**A-NUMERICAL RADIUS INEQUALITIES FOR SEMI-HILBERTIAN SPACE OPERATORS**

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**Abstract.** Let $A$ be a positive bounded operator on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. The semi-inner product $\langle x, y \rangle_A := \langle Ax, y \rangle$, $x, y \in \mathcal{H}$ induces a semi-norm $\| \cdot \|_A$ on $\mathcal{H}$. Let $\| T \|_A$ and $w_A(T)$ denote the $A$-operator semi-norm and the $A$-numerical radius of an operator $T$ in semi-Hilbertian space $(\mathcal{H}, \| \cdot \|_A)$, respectively. In this paper, we prove the following characterization of $w_A(T)$:

$$w_A(T) = \sup_{\alpha^2 + \beta^2 = 1} \left\| \frac{T + T^\sharp_A}{2} + \frac{\beta T - T^\sharp_A}{2i} \right\|_A,$$

where $T^\sharp_A$ is a distinguished $A$-adjoint operator of $T$. We then apply it to find upper and lower bounds for $w_A(T)$. In particular, we show that

$$\frac{1}{2} \left\| T \right\|_A \leq \max \left\{ \sqrt{1 - |\cos A T|}, \sqrt{2} \right\} w_A(T) \leq w_A(T),$$

where $|\cos A T|$ denotes the $A$-cosine of angle of $T$. Some upper bounds for the $A$-numerical radius of commutators, anticommutators, and products of semi-Hilbertian space operators are also given.

1. **Introduction and preliminaries**

Let $\mathbb{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and the corresponding norm $\| \cdot \|$. Let the symbol $I$ stand for the identity operator on $\mathcal{H}$. If $T \in \mathbb{B}(\mathcal{H})$, then we denote by $\mathcal{R}(T)$ the range of $T$, and by $\overline{\mathcal{R}(T)}$ the norm closure of $\mathcal{R}(T)$. Throughout this paper, we assume that $A \in \mathbb{B}(\mathcal{H})$ is a positive operator and that $P$ is the orthogonal projection onto $\overline{\mathcal{R}(A)}$. Recall that $A$ is called positive, denoted by $A \geq 0$, if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. Such an operator $A$ induces a positive semidefinite sesquilinear form $\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ defined by $\langle x, y \rangle_A = \langle Ax, y \rangle$, $x, y \in \mathcal{H}$. Denote by $\| \cdot \|_A$ the seminorm induced by $\langle \cdot, \cdot \rangle_A$, that is, $\| x \|_A = \sqrt{\langle x, x \rangle_A}$ for every $x \in \mathcal{H}$. It can be easily seen that $\| \cdot \|_A$ is a norm if and only if $A$ is an injective operator, and that $(\mathcal{H}, \| \cdot \|_A)$ is a complete space if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$. The semi-inner product $\langle \cdot, \cdot \rangle_A$ induces a semi-norm on a certain subspace of $\mathbb{B}(\mathcal{H})$. Namely, given $T \in \mathbb{B}(\mathcal{H})$, if there exists $c > 0$ such that $\| Tx \|_A \leq c \| x \|_A$ for all $x \in \overline{\mathcal{R}(A)}$, then it holds that

$$\| T \|_A := \sup_{x \in \overline{\mathcal{R}(A)}, x \neq 0} \frac{\| Tx \|_A}{\| x \|_A} = \inf \left\{ c > 0 : \| Tx \|_A \leq c \| x \|_A, x \in \mathcal{H} \right\} < \infty.$$
We set $\mathbb{B}^A(\mathcal{H}) := \{ T \in \mathbb{B}(\mathcal{H}) : \| T \|_A < \infty \}$. It can be seen that $\mathbb{B}^A(\mathcal{H})$ is not generally a subalgebra of $\mathbb{B}(\mathcal{H})$ and $\| T \|_A = 0$ if and only if $ATA = 0$. In addition, for $T \in \mathbb{B}^A(\mathcal{H})$, we have

$$\| T \|_A = \sup \left\{ |\langle Tx, y \rangle_A| : x, y \in \overline{\mathcal{R}(A)}, \| x \|_A = \| y \|_A = 1 \right\}.$$

An operator $T$ is called $A$-positive if $AT \geq 0$. Note that if $T$ is $A$-positive, then

$$\| T \|_A = \sup \left\{ \langle Tx, x \rangle_A : x \in \mathcal{H}, \| x \|_A = 1 \right\}.$$

For $T \in \mathbb{B}(\mathcal{H})$, an operator $R \in \mathbb{B}(\mathcal{H})$ is called an $A$-adjoint of $T$ if for every $x, y \in \mathcal{H}$, we have $\langle Tx, y \rangle_A = \langle x, Ry \rangle_A$, i.e., $AR = T^*A$. The existence of an $A$-adjoint operator is not guaranteed. In fact, an operator $T \in \mathbb{B}(\mathcal{H})$ may admit none, one or many $A$-adjoints. The set of all operators which admit $A$-adjoints is denoted by $\mathbb{B}_A(\mathcal{H})$. Note that $\mathbb{B}_A(\mathcal{H})$ is a subalgebra of $\mathbb{B}(\mathcal{H})$ which is neither closed nor dense in $\mathbb{B}(\mathcal{H})$. Moreover, the following inclusions $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}^A(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H})$ hold with equality if $A$ is injective and has a closed range.

If $T \in \mathbb{B}_A(\mathcal{H})$, the reduced solution of the equation $AX = T^*A$ is a distinguished $A$-adjoint operator of $T$, which is denoted by $T^{\sharp A}$; see [25]. Note that, $T^{\sharp A} = AT^*A$ in which $A^*$ is the Moore–Penrose inverse of $A$. It is useful that if $T \in \mathbb{B}_A(\mathcal{H})$, then $AT^{\sharp A} = T^*A$. An operator $T \in \mathbb{B}(\mathcal{H})$ is said to be $A$-selfadjoint if $AT$ is selfadjoint, i.e., $AT = T^*A$. Observe that if $T$ is $A$-selfadjoint, then $T \in \mathbb{B}_A(\mathcal{H})$. However it does not hold, in general, that $T = T^{\sharp A}$. For example, consider the operators $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$. Then simple computations show that $T$ is $A$-selfadjoint and $T^{\sharp A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq T$. More precisely, if $T \in \mathbb{B}_A(\mathcal{H})$, then $T = T^{\sharp A}$ if and only if $T$ is $A$-selfadjoint and $\mathcal{R}(T) \subseteq \overline{\mathcal{R}(A)}$. Notice that if $T \in \mathbb{B}_A(\mathcal{H})$, then $T^{\sharp A} \in \mathbb{B}_A(\mathcal{H})$, $(T^{\sharp A})^{\sharp A} = PTP$ and $((T^{\sharp A})^{\sharp A})^{\sharp A} = T^{\sharp A}$. In addition, $T^{\sharp A}T, TT^{\sharp A}$ are $A$-selfadjoint and $A$-positive and so we have

$$\| T^{\sharp A}T \|_A = \| TT^{\sharp A} \|_A = \| T \|_A = \| T^{\sharp A} \|_A^2.$$

Furthermore, if $T, S \in \mathbb{B}_A(\mathcal{H})$, then $(TS)^{\sharp A} = S^{\sharp A}T^{\sharp A}$, $\| TS \|_A \leq \| T \|_A \| S \|_A$ and $\| Tx \|_A \leq \| T \|_A \| x \|_A$ for all $x \in \mathcal{H}$.

For proofs and more facts about this class of operators, we refer the reader to [4, 5] and their references.

In recent years, several results covering some classes of operators on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ are extended to $(\mathcal{H}, \langle \cdot, \cdot \rangle_A)$ (see, e.g., [5, 7, 8, 13, 14, 18, 24, 30, 31, 35]).

The numerical radius of $T \in \mathbb{B}(\mathcal{H})$ is defined by

$$w(T) = \sup \left\{ |\langle Tx, x \rangle| : x \in \mathcal{H}, \| x \| = 1 \right\}.$$

This concept is useful in studying linear operators and has attracted the attention of many authors in the last few decades (e.g., see [11, 17, 22, 26, 36], and their references).
It is well known that $w(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$ such that for all $T \in \mathcal{B}(\mathcal{H})$,
\begin{equation}
\frac{1}{2} \|T\| \leq w(T) \leq \|T\|.
\end{equation}
(1.1)
The inequalities in (1.1) are sharp. The first inequality becomes an equality if $T^2 = 0$. The second inequality becomes an equality if $T$ is normal.

For more material about the numerical radius and other results on numerical radius inequality, see, e.g., [12, 17, 22, 23], and the references therein.

Some interesting numerical radius inequalities improving inequalities (1.1) have been obtained by several mathematicians (see, e.g., [1, 2, 10, 16, 21, 23, 32, 33]). Motivated by theoretical study and applications, there have been many generalizations of the numerical radius (e.g., see [3, 6, 9, 15, 17, 19, 20, 26, 27, 28, 29, 34]). One of these generalizations is the $A$-numerical radius of an operator $T \in \mathcal{B}(\mathcal{H})$ defined by
\begin{equation}
w_A(T) = \sup \left\{ \left| \langle Tx, x \rangle_A \right| : x \in \mathcal{H}, \|x\|_A = 1 \right\},
\end{equation}
see, e.g., [8].

Now, following by the Crawford number and the cosine of angle of an operator $T \in \mathcal{B}(\mathcal{H})$ introduced by Gustafson and Rao in [17], we introduce the following notations
\begin{equation}
c_A(T) = \inf \left\{ \left| \langle Tx, x \rangle_A \right| : x \in \mathcal{H}, \|x\|_A = 1 \right\},
\end{equation}
\begin{equation}
|\cos|_AT = \inf \left\{ \frac{\left| \langle Tx, x \rangle_A \right|}{\|Tx\|_A \|x\|_A} : x \in \mathcal{H}, \|Tx\|_A \|x\|_A \neq 0 \right\},
\end{equation}
and
\begin{equation}
|\sin|_AT = \sqrt{1 - |\cos|_AT^2}.
\end{equation}

The paper is organized as follows.

In Section 2, inspired by the numerical radius inequalities of bounded linear operators in [1], [2], [21], [23], [33] and by using some ideas of them, we first state a useful characterization of the $A$-numerical radius for $T \in \mathcal{B}_A(\mathcal{H})$ as follows:
\begin{equation}
w_A(T) = \sup \left\{ \left| \langle Tx, x \rangle_A \right| : x \in \mathcal{H}, \|x\|_A = 1 \right\},
\end{equation}
\begin{equation}
\alpha^2 + \beta^2 = 1 \left\| \frac{T + T^{\sharp_A}}{2} + \beta \frac{T - T^{\sharp_A}}{2i} \right\|_A.
\end{equation}
This expression was motivated by [23, Theorem 2.1]. We then apply it to find upper and lower bounds for the $A$-numerical radius of semi-Hilbertian space operators. Particularly, for $T \in \mathcal{B}_A(\mathcal{H})$ we prove that
\begin{equation}
w_A(T) \leq \frac{\sqrt{2}}{2} \sqrt{\|TT^{\sharp_A} + T^{\sharp_A} T\|_A} \leq \|T\|_{A},
\end{equation}
\begin{equation}
\frac{1}{2} \|T\|_A \leq \sqrt{\frac{w^2_A(T)}{2} + \frac{w_A(T)}{2} \sqrt{w^2_A(T) - c^2_A(T)}} \leq w_A(T),
\end{equation}
and
\begin{equation}
\frac{1}{2} \|T\|_A \leq \max \left\{ \left| \sin|_AT, \frac{\sqrt{2}}{2} \right\} w_A(T) \leq w_A(T).
\end{equation}
In Section 3, some upper bounds for the $A$-numerical radius of products of semi-Hilbertian space operators are given. In particular, for $T, S \in \mathcal{B}_A(\mathcal{H})$ we show that

$$w_A(TS) \leq w_A(T)\|S\|_A + \frac{1}{2}w_A\left( (TS)^{\sharp A} \pm T^{\sharp A}S \right) \leq 2w_A(T)\|S\|_A.$$  

In the last section we present some upper bounds for the $A$-numerical radius of commutators and anticommutators of semi-Hilbertian space operators. Particularly, for $T, S \in \mathcal{B}_A(\mathcal{H})$ we prove that

$$w_A(TS^{\sharp A} \pm ST^{\sharp A}) \leq \|T^{\sharp A}T + SS^{\sharp A}\|_A.$$  

Our results generalize recent numerical radius inequalities of bounded linear operators due to Kittaneh et al. [1, 2, 12, 22, 23, 33].

2. Upper and lower bounds of the $A$-numerical radius of operators

We start our work with the following lemmas. To establish the first lemma we use some ideas of [17, Theorem 1.3-1].

**Lemma 2.1.** Let $T \in \mathcal{B}_A(\mathcal{H})$ be an $A$-selfadjoint operator. Then

$$w_A(T) = \|T\|_A.$$  

**Proof.** Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. By the Cauchy-Schwarz inequality, we have

$$\left| \langle Tx, x \rangle_A \right| \leq \|Tx\|_A \|x\|_A \leq \|T\|_A,$$

and hence $w_A(T) = \sup \left\{ \left| \langle Tx, x \rangle_A \right| : x \in \mathcal{H}, \|x\|_A = 1 \right\} \leq \|T\|_A$.

Moreover, since $T$ is an $A$-selfadjoint operator, for every $y, z \in \mathcal{H}$ such that $\|y\|_A = \|z\|_A = 1$ we have

$$\langle T(y + z), y + z \rangle_A = \langle Ty, y \rangle_A + 2\text{Re}\langle Ty, z \rangle_A + \langle Tz, z \rangle_A$$

and

$$\langle T(y - z), y - z \rangle_A = \langle Ty, y \rangle_A - 2\text{Re}\langle Ty, z \rangle_A + \langle Tz, z \rangle_A.$$  

Consequently, we deduce

$$\text{Re}\langle Ty, z \rangle_A = \frac{1}{4}\left( \langle T(y + z), y + z \rangle_A - \langle T(y - z), y - z \rangle_A \right).$$

So, we obtain

$$\text{Re}\langle Ty, z \rangle_A \leq \frac{w_A(T)}{4}(\|y + z\|^2_A + \|y - z\|^2_A).$$

Then it follows from parallelogram law that

$$\left| \text{Re}\langle Ty, z \rangle_A \right| \leq \frac{w_A(T)}{4}(2\|y\|^2_A + 2\|z\|^2_A) = w_A(T). \quad (2.1)$$

Now, consider the polar decomposition $\langle Ty, z \rangle_A = e^{i\theta}\left| \langle Ty, z \rangle_A \right|$ with $\theta \in \mathbb{R}$. By replacing $z$ by $e^{i\theta}z$ in (2.1), we get $\left| \langle Ty, z \rangle_A \right| \leq \text{Re}\langle Ty, e^{i\theta}z \rangle_A \leq w_A(T)$. From this it follows that $\|T\|_A = \sup \left\{ \left| \langle Ty, z \rangle_A \right| : y, z \in \mathcal{H}, \|y\|_A = \|z\|_A = 1 \right\} \leq w_A(T)$ and consequently $w_A(T) = \|T\|_A$. \qed
Remark 2.2. Note that for an arbitrary operator $T$ of $\mathbb{B}_A(H)$, we have
\[
0 \leq \|T\|^2_A - w_A^2(T) \leq \inf_{\gamma \in \mathbb{C}} \left\{ \|T + \gamma I\|^2_A - c_A^2(T + \gamma I) \right\}.
\]
Indeed, if $x \in H$ with $\|x\|_A = 1$, then simple computations show that
\[
\|Tx\|^2_A - |\langle Tx, x \rangle_A|^2 = \|Tx - \gamma x\|^2_A - |\langle Tx - \gamma x, x \rangle_A|^2 \quad (\gamma \in \mathbb{C}),
\]
whence
\[
\|Tx\|^2_A - |\langle Tx, x \rangle_A|^2 \leq \|T - \gamma I\|^2_A - c_A^2(T - \gamma I) \quad (\gamma \in \mathbb{C}).
\]
Thus
\[
\|Tx\|^2_A - |\langle Tx, x \rangle_A|^2 \leq \inf_{\gamma \in \mathbb{C}} \left\{ \|T - \gamma I\|^2_A - c_A^2(T - \gamma I) \right\}.
\]
Taking the supremum in the above inequality over $x \in H$, $\|x\|_A = 1$, we deduce the desired inequality.

The second lemma is stated as follows.

Lemma 2.3. Let $T \in \mathbb{B}_A(H)$. For every $\theta \in \mathbb{R}$,
\[
w_A \left( \frac{e^{i\theta}T + (e^{i\theta}T)^{z_A}}{2} \right) = \left\| \frac{e^{i\theta}T + (e^{i\theta}T)^{z_A}}{2} \right\|_A.
\]

Proof. Let $\theta \in \mathbb{R}$. We have
\[
\left( \frac{e^{i\theta}T}2 \right)^{z_A} + \left( \frac{(e^{i\theta}T)^{z_A}}2 \right)^{z_A} = \left( \frac{(e^{i\theta}T)^{z_A}}2 \right)^{z_A} + \left( \frac{e^{i\theta}T}2 \right)^{z_A}.\text{ Hence}
\]
is an $A$-selfadjoint operator. So, by Lemma 2.1 we get
\[
w_A \left( \frac{e^{i\theta}T}2 \right)^{z_A} + \left( \frac{(e^{i\theta}T)^{z_A}}2 \right)^{z_A} = \left\| \frac{e^{i\theta}T}2 + \left( \frac{(e^{i\theta}T)^{z_A}}2 \right)^{z_A} \right\|_A. \quad (2.2)
\]
Since $w_A(R^{z_A}) = w_A(R)$ and $\|R^{z_A}\|_A = \|R\|_A$ for every $R \in \mathbb{B}_A(H)$, from (2.2) it follows that
\[
w_A \left( \frac{e^{i\theta}T + (e^{i\theta}T)^{z_A}}2 \right) = \left\| \frac{e^{i\theta}T + (e^{i\theta}T)^{z_A}}2 \right\|_A.
\]

We now state the third lemma, which will be used to prove Theorem 2.5.

Lemma 2.4. Let $T \in \mathbb{B}_A(H)$ and $x \in H$. Then
\[
\sup_{\theta \in \mathbb{R}} \left| \frac{e^{i\theta}T + (e^{i\theta}T)^{z_A}}2 x, x \right|_A = |\langle Tx, x \rangle_A|.
\]

Proof. Let $\theta \in \mathbb{R}$. We have
\[
\left| \frac{e^{i\theta}T + (e^{i\theta}T)^{z_A}}2 x, x \right|_A = \frac12 |e^{i\theta} \langle Tx, x \rangle_A + e^{-i\theta} \langle T^{z_A}x, x \rangle_A|
\]
\[
= \frac12 |e^{i\theta} \langle Tx, x \rangle_A + e^{-i\theta} \langle T^{z_A}x, x \rangle_A|.
\]
Thus
\[
\left| \left\langle \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2}, xx \right\rangle \right|_A = \left| \text{Re} \left( e^{i\theta} \langle Tx, x \rangle_A \right) \right|.
\] (2.3)

From this it follows that
\[
\left| \left\langle \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2}, xx \right\rangle \right| \leq \left| \langle Tx, x \rangle_A \right|,
\]
whence
\[
\sup_{\theta \in \mathbb{R}} \left| \left\langle \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2}, xx \right\rangle \right| \leq \left| \langle Tx, x \rangle_A \right|.
\] (2.4)

Now, if \( |\langle Tx, x \rangle_A| = 0 \), then from (2.3) we obtain
\[
\left| \left\langle \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2}, xx \right\rangle \right| = 0 \] and so
\[
\sup_{\theta \in \mathbb{R}} \left| \left\langle \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2}, xx \right\rangle \right| = 0 = |\langle Tx, x \rangle_A|.
\]

If \( |\langle Tx, x \rangle_A| \neq 0 \), then we put \( e^{i\theta_0} = \frac{\langle Tx, x \rangle_A}{\langle Tx, x \rangle_A} \). Therefore, by (2.3), we obtain
\[
\left| \left\langle \frac{e^{i\theta_0}T + (e^{i\theta_0}T)^{\sharp_A}}{2}, xx \right\rangle \right| = \left| \text{Re} \left( e^{i\theta_0} \langle Tx, x \rangle_A \right) \right| = |\langle Tx, x \rangle_A|.
\] (2.5)

From (2.4) and (2.5) it follows that
\[
\sup_{\theta \in \mathbb{R}} \left| \left\langle \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2}, xx \right\rangle \right| = |\langle Tx, x \rangle_A|.
\]

\[ \square \]

Now, we are in a position to state a useful characterization of the \( A \)-numerical radius for semi-Hilbertian space operators.

**Theorem 2.5.** Let \( T \in \mathbb{B}_A(\mathcal{H}) \). Then
\[
w_A(T) = \sup_{\theta \in \mathbb{R}} \left\| \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2} \right\|_A.
\]

**Proof.** Let \( \theta \in \mathbb{R} \). By Lemma 2.3 it follows that
\[
w_A \left( \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2} \right) = \left\| \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2} \right\|_A.
\]

Therefore, by Lemma 2.4 we conclude that
\[
\sup_{\theta \in \mathbb{R}} \left| \left\langle \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2}, xx \right\rangle \right|_A = \sup_{\theta \in \mathbb{R}} w_A \left( \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2} \right)
= \sup_{\theta \in \mathbb{R}} \sup_{\|x\|_A = 1} \left| \left\langle \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2}, xx \right\rangle \right|_A
= \sup_{\|x\|_A = 1} \left| \langle Tx, x \rangle_A \right| = w_A(T).
\]

\[ \square \]
Here we present one of the main results of this section.

**Theorem 2.6.** Let \( T \in \mathbb{B}_A(\mathcal{H}) \). Then for \( \alpha, \beta \in \mathbb{R} \),
\[
w_A(T) = \sup_{\alpha^2 + \beta^2 = 1} \left\| \alpha \frac{T + T^{\sharp A}}{2} + \beta \frac{T - T^{\sharp A}}{2i} \right\|_A.
\]

**Proof.** Let \( \theta \in \mathbb{R} \). Put \( \alpha = \cos \theta \) and \( \beta = -\sin \theta \). We have
\[
e^{i\theta}T + (e^{i\theta}T)^{\sharp A} = \frac{(\cos \theta + i \sin \theta)T + (\cos \theta - i \sin \theta)T^{\sharp A}}{2}
\]
\[
= \cos \theta \frac{T + T^{\sharp A}}{2} - \sin \theta \frac{T - T^{\sharp A}}{2i}
\]
\[
= \alpha T + \beta T^{\sharp A} + \beta T - T^{\sharp A}i.
\]
Therefore
\[
\sup_{\theta \in \mathbb{R}} \left\| \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp A}}{2} \right\|_A = \sup_{\alpha^2 + \beta^2 = 1} \left\| \alpha \frac{T + T^{\sharp A}}{2} + \beta \frac{T - T^{\sharp A}}{2i} \right\|_A,
\]
and hence, by Theorem 2.5, we obtain
\[
w_A(T) = \sup_{\alpha^2 + \beta^2 = 1} \left\| \alpha \frac{T + T^{\sharp A}}{2} + \beta \frac{T - T^{\sharp A}}{2i} \right\|_A.
\]

\( \square \)

As a consequence of Theorem 2.6, we have the following result.

**Corollary 2.7.** Let \( T \in \mathbb{B}_A(\mathcal{H}) \). Then
\[
\max \left\{ \left\| \frac{T + T^{\sharp A}}{2} \right\|_A, \left\| \frac{T - T^{\sharp A}}{2i} \right\|_A \right\} \leq w_A(T).
\]

**Proof.** By setting \( (\alpha, \beta) = (1, 0) \) and \( (\alpha, \beta) = (0, 1) \) in Theorem 2.6, the result follows. \( \square \)

The following result is another consequence of Theorem 2.6.

**Corollary 2.8.** Let \( T \in \mathbb{B}_A(\mathcal{H}) \). Then
\[
\frac{1}{2} \|T\|_A \leq w_A(T) \leq \|T\|_A.
\]

**Proof.** Clearly, \( w_A(T) \leq \|T\|_A \). On the other hand, by using Corollary 2.7, we get
\[
\|T\|_A = \left\| \frac{T + T^{\sharp A}}{2} + i \frac{T - T^{\sharp A}}{2i} \right\|_A \leq \left\| \frac{T + T^{\sharp A}}{2} \right\|_A + \left\| \frac{T - T^{\sharp A}}{2i} \right\|_A \leq 2w_A(T).
\]
Hence \( \frac{1}{2} \|T\|_A \leq w_A(T) \). \( \square \)

**Remark 2.9.** Corollary 2.8 has recently been proved by Baklouti et al. in [8]. Our approach here is different from theirs.

In the following theorem, we give a improvement of the second inequality in (2.6).
Theorem 2.10. Let $T \in \mathcal{B}_A(\mathcal{H})$. Then

$$w_A(T) \leq \frac{\sqrt{2}}{2} \sqrt{\|TT^*A + T^*A T\|_A} \leq \|T\|_A.$$ 

Proof. Put $M := \frac{T^*A + (T^*A)_A}{2}$ and $N := \frac{T^*A - (T^*A)_A}{2i}$. Then $T^*A = M + iN$. Also, simple computations show that

$$M^2 + N^2 = \left(\frac{T^*A + T^*A_T}{2}\right) = \left(\frac{T T^*A + T^*A T}{2}\right).$$

Since $\|R^*A\|_A = \|R\|_A$ for every $R \in \mathcal{B}_A(\mathcal{H})$, hence

$$\|M^2 + N^2\|_A = \frac{1}{2} \|TT^*A + T^*A T\|_A. \quad (2.7)$$

Now, let $x \in \mathcal{H}$ with $\|x\|_A = 1$. We have

$$|\langle x, Tx \rangle_A|^2 = |\langle T^*A x, x \rangle_A|^2$$

$$= |\langle (M + iN)x, x \rangle_A \langle x, (M + iN)x \rangle_A|$$

$$= \left|\langle (Mx, x) + i \langle Nx, x \rangle_A \langle x, Mx \rangle_A - i \langle x, Nx \rangle_A \right|^2$$

$$\leq \langle Mx, Mx \rangle_A + \langle Nx, Nx \rangle_A \quad \text{(by the Cauchy-Schwarz inequality)}$$

$$= \langle M^2 x, x \rangle_A + \langle N^2 x, x \rangle_A \quad \text{(since $M^*A = M$ and $N^*A = N$)}$$

$$= \langle (M^2 + N^2) x, x \rangle_A$$

$$\leq \|M^2 + N^2\|_A = \frac{1}{2} \|TT^*A + T^*A T\|_A \quad \text{(by (2.7))}$$

Hence

$$w_A^2(T) = \sup_{\|x\|_A = 1} |\langle x, Tx \rangle_A|^2 \leq \frac{1}{2} \|TT^*A + T^*A T\|_A,$$

or equivalently,

$$w_A(T) \leq \frac{\sqrt{2}}{2} \sqrt{\|TT^*A + T^*A T\|_A}. \quad (2.8)$$

Further, since $\|TT^*A\|_A = \|T^*A T\|_A = \|T\|_A^2$, by the triangle inequality we obtain

$$\frac{\sqrt{2}}{2} \sqrt{\|TT^*A + T^*A T\|_A} \leq \frac{\sqrt{2}}{2} \sqrt{\|TT^*A\|_A + \|T^*A T\|_A} = \|T\|_A.$$ 

Thus

$$\frac{\sqrt{2}}{2} \sqrt{\|TT^*A + T^*A T\|_A} \leq \|T\|_A. \quad (2.9)$$

By (2.8) and (2.9) we deduce the desired result. \qed

Next, we present another improvement of the second inequality in (2.6).
Theorem 2.11. Let $T \in \mathbb{B}_A(H)$. Then

$$w_A(T) \leq \frac{1}{2} \sqrt{\|TT^\sharp_A + T^\sharp_AT\|_A + 2w_A(T^2)} \leq \|T\|_A.$$ 

Proof. By Theorem 2.5, we have

$$w_A(T) = \sup_{\theta \in \mathbb{R}} \left\|e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}\right\|_A$$

$$= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\|(e^{i\theta}T)^{\sharp_A} + (e^{i\theta}T)^{\sharp_A}\right\|_A$$

(since $\|R\|_A = \|RR^{\sharp_A}\|_A$ for every $R \in \mathbb{B}_A(H)$)

$$= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \sqrt{\left\||(e^{i\theta}T)^{\sharp_A} + (e^{i\theta}T)^{\sharp_A}\right\|^2}$$

(since $\|R\|^2_A = \|RR^{\sharp_A}\|^2_A$ for every $R \in \mathbb{B}_A(H)$)

$$\leq \frac{1}{2} \sup_{\theta \in \mathbb{R}} \left\|TT^\sharp_A + T^\sharp_AT\right\|_A$$

(since $\|R^2\|_A = \|R\|_A$ for every $R \in \mathbb{B}_A(H)$)

$$\leq \frac{1}{2} \sqrt{\left\|TT^\sharp_A + T^\sharp_AT\right\|_A + 2\left\|\frac{e^{2i\theta}T^2 + (e^{2i\theta}T^2)^{\sharp_A}}{2}\right\|_A}$$

(by Theorem 2.5)

$$\leq \frac{1}{2} \sqrt{\|TT^\sharp_A\|_A + \|T^\sharp_AT\|_A + 2w_A(T^2)}$$

(since $\|RR^{\sharp_A}\|_A = \|R\|^2_A$ for every $R \in \mathbb{B}_A(H)$)

$$\leq \frac{\sqrt{2}}{2} \sqrt{\|T\|^2_A + w_A(T^2)}$$

(by Corollary 2.8)

$$\leq \frac{\sqrt{2}}{2} \sqrt{\|T\|^2_A + \|T\|^2_A} = \|T\|_A,$$

which proves the desired inequalities. □

In the following theorem, we establish an improvement of the first inequality in (2.6).
Theorem 2.12. Let \( T \in B_A(\mathcal{H}) \). Then
\[
\frac{1}{2} \| T \|_A \leq \frac{1}{2} \sqrt{\| TT^* + T^* T \|_A + 2c_A(T^2)} \leq w_A(T).
\]

Proof. Let \( x \in \mathcal{H} \) with \( \| x \|_A = 1 \). Suppose that \( \langle T^{*A} T x, x \rangle_A = e^{-i\theta} \langle T^{*A} T^{*A} x, x \rangle_A \) for some real number \( \theta \). Then, we have
\[
e^{2i\theta} \langle (T^{*A})^{*A} (T^{*A})^{*A} x, x \rangle_A = e^{2i\theta} \langle T^{*A} T^{*A} x, x \rangle_A = \langle T^{*A} T^{*A} x, x \rangle_A = \langle x, T^2 x \rangle_A.
\]

Thus
\[
e^{-2i\theta} \langle T^{*A} T^{*A} x, x \rangle_A = \langle x, T^2 x \rangle_A = e^{2i\theta} \langle (T^{*A})^{*A} (T^{*A})^{*A} x, x \rangle_A.
\]
So, by Theorem 2.5, we obtain
\[
4w_A^2(T) \geq \| e^{i\theta} T + (e^{i\theta} T)^{*A} \|^2_A
\]
\[
= \| (e^{i\theta} T)^{*A} + ((e^{i\theta} T)^{*A})^{*A} \|^2_A
\]
\[
= \| (e^{i\theta} T)^{*A} + ((e^{i\theta} T)^{*A})^{*A} \|_A
\]
\[
= \| T^{*A} (T^{*A})^{*A} + (T^{*A})^{*A} T^{*A} + e^{-2i\theta} T^{*A} T^{*A} + e^{2i\theta} (T^{*A})^{*A} (T^{*A})^{*A} \|_A
\]
\[
\geq \| \langle T^{*A} (T^{*A})^{*A} + (T^{*A})^{*A} T^{*A} + e^{-2i\theta} T^{*A} T^{*A} + e^{2i\theta} (T^{*A})^{*A} (T^{*A})^{*A} x, x \rangle_A
\]
\[
= \| \langle T^{*A} (T^{*A})^{*A} + (T^{*A})^{*A} T^{*A} x, x \rangle_A + e^{2i\theta} \langle (T^{*A})^{*A} (T^{*A})^{*A} x, x \rangle_A
\]
\[
= \| \langle T^{*A} (T^{*A})^{*A} + (T^{*A})^{*A} T^{*A} x, x \rangle_A + 2 \langle x, T^2 x \rangle_A \|
\]
\[
\geq \| \langle T^{*A} (T^{*A})^{*A} + (T^{*A})^{*A} T^{*A} x, x \rangle_A + 2c_A(T^2) \|
\]
\[
\geq \frac{1}{2} \sqrt{\| TT^{*A} + T^{*A} T \|_A + 2c_A(T^2)} \leq w_A(T).
\]

Taking the supremum over \( x \in \mathcal{H} \) with \( \| x \|_A = 1 \) in the above inequality we get
\[
\frac{1}{2} \sqrt{\| TT^{*A} + T^{*A} T \|_A + 2c_A(T^2)} \leq w_A(T), \tag{2.11}
\]

Furthermore, since \( T^{*A} T \) is an \( A \)-positive operator, from \( \| TT^{*A} + T^{*A} T \|_A \geq \| TT^{*A} \|_A = \| T \|^2_A \) it follows that
\[
\frac{1}{2} \sqrt{\| TT^{*A} + T^{*A} T \|_A + 2c_A(T^2)} \geq \frac{1}{2} \sqrt{\| TT^{*A} + T^{*A} T \|_A + 2c_A(T^2)} \geq \frac{1}{2} \| T \|_A. \tag{2.12}
\]
Let $\text{Theorem 2.13}$. Now, by (2.11) and (2.12) we conclude that

$$\frac{1}{2} \|T\|_A \leq \frac{1}{2} \sqrt{\|TT^*A + T^*A\|_A + 2c_A(T^2)} \leq w_A(T).$$

\[\square\]

Now, we give another improvement of the first inequality in (2.6).

**Theorem 2.13.** Let $T \in \mathcal{B}_A(\mathcal{H})$. Then

$$\frac{1}{2} \|T\|_A \leq \sqrt{\frac{w_A^4(T)}{2} + \frac{w_A(T)}{2} \sqrt{w_A^2(T) - c_A^2(T)}} \leq w_A(T).$$

**Proof.** Clearly, $\sqrt{\frac{w_A^4(T)}{2} + \frac{w_A(T)}{2} \sqrt{w_A^2(T) - c_A^2(T)}} \leq w_A(T)$. Now, let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Suppose that $\|Tx, x\|_A = e^{i\theta} \langle Tx, x \rangle_A$ for some real number $\theta$. Put $M := \frac{e^{i\theta} + e^{-i\theta}T}{2}$ and $N := \frac{e^{i\theta} - e^{-i\theta}T}{2i}$. Then $M + iN = e^{i\theta}T$ and

$$\langle Mx, x \rangle_A + i\langle Nx, x \rangle_A = e^{i\theta} \langle Tx, x \rangle_A = \|Tx, x\|_A \geq 0.$$

It follows from $\langle Nx, x \rangle_A = \text{Im}(\langle e^{i\theta}Tx, x \rangle_A) \in \mathbb{R}$ that

$$\langle e^{i\theta}Tx, x \rangle_A = \langle Mx, x \rangle_A, \quad \langle Nx, x \rangle_A = 0.$$

So, we have

$$\frac{1}{4} \|Tx\|_A^2 = \frac{1}{4} \left( \|e^{i\theta}Tx - \langle e^{i\theta}Tx, x \rangle_Ax\|_A^2 + \|\langle Tx, x \rangle_A\|^2 \right)$$

$$= \frac{1}{4} \left( \|Mx - \langle Mx, x \rangle_Ax + iNx\|_A^2 + \|\langle Tx, x \rangle_A\|^2 \right)$$

(since $\langle Nx, x \rangle_A = 0$)

$$\leq \frac{1}{4} \left( \|Mx - \langle Mx, x \rangle_A\|_A + \|Nx\|_A \right)^2 + \|\langle Tx, x \rangle_A\|^2$$

$$= \frac{1}{4} \left( \sqrt{\|Mx\|_A^2 - \|\langle Mx, x \rangle_A\|^2 + \|Nx\|_A} \right)^2 + \|\langle Tx, x \rangle_A\|^2$$

$$= \frac{1}{4} \left( \sqrt{\|Mx\|_A^2 - \|\langle e^{i\theta}Tx, x \rangle_A\|^2 + \|Nx\|_A} \right)^2 + \|\langle Tx, x \rangle_A\|^2$$

(since $\langle Mx, x \rangle_A = \langle e^{i\theta}Tx, x \rangle_A$)

$$\leq \frac{1}{4} \left( \sqrt{\|M\|_A^2 - \|\langle Tx, x \rangle_A\|^2 + \|N\|_A} \right)^2 + \|\langle Tx, x \rangle_A\|^2$$

$$\leq \frac{1}{4} \left( \sqrt{w_A^2(T) - \|\langle Tx, x \rangle_A\|^2 + w_A(T)} \right)^2 + \|\langle Tx, x \rangle_A\|^2$$

(since $\|M\|_A, \|N\|_A \leq w_A(e^{i\theta}T) = w_A(T)$)

$$= \frac{w_A^2(T)}{2} + \frac{w_A(T)}{2} \sqrt{w_A^2(T) - \|\langle Tx, x \rangle_A\|^2}.$$
Hence
\[ \frac{1}{2} \|Tx\|_A \leq \sqrt{\frac{w_A^2(T)}{2} + \frac{w_A(T)}{2} \sqrt{w_A^2(T) - |\langle Tx, x \rangle_A|^2}} \quad (\|x\|_A = 1), \quad (2.13) \]
which implies
\[ \frac{1}{2} \|Tx\|_A \leq \sqrt{\frac{w_A^2(T)}{2} + \frac{w_A(T)}{2} \sqrt{w_A^2(T) - c_A^2(T)}}. \]
Taking the supremum over \( x \in \mathcal{H} \) with \( \|x\|_A = 1 \) in the above inequality we get
\[ \frac{1}{2} \|T\|_A \leq \sqrt{\frac{w_A^2(T)}{2} + \frac{w_A(T)}{2} \sqrt{w_A^2(T) - c_A^2(T)}}. \]
\[ \square \]

We end this section with a considerable improvement of the first inequality in \((2.6)\).

**Theorem 2.14.** Let \( T \in \mathbb{B}_A(\mathcal{H}) \). Then
\[ \frac{1}{2} \|T\|_A \leq \max \left\{ \left| \sin \right|_A T, \frac{\sqrt{2}}{2} \right\} w_A(T) \leq w_A(T). \]

**Proof.** Obviously,
\[ \max \left\{ \left| \sin \right|_A T, \frac{\sqrt{2}}{2} \right\} w_A(T) \leq w_A(T). \]
Furthermore, by \((2.13)\) we have
\[ \frac{1}{2} \|Tx\|_A \leq \sqrt{\frac{w_A^2(T)}{2} + \frac{w_A(T)}{2} \sqrt{w_A^2(T) - |\langle Tx, x \rangle_A|^2}} \quad (\|x\|_A = 1), \]
and hence
\[ \frac{1}{2} \|Tx\|_A \leq \sqrt{\frac{w_A^2(T)}{2} + \frac{w_A(T)}{2} \sqrt{w_A^2(T) - \|Tx\|^2_A \cos^2_A T}}. \]
From this it follows that
\[ \|Tx\|^2_A - 2w_A^2(T) \leq 2w_A(T) \sqrt{w_A^2(T) - \|Tx\|^2_A \cos^2_A T}. \quad (2.14) \]
We consider two cases.

Case 1. \( \|Tx\|^2_A - 2w_A^2(T) \leq 0 \). Then we reach that \( \|Tx\|_A \leq \sqrt{2} w_A(T) \) and so
\[ \frac{1}{2} \|T\|_A \leq \frac{\sqrt{2}}{2} w_A(T). \quad (2.15) \]

Case 2. \( \|Tx\|^2_A - 2w_A^2(T) > 0 \). By \((2.14)\) it follows that
\[ \|Tx\|^4_A - 4 \|Tx\|^2_A w_A^2(T) + 4w_A^4(T) \leq 4w_A^2(T) - 4w_A^2(T) \|Tx\|^2_A \cos^2_A T. \]
Thus
\[ \|Tx\|^2_A \leq 4 \left( 1 - \cos^2_A T \right) w_A^2(T). \]
This yields
\[ \frac{1}{2} \|Tx\|_A \leq |\sin|_A Tw_A(T), \]
and hence
\[ \frac{1}{2} \|T\|_A \leq |\sin|_A Tw_A(T). \] (2.16)

Now, by (2.15) and (2.16) we obtain
\[ \frac{1}{2} \|T\|_A \leq \max \left\{ |\sin|_A, \frac{\sqrt{2}}{2} \right\} w_A(T). \]

\[ \square \]

3. Upper bounds for the $A$-numerical radius of products of operators

In this section, we derive upper bounds for the $A$-numerical radius of products of semi-Hilbertian space operators. Since for every $T, S \in \mathcal{B}_A(H)$ we have $\|TS\|_A \leq \|T\|_A \|S\|_A$, by the inequalities of (2.6) we obtain
\[ w_A(TS) \leq \|TS\|_A \leq 2\|T\|_A w_A(S) \leq 4 w_A(T) w_A(S). \] (3.1)

In the following theorems, we improve the inequalities 3.1. To achieve our goal, we need the following lemma.

**Lemma 3.1.** Let $T, S \in \mathcal{B}_A(H)$. Then
\[ w_A \left( (TS)^{z_A} \pm T^{z_A} S \right) \leq 2 w_A(T) \|S\|_A. \]

**Proof.** Let $\theta \in \mathbb{R}$. Since $\left( (R^{z_A})^{z_A} \right)^{z_A} = R^{z_A}$ for every $R \in \mathcal{B}_A(H)$, we have
\[
\frac{e^{i\theta} \left( (TS)^{z_A} + T^{z_A} S \right)^{z_A} + e^{i\theta} \left( ((TS)^{z_A} + T^{z_A} S)^{z_A} \right)^{z_A}}{2} = \frac{e^{i\theta} \left( T^{z_A} \right)^{z_A} \left( S^{z_A} \right)^{z_A} + e^{i\theta} S^{z_A} \left( T^{z_A} \right)^{z_A} + e^{-i\theta} S^{z_A} T^{z_A} + e^{-i\theta} T^{z_A} \left( S^{z_A} \right)^{z_A}}{2} = e^{-i\theta} T^{z_A} + \frac{\left( e^{-i\theta} T^{z_A} \right)^{z_A}}{2} \left( S^{z_A} \right)^{z_A} + S^{z_A} \frac{e^{-i\theta} T^{z_A} + \left( e^{-i\theta} T^{z_A} \right)^{z_A}}{2}. \] (3.2)
Therefore, by Theorem 2.5 and (3.2), we obtain

\[
w_A \left( \left( (TS)^{\sharp A} + T^{\sharp A}S \right)^{\sharp A} \right)
\]

\[
= \sup_{\theta \in \mathbb{R}} \left\| \frac{e^{i\theta} \left( ((TS)^{\sharp A} + T^{\sharp A}S)^{\sharp A} \right) + \left( e^{i\theta} \left( ((TS)^{\sharp A} + T^{\sharp A}S)^{\sharp A} \right) \right)^{\sharp A}}{2} \right\|_A
\]

\[
\leq \sup_{\theta \in \mathbb{R}} \left\| \frac{e^{-i\theta T^{\sharp A}} + \left( e^{-i\theta T^{\sharp A}} \right)^{\sharp A}}{2} \left( S^{\sharp A} \right)^{\sharp A} + S^{\sharp A} \frac{e^{-i\theta T^{\sharp A}} + \left( e^{-i\theta T^{\sharp A}} \right)^{\sharp A}}{2} \right\|_A
\]

\[
\leq \sup_{\theta \in \mathbb{R}} \left\| \frac{e^{-i\theta T^{\sharp A}} + \left( e^{-i\theta T^{\sharp A}} \right)^{\sharp A}}{2} \left( \| S^{\sharp A} \|_A + \| S^{\sharp A} \|_A \right) \right\|
\]

\[
= w_A(T) \| S \|_A.
\]

Hence

\[
\left. w_A \left( \left( (TS)^{\sharp A} + T^{\sharp A}S \right)^{\sharp A} \right) \right| \leq 2w_A(T) \| S \|_A. \tag{3.3}
\]

Since \( w_A(R^{\sharp A}) = w_A(R) \) for every \( R \in \mathcal{B}_A(\mathcal{H}) \), from (3.3) we obtain

\[
\left. w_A \left( (TS)^{\sharp A} + T^{\sharp A}S \right) \right| \leq 2w_A(T) \| S \|_A. \tag{3.4}
\]

Finally, by replacing \( S \) by \(-iS\) in (3.4), we reach that

\[
\left. w_A \left( (TS)^{\sharp A} - T^{\sharp A}S \right) \right| \leq 2w_A(T) \| S \|_A.
\]

\[
\square
\]

In the next theorem, we give a new upper bound for the \( A \)-numerical radius of products of semi-Hilbertian space operators.

**Theorem 3.2.** Let \( T, S \in \mathcal{B}_A(\mathcal{H}) \). Then

\[
\left. w_A(TS) \leq w_A(T) \| S \|_A + \frac{1}{2} w_A \left( (TS)^{\sharp A} \pm T^{\sharp A}S \right) \right| \leq 2w_A(T) \| S \|_A.
\]
Proof. The second inequality follows from Lemma 3.1. It is therefore enough to prove the first inequality. Let \( \theta \in \mathbb{R} \). By Lemma 2.3 we have

\[
\| \frac{e^{i\theta}TS + (e^{i\theta}TS)^{\sharp_A}}{2} \|_A = w_A \left( \frac{e^{i\theta}TS + (e^{i\theta}TS)^{\sharp_A}}{2} \right) 
\]

\[
= w_A \left( \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2} S + e^{-i\theta} S^{\sharp_A T^{\sharp_A}} - T^{\sharp_A} S \right) 
\]

\[
\leq w_A \left( \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2} S \right) + w_A \left( e^{-i\theta} S^{\sharp_A T^{\sharp_A}} - T^{\sharp_A} S \right) 
\]

\[
\leq \left\| \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2} S \right\|_A + \frac{1}{2} w_A \left( S^{\sharp_A T^{\sharp_A}} - T^{\sharp_A} S \right) \quad \text{(by Corollary 2.8)}
\]

\[
\leq \left\| \frac{e^{i\theta}T + (e^{i\theta}T)^{\sharp_A}}{2} S \right\|_A + \frac{1}{2} w_A \left( (TS)^{\sharp_A} - T^{\sharp_A} S \right) \quad \text{(by Theorem 2.5)}.
\]

Thus

\[
w_A(TS) = \sup_{\theta \in \mathbb{R}} \left\| \frac{e^{i\theta}TS + (e^{i\theta}TS)^{\sharp_A}}{2} \right\|_A \leq w_A(T)\|S\|_A + \frac{1}{2} w_A \left( (TS)^{\sharp_A} - T^{\sharp_A} S \right).
\]

(3.5)

Now, by replacing \( S \) by \(-iS\) in (3.5), we conclude that

\[
w_A(TS) \leq w_A(T)\|S\|_A + \frac{1}{2} w_A \left( (TS)^{\sharp_A} + T^{\sharp_A} S \right).
\]

(3.6)

By (3.5) and (3.6) we deduce the desired result. \(\square\)

As an immediate consequence of the preceding theorem, we have the following result.

**Corollary 3.3.** Let \( T, S \in \mathbb{B}_A(\mathcal{H}) \). If \((TS)^{\sharp_A} = T^{\sharp_A} S\), then

\[
w_A(TS) \leq w_A(T)\|S\|_A.
\]

In the following, for \( R \in \mathbb{B}_A(\mathcal{H}) \), let \( d_A(R) \) denote the \( A \)-numerical radius distance of \( R \) from the scalar operators, that is,

\[
d_A(R) = \inf \{ w_A(R + \zeta I) : \zeta \in \mathbb{C} \}.
\]

Next, we present a improvement of the second inequality in (3.1).

**Theorem 3.4.** Let \( T, S \in \mathbb{B}_A(\mathcal{H}) \). Then

\[
w_A(TS) \leq \|TS\|_A 
\]

\[
\leq \min \left\{ \|T\|_A (w_A(S) + d_A(S)), \|S\|_A (w_A(T) + d_A(T)) \right\} 
\]

\[
\leq 2 \min \left\{ \|T\|_A w_A(S), \|S\|_A w_A(T) \right\}.
\]
Proof. Using a compactness argument, let $\zeta_0 \in \mathbb{C}$ such that $d_A(T) = w_A(T + \zeta_0 I)$. If $\zeta_0 = 0$, then by the inequalities of (2.6) we get

$$w_A(TS) \leq \|TS\|_A \leq 2\|S\|_A w_A(T) = \|S\|_A(w_A(T) + d_A(T)). \quad (3.7)$$

Hence, we may assume that $\zeta_0 \neq 0$. Put $\zeta = \frac{\zeta_0}{|\zeta_0|}$. Then

$$w_A(TS) \leq \|TS\|_A \leq \|T\|_A \|S\|_A$$

$$= \left\| \frac{(\zeta T) + (\zeta T)^{\sharp A}}{2} + \frac{i(\zeta T) - (\zeta T)^{\sharp A}}{2i} \right\|_A \|S\|_A$$

$$\leq \left( \left\| \frac{(\zeta T) + (\zeta T)^{\sharp A}}{2} \right\|_A + \left\| \frac{(\zeta T) - (\zeta T)^{\sharp A}}{2i} \right\|_A \right) \|S\|_A$$

$$= \left( w_A(\zeta T) + w_A(\zeta T + \zeta_0 I) \right) \|S\|_A \quad \text{(by Corollary 2.7)}$$

$$= (w_A(T) + d_A(T)) \|S\|_A$$

Hence

$$w_A(TS) \leq \|TS\|_A \leq \left( w_A(T) + d_A(T) \right) \|S\|_A. \quad (3.8)$$

Since $d_A(T) \leq w_A(T)$, from (3.7) and (3.8) it follows that

$$w_A(TS) \leq \|TS\|_A \leq \|S\|_A \left( w_A(T) + d_A(T) \right) \leq 2w_A(T)\|S\|_A. \quad (3.9)$$

By a similar argument we have

$$w_A(TS) \leq \|TS\|_A \leq \|T\|_A \left( w_A(S) + d_A(S) \right) \leq 2w_A(S)\|T\|_A. \quad (3.10)$$

Now, by (3.9) and (3.10) we obtain the desired inequalities. □

We finish this section by another upper bound for the $A$-numerical radius of products of semi-Hilbertian space operators.

**Theorem 3.5.** Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then

$$w_A(TS) \leq \|TS\|_A \leq \left( w_A(T) + d_A(T) \right) \left( w_A(S) + d_A(S) \right) \leq 4w_A(T)w_A(S).$$

**Proof.** The fact that $d_A(R) \leq w_A(R)$ holds for every $R \in \mathbb{B}_A(\mathcal{H})$ implies that the third desired inequality.

Now, let $\zeta_0, \xi_0 \in \mathbb{C}$ such that $d_A(T) = w_A(T + \zeta_0 I)$ and $d_A(S) = w_A(S + \xi_0 I)$. As in the proof of Theorem 3.4 we may assume that $\zeta_0 \xi_0 \neq 0$. Put $\zeta = \frac{\zeta_0}{|\zeta_0|}$ and
Let $\xi = \frac{R_0}{\|B\|}$. Therefore, we have

$$w_A(TS) \leq \|TS\|_A$$

$$\leq \|T\|_A \|S\|_A$$

$$= \left\| \frac{(\xi T) + (\xi T)^{\dagger}}{2} + i\frac{(\xi T) - (\xi T)^{\dagger}}{2i} \right\|_A$$

$$\times \left\| \frac{(\xi S) + (\xi S)^{\dagger}}{2} + i\frac{(\xi S) - (\xi S)^{\dagger}}{2i} \right\|_A$$

$$\leq \left( \left\| \frac{(\xi T) + (\xi T)^{\dagger}}{2} \right\|_A + \left\| \frac{(\xi T) - (\xi T)^{\dagger}}{2i} \right\|_A \right)$$

$$\times \left( \left\| \frac{(\xi S) + (\xi S)^{\dagger}}{2} \right\|_A + \left\| \frac{(\xi S) - (\xi S)^{\dagger}}{2i} \right\|_A \right)$$

$$= \left( w_A(\xi T) + w_A(\xi T + \xi_0 I) \right) \left( w_A(\xi S) + w_A(\xi S + \xi_0 I) \right)$$

(by Corollary 2.7)

$$= (w_A(T) + d_A(T)) (w_A(S) + d_A(S)).$$

\[\square\]

4. Upper bounds for the $A$-numerical radius of commutators, and anticommutators of operators

In this section, we present some upper bounds for the $A$-numerical radius of commutators, and anticommutators of semi-Hilbertian space operators. To achieve the first main result in this section, we need the following lemma.

**Lemma 4.1.** Let $R \in \mathbb{B}_A(H)$. Then

$$\|R^{\dagger}A + RR^{\dagger}A\|_A \leq 2\left( w_A^2(R) + d_A^2(R) \right) \leq 4w_A^2(R).$$

**Proof.** Observe that, from $d_A(R) \leq w_A(R)$ we have $2\left( w_A^2(R) + d_A^2(R) \right) \leq 4w_A^2(R)$. It is therefore enough to prove the first inequality. Let $\xi_0 \in \mathbb{C}$ such that $d_A(R) = w_A(R + \xi_0 I)$. If $\xi_0 = 0$, then by employing Corollary 2.7 we have

$$\|R^{\dagger}A + RR^{\dagger}A\|_A = \left\| \frac{R + R^{\dagger}A}{2} \right\|^2 + 2\left\| \frac{R - R^{\dagger}A}{2i} \right\|^2$$

$$\leq 2\left\| \frac{R + R^{\dagger}A}{2} \right\|^2 + 2\left\| \frac{R - R^{\dagger}A}{2i} \right\|^2$$

$$\leq 2w_A^2(R) + 2w_A^2(R) = 2(w_A^2(R) + d_A^2(R)).$$
If \( \zeta_0 \neq 0 \), then put \( \zeta = \frac{w}{|\zeta_0|} \). A simple computation together with Corollary 2.7 gives

\[
\| R^{\sharp_A} R + RR^{\sharp_A} \|_A = \left\| 2 \left( \frac{(\zeta R) + (\zeta R)^{\sharp_A}}{2} \right)^2 + 2 \left( \frac{(\zeta R) - (\zeta R)^{\sharp_A}}{2i} \right)^2 \right\|_A
\leq 2 \left\| \frac{(\zeta R) + (\zeta R)^{\sharp_A}}{2} \right\|_A^2 + 2 \left\| \frac{(\zeta R) - (\zeta R)^{\sharp_A}}{2i} \right\|_A^2
= 2 \left\| \frac{(\zeta R) + (\zeta R)^{\sharp_A}}{2} \right\|_A^2 + 2 \left\| \frac{(\zeta (R + \zeta_0 I) - (\zeta (R + \zeta_0 I))^{\sharp_A}}{2i} \right\|_A^2
\leq 2w_A^2(\zeta R) + 2w_A^2(\zeta (R + \zeta_0 I))
= 2(w_A^2(R) + d_A^2(R)).
\]

The following result may be stated as well.

**Theorem 4.2.** Let \( T, S \in \mathcal{B}_A(\mathcal{H}) \). Then

\[
w_A(TS \pm ST) \leq \sqrt{\| TT^{\sharp_A} + T^{\sharp_A} T \|_A \| SS^{\sharp_A} + S^{\sharp_A} S \|_A}
\leq 2 \min \left\{ \| T \|_A \sqrt{w_A^2(S) + d_A^2(S)}, \| S \|_A \sqrt{w_A^2(T) + d_A^2(T)} \right\}
\leq 2\sqrt{2} \min \left\{ \| T \|_A w_A(S), \| S \|_A w_A(T) \right\}.
\]

**Proof.** Clearly,

\[
\min \left\{ \| T \|_A \sqrt{w_A^2(S) + d_A^2(S)}, \| S \|_A \sqrt{w_A^2(T) + d_A^2(T)} \right\}
\leq \sqrt{2} \min \left\{ \| T \|_A w_A(S), \| S \|_A w_A(T) \right\}.
\]

Now, let \( x \in \mathcal{H} \) with \( \|x\|_A = 1 \). By the Cauchy–Schwarz inequality, we have

\[
\left| \langle (TS \pm ST)x, x \rangle_A \right|^2 \leq \left| \langle TSx, x \rangle_A + \langle STx, x \rangle_A \right|^2
= \left( \| Sx \|_A \| T^{\sharp_A}x \|_A + \| Tx \|_A \| S^{\sharp_A}x \|_A \right)^2
\leq \left( \| Sx \|_A^2 + \| T^{\sharp_A}x \|_A^2 \right) \left( \| Sx \|_A^2 + \| S^{\sharp_A}x \|_A^2 \right)
= \left( T^{\sharp_A}T + TT^{\sharp_A} \right) \langle x, S^{\sharp_A}S + SS^{\sharp_A} \rangle_A \langle x, S^{\sharp_A}S + SS^{\sharp_A} \rangle_A
\leq \| T^{\sharp_A}T + TT^{\sharp_A} \|_A \| S^{\sharp_A}S + SS^{\sharp_A} \|_A.
\]

Thus

\[
\left| \langle (TS \pm ST)x, x \rangle_A \right| \leq \sqrt{\| TT^{\sharp_A} + T^{\sharp_A} T \|_A \| SS^{\sharp_A} + S^{\sharp_A} S \|_A}.
\]
Taking the supremum over \( x \in \mathcal{H} \) with \( \|x\|_A = 1 \) in the above inequality we get
\[
w_A(TS \pm ST) \leq \sqrt{\|TT^{z_A} + T^{z_A}T\|_A} \sqrt{\|SS^{z_A} + S^{z_A}S\|_A}. \tag{4.1}
\]
From (4.1) and Lemma 4.1 it follows that
\[
w_A(TS \pm ST) \leq \sqrt{\|TT^{z_A} + T^{z_A}T\|_A} \sqrt{\|SS^{z_A} + S^{z_A}S\|_A}
\leq \sqrt{\|TT^{z_A}\|_A + \|T^{z_A}T\|_A} \sqrt{2(w_A^2(S) + d_A^2(S))}
\leq 2\|T\|_A \sqrt{w_A^2(S) + d_A^2(S)},
\]
whence
\[
w_A(TS \pm ST) \leq \sqrt{\|TT^{z_A} + T^{z_A}T\|_A} \sqrt{\|SS^{z_A} + S^{z_A}S\|_A}
\leq 2\|T\|_A \sqrt{w_A^2(S) + d_A^2(S)}. \tag{4.2}
\]
Similarly,
\[
w_A(TS \pm ST) \leq \sqrt{\|SS^{z_A} + S^{z_A}S\|_A} \sqrt{\|TT^{z_A} + T^{z_A}T\|_A}
\leq 2\|S\|_A \sqrt{w_A^2(T) + d_A^2(T)}. \tag{4.3}
\]
Hence by (4.2) and (4.3) we deduce the desired result. \( \square \)

As a consequence of Lemma 4.1 and Theorem 4.2, we have the following result.

**Corollary 4.3.** Let \( T, S \in \mathbb{B}_A(\mathcal{H}) \). Then
\[
w_A(TS \pm ST) \leq 2\sqrt{w_A^2(T) + d_A^2(T)} \sqrt{w_A^2(S) + d_A^2(S)} \leq 4w_A(T)w_A(S).
\]

For the second main result in this section, we need the following lemma that is interesting on its own right.

**Lemma 4.4.** For \( T, S, R \in \mathbb{B}_A(\mathcal{H}) \) the following statements hold.
\[
\begin{align*}
(\mathrm{i}) \quad & w_A(TRT^{z_A}) \leq \|T\|_A^2 w_A(R), \\
(\mathrm{ii}) \quad & w_A(SRT^{z_A}) \leq \frac{1}{2} \|TT^{z_A} + SS^{z_A}\|_A \|R\|_A.
\end{align*}
\]

**Proof.** (i) Let \( x \in \mathcal{H} \) with \( \|x\|_A = 1 \). We have
\[
\left| \left\langle TRT^{z_A}x, x \right\rangle_A \right| = \left| \left\langle RT^{z_A}x, T^{z_A}x \right\rangle_A \right| \leq w_A(R) \|T^{z_A}x\|_A^2 \leq w_A(R) \|T\|_A^2.
\]
Now, by taking the supremum over all \( x \in \mathcal{H} \) with \( \|x\|_A = 1 \) we conclude that
\[
w_A(TRT^{z_A}) \leq \|T\|_A^2 w_A(R)
\]
(ii) Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. We have

$$\left| \langle SRT^{\sharp A} x, x \rangle_A \right| = \left| \langle T^{\sharp A} x, R^{\sharp A} S^{\sharp A} x \rangle_A \right|$$

$$\leq \|T^{\sharp A} x\|_A \|R^{\sharp A} S^{\sharp A} x\|_A$$

$$\leq \|T^{\sharp A} x\|_A \|S^{\sharp A} x\|_A \|R^{\sharp A}\|_A$$

$$\leq \frac{1}{2} \left( \|T^{\sharp A} x\|_A^2 + \|S^{\sharp A} x\|_A^2 \right) \|R\|_A$$

(by the arithmetic geometric mean inequality)

$$= \frac{1}{2} \left( \langle x, TT^{\sharp A} x \rangle_A + \langle x, SS^{\sharp A} x \rangle_A \right) \|R\|_A$$

$$= \frac{1}{2} \langle x, (TT^{\sharp A} + SS^{\sharp A}) x \rangle_A \|R\|_A$$

$$\leq \frac{1}{2} \|TT^{\sharp A} + SS^{\sharp A}\|_A \|R\|_A,$$

which, by taking the supremum over $x \in \mathcal{H}$, $\|x\|_A = 1$, implies that

$$w_A(SRT^{\sharp A}) \leq \frac{1}{2} \|TT^{\sharp A} + SS^{\sharp A}\|_A \|R\|_A.$$

Finally, we present the following result.

**Theorem 4.5.** Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then

$$w_A(TS^{\sharp A} \pm ST^{\sharp A}) \leq \|T^{\sharp A} T + SS^{\sharp A}\|_A.$$
Proof. Let $\theta \in \mathbb{R}$. We have
\[
\left\| \frac{e^{i\theta}(TS^{\sharp A} + ST^{\sharp A}) + (e^{i\theta}(TS^{\sharp A} + ST^{\sharp A}))^{\sharp A}}{2} \right\|_A
\]
\[
= \left\| \left( e^{i\theta}(TS^{\sharp A} + ST^{\sharp A}) \right)^{\sharp A} + \left( (e^{i\theta}(TS^{\sharp A} + ST^{\sharp A}))^{\sharp A} \right)^{\sharp A} \right\|_A
\]
(since $\|R^{\sharp A}\|_A = \|R\|_A$ for every $R \in \mathbb{B}_A(\mathcal{H})$)
\[
\leq w_A\left( (T^{\sharp A})^{\sharp A}(e^{i\theta}I + e^{-i\theta}I)S^{\sharp A} \right)
\]
(by Lemma 2.3)
\[
= w_A\left( S(e^{-i\theta}I + e^{i\theta}I)T^{\sharp A} \right)
\]
(since $w_A(R^{\sharp A}) = w_A(R)$ for every $R \in \mathbb{B}_A(\mathcal{H})$)
\[
\leq \frac{1}{2} \left\| TT^{\sharp A} + SS^{\sharp A} \right\|_A \left\| e^{-i\theta}I + e^{i\theta}I \right\|_A
\]
(by Lemma 4.4 (ii))
\[
= \left\| TT^{\sharp A} + SS^{\sharp A} \right\|_A.
\]
Thus
\[
\left\| \frac{e^{i\theta}(TS^{\sharp A} + ST^{\sharp A}) + (e^{i\theta}(TS^{\sharp A} + ST^{\sharp A}))^{\sharp A}}{2} \right\|_A \leq \left\| TT^{\sharp A} + SS^{\sharp A} \right\|_A,
\]
and so,
\[
\sup_{\theta \in \mathbb{R}} \left\| \frac{e^{i\theta}(TS^{\sharp A} + ST^{\sharp A}) + (e^{i\theta}(TS^{\sharp A} + ST^{\sharp A}))^{\sharp A}}{2} \right\|_A \leq \left\| TT^{\sharp A} + SS^{\sharp A} \right\|_A.
\]
Then, by Theorem 2.5, we get
\[
w_A(TS^{\sharp A} + ST^{\sharp A}) \leq \left\| T^{\sharp A}T + SS^{\sharp A} \right\|_A.
\]
Finally, by replacing $T$ by $iT$ in (4.4), we obtain
\[
w_A(TS^{\sharp A} - ST^{\sharp A}) \leq \left\| T^{\sharp A}T + SS^{\sharp A} \right\|_A,
\]
and the proof is completed. \qed

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A. ZAMANI

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