Generalized Wavefunctions for Correlated Quantum Oscillators I: Basic Formalism and Classical Antecedants.

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Abstract

In this first of a series of four articles, it is shown how a hamiltonian quantum dynamics can be formulated based on a generalization of classical probability theory using the notion of quasi-invariant measures on the classical phase space for our description of dynamics. This is based on certain distributions as probability amplitudes, and related distributional measures, rather than the familiar invariant Gibbs measures of classical statistical mechanics. The first quantization is by functorial analytic continuation of real probability amplitudes, mathematically effecting the introduction of correlation between otherwise independent subsystems, and whose physical consequence is the incorporation of Breit-Wigner resonances associated to Gamow vectors into our description of dynamics. The real probability amplitudes are probably of more formal than practical physical interest, but the demonstrate structure and the first quantization is functorial. The resulting representation of dynamics is indistinguishable from the rigged Hilbert space formulation of quantum mechanics, and this quantum dynamics admits a natural field theory interpretation. This basic formalism will be employed in subsequent installments of the series.

Key words: hamiltonian quantum dynamics, rigged Hilbert space, hamiltonian field theory

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1 Introduction

1.1 Motivation

This is the first installment of a four installment series of papers describing various aspects of a hamiltonian description of quantum fields [12]. We will create a quantum field theory over canonical position and momentum (phase space) variables, eventually adopting the familiar creation and destruction operator formalism when we deal with the dynamics of correlated (i.e., interacting) fields. We will incorporate Schrödinger’s equation into the theory, but extend the allowable solutions to the Schrödinger equation to include weak (distributional) solutions. This extension of the allowable solutions necessarily involves mathematical developments subsequent to von Neumann’s Hilbert space (HS) formulation of quantum mechanics [7]. Herein, we will avail ourselves of a variant of the rigged Hilbert space (RHS) formalism of Bohm and Gadella [18], which is a natural generalization of the Hilbert space formalism of von Neumann that affords us this enlarged solution set, and which retains the HS formalism of von Neumann for the description of stationary states. (See the reviews [456].) In this part, then, we will use a probabilistic description of hamiltonian dynamics on classical phase space, extended to a quantum theory by the introduction of multi-tiered notions of correlation. From the correlation emerges (Breit-Wigner) resonances, with self-consistent interactions and non-trivial dynamical time evolution. Mathematical necessity compels us to use the RHS formalism, in which our wave functions are “very well behaved”, i.e., functions of rapid decrease, generalizing the notion of Gaussian wave packet. One of the many interpretations of this is as a hamiltonian field theory.

The present series of papers is focused on four aspects of enlarging the RHS formalism of Bohm and Gadella:

- Adding solutions to concrete problems (which cannot even be respectably addressed in the conventional HS formalism) to the RHS’s formal achievements in the area of intrinsic irreversibility (including intrinsic irreversibility on the microphysical level [8910]) and causality [6].
- Demonstrating how it is possible using our variant of the RHS quantum formalisms to incorporate genuine mathematical hyperbolicity into quantum dynamics. This leads to notions of quantum dynamics which have many formal mathematical attributes analogous to the non-linear dynamics, chaos, fractals, etc., encountered in the contemporary literature for classical dynamical systems. This is categorically incompatible with von Neumann’s HS formalism. This will be covered in installment three [2].
- The variant of the RHS formalism which we adopt admits a field theory interpretation, and from the dynamics alone of 2, 3 and 4 correlated fields respectively we will deduce Yang-Mills structures. These ”generic” gauge field theories will
be covered in installment four [3].

- We will require a complex plane structure in our ring of scalars, making the algebraic field $\mathbb{C}$ unsuitable for several technical mathematical reasons, including uniqueness. This will be covered in length in installment two [1]. There we will deal with matching a substantial number of mathematical structures into a compatible and well defined whole.

The concrete problem addressed throughout these four papers is that of describing the dynamical evolution of coupled (correlated) quantum oscillators, which is intrinsically interesting from a physical perspective. This is a first description of this problem from a novel physical and mathematical perspective, and there are some quite deep conceptual concerns we will touch on—we will try to illuminate many of these, but there is no pretense of complete and definitive resolution on many key points. Certainly there is much of at least formal interest, and much structure will be revealed which is physically illuminating even if one does not take our variant of the RHS picture completely to heart, particularly the idiosyncratic real probability amplitude description whose analytic continuation introduces a symplectic structure and thereby works a quantization. Whether or not our developments are physically correct will have to await later judgment, but our conclusions are largely generic, with a very strong element of first principle deduction independent of the physical makeup of the fields or the nature of their interaction.

For instance, we will ultimately conclude that the gauge group of the Standard Model is very similar (even isomorphic) to the most general gauge structure one might expect for three fields, independent of the precise nature of those fields and determined solely by dynamics, but that the true gauge group must be exact. Thus, our developments will lead to essentially the same spectrum— isomorphic gauge groups—as the Standard Model, but there are differences in interpretation, such as dynamical symmetry breaking. If nature does not observe the precise “generic” gauge structure we predict for a given number of fields, then this suggests some special or non-generic physics is present, the possible existence of a greater or lesser number of fields, etc.

This first installment is concerned with establishing that a well defined mathematical formalism exists for our constructions, and in an aside we will show that analytic continuation of a real probability theory could possibly produce a structure which looks just like quantum theory, but our resulting theory admits a hamiltonian field theoretic interpretation (for correlated fields) whether or not one adopts this particular perspective.

The second installment will demonstrate the mathematical steps which must be taken in order to have mathematically well grounded complex spectra: there are quite well defined points of departure from the usual Hilbert space quantum formalism and also from the “standard” rigged Hilbert space (distributional) generalization, although this rigged Hilbert space (RHS) formalism must also be parallelized.
in many regards in order to accommodate the complex energies of Breit-Wigner resonances by mathematically well defined means.

The third installment illustrates how full fledged hyperbolic dynamics may be represented in the quantum mechanics and hamiltonian quantum field theory so formulated—the exponential decay of Breit-Wigner resonances is an example of exponential separation of trajectories, and may be associated with an increase in entropy.

The fourth and concluding installment of this series of papers ties up a lot of mathematical loose ends by defining a Clifford algebra structure in which our spinorial constructions are mathematically well defined, and demonstrating how a mathematically well defined gauge theory may be extracted from pretty much generic considerations of dynamics alone. This results in a largely generic construction, with the gauge group being dependent only on the number of fields, and recovers an exact gauge group in all cases, and the dynamical mechanisms for symmetry breaking are shown to exist beyond this most foundational and elemental gauge symmetry level. In this concluding paper we will show how the islands of relative stability we call elementary particles are related by an exact symmetry, but their full dynamical interactions are described by a larger semigroup which may produce unstable resonances. It will be shown that the gauge structure for four fields over canonical variables may be associated with structures in a non-trivial space-time with $(+,−,−,−)$ local signature, in a manner which suggests that some measure of Lorentz invariance may be a consequence of dynamics and even suggests that there is a dynamical basis for PCT. There is an association of hamiltonian dynamics of canonical variables, structures on non-trivial space-time, and possibly even to irreps of the inhomogeneous Lorentz group, although our group and Lie algebra representations are fundamental representations (i.e., spinorial) and not UIR’s.

So this is where a probabilistic description of canonical dynamics on phase space, in a formalism which includes multiple levels of correlation, will ultimately lead us. At a minimum, we will arrive at a fiber bundle description of all canonical dynamics possessing analytical interactions, which lead to stable correlated structures, and which can be phrased in a probability formalism, i.e., based on ensemble notions. This bundle structure will relate the various stable structures present in the dynamical system, while there is a larger dynamical structure associated with the production and subsequent dynamical evolution of resonances not amenable to the fiber bundle formalism, being based on semigroups rather than full group structures. See installment four [3]. Perhaps there is some particle physics in here, but, for instance, the basic results should be just as applicable to fluids or plasmas with long range interactions.
1.2 The present paper

The canonical transformations of physics are the symplectic transformations of mathematics. The second paper of this series will choose to represent the appropriate symplectic (semi-)group as the largest possible dynamical (semi-)group, generalizing those classical notions. In the theory of dynamical systems, these symplectic transformations are the dynamical transformations. See [13]. Classically, hamiltonian dynamical evolution is associated with the translation and reshaping of a region in phase space to which a dynamical system is localized, and this dynamical evolution is by symplectic (e.g., “area preserving”) maps. These are topologically transitive mappings.

We will use a sophisticated mathematical structure herein, whose implications we begin to relate in Section 2. For instance, we can replace the classical description of a dynamical system as a point in phase space with a more loosely localized region in phase space to which our dynamical system is localized by a “bump function” (function of rapid decrease, e.g., a generalized gaussian) which is properly normalized so that it may be regarded as a probability amplitude. We can then provide a similar “bump function” to represent sensitivities and capture probabilities, etc., of a measurement apparatus as another probability amplitude. (We might also use a more exotic generalized function, or distribution, rather than a simple function here.) There will be symplectic (=dynamical) transformations on the function space made up of all the mathematically acceptable “bump functions,” provided we give this function space a symplectic structure during its construction. (The detailed construction of this symplectic structure is the subject of our second paper.) Those symplectic transformations could possibly represent dynamical evolution of these classical (i.e., real) probability amplitudes within the context of a classical probability theory in which the outcome of a measurement is predicted by the scalar product of probability amplitudes, and this scalar product should be able to evolve with time: we are able to represent the time dependent interaction of a non-stationary dynamical system with a non-stationary measurement apparatus using this formalism. We will defer explicit consideration of the symplectic transformations until the second installment, and concentrate on how the nature of the spaces involved lends itself to a probability theory in this initial installment. For us, probabilities will be an intrinsic part of the construction, and not a subsequent interpretation.

The analytic continuation—a complex symplectic transformation—of just such a probabilistic description starting with real probability amplitudes is mathematically equivalent to representation of quantum resonances, including Breit-Wigner resonance poles (of the analytically continued S-matrix) belonging to one of the spaces of a Gel’fand triplet in our RHS formalism. Additionally, this accords—perhaps only loosely at times—with the physical interpretation of the RHS formalism in [4]. See Section 5.
Our quantum theory will be obtained then by a functorial analytic continuation of classical densities on phase space, and our measurement apparatus throughout will be associated with distributions mathematically dual to those densities.

2 Structure of the Spaces Involved

Working physicists virtually never use the full generality of the Hilbert space of von Nuemann’s formulation of quantum mechanics. Typically, they employ smooth functions and Riemann integrals rather than classes of Lebesgue square integrable functions. Although the smooth functions used by physicists usually do belong to the Hilbert space, they are almost invariably elements of the Schwartz space \( \mathcal{S} \) of functions of rapid decrease, so that, e.g., there are vanishingly small contributions to the probability attributable to infinities in energy or momentum or position. The rigged Hilbert space (RHS) formalism, which we shall use a variant of, adopts this usual practice of working physicists, and obtains thereby a variety of additional benefits, such as an enlarged class of solutions to the Schrödinger equation. We will be exploring the consequences of adopting an enlarged class of solutions to the Schrödinger equation in a RHS. (The “rigging” in rigged Hilbert space is based on a nautical metaphor: one thereby makes the bare Hilbert space “ready to sail”.) There are several excellent reviews on the RHS formalism which describe the Gadella diagrams we will make use of below [4,5], and one which addresses many issues including causality and the mathematics summarized in the Gadella diagrams (although not using the diagrams themselves) [6].

There has been a recent revival of interest in working in a quantum paradigm involving use of generalized wave functions in a rigged Hilbert space (RHS, which will be indicated generically by the Gel’fand triplet of spaces \( \Phi \subset \mathcal{H} \subset \Phi^\times \)). This RHS formalism involves applying a mathematically rigorous methodology of analytic continuation, and leads to the unification of the vector space representation of quantum systems and the representation of resonances by poles of the analytically continued \( S \)-matrix. In the RHS paradigm, resonances are represented by abstract Gamow vectors associated to Breit-Wigner poles of the analytically continued \( S \)-matrix, with exponential decay, and the time evolution is by semi-groups. These semi-groups provide the formalism with the ability to express the boundary conditions of an irreversible physical process without regeneration [8,9,10].

We follow the lead of Gadella [17], who refined the analysis issues in the function spaces relevant to our work, and showed how to obtain a rigorous analytic continuation based on necessary and sufficient conditions. Our concern, then, is erecting particular geometric structures in the types of spaces Gadella has shown us we must use. We begin by very tersely describing the interrelationships of the spaces involved in the RHS formalism, and also providing references for further inquiry. The RHS formalism involves working with a hierarchy of triplets of spaces, with
abstract spaces and function space realizations of the abstract spaces.

Distinguishing abstract spaces from their function space realizations, and the analytic continuation of those spaces, is an essential, if tedious part of the RHS methodology. This is physically motivated in part. There is considerable structure in the large number of vector spaces involved, and there is a significantly different physical content to each of the spaces: there is a difference between the abstract Hilbert space ($\mathcal{H}$) and its $L^2$ function space realization in the energy representation on the half-line, $L^2[0,\infty)$, and the space corresponding to the abstract $\mathcal{H}$ realized in a subspace of $L^2(\mathbb{R})$, also in the energy representation. Whether or not the energy spectrum is bounded from below has considerable practical physical implication as well as formal mathematical importance.

Formally, the RHS paradigm involves alternative topological completions of a pre-Hilbert space, the algebraic linear space $\Psi$, to form the Gel’fand triplet of spaces $\Phi \subset \mathcal{H} \subset \Phi^\times$. In the $\tau_\Phi$ topology set by a countable family of semi-norms we have one topological completion to form the topological vector space $\Phi$. Using the scalar product to define a norm, we define a Hilbert space topology, $\tau_{\mathcal{H}}$, and the completion of $\Psi$ in this topology is a Hilbert space $\mathcal{H}$ [25]. Dual to $\Phi$ is the conjugate space $\Phi^\times$, which has a weak-dual topology, $\tau^\times$. The resonances in the RHS formalism are obtained by analytic continuation, which proceeds from $\mathcal{H}^\times \longrightarrow \tilde{\Phi}^\times$ [17], where $\mathcal{H}^\times \cong \mathcal{H}$ (Riesz isomorphism), and $\tilde{\Phi}^\times$ is the complex extension (analytic continuation) of $\Phi^\times$. There are complex energy eigenvalues on $\tilde{\Phi}^\times$, indicating exponential decay for quantum systems represented by states there. In terms of (mathematically respectable) physical content, in the RHS formalism we have the analytically continued S-matrix (and its poles), the Lippman-Schwinger equation, Møller operators, etc. Although we will not deal with these explicitly, in installment two [1] we will effectively see Møller wave operators as dynamical (=symplectic) transformations, an algebraic Lippman-Schwinger equation, etc. As a point of distinction from prior work, previously the analytic continuation (complex extension) was of the absolutely continuous scattering spectrum, while the present work operates by complex extension (complex symplectic transformation) of an operator algebra in which the only spectrum initially apparent is discrete. See installment two [1]. We will not deal closely with representation issues, but will implicitly be working with both the operator algebra and its representations on both abstract and function spaces throughout. Even where we are not explicit in this series of articles, it is always implied that we intend eventually to work with representations, make use of spectral theorems, and so on.

Although we will not speak explicitly of dragging contours around in the complex plane, or making explicit use of the residue theorem, etc., nothing we will do will change the necessary and sufficient conditions for the complex extension [17], or any of the other major structural features of the RHS formalism. (See the reviews [54] for more details of that formalism than it is possible to recapitulate here.) In installment two [1], by rotations (and possibly other complex symplectic
transformations), we will shift the spectrum (i.e., the poles of the resolvent) from
the real axis out into the complex plane. Only a certain countable set of complex
symplectic transformations will produce the Breit-Wigner resonance poles.

The basic structure of the relationships of the abstract spaces and the associated
function spaces which occur in the RHS formalism are most easily seen in the
Gadella diagrams \[4,5\]. Conventionally one envisions a measurement process hav-
ing the preparation procedure ending at time \(t = 0\), with preparation of the in-state
\(\phi^+\) occurring during \(t \leq 0\) and observation of the effect \(\psi^-\) during \(t \geq 0\). The
Gadella diagram for representing the preparation of an in-state \(\phi^+\) during \(t \leq 0\) is

\[
\phi^+ \in \mathcal{H} \subset (\Phi_-)^\times \quad \exists \psi^G = |E - z^+_R >
\]

and the corresponding diagram for the effect \(\psi^-\) observed during \(t \geq 0\) is

\[
\psi^- \in \mathcal{H} \subset (\Phi_+)^\times \quad \exists \psi^G = |E - z^-_R >
\]

The basic ideas behind these stem from \[17\]; it or \[18\] should be consulted for
details of construction for the various spaces and mappings, as well as the careful
definition and meaning of the various vectors identified in the left and right hand
margins. Each level of each diagram contains a RHS or Gel’fand triplet of spaces,
and there are certain properties which each space inherits by virtue of its relative
position in a Gel’fand triplet.

Along the top line of each diagram are abstract spaces, the middle line gives the
function space realizations of the spaces in the top line conforming to the neces-
sary and sufficient mathematical conditions for performing analytic continuation in
a unique fashion (energy picture, physical energy bounded from below with bound-
ary conventionally taken as \(E = 0\)), and the bottom line shows the function spaces
resulting from the analytic continuation \[17\]. The lack of connecting link between
the \(L^2(\mathbb{R}_+^\times)\) realization of \(\mathcal{H}\) on the middle level and the subspace of \(L^2(\mathbb{R})\) on the
lower level reflects the lack of unique extension between these spaces.
The transformation
\[ U^+ : \Phi_+ \ni \psi \rightarrow \langle -E | \psi^- \rangle \in \mathcal{H}_+ \cap \mathcal{H}_+^2 \vert_{\mathbb{R}^+} . \]  
(1)
defines the space \( \Phi_+ \) in terms of the function space \( \mathcal{H}_+ \cap \mathcal{H}_+^2 \vert_{\mathbb{R}^+} \) (the well-behaved functions on the positive real line \( \mathbb{R}^+ \) which are also Hardy class functions from above) extends to a unitary transformation \( U^+ : \mathcal{H} \rightarrow L^2(\mathbb{R}^+) \), i.e., the Hilbert space of Lebesgue square integrable functions on the positive real line
\[ (U^+ f, U^+ g) \equiv \int_0^\infty dE \langle f | E^- \rangle \langle -E | g \rangle = \langle f | g \rangle , \quad f, g \in \mathcal{H} . \]  
(2)
The map \((U^+)^\times\) is the extension of \((U^+)^\dagger\). Similar relations hold for the transformation
\[ U^- : \Phi_- \ni \phi^+ \rightarrow \langle +E | \phi^+ \rangle \in \mathcal{H}_- \cap \mathcal{H}_-^2 \vert_{\mathbb{R}^+} . \]  
(3)
The hardy class functions have the remarkable property that their values on the entire real line \( \mathbb{R} \) are determined by their values on \( \mathbb{R}^+ \). In the second installment \([1]\), we will consider the symplectic transformations (=dynamical transformations), which form a superset of the unitary transformations, and their action on the spaces of states will be defined so that they obey the same rule as that laid out in equation (2), since that is the form which is necessary and sufficient for their action on the space to be symplectic (=dynamical) \([12]\). The significance of this is that one may define the maps in such a way that ones dynamics induce unitary transformations on Hilbert space, which we will see relates to the issue of whether or not the transformations are ergodic (installment three \([2]\)). We will note the diagrams are commutative, and refer further inquiries by the reader to the references cited.

Considering only the top diagram briefly for some further interpretation, we have an abstract (exponential formation culminating at \( t = 0 \)) Gamow vector \( \tilde{\psi}^G \) belonging to the rightmost abstract space \((\Phi)^\times\) on the top line being associated to a Breit-Wigner resonance pole in the the rightmost function spaces of the lowest line, which has a complex energy eigenvalue \( z^*_R = E_R + i\Gamma/2 \). As a practical matter, the physically prepared abstract state denoted \( \phi^+ \) is input into this hierarchy as a “very well behaved” element of the function space lying in the intersection of the Schwartz and Hardy class functions (from below) over the half line \( \mathcal{H}_- \cap \mathcal{H}_-^2 \vert_{\mathbb{R}^+} \), e.g., as a physically determined “very well behaved” energy distribution of a beam actually prepared in an accelerator \( \langle +E | \phi^+ \rangle = \phi^+(E) \). The \( \mathbb{R}_+ \) indicated with the spaces corresponds to the physical energy spectrum, which is bounded from below. The abstract prepared state \( \phi^+ \) (+ superscript physical notation) is an represented by an element of the space of Hardy class functions over the half line from below, \( \mathcal{H}_-^2 \vert_{\mathbb{R}^+} \) (− subscript mathematical notation).

The second diagram for \( t \geq o \) is similar, with an abstract Gamow vector of pure exponential decay associated with a Breit-Wigner resonance pole whose complex energy is \( z_R = E_R - i\Gamma/2 \), etc.
In the sequel, whether the given use of a rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$ is intended generically or as a particular Gel’fand triplet of spaces often will depend on the context of use. The spaces usually will be indicated as $\Phi$ or as $\Phi^\pm_\mathfrak{g}$, or as $\subset \mathcal{S} \cap \mathcal{H}^2$, etc., depending on whether one is concerned with a generic rigged Hilbert space structure (abstract or function space realized), its Lie algebra representation structure, or the space of “very well behaved” vectors in the intersection of the Schwartz and Hardy class functions (from above or below).

3 A classical system extended

For a real elliptic Hamiltonian (which the analytic continuation procedure must start with, see installment three [2])—such as the Hamiltonian of a system of two free oscillators—there exist real eigenfunctions. Indeed, this is the way the function space realizations of the vectors of quantum physics are usually encountered in the classroom for the first time, and it is only later that issues of time evolution and phase freedom for these special functions are discussed in detail. From a mathematical perspective, it would be reasonable to consider the abstract spaces $\Phi^\pm_\mathfrak{g}$ as real, and to also consider only the purely real subspaces of the associated (complex) function spaces identified in the preceding Gadella diagrams. See also [27].

In order to understand some aspects of the present treatment of this analytic continuation of a Lie algebra representation, it is useful to adopt an idiosynchratic perspective, and to consider extending the real representation of the real algebra to a representation of its complexification, both in terms of the abstract representation space of a group whose Lie algebra is $\mathfrak{g}$, $\Phi^\pm_\mathfrak{g}$, and the proper function space realization lying in the intersection of the Schwartz and Hardy class functions. At first, this must be regarded as idiosynchratic since the exponential map need not be a unitary transformation on a real Hilbert space if one follows the von Neumann model of construction, so that probabilities appear not to be Noetherian conserved quantities of the evolutionary flow there, etc. This associated real Hilbert space may not seem very interesting physically at first, but will offer more promise once it is clear we use a different Hilbert space than the one von Neumann constructs. Further, non-conservation of the probability of observing a resonance—essentially, its survival probability—suggests decay of the non-surviving system, which is not repugnant to us in any way. We require the conservation of the total probability, also including the probability of the system which comes into being as the product of the decay of the resonance system which does not survive. (We may not be able to show total probability conservation constructively, but we will be able to show the existence of the equilibrium towards which the system decays in the third installment of this series, from which conservation of total probability may be inferred.)

This idiosynchratic perspective arises simply from noting that there is nothing in Gadella’s construction which requires the top two levels of the Gadella diagrams
to involve complex spaces! There is a field of classical mechanics which can be called distribution density dynamics. (See, e.g., the appendix 14 to [19] and references therein.) There is thus a physical context in which the mathematically permissible choice of using real spaces on the top two lines of the Gadella diagrams makes sense. We can define a classical dynamical system on real spaces \( \Phi^{\pm} \) and \( \Phi^\times\), and the associated physical context is, in effect, an averaging over an infinite number of classical trajectories (e.g., in phase space) using a distribution density. Such a construction would have abstract spaces and function space realizations, corresponding to the top two levels of our idiosyncratic Gadella diagram. The transition from the middle to the bottom levels of the idiosyncratic diagram via analytic continuation represents the (functorial) first quantization of the classical distribution density dynamics method of description of a dynamical system, and results in the description of a dynamical system by a recognizable quantum theory which includes Breit-Wigner resonances [28].

Most recent physics using the RHS formalism has been preoccupied with the energy representation, since the primary recent interest has been in describing irreversible time evolution using the formalism’s ability to express the boundary conditions of an irreversible process, without regeneration [8,9,10]. There is also a momentum space picture of the Gamow states [20,21]. For our present idiosyncratic purposes, it is interesting to consider starting from a classical phase space, since when we do so many similarities appear between the RHS methodology advanced herein and a version of classical statistical mechanics.

By working with probability amplitudes in the RHS format, rather than with probability amplitudes on the constant energy surfaces in phase space, we are defining probability measures on phase space itself. When we cause those probability amplitudes to evolve dynamically, we avoid the many of the pitfalls of time evolving probabilities in the Boltzmann and Gibbs approaches based on discrete partitioning of phase space. (See, e.g., [22].)

In place of the energy representation of a prepared state \( \langle E|\phi^{\text{in}} \rangle = \langle +E|\phi^{+} \rangle \in \mathcal{I} \cap \mathcal{H}^2 \bigg|_{\mathbb{R}^+} \), we will introduce a similarly well behaved \( \langle \pi|\phi^{\text{in}} \rangle = \langle +\pi|\phi^{+} \rangle \in \mathcal{I} \cap \mathcal{H}^2 \), where \( \pi \) indicates the canonical phase space variables \( (p_x, p_y, q_x, q_y) \), and \( \mathcal{I} \cap \mathcal{H}^2 \) indicates real valued functions on \( (T \times \mathbb{R}^2) \oplus i \circ (T \times \mathbb{R}^2) \) (see [11]), which are understood to be properly normalized as probability amplitudes. For example, an equilibrium ideal gas in a container, \( \langle +\pi|\phi^{+} \rangle \) might have Maxwellian (gaussian) velocity distributions and a functions of compact support over the position as the components of a multi-component spinor (see [11]).

Similarly, in place of \( \langle E|\psi^{\text{out}} \rangle = \langle -E|\psi^{-} \rangle \in \mathcal{I} \cap \mathcal{H}^2 \bigg|_{\mathbb{R}^+} \) for the measurement resolution of our measuring apparatus (i.e., a measure on phase space), we will have \( \langle \pi|\psi^{\text{out}} \rangle = \langle -\pi|\psi^{-} \rangle \in \mathcal{I} \cap \mathcal{H}^2 \), meaning a real valued and very well behaved function of our \( \pi \)-variables, also normalized. Both of these sets of gener-
alized functions will be defined for restricted time domains just as the functions on the energy surfaces within these function spaces have restricted time domain of definition, and dynamical evolution (including dynamical time evolution) will be by semigroups, meaning we continue to have the ability to express the boundary conditions of an irreversible process referred to earlier. (Because we include distributions, we have the conceptual ability to include “distributional probability measures”, such as measures which will yield Boolean value 0 or 1 representing the outcome of a yes-no experiment.) Because of the hyperbolic structure inherent on properly constructed complex spaces, we later see it is possible to construct hyperbolic probability measures, e.g., the probability of observing a Breit-Wigner resonance decreases hyperbolically with time.

This formalism is also compatible with the notion of extended objects, because, e.g., δx ≠ 0 in general, so no point localization is assumed anywhere, although we do have the mathematical machinery (Dirac measures) to accommodate point localization if desired. The idiosyncratic perspective contemplates concurrent position and velocity measurements, and this is perfectly okay—we’re not in Hilbert space anymore! (This is made formally apparent in [18].) If we had a gaussian classical momentum distribution (e.g., a Maxwellian velocity distribution) and gaussian classical position distributions, then we would recover a classical equivalent of the Heisenberg uncertainty relation, based on the widths of the gaussians, even in the case that Planck’s constant and the full implications of the quantum mechanical complementarity principle are not incorporated into the system.

From this π-representation in terms of (p, q), analytic continuation takes us (p, q) → (p, ip, q, iq). Following a simple change of coordinates

\[ A = (q + ip)/\sqrt{2} \quad A^\dagger = (q - ip)/\sqrt{2} \]

so that now (p, ip, q, iq) → (A, iA, A^\dagger, iA^\dagger) provides a basis for the complexified phase space. We are thus an eyeblink away from the creation and destruction operator formalism used in the rest of this series of papers. In installment four, we will see how obtaining a real Witt basis from this basis enables the representation of either bosons or fermions without a change of basis (spinors are notoriously basis dependent).

Joint position and momentum probability distributions do not exist for any quantum state represented by an element of \( L^2(\mathbb{R}^n) \). (This is the motivation for the Wigner transform, for instance.) Both position and momentum cannot simultaneously be represented by continuous operators on \( L^2(\mathbb{R}^n) \). Similarly, on \( L^2(\mathbb{R}^n) \) it cannot be the case that both the creation and destruction operators are both continuous operators, and so, for instance, one has to use care in defining the coherent states. However, in the RHS formalism position and momentum are both represented by operators which are \( \tau_\Phi \)-continuous and \( \tau_\Phi \)-closed. Likewise, both creation and destruction operator are \( \tau_\Phi \)-continuous and \( \tau_\Phi \)-closed operators (see chapter two of [18]), and so these restrictions applicable to the HS formalism do not constrain us in a
RHS formalism. Thus, after the analytic continuation we change basis to the real Witt basis and think in terms of eigenvector probability densities of the creation and destruction operators, since they are $\tau_\Phi$-continuous $\tau_\Phi$-closed operators. Continuity and closedness in $\Phi$ and $\mathcal{H}$ need not agree, and this has far reaching consequences, which we are exploring.

4 Quasi-Invariant Measures

The space $\Phi^\times$ provides quasi-invariant measures for the space $\Phi$ in the Gel’fand triplet $\Phi \subset \mathcal{H} \subset \Phi^\times$ [14]. Because we use operators which are not symmetric, the left and right quasi-invariant measures will differ, e.g., in the resolution of the identity provided by the spectral theorem, the dyads $|\phi\rangle\langle\phi|$ are not symmetric, and in particular $|\phi\rangle$ and $\langle\phi|$ will be defined for different time domains as a consequence of our use of semigroups of dynamical time evolution. This will have physical consequences we will demonstrate in installment three [2]. In installment three [2], we will see that dynamical evolution can be hyperbolic, and these quasi-invariant measures are hyperbolically evolving as well, but that overall probability is conserved.

5 Physical Interpretation

In the present series of articles, we are exploring a dynamical system of oscillators in which there is correlation between the oscillators. We are using probability amplitudes rather than point particle localization, and in particular we allow the use of densities and distributions for those probability amplitudes. This involves us with not the $L^2$ functions but with a subset of the Hardy class functions which partition $L^2$, $L^2 = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$, according to the Paley-Wiener theorem.

Given the possibility of free oscillators, we expect some sort of direct sum structure (sort of like a Foch space), except our dynamical transformations may also mix the components. We will see in installment two [1] that these multi-component probability amplitudes are of mathematical necessity of a certain form and a sufficient form for them is put forward. On our phase space, and on the abstract and function space representations of it (using these densities and distributions), there is a symmetric form $Q$ and a skew-symmetric form $J$:

$$ Q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (5) $$

Each of these forms induces a scalar product, so that there are, in effect, three
possible scalar products. Assuming

\[
|\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \text{and} \quad |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]

we may represent these three scalar products as:

\[
\begin{align*}
(\psi|\phi) &= \psi_1^\times \circ \phi_1 + \psi_2^\times \circ \phi_2 \\
\langle \psi|\phi \rangle &= (Q\psi|\phi) = \psi_2^\times \circ \phi_1 + \psi_1^\times \circ \phi_2 \\
\{ \psi|\phi \} &= (J\psi|\phi) = \psi_2^\times \circ \phi_1 - \psi_1^\times \circ \phi_2
\end{align*}
\]

where \( \psi_2^\times \circ \phi_1 \) indicates a “single component” scalar product. We will return to these alternative scalar products (and their associations with alternative topologies) in installment four [3], but the important observation for now is the types of correlation shown in the above alternative scalar products. Thus, the complex unit scalar \( i \) is associated with a symplectic form \( J \), and is thus identified with a correlation mapping (a type of injective embedding in the dual) [12], and in fact a complex plane structure may be associated with both the symmetric and skew-symmetric correlations above, depending on whether one adopts an orthogonal or hyperbolic (Lobachevsky) geometry.

5.1 Collateral Implications

The foregoing “idiosyncratic perspective” suggests at once fairly straightforward and conceptually obvious modifications are likely possible to the Grad moment expansion [23,24], but such pursuits are very wide of our present concerns—possibly it will be taken up another day, establishing a connection between the present formalism and a generalized “thermodynamics”. In place of a moment expansion about the Maxwellian (e.g., gaussian) velocity distribution, we generalize to functions of rapid decrease, and similarly interpret the moments as thermodynamic quantities. This also has implications for the Meyer cluster expansion and its graphical representation by Feynman diagrams.

There are also implications for the theory of solution of differential equations. Thus, the second installment is concerned with the Lie-Poisson bracket of vector fields on phase space, and representations of this structure, and the fourth installment is concerned with the tensor algebra of phase space.

There is a breathtakingly direct analogy between a probabilistically oriented classical description and the mathematics of the present formalism. We find a description of the classical treatment in [19] Appendix 14, page 457:
“Jacobi realized that the (classical) Poisson brackets of the first integrals of any
hamiltonian system could be considered as a Poisson structure [reference in origi-
nal].

The construction of a Poisson structure on the dual space of a Lie algebra
leads to a new Lie algebra. This construction may then be repeated, leading to
a whole series of new (infinite dimensional) Poisson structures. More generally,
suppose that one is given any Poisson structure on a manifold. Then the space of
functions on that manifold carries the structure of a Lie algebra. This implies that
the dual space of this function space carries its own Poisson structure. Elements
of this dual space may be interpreted as distribution densities on the original
manifold. Thus, the space of distributions on a Poisson manifold (for example, on
a symplectic phase space) has a natural Poisson structure. This structure makes it
possible to apply the hamiltonian formalism to equations of Vlasov type, which
describe the evolution of distributions of particles in phase space under the action
of a field which is consistent with the particles themselves.”

Physically, this means we may be working in a paradigm whose classical (pre-
analytic continuation) analogue is based on ensemble notions, e.g., one is either
dealing with an ensemble of particles (perhaps even a field) or with an ensemble
of measurements of identical simple systems. In particular, both before and after
analytic continuation this construction includes a field theoretic treatment of a
large number of particles whose paths are locally hamiltonian, and whose interac-
tions are dealt with in a self consistent way [26]. There are both orthogonal and
symplectic correlations, and, in particular, there are both real symplectic and com-
plex symplectic correlations, corresponding to correlated classical dynamics and
to a complex dynamical correlation which admits oscillatory behavior in a formal-
ism which looks just like a quantum theory of resonances.

A Complex Spectral Theorem

From the starting point of a real semisimple Lie algebra, we undertake to show the
role of the complex covering algebra in the construction of a representation space
Φ. From the universal embedding algebra, one has the existence of a complete
set of commuting operators. Let us assume this c.s.c.o. of essentially self adjoint
operators is \{A_1, A_2, \ldots, A_N\}. Let A_i be the Hilbert space spectrum of the operator
A_i, \(i = 1, 2, \ldots, N\), and let \(\Lambda = \Lambda_1 \times A_2 \times \cdots \times \Lambda_N\) be the Cartesian product of the \(\Lambda_i\).
Then the general Gel’fand-Maurin Theorem (also called general Nuclear Spectral
Theorem) asserts there exists a rigged Hilbert space \(\Phi \subset \mathcal{H} \subset \Phi^\times\) such that there
exists a uniquely defined positive measure on \(\Lambda\) such that [4]:

(1) \(\Phi\) has a topology determined by a countable family of semi-norms.
(2) \(A_1, A_2, \ldots, A_N\) are esa and are \(\tau_0\)-continuous on \(\Phi\).
(3) For any \((\lambda_1, \lambda_2, \ldots, \lambda_N)\) in \(\Lambda\) there exists a generalized eigenvector in \(\Phi^\times\),
\[ |F_\lambda\rangle = |\lambda_1, \lambda_2, \ldots, \lambda_N\rangle \]

such that

(a) \( A_i^\times |F_\lambda\rangle = \lambda_i |F_\lambda\rangle \) for almost (with respect to \( \mu \)) all \( i = 1, 2, \ldots, N \). \( A_i^\times \) denotes the extension of \( A_i \) to \( \Phi^\times \), the dual space to \( \Phi \). If the \( A_i \) have no singular spectrum, “almost all” can be replaced by “all”.

(b) For any pair of vectors \( \phi, \psi \in \Phi \) and any well defined function \( f \) of \( N \) variables, one has

\[
(\phi, \psi) = \int_{\Lambda} \langle \phi | F_\lambda \rangle \langle F_\lambda | \psi \rangle \, d\mu(\lambda_1, \lambda_2, \ldots, \lambda_N) \tag{A.1}
\]

\[
(\phi, f(A_1, A_2, \ldots, A_N) \psi) = \int_{\Lambda} f(\lambda_1, \lambda_2, \ldots, \lambda_N) \langle \phi | F_\lambda \rangle \langle F_\lambda | \psi \rangle \, d\mu(\lambda_1, \lambda_2, \ldots, \lambda_N) \tag{A.2}
\]

The positive measure is unique up to equivalence of the null set. In general, the space \( \Phi \) of the RHS may not be unique \[29\].

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[25] Although all Hilbert spaces are isomorphic, the isomorphisms are not canonical or natural, so the spaces need not be strictly equivalent. In installment two [1], we will develop a Hilbert space physically inequivalent to the Hilbert space of von Neumann. The choice of topological completion still places us in the RHS paradigm. In effect, we will define a complex structure on the linear space $\Phi$ differently than is usually done, since there are mathematical issues with using the field of complex numbers.

[26] We have introduced the notion of correlation into a system in which long range interactions may figure. Our methods are based on quasi-invariant measures. The Gibbs ensembles based on invariant measures are also compatible with long range interactions.

[27] The extension of the $e^{-iHt}$ time evolution group on $\mathcal{H}$ to transformations of the same $U(1)$ form on $\Phi$ and $\Phi^\times$ is not a trivial matter, and more is involved than the generalization of the form of formal solutions to the Schrödinger equation. It is a
mathematically and physically reasonable step to do so (discussed in [4] and [5]), but one of the lessons of the present work is that even a time evolution operator of apparently “non-unitary” form on $\Phi \subset \Phi^\times$ could be ergodic and induce a unitary time evolution operator on $\mathcal{H}$ (in effect inducing the Schrödinger description of time evolution for pure states in $\mathcal{H}$). This will be addressed in the third installment of this series [2].

[28] We will not here address the issue of whether the bottom line of this idiosynchratic diagram can simultaneously serve as the bottom level of a conventional Gadella diagram, in effect determining the complex spaces above this bottom line.

[29] This statement of the complex spectral theorem stems from [5] and [4]. The present treatment is distinguishable in a variety of ways, and one of the consequences of those distinctions is that our work involves different left and right (quasi-)invariant measures. Our partition of unity is of the form $\mathbb{I} = \Sigma |F_\lambda\rangle\langle \tilde{F}_\lambda|$, where $|F_\lambda\rangle$ and $\langle \tilde{F}_\lambda|$ have different time domains of definition. This will be addressed in the second and third installments of this series [12], and is the result of attention to boundary conditions.