$C^\infty$ SCALING ASYMPTOTICS FOR THE SPECTRAL PROJECTOR OF THE LAPLACIAN

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ABSTRACT. This article concerns new off-diagonal estimates on the remainder and its derivatives in the pointwise Weyl law on a compact $n$-dimensional Riemannian manifold. As an application, we prove that near any non self-focal point, the scaling limit of the spectral projector of the Laplacian onto frequency windows of constant size is a normalized Bessel function depending only on $n$.

0. Introduction

Let $(M,g)$ be a compact, smooth, Riemannian manifold without boundary. We assume throughout that the dimension of $M$ is $n \geq 2$ and write $\Delta_g$ for the non-negative Laplace-Beltrami operator. Denote the spectrum of $\Delta_g$ by

$$0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots \uparrow \infty.$$ 

This article concerns the behavior of the Schwarz kernel of the projection operators

$$E_I : L^2(M) \to \bigoplus_{\lambda_j \in I} \ker(\Delta_g - \lambda_j^2),$$

where $I \subset [0, \infty)$. Given an orthonormal basis $\{\varphi_j\}_{j=1}^\infty$ of $L^2(M,g)$ consisting of real-valued eigenfunctions,

$$\Delta_g \varphi_j = \lambda_j^2 \varphi_j \quad \text{and} \quad \|\varphi_j\|_{L^2} = 1,$$

the Schwarz kernel of $E_I$ is

$$E_I(x,y) = \sum_{\lambda_j \in I} \varphi_j(x)\varphi_j(y).$$

The study of $E_{[0,\lambda]}(x,y)$ as $\lambda \to \infty$ has a long history, especially when $x = y$. For instance, it has been studied notably in [7, 8, 9, 10] for its close relation to the asymptotics of the spectral counting function

$$\# \{ j : \lambda_j \leq \lambda \} = \int_M E_{[0,\lambda]}(x,x)dv_g(x),$$

where $dv_g$ is the Riemannian volume form. An important result, going back to Hörmander [8, Thm 4.4], is the pointwise Weyl law (see also [11, 12]), which says that there exists $\varepsilon > 0$ so that if the Riemannian distance $d_g(x,y)$ between $x$ and $y$ is less than $\varepsilon$, then

$$E_{[0,\lambda]}(x,y) = \frac{1}{(2\pi)^n} \int_{|\xi| < \lambda} e^{i(\exp^{-1}(x),\xi)} \frac{d\xi}{\sqrt{|y|}} + R(x,y,\lambda).$$
The integral in (4) is over the cotangent fiber $T^*_yM$ and the integration measure is the quotient of the symplectic form $d\xi \wedge dy$ by the Riemannian volume form $dv_g = \sqrt{|g|} dy$. In Hörmander's original theorem, the phase function $\langle \exp^{-1}_y(x), \xi \rangle$ is replaced by any so-called adapted phase function and one still obtains that

$$\sup_{d_2(x,y) < \varepsilon} \left| \nabla^j_x \nabla^k_y R(x,y,\lambda) \right| = O(\lambda^{n-1+j+k})$$

as $\lambda \to \infty$, where $\nabla$ denotes covariant differentiation. The estimate (5) for $j = k = 0$ is already in [8, Thm 4.4], while the general case follows from the wave kernel method (e.g. as in §4 of [16] see also [3, Thm 3.1]).

Our main technical result, Theorem 1, shows that the remainder estimate (5) for $R(x,y,\lambda)$ can be improved from $O(\lambda^{n-1+j+k})$ to $o(\lambda^{n-1+j+k})$ under the assumption that $x$ and $y$ are near a non self-focal point (defined below). This paper is a continuation of [4] where the authors proved Theorem 2 for $j = k = 0$. An application of our improved remainder estimates is Theorem 1, which shows that we can compute the scaling limit of $E_{(\lambda,\lambda+1)}(x,y)$ and its derivatives near a non self-focal point as $\lambda \to \infty$.

**Definition 1.** A point $x \in M$ is non self-focal if the loopset

$$\mathcal{L}_x := \{ \xi \in S_x^*M : \exists t > 0 \text{ with } \exp_x(t \xi) = x \}$$

has measure 0 with respect to the natural measure on $T^*_xM$ induced by $g$. Note that $\mathcal{L}_x$ can be dense in $S_x^*M$ while still having measure 0 (e.g. for points on a flat torus).

**Theorem 1.** Let $(M,g)$ be a compact, smooth, Riemannian manifold of dimension $n \geq 2$, with no boundary. Suppose $x_0 \in M$ is a non self-focal point and consider a non-negative function $r_\lambda$ satisfying $r_\lambda = o(\lambda)$ as $\lambda \to \infty$. Define the rescaled kernel

$$E^{x_0}_{(\lambda,\lambda+1)}(u,v) := \lambda^{-(n-1)} E_{(\lambda,\lambda+1)} \left( \exp_{x_0} \left( \frac{u}{\lambda} \right), \exp_{x_0} \left( \frac{v}{\lambda} \right) \right).$$

Then, for all $k,j \geq 0$,

$$\sup_{|u|,|v| \leq r_\lambda} \left| \partial^j_u \partial^k_v \left( E^{x_0}_{(\lambda,\lambda+1)}(u,v) - \frac{1}{(2\pi)^n} \int_{S^*_{x_0}M} e^{i(u-v,\omega)} d\omega \right) \right| = o(1)$$

as $\lambda \to \infty$. The inner product in the integral over the unit sphere $S_{x_0}^*M$ is with respect to the flat metric $g(x_0)$ and $d\omega$ is the hypersurface measure on $S_{x_0}^*M$ induced by $g(x_0)$.

**Remark 1.** Theorem 1 holds for $\Pi_{(\lambda,\lambda+\delta)}$ with arbitrary fixed $\delta > 0$. The difference is that the limiting kernel is multiplied by $\delta$ and the rate of convergence in the $o(1)$ term depends on $\delta$.

**Remark 2.** One can replace the shrinking ball $B(x_0,r_\lambda)$ in Theorem 1 by a compact set $S \subset M$ in which for any $x, y \in S$ the measure of the set of geodesics joining $x$ and $y$ is zero (see Remark 3 after Theorem 2).

In normal coordinates at $x_0$, Theorem 1 shows that the scaling limit of $E^{x_0}_{(\lambda,\lambda+1)}$ in the $C^\infty$ topology is

$$E^{x_0}_{1}(u,v) = \frac{1}{(2\pi)^n} \int_{S^{n-1}} e^{i(u-v,\omega)} d\omega,$$
which is the kernel of the frequency 1 spectral projector for the flat Laplacian on $\mathbb{R}^n$. Theorem 1 can therefore be applied to studying the local behavior of random waves on $(M,g)$. More precisely, a frequency $\lambda$ monochromatic random wave $\varphi_\lambda$ on $(M,g)$ is a Gaussian random linear combination

$$\varphi_\lambda = \sum_{\lambda_j \in (\lambda,\lambda+1]} a_j \varphi_j \quad a_j \sim N(0,1) \ i.i.d,$$

of eigenfunctions with frequencies in $\lambda_j \in (\lambda,\lambda+1]$. In this context, random waves were first introduced by Zelditch in [20]. Since the Gaussian field $\varphi_\lambda$ is centered, its law is determined by its covariance function, which is precisely $E_{(\lambda,\lambda+1]}(x,y)$. In the language of Nazarov-Sodin [11] (cf [6, 14]), the estimate (6) means that frequency $\lambda$ monochromatic random waves on $(M,g)$ have frequency 1 random waves on $\mathbb{R}^n$ as their translation invariant local limits at every non self-focal point. This point of view is taken up in the forthcoming article [5].

**Theorem 2.** Let $(M,g)$ be a compact, smooth, Riemannian manifold of dimension $n \geq 2$, with no boundary. Let $K \subseteq M$ be the set of all non self-focal points in $M$. Then for all $k,j \geq 0$ and all $\varepsilon > 0$ there is a neighborhood $U = U(\varepsilon,k,j)$ of $K$ and constants $\Lambda = \Lambda(\varepsilon,k,j)$ and $C = C(\varepsilon,k,j)$ for which

$$\|R(x,y,\lambda)\|_{C^k(U) \times C^j(U)} \leq \varepsilon \lambda^{-1+j+k} + C\lambda^{-2+j+k}$$

(6)

for all $\lambda > \Lambda$. Hence, if $x_0 \in K$ and $U_\lambda$ is any sequence of sets containing $x_0$ with diameter tending to 0 as $\lambda \to \infty$, then

$$\|R(x,y,\lambda)\|_{C^k(U_\lambda) \times C^j(U_\lambda)} = o(\lambda^{-1+j+k}).$$

(7)

**Remark 3.** One can consider more generally any compact $S \subseteq M$ such that all $x,y \in S$ are mutually non-focal, which means

$$\mathcal{L}_{x,y} := \{\xi \in S^* \cdot M : \exists t > 0 \text{ with } \exp_x(t\xi) = y\}$$

has measure zero. Then, combining [12] Thm 3.3 with Theorem 2 for every $\varepsilon > 0$, there exists a neighborhood $U = U(\varepsilon,j)$ of $S$ and constants $\Lambda = \Lambda(\varepsilon,j,S)$ and $C = C(\varepsilon,j,S)$ such that

$$\sup_{x,y \in S} \left|\nabla_x \nabla_y^j R(x,y,\lambda)\right| \leq \varepsilon \lambda^{-1+2j} + C\lambda^{-2+2j}. $$

We believe that this statement is true even when the number of derivatives in $x,y$ is not the same but do not take this issue up here.

Our proof of Theorem 2 relies heavily on the argument for Theorem 1 in [4], which treated the case $j = k = 0$. That result was in turn was based on the work of Sogge-Zelditch [18, 19], who studied $j = k = 0$ and $x = y$. This last situation was also studied (independently and significantly before [4] [18, 19]) by Safarov in [12] (cf [13]) using a somewhat different method. The case $j = k = 1$ and $x = y$ is essentially Proposition 2.3 in [20]. We refer the reader to the introduction of [4] for more background on estimates like (6).
1. Proof of Theorem 2

Let $x_0$ be a non-self focal point. Let $I, J$ be multi-indices and set

$$\Omega := \lvert I \rvert + \lvert J \rvert.$$  

Using that $\int_{S^{n-1}} e^{i(u,w)} \, dw = (2\pi)^{n/2} J_{\frac{n-2}{2}}(|u|) |u|^{-\frac{n-2}{2}}$ for all $u \in \mathbb{R}^n$, we have

$$\frac{1}{(2\pi)^n} \int_{|\xi| \leq \lambda} e^{i(\exp^{-1}(x), \xi)} \frac{d\xi}{\sqrt{|g_y|}} = \int_0^\lambda \frac{\mu^{n-1}}{(2\pi)^{n/2}} \left( \frac{J_{\frac{n-2}{2}}(\mu d_g(x,y))}{(\mu d_g(x,y))^{\frac{n-2}{2}}} \right) \mu \, d\mu. \quad (8)$$

Choose coordinates around $x_0$. We seek to show that there exists a constant $c > 0$ so that for every $\varepsilon > 0$ there is an open neighborhood $U_\varepsilon$ of $x_0$ and a constant $c_\varepsilon$ so that we have

$$\sup_{x,y \in U_\varepsilon} \left\lvert \partial_x^I \partial_y^J E_\lambda(x, y) - \int_0^\lambda \frac{\mu^{n-1}}{(2\pi)^{n/2}} \partial_x^I \partial_y^J \left( \frac{J_{\frac{n-2}{2}}(\mu d_g(x,y))}{(\mu d_g(x,y))^{\frac{n-2}{2}}} \right) \mu \, d\mu \right\rvert \leq c \varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}. \quad (9)$$

Let $\rho \in \mathcal{S}(\mathbb{R})$ satisfy $\text{supp}(\tilde{\rho}) \subseteq (\text{inj}(M, g), \text{inj}(M, g))$ and

$$\tilde{\rho}(t) = 1 \quad \text{for all} \quad |t| < \frac{1}{2} \text{inj}(M, g). \quad (10)$$

We prove (9) by first showing that it holds for the convolved measure $\rho \ast \partial_x^I \partial_y^J E_\lambda(x, y)$ and then estimating the difference $|\rho \ast \partial_x^I \partial_y^J E_\lambda(x, y) - \partial_x^I \partial_y^J E_\lambda(x, y)|$ in the following two propositions.

**Proposition 3.** Let $x_0$ be a non-self focal point. Let $I, J$ be multi-indices and set $\Omega = \lvert I \rvert + \lvert J \rvert$. There exists a constant $c$ so that for every $\varepsilon > 0$ there exist an open neighborhood $U_\varepsilon$ of $x_0$ and a constant $c_\varepsilon$ so that we have

$$\left\lvert \rho \ast \partial_x^I \partial_y^J E_\lambda(x, y) - \int_0^\lambda \frac{\mu^{n-1}}{(2\pi)^{n/2}} \partial_x^I \partial_y^J \left( \frac{J_{\frac{n-2}{2}}(\mu d_g(x,y))}{(\mu d_g(x,y))^{\frac{n-2}{2}}} \right) \mu \, d\mu \right\rvert \leq c \varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega},$$

for all $x, y \in U_\varepsilon$.

**Proposition 4.** Let $x_0$ be a non-self focal point. There exists a constant $c$ so that for every $\varepsilon > 0$ there exist an open neighborhood $U_\varepsilon$ of $x_0$ and a constant $c_\varepsilon$ so that for all multi-indices $I, J$ we have

$$\sup_{x,y \in U_\varepsilon} \left\lvert \rho \ast \partial_x^I \partial_y^J E_\lambda(x, y) - \partial_x^I \partial_y^J E_\lambda(x, y) \right\lvert \leq c \varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}.$$

The proof of Proposition 4 hinges on the fact that $x_0$ is a non self-focal point. Indeed, for each $\varepsilon > 0$, Lemma 15 in [14] (which is a generalization of Lemma 3.1 in [15]) yields the existence of a neighborhood $O_\varepsilon$ of $x_0$, a function $\psi_\varepsilon \in C^\infty_c(M)$ and operators $B_\varepsilon, C_\varepsilon \in \Psi^0(M)$ supported in $O_\varepsilon$ satisfying both:

- $\text{supp}(\psi_\varepsilon) \subseteq O_\varepsilon$ and $\psi_\varepsilon = 1$ on a neighborhood of $x_0$, \quad (11)
- $B_\varepsilon + C_\varepsilon = \psi_\varepsilon^2$. \quad (12)

The operator $B_\varepsilon$ is built so that it is microlocally supported on the set of cotangent directions that generate geodesic loops at $x_0$. Since $x_0$ is non self-focal, the construction can be carried so that the principal symbol $b_0(x, \xi)$ satisfies $\|b_0(x, \xi)\|_{L^2(B_\varepsilon^c, M)} \leq \varepsilon$ for
all \( x \in M \). The operator \( C_\varepsilon \) is built so that \( U(t)C_\varepsilon^* \) is a smoothing operator for \( \frac{1}{2}\text{inj}(M, g) < |t| < \frac{1}{2} \). In addition, the principal symbols of \( B_\varepsilon \) and \( C_\varepsilon \) are real valued and their sub-principal symbols vanish in a neighborhood of \( x_0 \) (when regarded as operators acting on half-densities).

In what follows we use the construction above to decompose \( E_\lambda \), up to an \( O(\lambda^{-\infty}) \) error, as

\[
E_\lambda(x, y) = E_\lambda B_\varepsilon^*(x, y) + E_\lambda C_\varepsilon^*(x, y)
\]

for all \( x, y \) sufficiently close to \( x_0 \). This decomposition is valid since \( \psi_\varepsilon \equiv 1 \) near \( x_0 \).

1.1. **Proof of Proposition 3.** The proof of Proposition 3 consists of writing

\[
\rho \ast \partial^I_x \partial^J_y E_\lambda(x, y) = \int_0^\lambda \partial_\mu (\rho \ast \partial^I_x \partial^J_y E_\mu(x, y)) \, d\mu,
\]

and on finding an estimate for \( \partial_\mu (\rho \ast \partial^I_x \partial^J_y E_\mu(x, y)) \). Such an estimate is given in Lemma 5, which is stated for the more general case \( \partial_\mu (\rho \ast \partial^I_x \partial^J_y E_\mu Q^*(x, y)) \) with \( Q \in \{ \text{Id}, B_\varepsilon, C_\varepsilon \} \) that is needed in the proof of Proposition 4.

**Lemma 5.** Let \((M, g)\) be a compact, smooth, Riemannian manifold of dimension \( n \geq 2 \), with no boundary. Let \( Q \in \{ \text{Id}, B_\varepsilon, C_\varepsilon \} \) have principal symbol \( D_0^Q \). Consider \( \rho \) as in (10), and define

\[
\Omega = |I| + |J|.
\]

Then, for all \( x, y \in M \) with \( d_g(x, y) \leq \frac{1}{2} \text{inj}(M, g) \), all multi-indices \( I, J \), and all \( \mu \geq 1 \), we have

\[
\partial_\mu (\rho \ast \partial^I_x \partial^J_y E_\mu Q^*)(x, y) = \frac{\mu^{n-1}}{(2\pi)^n} \partial^I_x \partial^J_y \left( \int_{S^*_g M} e^{i\mu(\exp^{-1}(x, \omega)_g)} \left( D_0^Q(y, \omega) + \mu^{-1} D_{-1}^Q(y, \omega) \right) \frac{d\omega}{\sqrt{|g_y|}} \right) + W_{I,J}(x, y, \mu).
\]

Here, \( d\omega \) is the Euclidean surface measure on \( S^*_g M \), and \( D_{-1}^Q \) is a homogeneous symbol of order \(-1\). The latter satisfy

\[
D_{-1}^{B_\varepsilon}(y, \cdot) + D_{-1}^{C_\varepsilon}(y, \cdot) = 0 \quad \forall y \in O_\varepsilon,
\]

where \( O_\varepsilon \) is as in (11). Moreover, there exists \( C > 0 \) so that for every \( \varepsilon > 0 \)

\[
\sup_{x, y \in O_\varepsilon} \int_{S^*_g M} e^{i\exp^{-1}(x, \omega)_g} D_{-1}^Q(y, \omega) \frac{d\omega}{\sqrt{|g_y|}} \leq C \varepsilon.
\]

Finally, \( W_{I,J} \) is a smooth function in \((x, y)\) for which there exists \( C > 0 \) such that for all \( x, y \) satisfying \( d_g(x, y) \leq \frac{1}{2} \text{inj}(M, g) \) and all \( \mu > 0 \)

\[
|W_{I,J}(x, y, \mu)| \leq C \mu^{n-2+\Omega} \left( d_g(x, y) + (1 + \mu)^{-1} \right).
\]

**Remark 4.** Note that Lemma 5 does not assume that \( x, y \) are near an non self-focal point.
**Remark 5.** We note that Lemma 5 is valid for more general operators $Q$. Indeed, if $Q \in \Psi^k(M)$ has vanishing subprincipal symbol (when regarded as an operator acting on half-densities), then (14) holds with $D_k^Q(y,\omega)$ substituted by $\mu^k D_k^Q(y,\omega)$ and with $\mu^{-1} D_k^Q(y,\omega)$ substituted by $\mu^{-1} D_k^Q(y,\omega)$. Here, $D_k^Q$ is the principal symbol of $Q$ and $D_k^{Q'}$ is a homogeneous polynomial of degree $k-1$. In this setting, the error term satisfies $|W_{I,J}(x,y,\mu)| \leq C \mu^{n+k-2+\Omega} (d_g(x,y) + (1 + \mu)^{-1})$.

**Proof of Lemma 5.** We use that

$$\partial_{\mu}(\rho \ast EQ^*)(x,y,\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{it\lambda} \tilde{\rho}(t) U(t) Q^*(x,y) dt,$$

where $Q \in \Psi(M)$ is any pseudo-differential operator and $U(t) = e^{-it\sqrt{\Delta_g}}$ is the half-wave propagator. The argument from here is identical to that of [4, Proposition 12], which relies on a parametrix for the half-wave propagator for which the kernel can be controlled to high accuracy when $x$ and $y$ are close to the diagonal. The main corrections to the proof of [4, Proposition 12] are that $\partial_x \partial_y$ gives an $O(\mu^{n-3+\Omega})$ error in equations (54) and (60), and gives an $O(\mu^{n-1})$ error in (59). We must also take into account that $\partial_x \Theta(x,y)^{1/2}$ and $\partial_y \Theta(x,y)^{1/2}$ are both $O(d_g(x,y))$.

**Proof of Proposition 4.** Following the technique for proving [4, Proposition 7], we obtain Proposition 4 by applying Lemma 5 to $Q = Id$ (this gives $D_0^Q = 1$ and $D_{1-k}^Q = 0$) and integrating the expression in (14) from $\mu = 0$ to $\mu = \lambda$. One needs to choose $U_{\varepsilon}$ so that its diameter is smaller than $\varepsilon$, since this makes $\int_0^\lambda W_{I,J}(x,y,\mu) d\mu = O(\varepsilon \lambda^{n-1+\Omega} + \lambda^{n-2+\Omega})$ as needed. One also uses identity (3) to obtain the exact statement in Proposition 4.

**1.2. Proof of Proposition 4.** As in (13),

$$E_\lambda(x,y) = E_\lambda B_\varepsilon^*(x,y) + E_\lambda C_\varepsilon(x,y) + O \left( \lambda^{-\infty} \right)$$

for all $x,y$ sufficiently close to $x_0$. Proposition 4 therefore reduces to showing that there exist a constant $c$ independent of $\varepsilon$, a constant $c_\varepsilon = c_\varepsilon(I,J,x_0)$, and a neighborhood $U_{\varepsilon}$ of $x_0$ such that

$$\sup_{x,y \in U_{\varepsilon}} |\partial_x^I \partial_y^J E_\lambda B_\varepsilon^*(x,y) - \rho \ast \partial_x^I \partial_y^J E_\lambda B_\varepsilon^*(x,y)| \leq c \varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega},$$

(19)

and

$$\sup_{x,y \in U_{\varepsilon}} |\partial_x^I \partial_y^J E_\lambda C_\varepsilon(x,y) - \rho \ast \partial_x^I \partial_y^J E_\lambda C_\varepsilon(x,y)| \leq c \varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}.$$ 

(20)

Our proofs of (19) and (20) use that these estimates hold on diagonal when $|I| = |J| = 0$ (i.e. no derivatives are involved). This is the content of the following result, which was proved in [18] for $Q = Id$. Its proof extends without modification to general $Q \in \Psi^0(M)$.

**Lemma 6** (Theorem 1.2 and Proposition 2.2 in [18]). Let $Q \in \Psi^0(M)$ have real-valued principal symbol $q$. Fix a non-self focal point $x_0 \in M$ and write $\sigma_{\text{sub}}(Q Q^*)$ for
the subprincipal symbol of $QQ^*$ (acting on half-densities). Then, there exists $c > 0$ so that for every $\varepsilon > 0$ there exist a neighborhood $\mathcal{O}_\varepsilon$ and a constant $C_\varepsilon$ making

$$QE_\lambda Q^*(x, x) = (2\pi)^{-n} \int_{|\xi| \leq \lambda} \left( |q(x, \xi)|^2 + \sigma_{\text{sub}}(QQ^*)(x, \xi) \right) \frac{d\xi}{|g_x|} + R_Q(x, \lambda),$$

with

$$\sup_{x \in \mathcal{U}} |R_Q(x, \lambda)| \leq c \varepsilon \lambda^{n-1} + C_\varepsilon \lambda^{n-2}$$

for all $\lambda \geq 1$.

We prove relation (19) in Section 1.2.1 and relation (20) in Section 1.2.2.

1.2.1. Proof of relation (19). Define

$$g_{l, j}(x, y, \lambda) := \partial_x^l \partial_y^j E_\lambda B^*_\varepsilon(x, y) - \rho \ast \partial_x^l \partial_y^j E_\lambda B^*_\varepsilon(x, y).$$

Note that $g_{l, j}(x, y, \cdot)$ is a piecewise continuous function. We aim to find $c, c_\varepsilon$ and $\mathcal{U}_\varepsilon$ so that $x_0 \in \mathcal{U}_\varepsilon$ and

$$\sup_{x, y \in \mathcal{U}_\varepsilon} |g_{l, j}(x, y, \lambda)| \leq c \varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}. \tag{21}$$

By [4, Lemma 17], which is a Tauberian Theorem for non-monotone functions, relation (21) reduces to checking the following two conditions:

- $\mathcal{F}_{\lambda \to t}(g_{l, j})(x, y, t) = 0$ for all $|t| < \frac{1}{2} \text{inj}(M, g)$, \tag{22}
- $\sup_{x, y \in \mathcal{U}_\varepsilon} \sup_{s \in [0, 1]} |g_{l, j}(x, y, \lambda + s) - g_{l, j}(x, y, \lambda)| \leq c \varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}. \tag{23}$

By construction, $\mathcal{F}_{\lambda \to t}(\partial_\lambda g_{l, j})(x, y, t) = (1 - \hat{\rho}(t)) \partial_x^l \partial_y^j U(t) B^*_\varepsilon(x, y) = 0$ for all $|t| < \frac{1}{2} \text{inj}(M, g)$. Hence, since $\mathcal{F}_{\lambda \to t}(g_{l, j})$ is continuous at $t = 0$, we have (22). To prove (23) we write

$$\sup_{s \in [0, 1]} |g_{l, j}(x, y, \lambda + s) - g_{l, j}(x, y, \lambda)| \leq \sup_{s \in [0, 1]} |\partial_x^l \partial_y^j E_\lambda B^*_\varepsilon(x, y)| + \sup_{s \in [0, 1]} |\rho \ast \partial_x^l \partial_y^j E_\lambda B^*_\varepsilon(x, y)|. \tag{24}$$

The second term in (24) is bounded above by the right hand side of (23) by Lemma 5. To bound the first term, use Cauchy-Schwartz to get

$$\sup_{s \in [0, 1]} |\partial_x^l \partial_y^j [E_\lambda B^*_\varepsilon(x, y)]| = \sup_{s \in [0, 1]} \left| \sum_{\lambda_j \in \{\lambda, \lambda+1\}} \partial_x^l \varphi_j(x) \cdot \partial_y^j B_\varepsilon \varphi_j(y) \right| \leq \sum_{\lambda_j \in \{\lambda, \lambda+1\}} \left| \left( B_\varepsilon \partial_y^j [\partial_y^j B_\varepsilon] \right) \varphi_j(y) \right| \cdot |\partial_x^l \varphi_j(x)|. \tag{25}$$

Write $b_0$ for the principal symbol of $B_\varepsilon$. By construction, for all $y$ in a neighborhood of $x_0$, we have $\partial_y b_0(y, \xi) = 0$. Therefore, $\sigma_{\lambda, -1}([\partial_y^j B_\varepsilon]) = i^{2j} \{ \xi^j, b_0(y, \xi) \} = 0$ and
1.2.2. Proof of relation

This proves (23), which together with (22) allows us to conclude (21).

Next, define for each multi-index $K \in \mathbb{N}^n$ the order zero pseudo-differential operator

$$P_K := \partial^K \Delta_{g}^{-|K|/2}.$$ 

Using Cauchy-Schwarz and that $\partial^K \varphi_j = \lambda^{|K|} P_K \varphi_j$, we find

$$\sum_{\lambda_j \in (\lambda, \lambda+1]} |B_\varepsilon \partial^I_{\varphi_j} \cdot \partial^I_{\varphi_j}| \leq (\lambda + 1)^\Omega [(B_\varepsilon P_J) E_{(\lambda, \lambda+1]} (B_\varepsilon P_J)^\ast (y, y)]^{\frac{1}{2}} [P_I E_{(\lambda, \lambda+1]} P_I^\ast (x, x)]^{\frac{1}{2}}.$$ 

Again using the pointwise Weyl Law (see [19, Equation (2.31)]), we have $[P_I E_{(\lambda, \lambda+1]} P_I^\ast (x, x)]^{\frac{1}{2}}$ is $O(\lambda^\frac{n-1}{2})$. Next, since according to the construction of $B_\varepsilon$ we have

$$\sup_{x \in U_\varepsilon} \|b_0(x, \cdot)\|_{L^2(B_\varepsilon^2 M)} \leq \varepsilon$$

and $\partial_x b_0(x, \xi) = 0$ for $x$ in a neighborhood $U_\varepsilon$ of $x_0$, we conclude that

$$\sup_{x \in U_\varepsilon} \|\sigma_{\text{sub}}(B_\varepsilon P_J (B_\varepsilon P_J)^\ast (x, \cdot))\|_{L^2(B_\varepsilon^2 M)} \leq \varepsilon^2.$$ 

Proposition [5] therefore shows that there exists $c > 0$ making

$$\sup_{x, y \in U_\varepsilon} \left| (B_\varepsilon P_J) E_{(\lambda, \lambda+1]} (B_\varepsilon P_J)^\ast (y, y) \right|^{\frac{1}{2}} \leq c \lambda^{\frac{n-1}{2}}. \quad (25)$$

This proves (23), which together with (22) allows us to conclude (21).

1.2.2. Proof of relation (20). Write

$$\partial^I_{\varphi_j} E_{\lambda} C_\varepsilon^\ast (x, y) = \sum_{\lambda_j \leq \lambda} \lambda^{|I|} (P_I \varphi_j (x)) \cdot (C_\varepsilon P_J \varphi_j (y)) + \sum_{\lambda_j \leq \lambda} \lambda^{|I|} (P_I \varphi_j (x)) \cdot ([\partial^I, C_\varepsilon] \varphi_j (y)).$$ 

(26)

As before, $[\partial^I, C_\varepsilon] \in \Psi^{1/2}$. Hence, by the usual pointwise Weyl law, the second term in (26) and its convolution with $\rho$ are both $O(\lambda^{n-2+\Omega})$. Hence,

$$\sup_{x, y \in U_\varepsilon} \left| \partial^I_{\varphi_j} E_{\lambda} C_\varepsilon^\ast (x, y) - \rho \ast \partial^I_{\varphi_j} E_{\lambda} C_\varepsilon^\ast (x, y) \right| = \sup_{x, y \in U_\varepsilon} |V(x, y, \lambda) - \rho \ast V(x, y, \lambda)| + O(\lambda^{n+\Omega-2}),$$

where we have set

$$V(x, y, \lambda) := \partial^I E_{\lambda} (C_\varepsilon \partial^I)^\ast (x, y) = \sum_{\lambda_j \leq \lambda} \lambda^{|I|} (P_I \varphi_j (x)) \cdot (C_\varepsilon P_J \varphi_j (y)).$$
Define
\[ \alpha_{I,J}(x, y, \lambda) := V(x, y, \lambda) + \frac{1}{2} \sum_{\lambda_j \leq \lambda} \lambda_j^\Omega \left( |P_I \varphi_{\lambda_j}(x)|^2 + |C_\varepsilon P_J \varphi_{\lambda_j}(y)|^2 \right) \] (27)
\[ \beta_{I,J}(x, y, \lambda) := \rho \ast V(x, y, \lambda) + \frac{1}{2} \sum_{\lambda_j \leq \lambda} \lambda_j^\Omega \left( |P_I \varphi_{\lambda_j}(x)|^2 + |C_\varepsilon P_J \varphi_{\lambda_j}(y)|^2 \right). \] (28)

By construction, \( \alpha_{I,J}(x, y, \cdot) \) is a monotone function of \( \lambda \) for \( x, y \) fixed, and \( \alpha_{I,J}(x, y, \lambda) - \beta_{I,J}(x, y, \lambda) = V(x, y, \lambda) - \rho \ast V(x, y, \lambda) \). So we aim to show that
\[ \sup_{x,y \in \Omega} |\alpha_{I,J}(x, y, \lambda) - \beta_{I,J}(x, y, \lambda)| \leq c \varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}. \] (29)

We control the difference in (29) applying a Tauberian theorem for monotone functions [4, Lemma 16]. To apply it we need to show the following:

- There exists \( c > 0 \) and \( c_\varepsilon > 0 \) making
  \[ \int_{\lambda^{-\varepsilon}}^{\lambda^{+\varepsilon}} |\partial_\mu \beta_{I,J}(x, y, \mu)| \, d\mu \leq c \varepsilon \lambda^{n-1+\Omega} + c_\varepsilon \lambda^{n-2+\Omega}. \] (30)

- For all \( N \) there exists \( M_{\varepsilon,N} \) so that for all \( \lambda > 0 \)
  \[ |\partial_\lambda (\alpha_{I,J}(x, y, \cdot) - \beta_{I,J}(x, y, \cdot)) \ast \phi_\varepsilon(\mu)| \leq M_{\varepsilon,N} (1 + |\lambda|)^{-N}. \] (31)

In equation (31) we have set \( \phi_\varepsilon(\lambda) := \frac{1}{\varepsilon} \phi\left( \frac{\lambda}{\varepsilon} \right) \) for some \( \phi \in \mathcal{S}(\mathbb{R}) \) chosen so that \( \text{supp} \hat{\phi} \subseteq (-1, 1) \) and \( \hat{\phi}(0) = 1 \).

Relation (30) follows after applying Lemma 6 to the piece of the integral corresponding to the second term in (28) and from applying Lemma 5 together with Remark 5 to \( \rho \ast V = \rho \ast \partial^I E_\lambda Q^* \), where \( Q := C_\varepsilon \partial^J \) has vanishing subprincipal symbol.

To verify (31) note that \( \text{supp}(1 - \hat{\rho}) \subseteq \{ t : |t| \geq \text{inj}(M, g)/2 \} \) and \( \text{supp}(\hat{\phi}_\varepsilon) \subseteq \{ t : |t| \leq \frac{1}{\varepsilon} \} \). Observe that
\[ \partial_\lambda (\alpha_{I,J}(x, y, \cdot) - \beta_{I,J}(x, y, \cdot)) \ast \phi_\varepsilon(\lambda) = \mathcal{F}^{-1}_{-\varepsilon^{-\lambda}} \left( (1 - \hat{\rho}(t)) \hat{\phi}_\varepsilon(t) \partial^I U(t)(\partial^J C_\varepsilon)^*(x,y) \right)(\lambda). \]

By construction \( U(t)C_\varepsilon^* \) is a smoothing operator for \( \frac{1}{2} \text{inj}(M, g) < |t| < \frac{1}{\varepsilon} \). Thus, so is \( \partial^I U(t)(\partial^J C_\varepsilon)^* \) which implies (31). This concludes the proof of relation (20).

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