C\(^{1,1}\) regularity of geodesics in the space of volume forms

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Abstract
We prove a \(C^{1,1}\) estimate for solutions of a class of fully nonlinear equations introduced by Chen–He. As an application, we prove the \(C^{1,1}\) regularity of geodesics in the space of volume forms.

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1 Introduction
Let \((M, g)\) be a Riemannian manifold of real dimension \(n\). We use \(\nabla\) to denote the Levi-Civita connection. Recently, Chen–He [3] introduced the following function space

\[ \tilde{\mathcal{H}} = \{ \varphi \in C^\infty(M) \mid \Delta \varphi - b |\nabla \varphi|^2 + a(x) > 0 \}, \]

where \(b\) is a nonnegative constant and \(a(x)\) is a positive smooth function on \(M\). For any \(u_0, u_1 \in \tilde{\mathcal{H}}\), they also introduced the fully nonlinear equation

\[ u_{tt}(\Delta u - b |\nabla u|^2 + a(x)) - |\nabla u_t|^2 = f, \quad (1.1) \]

with boundary condition

\[ u(\cdot, 0) = u_0, \quad u(\cdot, 1) = u_1, \quad (1.2) \]

where \(f\) is a nonnegative function on \(M \times [0, 1]\). In [3], Chen–He solved the Eq. (1.1) with uniform weak \(C^2\) estimates, which also hold for the degenerate case (see also [9]).

When \(b = 0, a = 1\) and \(f = 0\), (1.1) becomes the geodesic equation in the space of volume forms on \((M, g)\). More specifically, in [7], Donaldson introduced a Weil–Peterson type metric on the space of volume forms (normalized) on any Riemannian manifold with fixed total volume. We write \(\mathcal{H}\) for this infinite dimensional space, which can be parameterized by the space of smooth functions.

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\[ \{ \varphi \in C^\infty(M) \mid 1 + \Delta \varphi > 0 \} . \]

For any \( \varphi \in \mathcal{H} \), the tangent space \( T_\varphi \mathcal{H} \) is \( C^\infty(M) \). And the metric is defined by

\[
\| \delta \varphi \|_\varphi^2 = \int_M |\delta \varphi|^2 (1 + \Delta \varphi) dV_g \quad \text{for} \quad \delta \varphi \in T_\varphi \mathcal{H}.
\]

For a path \( \Phi : [0, 1] \rightarrow \mathcal{H} \), the energy function is given by

\[
E(\Phi) = \int_0^1 \int_M |\dot{\Phi}|^2 (1 + \Delta \Phi) dV_g
\]

and the geodesic equation is

\[
\Phi_{tt}(1 + \Delta \Phi) - |\nabla \Phi_t|^2 = 0, \quad (1.3)
\]

with boundary condition

\[
\Phi(\cdot, 0) = \varphi_0, \quad \Phi(\cdot, 1) = \varphi_1,
\]

where \( \varphi_0, \varphi_1 \in \mathcal{H} \).

To solve this equation, for any \( \varepsilon > 0 \), Donaldson [7] introduced the following perturbed geodesic equation

\[
(\Phi_\varepsilon)_{tt}(1 + \Delta \Phi_\varepsilon) - |\nabla (\Phi_\varepsilon)_t|^2 = \varepsilon, \quad (1.4)
\]

with boundary condition

\[
\Phi_\varepsilon(\cdot, 0) = \varphi_0, \quad \Phi_\varepsilon(\cdot, 1) = \varphi_1. \quad (1.5)
\]

In [2], Chen–He solved this perturbed geodesic equation and proved weak \( C^2 \) estimate which is independent of \( \varepsilon \). Let \( \varepsilon \rightarrow 0 \), Chen–He proved that there is a unique weak geodesic \( \Phi \) connecting \( \varphi_0 \) and \( \varphi_1 \), and that the quantities \( \sup_{M \times [0, 1]} |\Phi|, \sup_{M \times [0, 1]} |\Phi_t|, \sup_{M \times [0, 1]} |\nabla \Phi|, \sup_{M \times [0, 1]} |\Phi_{tt}|, \sup_{M \times [0, 1]} |\Delta \Phi| \) are all bounded (see [2, Theorem 1.2, Corollary 5.3]). By the boundary condition (1.5), the quantity \( \sup_{\partial(M \times [0, 1])} |\nabla^2 \Phi| \) is also bounded. Hence, \( \Phi \) is \( C^{1,\alpha} \) for any \( \alpha \in (0, 1) \).

In general, it is well known that the weak geodesic \( \Phi \) is not \( C^2 \). Actually, in complex dimension 1, (1.3) becomes the geodesic equation in the space of Kähler metrics. And there are many examples which show that in general the weak geodesic in the space of Kähler metrics is not \( C^2 \) (see [5,6,10]). Recently, Chu–Tosatti–Weinkove [4] proved the \( C^{1,1} \) regularity of geodesics in the space of Kähler metrics. Hence, for (1.4), it was expected that \( \sup_{M \times [0, 1]} |\nabla^2 \Phi_\varepsilon| \leq C \), where \( C \) is independent of \( \varepsilon \). This implies that the weak geodesic \( \Phi \) is \( C^{1,1} \). In this paper, we prove the \( C^{1,1} \) regularity of geodesics in the space of volume forms.

**Theorem 1.1** Let \( (M, g) \) be a compact \( n \)-dimensional Riemannian manifold. For any two points \( \varphi_0, \varphi_1 \in \mathcal{H} \), the weak geodesic \( \Phi \) connecting them is \( C^{1,1} \).

As alluded to above, Theorem 1.1 is a consequence of [2, Theorem 1.2] and the \( C^{1,1} \) estimate for (1.4). More generally, for (1.1), Chen–He expected that \( \sup_{M \times [0, 1]} |\nabla^2 u| \) is bounded (see [3, Remark 2.15]). We prove the following \( C^{1,1} \) estimate, which confirms what Chen–He suggested.
Theorem 1.2 Let \((M, g)\) be a compact \(n\)-dimensional Riemannian manifold. Suppose that \(f\) is a positive smooth function on \(M \times [0, 1]\). For any smooth solution \(u\) of (1.1) satisfying
\[
u(t, \cdot) \in \mathcal{H} \text{ for } t \in [0, 1],
\]
there exists a constant \(C\) depending only on \(\sup_{M \times [0, 1]} |\nabla u|, \sup_{M \times [0, 1]} |u t t|, \sup_{M \times [0, 1]} |\Delta u|, \sup_{M \times [0, 1]} f, \sup_{M \times [0, 1]} |\nabla (f^\frac{1}{2})|, \sup_{M \times [0, 1]} |\nabla^2 (f^\frac{1}{2})|\), \(u_0, u_1, a, b\) and \((M, g)\), such that
\[
\sup_{M \times [0, 1]} |\nabla^2 u| \leq C. \tag{1.6}
\]

Combining this \(C^{1,1}\) estimate, [3, Theorem 1.1] and the approximation argument, we obtain the following corollary.

Corollary 1.3 Let \((M, g)\) be a compact \(n\)-dimensional Riemannian manifold. Suppose that \(f\) is a nonnegative function on \(M\) such that
\[
\sup_{M \times [0, 1]} \left( f + |(f^\frac{1}{2})_t| + |\nabla (f^\frac{1}{2})| + |f_t t| + |\nabla^2 (f^\frac{1}{2})| \right) \leq C
\]
for a constant \(C\). Then the Dirichlet problem (1.1) has a \(C^{1,1}\) solution.

We note that (1.1) also covers the Gursky–Streets equation when \(k = 1\) (see [8]). Thus, Corollary 1.3 shows the existence of \(C^{1,1}\) solutions to the Gursky–Streets equation \((k = 1)\).

2 Proof of Theorem 1.2

We use the same notations as in [3]. For \(r = (r_0, r_1, \ldots, r_{n+1})\), we write
\[
Q(r) = r_0 r_1 - \sum_{i=2}^{n+1} r_i^2 \text{ and } G(r) = \log Q(r).
\]

We denote the first and second derivatives of \(Q\) and \(G\) by
\[
Q^i = \frac{\partial Q}{\partial r_i}, \; Q^{i,j} = \frac{\partial^2 Q}{\partial r_i \partial r_j}, \; G^i = \frac{\partial G}{\partial r_i}, \; G^{i,j} = \frac{\partial^2 G}{\partial r_i \partial r_j}.
\]

For any point \(x_0 \in M\). Let \(\{e_i\}_{i=1}^n\) be a local orthonormal frame in a neighborhood of \(x_0\). In this paper, the subscripts of a function always denote the covariant derivatives. If we write \(r = (u_{tt}, B_u, u_{ti})\) and \(B_u = \Delta u - b|\nabla u|^2 + a(\lambda)\), then (1.1) can be written as
\[
Q(r) = Q(u_{tt}, B_u, u_{ti}) = u_{tt} B_u - |\nabla u_t|^2 = f. \tag{2.1}
\]
Since \(f > 0\) and \(u(\cdot, t) \in \mathcal{H}\) for \(t \in [0, 1]\), we have \(u_{tt} > 0\) and \(B_u > 0\). By [3, (2.8)], the linearized operator of \(Q\) is given by
\[
dQ(\psi) = u_{tt} (\Delta \psi - 2b(\nabla u, \nabla \psi)) + B_u \psi_{tt} - 2(\nabla u_t, \nabla \psi_t), \tag{2.2}
\]
where \((\cdot, \cdot)\) denotes the inner product. Clearly, the Eq. (2.1) is elliptic.

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2 Let \(\lambda_1(\nabla^2 u)\) be the largest eigenvalue of \(\nabla^2 u\). It is clear that
\[
|\nabla^2 u| \leq C|\Delta u| + C \max (\lambda_1(\nabla^2 u), 0). \tag{2.3}
\]
To prove Theorem 1.2, it suffices to prove \( \sup_{M \times [0,1]} \lambda_1(\nabla^2 u) \leq C \). Hence, we consider the following quantity

\[
H(x, t, \xi) = u_{\xi \xi} + |\nabla u|^2 + At^2,
\]

for \((x, t) \in M \times [0, 1], \xi \in T_x M\) a unit vector and \(A\) a constant to be determined later. Let \((x_0, t_0, \xi_0)\) be the maximum point of \(H\). Without loss of generality, we assume that \((x_0, t_0) \notin \partial(M \times [0, 1])\). Otherwise, by the boundary condition (1.2), we obtain (1.6) directly.

We choose a local orthonormal frame \(\{e_i\}_{i=1}^n\) near \(x_0\) such that

\[
e_1(x_0) = \xi_0.
\]

In a neighborhood of \((x_0, t_0)\), we define a new quantity by

\[
\tilde{H}(x, t) = H(x, t, e_1) = u_{11} + |\nabla u|^2 + At^2.
\]

Clearly, \(\tilde{H}\) still achieves its maximum at \((x_0, t_0)\). To prove Theorem 1.2, it suffices to prove \(u_{11}(x_0, t_0) \leq C\). By the maximum principle and (2.2), at \((x_0, t_0)\), we have

\[
0 \geq dQ(\tilde{H}) = dQ(u_{11}) + dQ(|\nabla u|^2) + 2AB_u,
\]

where \(B_u = \Delta u - b|\nabla u|^2 + a(x)\).

From now on, all the calculations will be carried out at \((x_0, t_0)\). For the first term of (2.4), using (2.2), we compute

\[
dQ(u_{11}) = u_{1tt} (\Delta u_{11} - 2b(\nabla u, \nabla u_{11})) + Bu_{1tt} - 2(\nabla u_t, \nabla u_{1tt}).
\]

Applying \(\nabla e_i \nabla e_i\) to the equation \(G(r) = \log f\) [the logarithm of (2.1)] and using the concavity of \(G\) (see [2,3,7]), we see that

\[
G^i(r_{i11}) = -G^{i,j}(r_{i1}) (r_{j1}) + \frac{f_{11}}{f} - \frac{|f_1|^2}{f^2} \geq \frac{f_{11}}{f} - \frac{|f_1|^2}{f^2},
\]

where \(r = (u_{1tt}, B_u, \nabla_i u_t)\). To obtain a lower bound for \(G^i(r_{i11})\), we need the following lemma.

**Lemma 2.1** (Lemma 3.1 of [1]). Let \(\Omega\) be a domain in \(\mathbb{R}^n\) and let \(\psi \in C^{1,1}(\bar{\Omega})\) be nonnegative. Then \(\sqrt{\psi} \in C^{0,1}(\Omega)\) and

\[
|(D\sqrt{\psi})(x)| \leq \max \left\{ \frac{|D\psi(x)|}{2 \text{dist}(x, \partial\Omega)}, 1 + \sup_{\Omega} \lambda_{\text{max}}(D^2\psi) \right\} \frac{1}{2}
\]

for almost all \(x \in \Omega\).

Using \(\partial M = \emptyset\) and Lemma 2.1 (taking \(\psi = f^\frac{1}{2}\)), we obtain

\[
|\nabla f^\frac{1}{2}| \leq C |\nabla (f^\frac{1}{2})| + C |\nabla^2 (f^\frac{1}{2})| + C,
\]

which implies

\[
|\nabla f|^2 \leq Cf^\frac{3}{2}.
\]

Combining this with (2.6), it is clear that

\[
G^i(r_{i11}) \geq \frac{2(f^\frac{1}{2})_{11}}{f^\frac{1}{2}} - \frac{|f_1|^2}{2f^2} \geq \frac{2|\nabla^2 (f^\frac{1}{2})|}{f^\frac{1}{2}} - \frac{|\nabla f|^2}{2f^2} \geq \frac{C}{f^\frac{1}{2}}.
\]
Recalling that $G(r) = \log Q(r)$ and $Q(r) = f$ [see (2.1)], it follows that
\[
Q^i (r_i)_{11} = Q(r)G^i (r_i)_{11} = fG^i (r_i)_{11} \geq -C\sqrt{f}.
\] (2.7)

By the commutation formula for covariant derivatives, $r = (u_{tt}, B_u, u_{ti})$, $B_u = \Delta u - b|\nabla u|^2 + a(x)$, $u_{tt} > 0$ and $b \geq 0$, it is clear that
\[
Q^i (r_i)_{11} = u_{tt}(B_u)_{11} + B_u u_{tt11} - 2 \sum_{i=1}^{n} u_{ti} u_{tti11}
\]
\[
= u_{tt}((\Delta u)_{11} - b(|\nabla u|^2)_{11} + a_{11}) + B_u u_{tt11} - 2 \sum_{i=1}^{n} u_{ti} u_{tti11}
\]
\[
\leq u_{tt}((\Delta u)_{11} + C|\nabla^2 u|)
\]
\[
- bu_{tt} \left( \sum_{i=1}^{n} |u_{ti}|^2 + 2(\nabla u, \nabla u_{11}) - C|\nabla u|^2 \right)
\]
\[
+ u_{tt}a_{11} + B_u u_{tt11} - 2(\nabla u_{11}, \nabla u_{tt}) + C|\nabla u_{tt}|^2
\]
\[
\leq dQ(u_{11}) + Cu_{tt}(|\nabla^2 u| + 1) + C|\nabla u_{tt}|^2,
\] (2.8)

where we used (2.5) in the last inequality. Combining (2.7) and (2.8), we obtain
\[
dQ(u_{11}) \geq -Cu_{tt}(|\nabla^2 u| + 1) - C|\nabla u_{tt}|^2 - C\sqrt{f}.
\] (2.9)

For the second term of (2.4), by [3, Proposition 2.9], we have
\[
dQ(|\nabla u|^2) = 2u_{tt} (\text{Ric}(\nabla u, \nabla u) - (\nabla u, \nabla a) + 2(\nabla f, \nabla u)
\]
\[
+ 2u_{tt} |\nabla^2 u|^2 + 2B_u |\nabla u|^2 - 4 \sum_{i,j=1}^{n} u_{ti} u_{tj} u_{ij}.
\] (2.10)

For the reader’s convenience, we give a proof of (2.10) here. Using (2.2), we compute
\[
dQ(|\nabla u|^2) = u_{tt} \left( \Delta(|\nabla u|^2) - 2b(\nabla u, \nabla(|\nabla u|^2)) \right)
\]
\[
+ B_u(|\nabla u|^2)_{tt} - 2(\nabla u_{tt}, \nabla(|\nabla u|^2))
\]
\[
= 2u_{tt} \left( |\nabla^2 u|^2 + (\nabla u, \nabla \Delta u) + \text{Ric}(\nabla u, \nabla u) \right)
\]
\[
- 2bu_{tt} (\nabla u, \nabla(|\nabla u|^2)) + 2B_u(\nabla u, \nabla u_{tt}) + 2B_u(|\nabla u_{tt}|^2)
\]
\[
- 2(\nabla u, \nabla(|\nabla u_{tt}|^2)) - 4 \sum_{i,j=1}^{n} u_{ti} u_{tj} u_{ij},
\] (2.11)

where for the second equality, we used
\[
(\nabla u_{tt}, \nabla(|\nabla u|^2))_{tt} = 2 \sum_{i,j=1}^{n} u_{ti} u_{tj} u_{tti} + 2 \sum_{i,j=1}^{n} u_{ti} u_{tj} u_{ttj}
\]
\[
= (\nabla u, \nabla(|\nabla u|^2)) + 2 \sum_{i,j=1}^{n} u_{ti} u_{tj} u_{ij}.
\]

Taking derivative of the Eq. (2.1), it is clear that
\[
u_{tt} (\nabla \Delta u - b\nabla(|\nabla u|^2) + \nabla a) + B_u \nabla u_{tt} - \nabla(|\nabla u_{tt}|^2) = \nabla f,
\]
which implies
\[2(\nabla u, \nabla f) - 2u_{tt}(\nabla u, \nabla a) = 2u_{tt}(\nabla u, \nabla u_t) - 2bu_{tt}(\nabla u, \nabla (|\nabla u|^2)) + 2B_u(\nabla u, \nabla u_{tt}) - 2(\nabla u, \nabla (|\nabla u_t|^2)).\] (2.12)

Combining (2.11) with (2.12), we obtain (2.10).

Using (2.10) and \(u_{tt} > 0\), we have
\[
dQ(|\nabla u|^2) \geq -Cu_{tt} - C|\nabla f| + 2u_{tt}|\nabla^2 u|^2 + 2B_u|\nabla u_t|^2 - 4n^2|\nabla u_t|^2|\nabla^2 u|,\] (2.13)

Recalling the Eq. (2.1) and \(f > 0\), we have
\[|\nabla u_t| = \sqrt{u_{tt}B_u - f} \leq \sqrt{u_{tt}B_u},\]
which implies
\[
4n^2|\nabla u_t|^2|\nabla^2 u| \leq 4n^2(\sqrt{u_{tt}B_u} |\nabla u_t|)(\sqrt{B_u} |\nabla u_t|) \leq u_{tt}|\nabla^2 u|^2 + 4n^4B_u|\nabla u_t|^2.\] (2.14)

Combining (2.13) and (2.14), it follows that
\[
dQ(|\nabla u|^2) \geq -Cu_{tt} - C|\nabla f| + u_{tt}|\nabla^2 u|^2 - CB_u|\nabla u_t|^2
\geq -Cu_{tt} - Cf^{1/2}|\nabla (f^{1/2})| + u_{tt}|\nabla^2 u|^2 - CB_u|\nabla u_t|^2
\geq u_{tt}(|\nabla^2 u|^2 - C) - C|\nabla u_t|^2 - C\sqrt{f},\] (2.15)

where we used \(B_u \leq C\) in the last inequality. Substituting (2.9) and (2.15) into (2.4), at \((x_0, t_0)\), we obtain
\[0 \geq u_{tt}(|\nabla^2 u|^2 - C|\nabla^2 u| - C) - C|\nabla u_t|^2 - C\sqrt{f} + 2AB_u.\] (2.16)

From the Eq. (2.1) and \(|B_u| + |u_{tt}| \leq C\), we have
\[C|\nabla u_t|^2 + C\sqrt{f} \leq Cu_{tt}B_u + C\sqrt{u_{tt}B_u} \leq C\sqrt{u_{tt}B_u} \leq CB_u + Cu_{tt}.\] (2.17)

Substituting (2.17) into (2.16), it follows that
\[0 \geq u_{tt}(|\nabla^2 u|^2 - C|\nabla^2 u| - C) + (2A - C)B_u.\]

Since \(u_{tt} > 0\) and \(B_u > 0\), after choosing \(A\) sufficiently large, we obtain \(u_{11}(x_0, t_0) \leq C\), as desired. \(\square\)

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