CRYSTAL BASE OF THE NEGATIVE HALF OF THE QUANTUM SUPERALGEBRA $U_q(\mathfrak{gl}(m|n))$  

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ABSTRACT. We construct a crystal base of $U_q(\mathfrak{gl}(m|n))^{-}$, the negative half of the quantum superalgebra $U_q(\mathfrak{gl}(m|n))$. We give a combinatorial description of the associated crystal $\mathcal{B}_{m|n}(\infty)$, which is equal to the limit of the crystals of the ($q$-deformed) Kac modules $K(\lambda)$. We also construct a crystal base of a parabolic Verma module $X(\lambda)$ associated with the subalgebra $U_q(\mathfrak{gl}_{0|n})$, and show that it is compatible with the crystal base of $U_q(\mathfrak{gl}(m|n))^{-}$ and the Kac module $K(\lambda)$ under the canonical embedding and projection of $X(\lambda)$ to $U_q(\mathfrak{gl}(m|n))^{-}$ and $K(\lambda)$, respectively.

1. Introduction

The crystal base theory for the quantized enveloping algebra $U_q(\mathfrak{g})$ associated to a symmetrizable Kac-Moody algebra $\mathfrak{g}$ has been one of the most important tools in the representation theory of $U_q(\mathfrak{g})$ [12], reflecting its fundamental combinatorial structure.

For a classical Lie superalgebra $\mathfrak{g}$, there have been several works on the crystal base of a representation of $U_q(\mathfrak{g})$, where the representation theory of $U_q(\mathfrak{g})$ is no longer parallel to that of a symmetrizable Kac-Moody algebra, and hence the existence of a crystal base is not easily expected. It is shown that an irreducible polynomial representation of $U_q(\mathfrak{g})$ has a crystal base when $\mathfrak{g}$ is a general linear Lie superalgebra $\mathfrak{gl}(m|n)$ [1], and a queer Lie superalgebra $\mathfrak{q}(n)$ [10]. When $\mathfrak{g}$ is an orthosymplectic Lie superalgebra $\mathfrak{osp}(m|n)$ with $m \geq 2$, the existence of a crystal base in the sense of [1] is shown in [20, 21] for a family of irreducible representations, which corresponds to the integrable highest weight representations of the classical Lie algebras from a viewpoint of super duality [4].

Let $U_q(\mathfrak{gl}(m|n))$ be the quantized enveloping algebra associated to $\mathfrak{gl}(m|n)$ [32], and let $U_q(\mathfrak{gl}(m|n))^{-}$ be its negative half. The purpose of this paper is to construct a crystal base of $U_q(\mathfrak{gl}(m|n))^{-}$.

Let $P$ be the integral weight lattice for $\mathfrak{gl}(m|n)$ and let $P^+$ be the set of dominant integral weights for the even subalgebra $\mathfrak{gl}(m|n)_0 = \mathfrak{gl}(m|0) \oplus \mathfrak{gl}(0|n)$. For $\lambda \in P^+$, let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{gl}(m|n))$-module with highest weight $\lambda$, which is finite-dimensional.

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A tensor power of the natural representation of $U_q(\mathfrak{gl}(m|n))$ is semisimple and an irreducible representation appearing here is called an irreducible polynomial representation, whose highest weight $\lambda$ is parametrized by $\mathcal{P}_{m|n}$, the set of $(m|n)$-hook partitions. In [11], it is shown that for $\lambda \in \mathcal{P}_{m|n}$, $V(\lambda)$ has a crystal base ($\mathcal{L}(\lambda), \mathcal{B}(\lambda)$), and an explicit combinatorial description of its crystal $\mathcal{B}(\lambda)$ is given. An important feature of ($\mathcal{L}(\lambda), \mathcal{B}(\lambda)$) is that it is a lower (resp. upper) crystal base of $V(\lambda)$ as a $U_q(\mathfrak{gl}(m|0))$-module (resp. $U_q(\mathfrak{gl}(0|n))$-module) so that the resulting crystal structure on $\mathcal{B}(\lambda)$ and hence on $\mathcal{B}(\mu) \otimes \mathcal{B}(\nu)$ for $\mu, \nu \in \mathcal{P}_{m|n}$ are more involved than the case of $\mathfrak{gl}(m+n)$ (cf. [11]).

As in the case of a symmetrizable Kac-Moody algebra [12], one may expect to construct a crystal of $U_q(\mathfrak{gl}(m|n))^{-}$ by taking a limit of $\mathcal{B}(\lambda)$ for $\lambda \in \mathcal{P}_{m|n}$. But the crystals $\mathcal{B}(\lambda)$ do not seem to admit naturally a directed system of $U_q(\mathfrak{gl}(m|n))$-crystals as $\lambda$ goes to infinity. Moreover, the upper crystal lattice of an integrable highest weight $U_q(\mathfrak{gl}(n|0))$-module is not compatible with the projection from $U_q(\mathfrak{gl}(0|n))^{-}$ onto it. These are two main differences from the case of symmetrizable Kac-Moody algebras.

In this paper, we use the crystal base of ($q$-deformed) Kac modules to construct a crystal base of $U_q(\mathfrak{gl}(m|n))^{-}$. A Kac module $K(\lambda)$ is the $U_q(\mathfrak{gl}(m|n))$-module induced from an integrable highest weight $U_q(\mathfrak{gl}(m|n)_0)$-module $V_{m|n}(\lambda) \otimes V_{0|n}(\lambda)$ with highest weight $\lambda = \lambda_+ + \lambda_- \in P^+$. It forms another important class of finite-dimensional indecomposable $U_q(\mathfrak{gl}(m|n))$-modules. It is shown in [12] that $K(\lambda)$ has a crystal base ($\mathcal{L}(K(\lambda)), \mathcal{B}(K(\lambda))$), and when $\lambda \in \mathcal{P}_{m|n}$ it is compatible with ($\mathcal{L}(\lambda), \mathcal{B}(\lambda)$) under the canonical projection from $K(\lambda)$ to $V(\lambda)$, that is, it preserves the crystal lattices and induces a morphism of $U_q(\mathfrak{gl}(m|n))$-crystals from $\mathcal{B}(K(\lambda))$ onto $\mathcal{B}(\lambda)$.

We remark that we use in this paper the generalized quantum group $U_{m|n}$ [17], which is isomorphic to $U_q(\mathfrak{gl}(m|n))$ as a $\mathbb{Q}(q)$-algebra under mild extensions, and which has a comultiplication equal to that of a usual quantum group. The irreducible polynomial modules and Kac modules together with their crystal bases are well-defined for $U_{m|n}$. Hence the problem is replaced by constructing a crystal base of $U_{m|n}^{-}$.

We first consider a directed system of $\mathcal{B}(K(\lambda))$ as crystals over $\mathfrak{gl}(m|n)_0$, and show that its limit $\mathcal{B}_{m|n}(\infty)$ has a well-defined abstract $U_{m|n}^{-}$-crystal structure (Theorem 5.7), though $\{ \mathcal{B}(K(\lambda)) \mid \lambda \in P^+ \}$ itself does not form a directed system of $U_{m|n}$-crystals. As a set, the limit $\mathcal{B}_{m|n}(\infty)$ can be identified with

$$\mathcal{B}(K_{m|n}) \times \mathcal{B}_{m|0}(\infty) \times \mathcal{B}_{0|n}(\infty), \tag{1.1}$$

where $\mathcal{B}(K_{m|n})$ is the crystal of $K(0)$ or the subalgebra $K_{m|n}$ of $U_{m|n}^{-}$ spanned by the set of PBW-type monomials in odd root vectors, and $\mathcal{B}_{m|0}(\infty)$ and $\mathcal{B}_{0|n}(\infty)$ are the crystals of the subalgebras $U_{m|0}^{-} \cong U_q(\mathfrak{gl}(m)|0)$ and $U_{0|n}^{-} \cong U_{-q^{-1}}(\mathfrak{gl}(n)|n)$ at $q = 0$, respectively. As to the (abstract) Kashiwara or crystal operators on $\mathcal{B}_{m|n}(\infty)$, the operators for $\mathfrak{gl}(m|1) \subset$
$gl(m|n)$ act on the first two components following explicit combinatorial rules, and those for $gl(0|n) \subset gl(m|n)$ act only on the last component as usual.

Let $A_t$ be the subring of $f(q) \in \mathbb{Q}(q)$ regular at $q = t$ ($t = 0, \infty$). Let

$$\mathcal{L}(\infty) = \mathcal{L}(K_{m|n}) \cdot \mathcal{L}_{m|0}(\infty) \cdot \mathcal{L}_{0|n}(\infty)$$

be an $A_0$-lattice of $U_{m|n}^{-}$, where $\mathcal{L}(K_{m|n})$ is the $A_0$-span of PBW-type monomials in odd root vectors, $\mathcal{L}_{m|0}(\infty)$ is the crystal lattice for $U_q(gl(m))^{-}$ at $q = 0$, and $\mathcal{L}_{0|n}(\infty)$ is the $A_0$-lattice corresponding to the $A_\infty$-dual crystal lattice of $U_q(gl(n))^{-}$ at $q = \infty$. Then we introduce the crystal operators on $U_{m|n}^{-}$ such that $(\mathcal{L}_{m|n}(\infty), \mathcal{B}_{m|n}(\infty))$ is a crystal base and the induced crystal is isomorphic to $\mathcal{B}_{m|n}(\infty)$ (Theorem 6.1).

To define the crystal operators on $U_{m|n}^{-}$, especially for $gl(m|0)$, we use the $U_{m|0}$-comodule structure on the algebra $B_q$ of $q$-bosons of type $A_{m-1}$, by which we realize the subalgebra $K_{m|n} \cdot U_{m|0}^{-} \cdot U_{m|n}^{-}$ as a $B_q$-module, and then apply the crystal base theory for $B_q$-modules in [12]. The crystal operators for $gl(0|n)$ are given by twisting the usual ones with respect to the dual bar involution on $U_{0|n}$.

The crystal base $(\mathcal{L}_{m|n}(\infty), \mathcal{B}_{m|n}(\infty))$ is compatible with the crystal bases of $K(\lambda)$ in the following sense: Let $X(\lambda)$ be a parabolic Verma module induced from an integrable highest weight $U_{0|n}$-module $V_{0|n}(\lambda_-)$ with highest weight $\lambda_-$. Instead of the projection from $U_{m|n}^{-}$ to $K(\lambda)$, we consider

$$U_{m|n}^{-} \xrightarrow{\pi_-} X(\lambda) \xrightarrow{\pi_+} K(\lambda),$$

where $\pi_+$ is the canonical projection and $\pi_-$ is the map obtained by applying the dual of the projection $U_{0|n}^{-} \rightarrow V_{0|n}(\lambda_-)$ to the projection $\pi_- : U_{m|n}^{-} \rightarrow X(\lambda)$. We show that $X(\lambda)$ has a crystal base (Theorem 6.2), and the diagram (1.2) is compatible with the crystal bases of $U_{m|n}^{-}$, $X(\lambda)$ and $K(\lambda)$ in the sense that it induces well-defined maps on their crystal lattices and the associated crystals at $q = 0$, and the crystal operators partially commute with the maps (Corollaries 6.3 and 6.4). The lattice $\mathcal{L}_{m|n}(\infty)$ is slightly different from the lattice spanned by the PBW-type basis in [7] which can be obtained by applying the dual bar involution on $U_{0|n}^{-}$ to $\mathcal{L}_{0|n}(\infty) \subset \mathcal{L}_{m|n}(\infty)$.

We further discuss the structure of $\mathcal{B}_{m|n}(\infty)$. We show that $\mathcal{B}_{m|n}(\infty)$ is connected if and only if $n = 1$, and in general it decomposes as follows:

$$\mathcal{B}_{m|n}(\infty) \cong (\mathcal{B}_{m|1}(\infty) \times \mathcal{B}_{0|n}(\infty)) \oplus 2^{m(n-1)},$$

as a $U_{m|n}$-crystal up to shift of weights (Theorem 6.8), where on the connected component $\mathcal{B}_{m|1}(\infty) \times \mathcal{B}_{0|n}(\infty)$, the crystal operators for $gl(m|1)$ act on the first component and those for $gl(0|n)$ act on the second component. We remark that the connected crystal $\mathcal{B}_{m|1}(\infty)$ for $U_{m|1}^{-}$ or for $U_q(gl(m|1))^{-}$ is isomorphic to the one constructed in [3]. Also it would be
interesting to find a categorical interpretation of the crystal structure on $B_m(\infty)$ from a viewpoint of [15].

The paper is organized as follows. In Section 2, we recall the notion of quantum superalgebras and generalized quantum groups of finite type $A$. In Section 3, we give necessary materials on the crystal bases for homogeneous generalized quantum groups. In Section 4, we review the crystal bases of irreducible polynomial representations and Kac modules. In Section 5, we construct a crystal $B_{m|n}(\infty)$ as a limit of the crystals of Kac modules. In Section 6, we show that $U^-_{m|n}$ has a crystal base whose crystal is isomorphic to $B_{m|n}(\infty)$ and it is compatible with ([12]). In Appendix A, we review the crystal base theory for the algebra of $q$-bosons, which plays a crucial role in this paper.

2. QUANTUM SUPERALGEBRA

2.1. Quantum superalgebra $U_q(\mathfrak{gl}(\epsilon))$. We assume the following notations:

- $\mathbb{Z}_+$: the set of non-negative integers,
- $k = \mathbb{Q}(q)$ where $q$ is an indeterminate, $k^\times = k \setminus \{0\}$,
- $\ell$: a fixed positive integer greater than 2,
- $\epsilon = (\epsilon_1, \cdots, \epsilon_\ell)$: a sequence with $\epsilon_i \in \{0, 1\}$ ($i = 1, \ldots, \ell$),
- $m = |\{ i \mid \epsilon_i = 0 \}|$, $n = |\{ i \mid \epsilon_i = 1 \}|$,
- $\epsilon_{m|n}$: a sequence with $\epsilon_1 = \cdots = \epsilon_m = 0$, $\epsilon_{m+1} = \cdots = \epsilon_{m+n} = 1$ ($m + n = \ell$),
- $\mathbb{I} = \{ 1 < 2 < \cdots < \ell \}$: a linearly ordered set with $\mathbb{Z}_2$-grading $\mathbb{I} = \mathbb{I}_{\uparrow} \cup \mathbb{I}_{\downarrow}$ such that
  $$\mathbb{I}_{\uparrow} = \{ i \mid \epsilon_i = 0 \}, \quad \mathbb{I}_{\downarrow} = \{ i \mid \epsilon_i = 1 \},$$
- $P$ : the free abelian group with a basis $\{ \delta_i \mid i \in \mathbb{I} \}$,
- $(\cdot | \cdot)$ : a symmetric bilinear form on $P$ such that $(\delta_i | \delta_j) = (-1)^{\epsilon_i \epsilon_j} \delta_{ij}$ ($i, j \in \mathbb{I}$),
- $P^\vee = \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$ with a basis $\{ \delta_i^\vee \mid i \in \mathbb{I} \}$ such that $(\delta_i, \delta_j^\vee) = \delta_{ij}$ ($i, j \in \mathbb{I}$),
- $I = \{ 1, \ldots, \ell - 1 \}$,
- $\alpha_i = \delta_i - \delta_{i+1} \in P$, $\alpha_i^\vee = \delta_i^\vee - (-1)^{\epsilon_i+1} \delta_{i+1}^\vee \in P^\vee$ ($i \in I$),
- $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$, $Q^+ = \sum_{i \in I} \mathbb{Z} \alpha_i$, $Q^- = -Q^+$,
- $I_{\text{even}} = \{ i \in I \mid (\alpha_i | \alpha_i) = \pm 2 \}$, $I_{\text{odd}} = \{ i \in I \mid (\alpha_i | \alpha_i) = 0 \}$.

The quantum superalgebra $U_q(\mathfrak{gl}(\epsilon))$ associated to $\epsilon$ or the Cartan matrix $A = (\langle \alpha_j, \alpha_i^\vee \rangle)_{i,j \in I}$ ([32]) is the associative $k$-algebra with 1 generated by $K_\mu, E_i, F_i$ for $\mu \in P$ and $i \in I$ satisfying

$$K_0 = 1, \quad K_\mu K_{\mu'} = K_{\mu + \mu'} \quad (\mu, \mu' \in P),$$

$$K_\mu E_i K^{-1}_\mu = q^{\langle \mu | \alpha_i \rangle} E_i, \quad K_\mu F_i K^{-1}_\mu = q^{-\langle \mu | \alpha_i \rangle} F_i \quad (i \in I, \mu \in P),$$

$$E_i F_j - (-1)^{\epsilon_i \epsilon_j} F_j E_i = (-1)^{\epsilon_i \epsilon_j} \delta_{ij} \frac{K_i - K^{-1}_i}{q - q^{-1}} \quad (i, j \in I),$$

$$E_i^2 = F_i^2 = 0 \quad (i \in I_{\text{odd}}),$$

$$E_i E_j - (-1)^{\epsilon_i \epsilon_j} E_j E_i = F_i F_j - (-1)^{\epsilon_i \epsilon_j} F_j F_i = 0 \quad (i, j \in I \text{ and } |i - j| \neq 0, 1),$$
\[ E_i^2 E_j - [2] E_i E_j E_i + E_j E_i^2 = 0 \quad (i \in I_{\text{even}} \text{ and } |i - j| = 1), \]
\[ F_i^2 F_j - [2] F_i F_j F_i + F_j F_i^2 = 0 \]
\[ [E_i, [E_{i-1}, E_i]_{(-1)^{p(i-1)+p(i)+p(i+1)}}^{(-1)^{p(i-1)+p(i)+p(i+1)}}]_{(1)^{p(i-1)+p(i)+p(i+1)}} = 0 \quad (i \in I_{\text{odd}}), \]
\[ [F_i, [F_{i-1}, F_i]_{(-1)^{p(i-1)+p(i)+p(i+1)}}^{(-1)^{p(i-1)+p(i)+p(i+1)}}]_{(1)^{p(i-1)+p(i)+p(i+1)}} = 0 \]

where \([a] = \frac{q^a - q^{-a}}{q - q^{-1}}\) for \(a \in \mathbb{Z}_+, \ p(i) = \varepsilon_i + \varepsilon_{i+1} (i \in I), \ [X, Y]_t = XY - tYX\) for \(t \in \mathbb{k}\), and \(K_i = K_{\varepsilon_i}\) for \(i \in I\). We write \(U_q(\mathfrak{gl}(m|n)) = U_q(\mathfrak{gl}(\epsilon))\) when \(\epsilon = \epsilon_{m|n}\).

2.2. Generalized quantum group \(U(\mathfrak{gl}(\epsilon))\). Let us recall the notion of the generalized quantum group of finite type \(A\) associated to \(\epsilon [17]\). Let

- \(q_i = (-1)^{\varepsilon_i} q_i^{(-1)^{\varepsilon_i}} (i \in I)\), that is,
\[
q_i = \begin{cases} 
q & \text{if } \varepsilon_i = 0, \\
q^{-1} & \text{if } \varepsilon_i = 1,
\end{cases} \quad (i \in I),
\]

- \(q(\cdot, \cdot)\): a symmetric biadditive function from \(P \times P\) to \(\mathbb{k}^\times\) given by
\[
q(\mu, \nu) = \prod_{i \in I} q_i^{(\mu_i, \delta_i^\vee) (\nu_i, \delta_i^\vee)}.
\]

**Definition 2.1.** We define \(U(\mathfrak{gl}(\epsilon))\) to be the associative \(\mathbb{k}\)-algebra with 1 generated by \(k_\mu, e_i, f_i\) for \(\mu \in P\) and \(i \in I\) satisfying

\[
k_0 = 1, \quad k_{\mu + \mu'} = k_\mu k_{\mu'} \quad (\mu, \mu' \in P),
\]
\[
k_\mu e_i k_{-\mu} = q(\mu, \alpha_i) e_i, \quad k_\mu f_i k_{-\mu} = q(\mu, \alpha_i)^{-1} f_i \quad (i \in I, \mu \in P),
\]
\[
e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \quad (i, j \in I),
\]
\[
e_i^2 = f_i^2 = 0 \quad (i \in I_{\text{odd}}),
\]

and

\[
e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0 \quad (i, j \in I \text{ and } |i - j| \neq 0, 1),
\]
\[
e_i^2 e_j - [2] e_i e_j e_i + e_j e_i^2 = 0 \quad (i \in I_{\text{even}} \text{ and } |i - j| = 1),
\]
\[
f_i^2 f_j - [2] f_i f_j f_i + f_j f_i^2 = 0 \quad (i \in I_{\text{even}} \text{ and } |i - j| = 1),
\]
\[
e_i e_{i-1} e_i e_{i+1} - e_i e_{i+1} e_i e_{i-1} + e_{i+1} e_i e_{i-1} e_i
\]
\[
- e_{i-1} e_i e_{i+1} e_i + (-1)^{\varepsilon_i} [2] e_i e_{i-1} e_{i+1} e_i = 0, \quad (i \in I_{\text{odd}}),
\]
\[
f_i f_{i-1} f_i f_{i+1} - f_i f_{i+1} f_i f_{i-1} + f_{i+1} f_i f_{i-1} f_i
\]
\[
- f_{i-1} f_i f_{i+1} f_i + (-1)^{\varepsilon_i} [2] f_i f_{i-1} f_{i+1} f_i = 0,
\]

where \(k_i = k_{\varepsilon_i}\) for \(i \in I\).

We write \(U(\mathfrak{gl}(m|n)) = U(\mathfrak{gl}(\epsilon))\) when \(\epsilon = \epsilon_{m|n}\).
Remark 2.2. The algebra $U(\mathfrak{gl}(\epsilon))$ is the subalgebra of the generalized quantum group $U(\epsilon)$ of affine type $A$ in [22, 23], which is denoted by $\overline{U}(\epsilon)$ or $\hat{U}(\epsilon)$. We follow the convention in [22, Definition 2.1] for its presentation, while $k_3$ corresponds to $\omega_j$ in [23, Definition 2.1].

The algebra $U(\mathfrak{gl}(\epsilon))$ is closely related to $U_q(\mathfrak{gl}(\epsilon))$ in the following sense. Let $\Sigma$ be the bialgebra over $k$ generated by $\sigma_j (j \in \mathbb{I})$ such that $\sigma_i \sigma_j = \sigma_j \sigma_i$ and $\sigma_i^2 = 1$ ($i, j \in \mathbb{I}$) where the comultiplication is given by $\Delta(\sigma_j) = \sigma_j \otimes \sigma_j$. Let $\Sigma$ act on $U_q(\mathfrak{gl}(\epsilon))$ by

$$\sigma_j K_\mu = K_\mu, \quad \sigma_j E_i = (-1)^{\epsilon_j(\delta_j|\alpha_i)}E_i, \quad \sigma_j F_i = (-1)^{\epsilon_j(\delta_j|\alpha_i)}F_i,$$

for $j \in \mathbb{I}$, $\mu \in P$ and $i \in I$ so that $U_q(\mathfrak{gl}(\epsilon))$ is a $\Sigma$-module algebra. Let $U_q(\mathfrak{gl}(\epsilon))[\sigma]$ be the semidirect product of $U_q(\mathfrak{gl}(\epsilon))$ and $\Sigma$. If we define $U(\mathfrak{gl}(\epsilon))[\sigma]$ in the same way, then there exists an isomorphism of $k$-algebras $\tau: U_q(\mathfrak{gl}(\epsilon))[\sigma] \to U(\mathfrak{gl}(\epsilon))[\sigma]$ such that $\tau(\sigma_j) = \sigma_j$ ($j \in \mathbb{I}$), $\tau(E_i) = e_iu_i$, $\tau(F_i) = f_iu_i$, and $\tau(K_i) = k_iw_i$ ($i \in I$) for some monomials $u_i, v_i, w_i$ in $\Sigma$ by [22, Theorem 2.7].

For example, when $\epsilon = \epsilon_{m|n}$, we have

$$U_q(\mathfrak{gl}(m|n))[\sigma] \xrightarrow{\tau} U(\mathfrak{gl}(m|n))[\sigma],$$

where

$$\tau(K_{\delta_j}) = \begin{cases} k_\delta_j & (1 \leq j \leq m), \\ k_\delta_j \sigma_j & (m < j \leq \ell), \end{cases}$$

$$\tau(E_i) = \begin{cases} e_i & (1 \leq i \leq m), \\ -e_i(\sigma_i \sigma_{i+1})^{\ell+m} & (m < i \leq \ell - 1), \end{cases}$$

$$\tau(F_i) = \begin{cases} f_i & (1 \leq i < m), \\ f_m \sigma_{m+1} & (i = m), \\ f_i(\sigma_i \sigma_{i+1})^{i+m+1} & (m < i \leq \ell - 1), \end{cases}$$

(see also [23, Proposition 4.4]).

Let $U(\mathfrak{gl}(\epsilon))^+$ (resp. $U(\mathfrak{gl}(\epsilon))^{-}$) be the subalgebra of $U(\mathfrak{gl}(\epsilon))$ generated by $e_i$ (resp. $f_i$) for $i \in I$, and let $U(\mathfrak{gl}(\epsilon))^0$ be the one generated by $k_\mu$ for $\mu \in P$. We have $U(\mathfrak{gl}(\epsilon)) \cong U(\mathfrak{gl}(\epsilon))^{-} \otimes U(\mathfrak{gl}(\epsilon))^0 \otimes U(\mathfrak{gl}(\epsilon))^+$ as a $k$-space, which follows from [32, Theorem 10.5.1] and (2.1). We call $U(\mathfrak{gl}(\epsilon))^+$ (resp. $U(\mathfrak{gl}(\epsilon))^{-}$) the positive (resp. negative) half of $U(\mathfrak{gl}(\epsilon))$. Note that $U(\mathfrak{gl}(\epsilon))^\pm = \bigoplus_{\alpha \in \mathbb{Q}^+} U(\mathfrak{gl}(\epsilon))^\pm_{\alpha}$, where

$$U(\mathfrak{gl}(\epsilon))^\pm_{\alpha} = \{ u \mid k_\mu u k_{-\mu} = q(\mu, \alpha)u \ (\mu \in P) \}. $$
We put $|x| = \alpha$ for $x \in \mathcal{U}(\mathfrak{gl}(e))_\alpha^\pm$. There is a Hopf algebra structure on $\mathcal{U}(\mathfrak{gl}(e))$, where the comultiplication $\Delta$, the antipode $S$ is given by
\[
\Delta(k_\mu) = k_\mu \otimes k_\mu, \quad \Delta(e_i) = 1 \otimes e_i + e_i \otimes k_i^{-1},
\]
\[
\Delta(f_i) = f_i \otimes 1 + k_i \otimes f_i, \quad S(k_i) = k_i^{-1}, \quad S(e_i) = -e_i k_i, \quad S(f_i) = -k_i^{-1} f_i,
\]
for $\mu \in P$ and $i \in I$. Let $\eta$ be the anti-involution on $\mathcal{U}(\mathfrak{gl}(e))$ defined by
\[
\eta(e_i) = q_i f_i k_i^{-1}, \quad \eta(f_i) = q_i^{-1} k_i e_i, \quad \eta(k_\mu) = k_{\mu^\pm} (i \in I, \mu \in P).
\]
Let $- : \mathcal{U}(\mathfrak{gl}(e)) \rightarrow \mathcal{U}(\mathfrak{gl}(e))$ be the involution of a $\mathbb{Q}$-algebra given by
\[
\bar{q} = q^{-1}, \quad \bar{e}_i = e_i, \quad \bar{f}_i = f_i, \quad \bar{k}_\mu = k_{\mu^\pm} (i \in I, \mu \in P).
\]
For $i \in I$, let $e_i'$ and $e_i''$ denote the $k$-linear maps on $\mathcal{U}(\mathfrak{gl}(e))^-$ defined by $e_i'(f_j) = \delta_{ij}$ and $e_i''(f_j) = \delta_{ij}$ for $j \in I$ and
\[
e_i'(uv) = e_i'(u)v + q(\alpha_i, |u|)ue_i'(v),
\]
\[
e_i''(uv) = e_i''(u)v + q(\alpha_i, |u|)^{-1}ue_i''(v),
\]
for $u, v \in \mathcal{U}(\mathfrak{gl}(e))^-$ with $u$ homogeneous.

2.3. Weight spaces and highest weight $\mathcal{U}(\mathfrak{gl}(e))$-modules. For a $\mathcal{U}(\mathfrak{gl}(e))$-module $V$ and $\mu \in P$, let
\[
V_\mu = \{ u \in V \mid k_\mu u = q(\mu, \nu) u \ (\nu \in P) \}
\]
be the $\mu$-weight space of $V$. For $u \in V_{\mu} \setminus \{0\}$, we call $u$ a weight vector with weight $\mu$ and put $\text{wt}(u) = \mu$. Note that $e_i V_\mu \subset V_{\mu+\alpha_i}$, and $f_i V_\mu \subset V_{\mu-\alpha_i}$ for $i \in I$.

Let $\mathcal{U}(\mathfrak{gl}(e))_{\geq 0}$ be the subalgebra generated by $k_\mu$ and $e_i$ for $\mu \in P$ and $i \in I$. For $\lambda \in P$, let $1_{\lambda}$ be the one-dimensional $\mathcal{U}(\mathfrak{gl}(e))_{\geq 0}$-module such that $e_i v_\lambda = 0$ and $k_\mu v_\lambda = q(\lambda, \mu) v_\lambda$ for $i \in I$ and $\mu \in P$. Let
\[
M_\varepsilon(\lambda) = \mathcal{U}(\mathfrak{gl}(e)) \otimes_{\mathcal{U}(\mathfrak{gl}(e))_{\geq 0}} 1_{\lambda}.
\]
We have $M_\varepsilon(\lambda) = \bigoplus_{\mu \in P} M_\varepsilon(\lambda)_\mu$, where the sum is over $\mu = \lambda - \sum_{i \in I} c_i \alpha_i$ with $c_i \in \mathbb{Z}_+$ and $\dim M_\varepsilon(\lambda)_\mu < \infty$. Let $V_\varepsilon(\lambda)$ be the maximal irreducible quotient of $M_\varepsilon(\lambda)$.

Remark 2.3. Let $V$ be a $\mathcal{U}(\mathfrak{gl}(e))$-module with $V = \bigoplus_{\mu \in P} V_\mu$. Then $V$ can be extended to a $\mathcal{U}(\mathfrak{gl}(e))[\sigma]$-module by defining $\sigma_j u = (-1)^{c_j} u$ for $j \in I$ and $u \in V_\mu$ with $\mu = \sum_i \mu_i \delta_i$. Let $V^\tau = \{ u^\tau \mid u \in V \}$ be a $\mathcal{U}(\mathfrak{gl}(e))$-module, where $xu^\tau = (\tau(x)u)^\tau$ for $x \in \mathcal{U}(\mathfrak{gl}(e))$ and $u \in V$. Then we have $V^\tau = \bigoplus_{\mu \in P \geq 0} V^\tau_\mu$, where
\[
V^\tau_\mu = \{ u \in V^\tau \mid K_\mu u = q^{|\mu|} u \ (\nu \in P) \}.
\]
2.4. The case of $\epsilon_{m|n}$. From now on, we assume that $\epsilon = \epsilon_{m|n}$ so that
\[
\mathbb{I}_0 = \{1, \ldots, m\}, \quad \mathbb{I}_1 = \{m+1, \ldots, m+n = \ell\},
\]
\[
I_{\text{even}} = I \setminus \{m\}, \quad I_{\text{odd}} = \{m\}.
\]
We let
\[
\begin{align*}
U = U(\mathfrak{gl}(\epsilon_{m|n})),
U^+ = U(\mathfrak{gl}(\epsilon_{m|n}))^+, \\
U_{m,n} : \text{the subalgebra generated by } k_\mu, e_k, f_k (\mu \in P, k \in I_{\text{even}}), \\\nU^+_{m,n} = U_{m,n} \cap U^+, \\
\mathcal{U}_{m|0} : \text{the subalgebra generated by } k_\delta, e_k, f_k (i \in \mathbb{I}_1, 1 \leq k < m), \\
\mathcal{U}_{0|n} : \text{the subalgebra generated by } k_\delta, e_k, f_k (j \in \mathbb{I}_1, m < k \leq \ell - 1), \\
M(\lambda) = M_{\epsilon_{m|n}}(\lambda), V(\lambda) = V_{\epsilon_{m|n}}(\lambda) \text{ for } \lambda \in P.
\end{align*}
\]
Let
\[
P^+ = \left\{ \lambda = \sum_{i \in I} \lambda_i \delta_i \in P \mid \lambda_i \geq \ldots \geq \lambda_m, \quad \lambda_{m+1} \geq \ldots \geq \lambda_{m+n} \right\}.
\]
For $\lambda \in P^+$, let
\[
\lambda_+ = \sum_{i \in \mathbb{I}_1} \lambda_i \delta_i, \quad \lambda_- = \sum_{j \in \mathbb{I}_1} \lambda_j \delta_j.
\]

2.5. PBW-type basis of $U^-$. Let
\[
\Phi^+ = \{ \delta_a - \delta_b \mid a < b \},
\]
\[
\Phi^+_{\mathcal{U}} = \{ \delta_a - \delta_b \mid a < b, \quad \epsilon_a = \epsilon_b \} = \{ \alpha \in \Phi^+ \mid (a|a) = \pm 2 \},
\]
\[
\Phi^+_{\mathcal{T}} = \{ \delta_a - \delta_b \mid a < b, \quad \epsilon_a \neq \epsilon_b \} = \{ \alpha \in \Phi^+ \mid (a|a) = 0 \}
\]
be the set of positive, even positive and odd positive roots of the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$, respectively (cf. [5]). We have $\Phi^+ = \Phi^+_{\mathcal{U}} \cup \Phi^+_{\mathcal{T}}$ and $\Phi^+_{\mathcal{U}} = \Phi^+_{m|0} \cup \Phi^+_{0|n}$, where
\[
\Phi^+_{m|0} = \{ \delta_a - \delta_b \mid 1 \leq a < b \leq m \} = \{ \alpha \in \Phi^+ \mid (a|a) = 2 \},
\]
\[
\Phi^+_{0|n} = \{ \delta_a - \delta_b \mid m < a < b \leq \ell \} = \{ \alpha \in \Phi^+ \mid (a|a) = -2 \}.
\]
For homogeneous $x, y \in U^-$, we let
\[
[x, y]_q = xy - q(|x|, |y|)^{-1}yx.
\]
Let $\beta = \alpha_i + \cdots + \alpha_j \in \Phi^+$ be given with $i \leq j$. We put
\[
\begin{align*}
f_\beta = \begin{cases}
ad_q(f_j) \circ \cdots \circ ad_q(f_{m+1}) \circ ad_q(f_i) \circ \cdots \circ ad_q(f_{m-1})(f_m) & \text{if } \beta \in \Phi^+_{\mathcal{U}}, \\
ad_q(f_j) \circ \cdots \circ ad_q(f_{j-1})(f_j) & \text{if } \beta \in \Phi^+_{\mathcal{U}|0}, \\
ad_q(f_j) \circ \cdots \circ ad_q(f_{i+1})(f_i) & \text{if } \beta \in \Phi^+_{0|n},
\end{cases}
\end{align*}
\]
where $ad_q(f_k)(u) = [f_k, u]_q$. 

Let be a linear order on such that for and if and only if the pair satisfies one of the following conditions (see Example 6.9):

1. if and only if the pair satisfies one of the following conditions (see Example 6.9):
   
   (1) if and only if the pair satisfies one of the following conditions (see Example 6.9):
   (2) if and only if the pair satisfies one of the following conditions (see Example 6.9):
   (3) if and only if the pair satisfies one of the following conditions (see Example 6.9):
   (4) if and only if the pair satisfies one of the following conditions (see Example 6.9):
   (5) if and only if the pair satisfies one of the following conditions (see Example 6.9):

Lemma 2.4. Assume such that . Then we have

\begin{align*}
[f_\beta, f_\alpha]_q = \begin{cases} 
(q^{-1} - q)f_\delta & \text{if } \alpha < \gamma < \delta < \beta \text{ and } \alpha + \beta = \gamma + \delta, \\
f_\gamma & \text{if } \gamma = \alpha + \beta \in \Phi^+, \\
0 & \text{otherwise},
\end{cases}
\end{align*}

and \( f_\alpha^2 = 0 \) if \( \alpha \in \Phi^+_T \).

Proof. It can be proved directly or by the relations in \( U_q(\mathfrak{gl}(m|n)) \) (cf. [19, (2.7)]) and (2.1). □

Proposition 2.5. Let

\[ B = \left\{ \prod_{\alpha \in \Phi^+} f_\alpha^{m_\alpha} \middle| m_\alpha \in \mathbb{Z}_{\geq 0} \left( \alpha \in \Phi^+_T \right), \ m_\alpha = 0, 1 \left( \alpha \in \Phi^+_T \right) \right\}, \]

where the product is taken in increasing order with respect to \( \prec \). Then \( B \) is a \( k \)-basis of \( \mathcal{U}^- \).

Proof. Let \( B \) be a PBW-type basis of \( U_q(\mathfrak{gl}(m|n)) \) given in [4, Sections 4 and 5] (cf. [25]) with respect to a total ordering \( \{ m < m - 1 < \cdots < 1 < m + 1 < \cdots < \ell - 1 \} \) on \( I \). We can check that

\[ \tau(B) = \left\{ \sigma \prod_{\alpha \in \Phi^+} f_\alpha^{m_\alpha} \middle| m_\alpha \in \mathbb{Z}_{\geq 0} \left( \alpha \in \Phi^+_T \right), \ m_\alpha = 0, 1 \left( \alpha \in \Phi^+_T \right) \right\}, \]

where \( \sigma \) is a monomial in \( \pm \sigma_i \)’s depending on \( (m_\alpha)_{\alpha \in \Phi^+} \). Using the diamond lemma [3], we can check that \( B \) is linearly independent. Therefore we conclude that \( B \) is a \( k \)-basis of \( \mathcal{U}^- \) by dimension argument. □

3. Crystal bases for homogeneous generalized quantum groups

3.1. Crystal base for \( U_q(\mathfrak{gl}(n)) \). Let

\[ U = U_q(\mathfrak{gl}_n) := U(\mathfrak{gl}(n|0)) = U_q(\mathfrak{gl}(n|0)). \]

Let us recall some necessary results on the crystal bases [12]. We keep the notations in Section 2.1. Let \( V(\lambda) = V_{n_0}^{n_0}(\lambda) \left( \lambda \in P^+ \right) \) for simplicity.
3.1.1. Crystal base of $V(\lambda)$ at $q = 0$. Let $V$ be a $U$-module with weight space decomposition $V = \bigoplus_{\mu \in P} V_\mu$ such that $e_i$ and $f_i$ act locally nilpotently.

Let $i \in I$ be given. For a weight vector $u \in V$, we have $u = \sum_{k \geq 0} f_i^{(k)} u_k$, where $f_i^{(k)} = f_i^k / [k]!$ with $[k]! = [1][2] \cdots [k]$, and $e_i u_k = 0$ for $k \geq 0$. Then we define the lower crystal operators

$$\tilde{e}^\text{low}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}^\text{low}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.$$ 

We also define the upper crystal operators

$$\tilde{e}^\text{up}_i u = \sum_{k \geq 1} q^{-l_k+2k-1} f_i^{(k-1)} u_k, \quad \tilde{f}^\text{up}_i u = \sum_{k \geq 0} q^l f_i^{(k+1)} u_k,$$

where $l_k = (\text{wt}(u_k) \alpha_i)$.

Let $A_0$ be the subring of $\mathbb{k}$ consisting of $f(q)/g(q)$ with $f(q), g(q) \in \mathbb{Q}[q]$ and $g(0) \neq 0$. For $\lambda \in P^+$, a lower crystal base of $V(\lambda)$ at $q = 0$ is a pair $(L^\text{low}(\lambda), B^\text{low}(\lambda))$ given by

$$L^\text{low}(\lambda) = \bigoplus_{r \geq 0, i_1, \ldots, i_r \in I} A_0 \tilde{e}^\text{low}_{i_1} \cdots \tilde{e}^\text{low}_{i_r} \nu_\lambda,$$

$$B^\text{low}(\lambda) = \{ \tilde{f}^\text{low}_{i_1} \cdots \tilde{f}^\text{low}_{i_r} \nu_\lambda \text{ (mod $qL^\text{low}(\lambda)$)} | r \geq 0, i_1, \ldots, i_r \in I \} \setminus \{0\},$$

where $\nu_\lambda$ is a highest weight vector in $V(\lambda)$. Then the pair $(L, B) = (L^\text{low}(\lambda), B^\text{low}(\lambda))$ is a crystal base of $V = V(\lambda)$ with respect to $\tilde{e}_i = \tilde{e}^\text{low}_i$ and $\tilde{f}_i = \tilde{f}^\text{low}_i (i \in I)$ in the following sense:

(C1) $L$ is an $A_0$-lattice of $V$ and $L = \bigoplus_{\mu \in P} L_\mu$, where $L_\mu = L \cap V_\mu$,

(C2) $B$ is a $\mathbb{Q}$-basis of $L/qL$,

(C3) $B = \bigcup_{\mu \in P} B_\mu$ where $B_\mu \subset (L/qL)_\mu$,

(C4) $\tilde{e}_i L \subset L, \tilde{f}_i L \subset L$ and $\tilde{e}_i B \subset B \cup \{0\}, \tilde{f}_i B \subset B \cup \{0\}$ for $i \in I$,

(C5) $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$ for $i \in I$ and $b, b' \in B$.

We define an upper crystal base $(L^\text{up}(\lambda), B^\text{up}(\lambda))$ of $V(\lambda)$ at $q = 0$ in the same way as in

using $\tilde{f}^\text{up}_i$ for $i \in I$, which also satisfies the above properties (C1)–(C5) with respect to $\tilde{e}_i = \tilde{e}^\text{up}_i$ and $\tilde{f}_i = \tilde{f}^\text{up}_i (i \in I)$.

Let $(\ , \ )_\lambda$ be a unique non-degenerate symmetric bilinear form on $V(\lambda)$ satisfying

$$(v_\lambda, v_\lambda)_\lambda = 1, \quad (e_i u, v)_\lambda = (u, f_i v)_\lambda, \quad (k_\mu u, v)_\lambda = (u, k_\mu v)_\lambda,$$

for $u, v \in V(\lambda), i \in I$ and $\mu \in P$. Then we have the following:

(1) $L^\text{up}(\lambda) = \{ u | (u, L^\text{low}(\lambda))_\lambda \subset A_0 \} \subset V(\lambda)$,

(2) $(\tilde{e}^\text{low}_i b, b')_\lambda = (b, \tilde{f}^\text{up}_i b')_\lambda$ for $i \in I, b \in B^\text{low}(\lambda)$ and $b' \in B^\text{up}(\lambda)$,

(3) $(\tilde{f}^\text{low}_i b, b')_\lambda = (b, \tilde{e}^\text{up}_i b')_\lambda$ for $i \in I, b \in B^\text{low}(\lambda)$ and $b' \in B^\text{up}(\lambda)$,

where we still denote the bilinear form $(\ , \ )_{\lambda|q=0}$ by $(\ , \ )_\lambda$. 


3.1.2. Crystal base of \( V(\lambda) \) at \( q = \infty \). Let \( \tau \) also denote a \( \mathbb{Q} \)-linear involution on \( V(\lambda) \) such that \( \tau = \overline{\tau} \) for \( v = xv \) with \( x \in U^- \), where \( \tau \) is given in (2.3). Let \( A_\infty = \overline{A}_0 \). Then \( (L^{\text{low}}(\lambda), B^{\text{low}}(\lambda)) \) can be viewed as a lower crystal base of \( V(\lambda) \) at \( q = \infty \), where the associated crystal operators are given by

\[
\tilde{e}_i^{\text{low}} := - \circ \tilde{e}_i^{\text{low}} \circ - , \quad \tilde{f}_i^{\text{low}} := - \circ \tilde{f}_i^{\text{low}} \circ - \quad (i \in I).
\]

Let \( \sigma_\lambda \) be a \( \mathbb{Q} \)-linear involution of \( V(\lambda) \) given by \( (\sigma_\lambda(u),v)_\lambda = (u,\overline{v})_\lambda \) for \( u, v \in V(\lambda) \). Indeed, we have \( \sigma_\lambda = \overline{\tau} \). Then \( (\sigma_\lambda(L^{\text{up}}(\lambda)), \sigma_\lambda(B^{\text{up}}(\lambda))) \) can be viewed as an upper crystal base of \( V(\lambda) \) at \( q = \infty \), where the associated crystal operators are given by

\[
\tilde{e}_i^{\text{up}} := \sigma_\lambda \circ \tilde{e}_i^{\text{up}} \circ \sigma_\lambda , \quad \tilde{f}_i^{\text{up}} := \sigma_\lambda \circ \tilde{f}_i^{\text{up}} \circ \sigma_\lambda \quad (i \in I).
\]

Indeed, for \( u = \sum_{k \geq 0} f_i^{(k)} u_k \) with \( e_i u_k = 0 \) \((k \geq 0)\), we have

\[
\tilde{e}_i^{\text{up}} u = \sum_{k \geq 1} q^{-k+1} f_i^{(k-1)} u_k, \quad \tilde{f}_i^{\text{up}} u = \sum_{k \geq 0} q^{k+1} f_i^{(k+1)} u_k,
\]

where \( l_k = (\text{wt}(u_k)|_i) \). Then it follows from the facts in case of \( q = 0 \) that

1. \( \sigma_\lambda(L^{\text{up}}(\lambda)) = \{ u \mid (u, L^{\text{low}}(\lambda))_\lambda \subset A_\infty \} \subset V(\lambda) \),
2. \( (\tilde{e}_i^{\text{low}} b, b')_\lambda = (b, \tilde{f}_i^{\text{up}} b')_\lambda \) for \( i \in I, b \in B^{\text{low}}(\lambda) \) and \( b' \in \sigma_\lambda(B^{\text{up}}(\lambda)) \),
3. \( (\tilde{f}_i^{\text{low}} b, b')_\lambda = (b, \tilde{e}_i^{\text{up}} b')_\lambda \) for \( i \in I, b \in B^{\text{low}}(\lambda) \) and \( b' \in \sigma_\lambda(B^{\text{up}}(\lambda)) \).

3.1.3. Crystal base of \( U^- \) at \( q = 0 \) and \( \infty \). Recall that \( U^- \) be the subalgebra of \( U \) generated by \( f_i \) for all \( i \in I \). Let \( i \in I \) be given. For homogeneous \( u \in U^- \), we have \( u = \sum_{k \geq 0} f_i^{(k)} u_k \) with \( e_i'(u_k) = 0 \) \((k \geq 0)\) (cf. (2.3)), and define

\[
\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.
\]

A crystal base of \( U^- \) at \( q = 0 \) is a pair \( (L(\infty), B(\infty)) \) given by

\[
L(\infty) = \sum_{r \geq 0, i_1, \ldots, i_r \in I} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_r},
\]

\[
B(\infty) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} 1 \pmod{qL(\infty)} \mid r \geq 0, i_1, \ldots, i_r \in I \} \setminus \{0\},
\]

which satisfies the conditions (C1)–(C5) where \( (L, B) = (L(\infty), B(\infty)) \) and \( V = U^- \) with weight space\( L(\infty) \) and with respect to\( B(\infty) \).

Let \( (\ , \ ) \) be a unique non-degenerate symmetric bilinear form on \( U^- \) satisfying \( (1, 1) = 1 \) and \( (e_i'u, v) = (u, f_i v) \) for \( u, v \in U^- \) and \( i \in I \). Then we have the following:

1. \( L(\infty) = \{ u \mid (u, L(\infty))_\lambda \subset A_0 \} \subset U^- \),
2. \( (\tilde{e}_i b, b') = (b, \tilde{f}_i b') \) for \( i \in I \) and \( b, b' \in B(\infty) \),

where we still denote by \( (\ , \ ) \) the bilinear form \( (\ , \ )_{q=0} \) on \( B(\infty) \). Then \( (L(\infty), B(\infty)) \) can be viewed as a crystal base of \( U^- \) at \( q = \infty \), where the associated crystal operators are given by

\[
\tilde{e}_i^- := - \circ \tilde{e}_i \circ - , \quad \tilde{f}_i^- := - \circ \tilde{f}_i \circ - \quad (i \in I).
\]
Let $\sigma$ be the $\mathbb{Q}$-linear involution on $U^-$ such that $(\sigma(u), v) = (u, -v)$ for $u, v \in U^-$. Another crystal base of $U^-$ at $q = \infty$ is the pair $(\sigma(L(\infty)), \sigma(B(\infty)))$ [16, 26], where the associated crystal operators are given by

$$\check{e}_i^\sigma := \sigma \circ \check{e}_i \circ \sigma, \quad \check{f}_i^\sigma := \sigma \circ \check{f}_i \circ \sigma \quad (i \in I).$$

**Lemma 3.1.** Let $u \in U^-_\alpha$ and $i \in I$ be given. Then $u$ can be written uniquely as

$$u = \sum_{k \geq 0} u_k f_i^{(k)},$$

where $e_i'(u_k) = 0$ for $k \geq 0$, and

$$\check{e}_i^\sigma u = \sum_{k \geq 1} q^{k-2k+2} u_k f_i^{(k-1)}, \quad \check{f}_i^\sigma u = \sum_{k \geq 0} q^{-l_k+2k} u_k f_i^{(k+1)},$$

where $l_k = (\text{wt}(u_k))(\alpha_i)$.

**Proof.** Let $*$ be the $\mathfrak{g}$-algebra anti-automorphism of $U$ such that $e_i^* = e_i$, $f_i^* = f_i$, and $k^*_\mu = k^-\mu$ for $i \in I$ and $\mu \in \Phi$. Then $\sigma(x) = q^{N(x)(\alpha)} x$ for $x \in U^-_\alpha$, where $N(\alpha) = \frac{1}{2}(\alpha | \alpha) + (\alpha | \rho)$ with $\rho$ the half sum of positive roots in $\Phi^+$. Also we have $\sigma(xy) = q^{(|x||y|)} \sigma(y) \sigma(x)$ for homogeneous $x, y \in U^-_\alpha$ [16, Proposition 3.2].

On the other hand, we have $(* \circ e_i' \circ *) (x) = q^{(\alpha | \alpha_i) + 2} e_i''(x)$ and $(- \circ e_i'' \circ -) (x) = e_i'(x)$ for $i \in I$ and $x \in U^-_\alpha$. Then we have $(e_i')^\sigma := \sigma \circ e_i' \circ \sigma = e_i'$. In particular, we have $\text{Ker}(e_i')^\sigma$.

Let $\sigma(u) = \sum_{k \geq 0} f_i^{(k)} v_k$ with $e_i'(v_k) = 0$ ($k \geq 0$). Then

$$\check{f}_i \sigma(u) = \sum_{k \geq 0} f_i^{(k+1)} v_k.$$

Applying $\sigma$ to $\sigma(u)$ and $\check{f}_i \sigma(u)$, we get

$$u = \sum_{k \geq 0} q^{(k-1)-(\alpha_i | v_k)} \sigma(v_k) f_i^{(k)},$$

$$\check{f}_i^\sigma u = \sum_{k \geq 0} q^{(k+1)-(\alpha_i | v_k)} \sigma(v_k) f_i^{(k+1)}.$$

Putting $u_k = q^{(k-1)-(\alpha_i | v_k)} \sigma(v_k) \in \text{Ker}(e_i')^\sigma = \text{Ker}(e_i')$, we get

$$u = \sum_{k \geq 0} u_k f_i^{(k)} , \quad \check{f}_i^\sigma u = \sum_{k \geq 0} q^{2k-(\alpha_i | v_k)} u_k f_i^{(k+1)}.$$

The proof for $\check{e}_i^\sigma u$ is similar. \hfill $\square$

Finally, let us review the relation between the crystal bases of $V(\lambda)$ and $U^-$ at $q = 0$ and $\infty$. Let $\pi_\lambda : U^- \rightarrow V(\lambda)$ be the canonical projection as a $U^-$-module. Then it induces a surjective $A_0$-linear map

$$\pi_\lambda : L(\infty) \rightarrow L^{\text{low}}(\lambda),$$

such that $\pi_\lambda(B(\infty)) = B(\lambda) \cup \{0\}$ and $\pi_\lambda(\check{f}_i b) = \check{f}_i^{\text{low}} \pi_\lambda(b)$ for $i \in I$ and $b \in B(\infty)$, where $\pi_\lambda$ is the induced map from $L(\infty)/qL(\infty)$ to $L^{\text{low}}(\lambda)/qL^{\text{low}}(\lambda)$. On the other hand, we have
an embedding $\pi^\vee_\lambda : V(\lambda)^\vee \to (U^-)^\vee$ where $V(\lambda)^\vee$ and $(U^-)^\vee$ are the restricted duals of $V(\lambda)$ and $U^-$. If we identify $V(\lambda)$ and $U^-$ with $V(\lambda)^\vee$ and $(U^-)^\vee$ by ( , )$\lambda$ and ( , ), respectively, then we have an $A_0$-linear map

$$
\pi^\vee_\lambda : L^{up}(\lambda) \to L(\infty),
$$

such that $\pi^\vee_\lambda(B^{up}(\lambda)) \subset B(\infty)$ and $\pi^\vee_\lambda(\tilde{e}_i^{up}b) = \tilde{e}_i \pi^\vee_\lambda(b)$ for $i \in I$ and $b \in B^{up}(\lambda)$, where $\pi^\vee_\lambda$ is the induced map from $L^{up}(\lambda)/qL^{up}(\lambda)$ to $L(\infty)/qL(\infty)$.

The map $\pi^\vee_\lambda$ is also compatible with the crystal bases at $q = \infty$ as follows.

**Lemma 3.2.** Under the above hypothesis, we have

$$
\pi^\vee_\lambda : \sigma_\lambda(L^{up}(\lambda)) \to \sigma(L(\infty)),
$$

where $\pi^\vee_\lambda(\sigma_\lambda(B^{up}(\lambda))) \subset \sigma(B(\infty))$ and $\pi^\vee_\lambda(\tilde{e}_i^{up}b) = \tilde{e}_i \pi^\vee_\lambda(b)$ for $i \in I$ and $b \in \sigma_\lambda(B^{up}(\lambda))$.

**Proof.** Recall that we have for $u \in V(\lambda)$ and $v \in U^-$

$$(\pi^\vee_\lambda(u), v) = (u, \pi_\lambda(v))_\lambda.$$

Let $u \in \sigma_\lambda(L^{up}(\lambda))$ be given. Since $\sigma_\lambda(L^{up}(\lambda)) = \{ u \mid (u, \overline{L^{low}(\lambda)})_\lambda \subset A_\infty \} \subset V(\lambda)$, we have

$$(\pi^\vee_\lambda(u), \overline{L(\infty)}) = (u, \pi_\lambda(L(\infty)))_\lambda = (u, \overline{\pi_\lambda(L(\infty))})_\lambda = (u, \overline{L^{low}(\lambda)})_\lambda \subset A_\infty.$$

Hence $\pi^\vee_\lambda(u) \in \sigma(L(\infty))$. Now we have $\pi^\vee_\lambda(\tilde{e}_i^{up}b) = \tilde{e}_i \pi^\vee_\lambda(b)$ for $i \in I$ and $b \in \sigma_\lambda(B^{up}(\lambda))$ since the following diagram is commutative

\[
\begin{array}{ccc}
L^{up}(\lambda) & \overset{\pi^\vee_\lambda}{\longrightarrow} & L(\infty) \\
\sigma_\lambda \downarrow & & \sigma \\
\sigma_\lambda(L^{up}(\lambda)) & \overset{\pi^\vee_\lambda}{\longrightarrow} & \sigma(L(\infty))
\end{array}
\]

The proof completes. \qed

3.2. Crystal bases for $U_{m|0}$ and $U_{0|n}$. In this subsection, we define the crystal operators for $U_{m|0}$ and $U_{0|n}$, which will be used in the later sections. For notational convenience, we assume in this subsection that $U_{0|n} = \mathcal{U}(\mathfrak{gl}(e_{0|n}))$, which is generated by $k_{\delta_j}, e_k, f_k$ ($1 \leq j \leq n$, $1 \leq k \leq n - 1$) by shifting the indices $j$ and $k$ (cf. Section 2.4).
3.2.1. $\mathcal{U}_{m|0}$. First, consider the case of $\mathcal{U}_{m|0} = U_q(\mathfrak{gl}_m)$. We let

\begin{equation}
(\mathcal{L}_{m|0}(\infty), \mathcal{R}_{m|0}(\infty)) = (L(\infty), B(\infty)),
\end{equation}

where $(L(\infty), B(\infty))$ is the crystal base of $U_q(\mathfrak{gl}_m)^-$ at $q = 0$ in Section 3.1.3.

For $\lambda \in P^+$, let $V_{m|0}(\lambda)$ and $V(\lambda)$ denote the irreducible representations of $\mathcal{U}_{m|0}$ and $U_q(\mathfrak{gl}_m)$ with highest weight $\lambda$, respectively, which coincide in this case. Let

\begin{equation}
(\mathcal{L}_{m|0}(\lambda), \mathcal{R}_{m|0}(\lambda)) = (L^{\text{low}}(\lambda), B^{\text{low}}(\lambda)),
\end{equation}

where $(L^{\text{low}}(\lambda), B^{\text{low}}(\lambda))$ is the lower crystal base of $V(\lambda)$ at $q = 0$ in Section 3.1.1. We simply denote by $\hat{e}_i$ and $\hat{f}_i$ for $i = 1, \ldots, m - 1$ the crystal operators in (3.5) and (3.6).

3.2.2. $\mathcal{U}_{0|n}$. Next, consider the case of $\mathcal{U}_{0|n}$. Let $q$ be an indeterminate. There exists an isomorphism of $\mathbb{Q}$-algebras

\begin{equation}
\psi : U_q(\mathfrak{gl}_n) \longrightarrow \mathcal{U}_{0|n},
\end{equation}

given by $\psi(q) = -q^{-1}$, $\psi(e_i) = e_i$, $\psi(f_i) = f_i$, and $\psi(k_\mu) = k_\mu$ for $i = 1, \ldots, n - 1$ and $\mu \in P$. We let

\begin{align*}
\mathcal{L}_{0|n}(\infty) &= \psi \circ \sigma(L(\infty)), \\
\mathcal{R}_{0|n}(\infty) &= \psi \circ \sigma(B(\infty)) \cup (-\psi \circ \sigma(B(\infty))),
\end{align*}

where $(\sigma(L(\infty)), \sigma(B(\infty)))$ is the crystal base of $U_q(\mathfrak{gl}_n)^-$ at $q = \infty$ in Section 3.1.3.

For $\lambda \in P^+$, let $V_{0|n}(\lambda)$ and $V(\lambda)$ denote the irreducible representations of $\mathcal{U}_{0|n}$ and $U_q(\mathfrak{gl}_n)$ with highest weight $\lambda$, respectively. There exists a unique $\mathbb{Q}$-linear isomorphism $\psi_\lambda : V(\lambda) \longrightarrow V_{0|n}(\lambda)$ such that $\psi_\lambda(v_\lambda) = v_\lambda$ and $\psi_\lambda(xv) = \psi(x)\psi_\lambda(v)$ for $x \in U_q(\mathfrak{gl}_n)$ and $v \in V(\lambda)$. Let

\begin{align*}
\mathcal{L}_{0|n}(\lambda) &= \psi_\lambda \circ \sigma_\lambda(L^{\text{up}}(\lambda)), \\
\mathcal{R}_{0|n}(\lambda) &= \psi_\lambda \circ \sigma_\lambda(B^{\text{up}}(\lambda)) \cup (-\psi_\lambda \circ \sigma_\lambda(B^{\text{up}}(\lambda))),
\end{align*}

where $(\sigma_\lambda(L^{\text{up}}(\lambda)), \sigma_\lambda(B^{\text{up}}(\lambda)))$ is the upper crystal base of $V(\lambda)$ at $q = \infty$ in Section 3.1.2.

Let us denote by $\hat{e}_i^\infty$ and $\hat{f}_i^\infty$ for $i = 1, \ldots, n - 1$ the crystal operators on $\mathcal{U}_{0|n}^-$ and $V_{0|n}(\lambda)$ induced from (3.8) and (3.9), respectively.

By Lemma 3.2 we have

\begin{equation}
\pi_\lambda^\vee : \mathcal{L}_{0|n}(\lambda) \longrightarrow \mathcal{L}_{0|n}(\infty),
\end{equation}

where $\pi_\lambda(\mathcal{R}_{0|n}(\lambda)) \subset \mathcal{R}_{0|n}(\infty)$ and $\pi_\lambda(\hat{e}_i^\infty b) = \hat{e}_i^\infty \pi_\lambda(b)$ for $i = 1, \ldots, n - 1$ and $b \in \mathcal{R}_{0|n}(\lambda)$.\]
3.2.3. Crystal operators for $U_{0|n}$. Let $u \in V_{0|n}(\lambda)$ be a weight vector. For $i = 1, \ldots, n - 1$, if $u = \sum_{k \geq 0} f_i^{(k)} u_k$ with $e_i u_k = 0$ ($k \geq 0$), then since $\sigma_\lambda(u) = \overline{u}$ we have

$$
\tilde{e}_i^\infty u = \sum_{k \geq 1} \pm q^{-l_k+2k-1} f_i^{(k-1)} u_k, \quad \tilde{f}_i^\infty u = \sum_{k \geq 0} \pm q^{l_k-2k-1} f_i^{(k+1)} u_k,
$$

where $\pm$ depends on the weight of $u_k$ and $k$, and $l_k = -\langle \text{wt}(u_k)|\alpha_i \rangle$ (recall the difference of symmetric bilinear forms on the weight lattices for $U_q(\mathfrak{gl}_n) = U_{n|0}$ and $U_{0|n}$). We have

$$
\mathcal{L}_{0|n}(\lambda) = \sum_{r \geq 0, i_1, \ldots, i_r \in I} A_{0} f_i^\infty \cdots f_i^\infty v_\lambda,
$$

$$
\mathcal{B}_{0|n}(\lambda) = \{ \tilde{f}_i^\infty \cdots \tilde{f}_i^\infty v_\lambda \ (\text{mod } q \mathcal{L}_{0|n}(\lambda)) \ | \ r \geq 0, i_1, \ldots, i_r \in I \} \setminus \{0\}.
$$

The pair $(L, B) = (\mathcal{L}_{0|n}(\lambda), \mathcal{B}_{0|n}(\lambda))$ satisfies the conditions (C1)–(C5) in Section 3.1.1 with respect to (3.11) except that (C2) and (C5) are replaced by

(C2') $B$ is a signed basis of $L/qL$, that is $B = B \cup -B$ where $B$ is a $\mathbb{Q}$-basis of $L/qL$,

(C5') $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = \pm b$ for $i \in I$ and $b, b' \in B$, where $\tilde{e}_i = \tilde{e}_i^\infty$ and $\tilde{f}_i = \tilde{f}_i^\infty$.

We may replace $\tilde{e}_i$ and $\tilde{f}_i$ ($i = 1, \ldots, n - 1$) for $(\mathcal{L}_{0|n}(\lambda), \mathcal{B}_{0|n}(\lambda))$ with

$$
\tilde{e}_i u = \sum_{k \geq 1} q^{-l_k+2k-1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} q^{l_k-2k-1} f_i^{(k+1)} u_k.
$$

More precisely, if we let $(\mathcal{L}'_{0|n}(\lambda), \mathcal{B}'_{0|n}(\lambda))$ be the pair defined as in (3.12) with respect to (3.13), then we have the following.

Lemma 3.3. Under the above hypothesis, we have

1. $(\mathcal{L}_{0|n}(\lambda), \mathcal{B}_{0|n}(\lambda)) = (\mathcal{L}'_{0|n}(\lambda), \mathcal{B}'_{0|n}(\lambda)),$

2. $\tilde{e}_i^\infty b \equiv \pm \tilde{e}_i b \ (\text{mod } q \mathcal{L}_{0|n}(\lambda))$ for $b \in \mathcal{B}_{0|n}(\lambda)$, $x = e, f$ and $i = 1, \ldots, n - 1$.

Proof. Let $i = 1, \ldots, n - 1$ be given. For $u \in \mathcal{L}_{0|n}(\lambda)$, let $u = \sum_{k \geq 0} f_i^{(k)} u_k$ with $e_i u_k = 0$ ($k \geq 0$). Then

$$
\tilde{f}_i^\infty u = \sum_{k \geq 0} c_k q^{l_k-2k-1} f_i^{(k+1)} u_k,
$$

where $c_k \in \{\pm 1\}$ for each $k$ depending on $k$ and $u_k$. Since $\mathcal{L}_{0|n}(\lambda)$ is invariant under $\tilde{e}_i^\infty$ and $\tilde{f}_i^\infty$, we have

$$
u_k \in \mathcal{L}_{0|n}(\lambda), \quad q^{l_k-2k-1} f_i^{(k+1)} u_k \in \mathcal{L}_{0|n}(\lambda) \quad (k \geq 0).
$$

This implies that $\tilde{f}_i u = \sum_{k \geq 0} q^{l_k-2k-1} f_i^{(k+1)} u_k \in \mathcal{L}_{0|n}(\lambda)$. Similarly, $\tilde{e}_i u \in \mathcal{L}_{0|n}(\lambda)$. Hence $\mathcal{L}_{0|n}(\lambda)$ is invariant under $\tilde{e}_i$ and $\tilde{f}_i$ for $i = 1, \ldots, n - 1$ and $\mathcal{L}'_{0|n}(\lambda) \subset \mathcal{L}_{0|n}(\lambda)$.

Let $b \in \mathcal{B}_{0|n}(\lambda)$ be given. We have $b \equiv (\tilde{f}_i^\infty)^k u \ (\text{mod } q \mathcal{L}_{0|n}(\lambda))$ for some $k \geq 0$ and $u \in \mathcal{B}_{0|n}(\lambda)$ with $\tilde{e}_i^\infty u = 0$. By the previous argument, we have $b \equiv \pm f_i^k u$ (mod $q \mathcal{L}_{0|n}(\lambda)$) and hence

$$
\tilde{f}_i b \equiv \pm (\tilde{f}_i^\infty)^{k+1} u \equiv \pm \tilde{f}_i^\infty b \ (\text{mod } q \mathcal{L}_{0|n}(\lambda)).
$$

This shows that $\mathcal{B}_{0|n}(\lambda) = \mathcal{B}'_{0|n}(\lambda)$. Since $\mathcal{B}_{0|n}(\lambda)$ is a signed $\mathbb{Q}$-basis of $\mathcal{L}_{0|n}(\lambda)/q \mathcal{L}_{0|n}(\lambda)$, we have $\mathcal{L}_{0|n}(\lambda) = \mathcal{L}'_{0|n}(\lambda)$ by Nakayama lemma. \qed
Similarly, we can replace $\tilde{e}_i^{\infty}$ and $\tilde{f}_i^{\infty}$ on $\mathcal{L}_{0|n}(\infty)$ by

$$
\tilde{e}_i u = \sum_{k \geq 1} q^{-i_k + 2k - 2} u_k f_i^{(k-1)}, \quad \tilde{f}_i u = \sum_{k \geq 0} q^{i_k - 2k} u_k f_i^{(k+1)},
$$

for $u = \sum_{k \geq 0} u_k f_i^{(k)} \in \mathcal{U}_n^*$ with $e_i(u_k) = 0$ ($k \geq 0$) and $l_k = -(\text{wt}(u_k)|\alpha_i)$ (cf. Lemma 3.1).

### 3.2.4. PBW-type bases for $\mathcal{U}_{m|0}$ and $\mathcal{U}_{0|n}$

Let $U$ be as in (3.1). Recall the $\mathbb{k}$-algebra automorphism $T_i : U \to U$ for $i = 1, \ldots, n - 1$, which is $T_{i+1}$ in [28 37.1.3], given by

$$
T_i(e_j) = \begin{cases} -f_i k_i & \text{if } i = j, \\ e_j & \text{if } |i - j| > 1, \\ e_i e_j - q^{-1} e_j e_i & \text{if } |i - j| = 1, 
\end{cases} \quad \quad T_i(f_j) = \begin{cases} -k_i^{-1} e_i & \text{if } i = j, \\ f_j & \text{if } |i - j| > 1, \\ f_j f_i - q f_i f_j & \text{if } |i - j| = 1. 
\end{cases}
$$

for $j = 1, \ldots, n - 1$. Let $i = (i_1, \ldots, i_N)$ be a reduced expression of the longest element $w_0$ of Weyl group for $\mathfrak{gl}_n$. For $1 \leq k \leq N$, let $\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_k) \in \Phi^+$ and $f_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(f_k)$, where $s_i$ denotes the simple reflection for $i$.

First consider the case of $\mathcal{U}_{m|0} = U_q(\mathfrak{gl}_m)$, and let $i = (i_1, \ldots, i_M)$ be the reduced expression of $w_0$ for $\mathfrak{gl}_m$ given by $i = i_1 \cdot i_2 \cdots \cdot i_{m-1}$, where $i_k = (m - 1, m - 2, \ldots, k)$ for $1 \leq k \leq m - 1$, and $i_k \cdot i_{k+1}$ denotes the concatenation of $i_k$ and $i_{k+1}$. Then the linear order $\beta_1 < \cdots < \beta_M$ is equal to the one on $\Phi^+_{m|0}$ in Section 2.5 and

$$
f_\beta = f_{\beta}(i) \quad (\beta \in \Phi^+_{m|0}),
$$

where $f_\beta$ is given in (2.6). Hence

$$
B_{m|0} := \left\{ f_{\beta}^{(c_\beta)} \cdots f_{i_M}^{(c_M)} \middle| c = (c_k) \in \mathbb{Z}_+^M \right\}
$$

is an $A_0$-basis of $\mathcal{L}_{m|0}(\infty)$, where $f_{\beta}^{(c_k)} = \frac{1}{\alpha^{(c_k)}} f_{\beta_k}^{c_k}$ for $1 \leq k \leq M$ [27].

Next, we consider the case of $\mathcal{U}_{0|n}$ and let $j = (j_1, \ldots, j_N)$ be the reduced expression of the longest element for $\mathfrak{gl}_n$ given by $j = j_{n-1} \cdot j_{n-2} \cdots \cdot j_1$, where $j_k = (1, 2, \ldots, k)$ for $1 \leq k \leq n - 1$. Then the induced linear order $\gamma_1 < \cdots < \gamma_N$ on $\Phi^+_{0|n}$ is equal to the one in Section 2.5 where $\gamma_k = s_{j_1} \cdots s_{j_{k-1}}(\alpha_k)$ for $1 \leq k \leq N$, and

$$
f_\gamma = \psi(f_j(j)) \quad (\gamma \in \Phi^+_{0|n}),
$$

where $f_\gamma$ is given in (2.6) and $\psi$ is the isomorphism in (6.7). We also denote by $\sigma$ the $Q$-linear involution on $\mathcal{U}_{0|n}$ defined by $\psi \circ \sigma \circ \psi^{-1}$. Then it follows from (3.8) that

$$
\sigma(B_{0|n}) := \left\{ \sigma f_{\gamma_1}^{(c_1)} \cdots f_{\gamma_N}^{(c_N)} \middle| c = (c_k) \in \mathbb{Z}_+^N \right\}
$$

is an $A_0$-basis of $\mathcal{L}_{0|n}(\infty)$. 

4. Crystal bases of polynomial modules and Kac modules

4.1. Polynomial representations $V(\lambda)$. Let $P_{\geq 0} = \sum_{i \geq 1} \mathbb{Z}_+ \delta_i$. Let $O_{\geq 0}$ be the category of $\mathcal{U}$-modules with objects $V$ such that $V = \bigoplus_{\mu \in P_{\geq 0}} V_{\mu}$ with $\dim V_{\mu} < \infty$. It is closed under taking submodules and tensor products.

Remark 4.1. Let $V \in O_{\geq 0}$ be given and consider $V^\tau$ (cf. Remark 2.3). Then we have $V^\tau \in O_{\text{int}}$ by [23 Proposition 4.6], where $O_{\text{int}}$ is the category of $U_q(\mathfrak{gl}(c))$-modules introduced in [1 Definition 2.2].

Let $\mathcal{P}$ be the set of all partitions $\lambda = (\lambda_i)_{i \geq 1}$ with $\lambda_1 \geq \lambda_2 \geq \ldots$. A partition $\lambda = (\lambda_i)_{i \geq 1} \in \mathcal{P}$ is called an $(m|n)$-hook partition if $\lambda_{m+1} \leq n$ (cf. [2]). Let $\mathcal{P}_{m|n}$ be the set of all $(m|n)$-hook partitions. For $\lambda \in \mathcal{P}_{m|n}$, let

$$\Lambda_\lambda = \lambda_1 \delta_1 + \cdots + \lambda_m \delta_m + \mu_1 \delta_{m+1} + \cdots + \mu_n \delta_{m+n},$$

where $(\mu_1, \ldots, \mu_n)$ is the conjugate of the partition $(\lambda_{m+1}, \lambda_{m+2}, \ldots)$. The map $\lambda \mapsto \Lambda_\lambda$ is injective, and hence we may identify $\lambda \in \mathcal{P}_{m|n}$ with $\Lambda_\lambda \in P_{\geq 0}$.

Proposition 4.2 ([23 Proposition 4.8]). Let $V$ be an irreducible $\mathcal{U}$-module in $O_{\geq 0}$. Then $V \cong V(\lambda)$ for some $\lambda \in \mathcal{P}_{m|n}$.

4.2. Crystal base of $V(\lambda)$. Let us recall the notion of crystal base of a $\mathcal{U}$-module $V$, which is introduced in [1] for a $U_q(\mathfrak{gl}(m|n))$-module $V^\tau$. Here we follow the convention in [23] for $\mathcal{U}$-modules.

Let $V$ be a $\mathcal{U}$-module which has a weight space decomposition. The crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ on $V$ for $i \in I$ are given by the ones in Section 5.2 for $i \neq m$ and

$$\tilde{e}_m u = \eta(f_m) u = q^{-1} k_m e_m u, \quad \tilde{f}_m u = f_m u \quad (u \in V).$$

Definition 4.3. A pair $(L, B)$ is a crystal base of $V$ if it satisfies the following conditions:

1. $L$ is an $A_0$-lattice of $V$ and $L = \bigoplus_{\mu \in P} L_{\mu}$, where $L_{\mu} = L \cap V_{\mu}$,
2. $B$ is a signed basis of $L/qL$, that is $B = B \cup -B$ where $B$ is a $Q$-basis of $L/qL$,
3. $B = \bigcup_{\mu \in P} B_{\mu}$ where $B_{\mu} \subset (L/qL)_{\mu}$,
4. $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$ and $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$ for $i \in I$,
5. $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = \pm b$ for $i \in I$ and $b, b' \in B$.

Note that a crystal base of a $U_q(\mathfrak{gl}(m|n))$-module is defined in [1], where $\tilde{E}_i$ and $\tilde{F}_i$ denote the crystal operators. The following lemma explains how it is related to a crystal base of a $\mathcal{U}$-module.

Lemma 4.4. Let $V$ be a $\mathcal{U}$-module in $O_{\geq 0}$. If $(L, B)$ is a crystal base of $V^\tau$ with the crystal operators $\tilde{E}_i$ and $\tilde{F}_i$ ($i \in I$) as a $U_q(\mathfrak{gl}(m|n))$-module, then $(L, B)$ is also a crystal base of $V$. Furthermore, for $b \in B$ and $i \in I$, we have $\tilde{f}_i b \equiv \pm \tilde{F}_i b \pmod{qL}$. 
Proof. The proof is similar to that of Lemma [3]. Let \((L, B)\) be a crystal base of \(V^\tau\) as a \(U_q(\mathfrak{gl}(m|n))\)-module in the sense of [1, Definition 2.4]. It is clear that \((L, B)\) satisfies the conditions (1)–(3) in Definition [3]. It suffices to check the conditions (4) and (5).

Suppose that \(u \in L\) is given. It is also clear that \(L\) satisfies the condition (4) when \(1 \leq i < m\). Suppose that \(i > m\). Let \(u \in L\) be given with \(u = \sum_{k \geq 0} f_i^{(k)} u_k\). By (2.1) and (1.2)], we have
\[
u = \sum_{k \geq 0} \pm f_i^{(k)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} \pm q^{k-2k-1} f_i^{(k+1)} u_k \in L,
\]
where \(\pm\) depends on the weight of \(u_k\) and \(k\) (cf. (3.13)). Since \(L\) is a crystal lattice for \(V^\tau\), we have \(u_k \in L\) and \(q^{k-2k-1} f_i^{(k+1)} u_k \in L\) \((k \geq 0)\). This implies that \(q^{k-2k-1} f_i^{(k+1)} u_k \in L\) for \(k \geq 0\), and
\[\tilde{f}_i u = \sum_{k \geq 0} q^{k-2k-1} f_i^{(k+1)} u_k \in L.
\]
Hence \(L\) is invariant under \(\tilde{f}_i\). The proof for \(i = m\) is the same.

Let us show that \(B\) satisfies the condition (5). It is clear when \(1 \leq i < m\). Suppose that \(m < i \leq n-1\). Let \(b \in B\) be given. By [1, Lemma 2.5] \(b \equiv \tilde{E}_i u \mod qL\) for some \(k \geq 0\) and \(u\) with \(\tilde{E}_i u = 0\). By the previous argument, we have \(b \equiv \pm \tilde{f}_i u \mod qL\) and hence \(\tilde{f}_i b \equiv \pm \tilde{f}_i b \mod qL\). This shows that \(B \cup \{0\}\) is invariant under \(\tilde{f}_i\). The proof for \(i = m\) is the same. \(\Box\)

Theorem 4.5. For \(\lambda \in \mathcal{P}_{m|n}\), let
\[
\mathcal{L}(\lambda) = \sum_{r \geq 0, i_1, \ldots, i_r \in I} A_0 x_{i_1} \cdots x_{i_r} v_\lambda,
\]
\[
\mathcal{B}(\lambda) = \{ \pm x_{i_1} \cdots x_{i_r} v_\lambda \mod q \mathcal{L}(\lambda) \mid r \geq 0, i_1, \ldots, i_r \in I \} \setminus \{0\},
\]
where \(v_\lambda\) is a highest weight vector in \(V(\lambda)\) and \(x = e, f\) for each \(ik\). Then \((\mathcal{L}(\lambda), \mathcal{B}(\lambda))\) is a crystal base of \(V(\lambda)\).

Proof. It follows from Lemma [1.2] and [1, Theorem 5.1]. \(\Box\)

An explicit combinatorial description of \(\mathcal{B}(\lambda)\) can be found in [1]. As in the case of crystal bases of \(U_q(\mathfrak{gl}(m|n))\)-modules [1, Proposition 2.8], we have a tensor product theorem for crystal bases of \(\mathcal{U}\)-modules as follows (see [23, Proposition 3.4] and [24, Section 2.2] for \(\mathcal{U}(\mathfrak{gl}(e))\)-modules). We remark that the map \(\tau\) in [23.1] does not preserve the comultiplications, so we may not obtain this directly from [1, Proposition 2.8].

Proposition 4.6. Let \(V_1, V_2 \in \mathcal{O}_{\geq 0}\) be given. Suppose that \((L_k, B_k)\) is a crystal base of \(V_i\) for \(k = 1, 2\). Then \((L_1 \otimes L_2, B_1 \otimes B_2)\) is a crystal base of \(V_1 \otimes V_2\), where \(B_1 \otimes B_2 \subseteq (L_1/qL_1) \otimes (L_2/qL_2) = (L_1 \otimes L_2)/(qL_1 \otimes L_2)\). Moreover, for \(i \in I, \tilde{e}_i\) and \(\tilde{f}_i\) act on \(B_1 \otimes B_2\) as follows:
(1) if $i = m$, then
\[ e_m(b_1 \otimes b_2) = \begin{cases} \bar{e}_m b_1 \otimes b_2, & \text{if } \langle \text{wt}(b_1), \alpha_m' \rangle > 0, \\ b_1 \otimes \bar{e}_m b_2, & \text{if } \langle \text{wt}(b_1), \alpha_m' \rangle = 0, \end{cases} \]
\[ f_m(b_1 \otimes b_2) = \begin{cases} \bar{f}_m b_1 \otimes b_2, & \text{if } \langle \text{wt}(b_1), \alpha_m' \rangle > 0, \\ b_1 \otimes \bar{f}_m b_2, & \text{if } \langle \text{wt}(b_1), \alpha_m' \rangle = 0, \end{cases} \]

(2) if $i < m$, then
\[ e_i(b_1 \otimes b_2) = \begin{cases} \bar{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \bar{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \]
\[ f_i(b_1 \otimes b_2) = \begin{cases} \bar{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \bar{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \]

(3) if $i > m$, then
\[ e_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \bar{e}_i b_2, & \text{if } \varphi_i(b_2) \geq \varepsilon_i(b_1), \\ s_i \bar{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1), \end{cases} \]
\[ f_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \bar{f}_i b_2, & \text{if } \varphi_i(b_2) > \varepsilon_i(b_1), \\ s_i \bar{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_2) \leq \varepsilon_i(b_1), \end{cases} \]

where $s_i = (-1)^{\langle \text{wt}(b_1), \alpha_i \rangle}$.

Here we put $\varepsilon_i(b) = \max\{k \geq 0 \mid \varepsilon_i b_k \neq 0\}$ and $\varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \neq 0\}$ for $b \in B_1, B_2$.

4.3. $q$-deformed Kac module $K(\lambda)$. Let $\mathcal{K} = \mathcal{K}_{m|n}$ be the subalgebra of $\mathcal{U}^-$ generated by $f_\alpha$ ($\alpha \in \Phi_1^+$). For $S \subset \Phi_1^+$ with $S = \{ \beta_1 < \ldots < \beta_r \}$, we put
\[ f_S = f_{\beta_1} \cdots f_{\beta_r}, \]
where we assume that $f_S = 1$ when $S = \emptyset$. Then $B_K = \{ f_S \mid S \subset \Phi_1^+ \}$ is a $k$-basis of $\mathcal{K}$, and
\[ \mathcal{U}^- \cong \mathcal{K} \otimes \mathcal{U}_{m,n}^- \cong \mathcal{K} \otimes \mathcal{U}_{m|0}^- \otimes \mathcal{U}_{0|n}^-, \]
as a $k$-space by Proposition 2.3 where the isomorphism is given by multiplication.

Let $\lambda \in P^+$ be given. Let $V_{m,n}(\lambda)$ be the irreducible $\mathcal{U}_{m,n}$-module with highest weight $\lambda$, and let $V_{m|0}(\lambda_\pm)$ (resp. $V_{0|n}(\lambda_-)$) the irreducible highest weight module over $\mathcal{U}_{m|0}$ (resp. $\mathcal{U}_{0|n}$) with highest weight $\lambda_\pm$ (resp. $\lambda_-$. Note that $V_{m,n}(\lambda) \cong V_{m|0}(\lambda_\pm) \otimes V_{0|n}(\lambda_-)$ as a $\mathcal{U}_{m|0} \otimes \mathcal{U}_{0|n}$-module since $V_{m,n} \cong U_{m|0} \otimes U_{0|n}$.

Let $\mathcal{P}$ be the subalgebra of $\mathcal{U}$ generated by $\mathcal{U}_{m,n}$ and $e_m$, and extend $V_{m,n}(\lambda)$ to a $\mathcal{P}$-module in an obvious way. Then we define
\[ K(\lambda) = \mathcal{U} \otimes_\mathcal{P} V_{m,n}(\lambda). \]
We call $K(\lambda)$ the $(q$-deformed) Kac module with highest weight $\lambda$. Let $1_\lambda$ denote a highest weight vector of $K(\lambda)$ with weight $\lambda$. Since $\mathcal{P} \cong \mathcal{U}_{m,-} \otimes \mathcal{U}_0 \otimes \mathcal{U}^+$ as a $\mathbb{k}$-space, we have as a $\mathbb{k}$-space
\begin{equation}
K(\lambda) \cong \mathcal{K} \otimes V_{m,n}(\lambda) \cong \mathcal{K} \otimes V_{m}(0) \otimes V_{0,n}(\lambda).\end{equation}

4.4. **Crystal base of $K(\lambda)$**. Let $\lambda \in P^+$ be given. The $\mathbb{k}$-linear map $e'_m$ on $\mathcal{U}^-$ in \([2.5]\) induces a $\mathbb{k}$-linear map on $K(\lambda)$ in \([19]\) Section 4.2.

We define a crystal base of $K(\lambda)$ in the same way as in Definition 4.3 where $\tilde{e}_i$ and $\tilde{f}_i$ on $V$ for $i \in I$ are given by the ones in Section 3.2 for $i \neq m$ and
\begin{equation}
\tilde{e}_m u = e'_m(u), \quad \tilde{f}_m u = f_m u.
\end{equation}

Then we have the following by \([19]\) Theorem 4.7 and Corollary 4.9.

**Theorem 4.7.** Let
\begin{equation}
\mathcal{L}(K(\lambda)) = \bigoplus_{r \geq 0, k_1, \ldots, k_r \in I} A_0 \tilde{x}_{k_1} \cdots \tilde{x}_{k_r} 1_\lambda,
\end{equation}
\begin{equation}
\mathcal{B}(K(\lambda)) = \{ \pm \tilde{x}_{k_1} \cdots \tilde{x}_{k_r} 1_\lambda \pmod{q \mathcal{L}(K(\lambda))} \mid r \geq 0, k_1, \ldots, k_r \in I \} \setminus \{0\},
\end{equation}
where $x = e, f$ for each $k_i$. Then $(\mathcal{L}(K(\lambda)), \mathcal{B}(K(\lambda)))$ is a crystal base of $K(\lambda)$.

**Proof.** We can check that $K(\lambda)^\tau$ is isomorphic to a Kac-module over $U_q(\mathfrak{gl}(m|n))$ \([19]\) Section 2.5]. It is shown \([19]\) Theorem 4.7 and Corollary 4.9] that the pair $(L, B)$ is a crystal base of $K(\lambda)^\tau$ in the sense of \([19]\) Definition 4.6, where
\begin{equation}
L = \bigoplus_{r \geq 0, k_1, \ldots, k_r \in I} A_0 \tilde{X}_{k_1} \cdots \tilde{X}_{k_r} 1_\lambda,
\end{equation}
\begin{equation}
B = \{ \pm \tilde{X}_{k_1} \cdots \tilde{X}_{k_r} 1_\lambda \pmod{q L} \mid r \geq 0, k_1, \ldots, k_r \in I \} \setminus \{0\},
\end{equation}
with $X = E, F$ for each $k_i$. By the same argument as in the proof of Lemma 4.4 \((L, B)$ is a crystal base of $K(\lambda)$, where we can replace $\tilde{E}_i$ and $\tilde{F}_i$ with $\tilde{e}_i$ and $\tilde{f}_i$, respectively. \hfill \Box

**Corollary 4.8.** Under the above hypothesis, we have as an $A_0$-module
\begin{equation}
\mathcal{L}(K(\lambda)) \cong \mathcal{L}(K(0)) \otimes \mathcal{L}_{m,n}(\lambda) \otimes \mathcal{L}_{0,n}(\lambda),
\end{equation}
where $\mathcal{L}(K(0))$ is the $A_0$-span of $f_S 1_0$ for $S \subset \Phi_+^\tau$.

**Proof.** It follows from \([19]\) Theorem 4.7 and the uniqueness of the crystal base of $K(\lambda)$ \([19]\) Theorem 4.10].

By Corollary 4.8 we may identify $\mathcal{L}(K(0))$ and $\mathcal{B}(K(0))$ with
\begin{equation}
\mathcal{L}(K) := \bigoplus_{S \subset \Phi_+^\tau} A_0 f_S, \quad \mathcal{B}(K) := \{ \pm f_S \pmod{q \mathcal{L}(K)} \mid S \subset \Phi_+^\tau \},
\end{equation}
as an $A_0$-module and its signed $\mathbb{Q}$-basis, respectively. We can also deduce the following from \([19]\) Theorem 4.11].
Theorem 4.9. Let $\lambda \in \mathcal{P}_{m,n}$ be given. Let $\pi_\lambda : K(\lambda) \to V(\lambda)$ be the $U$-module homomorphism such that $\pi_\lambda(1_\lambda) = v_\lambda$. Then

1. $\pi_\lambda(\mathcal{L}(K(\lambda))) = \mathcal{L}(\lambda)$,
2. $\mathcal{B}_\lambda(K(\lambda)) = \mathcal{B}(\lambda) \cup \{0\}$, where $\mathcal{B}_\lambda : \mathcal{L}(K(\lambda))/q^i\mathcal{L}(\lambda) \to \mathcal{L}(\lambda)/q^i\mathcal{L}(\lambda)$ is the induced $Q$-linear map,
3. $\mathcal{B}_\lambda$ restricts to a weight preserving bijection

\[ \pi_\lambda : \{ b \in \mathcal{B}(K(\lambda)) | \pi_\lambda(b) \neq 0 \} \to \mathcal{B}(\lambda), \]

which commutes with $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I$.

5. LIMIT OF CRYSTALS OF KAC-MODULES

5.1. $U$-crystals. Let us give some additional terminologies and conventions. A $U$-crystal is a set $B$ together with the maps $wt : B \to P, \varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}$ for $i \in I$ such that for $b \in B$,

1. $\varphi_i(b) = (wt(b), \alpha_i^\vee) + \varepsilon_i(b)$ (i ≠ m) and $\varphi_m(b) + \varepsilon_m(b) \in \{0, 1\}$,
2. $\tilde{e}_i(b) = \varepsilon_i(b) - 1, \varphi_i(b) = \varphi_i(b) + 1, wt(\tilde{e}_i(b)) = wt(b) + \alpha_i$ if $\tilde{e}_i b \in B$,
3. $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1, wt(\tilde{f}_i b) = wt(b) - \alpha_i$ if $\tilde{f}_i b \in B$,
4. $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b' \in B$,
5. $\tilde{e}_i b = \tilde{f}_i b = 0$ when $\varphi_i(b) = -\infty$,

where 0 is a formal symbol and $-\infty$ is the smallest element in $\mathbb{Z} \cup \{-\infty\}$ such that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$. As usual, a $U$-crystal becomes an $I$-colored oriented graph, where $b \overset{i}{\to} b'$ if and only if $b' = \tilde{f}_i b$ for $b, b' \in B$ and $i \in I$.

5.2. Crystal $\mathcal{B}(K(\lambda))$. Let us recall the crystal structure of $\mathcal{B}(K(\lambda))$ for $\lambda \in P^+$ [19]. By Corollary [13] we have a bijection

\[ \mathcal{B}(K(\lambda)) \xrightarrow{(1)} \mathcal{P}(\Phi_T^-) \times \mathcal{P}(\Phi_T^+) \times \mathcal{B}_{0|0}(\lambda_+) \times \mathcal{B}_{0|0}(\lambda_-), \]

\[ \mathfrak{f}_{S} \otimes \mathfrak{b}_+ \otimes \mathfrak{b}_- \longmapsto (-S, \mathfrak{b}_+, \mathfrak{b}_-) \]

where $\mathcal{P}(\Phi_T^-)$ is the power set of $\Phi_T^- = -\Phi_T^+$, and $-S = \{ -\beta | \beta \in S \}$ for $S \subseteq \Phi_T^+$. We identify $\mathcal{P}(\Phi_T^-)$ with $\mathcal{B}(K)$. 

Let \( < \) denote the linear order in Section 2.3 restricted on \( \Phi^+ \), and let \( <' \) be a linear order on \( \Phi^+ \) such that \( \alpha <' \beta \) if and only if \( (a > c) \) or \( (a = c, b > d) \). for \( \alpha, \beta \in \Phi^+ \) with \( \alpha = \delta_a - \delta_d \) and \( \beta = \delta_c - \delta_d \). For \( \alpha, \beta \in \Phi^+ \), we define \( \alpha < \beta \) (resp. \( \alpha <' \beta \)) if and only if \(-\alpha < -\beta \) (resp. \(-\alpha <' -\beta \)).

Let \( S \in \mathcal{P}(\Phi^+) \) be given with \( S = \{ \beta_1 \prec \ldots \prec \beta_r \} = \{ \beta_1' \prec' \ldots \prec' \beta_r' \} \). Then \( e_i S \) and \( f_i S \) for \( i \in I \) is given as follows:

**Case 1.** For \( i = m \), we have
\[
\hat{e}_m S = \begin{cases} S \setminus \{-a_m\} & \text{if } -a_m \in S, \\ 0 & \text{if } -a_m \not\in S, \end{cases} \quad \hat{f}_m S = \begin{cases} S \cup \{-a_m\} & \text{if } -a_m \not\in S, \\ 0 & \text{if } -a_m \in S. \end{cases}
\]

Here we understand 0 as the zero vector in \( \mathcal{L}(K) / q \mathcal{L}(K) \).

**Case 2.** Suppose that \( i \neq m \). First, we have for \( k = 1, \ldots, r \)
\[
\hat{e}_i \beta_k = \begin{cases} \beta_k + \alpha_i & \text{if } \beta_k + \alpha_i \in \Phi^-, \\ 0 & \text{otherwise}, \end{cases} \quad \hat{f}_i \beta_k = \begin{cases} \beta_k - \alpha_i & \text{if } \beta_k - \alpha_i \in \Phi^-, \\ 0 & \text{otherwise}. \end{cases}
\]

Next we identify \( S \) with \( \beta_1 \otimes \ldots \otimes \beta_i \) when \( i < m \) and with \( \beta_1' \otimes \ldots \otimes \beta_i' \) when \( i > m \). Then we define \( \hat{e}_i S \) and \( \hat{f}_i S \) by the tensor product rule given in Proposition 4.6. Indeed, we have
\[
\mathcal{P}(\Phi^-) \cong \bigsqcup_{\ell(\lambda) \leq m, \ell(\lambda') \leq n} \mathcal{B}_{m|0}(\lambda) \times \mathcal{B}_{0|n}(\lambda'),
\]
as a \((\mathcal{U}_{m|0}, \mathcal{U}_{0|n})\)-bicrystal, where \( \ell(\lambda) \) is the length and \( \lambda^t \) is the transpose of the partition \( \lambda \) (see [13, 18]).

Now, the crystal structure on \( \mathcal{B}(K(\lambda)) \) can be described as follows.

**Proposition 5.1** ([19 Proposition 5.1]). Let \( (S, b_+, b_-) \in \mathcal{B}(K(\lambda)) \) be given. For \( i \in I \) and \( x = e, f \), we have
\[
\bar{x}_i(S, b_+, b_-) = \begin{cases} (S', b'_+, b_-) & \text{if } i < m \text{ and } \bar{x}_i(S \otimes b_+) = S' \otimes b'_+, \\ (S'', b'_+, b''_+) & \text{if } i > m \text{ and } \bar{x}_i(S \otimes b_-) = S'' \otimes b''_+, \\ (\bar{x}_m S, b_+ b_-) & \text{if } i = m, \end{cases}
\]
where we assume that \( \bar{x}_i(S, b_+, b_-) = 0 \) if any of its components on the right-hand side is 0.

Recall from [13] that \( \mathcal{B}_{m|0}(\lambda_+) \) and \( \mathcal{B}_{0|n}(\lambda_-) \) can be viewed as subcrystals of \( \mathcal{B}_{m|0}(\infty) \otimes T_{\lambda_+} \) and \( \mathcal{B}_{0|n}(\infty) \otimes T_{\lambda_-} \), respectively as follows:
\[
\mathcal{B}_{m|0}(\lambda_+) \cong \{ b_+ \otimes t_{\lambda_+} \in \mathcal{B}_{m|0}(\infty) \otimes T_{\lambda_+} \mid \varepsilon_i^*(b_+) \leq \langle \lambda_+, \alpha_i^* \rangle \} \quad \text{for } 1 \leq i < m, 
\]
\[
\mathcal{B}_{0|n}(\lambda_-) \cong \{ b_- \otimes t_{\lambda_-} \in \mathcal{B}_{0|n}(\infty) \otimes T_{\lambda_-} \mid \varepsilon_i^*(b_-) \leq \langle \lambda_-, \alpha_i^* \rangle \} \quad \text{for } m < i \leq \ell - 1,
\]
where \(*\) denotes the involution on \( \mathcal{B}_{m|0}(\infty) \) and \( \mathcal{B}_{0|n}(\infty) \) [13 Theorem 2.1.1] and \( \varepsilon_i^*(b) = \varepsilon_i(b^*) \).
5.3. **Crystal** $\mathcal{B}(K(\infty))$. For $\lambda, \mu \in P^+$, we define $\lambda < \mu$ if and only if $\mu - \lambda = \nu \in P^+$.

Let $b = (S, b_+, b_-) \in \mathcal{B}(K(\lambda))$ be given. We observe that

$$ \tag{5.4} b_+ = Xv_{\lambda_+}, \quad S \otimes b_- = Y(S_0 \otimes v_{\lambda_-}), $$

where $X$ is a product of $\tilde{f}_i$’s for $i < m$, $Y$ is a product of $\tilde{f}_i$’s for $i > m$, and $S_0 \subset \Phi^+$ such that $\tilde{e}_i(S_0 \otimes v_{\lambda_-}) = 0$ for all $i > m$. Here we regard $S \in \mathcal{B}_0(n)(\eta) \subset \mathcal{B}(K)$ for some $\eta \in P^+$ as an element of a $U_0[n]$-crystal, and hence $S \otimes b_- \in \mathcal{B}_0(n)(\xi) \subset \mathcal{B}_0(n)(\eta) \otimes \mathcal{B}_0(n)(\lambda_-)$ for some $\xi \in P^+$. By Proposition 5.1, we may write

$$ \tag{5.5} b = (S, b_+, b_-) = Y(S_0, Xv_{\lambda_+}, v_{\lambda_-}). $$

For $\lambda < \mu$, we define a map

$$ \Theta_{\lambda, \mu} : \mathcal{B}(K(\lambda)) \longrightarrow \mathcal{B}(K(\mu)), $$

$$(S, b_+, b_-) \longrightarrow (S', b'_+, b'_-)$$

by $b'_+ = Xv_{\mu_+}$ and $S' \otimes b'_- = Y(S_0 \otimes v_{\mu_-})$ where $X$, $S_0$ and $Y$ are given in (5.4). Note that $\Theta_{\lambda, \mu}$ is a well-defined injective map, and for $\lambda < \mu < \nu$

$$ \Theta_{\mu, \nu} \circ \Theta_{\lambda, \mu} = \Theta_{\lambda, \nu}. $$

**Lemma 5.2.** Let $i, j \in I$ such that $i < m < j$. Then we have

1. $\tilde{f}_i \tilde{f}_j S = \tilde{f}_j \tilde{f}_i S$,
2. if $\tilde{f}_i S \neq 0$, then $\tilde{f}_j S \equiv S$ as elements of $U_0[1]$-crystals,
3. if $\tilde{f}_j S \neq 0$, then $\tilde{f}_i S \equiv S$ as elements of $U_m[0]$-crystals.

**Proof.** We may regard $\mathcal{B}(K) = \mathcal{P}(\Phi^+)$ as the set of $m \times n$ binary matrices, where $S \in \mathcal{P}(\Phi^+_T)$ is identified with the matrix $M = (m_{ab})$ $(1 \leq a \leq m < b \leq \ell)$ with $m_{ab} = 1$ if and only if $-\delta_a + \delta_b \in S$. Then we may apply the $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-bicrystal structure on $\mathcal{B}(K)$ (see [9, 18]).

**Lemma 5.3.** Under the above hypothesis, we have

1. if $\tilde{f}_i b \neq 0$ for some $i \neq m$, then $\Theta_{\lambda, \mu}(\tilde{f}_i b) = \tilde{f}_i \Theta_{\lambda, \mu}(b)$,
2. if $\tilde{f}_m b \neq 0$ and $\tilde{f}_m S_0 \neq 0$, then $\Theta_{\lambda, \mu}(\tilde{f}_m b) = \tilde{f}_m \Theta_{\lambda, \mu}(b) \neq 0$ for all $\lambda < \mu$,
3. if $\tilde{f}_m b \neq 0$ and $\tilde{f}_m S_0 = 0$, then there exists $M_b \in \mathbb{Z}_+$ such that $\tilde{f}_m \Theta_{\lambda, \mu}(b) = 0$ for $\mu$ with $\langle \mu, \alpha_m^\vee \rangle > M_b$.

**Proof.** (1) Let $b = (S, b_+, b_-)$ be given such that $\tilde{f}_i b \neq 0$. We have $b = Y(S_0, Xv_{\lambda_+}, v_{\lambda_-})$ by (5.4). If $i > m$, then it follows immediately from Proposition 5.1 and the definition of $\Theta_{\lambda, \mu}$ that $\Theta_{\lambda, \mu}(\tilde{f}_i b) = \tilde{f}_i \Theta_{\lambda, \mu}(b)$. So we assume that $i < m$.

Suppose that $\tilde{f}_i (S \otimes b_+) = (\tilde{f}_i S) \otimes b_+$, which is equivalent to

$$ \tag{5.7} \tilde{f}_i (S_0 \otimes Xv_{\lambda_+}) = (\tilde{f}_i S_0) \otimes Xv_{\lambda_+} $$

and $\tilde{f}_i (S_0 \otimes v_{\lambda_-}) = (\tilde{f}_i S_0) \otimes v_{\lambda_-}$ for all $S \in \mathcal{B}_0(n)(\eta)$.
since $S \equiv S_0$ by Lemma [5.23]. Since embedding of $\mathcal{B}_{m|0}(\lambda_+) \rightarrow \mathcal{B}_{m|0}(\mu_+)$ is $e$-strict (see [14] Lemma 7.1.2), we have $\varepsilon_i(b_+^i) = \varepsilon_i(b_+^i)$), and hence

$$f_i(S_0 \otimes Xv_{\mu_+}) = (f_iS_0) \otimes Xv_{\mu_+}.$$  

Since $f_i = Y(f_iS_0, Xv_{\lambda_+}, v_\lambda)$ by Lemma [5.2] and (5.1), it follows from Lemma [5.2] and (5.8) that

$$\Theta_{\lambda, \mu}(f_i b) = Y(f_i(S_0, Xv_{\mu_+}, v_{\mu_+})) = f_i(Y(S_0, Xv_{\mu_+}, v_{\mu_+})) = f_i\Theta_{\lambda, \mu}(b).$$

The proof for the case when $f_i(S \otimes b_+) = S \otimes (f_i b_+)$ is similar.

(2) Suppose that $b = Y(S_0, Xv_{\lambda_+}, v_{\lambda_+}) = (Y_1 S_0, Xv_{\lambda_+}, Y_2 v_{\lambda_+})$ for some $Y_1$ and $Y_2$, which are products of $f_i$'s for $i > m$.

Since $f_i b \neq 0$ and $f_i S_0 \neq 0$, that is, $-\alpha_m \notin S$ and $-\alpha_m \notin S_0$, we see from Propositions [4.6], [5.1] and the crystal structure on $\mathcal{B}(\Phi_+)$ that $Y_1 f_i S_0 = f_i Y_1 S_0$. Hence

$$f_i(Y_1 S_0, Xv_{\lambda_+}, Y_2 v_{\lambda_+}) = (f_i Y_1 S_0, Xv_{\lambda_+}, Y_2 v_{\lambda_+})$$

$$= Y(Y_1 f_i S_0, Xv_{\lambda_+}, Y_2 v_{\lambda_+})$$

and the equation (5.9) is also true when $\lambda_+ \text{ is replaced by } \mu_+$. Since $\varepsilon_i((f_i S_0) \otimes v_{\lambda_+}) = 0$ for all $i > m$, we have by (5.9)

$$\Theta_{\lambda, \mu}(f_i b) = \Theta_{\lambda, \mu}(Y(f_i S_0, Xv_{\lambda_+}, v_{\lambda_+})) = Y(f_i S_0, Xv_{\mu_+}, v_{\mu_+}) = f_i \Theta_{\lambda, \mu}(b).$$

(3) We may take $\varphi_{m+1}(v_{\mu_+}) = (\mu_+, \alpha_m^{\vee})$ to be large enough so that $Y(S_0 \otimes v_{\mu_+}) = (Y_1 S_0) \otimes (Y_2 v_{\mu_+})$ for some $Y_1$ and $Y_2$, which are products of $f_i$'s for $i > m$, and $Y_1$ has no factor $f_{m+1}$ (recall Proposition [4.6]). Since $f_i S_0 = 0$ and $Y_1$ has no factor $f_{m+1}$, we have

$$f_i Y_1 S_0 = Y_1 f_i S_0 = 0$$

and hence

$$f_i \Theta_{\lambda, \mu}(b) = f_i(Y_1 S_0, Xv_{\mu_+}, Y_2 v_{\mu_+}) = (f_i Y_1 S_0, Xv_{\mu_+}, Y_2 v_{\mu_+}) = 0.$$

□

**Remark 5.4.** Let $b \in \mathcal{B}(K(\lambda))$ be such that $f_i b \neq 0$. Due to Lemma [5.23], it may happen $f_i \Theta_{\lambda, \mu}(b) = 0$ for some $\mu > \lambda$, and hence we have $\Theta_{\lambda, \mu}(f_i b) \neq f_i \Theta_{\lambda, \mu}(b)$ in general.

Let $b^\lambda \in \mathcal{B}(K(\lambda))$ and $b^\mu \in \mathcal{B}(K(\mu)) \ (\lambda, \mu \in P^+)$ be given. We write $b^\lambda \sim b^\mu$ if there exists $\nu \in P^+$ such that $\lambda, \mu < \nu$ and $\Theta_{\lambda, \nu}(b^\lambda) = \Theta_{\mu, \nu}(b^\mu)$. It is not difficult to see that $\sim$ defines an equivalence relation on $\bigsqcup_{\lambda \in P^+} \mathcal{B}(K(\lambda))$.

**Lemma 5.5.** For $b^\lambda \in \mathcal{B}(K(\lambda))$ and $b^\mu \in \mathcal{B}(K(\mu))$, let $b^\lambda = Y^\lambda(S_0^\lambda, X^\lambda u_+, v_{\lambda_+})$, $b^\mu = Y^\mu(S_0^\mu, X^\mu u_+, v_{\mu_+})$ as in (5.5). Then $b^\lambda \sim b^\mu$ is equivalent to

$$S_0^\lambda = S_0^\mu, \quad X^\lambda u_+ = X^\mu u_+, \quad Y^\lambda u_- = Y^\mu u_-,$$

where $u_\pm$ denotes the highest weight elements 1 in $\mathcal{B}_{m|0}(\infty)$ and $\mathcal{B}_{0|n}(\infty)$, respectively.
\textbf{Proof.} Suppose }b^\lambda \sim b^\mu.\text{ Then there exists }\nu' > \lambda, \mu\text{ such that }\Theta_{\lambda,\nu'}(b^\lambda) = \Theta_{\mu,\nu'}(b^\mu).\text{ Choose }\nu \gg \nu'\text{ (that is, }\nu - \nu'\text{ is sufficiently large) such that }
\Theta_{\gamma,\nu'}(b^\gamma) = Y^\gamma(S_0^\gamma, X^\gamma v_{\nu_+}, v_{\nu_-}) = (S_0^\gamma, X^\gamma v_{\nu_+}, Y^\gamma v_{\nu_-}) \quad (\gamma = \lambda, \mu).

By \cite{[5,6]}, we have \(\Theta_{\lambda,\nu'}(b^\lambda) = \Theta_{\mu,\nu'}(b^\mu)\), that is, \((S_0^\lambda, X^\lambda v_{\nu_+}, Y^\lambda v_{\nu_-}) = (S_0^\mu, X^\mu v_{\nu_+}, Y^\mu v_{\nu_-})\). Hence we have \(X^\lambda u_+ = X^\mu u_+\) and \(Y^\lambda u_- = Y^\mu u_-\) by \cite{[5,3]}, and \(S_0^\lambda = S_0^\mu\). The proof of the converse is similar. \hfill \square

Let }\mathcal{B}(K(\infty))\text{ be the set of equivalence classes with respect to }\sim.\text{ For }b \in \mathcal{B}(K(\infty)),\text{ let us write }b = (b^\lambda)_{\lambda \in P^+},\text{ where }
\begin{enumerate}
\item \(b^\lambda \in \mathcal{B}(K(\lambda)) \cup \{0\}\) for all \(\lambda \in P^+\),
\item \(b^\mu \sim b^\nu\) for non-zero \(b^\mu\) and \(b^\nu\).
\end{enumerate}

Let }b = (b^\lambda)_{\lambda \in P^+} \in \mathcal{B}(K(\infty))\text{ be given. Note that }b^\lambda \neq 0\text{ for some }\lambda \in P^+.\text{ We define }wt(b) = wt(b^\lambda) - \lambda\text{ for some }\lambda \text{ with }b^\lambda \neq 0.\text{ For }i \in I,\text{ we define }\hat{e}_i b \text{ and }\hat{f}_i b\text{ as follows:}
\begin{enumerate}
\item \text{Suppose that }i \neq m.\text{ We define }\tilde{x}_i b = \left(\tilde{x}_i b^\lambda\right)_{\lambda \in P^+} (x = e, f)\text{, where we assume that }\tilde{x}_i b = 0\text{ if }\tilde{x}_i b^\lambda = 0\text{ for all }\lambda \in P^+.
\item \text{Suppose that }i = m.\text{ We define }\tilde{f}_m b\text{ to be the equivalence class of }\tilde{f}_m b^\mu\text{ if there exists }\mu \in P^+\text{ such that }\tilde{f}_m b^\nu \neq 0\text{ for all }\nu \geq \mu,\text{ and define }\tilde{f}_m b = 0\text{ otherwise. We define }\tilde{e}_m b = b'\text{ if }\tilde{f}_m b' = b\text{ for some }b \in \mathcal{B}(K(\infty)),\text{ and }\tilde{e}_m b = 0,\text{ otherwise.}
\end{enumerate}

Put
\begin{equation}
\varepsilon_i(b) = \max\{k | \hat{e}_i^k b \neq 0\}, \quad \varphi_i(b) = \begin{cases}
\langle wt(b), \alpha_i^\vee \rangle + \varepsilon_i(b) & \text{for }i \neq m,
\max\{k | \hat{f}_i^k b \neq 0\} & \text{for }i = m.
\end{cases}
\end{equation}

\textbf{Lemma 5.6.} \textit{Under the above hypothesis, }\mathcal{B}(K(\infty))\text{ is a well-defined }\mathcal{U}\text{-crystal.}

\textbf{Proof.} The well-definedness of }\hat{e}_i, \hat{f}_i\text{ for }i \neq m\text{ follows from Lemma 5.3(1), while }\tilde{e}_m, \tilde{f}_m\text{ are well-defined by Lemma 5.3(2) and (3). The other conditions are easy to check.} \hfill \square

Now, we give another description of \(\mathcal{B}(K(\infty))\). Let }b = (b^\lambda)_{\lambda \in P^+} \in \mathcal{B}(K(\infty))\text{ be given and choose }\lambda \in P^+\text{ such that }b^\lambda = Y(S_0^\lambda, X v_{\lambda_+}, v_{\lambda_-}) \neq 0.\text{ Then we define }
\kappa : \mathcal{B}(K(\infty)) \longrightarrow \mathcal{B}(K) \times \mathcal{B}_{m|0}(\infty) \times \mathcal{B}_{0|n}(\infty),
\begin{align*}
b \quad & \quad \mapsto (S, X u_+, Y u_-)
\end{align*}

where }u_\pm\text{ denotes the highest weight elements }1 \text{ in }\mathcal{B}_{m|0}(\infty)\text{ and }\mathcal{B}_{0|n}(\infty),\text{ respectively. It is well-defined by Lemma 5.3. Indeed }\kappa\text{ is a bijection since it is clearly surjective and injective by Lemma 5.5.}

On the other hand, we define a \(\mathcal{U}\)-crystal structure on \(\mathcal{B}(K) \times \mathcal{B}_{m|0}(\infty) \times \mathcal{B}_{0|n}(\infty)\) as follows: Let }b = (S, b_+, b_-)\text{ be given. We let }wt(b) = wt(S) + wt(b_+) + wt(b_-),\text{ and let }\varepsilon_i(b)
and \( \varphi_i(b) \) be as in (5.10). For \( i \in I \) and \( x = e, f \), we define

\[
\hat{x}_ib = \begin{cases} 
(S', b_{+}'), & \text{if } i < m \text{ and } \hat{x}_i(S \otimes b_{+}) = S' \otimes b_{+}', \\
(S, b_+), & \text{if } i > m, \\
(\hat{x}_mS, b_+), & \text{if } i = m,
\end{cases}
\]

where \( \hat{x}_i(S \otimes b_{+}) \) is defined by (A.2). Here we assume that \( \hat{x}_i b = 0 \) if any of its component on the right-hand side is 0.

**Theorem 5.7.** The map \( \kappa \) is an isomorphism of \( \mathcal{U} \)-crystals. Hence the crystal \( \mathcal{B}(K(\infty)) \)
is isomorphic to \( \mathcal{B}(K) \times \mathcal{B}_{m|0}(\infty) \times \mathcal{B}_{0|n}(\infty) \).

**Proof.** It is enough to show that \( \kappa \) is a homomorphism of \( \mathcal{U} \)-crystals. Let \( b \in \mathcal{B}(K(\infty)) \)
be given with \( \kappa(b) = (S, Xu_+, Yv_–) \) for some \( X \) and \( Y \). Choose a component \( b^\lambda = (S, Xv_\lambda, Yv_\lambda) \) in \( b \) such that \( \lambda \gg 0 \).

If \( i < m \), then we have \( \kappa(\hat{x}_ib) = \hat{x}_i\kappa(b) \) \((x = e, f)\) by (5.3). If \( i > m \), then by our choice of \( \lambda \), we may assume that \( \hat{x}_ib^\lambda = (S, Xu_+, Yv_–) = (S, Xv_\lambda, \hat{x}_iYv_\lambda) \), which implies \( \kappa(\hat{x}_ib) = \hat{x}_i\kappa(b) \) \((x = e, f)\) for \( b \in \mathcal{B}(K(\infty)) \), where we assume that \( \kappa(0) = 0 \). Therefore, \( \kappa \) is an isomorphism of \( \mathcal{U} \)-crystals. \( \square \)

6. **Crystal base of \( \mathcal{U}^- \)**

6.1. **Crystal base of \( \mathcal{U}^- \).** For \( i \in I \), we define \( \hat{e}_i \) and \( \hat{f}_i \) on \( \mathcal{U}^- \) as follows: Let \( u \in \mathcal{U}^- \) be given which is homogeneous.

1. Suppose that \( i < m \). We have \( u = \sum_{k \geq 0} f_i^{(k)}u_k \) with \( e_i'(u_k) = 0 \) \((k \geq 0)\), and then define

\[
\hat{e}_iu = \sum_{k \geq 1} f_i^{(k-1)}u_k, \quad \hat{f}_iu = \sum_{k \geq 0} f_i^{(k+1)}u_k,
\]

in the same way as in (3.3).

2. If \( i = m \), then we define

\[
\hat{e}_mu = e_m'(u), \quad \hat{f}_mu = f_mu,
\]

in the same way as in (4.5).

3. Suppose that \( i > m \). We have \( u = u_1u_2u_3 \) for unique \( u_1 \in K, u_2 \in \mathcal{U}_{m|0}^- \) and \( u_3 \in \mathcal{U}_{0|n}^- \) (up to scalar multiplication) by (4.3), and then define

\[
\hat{e}_iu = u_1u_2(\hat{e}_iu_3), \quad \hat{f}_iu = u_1u_2(\hat{f}_iu_3),
\]

where \( \hat{e}_iu_3 \) and \( \hat{f}_iu_3 \) are given in (3.14).

We define a crystal base of \( \mathcal{U}^- \) as in Definition 4.2 with respect to (5.1), (6.2), (6.3), and with weight space (2.2). Then we have the following, which is the main result in this paper.
Theorem 6.1. Suppose that \( m, n > 0 \). Let
\[
\mathcal{L}(\infty) = \mathcal{L}(K) \cdot \mathcal{L}_{m|0}(\infty) \cdot \mathcal{L}_{0|n}(\infty),
\]
\[
\mathcal{B}(\infty) = \mathcal{B}(K) \cdot \mathcal{B}_{m|0}(\infty) \cdot \mathcal{B}_{0|n}(\infty),
\]
where \( \cdot \) denotes the multiplication in \( \mathcal{U}^- \) and the induced one in \( \mathcal{L}(\infty)/q\mathcal{L}(\infty) \) respectively. Then \( (\mathcal{L}(\infty), \mathcal{B}(\infty)) \) is a crystal base of \( \mathcal{U}^- \), and as \( \mathcal{U} \)-crystals
\[
\mathcal{B}(\infty) \cong \mathcal{B}(K(\infty)).
\]

Proof. The pair \( (\mathcal{L}(\infty), \mathcal{B}(\infty)) \) clearly satisfies the conditions (1)–(3) in Definition \[4.3\] Let us check the conditions (4) and (5) in Definition \[4.3\] It is easy to check when \( i = m \) by Corollary \[4.8 \] and \[5.1 \].

Suppose that \( i < m \). It is enough to consider \( (\mathcal{L}(K) \cdot \mathcal{L}_{m|0}(\infty), \mathcal{B}(K) \cdot \mathcal{B}_{m|0}(\infty)) \) since \( e'_i(u) = 0 \) and \( f_i \) commutes with \( u \) for any \( u \in \mathcal{U}_{0|n}^- \). Let \( B_q = B_q(U_{m|0}) \) (see Appendix \[\mathcal{A}.1\]). Recall the \( k \)-linear isomorphism \[4.3\] given by multiplication. We can check that there is a well-defined action of \( B_q \) on \( \mathcal{U}^- \), where the actions of \( e_i', f_i, \) and \( k_\mu \) for \( i < m \) and \( \mu \in \mathcal{P} \) are given by \[2.3\], left multiplication, and conjugation, respectively. Moreover, \( K \cdot \mathcal{U}_{m|0}^- \) is a \( B_q \)-submodule of \( \mathcal{U}^- \) by Lemma \[2.4\] and
\[
e'_i(f_\beta) = \begin{cases} f_{\beta - \alpha_i} & \text{if } \beta - \alpha_i \in \Phi_1^+, \\ 0 & \text{otherwise,} \end{cases} \quad (i < m, \beta \in \Phi_1^+).
\]

On the other hand, we may regard \( K(0) \otimes \mathcal{U}_{m|0}^- \) as a \( B_q \)-module via \[\mathcal{A}.1\] since \( K(0) \) is a finite-dimensional \( \mathcal{U}_{m|0}^- \)-module and \( \mathcal{U}_{m|0}^- \) is a \( B_q \)-module. Note that \( K(0) = \mathcal{U} \oplus \mathcal{P} V_{m,n}(0) = \mathcal{K} \otimes V_{m,n}(0) \) by \[\mathcal{A}.1\], where \( V_{m,n}(0) \) is a trivial \( \mathcal{U}_{m,n} \)-module spanned by \( v_0 \).

We first claim that the \( k \)-linear isomorphism
\[
\phi : K(0) \otimes \mathcal{U}_{m|0}^- \longrightarrow K \cdot \mathcal{U}_{m|0}^-
\]
\[
(u_1 \otimes v_0) \otimes u_2 \\
\overline{\longrightarrow} \\
u_1 u_2
\]
is an isomorphism of \( B_q \)-modules. It suffices to show that the map \[6.4\] preserves the action of \( B_q \). Let \( \text{ad}_q : \mathcal{U} \longrightarrow \text{End}_k(\mathcal{U}) \) be the adjoint representation on \( \mathcal{U} \) with respect to \[2.3\], where
\[
\text{ad}_q(k_\mu)(u) = k_\mu u k_\mu^{-1},
\]
\[
\text{ad}_q(e_i)(u) = (e_i u - u e_i) k_i,
\]
\[
\text{ad}_q(f_i)(u) = f_i u - k_i u k_i^{-1} f_i,
\]
for \( i \in I, \mu \in \mathcal{P} \) and \( u \in \mathcal{U} \). We write \( \text{ad}_q(u)(x) = u \cdot x \) for simplicity.

Let \( u_1 \in \mathcal{K} \) and \( u_2 \in \mathcal{U}_{m|0}^- \) be homogeneous elements. First we have
\[
e'_i(u_1 u_2) = e'_i(u_1) u_2 + q(\alpha_i, u_1) u_1 e'_i(u_2) = e'_i(u_1) u_2 + (k_i \cdot u_1) e'_i(u_2),
\]
\[
e'_i((u_1 \otimes v_0) \otimes u_2) = (q^{-1} - q)(k_i e_i(u_1 \otimes v_0)) \otimes u_2 + (k_i (u_1 \otimes v_0)) \otimes e'_i(u_2),
\]
by \([2.5]\) and \([A.1]\), respectively. On the other hand, we have
\[
\begin{align*}
k_i(u_1 \otimes v_0) &= k_i u_1 k_i^{-1} \otimes v_0 \\
&= (k_i \cdot u_1) \otimes v_0,
\end{align*}
\]
\[
(q^{-1} - q) k_i e_i(u_1 \otimes v_0) = (q^{-1} - q)(k_i e_i u_1) \otimes v_0
= (q^{-1} - q)(k_i(e_i u_1 - u_1 e_i) k_i^{-1}) \otimes v_0
= (q^{-1} - q)k_i \left( k_i e_i^\prime(u_1) - k_i^{-1} e_i^\prime(u_1) \right) \otimes v_0
= (e_i^\prime(u_1) - k_i^2 e_i^\prime(u_1)) \otimes v_0
= e_i^\prime(u_1) \otimes v_0,
\]
where the last equation follows from the fact that \(e_i^\prime(f_{\beta}) = 0\) for \(i < m\) and \(\beta \in \Phi^+_\lambda\), which can be checked directly. Therefore \(\phi(e_i^\prime((u_1 \otimes v_0) \otimes u_2)) = e_i^\prime((u_1 u_2) = e_i^\prime\phi((u_1 \otimes v_0) \otimes u_2)).\)

Also we have
\[
\begin{align*}
f_i u_1 u_2 &= (f_i \cdot u_1) u_2 + k_i u_1 k_i^{-1} f_i u_2 = (f_i \cdot u_1) u_2 + (k_i \cdot u_1) f_i u_2, \\
f_i((u_1 \otimes v_0) \otimes u_2) &= (f_i(u_1 \otimes v_0)) \otimes u_2 + (k_i(u_1 \otimes v_0)) \otimes f_i u_2,
\end{align*}
\]
by \([A.1]\), where
\[
f_i(u_1 \otimes v_0) = (f_i u_1) \otimes v_0 = (f_i u_1 - k_i u_1 k_i^{-1} f_i) \otimes v_0 = (f_i \cdot u_1) \otimes v_0.
\]
This implies that \(\phi(f_i((u_1 \otimes v_0) \otimes u_2)) = f_i \phi((u_1 \otimes v_0) \otimes u_2).\) It is clear that \(\phi(k_i((u_1 \otimes v_0) \otimes u_2)) = k_i \phi((u_1 \otimes v_0) \otimes u_2).\) Therefore \(\phi\) is \(B_q\)-linear, which proves the claim.

Since \((\mathcal{L}(K(0)) \otimes \mathcal{L}_{m|0}(\infty), \mathcal{B}(K(0)) \otimes \mathcal{B}_{m|0}(\infty))\) is a crystal base of \(K(0) \otimes \mathcal{U}_{m|0}\) as a \(B_q\)-module by Theorem \([A.3]\) \((\mathcal{L}(K) \cdot \mathcal{L}_{m|0}(\infty), \mathcal{B}(K) \cdot \mathcal{B}_{m|0}(\infty))\) is a crystal base of \(\mathcal{K} \cdot \mathcal{U}_{m|0}\) as a \(B_q\)-module by \([6.4]\), and therefore satisfies the conditions Definition \([4.3]\) and \((4)\) and \(\mathcal{L}_{0|n}(\infty), \mathcal{B}_{0|n}(\infty)\) satisfies the conditions Definition \([4.3]\) and \((4)\) and by Section \([3.2.2]\) This completes the proof. \(\square\)

6.2. Parabolic Verma module. Let \(\mathcal{Q}\) be the subalgebra of \(\mathcal{U}\) generated by \(\mathcal{U}_{0|n}, \mathcal{U}^0\) and \(\mathcal{U}^+\). Let \(\lambda \in P^+\) be given. We extend \(V_{0|n}(\lambda_-)\) to a \(\mathcal{Q}\)-module by \(e_i v_{\lambda_-} = 0\) and \(k_{\mu} v_{\lambda_-} = q(\lambda, \mu) v_{\lambda_-}\) for \(i \leq m\) and \(\mu \in P\). Define a \(\mathcal{U}\)-module
\[
X(\lambda) = \mathcal{U} \otimes \mathcal{Q} V_{0|n}(\lambda_-).
\]
Let \(1_\lambda\) denote a highest weight vector of \(X(\lambda)\). Since \(\mathcal{Q} \cong \mathcal{U}_{0|n}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+\) as a \(\mathbb{k}\)-space, we have as a \(\mathbb{k}\)-space
\[
X(\lambda) \cong \mathcal{K} \otimes \mathcal{U}_{m|0}^- \otimes V_{0|n}(\lambda_-).
\]
We define $\tilde{e}_i, \tilde{f}_i$ ($i \in I$) on $X(\lambda)$ as in (6.1), (6.2) for $i \leq m$, and as in (6.13) for $i > m$. We may also define a crystal base of $X(\lambda)$ as in Definition 4.3.

Let us assume that $\mathcal{B}(K) \times \mathcal{B}_{m[0]}(\infty) \times \mathcal{B}_{0[n]}(\lambda_-)$ is a $U$-crystal, where for $b = (S, b_+, b_-)$, $i \in I$ and $x = e, f$,

$$\begin{align*}
wt(b) &= wt(S) + wt(b_+) + wt(b_-), \\
\tilde{x}_i b &= \begin{cases} (S', b'_+, b_-) & \text{if } i < m \text{ and } \tilde{x}_i (S \otimes b_+) = S' \otimes b'_+, \\
(S'', b_+, b'_-) & \text{if } i > m \text{ and } \tilde{x}_i (S \otimes b_-) = S'' \otimes b'_-, \\
(\tilde{x}_m S, b_+, b_-) & \text{if } i = m,
\end{cases}
\end{align*}$$

(6.6)

and $\varepsilon_i(b), \varphi_i(b)$ are defined as in (5.10). Here we assume that $\tilde{x}_i b = 0$ if any of its component on the right-hand side is 0.

**Theorem 6.2.** Let

$$\begin{align*}
\mathcal{L}(X(\lambda)) &= \sum_{r \geq 0, k_1, \ldots, k_r \in I} A_{0, k_1, \ldots, k_r} 1, \\
\mathcal{B}(X(\lambda)) &= \{ \pm \tilde{x}_{k_1} \cdots \tilde{x}_{k_r} 1 \lambda \pmod {q \mathcal{L}(X(\lambda))} \mid r \geq 0, k_1, \ldots, k_r \in I \} \setminus \{0\},
\end{align*}$$

where $x = e, f$ for each $k_i$. Then we have the following:

1. $(\mathcal{L}(X(\lambda)), \mathcal{B}(X(\lambda)))$ is a crystal base of $X(\lambda)$,
2. $\mathcal{L}(X(\lambda)) = (\mathcal{L}(K) \cdot \mathcal{L}_{m[0]}(\infty)) \otimes \mathcal{B}_{0[n]}(\lambda_-)$,
3. $\mathcal{B}(X(\lambda)) \cong (\mathcal{B}(K) \times \mathcal{B}_{m[0]}(\infty) \times \mathcal{B}_{0[n]}(\lambda_-)) \otimes T_{\lambda_+}$ as a $U$-crystal, and it is connected.

**Proof.** Let

$$\begin{align*}
\mathcal{L}' &= (\mathcal{L}(K) \cdot \mathcal{L}_{m[0]}(\infty)) \otimes \mathcal{B}_{0[n]}(\lambda_-), \\
\mathcal{B}' &= (\mathcal{B}(K) \cdot \mathcal{B}_{m[0]}(\infty)) \otimes \mathcal{B}_{0[n]}(\lambda_-) \otimes T_{\lambda_+}.
\end{align*}$$

We first claim that $(\mathcal{L}', \mathcal{B}')$ is a crystal base of $X(\lambda)$. It clearly satisfies the conditions Definition 4.3 (1)–(3). So it is enough to check the conditions Definition 4.3 (4)–(5), which is clear when $i = m$ by Corollary 4.8 and (5.1).

Suppose that $i < m$. We have shown in the proof of Theorem 6.1 that

$$(\mathcal{L}(K) \cdot \mathcal{L}_{m[0]}(\infty), \mathcal{B}(K) \cdot \mathcal{B}_{m[0]}(\infty))$$

is a crystal base of $K \cdot U_{m[0]}$ as a $B_q$-module. This implies the conditions Definition 4.3 (4) and (5). Suppose that $i > m$. It is enough to consider

$$\begin{align*}
((\mathcal{L}(K) \cdot 1) \otimes \mathcal{B}_{0[n]}(\lambda_-), (\mathcal{B}(K) \cdot 1) \otimes \mathcal{B}_{0[n]}(\lambda_-)),
\end{align*}$$

(6.7)

since $e_i$ and $f_i$ commute with $U_{m[0]}$. By Theorem 4.7 and Corollary 4.8, the pair (6.7) satisfies the conditions Definition 4.3 (4)–(5). Therefore $(\mathcal{L}', \mathcal{B}')$ is a crystal base of $X(\lambda)$, and as a $U$-crystal

$$\mathcal{B}' \cong (\mathcal{B}(K) \times \mathcal{B}_{m[0]}(\infty) \times \mathcal{B}_{0[n]}(\lambda_-)) \otimes T_{\lambda_+}.$$
Next, we claim that the $\mathcal{U}$-crystal $\mathcal{B}'$ is connected. Let

$$
i_{\lambda} : \mathcal{B}(K(\lambda)) \xrightarrow{\pi} \mathcal{B}', \quad (S, b_+, b_-) \rightarrow (S, b_+, b_-) \otimes \iota_{\lambda},$$

where we regard $\mathcal{B}_{m|0}(\lambda_+) \subset \mathcal{B}_{m|0}(\infty) \otimes T_{\lambda_+}$ by (5.3). Let $b_1, b_2 \in \mathcal{B}(K(\lambda))$ such that $\tilde{f}_i b_1 = b_2$ for some $i \in I$. Then it is straightforward to check that $\tilde{f}_i \iota_{\lambda}(b_1) = \iota_{\lambda}(b_2)$ by comparing Proposition 5.1 and (6.6).

Now let $b \in \mathcal{B}'$ be given. Note that $\mathcal{B}'$ depends not on $\lambda_+$, but only on $\lambda_-$ as an $I$-colored oriented graph. So we may assume that $\lambda_+ \gg 0$ so that $b = \iota_{\lambda}(b_0)$ for a unique $b_0 \in \mathcal{B}(K(\lambda))$. Since $\mathcal{B}(K(\lambda))$ is connected by Theorem 4.7, we have $b_0 = \tilde{x}_{k_1} \cdots \tilde{x}_{k_r} 1_\lambda$ for some $r \geq 0, k_1, \ldots, k_r \in I$, where $x = e, f$ for each $k_i$. By the previous arguments, we have

$$b = \iota_{\lambda}(b_0) = \iota_{\lambda}(\tilde{x}_{k_1} \cdots \tilde{x}_{k_r} 1_\lambda) = \tilde{x}_{k_1} \cdots \tilde{x}_{k_r} 1_\lambda.$$  

Therefore $\mathcal{B}'$ is connected.

Finally, we conclude by standard arguments that $(\mathcal{L}(X(\lambda)), \mathcal{B}(X(\lambda))) = (\mathcal{L}', \mathcal{B}')$, since $\mathcal{L}_\lambda = \mathcal{L}(X(\lambda)) = A_0 1_\lambda$ and $\mathcal{B}'$ is connected.

**Corollary 6.3.** A crystal base of $X(\lambda)$ is unique up to scalar multiplication.

Let

$$\mathcal{U}^{-} \xrightarrow{\pi_-} X(\lambda) \xrightarrow{\pi_+} K(\lambda).$$

be the canonical projections of $\mathcal{U}^{-}$-modules. By (1.3), (6.5) and (3.10), we have an embedding $\pi_- : X(\lambda) \rightarrow \mathcal{U}^-$ of a $k$-space so that we have

$$\mathcal{U}^- \xrightarrow{\pi_-} X(\lambda) \xrightarrow{\pi_+} K(\lambda).$$

The crystal bases for $\mathcal{U}^-$, $X(\lambda)$, and $K(\lambda)$ are compatible with (6.8) in the following sense.

**Corollary 6.4.** Under the above hypothesis, we have

1. $\pi_+(\mathcal{L}(X(\lambda))) = \mathcal{L}(K(\lambda))$,
2. $\mathcal{B}(X(\lambda)) = \mathcal{B}(K(\lambda)) \cup \{0\}$, where $\pi_+$ is the $\mathbb{Q}$-linear map induced from $\pi_+$,
3. $\pi_+$ restricts to a weight preserving bijection

$$\pi_+ : \{b \in \mathcal{B}(X(\lambda)) | \pi_+(b) \neq 0\} \rightarrow \mathcal{B}(K(\lambda)),$$

which commutes with $\tilde{e}_i, \tilde{f}_i$ for $i \in I$.

**Corollary 6.5.** Under the above hypothesis, we have

1. $\pi_-^\vee(\mathcal{L}(X(\lambda))) \subset \mathcal{L}(\infty)$,
2. $\mathcal{B}(X(\lambda)) \subset \mathcal{B}(\infty)$, where $\pi_-^\vee$ is the $\mathbb{Q}$-linear map induced from $\pi_-^\vee$. 

(3) \( \pi'_\nu \) restricts to a bijection

\[
\pi'_\nu : \mathcal{B}(X(\lambda)) \rightarrow \{ b = b_1b_2b_3 \in \mathcal{B}(\infty) \mid b_3 \otimes t_{\lambda-} \in \mathcal{B}_{0|n}(\lambda-) \},
\]

which commutes with \( \tilde{e}_i, \tilde{f}_i \) for \( i < m \), where each \( b \in \mathcal{B}(\infty) \) is uniquely written as \( b = b_1b_2b_3 \in \mathcal{B}(\infty) \) with \( b_1 \in \mathcal{B}(K), b_2 \in \mathcal{B}_{m|0}(\infty), b_3 \in \mathcal{B}_{0|n}(\infty) \).

**Remark 6.6.** Let \( b_0 \in \mathcal{B}(\infty) \) be given. Suppose \( b_0 = \pi'_\nu(b) \) for some \( b \in \mathcal{B}(X(\lambda)) \). Then we have \( \pi_+(\tilde{x}_ib) = \tilde{x}_i\pi_+(b) \) for \( i \in I \), where \( x = f \) for \( i < m \) and \( x = e, f \) for \( i \geq m \). Also, we have \( \pi'_\nu(\tilde{x}_ib) = \tilde{x}_i\pi'_\nu(b) \) for \( x = e, f \) and \( i < m \). On the other hand, we have \( \pi'_\nu(\tilde{x}_ib) = \tilde{x}_i\pi'_\nu(b) \), for \( x = e, f \) if \( i > m \) and \( \lambda_- \gg 0 \), or if \( i = m \) and \( \langle \phi(b), \alpha_{m+1}^\vee \rangle \gg 0 \) (see Examples 6.9 and 6.10).

### 6.3. Crystal \( \mathcal{B}(\infty) \)

Let us write \( \mathcal{B}_{m|1}(\infty) = \mathcal{B}(\infty) \).

**Proposition 6.7.** The crystal \( \mathcal{B}_{m|1}(\infty) \cong \mathcal{B}(K_{m|1}) \times \mathcal{B}_{m|0}(\infty) \) is connected. Moreover, \( \mathcal{B}_{m|1}(\infty) = \{ \tilde{f}_{k_1} \cdots \tilde{f}_{k_r}(\emptyset,1) \mid r \geq 0, k_1, \ldots, k_r \in I \} \setminus \{ \emptyset \} \).

**Proof.** We recall that \( \mathcal{B}(K_{m|1}) = \mathcal{B}(\Phi_{m|1}^-) \) where \( \Phi_{m|1}^- = \{ -\delta_a + \delta_{m+1} \mid 1 \leq a \leq m \} \). Let \( b = (S,b_+) \in \mathcal{B}(K_{m|1}) \times \mathcal{B}_{m|0}(\infty) \) be given. We claim that \( b = \tilde{f}_{k_1} \cdots \tilde{f}_{k_r}(\emptyset,1) \) for some \( r \geq 0 \) and \( k_1, \ldots, k_r \in I \). Let \( |b| = -\sum_{i \in I} c_i \alpha_i \) for some \( c_i \in \mathbb{Z}_+ \). We use induction on \( \sum_{i \in I} c_i \).

Suppose that \( S = \emptyset \). If \( b_+ \neq 1 \), then \( \tilde{e}_ib_+ \neq 0 \) for some \( i \in I \setminus \{ m \} \), and hence \( \tilde{e}_ib = \tilde{e}_i(\emptyset,b_+) = (\emptyset,\tilde{e}_ib_+) \neq 0 \). By induction hypothesis, \( \tilde{e}_ib \) is connected to \( (\emptyset,1) \) by \( \tilde{e}_j \)'s for \( j \in I \), and so is \( b \).

Suppose that \( S \neq \emptyset \). If \( -\alpha_m \in S \), then \( \tilde{e}_mS = S \setminus \{ -\alpha_m \} \neq \emptyset \), and hence \( \tilde{e}_mb = \tilde{e}_m(S,b_+) = (\tilde{e}_mS,b_+) \neq 0 \). If \( -\alpha_m \notin S \), then we may take \( \beta = -\delta_i + \delta_{m+1} \in S \) (\( 1 \leq i < m \)) such that \( i \) is the largest one. Then \( \tilde{e}_iS = (S \setminus \{ \beta \}) \cup \{ \beta + \alpha_1 \} \neq 0 \). Then we have

\[
\tilde{e}_ib = \begin{cases} 
(\tilde{e}_iS,b_+) & \text{if } \varphi_i(S) \geq \varepsilon_i(b_+), \\
(S,\tilde{e}_ib_+) & \text{if } \varphi_i(S) < \varepsilon_i(b_+). 
\end{cases}
\]

In both cases, \( \tilde{e}_ib \neq 0 \) due to our choice of \( i \). Therefore \( b \) is connected to \( (\emptyset,1) \) by \( \tilde{e}_j \)'s for \( j \in I \) by induction hypothesis.

Let \( \mathcal{B}_{m|1}(\infty) \times \mathcal{B}_{0|n}(\infty) \) be a \( \mathcal{U} \)-crystal, where for \( b = (b_1,b_2) \) and \( i \in I \),

\[
\text{wt}(b) = \text{wt}(b_1) + \text{wt}(b_2),
\]

\[
(6.9) \quad \tilde{x}_ib = \begin{cases} 
(\tilde{x}_ib_1,b_2) & \text{if } i \leq m, \\
(b_1,\tilde{x}_ib_2) & \text{if } i > m, 
\end{cases} \quad (x = e, f),
\]

and \( \varepsilon_i(b), \varphi_i(b) \) are given as in (5.10). We assume that \( \tilde{x}_ib = 0 \) if any of its component on the right-hand side is 0. Note that it follows from Proposition 6.7 and (6.9) that the \( \mathcal{U} \)-crystal \( \mathcal{B}_{m|1}(\infty) \times \mathcal{B}_{0|n}(\infty) \) is connected.
Theorem 6.8. We have the following:

1. Any connected component of \( \mathcal{B}_{m|n}(\infty) \) is isomorphic to \( \mathcal{B}_{m|1}(\infty) \times \mathcal{B}_{0|n}(\infty) \) as a \( \mathcal{U} \)-crystal up to shift of weight.

2. The number of connected components of \( \mathcal{B}_{m|n}(\infty) \) is \( 2^{m(n-1)} \).

Proof. Let \( X = \{-\delta_a + \delta_{m+1} | 1 \leq a \leq m \} \subset \Phi^\vee_T \) and \( Y = \Phi^\vee_T \setminus X \). Let \( \mathcal{C} = \{ S | S \subset Y \} \subset \mathcal{B}(K_{m|n}) \). As a \( \mathcal{U}_{m|0} \)-crystal, \( \mathcal{C} \) is isomorphic to a finite disjoint union of \( \mathcal{B}_{m|0}(\lambda) \) for \( \lambda \in P^+ \) (see the proof of Lemma 5.2). Then we have an isomorphism of \( \mathcal{U}_{m|1} \)-crystals,

\[
\begin{array}{c}
\mathcal{B}(K_{m|n}) & \longrightarrow & \mathcal{B}(K_{m|1}) \times \mathcal{C}, \\
S & \longrightarrow & (S \cap X, S \cap Y)
\end{array}
\]

where the \( \mathcal{U}_{m|1} \)-crystal structure on \( \mathcal{B}(K_{m|1}) \times \mathcal{C} \) is defined in the same manner as in Proposition 6.4 for \( 1 \leq i \leq m \).

Let us regard \( \mathcal{B}_{m|0} \subset P^+ \) (cf. 4.1). By (5.2), we get

\[
\mathcal{C} \cong \bigcup_{\ell(\lambda) \leq m} \bigcup_{\ell(\lambda') < n} \mathcal{B}_{m|0}(\lambda) \oplus m_{\lambda}
\]

as a \( \mathcal{U}_{m|0} \)-crystal, where \( m_{\lambda} = |\mathcal{B}_{0|n-1}(\lambda^i)| \) and \( B \oplus m \) denotes the crystal \( B \sqcup \cdots \sqcup B \) (\( m \) times) for a crystal \( B \) and \( m \geq 1 \). This implies

\[
\mathcal{C} \otimes \mathcal{B}_{m|0}(\infty) \cong \bigcup_{\ell(\lambda) \leq m} \bigcup_{\ell(\lambda') < n} \left( \mathcal{B}_{m|0}(\infty) \otimes T_{\text{wt}(b)} \right) \oplus m_{\lambda},
\]

by Corollary A.4. Therefore we have isomorphisms of \( \mathcal{U} \)-crystals

\[
\begin{align*}
\mathcal{B}_{m|n}(\infty) & \cong \mathcal{B}(K_{m|n}) \times \mathcal{B}_{m|0}(\infty) \times \mathcal{B}_{0|n}(\infty) \\
& \cong (\mathcal{B}(K_{m|1}) \times \mathcal{C}) \times \mathcal{B}_{m|0}(\infty) \times \mathcal{B}_{0|n}(\infty) \quad \text{by (6.10)} \\
& \cong \bigcup_{\ell(\lambda) \leq m} \bigcup_{\ell(\lambda') < n} \left( \mathcal{B}(K_{m|1}) \times (\mathcal{B}_{m|0}(\infty) \otimes T_{\text{wt}(b)}) \times \mathcal{B}_{0|n}(\infty) \right) \oplus m_{\lambda} \quad \text{by (6.12)} \\
& \cong \bigcup_{\ell(\lambda) \leq m} \bigcup_{\ell(\lambda') < n} \left( \mathcal{B}_{m|1}(\infty) \times \mathcal{B}_{0|n}(\infty) \right) \otimes T_{\text{wt}(b)} \oplus m_{\lambda}.
\end{align*}
\]

Note that \( \mathcal{B}_{m|1}(\infty) \times \mathcal{B}_{0|n}(\infty) \) is connected by Proposition 6.2. Also we see from the above decomposition that the number of connected components of \( \mathcal{B}_{m|n}(\infty) \) is \( |\mathcal{C}| = 2^{m(n-1)} \) by (6.11).

6.4. Examples. Let us give examples to describe the crystal operators \( f_i \) on \( \mathcal{B}(\infty) \cong \mathcal{B}(K) \times \mathcal{B}_{m|0}(\infty) \times \mathcal{B}_{0|n}(\infty) \).

Note that an explicit description of the crystal \( \mathcal{B}(\Phi) \) is given in Section 5.2, where we identify \( \mathcal{B}(\Phi) \) with \( \mathcal{B}(\Phi^\vee_T) \) or equivalently with the set of PBW-type monomials \( f_b \) in (4.2). Hence by using a well-known realization of \( \mathcal{B}_{m|0}(\infty) \) and \( \mathcal{B}_{0|n}(\infty) \) and applying Theorem
one may compute $\tilde{f}_i b$ ($b \in \mathcal{B}(\infty)$). Here, let us use the crystals of PBW bases (or the corresponding Lusztig data) for $\mathcal{B}_{m|0}(\infty)$ and $\mathcal{B}_{0|n}(\infty)$ with respect to \( (3.15) \) and \( (3.16) \), respectively, and a combinatorial description of $\tilde{f}_i$ on them (see [29, 31]).

Under the above identification, we may write $b = (S, b_+, b_-) \in \mathcal{B}(\infty)$ as
\[
S = (c_{st})_{\delta_s - \delta_t \in \Phi^+_T}, \quad b_+ = (c_{st})_{\delta_s - \delta_t \in \Phi^+_{m|0}}, \quad b_- = (c_{st})_{\delta_s - \delta_t \in \Phi^+_{0|n}},
\]
where $c_{st}$ is the multiplicity of $\delta_s - \delta_t$ or its root vector $f_{\delta_s - \delta_t}$ in $b$ with $c_{st} \in \{0, 1\}$ for $\delta_s - \delta_t \in \Phi^+_T$ and $c_{st} \in \mathbb{Z}^+$ otherwise.

**Example 6.9.** Let $m = 3$ and $n = 4$. Let $b = (S, b_+, b_-) \in \mathcal{B}(\infty)$ be given as in \( (6.13) \). We identify $b$ with the following array of $(c_{st})$:

\[
\begin{array}{cccccc}
\delta_3 - \delta_7 & \delta_2 - \delta_3 & \delta_1 - \delta_2 & c_{37} & c_{23} & c_{12} \\
\delta_3 - \delta_6 & \delta_2 - \delta_7 & \delta_1 - \delta_3 & c_{36} & c_{27} & c_{13} \\
\delta_3 - \delta_5 & \delta_2 - \delta_6 & \delta_1 - \delta_4 & c_{35} & c_{26} & c_{17} \\
\delta_3 - \delta_4 & \delta_2 - \delta_5 & \delta_1 - \delta_6 & \delta_4 - \delta_7 & c_{34} & c_{25} & c_{16} & c_{47} \\
\delta_3 - \delta_4 & \delta_1 - \delta_5 & \delta_4 - \delta_6 & \delta_5 - \delta_7 & c_{24} & c_{15} & c_{46} & c_{57} \\
\delta_3 - \delta_4 & \delta_1 - \delta_5 & \delta_3 - \delta_6 & \delta_5 - \delta_7 & c_{14} & c_{45} & c_{56} & c_{67}
\end{array}
\]

where the positive roots in $\Phi^+_T$ (resp. $\Phi^+_{3|0}$ and $\Phi^+_{0|4}$) are marked as $\otimes$ (resp. $\oplus$ and $\ominus$).

(The array of the roots on the left is equal to the one representing a convex order on the set of positive roots for $\mathfrak{gl}(m+n)$ associated to the longest element in the Weyl group of $\mathfrak{gl}(m|n)$ adapted to the oriented Dynkin diagram with a unique sink at $\delta_3 - \delta_4$.)

Let us compute $\tilde{f}_i b$ for
\[
b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix}
\]

**Case 1.** Suppose that $i = 3$. We have $\tilde{e}_3 b = 0$ since $c_{34} = 0$, and
\[
\tilde{f}_3 b = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix}
\]
Case 2. Suppose that $i = 1$. Let us first compute $\varphi_1(S)$. Let

$$
\sigma(S) = (+c_{24}, -c_{14}, +c_{25}, -c_{15}, \ldots, +c_{27}, -c_{17}) = (\sigma_1, \sigma_2, \ldots),
$$

be a finite sequence of $\pm$, where $\pm^a = (\pm, \ldots, \pm)$ ($a$ times) and $\pm^0 = \cdot$. We replace a pair $(\sigma_i, \sigma_j) = (+, -)$ by $(\cdot, \cdot)$, where $i < j$ and $\sigma_k = \cdot$ for $i < k < j$, and repeat this process until we get a sequence $\sigma^\text{red}(S)$ with no $-$ placed to the right of $+$. Then $\varphi_1(S)$ (resp. $\varepsilon_1(S)$) is the number of $+$'s (resp. $-$'s) in $\sigma^\text{red}(S)$ (cf. Section 5.2). In this case, we have $\varphi_1(S) = 2$ and $\varepsilon_1(S) = 0$ since $\sigma(S) = (+, \cdot, +, \cdot, +, -, \cdot, \cdot)$ and $\sigma^\text{red}(S) = (+, \cdot, +, \cdot, \cdot, \cdot, \cdot, \cdot)$.

Next, we compute $\varepsilon_1(b_+).$ Let

$$
\sigma(b_+) = (-c_{13}, +c_{23}, -c_{12}),
$$

and let $\sigma^\text{red}(b_+)$ be given in the same way as above. Then $\varepsilon_1(b_+)$ is the number of $-$'s in $\sigma^\text{red}(b_+)$. In this case, we have $\varepsilon_1(b_+) = 1$ since $\sigma(b_+) = (-, +, +, -, -)$ and $\sigma^\text{red}(b_+) = (-, \cdot, \cdot, \cdot, \cdot)$.

Therefore by (5.11), we have

$$
f_1b = (f_1S, b_+, b_-) =
\begin{pmatrix}
1 & 2 & 2 \\
0 & 0 & 1
\end{pmatrix},
$$

where the entry 1 at $\delta_2 - \delta_4$ corresponding to the leftmost $+$ in $\sigma^\text{red}(S)$ is moved to the place at $\delta_1 - \delta_4$. Here the bold-faced multiplicities denote the ones appearing in the sequences (6.15) and (6.16). Similarly, applying $f_1$ once again, we get

$$
f_1^2b = (f_1S, f_1b_+, b_-) =
\begin{pmatrix}
1 & 2 & 3 \\
0 & 0 & 1
\end{pmatrix},
$$

where $f_1b_+$ is given by adding 1 at $\delta_1 - \delta_2$ since there is no $+$ in $\sigma^\text{red}(b_+)$. 

Case 3. Suppose that \( i = 5 \). By (6.11), we have

\[
\begin{array}{cccc}
1 & 2 & 2 & \\
0 & 0 & 1 & \\
\end{array}
\]

\[
\tilde{f}_5 b = (S, b_+, \tilde{f}_5 b_-) =
\begin{array}{cccc}
1 & 1 & 0 & \\
0 & 1 & 1 & 1 \\
1 & 0 & 2 & 2 \\
0 & 1 & 1 & 1 \\
\end{array}
\]

where \( \tilde{f}_5 b_- \) is computed in the same way as \( f_1 b_+ \) in Case 2.

**Example 6.10.** Let us compare the crystal operators on \( \mathcal{B}(\infty) \) with the those on \( \mathcal{B}(X(\lambda)) \) and \( \mathcal{B}(K(\lambda)) \) for \( \lambda \in P^+ \) (6.6) and Proposition 5.1. Let \( b = (S, b_+, b_-) \in \mathcal{B}(\infty) \) given. Let \( m = 3 \), \( n = 4 \) and \( \lambda = 6 \delta_1 + 4 \delta_2 + \delta_3 + 3 \delta_4 + 2 \delta_5 - 4 \delta_7 \in P^+ \). Let \( b = (S, b_+, b_-) \) as in Example 6.9.

First consider \( \mathcal{B}(X(\lambda)) \). We claim that \( b_- \otimes t_{\lambda_-} \in \mathcal{B}_{\mathbb{Z}[t]}(\lambda_-) \). For example, to compute \( \varepsilon^*_5(b_-) \), we consider

\[
\varepsilon^*_5(b_-) = (-c_{57}, +c_{67}, -c_{56}) = (-, -, +, -), \quad \varepsilon^*_{5 \text{ red}}(b_-) = (-, -, \cdot, \cdot),
\]

where \( \varepsilon^*_{5 \text{ red}}(b_-) \) is given in the same way as in Example 6.9. Then \( \varepsilon^*_5(b_-) = 2 \), which is the number of \(-\)’s in \( \varepsilon^*_{5 \text{ red}}(b_-) \) (cf. [8, 29]). Similarly we have \( \varepsilon^*_4(b_-) = 1 \) and \( \varepsilon^*_{6}(b_-) = 1 \). By (5.3), we have \( b_-^\lambda := b_- \otimes t_{\lambda_-} \in \mathcal{B}_{\mathbb{Z}[t]}(\lambda_-) \), since \( \varepsilon^*_4(b_-) = 1 \), \( \varepsilon^*_5(b_-) = 2 \leq 2 \) and \( \varepsilon^*_{6}(b_-) = 1 \leq 4 \). Hence we may regard \( b' := (S, b_+, b_-^\lambda) \otimes t_{\lambda_+} \) as an element in \( \mathcal{B}(X(\lambda)) \), where we still identify \( b' \) with the array given in (6.14).

Let us compute \( \tilde{f}_i b' \). When \( 1 \leq i \leq m \), \( \tilde{f}_i b' \) is same as in Cases 1, 2 of Example 6.9 by (6.6).

Consider the case when \( i = 5 \). We first have \( \varphi_5(b_-^\lambda) = \varepsilon_5(b_-^\lambda) + \langle \text{wt}(b_-^\lambda), \alpha^\vee_5 \rangle = 1 \), since \( \varepsilon_5(b_-^\lambda) = \varepsilon_5(b_-) = 1 \) and \( \langle \text{wt}(b_-^\lambda), \alpha^\vee_5 \rangle = 0 \). To compute \( \varepsilon_5(S) \), we consider

\[
\sigma(S) = (+c_{15}, -c_{16}, +c_{25}, -c_{26}, +c_{35}, -c_{36}) = (\cdot, -, +, -, \cdot, \cdot), \quad \sigma^*_{5 \text{ red}}(S) = (\cdot, -, \cdot, \cdot, +, \cdot),
\]

which implies \( \varepsilon_5(S) = 1 \) and \( \varphi_5(S) = 1 \). Since \( \varphi_5(b_-^\lambda) = 1 \leq \varepsilon_5(S) = 1 \), we have by Proposition 4.6

\[
\begin{array}{cccc}
1 & 2 & 2 & \\
1 & 0 & 1 & \\
\end{array}
\]

\[
\tilde{f}_5 b' = (\tilde{f}_5 S, b_+, b_-^\lambda) =
\begin{array}{cccc}
0 & 1 & 0 & \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 2 \\
0 & 2 & 1 & 1 \\
\end{array}
\]
where the entry 1 at $\delta_3 - \delta_5$ corresponding to the leftmost + in $\sigma^{\text{red}}(S)$ is moved to the place at $\delta_3 - \delta_6$. Note that $f_5b'$ is not equal to $\tilde{f}_5 b$ in Example 6.9 (see Remark 6.6).

Next consider $\mathcal{B}(K(\lambda))$. We can check that $b_+^\lambda := b_+ \otimes t_{\lambda_+} \in \mathcal{B}_{m[0]}(\lambda_+)$ by (5.3), and may regard $b'' := (S, b_+^\lambda, b_-^\lambda)$ as an element in $\mathcal{B}(K(\lambda))$, where $b_+^\lambda = b_+ \otimes t_{\lambda_-}$. When $i \geq m$, $\tilde{f}_i b''$ is the same as $\tilde{f}_i b'$ by Proposition 5.1. When $i = 1$, $\tilde{f}_1 b''$ is the same as $\tilde{f}_1 b$ by (6.17). However, we have $\tilde{f}_1^2 b'' = (\tilde{f}_1^2 S, \tilde{f}_1 b_+^\lambda, b_-^\lambda) = 0$, since $\epsilon_1^1(\tilde{f}_1 b_+) = 3 > 2$ and hence $\tilde{f}_1 b_+^\lambda = 0$, while $\tilde{f}_1^2 b \neq 0$ (6.18).

\section*{Appendix A. The algebra $B_q(U)$ and tensor product rule}

\subsection*{A.1. The algebra $B_q(U)$}

Let $U$ be given as in (3.1). Let $B_q(U) = B_q$, be the associative $k$-algebra with 1 generated by $k_\mu$, $e_i'$, $f_i$ for $\mu \in P$ and $i \in I$ with the following relations:

\begin{equation*}
k_0 = 1, \quad k_\mu k_\mu' = k_\mu k_{\mu'} \quad (\mu, \mu' \in P),
\end{equation*}

\begin{equation*}
k_\mu e_i' k_\mu^{-1} = q^{(\mu, \alpha_i')} e_i', \quad k_\mu f_i k_\mu^{-1} = q^{- (\mu, \alpha_i')} f_i \quad (\mu \in P, i \in I),
\end{equation*}

\begin{equation*}
e_i' f_j = q^{(\alpha_j, \alpha_i')} f_j e_i' + \delta_{ij} \quad (i, j \in I),
\end{equation*}

\begin{equation*}
e_i' e_j' - e_j' e_i' = f_i f_j - f_j f_i = 0 \quad (|i - j| > 1),
\end{equation*}

\begin{equation*}
e_i'^2 - [2] e_i' e_i' + e_i'^2 = f_i^2 f_i - [2] f_i f_i f_i + f_i f_i^2 = 0 \quad (|i - j| = 1),
\end{equation*}

which is isomorphic to $B_q(\mathfrak{gl}_n)$ in [30]. We denote by $B_q^2$ the subalgebra of $B_q$ generated by $e_i'$ and $f_i$ for $i \in I$ [12]. Note that $U^-$ is a left $B_q$-module where the actions of $e_i'$, $f_i$, and $k_\mu$ are given by (2.5), left multiplication, and conjugation, respectively. Indeed, $U^-$ is irreducible since it is an irreducible $B_q^2$-module [12, Lemma 3.4.2, Corollary 3.4.9].

Let $\mathcal{O}$ be the category of $B_q$-modules $V$ such that $V$ has a weight space decomposition $V = \bigoplus_{\mu \in P} V_{\mu}$ with $\dim V_{\mu} < \infty$, and for any $v \in V$, there exists $l$ such that $e_i' \cdots e_i' v = 0$ for any $i_1, \ldots, i_l \in I$. Note that $U^-$ is the highest weight $B_q$-module in $\mathcal{O}$ with highest weight 0. The following is known in [12, Remark 3.4.10], [30, Propositions 2.3 and 2.4].

\begin{proposition}
The category $\mathcal{O}$ is semisimple, where each irreducible module in $\mathcal{O}$ is a highest weight module and it is isomorphic to $U^-$ as a $B_q^2$-module.
\end{proposition}

Let $V$ be a $B_q$-module in $\mathcal{O}$. For a weight vector $u \in V$ and $i \in I$, we have $u = \sum_{k \geq 0} f_i^{(k)} u_k$ with $e_i'(u_k) = 0$ for all $k \geq 0$, and define $\tilde{e}_i u$ and $\tilde{f}_i u$ as in (3.3). We may define a crystal base $(L, B)$ of $V$ satisfying the conditions (C1)–(C5) in Section 3.1 with respect to $\tilde{e}_i$ and $\tilde{f}_i$ ($i \in I$). For example, $(L(\infty), B(\infty))$ in (3.4) is a crystal base of $U^-$.\begin{corollary}
(12, Remark 3.5.1) Any $B_q$-module in $\mathcal{O}$ has a crystal base, which is isomorphic to a direct sum of $(L(\infty), B(\infty))$'s up to shift of weights.
\end{corollary}
Let $\Delta_B : B_q \rightarrow U \otimes B_q$ be a homomorphism of $k$-algebra given by
\[
\Delta_B(k_\mu) = k_\mu \otimes k_\mu,
\]
(A.1)
\[
\Delta_B(e_i') = (q^{-1} - q)k_i e_i \otimes 1 + k_i \otimes e_i',
\]
\[
\Delta_B(f_i) = f_i \otimes 1 + k_i \otimes f_i,
\]
for $\mu \in P$ and $i \in I$, which satisfies the coassociativity law \cite[Remark 3.4.11]{12} (see also \cite[Proposition 1.2]{30}). Let $V_1$ be a finite-dimensional $U$-module and let $V_2$ be a $B_q$-module in $\mathcal{O}$. Then $V_1 \otimes V_2$ is a $B_q$-module in $\mathcal{O}$ via (A.4).

A.2. Tensor product rule. The following is an analogue of \cite[Theorem 1]{12}.

**Theorem A.3.** (\cite[Remark 3.5.1]{12}) Let $V_1$ be a finite-dimensional $U$-module and let $V_2$ be a $B_q$-module in $\mathcal{O}$. Let $(L_k, B_k)$ be a crystal base of $V_k$ for $k = 1, 2$. Then $V_1 \otimes V_2$ is in $\mathcal{O}$, and the pair $(L_1 \otimes L_2, B_1 \otimes B_2)$ is a crystal base of $V_1 \otimes V_2$. Moreover, $\tilde{e}_i$ and $\tilde{f}_i$ ($i \in I$) act on $B_1 \otimes B_2$ by
\[
\tilde{e}_i(b_1 \otimes b_2) =\begin{cases}
\tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\
 b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2),
\end{cases}
\]
(A.2)
\[
\tilde{f}_i(b_1 \otimes b_2) =\begin{cases}
\tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
 b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2),
\end{cases}
\]
for $b_1 \otimes b_2 \in B_1 \otimes B_2$, where $\varphi_i(b_1) = \max\{ k \mid \tilde{f}_k b_1 \neq 0 \}$ and $\varepsilon_i(b_2) = \max\{ k \mid \tilde{e}_k b_2 \neq 0 \}$.

**Corollary A.4.** For $\lambda \in P^+$, we have
\[
B(\lambda) \otimes B(\infty) \cong \bigsqcup_{b \in B(\lambda)} B(\infty) \otimes T_{\lambda}(b),
\]
as a $U$-crystal.

**Proof.** Let $(B(\lambda) \otimes B(\infty))^h := \{ b^o \in B(\lambda) \otimes B(\infty) \mid \tilde{e}_i(b^o) = 0 \text{ for } i \in I \}$. By Theorem A.3 and Corollary A.2 we have
\[
B(\lambda) \otimes B(\infty) \cong \bigsqcup_{b^o \in (B(\lambda) \otimes B(\infty))^h} B(\infty) \otimes T_{\lambda}(b^o),
\]
since the connected component of $b^o \in (B(\lambda) \otimes B(\infty))^h$ is isomorphic to $B(\infty) \otimes T_{\lambda}(b^o)$. By (A.2), we have that $b^o = b_1 \otimes b_2 \in (B(\lambda) \otimes B(\infty))^h$ if and only if $b_1 = u_\lambda$ and $\varepsilon_i(b_2) \leq \langle \lambda, \alpha_i^\vee \rangle$ for all $i \in I$. Let $\ast$ denote the involution on $B(\infty)$ induced by an involution $\ast$ on $U$. \cite[Theorem 2.1.1]{13}. Note that $\text{wt}(b_2) = \text{wt}(b_2^\ast)$ and $\varepsilon_i(b_2) = \varepsilon_i^\ast(b_2^\ast) = \varepsilon_i((b')^\ast)$ for $i \in I$, where $\varepsilon_i^\ast(b') = \varepsilon_i((b')^\ast)$ for $b' \in B(\infty)$. Since $B(\lambda) = \{ b \otimes t_\lambda \in B(\infty) \otimes T_\lambda \mid \varepsilon_i(b) \leq \langle \lambda, \alpha_i^\vee \rangle \text{ for } i \in I \}$, we have a weight preserving bijection from $(B(\lambda) \otimes B(\infty))^h$ to $B(\lambda)$ sending $b^o = u_\lambda \otimes b_2$ to $b_2^\ast \otimes t_\lambda$. Therefore
\[
B(\lambda) \otimes B(\infty) \cong \bigsqcup_{b \in B(\lambda)} B(\infty) \otimes T_{\lambda}(b),
\]
For the reader’s convenience, we give a simple self-contained proof of Theorem A.3, which also works for the case associated with symmetrizable Kac-Moody algebras in [12] (cf. [28, 17.1]).

Lemma A.5. For integers \(a \geq b \geq 0\), we have
\[
\begin{bmatrix} a \\ b \end{bmatrix} = q^{-b} \begin{bmatrix} a-1 \\ b \end{bmatrix} + q^{a-b} \begin{bmatrix} a-1 \\ b-1 \end{bmatrix}.
\]
Here, for \(c, d \in \mathbb{Z}\),
\[
\begin{bmatrix} c \\ d \end{bmatrix} := \frac{[c]!}{[c-d]!d!} \quad \text{if } c \geq d \geq 0, \quad \text{and } \begin{bmatrix} c \\ d \end{bmatrix} := 0 \quad \text{otherwise}.
\]

Proof. It follows directly from the definition of \(q\)-binomial coefficient.

Lemma A.6. For \(a, b \in \mathbb{Z}_+\), we have
\[
\sum_{i=\max(0,b-a)}^{b} q^{(a-1)(b-i)} C_i(q) \begin{bmatrix} a \\ b-i \end{bmatrix} = q^{2ab},
\]
where \(C_i(q) = \prod_{i+1 \leq k \leq b} (q^{2k} - 1)\) for \(i < b\) and \(C_b(q) = 1\).

Proof. Put
\[
P(a, b) = \sum_{i=\max(0,b-a)}^{b} q^{(a-1)(b-i)} C_i(q) \begin{bmatrix} a \\ b-i \end{bmatrix}.
\]
Note that \(P(0, b) = 1\) for \(b \in \mathbb{Z}_+\). We use induction on \(a\). Suppose that \(P(a, b) = q^{2ab}\) for \(b \in \mathbb{Z}_+\). Since \(P(a+1, 0) = 1\), we may assume that \(b > 0\). By Lemma A.5, we have the following recursive formula:
\[
P(a+1, b) = P(a, b) + q^{2a} (q^{2b} - 1) P(a, b-1).
\]
Then by induction hypothesis, we have
\[
P(a+1, b) = q^{2ab} + q^{2a} (q^{2b} - 1) q^{2a(b-1)} = q^{2(a+1)b}.
\]
This completes the induction.

Proof of Theorem A.3. We divide the proof into the following steps.

Step 1. Let \(U = U_q(\mathfrak{sl}_2) = \langle e, f, k, k^{-1} \rangle\) and let \(B = B_q(U) = \langle e', f, k, k^{-1} \rangle\). Let
\[
V_1 = \bigoplus_{k=0}^{l} \mathbb{K} f^{(k)} v_1, \quad V_2 = \bigoplus_{k \geq 0} \mathbb{K} f^{(k)} v_2,
\]
be irreducible representations of \(U\) and \(B\) generated by \(v_1\) and \(v_2\), respectively, where \(ev_1 = 0\) and \(e' v_2 = 0\). Let
\[
L_1 = \bigoplus_{k=0}^{l} A_0 f^{(k)} v_1, \quad L_2 = \bigoplus_{k \geq 0} A_0 f^{(k)} v_2.
\]
be the crystal lattices of $V_1$ and $V_2$, respectively. For $0 \leq t \leq l$, we let

$$E_t = \sum_{i=0}^{t} a_{t,i}(q) f^{(i)} v_1 \otimes f^{(t-i)} v_2 \in L_1 \otimes L_2,$$

where $a_{t,i}(q) \in A_0$ is given by

$$a_{t,i}(q) = \prod_{0 \leq j \leq i-1} \frac{q^{s-t+1} - 1}{q^{2(s-t+1)} - 1}, \quad a_{t,0}(q) = 1.$$

We can check that $E_t$ is a highest weight vector of $V_1 \otimes V_2$, that is, $e'E_t = \Delta_{B_q}(e')(E_t) = 0$, and $\{E_t \mid 0 \leq t \leq l\}$ is a $k$-basis of Ker $e'$. Since $l-t+1 > 0$, we also have

$$E_t \equiv v_1 \otimes f^{(t)} v_2 \pmod{qL_1 \otimes L_2}.$$

By (A.3), we have for $s \geq 0$

$$\Delta_{B_q}(f^{(s)}) = \sum_{i=0}^{s} q^{-i(s-i)} f^{(s-i)} k^i \otimes f^{(i)},$$

and

$$\Delta_{B_q}(f^{(s)})(E_t) = \sum_{i=0}^{t} \sum_{j=0}^{s} a_{t,i}(q) q^{-j(s-j)} f^{(s-j)} k^j f^{(i)} v_1 \otimes f^{(j)} f^{(t-i)} v_2
= \sum_{i=0}^{t} \sum_{j=0}^{s} a_{t,i}(q) q^{(l-2i-s+j)} \begin{bmatrix} s-j+i \\ i \\ t-i+j \\ j \end{bmatrix} f^{(s-j+i)} v_1 \otimes f^{(t-i+j)} v_2.$$

Now, it is enough to show that

$$\Delta_{B_q}(f^{(s)})(E_t) \equiv \begin{cases} f^{(s)} v_1 \otimes f^{(t)} v_2 & \text{if } s \leq l-t, \\ f^{(l-t)} v_1 \otimes f^{(2t+s-l)} v_2 & \text{if } s > l-t, \end{cases} \pmod{qL_1 \otimes L_2}. \quad (A.3)$$

**Case 1.** $s \leq l-t$. In this case, we can check that

$$a_{t,i}(q) q^{(l-2i-s+j)} \begin{bmatrix} s-j+i \\ i \\ t-i+j \\ j \end{bmatrix} \in qA_0$$

if $(i, j) \neq (0, 0)$. Therefore $\Delta_{B_q}(f^{(s)})(E_t) \equiv f^{(s)} v_1 \otimes f^{(t)} v_2 \pmod{qL_1 \otimes L_2}$.

**Case 2.** $s > l-t$. It is enough to consider the case of $j \geq s+i-l$ since we have $f^{(s-j+i)} v_1 = 0$ if $s-j+i > l$. Thus we may write

$$\Delta_{B_q}(f^{(s)})E_t \equiv \begin{cases} f^{(s)} v_1 \otimes f^{(t)} v_2 & \text{if } s \leq l-t, \\ f^{(l-t)} v_1 \otimes f^{(2t+s-l)} v_2 & \text{if } s > l-t, \end{cases} \pmod{qL_1 \otimes L_2}. \quad (A.4)$$

$$= \sum_{i=0}^{l} \sum_{j=\max(0, s+i-l)}^{s} a_{t,i}(q) q^{(l-2i-s+j)} \begin{bmatrix} s-j+i \\ i \\ t-i+j \\ j \end{bmatrix} f^{(s-j+i)} v_1 \otimes f^{(t-i+j)} v_2
= \sum_{k=0}^{l} C(s, k) f^{(l-k)} v_1 \otimes f^{(t+s-l+k)} v_2,$$
Lemma A.6 that $q$ and that of $k$.

We use induction on $k$ which follows from (A.4) and the identity

$$f(\frac{A.6}{A.5})$$

where

$$d(\frac{A.6}{A.5})$$

Let us call an integer $d$ a minimal degree of $f(q) \in \mathbb{Q}(q)$ if $d$ is maximal such that $f(q) \in q^dA_0$. Let $d(s,k)$ be the minimal degree of $C(s,k)$. To prove (A.3), we show

$$d(s,k) = \begin{cases} 
0 & \text{if } k = t, \\
(s + k - l)(k - t) & \text{if } k > t, \\
(s + t + 1)(t - k) & \text{if } k < t.
\end{cases}$$

First, assume that $k \geq t$. Note that the minimal degree of

$$a_{t,i}(q)q^{(k-i)(i-l+s+k)} \left[ \frac{l-k}{i} \right] \left[ \frac{t-l+s+k}{t-i} \right]$$

is

$$i^2 + (2k-2t+1)i - (l-s-k)(k-t).$$

Since $(2k-2t+1) \geq 1$, the minimal value of (A.7) is $-(l-s-k)(k-t) \geq 0$ which appears only when $i = 0$ for $0 = \max(0,l-s-k) \leq i \leq \min(t,l-k)$. Therefore, (A.6) holds in this case. Next, assume that $k < t$. We need the following recursive formula for $C(s,k)$:

$$C(s+1,k) = \frac{1}{s+1}(q^{2k-l}[t+s-l+k+1]C(s,k) + [l-k]C(s,k+1)),$$

which follows from (A.3) and the identity

$$\Delta_{B_q}(f^{(s+1)}) = \frac{1}{s+1}\Delta_{B_q}f \cdot \Delta_{B_q}(f^{(s)}).$$

We use induction on $k$ to verify (A.6). Note that one can check directly from (A.5) and Lemma (A.6) that

$$C(s,0) = \frac{q^{l(t-s-l)}(s-t-l+1)}{(q^{2t-1}) \cdots (q^2-1)}, \quad d(s,0) = t(s + t - l + 1).$$

By (A.8), we have

$$C(s,k+1) = \frac{1}{l-k}((s+1)C(s+1,k) - q^{2k-l}[t+s-l+k+1]C(s,k)),$$

for $k < t$ and $s > l-t$. Since the minimal degree of $\frac{s+1}{l-k}C(s+1,k)$ is

$$Y = (l-k-s-1) + (s+1 + t-l+1)(t-k),$$

and that of $\frac{q^{2k-l}[t+s-l+k+1]}{l-k}C(s,k)$ is

$$Z = (l-k - (t+s-l+k+1)) + (2k-l) + (s+t-l+1)(t-k)$$

$$= (s+t-l+1)(t - (k+1)),$$
by induction hypothesis, we have $Y - Z = 2(t - k) > 0$, that is, $Y > Z$. Therefore, 
$d(s, k + 1) = Z = (s + t + l + 1)(t - (k + 1))$. We conclude that $C(s, k) ∈ qA_0$ if $k ≠ t$ by (A.6), and $C(s, t) ∈ 1 + qA_0$, which implies $Δ_{B_q}(f(s))E_l = f^{(l - t)}v_1 ⊗ f^{(2t + s - l)}v_2$ (mod $qL_1 ⊗ L_2$).

The proof of (A.3) completes by Case 1 and Case 2.

Step 2. Consider the case when $U = U_q(sl_n)$. Let us check that $(L, B)$ in Theorem A.3 satisfies the conditions (C1)–(C5) and (A.2). Note that $(V, L)$ and the $U_q( sl_n)$-submodule $B_q(U_i) = \{ e_i, f_i, k_i, k_i^{-1} \} ⊂ B_q(U)$. Consider the $U_1$-submodule $V'_1$ of $V_1$ generated by $v_1$ and the $B_q(U_i)$-submodule $V'_2$ of $V_2$ generated by $v_2$. If we let

$L'_1 = \bigoplus_{k=0}^{(w(v_1), ν_{v_1})} A_0 f_i^{(k)} v_1, \quad L'_2 = \bigoplus_{k=0}^{(w(v_2), ν_{v_2})} A_0 f_i^{(k)} v_2,$

$B'_1 = \{ f_i^{(k)} v_1 \ (\text{mod} \ qL'_1) | 0 ≤ k ≤ (w(v_1), ν_{v_1}) \}, \quad B'_2 = \{ f_i^{(k)} v_2 \ (\text{mod} \ qL'_2) | k ≥ 0 \},$

then $(L'_1, B'_1)$ is the crystal base of $V'_t$ for $t = 1, 2$. By Step 1, we have

$\tilde{x}(u_1 ⊗ u_2) ∈ L'_1 ⊗ L'_2 ⊂ L_1 ⊗ L_2 = L,$

$\tilde{x}(b_1 ⊗ b_2) ∈ B'_1 ⊗ B'_2 ⊂ \{ 0 \} ⊂ (B_1 ⊗ B_2) ⊂ \{ 0 \},$

($x = e, f$) together with (A.2). Thus (C4) is satisfied. The condition (C5) follows immediately from (A.2). □

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