Universal Single-Qubit Nonadiabatic Holonomic Quantum Gates on an Optomechanical System

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1. Introduction

Quantum geometric phases\textsuperscript{[1–3]} are very important resources for quantum computation. They have unique advantages of robustness in quantum computation due to their global geometric property in evolution process and thus attract much attention from both theoretical and experimental aspects.\textsuperscript{[4–13]} One of the important contributions in this field came from Zanardi and Rasetti,\textsuperscript{[4]} who proposed the adiabatic holonomic quantum computation (AHQC) by using the geometric phases. It is shown that AHQC can be used to implement the high-fidelity quantum gates as it is robust to small random perturbations of the path in the parameter space and the experimental imperfection.\textsuperscript{[13–19]} Up to now, several AHQC schemes have been proposed based on different physical systems, such as trapped ions,\textsuperscript{[20]} superconducting qubits,\textsuperscript{[21]} and semiconductor quantum dots.\textsuperscript{[22]} However, an adiabatic process may introduce more decoherence due to a long evolution time, while the decoherence leads to the decrease of fidelity. To solve this problem, the nonadiabatic holonomic quantum computation (NHQC) was proposed. The NHQC applications include the nonadiabatic geometric-phase-shift gate with nuclear magnetic resonance,\textsuperscript{[23]} universal nonadiabatic geometric quantum gates\textsuperscript{[24]} and some other theoretical schemes\textsuperscript{[25–35]} and the experimental realizations.\textsuperscript{[36–43]} These works show that the NHQC has the features of the built-in noise resilience, faster operation, less decoherence, and so on. Also, some related applications are completed with quantum state operation\textsuperscript{[35–39]} and quantum gate operation\textsuperscript{[35,46,60,61]} and platform to realize the quantum effects in the content of quantum optics\textsuperscript{[44]} and quantum information processing.\textsuperscript{[45,46]}

The fundamental study in this field includes the cooling of the mechanical resonator to its ground state,\textsuperscript{[47–49]} the strong coupling between the cavities and the mechanical resonator,\textsuperscript{[50,51]} the optomechanically induced transparency,\textsuperscript{[52–54]} and so on. Also, some related applications are completed with quantum state operation\textsuperscript{[35–39]} and quantum gate operation\textsuperscript{[35,46,60,61]} and platform to realize the quantum effects in the content of quantum optics\textsuperscript{[44]} and quantum information processing.\textsuperscript{[45,46]}

Here, we propose the first scheme to achieve a set of universal single-qubit nonadiabatic holonomic quantum gates on an optomechanical system which is composed of two optical cavities coupling with a mechanical oscillator. These gates work with the nonadiabatic geometric phases, including the noncommuting NOT gate, the phase gate, and the Hadamard gate. They are obtained in the computational basis of a single-excited state of the optomechanical system after a cyclic evolution of the system is finished. With these universal single-qubit gates, one can also achieve the quantum state transfer and the entanglement generation between two cavity modes. Our scheme owns all the good properties of the NHQC on a quantum system, such as the built-in noise resilience, faster operation, less decoherence, and the nonrequirement for the resource and time to remove the dynamical phases. Our scheme provides an interesting way for the quantum gates realized with the mechanical-motion degree of freedom and has the promising application in robust quantum computation and quantum information processing.

2. Basic Model for an Optomechanical System

The schematic diagram for an optomechanical system is shown in Figure 1. In this system, the two cavity modes couple to each other by the radiation pressure force via a mechanical oscillator and also are driven, respectively, by a laser in the red sideband resonating with the mechanical mode. After the linearization procedure, the Hamiltonian of this optomechanical system in the interaction picture is given by $\hbar = 1$\textsuperscript{[57,58]}

$$\hat{H}_i = \sum_{i=1,2} \delta_i \hat{a}_i^\dagger \hat{a}_i + G_i \hat{b}_i^\dagger \hat{b}_i + H.c.$$  \hspace{1cm} (1)

where $\hat{a}_i$ ($\hat{a}_i^\dagger$) $(i = 1, 2)$ and $\hat{b}_i$ ($\hat{b}_i^\dagger$) are the annihilation (creation) operators for the ith cavity with frequency $\omega_i$ and the mechanical...
oscillator with frequency $\omega_m$, respectively. $\delta_i = -\Delta_i - \omega_m$ with the detuning $\Delta_i = \omega_{ij} - \omega_i$ between the laser $\omega_{ij}$ and the cavity mode $\omega_i$. $G_i = e^{i \phi_i \omega_0 \sqrt{m_i}}$ is the effective coupling strength which depends on the single-photon coupling strength $g_0$, the intracavity photon number $n_i$, and the phase $\phi_i$ of the laser of frequency $\omega_{ij}$. Let us choose $\delta_i = 0$. $|g_i\rangle = |100\rangle$, $|g_2\rangle = |001\rangle$, and $|e\rangle = |010\rangle$ represent the single-excited states of the cavities 1 and 2 and the mechanical oscillator, respectively. We take $|g_1\rangle$ and $|g_2\rangle$ as the two computational basis states and $|e\rangle$ as the ancillary basis state to construct a single-qubit state subspace $S_i = \{|g_1\rangle, |e\rangle, |g_2\rangle\}$. In $S_i$, the Hamiltonian $\hat{H}_i$ shown in Equation (1) can be rewritten as

$$\hat{H}_i = G_0(t) \begin{bmatrix} \sin \frac{\theta}{2} e^{i\phi} & |e\rangle \langle g_1| - \cos \frac{\theta}{2} |e\rangle \langle g_2| & H.C. \end{bmatrix}$$

where $G_0(t) = \sqrt{G_1(t) + G_2(t)}$ with $G_1(t)$ and $G_2(t)$ being the Rabi frequencies defined by $G_1(t)/G_0(t) = \sin \frac{\theta}{2} e^{i\phi}$ and $G_2(t)/G_0(t) = -\cos \frac{\theta}{2}$, respectively, and $\phi = \phi_1 - \phi_2$. For simplification, we choose $\phi_2 = 0$ and $\phi = \phi_1$ in the following calculation. To ensure the positivity of $G_2$, $\frac{\pi}{2} \leq \theta \leq \pi$ should be satisfied.

The Hamiltonian $\hat{H}_i$ shown in Equation (2) can be turned into a $3 \times 3$ matrix form

$$\hat{H}_i = G_0(t) \begin{bmatrix} 0 & \sin \frac{\theta}{2} e^{i\phi} & 0 \\ \sin \frac{\theta}{2} e^{i\phi} & 0 & -\cos \frac{\theta}{2} \\ 0 & -\cos \frac{\theta}{2} & 0 \end{bmatrix}$$

Accordingly, $|g_1\rangle$, $|e\rangle$, and $|g_2\rangle$ become $|g_1\rangle = [1, 0, 0]^T$, $|e\rangle = [0, 1, 0]^T$, and $|g_2\rangle = [0, 0, 1]^T$, respectively. Here the superscript “$T$” means the transposition. The instantaneous eigenvectors of the Hamiltonian $\hat{H}_i$ shown in Equation (3) are given by

$$|E_0\rangle = \cos \frac{\theta}{2} |g_1\rangle + \sin \frac{\theta}{2} e^{i\phi} |g_2\rangle$$

$$|E_+\rangle = \frac{1}{\sqrt{2}} \left( \sin \frac{\theta}{2} e^{-i\phi} |g_1\rangle - \cos \frac{\theta}{2} |g_2\rangle + |e\rangle \right)$$

$$|E_-\rangle = \frac{1}{\sqrt{2}} \left( \sin \frac{\theta}{2} e^{-i\phi} |g_1\rangle - \cos \frac{\theta}{2} |g_2\rangle - |e\rangle \right)$$

and the corresponding eigenvalues are $E_0 = 0$, $E_+ = G_0(t)$, and $E_- = -G_0(t)$, respectively. In the dressed-state representation, the bright state $|b\rangle = \sin \frac{\theta}{2} e^{-i\phi} |g_1\rangle - \cos \frac{\theta}{2} |g_2\rangle$ and the dark state $|d\rangle = \cos \frac{\theta}{2} |g_1\rangle + \sin \frac{\theta}{2} e^{i\phi} |g_2\rangle$. The bright state couples to the excited state $|e\rangle$ and the dark state decouples from the state $|e\rangle$, respectively.

3. Universal Single-Qubit Nonadiabatic Holonomic Quantum Gates on an Optomechanical System

To implement the single-qubit gates on an optomechanical system by the nonadiabatic geometric dynamics, two conditions should be satisfied.[25] First, $\alpha(t) = \pi$ with $\int_0^\tau G_0(t')dt' = \alpha(t)$ is used to ensure the states undergo a cyclic evolution. Second, the parallel-transport condition with $\dot{\hat{H}}_i = \{\psi_1(t), \hat{H}_i|\psi_1(t)\rangle\} = 0$ is used to keep the zero dynamical phases. In this way, the total evolution phases are the purely geometric phases. The bright and the dark states evolve as

$$|\psi_1(t)\rangle = \hat{U}_i(t)|d\rangle = |d\rangle$$

$$|\psi_2(t)\rangle = e^{i\omega_0(t)} \hat{U}_i(t)|b\rangle = e^{i\omega_0(t)} \left[ \cos \alpha(t)|b\rangle - i \sin \alpha(t)|e\rangle \right]$$

where the evolution operator is $\hat{U}_i(t) = \exp(-i \int_0^\tau \hat{H}_i(t')dt')$. The inserted factor $e^{i\omega_0(t)}$ is used to ensure the cyclic evolution $|\psi_2(0)\rangle = |\psi_2(t)\rangle$ in the Hilbert space. According to Equation (5), the accumulated purely geometric phases during the evolution of the dark state and the bright state are $0$ and $\pi$, respectively.

The geometric dynamics process for the nonadiabatic quantum state transfer process is shown in Figure 2. The dark state keeps unchanged in the evolution under the driving of the Hamiltonian $\hat{H}_i$ with the basis $\{|g_1\rangle, |g_2\rangle\}$ and the bright state evolves along the longitude with the basis $\{|b\rangle, |e\rangle\}$. Projecting the dark–bright basis onto the computational subspace spanned by $\{|g_1\rangle, |g_2\rangle\}$, one makes a transformation of coordinates with the form

$$|\xi_1(t)\rangle = \sin \frac{\theta}{2} |\psi_2(t)\rangle + \cos \frac{\theta}{2} |d\rangle$$

$$|\xi_2(t)\rangle = -\cos \frac{\theta}{2} |\psi_2(t)\rangle + \sin \frac{\theta}{2} e^{-i\phi} |d\rangle$$
Therefore, the evolution operator becomes
\[
\hat{U}(\theta, \varphi) = \begin{bmatrix}
\cos \theta & \sin \theta e^{-i\varphi} \\
-\sin \theta & \cos \theta e^{-i\varphi}
\end{bmatrix}
\] (8)

where \( \theta \) and \( \varphi \) are the corresponding parameter values on the Bloch sphere. By changing the different values of the coupling strength, that is, \( \theta \) and \( \varphi \), one can get the NOT gate, the rotation gate, the Hadamard gate, and the phase-flip gate with \( (\theta, \varphi) = (\frac{\pi}{4}, 0), (\frac{\pi}{4}, \frac{\pi}{2}), (\frac{\pi}{4}, \pi), \) and \( (2\pi, 0) \), respectively.\(^{[38]}\) In addition, another phase gate can be obtained by the product of \( U(\frac{\pi}{4}, \frac{\pi}{2}) \) and \( U(\frac{\pi}{4}, 0) \), that is
\[
\begin{bmatrix}
0 & e^{-i\frac{\pi}{4}} \\
e^{i\frac{\pi}{4}} & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} =
\begin{bmatrix}
e^{-i\frac{\pi}{4}} & 0 \\
0 & e^{i\frac{\pi}{4}}
\end{bmatrix}
\] (9)

With these gates, one can obtain a set of universal single-qubit gates based on the subspace spanned by \( \langle g_1 | , g_2 | \rangle \). Besides, the NOT gate and the Hadamard gate can be efficiently applied to an optomechanical system for quantum information processing.

Now, let us discuss how to use the NOT gate and the Hadamard gate above to accomplish the quantum state transfer and the entanglement generation on the optomechanical system, respectively. For the quantum state transfer, with \( \varphi = 0 \) and \( \theta = \frac{\pi}{4} \), one can obtain the NOT gate
\[
U(\frac{3\pi}{4}, 0) = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\] (10)

If the initial quantum state is prepared in \( |g_1 \rangle \), one can accomplish the quantum state transfer between the two cavities under the driving of the NOT gate, that is
\[
-|g_2 \rangle = U(\frac{3\pi}{4}, 0) |g_1 \rangle
\] (11)

In the following, let us choose the square-shaped pulses as the driving fields to perform the quantum state transfer, where \( G_1(t), G_2(t), \) and \( G_3(t) \) are regarded as the constants over the pulse duration with \( G_1/2\pi = 2 \text{ MHz}, G_1/2\pi = 2 \text{ MHz}, \) and \( G_3/2\pi = 2\sqrt{2} \text{ MHz} \). We calculate the variation of the population and the fidelity shown in Figure 3a. The fidelity is defined as \( F = \langle \psi_{\text{ideal}} | \hat{G}_m | \rho(t) | \psi_{\text{ideal}} \rangle \), where \( \psi_{\text{ideal}} \) (\( \approx -|01 \rangle \)) represents the ideal final state and \( \hat{G}_m | \rho(0) | \) means a reduced density matrix obtained by tracing the mechanical oscillator degrees of freedom. When \( t = \pi/\hat{G}_o \approx 0.177 \mu s \) which equals to the evolution period, the complete population inversion indicates that the system achieves the quantum state transfer successfully and the system satisfies the cyclic evolution very well.

Also, one can generate a discrete-variable entangled state
\[
|\psi_{\text{ideal}} \rangle = \frac{(|00 \rangle + |01 \rangle + |10 \rangle + |11 \rangle)}{\sqrt{2}}
\]

between the two cavities with the Hadamard gate constructed based on Equation (8) with \( \varphi = 0 \) and \( \theta = \frac{\pi}{2} \). The process can be described by
\[
|\psi_{\text{ideal}} \rangle \approx U \left( \frac{5\pi}{4}, 0 \right) |g_1 \rangle
\] (12)

Here the Hadamard gate is
\[
U \left( \frac{5\pi}{4}, 0 \right) = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\] (13)

With the parameters \( G_1/2\pi \approx 2.6131 \text{ MHz}, \) \( G_1/2\pi \approx 1.0824 \text{ MHz}, \) and \( G_3/2\pi \approx 2.8284 \text{ MHz}, \) the process of the entanglement generation shown in Figure 3b can be realized. When \( t = \pi/\hat{G}_o \approx 0.177 \mu s \), the fidelity arrives 100% and the state becomes \( |\psi_{\text{ideal}} \rangle \).

3.1. Simulations and Fidelities

In a real case, a quantum system is affected by its environment. The performance of our scheme in its environment can be evaluated based on the fidelities of the NOT gate and the Hadamard gate. Due to the dissipation, the dynamics of the optomechanical system fulfills the following master equation
\[
\frac{d\hat{\rho}}{dt} = i[\hat{\rho}, \hat{H}_1] + \kappa_1 \hat{L}[\hat{a}_1] \hat{\rho} + \kappa_2 \hat{L}[\hat{a}_2]\hat{\rho} + \gamma_m \hat{D}[\hat{b}_m]\hat{\rho}
\] (14)

In Equation (14), \( \hat{\rho} \) is the reduced density operator for the optomechanical system and \( \hat{H}_1 \) is the Hamiltonian of the system. \( \gamma_m, \kappa_1, \) and \( \kappa_2 \) represent the mechanical damping rate and the decay rates of the cavities 1 and 2, respectively. The terms with the super-operators \( \hat{L} \) and \( \hat{D} \) have the Lindblad forms
\[
\hat{L}[\hat{a}]\hat{\rho} = (\hat{a} \hat{\rho} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \hat{\rho} - \hat{\rho} \hat{a} \hat{a}^\dagger)/2 + \kappa_1 (\hat{a}^\dagger \hat{\rho} \hat{a} - \hat{\rho} \hat{a}^\dagger)/2 + \kappa_2 (\hat{a} \hat{\rho} \hat{a}^\dagger - \hat{\rho} \hat{a}^\dagger \hat{a} + \hat{a}^\dagger \hat{a} \hat{\rho} - \hat{a} \hat{\rho} \hat{a}^\dagger)/2
\]
phonon number of the environment. Let us choose the parameters \( \kappa = \kappa_1 = \kappa_2 \) and \( \gamma_m \) in the range \( \kappa/2\pi = [5, 100] \times 10^{-4} \) MHz and \( \gamma_m/2\pi = [3, 57] \times 10^{-4} \) MHz, \( n_{th} = 100 \), the frequencies of the two cavities \( \omega_1/2\pi = 100 \) THz and \( \omega_2/2\pi = 100 \) THz. The frequency of the mechanical oscillator can be experimentally fabricated with gigahertz.\(^6\) Here, we choose it as \( \omega_m/2\pi = 10 \) MHz.

For the NOT gate, the influence of \( \kappa \) and \( \gamma_m \) on the fidelity of the quantum state transfer is shown in Figure 4a. The fidelity is inversely proportional to \( \kappa \) and \( \gamma_m \) and the maximum and the minimum fidelities are 0.96 and 0.56, respectively. For the Hadamard gate, as shown in Figure 4b, the maximum and the minimum fidelities become, accordingly, 0.95 and 0.50, respectively. In both cases, the fidelity tends to decrease monotonously with respect to \( \kappa \) and \( \gamma_m \). The higher the damping of the mechanical oscillator or the cavity mode is, the lower the fidelity one can obtain. Therefore, the preparation of the high-quality optomechanical system is helpful to implement the universal single-qubit holonomic gates.

The quantum gates on an optomechanical system in our schemes operate in a photon–phonon–photon way. In addition, our schemes can also work in a phonon–photon–phonon way. There is an example to describe their principles. Assuming that an optomechanical system is composed of two mechanical modes coupling to a common optical mode\(^{63}\) based on the Hamiltonian with the form similar to that of \( \hat{H} \) (Equation (1)) for \( \delta_i = 0 \). When we choose a single-excited state of the mechanical resonators and the cavity to construct the subspace, with the similar treatment above, there is no difficulty to obtain a set of universal single-qubit nonadiabatic holonomic quantum gates working in the phonon–photon–phonon configuration.

### 3.2. Summary

In summary, we have proposed the first scheme for the universal single-qubit quantum gates on an optomechanical system working with the nonadiabatic geometric phases. We have shown its typical application by changing the different coupling strengths to get various noncommute quantum gates: the NOT gate, the phase gate, and the Hadamard gate, and apply these gates to achieve the quantum state transfer and the two-cavity-mode entanglement generation in an optomechanical system. The result has shown that the quantum gates can have high fidelity against the negative influence of their dissipative environment. Our scheme is of all the good properties of the NHQC on a quantum system and can be extended to other hybrid optomechanical quantum systems.

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**Conflict of Interest**

The authors declare no conflict of interest.

**Keywords**

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