Tunneling Phenomena in Cosmology: Some Fundamental Problems

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ABSTRACT

The fundamental problem of how tunneling in thermal medium is completed is addressed, and a new time scale of order $1/\text{friction}$ for its termination, which is usually much shorter than the Hubble time, is pointed out. Enhanced non-linear resonance is responsible for this short time scale. This phenomenon occurs when the semiclassical periodic motion in a metastable potential well resonates with one of the environment harmonic oscillators coupled to its motion.

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The usual picture of how the first order phase transition proceeds in cosmology, for instance the electroweak phase transition which is relevant for the scenario of electroweak baryogenesis [1], is something like this; nucleation of the bubble of the true ground state (of broken symmetry) occurs with some probability in different parts of the universe. If the bubble formation does not take place strongly, the phase transition is terminated only when cosmological evolution changes the form of the potential (or more properly of the free energy) so that the symmetric phase is no longer the local minimum of the potential. This picture relies on the presumption that there is no inherent time scale for termination of the phase transition, or at least even if it exists, it is much larger than the Hubble time scale, which is the time for the potential change. We would like to examine this problem by working in a new formalism of the real-time description of the tunneling phenomena [2], [3]. Clearly, if the new time scale of the phase transition is much shorter than the Hubble time, one must reconsider the usual scenario. This might change the situation on, and even resurect the once failed GUT phase transition for the inflationary universe scenario [1].

Another important ingredient for our consideration is effect of cosmic environment; the order parameter that describes the state of the universe such as the homogeneous part of the Higgs field is necessarily coupled to matter fields that make up the thermal environment. One must then take into account the presence of thermal medium in discussion of the tunneling rate and its time evolution, and estimate how dissipation due to the environment interaction modifies the basic tunneling rate. There are already many works on the subject of tunneling in medium [5], and most past works deal with a system in equilibrium as a whole. The Euclidean technique such the bounce solution [6] is often used in this context [7], [8]. Our approach here is different, and we attempt to clarify dynamics of time evolution starting from an arbitrary initial state of the tunneling system, which can be either a pure or a mixed state. We find it more illuminating to use a real-time formalism instead of the Euclidean method much employed in the literature.

Our formalism is based on separation of a subsystem from thermal environment, and integrating out the environment variable including the interaction with the subsystem. In this picture dissipation seen in the behavior of the subsystem is due to our ignorance of huge environment degrees of freedom. Modeling of the environment and its interaction form with the subsystem is expected to be insensitive to the result one derives in this approach. We use the standard model of environment [9], [7].
This method is best suited to a (by itself) non-equilibrium system which is immersed in a larger thermal equilibrium state. This methodology has been used by us in a number of problems in cosmology [10], [11]. For instance, the relic abundance was calculated [11] from this approach and the usual estimate of thermally averaged Boltzmann rate was justified at high temperatures. At very low temperatures $T \ll M$ (the mass of relic particle) our calculation differs from the usual one; the Boltzmann suppressed number density $(MT/2\pi)^{3/2} e^{-M/T}$ at the freeze-out is replaced by some temperature power term. (There is some criticism against this calculation [12].) But numerical importance of this effect is presumably minor, although it is a theoretically interesting issue.

Here we consider the most basic problem of this kind, one dimensional subsystem described by a potential $V(q)$. This system is put in thermal medium. The potential is assumed to have some local minimum at $q = 0$, which is separated at $q = q_B$ of the barrier top from a global minimum. The environment part is modeled by infinitely many, continuously distributed harmonic oscillators whose coordinates are $Q(\omega)$. Its coupling to the tunneling system is given by a Hamiltonian,

$$ q \int_{\omega_c}^{\infty} d\omega c(\omega)Q(\omega). $$

(1)

The coupling strength is $c(\omega)$ and $\omega_c$ is some threshold frequency. Needless to say, one may imagine a generalized case in which the system variable $q$ is the order parameter for the first order phase transition, the homogeneous Higgs field, and the environment oscillator $Q(\omega)$ is a collection of various forms of matter fields coupled to the Higgs field.

The basic equation in our problem is

$$ \frac{d^2q}{dt^2} + \frac{dV}{dq} = - \int_{\omega_c}^{\infty} d\omega c(\omega)Q(\omega), \quad \frac{d^2Q(\omega)}{dt^2} + \omega^2 Q(\omega) = - c(\omega) q. $$

(2)

One may eliminate the environment variable $Q(\omega)$ to get the Langevin equation [13],

$$ \frac{d^2q}{dt^2} + \frac{dV}{dq} + 2 \int_0^t ds \alpha_1(t-s)q(s) = F_Q(t), $$

(3)

where $F_Q(t)$ is linear in initial values of environment variables, $Q_i(\omega)$ and $P_i(\omega)$;

$$ F_Q(t) = - \int_{\omega_c}^{\infty} d\omega c(\omega) \left( Q_i(\omega) \cos(\omega t) + \frac{P_i(\omega)}{\omega} \sin(\omega t) \right). $$

(4)

By taking the thermal bath of temperature $T = 1/\beta$ given by the density matrix,

$$ \rho_\beta(Q, Q') = \left( \frac{\omega}{\pi \coth(\beta \omega/2)} \right)^{1/2} \rho_\beta.$$
\[
\exp \left[ -\frac{\omega}{2 \sinh(\beta \omega)} \left( (Q^2 + Q'^2) \cosh(\beta \omega) - 2QQ' \right) \right], \quad (5)
\]
for each environment oscillator, the following correlation formula is obtained;

\[
\langle \{ F_Q(\tau), F_Q(s) \} \rangle_{\text{env}} = \int_{\omega_c}^{\infty} d\omega r(\omega) \cos(\omega(\tau - s)) \coth(\beta \omega / 2) \equiv \alpha_R(\tau - s), \quad (6)
\]
with

\[
r(\omega) = \frac{c^2(\omega)}{2\omega}. \quad (7)
\]

The kernel function \( \alpha_I \) in eq.(3) is given by

\[
\alpha_I(t) = -\int_{\omega_c}^{\infty} d\omega r(\omega) \sin(\omega t). \quad (8)
\]

The combination, \( \alpha_R(t) + i \alpha_I(t) \), is a sum of the real-time thermal Green’s function for \( Q(\omega) \)-oscillators, added with the weight \( c^2(\omega) \).

An often used simplification is the local, Ohmic approximation taking

\[
r(\omega) = \frac{\eta \omega}{\pi}, \quad (9)
\]
with \( \omega_c = 0 \), which amounts to

\[
\alpha_I(\tau) = \delta \omega^2 \delta(\tau) + \eta \delta'(\tau). \quad (10)
\]

This gives the local version of Langevin equation,

\[
\frac{d^2q}{dt^2} + \frac{dV}{dq} + \delta \omega^2 q + \eta \frac{dq}{dt} = 0. \quad (11)
\]

The parameter \( \delta \omega^2 \) is interpreted as a potential renormalization or a mass renormalization in the field theory analogy, since by changing the bare frequency parameter to the renormalized \( \omega_R^2 \) the term \( \delta \omega^2 q \) is cancelled by a counter term in the potential. On the other hand, \( \eta \) is the Ohmic friction coefficient. This local approximation breaks down both at early and at late times [14], but it is useful in many other cases.

Any mixture of quantum states is described by a density matrix,

\[
\rho = \sum_n w_n |n\rangle\langle n|, \quad (12)
\]
where \( |n\rangle \) is a state vector for pure quantum state and \( 0 \leq w_n \leq 1 \). The density matrix obeys the equation of motion;

\[
i\hbar \frac{\partial \rho}{\partial t} = [ H_{\text{tot}}, \rho ], \quad (13)
\]
where \( H_{\text{tot}} \) is the total Hamiltonian of the entire system. In configuration space this density matrix is given by its matrix elements,

\[
\rho(q, q'; Q(\omega), Q'(\omega)) = \langle q, Q(\omega) | \rho | q', Q'(\omega) \rangle.
\]  

(14)

Its Fourier transform with respect to relative coordinates, \( q - q', Q(\omega) - Q'(\omega) \), is called the Wigner function and denoted by \( f_W \):

\[
f_W(q, p; Q(\omega), P(\omega)) = \int_{-\infty}^{\infty} d\xi \prod dX(\omega) \exp[-i\xi p - i \int d\omega X(\omega) P(\omega)] \rho(q + \xi/2, q - \xi/2; Q(\omega) + X(\omega)/2, Q(\omega) - X(\omega)/2).
\]  

(15)

It is easy to show from eq. (13) that this quantity obeys

\[
\frac{\partial f_W}{\partial t} = -p \frac{\partial f_W}{\partial q} - \int d\omega \left( P(\omega) \frac{\partial f_W}{\partial Q(\omega)} + \omega^2 Q(\omega) \frac{\partial f_W}{\partial P(\omega)} \right) + c(\omega) \left( q \frac{\partial}{\partial P(\omega)} + Q(\omega) \frac{\partial}{\partial p} \right) f_W + \frac{1}{i\hbar} \left\{ V \left( q + \frac{i\hbar}{2} \frac{\partial}{\partial p} \right) - V \left( q - \frac{i\hbar}{2} \frac{\partial}{\partial p} \right) \right\} f_W.
\]  

(16)

In one dimensional quantum mechanics the semiclassical approximation is excellent when the potential barrier is large, and we assume that this is also true in the presence of the system-environment interaction. In the semiclassical \( \hbar \rightarrow 0 \) limit we have

\[
\frac{1}{i\hbar} \left\{ V \left( q + \frac{i\hbar}{2} \frac{\partial}{\partial p} \right) - V \left( q - \frac{i\hbar}{2} \frac{\partial}{\partial p} \right) \right\} f_W \rightarrow \frac{dV}{dq} \frac{\partial f_W}{\partial p}.
\]  

(17)

The resulting equation, being identical to the classical Liouville equation, has an obvious solution;

\[
f_W(q, p, Q, P) = \int dq_i dp_i \int dQ_i dP_i f_W^{(i)}(q_i, p_i, Q_i, P_i) \cdot \delta(q - q_{cl}) \delta(p - p_{cl}) \delta(Q - Q_{cl}) \delta(P - P_{cl}),
\]  

(18)

where \( q_{cl}(q_i, p_i, Q_i, P_i; t) \) etc. are the solution of (2), taken as the classical equation with the specified initial condition.

We consider the circumstance under which the tunneling system is initially in a state uncorrelated to the rest of environment. Thus we take the form of the density matrix, \( \rho^{(i)} = \rho_q^{(i)} \otimes \rho_Q^{(i)} \), to get the reduced Wigner function after the trivial
The problem of great interest is how further one can simplify the kernel function $K(q,p,q_i,p_i;t)$, using the kernel function of $Q_i(q,p,q_i,p_i,t)$. Expansion of $q_{cl}$ using $Q_i(q,p,q_i,p_i,t)$ leads to an integral transform \[2\] of the Wigner function, $f_{W}(q_i,p_i)$. Expansion of $q_{cl}$ in terms of $Q_i(q,p,q_i,p_i,t)$ is then justified. Thus, we use

\[
\delta(q - q_{cl}) = \int \frac{d\lambda_q}{2\pi} \exp[i\lambda_q(q - q_{cl})]
\approx \int \frac{d\lambda_q}{2\pi} \exp[i\lambda_q\left(q - q_{cl} - \int d\omega \left\{ Q_i(q)q_{cl}(Q_i) + P_i(q)q_{cl}(P_i) \right\}\right],
\]
valid to the first order of $Q_i(q), P_i(q)$. A similar expansion for $\delta(p - p_{cl})$ using $p_{cl}(0), p_{cl}(Q_i), p_{cl}(P_i)$, also holds.

An alternative justification of the expansion \[22\] is to keep $O[c(\omega)]$ terms consistently in the exponent since both $q_{cl}^{(Q_i)}(\omega)$ and $q_{cl}^{(P_i)}(\omega)$ are of this order.

Gaussian integral for the variables $Q_i(q), P_i(q)$ first and then for $\lambda_q, \lambda_p$ can be done explicitly with eq.\[22\], when one takes the thermal, hence Gaussian density matrix for the initial environment variable, $\rho_{Q}^{(i)}$. The result of this Gaussian integral leads to an integral transform \[2\] of the Wigner function, $f_{W}(i) \rightarrow f_{W}^{(R)}(f)$, using the kernel function of

\[
K(q,p,q_i,p_i;t) = \frac{\sqrt{\det J}}{2\pi} \exp\left[-\frac{1}{2}(q - q_{cl}^{(0)}, p - p_{cl}^{(0)}) J \left(q - q_{cl}^{(0)}, p - p_{cl}^{(0)}\right)\right],
\]
where the matrix elements of $J^{-1} = I_{ij}$ are given by

\[
I_{11} = \frac{1}{2} \int_{\omega_{c}}^{\infty} d\omega \coth\frac{\beta \omega}{2} \frac{1}{\omega} |z(\omega, t)|^2,
\]
\[
I_{22} = \frac{1}{2} \int_{\omega_{c}}^{\infty} d\omega \coth\frac{\beta \omega}{2} \frac{1}{\omega} |\bar{z}(\omega, t)|^2,
\]
\[
I_{12} = \frac{1}{2} \int_{\omega_{c}}^{\infty} d\omega \coth\frac{\beta \omega}{2} \frac{1}{\omega} \Re \{z(\omega, t)\bar{z}(\omega, t)\} = \frac{I_{11}}{2},
\]
where \( z(\omega,t) = q^{(Q)}_{cl}(\omega,t) + i\omega q^{(P)}_{cl}(\omega,t) \), and \( \dot{z}(\omega,t) = p^{(Q)}_{cl}(\omega,t) + i\omega p^{(P)}_{cl}(\omega,t) \).

Quantities that appear in the integral transform are determined by solving differential equations; the homogeneous Langevin equation for \( q^{(0)}_{cl} \) and an inhomogeneous linear equation for \( z(\omega,t) \) and \( \dot{z}(\omega,t) \),

\[
\frac{d^2 q^{(0)}_{cl}}{dt^2} + \left( \frac{dV}{dq} \right)_{q^{(0)}_{cl}} q^{(0)}_{cl}(s) = 0 , \tag{27}
\]

\[
\frac{d^2 z(\omega,t)}{dt^2} + \left( \frac{d^2 V}{dq^2} \right)_{q^{(0)}_{cl}} z(\omega,t) + 2 \int_0^t ds \alpha(t-s) z(\omega,s) = -c(\omega)e^{i\omega t} . \tag{28}
\]

A similar equation as for \( z(\omega,t) \) holds for \( \dot{z}(\omega,t) \). The initial condition is \( q^{(0)}_{cl}(t=0) = q_i, \quad p^{(0)}_{cl}(t=0) = p_i, \quad z(\omega,t=0) = 0, \quad \dot{z}(\omega,t=0) = 0 \).

The physical picture underlying the formula for the integral transform, eq.(19) along with (23), should be evident; the probability at a phase space point \((q,p)\) is dominated by the semiclassical trajectory \(q^{(0)}_{cl}\) (environment effect of dissipation being included in its determination by eq.(27)) reaching \((q,p)\) from an initial point \((q_i,p_i)\) whose contribution is weighed by the quantum mechanical probability \(f^{(i)}_W\) initially given. The contributing trajectory is broadened by the environment interaction with the width factor \(\sqrt{I_{ij}}\). The quantity \(I_{11}\) given by (24), for instance, is equal to \(\frac{(q - q^{(0)}_{cl})^2}{2}\); an environment driven fluctuation under the stochastic force \(F_Q(t)\).

We note that in the exactly solvable model of inverted harmonic oscillator potential the identical form of the integral transform was derived \[3\] without resort to the semiclassical approximation. Explicit form of the classical and the fluctuation functions \((q^{(0)}_{cl} \) and \(z(\omega,t))\) is given there.

To proceed further, one may separate the tunneling potential \(V(q)\) at the barrier top location, \(q = q_B\). We distinguish two cases of the potential type, depending on the value of \(V(\infty)\) relative to the local minimum value \(V_m\) at \(q = 0\) in the potential well. When \(V(\infty) < V_m\), the classical motion in the overbarrier region \(q > q_B\) is monotonic ending at \(q = \infty\), while for \(V(\infty) > V_m\) the motion is a damped oscillation towards \(q_0\) at the true minimum, unless the friction is large. If the friction is larger than a critical value of \(\approx 2\omega_\ast\) with \(\omega_\ast\) being the curvature of the potential at the global minimum, there occurs the overdamping such that \(q^{(0)}_{cl} \to q_0\) monotonically. A typical interesting case for \(V(\infty) > V_m\) is the asymmetric double well as may occur in the first order electroweak phase transition \[1\].

We consider here a few problems to illustrate consequences of our general formula of the integral transform. One problem is calculation of the barrier penetration fac-
tor; to determine the outgoing flux in the overbarrier region for the type of potential of \(V(\infty) < V_m\), assuming an initial energy eigenstate under the potential \(V\). The other problem is the tunneling probability for the asymmetric wine bottle type of potential.

Energy eigenstates are special among pure quantum states, since they evolve only with phase factors. Thus, if one takes for \(|n\rangle\langle n|\) in eq. (1) the energy eigenstate of the total Hamiltonian, the density matrix does not change with time. On the other hand, if one takes the energy eigenstate for the subsystem Hamiltonian, then the density matrix changes solely due to the environment interaction. It is thus best to use a pure eigenstate, or its superposition, of the subsystem Hamiltonian when one wishes to determine how the barrier penetration factor is modified in thermal medium.

We first discuss the stationary phase approximation for the initial state. Starting from the formula,

\[
f_W^{(i)}(q_i, p_i) = \int d\xi e^{-ip_i\xi} \sum_n w_n \psi_n^*(q_i - \frac{\xi}{2}) \psi_n(q_i + \frac{\xi}{2}) = \sum_n w_n \int d\xi \exp \left[ -ip_i\xi + \ln\psi_n^*(q_i - \frac{\xi}{2}) + \ln\psi_n(q_i + \frac{\xi}{2}) \right],
\]  

(29)

we locate the stationary point by \(\frac{\partial}{\partial \xi} = 0, \frac{\partial}{\partial p_i} = 0\) for each exponent factor. The partial derivative \(\frac{\partial}{\partial p_i} = 0\) is taken with the understanding that the rest of \(p_i\) integration contains smooth functions of \(p_i\). This leads to the stationary point at

\[
\xi = 0, \quad p_i = I_n(q_i)
\]

(30)

\[
I_n(x) = -i \frac{\psi_n^*(x) \dot{\psi}_n(x) - \dot{\psi}_n^*(x) \psi_n(x)}{|\psi_n(x)|^2}.
\]

(31)

Here \(I_n(x)\) is the usual flux factor at \(x\) for the pure state \(|n\rangle\). This gives the result for the initial density matrix,

\[
f_W^{(i)}(q_i, p_i) \approx \sum_n w_n |\psi_n(q_i)|^2 \frac{1}{2\pi} 2\pi \delta(p_i - I_n(q_i)).
\]

(32)

The barrier penetration factor is calculated from the flux formula,

\[
I(q, t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} p f_W^{(R)}(q, p; t).
\]

(33)

Thus, the semiclassical plus the stationary phase approximation gives

\[
I(q, t) \approx \int_{q_i}^{q_f} dq_i |\psi(q_i)|^2 \sqrt{\frac{1}{2\pi I_{11}}} \exp \left[ -\frac{(q - q_{cl}^{(0)})^2}{2I_{11}} \right] \cdot \left( p_{cl}^{(0)} + \frac{i}{2I_{11}} (q - q_{cl}^{(0)}) \right),
\]

(34)
for a pure initial state given by the wave function $\psi(x)$, where $q_*$ is the turning point in the overbarrier region.

In the region of $q \gg q_*$ there is always a classical trajectory that reaches the point $q$ of the Gaussian peak from an initial $q_i$ in the range $q_i > q_*$, and the entire region within the Gaussian width $\sqrt{\langle q \rangle}$ is fully covered by the integral. The factor outside the Gaussian exponent is well approximated by its peak value in the weak coupling case. Thus, one obtains

$$I(q, t) \approx |\psi(x_0)|^2 (\dot{x}_0) = |\psi(x_0)|^2 p_{cl}(x_0, t) \left( \frac{dq_{cl}(x_0, t)}{dx_0} \right)^{-1},$$

(35)

where $x_0(q, t)$ is determined from

$$q_{cl}(0) = x_0, p_i = I(x_0), Q_i(\omega) = 0, P_i(\omega) = 0; t = q.$$  

(36)

We now take the WKB wave function in the overbarrier region,

$$\psi(q_i) = \frac{T(E)}{\sqrt{p(q_i)}} \exp \left[ i \int_{x_i}^{x_0} dx p(x) \right], \quad p(x) = \sqrt{2(E - V(x))},$$

(37)

to get a factorized form of the flux [2],

$$I(q, t) \approx |T(E)|^2 f(q, t; E), \quad f = \frac{p_{cl}(x_0, t)}{p(x_0)} \left( \frac{dq_{cl}(x_0, t)}{dx_0} \right)^{-1}. \quad (38)$$

A feature that characterizes the classical trajectory $q_{cl}(0)$ is of special interest; its initial energy is

$$H_q(t = 0) \approx \frac{1}{2} p(q_i)^2 + V(q_i) = E. \quad (39)$$

This formula for the flux reproduces our previous result for a specific potential of the inverted harmonic oscillator [3]. $V(x) + \frac{1}{2} \delta \omega^2 x^2 = -\frac{1}{2} \omega_R^2 x^2$, with $\omega_R$ the renormalized curvature. The present approach actually improves our previous result;

$$f = \frac{\dot{g} x_0 + \dot{g} I(x_0)}{\dot{g} I(x_0) + g \omega^2_{cl} x_0} \rightarrow \frac{\dot{g} + \omega_B \dot{g}}{\omega_B (\dot{g} + \omega_B \dot{g})}, \quad (40)$$

where $\omega_B \approx \omega_R - \eta/2$ is a diagonalized frequency. The limiting formula of eq.(40), with $I(x_0) \rightarrow \omega_B x_0$ as $x_0 \rightarrow \infty$, is valid in the infinite $q$ limit, as derived in [3]. The function $g(t)$ is the homogeneous solution of the Langevin equation given explicitly in [3];

$$g(t) = N^2 \omega_B \sinh(\omega_B t) + 2 \int_{\omega_{c}}^{\infty} d\omega H(\omega) \sin(\omega t)$$

(41)

$$H(\omega) \approx \frac{r(\omega)}{(\omega^2 + \omega^2_{cl})^2 + (\pi r(\omega))^2},$$

(42)

$$N^2 = 1 - 2 \int_{\omega_c}^{\infty} d\omega \frac{H(\omega)}{\omega}.$$  

(43)
Both at early and late times the factor \( f \approx 1 \), deviating from unity only for the time range of order \( 1/\omega_B \).

According to the view of [7] the potential \( V(q) \) that determines the semiclassical penetration factor \( |T(E)|^2 \) should be expressed in terms of the renormalized parameters, \( \omega_R \) in this case. This explains the bulk of the suppression caused by dissipative environment interaction, as explained in [3].

For discussion of a more general case of finite \( V(\infty) < V_m \) we use the local, Ohmic approximation, which becomes excellent at late times. A potential that decreases fast as \( q \to \infty \) is assumed; \( \frac{dV}{dq} \to 0 \). The acceleration term can then be neglected when the friction satisfies \( \eta^2 \gg |\frac{dV}{dq}| \). This is a slow rolling approximation, and it always holds for \( q \) large enough. The classical equation is then solved as

\[
\eta \int_{q_*}^{q} dz \left( \frac{dV}{dz} \right)^{-1} = -t,
\]

which gives the factor

\[
f \approx -\frac{dV}{dq} \frac{1}{\eta p(\infty)}.
\] (45)

Thus, the tunneling probability decreases with time along with the decreasing slope of the potential. This result poses a curious question; the tunneling may not be terminated within a limited finite time. We shall encounter a similar situation for the asymmetric wine bottle potential.

We next consider the case in which the potential is very steep at both ends; \( V(\pm \infty) = \infty \). The tunneling rate, from the inner region at \( q < q_B \) into the outer region at \( q > q_B \), is an important measure of tunneling phenomena and is given by the flux at \( q = q_B \); \( \dot{P}(t) = -I(q_B, t) \), which is equal to

\[
- \int dq dp f_W(q_i, p_i) \left( \frac{p_{cl}^{(0)}}{2 I_{11}^{(0)}} \right) \frac{1}{\sqrt{2\pi I_{11}^{(0)}}} \exp \left[ -\frac{(q_B - q_{cl}^{(0)})^2}{2 I_{11}^{(0)}} \right].
\] (46)

On the other hand, the tunneling probability into the overbarrier region at \( q > x \) is given by

\[
P(x, t) = \int dq dp f_W(q_i, p_i) \int_x^{\infty} du \frac{1}{\sqrt{2\pi I_{11}^{(0)}}} \exp \left[ -\frac{(u - q_{cl}^{(0)})^2}{2 I_{11}^{(0)}} \right].
\] (47)

In both of the quantities, \( \dot{P}(t) \) and \( P(q_B, t) \) (the tunneling probability into \( q > q_B \)), it is essential to estimate how \( q_{cl}^{(0)} \) and \( I_{11} \) varies with time.
We consider an initial state localized in the potential well so that the dominant contribution in the \((q_i, p_i)\) phase space integration is restricted to \(q_i < q_B\). In the rest of discussion anharmonic terms play important roles, but at first we work out the harmonic approximation;

\[
V(q) + \frac{1}{2} \delta \omega^2 q^2 \approx \frac{1}{2} \omega_0^2 q^2 ,
\]

near the bottom of the well at \(q = 0\). In the Ohmic approximation the classical solution and the fluctuation is given by

\[
q_{cl}^{(0)} = \left( \cos \tilde{\omega}_0 t + \frac{\eta}{2\tilde{\omega}_0} \sin \tilde{\omega}_0 t \right) e^{-\eta t/2} q_i + \frac{\sin \tilde{\omega}_0 t}{\tilde{\omega}_0} e^{-\eta t/2} p_i ,
\]

\[
z(\omega, t) = \frac{c(\omega)}{\omega - \omega_0 - i\omega \eta} \left( e^{i\omega t} - \frac{\omega + \tilde{\omega}_0 - i\eta/2}{2\tilde{\omega}_0} e^{i\tilde{\omega}_0 t - \eta t/2} + \frac{\omega - \tilde{\omega}_0 - i\eta/2}{2\tilde{\omega}_0} e^{-i\tilde{\omega}_0 t - \eta t/2} \right),
\]

using \(\tilde{\omega}_0 = \sqrt{\omega_0^2 - \frac{\eta^2}{4}}\). In the rest of discussion we assume a small friction, \(\eta \ll \omega_0\).

Near \(\omega = \tilde{\omega}_0\) the fluctuation is approximately

\[
z(\omega, t) \approx \frac{i c(\omega)}{\omega + \tilde{\omega}_0 - i\eta/2} \left( t e^{i\tilde{\omega}_0 t} - \frac{1}{\tilde{\omega}_0} \sin \tilde{\omega}_0 t \right) e^{-\eta t/2}.
\]

This formula is valid at \(t < O[1/\eta]\). The appearance of the linear \(t\) term is a resonance effect. The resonance roughly contributes to \(I_{11}(t)\) by the amount, \(\eta t^2 e^{-\eta t} \times\) a smooth \(\omega\) integral which is cut off by a physical frequency scale. Thus, the width factor \(I_{11}\) initially increases with time until the time scale of order \(1/\eta\).

The width factor asymptotically behaves as

\[
I_{11}(t) = I_{11}(\infty) + O[e^{-\eta t/2}] ,
\]

\[
I_{11}(\infty) \approx \frac{1}{2 \omega_0} + \frac{1}{\omega_0} e^{\omega_0/T - 1} + \frac{\pi \eta}{3 \omega_0^3} T^2 .
\]

The asymptotic value of \(I_{11}(\infty)\) has the familiar zero point fluctuation of harmonic oscillator and in the last term the dominant finite temperature correction, valid for this Ohmic model at \(T \ll \omega_0\). In any event the probability rate \(\dot{P}(t)\) finally decreases to zero, along with

\[
p_{cl}^{(0)} + \frac{I_{11}}{2I_{11}} (q_B - q_{cl}^{(0)}) \to 0 .
\]

Moreover, the final tunneling probability \(P(q_B, \infty)\) has a finite value, and typically is very small for a large potential barrier. For instance, for the asymmetric wine
bottle potential shortly discussed,

\[ P(q_B, \infty) \approx \frac{1}{4} \sqrt{\frac{\omega_0}{2\pi V_h}} e^{-8V_h/\omega_0}, \tag{55} \]

with \( V_h \) the barrier height much smaller than \( \omega_0 \). This poses again a curious question; it appears that decay of a prepared metastable state localized in the potential well is never completed.

This simple picture is however valid only when one ignores anharmonic terms in the tunneling potential, but they must be there in order to give any realistic tunneling potential. The most important in the following discussion is effect of anharmonic terms in the equation for the fluctuation \( z(\omega, t) \). Presence of anharmonic terms gives a non-trivial periodicity in the coefficient function \( \left( \frac{d^2 V}{dq^2} \right)_{q_{cl}(0)} \) for (28), assuming a small friction \( \eta \ll \omega_0 \).

The homogeneous part of the \( z(\omega, t) \) solution which might exhibit the well known parametric resonance \[ \text{(14)} \] is closely related to the behavior for our inhomogeneous \( z(\omega, t) \) solution. It is then important to check whether the relevant parameter in the periodic coefficient function \( \left( \frac{d^2 V}{dq^2} \right)_{q_{cl}(0)} \) falls in the instability or the stability band. As is shown in \[ \text{(13)} \], unbounded exponential growth of \( \sqrt{I_{11}} \) does not take place. On the other hand, the power-law growth is observed in numerical computation. Moreover, we find that our \( z(\omega, t) \) solution belongs to the boundary between stability and instability bands. At the resonance frequency the enhancement factor due to the boundary effect is much larger than what one might expect from the harmonic case \[ \text{(51)} \]. We shall interpret this phenomenon as influenced in a subtle way by the parametric resonance, although it is not the parametric resonance itself.

It is best to discuss the resonance enhanced tunneling mechanism in concrete examples. We take the asymmetric wine bottle potential, which is described in the well and its vicinity region by

\[ V(q) \approx \frac{\lambda}{4} (q^2 - 2q_B q)^2. \tag{56} \]

The curvature parameters at two extrema of \( q = 0 \) and \( q = q_B \) are \( \omega_0^2 = 2\lambda q_B^2 \), \( \omega_B^2 = \lambda q_B^2 \), and the barrier height seen from the bottom of the well is \( V_h = \frac{\lambda}{4} q_B^4 = \omega_0^2 q_B^2 / 8 \).

The classical \( q_{cl}(0) \), and the fluctuation \( z(\omega, t) \) equations in the Ohmic approximation are written using rescaled variables, \( y = q_{cl}(0)/q_B \) and \( \tau = \omega_0 t/2 \),

\[ y'' + y(y - 1)(y - 2) + \frac{2\eta}{\omega_0} y' = 0, \tag{57} \]

\[ z'' + \left( 4 - 6 \left( y - \frac{1}{2} y^2 \right) \right) z + \frac{2\eta}{\omega_0} z' = -\frac{4c(\omega)}{\omega_0^2} e^{i2\omega\tau/\omega_0}, \tag{58} \]
where \( ' = d/d\tau \).

For a small friction one has approximate forms of solution in terms of the Jacobi's elliptic function;

\[
(y - 1)^2 = 1 + \sqrt{\epsilon} - 2\sqrt{\epsilon} \text{sn}^2 \left( \sqrt{1 + \sqrt{\epsilon}} \left( \tau + \int_{1}^{a} \frac{du}{\sqrt{(a^2 - u^2)(u^2 - b^2)}} \right), k \right),
\]

\[
p_{cl}^{(0)}(t) = 2\sqrt{2\epsilon V_h} \text{sn} \left( \sqrt{1 + \sqrt{\epsilon}} (\tau + \text{const}), k \right) \text{cn} \left( \sqrt{1 + \sqrt{\epsilon}} (\tau + \text{const}), k \right),
\]

\[
\tau = \frac{1}{2} \omega_0 t, \quad k = \sqrt{\frac{2\sqrt{\epsilon}}{1 + \sqrt{\epsilon}}}, \quad \epsilon = \frac{E_i}{V_h}.
\]

These formulas are valid for \( t \ll 1/\eta \). There are resonances at \( \tilde{\omega}_0, 2\tilde{\omega}_0, \cdots n\tilde{\omega}_0, \cdots \), where

\[
\tilde{\omega}_0 = \frac{\pi \omega_0}{2} \sqrt{1 + \sqrt{\epsilon}} \left( \int_{0}^{1} \frac{du}{\sqrt{(1 - u^2)(1 - k^2 u^2)}} \right)^{-1}.
\]

We found so far that these explicit solutions are not very illuminating, and numerically integrated the coupled \( q_{cl}^{(0)} \) and \( z(\omega, t) \) equations. In the \( \omega \) integral (24) for the width factor \( I_{11} \) the largest contribution is found to come from the fundamental resonance at \( \omega = \tilde{\omega}_0 \), then contribution from higher harmonics at \( n\tilde{\omega}_0 \) follows. Thus, phenomenon of a non-linear resonance occurs. At a time of \( O[1/\eta] \) the width factor \( I_{11} \) becomes maximal at its value much larger than in the harmonic case. The decrease observed at late times is due to the friction; we explicitly checked that for the zero friction, the fluctuation \( |z(\omega, t)/c(\omega)|^2 \) increases in time without a bound, with an averaged time power close to 4 at the resonance. Note that even if the effect of the friction is turned off, there exists an important environment effect here; the environment interaction drives the non-linear resonance oscillation. We refer to our paper \[16\] for detailed numerical results.

We numerically checked \[13\] the behavior of the most important part given by the product of two competing exponential factors,

\[
A = \exp \left[- \tanh \frac{\beta \omega_0 p_i^2 + \omega_0^2 q_i^2}{2} \right] \times \exp \left[- \frac{(q_B - q_{cl}^{(0)})^2}{2I_{11}} \right],
\]

the one for the initial state and the other for the kernel factor in the probability rate. Remarkably, the largest contribution comes, not from the dominant initial
component near the zero point energy, rather from the initially suppressed excited component. The first exponent in \[E_i\] is in proportion to \(E_i\), while the second one goes roughly like \(E_i^{-2}\), hence a maximum may appear somewhere away from the lowest energy of \(E_i = \omega_0/2\). In the example of \(\eta/\omega_0 = 0.0025\) the maximum product factor is of order \(10^{-3}\) at \(E_i \approx 4 \times \omega_0/2\), 6 orders of magnitudes larger than what one expects from the lowest energy state and also the asymptotic value of order \(10^{-36}\). A large value of the product factor of order unity suggests an interesting possibility of a rapid and violent termination of the tunneling.

The reason why one observes enhanced tunneling at resonant frequencies of \(\omega = n \tilde{\omega}_0\) is as follows. The semiclassical \(q\)--motion in the left potential well has a periodicity \(2\pi/\tilde{\omega}_0\) if one neglects the friction. The environment oscillators act as stochastic fluctuation to this motion in such a way that the effective potential including one environment oscillator,

\[
V(q) + q c(\omega) \left( Q_i(\omega) \cos(\omega t) + \frac{P_i(\omega)}{\omega} \sin(\omega t) \right),
\]

changes the potential barrier periodically. When the \(q\)--particle hits against the potential wall, the potential barrier may become smallest, hence the tunneling rate maximally enhanced, if one of environment oscillators has a period exactly equal to that of the classical motion, \(\omega = n \tilde{\omega}_0\). This is precisely the condition of the non-linear resonance we have been discussing. The resonance enhanced tunneling thus envisaged seems to have little connection to the stochastic resonance much discussed in the literature.

In summary, we gave a new mechanism of resonance enhanced tunneling, using the real-time semiclassical formalism. Our approach suggests a new time scale of order \(1/\eta\), the inverse friction, for completion of the first order phase transition. Evidently much has to be done to apply the idea here to realistic problems.

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