Commutators, commensurators, and \( \text{PSL}_2(\mathbb{Z}) \)

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Abstract

Let \( H < \text{PSL}_2(\mathbb{Z}) \) be a finite index normal subgroup which is contained in a principal congruence subgroup, and let \( \Phi(H) \neq H \) denote a term of the lower central series or the derived series of \( H \). In this paper, we prove that the commensurator of \( \Phi(H) \) in \( \text{PSL}_2(\mathbb{R}) \) is discrete. We thus obtain a natural family of thin subgroups of \( \text{PSL}_2(\mathbb{R}) \) whose commensurators are discrete, establishing some cases of a conjecture of Shalom.

1. Introduction

The Commensurability Criterion for Arithmeticity due to Margulis [12] [15, 16.3.3] says that among irreducible lattices in semi-simple Lie groups, arithmetic lattices are characterized as those that have dense commensurators. During the past decade, Zariski dense discrete subgroups of infinite covolume in semi-simple Lie groups, also known as thin subgroups [17], have gained a lot of attention. Heuristically, thin subgroups should be regarded as non-arithmetic, even though the essential difference between a thin group and a lattice is that the former has infinite covolume in the ambient Lie group. A question attributed to Shalom makes this heuristic precise in the following way.

Question 1.1 [8]. Suppose that \( \Gamma \) is a thin subgroup of a semisimple Lie group \( G \). Is the commensurator of \( \Gamma \) discrete?

Let \( X \) be the symmetric space of non-compact type associated to \( G \) and \( \partial X \) denote its Furstenberg boundary. Question 1.1 has been answered affirmatively in the following cases.

1. Let \( \Lambda_\Gamma \) denote the limit of \( \Gamma \) on \( \partial X \). If \( \Lambda_\Gamma \subseteq \partial X \), then the answer to Question 1.1 is affirmative ([5] for \( G = \text{PSL}_2(\mathbb{C}) \) and [13] in the general case).
2. If \( G = \text{PSL}_2(\mathbb{C}) \) and \( \Gamma \) is finitely generated, then the answer to Question 1.1 is affirmative [8, 13].

Question 1.1 thus remains unaddressed when

1. \( \Lambda_\Gamma = \partial X \), and further,
2. \( \Gamma \) is not a finitely generated subgroup of \( \text{PSL}_2(\mathbb{C}) \).

Curiously, Question 1.1 remains open even for thin subgroups \( \Gamma \) of the simplest non-compact simple Lie group \( G = \text{PSL}_2(\mathbb{R}) \), with \( \Gamma \) satisfying the condition \( \Lambda_\Gamma = \partial \mathbb{H}^2 = S^1 \). It is easy to see
that such a $\Gamma$ cannot be finitely generated. To see this last claim, if $\Gamma$ is thin then discreteness implies that $\Gamma$ must be free or the fundamental group of a closed surface. Since thin groups have infinite covolume, $\Gamma$ cannot be a closed surface group, and hence must be free. It follows that $\Sigma = \mathbb{H}^2/\Gamma$ must be a surface with boundary or punctures. If $\Sigma$ has infinitely many punctures or boundary components, or if $\Sigma$ has infinite genus, then clearly $\Gamma$ is not finitely generated. If $\Sigma$ has finite genus and only finitely many punctures and no boundary components, then $\Sigma$ has finite volume and so $\Gamma$ is not thin. If $\Sigma$ has a boundary component (that is, a flaring end), then $\Sigma$ admits a proper geodesically convex core, in which case the limit set $\Lambda_\Gamma$ will be properly contained in $\partial \mathbb{H}^2$.

In this paper, we shall study commensurators of certain natural infinite index subgroups of $\text{PSL}_2(\mathbb{Z})$. If $\Gamma < G$ is an arbitrary subgroup of a group $G$, we define $\text{Comm}_G(\Gamma) = \{ g \in G \mid [\Gamma : \Gamma^g \cap \Gamma] < \infty \}$ and $[\Gamma^g : \Gamma^g \cap \Gamma] < \infty \}$. Here, we use exponentiation notation to denote conjugation, so that $\Gamma^g = g^{-1} \Gamma g$.

1.1. The main result

In this paper, we will concentrate on thin subgroups of $\text{PSL}_2(\mathbb{R})$, which are of particular interest because of their intimate connections to hyperbolic geometry via the identification $\text{PSL}_2(\mathbb{R}) \cong \text{Isom}^+(\mathbb{H}^2)$.

For an integer $k \geq 2$, we will write $\Gamma(k) < \text{PSL}_2(\mathbb{Z})$ for the level $k$ principal congruence subgroup, which is to say the kernel of the map $\text{PSL}_2(\mathbb{Z}) \to \text{PSL}_2(\mathbb{Z}/k\mathbb{Z})$ given by reducing the entries modulo $k$.

As a matter of notation, we will write $\text{PSL}_2(\mathbb{Q}) \sqrt{Q}$ for the projectivization of the set of matrices in $\text{SL}_2(\mathbb{R})$ which differ from a matrix in $\text{GL}_2(\mathbb{Q})$ by a scalar matrix which is a square root of a rational number. That is, we have $A \in \text{PSL}_2(\mathbb{Q}) \sqrt{Q}$ if there is a representative of $A$ in $\text{SL}_2(\mathbb{R})$, a rational number $q \in \mathbb{Q}$, and a matrix $B \in \text{GL}_2(\mathbb{Q})$ such that

$$A = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{q} \end{pmatrix} \cdot B.$$ 

Since we may also write

$$A = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{q^{-1}} \end{pmatrix} \cdot B'$$

for some $B' \in \text{SL}_2(\mathbb{Q})$ and since the matrix

$$\begin{pmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{q^{-1}} \end{pmatrix}$$

normalizes $\text{SL}_2(\mathbb{Q})$, it is easy to see that $\text{PSL}_2(\mathbb{Q}) \sqrt{Q}$ forms a group which can be viewed as the join $\text{PSL}_2(\mathbb{Q}) \cdot \left\{ \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{q^{-1}} \end{pmatrix} \mid q \in \mathbb{Q} \right\} < \text{PSL}_2(\mathbb{R})$.

We need to consider $\text{PSL}_2(\mathbb{Q}) \sqrt{Q}$ because of the ambient group where the commensurator lives. We recall (see [15, p. 92] or [9, Ex. 6d]) that

$$\text{Comm}_{\text{PSL}_2(\mathbb{R})}(\text{PSL}_2(\mathbb{Z})) = \text{PSL}_2(\mathbb{Q}) \sqrt{Q}.$$
which is dense in $\text{PSL}_2(\mathbb{R})$. As a consequence, it is easy to see that if $\Gamma \leq \text{PSL}_2(\mathbb{Z})$ has finite index, then

$$\text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Gamma) = \text{PSL}_2(\mathbb{Q})\sqrt{\mathbb{Q}}.$$  

For an arbitrary group $G$, we recall the definition of the lower central series and the derived series of $G$. For the lower central series, we define $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$. The derived series is defined by $D_1(G) = G$ and $D_{i+1}(G) = [D_i(G), D_i(G)]$. We will often use the notation $G' = [G, G]$ for the derived subgroup of $G$. Observe that

$$G' = D_2(G) = \gamma_2(G),$$

and if $G$ is a free group, then for $i \geq 2$ we have that $\gamma_i(G)$ and $D_i(G)$ are both properly contained in $G$ as infinitely generated characteristic subgroups of infinite index.

The purpose of this article is to establish following result, which answers Question 1.1 in the affirmative for, perhaps, the most ‘arithmetically’ defined examples:

**Theorem 1.2.** Let $H \triangleleft \text{PSL}_2(\mathbb{Z})$ be a finite index normal subgroup with $H \leq \Gamma(k)$ for some $k \geq 2$, and let $\Phi(H)$ denote a term in the lower central series or derived series of $H$. Suppose furthermore that $\Phi(H) \neq H$, so that $[H : \Phi(H)] = \infty$. If

$$g \in \text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H)),$$

then we have that $g^2 \in \text{PSL}_2(\mathbb{Z})$. In particular, $\text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H))$ is discrete.

In Theorem 1.2, it is unclear to the authors how to weaken the hypothesis that $H$ be normal in $\text{PSL}_2(\mathbb{Z})$, as well as the assumption that $H$ be contained in a principal congruence subgroup. We will outline a conjectural picture shortly.

We remark that in order to establish a perfect analogy with Margulis’ arithmeticity theorem, one would like to show that $\Phi(H)$ has finite index in $\text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H))$. This, however, is simply not feasible. On the one hand, if $H = \Gamma(k)$ for some $k \geq 2$, then $\Phi(H)$ is normal in $\text{PSL}_2(\mathbb{Z})$, whence

$$\text{PSL}_2(\mathbb{Z}) < \text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H)).$$

On the other hand, the index of $\Phi(H)$ in $\text{PSL}_2(\mathbb{Z})$ is always infinite under the hypotheses of Theorem 1.2. Nevertheless, we obtain the following corollary to Theorem 1.2, which is the correct way to mend the analogy with the arithmeticity theorem.

**Corollary 1.3.** Let $\Phi(H)$ be as in Theorem 1.2. Then $\text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H))$ is commensurable with $\text{PSL}_2(\mathbb{Z})$. In particular, $\text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H))$ contains the normalizer of $\Phi(H)$ with finite index.

Corollary 1.3 follows from Theorem 1.2 by standard methods from the theory of lattices in Lie groups. Indeed, $\Phi(H)$ is normal in $H$, and the normalizer of $\Phi(H)$ is obviously contained in $\text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H))$. Note that $\Sigma = \mathbb{H}^2/H$ is a finite volume hyperbolic orbifold. Theorem 1.2 implies that

$$\mathbb{H}^2/\text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H))$$

is also a finite volume hyperbolic orbifold, and so is a quotient of $\Sigma$ by a finite group of isometries. It follows that $\text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H))$ is commensurable with $\text{PSL}_2(\mathbb{Z})$ and contains the normalizer of $\Phi(H)$ with finite index. The reader may compare these last remarks with Lemma 2.1 below, for instance.

In our notation, we will suppress which series for $H$ as well as which term we are considering, since this will not cause particular confusion. Observe that since $H$ has finite index in $\text{PSL}_2(\mathbb{Z})$
and since $\Phi(H)$ is a non-elementary normal subgroup of $H$, we have that on the level of limit sets,

$$\Lambda_{PSL_2(\mathbb{Z})} = \Lambda_H = \Lambda_{\Phi(H)} = S^1.$$ 

The reader may consult [1] for more details on Zariski density and limit sets, for instance. We have in particular that the group $\Phi(H)$ falls outside of the purview of extant results concerning the commensurators of thin subgroups.

The techniques used in the proof of Theorem 1.2 differ widely from those used in [8, 13]. These papers used an action on a topological space and discreteness of the commensurator stemmed from the fact that the commensurator preserved a ‘discrete geometric pattern’ (in the sense of Schwartz, cf. [2, 14, 18, 19]). In this paper we use an algebraic action from Chevalley–Weil theory and homological algebra as a replacement in order to conclude discreteness of the commensurator. Since the proof is somewhat tricky and consists of a number of modular pieces, we outline the steps involved (using the notation of Theorem 1.2 above).

(1) We discuss some basic facts about principal congruence groups in Section 1.2, and observe in Section 2.1 that the normalizer of $\Phi(H)$ lies in $PSL_2(\mathbb{Z})$.

(2) Section 2.2 introduces the first technical tool in the paper based on Chevalley–Weil theory. We prove (Lemma 2.3) that if $g \in PSL_2(\mathbb{R})$ conjugates $H < PSL_2(\mathbb{Z})$ to $H^g \neq H$ and both $H, H^g$ are contained in a common free subgroup $F$, then $g$ does not lie in the commensurator of $\Phi(H)$.

(3) Section 3 is devoted to proving that if $g \in PSL_2(\mathbb{R})$ commensurates $\Phi(H)$, then in fact $g^2 \in PSL_2(\mathbb{Q})\sqrt{Q}$ (Lemma 3.1). The proof, given in Section 3.1, uses ideas from invariant trace fields and quaternion division algebras and is number-theoretic in flavor.

(4) In Section 4, we introduce the last tool, which to the knowledge of the authors is completely novel: a partial action on homology. We call this partial action a pseudo-action and study it on homology classes carried by cusps to complete the proof of Theorem 1.2.

**A conjectural picture**

The proof of Theorem 1.2 draws on several different areas of mathematics, including hyperbolic geometry, homological algebra, non-commutative algebra, and Galois theory. In order to make all the arguments work, we were forced to adopt the hypothesis that $H$ be contained as a normal subgroup of a principal congruence subgroup. However, we expect the following more general statements to hold.

**Conjecture 1.4.** Let $H < PSL_2(\mathbb{Z})$ be a finite index subgroup with $b_1(H) \geq 1$. Then $Comm_{PSL_2(\mathbb{R})}(\Phi(H))$ is discrete provided that $H \neq \Phi(H)$. More generally, let $K$ be an infinite index normal subgroup of a lattice $\Gamma < PSL_2(\mathbb{R})$ such that $|K| = \infty$. Then $Comm_{PSL_2(\mathbb{R})}(K)$ is discrete. In all cases, we have that $Comm_{PSL_2(\mathbb{R})}(K)$ contains the normalizer of $K$ with finite index.

In the first statement of Conjecture 1.4, the Betti number assumption $b_1(H) \geq 1$ is simply to guarantee that the derived subgroup $[H, H] = H'$ has infinite index in $H$. It would also be a reasonable alternative assumption to require $H$ to be torsion-free. Normality of the infinite index subgroups is always assumed in order to make the limit set coincide with the full circle.

**1.2. Preliminaries on principal congruence subgroups**

In this subsection, we gather some well-known facts about principal congruence subgroups which will be useful in the sequel. We include proofs for the convenience of the reader and to keep the discussion as self-contained as possible.
Lemma 1.5. Let $k \geq 2$. Then $\Gamma(k)$ is a free group.

Proof. The quotient $H^2/\text{PSL}_2(\mathbb{Z})$ is the $(2, 3, \infty)$ hyperbolic orbifold. It follows that if $g \in \text{PSL}_2(\mathbb{Z})$ has finite order, then it is conjugate to (the image of) one of the two matrices
\[
A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
or
\[
B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}
\]
of orders 2 and 3, respectively. Let
\[
q: \text{PSL}_2(\mathbb{Z}) \to Q
\]
be a finite quotient. If $A$ and $B$ do not lie in the kernel of $q$, then $\ker q$ is torsion-free. Indeed, if $\ker q$ contains a torsion element, then this element would be conjugate to either $A$ or $B$, so that normality would imply that $A$ or $B$ lies in $\ker q$, contrary to the assumption. Since $A$ and $B$ are clearly non-trivial in $\text{PSL}_2(\mathbb{Z})/\Gamma(k)$, we see that $\Gamma(k)$ must be torsion-free. Since $\Gamma(k)$ is torsion-free and acts freely and properly discontinuously on $H^2$, and since $H^2/\text{PSL}_2(\mathbb{Z})$ is non-compact and orientable, we have that $H^2/\Gamma(k)$ is a non-compact 2-dimensional manifold with fundamental group $\Gamma(k)$. It follows that $\Gamma(k)$ is free. $\square$

Note that the modular surface $H^2/\text{PSL}_2(\mathbb{Z})$ has exactly one cusp. As an element of the orbifold fundamental group of the modular surface, the free homotopy class of the cusp is generated by the matrix
\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Lemma 1.6. Let $k \geq 2$. The hyperbolic manifold $H^2/\Gamma(k)$ has at least three cusps.

Proof. We may reduce to the case where $k$ is a prime, since if $p\mid k$ is a prime divisor of, $k$ then $\Gamma(k) < \Gamma(p)$. The map
\[
\text{PSL}_2(\mathbb{Z}) \to \text{PSL}_2(\mathbb{Z}/p\mathbb{Z})
\]
is surjective (as can be deduced from examining generating sets of $\text{PSL}_2(\mathbb{Z}/p\mathbb{Z})$), and its image has order
\[
p(p + 1)(p - 1)/2
\]
when $p$ is odd and order 6 when $p = 2$. The order of the matrix
\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]
in $\text{PSL}_2(\mathbb{Z}/p\mathbb{Z})$ is exactly $p$, so that we see that there are exactly
\[
(p + 1)(p - 1)/2
\]
distinct cusps in $H^2/\Gamma(p)$ when $p \geq 3$ and exactly three cusps when $p = 2$. The lemma now follows. $\square$

The following consequence is immediate from elementary surface topology.
Corollary 1.7. If \( k \geq 2 \), then any loop about a cusp in \( H^2/\Gamma(k) \) represents a non-trivial integral homology class. Moreover, two loops about distinct cusps in \( H^2/\Gamma(k) \) represent distinct integral homology classes.

Proof. For all \( k \geq 2 \), we have that \( \Sigma = H^2/\Gamma(k) \) has at least three cusps, as follows from Lemma 1.6. Choose a bi-infinite, simple, oriented geodesic \( \gamma \) on \( \Sigma \) that travels between two of the cusps of \( \Sigma \). If \( K \) denotes a small neighborhood of the union of the cusps, we have that \( H_1(\Sigma, K, \mathbb{Z}) \cong H_1(\Sigma, \mathbb{Z}) \), by relativized Poincaré duality (that is, Poincaré–Lefschetz duality), and algebraic intersection with \( \gamma \) represents a non-zero cohomology class in \( H_1(\Sigma, K, \mathbb{Z}) \). If \( \delta \) is a small oriented loop around one of the cusps of \( \Sigma \), then the algebraic intersection pairing of \( \delta \) and \( \gamma \) is \( \pm 1 \), so that in particular the homology class \( [\delta] \in H_1(\Sigma, \mathbb{Z}) \) is non-zero. Since \( \gamma \) connected an arbitrary pair of cusps, the first claim of the corollary follows.

If \( \delta_1 \) and \( \delta_2 \) encircle two distinct cusps of \( \Sigma \), then since there are three distinct cusps in \( \Sigma \), one can choose two geodesics \( \gamma_1 \) and \( \gamma_2 \) as above so that \( \gamma_i \) pairs non-trivially only with \( \delta_i \) under the Poincaré duality pairing, for \( i \in \{1, 2\} \). This shows that \( \delta_1 \) and \( \delta_2 \) represent distinct integral homology classes, whence the second claim of the corollary follows. \( \square \)

1.3. Powers and discreteness

We establish the following relatively straightforward fact about discrete subgroups of \( \text{PSL}_2(\mathbb{R}) \) which will be useful at the end of the proof of our main result.

Lemma 1.8. Let \( H \) be a Zariski dense subgroup of \( \text{PSL}_2(\mathbb{R}) \) and suppose that there is a finitely generated discrete subgroup \( \Gamma < \text{PSL}_2(\mathbb{R}) \) and an \( N > 0 \) such that for all \( h \in H \), we have \( h^N \in \Gamma \). Then \( H \) is discrete.

Proof. If \( H \) fails to be discrete, then its topological closure \( \overline{H} \) must have positive dimension. Since \( \text{PSL}_2(\mathbb{R}) \) is simple and since \( H \) is Zariski dense, we have that \( \overline{H} \) is necessarily equal to \( \text{PSL}_2(\mathbb{R}) \). It follows then that \( H \) must be topologically dense in \( \text{PSL}_2(\mathbb{R}) \). Since the condition \( |\text{tr} A| < 2 \) is open for \( A \in \text{PSL}_2(\mathbb{R}) \), it follows that \( H \) must contain elements whose traces form a dense subset of \( (-2, 2) \). It follows that \( H \) either contains an elliptic element of infinite order or elliptic elements of arbitrarily high orders. In the first case, we obtain that \( \Gamma \) also contains an elliptic element of infinite order, violating the discreteness of \( \Gamma \).

In the second case, choose a finite index subgroup \( \Gamma_0 \leq \Gamma \) which is torsion-free, which exists by Selberg’s lemma [16]. We therefore have a torsion element \( A \in H \) and a power of \( A \) that is non-trivial and which lies in \( \Gamma_0 \), a contradiction. \( \square \)

1.4. Commensurations of \( \text{PSL}_2(\mathbb{Z}) \) and rational matrices

In this section we record the following easy observation to which we have alluded already.

Lemma 1.9. Let
\[ g \in \text{PSL}_2(\mathbb{Q})^\sqrt{Q} = \text{Comm}_{\text{PSL}_2(\mathbb{R})}(\text{PSL}_2(\mathbb{Z})), \]
and let \( A \in \text{PSL}_2(\mathbb{Z}) \). Then \( A^g \in \text{PSL}_2(\mathbb{Q}) \).

What is technically meant by the statement of Lemma 1.9 is that for any representatives of \( A \) and \( g \) in \( \text{SL}_2(\mathbb{R}) \), the corresponding matrix \( A^g \) will have rational entries.
Proof of Lemma 1.9. Choose a representative for \( A \) in \( SL_2(\mathbb{Z}) \) and a representative \( g = \sqrt{q}B \), where \( q \in \mathbb{Q} \) and \( B \in GL_2(\mathbb{Q}) \). It is immediate that \( B^{-1}AB \) has rational entries. Since the representative for \( g \) differs from \( B \) by a scalar multiple of the identity, we see that \( A^g = B^{-1}AB \) has rational entries. □

The fact that the commensurator of \( PSL_2(\mathbb{Z}) \) requires the adjunction of square roots and is not simply \( PSL_2(\mathbb{Q}) \) is at times an annoying issue. We note that the occurrence of square roots is fundamentally a vestige of the fact that the center of \( SL_2(\mathbb{R}) \) is non-trivial.

We remark that it is possible to avoid the appearance of square roots by working inside of other Lie groups. For instance, one can consider \( PSL_2(\mathbb{Z}) \) as the group of integer points of the special orthogonal group \( SO^+(f) \), where

\[
f = xz - y^2.
\]

In this case, \( SO(2,1) \) has no center, and a general result of Borel [3] implies that the commensurator of the integral points is simply the group of rational points \( SO^+(f, \mathbb{Q}) \). Thus, one avoids the complications resulting from square roots.

The cost of passing to the \( 3 \times 3 \) matrix group is that for the upper half plane model or disk model of \( \mathbb{H}^2 \), the group \( SO^+(f, \mathbb{Q}) \) is less intuitive and calculations are more complicated. The relative simplicity of calculating inside of \( PSL_2(\mathbb{Q}) \), together with the particular properties of \( 2 \times 2 \) matrices, will be an important feature of the arguments in the sequel, especially in the proof of Proposition 3.2 and in Section 4. So, the potential gain from using a center-free Lie group is outweighed by computational and conceptual complications later on. We therefore opt to retain the \( 2 \times 2 \) matrix group setting for the purposes of this article.

2. Invariance under commensuration

Recall the notation that \( \Phi(H) \) is a term of the lower central series or derived series of \( H \). If \( \Phi(H) \neq H \), then we sometimes say that \( \Phi(H) \) is a proper term. In this section, we prove that elements

\[
g \in Comm_{PSL_2(\mathbb{R})}(\Phi(H))
\]

satisfying certain special properties must lie in \( PSL_2(\mathbb{Z}) \).

2.1. Normalization of Zariski dense subgroups

**Lemma 2.1.** Let \( G \) be a simple real Lie group and \( \Gamma \subset G \) be a Zariski dense discrete subgroup. Then the normalizer \( N_G(\Gamma) \) is discrete.

We restrict to real Lie groups, since there is a complication for complex Lie groups; namely, the unit complex numbers are Zariski dense in \( \mathbb{C} \), and are a positive dimensional closed subgroup of \( \mathbb{C} \). In the case of complex Lie groups, one needs to consider unbounded Zariski dense subgroups in order to avoid issues like this one. Since we are primarily interested in \( PSL_2(\mathbb{R}) \), we will simply rule out the generality of arbitrary complex Lie groups.

**Proof of Lemma 2.1.** Since \( \Gamma \subset G \) is Zariski dense, so is \( N_G(\Gamma) \). If \( N_G(\Gamma) \) is not discrete, then it must be all of \( G \), since a positive dimensional Zariski dense Lie subgroup of a simple real Lie group \( G \) is necessarily all of \( G \). This then forces \( G \) to admit a non-trivial Zariski dense normal subgroup \( \Gamma \), contradicting simplicity of \( G \). Hence \( N_G(\Gamma) \) is discrete. □

As a consequence, we have the following.
Lemma 2.2. Let $H \triangleleft \text{PSL}_2(\mathbb{Z})$ be a finite index normal subgroup. Suppose that $g \in \text{PSL}_2(\mathbb{R})$ satisfies $H^g = H$. Then $g \in \text{PSL}_2(\mathbb{Z})$. More generally, if $\Phi(H) = \Phi(H)^g$, then $g \in \text{PSL}_2(\mathbb{Z})$.

Proof. Let $G = \text{PSL}_2(\mathbb{R})$. Since $\text{PSL}_2(\mathbb{R})$ is simple and since $H$ is Zariski dense, it follows by Lemma 2.1 that $N_G(H)$ is discrete. We have that $H_1 = \langle g, H \rangle$ satisfies $H_1 \subset N_G(H)$ and hence $H_1$ is discrete.

Since $H$ is normal in $\text{PSL}_2(\mathbb{Z})$, we have

$$\text{PSL}_2(\mathbb{Z}) < N_G(H).$$

It follows that $\langle g, \text{PSL}_2(\mathbb{Z}) \rangle$ is a discrete subgroup of $\text{PSL}_2(\mathbb{R})$, by Lemma 2.1. It follows that $\langle g, \text{PSL}_2(\mathbb{Z}) \rangle$ is a discrete group of orientation preserving isometries of $\mathbb{H}^2$. A standard fact from hyperbolic geometry says that

$$X = \mathbb{H}^2/\text{PSL}_2(\mathbb{Z})$$

admits no further non-trivial orientation-preserving isometries and is therefore a minimal orbifold. Indeed, let $\iota$ be an orientation-preserving involution of $X$, so that $X/\langle \iota \rangle$ is the $(2, 3, \infty)$ triangular orbifold $\mathcal{T}$. The orbifold $\mathcal{T}$ has no symmetries, and thus $X$ admits no orientation preserving symmetries, as these would have descended to $\mathcal{T}$. Consequently, we see that if $\langle g, \text{PSL}_2(\mathbb{Z}) \rangle < \text{PSL}_2(\mathbb{R})$

is discrete, then $g \in \text{PSL}_2(\mathbb{Z})$.

The proof in the case where $\Phi(H) = \Phi(H)^g$

is identical, since $\Phi(H)$ is Zariski dense in $\text{PSL}_2(\mathbb{R})$ and characteristic in $H$, and hence normalized by $\text{PSL}_2(\mathbb{Z})$. \hfill $\square$

We note that it is at this point that we use the normality of $H$ in $\text{PSL}_2(\mathbb{Z})$. If $H$ were not normal, then we could pass to a finite-index subgroup of $H$ which was normal, though then $g$ might no longer normalize $H$.

2.2. An application of Chevalley–Weil theory

In this section, we prove the following lemma.

Lemma 2.3. Let $g \in \text{PSL}_2(\mathbb{R})$, let $H < \text{PSL}_2(\mathbb{Z})$ have finite index, and suppose that there is a free group $F < \text{PSL}_2(\mathbb{R})$ such that $H, H^g < F$. Suppose furthermore that $H$ is normal in $F$. If $H^g \neq H$, then $g$ does not commensurate $\Phi(H)$, provided that $\Phi(H)$ is proper.

For Lemma 2.3, the ambient group $\text{PSL}_2(\mathbb{R})$ is irrelevant. We have the following general fact, from which Lemma 2.3 will follow with some more work.

Lemma 2.4. Let $F$ be a finitely generated free group of rank at least two, and let $K_1, K_2 < F$ be distinct, isomorphic, finite index subgroups. Suppose furthermore that $K_2$ is normal in $F$. Then $K_1'$ and $K_2'$ are not commensurable. That is to say, $K_1' \cap K_2'$ has infinite index in $K_1'$. If $K_1$ is also normal in $F$, then $K_1' \cap K_2'$ has infinite index in $K_2'$ as well.

The reader may be dissatisfied with the apparent asymmetry of Lemma 2.4. In its application to commensurators of thin groups, the asymmetry disappears, however. In deducing Lemma 2.3 from Lemma 2.4, we set $H$ to be normal subgroup of finite index in a finite index-free subgroup of $\text{PSL}_2(\mathbb{Z})$, which for the sake of explicitness we assume to be $\Gamma(k)$ for some $k \geq 2$. We assume furthermore that $H^g \neq \Gamma(k)$. Of course, $H^g$ may fail to be normal in $\Gamma(k)$. The conclusion of
Lemma 2.4 will imply (via Lemma 2.6) that \( \Phi(H) \cap \Phi(H^g) \) has infinite index in \( \Phi(H^g) \), but says nothing about the index of \( \Phi(H) \cap \Phi(H^g) \) in \( \Phi(H) \). However, if
\[ g \in \text{Comm}_{PSL_2(\mathbb{R})}(\Phi(H)) \]
then we see that
\[ g^{-1} \in \text{Comm}_{PSL_2(\mathbb{R})}(\Phi(H)) \]
as well, and so we obtain that \( \Phi(H) \cap \Phi(H^{g^{-1}}) \) has infinite index in \( \Phi(H) \), which symmetrizes the conclusion somewhat. We note briefly that \( \Phi(H^g) = \Phi(H)g \).

Before proving Lemma 2.4, we recall the following classical fact about the homology of finite index subgroups of free groups. This is also called Gaschütz’s theorem [6, 7] and is a free-group version of a well-known theorem due to Chevalley and Weil [4]. We will not reproduce a proof of this result, though we indicate that it can easily be deduced from the Lefschetz fixed-point theorem:

**Theorem 2.5.** Let \( F_k \) be a free group of rank \( k \), and let \( N \triangleleft F_k \) be a finite index normal subgroup with \( Q = F_k/N \). Then as a \( \mathbb{Q}[Q] \)-module, we have an isomorphism
\[ H_1(N, \mathbb{Q}) \cong \tau^k \oplus \rho^{k-1}, \]
where \( \tau \) denotes the trivial representation of \( Q \), and \( \rho = \rho_{\text{reg}}/\tau \) denotes the quotient of the regular representation by the trivial representation.

The trivial isotypic component of \( H_1(N, \mathbb{Q}) \) is canonically isomorphic to \( H_1(F_k, \mathbb{Q}) \) via the transfer map. Note that if \( z \in H_1(N, \mathbb{Q}) \) is not in the image of the transfer map, then \( z \) is not fixed by some element of \( Q \). Moreover, if \( 1 \neq q \in Q \), then there is an element \( z \in H_1(N, \mathbb{Q}) \) such that \( q \cdot z \neq z \).

In this section and throughout the rest of the paper, when we refer to homology, we always mean the first homology (with coefficients that will be clear from context), unless otherwise noted.

**Proof of Lemma 2.4.** Since \( K_1 \) and \( K_2 \) are distinct, isomorphic, and both of finite index in \( F \), there can be no inclusion relations between \( K_1 \) and \( K_2 \). It follows that there exists an element \( a \in K_1 \setminus K_2 \). Let
\[ b \in K_1 \cap K_2, \]
and let \( x_b = [a, b] \). Observe that
\[ x_b \in K_1' \cap K_2. \]
Indeed, since \( a \) and \( b \) are in \( K_1 \), we have that \( x_b \in K_1' \). From the fact that \( x_b = b^{-1}b^a \) and the normality of \( K_2 \), we see that \( x_b \in K_2 \).

We may now compute the homology class \([x_b]\) of \( x_b \), as an element of \( H_1(K_2, \mathbb{Z}) \): we have
\[ [x_b] = a \cdot [b] - [b], \]
where \( a \cdot [b] \) denotes the image of \([b]\) under the action of \( a \), viewed as an element of \( F/K_2 \).

Since \( a \notin K_2 \), we have that \( a \) represents a non-trivial element of \( F/K_2 \). It follows that there is a homology class
\[ z \in H_1(K_2, \mathbb{Z}) \]
such that \( a \cdot z \neq z \), by Theorem 2.5. Note that replacing \( z \) by a non-zero integral multiple, we have
\[ a \cdot (nz) = n(a \cdot z) \neq nz. \]
We may therefore choose an $n \in \mathbb{Z} \setminus \{0\}$ and an element $b \in K_1 \cap K_2$ such that $[b] = nz$, since $K_1$ and $K_2$ have finite index in $F$.

With such a choice of $b$, we have that

$$[x_b] \in H_1(K_2, \mathbb{Z})$$

is a non-trivial homology class. Since $K_2$ is free, we see that no power of $x_b$ represents a trivial homology class of $K_2$. Therefore, for all $N \neq 0$, we have $x_b^N \in K_1'$ but $x_b^N \notin K_2'$. It follows that $K_1' \cap K_2'$ has infinite index in $K_1'$.

If $K_1$ is also normal in $F$, then one can switch the roles of $K_1$ and $K_2$ to conclude that $K_1' \cap K_2'$ has infinite index in $K_2'$ as well. \qed

Note that Lemma 2.4 establishes Lemma 2.3 for $\Phi(H) = H'$. We prove the following fact, which now immediately implies Lemma 2.3 and which will be useful in the sequel.

**Lemma 2.6.** Let $A$ and $B$ be commensurable, non-abelian, free subgroups of finite rank in an ambient group $G$. Write $\Phi(A)$ and $\Phi(B)$ for a proper term of the lower central series or derived series of $A$ and $B$, so that if $\Phi(A) = \gamma_i(A)$, then $\Phi(B) = \gamma_i(B)$, or if $\Phi(A) = D_i(A)$, then $\Phi(B) = D_i(B)$. In either case, the index $i \geq 2$ is the same for both $A$ and $B$.

Suppose that there exists an element $g \in A \cap B$ such that $g \in A'$ and such that $g \in B \setminus B'$. Then $\Phi(A)$ and $\Phi(B)$ are not commensurable.

**Proof.** By the definition of $g$, we have that $g^N \in A'$ for all $N$ and $g^N \notin B'$ for all $N \neq 0$. We deal with the two series separately, starting with the lower central series. The $n$th term of the lower central series of a group $H$ will be denoted by $\gamma_n(H)$, and the $n$th term of the derived series of a group $H$ will be denoted by $D_n(H)$.

Note that $g \in \gamma_2(A)$ and

$$g \in \gamma_1(B) \setminus \gamma_2(B).$$

Note also that there is an element $x \in A \cap B$ such that $[g, x] \in \gamma_3(A)$ and such that

$$[g, x] \in \gamma_2(B) \setminus \gamma_3(B).$$

Indeed, it suffices to choose an element $x$ lying in $A \cap B$ whose integral homology class in $B$ is non-zero, which exists since $A$ and $B$ are commensurable (cf. [11]). Note that $[g, x] \in A \cap B$.

By an easy induction, we can find elements $\{y_n\}_{n \geq 1} \subset A \cap B$ such that $y_n \in \gamma_{n+1}(A)$ and such that

$$y_n \in \gamma_n(B) \setminus \gamma_{n+1}(B).$$

Again, we can simply define $y_{n+1} = [y_n, x]$, where $x \in A \cap B$ represents a non-trivial homology class of $B$. For a finitely generated free group $F$, we have that $\gamma_n(F)/\gamma_{n+1}(F)$ is a finitely generated torsion-free abelian group for all $n \geq 1$ (see again [11]). It follows that for all $N \neq 0$, we have $y_n^N \in \gamma_{n+1}(A)$ and

$$y_n^N \in \gamma_n(B) \setminus \gamma_{n+1}(B),$$

whence no proper terms of the lower central series of $A$ and $B$ are commensurable.

We now consider the derived series, and begin with the same element $g$ as above. We consider the groups $H_1(A', \mathbb{Z})$ and $H_1(B', \mathbb{Z})$, both of which are infinitely generated free abelian groups. Since

$$g \in A' = D_2(A),$$

we have that $g$ represents a (possibly trivial) element of $H_1(A', \mathbb{Z})$. Since $g \notin B'$, we have that $g$ represents a non-trivial element of $H_1(B', \mathbb{Z})$. Thus, if $z \in H_1(B', \mathbb{Z})$ is a non-trivial homology class.
class, then for all sufficiently large \( N \), we have that \( g^N \cdot z - z \) represents a non-trivial element of \( H_1(B', \mathbb{Z}) \). This claim is easily checked using covering space theory, by choosing a finite wedge of circles whose fundamental group is \( B \) and taking the cover of the wedge corresponding to \( B' \). Then one simply uses the fact that \( H_1(B, \mathbb{Z}) \) acts properly discontinuously on the corresponding cover.

Since \( A \) and \( B \) are commensurable, we can choose an element \( x \in A' \cap B' \) such that \( x \) represents a non-trivial element of \( H_1(B', \mathbb{Z}) \). Indeed, we can choose any two element \( b_1, b_2 \) in a free basis for \( B \) and non-zero exponents \( n_1 \) and \( n_2 \) such that \( b_i^{n_i} \in A \) for \( i \in \{1, 2\} \). Then, the element \( x = [b_1^{n_1}, b_2^{n_2}] \) will represent a non-trivial element of \( H_1(B', \mathbb{Z}) \).

Note that for all \( N \), we have

\[
y = [g^N, x] \in D_3(A).
\]

On the other hand, for \( N \) sufficiently large we have that \( y \) represents a non-trivial element of \( H_1(B', \mathbb{Z}) \), and therefore,

\[
y \in D_2(B) \setminus D_3(B).
\]

Since \( H_1(B', \mathbb{Z}) \) is torsion-free, it follows that \( D_3(A) \) and \( D_3(B) \) are not commensurable.

An easy induction now shows that \( D_i(A) \) and \( D_i(B) \) are not commensurable for \( i \geq 2 \). Indeed, suppose that we have produced an element \( y_i \in A \cap B \) such that for all non-zero \( N \), we have \( y_i^N \in D_i(A) \) and

\[
y_i^N \in D_{i-1}(B) \setminus D_i(B).
\]

As in the case \( i = 2 \), it is straightforward to construct an element \( x \in D_i(A) \cap D_i(B) \) such that \( x \) represents a non-trivial element of \( H_1(D_i(B), \mathbb{Z}) \). Then for all \( N \) we have

\[
y_{i+1} = [y_i^N, x] \in D_{i+1}(A).
\]

For all sufficiently large \( N \), however, we have that \( y_{i+1} \) represents a non-trivial homology class in \( H_1(D_i(B), \mathbb{Z}) \). Using again the fact that \( H_1(D_i(B), \mathbb{Z}) \) is torsion-free, no power of \( y_{i+1} \) lies in \( D_{i+1}(B) \), so that \( D_{i+1}(A) \) and \( D_{i+1}(B) \) are not commensurable. \( \square \)

3. Integral commensurators

In this section, we establish the following fact.

**Lemma 3.1.** Let \( H < \text{PSL}_2(\mathbb{Z}) \) be a non-elementary subgroup and let \( g \in \text{Comm}_{\text{PSL}_2(\mathbb{R})}(H) \). Then \( g^2 \in \text{PSL}_2(\mathbb{Q})\sqrt{\mathbb{Q}} \).

Here, non-elementary simply means non-solvable. The proof of Lemma 3.1 is given in Section 3.1 below and draws from the theory of invariant trace fields and quaternion division algebras. We are grateful to Alan Reid for teaching us about these ideas.

3.1. Quaternion division algebras and commensurators

In this section, we give our first proof of Lemma 3.1.

We refer the reader to [10] for the relevant basics on invariant trace fields and quaternion division algebras. The proof of Proposition 3.2 below is guided and informed by the argument on page 118 of [10] proving Theorem 3.3.4 there (see especially Equations (3.8) and (3.9)).

**Proposition 3.2** (Reid). Let \( H \) be a non-elementary (not necessarily discrete) subgroup of \( \text{PSL}_2(\mathbb{C}) \) such that

\[
K := \mathbb{Q}(\text{tr} H) = \mathbb{Q}(\text{tr} H')
\]
for any subgroup \( H' \) of finite index in \( H \). Let \( B = A_0 H \) denote the quaternion algebra generated over \( K \). That is, \( B \) is obtained by taking finite linear combinations of elements of \( H \) over \( K \) [10, Section 3.2]. Let \( B^* \) denote the set of invertible elements of \( B \). Suppose that \( H \) and \( x H x^{-1} \) are commensurable, that is, \( x \in \text{Comm}_{PSL_2(C)}(H) \). Then the following conclusions hold.

1. \( x = ta \) where \( a \in B^* \) and \( t \) is a non-zero complex number.
2. \( t^2 \in K \) and so \( x^2 \in B^* \).

**Proof of Lemma 3.1 assuming Proposition 3.2.** Note that the hypothesis
\[
\mathbb{Q} = \mathbb{Q}(\text{tr} \ H) = \mathbb{Q}(\text{tr} \ H')
\]
in Proposition 3.2 is satisfied for arbitrary subgroups of \( PSL_2(\mathbb{Z}) \). Moreover, we have that
\[
B^* = PSL_2(\mathbb{Q})\sqrt{\mathbb{Q}}
\]
in our notation, whence the desired conclusion follows.

We now turn to the proof of Proposition 3.2.

**Proof of Proposition 3.2.** Let
\[
H_1 = H \cap x H x^{-1},
\]
so that \( H_1 \) has finite index in \( H \) and \( x H x^{-1} \). Hence,
\[
\mathbb{Q}(\text{tr} \ H_1) = \mathbb{Q}(\text{tr} \ H) = \mathbb{Q}(\text{tr} \ (x H x^{-1}))
\]
by hypothesis. Hence the quaternion algebras \( B = A_0 H \),
\[
A_0(x H x^{-1}) = xA_0 H x^{-1},
\]
and \( A_0 H_1 \) are all defined over \( K \) and hence are all equal.

It follows from the Skolem–Noether theorem [10, 2.9.8], there exists \( a \in B^* \) such that
\[
ag a^{-1} = x g x^{-1}
\]
for all \( g \in B \). Thus,
\[
a^{-1} x g = g a^{-1} x
\]
for all \( g \in B \).

We would like to conclude that
\[
a^{-1} x \in Z(B) = K,
\]
but we cannot immediately do this because \( a^{-1} x \) need not be in \( B \). However, after tensoring with \( \mathbb{C} \) over \( K \), the Equation (3.2) continues to hold:
\[
a^{-1} x g = g a^{-1} x
\]
for all \( g \in M_2(\mathbb{C}) \). Hence,
\[
a^{-1} x = t \in \mathbb{C}
\]
is a non-zero element and \( x = ta \) as required, which proves the first conclusion.

Finally, the equation
\[
\text{det}(x) = \text{det}(ta)
\]
gives us
\[
1 = t^2 \text{det}(a),
\]
and we have \( \det(a) \in K \) by assumption. It follows that \( t^2 \in K \), and so 
\[
x^2 = t^2 a^2 \in B^* ,
\]
which establishes the second part of the conclusion. \( \square \)

4. Homological pseudo-actions and completing the proof

In this section, we complete the proof of Theorem 1.2. Let \( H \triangleleft \text{PSL}_2(\mathbb{Z}) \) be a finite index normal subgroup which is contained in the principal congruence subgroup \( \Gamma(k) \). We have shown that if 
\[
g \in \text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H)),
\]
then \( g^2 \in \text{PSL}_2(\mathbb{Q})\sqrt{\mathbb{Q}} \) and hence 
\[
g^2 \in \text{Comm}_{\text{PSL}_2(\mathbb{R})}(\text{PSL}_2(\mathbb{Z})).
\]
Therefore, we have that \( H \cap H g^2 \) is a finite index subgroup of both \( H \) and \( H^g \). We wish to argue that \( H^g \subseteq \Gamma(k) \) as well, so that we can apply Lemma 2.2 or Lemma 2.3, depending on whether \( H = H^g \) or not. Once we achieve this goal, we can prove the main technical result of this section.

**Lemma 4.1.** Let \( H < \text{PSL}_2(\mathbb{Z}) \) be a finite index normal subgroup which is contained in \( \Gamma(k) \) for some \( k \geq 2 \), and let \( g \in \text{PSL}_2(\mathbb{Q})\sqrt{\mathbb{Q}} \) commensurate \( \Phi(H) \) for some proper term. Then \( g \in \text{PSL}_2(\mathbb{Z}) \).

The proof of Theorem 1.2 follows quickly from Lemma 4.1.

**Proof of Theorem 1.2, assuming Lemma 4.1.** Let \( H \) be as in the statement of the theorem and let \( g \in \text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H)) \). By Lemma 3.1, we see that 
\[
g^2 \in \text{PSL}_2(\mathbb{Q})\sqrt{\mathbb{Q}}.
\]
By Lemma 4.1, we have that \( g^2 \in \text{PSL}_2(\mathbb{Z}) \). It follows that the square of every element in 
\[
\text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H))
\]
lies in \( \text{PSL}_2(\mathbb{Z}) \), so that the group 
\[
\text{Comm}_{\text{PSL}_2(\mathbb{R})}(\Phi(H))
\]
is discrete by Lemma 1.8. \( \square \)

Thus, it remains to establish Lemma 4.1, which will occupy the remainder of this section.

4.1. Building homological pseudo-actions

Note that \( H \) and \( H^g \) both lie in \( \text{PSL}_2(\mathbb{Q}) \), as is checked by an easy computation. Indeed, \( H \) is a subgroup of \( \text{PSL}_2(\mathbb{Z}) \) and is therefore a subgroup of \( \text{PSL}_2(\mathbb{Q}) \). The group \( H^g \) lies in \( \text{PSL}_2(\mathbb{Q}) \) by Lemma 1.9, since \( g^2 \in \text{PSL}_2(\mathbb{Q})\sqrt{\mathbb{Q}} \). Let \( H = \langle x_1, \ldots, x_m \rangle \), where \( \{x_1, \ldots, x_n\} \) is a free basis for \( H \). We write \([x_i]\) for the homology class of \( x_i \). We have that 
\[
\{[x_1], \ldots, [x_n]\}
\]
generate the integral homology of \( H \), and for each \( N \geq 1 \), we have that 
\[
\{[x_1]^N, \ldots, [x_n]^N\}
\]
generate a finite index subgroup of the integral homology of $H$. Suppressing $N$ from the notation, we sometimes write $z_i = [x_i^N]$. For $y \in H^{\gamma^2}$ arbitrary, we consider the homology class of the commutator $[y, x_i^N]$, when it makes sense. Since $y \in H^{\gamma^2}$ and since $H$ and $H^{\gamma^2}$ are commensurable, there exist arbitrarily large values of $N$ for which

$$[y, x_i^N] \in (H^{\gamma^2})',$$

since we merely choose values of $N$ such that $x_i^N \in H \cap H^{\gamma^2}$.

On the other hand, we have that $x_i^N \in H$ for all $i$ and $N$. Similarly, there exist arbitrarily large values of $N$ such that $(x_i^N)^y \in H$ as well, independently of $i$, since $y \in \text{PSL}_2(\mathbb{Q})$ and hence $H$ and $H^{\gamma^2}$ are commensurable. For these values of $N$, we may make sense of the homology class $y \cdot z_i - z_i$ for all $i$, which is the homology class of $[y, x_i^N]$ in $H$. This defines a pseudo-action of $H^{\gamma^2}$ on the integral homology of $H$. We call it a pseudo-action since it is not defined for all $N$.

Suppose that $y \cdot z_i - z_i$ is non-zero for some $i$. Then we obtain

$$[y, x_i^N] \in H \setminus H'.$$

Then by Lemma 2.6, we see that the group $\Phi(H)$ and the group

$$\Phi(H^{\gamma^2}) = \Phi(H)^{\gamma^2}$$

are not commensurable. Note that the previous discussion was symmetric in $H$ and $H^{\gamma^2}$, so that we obtain elements of $H^{\gamma^2} \cap H'$, no powers of which lie in $(H^{\gamma^2})'$ unless the homological pseudo-action of $H$ on the integral homology of $H^{\gamma^2}$ is trivial.

Therefore, if $g$ commensurates $\Phi(H)$ for some proper term of the lower central series or derived series of $H$, we must have that $y \cdot z_i - z_i$ is the trivial integral homology class of $H$ for all $i$, whenever this homology class is defined. In particular, the pseudo-action of $y$ on the integral homology of $H$ is trivial.

### 4.2. Trivial homology pseudo-actions and parabolics

Let $\gamma \in H$ be a parabolic element fixing infinity, which exists because $H$ has finite index in $\text{PSL}_2(\mathbb{Z})$. Let $y \in H^{\gamma^2}$ be arbitrary, and suppose that the $y$-pseudo-action on the integral homology of $H$ is trivial.

We see that there is a positive integer $N$ such that the homology class of $\gamma^N$ is invariant under $y$. That is,

$$[(\gamma^N)^y] = [\gamma^N]$$

as homology classes of $H^2/H$. Since $H < \Gamma(k)$ for some $k \geq 2$, we see that each cusp of $H^2/H$ is homologically non-trivial, and no two distinct cusps represent the same homology class, as follows from Corollary 1.7.

It follows that the element $(\gamma^N)^y$ represents a power of a free homotopy class of a cusp of $H^2/H$, which is equal to the free homotopy class represented by $\gamma^N$. In particular, we have that $(\gamma^N)^y$ is a parabolic element of $\text{PSL}_2(\mathbb{Q})$, and the fixed point of $(\gamma^N)^y$ is in the $H$-orbit of infinity.

It follows that there is an element $h \in H$ such that

$$( (\gamma^N)^y )^h = (\gamma^N)^{yh}$$

stabilizes infinity. Since both $H$ and $H^{\gamma^2}$ are subgroups of $\text{PSL}_2(\mathbb{Q})$, it follows that $yh$ lies in the stabilizer of infinity in $\text{PSL}_2(\mathbb{Q})$, so that we have

$$y h = \begin{pmatrix} r & t \\ 0 & r^{-1} \end{pmatrix}$$

for some suitable $r, t \in \mathbb{Q}$. 
Writing
\[ \gamma^N = \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \]
for some suitable \( M \in \mathbb{Z} \), then we see that
\[ (\gamma^N)^y h = \begin{pmatrix} 1 & r^{-2}M \\ 0 & 1 \end{pmatrix}. \]
Since
\[ [(\gamma^N)^y h] = [(\gamma^N)^y] = [\gamma^N], \]
it follows that we must have \( r = 1 \), so that
\[ y h = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \]
Repeating the same argument for a parabolic element of \( H \) stabilizing 0, we find an element \( q \in H \) such that
\[ y q = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \]
for some suitable \( s \in \mathbb{Q} \). We now multiply \((y h)^{-1}\) and \((y q)\) to get
\[ \begin{pmatrix} 1 - ts & -t \\ s & 1 \end{pmatrix} \in H. \]
Since \( H < \Gamma(k) \), we see that \( t, s \in k\mathbb{Z} \). It follows that \( y \in \Gamma(k) \). Since \( y \in H^{g^2} \) was chosen arbitrarily, we see that \( H^{g^2} < \Gamma(k) \).

To conclude the discussion of the last two subsections, we can finally prove Lemma 4.1, which completes the proof of Theorem 1.2.

**Proof of Lemma 4.1.** The discussion in Sections 4.1 and 4.2 implies that \( H^g < \Gamma(k) \). Note here that we conjugate by \( g \) and not by \( g^2 \), since we already assume that \( g \in \text{PSL}_2(\mathbb{Q})\).

Note that \( H \) is normal in \( \text{PSL}_2(\mathbb{Z}) \) and therefore is normal in \( \Gamma(k) \) as well. Since \( \Gamma(k) \) is a free group and since \( H \) and \( H^g \) are isomorphic, if \( H \neq H^g \), then we can apply Lemma 2.3 to conclude that \( \Phi(H) \) and \( \Phi(H)^g \) are not commensurable. If \( H = H^g \), then Lemma 2.2 implies that \( g \in \text{PSL}_2(\mathbb{Z}) \). \[ \square \]

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