Stability of Schwarzschild-like solutions in $f(R, \mathcal{G})$ gravity models

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We study linear metric perturbations around a spherically symmetric static spacetime for general $f(R, \mathcal{G})$ theories, where $R$ is the Ricci scalar and $\mathcal{G}$ is the Gauss-Bonnet term. We find that unless the determinant of the Hessian of $f(R, \mathcal{G})$ is zero, even-type perturbations have a ghost for any multi-pole mode. In order for these theories to be plausible alternatives to General Relativity, the theory should satisfy the condition that the ghost is massive enough to effectively decouple from the other fields. We study the requirement on the form of $f(R, \mathcal{G})$ which satisfies this condition. We also classify the number of propagating modes both for the odd-type and the even-type perturbations and derive the propagation speeds for each mode.

I. INTRODUCTION

In cosmology, modified gravity models have been considered as a possible dynamical explanation for dark energy. These models have been constructed in order to give late time acceleration, however it is utterly important to see whether such modifications of gravity will also drastically change the behavior of gravity at small scales, e.g. in our solar system. Among these models, some made use of the extra-dimensions [1–4]. Others implemented a similar mechanism by introducing a modified kinetic-term Lagrangian for scalar field non-minimally coupled with gravity [5–10]. Most of these new theories of gravity try not to introduce gravitational ghosts in the spin-2 sector. Another possibility, which we consider in this paper, is to require the Lagrangian for the gravitational sector to be a function of the Lovelock scalars only [11–22].

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$$S = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} f(R, \mathcal{G}).$$

The $f(R)$ theories represent a subset of this general model [23–31] with an interesting phenomenology [32–36]. This Lagrangian can be rewritten in terms of Lagrange multipliers as

$$S = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} [F R + \xi \mathcal{G} - U(F, \xi)],$$

where one scalar field is coupled to $R$, the Ricci scalar, and the other to $\mathcal{G} \equiv R^2 - 4R_{\alpha \beta}R^{\alpha \beta} + R_{\alpha \beta \mu \nu}R^{\alpha \beta \mu \nu}$, the Gauss-Bonnet combination.

Recently some papers appeared which shed some light on this class of theories. In particular it was shown that on Friedmann-Lemaitre-Robertson-Walker (FLRW) backgrounds, there is a new (compared to General Relativity) gravitational scalar mode which propagates with a scale dependent speed of propagation. That is, its dispersion relation behaves like $\omega^2 \propto k^4$ for large values of the wave vector $k$ [18–21]. This behavior changes the effective gravitational constant for the matter perturbation at low redshifts. The reason for the appearance of this $k^4$ term was explained in Ref. [22] by studying another background, the Kasner spacetime. In fact, on this background, together with an odd mode, there are three even modes $\psi_i$. The determinant of the kinetic matrix for the even modes $A (\mathcal{L} \ni A_{ij} \psi_i \psi_j)$ vanishes in the FLRW limit. Then, in this limit, one of the fields can be integrated out from the Lagrangian, giving rise to a term proportional to $k^4$. This result shows that the number of the degrees of freedom for this theory depends on the background, in particular it depends on the symmetries of the background. Furthermore, on this same background, it was shown that, unless the Kasner manifold is very close to a FLRW one, a propagating ghost is always present. In fact, in the FLRW limit, it is this ghost that can be integrated out from the Lagrangian as its mass becomes infinitely large.

In this paper we will discuss the behavior of the perturbation about a spherically symmetric static vacuum background, whose metric can be written as

$$ds^2 = g^{00}_\mu dx^\mu dx^\nu = -A(r) dt^2 + \frac{dr^2}{B(r)} + \frac{r^2 dz^2}{1 - z^2} + r^2 (1 - z^2) d\varphi^2,$$

where $g^{00}$ is the determinant of the metric tensor.

In general, the perturbations of a static, spherically symmetric metric can be decomposed into spherical harmonics $Y_l^m(r, \varphi)$, where $l$ is the angular momentum and $m$ is the magnetic quantum number. The perturbation of the metric can be written as

$$ds^2 = g^{00}_\mu dx^\mu dx^\nu = -A(r) dt^2 + \frac{dr^2}{B(r)} + \frac{r^2 dz^2}{1 - z^2} + r^2 (1 - z^2) d\varphi^2 + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{r^2} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \right) Y_l^m(r, \varphi) dt^2 + \frac{dr^2}{B(r)} + \frac{r^2 dz^2}{1 - z^2} + r^2 (1 - z^2) d\varphi^2 + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{r^2} \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \varphi} \right) Y_l^m(r, \varphi) dt^2 + \frac{dr^2}{B(r)} + \frac{r^2 dz^2}{1 - z^2} + r^2 (1 - z^2) d\varphi^2.$$
where \( z \equiv \cos \theta \), and \( \theta, \phi \) are the standard spherical coordinates. So far, people have considered the way to constrain these theories by looking at the background solutions only (approximate or not), as this step is necessary to check if local gravity constraints are satisfied. However, it is important to understand, and this is the goal of this study of ours, whether these backgrounds are stable or not against linear perturbations, and what we can learn in terms of speed of propagation and ghost issues for the scalar gravitational modes. For such a theory, the background equations of motion read

\[
U = - \frac{4B \xi A'}{A r^2} + \frac{12B^2 \xi A'}{A r^2} + \frac{12B F'}{A r^2} - \frac{2B F A'}{A r} + \frac{2F}{r^2} + \frac{2B F}{r^2},
\]

\[
F'' = - \frac{2B \xi}{r^2 B} - \frac{2B}{2B} + B \xi'' - \frac{4G''}{r^2} - \frac{F A'}{B A r^2} - \frac{6G B A'}{2A} + \frac{F A'}{A r} + \frac{6B' \xi}{r^2},
\]

\[
R = \frac{\partial U}{\partial F},
\]

\[
G = \frac{\partial U}{\partial \xi},
\]

where \( ' \) stands for differentiation with respect to \( r \). We will see that, similarly to the Kasner background, also in the present case in general, there is one odd mode and three even modes (under the parity transformation, \( \theta \rightarrow \pi - \theta \) and \( \phi \rightarrow \phi + \pi \)). As for the even modes, the kinetic matrix in general has one negative eigenvalue, i.e. the theory does possess a ghost unless 1) the theory reduces to a subclass (to which \( f(R) \) belongs) which satisfies the equality \( f_{,R} R D_g - f_{,R G} = 0 \); or 2) the background manifold is more symmetrical than Schwarzschild, i.e. Minkowski or de Sitter. These theories then face the problem of having a propagating ghost for physical backgrounds in the scalar sector. It is however possible that for some models introduced to explain dark energy, this mode becomes so highly massive that it decouples from the relevant degrees of freedom at low energy.

### II. BRIEF REVIEW OF THE REGGE-WHEELER-ZERILLI FORMALISM

Before studying the metric perturbation of a spherically symmetric static spacetime for \( f(R,G) \) theories, let us briefly review the formalism developed by Regge, Wheeler [37], and Zerilli [38] to decompose the metric perturbations according to their transformation properties under two-dimensional rotations. Although Regge, Wheeler and Zerilli considered the perturbation of the Schwarzschild spacetime (namely GR), the formalism solely relies on the properties of spherical symmetry and can be applied to \( f(R,G) \) theories as well.

Let us denote the metric slightly perturbed from a spherically symmetric static spacetime by \( g_{\mu\nu} = g_{\mu\nu}^0 + h_{\mu\nu} \). Hence \( h_{\mu\nu} \) represent infinitesimal quantities. Then, under two-dimensional rotations on a sphere, \( h_{tt}, h_{tr} \) and \( h_{rr} \) transform as scalars, \( h_{ta} \) and \( h_{ra} \) transform as vectors and \( h_{ab} \) transforms as a tensor \((a,b)\) is either \( \theta \) or \( \phi \). Any scalar \( s \) can be decomposed into the sum of spherical harmonics as

\[
s(t, r, \theta, \phi) = \sum_{\ell, m} s_{\ell m}(t, r) Y_{\ell m}(\theta, \phi),
\]

Any vector \( V_a \) can be decomposed into a divergence part and a divergence-free part as follows:

\[
V_a(t, r, \theta, \phi) = \nabla_a \Phi_1 + E_a^b \nabla_b \Phi_2,
\]

where \( \Phi_1 \) and \( \Phi_2 \) are scalars and \( E_{ab} \equiv \sqrt{\det g} \epsilon_{ab} \) with \( \epsilon_{ab} \) being the two-dimensional metric on the sphere and \( \epsilon_{ab} \) being the totally anti-symmetric symbol with \( \epsilon_{\theta\phi} = 1 \). Here \( \nabla_a \) represents the covariant derivative with respect to the metric \( g_{ab} \). Since \( V_a \) is a two-component vector, it is completely specified by the quantities \( \Phi_1 \) and \( \Phi_2 \). Then we can apply the scalar decomposition [8] to \( \Phi_1 \) and \( \Phi_2 \) to decompose the vector quantity \( V_a \) into spherical harmonics.

Finally, any symmetric tensor \( T_{ab} \) can be decomposed as

\[
T_{ab}(t, r, \theta, \phi) = \nabla_a \nabla_b \Psi_1 + \gamma_{ab} \Psi_2 + \frac{1}{2} (E_a^c \nabla_c \Psi_3 + E_b^c \nabla_c \Psi_3),
\]

where \( \Psi_1, \Psi_2 \) and \( \Psi_3 \) are scalars. Since \( T_{ab} \) has three independent components, \( \Psi_1, \Psi_2 \) and \( \Psi_3 \) completely specify \( T_{ab} \). Then we can again apply the scalar decomposition [8] to \( \Psi_1, \Psi_2 \) and \( \Psi_3 \) to decompose the tensor quantity into spherical harmonics. We refer to the variables accompanied by \( E_{ab} \) by odd-type variables and the others by even-type variables.

What makes these decompositions useful is that in the linearized equations of motion (or equivalently, in the second order action) for \( h_{\mu\nu} \), odd-type perturbations and even-type ones completely decouple from each other, reflecting the invariance of the background spacetime under parity transformation. Therefore, we can study odd-type perturbations and even-type ones separately as we will do in the following.
III. PERTURBATION IN $f(R,G)$ THEORIES

A. The odd modes

Using the Regge-Wheeler formalism, the odd-type metric perturbations can be written as

$$h_{tt} = 0, \quad h_{tr} = 0, \quad h_{rr} = 0,$$

$$h_{1a} = \sum_{\ell,m} h_{0,\ell m}(t,r) E_{ab} \partial^b Y_{\ell m}(\theta, \varphi),$$

$$h_{ra} = \sum_{\ell,m} h_{1,\ell m}(t,r) E_{ab} \partial^b Y_{\ell m}(\theta, \varphi),$$

$$h_{ab} = \frac{1}{2} \sum_{\ell,m} h_{2,\ell m}(t,r) \left[ E_a^c \nabla_c \nabla_b Y_{\ell m}(\theta, \varphi) + E_b^c \nabla_c Y_b Y_{\ell m}(\theta, \varphi) \right].$$

Because of general covariance, not all the metric perturbations are physical in the sense that some of them can be set to vanish by using the gauge transformation $x^\mu \rightarrow x^\mu + \xi^\mu$, where $\xi^\mu$ are infinitesimal. For the odd-type perturbation, we can consider the following gauge transformation:

$$\xi_t = 0, \quad \xi_a = \sum_{\ell,m} \Lambda_{\ell m}(t,r) E_a b \nabla_b Y_{\ell m}.$$  \hspace{1cm} (15)

By $\Lambda_{\ell m}$, we can always set $h_{2,\ell m}$ to vanish (Regge-Wheeler gauge). By this procedure, $\Lambda_{\ell m}$ is completely fixed and there is no remaining gauge degrees of freedom.

Then, after substituting the metric into the action (2) and performing integrations by parts, we find that the action for the odd modes becomes

$$S_{\text{odd}} = \frac{M_0^2}{2} \sum_{\ell,m} \int dt dr \mathcal{L}_{\text{odd}} = \frac{M_0^2}{2} \sum_{\ell,m} \int dt dr \left[ A_1 \left( \dot{h} - \dot{h}_0 \right)^2 + A_2 h_0 \dot{h}_1 + A_3 h_0^2 - A_4 \dot{h}_1^2 \right],$$

omitting the suffixes $\ell$ and $m$ for the fields, and

$$A_1 = j^2 \frac{(r F - 4 B \xi')}{2r} \sqrt{\frac{B}{A}},$$

$$A_2 = \frac{4 A_1}{r},$$

$$A_3 = \frac{1}{r^2} \left[ 2 r A_1' + 2 A_1 + \frac{j^2 (j^2 - 2)}{2 \sqrt{A B}} \left( F - 2 B' \dot{\xi}' - 4 B \xi'' \right) \right],$$

$$A_4 = \frac{j^2 (j^2 - 2) (A F - 2 B \dot{A} \xi')}{r^2} \sqrt{\frac{B}{A}},$$

where $j^2 = \ell (\ell + 1)$. Since no time derivative of $h_0$ appears, varying with respect to $h_0$ yields a constraint equation. However, because of the presence of $h_0'$ in the action, the constraint results into a second order ordinary differential equation for $h_0$:

$$[A_1 (h_0' - \dot{h}_1)]' = A_3 h_0 + \frac{1}{2} A_2 \dot{h}_1,$$

which cannot be immediately solved for $h_0$. Hence, we take the following steps to overcome this obstacle.

Let us first rewrite the action as

$$\mathcal{L}_{\text{odd}} = A_1 \left( \dot{h}_1 - h_0' + 2 \frac{h_0}{r} \right)^2 - \frac{2 (A_1 + r A_1')}{r^2} h_0^2 + A_3 h_0^2 - A_4 \dot{h}_1^2.$$

so that all the terms containing $\dot{h}_1$ are inside the first squared term. Using a Lagrange multiplier $Q$, we rewrite Eq. (22) as follows

$$\mathcal{L}_{\text{odd}} = A_1 \left[ 2 Q \left( \dot{h}_1 - h_0' + 2 \frac{h_0}{r} \right) - Q^2 \right] - \frac{2 (A_1 + r A_1')}{r^2} h_0^2 + A_3 h_0^2 - A_4 \dot{h}_1^2.$$

where $Q$ is a Lagrange multiplier.
Then, both fields $h_0$ and $h_1$ can be integrated out by using their own equations of motion, which can be written as

$$h_1 = -\frac{A_1 \dot{Q}}{A_4},$$

(24)

$$h_0 = \frac{r}{2A_1 + 2r A'_1 - A_3 r^2} [ (r A'_1 + 2A_1) Q + r A_1 Q' ].$$

(25)

These relations link the physical modes $h_0$ and $h_1$ to the auxiliary field $Q$. Once $Q$ is known also $h_0$ and $h_1$ are. After substituting these expressions into the Lagrangian and performing an integration by parts for the term proportional to $Q' Q$, one finds the Lagrangian in the canonical form

$$L_{\text{odd}} = \frac{A^2_1}{A_4} \dot{Q}^2 - \frac{A^2_1 r^2}{A_3 r^2 - 2r A'_1 - 2A_1} (Q')^2 - \mu^2 Q^2,$$

(26)

where

$$\mu^2 = \frac{A_1 r^2 \left( r^2 A'_1 A'_3 - r^2 A''_1 A_3 + 2 A_1 A_3 + 4 A'_1^2 + A_3^2 r^2 - 2 A_1 A''_1 + 2 A_1 r A'_3 - 4 A'_1 r A_3 \right)}{(2 A_1 + 2 A'_1 r - A_3 r^2)^2}.$$  

(27)

From Eq. (26), we can derive the no ghost condition

$$A_4 \geq 0, \quad \text{or equivalently} \quad A F - 2 B \xi' A' \geq 0.$$  

For solutions proportional to $e^{i(\omega t - kr)}$ with large $k$ and $\omega$, we have the radial dispersion relation

$$\omega^2 = \frac{B (A F - 2 B \xi' A')}{(F - 4 B \xi'' - 2 B' \xi')} k^2,$$

where we made use of the background equations of motion. Finally the expression for the radial speed reads

$$c^2_{\text{odd}} = \left( \frac{dr_*}{d\tau} \right)^2 = \frac{(A F - 2 B \xi' A')}{A (F - 4 B \xi'' - 2 B' \xi')} ,$$

where we used the radial tortoise coordinate ($dr_*^2 = dr^2/B$) and the proper time ($d\tau^2 = A dt^2$). Therefore in order for the modes to be stable, one also requires

$$F - 4 B \xi'' - 2 B' \xi' \geq 0.$$  

B. Even modes

Now that we have got an idea how the action approach works for the odd modes we can tackle the more complicated problem of the even modes. In this case, the perturbed metric can be written as

$$h_{tt} = -A(r) \sum_{\ell, m} H_{0, \ell m}(t,r) Y_{\ell m}(\theta, \varphi),$$

(28)

$$h_{tr} = \sum_{\ell, m} H_{1, \ell m}(t,r) Y_{\ell m}(\theta, \varphi),$$

(29)

$$h_{rr} = \frac{1}{B(r)} \sum_{\ell, m} H_{2, \ell m}(t,r) Y_{\ell m}(\theta, \varphi),$$

(30)

$$h_{ra} = \sum_{\ell, m} a_{\ell m}(t,r) \partial_{\theta} Y_{\ell m}(\theta, \varphi) ,$$

(31)

and we use the gauge transformation to set $h_{ta}$ and $h_{ab}$ to vanish. In addition to the metric perturbations, we need to perturb also the extra scalar fields $F$ and $\xi$ as

$$F = F(r) + \sum_{\ell, m} \delta F_{\ell m}(t,r) Y_{\ell m}, \quad \text{and} \quad \xi = \xi(r) + \sum_{\ell, m} \delta \xi_{\ell m}(t,r) Y_{\ell m}.$$  

(32)
Then, the action at second order for the even modes, reads as follows

\[ S_{\text{even}} = \frac{M_{Pl}^2}{2} \sum_{\ell,m} \int dt dr \mathcal{L}_{\text{even}}, \]  

(33)

where

\[ \mathcal{L}_{\text{even}} = H'_0 (a_1 \delta \xi' + a_2 \delta F' + a_3 H_2 + j^2 a_4 \alpha + a_5 \delta \xi + a_6 \delta F) + j^2 H_0 (a_7 H_2 + a_8 \alpha + a_9 \delta \xi + a_{10} \delta F) \]

\[ + j^2 b_1 H'_2 + H_2 (b_2 \delta \xi' + b_3 \delta F' + b_4 H'_2 + j^2 b_5 \alpha + b_6 \delta \xi + b_7 \delta F) + c_1 H'_2 \]

\[ + H_2 [c_2 \delta \xi' + c_3 \delta F' + j^2 c_4 \alpha + \delta \xi (j^2 c_6 + c_7) + \delta F (j^2 c_8 + c_9)] + \dot{H}_2 (c_5 \delta \xi + c_8 \delta F) \]

\[ + j^2 (d_1 \delta \xi' + d_2 \alpha') + j^2 \delta \xi (d_3 \delta \xi' + d_4 \delta F' + d_5 \delta \xi + d_6 \delta F) + e_1 \delta F^2 + e_2 \delta F \delta \xi + f_1 \delta \xi^2, \]  

(34)

where \( a_i, b_i, c_i, d_i, e_i \) and \( f_1 \) are all functions of \( r \) only and their expressions are given in the Appendix. For simplicity, we omitted the subscripts \( \ell, m \) also for the even modes. In what follows, we will integrate out the fields \( H_0, H_1, \) and \( H_2 \).

We first integrate out the non-propagating field \( H_1 \), by using its own equation of motion

\[ H_1 = -\frac{1}{2j^2 b_1} (b_2 \delta \xi' + b_3 \delta F' + b_4 H'_2 + j^2 b_5 \alpha + b_6 \delta \xi + b_7 \delta F). \]

(35)

We note that the term proportional to \( H_0^2 \) in the action. Thus, the equation of motion for \( H_0 \)

\[ a_1 \delta \xi'' + a_2 \delta F'' + (a_5 + a'_1) \delta \xi + (a_6 + a_8) \delta F + j^2 a_4 \alpha' + a_3 H_2' + (a_5' - j^2 a_9) \delta \xi + (a_6' - j^2 a_{10}) \delta \xi \]

\[ + (a_5' - j^2 a_7) H_2 + (a_4' - a_8) j^2 \alpha = 0, \]

(36)

sets a constraint for the other fields. By looking at this equation, one may think that we cannot use it to directly substitute back any of the fields. However, we can remove the highest \( r \)-derivatives for the fields \( \delta F, \delta \xi, \alpha \) and \( H_2 \), by performing the following field redefinition

\[ j^2 a_4 \alpha = a_4 v_0 - a_3 H_2 - a_2 v'_1. \]

(37)

\[ \delta F = v_1 - \frac{a_1}{a_2} \delta \xi = v_1 - \frac{4(1 - B)}{r^2} \delta \xi. \]

(38)

Because of this field redefinition, Eq. (30) has no more second \( r \)-derivative for any of the fields, and no \( r \)-derivatives for the field \( H_2 \). Solving for \( H_2 \), we obtain

\[
\left(j^2 a_7 - \frac{a_8 a_3}{a_4}\right) H_2 = \left(a_6 + \frac{a_2 a_8}{a_4}\right) v'_1 + \left(a_5 - a'_1 + \frac{a'_2 a_1}{a_2} - \frac{a_1 a_6}{a_2}\right) \delta \xi' + a_4 v'_0 + (a'_6 - j^2 a_{10}) v_1 + (a'_4 - a_8) v_0
\]

\[ + \left[ a'_5 - j^2 a_9 + \frac{a'_1 (a'_5 - a_6)}{a_2} + \frac{a_4 (a'_5 - a_6)}{a_2} + \frac{a'_2 (a_6 - a'_5)}{a_2} - a'_5 + \frac{j^2 a_1 a_{10}}{a_2}\right] \delta \xi. \]

(39)

We can now substitute \( H_2 \) back into the action, so that also the field \( H_0 \) is automatically eliminated. We will find it convenient to finally perform the field substitution

\[ v_0 = v_2 (1 + 4 j^2)^{1/2}, \]

(40)

\[ \delta \xi = v_3 (1 + 4 j^2)^{1/2}, \]

(41)

where the dependence on \( j \) in Eqs. (10) to (11) is chosen such that, for large \( j \), the ghost conditions, which will be found below, become independent of \( j \). Now the Lagrangian takes the canonical form

\[ \mathcal{L}_{\text{even}} = \sum_{i,j=1}^{3} [K_{ij} (r,j) \dot{v}_i \dot{v}_j - L_{ij} (r,j) v'_i v'_j - D_{ij} (r,j) v'_i v_j - M_{ij} (r,j) v_i v_j], \]

(42)

where \( i, j \) run from 1 to 3, and the coefficients important for our discussion are given in appendix. All matrices are symmetric except for \( D_{ij} \), which is anti-symmetric. Now we can discuss the existence of ghosts and the speed of propagation for the modes. The no-ghost condition requires the matrix \( K \) to be positive definite, that is

\[ K_{33} > 0, \quad K_{22} K_{33} - K_{23}^2 > 0, \quad \det(K_{ij}) > 0. \]

(43)
We find that \( K_{22}K_{33} - K_{23}^2 \) is given by

\[
K_{22}K_{33} - K_{23}^2 = -\frac{16 (1 + 4j^2)^2 AB (2B - 2 - r B')^2 [4(B - 1) \xi' - r^2 F']^2}{r^2 \Delta^2} \leq 0, \tag{44}
\]

where

\[
\Delta = 24AB^2 \xi' - 12B^2 r A' \xi' - 8AB \xi' + 4r \xi' BA' - 4ABFr + 2BFA'r^2 - 2ABF'r^2 + BA'r^3 F' + j^2 (2AFr - 8AB^2 \xi'). \tag{45}
\]

Therefore, on this background, a ghost is always present. The determinant of the kinetic matrix is given by

\[
det(K_{ij}) = -32\sqrt{AB} \frac{(j^2 - 2)(1 + 4j^2)^2 r^2 (2B - 2 - r B')^2 (F - 2B^2 \xi' - 4B \xi'')}{j^2 \Delta^2}. \tag{46}
\]

This quantity gives us new information. Indeed, on backgrounds of exact solutions given by \( B = 1 + Cr^2 \) with \( C \) being constant (Minkowski or de Sitter solution), both the determinant and the ghost kinetic term vanish, which implies an effective reduction of the degrees of freedom similar to what occurs on FLRW background, where the missing degree constant (Minkowski or de Sitter solution), both the determinant and the ghost kinetic term vanish, which implies an equation in \( \omega \) as

\[
\omega^2 A_{ij} - k^2 D_{ij} = 0, \tag{47}
\]

which is a cubic equation in \( \omega^2 \). The three radial speeds of propagation we obtain are

\[
c_1^2 = c_2^2 = \frac{(2AB - 2A - r B A')}{(2B - 2 - r B')} A, \tag{48}
\]

\[
c_3^2 = \frac{AF - 2A' B \xi' - 4B \xi''}{A(F - 2B^2 \xi' - 4B \xi'')}. \tag{49}
\]

Two of the speeds of propagation reduce to unity for backgrounds with \( A = B \). The third one, which is identical to the one for the odd modes, depends directly on the profile of the two new scalar degrees of freedom \( F \) and \( \xi \) even if we set \( A = B \). The speed of propagation for large \( j \) does not have a simple analytical form. Although one can solve a cubic equation in \( \omega^2 \), the expression is too complicated to gain intuition from it.

### C. Discussion regarding the ghost

We have found that at least one ghost mode is always present around the spherically symmetric static background in vacuum. However, the existence of the ghost does not necessarily mean that the background spacetime is unstable due to the creation of the ghost and normal particle pairs. Our implicit assumptions are that there is an yet unknown complete theory (maybe string theory), which is well-defined at any energy scale and does not have any ghost and that the \( f(R, \mathcal{G}) \) theories are the derived effective theories, which are valid only below some cutoff scale \( M_{\text{cutoff}} \). What is generally thought is that the mass of the ghost is always heavier than \( M_{\text{cutoff}} \), such a ghost should not be regarded as a physical mode and must be integrated out to have more sensible effective theories. On the other hand, if the mass of the ghost becomes lighter than \( M_{\text{cutoff}} \) in certain situations, such theories do not make sense and must be ruled out from the list of the possible low energy effective gravitational theories.

In the models we study here, there are in general two ways out of which the ghost mode can become massive. One is due to symmetry. That is, as the background becomes more and more similar to either Minkowski, de Sitter or FLRW, its mass tends to infinity because its kinetic term vanishes. The other is due to the so-called chameleon mechanism. Thanks to the local value of \( R \) or \( \mathcal{G} \) much larger than the corresponding cosmological value, some modes may develop their masses. Our cutoff mass \( M_{\text{cutoff}} \) is not necessarily as large as Planck scale, but \( M_{\text{cutoff}}^{-1} \) must be sufficiently smaller than the experimentally relevant length scale \( L_{\exp} \).

Let us give approximate values for the masses of the modes, including the ghost mode. Assuming the background around a star to be very close to the standard GR case, then one has \( F \approx 1 \), or \( F' \approx 0 \), and \( \xi' \approx 0 \approx \xi'' \). Further we assume that the theories satisfy solar system constraints, that is, \( A \approx B \approx 1 - r_s/r \), (where \( r_s \) is the Schwarzschild
radius of the star. Under these assumptions, the leading contribution in the mass matrix is given by the terms originating from $U_{FF}, U_{F\xi}$, and $U_{\xi\xi}$,

$$M_{11} = \frac{1}{2} U_{FF} r^2, \quad M_{13} = -\left(1 + 4j^2\right)^{1/2} \frac{4r_s U_{FF} - r^3 U_{F\xi}}{r}, \quad M_{33} = \frac{1 + 4j^2}{2r^4} (U_{\xi\xi} r^6 - 8U_{F\xi r_s} r^3 + 16r_s^2 U_{FF}).$$

Then the discriminant equation, for low $k$, to solve is

$$\text{det}(m^2 A(r) K_{ij} - M_{ij}) = 0,$$  \hspace{1cm} (50)

where the factor $A(r)$ comes because of the choice of the proper time as time variable, and the elements $K_{ij}$ are given in Appendix. Equation (50) reduces to

$$m^2 \left[9\tilde{G} m^4 + 3(U_{\xi\xi} - \tilde{G} U_{FF}) m^2 - (U_{FF} U_{\xi\xi} - U_{F\xi}^2)\right] = 0,$$  \hspace{1cm} (51)

where we introduced $\tilde{G} \equiv 4G/3 \approx 16r_s^2/r^6$. It is also suggestive to rewrite the above equation in terms of the function $f(R, \tilde{G})$, by using the relation

$$\left(\begin{array}{cc} U_{FF} & U_{F\xi} \\ U_{F\xi} & U_{\xi\xi} \end{array}\right) = \left(\begin{array}{cc} f_{RR} & f_{RG} \\ f_{RG} & f_{GG} \end{array}\right)^{-1},$$  \hspace{1cm} (52)

as

$$m^2 \left[9\tilde{G} \det(f_{ij}) m^4 + 3(f_{RR} - \tilde{G} f_{GG}) m^2 - 1\right] = 0.$$  \hspace{1cm} (53)

One obvious solution of Eq. (51) is $m^2 = 0$, whose eigenvector is given by $E_0 \equiv [0, 1, 0]^T$. The kinetic term of this mode

$$E^i_0 A E_0 = \frac{2(r - r_s)^2 (1 + 4j^2) (j^2 - 2)}{j^2 [3r_s + (j^2 - 2)r]^2},$$

is positive for $\ell \geq 2$. Thus, this mode represents the GR standard contribution. The other two mass eigenvalues are given by

$$m^2_{\pm} = \frac{-U_{\xi\xi} + \tilde{G} U_{FF} \pm (U_{\xi\xi} + \tilde{G} U_{FF}) \sqrt{1 - \frac{4\tilde{G} U_{\xi\xi}}{(U_{\xi\xi} + \tilde{G} U_{FF})^2}}}{6\tilde{G}}.$$

In order to study which of these mass eigenvalues correspond to the ghost mode, let us perform a little more detailed analysis. First, we diagonalize the kinetic matrix $AK_{ij}$ by arranging linear combinations of fields $v_i = P_{ij} w_j$, which are explicitly expressed as

$$v_1 = \frac{\sqrt{6}}{3r} w_1 + \frac{2}{r \sqrt{6}} w_2,$$

$$v_2 = \frac{r j^2 + r_s}{\sqrt{6} \sqrt{1 + 4j^2} (r - r_s)} w_1 + \frac{r j^2 + 7r_s - 6r}{\sqrt{6} \sqrt{1 + 4j^2} (r - r_s)} w_2 + w_3,$$

$$v_3 = \frac{r^2}{2 \sqrt{6} \sqrt{1 + 4j^2} r_s} w_2.$$  \hspace{1cm} (54)

Then, the new diagonalized kinetic matrix $\tilde{K}_{ij} \equiv A(r) P_{ki} K_{kl} P_{lj}$ takes its diagonal elements

$$\tilde{K}_{11} = 1, \quad \tilde{K}_{22} = -1, \quad \tilde{K}_{33} = \frac{2(r - r_s)^2 (1 + 4j^2) (j^2 - 2)}{j^2 [(j^2 - 2)r + 3r_s]^2},$$

whereas the new symmetric mass matrix, $\tilde{M}_{ij} = P_{ki} M_{kl} P_{lj}$, satisfies $\tilde{M}_{ij} = 0$ for $i = 1, 2, 3$, and its non-zero components are

$$\tilde{M}_{11} = \frac{U_{FF}}{3}, \quad \tilde{M}_{12} = \frac{U_{F\xi}}{3\sqrt{G}}, \quad \tilde{M}_{22} = \frac{U_{\xi\xi}}{3G}.$$
At this point, one can easily see that \( \tilde{M}_{22} \) is the most enhanced mass-element in the Minkowski limit, \( r_s \to 0 \), which gives a divergent mass to the ghost mode. This is consistent with the notion that the ghost possesses a divergent mass in the Minkowski background. However, one possible natural hierarchy among the components of \( \tilde{M}_{ij} \) will be that \( \tilde{M}_{11}, \tilde{M}_{12} \) and \( \tilde{M}_{22} \) are the same order when \( \tilde{G} = O(H_0^4) \), where \( H_0 \) is the present value of the Hubble parameter. In this case, since the value of \( \tilde{G} \) around the local gravitational source is typically much larger than the cosmological value \( H_0^4 \), the simple limit \( \tilde{G} \to 0 \), which relatively enhances \( \tilde{M}_{22} \), is out of the relevant parameter range.

Since the GR mode \( w_3 \) completely decouples from the others, we shall concentrate on \( w_1 \) and \( w_2 \) below. They are still coupled through the off-diagonal mass matrix \( \tilde{M}_{12} \). The field transformation which keeps the kinetic matrix unchanged is given by \( w_i = Z_{ij} \bar{z}_j \), where \( Z_{11} = Z_{22} = \cos \beta \), and \( Z_{12} = Z_{21} = \sin \beta \). Then, the condition for the off-diagonal component of the new mass matrix to vanish yields

\[
\tanh 2\beta = -\frac{2\sqrt{g U_{FF}}}{U_{\xi\xi} + gU_{FF}}.
\]  

Therefore, \( 4gU_{\xi\xi}^2 < (U_{\xi\xi} + \tilde{G}U_{FF})^2 \) is required to have the mass matrix diagonalized. In terms of \( f \), this condition is equivalent to \(-4\tilde{G} \det(f_{ij}) < (f_{RR} - \tilde{G}f_{GG})^2 \). In this case, the new fields can be identified as independent decoupled massive modes, and \( 11 \) and \( 22 \) components of the diagonal mass matrix are identified with \( m_+^2 \) and \(-m_-^2 \), respectively. Namely, \( m_+^2 \) and \( m_-^2 \) are the squared masses of the non-ghost and ghost modes, respectively. This identification can be easily verified by considering the trivial case with \( U_{FF} = 0 \). The condition for the model to be applicable to an experiment on a scale \( L_{\text{exp}} \) will be \( |m_+^2| < M_{\text{cut off}}^2 \), with \( M_{\text{cut off}}^2 > L_{\text{exp}}^2 \). If we do not see any significant deviation from general relativity on this scale, a condition \( |m_+^2| > L_{\text{exp}}^2 \) has to be imposed as well. These conditions are less intuitive. If we are allowed to crudely identify \( M_{\text{cut off}}^2 \) with \( L_{\text{exp}}^2 \), the above conditions are simplified to

\[
\left| 9\tilde{G} \det(f_{ij}) \right| = \left| \frac{1}{m_+^2} \right| \lesssim L_{\text{exp}}^4, \quad 3(f_{RR} - \tilde{G}f_{GG}) = \left| \frac{1}{m_+^2} - \frac{1}{m_-^2} \right| \lesssim L_{\text{exp}}^2,
\]

where we used the fact that \( m_+^2 \) and \( m_-^2 \) are solutions of Eq. (53).

Next we consider the case with \( 4gU_{\xi\xi}^2 > (U_{\xi\xi} + \tilde{G}U_{FF})^2 \) (or equivalently \(-4\tilde{G} \det(f_{ij}) > (f_{RR} - \tilde{G}f_{GG})^2 \)), in which we cannot simultaneously diagonalize both the kinetic and mass matrices. In this case the eigenvalues \( m_2^2 \) obtained in Eq. (54) become complex. The complex nature of the solution means that those modes are classically unstable. Since the eigenvalues for \( m_2^2 \) are complex conjugate with each other, a unique mass scale \( \sqrt{|m_2^2|} \) must be much larger than the cutoff scale. Again, with the aid of Eq. (53), this requirement leads to \(-\tilde{G} \det(f_{ij}) \lesssim L_{\text{exp}}^2 \). This condition combined with \(-4\tilde{G} \det(f_{ij}) > (f_{RR} - \tilde{G}f_{GG})^2 \) is identical to the conditions of the previous case, given in (59). Thus, we conclude that the conditions (59) are the necessary and sufficient conditions for the model to be viable. When we apply these constraints on the model to the solar system, we need to plug in \( R \approx (4\pi/3)\rho_{\text{local}} \approx (4\pi/3) \times 10^{-24} g/cm^3 \approx (10^{26})^{-2} \) and \( \tilde{G} \approx 48 M_0^2 / r^4 \approx (5.7 \times 10^{16} \text{cm})^{-1} (1\text{AU}/r)^6 \) as the background values. In the following we present two examples, in which the expressions are simplified by assuming some hierarchy among the components of the mass matrix \( \tilde{M}_{ij} \).

The first case is the one in which the off-diagonal element \( \tilde{M}_{12} \) is suppressed, i.e. \( f_{RG} \) is negligible. If we set \( \tilde{M}_{12} = 0 \), the expressions for the mass eigenvalues simplify to give

\[
m_+^2 \approx -\frac{U_{FF}}{3}, \quad m_-^2 \approx \frac{U_{\xi\xi}}{3\tilde{G}} \approx -\frac{1}{3\tilde{G}f_{GG}}.
\]

The conditions that these masses are positive, \( f_{RR} > 0 \) and \( f_{GG} < 0 \), imply \( \det(f_{ij}) < 0 \). Such a situation is realized by considering the following type of toy models of dark energy:

\[
f(R,G) = R + f_R(R) + f_G(G),
\]

where the functions \( f_R(R) \) and \( f_G(G) \) have to be chosen such that \( f_{RR} > 0 \) and \( f_{GG} < 0 \). The correction \( f_R \) is of the kind discussed in Refs. 28, 29, 33, 40, whereas \( f_G \) is similar to the functions introduced in Refs. 14, 41. For concreteness, we specify the functions as

\[
f_R(R) = A_R R^{p+2}/(R^p + c_R R^{2p}), \quad f_G(G) = A_G G^{8n+2}/(G^{2n} + c_G G^{6n}),
\]

where \( \mu = O(H_0) \) and \( A_R, A_G, c_R, c_G \) are constant parameters of \( O(1) \). In these models, the corrections to the cosmological evolution become important only at around the present epoch \( \tilde{H}_0 \). When we consider local gravity in a

---

1 Here, \( f_G \) is a good approximation to the models introduced in 14 when \( \tilde{G} \gg H_0^4 \), as it happens in the vacuum Schwarzschild solution.
dense region, the values of $R$ and $G$ are much larger than the cosmological backgrounds: $R \gg H_0^2$ and $G \gg H_0^2$. In this case, we have $U_{,FF} = 1/f_{,RR} = O(\mu^{2p-2}R^{p+2})$ and $U_{,\xi} = 1/f_{,G\bar{G}} = O(\mu^{-8n-2}G^{2n+2})$, and the mass squared for each mode is given by

$$m^2_{\text{non-ghost}} = O \left( H_0^2 \left[ \frac{\rho_{\text{local}}}{\rho_c} \right]^{p+2} \right), \quad m^2_{\text{ghost}} = O \left( H_0^2 \left[ \frac{H_0^2 r_s}{\rho_c} \right]^{4n+2} \right),$$

(62)

where $\rho_c \approx 4 \times 10^{-30}$ g/cm$^3$ is the critical density of the universe. For the non-ghost mass, we have used the Einstein equation, which is a good approximation in the local region like the solar-system, to replace the Ricci scalar with the energy density $\rho_{\text{local}}$ of matter surrounding a star. If the system is exactly the vacuum, the non-ghost mode becomes massless. Therefore, for the toy model of Eq. (61), the non-ghost mode acquires mass through the chameleon mechanism. For example, if we put the values for the solar system, $r_s \approx 3$ km, $r \sim 1$ AU, $\rho_{\text{local}} \sim 10^{-2} g/cm^3$, then we have $m^2_{\text{non-ghost}} \approx (10^{22.7-2.77} cm)^{-2}$ and $m^2_{\text{ghost}} \approx (10^{6-88.4} cm)^{-2}$. One can easily make the ghost mode sufficiently massive, while we need to choose a relatively large power $p \geq 4$ to make also the non-ghost mode sufficiently massive, i.e. $m^2_{\text{non-ghost}} \gg (1\text{AU})^{-2}$.

Interestingly, unlike the standard chameleon mechanism, we do not need the matter to make the ghost mode very massive. However, the background value of $G$ is not always much higher than the cosmological value. For example the value of the Gauss-Bonnet term inside a star is typically negative (if evaluated, e.g. on a Schwarzschild interior solution) whereas it will be positive outside the (neutron) star. Therefore there should be a point where $G$ switches sign. At this point, the light ghost problem might arise. However, since the region of the surface of the star where $G = O(\mu^4)$ might be very thin, the variation of the mass in the current treatment, in which spatially homogeneous modes are assumed, becomes irrelevant. Hence, it is not clear if actually there is a problem of light ghost at this point. Another way around this problem consists of thinking of a theory in which the background value of $G$ is automatically guaranteed for $|f_{,RR}| > |G_{,G\bar{G}}|$. In contrast, when $|G_{,G\bar{G}}| > |f_{,RR}|$, the mass scale of the ghost mode is always lower than that of the non-ghost mode. This limit includes a natural situation in which $f_{,RR}, \mu^2 f_{,RG}$ and $\mu^4 f_{,G\bar{G}}$ are the same order with the typical energy scale $\mu$ satisfying $\mu^4 \ll G$. In this case, we have $|G_{,G\bar{G}}| \gg |f_{,RR}|$ and therefore the mass scale of the ghost mode is lower than that of the non-ghost mode. A more explicit expression for the masses are $m^2_{\text{non-ghost}} \approx f_{,G\bar{G}}/(3 \det(f_{,ij}))$ and $m^2_{\text{ghost}} \approx -1/(3G_{,G\bar{G}})$. The expression for $m^2_{\text{non-ghost}}$ is slightly different from the previous example in which we simply neglected $f_{,RG}$.

D. Case $\ell = 0$.

In the case with $j = 0$ neither tensor nor vector harmonics exist. Thus, the field $\alpha$ does not contribute any longer to the action. Furthermore also the term quadratic in the field $H_1$ disappears, together with the linear term in $H_0$. In this case the variation with respect to $H_1$ gives a constraint which can be solved for $H_2$ in terms of $\delta F$, $\delta \xi$, $\delta F'$ and $\delta \xi'$. At the same time also the contribution from $H_0$ in action (31) automatically cancels. Therefore we are left with an action in terms of $\delta F$, $\delta \xi$ only, i.e. there are only two degrees of freedom. In this case we find that the determinant of the new kinetic matrix $A^{(\ell=0)}$ can be written as follows

$$\det \left( A^{(\ell=0)}_{ij} \right) = -\frac{16 r^2 (2B - 2 - rB')^2}{AB (12B^2 - 2F - 4\xi' - rF')^2} < 0.$$

(63)

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2 Strictly speaking, one cannot take into account the effect of the matter $\rho_{\text{local}}$ in the present discussion since we have assumed vacuum throughout the analysis. However, if the gravity is dominated by the central star, which is true for the solar-system, we expect that the gravity from the surrounding matter can be neglected except for shifting the background values of $R$ in evaluating the matrix elements of $U_{,ij}$ or $f_{,ij}$, and our vacuum results can be used as the first approximation.
Therefore also in this case there is one (and only one) ghost degree of freedom. The speeds of propagation for both modes are equal to the expressions given in Eq. (48).

### E. Case \( \ell = 1 \).

In the case with \( j^2 = 2 \) the determinant in Eq. (49) identically vanishes, as the tensor harmonics do not exist. This leads to a reduction of the propagating degrees of freedom. In fact now the matrix \( A^{(\ell = 1)}_{ij} \) possesses an eigenvector \( N_2 \) with zero eigenvalue which satisfies

\[
A^{(\ell = 1)}_{ij} N_2 = 0.
\]

This vector can be written as

\[
v = (3AB\xi' - 4\xi' - r^2F'),\quad \Gamma = \frac{2AB(r F - 4B\xi')}{2AB(r F - 4B\xi')}, 1 \right)^t,
\]

where

\[
\Gamma = -4ABFr + 2BFA' + 2ABF' - 4B\xi' - 8A\xi' B A r B' - 32AB\xi' + 4A' B r \xi'.
\]

This suggests the field redefinition

\[
v_1 = Q_1 - \frac{3(4B\xi' - 4\xi' - r^2F')}{r^2\xi'} X,
\]

\[
v_2 = Q_2 + \frac{\Gamma}{2AB(r F - 4B\xi')} X,
\]

\[
v_3 = X.
\]

In this way also the couplings between \( X \) with \( Q_i \), and \( Q'_i \) vanish. The Lagrangian reduces to

\[
\mathcal{L} = \tilde{A}^{(\ell = 1)}_{ij} \dot{Q}_i \dot{Q}_j - \tilde{D}^{(\ell = 1)}_{ij} Q'_i Q'_j - \tilde{M}^{(\ell = 1)}_{ij} Q_i Q_j + C_1 X^2 + C_2 X Q_i,
\]

(with \( i, j = 1, 2 \)) so that by integrating out the field \( X \), only the mass term is affected. In this case we can find the ghost condition and the speed of propagation. In particular

\[
\det(\tilde{A}^{(\ell = 1)}_{ij}) = -\frac{144r^2AB(\xi')^2(2B - 2 - rB')^2}{\Gamma^2},
\]

and therefore also in this case one of the two propagating degrees of freedom is a ghost. The speeds of propagation for the two modes are equal and coincide with the expression given in Eq. (48).

### IV. SPECIAL CASES

In the theories of gravity discussed so far, we have studied the general function \( f(R, G) \) for which

\[
\Xi = \frac{\partial^2 f}{\partial R^2} \frac{\partial^2 f}{\partial G^2} - \left( \frac{\partial^2 f}{\partial R \partial G} \right)^2 \neq 0.
\]

However there is a special class for which \( \Xi \) identically vanishes, e.g. the \( f(R) \) theories. In general, \( \Xi \) vanishes when \( F \) and \( \xi \) are not independent, i.e. when \( \xi = \xi(F) \). When this happens, we have

\[
\delta \xi = \frac{\xi' F'}{F'} \delta F,
\]

and the independent scalar degrees of freedom reduce by one. In this case one has to follow the same procedure as we have done for the case \( \Xi \neq 0 \). In particular now

\[
\delta F = v_1,
\]

and \( \alpha \) is substituted by \( v_0 \) according to

\[
j^2 a_4 \alpha = a_4 v_0 - a_3 H_2 - \frac{a_4 (b_2 \xi' + b_3 F')}{4b_1 F'} v_1.
\]
by using the same relation (57). In order to remove higher r-derivatives from the action one performs another field
redefinition as
\[
\psi = v_1, \quad v_0 = (1 + 4j^2) v_2. \tag{74}
\]
We have two no-ghost conditions which must be satisfied in order to remove any ghost degree of freedom. In particular, the determinant of the 2×2 kinetic matrix \(A_{ij}\) becomes
\[
\det(A_{ij}) = \frac{(j^2 - 2)(1 + 4j^2)^2}{j^2} \frac{4AB [v^2 F' - 4(B - 1) \xi] (F - 2B' \xi' - 4B \xi'') \Gamma_2}{(F')^2 \Delta^2}, \tag{76}
\]
where
\[
\Gamma_2 = 3 r^2 F F' + 2 \xi' r^2 F' B' - 4 \xi' F - 16 \xi' B^2 \xi'' + 4 B r^2 F' \xi''
+ 16 \xi' B \xi'' + 24 \xi'^2 B B' - 8 \xi' F B' + 8 \xi'^2 B' - 16 \xi' B F' r + 4 \xi' B F, \tag{77}
\]
and \(\Delta\) is defined in Eq. (15). The other independent condition, say \(A_{11} \geq 0\), is rather complicated, but for large \(j\) is given by
\[
\lim_{j \to \infty} A_{11} = \frac{2 (r F' - 2 B' \xi') [v^2 F' - 4(B - 1) \xi]}{\sqrt{AB (F')^2 (r F - 4B \xi')}} \geq 0, \tag{78}
\]
so that in this case the ghost can be absent. As for the speeds of the two modes, one coincides with the expression
given in Eq. (15), whereas the other becomes
\[
\xi' = \frac{\Gamma_3}{A \Gamma_2}, \tag{79}
\]
where
\[
\Gamma_3 = 24 \xi'^2 B^2 A' + 8 \xi'^2 B A' B - 4 \xi' F A + 2 \xi' r^2 F' A' B - 8 \xi' r A' F B - 16 \xi' F' r A B + 4 \xi' F A B + 3 F' r^2 F A. \tag{80}
\]

V. CONCLUSION

We have studied linear perturbations around the static spherically symmetric spacetime for the modified gravity
theories whose Lagrangian consists of a general function \(f(R, G)\) of both the Ricci scalar \(R\), and the Gauss-Bonnet
scalar, \(\xi\). For the odd-type modes, there are two degrees of freedom (one dynamical variable). We derived the no
gradient instability condition. These conditions put constraints on the background quantities.

For the even-type modes, the picture is more interesting. We have found that there are, in total, four degrees of
freedom (corresponding to two dynamical fields) for monopole and dipole perturbations (\(\ell = 0, 1\)), which do not exist
in GR. Both for the monopole and dipole perturbations, one of these two new scalar modes is always a ghost. As
for the higher multipole perturbations, there are, in total, six degrees of freedom (corresponding to three dynamical
fields), four of which do not exist in GR. For Minkowski and (anti-)de Sitter backgrounds, one out of these three
fields is not independent from the others. Instead, for general backgrounds (including Schwarzschild), a ghost always
appears from the even-type perturbations with mass \(m_-\) given in Eq. (54), whereas \(m_+\) is the mass of the non-ghost
chameleon field. Finally the third dynamical field is massless describing the standard GR-mode. The necessary
and sufficient conditions that both the ghost and chameleon fields are sufficiently massive, i.e. their inverse mass
scale is smaller than \(L_{\text{exp}}\), the length scale of a determinate experiment, reduces to (69). When these conditions
hold, the ghost is massive enough to be treated as a non-propagating mode in the effective gravitational theory. In
order to satisfy both these conditions, some hierarchy among the \(f_{ij}\) coefficients is required, which means that some
engineering in choosing the function \(f(R, G)\) is needed.

We also found that, in the high frequency limit, the radial propagation speed of one field, among the three even-
type modes, coincides with the one for the odd-type mode, and the remaining two even-type modes have a common
propagation speed which is different from the former. The classification of the modes are summarized in Table I.

\[\text{These theories are equivalent to the double-scalar-tensor theories defined by } \mathcal{L} = FR + \xi G - U(F, \xi).\]
|                       | odd-type G                                                                 | odd-type S                                                                 | even-type G                                                               | even-type S                                                               |
|-----------------------|---------------------------------------------------------------------------|---------------------------------------------------------------------------|---------------------------------------------------------------------------|---------------------------------------------------------------------------|
| number of modes       | 0 (for $\ell = 0, 1$)                                                    | 0 (for $\ell = 0, 1$)                                                    | 2 (for $\ell = 0, 1$)                                                    | 1 (for $\ell = 0, 1$)                                                    |
|                       | 1 (for $\ell \geq 2$)                                                    | 1 (for $\ell \geq 2$)                                                    | 3 (for $\ell \geq 2$)                                                    | 2 (for $\ell \geq 2$)                                                    |
| ghost and gradient    | $AF - 2A'B\xi > 0$                                                       | $AF - 2A'B\xi > 0$                                                       | ghost present                                                             | constraints (see Sec. IIV)                                               |
| conditions            | $F - 4B\xi'' - 2B'\xi' > 0$                                              | $F - 4B\xi'' - 2B'\xi' > 0$                                              | massive if Eq. (59) is verified                                           |                                                                           |

Table I. Classification of the modes for the general $f(R, \mathcal{G})$ theories. G and S stand for the model that does or does not satisfy Eq. (81).

If the theory satisfies the condition:

$$\frac{\partial^2 f}{\partial R^2} \frac{\partial^2 f}{\partial G^2} - \left( \frac{\partial^2 f}{\partial R \partial G} \right)^2 = 0,$$

then the number of modes for the even-type perturbations reduces by one. In this case, the ghost is not necessarily present, i.e. we get non-trivial no-ghost condition which puts bound on the function form of $f(R, \mathcal{G})$ and also the background spacetime. For example, in $f(R)$ gravity theories, the no-ghost condition is satisfied once $\frac{\partial f}{\partial R} = F > 0$, and we find that the ghost does not exist in this case as expected.

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Appendix A: Coefficients in the action

We define here the different coefficients introduced in order to define the action. By calling \( \Theta \equiv r^2 \sqrt{A/B} \), we have

\[
\begin{align*}
a_1 &= -a_4 b_2/(4b_1), \\
a_2 &= -a_4 b_3/(4b_1), \\
a_3 &= -a_4 b_4/(4b_1), \\
a_4 &= \Theta B (r F - 4 B \xi^2) / r^3, \\
a_5 &= b_6 a_6/b_7, \\
a_6 &= -\Theta B A'/ (2A), \\
a_7 &= a_4 c_5/b_2, \\
a_8 &= -\Theta B (4 B \xi^2 - F r) (A' r - 2 A) / (2 Ar^4), \\
a_9 &= -2 \Theta B'/r^3, \\
a_{10} &= c_9, \\
b_1 &= \Theta B (F r - 4 B \xi^2) / (2 Ar^3), \\
b_2 &= 8 B \Theta (B-1) / (Ar^2), \\
b_3 &= -2 \Theta A', \\
b_4 &= -\Theta B (-F^2 r^2 - 4 \xi' - 2 Fr + 12 B \xi^2) / (Ar^2), \\
b_5 &= -2 b_1, \\
b_6 &= b_2 b_7/b_3, \\
b_7 &= -4 b_1 a_5/a_4, \\
c_1 &= -B \Theta (24 B \xi^2 A' - 4 \xi' A' - r^2 F^2 A' - 2 F r A' - 2 A F - 4 A r F') / (4 Ar^2), \\
c_2 &= 2 \Theta B A' (3B - 1) / (Ar^2), \\
c_3 &= -B \Theta (A' + 4 A) / (2Ar) \\
c_4 &= B \Theta (12 B \xi^2 A' - 2 A r F^2 - 2 A F - F r A') / (2 Ar^3), \\
c_5 &= 4 \Theta (B - 1) / (Ar^2), \\
c_6 &= d_5 c_5/b_2, \\
c_7 &= -\Theta (B A'' + 2 A B A'' - 2 A B A'' + 2 B^2 A A'' - AB A') / (A^2 r^2), \\
c_8 &= b_3 c_5/b_2, \\
c_9 &= \Theta / r^2, \\
c_{10} &= \Theta (4 A^2 B' - A B A'' - B^2 A'^2 + 3 A B B' A' + 2 B^2 A A'' - AB' A') / (Ar A^2), \\
d_1 &= b_1, \\
d_2 &= -\Theta B (2 B \xi^2 A' - A F) / (Ar^4), \\
d_3 &= -4 \Theta B^2 A'/ (Ar^3) \\
d_4 &= c_9 b_2/c_5, \\
d_5 &= 4 \Theta B^2 A'/ (Ar^4), \\
d_6 &= -2 B \Theta / r^3, \\
e_1 &= -\Theta U_{,FF}/2, \\
e_2 &= -\Theta U_{,\xi\xi}/2.
\end{align*}
\]
The elements of the kinetic matrix relevant for our discussion are

\[
K_{22} = \frac{1 + 4j^2}{j^2 (4c_5 a_d b_4 b_3^2 + a_8 b_2 b_4^2)} \left[ 2b_2^2 b_4^2 (b_1 a^4_8 a_4 + b_1 a_8 a_4' - 2b_1 a^2_8 - b_1 a^2_8 a_4) \\
- 8j^2 b_2^2 a_4 (a_6 c_5 b_4 b_4' - a_4 b_2 c_5 b_4 - 2c_5 b_2 b_4 a_4' + 2c_5 a_8 b_2 b_4) \right] \\
K_{33} = \frac{(1 + 4j^2) (b_2 b_3' - b_3 b_2')^2}{8j^2 b_2^2 b_3^2 r (4c_5 a_d b_4 b_3^2 + a_8 b_2 b_4^2)} \left[ -8j^4 b_2^3 c_5 (4a_4 r c_5 - c_3^2 b_1 b_2) \\
- 2j^2 b_1^2 (2a^2_2 c_3 b_2 r b_2' - 4a_4 c_3 b_2 b_4 a_4' r + 2a^2_2 c_3 b_2 b_4' r - b_2^2 b_4^2 a_8 c_5^2 + 8a_4 c_5 a_8 b_2 b_4 r - 2a_2^2 b_2 b_4 c_5 r) \\
+ b_2^2 b_1^2 r (b_1 a_8 a_4 - 2b_1 a^2_8 + b_1 a_4 a_8' - a_4 a_8 b_1') \right],
\]

(A37)

where

\[
b_2 b_3' - b_3 b_2' \propto r B' - 2B + 2,
\]

which vanishes on (anti-)de Sitter and on Minkowski.

For backgrounds close to the Schwarzschild one, the matrix \(K_{ij}\) reduces to

\[
K_{11} = \frac{(2r^2 j^6 + (10r_s - 7r) rj^4 + (14r_s^2 - 20r r_s + 6r^2) r^2 - r^2 s^2)^2}{j^2 (rj^2 - 2r + 3r_s)^2 (r - r_s)},
\]

\[
K_{12} = -\frac{r^2 \sqrt{1 + 4j^2} (j^2 - 2) (rj^2 + r_s)}{j^2 (rj^2 - 2r + 3r_s)^2},
\]

\[
K_{13} = -\frac{6 (r^2 j^6 + 2(2r - r) rj^4 + (7r_s^2 - 6r r_s) r^2 - 4r r_s + 4r_s^2) r_s \sqrt{1 + 4j^2}}{j^2 (rj^2 - 2r + 3r_s)^2 (r - r_s)},
\]

\[
K_{22} = \frac{2(r - r_s) r (1 + 4j^2) (j^2 - 2)^2}{j^2 (rj^2 - 2r + 3r_s)^2},
\]

\[
K_{23} = \frac{12r_s r}{r^2} K_{22},
\]

\[
K_{33} = \frac{12r_s}{r^2} K_{23},
\]

(A38)
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