NONLINEAR SCALAR MULTIPOINT BOUNDARY VALUE PROBLEMS AT RESONANCE

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Abstract. In this work we provide conditions for the existence of solutions to nonlinear boundary value problems of the form

\[ y(t + n) + a_{n-1}(t)y(t + n - 1) + \cdots + a_0(t)y(t) = g(t, y(t + m - 1)) \]

subject to

\[ \sum_{j=1}^{n} b_{ij}(0)y(j - 1) + \sum_{j=1}^{n} b_{ij}(1)y(j) + \cdots + \sum_{j=1}^{n} b_{ij}(N)y(j + N - 1) = 0 \]

for \( i = 1, \cdots, n \). The existence of solutions will be proved under a mild growth condition on the nonlinearity, \( g \), which must hold only on a bounded subset of \( \{0, \cdots, N\} \times \mathbb{R} \).

Keywords. Multipoint boundary value problems; Resonance; Lyapunov-Schmidt procedure; Brouwer’s degree

1. Introduction

In this paper we provide criteria for the solvability of nonlinear scalar multipoint boundary value problems of the form

(1) \[ y(t + n) + a_{n-1}(t)y(t + n - 1) + \cdots + a_0(t)y(t) = g(t, y(t + m - 1)) \]

subject to

(2) \[ \sum_{j=1}^{n} b_{ij}(0)y(j - 1) + \sum_{j=1}^{n} b_{ij}(1)y(j) + \cdots + \sum_{j=1}^{n} b_{ij}(N)y(j + N - 1) = 0 \]

for \( i = 1, \cdots, n \).

Throughout our discussion we will assume that \( g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous, \( m \) is fixed with \( 1 \leq m \leq n \), \( N \) is an integer greater than 2, the coefficients \( b_{ij}(\cdot) \) and \( a_0(\cdot), \cdots, a_{n-1}(\cdot) \) are real-valued with \( a_0(t) \neq 0 \) for all \( t \), and the boundary conditions are independent.

We focus on the solvability of nonlinear boundary value problems at resonance; that is, problems where the solution space of the associated linear homogeneous problem, (3), subject to boundary conditions, (4), is nontrivial. We will assume throughout that the solution space of this linear homogeneous problem is 1-dimensional.

In a vast majority of the literature on resonant boundary value problems, see [1, 6, 16, 17, 20, 22, 24, 25, 28, 30, 31, 33, 34], it is assumed that the nonlinearities of
Recently, there has been a large push to obtain existence results in cases where the nonlinearity of the difference (differential) equation is unbounded. There have been several results in this regard, most of which require \( g \) to satisfy a growth condition on intervals of the form \((-\infty, z_0]\) and \([z_0, \infty)\). For interested readers, we mention \([3, 4, 19, 21, 23, 26, 27]\). Our focus will be on the case where the nonlinearity is allowed to be unbounded, but must satisfy a mild growth condition on a bounded subset of \(\{0, \cdots, N\} \times \mathbb{R}\).

Those readers interested in results obtained for the case of nonresonant difference equations may consult \([10, 11, 13, 12, 14, 37, 38]\).

Our main result is Theorem 3.1, which establishes the existence of solutions to (1)-(2) under suitable interaction of the solution space of the linear homogeneous problem and the nonlinearity \(g\). We would like to remark that the result we obtain in Theorem 3.1 constitutes a significant generalization of the work found in \([18, 31]\).

In \([18]\), the author discusses the existence of solutions to (1)-(2) in the special case of nonlinear Sturm-Liouville problems with standard two-point linear boundary conditions. In \([31]\), the authors discuss the existence of solutions to (1)-(2) under the assumption of bounded nonlinearities that must also satisfy a limit condition at \(\pm \infty\). This limit assumption is quite standard, often referred to as a Landesman-Lazer type condition. In section 4, we give a detailed comparison between Theorem 3.1 and the work from \([18, 31]\).

Our main tool in the analysis of Theorem 3.1 will be the application of an alternative method in combination with Brouwer’s degree theory. The application of these ideas to discrete and continuous nonlinear boundary value problems is extensive. For those readers interested in fixed point methods, coincidence degree theory, the Lyapunov-Schmidt procedure or more general alternative methods, and their application to difference and differential equations, we suggest \([2, 6, 7, 8, 9, 17, 24, 27, 28, 29, 30, 32, 33, 34, 35, 36]\) and the references therein.

2. Preliminaries

The nonlinear boundary value problem (1)-(2) will be viewed as an operator problem. To help facilitate in the construction of this problem, we define

\[
A(t) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-1}(t)
\end{pmatrix},
\]

\[
f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \text{ by } f \left(\begin{array}{c}
t \\
x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n
\end{array}\right) = \left(\begin{array}{c}
0 \\
0 \\
\vdots \\
0
\end{array}\right),
\]

\[
f(t, x_m) = g(t, x_m)
\]
The nonlinear boundary value problem (1)-(2) is now equivalent to the nonlinear system

\[(3) \quad x(t + 1) = A(t)x(t) + f(t, x(t))\]

subject to boundary conditions

\[(4) \quad \sum_{i=0}^{N} B_i x(i) = 0.\]

The underlying function spaces for our operator problem are as follows:

\[X = \{ \phi : \{0, 1, 2, \ldots, N\} \to \mathbb{R}^n \mid \sum_{i=0}^{N} B_i \phi(i) = 0 \},\]

and

\[Z = \{ \phi : \{0, 1, 2, \ldots, N-1\} \to \mathbb{R}^n \}.\]

The topologies used on \(X\) and \(Z\) are that of the supremum norm. We use \(\| \cdot \|\) to denote both norms and we will use \(| \cdot |\) to denote the standard Euclidean norm.

We define operators as follows:

\[(\mathcal{L}x)(t) = x(t + 1) - A(t)x(t),\]

and

\[(\mathcal{F}x)(t) = f(t, x(t)).\]

Solving the nonlinear boundary value problem (1)-(2) is now equivalent to solving

\[(5) \quad \mathcal{L}x = \mathcal{F}(x).\]

**Remark 2.1.** It will be important to know that the very natural assumption regarding the independence of the boundary conditions is equivalent to the augmented matrix \([B_0, \ldots, B_N]\) having full row rank, and thus is also equivalent to \(\text{Ker}(\cap_{k=0}^{N} B_k^T) = \{0\}\). See Definition 2.6 and [32].

Crucial to the use of any alternative method is the construction of projections onto the kernel and image of \(\mathcal{L}\); to aid in the construction of these projections, we obtain a complete description of these spaces. The following characterization of kernel and image of \(\mathcal{L}\) can be found in [20].

Let

\[\Phi(t) = \begin{cases} I & \text{if } t = 0 \\ A(t-1)A(t-2) \cdots A(0) & \text{if } t = 1, 2, \ldots\end{cases}\]
We then have that $\Phi$ is the principal fundamental matrix solution to linear homogeneous problem
\begin{equation}
    x(t + 1) = A(t)x(t).
\end{equation}

For those readers interested in the general theory of difference equations, we suggest [5, 15].

**Proposition 2.2.** An element $h \in \mathbb{Z}$ is contained in the $\text{Im}(L)$ if and only if
\begin{equation}
    B_1\Phi(1)\Phi^{-1}(1)h(0) + \cdots + B_N\Phi(N)\sum_{i=0}^{N-1}\Phi^{-1}(i+1)h(i) \in \text{Ker}\left(\left(\sum_{i=0}^{N} B_i\Phi(i)\right)^T\right)^\perp.
\end{equation}

The proof of Proposition 2.2 is trivial and can be found in [20]. It follows easily from the variation of parameters formula,
\begin{equation}
    x(t) = \Phi(t)x(0) + \Phi(t)\sum_{i=0}^{t-1}\Phi^{-1}(i+1)h(i),
\end{equation}
and an application of the boundary conditions (4).

**Proposition 2.3.** $\text{Ker}\left(\sum_{i=0}^{N} B_i\Phi(i)\right)$ and the solution space of the linear homogeneous problem, (6), subject to the boundary conditions (4), have the same dimension.

**Proof.** Taking $h = 0$ in the variation of parameters formula, (7), and applying the boundary conditions, we have
\[\mathcal{L}x = 0\text{ if and only if } \exists u \in \mathbb{R}^n \text{ such that } x(\cdot) = \Phi(\cdot)u \text{ and } \sum_{i=0}^{N} B_i\Phi(i)u = 0.\]

Since we are assuming that the solution space of the linear homogeneous problem, (6), subject to boundary conditions, (4), is 1-dimensional, it follows from Proposition 2.3 that we may pick a vector $u \in \mathbb{R}^n$ which forms a basis for $\text{Ker}\left(\sum_{i=0}^{N} B_i\Phi(i)\right)$.

We define $S : \{0, 1, \cdots, N\} \rightarrow \mathbb{R}^n$ by
\[S(t) = \Phi(t)u.\]

It follows that a function $x \in \text{Ker}(\mathcal{L})$ if and only if $x(\cdot) = S(\cdot)\alpha$ for some $\alpha \in \mathbb{R}$.

Using the fact that $\text{Ker}\left(\sum_{i=0}^{N} B_i\Phi(i)\right)$ and $\text{Ker}\left(\left(\sum_{i=0}^{N} B_i\Phi(i)\right)^T\right)$ have the same dimension, we may also pick a vector $w \in \mathbb{R}^n$ which forms a basis for $\text{Ker}\left(\left(\sum_{i=0}^{N} B_i\Phi(i)\right)^T\right)$. We introduce the following notation which simplifies our
characterization of $Im(L)$. We define $\Psi^T : \{0, 1, 2, \cdots, N-1\} \rightarrow \mathbb{R}^n$ by

$$\Psi^T(t) = \sum_{i=t+1}^{N} w^T B_i \Phi(i) \Phi^{-1}(t+1).$$

We now have the following characterization of the $Im(L)$.

**Proposition 2.4.** An element $h \in Z$ is contained in the $Im(L)$ if and only if

$$\sum_{i=0}^{N-1} \Psi^T(i) h(i) = 0.$$ 

Having characterized the kernel and image of $L$, we are now in a position to construct the projections which will form the basis of the Lyapunov-Schmidt projection scheme. In this regard, we choose to follow [20, 32]. The proofs that the following operators, $P$ and $I - Q$, are projections onto the kernel and image of $L$, respectively, are simple consequences of our previous characterization of these spaces. Proofs may be found in [32].

**Definition 2.5.** Let $V : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the orthogonal projection onto $Ker \left( \sum_{i=0}^{N} B_i \Phi(i) \right)$. Define $P : X \rightarrow X$ by

$$[Px](t) = \Phi(t)Vx(0).$$

Then $P$ is a projection onto $Ker(L)$.

**Definition 2.6.** Define $Q : Z \rightarrow Z$ by

$$[Qh](t) = \Psi(t) \left( \sum_{j=0}^{N-1} |\Psi(j)|^2 \right)^{-1} \sum_{i=0}^{N-1} \Psi^T(i) h(i).$$

Then $I - Q$ is a projection onto $Im(L)$.

**Remark 2.7.** That $Q$ is well-defined is a consequence of Remark 2.1, see [32].

The following is the formulation of the alternative problem which we will use to analyze the nonlinear boundary value problem, (1), subject to boundary conditions, (2). It is often referred to as the Lyapunov-Schmidt projection scheme. This type of projection scheme has become quite standard in resonant boundary value problems, we include the proof simply for the convenience of the reader.

**Proposition 2.8.** Solving $Lx = F(x)$ is equivalent to solving the following system

$$x - Px = M_p(I - Q)F(x)$$

and

$$\sum_{i=0}^{N-1} [\Psi(i)]_n g(i, [x(i)]_m) = 0$$

where $M_p$ is $(L_{|Ker(P)})^{-1}$ and $[e]_k$ denotes the $k$th row of a vector $e$ in $\mathbb{R}^n$. 

Theorem 3.1. Suppose the following conditions hold:

We also define $s := \min_s \{ |S(t)|_m \}$, and $\alpha := \max_{t \in [0,...,N-1]} |[S(t)]_m|$, $\delta := \sup_{t \in [0,...,N-1]} |g(t,x)|$, and $p : \mathbb{R} \times Im(I - P) \to Im(I - P)$ by

$$p(\alpha, v) = M_p(I - Q)\mathcal{F}(\alpha S(\cdot) + v).$$

**Theorem 3.1.** Suppose the following conditions hold:

C1. $\mathcal{O}_0$ is empty
C2. There exists positive real numbers $c$ and $d$, with $c < d$, and functions $W_1, W_2, W_3, u_1, u_2$ and $v_2$ such that

if $x \in [c, d]$, then $W_1(t) < g(t, x)$ for $t \in \mathcal{O}_{++}$
if $x \in [-d, c]$, then $g(t, x) < U_1(t)$ for $t \in \mathcal{O}_{+-}$
if $x \in [c, d]$, then $g(t, x) < u_1(t)$ for $t \in \mathcal{O}_{+-}$
if $x \in [-d, c]$, then $w_1(t) < g(t, x)$ for $t \in \mathcal{O}_{-+}$
if $x \in [c, d]$, then $g(t, x) < W_2(t)$ for $t \in \mathcal{O}_{-+}$
if $x \in [-d, c]$, then $U_2(t) < g(t, x)$ for $t \in \mathcal{O}_{--}$
if $x \in [c, d]$, then $v_2(t) < g(t, x)$ for $t \in \mathcal{O}_{--}$
and
if $x \in [-d, c]$, then $g(t, x) < w_2(t)$ for $t \in \mathcal{O}_{--}$

C3. $d > \frac{c s_{\text{max}} + A ||g||_d (s_{\text{max}} + s_{\text{min}})}{s_{\text{min}}}$

C4. $J_2 \leq 0 \leq J_1$, where

\[
J_1 = \sum_{i=0}^{N-1} [\Psi(i)]_n K_1(i), \quad J_2 = \sum_{i=0}^{N-1} [\Psi(i)]_n K_2(i),
\]

and $K_1$ and $K_2$ are defined by

\[
K_1(t) = \begin{cases} 
W_1(t) & t \in \mathcal{O}_{++} \\
W_2(t) & t \in \mathcal{O}_{-+} \end{cases},
\]

and

\[
K_2(t) = \begin{cases} 
U_1(t) & t \in \mathcal{O}_{++} \\
U_2(t) & t \in \mathcal{O}_{-+} \end{cases}.
\]

Then there exists a solution to the nonlinear boundary value problem (1)-(2).

Proof. Define $H : \mathbb{R} \times \text{Im}(I - P) \to \mathbb{R} \times \text{Im}(I - P)$ by

(8) $H(\alpha, x) = \left( \sum_{i=0}^{N-1} [\Psi(i)]_n g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m) \right) \cdot v - p(\alpha, v)$

From Proposition 2.8, the zeros of $H$ are precisely the solutions of (1)-(2). We will show the existence of a solution to the nonlinear boundary value problem by showing that the Brouwer degree of $H$, $\deg(H, \Omega, 0)$, is nonzero for some appropriately chosen set $\Omega$. 

To this end, endow $\mathbb{R} \times \text{Im}(I - P)$ with the product topology and define

$$\Omega = \{(\alpha, v) \mid |\alpha| \leq \alpha^* \text{ and } \|v\| \leq r^*\},$$

where $\alpha^* = \frac{c + A\|g\|_d}{s_{\min}}$ and $r^* = A\|g\|_d$.

Define $Q : [0, 1] \times \overline{\Omega} \to \mathbb{R} \times \text{Im}(I - P)$ by

$$Q(\gamma, (\alpha, v)) = \left(1 - \gamma\right)\alpha + \gamma \sum_{i=0}^{N-1} \left[\Psi(i)\cdot g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m)\right] - \gamma p(\alpha, v) \cdot v,$$

It is evident that $Q$ is a homotopy between the identity mapping and $H$. In what follows, we will show that $Q(\gamma, (\alpha, v)) \neq 0$ for each $\gamma \in (0, 1)$ and every $(\alpha, v)$ in $\partial(\Omega) = \{(\alpha, v) \mid |\alpha| = \alpha^* \text{ and } \|v\| \leq r^* \text{ or } |\alpha| \leq \alpha^* \text{ and } \|v\| = r^*\}$, so that, by the invariance of the Brouwer degree under homotopy, $\text{deg}(H, \Omega, 0) = \text{deg}(I, \Omega, 0) = 1$.

In what follows, it will be useful to note that

$$\alpha^* s_{\max} + r^* = \left(\frac{c + A\|g\|_d}{s_{\min}}\right)s_{\max} + A\|g\|_d$$

(9)

$$= \frac{c s_{\max} + A\|g\|_d s_{\max} + A\|g\|_d s_{\min}}{s_{\min}} < d.$$

We now turn our attention to showing that $Q(\gamma, (\alpha, v)) \neq 0$ for each $\gamma \in (0, 1)$ and every $(\alpha, v) \in \partial(\Omega)$. We start by assuming $(\alpha, v) \in \partial(\Omega)$, with $|\alpha| \leq \alpha^*$ and $\|v\| = r^*$.

Since for every $i$, $|\alpha[S(i)]_m + [v(i)]_m| \leq \alpha^* s_{\max} + r^*$, we have, using (9), that $\alpha[S(i)]_m + [v(i)]_m \in [-d, d]$. It follows that

$$\|p(\alpha, v)\| = \|M_p(I - Q)\mathcal{F}(\alpha S(\cdot) + v)\|$$

$$\leq \|M_p(I - Q)\| \|\mathcal{F}(\alpha S(\cdot) + v)\|$$

$$= A \max_{\{0, \ldots, N-1\}} |f(i, \alpha S(i) + v(i))|$$

$$= A \max_{\{0, \ldots, N-1\}} |g(i, \alpha[S(i)]_m + [v(i)]_m)|$$

$$\leq A \|g\|_d$$

$$= r^*.$$

Thus, $\|p(\alpha, v)\| \leq r^* = \|v\|$ and it becomes clear that $Q(\gamma, (\alpha, v)) \neq 0$ for every $\gamma$ in $(0, 1)$, since $v - \gamma p(\alpha, v) \neq 0$.

We finish the proof by looking at the case when $(\alpha, v) \in \partial(\Omega)$ with $|\alpha| = \alpha^*$ and $\|v\| \leq r^*$. Combining the fact that $\|p(\alpha, v)\| \leq r^*$ with (9), we conclude that for each $i$ and for every $(\alpha, v) \in \partial(\Omega)$, $|\alpha[S(i)]_m + [p(\alpha, v)(i)]_m| \leq d$. 


Further, if $|\alpha| = \alpha^*$, then for all $i \in \mathcal{O}$, we have
\[
|\alpha[S(i)]_m + [p(\alpha, v)(i)]_m| \geq \alpha^* s_{\min} - \|p(\alpha, v)\|
\geq \alpha^* s_{\min} - A \|g\|_d
= \left( \frac{c + A \|g\|_d}{s_{\min}} \right) s_{\min} - A \|g\|_d
= c.
\]

Thus, we have shown that when $(\alpha, v) \in \partial(\Omega)$ with $|\alpha| = \alpha^*$ and $\|v\| \leq r^*$, then for all $i \in \mathcal{O}$, $|\alpha[S(i)]_m + [p(\alpha, v)(i)]_m| \in [c, d]$.

In fact, we have shown that if $\alpha = \alpha^*$ and $i \in \mathcal{O}_{++} \cup \mathcal{O}_{+-}$, then $\alpha[S(i)]_m + [p(\alpha, v)(i)]_m \in [c, d]$ and if $i \in \mathcal{O}_{+-} \cup \mathcal{O}_{-+}$, then $\alpha[S(i)]_m + [p(\alpha, v)(i)]_m \in [-d, -c]$. Similarly, if $\alpha = -\alpha^*$ and $i \in \mathcal{O}_{++} \cup \mathcal{O}_{+-}$, then $\alpha[S(i)]_m + [p(\alpha, v)(i)]_m \in [-d, -c]$ and if $i \in \mathcal{O}_{+-} \cup \mathcal{O}_{-+}$, then $\alpha[S(i)]_m + [p(\alpha, v)(i)]_m \in [c, d]$.

Using C2., we now conclude that when $\alpha = \alpha^*$,
\[
W_1(i) = g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m) \text{ for } i \in \mathcal{O}_{++}
\]
\[
w_1(i) = g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m) \text{ for } i \in \mathcal{O}_{+-}
\]
\[
g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m) < W_2(i) \text{ for } i \in \mathcal{O}_{-+}
\]
and
\[
g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m) < w_2(i) \text{ for } i \in \mathcal{O}_{-+}.
\]

Thus, since $\mathcal{O}_0$ is empty,
\[
\sum_{i=0}^{N-1} [\Psi(i)]_n g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m) = \sum_{i \in \mathcal{O}} [\Psi(i)]_n g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m)
\geq \sum_{i \in \mathcal{O}} [\Psi(i)]_n K_1(i)
= \sum_{i=0}^{N-1} [\Psi(i)]_n K_1(i)
= J_1
\geq 0.
\]

Similarly, we may conclude that if $\alpha = -\alpha^*$,
\[
\sum_{i=0}^{N-1} [\Psi(i)]_n g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m) = \sum_{i \in \mathcal{O}} [\Psi(i)]_n g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m)
< \sum_{i \in \mathcal{O}} [\Psi(i)]_n K_2(i)
= \sum_{i=0}^{N-1} [\Psi(i)]_n K_2(i)
= J_2
\leq 0.
\]
Corollary 3.3. It follows that \( \alpha \cdot \sum_{i=0}^{N-1} [\Psi(i)]_n g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m) > 0 \). Since
\[
(1 - \gamma)\alpha + \gamma \sum_{i=0}^{N-1} [\Psi(i)]_n g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m)
\]
would be 0 for some \( \gamma \in (0, 1) \) if and only if
\[
\alpha \cdot \sum_{i=0}^{N-1} [\Psi(i)]_n g(i, \alpha[S(i)]_m + [p(\alpha, v)(i)]_m) < 0,
\]
we have that \( Q(\gamma, (\alpha, v)) \) is nonzero in these cases. We now conclude, by the homotopy invariance of the Brouwer degree, that
\[
\deg(H, \Omega, 0) = \deg(I, \Omega, 0) = 1.
\]
The result now follows.
\( \square \)

Remark 3.2. If the inequalities of Theorem 3.1 are reversed; that is,

if \( x \in [c, d] \), then \( W_1(t) > g(t, x) \) for \( t \in \mathcal{O}_{++} \)
if \( x \in [-d, -c] \), then \( g(t, x) > U_1(t) \) for \( t \in \mathcal{O}_{++} \)
if \( x \in [c, d] \), then \( g(t, x) > U_1(t) \) for \( t \in \mathcal{O}_{++} \)
if \( x \in [-d, -c] \), then \( w_1(t) > g(t, x) \) for \( t \in \mathcal{O}_{--} \)
if \( x \in [c, d] \), then \( g(t, x) > W_2(t) \) for \( t \in \mathcal{O}_{+} \)
if \( x \in [-d, -c] \), then \( U_2(t) > g(t, x) \) for \( t \in \mathcal{O}_{++} \)
if \( x \in [c, d] \), then \( u_2(t) > g(t, x) \) for \( t \in \mathcal{O}_{--} \)

and

if \( x \in [-d, -c] \), then \( g(t, x) > w_2(t) \) for \( t \in \mathcal{O}_{--} \)

then provided \( J_1 \leq 0 \leq J_2 \), \([1]-[2]\) has a solution. The proof is essentially the same.

The following corollary isolates the special case in which \( [\Psi(i)]_n \) and \( [S(i)]_m \) have the same sign for all \( i = 0, \ldots, N - 1 \). This case is of special interest since it occurs in all ‘self-adjoint’ boundary value problems, Sturm-Liouville boundary value problems being a special case, specific cases of second-order periodic difference equations being another. It also happens in several other cases, as we will see in our example in section 5.

Corollary 3.3. Suppose the following conditions are satisfied:

C1*. \( \mathcal{O}_0 \) is empty.
C2*. \( [\Psi(i)]_n \cdot [S(i)]_m \geq 0 \) (or \( [\Psi(i)]_n \cdot [S(i)]_m \leq 0 \)) for all \( i = 0, \ldots, N - 1 \).
C3*. There exists positive real numbers \( c \) and \( d \), with \( c < d \), such that \( g(t, x) > 0 \) (or \( g(t, x) < 0 \)) for every \( x \in [c, d] \) and each \( t = 0, \ldots, N - 1 \) and \( g(t, x) < 0 \) (or \( g(t, x) > 0 \)) for every \( x \in [-d, -c] \) and each \( t = 0, \ldots, N - 1 \).
C4*. \( d > \frac{s_{\max} + A[\|g\|_d (s_{\max} + s_{\min})]}{s_{\min}} \)

Then the nonlinear boundary value problem \([1]-[4]\) has a solution.
Proof. It suffices to show that conditions C2. and C4. of Theorem 3.1 hold. We will assume \([\Psi(i)] \circ [S(i)]_m \geq 0\), that \(g(t, x) > 0\) for every \(x \in [c, d]\) and each \(t = 0, \cdots, N - 1\), and that \(g(t, x) < 0\) for every \(x \in [-d, -c]\) and each \(t = 0, \cdots, N - 1\). Since \([\Psi(i)] \circ [S(i)]_m \geq 0\), we have that \(O_{++}\) and \(O_{--}\) are empty. It follows that condition C2. of Theorem 3.1 reduces to

\((\text{NC2.})\) There exists positive numbers \(c\) and \(d\), with \(c < d\), and functions \(W_1, U_1, w_2\) and \(u_2\) such that

\[
\begin{align*}
\text{if } x &\in [c, d], \text{ then } W_1(t) < g(t, x) \text{ for } t \in O_{++} \\
\text{if } x &\in [-d, -c], \text{ then } g(t, x) < U_1(t) \text{ for } t \in O_{++} \\
\text{if } x &\in [c, d], \text{ then } w_2(t) < g(t, x) \text{ for } t \in O_{--} \quad \text{and} \\
\text{if } x &\in [-d, -c], \text{ then } g(t, x) < w_2(t) \text{ for } t \in O_{--}.
\end{align*}
\]

However, using C3**, NC2. is clearly satisfied by taking \(W_1 = U_1 = w_2 = w_2 = 0\). It then trivially follows that \(J_1 = J_2 = 0\), so that condition C4. of Theorem 3.1 is satisfied. This completes the proof for this case. The other cases are similar. □

The following corollary is an application of Theorem 3.1 to cases in which the nonlinearities satisfy a sublinear or ‘small’ linear growth condition.

**Corollary 3.4.** Suppose the following conditions are satisfied:

\(\text{C1**. Conditions C1, C2, and C4 of Theorem 3.1 hold.}\)

\(\text{C2**. There exist positive constants } M_1 \text{ and } M_2 \text{ such that } |g(t, x)| \leq M_1|x|^{\beta} + M_2 \) for every \(x \in [-d, d]\) and each \(t = 0, \cdots, N - 1\), where \(0 < \beta \leq 1\).

\(\text{C3**. } d > \frac{cs_{\max} + (K_1(1 - \beta) + K_2)(s_{\max} + s_{\min})}{s_{\min} - K_1\beta(s_{\max} + s_{\min})}, \text{ where } K_1 = AM_1, K_2 = AM_2, \text{ and we are assuming } s_{\min} - K_1\beta(s_{\max} + s_{\min}) > 0; \text{ that is, } K_1 < \frac{1}{\beta(s_{\min} + s_{\max})}.\)

Then the nonlinear boundary value problem, \((1) - (2)\), has at least one solution.

**Proof.** From (C2**.), we get \(\|g\|_d \leq M_1 d^\beta + M_2\). Thus,

\[
\frac{cs_{\max} + A \|g\|_d (s_{\max} + s_{\min})}{s_{\min}} \leq \frac{cs_{\max} + A(M_1 d^\beta + M_2)(s_{\max} + s_{\min})}{s_{\min}} = \frac{cs_{\max} + (K_1 d^\beta + K_2)(s_{\max} + s_{\min})}{s_{\min}}.
\]

Using (C3**.), we have \(\frac{cs_{\max} + (K_1(1 - \beta) + K_2)(s_{\max} + s_{\min})}{s_{\min} - K_1\beta(s_{\max} + s_{\min})} < d\), from which we conclude

\[
\begin{align*}
cs_{\max} + K_2(s_{\max} + s_{\min}) < &\, ds_{\min} - dK_1\beta(s_{\max} + s_{\min}) - K_1(1 - \beta)(s_{\max} + s_{\min}) \\
= &\, ds_{\min} - K_1(1 + \beta(d - 1))(s_{\max} + s_{\min}) \\
\leq &\, ds_{\min} - K_1(1 + (d - 1)\beta)(s_{\max} + s_{\min}) \\
= &\, ds_{\min} - K_1 d^\beta(s_{\max} + s_{\min}).
\end{align*}
\]
Rearranging, it follows that
\[
\frac{c_{s_{\text{max}}} + A \| g \|_d (s_{\text{max}} + s_{\text{min}})}{s_{\text{min}}} \leq \frac{c_{s_{\text{max}}} + (K_1 d^\beta + K_2)(s_{\text{max}} + s_{\text{min}})}{s_{\text{min}}} < d.
\]
\[\square\]

**Remark 3.5.** We would like to point out that if \( g \) is sublinear on all of \( \mathbb{R} \); that is, there exist positive numbers \( M_1, M_2 \) and a constant \( \beta, 0 \leq \beta < 1 \), such that \( |g(t, x)| \leq M_1 |x|^{\beta} + M_2 \) for every \( x \in \mathbb{R} \) and each \( t = 0, \cdots, N - 1 \), and there is a \( R > 0 \) such that

\[
\begin{align*}
&\text{if } x > R, \text{ then } W_1(t) < g(t, x) \text{ for } t \in \mathcal{O}_{++} \\
&\text{if } x < -R, \text{ then } g(t, x) < U_1(t) \text{ for } t \in \mathcal{O}_{++} \\
&\text{if } x > R, \text{ then } g(t, x) < u_1(t) \text{ for } t \in \mathcal{O}_{++} \\
&\text{if } x < -R, \text{ then } w_1(t) < g(t, x) \text{ for } t \in \mathcal{O}_{++} \\
&\text{if } x > R, \text{ then } g(t, x) < W_2(t) \text{ for } t \in \mathcal{O}_{--} \\
&\text{if } x < -R, \text{ then } u_2(t) < g(t, x) \text{ for } t \in \mathcal{O}_{--} \\
&\text{if } x > R, \text{ then } g(t, x) < U_2(t) \text{ for } t \in \mathcal{O}_{--} \\
&\text{if } x < -R, \text{ then } w_2(t) < g(t, x) \text{ for } t \in \mathcal{O}_{--}
\end{align*}
\]

(10) then \( C_3 \) of Theorem 3.1 holds, since \( \lim_{r \to \infty} \frac{R s_{\text{max}} + A \| g \|_r (s_{\text{max}} + s_{\text{min}})}{s_{\text{min}}} = 0 < 1 \). Thus, if conditions C1. and C4. of Theorem 3.1 are satisfied, then the nonlinear boundary value problem has a solution.

In fact, if \( g \) has ‘small’ linear growth; that is, \( |g(t, x)| \leq M_1 |x| + M_2 \) for every \( x \in \mathbb{R} \) and each \( t = 0, \cdots, N - 1 \) with

\[AM_1 \left( \frac{s_{\text{max}} + s_{\text{min}}}{s_{\text{min}}} \right) < 1,\]

then provided (10) holds we have that \( C_3 \). of Theorem 3.1 holds, since in this case

\[
\lim_{r \to \infty} \frac{R s_{\text{max}} + A \| g \|_r (s_{\text{max}} + s_{\text{min}})}{s_{\text{min}}} \leq AM_1 \left( \frac{s_{\text{max}} + s_{\text{min}}}{s_{\text{min}}} \right) < 1.\]

Thus, again, if conditions C1. and C4. of Theorem 3.1 are satisfied, then the nonlinear boundary value problem has a solution.

4. Comparison to previous results

In this section we show how Theorem 3.1 improves upon existing results in the literature.

**General Multipoint.** In \[31\] the authors look at the existence of solutions to (1)-(2). They obtain results by placing conditions on the nonlinearity, \( g \), which are much more restrictive than Theorem 3.1. Their main result, written in terms of the notation of this paper, is the following:
Theorem 4.1. Suppose \( 1 \) subject to boundary conditions \( 2 \) has a 1-dimensional solution space. If

1. \( g \) is independent of \( t \)
2. \( g(\pm\infty) := \lim_{x \to \pm\infty} g(x) \) exist
3. \( \mathcal{O}_0 \) is empty
4. \( L_1 L_2 < 0 \), where

\[
L_1 = g(+\infty) \cdot \sum_{\mathcal{O}_{++} \cup \mathcal{O}_{--}} [\Psi(i)]_n + g(-\infty) \cdot \sum_{\mathcal{O}_{+-} \cup \mathcal{O}_{-+}} [\Psi(i)]_n
\]

and

\[
L_2 = g(-\infty) \cdot \sum_{\mathcal{O}_{++} \cup \mathcal{O}_{--}} [\Psi(i)]_n + g(+\infty) \cdot \sum_{\mathcal{O}_{+-} \cup \mathcal{O}_{-+}} [\Psi(i)]_n
\]

Then there exists a solution to the nonlinear boundary value problem \( 1 \cdot 2 \).

Theorem 4.1 is a simple consequence of Theorem 3.1. To see this, suppose the conditions of Theorem 4.1 hold and assume \( L_2 < 0 < L_1 \). We will abuse notation, slightly, and use \( g \) to denote the time dependent function defined on \( \{0, \ldots, N\} \times \mathbb{R} \) by \( g(t, x) = g(x) \). Since \( g(\pm\infty) \) exist, we must have that \( g \) is bounded. Let \( \varepsilon > 0 \) and define the functions \( W_1, U_1, W_2, U_2, w_1, u_1, w_2 \) and \( u_2 \) in Theorem 3.1 as follows:

\[
W_1(t) = g(+\infty) - \varepsilon, \quad U_1(t) = g(-\infty) + \varepsilon, \quad W_2(t) = g(+\infty) + \varepsilon, \quad U_2(t) = g(-\infty) - \varepsilon,
\]

\[
w_1(t) = g(-\infty) - \varepsilon, \quad u_1(t) = g(+\infty) + \varepsilon, \quad w_2(t) = g(-\infty) + \varepsilon, \quad u_2(t) = g(+\infty) - \varepsilon.
\]

It is clear that for these functions there exists an \( R \), depending on \( \varepsilon \), such that \( 10 \) of Remark 3.2 holds.

Now, if we calculate \( J_1 = \sum_{i=0}^{N-1} [\Psi(i)]_n K_1(i) \), we get

\[
\sum_{\mathcal{O}_{++}} [\Psi(i)]_n (g(+\infty) - \varepsilon) + \sum_{\mathcal{O}_{--}} [\Psi(i)]_n (g(-\infty) - \varepsilon)
\]

\[
+ \sum_{\mathcal{O}_{+-}} [\Psi(t)]_n (g(+\infty) + \varepsilon) + \sum_{\mathcal{O}_{-+}} [\Psi(i)]_n (g(-\infty) + \varepsilon),
\]

or

\[
g(+\infty) \cdot \sum_{\mathcal{O}_{++} \cup \mathcal{O}_{--}} [\Psi(i)]_n + g(-\infty) \cdot \sum_{\mathcal{O}_{+-} \cup \mathcal{O}_{-+}} [\Psi(t)]_n - \sum_{\mathcal{O}} \|\Psi(i)\| \varepsilon dt.
\]

However, this is equal to \( L_1 - \sum_{\mathcal{O}} \|\Psi(i)\| \varepsilon \). Similarly, \( J_2 = L_2 + \sum_{\mathcal{O}} \|\Psi(i)\| \varepsilon \).

Since we are assuming \( L_2 < 0 < L_1 \), it is easy to see that for small enough \( \varepsilon \), \( J_2 < 0 < J_1 \). The case where \( L_1 < 0 < L_2 \) follows from Remark 3.2 by a similar argument. The result is now a consequence of Remark 3.2.

Remark 4.2. The above discussion shows that Theorem 3.1 is a substantial improvement of the result found in [3]. It is a generalization in two significant ways. Firstly, Theorem 3.1 does not require the nonlinearity, \( g \), to be bounded as is required in [3] where they impose that \( g(\pm\infty) \) exist. Secondly, the assumptions of
Theorem 3.1 are required only on a bounded interval. In Theorem 4.1 the existence of \( g(\pm\infty) \) requires very specific behavior of \( g \) on intervals of the form \([z_0, \infty)\) and \((-\infty, -z_0]\), for \( z_0 \) large.

**Sturm-Liouville.** In [18], the author proves the existence of solutions to non-linear Sturm-Liouville problems of the form

\[
\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \lambda x(t) = f(x(t)); \quad t \in \{a + 1, \cdots, b + 1\}
\]

subject to

\[
a_{11}x(a) + a_{12}x(a) = 0 \quad \text{and} \quad a_{21}x(b + 1) + a_{22}x(b + 1) = 0,
\]

where throughout it is assumed that \( f : \mathbb{R} \to \mathbb{R}, p : [0, 1] \to \mathbb{R} \) and \( q : [0, 1] \to \mathbb{R} \) are continuous, \( p(t) > 0 \) for all \( t \in [0, 1] \), \( a^2 + b^2, c^2 + d^2 > 0 \), and \( \lambda \) is an eigenvalue of the associated linear Sturm-Liouville problem.

Their main result is the following:

**Theorem 4.3.** Suppose \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( |f(x)| \leq M_1|x|^\beta + M_2 \), where \( M_1 \) and \( M_2 \) are nonnegative constants and \( \beta \in [0, 1) \). If there exist \( z^* \) such that

\[
\forall z > z^*, f(z) > 0 \quad \text{and} \quad \forall z < -z^*, f(z) < 0,
\]

then there exists a solution to (11) - (12).

Theorem 4.3 is also a consequence of Theorem 3.1. This follows from the fact that in the case of the Sturm-Liouville problem, because of the self-adjointness associated with it, \( [\Psi(i)]_2 \) and \( [S(i)]_1 \) (Theorem 3.1), may be chosen to be equal. The result is now a consequence of Corollary 3.3 and Remark 3.5.

5. **Example**

We now provide an example which shows the application of Theorem 3.1. Consider

\[
y(t+2) + y(t+1) + y(t) = g(y(t+1))
\]

subject to

\[
y(5) + y(8) + y(9) = 0
\]

and

\[
y(2) + y(8) + y(9) = 0
\]

Looking at equations (11) and (12), we see that \( n = m = 2 \). Writing this in system form, we have

\[
x(t+1) = Ax(t) + f(x(t))
\]

subject to

\[
B_2x(2) + B_5x(5) + B_8x(8) = 0,
\]

where

\[
x(t) = \begin{bmatrix} y(t) \\ y(t+1) \end{bmatrix},
\]

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

\[
B_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

\[
B_8 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]
and

\[
B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_8 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]

Since \( A \) is constant, it follows that \( \Phi(t) = A^t \). We then have that

\[
B_2 \Phi(2) + B_5 \Phi(5) + B_8 \Phi(8) = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}.
\]

If we choose \( \begin{bmatrix} 2 \\ -1 \end{bmatrix} \) as a basis for \( \text{Ker}(B_2 \Phi(2) + B_5 \Phi(5) + B_8 \Phi(8)) \), then we get

\[
S(t) = \begin{cases} \begin{bmatrix} 2 \\ -1 \end{bmatrix} & \text{if } t \equiv 0 \mod 3 \\
\begin{bmatrix} -1 \\ -1 \end{bmatrix} & \text{if } t \equiv 1 \mod 3 \\
\begin{bmatrix} -1 \\ 2 \end{bmatrix} & \text{if } t \equiv 2 \mod 3 \end{cases}.
\]

We now take \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) as a basis for \( \text{Ker} \left((B_2 \Phi(2) + B_5 \Phi(5) + B_8 \Phi(8))^T\right) \), which gives

\[
\Psi(t) = \begin{cases} \begin{bmatrix} 0 & 0 \end{bmatrix}^T & \text{if } t = 0, 1, 5, 6, 7 \\
\begin{bmatrix} 1 & 1 \end{bmatrix}^T & \text{if } t = 2 \\
\begin{bmatrix} 0 & -1 \end{bmatrix}^T & \text{if } t = 3 \\
\begin{bmatrix} -1 & 0 \end{bmatrix}^T & \text{if } t = 4 \end{cases}.
\]

Observe, \( \mathcal{O}_{++} = \{2\}, \mathcal{O}_{+-} = \emptyset, \mathcal{O}_{-+} = \emptyset, \mathcal{O}_{--} = \{3\} \), and \( \mathcal{O}_0 = \emptyset \).

Notice that \( [\Psi(t)]_2[S(t)]_2 \geq 0 \) for all \( t = 0 \cdots N - 1 \), so that Theorem 3.1 is applicable for an abundance of real-valued functions, \( g \), provided \( g \) is such that conditions C3*. and C4*. of Corollary 3.3 hold for some positive real numbers \( c \) and \( d \). We point out again, as in Remark 3.5, that if C3* holds eventually, then C4* is automatically satisfied.

Further, if the end behavior of \( g \) is not ‘sign-changing’, then (1)-(2) may still have a solution. It is of interest to note that this may happen for a \( g \) which satisfies \( \lim_{x \to \infty} g(\pm x) = \infty \) (or \( \lim_{x \to \infty} g(\pm x) = -\infty \)), and so is not of standard Landesman-Lazer form.
For a specific instance of this, fix $c > 2$ and let $g$ be a continuous function with

$$ g(x) = \begin{cases} 
\beta \ln(1 + |x|) & \text{if } x \in [-d, -c] \\
\gamma \ln(1 + x) & \text{if } x \in [c, d] 
\end{cases}, $$

where $d = e^\gamma (1 + c) - 1$, $0 < \beta < 1$, and $\gamma > \frac{\ln(1 + c)}{\ln(1 + c) - 1}$. We will assume that $g(x) < \beta \ln(1 + c)$ for $x \in [-c, 0]$ and $g(x) < \gamma \ln(1 + c)$ for $x \in [0, c]$.

Let $f(x) := e^x (1 + c) - 1 - (2c + 3A \ln(1 + c) x^2)$, where $A$ is as in the definition of Theorem 3.1 and choose $x_c$ such that if $x > x_c$, then $f(x) > 0$. If $\gamma > x_c$, then we have the following:

1. $g(-d) = \beta \ln(1 + d) < \ln(1 + d) = \gamma + \ln(1 + c) < \gamma \ln(1 + c) = g(c)$
2. $\frac{cs_{\text{max}} + A \|g\|_d (s_{\text{max}} + s_{\text{min}})}{s_{\text{min}}} = 2c + 3A \gamma \ln(1 + d)$
   $$< 2c + 3A \ln(1 + c) \gamma^2$$
   $$< e^\gamma (1 + c) - 1$$
   $$= d.$$

We now verify that conditions C1.-C4. of Theorem 3.1 are satisfied for this choice of $g$ and $c, d$. It has already been noted that $\mathcal{O}_0$ is empty, so that C1. holds. In this specific example, C2. reduces to the existence of numbers $w_1, w_2, u_1,$ and $u_2$ such that

- If $x \in [c, d]$, then $w_1 < g(x) := g(2, x)$
- If $x \in [-d, -c]$, then $g(2, x) := g(x) < u_1$
- If $x \in [c, d]$, then $u_2 < g(x) := g(3, x)$
- If $x \in [-d, -c]$, then $g(3, x) := g(x) < w_2$.

We take $w_1 = g(c), w_2 = g(-d), u_1 = g(-d), \text{ and } u_2 = g(c)$, so that C2. is clearly satisfied for these choices of $w_1, w_2, u_1, \text{ and } u_2$. (2) shows that condition C3. holds. Finally, if $J_1$ and $J_2$ are as in Theorem 3.1, then we have that $J_1 = v_1 - w_2 = g(c) - g(-d) > 0$ and $J_2 = u_1 - u_2 = g(-d) - g(c) < 0$ (see (1)). Thus, C4. is satisfied and we conclude, using Theorem 3.1, that there exists a solution to the nonlinear boundary value problem (1)-(2).

References

[1] L. Cesari and R. Kannan. An abstract existence theorem at resonance. *Proc. Amer. Math. Soc.*, 63:221–225, 1977.
[2] S. Chow and J. K. Hale. *Methods of Bifurcation Theory*. Springer, Berlin, 1982.
[3] P. Drábek. On the resonance problem with nonlinearity which has arbitrary linear growth. *J. Math. Anal. Appl.*, 127:435–442, 1987.
[4] P. Drábek. Landesman-Lazer type condition and nonlinearities with linear growth. *Czechooslovak Math. J.*, 40:70–86, 1990.
[5] S. Elaydi. *An introduction to difference equations*. Springer, 2005.
6. Debra L. Etheridge and Jesús Rodríguez. Periodic solutions of nonlinear discrete-time systems. Appl. Anal., 62:119–137, 1996.

7. R. E. Gaines and J. L. Mawhin. Coincidence Degree and Nonlinear Differential Equations (Lecture Notes in Mathematics). Springer-Verlag, 1977.

8. R. Iannacci and M.N. Nkashama. Nonlinear two point boundary value problems without Landesman-Lazer condition. Proc. Amer. Math. Soc., 106:943–952, 1989.

9. W. Jiang, B. Wang, and Z. Wang. Solvability of a second-order multi-point boundary-value problems at resonance on a half-line with $\text{dim ker } L = 2$. Electron. J. Differential Equations, 2011(120):1–11, 2011.

10. J. Henderson, S.K. Ntouyas, and I.K. Purnaras. Positive solutions for systems of three-point nonlinear discrete boundary value problems. Neural Parallel Sci. Comput., 16:209–223, 2008.

11. J. Henderson, S.K. Ntouyas, and I.K. Purnaras. Positive solutions for systems of nonlinear discrete boundary value problems. J. Difference Equ. Appl., 15(10):895–912, 2009.

12. J. Henderson and H. B. Thompson. Existence of multiple solutions for second-order discrete boundary value problems. Comput. Math. Appl., 43(10-11):1239–1248, 2002.

13. Johnny Henderson and Rodica Luca. Positive solutions for a system of second-order multi-point discrete boundary value problems. J. Difference Equ. Appl., 18(9):1575–1592, 2012.

14. S. Wang J. Liu and J. Zhang. Multiple solutions for boundary value problems of second order difference equations with resonance. J. Math. Anal. Appl., 374:187–196, 2011.

15. W. G. Kelley and A.C. Peterson. Difference Equations. Academic Press, New York, 1978.

16. E. M. Landesman and A. C. Lazer. Nonlinear perturbations of linear elliptic boundary value problems at resonance. J. Math. Mech. 19, 19:609–623, 1969.

17. A. C. Lazer and D. F. Leach. Bounded perturbations of forced harmonic oscillators at resonance. Ann. Mat. Pure Appl., 82:49–68, 1969.

18. R. Ma. Nonlinear discrete sturm-liouville problems at resonance. Nonlinear Anal., 67:3050–3057, 2007.

19. D. Maroncelli. Scalar multi-point boundary value problems at resonance. Differ. Equ. Appl., 7(4):449–468, 2015.

20. D. Maroncelli and J. Rodríguez. On the solvability of multipoint boundary value problems for discrete systems at resonance. J. Difference Equ. Appl., 20:24–35, 2013.

21. D. Maroncelli and J. Rodríguez. Existence theory for nonlinear Sturm-Liouville problems with unbounded nonlinearities. Differ. Equ. Appl., 6(4):455–466, 2014.

22. D. Maroncelli and J. Rodriguez. On the solvability of nonlinear impulsive boundary value problems. Topol. Methods Nonlinear Anal., 44(2):381–398, 2014.

23. D. Maroncelli and J. Rodríguez. Periodic behaviour of nonlinear, second-order, discrete dynamical systems. J. Difference Equ. Appl., 22(2):280–294, 2015.
[24] David Pollack and Padraic Taylor. Multipoint boundary value problems for discrete nonlinear systems at resonance. *Int. J. Pure Appl. Math.*, 63:311–326, 2010.

[25] J. Rodríguez. Nonlinear discrete Sturm-Liouville problems. *J. Math. Anal. Appl.*, 308, Issue 1:380–391, 2005.

[26] J. Rodríguez and Z. Abernathy. Nonlinear discrete Sturm-Liouville problems with global boundary conditions. *J. Difference Equ. Appl.*, 18:431–445, 2012.

[27] Jesús Rodríguez. An alternative method for boundary value problems with large nonlinearities. *J. Differ. Equ.*, 43:157–167, 1982.

[28] Jesús Rodríguez. Galerkin’s method for ordinary differential equations subject to generalized nonlinear boundary conditions. *J. Differential Equations*, 97:112–126, 1992.

[29] Jesús Rodríguez and Debra L. Etheridge. Periodic solutions of nonlinear second-order difference equations. *Adv. Difference Eqn.*, pages 173–192, 2005, no. 2.

[30] Jesús Rodríguez and D. Sweet. Projection methods for nonlinear boundary value problems. *J. Differ. Equ.*, 58:282–293, 1985.

[31] Jesús Rodríguez and Padraic Taylor. Scalar discrete nonlinear multipoint boundary value problems. *J. Math. Anal. Appl.*, 330:876–890, 2007.

[32] Jesús Rodríguez and Padraic Taylor. Weakly nonlinear discrete multipoint boundary value problems. *J. Math. Anal. Appl.*, 329:77–91, 2007.

[33] Jesús Rodriguez and Padraic Taylor. Multipoint boundary value problems for nonlinear ordinary differential equations. *Nonlinear Anal.*, 68:3465–3474, 2008.

[34] Jesús F. Rodríguez. Existence theory for nonlinear eigenvalue problems. *Appl. Anal.*, 87:293–301, 2008, no. 3.

[35] N. Rouche and J. Mawhin. *Ordinary differential equations*. Pitman, London, 1980.

[36] Z. Zhao and J. Liang. Existence of solutions to functional boundary value problems of second-order nonlinear differential equations. *J. Math. Anal. Appl.*, 373:614–634, 2011.

[37] Z. Guo Z. Zhou, J. Yu. Periodic solutions of higher-dimensional discrete systems. *Proc. Roy. Soc. Edinburgh Sect. A*, 134:1013–1022, 2004.

[38] G. Zhang and R. Medina. Three-point boundary value problems for difference equations. *Comput. Math. Appl.*, 48:1791–1799, 2004.

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