GRAVITY AS A HIGGS FIELD.

III. Nongravitational Deviations of Gravitational Fields.

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Abstract

In Parts I,II of the work (gr-qc/9405013, 9407032), we have shown that gravity is *sui generis* a Higgs field corresponding to spontaneous symmetry breaking when the fermion matter admits only the Lorentz subgroup of world symmetries of the geometric arena. From the mathematical viewpoint, the Higgs nature of gravity issues from the fact that different gravitational fields are responsible for nonequivalent representations of cotangent vectors to a world manifold by $\gamma$-matrices on spinor bundles. It follows that gravitational fields fail to form an affine space modelled on a linear space of deviations of some background field. In other words, even weak gravitational fields do not satisfy the superposition principle and, in particular, can not be quantized by usual methods. At the same time, one can examine superposable deviations $\sigma$ of a gravitational field $h$ so that $h + \sigma$ fail to be a gravitational field. These deviations get the adequate mathematical description in the framework of the affine group gauge theory in dislocated manifolds, and their Lagrangian densities differ from the familiar gravitational Lagrangian densities. They make contribution to the standard gravitational effects, e.g., modify Newton’s gravitational potential.

1 Introduction

In the naive manner, one usually describes deviations $\epsilon$ of a gravitational field $g$ as small deviations of ordinary physical field:

$$
g'_{\mu\nu} = g_{\mu\nu} + \epsilon_{\mu\nu},$$

$$g'_{\mu\nu} = g_{\mu\nu} - \epsilon_{\mu\nu} \approx g_{\mu\alpha}g_{\nu\beta}\epsilon^{\alpha\beta}. \tag{1}$$

These deviations however fail to be superposable even in the first order of $\epsilon$ if one does not ignore the geometric nature of gravity and its physical peculiarity as a Higgs field.

Gravitation theory is theory with spontaneous symmetry breaking since the fermion matter admits only the Lorentz subgroup of world symmetries of the geometric arena. In other words, this spontaneous symmetry breaking appears when one provides a world manifold with a spinor structure \[8, 10\].
Given a Minkowski space $M$ with the Minkowski metric $\eta$, let $\mathbb{C}_{1,3}$ be the complex Clifford algebra generated by elements of $M$. A spinor space $V$ is defined to be a minimal left ideal of $\mathbb{C}_{1,3}$ on which this algebra acts on the left. We have the representation

$$\gamma : M \otimes V \rightarrow V$$ (2)

of elements of the Minkowski space $M$ by Dirac’s matrices $\gamma$ on $V$.

Let us consider a bundle of complex Clifford algebras $\mathbb{C}_{3,1}$ over $X^4$. Its subbundles are both a spinor bundle $S_M \rightarrow X^4$ and the bundle $Y_M \rightarrow X^4$ of Minkowski spaces of generating elements of $\mathbb{C}_{3,1}$. To describe Dirac fermion fields on a world manifold, one must require that $Y_M$ is isomorphic to the cotangent bundle $T^*X$ of a world manifold $X^4$. It takes place if the structure group $GL_4$ of $LX$ is reducible to the Lorentz group $L = SO(3,1)$ and $LX$ contains a reduced $L$ subbundle $L^hX$ such that

$$Y_M = M^hX = (L^hX \times M)/L.$$ (3)

In this case, the spinor bundle $S_M$ is associated with the $L_s$-lift $P_h$ of $L^hX$:

$$S_M = S_h = (P_h \times V)/L_s.$$ (4)

For the sake of simplicity, we shall identify $P_h$ with $L^hX$.

In accordance with the well-known theorem, there is the 1:1 correspondence between the reduced subbubdles $L^hX$ of $LX$ and the tetrad gravitational fields $h$ identified with global sections of the quotient bundle

$$\Sigma := LX/L \rightarrow X^4.$$ (5)

This bundle is the 2-fold cover of the bundle of pseudo-Riemannian bilinear forms in cotangent spaces to $X^4$. Global sections of the latter are pseudo-Riemannian metrics $g$ on $X^4$.

Given a tetrad field $h$, let $\Psi^h$ be an atlas of $LX$ such that the corresponding local sections $z^h_\xi$ of $LX$ take their values into the reduced subbundle $L^hX$. With respect to an atlas $\Psi^h$ and a holonomic atlas $\Psi^T = \{\psi^T_\xi\}$ of $LX$, the tetrad field $h$ can be represented by a family of $GL_4$-valued tetrad functions

$$h_\xi = \psi^T_\xi \circ z^h_\xi,$$
$$dx^\lambda = h^\lambda_a(x)h_a,$$

which carry atlas (gauge) transformations between fibre bases $\{dx^\lambda\}$ and $\{h^a\}$ of $T^*X$ associated with $\Psi^T$ and $\Psi^h$ respectively. The well-known relation

$$g^{\mu\nu} = h^\mu_a h^\nu_b \eta^{ab}$$ (6)

takes place.
Given a tetrad field \( h \), one can define the representation
\[
\gamma_h : T^*X \otimes S_h \to (L^h \times (M \otimes V))/L = S_h,
\]
(7)
\[
\tilde{dx}^\lambda = \gamma_h(dx^\lambda) = h^\lambda_a(x)\gamma^a,
\]
of cotangent vectors to a world manifold \( X^4 \) by Dirac’s \( \gamma \)-matrices on elements of the spinor bundle \( S_h \).

Let \( A_h \) be a connection on \( S_h \) associated with a principal connection on \( L^hX \) and \( D \) the corresponding covariant differential. Given the representation (7), one can construct the Dirac operator
\[
\mathcal{D}_h = \gamma_h \circ D : J^1S_h \to T^*X \otimes V S_h \to V S_h
\]
on \( S_h \). Then, we can say that sections of the spinor bundle \( S_h \) describe Dirac fermion fields in the presence of the tetrad gravitational field \( h \).

The crucial point consists in the fact that, for different tetrad fields \( h \) and \( h' \), Dirac fermion fields are described by sections of spinor bundles associated with different reduced \( L \)-principal subbundles of \( LX \) and so, the representations \( \gamma_h \) and \( \gamma_{h'} \) (7) are not equivalent.

It follows that Dirac fermion field must be regarded only in a pair with a certain tetrad gravitational field \( h \). These pairs are represented by sections of the composite spinor bundle
\[
S \to \Sigma \to X^4
\]
(8)
where \( S \to \Sigma \) is a spinor bundle associated with the \( L \) principal bundle \( LX \to \Sigma \). In particular, every spinor bundle \( S_h \) (4) is isomorphic to restriction of \( S \) to \( h(X^4) \subset \Sigma \).

Since, for different tetrad fields \( h \) and \( h' \), the representations \( \gamma_h \) and \( \gamma_{h'} \) (7) are not equivalent, even weak gravitational fields, unlike matter fields and gauge potentials, fail to form an affine space modelled on a linear space of deviations of some background field. They thereby do not satisfy the superposition principle and can not be quantized by usual methods, for in accordance with the algebraic quantum field theory quantized fields must constitute a linear space. This is the common feature of Higgs fields. In algebraic quantum field theory, different Higgs fields correspond to nonequivalent Gaussian states of a quantum field algebra. Quantized deviations of a Higgs field can not change a state of this algebra and so, they fail to generate a new Higgs field.

At the same time, one can examine superposable deviations \( \sigma \) of a tetrad gravitational field \( h \) so that \( h + \sigma \) is not a tetrad gravitational field [4, 8]. In the coordinate form, these deviations read
\[
\tilde{h}^\mu_a = s^b_a h^\mu_b = (\delta^\mu_a + \sigma^\mu_a) h^\mu_b = s^\mu_\nu h^\nu_\alpha = (\delta^\mu_\nu + \sigma^\mu_\nu) h^\mu_a = h^\mu_a + \sigma^\mu_a,
\]
\[
\tilde{h}^\mu_a = g_{\mu\nu} \eta^{ab} \tilde{h}^\nu_b = s^a_\alpha h^\mu_\alpha, \quad \tilde{h}^\mu_a \tilde{h}^\nu_b \neq \delta^\mu_\nu, \quad \tilde{h}^\mu_a \tilde{h}^\mu_b \neq \delta^b_a.
\]
Note that the similar factors have been investigated by R.Percacci [2, 4].

3
In bundle terms, we can describe the deviations (9) as the special morphism $\Phi_2$ of the cotangent bundle. Given a gravitational field $h$ and the corresponding representation morphism $\gamma_h$ (7), the morphism $\Phi_2$ yields another $\gamma$-matrix representation
\[
\tilde{\gamma}_h = \gamma_h \circ \Phi_2,
\]
\[
\tilde{\gamma}_h(h^a) = s^b_a \gamma_h(h^b) = s^a_b \gamma^b,
\]
of cotangent vectors, but on the same spinor bundle $S_h$. Therefore, deviations (9) and their superposition $\sigma + \sigma'$ can be defined.

Let us note that, to construct a Lagrangian density of deviations $\epsilon$ of a gravitational field, one usually utilize a familiar Lagrangian density of a gravitational field $h' = h + \epsilon$ where $h$ is treated as a background field. In case of the deviations (9), one can not follow this method, for quantities $\tilde{\gamma}$ fail to be true tetrad fields. To overcome this difficulty, we use the fact that the morphisms $\Phi_2$ appears also in the dislocation gauge theory of the translation group. We therefore may apply the Lagrangian densities of this theory in order to describe deviations $\sigma$ (9). They differ from the familiar gravitational Lagrangian densities. In particular, they contain the mass-like term. Solutions of the corresponding field equations show that fields $\sigma$ make contribution to the standard gravitational effects. In particular, they lead to the "Yukawa type" modification of Newton's gravitational potential.

Note that a world manifold $X^4$ must satisfy the well-known global topological conditions in order that gravitational fields, space-time structure and spinor structure can exist. To summarize these conditions, we assume that $X^4$ is not compact and the linear frame bundle $LX$ over $X^4$ is trivial.

2 Deviations of tetrad fields

Let $\pi_P : P \rightarrow X$ be a principal bundle with a structure Lie group $G$ which acts freely and transitively on $P$ on the right:
\[
r_g : p \mapsto pg, \quad p \in P, \quad g \in G.
\]
Note that a principal bundle $P$ is also the general affine bundle modelled on the associated group bundle $\tilde{\pi} : \tilde{P} \rightarrow X$ with the standard fibre $G$ on which the structure group $G$ acts by the adjoint representation. The corresponding bundle morphism reads
\[
\tilde{P} \times P \ni (\tilde{p}, p) \mapsto \tilde{p}p \in P.
\]
Let $K$ be a closed subgroup of $G$. We have the composite manifold
\[
\pi_{\Sigma X} \circ \pi_{\Sigma P} : P \rightarrow P/K \rightarrow X
\]
where
\[
P_{\Sigma} := P \rightarrow P/K
\]
is a principal bundle with the structure group $K$ and

$$
\Sigma_K = P/K = (P \times G/K)/G
$$

is the $P$-associated bundle with the standard fiber $G/K$ on which the structure group $G$ acts on the left.

Let the structure group $G$ be reducible to its closed subgroup $K$. Recall the 1:1 correspondence

$$
\pi_{P\Sigma}(P_h) = (h \circ \pi_P)(P_h)
$$

between the global sections $h$ of the bundle $P/K \to X$ and the reduced $K$-principal subbundles $P_h$ of $P$ which consist with restrictions of the principal bundle $P_{\Sigma}$ to $h(X)$.

Let us consider the composite manifold

$$
Y = (P \times V)/K \to P/K \to X
$$

(12)

where the bundle

$$
Y_{\Sigma} := (P \times V)/K \to P/K
$$

is associated with the $K$-principal bundle $P_{\Sigma}$. Given a reduced subbundle $P_h$ of $P$, the associated bundle

$$
Y_h = (P_h \times V)/K
$$

is isomorphic to the restriction of $Y_{\Sigma}$ to $h(X) \subset \Sigma_K$.

Note that the manifold $(P \times V)/K$ possesses also the structure of the bundle

$$
Y = (P \times (G \times V))/G
$$

(13)

associated with the principal bundle $P$. Its standard fibre is $(G \times V)/K$ on which the structure group $G$ of $P$ (and its subgroup $K$) acts by the law

$$
G \ni g : (G \times V)/K \to (gG \times V)/K.
$$

It differs from action of the structure group $K$ of $P_{\Sigma}$ on this standard fibre. As a shorthand, we can write the latter in the form

$$
K \ni g : (G \times V)/K \to (G \times gV)/K.
$$

However, this action fails to be canonical and depends on existence and specification of a global section of the bundle $G \to G/K$. If the standard fibre $V$ of the bundle $Y_{\Sigma}$ carriers representation of the whole group $G$, these two actions are equivalent, otherwise in general case.

Let $\Phi$ be an isomorphism of the principal bundle $P$ over $\text{Id} X$. It is expressed as

$$
\Phi(p) = pf_s(p), \quad p \in P,
$$

(14)
where $f_s$ is a $G$-valued equivariant function

$$f_s(pg) = g^{-1}f(p)g, \quad g \in G,$$

on $P$. There is the 1:1 correspondence

$$s(\tilde{\pi}(p))p = pf_s(p)$$

between such functions and global sections $s$ of the corresponding group bundle $\tilde{P}$.

Every principal isomorphism $\Phi$ ([14]) yields the following morphism of the composite manifold ([12]):

$$\Phi_1 : (p \times v)/K \mapsto (pf_s(p) \times v)/K. \quad (15)$$

It is the morphism of $Y$ as the $P$-associated bundle ([13]). If the standard fibre $V$ of the bundle $Y_\Sigma$ admits a representation of the whole group $G$, every principal isomorphism $\Phi$ ([14]) of $P$ generates another morphism of the composite manifold ([12]):

$$\Phi_2 : (p \times v)/K \mapsto (p \times f_s(p)v)/K. \quad (16)$$

In comparison with ([15]), this is a morphism over $\text{Id} \Sigma$. If the function $f_s$ is $K$-valued, the morphisms ([15]) and ([16]) consist with each other and come to a familiar gauge morphism of the bundle $Y_\Sigma$.

In gravitation theory, we have the composite manifold

$$LX \rightarrow \Sigma \rightarrow X^4 \quad (17)$$

where $\Sigma$ is the quotient bundle ([5]), the associated composite spinor bundle $S$ ([8]) and the composite bundle

$$MX := (LX \times M)/L \rightarrow \Sigma \rightarrow X^4 \quad (18)$$

of Minkowski spaces.

Every principal isomorphism $\Phi$ of the linear frame bundle $LX$ yields the morphisms $\Phi_1$ ([14]) of the composite bundles $S$ and $MX$ and the morphism $\Phi_2$ ([16]) of the composite bundle $MX$.

Let $h$ be a section of the the quotient bundle $\Sigma$ ([5]). A principal isomorphism $\Phi$ ([14]) of $LX$ sends the reduced principal bundle $L^hX$ to some reduced principal bundle $L^{h'}X$. In other words, it transforms the tetrad field $h$ to the tetrad field

$$h'(x) = (\pi_{P\Sigma} \circ \Phi)(h^{-1}(x)).$$

The corresponding morphisms $\Phi_1$ of the composite bundles $S$ and $MX$ determines the bundle morphisms

$$\Phi_1 : S_h \rightarrow S_{h'}, \quad \Phi_1 : M^hX \rightarrow M^{h'}X \quad (19)$$

so that

$$\gamma_{h'} \circ \Phi_1 = \Phi_1 \circ \gamma_h$$

6
where $\gamma_h$ and $\gamma_{h'}$ are the representations \((7)\). Given an atlas \(\{z^h_\xi\}\) of the reduced principal bundle \(L^hX\), let us provide \(L^{h'}X\) and associated bundles with the atlas

\[ z^{h'}_\xi(x) = z^h_\xi(f_s(z^h_\xi(x))). \]  

(20)

With respect to these atlases, the morphisms \((19)\) read

\[ \Phi_1(h^a) = h^{\tilde{a}}, \quad \Phi_1(v_A(x)) = v'_{\tilde{A}}(x) \]

(21)

where \(\{h^a\}, \{v_A(x)\}, \{h^{\tilde{a}}\}\) and \(\{v'_{\tilde{A}}(x)\}\) are the corresponding bases of \(M^hX, S_h, M^{h'}X\) and \(S_{h'}\) respectively.

It should be noted that the bundles \(M^hX\) and \(M^{h'}X\) \((3)\) are isomorphic to the same cotangent bundle \(T^*X\), but provided with different Minkowski structures. Therefore, \(\Phi_1\) \((13)\) is an isomorphism of the cotangent bundle \(T^*X\). If \(h' \neq h\), there is however no isomorphism of the spinor bundle \(S_h\) so that the representations \(\gamma_h\) and \(\gamma_{h'}\) would be equivalent.

Let \(h\) be a section of the quotient bundle \(\Sigma\) and \(\Phi\) a principal isomorphism of \(LX\). In contrast with \(\Phi_1\), the corresponding morphism \(\Phi_2\) \((16)\) determines the morphism of \(M^hX\) to itself:

\[ \Phi_2 : (p \times e^a)/L \mapsto (p \times f_s(p)(e^a))/L, \quad p \in L^hX, \]

(22)

where \(\{e^a\}\) is the basis of the Minkowski space \(M\). Its coordinate expression relative to an atlas \(\{z^h_\xi\}\) is exactly \((1)\). The morphism \((22)\) does not alter the tetrad field \(h\), but transforms the cotangent bundle \(T^*X\) of a world manifold. We call it the deformation of the cotangent bundle. It is readily observed that, whenever \(h\), there is the 1:1 correspondence between these deformations and the section of the group bundle \(\tilde{L}X\).

Since deformations \((22)\) transform the cotangent bundle, but not the spinor bundle \(S_h\), one can say that they violate the correlation between the Lorentz structure and the spinor structure on a world manifold. As a consequence, the deformation \((22)\) yields another \(\gamma\)-matrix representation of cotangent vectors to a world manifold on the spinor bundle \(S_h\):

\[ \tilde{\gamma}_h = \gamma_h \circ \Phi_2 : (p \times (e^a \otimes v_A))/L \mapsto (p \times \gamma(f(p)e^a \otimes v_A))/L, \quad p \in L^hX, \]

(23)

where \(\{v_A\}\) is a basis of the standard fibre \(V\). The coordinate form of this representation is given by the expression \((10)\).

Thus, we can model the nongravitational deviations \((1)\) of a gravitational field by the deformations \((22)\) of the cotangent bundle. Since the representations \(\tilde{\gamma}_h\) and \(\gamma_h\) are defined on the same spinor bundle \(S_h\), these deviations exist

\[ s^a_{\; b} = \delta^a_{\; b} + \sigma^a_{\; b} \]

and their superposition \(\sigma + \sigma'\) can be defined.
In particular, the Dirac operator corresponding to the representation \( \tilde{\gamma}_h \) takes the form

\[
\tilde{D} = \tilde{\gamma}_h(dx^\mu)D_\mu \phi_h = h^a_\mu(x)s_a^b(x)\gamma^b D_\mu \phi_h = h^a_\mu(x)\gamma^a s^\nu_\mu(x)D_\nu \phi.
\] (24)

on sections \( \phi_h \) of the spinor bundle \( S_h \).

It should be noted that, given a holonomic atlas of \( LX \), it is the function \( s^\nu_\mu(x) \) which does not depend on a gravitational field, that is, the tetrad functions \( h^a_\mu \) and the deviations \( \sigma^\mu_\nu \) are independent dynamic variables.

Recall that, if the function \( f \) which determines the principal morphism \( \Phi \) is \( L \)-valued, the representations \( \tilde{\gamma}_h \) and \( \gamma_h \) are isomorphic. For an infinitesimal element \( \sigma \), we then have \( \sigma_{ab} = -\sigma_{ba} \).

Let us remark that the morphisms \( \Phi_1 \) and \( \Phi_2 \) are the equivalent transformations of the cotangent bundle regarded as the \( GL_4 \)-bundle. Therefore, if world symmetries are not broken (e.g., there are no fermion fields), the bundle \( T^*X \) "loses" the Lorentz structure and the transmutations

\[
M^h_x X = (p \times f(p)M)/L = (p \times f(p)T^*)/G
\]

\[
= (pf(p) \times T^*)/G \rightarrow (pf(p) \times M)/L = M^h'_x X
\]

of deviations \( \sigma \) of a gravitational field \( h \) into a new gravitational field \( h' \) may take place. Relative to the atlas (20), these transmutations take the coordinate form

\[
h^a_\mu = s_b^a h^b_\mu = \tilde{h}^a_\mu, \quad h^\mu_\alpha = \tilde{h}^\mu_\alpha.
\]

3 Deviations of metric fields

Without regard to fermion fields, one can choose metric functions \( g^{\mu\nu} \) as gravitational variables and examine their small deviations (1). However, if a space-time decomposition is considered, these deviations also fail to form a linear space in general.

Recall that, in virtue of the well-known theorems, if the structure group of \( LX \) is reducible to the structure Lorentz group, the latter, in turn, is reducible to its maximal compact subgroup \( SO(3) \). It follows that, for every reduced subbundle \( L^h X \), there exist a reduced subbundle \( F X \) of \( LX \) with the structure group \( SO(3) \) and the corresponding \( (3+1) \) space-time decomposition

\[
TX = FX \oplus T^0X
\]

of the tangent bundle of \( X^4 \) into a 3-dimensional spatial distribution \( FX \) and its time-like orthocomplement \( T^0X \). There is the 1:1 correspondence

\[
FX \mid \Omega = 0
\]

between the nonvanishing 1-forms \( \Omega \) on a manifold \( X \) and the 1-codimensional distributions on \( X \). Then, we get the following modification of the well-known theorem \( \text{[8]} \).
For every gravitational field $g$ on a world manifold $X^4$, there exists an associated pair $(FX, g^R)$ of a space-time distribution $FX$ generated by a tetrad 1-form

$$h^0 = h_\mu^0 dx^\mu$$

and a Riemannian metric $g^R$, so that

$$g^R = 2h^0 \otimes h^0 - g = h^0 \otimes h^0 + k$$

where $k$ is the Riemannian metric in the subbundle $FX$. Conversely, given a Riemannian metric $g^R$, every oriented smooth 3-dimensional distribution $FX$ with a generating form $\Omega$ is a space-time distribution compatible with the gravitational field $g$ given by expression (25) where

$$h_0 = \frac{\Omega}{|\Omega|}, \quad |\Omega|^2 = g^R(\Omega, \Omega) = g(\Omega, \Omega).$$

The triple $(g, FX, g^R)$ (25) sets up uniquely a space-time structure on a world manifold. The Riemannian metric $g^R$ in the triple (25) defines a $g$-compatible distance function on a world manifold $X^4$. Such a function brings $X^4$ into a metric space whose locally Euclidean topology is equivalent to the manifold topology on $X^4$.

Given a gravitational field $g$ and a $g$-compatible space-time distribution $FX$, let $k$ be a spatial part of the world metric $g$. If a world metric $g'$ results from some linear deviation

$$g' = g - \epsilon$$

of $g$, one can require the spatial parts $k'$ of $g'$ to be a linear deviation

$$k' = k + \epsilon_k$$

of $k$. It takes place if there exists a space-time distribution $FX$ compatible with both $g$ and $g'$. In this case, we have

$$k' = k + \epsilon + \frac{\Omega \otimes \Omega}{|g'(\Omega, \Omega)|^2} \epsilon(\Omega, \Omega),$$

$$\epsilon(\Omega, \Omega) = \epsilon^{\alpha\beta} \Omega_\alpha \Omega_\beta,$$

where $\Omega$ is a generating form of the distribution $FX$. For instance, given a triple $(g, FX, g^R)$, every linear deviation

$$g'^R = g^R - \epsilon^R$$

of the Riemannian metric $g^R$ in this triple involves the linear deviation

$$g' = g + \epsilon^R - 2\Omega \otimes \Omega \frac{\epsilon^R(\Omega, \Omega)}{|g^R(\Omega, \Omega)|^2}.$$
of the pseudo-Riemannian metric \( g \) in and its spatial part

\[
k' = k - \epsilon R - \Omega \otimes \Omega \frac{\epsilon R(\Omega, \Omega)}{|g^R(\Omega, \Omega)|^2}
\]

so that the triple \((g', FX, g'^R)\) is associated with the same distribution \( FX \).

Obviously, there are pseudo-Riemannian metrics \( g \) and \( g' \) which fail to admit the same space-time distribution. Their superposition is accompanied by superposition of space-time distributions which we face, e.g., in the case of gravitational singularities of the caustic type \([8, 9]\).

The deviations (9) also yields the corresponding nongravitational deviations of a metric field:

\[
\tilde{g}^{\mu\nu} = \tilde{h}_a^\mu \tilde{h}_b^\nu \eta^{ab} = s'^\alpha s'^\beta g^{\alpha\beta},
\]

\[
\tilde{g}_{\mu\nu} = \tilde{h}_a^\mu \tilde{h}_b^\nu \eta_{ab} = s'^\alpha s'^\beta g_{\alpha\beta},
\]

\[
\tilde{g}^{\mu\nu} - \tilde{g}^{\mu\alpha} \neq \delta^\nu_\alpha.
\]

The quantity \( \tilde{g} \) in this expression is not a world metric. In comparison with the relation (1), we have

\[
\tilde{g}^{\mu\nu} \approx g^{\mu\nu} + \sigma^{\mu\nu},
\]

\[
\tilde{g}_{\mu\nu} \approx g_{\mu\nu} + g_{\mu\alpha} g_{\nu\beta} \sigma^{\alpha\beta},
\]

for small deviations

\[
\sigma^{\mu\nu} = \sigma^a h^\mu_a h^{b\nu}.
\]

4 Dislocated manifolds

The deformations morphisms (22) of the cotangent bundle appear in the gauge theory of the translation group \([7, 8]\).

Let the tangent bundle \( TX \) be provided with the canonical structure of the affine tangent bundle. It is coordinatized by \((x^\mu, u^\lambda)\) where \( u^\alpha \neq \dot{x}^\alpha \) are the affine coordinates.

Every affine connection \( A \) on \( TX \) is brought into the sum

\[
A = \Gamma + \sigma = dx^\mu \otimes \frac{\partial}{\partial x^\mu} + \left( \Gamma^\alpha_\beta\mu u^\beta + \sigma^\alpha_\mu \right) \frac{\partial}{\partial u^\alpha}
\]

of a linear connection \( \Gamma \) and a soldering form

\[
\sigma = \sigma^\lambda_{\mu}(x) dx^\mu \otimes \frac{\partial}{\partial u^\lambda}
\]

which plays the role of a gauge translation potential.
In the conventional gauge theory of the affine group, one faces the problem of physical interpretation of both gauge translation potentials and sections \( u(x) \) of the affine tangent bundle \( TX \). In field theory, no fields possess the transformation law

\[ u(x) \rightarrow u(x) + a \]

under the Poincaré translations.

At the same time, one observes such fields in the gauge theory of dislocations [3] which is based on the fact that, in the presence of dislocations, displacement vectors \( u^k, k = 1, 2, 3 \), of small deformations are determined only with accuracy to gauge translations

\[ u^k \rightarrow u^k + a^k(x). \]

In this theory, gauge translation potentials \( \sigma^k_i \) describe the plastic distortion, the covariant derivatives

\[ D_i u^k = \partial_i u^k - \sigma^k_i \]

consist with the elastic distortion, and the strength

\[ \mathcal{F}_{ij}^k = \partial_i \sigma^k_j - \partial_j \sigma^k_i \]

is the dislocation density. Equations of the dislocation theory are derived from the gauge invariant Lagrangian density

\[ \mathcal{L} = \mu D_i u^k D^i u_k + \frac{\lambda}{2} D_i u^i D_m u^m - \epsilon \mathcal{F}_{ij}^k \mathcal{F}^{ij}_k \]  \hspace{1cm} (28)

where \( \mu \) and \( \lambda \) are the Lame coefficients of isotropic media. These equations however are not independent of each other since a displacement field \( u^k(x) \) can be removed by gauge translations and, thereby, it fails to be a dynamic variable.

In the spirit of the gauge dislocation theory, it was suggested that the gauge potentials of the Poincaré translations can describe new geometric structure (\textit{sui generis} dislocations) of a world manifold [1, 4].

Let the tangent bundle \( TX \) be provided with an affine connection [27]. We consider the following two morphisms:

- the morphism
  \[ \hat{o} : TX \rightarrow TTX \]
  which is the morphism defined by the connection \( A \) and restricted to the global zero section \( \hat{0} \) of \( TX \), that is,
  \[ \hat{o} = A \circ \hat{0} : \hat{0}(X) \times TX \rightarrow TTX; \]
• the geodesic morphism of $TX$ onto $X$:

$$\zeta : TX \ni u \to \zeta(x, u, 1) \in X, \quad x = \pi_X(u),$$

where $\zeta(x, u, s)$ is the geodesic defined by the linear part $\Gamma$ of the affine connection \[27\] through the point $x$ in the direction $u$. We shall call $\sigma$ the deformation field.

By dislocation of a world manifold $X$, we call the following bundle morphism over $\text{Id} \ X$:

$$\rho = T\zeta \circ \partial : TX \to TX,$$

$$\rho : \frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x^\mu} + (\Gamma^\alpha_{\beta\mu} u^\beta + \sigma^\alpha_{\mu}) \frac{\partial}{\partial u^\alpha} \to (\delta^\alpha_{\mu} + \sigma^\alpha_{\mu}) \frac{\partial}{\partial x^\alpha} = s^\alpha_{\mu} \frac{\partial}{\partial x^\alpha}. \tag{29}$$

Here, we use the relations

$$\zeta^\mu(x, \lambda u, 1) = \zeta^\mu(x, u, \lambda), \quad \lambda \in \mathbb{R},$$

$$\frac{\partial}{\partial u^\alpha} \zeta^\mu(x, u, 1)|_{u=0} = \delta^\mu_{\alpha},$$

and the expression

$$D_\mu u^\alpha|_{u=0} = (\partial_\mu u^\alpha + \Gamma^\alpha_{\beta\mu} u^\beta + \sigma^\alpha_{\mu})|_{u=0} = \sigma^\alpha_{\mu}$$

for the covariant derivatives of a displacement field $u$.

Let $Y \to X$ be a fibred manifold and $J^1Y$ the jet manifold of $Y$. The dislocation \[29\] gives rise to the morphism

$$J\rho : (x^\lambda, y^i, y^i_\lambda) \to (x^\lambda, y^i, s^\alpha_\lambda y^i_\alpha)$$

of $J^1Y$ over $\text{Id} \ Y$.

To define fields on a dislocated manifold, we therefore can replace sections $w(x)$ of $J^1Y \to X$ and $w(y)$ of $J^Y \to Y$ by sections

$$\tilde{w}(x) = (J\rho \circ w)(x), \quad \tilde{w}(y) = (J\rho \circ w)(y).$$

If $\phi$ is a section of the bundle $Y$, we have

$$\tilde{J^1}\phi = J\rho \circ J^1\phi,$$

$$\tilde{\phi}^i_\lambda = s^\alpha_\lambda(x) \partial_\alpha \phi^i(x).$$

Let $\Gamma$ be a connection on $Y$ and $D$ be the corresponding covariant differential. On a dislocated manifold, we get

$$\tilde{\Gamma}^i_\lambda(y) = s^\alpha_\lambda \Gamma^i_\alpha(y)$$

$$\tilde{D} = dx^\lambda \otimes \tilde{D}_\lambda = dx^\lambda \otimes s^\alpha_\lambda(x)(\partial_\alpha - \Gamma^i_\alpha(y) \partial_\lambda).$$
For instance, the Dirac operator on a dislocated manifold takes the form
\[ \tilde{L}_D = \gamma_h(dx^\lambda) \otimes \bar{D}_\lambda = s^\alpha \gamma_h(dx^\lambda) \otimes D_\alpha. \]

This operator looks like the Dirac operator \( [24] \) in the presence of deviations \( [9] \) of a tetrad gravitational field if the morphism \( [24] \) consists with the dual to the morphism \( [29] \). We therefore can apply Lagrangians of the field theory on dislocated manifolds to deviations \( [9] \).

A Lagrangian density of a scalar field \( \phi \) on the dislocated manifold reads
\[ L_{(m)} = \frac{1}{2} (g^\mu\nu s^\alpha_{\mu} s^\beta_{\nu} D_\alpha \phi D_\beta \phi - m^2 \phi^2) \sqrt{-g}. \]

Lagrangian densities \( L_{(g)} \) of the gravity and \( L_{(A)} \) of gauge potentials are constructed by means of the modified curvature
\[ \bar{R}^{ab}_{\mu\nu} = s^\epsilon_{\mu} s^\beta_{\nu} R^{ab}_{\epsilon\beta}, \]
and the modified strength
\[ \bar{F} = \bar{\rho} \circ F \circ J^1 A, \]
\[ \bar{F}^{m}_{\mu\nu} = s^\alpha_{\mu} s^\beta_{\nu} F^{m}_{\alpha\beta}. \]

The action functional and equations of motion of a point mass \( m_0 \) on the deformed manifold are given by expressions
\[ S = -m_0 \int (g_{\alpha\beta} s^\alpha_{\mu} s^\beta_{\nu} v^\mu v^\nu)^{1/2} ds, \]
\[ \frac{dv^\mu}{ds} + \bar{\Gamma}^\mu_{\alpha\beta} v^\alpha v^\beta = 0 \]
where \( v^\mu \) is the 4-velocity and the quantities \( \bar{\Gamma} \) look like the Christoffel symbols of the "metric"
\[ \bar{g}_{\mu\nu} = s^\alpha_{\mu} s^\beta_{\nu} g_{\alpha\beta}, \]
but the interval \( ds \) is defined by the true world metric \( g \).

Let us note that, on the dislocated manifold, a world metric and the volume form remain unchanged.

5 Gauge theory of deformation fields

The Lagrangian density \( L_{(\sigma)} \) of translation gauge potentials \( \sigma^\epsilon_{\mu} \) can not be built in the Yang-Mills form because the Lie algebra of the affine group does not admit an invariant nondegenerate bilinear form. To construct \( L_{(\sigma)} \), one can utilize the torsion
\[ F^{\alpha}_{\nu\mu} = D_\nu \sigma^\alpha_{\mu} - D_\mu \sigma^\alpha_{\nu}. \]
of the connection $\Gamma$ with respect to the soldering form $\sigma$.

The general form of a Lagrangian density $L(\sigma)$ is given by the expression

$$L(\sigma) = \frac{1}{2}[a_1 F_{\mu\nu} F^{\mu\nu} + a_2 F_{\mu\nu\sigma} F^{\mu\nu\sigma} + a_3 F_{\mu\nu} F^{\mu\nu} + a_4 \epsilon^{\mu\nu\sigma\gamma} F_{\mu\nu} F_{\gamma\delta} - \mu \sigma^{\mu\nu\sigma\mu} + \lambda \sigma^{\mu\nu} \sigma^{\mu\nu}] \sqrt{-g}$$

where $\epsilon^{\mu\nu\sigma\gamma}$ is the Levi-Civita tensor.

The mass-like term in $L(\sigma)$ is originated from the Lagrangian density (28) for displacement fields $u$ under the gauge condition $u = 0$.

It seems natural to require the component $t_{00}(\sigma)$ of a metric energy-momentum tensor of deformation fields $\sigma$ on the Minkowski space be positive. This condition implies the following constraints on the constants in $L(\sigma)$:

$$a_4 = 0, \quad a_1 \geq 0, \quad a_2 \geq 0, \quad a_3 + 2a_2 = 0, \quad \mu \geq 0, \quad \lambda \leq \frac{1}{4} \mu.$$ 

The Lagrangian density $L(\sigma)$ then takes the form

$$L(\sigma) = \frac{1}{2}[a_1 F_{\mu\nu} F^{\mu\nu} + a_2 F_{\mu\nu\sigma} F^{\mu\nu\sigma} - \mu \sigma^{\mu\nu\sigma\mu} + \lambda \sigma^{\mu\nu} \sigma^{\mu\nu}] \sqrt{-g}.$$ 

We here use the decomposition of the tensor $F^\lambda_{\mu\nu}$ in three irreducible parts

$$F^\lambda_{\mu\nu} = \tilde{F}^\lambda_{\mu\nu} + \frac{1}{3}(\delta^\lambda_{\nu} F_{\mu} - \delta^\lambda_{\mu} F_{\nu}) + \epsilon^\lambda_{\mu\nu\alpha} \tilde{F}^\alpha,$$

$$F_{\mu} = F^{\alpha}_{\mu\alpha}, \quad \tilde{F}^\alpha = \frac{1}{6} \epsilon^{\mu\nu\sigma} F_{\mu\nu\sigma},$$

where $F_{\mu}$ is the spur, $\tilde{F}^\alpha$ is the pseudo-spur, and $\tilde{F}$ is the spur-free part of the tensor $F$.

The total Lagrangian density includes Lagrangian densities $L(\text{m})$ of matter fields, $L(\text{A})$ of gauge potentials, and $L(\text{g})$ of a gravitational field. Matter sources of a deformation field $\sigma$ then are the following:

- the short canonical energy-momentum tensor of matter fields

  $$-\frac{\delta L(\text{m})}{\delta \sigma_{\mu\nu}} = -(s^{-1})_{\nu\beta} D_{\mu} \phi \frac{\partial L(\text{m})}{\partial D_{\beta} \phi} = -(s^{-1})_{\nu\beta} (T_{(\text{m})}^{\beta})_{\mu} + \delta_{\nu}^\beta L(\text{m})$$

  where $T_{(\text{m})}$ denotes a canonical energy-momentum tensor of matter fields;

- the short metric energy-momentum tensor $t(\text{A})$ of gauge potentials:

  $$-\frac{1}{\epsilon^2} \delta_{mn} \bar{g}^{\gamma\beta} s^\alpha_{\mu} F_{mn}^m F_{\alpha\beta} \sqrt{-g}$$

  where $\bar{g}^{\gamma\beta}$ is the "metric" (26);
• the curvature tensor

\[
\kappa^{-1} g_{\nu\alpha} s^\epsilon_{\gamma} R_{\mu
u}^{\alpha\gamma} \sqrt{-g}
\]

of a gravitational field.

Let us restrict ourselves to the case of a small field \(\sigma\). We neglect a gravitational field on the left-hand side of equations for \(\sigma\) and keep only \(\sigma\)-free terms in matter sources. Then, the Euler-Lagrange equations for a deformation field \(\sigma\) read

\[
\delta L(\sigma) \delta \sigma_{\mu\nu} = a_1 (\eta_{\mu\nu} \partial^\epsilon F^\alpha_{\epsilon\alpha} - \partial_{\mu} F^\alpha_{\nu\alpha}) + 2a_2 \partial^\epsilon (F_{\mu\nu} - F_{\nu\epsilon} + F_{\epsilon\nu}) - \mu \sigma_{\mu\nu} + \lambda \eta_{\mu\nu} \sigma^\alpha_{\alpha} = S_{\mu\nu},
\]

\[
S_{\mu\nu} = -(T_{(m)\nu\mu} + g_{\nu\mu} L_{(m)}) - \frac{1}{\epsilon^2} a_{mn} g^{\beta\gamma} F_{\mu\beta} F_{\nu\gamma} \sqrt{-g} + \kappa^{-1} R_{\mu\nu} \sqrt{-g}.
\] (30)

One can replace the gravitation term in the equation (30) by the right-hand side of the Einstein equations. Equations for \(\sigma\) then take the form

\[
\delta L(\sigma) \delta \sigma_{\mu\nu} = (t_{(m)\nu\mu} - T_{(m)\nu\mu}) - g_{\nu\mu}(L_{(m)} + \frac{1}{2} t_{(m)}) - g_{\mu\nu} L(\lambda).
\]

The equation (30) implies the equilibrium equation

\[
\partial^\nu \delta L(\sigma) \delta \sigma_{\mu\nu} = -\mu \partial^\nu \sigma_{\mu\nu} + \lambda \partial_{\mu} \sigma^\alpha_{\alpha} = \partial^\nu S_{\mu\nu}.
\] (31)

Note that the right-hand side of this equation is equal neither to zero nor to a gradient quantity in general. At the same time, this is a pure gradient quantity if matter sources of the field \(\sigma\) are gauge potentials and scalar fields. These facts result in the important condition

\[
\mu \neq 0, \quad \mu \neq 4\lambda.
\] (32)

Since equations (30) are linear, their solutions differ from each other in solutions of the free field equations. In the case of a free field \(\sigma\), equation (31) reads

\[
-\mu \partial^\nu \sigma_{\mu\nu} + \lambda \partial_{\mu} \sigma^\alpha_{\alpha} = 0.
\]

Taking into account this relation, one can bring equations (31) into the equations

\[
4a_2 \partial^\epsilon (w_{\mu\epsilon,\nu} + w_{\nu\epsilon,\mu} - w_{\nu\epsilon,\mu}) + 2a_1 (w_{\alpha,\nu\mu} - w_{\alpha,\mu\nu}) - \mu w_{\mu\nu} = 0,
\] (33)

\[
a_1 \left[ \frac{\lambda}{\mu} - 1 \right] \left[ \eta_{\mu\nu} \Box e - e_{,\mu\nu} \right] + 2a_1 (w_{\alpha,\nu\mu} + w_{\alpha,\mu\nu}) - \mu e_{\mu\nu} + \lambda \eta_{\mu\nu} \sigma = 0
\] (34)

where \(\Box = \partial^\alpha \partial_{\alpha}\) and

\[
e_{\mu\nu} = \frac{1}{2} (\sigma_{\mu\nu} + \sigma_{\nu\mu}), \quad w_{\mu\nu} = \frac{1}{2} (\sigma_{\mu\nu} - \sigma_{\nu\mu}), \quad e = \sigma^\alpha_{\alpha}.
\] (35)
It seems natural to choose the solution \( w = 0 \) of equations (33). Equations (34) then can be written in the form

\[
e_{\mu\nu} = \frac{\mu - \lambda}{3\mu} (\eta_{\mu\nu} e - \frac{3a_1}{\mu} e_{\mu\nu}),
\]
\[
\Box e + m^2 e = 0, \quad m^2 = \frac{\mu(\mu - 4\lambda)}{3a_1(\mu - \lambda)},
\]  

Equations (36) admit the following plane wave solutions

\[
e_{\mu\nu} = \frac{\mu - \lambda}{3\mu} \left[ \eta_{\mu\nu} + \frac{\mu - 4\lambda}{\mu - \lambda} \frac{p_{\mu} p_{\nu}}{p^2} \right] a(p) e^{ipx}, \quad p^2 = m^2.
\]

where the quantity \( m \) plays the role of a mass of deformation fields \( \sigma \). In virtue of the condition (32), this mass is not equal to zero. Equations (36) admit the following plane wave solutions

\[
e_{\mu\nu} = \frac{\mu - \lambda}{3\mu} \left[ \eta_{\mu\nu} + \frac{\mu - 4\lambda}{\mu - \lambda} \frac{p_{\mu} p_{\nu}}{p^2} \right] a(p) e^{ipx}, \quad p^2 = m^2.
\]

Now, let us consider a model of a small deformation field \( \sigma \) and a small gravitational field \( g = \eta + 2\varepsilon \) if their matter source is a motionless point mass \( M \). In this case, the right-hand side of equations (30) reads

\[
-\frac{1}{2} \eta_{\mu\nu} T_{(m)} = -\frac{1}{2} \eta_{\mu\nu} M \delta(r)
\]

where, by \((r, \phi, \theta)\), we denote spatial spherical coordinates.

Recalling the notations (35), we can rewrite equations (30) in the form

\[
-a_1 \left( e_{\nu,\alpha\mu} - e_{\mu,\alpha\nu} \right) + (4a_2 + \frac{a_1}{2}) \left( w_{\mu,\alpha\nu} - w_{\nu,\alpha\mu} \right) - 4a_2 \Box w_{\mu\nu} - \mu w_{\mu\nu} = 0,
\]

\[
a_1 \left[ \eta_{\mu\nu} (e_{\nu,\alpha\epsilon} - \Box e) - \frac{1}{2} (e_{\nu,\alpha\mu} + e_{\mu,\alpha\nu} + w_{\nu,\alpha\mu} + w_{\nu,\alpha\mu}) + e_{,\mu\nu} \right]
- \mu e_{\mu\nu} + \eta_{\mu\nu} \lambda e = -\frac{1}{2} \eta_{\mu\nu} M \delta(r).
\]

These equations admit the static spherically symmetric solution with the following nonzero components

\[
e_{rr} = -\frac{1}{\mu - \lambda} \left( 3\lambda e_{00} + \frac{1}{2} M \delta(r) \right),
\]
\[
e_{\theta\theta} = -e_{00} r^2, \quad e_{\phi\phi} = -e_{00} r^2 \sin^2 \theta,
\]
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} e_{00} \right) - m^2 e_{00} = -\frac{1}{6} \frac{\mu}{a_1(\mu - \lambda)} M \delta(r),
\]
\[
e_{00} = \frac{\mu M}{24\pi a_1(\mu - \lambda)} e^{-mr}
\]

where \( m \) is the mass (36).
Substituting this solution into equation (30), we obtain the modification of Newton’s gravitational potential

\[ \tilde{\epsilon} = \epsilon + \epsilon_{00} = -\frac{\kappa M}{8\pi r} \left( 1 - \frac{\kappa^{-1}\mu}{3a_1(\mu - \lambda)} e^{-mr} \right). \]

Such a "Yukawa type" modification of Newton’s gravitational potential is usually related to the hypothetical fifth fundamental force [5].

To contribute to standard gravitational effects, the fifth interaction must be as universal as gravity. Its matter source must contain a mass or other parts of the energy-momentum tensor. This interaction must be described by a massive classical field, though its mass is unusually small. A deformation field fits these conditions. For example, the mass (36) is expressed by means of constants of the Lagrangian density \( \mathcal{L}_\sigma \) where \( \mu \) and \( \lambda \) make the sense of coefficients of "elasticity" of a space-time.

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