BUNDLES OF VERLINDE SPACES AND GROUP ACTIONS

JAYA NN IYER

ABSTRACT. A Verlinde space of level $k$ is the space of global sections of the $k$-th power of the determinant line bundle on the moduli space $SU_C(r)$ of semi-stable bundles of rank $r$ on a curve $C$. The aim of this note is to make accessible some remarks on the action of the Theta group on the Verlinde spaces of higher level. This gives a decomposition of the bundle of Verlinde spaces over the moduli space of curves and we indicate how to compute the rank of the isotypical components in the decomposition.

CONTENTS

1. Introduction
2. The space $H^0(SU_C(r), \Theta_C)$ is a Heisenberg module
3. Parabolic case
4. A decomposition of the Verlinde bundles of higher level
5. A remark on the multiplicities of the isotypical components
6. References

1. Introduction

Let $C$ be a nonsingular connected projective curve defined over $\mathbb{C}$. The Jacobian variety $J(C)$ associated to the curve is a moduli space of rank one and degree zero bundles on the curve $C$. There is a natural polarization $\Theta_C$ on the Jacobian and one can associate the space $H^0(J(C), \Theta_C^k)$ of global sections of the $k$-th power of the line bundle $\Theta_C$, also called as the abelian theta functions. The Theta group $G(\Theta_C^k)$ was introduced by Mumford [Mu2] and he prescribed an action of this group on $H^0(J(C), \Theta_C^k)$ (more generally for sections of line bundles on abelian varieties, see §2.1). As an application, he obtained results on equations defining abelian varieties amongst many other moduli questions.

A higher rank analogue of $J(C)$ is the moduli space $U_C(r,0)$ of semi–stable bundles of rank $r$ and degree 0 and the moduli space $SU_C(r)$ of semi–stable vector bundles of rank $r$ and trivial determinant on $C$, introduced by Mumford, Narasimhan and Seshadri [Mu1], [Na-Se], [Se]. There is a polarization $\Theta$ on the moduli space $SU_C(r)$ called as the determinant bundle [Dr-Na]. The space $H^0(SU_C(r), \Theta^k)$ of global sections of $\Theta^k$ are called as the Verlinde spaces of level $k$. The sections are called the generalized theta functions.

---

*Mathematics Classification Number: 14C25, 14D05, 14D20, 14D21

*Keywords: Connections, moduli spaces, generalised theta functions, theta group.
An action of a theta group $G$ on the space $H^0(SU_C, \Theta)$ was prescribed in [BNR] and it was shown to be an irreducible $G$–module. We wish to investigate the $G$–action on the higher level Verlinde spaces.

We put this in the framework of families of these moduli spaces over the moduli space of curves. This is done to be able to compute the Chern classes of the bundle of the Verlinde spaces of level one and we hope that it finds applications on further questions on monodromy of the projective representations.

Suppose $\pi_C : C \rightarrow T$ is a smooth projective family of curves of genus $g$. We can associate to this family, the relative moduli space

$$\pi_S : SU_C(r) \rightarrow T$$

of semi–stable vector bundles of rank $r$ and trivial determinant. There is a relative polarization $\Theta$ on $SU_C(r)$, also called as the determinant bundle.

The Verlinde bundles

$$V_{r,k} := \pi_S^*(\Theta^k)$$

are known to be equipped with a projectively flat connection (i.e., a flat connection on the projectivization $\mathbb{P}(V_{r,k})$), also called as Hitchin’s connection (see [Fa1], [Hi]). We notice that $\Theta_C$ is not uniquely defined since we can tensor it by the pullback of any line bundle on $T$. This implies that the Verlinde bundles are defined up to taking tensor product with a line bundle on $T$.

Let $\gamma_{r,k} = \dim H^0(SU_C(t), \Theta^k_t)$ be the dimension of the space of sections of $\Theta^k_t$. Then, by [Be-La], [Fa2] we have the ‘Verlinde formula’:

$$\gamma_{r,k} = \left(\frac{r}{r+k}\right)^g \sum_{S \cup R = [1,r+k]} \prod_{s \in S} |2 \sin \frac{s-z}{r+k}|^{g-1}.$$

We show that there is a decomposition of the Verlinde bundle, of the form

$$\bigoplus_{\chi \in \hat{K}(\delta)_k} W_\chi \otimes F_\chi$$

over a suitable cover of $T$. Here $W_\chi$ is an irreducible Heisenberg representation (of higher weight) and $F_\chi$ is a vector bundle on an étale cover of $T$ over any point (Proposition 4.2). This is an application of Mumford’s Theorem [Mu3, Proposition 2, p.80] of theta groups, to the case of generalized theta functions.

We indicate how the rank of the bundles $F_\chi$ can be computed (section 5). This shows that the dimension of the isotypical components are different and the isotypical component corresponding to the trivial character is greater than the other components. This is in contrast with the abelian theta functions, where all the components are equi-dimensional (see [Iy1] Proposition 3.7, which is stated for level 2, but in fact it holds for any level).
As an application, we compute the Chern character of the level one Verlinde bundle in the rational Chow groups (Corollary 4.3).

The proof is via a study of the Heisenberg group representations [Mu2], [Iy]. We extend the action of the Heisenberg group to higher level Verlinde spaces to obtain our assertion. The action is prescribed in a more general set-up, i.e., for moduli of parabolic bundles. Our hope was to compute the multiplicities using degeneration of the moduli spaces with their polarizations and using the Factorisation theorems. It then becomes essential to consider moduli of parabolic bundles with a $G$-action on the space of generalized theta sections. The Factorization theorems were proved by Faltings, Narasimhan, Ramadas, Sun [Fa2], [Na-Ra], [Su] and many other mathematicians in computing the Verlinde formula in some cases. It seemed difficult for us to carry out the computations with a $G$–action though. We include §3 for the interested readers who might want to use this approach. Beauville, Laszlo, Sorger [Be1], [Be-La-So], Andersen-Masbaum [An-Ma] have treated special cases.

Acknowledgements: We thank H. Esnault for bringing our attention to the the Verlinde bundles, in summer 2004. We thank G. Masbaum for his interest on the contents and encouraging us to pursue it further.

2. THE SPACE $H^0(\mathcal{SU}_C(r), \Theta_C)$ IS A HEISENBERG MODULE

All the varieties are considered over the field of complex numbers.

2.1. Theta groups. We recall the definition of the Theta group introduced by Mumford and refer to [Mu2] for details.

Suppose $A$ is an abelian variety of dimension $g$ and let $L$ be an ample line bundle on $A$. Consider the translation map, for any $a \in A$:

$$t_a : A \to A, \ x \mapsto x + a.$$ 

Consider the group:

$$K(L) = \{a \in A : L \simeq t_a^* L\}$$

and the Theta group of $L$:

$$\mathcal{G}(L) = \{(a, \phi) : L \simeq \phi t_a^* L\}.$$ 

In particular there is a central extension:

$$1 \to \mathbb{C}^* \to \mathcal{G}(L) \to K(L) \to 0.$$ 

2.2. Heisenberg groups. Fix positive integers $\delta_1, \delta_2, \ldots, \delta_g$ such that $\delta_i$ divides $\delta_{i+1}$, for each $i$. The $g$–tuple $\delta = (\delta_1, \ldots, \delta_g)$ is called the type of $\delta$. 

Given a type $\delta$, write
\[ K_1(\delta) = \left( \frac{\mathbb{Z}}{\delta_1\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{\delta_g\mathbb{Z}} \right) \]
\[ \widehat{K_1(\delta)} = \text{Group of characters on } K_1(\delta) \]
\[ K(\delta) = K_1(\delta) \oplus \widehat{K_1(\delta)}. \]

The Heisenberg group $Heis(\delta)$ is the set
\[ \mathbb{C}^* \times K(\delta) \]
with a twisted group law: $(\alpha, x, l). (\beta, y, m) = (\alpha. \beta. m(x), x + y, l.m)$ \cite{Mu2}.

Consider the $\mathbb{C}$–vector space
\[ V(\delta) = \{ f : \frac{\mathbb{Z}}{\delta_1\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{\delta_g\mathbb{Z}} \rightarrow \mathbb{C} \} \]
and the action of $(\alpha, x, l) \in Heis(\delta)$ on $f \in V(\delta)$ is given as:
\[ (\alpha, x, l). f(y) = \alpha l(y). f(x + y). \]

Then we have

**Theorem 2.1.** The $\mathbb{C}$–vector space $V(\delta)$ is of dimension equal to $\delta_1. \delta_2... \delta_g$ and is the unique irreducible representation of the Heisenberg group $Heis(\delta)$ such that $\alpha \in \mathbb{C}^*$ acts by its natural character.

*Proof.* See \cite{Mu2} Proposition 1]. \hfill \Box

**Definition:** If $W$ is a representation of the Heisenberg group $Heis(\delta)$ such that $\alpha \in \mathbb{C}^*$ acts as multiplication by $\alpha^l$, then we say that $W$ is a $Heis(\delta)$–module of weight $l$.

We have the following result on higher weight $Heis(\delta)$–modules.

**Proposition 2.2.** The set of irreducible representations of the Heisenberg group $Heis(\delta)$ of weight $l$ is in bijection with the set of characters on the subgroup of $l$–torsion elements, $K(\delta)_l \subset K(\delta)$.

Moreover the dimension of any such representation is
\[ \frac{\delta_1...\delta_g}{(l, \delta_1)...(l, \delta_g)}. \]

If $\chi$ is a character on $K(\delta)_l$ and $W_\chi$ is the corresponding irreducible representation then $W_\chi \otimes \chi^{-1}$ is identified with the $Heis\left(\frac{\delta}{l}\right)$–representation $V\left(\frac{\delta}{l}\right)$ of weight 1. Here $\delta = \left( \frac{\delta_1}{(l, \delta_1)}, \ldots, \frac{\delta_g}{(l, \delta_g)} \right)$ and $(l, \delta_i)$ denotes the greatest common divisor of $l$ and $\delta_i$.

*Proof.* See \cite{Iy1} Proposition 3.2] when $l = 2$ and \cite{Iy2} Proposition 5.1] when $l > 2$. \hfill \Box
2.3. $H^0(\mathcal{SU}_C(r), \Theta_C)$ as a Heis(δ)–module of weight 1. Given a nonsingular projective curve $C$ of genus $g$ and integers $r, d$, the moduli space of semi–stable vector bundles of rank $r$ and degree $d$ is denoted by $\mathcal{U}_C(r, d)$. The moduli space of semi–stable bundles on $C$ of rank $r$ and trivial determinant is denoted by $\mathcal{SU}_C(r)$ and the ample polarization on it by $\Theta_C$ [Dr-Na]. The Jacobian $J^n_C$ parametrises degree $n$ line bundles on $C$, upto isomorphisms.

Notice that the subgroup $(J_C)_r$ of $r$–torsion points on $J_C$, acts on the moduli space $\mathcal{SU}_C(r)$

$$E \mapsto E \otimes l, \text{ for } l \in \text{Pic}^0(C) = J(C)_r$$

and it leaves $\Theta_C$ invariant [BNR, p.178].

Consider the commutative diagram (I):

$$\begin{array}{ccc}
\Theta^k_C & \overset{\phi}{\cong} & \Theta^k_C \\
\downarrow & & \downarrow \\
\mathcal{SU}_C(r) & \overset{\otimes l_r}{\longrightarrow} & \mathcal{SU}_C(r)
\end{array}$$

Here $l_r$ is the line bundle corresponding to a $r$–torsion point on $J(C)$.

Consider the group

$$G_k(\Theta_C) = \{(l_r, \phi) : \Theta^k_C \overset{\phi}{\cong} (\otimes l_r)^*\Theta^k_C\}.$$ 

Then there is an exact sequence:

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G_k(\Theta_C) \longrightarrow J(C)_r \longrightarrow 0$$

which is a central extension.

We recall the constructions in [BNR] which leads to a description of the vector space $H^0(\mathcal{SU}_C(r), \Theta_C)$.

Firstly, the moduli space $\mathcal{U}_C(r, d)$ is described as follows.

**Theorem 2.3.** There is a $r$–sheeted (ramified) covering $\pi : C' \longrightarrow C$ with $C'$ nonsingular and irreducible such that the rational map $\pi_* : J_C^\beta \longrightarrow \mathcal{U}_C(r, d)$ is dominant. The indeterminacy locus of $\pi_*$ is of codimension at least 2 and $\beta = d - \deg\pi_*(\mathcal{O}_{C'})$.

**Proof.** See [BNR] Theorem 1. □

Let $\sigma = (\det\pi_*\mathcal{O}_{C'})^{-1}$ be the line bundle and consider the norm map

$$Nm : J_C^{\deg\sigma} \longrightarrow J_C^{\deg\sigma}.$$ 

Let $P' = Nm^{-1}\sigma$ be the variety associated to the ramified covering $\pi : C' \longrightarrow C$. We denote $g = \text{genus of } C$ and $g' = \text{genus of } C'$. Then there is a commutative diagram (I) ([BNR Proposition 5.7, p. 178]):
Remark 2.4. Notice that P.3 and (II) imply that the pullback map is an isomorphism.

and satisfying:

P.1. the morphism \( i_s \) is an isogeny of degree \( r^{2g} \) and \( i_{SU} \) is the map given by tensor product. Further, \( (i_{SU})^* \Theta_U \simeq p_1^* \Theta_C \otimes p_2^* \Theta_J \), for the natural projections \( p_i \).

P.2. \( \pi_* \) induces a dominant (generically finite) rational map

\[
\pi_{*,is} : P' \longrightarrow SU_C(r).
\]

The indeterminacy locus of \( \pi_{*,is} \) is of codimension at least 2.

P.3. \( \Theta_{P'} = (\pi_{*,is})^*(\Theta_C) \) is a primitive line bundle (i.e., not a power of another line bundle) and is of type \( \delta = (1, 1, ..., 1, r, r, ..., r) \). Here \( r \) occurs \( g \)-times.

P.4. The subgroup \( (J_C)_r \) of \( r \)-torsion points of \( J_C \) acts on \( SU_C(r) \) and leaves the line bundle \( \Theta_C \) invariant. There is a \( G(\Theta_{P'}) \)-action on the sections of \( \Theta_{P'} \) such that the pullback map \( H^0(SU_C(r), \Theta_C) \longrightarrow H^0(P', \Theta_{P'}) \) is equivariant for this group and the pullback map is an isomorphism.

Consider the commutative diagram (II):

\[
P' \times J_C^{-1} \xrightarrow{i_s} J_C^{-1}
\]

\[
\downarrow \pi_{*,is} \quad \downarrow \pi_*
\]

\[
SU_C(r) \times J_C^{-1} \xrightarrow{i_{SU}} U(r, r(g - 1))
\]

Here \( l_r \in \text{Pic}^0(C)_r = J(C)_r \).

\textbf{Remark 2.4.} Notice that P.3 and (II) imply that \( G_1(\Theta_C) \simeq G(\Theta_{P'}) \). Indeed, the map given by \((l_r, \phi) \mapsto (l_r, (\pi_{*,is})*\phi)\) which is injective and hence an isomorphism. Further, this implies that the Weil pairing (given by the commutator map) on \( J(C)_r \), corresponding to the extension

\[
1 \longrightarrow \mathbb{C}^* \longrightarrow G_1(\Theta_C) \longrightarrow J(C)_r \longrightarrow 0
\]

is nondegenerate. Also, \( G_1(\Theta_C) \) acts on \( H^0(SU_C(r), \Theta_C) \) with weight 1 and is an irreducible representation.

\textbf{Remark 2.5.} The above mentioned remark can be extended to the following case: consider the moduli space \( SU_C(r, \eta) \) of semi-stable bundles with fixed determinant \( \eta \). Now \( l_r \in J(C)_r \) acts on \( SU_C(r, \eta) \) as \( E \mapsto E \otimes l_r \). Since \( \text{Pic} SU_C(r, \eta) = \mathbb{Z} \Theta_C \) (Dr-Na) any point \( l_r \) of \( J(C)_r \) corresponds to a finite order automorphism of \( SU_C(r, \eta) \), we have \( \Theta_C \simeq (\otimes l_r)^* \Theta_C \). As earlier we can form the group of automorphisms \( G_1(\Theta_C) \) of \( \Theta_C \). Further, there is a Weil form on \( J(C)_r \), given by the commutator map associated to the extension,

\[
1 \longrightarrow \mathbb{C}^* \longrightarrow G_1(\Theta_C) \longrightarrow J(C)_r \longrightarrow 0.
\]
This form is nondegenerate since $\Theta_C$ is primitive. In other words, $G_1(\Theta_C)$ can be identified with the standard Heisenberg group $Heis(\delta)$, where $\delta = (r, \ldots, r)$ and $r$ occurs $g$-times.

Remark 2.6. Since there is a (surjective) homomorphism

$$G_1(\Theta_C) \rightarrow G_k(\Theta_C), \ (x, \phi) \mapsto (x, \phi^k)$$

we see that $G_1(\Theta_C)$ acts on $H^0(SU_C(r, \eta), \Theta^k_C)$ and $\alpha \in \mathbb{C}^*$ acts as $\alpha \mapsto \alpha^k$, i.e., with weight $k$.

3. Parabolic case

Suppose $C$ is a nonsingular projective connected curve of genus $g$ and $E$ is a vector bundle on $C$. Fix a parabolic data $\Delta$:

$$S = \{x_i : 1 \leq i \leq n\} \subset C$$

is a finite set of $n$ distinct points, fix a positive integer $m$ and for each $x \in S$ associate a sequence of integers

$$0 < a_1(x) < a_2(x) < \ldots < a_{l_x+1}(x) < m$$

called weights $a(x) = (a_1(x), \ldots, a_{l_x+1}(x))$. The weights $a(x)$ have multiplicities $n(x) = (n_1(x), n_2(x), \ldots, n_{l_x+1}(x))$ associated to a flag of the fibre $E_x$

$$E(x) = F_0(E_x) \supset F_1(E_x) \supset \ldots \supset F_{l_x}(E_x) \supset F_{l_x+1}(E_x) = 0$$

such that $n_j(x) = \dim \left( \frac{F_{j-1}(E_x)}{F_j(E_x)} \right)$.

Consider the moduli space $SU_C(r, \Delta)$ of vector bundles of rank $r$ and trivial determinant and which are semi–stable with respect to the parabolic data $\Delta$. Then $SU_C(r, \Delta)$ is a projective variety ([Me-Se]). There is a parabolic theta line bundle $\Theta_\Delta$ on $SU_C(r, \Delta)$ which is ample ([Na-Ra, Theorem 1.(A)]).

We briefly recall the constructions (see also [Su]):

Consider the Quot-scheme $Q$ of coherent sheaves of rank $r$ and degree 0 over $C$ and trivial determinant, which are quotients of $O^{P(N)}(-N)$, with a fixed Hilbert polynomial $P$. Here $N$ is chosen large enough so that every $\Delta$–parabolic semi–stable vector bundle with Hilbert polynomial $P$ occurs as a point in $Q$.

Thus on $C \times Q$, there is a universal sheaf $F$, flat over $Q$ and denote the restriction on $x \times Q$ by $F_x$, for $x \in S$. Let

$$Flag_{n(x)}(F_x) \rightarrow Q$$

be the relative Flag scheme of type $n(x)$. Consider the fibre product

$$\mathcal{R} = \times_{x \in S} Flag_{n(x)}(F_x) \xrightarrow{pr} Q.$$

Let $\mathcal{R}^{ss} \subset \mathcal{R}$ denote the open subscheme of $\mathcal{R}$ whose points correspond to $\Delta$–parabolic semi–stable bundles with trivial determinant. The pullback of $F \rightarrow C \times Q$, under $Id \times pr$, to $C \times \mathcal{R}^{ss}$ is still denoted by $F$. 


Denote the quotients
\[ Q_{x,i} = \frac{F_x}{F_i(F_x)}. \]
The parabolic theta line bundle is defined as
\[ \Theta_\Delta = (\det R\pi_*(F))^m \otimes \bigotimes_{x \in S} ((\det F_x)^{m-a_{lx}+1} \otimes \bigotimes_{i=1}^{l_x} (\det Q_{x,i})^{a_{i+1(x)}-a_i(x)}). \]
Here \( \pi : C \times R^{ss} \to R^{ss} \) is the second projection and \( \det R\pi_*(F) = (\det \pi_*(F))^{-1} \otimes \det R^1\pi_*(F) \).

The variety \( \text{SU}_C(r, \Delta) \) is the ‘good quotient’ of \( R^{ss} \) under the action of \( SL(P(N)) \). The ample line bundle \( \Theta_\Delta \) descends to an ample line bundle on \( \text{SU}_C(r, \Delta) \) and is still denoted by \( \Theta_\Delta \).

**Remark 3.1.** Consider the open subscheme \( Q^0 \subset Q \) whose points correspond to semi-stable vector bundles (in the usual sense). Then \( SL(P(N)) \) acts on \( Q^0 \) and there are rational dominant maps

\[ q_1 : Q^0 \to SU_C(r) \]
\[ q_2 : SU_C(r, \Delta) \to SU_C(r) \]

**Remark 3.2.** Further, the ample line bundle \( \det R\pi_*(F) \) on \( Q^0 \) descends to the theta line bundle \( \Theta_C \) on \( SU_C(r) \). If \( m = 1 \), we write \( SU_C(r, \Delta) = SU_C(r) \).

3.1. **The space \( H^0(SU_C(r, \Delta), \Theta_\Delta) \) is a \( G_1(\Theta_C) \)-module.** Firstly, notice that the group \( J(C)_r \) acts on the moduli space \( SU_C(r, \Delta) \):

\[ E \mapsto E \otimes l_r \]
for a line bundle \( l_r \in \text{Pic}^0C = J(C)_r \).

In fact, there is a commutative diagram (III):
\[
\begin{array}{ccc}
SU_C(r, \Delta) & \xrightarrow{\otimes l_r} & SU_C(r, \Delta) \\
\downarrow q_2 & & \downarrow q_2 \\
SU_C(r) & \xrightarrow{\otimes l_r} & SU_C(r)
\end{array}
\]

**Lemma 3.3.** Suppose the indeterminacy of the map \( q_2 \) is of codimension at least 2. Then the vector space \( H^0(SU_C(r, \Delta), \Theta_\Delta) \) is a \( G_1(\Theta_C) \)-module of weight \( m \).

**Proof.** Since the indeterminacy of \( q_2 \) is of codimension at least 2, the pullback of \( \Theta_C \) defines a line bundle on \( SU_C(r, \Delta) \). Further, it follows from (2) that \( \Theta_\Delta = q^*\Theta_C^m \otimes M \), for some line bundle \( M \) on \( SU_C(r, \Delta) \) which is not a pullback from \( SU_C(r) \). Hence, given an element \((l_r, \phi) \in G_1(\Theta_C)\), there is an isomorphism
\[ \tilde{\phi} = q^* \phi^m \otimes \text{Id} : \Theta_{\Delta} \simeq (\otimes l_r)^* \Theta_{\Delta} \]

over \( SU_C(r, \Delta) \).

This gives an action of \( G_1(\Theta_C) \) on the space of sections \( H^0(SU_C(r, \Delta), \Theta_{\Delta}) \). In particular, the scalars act as \( \alpha \mapsto \alpha^m \). This proves our assertion.

\[ \square \]

**Corollary 3.4.** The vector space \( H^0(SU_C(r, \Delta), \Theta^k_{\Delta}) \) is a \( G_1(\Theta_C) \)–module of weight \( km \).

**Proof.** Indeed, as shown in Lemma 3.3, \( (l_r, \phi) \in G_1(\Theta_C) \) induces isomorphisms

\[ \Theta^k_{\Delta} \tilde{\phi}^k \simeq (\otimes l_r)^* \Theta^k_{\Delta} \]

over \( SU_C(r, \Delta) \). Thus \( \alpha \in \mathbb{C}^* \) acts on \( H^0(SU_C(r, \Delta), \Theta^k_{\Delta}) \) as \( \alpha \mapsto \alpha^{km} \).

\[ \square \]

Suppose \( \delta = (r, r, \ldots, r) \) with \( r \) occurring \( g \) times.

**Lemma 3.5.** Given a level \( r \)–structure on the Jacobian \( J(C) \), there is an isotypical decomposition

\[ H^0(SU_C(r, \Delta), \Theta^k_{\Delta}) \simeq \bigoplus_{\chi \in K(\delta)_{km}} n_{\chi}. \chi \]

where \( \chi \) is an irreducible representation of \( \text{Heis}(\delta) \) of weight \( km \). Moreover, \( \chi \otimes \chi^{-1} \) is identified with the \( \text{Heis}(\frac{\delta}{km}) \)–representation \( V(\frac{k}{km}) \) of weight 1.

**Proof.** A level \( r \)–structure \( h : J(C)_r \simeq K(\delta) \) is induced by an isomorphism

\[ G_1(\Theta_C) \simeq \text{Heis}(\delta). \]

This is true by Remark 2.4 and the arguments in [Mu2, p.318]): consider the subgroups \( h^{-1}(K_1(\delta)), h^{-1}(\tilde{K_1(\delta)}) \subset J(C)_r \). Consider their lifts which are level subgroups

\[ \tilde{K_1(\delta)}, \tilde{K_1(\delta)} \subset G_1(\Theta_C). \]

Construct \( f : G_1(\Theta_C) \rightarrow \text{Heis}(\delta) \) by mapping \( \tilde{K_1(\delta)} \) onto the subgroup \( \{(1, x, 0) : x \in K_1(\delta)\} \) and \( \tilde{K_1(\delta)} \) onto the subgroup \( \{(1, 0, l) : l \in \tilde{K_1(\delta)}\} \). Now extend multiplicatively to obtain an isomorphism \( G_1(\Theta_C) \simeq \text{Heis}(\delta) \).

Hence, by Remark 2.6 and Corollary 3.4, \( H^0(SU_C(r, \Delta), \Theta^k_{\Delta}) \) is now a \( \text{Heis}(\delta) \)–module of weight \( km \). By Proposition 2.2 there is an isotypical decomposition as asserted.

\[ \square \]

**Definition 3.6.** An isomorphism \( G_1(\Theta_C) \simeq \text{Heis}(\delta) \) is called a generalized theta structure.
4. A decomposition of the Verlinde bundles of higher level

4.1. The Verlinde bundles of level $km$. Fix a parabolic data $\Delta$ as in the previous section and satisfying the hypothesis in Lemma 3.3.

Consider a smooth projective family of curves with $n$–marked points

$\pi : \mathcal{C} \longrightarrow T$

of genus $g > 0$ and suppose $T$ is nonsingular.

**Remark 4.1.** We may assume that $T$ is the moduli space of nonsingular projective connected $n$–marked curves of genus $g$, with suitable level structures, so that there is a universal curve over $T$.

We can associate to (5), the following families:

(6) $\pi_J : \mathcal{J} \longrightarrow T$

is the family of Jacobian varieties of dimension $g$,

(7) $\pi_r : SU(r) \longrightarrow T$

is the family of moduli spaces of semi–stable vector bundles of rank $r$ and trivial determinant and

(8) $\pi_S : SU(r, \Delta) \longrightarrow T$

is the family of moduli spaces $SU_t(r, \Delta)$ of $\Delta$–parabolic semi–stable vector bundles on $\mathcal{C}_t$ of rank $r$ and trivial determinant.

There is a line bundle $\Theta^\Delta$ (resp. $\Theta$) on $SU(r, \Delta)$ (resp. $SU(r)$) such that $\Theta^\Delta$ restricts on any fibre $SU_t(r, \Delta)$ (resp. $SU_t(r)$) to the parabolic theta bundle $\Theta^\Delta_t$ (resp. $\Theta_t$) \[Dr-Na], [Na-Ra].

**Definition:** The vector bundles

$V_{r,km} = \pi_{S*}(\Theta^\Delta_k)$

are called as the Verlinde bundles of level $km$, for $k > 0$.

4.2. A decomposition of the Verlinde bundles. We denote

$\gamma_{r,km} = \frac{r^g}{(km, r)^g} \cdot \sum_{\chi \in K(\delta)_{km}} n_{\chi} = \text{rank} \ V_{r,km}$

Consider the group scheme $\mathcal{J}_r \longrightarrow T$ which is the kernel of the homomorphism

$\mathcal{J} \longrightarrow \mathcal{J}$

given by multiplication by $r$ on $\mathcal{J}$. There is an exact sequence

$1 \longrightarrow \mathbb{G}_{m,T} \longrightarrow \mathbb{G}_1(\Theta) \longrightarrow \mathcal{J}_r \longrightarrow 0$
where \( G_1(\Theta) \) represents the functor defining the automorphisms of \( \Theta \) over the sections of \( J_r \) (see also [Mu3, p.76], for similar constructions).

**Proposition 4.2.** Given a \( t_0 \in T \), there is an étale open cover \( U \to T \) of \( t_0 \), such that

\[
V_{r,km} \cong \bigoplus_{\chi \in \hat{K}(\delta)_{km}} W_{\chi} \otimes F_{\chi}
\]

over \( U \) and for some vector bundles \( F_{\chi} \) on \( U \).

**Proof.** Suppose \( T \) is the moduli space of nonsingular \( n \)-marked curves with level \( r \)-structure. Given a \( t_0 \in T \), a level \( r \)-structure can be lifted locally on \( T \) to a generalized theta structure, say over an open étale cover \( U \to T \), i.e., the group scheme \( G_1(\Theta) \) trivializes over \( U \) and is identified with \( \text{Heis}(\delta) \times U \). Hence \( \text{Heis}(\delta) \times U \) acts on the Verlinde bundle \( V_{r,km} \) with weight \( k \).

Now the proof is, by using Lemma 3.5 and the arguments in [Mu3, Proposition 2, p.80]: Since the subgroup \( K(\delta)_{km} \) is represented over \( T \), there is a vector bundle decomposition

\[
V_{r,km} \cong \bigoplus_{\chi \in \hat{K}(\delta)_{km}} W_{\chi}
\]

where \( W_{\chi} \) is a subbundle and is acted by the character \( \chi \).

Over \( U \), we know that \( W_{\chi} \) is acted upon by \( \text{Heis}(\frac{\delta}{k}) \) and hence

\[
W_{\chi} \cong W_{\chi} \otimes F_{\chi}
\]

where \( W_{\chi} \) is defined in section 2. and for some vector bundle \( F_{\chi} \) of rank \( n_{\chi} \).

This gives the required isomorphism

\[
(9) \quad V_{r,km} \cong \bigoplus_{\chi \in \hat{K}(\delta)_{km}} W_{\chi} \otimes F_{\chi}
\]

over \( U \). \( \square \)

**Corollary 4.3.** The Chern character of the Verlinde bundle \( V_{r,1} \) is written as

\[
\text{ch}(V_{r,1}) = \gamma_{r,1}.\text{ch}(L_S) \in CH^*(T)_\mathbb{Q}
\]

for some line bundle \( L_S \) on \( T \).

**Proof.** In the rational Grothendieck group \( K^0(T)_\mathbb{Q} \),

\[
V_{r,1} \cong L_S^{\otimes \gamma_{r,1}}
\]

where \( L_S = F_0 \) is a line bundle. This gives the assertion on the Chern characters in the rational Chow groups \( CH^*(T)_\mathbb{Q} \). \( \square \)
Remark 4.4. Since the moduli stack $\mathcal{M}_g$ of curves has $\text{Pic} \mathcal{M}_g = \mathbb{Z} \lambda$ ([Ar-Co]), where $\lambda$ is the first Chern class of the Hodge bundle $\pi_* \omega_{C/T}$, it follows that

$$L_S = l \cdot \lambda \in CH^*(T)_\mathbb{Q}, \text{ for some } l \in \mathbb{Q},$$

(we may assume $T \to \mathcal{M}_g$). In particular,

$$ch(V_{r,1}) = \gamma_{r,1} \cdot ch(l \cdot \lambda) \in CH^*(T)_\mathbb{Q}.$$ 

5. A REMARK ON THE MULTIPLICITIES OF THE ISOTYPICAL COMPONENTS

In this section, we indicate how the multiplicity $n_\chi$ of the representation $W_\chi$ which occurs in $H^0(SU_C(r, \eta), \Theta^k)$ (here $n_\chi$ are as defined in Lemma 3.5), can be computed. This was mentioned to us by A. Beauville.

Let $K \subset J(C)_r$ be any subgroup isomorphic to $\mu_s \times \mu_s$, $s \leq r$. Consider the moduli space $M_{SL(r)}^{\mu_s}$ of principal semistable $SL(r)^{\mu_s}$ bundles.

5.1. The multiplicities when $K = J(C)_r$. In this case we obtain the moduli space $M_{PGL(r)}$ of principal semi-stable $PGL(r)$-bundles. Further, fix a point $p \in C$ and denote $L = \mathcal{O}_C(d.p)$. Then we have [Be1],

$$M_{PGL(r)} = \cap_{0 \leq d < r} M_{PGL(r)}^d$$

and

$$M_{PGL(r)}^d = SU_C(r, L) / J(C)_r.$$ 

Suppose $\Theta'$ denotes the primitive line bundle on $M_{PGL(r)}^d$ (i.e., the first power of the determinant line bundle which descends to the quotient) and

$$\gamma_{r,k}^d = \dim H^0(SU_C(r, L), \Theta^k) \cap (J(C)_r).$$ 

By Remark 2.6 we can write

$$\gamma_{r,k}^d = \sum_{\chi \in J(C)_r} n_\chi^d \cdot \dim W_\chi.$$ 

Here $n_\chi^d$ is the multiplicity of $W_\chi$ which occurs in $H^0(SU_C(r, L), \Theta^k)$.

Lemma 5.1. Suppose $k$ is a multiple of $r$ and $r$ is odd or if $k$ is a multiple of $2r$ and $r$ is even. Then

$$n_{\chi^{\text{triv}}} = \dim H^0(M_{PGL(r)}^d, \Theta')$$

and

$$n_\chi = \frac{\gamma_{r,k}^d - n_{\chi^{\text{triv}}}}{r^{2g} - 1}, \chi \neq \chi^{\text{triv}}.$$
Proof. In \cite{Be-La-So}, it is shown that $\Theta^k_C$ descends down to the quotient $M^d_{PGL(r)}$ if $k = l.r$ and $r$ is odd or if $k = l.2r$ and $r$ is even. We note that the $J(C)_r$-invariant sections of $H^0(SU_C(r, L), \Theta^k_C)$ is the isotypical component $n^d_{\chi^{triv}} W_{\chi^{triv}}$ which is precisely the pullback of the space of sections $H^0(M^d_{PGL(r)}, \Theta^d)$. Further, by Proposition 2.2 it follows that $\dim W_{\chi} = 1$, for any $\chi \in \hat{J}(C)_r$. The assertion now follows from the equality

$$\gamma^d_{r,k} = \sum_{\chi \in \hat{K}} n^d_{\chi} \dim W_{\chi}$$

and noting that $n^d_{\chi}$ is constant, for any $\chi \neq \chi^{triv}$. \hfill \Box

Remark 5.2. In \cite[Proposition 3.4]{Be1}, when $r$ is a prime, $\dim H^0(M^d_{PGL(r)}, \Theta^d)$ is computed. Hence we get an explicit formula for the multiplicities $n_{\chi^{triv}}$ and $n_{\chi}$ in this case.

5.2. The multiplicities when $K \subset J(C)_r$. For a subgroup $K = \mu_s \times \mu_s \subset J(C)_r$, $s < r$, we consider the intermediate quotients

$$SU_C(r, L) \longrightarrow SU_C(r, L) \longrightarrow K \longrightarrow M^d_{PGL(r)}.$$  

As in \S 5.1 the disjoint union

$$M_{SU_C(r, L)} : = \coprod_{0 \leq d < r} SU_C(r, L) / K$$

is the moduli space of principal semi-stable $SU_C(r, L)$ bundles.  

Lemma 5.3. Given any integer $k$, there is a subgroup $K = \mu_s \times \mu_s \subset J(C)_r$, $s \leq r$, such that $\Theta^k_C$ descends down to the variety $SU_C(r, L) / K$ as a power of a primitive line bundle $\Theta'_K$.  

Proof. Notice that the degeneracy of the Weil form on $J(C)_r$ associated to the exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G_1(\Theta^k_C) \longrightarrow J(C)_r \longrightarrow 0$$

is a subgroup $K = \mu_s \times \mu_s \subset J(C)_r$ for some $s \leq r$. Hence there is a lift of $K$ in $G_1(\Theta^k_C)$ over $K$ which forms a descent data for the line bundle $\Theta^k_C$. \hfill \Box

As in \S 5.1 we denote for $L = \mathcal{O}(d, p)$ and $0 \leq d < r$,

$$\gamma^d_{r,k} = \dim H^0(SU_C(r, L), \Theta^k_C)$$

and $n^d_{\chi}$ is the multiplicity of $W_{\chi}$ which occurs in $H^0(SU_C(r, L), \Theta^k_C)$.  

Then

$$\gamma^d_{r,k} = \sum_{\chi \in \hat{K}} n^d_{\chi} \dim W_{\chi}.$$

Hence we write

$$\gamma_{r,k} := \sum_{0 \leq d < r} \gamma^d_{r,k} = \sum_{\chi \in \hat{K}} n_{\chi} W_{\chi}.$$
where \( n_\chi = \sum_{0 \leq d < r} n_\chi^d \).

**Lemma 5.4.** The multiplicities \( n_\chi \), for any \( \chi \in \hat{K} \), can be computed.

**Proof.** By Lemma 5.3, there is an \( s \), \( 0 \leq s \leq r \) and \( K = \mu_s \times \mu_s \subset J(C)_r \), such that \( \Theta^s \) descends to \( SL(r, L) \), as a power of a primitive line bundle \( \Theta^k \). By Remark 2.5, \( \Omega^0(SU_C(r, L), \Theta^k) \) is a \( G_{(\Theta_C)} \)-module of weight \( k \), for \( L = O(d.p) \) and \( 0 \leq d < r \). By conformal field theory ([S-Y]), we know the vector space dimension

\[
\sum_{0 \leq d < r} \dim H^0(M^d_{SL(r), \Theta^k}).
\]

As shown in Lemma 5.1, a similar argument gives the multiplicities \( n_\chi = \sum_{0 \leq d < r} n_\chi^d \), for any \( \chi \in \hat{K} \).

\( \square \)

**Remark 5.5.** If the dimensions of the individual vector spaces \( \Omega^0(M^d_{SL(r), \Theta^k}) \) are known then we would be able to compute the individual multiplicities \( n_\chi^d \).

5.3. **A remark on the multiplicities \( n_\chi \).** Let \( \gamma_{r,k} = \dim \Omega^0(SU_C(r), \Theta^k) \). Then by the Verlinde formula ([Be-La], [Pa2]), we have

\[
\gamma_{r,k} = \left( \frac{r}{r+k} \right)^g \sum_{S \cup R = [1, r+k], |S|=r} \prod_{z \in R} \left| 2 \sin \frac{\pi s - z}{r+k} \right|^{g-1}.
\]

Also, by Remark 2.6 we can write

\[
\gamma_{r,k} = \sum_{\chi \in (J(C)_r)_k} n_\chi \dim W_\chi
\]

\[
= \frac{r^g}{(r,k)^g} \sum_{\chi \in (J(C)_r)_k} n_\chi, \text{ by Proposition 2.2}
\]

Comparing the above two expressions, we get

\[
\sum_{\chi \in (J(C)_r)_k} n_\chi = \frac{(r,k)^g}{(r+k)^g} \sum_{S \cup R = [1, r+k], |S|=r} \prod_{z \in R} \left| 2 \sin \frac{\pi s - z}{r+k} \right|^{g-1}.
\]

(See also [Za], for the various aspects of the Verlinde formula.)

**References**

[An-Ma] Andersen, J. E., Masbaum, G. **Involutions on moduli spaces and refinements of the Verlinde formula**, Math. Ann. 314 (1999), no. 2, 291–326.

[Ar-Co] Arbarello, E., Cornalba, M. **The Picard groups of the moduli spaces of curves**, Topology 26 (1987), no. 2, 153–171.
The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India
E-mail address: jniyer@imsc.res.in