Extrinsic spin Hall effect in inhomogeneous systems

Takumi Funato and Hiroshi Kohno
Department of Physics, Nagoya University, Nagoya 464-8602, Japan
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Charge-to-spin conversion in inhomogeneous systems is studied theoretically. We consider free electrons subject to impurities with spin-orbit interaction and with spatially modulated distribution, and calculate spin accumulation and spin current induced by an external electric field. It is found that the spin accumulation is induced by the vorticity of electron flow through the side-jump and skew-scattering processes, and the differences of the two processes are discussed. The results can be put in a form of generalized spin diffusion equation with a spin source term given by the divergence of the spin Hall current. This spin source term reduces to the form of spin-vorticity coupling only when the spin Hall angle is independent of impurity concentration.

I. INTRODUCTION

Interconversion between charge and spin currents is one of the fundamental processes in spintronics. Among various methods, the direct and inverse spin Hall effects (SHE) via spin-orbit interaction (SOI) have been utilized most commonly\cite{1,2}. Recently, there are several experimental reports on the enhancement of charge-to-spin conversion in naturally oxidized (nox-) Cu\cite{3}. An et al.\ found an enhanced spin-orbit torque generation efficiency comparable to Pt in nox-Cu in the spin-torque ferromagnetic resonance (ST-FMR) experiment\cite{4}. Okano et al.\ measured charge-spin interconversion in nox-Cu by unidirectional spin Hall magnetoresistance (USMR) and spin pumping, and demonstrated its non-reciprocal character\cite{5}. A possibly related phenomenon was reported by Enoki et al., who observed enhanced weak antilocalization in nox-Cu\cite{6}. There are also reports on the enhancement of spin-orbit torques in nox-Co by Hibino et al.\ and Hasegawa et al.\\cite{7,8,9,10,11,12,13}.

Previous studies suggested that the spin-vorticity coupling (SVC)\cite{14,15} can be the main origin of the enhanced charge-to-spin conversion in nox-Cu. The SVC is the coupling between the electron spin and the vorticity of electron flow that arises in the rotating (i.e., non-inertial) frame of reference locally fixed to the lattice or the electrons. This effect has been demonstrated by several experiments, which include spin hydrodynamic generation in liquid metals\cite{16} and spin-current generation from surface acoustic waves in solids\cite{17}. However, applying the idea of SVC to the aforementioned phenomena in nox-Cu seems to contain a difficulty in the treatment of moving lattice in the non-inertial frame comoving with the electrons.

The purpose of this paper is to study charge-to-spin conversion phenomena that occur in inhomogeneous systems which allow electron flow with vorticity. We model the system by nearly free electrons placed under impurities with SOI and with spatially modulated distribution. We calculate spin accumulation and spin current in response to an applied electric field using Kubo formula and discuss their relation to the vorticity of the electron flow. We found a spin accumulation is induced by the vorticity of the electron flow via the side-jump and skew-scattering processes. The obtained results are precisely described by a generalized spin diffusion equation with a spin source term given by the divergence of the spin Hall (SH) current. The spin source term reduces to the vorticity of the electron flow if the SH angle is independent of the impurity concentration, which occurs for skew scattering in the present model. This may be considered as an “effective SVC” viewed in the laboratory (inertial) frame.

II. MODEL

We consider a nearly free electron system in the presence of SOI due to impurities. The Hamiltonian is given by

\begin{equation}
H = \sum_k \frac{k^2}{2m} \psi^\dagger_k \psi_k + H_{\text{imp}} + H_{\text{so}},
\end{equation}

where \( \psi_k = (\psi^\dagger_{k\uparrow}, \psi^\dagger_{k\downarrow}) \) and \( \psi_k \) are the electron creation (annihilation) operators. \( H_{\text{imp}} \) and \( H_{\text{so}} \) describe the coupling to the impurity potential and impurity SOI, respectively,

\begin{equation}
H_{\text{imp}} = \sum_{k,k'} V_{k-k'} \psi^\dagger_{k'} \psi_k,
\end{equation}

\begin{equation}
H_{\text{so}} = i\lambda_{\text{so}} \sum_{k,k'} V_{k-k'} (k' \times k) \cdot \psi^\dagger_{k'} \sigma \psi_k,
\end{equation}

where \( \lambda_{\text{so}} \) is the strength of SOI, and \( \sigma = (\sigma^x, \sigma^y, \sigma^z) \) are Pauli matrices. We assume a short-range impurity potential, \( V_{\text{imp}}(r) = u_i \sum_j \delta(r - R_j) \), where \( u_i \) is the strength of the potential and \( R_j \) is the position of \( j \)th impurity, and \( V_{k-k'} \) is the Fourier transform of \( V_{\text{imp}} \).

As a model of inhomogeneous systems, we consider a situation in which the impurity distribution has a slight spatial modulation. Specifically, we take \( n_i(r) = n_i + \delta n_i(Q) e^{iQ \cdot r} \) for the impurity concentration, where the first term is the “uniform” part and the second term is the “inhomogeneous” part with \( Q \) being the wave vector of the modulation. Using the probability density function.
of impurities,
\[ p_{\text{imp}}(r) = \frac{1}{\Omega} + \frac{\delta n_i(Q)}{n_i} e^{iQ \cdot r}, \]
(4)
where \( \Omega \) is the system volume, we average over the impurity positions as \( \langle V_k \rangle_{\text{av}} = \delta n_i(Q) u_{i k} Q_r, \langle V_k V_{k'} \rangle_{\text{av}} = n_i n_i' \delta_{k + k', 0} + \delta n_i(Q) n_i' \delta_{k + k', Q} \), and \( \langle V_k V_{k'} V_{k''} \rangle_{\text{av}} = n_i n_i' n_i'' \delta_{k + k' + k'', 0} + \delta n_i(Q) n_i' n_i'' \delta_{k + k' + k'', Q} \). In this paper, we consider the inhomogeneity of the impurities (\( \delta n_i(Q) \)) up to the first order, and the impurity SOI (\( \lambda_{\text{so}} \)) up to the second order. The impurity-averaged retarded/advanced Green function in the absence of inhomogeneity (i.e., averaged with the homogeneous part of \( p_{\text{imp}}(r) \)) is given by
\[ G_{R/A}(\epsilon) = \frac{1}{\epsilon + \mu - k^2/2m + i\gamma}, \]
(5)
where \( \gamma = \pi n_i u_i^2 N(\mu)/(1 + \frac{2}{3} \lambda_{\text{so}} k_F^2) \) is the damping rate, with the Fermi-level density of states (per spin) \( N(\mu) = \frac{m v_F^2}{2\pi^2} \), and the Fermi wave number \( k_F \).

We apply a spatially-modulated, time-dependent electromagnetic field, \( A_{\nu}(r, t) = A_{\nu}(q, \omega) e^{i(q \cdot r - \omega t)} \), where \( q \) is the wave vector, \( \omega \) is the frequency, and the four-vector notation \( A_{\nu} = (-\phi, A) \) has been used. The perturbation Hamiltonian is given by
\[ H_{\text{ext}} = -\hat{j}_{\nu, \nu}(\mathbf{q}) A_{\nu}(\mathbf{q}, \omega) e^{-i\omega t}, \]
(6)
where \( \hat{j}_{\nu, \nu} = (\hat{j}_e, \hat{j}_\nu) \) is the electric charge/current density operator. Throughout this paper, we assume that the spatial variations of \( n_i(r) \) and \( A_{\nu} \) are much slower compared to the electron mean free path \( l \), and the temporal variation of \( A_{\nu} \) is much slower compared to the electron damping rate \( \gamma \). These are expressed by \( q, Q \ll l^{-1} \) and \( \omega \ll \gamma \).

Spin and spin-current operators are given by
\[ \hat{j}^{\alpha}_{\nu, 0}(\mathbf{q}) = \delta^{\alpha \nu}(\mathbf{q}) = \sum_k \psi_k^\dagger \sigma^\alpha \psi_{k + \frac{1}{2}}, \]
(7)
\[ \hat{j}^{\alpha}_{\nu, i}(\mathbf{q}) = \sum_k v_i \psi_k^\dagger \sigma^\alpha \psi_{k + \frac{1}{2}} + \hat{j}^{\alpha \nu}_{\nu, i}(\mathbf{q}), \]
(8)
where \( \alpha (= x, y, z) \) specifies the spin direction, \( i (= x, y, z) \) specifies the current direction, and
\[ \hat{j}^{\alpha \nu}_{\nu, i}(\mathbf{q}) = -i \lambda_{\text{so}} \epsilon_{\alpha i j} \sum_{k, k'} V_{k' - k}(k' - k) \psi_{k'}^\dagger \sigma^\alpha \psi_{k + \frac{1}{2}}, \]
(9)
is the “anomalous” part of spin current (\( \epsilon_{\alpha i j} \) is the Levi-Civita symbol).

### III. CALCULATION

We are interested in the spin accumulation and spin current that arise in linear response to \( A_{\nu} \) (with wave vector \( q \)), and at the first order in \( \delta n_i \) (with wave vector \( Q \)),
\[ \langle \hat{j}^{\alpha}_{\nu, i}(q + Q) \rangle = \left[ K_{\nu \alpha}^{\text{ij}} + K_{\nu \alpha}^{\text{ss}} \right] A_{\nu}(q, \omega). \]
(10)
The suffix \( \delta \) indicates that this is first order in \( \delta n_i \). We will also present the terms zeroth-order in \( \delta n_i \), which will be denoted by \((\cdots)_0\). As indicated in Eq. (10), the response function arises from two processes. One, denoted by \( K_{\nu \alpha}^{\text{ij}} \), comes from the side-jump type processes, and the other, denoted by \( K_{\nu \alpha}^{\text{ss}} \), results from the skew-scattering type processes. We note that equilibrium spin currents do not arise.

#### A. Charge channel

Before proceeding, we first look at the charge and charge-current densities. Those in a homogeneous system (terms zeroth order in \( \delta n_i \) but first order in \( A_{\nu}(q, \omega) \)) are given by
\[ \langle \hat{\rho}_e \rangle_0 = -\sigma_c \frac{i q \cdot E}{D q^2 - i \omega}, \]
(11)
\[ \langle \hat{j}_{\nu, i} \rangle_0 = \frac{\sigma_c}{\epsilon^\nu} \left[ \tau_{ij} + \left( D_{q i} - D_{q j} \right) \right] E_j, \]
(12)
where \( \sigma_c = n_e e^2 \tau/m \) is the Boltzmann conductivity with the electron number density \( n_e = \frac{2m}{v_F^2} N(\mu) \), the Fermi velocity \( v_F \), and the relaxation time \( \tau = (2\gamma)^{-1} \). The terms first order in \( \delta n_i \) are calculated as
\[ \langle \hat{\rho}_e \rangle_\delta = \frac{\delta n_i(Q)}{n_i} \frac{1}{D(q + Q)^2 - i \omega} \times \left[ i(q_j + Q_j) - \frac{D_{q i} - D_{q j}}{D q^2 - i \omega} \right] E_j, \]
(13)
\[ \langle \hat{j}_{\nu, i} \rangle_\delta = -\delta n_i(Q) \frac{\sigma_c \epsilon_{\alpha i j} E_i D q^2 - i \omega}{n_i} \left[ \frac{D q (q + Q)(q + Q)}{D q^2 - i \omega} \right] E_j \]
(14)
\[ - D_{q i}(q + Q)(q + Q) \langle \hat{\rho}_e \rangle_0 - D_{q j}(q + Q)(q + Q) \langle \hat{\rho}_e \rangle_0. \]

#### B. Side-jump process

We now focus on the spin channel, and calculate spin accumulation and spin-current density. The side-jump type contributions are characterized by a \( \tau \)-independent SH conductivity, or the SH angle, \( \epsilon_{\alpha i j} \), inversely proportional to \( \tau \). The terms first order in \( \delta n_i \) are classified into two types, one due to the inhomogeneity of the impurity SOI, and the other due to the inhomogeneity of normal impurities. The former is expressed by the diagrams shown in Fig. 1(a), (b) and (c), and each
neous normal impurities are given in Fig. 1(d)–(f), and the former (latter) being zeroth (first) order in \( \delta n \). The latter is shown in Fig. 2. The diagrams (a) and (c) are due to the inhomogeneous distribution of SOI impurities, and those of (d) to (f) are due to the inhomogeneous distribution of normal impurities. The shaded and hatched regions represent impurity ladder vertex contributions of (a) to (c) are due to the inhomogeneous distribution, and the triangle with a dashed line represents the inhomogeneous part of the impurity distribution. The contributions of (a) to (c) are due to the inhomogeneous distribution of SOI impurities and those of (d) to (f) are due to the inhomogeneous distribution of normal impurities. The shaded and hatched regions represent impurity ladder vertex corrections, the former (latter) being zeroth (first) order in \( \delta n \). The latter is shown in Fig. 2. The diagrams (a) and (d) come from the “anomalous” part of electric current, and those of (b) and (e) come from the “anomalous” part of spin current. The upside-down diagrams are also considered in the calculation.

set of diagram gives

\[
K^{(a),\alpha}_{\mu j} = -e\lambda_0\delta n_1(Q)u^2_0\frac{\omega}{\pi}\varepsilon_{\alpha j l} \\
\times \sum_{k,k'}v_jk_lG^R_{k+k'}G^A_{k'-k} - (G^R_{k'+k} - G^A_{k'-k}) , \quad (15)
\]

\[
K^{(b),\alpha}_{\mu j} = -e\lambda_0\delta n_1(Q)u^2_0\frac{\omega}{\pi}\varepsilon_{\alpha i l} \\
\times \sum_{k,k'}v_jk_lG^R_{k+k'}G^A_{k'-k} - (G^R_{k'+k} - G^A_{k'-k}) , \quad (16)
\]

\[
K^{(c),\alpha}_{\mu j} = -e\lambda_0\delta n_1(Q)u^2_0\frac{\omega}{\pi}\sum_{k,k'}v_jk_l[k' \times (q + Q)] \\
- k \times q_\alpha G^R_{k+k'}G^A_{k'-k} + G^R_{k+k'}G^A_{k'-k} , \quad (17)
\]

where \( G^R_{ss'} = G^R_{k+s + \frac{1}{2}, s'} + \frac{1}{2} \) and \( G^R_{ss'} = G^R_{k+s + \frac{1}{2}, s'} \) with \( s, s' = \pm \). The contributions from the inhomogeneous normal impurities are given in Fig. 1(d)–(f), and further details are described in Appendix.

With these diagrams, and also by including ladder vertex corrections, the spin accumulation and spin-current density have been obtained as

\[
\langle \hat{\sigma}^{(a),\alpha}_{\delta j} \rangle_0 = -\frac{\alpha_{\text{SH}}}{e} \frac{\delta D(Q)q \cdot i(q + Q)}{D(q + Q)^2 - i\omega + \tau_{\text{sf}}^{-1}} ,
\]

where \( D = v_F^2/3 \) and \( \delta D(Q) = -\frac{\delta n_1(Q)}{n_1} D \) are the homogeneous and inhomogeneous parts, respectively, of the diffusion coefficient, and \( \tau_{\text{sf}} = \frac{2}{3}(\frac{\lambda^2}{\omega_0^2})^3/2 + \frac{1}{2} \). The spin relaxation time. These are to be added to those in the absence of inhomogeneity \( \delta n_1 \).

The spin accumulation \( \langle \hat{\sigma}^{(a),\alpha}_{\delta j} \rangle_0 \) [Eq. (18)] is due to the modulation of diffusion coefficient \( D \times \n_1^{-1} \) in the denominator of \( \langle \hat{\sigma}^{(a),\alpha}_{\delta j} \rangle_0 \) [Eq. (20)]. Note that they vanish when the electric field is uniform (\( q = 0 \)). In the spin current \( \langle \hat{\sigma}^{(a),\alpha}_{\delta j} \rangle_0 \) [Eq. (19)], the first term is the spin Hall current arising from the diffusion current charge term [the last term in Eq. (14)], and the second and third terms are the diffusion spin current whose dependence on \( \delta n_1 \) is through the modulation of spin accumulation \( \langle \hat{\sigma}^{(a),\alpha}_{\delta j} \rangle_0 \) [Eq. (18)] and the modulation of diffusion coefficient \( \delta D \), respectively. On the other hand, we found no spin Hall current originating from the first two terms of the charge current \( \langle \hat{c}_{e,i} \rangle_0 \) [Eq. (14)]. These results are consistent with the fact that the spin Hall conductivity due to side-jump process is independent of the impurity concentration, \( \alpha_{\text{SH}} \sigma_c \propto n_1^{\alpha} \). Finally, the “inhomogeneous contribution”, \( \langle \hat{\sigma}^{(a),\alpha}_{\delta j} \rangle_0 \) [Eqs. (18) and (19)] satisfy the same continuity equation (with spin-relaxation term),

\[
\partial_t \langle \hat{\sigma}^{(a),\alpha}_{\delta j} \rangle_0 + \nabla \cdot \langle \hat{\sigma}^{(a),\alpha}_{\delta j} \rangle_0 = -\frac{\langle \hat{\sigma}^{(a),\alpha}_{\delta j} \rangle_0}{\tau_{\text{sf}}},
\]

as that of the “homogeneous contribution”. 

\[=\] \[
\text{FIG. 2. The hatched rectangle (left-hand side) represents the inhomogeneous part (first order in } \delta n_1 \text{) of the four-point impurity-ladder vertex. Only two typical diagrams are shown on the right-hand side, but diagrams without the shaded rectangle (the homogeneous part of impurity-ladder diagrams, see Fig. 1) on either or both sides should also be included.} \]
the diagrams in Fig. 3(a), the first of which leads to the inhomogeneity comes either from the impurity SOI (a) and from the normal impurities (b). The upside-down diagrams are also considered in the calculation.

C. Skew scattering process

The skew-scattering type contributions are characterized by the diagrams as shown in Fig. 3. Again, the terms first order in $\delta n_i$ are classified into two types according to whether the inhomogeneity comes from the SOI impurities or the normal impurities. The response functions of the former type (SOI inhomogeneity) are expressed by the diagrams in Fig 3(a), the first of which leads to

\[
K_{ij}^{\alpha \alpha} = e\lambda_{so}^{\alpha}i\omega \sum_{k,k'} v_{ij}^k [k \times k']^\alpha
\times \left[ G_{k'}^{R} \left( \frac{\omega}{2} \right) - G_{k'}^{A} \left( \frac{\omega}{2} \right) \right] G_{k'}^{R} + G_{k'}^{A} G_{k'}^{R}
\]

The latter type (normal-scattering inhomogeneity) is described by the diagrams in Fig 3(b). Details can be found in Appendix.

These diagrams, with ladder vertex corrections included, lead to the following expression of the spin accumulation and spin-current density,

\[
\langle \hat{\sigma}^{\alpha} \rangle_{S} = -\frac{\alpha_{so}^{\alpha}}{e} \frac{1}{D(q+Q)^2 - \omega^2 + \tau_{sf}^{-1}} \\
\times \left\{ \left[ i(q+Q) \times \langle \hat{\gamma} \rangle_{\alpha} \right]_{\alpha} + \delta Diq \cdot i(q + Q) \mid i(q + Q) \rangle \delta \right\}.
\]

\[
\langle \hat{\gamma}^{\alpha \beta} \rangle_{S} = \alpha_{so}^{\alpha} \epsilon_{\alpha \beta} - Diq_{i} \langle \hat{\sigma}^{\alpha \beta} \rangle_{S} - \delta D(q) \delta \langle \hat{\sigma}^{\alpha \beta} \rangle_{S},
\]

where $\alpha_{so}^{\alpha} = \frac{2e}{\hbar^2} k_F^2 \lambda_{so}(\mu) u_i$ is the SH angle due to skew scattering. These results are to be added to the homogeneous contributions.

\[
\langle \hat{\sigma}^{\alpha} \rangle_{S} = -\frac{\alpha_{so}^{\alpha}}{e} \frac{[i q \times E]_{\alpha}}{Dq^2 - \omega^2 + \tau_{sf}^{-1}},
\]

\[
\langle \hat{\gamma}^{\alpha \beta} \rangle_{S} = \alpha_{so}^{\alpha} \epsilon_{\alpha \beta} \langle \hat{\sigma}^{\alpha \beta} \rangle_{S}.
\]

which have exactly the same form as $\langle \hat{\sigma}^{\alpha} \rangle_{S}$ and $\langle \hat{\gamma}^{\alpha \beta} \rangle_{S}$ [Eqs. (20) and (21)] except for the coefficient ($\alpha_{so}^{\alpha}$ instead of $\alpha_{so}$).

The contributions $\langle \hat{\sigma}^{\alpha} \rangle_{S}^{\alpha} = \langle \hat{\gamma}^{\alpha \beta} \rangle_{S}^{\alpha}$ are written with the charge current $\langle \hat{\gamma}^{\alpha \beta} \rangle_{S}^{\alpha}$ [Eq. (14)]. Unlike the side-jump contribution, the spin accumulation $\langle \hat{\sigma}^{\alpha} \rangle_{S}^{\alpha}$ remains finite at $q = 0$. Thus a uniform electric field $E$ induces a spin accumulation through the inhomogeneous skew scattering. In Eq. (25), the first term is the spin Hall current arising from the charge current $\langle \hat{\gamma}^{\alpha \beta} \rangle_{S}^{\alpha}$ [Eq. (14)]. These results are consistent with the fact that the SH angle due to skew scattering is independent of the impurity concentration, thus $\alpha_{so}^{\alpha} \sigma_c \propto n_i^{-1}$. Finally, the modulated parts satisfy the same (continuity) equation as Eq. (22),

\[
\partial_t \langle \hat{\sigma}^{\alpha} \rangle_{S} + \nabla \cdot \langle \hat{\gamma}^{\alpha \beta} \rangle_{S}^{\alpha} = -\frac{\langle \hat{\sigma}^{\alpha} \rangle_{S}^{\alpha}}{\tau_{sf}}.
\]

Therefore, the total spin accumulation, $\langle \hat{\sigma}^{\alpha} \rangle_{S} = \langle \hat{\sigma}^{\alpha} \rangle_{S}^{\alpha} + \langle \hat{\gamma}^{\alpha \beta} \rangle_{S}^{\alpha}$ and the total spin current (defined similarly) also satisfy the same equation.

D. Extrinsic Rashba process

Because of the modulated distribution of SOI impurities, the SOI Hamiltonian survives the impurity average,

\[
\langle H_{so} \rangle_{av} = \lambda_{so}^{\alpha} \sum_k (iQ \times k) \cdot \psi_{k+Q}^\dagger \sigma \psi_k.
\]

This may be called “extrinsic Rashba SOI” since it arises from the inversion symmetry breaking by the perturbation of the SOI. However, the contribution to the spin accumulation and spin current, shown in Fig. 4, turned out to vanish. This is in agreement with the well-known fact that the Edelstein effect vanishes in such perturbative calculation. To obtain the spin accumulation due to the Edelstein effect via the extrinsic Rashba SOI, we need to treat it nonperturbatively. If we neglect the momentum change ($\psi_{k+Q}^\dagger \sigma \psi_k \rightarrow \psi_{k+Q}^\dagger \sigma \psi_k$), the spin accumulation is obtained as

\[
\langle \hat{\sigma}^{\alpha} \rangle = -\lambda_{so} \frac{3\sigma_c}{eV_F} \nabla \delta n_i(r) \times E \rangle_{\alpha}
\]

in three dimensions. This spin accumulation will also contribute to the charge-to-spin conversion. More faithful analysis to the model [Eq. (29)] will be presented elsewhere.

IV. DISCUSSION

In the present model, there are two origins of spatial modulation of the currents, impurity distribution (with
wave vector $\mathbf{Q}$ and the applied electric field (with wave vector $\mathbf{q}$).

Let us first consider the case that a uniform electric field ($\mathbf{q} = 0$) is applied to the inhomogeneous system ($\mathbf{Q} \neq 0$). The spin accumulation and spin current are given by

$$
\langle \delta^\alpha (\mathbf{Q} ) \rangle_\delta = - \frac{\alpha_{\mathbf{SH}}^{\mathbf{ss}}}{-e} \frac{[i \mathbf{q} \times \langle \mathbf{j}_e(\mathbf{Q} ) \rangle_\alpha]}{D \mathbf{Q}^2 - i \omega + \tau_{\mathbf{sf}}},
$$

$$
\langle \delta^\alpha (\mathbf{Q} ) \rangle_\delta = \alpha_{\mathbf{SH}}^{\mathbf{ss}} \epsilon_{\alpha ij} \frac{[i \mathbf{q} \times \langle \mathbf{j}_e(\mathbf{Q} ) \rangle_\alpha]}{-e} - D \mathbf{Q} [\langle \sigma^\alpha (\mathbf{Q} ) \rangle_\delta] - \alpha_{\mathbf{SH}}^{\mathbf{sj}} \epsilon_{\alpha ij} D \mathbf{Q} \mathbf{j}_s [\langle \rho_e(\mathbf{Q} ) \rangle_\delta],
$$

where $\langle \rho_e(\mathbf{Q} ) \rangle_\delta$ and $\langle \mathbf{j}_e(\mathbf{Q} ) \rangle_\delta$ are given, respectively, by Eqs. (13) and (14) with $\mathbf{q} = 0$. They satisfy the spin diffusion equation,

$$
(D \nabla^2 - \partial_t - \tau_{\mathbf{sf}}^{-1}) \langle \sigma^\alpha \rangle = \alpha_{\mathbf{SH}}^{\mathbf{ss}} \Omega_e^\alpha,
$$

where $\Omega_e = \frac{1}{2} \nabla \times \mathbf{j}_e$ is the vorticity of the electron flow. The right-hand side shows that the vorticity acts as a spin source via SHE. This may be considered as the “effective SVC” in laboratory (inertial) frame, and this is caused by the spin-orbit coupling, or more specifically, by the skew-scattering process. The side-jump contribution is absent because it is independent of impurity concentration, namely, $\langle \delta^\alpha \rangle_0^{(\mathbf{ss})}$ is spatially uniform (if $\mathbf{E}$ is uniform) even if $n_i(\mathbf{r})$ is inhomogeneous.

When a spatially modulated electric field ($\mathbf{q} \neq 0$) is applied to a uniform system ($\mathbf{Q} = 0$), the spin diffusion equation is given by

$$
(D \nabla^2 - \partial_t - \tau_{\mathbf{sf}}^{-1}) \langle \sigma^\alpha \rangle = \alpha_{\mathbf{SH}}^{\mathbf{sj}} \mathbf{E} |\mathbf{E}|, \tag{34}
$$

with $\alpha_{\mathbf{SH}}^{\mathbf{sj}} = \alpha_{\mathbf{SH}}^{\mathbf{ss}} + \alpha_{\mathbf{SH}}^{\mathbf{sj}}$. The right-hand side is the spin source term, which can also be written as $\alpha_{\mathbf{SH}} \Omega_e^\alpha$ in terms of the vorticity $\Omega_e = \frac{1}{2} \nabla \times \mathbf{j}_e$ of the electric current $\mathbf{j}_e = \sigma_e \mathbf{E}$ generated now by the nonuniform electric field. Therefore, we can see a strong connection between the vorticity of the electron flow and the spin accumulation independently of their origin.

For general cases with $\mathbf{q} \neq 0$ and $\mathbf{Q} \neq 0$, the sum of the inhomogeneous contribution, Eqs. (18), (19), (24) and (25), and the homogeneous contribution, $\langle \delta^\alpha \rangle_0^{(\mathbf{ss})}$ and $\langle \delta^\alpha \rangle_0^{(\mathbf{ss})}$, leads to

$$
\langle \delta^\alpha \rangle = - \Pi(\mathbf{q}) \frac{\alpha_{\mathbf{SH}}^{\mathbf{ss}}}{-e} [i \mathbf{q} \times \langle \mathbf{j}_e(\mathbf{Q} + \mathbf{q} + \mathbf{Q} ) \rangle_\alpha] + \Pi(\mathbf{Q} + \mathbf{q} + \mathbf{Q} ) \frac{\alpha_{\mathbf{SH}}^{\mathbf{ss}}}{-e} \delta D(\mathbf{Q} ) [i \mathbf{q} \times \langle \mathbf{j}_e(\mathbf{Q} ) \rangle_\alpha],
$$

$$
\langle \delta^\alpha \rangle = \alpha_{\mathbf{SH}}^{\mathbf{ss}} \epsilon_{\alpha ij} \frac{[i \mathbf{q} \times \langle \mathbf{j}_e(\mathbf{Q} + \mathbf{q} + \mathbf{Q} ) \rangle_\alpha]}{-e} - \alpha_{\mathbf{SH}}^{\mathbf{sj}} \epsilon_{\alpha ij} D \mathbf{j}_s [\langle \hat{\rho}_e(\mathbf{Q} + \mathbf{q} + \mathbf{Q} ) \rangle_\alpha] - [D + \delta D(\mathbf{Q} )] [i \mathbf{q} \times \langle \mathbf{j}_e(\mathbf{Q} + \mathbf{q} + \mathbf{Q} ) \rangle_\alpha],
$$

where $\Pi(\mathbf{q}) = (D \mathbf{q}^2 - i \omega + \tau_{\mathbf{sf}}^{-1})^{-1}$ is the spin diffusion propagator. The first terms in Eqs. (35) and (36) are zeroth order in $\delta n_i$, and the remaining terms are first order. Noting that the wave vectors correspond to spatial gradients $\nabla$ in real space ($\mathbf{q}$ acting on the electric field, and $\mathbf{Q}$ acting on the impurity concentration), the spin accumulation (35) and spin current (36) are expressed in real space as

$$
\langle \delta^\alpha (\mathbf{r} ) \rangle = - \frac{\alpha_{\mathbf{SH}}^{\mathbf{ss}}}{-e} \Pi(\nabla) \left[ 1 + \nabla \cdot (\delta D(\mathbf{Q} + \mathbf{q} + \mathbf{Q} ) \nabla) \right] \nabla \times \langle \mathbf{j}_e(\mathbf{r} ) \rangle_0 - \Pi(\nabla) \frac{\alpha_{\mathbf{SH}}^{\mathbf{ss}}}{-e} \nabla \times \langle \mathbf{j}_e(\mathbf{r} ) \rangle_\delta,
$$

$$
\langle \delta^\alpha (\mathbf{r} ) \rangle = \epsilon_{\alpha ij} \left[ \frac{\alpha_{\mathbf{SH}}^{\mathbf{ss}}}{-e} \langle \mathbf{j}_e(\mathbf{r} ) \rangle_0 + \frac{\alpha_{\mathbf{SH}}^{\mathbf{ss}}}{-e} \langle \mathbf{j}_e(\mathbf{r} ) \rangle_\delta \right] - \frac{\alpha_{\mathbf{SH}}^{\mathbf{sj}}}{-e} \epsilon_{\alpha ij} D \mathbf{j}_s [\langle \hat{\rho}_e(\mathbf{r} ) \rangle_\delta - D(\mathbf{r} ) \nabla \mathbf{j}_s [\langle \sigma^\alpha (\mathbf{r} ) \rangle_0 - D \nabla \mathbf{j}_s [\langle \sigma^\alpha (\mathbf{r} ) \rangle_\delta],
$$

FIG. 4. Rashba type contribution to the spin current and spin accumulation which are first order in the impurity potential strength $u_i$. The diagrams (a) and (b) come from the anomalous part of the electric current and spin current operators, respectively. The diagrams without vertex corrections (i.e., without shaded regions) in (a), (b) and (c) are canceled out. The diagrams with left vertex corrections in (a) and (c) (i.e., without shaded regions) in (a), (b) and (c) are canceled out, and so are the diagrams with right vertex corrections in (b) and (c). The diagram having vertex corrections on both sides in (c) vanishes up to $q^7$. 

\[ \]
where \( \dot{\Pi}(\nabla) = (-D\nabla^2 + \partial_t + \tau^{-1}_s) \) is the spin diffusion propagator expressed in real space and \( D(r) = D + \delta D(r) \) is the diffusion coefficient that includes the effects of in-homogeneity, \( \delta D(r) = \delta D(Q) e^{iQ \cdot r} \).

In the following, we drop the brackets \( \langle \cdots \rangle \) and simply write as \( \sigma(r) = \langle \sigma(r) \rangle \), etc. Quantities with explicit dependence on \( r \) include the spatially-modulated parts up to the first order in \( \delta n_i(r) \). The spin accumulation \( \langle \sigma \rangle \) and spin current \( j^\alpha \) satisfy the generalized spin diffusion equation,

\[
\left\{ \nabla \cdot [D(r)\nabla] - \partial_t - \tau^{-1}_s \right\} \sigma^\alpha(r) = \nabla \cdot j^\alpha_{SH}(r),
\]

where \( j^\alpha_{SH}(r) \) is the spin Hall current,

\[
j^\alpha_{SH,\alpha}(r) = \frac{\alpha_{SH}(r)}{-e} \epsilon_{\alpha ij} j_\alpha(r),
\]

and \( j_\alpha(r) \) is the charge current,

\[
j_\alpha(r) = \sigma_c(r) E(r) - D(r) \nabla \rho_c(r),
\]

which of course satisfies the continuity equation, \( \partial_t \rho_c(r) + \nabla \cdot j_\alpha(r) = 0 \). The right-hand side of Eq. (39) is the spin source term coming from the divergence of the spin Hall current. This can be written in the form of "effective SVC" only if the SH angle is constant without spatial modulation, \( \nabla \cdot j^\alpha_{SH}(r) = \alpha_{SH} (\nabla \times j_\alpha(r)) / (-e) \).

Finally, if we define the total spin current by the sum of the drift and diffusion spin currents,

\[
j^\alpha_s(r) = j^\alpha_{SH}(r) - D(r) \nabla \sigma^\alpha(r),
\]

Eq. (39) reduces to the spin continuity equation,

\[
\partial_t \sigma^\alpha(r) + \nabla \cdot j^\alpha_s(r) = -\frac{\sigma^\alpha(r)}{\tau_s}. \quad (43)
\]

The obtained equations may be used widely for phenomenological analyses of inhomogeneous systems; Eq. (39) for the determination of spin accumulation (spin chemical potential), and Eq. (42) for boundary conditions imposed on the spin current.

V. CONCLUSION

We studied extrinsic spin Hall effect in systems with inhomogeneously distributed impurities. We found that the spin accumulation is induced by the inhomogeneity via the side-jump and skew-scattering processes. The results satisfy a generalized spin diffusion equation with a spin source term, which is expressed as the divergence of spin Hall current and reduces to the "effective SVC in the laboratory (inertial) frame" if the SH angle is homogeneous. It remains a challenge for future research to study the Edelstein effect due to the "extrinsic Rashba SOI" with non-perturbative treatment. It would be interesting to apply the obtained spin diffusion equation to the experiments of spin-current injection from surface oxidized copper. These are left as future studies.

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Appendix A: Ladder vertex corrections

In this Appendix, we calculate ladder vertex corrections due to impurities. The four-point vertex \( \Gamma_{ab,cd} \) in the absence of impurity inhomogeneity (zeroth order in \( \delta n_i \)), shown in Fig. 5, is given by

\[
\Gamma_{ab,cd}(q) = \Gamma_c(q) \delta_{ab} \delta_{cd} + \Gamma_s(q) \sigma_{ab} \cdot \sigma_{cd}, \quad (A1)
\]

where \( a-d \) are spin indices, and

\[
\Gamma_c(q) = \frac{1}{4\pi N(\mu)\tau^2} \frac{1}{D(q^2 - i\omega)}, \quad (A2)
\]

\[
\Gamma_s(q) = \frac{1}{4\pi N(\mu)\tau^2} \frac{1}{D(q^2 - i\omega + \tau_s^{-1})}. \quad (A3)
\]

The four-point vertex first order in \( \delta n_i \) is shown in Fig. 7, as

\[
\delta \Gamma_{ab,cd}(Q) = \delta \Gamma_c(q) \delta_{ab} \delta_{cd} + \delta \Gamma_s(q) \sigma_{ab} \cdot \sigma_{cd}, \quad (A4)
\]

where \( \delta \Gamma_{c(s)} \) is given by

\[
\delta \Gamma_{c(s)}(Q) = 4\Gamma_{c(s)}(q + Q) \Gamma_{c(s)}(q) \frac{\delta n_i(Q)}{n_i} \frac{\gamma}{\pi N(\mu)} \frac{1}{\pi N(\mu)}
\]

\[
\times \left[ I_0(q + Q) I_0(q) - Y_0(q) \right], \quad (A5)
\]

where \( I_0 \) and \( Y_0 \) are given in Appendix C. Thus,

\[
\delta \Gamma_c(Q) = \frac{\delta n_i(Q)}{n_i} \frac{\gamma}{\pi N(\mu)} \frac{1}{D(q + Q)^2 - i\omega} \frac{1}{\tau(D^2 - i\omega)^2}, \quad (A6)
\]

\[
\delta \Gamma_s(Q) = \frac{\delta n_i(Q)}{n_i} \frac{\gamma}{\pi N(\mu)} \frac{1}{D(q + Q)^2 - i\omega + \tau_s^{-1}}
\]

\[
\times \frac{Dq \cdot (q + Q)}{\tau(D^2 - i\omega) + \tau_s}. \quad (A7)
\]

Next, the three-point vertices, \( \Lambda^c \) and \( \Lambda^s \), shown in
\[ \Gamma_{ab,cd}(q) = \begin{array}{c|c} \alpha + \beta & \gamma + \delta \\ \hline a & b \\ \hline c & d \end{array} \]

\[ = \begin{array}{c} + \begin{array}{c|c} \alpha + \beta & \gamma + \delta \\ \hline a & b \\ \hline c & d \end{array} \\ + \begin{array}{c} \end{array} \end{array} \]

FIG. 5. Four-point vertex in the ladder approximation due to homogeneously distributed impurities. The circle with a cross represents the sum of spin-flip and spin-nonflip vertices. \( a-d \) represent spin indices.

The three-point vertices, \( \Pi^{(L)} \) and \( \Pi^{(R)} \), shown in Fig. 8 are calculated as

\[ \Pi^{(L)}(q) = 2\Gamma_{\mu}(q) \frac{\delta n_i(Q)}{n_i} \frac{\gamma}{\pi N(\mu)} \left[ I_0(q)I_{\mu}(q + Q) - Y_{\mu}(Q) \right] + \delta \Gamma_{\mu}(q + Q) \]  

(A10)

\[ = \begin{cases} \frac{\delta n_i(Q)}{n_i} \frac{\gamma}{\pi N(\mu)} \left( 1 - \frac{\tau_{sf}}{\tau} \right)^2 \frac{Dq \cdot (q + Q)}{\tau(Dq^2 - i\omega)} + \frac{1}{\tau} \right), & (\mu = 0), \\
\frac{\delta n_i(Q)}{n_i} \frac{\gamma}{\pi N(\mu)} \left( 1 - \frac{\tau_{sf}}{\tau} \right)^{-1} \frac{Dq \cdot (q + Q)}{\tau(Dq^2 - i\omega) + \frac{i}{\tau} \tau}, & (\mu = i), \end{cases} \]  

(A11)

\[ \Pi^{(R)}(q) = 2\Gamma_{\nu}(q + Q) \frac{\delta n_i(Q)}{n_i} \frac{\gamma}{\pi N(\mu)} \left[ I_0(q + Q)I_{\nu}(q) - Y_{\nu}(Q) \right] + \delta \Gamma_{\nu}(q) \]  

(A12)

\[ = \begin{cases} \frac{\delta n_i(Q)}{n_i} \frac{\gamma}{\pi N(\mu)} \left( \frac{Dq \cdot (q + Q)}{\tau(Dq^2 - i\omega)} \right), & (\nu = 0), \\
\frac{\delta n_i(Q)}{n_i} \frac{\gamma}{\pi N(\mu)} \left( Dq(q + Q) - \frac{Di q_j (q + Q)}{\tau(Dq^2 - i\omega)} \right), & (\nu = j). \end{cases} \]  

(A13)

Appendix B: Response functions

The response functions of side-jump process contributed by the inhomogeneous scattering due to impurities (see Fig. 9) are derived as

\[ L_{\mu,\nu}^{(\alpha)} = -\frac{\epsilon_i \omega}{2\pi \sigma_{sh}} \frac{\delta n_i(Q)}{n_i} \frac{\gamma}{\pi N(\mu)} \epsilon_{\alpha l m} \tau \left[ \frac{i q_l (q + Q) I_{\mu}(q) - i(q_l + Q_l)I_{\mu}(q + Q)I_{\nu}(q)}{\tau(Dq^2 - i\omega)} \right]. \]  

(B1)

\[ L_{\mu,\nu}^{(d,\alpha)} = -\frac{\epsilon_i \omega}{2\pi \sigma_{sh}} \frac{\delta n_i(Q)}{n_i} \frac{\gamma}{\pi N(\mu)} \epsilon_{\alpha l j} \left[ I_{\mu}(q + Q)I_{\nu}(q) - Y_{\mu l}(Q) \right]. \]  

(B2)

FIG. 6. Four-point vertex in the ladder approximation which is first order in the inhomogeneity of normal impurity scattering.
FIG. 7. Three-point vertices with ladder vertex corrections of Fig. 5.

\[
\delta_{ab}\Lambda^{c}_{\nu}(q) = \begin{array}{c}
\text{a} \\
\text{b}
\end{array} \quad \text{and} \quad \sigma^{\alpha}_{ab}\Lambda^{\alpha}_{\nu}(q) = \begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\]

FIG. 8. Left and right three-point vertices in the first order of \( \delta n_i \).

\[
\Pi^{(L)}_{\mu}(Q) = \quad + \quad + \quad \Pi^{(R)}_{\nu}(Q) = \quad + \quad +
\]

With the ladder vertex corrections included, the response functions are expressed as

\[
L^{(c),\alpha}_{ij}(\omega = -\frac{i\omega}{2\pi} \delta n_i(Q) \frac{\gamma}{\pi N(\mu)} \epsilon_{\alpha i j} \left[ I_i(q + Q)I_j(q) - Y_{\mu}(\omega, Q) \right],
\]

\[
B_4 \quad + \quad i(q_i + Q_i) \left\{ (I_{\mu m}I_0 - I_{\mu j}I_m) q_i + Q_i I_j(q) - \left( Y_{\mu m}(Q)I_i(q) - Y_{\mu j}(Q)I_m(q) \right) \right\},
\]

With the ladder vertex corrections included, the response functions are expressed as

\[
K^{(a-c),\alpha}_{ij} = \left[ 1 + \Lambda_{ij}(q + Q) \right] \left[ L^{(c),\alpha}_{ij}(Q) + \left[ L^{(a),\alpha}_{ij}(Q) + \frac{\delta n_i(Q)}{n_i} \right] \frac{\gamma}{\pi N(\mu)} \epsilon^{\alpha}_{ij} \left( I_i(q + Q)I_j(q) - Y_{\mu}(\omega, Q) \right) \right],
\]

\[
K^{(a-c),\alpha}_{ij} = \frac{\delta n_i(Q)}{n_i} \left[ X^{(a),\alpha}_{ij}(Q) + X^{(b),\alpha}_{ij}(Q) \right] + \left[ L^{(c),\alpha}_{ij}(Q) + \frac{\delta n_i(Q)}{n_i} \right] \frac{\gamma}{\pi N(\mu)} \epsilon^{\alpha}_{ij} \left( I_i(q + Q)I_j(q) - Y_{\mu}(\omega, Q) \right) \right],
\]

FIG. 9. Contributions to the response function \( L^{(c),\alpha}_{ij} \). The self-energy type and side-jump type parts can also be located at the lower line of the diagram. All such possible patterns are considered in the calculation.
FIG. 10. Contributions to the response function \( L_{\mu\nu}^{(a)} \). As in the case of Fig. 3, all possible patterns as to the position of the self-energy type part are considered in the calculation.

\[
K_{ij}^{(d-f),\alpha} = \left[ 1 + \Lambda_0^\alpha(q + Q) \right] \left( L_{ij}^{(d),\alpha} + L_{ij}^{(f),\alpha} \right) + \Lambda_i^\alpha(q) + \left[ X_{ij}^{(b),\alpha}(q + Q) + X_{ij}^{(c),\alpha}(q + Q) \right] \Pi_{ij}^{(R)}(Q)
\]

The response functions are given as

\[
\langle \sigma^\alpha \rangle_{\delta}^{\text{so}(a-c)} = -\alpha_{SH} \frac{\delta n_i(Q)}{n_i} \frac{\sigma_c}{-e} \left[ \frac{i(q + Q) \times E}{\alpha} \right] \frac{\delta n_i(Q)}{n_i} \frac{\sigma_c}{-e} \frac{e_{\text{al}m}iQ_l}{Dq_m^2 - i\omega + \tau_f^{-1}} \frac{Di}{q_m \cdot E},
\]

\[
\langle j_{n,\delta}^{\text{so}(a-c)} \rangle = \alpha_{SH} \frac{\delta n_i(Q)}{n_i} \frac{\sigma_c}{-e} \left[ \frac{j_{q,\delta}(q)}{-e} \right] \frac{1}{D(q + Q)^2 - i\omega + \tau_f^{-1}} \left[ \frac{i(q + Q) \times E}{\alpha} \right] \frac{\delta n_i(Q)}{n_i} \frac{\sigma_c}{-e} \frac{e_{\text{al}m}iQ_l}{Dq_m^2 - i\omega + \tau_f^{-1}} \frac{Di}{q_m \cdot E},
\]

Equations [(9)] and [(10)] are due to the inhomogeneity of SOI impurities, and Eqs. [(11)] and [(12)] are due to the inhomogeneity of normal impurities.

The response function \( L_{\mu\nu}^{(ss)} \) due to skew scattering from the inhomogeneous impurities, shown in Fig. 10, is expressed as

\[
L_{\mu\nu}^{(ss)} = i\omega \alpha_{SH} \frac{\delta n_i(Q)}{n_i} \frac{\sigma_c}{-e} \left( \frac{3}{\sqrt{2}} \right) \left( \frac{\gamma}{\pi N(\mu)} \right)^3 \epsilon_{\text{al}m} \left\{ I_{\mu}(q + Q)I_{\nu}(q) - Y_{\mu\nu}(Q) \right\} I_{\nu m}(q) + I_{\mu}(q + Q)I_{\nu}(q) - Y_{\mu m}(Q)
\]

With the ladder vertex corrections included, the response functions are given as

\[
K_{0j}^{(a),\alpha} = i\omega \alpha_{SH} \frac{\delta n_i(Q)}{n_i} \frac{\sigma_c}{-e} \left( \frac{3}{\sqrt{2}} \right) \left( \frac{\gamma}{\pi N(\mu)} \right)^3 \epsilon_{\text{al}m} \left[ 1 + \Lambda_0^\alpha(q + Q) \right] I_i(q + Q)I_{mj}(q) + I_{mj}(q)\Lambda_0^\alpha(q + Q)
\]

\[
K_{ij}^{(a),\alpha} = i\omega \alpha_{SH} \frac{\delta n_i(Q)}{n_i} \frac{\sigma_c}{-e} \left( \frac{3}{\sqrt{2}} \right) \left( \frac{\gamma}{\pi N(\mu)} \right)^3 \epsilon_{\text{al}m} \left\{ I_i(q + Q)I_{mj}(q) + I_{mj}(q)\Lambda_0^\alpha(q + Q) \right\} + \Lambda_i^\alpha(q + Q)I_{ij}(q + Q) + \Lambda_i^\alpha(q + Q)\left[ X_{ij}^{(b),\alpha}(q + Q) + X_{ij}^{(c),\alpha}(q + Q) \right] \Pi_{ij}^{(R)}(Q)
\]

\[
K_{0j}^{(b),\alpha} = \left[ 1 + \Lambda_0^\alpha(q + Q) \right] I_{0j}^{(a)} + \Lambda_0^\alpha(q + Q)\Pi_{0j}^{(R)}(Q)\X_{ij}^{(a),\alpha}(q),
\]

\[
K_{ij}^{(b),\alpha} = I_{ij}^{(a),\alpha} + \left[ I_{0j}^{(a),\alpha} + X_{ij}^{(b),\alpha}(q + Q)\Pi_{ij}^{(R)}(Q) + \Lambda_i^\alpha(q + Q)\left[ X_{ij}^{(a),\alpha}(q + Q) + X_{ij}^{(c),\alpha}(q + Q) \right] \Pi_{ij}^{(R)}(Q) \right] \X_{ij}^{(b),\alpha}(q) + \left[ X_{ij}^{(b),\alpha}(q + Q) + X_{ij}^{(c),\alpha}(q + Q) \right] \Pi_{ij}^{(R)}(Q) \X_{ij}^{(c),\alpha}(q).
\]
The spin accumulation and spin current coming from each contributions are derived as

\[
\langle \sigma^{\alpha} \rangle_{\delta} = -\alpha_{SH}^{\alpha} \frac{n_i(Q)}{n_i} e^{i \omega} \delta n_i(Q) - e^{i \omega} \frac{D(q + Q)^2 - i \omega + \tau_s^2}{Dq^2 - i \omega} \frac{Dq \cdot i q \cdot E}{Dq^2 - i \omega},
\]

(B18)

\[
\langle j_{s,i}^{\alpha} \rangle_{\delta} = -\alpha_{SH}^{\alpha} \frac{n_i(Q)}{n_i} e^{i \omega} \delta n_i(Q) e^{i \omega} \frac{D(q + Q)^2 - i \omega + \tau_s^2}{Dq^2 - i \omega} \frac{Dq \cdot i q \cdot E}{Dq^2 - i \omega},
\]

(B19)

Equations (B18) and (B19) are due to the inhomogeneity of SOI impurities, and Eqs. (B20) and (B21) are due to the inhomogeneity of normal impurities.

The response functions of the “extrinsic Rashba process” (Sec. IID) are derived from the diagrams shown in Fig. 4

\[
K_{\mu j}^{\alpha} = e^{\lambda_{\omega}} \delta n_i(Q) u_i \frac{e^{i \omega} \epsilon_{aij} Q}{\sum_k v_k G_{++}^{R} G_{+-}},
\]

(B22)

\[
K_{i \nu}^{\alpha} = -e^{\lambda_{\omega}} \delta n_i(Q) u_i \frac{e^{i \omega} \epsilon_{aij} Q}{\sum_k v_k G_{+-}^{R} G_{++}},
\]

(B23)

\[
K_{\mu \nu}^{\alpha} = e^{\lambda_{\omega}} \delta n_i(Q) u_i \frac{e^{i \omega} \epsilon_{aij} Q}{\sum_k v_k v_\nu G_{++}^{R} G_{+-} \left[ \left( k + \frac{q}{2} \right) m G_{+-}^{R} + \left( k - \frac{q}{2} \right) m G_{++}^{R} \right]}.
\]

(B24)

With the ladder vertex correction included, their sum turns out to vanish.

We define the response functions of charge and current densities as \( \langle j_{c}\mu (q + Q) \rangle_{\delta} = K_{c j}^{\mu} A_j(q, \omega) \). The response functions of charge and current densities without ladder vertex corrections (see Fig. 14) are derived as

\[
L_{\mu \nu}^{cc} = e^{\lambda_{\omega}} \delta n_i(Q) u_i i \omega \frac{\epsilon_{aij} Q}{n_i n_i} \left[ I_{\mu} (q + Q) I_{\nu} (q) - Y_{\mu \nu} (Q) \right].
\]

(B25)

With the ladder vertex corrections included, the response functions (see Fig. 11) are derived as

\[
K_{0 j}^{cc} = \left[ 1 + \Lambda_0^{0} (q + Q) \right] L_{0 j}^{cc} + e^{\lambda_{\omega}} \frac{i \omega}{\pi} I_0 (q + Q) \Pi_j^{(R)} (Q)
\]

(B26)

\[
K_{ij}^{cc} = L_{0 j}^{cc} + e^{\lambda_{\omega}} \pi \frac{i \omega}{\pi} I_0 (q + Q) \Pi_j^{(R)} (Q).
\]

(B27)

Appendix C: Integrals

The integrals \( I_{\mu \nu} (q) \) defined by

\[
I_{\mu \nu} (q) \equiv \sum_k v_\mu v_\nu G_{k+}^{R} G_{k-},
\]

(C1)

are calculated as

\[
I_0 (q) = I_{00} (q) = \frac{\pi N(\mu)}{\gamma} \left[ 1 - \tau (Dq^2 - i \omega) \right],
\]

(C2)

\[
I_i (q) = I_{i0} (q) = -Di_q \frac{\pi N(\mu)}{\gamma},
\]

(C3)

\[
I_{ij} (q) = \delta_{ij} \frac{\pi N(\mu)}{\gamma} \left[ 1 - \tau (Dq^2 - i \omega) \right].
\]

(C4)

The integrals \( Y_{\mu \nu} \) defined by

\[
Y_{\mu \nu} (q, Q) \equiv i \pi N(\mu) \sum_k v_\mu v_\nu G_{k+}^{R} G_{k-} - (G_{k+}^{R} - G_{k-}^{A}),
\]

(C5)
are calculated as

\[ Y_0(q, Q) = \left( \frac{\pi N(\mu)}{\gamma} \right)^2 \left[ 1 - \tau (Dq^2 - i\omega) - \tau [D(q + Q)^2 - i\omega] - \tau Dq \cdot (q + Q) \right]. \tag{C6} \]

\[ Y_i(q, Q) = -\left( \frac{\pi N(\mu)}{\gamma} \right)^2 Di(2q_i + Q_i), \tag{C7} \]

\[ Y_{ij}(q, Q) = \delta_{ij} \frac{v_F^2}{3} \left( \frac{\pi N(\mu)}{\gamma} \right)^2 \left[ 1 - \tau (Dq^2 - i\omega) - \tau [D(q + Q)^2 - i\omega] - \tau Dq \cdot (q + Q) \right]. \tag{C8} \]

Side-jump type response functions are derived as

\[ X_{ij}^{(a), \alpha}(q) = e^{i\omega \frac{\alpha_{ij}}{2\pi} \epsilon_{\alpha \alpha} I_{\mu l}(q)}. \tag{C9} \]

\[ X_{ij}^{(b), \alpha}(q) = -e^{i\omega \frac{\alpha_{ij}}{2\pi} \epsilon_{\alpha \alpha} I_{\mu l}(q)}, \tag{C10} \]

\[ X_{ij}^{(c), \alpha}(q) = -e^{i\omega \frac{\alpha_{ij}}{2\pi} \epsilon_{\alpha \alpha} I_{\mu l}(q)} \frac{1}{\epsilon_{\alpha \alpha}} \left[ -e^{i\omega \frac{\alpha_{ij}}{2\pi} \epsilon_{\alpha \alpha} I_{\mu l}(q)} \right]. \tag{C11} \]

Skew-scattering type response functions are derived as

\[ X_{\mu \nu}^{(ss), \alpha}(q) = i\omega \frac{\alpha_{ss} \epsilon_{\alpha \alpha}}{v_F^2} \left( \frac{\gamma}{\pi N(\mu)} \right)^2 \epsilon_{\alpha \alpha} I_{\mu l}(q) I_{\nu m}(q). \tag{C12} \]

1. M. I. D’yakonov and V. I. Perel’, JETP Lett. 13, 467 (1971).
2. J. E. Hirsch, Phys. Rev. Lett. 83, 1834 (1999).
3. S. Murakami, N. Nagaosa, and S. C. Zhang, Science 301, 1348 (2003).
4. J. Sinova, D. Culcer, Q. Niu, N. A. Sinitsyn, T. Jungwirth, and A. H. MacDonald, Phys. Rev. Lett. 92, 126603 (2004).
5. Y. K. Kato, R. C. Myers, A. C. Gossard, and D. D. Awschalom, Science 306, 1910 (2004).
6. E. Saitoh, M. Ueda, and H. Miyajima, Appl. Phys. Lett. 88, 182509 (2006).
7. H. An, Y. Kageyama, Y. Kanno, N. Enishi, and K. Ando, Nat. Commun. 7, 13069 (2016).
8. R. Enoki, H. Gamou, M. Kohda, and J. Nitta, Appl. Phys. Express 11, 033001 (2018).
9. G. Okano, M. Matsuo, Y. Ohnuma, S. Maekawa, and Y. Nozaki, Phys. Rev. Lett. 122, 217701 (2019).
10. Y. Hibino, T. Hirai, K. Hasegawa, T. Koyama, and D. Chiba, Appl. Phys. Rev. 111, 132404 (2017).
11. K. Hasegawa, Y. Hibino, M. Suzuki, T. Koyama, and D. Chiba, Phys. Rev. B 98, 020405(R) (2018).
12. M. Matsuo, J. Ieda, K. Harii, E. Saitoh, and S. Maekawa, Phys. Rev. B 87, 180402(R) (2013).
13. M. Matsuo, Y. Ohnuma, and S. Maekawa, Phys. Rev. B 96, 020401(R) (2017).
14. R. Takahashi, M. Matsuo, M. Ono, K. Harii, H. Chudo, S. Okaya, J. Ieda, S. Takahashi, S. Maekawa, and E. Saitoh, Nature Physics 12, 52 (2016).
15. D. Kobayashi, T. Yoshikawa, M. Matsuo, R. Iguchi, S. Maekawa, E. Saitoh, and Y. Nozaki, Phys. Rev. Lett. 119, 077202 (2017).
16. K. Hosono, A. Yamaguchi, Y. Nozaki, and G. Tatara, Phys. Rev. B 83, 144428 (2011).
17. V. M. Edelstein, Solid State Commun. 73, 233 (1990).