An Algebraic System for Constructing Cryptographic Permutations over Finite Fields

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Abstract

In recent years a new class of symmetric-key primitives over \( \mathbb{F}_p \) that are essential to Multi-party computation and Zero-knowledge proofs based protocols have emerged. Towards improving the efficiency of such primitives, a number of new block ciphers and hash functions over \( \mathbb{F}_p \) were proposed. A number of these constructions show that following alternative design strategies to the classical substitution-permutation network (SPN) and Feistel networks, leads to more efficient cipher and hash function designs over \( \mathbb{F}_p \).

In view of these efforts, in this work we build an algebraic framework that allows the systematic exploration of viable and efficient design strategies for constructing symmetric-key (iterative) permutations over \( \mathbb{F}_p \). We first identify iterative polynomial dynamical systems over finite fields as the central building block of almost all block cipher design strategies. We propose a generalized triangular polynomial dynamical system (GTDS), and based on the GTDS we provide a generic definition of an iterative (keyed) permutation over \( \mathbb{F}_p^n \).

Our GTDS-based generic definition is able to describe the three most well-known design strategies, namely SPNs, Feistel networks and Lai–Massey. Consequently, the block ciphers that are constructed following these design strategies can also be instantiated from our generic definition. Most notably, our generic definition allows for instantiations of novel and efficient (keyed) permutations. For example, we show that the partial SPN-based permutations and the recently proposed Griffin design, which does not explicitly follow Feistel or SPN, can be described using the generic GTDS-based definition.

We further provide generic security analysis of GTDS-based iterative permutations with minimal assumptions. We show that GTDS-based permutations are able to achieve exponential degree growth - a property desirable for constructing efficient cryptographic permutations. In addition, we prove a general upper bound on the differential uniformity of the GTDS. Given that the upper bound on differential uniformity is small enough, we also prove that the distribution of two sampled values by a two round block cipher is \( \epsilon \)-close to the uniform distribution, i.e., the block cipher is \( \epsilon \)-close to being pairwise independent.

Finally, we provide the discrepancy analysis, a technique used to measure the (pseudo-)randomness of a sequence, for analyzing the randomness of the sequence generated by the generic permutation described by GTDS.
1 Introduction

Constructing (keyed and unkeyed) permutations is at the center of designing some of the most broadly used cryptographic primitives like block ciphers and hash functions. After half a century of research, Feistel and Substitution-Permutation Networks (SPN) have emerged as the two dominant iterative design strategies for constructing keyed permutations or block ciphers. Another notable, although not much used design strategy is the Lai–Massey. Altogether, SPN, Feistel and Lai–Massey are at the core of some of most well-known block ciphers such as AES [DBN+01; DR20], DES, CLEFIA [SSA+07], IDEA [LM91], etc.

In the past few years, a new class of symmetric-key cryptographic functions (block ciphers, hash functions and stream ciphers) that are essential in privacy preserving cryptographic protocols based on Multi-party computation and Zero-knowledge proofs, have emerged. For efficiency reasons these primitives are designed over \( \mathbb{F}_p \) as opposed to the classical symmetric primitives over \( \mathbb{F}_{2^n} \) (typically for small \( n \) e.g. \( \leq 8 \)). Following the classical approaches, a number of such symmetric-key functions were constructed either by utilizing the SPN or Feistel design principles. However, current research suggests that these traditional strategies are not the best choices for efficient primitives over \( \mathbb{F}_p \). For example, the partial SPN-based hash function Poseidon [GKR+21] performs more efficiently than the generalized unbalanced Feistel-based construction GMiMCHash [AGP+19]. Another recently proposed design - Griffin [GHR+22], follows neither SPN nor Feistel, and is more efficient than GMiMCHash and Poseidon.

An important and relevant question here is thus: **What is the space of possible design strategies for constructing (efficient) symmetric-key cryptographic permutations/functions over \( \mathbb{F}_p \)?**

Moreover, given that such new cryptographic functions are inherently algebraic by design, their security is dictated by algebraic cryptanalytic techniques. For example, algebraic attacks (interpolation, Gröbner basis, GCD, etc.) [EGL+20; ACG+19; AGR+16; RAS21] are the main attack vectors in determining the security of GMiMC, Poseidon, MiMC [AGR+16], etc.

Hence, a well-defined generic algebraic design framework will prescribe a systematic approach towards exploring viable and efficient design strategies over \( \mathbb{F}_p \). Such generic framework will allow the design of new symmetric-key primitives and will possibly shed new light into the algebraic properties of SPN- and Feistel-based designs, among others, over \( \mathbb{F}_p \). A “good” generic framework should ultimately allow instantiation of primitives over \( \mathbb{F}_q \) where \( q = p^n \) for arbitrary primes \( p \) and naturally encompass existing classical design strategies, such as SPN, Feistel and Lai–Massey.

The primary aim of this work is to find such a general framework which describes iterative algebraic systems for constructing keyed or unkeyed permutations.

1.1 Our Results

In this paper we first discuss (Section 2) that so-called orthogonal systems are the only polynomial systems suitable to represent (keyed) permutations and henceforth block
ciphers over finite fields.

We then propose a novel algebraic system (in Section 3) that is the foundation for constructing generic iterative permutations. More specifically, we construct a polynomial dynamical system over a finite field $\mathbb{F}_q$ (where $q = p^n$ where $p$ and $n \geq 1$) that we call Generalized Triangular Dynamical System (GTDS). We then provide a generic definition of iterative (keyed) permutations using the GTDS and a linear/affine permutation. We show (in Section 4) that our GTDS-based definition of iterative permutations is able to describe the SPN, different types of Feistel networks and the Lai–Massey construction. Consequently, different block ciphers that are instantiations of these design strategies can also be instantiated from the GTDS-based permutation.

Beyond encompassing these well-known design strategies, our framework provides a systematic way to study different algebraic design strategies and security of permutations (with or without key). This is extremely useful in connection with the recent design efforts for constructing block ciphers and hash function over $\mathbb{F}_p$, where $p$ is a large prime. For example, GTDS already covers the recently proposed partial SPN design strategy [GLR+20] used in designing block ciphers and hash functions [GKR+21].

Our GTDS-based definition of iterative permutations allows for instantiations of new (keyed) permutations. For example, the recently proposed construction Griffin can also be instantiated from our generic definition of an iterative permutation. Moreover, using our generic definition we propose a generalization (Section 4.3) of the Lai–Massey design strategy.

We provide a generic analysis (in Section 5.1, Section 5.2 and Section 5.3) of the GTDS-based iterative permutation from a cryptographic perspective. Mathematically, the previously known triangular dynamical system (TDS) by Ostafe and Shparlinski [OS10a] is a special case of our GTDS. In general, while TDS shows polynomial degree growth, the GTDS is able to achieve exponential degree growth (Section 5.1). This is a more desirable property for constructing efficient cryptographic functions. Further, (Section 5.2) we derive a upper bound for the differential uniformity of the GTDS. We also derive mild (but cryptographically relevant) assumptions which ensure that the upper bound is non-trivial. In Section 5.3 we then apply the non-trivial upper bounds on differential uniformity to establish that a two-round block cipher (derived from our generic GTDS-based definition) is $\epsilon$-close to pairwise independence - a notion newly introduced in [LTV21a].

Finally, we consider GTDS-based generic permutation as pseudo-random number generators and compute bounds on the discrepancy (in Section 6) of a sequence generated by such generic block cipher.

\footnote{The earliest version of this paper appeared before the Griffin, which we learned by communicating with the authors.}
2 Block Ciphers and Permutation Polynomials

In general a (block) cipher can be described as a pair of key ed mappings

\[ F : M \times K \to C, \quad F^{-1} : C \times K \to C, \quad (1) \]

where \( M, K \) and \( C \) denote the message, key and cipher domain and such that \( F^{-1}(\_), k) \circ F(\_, k) = \text{id}_M \) for all \( k \in K \). In practice the domains \( M, K \) and \( C \) are finite, thus by [Bar09, Theorem 72] any cipher can be modeled as a mapping between vector spaces over finite fields. In this work we will assume that \( M = C = \mathbb{F}_q^n \) and \( K = \mathbb{F}_q^{n \times r} \), where \( r, n \geq 1, q \) is a prime power and \( \mathbb{F}_q \) is the field with \( q \) elements. For a block cipher we also require that \( F \) is a keyed permutation over \( \mathbb{F}_q^n \), i.e., for all \( k \in \mathbb{F}_q^{n \times r} \) the function \( F(\_, k) \) is a permutation. Note that for any function \( F : \mathbb{F}_q^n \to \mathbb{F}_q^n \) we can find via interpolation a unique polynomial \( P \in \mathbb{F}_q[x_1, \ldots, x_n] \) with degree less than \( q \) in each variable such that \( F(x) = P(x) \) for all \( x \in \mathbb{F}_q^n \). Therefore, we will also interpret all ciphers as vectors of polynomial valued functions. We recall the formal (algebraic) notion of permutation inducing polynomial vectors.

**Definition 2.1** ([LN97, 7.34., 7.35. Definition]). Let \( \mathbb{F}_q \) be a finite field.

1. A polynomial \( f \in \mathbb{F}_q[x_1, \ldots, x_n] \) is called a permutation polynomial if the equation \( f(x_1, \ldots, x_n) = \alpha \) has \( q^n - 1 \) solutions in \( \mathbb{F}_q \) for each \( \alpha \in \mathbb{F}_q \).

2. A system of polynomials \( f_1, \ldots, f_m \in \mathbb{F}_q[x_1, \ldots, x_n] \), where \( 1 \leq m \leq n \), is said to be orthogonal if the system of equations \( f_1(x_1, \ldots, x_n) = \alpha_1, \ldots, f_m(x_1, \ldots, x_n) = \alpha_m \) has exactly \( q^n - m \) solutions in \( \mathbb{F}_q^n \) for each \( (\alpha_1, \ldots, \alpha_m) \in \mathbb{F}_m^n \).

Over \( \mathbb{F}_2 \), the permutation polynomials are known as balanced functions [BCD11] in the cryptography/computer science literature.

**Remark 2.2.** It is immediate from the definition that every subset of an orthogonal system is also an orthogonal system. In particular, every polynomial in an orthogonal system is also a multivariate permutation polynomial. If for an orthogonal system \( m = n \), then the orthogonal system induces a permutation on \( \mathbb{F}_q^n \). Moreover, if we restrict orthogonal system to the \( \mathbb{F}_q \)-algebra of polynomial valued functions \( \mathbb{F}_q[x_1, \ldots, x_n]/(x_1^q - x_1, \ldots, x_n^q - x_n) \), that is the polynomials with degree less than \( q \) in each variable, then the orthogonal systems of size \( n \) form a group under composition. If we denote this group with \( \text{Orth}_n(\mathbb{F}_q) \), then one can establish the following isomorphisms of groups \( \text{Orth}_n(\mathbb{F}_q) \cong \text{Sym}(\mathbb{F}_q^n) \cong \text{Sym}(\mathbb{F}_{q^n}) \), where \( \text{Sym}(\_) \) denotes the symmetric group (cf. [LN97, 7.45. Corollary]).

Since one of our main interests is in keyed permutations let’s extend the definition of orthogonal systems. In general, we will denote with \( x \) plain text variables and with \( y \) key variables.

**Definition 2.3.** Let \( \mathbb{F}_q \) be a finite field.
(1) Let $F : \mathbb{F}_q^n \times \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$ be a function. We call $F$ a keyed permutation, if for any fixed $y \in \mathbb{F}_q^n$ the function $F(\cdot, y) : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$ induces a permutation.

(2) Let $f_1, \ldots, f_n \in \mathbb{F}_q[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be polynomials. We call $f_1, \ldots, f_n$ a keyed orthogonal system, if for any fixed $(y_1, \ldots, y_n) \in \mathbb{F}_q^n$ the system $f_1, \ldots, f_n$ is an orthogonal system.

**Remark 2.4.** (1) Note that in our definition we allow for trivial keyed permutations. In particular, every permutation $F : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$ is also a keyed permutation.

(2) A keyed orthogonal system is also an orthogonal system in $\mathbb{F}_q[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Suppose we are given a keyed orthogonal system $f_1, \ldots, f_n \in \mathbb{F}_q[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and equations $f_i(x, y) = \alpha_i$, where $\alpha_i \in \mathbb{F}_q$. If we fix $y$ then we find a unique solution for $x$. There are $q^n$ possible choices for $y$ so the system has $q^n$ solutions. On the other hand, a system of $n$ polynomials in $2n$ variables is orthogonal if it has $q^{2n-n} = q^n$ solutions. Hence, our definition of keyed orthogonal systems does not induce any essentially new structure, it is merely semantic.

As intuition suggests keyed orthogonal systems are well-behaved under iteration. We state the following theorem for completeness.

**Theorem 2.5.** Let $\mathbb{F}_q$ be a finite field. The keyed system $f_1, \ldots, f_n \in \mathbb{F}_q[x_1, \ldots, x_n, y_1, \ldots, y_n]$ is keyed orthogonal for a fixed $y = (y_1, \ldots, y_n)$ if and only if the system $g_1(f_1, \ldots, f_n, y_1, \ldots, y_n), \ldots, g_n(f_1, \ldots, f_n, y_1, \ldots, y_n) \in \mathbb{F}_q[x_1, \ldots, x_n, y_1, \ldots, y_n]$ is keyed orthogonal for every keyed orthogonal system $g_1, \ldots, g_n \in \mathbb{F}_q[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

**Proof.** $\Rightarrow$: Simply choose $g_i = x_i$.

$\Rightarrow$: Suppose we are given a system of equations

$$g_1(f_1, \ldots, f_n, y_1, \ldots, y_n) = \beta_1,$$

$$\ldots$$

$$g_n(f_1, \ldots, f_n, y_1, \ldots, y_n) = \beta_n,$$

where $\beta_1, \ldots, \beta_n \in \mathbb{F}_q$ and $\{f_i\}_{1 \leq i \leq n}$ and $\{g_i\}_{1 \leq i \leq n}$ are keyed orthogonal systems. Fix $y \in \mathbb{F}_q^n$ and substitute $\hat{x}_i = f_i$, then the equations $g_i(\hat{x}_1, \ldots, \hat{x}_n, y) = \beta_i$ have a unique solution for the $\hat{x}_i$’s. Since $y$ is fixed also the equations $\hat{x}_i = f_i$ admit a unique solution. Therefore, the composition of keyed orthogonal systems is again keyed orthogonal.

In practice keyed orthogonal systems are usually derived from orthogonal systems by a simple addition of the key variables before or after evaluation of a function.

**Example 2.6.** If $F : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$ is a permutation, then

$$F(x + y) \quad \text{and} \quad F(x) + y$$

are keyed permutations.
We want to finish with a warning and a counterexample: Not every orthogonal system of \( n \) polynomials in \( 2n \) variables induces a keyed orthogonal system of \( n \) polynomials!

**Example 2.7.** Consider the polynomial \( P \in \mathbb{F}_3[x,y] \) defined as
\[
P(x,y) = x^2 + y.
\]
It can be checked by hand that \( P \) is a permutation polynomial. However, if we fix \( y \) then we do not obtain a univariate permutation polynomial, because squaring does not induce a permutation over \( \mathbb{F}_3 \).

### 3 Generalized Triangular Dynamical Systems

We propose the generalized triangular dynamical system (GTDS) as the main ingredient when designing a block cipher. The GTDS is also the main ingredient in unifying different design principles proposed in the literature such as SPN and Feistel networks.

**Definition 3.1** (Generalized triangular dynamical system). Let \( \mathbb{F}_q \) be a finite field, and let \( n \geq 1 \). For \( 1 \leq i \leq n \) let \( p_i \in \mathbb{F}_q[x] \) be permutation polynomials, and for \( 1 \leq i \leq n-1 \) let \( g_i, h_i \in \mathbb{F}_q[x_{i+1}, \ldots, x_n] \) be polynomials such that the polynomials \( g_i \) do not have zeros over \( \mathbb{F}_q \). Then we define a generalized triangular dynamical system \( \mathcal{F} = \{f_1, \ldots, f_n\} \) as follows
\[
\begin{align*}
f_1(x_1, \ldots, x_n) &= p_1(x_1) \cdot g_1(x_2, \ldots, x_n) + h_1(x_2, \ldots, x_n), \\
f_2(x_1, \ldots, x_n) &= p_2(x_2) \cdot g_2(x_3, \ldots, x_n) + h_2(x_3, \ldots, x_n), \\
&\quad \vdots \\
f_{n-1}(x_1, \ldots, x_n) &= p_{n-1}(x_{n-1}) \cdot g_{n-1}(x_n) + h_{n-1}(x_n), \\
f_n(x_1, \ldots, x_n) &= p_n(x_n).
\end{align*}
\]

**Proposition 3.2.** A generalized triangular dynamical system is an orthogonal system.

**Proof.** Suppose for \( 1 \leq i \leq n \) we are given equations
\[
f_i(x_i, \ldots, x_n) = \alpha_i,
\]
where \( \alpha_i \in \mathbb{F}_q \). To solve the system we work upwards. The last polynomial \( f_n \) is a permutation polynomial, so we can find a unique solution \( \beta_n \) for \( x_n \). We plug this solution into the next equation, i.e., \( f_{n-1}(x_{n-1}, \beta_n) = p_{n-1}(x_{n-1}) \cdot f_n(\beta_n) + h(\beta_n) \). To solve for \( x_{n-1} \) we subtract \( h(\beta_n) \), divide by \( g(\beta_n) \) and invert \( p_{n-1} \). Iterating this procedure we can find a unique solution for all \( x_i \).

**Corollary 3.3.** The inverse system \( \mathcal{F}^{-1} = \{\bar{f}_1, \ldots, \bar{f}_n\} \) to the generalized triangular
dynamical system \( \mathcal{F} = \{f_1, \ldots, f_n\} \) is given by
\[
\tilde{f}_1(x_1, \ldots, x_n) = p_1^{-1} \left( (x_1 - h_1(\tilde{f}_2, \ldots, \tilde{f}_n)) \cdot \left( g_1(\tilde{f}_2, \ldots, \tilde{f}_n) \right)^{q-2} \right)
\]
\[
\tilde{f}_2(x_1, \ldots, x_n) = p_2^{-1} \left( (x_2 - h_2(\tilde{f}_3, \ldots, \tilde{f}_n)) \cdot \left( g_2(\tilde{f}_3, \ldots, \tilde{f}_n) \right)^{q-2} \right)
\]
\[
\vdots
\]
\[
\tilde{f}_{n-1}(x_1, \ldots, x_n) = p_{n-1}^{-1} \left( (x_{n-1} - h_{n-1}(\tilde{f}_n)) \cdot \left( g_{n-1}(\tilde{f}_n) \right)^{q-2} \right)
\]
\[
\tilde{f}_n(x_1, \ldots, x_n) = p_n^{-1}(x_n).
\]

Proof. If we consider \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) in \( \mathbb{F}_q[x_1, \ldots, x_n]/(x_1^q - x_1, \ldots, x_n^q - x_n) \), then it is easy to see that \( \mathcal{F}^{-1} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{F}^{-1} = \text{id.} \)

Note that the Triangular Dynamical System introduced by Ostafe and Shparlinski [OS10a] is a special case of our GTDS. In particular, if we choose \( p_i(x_i) = x_i \) for all \( i \) and impose the condition that each polynomial \( g_i \) has a unique leading monomial, i.e.,
\[
g_i(x_{i+1}, \ldots, x_n) = x_{i+1}^{s_{i+1}} \cdots x_n^{s_n} + \tilde{g}_i(x_{i+1}, \ldots, x_n),
\]
where
\[
\deg (\tilde{g}) < s_{i+1} + \cdots + s_n
\]
and
\[
\deg (h_i) \leq \deg (g_i)
\]
for \( i = 1, \ldots, n - 1 \), then we obtain the original triangular dynamical systems. Notice that under iteration these systems exhibit a property highly uncommon for general polynomial dynamical systems: polynomial degree growth, see [OS10a, § 2.2].

3.1 GTDS and (Keyed) Permutations

Every keyed permutation or block cipher (in cryptography) is constructed using an iterative structure where round functions are iterated a fixed number of times. Using the GTDS we first define such a round function. In this section \( n \in \mathbb{N} \) denotes the number of field elements constituting a block and \( r \in \mathbb{N} \) denotes the number of rounds of an iterative permutation.

**Definition 3.4 (Round function).** Let \( \mathbb{F}_q \) be a finite field, let \( n \geq 1 \) be an integer, let \( A \in \mathbb{F}_q^{n \times n} \) be an invertible matrix, and let \( b \in \mathbb{F}_q^n \) be a vector. Then, the affine mixing layer is described by the map
\[
\mathcal{L} : \mathbb{F}_q^n \to \mathbb{F}_q^n, \quad x \mapsto A \cdot x + b,
\]
and the key addition is described by the map
\[
\mathcal{K} : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{F}_q^n, \quad (x, k) \mapsto x + k.
\]
We abbreviate $K_k = K_{(\_ , k)}$. Let $F \subset \mathbb{F}_q[x_1,\ldots,x_n]$ be a GTDS. Then the round function of a block cipher is defined as the following composition

$$R : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{F}_q^n, \quad (x, k) \mapsto K_k \circ L \circ \tilde{F}(x).$$

where $\tilde{F} = F$ or $\tilde{F}$ is a composition of two or more $F$ and $L$. We also abbreviate $R_k = R(\_ , k)$.

It is obvious that $R$ is keyed permutation, hence it also is a keyed orthogonal system of polynomials in the sense of Definition 2.3. Now we can introduce our generalized notion of block ciphers which encompasses almost all existing block ciphers.

**Definition 3.5** (An algebraic description of keyed permutations). Let $\mathbb{F}_q$ be a finite field, let $n, r \geq 1$ be integers, and let $K \in \mathbb{F}_q^{n \times (r+1)}$ be a matrix. We index the columns of $K$ by $0,\ldots,r$, then the $i$th column $k_i$ denotes the $i$th round key. Let $K : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{F}_q^n$ be the key addition function, and let $R^{(1)},\ldots,R^{(r)} : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{F}_q^n$ be the round functions. Then a block cipher is defined as the following composition

$$C_r : \mathbb{F}_q^n \times \mathbb{F}_q^{n \times (r+2)} \to \mathbb{F}_q^n, \quad (x, K) \mapsto R^{(r)}_k \circ \cdots \circ R^{(1)}_k \circ K_{k_0}(x).$$

We abbreviate $C_{r,K} = C_r(\_ , K)$, and if the round functions are clear from context or identical, then we also abbreviate $R^r_k = R^{(r)}_k \circ \cdots \circ R^{(1)}_k$.

For the remaining parts of the paper a keyed permutation or a block cipher should be understood as a function described as in Definition 3.5, unless specified otherwise.

Note that the round $l \geq 0$ of an iterative construction that is described by the GTDS can be written as a vector of polynomial functions i.e. $(f_1^{(l)},\ldots,f_n^{(l)})$. We stress that a generic definition of an iterative block cipher may only use the notion of round key(s) (as defined with $K$ in Definition 3.5) and does not require explicit definition of key scheduling function. For example, defining $h_i(l)$ with different additive constants $c_i^{(l)}$ for different rounds $l$ captures the notion of round constants in iterative block cipher or (cryptographic) permutations.

**Remark 3.6.** We stress that a generic definition of an iterative block cipher may only use the notion of round key(s) (as defined with $K$ in Definition 3.5) and does not require explicit definition of key scheduling function. This is because specific definition of key scheduling that can depend on the input key size and specific instantiation of the iterative block cipher, is not necessary for generic definition and analysis. In cryptography literature generic definition, (security) analysis and security proofs of iterative block ciphers (e.g. SPN, Even-Mansour etc.) only use with the notion of round keys [LPS12; DDK+14; CS15].

4 Instantiating Block Ciphers

In this section we will show that the GTDS-based algebraic definition of iterative permutation is able to describe different design strategies.
We note with respect to GTDS that well-known design strategies such as SPN, partial SPN, Feistel, generalized Feistel and Lai–Massey are constructed with trivial polynomials $g_i$ with no zeros, namely $g_i = 1$.

### 4.1 Feistel Networks

For simplicity we show how GTDS-based algebraic definition can describe two-branch Feistel, and unbalanced Feistel with expanding round function. It is straightforward to show that GTDS-based algebraic definition can describe other types of Feistel networks such as unbalanced Feistel with expanding round functions, Nyberg’s GFN, etc.

#### 4.1.1 Balanced Feistel

The balanced Feistel network invented by Horst Feistel is one of the oldest and most studied constructions in modern cryptography. The GTDS

\[
\begin{align*}
    f_1(x, y) &= x + f(y), \\
    f_2(x, y) &= y,
\end{align*}
\]

where $f \in \mathbb{F}_q[x]$, and the affine layer is the swap permutation $(x, y) \mapsto (y, x)$ describes the two branch Feistel network.

#### 4.1.2 Unbalanced Feistel

Let $n > 1$, and let $f \in \mathbb{F}_q[x]$ be any function represented by a polynomial. Then for $l \geq 0$, the unbalanced Feistel network with expanding round function is defined as

\[
\begin{pmatrix}
    x_1^{(l)} \\
    \vdots \\
    x_{n-1}^{(l)} \\
    x_n^{(l)}
\end{pmatrix}
\mapsto
\begin{pmatrix}
    x_1^{(l)} \\
    \vdots \\
    x_{n-1}^{(l)} \\
    x_n^{(l)} + f(x_n^{(l)})
\end{pmatrix}
= 
\begin{pmatrix}
    x_1^{(l+1)} \\
    \vdots \\
    x_{n-1}^{(l+1)} \\
    x_n^{(l+1)}
\end{pmatrix}.
\]

The GTDS

\[
\begin{align*}
    f_i(x_1, \ldots, x_n) &= x_i + f(x_n), \\
    f_n(x_1, \ldots, x_n) &= x_n,
\end{align*}
\]

where $1 \leq i \leq n - 1$ together with the linear transformation $\mathcal{L}$ defined by a shift permutation

\[
(x_1, \ldots, x_{n-1}, x_n) \mapsto (x_n, x_1, \ldots, x_{n-1}).
\]

describes the UFN with expanding round function.

The total degree growth of Feistel ciphers that only use swap or shift permutations is described by Proposition 5.4 and Corollary 5.5.
4.2 Substitution-Permutation Networks

In [KL20, §7.2.1] a handy description of substitution-permutation networks (SPN) was given. Let \( S \in \mathbb{F}_q[x] \) be a permutation polynomial, the so called S-box. Then the round function of a SPN consists of three parts:

1. Addition of the round keys.
2. Application of the S-box, i.e.,
   \[
   (x_1, \ldots, x_n) \mapsto (S(x_1), \ldots, S(x_n)).
   \] (9)
3. Permutation and mixing of the blocks.

The mixing in the last step is usually done via linear/affine transformations. In this case the GTDS of a SPN reduces to

\[
 f_i(x_1, \ldots, x_n) = S(x_i),
\]

where 1 ≤ \( i \) ≤ \( n \). If the last step is not linear then one either must introduce additional GTDS as round functions or modify the GTDS in Equation (10).

Observe that the degree growth of such a SPN is described by Proposition 5.8.

4.2.1 AES-128

At the time of writing the most famous SPN is the AES family [DBN+01]. If we use the description of AES-128 given in [BPW06], then it is easy to see that AES-128 is also covered by our definition of block ciphers. AES-128 is defined over the field \( \mathbb{F} = \mathbb{F}_{2^8} \) and has 16 blocks, i.e., it is a keyed permutation over \( \mathbb{F}^{16} \). The AES-128 S-box is given by

\[
 S : \mathbb{F} \to \mathbb{F},
\]

\[
 x \mapsto 05x^{254} + 09x^{253} + F9x^{251} + 25x^{247} + F4x^{239} + x^{223} + B5x^{191} + 8Fx^{127} + 63,
\]

and the GTDS of AES-128 is given by Equation (9).

Let’s now describe the permuting and mixing of the blocks via linear transformations. The \texttt{ShiftRows} operations can be described with the block matrix

\[
 D_{SR} = \begin{pmatrix}
 D_{SR_0} & 0 & 0 & 0 \\
 0 & D_{SR_1} & 0 & 0 \\
 0 & 0 & D_{SR_2} & 0 \\
 0 & 0 & 0 & D_{SR_3}
\end{pmatrix} \in \mathbb{F}^{16 \times 16},
\]

where

\[
 D_{SR_t} = \left( \Delta_{i,(j-t) \mod 4} \right) \in \mathbb{F}^{4 \times 4}.
\]
The MixColumns operation can be described as the following tensor product

\[ D_{MC} = \begin{pmatrix} 02 & 03 & 01 & 01 \\ 01 & 02 & 03 & 01 \\ 01 & 01 & 02 & 03 \\ 03 & 01 & 01 & 02 \end{pmatrix} \otimes I_4 \in \mathbb{F}_{16}^{16 \times 16}, \]  

where the entries in the left matrix are hexadecimal representations of field elements. The linear mixing layer \( \mathcal{L} \) of AES-128 can now be represented by the following matrix \( D \)

\[ D = P \cdot D_{MC} \cdot D_{SR} \cdot P, \]  

where \( P \in \mathbb{F}_{16}^{16 \times 16} \) denotes the transposition matrix. In the last round the MixColumns operation is dropped, hence \( \tilde{\mathcal{L}} \) is represented by \( \tilde{D} \)

\[ \tilde{D} = P \cdot D_{SR} \cdot P. \]  

In a similar fashion we can also describe the key schedule of AES-128.

### 4.2.2 Partial SPN

In a partial SPN the S-box is only applied to some of the input variables not all of them. This construction was proposed for ciphers like LowMC [ARS+15], the Hades design strategy [GLR+20] and the Poseidon family [GKR+21] that are efficient in the MPC setting. Clearly, any partial SPN is also covered by the GTDS.

### 4.3 Lai–Massey Ciphers and GTDS

#### 4.3.1 Lai–Massey

Another well-known design strategy for block ciphers is the Lai–Massey design which was first introduced in [Lai92]. For two branches let \( f \in \mathbb{F}_q[x] \) be a polynomial, then the round function of the Lai–Massey cipher is defined as

\[ \mathcal{F}_{LM} : (x, y) \mapsto (x + f(x - y), y + f(x - y)). \]

Since the difference between the branches is invariant under application of \( \mathcal{F}_{LM} \) it is possible to invert the construction. At the first look it may appear that the Lai–Massey can not be described with GTDS. However, a careful analysis shows one round of Lai–Massey is in fact a composition of a Feistel Network and two linear permutations. We consider the following triangular dynamical systems

\[ \mathcal{F}_1(x, y) = \begin{pmatrix} x - y \\ y \end{pmatrix}, \quad \mathcal{F}_2(x, y) = \begin{pmatrix} x \\ y + F(x) \end{pmatrix}, \quad \mathcal{F}_3(x, y) = \begin{pmatrix} x + y \\ y \end{pmatrix}. \]

Then, it is easily checked that \( \mathcal{F}_{LM} = \mathcal{F}_3 \circ \mathcal{F}_2 \circ \mathcal{F}_1. \)
4.3.2 Generalized Lai–Massey

Recently, a generalization of the Lai–Massey was proposed in [GOP+21, §3.3] by Grassi et al. It is based on the following observation: If one is given field elements $\omega_1, \ldots, \omega_n \in \mathbb{F}_q$ such that $\sum_{i=1}^{n} \omega_i = 0$, then the mapping

$$
\begin{pmatrix}
    x_1 \\
    \vdots \\
    x_n
\end{pmatrix} \mapsto
\begin{pmatrix}
    x_1 + g(\sum_{i=1}^{n} \omega_i x_i) \\
    \vdots \\
    x_n + g(\sum_{i=1}^{n} \omega_i x_i)
\end{pmatrix}
$$

(19)

is invertible for any polynomial $f \in \mathbb{F}_q[x]$.

We will use this observations to propose an even more general version of the Lai–Massey from the GTDS and linear permutations.

**Definition 4.1** (Generalized Lai–Massey). Let $\mathbb{F}_q$ be a finite field, and let $n \geq 2$ and $1 \leq N \leq n - 1$ be integers. Let $\omega_1, \ldots, \omega_n \in \mathbb{F}_q$ be such that $\sum_{i=1}^{n} \omega_i = 0$ and assume that there is $2 \leq m \leq n$ such that $\omega_i \neq 0$ for at least one $1 \leq i \leq m - 1$ and $i = m$ and $\omega_i = 0$ for all $i > m$. For $1 \leq i \leq n$ let $p_i \in \mathbb{F}_q[x]$ be permutation polynomials, and let $g \in \mathbb{F}_q[x, x_{m+1}, \ldots, x_n]$ be a polynomial. Then we define the generalized Lai–Massey $F_{LM} = \{f_1, \ldots, f_n\}$ as follows

$$
\begin{align*}
    f_1(x_1, \ldots, x_n) &= p_1(x_1) + g \left( \sum_{i=1}^{m} \omega_i p_i(x_i), x_{m+1}, \ldots, x_n \right), \\
    \vdots \\
    f_m(x_1, \ldots, x_n) &= p_m(x_m) + g \left( \sum_{i=1}^{m} \omega_i p_i(x_i), x_{m+1}, \ldots, x_n \right), \\
    f_{m+1}(x_1, \ldots, x_n) &= p_{m+1}(x_{m+1}), \\
    \vdots \\
    f_n(x_1, \ldots, x_n) &= p_n(x_n).
\end{align*}
$$

**Remark 4.2.** If $n \equiv 0 \mod 2$, then it is evident from the first equation in the proof of [GOP+21, Proposition 5] that Grassi et al.’s generalized Lai–Massey permutation is also covered by Definition 4.1 and a linear transformation.

For completeness we establish that the generalized Lai–Massey is indeed invertible.

**Lemma 4.3.** Let $\mathbb{F}_q$ be a finite field. The generalized Lai–Massey is an orthogonal system.

**Proof.** Suppose we are given equations $f_i(x_1, \ldots, x_n) = \alpha_i$, where $\alpha_i \in \mathbb{F}_q$. For $i = m+1, \ldots, n$ we simply invert $p_i$ to solve for $x_i$. For $i = 1, \ldots, m$ we compute $\sum_{i=1}^{m} \omega_i f_i = \sum_{i=1}^{m} \omega_i p_i(x_i) = \sum_{i=1}^{m} \omega_i \alpha_i = \alpha$. Now we plug $\alpha$ and the solutions for $x_{m+1}, \ldots, x_n$ into the polynomial $g$ in the first $m$ equations, rearrange them, and invert the univariate permutation polynomials to obtain an unique solution. □
Before we prove the reduction we explain some of the rationale behind Definition 4.1. Usually, in the Lai–Massey the polynomial $g$ is added to all the branches, but our definition allows the concatenation of two independent Lai–Massey permutations

$$
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 \\
\end{pmatrix} \mapsto \begin{pmatrix}
    x_1 + g_1(x_1 - x_2) \\
    x_2 + g_1(x_1 - x_2) \\
    x_3 + g_2(x_3 - x_4) \\
    x_4 + g_2(x_3 - x_4) \\
\end{pmatrix}.
$$

(20)

Further, it is possible to construct intertwined Lai–Massey permutations

$$
\begin{pmatrix}
    x_1 \\
    \vdots \\
    x_m \\
    x_{m+1} \\
    \vdots \\
    x_n \\
\end{pmatrix} \mapsto \begin{pmatrix}
    x_1 + g_1\left(\sum_{i=1}^m \omega_i x_i, \sum_{i=m+1}^n \omega'_i x_i\right) \\
    \vdots \\
    x_m + g_1\left(\sum_{i=1}^m \omega_i x_i, \sum_{i=m+1}^n \omega'_i x_i\right) \\
    x_{m+1} + g_2\left(\sum_{i=m+1}^n \omega'_i x_i\right) \\
    \vdots \\
    x_n + g_2\left(\sum_{i=m+1}^n \omega'_i x_i\right) \\
\end{pmatrix}.
$$

(21)

By now it should be a surprise to the reader that we will use the same strategy as for the two branch Lai–Massey to derive the generalized one as composition of several GTDS.

**Theorem 4.4.** Let $\mathbb{F}_q$ be a finite field. The generalized Lai–Massey can be constructed with generalized triangular dynamical systems.

**Proof.** The first dynamical system is the application of the univariate permutation polynomials to the first $m$ branches

$$\mathcal{F}_1 : \begin{pmatrix}
    x_1 \\
    \vdots \\
    x_m \\
    x_{m+1} \\
    \vdots \\
    x_n \\
\end{pmatrix} \mapsto \begin{pmatrix}
    p_1(x_1) \\
    \vdots \\
    p_m(x_m) \\
    x_{m+1} \\
    \vdots \\
    x_n \\
\end{pmatrix}.$$

In the second one we construct the sum with the $\omega'_i$'s

$$\mathcal{F}_2 : \begin{pmatrix}
    x_1 \\
    \vdots \\
    x_m \\
    x_{m+1} \\
    \vdots \\
    x_n \\
\end{pmatrix} \mapsto \begin{pmatrix}
    x_1 \mapsto \begin{cases}
        \omega_i x_i, & \omega_i \neq 0, \\
        x_i, & \omega_i = 0
    \end{cases} \\
    x_i \mapsto \sum_{i=1}^m \omega_i x_i \\
    x_{m+1} \\
    \vdots \\
    x_n \\
\end{pmatrix}.$$
In the third one we add the polynomial $g$ to the first $m - 1$ branches, though we have to do a case distinction whether $\omega_i \neq 0$ or not,

$$F_3 : \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_n \end{pmatrix} \mapsto \left\{ \begin{array}{ll} x_i + \omega_i g(x_m, x_{m+1}, \ldots, x_n), & \omega_i \neq 0, \\ x_i + g(x_m, x_{m+1}, \ldots, x_n), & \omega_i = 0 \end{array} \right\}_{1 \leq i \leq m-1}$$

Then we add the polynomial $g$ to the $m$th branch and cancel the factors $\omega_i$ whenever necessary

$$F_4 : \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_n \end{pmatrix} \mapsto \left\{ \begin{array}{ll} \omega_i^{-1} x_i, & \omega_i \neq 0, \\ x_i, & \omega_i = 0 \end{array} \right\}_{1 \leq i \leq m-1} \omega_m^{-1} \left( x_m - \sum_{1 \leq i \leq m-1} x_i \right) = 0.$$  

Lastly, we apply the univariate permutation polynomials to the remaining branches

$$F_5 : \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ p_{m+1}(x_{m+1}) \\ \vdots \\ p_n(x_n) \end{pmatrix}.$$  

Now it follows from a simple calculation that indeed $F_5 \circ \cdots \circ F_1$ implements the generalized Lai–Massey construction.

\[\Box\]

### 4.4 Constructions with Non-Trivial Polynomials with No Zeros

Recall that for $1 \leq i \leq n - 1$ the $i$th branch in a GTDS is given by

$$f_i(x_1, \ldots, x_n) = p_i(x_i) \cdot g_i(x_{i+1}, \ldots, x_n) + h_i(x_{i+1}, \ldots, x_n),$$  \hspace{1cm} (22)

where $g_i$ is a polynomial that does not have any zeros. All constructions we have investigated so far have one thing in common, they all use trivial $g_i$’s, that is $g_i = 1$. Therefore, it is now time to cover constructions that have non-trivial $g_i$’s.
4.4.1 Reinforced Concrete

Reinforced Concrete [BGK+21] is proposed as hash function for efficient implementation of proof systems. The Reinforced Concrete permutation over $\mathbb{F}_p^3$ where $p \geq 2^{64}$ is a prime and consists of three small permutations. The first permutation is the mapping \textit{Bricks}

\[ \text{Bricks}: \mathbb{F}_p^3 \rightarrow \mathbb{F}_p^3, \]

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1^d \\ x_2 \cdot (x_3^2 + \alpha_1 \cdot x_1 + \beta_1) \\ x_3 \cdot (x_2^2 + \alpha_2 \cdot x_2 + \beta_2) \end{pmatrix}, \tag{23} \]

where $d = 5$, note that the prime must be suitable chosen such that $\gcd (d, p - 1) = 1$ else the first component does not induce a permutation, and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}_p$ such that $\alpha_i^2 - 4\beta_i$ is not quadratic residue module $p$, then the quadratic polynomials do not have any zeros over $\mathbb{F}_p$. The second permutation is the mapping \textit{Concrete} which is given by an maximum distance separable matrix (MDS) and a constant

\[ \text{Concrete}: \mathbb{F}_p^3 \rightarrow \mathbb{F}_p^3, \]

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + c, \tag{24} \]

where $c \in \mathbb{F}_p^3$ is a constant. The third permutation is the function \textit{Bars} which is a simple S-box

\[ \text{Bars}: \mathbb{F}_p^3 \rightarrow \mathbb{F}_p^3, \]

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} \text{Bar}(x_1) \\ \text{Bar}(x_2) \\ \text{Bar}(x_3) \end{pmatrix}, \tag{25} \]

where $\text{Bar}: \mathbb{F}_p \rightarrow \mathbb{F}_p$ is a univariate permutation. Unlike all the other mappings $\text{Bar}$ does not admit a simple polynomial description, therefore for more details on it we refer the reader to [BGK+21, §VI.C]. Clearly, the mappings \textit{Bricks}, \textit{Concrete} and \textit{Bars} are covered by the GTDS. The Reinforced Concrete permutation is now given by

\[ \text{RC} = \text{Concrete}^{(8)} \circ \text{Bricks} \circ \text{Concrete}^{(7)} \circ \text{Bricks} \circ \text{Concrete}^{(6)} \]

\[ \circ \text{Bricks} \circ \text{Concrete}^{(5)} \circ \text{Bricks} \circ \text{Concrete}^{(4)} \circ \text{Bricks} \]

\[ \circ \text{Concrete}^{(3)} \circ \text{Bricks} \circ \text{Concrete}^{(2)} \circ \text{Bricks} \circ \text{Concrete}^{(1)}. \tag{26} \]

4.4.2 Horst & Griffin

The \textit{Horst} scheme [GHR+22] was introduced as generalization of the Feistel scheme. One can recover it from the GTDS by choosing all univariate permutation polynomials to be the identity. Consequently, the permutation $\text{Griffin-}\pi$ [GHR+22] that is based on the \textit{Horst} construction is also covered by the GTDS framework.
4.5 Other Block Ciphers with Triangular Structures

All ciphers we have discussed so far can be summarized as key-alternating ciphers. However, triangular dynamical systems as in Definition 3.1 also appear in different constructions. One example is the cipher Ciminion [DGG+21] which utilizes a Farfalle-like construction. The round function of Ciminion is the combination of a linear transformation and the Toffoli-gate, see [Tof80],

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z + x \cdot y \end{pmatrix}. \tag{27}
\]

Clearly, the Toffoli-gate is covered by Definition 3.1. Though, Ciminion is not covered by our definition of block ciphers, but we still believe that the algebraic description of Ciminion is closely related to the unified algebraic description.

5 Analysis of GTDS-based Permutations

5.1 Degree Growth of Block Ciphers

For algebraic security analysis of block ciphers it is important to understand the polynomial degree growth over \( \mathbb{F}_q[x_1, \ldots, x_n] \) of the polynomials that represent the branches. During this section we will ignore the keys, we only investigate the degree growth in the input variables of a permutation described by a block cipher.

Before we start also recall that any multivariate function over a finite field can be viewed as polynomial in \( \mathbb{F}_q[X] = \mathbb{F}_q[x_1, \ldots, x_n]/(x_1^q - x_1, \ldots, x_n^q - x_n) \). If one iterates a polynomial vector over \( \mathbb{F}_q[X] \) one can split the degree growth into two stages. Until the first degree overflow in one of the input variables occurs, i.e., until a non-trivial reduction modulo \((x_1^q - x_1, \ldots, x_n^q - x_n)\) occurs, we say that the system is in the exponential degree growth phase. During this phase the degree growth over \( \mathbb{F}_q[X] \) is identical to the growth over the usual polynomial ring \( \mathbb{F}_q[x_1, \ldots, x_n] \). This is the scenario we will study in this section. After the first degree overflow we say that the degree growth is in the saturation phase. Note that for precise security claims it is important to understand the degree growth in the saturation phase. To the best of our knowledge previous research on degree growth [BCD11; EGL+20; CGG+22] in the saturation phase has focused on SPN-like constructions or univariate constructions over \( \mathbb{F}_2 \) and \( \mathbb{F}_{2^n} \).

In order to keep the degree growth analysis generic as well as obtain interesting result we need to impose some conditions that are cryptographically sensible, on the polynomials defining the GTDS. We introduce the notion of well-behaved GTDS to describe the polynomial degree growth of an iterative (keyed or unkeyed) permutation.

Definition 5.1 (Well-behaved GTDS). Let \( \mathbb{F}_q \) be a finite field, let \( \mathcal{C}_r \) be a block cipher with \( n > 1 \) blocks and \( r \) rounds, let \( \mathcal{R}^{(1)}, \ldots, \mathcal{R}^{(r)} \) denote the \( r \) round functions. Assume that for all \( 1 \leq l \leq r \) one of the following cases is satisfied.
(Case 1) In the $l$th round we have for all $1 \leq i \leq n$ that

$$\deg(h_i^{(l)}) \leq \deg(g_i^{(l)}) + \deg(p_i^{(l)})$$

and $g_i^{(l)}$ has a unique leading monomial, i.e.,

$$g_i^{(l)}(x_{i+1}, \ldots, x_n) = x_{i+1}^{s_{i,n}} + g_i^{(l)}(x_{i+1}, \ldots, x_n),$$

for some $g_i^{(l)} \in \mathbb{F}_q[x_{i+1}, \ldots, x_n]$ with $\deg(g_i^{(l)}) < s_{i,n} + s_{i,n}$. Then we define

$$S_i = \begin{pmatrix} d_1 & s_{1,2} & s_{1,3} & \cdots & s_{1,n} \\ 0 & d_2 & s_{2,3} & \cdots & s_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix} \in \mathbb{Z}^{n \times n},$$

where $d_i = \deg(p_i^{(l)})$ for all $1 \leq i \leq n$.

(Case 2) In the $l$th round we have for all $1 \leq i \leq n$ that

$$\deg(h_i^{(l)}) > \deg(g_i^{(l)}) + \deg(p_i^{(l)})$$

and $h_i^{(l)}$ has a unique leading monomial, i.e.,

$$h_i^{(l)}(x_{i+1}, \ldots, x_n) = x_{i+1}^{t_{i,n}} + h_i^{(l)}(x_{i+1}, \ldots, x_n),$$

for some $h_i^{(l)} \in \mathbb{F}_q[x_{i+1}, \ldots, x_n]$ with $\deg(h_i^{(l)}) < t_{i,n} + t_{i,n}$. Then we define

$$T_i = \begin{pmatrix} 0 & t_{1,2} & t_{1,3} & \cdots & t_{1,n} \\ 0 & 0 & t_{2,3} & \cdots & t_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix} \in \mathbb{Z}^{n \times n}.$$

To convince the reader that conditions in the above definition are cryptographically relevant we give the examples that any SPN satisfies (Case 1) and any unbalanced Feistel network with expanding round function satisfies (Case 2), for more details see Sections 4.1 and 4.2.

Let’s first describe a general upper bound for the degree growth of a block cipher. Suppose we are given a polynomial vector $v \in \mathbb{F}_q[x_1, \ldots, x_n]^n$ and a matrix $A \in \mathbb{F}_q^{n \times n}$, we would like to bound the degrees of $A \cdot v$. Suppose the entries of $d \in \mathbb{Z}^n$ correspond to the degrees of the polynomials in $v$. Let $B = \{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{F}_q^n$, i.e., $e_i$ has entry 1 in the $i$th component and else zeros, then we define the map

$$\psi : \mathbb{F}_q^{n \times n} \times \mathbb{Z}^n \rightarrow \mathbb{Z}^n, \quad (A, d) = \left( \max_{e_j \in B} \left\{ e_j^i \cdot d \right\} \right)_{1 \leq i \leq n}. \tag{28}$$
By definition of \( \psi \) we certainly have that \( \deg(A \cdot v) \leq \psi(A, d) \). Now it is straightforward to upper bound the degree growth of block ciphers.

**Proposition 5.2.** Let \( \mathbb{F}_q \) be a finite field, let \( C_r \) be a block cipher with \( n > 1 \) blocks and \( r \) rounds that satisfies Definition 5.1, and let \( A_1, \ldots, A_r \in \mathbb{F}^{n \times n}_q \) be the matrices that define the affine layers of the round functions. Let \( d_l \in \mathbb{Z}^n \) denote the degree vector in the \( l \)th round with \( d_0 = (1, \ldots, 1)^T \). Then the total degree growth of \( C_r \) can be upper bounded via the following iteration

\[
d_l \leq \begin{cases} 
\psi(A_l, S_l \cdot d_{l-1}), & \text{(Case 1)} \\
\psi(A_l, T_l \cdot d_{l-1}), & \text{(Case 2)}
\end{cases}
\]

**Proof.** The claim is a generalization of [OS10a, § 2.2] to block ciphers. Clearly, in the \( l \)th round \( S_l \cdot d_{l-1} \) respectively \( T_l \cdot d_{l-1} \) describes the degree growth after applying the \( l \)th GTDS. Now the components are mixed with the matrix \( A_l \) and \( \psi \) provides an upper bound of the total degree after the mixing. \( \square \)

We provide a simple example of GTDS and linear transformations where the actual degree is strictly smaller than the upper bound from Proposition 5.2.

**Example 5.3.** We consider the following iterated mappings

\[
\begin{align*}
x \mapsto x^2 \\ y^3 \mapsto x^2 + y^3 \\ (x^2 + y^3)^2 \mapsto y^6 \\ x^4 + 2x^2y^3 \mapsto y^6
\end{align*}
\]

Each step can be described via either a GTDS or a linear mapping. Applying Proposition 5.2 we obtain that \( d_4 = (5, 6) < (6, 6)^T \).

We will now study two cases for the affine layers where we can prove an exact iterated formula for the degree growth. For the first case we need the following function

\[
\phi : \mathbb{F}_q \to \mathbb{Z}, \quad x \mapsto \begin{cases} 
1, & x \neq 0 \\
0, & x = 0.
\end{cases}
\]

If we are given a matrix \( A \in \mathbb{F}_q^{n \times n} \), then we denote with \( \phi(A) \) the application of \( \phi \) to all entries of \( A \).

**Proposition 5.4.** Let \( \mathbb{F}_q \) be a finite field, let \( C_r \) be a block cipher with \( n > 1 \) blocks and \( r \) rounds that satisfies Definition 5.1, and let \( A_1, \ldots, A_r \in \mathbb{F}_q^{n \times n} \) be the matrices that define the affine layers of the round functions. If in the round \( l \geq 1 \) the matrix \( A_l \) only has one non-zero entry in each row, then with \( d_0 = (1, \ldots, 1)^T \) the total degree growth of \( C_r \) after \( l \) iterations is given by

\[
d_l = \phi(A_l) \cdot \begin{cases} 
S_l \cdot d_{l-1}, & \text{(Case 1)} \\
T_l \cdot d_{l-1}, & \text{(Case 2)}
\end{cases}
\]
Proof. The claim is a generalization of [OS10a, § 2.2] to a single round of a block cipher. If the degree after \( l \) iterations is \( d_{l-1} \), then \( S_l \cdot d_{l-1} \) computes the degree of the application of the \( l \)th GTDS. Moreover, if an invertible matrix \( A \in \mathbb{F}_q^{n \times n} \) has only one non-zero entry in each row, then \( A \cdot v \) simply reorders the elements of \( v \) and multiplies them by a constant. Then, for an integer vector \( d \in \mathbb{Z}^n \) we have that \( \psi(A, d) = \phi(A) \cdot d \). Now the claim follows. \( \square \)

**Corollary 5.5.** Let \( \mathbb{F}_q \) be a finite field, let \( C_r \) be a block cipher with \( n > 1 \) blocks and \( r \) rounds that satisfies Definition 5.1, and let \( A_1, \ldots, A_r \in \mathbb{F}_q^{n \times n} \) be the matrices that define the affine layers of the round functions such that \( A_l \) has only one non-zero entry in each row for all \( 1 \leq l \leq r \). Then with \( d_0 = (1, \ldots, 1)^T \) the total degree growth of \( C_r \) is given via the following iteration

\[
d_l = \phi(A_l) \cdot \begin{cases} S_l \cdot d_{l-1}, & \text{(Case 1)} \\ T_l \cdot d_{l-1}, & \text{(Case 2)} \end{cases}
\]

In particular, if we have that \( A_l = A \) and \( S_l = S \) or \( T_l = T_l \) for all \( 1 \leq l \leq r \), then the degree growth formula becomes

\[
d_l = \begin{cases} (\phi(A) \cdot S)^l \cdot d_0, & \text{(Case 1)} \\ (\phi(A) \cdot T)^l \cdot d_0, & \text{(Case 2)} \end{cases}
\]

In practice one is also interested in the degree growth of individual components.

**Corollary 5.6.** In the situation of Corollary 5.5, in each branch the degree in the \( i \)th variable of some monomial of maximal degree after \( l \) iterations can be computed via

\[
d_{l,x_i} = \begin{cases} (\phi(A) \cdot S)^l \cdot e_i, & \text{case (1)}, \\ (\phi(A) \cdot T)^l \cdot e_i, & \text{case (2)} \end{cases}
\]

**Remark 5.7.** Denote with \( f_j^{(l)} \) the polynomial representing branch \( j \) after \( l \) iterations of a block cipher \( C_r \). Note that in general

\[
\deg_{x_i} \left( f_j^{(l)} \right) \geq d_{l,x_i,j}
\]

for all \( i, j \) and \( l \) and the inequality can be strict. Therefore, we stress that an individual analysis of the degree growth in each variable must be done for every instance of the block cipher.

For the second case we allow mixing in the affine layers. Though, when two branches are mixed we have to make sure that the monomials of maximal degree cannot cancel each other.

**Proposition 5.8.** Let \( \mathbb{F}_q \) be a finite field, let \( C_r \) be a block cipher with \( n > 1 \) blocks and \( r \) rounds that satisfies Definition 5.1, let \( R^{(1)}, \ldots, R^{(r)} \) denote the \( r \) round functions of \( C \), and let \( A_1, \ldots, A_r \in \mathbb{F}_q^{n \times n} \) be the matrices that define the affine layers \( L^{(1)}, \ldots, L^{(r)} \) of
the round functions. For \( 1 \leq l \leq r \) we denote with \( f_{i}^{(l)} \in \mathbb{F}_{q}[x_{1}, \ldots, x_{n}] \) the polynomial representing the \( i \)th branch of \( \left( L^{(l)} \right)^{-1} \circ \mathcal{R}^{(l)} \circ \cdots \circ \mathcal{R}^{(1)} \). If in the round \( l \geq 1 \) the polynomials \( f_{1}^{(l)}, \ldots, f_{n}^{(l)} \) have monomials \( m_{1}^{(l)}, \ldots, m_{n}^{(l)} \) which are linearly independent and such that \( \deg \left( m_{i}^{(l)} \right) = \deg \left( f_{i}^{(l)} \right) \) for all \( 1 \leq i \leq n \), then with \( d_{0} = (1, \ldots, 1)^{T} \) the total degree growth of \( \mathcal{C}_{r} \) after \( l \) iterations is given by

\[
\mathbf{d}_{l} = \begin{cases} 
\psi(\mathbf{A}_{i}, \mathbf{S}_{i} \cdot \mathbf{d}_{l-1}), & \text{(Case 1),} \\
\psi(\mathbf{A}_{i}, \mathbf{T}_{l} \cdot \mathbf{d}_{l-1}), & \text{(Case 2).}
\end{cases}
\]

**Proof.** For the \( i \)th branch of \( \mathcal{R}^{(l)} \circ \cdots \circ \mathcal{R}^{(1)} \) we can distinguish between three cases. If there is no mixing at all, then only one entry of the \( i \)th row of \( \mathbf{A}_{i} \) is non-zero, in this case obviously \( \psi \) properly encompasses the degree growth. If there is mixing between the branches \( 1 \leq s < t \leq n \) but either \( \deg \left( f_{s}^{(l)} \right) < \deg \left( f_{t}^{(l)} \right) \) or \( \deg \left( f_{s}^{(l)} \right) > \deg \left( f_{t}^{(l)} \right) \), then also in this case \( \psi \) properly encompasses the degree growth. The only non-trivial case is when there is mixing between the branches \( 1 \leq s < t \leq n \) and \( \deg \left( f_{s}^{(l)} \right) = \deg \left( f_{t}^{(l)} \right) \). By assumption there are monomials \( m_{s}^{(l)} \) and \( m_{t}^{(l)} \) which are present in \( f_{s}^{(l)} \) and \( f_{t}^{(l)} \), that are linearly independent, and such that \( \deg \left( m_{s}^{(l)} \right) = \deg \left( f_{s}^{(l)} \right) \) and \( \deg \left( m_{t}^{(l)} \right) = \deg \left( f_{t}^{(l)} \right) \). Due to their linear independence they are present in any non-trivial linear combination \( a_{s} \cdot f_{s}^{(l)} + a_{t} \cdot f_{t}^{(l)} \). Thus, also in this case \( \psi \) properly encompasses the degree growth. Obviously, the second and the third case naturally generalize when more than two branches are mixed. \( \square \)

One could extend Corollary 5.6 also to the situation of Proposition 5.8 though the degree growth then depends on how the maximum is chosen in \( \psi \).

**Remark 5.9.** Our generic degree growth analysis is generic makes minimal assumptions on the polynomials. The degree growth analysis analysis can be refined further when a specific instantiation of GTDS and linear transformations \( \mathbf{A}_{i} \) are considered. An example of such analysis specific to instantiations can be found in [BCP22].

### 5.2 Bounding the Differential Uniformity of the GTDS

Differential cryptanalysis [BS91] and its variants are one of the most widely used attack vectors in modern cryptography. It is based on the observation that certain input differences can propagate through the rounds of a block cipher with high probability. The key measure to quantify whether a function is weak to differential cryptanalysis is the so called differential uniformity. In this section we will prove upper bounds for the differential uniformity of the GTDS under minimal assumptions on the polynomials \( p_{i}, g_{i} \) and \( h_{i} \). We start with the definition of differential uniformity.

**Definition 5.10.** Let \( \mathbb{F}_{q} \) be a finite field, and let \( f : \mathbb{F}_{q}^{n} \to \mathbb{F}_{q}^{m} \) be a function.
(1) The differential distribution table of $f$ at $a, b \in \mathbb{F}_q^n$ and $b \in \mathbb{F}_q^m$ is defined as
\[
\delta_S(a, b) = \left| \{ x \in \mathbb{F}_q^n \mid S(x + a) - S(x) = S(b) \} \right|.
\]

(2) The differential uniformity of $S$ is defined as
\[
\delta(S) = \max_{a \in \mathbb{F}_q^n \setminus \{0\}, \ b \in \mathbb{F}_q^m} \delta_S(a, b).
\]

Let’s now compute an upper bound for the differential uniformity of a GTDS.

**Proposition 5.11.** Let $\mathbb{F}_q$ be a finite field, let $n \geq 1$ be an integer, and let $F : \mathbb{F}_q^n \to \mathbb{F}_q^n$ be a GTDS. Let $p_1, \ldots, p_n \in \mathbb{F}_q[x]$ be the univariate permutation polynomials of the GTDS, and let $\Delta x, \Delta y \in \mathbb{F}_q^n$ be such that $\Delta x \neq 0$. Then the differential distribution table of $F$ at $\Delta x$ and $\Delta y$ is bounded by
\[
\delta_F(\Delta x, \Delta y) \leq \prod_{i=1}^{n-1} \begin{cases} \deg (p_i), & \Delta x_i \neq 0, \ \deg (p_i) > 1, \\
q, & \Delta x_i \neq 0, \ \deg (p_i) = 1, \\
q, & \Delta x_i = 0 \end{cases} \cdot \begin{cases} \delta(p_n), & \Delta x_n \neq 0, \\
q, & \Delta x_n, \Delta y_n = 0, \\
0, & \Delta x_n = 0, \Delta y_n \neq 0 \end{cases}
\]

**Proof.** Suppose we are given the differential equation
\[
F(x + \Delta x) - F(x) = \Delta y,
\]
then the last component of the differential equation only depends on the variable $x_n$, i.e.,
\[
p_n(x_n + \Delta x_n) - p_n(x_n) = \Delta y_n.
\]
If $\Delta x_n \neq 0$, then this equation has at most $\delta(p_n)$ many solutions. If $\Delta x_n = \Delta y_n = 0$, then this equation has $q$ many solutions for $x_n$. Lastly, if $\Delta x_n = 0$ and $\Delta y_n \neq 0$, then there cannot be any solution for $x_n$.

Now suppose we have a solution for the last component, say $\hat{x}_n \in \mathbb{F}_q$. Then, we can substitute it in Equation (30) into the $(n-1)^{th}$ component
\[
f_{n-1}(x_1 + \Delta x_1, \ldots, x_{n-1} + \Delta x_{n-1}, \hat{x}_n + \Delta x_n) - f_{n-1}(x_1, \ldots, x_{n-1}, \hat{x}_n) = \Delta y_{n-1}.
\]
Since $\hat{x}_n$ is a field element we can reduce this equation to
\[
\alpha \cdot p_{n-1}(x_{n-1} + \Delta x_{n-1}) - \beta \cdot p_{n-1}(x_{n-1}) + \gamma = \Delta y_{n-1},
\]
where $\alpha, \beta, \gamma \in \mathbb{F}_p$ and $\alpha, \beta \neq 0$. For $\Delta x_i \neq 0$, if $\alpha \neq \beta$, then this equation has at most $\deg (p_{n-1})$ solutions. If $\alpha = \beta$ and $\deg (p_{n-1}) > 1$, then we obtain a univariate equation of degree $\deg (p_{n-1}) - 1$ which obviously has less than $\deg (p_{n-1})$ many solutions. If $\alpha = \beta$ and $\deg (p_{n-1}) = 1$, then only constant terms remain in the equation. Thus, depending
on the constant terms it has either 0 or \( q \) many solutions for \( x_{n-1} \). So depending on the
scenario for \( \Delta x \) and \( \Delta y \) we either have at most \( \deg (p_{n-1}) \cdot \delta (p_n) \), \( \deg (p_{n-1}) \cdot q \), \( q^2 \) or 0
many solutions in the variables \( x_{n-1} \) and \( x_n \) for the last two branches. For \( \Delta x_{n-1} = 0 \),
note that in principle it can happen that \( \alpha = \beta \) and \( \Delta y_{n-1} = 0 \). In this case the
equation has \( q \) many solutions for \( x_{n-1} \), thus we have to use this scenario as worst case
upper bound.

Inductively, we now work upwards through the branches to derive the claim. \( \square \)

If all the \( p_i \)'s are non-linear, then one can ensure that the upper bound is non-trivial.

**Corollary 5.12.** Let \( \mathbb{F}_q \) be a finite field, let \( n \geq 1 \) be an integer, and let \( \mathcal{F} : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n \) be
a GTDS. Let \( p_1, \ldots, p_n \in \mathbb{F}_q[x] \) be the univariate permutation polynomials of the GTDS,
and let \( \Delta x, \Delta y \in \mathbb{F}_q^n \) be such that \( \Delta x \neq 0 \). If for all \( 1 \leq i \leq n \) one has that \( \deg (p_i) > 1 \),
then
\[
\delta (\mathcal{F}) \leq q^{n-1} \cdot \max \left\{ \max _{1 \leq i \leq n-1} \{ \deg (p_i) \}, \delta (p_n) \right\} .
\]

In particular,
\[
\Pr [\mathcal{F} : \Delta x \rightarrow \Delta y] \leq \frac{\max \{ \max _{1 \leq i \leq n-1} \{ \deg (p_i) \}, \delta (p_n) \}}{q}.
\]

**Proof.** Choose \( \Delta x \) such that it only has one non-zero entry, then apply Proposition 5.11.
The second claim follows from the first and division by \( q^n \). \( \square \)

We conclude this section with another special case of Proposition 5.11.

**Corollary 5.13.** Let \( \mathbb{F}_q \) be a finite field, let \( n \geq 1 \) be an integer, and let \( \mathcal{F} : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n \) be
a GTDS. Let \( p, p^{-1} \in \mathbb{F}_q[x] \) be non-linear permutation polynomials such that \( p \circ p^{-1} (x) = p^{-1} \circ p (x) = x \), and assume that for all \( 1 \leq i \leq n \) the \( i \)th univariate permutation
polynomial of \( \mathcal{F} \) is either equal to \( p \) or \( p^{-1} \). Then
\[
\delta (\mathcal{F}) \leq q^{n-1} \cdot \min \left\{ \deg (p), \deg (p^{-1}) \right\} .
\]

In particular,
\[
\Pr [\mathcal{F} : \Delta x \rightarrow \Delta y] \leq \frac{\min \{ \deg (p), \deg (p^{-1}) \}}{q}.
\]

**Proof.** For a bijection \( f : \mathbb{F}_q \rightarrow \mathbb{F}_q \) it is well-known that if \( \delta (f) \leq d \) then also \( \delta (f^{-1}) \leq d \)
(cf. [Nyb94, Proposition 2]). Obviously, for non-linear permutation polynomials one has
that \( \delta (p), \delta (p^{-1}) \leq \min \{ \deg (p), \deg (p^{-1}) \} \). Now we can apply Corollary 5.12. \( \square \)

### 5.3 Pairwise Independence of a Two Round Block Cipher

Up to date there is still no theoretical security proof for a concrete instantiations of
block ciphers. As a remedy the authors of [LTV21a] propose to study how well the
distribution of \( t \) points sampled by a block cipher is to the uniform distribution via the
so-called statistical distance (or total variation). Given that the statistical distance for
t points is small to the uniform distribution one already has resilience against a wide class of differential attacks. E.g., for $t = 2$ it already implies resilience to linear (in characteristic 2), differential and truncated differential attacks. In [LTV21a] the case $t = 2$ over binary fields was proven for substitution-permutation networks that use $x^3$ or $x^{-1}$ as degree increasing function. In this section we will investigate the case $t = 2$ for GTDS for which Proposition 5.11 provides a non-trivial upper bound. For starters let’s introduce the formal definition of statistical distance.

**Definition 5.14.** Let $\mu$ and $\nu$ be two probability distributions on a finite set $\Omega$. The statistical distance (or total variation distance) between $\mu$ and $\nu$ is defined as

$$d_{TV}(\mu, \nu) = \frac{1}{2} \cdot \sum_{x \in \Omega} |\mu(x) - \nu(x)|.$$

The following identity is useful to compute the statistical distance.

**Lemma 5.15** (cf. [LPW09, Remark 4.3]). Let $\mu$ and $\nu$ be two probability distributions on a finite set $\Omega$. Then

$$d_{TV}(\mu, \nu) = \sum_{x \in \Omega} \mu(x) - \nu(x) \cdot \mu(x) \geq \nu(x).$$

Now we recall the technical definition of $t$-wise independence of a block cipher.

**Definition 5.16** (cf. [LTV21a, Definition 3]). We say that a permutation family $F : \mathbb{F}_q^n \times \mathbb{F}_q^m \rightarrow \mathbb{F}_q^n$ is $\epsilon$-close to $t$-wise independent if for all distinct $x_1, \ldots, x_t \in \mathbb{F}_q^n$, and a uniformly random key $k \in \mathbb{F}_q^m$, the distribution of $(F_k(x_1), \ldots, F_k(x_t))$ has statistical distance at most $\epsilon$ from that of $t$ uniformly sampled distinct vectors in $\mathbb{F}_q^n$.

Given that the statistical distance to the uniform distribution is less than $1/2$, then one can easily upper bound the statistical distance of an iterated cipher.

**Lemma 5.17** (MPR Amplification Lemma, [MPR07, Lemma 1]). Let $F, G : \mathbb{F}_q^n \times \mathbb{F}_q^m \rightarrow \mathbb{F}_q^n$ be $\epsilon$- and $\delta$-close to $t$-wise independent permutation families. Then, the permutation family $F \circ G$ such that $(F \circ G)_{k_1||k_2} = F_{k_1} \circ G_{k_2}(x)$ is $2\epsilon\delta$-close to $t$-wise independent.

In [LTV21a, Lemma 2] pairwise independence of Key Alternating Ciphers (KAC) is established via the distribution of the differential equation

$$\Delta y = F(x + \Delta x) - F(x). \quad (31)$$

Before, we state the technical result let’s shortly revisit the notion of (KAC). A KAC is a family of functions indexed by $r + 1$ sub-keys $k_0, \ldots, k_r \in \mathbb{F}_q^n$ and $r$ permutations

$$P_i : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$$

such that

$$F_{k_0}^{(0)}(x) = x + k_0$$

$$F_{k_0, \ldots, k_i, P_1, \ldots, P_t}^{(i)}(x) = P_t \left( F_{k_0, \ldots, k_{i-1}, P_1, \ldots, P_{i-1}}^{(i-1)}(x) \right) + k_i. \quad (32)$$
Clearly, our notion of block ciphers is covered by KAC. For completeness, we mention that [LTV21a, Lemma 2] is proven for univariate KACs in characteristic 2, though the proof in the extended version of the paper [LTV21b, Lemma 2.7] generalizes naturally to multivariate KACs over arbitrary finite fields.

**Lemma 5.18** (cf. [LTV21a, Lemma 2], [LTV21b, Lemma 2.7]). Let $\mathbb{F}_q$ be a finite field. Assume that the KAC $F : \mathbb{F}_q^n \times \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$ has the property that for any input difference $\Delta x \in \mathbb{F}_q^n \setminus \{0\}$, the distribution of

$$\Delta y = F_k(x + \Delta x) - F_k(k)$$

is $\epsilon$-close to uniform (where the randomness of the distribution is taken over $x$ and $k$). Then, the KAC $F$ is $\epsilon$-close to pairwise independent.

To apply this lemma one either needs to know the distribution of the differential equation at every point or one needs suitable upper bounds. E.g., for the patched inverse function $x^{2^n-2}$ over $\mathbb{F}_{2^n}$ this distribution was computed in [Nyb94]. In [LTV21b, Theorem A.1] the authors then establish that a two round SPN based on the patched inverse function is $\frac{1}{2^n}$-close to uniform. Analog we will know establish a similar result for two round block ciphers that have non-linear univariate permutation polynomials in the GTDS in every round.

**Theorem 5.19.** Let $\mathbb{F}_q$ be a finite field, let $n \geq 1$ and let $C : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^{n \times 3}$ be a two round block cipher with independent round keys. Denote $p_i(j) \in \mathbb{F}_q[x]$ with the univariate permutation polynomial in the $i$th branch and $j$th round. Assume that there exists an integer $1 < e < q$ such that for all $1 \leq i \leq n-1$ and $j = 1, 2$ we have that $1 < \deg (p_i^j) \leq e$ and $\delta (p_n^{(1)}), \delta (p_n^{(2)}) \leq e$. Let

$$\rho(e, n, q, i) = \left( \frac{e}{q} \right)^{n-i} \cdot \left( \frac{n}{\lfloor \frac{n}{2} \rfloor} \right) \cdot \left( \left( \frac{e \cdot (q-1)}{q} \right)^n - 1 \right) \cdot \frac{e \cdot (q-1)}{e \cdot (q-1) - q},$$

and

$$\epsilon = \sum_{i=0}^{n-1} \binom{n}{i} \cdot \frac{(q-1)^{n-i}}{q^n - 1} \cdot \begin{cases} \rho(e, n, q, \text{wt}^*(\gamma)) \cdot \frac{1}{q-1}, & \text{if } \rho(e, n, q, i) \geq \frac{1}{q-1}; \\ 0, & \text{else}. \end{cases}$$

Then $C$ is $\epsilon$-close to pairwise independent.

**Proof.** We split the cipher into two rounds $C = F_1 \circ F_2$. Under the assumption that the round keys are independent the probability that an input difference $\delta \neq 0$ results in an output difference $\delta'$ is

$$\sum_{\gamma \neq 0} \mathbb{P}[F_1 : \delta \rightarrow \gamma] \cdot \mathbb{P}[F_2 : \gamma \rightarrow \delta'].$$

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We denote with \( \text{wt}^* : \mathbb{R}_q^n \to \mathbb{Z} \) the function that counts the number of zeros in the input vector. By our assumptions we can apply Proposition 5.11 to obtain the following bound on the first probability
\[
P[\mathcal{F}_1 : \mathbf{\delta} \to \mathbf{\gamma}] \leq \left( \frac{e}{q} \right)^{n - \text{wt}^*(\mathbf{\delta})}.
\]
For the second probability we will do a case distinction of \( \text{wt}^*(\mathbf{\gamma}) \)
\[
\sum_{\gamma \neq 0} P[\mathcal{F}_1 : \mathbf{\delta} \to \mathbf{\gamma}] \cdot P[\mathcal{F}_2 : \mathbf{\gamma} \to \mathbf{\delta}'] \leq \left( \frac{e}{q} \right)^{n - \text{wt}^*(\mathbf{\gamma})} \cdot \sum_{i=0}^{n-1} \sum_{\mathbf{\gamma} \in \mathbb{R}_q^n \setminus \{0\}} P[\mathcal{F}_2 : \mathbf{\gamma} \to \mathbf{\delta}'] = (*) .
\]

For a fixed \( i \) there exist \( \binom{n}{i} \cdot (q-1)^{n-i} \) many vectors \( \mathbf{\gamma} \in \mathbb{R}_q^n \setminus \{0\} \) such that \( \text{wt}^*(\mathbf{\gamma}) = i \). Moreover, by Pascal’s triangle we can bound the binomial coefficient by \( \binom{n}{i} \leq \binom{n}{\lfloor n/2 \rfloor} \) for all \( 0 \leq i \leq n-1 \), and for the probability we again apply Proposition 5.11. This yields
\[
(*) \leq \left( \frac{e}{q} \right)^{n - \text{wt}^*(\mathbf{\delta})} \cdot \sum_{i=0}^{n-1} \binom{n}{\lfloor n/2 \rfloor} \cdot (q-1)^{n-i} \cdot \left( \frac{e}{q} \right)^{n-i}
\]
\[
= \left( \frac{e}{q} \right)^{n - \text{wt}^*(\mathbf{\delta})} \cdot \binom{n}{\lfloor n/2 \rfloor} \cdot \left( \frac{e \cdot (q-1)}{q} \right)^n \cdot \sum_{i=0}^{n-1} \left( \frac{q}{e \cdot (q-1)} \right)^i
\]
\[
\leq \left( \frac{e}{q} \right)^{n - \text{wt}^*(\mathbf{\delta})} \cdot \binom{n}{\lfloor n/2 \rfloor} \cdot \left( \frac{e \cdot (q-1)}{q} \right)^n \cdot \frac{1 - \left( \frac{q}{e \cdot (q-1)} \right)^n}{1 - \frac{q}{e \cdot (q-1)}}
\]
\[
= \left( \frac{e}{q} \right)^{n - \text{wt}^*(\mathbf{\delta})} \cdot \binom{n}{\lfloor n/2 \rfloor} \cdot \left( \frac{e \cdot (q-1)}{q} \right)^n \cdot \frac{(e \cdot (q-1))^n - q^n}{(e \cdot (q-1))^n} \cdot \frac{e \cdot (q-1)}{e \cdot (q-1) - q}
\]
\[
= \left( \frac{e}{q} \right)^{n - \text{wt}^*(\mathbf{\delta})} \cdot \binom{n}{\lfloor n/2 \rfloor} \cdot \left( \frac{e \cdot (q-1)}{q} \right)^n \cdot \frac{e \cdot (q-1)}{e \cdot (q-1) - q}
\]
\[
=: \rho(e, n, q, \text{wt}^*(\mathbf{\gamma})).
\]
The second inequality is a simply rearrangement of the terms, the third one is an application of the sum formula of the geometric sum, and the last two inequalities follow by canceling and rearranging the terms. With Lemma 5.15 we can now estimate the statistical distance between the differential equation and the uniform distribution with
\[
\sum_{\mathbf{\delta} \in \mathbb{R}_q^n \setminus \{0\}} \left\{ \rho(e, n, q, \text{wt}^*(\mathbf{\gamma})) - \frac{1}{q^n-1}, \text{ if } \rho(e, n, q, \text{wt}^*(\mathbf{\gamma})) \geq \frac{1}{q^n-1}, \right. \\
\left. 0, \text{ else.} \right.
\]
\[
= \sum_{i=0}^{n-1} \binom{n}{i} \cdot \frac{(q-1)^{n-i}}{q^n-1} \cdot \left\{ \rho(e, n, q, i) - \frac{1}{q^n-1}, \text{ if } \rho(e, n, q, i) \geq \frac{1}{q^n-1}, \right. \\
\left. 0, \text{ else.} \right.
\]
\[
=: \epsilon.
\]
The final claim now follows from Lemma 5.18.

Remark 5.20.  (1) In principle one could estimate $\epsilon$ directly with the degrees, though due to the combinatorial structure one cannot derive a "simple" formula for the estimation.

(2) For large $q$ we can approximate $\rho$ with

$$\rho(e, n, q, i) \approx \left(\frac{e}{q}\right)^{n-i} \cdot \left(\frac{n}{\lfloor \frac{n}{2} \rfloor}\right) \cdot \frac{e \cdot (e^n - 1)}{e - 1}.$$

(3) As we will see in Section 4.2 any SPN is covered by the GTDS, and if every component is non-linear, then our bound is applicable.

Of course, we can now apply the MPR Amplification Lemma bound the distance of a $2r$-round block cipher to the uniform distribution.

Corollary 5.21. In the situation of Theorem 5.19, the $2r$-round block cipher $C$ is $2^{r-1} e^r$-close to pairwise independent.

In Table 1 we computed the bound of Theorem 5.19 for some sample values to showcase how it behaves in practice. As one can observe from the table (or from the formula for $\epsilon$) the degree bound $e$ has to be small relative to $q$, else the estimation will be greater than 1. E.g., for $q = 2^8$ the degree bound $e = 10$ is also the largest integer such that $\epsilon < 1/2$.

| $q$  | $n$ | $e$  | $\epsilon$     |
|------|-----|------|----------------|
| $2^8$| 2   | 10   | 0.3970         |
| $2^8$| 5   | 10   | 0.1568         |
| $2^8$| 2   | 100  | $> 1$          |
| $2^8$| 5   | 100  | $> 1$          |
| $2^{64}$| 2 | 10   | $7.8 \cdot 10^{-35}$ |
| $2^{64}$| 5 | 10   | $8.4 \cdot 10^{-86}$ |
| $2^{64}$| 2 | 100  | $6.1 \cdot 10^{-31}$ |
| $2^{64}$| 5 | 100  | $5.0 \cdot 10^{-76}$ |

Table 1: Sample values for $\epsilon$-close to pairwise independent of two round block ciphers.

We finish this section with an example of a GTDS that satisfies the assumptions of Proposition 5.11 and Theorem 5.19.

Example 5.22. Let $q$ be a 64-bit prime, let $\mathbb{F}_q$ be the field with $q$ elements, let $d \in \{3, 5, 7, 11\}$ be the smallest integer such that $\gcd(d, q-1) = 1$, and let $e \cdot d \equiv 1 \mod p-1$. 

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We consider the GTDS
\[
F : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1^d \\ \vdots \\ x_n^{d_{n-1}} \end{pmatrix}.
\]

For power functions one always has that \(\delta(x^d) \leq d\). Moreover, for a bijection \(f : \mathbb{F}_q \to \mathbb{F}_q\) it is well-known that if \(\delta(f) \leq d\) then also \(\delta(f^{-1}) \leq d\) (cf. [Nyb94, Proposition 2]). Hence, also \(\delta(x^e) \leq d\). So Proposition 5.11 and Theorem 5.19 are applicable and non-trivial.

6 Discrepancy Analysis

One crucial property of cryptographic functions is that they are able to produce (pseudo-)random outputs or sequences. There is no general method to quantify how well a generated sequence resembles a random sequence. However, several existing measures try to quantify how well a sequence of outputs (produced from an underlying function) resembles the properties of a random sequence. One of these measures is the so-called discrepancy of a sequence, which quantifies the uniformity of the distribution of a sequence. Discrepancy analysis is used in computer science and mathematics literature for measuring the pseudo-randomness of sequence (e.g. cf. [Nie78, § 6], [DT97, Chapter 3 § 3.4]).

In this section we compute the discrepancy of a sequence generated by a fixed key or unkeyed permutation (as in Definition 3.5) based on GTDS. To the best of our knowledge this was never done before. In [OS10a] the discrepancy of a sequence generated by a triangular dynamical system was first computed. Conceptually, we will mimic this proof and adapt it to block ciphers. It is worth mentioning that the discrepancy of arbitrary iterated polynomial dynamical system has been investigated in [OPS10], but our bound will be an improvement from this generic bound.

At first we collect some additional notation.

**Definition 6.1.** Let \(\mathbb{F}_q\) be a finite field, and let \(C\) be a block cipher of size \(n > 1\). We say that \(C\) is a \((n)\)-uniform block cipher if in every round the same round function is used.

We say that a \((n)\)-uniform block cipher \(C\) is strictly increasing, if for all \(l \geq 0\) the sequence of degree vectors \((d_l)_{l \geq 0}\) is strictly increasing, i.e., \(d_{l+1} > d_l\).

If for a strictly increasing \((n)\)-uniform block cipher \(C\) we in addition have that
\[
\max_{1 \leq i \leq n} \deg(f^{(l)}_i) = d^l, \quad l \geq 0,
\]
then we call \(C\) a \((n, d)\)-uniform block cipher.

For our proof we need an equivalent of [OS10a, Cor. 2], unfortunately we were not able to find a proof in full generality therefore we introduce it as assumption.
Assumption 6.2. Let $\mathbb{F}_q$ be a finite field, and let $C$ be a strictly increasing $(n)$-uniform block cipher. Let $A \in \mathbb{F}_q^{n \times n}$ be the matrix which represents the affine layer of $C$, and denote with $f_1^{(l_1)}, \ldots, f_n^{(l)} \in \mathbb{F}_q[x_1, \ldots, x_n]$ the polynomials representing the branches after $l$ rounds. Assume that for any $\nu \geq 1$ there is a constant $l_0$ depending only on the matrices $S$ or $T$ (from Definition 5.1), $A$ and $\nu$ such that for any integers $l_1, m_1, \ldots, l_\nu, m_\nu \geq l_0$ and any non-zero vector $a = (a_1, \ldots, a_n) \in \mathbb{F}_q^n$,

$$ F_{a, l_1, m_1, \ldots, l_\nu, m_\nu} = \sum_{i=1}^n a_i \sum_{j=1}^\nu \left( f_i^{(l_j)} - f_i^{(m_j)} \right), $$

is a non-constant polynomial of degree

$$ \deg \left( F_{a, l_1, m_1, \ldots, l_\nu, m_\nu} \right) \leq \max_{1 \leq i \leq n} \deg \left( f_i^{(l_j)} \right), $$

where

$$ l = \max \{ l_1, m_1, \ldots, l_\nu, m_\nu \} $$

unless the components of the vectors

$$ (l_1, \ldots, l_\nu) \quad \text{and} \quad (m_1, \ldots, m_\nu) $$

are permutations of each other.

Luckily, for many instances of our interest the assumption turns out to be true.

Example 6.3. (1) Suppose we are in the scenario of (Case 1) of Definition 5.1 and that the matrix of the affine layer only reorders the elements. Let $<$ be the degree lexicographic monomial order. Further assume that for all $k$ and all $i$ we can strictly order the leading monomials of the $f_i^{(k)}$. Then the leading monomials in every round are linearly independent, and by strict degree growth over the rounds they cannot be canceled. E.g., let $g_i(x) = x^n + a_i$ be polynomials with no zeros over $\mathbb{F}_q$, and let $\deg(h_i) < \deg(g_i)$. We consider the GTDS

$$ f_1(x_1, \ldots, x_n) = x_1 \cdot g_1(x_2) + h_1(x_2, \ldots, x_n), $$

$$ \ldots $$

$$ f_{n-1}(x_1, \ldots, x_n) = x_{n-1} \cdot g_{n-1}(x_n) + h_1(x_n), $$

$$ f_n(x_1, \ldots, x_n) = x_n, $$
together with the shift permutation as linear layer. Via induction it is easy to prove that in every round the leading monomials are strictly ordered.

(2) For SPNs like AES-128, after application of the first S-box all leading monomials are linearly independent. If the linear layer is applied, then it still must be possible to mark any branch with a unique monomial which was a leading monomial before
the linear layer. Thus, after application of the linear layer the polynomials still have monomials of maximal degree that are linearly independent. By iteration of this argument we can conclude that in every round some set of monomials of maximal degree is linearly independent and by strict degree growth they cannot be canceled.

(3) Any Feistel network with contracting round function. In this construction it is again possible to strictly order the leading monomials in one round, together with strict degree growth the assumption follows.

6.1 Construction of Polynomial Pseudorandom Number Generators with Iterative Permutation

Let \( p \) be a prime, and suppose we are given a \((n)\)-uniform block cipher \( C \) over \( \mathbb{F}_p \). Denote with \( f_1^{(l)}, \ldots, f_n^{(l)} \in \mathbb{F}_p[x_1, \ldots, x_n] \) the polynomials representing the branches after \( l \geq 0 \) rounds. We consider the sequence defined by a recurrence congruence modulo a prime \( p \) of the form

\[
u_{l+1} = f_i^{(l)}(u_{1,0}, \ldots, u_{n,0}) \mod p, \quad l \geq 0,
\]

with some initial values \( u_{1,0}, \ldots, u_{n,0} \in \mathbb{F}_p \). Alternatively, if \( R \) is the round function of \( C \) then we can rewrite this relation using vector notation

\[
u_{l+1} = R(u_l) = R^{(l)}(u_0).
\]

Clearly, the sequence \( u_{k+1} \) becomes periodic with some period \( T \leq p^n \). Without loss of generality we assume that

\[
u_{k+T} = \nu_k, \quad k \geq 0.
\]

6.2 Exponential Sums

Let

\[
e_p(z) = \exp \left( \frac{2\pi i z}{p} \right).
\]

A key ingredient in the proof of the discrepancy bound is the Weil bound on exponential sums (cf. [LN97, Chapter 5]). We present it in the same form as in [OS10a, Lemma 3].

Lemma 6.4. For any non-constant polynomial \( F \in \mathbb{F}_p[x_0, \ldots, x_n] \) of total degree \( D \) we have the bound

\[
\left| \sum_{x_1, \ldots, x_n=1}^p e_p(F(x_0, \ldots, x_n)) \right| < Dp^{n+\frac{1}{2}}.
\]

Assume that \( (u_l)_{l \geq 0} \) is generated by Equation (33), for an integer vector \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \) we introduce the exponential sum

\[
S_{a}(N) = \sum_{m=0}^{N-1} e_p \left( \sum_{i=0}^n a_i u_{n,i} \right).
\]
To bound the absolute value of this exponential sum we follow the scheme introduced in [NS99; NS01]. The next theorem is the extension of [OS10a, Theorem 4] to block ciphers. Note that due to Assumption 6.2 our result is better than [OPS10, Theorem 3] which holds for any polynomial dynamical system.

**Theorem 6.5.** Let $p$ be a prime, and let $C$ be a $(n, d)$-uniform block cipher over $\mathbb{F}_p$ which satisfies Assumption 6.2. Let the sequence $(u_k)_{k \geq 0}$ be generated by Equation (33) and assume that it is purely periodic with period $T$. Then for any fixed integer $\nu \geq 1$, and any positive integer $N \leq T$, the bound

$$
\max_{\gcd(a_1, \ldots, a_n, p) = 1} |S_a(N)| = O \left( p^{\frac{1}{2} + \frac{\nu}{d}} (\log p)^{-\frac{1}{2}} \right)
$$

holds and the implied constant depends only on $d$, $n$ and $\nu$.

**Proof.** Let $n' = n - 1$, so we can model $C$ over a polynomial ring with $n' + 1$ variables. (This will make the application of Lemma 6.4 a little bit easier.) Select any $a = (a_0, \ldots, a_n) \in \mathbb{Z}^n$ with $\gcd(a_0, \ldots, a_{n'}, p) = 1$.

We can proceed similar as in the proof of [OS10a, Theorem 4] to obtain the inequalities

$$
(W - 2^\nu K)
\leq
W^{2^\nu}
= N^{2^\nu - 1} \sum_{l_1, m_1, \ldots, l_{\nu}, m_{\nu} = k_0}^{k_0} \mathbb{E}_p \left( \sum_{i=0}^{n'} a_i \sum_{j=1}^{\nu} f_i^{(l_i)}(w) - f_i^{(m_j)}(w) \right),
$$

where $k_0$ is the same as in Assumption 6.2 and $K \geq k_0$. For $O(K^\nu)$ vectors

$$(l_1, \ldots, l_{\nu}) \quad \text{and} \quad (m_1, \ldots, m_{\nu})$$

are permutations of each other, we estimate the inner sum trivially as $p^{n'+1}$. For the other $O(K^\nu)$ vectors, we combine Assumption 6.2 and Lemma 6.4 getting the upper bound $d^K p^{n'+1}$ for the inner sum for at most $K^2$ sums. Hence,

$$
W^{2^\nu} \leq K^\nu N^{2^\nu - 1} p^{n' + 1} + d^K K^{2^\nu} N^{2^\nu - 1} p^{n' + 1}.
$$

Thus, we yield that

$$
S_a(N) = O \left( N^{-\frac{1}{2}} N^{1 - \frac{1}{d}} p^{\frac{n'+1}{d}} + d^{\frac{1}{2}} N^{1 - \frac{1}{d}} p^{\frac{2n'+1}{d}} + K \right).
$$

We choose

$$
K = \left\lceil \frac{1}{2} \log_d p \right\rceil,
$$

and assume that $p$ is large enough, so $K \geq k_0$. Using $K \geq \left\lceil \frac{1}{2} \log_d p \right\rceil$ we see that the first summand can be bounded by $N^{1 - \frac{1}{d}} p^{\frac{n'+1}{d}} (\log p)^{-\frac{1}{2}}$. For the second summand we use that $K \leq \frac{1}{2} \log_d p + 1$, then we can estimate the summand by $d^{\frac{1}{2}} N^{1 - \frac{1}{d}} p^{\frac{2n'+1}{d} - \frac{1}{d}} = d^{\frac{1}{2}} N^{1 - \frac{1}{d}} p^{\frac{2n'+1}{d}}$. Bounding the third summand is trivial, so the claim follows.  

\hfill $\square$
It is immediate that for any fixed $\epsilon > 0$ there is $\delta > 0$ such if $T \geq N \geq p^\epsilon$ then
\[
\max_{\gcd(a_1, \ldots, a_n, p) = 1} |S_a(N)| = O \left( N^{1-\delta} (\log p)^{-\frac{1}{2}} \right).
\] (38)

### 6.3 Discrepancy

For a sequence $\Gamma$ of $N$ points contained in the $n$-dimensional unit cube $[0,1)^n$ the discrepancy is a measure to quantify its statistical uniformity. It is defined as, see [DT97, Definition 1.5],
\[
D(\Gamma) = \sup_{I \subset [0,1)^n} \left| \frac{A(I, N, \Gamma)}{N} - \lambda_n(I) \right|,
\] (40)

where $A(I, N, \Gamma)$ denotes the number of points of the sequence $\Gamma$ up to index $N$ contained in the interval $I$ and $\lambda_n(I)$ denotes the $n$-dimensional Lebesgue measure of $I$. In general, pseudorandom sequences should have a low discrepancy, cf. [Nie78; Nie92].

A fundamental result in the theory of low discrepancy sequences is the Koksma-Szüsz inequality, cf. [DT97, Theorem 1.21], which we present in the same form as [OS10a, Lemma 5].

**Lemma 6.6.** For any integer $L > 1$ and any sequence $\Gamma$ of $N$ points as in Equation (39) the discrepancy $D(\Gamma)$ satisfies the following bound:
\[
D_N(\Gamma) < O \left( \frac{1}{L} + \frac{1}{N} \sum_{|a_1|, \ldots, |a_n| \leq L} \prod_{j=1}^{n} \frac{1}{|a_j| + 1} \sum_{l=0}^{N-1} \exp \left( 2\pi i \sum_{j=0}^{n} a_j \gamma_j, l \right) \right)
\]

Combining Theorem 6.5 and Lemma 6.6 and taking $L = \lfloor \log p \rfloor$ we obtain the following bound which is the extension of [OS10a, Theorem 6] to block ciphers.

**Theorem 6.7.** Let $p$ be a prime, and let $C$ be a $(n,d)$-uniform block cipher over $\mathbb{F}_p$ which satisfies Assumption 6.2. Let the sequence $(u_l)_{l=0}^N$ be generated by Equation (33) and assume that it is purely periodic with period $T$. Then for any fixed integer $\nu \geq 1$, and any positive integer $N \leq T$, the discrepancy $D_N$ of the sequence
\[
\left( \frac{u_{l,1}}{p}, \ldots, \frac{u_{l,n}}{p} \right), \quad l = 0, \ldots, N - 1,
\]
satisfies the bound
\[
D_N = O \left( p^{\nu} N^{-\frac{d}{2d}} (\log p)^{-\frac{1}{2}} (\log \log p)^n \right)
\]
and the implied constant depends only on $d$, $n$ and $\nu$. 

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6.4 Average Discrepancy

Theorem 6.7 has one big disadvantage, the discrepancy depends on the period of the initial value which is general not known and can be far from \( p^n \). However, we can use the method introduced in [NS02] to estimate the discrepancy on average over all input values. This method has also successfully been applied to triangular dynamical systems with polynomial degree growth and closely related systems, see [Ost10a; OS10b; Ost10b; GOS14; GGO16].

For a vector \( a \in \mathbb{F}_p^n \) and integers \( c, M, N \) with \( M, N \geq 1 \), we introduce

\[
V_{a,c}(M, N) = \sum_{a \in \mathbb{F}_p^n} \left| \sum_{l=0}^{N-1} e_p \left( \sum_{j=1}^n a_j f_j^{(l)}(a) \right) e_M(cn) \right|^2.
\]

(41)

**Lemma 6.8.** Let \( p \) be a prime, let \( n \geq 2 \), and let \( C \) be a \((n,d)\)-uniform block cipher over \( \mathbb{F}_p \) which satisfies Assumption 6.2. Then for any positive integers \( c, M, N \) and any non-zero vector \( a \in \mathbb{F}_p^n \) we have

\[
V_{a,c}(M, N) = O(A(N, p)),
\]

where

\[
A(N, p) = \begin{cases} 
Np^n, & N \leq \log_d p, \\
N^2p^n(\log p)^{-1} & N > \log_d p,
\end{cases}
\]

and the implied constant depends only on \( d \) and \( n \).

**Proof.** Again, let \( n' = n - 1 \). We proceed in the same way as in the proof of [Ost10a, Lemma 3], we only replace the application of [Ost10a, Lemma 2] by Assumption 6.2. This yields

\[
V_{a,c}(M, N) = O \left( N^2 \left( K^{-1} p^{n'+1} + d^K p^{n'} \right) \right).
\]

Thus, selecting

\[
K = \min \{ N, \lceil \log_d p \rceil \}
\]

and taking into account that \( N^{-1} p^{n'+1} \leq d^N p^{n'} \) for \( N \leq \log_d p \), we obtain the desired result. \( \square \)

As in Theorem 6.7 we now obtain the following bound on the average discrepancy, again we choose \( L = \lceil \log p \rceil \).

**Theorem 6.9.** Let \( 0 < \epsilon < 1 \), let \( p \) be a prime, and let \( C \) be a \((n,d)\)-uniform block cipher over \( \mathbb{F}_p \) which satisfies Assumption 6.2. Let the sequence \((u_i)_{i \geq 0}\) be generated by Equation (33). Then for all initial values \( v \in \mathbb{F}_p^n \) except at most \( O(\epsilon p^n) \) of them, and any positive integer \( N \leq p^n \), the discrepancy \( D_N(v) \) of the sequence

\[
\left( \frac{C_n(v)}{p} \right), \quad n = 0, \ldots, N,
\]

is
satisfies the bound
\[ D_N(v) = O\left( \epsilon^{-1} B(N,p) \right), \]
where
\[ B(N,p) = \begin{cases} N^{-\frac{1}{2}} (\log N)^n \log \log p, & N \leq \log_d p, \\ (\log p)^{-\frac{1}{2}} (\log N)^n \log \log p, & N > \log_d p, \end{cases} \]
and the implied constant depends only on \( d \) and \( n \).

Proof. Analog to [Ost10a, Theorem 5]. \( \square \)

6.5 Comparison with The Generic Bound

In [OPS10] the discrepancy of arbitrary multivariate polynomial dynamical systems was computed. Since this bound applies to any dynamical system with strict degree growth we refer to it as generic bound. The respective bound for exponential sums and discrepancy is
\[ \max_{\gcd(a_1,\ldots,a_n,p)=1} |S_a(N)| = O\left( N^{\frac{1}{2}} p^{\frac{\nu}{2}} (\log(p))^{-\frac{1}{2}} \right), \]
\[ D_N = O\left( p^\frac{\nu}{2} N^{-\frac{1}{2}} (\log \log p)^n \right). \]

For \( \nu = 1 \) our discrepancy bound is equal to the generic bound. For \( \nu > 1 \), we see that if \( N \leq p^n \) (which is always the case) then
\[ p^\frac{\nu}{2} N^{-\frac{1}{2}} \leq p^\frac{\nu}{2} N^{-\frac{1}{2}}. \]
I.e., our bound improves the trivial bound.

7 Discussion

7.1 Algebraic Frameworks Beyond the GTDS

It is worth noting that our GTDS framework is not the first attempt to unify block cipher design strategies. In [YPL11] the quasi-Feistel cipher idea was introduced. It provides an unified framework for Feistel and Lai–Massey ciphers. While our approach utilizes the full algebraic structure of finite fields, the authors of [YPL11] use a contrarian approach by requiring as little algebraic structure as possible. In particular, they demonstrate that invertible Feistel- and Lai–Massey ciphers can be instantiated over quasigroups (cf. [Smi06]). Furthermore, this little algebraic structure is already sufficient to prove theoretical security bounds in the Luby-Rackoff model for quasi-Feistel ciphers.
7.2 Hash Functions

Our analysis and discussion have been focused on iterative permutations. For most instantiations of known hash functions, a (fixed key) permutation is used to build the compression function. Then, iterating the compression function a hash function is built over an arbitrary domain. Our generic description of (keyed) permutations may be viewed as a vector of functions over $\mathbb{F}_q^n$. Thus, such permutations can be used to define a hash function $H : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^t$ where the domain of $H$ is of arbitrary length over $\mathbb{F}_q$ and the hash value is of length $t > 0$ over $\mathbb{F}_q$. For example, an instantiation of GTDS-based permutations can used in a Sponge mode to define such a hash function. Thus, all our analysis can be easily extrapolated to hash functions.

7.3 Beyond Permutations

The different conditions on the polynomials defining the GTDS are imposed to ensure that the resulting system is invertible. However, these conditions can be dropped if the goal is not to construct a permutation but possibly a pseudo-random function. Potentially, such a GTDS (without the necessary constraints for invertibility) can used to construct PRFs over $\mathbb{F}_q$ and is an interesting direction for future work.

References

[ACG+19] Martin R. Albrecht, Carlos Cid, Lorenzo Grassi, Dmitry Khovratovich, Reinhard Lüftenegger, Christian Rechberger, and Markus Schofnegger. “Algebraic Cryptanalysis of STARK-Friendly Designs: Application to MARVELlous and MiMC”. In: Advances in Cryptology – ASIACRYPT 2019. Ed. by Steven D. Galbraith and Shiho Moriai. Vol. 11923. Lecture Notes in Computer Science. Cham: Springer International Publishing, 2019, pp. 371–397. isbn: 978-3-030-34618-8. doi: 10.1007/978-3-030-34618-8_13.

[AGP+19] Martin R. Albrecht, Lorenzo Grassi, Léo Perrin, Sebastian Ramacher, Christian Rechberger, Dragos Rotaru, Arnab Roy, and Markus Schofnegger. “Feistel Structures for MPC, and More”. In: Computer Security – ESORICS 2019. Ed. by Kazue Sako, Steve Schneider, and Peter Y. A. Ryan. Cham: Springer International Publishing, 2019, pp. 151–171. isbn: 978-3-030-29962-0. doi: 10.1007/978-3-030-29962-0_8.

[AGR+16] Martin Albrecht, Lorenzo Grassi, Christian Rechberger, Arnab Roy, and Tyge Tiessen. “MiMC: Efficient Encryption and Cryptographic Hashing with Minimal Multiplicative Complexity”. In: Advances in Cryptology – ASIACRYPT 2016. Ed. by Jung Hee Cheon and Tsuyoshi Takagi. Vol. 10031. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 2016, pp. 191–219. isbn: 978-3-662-53887-6. doi: 10.1007/978-3-662-53887-6_7.
[ARS+15] Martin R. Albrecht, Christian Rechberger, Thomas Schneider, Tyge Tiessen, and Michael Zohner. “Ciphers for MPC and FHE”. In: Advances in Cryptology – EUROCRYPT 2015. Ed. by Elisabeth Oswald and Marc Fischlin. Berlin, Heidelberg: Springer Berlin Heidelberg, 2015, pp. 430–454. ISBN: 978-3-662-46800-5. DOI: 10.1007/978-3-662-46800-5_17.

[Bar09] Gregory V. Bard. Algebraic Cryptanalysis. 1st ed. Boston, MA: Springer US, 2009. ISBN: 978-0-387-88756-2. DOI: 10.1007/978-0-387-88757-9.

[BCD11] Christina Boura, Anne Canteaut, and Christophe De Cannière. “Higher-Order Differential Properties of KECCAK and Luffa”. In: Proceedings of the 18th International Conference on Fast Software Encryption. FSE’11. Lyngby, Denmark: Springer-Verlag, 2011, pp. 252–269. ISBN: 9783642217012. DOI: 10.1007/978-3-642-21702-9_15.

[BCP22] Clémence Bouvier, Anne Canteaut, and Léo Perrin. On the Algebraic Degree of Iterated Power Functions. Cryptology ePrint Archive, Paper 2022/366. https://eprint.iacr.org/2022/366. 2022.

[BGK+21] Mario Barbara, Lorenzo Grassi, Dmitry Khovratovich, Reinhard Lüftenegger, Christian Rechberger, Markus Schofnegger, and Roman Walch. Reinforced Concrete: Fast Hash Function for Zero Knowledge Proofs and Verifiable Computation. Cryptology ePrint Archive, Report 2021/1038. https://ia.cr/2021/1038. 2021.

[BPW06] Johannes Buchmann, Andrei Pyshkin, and Ralf-Philipp Weinmann. “A Zero-Dimensional Gröbner Basis for AES-128”. In: Fast Software Encryption. Ed. by Matthew Robshaw. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006, pp. 78–88. ISBN: 978-3-540-36598-3. DOI: 10.1007/11799313_6.

[BS91] Eli Biham and Adi Shamir. “Differential Cryptanalysis of DES-like Cryptosystems”. In: Advances in Cryptology-CRYPTO’90. Ed. by Alfred J. Menezes and Scott A. Vanstone. Berlin, Heidelberg: Springer Berlin Heidelberg, 1991, pp. 2–21. ISBN: 978-3-540-38424-3. DOI: 10.1007/3-540-38424-3_1.

[CGG+22] Carlos Cid, Lorenzo Grassi, Aldo Gunsing, Reinhard Lüftenegger, Christian Rechberger, and Markus Schofnegger. “Influence of the Linear Layer on the Algebraic Degree in SP-Networks”. In: IACR Transactions on Symmetric Cryptology 2022.1 (Mar. 2022), pp. 110–137. DOI: 10.46566/tosc.v2022.i1.110-137.

[CS15] Benoit Cogliati and Yannick Seurin. “On the Provable Security of the Iterated Even-Mansour Cipher Against Related-Key and Chosen-Key Attacks”. In: Advances in Cryptology - EUROCRYPT 2015 - 34th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Sofia, Bulgaria, April 26-30, 2015, Proceedings, Part I. Ed. by Elisabeth Oswald and Marc Fischlin. Vol. 9056. Lecture Notes in Computer Science. Springer, 2015, pp. 584–613. DOI: 10.1007/978-3-662-46800-5_23.
[DBN+01] Morris Dworkin, Elaine Barker, James Nechvatal, James Foti, Lawrence Bassham, E. Roback, and James Dray. Advanced Encryption Standard (AES). Nov. 2001. DOI: 10.6028/NIST.FIPS.197.

[DDK+14] Itai Dinur, Orr Dunkelman, Nathan Keller, and Adi Shamir. “Cryptanalysis of Iterated Even-Mansour Schemes with Two Keys”. In: Advances in Cryptology – ASIACRYPT 2014. Ed. by Palash Sarkar and Tetsu Iwata. Berlin, Heidelberg: Springer Berlin Heidelberg, 2014, pp. 439–457. ISBN: 978-3-662-45611-8. DOI: 10.1007/978-3-662-45611-8_23.

[DGG+21] Christoph Dobraunig, Lorenzo Grassi, Anna Guinet, and Daniël Kuijsters. “Ciminion: Symmetric Encryption Based on Toffoli-Gates over Large Finite Fields”. In: Advances in Cryptology – EUROCRYPT 2021. Ed. by Anne Canteaut and François-Xavier Standaert. Cham: Springer International Publishing, 2021, pp. 3–34. ISBN: 978-3-030-77886-6. DOI: 10.1007/978-3-030-77886-6_1.

[DR20] Joan Daemen and Vincent Rijmen. The Design of Rijndael: AES - The Advanced Encryption Standard. 2nd ed. Information Security and Cryptography. Springer Berlin, Heidelberg, 2020. ISBN: 978-3-662-60768-8. DOI: 10.1007/978-3-662-60769-5.

[DT97] Michael Drmota and Robert F. Tichy. Sequences, Discrepancies and Applications. 1st ed. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 1997. ISBN: 3-540-62606-9. DOI: 10.1007/BFb0093404.

[EGL+20] Maria Eichlseder, Lorenzo Grassi, Reinhard Lüftenegger, Morten Øygarden, Christian Rechberger, Markus Schofnegger, and Qingju Wang. “An Algebraic Attack on Ciphers with Low-Degree Round Functions: Application to Full MiMC”. In: Advances in Cryptology – ASIACRYPT 2020: 26th International Conference on the Theory and Application of Cryptology and Information Security, Daejeon, South Korea, December 7–11, 2020, Proceedings, Part I. Daejeon, Korea (Republic of): Springer-Verlag, 2020, pp. 477–506. ISBN: 978-3-030-64836-7. DOI: 10.1007/978-3-030-64837-4_16.

[GGO16] Domingo Gómez-Pérez, Jaime Gutierrez, and Alina Ostafe. “Common composites of triangular polynomial systems and hash functions”. In: J. Symb. Computat. 72 (2016), pp. 182–195. ISSN: 0747-7171. DOI: 10.1016/j.jsc.2015.02.005.

[GHR+22] Lorenzo Grassi, Yonglin Hao, Christian Rechberger, Markus Schofnegger, Roman Walch, and Qingju Wang. A New Feistel Approach Meets Fluid-SPN: Griffin for Zero-Knowledge Applications. Cryptology ePrint Archive, Report 2022/403. https://ia.cr/2022/403. 2022.

[GKR+21] Lorenzo Grassi, Dmitry Khovratovich, Christian Rechberger, Arnab Roy, and Markus Schofnegger. “Poseidon: A New Hash Function for Zero-Knowledge Proof Systems”. In: 30th USENIX Security Symposium (USENIX Security 21). USENIX Association, Aug. 2021, pp. 519–535. ISBN: 978-1-939133-24-3. URL: https://www.usenix.org/conference/usenixsecurity21/presentation/grassi.
| Reference | Title | Authors | Details |
|-----------|-------|---------|---------|
| [OS10b]   | Pseudorandom numbers and hash functions from iterations of multivariate polynomials | Alina Ostafe and Igor E. Shparlinski | In: Cryptogr. Commun. 2.1 (Apr. 2010), pp. 49–67. DOI: 10.1007/s12095-009-0016-0. |
| [OS12]    | On the power generator and its multivariate analogue | Alina Ostafe and Igor E. Shparlinski | In: J. Complexity 28.2 (2012), pp. 238–249. ISSN: 0885-064X. DOI: 10.1016/j.jco.2011.10.010. |
| [Ost10a]  | Multivariate permutation polynomial systems and nonlinear pseudorandom number generators | Alina Ostafe | In: Finite Fields Th. App. 16.3 (May 2010), pp. 144–154. ISSN: 1071-5797. DOI: 10.1016/jffa.2009.12.003. |
| [Ost10b]  | Pseudorandom Vector Sequences Derived from Triangular Polynomial Systems with Constant Multipliers | Alina Ostafe | In: Arithmetic of Finite Fields. Ed. by M. Anwar Hasan and Tor Helleseth. Berlin, Heidelberg: Springer Berlin Heidelberg, 2010, pp. 62–72. ISBN: 978-3-642-13797-6. DOI: 10.1007/978-3-642-13797-6_5. |
| [RAS21]   | Interpolation Cryptanalysis of Unbalanced Feistel Networks with Low Degree Round Functions | Arnab Roy, Elena Andreeva, and Jan Ferdinand Sauer | In: Selected Areas in Cryptography. Ed. by Orr Dunkelman, Michael J. Jacobson Jr., and Colin O’Flynn. Vol. 12804. Lecture Notes in Computer Science. Cham: Springer International Publishing, 2021, pp. 273–300. ISBN: 978-3-030-81652-0. DOI: 10.1007/978-3-030-81652-0_11. |
| [Smi06]   | An Introduction to Quasigroups and Their Representations | Jonathan D.H. Smith | Studies in Advanced Mathematics. New York: Chapman & Hall / CRC Press, 2006. ISBN: 978-1-58488-537-5. DOI: 10.1201/9781420010633. |
| [SSA+07]  | The 128-Bit Blockcipher CLEFIA (Extended Abstract) | Taizo Shirai, Kyoji Shibutani, Toru Akishita, Shiho Moriai, and Tetsu Iwata | In: Fast Software Encryption. Ed. by Alex Biryukov. Vol. 4593. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007, pp. 181–195. ISBN: 978-3-540-74619-5. DOI: 10.1007/978-3-540-74619-5_12. |
| [Tof80]   | Reversible computing | Tommaso Toffoli | In: Automata, Languages and Programming. Ed. by Jaco de Bakker and Jan van Leeuwen. Berlin, Heidelberg: Springer Berlin Heidelberg, 1980, pp. 632–644. ISBN: 978-3-540-39346-7. DOI: 10.1007/3-540-100032_104. |
| [YPL11]   | On Lai–Massey and quasi-Feistel ciphers | Aaram Yun, Je Hong Park, and Jooyoung Lee | In: Designs, Codes and Cryptography 58.1 (Jan. 2011), pp. 45–72. ISSN: 1573-7586. DOI: 10.1007/s10623-010-9386-8. |