Tangles are decided by weighted vertex sets

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joint work with Christian Elbracht and Maximilian Teegen

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![Diagram of a separation](image)

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A \( k \)-tangle is a set \( \tau \) of separations of order \( < k \) such that:
- \( \tau \) contains exactly one of \((A, B)\) and \((B, A)\) for all \((A, B)\) of order \( < k \)

Motivation

A large cluster in a graph orients all the low order separations of a graph, the second condition (tangle property) ensures that we point to something substantial.
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grids

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Do we always have such a decider set? Maybe.
We can show that weighted deciders exist:

Theorem (Elbracht, K, Teegen, 2020)

Let $G = (V, E)$ be a finite graph and $\tau$ a $k$-tangle in $G$.
Then there exists a function $w : V \rightarrow \mathbb{N}$ such that a separation $(A, B)$ of $G$ of order $< k$ lies in $\tau$ if and only if $w(A) < w(B)$, where $w(U) := \sum_{u \in U} w(u)$ for $U \subseteq V$. 

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How did we prove this?
First observation

The separations come with a natural partial order:

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Let \((A, B)\) and \((C, D)\) be two distinct maximal separations in a \(k\)-tangle. Consider the separators \(A \cap B\) and \(C \cap D\).

The separation \((A \cup C, B \cap D)\) cannot be of order \(< k\).

Taken together, the separator vertices are 'more often right than wrong':

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|B \cap (C \cap D)| + |D \cap (A \cap B)| - (|A \cap (C \cap D)| + |C \cap (A \cap B)|) > 0
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- Enumerate the maximal separations of $\tau$ as $(A_1, B_1), \ldots, (A_m, B_m)$.
- Define an $m \times m$ matrix $M$ where $m_{ij}$ specifies how well $A_j \cap B_j$ decides $(A_i, B_i)$, by setting
  \[ m_{ij} := |B_i \cap (A_j \cap B_j)| - |A_i \cap (A_j \cap B_j)|. \]
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Now if $x$ is a vector of weights for the separators, then

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so $Mx$ is the vector of the ‘net scores’ of the $(A_i, B_i)$ in the weighting $x$. 

Tangles are decided by weighted vertex sets.
We need to find a weight vector $x \geq 0$ with $Mx > 0$. 

By the second observation we have $m_{ij} + m_{ji} > 0$, so $M + M^T$ has positive off-diagonal entries.

Lemma (Farkas' Lemma) Given $A$ and $b$, either $Ax \geq b$ has a solution with $x \geq 0$, or there exists a $y \geq 0$ such that $A^T y \leq 0$ and $b^T y > 0$.

Apply Farkas with $A = M$ and $b = (1, \ldots, 1)^T$. Two possible outcomes: there is $x \geq 0$ with $Mx \geq (1, \ldots, 1)^T > 0$. there is $y \geq 0$ with $M^T y \leq 0$ and $y \neq 0$.

But then $0 \leq (M + M^T)y \leq My$. 

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\[ \text{Diagram:} \]

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- there is \( x \geq 0 \) with \( Mx \geq (1, \ldots, 1)^T > 0 \).
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Similar to tangles, we can define $k$-edge-tangles of a graph $G = (V, E)$:

For a cut $(A, B)$, a bipartition of $V$, we define its order as the number of cross-edges $|E(A, B)|$.

A $k$-edge-tangle is a set $\tau$ of cuts of order $< k$ such that:

1. For every cut $(A, B)$ of order $< k$, either $(A, B)$ or $(B, A)$ is in $\tau$.
2. If $\tau$ contains $(A_1, B_1)$, $(A_2, B_2)$, and $(A_3, B_3)$, then $A_1 \cup A_2 \cup A_3 \neq V$.
3. For every $(A, B)$ in $\tau$ there are at least $k$ edges incident with vertices in $B$. 

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Similar to tangles, we can define \textit{k-edge-tangles} of a graph $G = (V, E)$: For a \textit{cut} $(A, B)$, a bipartition of $V$, we define as \textit{order} the number of cross-edges $|E(A, B)|$.
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**Theorem (Elbracht, K, Teegen, 2020)**

Let $G = (V, E)$ be a finite (multi-)graph and $\tau$ a $k$-edge-tangle in $G$. Then there exists a function $w : V \rightarrow \mathbb{N}$ such that a cut $(A, B)$ of $G$ of order $< k$ lies in $\tau$ if and only if $w(A) < w(B)$. 

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The edge-tangle result does not extend to hypergraphs.

Consider the hypergraph $H = (V,E)$ where $V$ is the set of 3-element subsets of \{1,...,7\}; for each $i \in \{1,...,7\}$ we have a hyperedge $e_i = \{v \in V | i \in v\}$.

In this hypergraph we have an edge-tangle $\tau = \{(A,B) | \exists e_i \in E \text{ with } B \supset e_i\}$.

This edge tangle has no weighted decider: consider

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Thank you!