An electrostatic depiction of the validity of the Riemann Hypothesis and a formula for the N-th zero at large N

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Abstract

We construct a vector field $\vec{E}$ from the real and imaginary parts of an entire function $\xi(z)$ which arises in the quantum statistical mechanics of relativistic gases when the spatial dimension $d$ is analytically continued into the complex $z$ plane. This function is built from the $\Gamma$ and Riemann $\zeta$ functions and is known to satisfy the functional identity $\xi(z) = \xi(1-z)$. $\vec{E}$ satisfies the conditions for a static electric field. The structure of $\vec{E}$ in the critical strip is determined by its behavior near the Riemann zeros on the critical line $\Re(z) = 1/2$, where each zero can be assigned a $\oplus$ or $\ominus$ vorticity of a related pseudo-magnetic field. Using these properties, we show that a hypothetical Riemann zero that is off the critical line leads to a frustration of this “electric” field. We formulate our argument more precisely in terms of the potential $\Phi$ satisfying $\vec{E} = -\vec{\nabla}\Phi$ and construct $\Phi$ explicitly. One outcome of our analysis is a formula for the $n$-th zero on the critical line for large $n$ expressed as the solution of a simple transcendental equation. Riemann’s counting formula for the number of zeros on the entire critical strip can be derived from this formula. Our result is much stronger than Riemann’s counting formula, since it provides an estimate of the $n$-th zero along the critical line. This provides a simple way to estimate very high zeros to very good accuracy, and we estimate the $10^{10^6}$-th one.
I. INTRODUCTION

Riemann’s zeta function[1] was originally defined as the infinite series[2].

\[ \zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z} \quad \Re(z) > 1 \quad (1) \]

It can be analytically continued throughout the complex \( z \) plane. One manner is based on the integral below eq. (3), which extends the function to \( \Re(z) > 0 \), combined with the functional relation that relates \( \zeta(z) \) to \( \zeta(1 - z) \). We will refer to roots \( \rho \) of the equation \( \zeta(\rho) = 0 \) as “Riemann zeros”, or simply as “zeros”. There are trivial zeros for \( z \) equal to any negative even integer. It has been proven that there are no zeros along the line \( \Re(z) = 1 \), which is equivalent to proving the Prime Number Theorem, as proven by Hadamard and de la Vallée Poussin. It is also known that there are an infinite number of zeros along the “critical line” \( \Re(z) = \frac{1}{2} \), which was proven by Hardy. The Riemann Hypothesis (RH) is the statement that the latter are the only zeros within the “critical strip” \( 0 \leq \Re(z) \leq 1 \). Riemann’s major result was an explicit formula, expressed in terms of these zeros, which describes the distribution of prime numbers. However the RH, and other several other statements in the original paper, are still unresolved[1]. We refer the reader to Conrey’s article for a short, but excellent introduction to the RH itself[3].

An important aspect of this subject concerns the counting of zeros. Riemann estimated that the number \( N \) of zeros on the strip, i.e. with \( 0 \leq \Re(\rho) \leq 1 \), and \( 0 \leq \Im(\rho) \leq T \), as

\[ N(T) \approx \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} \quad (2) \]

There are known corrections to the above formula, which we will rederive below. Riemann did not provide a proof, but it was eventually proven by von Mangoldt about 45 years later. It has never been proven that the above formula applies to the counting of zeros on the critical line. The reason being that this would nearly prove
the RH, or at the very least certainly would follow from the RH. We will present a
proof of this below, and more, by deriving a simple formula for the \( N \)-th zero on the
critical line for large \( N \), eqs. (20, 29) below.

There have been a number of elaborate attempts to prove the RH using physical
ideas, but unfortunately without success as far as an actual proof of the RH. (One
was proposed by us.) Many of these approaches attempt to explain the explicit values
of the Riemann zeros, based for instance on the Hilbert-Pólya idea that the zeros are
the spectrum of some as yet unknown hamiltonian. For an extensive review, we refer
to [4] and references therein. Some more recent work not reviewed there is by Sierra
and collaborators [5, 6]. These ideas are very interesting, but may perhaps soon be
exhausted since they have not yet furnished a proof, although it certainly cannot be
ruled out that they may eventually do so. In any case, we were led to suspect that
the resolution lies in traditional real and complex analysis rather than physics, nor
arithmetic. Based on reading his original paper, Riemann appeared to be confident
of this point of view [7].

Incidentally, it may be argued that the area of physics where Riemann’s zeta func-
tion plays the most prominent role is in the quantum statistical mechanics of gases.
Although we will not be invoking ideas from statistical physics to study the details
of the RH here, and furthermore, the following discussion is not necessary for our
purposes, it is nevertheless instructive to use this connection as a way of describing
some of the important properties of \( \zeta(z) \). It is this connection that initiated our
interest in the problem, however we should state from the outset that we will not
bring any methods based intrinsically on physics to bear on the problem.

Consider a gas of massless, relativistic bosons with single-particle energy \( E_k = |k| \)
in \( d \) spatial dimensions, where \( k \) is the momentum vector, at a temperature \( T = 1/\beta \)
and zero chemical potential. The free energy density, which is minus the pressure, is
given by the well-known formula which can be found in any elementary textbook on
quantum statistical physics:

\[
\mathcal{F} = \frac{1}{\beta} \int \frac{d^d k}{(2\pi)^d} \log \left(1 - e^{-\beta|k|}\right) = -\frac{1}{\beta^{d+1}} \frac{\Gamma(d)\zeta(d+1)}{2^{d-1}\pi^{d/2}\Gamma(d/2)}, \quad \Re(d) > 0 \quad (3)
\]

Here \( \Gamma \) is the standard Euler gamma function satisfying \( \Gamma(z+1) = z\Gamma(z) \). When the boson in question is one polarization of a physical photon, then the above formula leads to Planck’s black body spectrum, and this discovery in fact marked the birth of Quantum Mechanics; \( \hbar \) was first determined in this way. In performing the above integral we used

\[
\int d^d k = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int dk \ k^{d-1}, \quad \text{where} \quad k = |k|.
\]

In many other physical situations, in order to regularize divergent integrals, one analytically continues certain functions of the dimension \( d \) into the complex plane in a procedure referred to as “dimensional regularization”, and we will do the same here in this Introduction. However our analysis of the RH will not rely on this in any way whatsoever.

In the path integral approach to quantum field theory at finite temperature, the above free energy corresponds to the logarithm of a functional integral over a scalar field which is a function of \( d + 1 \) dimensional spatial coordinates after analytically continuing to euclidean time \( t \to -i\tau \) and compactifying \( \tau \) so that it lives on a circle of circumference \( \beta \). (For \( d = 1 \) this would be an infinitely long cylinder.) The same path integral can be viewed as a zero temperature quantum mechanical problem where the compact \( \tau \) direction is now regarded as a spatial coordinate and time is an infinite line, i.e. euclidean time and one spatial coordinate are interchanged in comparison to the finite temperature picture. The path integral defining \( \mathcal{F} \) now corresponds to a Casimir energy, i.e. the ground state energy density with one compactified spatial coordinate. This leads to a very different divergent expression which must be regulated using the \( \zeta \) function:

\[
\mathcal{E}_0 = \frac{1}{2\beta} \sum_{n \in \mathbb{Z}} \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \sqrt{k^2 + (2\pi n\beta)^2} = -\frac{1}{\beta^{d+1}} \pi^{d/2} \Gamma(-d/2)\zeta(-d) \quad (4)
\]

That these two expressions must be equal, which is to say, \( \mathcal{F} = \mathcal{E}_0 \), leads to
the primary functional identity satisfied by the $\zeta$ function. Namely, define $\chi(z) = \frac{\pi^{-z/2}\Gamma(z/2)}{\sqrt{\pi}}\zeta(z)$. Using the doubling formula for the $\Gamma$ function $\Gamma(d) = 2^{d-1}\Gamma(d/2)\Gamma((d+1)/2)/\sqrt{\pi}$, one has $-\beta^{d+1}F = \chi(d+1)$ and $-\beta^{d+1}E_0 = \chi(-d)$. These two expressions are equal due to the functional identity $\chi(z) = \chi(1-z)$. The latter identity was known to Riemann, but obviously based on different arguments.

The fact that the above functional identity can be obtained by this very different quantum mechanical argument indicates its non-triviality. This leads one to think that this identity plays a central role in establishing the validity of the RH. For instance, the trivial zeros at $z = -2n$ follow simply from this functional identity since it implies $\zeta(-2n) = \pi^{-2n-1/2}\Gamma(n+1/2)\zeta(1+2n)/\Gamma(-n)$ and $\Gamma(z)$ has a pole at $z = -n$. From this functional relation one can also obtain results such as $\zeta(4) = \pi^4/90$, relevant for black body physics, and similar expressions involving the Bernoulli numbers for all $z$ equal to a positive even integer. Thus many of the most important properties of $\zeta$ follow simply from the functional identity for $\chi$.

The function $\zeta(z)$ has only one pole, a simple pole at $z = 1$. Incidentally, this pole is the reason why Bose-Einstein condensation is well known to be impossible in two dimensions, based on the work of Coleman, Mermin and Wagner. The density of non-relativistic bosons with $E_k = k^2/2m$ is

$$n = \int \frac{d^d k}{(2\pi)^d} \frac{1}{e^{\beta k^2/2m} - 1} = \left( \frac{m}{2\pi^2} \right)^{d/2} \zeta(d/2)$$

and diverges when $d = 2$.

It is thus convenient to multiply $\chi(z)$ by $z(z-1)$ to remove this pole. We therefore define, as Riemann did, the function

$$\xi(z) \equiv \frac{1}{2}z(z-1)\chi(z) = \frac{1}{2}z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z)$$

The function $\xi(z)$ has the same zeros as $\zeta(z)$, and also satisfies the important identity

$$\xi(z) = \xi(1-z)$$
\( \xi(z) \) is an entire function, which is to say it is single valued, analytic, and differentiable everywhere in the complex plane.

Having made these introductory remarks, some of which are admittedly superfluous, let us now summarize the main work of this article. The \( \zeta \) function is difficult to visualize because it is a map from the complex plane into a complex function, and requires a four dimensional plot to display all of its properties. In order to visualize the function, we construct a vector field \( \vec{E} \) from the real and imaginary parts of \( \xi(z) \). By virtue of the Cauchy-Riemann equations this field satisfies the conditions for a static electric field with no charge sources, namely it has zero divergence and curl. We emphasize that our analysis is not based on physical principles but only on mathematics; the analogy with electrostatics is simply a useful one, and guided our investigation. We will thus refer to \( \vec{E} \) as the “electric field”. We apologize to mathematician readers who may find this a distraction, but they may readily ignore the terminology; we included this electrostatic visualization since it was instrumental to our initial understanding of the problem and may perhaps lead to further developments. Our main statements are sufficiently self-evident that we will not present them as theorems here; our presentation will be in the style of most of current mathematical physics, of which we are more accustomed to.

The vector field \( \vec{E} \) can be expressed as the gradient of an electric potential \( \Phi \), \( \vec{E} = -\vec{\nabla}\Phi \). Since \( \Phi \) is a real function, this provides a more economical way to visualize the RH. This is described in the next section, where it is shown that the RH follows from some simple properties of \( \Phi \) along the line \( \Re(z) = 1 \), which we refer to as a “regular alternating” property of a real function.

In order to establish this property requires some analysis of the detailed properties of \( \xi(z) \), which are presented in sections III and IV. A by-product of this analysis is a characterization of the zeros that is stronger than Riemann’s counting estimate, namely, that the n-th zero is of the form \( \rho_n = \frac{1}{2} + iy_n \) where \( y_n \) satisfies the simple
transcendental equation (20) or the slightly improved eq. (29). This is the main analytic result of our work. This new formula works extremely well, as Table I below shows.

II. ELECTROSTATIC ANALOGY AND VISUALIZATION OF THE RH.

Let us define the real and imaginary parts of \( \xi(z) \) as

\[
\xi(z) = u(x, y) + i v(x, y)
\]

where \( z = x + iy \). The Cauchy-Riemann equations, \( \partial_x u = \partial_y v \) and \( \partial_y u = -\partial_x v \), are satisfied everywhere since \( \xi \) is an entire function. Consequently, both \( u \) and \( v \) are harmonic functions, i.e. solutions of the Laplace equation \( \vec{\nabla}^2 u = (\partial_x^2 + \partial_y^2)u = 0 \)

\( \vec{\nabla}^2 v = 0 \), although they are not completely independent. Let us define \( u \) or \( v \) contours as the curves in the \( x, y \) plane corresponding to \( u \) or \( v \) equal to a constant, respectively. The critical line is a \( v = 0 \) contour since \( \xi \) is real along it. As a consequence of the Cauchy-Riemann equations, \( \vec{\nabla}u \cdot \vec{\nabla}v = 0 \). Thus, where the \( u, v \) contours intersect, they are necessarily perpendicular, and this is one aspect of their dependency. A Riemann zero occurs wherever the \( u = 0 \) and \( v = 0 \) contours intersect.

From the symmetry eq. (7) and \( \xi(z)^* = \xi(z^*) \) it follows that

\[
u(x, y) = u(1 - x, y), \quad v(x, y) = -v(1 - x, y) \]

(9)

This implies that the \( v \) contours do not cross the critical line except for \( v = 0 \). All the \( u \) contours on the other hand are allowed to cross it by the above symmetry. Away from the \( v = 0 \) points on the line \( \Re(z) = 1 \), since the \( u \) and \( v \) contours are perpendicular, the \( u \) contours generally cross the critical line and span the whole strip due to the symmetry eq. (9). The \( u \) contours that do not cross the critical line must be in the vicinity of the \( v = 0 \) contours, again by the perpendicularity of
their intersections. Putting all these facts together, the \( u, v \) contours on the critical strip have the properties shown in Figure 1. This demonstrates that Riemann zeros indeed exist on the critical line. Hardy proved that there are infinitely many of such zeros, and we will refer to this fact later.

Introduce the vector field

\[
\vec{E} = E_x \hat{x} + E_y \hat{y} \equiv u(x, y) \hat{x} - v(x, y) \hat{y}
\]

where \( \hat{x} \) and \( \hat{y} \) are unit vectors in the \( x \) and \( y \) directions. One purpose of this article is to map out the properties of this field and describe their implications for the RH.

This field has zero divergence and curl as a consequence of the Cauchy-Riemann equations

\[
\nabla \cdot \vec{E} = 0, \quad \nabla \times \vec{E} = 0,
\]

FIG. 1: Constant \( u \) contours are dashed lines (red on-line) and \( v \) contours are solid lines (blue on-line). Riemann zeros occur where a \( u = 0 \) contour spans the entire strip and crosses the critical line.
which are defined everywhere since $\xi$ is entire. Thus it satisfies the conditions of a static electric field with no charged sources. We will continue to make this analogy and refer to $\vec{E}$ as the “electric field”. However we wish to emphasize that although such analogies will be invoked in the sequel, $\vec{E}$ is not a physically realized electric field here, in that we do not need to specify what kind of charge distribution would give rise to such a field. All of our subsequent arguments will be based only on the mathematical identities expressed in eq. (11), and our reference to electrostatics is simply a useful analogy, as stated in the Introduction. Since the divergence of $\vec{E}$ equals zero everywhere, the hypothetical electric charge distribution that gives rise to $\vec{E}$ should be thought of as existing at infinity. Alternatively, since $u$ and $v$ are harmonic functions, one can view them as being determined by their values on the boundary of the critical strip. Although less meaningful, it is useful to also consider a “magnetic field” $\vec{B} = u(x,y) \hat{x} + v(x,y) \hat{y}$. Here $\vec{\nabla} \times \vec{B} \neq 0$, so it is analogous to a magnetic field with non-zero currents as sources. However it is not a proper magnetic field since $\vec{\nabla} \cdot \vec{B} \neq 0$; nevertheless it will be useful merely for establishing some terminology.

As we now argue, the main properties of the above $\vec{E}$ field on the critical strip are determined by its behavior near the Riemann zeros on the critical line combined with the behavior near $\Re(z) = 1$. The $\vec{E}$ field can be expressed as a gradient of an electric potential $\vec{E} = -\vec{\nabla}\Phi$, and satisfies the usual properties of a physical electric field with no sources. In particular, electric field lines do not cross. We will reconstruct $\Phi$ below, which leads to a more precise argument.

Any Riemann zero on the critical line arises from a $u = 0$ contour that crosses the full width of the strip and thus intersects the vertical $v = 0$ contour. On the $u = 0$ contour, $E_x = 0$, whereas on the $v = 0$ contour of the critical line itself, $E_y = 0$. Furthermore, $E_y$ changes direction as one crosses the critical line. Finally, taking into account that $\vec{E}$ has zero curl, one can easily see that there are only two ways that
all these conditions can be satisfied near the Riemann zero. Simply using the fact that the $y$ component of $\vec{B}$ has the opposite sign of $\vec{E}$, one can see that the $\vec{B}$ field encircles the Riemann zero in either the clockwise or anti-clockwise direction. Thus each Riemann zero can be assigned a vorticity $\oplus$ or $\ominus$ where $\oplus$ refers to clockwise. These properties are sketched in Figure 2. In short, Riemann zeros on the critical strip are manifestly consistent with the necessary properties of $\vec{E}$. The vector plots of $\vec{E}$ and $\vec{B}$ for the actual function $\xi(z)$ in the vicinity of the first Riemann zero at $z = 1/2 + 14.1347i$ are shown in Figures 3, 4, and confirm these simple arguments.

In this picturesque analogy, which will not be mathematically important, the Riemann zeros along the critical line behave like very small regions with nearly constant potential $\Phi$. Since physically $\Phi$ is constant inside a conductor, the Riemann zeros are analogous to very thin conducting wires that penetrate the strip perpendicularly to it, and are electrically neutral. The wires can also be thought to carry a current that gives rise to $\vec{\nabla} \times \vec{B} \neq 0$, however as stated earlier, the analogy with a bonafide physical magnetic field is not correct, since $\vec{\nabla} \cdot \vec{B} \neq 0$, and $\vec{\nabla} \times \vec{B}$ is not singular at the Riemann zeros. In other words, there is no analog of Ampere’s law for $\vec{B}$.

![FIG. 2: Sketches of the “electric” and “magnetic” fields $\vec{E}$, $\vec{B}$, in the vicinity of a Riemann zero along the critical line identified as a $\oplus$ vortex. A zero of type $\ominus$ has the direction of all arrows reversed.](image)
FIG. 3: The field $\vec{E}$ in the vicinity of the first Riemann zero $z = 1/2 + 14.1347i$. This zero has $\ominus$ vorticity.

FIG. 4: The field $\vec{B}$ in the vicinity of the Riemann zero $z = 1/2 + 14.1347i$.

We now turn to the global properties of $\vec{E}$ along the entire critical strip. It is not difficult to show that the vorticity of the Riemann zeros on the critical line, $\oplus$ or $\ominus$, alternate in sign as one moves along it. Otherwise, the curl of $\vec{E}$ would not be zero in a region between two consecutive zeros. Thus there is a form of quasi-
periodicity along the critical line, in the sense that zeros alternate between being even and odd, like the integers, and also analogous to the zeros of \( \sin(x) \) at \( x = \pi n \) where \( e^{i\pi n} = (-1)^n \). Also, along the nearly horizontal \( v = 0 \) contours that cross the critical line, \( \vec{E} \) is in the \( x \) direction. This leads to the pattern in Figure 5. One aspect of the rendition of this pattern is that it implicitly assumes that the \( v = 0 \) and \( u = 0 \) points along the line \( \Re(z) = 1 \) alternate, namely, between two consecutive \( v = 0 \) points along this line, there is only one \( u = 0 \) point, which is consistent with the knowledge that there are no zeros of \( \xi \) along the line \( \Re(z) = 1 \). This fact will be clearer when we reformulate our argument in terms of the potential \( \Phi \) below.

![Figure 5](image-url)

**FIG. 5:** The field \( \vec{E} \) (in green on-line) in the vicinity of two consecutive Riemann zeros \( \rho_1, \rho_2 \) on the critical line.

Let us now consider the possibility of a Riemann zero elsewhere on the critical strip, i.e. off of the critical line. The only location this could occur is along a \( v = 0 \) contour that intersects the critical line. In order for a Riemann zero to exist there requires a \( u = 0 \) contour to intersect the \( v = 0 \) contour, and recall they must be
perpendicular at the intersection. Such a $u$ contour, of the kind that does not cross the critical line, is shown in Figure 1. In comparison with the zeros on the critical line, the $u$ and $v$ contours are nearly rotated by $90^\circ$. This leads to a “frustration” of the electric field. Namely, as Figure 4 indicates, $\vec{E}$ wants to be horizontal at such a location, but if there exists a $u = 0$ contour here, this would imply the electric field would be vertical along it, as shown in Figure 6. Since the pattern in Figure 5 very likely repeats itself all along the critical strip because of the known infinity of zeros along it, this frustration appears to be inconsistent, and this suggests that such a Riemann zero would appear to be impossible, although this is not yet a proof. One needs to establish that the pattern in Figure 5 indeed regularly repeats itself all along the critical strip.

The RH is equivalent to the statement that all $u = 0$ contours cross the critical line and span the entire strip. We have argued that this follows from the existence of the infinite number of zeros along the critical line, the identity eq. (7), and the global properties of the field $\vec{E}$ which are a consequence of the existence of these zeros. To strengthen this picture, one needs to more accurately define the notion of “frustration” in this context. Below, a more concrete analysis will be based on the potential $\Phi$. A related remark is that the properties of $\vec{E}$ depicted in Figure 5 require that the $v = 0$ and $u = 0$ points along the line $\Re(z) = 1$ are well separated, and this is consistent with the proven fact that there are no roots to the equation $\zeta(1 + iy) = 0$, which can be used to prove the Prime Number Theorem.

A mathematically integrated version of the above arguments, which has the advantage of making manifest the dependency of $u$ and $v$, can be formulated in terms of the electric potential $\Phi$ which is a single real function. Although it contains the same information as the above argument, it is more economical, and more importantly, because it does not rely on properly defining the notion of “frustration” in order to make the argument. By virtue of $\nabla \cdot \vec{E} = 0$, $\Phi$ is also a solution of Laplace’s equation.
\[ \partial_z \partial_{\bar{z}} \Phi = 0 \] where \( \bar{z} = z^* \). The general solution is that \( \Phi \) is the sum of a function of \( z \) and another function of \( \bar{z} \). Since \( \Phi \) must be real,

\[ \vec{E} = -\vec{\nabla} \Phi, \quad \Phi(x, y) = \frac{1}{2} (\varphi(z) + \varphi(\bar{z})) \] (12)

where \( \varphi(\bar{z}) = \varphi(z)^* \). Clearly \( \Phi \) is not analytic, whereas \( \varphi \) is; it is useful to work with \( \Phi \) since we only have to deal with one real function. Comparing the definitions of \( \vec{E} \) and \( \xi \) in terms of \( u, v \), one finds

\[ u = -\left( \partial_z \varphi + \partial_{\bar{z}} \varphi \right)/2 \quad \text{and} \quad v = -i(\partial_z \varphi - \partial_{\bar{z}} \varphi)/2. \]

The latter implies

\[ \xi(z) = -\frac{\partial \varphi(z)}{\partial z} \] (13)

This equation can be integrated because \( \xi \) is entire. Using an integral representation for \( \xi(z) \) derived in Riemann’s original paper, one can show that up to an irrelevant additive constant,

\[ \varphi(z) = -8 \int_1^\infty d[t^{3/2}g'(t)] \frac{t^{-1/4}}{\log t} \sinh \left[ \frac{1}{2}(z - \frac{1}{2}) \log t \right] \] (14)

where \( g'(t) \) is the \( t \)-derivative of the function \( g(t) = \frac{1}{2} (\theta_3(0, e^{-\pi t}) - 1) = \sum_{n=1}^{\infty} e^{-n^2 \pi t} \), and \( \theta_3 \) is one of the four elliptic theta functions.

Let us now consider the \( \Phi = \text{constant} \) contours in the critical strip. Using the integral representation eq. (14), one finds the symmetry \( \Phi(x, y) = -\Phi(1-x, y) \). One
sees then that the $\Phi \neq 0$ contours do not cross the critical line, whereas the $\Phi = 0$ contours can and do. Since $\varphi$ is imaginary along the critical line, the latter is also a $\Phi = 0$ contour. Thus the $\Phi$ contours have the same structure as those for $\psi$ shown in Figure 1.

All Riemann zeros $\rho$ necessarily occur at isolated points, which is a property of entire functions. This is clear from the factorization formula $\xi(z) = \xi(0) \prod_{\rho} (1 - z/\rho)$, conjectured by Riemann, and later proved by Hadamard. At $\rho$, $\vec{\nabla}\Phi = 0$. Such isolated zeros occur when two $\Phi$ contours intersect, which can only occur if the two contours correspond to the same value of $\Phi$ since $\Phi$ is single-valued. The argument is simple: $\vec{\nabla}\Phi$ is perpendicular to the $\Phi$ contours, however as one approaches $\rho$ along one contour, one sees that it is not in the same direction as inferred from the approach from the other contour. The only way this could be consistent is if $\vec{\nabla}\Phi = 0$ at $\rho$.

With these properties of $\Phi$, we can now begin to understand the location of the known Riemann zeros. Since the $\Phi = 0$ contours intersect the critical line, which is also a $\Phi = 0$ contour, a zero exists at each such intersection, and we know there are an infinite number of them. The contour plot in Figure 7 for the actual function $\Phi$ constructed above verify these statements.

A hypothetical Riemann zero off of the critical line would then necessarily correspond to an intersection of two $\Phi \neq 0$ contours. For simplicity, let us assume that only two such contours intersect, since our arguments can be easily extended to more of such intersections. Such a situation is depicted in Figure 8. Although it does not look very natural, it is not ruled out as of yet. It implies that on the line $\Re(z) = 1$, specifically $z = 1 + iy$, $\Phi$ takes on the same non-zero value at four different values of $y$ between consecutive zeros, i.e. roots of the equation $f(y) = 0$, where $f(y) \equiv \Phi(1, y) = \Re(\varphi(1 + iy))$. Thus, the real function $f(y)$ would have to have 3 extrema between two consecutive zeros. Figure 7 suggests that this does not occur.
FIG. 7: Contour plot of the potential $\Phi$ in the vicinity of the first Riemann zero at $z = 1/2 + 14.1347i$. The horizontal, vertical directions are the $x, y$ directions where $z = x + iy$. The critical line and nearly horizontal line are $\Phi = 0$ contours and they intersect at the zero.

In order to begin to prove it, let us define a “regular alternating” real function $h(y)$ of a real variable $y$ as a function that alternates between positive and negative values in the most regular manner possible: between two consecutive zeros $h(y)$ has only one maximum, or minimum. For example, the $\sin(y)$ function is obviously regular alternating. By the above argument, if $f(y)$ is regular alternating, then two $\Phi \neq 0$ contours cannot intersect and there are no Riemann zeros off the critical line.

To summarize this section, based on the symmetry eq. (7), and the existence of the known infinity of Riemann zeros along the critical line, we have argued that $\vec{E}$ and $\Phi$ satisfy a regular repeating pattern all along the critical strip, and the RH would follow from such a repeating pattern. In order to go further, one needs to investigate the detailed properties of the function $\xi$, in particular it’s large $y$ asymptotic behavior, and establish its repetitive behavior. This is the subject of the next section, which is more constructive.
FIG. 8: A sketch of the contour plot of the potential $\Phi$ in the vicinity of a hypothetical Riemann zero off of the critical line. Such a zero occurs where the contours intersect. $\rho_n$ and $\rho_{n+1}$ are consecutive zeros on the line.

III. ANALYSIS AND AN ASYMPTOTIC FORMULA FOR THE $N$-TH RIEMANN ZERO.

To establish that $f(y)$ of the last section is a regular alternating function, we need some precise properties of the function $\xi$. In this section we describe an approximation that proved useful, however it should be said that we do not here rigorously control this approximation by estimating errors. One justification is that the approximation of the next section is a more controlled one, and will actually lead to stronger results. We present the approximation of this section primarily because it motivated the analysis of the next one.
Differentiating eq. (14), and using the summation formula for $g(t)$, one can show

$$\xi(z) = \lim_{N \to \infty} \xi^{(N)}(z) = \lim_{N \to \infty} \sum_{n=1}^{N} \xi_n(z)$$

$$\xi_n(z) = n^2 \pi \left[ 4e^{-\pi n^2} - z E_{-\frac{1}{4}}(\pi n^2) + (z - 1)E_{-\frac{3}{2}}(\pi n^2) \right]$$

where $E_\nu(r) = \int_{1}^{\infty} dt e^{-rt} t^{-\nu}$ is the standard exponential-integral function. The nature of this approximation is that the roots $\rho$ of $\xi^{(N)}(\rho) = 0$ provide a very good approximation to the smaller Riemann zeros for large enough $N$. However small values of $N$ are actually sufficient to a good degree of accuracy. For instance, the first root for $\xi^{(3)}$ coincides with the first Riemann zero to 15 digits, and it’s sixth root is correct to 8 digits. Furthermore $\xi_{n+1}$ is much smaller than $\xi_n$ because of the $e^{-n^2 \pi t}$ suppression in the integrand for $E_\nu(\pi n^2)$, so that $\xi = \xi^{(N)}$ for finite $N$, even $N = 1$, is a good approximation.

We need certain aspects of the asymptotic behavior in the critical strip as $y \to \infty$ for the various exponential-integral functions that appear in the above equation. The function $E_\nu(r)$ has a representation in terms of a continued fraction:

$$E_\nu(r) = r^{\nu-1} \Gamma(1-\nu) - \frac{e^{-r}}{1 - \nu - \frac{(1-\nu)r}{2 - \nu - \frac{(1-\nu)r}{2 - \nu + \frac{(1-\nu)r}{2 - \nu + \cdots}}}}$$

The Riemann zeros for large $y$ arise from the part of $E_\nu(r)$ which oscillates around zero, which is the first term. This will be clearer below, and verified more rigorously in the next section. We thus truncate $E_\nu(r)$ and define $\tilde{E}$ as this oscillatory part, i.e. $\tilde{E}_\nu(r) = r^{\nu-1} \Gamma(1-\nu)$. Using Stirling’s formula $\Gamma(\nu) \approx \sqrt{2\pi \nu^{\nu-1/2} e^{-\nu}}$, one can easily derive for large $y$

$$\tilde{E}_{\alpha - iy}(\pi n^2) \approx \frac{1}{n^2 e} \sqrt{\frac{y}{\pi}} \left( \frac{y}{2\pi n^2 e} \right)^{-\alpha} e^{-\pi y/4} \exp \left[ \frac{iy}{2} \log \left( \frac{y}{2\pi n^2 e} \right) + i\pi(1/2 - \alpha)/2 \right]$$

Let us first apply this approximation to $\xi_n$ on the critical line. One has

$$\xi_n(\frac{1}{2} + iy) = n^2 \pi \left[ 4e^{-n^2 \pi} - 2y \Im(E_{-\frac{1}{4} - iy}(\pi n^2)) - \Re(E_{-\frac{3}{2} - iy}(\pi n^2)) \right]$$
To a good approximation $\xi(z)$ is dominated by $\xi_1(z)$, and the latter is dominated by the second term for large $y$, i.e. $\xi(1/2 + iy) \approx -2\pi y \Im(E_{-1/2 - i y}(\pi))$. As we did for $E_{\nu}(r)$, we define $\tilde{\xi}$ as the oscillatory part of $\xi$. The imaginary part is easily read off from eq. (17):

$$\tilde{\xi}(1/2 + iy) \approx \frac{1}{e} \sqrt{\frac{y}{\pi}} \left(\frac{y}{2\pi e}\right)^{1/4} e^{-\pi y/4} \sin\left(\frac{y}{2} \log\left(\frac{y}{2\pi e}\right) + \frac{3\pi}{8}\right)$$

(19)

The above expression leads to a simple approximation of the $n$-th zero on the critical line, $\rho_n = \frac{1}{2} + iy_n$, because of its log-periodic behavior. It occurs when the argument of the sine function in the above equation is $m\pi$ where $m$ is an integer. Since $m = 1$ is close to the second zero on the critical line at $\Im(\rho) \approx 21$, in order to conform with the standard counting, the $n$-th zero occurs at $m = n - 1$. Asymptotically $y_n$ is thus a solution of the transcendental equation

$$n = \frac{y_n}{2\pi} \log\left(\frac{y_n}{2\pi e}\right) + \frac{11}{8}$$

(20)

The above formula is closely related to the Gram points $g_n$, which are solutions to $\vartheta(g_n) = n\pi$ where $\vartheta$ is the Riemann-Siegel $\vartheta$ function. Using the asymptotic expansion of $\vartheta$, one can show that for large $n$, $g_n \approx y_n$. Gram’s Law is the tendency for Riemann zeros to lie between consecutive Gram points, but it is known to fail for about $1/4$ of all Gram intervals. Our result is essentially different, in that the eq. (20) is an asymptotic formula for the actual zeros and is thus somewhat stronger than Gram’s criterion, which in any case is known to be violated. We will present an important correction to the above formula in the next section in a more rigorous fashion.

The number $N$ of zeros with $y < T$ on the critical line is clearly given by the above formula with $n \to N + 1/2$ and $y_n \to T$, which gives

$$N(T) \approx \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + \frac{7}{8}$$

(21)
and agrees with Riemann’s estimate eq. (2) for the zeros on the entire critical strip. (Riemann knew about the additional $7/8$.) However equation (20) is a much stronger result than the counting formula eq. (2) since it provides an estimate of each zero on the critical line. Table I shows how well the formula works for various $n$ up to $10^{22} + 1$, where for the latter the fractional error is about $10^{-22}$. This confirms the validity of the approximations we have made. Tables of zeros were calculated to high accuracy by Odlyzko, which in part relies on the Gram points, and can be found at his website [12]. The two adjacent zeros at 1 million indicates that eq. (20) can accurately distinguish nearby zeros. Here is our estimate of the $10^{10^6}$-th zero:

$$\rho = \frac{1}{2} + i \times 2.72877125379720787388146263022827376095518195769921562\times 10^{999994}$$

(22)

and is expected to be exact to the number of digits shown.

Based on the above approximations, one can begin to understand why there are no zeros off the critical line at large $y$ using this approximation, since $\tilde{\xi}(a + iy)$ is dominated by the terms

$$iy \Re \left( \tilde{E}_{a+1+iy} - \tilde{E}_{a-1+iy} \right) + y \Im \left( \tilde{E}_{a+1+iy} - \tilde{E}_{a-1+iy} \right)$$

(23)

where all $\tilde{E}_\nu = E_\nu(\pi)$. When $a = \frac{1}{2}$, the first term vanishes. When $a \neq \frac{1}{2}$, the $\alpha$’s in eq. (17) are mismatched, and the real and imaginary parts cannot both be zero, in other words $\tilde{\xi}(a + iy)$ is not proportional to a sine or cosine, in contrast to eq. (19). Again, a more rigorous argument is given in the next section.

Finally, let us return to what motivated the above analysis, which was to show that the function $f(y)$ defined in the last section, namely the electric potential along the line $\Re(z) = 1$, is a regular alternating function. If $f(y)$ is regular alternating, then so is its derivative. The latter is easier to work with since $\partial_y f(y) = \Im(\xi(1+iy))$.
and we can use the above formulas. One finds
\[
\begin{align*}
\partial_y f(y) &= \sum_{n=1}^{\infty} F_n(y) \\
F_n(y) &= \pi n^2 \left[ y \Re \left( E_{-\frac{1+i y}{2}}(\pi n^2) - E_{\frac{i y}{2}}(\pi n^2) \right) - \Im (E_{\frac{i y}{2}}(\pi n^2)) \right]
\end{align*}
\] (24) (25)

Keeping only the \( n = 1 \) term for the same reasons as before, for large \( y \), \( \partial_y f(y) \approx \pi y \Re \left( E_{-\frac{1+i y}{2}}(\pi) \right) \). The properties of this exponential integral function are well understood, and it is indeed regular alternating. One way to see this is to examine the oscillatory part:
\[
\begin{align*}
\partial_y \tilde{f}(y) &\approx \pi y \Re \left( \tilde{E}_{-\frac{1+i y}{2}}(\pi) \right) \propto y^2 e^{-\pi y/4} \cos \left( \frac{y}{2} \log \left( \frac{y}{2\pi e} \right) \right)
\end{align*}
\] (26)
which is a regular alternating function because of the log-periodic factor 
\[ \cos\left(\frac{y \log(y)}{2}\right). \]

IV. A FLUCTUATING CORRECTION.

The approximations of the last section had the advantage that they preserve
the symmetry eq. 7 at each \( \xi^{(N)} \), and we showed that \( \xi^{(1)} \) led to a very good
approximation of the \( n \)-th zero, in spite of the fact that we did not rigorously control
the effect of the \( \xi_{n>1} \) corrections to \( \xi \). This suggested to us that an accurate formula
for the \( n \)-th zero can be obtained from \( \xi \) itself, or equivalently from \( \chi(z) \), which
captures the \( \xi_{n>1} \) corrections. We obtain such a formula in this section in a much
simpler and well-controlled approximation, which leads to a small but imp ortant

correction to eq. 20.

As in the Introduction, define \( \chi(z) = \pi^{-z/2}\Gamma(z/2)\zeta(z) \), which also has the sym-
metry \( \chi(z) = \chi(1-z) \). It has a pole at \( z = 1 \), but this does not affect the
analysis of the zeros for large \( \Im(z) \) on the critical strip. Let us represent \( \chi(z) \) as
\[ \chi(z) = \pi^{-z/2}\Gamma(z/2)e^{i \arg \zeta(z)}|\zeta(z)|. \]
Define \( \hat{\chi}(z) \) as \( \chi(z) \) with the Stirling formula approx-
ation to \( \Gamma(z/2) \). This approximation breaks the \( z \to 1-z \) symmetry, but
since we know it exists, we can easily restore it:

\[ \chi(z) \approx \frac{1}{2} [\hat{\chi}(z) + \hat{\chi}(1-z)] \]

Similarly to the last section, one can show that for large \( y \),

\[ \hat{\chi}(a + iy) = \sqrt{2} \pi^{(1-a)/2} (y/2)^{(a-1)/2} e^{-y\pi/4} |\zeta(a + iy)| \]
\[ \times \exp \left[ i \left( \frac{y}{2} \log \left( \frac{y}{2\pi e} \right) + \frac{(a - 1)\pi}{4} + \arg \zeta(a + iy) + O(1/y) \right) \right] \]

It is not difficult to see that for \( a \neq \frac{1}{2} \), this approximation \( \chi(a + iy) = \frac{1}{2}(\hat{\chi}(a + iy) + \hat{\chi}(1 - a - iy)) \) for large \( y \) has no zeros, since its real and imaginary parts, which
just involve sine’s and cosine’s of the arg-function of the exponential in eq. (28),
cannot simultaneously be zero, for the same reason that \(\sin(\theta)\) and \(\cos(\theta)\) are not
simultaneously zero for any \(\theta\). For \(a = 1\), this is equivalent to proving the Prime
Number Theorem. Thus for large \(y\), there are no Riemann zeros off of the critical
line. On the other hand, when \(a = \frac{1}{2}\), the function \(\chi(\frac{1}{2} + iy)\) is proportional to a single
cosine, namely the cosine of the arg-function of the above expression. Thus \(\chi(\frac{1}{2} + iy)\)
is zero when the argument of this cosine is \((m - \frac{1}{2})\pi\), where \(m\) is an integer. As in
the last section, in order to conform with the standard definition of the \(n\)-th zero
by comparing with the lowest zeros, one must chose \(m = n - 1\), and one finds that
the \(n\)-th Riemann zero is of the form \(\rho_n = \frac{1}{2} + iy_n\), where \(y_n\) satisfies the following
equation:

\[
n = \frac{y_n}{2\pi} \log \left(\frac{y_n}{2\pi e}\right) + \frac{11}{8} + \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + iy_n\right) + O(1/y_n)
\]  

(29)

Thus, the well-controlled approximation of this section simply led to a small fluctu-
ating correction to the equation (20) since \(-1 < \frac{1}{\pi} \arg \zeta < 1\). As in the last section,
the number of zeros along the critical line with \(\Im(y) < T\) is given by the above
expression with \(n = N + \frac{1}{2}\) and \(y_n = T\):

\[
N(T) = \frac{T}{2\pi} \log \left(\frac{T}{2\pi e}\right) + \frac{7}{8} + \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + iT\right) + O(1/T)
\]  

(30)

The result above was obtained by Riemann for the zeros on the entire critical strip [3].
However, eq. (29) is a stronger result since it is for zeros on the critical line and is a
formula for each individual zero.

Note that when \(y_n\) is an actual zero, the \(\arg \zeta\) term is zero in eq. (29), which
explains why our previous approximation eq. (20) works so well without it. However
eq. (29) is not exact, and consequently it actually improves the estimates of the
Riemann zeros of the last section. It is more difficult to solve numerically, but
can be solved by iteration. To illustrate the improvement compared to Table II for
$n = 10,000$, the solution to eq. (29) is $y_n = 9877.78272$ whereas the exact result is $y_n = 9877.78265$, which is an improvement by 4 decimal places.

Let us return to the statement that the validity of the RH is equivalent to the property that $f(y)$ of the last section is a regular alternating function. This condition can be made more rigorous by the approximations of this section. Away from the pole at $z = 1$, as in section II we can define an electric field $\vec{E}'$ from the real and imaginary parts of $\chi(z)$, which also has zero divergence and curl, and $\vec{E}' = -\vec{\nabla}\Phi'$. The same arguments as in section II apply, and the RH follows if $\Im(\chi(1 + iy))$ is a regular alternating function. For large $y$, this follows from the log-periodic behaviour in eq. (28).

V. CONCLUDING REMARKS

In summary, our analogy with the electric field $\vec{E}$ and potential $\Phi$ suggested a very regular pattern of the function $\xi(z)$ in the critical strip. We argued that the RH follows if the real function $\Phi$ on the line $\Re(z) = 1$, or equivalently $\Im(\xi(1 + iy))$, is what we referred to as a regular alternating function of $y$. This motivated the analysis of the last two sections, which revealed such a pattern, more specifically the log-periodic behavior we derived in eq. (19) and (28). This led to the simple formulas (20) and (29) for the $n$-th Riemann zero at large $n$. The approximations in section IV that led to eq. (29) are well controlled, more than those of section III.

The formulas (20 and 29) are for zeros on the critical line, and Riemann’s counting formula eq. (2) for the zeros on the entire strip is a consequence of it, including the corrections in eq. (30). We also provided additional analysis in the last two sections showing that there are no zeros off of the critical line with sufficiently large imaginary part. Our work thus indicates an extreme regularity of the zeros with large imaginary part. Since it is already known that there are no zeros off the line for $y < 100,000$, 

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nor for even much larger $y[3,12]$, this would seem to establish the RH.

We have only included the leading approximations to the main formulas. It would be interesting to understand the sub-leading corrections that modify eqs. (20) and (29), although we have shown that they are very small. Also, since we now have a simple approximation to the high zeros, it would be interesting to relate it to Montgomery’s pair correlation conjecture[13], which is the same as that of random Hermitian matrices in the GUE universality class[14].

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In the mathematics literature the complex variable $z$ is conventionally denoted as $s = \sigma + it$ in this context. Here the notation will be $s = z = x + iy$ since this makes the vectorial aspects we will consider more natural.

[1] H. M. Edwards, *Riemann’s Zeta Function*, Dover Publications Inc. 1974.

[2] J. B. Conrey, *The Riemann Hypothesis*, Notices of the AMS **50** (2003) 342.

[3] D. Schumayer and D. A. W. Hutchinson, *Physics of the Riemann Hypothesis*, Rev. Mod. Phys. **83** (2011) 307 [arXiv:1101.3116].

[4] G. Sierra and P. K. Townsend, *Landau levels and Riemann zeros*, Phys. Rev. Lett. **101** (2008) 110201.

[5] G. Sierra and J. Rodriguez-Laguna, *The $H=xp$ model revisited and the Riemann zeros*, Phys. Rev. Lett. **106** (2011) 200201.

[6] S. Coleman, *There are no Goldstone bosons in two dimensions*, Commun. Math. Phys. **31**, 259 (1973).

[7] N. D. Mermin and H. Wagner, *Absence of ferromagnetism or antiferromagnetism in one- or two-dimensional isotropic heisenberg models*, Phys. Rev. Lett. **17**, 1133 (1966).

[8] A. Odlyzko, *Tables of zeros of the Riemann zeta function*, www.dtc.umn.edu/~odlyzko/zeta-articles/.

[9] H. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic number theory, Proc. Sympos. Pure Math. XXIV, Providence, R.I.: American Mathematical
Society, pp. 181193, 1973.

[14] F. Dyson, *Correlations between eigenvalues of a random matrix*, Comm. Math. Phys. 19 (1970) 235.

[15] J. Derbyshire, *PRIME OBSESSION. Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*, Joseph Henry Press, Washington, D.C. 2008.