First-Order Perturbation Analysis of the SECSI Framework for the Approximate CP Decomposition of 3-D Noise-Corrupted Low-Rank Tensors

Sher Ali Cheema, Student Member, IEEE, Emilio Rafael Balda, Yao Cheng, Member, IEEE, Martin Haardt, Senior Member, IEEE, Amir Weiss, Student Member, IEEE, and Arie Yeredor, Senior Member, IEEE

Abstract—The Semi-Algebraic framework for the approximate Canonical Polyadic (CP) decomposition via Simultaneous matrix diagonalization (SECSI) is an efficient tool for the computation of the CP decomposition. The SECSI framework reformulates the CP decomposition into a set of joint eigenvalue decomposition (JEVD) problems. Solving all JEVDs, we obtain multiple estimates of the factor matrices and the best estimate is chosen in a subsequent step by using an exhaustive search or some heuristic strategy that reduces the computational complexity. Moreover, the SECSI framework retains the option of choosing the number of JEVDs to be solved, thus providing an adjustable complexity-accuracy trade-off. In this work, we provide an analytical performance analysis of the SECSI framework for the computation of the approximate CP decomposition of a noise corrupted low-rank tensor, where we derive closed-form expressions of the relative mean square error for each of the estimated factor matrices. These expressions are obtained using a first-order perturbation analysis and are formulated in terms of the second-order moments of the noise, such that apart from a zero mean, no assumptions on the noise statistics are required. Simulation results exhibit an excellent match between the obtained closed-form expressions and the empirical results. Moreover, we propose a new Performance Analysis based Selection (PAS) scheme to choose the final factor matrix estimate. The results show that the proposed PAS scheme outperforms the existing heuristics, especially in the high SNR regime.

Index Terms—Perturbation analysis, higher-order singular value decomposition (HOSVD), tensor signal processing.

I. INTRODUCTION

The Canonical Polyadic (CP) decomposition of $R$-way arrays is a powerful tool in multi-linear algebra. It allows to decompose a tensor into a sum of rank-one components. There exist many applications where the underlying signal of interest can be represented by a trilinear or multilinear CP model. These range from psychometrics and chemometrics over array signal processing and communications to biomedical signal processing, image compression or numerical mathematics [1]–[3]. In practice, the signal of interest in these applications is contaminated by the noise. Therefore, we only compute an approximate CP decomposition of the noisy signal.

Algorithms for the computation of an approximate CP decomposition from noisy observations are often based on Alternating Least Squares (ALS). These algorithms compute the CP decomposition in an iterative manner procedure [4], [5]. The main drawbacks of ALS-based algorithms is that the number of required iterations may be very large, and convergence is not guaranteed. Moreover, ALS based algorithms are less accurate in ill-conditioned scenarios, especially if the columns of the factor matrices are highly correlated. Alternatively, semi-algebraic solutions, where the CP decomposition is rephrased into a generic problem such as the Joint Eigenvalue Decomposition (JEVD) (also called Simultaneous Matrix Diagonalization (SMD)), have been proposed in the literature [2]. The link between the CP decomposition and the JEVD is discussed in [6] where it has been shown that the canonical components can be obtained from a simultaneous matrix diagonalization by congruence. A Semi-algebraic framework for CP decompositions via Simultaneous matrix diagonalization (SECSI) was presented in [7]–[9] for $R = 3$ dimensional tensors and was extended for tensors with $R > 3$ dimensions using the concept of generalized unfoldings (SECSI-GU) in [10]. The SECSI concept facilitates a distributed implementation on a parallel JEVDs to be solved depending upon the accuracy and the computational complexity requirements of the system. By solving all JEVDs, multiple estimates of the factor matrices are obtained. The selection of the best factor matrices from the resulting estimates can be obtained either by using an exhaustive search based best matching scheme or by using heuristic selection schemes with a reduced computational complexity. Several schemes with different accuracy-complexity trade-off points are presented in [9]. Thus, the SECSI framework results in more reliable estimates and also offers a flexible accuracy-complexity trade-off.

An analytical performance assessment of the semi-algebraic algorithms to compute an approximate CP decomposition is of considerable research interest. In the literature, the performance of the CP decomposition is often evaluated using Monte-Carlo simulations. To the best of our knowledge, there exists no analytical performance analysis of an approximate CP decomposition of noise-corrupted low-rank tensors in the literature. In this work, a first-order perturbation analysis of the SECSI framework is carried out, where apart from zero-mean and finite second order moments, no assumptions about the noise are required. The SECSI framework performs three
distinct step to compute the approximate CP decomposition of a noisy tensor, as summarized in Fig. [1]. First, the truncated higher order singular value decomposition (HOSVD) is used to suppress the noise. In the second step, several JEVDs are constructed from the core tensor. This results in several estimates of the factor matrices. Lastly, the best factor matrices are selected from these estimates by applying the best matching scheme or an appropriate heuristic scheme that has a lower computational complexity [9]. Hence, a perturbation analysis for each of the steps is required for the overall performance analysis of the SECSI framework.

In [11], we have already presented a first-order perturbation analysis of low-rank tensor approximations based on the truncated HOSVD. We have also performed the perturbation analysis of JEVD algorithms which are based on the indirect least squares (LS) cost function in [12]. In this paper, we extend our work to the overall performance analysis of the SECSI framework for 3-D tensors. Finally, we present closed-form expressions for the relative Mean Square Factor Error (rMSFE) for each of the estimates of the three factor matrices. These expressions are asymptotic in the SNR and are expressed in terms of the covariance matrix of the noise. Furthermore, we devise a new heuristic approach based on the performance analysis results to select the best estimates that we call Performance Analysis based Selection (PAS) scheme.

The remainder of the paper is organized as follows. In Section [II], we provide the data model. We perform the first-order perturbation analysis of the SECSI framework in terms of known noisy tensor in Section [III]. The closed-form rMSFE expressions for each of the factor matrices for the first JEVD (corresponding to the first column of Fig. [1]) are presented in Section [IV]. The results are extended for each of the factor matrices resulting from the second JEVD (corresponding to the second column of Fig. [1]) in Section [VII]. The results of the remaining JEVDs in Fig. [1] are obtained via permutations of the previous results that are discussed in Section [VI]. In Section [VII], we propose a new estimates selection scheme that is based on the performance analysis results. The simulation results are discussed in Section [VIII] and the conclusions are provided in Section [VIII].

Notation: For the sake of notation, we use a, a, A, and A for a scalar, column vector, matrix, and tensor, respectively, where, A(i, j, k) defines the element (i, j, k) of a tensor A. The same applies to a matrix A(i, j) and a vector a(i). The superscripts \(^{-1}\), \(^{+}\), \(^{*}\), \(^{T}\), and \(^{H}\) denote the matrix inverse, Moore-Penrose pseudo inverse, conjugate transposition, and conjugate transposition, respectively. We also use the notation \(\mathbb{E}\{\cdot\}\), \(\text{tr}\{\cdot\}\), \(\odot\), \(\circ\), \(\|\cdot\|_{F}\), and \(\|\cdot\|_{2}\) for the expectation, trace, Kronecker product, the Khatri-Rao (column-wise Kronecker) product, Frobenius norm, and 2-norm operators, respectively. Moreover, we define an operator \(\text{stack}\{\cdot\}\) that arranges the K vectors or matrices as

\[
\text{stack}\{A^{(k)}\} = \begin{bmatrix} A_1 \\ \vdots \\ A_K \end{bmatrix} \quad \forall k = 1, 2, \cdots, K 
\]  

For a matrix \(A = [a_1, a_2, \ldots, a_M] \in \mathbb{C}^{N \times M}\), the operator \(\text{vec}\{\cdot\}\) defines the vectorization operation as \(\text{vec}\{A\}^T = [a_1^T, a_2^T, \ldots, a_M^T]\). This operator has the property

\[
\text{vec}\{A \cdot X \cdot B\} = (B^T \odot A) \cdot \text{vec}\{X\}, \quad (2)
\]

where \(A, X, B\) are matrices with proper dimensions. Moreover, we define the diagonalization operator \(\text{diag}(\cdot)\) as in Matlab. Note that, when this operator is applied to a vector, the output is a diagonal matrix, and when applied to a matrix, the output is a column vector. Therefore, for the sake of clarity, we use the notation \(\text{Diag}(\cdot)\) when applying this operator to a vector, and \(\text{diag}(\cdot)\) when applying it to a matrix. Furthermore, we define the operators \(\text{Ddiag}(\cdot)\) and \(\text{Off}(\cdot)\) as

\[
\text{Ddiag}(X) = \text{Diag}(\text{diag}(X)) \in \mathbb{C}^{N \times N}, \\
\text{Off}(X) = X - \text{Diag}\left(\text{diag}(X)\right) \in \mathbb{C}^{N \times N},
\]

where \(X\) is a square matrix matrix of size \(N \times N\). Note that the \(\text{Ddiag}(\cdot)\) operator sets all the off-diagonal elements of \(X\) to zero, while the \(\text{Off}(\cdot)\) operator sets the diagonal elements of \(X\) to zero.

Let \(\mathcal{A} \in \mathbb{C}^{M_1 \times M_2 \times \cdots \times M_R}\) be a \(R\)-way tensor, where \(M_r\) is the size along the \(r\)-th dimension and \(|\mathcal{A}|_{(r)}\) denote the \(r\)-mode unfolding of \(\mathcal{A}\) which is performed according to the reverse cyclical order [13]. The \(r\)-mode product of a tensor \(\mathcal{A}\) with a matrix \(B \in \mathbb{C}^{N \times M_r}\) (i.e., \(\mathcal{A} \times_r B\)) is defined as

\[
C = \mathcal{A} \times_r B \iff |C|_{(r)} = B \cdot |\mathcal{A}|_{(r)},
\]

where \(C\) is a tensor with the corresponding dimensions. Moreover, if \(C = \mathcal{A} \times_1 X^{(1)} \times_2 X^{(2)} \times_3 \cdots \times_R X^{(R)}\), then

\[
|C|_{(r)} = X^{(r)} \cdot |\mathcal{A}|_{(r)}, \\
\left( X^{(r+1)} \otimes X^{(r+2)} \otimes \cdots \times_R X^{(1)} \otimes \cdots \otimes X^{(r-1)} \right)^T,
\]

where \(X^{(r)}\) for \(r = 1, 2, \ldots, R\) are matrices with the corresponding dimensions. For the sake of notational simplicity, we define the following notation for \(r\)-mode products

\[
\mathcal{A} \times_r^{(R)} X^{(r)} = \mathcal{A} \times_1 X^{(1)} \times_2 X^{(2)} \times_3 \cdots \times_R X^{(R)}.
\]

In addition, \(\|\cdot\|_H\) denotes the higher order norm of a tensor, defined as

\[
\|\mathcal{A}\|_H = \| |\mathcal{A}|_{(r)}\|_F = \|\text{vec}\{|\mathcal{A}|_{(r)}\}\|_2 \quad \forall r = 1, 2, \ldots, R.
\]

Moreover, the space spanned by the \(r\)-mode vectors is termed \(r\)-space of \(\mathcal{A}\), the rank of \(|\mathcal{A}|_{(r)}\) is the \(r\)-rank of \(\mathcal{A}\). Note that in general, the \(r\)-ranks (also referred to as the multilinear ranks) of a tensor \(\mathcal{A}\) can all be different. Furthermore, the tensor rank refers to the smallest possible \(r\) such that a tensor can be written as the sum of \(r\) rank-one tensors. Note that the tensor rank is not directly related to the multilinear ranks. Furthermore, \(e_{d,k}\) denotes the \(k\)-th standard basis column vector of size \(d\).
The SECSI framework can also be applied in the degenerate case where only a noise-corrupted version of the noiseless tensor $X$ is known, and the goal of the SECSI framework is to estimate the tensor reconstruction error induced by the low-rank approximation.

We consider the non-degenerate case first where $X$ is the noiseless low-rank tensor $\approx U_1 \times U_2 \times U_3$. The CP decomposition of such a low-rank noiseless tensor is given by

$$X_0 = T_3,1 \times U_1 \times U_2 \times U_3,$$

where $U^{(r)} \in \mathbb{C}^{M_1 \times M_2}$ for $r = 1, 2, 3$ is the factor matrix in the $r$-th mode and $T^{(3)}$ is the 3-way identity tensor of size $M_1 \times M_2 \times M_3$. In this work, we assume that the factor matrices are known, and the goal of the SECSI framework is to estimate them. For future reference, the SVD of the $r$-mode unfolding of the noiseless low-rank tensor $X_0$ in $\mathbb{C}^{M_1 \times M_2 \times M_3}$ is given as

$$[X][r] = U_r \cdot \Sigma_r \cdot V_r^H,$$

where $U^{(r)} \in \mathbb{C}^{M_1 \times \Sigma_r \times M_r}$, $\Sigma_r \in \mathbb{R}^{\Sigma_r \times M_r}$, and $V^{(r)} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$. The CP decomposition of such a low-rank noiseless tensor $X_0$ is superimposed by a zero-mean additive noise tensor $N \in \mathbb{C}^{M_1 \times M_2 \times M_3}$. The SVD of the $r$-mode unfolding of the observed noisy tensor $X$ is given as

$$[X][r] = \hat{U}_r \cdot \hat{\Sigma}_r \cdot \hat{V}_r^H,$$

where $\hat{U}_r \in \mathbb{C}^{M_1 \times \hat{\Sigma}_r \times M_r}$ and $\hat{V}_r \in \mathbb{C}^{M_1 \times M_2 \times M_3}$. The CP decomposition of such a low-rank noiseless tensor $X_0$ is superimposed by a zero-mean additive noise tensor

$$X = X_0 + N \in \mathbb{C}^{M_1 \times M_2 \times M_3}.$$

be the observed noisy tensor where the desired signal component $X_0$ is superimposed by a zero-mean additive noise tensor $N \in \mathbb{C}^{M_1 \times M_2 \times M_3}$. The SVD of the $r$-mode unfolding of the observed noisy tensor $X$ is given as

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where $\hat{U}_r \in \mathbb{C}^{M_1 \times \hat{\Sigma}_r \times M_r}$ and $\hat{V}_r \in \mathbb{C}^{M_1 \times M_2 \times M_3}$. The CP decomposition of such a low-rank noiseless tensor $X_0$ is superimposed by a zero-mean additive noise tensor

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$$X = X_0 + N \in \mathbb{C}^{M_1 \times M_2 \times M_3}.$$
The perturbation present in the r-mode signal subspace estimate $\hat{U}_r$ is given by \[14\]
\[
\Delta U_r = U_r \cdot U_r^\text{H} \cdot [N]_{(r)} \cdot \Sigma_r^{-1} + O(\Delta^2)
\]
\[
= \Gamma_r \cdot [N]_{(r)} \cdot \Sigma_r^{-1} + O(\Delta^2),
\]
(10)
where $\Gamma_r \triangleq U_r \cdot U_r^\text{H}$ and all higher order terms are contained in $O(\Delta^2)$. Furthermore, we use this result to expand the expression for the truncated core tensor $\hat{S}[s] = \chi^3_r \cdot \hat{U}_r^\text{H}$ as
\[
\hat{S}[s] + \Delta \hat{S}[s] = (\chi_0 \cdot N) \chi^3_r \left( U_r^\text{H} + \Delta U_r^\text{H} \right) + O(\Delta^2)
\]
\[
= \chi_0 \chi^3_r U_r^\text{H} + N \chi^3_r U_r^\text{H}
\]
\[
+ \chi_0 \cdot \Delta U_r^\text{H} \times \chi^2_r U_r^\text{H} \times \chi^3_r U_r^\text{H}
\]
\[
+ \chi_0 \cdot \Delta U_r^\text{H} \times \chi^2_r \Delta U_r^\text{H} \times \chi^3_r U_r^\text{H}
\]
\[
+ \chi_0 \cdot \Delta U_r^\text{H} \times \chi^2_r \Delta U_r^\text{H} \times \chi^3_r U_r^\text{H} + O(\Delta^2).
\]
Note that all terms that include products of more than one “$\Delta$ term” are included in $O(\Delta^2)$. Moreover, $N$ is also considered as a “$\Delta$ term”. In the noiseless case, the truncated core tensor $\hat{S}[s]$ is equal to $\hat{S}[s] \triangleq \chi^3_r \cdot \hat{U}_r^\text{H}$. Using the definitions in eq. \[4\] and eq. \[10\], the above expression simplifies to
\[
\Delta \hat{S}[s] = N \chi^3_r U_r^\text{H} + O(\Delta^2).
\]
\[11\]
B. Perturbation of the JEVD Estimates

For a 3-way array, we can construct up to 6 JEVD problems in the SECSI framework \[9\] that are obtained from
\[
\hat{S}_1[s] = \hat{S}[s] \times_1 \hat{U}_1[s] \in \mathbb{C}^{M_1 \times d \times d}
\]
\[12\]
\[
\hat{S}_2[s] = \hat{S}[s] \times_2 \hat{U}_2[s] \in \mathbb{C}^{n \times M_2 \times d}
\]
\[13\]
\[
\hat{S}_3[s] = \hat{S}[s] \times_3 \hat{U}_3[s] \in \mathbb{C}^{d \times d \times M_3},
\]
\[14\]
respectively. As an example, let us consider the two JEVD problems constructed from eq. \[14\]. Here the noisy estimate \[\hat{S}_3\] can be expressed as
\[
\hat{S}_3[s] = \hat{S}_3[s] + \Delta \hat{S}_3[s].
\]
\[15\]
Using eq. \[14\], we get
\[
\hat{S}_3[s] + \Delta \hat{S}_3[s] = (\hat{S}[s] + \Delta \hat{S}[s]) \times_3 \left( U_3[s] + \Delta U_3[s] \right)
\]
\[
= \hat{S}_3[s] + \Delta \hat{S}[s] \times_3 U_3[s] + \hat{S}[s] \times_3 \Delta U_3[s] + O(\Delta^2).
\]
Therefore, we have
\[
\Delta \hat{S}_3[s] = \hat{S}[s] \times_3 \Delta U_3[s] + \hat{S}[s] \times_3 \Delta U_3[s] + O(\Delta^2).
\]
\[16\]
According to \[7\], \[9\], we define the 3-mode slices of $\hat{S}_3$ as
\[
\hat{S}_{3,k} \triangleq \hat{S}_3[s] \times_3 e_k^T, \quad k = 1, 2, \ldots, M_3,
\]
\[17\]
where $\hat{S}_{3,k}$ represents the $k$-th slice (along the third dimension) of $\hat{S}_3$ in eq. \[14\]. As explained in \[9\], these slices satisfy
\[
\text{Diag} \left\{ F^{(3)}(k,:) \right\} \approx T_1^{-1} \cdot \hat{S}_{3,k} \cdot T_2^{-1},
\]
\[18\]
where $T_1$ and $T_2$ are transformation matrices obtained by solving the associated JEVD problems. In the noiseless case, the factor matrices are related to the signal subspaces via these transformation matrices as $F^{(r)} = U_r \cdot T_r$, $r = 1, 2, 3$. Defining the perturbation in the $k$-th slice, we get
\[
\hat{S}_{3,k} \triangleq S_{3,k} + \Delta S_{3,k}, \quad \forall k = 1, 2, \ldots, M_3.
\]
\[19\]
Using eq. \[17\], this results in
\[
\Delta S_{3,k} = \Delta \hat{S}_3[s] \times_3 e_k^T + O(\Delta^2).
\]
\[20\]
According to \[9\], we select the slice of $\hat{S}_3[s]$ with the lowest condition number, i.e., $\hat{S}_{3,p}$ where $p = \text{argmin}_k \{ \text{cond}\left( \hat{S}_{3,k} \right) \}$ and cond($\cdot$) denotes the condition number operator. This leads to two sets of matrices, namely the right-hand-side (rhs) set and the left-hand-side (lhs) set that are defined as
\[
\hat{S}_{3,k}^{\text{rhs}} \triangleq \hat{S}_{3,k} \cdot \hat{S}_{3,p}^{-1}, \quad \forall k = 1, 2, \ldots, M_3
\]
\[21\]
\[
\hat{S}_{3,k}^{\text{lhs}} \triangleq \left( \hat{S}_{3,p} \cdot \hat{S}_{3,k} \right)^T, \quad \forall k = 1, 2, \ldots, M_3.
\]
\[22\]
As an example, we compute the perturbation in eq. \[21\]. To this end, let us obtain the perturbation in $\hat{S}_{3,3,k}$ as
\[
\hat{S}_{3,3,k}^{\text{rhs}} \triangleq \hat{S}_{3,3,k}^{\text{rhs}} + \Delta \hat{S}_{3,3,k}^{\text{ rhs}}, \quad \forall k = 1, 2, \ldots, M_3.
\]
\[23\]
Using this definition, we now expand eq. \[21\], as
\[
\hat{S}_{3,3,k}^{\text{rhs}} + \Delta \hat{S}_{3,3,k}^{\text{ rhs}} = (S_{3,k} + \Delta S_{3,k}) \cdot (S_{3,p} + \Delta S_{3,p})^{-1} + O(\Delta^2)
\]
\[
\text{Using the Taylor’s expansion of the matrix inverse, we get}
\]
\[
(S_{3,p} + \Delta S_{3,p})^{-1} = S_{3,p}^{-1} - \Delta S_{3,p} \cdot S_{3,p}^{-1} + O(\Delta^2).
\]
\[24\]
According to eq. \[23\], the perturbation in the slices $\hat{S}_{3,3,k}^{\text{rhs}}, \forall k = 1, 2, \ldots, M_3$ is given by
\[
\Delta S_{3,k} = \hat{S}_{3,k}^{\text{rhs}} \cdot S_{3,p}^{-1} - S_{3,k} \cdot \Delta S_{3,p} \cdot S_{3,p}^{-1} + O(\Delta^2).
\]
\[25\]
Using the results in eq. \[18\], it is easy to show that the two sets of matrices in eq. \[21\] and eq. \[22\] correspond to the following JEVD problems
\[
\hat{S}_{3,k}^{\text{rhs}} \approx T_1 \cdot \hat{D}_{3,k} \cdot T_1^{-1}, \quad \forall k = 1, 2, \ldots, M_3,
\]
\[26\]
\[
\hat{S}_{3,k}^{\text{lhs}} \approx T_2 \cdot \hat{D}_{3,k} \cdot T_2^{-1}, \quad \forall k = 1, 2, \ldots, M_3,
\]
\[27\]
respectively, where the diagonal matrices $\hat{D}_{3,k}$ are defined as
\[
\hat{D}_{3,k} \triangleq \text{Diag} \left\{ F^{(3)}(k,:) \right\} \cdot \text{Diag} \left\{ F^{(3)}(p,:) \right\}^{-1}.
\]
joint diagonalization algorithms based on the indirect least squares (LS) cost function such as Sh-Rt [15], JDTM [16], or the coupled JEVD [17]. In [12], we have presented a first order perturbation analysis of JEVD algorithms that are based on the indirect LS cost function. We can use these results to obtain analytical expressions for the perturbation in the estimates \( \hat{T}_1, \hat{T}_2, \) and \( \hat{D}_{3,k}, \forall k = 1, 2, \ldots, M_3. \) To this end, the perturbations in the \( \hat{T}_1, \hat{T}_2, \) and \( \hat{D}_{3,k} \) estimates are defined as

\[
\hat{T}_1 \triangleq T_1 + \Delta T_1
\]

\[
\hat{T}_2 \triangleq T_2 + \Delta T_2
\]

\[
\hat{D}_{3,k} \triangleq D_{3,k} + \Delta D_{3,k}, \quad \forall k = 1, 2, \ldots, M_3.
\]

According to [12], we also define the following matrices

\[
B_0 = J_{(d)} \cdot (T_1^T \otimes T_1^{-1})
\]

\[
s_k = \text{vec}\left\{ \Delta S_{3,k}^{\text{rhs}} \right\}
\]

\[
A_k = J_{(d)} \cdot \left( (I_M \otimes T_1^{-1} \cdot S_{3,k}^{\text{rhs}}) - (D_{3,k} \otimes T_1^{-1}) \right),
\]

where \( J_{(d)} \in \{0,1\}^{d \times d^2} \) is a selection matrix that satisfies the relation \( \text{vec}\{\text{Off}(X)\} = J_{(d)} \cdot \text{vec}(X) \), for any given \( X \in C^{d \times d} \). After defining the quantities

\[
A = \begin{bmatrix} A_1 & A_2 & \cdots & A_{M_3} \end{bmatrix}, \quad B = I_{M_3} \otimes B_0, \quad s = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_{M_3} \end{bmatrix},
\]

we can use the results obtained in [12]. This leads to

\[
\text{vec}\{\Delta T_1\} = -A^+ \cdot B \cdot s + O(\Delta^2)
\]

\[
\Delta D_{3,k} = \text{Diag}\left(T_1^{-1} \cdot \Delta S_{3,k}^{\text{rhs}} \cdot T_1\right) + O(\Delta^2),
\]

for the rhs JEVD problem.

C. Perturbation of the Factor Matrix Estimates

Using the SECSI framework for a 3-way tensor, we can get up to six different estimates for each factor matrix. The factor matrix \( F^{(1)} \) can be estimated from the transform matrix \( \hat{T}_1 \) via \( \hat{F}^{(1)} = U_1^{(s)} \cdot \hat{T}_1 \) by using the result of the JEVD defined in eq. (25). Expanding this equation leads to

\[
\hat{F}^{(1)} = \left( U_1^{(s)} + \Delta U_1^{(s)} \right) \cdot (T_1 + \Delta T_1) + O(\Delta^2)
\]

\[
= F^{(1)} + \Delta U_1^{(s)} \cdot T_1 + U_1^{(s)} \cdot \Delta T_1 + O(\Delta^2).
\]

Again, we express the perturbation in \( \hat{F}^{(1)} \) as

\[
\hat{F}^{(1)} = F^{(1)} + \Delta F^{(1)}. \tag{37}
\]

This results in an expression for the perturbation \( \Delta F^{(1)} \) in the first factor matrix as

\[
\Delta F^{(1)} = \Delta U_1^{(s)} \cdot T_1 + U_1^{(s)} \cdot \Delta T_1 + O(\Delta^2). \tag{38}
\]

Using the result of the JEVD defined in eq. (25), a scaled version of the \( k \)th row of the factor matrix \( \hat{F}^{(3)} \) can be estimated from the diagonal of \( D_{3,k}, \forall k = 1, 2, \ldots, M_3 \), according to eq. (27) as

\[
\hat{F}^{(3)}(k,:) = \text{diag}\left(D_{3,k} \right)^T
\]

\[
= \text{diag}\left(D_{3,k} + \Delta D_{3,k}\right)^T
\]

\[
= \hat{F}^{(3)}(k,:) + \text{diag}\left(\Delta D_{3,k}\right)^T + O(\Delta^2),
\]

where the perturbation in \( \hat{F}^{(3)}(k,:) \) is defined as

\[
\hat{F}^{(3)}(k,:) = F^{(3)}(k,:) + \Delta F^{(3)}(k,:),
\]

which results in

\[
\Delta F^{(3)}(k,:) = \text{diag}\left(\Delta D_{3,k}\right)^T + O(\Delta^2).
\]

To get an expression of the corresponding factor matrix estimate \( \hat{F}^{(3)} \), we take into account eq. (27) via \( \hat{F}^{(3)} = \hat{F}^{(3)} \circ \hat{F}^{(3)} \cdot \text{Diag}\left(F^{(3)}(p,:\right) \cdot \hat{F}^{(3)}(p,:), \hat{F}^{(3)}(p,:), \hat{F}^{(3)}(p,:), \hat{F}^{(3)}(p,:), \hat{F}^{(3)}(p,:), \hat{F}^{(3)}(p,:). \) This leads to

\[
\hat{F}^{(3)} = F^{(3)} + \Delta F^{(3)}, \tag{39}
\]

where

\[
\Delta F^{(3)} = \Delta \hat{F}^{(3)} \cdot \text{Diag}\left(F^{(3)}(p,:), \hat{F}^{(3)}(p,:), \hat{F}^{(3)}(p,:), \hat{F}^{(3)}(p,:), \hat{F}^{(3)}(p,:), \hat{F}^{(3)}(p,:) \right). \tag{40}
\]
Using this equation, the perturbation of the $k$th row of $\Delta F^{(3)}$ is obtained as
\[
\Delta F^{(3)}(k,:) = \text{diag}(\Delta D_{3,k})^T \cdot \text{Diag}(F^{(3)}(p,:)) + O(\Delta^2).
\]
(41)

The factor matrix $F^{(2)}$ can be estimated via a LS fit. To this end, we define the LS estimate of $F^{(2)}$ as
\[
\hat{F}^{(2)} = [\mathcal{X}]_{(2)} \cdot (F^{(3)} \circ \hat{F}^{(1)})^T + T.
\]

Using equations (37) and (39), we get:
\[
\hat{F}^{(2)} = \left[\mathcal{X}_0 + \mathcal{N}\right]_{(2)} \cdot \left(F^{(3)} + \Delta F^{(3)} \circ (F^{(1)} + \Delta F^{(1)})\right)^T.
\]
Since $\hat{F}^{(2)} = F^{(2)} + \Delta F^{(2)}$, we finally calculate $\Delta F^{(2)}$ as
\[
\Delta F^{(2)} = \hat{F}^{(2)} - F^{(2)} = \left[\mathcal{X}_0 + \mathcal{N}\right]_{(2)} \cdot \left(F^{(3)} + \Delta F^{(3)} \circ (F^{(1)} + \Delta F^{(1)})\right)^T - F^{(2)} + O(\Delta^2).
\]

Using the Taylor expansion, we get
\[
\Delta F^{(2)} = \left[\mathcal{X}_0\right]_{(2)} \cdot \left[F^{(3)} \circ F^{(1)} + (\Delta F^{(3)} \circ F^{(1)}) + \frac{(F^{(3)} \circ \Delta F^{(1)})}{2} + \left[F^{(3)} \circ F^{(1)}\right]^T + \mathcal{N}_{(2)} \cdot \left[F^{(3)} \circ F^{(1)}\right]^T + O(\Delta^2)\right.
\]
\[
\left. - F^{(2)} + O(\Delta^2) \right] = \left[\mathcal{X}_0\right]_{(2)} \cdot \left[F^{(3)} \circ F^{(1)} + (\Delta F^{(3)} \circ F^{(1)}) + \frac{(F^{(3)} \circ \Delta F^{(1)})}{2} + \left[F^{(3)} \circ F^{(1)}\right]^T + \mathcal{N}_{(2)} \cdot \left[F^{(3)} \circ F^{(1)}\right]^T + O(\Delta^2)\right].
\]

In the same manner, another set of factor matrix estimates can be obtained by solving the lhs of the JVD problem for $T_2$ and $D_{3,k}$, $k = 1, 2, \ldots, M_3$ in eq. (26). This leads to the two sets of estimates (lhs and lhs) in the third mode, as summarized in Table 1. Note that the obtained expressions can be directly used for the first and second mode estimates obtained from eq. (12) and eq. (13), respectively, since such estimates can be derived by applying the SECSI framework to a permuted version of $\mathcal{X}$. For example, we can obtain the first mode estimates, corresponding to eq. (12), by applying the SECSI framework on the third mode of permute ($\mathcal{X}$, $[2, 3, 1]$), where permute ($\mathcal{A}$, ORDER) rearranges the dimensions of $\mathcal{A}$ so that they are in the order specified by the vector ORDER (as defined in Matlab). In the same manner, the second mode estimates, corresponding to eq. (13), are obtained by using the SECSI framework on the third mode of permute ($\mathcal{X}$, $[1, 3, 2]$). Therefore, we obtain a total of six estimates for each factor matrix (two from each mode), as shown in Fig. 7. To select the final estimates, we can use best matching scheme or any low-complexity heuristic alternative that has been discussed in [9].

IV. CLOSED-FORM RMSFE EXRESSIONS

In this section, we present an analytical rMSFE expression for each of the factor matrices estimates. We first introduce some definitions which will be used subsequently to derive the analytical expressions. Additionally, Theorem 1 will be used to resolve the scaling ambiguity in the factor matrices of the CP decomposition.

A. Preliminary Definitions

Let $P^{(r)}_{(M_1, M_2, M_3)} \in \{0, 1\}^{(M_1 \cdot M_2 \cdot M_3) \times (M_1 \cdot M_2 \cdot M_3)}$ be the $r$-to-1 mode permutation matrix of any third order tensor $Z \in \mathbb{C}^{M_1 \times M_2 \times M_3}$. This means that the permutation matrix $P^{(r)}_{(M_1, M_2, M_3)}$ satisfies the property
\[
\text{vec} \{[Z]_{(r)}\} = P^{(r)}_{(M_1, M_2, M_3)} \cdot \text{vec} \{[Z]_{(1)}\}.
\]
(43)

Note that $P^{(1)}_{(M_1, M_2, M_3)} = I_{(M_1 \cdot M_2 \cdot M_3)}$. In the same manner, $\hat{P}^{(2)}_{(M_1, M_2, M_3)} \in \{0, 1\}^{(M_1 \cdot M_2) \times (M_1 \cdot M_2)}$ be the permutation matrix that satisfies the following relation for any $Z \in \mathbb{C}^{M_1 \times M_2}$
\[
\text{vec} \{Z^T\} = Q^{(2)}_{(M_1, M_2, M_3)} \cdot \text{vec} \{Z\}.
\]
(44)

Additionally, let $n_r \triangleq \text{vec} \{[N]_{(r)}\}$ be the $r$-mode noise vector, with $R^{(r)}_{dn} \triangleq \mathbb{E}\{n_r n_r^T\}$ and $C^{(r)}_{dn} \triangleq \mathbb{E}\{n_r n_r^T\}$ being the corresponding $r$-mode covariance and pseudo-covariance matrices, respectively. Note that the these covariance matrices $R^{(r)}_{dn}$ are permuted versions of $R^{(1)}_{dn}$ since $n_r = P^{(r)}_{(M_1, M_2, M_3)} \cdot n_1$. This property is also satisfied for the pseudo-covariance matrices $C^{(r)}_{dn}$.

Let $W_{(d)} \in \{0, 1\}^{d^2 \times d^2}$ be the diagonal elements selection matrix defined as
\[
\text{vec} \{\text{Diag} \{Z\}\} = W_{(d)} \cdot \text{vec} \{Z\} \in \mathbb{C}^{d \times 1},
\]
(45)
where $Z \in \mathbb{C}^{d \times d}$ is a square matrix. Note that $W_{(d)}$ is simply $I_{d^2} - J_{(d)}$, where $J_{(d)}$ has already been defined below eq. (33). Likewise, let $W^\text{red}_{(d)} \in \{0, 1\}^{d \times d}$ be the reduced dimensional diagonal elements selection matrix that selects only the diagonal elements i.e.,
\[
\text{diag} \{Z\} = W^\text{red}_{(d)} \cdot \text{vec} \{Z\} \in \mathbb{C}^{d \times 1}
\]
(46)
for any square matrix $Z \in \mathbb{C}^{d \times d}$.

Theorem 1. Let $Z \in \mathbb{C}^{d \times d}$ and $\Delta Z \in \mathbb{C}^{d \times d}$ be two matrices where the norm of each column in $\Delta Z$ is much smaller than the corresponding column in $Z$. Let $\tilde{P} \in \mathbb{C}^{d \times d}$ be a diagonal matrix that introduces a scaling ambiguity in $(Z + \Delta Z)$. Then, a diagonal matrix $P_{opt}$ that resolve this scaling ambiguity can be expressed as
\[
P_{opt} = \arg\min_{\tilde{P} \in \mathcal{M}_D(d)} \|Z - (Z + \Delta Z) \cdot \tilde{P} \cdot P\|_F^2,
\]
where $\mathcal{M}_D(d)$ is the set of $d \times d$ diagonal matrices. Thus, the following relation holds
\[
Z - (Z + \Delta Z) \cdot \tilde{P} \cdot P_{opt} = Z \cdot \text{Diag} \{(Z^H \cdot \Delta Z) \cdot K^{-1} - \Delta Z + O(\Delta^2),
\]
where $K = \text{Ddiag}(Z^\top \cdot Z) \in \mathbb{R}^{d \times d}$ is a diagonal matrix.

Proof. cf. Appendix A

B. Factor Matrices rMSFE Expressions

In this section, we derive closed-form rMSFE expressions for the three factor matrices. As an example, we consider these closed-form rMSFE expressions using the JEV of the rhs tensor slices in eq. (23). In Section IV C, we also present the results for the lhs tensor slices in eq. (26). The rMSFE in the r-th factor matrix is defined as

$$
\text{rMSFE}^{(r)} = \mathbb{E} \left\{ \min_{\hat{P}^{(r)} \in \mathcal{M}_{PD}(d)} \frac{\| F^{(r)} - \hat{F}^{(r)} \cdot P^{(r)} \|^2_F}{\| F^{(r)} \|^2_F} \right\},
$$

where $\mathcal{M}_{PD}(d)$ is the set of $d \times d$ permuted diagonal matrices (also called monomial matrices), i.e., the matrices $P^{(r)}$ correct the permutation and scaling ambiguity that is inherent in the estimation of the loading matrices in eq. (9) and $F^{(r)}$ is the true factor matrix. The goal of the SECSI framework is to estimate these factor matrices up to the inevitable ambiguities, and our goal in the performance analysis is to predict the resulting relative errors, assuming (for the sake of the analysis) that the true matrices are known and that these ambiguities are resolved. Consequently, the rMSFE measures how accurately the actual CP model can be estimated from the noisy observations. After correcting the permutation ambiguity, the factor matrix estimates that we get from Monte-Carlo simulations can be approximated as

$$
\hat{F}^{(r)} = (F^{(r)} + \Delta F^{(r)}) \cdot \hat{P}^{(r)},
$$

where $\Delta F^{(r)}$ represents the perturbation that can be obtained from the performance analysis and $\hat{P}^{(r)}$ is a diagonal matrix modeling the scaling ambiguity. By using this relation, we rewrite the rMSFE definition in eq. (47) as

$$
\text{rMSFE}^{(r)} = \mathbb{E} \left\{ \frac{\| F^{(r)} - (F^{(r)} + \Delta F^{(r)}) \cdot \hat{P}^{(r)} \cdot P^{(r)} \|^2_F}{\| F^{(r)} \|^2_F} \right\},
$$

where $P^{(r)}$ is the optimal column scaling matrix that can be obtained from Theorem I and the definitions in eq. (45) and eq. (44), to get

$$
\text{vec} \left\{ F^{(r)} - (F^{(r)} + \Delta F^{(r)}) \cdot \hat{P}^{(r)} \cdot P^{(r)} \right\} 
\approx \text{vec} \left\{ F^{(r)} \cdot \text{Ddiag} \left( F^{(r)H} \cdot \Delta F^{(r)} \cdot K_r^{-1} - \Delta F^{(r)} \right) \right\} 
\approx \text{vec} \left\{ F^{(r)} \cdot (I_d \otimes F^{(r)}) \cdot (K_r^{-1} \otimes I_d) \right\} 
= \text{vec} \left\{ F^{(r)} \cdot (I_d \otimes F^{(r)}) \cdot (K_r^{-1} \otimes I_d) \right\} 
= \text{vec} \left\{ F^{(r)} \cdot (I_d \otimes F^{(r)} \cdot (K_r^{-1} \otimes I_d \cdot W_{(d)})) \right\}.
$$

where $K_r = \text{Ddiag}(F^{(r)H} \cdot F^{(r)})$. Note that the resulting expression contains the vectorization of the perturbation in the respective factor matrix estimates. As shown in Section III-C, the perturbations in the three factor matrix estimates (eq. (38), eq. (42), and eq. (40)) are a function of different perturbations, i.e., $\Delta U^{[s]}_r$ in eq. (10), $\Delta S^{[s]}_r$ in eq. (11), $\Delta S^{[s]}_n$ in eq. (16), $\Delta S^{[s]}_{n,k}$ in eq. (29), $\Delta S^{[s]}_{k,k}$ in eq. (24), and $\Delta D_{3,k}$ in eq. (36). Therefore, to get final closed-form rMSFE expressions for the three factor matrices in eq. (49), we vectorize all of the perturbation expressions. We start by applying the vectorization operator to $\Delta U^{[s]}_r$ in eq. (10) and use the r-to-1 mode permuation matrix in eq. (43) to get

$$
\text{vec} \left\{ \Delta U^{[s]}_r \right\} = (\Sigma^{[-1]}_r \cdot V^{[s]T}_r \otimes \Gamma^{[s]}_r) \cdot n_1 + O(\Delta^2).
$$

Next, we vectorize the 1-mode unfolding of $\Delta S^{[s]}_n$ in eq. (11), by using the definition in eq. (43) to obtain

$$
\text{vec} \left\{ [\Delta S^{[s]}_n]_{(1)} \right\} = \text{vec} \left\{ \left[ n^3 \times \times U_{r}^{[s]H} \right] \right\} + O(\Delta^2)
= (U^{[s]H}_2 \otimes U^{[s]H}_3 \otimes U^{[s]H}_1) \cdot n_1 + O(\Delta^2).
$$

In the same manner, we expand $\text{vec} \left\{ [\Delta S^{[s]}_n]_{(1)} \right\}$ using eq. (50) and eq. (51) to get

$$
\text{vec} \left\{ [\Delta S^{[s]}_n]_{(1)} \right\} = \text{vec} \left\{ \left[ \times \times \times U_3^{[s]H} \right] \right\} + O(\Delta^2)
= (P^{(3)}_{(d,d,3)}) \cdot \text{vec} \left\{ [\Delta S^{[s]}_n]_{(3)} \right\}
= (P^{(3)}_{(d,d,3)}) \cdot \text{vec} \left\{ [\Delta S^{[s]}_n]_{(3)} \right\}
= (P^{(3)}_{(d,d,3)}) \cdot \text{vec} \left\{ [\Delta S^{[s]}_n]_{(3)} \right\}
= \left( [\Delta S^{[s]}_n]_{(3)} \cdot \Sigma^{[-1]}_3 \cdot V^{[s]T}_3 \otimes \Gamma^{[s]}_3 \right) \cdot P^{(3)}_{(d,d,3)} \cdot L_0 \cdot n_1 + O(\Delta^2)
= L_1 \cdot n_1 + O(\Delta^2),
$$

where

$$
L_1 = P^{(3)}_{(d,d,3)} \cdot \left( I_d \otimes U^{[s]}_3 \right) \cdot P^{(3)}_{(d,d,3)} \cdot L_0 \cdot n_1 + P^{(3)}_{(d,d,3)}
$$

Furthermore, we can use this result to obtain an expression for the vectorization of each slice $\Delta S^{[s]}_{3,k}$, $\forall k = 1, 2, \ldots, M_3$ of
\( \Delta S^{[s]} \). Using eq. (20) and eq. (52), we derive an expression for \( \vec{\{\Delta S_{3,k}\}} \) as

\[
\vec{\{\Delta S_{3,k}\}} = \vec{\{\Delta S_3^{[s]} \times_3 e_{M_3,k}^T\}}
= \vec{\{\Delta S_3^{[s]} \times_3 e_{M_3,k}^T(1)\}}
= \vec{\{\Delta S_3^{[s]}(1) \cdot (I_d \otimes e_{M_3,k}^T)^T\}} + O(\Delta^2)
= (I_d \otimes e_{M_3,k}^T \otimes I_d) \cdot L_1 \cdot n_1
= L_2^{(k)} \cdot n_1 + O(\Delta^2),
\]

(54)

where

\[
L_2^{(k)} \triangleq (I_d \otimes e_{M_3,k}^T \otimes I_d) \cdot L_1
\]

(55)

for \( k = 1, 2, \ldots, M_3 \). Next, we use this result and eq. (11) to expand the vectorization of eq. (24). This leads to

\[
\vec{\{\Delta S_{3,k}^{\text{rhs}}\}} =
\vec{\{\Delta S_{3,k} \cdot S_{3,p}^{-1} \cdot S_{3,k} \cdot S_{3,p} \cdot \Delta S_{3,p} \cdot S_{3,p}^{-1}\}} + O(\Delta^2)
= (S_{3,p}^{-T} \otimes I_d) \cdot \vec{\{\Delta S_{3,k}\}} - (S_{3,p}^{-T} \otimes S_{3,k} \cdot S_{3,p}^{-1}) \cdot \vec{\{\Delta S_{3,p}\}} + O(\Delta^2)
= (S_{3,p}^{-T} \otimes I_d) \cdot L_2^{(k)} \cdot n_1 - (S_{3,p}^{-T} \otimes S_{3,k} \cdot S_{3,p}^{-1}) \cdot L_2^{(p)} \cdot n_1
+ O(\Delta^2)
= L_3^{(k)} \cdot n_1 + O(\Delta^2),
\]

(56)

where

\[
L_3^{(k)} \triangleq (S_{3,p}^{-T} \otimes I_d) \cdot L_2^{(k)} - (S_{3,p}^{-T} \otimes S_{3,k} \cdot S_{3,p}^{-1}) \cdot L_2^{(p)} .
\]

(57)

Next, we stack the column vectors \( s_k = \vec{\{\Delta S_{3,k}^{\text{rhs}}\}} \) into the vector \( s \), as defined in eq. (34), and use the previous result to obtain

\[
s = L_3 \cdot n_1 + O(\Delta^2),
\]

(58)

where \( L_3 = \begin{bmatrix} L_3^{(1)} & L_3^{(2)} & \cdots & L_3^{(M_3)} \end{bmatrix}^T \). This expression for \( s \) is used to expand eq. (35) into

\[
\vec{\{\Delta T_1\}} = -A^+ \cdot B \cdot s + O(\Delta^2)
= -A^+ \cdot B \cdot L_3 \cdot n_1 + O(\Delta^2).
\]

(59)

Since \( \text{diag}(\text{Ddiag}(Z)) = \text{diag}(Z) \) for any matrix \( Z \), we use the matrix \( W_{(d)}^{\text{red}} \) (defined in eq. (46)) and eq. (36) to vectorize eq. (41) as

\[
\vec{\{\Delta F^{(3)}(k, :)}\} = \vec{\{\text{diag}(\Delta D_{3,k})^T \cdot \text{diag}(F^{(3)}(p, :))\}}
= \vec{\{\text{diag}(\text{Ddiag}(T_1^{-1} \cdot \Delta S_{3,k}^{\text{rhs}} \cdot T_1)^T \cdot \text{diag}(F^{(3)}(p, :))\}}
+ O(\Delta^2)
= \vec{\{\text{diag}(T_1^{-1} \cdot \Delta S_{3,k}^{\text{rhs}} \cdot T_1)^T \cdot \text{diag}(F^{(3)}(p, :))\}} + O(\Delta^2)
= \vec{\{\text{diag}[T_1^{-1} \cdot \Delta S_{3,k}^{\text{rhs}} \cdot T_1 \cdot \text{diag}(F^{(3)}(p, :))\}} + O(\Delta^2)
= W_{(d)}^{\text{red}} \cdot \vec{\{\text{diag}(F^{(3)}(p, :)) \cdot T_1^{-1} \cdot \text{diag}(\Delta S_{3,k}^{\text{rhs}})\}} + O(\Delta^2)
= W_{(d)}^{\text{red}} \cdot \left(\text{diag}(F^{(3)}(p, :)) \cdot T_1^{-1}\right) \cdot L_3^{(k)} \cdot n_1 + O(\Delta^2)
= L_4^{(k)} \cdot n_1 + O(\Delta^2),
\]

(60)

where

\[
L_4^{(k)} \triangleq W_{(d)}^{\text{red}} \cdot \left(\text{diag}(F^{(3)}(p, :)) \cdot T_1^{-1}\right) \cdot L_3^{(k)}.
\]

(61)

The expression for each row of \( \Delta F^{(3)} \) in eq. (60) can be used to obtain an expression for the vectorization of \( \Delta F^{(3)} \). To this end, we use the permutation matrix \( Q_{(M_3,d)} \), defined in eq. (44), to express \( \vec{\{\Delta F^{(3)}\}} \) as

\[
\vec{\{\Delta F^{(3)}\}} = Q_{(M_3,d)}^T \cdot \vec{\{\Delta F^{(3)}(k, :)}\}
= Q_{(M_3,d)}^T \cdot \text{stack}\{\vec{\{\Delta F^{(3)}(k, :)\}}\}
= L_5 \cdot n_1 + O(\Delta^2),
\]

(62)

where \( k = 1, 2, \ldots, M_3 \) and

\[
L_5 = Q_{(M_3,d)}^T \cdot \text{stack}\{L_4^{(k)}\}
\]

(63)

Then we insert eq. (62) in eq. (49) to get

\[
\vec{\{F^{(3)} - (F^{(3)} + \Delta F^{(3)}) \cdot \hat{F}^{(3)} \cdot P^{(3)}_{\text{opt}}\}} \approx L_{F_3} \cdot n_1,
\]

(64)

where

\[
L_{F_3} = \left( I_d \otimes F^{(3)} \right) \cdot \left( K_{S}^{-1} \otimes I_d \right) \cdot W_{(d)}
\]

(65)

The final closed-form rMSFE\(_{F_3} \) expression can be approximated to

\[
r\text{MSFE}_{F_3} = \frac{\text{tr}\left(L_{F_3} \cdot R_{kn}^{(1)} \cdot L_{F_3}^H \right)}{\left\| F^{(3)} \right\|^2}.
\]

(66)
Similarly, vectorization of the perturbation in the first factor matrix estimate $\Delta F^{(1)}$ is performed by using eq. (38), eq. (50), and eq. (59) as

$$\text{vec} \{ \Delta F^{(1)} \} = \text{vec} \left \{ \Delta U_{1}^{[b]} \cdot T_{1} + U_{1}^{[b]} \cdot \Delta T_{1} \right \} + \mathcal{O}(\Delta^2)$$

$$= \left ( I_{d} \otimes U_{1}^{[b]} \right ) \cdot \text{vec} \left \{ \Delta T_{1} \right \} + \left ( T_{1}^{T} \otimes I_{M_{1}} \right ) \cdot \text{vec} \left \{ \Delta U_{1}^{[b]} \right \} + \mathcal{O}(\Delta^2)$$

$$= \left ( I_{d} \otimes U_{1}^{[b]} \right ) \cdot L_{4} \cdot n_{1} + \left ( T_{1}^{T} \otimes I_{M_{1}} \right ) \cdot \left ( \Sigma_{1}^{[s]-1} \cdot V_{1}^{[s]T} \otimes I_{1}^{[n]} \right ) \cdot n_{1} + \mathcal{O}(\Delta^2)$$

$$= L_{6} \cdot n_{1} + \mathcal{O}(\Delta^2),$$

(67)

where

$$L_{6} = \left ( I_{d} \otimes U_{1}^{[b]} \right ) \cdot L_{4} + \left ( T_{1}^{T} \cdot \Sigma_{1}^{[s]-1} \cdot V_{1}^{[s]T} \otimes I_{1}^{[n]} \right ) \cdot n_{1}.$$

(68)

By inserting eq. (67) in eq. (49), we get

$$\text{vec} \left \{ F^{(1)} - (F^{(1)} + \Delta F^{(1)}) \cdot \hat{P}^{(1)} \cdot P_{\text{opt}} \right \} \approx L_{F_{1}} \cdot n_{1},$$

(69)

where

$$L_{F_{1}} = \left ( I_{d} \otimes F^{(1)} \right ) \cdot (K_{1}^{-1} \otimes I_{d}) \cdot W_{(d)},$$

$$= \left ( I_{d} \otimes F^{(1)H} \right ) \cdot L_{6} - L_{6}.$$

(70)

Now we get the closed-form expression for the first factor matrix rMSFE_{F_{1}} by

$$\text{rMSFE}_{F_{1}} = \frac{\text{tr} \left ( L_{F_{1}} \cdot R_{mn}^{(1)} \cdot L_{F_{1}}^{H} \right )}{\left \| F^{(1)} \right \|_{F}^{2}}.$$

(71)

Finally, the vectorization of the perturbation in the second factor matrix estimate $\Delta F^{(2)}$ is achieved by using the obtained expressions for the vectorization of $\Delta F^{(1)}$ (eq. (67)) and $\Delta F^{(3)}$ (eq. (62)) to vectorize $\Delta F^{(2)}$ as

$$\text{vec} \left \{ \Delta F^{(2)} \right \}$$

$$= \text{vec} \left \{ H \left [ N \right ] \right \} \cdot \left [ F^{(2)} \cdot F^{(1)} \right ]^{T} + \mathcal{O}(\Delta^2)$$

$$= \text{vec} \left \{ H \left [ F^{(3)} \cdot F^{(3)} \right ] \right \} \cdot \left [ (F^{(3)} \cdot F^{(3)})^{T} \otimes I_{M_{2}} \right ] \cdot \hat{P}^{(2)}_{(M_{1},M_{2},M_{3})} \cdot n_{1}$$

$$- \text{vec} \left \{ H \left [ F^{(3)} \cdot F^{(3)} \right ] \right \} \cdot \left [ (F^{(3)} \cdot F^{(3)})^{T} \otimes I_{M_{2}} \right ] \cdot Q_{(M_{1},M_{2},d)} \cdot n_{1}$$

$$+ \mathcal{O}(\Delta^2).$$

(72)

The following relation for two matrices $X = [x_{1}, x_{2}, \ldots, x_{d}] \in \mathbb{C}^{M_{2} \times d}$ and $Y = [y_{1}, y_{2}, \ldots, y_{d}] \in \mathbb{C}^{M_{2} \times d}$ can easily be verified for the vectorization of Khatri-Rao products.

$$\text{vec} \left \{ X \cdot Y \right \} = G(X, M_{2}) \cdot \text{vec} \left \{ Y \right \} = H(Y, M_{1}) \cdot \text{vec} \left \{ X \right \},$$

(73)

where

$$G(X, M_{2}) \triangleq \begin{bmatrix} (x_{1} \otimes I_{M_{2}}) \cdot (e_{d,1}^{T} \otimes I_{M_{2}}) \\ \vdots \\ (x_{d} \otimes I_{M_{2}}) \cdot (e_{d,d}^{T} \otimes I_{M_{2}}) \end{bmatrix}$$

$$H(Y, M_{1}) \triangleq \begin{bmatrix} (I_{M_{1}} \otimes y_{1}) \cdot (e_{d,1}^{T} \otimes I_{M_{1}}) \\ \vdots \\ (I_{M_{1}} \otimes y_{d}) \cdot (e_{d,d}^{T} \otimes I_{M_{1}}) \end{bmatrix}.$$
versions of the input tensor $\mathbf{X}$. In the same manner, the rMSFE\(^{(r)}\) for the 1-mode and 2-mode rhs estimates can also be approximated, by applying this performance analysis framework on the permuted versions of the noiseless input tensor $\mathbf{X}_0$, as shown in Table II.

| rMSFE\(^{(r)}\) | 3-mode | 2-mode | 1-mode |
|----------------|--------|--------|--------|
| $\mathbf{X}$ | $\text{perm}(\mathbf{X} [2,3,1])$ | $\text{perm}(\mathbf{X} [1,3,2])$ |
| rMSFE\(^{(1)}\) | rMSFE\(_F^1\) | rMSFE\(_F^1\) |
| rMSFE\(^{(2)}\) | rMSFE\(_F^2\) | rMSFE\(_F^2\) |
| rMSFE\(^{(3)}\) | rMSFE\(_F^3\) | rMSFE\(_F^3\) |

Table II: rMSFE\(^{(r)}\) from eq. (47) approximation, using eqs. (71), (80), (66), for the different $r$-modes of SECSI

V. EXTENSIONS

A. Extension to the 3-mode LHS

In this section, we extend the obtained results for the rhs estimates to the lhs estimates. We first describe the main differences between the rhs and lhs third mode estimates, and later redefine the performance analysis framework accordingly. Note that, since there are no significant changes for computing the rhs estimates, when compared to the lhs case, we can reuse most of the expressions obtained in Section IV for this extension to the lhs.

For computing the lhs estimates, the matrices $S_{3,k}^{lhs}$ from eq. (26) are used as input to the JEVD problem, instead of the matrices $S_{3,k}^{rhs}$ from eq. (25). This leads to a redefinition of the matrices $K^{(k)}_{3}$ and $L^{(k)}_{3}$ that appear in the vectorization of $S_{3,k}^{rhs}$ (eq. (56)), in the rhs case, and now are used to express the vectorization of $S_{3,k}^{lhs}$ as $\text{vec} \{S_{3,k}^{lhs}\} = L^{(k)}_{3} \otimes n_1$, where

$$L^{(k)}_{3,lhs} = Q^{T}_{1,3} \cdot (I_d \otimes S_{3,p}^{-1}) \cdot L^{(k)}_{2} - (S_{3,k}^{T} \otimes S_{3,p}^{-1} \otimes S_{3,p}^{-1}) \cdot L^{(p)}_{2},$$

(81)

The calculation of a JEVD for the slices $S_{3,k}^{lhs}$ in eq. (26) results in the estimation of the transformation matrix $T_{2}$, instead of $T_{1}$. Therefore, this leads to a different way of estimating the factor matrices, as defined in Table I. For instance, $F^{(1)}$ is estimated via a LS-fit and $F^{(2)}$ is now estimated from $T_{2}$.

These changes lead to a redefinition of $L_{6}$ that is similar to the definition of $L_{7}$ for the rhs (eq. (77)). Therefore, we refer to it as

$$L^{(k)}_{7,lhs} = \left( I_d \otimes U^{(n)}_{2} \right) \cdot L_{4} + \left( T^{T}_{2} \otimes \Sigma_{2}^{-1} \otimes V^{T}_{2} \otimes \Gamma^{(n)}_{2} \right).$$

(82)

In the same manner, $L_{7}$ for the rhs is also redefined for the lhs as

$$L^{(k)}_{6,lhs} = \left( F^{(2)} \otimes F^{(3)} \right) \otimes I_{M_{1}} \cdot P^{(1)}_{(M_{1},M_{2},M_{3})}$$

$$- \left( F^{(2)} \otimes F^{(3)} \right) \otimes F^{(1)} \cdot Q_{(M_{2},M_{3},d)}$$

$$\cdot \left( G(F^{(2)},M_{3}) \cdot L_{5} + H(F^{(3)},M_{2}) \cdot L_{7} \right).$$

(83)

Finally, a summary of the third mode rhs and lhs performance analysis is shown in Table III. Note that only steps 4 (compute $\Delta \mathbf{F}^{(3)}(k,:)$, eq. (81)) and 8 (compute $\Delta \mathbf{F}^{(2)}$, eq. (83)) are changed from the rhs to the lhs performance analysis. Moreover the lhs rMSFE\(^{(r)}\) expressions in the other modes (i.e., 1-mode and 2-mode) can be approximated in the same manner as in Section IV by applying this first order performance analysis to permuted versions of the noiseless tensor $\mathbf{X}_0$, as shown in Table II.

B. Extension to the underdetermined (degenerate) case

In this work, we have assumed the non-degenerate case, but the results can also be applied to the underdetermined (degenerate) case. The decomposition is underdetermined in mode $d > M_{n}$. The SECSI framework is still applicable if the problem is underdetermined in up to one mode \cite{9}. For example, let the tensor rank of the noiseless tensor be greater than any one of the dimension, i.e., $M_{1} < d \leq \min\{M_{2},M_{3}\}$. In this case, $F^{(1)} \in \mathbb{C}^{M_{1} \times d}$ is a flat matrix. But $U_{1}$ has the dimension of $M_{1} \times M_{1}$. Therefore, $T_{1}$ does not exist. However, the diagonalization problems $S_{rh}^{lhs}$ and $S_{rh}^{rhs}$ in Fig. 1 can still be solved and yield two estimates for $F^{(1)}$, one for $F^{(2)}$, and one for $F^{(3)}$.

VI. A PERFORMANCE ANALYSIS BASED FACTOR MATRIX ESTIMATES SELECTION SCHEME

In this section we propose a new estimates selection scheme for the SECSI framework. We refer to this new scheme as Performance Analysis Based Selection (PAS). Unlike other selection schemes for the final matrix estimates proposed in \cite{9}, such as CON-PS (conditioning number - paired solutions), REC-PS (reconstruction error - paired solutions), and BM (best matching), which perform a heuristic selection or an exhaustive search (BM) to select the final estimates, this new selection scheme allows us to directly select one estimate per factor matrix (instead of performing a search among all the possible estimate combinations). For instance, the BM scheme tests all possible combinations of the estimates of the loading matrices in an exhaustive search. It therefore requires the reconstruction of $(R \cdot (R - 1))^T$ tensors \cite{9}, which grows...
rapidly with $R$ and already reaches 216 for $R = 3$, whereas the PAS scheme requires only 18 estimates for $R = 3$.

All the analytical expressions in the performance analysis are computed from the noiseless estimates (such as $X_0$, $U^{[s]}$, $S^{[s]}$, etc.). For the PAS scheme, we approximate the noiseless quantities with the noisy ones as $X_0 \approx \hat{X}$, $U^{[s]} \approx \hat{U}^{[s]}$, $S^{[s]} \approx \hat{S}^{[s]}$, etc. Then, we use the corresponding performance analysis and assume perfect knowledge of the second order moments of the noise (i.e., $R^{(r)}_{\text{in}}$ and $C^{(r)}_{\text{in}}$), to estimate rMSFE$(r)$ for all $r = 1, 2, 3$, in all the $r$-mode lhs and rhs solutions. Finally, we select the estimates that correspond to the smallest estimated rMSFE$(r)$ values, for all $r = 1, 2, 3$. Note that estimating the noise variance has no effect on this scheme, since all the estimated rMSFE$(r)$ values are multiplied by the same $\sigma_N^2$ factor.

VII. SIMULATION RESULTS

In this section, we validate the resulting analytical expressions with empirical simulations. First we compare the results for the proposed analytical framework to the empirical results. Then, we compare the performance of the PAS selection scheme with other estimates selection schemes.

A. Performance Analysis Simulations

We define three simulation scenarios, where the properties of the noiseless tensor $X_0$ and the number of trials used for every point of the simulation are stated in Table IV.

| Scenario | Size | $d$ | $F^{(r)}$ Correlation | Trials |
|----------|------|-----|----------------------|--------|
| I        | $5 \times 5 \times 5$ | 4   | (none)               | 10,000 |
| II       | $5 \times 8 \times 7$ | 4   | $r = 1$              | 10,000 |
| III      | $3 \times 15 \times 70$ | 3   | (none)               | 5,000  |

Table IV: Scenarios for varying SNR simulations.

In scenario I and scenario III, we have used real-valued tensors while complex-valued tensors are used for scenario II. Moreover, to further illustrate the robustness of our performance analysis results, we have used different JEVs algorithms for both scenarios. We employ the JDTM algorithm [16] for scenario I while the coupled JEVG [17] is employed for scenario II. Note that both algorithms are based on the indirect LS cost function [12]. For every trial, the noise tensor $\mathcal{N}$ is randomly generated and has zero-mean Gaussian entries of variance $\sigma_N^2 = \|X_0\|_2^2/(\text{SNR} \cdot M)$. Moreover, the noiseless tensor $X_0$ is fixed in each the experiment and has zero-mean uncorrelated Gaussian entries on its factor matrices $F^{(r)}$. We plot several realizations of the experiments (therefore different $X_0$) on top of each other, to provide better insights about the performance of the tested algorithms. This simulation setup is selected, since the derived analytical expression depicts the rMSFE for a known noiseless tensor $X_0$ over several noise trials. Moreover, every realization of the noiseless tensor $X_0$ is given by $X_0 = L_{3,d} \times 1 F^{(1)} \times 2 F^{(2)} \times 3 F^{(3)}$, where the factor matrices $F^{(r)} \in \mathbb{R}^{M_r \times d}$ have uncorrelated Gaussian entries for all $r = 1, 2, 3$ for scenario I. Furthermore, for scenario II, we also introduce correlation in the factor matrix $F^{(1)}$. In this scenario, the factor matrices $F^{(2)}$ and $F^{(3)}$ are randomly drawn but $F^{(1)}$ is fixed along the experiments as

$$F^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0.95 & 0.95 & 0.95 \\ 1 & 0.95 & 1 & 1 \\ 1 & 1 & 0.95 & 1 \\ 0.95 & 1 & 1 & 1 \end{bmatrix}. \quad (84)$$

We depict the results in the form of the Total rMSFE (TrMSFE) since it reflects the total factor matrix estimation accuracy of the tested algorithms. The TrMSFE is defined as $\text{TrMSFE} = \sum_{r=1}^{3} r\text{MSFE}^{(r)}$ where $r\text{MSFE}^{(r)}$ is the same as in eq. [47].

It is evident from the results in Fig. 2 that the analytical results from the proposed first-order perturbation analysis match well with the Monte-Carlo simulations for both real and complex valued tensors. Moreover, the results also show an excellent match to the empirical simulations for scenario II where we have an asymmetric tensor and also have high correlation in the first factor matrix.

In Fig. 3, we show the analytical and empirical results for the asymmetric scenario III where the results are shown for
6 estimates of each factor matrix obtained by solving all of the $r$-modes for a SNR = 50 dB. The estimates are arranged according to Fig. [1] The results show an excellent match for empirical and analytical results for all of 6 estimates for each factor matrix. Moreover, the estimates $F_{1V}^{(1)}$, $F_{1V}^{(2)}$, $F_{1V}^{(2)}$, $F_{1V}^{(3)}$, and $F_{1V}^{(3)}$ are the best estimates for each of the factor matrices. Note that all of these estimates are obtained by solving the JEVD problem for the 3-mode in eq. (25) and eq. (26). This results from the fact that JEVD for this mode has the highest number of slices (70) which results in a better accuracy.

### B. SECSI framework based on PAS scheme

In this section, we compare the performance of the PAS scheme with other selection schemes, such as the CON-PS, REC-PS, and BM schemes from [9] for two simulation scenarios in Table [V].

| Scenario | Size         | SNR | $d$ | $F^{(r)}$ Correlation | Trials |
|----------|--------------|-----|-----|-----------------------|--------|
| IV       | $5 \times 5 \times 5$ | 50 dB | 3   | (none)                | 10,000 |
| V        | $3 \times 15 \times 70$ | 40 dB | 3   | (none)                | 5,000  |

Table [V]: Scenarios for fixed SNR simulations.

In these simulation setups, the noise tensor $\mathcal{N}$ as well as the noiseless tensor $\mathcal{X}_0$ are randomly generated at every trial. Since the noise variance has no effect on the selected estimate, for the real-valued uncorrelated noise with equal variance scenario, we use $R_{nm}^{(r)} = I_M$ and $C_{nm}^{(r)} = I_M$, for all $r = 1, 2, 3$, as input for this PAS scheme, regardless of the value of $\sigma_n^2$. Moreover, to provide more insights about the performance, we also define a naive selection scheme denoted as DUMMYR, where the final estimates are randomly selected among the 6 possible estimates available per factor matrix. We use the complementary cumulative density function (CCDF) of the TrMSFE to illustrate the robustness of the selected strategy [9].

The results are shown in Fig. [4] for two scenarios with different tensor sizes. The results show that the proposed PAS scheme outperforms the other schemes especially in scenario IV where the tensor size is small and the SNR is also high. This is to be expected, since the expressions used to build the PAS scheme are based on the high SNR assumption. Nevertheless, we also observe that even with the higher dimensional tensor and relatively low SNR, as shown in Fig. [4]b, we observe that the PAS scheme performs slightly better than the CON-PS, REC-PS, and BM schemes.

### VIII. CONCLUSIONS

In this work, a first-order perturbation analysis of the SECSI framework for the approximate CP decomposition of 3-D noise-corrupted low-rank tensors is presented, where we provide closed-form expressions for the relative mean square error for each of the estimated factor matrices. The derived expressions are formulated in terms of the second-order moments of the noise, such that apart from a zero mean, no assumptions on the noise statistics are required. The simulation results depict the excellent match between the closed-form expressions and the empirical results for both real and complex valued data. Moreover, these expressions can also be used for an enhancement in the factor matrix estimates selection step of the SECSI framework. The new PAS selection scheme outperforms the existing schemes especially at high SNR values.

### APPENDIX A

#### PROOF OF THEOREM [1]

The term $(Z + \Delta Z) \cdot \hat{P}$ can be interpreted as the estimate of $Z$ up to a scaling of its column, given by

$$\hat{Z} = (Z + \Delta Z) \cdot \hat{P} = \hat{Z} + \Delta \hat{Z}.$$ 

Therefore, $\hat{P}$ can be expressed as $\hat{P} = D\text{diag} \left( Z^H \cdot \hat{Z} \cdot K^{-1} \right)$. To resolve the scaling ambiguity, we compute a diagonal matrix $P_{\text{opt}}$ that satisfies $P_{\text{opt}} = D\text{diag}(Z^H \cdot \hat{Z} \cdot K^{-1})^{-1}$. This result leads to

$$\hat{P} \cdot P_{\text{opt}} = \hat{P} \cdot D\text{diag} \left( Z^H \cdot \hat{Z} \cdot K^{-1} \right)^{-1} = \hat{P} \cdot D\text{diag} \left( Z^H \cdot (Z + \Delta Z) \cdot \hat{P} \cdot K^{-1} \right)^{-1} = \left( I_d + D\text{diag}(Z^H \cdot \Delta Z) \cdot K^{-1} \right)^{-1}$$

Let us define the diagonal matrix $U \triangleq D\text{diag} \left( Z^H \cdot \Delta Z \right) \cdot K^{-1}$ of size $d \times d$ with diagonal elements $U(i, i)$ for all $i = 1, 2, \ldots, d$. Therefore, $(I_d + U)^{-1}$ is also a diagonal matrix, with diagonal elements $(1 + U(i, i))^{-1}$ for all $i = 1, 2, \ldots, d$. Since the elements $U(i, i)$ are first-order terms, we use the well known Taylor expansion $(1 + x)^{-1} = \ldots$
1 - x + O(Δ^2) which holds for any first-order scalar term x. This leads to \((I_d + U)^{-1} = (I_d - U) + O(Δ^2)\). Therefore, we get 
\[
(I_d + \text{Ddiag}(Z^H \cdot ΔZ) \cdot K^{-1}) = I_d - \text{Ddiag}(Z^H \cdot ΔZ) \cdot K^{-1} + O(Δ^2),
\]
and thus
\[
Z - (Z + ΔZ) \cdot P \cdot P_{\text{opt}} = Z - (Z + ΔZ) \cdot \left[I_d - \text{Ddiag}(Z^H \cdot ΔZ) \cdot K^{-1}\right] + O(Δ^2)
\]
\[
= Z \cdot \text{Ddiag}(Z^H \cdot ΔZ) \cdot K^{-1} - ΔZ + O(Δ^2),
\]
which proves this theorem.

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