On the multi-robber damage number

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Abstract
We study a variant of the Cops and Robbers game on graphs in which the robbers damage the visited vertices, aiming to maximise the number of damaged vertices. For that game with one cop against $s$ robbers a conjecture was made in [3] that the cop can save three vertices from being damaged as soon as the maximum degree of the base graph is at least $\left(\binom{s}{2}\right) + 2$.

We are able to verify the conjecture and prove that it is tight once we add the assumption that the base graph is triangle-free. We also study the game without that assumption, disproving the conjecture in full generality and further attempting to locate the smallest maximum degree of a base graph which guarantees that the cop can save three vertices against $s$ robbers. We show that this number is between $2\left(\binom{s}{2}\right) - 3$ and $2\left(\binom{s}{2}\right) + 1$.

1 Introduction

In many variants of the game of Cops and Robbers on graphs, see e.g. [1] for an overview, multiple cops play against a single robber. In [4], a variant of the game that gives the robber a more active role than simply evading the cops was introduced: the robber tries to damage as many vertices as possible, by visiting them before getting captured, and the cops attempt to minimize this damage. While the damage variant was originally studied with one cop and one robber, it was later extended to multiple cops in [2]. Unlike the original game, in the damage variant it makes sense to introduce multiple robbers, which was done recently in [3].

For a formal definition of the problem, let $G$ be a graph and $s$ a positive integer. In round 0 the cop first chooses an initial vertex to occupy, and

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then each of the $s$ robbers chooses an initial vertex. Each subsequent round consists of a turn for the cop followed by a turn for the robbers, where each individual has the opportunity to (but is not required to) move to a neighboring vertex on their turn. If the cop ever occupies the same vertex as a robber, that robber is removed from the game. All information about the locations and movements of everyone is publicly available.

Now it remains to add the concept of damage to the setup. If a vertex $v$ is occupied by a robber at the end of a round and the robber is not caught in the following round, then $v$ becomes damaged. In this version of the game, the cop is trying to minimize the number of damaged vertices, while the robbers play to damage as many vertices as possible. The $s$-robin damage number of $G$, denoted $\text{dmg}(G; s)$, is the number of damaged vertices when both the cop and the $s$ robbers play optimally on $G$.

We consider the setting of one cop and multiple robbers. We say that the cop saves $k$ vertices, if $\text{dmg}(G, s) \leq n - k$. In other words, assuming optimal play from both the cop and robber, we have that $k$ out of the $n$ vertices do not get damaged. How many vertices can the cop save?

As was observed by Carlson, Halloran and Reinhart [3], it is always possible for a single cop to save two vertices (assuming the graph has at least one edge): if the cop simply patrols a fixed edge of the graph, none of its endpoints can ever get damaged. So we trivially have that two vertices can always get saved. This begs the question: When is it possible to save three vertices?

This question was studied thoroughly in [3] for the case of $s = 2$ robbers. They showed that in any graph $G$ with $\Delta(G) \geq 3$ (that is, there is at least one vertex of degree at least 3) the cop can indeed save three vertices. They used this insight to give a full characterization of all the graphs where the cop can save three vertices against 2 robbers. For the case of more than two robbers, they proposed the following conjecture.

**Conjecture 1.1** ([3]). For all $s > 2$, if $G$ is a graph with $\Delta(G) \geq \left(\frac{s}{2}\right) + 2$, then the cop can save three vertices against $s$ robbers.

It turns out that this conjecture is “almost true”, as we are able to prove it with an additional condition, that the graph $G$ is triangle-free. In the general case where $G$ is not triangle-free, we have examples that falsify the conjecture. But it becomes true and (asymptotically) tight again, if the corresponding bound $\left(\frac{s}{2}\right) + 2$ is multiplied by a factor of 2.

**Definition 1.2.** For $s \geq 1$, let $\Delta_s$ denote the smallest integer such that for all graphs $G$ with $\Delta(G) \geq \Delta_s$, the cop can save three vertices against $s$ robbers.

Further, let $\Delta'_s$ denote the smallest integer such that for all triangle-free graphs $G$ with $\Delta(G) \geq \Delta'_s$, the cop can save three vertices against $s$ robbers.
In the following section we prove that
\[ \Delta'_s = \binom{s}{2} + 2, \text{ and} \]
\[ 2\binom{s}{2} - 3 \leq \Delta_s \leq 2\binom{s}{2} + 1. \]

2 Which degree is sufficient to protect three vertices?

2.1 Upper bounds

Lemma 2.1. If the graph \( G \) contains an induced star \( K_{1,t} \) with \( t \geq \binom{s}{2} + 2 \), then the cop can save 3 vertices against \( s \) robbers.

Proof. Assume \( G \) contains an induced star \( K_{1,t} \). We have to give a strategy for the cop which saves three vertices. Let \( v \) be the central vertex of the star and let \( W = \{w_1, \ldots, w_t\} \) be the set of its leaves. Recall that every round consists out of a cop move followed by the robbers moves. The strategy of the cop is to guard the central vertex \( v \). Note that any robber that never enters any of the vertices of \( W \) can obviously never damage any of them.

Let \( i_1 \) be the index of the first round where a robber enters \( N(v) \). Then the cop immediately moves to catch this robber in round \( i_1 + 1 \) and afterwards moves back to \( v \) in round \( i_1 + 2 \). Now, the cop repeats the same strategy: We let \( i_2 \) be the first index with \( i_2 \geq i_1 + 2 \) such that a robber enters \( N(v) \) in round \( i_2 \). Then the cop catches this robber and afterwards moves back to vertex \( v \), and so on.

We claim that this strategy guarantees that vertex \( v \) and at least two vertices from \( W \) do not get damaged. Indeed, vertex \( v \) stays undamaged, because there are no two consecutive rounds where \( v \) is not occupied by the cop. Furthermore, the only rounds where some vertex in \( W \) can get damaged are by definition the rounds \( i_1, i_1 + 1, i_2, i_2 + 1, i_3, i_3 + 1 \), and so on. Observe that after round \( i_r + 1 \) the last remaining robber is caught (of the robbers that ever enter \( W \)). Furthermore, for all \( j = 1, \ldots, r \), we have that in the rounds \( i_j, i_j + 1 \) at most \( s - j \) robbers are not caught. In these two rounds, each one of them can damage at most one vertex from \( W \), because no two vertices of \( W \) are connected by an edge. In total, the robbers damage at most \( (s - 1) + (s - 2) + \cdots + 1 = \binom{s}{2} \) vertices, so at least 2 vertices from \( W \) survive. Therefore the cop saved three vertices. \( \square \)

The strategy described in the proof of Lemma 2.1 will be useful in other situations too, so we give it a name. We define the strategy GUARD\((v)\) to be
the strategy for the cop of initially starting at vertex $v$ and waiting until a robber enters $N(v)$, then catching this robber, and then immediately going back to $v$. (Even if there is a second robber in $N(v)$ that could potentially be caught, the cop still moves back to $v$.)

**Theorem 2.2.** For all $s \geq 2$, we have $\Delta'_s \leq \binom{s}{2} + 2$.

**Proof.** Let $t := \binom{s}{2} + 2$. If $G$ is triangle-free and $\Delta(G) \geq t$, then $G$ contains an induced star $K_{1,t}$. By Lemma 2.1 this proves that the cop can save three vertices.

**Theorem 2.3.** For all $s \geq 2$, we have $\Delta_s \leq 2\binom{s}{2} + 1$.

**Proof.** Let $s \geq 2$. We have to prove that for all graphs $G$ with $\Delta(G) \geq 2\binom{s}{2} + 1$, the cop can save three vertices. For this purpose, assume $G$ contains a vertex $v$ of degree at least $2\binom{s}{2} + 1$. The general approach of the cop is to apply the strategy $\text{guard}(v)$ until only two robbers remain.

To be more precise, if $s > 2$, let $i$ be the first round such that the cop got out of vertex $v$ to catch a robber in round $i - 1$ and returned to vertex $v$ in round $i$, and now it is the turn of at most two remaining robbers to move in round $i$. If $s = 2$, then let $i$ be one.

We claim that in this exact situation, in round $i$ where the cop has moved (and finds himself in $v$) and the robbers have not yet moved, there are at least three vertices in $N(v)$ which are not yet damaged and which are not currently occupied by a robber.

Indeed, observe that previous to that point, whenever the cop catches a robber and goes back to $v$, the $k$ robbers that are still not captured can enter at most $2k$ vertices from $N(v)$ in these two rounds. Furthermore, $v$ itself is never damaged. It then follows that the number of vertices in $N(v)$ which were entered by a robber is at most $2(s - 1) + 2(s - 2) + \cdots + 2 \cdot 2 = 2\binom{s}{2} - 2$.

So at that point there are at least three vertices in $N(v)$ which are not damaged and not occupied by a robber.

It then follows from the result of Carlson, Halloran and Reinhart [3, Proposition 2.7] that playing on $v$ and these three neighboring vertices, the cop can protect at least three vertices against the remaining two robbers. □

### 2.2 Lower bounds

The goal of this subsection is to give examples of graphs with high degree, where the cop can not save three vertices. In particular, to show lower bounds for $\Delta'_s$ and $\Delta_s$, we introduce the graphs $G'$ and $\hat{G}$ which are depicted in Figure 1. The graph $G'$ consists out of vertices $v_1$ and $v_2$ and a total of $\ell$ internally vertex-disjoint paths $P_1, \ldots, P_\ell$ of length 7 from $v_1$ to
If $N(v_1) = \{w_1, \ldots, w_\ell\}$ and $N(v_2) = \{u_1, \ldots, u_\ell\}$, then $G$ is created from $G'$, with an additional assumption that $\ell$ is even, by adding the edges 
\{w_1, w_2\}, \{w_3, w_4\}, \ldots, \{w_{\ell-1}, w_\ell\} and \{u_1, u_2\}, \{u_3, u_4\}, \ldots, \{u_{\ell-1}, u_\ell\}.

We now proceed to prove some helpful auxiliary statements which establish that the robbers can achieve certain goals when playing on the graphs $G$ and $G'$. A great cycle in $G$ is a cycle of length 14 which contains both $v_1$ and $v_2$ and has no chord. In other words, a great cycle in $G'$ is created by any two paths $P_i, P_j$ with $i \neq j$, and a great cycle in $G$ is created by any two paths $P_i, P_j$ with $\lceil i/2 \rceil \neq \lceil j/2 \rceil$.

We will say that a robber is cautious if his first priority is not to get caught by the cop. When we say that a cautious robber is playing according to a certain strategy, we mean that he plays according to the strategy unless that takes him within the cop’s neighborhood, in which case he suspends the strategy and avoids getting caught.

**Lemma 2.4.** If a cautious robber finds himself on a great cycle $C$ of $G$ (respectively $G'$), he can stay on it indefinitely.

**Proof.** No matter on which vertex $v \in V(C)$ the robber is, when the cop gets into the neighborhood of $v$ there is always a vertex on $C$ that the robber can move to and stay out of cop’s reach. \qed

**Lemma 2.5.** Assume that there are at least three robbers positioned on...
vertices of \( G \) (respectively \( G' \)), such that they are all at distance at least two from the cop's position. If \( C \) is a great cycle, then the robbers have a strategy such that no robber gets caught and at least two robbers can enter \( C \) and stay on it indefinitely.

Proof. First we show that two robbers, \( R_1 \) and \( R_2 \), can make sure that one of them gets onto \( C \). Suppose w.l.o.g. that \( R_1 \) is not further away from \( v_1 \) than \( R_2 \). Then \( R_1 \) will cautiously go towards \( v_1 \), and \( R_2 \) will cautiously go towards \( v_2 \). Clearly, looking at the base graph structure, no matter where the cop is at least one of the robbers will manage to reach his target, thus getting on \( C \).

Now we apply this reasoning twice. First we pick arbitrary two robbers and get one of them on \( C \). Note that he can stay on it indefinitely without getting caught, by Lemma 2.4. If no other robber is on \( C \) at that point, we observe two such robbers and get one of them to \( C \). □

Given a great cycle \( C \) of either \( G \) or \( G' \) and three robbers \( R_1, R_2, R_3 \), we are going define a strategy \( \text{CYCLE-ATTACK}(C, R_1, R_2, R_3) \). The idea behind this strategy is that the robbers either manage to damage all vertices of \( C \), or do a considerable damage to the rest of the graph, all this while none of the robbers are caught.

Formally, the strategy \( \text{CYCLE-ATTACK}(C, R_1, R_2, R_3) \) is defined as follows. All three robbers remain cautious for the whole duration of the strategy. As soon as all vertices of \( C \) are damaged, the \( \text{CYCLE-ATTACK} \) is concluded.

In Stage 1, the robbers \( R_1 \) and \( R_2 \) enter the cycle \( C \), on which they are going to stay for the remainder of the \( \text{CYCLE-ATTACK} \). Then in Stage 2, the robber \( R_1 \) attempts to go clockwise (CW) around \( C \) while the robber \( R_2 \) attempts to go counter-clockwise (CCW) around \( C \).

If during Stage 2 the robbers \( R_1 \) and \( R_2 \) are both halted while failing to damage all the vertices of \( C \), Stage 3 starts – the robbers \( R_1 \) and \( R_2 \) now swap their directions, \( R_1 \) going CCW on \( C \) and \( R_1 \) going CW on \( C \) for the remainder of the \( \text{CYCLE-ATTACK} \). If after that they again both stop their rotational movement before all the vertices of \( C \) are damaged, Stage 4 starts and the robber \( R_3 \) comes into play. Suppose that w.l.o.g. at that point the cop is closer to \( v_2 \) than to \( v_1 \). The robber \( R_3 \) goes to \( v_1 \), and then for every \( i \) he attempts to go down the path \( P_i \) from \( v_1 \) towards \( v_2 \) for as long as possible while staying cautious, returning to \( v_1 \) between each two paths. Once he is done with all the paths \( P_i \), the Stage 4 and the whole \( \text{CYCLE-ATTACK} \) is concluded.

Lemma 2.6. Let \( C \) be a great cycle of \( G \) (respectively \( G' \)). If there are at least three robbers positioned at distance at least two from the cop’s position,
then they can execute the strategy CYCLE-ATTACK\((C, R_1, R_2, R_3)\) in finitely many moves without a single one of them getting caught.

Proof. Stage 1 can be completed by Lemma 2.5. If in Stage 2 the robbers \(R_1\) and \(R_2\) stop moving around the cycle, that means that the cop is at distance at most two from each of them. Note that at most three vertices of \(C\) can be between them at that point.

Once Stage 3 starts the robbers \(R_1\) and \(R_2\) will clearly be able to meet each other on \(C\) after changing the orientation of their rotational movement. If after that they are both halted, at most three vertices on \(C\) can remain undamaged. Note that from that point on, if the cop wants to prevent \(C\) from being damaged to the last vertex he must remain tied up within distance one from at least one of those three vertices to the end of the CYCLE-ATTACK.

During Stage 4, as long as the cop remains at distance at most one to at least one undamaged vertex of \(C\) he will not being able to enter the neighborhood of \(v_1\). Hence, the robber \(R_3\) will be able to repeatedly find an unobstructed route to \(v_1\), before starting the traverse for each of the paths \(P_i\).

Clearly, the duration of each of the stages is limited and thus the CYCLE-ATTACK must conclude.

Lemma 2.7. If CYCLE-ATTACK\((C, R_1, R_2, R_3)\) is performed on \(G\) (respectively \(G'\)) for its every great cycle \(C\), then all of its undamaged vertices are contained in the closed neighborhood of a single vertex.

Proof. Let us assume that CYCLE-ATTACK\((C, R_1, R_2, R_3)\) was performed on \(G\) or \(G'\) for its every great cycle \(C\).

Suppose first that a vertex \(x\) that is neither in \(N[v_1]\) nor in \(N[v_2]\) remains undamaged. Due to the symmetry of the base graph w.l.o.g. we can assume that \(x \in V(P_1)\). As the whole closed neighborhood of \(x\) belongs to \(V(P_1)\), the cop must have stayed on \(P_1\) in Stage 4 of every CYCLE-ATTACK\((C, R_1, R_2, R_3)\) where \(C\) contained \(P_1\). If in any of those Stages 4 the cop got to either \(u_1\) or \(w_1\), then all the undamaged vertices of the base graph must be, respectively, either in \(N[u_1]\) or in \(N[w_1]\). On the other hand, if in all of those Stages 4 the cop remained out of both \(N[v_1]\) and \(N[v_2]\), then all the undamaged vertices of the base graph must be consecutive vertices of \(P_1\), at most three of them, which are clearly within a closed neighborhood of a vertex.

The case that remains to be considered is when all of the undamaged vertices are in \(N[v_1] \cup N[v_2]\). Clearly, it cannot be that there are undamaged vertices in both \(N[v_1]\) and \(N[v_2]\), as the cop cannot simultaneously be at both of the neighborhoods during the CYCLE-ATTACK’s. Hence, the undamaged vertices are either all within \(N[v_1]\) or all within \(N[v_2]\).
We are now ready to prove the two main theorems of this section.

**Theorem 2.8.** For all \( s \geq 2 \), we have \( \Delta'_s \geq \binom{s}{2} + 2 \).

**Proof.** We will show that \( s \) robbers have a strategy to damage all but at most two vertices, when playing on \( G' \) with \( \ell = \binom{s}{2} + 1 \) paths.

If \( s > 2 \), arbitrarily chosen three robbers \( R_1, R_2, R_3 \) will first perform the \textsc{cycle-attack}(\( C, R_1, R_2, R_3 \)) for every great cycle \( C \) of \( G' \). By Lemma 2.7, then all of the undamaged vertices are contained in the closed neighborhood of a single vertex. Lemma 2.6 ensures that the three involved robbers will not be caught in the process, while the remaining robbers can clearly remain cautious.

Let us first suppose that all the undamaged vertices are in the closed neighborhood of either \( v_1 \) or \( v_2 \), w.l.o.g. assume they are all in \( N[v_2] \). That means there are at most \( \binom{s}{2} + 2 \) undamaged vertices. For the remainder of the game, the \( k \) robbers that are not captured yet repeatedly perform the following strategy that we call \textsc{all-out-attack}(\( k \)).

First of all, if at any point any of the robbers gets within distance two of the cop while the cop is outside of \( N[v_2] \), “an intermission” is activated – all the robbers become cautious and attempt walking towards undamaged vertices in \( N(v_2) \), trying to take different routes that avoid the cop. If during that action the cop gets to \( v_2 \), all the robbers resume following the strategy, if needed going to \( v_1 \) first.

The strategy starts with all the robbers gathering on \( v_1 \), and each of them picking a different path \( P_i \) such that \( u_i \) is undamaged. Then they simultaneously go down their picked paths towards \( u_i \). Even if a robber gets within distance two from the cop who is in \( N[v_2] \), the robber proceeds towards \( u_i \) risking getting caught. Every robber that gets to \( u_i \) turns back immediately and goes back to \( v_1 \). Once all the robbers that are not captured reach \( v_1 \), the \textsc{all-out-attack}(\( k \)) is complete.

Note that during the \textsc{all-out-attack}(\( k \)) that is completed at most one robber gets caught and at least \( k - 1 \) undamaged vertices in \( N[v_2] \) get damaged. If an intermission is activated, either an undamaged vertex will get damaged without any robbers caught, or the robbers will manage to get back to the main course of the attack and progress to its later stage. Hence, there will be a limited number of intermissions.

All in all, the \textsc{all-out-attack}(\( k \)) will get completed for \( k = s, (s-1), \ldots, 2 \), damaging at least \( (s-1) + (s-2) + \cdots + 1 = \binom{s}{2} \) undamaged vertices. That means all but \( \binom{s}{2} + 2 - \binom{s}{2} = 2 \) vertices are damaged, and we are done.

In the case when all the undamaged vertices are in the closed neighborhood of a vertex \( x \) that is neither \( v_1 \) nor \( v_2 \), it is enough to perform an attack analogous to the all out attack above, but with just two robbers on two
internally disjoint paths towards $x$, and the number of undamaged vertices will get reduced to 2.

**Theorem 2.9.** For all $s \geq 2$, we have $\Delta_s \geq 2\binom{s}{2} - 3$.

**Proof.** The statement trivially holds for $s = 2$ and $s = 3$, so from now on we can assume that $s > 3$. We will show that $s$ robbers have a strategy to damage all but at most two vertices, when playing on $G$ with $\ell = 2\binom{s}{2} - 4$, for which we have $\Delta(G) = 2\binom{s}{2} - 4$.

Arbitrarily chosen three robbers $R_1, R_2, R_3$ will first perform the cycle-attack $(C, R_1, R_2, R_3)$ for every great cycle $C$ of $G$. By Lemma 2.7 then all of the undamaged vertices are contained in the closed neighborhood of a single vertex. Lemma 2.6 ensures that the three involved robbers will not be caught in the process, while the remaining robbers can clearly remain cautious.

Let us first suppose that all the undamaged vertices are in the closed neighborhood of either $v_1$ or $v_2$, w.l.o.g. assume they are all in $N[v_2]$. That means there are at most $2\binom{s}{2} - 3$ undamaged vertices. From that point on, the $k$ robbers that are not captured yet repeatedly perform the following strategy that we call **all-out-attack-2**$(k)$, until there are only three robbers left.

First of all, if at any point any of the robbers gets within distance two of the cop while the cop is outside of $N[v_2]$, “an intermission” is activated – all the robbers become cautious and attempt walking towards undamaged vertices in $N(v_2)$, trying to take different routes that avoid the cop. If during that action the cop gets to $v_2$, all the robbers resume following the strategy, if needed going to $v_1$ first.

The strategy starts with all the robbers gathering on $v_1$, and each of them picking a different pair of paths $P_{2i-1}$, $P_{2i}$ such that $u_{2i-1}$ and $u_{2i}$ are undamaged. Then they simultaneously go down $P_{2i-1}$ towards $u_{2i-1}$. Even if a robber gets within distance two from the cop who is in $N[v_2]$, the robber proceeds towards $u_{2i-1}$ risking getting caught. Every robber that gets to $u_{2i-1}$, then visits $u_{2i}$ and goes back to $v_1$. Once all the robbers that are not captured reach $v_1$, the **all-out-attack-2**$(k)$ is complete.

Note that during the **all-out-attack-2**$(k)$ that is completed at most one robber gets caught and at least $2(k - 1)$ undamaged vertices in $N[v_2]$ get damaged. If an intermission is activated, either an undamaged vertex will get damaged without any robbers caught, or the robbers will manage to get back to the main course of the attack and progress to its later stage. Hence, there will be a limited number of intermissions.

All in all, the **all-out-attack-2**$(k)$ will get completed for $k = s, (s - 1), \ldots, 4$, damaging at least $2(s - 1) + 2(s - 2) + \cdots + 2 \cdot (4 - 1) = 2\binom{s}{2} - 6$ undamaged vertices. That means all but $2\binom{s}{2} - 3 - 2\binom{s}{2} - 6 = 3$ vertices are damaged.
damaged.
The final three undamaged vertices are forming a triangle. The remaining three robbers can target one undamaged vertex each, approaching them on three disjoint paths. If they remain cautious on the approach, one more vertex will clearly get damaged, completing the proof in this case.

In the case when all the undamaged vertices are in the closed neighborhood of a vertex \( x \) that is neither \( v_1 \) nor \( v_2 \), we again have a considerably simpler situation to analyse. No matter where \( x \) is on the graph, at most two coordinated attacks by three robbers towards the neighbors of \( x \) are needed to reduce the total number of undamaged vertices to at most two.

\[ \square \]

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