Fractional quantum Hall effect driven by multi-particle correlations in edge current

Jongbae Hong
School of Physics and Astronomy, Seoul National University, Seoul 08826 Korea
Asia Pacific Center for Theoretical Physics, Pohang, Gyeongbuk 37673 Korea
(Dated: May 25, 2021)

The fractional quantum Hall effect has been considered as a puzzling quantum many-body phenomenon that has yet to be fully explained. The plateau width and excitation energy gap are particularly problematic. We report here that those two are determined by degrees of multi-particle correlations among the skipping electrons forming the edge current flowing in incompressible strips (ISs). Consideration of the total angular momentum of correlated skipping electrons and their images, which are introduced to eliminate the confining potential within the IS, yields additional Zeeman energies that hierarchically split the Landau levels (LLs) by correlation order. This level splitting produces all the odd-denominator plateaus and explains the occurrence of fractional charges, while the split distances representing the correlation strengths determine both plateau widths and excitation energy gaps. With such a scheme, we explicitly reproduce an experimental Hall resistivity curve for the lowest LL and reveal the characteristics of the half-filling state.

Two-dimensional electron systems often exhibit marvelous phenomena (1–3), with one being the fractional quantum Hall effect (FQHE) (2). It is well known that in a clean two-dimensional system at low temperature under a strong perpendicular magnetic field, many plateaus are observed at various fractional filling ν in the Hall resistivity. Most fractions have odd denominators except for special cases (4). Even though the Laughlin wave function (5) provides the basic idea behind the effect, and the composite fermion theory (6) picks up most of the plateau heights that have appeared in experimental Hall resistivities, such works neither mention the plateau widths nor calculate the values of the excitation energy gap. For a comprehensive understanding of the FQHE, we must be able to explain existing experimental results such as the plateau widths in Hall resistivity (7), the excitation energy gaps (8), and the fractional charges (9–12), as well as provide predictions for field-dependent chemical potentials and effective g-factors. Clarifying all of these issues via edge current dynamics is the purpose of this study.

Since Hall resistivity is the ratio of transverse Hall bias to longitudinal current, certain pieces of information, i.e., where and how the longitudinal current flows inside a Hall bar, are essential to understand the FQHE. Recently, a longitudinal current distribution was reported in a fractional quantum Hall system (FQHS) around filling fraction υ = 2/3 by measuring transverse potential profiles using a scanning probing microscope (13). The result revealed two important features about the longitudinal current, namely that (i) it flows macroscopically over a width of about 5 μm, three orders of magnitude wider than the magnetic length scale ℓB = (ℏc/eB)2, and also that (ii) it flows symmetrically on the left and right sides of the Hall bar.

The first feature is explainable by introducing stacks of multiple incompressible strips (ISs) in which the current flows, as sketched in the inset of Fig. 1A. The alternating arrangement of IS and compressible strip (CS) near the boundary of the Hall bar has been predicted by theoretical studies (14, 15) and observed by experiments (16–18) for an integral quantum Hall system (IQHS). Figure 1 illustrates these strips, distinguished by their electron density profiles. The IS is insulating and maintains a constant density, while the CS is conducting and maintains a constant potential (14, 15).

To explain the second feature, i.e., the symmetry of the longitudinal current, one needs information on how the longitudinal current actually flows within each IS. Under a strong perpendicular magnetic field, electrons perform a skipping motion at the IS edge and a deformed cycloidal motion away from the edge, as illustrated in Fig. 1B. The skipping motion is faster than the deformed cycloidal motion, and the direction of drift is opposite at the opposite side of the strip. The difference in the speed and orientation of the guiding center implies the presence of a well-type transverse potential VC(y) (red line in Fig. 1B) in equilibrium because the slope of the potential reflects the electron group velocity. The potential wall at the boundary of the IS is basically formed by free electrons accumulated at the edge of adjacent CSs. Figure 1C shows that formations of tilted VC(y), Hall bias VH, and symmetric currents (two long arrows) on both sides of the IS loop are self-consistently connected out of equilibrium.

We simplify the problem by replacing the potential VC(y) with an image electron of the same spin orientation, as depicted in Fig. 1D, where we introduce y-dependent electric permittivity ε(y) on the image side to fit VC(y). This replacement ensures electron wave function nodes at the boundaries of the IS via the Pauli principle, which in essence bounce electrons back at the boundary. Thus, an electron and its image performing skipping or deformed cycloidal motion (Fig. 1E) constitute the basic transport entity (BTE) of the edge current in the IS. We neglect the effect of Coulomb interaction between an electron and its image under the strong magnetic field in this study.

Now, we concentrate on a single BTE (Fig. 1E). A Zeeman energy −μ1 · B, where μ1 is the magnetic moment of a BTE, is formed due to the total angular momentum
$\vec{J}_2'$ that is expressed by $\vec{J}_2' = (\vec{L}_1' + \vec{s}_1') + (\vec{L}_1' + \vec{s}_1')$, where $\vec{L}_1'$ and $\vec{s}_1'$ denote the orbital and spin angular momenta of the skipping electron (image), respectively. The vector sum is achieved via $jj$-coupling with the condition $s_1^{(e,i)} = \uparrow$ for the fully polarized lowest Landau level (LLL) (19). Since the skipping motion in $V_C(y)$ may be described by a linear combination of $p_x$ and $p_y$ orbitals, the quantum number of $\vec{L}_1'^{e,i}$ is $\ell_1 = 1$, and the eigenvalues of the $z$-component ($\vec{L}_1'^{e,i} + \vec{s}_1'^{e,i}$) are given by $m_1^{e,i} = -1 + \frac{1}{2}, 0 + \frac{1}{2}, 1 + \frac{1}{2}$ in units of $\hbar$. Combining $m_1^{e,i}$ and $m_1^{i}$ gives the eigenvalues of $\vec{J}_2'$ as $m_{j2} = -1, 0, 1$ considering inversion symmetry. For example, $m_{j2} = 3$ obtained by adding $m_1^{e,i} = 1 + \frac{1}{2}$ and $m_1^{i} = 1 + \frac{1}{2}$ is excluded. We sketch two correlated BTEs comprising two electrons and two images in Fig. 1F. Note that such correlations likely form only among electrons drifting with equal group velocity, as correlation is strongly affected by inter-particle distance. Addition of two angular momenta described by $p$ orbitals and two up spins for the two electrons (images) comprising two correlated BTEs, $\vec{L}_2^{e,i} + \vec{s}_2^{e,i}$, gives the $z$-component eigenvalues $m_{j2} = -2, -1, 0, 1, 2$ following the discussion given above. One can easily picture the semiclassical motion of $n$ correlated BTEs in the same manner and obtain $m_{j2} = -n, \cdots, n$ for $\vec{J}_2^{n}$.

The total angular momentum of $n$ correlated BTEs, $\vec{J}_n$, contributes an additional Zeeman energy, $-\mathcal{N}\cdot \vec{B}/n$ where $\mathcal{N} = (\alpha_n/\hbar)\vec{J}_n$ with gyromagnetic ratio $\alpha_n/\hbar$, on average to each real electron in the $n$ correlated BTEs. The effective Hamiltonian in the single-particle description is then written as

$$H = \sum_i \left[ \frac{(\vec{p}_i + e\vec{A}_i)^2}{2m^*_e} - g^* \mu_B \frac{\vec{s}_i}{\hbar} \cdot \vec{B} \right] - \sum_{n=1}^{\infty} \frac{\vec{\mathcal{N}}_n}{n} \cdot \vec{B} ,$$

where $i$ runs over all electrons in the IS edges, $m^*_e$ is the cyclotron effective mass, $\vec{p}$ and $\vec{A}$ are the electron momentum and external vector potential, respectively, $g^*$ is the effective $g$-factor, $\mu_B = e\hbar/2m_0$ is the Bohr magneton with bare electron mass $m_0$, and $\vec{s}_i$ is the electron spin operator. Setting $\vec{B} = B\hat{z}$ gives the energy spectrum for

---

**FIG. 1:** Semiclassical picture of electron motion in a FQHS. (A) Electron concentration in the left half of a Hall bar of width $L$ is presented by the dark blue line (15). ISs (yellow), CSs (white), and a depletion region (blue) are shown. The inset depicts a schematic of ISs in the region of Hall current (brown) (19). (B) Electron motions and well-type potential within an IS from (A). (C) Formation of left–right symmetric longitudinal currents (longer arrows), tilted potentials, and a Hall bias $V_H$ in an IS loop out of equilibrium. Hot spots (red dots) are observed at the drain and source points (19). (D) A BTE (dashed ellipse) comprising an electron (blue) and its image (gray). The symbol $\epsilon(y)$ denotes a $y$-dependent electric permittivity on the image side. (E) Skipping and deformed cycloidal motions of a single BTE with two up spins in an IS. (F) Skipping motion of two correlated BTEs with four up spins.

---

**FIG. 2:** Schematics of LLL splitting by the Zeeman interactions of $n$ correlated electrons in the IS edge. (A) $\nu = p/3$ and $1/2$. The half-filling state is the limit of the splitting hierarchy. (B) $\nu = p/5$. (C) $\nu = p/7$. The correlation order $n$ and corresponding effective electron charge (green) in the edge state are shown. The $\delta_i$ parameters denote the split by $n$-th order correlation.
a given filling factor $\nu$ as

$$E_{m,j}^\nu = \hbar \omega_c \left( \frac{1}{2} - \zeta_\nu - \sum_{n=1}^{\infty} \delta_n^\nu m_j n \right),$$

where $\omega_c = eB/m_c^*\zeta_\nu = (|q_0^*|/4)(m_c^*/m_0)$, and $\delta_n^\nu = (\alpha_0^n/\mu_B^2)/2n$ with $\mu_B^2 = \hbar e/2m_c^*$. The $\delta_n^\nu$ represents the amount of magnetic moment per electron in a group of correlated $n$ electrons expressed in units of effective Bohr magneton at filling $\nu$, and carries information on the strength of $n$-particle correlation.

We now discuss the level splits given in the energy spectrum. Normal level splitting according to the correlation order is sketched in Fig. 2A, which shows that $n$ correlated electrons yield $(2n + 1)$ split levels and also where the odd-denominator fillings start to appear. Half-filling is special, and we discuss it at the end of this study. We show below that the $n$-particle correlation produces plateaus at odd-denominator fillings $\nu = p/(2n + 1)$ where $p = 1, \cdots , 2n$, implying that in the FQHE, plateau heights $h/\nu e^2$ are automatically quantized by multi-particle correlations.

Next, we turn to the fractional charges $(\delta)$ in terms of the level splitting shown in Fig. 2. The first split levels equally share the underlying LLL states of degeneracy $D_L = e(BA)/h$, where $A$ is the area of the IS region. Thus, each level in the leading split in Fig. 2A has degeneracy $D_L = \frac{1}{2} D_L$, where the subscript $F$ denotes the first splitting. This indicates that the edge current through the IS at filling $\nu = p/3$ has a fractional charge $e/3$ as the lowest-order effective charge. However, at fillings $\nu = p/5$ and $p/7$, for example, the normal splitting of Fig. 2A is not allowed because a strong requirement for the IS, i.e., being insulating, is not satisfied in the leading split. The allowed split number in the leading split for $\nu = p/5$ is five, as illustrated in Fig. 2B, with each level having degeneracy $D_L = \frac{1}{2} D_L$ and effective charge $e/5$. Figure 2C shows the case for $\nu = p/7$. This analysis argues that the edge current at filling $\nu = p/(2n + 1)$ has a lowest-order effective charge of $e/(2n + 1)$. Higher-order fractional charges due to mixed correlations may exist, as shown on the right side of Fig. 2.

We now obtain the Hall resistivity, we first set the chemical potential $\mu_0$ as linearly increasing over the range of magnetic field and drop it at the transition point in each plateau region, as shown in Fig. 3B, where the amount of drop is equal to the corresponding excitation energy gap $(21)$. We extrapolate $\mu_0(B)$ at zero magnetic field to the limiting value $\mu_0^{1/2}$ for the sequence $\nu = q/(2q - 1)$ and $\mu_0^{1/4}$ for $\nu = q/(2q + 1)$. Determining $\mu_0^{1/2}$ and $\mu_0^{1/4}$ is explained in the Supplementary Materials. The piece-wise linear behavior of $\mu$ has been experimentally observed in both an IQHS study (22) and in a FQHS study (23). Since the mathematical role of $|g^*|/\mu_B B$ is similar to that of the chemical potential, determining $|g^*|$ is rather simple once $\mu$ is determined. The most crucial parameters are $\delta_n$ that contain the degrees of $n$-particle correlation. It turns out that the entire Hall resistivity curve and excitation energy gaps follow from only a few parameters, $\delta_n$, for each filling.

We plot our theoretical $\rho_{xy}^\nu$ and $E_{x}^\nu$ superimposed on previous experimental data (7, 8) in Fig. 3C and 3D, respectively. We used $|g^*|$ and $\mu(B)$ given in Fig. 3C for the Hall resistivity and $\delta_n^\nu$ given in Table I for both. We set $T = 10$ mK, $\hbar \omega_c = (1.9613 \text{meV/T})B$ for $m_c^*/m_0 = 0.659$, which is close to the value of GaAs. We added Hall resistivities for $\nu = \pm \frac{5}{3}, \pm \frac{1}{3}$, and 1 for a continuation of the curve even though they belong to different sequences. Detailed procedures and further explanations are given in the Supplementary Materials. Agreement with the experimental data over the entire magnetic field range is remarkable. It is noteworthy that the plateau width and excitation energy gap are connected by $\delta_n$, and also that the flattening near half-filling in Fig. 3D stems from vanishing $\Delta_\nu$ as $\nu \rightarrow 1/2$. We restrict ourselves here to filling sequences $\nu = q/(2q \pm 1)$; extension to other sequences such as $\nu = q/(4q \pm 1)$ or $(3q \pm 1)/(4q \pm 1)$ (24) is an open problem.

Finally, we discuss the half-filling region (25). We obtained the ideal Hall resistivity once $\delta_n$ is systematically given as $\delta_n = 0.3446/[3 \cdot 5 \cdots (2n + 1)]$ (Table I). The $\delta_n$ parameters are then rewritten as $\delta_n = (g_n/(2n)) e^*/e$ using $g_n = g_n \mu_B$ with $\mu_B = \hbar c_n^*/2m_c^*$ and Kohn’s theorem (26) asserting that the cyclotron mass is unaffected by interactions, i.e., $m_n^* = m_c^*$, where $e_n^*$ and $m_n^*$ are the single electron effective charge and mass in $n$ correlated BTEs, respectively. Since we showed $e_n^*/e = 1/[3 \cdot 5 \cdots (2n + 1)]$ in Fig. 2A, the numerator 0.3446 corresponds to $(g_n/(2n))$, which contains the effect of $n$-particle correlation. This argument implies that all correlation orders have the same correlation strength, which is equivalent to ideal particles. The half-filling regions in Fig. 3B and 3C (straight orange lines) may contain infinitely many piece-wise linear structures and plateaus, respectively, even though they are not explicitly shown in this study.

In conclusion, we stress that the plateau heights $h/\nu e^2$ are automatically quantized with odd-denominator fractions in $\nu$ just by the presence of multi-particle correlations (Fig. 2), and also that the plateau width and ex-
excitation energy gap are connected via the strengths of multi-particle correlations $\delta_n$ ($\Delta_\nu$ in Table I). Extension to $\nu > 1$ remains challenging, and is relegated to a separate work.

We plot our theoretical $\rho_{xy}^\nu$ and $E_{\text{exc}}^\nu$ superimposed on previous experimental data (7, 8) in Fig. 3C and 3D, respectively. We used $g^*$ and $\mu$ given in Fig. 3C for the Hall resistivity and $\delta_n^\nu$ given in Table I for both. We set $T = 10$ mK and $\hbar \omega_c = (1.9613$ meV/T)$B$ for $m^*_n/m_0 = 0.059$, which is close to the value of GaAs. Our detailed procedure and further explanations are given in the Supplementary Materials. We added Hall resistivities for $\nu = \frac{2}{3}, \frac{4}{5}, 1$ for a continuation of the curve even though they belong to different sequences. The parameter values used for these fillings are given in the Supplementary Materials. Agreement with the experimental data over the entire magnetic field range is remarkable. It is noteworthy that the plateau width and excitation energy gap are connected by $\delta_n$, and also that the flattening near half-filling in Fig. 3D stems from vanishing $\Delta_\nu$ as $\nu \rightarrow 1/2$. We restrict ourselves here to filling sequences $\nu = q/(2q \pm 1)$; extension to other sequences such as $\nu = q/(4q \pm 1)$ or $(3q \pm 1)/(4q \pm 1)$ is an open problem.

Finally, we discuss the half-filling region (25). We obtained the ideal Hall resistivity once $\delta_n$ is systematically given as $\delta_n = 0.3446/[3 \cdot 5 \cdots (2n + 1)]$ (Table I). The $\delta_n$ parameters are then rewritten as $\delta_n = (g_n/2n)(e_n^*/e)$ using $\alpha_n = g_n\mu n_B$ with $\mu n_B = e_n^*/2m_n^*$ and Kohn’s theorem (26) asserting that the cyclotron mass is unaffected by interactions, i.e., $m_n^* = m_e^*$, where $e_n^*$ and $m_n^*$ are the single electron effective charge and mass in $n$ correlated BTEs, respectively. Since we showed $e_n^*/e = 1/[3 \cdot 5 \cdots (2n + 1)]$ in Fig. 2A, the numerator

![Figure 3: Hall resistivity curve and excitation energy gaps.](image-url)

- (A) Illustration of the excitation energy gap $E_G$ at filling $\nu = \frac{1}{2}$ from Fig. 2. (B) Chemical potential and $|g^*|$ with $\mu_{1/2}^1 = 6.420$ meV and $\mu_{1/4}^2 = 4.140$ meV. They are matched to the Hall resistivity intervals in (C) according to color. The green region belongs to the (3$\nu$ ± 1)/(4$\nu$ ± 1) sequences including $\frac{1}{2}$ and $\frac{5}{4}$ fillings. (C) Theoretical Hall resistivity superimposed on experimental data. The thick gray line behind the colored lines is the experimental data (7), the black dashed line is our result for $\nu = \frac{1}{2}$, and the red dashed line is its extrapolation. (D) Excitation energy gaps. Our results (red dots) are superimposed on experimental data (black squares) (8). The blue line is our prediction.
0.3446 corresponds to \((g_n/2n)\), which contains the effect of \(n\)-particle correlation. This argument implies that all correlation orders have the same correlation strength. In other words, there is no correlation fluctuation at half-filling (which is equivalent to ideal particles), and this is what produces the classical Hall resistivity (black dashed line in Fig. 3C) and constant chemical potential (Fig. 3B). The half-filling regions in Fig. 3B and 3C (straight orange lines) may contain infinitely many piece-wise linear structures and plateaus, respectively, even though they are not explicitly shown in this study. In conclusion, we stress that the plateau heights \(h/νe^2\) are automatically quantized with odd denominator fractions in \(ν\) just by the presence of multi-particle correlations (Fig. 2), and also that the plateau width and excitation energy gap are connected via the strengths of multi-particle correlations \(δ_ν\) (Table I). Extension to \(ν > 1\) remains challenging, which we relegate to a separate work.

We thank A. Gauss, R. Haug, T. Toyoda, J. Weis, and particularly S.-J. Rey for valuable discussions.

Funding: This work was partially supported by grants from the National Research Foundation of Korea (2017R1D1A1A02017587) and by the Open KIAS Center.

Table I: Values of \(|g^*|\), \(δ_n\), and the magnetic field along with energy gap parameter \(Δ_ν\). \(λ=34.46\).

| \(ν\) | 1/3 | 2/5 | 3/7 | 1/2 | 4/7 | 3/5 | 2/3 | 5/7 | 4/5 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| \(|g^*|\) | 24.68 | 23.51 | 22.41 | 22.20 | 22.14 | 22.10 | 20.34 | 20.75 | 18.58 | 5.22 |
| \(10^2 \cdot δ_1\) | 5.410 | 5.830 | 9.050 | \(λ/3\) | 11.73 | 12.06 | 12.30 | 13.40 | 14.30 | 3.250 |
| \(10^2 \cdot δ_2\) | 0.810 | 1.295 | 2.137 | \(λ/15\) | 2.540 | 3.100 | 2.400 | 2.400 | 3.000 | 0.6535 |
| \(10^2 \cdot δ_3\) | 0.071 | 0.076 | 0.600 | \(λ/105\) | 0.657 | 0.350 | 0.110 | 0.560 | 0.300 | 0.0756 |
| \(10^2 \cdot δ_4\) | 0.000 | 0.000 | 0.013 | \(λ/945\) | 0.024 | 0.014 | 0.000 | 0.000 | 0.000 | 0.000 |

| \(Δ_ν\) | \(δ_1\) | 0 | 0 | 0 | 0 | 0 | \(δ_1\) | – | – | – |
|---|---|---|---|---|---|---|---|---|---|---|
| – | \(δ_2\) | 0 | 0 | 0 | \(δ_2\) | – | – | – | – | – |
| – | \(δ_3\) | \(δ_3\) | 0 | \(δ_3\) | – | – | – | – | – | – |
| – | \(δ_4\) | \(δ_4\) | \(δ_4\) | \(δ_4\) | – | – | – | – | – | – |

| \(B(T)\) | 28.50 | 23.75 | 22.12 | 19.00 | 16.57 | 15.71 | 14.10 | 13.30 | 12.00 | 9.50 |

[1] K. von Klitzing, G. Dorda, M. Pepper, Phys. Rev. Lett. 45, 494-497 (1980).
[2] D. C. Tsui, H. L. Stormer, A. C. Gossard, Phys. Rev. Lett. 48, 1559-1562 (1982).
[3] J. G. Bednorz, K. A. Müller, Zeit. für Phys. B 64 (2), 189-193 (1986).
[4] R. B. Laughlin, Phys. Rev. Lett. 50, 1395-1398 (1983).
[5] J. K. Jain, Phys. Rev. Lett. 63, 199-202 (1989).
[6] J. P. Eisenstein, H. L. Stormer, Science 248, 1510-1516 (1990).
[7] R. R. Du et al., Phys. Rev. Lett. 70, 2944-2947 (1993).
[8] R. de-Picciotto et al., Nature 399, 162-164 (1997).
[9] L. Saminadayar, Y. Jin, B. Etienne, C. Giannetti, Phys. Rev. Lett. 79, 2526-2529 (1997).
[10] M. Reznikov et al., Nature 399, 238-241 (1999).
[11] Y. C. Chung, M. Heiblum, V. Umansky, Phys. Rev. Lett. 91, 216804 (2003).
[12] A. Gauss, thesis, University of Stuttgart (2019).
[13] D. B. Chklovskii, B. I. Shklovskii, L. I. Glazman, Phys. Rev. B 46, 4026-4034 (1992).
[14] K. Lier, R. R. Gerhardts, Phys. Rev. B 50, 7757-7767 (1994).
[15] K. Lai et al., Phys. Rev. Lett. 107, 176809 (2011).
[16] H. Ito et al., Phys. Rev. Lett. 107, 256803 (2011).
[17] M. E. Suddards, A. Baumgartner, M. Henini, C. J. Mellow, New J. Phys. 14, 083015 (2012).
[18] I. V. Kukushkin, K. v. Klitzing, K. Eberl, Phys. Rev. Lett. 82, 3665-3668 (1999).
[19] T. Toyoda, Sci. Rept. 8, 12741 (2018).
[20] V.S. Kharapai et al., Phys. Rev. Lett. 99, 086802 (2007).
[21] Y. Y. Wei, J. Weis, K. v. Klitzing, K. Eberl, Appl. Phys. Lett. 71, 2514-2516 (1997).
[22] V.S. Kharapai et al., Phys. Rev. Lett. 100, 196805 (2008).
[23] B. I. Halperin, P. A. Lee, N. Read, Phys. Rev. B 47, 7312-7343 (1993).
[24] W. Kohn, Phys. Rev., 123, 1242-1244 (1961).
1 Chemical potential

To obtain the Hall resistivity, we first determine the chemical potential $\mu$. We set $\mu$ linearly increasing and drop at transition points over the range of magnetic fields in each plateau. The amount of drop is set to the excitation energy gap, as reported in an experimental study (21). In this study, the amount of drop or jump at the transition point is determined from the data given in Ref. (8). The linearly increasing and drop behavior of the chemical potential has been observed experimentally in both an integral quantum Hall system (IQHS) (22) and fractional quantum Hall system (FQHS) (23). Therefore, we choose the chemical potential $\mu(B)$ as a piece-wise linear function of the external magnetic field $B$ of the form

$$\mu_\nu(B) = (\kappa_\nu B + \mu^0_\nu) \text{ meV} \quad \text{for} \quad B = (B^\nu_L, B^\nu_U),$$

where the superscript labels each Hall plateau of filling factor $\nu$, $\kappa_\nu$ is a constant coefficient, $\mu^0_\nu$ denotes the chemical potential at zero magnetic field, and $B^\nu_{L,U}$ are the lower and upper magnetic fields corresponding to the transition points associated with the plateau at filling $\nu$. 

For the FQHS under consideration, we predict that the value of \( \mu_0^{\nu} \) for the sequence \( \nu = q/(2q - 1) \) is that at half filling, which is given by \( \mu_{1/2}^{0} = \xi(\frac{1}{2} - \zeta_{1/2})B_{1/2} \), where \( \xi = e\hbar/m_e^* \), due to vanishing correlation energy at half filling (See Fig. 2A). A different \( \mu_0^{\nu} \) is needed for the sequence \( \nu = q/(2q + 1) \), as shown in Fig. 3B, and we chose \( \mu_{1/4}^{0} = \xi(\frac{1}{2} - \zeta_{1/4})B_{1/4} - E_{1/4}^{\text{corr}} \), where \( E_{1/4}^{\text{corr}} \) denotes the correlation energy at quarter filling, which is given by \( E_{1/4}^{\text{corr}} = \xi(\delta_1 - \delta_2 - \delta_3 + \cdots)B_{1/4} \). The signs in front of \( \delta_n \) are determined by finding the location of \( \nu = \frac{1}{4} \) in Fig. 2A, and the values of \( \delta_n \) for \( \nu = \frac{1}{4} \) are given below in this paragraph. Specifically, \( \mu_{1/2}^{0} = 6.420 \text{ meV} \) with \( E_{1/2}^{\text{corr}} = 0 \), \( \xi = 1.9613 \text{ meV/T} \), and \( \zeta_{1/2} = 0.32772 \) at \( B_{1/2} = 28.5 \text{ T} \). In contrast, we obtain \( \mu_{1/4}^{0} = 4.0182 \text{ meV} \) with \( \zeta_{1/4} = 0.3890 \) at \( B_{1/2} = 38.0 \text{ T} \) and \( E_{1/4}^{\text{corr}} = 4.25456 \text{ meV} \) considering up to \( \delta_3 \). But we choose \( \mu_{1/4}^{0} = 4.140 \text{ meV} \) to recover the Hall resistivity of \( \nu = 1/2 \). We understand that the Hall resistivities at the reference points such as \( \nu = 1/2, 1/4, \) and \( 3/4 \) corresponding to filling sequence \( \nu = q/(2q \pm 1) \), \( \nu = q/(4q \pm 1) \), and \( \nu = (3q \pm 1)/(4q \pm 1) \), respectively must have the same as the one (blue line) for \( \nu = 1/2 \) shown in Fig. 1S below. We obtain the same straight line Hall resistivity using \( \mu = 6.420 \text{ meV} \), \( \zeta = 0.32772 \), and \( \delta_n = 0.3446/[3 \cdot 5 \cdot \cdots (2n + 1)] \) for \( \nu = 1/2 \), \( \mu = 4.140 \text{ meV} \), \( \zeta = 0.3890 \), and \( \delta_n = 0.222/[3 \cdot 5 \cdot \cdots (2n + 1)] \) for \( \nu = 1/4 \), and \( \mu = 7.9863 \text{ meV} \), \( \zeta = 0.2854 \), and \( \delta_n = 0.431/[3 \cdot 5 \cdot \cdots (2n + 1)] \) for \( \nu = 3/4 \).

Since we fixed \( \mu_\nu \) for \( \nu = 1/2 \) and \( 1/4 \) and the amounts of drop at the transition points from the experimental data of Ref. (8), the values of slope, \( \kappa_\nu \), are determined. We show \( \kappa_\nu \) and \( \mu_\nu \) in Table S1, and \( \mu(B) \) is explicitly shown in Fig. 3B in the main text.
In Table S1, we included the values for $\nu = 5/7, 4/5$, and 1 in the last three columns to obtain Hall resistivity curves. The former two fillings, $\nu = 5/7$ and $4/5$, belong to different sequences $\nu = (3q - 1)/(4q - 1)$ and $\nu = (3q + 1)/(4q + 1)$, respectively. Hence, we set $\mu_0^0$ for the sequence $\nu = (3q - 1)/(4q - 1)$ the chemical potential at $1/2$ filling, $\mu_{1/2}^0$, and for the sequence $\nu = (3q + 1)/(4q + 1)$ the chemical potential at $3/4$ filling, $\mu_{3/4}^0$, which we set to $7.9863$ meV.

Filling $\nu = 1$, which is a member of the integral quantum effect, is special because it is the highest energy level among the split levels shown in Fig. 2 and Fig. 3A, and the energy excitation involves the next Landau level ($N = 0, \sigma = -1$) unlike other fillings in the lowest Landau level (LLL). Therefore, $\nu = 1$ may belong to another class, and we exclude it from the sequence $\nu = q/(2q - 1)$. We chose $\mu_1^0 = 9.30$ meV for $\nu = 1$, which gives a relatively small effective $g$-factor, $|g^*| = 5.22$.

Experimental data for excitation energy gaps at fillings $5/7, 4/5$, and 1 are needed to predict more reliable values for the above parameters as well as $\delta_n$ for these fillings. Therefore, we did not present chemical potentials, effective $g$-factors (Fig. 3B), and excitation energy gaps.

| $\nu$ | $1/3$ | $2/5$ | $3/7$ | $1/2$ | $4/7$ | $3/5$ | $2/3$ | $5/7$ | $4/5$ | 1 |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---|
| $10^2 \cdot \kappa_{1\nu}$ | 7.495 | 9.059 | 10.049 | 0 | 0.9075 | 2.612 | 6.352 | 7.800 | 3.189 | 9.500 |
| $\mu_1^0$ | 4.14 | 4.14 | 4.14 | 6.42 | 6.42 | 6.42 | 6.42 | 6.42 | 7.99 | 9.30 |
| $|g^\ast_1|$ | 24.68 | 23.51 | 22.41 | 22.22 | 22.14 | 22.10 | 20.34 | 20.75 | 18.58 | 5.22 |
| $10^2 \cdot \zeta_{1\nu}$ | 36.41 | 34.68 | 33.05 | 32.77 | 32.65 | 32.60 | 30.00 | 30.60 | 27.20 | 7.70 |
| $10^2 \cdot \delta_1$ | 5.410 | 5.830 | 9.050 | $\lambda/3$ | 11.73 | 12.06 | 12.30 | 13.40 | 14.30 | 3.250 |
| $10^2 \cdot \delta_2$ | 0.810 | 1.295 | 2.137 | $\lambda/15$ | 2.540 | 3.100 | 2.400 | 2.400 | 3.000 | 0.6535 |
| $10^2 \cdot \delta_3$ | 0.071 | 0.076 | 0.600 | $\lambda/105$ | 0.657 | 0.350 | 0.110 | 0.560 | 0.300 | 0.0756 |
| $10^2 \cdot \delta_4$ | 0.000 | 0.000 | 0.013 | $\lambda/945$ | 0.024 | 0.014 | 0.000 | 0.000 | 0.000 | 0.000 |

Table S1: Values of $\kappa, \mu_0^0, |g^\ast|, \zeta$, and $\delta_n$ ($\lambda=34.46$).
(Fig. 3D) for these fillings in the text. However, we presented Hall resistivity curves for these fillings obtained using Table S1 for continuation (Fig. 3C).

2 Hall resistivity calculation

We obtain the Hall resistivity curves separately for each plateau region because each plateau state has different characteristics in $\kappa$, $|g^*|$, and $\delta_n$ representing $n$-body correlation, and the transition between adjacent plateau states is considered as a quantum phase transition. For the LLL, we set $\vec{B} = B\hat{z}$, which leads to the $z$-component of total angular momentum as $J^z_n = m_{jn}\hbar$, where $m_{jn} = -n, -n + 1, \ldots, n - 1, n$. The Hall resistivity formula is then given by

$$\rho'_{xy} = \frac{(h/e^2)}{\sum_{\{m_{jn}\}} [1 + \exp\{(E^\nu_{m_j} - \mu_\nu)/k_BT\}]^{-1},}$$

where $E^\nu_{m_j} = \hbar\omega_c \left(\frac{1}{2} - \zeta_\nu - \sum_{n=1}^\infty \delta^\nu_n m_{jn}\right)$ with $\sum_{\{m_{jn}\}}$ abbreviates the sum over correlated states

$$\sum_{\{m_{jn}\}} = \frac{1}{3 \times 5 \times 7 \cdots} \sum_{m_{j_1} = -1}^{+1} \sum_{m_{j_2} = -2}^{+2} \sum_{m_{j_3} = -3}^{+3} \cdots,$$

in which the prefactor is to render the upper bound of Fermi distribution function unity.

We illustrate the explicit expression of $\rho'_{xy}$ for the second-order truncation considering up to two-particle correlation, namely $n = 1$ and 2. The corresponding Hall resistivity expression is explicitly written as

$$\rho'_{xy} = \frac{3 \times 5(h/e^2)}{\sum_{m_{j_1} = -1}^{+1} \sum_{m_{j_2} = -2}^{+2} [1 + e^{\{\hbar\omega_c(0.5 - \zeta_\nu - m_{j_1}\delta_1^\nu - m_{j_2}\delta_2^\nu) - \mu_\nu\}/k_BT}]^{-1},}$$

where all the parameters except chemical potential and temperature appear in the energy spec-
The explicit expression of the denominator is given by

\[
(1 + \alpha e^{-1*\delta_1*\gamma}e^{-2*\delta_2*\gamma})^{-1} + (1 + \alpha e^{-1*\delta_1*\gamma}e^{-1*\delta_2*\gamma})^{-1}
+ (1 + \alpha e^{0*\delta_1*\gamma}e^{-2*\delta_2*\gamma})^{-1} + (1 + \alpha e^{0*\delta_1*\gamma}e^{-1*\delta_2*\gamma})^{-1}
+ (1 + \alpha e^{0*\delta_1*\gamma}e^{-2*\delta_2*\gamma})^{-1} + (1 + \alpha e^{0*\delta_1*\gamma}e^{-2*\delta_2*\gamma})^{-1}
+ (1 + \alpha e^{0*\delta_1*\gamma}e^{-2*\delta_2*\gamma})^{-1} + (1 + \alpha e^{0*\delta_1*\gamma}e^{-2*\delta_2*\gamma})^{-1}
+ (1 + \alpha e^{1*\delta_1*\gamma}e^{-2*\delta_2*\gamma})^{-1} + (1 + \alpha e^{1*\delta_1*\gamma}e^{-2*\delta_2*\gamma})^{-1}
+ (1 + \alpha e^{1*\delta_1*\gamma}e^{0*\delta_2*\gamma})^{-1} + (1 + \alpha e^{1*\delta_1*\gamma}e^{1*\delta_2*\gamma})^{-1}
+ (1 + \alpha e^{1*\delta_1*\gamma}e^{0*\delta_2*\gamma})^{-1} + (1 + \alpha e^{1*\delta_1*\gamma}e^{1*\delta_2*\gamma})^{-1},
\]

where \( \alpha = e^{(\hbar\omega_c(0.5-\zeta)-\mu)/k_BT} \) with \( \zeta = (|g^*|/4)(m_c^*/m_0) \), and \( \gamma = \hbar\omega_c/k_BT \). We omit the superscript \( \nu \) in \( \zeta \) and \( \delta \) for convenience. If one considers up to four-particle correlation, the denominator has 945 terms.

We obtain the Hall resistivity curve in Fig. 3C as follows. At each plateau region of filling fraction \( \nu \), we first calculate \( \rho_{xy}^\nu \) by using the chemical potential, parameter value \( \zeta(g^*) \), and \( \delta_n \) given in the corresponding column of Table S1. Then, we obtain the blue line for a given \( \nu \) in Fig. S1 that best fits the corresponding region in the precision experiment (7) and perform the same process for the adjacent plateau state. Next, we choose the transition point at which two neighboring Hall resistivity curves meet. In this way, we obtain the Hall resistivity curve \( \rho_{xy}(B) \) over the entire domain of the magnetic field, \( B = 10 - 30T \), as shown in the lowermost panel in Fig. S1.

It is notable that the calculated Hall resistivity curve for filling \( \nu = 1/3 \) shows an exact matching to the plateau heights of fillings 1/3 and 2/3 (first panel), and the curve for \( \nu = 2/5 \) matches the heights of fillings 2/5, 3/5, and 4/5 (second panel). One can control the central position of a plateau by varying the effective \( g \)-factor once the chemical potential is determined, and can do so the plateau width by varying \( \delta_n \).
Fig. S1: Hall resistivity curves for each plateau (upper three panels) and their sum (bottom panel). The thick gray line is experimental data (7), and the blue line drawn over the entire domain of the magnetic field is the calculated Hall resistivity curve for a given filling $\nu$. 