A4-based see-saw model for realistic neutrino masses and mixing

Soumita Pramanick*, Amitava Raychaudhuri†

Department of Physics, University of Calcutta, 92 Acharya Prafulla Chandra Road, Kolkata 700009, India

Abstract

We present an A4-based model where neutrino masses arise from a combination of see-saw mechanisms. The model is motivated by several small mixing and mass parameters indicated by the data. These are $\theta_{13}$, the solar mass splitting, and the small deviation of $\theta_{23}$ from maximal mixing ($= \pi/4$). We take the above as indications that at some level the small quantities are well-approximated by zero. In particular the mixing angles, to a zero order, should be either 0 or $\pi/4$. Accordingly, in this model the Type-II see-saw dominates and generates the larger atmospheric mass splitting and sets $\theta_{23} = \pi/4$. The other mixing angles are vanishing as is the solar splitting. We show how the A4 assignment for the lepton doublets leads to this form. We also specify the A4 properties of the right-handed neutrinos which result in a smaller Type-I see-saw contribution that acts as a perturbation and shifts the angles $\theta_{12}$ and $\theta_{13}$ into the correct range and the desired value of $\Delta m^2_{\text{solar}}$ is produced. The A4 symmetry results in relationships between these quantities as well as with a small deviation of $\theta_{23}$ from $\pi/4$. If the right-handed neutrino mass matrix, $M_R$, is chosen real then there is no leptonic CP-violation and only Normal Ordering is admissible. If $M_R$ is complex then Inverted Ordering is also allowed with the proviso that the CP-phase, $\delta$, is large, i.e., $\sim \pi/2$ or $-\pi/2$. The preliminary results from NO$\nu$A favouring Normal Ordering and $\delta$ near $-\pi/2$ imply quasi-degenerate neutrino masses in this model.

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I Introduction

Many neutrino oscillation experiments have established that neutrinos are massive and non-degenerate and that the flavour eigenstates are not identical with the mass eigenstates. For the three neutrino paradigm the two independent mass square splittings are the solar ($\Delta m^2_{\text{solar}}$) and the atmospheric ($\Delta m^2_{\text{atmos}}$). The mass and flavour bases are related through the Pontecorvo, Maki, Nakagawa, Sakata – PMNS – matrix usually parametrized as:

$$U = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\ -c_{23} s_{12} + s_{23} s_{13} c_{12} e^{i\delta} & c_{23} c_{12} + s_{23} s_{13} s_{12} e^{i\delta} & -s_{23} c_{13} \\ -s_{23} s_{12} - c_{23} s_{13} c_{12} e^{i\delta} & s_{23} c_{12} - c_{23} s_{13} s_{12} e^{i\delta} & c_{23} c_{13} \end{pmatrix},$$

where $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$.

The recent measurement of a non-zero value for $\theta_{13}$ [1], which is small compared to the other mixing angles, has led to a flurry of activity in developing neutrino mass models which incorporate this feature. Earlier we had demonstrated [2] that a direction which bears exploration is whether two small quantities...
in the neutrino sector, namely, $\theta_{13}$ and the ratio, $R \equiv \Delta m^2_{solar}/\Delta m^2_{atmos}$, could in fact be related to each other, both resulting from a small perturbation. Subsequently we had shown [3] that it is possible to envisage scenarios where only the larger $\Delta m^2_{atmos}$ and $\theta_{23} = \pi/4$ are present in a basic structure of neutrino mass and mixing and the rest of the quantities, namely, $\theta_{13}, \theta_{12}$, the deviation of $\theta_{23}$ from $\pi/4$, and $\Delta m^2_{solar}$ all have their origin in a smaller see-saw induced perturbation. Obviously, this gets reflected in constraints on the measured quantities. A vanishing $\theta_{13}$ follows rather easily from certain symmetries and indeed many of the newer models are based on perturbations of such structures [5, 6].

Encouraged by the success of this program we present here a model based on the group $A_4$ which relies on the see-saw mechanism [7] in which the lightest neutrino mass, $m_0$, the see-saw scale and one other parameter determine $\theta_{13}$, $R$, $\theta_{12}$, and the deviation of $\theta_{23}$ from $\pi/4$. If this last parameter is complex then the CP-phase $\delta$ is also a prediction. Here, the atmospheric mass splitting is taken as an input which together with the lightest neutrino mass completely defines the unperturbed mass matrix generated by the Type-II see-saw. The size of the perturbation is determined by the Type-I see-saw and is of the form $m^2_D/m_R$ where $m_D$ and $m_R$ respectively are the scale of the Dirac and right-handed Majorana mass terms.

After a brief summary of the $A_4$ group properties and the structure of the model in the following section we describe the implications of the model in the next section. The comparison of this model with the experimental data appears next. We end with our conclusions. The model has a rich scalar field content. In an Appendix we discuss the $A_4$ invariant scalar potential and the conditions under which the desired potential minimum can be realized.

It should be noted that neutrino mass models based on $A_4$ have also been investigated earlier [8, 9, 10]. In a majority of them the neutrino mass matrix is obtained from a Type-II see-saw and the earlier emphasis was on obtaining tribimaximal mixing. Recent work has focussed on obtaining more realistic mixing patterns [11] sometimes taking recourse to breaking of $A_4$ symmetry [12]. Our work is unique in two respects. Firstly, it uses a combination of Type-II and Type-I see-saw mechanisms where the former yields mixing angles which are either vanishing ($\theta_{12}$ and $\theta_{13}$) or maximal – i.e., $\pi/4$ – ($\theta_{23}$) while keeping the solar splitting absent. This kind of a scenario has not been considered before. The Type-I see-saw acting as a perturbation results in a non-zero CP-phase and realistic mixing angles while at the same time creating the correct solar splitting. Secondly, all this is accomplished keeping the $A_4$ symmetry intact.

**II The Model**

$A_4$ is the group of even permutations of four objects comprising of 12 elements which can be generated using the two basic permutations $S$ and $T$ satisfying $S^2 = T^3 = (ST)^3 = I$. The group has four inequivalent irreducible representations one of 3 dimension and three of 1 dimension denoted by 1, $1'$ and $1''$. The one-dimensional representations are all singlets under $S$ and transform as 1, $\omega$, and $\omega^2$ under $T$ respectively, where $\omega$ is the cube root of unity. Thus $1' \times 1'' = 1$.

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1 Early work on neutrino mass models where some variables are much smaller than others can be found in [4].
| Fields                  | Notations | $A_4$ | $SU(2)_L \ (Y)$ | $L$ |
|------------------------|-----------|-------|-----------------|-----|
| Left-handed leptons    | $(\nu_i, l_i)_L$ | 3     | 2 \ (-1)        | 1   |
| Right-handed charged leptons | $l_{1R}$   | 1     |                 |     |
|                         | $l_{2R}$   | 1'    | 1 \ (-2)        | 1   |
|                         | $l_{3R}$   | 1''   |                 |     |
| Right-handed neutrinos | $N_{iR}$   | 3     | 1 \ (0)         | -1  |

Table 1: The fermion content of the model. The transformation properties under $A_4$ and $SU(2)_L$ are shown. The hypercharge of the fields, $Y$, and their lepton number, $L$, are also indicated.

For the three-dimensional representation

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (2)$$

This representation satisfies the product rule

$$3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3 \oplus 3. \quad (3)$$

The two triplets arising from the product of $3_a \equiv a_i$ and $3_b \equiv b_i$, where $i = 1, 2, 3$, can be identified as $3_c \equiv c_i$ and $3_d \equiv d_i$ with

$$c_i = \left( \frac{a_2b_3 + a_3b_2}{2}, \frac{a_3b_1 + a_1b_3}{2}, \frac{a_1b_2 + a_2b_1}{2} \right), \text{ or, } c_i \equiv \alpha_{ijk}a_jb_k,$$

$$d_i = \left( \frac{a_2b_3 - a_3b_2}{2}, \frac{a_3b_1 - a_1b_3}{2}, \frac{a_1b_2 - a_2b_1}{2} \right), \text{ or, } d_i \equiv \beta_{ijk}a_jb_k, \ (i, j, k, \text{ are cyclic}). \quad (4)$$

In this notation the one-dimensional representations in the $3 \otimes 3$ product can be written as:

$$1 = a_1b_1 + a_2b_2 + a_3b_3 \equiv \rho_{1ij}a_ib_j, \quad 1' = a_1b_1 + \omega^2a_2b_2 + \omega a_3b_3 \equiv \rho_{3ij}a_ib_j, \quad 1'' = a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3 \equiv \rho_{2ij}a_ib_j. \quad (5)$$

Further details of the group $A_4$ are available in the literature \[8\] [9].

In the proposed model the left-handed lepton doublets of the three flavours are assumed to form an $A_4$ triplet while the right-handed charged leptons are taken as $1(e_R), \ 1' (\mu_R), \text{ and } 1'' (\tau_R)$ under $A_4$. The remaining leptons, the right-handed neutrinos, form an $A_4$ triplet\[2\] The lepton content of the model

\[2\]We closely follow the notation of [8].
### Table 2: The scalar content of the model. The transformation properties under $A_4$ and $SU(2)_L$ are shown. The hypercharge of the fields, $Y$, their lepton number, $L$, and the vacuum expectation values are also indicated.

| Purpose                        | Notations | $A_4$ | $SU(2)_L$ | $L$ | $vev$         |
|--------------------------------|-----------|-------|-----------|-----|---------------|
| Charged fermion mass           | $\Phi = \begin{pmatrix} \phi_1^+ & \phi_1^0 \\ \phi_2^+ & \phi_2^0 \\ \phi_3^+ & \phi_3^0 \end{pmatrix}$ | 3     | 2 (1)     | 0   | $\langle \Phi \rangle = \frac{v}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ |
| Neutrino Dirac mass            | $\eta = (\eta^0, \eta^-)$ | 1     | 2 (-1)    | 2   | $\langle \eta \rangle = (0, u)$ |
| Type-I see-saw mass            | $\hat{\Delta}^L = \begin{pmatrix} \hat{\Delta}_1^{++} & \hat{\Delta}_1^+ &\hat{\Delta}_1^0 \\ \hat{\Delta}_2^{++} & \hat{\Delta}_2^+ &\hat{\Delta}_2^0 \\ \hat{\Delta}_3^{++} & \hat{\Delta}_3^+ &\hat{\Delta}_3^0 \end{pmatrix}$ | 3     | 3 (2)     | -2  | $\langle \hat{\Delta}^L \rangle = v_L \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ |
| Type-I see-saw mass            | $\Delta^L_\zeta = (\Delta^{++}_\zeta, \Delta^+_\zeta, \Delta^0_\zeta)^L$ | 1     | 3 (2)     | -2  | $\langle \Delta^L_\zeta \rangle = (0, 0, u_L)$ |
| Right-handed neutrino mass     | $\hat{\Delta}^R = \begin{pmatrix} \hat{\Delta}_1^0 \\ \hat{\Delta}_2^0 \\ \hat{\Delta}_3^0 \end{pmatrix}^R$ | 3     | 1 (0)     | 2   | $\langle \hat{\Delta}^R \rangle = v_R \begin{pmatrix} 1 \\ \omega^2 \end{pmatrix}$ |
| Right-handed neutrino mass     | $\Delta^R_3 = (\Delta_3^0)^R$ | $1''$ | 1 (0)     | 2   | $\langle \Delta^R_3 \rangle = u_R$ |

Note that the right-handed neutrinos are assigned lepton number -1. This choice is made to ensure that the neutrino Dirac mass matrix takes a form proportional to the identity matrix, as we remark in the following. The assignment of $A_4$ quantum numbers of the leptons is by no means unique. The entire list of options for this have been catalogued in [13]. Our choice corresponds to class B of [13]. We do not discuss the extension of this model to the quark sector.

All lepton masses arise from $A_4$-invariant Yukawa-type couplings. This requires several scalar fields which develop appropriate vacuum expectation values ($vev$). To generate the charged lepton masses, one uses an $SU(2)_L$ doublet $A_4$ triplet of scalar fields $\Phi_i$ ($i = 1, 2, 3$). The Type-II see-saw for left-handed neutrinos is treated with the $A_4$ and $SU(2)_L$ properties as well as the lepton number assignments is shown in Table II. Note that the right-handed neutrinos are assigned lepton number -1. This choice is made to ensure that the neutrino Dirac mass matrix takes a form proportional to the identity matrix, as we remark in the following. The assignment of $A_4$ quantum numbers of the leptons is by no means unique. The entire list of options for this have been catalogued in [13]. Our choice corresponds to class B of [13]. We do not discuss the extension of this model to the quark sector.

For $A_4$-based models dealing with the quark sector see, for example, [14] and [15].

Alternate $A_4$ models address this issue by separating the $SU(2)_L$ and $A_4$ breakings [9]. The former proceeds through the conventional doublet and triplet scalars which do not transform under $A_4$. The $A_4$ breaking is triggered through the $vev$ of $SU(2)_L$ singlet ‘flavon’ scalars which transform non-trivially under $A_4$. While economy is indeed a virtue here, one pays a price in the form of the effective dimension-5 interactions which have to be introduced to couple the fermion fields simultaneously to the two types of scalars.
handed neutrino masses requires $SU(2)_L$ triplet scalars. The product rule in eq. (3) indicates that these could be in the triplet ($\hat{\Delta}^L_3$), or the singlet 1, 1’, 1” ($\hat{\Delta}^L_\zeta$, $\zeta = 1, 2, 3$) representations of $A_4$. As discussed in the following, all of these are required to obtain the dominant Type-II see-saw neutrino mass matrix of the form of our choice. The Type-I see-saw results in a smaller contribution whose effect is included perturbatively. For the Dirac mass matrix of the neutrinos an $A_4$ singlet $SU(2)_L$ doublet $\eta$, with lepton number -1, is introduced\(^5\). The right-handed neutrino mass matrix also arises from Yukawa couplings which respect $A_4$ symmetry\(^6\). The scalar fields required for this are all $SU(2)_L$ singlets and under $A_4$ they transform as triplet ($\hat{\Delta}^R_3$) or the singlet 1” ($\hat{\Delta}^R_3$). The scalar fields of the model, their transformation properties under the $A_4$ and $SU(2)_L$ groups, their lepton numbers and vacuum expectation values are summarized in Table 2.

The Type-I and Type-II mass terms for the neutrinos as well as the charged lepton mass matrix arise from the $A_4$ and $SU(2)_L$ conserving Lagrangian\(^7\).

$$L_{mass} = y_{ij} \rho_{ijk} \bar{l}_i \bar{d}_j \Phi_k^0 \quad \text{(charged lepton mass)}$$

$$+ f \rho_{ijk} \bar{v}_i N_{Rj} \eta^0 \quad \text{(neutrino Dirac mass)}$$

$$+ \frac{1}{2} (\hat{Y}^L \alpha_{ijk} v^T_{Lj} C^{-1} v_{Lj} \hat{\Delta}_k^L + Y^L \rho_{\xi ij} v^T_{Lj} C^{-1} v_{Lj} \hat{\Delta}_\xi^L) \quad \text{(neutrino Type-II see-saw mass)}$$

$$+ \frac{1}{2} (\hat{Y}^R \alpha_{ijk} N^T_{Rj} C^{-1} N_{Rj} \hat{\Delta}_k^R + Y^R \rho_{\zeta ij} N^T_{Rj} C^{-1} N_{Rj} \hat{\Delta}_\zeta^R) \quad \text{(rh neutrino mass) + h.c.} \quad (6)$$

The scalar fields in the above Lagrangian get the following vers (suppressing the $SU(2)_L$ part):

$$\langle \Phi^0 \rangle = \frac{v}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \langle \hat{\Delta}_0^L \rangle = v_L \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \langle \hat{\Delta}_1^L \rangle = \langle \hat{\Delta}_2^L \rangle = \langle \hat{\Delta}_3^L \rangle = u_L,$$

$$\langle \eta^0 \rangle = u, \quad \langle \hat{\Delta}_0^R \rangle = v_R \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}, \quad \langle \hat{\Delta}_1^R \rangle = \langle \hat{\Delta}_2^R \rangle = \langle \hat{\Delta}_3^R \rangle = u_R. \quad (7)$$

The scalar potential involving the fields listed in Table 2 has many terms and is given in an Appendix. There we indicate the conditions under which the scalars achieve the vers listed in eqs. (7) and (8).

This results in the charged lepton mass matrix and the left-handed neutrino Majorana mass matrix of the following forms:

$$M_{\mu\tau} = \frac{v}{\sqrt{3}} \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1 & \omega y_2 & \omega^2 y_3 \\ y_1 & \omega^2 y_2 & \omega y_3 \end{pmatrix}, \quad M_{\nu L} = \begin{pmatrix} Y^L_1 + 2 Y^L_2 u_L \\ 0 \\ 0 \end{pmatrix} u_L \begin{pmatrix} 0 \\ (Y^L_1 - Y^L_2) u_L \\ \hat{Y}^L \nu L / 2 \end{pmatrix}. \quad (9)$$

where we have chosen $Y^L_2 = Y^L_3$. In the above the Yukawa couplings satisfy $y_1 v = m_e$, $y_2 v = m_\mu$, $y_3 v = m_\tau$. The Type-II see-saw generates, $M_{\nu L}$, the dominant contribution to the neutrino mass matrix. In the absence of the solar splitting this involves just two masses $m_1^{(0)}$ and $m_3^{(0)}$. To obtain the requisite

\(^5\) The assignment of opposite lepton numbers to $\nu_L$ and $N_R$ forbids their Yukawa coupling with $\Phi$ and the Dirac mass matrix can be kept proportional to the identity.

\(^6\) Since the right-handed neutrinos are $SU(2)_L$ singlets, in principle, one can include direct Majorana mass terms for them. These dimension three terms would break $A_4$ softly.

\(^7\) Note that the Dirac mass terms are also $L$ conserving.
structure one must identify $3Y_1^L u_L = 2[m_1^{(0)} - m_3^{(0)}]$, $3Y_2^L u_L = m_1^{(0)} + m_3^{(0)}$, and $\hat{Y}^L v_L = 2[m_1^{(0)} + m_3^{(0)}]$. The neutrino Dirac mass matrix and the mass matrix of the right-handed neutrinos are:

$$M_D = f u \mathbb{1}, \quad M_{\nu R} = \begin{pmatrix} Y_3^R u_R & \hat{Y}^R v_R \omega/2 & \hat{Y}^R v_R \omega^2/2 \\ \hat{Y}^R v_R \omega/2 & Y_3^R u_R \omega^2 & \hat{Y}^R v_R /2 \\ \hat{Y}^R v_R /2 & \hat{Y}^R v_R /2 & Y_3^R u_R \omega^2 \end{pmatrix}. \quad (10)$$

The two unknown combinations appearing in $M_R$ above are expressed as $Y_3^R u_R \equiv (2a + b)$ and $\hat{Y}^R v_R \equiv 2(b - a)$.

The mass matrices in Eq. (9) can be put in a more tractable form by using two transformations, the first being $U_L$ on the left-handed fermion doublets and the other $V_R$ on the right-handed neutrino singlets. $U_L$ and $V_R$ are given by

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & 1 \\ 1 & \omega & \omega^2 \end{pmatrix} = V_R. \quad (11)$$

No transformation is applied on the right-handed charged leptons. In the new basis, which we call the flavour basis, the charged lepton mass matrix is diagonal and the entire lepton mixing resides in the neutrino sector. The mass matrices now are:

$$M_{\text{flavour}}^{\mu \tau} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, \quad M_{\nu L}^{\text{flavour}} = \frac{1}{2} \begin{pmatrix} 2m_1^{(0)} & 0 & 0 \\ 0 & m_+ & -m_- \\ 0 & -m_- & m_+ \end{pmatrix}, \quad (12)$$

and

$$M_D = f u \mathbb{1}, \quad M_{\nu R}^{\text{flavour}} = \frac{1}{2} \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & b \end{pmatrix}. \quad (13)$$

Here $m^\pm = m_3^{(0)} \pm m_1^{(0)}$. $m_-$ is positive for normal ordering (NO) of masses while it is negative for inverted ordering (IO). We use the notation $m_D = f u$.

III Model implications

The $A_4$ model we have presented results in the four mass matrices in eqs. (12) and (13). The lepton mixing and CP-violation will be determined, in this basis, entirely by the neutrino sector on which we focus from here on.

The left-handed neutrino mass matrix $M_{\nu L}^{\text{flavour}}$, obtained via a Type-II see-saw, dominates over the Type-I see-saw contribution from the mass matrices in eq. (13). The contribution from the latter is included using perturbation theory.

In the `mass basis' the left-handed neutrino mass matrix is diagonal. The columns of the diagonalising matrix are the unperturbed flavour eigenstates in this basis. We find from $M_{\nu L}^{\text{flavour}}$:

$$M^0 = M_{\nu L}^{\text{mass}} = U^{\text{0T}} M_{\nu L}^{\text{flavour}} U^0 = \begin{pmatrix} m_1^{(0)} & 0 & 0 \\ 0 & m_1^{(0)} & 0 \\ 0 & 0 & m_3^{(0)} \end{pmatrix}, \quad (14)$$
the orthogonal matrix, \( U^0 \), being
\[
U^0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\] (15)

From eqs. (14), (1) and (15) it is seen that the solar splitting is absent, \( \theta_{12} = 0 \), \( \theta_{13} = 0 \), \( \delta = 0 \), and \( \theta_{23} = \pi/4 \).

Before proceeding with the analysis we would like to remark on the right-handed neutrino Majorana mass matrix in eq. (13), \( M^\nu_R \), which follows from the \( A_4 \) symmetric Lagrangian. It has a four-zero texture. This has the virtue of being of a form of \( M^\nu_R \) with the most number of texture zeros. For the see-saw to be operative, the matrix has to be invertible. This eliminates matrices with five texture zeros in the flavour basis. Of the 15 possibilities with four texture zeros there are only two which are invertible and also meet the requirements of the model (i.e., result in a non-zero \( \theta_{12} \), \( \theta_{13} \), and shift \( \theta_{23} \) from \( \pi/4 \)). These are:
\[
M_1 = \frac{1}{2} \begin{pmatrix}
0 & a & 0 \\
a & 0 & 0 \\
0 & 0 & b
\end{pmatrix},
M_2 = \frac{1}{2} \begin{pmatrix}
0 & 0 & a \\
0 & b & 0 \\
a & 0 & 0
\end{pmatrix}.
\] (16)

Note, \( M_1 \leftrightarrow M_2 \) under 2 \( \leftrightarrow \) 3 exchange\(^8\). The results from these two alternatives are very similar except for a few relative signs in the interrelationships among \( \theta_{13} \), \( \theta_{12} \), and \( \theta_{23} \). The \( M^\nu_R \) in eq. (13) is of the form of \( M_1 \). We remark in the end about the changes which entail if the \( M_2 \) alternative is used.

Taking \( a \) and \( b \) as complex we express \( M^\nu_R \) as:
\[
M^\nu_R = m_R \begin{pmatrix}
x e^{-i\phi_1} & 0 & 0 \\
0 & 0 & y e^{-i\phi_2} \\
0 & y e^{i\phi_2} & 0
\end{pmatrix},
\] (17)
where \( x, y \) are dimensionless real constants of \( O(1) \) and \( m_R \) sets the mass-scale. No generality is lost by keeping the Dirac mass real.

In the flavour basis, the Type-I see-saw contribution, which we treat as a perturbation, is:
\[
M'_{\text{flavour}} = \left[ M_D^T (M^\nu_R)^{-1} M_D \right] = \frac{m_D^2}{x y m_R} \begin{pmatrix}
0 & y e^{i\phi_1} & 0 \\
y e^{-i\phi_1} & 0 & 0 \\
0 & 0 & x e^{i\phi_2}
\end{pmatrix}.
\] (18)

In the mass basis it is:
\[
M'_{\text{mass}} = U^0 M'_{\text{flavour}} U^0 = \frac{m_D^2}{\sqrt{2} x y m_R} \begin{pmatrix}
0 & y e^{i\phi_1} & -y e^{i\phi_1} \\
y e^{-i\phi_1} & x e^{i\phi_2} & x e^{i\phi_2} \\
-y e^{i\phi_1} & x e^{i\phi_2} & x e^{i\phi_2}
\end{pmatrix}.
\] (19)

\(^8\)In the Lagrangian in eq. (13) the replacement \( \Delta^R_3 \rightarrow \Delta^R_2 \), where \( \Delta^R_2 \) transforms like a 1’ under \( A_4 \), yields a \( M^\nu_R \) of the form of \( M_2 \).
IV Results

After having presented the group-theoretic underpinnings of the model we now indicate its predictions which could be tested in the near future. As noted, from eq. (15) one has $\theta_{23} = \pi/4$ and the other mixing angles are vanishing. Further, once a choice of $m_0$, the lightest neutrino mass, is made, depending on the mass ordering either $m_1^{(0)}$ or $m_3^{(0)}$ is determined. The remaining one of these two is fixed so that the atmospheric mass splitting is correctly reproduced. The solar mass splitting, $\theta_{12}$, $\theta_{13}$ and the deviation of $\theta_{23}$ from maximality are all realized through the first order perturbation, which results in inter-relationships between them. These offer a scope of subjecting the model to experimental probing. From eq. (14) it is seen that to obtain the solar mixing parameters one must take recourse to degenerate perturbation theory.

IV.1 Data

From global fits the currently favoured $3\sigma$ ranges of the neutrino mixing parameters are [16, 17]
\[
\Delta m^2_{21} = (7.03 - 8.09) \times 10^{-5} \text{eV}^2, \quad \theta_{12} = (31.30 - 35.90)^\circ, \\
|\Delta m^2_{31}| = (2.325 - 2.599) \times 10^{-3} \text{eV}^2, \quad \theta_{23} = (38.4 - 53.3)^\circ, \\
\theta_{13} = (7.87 - 9.11)^\circ, \quad \delta = (0 - 360)^\circ.
\] (20)

Here, $\Delta m^2_{ij} = m_i^2 - m_j^2$, so that $\Delta m^2_{31} > 0 \ (< 0)$ for normal (inverted) ordering. The data indicate two best-fit points for $\theta_{23}$ in the first and second octants. Later, we also use the recent preliminary T2K hints [18] of $\delta$ being near $-\pi/2$.

IV.2 Real $M_R \ (\phi_1 = 0 \text{ or } \pi, \phi_2 = 0 \text{ or } \pi)$

$M_R$ is real if the phases $\phi_{1,2}$ in eq. (17) are 0 or $\pi$. For notational simplicity, instead of retaining these phases we allow $x \ (y)$ to be of either sign, thus capturing the possibilities of $\phi_1 \ (\phi_2)$ being 0 or $\pi$.

In the real limit eq. (19) becomes

\[
M'^\text{mass} = \frac{m_D^2}{\sqrt{2} x y m_R} \begin{pmatrix}
0 & y & -y \\
y & \frac{x}{\sqrt{2}} & \frac{-y}{\sqrt{2}} \\
-y & \frac{x}{\sqrt{2}} & \frac{y}{\sqrt{2}} 
\end{pmatrix}.
\] (21)

The effect of this perturbation on the degenerate solar sector is obtained from the following $2 \times 2$ submatrix of the above,

\[
M'^\text{mass}_{2\times2} = \frac{m_D^2}{\sqrt{2} x y m_R} \begin{pmatrix}
0 & y \\
y & x/\sqrt{2}
\end{pmatrix}.
\] (22)

This yields

\[
\tan 2\theta_{12} = 2\sqrt{2} \left(\frac{y}{x}\right)
\] (23)

If $y/x = 1$, i.e., $Y^R = 0$ in eq. (10), $\theta_{12}$ assumes the tribimaximal value, which though consistent at $3\sigma$ is disallowed by the data at $1\sigma$. From the data, $\tan 2\theta_{12} > 0$ always, implying $x$ and $y$ have to be
either both positive or both negative. In other words, $\phi_1 = \phi_2$ and can be either 0 or $\pi$. The fitted range of $\theta_{12}$ translates to

$$0.682 < \frac{y}{x} < 1.075 \text{ at } 3\sigma.$$  \hspace{1cm} (24)

Eqn. (22) also implies

$$\Delta m^2_{\text{solar}} = \frac{m_D^2}{x y m_R} m_1^{(0)} \sqrt{x^2 + 8y^2}.$$  \hspace{1cm} (25)

Including the first-order corrections from eq. (21) the wave function for the non-degenerate state, $|\psi_3\rangle$, becomes:

$$|\psi_3\rangle = \begin{pmatrix} -\kappa \\ \sqrt{2}(1 - \frac{\kappa}{\sqrt{2}} x y) \\ \sqrt{2}(1 + \frac{\kappa}{\sqrt{2}} x y) \end{pmatrix},$$  \hspace{1cm} (26)

with

$$\kappa \equiv \frac{m_D^2}{\sqrt{2} x y m_R}.$$  \hspace{1cm} (27)

For $x > 0$ the sign of $\kappa$ is the same as that of $m^-$. Comparing with the third column of eq. (1) one then has

$$\sin \theta_{13} \cos \delta = -\kappa = -\frac{m_D^2}{\sqrt{2} x y m^-}.$$  \hspace{1cm} (28)

In the case of normal ordering $x > 0$ implies $\delta = \pi$ while for inverted ordering $\delta = 0$, CP remaining conserved in both cases\textsuperscript{9}. If $x < 0$ NO (IO) gives $\delta = 0 (\pi)$. From eqs. (28), (24), and (25) one can write,

$$\Delta m^2_{\text{solar}} = -\text{sgn}(x) \ m^- m_1^{(0)} \frac{4 \sin \theta_{13} \cos \delta}{\sin 2\theta_{12}},$$  \hspace{1cm} (29)

which relates the solar sector with $\theta_{13}$. Once the neutrino mass splittings, and the angles $\theta_{12}$, and $\theta_{13}$ are given, eq. (29) fixes the lightest neutrino mass, $m_0$.

It can be checked that eq. (29) does not permit inverted ordering. To this end, one defines $z \equiv m^- m_1^{(0)} / \Delta m^2_{\text{atmos}}$ and $\tan \xi \equiv m_0 / \sqrt{\Delta m^2_{\text{atmos}}}$. Note that $z$ is positive for both mass orderings and one has:

$$z = \sin \xi / (1 + \sin \xi) \text{ (normal ordering)},$$

$$z = 1 / (1 + \sin \xi) \text{ (inverted ordering)}.$$  \hspace{1cm} (30)

This implies $0 \leq z \leq 1/2$ for NO while $1/2 \leq z \leq 1$ for IO. In both cases $z$ approaches $1/2$ as $m_0 \to$ large, i.e., one tends towards quasi-degeneracy. From eq. (29)

$$z = \left( \frac{\Delta m^2_{\text{solar}}}{\Delta m^2_{\text{atmos}}} \right) \left( \frac{\sin 2\theta_{12}}{4 \sin \theta_{13} |\cos \delta|} \right).$$  \hspace{1cm} (31)

Bearing in mind that for real $M_R$ one has $|\cos \delta| = 1$ and using the measured values of the other oscillation parameters one finds $z \sim 10^{-2}$. This excludes the inverted mass ordering option.

\textsuperscript{9} The mixing angles $\theta_{ij}$ are kept in the first quadrant and $\delta$ ranges from $-\pi$ to $\pi$, as is the convention.

\textsuperscript{10} It is readily seen from eq. (23) that $-\text{sgn}(x) m^- \cos \delta$ is always positive, ensuring $\Delta m^2_{\text{solar}} > 0$.  

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Further, eq. (20) implies:
\[
\tan \theta_{23} \equiv \tan(\pi/4 - \omega) = 1 - \frac{\kappa x}{\sqrt{2} y} \left(1 + \frac{\kappa x}{\sqrt{2} y}\right). \tag{32}
\]

Taken together with eqs. (23) and (28) one has
\[
\tan \omega = \frac{\kappa x}{\sqrt{2} y} = -\frac{2 \sin \theta_{13} \cos \delta}{\tan 2\theta_{12}}. \tag{33}
\]

When \(\omega\) is positive (negative), i.e., \(\delta = \pi (0)\), we get \(\theta_{23}\) in the first (second) octant. This corresponds to \(x > 0 (x < 0)\) for NO, the allowed ordering for real \(M_R\).

We are now in a position to state the consequences of this model for real \(M_R\). There are three independent input parameters, namely, \(m_0, \kappa,\) and \(y/x\) which determine \(\theta_{12}, \theta_{13}, \theta_{23},\) and \(\Delta m^2_{\text{solar}}\) for NO. For real \(M_R\) inverted ordering is not permitted.

In Fig. 1 the main consequences of this model for real \(M_R\) are displayed. The region inside the blue dot-dashed box is the \(3\sigma\) range of \(\sin \theta_{13}\) and \(\tan 2\theta_{12}\) from the global fits, the best-fit point being the black dot. From the data in Sec. IV.1 for both octants \(\omega_{\text{min}} = 0\) at \(3\sigma\) and \(\omega_{\text{max}} = 6.6^\circ (-8.3^\circ)\) for the first (second) octant. For the \(\omega_{\text{max}}\) for the first (second) octant eq. (33) of this model corresponds to the green solid (dashed) straight line, the area below being excluded. Further, for real \(M_R\), as \(|\cos \delta| = 1\), from eq. (33) we get \(|\omega| \geq 5.14^\circ\) for both octants. So far, we have not considered the solar mass splitting. Once \(\Delta m^2_{\text{solar}}\) and \(|\Delta m^2_{\text{atmos}}|\) are specified, the \(z\) (or equivalently \(m_0\)) that produces the correct solar splitting for any chosen point in the plane is determined by eq. (31). In this way, using the \(3\sigma\) ranges of \(\theta_{13}\) and \(\theta_{12}\) one finds \(z_{\text{max}} = 6.03 \times 10^{-2}\), corresponding to \((m_0)_{\text{max}} = 3.10\) meV. The consistency of eq. (31) with eq. (33) at \(\omega_{\text{max}}\) sets \(z_{\text{min}} = 4.01 \times 10^{-2}\) \((3.88 \times 10^{-2})\) for the first (second) octant which translates to \((m_0)_{\text{min}} = 2.13\) \((2.06)\) meV. As an example, choosing \(m_0 = 2.5\) meV and taking the best-fit points of the solar and atmospheric mass splittings eq. (29) gives the red dotted curve in Fig. 1.

Figure 1: The area inside blue dot-dashed box in the \(\sin \theta_{13} - \tan 2\theta_{12}\) plane is allowed by the experimental data at \(3\sigma\). The best-fit point is shown as a black dot. The red dotted curve gives the best-fit solar splitting – from eq. (27) – for \(m_0 = 2.5\) meV. Using eq. (33) for \(\theta_{23}\) the area excluded at \(3\sigma\) is below the green solid (dashed) straight line for the first (second) octant. Only normal ordering is allowed for real \(M_R\).
IV.3 Complex $M_R$

The shortcomings of the real $M_R$ case – no CP-violation, inverted ordering disallowed – can be overcome when $M_R$ is complex. One then has, as in eq. (19),

$$M^{\text{mass}} = \frac{m_D^2}{\sqrt{2}xy} \begin{pmatrix} 0 & ye^{i\phi_1} & -ye^{i\phi_1} \\ ye^{i\phi_1} & \frac{x e^{i \phi_2}}{\sqrt{2}} & \frac{x e^{i \phi_2}}{\sqrt{2}} \\ -ye^{i\phi_1} & \frac{x e^{i \phi_2}}{\sqrt{2}} & \frac{x e^{i \phi_2}}{\sqrt{2}} \end{pmatrix}.$$  \hspace{1cm} (34)

$x$ and $y$ are now both positive. Since $M'$ is not hermitian any more the hermitian combination $(M^0 + M')^\dagger (M^0 + M')$ is considered, treating $(M^0)^\dagger M^0$ as the unperturbed piece and $(M^0 M' + M'^\dagger M^0)$ as the perturbation at lowest order. The zero-order eigenvalues are $(m_i^{(0)})^2$. Written as a $3 \times 3$ hermitian matrix the perturbation is

$$(M^0 M' + M'^\dagger M^0)^{\text{mass}} = \frac{m_D^2}{\sqrt{2}xy} \begin{pmatrix} 0 & 2m_1^{(0)} y \cos \phi_1 & -y f(\phi_1) \\ 2m_1^{(0)} y \cos \phi_1 & \frac{2}{\sqrt{2}} m_1^{(0)} x \cos \phi_2 & \frac{1}{\sqrt{2}} x f(\phi_2) \\ -y f^*(\phi_1) & \frac{1}{\sqrt{2}} x f^*(\phi_2) & \frac{1}{\sqrt{2}} m_3^{(0)} x \cos \phi_2 \end{pmatrix}.$$  \hspace{1cm} (35)

Above

$$f(\varphi) = m^+ \cos \varphi - im^- \sin \varphi.$$  \hspace{1cm} (36)

Eqn. (35) provides the basis for the remaining calculation.

In a manner similar to the real $M_R$ case, from eq. (35) we get

$$\tan 2\theta_{12} = 2\sqrt{2} \frac{y}{x} \frac{\cos \phi_1}{\cos \phi_2}.$$  \hspace{1cm} (37)

Since $\tan 2\theta_{12}$ remains positive at $3\sigma$, $\cos \phi_1$ and $\cos \phi_2$ must be of the same sign. The allowed possibilities for these phases are shown in Table 3. We can take over the limits in eq. (24) which now apply on $(y/x)(\cos \phi_1/\cos \phi_2)$.

In place of eq. (26) we now have at first order:

$$|\psi_3\rangle = \begin{pmatrix} -\frac{1}{\sqrt{2}} [1 - \frac{\kappa}{\sqrt{2}} y f(\phi_1)/m^+] \\ \frac{1}{\sqrt{2}} [1 + \frac{\kappa}{\sqrt{2}} y f(\phi_2)/m^+] \end{pmatrix}.$$  \hspace{1cm} (38)

Since $x, y$ are now positive quantities, the sign of $\kappa$ is determined by that of $m^-$, i.e., $\kappa$ is positive (negative) for normal (inverted) ordering. From eqs. (1) and (38)

$$\sin \theta_{13} \cos \delta = -\kappa \cos \phi_1,$$

$$\sin \theta_{13} \sin \delta = -\kappa \frac{m^-}{m^+} \sin \phi_1.$$  \hspace{1cm} (39)

Using eq. (39) one can immediately relate the quadrant of $\delta$ with that of $\phi_1$ for both orderings. These are also presented in Table 3. It can be seen that a near-maximal $\delta = -\pi/2 - \epsilon$ is obtained for normal (inverted) ordering if $\phi_1 \sim -\pi/2 - \epsilon$ ($-\pi/2 + \epsilon$).
| $\phi_1$ quadrant | $\phi_2$ quadrant | Normal Ordering |  | Inverted Ordering |  |
|------------------|------------------|-----------------|-----------------|-----------------|-----------------|
| 0 - $\pi$/2      | 0 - $\pi$/2 or -$\pi$/2 - 0 | -$\pi$ - -$\pi$/2 | 0 - $\pi$/4 | -$\pi$/2 - 0 | $\pi$/4 - $\pi$/2 |
| $\pi$/2 - $\pi$  | $\pi$/2 - $\pi$ or -$\pi$ - -$\pi$/2 | -$\pi$/2 - 0 | $\pi$/4 - $\pi$/2 | -$\pi$ - -$\pi$/2 | 0 - $\pi$/4 |
| -$\pi$ - -$\pi$/2 | $\pi$/2 - $\pi$ or -$\pi$ - -$\pi$/2 | 0 - $\pi$/2 | $\pi$/4 - $\pi$/2 | $\pi$/2 - $\pi$ | 0 - $\pi$/4 |
| -$\pi$/2 - 0     | 0 - $\pi$/2 or -$\pi$/2 - 0 | $\pi$/2 - $\pi$ | 0 - $\pi$/4 | 0 - $\pi$/2 | $\pi$/4 - $\pi$/2 |

Table 3: The options for the phase $\phi_1$ in $M_R$ and the consequent ranges of the other phase $\phi_2$ in $M_R$, the leptonic CP-phase $\delta$, and the octant of $\theta_{23}$ for both mass orderings. All angles are in radians. For inverted ordering or quasi-degeneracy $\delta \sim \pi/2$ or $-\pi/2$.

In addition, for $\theta_{23}$ eq. [38] implies

$$\tan \theta_{23} = \frac{1 - \frac{1}{\sqrt{2}} \cos \phi_2}{1 + \frac{1}{\sqrt{2}} \cos \phi_2}. \quad (40)$$

The deviation from maximality, $\omega$, can be obtained from the above and using eqs. (37) and (39) expressed as

$$\tan \omega = -\frac{2 \sin \theta_{13} \cos \delta}{\tan 2\theta_{12}}, \quad (41)$$

which has the same form as eq. (33) for the real $M_R$ case except that now $\cos \delta$ can deviate from $\pm 1$. The octant of $\theta_{23}$ for different choices of $\phi_1$ quadrants is given in Table 3 for both mass orderings.

Substituting for $m_D^2/m_R$ in terms of $\sin \theta_{13} \cos \delta$, using eq. (39) one has from eq. (35)

$$\Delta m_{solar}^2 = -\text{sgn}(\cos \phi_2) \ m^{-1}_m (0) \frac{4 \sin \theta_{13} \cos \delta}{\sin 2\theta_{12}}, \quad (42)$$

which is reminiscent of eq. (29) for real $M_R$. Keeping in mind that $\cos \phi_1 / \cos \phi_2$ must be positive and using eq. (39) it is easy to see that $\Delta m_{solar}^2 > 0$ always. Since the sign of $\omega$ – i.e., the octant of $\theta_{23}$ – depends on the quadrant of only $\cos \phi_1$, irrespective of the mass ordering one can accommodate both octants while meeting the solar splitting requirement.

As in eq. (31) one again has

$$|\cos \delta| = \left( \frac{\Delta m_{solar}^2}{\Delta m_{atmos}^2} \right) \frac{\sin 2\theta_{12}}{4 \sin \theta_{13} z}, \quad (43)$$

with the further proviso that $|\cos \delta|$ can now be anywhere between zero and unity. This freedom removes the bar which applied on inverted ordering for real $M_R$.

Here we use $m_0, \theta_{13}$, and $\theta_{12}$ as inputs to fix the model parameters. Eqs. (41) and (43) then determine $\theta_{23}$ and $\delta$ respectively, as shown in Figs. 2 and 3. One can also obtain $|m_{\nu_e,\nu_e}|$, which determines the neutrino-less double-beta decay rate, in terms of the mass eigenvalues and the mixing angles. In these figures the green (pink) curves are for NO (IO).

In the left panel of Fig. 2 the dependence of $\theta_{23}$ on $m_0$ is presented while the right panel shows $|m_{\nu_e,\nu_e}|$ again as a function of $m_0$. The thick lines delimit the $3\sigma$ allowed regions while the thin lines correspond
to the best-fit values of input parameters. The solid (dashed) curves are for the first (second) octant of $\theta_{23}$. The thick and thin curves for IO overlap and cannot be distinguished. As expected from eq. (41) $\theta_{23}$ is symmetric about $\pi/4$. The experimental 3$\sigma$ bounds on $\theta_{23}$ for both octants determine a minimum permitted $m_0$ for NO. For IO there is no such lower bound. Planned experiments to measure the neutrino mass [19] are sensitive to $m_0$ not less than 200 meV. From Fig. 2 it is seen that at such a scale the two mass orderings have close predictions, which is a reflection of quasi-degeneracy.

In the left panel of Fig. 3 we show the dependence of $\delta$ on $m_0$ for both NO and IO. The line-type conventions are the same as in Fig. 2. As noted in Table 3 and eq. (39), $\delta$ can be in any of the four quadrants depending on the choice of $\phi_1$. Eq. (43) indicates that for all these four cases, namely, $\pm \delta$ and $\pm (\pi - \delta)$, the dependence of $|\cos \delta|$ on $m_0$ is identical for a chosen mass ordering. Keeping this in mind, Fig. 3 (left panel) has been plotted with $\delta$ in the first quadrant. For $m_0$ smaller than $\sim 10$ meV, $\delta$ differs significantly for the two orderings. As expected from Fig. 1 the real $M_R$ limit, i.e., $\delta = 0$, is obtained only for NO.

The variation of $\delta$ with $\sin^2 2\theta_{13}$ obtained from eq. (42) for both mass orderings for two representative values of $m_0 = 0.5$ eV (solid curves) and 2.5 meV (dashed curves) is shown in the right panel of Fig. 3. Here the best-fit values of the two mass splittings and $\theta_{12}$ have been used. The allowed range of $\sin^2 2\theta_{13}$ from the global fits at 3$\sigma$ (1$\sigma$) is bounded by the blue solid (dot-dashed) vertical lines. Note that in all cases there are solutions for $\delta$ in every quadrant. For IO $\delta$ remains close to $\pm \pi/2$ for all $m_0$. For NO, with $m_0 = 0.5$ eV, which is in the quasi-degenerate region, $\delta$ is the same as for IO while for $m_0 = 2.5$ meV one finds $\delta$ around 0 or $\pm \pi$ and that too for a limited range of $\sin^2 2\theta_{13}$.

In this panel we have also shown 90% C.L. exclusion limits in the $\sin^2 2\theta_{13} - \delta$ plane – dotted curves – identified by the T2K collaboration. The regions to the left of the curves are disfavoured. Notice that $\delta = -\pi/2$ is preferred, which in our model is consistent with IO for all masses but a limited range of $\sin^2 2\theta_{13}$ while for NO though the full range of the latter is consistent one must have $m_0 \geq 100$ meV. More precise measurements of neutrino parameters will test this model closely.

Finally, it is noteworthy that the phase $\phi_2$ enters only in three places: in the combination $x \cos \phi_2/y$
in the expressions for $\tan 2\theta_{12}$ and $\tan \theta_{23}$ - eqs. (37) and (40), and as $\text{sgn}(\cos \phi_2)$ in the formula for the solar splitting - eq. (42). So, its effect can be entirely subsumed by redefining $\cos \phi_2/y \rightarrow 1/y$ and permitting $y$ to be both positive and negative. Therefore for complex $M_R$ the free input parameters are really $m_0$, $\kappa$, $y/x$ and $\phi_1$ which determine the three mixing angles, the solar mass splitting, and the CP-phase $\delta$.

Before concluding, we would like to make a comment on our choice of $M_R$. In eq. (16) two four-zero textures, $M_1$ and $M_2$, were identified both of which could be admissible for $M_R$. We had chosen $M_1$ for this work. If instead, we had chosen $M_2$ then the discussion would go through essentially unchanged except for the replacement $\kappa \rightarrow -\kappa$.

V Conclusions

We have proposed a model for neutrino masses and lepton mixing which relies on an underlying $A_4$ symmetry. All masses are generated from $A_4$ invariant Yukawa couplings. There are contributions to the neutrino masses from both Type-I and Type-II see-saw terms of which the latter is dominant. It generates the atmospheric mass splitting and keeps the mixing angles either maximal, e.g., $\pi/4$ for $\theta_{23}$, or vanishing, for $\theta_{13}$ and $\theta_{12}$. The solar splitting is absent. The Type-I see-saw contribution, which is treated perturbatively, results in $\theta_{13}$, $\theta_{12}$, and $\theta_{23}$ consistent with the global fits and generates the solar splitting. Both octants of $\theta_{23}$ are permitted. Testable relationships between these quantities, characteristic of this model, are derived. As another example, inverted ordering of neutrino masses is correlated with a near-maximal CP-phase $\delta$ and allows arbitrarily small neutrino masses. For normal ordering $\delta$ can vary over the entire range and approaches maximality in the quasi-degenerate limit. The lightest neutrino mass cannot be lower than a few meV in this case.
While this paper was being finalised, NOνA announced [20] their preliminary results based on the equivalent of $2.74 \times 10^{20}$ p.o.t. With $\sin^2 \theta_{23} = 0.50$ the data favour NO and at 90% CL indicate $\delta$ between $-\pi$ to 0 with a preference for $\delta \sim -\pi/2$. As seen from Fig. 3 this is consistent with our model, with $\delta \sim -\pi/2$ favouring $m_0$ in the quasi-degenerate regime, i.e., $m_0 \geq \mathcal{O}(0.1 \text{ eV})$. If this result is confirmed by further analysis then the model will require neutrino masses to be in a range to which ongoing experiments will be sensitive [21].

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**Appendix: The scalar potential minimization**

In this Appendix we discuss the nature of the scalar potential of the model in some detail. We also identify the conditions which must be satisfied by the parameters of the potential so that the vevs take the values considered in the model. Needless to say, the conditions ensure that the potential is locally minimized by this choice. To check whether it is also a *global* minimum is beyond the scope of this paper.

The scalars listed in Table 2 have fields transforming under $A_4$, $SU(2)_L$, and $U(1)_Y$ which also carry a lepton number. The scalar potential has to be of the most general quartic form which is a singlet under all these symmetries. Below we include all terms that are permitted by the symmetries. Invariance under $SU(2)_L$, $U(1)_Y$ and lepton number are easy to verify.

### A.1 $A_4$ invariants: Notation and generalities

Here we give a brief account of our notation and the $A_4$-invariant terms. First recall that there are scalars which transform as $1, 1', 1''$, and 3 under $A_4$. One must include in the potential up to quartics in these fields which give rise to $A_4$ singlets. The product rules of the one-dimensional representations $1, 1'$ and $1''$ are simple, it is the $A_4$ triplet which requires some discussion. To this end consider two $A_4$ triplet fields $A \equiv (a_1, a_2, a_3)^T$ and $B \equiv (b_1, b_2, b_3)^T$ where $a_i, b_i$ may have $SU(2)_L \times U(1)_Y$ transformation properties which we suppress here. As noted in eq. (3), combining $A$ and $B$ one can obtain

$$3_A \otimes 3_B = 1 \oplus 1' \oplus 1'' \oplus 3 \oplus 3 \; .$$  \hspace{1cm} (A.1)

We denote the irreducible representations on the right-hand-side by $O_1(A, B)$, $O_2(A, B)$, $O_3(A, B)$, $T_3(A, B)$ and $T_3(A, B)$, respectively, where, as noted in eqs. (4) [5]

$$O_1(A, B) \equiv \; 1 = a_1 b_1 + a_2 b_2 + a_3 b_3 \equiv \rho_{1ij} a_i b_j \; ,$$

$$O_2(A, B) \equiv \; 1' = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \equiv \rho_{3ij} a_i b_j \; ,$$

$$O_3(A, B) \equiv \; 1'' = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \equiv \rho_{2ij} a_i b_j \; ,$$  \hspace{1cm} (A.2)

and

$$T_3(A, B) \equiv \; 3 = \left( \frac{a_2 b_3 + a_3 b_2}{2}, \frac{a_3 b_1 + a_1 b_3}{2}, \frac{a_1 b_2 + a_2 b_1}{2} \right)^T \; .$$

\footnote{For example, the global minima of the relatively simple case of one $A_4$ triplet $SU(2)_L$ doublet scalar multiplet have been identified in [22] and used in the context of a model for leptons in [23].}
\[ T_a(A, B) \equiv 3 = \left( \frac{a_2 b_3 - a_3 b_2}{2}, \frac{a_3 b_1 - a_1 b_3}{2}, \frac{a_1 b_2 - a_2 b_1}{2} \right)^T. \]  

Note that \( O_3(A^\dagger, A) = [O_2(A^\dagger, A)]^\dagger \) and \( T_a(A, A) = 0 \).

The scalar potential can be written in this notation keeping in mind the following:

- No two scalar multiplets have all quantum numbers the same. So terms in the potential cannot be related by replacing any one field by some other.
- Neither is there a scalar which is a singlet under all the symmetries.

Thus the scalar potential can consist of terms of the following forms (displaying only \( A^4 \) behaviour):

1. Quadratic: \( W^\dagger W \),
2. Cubic: \( X, X_j^\prime X''_k, X_i^r X_j^l X_k, X_i' X_j' X_k', X''_i X''_j X_k'', O_1(Y_i, Y_j) X_k, O_2(Y_i, Y_j) X_k'', O_3(Y_i, Y_j) X_k' \),
3. Quartic:
   \[ (W_i^\dagger W_i)(W_j^\dagger W_j), (X_i X_j)(X_k X_l), (X_i X_j)(X_k' X_l'), (X_i' X'_j)(X_k' X_l'), (X''_i X''_j)(X_k'' X_l''), \]
   \[ O_1(Y_i, Y_j) X_k X_l, O_1(Y_i, Y_j) X_k' X_l', O_2(Y_i, Y_j) X_k X_l', O_2(Y_i, Y_j) X_k' X_l'', \]
   \[ O_3(Y_i, Y_j) X_k' X_l', O_3(Y_i, Y_j) X_k X_l', \]
   \[ O_1(Y_i, Y_j) O_1(Y_k, Y_l) O_2(Y_k, Y_i) O_3(Y_k, Y_l) O_3(Y_k, Y_l) O_3(Y_k, Y_i) O_2(Y_k, Y_i) O_3(Y_k, Y_l), \]
   \[ O_1(T_a(Y_i, Y_j), T_a(Y_k, Y_l)) O_1(T_a(Y_i, Y_j), T_a(Y_k, Y_l)), O_1(T_a(Y_i, Y_j), T_a(Y_k, Y_l)) O_1(T_a(Y_i, Y_j), T_a(Y_k, Y_l)). \]

In the above \( W \) is any field, \( X, X' \), and \( X'' \) stand for generic fields transforming as 1, 1', and 1'' under \( A^4 \) while \( Y \) is a generic \( A^4 \) triplet field. We have not separately listed the invariants formed using \( X^\dagger, X'^\dagger, X''^\dagger \), and \( Y^\dagger \).

Because of the large number of scalar fields in our model – e.g., \( SU(2)_L \) singlets, doublets, and triplets – the scalar potential has many terms. To simplify this discussion, we exclude cubic terms in the fields and take all couplings in the potential to be real. For ease of presentation, we list the potential in separate pieces: (a) those restricted to any one \( SU(2)_L \) sector, and (b) those coupling scalars of different \( SU(2)_L \) sectors. Since the \( vev \) of the \( SU(2)_L \) singlets, which are responsible for the right-handed neutrino mass, are much larger than that of the other scalars, in the latter category we keep only the terms which couple the singlet fields to either the doublet or the triplet sectors.

### A.2 \( SU(2)_L \) Singlet Sector:

In the \( SU(2)_L \) singlet scalar sector there is an \( A^4 \) triplet \( \hat{\Delta}^R \) and another scalar \( \Delta_3^R \) that transforms as a 1''. Eq. (A.11) shows that two \( \hat{\Delta}^R \) triplets can combine to give different \( A^4 \) irreducible representations. For this purpose we introduce the notations:

\[ O_1^{ss} \equiv O_1(\hat{\Delta}^{R\dagger}, \hat{\Delta}^R); \quad O_2^{ss} \equiv O_2(\hat{\Delta}^{R\dagger}, \hat{\Delta}^R); \quad T_s^{ss} \equiv T_s(\hat{\Delta}^R, \hat{\Delta}^R). \]  

(A.4)
Generically, we will use the notation \( \tilde{O}_i \) or \( \tilde{T}_{s,a} \) if the second A4 triplet field in the argument is replaced by its hermitian conjugate. For example, here

\[
\tilde{O}_3^s \equiv O_3(\hat{\Delta}^R, \hat{\Delta}^R) \text{ and } \tilde{T}_s^{ss} \equiv T_s(\hat{\Delta}^R, \hat{\Delta}^R).
\] (A.5)

We will also require the combinations:

\[
\tilde{O}_2^s \equiv O_2(\hat{\Delta}^R, T_s^{ss†}).
\] (A.6)

From the A4 singlet \( \Delta_R^3 \) one can make the combination

\[
Q_{ss}^3 \equiv \Delta_R^3 \Delta_R^3,
\] (A.7)

which is obviously a singlet under all the symmetries.

Using this notation the most general scalar potential of this sector is given by:

\[
V_{\text{singlet}} = m_2^2 \Delta_R^3 Q_{ss}^3 + m_2^2 O_2^{ss} + \frac{1}{2} \lambda_1^s \left[ Q_1^{ss} \right]^2 + \frac{1}{2} \lambda_2^s \left\{ \left( O_1^{ss} \right)^2 + (O_2^{ss})^\dagger O_2^{ss} + O_1(T_s^{ss}, T_s^{ss†}) \right\}
+ \frac{1}{2} \lambda_3 \left[ O_3^{ss} \Delta_R^3 + \text{h.c.} \right] + \lambda_5^s \left[ \bar{O}_3^{ss} \Delta_R^3 \Delta_R^3 + \text{h.c.} \right].
\] (A.8)

In the above, we have taken \( \lambda_3^s \) as the common coefficient of the different A4-singlets that can be obtained from the combination of two \( \hat{\Delta}^R \) and two \( (\hat{\Delta}^R)^† \) fields. We also follow a similar principle for the fields with other \( SU(2)_L \) behaviour.

### A.3 \( SU(2)_L \) Doublet Sector:

The \( SU(2)_L \) doublet scalar sector comprises of the two fields \( \Phi \) and \( \eta \) transforming as 3 and 1 under A4 respectively. Recall that \( \Phi \) and \( \eta \) have opposite hypercharge. In analogy to the singlet sector we denote the required A4 triplet \( \Phi \) combinations as:

\[
O_1^{dd} \equiv O_1(\Phi^†, \Phi); \quad O_2^{dd} \equiv O_2(\Phi^†, \Phi); \quad T_s^{dd} \equiv T_s(\Phi, \Phi),
\] (A.9)

and from the A4 singlet \( \eta \)

\[
Q_{\eta}^{dd} \equiv \eta^† \eta.
\] (A.10)

The potential for this sector is:

\[
V_{\text{doublet}} = m_2^2 Q_\eta^{dd} + m_2^2 O_1^{dd} + \frac{1}{2} \lambda_1^d \left[ Q_\eta^{dd} \right]^2 + \frac{1}{2} \lambda_2^d \left\{ \left( O_1^{dd} \right)^2 + (O_2^{dd})^\dagger O_2^{dd} \right\}
+ O_1(T_s^{dd}, T_s^{dd†}) \right\} \right] + \frac{1}{2} \lambda_3^d \left[ Q_\eta^{dd} O_1^{dd} \right].
\] (A.11)

### A.4 \( SU(2)_L \) Triplet Sector:

The \( SU(2)_L \) triplet sector consists of four fields, viz, \( \hat{\Delta}_L^1 \), \( \Delta_L^1 \), \( \Delta_L^2 \) and \( \Delta_L^3 \) transforming as 3, 1, 1', 1'' under A4.
We define
\[ O_i^{tt} \equiv O_i(\hat{\Delta}^{L\dagger}, \hat{\Delta}^{L}); \quad O_i^{tt} \equiv O_2(\hat{\Delta}^{L\dagger}, \hat{\Delta}^{L}); \quad T_s^{tt} \equiv T_s(\hat{\Delta}^{L}, \hat{\Delta}^{L}), \]
\[ Q_i^{tt} \equiv \Delta_i^{L\dagger} \Delta_i^{L}, \quad (i = 1, 2, 3), \]
and
\[ \varrho_i^{tt} \equiv O_i(\hat{\Delta}^{L}, T_s^{tt}) \quad (i = 1, 2, 3). \]

The scalar potential for this sector:
\[ V_{\text{triplet}} = \sum_{i=1}^{3} m_i^{2} Q_i^{tt} + m_i^{2} O_i^{tt} + \frac{1}{2} \sum_{i=1}^{3} \lambda_i^{1t} [Q_i^{tt}]^2 + \frac{1}{2} \sum_{k<j, k=1}^{2} \lambda_{k,j}^{t} Q_{j}^{tt} Q_{k}^{tt} \]
\[ + \frac{1}{2} \lambda_{5}^{t} \{[O_i^{tt}]^2 + \{O_j^{tt}\}^{t} O_2^{tt} + O_1^{tt} (T_s^{tt}, T_s^{tt})\} + \frac{1}{2} \sum_{i=1}^{3} \lambda_i^{tt} [Q_i^{tt} O_i^{tt}] \]
\[ + \lambda_{5}^{t} \varrho_i^{tt} \Delta_i^{L} + \text{h.c.} + \lambda_{6}^{t} \varrho_i^{tt} \Delta_i^{L} + \text{h.c.} + \lambda_{7}^{t} \varrho_i^{tt} \Delta_i^{L} + \text{h.c.} \]
\[ + \sum_{i=1}^{3} \lambda_{6}^{tt} \{\varrho_i^{tt} \Delta_i^{L} \Delta_i^{L} + \text{h.c.}\} + \lambda_{8}^{tt} \varrho_i^{tt} \Delta_i^{L} \Delta_i^{L} + \text{h.c. + cyclic} \] .

\[ \text{(A.15)} \]

A.5 Inter-sector terms:

So far, we have listed the terms in the potential that involve scalar fields which belong to any one of three sectors: singlets, doublets, or triplets of SU(2)\(_L\). Besides these, there will also be terms in the scalar potential which involve fields from multiple sectors. Below we list the terms which arise from couplings of the singlet sector with either the doublet or the triplet sector. The other inter-sector terms – doublet-triplet type – are dropped. Since the vacuum expectation values of the singlet fields are by far the largest this is not an unreasonable approximation.

A.5.1 Inter-sector Singlet-Doublet terms:

It is useful to define,
\[ \tilde{T}_{ss}^{ss} \equiv T_s(\hat{\Delta}^{R}, \hat{\Delta}^{R\dagger}), \quad \text{and} \quad \tilde{T}_{ss}^{dd} \equiv T_s(\Phi, \Phi^{\dagger}), \]
\[ \text{(A.16)} \]
and
\[ O_{1S}^{sd} \equiv O_1(\tilde{T}_{ss}^{dd}, \tilde{T}_{ss}^{ss}), \quad \varrho_{3}^{sd} \equiv O_3(\hat{\Delta}^{R}, \hat{\Delta}^{R\dagger}) \] .
\[ \text{(A.17)} \]

For simplicity, we do not keep the combinations \( \tilde{T}_{ss}^{ss} \equiv T_a(\hat{\Delta}^{R}, \hat{\Delta}^{R\dagger}) \) and \( \tilde{T}_{ss}^{dd} \equiv T_a(\Phi, \Phi^{\dagger}) \).

In terms of the above:
\[ V_{sd} = \frac{1}{2} \lambda_{1}^{sd} \left[ Q_3^{ss} Q_7^{dd} \right] + \frac{1}{2} \lambda_{2}^{sd} \left[ Q_3^{ss} O_1^{dd} \right] + \frac{1}{2} \lambda_{3}^{sd} \left[ Q_9^{dd} O_1^{ss} \right] + \lambda_{4}^{sd} \left[ \{ \varrho_{3}^{sd} \}^{\dagger} \hat{\Delta}^{R} + \text{h.c.} \right] \]
\[ + \frac{1}{2} \lambda_{5}^{sd} \left[ O_1^{dd} O_1^{ss} + \{ O_2^{ss} \}^{\dagger} O_2^{dd} + \{ O_2^{dd} \}^{\dagger} O_2^{ss} + O_{1S}^{sd} \right] . \]
\[ \text{(A.18)} \]

Here, in the last term, we have made the simplifying assumption that there is a common coupling \( \lambda_{5}^{sd} \) for the terms in the potential which arise from various combinations of \( (\Phi^{\dagger} \Phi)(\hat{\Delta}^{R\dagger}\hat{\Delta}^{R}) \), each of the four fields being A4 triplets.
A.5.2 Inter-sector Singlet-Triplet terms:

For this case the following combinations arise:
\[
O^{ts}_i \equiv O_i(\hat{\Delta}_i^R, \hat{\Delta}_i^L) \quad (i = 1, 2, 3); \quad O^{ts}_3 \equiv O_3(\hat{T}_3^L, \hat{T}_3^R);
\]
\[
\hat{O}^{ts}_i \equiv O_i(\hat{T}_i^L, \hat{T}_i^R) \quad (i = 1, 2, 3); \quad \hat{O}^{ts}_3 \equiv O_3(\hat{\tilde{T}}_3^L, \hat{\tilde{T}}_3^R) .
\]
(A.19)

The intersector potential for this case is given by:
\[
V_{ts} = \frac{1}{2} \sum_{i=1}^{3} \lambda_{1i}^{ts} \left[ O_{i3}^s Q_{i1}^U \right] + \frac{1}{2} \lambda_{2}^{ts} \left[ Q_{3s}^s O_{11}^U \right] + \frac{1}{2} \sum_{i=1}^{3} \lambda_{3i}^{ts} \left[ O_{i1}^U O_{i1}^s \right] \\
+ \frac{1}{2} \lambda_{4}^{ts} \left[ O_{1}^U O_{1}^{ss} + \{ O_{2}^{ss} \} O_{2}^U + \{ O_{2}^U \} O_{2}^{ss} + O_{1S}^{ts} \right] \\
+ \sum_{i=1}^{3} \lambda_{5i}^{ts} \left[ \hat{O}^{ts}_i \Delta_i^L \right] + h.c. \right] + \lambda_{6}^{ts} \left[ \hat{O}^{ts}_3 \Delta_3^L \right] + h.c. \right] \\
+ \lambda_{7}^{ts} \left[ O_{1}^{ts} \Delta_5^L \Delta_3^R + h.c. \right] + \lambda_{8}^{ts} \left[ O_{2}^{ts} \Delta_1^L \Delta_3^R + h.c. \right] + \lambda_{9}^{ts} \left[ O_{3}^{ts} \Delta_2^L \Delta_3^R + h.c. \right] \\
+ \lambda_{10}^{ts} \left[ \hat{O}^{ts}_3 \Delta_3^R \Delta_3^L + h.c. \right] + \lambda_{11}^{ts} \left[ \hat{O}^{ts}_1 \Delta_3^R \Delta_3^L + h.c. \right] + \lambda_{12}^{ts} \left[ \hat{O}^{ts}_3 \Delta_3^R \Delta_3^L + h.c. \right] .
\]
(A.20)

A.6 The minimization conditions:

After having presented the scalar potential we now seek to find the conditions under which the vev we have used in the model – see eqs. (7) and (8) and Table 2 – constitute a local minimum. For ready reference the vev are:
\[
\langle \Phi^0 \rangle = \frac{v}{\sqrt{3}} \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right), \quad \langle \hat{\Delta}^L \rangle = v_L \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \quad \langle \hat{\Delta}^R \rangle = v_R \left( \begin{array}{c} 1 \\ \omega^2 \\ \omega \end{array} \right), \quad \langle \hat{\Delta}^R \rangle = v_R \left( \begin{array}{c} 1 \\ \omega^2 \\ \omega \end{array} \right), \quad \langle \hat{\Delta}^R \rangle = v_R .
\]
(A.21)
\[
\langle \Delta^L \rangle = u_L, \quad \langle \Delta^L \rangle = \langle \Delta^L \rangle = \langle \Delta^L \rangle = u_L, \quad \langle \Delta^R \rangle = u_R .
\]
(A.22)

where the SU(2)_{L} nature of the fields is suppressed.

It can be seen from eq. (A.21) that the A4 triplet fields – \hat{\Delta}^L, R and \Phi – acquire vev which have been shown to be global minima in [22]. While this is certainly encouraging, that result is for one A4 triplet in isolation. Here there are many other fields and so it is not straight-forward to directly extend the results of [22].

In the following we list, sector by sector, the conditions under which the vev in eqs. (A.21) and (A.22) correspond to a minimum.

A.6.1 SU(2)_{L} singlet sector:

The vev of the singlet fields \hat{\Delta}^R and \Delta^R_{3} are much larger than those of the SU(2)_{L} doublet and triplet scalars. So, the contributions to the minimization equations from the inter-sector terms can be neglected.
Using the singlet sector potential in eq. (A.8) and the vev in eqs. (A.21) and (A.22) we get (bearing in mind \( v_R \) is real):

\[
\frac{\partial V_{\text{singlet}}|_{\min}}{\partial u_R} = 0 \Rightarrow u_R \left[ m_\Delta^2 + \frac{3}{2} \lambda_5^2 v_R^2 \right] + 3 v_R^2 \left( \lambda_4^2 v_R + 2 \lambda_5^2 u_R \right) = 0 , \quad (A.23)
\]

and

\[
\frac{\partial V_{\text{singlet}}|_{\min}}{\partial v_{Ri}} = 0
\]

\[
\Rightarrow v_R \left[ m_\Delta^2 + 4 \lambda_2^2 v_R^2 + \frac{\lambda_5^2}{2} u_R^2 + \lambda_4^2 u_R + 2 \lambda_5^2 v_R^2 \right] = 0 . \quad (A.24)
\]

### A.6.2 \( SU(2)_L \) doublet sector:

In this sector we have to include the contributions from both the doublet sector itself – eq. (A.11) – as well as the inter-sector terms in eq. (A.18). We define \( V_\phi = V_{\text{doublet}} + V_{sd} \).

In order that the potential minimum corresponds to the vev in eqs. (A.21) and (A.22) we must have:

\[
\frac{\partial V_\phi|_{\min}}{\partial u} = 0 \Rightarrow u \left[ 2m_{\eta}^2 + 2 \lambda_1^d u^* v + \lambda_3^s u^*_R u_R + 3 \lambda_3^{sd} v_R^2 \right] = 0 . \quad (A.25)
\]

and

\[
\frac{\partial V_\phi|_{\min}}{\partial v_i^*} = 0
\]

\[
\Rightarrow \frac{v}{\sqrt{3}} \left[ m_\Phi^2 + \frac{4 \lambda_2^d}{3} \left( v^* v \right) + \frac{2}{3} \lambda_3^{sd} u_R^* u_R + \frac{5}{4} \lambda_5^{sd} v_R^2 \right] = 0 . \quad (A.26)
\]

Notice that one has to resort to some degree of fine-tuning to satisfy eqs. (A.25) and (A.26) which involve both \( SU(2)_L \) doublet and singlet vev of quite different magnitudes.

### A.6.3 \( SU(2)_L \) triplet sector:

Using eqs. (A.8) and (A.20) we define \( V_\phi = V_{\text{triplet}} + V_{ts} \).

In this sector there are a plethora of couplings. To ease the presentation we choose

\[
m_\Delta^2 = m_\Delta^L = m_\Delta^R = m_\Delta^L ; \quad \lambda_{11}^t = \lambda_{12}^t = \lambda_{13}^t = \lambda_{a}^t ; \quad \lambda_{41}^t = \lambda_{42}^t = \lambda_{43}^t = \lambda_{b}^t
\]

\[
\lambda_{221}^t = \lambda_{222}^t = \lambda_{231}^t = \lambda_{c}^t ; \quad \lambda_{81}^t = \lambda_{82}^t = \lambda_{83}^t = \lambda_{d}^t ; \quad \lambda_{50}^t = \lambda_{51}^t = \lambda_{52}^t = \lambda_{g}^t
\]

\[
\lambda_{11}^s = \lambda_{12}^s = \lambda_{13}^s = \lambda_{a}^s ; \quad \lambda_{31}^s = \lambda_{32}^s = \lambda_{33}^s = \lambda_{b}^s ; \quad \lambda_{51}^s = \lambda_{52}^s = \lambda_{53}^s = \lambda_{c}^s
\]

\[
\lambda_{10}^s = \lambda_{11}^s = \lambda_{12}^s = \lambda_{d}^s ; \quad \lambda_{75}^s = \lambda_{8}^s = \lambda_{9}^s = \lambda_{f}^s . \quad (A.27)
\]
For the minimization of \( V_T \) so as to arrive at the vev in eqs. \( \text{(A.21)} \) and \( \text{(A.22)} \) one must satisfy:

\[
\frac{\partial V_T}{\partial u_L} \bigg|_{\text{min}} = 0
\]
\[
\Rightarrow u_L \left[ m_{\Delta L}^2 + (\lambda_{t}^u + \lambda_{c}^u) u_L^* u_L + \frac{1}{2} \lambda_{b}^t v_L^* v_L + \frac{1}{2} \lambda_{t}^s u_R^* u_R + \frac{3}{2} \lambda_{t}^s v_R^2 \right]
+ 2v_L^2 u_L^*(\lambda_{t}^d + \lambda_{e}^u) + v_L v_R \left[ -\frac{1}{2} \lambda_{d}^s v_R + \lambda_{t}^s u_R^* + \lambda_{t}^s u_R \right] = 0. \tag{A.28}
\]

Again:

\[
\frac{\partial V_T}{\partial v_L^*} \bigg|_{\text{min}} = 0
\]
\[
\Rightarrow v_L \left[ m_{\Delta L}^2 + \frac{3}{2} \lambda_{b}^t u_L^* u_L + 2\lambda_{b}^t v_L^* v_L + \frac{1}{2} \lambda_{d}^s u_R^* u_R + \frac{3}{2} \lambda_{e}^s v_R^2 \right]
+ u_L \left[ 6u_L v_L^*(\lambda_{d}^u + \lambda_{e}^u) - \frac{3}{2} \lambda_{d}^s v_R^2 + 3\lambda_{t}^s u_R^* v_R + 3\lambda_{c}^s u_R v_R \right] = 0. \tag{A.29}
\]

Also we have

\[
\frac{\partial V_T}{\partial v_{L2}^*} = \frac{\partial V_T}{\partial v_{L3}^*} = 0
\]
\[
\Rightarrow v_L v_R \left[ -\frac{1}{4} \lambda_{4}^s v_R + \lambda_{5}^s (u_R^* + u_R) \right] = 0. \tag{A.30}
\]

Here again fine-tuning is required to ensure that eqs. \( \text{(A.28)} \) - \( \text{(A.30)} \) are satisfied.

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