Killing-Yano Symmetry for a Class of Spacetimes Admitting Parallel Null 1-Planes

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Abstract

A possible generalization of plane fronted waves with parallel rays (gpp-wave) fall into a more general class of metrics admitting parallel null 1-planes. For gpp-wave metric, the zero-curvature condition is given, the Killing-Yano tensors of order two and three are found and the corresponding Killing tensors are constructed. Henceforth, the compatibility between geometric duality and non-generic symmetries is presented.

PACS 04.20-q Classical general relativity
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1 Introduction

The plane fronted waves with parallel rays (pp-wave) is one of those certain metrics that receive much attention in the literature, as well as its generalized forms which were studied from various aspects. On the other hand, Walker’s metric is also interesting in its own right. For instance, in four dimensions, for each field of null 1-plane, it is possible to associate a ”classical” massless particle, with its four momentum vector to be the basis of the 1-planes. Furthermore, the recurrence vector of such a plane was interpreted to represent the electromagnetic potential. It has been shown that the canonical form of Walker’s metric can be brought into a simpler form, by an appropriate coordinate transformation and the resulting form can always be diagonalized within the components related to the space part of the metric. Kundt’s metric, and some of its generalizations, as will be noted in this paper, fall into a subclass of Walker’s metric.

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Recently, the generic and especially non-generic symmetries [7, 8, 9, 10, 11] of the spinning space [12, 13, 14, 15] have been investigated by several authors. The Killing-Yano (KY) tensors of rank two or higher order [16, 17, 18, 19, 20, 21] play an important role for new non-generic symmetries as well as the generalized KY tensors [22], which are deeply connected to generalized Runge-Lenz symmetry. Also recently, it was discovered that there is a connection between Lax tensors [23, 24] and KY tensors of order three.

Within the same context of the Killing and KY tensors, the notion of geometric duality was defined and applied on Taub-NUT and Kerr-Newman spacetimes [25]. The direct way to construct the dual metrics is to calculate the Killing tensors [26]. An alternative way is to find the KY tensor and contract two of them to find Killing tensors [25, 28]. However, finding an explicit spacetime with a physical significance admitting higher rank KY tensors is not an easy task, not only because of the complexity of the calculations, but also earlier it was found that not all metrics admit KY tensors of higher order [29]. Furthermore, it is worthwhile to compare the dual metrics obtained from the Killing tensors that are found by solving their defining equations [26], with those to be investigated in this paper. Henceforth, we believe to have a fuller understanding of geometric duality.

The plan of our paper is the following: In Sec. 2 we give a brief review of non-generic supercharges and geometric duality. In Sec. 3 we give the generalized form of the pp-wave metric. In Sec. 4 we present the KY and the Killing tensors. The dual metrics will also be given if they exist. Section 5 is devoted to concluding remarks. Finally, in the Appendix, we have presented the pure radiation condition and the Petrov classification of the background metric admitting KY tensors.

2 Killing-Yano tensors and the Dual Metrics

2.1 Killing-Yano tensors and non-generic supercharges

An \( n \)th rank Killing-Yano tensor \( f_{\nu_1\nu_2\cdots\nu_n} \), is an antisymmetric tensor fulfilling the following equations:

\[
f_{\nu_1\nu_2\cdots(\nu_n;\lambda)} = 0,
\]

where semicolon denotes the covariant derivative.

The spinning particle model was constructed to be supersymmetric [12, 13, 14, 15], therefore independent of the form of the metric there are always four independent generic symmetries [7, 11]. The existence of Killing-Yano tensors of valence \( n \) are related to non-generic supersymmetries described by the supercharge \( Q_f = f_{\nu_1\nu_2\cdots\nu_n} \Pi^{\nu_1}\psi^{\nu_2} \cdots \psi^{\nu_n} \), which is a superinvariant: \( \{Q_0, Q_f\} = 0 \), where \( Q_0 = \Pi_\mu \psi^\mu \). Here, \( \Pi_\mu \) is the covariant momenta and \( \psi^\mu \) are odd Grassmann variables. The KY equation and the Jacobi identities guarantee that it is also a constant of motion: \( \{Q_f, H\} = 0 \), with \( H = \frac{1}{2}g^{\mu\nu}\Pi_\mu\Pi_\nu \), and with the appropriate definitions of the brackets.
2.2 Killing tensors and dual spacetimes

A Killing tensor of valence two is defined through the equation

$$K_{(\mu\nu;\alpha)} = 0.$$  \tag{2}$$

Killing-Yano tensors of any valence can be considered as the square root of the Killing tensors of valence two in the sense that, their appropriate contractions yield

$$K_{\mu\nu} = g^{\alpha\beta} f_{\mu\alpha} f_{\beta\nu}$$ \tag{3}$$
or for valence three it can be written as

$$K_{\mu\nu} = g^{\rho\delta} g^{\beta\gamma} f_{\mu\rho\alpha} f_{\gamma\delta\nu}.$$ \tag{4}$$

It has been shown in detail in reference [25] that $K_{\mu\nu}$ and $g_{\mu\nu}$ are reciprocally the contravariant components of the Killing tensors with respect to each other. If $K_{\mu\nu}$ is non-degenerate, then through the relation

$$K^\mu_{\mu} k_{\alpha\nu} = \delta^\mu_\nu,$$ \tag{5}$$

the second rank non-degenerate tensor $k_{\mu\nu}$, can be viewed as the metric on the "dual" space. Furthermore, the notion of geometric duality extends to that of phase space. The constant of motion $K = \frac{1}{2} K_{\mu\nu} p_{\mu} p_{\nu}$, generates symmetry transformations on the phase space linear in momentum: $\{ x^\mu, K \} = K_{\mu\nu} p_{\nu}$. The Poisson brackets satisfy $\{ H, K \} = 0$, where $H = \frac{1}{2} g_{\mu\nu} p_{\mu} p_{\nu}$. Thus, in the phase space there is a reciprocal model with constant of motion $H$ and the Hamiltonian $K$.

3 Subclasses of Parallel Null 1-Planes

The well-known pp-wave metric is given in the form [1]:

$$ds^2 = 2 dv du + dx^2 + dy^2 + H(x, y, u) du^2.$$ \tag{6}$$

One possible generalization of this metric can be written as:

$$ds^2 = 2 dv du + A(x, y, u) [dx^2 + dy^2] + H(v, x, y, u) du^2$$ \tag{7}$$

and, hereafter we shall refer it as the gpp-wave metric. With such a generalization it falls into a class of spacetimes admitting a parallel field of null 1-planes. This latter, consists in a recurrent field of null vectors. If $l_\mu$ is a basis for the plane we have:

$$\nabla_\nu l_\mu = \kappa_\nu l_\mu, \quad l_\mu l^\mu = 0, \quad l_\mu \neq 0$$ \tag{8}$$

where $\kappa_\mu$ is the recurrence vector of the plane. It can be seen that such a vector field is geodesic and non-rotating. From (8) and the Ricci identity we have

$$l^\nu R_{\mu\nu \alpha\beta} = l_\mu t_{\alpha\beta}.$$ \tag{9}$$
where \( t_{\alpha\beta} = \partial_\alpha \kappa_\beta - \partial_\beta \kappa_\alpha \). Also from (8) one can observe that the principal null vector \( l_\mu = \delta_\mu^4 = \partial_\mu u \), is hypersurface-orthogonal and the recurrence vector for the PN1P is \( \kappa_\mu = -\Gamma^4_\mu_4 \).

The canonical form of the metric admitting parallel field of null 1-planes is given by Walker [4], and that also can be brought into a simpler form, by an appropriate the coordinate transformation [6]. Then the resulting form can always be diagonalized within the metric functions \( g_{22}, g_{33} \) and \( g_{23} \). As such, the simplified form becomes:

\[
\begin{align*}
    ds^2 &= 2 dv du + A(x, y, u) dx^2 + B(x, y, u) dy^2 + H(v, x, y, u) du^2
\end{align*}
\]

where \( A(x, y, u), B(x, y, u) \) and \( H(v, x, y, u) \) are functions of their arguments. This metric reduces to (7), when \( B = A \).

In the following analysis, the scalar curvature for (7) will prove to be important. It is calculated to be:

\[
R = \frac{1}{A^3}[A_x^2 + A_y^2 - A(A_{xx} + A_{yy})].
\]

4 Solutions to the KY equations, Killing tensors and the dual metrics

In this section the KY equations for rank two and three will be investigated. There are 24 independent equations, to be solved for the six independent components of the Killing-Yano tensor of rank two and 15 independent equations corresponds for four independent components of the KY tensor of rank three respectively. We will look for the subclasses of the gpp-wave metric admitting KY tensors with each and every number of surviving components ranging from one to six for rank two, and from one to four for rank three. We will eliminate all flat background solutions. Having in mind the importance of the manifolds with scalar curvature [30] we will investigate the \( R = 0 \) and \( R \neq 0 \) cases separately. Using the method of separation of variables, the solution to the \( R = 0 \) equation in (11) is obtained as:

\[
A(x, y, u) = a(u) \exp\left\{ \frac{c_1}{2} \left[ (y - c_2)^2 - (x - c_2)^2 \right] \right\},
\]

where \( c_1 \) and \( c_2 \) are constants and \( a \) depends only on \( u \). The \( g_{44} \) component of the metric plays an important role in solving the KY equations. The consistency conditions of KY equations leads us to the form of \( H(x, y, u, v) \) as

\[
H(x, y, u) = v h_1(x, y, u) + h_2(x, y, u).
\]

4.1 Second rank KY tensors

In the following we will present the KY tensor \( f_{\mu\nu} \), the corresponding Killing tensor \( K_{\mu\nu} \) obtained by using (3), the dual metric \( k_{\mu\nu} \) if it exists, and the form of the
metric, in that order. All solutions are for \( R = 0 \), unless otherwise stated.

**One component solutions:**

\( f_{14} = c \) \hspace{1cm} (14)

Then

\[ K_{14} = c^2, \quad K_{44} = c^2 H(u,v). \] \hspace{1cm} (15)

Here, \( c \) is an arbitrary constant. The metric function \( A \) is independent of \( u \), and \( H(v, u) = v h_1(u) + h_2(u) \).

\( f_{23} = c A(x, y) \) \hspace{1cm} (16)

and

\[ K_{22} = K_{33} = -c^2 A(x, y). \] \hspace{1cm} (17)

Here, \( c \) is an arbitrary constant and \( A \) and \( H \) are as above.

\( f_{24} = A(u)^{1/2} Q(u) \) \hspace{1cm} (18)

and we obtain

\[ K_{44} = -Q(u)^2, \] \hspace{1cm} (19)

where

\[ Q(u) = \exp \left[ -\frac{1}{2} \int h_1(u) \, du \right]. \] \hspace{1cm} (20)

Now, \( A \) is a function of only \( u \) and \( H(v, x, y, u) = v h_1(u) + h_2(x, y, u) \).

\( f_{34} \) the solution is as above.

\( f_{23} = g(u)^{3/2} s(x, y) \) \hspace{1cm} (21)

Then

\[ K_{22} = K_{33} = -g(u)^2 s(x, y), \] \hspace{1cm} (22)

with \( A(x, y, u) = g(u)s(x, y) \), where \( s(x, y) \) is an arbitrary function of its arguments and \( H(v, u) = v h_1(u) + h_2(u) \). This is the only solution for a second rank KY tensor with a non-zero scalar curvature.

**Two component solutions:**

\( f_{14} = c_3, \quad f_{23} = c_4 A(x, y) \) \hspace{1cm} (23)

so

\[ K_{14} = c_3^2, \quad K_{22} = K_{33} = -c_3^2 A(x, y), \quad K_{44} = c_3^2 H(v, u) \] \hspace{1cm} (24)

and

\[ k_{14} = 1/c_3^2, \quad k_{22} = k_{33} = -A(x, y)/c_3^2 \quad k_{44} = H(v, u)/c_3^2. \] \hspace{1cm} (25)

Here, \( c_3 \) and \( c_4 \) are constants, \( A \) is independent of \( u \), and \( H(v, u) = v h_1(u) + h_2(u) \). This is the only case where we have the compatibility between the non-generic symmetries and the dual space.
b) \[ f_{23} = A^\frac{1}{2}(u), \quad f_{24} = A(u)^\frac{1}{2}P(u). \] (26)

We find
\[ K_{22} = K_{33} = -A(u)^2, \quad K_{34} = -A(u)P(u), \quad K_{44} = -P(u)^2, \] (27)

where
\[ P(u) = Q(u) \left[ \frac{\epsilon}{2} \int \left\{ h3(u)Q(u)^{-1} \right\} du + c \right] . \] (28)

with \( Q(u) \) is as in (20), \( c \) is an integration constant and here \( \epsilon = -1 \). The metric function \( A \) depends only on \( u \) and \( H(v, y, u) = vh1(u) + yh3(u) \).

c) \[ f_{23} = A(u)^\frac{1}{2}, \quad f_{34} = A(u)^\frac{1}{2}P(u), \] (29)

and
\[ K_{22} = K_{33} = -A(u)^2, \quad K_{34} = -A(u)P(u), \quad K_{44} = -P(u)^2, \] (30)

where \( P(u) \) is as in (28), with \( \epsilon = 1 \), \( A \) is a function of \( u \) and \( H = vh1(u) + xh3(u) \).

d) \[ f_{24} = f_{34} = A(u)^\frac{1}{2}Q(u), \] (31)

so
\[ K_{44} = -2Q(u)^2. \] (32)

Here, \( A \) depends only on \( u \), and \( H(v, x, y, u) = v h1(u) + h2(x, y, u) \).

Three component solutions:
a) We have \( f_{23}, f_{24}, f_{34} \) as surviving components. From the integrability conditions two cases can be distinguished in regards to the metric function \( h2 \).

Case a1) \( h2 \) is only a function of \( u \).
\[ f_{23} = A^\frac{1}{2}(u)r(u), \quad f_{24} = -ys(u), \quad f_{34} = xs(u). \] (33)

Then
\[ K_{22} = K_{33} = -A(u)^2r(u)^2, \quad K_{24} = x A(u)^\frac{1}{2}r(u)s(u), \]
\[ K_{34} = y A(u)^\frac{1}{2}r(u)s(u), \quad K_{44} = -s(u)^2(x^2 + y^2)/A(u), \] (34)

where
\[ r(u) = \int A(u)^{-1}Q(u) du, \quad s(u) = A(u)^\frac{1}{2}Q(u) \] (35)

with \( Q(u) \) as in (20), \( A \) depending only on \( u \) and \( H(u, v) = vh1(u) + h2(u) \).

Case a2) \( h2 \) is quadratic in \( x \) and \( y \). The components are the same as (33) and (34), with \( H(u, x, y, v) = vh1(u) + h3(u)(x^2 + y^2) \). The functions \( r(u), s(u) \) are as in (33) and \( A(u) \) is subject to the solutions of
\[ \frac{A(u)_{,u}}{A(u)} - h1(u) + 2A^\frac{1}{2}(u)h3(uy)\frac{r(u)}{s(u)} = 0. \] (36)
b) \[ f_{14} = c, \quad f_{24} = -c x A(u),_{u}/2, \quad f_{34} = -c y A(u),_{u}/2. \] (37)

We have

\[ K_{14} = c^2, \quad K_{24} = -\frac{1}{2} c^2 x A(u),_{u}, \quad K_{34} = -\frac{1}{2} c^2 y A(u),_{u}, \]

\[ K_{44} = c^2 \left[ H(v, x, y, u) - (x^2 + y^2) A(u),_{u}/2 \right]. \] (38)

In this case \( A \) depends on \( u \), \( H(v, x, y, u) = vh_1(u) + h_3(u)(x^2 + y^2) \), and \( h_1, h_3, A \) are related as

\[ 2 A(u),_{uu} - A(u),_{u}^2 + A(u),_{u} A(u)h_1(u) + 4 A(u) h_3(u) = 0. \] (39)

### 4.2 Third rank KY tensors

*Two component solution:*

a) \[ f_{124} = A(u),^{\frac{1}{2}}, \quad f_{234} = \frac{1}{3} y A(u),^{\frac{3}{2}}. \] (40)

Then

\[ K_{14} = 2, \quad K_{22} = 2 A(u), \quad K_{34} = -y A(u),_{u}, \]

\[ K_{44} = 2 H(v, x, u) - y^2 A(u),_{u}^2/2A(u). \] (41)

The metric functions are \( A(u) \) and \( H = vh_1(u) + h_2(x, u) \), with \( A(u) \) and \( h_1(u) \) related as:

\[ 2 A(u),_{uu} - A(u),_{u}^2 + A(u),_{u} A(u)h_1(u) = 0. \] (42)

Such a spacetime is pure radiative, and is not conformally flat.

b) \[ f_{134} = A(u),^{\frac{1}{2}}, \quad f_{234} = -\frac{1}{3} x A(u),^{\frac{3}{2}}. \] (43)

We obtain

\[ K_{14} = 2, \quad K_{24} = -x A(u),_{u}, \quad K_{33} = 2 A(u), \]

\[ K_{44} = 2 H(v, y, u) - x^2 A(u),_{u}^2/2A(u). \] (44)

The metric function \( A \) depends on \( u \), \( H = vh_1(u) + h_2(y, u) \), with \( A(u) \) and \( h_1(u) \) are related as in (42). Such a spacetime is pure radiative, and is not conformally flat.

c) If \( R \neq 0 \) we have the following solution:

\[ f_{123} = u A(x, y, u), \quad f_{234} = r(u) A(x, y, u). \] (45)

Then

\[ K_{11} = -2 u^2, \quad K_{14} = -2 u r(u), \]

\[ K_{22} = K_{33} = 2 u A(x, y, u) \left[ u H(u, v) - 2 r(u) \right], \quad K_{44} = -2 r(u)^2 \] (46)
We have \( A(x, u, y) = u s(x, y) \), where \( s(x, y) \) is an arbitrary function of its arguments and \( H = \frac{u}{u} + h2(u) \). The functions \( r(u) \) and \( h2(u) \) are related as:

\[
2u r(u),u + r(u) - u^2 h2(u),u - u h2(u) = 0. \tag{47}
\]

This metric is not conformally flat.

5 Concluding remarks

In this paper, we have presented a possible generalization of a pp-wave metric which manifestly proved to fall into a subclass of metric admitting parallel null 1-planes. The existence of the KY tensors of order two and three for this qpp-wave metric has been investigated. Our work was divided in two parts with respect to when the scalar curvature is zero or not because those cases arose from the consistency conditions of KY equations. A subclass of this metric admitting null scalar curvature was found. The existence of third rank KY tensors suggests that some subclasses has new supercharges. We observe that Killing tensors generated by KY tensors of order three are not invertible. Nevertheless, we have found a subclass for which the geometric duality and non-generic symmetries are compatible. We have also presented the Petrov types and the pure radiation condition. The results obtained in this paper strongly encourage us to investigate the existence of the generalized KY tensors recently introduced by Collinson.

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Appendix

If \( H \) is as in (13) and in general \( R \neq 0 \), then the Einstein tensor for (7) can be found to be of the form:

\[
G_{\mu\nu} = l_\mu k_\nu + l_\nu k_\mu \tag{48}
\]

where \( k_\mu \) are:

\[
k_1 = \frac{1}{2A^2} \left[ A(A_{xx} + A_{yy}) - (A^2_x + A^2_y) \right], \tag{49}
k_2 = \frac{-1}{2A^2} (A A_{xu} - A_{u} A_x - A^2 h_{1,x}), \tag{50}
k_3 = \frac{-1}{2A^2} (A A_{yu} - A_{u} A_y - A^2 h_{1,y}). \tag{51}
\]
\[ k_4 = \frac{-1}{4A^3} \{ H \left[ A(A_{xx} + A_{yy}) - (A^2_x + A^2_y) \right] - A \left[ 2A A_{uu} - A^2_{u} + AA_A h1 + A (H_{xx} + H_{yy}) \right] \}. \]  

One can observe that, when \( h1 \) is independent of \( x \) and \( y \) and \( R = 0 \) we have pure radiation; i.e., \( G_{\mu\nu} = \rho(\sigma^\sigma)l_{\mu}l_{\nu} \), where \( \rho(\sigma^\sigma) \) is the energy density.

It is worthwhile to study the algebraic properties of the conformal Weyl tensor with respect to the principal null vector \( l_\mu \). With \( l_\mu = \delta_\mu^1 \) the following characterization for different Petrov types (PT) are obtained:

For \( R = 0 \):
\[ H = vh1(u) + h2(x, y, u), \quad \Leftrightarrow \quad l^\sigma C_{\mu\sigma\alpha\beta} \neq 0, \quad l^\sigma l_{[\mu} C_{\nu]\sigma\alpha\beta} = 0 \quad \Leftrightarrow \quad \text{type N.} \]

\[ H = vh1(x, y, u) + h2(x, y, u), \quad \Leftrightarrow \quad l^\sigma l_{[\mu} C_{\nu]\sigma\alpha\beta} \neq 0, \quad l^\sigma l_{[\mu} C_{\nu]\sigma\alpha\beta} = 0 \quad \Leftrightarrow \quad \text{type III.} \]

For \( R \neq 0 \):
\[ H = vh1(x, y, u) + h2(x, y, u), \quad \Leftrightarrow \quad l^\sigma l_{[\mu} C_{\nu]\sigma\alpha\beta} \neq 0, \quad l^\sigma l_{[\mu} C_{\nu]\sigma\alpha\beta} = 0 \quad \Leftrightarrow \quad \text{type II or D.} \]

If \( R = 0 \), \( h1 \) is depending only on \( u \) and \( h2 \) is depending on \( u, x, y \) we present the equations for the Weyl tensor to be zero:

\[ h_{2,xx} - h_{2,yy} - c_1 c_2 (h_{2,x} + h_{2,y}) + c_1 (x h_{2,x} + y h_{2,y}) = 0, \]
\[ 2 h_{2,xy} + c_1 c_2 (h_{2,x} - h_{2,y}) - c_1 (y h_{2,x} + x h_{2,y}) = 0, \]
\[ h_{2,xx} - h_{2,yy} - c_1 ((c_2 - x) h_{2,x} - (c_2 - y) h_{2,y}) = 0. \]

Obviously if \( h2 \) depends only on \( u \) the above equations are fulfilled.

Earlier, Petrov types for PN1P have been investigated, in terms of a second rank tensor involving the Ricci tensor and a bivector [5]. It can be observed that, when the metric function \( H \) is as in (13), then \( t_{\alpha\beta} \) in (4) vanishes. Petrov types of spacetimes \( v^\mu R_{\mu\alpha\beta} = 0 \), where \( v^\mu \) is not necessarily null, have also been investigated from a more general point of view [11]. Here, we have made a further analysis on PT to give the explicit forms of the metric functions.

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