ON UNITARITY OF SOME REPRESENTATIONS OF CLASSICAL $p$-ADIC GROUPS I

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Abstract. In the case of $p$-adic general linear groups, each irreducible representation is parabolically induced by a tensor product of irreducible representations supported by cuspidal lines. One gets in this way a parameterization of the irreducible representations of $p$-adic general linear groups by irreducible representations supported by cuspidal lines. It is obvious that in this correspondence an irreducible representation of a $p$-adic general linear group is unitarizable if and only if all the corresponding irreducible representations supported by cuspidal lines are unitarizable.

C. Jantzen has defined in [18] an analogue of such correspondence for irreducible representations of classical $p$-adic groups. It would have interesting consequences if one would know that the unitarizability is also preserved in this case. A purpose of this paper and its sequel, is to give some very limited support for possibility of such preservation of the unitarizability. More precisely, we show that if we have an irreducible unitarizable representation $\pi$ of a classical $p$-adic group whose one attached representation $\pi_L$ supported by a cuspidal line $L$ has the same infinitesimal character as the generalized Steinberg representation supported by that cuspidal line, then $\pi_L$ is unitarizable.

1. Introduction

Although in the case of classical $p$-adic groups a number of important irreducible unitary representations is known and some important classes of irreducible unitary representations of these groups are classified, we do not know much about the general answer of the unitarizability in this case. Since the case of general linear groups is well-understood, we shall start with description of the unitarizability in the case of these groups, the history related to this and what this case could suggest us regarding the unitarizability for the classical $p$-adic groups.

Fix a local field $F$. We shall use a well-known notation $\times$ of Bernstein and Zelevinsky for parabolic induction of two representations $\pi_i$ of $GL(n_i, F)$:

$$\pi_1 \times \pi_2 = \text{Ind}_{GL(n_1, F) \times GL(n_2, F)}^{GL(n_1 + n_2, F)}(\pi_1 \otimes \pi_2)$$

(the above representation is parabolically induced from a suitable parabolic subgroup containing upper triangular matrices whose Levi factor is naturally isomorphic to the direct product $GL(n_1, F) \times GL(n_2, F)$). Denote by $\nu$ the character $|\det|_F$ of a general linear
group. Let \( D_u = D_u(F) \) be the set of all the equivalence classes of the irreducible square integrable (modulo center) representations of all \( GL(n, F) \), \( n \geq 1 \). For \( \delta \in D_u \) and \( m \geq 1 \) denote by

\[
u^{(m-1)/2} \delta \times \nu^{(m-1)/2 - 1} \delta \times \ldots \times \nu^{-(m-1)/2} \delta.
\]

This irreducible quotient is called a Speh representation. Let \( B_{\text{rigid}} \) be the set of all Speh representations, and

\[
B = B(F) = B_{\text{rigid}} \cup \{ \nu^\alpha \sigma \times \nu^{-\alpha} \sigma; \sigma \in B_{\text{rigid}}, 0 < \alpha < 1/2 \}.
\]

Denote by \( M(B) \) the set of all finite multisets in \( B \). Then the following simple theorem solves the unitarizability problem for the archimedean and the non-archimedean general linear groups in a uniform way:

**Theorem 1.1.** ([35], [42]) A mapping

\[
(\sigma_1, \ldots, \sigma_k) \mapsto \sigma_1 \times \ldots \times \sigma_k
\]

defined on \( M(B) \) goes into \( \cup_{n \geq 0} \hat{GL}(n, F) \), and it is a bijection.

The above theorem was first proved in the \( p \)-adic case (in [35]). Since the claim of the theorem makes sense also in the archimedean case, immediately became evident that the theorem extends also to the archimedean case, with the same strategy of the proof (the main ingredients of the proof were already present in that time, although one of them was announced by Kirillov, but the proof was not complete in that time). One can easily get an idea of the proof from [34] (there is considered the \( p \)-adic case, but exactly the same strategy holds in the archimedean case). Vogan’s classification in the archimedean case (Theorem 6.18 of [45]) gives a very different description of the unitary dual (it is equivalent to Theorem 1.1 but it is not obvious to see that it is equivalent).

The proof of Theorem 1.1 in [35] is based on a very subtle Bernstein-Zelevinsky theory based on the derivatives ([46]), and on the Bernstein’s paper ([7]). Among others, the Bernstein’s paper ([7]) proves a fundamental fact about distributions on general linear groups. It is based on the geometry of these groups (a key idea of that paper can be traced to the Kirillov’s paper [20], which is motivated by a result of the Gelfand-Naimark book ([11])).

We presented in [36] what we expected to be the answer to the unitarizability question for general linear groups over a local non-archimedean division algebra \( A \). For \( \delta \in D(A) \) denote by \( \nu_\delta := \nu^{s_\delta} \), where \( s_\delta \) is the smallest non-negative number such that \( \nu^{s_\delta} \delta \times \delta \) reduces. Introduce \( u(\delta, n) \) in the same way as above, except that we use \( \nu_\delta \) in the definition of \( u(\delta, n) \) instead of \( \nu \). Then the expected answer is the same as in the Theorem 1.1, except that one replace \( \nu \) by \( \nu_\delta \) in the definition of \( B(A) \). We have reduced in [36] a proof of the expected answer to two expected facts.
The first of these two expected facts was proved by J. Badulescu and D. Renard (\cite{4}), while the second one was proved by V. Sécherée (\cite{32}). These two proofs were not simple (the first one required global methods, while the second one required knowledge of a complete theory of types for these groups).

J. Badulescu gave in \cite{3} another very simple (local) proof of his and Renard’s result. Relaying on the Jacquet module methods, E. Lapid and A. Mínguez gave in \cite{21} (among others) another proof of the Sécherée result. The proof of Lapid and Mínguez (based on Jacquet modules) is surprisingly simple in comparison with the Sécherée’s proof. Their proof also reproves the Bernstein’s results of the irreducibility of the unitary parabolic induction in the field case (the Sécherée’s proof uses it). Actually, the characteristics of the both new proofs is that they use very standard non-unitary theory (both proofs are based on the reducibility point between two irreducible cuspidal representations of general linear groups, i.e. when $\rho \times \rho'$ reduces for $\rho$ and $\rho'$ irreducible cuspidal representations).

Further, one can find in the appendix of \cite{21} simple proofs of the facts of the non-unitary theory necessary for the unitarizability of general linear groups over $\mathcal{A}$ (see also \cite{44}). These proofs rely only on the above mentioned reducibility point between irreducible cuspidal representations and some standard facts of the representation theory of reductive $p$-adic groups (obtained mainly in 1970-es).

Therefore now we have a relatively simple solution of the unitarizability problem for (split and non-split) $p$-adic general linear groups, based only on the knowledge of the reducibility point among two cuspidal representations (which is $\pm 1$ in the split case, while in the non-split case we have finitely many possibilities). It is very remarkable that we have such a relative simple approach to the irreducible unitary representations in this case. These representations are basic ingredients of some very important unitary representations, like the representations in the spaces of the square integrable automorphic forms, and their knowledge can be quite useful (see \cite{23}, or \cite{15} or \cite{16}).

A natural question is if we can have an approach to the unitarizability in the case of classical $p$-adic groups based only on the reducibility point. These reducibilities can be $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and known examples of the unitarizability tell us that the unitarizability answers are different (for reducibilities $0, \frac{1}{2}, 1$, see \cite{30}, \cite{24} for rank two case, and \cite{22}, \cite{29} for the generic and the unramified case in any rank). In the case of the general linear groups, unitarizability answers are essentially always the same.

\footnote{Here is the problem in proving of the irreducibility of the unitary parabolic induction that the Kirillov’s strategy (which was used by Bernstein in the field case) does not work for the non-commutative division algebras.}

\footnote{Thanks to the work of J. Arthur, C. Mœglin and J.-L. Waldspurger, in the characteristic zero case we have now classification of irreducible cuspidal representations. Their parameters give the reducibility points with irreducible cuspidal representations of general linear groups.}
We shall try to explain a possible strategy for such approach based only on the reducibility point (note that now it is not clear if such strategy is possible; known examples do not provide a counterexample). For motivation of our approach, we shall again recall of a well known case of the general linear groups (despite the situation with unitarizability in the case of these two series of groups is very different\(^3\)).

When one deals with the parabolically induced representations of \(p\)-adic general linear groups, one very useful simplification is a reduction to the cuspidal lines. For an irreducible representation \(\pi\) of \(GL(n, F), n \geq 1\), and a cuspidal \(\mathbb{Z}\)-line \(\mathcal{L}\) along \(\rho\) (i.e. the set \(\{\nu^k \rho; k \in \mathbb{Z}\}\)), there exist a unique irreducible representation \(\pi_{\mathcal{L}}\) supported in \(\mathcal{L}\) and an irreducible representation \(\pi'_{\mathcal{L}}\) supported out of \(\mathcal{L}\) such that

\[
\pi \leftrightarrow \pi'_{\mathcal{L}} \times \pi_{\mathcal{L}}.
\]

Denote by \((\pi_{\mathcal{L}_1}, \ldots, \pi_{\mathcal{L}_k})\) the sequence of all \(\pi_{\mathcal{L}}\)’s which are representations of \(GL(m, F)\)’s, with \(m \geq 1\). Then the correspondence

\[
\pi \leftrightarrow (\pi_{\mathcal{L}_1}, \ldots, \pi_{\mathcal{L}_k})
\]

reduces some of the most basic data about general parabolically induced representations (like for example the Kazhdan-Lusztig multiplicities) to the corresponding data for the representations supported by single cuspidal lines.

When one considers unitarizability of the irreducible representations, we can have many complementary series. This is a reason that for the unitarizability questions, it is more convenient to consider parameterization as in (1.2), but with respect to the cuspidal \(\mathbb{R}\)-lines (i.e. the sets \(\{\nu^k \rho; k \in \mathbb{R}\}\)). Then holds

\[
\pi \text{ is unitarizable} \iff \text{all } \pi_{\mathcal{L}_i} \text{ are unitarizable},
\]

This fact follows directly from the simple fact that

\[
\pi \cong \pi_{\mathcal{L}_1} \times \ldots \times \pi_{\mathcal{L}_k}.
\]

In the case of the classical \(p\)-adic groups, C. Jantzen has defined correspondence of type (1.2). Here instead of cuspidal lines \(\mathcal{L}\), one needs to consider union of the line and its contragredient line, i.e. \(\mathcal{L} \cup \tilde{\mathcal{L}}\). A reduction of the unitarizability problem for the classical \(p\)-adic groups is reduced in \([43]\) to the case when all the lines that show up satisfy \(\mathcal{L} = \tilde{\mathcal{L}}\). Because of this, we shall in the sequel consider only the representations \(\pi\) of the classical \(p\)-adic groups for which only the selfcontragredient lines show up.

The multiplication \(\times\) between representations of general linear groups defined by parabolic induction has a natural generalization to a multiplication

\[
\times
\]
between representations of general linear and classical groups (see the second section of this paper). Fix an irreducible cuspidal representation $\sigma$ of a classical group. One says that an irreducible representation $\tau$ of a classical group is supported in $\mathcal{L} \cup \{\sigma\}$ if $\tau \hookrightarrow \rho_1 \times \ldots \times \rho_k \times \sigma$ for some $\rho_i \in \mathcal{L}$. Now one defines the correspondence (1.2) in a similar way as above: for an irreducible representation $\pi$ of a classical group and a selfcontragredient cuspidal $\mathbb{Z}$-line $\mathcal{L}$ along $\rho$, there exists a unique irreducible representation $\pi_{\mathcal{L}}$ of a classical group supported in $\mathcal{L} \cup \{\sigma\}$ and an irreducible representation $\pi_{\mathcal{L}}^c$ of a general linear group supported out of $\mathcal{L}$ such that

\begin{equation}
\pi \leftrightarrow \pi_{\mathcal{L}}^c \times \pi_{\mathcal{L}}.
\end{equation}

C. Jantzen has shown that then the correspondence

\begin{equation}
\pi \leftrightarrow (\pi_{\mathcal{L}_1}, \ldots, \pi_{\mathcal{L}_k})
\end{equation}

reduces again some of the most basic data from the non-unitary theory about general parabolically induced representations (like for example the Kazhdan-Lusztig multiplicities) to the corresponding data for such representations supported in single cuspidal lines. While in the case of general linear groups this is obvious, here it is not that simple ((1.4) does not make sense for the classical groups).

Regarding unitarizability, it would be very important to know if analogue of (1.3) holds here. Known cases give some limited evidence for this. From the other side, we do not know much reason why this should hold.

If we would know that (1.3) holds for the correspondence (1.6) (i.e. for the classical $p$-adic groups), then this would give a reduction of the general unitarizability to the unitarizability for the irreducible representations supported in single cuspidal lines. Further question would be to try to see if the unitarizability for the irreducible representations supported by a single cuspidal line depends only on the reducibility point (i.e., not on particular representations involved which have that reducibility).

This paper and its sequel gives some additional very limited evidence for (1.3) in the case of the classical $p$-adic groups.

Generalized Steinberg representations are defined and studied in [39]. The aim of this paper and its sequel is to prove the following

**Theorem 1.2.** Suppose that $\pi$ is an irreducible unitary representation of a classical group, and suppose that the infinitesimal character of some $\pi_{\mathcal{L}}$ is the same as the infinitesimal character of a generalized Steinberg representation. Then $\pi_{\mathcal{L}}$ is unitarizable.

\[4\] In the paper that follows this one, we shall denote $\pi_{\mathcal{L}}$ differently (closer to the notation of [18]).

\[5\] Known examples of the unitarizability at the generalised rank $\leq 3$ support independence. Also the unitarizability at the generic case supports this.

\[6\] Actually, we prove that $\pi_{\mathcal{L}}$ is equivalent to the generalized Steinberg representation, or its Aubert dual.
For the case when \( \pi = \pi_\mathcal{L} \) (i.e. when \( \pi \) is supported by a single cuspidal line), this theorem is proved in [13] and [14]. Our first idea to prove the above theorem was to use the strategy of that two papers, combined with the methods of [18]. While we were successful in extending [13], we were not for [14]. This was a reason for search of a new (uniform) proof for [13] and [14], which is very easy to extend to the proof of the above theorem (using [18]). This new proof is based on the following fact.

**Proposition 1.3.** Fix an irreducible unitarizable cuspidal representations \( \rho \) and \( \sigma \) of a general linear and a classical group respectively. Suppose that \( \nu^\alpha \rho \rtimes \sigma \) reduces for some \( \alpha \in \frac{1}{2}\mathbb{Z}, \alpha > 0 \). Let \( \gamma \) be an irreducible subquotient of

\[
\nu^{\alpha+n} \rho \times \nu^{\alpha+n-1} \rho \times \cdots \times \nu^\alpha \rho \rtimes \sigma,
\]

different from the generalized Steinberg representation and its Aubert involution. Then there exists an irreducible selfcontragredient unitarizable representation \( \pi \) of a general linear group with support in \( \{ \nu^k \rho; k \in \frac{1}{2}\mathbb{Z} \} \), such that the length of

\[
\pi \rtimes \gamma
\]

is at least 5, and that the multiplicity of \( \pi \otimes \gamma \) in the Jacquet module of \( \pi \rtimes \gamma \) is at most 4.

The main aim of this paper is to prove the above proposition (at the end of this paper we show how one gets easily the main results of [13] and [14] from the above proposition).

Now we shall recall a little bit of the history of the unitarizability of the irreducible representations which have the same infinitesimal character as a generalized Steinberg representation. The first case is the case of the Steinberg representations. The question of their unitarizability in this case came from the question of cohomologically non-trivial irreducible unitary representations. Their non-unitarizability or unitarizability was proved by W. Casselman ([9]). His proof of the non-unitarizability relays on the study of the Iwahori-Matsumoto Hecke algebra. The importance of this non-unitarizability is very useful in considerations of the unitarizability in low ranks, since it implies also non-existence of some complementary series (it also reproves the classical result of Kazhdan from [19] in the \( p \)-adic case).

A. Borel and N. Wallach observed that the Casselman’s non-unitarizability follows from the Howe-Moore theorem about asymptotics of the matrix coefficients of the irreducible unitary representations ([17]) and the Casselman’s asymptotics of the matrix coefficients of the admissible representations of reductive \( p \)-adic groups ([8]). Neither of that two methods can be used for the case of the generalized Steinberg representation. This was a motivation to write papers [13] and [14]. The strategy of that two papers was that the considered non-unitarity representations tensored by unitary representations and then

\[7\]Although it is easy to extend, we leave this extension for the following paper (we shall present there a little bit reformulated interpretation of the Jantzen’s main results from [18], which makes this extension technically a little bit simpler)
parabolically induced, easily produces a non-semisimplicity. The semisimplicity of parabolically induced representation has very well-known consequences (after application of the Frobenius reciprocity). Therefore if these consequences are missing, then we can conclude the non-unitarizability.

The difference between methods of \[13\] and \[14\], and our approach in this paper, is that in \[13\] and \[14\] the consequences were considered for a specific irreducible subquotient of the parabolically induced representation, while in this paper our main (and the only) concern is the length of the parabolically induced representation.

The discussion with M. Hanzer, C. Jantzen, E. Lapid and A. Moy were useful during the writing of this paper. We are thankful to them.

We shall now briefly review the contents of the paper. The second section brings the notation that we use in the paper, while the third one describes the irreducible representations that we shall consider and states the main result of the paper. The fourth section proves Proposition \[13\] in a special case (when the essentially square integrable representation of a general linear group with the lowest exponent that enters the Langlands parameter is non-cuspidal, and the tempered representation of the classical group which enters the Langlands parameter is cuspidal). The following section considers the situation as in the previous section, except that the essentially square integrable representation of a general linear group with the lowest exponent that enters the Langlands parameter is now cuspidal. Actually, we could handle these two cases as a single case. Nevertheless we split it, since the first case is a simpler one, and it is convenient to consider it first. The sixth section handles the remaining case, when the tempered representation of a classical group which enters the Langlands parameter is not cuspidal. This case is obtained from the previous two sections by a simple application of the Aubert involution. At the end of this section we get the main results of \[13\] and \[14\] as a simple application of Proposition \[13\].

### 2. Notation and Preliminaries

Now we shall briefly introduce the notation that we shall use in the paper. One can find more details in \[38\] (or \[13\]).

We fix a local non-archimedean field \(F\) of characteristic different from two. We denote by \(|\quad|_F\) the normalized absolute value on \(F\).

For the group \(\mathcal{G}\) of \(F\)-rational points of a connected reductive group over \(F\), we denote by \(\mathcal{R}(\mathcal{G})\) the Grothendieck group of the category \(\text{Alg}_{f.l.}(\mathcal{G})\) of all smooth representations of \(\mathcal{G}\) of finite length. We denote by s.s. the semi simplification map \(\text{Alg}_{f.l.}(\mathcal{G}) \to \mathcal{R}(\mathcal{G})\). The irreducible representations of \(\mathcal{G}\) are also considered as elements of \(\mathcal{R}(\mathcal{G})\).

We have a natural ordering \(\leq\) on \(\mathcal{R}(\mathcal{G})\) determined by the cone \(\text{s.s.}(\text{Alg}_{f.l.}(\mathcal{G}))\).
If $s.s.(\pi_1) \leq s.s.(\pi_2)$ for $\pi_i \in \mathrm{Alg}_{f.l.}(\mathcal{G})$, then we write simply $\pi_1 \leq \pi_2$.

Now we go to the notation of the representation theory of general linear groups (over $F$), following the standard notation of the Bernstein-Zelevinsky theory ([46]). Denote 

$$\nu : GL(n, F) \to \mathbb{R}^\times, \quad \nu(g) = |\det(g)|_F.$$ 

The set of equivalence classes of all irreducible essentially square integrable modulo center representations of all $GL(n, F)$, $n \geq 1$, is denoted by $D$.

For $\delta \in D$ there exists a unique $e(\delta) \in \mathbb{R}$ and a unique unitarizable representation $\delta^u$ (which is square integrable modulo center), such that

$$\delta \cong \nu^{e(\delta)}\delta^u.$$ 

The subset of cuspidal representations in $D$ is denoted by $\mathcal{C}$.

For smooth representations $\pi_1$ and $\pi_2$ of $GL(n_1, F)$ and $GL(n_2, F)$ respectively, $\pi_1 \times \pi_2$ denotes the smooth representation of $GL(n_1 + n_2, F)$ parabolically induced by $\pi_1 \otimes \pi_2$ from the appropriate maximal standard parabolic subgroup (for us, the standard parabolic subgroups will be those parabolic subgroups which contain the subgroup of the upper triangular matrices). We use the normalized parabolic induction in the paper.

We consider

$$R = \bigoplus_{n \geq 0} \mathcal{R}(GL(n, F))$$

as a graded group. The parabolic induction $\times$ lifts naturally to a $\mathbb{Z}$-bilinear mapping $R \times R \to R$, which we denote again by $\times$. This $\mathbb{Z}$-bilinear mapping factors through the tensor product, and the factoring homomorphism is denoted by $m : R \otimes R \to R$.

Let $\pi$ be an irreducible smooth representation of $GL(n, F)$. The sum of the semi simplifications of the Jacquet modules with respect to the standard parabolic subgroups which have Levi subgroups $GL(k, F) \times GL(n-k, F)$, $0 \leq k \leq n$, defines an element of $R \otimes R$ (see [46] for more details). The Jacquet modules that we consider in this paper are normalized. We extend this mapping additively to the whole $R$, and denote the extension by $m^* : R \to R \otimes R$.

In this way, $R$ is a graded Hopf algebra.

For an irreducible representation $\pi$ of $GL(n, F)$, there exist $\rho_1, \ldots, \rho_k \in \mathcal{C}$ such that $\pi$ is isomorphic to a subquotient of $\rho_1 \times \cdots \times \rho_k$. The multiset of equivalence classes $(\rho_1, \ldots, \rho_k)$ is called the cuspidal support of $\pi$.

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8These are irreducible representations which become square integrable modulo center after twist by a (not necessarily unitary) character of the group.
Denote by $M(D)$ the set of all finite multisets in $D$. We add multisets in a natural way:

$$(\delta_1, \delta_2, \ldots, \delta_k) + (\delta'_1, \delta'_2, \ldots, \delta'_{k'}) = (\delta_1, \delta_2, \ldots, \delta_k, \delta'_1, \delta'_2, \ldots, \delta'_{k'}).$$

For $d = (\delta_1, \delta_2, \ldots, \delta_k) \in M(D)$ take a permutation $p$ of $\{1, \ldots, k\}$ such that

$$e(\delta_{p(1)}) \geq e(\delta_{p(2)}) \geq \cdots \geq e(\delta_{p(k)}).$$

Then the representation

$$\lambda(d) := \delta_{p(1)} \times \delta_{p(2)} \times \cdots \times \delta_{p(k)}$$

(called the standard module) has a unique irreducible quotient, which is denoted by $L(d)$.

The mapping $d \mapsto L(d)$ defines a bijection between $M(D)$ and the set of all equivalence classes of irreducible smooth representations of all the general linear groups over $F$. This is a formulation of the Langlands classification for general linear groups. We can describe $L(d)$ as a unique irreducible subrepresentation of $\delta_{p(k)} \times \delta_{p(k-1)} \times \cdots \times \delta_{p(1)}$.

The formula for the contragredient is

$$L(\delta_1, \delta_2, \ldots, \delta_k)^\sim \cong L(\tilde{\delta}_1, \tilde{\delta}_2, \ldots, \tilde{\delta}_k).$$

A segment in $\mathcal{C}$ is a set of the form

$$[\rho, \nu^k \rho] = \{\rho, \nu \rho, \ldots, \nu^k \rho\},$$

where $\rho \in \mathcal{C}$, $k \in \mathbb{Z}_{\geq 0}$. We shall denote a segment $[\nu^{k'} \rho, \nu^{k''} \rho]$ also by $[k', k''](\rho)$, or simply by $[k', k'']$ when we fix $\rho$ (or it is clear from the context which $\rho$ is in question). We denote $[k, k](\rho)$ simply by $[k](\rho)$.

The set of all such segments is denoted by $\mathcal{S}$.

For a segment $\Delta = [\rho, \nu^k \rho] \in \mathcal{S}$, the representation

$$\nu^k \rho \times \nu^{k-1} \rho \times \cdots \times \nu \rho \times \rho$$

contains a unique irreducible subrepresentation, which is denoted by $\delta(\Delta)$ and a unique irreducible quotient, which is denoted by $s(\Delta)$. 
The representation $\delta(\Delta)$ is an essentially square integrable representation modulo center. In this way we get a bijection between $S$ and $D$. Further, $s(\Delta) = L(\rho, \nu^1 \rho, \ldots, \nu^k \rho)$ and

\begin{equation}
 m^*(\delta([\rho, \nu^k \rho])) = \sum_{i=-1}^{k} \delta([\nu^{i+1} \rho, \nu^i \rho]) \otimes \delta([\rho, \nu^i \rho]),
\end{equation}

\begin{equation}
 m^*(s([\rho, \nu^k \rho])) = \sum_{i=-1}^{k} s([\rho, \nu^i \rho]) \otimes s([\nu^{i+1} \rho, \nu^k \rho]).
\end{equation}

Using the above bijection between $D$ and $S$, we can express Langlands classification in terms of finite multisets $M(S)$ in $S$:

$L(\Delta_1, \ldots, \Delta_k) := L(\delta(\Delta_1), \ldots, \delta(\Delta_k))$.

The Zelevinsky classification tells that

$s(\Delta_{p(1)} \times \Delta_{p(2)} \times \cdots \times \Delta_{p(k)})$,

has a unique irreducible subrepresentation, which is denoted by

$Z(\Delta_1, \ldots, \Delta_k)$

($p$ is as above).

Since the ring $R$ is a polynomial ring over $D$, the ring homomorphism $\pi \mapsto \pi^t$ on $R$ determined by the requirement that $\delta(\Delta) \mapsto s(\Delta)$, $\Delta \in S$, is uniquely determined by this condition. It is an involution, and is called the Zelevinsky involution. It is a special case of an involution which exists for any connected reductive group, called the Aubert involution. This extension we shall also denote by $\pi \mapsto \pi^t$. A very important property of the Zelevinsky involution, as well as of the Aubert involution, is that it carries irreducible representations to the irreducible ones\footnote{In the case of the Aubert involution, up to a sign.} ([2], Corollaire 3.9; also [31]).

The Zelevinsky involution $^t$ on the irreducible representations can be introduced by the requirement

$L(a)^t = Z(a)$,

for any multisegment $a$. Then we define $^t$ on the multisegments by the requirement

$Z(a)^t = Z(a^t)$.

For $\Delta = [\rho, \nu^k \rho] \in S$, let

$\Delta^- = [\rho, \nu^{k-1} \rho]$,

and for $d = (\Delta_1, \ldots, \Delta_k) \in M(S)$ denote

$d^- = (\Delta_1^-, \ldots, \Delta_k^-)$.

Then the ring homomorphism $D : R \rightarrow R$ defined by the requirement that $\delta(\Delta)$ goes to $\delta(\Delta) + \delta(\Delta^-)$ for all $\Delta \in S$, is called the derivative. This is a positive mapping. Let $\pi \in R$
and \( \mathcal{D}(\pi) = \sum \mathcal{D}(\pi)_n \), where \( \mathcal{D}(\pi)_n \) is in the \( n \)-th grading group of \( R \). If \( k \) is the lowest index such that \( \mathcal{D}(\pi)_k \neq 0 \), then \( \mathcal{D}(\pi)_k \) is called the highest derivative of \( \pi \), and denoted by \( \text{h.d.}(\pi) \). Obviously, the highest derivative is multiplicative (since \( R \) is an integral domain).

Further
\[
\text{h.d.}(Z(\Delta_1, \ldots, \Delta_k)) = Z(\Delta_1^{-1}, \ldots, \Delta_k^{-1})
\]
(see [46]).

We now very briefly recall basic notation for the classical \( p \)-adic groups. We follow the notation of [38].

Denote by \( J_n \) the \( n \times n \) matrix which has on the second diagonal 1’s, and at the remaining places 0’s. The identity \( n \times n \) matrix is denoted by \( I_n \). For a \( 2n \times 2n \) matrix \( S \) with entries in \( F \), set
\[
\times S = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} \text{ } tS \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix}.
\]

Note that \( \times (S_1S_2) = \times S_2 \times S_1 \). Then the group \( \text{Sp}(n, F) \) is the set of all \( 2n \times 2n \) matrices over \( F \) which satisfy \( \times S^2 = I_{2n} \). Denote by \( \text{SO}(2n+1, F) \) the group of all \( (2n+1) \times (2n+1) \) matrices \( X \) of determinant one with entries in \( F \), which satisfy \( ^tXX = I_{2n+1} \), where \( ^tS \) denotes the transposed matrix with respect to the second diagonal.

For the groups from the above series, we fix always the maximal split torus consisting of all the diagonal matrices in the group, and the minimal parabolic subgroup consisting of all the upper triangular matrices.

In general, for the non-split orthogonal groups, we follow [28]. One fixes a Witt tower \( \mathcal{V} \) of orthogonal vector spaces starting with an anisotropic space, and consider the group of isometries of \( \mathcal{V} \) of determinant 1.

The results of this paper hold also for the unitary groups, with essential same proofs (we need only to put everywhere hermitian contragredient instead of usual contragredient in the claims as well as in the proofs).

In the case of split even orthogonal groups we need to consider orthogonal groups instead of special orthogonal groups. We do not follow this series of groups in this paper, although we expect that the results of this paper hold also in this case, with essential the same proofs (but we have not checked them).

The group of split rank \( n \) will be denoted by \( S_n \) or \( S_n(F) \) (for some other purposes a different indexing may be more convenient).

The direct sum of Grothendieck groups \( \mathcal{R}(S_n), n \geq 0 \), is denoted by \( R(S) \).

For \( 1 \leq k \leq n \), we denote by \( P_{(k)} \) the standard maximal parabolic subgroup of \( S_n \) whose Levi factor is a direct product of \( GL(k, F) \) and of a classical group \( S_{n-k} \). We take \( P_{(0)} \) to be \( S_n \). Using the parabolic induction, in an analogous way as for the case of general
linear groups, one defines $\pi \rtimes \sigma := \text{Ind}_{P_{(k)}}^{S_{k+m}}(\pi \otimes \sigma)$ for smooth representations $\pi$ and $\sigma$ of $GL(k, F)$ and $S_m$ respectively.

An irreducible representation of a classical group will be called weakly real if it is a subquotient of a representation of the form

$$\nu^{r_1}\rho_1 \times \ldots \times \nu^{r_k}\rho_k \rtimes \sigma,$$

where $\rho_i \in C$ are selfcontragredient, $r_i \in \mathbb{R}$ and $\sigma$ is an irreducible cuspidal representation of a classical group.

The following theorems reduce the unitarizability problem for classical $p$-adic groups to the weakly real case (see [43]).

**Theorem 2.1.** If $\sigma$ is an irreducible unitarizable representation of some $S_q$, then there exist an irreducible unitarizable representation $\pi$ of a general linear group and a weakly real irreducible unitarizable representation $\sigma'$ of some $S_{q'}$ such that

$$\sigma \cong \pi \rtimes \sigma'.$$

Denote by $C_u$ the set of all unitarizable classes in $C$. The above reduction can be made more precise:

**Theorem 2.2.** Let $C'_u$ be a subset of $C_u$ satisfying $C'_u \cap \widetilde{C}_u = \emptyset$, such that $C'_u \cup \widetilde{C}_u$ contains all $\rho \in C_u$ which are not self dual. Denote

$$C' = \{\nu^\alpha\rho; \alpha \in \mathbb{R}, \; \rho \in C'_u\}.$$

Then there exists an irreducible representation $\theta$ of a general linear group with support contained in $C'$, and a weakly real irreducible representation $\pi'$ of some $S_{n'}$ such that

$$\pi \cong \theta \rtimes \pi'.$$

Moreover, $\pi$ determines such $\theta$ and $\pi'$ up to an equivalence. Further, $\pi$ is unitarizable (resp. Hermitian) if and only both $\theta$ and $\pi'$ are unitarizable (resp. Hermitian).

As in the case of general linear groups, one lifts $\rtimes$ to a mapping $R \times R(S) \to R(S)$ (again denoted by $\rtimes$). Factorization through $R \otimes R(S)$ is denoted by $\mu$. In this way $R(S)$ becomes an $R$-module.

We denote by $s_{(k)}(\pi)$ the Jacquet module of a representation $\pi$ of $S_n$ with respect to the parabolic subgroup $P_{(k)}$. For an irreducible representation $\pi$ of $S_n$, the sum of the semi simplifications of $s_{(k)}(\pi)$, $0 \leq k \leq n$, is denoted by

$$\mu^*(\pi) \in R \otimes R(S).$$

We extend $\mu^*$ additively to $\mu^* : R(S) \to R \otimes R(S)$. With this comultiplication, $R(S)$ becomes an $R$-comodule.
Further, $R \otimes R(S)$ is an $R \otimes R$-module in a natural way (the multiplication is denoted by $\times$). Let $\sim : R \to R$ be the contragredient map and $\kappa : R \otimes R \to R \otimes R$, $\sum x_i \otimes y_i \mapsto \sum y_i \otimes x_i$. Denote

$$M^* = (m \otimes \text{id}_R) \circ (\sim \otimes m^*) \circ \kappa \circ m^*.$$ 

Then (2.8)

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma)$$

for $\pi \in R$ and $\sigma \in R(S)$ (or for admissible representations $\pi$ and $\sigma$ of $GL(n, F)$ and $S_m$ respectively).

A direct consequence of this is the following simple fact. Let $\tau$ be a representation of some $GL(m, F)$ and $m^*(\tau) = \sum x \otimes y$. Then the sum of the irreducible subquotients of the form $\ast \otimes 1$ in $M^*(\tau)$ will be denoted by

$$M_{GL}(\tau),$$

and this is equal to the following sum

(2.9) $$\sum x \times \tilde{y} \otimes 1.$$ 

Further, the sum of the irreducible subquotients of the form $1 \otimes \ast$ in $M^*(\tau)$ is

(2.10) $$1 \otimes \tau.$$ 

A direct consequence of the formulas (2.8) and (2.7) is the following formula:

$$M^*(\delta(\nu^a \rho, \nu^c \rho)) = \sum_{s=0}^{c} \sum_{t=1}^{c} \delta(\nu^{-s} \rho, \nu^{-a} \rho) \times \delta(\nu^{t+1} \rho, \nu^c \rho) \otimes \delta(\nu^{s+1} \rho, \nu^t \rho).$$

Set

$$D_+ = \{ \delta \in D; e(\delta) > 0 \}.$$ 

Let $T$ be the set of all equivalence classes of tempered representations of $S_n$, for all $n \geq 0$. For $((\delta_1, \delta_2, \ldots, \delta_k), \tau) \in M(D_+) \times T$ take a permutation $p$ of $\{1, \ldots, k\}$ such that

$$\delta_{p(1)} \geq \delta_{p(2)} \geq \cdots \geq \delta_{p(k)}.$$ 

Then the representation

$$\delta_{p(1)} \times \delta_{p(2)} \times \cdots \times \delta_{p(k)} \rtimes \tau$$

has a unique irreducible quotient, which is denoted by

$$L(\delta_1, \delta_2, \ldots, \delta_k; \tau).$$

The mapping

$$((\delta_1, \delta_2, \ldots, \delta_k), \tau) \mapsto L(\delta_1, \delta_2, \ldots, \delta_k; \tau)$$

defines a bijection from the set $M(D_+) \times T$ onto the set of all equivalence classes of the irreducible smooth representations of all $S_n$, $n \geq 0$. This is the Langlands classification for classical groups.
For $\Delta \in S$ define $\nu(\Delta)$ to be $e(\delta(\Delta))$. Let
\[ S_+ = \{\Delta \in S; \nu(\Delta) > 0\}. \]
In this way we can define in a natural way the Langlands classification $(a, \tau) \mapsto L(a; \tau)$ using $M(S_+) \times T$ for the parameters.

Let $\tau$ and $\omega$ be irreducible representations of $GL(p, F)$ and $S_q$, respectively, and let $\pi$ an admissible representation of $S_{p+q}$. Then a special case of the Frobenius reciprocity tells us
\[ \text{Hom}_{S_{p+q}}(\pi, \tau \otimes \omega) \cong \text{Hom}_{GL(p,F) \times S_q}(s(\rho)(\pi), \tau \otimes \omega), \]
while the second second adjointness implies
\[ \text{Hom}_{S_{p+q}}(\tau \otimes \omega, \pi) \cong \text{Hom}_{GL(p,F) \times S_q}(\tilde{\tau} \otimes \omega, s(\rho)(\pi)). \]

We could write down the above formulas for the parabolic subgroups which are not necessarily maximal.

In the rest of the paper we fix an irreducible unitarizable cuspidal representations $\rho$ and $\sigma$ of $GL(p, F)$ and $S_q$ respectively, such that $\rho$ is unitarizable and that
\[ \nu^{\alpha_{\rho,\sigma}} \rho \times \sigma \]
reduces for some $\alpha_{\rho,\sigma} > 0$. Then $\rho$ is self dual. We shall assume that
\[ \alpha_{\rho,\sigma} \in (1/2)\mathbb{Z}. \]
Actually, from the recent work of J. Arthur, C. Mœglin and J.-L. Waldspurger, this assumption is known to hold if $\text{char}(F) = 0$.

In the rest of the paper we shall denote the reducibility point $\alpha_{\rho,\sigma}$ simply by $\alpha$.

In this paper we shall deal with irreducible sub quotients of
\[ \nu^{\alpha+n} \rho \times \nu^{\alpha+n-1} \rho \times \cdots \times \nu^{\alpha+1} \rho \times \nu^\alpha \rho \times \sigma \]
($\rho$, $\sigma$ and $\alpha$ are as above). The above representation has a unique irreducible subrepresentation, which is denoted by
\[ \delta([\nu^\alpha \rho, \nu^{\alpha+n} \rho]; \sigma) \quad (n \geq 0). \]
This subrepresentation is square integrable and it is called a generalized Steinberg representation. We have
\[ \mu^* \left( \delta([\nu^\alpha \rho, \nu^{\alpha+n} \rho]; \sigma) \right) = \sum_{k=-1}^{n} \delta([\nu^{\alpha+k+1} \rho, \nu^{\alpha+n} \rho]) \otimes \delta([\nu^\alpha \rho, \nu^{\alpha+k} \rho]; \sigma), \]
\[ \delta([\nu^\alpha \rho, \nu^{\alpha+n} \rho]; \sigma)^\sim \cong \delta([\nu^\alpha \rho, \nu^{\alpha+n} \rho]; \overline{\sigma}). \]
Further applying the Aubert involution, we get

$$
\mu^*(L(\nu^{\alpha+n}\rho, \ldots, \nu^{\alpha+1}\rho, \nu^{\alpha}\rho; \sigma)) = \sum_{k=-1}^{n} L(\nu^{-(\alpha+n)}\rho, \ldots, \nu^{-(\alpha+k+2)}\rho, \nu^{-(\alpha+k+1)}\rho) \otimes L(\nu^{\alpha+k}\rho, \ldots, \nu^{\alpha+1}\rho, \nu^{\alpha}\rho; \sigma).
$$

3. IRREDUCIBLE SUB QUOTIENTS AND THE MAIN CLAIM

We fix an irreducible cuspidal self contragredient representation $\rho$ of $GL(p, F)$ and an irreducible cuspidal representation $\sigma$ of $S_q$ for which there exists $\alpha \in \frac{1}{2}\mathbb{Z}$, $\alpha > 0$ such that

$$
\nu^{\alpha}\rho \rtimes \sigma
$$

reduces.

We say that a sequence of segments $\Delta_1, \ldots, \Delta_l$ is decreasing if $c(\Delta_1) \geq \cdots \geq c(\Delta_l)$.

Now we recall of Lemma 3.1 from [13] which we shall use several times in this paper:

**Lemma 3.1.** Let $n \geq 1$. Fix an integer $b$ satisfying $0 \leq b \leq n - 1$. Let $\Delta_1, \ldots, \Delta_k$ be a sequence of decreasing non-empty segments such that

$$
\Delta_1 \cup \ldots \cup \Delta_k = \{\nu^{\alpha+b+1}\rho, \ldots, \nu^{\alpha+n-1}\rho, \nu^{\alpha+n}\rho\}.
$$

Let $\Delta_{k+1}, \ldots, \Delta_l, k < l$, be a sequence of decreasing segments satisfying

$$
\Delta_{k+1} \cup \ldots \cup \Delta_l = \{\nu^{\alpha}\rho, \nu^{\alpha+1}\rho, \ldots, \nu^{\alpha+b}\rho\},
$$

such that $\Delta_{k+1}, \ldots, \Delta_{l-1}$ are non-empty. Let

$$
a = (\Delta_1, \ldots, \Delta_{k-1}),
$$

$$
b = (\Delta_{k+2}, \ldots, \Delta_{l-1}).
$$

Then in $R(S)$ we have:

1. If $k + 1 < l$, then

$$
L(a + (\Delta_k)) \rtimes L((\Delta_{k+1}) + b; \delta(\Delta_l; \sigma)) = L(a + (\Delta_k, \Delta_{k+1}) + b; \delta(\Delta_l; \sigma)) + L(a + (\Delta_k \cup \Delta_{k+1}) + b; \delta(\Delta_l; \sigma)).
$$

2. If $k + 1 = l$, then

$$
L(a + (\Delta_k)) \rtimes \delta(\Delta_{k+1}; \sigma) = L(a + (\Delta_k); \delta(\Delta_{k+1}; \sigma)) + L(a; \delta(\Delta_k \cup \Delta_{k+1}; \sigma)). \quad \square
$$
We assume

\[ n \geq 1, \]

and consider irreducible subquotients of \( \nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^\alpha \rho \times \sigma \). Each irreducible subquotient can be written as

\[ \gamma = L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}; \sigma)) \]

for some \( k \geq 0 \), where \( \Delta_1, \ldots, \Delta_k \) is a sequence of decreasing segments such that

\[ \Delta_1 \cup \ldots \cup \Delta_k \cup \Delta_{k+1} = \{ \nu^\alpha \rho, \ldots, \nu^{\alpha+n} \rho \}, \]

and that \( \Delta_1, \ldots, \Delta_k \) are non-empty.

**Remark 3.2.** Observe that

\[ (\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^\alpha \rho \times \delta(\nu^\alpha \rho; \sigma))^t = \nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^\alpha \rho \times L(\nu^\alpha \rho; \sigma). \]

Irreducible subquotients of \( \nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^\alpha \rho \times \delta(\nu^\alpha \rho; \sigma) \) satisfy \( \Delta_{k+1} \neq \emptyset \), while irreducible subquotients of \( \nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^\alpha \rho \times L(\nu^\alpha \rho; \sigma) \) satisfy \( \Delta_{k+1} = \emptyset \).

From this directly follows that the Aubert involution is a bijection between the irreducible subquotients for which \( \Delta_{k+1} \neq \emptyset \) and the irreducible subquotients for which \( \Delta_{k+1} = \emptyset \).

The aim of this paper is to prove the following

**Proposition 3.3.** Let \( \gamma \) be an irreducible subquotient of \( \nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^\alpha \rho \times \sigma \), different from

\[ L(\nu^\alpha \rho, \nu^{\alpha+1} \rho, \ldots, \nu^{\alpha+n} \rho; \sigma) \]

and \( \delta([\nu^\alpha \rho, \nu^{\alpha+n} \rho]; \sigma) \).

Then there exists an irreducible selfcontragredient unitarizable representation \( \pi \) of a general linear group with support in \([-\alpha - n, \alpha + n]_{(\rho)} \), such that the length of

\[ \pi \rtimes \gamma \]

is at least 5, and that

\[ 5 \cdot \pi \otimes \gamma \leq \mu^*((\pi \rtimes \gamma)). \]

We shall consider \( \gamma \) as in the proposition, and write \( \gamma = L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}; \sigma)) \) as above (recall, \( \Delta_1, \ldots, \Delta_k \) are non-empty and decreasing, and additionally \( \Delta_k, \Delta_{k+1} \) are decreasing if \( \Delta_{k+1} \neq \emptyset \)). Since \( \gamma \) is different from \( \delta([\nu^\alpha \rho, \nu^{\alpha+n} \rho]; \sigma) \), we have

\[ k \geq 1, \]

and since \( \gamma \) is different from \( L(\nu^\alpha \rho, \nu^{\alpha+1} \rho, \ldots, \nu^{\alpha+n} \rho; \sigma) \) we have

\[ \Delta_{k+1} \neq \emptyset \text{ or } \Delta_{k+1} = \emptyset \text{ and } \text{card}(\Delta_i) > 1 \text{ for some } 1 \leq i \leq k. \]

We shall first study \( \gamma \) for which \( \Delta_{k+1} = \emptyset \). We split our study into two cases.
4. The case of \( \text{card}(\Delta_k) > 1 \) and \( \Delta_{k+1} = \emptyset \)

Denote

\[ \Delta_k = [\nu^\alpha \rho, \nu^\epsilon \rho], \]
\[ \Delta_u = [\nu^{-\alpha} \rho, \nu^\alpha \rho], \]
\[ \Delta = \Delta_k \cup \Delta_u = [\nu^{-\alpha} \rho, \nu^\epsilon \rho]. \]

Then

\[ \alpha < c. \]

Denote

\[ a = (\Delta_1, \Delta_2, \ldots, \Delta_{k-1}), \]
\[ a_1 = (\Delta_1, \Delta_2, \ldots, \Delta_{k-2}), \quad \text{if } a \neq \emptyset. \]

For \( L(a, \Delta_k) \rtimes \sigma \) in the Grothendieck group we have

\[ L(a, \Delta_k) \rtimes \sigma = L(a + (\Delta_k); \sigma) + L(a; \delta(\Delta_k; \sigma)). \tag{4.11} \]

We shall denote \( L(a + (\Delta_k); \sigma) \) below simply by \( L(a, \Delta_k; \sigma). \)

Our first goal in this sections is to prove:

**Lemma 4.1.** The representation \( \delta(\Delta_u) \rtimes L(a, \Delta_k; \sigma) \) is of length at lest 5 if \( \text{card}(\Delta_k) > 1 \).

We know from [40] that \( \delta(\Delta_u) \rtimes \sigma \) reduces, and it reduces into two non-equivalent irreducible pieces. Denote them by \( \tau((\Delta_u)_+; \sigma) \) and \( \tau((\Delta_u)_-; \sigma) \). Now Proposition 4.2 [43] implies

\[ L(a, \Delta_k; \tau((\Delta_u)_+; \sigma), L(a, \Delta_k; \tau((\Delta_u)_-; \sigma) \leq \delta(\Delta_u) \rtimes L(a, \Delta_k; \sigma). \tag{4.12} \]

Therefore, \( \delta(\Delta_u) \times L(a, \Delta_k; \sigma) \) has length at least two.

Now we shall recall of a simple Lemma 4.2 from [13]:

**Lemma 4.2.** If \( |\Delta_k| > 1 \), then we have

\[ L(a + (\Delta)) \times \nu^\alpha \rho \leq \delta(\Delta_u) \times L(a + (\Delta_k)), \]

and the representation on the left hand side is irreducible. \( \square \)

The above lemma now implies

\[ L(a, \Delta) \times \nu^\alpha \rho \rtimes \sigma \leq \delta(\Delta_u) \times L(a, \Delta_k) \rtimes \sigma = \delta(\Delta_u) \rtimes L(a, \Delta_k; \sigma) + \delta(\Delta_u) \rtimes L(a; \delta(\Delta_k; \sigma)). \]

Since by Proposition 4.2 of [43],

\[ L(a, \Delta, \nu^\alpha \rho; \sigma) \]

is a sub quotient of \( L(a, \Delta) \times \nu^\alpha \rho \rtimes \sigma \), it is also a sub quotient of

\[ \delta(\Delta_u) \rtimes L(a, \Delta_k; \sigma) + \delta(\Delta_u) \rtimes L(a; \delta(\Delta_k; \sigma)). \]
Suppose
\[ L(a, \Delta, \nu^\alpha \rho; \sigma) \leq \delta(\Delta_u) \times L(a; \delta(\Delta_\kappa; \sigma)). \]
Then
\[ L(a, \Delta, \nu^\alpha \rho; \sigma) \leq \lambda(a) \times \tau = \lambda(a, \tau), \]
where \( \tau \) is some irreducible sub representation of \( \delta(\Delta_u) \times \delta(\Delta_\kappa; \sigma) \). Basic facts about irreducible sub quotients of standard modules in the Langlands classification imply that this is not possible (since \( \alpha > 0 \)). Therefore
\[(4.13) \quad L(a, \Delta, \nu^\alpha \rho; \sigma) \leq \delta(\Delta_u) \times L(a, \Delta_\kappa; \sigma).\]
Now we know that \( \delta(\Delta_u) \times L(a, \Delta_\kappa; \sigma) \) has length at least three.

From [41] we know that \( \delta(\Delta) \times \sigma \) has two nonequivalent irreducible square integrable sub representations, and that they are square integrable. They are denoted there by \( \delta(\Delta_\pm; \sigma) \) and \( \delta(\Delta_\mp; \sigma) \). This and [43] imply
\[
L(a, \nu^\alpha \rho; \delta(\Delta_\pm; \sigma)) \leq L(a, \nu^\alpha \rho) \times \delta(\Delta_\pm; \sigma) \leq L(a) \times \nu^\alpha \rho \times \delta(\Delta \mp \sigma) \leq \\
L(a) \times \delta(\Delta_\kappa) \times \delta(\Delta_u) \times \sigma.
\]
Therefore
\[
L(a, \nu^\alpha \rho; \delta(\Delta_\pm; \sigma)) \leq L(a) \times \delta(\Delta_\kappa) \times \delta(\Delta_u) \times \sigma.
\]
If \( a = \emptyset \), then formally
\[(4.14) \quad L(a, \nu^\alpha \rho; \delta(\Delta_\pm; \sigma)) \leq L(a, \Delta_\kappa) \times \delta(\Delta_u) \times \sigma \]
since then \( L(a) \times \delta(\Delta_\kappa) = L(\Delta_\kappa) = L(a, \Delta_\kappa) \).

Now we shall show that \((4.14)\) holds also if \( a \neq \emptyset \). In this case we have
\[
L(a, \nu^\alpha \rho; \delta(\Delta_\pm; \sigma)) \leq L(a) \times \delta(\Delta_\kappa) \times \delta(\Delta_u) \times \sigma = \\
L(a, \Delta_\kappa) \times \delta(\Delta_u) \times \sigma + L(a_1, \Delta_{k-1} \cup \Delta_\kappa) \times \delta(\Delta_u) \times \sigma
\]
(here we have used that \( L(a) \times \delta(\Delta_\kappa) = L(a, \Delta_\kappa) + L(a_1, \Delta_{k-1} \cup \Delta_\kappa) \), which is easy to prove). Suppose
\[
L(a, \nu^\alpha \rho; \delta(\Delta_\pm; \sigma)) \leq L(a_1, \Delta_{k-1} \cup \Delta_\kappa) \times \delta(\Delta_u) \times \sigma.
\]
Observe that \( L(a, \nu^\alpha \rho; \delta(\Delta_\pm; \sigma)) \hookrightarrow L(\tilde{a}, \nu^{-\alpha} \rho) \times \delta(\Delta_\pm; \sigma) \hookrightarrow L(\tilde{a}, \nu^{-\alpha} \rho) \times \delta(\Delta \mp \sigma). \) This implies that the Langlands quotient \( L(a, \nu^\alpha \rho; \delta(\Delta_\pm; \sigma)) \) has in the GL-type Jacquet module an irreducible sub quotient
\[
\beta
\]
which has exponent \( c \) in its Jacquet module, but has not \( c + 1 \).

Observe that
\[
s_{GL}(L(a_1, \Delta_{k-1} \cup \Delta_\kappa) \times \delta(\Delta_u) \times \sigma) \leq s_{GL}(L(a_1) \times \delta(\Delta_{k-1} \cup \Delta_\kappa) \times \delta(\Delta_u) \times \sigma) = \\
M^*_{GL}(L(a_1)) \times M^*_{GL}(\delta(\Delta_{k-1} \cup \Delta_\kappa)) \times M^*_{GL}(\delta(\Delta_u)) \times (1 \otimes \sigma).
\]
We cannot get any of exponents $c$ and $c + 1$ from $M_{GL}^*(\langle L(a_1) \rangle) \text{ or } M_{GL}^*(\delta(\Delta_u))$ (consider the supports). Therefore if some of them shows up, it must come from

$$M_{GL}^*(\delta(\Delta_{k-1} \cup \Delta_k)) = M_{GL}^*(\delta([a, d])) = \sum_{x=-a}^{d} \delta([-x, -\alpha]) \times \delta([x + 1, d]),$$

where $c + 1 \leq d$. Now the above formula for $M_{GL}^*(\delta([a, d]))$ implies that whenever we have in the support $c$, we must have it in a segment which ends with $d$, and therefore, we must have in the support $c + 1$. Therefore, $\beta$ cannot be a sub quotient of $L(a_1, \Delta_{k-1} \cup \Delta_k) \times \delta(\Delta_u) \rtimes \sigma$. This contradiction implies

$$L(a, \nu^\alpha \rho; \delta(\Delta; \sigma)) \not\subseteq L(a_1, \Delta_{k-1} \cup \Delta_k) \times \delta(\Delta_u) \rtimes \sigma.$$  

Now the following relation (which we have already observed above)

$$L(a, \nu^\alpha \rho; \delta(\Delta; \sigma)) \leq L(a, \delta(\Delta)) \times \delta(\Delta_u) \rtimes \sigma + L(a_1, \Delta_{k-1} \cup \Delta_k) \times \delta(\Delta_u) \rtimes \sigma$$

implies

$$L(a, \nu^\alpha \rho; \delta(\Delta; \sigma)) \leq L(a, \Delta_k) \times \delta(\Delta_u) \rtimes \sigma.$$  

Therefore (in both cases) we have

$$L(a, \nu^\alpha \rho; \delta(\Delta; \sigma)) \leq L(a, \Delta_k) \times \delta(\Delta_u) \rtimes \sigma = \delta(\Delta_u) \rtimes L(a, \Delta_k; \sigma) + \delta(\Delta_u) \rtimes L(a; \delta(\Delta_k; \sigma)),$$

Suppose

$$L(a, \nu^\alpha \rho; \delta(\Delta; \sigma)) \leq \delta(\Delta_u) \rtimes L(a; \delta(\Delta_k; \sigma)).$$

One directly sees that in the GL-type Jacquet module of the left hand side we have an irreducible term in whose support appears exponent $-\alpha$ twice times.

Observe $\delta(\Delta_u) \rtimes L(a; \delta(\Delta_k; \sigma)) \leq \delta(\Delta_u) \rtimes L(a) \rtimes \delta(\Delta_k; \sigma)$. For $\delta(\Delta_u) \rtimes L(a) \rtimes \delta(\Delta_k; \sigma)$, the exponent $-\alpha$ which cannot come neither from $M_{GL}^*(L(a))$ nor from $\mu^*(\delta(\Delta_k; \sigma))$. Therefore, it must come from

$$M_{GL}^*(\delta(\Delta_u)) = \sum_{x=-\alpha}^{\alpha} \delta([-x, \alpha]) \times \delta([x + 1, \alpha]).$$

This implies that we can have the exponent $-\alpha$ at most once in the GL-part of Jacquet module of the right hand side. This contradiction implies that the inequality which we have supposed is false. This implies

$$L(a, \nu^\alpha \rho; \delta(\Delta; \sigma)) \leq \delta(\Delta_u) \rtimes L(a, \Delta_k; \sigma).$$

Therefore, $\delta(\Delta_u) \rtimes L(a, \Delta_k; \sigma)$ has length at least five. This completes the proof of the lemma.

The second aim of this section is to prove the following
Lemma 4.3. The multiplicity of 
\[ \delta(\Delta_u) \otimes L(a, \Delta_k; \sigma) \]
in 
\[ \mu^*(\delta(\Delta_u) \rtimes L(a, \Delta_k; \sigma)) \]
is at most 4 if \( \text{card}(\Delta_k) > 1 \).

Proof. Denote \( \beta := \delta(\Delta_u) \otimes L(a, \Delta_k; \sigma) \). Recall
\[
M^*(\delta(\Delta_u)) = M^*(\delta([-\alpha, \alpha])) = \sum_{x=\alpha-1}^{\alpha} \delta([-x, \alpha]) \otimes \delta([x, y])
\]
Now if we take from \( \mu^*(L(a, \Delta_k; \sigma)) \) the term \( 1 \otimes L(a, \Delta_k; \sigma) \), to get \( \beta \) for a sub quotient we need to take from \( M^*(\delta(\Delta_u)) \) the term \( \delta(\Delta_u) \otimes 1 \), which shows up there two times. This gives multiplicity two of \( \beta \).

Now we consider terms from \( \mu^*(L(a, \Delta_k; \sigma)) \) different from \( 1 \otimes L(a, \Delta_k; \sigma) \) which can give \( \beta \) after multiplication with a term from \( M^*(\delta(\Delta_u)) \) (then a term from \( M^*(\delta(\Delta_u)) \) that can give \( \beta \) for a sub quotient is obviously different from \( \delta(\Delta_u) \otimes 1 \), which implies that we have \( \nu^\alpha \rho \) in the support of the left hand side tensor factor). The above formula for \( M^*(\delta(\Delta_u)) \) and the set of possible factors of \( L(a, \Delta_k; \sigma) \) (which is \( \nu^{\pm \alpha} \rho, \nu^{\pm(\alpha+1)}, \ldots \)) imply that we need to have \( \nu^{-\alpha} \rho \) on the left hand side of the tensor product of that term from \( \mu^*(L(a, \Delta_k; \sigma)) \).

For such a term from \( \mu^*(L(a, \Delta_k; \sigma)) \), considering the support we see that we have two possible terms from \( M^*(\delta(\Delta_u)) \). They are \( \delta([-\alpha + 1, \alpha]) \otimes [-\alpha] \) and \( \delta([-\alpha + 1, \alpha]) \otimes [\alpha] \). Each of them will give multiplicity at most one (use the fact that here on the left and right hand side of \( \otimes \) we are in the regular situation). □

5. The case of \( \text{card}(\Delta_k) = 1 \) and \( \Delta_{k+1} = \emptyset \)

As we already noted, we consider the case when \( \text{card}(\Delta_i) > 1 \) for some \( i \). Denote maximal such index by \( k_0 \). Clearly,
\[ k_0 < k. \]

Write
\[ \Delta_{k_0} = [\nu^{\alpha+k-k_0} \rho, \nu^c \rho] = [\nu^{\alpha'} \rho, \nu^c \rho], \]
\[ \Delta_u = [\nu^{-\alpha'} \rho, \nu^c \rho], \]
\[ \Delta = [\nu^{-\alpha} \rho, \nu^c \rho], \]
\[ \Delta_1 = [\alpha, \alpha' - 1], \]
\[ b = [\alpha, \alpha' - 1]^t = ([\alpha], [\alpha + 1], \ldots, [\alpha' - 1]) \neq \emptyset. \]

Then
\[ \alpha' < c. \]

Let
\[ a = (\Delta_1, \Delta_2, \ldots, \Delta_{k_0-1}), \]
\[ a_1 = (\Delta_1, \Delta_2, \ldots, \Delta_{k_0 - 2}), \quad \text{if } a \neq \emptyset. \]

Then
\[ (\Delta_1, \ldots, \Delta_k) = (a, \Delta_{k_0}, b). \]

We shall study \( L(a, \Delta_{k_0}, b) \rtimes \sigma \). The previous lemma implies that in the Grothendieck group we have
\[ L(a, \Delta_{k_0}, b) \rtimes \sigma = L(a; \Delta_{k_0}, b; \sigma) + L(a; \Delta_{k_0}, \nu^{\alpha^\prime - 1}, \ldots, \nu^{\alpha + 1}; \delta(\nu^\alpha; \sigma)). \]

Our first goal in this section is to prove the following

**Lemma 5.1.** Then length of the representation \( \delta(\Delta_u) \rtimes L(a, \Delta_{k_0}, b; \sigma) \) is at least 5 if \( k_0 < k \) and \( \text{card}(\Delta_{k_0}) > 1 \).

First we get that we have two non-equivalent subquotients
\[ L(a, \Delta_{k_0}, b, \tau((\Delta_u)_{\pm}; \sigma)) \leq \delta(\Delta_u) \rtimes L(a, \Delta_{k_0}, b; \sigma) \]
in the same way as in the previous section. Therefore, the length is at least two.

Now we shall prove the following simple

**Lemma 5.2.** If \( |\Delta_k| = 1 \), then we have
\[ L(a, \Delta_{k_0} \cup \Delta_u, b, \nu^{\alpha^\prime} \rho) \leq \delta(\Delta_u) \rtimes L(a, \Delta_{k_0}, b). \]

*Proof.* Since in general \( L(\Delta'_1, \Delta'_2, \ldots, \Delta'_m) = Z(\Delta'_1, \Delta'_2, \ldots, \Delta'_m) \), it is enough to prove the lemma for the Zelevinsky classification.

The highest (non-trivial) derivative of \( s(\Delta_u) \times Z(a, \Delta_{k_0}, b) = s(\Delta_u^{-}) \times Z(a^{-}, \Delta^{-}_{k_0}) \). One can easily see that one subquotient of the last representation is \( Z(a^{-}, \Delta^{-}) \). Therefore, there must exist an irreducible subquotient of \( s(\Delta_u) \times Z(a, \Delta_{k_0}, b) \) whose highest derivative is \( Z(a^{-}, \Delta^{-}) \). The support and highest derivative completely determine the irreducible representation. One directly sees that this representation is \( Z(a, \Delta_{k_0} \cup \Delta_u, b, \nu^{\alpha^\prime} \rho) \). The proof is now complete. \( \square \)

The above lemma implies
\[ L(a, \Delta_{k_0} \cup \Delta_u, b, \nu^{\alpha^\prime} \rho; \sigma) \leq L(a, \Delta_{k_0} \cup \Delta_u, b, \nu^{\alpha^\prime} \rho) \rtimes \sigma \leq \delta(\Delta_u) \rtimes L(a, \Delta_{k_0}, b) \rtimes \sigma. \]

By Lemma 5.1 we have for the right hand side
\[ \delta(\Delta_u) \times L(a, \Delta_{k_0}, b) \rtimes \sigma = \delta(\Delta_u) \times L(a, \Delta_{k_0}, b; \sigma) + \delta(\Delta_u) \times L(a; \Delta_{k_0}, \nu^{\alpha^\prime - 1}, \ldots, \nu^{\alpha + 1}; \delta(\nu^\alpha; \sigma)). \]

This implies
\[ L(a, \Delta_{k_0} \cup \Delta_u, b, \nu^{\alpha^\prime} \rho; \sigma) \leq \delta(\Delta_u) \times L(a, \Delta_{k_0}, b; \sigma) + \delta(\Delta_u) \times L(a; \Delta_{k_0}, \nu^{\alpha^\prime - 1}, \ldots, \nu^{\alpha + 1}; \delta(\nu^\alpha; \sigma)). \]

Suppose
\[ L(a, \Delta_{k_0} \cup \Delta_u, b, \nu^{\alpha^\prime} \rho; \sigma) \leq \delta(\Delta_u) \times L(a; \Delta_{k_0}, \nu^{\alpha^\prime - 1}, \ldots, \nu^{\alpha + 1}; \delta(\nu^\alpha; \sigma)). \]
Using the properties of the irreducible subquotients of the standard modules in the Langlands classification, we now conclude in the same way as in the last section that this cannot be the case (the sum of all exponents on the left hand side which are not coming from the tempered representation of the classical group is the same as the sum of exponents of cuspidal representations which show up in the segments of $a$, in $\Delta_{k_0}$, and $\alpha' - 1, \alpha' - 2, \ldots, \alpha' + 1$, while the corresponding sum of the standard modules which come from the right hand side is the sum of exponents of cuspidal representations which show up in the segments of $a$, in $\Delta_{k_0}$, and $\alpha' - 1, \alpha' - 2, \ldots, \alpha' + 1$, which is strictly smaller (for $\alpha > 0$) then we have on the left hand side).

This implies
\begin{equation}
L(a, \Delta_{k_0} \cup \Delta_u, b, \nu^{\alpha'} \rho; \sigma) \leq \delta(\Delta_u) \times L(a, \Delta_{k_0}, b; \sigma).
\end{equation}
Therefore, $\delta(\Delta_u) \times L(a, \Delta_{k_0}, b; \sigma)$ has length at least three.

The following (a little bit longer) step will be to show that $\delta(\Delta_u) \times L(a, \Delta_{k_0}, b; \sigma)$ has two additional irreducible subquotients.

We start this step with an observe that
\begin{equation}
L(a, \nu^{\alpha'} \rho, b; \delta(\Delta_\pm; \sigma)) \leq L(a) \times L(b) \times \nu^{\alpha'} \rho \times \delta(\Delta) \times \delta(\Delta_u) \times \sigma \leq L(a) \times L(b) \times \delta(\Delta_{k_0}) \times \delta(\Delta_u) \times \sigma
\end{equation}
If $a = \emptyset$, then formally
\begin{equation*}
L(a, b, \nu^{\alpha'} \rho; \delta(\Delta_\pm; \sigma)) \leq L(a, \Delta_{k_0}) \times L(b) \times \delta(\Delta_u) \times \sigma
\end{equation*}
since $L(a) \times \delta(\Delta_{k_0}) = L(\Delta_{k_0}) = L(a, \Delta_{k_0})$.

We shall now show that the above inequality holds also if $a \neq \emptyset$. Then staring with (5.18) we get
\begin{align*}
L(a, b, \nu^{\alpha'} \rho; \delta(\Delta_\pm; \sigma)) &\leq L(a) \times \delta(\Delta_{k_0}) \times L(b) \times \delta(\Delta_u) \times \sigma = L(a, \Delta_{k_0}) \times L(b) \times \delta(\Delta_u) \times \sigma + L(a_1, \Delta_{k_0 - 1} \cup \Delta_{k_0}) \times L(b) \times \delta(\Delta_u) \times \sigma.
\end{align*}
Suppose
\begin{align*}
L(a, b, \nu^{\alpha'} \rho; \delta(\Delta_\pm; \sigma)) &\leq L(a_1, \Delta_{k_0 - 1} \cup \Delta_{k_0}) \times L(b) \times \delta(\Delta_u) \times \sigma.
\end{align*}
Observe that $L(a, \nu^{\alpha'} \rho, b; \delta(\Delta_\pm; \sigma)) \rightarrow L(\tilde{a}, \nu^{-\alpha'} \rho, \tilde{b}) \times \delta(\Delta_\pm; \sigma) \rightarrow L(\tilde{a}, \nu^{-\alpha'} \rho, \tilde{b}) \times \delta(\Delta) \times \sigma$.
This implies that the Langlands quotient has in the GL-type Jacquet module an irreducible subquotient which has exponent $c$ in its Jacquet module, but does not have $c + 1$.

Observe that
\begin{align*}
s_{GL}(L(a_1, \Delta_{k_0 - 1} \cup \Delta_{k_0}) \times L(b) \times \delta(\Delta_u) \times \sigma) &\leq s_{GL}(L(a_1) \times \delta(\Delta_{k_0 - 1} \cup \Delta_{k_0}) \times L(b) \times \delta(\Delta_u) \times \sigma) = M^*_{GL}(L(a_1)) \times M^*_{GL}(\delta(\Delta_{k_0 - 1} \cup \Delta_{k_0})) \times M^*_{GL}(L(b)) \times M^*_{GL}(\delta(\Delta_u)) \times (1 \otimes \sigma).
\end{align*}
We cannot get any one of exponents \(c\) and \(c + 1\) from \(M'_{GL}(L(a_1))\) or \(M'_{GL}(\delta(\Delta_u))\) or \(M'_{GL}(L(b))\) (consider support). Therefore, it must come from

\[
M'_{GL}(\delta(\Delta_{k_0 - 1} \cup \Delta_{k_0})) = M'_{GL}(\delta([\alpha', d])) = \sum_{x=\alpha'-1}^{d} \delta([-x, -\alpha']) \times \delta([x + 1, d]),
\]

where \(c + 1 \leq d\). The above formula for \(M'_{GL}(\delta([\alpha', d]))\) implies that whenever we have in the support \(c\), it must come from a segment which ends with \(d\), and therefore, we must have in the support also \(c + 1\). Therefore, we cannot have only \(c\). In this way we have proved that (in both cases)

\[
L(a, \nu^{\alpha'} \rho, b; \delta(\Delta_{\pm}; \sigma)) \leq L(a, \Delta_{k_0}) \times L(b) \times \delta(\Delta_u) \times \sigma = L(a, \Delta_{k_0}, b) \times \delta(\Delta_u) \times \sigma + L(a, [\alpha' - 1, c]) \times L([\alpha, \alpha' - 2]^{\dagger}) \times \delta(\Delta_u) \times \sigma.
\]

Suppose

\[
L(a, \nu^{\alpha'} \rho, b; \delta(\Delta_{\pm}; \sigma)) \leq L(a, [\alpha' - 1, c]) \times L([\alpha, \alpha' - 2]^{\dagger}) \times \delta(\Delta_u) \times \sigma.
\]

Observe that

\[
L(a, \nu^{\alpha'} \rho, b; \delta(\Delta_{\pm}; \sigma)) \hookrightarrow \delta([\Delta_1] \times \ldots \times \delta([\Delta_{k_0 - 1}] \times \nu^{-\alpha'} \rho \times \nu^{-\alpha' + 1} \rho \times \ldots \times \nu^{-\alpha} \rho \times \delta(\Delta_{\pm}; \sigma)),
\]

which implies (because of unique irreducible subrepresentation of the right hand side)

\[
L(a, \nu^{\alpha'} \rho, b; \delta(\Delta_{\pm}; \sigma)) \hookrightarrow L(\tilde{a}) \times \delta([-\alpha', -\alpha])^{\dagger} \times \delta([-\alpha', \sigma] \times \sigma)
\]

\[
\hookrightarrow L(\tilde{a}) \times \delta([-\alpha', -\alpha])^{\dagger} \times \delta([-\alpha', \sigma] \times \sigma).
\]

Therefore, we have in the Jacquet module of the left hand side the irreducible representation

\[
L(\tilde{a}) \otimes \delta([-\alpha', -\alpha])^{\dagger} \times \delta([-\alpha', \sigma] \times \sigma).
\]

Now we shall examine how we can get this from

\[
\mu^{*}(L(a) \times \delta([\alpha' - 1, c]) \times L([\alpha, \alpha' - 2]^{\dagger}) \times \delta(\Delta_u) \times \sigma)
\]

a term of the form \(\beta \otimes \gamma\) such that the support of \(\beta\) is the same as of \(\tilde{a}\). Grading and disjointness of supports "up to a contragredient" imply that we need to take \(\beta\) from \(M^{*}(L(a))\) (we must take \(L(\tilde{a}) \otimes 1\)). This implies (using transitivity of Jacquet modules) that we need to have

\[
\delta([-\alpha', -\alpha])^{\dagger} \times \delta([-\alpha', \sigma] \otimes \sigma \leq \mu^{*}(\delta([\alpha' - 1, c]) \times L([\alpha, \alpha' - 2]^{\dagger}) \times \delta(\Delta_u) \times \sigma),
\]

which implies

\[
\delta([-\alpha', -\alpha])^{\dagger} \times \delta([-\alpha', \sigma] \otimes \sigma \leq M^{*}_{GL}(\delta([\alpha' - 1, c]) \times L([\alpha, \alpha' - 2]^{\dagger}) \times \delta(\Delta_u) \otimes \sigma.
\]

Observe that in the multisegment that represents the left hand side, we have \([-\alpha']\). In particular, we have a segment which ends with \(-\alpha'\).
Such a segment (regarding ending at $-\alpha'$) we cannot get from $M_{GL}^*(L([\alpha, \alpha'-2]t))$ (because of the support). Neither we can get it from $M_{GL}^*(\delta(\Delta_u))$ because of the formula:

$$M_{GL}^*(\delta(\Delta_u)) = \sum_{x=-\alpha'-1}^{\alpha'} \delta([-x, \alpha']) \times \delta([x+1, \alpha']).$$

The only possibility is $M_{GL}^*(\delta([\alpha'-1, c]))$. But segments coming from this term end with $c$ or $-\alpha' + 1$. So we cannot get $-\alpha'$ for end.

Therefore, we have got a contradiction.

This implies

$$L(a, \nu^{\alpha'} \rho, b; \delta(\Delta_\pm; \sigma)) \leq L(a, \Delta_{k_0}, b) \times \delta(\Delta_u) \times \sigma.$$

$$= \delta(\Delta_u) \times L(a, \Delta_{k_0}, b; \sigma) + \delta(\Delta_u) \times L(a, \Delta_{k_0}, [\alpha + 1, \alpha'-1]^t; \delta([\alpha]; \sigma)).$$

Suppose

$$L(a, \nu^{\alpha'} \rho, b; \delta(\Delta_\pm; \sigma)) \leq \delta(\Delta_u) \times L(a, \Delta_{k_0}, [\alpha + 1, \alpha'-1]^t; \delta([\alpha]; \sigma)).$$

One directly sees (using the Frobenius reciprocity) that in the GL-type Jacquet module of the left hand side we have an irreducible term in whose support appears exponent $-\alpha$ two times.

This cannot happen on the right hand side. To see this, observe that the right hand side is

$$\leq \delta(\Delta_u) \times L(a, \Delta_{k_0}, [\alpha + 1, \alpha'-1]^t) \times \delta([\alpha]; \sigma)).$$

Observe that we cannot get $-\alpha$ from $L(a, \Delta_{k_0}, [\alpha + 1, \alpha'-1]^t)$ (consider support, and its contragredient). We cannot get it from $\delta([\alpha]; \sigma))$ (since $\mu^*(\delta([\alpha]; \sigma)) = 1 \otimes \delta([\alpha]; \sigma) + [\alpha] \otimes \sigma$).

From the formula for $M_{GL}^*(\delta(\Delta_u)))$ we see that we can get $-\alpha$ at most once (since it is negative).

Therefore, this inequality cannot hold. This implies

(5.19) $$L(a, \nu^{\alpha'} \rho, b; \delta(\Delta_\pm; \sigma)) \leq \delta(\Delta_u) \times L(a, \Delta_{k_0}, b; \sigma).$$

Therefore, $\delta(\Delta_u) \times L(a, \Delta_{k_0}, b; \sigma)$ has length at least five. The proof of the lemma is now complete.

Our second goal in this section is to prove

**Lemma 5.3.** The multiplicity of

$$\delta(\Delta_u) \otimes L(a, \Delta_{k_0}, b; \sigma)$$

in

$$\mu^*(\delta(\Delta_u) \times L(a, \Delta_{k_0}, b; \sigma))$$

is at most 4 if $k_0 < k$ and $\text{card}(\Delta_{k_0}) > 1$. 
Proof. Denote
\[ \beta := \delta(\Delta_u) \otimes L(a, \Delta_{k_0}, b; \sigma). \]
If we take from \( \mu^*(L(a, \Delta_{k_0}, b; \sigma)) \) the term \( 1 \otimes L(a, \Delta_{k_0}, b; \sigma) \), to get \( \beta \) for a sub quotient, we need to take from \( M^*(\delta(\Delta_u)) \) the term \( \delta(\Delta_u) \otimes 1 \) (we can take it two times - see the above formula for \( M^*(\delta(\Delta_u)) \)). In this way we get multiplicity two.

Now we consider in \( \mu^*(L(a, \Delta_{k_0}, b; \sigma)) \) terms different from \( 1 \otimes L(a, \Delta_{k_0}, b; \sigma) \) which can give \( \beta \) for a sub quotient.

Observe that by Lemma 3.1
\[ L(a, \Delta_{k_0}, b; \sigma) \leq L(a, [\alpha' + 1, c]) \times L([\alpha, \alpha']^t; \sigma). \]
Now the support forces that from \( M^*(L([\alpha, \alpha']^t; \sigma)) \) we must take \( 1 \otimes L(a, [\alpha' + 1, c]) \). The only possibility which would not give a term of the form \( 1 \otimes - \) is to take from \( M^*(L([\alpha, \alpha']^t; \sigma)) \) the term \( \sum_{i=0}^{\alpha'-\alpha+1} L([\alpha' - i + 1, \alpha']^t) \otimes L([\alpha, \alpha' - i]^t; \sigma) \).

Now we need to take from \( M^*(\Delta_u) \) a term of form \( \delta([\alpha', \alpha']) \otimes - \), for which we have two possibilities (analogously as in the proof of former corresponding lemma; use the formula for \( M^*(\Delta_u) \)). Since on the left and right hand side of \( \otimes \) we have regular representations (which are always multiplicity one), we get in this way at most two additional multiplicities. Therefore, the total multiplicity is at most 4.

\[ \square \]

6. Concluding remarks

A direct consequence of the claims that we have proved in the last two sections is the following

**Corollary 6.1.** Let \( \Delta_{k+1} \neq \emptyset \) and \( k \geq 1 \). Consider
\[ L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}; \sigma))^t = L(\Delta'_1, \ldots, \Delta'_k; \sigma). \]
Then \( \text{card}(\Delta'_i) > 1 \) for some \( i \). Denote maximal such index by \( k'_0 \). Write
\[ \Delta_{k_0} = [\nu^{\alpha+k-k_0}, \nu^c], \quad \Delta_{k_0}' = [\nu^{\alpha'}, \nu^c]. \]
Denote
\[ \Delta_u = [\nu^{-\alpha'}, \nu^c]. \]
Then

1. The length of \( \delta(\Delta_u)^t \times L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}; \sigma)) \) is at least 5.
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(2) The multiplicity of \( \delta(\Delta_u)^t \otimes L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}; \sigma)) \) in \\
\( \mu^*(\delta(\Delta_u)^t \rtimes L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}; \sigma))) \)

is most 4.

Proof. Denote \\
\( \tau = L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}; \sigma)) \).

Now by Lemmas 4.1 and 5.1 we know that \\
\( \delta(\Delta_u) \rtimes \pi^t \)

This implies that \\
\( \delta(\Delta_u)^t \rtimes \pi \)

has length \( \geq 5 \).

Further, Lemmas 4.3 and 5.3 imply that the multiplicity of \( \delta(\Delta_u) \otimes \pi^t \) in \( \mu^*(\delta(\Delta_u) \rtimes \pi^t) \) is at most 4. This implies that the multiplicity of \( \delta(\Delta_u)^t \otimes \pi \cong \delta(\Delta_u)^t \otimes \pi \) in \( \mu^*(\delta(\Delta_u)^t \rtimes \pi) \) is \( \leq 4 \). This completes the proof of the corollary. \( \square \)

This corollary, together with Lemmas 4.1, 4.3, 5.1 and imply Proposition 3.3.

In the sequel of this paper we shall show how Proposition 3.3 implies in a simple way Theorem 1.2.

At the end of the paper we shall given another proof of the following result of Hanzer, Jantzen and Tadić, which (applying [12]) is a special case of Theorem 1.2:

**Theorem 6.2.** If \( \gamma \) is an irreducible sub quotient of \\
\( \nu^{\alpha+n} \rho \times \nu^{\alpha+n-1} \rho \times \cdots \times \nu^\alpha \rho \times \sigma \)

different from \( L([\alpha, \alpha+n]^{(\rho)}; \sigma) \) and \( L([\alpha+n]^{(\rho)}, [\alpha+n-1]^{(\rho)}, \ldots, [\alpha]^{(\rho)}, \sigma), \) then \\
\( L(\Delta_1, \ldots, \Delta_k; \delta(\Delta_{k+1}; \sigma)) \)

is not unitarizable.

Proof. Chose \( \pi \) as in Proposition 3.3 Suppose that \( \gamma \) is unitarizable. Then \( \pi \rtimes \gamma \) is unitarizable. Let \( \tau \) be a sub quotient of \( \pi \rtimes \gamma \). Then \( \tau \hookrightarrow \pi \rtimes \gamma \). Now the Frobenius reciprocity implies that \( \pi \otimes \gamma \) is in the Jacquet module of \( \tau \).

We know that \( \pi \rtimes \gamma \) has length \( \geq 5 \). This (and unitarizability) implies that there are (at least) 5 different irreducible subrepresentations of \( \pi \rtimes \gamma \). Denote them by \( \tau_1, \ldots, \tau_5 \). Then \\
\( \tau_1 \oplus \cdots \oplus \tau_5 \hookrightarrow \pi \rtimes \gamma \).

Since the Jacquet functor is exact, the first part of the proof implies that the multiplicity of \( \pi \otimes \gamma \) in the Jacquet module of \( \pi \rtimes \gamma \) is at least 5. This contradicts to the second claim of Proposition 3.3. The proof is now complete. \( \square \)
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