An integral bilinear form and related forms on abelian groups as hexagon cocycles

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Abstract

Hexagon relations are algebraic realizations of four-dimensional Pachner moves, and there are hexagon relations admitting nontrivial cohomologies and leading thus to piecewise linear (PL) 4-manifold invariants. We show that some—but not all!—of the known nontrivial cohomologies can be obtained from a single integral bilinear form corresponding to a PL 4-manifold by using a Frobenius homomorphism for a half of ‘color’ variables (or different Frobenius homomorphisms for both halves). This form can be regarded as a sophisticated analogue of the manifold’s intersection form.

1 Introduction

1.1 Generalities

A triangulation of a piecewise linear (PL) 4-manifold can be transformed into its any other triangulation by a finite sequence of Pachner moves [8, 7]. Hence, it is natural to expect that a PL 4-manifold invariant can be constructed if we have an algebraic realization of Pachner moves—informally speaking, such formulas whose structure corresponds to these moves naturally. Such formulas are often called hexagon relations (or, more generally, (n+2)-gon relations for n-manifolds, see, for instance, [3]).

A fruitful version of hexagon relation has been proposed in [5, 6]. In it, a two-component ‘color’ (x_t, y_t) is ascribed to every tetrahedron t—3-face of a triangulation. Thus, there appear ten variables on the 3-faces of each pentachoron (4-simplex) u and it is required that these ten variables obey five relations. Moreover, the variables x_t, y_t in [5, 6] belonged to a field, and the relations were linear. As we will see below in this paper, this construction can work productively also, at least, for modules over ring Z, that is, abelian groups.

The most interesting manifold invariants appear if we use cohomology of our hexagon relations (in perfect analogy with the fact that powerful invariants of knots and their higher analogues are obtained from quandle cohomology [1]). In [6], we have calculated some ‘polynomial’ cohomologies; one can see that they are very nontrivial, and give nontrivial manifold invariants.

The nature of these polynomial cohomologies looks, at the moment, quite mysterious. At the same time, there is a hope that it can be clarified; compare,
for instance, the study of polynomial quandle cocycles in [2] and references therein. A natural desire is also to relate our cohomologies to known algebraic structures appearing in the study of 4-manifolds, or at least find parallels between them.

1.2 What we do in the present paper

We bring to light some parallels between one cocycle appearing in our work and the intersection form of 4-manifolds (see textbooks [4, 9]). We construct a similar Z-bilinear form in the framework of our theory. Further, we show that some of nontrivial polynomial hexagon cocycles in finite characteristics found in [6] can be derived from this single Z-bilinear form—to be exact, its modifications involving finite fields—using manipulations with Frobenius homomorphisms.

*Important remark.* Not all polynomial cocycles could be obtained that way, at least by now!

1.3 Notational conventions

Typically, we denote 2-simplices—triangles—by the letter $s$, 3-simplices—tetrahedra—by the letter $t$, and 4-simplices—pentachora—by the letter $u$.

All vertices of any triangulated object are assumed to be *numbered*. A triangle $s = ijk$ has vertices whose numbers are $i, j$ and $k$.

Moreover, when we denote a simplex by its vertices, these go, by default, in the *increasing* order. For the above triangle $s$, this means that $i < j < k$.

1.4 The content of the paper by sections

Below,

- in Section 2 we recall, very briefly, the notion of permitted colorings of 3-faces satisfying the full hexagon. One small new moment is that we formulate all this in terms of general abelian groups,
- in Section 3 we introduce our Z-bilinear hexagon cocycle,
- in Section 4 we describe two kinds of manifold invariants: the ‘probabilities’ of the values that our bilinear ‘action’ can take on a manifold, and just the action itself understood as an integral bilinear form taken to within invertible Z-linear transformations and adding zero direct summands,
- in Section 5 we recall polynomial cocycles over finite fields from [6] and show how some of these can be reproduced using our bilinear form and ‘Frobenius tricks’,
- in Section 6 we explain why the usual intersection form of a 4-manifold can be seen as an analogue of our Z-bilinear ‘action’,
- and finally, in Section 7 we give a very brief discussion of our results and some related intriguing questions.
2 Hexagon relation

2.1 Colorings by an abelian group

Let $G$ be an abelian group. We will color tetrahedra—that is, triangulation 3-faces—by pairs $(x_t, y_t)$ of elements $x_t, y_t \in G$, and call this simply $G$-coloring. For a pentachoron $u = ijklm$, we introduce two vector columns of height 5, corresponding to its 3-faces going in the inverse lexicographic order:

$$
\begin{align*}
  x_u &= \begin{pmatrix}
  x_{ijkl} \\ x_{iklm} \\ x_{ijlm} \\ x_{ijkm} \\ x_{ijkl}
\end{pmatrix}, \\
  y_u &= \begin{pmatrix}
  y_{ijkl} \\ y_{iklm} \\ y_{ijlm} \\ y_{ijkm} \\ y_{ijkl}
\end{pmatrix}.
\end{align*}
$$

We also take the following matrix with integer entries from [Eq. (5) and (6)]:

$$
\mathcal{R} = \begin{pmatrix}
  0 & -2 & 1 & 1 & -2 \\
  -1 & 2 & -2 & 0 & 1 \\
  -1 & 3 & -2 & -1 & 2 \\
  0 & 1 & -1 & 0 & 0
\end{pmatrix}.
$$

**Definition 1.** A coloring of (the 3-faces of) a pentachoron $u$ is called permitted if the following relation holds:

$$
y_u = \mathcal{R} x_u.
$$

**Definition 2.** A coloring of a triangulated piecewise linear 4-manifold $M$ is called permitted if it induces permitted colorings on all its pentachora.

Permitted colorings of the initial and final configurations of any Pachner move are in good correspondence with each other. We say that they satisfy full set theoretic hexagon. For exact formulations, the reader is referred to [Sections 2 and 3].

Remark. The fact that we are using here an arbitrary abelian group instead of a field in [6] brings nothing significantly new into our definitions and reasonings.

2.2 Double colorings

One case of special importance is double coloring. This is, by definition, a coloring in the sense of Subsection 2.1 with $G$ being a direct sum of two abelian groups:

$$
G = A \oplus B.
$$

In this case, we will use the following notations for the pairs of elements of each group:

$$(x_t, y_t) \in A^{\oplus 2}, \quad (\xi_t, \eta_t) \in B^{\oplus 2}.$$
Similarly to (1), we introduce four columns:

\[
    \begin{pmatrix}
        x_{jklm} \\
        x_{iklm} \\
        x_{ijlm} \\
        x_{ijkl}
    \end{pmatrix}, \quad
    \begin{pmatrix}
        y_{jklm} \\
        y_{iklm} \\
        y_{ijlm} \\
        y_{ijkl}
    \end{pmatrix}, \quad
    \begin{pmatrix}
        \xi_{jklm} \\
        \xi_{iklm} \\
        \xi_{ijlm} \\
        \xi_{ijkl}
    \end{pmatrix}, \quad \text{and} \quad
    \begin{pmatrix}
        \eta_{jklm} \\
        \eta_{iklm} \\
        \eta_{ijlm} \\
        \eta_{ijkl}
    \end{pmatrix},
\]

and a permitted double coloring of a pentachoron is of course such that

\[
y_u = R x_u, \quad \eta_u = R \xi_u.
\] (2)

3 A hexagon cocycle in the form of a \(\mathbb{Z}\)-bilinear form

**Theorem 1.** The following bilinear form:

\[
    \Phi_u(x_u, \xi_u) = (x_{jklm} + y_{jklm}) \otimes (\xi_{ijkl} + \eta_{ijkl}) \\
    = (x_{jklm} - 2x_{iklm} + x_{ijlm} + x_{ijkm} - 2x_{ijkl}) \otimes (\xi_{iklm} - \xi_{ijlm} + \xi_{ijkl})
\] (3)

is a nontrivial hexagon 4-cocycle. Here \(u = ijk lm\), and the form \(\Phi\) depends thus on a pair “permitted A-coloring, permitted B-coloring” (relations (2) are of course implied) and takes values in the abelian group \(A \otimes B\).

**Proof.** Direct calculation. \(\square\)

**Important remark.** Recall (Subsection 1.3) that our notational conventions imply that \(i < j < k < l < m\), so \(ijkl\) may be called the front 3-face of pentachoron \(u = ijk lm\), while \(jklm\)—its rear 3-face. The first line of (3) shows that \(\Phi_u\) is the product of two quantities belonging to these two 3-faces. This is especially interesting when compared with the manifold’s intersection form, see Section 6 below.

4 Invariants

4.1 The action

For a triangulated oriented 4-manifold \(M\), maybe with boundary, we introduce the following action, depending on a permitted coloring of \(M\):

\[
    S = \sum_u \epsilon_u \Phi_u(x_u, \xi_u),
\] (4)

where the sum goes over all triangulation pentachora \(u\), and \(\epsilon_u = 1\) if the orientation of \(u\) determined by the increasing order of its vertices coincides with its orientation induced from \(M\), and \(\epsilon_u = -1\) otherwise.

Below, in Subsections 1.2 and 1.3 we specialize this action for two interesting cases.
4.2 Subgroups of permitted colorings and probabilities of action values

Theorem 2. Let $A$ and $B$ be as in Subsection 2.2 and, moreover, let them be finite, and let a subgroup $F \subset A \oplus B$ be given. For each permitted $F$-coloring, action $S$ takes some value $v \in A \otimes B$. Define the probability of value $v$ as

$$P(v) = \frac{\#(S = v)}{\#(\text{all permitted colorings})},$$

(5)

where $\#$ means the cardinality of a set; so, $\#(S = v)$ is the number of those permitted colorings where $S = v$, compare [6, (19)].

Then, the probabilities $P(v)$ for all $v \in A \otimes B$ are invariants of the piecewise linear manifold $M$.

Proof. It repeats the proof of [6, Theorem 4(i)]. \qed

We will see in Section 5 how to choose subgroup $F$ in order to obtain some of polynomial actions in [6].

4.3 Integral bilinear form

Set now simply $A = B = \mathbb{Z}$, and the subgroup $F = G = A \oplus B$. Then action $S$ given by (4) becomes an integral bilinear form.

Theorem 3. Let bilinear form $S$ (depending on a pair of permitted integral colorings) be defined according to (4), (3) with integer variables $x_t$ and $\xi_t$. Then $S$, taken to within a zero direct summand and an invertible $\mathbb{Z}$-linear transformation, is a piecewise linear 4-manifold invariant.

Proof. First, we analyze what happens under any Pachner move. Everything goes in, essentially, the same way as in [6, Section 3] (although, there was a field in [6] instead of our current $\mathbb{Z}$):

• for a Pachner move 3–3, there is a bijective—and certainly $\mathbb{Z}$-linear—correspondence between the permitted $\mathbb{Z}$-colorings before and after this move,

• for a move 2–4, there appears one additional ‘$\mathbb{Z}$-degree of freedom’: to each permitted coloring before the move there correspond colorings after the move, parameterized by $\mathbb{Z}$,

• for a move 1–5, there appear, in a similar way, four additional ‘$\mathbb{Z}$-degrees of freedom’,

• and for all Pachner moves, the value of $S$ remains the same, after applying this correspondence, due to the cocycle property.

Finally, if we choose a different basis in the space (to be exact, free $\mathbb{Z}$-module) of permitted colorings, this will correspond, of course, to an invertible $\mathbb{Z}$-linear transformation applied to the form $S$. \qed
**Theorem 4.** For a closed oriented triangulated 4-manifold $M$, the form $S$ of Theorem 3 is symmetric.

**Proof.** To see this, we subtract from (3) the same expression, but with interchanges $x_t \leftrightarrow \xi_t$, $y_t \leftrightarrow \eta_t$. It turns out that the result is a coboundary, in the sense of hexagon cohomology. Below we write this coboundary even in two different ways:

\[
(\xi_{jklm} + \eta_{jklm})(\xi_{ijkl} + \eta_{ijkl}) - (\xi_{ijkl} + \eta_{ijkl})(\xi_{jklm} + \eta_{jklm}) + \ldots
= (\xi_{jklm} + \eta_{jklm})\eta_{jklm} - (\xi_{ijkl} + \eta_{ijkl})\eta_{ijkl} + \ldots
\]

Here the omission points mean, in both cases, three obvious similar summands, corresponding to the remaining 3-faces $ijlm$, $iklm$ and $ijkl$, and coming with alternating signs.

Thus, the difference between $S$ and the same $S$, but with the mentioned ‘Latin–Greek’ interchanges, consists of summands belonging to triangulation tetrahedra, and, moreover, for a closed $M$ they all cancel away. \qed

**Remark.** Of course, if we know the integral bilinear form $S$ of this Subsection, we can get also all the forms mentioned above and involving abelian groups, by using the tensor product operation. The reader will hopefully have no difficulty in writing the exact formulas.

5 Particular case: polynomial cocycles over finite fields

5.1 Introducing dependence of $\xi_t$ on $x_t$ using Frobenius homomorphism

We continue to use the notations of Subsection 2.2. Let $A = B = \mathbb{F}_{p^n}$, that is, the finite field of $p^n$ elements, and let $F \subset A \oplus B$ be generated by such pairs $(a, b) = (x_t, \xi_t)$ that $b = a^{p^m}$, where $m \in \mathbb{N}_0 = \{0, 1, 2, \ldots \}$. Also, we replace the tensor product in the definition (3) of $\Phi_u$ by the usual multiplication in $\mathbb{F}_{p^n}$. Then, it is easy to see that the so defined $\Phi_u$ yields in the standard way the action $S = \sum_u \epsilon_u \Phi_u$ whose probabilities (5) are manifold invariants, and this $\Phi_u$ is a polynomial of degree $p^m + 1$ in the five variables $x_t$, where $t \subset u$.

**Important remark.** Probabilities $P(v)$ for $v \in A \otimes B$ may be more informative than those defined according to the previous paragraph (and also used in [6]), because $A \otimes B = A \otimes_{\mathbb{Z}} B$ is typically greater that just the field $\mathbb{F}_{p^n} = A \otimes_{\mathbb{F}_{p^n}} B$ where the action defined in the previous paragraph takes values.

**Example 1.** Let $p = 2$, $n$ any natural number, and $m = 0$. Then (3) becomes a homogeneous quadratic polynomial in the variables $x_t$ in characteristic 2, namely

\[
(x_{jklm} + x_{ijkl} + x_{ijkl})(x_{iklm} + x_{ijlm} + x_{ijkl}).
\]

This is the same polynomial as appears in [6, Eq. (22)].
Example 2. Let again \( p = 2 \), \( n \) any natural number, but now \( m = 1 \). Then (3) becomes a homogeneous cubic polynomial, namely
\[
(x_{jklm} + x_{ijlm} + x_{ijkl})(x^2_{iklm} + x^2_{ijlm} + x^2_{ijkl}).
\]
This is the same polynomial as appears in [6, Eq. (24)].

5.2 Example of a cocycle that could not be obtained by the Frobenius trick

The following cubic cocycle in characteristic 2 could not (as yet) be obtained by the author using the above ‘Frobenius’ or any similar trick:
\[
x_{iklm}x_{ijkl} + x_{iklm}x_{ijlm}x_{ijkl} + x_{jklm}x_{ijlm}x_{ijkl} + x_{ijkl}x_{iklm}x_{ijkl}.
\]
This cocycle appears, and in more elegant notations, in [6, Eq. (23)].

5.3 Double Frobenius trick

One obvious generalization of the ‘Frobenius trick’ of Subsection 5.1 is to do it on both ‘Latin’ and ‘Greek’ variables. Namely, let now introduce variables \( \xi_t \) over the field \( A = B = F_{p^n} \), and let \( F \subset A \oplus B \) consist of pairs \( (a, b) = (x_t, \xi_t) \) parameterized by variables \( c = \xi_t \) in such way that \( a = c^{m_1} \) and \( b = c^{m_2} \), where \( m_1, m_2 \in \mathbb{N}_0 \).

For a single pentahoron \( u \), we consider the five \( x_t, t \subset u \), as independent variables, and for the whole manifold \( M \), variables \( x_t \) are assumed to have the same linear dependencies as either \( x_t \) or \( \xi_t \). Recall that these dependencies arise due to (2) and the fact the pair \( (x_t, y_t) \) or \( (\xi_t, \eta_t) \) for any tetrahedron \( t \) must be the same regardless of the pentahoron \( u \supset t \) where we used formula (2) (and there are of course two such \( u \) for a non-boundary \( t \)).

Example 3. Let again \( p = 2 \), \( n \) any natural number, \( m_1 = 1 \) and \( m_2 = 2 \). Then (3) becomes a homogeneous polynomial of the sixth degree, namely
\[
(x_{jklm} + x_{ijlm} + x_{ijkl})(x^4_{iklm} + x^4_{ijlm} + x^4_{ijkl}).
\]
The reader will hopefully have no problem in finding this polynomial also in an unnumbered formula in [6, Subsection 5.1].

Still, as of now, neither cocycle (6) nor many other cocycles from [6] could be obtained using anything like these tricks.

6 Intersection form as a simple analogue of our construction

This time, we color 2-faces \( s = ijk \), \( i < j < k \) (remember Subsection 1.3), by elements \( x_{ijk} \in \mathbb{Z} \).
Permitted colorings are such where values $x_{ijk}$ make a cocycle, that is,

$$x_{ijk} - x_{ijl} + x_{ikl} - x_{jkl} = 0$$

for any triangulation tetrahedron $ijkl$.

In analogy with Subsection 2.2, we consider now double colorings $(x_s, \xi_s)$, where $\xi_{ijk} \in \mathbb{Z}$ also make a cocycle. A simple calculation shows that the bilinear form

$$\varphi_{ijklm} = x_{ijk} \xi_{klm}$$

well-known from the theory of the cup product, is (of course!) a hexagon cocycle. We introduce then the following simple ‘action’ (with the same $\epsilon_u$ as in (4)):

$$S = \sum_u \epsilon_u \varphi_u,$$

which is nothing but the well-known intersection form \[4, 9\] of a 4-manifold.

7 Discussion of results
Our integral bilinear form in Subsection 4.3 may actually simply coincide with the usual intersection form. At the moment, there are neither calculations that could disprove this conjecture, nor a theory that would prove it.

Anyhow, what looks most interesting is that our form, together with ‘Frobenius tricks’ from Section 5, yields some but not all polynomial cocycles found in \[6\]. So, the nature of the remaining cocycles looks very intriguing.

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