MEAN CURVATURE FLOW OF HIGHER CODIMENSION
IN HYPERBOLIC SPACES

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Abstract. In this paper we investigate the convergence for the mean curvature flow of closed submanifolds with arbitrary codimension in space forms. Particularly, we prove that the mean curvature flow deforms a closed submanifold satisfying a pinching condition in a hyperbolic space form to a round point in finite time.

1. Introduction

In this paper, we study the convergence of the mean curvature flow of submanifolds in space forms. Let $F : M^n \to \mathbb{R}^{n+d}(c)$ be a smooth immersion from an $n$-dimensional closed Riemannian manifold $M^n$ to an $(n+d)$-dimensional complete simply connected space form $\mathbb{R}^{n+d}(c)$ with constant sectional curvature $c$. Consider a one-parameter family of smooth immersions $F : M \times [0, T) \to \mathbb{R}^{n+d}(c)$ satisfying

\[
\begin{align*}
\frac{\partial}{\partial t} F(x, t) &= H(x, t), \\
F(x, 0) &= F(x),
\end{align*}
\]

where $H(x, t)$ is the mean curvature vector of $F_t(M)$ and $F_t(x) = F(x, t)$. We call $F : M \times [0, T) \to \mathbb{R}^{n+d}(c)$ the mean curvature flow with initial value $F$.

The mean curvature flow was proposed by Mullins [17] to describe the formation of grain boundaries in annealing metals. In [3], Brakke introduced the motion of a submanifold by its mean curvature in arbitrary codimension and constructed a generalized varifold solution for all time. For the classical solution of the mean curvature flow, most works have been done on hypersurfaces. Huisken [11, 12] showed that if the initial hypersurface in a Riemannian manifold is uniformly convex, then the mean curvature flow converges to a round point in finite time. Later, Huisken [13] extend this result to hypersurfaces satisfying a pinching condition in a sphere. Many other beautiful results have been obtained, and there are various approaches to study the mean curvature flow of hypersurfaces (see [6, 7], etc.). For the mean curvature flow of submanifolds in higher codimension, some special cases have been studied, see [19, 20, 21, 22, 23, 24] etc. for example. Recently, Andrews-Baker [1] proved a convergence theorem for the mean curvature flow of closed submanifolds satisfying a pinching condition in the Euclidean space. In [2], Baker proved a convergence result for the mean curvature flow of submanifolds in a sphere. In this paper, we study the mean curvature flow of closed submanifolds in hyperbolic...
spaces and extend the convergence result in \cite{1, 2} to the mean curvature flow of arbitrary codimension in space forms.

**Theorem 1.1.** Let $F : M^n \to \mathbb{F}^{n+d}(c)$ be a smooth closed submanifold in a hyperbolic space with constant curvature $c < 0$. Assume $F$ satisfies
\begin{equation}
|A|^2 \leq \begin{cases} 
\frac{1}{4n}|H|^2 + \frac{2}{n}c, & n = 2, 3, \\
\frac{1}{n-1}|H|^2 + 2c, & n \geq 4.
\end{cases}
\end{equation}
Then the mean curvature flow with $F$ as initial value converges to a round point in finite time.

As an immediate consequence of Theorem 1.1, we obtain the following differentiable sphere theorem.

**Corollary 1.2.** Let $F : M^n \to \mathbb{F}^{n+d}(c)$ be a smooth closed submanifold in a hyperbolic space with constant curvature $c < 0$. Assume $F$ satisfies
\begin{equation}
|A|^2 \leq \begin{cases} 
\frac{1}{4n}|H|^2 + \frac{2}{n}c, & n = 2, 3, \\
\frac{1}{n-1}|H|^2 + 2c, & n \geq 4.
\end{cases}
\end{equation}
Then $M$ is diffeomorphic to the unit $n$-sphere.

**Remark 1.3.** This differentiable sphere theorem was also obtained by Gu and Xu \cite{10, 25} provided the submanifold is simply connected. In fact, they proved the sphere theorem for submanifolds in a Riemannian manifold by using a different method. For more sphere theorems of submanifolds, we refer the readers to \cite{1, 2, 9, 10, 16, 18, 25, 28}, etc.

Combining Theorem 1.1 and the convergence results in \cite{1, 2}, we obtain the following theorem.

**Theorem 1.4.** Let $F : M^n \to \mathbb{F}^{n+d}(c)$ be a smooth closed submanifold in a complete simply connected space form with $|H|^2 + n^2c > 0$. Assume $F$ satisfies
\begin{equation}
|A|^2 \leq \begin{cases} 
\frac{1}{4n}|H|^2 + \frac{1}{7n-4+\text{sgn}(c)(n-4)}c, & n = 2, 3, \\
\frac{1}{n-1}|H|^2 + 2c, & n \geq 4.
\end{cases}
\end{equation}
Then either $F_t(M)$ converges to a round point in finite time, or $c > 0$ and $F_t(M)$ converges to a total geodesic sphere in $\mathbb{F}^{n+d}(c)$ as $t \to \infty$.

**Remark 1.5.** For $c > 0$, $|H|^2 + n^2c > 0$ is automatically satisfied. For $c = 0$, $|H|^2 + n^2c > 0$ is equivalent to that the mean curvature is nowhere vanishing. For $c < 0$, $|H|^2 + n^2c > 0$ is implied by condition (1.3).

**Remark 1.6.** For $c > 0$, the maximal existence time of the mean curvature flow may be finite or infinite. For $c \leq 0$, the mean curvature flow with a closed initial submanifold always has finite maximal existence time.

2. Basic equations

Let $F : M \times [0, T) \to \mathbb{F}^{n+d}(c)$ be a smooth mean curvature flow with initial closed immersion $F_0 : M \to \mathbb{F}^{n+d}(c)$. Denote by $g(t)$ and $d\mu_t$ the induced metric and the volume form on $M$. Let $A$ and $H$ be the second fundamental form and
the mean curvature vector of $M$ in $\mathbb{R}^{n+d}(c)$, respectively. We shall make use of the following convention on the range of indices.

$$1 \leq i, j, k, \cdots \leq n, \quad 1 \leq A, B, C, \cdots \leq n + d \quad \text{and} \quad n + 1 \leq \alpha, \beta, \gamma, \cdots \leq n + d.$$ 

As in [1, 3], we consider the evolution on the spatial tangent bundle. Choose a local orthonormal frame $\{e_i\}$ for the spatial tangent bundle and a local orthonormal frame $\{\nu_\alpha\}$ for the normal bundle. Let $\{\omega_i\}$ be the dual frame of $\{e_i\}$. Then $A$ and $H$ can be written as

$$A = \sum_{i,j,\alpha} h_{ij\alpha} \omega_i \otimes \omega_j \otimes \nu_\alpha = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j, \quad H = \sum_{\alpha} H_\alpha \nu_\alpha.$$ 

We have the following evolution equations.

$$\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2R_1 + 4c|H|^2 - 2nc|A|^2,$$  \hfill (2.1)

$$\frac{\partial}{\partial t} |H|^2 = \Delta |H|^2 - 2|\nabla H|^2 + 2R_2 + 2nc|H|^2,$$  \hfill (2.2)

where

$$R_1 = \sum_{\alpha, \beta} \left( \sum_{i,j} h_{ij\alpha} h_{ij\beta} \right)^2 + |R^\perp|^2,$$  \hfill (2.3)

$$|R^\perp|^2 = \sum_{i,j,\alpha, \beta} \left( \sum_p \left( h_{ipa} h_{jp\beta} - h_{jpa} h_{ip\beta} \right) \right)^2,$$  \hfill (2.4)

$$R_2 = \sum_{i,j} \left( \sum_{\alpha} H_\alpha h_{ij\alpha} \right)^2.$$  \hfill (2.5)

The contracted form of Simons’ identity for traceless second fundamental form $\hat{A} := A - \frac{1}{n} g \otimes H$ is

$$\frac{1}{2} \Delta |\hat{A}|^2 = \hat{h}_{ij} \nabla_i \nabla_j H + |\nabla \hat{A}|^2 + Z + nc|\hat{A}|^2.$$  \hfill (2.6)

Here

$$Z = -R_1 + \sum_{i,j,\alpha, \beta} H_\alpha h_{ipa} h_{ij\beta} h_{pj\beta}.$$  \hfill (2.7)

We also have the following inequality.

$$|\nabla A|^2 \geq \frac{3}{n+2} |\nabla H|^2.$$  \hfill (2.8)

3. Preserved curvature pinching condition

Now we prove that the pinching condition (1.2) is preserved under the mean curvature flow with arbitrary codimension in the hyperbolic space.

**Lemma 3.1.** For $c < 0$ and $n \geq 2$, if the initial immersion satisfies (1.2), then this condition is preserved along the mean curvature flow.
Proof. We consider $Q = |A|^2 - \alpha|H|^2 - \beta c$, where the constants

$$
\alpha \leq \begin{cases} 
\frac{4}{n-1}, & n = 2, 3, \\
\frac{n}{n-1}, & n \geq 4,
\end{cases}
\quad \text{and} \quad
\beta \geq \begin{cases} 
\frac{2}{2}, & n = 2, 3, \\
2, & n \geq 4.
\end{cases}
$$

By (2.11) and (2.2) we have

$$
\frac{\partial}{\partial t} Q = \Delta Q - 2(|\nabla A|^2 - \alpha |\nabla H|^2)
+ 2R_1 - 2\alpha R_2 - 2nc|\dot{A}|^2 - 2n \left(\alpha - \frac{1}{n}\right) c|H|^2.
$$

(3.1)

We only have to show that if $Q = 0$ at a point $x \in M$, then

$$
2R_1 - 2\alpha R_2 - 2nc|\dot{A}|^2 - 2n \left(\alpha - \frac{1}{n}\right) c|H|^2 \leq 0
$$

holds at $x$. We also have $H \neq 0$ at $x$. Choose $\left\{\nu_a\right\}$ such that $\nu_{n+1} = \frac{H}{|H|}$. Let

$A_H = \sum_{i,j} h_{i,j,n+1} \omega_i \otimes \omega_j$. Set $\dot{A}_H = A_H - \frac{H}{n} \text{Id}$ and $|\dot{A}_H|^2 = |\dot{A}|^2 - |A_H|^2$.

We replace $|H|^2$ with $\frac{|\dot{A}|^2 - \beta c}{\alpha - \frac{2}{n}}$. Then

$$
2R_1 - 2\alpha R_2 - 2nc|\dot{A}|^2 - 2n \left(\alpha - \frac{1}{n}\right) c|H|^2
\leq 2|\dot{A}_H|^2 - 2 \left(\alpha - \frac{1}{n}\right) |\dot{A}_H|^2 |H|^2 + \frac{2}{n} |\dot{A}_H|^2 |H|^2 - \frac{2}{n} \left(\alpha - \frac{1}{n}\right) |H|^4
+ 8 |\dot{A}_H|^2 |\dot{A}|^2 + 3 |\dot{A}|^2 - 2nc(|\dot{A}_H|^2 + |\dot{A}|^2) - 2n \left(\alpha - \frac{1}{n}\right) c|H|^2
\leq \left(6 - \frac{2}{n(\alpha - \frac{1}{n})}\right) |\dot{A}_H|^2 |\dot{A}|^2 + \left(3 - \frac{2}{n(\alpha - \frac{1}{n})}\right) |\dot{A}|^4
+ \left(2\beta - 4n + \frac{2\beta}{n(\alpha - \frac{1}{n})}\right) c|\dot{A}_H|^2 + 4 \left(\frac{\beta}{n(\alpha - \frac{1}{n})} - n\right) c|\dot{A}|^2
- 2\beta \left(\frac{\beta}{n(\alpha - \frac{1}{n})} - n\right) c^2.
$$

(3.2)

By the definition of $\alpha$ and $\beta$, we know that the right hand side of (3.2) is nonpositive for $n \geq 2$. This completes the proof of the lemma. \qed

For $\epsilon > 0$, set

$$
\alpha_{\epsilon} = \begin{cases} 
\frac{4}{3n+\epsilon n}, & n = 2, 3, \\
\frac{1}{n-1+\epsilon}, & n \geq 4,
\end{cases}
\quad \text{and} \quad
\beta_{\epsilon} = \begin{cases} 
\frac{2}{2}(1+\epsilon), & n = 2, 3, \\
2(1+\epsilon), & n \geq 4.
\end{cases}
$$

If the initial immersion satisfies $|A|^2 < \frac{4}{3n}|H|^2 + \frac{\epsilon}{c} c$ for $n = 2, 3$, and $|A|^2 < \frac{1}{n-1}|H|^2 + 2c$ for $n \geq 4$, then there exists an $\epsilon > 0$ such that $|A|^2 \leq \alpha_{\epsilon}|H|^2 + \beta_{\epsilon} c$ holds on $M_0$. From the proof of Lemma 3.3 this inequality also holds for $t > 0$. On the other hand, if $|A|^2 = \frac{4}{3n}|H|^2 + \frac{\epsilon}{c} c$ for $n = 2, 3$, or $|A|^2 = \frac{1}{n-1}|H|^2 + 2c$ for $n \geq 4$ holds somewhere on $M_0$, then by the maximum principle, we see that either the equality holds everywhere on $M_0$, or the strict inequality holds everywhere for $t > 0$. For the first case, we have $\nabla A = 0$ and $A_t = 0$ on $M_0$. By (3), $M_0$ lies in an $(n + 1)$-dimensional total geodesic submanifold of $\mathbb{R}^{n+d}(c)$. Since $\nabla A = 0$, from Theorem 4 of [15], $M_0$ is either locally isometric to an Euclidean space, or...
locally isometric to a product $\mathbb{F}^k(c_1) \times \mathbb{F}^{n-k}(c_2)$ for some $c_1 > 0$, $c_2 < 0$ and $k = 0, \cdots , n$. Since $M_0$ is closed, we see that $M_0$ is a totally umbilical sphere. Then $|A|^2 \leq \alpha |H|^2 + \beta c$ holds on $M_0$ for some $\epsilon > 0$. For the second case, we see that after a short time, we also have $|A|^2 \leq \alpha |H|^2 + \beta c$ for some $\epsilon > 0$. Hence, we may assume that $|A|^2 \leq \alpha |H|^2 + \beta c$ for some $\epsilon \in (0, 1)$ and $t \geq t_0 > 0$.

4. Pinching of $\hat{A}$ along the mean curvature flow

Assume that $c < 0$. We prove a pinching estimate for the traceless second fundamental form, which guarantees that $M_t$ becomes spherical along the mean curvature flow.

**Theorem 4.1.** There are positive constants $C_0$ and $\sigma_0$ independent of $t$ such that

$$|\hat{A}|^2 \leq C_0 |H|^{2-\sigma_0}$$

holds along the mean curvature flow.

**Proof.** We consider the function $f_\sigma = \frac{|\hat{A}|^2}{(\alpha |H|^2 + \beta c)}$, where $\sigma \in (0, 1)$ and

$$a = \begin{cases} 
\frac{1}{3n + ne}, & n = 2, 3 \\
\frac{1}{n(n-1+\epsilon)}, & n \geq 4.
\end{cases}$$

Notice that

$$a|H|^2 + \beta c - \left( \alpha c - \frac{1}{n} \right) |H|^2 + \beta c$$

for $n = 2, 3$, and

$$a|H|^2 + \beta c - \left( \alpha c - \frac{1}{n} \right) |H|^2 + \beta c$$

for $n \geq 4$. So $f_\sigma$ is well-defined. From (4.2) and (4.3) we also have

$$a|H|^2 + \beta c \geq b|H|^2,$$

where

$$b = \begin{cases} 
\frac{\epsilon}{3n + ne}, & n = 2, 3 \\
\frac{\epsilon}{n(n-1+\epsilon)}, & n \geq 4.
\end{cases}$$
By a similar computation as in [1], we have

$$\frac{\partial}{\partial t} f_{\sigma} = \Delta f_{\sigma} + \frac{2a(1-\sigma)}{|A|^2 + \beta_c c} \langle \nabla |H|^2, \nabla f_{\sigma} \rangle$$

$$- \frac{2}{(a|H|^2 + \beta_c c)^{1-\sigma}} \langle \nabla A, \frac{1}{n} |\nabla H|^2 - \frac{a|\dot{A}|^2}{a|H|^2 + \beta_c c} |\nabla H|^2 \rangle$$

$$- \frac{4a^2\sigma(1-\sigma)}{a|H|^2 + \beta_c c^2} f_{\sigma} |H|^2 \langle \nabla |H|^2, \nabla f_{\sigma} \rangle - \frac{2a\sigma f_{\sigma}}{a|H|^2 + \beta_c c} |\nabla H|^2$$

$$+ \frac{2}{a|H|^2 + \beta_c c^{1-\sigma}} \left( R_1 - \frac{1}{n} \frac{aR_2 |\dot{A}|^2}{a|H|^2 + \beta_c c} - nc|\dot{A}|^2 - \frac{2a\sigma R_2 f_{\sigma}}{a|H|^2 + \beta_c c} \right)$$

$$+ \frac{2a\sigma R_2 f_{\sigma}}{a|H|^2 + \beta_c c}.$$

By (2.8), we have

$$|\nabla A|^2 - \frac{1}{n} |\nabla H|^2 - \frac{a|\dot{A}|^2}{a|H|^2 + \beta_c c} |\nabla H|^2$$

$$\geq \left( \frac{3}{n+2} - \frac{1}{n} - a \left( \frac{(\alpha - \frac{1}{n})|H|^2 + \beta_c c}{a|H|^2 + \beta_c c} \right) \right) |\nabla H|^2$$

$$\geq \left( \frac{3}{n+2} - \frac{1}{n} - a \right) |\nabla H|^2$$

$$:= \epsilon_{\nabla} |\nabla H|^2.$$

Here $\epsilon_{\nabla}$ is a positive constant for $n \geq 2$.

We also have the following estimate.

$$R_1 - \frac{1}{n} R_2 - \frac{aR_2 |\dot{A}|^2}{a|H|^2 + \beta_c c}$$

$$\leq R_1 - \frac{1}{n} R_2 - \frac{2|\dot{A}|^2}{|H|^2}$$

$$= R_1 - \frac{2|\dot{A}|^2}{|H|^2}$$

$$\leq |\dot{A}|^4 - 2 \left( \frac{|A|^2}{|H|^2} - \frac{2}{n} \right) |\dot{A}|^2 |H|^2 - \frac{2}{n} \left( \frac{|A|^2}{|H|^2} - \frac{1}{n} \right) |H|^4$$

$$- 4 |\dot{A}|^2 |\dot{A}|^2 - 3 \frac{3}{2} |\dot{A}|^4$$

$$\leq 0.$$

In (4.7) we have used the pinching condition $|A|^2 \leq \alpha_c |H|^2 + \beta_c c < \alpha_c |H|^2$ for $\epsilon \in (0, 1)$. 
By (4.4), we have

\[ -nc|\dot{A}|^2 - \frac{an(1-\sigma)c|\dot{A}|^2|H|^2}{a|H|^2 + \beta c} \leq -nc|\dot{A}|^2 - \frac{an(1-\sigma)c|\dot{A}|^2(a|H|^2 + \beta c)}{b(a|H|^2 + \beta c)} \leq -nc|\dot{A}|^2 - \frac{anc}{b}|\dot{A}|^2 := b|\dot{A}|^2. \]  

(4.8)

For the last term of right hand side of (4.5), we have by (4.4)

\[ \frac{2a\sigma_2f_\sigma}{a|H|^2 + \beta c} \leq \frac{2a\sigma|H|^2|A|^2f_\sigma}{a|H|^2 + \beta c} \leq \frac{2a\sigma}{b}|A|^2f_\sigma := \tilde{b}\sigma|A|^2f_\sigma. \]

(4.9)

Combining (4.5), (4.6), (4.7), (4.8) and (4.9), we have

\[ \frac{\partial}{\partial t} f_\sigma \leq \triangle f_\sigma + \frac{2a(1-\sigma)}{a|H|^2 + \beta c} (\nabla|H|^2, \nabla f_\sigma) - \frac{2\epsilon}{(a|H|^2 + \beta c)^{1-\sigma}} |\nabla H|^2 + 2\beta f_\sigma + b\sigma|A|^2f_\sigma. \]

(4.10)

To deal with the last term of the right hand side of (4.10), we need the following estimate.

**Proposition 4.2.** There exists a positive constant \( \varepsilon \) independent of \( t \) such that

\[ Z + nc|\dot{A}|^2 \geq \varepsilon|\dot{A}|^2(a|H|^2 + \beta c) \]  

holds for \( t \geq t_0 \).

**Proof.** By the argument in the proof of Lemma 5.4 in [2], we only have to show

\[ -\frac{\beta c}{\alpha c - \frac{1}{n}} \left( \frac{1}{n} - \frac{n-2}{2n(n-1)} \right) + n \leq 0. \]

This is true by our choice of \( \alpha c \) and \( \beta c \). \( \square \)

**Proposition 4.3.** For any \( \eta > 0, p \geq 2 \) and \( t \geq t_0 \), we have

\[ \int_{M_t} f_\sigma^p(a|H|^2 + \beta c)d\mu_t \leq \frac{2p\eta + 5}{b\varepsilon} \int_{M_t} \frac{f_\sigma^{p-1}}{(a|H|^2 + \beta c)^{1-\sigma}} |\nabla H|^2d\mu_t + \frac{2(p-1)}{b\eta\varepsilon} \int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2d\mu_t. \]

(4.12)
Proof. We have the following estimate.

\[
\begin{align*}
2 & \int_{M_t} \frac{f_{\sigma}^{p-1}}{(a|H|^2 + \beta_c)^{1-\sigma}} Z d\mu_t + 2nc \int_{M_t} \frac{f_{\sigma}^{p-1}}{(a|H|^2 + \beta_c)^{1-\sigma}} |\dot{A}|^2 d\mu_t \\
& \leq 2(p-1) \int_{M_t} \frac{f_{\sigma}^{p-2}}{(a|H|^2 + \beta_c)^{1-\sigma}} |\nabla f_{\sigma}| |\dot{A}||\nabla H| d\mu_t \\
& + \frac{2(n-1)}{n} \int_{M_t} \frac{f_{\sigma}^{p-1}}{(a|H|^2 + \beta_c)^{1-\sigma}} |\nabla H|^2 d\mu_t \\
& + 4 \int_{M_t} \frac{f_{\sigma}^{p-1}}{(a|H|^2 + \beta_c)^{2-\sigma}} |H||\dot{A}||\nabla H|^2 d\mu_t \\
& + 4(1-\sigma)(p-2) \int_{M_t} \frac{f_{\sigma}^{p-1}}{(a|H|^2 + \beta_c)} |H||\nabla H||\nabla f_{\sigma}| d\mu_t \\
& + 4 \int_{M_t} \frac{f_{\sigma}^{p}}{(a|H|^2 + \beta_c)^2} |H|^2 |\nabla H|^2 d\mu_t.
\end{align*}
\]

(4.13)

Since $|\dot{A}|^2 \leq f_{\sigma}(a|H|^2 + \beta_c)^{1-\sigma}$ and $f_{\sigma} \leq (a|H|^2 + \beta_c)^\sigma$, by choosing $\sigma \in (0,1)$ we have the following estimates.

\[
\begin{align*}
2 & \int_{M_t} \frac{f_{\sigma}^{p-2}}{(a|H|^2 + \beta_c)^{1-\sigma}} |\nabla f_{\sigma}| |\dot{A}||\nabla H| d\mu_t \\
& \leq \frac{1-\sigma}{\eta} \int_{M_t} \frac{f_{\sigma}^{p-2}}{(a|H|^2 + \beta_c)^{1-\sigma}} |\nabla f_{\sigma}| d\mu_t + (p-1)\eta \int_{M_t} \frac{f_{\sigma}^{p-1}}{(a|H|^2 + \beta_c)^{1-\sigma}} |\nabla H|^2 d\mu_t \\
& + 4 \int_{M_t} \frac{f_{\sigma}^{p-1}}{(a|H|^2 + \beta_c)^{2-\sigma}} |H||\dot{A}||\nabla H|^2 d\mu_t \\
& \leq \frac{4}{b} \int_{M_t} \frac{f_{\sigma}^{p-1}}{(a|H|^2 + \beta_c)^{1-\sigma}} |\nabla H|^2 d\mu_t \\
& + 4(p-2) \int_{M_t} \frac{f_{\sigma}^{p-1}}{(a|H|^2 + \beta_c)} |H||\nabla H||\nabla f_{\sigma}| d\mu_t \\
& \leq \frac{(p-2)}{b\eta} \int_{M_t} \frac{f_{\sigma}^{p-2}}{(a|H|^2 + \beta_c)} |\nabla f_{\sigma}|^2 d\mu_t \\
& + 2(p-2)\eta \int_{M_t} \frac{f_{\sigma}^{p-1}}{(a|H|^2 + \beta_c)^{1-\sigma}} |\nabla H|^2 d\mu_t,
\end{align*}
\]

(4.15)

\[
\begin{align*}
4 & \int_{M_t} \frac{f_{\sigma}^{p}}{(a|H|^2 + \beta_c)^2} |H|^2 |\nabla H|^2 d\mu_t \\
& \leq \frac{4}{b} \int_{M_t} \frac{f_{\sigma}^{p-1}}{(a|H|^2 + \beta_c)^{1-\sigma}} |\nabla H|^2 d\mu_t.
\end{align*}
\]

(4.16)

In (4.15), (4.16) and (4.17) we have used (4.14).

By (4.11), we have

\[
\begin{align*}
2 & \int_{M_t} \frac{f_{\sigma}^{p-1}}{(a|H|^2 + \beta_c)^{1-\sigma}} (Z + 2nc|\dot{A}|^2) d\mu_t \\
& \geq 2\varepsilon \int_{M_t} f_{\sigma}(a|H|^2 + \beta_c) d\mu_t.
\end{align*}
\]

(4.18)
Combining (4.13)–(4.18), we obtain
\[
2\varepsilon \int_{M_t} f^p_\sigma (a|H|^2 + \beta_c c) d\mu_t \leq \frac{3p\eta + 10}{b} \int_{M_t} \frac{f^{p-1}_\sigma}{(a|H|^2 + \beta_c c)^{1-\sigma}} |\nabla H|^2 d\mu_t + \frac{3(p-1)}{b\eta} \int_{M_t} f^{p-2}_\sigma |\nabla f_\sigma|^2 d\mu_t.
\]
(4.19)

Dividing through by $2\varepsilon$ completes the proof.

Now we show that the $L^p$-norm of $f_\sigma$ is bounded for sufficiently high $p$.

**Lemma 4.4.** For any $p \geq \max\{2, \frac{8}{b \varepsilon c} + 1\}$ and $\sigma \leq \min\left\{\frac{b^2 \varepsilon c}{10b\varepsilon c}, \frac{b^2 \varepsilon c}{4b\varepsilon c}, \frac{7}{2}\right\}$, there exist a constant $C$ independent of $t$ such that for all $t \in [0, T_{\max})$ where $T_{\max} < \infty$, we have
\[
\left( \int_{M_t} f^p_\sigma d\mu_t \right)^{\frac{1}{p}} \leq C.
\]
(4.20)

**Proof.** For $t \geq t_0$, form (4.10), we have
\[
\frac{\partial}{\partial t} \int_{M_t} f^p_\sigma d\mu_t \leq \int_{M_t} pf^{p-1}_\sigma \frac{\partial}{\partial t} f_\sigma d\mu_t
\leq -p(p-1) \int_{M_t} f^{p-2}_\sigma |\nabla f_\sigma|^2 d\mu_t
+ 4(1-\sigma)p \int_{M_t} \frac{f^{p-1}_\sigma}{a|H|^2 + \beta_c c} |H||\nabla |H|||\nabla f_\sigma|d\mu_t
- 2p\varepsilon \int_{M_t} \frac{f^{p-1}_\sigma}{(a|H|^2 + \beta_c c)^{1-\sigma}} |\nabla H|^2 d\mu_t
+ 2\tilde{b}p \int_{M_t} f^p_\sigma d\mu_t + \tilde{b}\sigma p \int_{M_t} |A|^2 f^p_\sigma d\mu_t.
\]
(4.21)

As in (4.10), we have
\[
4(1-\sigma)p \int_{M_t} \frac{f^{p-1}_\sigma}{a|H|^2 + \beta_c c} |H||\nabla |H|||\nabla f_\sigma|d\mu_t
\leq \frac{2p}{b\mu} \int_{M_t} f^{p-2}_\sigma |\nabla f_\sigma|^2 d\mu_t + 2\mu \int_{M_t} \frac{f^{p-1}_\sigma}{(a|H|^2 + \beta_c c)^{1-\sigma}} |\nabla H|^2 d\mu_t.
\]
(4.22)

Substituting (4.22) to (4.21), letting $\mu = \frac{4}{b(p-1)}$ and $p \geq \max\{2, \frac{8}{b \varepsilon c} + 1\}$ we obtain
\[
\frac{\partial}{\partial t} \int_{M_t} f^p_\sigma d\mu_t \leq -\frac{p(p-1)}{2} \int_{M_t} f^{p-2}_\sigma |\nabla f_\sigma|^2 d\mu_t
- p\varepsilon \int_{M_t} \frac{f^{p-1}_\sigma}{(a|H|^2 + \beta_c c)^{1-\sigma}} |\nabla H|^2 d\mu_t
+ 2\tilde{b}p \int_{M_t} f^p_\sigma d\mu_t + \frac{\tilde{b}\sigma c p}{b} \int_{M_t} f^p_\sigma (a|H|^2 + \beta_c c) d\mu_t.
\]
This together with (4.12) implies

\[ \frac{\partial}{\partial t} \int_{M_t} f_\sigma^p d\mu_t \leq -p(p-1) \left( \frac{1}{2} - \frac{2\bar{b}\alpha_c}{b^2\eta_c} \right) \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu_t \]

\[ - \left( p\sqrt{\eta_c} - \frac{(2m+5)b\alpha_c\bar{p}}{b^2\varepsilon} \right) \int_{M_t} \frac{f_\sigma^{p-1}}{a(H^2 + \beta c)^{1-\sigma}} |\nabla H|^2 d\mu_t \]

\[ + 2\bar{b} p \int_{M_t} f_\sigma^p d\mu_t. \]

Now we pick \( \eta = \frac{4\bar{b}\alpha_c}{b^2 c} \) and let \( \sigma \leq \min \left\{ \frac{b^2 \sqrt{\alpha_c}}{108\alpha_c}, \frac{b^2 \varepsilon \sqrt{\varepsilon}}{4\bar{b}\alpha_c}, \sqrt{\frac{1}{2}} \right\} \). Then (4.23) reduces to

\[ \frac{\partial}{\partial t} \int_{M_t} f_\sigma^p d\mu_t \leq 2\bar{b} p \int_{M_t} f_\sigma^p d\mu_t. \]

This implies

\[ \int_{M_t} f_\sigma^p d\mu_t \leq e^{2\bar{b}pt} \int_{M_{t_0}} f_\sigma^p d\mu_t. \]

If \( t \in [0, t_0] \), by the smoothness of the mean curvature flow we see that \( \int_{M_t} f_\sigma^p d\mu_t \) is bounded. For \( t \geq t_0 \), we only have to show that \( T_{\text{max}} \) is finite.

**Lemma 4.5.** The maximal existence time \( T_{\text{max}} \) of the mean curvature flow is finite.

**Proof.** Fixed a point \( y \in \mathbb{F}^{n+d}(c) \) and let \( r \) be the distance function on \( \mathbb{F}^{n+d}(c) \) from \( y \). Denote also by \( r \) the composition \( r \circ F_t \). We may assume that \( r > 0 \) on \( M_t \) for \( t \in [0, T_{\text{max}}] \). In fact, if \( d = 1 \), we may choose \( y \) such that it is outside of a geodesic ball in \( \mathbb{F}^{n+1}(c) \) that encloses \( M_0 \). By the maximum principle we see that \( y \) doesn’t lies in any \( M_t \). If \( d > 1 \), then the Hausdorff dimension of \( F(M \times [0, T_{\text{max}}]) \) is no more than \( n + 1 \). So we can also pick a point \( y \) such that it doesn’t lies in any \( M_t \). In both cases, we have \( r > 0 \) on each \( M_t \).

From (4.1), we know that

\[ \triangle r = \langle H, \partial_r \rangle + c_\varepsilon(r)(n - |\partial_r|^2). \]

Here \( c_\varepsilon(r) = \frac{\varepsilon \cosh(\sqrt{-c}r)}{\sinh(\sqrt{-c}r)} = \frac{\varepsilon (\sqrt{c}e^{-r} - e^{-r})}{e^{\sqrt{c}r} - e^{-\sqrt{c}r}} \). \( \partial_r \) is the gradient of \( r \) in \( \mathbb{F}^{n+d}(c) \), and \( \partial_r^T \) is the tangent part of \( \partial_r \) to \( M_t \). Clearly we have \( c_\varepsilon(r) \geq \sqrt{-c} \) and \( |\partial_r^T|^2 \leq 1 \).

On the other hand, since \( F_t \) satisfies (1.1), we have

\[ \frac{\partial}{\partial t} r = \langle H, \partial_r \rangle. \]

Combining (4.25) and (4.26) we obtain

\[ \frac{\partial}{\partial t} r = \triangle r - c_\varepsilon(r)(n - |\partial_r^T|^2). \]

Suppose that \( r(0) < R \). By the maximum principle we see that

\[ r(t) < R - (n - 1)\sqrt{-c}t. \]

Then \( T_{\text{max}} < \frac{R}{(n-1)\sqrt{-c}} \), i.e., the maximal existence time of the mean curvature flow is finite. \( \square \)
By Proposition 4.5 we finish the proof of Lemma 4.4.

Now we can proceed as in [11] or [14] via a Stampacchia iteration procedure to complete the proof of Theorem 4.1.

5. A gradient estimate for the mean curvature

We establish a gradient estimate for the mean curvature flow, which will be used to compare the mean curvature at different points of the submanifold. We also assume that \( c < 0 \).

**Theorem 5.1.** For every \( \eta > 0 \), there exists a constant \( C_\eta \) independent of \( t \) such that for all \( t \in [0, T_{\max}) \), there holds

\[
|\nabla H|^2 \leq \eta |H|^4 + C_\eta.
\]

**Proof.** By direct computation, we have

\[
\frac{\partial}{\partial t} |H|^4 \geq \triangle |H|^4 - 12 |H|^2 |\nabla H|^2 + \frac{4}{n} |H|^6 + 4nc |H|^4,
\]

\[
\frac{\partial}{\partial t} |\nabla H|^2 \leq \triangle |\nabla H|^2 + C_1 |H|^2 |\nabla A|^2 + C_2 |\nabla A|^2,
\]

for constants \( C_1 \) and \( C_2 \) independent of \( t \).

We also have the following estimate for sufficiently large positive constants \( N_1 \) and \( N_2 \) independent of \( t \).

\[
\frac{\partial}{\partial t} \left( (N_1 + N_2 |H|^2)|\hat{A}|^2 \right) \leq \triangle \left( (N_1 + N_2 |H|^2)|\hat{A}|^2 \right) - \frac{4(n-1)}{3n} (N_2 - 1)|H|^2 |\nabla A|^2
\]

\[- \frac{4(n-1)}{3n} (N_1 - C(N_2))|\nabla A|^2
\]

\[- C_2 (N_1, N_2)|\hat{A}|^2(|H|^4 + 1) - 2nc N_1 |\hat{A}|^2.
\]

In (5.3), \( C(N_2) \) and \( C(N_1, N_2) \) are constants depending on \( N_2 \) and \( N_1 \). \( N_2 \) respectively. Consider the function \( f = |\nabla H|^2 + (N_1 + N_2 |H|^2)|\hat{A}|^2 - \eta |H|^4 \). From (5.2), (5.3), and (5.4), we have

\[
\frac{\partial}{\partial t} f \leq \triangle f - \frac{4(n-1)}{3n} (N_2 - 1)|H|^2 |\nabla A|^2 - \frac{4(n-1)}{3n} (N_1 - C_3(N_2))|\nabla A|^2
\]

\[+ C_4(N_1, N_2)|\hat{A}|^2(|H|^4 + 1) + 12\eta |H|^2 |\nabla H|^2 - \frac{4\eta}{n} |H|^6 - 4nc\eta |H|^4.
\]

Here we have consumed \( C_1 |H|^2 |\nabla A|^2 + C_2 |\nabla A|^2 \) by firstly choosing sufficiently large \( N_2 \) and secondly choosing sufficiently large \( N_1 \). Notice that \( |\nabla H|^2 \leq n |\nabla A|^2 \).

We can choose larger \( N_2 \) and \( N_1 \) depending on \( \eta \) to consume \( 12\eta |H|^2 |\nabla H|^2 \) and make the second, third terms of the right hand side of (5.5) negative. Since \( |\hat{A}|^2 \leq |A|^2 \) for \( t \geq t_0 \) and \( |\hat{A}|^2 \) is uniformly bounded for \( t \in [0, t_0] \), using Young’s inequality we get

\[
C_4(N_1, N_2)|\hat{A}|^2(|H|^4 + 1) - 4nc\eta |H|^4 \leq \frac{4\eta}{n} |H|^6 + C_\eta.
\]
Here $C_\eta$ is a constant depending on $\eta$ and other quantities but independent of $t$. Then we obtain
\[
\frac{\partial f}{\partial t} \leq \Delta f + C_5.
\]
Notice that $T_{\text{max}}$ is finite. Then the theorem follows from the maximum principle and the definition of $f$. \qed

6. Convergence of MCF in a hyperbolic space

**Theorem 6.1.** Let $F : M^n \to \mathbb{H}^{n+d}(c)$ be a smooth closed submanifold, where $n \geq 2$ and $c < 0$. Assume $F$ satisfies
\[
|A|^2 \leq \begin{cases} \frac{4}{n-1}|H|^2 + \frac{2}{n}c, & n = 2, 3, \\ \frac{1}{n-1}|H|^2 + 2c, & n \geq 4. \end{cases}
\]
Then $F_t(M)$ converges to a round point in finite time.

**Proof.** By the curvature estimate in [5], we see that
\[
K_{\min}(x) \geq \frac{1}{2} \left( \frac{1}{n-1} - \alpha \right)|H|^2(x) + \frac{1}{2}(2 - \beta) c.
\]
By our choices of $\alpha$ and $\beta$, and the preserved pinching condition, we see that there exists a positive constant $\epsilon_0$ independent of $t$ such that
\[
K_{\min}(x) \geq \epsilon_0 |H|^2.
\]
Since $T_{\text{max}}$ is finite, $\max_{M_t} |A|^2 \to \infty$ as $t \to T_{\text{max}}$. By similar arguments as in [11, 12] we have $\max_{M_t} |H| / \min_{M_t} |H| \to 1$ as $t \to T_{\text{max}}$, and $M_t$’s converge to a single point $o$ as $t \to T_{\text{max}}$. If we take a rescaling around $o$ (since $\mathbb{H}^{n+d}(c)$ can be consider as a linear space that isomorphic to $\mathbb{R}^{n+d}$) such that the total area of the expanded submanifolds are fixed, then the rescaled immersions converge to a totally umbilical immersion as $t \to T_{\text{max}}$. \qed

When $p = 1$ and $n = 3$, we have the following proposition.

**Proposition 6.2.** Let $F : M^3 \to \mathbb{H}^4(c)$ be a smooth closed hypersurface in a hyperbolic space with constant curvature $c < 0$. Assume $F$ satisfies
\[
|A|^2 \leq \frac{1}{2}|H|^2 + 2c.
\]
Then the mean curvature flow with $F$ as initial value converges to a round point in finite time.

**Proof.** If $d = 1$, then $A_{I} = 0$. If we take $\alpha \leq \frac{1}{n-1}$ and $\beta \geq 2$ for all $n \geq 2$, the left hand side of (3.2) is nonpositive. Hence $|A|^2 \leq \frac{1}{n-1}|H|^2 + 2c$ is preserved along the mean curvature flow for all $n \geq 2$. When $n = 3$, if we set $a = \frac{1}{2(2+\epsilon)}$, then $\epsilon_\nabla = \frac{2}{3} - \frac{1}{3} - a > 0$, and Theorem [4.1] also holds. Then Theorem [5.1] follows. By a similar argument as in the proof of Theorem [6.1] we get the convergence of the mean curvature flow. \qed
Remark 6.3. When $d = 1$ and $n = 3$, the pinching condition $|A|^2 \leq \frac{1}{2}|H|^2 + 2c$ is better than condition (6.1). In fact, we have
$$\frac{1}{2}|H|^2 + 2c - \frac{4}{9}|H|^2 - \frac{4}{9}c = \frac{1}{18}|H|^2 + \frac{1}{2}c > 0.$$ Here we have used the fact that $|H|^2 + 9c > 0$, which is implied by $|A|^2 \leq \frac{1}{2}|H|^2 + 2c$.

References

[1] B. Andrews and C. Baker: Mean curvature flow of pinched submanifolds to spheres, J. Differential Geom. 85(2010), 357-395.
[2] C. Baker: The mean curvature flow of submanifolds of high codimension, arXiv: math.DG/1104.4409.
[3] K. Brakke: The motion of a surface by its mean curvature, Princeton, New Jersey: Princeton University Press, 1978.
[4] F. J. Carreras, F. Giménez and V. Miquel: Immersions of compact riemannian manifolds into a ball of a complex space form, Math. Z. 225(1997), 103-113.
[5] B. Y. Chen: Some pinching and classification theorems for minimal submanifolds, Arch. Math. 60(1993), 568-578.
[6] Y. G. Chen, Y. Giga and S. Goto: Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Differential Geom. 33(1991), 749-786.
[7] L. C. Evans and J. Spruck: Motion of level sets by mean curvature, I, J. Differential Geom. 33(1991), 635-681.
[8] J. Erbacher: Reduction of the codimension of an isometric immersion, J. Differential Geom. 5(1971), 333-340.
[9] H. P. Fu and H. W. Xu: Vanishing and topological sphere theorems for submanifolds in a hyperbolic space, Intern. J. Math. 19(2008), 811-822.
[10] J. R. Gu and H. W. Xu: The sphere theorems for manifolds with positive scalar curvature, arXiv: math.DG/1102.2424.
[11] G. Huisken: Flow by mean curvature of convex surfaces into spheres, J. Differential Geom. 20(1984), 237-266.
[12] G. Huisken: Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature, Invent. Math. 84(1986), 463-480.
[13] G. Huisken: Deforming hypersurfaces of the sphere by their mean curvature, Math. Z. 195(1987), 205-219.
[14] G. Huisken and C. Sinestrari: Mean curvature flow singularities for mean convex surfaces, Calc. Var. 8(1999), 1-14.
[15] B. Lawson: Local rigidity theorems for minimal hypersurfaces, Ann. Math. 89(1969), 179-185.
[16] K. F. Liu, H. W. Xu, F. Ye and E. T. Zhao: The extension and convergence of mean curvature flow in higher codimension, arXiv: math.DG/1104.0971.
[17] W. W. Mullins: Two-dimensional motion of idealized grain boundaries, J. Appl. Phys. 27(1956), 900-904.
[18] K. Shiohama and H. W. Xu, The topological sphere theorem for complete submanifolds, Compositio Math. 107(1997), 221-232.
[19] K. Smoczyk: Longtime existence of the Lagrangian mean curvature flow, Calc. Var. 20(2004), 25-46.
[20] K. Smoczyk: Mean curvature flow in higher codimension - Introduction and survey, arXiv: math.DG/1104.3222v2.
[21] K. Smoczyk and M. T. Wang: Mean curvature flows for Lagrangian submanifolds with convex potentials, J. Differential Geom. 62(2002), 243-257.
[22] M. T. Wang: Mean curvature flow of surfaces in Einstein four-manifolds, J. Differential Geom. 57(2001), 301-338.
[23] M. T. Wang: Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension, Invent. math. 148(2002), 525-543.
[24] M. T. Wang: Lectures on mean curvature flows in higher codimensions, Handbook of geometric analysis. No. 1, 525-543, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, 2008.
[25] H. W. Xu and J. R. Gu: An optimal differentiable sphere theorem for complete manifolds, 
Math. Res. Lett. 17(2010), 1111-1124.
[26] H. W. Xu, F. Ye and E. T. Zhao: Extend mean curvature flow with finite integral curvature, 
to appear in Asian J. Math. 2011.
[27] H. W. Xu, F. Ye and E. T. Zhao: The extension for mean curvature flow with finite integral 
curvature in Riemannian manifolds, to appear in Sci. China Math. 2011.
[28] H. W. Xu and E. T. Zhao: Topological and differentiable sphere theorems for complete 
submanifolds, Comm. Anal. Geom. 17(2009), 565-585.

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