has no dependence on it.
### TABLE VI: Fit to 11 decay modes and cross section data for $e^+ e^- \rightarrow \pi^0 \pi^0 \gamma$.

| Parameter | Central value | Error  |
|-----------|---------------|--------|
| $g_{\rho \pi \pi}$  | 5.9484        | 0.0668 |
| $g_\rho$     | 4.9618        | 0.0819 |
| $g_\omega$   | 16.907        | 0.6625 |
| $g_{\omega \rho \pi}$ (GeV$^{-1}$) | 11.486    | 0.4951 |
| $g_{\rho' \pi \pi}$ | 4.5103    | 1.0371 |
| $g_{\omega \rho' \pi}$ (GeV$^{-1}$) | 3.1363    | 1.7702 |
| $g_{3\pi}$ (GeV$^{-3}$)      | -53.612     | 6.8932 |
| $g_{\rho'}$   | 12.472        | 1.2437 |
| $\theta$ (in $\pi$ units) | 0.8697    | 0.0452 |

### TABLE VII: Fit to the 11 decay modes and all the cross section data.

| Parameter | Central value | Error  |
|-----------|---------------|--------|
| $g_{\rho \pi \pi}$  | 5.9486        | 0.0755 |
| $g_\rho$     | 4.9622        | 0.0928 |
| $g_\omega$   | 16.652        | 0.4726 |
| $g_{\omega \rho \pi}$ (GeV$^{-1}$) | 11.314    | 0.383  |
| $g_{\rho' \pi \pi}$ | 5.4999    | 1.0597 |
| $g_{\omega \rho' \pi}$ (GeV$^{-1}$) | 3.4774    | 0.96262|
| $g_{3\pi}$ (GeV$^{-3}$)      | -54.338     | 6.6739 |
| $g_{\rho'}$   | 12.918        | 1.1907 |
| $\theta$ (in $\pi$ units) | 0.8715    | 0.0512 |

### A. The $e^+ e^- \rightarrow \pi \omega \rightarrow 4\pi$ cross section

As an application of the results, we compute the $e^+ e^- \rightarrow 4\pi$ cross section for the so-called omega channel, and compare with the data measured by BABAR [13], we do not consider the recent measurement from SND [14] since explicit data is not provided. This process has been considered in previous studies to test models viability to account for the
observed data, study isospin symmetry breaking effects as compared with the analog in tau decays [47, 48] and to determine the magnetic dipole moment of the $\rho$ [49]. Here, we do not fit the data but use the parameters found as listed in Tables VI and VII to describe it. The $\omega$ channel is depicted in Fig. 13. Let us set the notation for the momenta of the process as follows: $e^+(k_+) e^-(k_-) \rightarrow \pi^+(p_1) \pi^0(p_2) \pi^-(p_3) \pi^0(p_4)$. Then, we can write the amplitude as:

$$
\mathcal{M}_{e^+e^-\rightarrow 4\pi} = \frac{e}{q^2} \left( G_\rho + e^{i\theta} G_{\rho'} \right) D_\omega(q - p_4) A((q - p_4)^2) \epsilon_{\sigma\alpha\eta\beta} \epsilon_{\mu\gamma\chi\sigma} q^\gamma p_1^\alpha p_2^\eta p_3^\beta p_4^\chi l^\mu ,
$$

(20)

where

$$
G_\rho = \frac{g_{\omega\rho\pi}}{g_\rho} m_\rho^2 D_\rho(q), \quad G_{\rho'} = \frac{g_{\omega\rho'\pi}}{g_{\rho'}} m_{\rho'}^2 D_{\rho'}(q).
$$

(21)

FIG. 8: Correlation matrix for the couplings considering 11 decay modes and data for $e^+e^-\rightarrow \pi^0\pi^0\gamma$ cross section. See text for details.

FIG. 9: Correlation matrix for the couplings considering 11 decay modes and data for $e^+e^- \rightarrow \pi^0\pi^0\gamma$ and $e^+e^- \rightarrow \omega \rightarrow 3\pi$ cross sections. See text for details.

The Bose-Einstein symmetry, applied to the neutral pions, leads to an additional contribution by the momentum exchange of the neutral pions in all diagrams. The corresponding amplitude is similar to Eq.(20) by exchanging $p_4 \leftrightarrow p_2$.

In Fig. 14 we plot the cross section for the parameters values on Table VI (dashed line) and Table VII (solid line). Experimental data from BABAR [13] are shown in circle symbols. Scaled data from the SND [4, 10, 11] and CMD2 [12] obtained from the $e^+e^- \rightarrow \pi\pi\gamma$ cross section measurements are also displayed (see detailed description inside the Figure). We observe that there is a proper description of the data using either parameters data set. The $\rho'$ contribution is shown and a non-trivial role of the interference is important ($\theta$ phase).
FIG. 10: All the coupling parameters involved in the analysis, with their corresponding errors as a function of the data set considered for the fit. See Fig. 12 for the labeling on the $x$ axis.

to properly account for the data. The non-resonant contribution ($g_{3\pi}$) coming from the $\omega$ decay is also shown (dotted line).

XI. DISCUSSION

We have explored the role of the $\rho'$ in low energy observables by performing a global analysis of a set of decay modes and cross sections. In a first step, we determined the parameters of the model involving the light mesons, from 10 decay modes which are insensitive to the $\rho'$. This provided the ground to set the expected region for these parameters and compare their behavior as observables sensitive to the $\rho'$ were included. The incorporation
of the \( \omega \to 3\pi \) decay, exhibited a strong departure of the \( g_{\rho\omega\pi} \) coupling from the previously obtained values, the other couplings involved, namely \( g_{\rho}, g_{\rho\pi}\pi \) and \( g_{\omega} \), also reflected this tension (jump on these parameters in Fig. 10 at \( x \)-axis value 2). Extending the description by incorporating the \( \rho' \) and a contact term as prescribed by the WZW anomaly brought this parameter to peace with the previous data results (in Fig. 10, \( x \)-axis value 3). Upon the incorporation of the data from the \( e^+ e^- \to 3\pi \) (as measured by SND, CMD2, BABAR and BES III), and \( e^+ e^- \to \pi^0 \pi^0 \gamma \) (as measured by SND and CDM2) cross section data, we were able to further restrict the \( \rho' \) parameters validity region. The analysis exhibited the sensitivity to the relative phase and the \( \rho' \) parameters, for these last the \( A_1 \) combination restriction was very useful to bring the parameters within the physically expected region. The behavior of the \( g_{\omega\rho'\pi} \) and \( g_{3\pi} \) parameters reflected a process dependence as they favored different values for \( e^+ e^- \to 3\pi \) compared to \( e^+ e^- \to \pi^0 \pi^0 \gamma \) cross section. This may be due to the different precision and energy scanned by each process. In addition, since the \( g_{3\pi} \)
FIG. 12: Label for the $x$ axis in Figures [10] and [11]. In 2 and 3, the (+) symbol means in addition to the case at 1 (That is 10 decay modes). In 4 to 11, the (+) symbol means in addition to the case at 3 (That is 11 decay modes). Global, means considering all the previous cross section data and case at 3.

is a non-resonant contribution, it can be affected by the background subtraction procedure followed by the experiments. All over, we observed that the behavior of the $g_{\rho\omega\pi}$ coupling was very stable, upon the inclusion of the $\rho'$ and the contact term, for all data involved, favoring a value of $g_{\rho\omega\pi} = 11.314 \pm 0.383$ GeV$^{-1}$. This is important as this parameter has implications on other observables related with precision physics [50–53]. As an application of the results, we computed the $e^+ e^- \rightarrow 4\pi$ cross section for the so-called omega channel, measured by BABAR and find a good description of the data considering the parameters found, exhibiting the importance of the $\rho'$ parameters and the relative phase to properly describe the data. This channel plays an important role to extract further information from the total $4\pi$ process [49], as it provides the dominant contribution at low energies.

This analysis has exhibited the importance of the $\rho'$ meson and provided a reliable region for its parameters. This is not an exhaustive analysis since more processes such as $\tau$ decays were not included, but provides the ground to extend them (as we plan to do) while already points out to definite regions for the parameters that can be useful to describe
FIG. 13: The $e^+e^- \to \pi\omega \to 4\pi$ cross section due to the $\omega$ channel.

other processes.

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FIG. 14: The $e^+ e^- \rightarrow \pi \omega \rightarrow 4\pi$ cross section driven by the $\omega$ channel. Symbols correspond to experimental data. Lines correspond to the evaluation in the model considering the parameters determination from the different data sets, Table VI (dashed line) and VII (solid line). The $\rho'$ (dot-dashed line) and contact contribution (dotted line) are also shown.

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