MINIMALITY OF THE BOUNDARY OF A RIGHT-ANGLED COXETER SYSTEM

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Abstract. In this paper, we show that the boundary \( \partial \Sigma(W,S) \) of a right-angled Coxeter system \((W,S)\) is minimal if and only if \( W_\Sigma \) is irreducible, where \( W_\Sigma \) is the minimum parabolic subgroup of finite index in \( W \). We also provide several applications and remarks. In particular, we show that for a right-angled Coxeter system \((W,S)\), the set \( \{w^\infty \mid w \in W, o(w) = \infty \} \) is dense in the boundary \( \partial \Sigma(W,S) \).

1. Introduction and preliminaries

The purpose of this paper is to study dense subsets of the boundary of a Coxeter system. A Coxeter group is a group \( W \) having a presentation \( \langle S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S \rangle \), where \( S \) is a finite set and \( m : S \times S \to \mathbb{N} \cup \{\infty\} \) is a function satisfying the following conditions:

1. \( m(s,t) = m(t,s) \) for each \( s, t \in S \),
2. \( m(s,s) = 1 \) for each \( s \in S \), and
3. \( m(s,t) \geq 2 \) for each \( s, t \in S \) with \( s \neq t \).

The pair \((W,S)\) is called a Coxeter system. If, in addition,

4. \( m(s,t) = 2 \) or \( \infty \) for each \( s, t \in S \) with \( s \neq t \),

then \((W,S)\) is said to be right-angled. Let \((W,S)\) be a Coxeter system. Then \( W \) has the word metric \( d_\ell \) defined by \( d_\ell(w,w') = \ell(w^{-1}w') \) for each \( w, w' \in W \), where \( \ell(w) \) is the word length of \( w \) with respect to \( S \). For a subset \( T \subset S \), \( W_T \) is defined as the subgroup of \( W \) generated by \( T \), and is called a parabolic subgroup. If \( T \) is the empty set, then \( W_T \) is the trivial group. A subset \( T \subset S \) is called a spherical subset of \( S \) if the parabolic subgroup \( W_T \) is finite.

Every Coxeter system \((W,S)\) determines a Davis complex \( \Sigma(W,S) \) which is a CAT(0) geodesic space \([6, 7, 8, 20]\). Here the 1-skeleton of \( \Sigma(W,S) \) is the Cayley graph of \( W \) with respect to \( S \). The natural action of \( W \) on \( \Sigma(W,S) \) is proper, cocompact and by isometries. If \( W \) is infinite, then \( \Sigma(W,S) \) is noncompact and \( \Sigma(W,S) \) can be compactified by adding its ideal boundary \( \partial \Sigma(W,S) \) \([4, \text{ 11} \ §4]\).
This boundary $\partial \Sigma(W, S)$ is called the boundary of $(W, S)$. We note that the natural action of $W$ on $\Sigma(W, S)$ induces an action of $W$ on $\partial \Sigma(W, S)$ by homeomorphisms.

A subset $A$ of a space $X$ is said to be dense in $X$ if $\overline{A} = X$. A subset $A$ of a metric space $X$ is said to be quasi-dense if there exists $N > 0$ such that each point of $X$ is $N$-close to some point of $A$. Suppose that a group $G$ acts on a compact metric space $X$ by homeomorphisms. Then $X$ is said to be minimal if every orbit $Gx$ is dense in $X$.

For a negatively curved group $\Gamma$ and the boundary $\partial \Gamma$ of $\Gamma$, it is known that each orbit $\Gamma \alpha$ is dense in $\partial \Gamma$ for any $\alpha \in \partial \Gamma$, that is, $\partial \Gamma$ is minimal [9]. We note that Coxeter groups are nonpositive curved groups and not negatively curved groups in general. Indeed, there exist examples of Coxeter systems whose boundaries are not minimal (cf. [14], [16]). The purpose of this paper is to investigate when the boundary of a Coxeter system is minimal.

In [14, Theorem 1], we have obtained a sufficient condition of a Coxeter system $(W, S)$ such that some orbit of the Coxeter group $W$ is dense in the boundary $\partial \Sigma(W, S)$. After some preliminaries in Section 2, we first show that the boundary of such a Coxeter system is minimal; that is, we prove the following theorem in Section 3.

**Theorem 1.** Let $(W, S)$ be a Coxeter system. Suppose that $W^\ell(s_0)$ is quasi-dense in $W$ with respect to the word metric and $o(s_0t_0) = \infty$ for some $s_0, t_0 \in S$, where $o(s_0t_0)$ is the order of $s_0t_0$ in $W$. Then

1. $\partial \Sigma(W, S)$ is minimal, and
2. $\{w^\infty \mid w \in W, o(w) = \infty\}$ is dense in $\partial \Sigma(W, S)$.

Here $W^\ell(s_0) = \{w \in W \mid \ell(wt) > \ell(w)\} \setminus \{1\}$ and $w^\infty$ is the point of $\partial \Sigma(W, S)$ to which the sequence $\{w^i \mid i \in \mathbb{N}\} \subset \Sigma(W, S)$ converges in $\Sigma(W, S) \cup \partial \Sigma(W, S)$.

In Sections 4 and 5, we investigate right-angled Coxeter groups and we prove the following main theorem.

**Theorem 2.** For a right-angled Coxeter system $(W, S)$, the boundary $\partial \Sigma(W, S)$ is minimal if and only if $W_S$ is irreducible.

Here for $T \subset S$, $W_T$ is said to be irreducible if $W_T$ does not split as a product $W_{T_1} \times W_{T_2}$ for any nonempty subsets $T_1$ and $T_2$ of $T$, and $W_S$ is the minimum parabolic subgroup of finite index in $(W, S)$, that is, for the irreducible decomposition $W = W_{S_1} \times \cdots \times W_{S_n}$, $\mathcal{S} = \bigcup \{S_i \mid W_{S_i} \text{ is finite}\}$ [11].

We provide several applications of Theorem 2 in Sections 5 and 6. In particular, we show the following corollary.

**Corollary 3.** For a right-angled Coxeter system $(W, S)$, the set $\{w^\infty \mid w \in W, o(w) = \infty\}$ is dense in the boundary $\partial \Sigma(W, S)$.

In Section 6, we give some remarks on dense subsets of boundaries of CAT(0) groups.

2. **Lemmas on Coxeter Groups**

In this section, we show some lemmas for (right-angled) Coxeter groups which are used later.

We first give some definitions.
Definition 2.1. Let \((W, S)\) be a Coxeter system and \(w \in W\). A representation \(w = s_1 \cdots s_l\) (\(s_i \in S\)) is said to be reduced, if \(\ell(w) = l\), where \(\ell(w)\) is the minimum length of a word in \(S\) which represents \(w\).

Definition 2.2. Let \((W, S)\) be a Coxeter system. For each \(w \in W\), we define \(S(w) = \{s \in S | \ell(ws) < \ell(w)\}\). For a subset \(T \subset S\), we also define \(W^T = \{w \in W | S(w) = T\}\).

The following lemma is known.

Lemma 2.3 ([1], [3] p.37], [18]). Let \((W, S)\) be a Coxeter system.

1. Let \(w \in W\) and let \(w = s_1 \cdots s_l\) be a representation. If \(\ell(w) < l\), then \(w = s_1 \cdots s_i \cdots s_j \cdots s_l\) for some \(1 \leq i < j \leq l\).
2. For each \(w \in W\) and \(s \in S\), \(\ell(ws)\) equals either \(\ell(w) + 1\) or \(\ell(w) - 1\), and \(\ell(sw)\) also equals either \(\ell(w) + 1\) or \(\ell(w) - 1\).
3. For each \(w \in W\), \(S(w)\) is a spherical subset of \(S\); i.e., \(W_{S(w)}\) is finite.

We can obtain the following lemma from Lemma 2.3 (3).

Lemma 2.4. Let \((W, S)\) be a Coxeter system and let \(T\) be a maximal spherical subset of \(S\). Then \(W^T\) is quasi-dense in \(W\).

Proof. Let \(w \in W\). There exists an element \(w'\) of longest length in the coset \(wW_T\). Then we show that \(S(w') = T\).

Let \(t \in T\). Since \(w't \in w'W_T = wW_T\) and \(w'\) is the element of longest length in \(wW_T\), \(\ell(w't) < \ell(w')\), i.e., \(t \in S(w')\). Thus \(T \subset S(w')\). Now \(T\) is a maximal spherical subset of \(S\) and \(S(w')\) is a spherical subset of \(S\) by Lemma 2.3 (3). Hence \(S(w') = T\) and \(w' \in W^T\).

Here, \(d_t(w, w') \leq \max\{\ell(v) | v \in W_T\}\). Hence \(W^T\) is quasi-dense in \(W\).

Lemma 2.5 ([13] Lemma 2.3 (3)]). Let \((W, S)\) be a Coxeter system and \(s, t \in S\) such that \(o(st) = \infty\). Then \(W^{\{s\}} \subset W^{\{t\}}\).

Next, we provide some lemmas for right-angled Coxeter groups. We note that right-angled Coxeter groups are rigid; that is, a right-angled Coxeter group determines its Coxeter system uniquely up to isomorphism ([21]).

By a consequence of Tits’ solution to the word problem ([23], [5] p.50)), we can obtain the following lemma (cf. [12] Lemma 5).

Lemma 2.6. Let \((W, S)\) be a right-angled Coxeter system, let \(w \in W\), let \(w = s_1 \cdots s_l\) be a reduced representation and let \(t, t' \in S\). If \(tw = t(s_1 \cdots s_l)\) is reduced and \(twt' = w\), then \(t = t'\) and \(ts_i = s_it\) for any \(i \in \{1, \ldots, l\}\).

Using Lemma 2.6, we prove the following lemma.

Lemma 2.7. Let \((W, S)\) be a right-angled Coxeter system, let \(U\) be a spherical subset of \(S\), let \(s_0 \in S \setminus U\) and let \(T = \{t \in U | o(st) = 2\}\). Then \(W^U s_0 \subset W^{T \cup\{s_0\}}\).

Proof. Let \(w \in W^U\). To prove that \(w_{s_0} \in W^{T \cup\{s_0\}}\), we show that \(S(w_{s_0}) = T \cup\{s_0\}\). We note that \(\ell(w_{s_0}) = \ell(w) + 1\) since \(s_0 \not\in U = S(w)\). Hence \(s_0 \in S(w_{s_0})\). Also for each \(t \in T\), by the definition of \(T\), \(\ell(w_{s_0}t) = \ell(wts_0) < \ell(w_{s_0})\), and \(t \in S(w_{s_0})\). Thus \(T \cup\{s_0\} \subset S(w_{s_0})\). Next, we show that \(S(w_{s_0}) \subset T \cup\{s_0\}\). Let \(t \in S(w_{s_0})\). Then \(\ell(w_{s_0}t) < \ell(w_{s_0})\). If \(w = a_1 \cdots a_l\) is a reduced representation, then by Lemma 2.3 (1),

\[
    w_{s_0}t = (a_1 \cdots a_l)s_0t = (a_1 \cdots a_i \cdots a_l)s_0
\]
for some \(i \in \{1, \ldots, l\}\), or \(t = s_0\). By Lemma 2.5, we obtain that \(s_0t = ts_0\).

This implies that if \(t \neq s_0\), then \(\ell(wt) < \ell(w)\), i.e., \(t \in S(w) = U\). Since \(t \in U\) and \(s_0t = ts_0\), \(t \in T\). Hence \(S(ws_0) \subset T \cup \{s_0\}\). Thus \(S(ws_0) = T \cup \{s_0\}\) and \(ws_0 \in W \cup \{s_0\}\). We obtain that \(Ww_0 \subset W \cup \{s_0\}\).

It is well-known that a Coxeter system \((W, S)\) is irreducible if and only if the underlying graph of its Coxeter graph is connected ([1], [5, p.23], [18, p.30]). If the Coxeter system \((W, S)\) is right-angled, then the underlying graph of its Coxeter graph is the graph \(\Gamma_\infty(W, S)\), where \(\Gamma_\infty(W, S)\) is defined as follows: the vertex set of \(\Gamma_\infty(W, S)\) is \(S\) and for \(s, t \in S\), \(\{s, t\}\) spans an edge in \(\Gamma_\infty(W, S)\) if and only if \(m(s, t) = \infty\). Hence we obtain the following lemma.

**Lemma 2.8 (cf. [1], [5], [18]).** For a right-angled Coxeter system \((W, S)\), the following statements are equivalent:

1. \((W, S)\) is irreducible.
2. \(\Gamma_\infty(W, S)\) is connected.
3. For each \(a, b \in S\) with \(a \neq b\), there exists a sequence \(\{a = s_1, s_2, \ldots, s_n = b\}\) \(\subset S\) such that \(o(s_is_{i+1}) = \infty\) for any \(i \in \{1, \ldots, n - 1\}\).

### 3. Minimality of the Boundary of a Coxeter System

In this section, we show an extension of a result in [14] on minimality of the boundary of a Coxeter system.

**Theorem 3.1.** Let \((W, S)\) be a Coxeter system. Suppose that \(W^{\{s_0\}}\) is quasi-dense in \(W\) and \(o(s_0t_0) = \infty\) for some \(s_0, t_0 \in S\). Then

1. \(\partial \Sigma(W, S)\) is minimal, and
2. \(\{w^\infty \mid w \in W, o(w) = \infty\}\) is dense in \(\partial \Sigma(W, S)\).

**Proof.** Suppose that \(W^{\{s_0\}}\) is quasi-dense in \(W\) and \(o(s_0t_0) = \infty\) for some \(s_0, t_0 \in S\). Then we show that \(W\gamma\) is dense in \(\partial \Sigma(W, S)\) for any \(\gamma \in \partial \Sigma(W, S)\).

Let \(\gamma \in \partial \Sigma(W, S)\) and let \(\{v_i\} \subset W\) be a sequence which converges to \(\gamma\) in \(\Sigma(W, S) \cup \partial \Sigma(W, S)\). Since \(W^{\{s_0\}}\) is quasi-dense in \(W\), there exists a number \(N > 0\) such that for each \(v \in W\), \(d_i(v, x) \leq N\) for some \(x \in W^{\{s_0\}}\). Hence for each \(v \in W\), there exists \(u \in W\) such that \(\ell(u) \leq N\) and \(vu \in W^{\{s_0\}}\). For each \(i\), there exists \(u_i \in W\) such that \(\ell(u_i) \leq N\) and \((v_i)^{-1}u_i \in W^{\{s_0\}}\). We note that the set \(\{u \in W \mid \ell(u) \leq N\}\) is finite because \(S\) is finite. Hence \(\{u_i \mid i \in \mathbb{N}\}\) is finite, and there exist \(u \in W\) and a sequence \(\{i_j \mid j \in \mathbb{N}\} \subset \mathbb{N}\) such that \(u_i = u\) for any \(j \in \mathbb{N}\). Then for each \(j \in \mathbb{N}\), \((v_{i_j})^{-1}u_{i_j} = (v_{i_j})^{-1}u \in W^{\{s_0\}}\) and \((v_{i_j})^{-1}u \in W^{\{t_0\}}\).

By Lemma 2.5, since \(o(s_0t_0) = \infty\). Hence \(t_0u^{-1}v_j \in (W^{\{t_0\}})^{-1}\). The sequence \(\{t_0u^{-1}v_j \mid j \in \mathbb{N}\}\) converges to \(t_0u^{-1}\gamma\), since \(\{v_j \mid j \in \mathbb{N}\}\) converges to \(\gamma\). Here we recall the proof of [14, Theorem 4.1]. If we put \(x_j = t_0u^{-1}v_j\) and \(\alpha = t_0u^{-1}\gamma\), then the sequence \(\{x_j\} \subset (W^{\{t_0\}})^{-1}\) converges to \(\alpha\). By the proof of [14, Theorem 4.1], we obtain that \(W\alpha\) is dense in \(\partial \Sigma(W, S)\); that is, \(Wt_0u^{-1}\gamma\) is dense in \(\partial \Sigma(W, S)\).

Hence \(W\gamma\) is dense in \(\partial \Sigma(W, S)\), since \(Wt_0u^{-1} = W\). Thus every orbit \(W\gamma\) is dense in \(\partial \Sigma(W, S)\) and \(\partial \Sigma(W, S)\) is minimal.

The minimality of \(\partial \Sigma(W, S)\) implies that the set \(\{w^\infty \mid w \in W, o(w) = \infty\}\) is dense in \(\partial \Sigma(W, S)\) (see Proposition 6.2). 

\(\square\)
Here we have a question whether conversely if \( \partial \Sigma(W, S) \) is minimal, then \( W^{\{s_0\}} \) is quasi-dense in \( W \) and \( o(s_0t_0) = \infty \) for some \( s_0, t_0 \in S \). The answer to this question is no in general.

For example, let \( S = \{ s_1, s_2, s_3 \} \) and let
\[
W = \langle S \mid s_1^3 = s_2^3 = s_3^3 = (s_1s_2)^4 = (s_2s_3)^4 = (s_3s_1)^4 = 1 \rangle.
\]
Then \( W \) is a negatively curved group and the boundary \( \partial \Sigma(W, S) \) is minimal. On the other hand, there do not exist \( s_0, t_0 \in S \) such that \( o(s_0t_0) = \infty \).

In Section 5, we will show that in the case \((W, S)\) is right-angled, the answer to this question is yes.

4. Key lemma

In this section, we prove the following lemma, which plays a key role in the proof of the main theorem.

**Lemma 4.1.** Let \((W, S)\) be a right-angled Coxeter system such that \( W \) is infinite. If \( W \) is irreducible, then \( W^{\{s_0\}} \) is quasi-dense in \( W \) for some \( s_0 \in S \).

**Proof.** We suppose that \( W^{\{s\}} \) is not quasi-dense in \( W \) for any \( s \in S \). Then we show that \( W \) is not irreducible.

Let \( s_0 \in S \), let \( T_1 = \{ t \in S \mid o(s_0t) = 2 \} \) and let \( S_1 = S \setminus T_1 \). If \( T_1 = \emptyset \), then \( o(s_0s) = \infty \) for each \( s \in S \setminus \{ s_0 \} \); hence \( W^{\{s_0\}} \) is quasi-dense in \( W \), which contradicts the assumption. Thus \( T_1 \neq \emptyset \). If \( S_1 = \{ s_0 \} \), then \( W = W^{\{s_0\}} \times W_{T_1} \); i.e., \( W \) is not irreducible. We suppose that \( S_1 \neq \{ s_0 \} \).

Let \( s_1 \in S_1 \setminus \{ s_0 \} \), let \( T_2 = \{ t \in T_1 \mid o(s_1t) = 2 \} \) and let \( S_2 = S \setminus T_2 = S_1 \cup (T_1 \setminus T_2) \). We note that \( o(s_1t) = 2 \) for each \( i \in \{0, 1\} \) and \( t \in T_2 \), i.e., \( W^{\{s_0, s_1\}} \times W_{T_2} \). Since \( s_1 \in S_1 \setminus \{ s_0 \} \), we obtain that \( o(s_0s_1) = \infty \) and \( W^{\{s_0, s_1\}} \) is irreducible.

Now we show that \( T_2 \neq \emptyset \). Suppose that \( T_2 = \emptyset \). This means that \( o(s_1t) = \infty \) for any \( t \in T_1 \). Let \( U \) be a maximal spherical subset of \( S \) such that \( s_0 \in U \). Then \( o(uv) = 2 \) for each \( u, v \in U \) with \( u \neq v \), because \((W, S)\) is right-angled and \( W_U \) is finite. Hence \( o(s_0u) = 2 \) for any \( u \in U \), since \( s_0 \in U \) if \( U \). This means that \( U \subset T_1 \cup \{ s_0 \} \). Hence \( o(s_1u) = \infty \) for any \( u \in U \), because \( o(s_1t) = \infty \) for any \( t \in T_1 \) and \( o(s_0s_1) = \infty \). Thus \( W_U \setminus S \subset W^{\{s_1\}} \) by Lemma 2.4. Here by Lemma 2.4, \( W_U \subset W_{T_2} \). This contradicts the assumption. Thus we obtain that \( T_2 \neq \emptyset \).

If \( S_2 = \{ s_0, s_1 \} \), then \( W = W^{\{s_0, s_1\}} \times W_{T_2} \) and \( W \) is not irreducible. We suppose that \( S_2 \neq \{ s_0, s_1 \} \). Let \( s_2 \in S_2 \setminus \{ s_0, s_1 \} \), let \( T_3 = \{ t \in T_2 \mid o(s_2t) = 2 \} \) and let \( S_3 = S \setminus T_3 = S_2 \cup (T_2 \setminus T_3) \).

By induction, we define \( s_k, T_{k+1}, S_{k+1} \) as follows: Let
\[
\begin{align*}
s_k & \in S \setminus \{ s_0, \ldots, s_{k-1} \} , \\
T_{k+1} & = \{ t \in T_k \mid o(s_kt) = 2 \} \quad \text{and} \\
S_{k+1} & = S \setminus T_{k+1}.
\end{align*}
\]
Then \( W^{\{s_0, s_1, \ldots, s_k\}} \setminus T_k = W^{\{s_0, s_1, \ldots, s_k\}} \times W_{T_{k+1}} \). If \( S_{k+1} \setminus \{ s_0, s_1, \ldots, s_k \} = \emptyset \), then \( W = W_{S_{n+1}} \times W_{T_{k+1}} \); i.e., \( W \) is not irreducible. Here we note that \( T_{k+1} \subset T_k \subset \cdots \subset T_2 \subset T_1 \). If \( T_k \neq \emptyset \) for each \( k \), then by the finiteness of \( S \), there exists a number \( n \) such that \( W = W_{S_n} \times W_{T_n} \); hence \( W \) is not irreducible.

We prove the following statements by induction on \( k \).
(i) $T_k \neq \emptyset$.
(ii) $W_{\{s_0, \ldots, s_{k-1}\}}$ is irreducible.
(iii) There exists a spherical subset $U_k \subset T_k$ such that $W^{U_k \cup \{s_i\}}$ is quasi-dense in $W$ for each $i \in \{0, \ldots, k-1\}$.

We first consider the case $k = 2$. The statement (i) $T_2 \neq \emptyset$ was proved in the above. Also (ii) holds, since $W_{\{s_0, s_1\}} = W_{\{s_0\}} * W_{\{s_1\}}$ is irreducible. We show that the statement (iii) holds. Let $U$ be a maximal spherical subset of $S$ such that $s_0 \in U$. Then $W^U$ is quasi-dense in $W$ by Lemma 2.4. Let $U_2 = U \cap T_2$. We note that $U_2 = \{t \in U | o(s_1 t) = 2\}$. By Lemma 2.7, $W^U s_1 \subset W^{U_2 \cup \{s_1\}}$. Hence $W^{U_2 \cup \{s_1\}}$ is quasi-dense in $W$. (This implies that $U_2 \neq \emptyset$ by the assumption.) Also $W^{U_2 \cup \{s_0\}}$ is quasi-dense in $W$, since $W^{U_2 \cup \{s_1\}} s_0 \subset W^{U_2 \cup \{s_0\}}$ by Lemma 2.7. Thus (iii) holds.

We suppose that (i), (ii) and (iii) hold for some $k \geq 2$. Then we prove that (i), (ii) and (iii) hold.

(i): We show that $T_{k+1} \neq \emptyset$. Suppose that $T_{k+1} = \emptyset$. If $o(s_k s_j) = 2$ for any $i \in \{0, \ldots, k\}$, then $s_k \in T_k$, which contradicts the definition of $s_k$. Hence $o(s_k s_j) = \infty$ for some $j \in \{0, \ldots, k\}$ and $t \in T_k$. Here $U_k \subset T_k$ and $o(s_k t) = \infty$ for any $t \in T_k$. Hence $W^{U_k \cup \{s_j\}} s_k \subset W^{U_k \cup \{s_j\}}$ by Lemma 2.7. By (iii), $W^{U_k \cup \{s_j\}}$ is quasi-dense in $W$. Thus $W^{s_k}$ is also quasi-dense in $W$, which contradicts the assumption. Hence $T_{k+1} \neq \emptyset$.

(ii): We show that $W_{\{s_0, \ldots, s_{k-1}, s_k\}}$ is irreducible. Now $o(s_k s_j) = \infty$ for some $j \in \{0, \ldots, k\}$ by the above argument. Also $W_{\{s_0, \ldots, s_{k-1}\}}$ is irreducible by the hypothesis (ii). Hence $W_{\{s_0, \ldots, s_{k-1}, s_k\}}$ is irreducible.

(iii): By (iii), there exists a spherical subset $U_k \subset T_k$ such that $W^{U_k \cup \{s_i\}}$ is quasi-dense in $W$ for each $i \in \{0, \ldots, k\}$. We define $U_{k+1} = U_k \cap T_{k+1}$, i.e., $U_{k+1} = \{t \in U_k | o(s_k t) = 2\}$. Here $o(s_k s_j) = \infty$ for some $j \in \{0, \ldots, k\}$ by the above argument. Then $W^{U_{k+1} \cup \{s_k\}} s_k \subset W^{U_{k+1} \cup \{s_k\}}$ by Lemma 2.7. Hence $W^{U_{k+1} \cup \{s_k\}}$ is quasi-dense in $W$, since $W^{U_{k+1} \cup \{s_k\}}$ is so. Finally we show that $W^{U_{k+1} \cup \{s_k\}}$ is quasi-dense in $W$ for each $i \in \{0, \ldots, k-1\}$. We note that $W_{\{s_0, \ldots, s_{k-1}, s_k\}}$ is irreducible by (iii). Hence for each $j_0 \in \{0, \ldots, k-1\}$, there exists a sequence $\{s_k = a_0, a_1, \ldots, a_m = s_{j_0}\} \subset \{s_i | i = 0, 1, \ldots, k\}$ such that $o(s_i a_{i+1}) = \infty$ by Lemma 2.8. Then by Lemma 2.7,

$$W^{U_{k+1} \cup \{s_k\}} a_1 a_2 \cdots a_m \subset W^{U_{k+1} \cup \{s_i\}} a_2 \cdots a_m \subset \cdots \subset W^{U_{k+1} \cup \{a_m\}} = W^{U_{k+1} \cup \{s_{j_0}\}},$$

because $o(s_i u) = 2$ for any $i \in \{0, 1, \ldots, k-1\}$ and $u \in U_{k+1}$. Thus $W^{U_{k+1} \cup \{s_{j_0}\}}$ is quasi-dense in $W$. Hence (iii) holds.

Thus by the induction on $k$, we can define $s_{k-1}, T_k, S_k$ which satisfy (i), (ii) and (iii). Since $S$ is finite, there exists a number $n$ such that $S_n = \{s_0, s_1, \ldots, s_{n-1}\}$ and $W = W_{S_n} \times W_{T_n}$, where $T_n \neq \emptyset$. Therefore $W$ is not irreducible.

5. Dense subsets of the boundary of a right-angled Coxeter group

We obtain the following main theorem from Theorem 3.1 and Lemma 4.1

**Theorem 5.1.** Let $(W, S)$ be a right-angled Coxeter system such that $W$ is infinite. Then the following statements are equivalent:

(1) $\partial \Sigma(W, S)$ is minimal.
(2) \( W_\mathcal{S} \) is irreducible.
(3) \( W^{(s_0)} \) is quasi-dense in \( W \) and \( o(s_0 t_0) = \infty \) for some \( s_0, t_0 \in S \).
(4) There does not exist a finite-index subgroup of \( W \) which splits as a product \( W_1 \times W_2 \) where each \( W_i \) is infinite.

**Proof.** (3) \( \Rightarrow \) (1): If the statement (3) holds, then \( \partial \Sigma(W, S) \) is minimal by Theorem 5.1.

(1) \( \Rightarrow \) (2): Suppose that \( W_\mathcal{S} \) is not irreducible. Let \( W_\mathcal{S} = W_{S_1} \times W_{S_2} \), where \( W_{S_1} \) and \( W_{S_2} \) are infinite. Then \( \partial \Sigma(W, S) = \partial \Sigma(W_\mathcal{S}, \mathcal{S}) \) and \( \Sigma(W_\mathcal{S}, \mathcal{S}) = \Sigma(W_{S_1}, S_1) \times \Sigma(W_{S_2}, S_2) \). Here by [14] Theorem 4.3, \( \partial \Sigma(W_{S_1}, S_1) \) is \( \Sigma \)-invariant, that is, \( W \partial \Sigma(W_{S_1}, S_1) = \partial \Sigma(W_{S_1}, S_1) \). Thus for \( \alpha \in \partial \Sigma(W_{S_1}, S_1) \), \( W\alpha \subset \partial \Sigma(W_{S_1}, S_1) \). Hence \( \partial \Sigma(W, S) \) is not minimal. In Section 6, we will give a more general proof (Theorem 6.4).

(2) \( \Rightarrow \) (3): Suppose that \( W_\mathcal{S} \) is irreducible. By Lemma 4.1, \( (W_\mathcal{S})^{(s_0)} = W^{(s_0)} \cap W_\mathcal{S} \) is quasi-dense in \( W_\mathcal{S} \) for some \( s_0 \in \bar{S} \). Here \( W = W_\mathcal{S} \times W_{S_0} \) and \( W_{S_0} \) is finite (see [11]). Hence \( W^{(s_0)} \) is quasi-dense in \( W \). Since \( W_\mathcal{S} \) is irreducible, \( o(s_0 t_0) = \infty \) for some \( t_0 \in \bar{S} \) by Lemma 2.8. Thus the statement (3) holds.

(4) \( \Rightarrow \) (2): If \( W_\mathcal{S} \) is not irreducible, then \( W_\mathcal{S} \) splits as a product \( W_\mathcal{S} = W_{A_1} \times W_{A_2} \) for some \( A_1 \subset \bar{S} \), where each \( W_{A_i} \) is infinite. Here \( W_\mathcal{S} \) is a finite-index subgroup of \( W \).

(1) \( \Rightarrow \) (4): We obtain this implication from Theorem 6.4. 

The following question appears in [15].

**Question 5.2.** Let \((W, S)\) be a Coxeter system. Is it the case that if \((W, S)\) is an irreducible Coxeter system, then \( W \partial \Sigma(W_T, T) \) is dense in \( \partial \Sigma(W, S) \) for any subset \( T \subset S \) such that \( W_T \) is infinite?

Theorem 5.1 implies that the answer to Question 5.2 is yes for right-angled Coxeter groups. Moreover, as an application of Theorem 5.1, we obtain the following corollary.

**Corollary 5.3.** Let \((W, S)\) be a right-angled Coxeter system and let \( T \subset S \). Then the following statements are equivalent:

1. \( W \partial \Sigma(W_T, T) \) is dense in \( \partial \Sigma(W, S) \).
2. If \( W = W_{S_1} \times \cdots \times W_{S_n} \) is the irreducible decomposition of \( W \), then \( W_{S_i \cap T} \) is infinite for each \( i \in \{1, \ldots, n\} \) such that \( W_{S_i} \) is infinite.

**Proof.** (1) \( \Rightarrow \) (2): Let \( W = W_{S_1} \times \cdots \times W_{S_n} \) be the irreducible decomposition of \( W \). Suppose that there exists \( i_0 \in \{1, \ldots, n\} \) such that \( W_{S_{i_0}} \) is infinite and \( W_{S_{i_0} \cap T} \) is finite. Let \( A_1 = S \setminus S_{i_0} \) and \( A_2 = S_{i_0} \). Then \( W = W_{A_1} \times W_{A_2} \), \( W_{A_2} \) is infinite and \( W_{A_2 \cap T} \) is finite. We note that \( \partial \Sigma(W_{A_1}, A_1) \) is \( W \)-invariant by [11] Theorem 4.3. Since \( W_T = W_{A_1 \cap T} \times W_{A_2 \cap T} \) and \( W_{A_2 \cap T} \) is finite, \( \partial \Sigma(W_T, T) \subset \partial \Sigma(W_{A_1}, A_1) \). Thus \( W \partial \Sigma(W_T, T) \subset W \partial \Sigma(W_{A_1}, A_1) = \partial \Sigma(W_{A_1}, A_1) \). Since \( W_{A_2} \) is infinite and

\[
\partial \Sigma(W, S) = \partial \Sigma(W_{A_1}, A_1) \ast \partial \Sigma(W_{A_2}, A_2),
\]

\( W \partial \Sigma(W_T, T) \) is not dense in \( \partial \Sigma(W, S) \).

(2) \( \Rightarrow \) (1): Let \( W = W_{S_1} \times \cdots \times W_{S_n} \) be the irreducible decomposition of \( W \). Suppose that (2) holds. Then we prove that (1) holds by induction on \( n \).
We first consider the case $n = 1$. Then $W = W_{S_1}$ is irreducible. Since $W_{S_1 \cap T}$ is infinite, $\partial \Sigma(W_T, T) \neq \emptyset$. Hence $W \partial \Sigma(W_T, T)$ is dense in $\partial \Sigma(W, S)$ by Theorem 5.1

Next we consider the case $n > 1$. Let $A_1 = S_1 \cup \cdots \cup S_{n-1}$ and $A_2 = S_n$. Then $W = W_{A_1} \times W_{A_2}$ and $W_T = W_{A_1 \cap T} \times W_{A_2 \cap T}$. Here

$$W \partial \Sigma(W_T, T) = W \partial \Sigma(W_{A_1 \cap T}, A_1 \cap T) \ast \partial \Sigma(W_{A_2 \cap T}, A_2 \cap T)$$

$$\cup W_{A_1} \partial \Sigma(W_{A_1 \cap T}, A_1 \cap T) \ast W_{A_2} \partial \Sigma(W_{A_2 \cap T}, A_2 \cap T).$$

By the inductive hypothesis, $W_{A_i} \partial \Sigma(W_{A_i \cap T}, A_i \cap T)$ is dense in $\partial \Sigma(W_{A_i}, A_i)$ for each $i = 1, 2$. Since

$$\partial \Sigma(W, S) = \partial \Sigma(W_{A_1}, A_1) \ast \partial \Sigma(W_{A_2}, A_2),$$

we obtain that $W \partial \Sigma(W_T, T)$ is dense in $\partial \Sigma(W, S)$. \qed

Also we obtain the following corollary from Theorem 5.1. We give a proof in Section 6.

**Corollary 5.4.** For a right-angled Coxeter system $(W, S)$, the set $\{w^\infty | w \in W, o(w) = \infty\}$ is dense in the boundary $\partial \Sigma(W, S)$.

6. **Remarks on dense subsets of boundaries of CAT(0) groups**

In this section, we investigate dense subsets of boundaries of CAT(0) groups. The definitions and basic properties of CAT(0) spaces and their boundaries can be found in [2]. A group $\Gamma$ is called a CAT(0) group if $\Gamma$ acts geometrically (i.e., properly and cocompactly by isometries) on some CAT(0) space. For example, a Coxeter group $W$ acts geometrically on the Davis complex $\Sigma(W, S)$, which is a CAT(0) space, and every Coxeter group is a CAT(0) group.

We pose the following open problem.

**Question 6.1.** Suppose that a group $\Gamma$ acts geometrically on a CAT(0) space $X$. Is it the case that the set $\{\gamma^\infty | \gamma \in \Gamma, o(\gamma) = \infty\}$ is dense in the boundary $\partial X$?

Here we note that $\gamma^\infty$ is the point of the boundary $\partial X$ to which the sequence $\{\gamma^i x_0 | i \in \mathbb{N}\} \subset X$ converges in $X \cup \partial X$, where $x_0 \in X$ and $\gamma^\infty$ does not depend on the point $x_0$.

We introduce some relations between this question and the minimality of boundaries of CAT(0) groups.

We first show the following proposition.

**Proposition 6.2.** Suppose that a group $\Gamma$ acts geometrically on a CAT(0) space $X$. If there exists $\delta \in \Gamma$ such that $o(\delta) = \infty$ and $\Gamma \delta^\infty$ is dense in the boundary $\partial X$, then the set $\{\gamma^\infty | \gamma \in \Gamma, o(\gamma) = \infty\}$ is dense in $\partial X$. Hence, if the boundary $\partial X$ is minimal, then the set $\{\gamma^\infty | \gamma \in \Gamma, o(\gamma) = \infty\}$ is dense in $\partial X$.

**Proof.** Suppose that $\delta \in \Gamma$ such that $o(\delta) = \infty$ and $\Gamma \delta^\infty$ is dense in $\partial X$. Let $\alpha \in \partial X$. Since $\Gamma \delta^\infty$ is dense in $\partial X$, there exists a sequence $\{\gamma_i\} \subset \Gamma$ such that $\{\gamma_i \delta^\infty\}$ converges to $\alpha$ in $\partial X$. Here for $x_0 \in X$ and each $i$, the sequence $\{(\gamma_i \delta \gamma_i^{-1})^j x_0\}_j$ converges to $\gamma_i \delta^\infty$ in $X \cup \partial X$. Hence $\{(\gamma_i \delta \gamma_i^{-1})^\infty = \gamma_i \delta^\infty\}$ converges to $\alpha$ in $\partial X$. Thus $\{\gamma^\infty | \gamma \in \Gamma, o(\gamma) = \infty\}$ is dense in $\partial X$.

Now we suppose that the boundary $\partial X$ is minimal. It is known that every CAT(0) group has an element of infinite order ([22, Theorem 11]). Let $\delta \in \Gamma$ with $o(\delta) = \infty$. Then $\Gamma \delta^\infty$ is dense in $\partial X$ because $\partial X$ is minimal. Hence, by the above argument, the set $\{\gamma^\infty | \gamma \in \Gamma, o(\gamma) = \infty\}$ is dense in the boundary $\partial X$. \qed
Let $\Gamma$ be a group.

**Proposition 6.3.** Suppose that a group $\Gamma = G_1 \times G_2$ acts geometrically on a CAT(0) space $X$, where $G_1$ and $G_2$ are infinite. Then $X$ contains a quasi-dense subspace $X' = X_1 \times X_2$ and there exists a product subgroup $G_1' \times G_2'$ of finite index in $\Gamma$ such that $X_1$ is the convex hull $C(G_1' x_0)$ for some $x_0 \in X$ and $G_2'$ acts geometrically on $X_2$ by projection.

**Proof.** By [13, Lemma 2.1], there exist subgroups $G_1 \times A_1$ and $G_2 \times A_2$ of finite index in $G_1$ and $G_2$ respectively such that $G_1$ and $G_2$ have finite center and $A_i$ is isomorphic to $\mathbb{Z}^{n_i}$ for some $n_i$ ($i = 1, 2$).

In the case that $A_i$ is not trivial for some $i \in \{1, 2\}$, we put $\Gamma_1' = A_i$ and $\Gamma_2' = G_1 \times G_2 \times A_{3-i}$. Then by [3, Proposition 1.1] and [4, Theorem II.7.1], the proposition holds.

In the case that $A_1$ and $A_2$ are trivial, we put $\Gamma_1' = G_1$ and $\Gamma_2' = G_2$. Here $G_1$ and $G_2$ have finite center. By [13, Theorem 2] and [19, Corollary 10], the proposition holds. Here concerning the condition in [19, Corollary 10], we note that if the CAT(0) group $\Gamma$ has finite center, then there does not exist a $\Gamma$-fixed point in the boundary $\partial X$ (cf. [17, Lemma 3.2]).

Concerning the nonminimality of boundaries of CAT(0) groups, using Proposition 6.3 we show the following theorem.

**Theorem 6.4.** Suppose that a group $\Gamma$ acts geometrically on a CAT(0) space $X$. If $\Gamma$ contains a finite-index subgroup $\Gamma_1 \times \Gamma_2$ where $\Gamma_1$ and $\Gamma_2$ are infinite, then the boundary $\partial X$ is not minimal.

**Proof.** Let $\Gamma_1 \times \Gamma_2$ be a finite-index subgroup of $\Gamma$, where $\Gamma_1$ and $\Gamma_2$ are infinite. Then $\Gamma_1 \times \Gamma_2$ acts geometrically on $X$. By Proposition 6.3 $X$ contains a quasi-dense subspace $X_1 \times X_2$ and there exists a product subgroup $\Gamma_1' \times \Gamma_2'$ of finite index in $\Gamma$ such that $X_1$ is the convex hull $C(\Gamma_1' x_0)$ for some $x_0 \in X$ and $\Gamma_2'$ acts geometrically on $X_2$ by projection.

To prove that $\partial X$ is not minimal, we show that $\Gamma(\partial X_1)$ is not dense in $\partial X$.

Since $\Gamma_1' \times \Gamma_2'$ is a subgroup of finite index in $\Gamma$, there exist a number $n$ and $\{d_1, \ldots, d_n\} \subset \Gamma$ such that $\Gamma = \bigcup_{i=1}^n d_i(\Gamma_1' \times \Gamma_2')$.

Since $X_1 = C(\Gamma_1' x_0)$ is $\Gamma_1'$-invariant, $\Gamma_1'(\partial X_1) = \partial X_1$. For each $\gamma_2 \in \Gamma_2'$, $\gamma_2 X_1$ and $X_1$ are parallel by the proof of the splitting theorems ([3, 4, 17, 19]); hence $\gamma_2(\partial X_1) = \partial X_1$, that is, $\Gamma_2'(\partial X_1) = \partial X_1$. Thus $(\Gamma_1' \times \Gamma_2')(\partial X_1) = \partial X_1$.

Hence

$$
\Gamma(\partial X_1) = \left( \bigcup_{i=1}^n d_i(\Gamma_1' \times \Gamma_2') \right)(\partial X_1)
= \bigcup_{i=1}^n (\delta_i(\partial X_1))
= \bigcup_{i=1}^n (\delta_i(\partial X_1)).
$$
Here we note that $\Gamma(\partial X_1) = \bigcup_{i=1}^{n}(\delta_i(\partial X_1))$ is closed. Hence
\[
\dim(\partial X_1) = \dim \bigcup_{i=1}^{n}(\delta_i(\partial X_1)) = \dim \partial X_1
\]
\[
< \dim(\partial X_1 \times [0,1]) \leq \dim(\partial X_1 \ast \partial X_2) = \dim \partial X.
\]
Here we note that $\dim \partial X$ is finite, because the boundary of a cocompact proper CAT(0) space is finite-dimensional ([22, Theorem 12]).

Thus $\Gamma(\partial X_1)$ is not dense in $\partial X$. This implies that $\partial X$ is not minimal. □

The referee has pointed out that the converse of Theorem 6.4 (if $\Gamma$ does not contain a finite-index subgroup $\Gamma_1 \times \Gamma_2$ where $\Gamma_1$ and $\Gamma_2$ are infinite, then the boundary $\partial X$ is minimal) will not be true in general and that a counterexample will be supplied by the theory of lattices in semisimple groups, since an irreducible lattice on a product of two hyperbolic planes does not factor (with infinite factors) (cf. [10]).

On the other hand, Theorem 5.1 implies that the converse of Theorem 6.4 holds for right-angled Coxeter groups and their boundaries.

Let $A$ be the set of all infinite CAT(0) groups $\Gamma$ such that for any $\Gamma(0)$ space $X$ on which $\Gamma$ acts geometrically, the set $\{\gamma^\infty | \gamma \in \Gamma, o(\gamma) = \infty\}$ is dense in $\partial X$.

Now we show the following proposition.

**Proposition 6.5.** Suppose that $\Gamma_1, \ldots, \Gamma_n \in A$ and that each $\Gamma_i$ does not contain a finite-index subgroup $\Gamma_1 \times \Gamma_2$ such that $\Gamma_1$ and $\Gamma_2$ are infinite. Then $\Gamma_1 \times \cdots \times \Gamma_n \in A$.

**Proof.** We note that each $\Gamma_i$ is either isomorphic to $Z$ or has finite center by [13, Lemma 2.1]. Hence we can suppose that for some number $k$, $\Gamma_i$ is isomorphic to $Z$ for each $i \leq k$ and $\Gamma_i$ has finite center for each $i > k$.

We prove that $\Gamma \in A$ by induction on $n$.

In the case $n = 1$, it is obvious.

We consider the case $n = 2$. Suppose that $\Gamma = \Gamma_1 \times \Gamma_2$ acts geometrically on a CAT(0) space $X$. By Proposition 6.3 $X$ contains a quasi-dense subspace $X_1 \times X_2$ such that $X_1 = C(\Gamma_1 x_0)$ for some $x_0 \in X$ and $\Gamma_2$ acts geometrically on $X_2$ by projection. Let $\alpha \in \partial X$. Here
\[
\partial X = \partial X_1 \ast \partial X_2 = (\partial X_1 \times \partial X_2 \times [-\pi, \pi]) / \sim.
\]
Hence $\alpha = [\alpha_1, \alpha_2, \theta]$ for some $\alpha_1 \in \partial X_1$, $\alpha_2 \in \partial X_2$ and $\theta \in [-\pi, \pi]$. Now $\{\gamma^\infty | \gamma \in \Gamma_1, o(\gamma) = \infty\}$ is dense in $\partial X_1$ and $\{\delta^\infty | \delta \in \Gamma_2, o(\delta) = \infty\}$ is dense in $\partial X_2$.

Hence there exist sequences $\{\gamma_i\} \subset \Gamma_1$ and $\{\delta_i\} \subset \Gamma_2$ such that $\{\gamma_i^\infty\}$ converges to $\alpha_1$ and $\{\delta_i^\infty\}$ converges to $\alpha_2$. Since $\langle \gamma_i, \delta_i \rangle$ is isomorphic to $Z \times Z$, by the Flat Torus Theorem ([4, Theorem II.7.1]), $\langle \gamma_i, \delta_i \rangle$ acts geometrically on some convex hull $C(\langle \gamma_i, \delta_i \rangle x_i)$ which is isometric to the Euclidean plane. Here $C(\langle \gamma_i, \delta_i \rangle x_i) \subset X_1 \times X_2$ and
\[
\{\gamma_i^{-\infty}, \gamma_i^\infty, \delta_i^{-\infty}, \delta_i^\infty\} \subset \partial(C(\langle \gamma_i, \delta_i \rangle x_i)).
\]
Then there exists a sequence $\{a_{ij}\} \subset \langle \gamma_i, \delta_i \rangle$ such that $\{a_{ij}^\infty\}$ converges to $[\gamma_i^\infty, \delta_i^\infty, \theta]$. Here the sequence $\{[\gamma_i^\infty, \delta_i^\infty, \theta]\}_i$ converges to $\alpha$. Hence
\[
\alpha \in \{a_{ij}^\infty | i, j \in N\} \subset \{\gamma^\infty | \gamma \in \Gamma, o(\gamma) = \infty\}.
\]
Thus $\{\gamma^\infty | \gamma \in \Gamma, o(\gamma) = \infty\}$ is dense in $\partial X$ and $\Gamma = \Gamma_1 \times \Gamma_2 \in A$. 

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We consider the case \( n > 2 \). Suppose that \( \Gamma = \Gamma_1 \times \cdots \times \Gamma_{n-1} \times \Gamma_n \) acts geometrically on a CAT(0) space \( X \). Let \( \Gamma_1 = \Gamma_1 \times \cdots \times \Gamma_{n-1} \) and \( \Gamma_2 = \Gamma_n \). Here we can suppose that \( \Gamma_2 \) has finite center or that each \( \Gamma_i \) is isomorphic to \( \mathbb{Z} \) for \( i = 1, \ldots, n \). By the inductive hypothesis and the same argument as the proof in the case \( n = 2 \), we obtain that \( \{ \gamma^\infty \mid \gamma \in \Gamma, \ o(\gamma) = \infty \} \) is dense in \( \partial X \) and \( \Gamma = \Gamma_1 \times \cdots \times \Gamma_{n-1} \times \Gamma_n \in \mathcal{A} \).

Suppose that a group \( \Gamma \) acts geometrically on a CAT(0) space \( X \). Then there exists a finite-index subgroup \( \Gamma_1 \times \cdots \times \Gamma_n \) of \( \Gamma \) such that each \( \Gamma_i \) is infinite and each \( \Gamma_i \) does not contain a finite-index subgroup \( \Gamma_i \times \Gamma_i \) where \( \Gamma_i \) and \( \Gamma_i \) are infinite. Here the decomposition process terminates and \( n \) is finite. Indeed each \( \Gamma_i \) is a CAT(0) group by [17, Theorem 9.1] and there exists \( \gamma_i \in \Gamma_i \) with \( o(\gamma_i) = \infty \) by [22, Theorem 11]. Then \( \langle \gamma_1, \ldots, \gamma_n \rangle \subset \Gamma \) is isomorphic to \( \mathbb{Z}^n \). Here such an \( n \) is finite, because every abelian subgroup of a CAT(0) group is finitely generated (see Corollary II.7.6).

Hence Proposition 6.5 implies that Question 6.1 is equivalent to the following question.

**Question 6.6.** For an infinite CAT(0) group \( \Gamma \) which does not contain a finite-index product subgroup of two infinite subgroups, does \( \Gamma \in \mathcal{A} \)?

Finally, we prove Corollary 5.4. Concerning Question 6.1, we obtain a positive answer for right-angled Coxeter groups and their boundaries.

**Proof of Corollary 5.4.** Let \((W, S)\) be a right-angled Coxeter system and let \( W = W_{S_1} \times \cdots \times W_{S_n} \) be the irreducible decomposition of \( W \). We may suppose that \( W_{S_i} \) is infinite for any \( i \leq k \) and \( W_{S_i} \) is finite for any \( i > k \) for some number \( k \). Then \( \hat{S} = S_1 \cup \cdots \cup S_k \) and

\[
\Sigma(W, S) = \Sigma(W_{S_1}, S_1) \times \cdots \times \Sigma(W_{S_k}, S_k) \times \Sigma(W_{\hat{S} \setminus S}, \hat{S} \setminus \hat{S}),
\]

where \( \Sigma(W_{\hat{S} \setminus S}, \hat{S} \setminus \hat{S}) \) is bounded, since \( W_{\hat{S} \setminus S} \) is finite. Hence,

\[
\partial \Sigma(W, S) = \partial \Sigma(W_{S_1}, S_1) \times \cdots \times \partial \Sigma(W_{S_k}, S_k).
\]

Here each Coxeter system \((W_{S_i}, S_i)\) is irreducible and right-angled and \( \partial \Sigma(W_{S_i}, S_i) \) is minimal by Theorem 5.1. Thus for each \( i \in \{1, \ldots, k\} \), the set \( \{ w^\infty \mid w \in W_i, \ o(w) = \infty \} \) is dense in \( \partial \Sigma(W_{S_i}, S_i) \) by Proposition 6.2. By a similar argument to the proof of Proposition 6.5, we obtain that the set \( \{ w^\infty \mid w \in W, \ o(w) = \infty \} \) is dense in the boundary \( \partial \Sigma(W, S) \).

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