PROJECTIVE DIMENSION OF HYPERGRAPHS

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ABSTRACT. Given certain a square-free monomial ideal $I$ in a polynomial ring $R$ over a field $K$, we compute the projective dimension of $I$. Specifically, we focus on the cases where the 1-skeleton of a hypergraph is either a string or a cycle. We investigate what the impact on the projective dimension is when higher dimensional edges are removed. We prove that the higher dimensional edge either has no impact on the projective dimension or the projective dimension only goes up by one with the extra higher dimensional edge.

1. INTRODUCTION

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field $K$. The minimal free resolution of $R/I$ for an ideal $I \subset R$ is an exact sequence of the form

$$0 \to \bigoplus_{j} S(-j)^{\beta_{j,0}(R/I)} \to \cdots \to \bigoplus_{j} S^{\beta_{1,j}(R/I)} \to R \to R/I \to 0$$

The exponents $\beta_{i,j}(R/I)$ are invariants of $R/I$, called the Betti numbers of $R/I$. In general, finding Betti numbers is still a wide open question. The projective dimension of $R/I$, denoted $pd(R/I)$, which is defined as follows

$$pd(R/I) = \max\{i \mid \beta_{i,j}(R/I) \neq 0\}.$$  

We also recall that the (Castelnuovo-Mumford) regularity of $R/I$, denoted $reg(R/I)$, which is defined as follows

$$reg(R/I) = \max\{j-i \mid \beta_{i,j}(R/I) \neq 0\}.$$  

Those two invariants play important roles in algebraic geometry, commutative algebra, and combinatorial algebra. In general, one finds the graded minimal free resolution of an ideal to obtain those invariants, but the computation can be difficult and computationally expensive.

Kimura, Terai and Yoshida define the dual hypergraph of a square-free monomial ideal in order to compute its arithmetical rank [14] (see Definition 2.1 for definition of a hypergraph). Since then, there are a couple of papers using this combinatorial object to study various properties, for example, [9] and [18]. In particular, Lin and Mantero use it to show that ideals with the same dual hypergraph have the same Betti numbers and projective dimension [15] (Theorem 2.3(1)), which has found use in other papers, such as in [13].

The focus of this work is to use hypergraph properties to compute the projective dimension of a square-free monomial ideal without finding the minimal free resolution of the ideal, which has different focus on various work by others, for example, the recent work of Eagon, Miller, and Ordog [5]. More precisely, we find the projective dimension of a hypergraph when its 1-skeleton is a string or cycle. This extends the work of Lin and Mantero in [15]. One of the main tools is built using the result of...
Lin and Mapes in [17], one can remove a higher dimensional edge of a hypergraph without impacting its projective dimension (Corollary 4.4 [17]). We also establish a new technique for computing the projective dimension using bounds on sub-ideals, which is inspired by methods from [6], namely Betti splittings (Lemma 2.10). We then proceed with our results concerning higher dimensional edges on strings and cycles in Section 3 Section 4. Through out this paper, ideals are monomial ideals in a polynomial ring $R$ over the field $K$.

2. Preliminaries

2.1. Hypergraph of a square-free monomial ideal. Kimura, Terai, and Yoshida associate a square-free monomial ideal with a hypergraph in [14], see Definition 2.1. Note that this construction different then the constructions associating ideals to hypergraphs coming from the study of edge ideals. In particular relative to edge ideals, the hypergraph of Kimura, Terai, and Yoshida might be more aptly named the “dual hypergraph”. The construction of dual hypergraph is first introduced by Berge in [1]. In the edge ideal case, one associates a square-free monomial with a hypergraph by setting variables as vertices and each monomial corresponds to an edge of the hypergraph (see for example [8]). In the following definition, we actually associate variables with edges of the hypergraph and vertices with the monomial generators of the ideal, and in practice this is the dual hypergraph of the hypergraph in the edge ideal construction.

Definition 2.1. Let $I$ be a square-free monomial ideal in a polynomial ring with $n$ variables with minimal monomial generating set $\{m_1, \ldots, m_\mu\}$. Let $V$ be the set $\{1, \ldots, \mu\}$. We define $\mathcal{H}(I)$ (or $\mathcal{H}$ when $I$ is understood) to be the hypergraph associated to $I$ which is defined as $\{\{j \in V : x_i|m_j\} : i = 1, 2, \ldots, n\}$. Moreover $\mathcal{H}$ is separated if in addition for every $1 \leq j_1 < j_2 \leq \mu$, there exist edges $F_1$ and $F_2$ in $\mathcal{H}$ so that $j_1 \in F_1 \cap (V - F_2)$ and $j_2 \in F_2 \cap (V - F_1)$.

Note that when a hypergraph is separated then its vertices correspond to a minimal generating set of the monomial ideal.

Example 2.2. Let $I = (m_1 = abk, m_2 = bel, m_3 = dklm, m_4 = dekn, m_5 = efgn, m_6 = ghmn, m_7 = hikl, m_8 = ijk)$ the Figure 1 is the hypergraph associated to $I$ via the Definition 2.1 where

\[
\mathcal{H}(I) = \{a = \{1\}, b = \{1, 2\}, c = \{2, 3\}, d = \{3, 4\}, e = \{4, 5\}, f = \{5\}, g = \{5, 6\}, h = \{6, 7\}, i = \{7, 8\}, j = \{8\}, k = \{1, 3, 4, 7, 8\}, l = \{2, 3, 7\}, m = \{3, 6\}, n = \{4, 5, 6\}\.
\]
Some important terminology regarding these hypergraphs is the following. We say a vertex \( i \in V \) of \( \mathcal{H} \) is an open vertex if \( \{i\} \) is not in \( \mathcal{H} \), and otherwise \( i \) is closed. In Figure 1 we can see that the vertices labeled by \( a, f \) and \( j \) are all closed, and the rest are open. Moreover, a hypergraph \( \mathcal{H} \) with \( V = [\mu] \) is a string if \( \{i, i + 1\} \) is in \( \mathcal{H} \) for all \( i = 1, \ldots, \mu - 1 \), and the only edges containing \( i \) are \( \{i - 1, i\} \) and possibly \( \{i\} \). We say that a string is an open string if all vertices other than 1 and \( \mu \) are open (note to be separated 1 and \( \mu \) must be closed). Also, \( \mathcal{H} \) is a \( \mu \)-cycle if \( \mathcal{H} = \mathcal{H} \cup \{\mu, 1\} \) where \( \mathcal{H} \) is a string. We say a cycle is an open cycle if all the vertices are open. Let \( \mathcal{H}^i = \{F \in \mathcal{H} : |F| \leq i + 1\} \) denote the \( i \)-th dimensional subhypergraph of \( \mathcal{H} \) where \( |F| \) is the cardinality of the \( F \). We call \( \mathcal{H}^1 \), the 1-skeleton of \( \mathcal{H} \).

Recently there has been a number of results concerning determining both the projective dimension and the regularity of square-free monomial ideals from the associated hypergraph. As this paper focuses more on the projective dimension, we include the statements of some of results that are useful for the rest of the paper here (these appear separately in the literature but we list them all here as part of one statement). We write \( \text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}(I)) = \text{pd}(R/I) \) and \( \text{reg}(\mathcal{H}) = \text{reg}(\mathcal{H}(I)) = \text{reg}(R/I) \) where \( \mathcal{H}(I) \) is the hypergraph obtained from a square-free monomial ideal \( I \). We do this for the Betti numbers as well.

**Theorem 2.3.**

1. (cf. Proposition 2.2 [15]) If \( I_1 \) and \( I_2 \) are square-free monomial ideals associated to the same separated hypergraph \( \mathcal{H} \), then the total Betti numbers of two ideals coincide.
2. (cf. Corollary 4.4 [17]) Let \( F \) be an edge on the hypergraph \( \mathcal{H} \). If \( F \) is an union of other edges of \( \mathcal{H} \), then \( \text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}(F)) \).
3. (cf. Theorem 2.9 (c) [16]) If \( \mathcal{H}' \subseteq \mathcal{H} \) are hypergraphs with \( \mu(\mathcal{H}') = \mu(\mathcal{H}) \), then \( \text{pd}(\mathcal{H}') \leq \text{pd}(\mathcal{H}) \) where \( \mu(*) \) denotes the number of vertices of \(*\).
4. (cf. Corollary 3.8 [15], Theorem 1.1 [2]) An open string hypergraph with \( \mu \) vertices has projective dimension \( \mu - \left\lceil \frac{n}{3} \right\rceil \) and regularity \( \left\lfloor \frac{\mu}{3} \right\rfloor \).
5. (cf. Theorem 7.7.34 [11]) An open string hypergraph \( \mathcal{H} \) with \( \mu \) vertices has \( \beta_{\mu - \left\lfloor \frac{\mu}{3} \right\rfloor}(\mathcal{H}) \neq 0 \).
6. (cf. Corollary 2.2 [12]) A hypergraph \( \mathcal{H} \) with \( n_1 + n_2 \) vertices that is a disjoint union of open string hypergraphs, \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) with \( n_1 \) and \( n_2 \) vertices has \( \beta_{n_1 + n_2 - \left\lceil \frac{n_1}{3} \right\rceil - \left\lceil \frac{n_2}{3} \right\rceil}(\mathcal{H}) \neq 0 \).
7. (cf. Corollary 7.6.30 [11]) If \( \mathcal{H} \) is a cycle having only \( \mu \) open vertices then \( \text{pd}(\mathcal{H}) = \mu - 1 - \left\lceil \frac{\mu^2 - 2}{3} \right\rceil \).
8. (cf. Corollary 20.19 [3]) If \( I = m'I \) where \( m \) is a monomial of degree \( r \) then \( \text{pd}(R/I') = \text{pd}(R/I'), \text{reg}(R/I) = r + \text{reg}(R/I') \).
9. (cf. Corollary 20.19 [3]) Let \( m \) be a monomial of degree \( r \), then
\[
\text{reg}(R/(I, m)) \leq \max\{\text{reg}(R/I), \text{reg}(R/(I : m) + r - 1) \}.
\]

**Remark 2.4.** Note that Theorem 2.3 part (1) allows us to talk about the projective dimension of a hypergraph rather than an ideal. We will use \( \text{pd}(\mathcal{H}(I)) \) in the place of \( \text{pd}(R/I) \) throughout the paper. If an edge is an union of other edges in a hypergraph, and we are considering the projective dimension of the hypergraph, we can just ignore or remove the edge using Theorem 2.3 part (2). For example, in Figure 1 we can remove the edge \( k = \{1, 3, 4, 7, 8\} = \{1\} \cup \{3, 4\} \cup \{7, 8\} \) and \( n = \{4, 5, 6\} = \{4, 5\} \cup \{5, 6\} \). If a hypergraph \( \mathcal{H} \) is an union of two disconnected hypergraphs \( G_1 \) and \( G_2 \), we have \( \text{pd}(\mathcal{H}) = \text{pd}(G_1) + \text{pd}(G_2) \) by Proposition 2.2.8 of [11] and \( \text{reg}(\mathcal{H}) = \text{reg}(G_1) + \text{reg}(G_2) \) by Lemma 3.2 of [10]. Moreover, one can compute the Betti numbers using the Betti numbers of \( G_1 \) and \( G_2 \) if both \( G_1 \) and \( G_2 \) are open strings by Corollary 2.2 [12].

2.2. **Colon Ideals - Key tool.** One technique that is used in [15] and [16] which we will need here, is using the short exact sequences obtained by looking at colon ideals. Specifically there are two types
of colon ideals that we are interested in, and we explain below what each operation looks like on the associated hypergraphs.

**Definition 2.5.** Let $H$ be a hypergraph, and $I = I(H)$ be the standard square-free monomial ideal associated to it in the polynomial ring $R$. Let $G(I) = \{m_1, \ldots, m_\mu\}$ be the minimal generating set of $I$. Let $F$ be a edge in $H$ and let $x_F \in R$ be the variable associated to $F$. Also let $v$ be a vertex in $H$ and $m_v \in I$ be the monomial generator associated to it.

- The hypergraph $H_v : v = Q_v$ is the hypergraph associated to the ideal $I_v : m_v$, where $I_v = G(I) \backslash m_v$, and $H_v = H(I_v)$ is the hypergraph associated to the ideal $I_v$.
- The hypergraph $H : F$, obtained by removing $F$ in $H$, is the hypergraph associated to the ideal $I : x_F$.
- The hypergraph $(H, x_F)$, obtained by adding a vertex corresponding to the variable $x_F$ in $H$, is the hypergraph associated to the ideal $(I, x_F)$.

The following result appearing in [16] will be very useful to us in this paper. We put them here for the self-containment of this work and for the reader's convenience.

**Theorem 2.6.** [16] Let $H$ be a 1-dimensional hypergraph, $w$ a vertex with degree at least 3 in $H$, and $S$ be a branch departing from $w$ with $\mu$ vertices. Suppose all the vertices of $S$ are open except the end vertex, and let $E$ be the edge connecting $w$ to $S$. Then $pd(H) = pd(H')$, where $H'$ is the following hypergraph: (a) if $n \equiv 1 \mod 3$, then $H' = H : E$; (b) if $n \equiv 2 \mod 3$, then $H' = H_w$.

### 2.3. Splittings - Key Tool

**Definition 2.7.** [4] A monomial ideal $I$ is **splittable** if $I$ is the sum of two nonzero monomial ideals $J$ and $K$, i.e. $I = J + K$, such that

1. The generating set $G(I)$ of $I$, is the disjoint union of $G(J)$ and $G(K)$.
2. There is a splitting function

\[
G(J \cap K) \rightarrow G(J) \times G(K)
\]

\[
w \rightarrow (\psi(w), \phi(w))
\]

satisfying

(a) (S1) for all $w \in G(J \cap K)$, $w = \text{lcm}(\psi(w), \phi(w))$.

(b) (S2) for every subset $S \subseteq G(J \cap K)$, both $\text{lcm}(\psi(S))$ and $\text{lcm}(\phi(S))$ strictly divide $\text{lcm}(S)$.

If $J$ and $K$ satisfy the above properties they are called a **splitting** of $I$.

Now the key reason we are interested in splittings is the following result by both Eliahou-Kervaire and separately Fatabbi.

**Theorem 2.8.** (Eliahou-Kervaire [4] Fatabbi [6]) Suppose $I$ is a splittable monomial ideal with splitting $I = J + K$. Then for all $i, j \geq 0$

\[
\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K).
\]

It is important to note that not all monomial ideals admit splittings. What is interesting is that there are sometimes monomial ideals that can be decomposed into a sum of ideals $J$ and $K$ which satisfy the conclusions of the previous theorem. This motivates the following definition by Francisco, Ha, and Van Tuyl in [7].

**Definition 2.9.** Let $I, J$ and $K$ be monomial ideals such that $G(I)$ is the disjoint union of $G(J)$ and $G(K)$. Then $I = J + K$ is a **Betti splitting** if

\[
\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)
\]
for all $i \in \mathbb{N}$ and all (multi)degrees $j$.

One complication however is that if one wants to use the existence of a Betti splitting to prove something about a resolution, one must first know something about the resolution in question. The key for us will be in dissecting the proof of Fatabbi in order to prove that in some special cases, which may fail condition (S2) in Definition 2.7, that a similar formula for (some) Betti numbers holds.

The following lemma is an adaptation of the proof of Fatabbi in a special case where we do not have a splitting. In this case we can show that the necessary conditions hold at the end of the resolution, so that we get a formula like that of Theorem 2.8 for the last Betti numbers. In particular this allows us to prove statements about projective dimension.

**Lemma 2.10.** Let $I$ be a monomial ideal and $I = J + K$ in the ring $R = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$. Suppose we have the following conditions on projective dimension

1. $\text{pd}(R/J) < q$
2. $\text{pd}(R/K) = q$.

and $\text{reg}(R/K) < r$. If $\beta_{q,q+r}(R/J \cap K) \neq 0$, then $\text{pd}(R/I) = q + 1$.

**Proof.** We consider the short exact sequence

$$0 \to J \cap K \to J \oplus K \to I \to 0.$$  

Let $\alpha(w) = (w, w)$ be the map from $J \cap K$ to $J \oplus K$ and $\pi(u, v) = u - v$ be the map from $J \oplus K$ to $I$. There is an induced homology sequence

$$\cdots \to \text{Tor}^R_{q+1}(J \cap K, \mathbb{K}) \to \text{Tor}^R_{q+1}(J, \mathbb{K}) \oplus \text{Tor}^R_{q+1}(K, \mathbb{K})$$

$$\to \text{Tor}^R_{q+1}(I, \mathbb{K}) \to \text{Tor}^R_{q}(J \cap K, \mathbb{K})$$

$$\to \text{Tor}^R_{q}(J, \mathbb{K}) \oplus \text{Tor}^R_{q}(K, \mathbb{K}) \to \cdots$$

Suppose we have $\text{pd}(R/J) < q$, $\text{pd}(R/K) = q$, and $\text{pd}(R/(J \cap K)) = q$, then the short exact sequence gives $\text{pd}(R/I) \leq \max\{\text{pd}(R/(J \cap K)) + 1, \text{pd}(R/J \oplus K)\} \leq q + 1$. The homology sequence becomes

$$0 \to 0 \to \text{Tor}^R_{q+1}(I, \mathbb{K}) \to \text{Tor}^R_{q}(J \cap K, \mathbb{K}) \to \text{Tor}^R_{q}(K, \mathbb{K}) \to \cdots$$

Moreover, the $q + r$ graded piece is the following:

$$0 \to \text{Tor}^R_{q+1}(I, \mathbb{K})_{q+r} \to \text{Tor}^R_{q}(J \cap K, \mathbb{K})_{q+r} \to \text{Tor}^R_{q}(K, \mathbb{K})_{q+r} \to \cdots$$

Now using our assumption that $\text{reg}(R/K) < r$ and $\text{pd}(R/K) = q$ then we have $\text{Tor}^R_{q}(K, \mathbb{K})_{q+r} = 0$. This shows $\text{Tor}^R_{q+1}(I, \mathbb{K})_{q+r} \cong \text{Tor}^R_{q}(J \cap K, \mathbb{K})_{q+r} \neq 0$ by the fact that $\beta_{q,q+r}(R/J \cap K) \neq 0$ and hence $\text{pd}(R/I) = q + 1$. \qed

**Remark 2.11.** The above lemma can be translated in terms of associated hypergraphs. Let $\mathcal{H} = \mathcal{H}(I)$ be a hypergraph with underlying vertex set $V$, and let $V_1$ and $V_2$ be a partition of the vertices of $\mathcal{H}$ such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2$ is empty. Now define $I_i$ to be the ideal generated by the generators of $I$ indexed by the elements in $V_i$ for $i = 1, 2$. Let $G_i = \mathcal{H}(I_i)$ for $i = 1, 2$ and $\mathcal{H}(I_1 \cap I_2)$ be the hypergraphs corresponding the ideals $I_i$ for $i = 1, 2$ and the ideal $I_1 \cap I_2$. Suppose $\text{pd}(G_1) < q$, $\text{pd}(G_2) = q$, $\text{pd}(\mathcal{H}(I_1 \cap I_2)) = q$, and $\text{reg}(G_2) < r$ and $\text{reg}(\mathcal{H}(I_1 \cap I_2)) = r$, then $\text{pd}(\mathcal{H}) = q + 1$.

The next lemma deals with the intersection ideal in the special case where $G_1$ will correspond to one vertex of a larger hypergraph $\mathcal{H}$.  

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Lemma 2.12. Let $\mathcal{H}$ be a hypergraph consisting of a string with vertices $\{w_1, \ldots, w_\mu\}$ such that $w_i's$ are open, and a $k$-edge $F$ consisting of vertices $\{v_1, \ldots, v_k\}$ where $v_1 = w_1$ and $v_k = w_\mu$. Also assume $v_2$ is not $w_2$. Let $G_1 = \{v_2\}$ and $G_2 = \mathcal{H}_{v_2}$ (which is just $\mathcal{H}$ with $v_2$ removed) and denote $I_i = I(G_i)$ as the ideals corresponding to $G_i$, then $I_1 \cap I_2 = m_{v_2} \Gamma$ where $m_{v_2}$ is the monomial corresponding to the vertex $v_2$ and $\mathcal{H}(\Gamma)$ has 4 isolated vertices and 2 strings one of length:

- $n_2 - 3$ when $n_2 > 2$, or
- $0$ if $n_2 = 2$

and the other of length:

- $k - 3 + n_k - 1 + n_3 - 2 + \sum_{i=4}^{k-1} n_i$ when $k > 3$, or
- $n_3 - 3$ when $k = 3$ and when $n_3 > 2$, or
- $0$ when $k = 3$ and $n_3 = 2$

(where $n_i$ is the number of vertices between $v_{i-1}$ and $v_i$ not including endpoints, note there is no $n_1$ in this convention).

Proof. To see this consider the hypergraph $\mathcal{H}$, for notational convenience, let us denote the vertex neighboring $v_1$ as $w_0$, the vertices neighboring $v_2$ as $w_{\beta_1}$ and $w_{\beta_2}$, and the vertex neighboring $v_k$ as $w_\gamma$. Now removing $v_2$ from $\mathcal{H}$ leaves us with a hypergraph on the same vertex set excluding $v_2$ and all vertices remain open except $w_{\beta_1}$ and $w_{\beta_2}$ which become closed, together with a $k - 1$ edge $F'$ that has $\{v_1, v_3, \ldots, v_k\}$ as its vertex set (note this also describes the hypergraph $\mathcal{H}_{v_2}$).

Now we consider the intersection with the ideal generated by $G_1$. A first step towards finding these new generators is to multiply each generator for $G_2$ by the monomial $m_{v_2}$ corresponding to $v_2$ in the original $\mathcal{H}$. The result on hypergraphs is now we get a hypergraph, which we will denote as $m_{v_2} \mathcal{H}_{v_2}$, consisting of 2 strings of only open vertices where one string is the part of $\mathcal{H}$ consisting of $v_1$ to $w_{\beta_1}$ and the other is $w_{\beta_2}$ to $v_k$, and all the open vertices are in an edge corresponding to $m_{v_2}$. Note that the spacing measurements for $m_{v_2} \mathcal{H}_{v_2}$ are the same as for $\mathcal{H}_{v_2}$. The issue here is that $m_{v_2} \mathcal{H}_{v_2}$ is not separated (i.e. the generators are not minimal). Denote the vertices neighboring each $w_{\beta_i}$ as $w_{\lambda_i}$. It is easy to see that removing vertices $w_{\alpha}, w_{\beta_1}, w_{\beta_2}$, and $w_\gamma$ from $m_{v_2} \mathcal{H}_{v_2}$ produces the desired separated hypergraph corresponding to $I_1 \cap I_2$.

Notice that when $n_2 = 2$ then $w_0 = w_{\beta_1}$ and similarly if $k = 3$ and $n_3 = 2$ then $w_{\beta'_2} = w_\gamma$. It is easy to see then that we get 4 isolated vertices corresponding to the original $v_1, w_{\beta_1}, w_{\beta_2}$ and $v_k$. And the remaining chains of open strings reflect removing $v_1, w_2, w_{\beta_1}$ and $w_{\beta_2}$ from the chain of length $n_2 + 1$ so the result is a chain of length $n_2 - 3$ when $n_2 > 2$, or 0 when $n_2 = 2$. Similarly, after removing $v_k, w_{\beta_2}, w_{\beta'_2}$ and $w_\gamma$ from a chain of length $k - 2 + n_3 + \sum_{i=4}^{k} n_i$ results a chain of length $k - 3 + n_3 - 2 + n_k - 1 + \sum_{i=4}^{k-1} n_i$ when $k > 3$, or $n_3 - 3$ when $k = 3$ and $n_3 > 2$, or 0 when $k = 3$ and $n_3 = 2$.

3. Strings with higher dimensional edges

In this section we are primarily interested in finding the projective dimension of a square-free monomial ideal such that its hypergraph is a string with higher dimensional edges attached to it. Our primary object is describe below and we fix the notations now for the easy reference later.

Notation 3.1.

(1) When we say $\mathcal{H}$ is a string together with an edge consisting of $k$ vertices, we mean that an edge $F$ with $k$ vertices $\{v_1, \ldots, v_k\}$ is attached to a string where we use the notation $\{w_1, \ldots, w_\mu\}$ to denote the vertices in the string, $\mathcal{H}_{S_\mu}$. 

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(2) We use \( n_i \) to represent the number of vertices from \( w_i \) to \( v_i \) including \( w_i \), \( n_i \) to represent the vertices between \( v_i-1 \) and \( v_i \) for \( i = 2, \ldots, k \), and \( n_{k+1} \) to represent the number of vertices between \( v_k \) and \( w_\mu \) including \( w_\mu \).

(3) We write \( n_i = 3l_i + r_i \) where \( l_i \) are some non-negative integers, and \( 0 \leq r_i \leq 2 \) for \( i = 1, \ldots, k+1 \).

**Example 3.2.** The hypergraph shown in Figure 3.5 shows the string hypergraph with a higher dimensional edge with the notation outlined above. In this case, \( \mu = 11 \), \( k = 4 \), \( n_1 = 1 \), \( n_2 = n_3 = n_4 = 2 \), and \( n_5 = 0 \).

![Figure 2](image_url)

Because of Theorem 2.3(2), we will primarily focus on the higher dimensional edges which are not unions of two or more edges. The vertex \( w_i \) is assumed to be open for all \( i \) unless otherwise stated. We find the projective dimensions by considering three different cases: \( \sum_{i=1}^{k+1} r_i < 2k \), \( \sum_{i=1}^{k+1} r_i \geq 2k + 1 \), and \( \sum_{i=1}^{k+1} r_i = 2k \) in three propositions. We conclude this section with Theorem 3.11 which covers all the previous results. Example 3.2 is an example of the cases when \( \sum_{i=1}^{k+1} r_i < 2k \).

The next two propositions will deal with the case when \( \sum_{i=1}^{k+1} r_i < 2k \), or \( \sum_{i=1}^{k+1} r_i \geq 2k + 1 \). In these two cases we will show that for a hypergraph \( \mathcal{H} \) satisfying the hypotheses of the propositions, that the projective dimension will be the same as for \( \mathcal{H}_{S_\mu} \).

**Proposition 3.3.** We adapt Notation 3.7. If \( \sum_{i=1}^{k+1} r_i < 2k \), then \( \text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}_{S_\mu}) \).

**Proof.** Notice that \( \mu = \sum_{i=1}^{k+1} n_i + k \) and \( \text{pd}(\mathcal{H}_{S_\mu}) = \mu - \left\lfloor \frac{\mu}{3} \right\rfloor = \sum_{i=1}^{k+1} n_i + k - \left\lfloor \frac{\sum_{i=1}^{k+1} n_i + k}{3} \right\rfloor \). Now consider the short exact sequence

\[
0 \hookrightarrow (\mathcal{H}, x_F) \twoheadrightarrow \mathcal{H} \twoheadrightarrow (\mathcal{H} : F) \hookrightarrow 0
\]

where \( x_F \) is the variable corresponding to \( F \). We first observe that \( (\mathcal{H} : F) = \mathcal{H}_{S_\mu} \) hence \( \text{pd}(\mathcal{H}_{S_\mu}) \leq \text{pd}(\mathcal{H}) \). Let \( \mathcal{H}_{V_F} = \mathcal{H}_{S_\mu} \cap (V \setminus V_F) \) be the hypergraph obtained by removing all the vertices \( v_1, \ldots, v_k \). Then \( \mathcal{H}_{V_F} \cup \{ x_F \} = (\mathcal{H}, x_F) \). Notice that \( \mathcal{H}_{V_F} \) is union of \( k + 1 \)-string such that each string has \( n_i \) vertices so by Theorem 2.3(4) we get that \( \text{pd}(\mathcal{H}_{V_F}) = \sum_{i=1}^{k+1} (n_i - \left\lfloor \frac{n_i}{3} \right\rfloor) \). Once we show that \( \text{pd}(\mathcal{H}_{V_F}) < \text{pd}(\mathcal{H}_{S_\mu}) \), then by the short exact sequence, we have

\[
\text{pd}(\mathcal{H}_{S_\mu}) \leq \text{pd}(\mathcal{H}) \leq \max\{\text{pd}(\mathcal{H}_{S_\mu}), \text{pd}(\mathcal{H}_{V_F} + 1)\} = \text{pd}(\mathcal{H}_{S_\mu}).
\]

To show that \( \text{pd}(\mathcal{H}_{V_F}) < \text{pd}(\mathcal{H}_{S_\mu}) \) it is sufficient to show that

\[
\sum_{i=1}^{k+1} (n_i - \left\lfloor \frac{n_i}{3} \right\rfloor) < \sum_{i=1}^{k+1} n_i + k - \left\lfloor \frac{\sum_{i=1}^{k+1} n_i + k}{3} \right\rfloor.
\]
which is equivalent to show
\[
\left\lceil \frac{\sum_{i=1}^{k+1} n_i + k}{3} \right\rceil < \sum_{i=1}^{k+1} \left\lceil \frac{n_i}{3} \right\rceil + k
\]
or
\[
\left\lceil \frac{\sum_{i=1}^{k+1} r_i + k}{3} \right\rceil < k,
\]
which is true by the assumption \(\sum_{i=1}^{k+1} r_i < 2k\). \(\square\)

**Proposition 3.4.** We adapt Notation 3.7. If \(\sum_{i=1}^{k+1} r_i \geq 2k + 1\), then \(pd(H) = pd(H_{S_n})\).

**Proof.** We first notice that \(pd(H_{S_n}) = \sum_{i=1}^{k+1} n_i + k - \left\lceil \frac{\sum_{i=1}^{k+1} n_i + k}{3} \right\rceil = \sum_{i=1}^{k+1} (n_i - \left\lceil \frac{n_i}{3} \right\rceil)\) because \(2k + 2 \geq \sum_{i=1}^{k+1} r_i \geq 2k + 1\) and by Theorem 2.3(4). Since \(\sum_{i=1}^{k+1} r_i \geq 2k + 1\), \(r_i < 3\), and \(k > 1\), we have at most one \(r_i\) is equal to 1. We may assume \(r_1 = 2\). Let \(H_{v_1} = H_v\) be the hypergraph where we remove the vertex \(v_1 = v\) from \(H\) and let \(H_v : v_1 = Q_v\) be the hypergraph \(H(I_v : m_v)\) where \(I_v = I(H_v)\) and \(m_v\) is the monomial corresponding to the vertex \(v\). We have a short exact sequence
\[
0 \leftarrow H \leftarrow H_v \leftarrow Q_v \leftarrow 0.
\]
We claim that in this case \(pd(H_{S_n}) = pd(H_v) > pd(Q_v)\). Note with this claim, and the facts that \(pd(H_v) \leq \max\{pd(Q_v), pd(H)\}\) and
\[
\text{pd}(H) \leq \max\{\text{pd}(H_v), \text{pd}(Q_v) + 1\} = \text{pd}(H_v),
\]
we conclude that \(pd(H_{S_n}) = pd(H_v) = pd(H)\).

To see the proof of the claim, we use induction on \(k\). When \(k = 2\), \(H_v\) is a union of two strings of length \(n_2 + n_3 + 1\) and \(n_1\). When \(n_2 \geq 2\) and \(n_3 \geq 2\), the string of length \(n_2 + n_3 + 1\) has two open strings with \(n_2 - 1\) and \(n_3 - 1\) open vertices. When \((n_2 = 1\text{ and } r_3 = 2),\ (n_3 = 1\text{ and } r_2 = 2),\) the string of length \(n_2 + n_3 + 1\) has exactly 3 closed vertices at the ends of string and all other vertices are open. By the work of [15], Theorem 2.3(4), we have either
\[
pd(H_v) = n_1 - \left\lceil \frac{n_1}{3} \right\rceil + n_2 + n_3 + 1 - 2 - \left\lceil \frac{n_2 - 2}{3} \right\rceil - \left\lceil \frac{n_3 - 2}{3} \right\rceil + 1
\]
when \(n_2 \geq 2\), \(n_3 \geq 2\), \(r_2 = 2\) and \(r_3 = 2\), or
\[
pd(H_v) = n_1 - \left\lceil \frac{n_1}{3} \right\rceil + n_2 + n_3 + 1 - 2 - \left\lceil \frac{n_2 - 2}{3} \right\rceil - \left\lceil \frac{n_3 - 2}{3} \right\rceil
\]
when \(n_2 \geq 2\), \(n_3 \geq 2\), \((r_2 = 1\text{ and } r_3 = 2)\) or \((r_2 = 2\text{ and } r_3 = 1)\), or
\[
pd(H_v) = n_1 - \left\lceil \frac{n_1}{3} \right\rceil + n_2 + n_3 + 1 - 1 - \left\lceil \frac{n_2 - 2}{3} \right\rceil
\]
= \(\sum_{i=1}^{3} (n_i - \left\lceil \frac{n_i}{3} \right\rceil) = pd(H_{S_n})\).
when \( n_2 = 1 \) and \( r_3 = 2 \), or
\[
\text{pd}(\mathcal{H}_v) = n_1 - \left\lfloor \frac{n_1}{3} \right\rfloor + n_2 + n_3 + 1 - \left\lfloor \frac{n_2 - 2}{3} \right\rfloor
\]
\[
= \sum_{i=1}^{3} (n_i - \left\lfloor \frac{n_i}{3} \right\rfloor) = \text{pd}(\mathcal{H}_{S_\mu})
\]
when \( n_3 = 1 \) and \( r_2 = 2 \). On the other hand, \( Q_v \) is a union of two isolated vertices and two strings of length, \( n_1 - 2, n_2 - 2 + n_3 + 1 \), hence we have
\[
\text{pd}(Q_v) = 2 + n_1 - 2 - \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + n_2 + n_3 - 1 - \left\lfloor \frac{n_2 + n_3 - 1}{3} \right\rfloor
\]
\[
\leq \sum_{i=1}^{3} (n_i - \left\lfloor \frac{n_i}{3} \right\rfloor) - 1 < \text{pd}(\mathcal{H}_v).
\]
For the second inequality above, we use the fact that \( r_2 + r_3 \geq 3 \).

For the case when \( k > 2 \), we use the same exact sequence. Here the hypergraph \( \mathcal{H}_v \) is a union of a string of \( n_1 \) vertices and a string of length \( \mu' = \sum_{i=2}^{k+1} n_i + k - 1 \) with a \( k - 2 \)-dimensional edge of \( k - 1 \) vertices such that \( \sum_{i=2}^{k+1} r_i \geq 2(k - 1) + 1 \). By the induction hypothesis and Theorem 2.3 (4),
\[
\text{pd}(\mathcal{H}_v) = n_1 - \left\lfloor \frac{n_1}{3} \right\rfloor + \text{pd}(\mathcal{H}_{S_\mu}) = \sum_{i=1}^{k+1} (n_i - \left\lfloor \frac{n_i}{3} \right\rfloor) = \text{pd}(\mathcal{H}_{S_\mu}).
\]
On the other hand, \( Q_v \) is a union of two isolated closed vertices and two strings of length \( n_1 - 2 \) and \( \sum_{i=2}^{k+1} n_i - 2 + k - 1 \). Hence we have
\[
\text{pd}(Q_v) = 2 + n_1 - 2 - \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + n_2 + n_3 - 1 - \left\lfloor \frac{n_2 + n_3 - 1}{3} \right\rfloor
\]
\[
= n_1 - \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + \sum_{i=2}^{k+1} n_i + k - 3 - \left\lfloor \sum_{i=2}^{k+1} n_i + k \right\rfloor
\]
\[
< \sum_{i=1}^{k+1} (n_i - \left\lfloor \frac{n_i}{3} \right\rfloor) = \text{pd} \mathcal{H}_v
\]
when \( r_1 = 2 \) and \( 2k + 2 \geq \sum_{i=1}^{k+1} r_i \geq 2k + 1 \). We conclude \( \text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}_v) = \text{pd}(\mathcal{H}_{S_\mu}) \).

Now we want to deal with the case when \( \sum_{i=1}^{k+1} r_i = 2k \). In this case we get two different outcomes, and it will be necessary to prove a number of lemmas that will allow us to work with the special case.

First we will need to to prove some results about when the spacing measured by the \( n_i \) is equivalent to 2 modulo 3. The following lemma deals with the case where one of the ends of the string coincides with a vertex from \( F \).

**Lemma 3.5.** We adapt Notation 3.1. Suppose the end vertices of the string are \( w_1 \) and \( w_\mu \) where \( w_1 \) is closed and \( w_\mu \) is open, and \( w_\mu = v_k \). If \( n_i = 2 + 3l_i \) for \( i = 1, \ldots, k \), then the projective dimension of \( \mathcal{H} \) is
\[
2k + 2 \sum_{i=1}^{k} l_i = \sum_{i=1}^{k} n_i + k - \left\lfloor \sum_{i=1}^{k} n_i + k \right\rfloor
\]
and
\[
\text{reg}(\mathcal{H}) \leq k + \sum_{i=1}^{k} l_i.
\]
Proof. We use induction on $k$. When $k = 1$, we have $w_\mu = v_1$. In this case the edge has only one vertex $v_1$ which forces $w_\mu = v_1$ to become a closed vertex. Also, $\mathcal{H}$ becomes a string of length $\mu = n_1 + 1$, so by Theorem 2.3 (4), \(pd(\mathcal{H}) = n_1 + 1 - \left\lfloor \frac{n_1 + 1}{3} \right\rfloor = 2 + 2l_1\) and \(\text{reg}(\mathcal{H}) = \left\lfloor \frac{n_1 + 1}{3} \right\rfloor = 1 + l_1\).

For the induction step, we consider the short exact sequence
\[
0 \to \mathcal{H} \to \mathcal{H}_{v_1} \to \mathcal{Q}_{v_1} \to 0.
\]
Since $k > 1$, the vertex $v_1$ corresponds to a monomial of degree 3. Notice that $\mathcal{H}_{v_1}$ is the union of a string of length $n_1$ and a hypergraph with exactly the same structure of $\mathcal{H}$ (i.e. closed vertex on one end of string and an open vertex coinciding with a vertex of the higher dimensional edge at the other end) such that it has an edge with $k - 1$ vertices. By Theorem 2.3 (4) and induction hypothesis, we have
\[
\text{pd}(\mathcal{H}_{v_1}) = n_1 - \left\lfloor \frac{n_1}{3} \right\rfloor + 2(k - 1) + 2 \sum_{i=2}^{k} l_i = 2k + 2 \sum_{i=1}^{k} l_i
\]
and
\[
\text{reg}(\mathcal{H}_{v_1}) \leq \left\lfloor \frac{n_1}{3} \right\rfloor + k - 1 + \sum_{i=2}^{k} l_i = k + \sum_{i=1}^{k} l_i.
\]

Moreover, the hypergraph $\mathcal{Q}_{v_1}$ is a union of three isolated vertices and two strings of length $n_2 - 2$ and $n_2 - 2 + n_k - 1 + k - 2 + \sum_{i=3}^{k-1} n_i$. Therefore by Theorem 2.3 (4) again, we have
\[
\text{pd}(\mathcal{Q}_v) = 3 + n_1 - 2 - \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + \sum_{i=2}^{k} n_i + k - 5 - \left\lfloor \sum_{i=2}^{k} \frac{n_i + k - 5}{3} \right\rfloor
\]
\[
= 2k + 2 \sum_{i=1}^{k} l_i - 1
\]
and
\[
\text{reg}(\mathcal{Q}_v) = \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + \left\lfloor \sum_{i=2}^{k} \frac{n_i + k - 5}{3} \right\rfloor = k + \sum_{i=1}^{k} l_i - 2.
\]

Using the short exact sequence on the projective dimension and Theorem 2.3 (9), we have
\[
\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}_{v_1}) = 2k + 2 \sum_{i=1}^{k} l_i
\]
and
\[
\text{reg}(\mathcal{H}) \leq \max\{k + \sum_{i=1}^{k} l_i, k + \sum_{i=1}^{k} l_i - 2 + 3 - 1\} = k + \sum_{i=1}^{k} l_i.
\]

\[
\square
\]

In the next lemma we will need to adapt our notation a bit as here we need to consider a specific case which is necessary for the proof of Lemma 3.7. Specifically this will address $\mathcal{H}_{v_2}$ in Lemma 3.7 where $\mathcal{H}$ is a string together with an edge consisting of $k$ vertices.

Lemma 3.6. Let $\mathcal{H}$ be a hypergraph such that it has two strings attached with an edge of $F$ consisting of $k - 1$ vertices, $v_1, \ldots, v_{k-1}$, and $k > 2$. Suppose the end vertices of the first string are $w_1$ and $w_t$ with $w_1$ open and $w_t$ closed. The vertices of the second string are $u_1$ and $u_s$ with $u_1$ closed and $u_s$ open. Moreover, let $w_1 = v_1$ and $u_s = v_{k-1}$. Assume that there are $n_2$ vertices between $v_1$ and $v_2$ including $u_1$ and $n_i$ vertices between $v_{i-1}$ and $v_i$ for $i = 3, \ldots, k - 1$. Let $n_1 = t - 1$ and if $n_i = 2 + 3l_i$
for all \( i = 1, \ldots, k - 1 \) where \( l_i \) are some non-negative integers. Then projective dimension of \( \mathcal{H} \) is 
\[
2(k - 1) + 2 \sum_{i=1}^{k-1} l_i \quad \text{and} \quad \text{reg}(\mathcal{H}) \leq k - 1 + \sum_{i=1}^{k-1} l_i.
\]

Proof. We use induction on \( k \). When \( k = 3 \), \( \mathcal{H} \) is a string of length \( n_1 + n_2 + 2 \). By Theorem 2.3(4)
\[
\text{pd}(\mathcal{H}) = n_1 + n_2 + 2 - \left\lfloor \frac{n_1 + n_2 + 2}{3} \right\rfloor = 4 + 2(l_1 + l_2) \quad \text{and} \quad \text{reg}(\mathcal{H}) = \left\lfloor \frac{n_1 + n_2 + 2}{3} \right\rfloor = 2 + l_1 + l_2.
\]
For the induction step, we consider the short exact sequence
\[
0 \rightarrow \mathcal{H} \leftarrow \mathcal{H}_{v_1} \leftarrow \mathcal{H}_{v_1} : v_1 = Q_{v_1} \rightarrow 0.
\]
The proof is almost identical to the Lemma 3.5 except that \( v_1 \) corresponds to a monomial of degree 2, \( \mathcal{H}_{v_1} \) is a union of a string and a hypergraph satisfying the assumptions of Lemma 3.5, and \( \text{reg}(Q_{v_1}) \leq k - 1 + \sum_{i=1}^{k-1} l_i - 1 \). Hence we have \( \text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}_{v_1}) = 2(k - 1) + 2 \sum_{i=1}^{k-1} l_i \) and \( \text{reg}(\mathcal{H}) \leq \max\{k - 1 + \sum_{i=1}^{k-1} l_i, k - 1 + \sum_{i=1}^{k-1} l_i - 1 + 2 - 1\} = k - 1 + \sum_{i=1}^{k-1} l_i \) by Theorem 2.3(9) \( \square \)

Now we will use the splitting type result in Lemma 2.10 to finish our necessary results for the hypergraphs which are a string together with an edge consisting of \( k \) vertices, where the spacing between the vertices of the edge are equivalent to 2 modulo 3.

**Lemma 3.7.** We adapt Notation 3.1. Assume \( w_1 = v_1 \) and \( w_{\mu} = v_k \). If \( n_i = 2 + 3l_i \) for \( l_i \). Then the projective dimension of \( \mathcal{H} \) is 
\[
2(k - 1) + 2 \sum_{i=2}^{k} l_i + 1.
\]

Proof. By making vertices \( v_1, \ldots, v_k \) become closed, we obtain a hypergraph \( \mathcal{H}' \) and \( \text{pd}(\mathcal{H}) \leq \text{pd}(\mathcal{H}') \) by Theorem 2.3(3). Let \( \mathcal{H}'' \) be the hypergraph obtained from \( \mathcal{H}' \) by removing the higher dimensional edge. Then by Theorem 2.3(2) we have that \( \text{pd}(\mathcal{H}') = \text{pd}(\mathcal{H}'') \) because all the vertices \( v_1, \ldots, v_k \) are closed in \( \mathcal{H}' \). Note that \( \mathcal{H}'' \) is a string of length \( k + \sum_{i=2}^{k} n_i \) with \( k - 1 \) open strings of \( n_2, \ldots, n_k \) open vertices. By Theorem 3.4 in [13], the projective dimension is the sum of projective dimension of each open string plus 1, hence we get
\[
\text{pd}(\mathcal{H}'') = \sum_{i=2}^{k} n_i - \sum_{i=2}^{k} \left\lfloor \frac{n_i}{3} \right\rfloor + 1 = 2(k - 1) + 2 \sum_{i=2}^{k} l_i + 1.
\]
Thus, we obtain \( \text{pd}(\mathcal{H}) \leq 2(k - 1) + 2 \sum_{i=2}^{k} l_i + 1 \).

Now we consider \( \mathcal{H} = \{v_2\} \cup \mathcal{H}_{v_2} \) and we will show it satisfies the condition of Remark 2.11 with \( V_1 = \{v_2\} \) and \( V_2 \) as the vertex set of \( \mathcal{H}_{v_2} \). Denote \( I_1 \) as the ideal generated by the generators of \( I(\mathcal{H}) \) corresponding to \( V_1 \), and \( G_1 = \{v_2\} = \mathcal{H}(I_1) \) and \( G_2 = \mathcal{H}_{v_2} = \mathcal{H}(I_2) \). First notice that \( \text{pd}(G_1) = 1 \) and \( \text{reg}(G_1) = 2 \), since the degree of the generator corresponding to \( v_2 \) is 3. Moreover \( G_2 \) satisfies the condition of Lemma 3.6 hence \( \text{pd}(G_2) = 2(k - 1) + 2 \sum_{i=2}^{k} l_i = q \) and \( \text{reg}(G_2) \leq k - 1 + \sum_{i=2}^{k} l_i = r - 1 \). By Lemma 2.12 \( I_1 \cap I_2 = m_{v_2} I' \) where \( \mathcal{H}(I') \) has 4 isolated vertices and two strings of length \( n_2 - 3 \) and \( k - 3 + n_k - 1 + n_3 - 2 + \sum_{i=3}^{k-1} n_i \). Then by Theorem 2.3(4) and (8)
\[
\text{pd}(\mathcal{H}(I_1 \cap I_2)) = 4 + n_2 - 3 - \left\lfloor \frac{n_2 - 3}{3} \right\rfloor + k - 6 + \sum_{i=3}^{k} n_i - \left\lfloor \frac{k - 6 + \sum_{i=3}^{k} n_i}{3} \right\rfloor
\]
\[
= 2(k - 1) + 2 \sum_{i=2}^{k} l_i = q
\]
and

\[ \text{reg}(\mathcal{H}(I_1 \cap I_2)) = 3 + \left\lfloor \frac{n_2 - 3}{3} \right\rfloor + \left\lfloor \frac{k - 6 + \sum_{i=3}^{k} n_i}{3} \right\rfloor \]

\[ = 3 + l_2 + k - 3 + \sum_{i=3}^{k} l_i = k + \sum_{i=2}^{k} l_i = r. \]

Moreover, by Theorem 2.3 (5) and (6), we have \( \beta_{q,q+r}(\mathcal{H}(I_1 \cap I_2)) \neq 0 \). Hence by Lemma 2.10, \( \text{pd}(\mathcal{H}) = 2(k - 1) + 2 \sum_{i=2}^{k} l_i + 1. \)

The following example offers a view of the "splitting" in the proof of Lemma 3.7.

**Example 3.8.** Let \( \mathcal{H} \) be the whole hypergraph in the left side of Figure 3. Let \( V_1 \) be the \( w_4 \) (black part) of \( \mathcal{H} \) and \( V_2 = \{w_1, w_2, w_3, w_5, w_6, w_7\} \) (blue part) of \( \mathcal{H} \). Let \( G_1 \) and \( G_2 \) be the hypergraphs associated to the vertex sets \( V_1 \) and \( V_2 \). Then the edges \( \{w_3, w_4\}, \{w_4, w_5\}, \) and \( \{w_1, w_4, w_7\} \) (the purple part) are the shared edges or edges of \( G_1 \) and \( G_2 \). In the right of Figure 3 we show the hypergraphs for \( G_1 \) and \( G_2 \) separately.

**Figure 3.**

Now with Lemma 3.7 we are ready to address the case when the sum of the \( n_i \) is \( 2k \) modulo 3. In this case Lemma 3.9 will be an instance of the special sub-case, and Proposition 3.10 will give the general result.

**Lemma 3.9.** We adapt Notation 3.1. If \( r_1 = 1 = r_{k+1} \), and \( r_i = 2 \) for all \( 1 < i < k+1 \), then \( \text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}_{S_p}) + 1. \)

**Proof.** First notice that \( \text{pd}(\mathcal{H}_{S_p}) = \sum_{i=1}^{k+1} n_i + k - \left\lfloor \frac{\sum_{i=1}^{k+1} n_i + k}{3} \right\rfloor. \)

Let \( x_E \) be the variable corresponding to the edge \( E_x \) that connecting \( v_1 \) and the vertex of \( w_{n_1} \) and \( y_E \) be the variable corresponding to the edge \( E_y \) that connecting \( v_k \) and the vertex of \( w_{k+\sum_{i=1}^{k+1} n_i+1} \). We consider the short exact sequences:

\[ 0 \leftarrow (\mathcal{H}, x_E) \leftarrow \mathcal{H} \leftarrow (\mathcal{H} : E_x) \leftarrow 0, \]

and

\[ 0 \leftarrow ((\mathcal{H} : E_x), y_E) \leftarrow (\mathcal{H} : E_x) \leftarrow ((\mathcal{H} : E_x) : E_y) \leftarrow 0. \]
Notice that \((\mathcal{H} : E_x) : E_y\) is a union of two isolated vertices, two strings of length \(n_1 - 2\) and \(n_{k+1} - 2\), and a hypergraph satisfies assumptions of Lemma 3.7. Now with assumptions of \(r_i\)'s, we have

\[
\text{pd}((\mathcal{H} : E_x) : E_y) = 2 + n_1 - 2 - \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + n_{k+1} - 2 - \left\lfloor \frac{n_{k+1} - 2}{3} \right\rfloor + 2(k - 1) + 2 \sum_{i=2}^{k} l_i + 1
\]

\[
= \sum_{i=1}^{k+1} n_i + k + 1 - \left\lfloor \frac{k+1}{3} \right\rfloor = \text{pd}(\mathcal{H}_{S_\mu}) + 1.
\]

Since \((\mathcal{H}, x_E)\) is a union of an isolated vertex, a string of length \(n_1 - 1\), and a string of length \(\sum_{i=1}^{k+1} n_i + k - 1\) with a \(k - 1\)-dimensional edge such that \(n_2, \ldots, n_{k+1}\) are the numbers of vertices between vertices of \(F\) and \(\sum_{i=2}^{k+1} r_i = 2(k - 1) + 1\). By Lemma 3.4 and Theorem 2.3(4)

\[
\text{pd}(\mathcal{H}, x_E) = 1 + n_1 - 1 - \left\lfloor \frac{n_1 - 1}{3} \right\rfloor + \sum_{i=2}^{k+1} n_i + k - 1 - \left\lfloor \frac{k+2}{3} \right\rfloor < \sum_{i=1}^{k+1} n_i + k + 1 - \left\lfloor \frac{k+1}{3} \right\rfloor.
\]

Moreover, \((\mathcal{H} : E_x), y_E)\) is a union of two isolated vertices, two strings of length \(n_1 - 2\) and \(n_{k+1} - 1\) and a hypergraph satisfies the assumptions of Lemma 3.5 with a \(k - 1\) edge. Then by Lemma 3.5 and Theorem 2.3(4)

\[
\text{pd}((\mathcal{H} : E_x), y_E) = 2 + n_1 - 2 - \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + n_{k+1} - 1 - \left\lfloor \frac{n_{k+1} - 1}{3} \right\rfloor + \sum_{i=2}^{k} n_i + k - 1 - \left\lfloor \frac{k}{3} \right\rfloor < \sum_{i=1}^{k+1} n_i + k + 1 - \left\lfloor \frac{k+1}{3} \right\rfloor = \text{pd}((\mathcal{H} : E_x) : E_y).
\]

Since \(\text{pd}((\mathcal{H} : E_x) : E_y) > \text{pd}((\mathcal{H} : E_x), y_E)\), we have

\[
\text{pd}(\mathcal{H} : E_x) \leq \max\{\text{pd}((\mathcal{H} : E_x) : E_y), \text{pd}((\mathcal{H} : E_x), y_E)\} = \text{pd}((\mathcal{H} : E_x) : E_y)
\]

and \(\text{pd}((\mathcal{H} : E_x) : E_y) \leq \max\{\text{pd}(\mathcal{H} : E_x), \text{pd}((\mathcal{H} : E_x), y_E) - 1\}. \) This shows \(\text{pd}(\mathcal{H} : E_x) = \text{pd}((\mathcal{H} : E_x) : E_y)\). Similarly \(\text{pd}(\mathcal{H}, x_E) < \text{pd}(\mathcal{H} : E_x)\) gives \(\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H} : E_x) = \text{pd}(\mathcal{H}_{S_\mu}) + 1\).

We are now finally ready for the case, \(\sum_{i=1}^{k+1} r_i = 2k\).

**Proposition 3.10.** We adapt Notation 3.1 and assume \(\sum_{i=1}^{k+1} r_i = 2k\) with \(r_1 = 2\). If \(r_i \neq 0\) for all \(i\), then \(\text{pd}(\mathcal{H}) = (\text{pd}(\mathcal{H}_{S_\mu}) + 1, otherwise \text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}_{S_\mu})\).

**Proof.** We consider the short exact sequence

\[
0 \leftarrow \mathcal{H} \leftarrow \mathcal{H}_{v_1} \leftarrow \mathcal{Q}_{v_1} \leftarrow 0.
\]
We will show \( \text{pd}(\mathcal{H}_{v_1}) > \text{pd}(\mathcal{Q}_{v_1}) \). Apply the projective dimension on the short exact sequence, we have

\[
\text{pd}(\mathcal{H}_{v_1}) \leq \max\{\text{pd}(H), \text{pd}(\mathcal{Q}_{v_1})\} = \text{pd}(H)
\]

and

\[
\text{pd}(H) \leq \max\{\text{pd}(\mathcal{H}_{v_1}), \text{pd}(\mathcal{Q}_{v_1}) + 1\} = \text{pd}(\mathcal{H}_{v_1}).
\]

Moreover, we will show that when \( r_i \neq 0 \) for all \( i \), then \( \text{pd}(\mathcal{H}_{v_1}) = \text{pd}(H_{S_n}) + 1 \), and otherwise \( \text{pd}(\mathcal{H}_{v_1}) = \text{pd}(H_{S_n}) \). These two claims will prove the proposition. We proceed by induction on \( k \) for both claims.

When \( k = 3 \), observe that by Definition 2.6 and Discussion 2.8 in [16], \( \mathcal{Q}_{v_1} \) is a union of two isolated vertices, and two strings of length \( n_1 - 2 \) and \( n_2 - 2 + n_3 + 2 + n_4 \). Hence by Theorem 2.6 (4)

\[
\text{pd}(\mathcal{Q}_{v_1}) = 2 + n_1 - 2 - \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + n_2 + n_3 + n_4 - \left\lfloor \frac{n_2 + n_3 + n_4}{3} \right\rfloor < \sum_{i=1}^{4} n_i + 3 - \left\lfloor \sum_{i=1}^{4} n_i + 3 \right\rfloor
\]

Notice \( H_{v_1} \) is a union of a string of length \( n_1 \) and a cycle of length \( n_3 + 2 \) with two branches of length \( n_2 \) and \( n_4 \). By Theorem 2.6

\[
\text{pd}(H_{v_1}) = n_1 - \left\lfloor \frac{n_1}{3} \right\rfloor + n_2 - \left\lfloor \frac{n_2}{3} \right\rfloor + n_3 + 1 - \left\lfloor \frac{n_3}{3} \right\rfloor + n_4 - \left\lfloor \frac{n_4}{3} \right\rfloor = \sum_{i=1}^{4} n_i + 3 - \left\lfloor \sum_{i=1}^{4} n_i + 3 \right\rfloor = \text{pd}(H_{S_n}) + 1
\]

when \( r_i \neq 0 \) for all \( i \). When \( r_i = 0 \) for some \( i \), we may assume \( r_1 = 2 \) and \( r_3 = 2 \) by symmetry. Hence by Theorem 2.6 again,

\[
\text{pd}(H_{v_1}) = n_1 - \left\lfloor \frac{n_1}{3} \right\rfloor + n_2 - \left\lfloor \frac{n_2}{3} \right\rfloor + n_3 + 1 - \left\lfloor \frac{n_3 + 1}{3} \right\rfloor + n_4 - \left\lfloor \frac{n_4}{3} \right\rfloor = \sum_{i=1}^{4} n_i + 3 - \left\lfloor \sum_{i=1}^{4} n_i + 3 \right\rfloor = \text{pd}(H_{S_n}).
\]

In both cases, we have \( \text{pd}(H_{v_1}) > \text{pd}(\mathcal{Q}_{v_1}) \) and this concludes the case when \( k = 3 \).

Now suppose \( F \) is a \( k \)-dimensional edge with \( k + 1 \) vertices, \( v_1, ..., v_{k+1} \). Suppose \( r_i \neq 0 \) for all \( i \) then again by Definition 2.6 and discussion 2.8 in [16], \( \mathcal{Q}_{v_1} \) is either

1. a union of two isolated vertices, a string of length \( n_1 - 2 \) and a string of length \( \sum_{i=2}^{k+2} n_i + k \) when \( n_2 \geq 2 \), or
2. \( \mathcal{Q}_{v_1} \) is a union of two isolated vertices, a string of length \( n_1 - 2 \) and a string of length \( \sum_{i=3}^{k+2} n_i + k - 1 \) when \( n_2 = 1 \).

For the latter case, \( \sum_{i=3}^{k+2} n_i = 2k - 1 \), then by Theorem 2.3 (4)

\[
\text{pd}(\mathcal{Q}_{v_1}) = 2 + n_1 - 2 - \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + \sum_{i=3}^{k+2} n_i + k - 1 - \left\lfloor \sum_{i=3}^{k+2} n_i + k - 1 \right\rfloor < \sum_{i=1}^{k+2} n_i + 2 - \left\lfloor \sum_{i=1}^{k+2} n_i + k + 1 \right\rfloor.
\]
For the first case, by Theorem 2.3 (4), we have
\[
\text{pd}(Q_{v_1}) = 2 + n_1 - 2 - \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + \sum_{i=2}^{k+2} n_i + k - 2 - \left\lfloor \sum_{i=2}^{k+2} n_i + k - 2 \right\rfloor
\]
\[< \sum_{i=1}^{k+2} n_i + k + 2 - \left\lfloor \sum_{i=1}^{k+2} n_i + k + 1 \right\rfloor \]
with the fact \( r_1 + \ldots + r_{k+2} = 2k + 2 \) and \( r_1 = 2 \).

Now in both cases consider that \( H_{v_1} \) is a union of a string of length \( n_1 \) and a string of length \( \sum_{i=2}^{k+2} n_i + k \) with a \( k \) - 1-dimensional edge. Notice that \( \sum_{i=2}^{k+2} r_i = 2k \) and \( r_i \neq 0 \) for all \( i \) such that \( 2 \leq i \leq k + 2 \). By induction and Theorem 2.3 (4), we have
\[
\text{pd}(H_{v_1}) = n_1 - \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + \sum_{i=2}^{k+2} n_i + k - 1
\]
\[= \text{pd}(\mathcal{H}_{S_{\mu}}) + 1
\]
\[= \sum_{i=1}^{k+2} n_i + k + 2 - \left\lfloor \sum_{i=1}^{k+2} n_i + k + 1 \right\rfloor \]
where we use the fact that \( r_1 + \ldots + r_{k+2} = 2k + 2 \) and \( r_1 = 2 \). Therefore \( \text{pd}(Q_{v_1}) < \text{pd}(H_{v_1}) \), and \( \text{pd}(H) = \text{pd}(H_{v_1}) = \text{pd}(\mathcal{H}_{S_{\mu}}) + 1 \), satisfying the 2 claims.

Now suppose \( r_i = 0 \) for some \( i > 1 \) then \( r_j = 2 \) for all \( j \neq i \). Notice again by the discussions in [16] that \( Q_{v_1} \) is either

1. a union of two isolated vertices, a string of length \( n_1 - 2 \) and a string of length \( \sum_{i=2}^{k+2} n_i + k - 2 \) when \( n_2 \geq 2 \), or
2. \( Q_{v_1} \) is a union of two isolated vertices, a string of length \( n_1 - 2 \) and a string of length \( \sum_{i=3}^{k+2} n_i + k - 2 \) when \( n_2 = 0 \).

For the later case, we have \( \sum_{i=3}^{k+2} r_i = 2k \). By Theorem 2.3 (4),
\[
\text{pd}(Q_{v_1}) = 2 + n_1 - 2 - \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + \sum_{i=3}^{k+2} n_i + k - 2 - \left\lfloor \sum_{i=3}^{k+2} n_i + k - 2 \right\rfloor
\]
\[< \sum_{i=1}^{k+2} n_i + k + 1 - \left\lfloor \sum_{i=1}^{k+2} n_i + k + 1 \right\rfloor .
\]

For the first case, by Theorem 2.3 (4), we have
\[
\text{pd}(Q_{v_1}) = 2 + n_1 - 2 - \left\lfloor \frac{n_1 - 2}{3} \right\rfloor + \sum_{i=2}^{k+2} n_i + k - 2 - \left\lfloor \sum_{i=2}^{k+2} n_i + k - 2 \right\rfloor
\]
\[< \sum_{i=1}^{k+2} n_i + k + 1 - \left\lfloor \sum_{i=1}^{k+2} n_i + k + 1 \right\rfloor \]
with the fact that \( r_1 + \ldots + r_{k+2} = 2k + 2 \) and \( r_1 = 2 \). Similar to the case where \( r_i \) is never 0, we get that \( H_{v_1} \) is a union of a string of length \( n_1 \) and a string of length \( \sum_{i=2}^{k+2} n_i + k \) with a \( k \) - 1-dimensional edge. Notice that \( \sum_{i=2}^{k+2} r_i = 2k \) and \( r_i = 0 \) for some \( 2 \leq i \leq k + 2 \). So by induction and Theorem 2.3
we have
\[
\text{pd}(H_{v_1}) = n_1 - \left\lfloor \frac{n_1}{3} \right\rfloor + \sum_{i=2}^{k+2} n_i + k - \left\lfloor \frac{\sum_{i=2}^{k+2} n_i + k}{3} \right\rfloor
\]
\[
= \text{pd}(S_\mu)
\]
\[
= \sum_{i=1}^{k+2} n_i + k + 1 - \left\lfloor \frac{\sum_{i=1}^{k+2} n_i + k + 1}{3} \right\rfloor
\]
where we use the fact that \( r_1 + \ldots + r_{k+2} = 2k + 2 \) and \( r_1 = 2 \). Hence \( \text{pd}(Q_{v_1}) < \text{pd}(H_{v_1}) \), and \( \text{pd}(H) = \text{pd}(H_{v_1}) = \text{pd}(H_{S_\mu}) \), thus finishing the proof.

Now tying together Proposition 3.3, Proposition 3.4, and Proposition 3.10 we can prove the following result.

**Theorem 3.11.** Let \( H_{S_\mu} \) be a string hypergraph such that it has \( \mu \) vertices with all vertices are open except the end vertices. We adapt Notation 3.7. If \( r_1 + \ldots + r_{k+1} = 2k \) and \( r_i \neq 0 \) for all \( i \), then \( \text{pd}(H) = \text{pd}(H_{S_\mu}) + 1 \) otherwise \( \text{pd}(H) = \text{pd}(H_{S_\mu}) = \mu - \left\lfloor \frac{4}{3} \right\rfloor \).

**Proof.** The final equality is coming from Theorem 2.3(4). For the cases when \( r_1 + \ldots + r_{k+1} < 2k \) or \( r_1 + \ldots + r_{k+1} > 2k \), we have \( \text{pd}(H) = \text{pd}(H_{S_\mu}) \) by Proposition 3.3 and Proposition 3.4. We are left to consider the case, \( r_1 + \ldots + r_{k+1} = 2k \). Notice the assumptions of Lemma 3.9 gives \( r_1 + \ldots + r_{k+1} = 2k \) and \( r_i \neq 0 \) for all \( i \). Hence by Lemma 3.9 and Proposition 3.10, the conclusion follows.

**Remark 3.12.** With the theorem above, one can easily compute the projective dimension of a string with extra edges. We first remove all the edges that are union of other edges using Theorem 2.3(2). If we are left with one higher dimensional edge, we can use Theorem 3.11 and Theorem 2.3(4) to compute the projective dimension. The example below illustrates the process.

**Example 3.13.** Let \( H \) be the hypergraph shown in Figure 4. We can remove all red edges using Theorem 2.3(2) and remove the blue edge using Theorem 3.11. The projective dimension of the hypergraph is \( 11 - \left\lfloor \frac{11}{3} \right\rfloor = 8 \).

**FIGURE 4.**

4. CYCLES WITH HIGHER DIMENSIONAL EDGES

Now we examine the case where we have added a higher dimensional edge to an open cycle.

**Notation 4.1.**

1. When we say \( H \) is an open cycle together with an edge consisting of \( k \) vertices, we mean that an edge \( F \) with \( k \) vertices \( \{v_1, \ldots, v_k\} \) is attached to an open cycle where we use the notation \( \{w_1, \ldots, w_\mu\} \) to denote the vertices in the cycle, \( H_{C_\mu} \).
(2) We use $n_1$ to represent the number of vertices from $w_1$ to $v_1$ including $w_1$, $n_i$ to represent the vertices between $v_{i-1}$ and $v_i$ for $i = 2, \ldots, k$, and $n_{k+1}$ to represent the number of vertices between $v_k$ and $w_\mu$ including $w_\mu$.

(3) We write $n_i = 3l_i + r_i$ where $l_i$ are some non-negative integers, and $0 \leq r_i \leq 2$ for $i = 1, \ldots, k + 1$.

We start by showing that the induced hypergraph of $H_{C_\mu}$ on the complement of $V_F$ has smaller projective dimension than that of $H_{C_\mu}$ when the sum of the $n_i$ modulo 3 is less than $2k - 1$. This is necessary in the proof of Theorem 4.4.

**Lemma 4.2.** We adapt Notation 4.7. Let $H_{V_F} = H_{C_\mu} \cap (V \setminus V_F)$ be the hypergraph obtained by removing all the vertices $v_1, \ldots, v_k$. If $r_1 + \ldots + r_k < 2k - 1$, then $pd(H_{V_F}) < pd(H_{C_\mu})$.

**Proof.** Notice that $\mu = \sum_{i=1}^k n_i + k$ and
\[
pd(H_{C_\mu}) = \mu - 1 - \left\lfloor \frac{\mu - 2}{3} \right\rfloor
= \sum_{i=1}^k n_i + k - 1 - \left\lfloor \frac{\sum_{i=1}^k n_i + k - 2}{3} \right\rfloor
\]
by Theorem 2.3(7). On the other hand, $pd(H_{V_F}) = \sum_{i=1}^k (n_i - \left\lfloor \frac{n_i}{3} \right\rfloor)$ because $H_{V_F}$ is union of $k$ strings such that each string has $n_i$ vertices. Notice that this notation allows that $n_i$ can be 0 for some $i$. It is sufficient to show that
\[
\sum_{i=1}^k (n_i - \left\lfloor \frac{n_i}{3} \right\rfloor) < \sum_{i=1}^k n_i + k - 1 - \left\lfloor \frac{\sum_{i=1}^k n_i + k - 2}{3} \right\rfloor
\]
which is equivalent to showing
\[
\left\lfloor \frac{\sum_{i=1}^k n_i + k - 2}{3} \right\rfloor < \sum_{i=1}^k \left\lfloor \frac{n_i}{3} \right\rfloor + k - 1
\]
or
\[
\left\lfloor \frac{\sum_{i=1}^k r_i + k - 2}{3} \right\rfloor < k - 1,
\]
which is true by the assumption $\sum_{i=1}^k r_i < 2k - 1$. \hfill $\square$

Next, we show that when the sum of the $n_i$ modulo 3 is greater than $2k - 1$ that the projective dimension of $H$ is the same as for the underlying cycle.

**Lemma 4.3.** We adapt Notation 4.7. If $\sum_{i=1}^k r_i \geq 2k - 1$, then $pd(H) = pd(H_{C_\mu})$.

**Proof.** By Theorem 2.3(7), we have
\[
pd(H_{C_\mu}) = \sum_{i=1}^k n_i + k - 1 - \left\lfloor \frac{\sum_{i=1}^n n_i + k - 2}{3} \right\rfloor
= \sum_{i=1}^k n_i + k - 1 - \left\lfloor \frac{\sum_{i=1}^k n_i + k - 1}{3} \right\rfloor
\]
when $2k \geq \sum_{i=1}^k r_i \geq 2k - 1$. Since $\sum_{i=1}^k r_i \geq 2k - 1$, $r_i < 3$, and $k > 2$, we have at most one $r_i$ such that $r_i = 1$. We may assume $r_k = 1$ if there is one otherwise $r_i = 2$ for all $i$. Let $r_1 = 2$ and $r_2 = 2$ and
vertex \( v_1 \) is the vertex between between those vertices. Let \( H_{v_1} = H_v \) be the hypergraph removing the vertex \( v_1 = v \) from \( H \) and let \( H_v : v_1 = Q_v. \) We have a short exact sequence

\[
0 \leftarrow H \leftarrow H_v \leftarrow Q_v \leftarrow 0.
\]

We will show that \( \text{pd}(H_v) = \sum_{i=1}^{k} n_i + k - 1 - \left\lfloor \frac{\sum_{i=1}^{k} n_i + k - 1}{3} \right\rfloor > \text{pd}(Q_v). \) Then with the facts that \( \text{pd}(H_v) \leq \max\{\text{pd}(Q_v), \text{pd}(H)\} \) and

\[
\text{pd}(H) \leq \max\{\text{pd}(H_v), \text{pd}(Q_v) + 1\},
\]

we conclude that \( \text{pd}(H) = \text{pd}(H_v) = \text{pd}(H_{C_{\mu}}). \)

Notice that \( H_v \) is a string of length \( \mu - 1 = \sum_{i=1}^{k} n_i + k - 1 \) attached with a \( k - 2 \)-dimensional edge of \( k - 1 \) vertices. Moreover, \( \sum_{i=1}^{k} n_i \geq 2k - 1 = 2(k - 1) + 1. \) By Proposition 3.4

\[
\text{pd}(H_v) = \text{pd}(H_{S_{\mu-1}}) = \sum_{i=1}^{k} n_i + k - 1 - \left\lfloor \frac{\sum_{i=1}^{k} n_i + k - 1}{3} \right\rfloor.
\]

On the other hand, \( Q_v \) is the union of two closed vertices and a string of length \( n_1 - 2 + \sum_{i=2}^{k} n_i + k - 1 - 2 \) hence by Theorem 2.3(4)

\[
\text{pd}(Q_v) = 2 + \sum_{i=1}^{k} n_i + k - 5 - \left\lfloor \frac{\sum_{i=2}^{k} n_i + k - 5}{3} \right\rfloor
= \sum_{i=1}^{k} n_i + k - 1 - \left\lfloor \frac{\sum_{i=1}^{k} n_i + k - 2}{3} \right\rfloor - 1 < \text{pd}(H_v)
\]

when \( r_1 = r_2 = 2 \) and \( 2k \geq \sum_{i=1}^{k} r_i \geq 2k - 1. \)

Now we can show that the projective dimension is always preserved \( H \) and \( \text{pd}(H_{C_{\mu}}) \) differ by a single higher dimensional edge \( F. \)

**Theorem 4.4.** Let \( H_{C_{\mu}} \) be the cycle hypergraph with \( \mu \) open vertices. Let \( H = H_{C_{\mu}} \cup F \) where \( F \) is a \( k - 1 \)-dimensional edge with \( k \) vertices. Then \( \text{pd}(H) = \text{pd}(H_{C_{\mu}}) = \mu - 1 - \left\lfloor \frac{\mu - 2}{3} \right\rfloor. \)

**Proof.** The second equality is coming from Theorem 2.3(7) Let \( V_F = \{v_1, \ldots, v_k\}, \) which is a subset of the vertex set of \( H_{C_{\mu}}, \) be the vertex set of \( F. \) Let \( n_i = 3l_i + r_i \) be the spacing between the \( v_i \) and defined as before. If \( \sum_{i=1}^{k} r_i \geq 2k - 1 \) then the theorem holds by Lemma 4.3 So we need only consider the case when \( \sum_{i=1}^{k} r_i < 2k - 1. \)

When \( \sum_{i=1}^{k} r_i < 2k - 1, \) we will use Lemma 4.2 and the short exact sequence

\[
0 \leftarrow (H, x_F) \leftarrow (H) \leftarrow (H : F) = H_{C_{\mu}} \leftarrow 0
\]

where \( x_F \) is the variable corresponding to the edge \( F. \) The short exact sequence gives \( \text{pd}(H) \leq \max\{\text{pd}(H_{C_{\mu}}), \text{pd}(H, x_F)\}. \) Since \( (H, x_F) \) is the union of \( H_{V_F} \) and an isolated vertex presenting the variable of \( x_F, \) we have \( \text{pd}(H, x_F) = \text{pd} H_{V_F} + 1 \leq \text{pd} H_{C_{\mu}} \) by Lemma 4.2 By Theorem 2.3(3) we have to \( \text{pd} H_{C_{\mu}} \leq \text{pd} H. \) Therefore we have

\[
\text{pd} H \leq \max\{\text{pd}(H_{C_{\mu}}), \text{pd}(H, x_F)\} = \text{pd} H_{C_{\mu}} \leq \text{pd} H.
\]

\[\square\]

**Remark 4.5.** As before, one can compute the projective dimension of an open cycle with extra edges either using Theorem 2.3(2) or above theorem.
Remark 4.6. It is natural to conjecture that given an open cycle, no matter how many higher dimensional edges are on the cycle the projective dimension of the hypergraph is the same as the projective dimension of the open cycle. There are more cases that one needs to consider for the proof of the conjecture. In particular, in Theorem 3.11 we have a case for strings where the projective dimension does not stay the same once a higher dimensional edge is removed. The proof of the case when the projective dimension jumped up by one requires a lot of steps. In light of this, new tools must be developed in order to prove the conjecture for cycles.

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