Given a finite alphabet $\Lambda$, and a not necessarily finite type subshift $X \subseteq \Lambda^\infty$, we introduce a partial action of the free group $\mathbb{F}(\Lambda)$ on a certain compactification $\Omega_X$ of $X$, which we call the spectral partial action.

The space $\Omega_X$ has already appeared in many papers in the subject, arising as the spectrum of a commutative $C^*$-algebra usually denoted by $D_X$. Since the descriptions given of $\Omega_X$ in the literature are often somewhat terse and obscure, one of our main goals is to present a sensible model for it which allows for a detailed study of its structure, as well as of the spectral partial action, from various points of view, including topological freeness and minimality.

We then apply our results to study certain $C^*$-algebras associated to $X$, introduced by Matsumoto and Carlsen. Most of the results we prove are already well known, but our proofs are hoped to be more natural and more in line with mainstream techniques used to treat similar $C^*$-algebras. The clearer understanding of $\Omega_X$ provided by our model in turn allows for a fine tuning of some of these results, including a necessary and sufficient condition for the minimality of the Carlsen-Matsumoto $C^*$-algebra $O_X$, generalizing a similar result of Thomsen.

1. Introduction.

The theory of $C^*$-algebras associated to subshifts has a long and exciting history, having been initiated by Matsumoto in [24], later receiving invaluable contributions from many other authors, notably Carlsen, Silvestrov, and Thomsen. Accounts of this history may be found in [9] and [7], to which the interested reader is referred.

For now it suffices to say that, given a finite alphabet $\Lambda$, and a closed subset $X \subseteq \Lambda^N$ which is invariant under the left shift

$$S : \Lambda^N \to \Lambda^N,$$

in which case one says that the pair $(X, S)$ is a (one sided) subshift, Matsumoto initiated a study of certain $C^*$-algebras associated to $X$, whose algebraic properties reflect certain important dynamical properties of the subshift itself, and whose $K$-theory groups provide new invariants for subshifts.

The extensive literature in this field (see for instance the list of references in [8]) contains a lot of information about the structure of these algebras, such as faithful representations, nuclearity, characterization of simplicity, computation of $K$-theory groups and a lot more. It is therefore a perilous task to attempt to add anything else to the wealth of results currently available, a task we hope to be undertaking in a responsible manner.

The motivation that brought us to revisit the theory of $C^*$-algebras associated to subshifts, and the justification for writing the present paper, is twofold: firstly we are able to offer a sensible description of a certain topological space, which we will denote by $\Omega_X$ (for the cognoscenti, we are referring to the spectrum of the ill-fated commutative algebra $D_X$ appearing in most papers on the subject), and which has evaded all attempts at analysis, except maybe for some somewhat obscure projective limit descriptions given in [25] and [6: Section 2.1]. See also [30].
Secondly we will introduce a partial action of the free group $\mathbb{F} = \mathbb{F}(\Lambda)$ on $\Omega_X$, called the *spectral partial action*, whose associated crossed product is the Carlsen-Matsumoto C*-algebra $O_X$. Given that description, we may recover many known results for $O_X$, as well as give the first necessary and sufficient condition for simplicity which applies for all subshifts, including those where the shift map is not surjective.

Our study of the space $\Omega_X$ is perhaps the single most important contribution we have to offer. The method we adopt is essentially the same one used by the second named author and M. Laca in the analysis of Cuntz-Krieger algebras for infinite matrices [17], the crucial insight being the introduction of a partial action of the free group $\mathbb{F}$. To be more precise, for each letter $a$ in the alphabet $\Lambda$, consider the subsets $F_a$ and $Z_a$ of $X$, given by

$$F_a = \{ y \in X : ay \in X \}, \text{ and}$$

$$Z_a = \{ x \in X : x = ay, \text{ for some infinite word } y \}.$$

It is then evident that the assignment

$$\theta_a : y \in F_a \rightarrow ay \in Z_a$$

is a continuous bijective map. By iterating the $\theta_a$ and their inverses, we obtain partially defined continuous bijective maps on $X$, thus forming what is known as a partial action of the group $\mathbb{F}$ on $X$.

Partial actions on topological spaces are often required to map open sets to open sets, but except for the case of shifts of finite type, the above partial action does not satisfy this requirement since the $F_a$ may fail to be open. One may nevertheless use this badly behaved partial action to build a bona fide partial representation on the (typically non-separable) Hilbert space $\ell^2(X)$. At this point it is evident that we have grossly violated the topology of $X$, but alas a new commutative C*-algebra is born, generated by the range projections of all partial isometries in our partial representation, and with it a new topological space is also born, namely the spectrum of said algebra.

This algebra, usually denoted $D_X$, turns out to be as well known for experts in the field, as it is dreaded. Nevertheless it plays a well known role in the theory of partial representations and, like many other commutative algebras arising from partial representations, its spectrum $\Omega_X$ may be seen as a subspace of the Bernoulli space $2^F$, which is moreover invariant under the well known Bernoulli partial action [16: 5.10].

Based on the algebraic relations possessed by this representation we may deduce certain special properties enjoyed by the elements of $2^F$ which lie in $\Omega_X$, and if we see $2^F$ as the set of all subsets of the Cayley graph of $\mathbb{F}$, such properties imply that these elements must have the aspect of a *river basin*, in the sense that there is a main river consisting of an infinite (positive) word, together with its tributaries. Not only must the main river
form an element of the subshift $X$, but if we start anywhere in any tributary, and if we decide to travel downstream (i.e. following the edges of the Cayley graph corresponding to positive generators), we will pick up another infinite word, which will merge into the main river, and which will also consist of an infinite word belonging to $X$.

Although we do not believe it is possible to find a complete set of properties characterizing $\Omega_X$, we at least know that it contains a dense copy of $X$ (not necessarily with the same topology), which may be precisely characterized and which allows for a reasonably good handle on the other, more elusive elements of $\Omega_X$. Seen from this perspective, $\Omega_X$ appears slightly friendlier and we are in turn able to explore it quite efficiently.

While the methods most often used in the literature for analyzing $O_X$ are based on a rather technical study of a certain AF-subalgebra (see e.g. [24: Corollary 3.9] and [31: Theorem 4.14]), our arguments are rooted in the dynamical properties of the spectral partial action. In particular our description of $\Omega_X$ is concrete enough to allow us to find sensible necessary and sufficient conditions for this partial action to be topologically free and minimal. Our condition for minimality, for instance, is a lot similar to the cofinality
condition which has played an important role in characterizing simplicity for graph algebras [21: Theorem 6.8], [20: Corollary 3.11]. These two crucial properties, namely topological freeness and minimality, have been extensively used to characterize simplicity, and thus our treatment of $O_X$ is done in the same footing as for several other better behaved C*-algebras.

We believe this new picture for the hitherto intractable spectrum of $D_X$ will allow for further advances in the understanding of subshifts as well as of Carlsen-Matsumoto C*-algebras.

2. Subshifts.

We begin by fixing a nonempty finite set $\Lambda$ which will henceforth be called the alphabet, and whose elements will be called letters.

Any finite sequence of letters will be called a finite word, including the empty word, namely the word with length zero, which will be denoted by $\emptyset$. The set of all finite words will be denoted by $\Lambda^*$. Infinite sequences of letters will also be considered and we shall call them infinite words.

The best way to formalize the notion of sequences, twice referred to above, is by resorting to the Cartesian product $\Lambda \times \Lambda \times \ldots \times \Lambda$, whose elements are therefore denoted by something like $(x_1, x_2, \ldots, x_n)$.

We shall however choose a more informal notation, denoting such a sequence by

$$x_1 x_2 \ldots x_n,$$

or by

$$x_1 x_2 x_3 \ldots$$

in the infinite case. This is compatible with our point of view according to which sequences are viewed as words.

If $\alpha$ is a finite word and $x$ is a finite or infinite word, then we will write $\alpha x$ for the concatenation of $\alpha$ and $x$, namely the word obtained by juxtaposing $\alpha$ and $x$ together.

The length of a finite word $\alpha$, denoted $|\alpha|$, is the number of letters in it.

Assigning the discrete topology to $\Lambda$, the set of all infinite words, namely $\Lambda^N$, becomes a compact topological space with the product topology by Tychonoff’s Theorem. The map

$$S : \Lambda^N \to \Lambda^N$$

given by

$$S(x_1 x_2 x_3 x_4 \ldots) = x_2 x_3 x_4 x_5 \ldots,$$  \hspace{1cm} (2.1)

for every $x = x_1 x_2 x_3 x_4 \ldots \in \Lambda^N$, is called the (left) shift. It is easy to see that $S$ is continuous.

Given a nonempty closed subset $X \subseteq \Lambda^N$ such that

$$S(X) \subseteq X,$$
we may consider the restriction of $S$ to $X$, and then the pair $(X, S|_X)$ is called a (one-sided) subshift. Sometimes we will also say that $X$ itself is a subshift, leaving the shift map to be deduced from the context.

There are many concrete situations in Mathematics where subshifts arise naturally such as in dynamical systems, Markov chains, maps of the interval, billiards, geodesic flows, complex dynamics, information theory, automata theory and matrix theory. The present work is dedicated to studying subshifts from the point of view or partial dynamical systems [16].

In what follows let us give an important example of subshifts. A finite word $\alpha$ is said to occur in an infinite word $x = x_1 x_2 x_3 x_4 \ldots$, if there are integer numbers $n \leq m$, such that

$$\alpha = x_n x_{n+1} \ldots x_m.$$  

In other words, $\alpha$ may be found within $x$ as a contiguous block of letters. By default we consider the empty word $\emptyset$ as occurring in any infinite word.

Given an arbitrary subset $F \subseteq \Lambda^*$, appropriately called the set of forbidden words, let

$$X = X_F$$

be the set of all infinite words $x$ such that no member of $F$ occurs in $x$. It is easy to see that $S(X_F) \subseteq X_F$, and a simple argument shows that $X_F$ is closed in $\Lambda^N$, hence $X_F$ is a subshift. If a subshift $X$ coincides with $X_F$, for some finite set $F \subseteq \Lambda^*$, then we say that $X$ is a subshift of finite type.

A notable example of a shift which is not of finite type, and which will often be used as a counter-example below, is as follows: over the alphabet $\Lambda = \{0, 1\}$, consider as forbidden all words of the form

$$01^{2n+1}0 = 01\ldots10,$$

where $n \geq 0$. Thus any odd string of 1’s delimited by two 0’s is forbidden. The subshift defined by this set of forbidden words is called the even shift. Clearly, an infinite word $x$ lies in the space of the even shift if and only if, anytime a contiguous block of 1’s occurring in $x$ is delimited by 0’s, there is an even number of said 1’s. It should be noted that an infinite word beginning with an odd (sic) number of ones and followed by a zero is not immediately excluded.

Given a subshift $X$, the language of $X$ is defined to be the subset

$$\mathcal{L}_X \subseteq \Lambda^*$$  \hspace{1cm} (2.2)

formed by all finite words which occur in some $x \in X$.

For future reference we cite here a well known result in Symbolic Dynamics:

2.3. Proposition. [23: Proposition 1.3.4] For any subshift $X$ one has that $X = X_F$, where $F = \Lambda^* \setminus \mathcal{L}_X$.

The following notions will prove to be of utmost importance in what follows:
2.4. Definition. Let $\Lambda$ be a finite alphabet, and let $X \subseteq \Lambda^\mathbb{N}$ be a subshift. For each $\alpha$ in $\Lambda^*$, the follower set $F_\alpha$, and the cylinder set $Z_\alpha$ are defined by

$$F_\alpha = \{ y \in X : \alpha y \in X \}, \text{ and}$$

$$Z_\alpha = \{ x \in X : x = \alpha y, \text{ for some infinite word } y \},$$

It is well known that the $Z_\alpha$ form a basis for the product topology on $X$, consisting of compact open subsets.

Thus the follower set of $\alpha$ is the set of all infinite words which are allowed to follow $\alpha$, while the cylinder $Z_\alpha$ is the set of words which begin with the prefix $\alpha$. If $|\alpha| = n$, then clearly $S^n(Z_\alpha) = F_\alpha$, and the restriction of $S^n$ to $Z_\alpha$ gives a bijective map

$$S^n : Z_\alpha \to F_\alpha.$$ 

Since $Z_\alpha$ is compact, the above is a homeomorphism from $Z_\alpha$ to $F_\alpha$. In particular it follows that $F_\alpha$ is also compact, although it is not necessarily open as we shall now see in the following well known result whose precise statement we have not been able to locate in the literature.

2.5. Proposition. Given a finite alphabet $\Lambda$, and a subshift $X \subseteq \Lambda^\mathbb{N}$, the following are equivalent:

(i) $F_\alpha$ is open in $X$ for every finite word $\alpha$ in $\Lambda^*$,

(ii) $F_\alpha$ is open in $X$ for every letter $a$ in $\Lambda$,

(iii) $S$ is an open mapping on $X$,

(iv) $X$ is a subshift of finite type.

Proof. (i) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (iii). Given an open subset $U \subseteq X$, we have for every $a$ in $\Lambda$, that $U \cap Z_a$ is open relative to $Z_a$, and since $S$ is a homeomorphism from $Z_a$ to $F_a$, we conclude that $S(U \cap Z_a)$ is open relative to $F_a$, which in turn is open relative to $X$ by hypothesis. So $S(U \cap Z_a)$ is open relative to $X$, whence

$$S(U) = S\left( \bigcup_{a \in A} U \cap Z_a \right) = \bigcup_{a \in A} S(U \cap Z_a)$$

is open in $X$.

(iii) $\Rightarrow$ (i). As already seen $Z_\alpha$ is always open and $F_\alpha = S^n(Z_\alpha)$, where $n = |\alpha|$.

(iii) $\Leftrightarrow$ (iv) This is a well known classical result in Symbolic Dynamics. See [28: Theorem 1], [19: Theorem 1] and [22: Theorem 3.35].
3. Circuits.

One of the most important aspects of subshifts to be discussed in this work is the question of \textit{topological freeness} (to be defined later), and which requires a careful understanding of \textit{circuits}. We will therefore set this section aside to discuss this concept. Besides the essential facts about circuits we shall need later, we will also present some interesting, and perhaps unknown facts which came up in our research.

In order to motivate the concept of circuits let us first discuss the case of Markov subshifts. So, for the time being, we will let $X$ be a Markov subshift, which means that $X = X_F$, where the set $F$ of forbidden words contains only words of length two. Setting

$$a_{ij} = \begin{cases} 0, & \text{if the word } 'ij' \text{ lies in } F, \\ 1, & \text{otherwise,} \end{cases}$$

for every $i, j \in \Lambda$, the resulting 0-1 matrix $A = (a_{ij})_{i,j \in \Lambda}$ is called the \textit{transition matrix} for $X$. Excluding the uninteresting case in which some letter of the alphabet is never used in any infinite word in $X$, the transition matrix has no zero rows.

Let $Gr(A)$ be the graph having $A$ as its adjacency matrix, so that its vertices are the elements of $\Lambda$, while there is one edge from vertex $i$ to vertex $j$ when $a_{ij} = 1$, and none otherwise. The elements of $X$ may then be thought of as infinite paths in $Gr(A)$, namely infinite sequences of vertices (as opposed to edges, another popular concept) in which two successive vertices are joined by an edge.

3.1. Definition. Given a Markov subshift $X$, with transition matrix $A$, we will say that a \textit{circuit}\footnote{Also known as a \textit{loop}, a \textit{cycle}, or a \textit{closed path}.} in $Gr(A)$ is a finite path

$$\gamma = x_1 x_2 \ldots x_n,$$

such that $a_{x_n, x_1} = 1$.

For each circuit $\gamma$, the infinite periodic path

$$\gamma^\infty = \gamma \gamma \gamma \ldots$$

lies in $X$. The question of topological freeness, already alluded to (but not yet defined), will be seen to be closely related the the non-existence of circuits $\gamma$ such that $\gamma^\infty$ is an isolated point of $X$. If indeed $\gamma^\infty$ is not isolated, that is, if $\{\gamma^\infty\}$ is not an open set, we have in particular that,

$$\{\gamma^\infty\} \neq Z_\gamma.$$

This is to say that $Z_\gamma$ must contain at least one point other than $\gamma^\infty$. In other words, it must be possible to prolong the finite word $\gamma$ in such a way as to obtain an infinite path distinct from $\gamma^\infty$. Thus the relevance of an \textit{exit} for $\gamma$, namely a letter $x_i$ in the above expression for $\gamma$ which may be followed by at least one letter other than $x_{i+1}$ (in case $i = n$, we take $x_{i+1}$ to mean $x_1$).

In a general subshift, not necessarily of finite type, the above considerations do not make any sense since there is no underlying graph, but they may nevertheless be reinterpreted, as we will now see.
3.2. Definition. Let $\Lambda$ be a finite alphabet, and let $X \subseteq \Lambda^\mathbb{N}$ be a subshift.

(a) We will say that a finite word $\gamma \in \Lambda^*$ is a **circuit** (relative to $X$), provided the infinite periodic word $\gamma^\infty = \gamma \gamma \gamma \ldots$ lies in $X$.

(b) We will say that an infinite word $y \neq \gamma^\infty$ is an **exit** for a given circuit $\gamma$, if $\gamma y$ lies in $X$.

Thus, to say that a circuit $\gamma$ has an exit is equivalent to saying that the follower set $F_\gamma$ has at least one element other than $\gamma^\infty$. If $\gamma$ is a circuit then $\gamma^n := \gamma \gamma \gamma \ldots$ is also a circuit for any $n \in \mathbb{N}$. However, even if $\gamma$ has an exit, there is no reason why $\gamma^n$ would also have an exit (unless $X$ is a Markov subshift).

An example of this situation is the shift (of finite type) $X_F$, on the alphabet \{0, 1\}, where $F$ consists of a single forbidden word, namely ‘001’. Evidently $\gamma = \mathord{'}0\mathord{'}$ is a circuit, which admits the word $y = \mathord{'}111\ldots\mathord{'}$ as an exit. However $\gamma^2 = \mathord{'}00\mathord{'}$ clearly has no exit.

Having an exit is therefore no big deal. A much more impressive property of a circuit $\gamma$ is for $\gamma^n$ to have an exit for every $n$.

3.3. Proposition. Let $\Lambda$ be a finite alphabet and let $X \subseteq \Lambda^\mathbb{N}$ be a subshift. Given a circuit $\gamma$, the following are equivalent:

(i) For each $n \in \mathbb{N}$, one has that $\gamma^n$ has an exit,

(ii) There is an infinite word $z$ which is an exit for $\gamma^n$, for all $n \in \mathbb{N}$.

Proof. The crucial difference between (i) and (ii), as the careful reader would have already noticed, is that in (i) it is OK for each $\gamma^n$ to have a different exit, while in (ii) it is required that there is one single exit which works for all $\gamma^n$.

It is obvious that (ii) $\Rightarrow$ (i), so let us focus on the converse. Assuming that $\gamma^n$ has an exit for each $n$, we have that the follower set $F_{\gamma^n}$ contains at least one element besides $(\gamma^n)^\infty = \gamma^\infty$. We claim that in fact $F_{\gamma^n}$ contains at least one point $z$ which does not lie in the cylinder $Z_\gamma$.

Notice that, even though $\gamma^\infty$ lies in $F_{\gamma^n}$, it cannot be taken as the $z$ above since it lies in the cylinder $Z_\gamma$, while $z$ should not.

To prove the claim, choose $y \in F_{\gamma^n}$, with $y \neq \gamma^\infty$, and write

$$y = \alpha_1 \alpha_2 \alpha_3 \ldots,$$

where each $\alpha_i$ is a finite word with the same length as $\gamma$. If $\alpha_1 \neq \gamma$, then $y$ is not in $Z_\gamma$, and there is nothing to be done. Otherwise let $k$ be the smallest index such that $\alpha_k \neq \gamma$, so $k > 1$, and $\alpha_1 = \alpha_2 = \ldots = \alpha_{k-1} = \gamma$. We may then write

$$y = \gamma^{k-1} z,$$

where $z = \alpha_k \alpha_{k+1} \ldots$ is therefore an infinite word not in $Z_\gamma$. Since $y$ is in the follower set of $\gamma^n$, we then have that

$$X \ni \gamma^n y = \gamma^n \gamma^{k-1} z = \gamma^{k-1} \gamma^n z.$$
Recall that $X$ is invariant under the shift. So, after applying $S$ to the above element $|\gamma|(k-1)$ times, we conclude that $\gamma^n z$ lies in $X$, so $z \in F_\gamma^n \setminus Z_\gamma$, as desired.

Since $F_\gamma^n$ is closed and $Z_\gamma$ is open, we have that $F_\gamma^n \setminus Z_\gamma$ is closed, and nonempty as seen above. Observing that the $F_\gamma^n$ are decreasing with $n$, we have by compactness of $X$ that

$$\bigcap_{n \in \mathbb{N}} F_\gamma^n \setminus Z_\gamma \neq \emptyset.$$  

Any element $z$ chosen in the above intersection is therefore not equal to $\gamma^\infty$, because it is not in $Z_\gamma$, and it is therefore an exit for $\gamma^n$, for all $n \in \mathbb{N}$. \hfill \Box

3.4. **Definition.** Let $\Lambda$ be a finite alphabet, and let $X \subseteq \Lambda^\mathbb{N}$ be a subshift. If $\gamma$ is a circuit relative to $X$, we will say that an infinite word $y \neq \gamma^\infty$ is a **strong exit** for $\gamma$, if $\gamma^n y \in X$, for all $n \in \mathbb{N}$.

It follows from (3.3) that a subshift in which all circuits have an exit, also satisfies the apparently stronger property that all circuits have a strong exit (think about it).

An interesting example of such a subshift is as follows:

3.5. **Proposition.** Let $X$ be the even shift. Then any circuit $\gamma$ relative to $X$ has a strong exit.

*Proof.* Let us first assume that

$$\gamma = 1^k = 1\ldots1_k$$

for some $k \geq 1$. Then, no matter how big is $n$, we may always exit $\gamma^n$ via an infinite string of 0’s. All other circuits $\gamma$ have ‘0’ somewhere, say $\gamma = \gamma’0\gamma''$. Observing that

$$\gamma^n \gamma'01111\ldots$$

lies in the space of the even shift, one may always exit $\gamma^n$ via the infinite word $\gamma’01111\ldots$, which is therefore a strong exit for $\gamma$. \hfill \Box

In section (12) we will carefully study topological freeness for the partial dynamical systems we shall encounter along the way. However we warn the reader that the nice property of the even shift proven above (existence of strong exits for all circuits) will be seen to be still insufficient for our purposes.

4. **The standard partial action associated to a subshift.**

Throughout this section we will fix a finite alphabet $\Lambda$ and a subshift $X \subseteq \Lambda^\mathbb{N}$.

As already seen, the shift restricts to a homeomorphism from $Z_a$ to $F_a$, for each $a$ in $\Lambda$. The inverse of this homeomorphism is clearly given by the map

$$\theta_a : F_a \rightarrow Z_a,$$

defined by

$$\theta_a(y) = ay, \quad \forall y \in F_a.$$
4.1. Definition. Let $\mathbb{F} = \mathbb{F}(\Lambda)$ be the free group on $\Lambda$ and let
\[
\theta = \{ \{ X_g \}_{g \in \mathbb{F}}, \{ \theta_g \}_{g \in \mathbb{F}} \}
\]
be the unique semi-saturated partial action of $\mathbb{F}$ on $X$ assigning $\theta_a$ to each $a$ in $\Lambda$, given by [16: 4.10]. Henceforth $\theta$ will be referred to as the *standard partial action* associated to $X$.

Incidentally, to say that $\theta$ is *semi-saturated* is to say that
\[
\theta_{gh} = \theta_g \circ \theta_h,
\]
whenever $|gh| = |g| + |h|$, where $|\cdot|$ is the usual length function on $\mathbb{F}$. See [16: 4.9].

For the case of Markov subshifts, in fact for a generalization thereof, the standard partial action was first studied in [17].

Recall from [16: 5.1] that, for a partial action on a topological space, it is usually required that the $X_g$ be open sets. However, since $X_a^{-1} = \text{dom}(\theta_a) = F_a, \forall a \in \Lambda$, that requirement is not fulfilled for our $\theta$ unless $X$ is a subshift of finite type by (2.5). Rather than a nuisance, this is the first indication that a non-finite type subshift conceals another partial action which will be seen to be crucial for the analysis we will carry out later.

When considering $\theta$, it is therefore best to think of it as a partial action in the category of sets (as opposed to topological spaces).

In what follows let us give a simple description for $\theta$, but first let us introduce some notation. Denote by $\mathbb{F}_+$ the subsemigroup of $\mathbb{F}$ generated by $\Lambda \cup \{1\}$, so that $\mathbb{F}_+$ may be identified with the set $\Lambda^*$ of all finite words in $\Lambda$. Under this identification we shall see the empty word $\emptyset$ as the unit of $\mathbb{F}$.

4.2. Proposition. For every $g$ in $\mathbb{F}$ one has:

(i) If $g \not\in \mathbb{F}_+\mathbb{F}_+^{-1}$, then $X_g$ is the empty set.

(ii) If $g = \alpha\beta^{-1}$, with $\alpha, \beta \in \mathbb{F}_+$, and $|g| = |\alpha| + |\beta|$, then
\[
X_g = \{ \alpha y \in X : y \in F_\alpha \cap F_\beta \}.
\]

(iii) If $g$ is as in (ii), and $x \in X_{g^{-1}}$, write $x = \beta y$, for some $y \in F_\beta \cap F_\alpha$. Then
\[
\theta_g(x) = \theta_{\alpha\beta^{-1}}(\beta y) = \alpha y.
\]

Proof. Given $g$ in $\mathbb{F}$, write
\[
g = c_1c_2\ldots c_n,
\]
with $c_i \in \Lambda \cup \Lambda^{-1}$ in reduced form, meaning that $c_{i+1} \neq c_i^{-1}$, for every $i$. Since $\theta$ is semi-saturated we have that
\[
\theta_g = \theta_{c_1} \circ \theta_{c_2} \circ \cdots \circ \theta_{c_n}.
\]
If \( g \notin \mathbb{F}_+\mathbb{F}_+^{-1} \) then there is some \( i \) such that \( c_i \in \Lambda^{-1} \), and \( c_{i+1} \in \Lambda \), say \( c_i = a^{-1} \) and \( c_{i+1} = b \), for some \( a,b \in \Lambda \). Therefore \( a \neq b \), and then

\[
\theta_{c_i} \circ \theta_{c_{i+1}} = \theta_{a}^{-1}\theta_{b}
\]

is the empty map, because after inserting \( b \) as the prefix of an infinite word \( x \), we are left with a word which does not begin with the letter \( a \)! Under these conditions we then have that \( \theta_g \) is the empty map, hence \( X_g \) is the empty set. This proves (i).

Given a finite word \( \alpha \), it is easy to see that \( X_{\alpha^{-1}} = F_{\alpha}, X_{\alpha} = Z_{\alpha}, \) and

\[
\theta_\alpha(x) = \alpha x, \quad \forall x \in X_{\alpha^{-1}}.
\]

If \( g \) is as in (ii) we have, again by semi-saturatedness that \( \theta_g = \theta_\alpha \circ \theta_\beta^{-1} \), from where (ii) and (iii) follow easily. \( \square \)

We should remark that the sets \( X_g \), appearing in (4.2.ii) above, have also played an important role in Carlsen’s study of subshifts [8: Definition 1.1.3].

Due to (4.2.ii) there will be numerous situations below in which we will consider group elements of the form \( g = \alpha\beta^{-1}, \) with \( \alpha, \beta \in \mathbb{F}_+ \), and

\[
|g| = |\alpha| + |\beta|.
\]

To express this fact we will simply say that \( g \) is in reduced form. This is clearly equivalent to saying that the last letter of \( \alpha \) is distinct from the last letter of \( \beta \), so that no cancellation takes place.

Observe also that “reduced form” is an attribute of the presentation of \( g \) as a product of two elements, rather than of \( g \) itself. Any element of \( \mathbb{F}_+\mathbb{F}_+^{-1} \) may be written as \( g = \alpha\beta^{-1}, \) with \( \alpha, \beta \in \mathbb{F}_+ \), and upon canceling as many final letters of \( \alpha \) and \( \beta \) as necessary, one is left with a reduced form presentation of \( g \).

5. A partial representation associated to the subshift.

Throughout this section we will again fix a finite alphabet \( \Lambda \) and a subshift \( X \subseteq \Lambda^\mathbb{N} \).

Recall from [16: 9.1 & 9.2] that a *-partial representation of a group \( G \) in a unital *-algebra \( B \) is a map \( u : G \to B \) satisfying

(PR1) \( u_1 = 1, \)

(PR2) \( u_g u_h u_{h^{-1}} = u_{gh} u_{h^{-1}}. \)

(PR3) \( u_{g^{-1}} = (u_g)^*, \)

for every \( g \) and \( h \) in \( G \).

Consider the complex Hilbert space \( \ell^2(X) \), with its canonical orthonormal basis \( \{\delta_x\}_{x \in X} \). Here we shall consider *-partial representations of \( \mathbb{F} = \mathbb{F}(\Lambda) \) in the algebra of bounded linear operators on \( \ell^2(X) \), and we will refer to these simply as partial representations of \( \mathbb{F} \) on \( \ell^2(X) \).
5.1. Proposition. For each $g$ in $\mathbb{F}$, denote by $u_g$ the unique bounded linear operator

$$u_g : \ell^2(X) \to \ell^2(X),$$

such that for each $x$ in $X$,

$$u_g(\delta_x) = \begin{cases} \delta_{\theta_g(x)}, & \text{if } x \in X_{g^{-1}}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta$ is the standard partial action associated to $X$. Then the correspondence $g \mapsto u_g$ is a semi-saturated partial representation of $\mathbb{F}$ on $\ell^2(X)$.

Proof. Given $g$ and $h$ in $\mathbb{F}$, and $x$ in $X$, we must prove that

$$u_g u_h u_{h^{-1}}(\delta_x) = u_{gh} u_{h^{-1}}(\delta_x). \tag{5.1.1}$$

Suppose first that $x \in X_h \cap X_{g^{-1}}$. Then the left-hand-side above equals

$$u_g u_h u_{h^{-1}}(\delta_x) = u_g u_h(\delta_{\theta_{h^{-1}}(x)}) = u_g(\delta_x) = \delta_{\theta_g(x)}.$$

Observing that

$$\theta_{h^{-1}}(x) \in \theta_{h^{-1}}(X_h \cap X_{g^{-1}}) = X_{h^{-1}} \cap X_{(gh)^{-1}},$$

the right-hand-side of (5.1.1) equals

$$u_{gh} u_{h^{-1}}(\delta_x) = u_{gh}(\delta_{\theta_{h^{-1}}(x)}) = \delta_{\theta_{gh}(\theta_{h^{-1}}(x))} = \delta_{\theta_g(x)},$$

proving (5.1.1) in the present case.

Suppose now that $x \notin X_h \cap X_{g^{-1}}$. In case $x \notin X_h$, then $u_{h^{-1}}(\delta_x) = 0$, and (5.1.1) follows trivially. So we are left with the case that $x \in X_h \setminus X_{g^{-1}}$. Under this assumption we have

$$u_g u_h u_{h^{-1}}(\delta_x) = u_g u_h(\delta_{\theta_{h^{-1}}(x)}) = u_g(\delta_x) = 0.$$

Moreover notice that $y := \theta_{h^{-1}}(x) \notin X_{(gh)^{-1}}$, since otherwise

$$x = \theta_h(y) \in \theta_h(X_{h^{-1}} \cap X_{(gh)^{-1}}) = X_h \cap X_{g^{-1}},$$

contradicting our assumptions. Therefore,

$$0 = u_{gh}(\delta_y) = u_{gh}(\delta_{\theta_{h^{-1}}(x)}) = u_{gh} u_{h^{-1}}(\delta_x),$$

showing that the right-hand-side of (5.1.1) also vanishes. This proves (5.1.1), also known as (PR2), and we leave the easy proofs of (PR1) and (PR3), as well as the fact that $S$ is semi-saturated, for the reader. \qed
Either analyzing the definition of \( u_g \) directly, or as a consequence of the above result (see [16: 9.8.i]), we have that \( u_g \) is a partial isometry for every \( g \) in \( \mathbb{F} \), with initial space \( \ell^2(X_{g^{-1}}) \) and with final space \( \ell^2(X_g) \).

More specifically, recall from (4.2.i) that \( \theta_g \) is the empty map when \( g \) is not in \( \mathbb{F}_+ \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \), in which case
\[
    u_g = 0. \tag{5.2}
\]

On the other hand, assuming that \( g \) lies in \( \mathbb{F}_+ \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \mathbb{F}_+^{-1} \), we may write \( g = \alpha \beta^{-1} \), in reduced form, with \( \alpha, \beta \in \mathbb{F}_+ \), whence
\[
    u_g = u_\alpha u_\beta^*,
\]
by semi-saturatedness. Based on the description of \( \theta \) given in (4.2.ii & iii), we then have for every \( x \) in \( X \) that
\[
    u_{\alpha \beta^{-1}}(\delta_x) = \begin{cases} 
        \delta_{\alpha y}, & \text{if } x = \beta y, \text{ for some } y \in F_\alpha \cap F_\beta, \\
        0, & \text{otherwise}. 
    \end{cases}
\]

Still under the assumption that \( g = \alpha \beta^{-1} \), in reduced form, we have by (4.2.ii) that the final space of \( u_g \) is
\[
    \ell^2(X_g) = \text{span}\{\delta_{\alpha y} : y \in F_\alpha \cap F_\beta\}. \tag{5.3}
\]

A relevant remark is that the final projections \( u_g u_g^* \) are then seen to be diagonal operators relative to the canonical orthonormal basis. In particular these commute with each other, a well known fact from the general theory of partial representations [16: 9.8.iv].

A special case of interest is when \( g = \alpha \in \mathbb{F}_+ \), in which case we have that
\[
    u_\alpha(\delta_x) = \begin{cases} 
        \delta_{\alpha x}, & \text{if } x \in F_\alpha, \\
        0, & \text{otherwise}. 
    \end{cases}
\]

From this we see that the initial space of \( u_\alpha \) is \( \ell^2(F_\alpha) \), and its final space is \( \ell^2(Z_\alpha) \).

The adjoint of \( u_\alpha \) may be described by
\[
    u_\alpha^*(\delta_y) = u_{\alpha^{-1}}(\delta_y) = \begin{cases} 
        \delta_x, & \text{if } y \in Z_\alpha, \text{ and } y = \alpha x, \text{ with } x \in F_\alpha, \\
        0, & \text{otherwise}. 
    \end{cases}
\]

5.4. Proposition. Let \( T \) be the operator on \( \ell^2(X) \) defined by
\[
    T(\delta_x) = \delta_{S(x)},
\]
for every \( x \) in \( X \), where \( S \) is the shift map introduced in (2.1).

Proof. Given any \( y \) in \( X \), let \( b \) be the first letter of \( y \), and write \( y = bx \), for some infinite word \( x \), necessarily in \( F_b \). Then \( u_b^*(\delta_y) = \delta_x \), while \( u_a^*(\delta_y) = 0 \), for all \( a \neq b \). This implies that
\[
    T(\delta_y) = \sum_{a \in \Lambda} u_a^*(\delta_y) = \delta_x = \delta_{S(y)}. \tag*{\Box}
\]

The operator \( T \) may therefore be interpreted as the manifestation of the shift \( S \) at the level of operators on \( \ell^2(X) \). Since each \( u_a \) is a partial isometry, it is clear that \( \|T\| \leq |\Lambda| \).
6. C*-algebras associated to subshifts.

Throughout this section we will fix a finite alphabet \( \Lambda \) and a subshift \( X \subseteq \Lambda^\mathbb{N} \). Our goal here is to describe two important C*-algebras that have been extensively studied in association with a subshift.

6.1. Proposition. Let \( \pi \) be the representation of \( C(X) \) on \( \ell^2(X) \) defined on the canonical orthonormal basis by

\[
\pi(f)\delta_x = f(x)\delta_x, \quad \forall f \in C(X), \quad \forall x \in X.
\]

Then the following three sets generate the same C*-algebra of operators on \( \ell^2(X) \):

(i) \( u(\mathbb{F}) = \{u_g : g \in \mathbb{F}\} \),
(ii) \( u(\Lambda) = \{u_a : a \in \Lambda\} \),
(iii) \( \pi(C(X)) \cup \{T\} \).

Proof. Denote the C*-algebras generated by the sets in (i), (ii) and (iii) by \( B_{\mathbb{F}} \), \( B_\Lambda \) and \( B_T \), respectively.

Since \( u \) is semi-saturated, for every \( g \) in \( \mathbb{F} \), one has that \( u_g \) may be written as a product of elements in \( u(\Lambda) \cup u(\Lambda)^* \). Therefore \( B_\mathbb{F} \subseteq B_\Lambda \).

For every \( \alpha \) in \( \Lambda^* \), the cylinder \( Z_\alpha \) is a clopen subset of \( X \), so its characteristic function, which we denote by \( 1_\alpha \), is a continuous function. Moreover \( \pi(1_\alpha) \) is the orthogonal projection onto \( \ell^2(Z_\alpha) \), hence it coincides with the final projections of \( u_\alpha \), so

\[
\pi(1_\alpha) = u_\alpha u_\alpha^* \in B_\mathbb{F}.
\]

It is evident that the set \( \{1_\alpha : \alpha \in \Lambda^*\} \) separates points of \( X \), so by virtue of the Stone-Weierstrass Theorem, it generates \( C(X) \), as a C*-algebra. Therefore

\[
\pi(C(X)) \subseteq B_\mathbb{F}.
\]

Since \( T \in B_\mathbb{F} \), by definition, we then conclude that

\[
B_T \subseteq B_\mathbb{F}.
\]

Given \( a \) in \( \Lambda \), notice that

\[
\pi(1_a)T^* = \sum_{b \in \Lambda} u_a u_a^* u_b = u_a,
\]

proving that \( u_a \) lies in the C*-algebra generated by \( \pi(C(X)) \) and \( T \), whence \( B_\Lambda \subseteq B_T \).

We have therefore shown that

\[
B_\mathbb{F} \subseteq B_\Lambda \subseteq B_T \subseteq B_\mathbb{F},
\]

whence the conclusion. \( \Box \)
6.2. Definition. The Matsumoto C*-algebra associated to a given subshift \( X \), henceforth denote by \( \mathcal{M}_X \), is the closed *-algebra of operators on \( \ell^2(X) \) generated by any one of the sets described in the statement of (6.1).

The algebra defined above was first introduced by Matsumoto in [26: Lemma 4.1] under the notation \( \mathcal{O}_X \). However, due to the existence of several different algebras associated with subshifts usually denoted by \( \mathcal{O}_X \) in the literature, we would rather use a new notation. See also remark (6.5), below.

Let \( \lambda \) denote the left regular representation of the free group \( \mathbb{F} = \mathbb{F}(\Lambda) \) on \( \ell^2(\mathbb{F}) \). Thus, denoting the canonical orthonormal basis of \( \ell^2(\mathbb{F}) \) by \( \{ \delta_g \}_{g \in \mathbb{F}} \), one has that

\[
\lambda_g(\delta_h) = \delta_{gh}, \quad \forall g, h \in \mathbb{F}.
\]

Regarding the partial representation \( u \) introduced in (5.1), we may define a new partial representation \( \tilde{u} \) of \( \mathbb{F} \) on \( \ell^2(X) \otimes \ell^2(\mathbb{F}) \), by tensoring \( u \) with \( \lambda \), namely

\[
\tilde{u}_g = u_g \otimes \lambda_g, \quad \forall g \in \mathbb{F}.
\]

6.3. Proposition. Let \( \tilde{T} \) be the operator on \( \ell^2(X) \otimes \ell^2(\mathbb{F}) \) given by \( \tilde{T} = \sum_{a \in \Lambda} \tilde{u}_a^* \). Then the following three sets generate the same C*-algebra of operators on \( \ell^2(X) \otimes \ell^2(\mathbb{F}) \):

(i) \( \tilde{u}(\mathbb{F}) \),
(ii) \( \tilde{u}(\Lambda) \),
(iii) \( (\pi(C(X)) \otimes 1) \cup \{ \tilde{T} \} \).

Proof. Denote the C*-algebras generated by the sets in (i), (ii) and (iii) by \( \tilde{B}_\mathbb{F}, \tilde{B}_\Lambda \) and \( \tilde{B}_T \), respectively. That \( \tilde{B}_\mathbb{F} \subseteq \tilde{B}_\Lambda \) follows, as above, from the fact that \( \tilde{u} \) is semi-saturated. For \( \alpha \) in \( \Lambda^* \) one has

\[
\pi(1_\alpha) \otimes 1 = u_\alpha u_\alpha^* \otimes 1 = (u_\alpha \otimes \lambda_\alpha)(u_\alpha \otimes \lambda_\alpha)^* = \tilde{u}_\alpha \tilde{u}_\alpha^* \in \tilde{B}_\mathbb{F}.
\]

This, plus the fact that \( C(X) \) is generated by \( \{1_\alpha : \alpha \in \Lambda^*\} \), gives \( \pi(C(X)) \otimes 1 \subseteq \tilde{B}_\mathbb{F} \). Since \( \tilde{T} \in \tilde{B}_\mathbb{F} \), by definition, we then conclude that \( \tilde{B}_T \subseteq \tilde{B}_\mathbb{F} \). Given \( a \) in \( \Lambda \), notice that

\[
(\pi(1_a) \otimes 1) \tilde{T}^* = (u_a u_a^* \otimes 1) \sum_{b \in \Lambda} u_b \otimes \lambda_b = \sum_{b \in \Lambda} u_a u_a^* u_b \otimes \lambda_b = u_a \otimes \lambda_a = \tilde{u}_a,
\]

proving that \( \tilde{u}_a \) lies in the C*-algebra generated by \( \pi(C(X)) \otimes 1 \) and \( \tilde{T} \), whence \( \tilde{B}_\Lambda \subseteq \tilde{B}_T \), concluding the proof. \( \square \)

6.4. Definition. The Carlsen-Matsumoto C*-algebra associated to a given subshift \( X \), henceforth denote by \( \mathcal{O}_X \), is the closed *-algebra of operators on \( \ell^2(X) \otimes \ell^2(\mathbb{F}) \) generated by any one of the sets described in the statement of (6.3).

The algebra defined above was studied in [10], [7], [8] (where it was also denoted by \( \mathcal{O}_X \)). Its definition in the above mentioned references is not the one given in (6.4), but we will prove in (9.5) that the two definitions lead to the same algebra. We nevertheless note that the description of \( \mathcal{O}_X \) given by (6.3.iii) is closely related to that given in [10].
6.5. Remark. Let us clarify the relationship between our notation and the one used in the literature. According to [10] there are at least three possibly non-isomorphic C*-algebras associated to a subshift in the literature, some of them defined only for two-sided subshifts. These are:

(a) the C*-algebra $\mathcal{O}_\Lambda$ defined in [24],
(b) the C*-algebra $\mathcal{O}_\Lambda$ defined in [9],
(c) the C*-algebra $\mathcal{O}_X$ defined in [7].

We shall not be concerned with the first algebra above, while the second and third ones are respectively being denoted by $\mathcal{M}_X$ and $\mathcal{O}_X$ in this work.

Notice that, although the notation $\mathcal{O}_X$ emphasizes the dynamical system $(S,X)$, the construction of this algebra was actually made based on the specific way in which $X$ is represented as a space of infinite words, as well as on the syntactic rules of inserting a letter ahead of an infinite word. It is then legitimate to ask what is the precise relationship between $\mathcal{O}_X^1$ and $\mathcal{O}_X^2$, in case $X_1$ and $X_2$ are conjugate subshifts. This question has been addressed for the first time in [27] where it was proved that $\mathcal{M}_X^1$ is Morita equivalent to $\mathcal{M}_X^2$ under suitable hypothesis. Later Carlsen [8: Theorems 1.3.1 & 1.3.5] found a proof of the fact that $\mathcal{O}_X^1$ and $\mathcal{O}_X^2$ are Morita equivalent without any additional hypotheses. In [7: Section 8], having realized $\mathcal{O}_X$ as a Cuntz-Pimsner algebra, Carlsen finally proved that $\mathcal{O}_X^1$ and $\mathcal{O}_X^2$ are actually isomorphic. A further, much simpler proof of this result was found by Carlsen and Silvestrov [10: Section 11], based on the description of $\mathcal{O}_X$ as a crossed product by an endomorphism and a transfer operator [14].

The following proof, based on [10], is perhaps the simpler possible proof of the invariance of Matsumoto’s algebras, given that it only needs the original description of these algebras.

6.6. Proposition. Suppose that $X_1$ and $X_2$ are conjugate subshifts. Then

$$\mathcal{M}_{X_1} \simeq \mathcal{M}_{X_2}.$$  

Proof. Let $\varphi : X_1 \to X_2$ be a homeomorphism such that $\varphi \circ S_1 = S_2 \circ \varphi$, where $S_1$ and $S_2$ are the shift maps on $X_1$ and $X_2$, respectively. Define a unitary operator $U : \ell^2(X_1) \to \ell^2(X_2)$, by setting

$$U(\delta_x) = \delta_{\varphi(x)}, \quad \forall x \in X_1.$$  

Decorating all of the ingredients of (6.1) with subscripts to indicate whether we are speaking of $X_1$ or $X_2$, it is easy to see that

$$U \pi_1(C(X_1)) U^* = \pi_2(C(X_2)), \quad \text{and} \quad U T_1 U^* = T_2.$$  

By (6.1) we then see that $U \mathcal{M}_{X_1} U^* = \mathcal{M}_{X_2}$, so $\mathcal{M}_{X_1}$ and $\mathcal{M}_{X_2}$ are in fact spatially isomorphic. □

Even though the above result is probably well known to the experts, we have not been able to locate it in the literature with this exact formulation.

Given our reliance on the free group, whose rank is definitely not an invariant of the subshift, our method does not seem appropriate to prove Carlsen’s invariance Theorem [7: Theorem 8.6].
7. The spectrum.

As always we fix a finite alphabet $\Lambda$ and a subshift $X \subseteq \Lambda^\mathbb{N}$. There is an important subalgebra of $\mathcal{M}_X$ which has played a crucial role in virtually every attempt to study subshifts from the point of view of C*-algebras (see the references given in the introduction), and which we would now like to describe.

7.1. Definition. We will denote by $\mathcal{D}_X$ the closed *-algebra of operators on $\ell^2(X)$ generated by the final projections

$$e_g := u_g u_g^*,$$

for all $g$ in $\mathbb{F}$.

Recall from [16: 9.8.iv] that the $e_g$ commute with each other, so $\mathcal{D}_X$ is a commutative C*-algebra, which is moreover unital because $e_1 = 1$.

Since $\tilde{u}$ is also a partial representation, the final projections

$$\tilde{e}_g := \tilde{u}_g \tilde{u}_g^*,$$

likewise commute with each other, and hence generate a commutative C*-algebra. However notice that

$$\tilde{e}_g = (u_g \otimes \lambda_g)(u_g \otimes \lambda_g)^* = u_g u_g^* \otimes 1 = e_g \otimes 1,$$

from where one concludes that the C*-algebra generated by the $\tilde{e}_g$ is nothing but $\mathcal{D}_X \otimes 1$.

This section is dedicated to studying the spectrum of $\mathcal{D}_X$, henceforth denoted by

$$sp(\mathcal{D}_X).$$

The full description of this space requires some machinery still to be developed, but we may easily give examples of some of its elements, as follows. Observing that the multiplication of two diagonal operators is done by simply multiplying the corresponding diagonal entries, we see that the assignment of a given diagonal entry to an operator defines a multiplicative linear functional on the set of all diagonal operators. In what follows we will refer to diagonal entries indirectly, as eigenvalues relative to eigenvectors taken from the canonical basis.

7.3. Definition. Given any $x$ in $X$, let $\varphi_x$ be the unique linear functional on $\mathcal{D}_X$ such that

$$a(\delta_x) = \varphi_x(a) \delta_x, \quad \forall a \in \mathcal{D}_X.$$

As observed above, each $\varphi_x$ is a character on $\mathcal{D}_X$, hence an element of $sp(\mathcal{D}_X)$. We will see that not every element of $sp(\mathcal{D}_X)$ is of the form $\varphi_x$, but the $\varphi_x$ nevertheless form a large subset of $sp(\mathcal{D}_X)$ in the following sense:

7.4. Proposition. The subset of $sp(\mathcal{D}_X)$ formed by all the $\varphi_x$ is a dense set.
Proof. Assume by way of contradiction that the closure of
\[ \{ \varphi_x : x \in X \}, \]
which we denote by \( C \), is a proper subset of \( sp(D_X) \). Picking a point \( \varphi \) outside \( C \) we may invoke Urysohn’s Lemma to find a continuous complex valued function \( f \) on \( sp(D_X) \) which vanishes on \( C \), and such that \( f(\varphi) = 1 \). By Gelfand’s Theorem we have that \( f \) is the Gelfand transform of some \( a \) in \( D_X \), and then for every \( x \) in \( X \) we have
\[ \varphi_x(a) = f(\varphi_x) = 0. \]
This implies that all diagonal entries of \( a \) are zero, and since \( a \) is itself a diagonal operator, we deduce that \( a = 0 \), and hence also that \( f = 0 \), a contradiction. □

We thus get a map
\[ \Phi : x \in X \mapsto \varphi_x \in sp(D_X), \]
whose range is dense in \( sp(D_X) \).

A crucial question in this subject is whether or not \( \Phi \) is continuous. Should this be the case, the compactness of \( X \) would imply that \( \Phi \) is onto, and the mystery surrounding \( sp(D_X) \) would be immediately dispelled. However, we will see later in (7.19) that for subshifts not of finite type \( sp(D_X) \) is strictly bigger than the range of \( \Phi \) and this can only happen if \( \Phi \) is discontinuous!

In order to describe the whole of \( sp(D_X) \) it is useful to recall that \( D_X \) is generated, as a C*-algebra, by the projections
\[ e_g := u_g u_g^*, \]
so a character \( \varphi \) on \( D_X \) is pinned down as soon as we know the numbers \( \varphi(e_g) \), which necessarily lie in \( \{0,1\} \), for every \( g \) in \( \mathbb{F} \).

To be precise, given \( \varphi \in sp(D_X) \), consider the element \( \xi_\varphi \in 2^\mathbb{F} \) given by
\[ \xi_\varphi(g) = \varphi(e_g), \quad \forall g \in \mathbb{F}. \]

Identifying \( 2^\mathbb{F} \) with the set of all subsets of \( \mathbb{F} \) as usual, we may think of \( \xi_\varphi \) as the subset of \( \mathbb{F} \) given by
\[ \xi_\varphi = \{ g \in \mathbb{F} : \varphi(e_g) = 1 \}. \quad (7.5) \]

**7.6. Proposition.** Considering \( 2^\mathbb{F} \) as a topological space with the product topology, the mapping
\[ \Psi : \varphi \in sp(D_X) \mapsto \xi_\varphi \in 2^\mathbb{F}, \]
is a homeomorphism from \( sp(D_X) \) onto its range.

Proof. As already observed, each \( \varphi \) in \( sp(D_X) \) is characterized by its values on the generating idempotents, so \( \Psi \) is seen to be one-to-one. It is evident that \( \Psi \) is continuous, and since \( sp(D_X) \) is compact, we have that \( \Psi \) is a homeomorphism onto its image. □
7.7. Definition. The range of \( \Psi \), which will henceforth be denoted by 

\[
\Omega_X = \Psi(sp(D_X)),
\]

will be referred to as the \textit{spectrum} of the subshift \( X \).

As seen in (7.6), we have that \( \Omega_X \) is homeomorphic to \( sp(D_X) \), and we will take the former as our main model to study the latter. Since \( sp(D_X) \) contains a dense copy of the set\(^2 X \) by (7.4), we have that \( \Omega_X \) also contains a dense copy of \( X \). However we will see that for subshifts not of finite type, \( \Omega_X \) is strictly bigger than \( X \).

We will generally prefer to regard a given \( \xi \) in \( 2^F \) as a subset of \( F \), in the spirit of (7.5), rather than as a \( \{0,1\} \)-valued function on \( F \).

Here is the list of properties, alluded to in the introduction, that every \( \xi \) in \( \Omega_X \) must satisfy:

7.8. Proposition. For any \( \xi \) in \( \Omega_X \) one has that:

(i) \( 1 \in \xi \),

(ii) \( \xi \) is convex,

(iii) for every \( g \in \xi \), there exists a unique \( a \) in \( \Lambda \), such that \( ga \in \xi \),

(iv) if \( g \in \xi \), and \( \alpha \) is a finite word in \( \Lambda \) such that \( g\alpha \in \xi \), then \( \alpha \) lies in \( L_X \).

(v) \( \xi \subseteq F_+ F_+^{-1} \).

\textit{Proof.} Let \( \varphi \) be the character on \( D_X \) such that \( \xi = \xi \varphi \), so that

\[
\xi = \{ g \in F : \varphi(e_g) = 1 \}.
\]

Since \( e_1 = 1 \), one has that \( \varphi(e_1) = 1 \), so \( 1 \in \xi \), proving (i).

Recall from \[16: 14.19\] that to say that \( \xi \) is \textit{convex} is to say that, whenever \( g, h \in \xi \), then the \textit{segment} joining \( g \) and \( h \), namely

\[
\overline{gh} := \{ k \in F : |g^{-1}h| = |g^{-1}k| + |k^{-1}h| \}
\]

is contained in \( \xi \). Assuming that \( g \) and \( h \) are in \( \xi \), and that \( k \) is in \( \overline{gh} \), set

\[
s = g^{-1}k, \quad \text{and} \quad t = k^{-1}h.
\]

We then have that \( |st| = |s| + |t| \), so \( u_{st} = u_s u_t \), by semi-saturatedness. Employing \[16: 14.5\] we conclude that \( e_{st} \leq e_s \), which is to say that

\[
e_{g^{-1}h} \leq e_{g^{-1}k}.
\]

Conjugating the left-hand-side of this inequation by \( u_g \), we get by \[16: 9.8.iii\] that

\[
u_g e_{g^{-1}h} u_g^{-1} = e_h u_g u_g^{-1} = e_h e_g.
\]

\(^2\) We use \textit{set} as opposed to \textit{topological space} to highlight the fact that it is not necessarily homeomorphic to \( X \), as \( \Phi \) may be discontinuous.
Doing the same relative to the right-hand-side leads to
\[ u_g e_g^{-1} k u_g^{-1} = e_k e_g, \]
so we deduce that \( e_h e_g \leq e_k e_g \), whence
\[ \varphi(e_h e_g) \leq \varphi(e_k e_g). \]

Having assumed that \( g, h \in \xi \), we see that \( \varphi(e_g) = \varphi(e_h) = 1 \), so the left-hand-side above evaluates to \( 1 \). The same is therefore true for the right-hand-side, which implies that \( \varphi(e_k) = 1 \), from where it follows that \( k \in \xi \), as desired. This proves (ii).

In order to prove (iii), pick any \( g \in \xi \). Observing that \( X \) is the disjoint union of the cylinders \( Z_a \), as \( a \) range in \( \Lambda \), and that \( e_a \) is the orthogonal projection onto \( \ell^2(Z_a) \), we see that
\[ \sum_{a \in \Lambda} e_a = 1. \]

Conjugating the above identity by \( u_g \), we deduce that
\[ e_g = u_g u_g^{-1} = u_g \left( \sum_{a \in \Lambda} e_a \right) u_g^{-1} = \sum_{a \in \Lambda} u_g e_a u_g^{-1} = \sum_{a \in \Lambda} e_g a u_g u_g^{-1} = \sum_{a \in \Lambda} e_g a e_g. \]
Since \( g \in \xi \), we have that \( \varphi(e_g) = 1 \), so
\[ 1 = \varphi(e_g) = \sum_{a \in \Lambda} \varphi(e_g a e_g) = \sum_{a \in \Lambda} \varphi(e_g a). \]

Each \( e_g a \) is idempotent so \( \varphi(e_g a) \) is either 0 or 1. We then see that there exists a unique \( a \) in \( \Lambda \) such that \( \varphi(e_g a) = 1 \), meaning that \( g a \in \xi \), hence proving (iii).

Supposing, as in (iv), that \( g \) and \( g a \) lie in \( \xi \), observe that
\[ e_g e_g a = u_g u_g^{-1} e_g a = u_g e_a u_g^{-1} = e_a. \]
Since \( \varphi(e_g e_g a) = 1 \), by hypothesis, it follows that \( e_g e_g a \neq 0 \), whence also \( e_a \neq 0 \). Observing that \( e_a \) is the orthogonal projection onto \( \ell^2(Z_a) \), one deduces that \( Z_a \) is nonempty, which implies that \( a \) is a word in the language \( L_X \).

Regarding (v), let \( g \in \xi \). Then \( \varphi(e_g) = 1 \), so \( e_g \) is nonzero and hence neither is \( u_g \). It then follows from (5.2) that \( g \) lies in \( F_+F_+^{-1} \).

It should be stressed that properties (7.8.i-v), which we have seen to hold for every \( \xi \) in \( \Omega_X \), are not enough to characterize the elements in \( \Omega_X \). Although it would be highly desirable to find a set of properties giving such a precise characterization, we have not been able to succeed in this task.

Nevertheless, recalling that the image of \( X \) under \( \Phi \) is dense in \( sp(D_X) \) by (7.4), we already have a somewhat satisfactory description of \( \Omega_X \), as the closure of \( X \), or rather, of its image under the following composition of maps
\[ X \xrightarrow{\Phi} sp(D_X) \xrightarrow{\Psi} \Omega_X \subseteq 2^F. \]

The map described above will acquire a special relevance in what follows, so it deserves a special notation:
7.9. Definition. We will denote by

$$\Xi : X \to \Omega_X$$

the map given by the above composition, namely $\Xi = \Psi \circ \Phi$.

Unraveling the appropriate definitions it is easy to see that

$$\Xi(x) = \{ g \in \mathbb{F} : e_g(\delta_x) = \delta_x \}.$$  \hfill (7.8.v)

Notice that to say that $e_g(\delta_x) = \delta_x$ is the same as saying that $\delta_x$ lies in the final space of $u_g$, which we have seen to be $\ell^2(X_g)$. Thus we may alternatively describe $\Xi(x)$ as

$$\Xi(x) = \{ g \in \mathbb{F} : x \in X_g \}. \hfill (7.10)$$

We will now give a further, more detailed, description of $\Xi(x)$.

7.11. Proposition. Given $x$ in $X$, let

$$\xi_x = \Xi(x).$$

Then $\xi_x$ consists precisely of the elements $g$ in $\mathbb{F}$ such that the following conditions hold:

(i) $g$ may be written in reduced form as $\alpha \beta^{-1}$, with $\alpha, \beta \in \mathbb{F}_+$,

(ii) $\alpha$ is a prefix of $x$ and, writing $x = \alpha y$, one has that $y \in F_\alpha \cap F_\beta$.

Proof. Given $g \in \xi_x$ we may use (7.8.v) to write $g = \alpha \beta^{-1}$, and we may clearly assume that (i) above holds. By (7.10) we have that $x \in X_g$, and hence (ii) follows from (4.2.ii).

Conversely, assuming that $g$ satisfies (i) and (ii), we have again by (4.2.ii) that $x$ lies in $X_g$, proving that $g \in \xi_x$. \hfill $\square$

We shall often use the above result in the special case that $\alpha = \varnothing$, in which case it reads:

7.12. Corollary. Given $x$ in $X$, and $\beta$ in $\Lambda^*$, one has that

$$\beta^{-1} \in \xi_x \iff x \in F_\beta.$$  \hfill (7.10)

It is instructive to view these elements within the Cayley graph of $\mathbb{F}$.
Given any \( x \) in \( X \), the goal is to mark the vertices of the Cayley graph of \( \mathbb{F} \) corresponding to the elements of \( \xi_x \). Due to (7.8.i), we must always mark the unit group element. Thereafter, beginning at the unit group element we mark all vertices according to the successive letters of \( x = x_1x_2x_3 \ldots \), thus forming the stem of \( \xi_x \). We then choose an integer \( n \) and focus on the \( n \)th vertex along this path, letting

\[
\alpha = x_1x_2 \ldots x_n, \quad \text{and} \quad y = x_{n+1}x_{n+2} \ldots,
\]

so that \( x = \alpha y \), and \( y \in F_\alpha \). Choosing a finite word \( \beta \), we then back up starting at the vertex chosen above, along the letters of \( \beta \). However, before we do this, we must make sure that the infinite word \( \beta y \) lies in \( X \), which the same as saying that \( y \in F_\beta \). It is also best to choose \( \beta \) such that \( \beta_n \neq x_n \), since this will avoid stepping on a vertex we have already traversed, guaranteeing (7.11.i).

The group element \( g = \alpha \beta^{-1} \) is therefore an element of \( \xi_x \) and, as seen in (7.11), all elements of \( \xi_x \) arise in this way.

One may also think of the stem of \( \xi_x \) as a river, the \( \beta \)'s considered in the above diagram being its tributaries, while \( \xi_x \) consists of the whole river basin.

In fact, not only the \( \xi_x \), but every \( \xi \) in \( \Omega_X \) has an interpretation as a river basin, but first we need to identify the appropriate rivers.

**7.13. Proposition.** Let \( \xi \) be in \( \Omega_X \) and let \( g \in \xi \). Then there exists a unique \( x \) in \( X \) such that

\[
\xi \cap g \mathbb{F}_+ = \{g\alpha : \alpha \text{ is a prefix of } x\}.
\]

**Proof.** By induction and (7.8.iii), there exists an infinite sequence \( x = x_1x_2x_3x_4 \ldots \), with \( x_i \in \Lambda \), such that

\[
gx_1x_2 \ldots x_n \in \xi, \quad \forall n \in \mathbb{N}.
\]

Using (7.8.iv) we have that \( x_1x_2 \ldots x_n \in \mathcal{L}_X \), for every \( n \), so it follows from (2.3) that \( x \in X \). The inclusion "\( \supset \)" relative to the sets in the statement then clearly holds.

On the other hand, given any \( g\alpha \in \xi \cap g \mathbb{F}_+ \), we claim that \( \alpha \) is a prefix of \( x \). In order to prove this, suppose otherwise, and let \( \alpha' \) be the shortest prefix of \( \alpha \) which is not a prefix of \( x \). By (7.8.ii) we have that \( g\alpha' \in \xi \), which is to say that we may assume without loss of generality that \( \alpha \) is already minimal. Write \( \alpha = y_1y_2 \ldots y_n \), with \( y_i \) in \( \Lambda \), so that

\[
\beta := y_1y_2 \ldots y_{n-1}
\]

is a prefix of \( x \) by minimality. We then have that \( \beta x_n \) is a prefix of \( x \), whence \( g\beta x_n \) lies in \( \xi \). But \( g\alpha = g\beta y_n \) also lies in \( \xi \), so \( x_n = y_n \), by (7.8.iii), whence

\[
\alpha = \beta y_n = \beta x_n
\]

is a prefix of \( x \), a contradiction. \( \square \)

The following concept is reminiscent of [17: Definition 5.5].

**7.14. Definition.** Given \( \xi \) in \( \Omega_X \) and \( g \) in \( \xi \), the unique \( x \) in \( X \) satisfying the conditions of (7.13) will be called the stem of \( \xi \) at \( g \), and it will be denoted by \( \sigma_g(\xi) \). In the special case that \( g = 1 \), we will refer to \( x \) simply as the stem of \( \xi \), denoting it by \( \sigma(\xi) \).
Given $x$ in $X$, we have that all prefixes of $x$ lie in $\xi_x$ by (7.11). From this it immediately follows that the stem of $\xi_x$ is precisely $x$. In symbols
\[ \sigma(\xi_x) = x. \]

This shows the following:

**7.15. Proposition.** The stem, viewed as a map
\[ \sigma : \Omega X \mapsto X, \]
is a left inverse for $\Xi$. Consequently $\Xi$ is one-to-one and $\sigma$ is onto.

We have already hinted at the fact that the map
\[ \Phi : x \in X \mapsto \varphi_x \in sp(D_X) \]
may not be continuous, in which case neither is $\Xi$. Fortunately, not all of the maps in sight are discontinuous:

**7.16. Proposition.** The stem defines a continuous mapping from $\Omega X$ to $X$.

**Proof.** Given a net $\{\xi_i\}_i$ in $\Omega X$ converging to some $\xi$, let
\[ x_i = \sigma(\xi_i), \quad \text{and} \quad x = \sigma(\xi). \]

In order to prove the statement we need to show that $\{x_i\}_i$ converges to $x$.

By the definition of the product topology on $X \subseteq \Lambda^N$, given any neighborhood $U$ of $x$, we may find a cylinder $Z_\alpha$, with
\[ x \in Z_\alpha \subseteq U. \]
It follows that $\alpha$ is a prefix of $x$, whence $\alpha$ belongs to $\xi$.

By the definition of the product topology on $\Omega X \subseteq 2^F$, for every $g$ in $F$, the function
\[ \eta \in \Omega X \mapsto [g \in \eta] \in \{0, 1\}, \tag{7.16.1} \]
where the brackets correspond to Boolean value, is continuous. Therefore
\[ 1 = [\alpha \in \xi] = \lim_i [\alpha \in \xi_{x_i}], \]
which means that $\alpha \in \xi_{x_i}$ for all sufficiently large $i$. Consequently $\alpha$ is a prefix of $x_i$, so
\[ x_i \in Z_\alpha \subseteq U. \]
This concludes the proof. \qed
Although $\sigma$ is onto, it might not be one-to-one. This is to say that there may be many different elements in $\Omega_X$ with the same stem. An example of this situation is obtained by taking any $\xi$ together with $\xi_x$, where $x = \sigma(\xi)$. The following result further explores the relationship between these two elements.

**7.17. Proposition.** Given $\xi$ in $\Omega_X$, let $x = \sigma(\xi)$. Then $\xi \subseteq \xi_x$.

*Proof.* Given $g$ in $\xi$, we may write $g = \alpha \beta^{-1}$, with $\alpha, \beta \in \mathbb{F}_+$ by (7.8.v), and we may clearly assume that $g$ is in reduced form, so (7.11.i) holds. By (7.8.ii) we have that $\alpha \in \xi \cap \mathbb{F}_+$, so $\alpha$ is a prefix of $x$.

Write $\alpha = x_1x_2\ldots x_n$, and $x = \alpha x_{n+1}x_{n+2}\ldots$, and notice that, for every integer $k > n$, one has that $\alpha x_{n+1}\ldots x_k$ is a prefix of $x$, so

$$\xi \ni \alpha x_{n+1}\ldots x_k = \alpha \beta^{-1}\beta x_{n+1}\ldots x_k = g \beta x_{n+1}\ldots x_k.$$  

From (7.8.iv) it follows that $\beta x_{n+1}\ldots x_k$ lies in $\mathcal{L}_X$, for every $k$, whence the infinite word

$$z = \beta x_{n+1}x_{n+2}\ldots$$

lies in $X$ by (2.3). We deduce that the infinite word

$$y = x_{n+1}x_{n+2}\ldots$$

lies in $F_{\beta}$ and clearly also in $F_{\alpha}$. This proves (7.11.ii), so

$$g = \alpha \beta^{-1} \in \xi_x.$$  

As a consequence we see that the river basin picture of the $\xi_x$ also applies to a general element $\xi$ in $\Omega_X$. That is, if $x = \sigma(\xi)$, then $\xi$ contains the whole “river” $x$, while it is contained in the “river basin” $\xi_x$ by (7.17).

In particular, conditions (7.11.i-ii), which are necessary and sufficient for a given group element $g$ to lie in $\xi_x$, are seen to still be necessary for membership in $\xi$, as long as we take $x$ to be the stem of $\xi$. By this we mean that when $g \in \xi$, then necessarily $g \in \xi_x$, whence said conditions hold.

Incidentally, there is a special case in which these conditions are also sufficient, as we shall now see.

**7.18. Proposition.** Given $\xi$ in $\Omega_X$, suppose that $\alpha$ and $\beta$ are elements of $\mathbb{F}_+$ satisfying conditions (7.11.i-ii) relative to $x = \sigma(\xi)$. Suppose, in addition, that the element $y$ referred to in condition (7.11.ii) actually lies in the interior of $F_\beta$. Then $\alpha \beta^{-1} \in \xi$.

*Proof.* Since $\Phi(X)$ is dense in $sp(D_X)$, we have that $\Xi(X)$ is dense in $\Omega_X$, so we may write $\xi = \lim_i \xi_{x_i}$, with $x_i$ in $X$. By hypothesis we have that $\alpha$ is a prefix of $x$, so $\alpha \in \xi$, and by the continuity of the maps described in (7.16.1), we have that $\alpha \in \xi_{x_i}$, for all sufficiently large $i$. It follows that $\alpha$ is a prefix of $x_i$, so

$$x_i = \alpha y_i.$$
for some infinite word $y_i$, necessarily belonging to $F_\alpha$. Since $x_i$ converges to $x$ by (7.16), we have that $y_i$ converges to $y$, which belongs to the interior of $F_\beta$, by hypothesis. So the $y_i$ lie in $F_\beta$, again for all sufficiently large $i$. We conclude that $\alpha\beta^{-1}$ satisfies (7.11.ii) relative to all such $x_i$, whence $\alpha\beta^{-1} \in \xi_{x_i}$. By continuity of Boolean values we then have

\[
[\alpha\beta^{-1} \in \xi] = \lim_i [\alpha\beta^{-1} \in \xi_{x_i}] = 1,
\]

so $\alpha\beta^{-1} \in \xi$. □

This result has the following important consequence (see also [8: Remark 1.1.5]):

7.19. Theorem. Given a subshift $X$, consider the mapping

\[\Xi: x \in X \mapsto \xi_x \in \Omega_X,\]

already defined in (7.9). Then the following are equivalent:

(i) $X$ is a subshift of finite type,

(ii) $\Xi$ is onto,

(iii) $\Xi$ is continuous,

(iv) $\Xi$ is a homeomorphism.

Proof. (i) $\Rightarrow$ (ii). Given $\xi$ in $\Omega_X$, let $x$ be its stem, and we claim that fact $\xi = \xi_x$. On the one hand we have that $\xi \subseteq \xi_x$ by (7.17). In order to prove the reverse inclusion, pick $g$ in $\xi_x$. By (7.11) we may find a decomposition $g = \alpha\beta^{-1}$ satisfying conditions (7.11.i-ii) so that, among other things, $x = \alpha y$, with $y \in F_\beta$.

Since $X$ is of finite type, we have by (2.5) that $F_\beta$ is open, so $y$ automatically lies in the interior of $F_\beta$, and we deduce from (7.18) that $g = \alpha\beta^{-1} \in \xi$. This shows that $\xi = \xi_x$, and hence that $\Xi$ is onto.

(ii) $\Rightarrow$ (iii). Recall from (7.15) that $\sigma$ is one-to-one. If we assume, in addition, that $\Xi$ is onto, then $\Xi$ is an invertible map whose left-inverse $\sigma$ is also its two-sided inverse and hence invertible. Since $\sigma$ is a continuous invertible map defined on a compact set, it must be a homeomorphism, so its inverse, namely $\Xi$, is then seen to be continuous.

(iii) $\Rightarrow$ (i). Observe that (7.12) allows for the following description of $F_\beta$:

\[F_\beta = \{x \in X : \beta^{-1} \in \xi_x\} = \Xi^{-1}(\{\xi \in \Omega_X : \beta^{-1} \in \xi\}),\]

which is then the inverse image of an open set under the continuous mapping $\Xi$. We conclude that $F_\beta$ is open and then (i) follows from (2.5).

We have not taken (iv) into account so far, but it is clear that it implies either (ii) or (iii), while (ii)+(iii) easily implies (iv) since $\Xi$ is one-to-one by (7.15). □
8. The spectral partial action.

Recall from (4.1) that $\theta$ is a partial action of $\mathbb{F}$ on $X$ which might however not be a topological partial action in the sense that not all of the $X_g$ are open sets. To supersede this badly behaved partial action we will now show that there exists a fully compliant topological partial action of $\mathbb{F}$ on $\Omega_X$ extending $\theta$. The idea will be to build an algebraic partial action on $D_X$ and then consider the corresponding action at the level of the spectrum.

For each $g$ in $\mathbb{F}$, let $D_g$ be the two-sided ideal of $D_X$ generated by $e_g$, namely

$$D_g = e_g D_X,$$

and let $\tau_g$ be the map from $D_{g^{-1}}$ to $D_g$ given by

$$\tau_g(a) = u_g au_{g^{-1}}, \quad \forall a \in D_{g^{-1}}. \quad (8.1)$$

As in [16:10.1] one may show that

$$\tau = (\{D_g\}_{g \in \mathbb{F}}, \{\tau_g\}_{g \in \mathbb{F}}) \quad (8.2)$$

is a C*-algebraic partial action (see [16:11.4]) of $\mathbb{F}$ on $D_X$. In fact [16:10.1] is proved in a purely algebraic context, but the proof given there carries over to the C*-algebraic setting.

By [16:11.6], there exists a topological partial action

$$\vartheta = (\{\Omega_g\}_{g \in \mathbb{F}}, \{\vartheta_g\}_{g \in \mathbb{F}})$$

of $\mathbb{F}$ on $\Omega_X$ (here identified with the spectrum of $D_X$) linked to $\alpha$ via the fact that $D_g$ consists of the elements of $D_X$ (here identified with the algebra $C(\Omega_X)$ of all continuous complex valued functions on $\Omega_X$ by Gelfand’s Theorem) vanishing off $\Omega_g$, plus the relation

$$\tau_g(f)|_\xi = f(\vartheta_{g^{-1}}(\xi)),$$

for every $g \in \mathbb{F}$, $f \in D_{g^{-1}}$, and $\xi \in \Omega_g$.

8.3. Definition. We will refer to the partial action $\vartheta$ introduced above as the spectral partial action associated to $X$.

As we shall see, the spectral partial action may be seen as a the restriction of the partial Bernoulli action of $\mathbb{F}$ [16:5.12].

8.4. Proposition. Regarding the spectral partial action associated to $X$, one has that

(i) $\Omega_g = \{\xi \in \Omega_X : g \in \xi\}$,
(ii) $\vartheta_g(\xi) = g\xi = \{gh : h \in \xi\}$,

for each $g$ in $\mathbb{F}$, and for each $\xi \in \Omega_{g^{-1}}$. 

Proof. Under the well known correspondence between closed two-sided ideals in $C(\Omega_X)$ and open sets in $\Omega_X$, notice that the ideal generated by an idempotent element corresponds to the support of the latter. In case of the idempotent $e_g$, its support, initially viewed in $sp(D_X)$, corresponds to the set of all characters $\varphi$ such that $\varphi(e_g) = 1$. Identifying $sp(D_X)$ and $\Omega_X$ by (7.7), and noting that $\varphi(e_g) = 1$ (7.5) $\iff g \in \xi_\varphi$, for all characters $\varphi$, one sees that (i) follows.

Given $g$ in $\mathbb{F}$, and $\xi \in \Omega_{g^{-1}}$, let $\varphi$ be the character on $D_X$ corresponding to $\xi$ under $\Psi$, namely such that $\xi = \xi_\varphi$. Since $e_{g^{-1}} \in \xi$, by (8.4.i), we have that $\varphi(e_{g^{-1}}) = 1$. Moreover, $\vartheta_g(\xi)$ will correspond to a character $\psi$, such that $\psi(e_g) = 1$, and

$$\psi(a) = \varphi(\tau_{g^{-1}}(a)), \quad \forall a \in D_g.$$  

For any $a$ in $D_X$, regardless of whether $a$ is in $D_g$ or not, we have that $ae_g \in D_g$, whence

$$\psi(a) = \psi(e_g)\psi(a) = \psi(e_\varphi) = \varphi(\tau_{g^{-1}}(e_\varphi)).$$

We then have that $\vartheta_g(\xi) = \xi_\psi$, so for any $h$ in $\mathbb{F}$, one has that

$$h \in \vartheta_g(\xi) \iff h \in \xi_\psi \iff \varphi(e_h) = 1. \quad (8.4.1)$$

Incidentally notice that

$$\psi(e_h) = \varphi(\tau_{g^{-1}}(e_he_g)) = \varphi(u_{g^{-1}}e_he_gu_g) = \varphi(e_{g^{-1}}e_ge_h) = \varphi(e_{g^{-1}h}),$$

because $\varphi(e_{g^{-1}}) = 1$. Focusing on (8.4.1), we then have that

$$\psi(e_h) = 1 \iff \varphi(e_{g^{-1}h}) = 1 \iff g^{-1}h \in \xi \iff h \in g\xi,$$

proving that $\vartheta_g(\xi) = g\xi$, as desired. \hfill \Box

We then have partial dynamical systems

$$\theta = \{X_g\}_{g \in \mathbb{F}}, \{\theta_g\}_{g \in \mathbb{F}}, \quad \text{and} \quad \vartheta = \{\Omega_g\}_{g \in \mathbb{F}}, \{\vartheta_g\}_{g \in \mathbb{F}} \quad (8.5)$$

on $X$ and $\Omega_X$, respectively, and it is interesting to notice that these sets are related to each other by the maps

$$\Xi : X \to \Omega_X, \quad \text{and} \quad \sigma : \Omega_X \to X,$$

introduced in (7.9) and (7.14), respectively.

8.6. Proposition. Both $\Xi$ and $\sigma$ are equivariant maps [16:2.7] relative to the partial dynamical systems in (8.5).
Proof. Let us first prove that
\[ \Xi(X_g) \subseteq \Omega_g, \quad \forall g \in \mathbb{F}. \]
Given \( x \in X_g \), we have by (7.10) that \( g \in \Xi(x) \), whence \( \Xi(x) \in \Omega_g \), by (8.4.i), proving the above inclusion. We next must show that
\[ \Xi(\theta_g(x)) = \vartheta_g(\Xi(x)), \quad (8.6.1) \]
for all \( g \) in \( \mathbb{F} \), and all \( x \in X_{g^{-1}} \). Given such an \( x \), let \( y = \theta_g(x) \). Then evidently \( y \in X_g \), and for any \( h \in \mathbb{F} \), one has that
\[ h \in \Xi(y) \quad (7.10) \quad y \in X_h \iff y \in X_h \cap X_g \iff \theta_g(x) \in X_h \cap X_g \iff \iff x \in \theta_{g^{-1}}(X_h \cap X_g) = X_{g^{-1}h} \cap X_{g^{-1}} \iff x \in X_{g^{-1}h} \quad (7.10) \quad g^{-1}h \in \Xi(x). \]
From this we see that
\[ g^{-1}\Xi(y) = \{g^{-1}h : h \in \Xi(y)\} = \Xi(x), \]
so
\[ \Xi(\theta_g(x)) = \Xi(y) = g\Xi(x) = g\Xi(x) = \vartheta_g(\Xi(x)), \]
showing (8.6.1), and hence concluding the proof of the equivariance of \( \Xi \).

To prove that
\[ \sigma(\Omega_g) \subseteq X_g, \quad \forall g \in \mathbb{F}, \]
notice that if \( g \) is not in \( \mathbb{F}_+\mathbb{F}_-^{-1} \), then \( g \notin \xi \) for every \( \xi \in \Omega_X \), by (7.8.v). It follows that \( \Omega_g \) is empty and then there is nothing to prove. Otherwise write \( g = \alpha\beta^{-1} \), with \( \alpha, \beta \in \mathbb{F}_+ \), in reduced form. Given \( \xi \) in \( \Omega_g \), we have that
\[ g \in \xi \subseteq \xi_x, \]
where \( x = \sigma(\xi) \). It then follows from (7.11) that \( x = \alpha y \), where \( y \in F_\alpha \cap F_\beta \), which implies that \( x \in X_g \), by (4.2.ii). This proves that \( \sigma(\Omega_g) \subseteq X_g \), as desired.

We must finally prove that
\[ \sigma(\vartheta_g(\xi)) = \theta_g(\sigma(\xi)), \]
for all \( g \) in \( \mathbb{F} \), and all \( \xi \in \Omega_{g^{-1}} \). Again there is nothing to do unless \( g \) lies in \( \mathbb{F}_+\mathbb{F}_-^{-1} \), so we may assume that \( g = \alpha\beta^{-1} \), with \( \alpha, \beta \in \mathbb{F}_+ \), in reduced form.

Since \( g^{-1} = \beta\alpha^{-1} \in \xi \), we have that \( \beta \in \xi \), by (7.8.ii) so also \( \beta \in \xi \cap \mathbb{F}_+ \). By (7.13) the latter set consists precisely of the prefixes of the stem of \( \xi \), which we will denote by \( x \) from now on. In particular we have that \( x = \beta y \), for some infinite word \( y \), and we have by (4.2.iii) that
\[ \theta_g(\sigma(\xi)) = \theta_{\alpha\beta^{-1}}(\beta y) = \alpha y. \]

On the other hand, observe that if \( \gamma \) is any prefix of \( y \), then \( \beta \gamma \) is a prefix of \( x \), so \( \beta \gamma \) lies in \( \xi \). Consequently
\[ \alpha \gamma = g\beta \gamma \in g\xi = \vartheta_g(\xi). \]
Since \( \alpha \gamma \in \mathbb{F}_+ \) for any such \( \gamma \), one sees that \( \alpha \gamma \) is a prefix of the stem of \( \vartheta_g(\xi) \), from where it follows that
\[ \sigma(\vartheta_g(\xi)) = \alpha y = \theta_g(\sigma(\xi)). \]

In case \( X \) is a subshift of finite type the above result may be combined with (7.19) leading up to the following:
8.7. Proposition. If $X$ is a subshift of finite type then the maps
\[ \sigma : \Omega_X \to X, \quad \text{and} \quad \Xi : X \to \Omega_X \]
are mutually inverse equivariant homeomorphisms, whence the spectral partial action and
the standard partial action are equivalent.

Let us conclude this section with an important technical result, regarding membership
of an element of the form $g\beta^{-1}$, when we already know that $g$ is a member of a given $\xi$.
This result actually belongs in section (7), but it was delayed up to now since its proof is
greatly facilitated by the existence of the spectral action.

8.8. Proposition. Given $\xi$ in $\Omega_X$, pick $g \in \xi$, and let $\beta$ be a finite word. Regarding the
statements:
(a) $g\beta^{-1} \in \xi$, and
(b) $\sigma_g(\xi) \in F_\beta$,
\[ x_1 \ x_2 \ x_3 \ \ldots \]
\[ h = g\beta^{-1} \]
one has that (a) $\Rightarrow$ (b). Moreover, if $\xi = \xi_x$ for some $x$ in $X$, then (b) $\Rightarrow$ (a) as well.

Proof. Let $h = g\beta^{-1}$, let $x = \sigma_g(\xi)$, and write
\[ x = x_1x_2x_3\ldots, \quad \text{and} \quad \beta = \beta_1\beta_2\ldots\beta_m. \]

Notice that $h\beta_1$ lies in the segment joining $h$ and $g$ so, supposing (a), we have by
(7.8.ii) that $h\beta_1 \in \xi$. Likewise
\[ h\beta_1\beta_2\ldots\beta_i \in \xi, \]
for all $i$. Furthermore, for every integer $j$ we have that
\[ h\beta_1\beta_2\ldots\beta_mx_1x_2\ldots x_j = gx_1x_2\ldots x_j \in \xi; \]
by the definition of the stem of $\xi$ at $g$. It then follows that $\sigma_h(\xi) = \beta x$, so in particular
$\beta x \in X$, whence $x \in F_\beta$, proving (b).

In order to prove the last sentence of the statement, let $\xi = \xi_x$ for some $x$ in $X$, and
suppose that (b) holds. In the special case that $g = 1$, we have that
\[ x = \sigma(\xi_x) = \sigma_g(\xi) \in F_\beta, \]
so $\beta^{-1}$ belongs to $\xi$ by (7.12), proving (a).

Dropping the assumption that $g$ is the unit group element, let us deal with the general
case. Observing that $g \in \xi_x$, we have by (7.10) that $x \in X_g$, so
\[ y := \theta_{g^{-1}}(x) \]
is well defined. Moreover, by (8.6)
\[ g\xi_y = \vartheta_g(\xi_y) = \xi_{\vartheta_g(y)} = \xi_x. \]
In other words, \( \xi_x \) is obtained by left-translating \( \xi_y \) by \( g \). It then easily follows that
\[ \sigma(\xi_y) = \sigma_g(\xi_x) \in F_\beta, \]
so the first case treated above (i.e. \( g = 1 \)) applies for \( \xi_y \), and we deduce that \( \beta^{-1} \in \xi_y \), whence
\[ g\beta^{-1} \in g\xi_y = \xi_x. \]
\[ \square \]

The above result plays a crucial role in understanding the elements of \( \Omega_X \) from the point of view of their stem. By this we mean that, once the stem of \( \xi \) is marked in the Cayley graph of \( \mathbb{F} \), and we wish to mark the remaining group elements in \( \xi \), we know from (7.8.v) that we need only worry about elements of the form \( g = \alpha \beta^{-1} \). If we are careful to take the reduced form of \( g \), then a necessary condition for it to be marked is that \( \alpha \) also be marked, in which case \( \alpha \) must be a prefix of the stem.

We then must decide whether or not to mark \( g \) itself, and this is precisely where (8.8) intervenes: in case \( \xi \) is some of the \( \xi_x \), then we should mark \( g \) if and only if the resulting stem at \( g \), namely
\[ \beta_1 \beta_2 \cdots \beta_m x_{n+1} x_{n+2} \cdots \]
lies in \( X \) (which is to say that \( x_{n+1} x_{n+2} \cdots \) lies in \( F_\beta \)). When \( \xi \) is not necessarily a \( \xi_x \), then (8.8) does not give a definite answer, except that marking \( g \) is forbidden in case the above infinite word does not lie in \( X \).

At this point it is perhaps useful to discuss an example: it is well known that the even shift is not of finite type\(^3\), and hence (7.19) predicts the existence of elements in \( \Omega_X \) beyond the range of \( \Xi \), meaning not of the form \( \xi_x \). In what follows we will concretely exhibit an example of such anomalous elements.

Let \( X \) be the even shift, and for each \( n \), consider the infinite word
\[ x_n = 1^{2n+1} 0^\infty = 1 \ldots 1 0000 \ldots \]

\[ 2n+1 \]
\[ \text{This will also follow from the analysis we are about to undertake.} \]
Since $\Omega_X$ is compact, there exists a subsequence, say $\{y_k\}_k = \{x_{n_k}\}_k$, such that $\{\xi_{y_k}\}_k$ converges to some $\xi \in \Omega_X$. Our next goal will be to prove that $\xi$ is not of the form $\xi_x$, for any $x$ in $X$.

The fact that $\{x_n\}_n$, and hence also $\{y_k\}_k$ converges to the infinite word

$$1^\infty = 1111\ldots$$

relative to the topology of $X$, does not imply that $\xi_{y_k}$ converges to $\xi_1^\infty$, as the correspondence $x \to \xi_x$ is not known to be continuous (it will soon be evident that it is discontinuous at $1^\infty$). Nevertheless, the continuity of the stem (7.16) implies that

$$\sigma(\xi) = \lim_k \sigma(\xi_{y_k}) = \lim_k y_k = 1^\infty.$$

Therefore, if $\xi = \xi_x$, for some $x$, then

$$1^\infty = \sigma(\xi) = \sigma(\xi_x) = x.$$

So, to prove that $\xi$ is not equal to any $\xi_x$ we therefore only need to verify that $\xi \neq \xi_1^\infty$.

Observing that $y_k$ is not in the follower set of the finite word

$$\beta = '0',$$

we have by (7.12) that $\beta^{-1}$ is not in $\xi_{y_k}$. By (7.16.1) it follows that

$$[\beta^{-1} \in \xi] = \lim_k [\beta^{-1} \in \xi_{y_k}] = 0,$$

so $\beta^{-1} \notin \xi$. Nevertheless, $1^\infty$ does belong to the follower set of $\beta$, hence $\beta^{-1} \in \xi_1^\infty$, again by (7.12). This proves that $\xi \neq \xi_1^\infty$, so $\xi$ is not in the range of $\Xi$, whence $\Xi$ is not onto, and we then deduce from (7.19) that $\Xi$ is not continuous. We also recover the well known fact that the even shift is not of finite type.

This example also illustrates that the implication “(b) $\Rightarrow$ (a)” in (8.8) may indeed fail, since the stem of $\xi$ lies in $F_\beta$, and yet $\beta^{-1}$ is not in $\xi$.

9. Partial crossed product description of $O_X$.

As always we fix a finite alphabet $\Lambda$ and a subshift $X \subseteq \Lambda^\mathbb{N}$. In this section we plan to prove that $O_X$ is isomorphic to the crossed product of $C(\Omega_X)$ (also known as $D_X$) by the spectral partial action of the free group $\mathbb{F}$. In symbols

$$O_X \simeq C(\Omega_X) \rtimes_\vartheta \mathbb{F}. \quad (9.1)$$

We will also show that the associated semi-direct product Fell bundle is amenable [12], whence the full and reduced crossed products coincide.

Regarding the partial action $\tau$ of $\mathbb{F}$ on $D_X$ introduced in (8.2), recall that $\vartheta$ was defined as the partial action on the spectrum of $D_X$ induced by $\tau$. Thus, moving in the opposite direction, the partial dynamical system induced by $\vartheta$ on $C(\Omega_X)$ is equivalent to $\tau$. In order to prove (9.1), it therefore suffices to prove that

$$O_X \simeq D_X \rtimes_\tau \mathbb{F}.$$

9.2. Proposition. There exists a surjective *-homomorphism

$$\varphi : D_X \rtimes_\tau \mathbb{F} \to O_X,$$

such that $\varphi(a\delta_g) = (a \otimes 1)\tilde{u}_g = au_g \otimes \lambda_g$, for all $g$ in $\mathbb{F}$, and every $a \in D_g$. 
Proof. We claim that the pair \((j, \tilde{u})\) is a covariant representation (see [16: 9.10]) of \(\tau\) in \(O_X\), where

\[ j : a \in D_X \mapsto a \otimes 1 \in O_X. \]

To see this we pick any \(g\) in \(\mathbb{F}\), and \(a\) in \(D_{g^{-1}}\), and compute

\[
\tilde{u}_g j(a) \tilde{u}_g^{-1} = (u_g \otimes \lambda_g)(a \otimes 1)(u_{g^{-1}} \otimes \lambda_{g^{-1}}) = u_g a u_{g^{-1}} \otimes 1 \overset{(8.1)}{=} \tau_g(a) \otimes 1 = j(\tau_g(a)).
\]

This shows that indeed \((j, \tilde{u})\) is a covariant representation, so the existence of \(\varphi\) follows from [16: 13.1]. Given \(\alpha\) in \(\Lambda^*\), notice that \(e_\alpha \in D_\alpha\), so

\[
\varphi(e_\alpha \delta_\alpha) = e_\alpha u_\alpha \otimes \lambda_\alpha = u_\alpha \otimes \lambda_\alpha = \tilde{u}_\alpha.
\]

This shows that every \(\tilde{u}_\alpha\) lies in the range of \(\varphi\), whence \(\varphi\) is onto. \(\square\)

In order to find a map in the opposite direction, let us consider the standard conditional expectation \(E\) from the algebra of all bounded operators on \(\ell^2(X) \otimes \ell^2(\mathbb{F})\) onto the subalgebra of all diagonal operators relative to the standard orthonormal basis. Thus, if \(t\) is any bounded operator on \(\ell^2(X) \otimes \ell^2(\mathbb{F})\), then \(E(t)\) is the operator whose off diagonal entries are zero, and whose diagonal entries are the same as those of \(t\).

9.3. Lemma. Let \(t = \tilde{e}_{h_1} \tilde{e}_{h_2} \ldots \tilde{e}_{h_n} \tilde{u}_g\), where \(h_1, h_2, \ldots, h_n, g \in \mathbb{F}\). Then

\[
E(t) = \begin{cases}
t, & \text{if } g = 1, \\
0, & \text{otherwise.}
\end{cases}
\]

Proof. We have

\[
E(t) = E(\tilde{e}_{h_1} \tilde{e}_{h_2} \ldots \tilde{e}_{h_n} \tilde{u}_g) = \tilde{e}_{h_1} \tilde{e}_{h_2} \ldots \tilde{e}_{h_n} E(\tilde{u}_g),
\]

because the \(\tilde{e}_h\)'s are diagonal operators by (7.2). When \(g = 1\) we have that \(\tilde{u}_g = 1\), so it is easy to see that \(E(t) = t\). When \(g \neq 1\), notice that for all \(x \in X\), and \(h \in \mathbb{F}\),

\[
\langle \tilde{u}_g(\delta_x \otimes \delta_h), \delta_x \otimes \delta_h \rangle = \langle u_g(\delta_x), \delta_x \rangle \langle \delta_{gh}, \delta_h \rangle = 0,
\]

since \(gh \neq h\). Thus all diagonal entries for \(\tilde{u}_g\) vanish, whence \(E(\tilde{u}_g) = 0\), concluding the proof. \(\square\)

Observe that the elements of the form \(\tilde{e}_{h_1} \tilde{e}_{h_2} \ldots \tilde{e}_{h_n} \tilde{u}_g\) span a *-subalgebra of \(O_X\) by [16: 9.8]. It is therefore a dense subalgebra, and (9.3) then implies that \(O_X\) is invariant under \(E\). Again by (9.3) one sees that the

\[
E(O_X) \subseteq D_X \otimes 1,
\]

so we may see \(E\) as a conditional expectation from \(O_X\) to \(D_X \otimes 1\).

In our next result we will refer to the reduced partial crossed product of \(D_X\) by \(\mathbb{F}\) under \(\tau\) (see [16: 17.10]), which we will denote by \(D_X \rtimes^\text{red}_\tau \mathbb{F}\).
9.4. Proposition. There exists a \(*\)-homomorphism

\[ \psi : \mathcal{O}_X \rightarrow \mathcal{D}_X \rtimes^\text{red}_\tau \mathbb{F}, \]

such that \( \psi(\tilde{u}_\alpha) = e_\alpha \delta_\alpha \), for all \( \alpha \) in \( \Lambda^* \).

Proof. For each \( g \) in \( \mathbb{F} \), let \( B_g \) be the closed linear subspace of \( \mathcal{O}_X \) given by

\[ B_g = (D_g \otimes 1)\tilde{u}_g = (D_g u_g \otimes \lambda_g). \]

We claim that \( B_g B_h \subseteq B_{gh} \), for all \( g, h \in \mathbb{F} \). In fact, given \( a \in D_g \), and \( b \in D_h \), we have

\[ (au_g \otimes \lambda_g)(bu_h \otimes \lambda_h) = au_g b u_h \otimes \lambda_{gh}, \tag{9.4.1} \]

while

\[ au_g b u_h = e_g au_g b e_g^{-1} u_h = u_g a u_g b u_g^{-1} u_g u_h = u_g \tau_g^{-1}(a) b u_g^{-1} u_{gh} = \tau_g(\tau_g^{-1}(a)b) u_{gh} \in \tau_g(D_g^{-1} \cap D_h)u_{gh} \subseteq D_{gh} u_{gh}. \]

Therefore the element described in (9.4.1) lies in \( B_{gh} \), proving the claim. We leave it for the reader to prove that \( (B_g)^* = B_g^{-1} \), as well as that \( \sum_{g \in \mathbb{F}} B_g \) is a dense subspace of \( \mathcal{O}_X \), which, when combined with the conditional expectation \( E \), provided above, verifies all of the assumptions of \([16: 19.1]\), which in turn provides for the desired map \( \psi \). \( \Box \)

Combining the above with (9.2) we arrive at the main result of this section:

9.5. Theorem. Let \( \Lambda \) be a finite alphabet, and let \( X \subseteq \Lambda^N \) be a subshift. Then:

(i) The semi-direct product bundle \([16: 16.6]\) corresponding to the spectral partial action satisfies the approximation property \([16: 20.4]\), and hence is amenable \([16: 20.1]\).

(ii) The Carlsen-Matsumoto \( C^*\)-algebra \( \mathcal{O}_X \) is naturally isomorphic to both the full and the reduced crossed product of \( C(\Omega_X) \) by the free group \( \mathbb{F}(\Lambda) \) under the spectral partial action. In symbols

\[ \mathcal{O}_X \simeq C(\Omega_X) \rtimes_{\varnothing} \mathbb{F} \]

\[ \simeq C(\Omega_X) \rtimes_{\varnothing}^\text{red} \mathbb{F}. \]

Proof. The first point follows from \([13: \text{Theorems }4.1 \& 6.3]\) (see also \([16: 20.13]\)).

As for (ii), regarding the maps \( \varphi \) and \( \psi \) provided by (9.2) and (9.4), it is easy to see that the composition \( \psi \circ \varphi \) is the regular representation \([16: 17.6]\) of \( C(\Omega_X) \rtimes_{\varnothing} \mathbb{F} \), which is an isomorphism by (i). In particular \( \varphi \) is one-to-one, but since we already saw that it is onto in (9.2), we deduce that it is an isomorphism. If both \( \varphi \) and the composition \( \psi \circ \varphi \) are isomorphisms, then so is \( \psi \), whence \( \mathcal{O}_X \) is isomorphic to the reduced crossed product as well. \( \Box \)

In the introduction of \([7]\) Carlsen argues that, contrary to \( \mathcal{O}_X \), Matsumoto’s algebra \( \mathcal{M}_X \) does not have good universal properties and hence they should be considered as the class of \textit{reduced} \( C^*\)-algebras associated with subshifts. Interpreting the term \textit{reduced} as one usually does when speaking of crossed products, the above result seems to indicate that \( \mathcal{O}_X \) should be seen as both full and reduced \( C^*\)-algebras and that \( \mathcal{M}_X \) is just an epimorphic image of \( \mathcal{O}_X \).
10. Comparison with Carlsen’s description of $O_X$.

As always we fix a finite alphabet $\Lambda$ and a subshift $X \subseteq \Lambda^\mathbb{N}$.

In this section we will prove the fact, already hinted at after (6.4), that $O_X$ is isomorphic to the algebra introduced by Carlsen in [7: Definition 5.1].

We have already observed that the $e_g$ are diagonal operators relative to the canonical orthonormal basis of $\ell^2(X)$. On the other hand, the algebra of all diagonal operators is clearly isomorphic to $\ell^\infty(X)$, so we may see $D_X$ as a subalgebra of $\ell^\infty(X)$.

10.1. Proposition. For all finite words $\alpha$ and $\beta$ in $\Lambda^*$, let

$$C(\beta, \alpha) = \{\alpha y \in X : y \in F_\alpha \cap F_\beta\}.$$ 

Then $D_X$ coincides with the closed *-subalgebra of $\ell^\infty(X)$ generated by the characteristic functions $1_{C(\beta, \alpha)}$, as $\alpha$ and $\beta$ range in $\Lambda^*$.

Proof. In case $\alpha$ and $\beta$ are such that $|\alpha\beta^{-1}| = |\alpha| + |\beta^{-1}|$, that is, when $g := \alpha\beta^{-1}$ is in reduced form, we have seen in (4.2.ii) that $X_g = C(\beta, \alpha)$. Since $e_g$ is the final projection of $u_g$, whose range is $\ell^2(X_g)$ by (5.3), we have that $e_g$ coincides with $1_{C(\beta, \alpha)}$ (up to the above identification of diagonal operators and bounded functions). For general $\alpha$ and $\beta$, let $\gamma$ be the longest common suffix of $\alpha$ and $\beta$, so we may find $\alpha'$ and $\beta'$ such that

$$\alpha = \alpha'\gamma, \quad \beta = \beta'\gamma,$$

and $\alpha'\beta'^{-1}$ is the reduced form for the group element $g := \alpha\beta^{-1}$. We leave it for the reader to check that

$$C(\beta, \alpha) = C(\beta', \alpha') \cap C(\emptyset, \alpha),$$

from where it follows that

$$1_{C(\beta, \alpha)} = 1_{C(\beta', \alpha')}1_{C(\emptyset, \alpha)} = e_\alpha e_{\alpha^{-1}} = e_{\alpha\beta^{-1}}e_\alpha.$$ (10.1.1)

We therefore conclude that the algebra generated by all of the $1_{C(\beta, \alpha)}$ is the same as the algebra generated by all of the $e_g$. $\square$

As a consequence we see that $D_X$ is the same as the algebra denoted $\tilde{D}_X$ studied in [7: Definition 4.1], or the algebra denoted $D_X$ studied in [10: Lemma 7], and which also appears in many other papers dealing with $C^*$-algebras associated with subshifts (see the references given in the introduction).

10.2. Theorem. For every subshift $X$, denote by $O'_X$ the $C^*$-algebra introduced by Carlsen in [7: Definition 5.1]. Then there is an isomorphism

$$\varphi : O'_X \to O_X,$$

such that $\varphi(S_\alpha) = \tilde{u}_\alpha$, for all $\alpha$ in $\Lambda^*$, where $S_\alpha$ is the partial isometry given in [7: Definition 5.3].
Proof. For each $\alpha \in \Lambda^*$, let

$$T_\alpha = \tilde{u}_\alpha = u_\alpha \otimes \lambda_\alpha,$$

and notice that for any given $\alpha, \beta \in \Lambda^*$, we have

$$T_\alpha T_\beta^* T_\alpha^* = \tilde{u}_\alpha \tilde{e}_{\beta^{-1}} \tilde{u}_{\alpha^{-1}} \overset{[16.9.8.iii]}{=} \tilde{e}_{\alpha \beta^{-1}} \tilde{u}_{\alpha^{-1}} =$$

$$= \tilde{e}_{\alpha \beta^{-1}} \tilde{\epsilon}_\alpha = (e_{\alpha \beta^{-1}} \epsilon_\alpha) \otimes 1 \overset{(10.1.1)}{=} 1_{C(\beta, \alpha)} \otimes 1.$$

Using the universal property of $O'_X$ [7: Remark 7.3], we see that there exists a *-homomorphism $\varphi : O'_X \to O_X$ sending each $S_\alpha$ to $T_\alpha$, and which is therefore necessarily onto.

In order to complete the proof it now suffices to prove that $\varphi$ is injective, and for this we will employ [10: Theorem 13], which demands that we build a suitable action of the circle group $\mathbb{T}$ on $O_X$. Consider the unique group homomorphism

$$\epsilon : \mathbb{F} \to \mathbb{Z},$$

such that $\epsilon(a) = 1$, for every $a \in \Lambda$, so that in particular $\epsilon(\alpha) = |\alpha|$, if $\alpha$ is a finite word. Moreover, for each $z$ in $\mathbb{T}$, let $V_z$ be the unitary operator defined on the canonical orthonormal basis of $\ell^2(X) \otimes \ell^2(\mathbb{F})$ by

$$V_z(\delta_x \otimes \delta_g) = z^{\epsilon(\alpha)} \delta_x \otimes \delta_g, \quad \forall x \in X, \quad \forall g \in \mathbb{F}.$$\n
We then claim that

$$V_z T_\alpha V_z^{-1} = z^{|\alpha|} T_\alpha, \quad \forall z \in \mathbb{T}, \quad \forall \alpha \in \Lambda^*.$$ (10.2.1)

To prove it we compute on a general element $\delta_x \otimes \delta_g$ of the canonical orthonormal basis:

$$V_z T_\alpha V_z^{-1}(\delta_x \otimes \delta_g) = z^{-\epsilon(\alpha)} V_z (u_\alpha \otimes \lambda_\alpha)(\delta_x \otimes \delta_g) = z^{-\epsilon(\alpha)} V_z (u_\alpha(\delta_x) \otimes \delta_{\alpha g}) =$$

$$= z^{-\epsilon(\alpha) + \epsilon(\alpha g)} (u_\alpha(\delta_x) \otimes \delta_{\alpha g}) = z^{\epsilon(\alpha)} T_\alpha (\delta_x \otimes \delta_g).$$

This proves (10.2.1), so it follows that the formula

$$\gamma_z(a) = V_z a V_z^{-1}, \quad \forall x \in \mathbb{T}, \quad \forall a \in O_X,$$

defines a strongly continuous action of $\mathbb{T}$ on $O_X$ satisfying point (2) of [10: Theorem 13], whence $\varphi$ is injective. \qed
11. The topology on the spectrum.

The topology on $\Omega_X$, being induced from the product topology of $2^F$, admits a basis formed by the open sets

$$U_{g_1,g_2,\ldots,g_n;h_1,h_2,\ldots,h_m} = \left\{ \eta \in \Omega_X : \begin{array}{l} g_1 \in \eta, \ g_2 \in \eta, \ldots, \ g_n \in \eta \\ h_1 \notin \eta, \ h_2 \notin \eta, \ldots, \ h_m \notin \eta \end{array} \right\}, \quad (11.1)$$

where $g_1,g_2,\ldots,g_n; h_1,h_2,\ldots,h_m$ range in $F$. Given the special nature of elements of $\Omega_X$, we may restrict to sets of a somewhat special nature, as follows.

11.2. Proposition. Given $\alpha, \beta_1,\beta_2,\ldots,\beta_n, \gamma_1,\gamma_2,\ldots,\gamma_m \in F_+$, consider the subset of $\Omega_X$ given by

$$V_{\alpha;\beta_1,\beta_2,\ldots,\beta_n;\gamma_1,\gamma_2,\ldots,\gamma_m} = \left\{ \eta \in \Omega_X : \begin{array}{l} \alpha \in \eta, \\
\alpha\beta_i^{-1} \in \eta, \text{ for } i = 1,\ldots,n, \\
\alpha\gamma_j^{-1} \notin \eta, \text{ for } j = 1,\ldots,m \end{array} \right\},$$

Then the collection consisting of all sets of the above form is a basis for the topology of $\Omega_X$.

Proof. Let $A$ be an open subset of $\Omega_X$, and let $\xi \in A$. It suffices to prove that there is an open set $V$ of the form described in the statement with $\xi \in V \subseteq A$.

By the definition of the product topology on $2^F$ there are $g_1,g_2,\ldots,g_n; h_1,h_2,\ldots,h_m \in F$, as in (11.1), such that

$$\xi \in U_{g_1,g_2,\ldots,g_n;h_1,h_2,\ldots,h_m} \subseteq A. \quad (11.2.1)$$

Since the elements of $\Omega_X$ only contain group elements of the form $\alpha\beta^{-1}$, with $\alpha, \beta \in F_+$ by (7.8.v), we may ignore the $g_i$ and the $h_j$ which are not of this form without affecting (11.2.1). We may therefore assume that

$$g_i = \alpha_i\beta_i^{-1}, \quad \text{and} \quad h_j = \mu_j\gamma_j^{-1},$$

in reduced form, with $\alpha_i, \beta_i, \mu_j, \gamma_j \in F_+$. Let $\alpha$ be any prefix of the stem of $\xi$, long enough so that

$$|\alpha| \geq \max \{ |\alpha_1|, |\alpha_2|,\ldots,|\alpha_n|, |\mu_1|, |\mu_2|,\ldots,|\mu_m| \}, \quad (11.2.2)$$

and observe that since $\alpha \in \xi$, then

$$\xi \in U_{\alpha;g_1,g_2,\ldots,g_n;h_1,h_2,\ldots,h_m} \subseteq U_{g_1,g_2,\ldots,g_n;h_1,h_2,\ldots,h_m} \subseteq A.$$

The reader is asked to compare the display above with (11.2.1), paying special attention to the important detail that $\alpha$ was inserted ahead of the $g_i$’s in the subscripts of the first occurrence of $U$ above. The proof will consist in showing that this occurrence of $U$ coincides with the set displayed in the statement for suitable choices of $\beta_i$ and $\gamma_j$. 

Notice that, by assumption the $g_i \in \xi$, so (7.8.ii) implies that $\alpha_i \in \xi$, and then the $\alpha_i$ are necessarily prefixes of the stem of $\xi$ by (7.13). Consequently the $\alpha_i$ are also prefixes of the long $\alpha$ chosen above, and we may find suitable elements $\delta_i$ in $\mathbb{F}_+$, such that

$$\alpha = \alpha_i \delta_i,$$

for all $i$. Letting $\beta'_i = \beta_i \delta_i$, observe that

$$\alpha \beta'^{-1}_i = \alpha \delta^{-1}_i \beta^{-1}_i = \alpha_i \beta^{-1}_i = g_i, \quad \text{(11.2.3)}$$

so, upon replacing each $\beta_i$ by $\beta'_i$, we may suppose that $g_i = \alpha \beta^{-1}_i$. It should however be noticed that this presentation of $g_i$ is no longer in reduced form.

Next we should treat the $h_j$. Firstly, let us consider those $h_j$ whose corresponding $\mu_j$ is not a prefix of the stem of $\xi$. We then claim that, for any $\eta$ in $\Omega_X$, one has that

$$\alpha \in \eta \Rightarrow \mu_j \notin \eta.$$

Otherwise, if both $\alpha$ and $\mu_j$ lie in $\eta$, then both would be prefixes of the stem of $\eta$, in which case $\mu_j$ would be a prefix of $\alpha$, since the former it is shorter than the latter by (11.2.2). This would entail that $\mu_j$ is a prefix of the stem of $\xi$, contradicting our assumptions, and hence proving our claim. By (7.8.ii) we have that

$$\mu_j \notin \eta \Rightarrow \mu_j \gamma^{-1}_j \notin \eta,$$

so a combination of these two implications yields

$$\alpha \in \eta \Rightarrow h_j^{-1} \notin \eta.$$

We then see that $h_j$ may be deleted from the list of subscripts of our

$$U_{\alpha, g_1, g_2, \ldots, g_n; h_1, h_2, \ldots, h_m}, \quad \text{(11.2.4)}$$

since the condition “$\alpha \in \eta$” in the first line of (11.1) already gives “$h_j^{-1} \notin \eta$”, in the second.

After deleting such $h_j$’s, we may assume that $\mu_j$ is a prefix of the stem of $\xi$, for every $j$, and, again by (11.2.2), $\mu_j$ is necessarily a prefix of $\alpha$. Arguing as in (11.2.3) we may then stretch each $\mu_j$ and $\gamma_j$ by the same amount, and hence assume that the $\mu_j$ all coincide with $\alpha$, so that $h_j = \alpha \gamma^{-1}_j$.

The description of the set in (11.2.4) is then identical to the description of the set displayed in the statement, so we have concluded the proof. \[\square\]

Up to the statement of the above result, whenever we considered an expression of the form “$\alpha \beta^{-1}$”, this was supposed to be in reduced form. However the reader should be warned that this is no longer the case, especially after (11.2.3), were we deliberately gave up on reduced forms in exchange for working with a single $\alpha$.

There is a further simplification which may be bestowed upon the general form of the open sets described in (11.2), provided we are concerned with neighborhoods of points in the range of $\Xi$, namely we may do away with the $\gamma_j$. Making this idea precise is our next goal.
11.3. Proposition. Given \( x \) in \( X \), let \( \xi_x = \Xi(x) \). Then the collection of all sets of the form

\[
V_{\alpha; \beta_1, \beta_2, \ldots, \beta_n} = \left\{ \eta \in \Omega_X : \alpha \in \eta, \alpha \beta_i^{-1} \in \eta, \text{ for } i = 1, \ldots, n \right\},
\]

for \( \alpha, \beta_1, \beta_2, \ldots, \beta_n \in \mathbb{F}_+ \), which moreover contains \( \xi_x \), forms a neighborhood base for \( \xi_x \).

Proof. Needless to say, the above sets are special cases of the sets in (11.2), corresponding to taking \( m = 0 \), meaning that the conditions “\( \alpha \gamma_j^{-1} \notin \eta \)” are now absent.

In order to prove the statement, we must show that for every open set \( U \) containing \( \xi_x \), there are \( \alpha, \beta_1, \beta_2, \ldots, \beta_n \in \mathbb{F}_+ \) such that

\[
\xi_x \in V_{\alpha; \beta_1, \beta_2, \ldots, \beta_n} \subseteq U.
\]

Using (11.2) we may clearly suppose that

\[
U = V_{\alpha; \beta_1, \beta_2, \ldots, \beta_n; \gamma_1, \gamma_2, \ldots, \gamma_m},
\]

for suitable \( \alpha, \beta_1, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2, \ldots, \gamma_m \in \mathbb{F}_+ \). Observing that \( \xi_x \) lies in \( U \), we have that \( \alpha \in \xi_x \), while \( \alpha \gamma_j^{-1} \notin \xi_x \), for every \( j = 1, \ldots, m \).

![Diagram](image)

It follows that \( \alpha \) is a prefix of the stem of \( \xi_x \), also known as \( x \), so we may write \( x = \alpha y \) for some infinite word \( y \). The stem of \( \xi_x \) at \( \alpha \) is therefore \( y \) and the fact, noted above, that \( \alpha \gamma_j^{-1} \notin \xi_x \), together with (8.8) leads to the conclusion that \( y \) is not in the follower set of \( \gamma_j \), which is to say that

\[
\gamma_j y \notin X.
\]

By (2.3) we then deduce that \( \gamma_j y \) has some prefix which is a forbidden word, namely a finite word not in the language \( \mathcal{L}_X \). By increasing its length we may suppose that this forbidden prefix is of the form

\[
\gamma_j \delta_j
\]

where \( \delta_j \) is a prefix of \( y \). Denote by \( \delta \) the longer among the \( \delta_j \), and set

\[
\alpha' = \alpha \delta, \quad \text{and} \quad \beta'_i = \beta_i \delta,
\]

for every \( i = 1, \ldots, n \). It follows that

\[
\alpha' \beta'^{-1}_i = \alpha \beta^{-1}_i,
\]
and the proof will be concluded once we show that
\[ \xi_x \in V_{\alpha'; \beta'_1, \beta'_2, \ldots, \beta'_n} \subseteq V_{\alpha; \beta_1, \beta_2, \ldots, \beta_n; \gamma_1, \gamma_2, \ldots, \gamma_m}. \tag{11.3.1} \]

We leave the easy “\( \in \)” for the reader to check and concentrate on the “\( \subseteq \)”. We thus pick \( \eta \) in the set appearing in the left-hand-side above, and we note that
(a) \( \alpha' \in \eta \),  
(b) \( \alpha' \beta_j^{-1} \in \eta \),  
for every \( j = 1, \ldots, m \). By (a) and (7.8.ii) we have that \( \alpha \in \eta \), and clearly
\[ \alpha \beta_i^{-1} = \alpha' \beta_j^{-1} \in \eta. \]

Thus, in order to prove that \( \eta \) lies in the set appearing in the right-hand-side of (11.3.1), we must only check that \( \alpha \gamma_j^{-1} \notin \eta \). For this, observe that
\[ \alpha \gamma_j^{-1} = \alpha \delta(\gamma_j \delta)^{-1} = \alpha' (\gamma_j \delta)^{-1}. \]

Assuming by contradiction that this element belongs to \( \eta \), we have by (a) and (8.8) that
\[ \sigma_{\alpha'}(\eta) \in F_{\gamma_j \delta}. \]

Letting \( z = \sigma_{\alpha'}(\eta) \), this means that \( \gamma_j \delta z \in X \), but \( \gamma_j \delta \) admits the forbidden word \( \gamma_j \delta_j \) as a prefix, a contradiction. This proves our claim that \( \alpha \gamma_j^{-1} \notin \eta \), and hence that
\[ \eta \in V_{\alpha; \beta_1, \beta_2, \ldots, \beta_n; \gamma_1, \gamma_2, \ldots, \gamma_m}, \]
showing (11.3.1). \( \square \)

Speaking of a neighborhood of the form \( V_{\alpha; \beta_1, \beta_2, \ldots, \beta_n} \), as above, notice that for every \( \eta \) in this set, one automatically has that
\[ \alpha \beta^{-1} \in \eta, \]
for \( \beta = \emptyset \), as well as for \( \beta = \alpha \), regardless of whether or not the words \( \emptyset \) and \( \alpha \) are among the \( \beta_i \)'s. Therefore, should one so wish, these two words may be added to the \( \beta_i \)'s without altering the resulting neighborhood. That is
\[ V_{\alpha; \beta_1, \beta_2, \ldots, \beta_n} = V_{\alpha; \emptyset, \alpha, \beta_1, \beta_2, \ldots, \beta_n}. \]
12. Topological freeness.

Recall from [18: Definition 2.1] (see also [3] and [16: Section 29]) that a topological partial action
\[ \rho = (\{Y_g\}_{g \in G}; \{\rho_g\}_{g \in G}) \]
of a group \( G \) on a space \( Y \) is said to be topologically free if, for every \( g \neq 1 \), the set of fixed points for \( \rho_g \), namely
\[ \text{Fix}_g = \{ x \in Y_{g^{-1}} : \rho_g(x) = x \}, \]
has empty interior.

In this section we will give necessary and sufficient conditions for a general subshift to have a topologically free spectral partial action. However, let us begin by reviewing the well known Markov case.

Observing that Markov subshifts are of finite type, we see that the spectral partial action is equivalent to the standard partial action by (8.7).

The following result, inspired by condition (I) of [11], characterizes topological freeness in terms of circuits and exits. It was first proved for row-finite matrices in [20: Lemma 3.4] in the language of groupoids. A generalization for infinite matrices was given in [17: Proposition 12.2].

12.2. Proposition. Let \( X \) be a Markov subshift. Then the standard partial action \( \theta \) of \( F \) on \( X \) (which, by (8.7) is equivalent to the spectral partial action) is topologically free if and only if every circuit has an exit.

For subshifts not of finite type, the situation is a lot more delicate. Even if all circuits have strong exits (see (3.4)), the spectral partial action may fail to be topologically free.

To see this, consider the even shift. In section (2) we have proven that all of its circuits have strong exits, and yet its associated spectral partial action is not topologically free for the following reason: consider the infinite word
\[ 1^\infty = 11111\ldots, \]
which is fixed by \( \theta_g \), where \( g \) is the element\(^4\) of \( F \) corresponding to the word ‘1’. By (8.6) we have that \( \Xi \) is equivariant, so \( \xi_{1^\infty} \) is fixed by \( \psi_g \).

We will show that \( \vartheta \) is not topologically free by showing that \( \xi_{1^\infty} \) is an isolated point in \( \Omega_X \) (even though \( 1^\infty \) is not isolated in \( X \)). In fact, let
\[ \beta_1 = 01, \quad \text{and} \quad \beta_2 = 011, \]
and consider the open set \( V_{\vartheta;\beta_1,\beta_2} \) described in (11.3). Notice that for any \( x \) in \( X \) one has that
\[ \xi_x \in V_{\vartheta;\beta_1,\beta_2} \iff \beta_1^{-1}, \beta_2^{-1} \in \xi_x \ \overset{(8.8)}{\iff} x \in F_{\beta_1} \cap F_{\beta_2}. \]

\(^4\) Warning: this is one of the generators of \( F \) and not the unit group element!
We leave it for the reader to check that the rules of the even shift imply that \( F_{\beta_1} \cap F_{\beta_2} = \{1^\infty\} \), so the only \( \xi_x \) in \( V_{\emptyset;\beta_1,\beta_2} \) is \( \xi_1^\infty \). Since the set of all \( \xi_x \) is dense in \( \Omega_X \) by (7.4), and since \( V_{\emptyset;\beta_1,\beta_2} \) is open, a simple exercise in Topology gives that

\[
V_{\emptyset;\beta_1,\beta_2} = \{\xi_1^\infty\}. \tag{12.3}
\]

The above is then an open set of fixed points, whence \( \vartheta \) is not topologically free.

We have therefore proven:

**12.4. Proposition.** The spectral partial action associated to the even shift is not topologically free.

One might wonder if the standard (as opposed to spectral) partial action \( \theta \) for the even shift is topologically free. Although we believe this is not a well posed question, since the standard partial is not topological (the \( X_g \) are not all open), one might decide to ignore this and insist in checking whether the interior of any set of fixed points is empty. In this case the answer is easily seen to be positive, so in this sense the standard partial of the even shift is topologically free.

Before we give the appropriate characterization of topological freeness for the spectral partial action of general subshifts, let us understand their fixed points a little better. The following result is entirely similar to known results for Markov subshifts, but we give a full proof, which takes no more than a few lines, for the convenience of the reader:

**12.5. Proposition.** Let \( \Lambda \) be a finite alphabet, let \( X \subseteq \Lambda^\mathbb{N} \) be a subshift, and denote by \( \theta \) the standard partial action of \( \mathbb{F} \) on \( X \). Given \( g \in \mathbb{F} \setminus \{1\} \), let \( x \in X \) be a fixed point for \( g \). Then

(i) \( g \) admits a decomposition in reduced form as \( \nu\alpha^{\pm 1}\nu^{-1} \), with \( \alpha,\nu \in \mathbb{F}_+ \),

(ii) \( x = \nu\alpha^\infty \), whence \( \alpha \) is a circuit, and \( x \) is the unique fixed point for \( g \).

**Proof.** Since \( x \) lies in the domain of \( \theta_g \), that domain is nonempty, whence \( g \in \mathbb{F}_+ \mathbb{F}_-^{-1} \) by (4.2.i). We may therefore write \( g \) in reduced form as \( \mu\nu^{-1} \), with \( \mu,\nu \in \mathbb{F}_+ \). Since \( x \in X_{g^{-1}} \), we have by (4.2.ii) that \( x = \nu y \), for some \( y \in F_\nu \cap F_\mu \), and then

\[
\nu y = x = \theta_{\mu\nu^{-1}}(x) = \mu y.
\]

Therefore either \( \nu \) is a prefix of \( \mu \), or vice-versa. We assume without loss of generality that \( |\mu| > |\nu| \) (one cannot have \( |\mu| = |\nu| \) because \( g \neq 1 \)), and so we may write \( \mu = \nu\alpha \), for some \( \alpha \in \Lambda^* \). It then follows from the above that \( \nu y = \nu \alpha y \), so \( y = \alpha y \), whence \( y = \alpha^\infty \). Summarizing we have

\[
g = \mu\nu^{-1} = \nu\alpha\nu^{-1}, \quad \text{and} \quad x = \nu y = \nu\alpha^\infty.
\]

completing the proof. \( \square \)

We may now present the main result of this section. Contrary to one might expect, this result does not explicitly mention the existence of exits for circuits, but please see the remarks after the proof below for an interpretation in terms of exits for circuits.
12.6. Theorem. Let $\Lambda$ be a finite alphabet and let $X \subseteq \Lambda^\mathbb{N}$ be a subshift. Then the following are equivalent:

(i) the spectral partial action $\vartheta$ associated to $X$ is topologically free,

(ii) for every $\beta_1, \beta_2, \ldots, \beta_n$ in $\Lambda^*$, and for every circuit $\gamma$ such that

$$\gamma^\infty \in \bigcap_{i=1}^n F_{\beta_i},$$

one has that $\bigcap_{i=1}^n F_{\beta_i}$ contains some element other than $\gamma^\infty$.

Proof. (i) $\Rightarrow$ (ii). Pick $\beta_1, \beta_2, \ldots, \beta_n$ in $\Lambda^*$, and let $\gamma$ be a circuit such that $\gamma^\infty \in \bigcap_{i=1}^n F_{\beta_i}$.

Notice that $\gamma^\infty$ is a fixed point for $\theta_\gamma$, so $\xi_{\gamma^\infty}$ is a fixed point for $\vartheta_\gamma$ by (8.6). Also observe that $\beta_i^{-1} \in \xi_{\gamma^\infty}$, for all $i$, by (7.12), so

$$\xi_{\gamma^\infty} \in V_{\emptyset; \beta_1, \beta_2, \ldots, \beta_n}.$$

Should $\xi_{\gamma^\infty}$ be the only element of this set, the singleton $\{\xi_{\gamma^\infty}\}$ would be an open set consisting of fixed points for $\vartheta_\gamma$, contradicting (i). Thus

$$V_{\emptyset; \beta_1, \beta_2, \ldots, \beta_n} \setminus \{\xi_{\gamma^\infty}\}$$

is not empty, hence it contains some $\xi_x$, for $x$ in $X$, as $\Xi(X)$ is dense in $\Omega_X$. Thus necessarily $x \neq \gamma^\infty$, and since each $\beta_i^{-1}$ belongs to $\xi_x$, we have that $x \in F_{\beta_i}$, again by (7.12). This proves (ii).

(ii) $\Rightarrow$ (i). Arguing by contradiction, let $g \in \mathbb{F} \setminus \{1\}$, and let $U$ be a nonempty open subset of $\Omega_{g^{-1}}$ consisting of fixed points for $\vartheta_g$. Since $\Xi(X)$ is dense in $\Omega_X$ by (7.4), there exists some $x$ in $X$ such that $\xi_x \in U$, whence $\xi_x$ is fixed by $\vartheta_g$.

By (8.6) we conclude that $x = \sigma(\xi_x)$ is fixed by $\theta_g$, so (12.5.i) provides $\alpha, \nu \in \mathbb{F}_+$, such that $g = \nu \alpha^{\pm 1} \nu^{-1}$, in reduced form. Upon replacing $g$ by $g^{-1}$ if necessary (and keeping $U$ unaltered), we may assume without loss of generality that $g = \nu \alpha \nu^{-1}$. Observing that $\vartheta$ is semi-saturated, we have that

$$\vartheta_g = \vartheta_\nu \circ \vartheta_\alpha \circ \vartheta_{\nu^{-1}},$$

so in particular $\Omega_{g^{-1}} \subseteq \Omega_\nu$. Setting $W = \vartheta_{\nu^{-1}}(U)$ we then have that $W \subseteq \Omega_{\alpha^{-1}}$, and it is clear that $\vartheta_\alpha$ is the identity on $W$.

The upshot of the above argument is that, if $\vartheta$ is not topologically free, then there exists a finite word $\alpha$, and a nonempty open set $W$ consisting of fixed points for $\vartheta_\alpha$. Employing (7.4) once more, there exists some $x$ in $X$ such that $\xi_x \in W$. Therefore $\xi_x$ is fixed by $\vartheta_\alpha$, whence $x = \sigma(\xi_x)$ is fixed by $\theta_\alpha$, and we deduce from (12.5.ii) that $x = \alpha^\infty$.

Using the special neighborhood base of $\xi_x$ provided by (11.3), we may then find finite words $\mu, \beta_1, \beta_2, \ldots, \beta_n \in \Lambda^*$, such that

$$\xi_x \in V_{\mu; \beta_1, \beta_2, \ldots, \beta_n} \subseteq W. \quad (12.6.1)$$
In particular we have that \( \mu \) lies in \( \xi_x \), which in turn implies that \( \mu \) is a prefix of \( x \) by (7.11). Being a prefix of \( x = \alpha^\infty \) therefore implies that \( \mu \) is a prefix of \( \alpha^n \), for some \( n \), whence there exists a finite word \( \gamma \) such that \( \alpha^n = \mu \gamma \). Setting \( \mu' = \mu \gamma \), and \( \beta_i' = \beta_i \gamma \), we have that

\[
\xi_x \in V_{\mu'; \beta_1', \beta_2', ..., \beta_n'} \subseteq V_{\mu; \beta_1, \beta_2, ..., \beta_n} \subseteq W.
\]

Replacing \( \mu \) by \( \mu' \), and each \( \beta_i \) by the corresponding \( \beta_i' \), we then may assume without loss of generality that (12.6.1) reads

\[
\xi_x \in V_{\alpha^n; \beta_1, \beta_2, ..., \beta_n} \subseteq W.
\]

Among other things we then have that both \( \alpha^n \) and \( \alpha^n \beta_i^{-1} \) lie in \( \xi_x \), for all \( i \), so (8.8) gives

\[
\sigma_{\alpha^n}(\xi_x) \in F_{\beta_i}.
\]

Recalling that \( x = \alpha^\infty \), and staring at the definition of the stem, will make it clear that \( \sigma_{\alpha^n}(\xi_x) = \alpha^\infty \). So

\[
\alpha^\infty \in \bigcap_{i=1}^{n} F_{\beta_i}.
\]

Noticing that \( \alpha^\infty \in F_{\alpha^n} \), we may soup up the above conclusion by writing

\[
\alpha^\infty \in \bigcap_{i=1}^{n} F_{\beta_i} \cap F_{\alpha^n},
\]

and then we may use hypothesis (ii) to produce an infinite word \( y \neq \alpha^\infty \) belonging to \( \bigcap_{i=1}^{n} F_{\beta_i} \), as well as to \( F_{\alpha^n} \). The infinite word

\[
z = \alpha^n y
\]

thus lies in \( X \) and the stem of \( \xi_z \) at \( \alpha^n \) is clearly \( y \). For that reason, and using (8.8), we have that \( \alpha^n \beta_i^{-1} \in \xi_z \), for all \( i \), whence

\[
\xi_z \in V_{\alpha^n; \beta_1, \beta_2, ..., \beta_n} \subseteq W,
\]

so \( \xi_z \) is a fixed point for \( \vartheta_{\alpha} \), whence \( z \) is a fixed point for \( \theta_{\alpha} \). However the only such fixed point is \( x \), which is manifestly different from \( z \). We have thus reached a contradiction, hence proving that \( \vartheta \) is topologically free.

When one compares our last result to the well known characterization of topological freeness for Markov subshifts given in (12.2), one might wonder what happened to the role of exits for circuits. Although property (12.6.ii) does not explicitly mention exits, it is closely related to that concept. To see this let \( \gamma \) be a circuit. Then evidently \( \gamma^\infty \in F_{\gamma^n} \), for every \( n \), so the existence of an element \( x \) in \( F_{\gamma^n} \), other than \( \gamma^\infty \), as provided by (12.6.ii), gives an exit for \( \gamma^n \). If this indeed holds for every \( n \), then \( \gamma \) has a strong exit by (3.3).

We therefore see that (12.6.ii) implies that all circuits have strong exits. However we should not forget that the existence of strong exits by itself is not enough to guarantee topological freeness for the spectral partial action, as the example of the even shift in (3.5) above shows.
13. Minimality.

Recall from [18: Definition 2.8] that a topological partial action
\[
\rho = (\{Y_g\}_{g \in G}, \{\rho_g\}_{g \in G})
\]
of a group \(G\) on a space \(Y\) is said to be minimal if there are no nontrivial \(\rho\)-invariant closed subsets. This is clearly equivalent to saying that for every \(y\) in \(Y\), the orbit of \(y\), namely the set
\[
\text{Orb}(y) = \{\rho_g(y) : g \in G, \ Y_g^{-1} \ni y\},
\]
is dense in \(Y\).

Minimality is well understood for Markov subshifts as well as for many other partial dynamical systems related to it. See, for example [20] and [21].

For the purpose of comparison, let us state the following well known result.

13.1. Proposition. Let \(X\) be a Markov subshift with transition matrix \(A\). Then the standard partial action \(\theta\) of \(\mathbb{F}\) on \(X\) (which, by (8.7) is equivalent to the spectral partial action) is minimal if and only if, for any vertex \(a\), and any infinite path \(x\) in \(\text{Gr}(A)\), there exists a finite path starting in \(a\) and ending in some vertex of \(x\).

The property described in the above result is sometimes referred to as cofinality. Motivated by this concept it is natural to consider the following property applicable for general subshifts.

13.2. Definition. Let \(\Lambda\) be a finite alphabet and let \(X \subseteq \Lambda^\mathbb{N}\) be a subshift. We shall say that an infinite word \(x\) in \(X\) may be reached from a finite word \(\beta\) in \(\mathcal{L}_X\), when there exists a finite word \(\gamma\), and a prefix \(\alpha\) of \(x\), such that, upon writing \(x = \alpha y\), one has that \(\beta \gamma y \in X\).

We shall moreover say that \(X\) is cofinal if, for every \(\beta \in \mathcal{L}_X\), and every \(x\) in \(X\), one has that \(x\) may be reached from \(\beta\).

One could think of \(\gamma\) as being the bridge allowing one to travel from \(\beta\) in order to reach \(x\) at \(\alpha\). A quicker way to express (13.2) is to say that
\[
\exists \gamma \in \Lambda^*, \exists n \in \mathbb{N} : S^n(x) \in F_{\beta\gamma}.
\]
Substituting the above notion of cofinality for the classical notion, one might be tempted to conjecture a generalization of Proposition (13.1) above to arbitrary subshifts, but once more the even shift stands as a counter-example. It is easy to see that the even shift satisfies the property just mentioned, because, given any \( x \) and \( \beta \), as in the above diagram, one could simply take \( \alpha = \emptyset \), and then either

\[
\gamma = \emptyset, \quad \text{or} \quad \gamma = \text{‘1’}
\]

will always work as a bridge, as one of them will correctly adjust the parity of the amount of 1’s between the last ‘0’ of \( \beta \) and the first ‘0’ of \( x \).

However the spectral partial action for the even shift is not minimal. To see this let us return to the situation presented in (12.3), when we have verified that the open set \( V_{\emptyset; \beta_1, \beta_2} \) consists of a single point, namely \( \xi_{1\infty} \). Should the spectral partial action for the even shift be minimal, then \( \text{Orb}(\xi_x) \) should intersect \( V_{\emptyset; \beta_1, \beta_2} \), for every \( x \) in \( X \), but this is not the case, e.g. for the element \( x = \text{‘00000...’} \). In fact, if \( \vartheta_g(\xi_{0\infty}) \in V_{\emptyset; \beta_1, \beta_2} \), for some \( g \) in \( \mathbb{F} \), then \( \vartheta_g(\xi_{0\infty}) = \xi_{1\infty} \), whence by (8.6) one has that \( \theta_g(0\infty) = 1\infty \), and this is manifestly impossible.

For future reference let us highlight the conclusion reached above:

13.5. Proposition. The spectral partial action associated to the even shift is not minimal.

As in the previous section, one might decide to ignore that the standard partial action for the even shift is not a topological partial action and ask whether or not it is minimal in the sense that all orbits are dense. The answer is then easily seen to be positive, so in this sense the standard partial for the even shift is minimal.

Even though cofinality for a subshift does not imply minimality for the spectral partial action, as in the Markov case, the former concept is quite relevant in our study of minimality, having the following dynamical interpretation.

13.6. Proposition. Let \( X \) be a subshift. Then the following are equivalent:

(i) \( X \) is cofinal,
(ii) for every \( x \) in \( X \), and for every \( \beta \) in \( \mathcal{L}_X \), there is some \( g \) in \( \mathbb{F} \) such that both \( g \) and \( g\beta \) lie in \( \xi_x \),
(iii) for every \( x \) in \( X \), and for every \( \beta \) in \( \mathcal{L}_X \), one has that \( \text{Orb}(\xi_x) \cap \Omega_{\beta} \neq \emptyset \).

Proof. (i) \( \Rightarrow \) (ii). Given \( x \) and \( \beta \), as in (ii), choose \( \alpha, \gamma \) and \( y \), as in (i), so that \( x = \alpha y \), and \( \beta \gamma y \in X \). We then have that \( \sigma_{\alpha}(\xi_x) = y \), so the fact that \( y \in F_{\gamma} \) yields \( \alpha \gamma^{-1} \in \xi_x \), by (8.8). Similarly, the fact that \( y \in F_{\beta \gamma} \) yields \( \alpha(\beta \gamma)^{-1} \in \xi_x \). It is then clear that \( g = \alpha(\beta \gamma)^{-1} \) satisfies the conditions of (ii).

(ii) \( \Rightarrow \) (i). Given \( \beta \) in \( \mathcal{L}_X \), and \( x \) in \( X \), choose \( g \) in \( \mathbb{F} \) such that \( g, g\beta \in \xi_x \). By (7.11) we may write \( g\beta = \alpha \gamma^{-1} \) in reduced form, with \( \alpha, \gamma \in \mathbb{F}^+ \), such that \( x = \alpha y \), for some infinite word \( y \in F_{\alpha} \cap F_{\gamma} \). Observing that \( \xi_x \) contains both \( \alpha \) and

\[
g = g\beta \beta^{-1} = \alpha \gamma^{-1} \beta^{-1} = \alpha(\beta \gamma)^{-1},
\]
we deduce from (8.8) that \( y = \sigma(x) \in F_{\beta\gamma} \), whence \( \beta\gamma y \in X \). This shows that \( x \) may be reached from \( \beta \).

The equivalence of (ii) and (iii) follows from the next simple result which we will also use later in a slightly more general situation.

**13.7. Proposition.** Let \( X \) be a subshift and let \( \xi \in \Omega_X \). Given \( g \in F \), and \( \beta \in L_X \), the following are equivalent:

(i) both \( g \) and \( g\beta \) lie in \( \xi \),

(ii) \( \xi \in \Omega_g \), and \( \vartheta_{g^{-1}}(\xi) \in \Omega_\beta \).

**Proof.** Notice that \( g \in \xi \), if and only if \( \xi \in \Omega_g \), by (8.4.i), and in this case

\[
g\beta \in \xi \iff \beta \in g^{-1}\xi = \vartheta_{g^{-1}}(\xi) \iff \vartheta_{g^{-1}}(\xi) \in \Omega_\beta.
\]

Recall that \( \vartheta \) is minimal if and only if, for every \( \xi \) in \( \Omega_X \), and for every nonempty open set \( U \subseteq \Omega_X \), one has that

\[
\text{Orb}(\xi) \cap U \neq \emptyset.
\]

Expressing cofinality in terms of (13.6.iii) may thus be interpreted as a weak form of minimality in the sense that the above holds when \( \xi \) has the form \( \xi_x \), and \( U \) has the form \( \Omega_\beta \). One therefore has a clear perspective that cofinality is a weaker property than minimality, and in fact strictly weaker, as the example of the even shift shows.

If we are to find a set of properties related to cofinality, characterizing minimality for the spectral partial action, we must therefore significantly strengthen the notion of cofinality. We will in fact do this in two directions, introducing the notions of collective cofinality and strong cofinality. The first will lead to Theorem (13.12), ensuring that (13.8) holds for an arbitrary \( U \), but still under the restriction that \( \xi = \xi_x \), while the second will show up in Theorem (13.16), yielding (13.8) for an arbitrary \( \xi \), but under the restriction that \( U = \Omega_\beta \). Together these two will eventually prove \( \vartheta \) to be minimal.

**13.9. Definition.** Let \( \Lambda \) be a finite alphabet and let \( X \subseteq \Lambda^N \) be a subshift. Given a finite set of finite words \( B \subseteq L_X \), and an infinite word \( x \in X \), we shall say that \( x \) may be collectively reached from \( B \), when there exists a finite word \( \gamma \), and a prefix \( \alpha \) of \( x \), such that, upon writing \( x = \alpha y \), one has that \( \beta\gamma y \in X \), for all \( \beta \in B \).
It should be stressed that the bridge $\gamma$ above is supposed to be the same for all $\beta$. In fact an infinite word $x$ may be reached individually from both $\beta_1$ and $\beta_2$, but not collectively by the set $\{\beta_1, \beta_2\}$, as is the case of the even shift with

$$\beta_1 = 01, \quad \beta_2 = 011, \quad \text{and} \quad x = 00000\ldots$$

In case a word $x$ is collectively reached from a subset $B \subseteq L_X$, as in the diagram above, notice that $\gamma y$ lies in the follower set $F_\beta$, for all $\beta$ in $B$, and in particular the intersection of the $F_\beta$ is nonempty. This motivates an extension of the notion of follower sets:

13.10. **Definition.** Given a finite subset $B \subseteq L_X$, we will say that the follower set of $B$ is the set $F_B$ defined by

$$F_B = \bigcap_{\beta \in B} F_\beta.$$  

13.11. **Definition.** We will say that a subshift $X$ is collectively cofinal if, for every finite set $B \subseteq L_X$, with $F_B$ nonempty, and every $x$ in $X$, one has that $x$ may be collectively reached from $B$.

The first relationship between collective cofinality and minimality is as follows:

13.12. **Theorem.** Let $X$ be a subshift. Then the following are equivalent:

(i) for every $x$ in $X$, the orbit of $\xi_x$ is dense in $\Omega_X$,

(ii) $X$ is collectively cofinal.

**Proof.** (i) $\Rightarrow$ (ii). Given any finite subset $B \subseteq L_X$, such that $F_B$ is nonempty, say $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$, we claim that

$$V_{\emptyset; \beta_1, \beta_2, \ldots, \beta_n} \neq \emptyset.$$  

To see this, pick $y$ in $F_B$, and notice that $\beta^{-1} \in \xi_y$, for every $\beta$ in $B$, by (7.12), whence $\xi_y$ is an element in the above set, proving it to be nonempty.
Having already chosen $B$, let us also choose any $x$ in $X$, and our task is to show that $x$ may be collectively reached from $B$. By hypothesis the orbit of $\xi_x$ is dense in $\Omega_X$, so there exists $g$ in $\mathbb{F}$ such that $\xi_x \in \Omega_{g^{-1}}$, and
\[
\vartheta_g(\xi_x) \in V_{\emptyset; \beta_1, \beta_2, \ldots, \beta_n}.
\] (13.12.1)
It follows that $g^{-1} \in \xi_x$, by (8.4.i), whence we may use (7.11) to write $g^{-1} = \alpha \gamma^{-1}$, in reduced form, with $\alpha, \gamma \in \mathbb{F}_+$, and moreover write $x = \alpha y$, where $y \in F_\alpha \cap F_\gamma$. Therefore
\[
\vartheta_g(\xi_x) = \xi_{\theta_g(x)} = \xi_{\gamma y}.
\]
It then follows from (13.12.1) that $\beta_i^{-1} \in \xi_{\gamma y}$, for every $i$, so $\gamma y \in F_{\beta_i}$, by (7.12), which is the same as saying that $\beta_i \gamma y \in X$. This shows that $x$ may be collectively reached from $B$, and hence proves (ii).

(ii) $\Rightarrow$ (i). Given any $x$ in $X$, we must show that the orbit of $\xi_x$ is dense. Since $\Xi(X)$ is already known to be dense in $\Omega_X$, it is enough to prove that
\[
\xi_x \in \overline{\text{Orb}(\xi)}, \quad \forall z \in X.
\]

Given any $z$ in $X$, let $U$ be an arbitrary neighborhood of $\xi_z$. By (11.3) there are finite words $\alpha, \beta_1, \beta_2, \ldots, \beta_n$, such that
\[
\xi_z \in V_{\alpha; \beta_1, \beta_2, \ldots, \beta_n} \subseteq U,
\] (13.12.2)
and all we need to do is prove that
\[
\text{Orb}(\xi_x) \cap V_{\alpha; \beta_1, \beta_2, \ldots, \beta_n} \neq \emptyset.
\] (13.12.3)

Observing that $V_{\alpha; \beta_1, \beta_2, \ldots, \beta_n} \subseteq \Omega_\alpha$, and that
\[
\vartheta_{\alpha^{-1}}(V_{\alpha; \beta_1, \beta_2, \ldots, \beta_n}) = V_{\emptyset; \alpha, \beta_1, \beta_2, \ldots, \beta_n},
\]
notice that any orbit intersecting $V_{\emptyset; \alpha, \beta_1, \beta_2, \ldots, \beta_n}$ will also intersect $V_{\alpha; \beta_1, \beta_2, \ldots, \beta_n}$. So we may assume without loss of generality that $\alpha = \emptyset$.

Letting $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$, we claim that $F_B$ is nonempty. In fact, by (13.12.2) we have that $\beta_i^{-1} \in \xi_z$, for all $i$, so $z \in F_{\beta_i}$ by (7.12), whence also $z \in F_B$, proving our claim.

By hypothesis $x$ may be collectively reached from $B$, so there are finite words $\mu$ and $\gamma$, such that $x = \mu y$, and $\beta \gamma y \in X$, for all $\beta$ in $B$. Noticing that for all such $\beta$,
\[
\sigma_\mu(\xi_x) = y \in F_{\beta \gamma},
\]
we have that
\[
\mu(\beta \gamma)^{-1} \in \xi_x, \quad \text{(13.12.4)}
\]
by (8.8). The fact that $\beta \gamma y \in X$ implies that $\gamma y \in X$ as well, so a similar argument gives
\[
\mu(\alpha)^{-1} \in \xi_x, \quad \text{whence } \xi_x \in \Omega_{\mu^{-1}},
\]
and
\[
\vartheta_{\gamma \mu^{-1}}(\xi_x) = \gamma \mu^{-1} \xi_x \quad \text{by (13.12.4)} \beta^{-1}.
\]
Consequently $\vartheta_{\gamma \mu^{-1}}(\xi_x) \in V_{\emptyset; \beta_1, \beta_2, \ldots, \beta_n}$, proving (13.12.3). □
The reader is invited to check that the only reason why (13.12.i) must be stated just for \( \xi_x \), rather than for a general \( \xi \) in \( \Omega_X \), is that the above proof has used the crucial implication “(b) \( \Rightarrow \) (a)” of (8.8), which does not hold in general.

In the presence of collective cofinality, topological freeness for the spectral partial action may be characterized in a very simple way:

13.13. **Proposition.** Let \( X \) be a subshift, and consider the following statements:

(a) The spectral partial action associated to \( X \) is topologically free,

(b) \( X \) has at least one point which is not eventually periodic (by eventually periodic we mean a word of the form \( \alpha \gamma^{\infty} \)).

Then (a) implies (b). If \( X \) is collectively cofinal, then also (b) implies (a).

**Proof.** (a) \( \Rightarrow \) (b). It is easy to see that there is only a countable number of eventually periodic infinite words (even within the full shift \( \Lambda^\infty \)). Assuming by contradiction that every \( x \) in \( X \) is eventually periodic, one then has that \( X \) is countable, and a standard application of Baire’s category Theorem implies that \( X \) has at least one isolated point, say \( x \). Since \( x \) is eventually periodic let us write \( x = \alpha \gamma^{\infty} \), for some circuit \( \gamma \).

To say that \( x \) is isolated (relative to the topology of \( X \)), is to say that \( \{x\} \) coincides with some cylinder \( Z_\mu \), where \( \mu \) is necessarily a prefix of \( x = \alpha \gamma^{\infty} \). By increasing the size of \( \mu \) up to \( |\alpha| \) plus some multiple of \( |\gamma| \), we may suppose that \( \mu = \alpha \gamma^n \), for some integer \( n \).

Thus the only infinite word in \( X \) that one may produce by starting with \( \alpha \gamma^n \) is \( \alpha \gamma^{\infty} \), which may also be expressed by saying that \( F_{\alpha \gamma^n} = \{\gamma^{\infty}\} \). This goes against (12.6.ii), and hence also against (a), thus bringing about a contradiction, proving that \( X \) must have some non eventually periodic point.

(b) \( \Rightarrow \) (a). We will prove (a) by verifying (12.6.ii). So let \( \nu \) be a circuit and let \( B \) be a finite set of finite words such that \( \nu^{\infty} \in F_B \). Our task is then to find some element in \( F_B \) other than \( \nu^{\infty} \).

For this, pick any non eventually periodic element \( x \) of \( X \) and, using that \( X \) is collectively cofinal, choose a prefix \( \alpha \) of \( x \), and a finite word \( \gamma \), such that, upon writing \( x = \alpha y \), one has that \( \beta \gamma y \in X \), for all \( \beta \) in \( B \). This implies that \( \gamma y \) lies in \( F_B \), and since it is clearly not eventually periodic, one necessarily has that \( \gamma y \neq \nu^{\infty} \), thus verifying (12.6.ii). \( \square \)

Having understood the dynamical meaning of collective cofinality in (13.12), let us now discuss yet another version of cofinality. In order to motivate this notion let us refer back to diagram (13.3). One might see the question of reaching \( x \) from \( \beta \) as an attempt to bringing \( x \) into the follower set of \( \beta \) by deleting a few letters from the beginning of \( x \), namely the prefix \( \alpha \), and then inserting new letters in its place, namely \( \gamma \), after which the resulting word \( \gamma y \) is supposed to lie in \( F_\beta \). If we have to pay a price for each deleted, as well as for each inserted letter, then the cheapest situation is obviously when \( x \) itself lies in the follower set of \( \beta \), namely when \( \alpha = \gamma = \emptyset \) are enough to do the job. Beyond this ideal situation we have:

13.14. **Definition.** Let \( \beta \in L_X \), and let \( x \in X \). We shall say that the **cost** of reaching \( x \) from \( \beta \) is the minimum value of \( |\alpha| + |\gamma| \), where \( \alpha \) and \( \gamma \) are as in (13.2). In symbols,

\[
\text{Cost}(\beta, x) = \min\{|\alpha| + |\gamma| : x = \alpha y, \beta \gamma y \in X\}.
\]

If no such \( \alpha \) and \( \gamma \) exist, we shall say that the cost is infinite.
In a cofinal subshift, given $\beta$ in $\mathcal{L}_X$, one may reach any $x$ in $X$, but perhaps at an increasingly high cost. This motivates the following:

**13.15. Definition.** Let $X$ be a subshift. We shall say that $X$ is *strongly cofinal* if

$$\sup_{x \in X} \text{Cost}(\beta, x) < \infty,$$

for every $\beta$ in $\mathcal{L}_X$.

Let us pause for a moment to give an example of a cofinal subshift which is not strongly cofinal. Our staple counter-example, namely the even shift, will not serve us now since it is both cofinal and strongly cofinal. In fact, as we already mentioned in (13.4), when trying to reach an infinite word $x$ from a finite word $\beta$, it suffices to take $\alpha = \emptyset$, and either $\gamma = \emptyset$ or $\gamma = '1'$, which means that Cost($\beta, x$) is at most 1.

However, a subshift based on a similar principle as the even shift will provide the counter-example sought. Consider the alphabet $\Lambda = \{0, 1\}$, and let us take the following set $F$ of forbidden words:

$$F = \{01^n0 : n \text{ is not a power of 2}\}.$$

If $X = X_F$ is the corresponding subshift, then an infinite word $x$ lies in $X$ if and only if, anytime a contiguous block of 1’s occurring in $x$ is delimited by 0’s, the amount of said 1’s is a power of 2.

We leave it for the reader to check that $X$ is a duly cofinal subshift, but that reaching the infinite word $x = 1^n0^\infty$ from the finite word $\beta = '0'$ involves unbounded costs. Hence $X$ is not strongly cofinal.

Exploring the relationship between strong cofinality and the dynamical properties of the spectral partial action is our next goal.

**13.16. Theorem.** Let $X$ be a subshift. Then the following are equivalent:

(i) For every $\beta$ in $\mathcal{L}_X$, and for every $\xi$ in $\Omega_X$, one has that $\text{Orb}(\xi) \cap \Omega_\beta$ is nonempty,

(ii) $X$ is strongly cofinal.

*Proof. (i) $\Rightarrow$ (ii).* We will in fact prove that the negation of (ii) implies the negation of (i). We therefore suppose that

$$\sup_{x \in X} \text{Cost}(\beta, x) = \infty,$$

for some $\beta$ in $\mathcal{L}_X$. Then, for every natural number $n$, there is some $z_n$ in $X$, with $\text{Cost}(\beta, z_n) \geq n$. By compactness we may then find a subsequence $x_k = z_{n_k}$, such that $\{\xi_{x_k}\}_k$ converges to some $\xi$ in $\Omega_X$. We will accomplish our task by proving that

$$\text{Orb}(\xi) \cap \Omega_\beta = \emptyset.$$  \hspace{1cm} (13.16.1)

Arguing by contradiction, suppose that there exists some $h$ in $F$ such that $\xi \in \Omega_{h^{-1}}$, and $\vartheta_h(\xi) \in \Omega_\beta$. By (13.7) we deduce that both $h^{-1}$ and $h^{-1}\beta$ lie in $\xi$. In order to simplify our notation, we shall make the change of variables $g = h^{-1}\beta$, whence

$$g, \ g\beta^{-1} \in \xi.$$
Using (7.8.v) we may write \( g = \alpha \gamma^{-1} \) in reduced form, so that \( \alpha \in \xi \), by (7.8.ii). Employing (7.16.1) we then have that

\[
\alpha, \ g, \ g\beta^{-1} \in \xi_{x_k},
\]

for all large enough \( k \). Focusing on the fact that

\[
\alpha \gamma^{-1} = g \in \xi_{x_k},
\]

if follows from (7.11) that \( x_k = \alpha y_k \), where \( y_k \in F_\alpha \cap F_\gamma \). Observing that the stem of \( \xi_{x_k} \) at \( g \) is \( \gamma y_k \), and that \( g\beta^{-1} \in \xi_{x_k} \), we conclude from (8.8) that \( \beta \gamma y_k \in X \). This implies not only that \( x_k \) may be reached from \( \beta \), but also that

\[
\text{Cost}(\beta, x_k) \leq |\alpha| + |\gamma|.
\]

Since neither \( \alpha \) nor \( \gamma \) depend on \( k \), we arrive at a contradiction with the fact that \( \lim_{k \to \infty} \text{Cost}(\beta, x_k) = \infty \). This proves (13.16.1), as desired.

(ii) \( \Rightarrow \) (i). Given \( \beta \) in \( L_X \), and \( \xi \) in \( \Omega_X \), we must prove that \( \text{Orb}(\xi) \cap \Omega_\beta \neq \emptyset \). Writing \( \xi = \lim_n \xi_{x_n} \), with \( x_n \in X \), by hypothesis we may reach every \( x_n \) from \( \beta \) with bounded cost, meaning that for every \( n \), there are finite words \( \alpha_n \) and \( \gamma_n \), such that \( x_n = \alpha_n y_n \), and \( \beta \gamma_n y_n \in X \), and moreover \( |\alpha_n| + |\gamma_n| \leq M \), where \( M \) is a fixed constant.

As we are working with a finite alphabet, there are finitely many words of any given length. Therefore the \( \alpha_n \) and \( \gamma_n \) just obtained must necessarily repeat infinitely often. We may then choose a subsequence \( \{x_{n_k}\}_k \), such that \( \alpha_{n_k} = \alpha \), and \( \gamma_{n_k} = \gamma \), for all \( k \). Therefore \( x_{n_k} = \alpha y_{n_k} \), and \( \beta \gamma y_{n_k} \in X \). We then have that

\[
\alpha \in \xi_{x_{n_k}}, \quad \text{and} \quad \sigma_\alpha(\xi_{x_{n_k}}) = y_{n_k} \in F_\beta \gamma,
\]

whence \( \alpha(\beta \gamma)^{-1} \in \xi_{x_{n_k}} \) by (8.8). A similar reasoning, based on the fact that \( y_{n_k} \) also lies in \( F_\gamma \), gives \( \alpha \gamma^{-1} \in \xi_{x_{n_k}} \), so by the continuity of Boolean values (7.16.1), we have that

\[
\alpha \gamma^{-1} \in \xi, \quad \text{and} \quad \alpha(\beta \gamma)^{-1} \in \xi.
\]

Setting \( g = \alpha(\beta \gamma)^{-1} \), we have that

\[
g \beta = \alpha \gamma^{-1} \beta^{-1} \beta = \alpha \gamma^{-1} \in \xi,
\]

whence \( \vartheta_g^{-1}(\xi) \in \Omega_\beta \) by (13.7). This shows that the orbit of \( \xi \) intersects \( \Omega_\beta \), concluding the proof. \( \square \)

From (13.12) and (13.16) it is clear that a subshift whose associated spectral partial action is minimal must necessarily be both collectively cofinal and strongly cofinal. Our next major goal will be to prove that these two properties in turn characterize minimality for the spectral partial action. However there is an important technical tool we still need to develop before proving this main result.
13.17. Lemma. Let $X$ be a collectively cofinal subshift. Then for every finite set $B \subseteq \mathcal{L}_X$, with $F_B$ nonempty, there are $\mu$ and $\nu$ in $\mathcal{L}_X$, such that

$$F_\mu \subseteq F_{B\nu},$$

where $B\nu$ evidently means the set $\{\beta\nu : \beta \in B\}$.

Proof. Let $B$ be as in the statement. By hypothesis any $x$ in $X$ may be collectively reached from $B$, so we may pick $\alpha$, $\gamma$ and $y$, as in (13.9), so that $y \in F_\alpha \cap F_{B\gamma}$, and $x = \theta_\alpha(y)$, whence $x \in \theta_\alpha(F_\alpha \cap F_{B\gamma})$. This implies that

$$X = \bigcup_{\alpha, \gamma \in \Lambda^*} \theta_\alpha(F_\alpha \cap F_{B\gamma}).$$

Observing that $F_\alpha \cap F_{B\gamma}$ is compact, and hence that the sets in the above union are closed in $X$, we may employ Baire's category Theorem producing $\alpha$ and $\gamma$ in $\Lambda^*$ such that $\theta_\alpha(F_\alpha \cap F_{B\gamma})$ has a nonempty interior. En passant we stress that both $\alpha$ and $\gamma$ must lie in $\mathcal{L}_X$, or else $F_\alpha \cap F_{B\gamma}$ is the empty set. Therefore, by the definition of the product topology on $X$, there exists some $\mu$ in $\mathcal{L}_X$ such that

$$Z_\mu \subseteq \theta_\alpha(F_\alpha \cap F_{B\gamma}).$$

Assuming without loss of generality that $|\mu| > |\alpha|$, and noticing that the range of $\theta_\alpha$ is contained in $Z_\alpha$, we have that $Z_\mu \subseteq Z_\alpha$, so $\alpha$ must be a prefix of $\mu$, and we may then write $\mu = \alpha\delta$, for some finite word $\delta \in \mathcal{L}_X$. We will now conclude the proof by showing that the inclusion in the statement holds once we choose $\nu = \gamma\delta$.

To prove this, let $y \in F_\mu$, so

$$x := \mu y \in Z_\mu \subseteq \theta_\alpha(F_\alpha \cap F_{B\gamma}),$$

and then we may write $x = \alpha z$, for some $z \in F_\alpha \cap F_{B\gamma}$. Since

$$\alpha\delta y = \mu y = x = \alpha z,$$

it follows that $\delta y = z$, so for any $\beta$ in $B$, we have that

$$\beta\nu y = \beta\gamma\delta y = \beta\gamma z \in X.$$  

This shows that $y \in F_{B\nu}$, as desired, concluding the proof. $\square$

Let us now give a dynamical interpretation of the conclusion of the above result.

13.18. Proposition. Let $B = \{\beta_1, \beta_2, \ldots, \beta_n\} \subseteq \mathcal{L}_X$, be a nonempty finite set and let $\mu, \nu \in \mathcal{L}_X$, be such that $F_\mu \subseteq F_{B\nu}$, precisely as in the conclusion of (13.17). Then $\Omega_{\mu^{-1}}$ is contained in the domain of $\theta_\nu$ (also known as $\Omega_{\nu^{-1}}$), and

$$\theta_\nu(\Omega_{\mu^{-1}}) \subseteq V_{\emptyset; \beta_1, \beta_2, \ldots, \beta_n}.$$
Proof. We will first prove that

\[ \Xi(X) \cap \Omega_{\mu}^{-1} \subseteq \Omega_{\nu}^{-1}. \tag{13.18.1} \]

We thus pick any \( x \) in \( X \) such that \( \xi_x \in \Omega_{\mu}^{-1} \). Notice that

\[ \xi_x \in \Omega_{\mu}^{-1} \quad \xRightarrow{(8.4.i)} \quad \mu^{-1} \in \xi_x \quad \xRightarrow{(7.12)} \quad \xi_{\mu^{-1}} \in \xi_x \quad \xRightarrow{(7.12)} \quad x \in F_{\mu}. \]

So \( x \in F_{\mu} \), and by hypothesis we then have that \( x \in F_{\nu} \). Since \( B \) is nonempty, we may pick any \( \beta \) in \( B \), and so deduce that \( x \in F_{\beta \nu} \), whence \( \beta \nu x \in X \). This implies that \( x \in F_{\nu} \), and a reasoning similar to the above shows that \( \xi_x \in \Omega_{\mu}^{-1} \), proving (13.18.1). A trivial exercise in Topology, using that \( \Xi(X) \) is dense, \( \Omega_{\mu}^{-1} \) is open, and \( \Omega_{\nu}^{-1} \) is closed, now shows that \( \Omega_{\mu}^{-1} \subseteq \Omega_{\nu}^{-1} \), as required.

We will next show that \( \vartheta_{\nu}(\Xi(X) \cap \Omega_{\mu}^{-1}) \subseteq V_{\emptyset; \beta_1, \beta_2, \ldots, \beta_n}. \tag{13.18.2} \)

We thus again pick any \( x \) in \( X \) such that \( \xi_x \in \Omega_{\mu}^{-1} \). Then

\[ \vartheta_{\nu}(\xi_x) = \xi_{\vartheta_{\nu}(x)} = \xi_{\nu x}. \]

In addition, as seen above, \( x \in F_{B \nu} \), so \( \beta \nu x \in X \), for every \( \beta \) in \( B \), and we see that \( \nu x \in F_{\beta} \). We then have by (7.12) that \( \beta^{-1} \in \xi_{\nu x} \), whence \( \xi_{\nu x} \in V_{\emptyset; \beta_1, \beta_2, \ldots, \beta_n} \), proving (13.18.2). The same “dense plus open plus closed” argument above, but now also using that \( \vartheta_{\nu} \) is continuous, leads one from (13.18.2) to the last conclusion in the statement. \( \square \)

We have now come to the main result of this section:

13.19. Theorem. Let \( \Lambda \) be a finite alphabet and let \( X \subseteq \Lambda^\mathbb{N} \) be a subshift. Also let \( \vartheta \) be the spectral partial action of the free group \( \mathbb{F}(\Lambda) \) on \( \Omega_X \) introduced in (8.3). Then a necessary and sufficient condition for \( \vartheta \) to be minimal is that \( X \) be both collectively cofinal and strongly cofinal.

Proof. As already observed, the necessity of the above conditions follows immediately from (13.12) and (13.16).

Conversely, suppose that \( X \) is both collectively cofinal and strongly cofinal. In order to prove that \( \vartheta \) is minimal, given any \( \xi \) in \( \Omega_X \), we must show that the orbit of \( \xi \) is dense. Arguing as in the proof of “(ii) \Rightarrow (i)” in (13.12), it suffices to show that if \( B = \{ \beta_1, \beta_2, \ldots, \beta_n \} \) is a subset of \( \mathcal{L}_X \) such that \( F_B \) is nonempty, then

\[ \text{Orb}(\xi) \cap V_{\emptyset; \beta_1, \beta_2, \ldots, \beta_n} \neq \emptyset. \]

Using (13.17) and (13.18), pick \( \mu \) and \( \nu \) in \( \mathcal{L}_X \), such that

\[ \vartheta_{\nu}(\Omega_{\mu}^{-1}) \subseteq V_{\emptyset; \beta_1, \beta_2, \ldots, \beta_n}. \]

Thus, to show that the orbit of \( \xi \) intersects \( V_{\emptyset; \beta_1, \beta_2, \ldots, \beta_n} \), it is enough to show that it intersects \( \Omega_{\mu}^{-1} \), or even \( \Omega_{\mu} \), since \( \vartheta_{\mu^{-1}}(\Omega_{\mu}) = \Omega_{\mu}^{-1} \).

In order to prove the latter fact, that is, that \( \text{Orb}(\xi) \cap \Omega_{\mu} \) is nonempty, we just have to note that it follows immediately from strong cofinality and (13.16). \( \square \)
Of course one could put together the notions of collective cofinality and strong cofinality, to form a new high powered notion, as follows:

13.20. Definition. Let $X$ be a subshift.

(a) Given a finite subset $B$ of $L_X$, we shall say that the cost of reaching a given $x$ in $X$ from $B$ is the minimum value of $|\alpha| + |\gamma|$, where $\alpha$ and $\gamma$ are as in (13.9). In symbols,

$$\text{Cost}(B, x) = \min\{|\alpha| + |\gamma| : x = \alpha y, \beta \gamma y \in X, \text{ for all } \beta \in B\}.$$ 

If no such $\alpha$ and $\gamma$ exist, we shall say that the cost if infinite.

(b) We shall say that $X$ is \textit{hyper cofinal} if

$$\sup_{x \in X} \text{Cost}(B, x) < \infty,$$

for every finite subset $B$ of $L_X$ such that $F_B$ is nonempty.

It is evident that hyper cofinality implies both collective cofinality and strong cofinality, but the reverse implication is not so obvious to see. Nevertheless it is true, as we shall now prove.

13.21. Proposition. Let $X$ be a subshift. Then the following are equivalent:

(i) $X$ is is both collectively cofinal and strongly cofinal,

(ii) $X$ is hyper cofinal,

(iii) the spectral partial action associated to $X$ is minimal.

Proof. The equivalence of (i) and (iii), repeated here just for emphasis, is precisely the content of (13.19). As we have already seen (ii) implies (i) for obvious reasons, so it suffices to show that (iii) implies (ii).

So, assuming that $\vartheta$ is minimal, let $B = \{\beta_1, \beta_2, \ldots, \beta_n\} \subseteq L_X$ be such that $F_B$ is nonempty. Consider the open subset

$$V := V\emptyset; \beta_1, \beta_2, \ldots, \beta_n \subseteq \Omega_X,$$

which is nonempty by the short argument used in the beginning of the proof of (13.12). By minimality, for every $\xi$ in $\Omega_X$, one has that $\text{Orb}(\xi)$ is dense, so it must intersect $V$, meaning that there exists some $g$ in $\mathbb{F}$, such that $\xi \in \Omega_{g^{-1}}$, and $\vartheta_g(\xi) \in V$. It follows that $\xi \in \vartheta_{g^{-1}}(V \cap \Omega_g)$, so we conclude that

$$\Omega_X = \bigcup_{g \in \mathbb{F}} \vartheta_{g^{-1}}(V \cap \Omega_g).$$

This is an open covering of the compact space $\Omega_X$, so there exists a finite collection $g_1, g_2, \ldots, g_m$ of elements in $\mathbb{F}$ such that

$$\Omega_X = \bigcup_{i=1}^m \vartheta_{g_i^{-1}}(V \cap \Omega_{g_i}).$$
Given any \( x \) in \( X \), we may then find some \( i \leq m \), such that \( \xi_x \in \vartheta_{g_i^{-1}}(V \cap \Omega_{g_i}) \), so in particular \( \xi_x \in \Omega_{g_i} \) and \( \vartheta_{g_i}(\xi_x) \in V \). Writing \( g_i = \gamma_i\alpha_i^{-1} \), in reduced form, we then have that \( \alpha_i \) is a prefix of \( x \) and, upon writing \( x = \alpha_i y \), one has that

\[
\vartheta_{g_i}(\xi_x) = \xi_{\vartheta_{g_i}(x)} = \xi_{\gamma_i y} \in V = V_{\varnothing, \beta_1, \beta_2, \ldots, \beta_n}.
\]

Consequently \( \beta_j^{-1} \in \xi_{\gamma_i y} \), for all \( j \leq n \), whence \( \gamma_i y \in F_{\beta_j} \) by (7.12), which is to say that \( \beta_j \gamma_i y \) lies in \( X \). This shows that \( x \) may be reached from \( B \), and moreover that

\[
\text{Cost}(B, x) \leq |\alpha_j| + |\gamma_j|.
\]

Since the \( j \) above may take only finitely many values, we see that the cost of reaching any \( x \) from the given \( B \) is bounded, whence \( X \) is hyper cofinal. \( \square \)

Not all subshifts are surjective\(^5\), but when they are, there is another property (to be described in our next result) even stronger than hyper cofinality, which is still equivalent to minimality. It says that one may reach infinite words \textit{at their source} with bounded cost. This property is essentially the same as the condition appearing in [31: Theorem 4.20]. See section (14) for a more thorough discussion of this condition.

**13.22. Proposition.** Let \( X \) be a subshift and suppose that the shift map \( S : X \to X \) is surjective. Then the spectral partial action associated to \( X \) is minimal if and only if, for every finite subset \( B \subseteq \Lambda^* \), with \( F_B \) nonempty, there exists a constant \( M > 0 \), such that, for every \( x \) in \( X \), one may find \( \gamma \in \Lambda^* \) with \( |\gamma| \leq M \), and such that \( \beta \gamma x \in X \), for all \( \beta \in B \). In other words,

\[
\sup_{x \in X} \min\{ |\gamma| : \beta \gamma x \in X, \text{ for all } \beta \in B \} < \infty.
\]

\(^5\) If a subshift is built from a two-sided shift, as in many papers on the subject (e.g. [9]), then the shift map \( S \) is surjective, but there are many non-surjective (one-sided) subshifts as well.
Proof. It is evident that the condition given implies hyper cofinality, and hence minimality by (13.21). Conversely, suppose that $X$ is minimal, hence also hyper cofinal by (13.21). Given any finite subset $B$ of $\mathcal{L}_X$, let

$$n = \sup_{x \in X} \text{Cost}(B, x).$$

Fixing $x$ in $X$, we may use the fact that $S$ is surjective to find $z$ in $X$ such that $S^n(z) = x$. It is then clear that $z$ is of the form $\mu x$, with $|\mu| = n$. Spelling out the fact that $z$ may be reached from $B$ with cost no more than $n$, there are $\alpha$ and $\gamma$ in $\Lambda^*$, such that $\alpha$ and $\gamma$ are prefixes of $z$, and their concatenation $\gamma \gamma y$ is in $X$, for every $\beta$ in $B$, and also $|\alpha| + |\gamma| \leq n$.

Of crucial importance is whether we have reached $z$ before or after the end of the prefix $\mu$. Observing that $|\alpha| \leq |\alpha| + |\gamma| \leq n = |\mu|$, the answer to the above dilemma is before! Since both $\alpha$ and $\mu$ are prefixes of $z$, we deduce that $\alpha$ is a prefix of $\mu$, and we may then write $\mu = \alpha \delta$, for some finite word $\delta$. It is then clear that $y = \delta x$, and

$$X \ni \beta \gamma y = \beta \gamma \delta x = \beta \gamma' x,$$

where $\gamma' = \gamma \delta$. Noticing that

$$|\gamma'| = |\gamma| + |\delta| \leq |\alpha| + |\gamma| + |\mu| \leq 2n,$$

the proof is concluded. $\square$
14. Applications.

In this short section we will draw a few conclusions about the Carlsen-Matsumoto C*-algebra which may be derived from the work we did so far. Some of these results are already known, but we may recover them easily given the many tools available to treat partial crossed product algebras.

14.1. Theorem. (cf. [10: Theorem 17]) For every subshift $X$, one has that $\mathcal{O}_X$ is a nuclear C*-algebra.

Proof. By (9.5) we have that $\mathcal{O}_X$ is the reduced crossed product relative to a partial dynamical system satisfying the approximation property. The result is then an immediate application of [16: 25.10]. □

The first description of $\mathcal{O}_X$ as a groupoid C*-algebra was given in [6]. We may recover it here due to its description as a partial crossed product:

14.2. Theorem. Let $\mathcal{G}_X$ be the transformation groupoid associated to the spectral partial action for a given subshift $X$. Then $\mathcal{G}_X$ is a second countable, Hausdorff, étale, amenable groupoid and

$$C^*(\mathcal{G}_X) \simeq \mathcal{O}_X.$$  

Proof. We refer the reader to [1: Section 2] for the construction of the transformation groupoid relative to a partial action. Since the acting group, namely $\mathbb{F}$, is discrete, it is clear that $\mathcal{G}_X$ is étale. It is also easy to see that $\mathcal{G}_X$ is Hausdorff and, based on the fact that $\Omega_X$ is metrizable and $\mathbb{F}$ is countable, $\mathcal{G}_X$ is seen to be second countable. That $C^*(\mathcal{G}_X)$ is isomorphic to the crossed product $C(\Omega_X) \rtimes_\theta \mathbb{F}$, and hence also to $\mathcal{O}_X$, follows from [1: Theorem 3.3]. Finally, since $\mathcal{O}_X$ is nuclear, we deduce from [5: Theorem 5.6.18] that $\mathcal{G}_X$ is amenable. □

Our next result involves a weaker alternative to the condition (I) introduced by Matsumoto in [25]. Under condition (I) the result below has essentially been proved by Matsumoto and Carlsen in [9: Lemma 2.3]. See also [10: Theorem 16].

14.3. Theorem. Let $X$ be a subshift. Then the following are equivalent:

(i) $X$ satisfies (12.6.ii), that is, for every $\beta_1, \beta_2, \ldots, \beta_n$ in $\Lambda^*$, and for every circuit $\gamma$ such that $\gamma^\infty \in \bigcap_{i=1}^n F_{\beta_i}$, one has that $\bigcap_{i=1}^n F_{\beta_i}$ contains some element other than $\gamma^\infty$,

(ii) the spectral partial action associated to $X$ is topologically free,

(iii) $\mathcal{G}_X$ is an essentially principal groupoid (see [29: Definition 3.1]),

(iv) every nontrivial closed two-sided ideal in $\mathcal{O}_X$ has a nontrivial intersection with $\mathcal{D}_X$,

(v) every *-homomorphism defined on $\mathcal{O}_X$ is injective, provided it is injective on $\mathcal{D}_X$.

In case the equivalent conditions above hold, then $\mathcal{O}_X$ is naturally isomorphic to $\mathcal{M}_X$.

Proof. The equivalence between (i) and (ii) is precisely the content of (12.6), while the equivalence between (ii) and (iii) is evident. That (ii) implies (iv) was proved in [18: Theorem 2.6] (see also [15: Theorem 4.4]), for the reduced crossed product, which is isomorphic to $\mathcal{O}_X$ by (9.5). A standard argument equating an ideal with the kernel of a *-homomorphism proves that (iv) and (v) are equivalent.
To close the cycle it suffices to prove that (iv) implies (iii), and this follows from [4: Proposition 5.5].

Addressing the last sentence in the statement, it is easy to see that the pair \((i, u)\) is a covariant representation of the spectral partial action in \(\mathcal{M}_X\), where \(i\) denotes the inclusion of \(C(\Omega_X) = \mathcal{D}_X\) in \(\mathcal{M}_X\). By the universal property of the crossed product [16: 13.1] there is an obviously surjective *-homomorphism

\[ \varphi : C(\Omega_X) \rtimes \theta \mathbb{F} \to \mathcal{M}_X, \]

extending the above covariant pair. Since \(\varphi\) restricts to the identity map on \(C(\Omega_X)\), it is injective there. The kernel of \(\varphi\) therefore has trivial intersection with \(C(\Omega_X) = \mathcal{D}_X\), whence this kernel itself is trivial by (iv). Therefore \(\varphi\) is one-to-one and the result follows from the identification between the crossed product and \(\mathcal{O}_X\) given by (9.5). □

It is curious to compare (12.6.ii) with condition (I) from [25] already referred to. Observe that the former is essentially saying that there is no \(l\)-past equivalence class containing a single periodic point, while the latter says that there is no such class containing a single point (periodic or not). Seen from this point of view, one realizes that condition (I) is stronger than (12.6.ii), and therefore (14.3) is a stronger result than [10: Theorem 16], say. The following example shows that there is a real difference between the above conditions.

**14.4. Example.** There exists a subshift satisfying condition (12.6.ii) but not condition (I).

**Proof.** Let \(\Sigma = \{1, 2\}\) and let

\[ Y = \Sigma^N = \{1, 2\}^N \]

be the full shift. Choose any non-periodic point \(z\) in \(Y\), and let us construct another subshift on the new alphabet \(\Lambda = \{0, 1, 2\}\). For this we let \(\mathcal{F}\) be the following set of forbidden words:

\[ \mathcal{F} = \{a0 : a \in \Lambda\} \cup \{0\alpha : \alpha \in Y, \ \alpha \ \text{is not a prefix of} \ z\}, \]

and we claim that the subshift

\[ X = Y_\mathcal{F} \]

satisfies the requirements in the statement.

Since all forbidden words have a ‘0’ somewhere, any infinite word not involving ‘0’ is allowed, meaning that \(Y \subseteq X\). Another interesting element of \(X\) is the word \(0z\), which narrowly escapes being ruled out! Other than that, there is nothing else in \(X\), meaning that

\[ X = Y \cup \{0z\}, \]

as the reader may easily verify. Denoting by

\[ \Lambda_l(x) = \{\alpha \in \Lambda^* : |\alpha| = l, \ \alpha x \in X\}, \ \forall x \in X, \ \forall l \in \mathbb{N}, \]

...
notice that
\[ \Lambda_1(z) = \{0, 1, 2\}, \quad \text{and} \]
\[ \Lambda_1(x) = \{1, 2\}, \quad \forall x \neq z. \]
So the 1-\textit{past equivalence class} of \( z \), in the sense of [25], consists only of \( z \), and we thus see that \( X \) does not satisfy condition (I).

In order to prove that \( X \) satisfies condition (12.6.ii), let \( \gamma \) be a circuit in \( X \), and suppose that \( \gamma^\infty \) lies in \( F_B \), for some finite set \( B \) of finite words. Since we have chosen \( z \) not periodic, we necessarily have that \( z \neq \gamma^\infty \).

Given \( \beta \) in \( B \) we then have that \( \beta \gamma^\infty \) is in \( X \), so \( \beta \) cannot involve ‘0’. It then follows that \( x \in F_\beta \), for every \( x \) in \( Y \), whence \( Y \subseteq F_B \). So there are plenty of elements in \( F_B \) other than \( \gamma^\infty \) to choose from, proving that \( X \) satisfies condition (12.6.ii).

\[ \square \]

Our next Theorem is related to various known simplicity results for \( C^* \)-algebras of subshifts.

14.5. \textbf{Theorem.} Let \( X \) be a subshift. Then the following conditions are equivalent:

(i) \( \mathcal{O}_X \) is simple,

(ii) \( X \) is both collectively cofinal and strongly cofinal, and it satisfies condition (12.6.ii),

(iii) \( X \) is hyper cofinal and contains at least one non eventually periodic point.

\textit{Proof.} (i) \( \Rightarrow \) (ii). By (14.2) and [4: Theorem 5.1] one has that \( \mathcal{G}_X \) is an essentially principal and minimal groupoid. It then follows that the spectral partial action is topologically free and minimal, so the conclusion follows from (12.6) and (13.19).

(ii) \( \Rightarrow \) (i). The spectral partial action is topologically free by (12.6), and minimal by (13.19). Therefore the reduced crossed product \( D_X \rtimes^{\text{red}} \mathbb{F} \) is simple by [18: Corollary 2.9], and hence so is \( \mathcal{O}_X \) by (9.5).

(ii) \( \Leftrightarrow \) (iii) We have seen in (13.21) that \( X \) is both collectively cofinal and strongly cofinal if and only if it is hyper cofinal. In this case the existence of a non eventually periodic point is equivalent to topological freeness of the spectral partial action by (13.13), hence also to condition (12.6.ii).

Of course the last part of (14.5.ii) could be interchanged with the last part of (14.5.iii), the above choice happening to be purely personal.

Similar simplicity results for \( C^* \)-algebras associated to subshifts are to be found in many works in the literature, such as [25: Corollary 6.11], [9: Proposition 2.6], [2: Proposition 4.3], and [31: Theorem 4.20].

We nevertheless feel that the result above is worth the trouble of proving it, mainly because it gives necessary and sufficient conditions for simplicity. These conditions also have a distinctively geometrical flavor at the same time that they are closer to the cofinality hypothesis of classical results about graph algebras, especially if compared with the intricate condition of “irreducibility in l-past equivalence” present in [25] and [9].

Regarding [2: Proposition 4.3], one should be aware that it only holds for subshifts of finite type because otherwise the shift map is not open, as we have seen in (2.5). In particular the statement that \( \mathcal{O}_X \) is simple when \( X \) is the even shift, given in the second
paragraph of page 222, is not correct. As we have seen in (13.5), the spectral partial action for the even shift is not minimal, so it fails to be hyper cofinal by (13.21), whence $O_X$ is not simple by (14.5).

Among the references cited above, the only one providing for necessary and sufficient conditions is [31: Theorem 4.20], which however only applies to surjective subshifts. The condition given there is essentially the same as the one described in (13.22), hence equivalent to minimality in the surjective case, but it is much too strong in general, even for Markov subshifts. For example, the Markov subshift on the alphabet $\Lambda = \{1, 2, 3\}$, corresponding to the graph

![Graph](image)

(recall that our subshifts consist of infinite paths formed by vertices, as opposed to edges) is non-surjective: since ‘1’ is a source, any infinite word beginning with ‘1’ is not in the range of $S$. The associated Carlsen-Matsumoto algebra (which in this case is well known to be the Cuntz-Krieger algebra for the adjacency matrix of the above graph) is simple by (14.5), as the reader may easily verify. However its associated subshift does not satisfy Thomsen’s condition, namely the condition given in (13.22), e.g. for

$$B = \{\text{‘2’}\}, \quad \text{and} \quad x = 1222\ldots,$$

simply because $x$ is not in any follower set whatsoever.

References

[1] F. Abadie, “On partial actions and groupoids”, Proc. Amer. Math. Soc., 132, (2003), 1037–1047.
[2] C. Anantharaman-Delaroche, “Purely infinite C*-algebras arising form dynamical systems”, Bull. Soc. Math. France, 125 (1997), 199–225.
[3] R. J. Archbold and J. Spielberg, “Topologically free actions and ideals in discrete dynamical systems”, Proc. Edinburgh Math. Soc., 37 (1993), 119–124.
[4] J. Brown, L. O. Clark, C. Farthing and A. Sims, “Simplicity of algebras associated to étale groupoids”, Semigroup Forum, 88 (2014), 433–452.
[5] N.P. Brown and N. Ozawa, “C*-algebras and finite-dimensional approximations”, Graduate Studies in Mathematics, 88, American Mathematical Society, Providence, RI, 2008.
[6] T. M. Carlsen, “Operator algebraic applications in symbolic dynamics”, Ph.D. thesis, University of Copenhagen (2004).
[7] T. M. Carlsen, “Cuntz-Pimsner C*-algebras associated with subshifts”, Internat. J. Math., 19 (2008), 47–70.
[8] T. M. Carlsen, “C*-algebras associated to shift spaces”, Notes for the Summer School Symbolic Dynamics and Homeomorphisms of the Cantor set, University of Copenhagen, 23–27 June 2008.
[9] T. M. Carlsen and K. Matsumoto, “Some remarks on the C*-algebras associated with subshifts”, Math. Scand., 95 (2004), 145–160.
[10] T. M. Carlsen and S. Silvestrov, “C*-crossed products and shift spaces”, Expo. Math., 25 (2007), 275–307.
[11] J. Cuntz and W. Krieger, “A class of C*-algebras and topological Markov chains”, Invent. Math., 63 (1981), 25–40.
[12] R. Exel, “Amenability for Fell bundles”, J. reine angew. Math., 492 (1997), 41–73.
[13] R. Exel, “Partial representations and amenable Fell bundles over free groups”, Pacific J. Math., 192 (2000), 39–63.
[14] R. Exel, “A new look at the crossed-product of a C*-algebra by an endomorphism”, Ergodic Theory Dynam. Systems, 23 (2003), 1733–1750.
[15] R. Exel, “Non-Hausdorff étale groupoids”, Proc. Amer. Math. Soc., 139 (2011), 897–907.
[16] R. Exel, “Partial Dynamical Systems, Fell Bundles and Applications”, to be published in a forthcoming NYJM book series. Available from http://mtm.ufsc.br/~exel/papers/pdynsysfellbun.pdf.
[17] R. Exel and M. Laca, “Cuntz-Krieger algebras for infinite matrices”, J. reine angew. Math., 512 (1999), 119–172.
[18] R. Exel, M. Laca and J. Quigg, “Partial dynamical systems and C*-algebras generated by partial isometries”, J. Operator Theory, 47 (2002), 169–186.
[19] S. Ito and Y. Takahashi, “Markov subshifts and realization of β-expansions”, J. Math. Soc. Japan, 26 (1974), 33–55.
[20] A. Kumjian, D. Pask and I. Raeburn, “Cuntz-Krieger algebras of directed graphs”, Pacific J. Math, 184 (1998), 161–174.
[21] A. Kumjian, D. Pask, I. Raeburn, and J. Renault, “Graphs, groupoids, and Cuntz-Krieger algebras”, J. Funct. Anal., 144 (1997), 505–541.
[22] P. Kúrka, “Topological and symbolic dynamics”, Cours Spécialisés, 11. Société Mathématique de France, Paris, 2003. xii+315 pp.
[23] D. Lind and B. Marcus, “An introduction to symbolic dynamics and coding”, Cambridge Univ. Press, 1999.
[24] K. Matsumoto, “On C*-algebras associated with subshifts”, Internat. J. Math., 8 (1997), 357–374.
[25] K. Matsumoto, “Dimension groups for subshifts and simplicity of the associated C*-algebras”, J. Math. Soc. Japan, 51 (1999), 679–698.
[26] K. Matsumoto, “On automorphisms of C*-algebras associated with subshifts”, J. Operator Theory, 44 (2000), 91–112.
[27] K. Matsumoto, “Stabilized C*-algebras constructed from symbolic dynamical systems”, Ergodic Theory Dynam. Systems, 20 (2000), 821–841.
[28] W. Parry, “Symbolic dynamics and transformations of the unit interval”, Trans. Amer. Math. Soc., 122 (1966), 368–378.
[29] J. Renault, “Cartan subalgebras in C*-algebras”, Irish Math. Soc. Bull., 61 (2008), 29–63.
[30] D. Royer, “Representações parciais de grupos”, Master Thesis, Universidade Federal de Santa Catarina, 2001.
[31] K. Thomsen, “Semi-étale groupoids and applications”, Ann. Inst. Fourier (Grenoble), 60 (2010), 759–800.