Universality Class of Bak-Sneppen Model on Scale-Free Network

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Abstract

We study the critical properties of the Bak-Sneppen coevolution model on scale-free networks by Monte Carlo method. We report the distribution of the avalanche size and fractal activity through the branching process. We observe that the critical fitness $f_c(N)$ depends on the number of the node such as $f_c(N) \sim 1/\log(N)$ for both the scale-free network and the directed scale-free network. Near the critical fitness many physical quantities show power-law behaviors. The probability distribution $P(s)$ of the avalanche size at the critical fitness shows a power-law like $P(s) \sim s^{-\tau}$ with $\tau = 1.80(3)$ regardless of the scale-free network and the directed scale-free network. The probability distribution $P_f(t)$ of the first return time also shows a power-law such as $P_f(t) \sim t^{-\tau_f}$. The probability distribution of the first return time has two scaling regimes. The critical exponents $\tau_f$ are equivalent for both the scale-free network and the directed scale-free network. We obtain the critical exponents as $\tau_{f1} = 2.7(1)$ at $t < t_c$ and $\tau_{f2} = 1.72(3)$ at $t > t_c$ where the crossover time $t_c \sim 100$. The Bak-Sneppen model on the scale-free network and directed scale-free network shows a unique universality class. The critical exponents are different from the mean-field results. The directionality of the network does not change the universality on the network.

Key words: Universality class, Bak-Sneppen model, scale-free network, coevolution, self-organized criticality
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1 Introduction

The concepts of self-organized criticality (SOC) has been widely applied in nonequilibrium systems such as biological evolution, slowly driven systems and economic systems[1,2,3,4,5,6,7,8]. In the SOC there is no characteristic scale of the system. The physical quantities show the power-law behaviors. The Bak-Sneppen (BS) model is a typical model showing the SOC[9,10]. The BS model is a simple coevolution model of the biological species evolving with fitness. The avalanche size distribution $P(s)$ is characterized by a power-law like $P(s) \sim s^{-\tau}$ where $\tau$ is an avalanche critical exponent[3,10]. The critical exponent $\tau$ depends on the dimensionality of the lattice. In the lattice BS model the directionality of the update of the fitness influences the critical exponents of the avalanche size distribution. The critical exponents of the fully anisotropic BS model are different to those of the isotropic BS model[11,12].

Recently, complex networks have been found in many systems such as WWW, social networks, biological networks, etc.[13,14,15,16,17]. Such networks is called scale-free network (SFN). The degree distribution of the SFN follow the power-law like $P(k) \sim k^{-\gamma}$ with $\gamma > 2$ where $k$ is a degree of the node on the SFN[14,15,16,17]. In the SFN there are hubs with large degrees. The degree distribution and the existence of the hubs contribute to the dynamics on the SFN. Moreno and Vazquez reported the critical fitness and the critical exponent $\tau$ of the BS model on Barabasi-Albert (BA) network. They observed that the critical exponent $\tau$ is close to the mean-field value $3/2$[18]. Lee and Kim also reported the critical fitness $f_c(N) \sim 1/\langle(k+1)^2 \rangle$ on the SFN with the size $N$ for $2 < \gamma \leq 3$. They observed two power-law regimes of the avalanche size distribution[19]. Masuda et al. obtained the analytic solution of the critical fitness $f_c \sim 1/\langle(k^2)/\langle k \rangle + 1 \rangle$ and $\tau = 3/2$[20]. In this article we study the BS model on the SFN and DSFN by Monte Carlo method. We observe that the critical fitness shows the logarithmic dependence of the total number of the node on the SFN. The avalanche size distribution follows the power-law. The critical exponent $\tau$ is independent of the directionality of the SFN. We observe the two scaling regimes in the probability distribution of the first return time. We observe that the BS model on the SFN and DSFN belongs to the same universality class. In section 2 we introduce the BS model and SFN. In section 2 we report the critical exponents of the BS model on the SFN and DSFN. We give the concluding remarks in section 4.

2 Bak-Sneppen model on scale-free network

Consider a scale-free network with a degree distribution $p(k) \sim k^{-\gamma}$ with a exponent $\gamma = 3.0$. The SFN has the different geometrical properties in
Fig. 1. The probability density of the fitness $p(f)$ and the least fitness $p(f_{\text{min}})$ on the SFN with the total number of nodes $N = 512$ for the Bak-Sneppen model. We obtain the critical fitness $f_c = 0.10413(5)$.

comparison to the regular lattice, small-world network, and random network. The SFN shows the scale-free behavior of the node degree distribution and small-world behaviors. We consider a scale-fee network and a directed scale-free network. We construct the SFN by Barabasi-Albert algorithm: (i) put $m_o = 3$ initial nodes. (ii) put a new node of $m = 3$ links with connecting probability $p_i = k_i / \sum j k_j$ where $k_i$ is the degree of the node (the number of links connecting to node $i$). (iii) repeat step 2 until the total number of nodes are $N = 10^5$. The directed SFN is obtained by applying a random direction of each links of the SFN.

We apply the BS coevolution model on the networks. The BS model is a simple coevolution model which shows punctuated equilibrium and the self-organized criticality. Let’s assign random fitness $f_i$ (a uniform real random number $0 < f_i < 1$) on the node of the network. Search a node with the least fitness. Update the fitness of the node with the least fitness and its nearest neighbor nodes. Repeat these steps. The system goes to a steady-state. At the steady-state the probability density that the system lies $f < f_c$ vanishes and is uniform above $f_c$ where $f_c$ is a critical fitness. We show the probability density $P(f)$ of the fitness and the probability density $P(f_{\text{min}})$ of the least fitness with the total number of nodes $N = 512$ on the SFN in Fig. 1 and DSFN in Fig. 2. The probability density $P(f)$ of the fitness is uniform at
Fig. 2. The probability density of the fitness $p(f)$ and the least fitness $p(f_{\text{min}})$ on the DSFN with the total number of nodes $N = 512$ for the Bak-Sneppen model. We obtain the critical fitness $f_c = 0.1349(5)$.

$f > f_c$. The fluctuations of the probability density $P(f)$ are very large at $f > f_c$ on the DSFN because of the directed connections of the links. The probability density $P(f_{\text{min}})$ of the least fitness drops to zero at the critical fitness.

We obtained the critical fitness as $f_c = 0.10413(5)$ on the SFN and $f_c = 0.1349(5)$ on the DSFN for $N = 512$. In the SFN the critical fitness depends on the total number of nodes $N$. In the thermodynamic limit $N \to \infty$, the critical fitness approaches logarithmically to zero. In Fig. 3 we present the critical fitness as a function of the number of the nodes $N$. We observed $f_c \sim 1/\ln N$ and $f_c = 0$ at $N \to \infty$ for both the SFN and DSFN. This result for the SFN is consistent with the observations of Mareno and Vazquez[18]. Masuda et al. derived the critical fitness of isotropic BS model such as $f_c \sim 1/(< k^2 > / < k > +1)$ for the SFN. In the finite number of node $N$, the average number of nodes for the SFN is given as $< k^2 > \sim \ln N$ and $< k > \sim$ constant. Our result of the critical fitness is consistent with the predictions of Masuda et al.[20]. The critical fitness for the DSFN also follows the logarithmic dependence. So, we observe the critical behaviors at the finite critical point $f_c(N)$ in the simulation.
Fig. 3. The critical fitness vs the number of the nodes on the network. We observe $f_c \sim 1/\log(N)$. At $N \to \infty$ $f_c$ goes to zero.

3 Critical Exponents and Universality Class

In the BS model the branching processes simplify the dynamic evolutions. Consider an $f_o$-avalanche. In $f_o$-avalanche all the sites with $f_i > f_o$ are treated as vacuum sites. The sites with $f_i < f_o$ are active sites. In the branching process we first create a random number, chosen from the uniform distribution, at a randomly selected node on the network. All other sites are vacuum nodes. At each time step, the site of the minimal fitness with $f_i < f_o$ activates until there are no more nodes with $f_i < f_o$. If there is no an active site anymore, the $f_o$-avalanche is finished.

We measure the distribution of the avalanche size by the branching process. In the lattice BS model the probability distribution $P(s, f_o)$ of the $f_o$-avalanche of the size $s$ is described by the scaling function[3]

$$P(s, f_o) = s^{-\tau} g(s(f_c - f_o)^{1/\sigma})$$

(1)

where $\tau$ and $\sigma$ are critical exponents and $g(x)$ is a scaling function. The scaling function is given by $g(x) \to 0$ for $x \gg 1$ and $g(x) \to \text{constant}$ for $x \to 0$. At the critical fitness $f_o \to f_c$, the probability distribution of the avalanche
In Fig. 4 we present the probability distribution of the avalanche size as a function of the avalanche size $s$ at the critical fitness on the SFN and DSFN. The probability distribution $P(s)$ follows the power-law as

$$P(s) \sim s^{-\tau}$$  \hspace{1cm} (2)

We plot the histogram of the avalanche size distribution by exponentially increasing bins to reduce the fluctuations of the data[21]. We obtain the critical exponent $\tau = 1.80(3)$ both the SFN and DSFN. At the large avalanche size the probability distribution $P(s)$ decays according to the exponential function. The observed critical exponent $\tau$ is different from the mean-field value $\tau = 3/2$ and bigger than the exponents of the lattice BS model. The scale-free network contributes the enhancement of the critical exponent $\tau$. In our results we observe only a scaling regime at the range $1 < s < 100$. At the large $s$ the avalanche size distribution deviates from the power-law and follow exponential decays. We can not observe two scaling regimes reported by Lee and Kim[19]. The critical exponents $\tau$ are the same for both the SFN and DSFN. The directionality of the network does not change the critical behavior of the probability distribution of the avalanche size.

In Fig. 5 we present the probability distribution of the first return time. Con-
Fig. 5. The log-log plot of the probability distribution function $P(s)$ of the first return time as a function of the time $t$ at the critical fitness on SFN and DSFN.

Consider an active site at a time $t'$. This site is active again at a time $t' + t$. We define the time interval $t$ as the first return time (or waiting time). In the random process the probability distribution of the first return time is Poisson distribution. However, in the SOC system the probability distribution $P_f(t)$ of the first return time follows a power-law like

$$P_f(t) \sim t^{-\tau_f}$$

(3)

where $\tau_f$ is a critical exponent. In Fig. 5 we can observe the obvious two scaling regimes with the crossover time $t_c \sim 100$ for both the SFN and DSFN. We obtain the critical exponent $\tau_{f1} = 2.7(1)$ at the early-time regime and $\tau_{f2} = 1.72(3)$ at the late-time regime for both the SFN and DSFN. The critical exponent $\tau_{f2}$ is greater than the exponent $\tau_f = 1.58$ on one dimensional lattice and $\tau_f = 1.28$ on two dimensional lattice.

The observed critical exponents on the SFN and DSFN are different from the mean-field predictions and those of the lattice BS model. We conclude that the BS model on the SFN and DSFN belongs to a unique universality class. The directionality of the network does not change the critical exponents.
4 Conclusion

We consider the Bak-Sneppen coevolution model on the SFN and DSFN. We obtain the critical fitness, the probability distribution of the avalanche size, and the probability distribution of the first return time by the Monte Carlo simulations. We observe that the critical fitness depend on the total number of the nodes for both the SFN and DSFN. The critical fitness goes to zero according to $f_c \sim 1/\log(N)$. The avalanche size distribution decays according to the power-law at the early scaling regime. The critical exponent of avalanche size is greater than that of the lattice BS model. The probability distribution of the first return time has two scaling regimes separated by the crossover time $t_c \sim 100$. We have found that the Bak-Sneppen model on the SFN and DSFN belongs to a new universality class. The directionality of the networks does not change the universality for the Bak-Sneppen model on the SFN.

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