TOPICAL REVIEW

Zitterbewegung (trembling motion) of electrons in semiconductors: a review

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Abstract

We review recent research on Zitterbewegung (ZB, trembling motion) of electrons in semiconductors. A brief history of the subject is presented, the trembling motion in semi-relativistic and spin systems is considered and its main features are emphasized. ZB of charge carriers in monolayer and bilayer graphene as well as in carbon nanotubes is elaborated in some detail. We describe the effects of an external magnetic field on ZB using monolayer graphene as an example. The nature of electron ZB in crystalline solids is explained. We also review various simulations of the trembling motion in a vacuum and in semiconductors, and mention ZB-like wave phenomena in sonic and photonic periodic structures. An attempt is made to quote all the relevant literature on the subject.

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1. Introduction and brief history

This review article is concerned with a somewhat mysterious phenomenon known in the literature under the German name of ‘Zitterbewegung’ (trembling motion). Both the phenomenon and its name were conceived by Erwin Schrödinger who, in 1930, published the paper ‘Ueber die kraeftefreie Bewegung in der relativistischen Quantenmechanik’, in which he observed that in the Dirac equation, describing relativistic electrons in a vacuum, the $4 \times 4$ operators corresponding to components of relativistic velocity do not commute with the free-electron Hamiltonian [1]. In consequence, the electron velocity is not a constant of the motion in the absence of external fields. Such an effect must be of a quantum nature as it does not obey Newton’s first law of classical motion. Schrödinger calculated the resulting time dependence of the electron velocity and position, concluding that, in addition to classical motion, they experience very fast periodic oscillations which he called Zitterbewegung (ZB). Schrödinger’s idea stimulated numerous theoretical investigations but no experiments since the predicted frequency $\hbar \omega \approx 2m_0c^2 \approx 1$ MeV and the amplitude of about $\lambda_c = \hbar/m_0c \approx 3.86 \times 10^{-3}$ Å are not accessible to current experimental techniques. Huang [2] put the theory on a more physical basis by calculating averages of velocity and position operators. It was recognized that the ZB is due to an interference of states corresponding to the positive and negative electron energies resulting from the Dirac equation [3–5]. Lock [6] showed that, if an electron is
represented by a wavepacket, its ZB has a transient character, i.e. it disappears with time.

It was conceived years later that the trembling electron motion should occur also in crystalline solids if their band structure could be represented by a two-band model reminiscent of the Dirac equation. The first paper in this vein, published in 1970 by Lurie and Cremer [7], was concerned with superconductors, in which the energy–wavevector dependence is similar to the relativistic relation. A similar approach was applied to semiconductors 20 years later using a model of two energy bands [8–11]. However, intense interest in ZB of electrons in semiconductors began only in 2005. Zawadzki [12] used a close analogy between the \( \mathbf{k} \cdot \mathbf{p} \) theory of energy bands in narrow-gap semiconductors (NGS) and the Dirac equation for relativistic electrons in a vacuum to show that one should deal with the electron ZB in NGS which would have much more favorable frequency and amplitude characteristics than those in a vacuum. On the other hand, Schliemann et al [13] demonstrated that the spin splitting of energies linear in \( k \), caused by the inversion asymmetry in semiconductor systems (the Bychkov–Rashba splitting), also leads to a ZB-type of motion if the electron wavepacket has a non-vanishing initial momentum. The above contributions triggered a wave of theoretical considerations for various semiconductor and other systems. It was recognized that the phenomenon of ZB occurs every time one deals with two or more interacting energy bands [14–16].

It was shown that, indeed, when the electron is represented by a wavepacket, the ZB has a transient character [17]. Considering graphene in a magnetic field it was demonstrated that, if the electron spectrum is discrete, ZB contains many frequencies and it is sustained in time [18]. It was pointed out that the trembling electrons should emit electromagnetic radiation if they are not in their eigenstates [19]. The physical origin of ZB was analyzed and it turned out that, at least in its ‘classical’ solid state version analogous to the ZB in a vacuum, the trembling motion represents simply oscillations of electron velocity due to the energy conservation as the particle moves in a periodic potential [21].

As mentioned above, in a vacuum the ZB characteristics are not favorable. In solids, the ZB characteristics are much better but it is difficult to observe the motion of a single electron. However, recently Gerritsma et al [20] simulated experimentally the Dirac equation and the resulting electron ZB with the use of trapped ions and laser excitations. The power of the simulation method is that one can adjust experimentally the essential parameters of the Dirac equation, \( m_0 c^2 \) and \( c \), and thus achieve more favorable values of the ZB frequency and amplitude. The experimental results obtained by Gerritsma et al agreed well with the predictions of Zawadzki and Rusin [21]. Interestingly, it turned out that analogues of ZB can occur also in classical wave propagation phenomena. Several predictions were made, but in two systems, namely macroscopic sonic crystals [22] and photonic superlattices [23], the ZB-like effects were actually observed. Finally, there has been growing recognition that the mechanisms responsible for ZB in solids are related to their other properties, for example to the electric conductivity.

Thus the subject of our interest is not only quickly developing but also quite universal. From an obscure, perplexing and somewhat marginal effect that would probably never be observed, the Zitterbewegung has grown into a universal, almost ubiquitous phenomenon that has been experimentally simulated in its quantum form and directly observed in its classical version. Our review summarizes the first five years of intensive development, which can be characterized as the ’Sturm und Drang’ period, to use another pertinent German term. We concentrate mostly on the ZB in semiconductors but mention other systems, in particular different ZB simulations by trapped ions and atoms, as these seem to be most promising for future experimental observations. We also briefly review important papers describing the ZB of free relativistic electrons in a vacuum since they inspired early considerations concerning solids. The subject of Zitterbewegung has been until now almost exclusively theoretical. Below we quote mostly derivations and figures from our own papers, not because we believe that they are the only important ones, but because of copyright restrictions.

The review is organized in the following way. In section 2 we present descriptions of ZB for semi-relativistic, spin, and nearly free electron Hamiltonians and quote papers on other model systems. Section 3 treats the trembling motion in bilayer graphene, monolayer graphene and carbon nanotubes. In section 4 we consider the ZB in the presence of an external magnetic field and quote related works. Section 5 is concerned with the origin of ZB in crystalline solids. There follows the short section 6, in which we mention work relating ZB to calculations of electric conductivity. In section 7 we describe papers on the ZB of free relativistic electrons, necessary to understand the trembling motion in solids. Section 8 contains a very brief introduction to simulations of the Dirac equation and the resulting ZB in the absence of fields and in a magnetic field. The section is completed by summaries of related papers. In section 9 we describe wave ZB-like effects in non-quantum periodic systems. The review is terminated by discussion and conclusions.

2. ZB in model systems

We begin our considerations of electron ZB in semiconductors by using the so-called relativistic analogy [12]. In this way we can follow simultaneously the procedure of Schrödinger and derive corresponding relations for narrow-gap semiconductors. It was noted in the past that the \( E(k) \) relation between the energy \( E \) and the wavevector \( k \) for electrons in NGS is analogous to that for relativistic electrons in a vacuum \([11, 24, 25]\). The semi-relativistic phenomena appear at electron velocities of \( v \approx 10^7 \text{ cm s}^{-1} \), much lower than the light velocity \( c \). The reason is that the maximum velocity \( u \) in semiconductors, which plays the role of \( c \) in a vacuum, is about \( 10^4 \text{ cm s}^{-1} \). To be more specific, we use the \( \mathbf{k} \cdot \mathbf{p} \) approach to InSb-type semiconductors [26]. Taking the limit of large spin–orbit energy, the resulting dispersion relation for
the conduction and the light-hole bands is \( E = \pm E_p \), where

\[
E_p = \left( \frac{E_c}{2} + \frac{E_\omega p^2}{2m_0} \right)^{1/2}. \tag{1}
\]

Here \( E_\omega \) is the energy gap and \( m_0^*= \) the effective mass at the band edge. This expression is identical to the relativistic relation for electrons in a vacuum, with the correspondence \( 2m_0c^2 \rightarrow E_\omega \) and \( m_0 \rightarrow m_0^* \). The electron velocity \( v \) in the conduction band described by (1) reaches a saturation value as \( p \) increases. This can be seen directly by calculating \( v_i = \partial E_\omega / \partial p_i \) and taking the limit of large \( p_i \), or by using the analogy \( c = (2m_0c^2/2m_0)^{1/2} \rightarrow (E_\omega /2m_0^{*3/2})^{1/2} = u \).

Taking the experimental parameters \( E_\omega \) and \( m_0^* \) one calculates a very similar value of \( u \approx 1.3 \times 10^8 \text{ cm} \text{s}^{-1} \) for different semiconductor compounds. Now we define an important quantity

\[
\lambda_z = \frac{\hbar}{m_0^* u} \tag{2}
\]

which we call the length of ZB for reasons given below. We note that it corresponds to the Compton wavelength \( \lambda_c = \hbar/m_0 c \) for electrons in a vacuum.

Next we consider the band Hamiltonian for NGS. It is derived within the model including \( \Gamma_6 \) (conduction), \( \Gamma_8 \) (light and heavy hole), and \( \Gamma_7 \) (split-off) bands and it represents an \( 8 \times 8 \) operator matrix [26]. We assume, as before, \( \Delta \gg E_\omega \) and omit the free-electron terms since they are negligible for NGS. The resulting \( 6 \times 6 \) Hamiltonian has \( \pm E_\omega /2 \) terms on the diagonal and \( \tilde{p}_i \), terms off the diagonal, just like in the Dirac equation for free electrons. However, the three \( 6 \times 6 \) matrices \( \tilde{a}_i \), multiplying the momentum components \( \tilde{p}_i \) do not have the properties of \( 4 \times 4 \) Dirac matrices, which considerably complicates calculations. For this reason, with only a slight loss of generality, we take \( \tilde{p}_i \neq 0 \) and \( \tilde{p}_z = \tilde{p}_r = 0 \). In the \( \tilde{a}_i \) matrix, two rows and columns corresponding to the heavy holes contain only zeros and can be omitted. The remaining Hamiltonian for the conduction and the light-hole bands reads

\[
\hat{H} = u\tilde{a}_3 \tilde{p}_z + \frac{1}{2}E_\omega \tilde{b}, \tag{3}
\]

where \( \tilde{a}_3 \) and \( \tilde{b} \) are the well-known \( 4 \times 4 \) Dirac matrices [27].

The Hamiltonian (3) has the form appearing in the Dirac equation and in the following we can use the procedures of relativistic quantum mechanics (RQM). The electron velocity is \( \tilde{v} = (1/\hbar)\tilde{[\hat{z}, \hat{H}]} = u\tilde{a}_3 \). The eigenvalues of \( \tilde{a}_3 \) are \( \pm 1 \), so that the eigenvalues of \( \tilde{v} \) are, paradoxically, \( \pm u \). In order to determine \( \tilde{a}_3(t) \) we calculate \( \tilde{a}_3(t) \) by commuting \( \tilde{a}_3(t) \) with \( \hat{H} \) and integrating the result with respect to time. This gives \( \tilde{z}(t) \) and we calculate \( \tilde{z}(t) \) by integrating again. The final result is

\[
\tilde{z}(t) = \tilde{z}(0) + \frac{u^2 \tilde{p}_z}{\hat{H}} + i\hbar u \frac{\tilde{a}_3(0) - u \tilde{p}_z/\hat{H}}{2}\hat{A}_0 \left[ e^{-(2i\hat{H}t)/\hbar} - 1 \right] \tag{4}
\]

where \( \hat{A}_0 = \tilde{a}_3(0) - u \tilde{p}_z/\hat{H} \). This first two terms of (4) represent the classical electron motion. The third term describes time-dependent oscillations with the frequency of \( \omega_Z \approx E_\omega /\hbar \). Since \( \hat{A}_0 \approx 1 \), the amplitude of oscillations is \( hu/2\hat{H} \approx h/2m_0^*u = \lambda_Z/2 \).

In RQM the analogous oscillations are called ZB, which explains the name given above to \( \lambda_z \). The expression obtained by Schrödinger for ZB of free relativistic electrons in a vacuum is identical to that given by (4) with the use of the above relativistic analogy. In RQM it is demonstrated that ZB is a result of interference between states of positive and negative electron energies [3–5]. Clearly, one can say the same of ZB in semiconductors calculated according to the above model. However, as we show below, the origin of ZB in crystalline solids can be interpreted in more physical terms. The magnitude of \( \lambda_Z \) is essential.

Next, we briefly consider another example of ZB proposed by Schliemann et al [13]. It is based on the so-called Bychko–Rashba (BR) spin splitting caused by structure inversion asymmetry in two-dimensional semiconductor heterostructures [28]. The interaction describing this splitting is

\[
\hat{H}_{\text{BR}} = \frac{\alpha}{\hbar} (p_x \sigma_y - p_y \sigma_x), \tag{5}
\]

where \( p \) is the momentum of an electron confined in two-dimensional geometry, and \( \sigma \) is the vector of Pauli matrices. The coefficient \( \alpha \) is to be calculated using details of the structure [29]. One can easily solve the eigenenergy equation and obtain the spin energies \( E = \pm \omega k \). The important difference with the case considered above is that here at \( k = 0 \) there is \( \Delta E = 0 \), i.e. there is no gap. The complete Hamiltonian is \( \hat{H} = p^2 /2m^* + \hat{H}_{\text{BR}} \), where \( m^* \) is the effective mass. The position operator in the Heisenberg picture has the standard form

\[
\hat{r}(t) = e^{i\hat{p}(t)/\hbar} \hat{r}(0) e^{-i\hat{p}(t)/\hbar}. \tag{6}
\]

One calculates \( \hat{r}(t) \) explicitly using the Hamiltonian and averages it employing a Gaussian wavepacket of width \( d \) centered at the wavevector \( k_{0y} = 0 \) and \( k_{0y} \neq 0 \). In the case \( d k_{0y} \gg 1 \), Schliemann et al obtained

\[
\langle \psi | \hat{r}(t) | \psi \rangle = \frac{1}{2k_{0y}} \left[ 1 - \cos \left( \frac{2\alpha k_{0y}t}{\hbar} \right) \right]. \tag{7}
\]

The above result describes ZB with the frequency given by \( \hbar \omega_Z = 2\alpha k_{0y} \), where \( \hbar \omega_Z \) is the excitation energy between the two branches of the BR energies at \( k = k_{0y} \). It is seen that ZB is absent for \( k_{0y} = 0 \). Similar results for ZB are obtained if, instead of the spin splitting due to structure inversion asymmetry, one uses a two-dimensional version of the spin
calculates the displacement matrix of the position operator $\hat{\psi}$ when femtosecond pulse technology emerged. Thus, in a more realistic picture the electrons are described by wavepackets: 

$$
\psi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}t^2(k_z - k_{\alpha})^2\right) \times \exp(i\mathbf{k}_z \mathbf{z})dk_z \left(\begin{array}{c} 1 \\ 0 \end{array}\right). 
$$

The electron displacement is calculated as an average of the position operator $\hat{z}$ over the above wavepacket, see figure 1 and [31]. The essential result is that, in agreement with Lock general predictions [6], the ZB oscillations of the average electron position have a transient character, i.e. they disappear with time on a femtosecond scale. The frequency of oscillations is $\omega_{\alpha}(t)$ and spin $v_{\alpha}(t)$, their time evolutions with the resulting ZB-like effects. It was also shown that the average angular momentum and spin vector undergo a transient precession due to the interference of the light- and heavy-holes’ energies: $h\omega(t) = E_l(t) - E_h(t)$, and they increase in time as the holes are accelerated to higher $k$ values by the electric field.

Bernardez et al [34] described the ZB in spin space caused by a new kind of spin–orbit interaction resulting from an inter-subband coupling in symmetric quantum wells. In this case the ZB is characterized by cycloidal electron trajectories.

Winkler et al [15] considered the oscillatory dynamics of Heisenberg observables such as position $\mathbf{r}(t)$, velocity $\mathbf{v}(t)$, orbital angular momentum $\mathbf{L}(t)$ and spin $\mathbf{S}(t)$ in a variety of different systems described by the BR [28], Luttinger [35] and Kane [26] Hamiltonians. They illustrated similarities between their time evolutions with the resulting ZB-like effects.

Demikhovskii et al [36] described a 3D hole system having the effective spin $3/2$ with the use of wavepackets. For $dk_0 > 1$, where $k_0$ is the average packet wavevector and $d$ is the packet width, the initial wavepacket splits into two parts and the packet’s center experiences the transient ZB. It was also shown that the average angular momentum and spin vector undergo a transient precession due to the interference of the light- and heavy-hole states.

Cserti and David [14] observed that the Hamiltonians mentioned above and describing the BR and the Dresselhaus spin splitting, monolayer and bilayer graphene, nearly free electrons, electrons in superconductors, etc, can be represented in the general form

$$
\hat{H} = \epsilon(p) \mathbf{1} + \Omega^T \hat{S},
$$

where $V = V_0 - V_q$ are the Fourier coefficients in the expansion of $V(r)$, and $E_k = h^2 k^2/2m_0$ is the free-electron energy. The $2 \times 2$ quantum velocity $\hat{v}_z$ can now be calculated and the acceleration $\hat{a}_z$ is computed in the standard way. Finally, one calculates the displacement matrix $\hat{z}$.

Since the ZB is by its nature not a stationary state but a dynamical phenomenon, it is natural to study it with the use of wavepackets. These became a practical instrument when femtosecond pulse technology emerged. Thus, in a more realistic picture the electrons are described by wavepackets: 

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where the one-particle dispersion is described by $\epsilon(p)$ and the second term has the form of an effective magnetic field $\Omega(p)$ coupled to the spin $\hat{S}$. Here $T$ stands for the transpose of a vector, while $I$ is the unit vector in spin space. One can use the Hamiltonian (11) to calculate time-dependent position operator in the Heisenberg picture and show that it consists in general of the ‘classical’ (mean) part and the ZB part. For the BR coupling, see (11), and for monolayer graphene, see below, the ZB can be interpreted as a consequence of conservation of the total angular momentum $J_z = L_z + S_z$, where $L = r \times p$ is the orbital angular momentum. This is in analogy to the results discussed in section 5, where ZB is shown to be a consequence of the energy conservation.

David and Cserti [37] considered a general multi-band Hamiltonian and showed that in this case one deals with a trembling motion which is a superposition of trembling motions corresponding to all possible differences of energy eigenvalues. It was also shown, following the remarks of [38, 39], that the ZB amplitudes in the position operator are related to the Berry connection matrix appearing in the expression of the Berry phase. We may add that a good example of multi-frequency ZB motion is given by carriers in graphene in a magnetic field (see [18], and figure 5), where the role of different eigenenergies is represented by different Landau levels.

Wilamowski et al [40] investigated microwave absorption in asymmetric Si quantum wells in an external magnetic field and detected a spin-dependent component of the Joule heating at the spin resonance. The observation was explained in terms of the BR spin splitting due to the structure inversion asymmetry with the resulting current-induced spin precession and the ZB at the Larmor frequency.

3. ZB in graphene

Now we study in some detail the ZB of mobile charge carriers in three modern materials: bilayer graphene, monolayer graphene and carbon nanotubes [17].

3.1. Bilayer graphene

We first present the results for bilayer graphene since they can be obtained in analytical form, which allows one to see directly important features of the trembling motion. The two-dimensional Hamiltonian for bilayer graphene is well approximated by [41]

$$\hat{H}_B = -\frac{1}{2m^*} \begin{pmatrix} 0 & (\hat{p}_x + i\hat{p}_y)^2 \\ (\hat{p}_x - i\hat{p}_y)^2 & 0 \end{pmatrix},$$

(12)

where $m^* = 0.054 m_0$. The energy spectrum is $E_k = \pm E_k$, where $E_k = \hbar^2 k^2/2m^*$, i.e. there is no energy gap between the conduction and valence bands. The position operator in the Heisenberg picture is a $2 \times 2$ matrix $\bar{x}(t) = \exp(i\hat{H}_B t/\hbar)\bar{x}(0)\exp(-i\hat{H}_B t/\hbar)$. One calculates

$$x_{11}(t) = x(0) + \frac{k_y}{k_z} \left[ 1 - \cos \left( \frac{\hbar k_z^2 t}{m^*} \right) \right],$$

(13)

where $k^2 = k_x^2 + k_y^2$. The third term represents the ZB with frequency $\omega_Z = 2\hbar^2 k^2/2m^*$, corresponding to the energy difference between the upper and lower energy branches for a given value of $k$. We want to calculate ZB of a charge carrier represented by a two-dimensional wavepacket centered at $k_0 = (0, k_y)$ and characterized by the width $d$. An average of $\bar{x}_{11}(t)$ is a two-dimensional integral which can be calculated analytically:

$$\bar{x}_{11}(t) = \langle \psi(r) | \bar{x}(t) | \psi(r) \rangle = \bar{x}_c + \bar{x}_Z(t),$$

(14)

where $\bar{x}_c = (1/k_0) [1 - \exp(-d^2 k_0^2)]$, and

$$\bar{x}_Z(t) = \frac{1}{k_0} \left[ \exp \left( \frac{-\delta^4 k_0^4}{d^4 + \delta^4} \right) \cos \left( \frac{\delta^4 d^2 k_0^2}{d^4 + \delta^4} \right) - \exp(-d^2 k_0^2) \right],$$

(15)

in which $\delta = \sqrt{\hbar t/m^*}$ contains the time dependence. In figure 2 a we show the ZB of the electron position $\bar{x}_{11}$ as given in (14) and (15).

We enumerate the main features of ZB following from (14) and (15). First, in order to have ZB in the direction $x$ one needs an initial transverse momentum $h k_{0y}$. Second, the ZB frequency depends only weakly on the packet width: $\omega_Z = (\hbar k_{0y}^2/m^*) (d^4/(d^4 + \delta^4))$, while its amplitude is strongly dependent on the width $d$. Third, the ZB has a transient character since it is attenuated by the exponential term. For small $t$ the amplitude of $\bar{x}_Z(t)$ diminishes as $\exp(-\Gamma_Z^2 t^2)$ with

$$\Gamma_Z = \frac{\hbar k_{0y}}{m^* d^2}.$$  

(16)

Fourth, as $t$ (or $\delta$) increases, the cosine term tends to unity and the first term in (15) cancels out with the second term, which
illustrates the Riemann–Lebesgue theorem (see [6]). After the oscillations disappear, the charge carrier is displaced by the amount \( \bar{x}_c \), which is a ‘remnant’ of ZB. Fifth, for very wide packets (\( d \to \infty \)) the exponent in (15) tends to unity, the oscillatory term is \( \cos(\delta (k_y^2 \bar{r}) \omega Z \tau) \) and the last term vanishes. In this limit one recovers undamped ZB oscillations.

Next, we consider other quantities related to ZB, beginning with the current. This is given by the velocity multiplied by charge. The velocity is simply \( \bar{v}_c = \delta \bar{x}_c / \delta t \), where \( \bar{x}_c \) is given by (15). The calculated current is plotted in figure 2(b), its oscillations are a direct manifestation of ZB. The transient character of ZB is accompanied by a temporal spreading of the wavepacket. In fact, the question arises as to whether the attenuation of ZB is not simply caused by the spreading of the packet. The calculated packet width \( \Delta R \) is plotted versus time in figure 2(c). It is seen that during the initial 80 fs the packet’s width increases only twice compared to its initial value, while the ZB disappears almost completely. We conclude that the spreading of the packet is not the main cause of the transient character of ZB. Looking for physical reasons behind the transient character of ZB we decompose the total wavefunction \( \psi(r, t) \) into the positive (p) and negative (n) components \( \psi_p(r, t) \) and \( \psi_n(r, t) \). We have

\[
\langle p|\psi(t)\rangle = e^{-iE/t}\langle 0|\psi(0)|p\rangle = e^{-iE/t}\langle p|\psi(0)|p\rangle + e^{-iE/t}\langle n|\psi(0)|n\rangle, \tag{17}
\]

where \( |p\rangle \) and \( |n\rangle \) are the eigenfunctions of the Hamiltonian (12) in \( k \) space corresponding to positive and negative energies, respectively. Further

\[
\langle k|p\rangle = \frac{1}{\sqrt{2}} \left( \frac{1}{k_x^2/k^2} \right) \delta(k-k), \tag{18}
\]

\[
\langle k|n\rangle = \frac{1}{\sqrt{2}} \left( -\frac{1}{k_x^2/k^2} \right) \delta(k-k). \tag{19}
\]

After some manipulations one obtains

\[
\psi_p(r, t) = \frac{d}{4\pi \sqrt{\pi}} \int d^2k e^{i\delta (k_x^2 + (k_y - k_0)^2) - \frac{1}{k_x^2/k^2}} \times \left( 1 \right). \tag{20}
\]

The function \( \psi^*(r, t) \) is given by the identical expression with changed signs in front of \( E \) and \( k_x^2/k^2 \) terms. We have \( \psi(r, t) = \psi_p^*(r, t) + \psi_p^*(r, t) + \psi_p^2(r, t) \) and \( \langle p|\psi|p\rangle = 0 \). Now, one can calculate the average values of \( \bar{x} \) and \( \bar{y} \) using the positive and negative components in the above sense. We have

\[
\bar{x}(t) = \int (\psi^n + \psi^p)^2 x (\psi^n + \psi^p) d^2 r, \tag{21}
\]

so that we deal with four integrals. A direct calculation gives

\[
\int |\psi|^2 x d^2 r + \int |\psi^n|^2 x d^2 r = \bar{x}_c, \tag{22}
\]

\[
\int \psi^n x \psi^p d^2 r + \int \psi^p x \psi^n d^2 r = \bar{x}_Z(t), \tag{23}
\]

where \( \bar{x}_c \) and \( \bar{x}_Z(t) \) have been defined in (14). Thus the integrals involving only the positive and only the negative components give the constant shift due to ZB, while the mixed terms lead to the ZB oscillations. All terms together reconstruct the result (14). Next we calculate the average value \( \bar{y} \). There is no symmetry between \( \bar{x} \) and \( \bar{y} \) because the wavepacket is centered around \( k_x = 0 \) and \( k_y = k_{0y} \). The average value \( \bar{y} \) is again given by four integrals. However, now the mixed terms vanish, while the integrals involving the positive and negative components give

\[
\int |\psi|^2 y d^2 r = \frac{\hbar k_{0y}}{2m^* t}, \tag{24}
\]

\[
\int |\psi|^2 y d^2 r = -\frac{\hbar k_{0y}}{2m^* t}. \tag{25}
\]

This means that the ‘positive’ and ‘negative’ sub-packets move in opposite directions with the velocity \( v = \hbar k_{0y}/m^* \). Each of these packets has the initial width \( d \) and it (slowly) spreads in time. After the time \( \Gamma_Z^{-1} = d/v \) the distance between the two packets equals \( d \), so the integrals (22) are small, resulting in the diminishing ZB amplitude. This reasoning gives the decay constant \( \Gamma_Z = \hbar k_{0y}/m^* d \), which is exactly what we determined above from the analytical results (see (16)). Thus, the transient character of the ZB oscillations is due to the increasing spatial separation of the sub-packets corresponding to the positive and negative energy states. This confirms our previous conclusion that it is not the packet’s slow spreading that is responsible for the attenuation. The separation of sub-packets with the resulting decay of ZB turns out to be a general feature of this phenomenon.

### 3.2. Monolayer graphene

Now we turn to monolayer graphene. The two-dimensional Hamiltonian describing its band structure is [42–45]

\[
\hat{H}_M = u \begin{pmatrix} 0 & \hat{p}_x + i \hat{p}_y \\ \hat{p}_x - i \hat{p}_y & 0 \end{pmatrix}, \tag{26}
\]

where \( u \approx 1 \times 10^8 \text{ cm s}^{-1} \). The resulting energy dispersion is linear in momentum: \( \varepsilon = \pm u k \), where \( k = \sqrt{k_x^2 + k_y^2} \). The quantum velocity in the Schrödinger picture is \( \hat{v}_i = \partial \hat{H}_M / \partial \hat{p}_i \); it does not commute with the Hamiltonian (26). In the Heisenberg picture we have \( \hat{v}(t) = \exp(i\hat{H}_M t/\hbar) \hat{v} \exp(-i\hat{H}_M t/\hbar) \). Using (26) one calculates

\[
v_i^{(1)} = \frac{u \hbar}{k} \sin(2\omega k t). \tag{27}
\]

The above equation describes the trembling motion with the frequency \( \omega_Z = \frac{2uK}{k} \), determined by the energy difference between the upper and lower energy branches for a given value of \( k \). As before, ZB in the direction \( x \) occurs only if there is a non-vanishing momentum \( \hbar k_x \). One calculates an average velocity (or current) taken over a two-dimensional wavepacket with nonzero initial momentum \( k_{0y} \). The results for the current \( j_x = e \bar{v}_x \) are plotted in figure 3 for different realistic packet widths \( d \). It is seen that the ZB frequency does not depend on \( d \) and is nearly equal to \( \omega_Z \) given above for the plane wave.
On the other hand, the amplitude of ZB does depend on $d$ and we deal with decay times of the order of femtoseconds. For small $d$ there are almost no oscillations, for very large $d$ the ZB oscillations are undamped. These conclusions agree with our analytical results for bilayer graphene. The behavior of ZB depends quite critically on the values of $k_0$, and $d$, which is reminiscent of the damped harmonic oscillator. In the limit $d \rightarrow \infty$ the above results for the electric current resemble those of Katsnelson [46] for ZB in graphene obtained with the use of a plane wave representation.

Maksimova et al [47] investigated dynamics of wavepackets in monolayer graphene for different pseudo-spin polarizations with the resulting ZB. For specific packet components and their relative phases a ‘longitudinal ZB’ can take place, but its intensity is weak.

Martinez et al [48] considered a creation of electron–hole pairs by a constant electric field in the plane of a monolayer graphene sheet. They showed that, as the pairs undergo the ZB in opposite directions, a Hall-like separation of the charge grapheme sheet. They showed that, as the pairs undergo the ZB in monolayer graphene for different pseudo-spin polarizations $\frac{\partial}{\partial t}|k_x|^2$, which is reminiscent of the damped harmonic oscillator. In the limit $d \rightarrow \infty$ the above results for the electric current resemble those of Katsnelson [46] for ZB in graphene obtained with the use of a plane wave representation.

Martinez et al [48] considered a creation of electron–hole pairs by a constant electric field in the plane of a monolayer graphene sheet. They showed that, as the pairs undergo the ZB in opposite directions, a Hall-like separation of the charge occurs, giving a measurable dipole moment. We note that it is not the time-dependent motion but the ZB shift at large times which is responsible for the charge separation, see (13) and figure 2.

Englman and Vertesi [39] calculated a ZB-related electron current in monolayer graphene in the adiabatic approximation and related it to the Berry phase.

3.3. Carbon nanotubes

Next, we consider monolayer graphene sheets rolled into single semiconducting carbon nanotubes (CNTs) [17, 49]. The band Hamiltonian in the vicinity of the $K$ point is [50]

$$\hat{H}_{\text{CNT}} = \hbar \left( \begin{array}{cc} 0 & \hbar k_{nv} - i \hbar \hat{p}_\gamma \\ \hbar k_{nv} + i \hbar \hat{p}_\gamma & 0 \end{array} \right).$$

(28)

This Hamiltonian is similar to (26) except that, because of the periodic boundary conditions, the momentum $p_x$ is quantized and takes discrete values $\hbar k_n = h|k_{nv}|$, where $k_{nv} = \frac{2\pi}{L}(n - \nu/3)$, $n = 0, \pm 1, \ldots$, $\nu = \pm 1$, and $L$ is the length of circumference of the CNT. As a result, the free-electron motion can occur only in the direction $y$, parallel to the tube axis. The geometry of the CNT has important consequences. There exists an energy gap $E_g = \frac{2\hbar|k_{nv}|}{u}$ and the effective mass at the band edge $m^*_n = \frac{\hbar|k_{nv}|}{u}$. For $\nu = \pm 1$ there always exists a non-vanishing value of the quantized momentum $\hbar k_{nv}$. Finally, for each value of $k_{nv}$ there exists $k_{n,-\nu} = -k_{nv}$ resulting in the same subband energy $E = \pm E_g$, where

$$E = \hbar u \left( k_{nv}^2 + k_{\nu}^2 \right).$$

(29)

The time-dependent velocity $\hat{v}_x(t)$ and the displacement $\hat{y}(t)$ can be calculated for the plane electron wave in the usual way and they exhibit the ZB oscillations (see [49]). For small momenta $k$, the ZB frequency is $\hbar\omega_{Z} = E_g$ and the ZB length is $\lambda_Z \approx 1/[k_{nv}]$. We are again interested in the displacement $\hat{y}(t)$ of a charge carrier represented by a one-dimensional wavepacket analogous to that described in (10). The average displacement is $\hat{y}(t) = \hat{y}_Z(t) - \hat{y}_{sh}$, where

$$\hat{y}_Z(t) = \frac{\hbar^2 d^2 u^2 k_{nv}}{2\sqrt{\pi}} e^{-\frac{2E_t}{h}} e^{-\frac{\nu^2}{2}}$$

(30)

and $\hat{y}_{sh} = 1/2\sqrt{\pi} \text{d}g(b)[1 - \Phi(b)] \exp(b^2)$, where $b = k_{nv}d$ and $\Phi(x)$ is the error function. The ZB oscillations of $\hat{y}(t)$ are plotted in figure 4. It is seen that, after the transient ZB oscillations disappear, there remains a shift $\hat{y}_{sh}$. Thus the ZB separates spatially the charge carriers that are degenerate in energy but characterized by $n, \nu$ and $-n, \nu$ quantum numbers. The current is proportional to $\tilde{v}_x = \partial \hat{y}/\partial t$, so that the currents related to $\nu = 1$ and $-1$ cancel each other. To have a

![Figure 3](Image.png)

**Figure 3.** Oscillatory electric current in the $x$ direction caused by the ZB in monlayer graphene versus time, calculated for a Gaussian wavepacket with $k_0 = 1.2 \times 10^6 \text{m}^{-1}$ and various packet widths $d$. The transient character of ZB is clearly seen. After [17].

![Figure 4](Image.png)

**Figure 4.** ZB of two charge carriers in the ground subband of a single carbon nanotube of $L = 200 \text{ A}$ versus time (logarithmic scale), calculated for Gaussian wavepackets of two different widths $d$ and $k_0 = 0$. After the ZB disappears a constant shift remains. The two carriers are described by different quantum numbers $\nu$. At higher times the amplitude of ZB oscillations decays as $t^{-1/2}$. After [17].
The basic energy is $k_n$. The reason is that we consider the situation with each other. Thus, the slow damping of ZB is due only to magnetic field [18]. The magnetic field is known to cause described above for no external potentials. Now we consider the trembling motion of charge carriers in solids has been experimentally, see section 8. We add that for $k_{0y} ≠ 0$ the subpackets run away from each other and the decay time remains almost unchanged.

One can show that we again deal here with two subpackets which, however, for $k_{0y} = 0$ do not run away from each other. Thus, the slow damping of ZB is due only to the slow broadening of the subpackets. We emphasize the slow decay, as illustrated in figure 4, because it is confirmed experimentally, see section 8. We add that for $k_{0y} ≠ 0$ the subpackets run away from each other and the decay time is much faster.

4. ZB in a magnetic field

The trembling motion of charge carriers in solids has been described above for no external potentials. Now we consider the trembling motion of electrons in the presence of an external magnetic field [18]. The magnetic field is known to cause no interband electron transitions, so the essential features of ZB are expected not to be destroyed. On the other hand, introduction of an external field provides an important parameter affecting the ZB behavior. This case is special because the electron spectrum is fully quantized. We consider a graphene monolayer in an external magnetic field parallel to the $z$ axis. The Hamiltonian for electrons and holes at the $K_1$ point is [42, 43]

$$\hat{H} = u \left( \begin{array}{cc} 0 & \hat{\pi}_y - i \hat{\pi}_x \\ \hat{\pi}_y + i \hat{\pi}_x & 0 \end{array} \right),$$

(31)

where $u ≈ 1 \times 10^8$ cm s$^{-1}$ is the characteristic velocity, $\hat{\pi} = \hat{p} - q \hat{A}$ is the generalized momentum, in which $\hat{A}$ is the vector potential, and $q$ is the electron charge. Using the Landau gauge, we take $\hat{A} = (-By, 0, 0)$, and for an electron $q = -e$ with $e > 0$. We take the wavefunction in the form $\Psi(x, y) = e^{i k_n x} \Phi(y)$. Introducing the magnetic radius $L = \sqrt{\hbar/eB}$, the variable $\xi = y/L - k_n L$, and defining the standard raising and lowering operators for the harmonic oscillator $\hat{a} = (\xi + \partial/\partial \xi)/\sqrt{2}$ and $\hat{a}^\dagger = (\xi - \partial/\partial \xi)/\sqrt{2}$, the Hamiltonian becomes

$$\hat{H} = -\hbar \Omega \left( \begin{array}{cc} 0 & \hat{a}^\dagger \\ \hat{a} & 0 \end{array} \right),$$

(32)

where the frequency is $\Omega = \sqrt{2u}/L$. Next one determines the eigenstates and eigenenergies of the Hamiltonian $\hat{H}$. The energy is $E_{ns} = s \hbar \Omega \sqrt{n}$. Here $n = 0, 1, \ldots$, and $s = ±1$ for the conduction and valence bands, respectively. The above energies were confirmed experimentally. The complete wavefunction is

$$|n⟩ = |n k_s⟩ = \frac{e^{i k_n x}}{\sqrt{4\pi}} \left( -s|n - 1⟩ \right),$$

(33)

where $|n⟩$ are the harmonic oscillator functions.

We want to calculate the velocity of charge carriers described by a wavepacket. We first calculate matrix elements $\langle f|n⟩$ between an arbitrary two-component function $f = (f^u, f^l)$ and eigenstates (33). A straightforward manipulation gives $\langle f|n⟩ = -s F_{n-1}^u + F_n^l$, where

$$F_n^l(k_s) = \frac{1}{\sqrt{2L} c_n} \int g^l(k_s, y)e^{-\frac{\pi i}{2} |n⟩} H_n(\xi) d\xi,$$

(34)

in which

$$g^l(k_s, y) = \frac{1}{\sqrt{2\pi}} \int f^l(x, y)e^{i k_s x} dx.$$

(35)

The superscript $j = u, l$ stands for the upper and lower components of the function $f$. The Hamilton equations give the velocity components: $\tilde{v}_i(0) = \partial H/\partial \tilde{p}_i$, with $i = x, y$. We want to calculate averages of the time-dependent velocity operators $\tilde{v}_i(t)$ in the Heisenberg picture taken on the function $f$. The averages are

$$\tilde{v}_i(t) = \sum_{n, n'} e^{i E_{n'}/\hbar} \langle f|n'⟩|v_i(0)|n⟩ |n⟩ |f⟩ e^{-i E_n t}/\hbar,$$

(36)

where the energies and eigenstates are given in (33). The summation in (36) goes over all the quantum numbers: $n, n', s, s', k_c, k'_c$. The only non-vanishing matrix elements of
the velocity components are for the states \( n' = n \pm 1 \). One finally obtains after some manipulation

\[
\vec{v}_j(t) = u \sum_{n=0}^{\infty} V^+_n \sin(\alpha^+_n t) + u \sum_{n=0}^{\infty} V^-_n \sin(\alpha^-_n t)
+ iu \sum_{n=0}^{\infty} A^+_n \cos(\alpha^+_n t) + iu \sum_{n=0}^{\infty} A^-_n \cos(\alpha^-_n t), \tag{37}
\]

\[
\vec{v}_k(t) = u \sum_{n=0}^{\infty} B^+_n \cos(\alpha^+_n t) + u \sum_{n=0}^{\infty} B^-_n \cos(\alpha^-_n t)
+ iu \sum_{n=0}^{\infty} T^+_n \sin(\alpha^+_n t) + iu \sum_{n=0}^{\infty} T^-_n \sin(\alpha^-_n t), \tag{38}
\]

where \( V^\pm_n, T^\pm_n, A^\pm_n \) and \( B^\pm_n \) are given by combinations of \( U^\alpha_\beta \) integrals

\[
U^\alpha_\beta = \int F^\alpha_{\vec{k}}(k_x) F^\beta_{\vec{k}}(k_x) \, dk_x. \tag{39}
\]

The superscripts \( \alpha \) and \( \beta \) refer to the upper and lower components, see [18]. The time-dependent sine and cosine functions come from the exponential terms in (36). The frequencies in (37) and (38) are \( \omega^+_n = \Omega(\sqrt{n+1} - \sqrt{n}) \), \( \omega^-_n = \Omega(\sqrt{n+1} + \sqrt{n}) \), where \( \Omega \) is given in (32). The frequencies \( \omega^+_n \) correspond to the intraband energies while the frequencies \( \omega^-_n \) correspond to the interband energies, see figure 5. The interband frequencies are characteristic of the ZB. The intraband (cyclotron) energies are due to the band quantization by the magnetic field and they do not appear in field-free situations.

Final calculations were carried out for a two-dimensional Gaussian wavepacket centered around the wavevector \( \vec{k}_0 = (k_{0x}, 0) \) and having two non-vanishing components. In this case one can obtain analytical expressions for \( U^\alpha_\beta \). The main frequency of oscillations is \( \omega_0 = \Omega \), which can be interpreted either as \( \omega^+_0 = \Omega(\sqrt{n+1} - \sqrt{n}) \) or \( \omega^-_0 = \Omega(\sqrt{n+1} + \sqrt{n}) \) for \( n = 0 \). The frequency \( \omega^+_0 \) belongs to the intraband (cyclotron) set, while \( \omega^-_0 \) belongs to the interband set (see figure 5). The striking feature is that ZB is manifested by several frequencies simultaneously. This is a consequence of the fact that in graphene the energy distances between the Landau levels diminish with \( n \), which results in different values of \( \omega^+_n \) and \( \omega^-_n \) for different \( n \). It follows that it is the presence of an external quantizing magnetic field that introduces various frequencies into ZB. It turns out that, after the ZB oscillations seemingly die out, they actually reappear at higher times. Thus, for all \( k_{0x} \) values (including \( k_{0x} = 0 \)), the ZB oscillations have a permanent character, that is they do not disappear in time. This feature is due to the discrete character of the electron spectrum caused by a magnetic field. The above property is in sharp contrast to the no-field cases considered above, in which the spectrum is not quantized and the ZB of a wavepacket has a transient character. In mathematical terms, due to the discrete character of the spectrum, averages of operator quantities taken over a wavepacket are sums and not integrals. The sums do not obey the Riemann–Lebesgue theorem for integrals which guaranteed the damping of ZB in time for a continuous spectrum (see [6]).

Finally, one calculates the displacements \( \vec{x}(t) \) and \( \vec{y}(t) \) of the wavepacket. To this end we integrate (37) and (38) with respect to time using the initial conditions \( x_0 = \vec{x}(0) = 0 \) and \( y_0 = \vec{y}(0) = k_x L^2 \). The results are plotted in figure 6 in the form of \( x \)-\( y \) trajectories for different initial wavevectors \( k_{0x} \).

The direction of movement is clockwise and the trajectories span early times (1 ps) after the creation of a wavepacket.

All in all, the presence of a quantizing magnetic field has the following important effects on the trembling motion. (1) For \( B \neq 0 \) the ZB oscillations are permanent, while for \( B = 0 \) they are transient. The reason is that for \( B \neq 0 \) the electron spectrum is discrete. (2) For \( B \neq 0 \) many ZB frequencies appear, whereas for \( B = 0 \) only one ZB frequency exists. (3) For \( B \neq 0 \) both interband and intraband (cyclotron) frequencies appear in ZB; for \( B = 0 \) there are no intraband frequencies. (4) Magnetic field intensity changes not only the ZB frequencies but the entire character of the ZB spectrum.

The ZB should be accompanied by electromagnetic dipole radiation emitted by the trembling electrons. The oscillations \( \vec{r}(t) \) are related to the dipole moment \( -e \vec{r}(t) \), which couples to the electromagnetic radiation. One can calculate the emitted electric field from the electron acceleration \( \vec{r}(t) \) and takes its Fourier transform to determine the emitted frequencies. In figure 7 we plot the calculated intensities of various emitted lines. The strong peak corresponds to oscillations with the basic frequency \( \omega = \Omega \). The peaks on the high-frequency side correspond to the interband excitations and are characteristic of ZB. The peaks on the lower-frequency side correspond to the intraband (cyclotron) excitations. In the absence of ZB the emission spectrum would contain only the intraband (cyclotron) frequencies. Thus the intraband frequencies \( \omega^+_n \) shown in figure 7 are a direct signature of the trembling motion. It can be seen that the \( \omega^+_n \) peaks are not much weaker than the central peak at \( \omega = \Omega \), which means that there exists a reasonable chance to observe them. Generally speaking, the excitation of the system is due to the nonzero momentum \( \hbar k_{0x} \) given to the electron. It can be provided by accelerating the electron in the band or by exciting the electron with a nonzero momentum by light from the valence band to the conduction band.

The electron can emit light because the Gaussian wavepacket is not an eigenstate of the system described by the Hamiltonian (31). The energy of the emitted light is provided by the initial kinetic energy related to the momentum \( \hbar k_{0x} \). Once this energy is completely used, the emission will cease. Radiation emitted by the trembling electrons in monolayer graphene excited by femtosecond laser pulses is described in [19]. This problem is not trivial since it is difficult to prepare an electron in a solid in the form of a Gaussian wavepacket. On the other hand, the formation of a light wavepacket is mastered by the present techniques. It was shown that, when the Landau levels are broadened by scattering or defects, the light emission is changed from sustained to decaying in time.

Schliemann [51] described the time dependence of the cyclotron motion in monolayer graphene in the presence of a magnetic field using the semiclassical approximation for high carrier energies. He showed that the cyclotron motion is perturbed by interband ZB contributions of higher frequencies.

Krueckl and Kramer [52] described the time propagation of an initially concentrated wavepacket in monolayer graphene.
Figure 6. ZB trajectories of an electron at the $K_1$ point of the Brillouin zone in monolayer graphene at $B = 20$ T during the first picosecond for various values of $k_0$. After [18].

Figure 7. Intensity spectrum versus frequency during the first 20 ps of motion of an electron described by a Gaussian wavepacket having $k_0 = 0.035 \text{ Å}^{-1}$ in monolayer graphene. After [18].

Romera and de los Santos [53] studied monolayer graphene in a magnetic field, concentrating on the collapse–revival pattern of ZB oscillations.

Wang et al [54] carried out a study similar to the one described above, but for bilayer graphene in a magnetic field. This system is somewhat different from monolayer graphene since the Landau levels are nearly uniformly spaced due to the quadratic dependence of the positive and negative energies on momentum, see (12). Also, the laser pulse was assumed to contain only one frequency, $\omega_L$. The authors estimated that in high quality bilayer graphene samples the stimulated ZB electric field can be of the order of volts per meter and the corresponding coherence times of tens of femtoseconds.

Zulicke et al [55] investigated the influence of ZB on the cyclotron motion, considering the so-called Landau–Rashba Hamiltonian which, in addition to the 2D motion in a magnetic field, contains also the BR spin–orbit term due to the structure inversion asymmetry. The latter is a source of ZB, see [13].

Demikhovskii et al [56] studied 2D electron dynamics in the presence of BR spin splitting. It was shown that in this case one deals with two spin sub-packets propagating with unequal group velocities. As the sub-packets go apart, their weakening interference is responsible for a transient character of ZB in time. It was also demonstrated that in the presence of an external magnetic field the spin sub-packets rotate with different cyclotron frequencies.

in a perpendicular magnetic field. A collapse–revival pattern of ZB was investigated and the effect of impurities (disorder) on the packet dynamics was analyzed. It turned out that ZB ‘survives’ the perturbation by impurities.
5. Nature of ZB in solids

In spite of the great interest in the phenomenon of ZB, its physical origin has remained mysterious. As mentioned above, it was recognized that the ZB in a vacuum is due to an interference of states corresponding to positive and negative electron energies. Since the ZB in solids was treated by the two-band Hamiltonian similar to the Dirac equation, its interpretation was also similar. This did not explain its origin, it only provided a way to describe it. For this reason we consider the fundamentals of electron propagation in a periodic potential trying to elucidate the nature of electron ZB in solids. The physical origin of ZB is essential because it resolves the question of its observability. The second purpose is to decide whether the two-band k · p model of the band structure, used to describe the ZB in solids, is adequate.

One should keep in mind that we described above various kinds of ZB. Every time one deals with two interacting energy bands, an interference of the lower and upper states results in electron oscillations. In particular, one deals with ZB related to the BR-type spin subbands [13] or to the Luttinger-type light- and heavy-hole subbands [15, 33]. However, the problem of interest here is the simplest electron propagation in a periodic potential. The trembling motion of this type was first treated in [12, 17]. It is often stated that an electron moving in a periodic potential behaves like a free particle characterized by an effective mass m*. The above picture suggests that, if there are no external forces, the electron moves in a crystal with a constant velocity. This, however, is clearly untrue because the electron velocity operator \( \hat{v} = \frac{\hat{p}}{m_0} \) does not commute with the Hamiltonian \( \hat{H} = \hat{p}^2/2m_0 + V(r) \), so that \( \hat{v} \) is not a constant of the motion. In reality, as the electron moves in a periodic potential, it accelerates or slows down keeping its total energy constant. This situation is analogous to that of a rollercoaster: as it goes down losing its potential energy, its velocity (i.e. its kinetic energy) increases, and when it goes up its velocity decreases.

We first consider the trembling frequency \( \omega_z \) [21]. The latter is easy to determine if we assume, in the first approximation, that the electron moves with a constant average velocity \( \bar{v} \) and the period of the potential is \( a \), so \( \omega_x \approx 2\pi /a \). Using typical values for GaAs: \( a = 5.66 \text{ Å}, \bar{v} = 2.3 \times 10^5 \text{ cm s}^{-1} \), one obtains \( \hbar \omega_z = 1.68 \text{ eV} \), i.e. the interband frequency since the energy gap is \( E_g \approx 1.5 \text{ eV} \). The interband frequency is in fact typical for the ZB in solids.

Next, we describe the velocity oscillations classically, assuming for simplicity a one-dimensional periodic potential of the form \( V(z) = V_0 \sin(2\pi z/a) \). The first integral of the motion expressing the total energy is \( E = m_0 \bar{v}^2/2 + V(z) \). Thus the velocity is

\[
d\bar{z}/dt = \sqrt{\frac{2E}{m_0} \left[ 1 - \frac{V(z)}{E} \right]}^{1/2}. \tag{40}
\]

One can now separate the variables and integrate each side in the standard way. In the classical approach \( V_0 \) must be smaller than \( E \). In general, the integration of equation (40) leads to elliptical integrals. However, trying to obtain an analytical result we assume \( V_0(z) \approx E/2 \), expand the square root retaining the first two terms and solve the remaining equation by iteration taking in the first step a constant velocity \( v_o = (2E/m_0)^{1/2} \). This gives \( z = v_o t \) and

\[
v_z(t) \approx v_o - \frac{v_o V_0}{2E} \sin \left( \frac{2\pi v_o t}{a} \right). \tag{41}
\]

Thus, as a result of the motion in a periodic potential, the electron velocity oscillates with the expected frequency \( \omega_z = 2\pi v_o/a \) around the average value \( v_o \). Integrating with respect to time we get an amplitude of ZB: \( \Delta z = V_0 a/(4\pi E) \). Taking again \( V_0 \approx E/2 \), and estimating the lattice constant to be \( a \approx \hbar p_c/(m_0 E_g) \) (see Luttinger and Kohn [57]), we have finally \( \Delta z \approx \hbar p_c/(8\pi m_0 E_g) \), where \( p_c \) is the interband matrix element of momentum. This should be compared with an estimation obtained previously from the two-band k · p model [12]: \( \Delta z \approx \lambda_z = \hbar/m^* u = \hbar (2/m^* E_g)^{1/2} \approx 2\hbar p_c/(m_0 E_g) \). Thus the classical and quantum results depend in the same way on the fundamental parameters, although the classical approach makes no use of the energy band structure. We conclude that the ZB in solids is simply due to the electron velocity oscillations assuring the energy conservation during motion in a periodic potential.

Now we describe ZB using a rigorous quantum approach. We employ the Kronig–Penney delta-like potential since it allows one to calculate explicitly the eigenenergies and eigenfunctions [58, 59]. In the extended zone scheme the Bloch functions are \( \psi_k(z) = e^{ikz} A_k(z) \), where

\[
A_k(z) = e^{-ikz} C_k \left\{ \left| e^{i\alpha} \sin(\beta_k z) + \sin(\beta_k (a - z)) \right| \right\}, \tag{42}
\]

in which \( k \) is the wavevector, \( C_k \) is a normalizing constant and \( \beta_k = \sqrt{2m_0 E}/\hbar \) is a solution of the equation

\[
Z \sin(\beta_k a) + \cos(\beta_k a) = \cos(ka), \tag{43}
\]

with \( Z > 0 \) being the effective strength of the potential. In the extended zone scheme, the energies \( E(k) \) are discontinuous functions for \( k = n\pi/a \), where \( n = \ldots, -1, 0, 1, \ldots \). In the Heisenberg picture the time-dependent velocity averaged over a wavepacket \( f(z) \) is

\[
\langle \dot{V}(t) \rangle = \frac{\hbar}{m_0} \int d\bar{k} d'k' V(k) \overline{\langle f|\bar{\psi}(\bar{k})|k'\rangle \langle k'|f \rangle} e^{i(E_k - E_{k'})t/\hbar}, \tag{44}
\]

where \( |k\rangle \) and \( |f\rangle \) are the Bloch states. The matrix elements of momentum \( \langle \bar{k}'|\bar{\psi}|k\rangle = \hbar \delta_{\bar{k}+k,k'} K(k,k') \) are calculated explicitly. The wavepacket \( f(z) \) is taken in a Gaussian form of the width \( d \) and centered at \( k_0 \). Figure 8 shows results for the electron ZB, as computed for a superlattice. The electron velocity and position are indicated. It is seen that for a superlattice with the period \( a = 200 \text{ Å} \) the ZB displacement is about ±50 Å, i.e. a fraction of the period, in agreement with the rough estimations given above. The period of oscillations is of the order of several picoseconds.

The oscillations of the packet velocity calculated directly from the periodic potential have many similarities to those computed on the basis of the two-band k · p model. The
Kronig–Penney parameter is \( Z \) calculated using: (a) real LK functions. It is seen that the two-band \( k \) energy band structure is more universal since it also includes trembling motion of the electron. The procedure based on the energy band structure is more universal since it also includes cases like the Rashba-type spin subbands or the Luttinger-type light- and heavy-hole subbands which do not exhibit an energy gap and do not seem to have a direct classical interpretation.

The distinctive character of the situation we considered is that it has a direct spatial interpretation and it is in analogy to the situation first considered by Schrödinger for a vacuum.

The main conclusion of the above considerations is that the electron ZB in crystalline solids is not an obscure and marginal phenomenon but the basic way of electron propagation in a periodic potential. The ZB oscillations of electron velocity are simply due to the total energy conservation. The trembling motion can be described either as a mode of propagation in a periodic potential or, equivalently, by the two-band \( k \cdot p \) model of band structure. The latter gives very good results because, using the effective mass and the energy gap, it reproduces the main features of the periodic potential. According to the two-band model, the ZB is related to the interference of positive and negative energy components, while the direct periodic potential approach reflects the real character of this motion. The established nature of ZB indicates that the latter should certainly be observable.

It should be mentioned that in their early paper Ferrari and Russo [9] wrote: ‘the motion of Zitterbewegung and the resulting formalism . . . is applied to describe the acceleration of a non-relativistic electron moving in a crystal, due to the periodic force experienced . . . The resulting Zitterbewegung is a real effect just because it follows from a real force.’

6. Transport

In this section we mention papers that relate the ZB phenomenon to the calculations of electron transport in semiconductors.

**Figure 8.** Calculated electron ZB velocities and displacement in a superlattice versus time. The packet width is \( d = 400 \) Å, the Kronig–Penney parameter is \( Z = 1.5\pi \), the superlattice period is \( a = 200 \) Å. (a) Packet centered at \( k_0 = \pi/a \); (b) and (c) packet centered at \( k_0 = 0.75\pi/a \). The dashed lines indicate motions with average velocities. After [21].

**Figure 9.** ZB of electron velocity in a periodic lattice versus time. Solid line: the Kronig–Penney model. Dashed line: the two-band \( k \cdot p \) model. Inset: calculated bands for the Kronig–Penney (solid line) and the two-level \( k \cdot p \) model (dashed line) in the vicinity of \( k = \pi/a \). The wavepacket \( f (k) \) centered at \( k_0 = 0.75\pi/a \) is also indicated (not normalized). After [21].

\[
\hat{H}_{kp} = \begin{pmatrix} h^2 q^2 / 2m + E_1 & h q P_{12} / m \\ h q P_{21} / m & h^2 q^2 / 2m + E_2 \end{pmatrix}, \quad (45)
\]

where \( E_1 \) and \( E_2 \) are the energies at band extrema, \( P_{12} = \hbar / m (u_{k0}, \partial / \partial x | u_{k0}) \), and \( q = k - \pi/a \). The band gap \( E_0 = E_2 - E_1 \) and the matrix elements \( P_{12} \) are calculated from the same Kronig–Penney potential, see inset of figure 9. Apart from the small free-electron terms on the diagonal, equation (45) simulates the \( 1 + 1 \) Dirac equation for free relativistic electrons in a vacuum.

In figure 9 we compare the ZB oscillations of velocity calculated using: (a) real \( E(k) \) dispersions resulting from the Kronig–Penney model and the corresponding Bloch functions of (42); (b) two-band \( E(k) \) dispersions and the corresponding LK functions. It is seen that the two-band \( k \cdot p \) model gives an excellent description of ZB for instantaneous velocities. This agreement demonstrates that the theories based on (a) the periodic potential and (b) the band structure, describe the same trembling motion of the electron. The procedure based on the average velocities. After [21].
Katsnelson [46] used the Kubo and Landauer formalism to explain the observed finite minimum of the zero-temperature conductivity of monolayer and bilayer graphene at the vanishing carrier density. He showed that it is the ZB (interband) term in the current that is responsible for this unusual behavior of conductivity in such extreme conditions.

Trauzettel et al [60] discussed photon-assisted electron transport in ballistic graphene related to electron ZB in this material and concluded that, while the considered setup is potentially relevant to the detection of ZB, the fundamental signature of ZB needs more precise identification.

Castro Neto et al [61] describing electronic properties of graphene considered possible manifestations of ZB in electron transport due to confinement of electron motion.

Cserti and David [62] showed recently that the charge conductivity of the impurity-free conductor can be expressed by non-diagonal amplitudes of ZB, while the Berry curvature and the Chern number are related to the diagonal ZB parts. The developed method was applied to calculate the electric conductivity of various systems.

7. Relativistic electrons in a vacuum

The subject of ZB for free relativistic electrons is vast and we cannot possibly do justice to it. We mention below a few papers which contributed to the understanding of ZB in solids and its simulations in other systems. The main idea of Schrödinger’s pioneering work is given in (4) because the initial equation (3) is the same as the Dirac equation (DE) with changed parameters. Details of the original Schrödinger derivations were given by Barut and Bracken [1]. Considerations showing that the ZB is caused by the interference of electron states related to positive and negative electron energies are quoted in most books on relativistic quantum mechanics, see, e.g., [3–5]. Feschbach and Villars [63] argued that, in addition to the so-called Darwin term, the spin–orbit term in the standard $v^2/c^2$ expansion of the DE can also be related to ZB.

Huang [2] went beyond the operator considerations of ZB, calculating averages of the electron position and angular momentum with the use of wavepackets. According to this treatment the electron magnetic moment may be viewed as a result of ZB, see also [64]. Huang did not predict the transient character of ZB since he assumed a very narrow packet in $k$ space (see [6]). Foldy and Wouthuysen [65] (see also [64, 66]) found a unitary transformation that separates the states of positive and negative electron energies in the free-electron DE. They showed that such states do not exhibit the ZB.

Lock remarked that, in order to talk seriously about observing ZB, one should consider a localized electron since ‘it seems to be of limited practicality to speak of rapid fluctuations in the average position of a wave of infinite extent’. He then showed that, if an electron is represented by a localized wavepacket, its ZB is transient, i.e. it decays in time. This prediction was subsequently confirmed by many descriptions (beginning with [17, 67] and in experimental simulation [20]).

Lock further showed that, if the electron spectrum is discrete, the resulting ZB is sustained in time. This property was confirmed for graphene in the presence of an external magnetic field [18], as well as for relativistic electrons in a vacuum when the spectrum is quantized into Landau levels [68].

It was pointed out, see, e.g. [4, 69], that according to the DE not only the velocity and position operators experience ZB, but also the angular momentum $\hat{L}$, the spin $\hat{S}$, and the operator $\hat{p}$ exhibit the trembling time dependence. On the other hand, the total angular momentum $\hat{J} = \hat{L} + \hat{S}$ is a constant of the motion, which can be shown directly by its vanishing commutator with the free-electron Hamiltonian $\hat{H}_D$.

Braun et al [70] used a split-operator technique to solve numerically the 3D time-dependent DE. Gaussian wavepackets were employed to calculate the transient ZB in the position and spin of free relativistic electrons for different packet widths. The authors remark that ‘the Zitterbewegung can be found only if the initial velocity (or wavevector) $k_0$ is nonzero’. Very good numerical approximations to the exact solutions were found but they require powerful computers.

Thaller [67] computed and simulated the time behavior of relativistic Gaussian wavepackets according to the one-dimensional DE. For a packet with vanishing average momentum, the packet position shows ZB that decays with time very slowly. For a non-vanishing average momentum the decay of ZB is much faster. This is caused by the fact that the ZB arises due to an interference of positive and negative energy sub-packets which in this case move in opposite directions and cease to overlap relatively quickly. This process is explained in some detail in (24) and (25) for electrons in bilayer graphene.

Krekora et al [71] studied pair creation in a vacuum and stated that ‘quantum theory prohibits the occurrence of Zitterbewegung for an electron’. This conclusion was contradicted by the analysis of Wang and Xiong [72]. Arunagiri [73] proposed to circumvent the difficulty preventing observability of ZB due to pair creation by localizing the electron in the presence of a magnetic field and using monolayer graphene as a model of massless fermions. Barut and Malin [74] considered the problem of filled negative energies in the DE and its effect on electron localization.

Barut and Thacker [75] treated the ZB of relativistic electrons in a vacuum in the presence of an external magnetic field. This description suffered from a few deficiencies, as explained in [68]. Bermudez et al [76] treated the problem of time-dependent relativistic Landau states by mapping the relativistic model of electrons in a magnetic field onto a combination of the Jaynes–Cummings and anti-Jaynes–Cummings interactions. For simplicity the $p_z = 0$ restriction was assumed. Three regimes of high (macroscopic), small (microscopic) and intermediate (mesoscopic) Landau quantum numbers $n$ were considered. In all the cases only one interband frequency contributed to the ZB because the authors did not use wavepackets to calculate average values. The same problem was recently tackled by Rusin and Zawadzki [68], who showed that the quantization of the electron spectrum into the Landau levels has strong effects on the ZB. The trembling motion becomes a multi-frequency phenomenon and in two dimensions is not transient, as opposed to the no-field case. In practice, however, for magnetic fields available in terrestrial conditions the decisive ratio $\hbar(eB/m_0)/2mc^2$ is very small, so the magnetic effects in the ZB are insignificant. The only
promising way to see the magnetic effects in ZB is to carry out simulations. Such a simulation was proposed in [68], see section 8.

8. Simulations

As we said above, the electron ZB in a vacuum or in a solid is difficult to observe. The characteristics of electron ZB in semiconductors are much more favorable than in a vacuum but it is difficult to follow the motion of a single electron; one would need to follow the motion of many electrons moving in phase. Recently, however, there appeared many propositions to simulate the DE and the resulting phenomena with the use of other systems. We want to enumerate these propositions below, but we are not in a position to explain all the underlying ideas. It will suffice to say that many (not all) ideas make use of trapped atoms or ions interacting with laser light. There are two essential advantages of such simulations. First, it is possible to follow the interaction of laser light with a few or even single atoms or ions. Second, when simulating the DE it is possible to modify its two basic parameters, \( m_0e^2 \) and \( c \), in order to make the ZB frequency much lower and its amplitude much larger than in a vacuum. In consequence, they become measurable with current experimental techniques.

As a matter of example we will briefly consider a simulation of the DE with the use of the Jaynes–Cummings model [77] known from quantum and atomic optics, see [78–80]. The DE contains electron momenta, so the essential task is to simulate \( \hat{p}_1 \). The common types of light interactions with ions and vibronic levels are used to that purpose: a carrier interaction \( \hat{H}_c = \hbar \Omega (\sigma^+ e^{i\phi} + \sigma^- e^{-i\phi}) \), the Jaynes–Cummings (JC) interaction \( \hat{H}_{JC}^{\phi} = \hbar \eta \Omega (\sigma^+ \hat{a} e^{i\phi} + \sigma^- \hat{a}^\dagger e^{-i\phi}) \), and the anti-Jaynes–Cummings (AJC) interaction \( \hat{H}_{AJC}^{\phi} = \hbar \eta \Omega (\sigma^+ \hat{a}^\dagger e^{i\phi} + \sigma^- \hat{a} e^{-i\phi}) \). Here \( \sigma^\pm = \sigma_x \pm i\sigma_y \) are the raising and lowering ionic spin-1/2 operators, \( \hat{a} \) and \( \hat{a}^\dagger \) are the creation and annihilation operators associated with the motional states of the ion, \( \eta \) is the so-called Lamb–Dicke parameter, \( \Omega \) and \( \bar{\Omega} \) are the Rabi frequencies. The basic idea is to use proper light phases in \( \hat{H}_c \) and \( \hat{H}_{JC}^{\phi} \) in order to obtain the momentum from the relation \( \hat{p}_1 = i\hbar (\hat{a}^\dagger - \hat{a})/\Delta \), which results in \( \hat{H}_{JC}^{\phi} = \pm i \hbar \bar{\Omega} \sigma_y (\hat{a}^\dagger - \hat{a}) \). One can show that the simulated parameters of the DE are

\[
c \rightarrow 2 \eta \Delta \bar{\Omega}, \quad m_0e^2 \rightarrow \hbar \Omega,
\]

where \( \Delta \) is the spread in the position of the ground ion wavefunction. If the dynamics created by the \( 1 \pm 1 \) DE is to be reproduced in an experiment with a four-level ion system using Raman beams, it requires 14 pairs of Raman lasers. One needs to control their phases independently.

In figure 10 we show experimental results of Gerritsma et al [20], who simulated for the first time the \( 1 \pm 1 \) DE with the resulting one-dimensional ZB using \( ^{40}\text{Ca}^+ \) trapped ions. It can be seen that the results agree very well with the predictions of [21], see our figure 8(c). The reason for this agreement is that the theory of [21], while concerned with solids, also uses an effective DE, see (45) and the inset of figure 9. Gerritsma et al showed that, if the wavepacket does not have the initial momentum, the decay time of ZB is much slower than that seen in figure 10. This agrees with theoretical results for carbon nanotubes, as shown in our figure 4, see also [67].

The problem of ZB for free relativistic electrons in a magnetic field was recently described by Rusin and Zawadzki [68]. The main experimental problem in investigating the ZB phenomenon in an external magnetic field was recently described by Rusin and Zawadzki [68]. The main experimental problem in investigating the ZB phenomenon in an external magnetic field was recently described by Rusin and Zawadzki [68]. The main experimental problem in investigating the ZB phenomenon in an external magnetic field was recently described by Rusin and Zawadzki [68].

![Figure 10](image-url)

**Figure 10.** (a) ZB for a state with nonzero average momentum. The solid curve represents a numerical simulation. (b) Measured (filled areas) and numerically calculated (solid lines) probability distributions \( |\psi(x)|^2 \) at the times \( t = 0, 75 \) and 150 \( \mu s \) (as indicated by the arrows in (a)). The probability distribution corresponding to the state \( |1\rangle \) is inverted for clarity. The vertical solid line represents \( \langle x \rangle \) as plotted in (a). The two dashed lines are the expectation values for the positive and negative energy parts of the spinor. Error bars \( 1\sigma \). After [20]. (This figure is in colour only in the electronic version)
Positions are given in $L$ in time. After $\kappa$, energy $\kappa$ seen that, as $\kappa$ of $\kappa$, are similar to those obtained for graphene in a magnetic field, and the light velocity is replaced by the lattice recoil velocity equal to the wavelength of light producing the optical lattice DE with the resulting ZB. The characteristic ZB amplitude is that such a system in a tripod configuration simulates the ultra-cold neutral atoms in an optical lattice. One can show accessible characteristics: the amplitude of tens of nanometers of a few centimeters per second. The simulated ZB has are comparable. Qualitatively, the results shown in figure 11 high magnetic fields the interband and intraband components contribute to the spectrum. ZB becomes richer. This means that more interband and intraband frequency components correspond to the periodic potential of such crystals. These ZB-like effects result from the wave propagation in 2D macroscopic sonic crystals made of steel cylinders immersed in water. For acoustic waves, the band structure $\omega(q)$ of such crystals resembles the relativistic two-band dispersion. It is par excellence a quantum phenomenon \textit{par excellence} since it predicts electron behavior that goes beyond Newton’s first law of classical motion. However, a similar behavior is predicted and also observed in the propagation of acoustic and light waves in periodic systems. These ZB-like effects result from the wave nature of phonons and photons and they are in principle of a non-quantum nature (they do not involve the Planck constant). Clearly, they do not go beyond Newton’s first law of motion, similarly to the electron ZB effects in solids which, as we demonstrated above, are related to the periodic potential of the crystal lattice. We emphasize here that the acoustic and light effects mentioned below do not simulate the relativistic quantum mechanics (as sometimes presented), but represent the wave ZB-like effects in other systems.

Vaishnav and Clark [38] proposed to observe the ZB with ultra-cold neutral atoms in an optical lattice. One can show that such a system in a tripod configuration simulates the DE with the resulting ZB. The characteristic ZB amplitude is equal to the wavelength of light producing the optical lattice and the light velocity is replaced by the lattice recoil velocity of a few centimeters per second. The simulated ZB has accessible characteristics: the amplitude of tens of nanometers and frequencies in the range of megahertz. Interestingly, the transient ZB occurs also for a vanishing average momentum of the wavepacket. Also, the ZB can be viewed as a measurable consequence of the momentum-space Berry phase.

Zhang et al [81] extended the idea of Vaishnav and Clark [38] for the tripod scheme proposing that one can use vibrating mirrors to modulate the laser light and simulate the time-dependent DE in order to influence the amplitude, frequency and decay time of the resulting ZB.

Merkel et al [82] described the ZB of ultra-cold atoms moving in one dimension and interacting with laser beams in a tripod system. It was shown explicitly that in this case the decay time of ZB is inversely proportional to the $k$ spread of a wavepacket and that the oscillation amplitude decreases as $t^{-1/2}$. These features agree with the analytical results shown above for the ZB of electrons in bilayer graphene, cf (14)–(15).

Song and Foreman [83] proposed to create the atomic ZB using trapped cold atoms in an Abelian vector potential in a tripod configuration. It was shown that, in a purely 1D potential, one can still achieve a time-dependent velocity operator if the scalar potential does not commute with the vector potential. The predicted amplitude of transient ZB is around 0.2 $\mu$m and the frequency around 1 ms$^{-1}$.

Braun [84] showed how a single harmonically trapped cold atom in a real spatially tailored magnetic field can be used to simulate the BR and the linear Dresselhaus spin coupling with the resulting ZB.

### 9. ZB-like wave effects

In its original version proposed by Schrödinger the ZB is a quantum phenomenon \textit{par excellence} since it predicts electron behavior that goes beyond Newton’s first law of classical motion. However, a similar behavior is predicted and also observed in the propagation of acoustic and light waves in periodic systems. These ZB-like effects result from the wave nature of phonons and photons and they are in principle of a non-quantum nature (they do not involve the Planck constant). Clearly, they do not go beyond Newton’s first law of motion, similarly to the electron ZB effects in solids which, as we demonstrated above, are related to the periodic potential of the crystal lattice. We emphasize here that the acoustic and light effects mentioned below do not simulate the relativistic quantum mechanics (as sometimes presented), but represent the wave ZB-like effects in other systems.

Zhang and Liu [22] investigated experimentally acoustic wave propagation in 2D macroscopic sonic crystals made of steel cylinders immersed in water and having a lattice constant $a = 1.5$ mm. The band structure of such crystals for acoustic waves has no energy gap and almost linear $E(q)$ dependence at the $K$ point of the Brillouin zone. As a consequence, the acoustic wave propagation near the $K$ point can be described by a $2 \times 2$ set of equations resembling the Dirac-like Hamiltonian for monolayer graphene. The observed effects in the experimental behavior of the transmitted acoustic waves were interpreted as a classical analogue of ZB.

Wang et al [85] described sound propagation in sonic crystals consisting of square arrays of steel cylinders immersed in water. For acoustic waves, the band structure $\omega(q)$ of such systems resembles the relativistic two-band dispersion. It is
predicted that the time evolution of an acoustic wavepacket (pressure intensity) should exhibit a transient ZB-like effect with an initial amplitude of about one lattice constant and frequency \( \omega_z = \omega_0(q_0) - \omega_1(q_0) \), equal to the frequency difference of the bands in question. The transient character of ZB is due to a weakening interference of sub-packets as they move apart. All these features resemble very closely the electron ZB in crystalline solids.

Zhang [86] simulated numerically the photon transport in 2D photonic crystals made of cylinders immersed in air. In such a crystal, the band structure for photons near the electron ZB in crystalline solids is due to a weakening interference of sub-packets as the difference of the bands in question. The transient character of ZB is due to a weakening interference of sub-packets as they move apart. All these features resemble very closely the electron ZB in crystalline solids.

Wang et al. [87] used the fact that in a homogeneous optical medium consisting of three slabs characterized by negative–zero–positive refractive indices there exist two optical passbands for photons with linear zero–positive refractive indices. There exist two optical passbands for photons with linear zero–positive refractive indices. Inside and outside of it, exhibits the ZB behavior. The photon propagation in photonic crystals bears many similarities to the sound propagation in sonic crystals, see [22].

Dreisow et al. [23], following the suggestion of Longhi [89], realized recently an optical binary waveguide system which was shown to have for photons the relativistic–like dispersion \( E(k) = \sqrt{\sigma^2 + \kappa^2 k^2} \). The energy gap is \( 2\hbar\sigma \), where \( \sigma \) is the mismatch of propagation constants and \( \kappa \) is the coupling rate between two adjacent waveguides. Description of light propagation in such a system is analogous to the DE with the time variable replaced by one of the spatial variables.

The resulting trembling motion was observed as a spatial oscillatory motion of an optical beam with the frequency \( \omega_z = 2\sigma \) and amplitude \( R_z = \kappa/(2\sigma) \). The experiments were carried out for highly relativistic (small \( \sigma \)) and weakly relativistic (higher \( \sigma \)) regimes.

The ZB-like wave phenomena in periodic structures are very similar to the electronic ZB in crystalline solids: they are characterized by the interband frequency, they result from an interference of states related to the positive and negative energies and they decay in time. Paradoxically, these wave phenomena seem to be easier to observe than their ‘older’ electronic analogues.

10. Discussion and conclusions

An important recognition won after the considerable effort of the last five years is that the Zitterbewegung is not a marginal, obscure and probably unobservable effect of interest to a few esoteric theorists, but a real and universal phenomenon that often occurs in both quantum and non-quantum systems. Clearly, the ZB in a vacuum proposed by Schrödinger [1] stands out as an exception since it is supposed to occur without any external force. However, in its original form it will probably not be directly observable for years to come and one has to resort to its simulations. A proof-of-principle of such simulations was recently carried out, see figure 10 and [20]. On the other hand, manifestations of ZB in crystalline solids and other periodic systems turned out to be quite common and they are certainly observable. A universal background underlying the phenomenon of ZB in any system (including a vacuum) is an interference of states belonging to positive and negative energies (in a generalized sense, see below). The positive and negative energies usually belong to bands but they can also be discrete levels, as shown for electrons in graphene in a magnetic field, see figure 5.

As we said above, the ZB amplitude is around \( \lambda_Z = \hbar/(m_e^*\mu) \), which we called the length of Zitterbewegung. Let us suppose that we confine an electron to the dimension \( \Delta_z \approx \lambda_Z/2 \). Then the uncertainty of momentum is \( \Delta p_z \geq \hbar/\Delta_z \) and the resulting uncertainty of energy \( \Delta E \approx (\Delta p_z)^2/(2m_e^*) \) becomes \( \Delta E \approx 2m_e^*\mu^2 \). Thus an electron confined to \( \Delta_z \approx \lambda_Z/2 \) has an uncertainty of energy larger than the gap. For electrons in a vacuum the restriction \( \Delta_z \approx \lambda_Z/2 \) is not significant, but for electrons in narrow-gap semiconductors the restriction \( \Delta_z \approx \lambda_Z/2 \) should be taken seriously, since \( \lambda_Z \) is of the order of tens of angstroms, so that this confinement is not difficult to realize experimentally by quantum wells or magnetic fields. The question arises of what happens if the electron is confined to \( \Delta_z < \lambda_Z/2 \), so that the trembling motion is strongly perturbed by the confinement. We showed above, see figure 7, that an electron in a magnetic field radiates interband ZB frequencies, and their contribution to the motion increases with the increasing field, see also [68]. It is possible that this effect is just a manifestation of the perturbation of the trembling motion by magnetic confinement. Also, it was shown that an effective one-band semi-relativistic Hamiltonian in a narrow-gap semiconductor contains the so-called Darwin term which can be traced back to the ZB. The Darwin term can lead to measurable effects for ground impurity states [12].

An important question arises: what should be called ‘Zitterbewegung’? It seems that the signature of the ZB phenomenon is its interband frequency, in which the term interband has the meaning ‘between interacting bands’. Thus, for example, the ZB resulting from the Bychkov–Rashba spin splitting (or the so-called linear Dresselhaus spin splitting) is not characterized by a truly interband frequency, since in this situation there is no gap, but the frequency corresponding to the energy difference between the two spin branches of the same band: \( \hbar\omega_{z} = E_\uparrow - E_\downarrow \), see (7) and [13]. Another illustration is the ZB of holes in the valence bands of \( \Gamma_8 \) symmetry [35], where the ZB frequency is given by the energy difference of light- and heavy-hole bands [15, 33, 36]. Finally, an instructive example is provided by graphene in a magnetic field (see figure 5), where the electron motion contains both intraband and interband frequencies. We believe that only the interband contributions should be called ZB, while the intraband ones are simply the cyclotron components. It appears that the second signature of ZB is the actual motion which, for
instance, distinguishes it from the Rabi oscillations. The above considerations indicate that an unambiguous definition of ZB is not obvious.

If an electron is prepared in the form of a wavepacket, and if the electron spectrum is not completely quantized, the ZB has a transient character, i.e. it decays in time. This was predicted by Lock [6] on the basis of the Riemann–Lebesgue lemma, and was confirmed by many specific calculations, see, e.g., figures 2 and 3, as well as by observations [20, 22]. One can show that the decay time is inversely proportional to the momentum spread \( \Delta k \) of the wavepacket, see (16) and [17, 82]. Physically, the transient character of ZB comes about as a result of the waning interference of the two sub-packets belonging to positive and negative energies as they go apart because of different speeds, see (24) and (25) and [17, 85]. The decay time is usually much longer in one-dimensional systems, see figure 4 and [17, 82]. On the other hand, if the electron spectrum is discrete, ZB persists in time, sometimes in the form of collapse–revival patterns [53]. In general, the wavepacket should have a non-vanishing initial momentum in one direction to exhibit the ZB in the perpendicular direction see (7) and (13), but this is not always the case [47].

We described above the phenomena related to photons and phonons in the separate section 9 because, in our opinion, they are not simulations of the relativistic quantum mechanics but represent ZB-like effects of their own. They are non-quantum wave effects in periodic structures. Along with the quantum ZB effects for electrons they have in common the Floquet and Mathieu descriptions of eigenvalues and eigenfunctions of the second-order differential equations with periodic potentials, see, e.g., [90]. Quantum and non-quantum phenomena can be quite similar because of the wave character of quantum mechanics. It is an important success of the efforts concerned with the electronic ZB that they have led to discoveries in non-electronic areas. It appears that, in fact, the non-electronic ZB-like effects are easier to observe than the ‘original’ electronic ones. Finally, it is important that photons and phonons in the ZB-like wave phenomena do not obey the Pauli exclusion principle for fermions.

Clearly, one should ask the question about possible observation of Zitterbewegung in solids. Two different ways were proposed to observe the trembling electrons. The first is to detect an ac current related to the ZB velocity, see, e.g., figure 3. One needs a current meter sensitive to the ZB frequency. Then, even if the electrons do not move in phase so that the net current averages to zero, the meter should detect a clear increase of noise at the frequency \( \omega \lambda \). The second possible way to observe the ZB is to detect electromagnetic radiation emitted by the trembling electrons, see figure 7 and [18]. The emission is possible because, if the electrons are prepared in form of wavepackets they respond to light wavepackets, they are not in their eigenstates. The proposed ZB should not be confused with the Bloch oscillations of charge carriers in superlattices. The Bloch oscillations are basically a one-band phenomenon and they require an external electric field driving electrons all the way to the Brillouin zone boundary. On the other hand, the ZB needs at least two bands and it is a no-field phenomenon. Narrow-gap superlattices can provide a suitable system for its observation. In the near future one can expect theoretical predictions of ZB in new systems as well as observations of ZB-like wave effects. A real challenge remains: direct experimental evidence for the electron Zitterbewegung in semiconductors.

**Appendix A**

In this appendix we show that the ZB length \( \lambda \lambda \) defined in (2) can be measured directly. We write (1) in the form [12]

\[
E = \pm \hbar u \left( \lambda \lambda^2 + k^2 \right)^{1/2}, \tag{A.1}
\]

where \( u = (E_g/2m^*_)1/2 \). For \( k^2 > 0 \) this formula describes the conduction and light-hole bands. But for imaginary values of \( k \) there is \( k^2 < 0 \) and (A.1) describes the dispersion in the energy gap. This region is classically forbidden but it can become accessible through quantum tunneling. Figure A.1 shows the data for the dispersion in the gap of InAs, obtained by Parker and Mead [91] from tunneling experiments with double Schottky barriers. The solid line indicates the fit using (A.1). The value of \( \lambda \lambda \) is determined directly by the \( k_0 \) for which the energy is zero: \( \lambda \lambda^2 = k_0^2 \). The fit gives \( \lambda \lambda \approx 41.5 \text{ Å} \) and \( u \approx 1.33 \times 10^8 \text{ cm s}^{-1} \), in good agreement with the estimation for InAs given in section 2. Similar data for GaAs give \( \lambda \lambda \) between 10 Å [92] and 13 Å [93], again in good agreement with the estimation quoted in section 2.

![Figure A.1](https://example.com/figure.png)

Figure A.1. Energy–wavevector dependence in the forbidden gap of InAs. Various symbols show the experimental data of Parker and Mead [91] for five InAs samples, the solid line is a theoretical fit using (A.1). The determined parameters are \( \lambda \lambda = 41.5 \text{ Å} \) and \( u = 1.33 \times 10^8 \text{ cm s}^{-1} \). After [12].

**Appendix B**

We briefly discuss here the classical electron velocity and mass for a linear energy band of monolayer graphene, as they are often subjects of misunderstandings. Let us consider
the conduction band and take $p \geq 0$, where the pseudo-momentum is $p = \hbar k$. Then the band dispersion is $E = up$ and the classical velocity is

$$
v_i = \frac{\partial E}{\partial p_i} = \frac{dE}{dp} \frac{\partial p_i}{\partial p} = \frac{dE}{dp} p_i = \frac{dE}{dp} \delta_{ij} p_j, \quad (B.1)
$$

where $\delta_{ij}$ is the Kronecker delta function and we use the sum convention over the repeated index $j = 1, 2$. The electron mass tensor $m$ relating the velocity to pseudo-momentum is defined by $\hat{m} v = p$. Then the inverse mass tensor $(1/m)$ is defined by

$$
(1/m)_{ij} = \frac{dE}{dp} \delta_{ij}, \quad (B.2)
$$

Equating (B.1) with (B.2) we obtain

$$
(1/m)_{ij} = \frac{dE}{dp} \delta_{ij}. \quad (B.3)
$$

Thus the inverse mass tensor is a scalar: $1/m = (dE/dp)(1/p)$. Using the initial band dispersion one has $dE/dp = u$, so that $m = p/u = E/u^2$. This equality can be seen in two ways. First, it gives

$$
E = m u^2, \quad (B.4)
$$

which is analogous to the Einstein relation between the particle energy and mass. Second, the formula

$$
m = \frac{E}{u^2} \quad (B.5)
$$

states that the mass vanishes at $E = 0$ (or $p = 0$), but it is nonzero for $E > 0$ (or $p > 0$). These relations hold also for a more general ‘semi-relativistic’ case of narrow-gap semiconductors described by (1), see [11, 49].

One should add that, if one defined the mass by the relation of the force to the acceleration, $Ma = F$, the inverse mass would be given by the second derivative of the energy with respect to momentum. For the linear band of graphene: $E = up$, such a mass would be infinitely large for all energies, so it is not a useful quantity.

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