PERIODS OF COMPLETE INTERSECTION ALGEBRAIC CYCLES

ROBERTO VILLAFLOR LOYOLA

Abstract. In this article we compute periods of complete intersection algebraic cycles inside smooth degree $d$ hypersurfaces of $\mathbb{P}^{n+1}$, of even dimension $n$. This is done by determining the Poincaré dual of the given algebraic cycle. As an application, we prove that the locus of general hypersurfaces containing two linear cycles whose intersection is of dimension less than $\frac{n}{2} - \frac{d}{2}$, corresponds to the Hodge locus of any integral combination of such linear cycles. This proves variational Hodge conjecture for those algebraic cycles.

Background

This article deals with two different problems, variational Hodge conjecture for smooth hypersurfaces of the projective space, and the computation of periods of algebraic cycles inside smooth hypersurfaces of the projective space.

Variational Hodge conjecture was proposed by Grothendieck in 1966, as a weaker version of Hodge conjecture (see [Gro66, page 103]). While Hodge conjecture claims that every Hodge cycle inside a smooth projective variety is an algebraic cycle. Variational Hodge conjecture claims that in all proper families of smooth projective varieties with connected base, a flat section of its de Rham cohomology bundle is an algebraic cycle at one point if and only if it is an algebraic cycle everywhere. In other words, flat deformations of an algebraic cycle remain algebraic inside the deformed smooth projective variety. In 1972, Bloch proved variational Hodge conjecture for deformations of algebraic cycles supported in local complete intersections which are semi-regular inside the corresponding smooth projective variety (see [Blo72]). Semi-regularity is a strong condition, difficult to check in concrete examples (see [DK16] for a discussion about examples of semi-regular varieties). In 2003, Otwinowska considered variational Hodge conjecture for algebraic cycles inside smooth degree $d$ hypersurfaces $X$ of the projective space $\mathbb{P}^{n+1}$ of even dimension $n$. In this context, she proved that variational Hodge conjecture is satisfied for algebraic cycles supported in one $\frac{n}{d}$-dimensional complete intersection $Z$ of $\mathbb{P}^{n+1}$ contained in $X$, and $d >> 0$ (see [Otw03]). This result was improved by Dan in 2014, who removed the condition on the degree $d$ provided that $\text{deg}(Z) < d$ (see [Dan14]). It is not known if the complete intersection subvarieties considered by Otwinowska and Dan are semi-regular inside the corresponding hypersurface.
The computation of periods of algebraic cycles was considered by Deligne in 1982. He proved that up to some constant power of $2\pi\sqrt{-1}$, the periods of algebraic cycles belong to the field of definition of the variety and the corresponding algebraic cycle (see [Del82]). This problem was reconsidered in 2014 by Movasati, who explained how explicit computations of periods of algebraic cycles can be used to prove variational Hodge conjecture (see [Mov17b]). Formulas for periods of algebraic cycles were computed in [MV17] for the case of linear cycles inside Fermat varieties. These periods were used to prove variational Hodge conjecture for some combinations of linear cycles inside Fermat varieties of small dimension and degree. These combinations of linear cycles are not supported in one complete intersection cycle, and so this result does not follow from Otwinowska or Dan’s work (they are not known to be semi-regular inside Fermat variety either). In [Ser18], Sertöz implemented an algorithm for approximating periods of arbitrary Hodge cycles inside hypersurfaces. Using this algorithm he performed reliable computations of the Picard rank of certain K3 surfaces.

1. Introduction

Let us explain what we mean by periods of algebraic cycles inside smooth hypersurfaces. Consider the even dimensional smooth hypersurface of the complex projective space

$$X = \{ F = 0 \} \subseteq \mathbb{P}^{n+1},$$

given by a homogeneous polynomial with $\deg F = d$. Every $\frac{n}{2}$-dimensional subvariety $Z$ of $X$ determines an algebraic cycle

$$[Z] \in H_n(X, \mathbb{Z}).$$

Recalling from Griffiths’ work [Gri69], each piece of the Hodge filtration is generated by the differential forms

$$\omega_P := \text{res} \left( \frac{P \Omega}{\Omega^{q+1}} \right) \in F^{n-q}H^d_{\text{prim}}(X),$$

for $P \in \mathbb{C}[x_0, \ldots, x_{n+1}]_{d(q+1)-n-2}$, where

$$\Omega := \sum_{i=0}^{n+1} x_i \frac{d}{dx_i}(dx_0 \wedge \cdots \wedge dx_{n+1}) = \sum_{i=0}^{n+1} (-1)^i x_i \hat{dx}_i,$$

and $\text{res} : H^d_{\text{prim}}(\mathbb{P}^{n+1} \setminus X) \to H^d_{\text{prim}}(X)$ is the residue map.

Remark 1. Whenever we are considering a set of 1-forms $\{ y_i : i = 1, \ldots, k \}$ we will use the notation

$$\hat{y}_i := y_1 \wedge \cdots \wedge \hat{y}_i \cdots \wedge y_k.$$

This notation will be highly used in §4.

We say that

$$\int_Z \omega_P \in \mathbb{C}$$

is a period of $Z$. Notice that, since $Z$ is a projective variety of positive dimension, it intersects every divisor of $X$, so it is impossible to find an affine chart of $X$ where to compute the periods of $Z$. Since we are integrating over an algebraic cycle (consequently a Hodge cycle) we just care about the $(\frac{n}{2}, \frac{n}{2})$-part of $\omega_P$. Thus, we will fix $q = \frac{n}{2}$, and we will work with $\omega_P$ as an element of the quotient
\[ F^n H^2_{dR}(X)/F^{n+1} H^2_{dR}(X) \simeq H^{n+2}(X, \Omega^2_X) \simeq H^{n+2}(X, \Omega^2_X). \] After Carlson-Griffiths’ work [CG80 page 7], we know

\[ \omega_P = \frac{1}{4!} \left\{ \frac{P \Omega_f}{F_f} \right\} \in H^{n+2}(U, \Omega^2_X). \]

Where \( U \) is the Jacobian covering of \( X \). For \( J = (j_0, \ldots, j_{\frac{n}{2}}) \),

\[ F_J := F_{j_0} \cdots F_{j_{\frac{n}{2}}}, \]

where \( F_i := \frac{\partial P}{\partial x_i} \) for every \( i = 0, \ldots, n+1 \), and

\[ \Omega_J := \left( \cdots \frac{\partial^2 \phi}{\partial x_{j_0} \partial x_{j_{\frac{n}{2}}} \cdots} (\Omega) \cdots \right) = (-1)^{j_0 + \cdots + j_{\frac{n}{2}}} + (\frac{n}{2} + 2) \sum_{i=0}^{\frac{n}{2}} (-1)^i x_{k_i} \frac{\partial \phi}{\partial x_{k_i}}, \]

for \( (k_0, \ldots, k_{\frac{n}{2}-1}) \) the multi-index obtained from \( (0,1, \ldots, n+1) \) by removing the entries of \( J \). We will usually write \( \omega_P \) in Čech cohomology as in (1.1), but we will denote the period by abuse of notation as \( \int_Z \omega_P \in \mathbb{C} \), letting it be understood that we are identifying \( \omega_P \) with its image under the isomorphism \( H^{n+2}(U, \Omega^2_X) \simeq H^{n+2}(X) \subseteq H^n_{dR}(X) \).

The main result of this article is the computation of periods of algebraic cycles \([Z] \in H_n(X, \mathbb{Z})\) for \( Z \subseteq X \) a complete intersection inside \( \mathbb{P}^{n+1} \).

**Theorem 1.** Let \( X \subseteq \mathbb{P}^{n+1} \) be a smooth degree \( d \) hypersurface of even dimension \( n \) given by \( X = \{ F = 0 \} \). Suppose that \( Z := \{ f_1 = \cdots = f_{\frac{n}{2}+1} = 0 \} \subseteq X \) is a complete intersection inside \( \mathbb{P}^{n+1} \) (i.e. \( I(Z) = \langle f_1, \ldots, f_{\frac{n}{2}+1} \rangle \subseteq \mathbb{C}[x_0, \ldots, x_{n+1}] \)) and \( F = f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1} \).

Define \( H = (h_0, \ldots, h_{n+1}) := (f_1, g_1, f_2, g_2, \ldots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1}) \).

Then

\[ \int_Z \omega_P = \frac{2\pi \sqrt{-1}}{4!} c \cdot (d-1)^{n+2} \cdot d_1 \cdots d_{\frac{n}{2}+1}, \]

where \( d_i = \deg f_i \), \( \omega_P \) is given by (1.1), and \( c \in \mathbb{C} \) is the unique number such that

\[ P \cdot \det(\text{Jac}(H)) \equiv c \cdot \det(\text{Hess}(F)) \mod J^F. \]

Where \( J^F := \langle F_0, \ldots, F_{n+1} \rangle \subseteq \mathbb{C}[x_0, \ldots, x_{n+1}] \) is the Jacobian ideal associated to \( F \).

**Remark 2.** Theorem 1 essentially says that the Poincaré dual of the algebraic cycle \([Z] \in H_n(X, \mathbb{Z})\) is given (up to a non-zero constant factor) by

\[ [Z]^d = \text{res} \left( \frac{\det(\text{Jac}(H)) \Omega}{F_{\frac{n}{2}+1}} \right) \in H^{n+2}(X)_{\text{prim}}. \]

This follows from [CG80 Theorem 3]. It is also implicit in the statement of Theorem 1 that the Jacobian ring

\[ R^F := \mathbb{C}[x_0, \ldots, x_{n+1}] / J^F, \]

satisfies \( R^F_{(d-2)(n+2)} = \mathbb{C} \cdot \det(\text{Hess}(F)) \). This is consequence of a classical Theorem due to Macaulay (see Theorem 3), which implies that \( R^F \) is an Artinian Gorenstein ring of socle \( (d-2)(n+2) \). We will briefly discuss Artinian Gorenstein rings in [2].
Using Theorem 1 we can prove variational Hodge conjecture for combinations of linear cycles inside Fermat varieties. This problem was treated in [MV17] with computer assistance, and so variational Hodge conjecture was verified for small degree and dimension. We generalize this result to arbitrary degree and dimension in the following way (the key ingredient not appearing in [MV17] is the explicit description of the Poincaré dual of each linear cycle).

**Theorem 2.** Let $X \subseteq \mathbb{P}^{n+1}$ be the Fermat variety of even dimension $n$ and degree $d \geq 2 + \frac{n}{2}$. Let $\mathbb{P}_\delta, \hat{\mathbb{P}}_\delta \subseteq X$ be the two linear subvarieties such that $\mathbb{P}_\delta \cap \hat{\mathbb{P}}_\delta = \mathbb{P}^m$ given by

$$
\mathbb{P}^m := \{ x_{n-2m} - \zeta_2 \delta_2 x_{n-2m+1} = \cdots = x_n - \zeta_2 \delta_2 x_{n+1} = 0 \},
$$

$$
\mathbb{P}_\delta := \{ x_0 - \zeta_2 \delta_1 x_1 = \cdots = x_{n-2m-2} - \zeta_2 \delta_2 x_{n-2m-1} = 0 \} \cap \mathbb{P}^m,
$$

$$
\hat{\mathbb{P}}_\delta := \{ x_0 - \frac{1}{2} \zeta_2 \delta_1 x_1 = \cdots = x_{n-2m-2} - \frac{1}{2} \zeta_2 \delta_2 x_{n-2m-1} = 0 \} \cap \mathbb{P}^m,
$$

where $\zeta_2 \in \mathbb{C}$ is a primitive 2d-root of unity, and $\alpha_0, \alpha_2, \ldots, \alpha_{n-2m-2} \in \{ 3, 5, \ldots, 2d-1 \}$. Then, for $m < \frac{n}{2} - \frac{d}{2}$, $a, b \in \mathbb{Z} \setminus \{ 0 \}$ and $\delta := a[\mathbb{P}_\delta] + b[\hat{\mathbb{P}}_\delta] \in H_n(X, \mathbb{Z})$ we have

$$
V_\delta = V[\mathbb{P}_\delta \cap \hat{\mathbb{P}}_\delta],
$$

and the Hodge locus $V_\delta$ is smooth and reduced (see [3] Definition 3 for the definition of the Hodge locus). In particular, variational Hodge conjecture holds for $\delta$ in these cases. On the other hand, for $m \geq \frac{n}{2} - \frac{d}{2}$, the Zariski tangent space of $V_\delta$ has dimension strictly bigger than the dimension of $V[\mathbb{P}_\delta \cap \hat{\mathbb{P}}_\delta]$ (which is smooth and reduced, see [3] Proposition 3).

**Remark 3.** In [8] we argue how Theorem 2 implies variational Hodge conjecture for a general hypersurface $X$ containing two linear subvarieties $\mathbb{P}_\delta, \hat{\mathbb{P}}_\delta \subseteq X$ with $\mathbb{P}_\delta \cap \hat{\mathbb{P}}_\delta = \mathbb{P}^m$, $m < \frac{n}{2} - \frac{d}{2}$, and $\delta = a[\mathbb{P}_\delta] + b[\hat{\mathbb{P}}_\delta] \in H_n(X, \mathbb{Z})$.

**Remark 4.** After the algebraicity of the locus of Hodge cycles proved by Cattani, Deligne and Kaplan [CDK95], we can state variational Hodge conjecture in the following local analytic format:

“If $\delta_0 \in H_n(X_0, \mathbb{Q})$ is an algebraic cycle, then $\delta_t \in H_n(X_t, \mathbb{Q})$ is an algebraic cycle for every $t \in V_{\delta_0}$.”

This version of variational Hodge conjecture is the one we are always referring to, in particular in Theorem 2.

2. Artinian Gorenstein rings

As part of the algebraic background we need, we will state in this section some results about Artinian Gorenstein rings. We begin with a classical result due to Macaulay (for a proof see [Voi03, Theorem 6.19]).

**Theorem 3** (Macaulay [Mac16]). Given $f_0, \ldots, f_{n+1} \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ homogeneous polynomials with $\deg(f_i) = d_i$ and

$$
\{ f_0 = \cdots = f_{n+1} = 0 \} = \emptyset \subseteq \mathbb{P}^{n+1}.
$$

Letting

$$
R := \frac{\mathbb{C}[x_0, \ldots, x_{n+1}]}{(f_0, \ldots, f_{n+1})},
$$

then for $\sigma := \sum_{i=0}^{n+1} (d_i - 1)$, we have that
(i) $\dim \mathcal{C} R_\sigma = 1$.
(ii) For every $0 \leq i \leq \sigma$ the multiplication map
\[ R_i \times R_{\sigma-i} \rightarrow R_\sigma \]
is a perfect pairing.
(iii) $R_e = 0$ for $e > \sigma$.

**Definition 1.** Let $n \in \mathbb{N}$, and $I \subseteq \mathbb{C}[x_0, \ldots, x_{n+1}]$ an ideal. We say that $R := \mathbb{C}[x_0, \ldots, x_{n+1}]/I$ is Artinian Gorenstein if it satisfies items (i), (ii), (iii) of Macaulay Theorem \[3\] for some $\sigma \in \mathbb{N}$. We say $\sigma$ is the socle of $R$.

**Notation 1.** Despite Artinian Gorenstein property is reserved for rings, we will also say that $I$ is Artinian Gorenstein of socle $\sigma$, when $R = \mathbb{C}[x_0, \ldots, x_{n+1}]/I$ is.

**Remark 5.** Note that if $I$ is Artinian Gorenstein of socle $\sigma$, and $P \in \mathbb{C}[x_0, \ldots, x_{n+1}]\mu \setminus I_\mu$, then the quotient ideal
\[ (I : P) := \{Q \in \mathbb{C}[x_0, \ldots, x_{n+1}] : PQ \in I\}, \]
is Artinian Gorenstein of socle $\sigma - \mu$. Note also that if $I_1 \subseteq I_2$ are two Artinian Gorenstein ideals of the same socle, then $I_1 = I_2$.

We close this section with a proposition we will use in the proof of Theorem \[2\]

**Proposition 1.** Consider the polynomial ring $\mathbb{C}[x_0, \ldots, x_{2r-1}]$, and the ideal $I := \langle x_0^{d-1}, \ldots, x_{2r-1}^{d-1} \rangle$. Let $d \geq 3$, and $\beta_1, \beta_2, c_1, c_2 \in \mathbb{C}^\times$ with $\beta_1 \neq \beta_2$. For $i = 1, 2$, define
\[ R_i := c_i \cdot \prod_{j=1}^{r} \frac{(x_{2j-2}^{d-1} - (\beta_j x_{2j-1}^{d-1}))}{(x_{2j-2} - \beta_j x_{2j-1})}. \]
Then
\[ (I : R_1)_e \cap (I : R_2)_e = (I : R_1 + R_2)_e, \]
if and only if $e \neq (d-2) \cdot r$.

**Proof** First of all, note that $(I : R_1), (I : R_2)$ and $(I : R_1 + R_2)$ are Artinian Gorenstein ideals of socle $(d-2) \cdot r$. In consequence,
\[ (I : R_1) \cap (I : R_2) \neq (I : R_1 + R_2). \]
Otherwise, we would have $(I : R_1 + R_2) \subseteq (I : R_1)$, which implies $(I : R_1) = (I : R_1 + R_2) = (I : R_2)$ a contradiction. Therefore, in order to prove the proposition, it is enough to prove \[2.1\] for $e \neq (d-2) \cdot r$. If $e > (d-2) \cdot r$, the equality \[2.1\] is trivial since $(d-2) \cdot r$ is the socle of the three ideals. If $e < (d-2) \cdot r$, we claim \[2.1\] reduces to the case $e = (d-2) \cdot r - 1$. In fact, if we assume \[2.1\] fails for some $e < (d-2) \cdot r$, we can choose
\[ (2.2) \quad p \in (I : R_1 + R_2)_e \setminus (I : R_1)_e. \]
Since $(I : R_1)$ is Artinian Gorenstein of socle $(d-2) \cdot r$, we can find a monomial $x^i = x_0^{i_0} \cdots x_{2r-1}^{i_{2r-1}} \in \mathbb{C}[x_0, \ldots, x_{2r-1}]_{(d-2) \cdot r - e}$ such that
\[ (2.3) \quad x^i \cdot p \in (I : R_1 + R_2)_{(d-2) \cdot r} \setminus (I : R_1)_{(d-2) \cdot r}. \]
Since $\deg(x^i) > 0$, there exist some $i_j > 0$, then \[2.2\] and \[2.3\] imply that
\[ \frac{x^i}{x_j} \cdot p \in (I : R_1 + R_2)_{(d-2) \cdot r - 1} \setminus (I : R_1)_{(d-2) \cdot r - 1}, \]
and so (2.1) would fail for \( e = (d - 2) \cdot r - 1 \), as claimed. Therefore, we just consider the case \( e = (d - 2) \cdot r - 1 \). It is enough to show that \((I : R_1 + R_2)_e \subseteq (I : R_1)_e \cap (I : R_2)_e \). Take \( p \in (I : R_1 + R_2)_e \). Without loss of generality we may assume it can be written as

\[
p = \sum_{k \text{ even}} \sum_{l=0}^{d-3} x_k^{d-3-l} p_{k,l},
\]

where each \( p_{k,l} \) does not depend on \( x_k \) and \( x_{k+1} \), and is a \( \mathbb{C} \)-linear combination of monomials of the form \( x_k^{3p} \cdot x_{k+1}^{3p} \cdot x_{k+2}^{3p} \cdots x_{2r-1}^{3p} \) with \( i_{2j-2} + i_{2j-1} = d - 2 \), for all \( j \in \{1, \ldots, r \} \setminus \{ \frac{d}{2} + 1 \} \). For every \( k \) and \( l \), and \( i = 1, 2 \), there exist a constant \( a_{k,l,i} \in \mathbb{C} \) such that

\[
p_{k,l} \left( x_k^{d-3} + x_k^{d-3} (\beta_i x_{k+1}) + \cdots + (\beta_i x_{k+1})^{d-2} \right) = a_{k,l,i} \left( x_k \cdots x_{2r-1} \right)^{d-2} (x_k x_{k+1})^{d-2},
\]

modulo \( \langle x_0^{d-1}, \ldots, x_r^{d-1}, x_{r+1}^{d-1}, \ldots, x_{2r-1}^{d-1} \rangle \). Then

\[
p R_i \equiv (x_0 \cdots x_{n-2m-1})^{d-2} \sum_{k \text{ even}} \left( \frac{1}{x_k} \sum_{l=0}^{d-3} a_{k,l,i} \beta_1^{l+1} + \frac{1}{x_{k+1}} \sum_{l=0}^{d-3} a_{k,l,i} \beta_1^l \right),
\]

modulo \( I \). Since \( p \cdot (R_1 + R_2) \in I \) we conclude that

\[
\sum_{l=0}^{d-3} a_{k,l,1} \beta_1^{l+1} + \sum_{l=0}^{d-3} a_{k,l,2} \beta_2^{l+1} = \sum_{l=0}^{d-3} a_{k,l,1} \beta_1^l + \sum_{l=0}^{d-3} a_{k,l,2} \beta_2^l = 0.
\]

Since \( \beta_1 \neq \beta_2 \), this implies

\[
\sum_{l=0}^{d-3} a_{k,l,1} \beta_1^l = \sum_{l=0}^{d-3} a_{k,l,2} \beta_2^l = 0,
\]

and so \( p R_i \in I \) for \( i = 1, 2 \). 

\[\square\]

3. Preliminaries on periods

In this section we prove some preliminary results about periods. We begin by computing periods of top forms over the projective space \( \mathbb{P}^{n+1} \). By a top form we mean an element of \( H^{n+1,n+1}(\mathbb{P}^{n+1}) \) seen as an element of the Čech cohomology group \( H^{n+1}(\mathcal{U}, \Omega^{n+1}_{\mathbb{P}^{n+1}}) \) with respect to some affine open cover \( \mathcal{U} \) of \( \mathbb{P}^{n+1} \).

**Proposition 2** (Periods of top forms over the projective space). Let \( f_0, \ldots, f_{n+1} \in \mathbb{C}[x_0, \ldots, x_{n+1}] \) homogeneous polynomials of the same degree \( l > 0 \), such that

\[
\{ f_0 = \cdots = f_{n+1} = 0 \} = \emptyset \subseteq \mathbb{P}^{n+1}.
\]

They define the finite morphism \( f : \mathbb{P}^{n+1} \to \mathbb{P}^{n+1} \) given by

\[
f(x_0 : \cdots : x_{n+1}) := (f_0 : \cdots : f_{n+1}).
\]

Let \( \mathcal{U}_f = \{ V_i \}_{i=0}^{n+1} \) be the open covering associated to \( f \), i.e. \( V_i = \{ f_i \neq 0 \} \). Then the top form

\[
\frac{\Omega_f}{f_0 \cdots f_{n+1}} := \frac{\sum_{i=0}^{n+1} (-1)^i f_i \Omega_f}{f_0 \cdots f_{n+1}} \in H^{n+1}(\mathcal{U}_f, \Omega^{n+1}_{\mathbb{P}^{n+1}}),
\]

where \( \Omega_f \) is the top form associated to the morphism \( f \).
has period
\[
\int_{\mathbb{C}^{n+1}} \frac{\Omega_f}{x_0 \cdots x_{n+1}} = l^{n+1} \cdot (-1)^{\left(\frac{n+2}{2}\right)} (2\pi \sqrt{-1})^{n+1}.
\]

**Proof** The form \( \frac{\Omega}{x_0 \cdots x_{n+1}} \in H^{n+1}(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}) \) corresponds to a form \( \omega \in H^{2n+2}_{dR}(\mathbb{P}^{n+1}) \).
We determine this element via the natural isomorphism in hypercohomology
\[
H^{n+1}(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}) \simeq \mathbb{H}^{2n+2}(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}) \simeq \mathbb{H}^{2n+2}(\mathbb{P}^{n+1}, \Omega^{n+1}_{(\mathbb{P}^{n+1})^m}) \simeq H^{2n+2}_{dR}(\mathbb{P}^{n+1}).
\]

Let \( \{a_i\}_{i=0}^{n+1} \) be a partition of unity subordinated to the standard covering \( \{U_i\}_{i=0}^{n+1} \) of \( \mathbb{P}^{n+1} \). Computing \( \omega \) in terms of this partition of unity, we see that
\[
\text{Supp } \omega \subseteq U_0 \cap \cdots \cap U_{n+1}.
\]

In fact, taking the standard coordinates of \( U_0 \) given by \((z_1, \ldots, z_{n+1}) = \left(\frac{x_1}{x_0}, \ldots, \frac{x_{n+1}}{x_0}\right) \in \mathbb{C}^{n+1}\) we can write
\[
\int_{\mathbb{C}^{n+1}} \omega = (n+1)!(-1)^{n+1} \int_{\mathbb{C}^{n+1}} da_1 \wedge \cdots \wedge da_{n+1} \wedge \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_{n+1}}{z_{n+1}}.
\]
Furthermore, we can assume that \( a_1, \ldots, a_{n+1} \) are \( C^\infty \) functions defined in \( \mathbb{C}^{n+1} \) such that
\[
a_i = \begin{cases} 0 & \text{if } |z_i| \leq 1 \\ 1 & \text{if } |z_i| \geq 2 \end{cases}
\]
and
\[
a_1 + \cdots + a_{n+1} = 1 \text{ if } \exists j \in \{1, \ldots, n+1\} : |z_j| \geq 2.
\]

Applying Stokes theorem several times we obtain
\[
\int_{\mathbb{C}^{n+1}} \frac{\Omega}{x_0 \cdots x_{n+1}} = (-1)^{\left(\frac{n+2}{2}\right)} \int_{\mathbb{C}^{n+1}} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_{n+1}}{z_{n+1}} = (-1)^{\left(\frac{n+2}{2}\right)} (2\pi \sqrt{-1})^{n+1}.
\]

Pulling back this form by \( f \), it follows that
\[
\int_{\mathbb{C}^{n+1}} \frac{\Omega f}{x_0 \cdots x_{n+1}} = \text{deg}(f) \cdot \int_{\mathbb{C}^{n+1}} \frac{\Omega}{x_0 \cdots x_{n+1}} = \text{deg}(f) \cdot (-1)^{\left(\frac{n+2}{2}\right)} (2\pi \sqrt{-1})^{n+1}.
\]

Since \( f \) is defined by a base point free linear system, the fiber of \( f \) is generically reduced and corresponds to \( l^{n+1} \) points by Bézout’s theorem.

**Remark 6.** The sign appearing in the formula comes from the identification \( H^{n+1}(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}) \simeq H^{2n+2}_{dR}(\mathbb{P}^{n+1}) \) via the isomorphism in hypercohomology. We have adopted Carlson and Griffiths’ convention for the differential in the total complex \( D := (d + (-1)^k \delta)_C^{2n+2-k}(\mathbb{P}^{n+1}, \Omega^k_{(\mathbb{P}^{n+1})^m}) \), see [CG80, page 9]. This sign was already pointed out by Deligne in [DeS2 page 6].

**Remark 7.** In general, any element of \( H^{n+1}(\mathcal{U}, \Omega^{n+1}_{\mathbb{P}^{n+1}}) \) is of the form
\[
\omega = \frac{P \Omega}{f_0^{\alpha_0} \cdots f_{n+1}^{\alpha_{n+1}}},
\]
where \( \alpha_0, \ldots, \alpha_{n+1} \in \mathbb{Z}_{>0} \) with \( l(\alpha_0 + \cdots + \alpha_{n+1}) = \text{deg}(P) + n + 2 \). Using Macaulay’s Theorem [3] applied to \( \langle f_0, \ldots, f_{n+1} \rangle \subseteq \mathbb{C}[x_0, \ldots, x_{n+1}] \), we obtain that
\[
P = \sum_{l(\beta_0 + \cdots + \beta_{n+1}) = \text{deg}(P) - (n+2)} f_0^{\beta_0} \cdots f_{n+1}^{\beta_{n+1}} P_{\beta},
\]
with $\deg(P_0) = (l - 1)(n + 2)$. This reduces the problem of computing periods of top forms over $\mathbb{P}^{n+1}$ with respect to the cover $\mathcal{U}_f$, to forms
\[(3.1) \quad \frac{P_0 \Omega}{f_0 \cdots f_{n+1}} \in H^{n+1}(\mathcal{U}_f, \Omega_{P^{n+1}}^{n+1}),\]
with $\alpha_0, ..., \alpha_{n+1} \in \mathbb{Z}$ such that $\alpha_0 + ... + \alpha_{n+1} = n + 2$ and $\deg(P_0) = (l - 1)(n + 2)$. If some $\alpha_i$ is non-positive, (3.1) represents an exact top form of $\mathbb{P}^{n+1}$. Therefore, the following are the forms which may have non-trivial periods
\[\frac{Q \Omega}{f_0 \cdots f_{n+1}} \in H^{n+1}(\mathcal{U}_f, \Omega_{P^{n+1}}^{n+1}),\]
with $\deg(Q) = (l - 1)(n + 2)$.

**Corollary 1 (Periods of top forms over the projective space II).** For every homogeneous polynomial $Q \in \mathbb{C}[x_0, ..., x_{n+1}]$, 
\[\int_{\mathbb{P}^{n+1}} \frac{Q \Omega}{f_0 \cdots f_{n+1}} = c \cdot l^{n+2} \cdot (-1)^{\left(\frac{n+2}{2}\right)}(2\pi \sqrt{-1})^{n+1},\]
where $c \in \mathbb{C}$ is the unique number such that $Q \equiv c \cdot \det(Jac(f)) \mod \langle f_0, ..., f_{n+1} \rangle$.

**Proof** Considering $f$ and $\Omega_f$ as in Proposition 2. It is easy to see (using Euler’s identity) that
\[(3.2) \quad \Omega_f = l^{-1} \det(Jac(f)) \Omega,\]
where $Jac(f) = \left(\frac{\partial f_i}{\partial x_j}\right)_{0 \leq i,j \leq n+1}$ is the Jacobian matrix of $f$. The rest follows from item (i) of Macaulay’s Theorem and Proposition.

In order to compute periods of complete intersection algebraic cycles, we will compute periods of smooth hyperplane sections of a given projectively smooth variety $X$ (by hyperplane section, we mean that in some projective embedding it corresponds to the intersection of a hyperplane with $X$). In fact, for $Y \hookrightarrow X$ a smooth hypersurface given by $\{F = 0\}$, we will give an explicit description of the isomorphism
\[H^n(Y, \Omega^n_Y) \cong H^{n+1}(X, \Omega^{n+1}_X),\]
\[\omega \mapsto \bar{\omega}\]
together with the relation between periods, i.e. the number $a \in \mathbb{C}$ such that
\[\int_X \bar{\omega} = a \int_Y \omega.\]

For this purpose recall the long exact sequence
\[\cdots \rightarrow H^{k+1}_{\text{dR}}(X) \rightarrow H^{k+1}_{\text{dR}}(U) \xrightarrow{\text{res}} H^{k}_{\text{dR}}(Y) \xrightarrow{\tau} H^{k}_{\text{dR}}(X) \rightarrow \cdots,\]
induced by Poincaré residue sequence
\[0 \rightarrow \Omega^n_X \rightarrow \Omega^n_X(\log Y) \xrightarrow{\text{Res}} j_* \Omega^{n-1}_Y \rightarrow 0.\]
Since $H^{2n+1}_{\text{dR}}(U) = H^{2n+2}_{\text{dR}}(U) = 0$, the coboundary map is an isomorphism
\[H^{2n}_{\text{dR}}(Y) \cong H^{2n+2}_{\text{dR}}(X).\]
Noting that these vector spaces are one dimensional, and that $\tau$ preserves Hodge filtrations (since it is the cup product with the polarization), we obtain the desired isomorphism

$$H^n(Y, \Omega^n_Y) \cong H^{n+1}(X, \Omega^{n+1}_X).$$

**Proposition 3** (Explicit description of the coboundary map [3.3]). Let $X \subseteq \mathbb{P}^N$ be a smooth complete intersection of dimension $n+1$, and $Y \subseteq X$ a smooth hyperplane section given by $\{F = 0\} \cap X$, for some homogeneous $F \in \mathbb{C}[x_0, \ldots, x_N]_d$. Let $\omega \in C^n(X, \Omega^n_X)$ such that $\omega|_Y \in H^n(Y, \Omega^n_Y)$. For any $\overline{\omega} \in C^n(X, \Omega^{n+1}_X(\log Y))$ such that

$$\overline{\omega} \equiv \omega \wedge \frac{dF}{F} \pmod{C^n(X, \Omega^{n+1}_X)},$$

define

$$\bar{\omega} := (-1)^{n+1} \delta(\overline{\omega}) \in C^{n+1}(X, \Omega^{n+1}_X).$$

Then, $\bar{\omega} \in H^{n+1}(X, \Omega^{n+1}_X)$ is uniquely determined by $\omega|_Y \in H^n(Y, \Omega^n_Y)$ and

$$\int_X \bar{\omega} = \frac{(-1)^{n+1} \cdot 2\pi \sqrt{-1}}{d} \int_Y \omega.$$  

**Proof** The map defined in the statement of the proposition is the coboundary map $\tau$, i.e. $\tau(\omega) = \bar{\omega}$. It is left to prove the period relation. Since $\tau$ is an isomorphism of one-dimensional vector spaces, there exist a constant $a_{X,Y} \in \mathbb{C}^\times$ such that

$$\int_X \tau(\omega) = a_{X,Y} \int_Y \omega,$$

for every $\omega \in H^n(Y, \Omega^n_Y)$. Since $X$ and $Y$ are complete intersections, we can extend $\omega$ and $\tau(\omega)$ to $\mathbb{P}^N$ (by Lefschetz hyperplane section theorem). If $X$ is complete intersection of type $(d_1, \ldots, d_k)$, then $[X] = d_1 \cdots d_k [\mathbb{P}^{n+1}] \in H_{2n+2}(\mathbb{P}^N, \mathbb{Z})$ and $[Y] = d_1 \cdots d_k [\mathbb{P}^n] \in H_{2n}(\mathbb{P}^N, \mathbb{Z})$. Then

$$d_1 \cdots d_k \int_{\mathbb{P}^{n+1}} \tau(\omega) = a_{X,Y} d_1 \cdots d_k \int_{\mathbb{P}^n} \omega.$$

In other words

$$a_{X,Y} = \frac{a_{\mathbb{P}^{n+1}, \mathbb{P}^n}}{d}.$$

In order to compute $a_{\mathbb{P}^{n+1}, \mathbb{P}^n}$ we suppose $\mathbb{P}^n = \{x_{n+1} = 0\}$, we take $\omega \in C^n(\mathcal{U}, \mathbb{P}^{n+1})$, where $\mathcal{U}$ is the standard open covering of $\mathbb{P}^{n+1}$, and

$$\omega_J = \sum_{i=0}^{n} (-1)^{i} x_j \overline{d_{j_i}}, \text{ for } |J| = n.$$  

Then

$$\overline{\omega}_J = \begin{cases} \sum_{i=0}^{n+1} (-1)^{i} x_{j_i} \overline{d_{j_i}} & \text{if } J = (0, \ldots, n), \\ 0 & \text{otherwise.} \end{cases}$$

In consequence

$$\overline{\omega}_{0\cdots n} = \sum_{i=0}^{n+1} (-1)^{i} x_i \overline{d_i}.$$  

It follows from Proposition [2] that $a_{\mathbb{P}^{n+1}, \mathbb{P}^n} = (-1)^{n+1} \cdot 2\pi \sqrt{-1}$. \hfill $\blacksquare$
4. Proof of Theorem 1

Let \( X \subseteq \mathbb{P}^{n+1} \) be a smooth degree \( d \) hypersurface of even dimension \( n \). Given the complete intersection \( Z \subseteq X \) of dimension \( \frac{n}{2} \), we construct a chain of subvarieties

\[
Z = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots \subseteq Z_{\frac{n}{2}+1} = \mathbb{P}^{n+1},
\]

where each \( Z_i \) is a hyperplane section of \( Z_{i+1} \). In order to prove Theorem 1 we will apply inductively the coboundary map, to reduce the computation of the period of \( Z \) to the computation of a period of \( \mathbb{P}^{n+1} \). Since both sides of (1.3) are continuous with respect to the parameters

\[
(f_1, g_1, \ldots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1}) \in \bigoplus_{i=1}^{\frac{n}{2}+1} \mathbb{C}[x_0, \ldots, x_{n+1}]_{d_i} \oplus \mathbb{C}[x_0, \ldots, x_{n+1}]_{d-d_i},
\]

such that \( F := f_1 g_1 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1} \). It is enough to prove Theorem 1 for a generic \((f_1, g_1, \ldots, f_{\frac{n}{2}+1}, g_{\frac{n}{2}+1})\). This is why we can assume each \( Z_{i-1} \) is a smooth hyperplane section of \( Z_i \), for \( l = 1, \ldots, \frac{n}{2} + 1 \), as in the hypothesis of Proposition 3.

Let \( P \in \mathbb{C}[x_0, \ldots, x_{n+1}]_{(d-2)(\frac{n}{2}+1)} \), and

\[
\omega := \omega_P \in H^\frac{n}{2}(X, \Omega^\frac{n}{2}_X),
\]

as in (1.11). Using Proposition 3 we construct inductively

\[
\omega^{(0)} := \omega|_Z \in H^\frac{n}{2}(Z, \Omega^\frac{n}{2}_Z) \text{ and } Z_0 := Z.
\]

Then for \( l = 1, \ldots, \frac{n}{2} + 1 \) we define

\[
\omega^{(l)} := \omega^{(l-1)} \in H^\frac{n}{2}+l(Z_l, \Omega^\frac{n}{2}+l_{Z_l}) \text{ and } Z_l := \{f_{l+1} = \cdots = f_{\frac{n}{2}+1} = 0\} \subseteq \mathbb{P}^{n+1}.
\]

Observe that \( Z_{\frac{n}{2}+1} = \mathbb{P}^{n+1} \).

**Lemma 1.** For each \( l = 0, \ldots, \frac{n}{2} + 1 \) and \( J = (j_0, \ldots, j_{\frac{n}{2}+l}) \) with \( 0 \leq j_0 < \cdots < j_{\frac{n}{2}+l} \leq n + 1 \)

\[
(\omega^{(l)})_J = (-1)^{\frac{n}{2}+l} \cdot F_{J^1} \cdot \sum_{m=1}^{l} \left(-1\right)^{m-1} \frac{dg_m}{d} \bigwedge_{r=0}^{l-m} dx_k \bigwedge_{t=1}^{l} \frac{df_t}{d} + (-1)^{l} \sum_{p=0}^{l} \left(-1\right)^p x_k \bigwedge_{s=1}^{l} \frac{dg_s}{d} \bigwedge_{r=0}^{l-m} dx_k \bigwedge_{t=1}^{l} \frac{df_t}{d} + (-1)^{\frac{n}{2}+l} \sum_{q=1}^{l} \frac{dg_q}{d} \bigwedge_{r=0}^{l-m} dx_k \bigwedge_{t=1}^{l} \frac{df_t}{d},
\]

where \( K = (k_0, \ldots, k_{\frac{n}{2}+l}) \) is obtained from \((0, 1, \ldots, n+1)\) by removing the entries of \( J \) (the notation \( \frac{dg_m}{d} := \frac{dg_m}{d} \wedge \cdots \wedge \frac{dg_0}{d} \wedge \frac{dg_{n+1}}{d} \)) and analogously for \( dx_k \) and \( \frac{df_t}{d} \), was already set in Remark 1.1.

**Proof** We proceed by induction on \( l \):
Assuming it is true for $l$, then

$$(\omega^{(l)})_J \wedge \frac{df_{l+1}}{f_{l+1}} \equiv \varpi_J \mod C_p^{Z_{l+1}, Z_{l+1}}(\log Z_l)),$$

where $\varpi_J \in C_p^{Z_{l+1}, Z_{l+1}}(\log Z_l))$ is given by

$$\varpi_J = \left(1 \left(\frac{\hat{s}+1}{2}\right)^{l+1}, \left(\frac{\hat{s}+1}{2}\right)^{l+1}, \cdots, \left(\frac{\hat{s}+1}{2}\right)^{l+1}\right) Pd^d_1 \cdots d^d_{l+1}$$

Applying $\hat{\delta}$ we get

$$\omega_{J(l+1)}^{(l+1)} = \left(-1\right)^l \left(\frac{\hat{s}+1}{2}\right)^{l+1}, \left(\frac{\hat{s}+1}{2}\right)^{l+1}, \cdots, \left(\frac{\hat{s}+1}{2}\right)^{l+1}\right) Pd^d_1 \cdots d^d_{l+1}$$

Computing $\Omega_f$ (as in [1,2]) we get

$$(\omega^{(0)})_{J_0 \cdots J_p} = (\omega)_{J_0 \cdots J_p} = \left(-1\right)^l \left(\frac{\hat{s}+1}{2}\right)^{l+1}, F_J \left(\sum_{p=0}^{\hat{s}} \left(-1\right)^p x_k \cdot \frac{dx_k}{p!} \right).$$

Assuming it is true for $l$, then

$$(\omega^{(l)})_J \wedge \frac{df_{l+1}}{f_{l+1}} \equiv \varpi_J \mod C_p^{Z_{l+1}, Z_{l+1}}(\log Z_l)),$$
Replacing $F$ 

Replacing $F = f_1g_1 + \cdots + f_jg_{j+1}$ in the first three expressions.

Proof of Theorem [1] Using Lemma [1] for $l = \frac{n}{2} + 1$ we get

\[
(\omega(\frac{n}{2} + 1))_{0, n+1} = \frac{(-1)^{\binom{n+2}{2}} \cdot P \cdot \det(Jac(H))}{\frac{n}{2}! \cdot F_0 \cdots F_{n+1}} \sum_{k=0}^{n+1} (-1)^k h_k \hat{d}h_k e_k.
\]

where $e_k = deg(h_k)$. Replacing $e_i, h_i = \sum_{j=0}^{n+1} \frac{\partial h_i}{\partial x_j} x_j$ and $dh_i = \sum_{j=0}^{n+1} \frac{\partial h_i}{\partial x_j} dx_j$ we obtain

\[
(\omega(\frac{n}{2} + 1))_{0, n+1} = \frac{(-1)^{\binom{n+2}{2}} \cdot \det(Jac(H))}{\frac{n}{2}! \cdot F_0 \cdots F_{n+1}} \sum_{k=0}^{n+1} (-1)^k x_k \hat{d}x_k.
\]

The theorem follows from Corollary [3] and Proposition [3].
5. Hodge locus

Before going to the applications of Theorem \ref{periods} let us recall the Hodge locus associated to a Hodge cycle inside a smooth degree $d$ hypersurface of the projective space $\mathbb{P}^{n+1}$, of even dimension $n$.

**Definition 2.** Let $\pi : X \to T$ be the family of smooth degree $d$ hypersurfaces of $\mathbb{P}^{n+1}$, of even dimension $n$. Given a fixed parameter $0 \in T$, and a Hodge cycle $\delta_0 \in H_n(X_0, \mathbb{Z})$. Since $\pi$ is a locally trivial fibration, we can extend $\delta_0$ to a polydisc around $0 \in T$ by parallel transport. If we denote this extension by $\delta_t \in H_n(X_t, \mathbb{Z})$, the Hodge locus associated to $\delta$ is

$$V_{\delta} := \{ t \in (T, 0) : \delta_t \text{ a Hodge cycle of } X_t \},$$

where $(T, 0)$ denotes the germ of neighbourhoods of $0 \in T$ in the analytic topology, and by Hodge cycle we mean $\delta^{pd}_t \in H^n(X_t, \mathbb{Z}) \cap H^{2, n}_{dR}(X_t)$. Considering $\omega_1, \ldots, \omega_k \in H^n_{dR}(X/T)$ such that they form a basis for $F^{d+1}H^n_{dR}(X_t)$ for every $t$ in a neighbourhood of $0 \in T$, we can induce an structure of analytic space in the Hodge locus as

$$\mathcal{O}_{V_{\delta}} = \frac{O_{(T, 0)}}{(\int_{\delta_t}^{\omega_1}, \ldots, \int_{\delta_t}^{\omega_k})}.$$

This structure might be non-reduced, see for instance \cite[p. 154, Exercise 2]{Voi03}.

We will close this section with a restatement of a well known fact relating periods of a Hodge cycle, to the Zariski tangent space of its associated Hodge locus.

**Proposition 4.** Let $T \subseteq \mathbb{C}[x_0, \ldots, x_{n+1}]_d$ be the parameter space of smooth degree $d$ hypersurfaces of $\mathbb{P}^{n+1}$, of even dimension $n$. For $t \in T$, let $X_t = \{ F = 0 \} \subseteq \mathbb{P}^{n+1}$ be the corresponding hypersurface. For every Hodge cycle $\delta \in H_n(X_t, \mathbb{Z})$, we can compute the Zariski tangent space of its associated Hodge locus $V_{\delta}$ as

$$T_{t}V_{\delta} = \left\{ P \in \mathbb{C}[x_0, \ldots, x_{n+1}]_d : \int_{\delta} \text{res} \left( \frac{PQ\Omega}{F^{d+1}} \right) = 0, \forall Q \in \mathbb{C}[x_0, \ldots, x_{n+1}]_d \right\}.$$

**Proof** We know from Voisin \cite[Lemma 5.16]{Voi03}, that

$$T_{t}V_{\delta} = \text{Ker } \nabla_{t}(\lambda_{t}),$$

where $\lambda \in H^n(\Delta)$ is a local section (on a polydisc $\Delta$ around $t$) of the local system $H^n := R^n_{\pi, \mathbb{Z}}$ such that $\lambda_t = \delta^{pd}_t$. The result follows since the map $\nabla_{t}$ is induced by the infinitesimal variations of Hodge structures. This map is well known in the case of hypersurfaces and corresponds with

$$\nabla_{t} : H^{2, n}_{dR}(X_t) \times T_{t} T \to H^{2-d, n+1}_{dR}(X_t)^*,$$

given by the multiplication map (see \cite[Theorem 6.17]{Voi03})

$$(\nabla_{t}(\lambda_{t}, P))(\text{res} \left( \frac{Q\Omega}{F^{d+1}} \right)) \equiv \int_{\delta} \text{res} \left( \frac{PQ\Omega}{F^{d+1}} \right).$$

Notice we have identified $P \in \mathbb{C}[x_0, \ldots, x_{n+1}]_d \simeq T_{t} T$. \hfill $\blacksquare$
6. Some consequences of Theorem 1

**Definition 3.** We will say that a Hodge cycle \( \delta \in H^{n}(X, \mathbb{Z}) \) is of *complete intersection type* if
\[
\delta = \sum_{i=1}^{k} n_{i} [Z_{i}],
\]
for \( Z_{1}, ..., Z_{k} \subseteq X \) a set of \( \frac{d}{2} \)-dimensional subvarieties that are complete intersection inside \( \mathbb{P}^{n+1} \), given by
\[
Z_{i} = \{ f_{i,1} = \cdots = f_{i, \frac{d}{2}+1} = 0 \},
\]
for every \( i = 1, ..., k \), such that there exist \( g_{i,1}, ..., g_{i,k} \in \mathbb{C}[x_{0}, ..., x_{n+1}] \) with
\[
F = \sum_{j=1}^{\frac{d}{2}+1} f_{i,j} g_{i,j}.
\]

For every such Hodge cycle, we define its *associated polynomial*
\[
P_{\delta} := \sum_{i=1}^{k} d_{i} \cdot n_{i} \cdot \det(Jac(H_{i})) \in R_{(d-2)(\frac{d}{2}+1)}^{F},
\]
where \( d_{i} := \deg Z_{i} \), and \( H_{i} := (f_{i,1}, g_{i,1}, ..., f_{i, \frac{d}{2}+1}, g_{i, \frac{d}{2}+1}) \).

**Remark 8.** Theorem 1 tells us that in order to compute the periods of a complete intersection type cycle \( \delta \) it is enough to know its associated polynomial \( P_{\delta} \). In fact, we are determining the Poincaré dual of the cycle \( \delta \)
\[
\delta^{pd} = \text{res} \left( \frac{P_{\delta} \Omega}{F^{\frac{d}{2}+1}} \right) \in H^{\frac{d}{2}}(X, \Omega^{\frac{d}{2}}_{\frac{d}{2}}).
\]
In the sense that it satisfies (up to some non-zero constant factor) the relation
\[
\int_{\delta} \omega = \int_{X} \omega \wedge \text{res} \left( \frac{P_{\delta} \Omega}{F^{\frac{d}{2}+1}} \right), \quad \forall \omega \in H^{n}_{dR}(X).
\]

**Corollary 2.** Let \( X \subseteq \mathbb{P}^{n+1} \) be a smooth hypersurface given by
\[
X = \{ F = 0 \}.
\]
If \( \delta \in H_{d}(X, \mathbb{Z}) \) is a complete intersection type algebraic cycle, then
\( P_{\delta} \in J^{F} \) if and only if \( \delta = \alpha \cdot [X \cap \mathbb{P}^{\frac{d}{2}+1}] \) in \( H_{n}(X, \mathbb{Q}) \), for some \( \alpha \in \mathbb{Q} \).

**Proof** Applying Macaulay’s Theorem 3 to the Jacobian ring \( R^{F} \), we see that \( P_{\delta} \in J^{F} \), if and only if, \( P \cdot P_{\delta} \in J^{F} \) for all \( P \in \mathbb{C}[x_{0}, ..., x_{n+1}] \). Theorem 1 says this is equivalent to the vanishing of all periods for \( \omega \in H^{*}_{dR}(X)_{\text{prim}}. \) By Poincaré duality we conclude this is equivalent to \( \delta = \alpha \cdot [X \cap \mathbb{P}^{\frac{d}{2}+1}] \) in \( H_{n}(X, \mathbb{Q}) \), for some \( \alpha \in \mathbb{Q} \).

**Remark 9.** Another observation we can derive from Theorem 1 is that each period is of the form \( (2\pi \sqrt{-1})^{\frac{d}{2}} \) times a number in a number field \( k \), where \( k \) is the smallest number field such that \( f_{1}, g_{1}, ..., f_{\frac{d}{2}+1}, g_{\frac{d}{2}+1} \in k[x_{0}, ..., x_{n+1}] \). We mention that the periods belong to the same field where we can decompose \( F \) as \( f_{1}g_{1} + \cdots + f_{\frac{d}{2}+1}g_{\frac{d}{2}+1} \). This was already mentioned in Deligne’s work about absolute Hodge cycles (see [Del82, Proposition 7.1]).
One of the main ingredients of the proof of Theorem[2] is the description of the Zariski tangent space of the Hodge locus $V_{\delta}$ as the degree $d$ part of the quotient ideal $(J^F : P_{\delta})$.

**Corollary 3.** Let $T$ be the parameter space of smooth degree $d$ hypersurfaces of $\mathbb{P}^{n+1}$, of even dimension $n$. For $t \in T$, let $X_t = \{ F = 0 \} \subseteq \mathbb{P}^{n+1}$ be the corresponding hypersurface. If $\delta \in H_n(X_t, \mathbb{Z})$ is a complete intersection type algebraic cycle, then

$$T_t V_{\delta} = (J^F : P_{\delta})_d.$$  

**Proof** By Proposition[3] and Theorem[1] we have

$$T_t V_{\delta} = \{ P \in \mathbb{C}[x_0, \ldots, x_{n+1}]_d : P \cdot Q \cdot P_{\delta} \in J^F, \forall Q \in \mathbb{C}[x_0, \ldots, x_{n+1}]_d \mathbb{P}^{2n-2} \}.$$  

By item (ii) of Macaulay’s Theorem[3] applied to the Jacobian ring $R^F$, we conclude

$$T_t V_{\delta} = \{ P \in \mathbb{C}[x_0, \ldots, x_{n+1}]_d : P \cdot P_{\delta} \in J^F \} = (J^F : P_{\delta})_d.$$  

In order to prove Theorem[2] we will use Corollary[3] for $t = 0 \in T$ corresponding to the Fermat variety, and $\delta = [\mathbb{P}^2] \in H_n(X_0, \mathbb{Z})$ a linear cycle inside it.

**Corollary 4.** Let

$$X = \{ x_0^d + \cdots + x_{n+1}^d = 0 \}$$

be the Fermat variety. For $\alpha_0, \alpha_2, \ldots, \alpha_n \in \{ 1, 3, \ldots, 2d - 1 \}$ consider

$$(6.1) \quad \mathbb{P}^2_{\alpha} := \{ x_0 - \zeta_{2d}^{\alpha_0} x_1 = \cdots = x_n - \zeta_{2d}^{\alpha_n} x_{n+1} = 0 \},$$

and $\delta := [\mathbb{P}^2_{\alpha}]$. Its associated polynomial is

$$P_{\delta} = d^2 + 1 \cdot \frac{\zeta_{2d}^{\alpha_0 + \cdots + \alpha_n} (d-2)}{\sum_{j=1}^{d-2} x_{2j-2} - x_{2j-1}}.$$  

**Proof** Computing the Jacobian matrix of $H$ as in Theorem[1] we see it is diagonal by $2 \times 2$ blocks, and each block has determinant

$$ \frac{d((\zeta_{2d}^{\alpha_0 + \alpha_2}) - x_{2j-2} x_{2j-1})}{x_{2j-2} - x_{2j-1}}.$$  

We close this section by computing the periods of linear cycles inside Fermat varieties. This was the main theorem in [MV17, Theorem 1]. Consider the following set

$$I_{(d-2)(\frac{d+1}{2})} := \{ (i_0, \ldots, i_{n+1}) \in \{ 0, \ldots, d-2 \}^{n+2} : i_0 + \cdots + i_{n+1} = (d-2)(\frac{n}{2} + 1) \},$$

we define for every $i \in I_{(d-2)(\frac{d+1}{2})}$

$$\omega_i := res \left( \frac{x^i \Omega}{F_{\frac{d+1}{2}}^i} \right) = \frac{1}{\frac{d+1}{2}!} \left\{ \frac{x^i \Omega_j}{F_j} \right\}_{|j|=\frac{d+1}{2}} \in H^1_{\mathbb{Q}}(X, \Omega_X^{\frac{d+1}{2}}).$$

From Griffiths’ work [Griff69] we know these forms are a basis for $H^1_{\mathbb{Q}}(X)_{prim}$. 

Corollary 5 ([MV17]). For $X \subseteq \mathbb{P}^{n+1}$ the Fermat variety, $\mathbb{P}^T_\alpha \subseteq X$ as in (6.1) for $\alpha_0 = \cdots = \alpha_n = 1$, and $i \in I_{(d-2)(\frac{n}{2} + 1)}$ we have

$$\int_{\mathbb{P}^T_\alpha} \omega_i = \left\{ \begin{array}{ll}
\frac{(2\pi)^{-\frac{n}{2}+1}}{d^{\frac{n}{2}+1}} \zeta_{2d}^{\frac{n}{2}+1+i_0+i_2+i_3+\cdots+i_n} & \text{if } i_{2l-2} + i_{2l-1} = d - 2, \forall l = 1, \ldots, \frac{n}{2} + 1, \\
0 & \text{otherwise.}
\end{array} \right.$$ 

Proof By Theorem 1 we just need to compute $\text{dim} V_\alpha$. The following result is due to Movasati [Mov17a, Propositions 17.7 and 17.8].

By Proposition 5.

$$x^i P_\alpha = d^\frac{n}{2}+1 \sum_{j=1}^{\frac{n}{2}+1} x^i \prod_{l=0}^{d-2} x^{d-2-l} \zeta_{2d}^{d-2-l} \zeta_{2d}^{d-2-l} \cap \mathbb{P}^T_\alpha \equiv \mathbb{P}^T_\alpha \cap \mathbb{P}^T_\alpha (x_0 \cdots x_{n+1})^{d-2} \equiv \mathbb{P}^T_\alpha \cap \mathbb{P}^T_\alpha (x_0 \cdots x_{n+1})^{d-2}.$$

if there exist $l_j \in \{0, \ldots, d-2\}$ such that $l_j + i_{2j-1} = d - 2$ and $d - 2 - l_j + i_{2j-2} = d - 2$, for every $j = 1, \ldots, \frac{n}{2} + 1$. And is zero otherwise. This condition is equivalent to $l_j = i_{2j-2}$ and $i_{2j-2} + i_{2j-1} = d - 2$. The desired result follows from the computation of the Hessian matrix for Fermat

$$\det(\text{Hess}(F)) = d^{(d-1)} (x_0 \cdots x_{n+1})^{d-2}.$$

7. Proof of Theorem 2

Let $T$ be the parameter space of smooth degree $d$ hypersurfaces of $\mathbb{P}^{n+1}$, of even dimension $n$. Let $0 \in T$ be the point corresponding to the Fermat variety $X_0 = \{x_0^d + \cdots + x_{n+1}^d = 0\}$. Letting

$$\mathbb{P}^{n-m} := \{x_0 - \zeta_2 x_{2m+1} = \cdots = x_{n-2} - \zeta_2 x_{n-1} = 0\},$$

$$\mathbb{P}^T := \{x_0 - \zeta_2 x_1 = \cdots = x_{n-2} - \zeta_2 x_{n-1} = 0\} \cap \mathbb{P}^{n-m},$$

$$\mathbb{P}^T := \{x_0 - \zeta_2 x_1 = \cdots = x_{n-2} - \zeta_2 x_{n-1} = 0\} \cap \mathbb{P}^{n-m},$$

where $\alpha_0, \alpha_2, \ldots, \alpha_{n-2} \in \{3, \ldots, 2d-1\}$. Then

$$\mathbb{P}^m := \mathbb{P}^T \cap \mathbb{P}^T = \{x_0 = x_1 = \cdots = x_{n-2} = 0\} \cap \mathbb{P}^{n-m}.$$

The following result is due to Movasati [Mov17a, Propositions 17.7 and 17.8].

Proposition 5. $\text{dim} V_{[\mathbb{P}^T]} \cap V_{[\mathbb{P}^T]} = \text{dim} T_0 V_{[\mathbb{P}^T]} \cap T_0 V_{[\mathbb{P}^T]}$.

Movasati’s proof of Proposition 5 consists in computing explicitly both sides of the equality. Since variational Hodge conjecture holds for linear cycles (see [Dan14, Mov17b, or MV17]), $V_{[\mathbb{P}^T]} \cap V_{[\mathbb{P}^T]}$ corresponds to the locus of hypersurfaces containing two linear cycles intersecting each other in a $m$ dimensional linear subvariety. Knowing this, it is easy to compute its dimension as a fibration over the Grassmannian of pairs of $\frac{n}{2}$-dimensional linear subvarieties of $\mathbb{P}^{n+1}$ intersecting each other in a $m$-dimensional linear subvariety. In fact, (this computation can be found in [Mov17a, Proposition 17.8])

$$\text{Codim } V_{[\mathbb{P}^T]} \cap V_{[\mathbb{P}^T]} = 2 \left( \frac{\frac{n}{2} + d}{d} - 2 \left( \frac{n}{2} + 1 \right)^2 \right) - \left( \frac{m + d}{d} \right) + (m + 1)^2.$$
On the other hand, it is also easy to compute the codimension of \( T_0V_{[p^*]} \cap T_0V_{[\hat{p}^*]} \) (see [Mov17a, Proposition 17.7]) and coincides with (7.1)

\[
\text{Codim } T_0V_{[p^*]} \cap T_0V_{[\hat{p}^*]} = 2\text{Codim } T_0V_{[p^*]} - \text{Codim } T_0V_{[\hat{p}^*]} + T_0V_{[\hat{p}^*]} = 2\left(\frac{n}{2} + d\right) - 2\left(\frac{n}{2} + 1\right)^2 - \left(m + d\right)(m + 1)^2.
\]

**Remark 10.** After Proposition 9, Theorem 2 is reduced to show that

\[
T_0V_{[p^*]} \cap T_0V_{[\hat{p}^*]} = T_0V_\delta,
\]

if and only if \( m < \frac{d}{2} - \frac{d-2}{d-2} \). By Corollaries 3 and 14, this is equivalent to the following algebraic equality

(7.2) \( (J^F : P_1)_d \cap (J^F : P_2)_d = (J^F : P_1 + P_2)_d \).

Where \( P_1 = R_1Q \), \( P_2 = R_2Q \),

\[
Q := \prod_{k \geq n - 2m \text{ even}} \frac{(x_k^{d-1} - (\zeta_{2d}x_{k+1})^{d-1})}{(x_k - \zeta_{2d}x_{k+1})},
\]

\[
R_1 := c_1 \cdot \prod_{k < n - 2m \text{ even}} \frac{(x_k^{d-1} - (\zeta_{2d}x_{k+1})^{d-1})}{(x_k - \zeta_{2d}x_{k+1})},
\]

and

\[
R_2 := c_2 \cdot \prod_{k < n - 2m \text{ even}} \frac{(x_k^{d-1} - (\zeta_{2d}x_{k+1})^{d-1})}{(x_k - \zeta_{2d}x_{k+1})},
\]

for some \( c_1, c_2 \in \mathbb{C}^\times \).

**Proof of Theorem 2** After Remark 10, we have reduced the proof to prove (7.2). We claim that

(7.3) \( (J^F : P_1)_e \cap (J^F : P_2)_e = (J^F : P_1 + P_2)_e \),

if and only if \( e < (d-2)(\frac{n}{2} - m) \) or \( e > (d-2)(\frac{n}{2} + 1) \). In fact, for \( e > (d-2)(\frac{n}{2} + 1) \) the claim follows from the fact that \((d-2)(\frac{n}{2} + 1)\) is the socle of the three ideals appearing in (7.2). Suppose there exist \( e < (d-2)(\frac{n}{2} - m) \) and \( q \in (J^F : P_1 + P_2)_e \setminus (J^F : P_1)_e \).

Write

\[
qQ = \sum_i \hat{x}^i p_i(x_0, ..., x_{n-2m-1}),
\]

where \( i \) runs over the set of multi-indexes \( i = (i_{n-2m}, ..., i_{n+1}) \in \mathbb{Z}_{\geq 0}^{2m+2} \) and each \( \hat{x}^i := x_{n-2m}^{i_{n-2m}} \cdots x_{n+1}^{i_{n+1}} \) is a monomial depending just on \( x_{n-2m}, ..., x_{n+1} \), while \( p_i \in \mathbb{C}[x_0, ..., x_{n-2m-1}] \). Since \( Q \) depends only on \( x_{n-2m}, ..., x_{n+1} \), we have that

\[
\deg p_i \leq \deg q \quad \forall i \in \mathbb{Z}_{\geq 0}^{2m+2},
\]

and

\[
\sum_i \hat{x}^i p_i \cdot (R_1 + R_2) = q \cdot (P_1 + P_2) \in J^F.
\]

Since \( J^F \) is a monomial ideal, this implies

\[
p_i \cdot (R_1 + R_2) \in J^F,
\]
for all $i$ such that $\hat{x}^i \notin J^F$. On the other hand, if all those $i$ such that $\hat{x}^i \notin J^F$ satisfy that $p_iR_1 \in J^F$, then
\[ qP_1 = qQR_1 = \sum_i \hat{x}^i p_i R_1 \in J^F, \]
a contradiction with our choice of $q$. We conclude that there exist some $i$ such that $p_iR_1 \notin J^F$, $p_i \cdot (R_1 + R_2) \in J^F$ and $\deg p_i \leq \deg q < (d - 2)(\frac{n}{2} - m)$, which contradicts Proposition 1 for $r = \frac{n}{2} - m$.

Finally, if $(d - 2)(\frac{n}{2} - m) \leq e \leq (d - 2)(\frac{n}{2} + 1)$, we know from Proposition 1 for $r = \frac{n}{2} - m$, that there exist some $p \in \mathbb{C}[x_0, ..., x_{n-2m}]$ such that
\[ p \in (J^F : R_1 + R_2)(d - 2)(\frac{n}{2} - m) \setminus (J^F : R_1)(d - 2)(\frac{n}{2} - m), \]
and so
\[ p \in (J^F : P_1 + P_2)(d - 2)(\frac{n}{2} - m) \setminus (J^F : P_1)(d - 2)(\frac{n}{2} - m). \]
Since $(J^F : P_1)$ is Artinian Gorenstein with socle $(d - 2)(\frac{n}{2} + 1)$, we conclude there exist some $q \in \mathbb{C}[x_0, ..., x_{n+1}]$, $(d - 2)(\frac{n}{2} - m)$ such that
\[ pq \in (J^F : P_1 + P_2)_e \setminus (J^F : P_1)_e, \]
as desired. 

8. Final remarks

The following argument, which improves Theorem 2 to general hypersurfaces containing two $\frac{n}{2}$-dimensional linear cycles intersecting in a $m$-dimensional linear cycle with $m < \frac{n}{2} - \frac{d}{d-2}$, is due to Movasati: Let $\pi : X \to T$ be the family of smooth degree $d$ hypersurfaces of $\mathbb{P}^{n+1}$, of even dimension $n$. Consider
\[ W := \{ t \in T : X_t \text{ contains two linear cycles } \mathbb{P}^\frac{n}{2}, \mathbb{P}^\frac{n}{2} \text{ with } \mathbb{P}^\frac{n}{2} \cap \mathbb{P}^\frac{n}{2} = \mathbb{P}^m \}. \]
Let $\mathcal{H}$ be the relative Hilbert scheme of triples $(X_t, \mathbb{P}^\frac{n}{2}, \mathbb{P}^\frac{n}{2})$, where $X_t$ is a smooth degree $d$ hypersurface of $\mathbb{P}^{n+1}$ containing two linear subvarieties $\mathbb{P}^\frac{n}{2}, \mathbb{P}^\frac{n}{2} \subseteq X_t$ such that $\mathbb{P}^\frac{n}{2} \cap \mathbb{P}^\frac{n}{2} = \mathbb{P}^m$. Consider the map
\[ \Phi : \mathcal{H} \to Hom_C(\mathbb{C}[x_0, ..., x_{n+1}], \mathbb{C}[x_0, ..., x_{n+1}]^{\ast}_{d \frac{n}{2} - n - 2}), \]
given by
\[ \Phi(X_t, \mathbb{P}^\frac{n}{2}, \mathbb{P}^\frac{n}{2}) = \left[ \int_{a[\mathbb{P}^\frac{n}{2}] + b[\mathbb{P}^\frac{n}{2}]} \text{res} \left( \frac{PQ\Omega}{F^{\frac{n}{2}+1}} \right) \right]_{P,Q}. \]
This map is regular, then it is continuous in the Zariski topology of $\mathcal{H}$. By Proposition 3 we already know that
\[ T_0 V_{a[\mathbb{P}^\frac{n}{2}] + b[\mathbb{P}^\frac{n}{2}]} = \text{Ker } \Phi(X_t, \mathbb{P}^\frac{n}{2}, \mathbb{P}^\frac{n}{2}). \]
This implies that each subset of $\mathcal{H}$ where Ker $\Phi$ has constant dimension is a locally closed subset in Zariski topology. Theorem 2 implies that
\[ \dim \mathcal{H} = \dim T_0 V_{a[\mathbb{P}^\frac{n}{2}] + b[\mathbb{P}^\frac{n}{2}]} \]
Therefore, the point $(X_0, \mathbb{P}^\frac{n}{2}, \mathbb{P}^\frac{n}{2})$ corresponding to the Fermat variety together with its two linear subvarieties, is a smooth point of $\mathcal{H}$. Furthermore Ker $\Phi$ has
constant dimension in a polydisc around $(X_0, \mathbb{P}^a, \mathbb{P}^b) \in \mathcal{H}$, then the same holds in a Zariski neighbourhood of this point. Therefore

$$\dim \mathcal{H} = \dim T_V a[\mathbb{P}^a] + b[\mathbb{P}^b] \quad \forall t \in U,$$

for $U$ a Zariski open set of $W$ (not necessarily a neighbourhood of the Fermat variety), and variational Hodge conjecture holds for $\delta = a[\mathbb{P}^a] + b[\mathbb{P}^b] \in H^a_n(X, \mathbb{Z})$ and $t \in U$.

**Remark 11.** Considering $\delta = a[\mathbb{P}^a] + b[\mathbb{P}^b] \in H^a_n(X, \mathbb{Z})$. After Theorem 2, variational Hodge conjecture for $\delta$ is open in the following cases: $(d, m) = (3, \frac{n}{2} - 3)$, $(3, \frac{n}{2} - 2)$, $(4, \frac{n}{2} - 2)$ and $m = \frac{n}{2} - 1$ with $a \neq b$. Note that the cases $m = \frac{n}{2} - 1$ with $a = b$, and $m = \frac{n}{2}$ are both complete intersection algebraic cycles, where variational Hodge conjecture holds by [Dan14]. It would be interesting to determine whether for the remaining cases the corresponding Hodge locus $V_\delta$ is smooth and reduced. This problem has been considered by Movasati for small degree and dimension in [Mov17a, Theorem 18.2 and Theorem 18.3]. Movasati proves that in several cases $V_\delta$ is not smooth and reduced, but he also provides interesting examples, such as $(n, d, m, a, b) = (6, 3, 1, 1, -1)$, where $V_\delta$ is possibly smooth and reduced. In such cases $V_\delta$ must be strictly bigger than $V_{[\mathbb{P}^a]} \cap V_{[\mathbb{P}^b]}$, and it would be very interesting to study this phenomena, and determine if it is due to the existence of some new algebraic cycle (homologous to $\delta$) with higher deformation space.

9. Acknowledgements

Part of this work was developed during my Ph.D. at IMPA between 2016 and 2018. The final version of this article was written during my short stay at Harvard CMSA. I thank both institutes for providing such stimulating environments to work. I am deeply grateful to my advisor Hossein Movasati, for all his comments and contributions to this article, and for constantly sharing his knowledge with me during my doctorate.

References

[Blo72] Spencer Bloch. Semi-regularity and de Rham cohomology. *Invent. Math.*, 17:51–66, 1972.
[CDK95] Eduardo H. Cattani, Pierre Deligne, and Aroldo G. Kaplan. On the locus of Hodge classes. *J. Amer. Math. Soc.*, 8(2):483–506, 1995.
[CG80] James A. Carlson and Phillip A. Griffiths. Infinitesimal variations of Hodge structure and the global Torelli problem. *Journees de geometrie algebrique*, Angers/France 1979, 51-76, 1980.
[Dan14] A. Dan. Noether-Lefschetz locus and a special case of the variational Hodge conjecture. *ArXiv e-prints*, April 2014.
[Del82] P. Deligne. *Hodge Cycles on Abelian Varieties*, pages 9–100. Springer Berlin Heidelberg, Berlin, Heidelberg, 1982.
[DK16] Ananyo Dan and Inder Kaur. Semi-regular varieties and variational Hodge conjecture. *C. R. Math. Acad. Sci. Paris*, 354(3):297–300, 2016.
[Gri69] Phillip A. Griffiths. On the periods of certain rational integrals. I, II. *Ann. of Math.* (2) 90 (1969), 460-495; *ibid.* (2), 90:496–541, 1969.
[Gro66] Alexander Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.
[Mac16] F.S. Macaulay. *The algebraic theory of modular systems*. Cambridge tracts in mathematics and mathematical physics. University press, 1916.
[Mov17a] H. Movasati. *A Course in Hodge Theory: with Emphasis on Multiple Integrals*. Available at author’s webpage, 2017.
[Mov17b] Hossein Movasati. Gauss-Manin connection in disguise: Noether-Lefschetz and Hodge loci. *Asian Journal of Mathematics*, 2017.

[ MV17] H. Movasati and R. Villaflor Loyola. Periods of linear algebraic cycles. *ArXiv e-prints*, April 2017.

[Otw03] Ania Otwinowska. Composantes de petite codimension du lieu de Noether-Lefschetz: un argument asymptotique en faveur de la conjecture de Hodge pour les hypersurfaces. *J. Algebraic Geom.*, 12(2):307–320, 2003.

[Ser18] E. C. Sertöz. Computing Periods of Hypersurfaces. *ArXiv e-prints*, March 2018.

[Voï03] Claire Voisin. *Hodge theory and complex algebraic geometry. II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003. Translated from the French by Leila Schneps.

**Instituto de Matemática Pura e Aplicada, IMPA, Estrada Dona Castorina, 110, 22460-320, Rio de Janeiro, RJ, Brazil**

*E-mail address: rvilla@impa.br*