Is Bootstrap Really Helpful in Point Process Statistics?

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Martin Snethlage
Graduiertenkolleg “Räumliche Statistik”
TU Bergakademie Freiberg
Bernhard-von-Cotta-Str. 2
D-09596 Freiberg
snethlag@grad.tu-freiberg.de

Abstract

There are some papers which describe the use of bootstrap techniques in point process statistics. The aim of the present paper is to show that the form in which bootstrap is used there is dubious. In case of variance estimation of pair correlation function estimators the used bootstrap techniques lead to results which can be obtained simpler without simulation; furthermore, they differ from the desired results. The problem to obtain confidence regions for the intensity function of inhomogeneous Poisson processes can be easily solved without bootstrap techniques.

Key words: Bootstrap, confidence region, intensity function, pair correlation function, Poisson point process, variance estimator

1 Introduction

Recently, bootstrap is a popular tool in many branches of statistics, also for stochastic processes. Thus it is natural to ask whether bootstrap techniques could be helpful also in point process statistics. Indeed, some authors have developed statistical procedures using bootstrap techniques, see e.g. [1], [2], [3] and [7]. All these papers deal with the estimation of the accuracy of estimators of point process characteristics. In the first three papers estimation of variance of pair correlation function estimators is treated. The last one presents a procedure to determine confidence regions for the intensity function of an inhomogeneous Poisson process.

The fundamental idea of bootstrap to resample given data to obtain ‘new’ pseudo data appears also in statistics of stochastic processes, in particular in the analysis of time series, see e.g. [6]. In some variants of the method, called the blockwise bootstrap, the time series is partitioned into several parts, which are then resampled. A similar idea is also applied in [3] in the statistical analysis of a planar random set. Clearly, the partitioning procedure can also be adapted to point process statistics. However, partition can destroy point structures or add new artificial structures to
the point pattern. In the one-dimensional case the error resulting from this loss of
information may be still acceptable, but in higher dimensions it will be serious. Thus
in spatial point process statistics another method is used which is quite similar to the
application of bootstrap in case of classical statistics: the points of the process (includ-
ing their places, which are assumed to be pairwise different) are resampled. The
pseudo pattern then consists of \( n \) points \( x_1^*, \ldots, x_n^* \) which are obtained by sampling
randomly with replacement \( n \) times from the original data \( \{x_1, \ldots, x_n\} \).

Naturally, the pseudo patterns generated by this method have always multiple
points. Thus they have a character different to that of the original, which does not
have multiple points.

Consequently, it would be surprising if quantities of such point processes would
produce good estimators for quantities of the original point process.

This paper analyses the pointwise resampling technique for some examples of
point process statistics. Section 2 discusses the main ideas of the paper [2] which
presents a procedure for estimating the standard error of an estimator of the pair
correlation function. In Section 3 a method (drawn from [3]) to determine confi-
dence intervals for the intensity function of an inhomogeneous Poisson process is
considered. Finally, an easier method is presented which yields confidence regions
for the intensity function of an inhomogeneous Poisson process without bootstrap.

2 Variance of estimators of the product density

This part discusses the main ideas of [2], where bootstrap techniques are used to
approximate the standard error of a pair correlation function estimator. The calcula-
tions are presented in an abridged form; the complete calculations are given in the
Appendix.

2.1 Fundamentals

Let \( \Phi \) be a stationary and isotropic point process, see, for example, [9] for definitions.
A standard second order characteristic of \( \Phi \) is the product density function \( \rho^{(2)}(r) \).
This function can be interpreted heuristically as follows. If \( B_1 \) and \( B_2 \) are two
infinitesimally small disjoint Borel sets of volumes \( dV_1 \) and \( dV_2 \) and if \( x_1 \in B_1 \) and
\( x_2 \in B_2 \) are points of distance \( \|x_1 - x_2\| = r \) then \( \rho^{(2)}(r)dV_1dV_2 \) is the probability
that \( \Phi \) has a point in each of \( B_1 \) and \( B_2 \). A simple estimator of \( \rho^{(2)} \) without any
border correction is given by

\[
\hat{\rho}(r) = \frac{1}{2\pi \nu(W)} \sum_{x,y \in \Phi \cap W} K(r - \|x - y\|).
\]

The summation goes over all point pairs with different members, \( W \) denotes the
window of observation and \( K \) is a kernel function.
This situation can be generalized to the case of any ‘two-point estimator’

\[ \hat{\theta} = \sum_{x, y \in \Phi} f(x, y) \]  

with \( f \) being symmetrical in its arguments and of the form

\[ f(x, y) = 1_{W}(x)1_{W}(y)h(x, y) \]

with some function \( h \). As the special form of \( f \) leading to \( \hat{\rho}(r) \) is unimportant, the following calculations are carried out for a general \( \hat{\theta} \).

The quantity of interest is the variance of \( \hat{\theta} \) which is given by

\[ V\hat{\theta} = \mathbb{E}\hat{\theta}^2 - (\mathbb{E}\hat{\theta})^2 = s_4 + 4s_3 + 2s_2 - (\mathbb{E}\hat{\theta})^2 \]  

with

\[ s_i = \int \theta^{(i)}(x_1, \ldots, x_i)f(x_1, x_2)f(x_{i-1}, x_i)dx_1 \ldots dx_i, \]

where \( \theta^{(i)} \) is the \( i \)th order product density function of \( \Phi \), see the Appendix.

### 2.2 Bootstrap version of \( \hat{\theta} \)

Assume that a sample of \( \Phi \) is given which consists of \( n \) points \( x_1, \ldots, x_n \) in the observation window \( W \). It is resampled \( N \) times to obtain \( N \) ‘new’ point patterns. Each pseudo pattern consists of \( n \) points \( x_1^*, \ldots, x_n^* \) which are obtained by sampling randomly with replacement \( n \) times from \( \{x_1, \ldots, x_n\} \). Thus it happens that in the pseudo samples some points of the original point pattern do not occur while others occur twice or even more. Let the number of occurrences of \( x_i \) in the \( k \)th sample be \( w_k(i) \). Then the \( k \)th sample can be represented by the vector \( w_k = (w_k(1), \ldots, w_k(n)) \) which has a multinomial distribution. This distribution depends only on \( n \). In the limiting case \( n \to \infty \) the components \( w_k(i) \) of \( w_k \) are independent and Poisson distributed with mean \( \mu = 1 \).

The bootstrap estimate for the \( k \)th pseudo sample is

\[ \hat{\theta}_k^* = \sum_{i,j=1 \atop i \neq j}^{n} f(x_i, x_j)w_k(i)w_k(j), \quad k = 1, \ldots, N \]

where the summation goes over all pairs \((i, j)\) with \( i \neq j \). The variance of \( \hat{\theta} \) is estimated by the usual variance estimator corresponding to the \( \hat{\theta}_k^* \)'s,

\[ v_N^* = \frac{1}{N-1} \sum_{k=1}^{N} \left( \hat{\theta}_k^* - \frac{1}{N} \sum_{i=1}^{N} \hat{\theta}_i^* \right)^2. \]
Since the $\hat{\theta}_k^*$ are (conditionally on $x_1, \ldots, x_n$) independent and identically distributed, it is

$$\lim_{N \to \infty} \hat{v}_N^* = \mathbf{V}\hat{\theta}_1^*$$

$$= \mathbf{E}\hat{\theta}_1^{*2} - (\mathbf{E}\hat{\theta}_1^*)^2$$

$$= \alpha_4 \sum_{i, j, k, l=1}^{n} f(x_i, x_j)f(x_k, x_l)$$

$$+ 4\alpha_3 \sum_{i, j, k=1}^{n} f(x_i, x_j)f(x_i, x_k)$$

$$+ 2\alpha_2 \sum_{i, j=1}^{n} (f(x_i, x_j))^2$$

with

$$\alpha_4 = \left[ \mathbf{E}w_1(1)w_1(2)w_1(3)w_1(4) - (\mathbf{E}w_1(1)w_1(2))^2 \right]$$

$$\alpha_3 = \left[ \mathbf{E}(w_1(1))^2w_1(2)w_1(3) - (\mathbf{E}w_1(1)w_1(2))^2 \right]$$

$$\alpha_2 = \left[ \mathbf{E}(w_1(1)w_1(2))^2 - (\mathbf{E}w_1(1)w_1(2))^2 \right],$$

where the expectations are conditionally on fixed $x_1, \ldots, x_n$. All the $\alpha_i$ can be calculated numerically and depend only on $n$ (see the Appendix). Thus the result of the whole bootstrap procedure for $N \to \infty$ can be simply obtained by direct computation.

### 2.3 Expectation of $\hat{v}_N^*$

The futility of $\hat{v}_N^*$ is demonstrated by the fact that it does neither estimate what is hoped (the variance of $\hat{\theta}$) nor a multiple with a fixed factor. To show this, the unconditional expectation of $\hat{v}_N^*$ is determined, see the Appendix. Since the result is not very transparent, here an approximation is given which makes it possible to characterize the quality of $\hat{v}_N^*$.

Assume that the $w_k(i)$ are independent and Poisson distributed with parameter $\mu = 1$; this simplifying assumption is exact in the limiting case $n \to \infty$, see above. This leads to a result which is close to the exact value for large $n$ and is easy to interpret. By the way, the simplification is equivalent to replacement of $n$ by $n^*$ in each pseudo sample where $n^*$ is a Poisson distributed number with mean $\mu = n$. In this scheme each pseudo sample consists of a random number of points.

The result is

$$\lim_{N \to \infty} \mathbf{E}\hat{v}_N^* = \mathbf{E} \lim_{N \to \infty} \hat{v}_N^* = 4s_3 + 6s_2,$$  

(4)
see the Appendix, while the desired result, given by (2), is

\[ V \hat{\theta} = s_4 + 4s_3 + 2s_2 - \left( E \hat{\theta} \right)^2. \]

**Remark:** The formulae suggest that the bootstrap result (4) can considerably differ from the true variance of \( \hat{\theta} \). Nevertheless, the bootstrap procedure may make sense. In some cases \( s_4 \) converges to \( \left( E \hat{\theta} \right)^2 \) with growing \( W \) and \( s_3 \) is small compared with \( s_2 \). Then the bootstrap result (4) may approximate three times the true variance, see [2].

### 3 Confidence regions for the intensity function of an inhomogeneous Poisson process

The paper [3] presents a procedure which uses bootstrap techniques to determine confidence regions for the intensity function of an inhomogeneous Poisson process. The confidence regions are estimated using a kernel estimator. The following discusses the main idea of that paper and shows that, as above, it is not necessary to carry out the bootstrap procedure.

#### 3.1 Fundamentals

For simplicity, the following calculations are carried out for an one-dimensional point process, but they could be easily generalized to higher-dimensional processes.

Consider an inhomogeneous Poisson point process \( \Phi \) with unknown intensity function \( \lambda(x) \) in the interval \((0, 1)\), with points \( 0 < x_1 \leq x_2 \leq \ldots \leq x_n < 1 \). A kernel estimator for \( \lambda(x) \) is used as

\[ \hat{\lambda}(x) = \frac{1}{h} \sum_{i=1}^{n} K \left( \frac{x - x_i}{h} \right), \quad x \in (0, 1), \]

where \( K \) is a kernel function and \( h \) bandwidth (see, for example, [4]).

Define

\[ T(x) = \frac{\hat{\lambda}(x) - E\hat{\lambda}(x)}{\sqrt{\hat{\lambda}(x)}}, \quad 0 < x < 1, \]

and, for a given \( \alpha \) with \( 0 \leq \alpha \leq 1 \),

\[ t_\alpha(x) = \min_{t \in \mathbb{R}^+} \{ t : P \{ |T(x)| \leq t \} \geq 1 - \alpha \}, \quad 0 < x < 1. \]

Then an estimate of a confidence region for \( \lambda(x) \) of level \( 1 - \alpha \) is the interval

\[ C(x) = \left[ \hat{\lambda}(x) - t_\alpha(x)\sqrt{\hat{\lambda}(x)}, \hat{\lambda}(x) + t_\alpha(x)\sqrt{\hat{\lambda}(x)} \right], \quad 0 < x < 1, \]

where the left border is set on 0 if it is negative.
3.2 Bootstrap versions

Since the distribution of $T$ is not available (because the intensity function is unknown) it is approximated by simulation of pseudo data, see [3]. A set of pseudo data is obtained by drawing $x_1^*, \ldots, x_n^*$ by sampling randomly with replacement $n^*$ times from $\{x_1, \ldots, x_n\}$, where $n^*$ has a Poisson distribution with mean $n$ (this is method 2 in [3] and similar to the simplified case in Section 2.3). The number of occurrences of $x_i$ in the $k$th sample is a random variable, denoted as above by $w_k(i)$. All the $w_k(i)$ are independent and Poisson distributed with mean $\lambda = 1$ for $i = 1, \ldots, n$.

For given $\alpha$ with $0 \leq \alpha \leq 1$ the bootstrap versions of the quantities defined above are

\[ \hat{\lambda}_k^*(x) = \frac{1}{h} \sum_{i=1}^{n} K \left( \frac{x - x_i}{h} \right) w_k(i), \]
\[ T_k^*(x) = \frac{\hat{\lambda}_k^*(x) - \hat{\lambda}(x)}{\sqrt{\hat{\lambda}_k^*(x)}}, \quad x \in (0, 1), \]
\[ t_{\alpha}^*(x) = \min \{ t : P^* \{ |T^*(x)| \leq t \} \geq 1 - \alpha \}, \]
\[ C^*(x) = \left[ \hat{\lambda}(x) - t_{\alpha}^*(x) \sqrt{\hat{\lambda}(x)} \right] \left[ \hat{\lambda}(x) + t_{\alpha}^*(x) \sqrt{\hat{\lambda}(x)} \right], \]

where $P^*(\cdot) = P(\cdot | \{x_1, \ldots, x_n\})$ is the distribution conditionally on $\{x_1, \ldots, x_n\}$.

The determination of $t_{\alpha}^*(x)$ can be carried out by simulation. However, a faster and simpler possibility uses the well-known fact that the sum of independent Poisson distributed random variables is also Poisson distributed. It is demonstrated here for the simple rectangular kernel function

\[ K(x) = \frac{1}{2} \cdot 1_{[-1,1]}(x). \]

For other kernels, similar calculations are possible. Let $p(x)$ be the number of observed points in the interval $[x - h, x + h]$. Then its bootstrap version $P^*(x)$ is a random variable which is Poisson distributed with mean $p(x)$. Its cumulative distribution function is denoted by $F^*$. Thus, for given $\alpha$,

\[ t_{\alpha}^*(x) = \min \{ t : P^* \{ |T^*_1(x)| \leq t \} \geq 1 - \alpha \} \]

\[ = \min \{ t : F^* \left\{ \sum_{i=1}^{p(x)} w_1(i) - a_x(t) \leq b_x(t) \right\} \geq 1 - \alpha \} \]

\[ = \min \{ t : F^*(a_x(t) + b_x(t)) - F^*(a_x(t) - b_x(t)) \geq 1 - \alpha \}, \]

with

\[ a_x(t) = p(x) + ht^2, \]
\[ b_x(t) = t \sqrt{2hp(x) + h^2t^2} \]
The calculations are of an elementary nature using that \( \hat{\lambda}_k^* \) is equal to \( \frac{1}{2h} \sum_{i=1}^{p(x)} w_k(i) \).

3.3 Confidence regions without bootstrap

Of course, in case of an inhomogeneous Poisson process it is easy to build confidence regions without the bootstrap methodology. Assume that the intensity function \( \lambda(x) \) is approximately linear in the interval \( [x-h, x+h] \). Then \( 2h\hat{\lambda}(x) \) using the rectangular kernel is Poisson distributed with mean \( 2h\lambda(x) \). (The rectangular kernel could be replaced by another kernel; then the corresponding calculations become a bit more difficult.) Therefore, known confidence regions for the Poisson parameter can be used, see for example [8]. Thus it is easy to build the desired confidence region for \( \lambda(x) \).

This result corresponds to a general observation. If a parametric statistic problem is given, then parametric estimators lead usually to better results than bootstrap techniques.

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Appendix

Here the derivation of some equations of Section 3 is given.

Equation (2)

For

\[
\hat{\theta} = \sum_{x,y \in \Phi} \neq f(x,y) = \sum_{x,y \in \Phi} \neq 1_W(x)1_W(y)h(x,y)
\]

it is

\[
E\hat{\theta}^2 = E\left( \sum_{x,y \in \Phi} \neq f(x,y) \right)^2
\]

\[
= E \sum_{w,x,y,z \in \Phi} \neq f(w,x)f(y,z)
\]

\[
+ 4E \sum_{x,y,z \in \Phi} \neq f(x,y)f(x,z)
\]

\[
+ 2E \sum_{x,y \in \Phi} \neq (f(x,y))^2
\]

\[
= \int g^{(4)}(x_1,x_2,x_3,x_4)f(x_1,x_2)f(x_3,x_4)dx_1dx_2dx_3dx_4
\]
\[ + 4 \int \varphi^{(3)}(x_1, x_2, x_3)f(x_1, x_2)f(x_1, x_3) dx_1 dx_2 dx_3 \\
+ 2 \int \varphi^{(2)}(x_1, x_2)(f(x_1, x_2))^2 dx_1 dx_2 \\
= s_4 + 4s_3 + 2s_2 \]

with
\[ s_i = \int \varphi^{(i)}(x_1, \ldots, x_i)f(x_1, x_2)f(x_{i-1}, x_i) dx_1 \ldots dx_x. \]

**Equation (3)**

For \( \hat{\theta}_1 \)
\[
\hat{\theta}_1 = \sum_{i, j=1}^n \neq f(x_i, x_j)w_1(i)w_1(j) \]

it is
\[
E\hat{\theta}_1^2 = E\hat{\theta}_1^2 - E\hat{\theta}_1 \hat{\theta}_2 \\
= E \sum_{i, j, k, l=1}^n \neq f(x_i, x_j)f(x_k, x_l)w_1(i)w_1(j)w_1(k)w_1(l) \\
+ 4E \sum_{i, j, k=1}^n \neq f(x_i, x_j)f(x_i, x_k)(w_1(i))^2w_1(j)w_1(k) \\
+ 2E \sum_{i, j=1}^n \neq (f(x_i, x_j))^2(w_1(i)w_1(j))^2 \\
= \sum_{i, j, k, l=1}^n \neq f(x_i, x_j)f(x_k, x_l)Ew_1(i)w_1(j)w_1(k)w_1(l) \\
+ 4 \sum_{i, j, k=1}^n \neq f(x_i, x_j)f(x_i, x_k)E(w_1(i))^2w_1(j)w_1(k) \\
+ 2 \sum_{i, j=1}^n \neq (f(x_i, x_j))^2E(w_1(i)w_1(j))^2 \\
= Ew_1(1)w_1(2)w_1(3)w_1(4) \sum_{i, j, k, l=1}^n \neq f(x_i, x_j)f(x_k, x_l) \\
+ 4E(w_1(1))^2w_1(2)w_1(3) \sum_{i, j, k=1}^n \neq f(x_i, x_j)f(x_i, x_k) \\
+ 2E(w_1(1)w_1(2))^2 \sum_{i, j=1}^n \neq (f(x_i, x_j))^2 \\
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\[ \mathbb{E} \hat{\theta}_1^* \hat{\theta}_2^* = \mathbb{E} \sum_{i,j,k,l=1}^n f(x_i, x_j) f(x_k, x_l) w_1(i) w_1(j) w_2(k) w_2(l) \\
+ 4\mathbb{E} \sum_{i,j,k=1}^n f(x_i, x_j) f(x_i, x_k) w_1(i) w_1(j) w_2(i) w_2(k) \\
+ 2\mathbb{E} \sum_{i,j=1}^n (f(x_i, x_j))^2 w_1(i) w_1(j) w_2(i) w_2(j) \\
= (\mathbb{E} w_1(1) w_1(2))^2 \sum_{i,j,k,l=1}^n f(x_i, x_j) f(x_k, x_l) \\
+ 4 (\mathbb{E} w_1(1) w_1(2))^2 \sum_{i,j,k=1}^n f(x_i, x_j) f(x_i, x_k) \\
+ 2 (\mathbb{E} w_1(1) w_1(2))^2 \sum_{i,j=1}^n (f(x_i, x_j))^2. \]

This yields

\[
\lim_{N \to \infty} \hat{v}_N^* = \left[ \mathbb{E} w_1(1) w_1(2) w_1(3) w_1(4) - (\mathbb{E} w_1(1) w_1(2))^2 \right] \\
\cdot \sum_{w,x,y,z \in \Phi} f(w, x) f(y, z) \\
+ 4 \left[ \mathbb{E} (w_1(1))^2 w_1(2) w_1(3) - (\mathbb{E} w_1(1) w_1(2))^2 \right] \\
\cdot \sum_{x,y,z \in \Phi} f(x, y) f(x, z) \\
+ 2 \left[ \mathbb{E} (w_1(1) w_1(2))^2 - (\mathbb{E} w_1(1) w_1(2))^2 \right] \\
\cdot \sum_{x,y \in \Phi} (f(x, y))^2 \\
= \alpha_4 \sum_{i,j,k,l=1}^n f(x_i, x_j) f(x_k, x_l) \\
+ 4\alpha_3 \sum_{i,j,k=1}^n f(x_i, x_j) f(x_i, x_k) \\
+ 2\alpha_2 \sum_{i,j=1}^n (f(x_i, x_j))^2
\]

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with
\[
\begin{align*}
\alpha_4 &= \frac{(-4n^2 + 10n - 6)}{n^3} \\
\alpha_3 &= \frac{(n^3 - 7n^2 + 12n - 6)}{n^3} \\
\alpha_2 &= \frac{(3n^3 - 11n^2 + 14n - 6)}{n^3}
\end{align*}
\]

Equation (6)  

The expectation value of $\hat{v}^*$ is
\[
E\hat{v}^* = \alpha_4 \int \varrho^{(4)}(x_1, x_2, x_3, x_4)f(x_1, x_2)f(x_3, x_4)dx_1dx_2dx_3dx_4 \\
+ 4\alpha_3 \int \varrho^{(3)}(x_1, x_2, x_3)f(x_1, x_2)f(x_1, x_3)dx_1dx_2dx_3 \\
+ 2\alpha_2 \int \varrho^{(2)}(x_1, x_2)(f(x_1, x_2))^2dx_1dx_2 \\
= s_4\alpha_4 + 4s_3\alpha_3 + 2s_2\alpha_2
\]

In the limiting case ($n \to \infty$) it is
\[
E\hat{v}^* = 4s_3 + 6s_2
\]

(see Equation (6)).

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