ON TRIANGULABLE TENSOR PRODUCTS OF $B$-PAIRS AND TRIANGULINE REPRESENTATIONS

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Abstract. We show that if $V$ and $V'$ are two $p$-adic representations of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ whose tensor product is trianguline, then $V$ and $V'$ are both potentially trianguline.

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Introduction

The notion of a trianguline representation of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ was introduced by Colmez [Col08] in the context of his work on the $p$-adic local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$. Examples of trianguline representations include the semi-stable representations of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ as well as the $p$-adic representations of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ attached to overconvergent cuspidal eigenforms of finite slope (theorem 6.3 of [Kis03] and proposition 4.3 of [Col08]). The category of all trianguline representations of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is stable under direct sums, tensor products, extensions, and duals. We refer the reader to the book [BC09] and the survey [Ber11] for a detailed discussion of trianguline representations. Let us at least mention the following analogue of the Fontaine-Mazur conjecture: if $V$ is an irreducible 2-dimensional $p$-adic representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that is unramified at $\ell$ for almost all $\ell \neq p$, and whose restriction to a decomposition group at $p$ is trianguline, then $V$ is a twist of the Galois representation attached to an overconvergent cuspidal eigenform of finite slope. This conjecture is a theorem of Emerton (§1.2.2 of [Eme11]) under additional technical hypothesis on $V$. The trianguline property is in general a condition at $p$ reflecting (conjecturally at $p$...
least) the fact that a \( p \)-adic representation comes from a \( p \)-adic automorphic form. This theme is pursued, for example, in [Han17, Ber17] and [Con17].

If \( K \) is a finite extension of \( \mathbb{Q}_p \), we also have the notion of a trianguline representation of \( G_K = \text{Gal}(\mathbb{Q}_p/K) \). We say that a representation \( V \) of \( G_K \) is potentially trianguline if there exists a finite extension \( L/K \) such that the restriction of \( V \) to \( G_L \) is trianguline. The goal of this article is to prove the following theorem.

**Theorem A.** If \( V \) and \( V' \) are two non-zero \( p \)-adic representations of \( G_{\mathbb{Q}_p} \) whose tensor product is trianguline, then \( V \) and \( V' \) are both potentially trianguline.

We now give more details about the contents of this article. The definition of “trianguline” can be given either in terms of \((\varphi,\Gamma)\)-modules over the Robba ring, or in terms of \( B \)-pairs. In this article, we use the theory of \( B \)-pairs, which was introduced in [Ber08]. We remark in passing that \( B \)-pairs are the same as \( G_K \)-equivariant bundles on the Fargues-Fontaine curve [FP18]. Let \( K \) be a finite extension of \( \mathbb{Q}_p \). Let \( B_{\text{dr}}^+, B_{\text{dr}} \) and \( B_\varphi = (B_{\text{cris}})^{\varphi=1} \) be some of Fontaine’s rings of \( p \)-adic periods [Fon94]. A \( B \)-pair is a pair \( W = (W_\varphi, W_{\text{dr}}^+) \) where \( W_\varphi \) is a free \( B_\varphi \)-module of finite rank endowed with a continuous semi-linear action of \( G_K \), and \( W_{\text{dr}}^+ \) is a \( G_K \)-stable \( B_{\text{dr}}^+ \)-lattice in \( W_{\text{dr}} = B_{\text{dr}} \otimes B_\varphi W_\varphi \). If \( V \) is a \( p \)-adic representation of \( G_K \), then \( W(\chi) = (B_\varphi \otimes \mathbb{Q}_p V, B_{\text{dr}}^+ \otimes \mathbb{Q}_p V) \) is a \( B \)-pair. If \( E \) is a finite extension of \( \mathbb{Q}_p \), the definition of \( B \)-pairs can be extended to \( E \)-linear objects, and we get objects called \( B_{\text{dr}}^{\otimes E} \)-pairs in [BC10] or \( E-B \)-pairs of \( G_K \) in [Nak09]. They are pairs \( W = (W_\varphi, W_{\text{dr}}^+) \) where \( W_\varphi \) is a free \( E \otimes \mathbb{Q}_p \) \( B_\varphi \)-module of finite rank endowed with a continuous semi-linear action of \( G_K \), and \( W_{\text{dr}}^+ \) is a \( G_K \)-stable \( E \otimes \mathbb{Q}_p B_{\text{dr}}^+ \)-lattice in \( W_{\text{dr}} = (E \otimes \mathbb{Q}_p B_{\text{dr}}) \otimes_{E \otimes \mathbb{Q}_p B_\varphi} W_\varphi \). Note that the action of \( G_K \) is \( E \)-linear.

We say (definition 1.15 of [Nak09]) that a \( B_{\text{dr}}^{\otimes E} \)-pair \( W \) is split triangulable if \( W \) is a successive extension of objects of rank 1, triangulable if there exists a finite extension \( F/E \) such that the \( B_{\text{dr}}^{\otimes E} \)-pair \( F \otimes_E W \) is split triangulable, and potentially triangulable if there exists a finite extension \( L/K \) such that the \( B_{\text{dr}}^{\otimes E} \)-pair \( W|_{G_L} \) is triangulable. If \( V \) is a \( p \)-adic representation of \( G_K \), we say that \( V \) is trianguline if \( W(V) \) is triangulable.

Let \( \Delta \) be a set of rank 1 \( E \otimes \mathbb{Q}_p B_\varphi \)-representations of \( G_K \). We say that a \( B_{\text{dr}}^{\otimes E} \)-pair is split \( \Delta \)-triangulable if it is split triangulable, and the rank 1 \( E \otimes \mathbb{Q}_p B_\varphi \)-representations of \( G_K \) that come from the triangulable are all in \( \Delta \). Let \( \Delta(\mathbb{Q}_p) \) be the set of rank 1 \( E \otimes \mathbb{Q}_p B_\varphi \)-representations of \( G_K \) that extend to \( G_{\mathbb{Q}_p} \). Theorem A then results from the following more general result (theorem 5.4), applied to \( K = \mathbb{Q}_p \).

**Theorem B.** If \( X \) and \( Y \) are two non-zero \( B_{\text{dr}}^{\otimes E} \)-pairs whose tensor product is \( \Delta(\mathbb{Q}_p) \)-triangulable, then \( X \) and \( Y \) are both potentially triangulable.

The proof of theorem B relies on the study of \( E \otimes \mathbb{Q}_p B_\varphi \)-representations of \( G_K \) as well as on the study of the slopes, weights and cohomology of \( B_{\text{dr}}^{\otimes E} \)-pairs. The ring \( E \otimes \mathbb{Q}_p B_\varphi \)
has many non-trivial units, which makes the study of $B_{|K}^{\otimes E}$-pairs more difficult than when $E = \mathbb{Q}_p$. Note finally that some of the results of this article already appear in [DM13].

1. Reminders and complements

If $K$ is a finite extension of $\mathbb{Q}_p$, let $G_K = \text{Gal}(\overline{\mathbb{Q}_p}/K)$. Let $E$ be a finite Galois extension of $\mathbb{Q}_p$ such that $K \subset E$, and let $\Sigma = \text{Gal}(E/\mathbb{Q}_p)$. Let $E_0$ be the maximal unramified extension of $\mathbb{Q}_p$ inside $E$. Let $B_{dR}^+, B_{dR}, B_{\text{cris}}^+$ and $B_{\text{cris}}$ be Fontaine’s rings of $p$-adic periods (see for instance [Fon94]). They are all equipped with an action of $G_{\mathbb{Q}_p}$, and $B_{dR}^+$ and $B_{\text{cris}}$ have in addition a Frobenius map $\varphi$. Let $B_e = (B_{\text{cris}})^{\varphi=1}$ and $B_{e,E} = E \otimes_{\mathbb{Q}_p} B_e$. The group $G_{\mathbb{Q}_p}$ acts $E$-linearly on $B_{e,E}$.

**Proposition 1.1.** The ring $B_{e,E}$ is a principal ideal domain.

**Proof.** The ring $B_{e,E}$ is a Bézout domain; for $E = \mathbb{Q}_p$ this is shown in proposition 1.1.9 of [Ber08], and the same argument is used to show the general case in lemma 1.6 of [Nak09].

By theorem 6.5.2 of [FF18], the ring $B_e$ is a principal ideal domain, and therefore $B_{e,E}$ is a principal ideal domain as well, since it is a quotient of the polynomial ring $B_e[X]$, and thus Noetherian.

Recall that a $B_{|K}^{\otimes E}$-pair is a pair $W = (W_e, W_{dR}^+]$ where $W_e$ is a free $B_{e,E}$-module of finite rank endowed with a continuous semi-linear action of $G_K$, and $W_{dR}^+$ is a $G_K$-stable $E \otimes_{\mathbb{Q}_p} B_{dR}^+$-lattice in $W_{dR} = (E \otimes_{\mathbb{Q}_p} B_{dR}) \otimes_{B_{e,E}} W_e$.

**Proposition 1.2.** If $W_e$ is a $B_{e,E}$-representation of $G_K$, then $(E \otimes_{\mathbb{Q}_p} B_{dR}) \otimes_{B_{e,E}} W_e$ admits an $E \otimes_{\mathbb{Q}_p} B_{dR}^+$-lattice stable under $G_K$.

**Proof.** See §3.5 of [Fon04]. The same argument gives an $E \otimes_{\mathbb{Q}_p} B_{dR}^+$-lattice instead of a $B_{dR}^+$-lattice if one starts from an $E \otimes_{\mathbb{Q}_p} B_{dR}$-representation.

Recall that Nakamura has classified the $B_{|K}^{\otimes E}$-pairs of rank 1, under the assumption that $E$ contains the Galois closure of $K$. Given a character $\delta : K^\times \rightarrow E^\times$, he constructs in §1.4 of [Nak09] a rank 1 $B_{|K}^{\otimes E}$-pair $W(\delta)$, that we denote by $B(\delta)$, and proves that every rank 1 $B_{|K}^{\otimes E}$-pair is of this form for a unique $\delta$. We have $B(\delta_1) \otimes B(\delta_2) = B(\delta_1 \delta_2)$ (§1.4 of [Nak09]). We denote by $B(\delta)_e$ the $B_{e,E}$-component of $B(\delta)$.

Recall (see for instance §2 of [BC10] or §1.3 of [Nak09]) that $B_{|K}^{\otimes E}$-pairs have slopes. This comes from the equivalence of categories between $B_{|K}^{\otimes E}$-pairs and $(\varphi, \Gamma)$-modules over the Robba ring, and Kedlaya’s constructions and results for $\varphi$-modules over the Robba ring (see [Ked04]). In particular, one can define the notion of isoclinic (pure of a certain slope) $B_{|K}^{\otimes E}$-pairs. For example, if $V$ is an $E$-linear representation of $G_K$, then $W(V) = (B_{e,E} \otimes E V, (E \otimes_{\mathbb{Q}_p} B_{dR}^+) \otimes_E V)$ is pure of slope 0, and every such $B_{|K}^{\otimes E}$-pair is of this form (proposition 2.2 of [BC10]).
We have the following slope filtration theorem (see theorem 2.1 of [BC10]).

**Theorem 1.3.** If $W$ is a $B_K^{\otimes E}$-pair, then there is a canonical filtration $\{0\} = W_0 \subset W_1 \subset \cdots \subset W_\ell = W$ by sub $B_K^{\otimes E}$-pairs such that

1. for every $1 \leq i \leq \ell$, the quotient $W_i/W_{i-1}$ is isoclinic;
2. if $s_i$ is the slope of $W_i/W_{i-1}$, then $s_1 < s_2 < \cdots < s_\ell$.

The following proposition gathers the results that we need concerning slopes of $B_K^{\otimes E}$-pairs. Recall that $\Hom(X,Y) = (\Hom_{E\otimes Q_p B_e}(X_\cris, Y_\cris), \Hom_{E\otimes Q_p B_{dR}^+}(X_\dR, Y_\dR))$.

**Proposition 1.4.** If $X$ is pure of slope $s$ and $Y$ is pure of slope $t$, then

1. $\Hom(X,Y)$ is pure of slope $t - s$ and $X \otimes Y$ is pure of slope $s + t$;
2. if $X$ and $Y$ have the same rank and $X \subset Y$ and $s = t$, then $X = Y$;
3. if $Y$ is a direct summand of $X$, then $s = t$.

**Proof.** For (1), see theorem 6.10 and proposition 5.13 of [Ked04]. For (2), we can take determinants and assume that $X$ and $Y$ are of rank 1. The claim is then proposition 2.3 of [Ber08]. Item (3) follows from the fact that if $X = Y \oplus Z$, then the set of slopes of $X$ is the union of those of $Y$ and $Z$ (proposition 5.13 of [Ked04]).

2. The ring $B_{e,E}$

Recall that $B_{e,E} = E \otimes Q_p B_e$. In this section, we determine the units of $B_{e,E}$ and study the rank 1 $B_{e,E}$-representations of $G_E$. Let $q = p^h$ be the cardinality of the residue field of $\mathcal{O}_E$, so that $E_0 = Q_p$. Let $\varphi_E : E \otimes_{E_0} B_{\cris} \to E \otimes_{E_0} B_{\cris}$ be the map $\Id \otimes \varphi^h$.

**Proposition 2.1.** We have an exact sequence

$$0 \to E \to B_{e,E} \to (E \otimes Q_p B_{dR})/(E \otimes Q_p B_{dR}^+) \to 0.$$

**Proof.** This follows from tensoring by $E$ the usual fundamental exact sequence $0 \to Q_p \to B_e \to B_{dR}/B_{dR}^+ \to 0$ (proposition 1.17 of [BK90]).

**Proposition 2.2.** The natural map $B_{e,E} \to (E \otimes_{E_0} B_{\cris})^{\varphi_E=1}$ is an isomorphism.

**Proof.** Since $\varphi_E$ is $E$-linear, we have $(E \otimes_{E_0} B_{\cris})^{\varphi_E=1} = E \otimes_{E_0} B_{\cris}^{\varphi^h=1}$ and it is therefore enough to prove that $B_{\cris}^{\varphi^h=1} = Q_{p^h} \otimes_{Q_p} B_{\cris}^{\varphi=1}$. The group $\Gal(Q_{p^h}/Q_p)$ acts $Q_{p^h}$-semilinearly on $B_{\cris}^{\varphi=1}$ via $\varphi$, and the claim follows from Galois descent (Speiser’s lemma).

**Remark 2.3.** The isomorphism of proposition 2.2 is $G_E$-equivariant. In addition, if $g \in G_{Q_p}$ acts by $\Id \otimes g$ on $E \otimes Q_p B_e$, then it acts by $\Id \otimes g \varphi^{-n(g)}$ on $(E \otimes_{E_0} B_{\cris})^{\varphi_E=1}$ (where $n(g)$ is defined below).
Let $\pi$ be a uniformizer of $\mathcal{O}_E$, and let $\chi_\pi$ denote the Lubin-Tate character $\chi_\pi : G_E \to \mathcal{O}_E^\times$ attached to $\pi$. For each $\tau \in \Sigma = \text{Gal}(E/Q_p)$, let $u(\tau)$ be the element of $\{0, \ldots, h-1\}$ such that $\tau = \varphi^{u(\tau)}$ on $E_0$. Let $t_\tau \in E \otimes_{E_0} B^{+}_\text{cris}$ denote the element constructed in §5 of [Ber16], where (in the notation of [Ber16]) we take $F = E$. We have $t_\tau = (\tau \otimes \varphi^{u(\tau)})(t_{\text{Id}})$. The element $t_{\text{Id}}$ is also denoted by $t_\pi$ in [Ber16], and it is the same as the element $t_E$ constructed in §9 of [Col02]. The usual $t$ of $p$-adic Hodge theory is $t = t_{Q_p}$ for $\pi = p$.

For each $\sigma \in \Sigma$, we have a map $E \otimes_{E_0} B^{+}_\text{cris} \to B^{+}_{\text{dR}}$ given by $x \mapsto (\sigma \otimes \varphi^{u(\sigma)})(x)$, followed by the natural injection of $E \otimes_{E_0} B^{+}_\text{cris}$ in $B^{+}_{\text{dR}}$ (theorem 4.2.4 of [Font94]).

**Proposition 2.4.** Let the notation be as above.

1. We have $\varphi_E(t_\tau) = (\tau(\pi) \cdot t_\tau)$ and $g(t_\tau) = (\tau(\chi_\pi(g))) \cdot t_\tau$ if $g \in G_E$;

2. the $t$-adic valuation of the $\sigma$-component of the image of $t_\tau$ via the map $E \otimes_{E_0} B^{+}_\text{cris} \to E \otimes_{Q_p} B^{+}_{\text{dR}}$ given by $x \mapsto \{(\sigma \otimes \varphi^{u(\sigma)})(x)\}_{\sigma \in \Sigma}$ is 1 if $\sigma = \tau^{-1}$ and 0 otherwise;

3. there exists $u \in (E \cdot \hat{Q}_p^\times)^{\times}$ such that $\prod_{\tau \in \Sigma} t_\tau = u \cdot t$ in $E \otimes_{E_0} B^{+}_\text{cris}$.

**Proof.** Since $t_\tau = (\tau \otimes \varphi^{u(\tau)})(t_{\text{Id}})$, it is enough to check (1) for $\tau = \text{Id}$. The corresponding statement is at the end of §3 of [Ber16] (page 3578). Likewise, (2) follows from the case $\tau = \text{Id}$. That case now follows from (1) and the fact that the Hodge-Tate weight of $\chi_\pi$ is 1 at $\sigma = \text{Id}$ and 0 at $\sigma \neq \text{Id}$. Finally, we have $N_{E/Q_p}(\chi_\pi) = \chi_{\text{cycl}} \eta$ where $\eta : G_E \to Q_p^\times$ is unramified, and by (1), this implies (3).

Note that $t^{-1}_\tau \in E \otimes_{E_0} B^{+}_\text{cris}$ since $t_\tau$ divides $t$ in $B^{+}_\text{cris}$ by (3) of proposition 2.4.

**Proposition 2.5.** If $n = \{n_\tau\}_{\tau \in \Sigma}$ is a tuple of integers whose sum is 0, then there exists $u_n \in (E \cdot \hat{Q}_p^\times)^{\times}$ such that $u = \prod_{\tau \in \Sigma} t_{\tau}^{n_\tau} u_n$ belongs to $B^{+}_{n,E}$. The element $u_n$ is then a unit of $B^{+}_{n,E}$ and every unit of $B^{+}_{n,E}$ is of this form up to multiplication by $E^\times$.

**Proof.** Let $w = \varphi_E(\prod_{\tau \in \Sigma} t_{\tau}^{n_\tau})/\prod_{\tau \in \Sigma} t_{\tau}^{n_\tau} = \prod_{\tau \in \Sigma} (\tau(\pi))^{n_\tau}$ by (1) of proposition 2.4. Since $\sum_{\tau \in \Sigma} n_{\tau} = 0$, we have $w \in \mathcal{O}_E^\times$. There exists $u_n \in (E \cdot \hat{Q}_p^\times)^{\times}$ such that $\varphi_E(u_n)/u_n = w^{-1}$, and then $u = \prod_{\tau \in \Sigma} t_{\tau}^{n_\tau} u_n$ belongs to $B^{+}_{n,E}$. The inverse of $u$ is $\prod_{\tau \in \Sigma} t_{\tau}^{-n_\tau} u_n$ which also belongs to $B^{+}_{n,E}$, so that $u \in B^{+}_{n,E}$. We now show that every $u \in B^{+}_{n,E}$ is of this form. Let $n_\tau$ be the $t$-adic valuation in $B^{+}_{\text{dR}}$ of the $\tau^{-1}$-component $u_{\tau^{-1}} = (\tau^{-1} \otimes \text{Id})(u)$ of the image of $u \in E \otimes_{Q_p} B^{+}_{e,E}$ in $E \otimes_{Q_p} B^{+}_{\text{dR}} = \prod_{\sigma \in \Sigma} B^{+}_{\text{dR}}$. Note that $u_\sigma \in B^{+}_{e,E}$ for all $\sigma \in \Sigma$ and that $\prod_{\sigma \in \Sigma} u_\sigma \in (B^{+}_{e,E})^\Sigma = B^{+}_{e,E}$. We have $B^{+}_n = Q_p^\times$ by lemma 1.1.8 of [Ber08], so that $\sum_{\tau \in \Sigma} n_{\tau} = 0$. By (2) of proposition 2.4, the element $u \cdot \prod_{\tau \in \Sigma} t_{\tau}^{-n_\tau} u_n^{-1}$ belongs to $(E \otimes_{Q_p} B^{+}_{\text{dR}}) \cap B^{+}_{n,E}$, and $(E \otimes_{Q_p} B^{+}_{\text{dR}}) \cap B^{+}_{n,E} = E^\times$ by proposition 2.4.

Recall that an $E$-linear representation is crystalline or de Rham if the underlying $Q_p$-linear representation is crystalline or de Rham. We say that a character $\delta : G_E \to E^\times$ is
$B_{e,E}$-admissible if there exists $y \in B_{e,E} \setminus \{0\}$ such that $\delta(g) = g(y)/y$. Such a character is then crystalline, hence also de Rham.

**Proposition 2.6.** If $y \in B_{e,E} \setminus \{0\}$ is such that $y \cdot B_{e,E}$ is stable under $G_E$, then $y \in B_{e,E}^\times$ and there exists $n_\tau \in \mathbb{Z}$ with $\sum_{\tau \in \Sigma} n_\tau = 0$ and $y_0 \in (E \cdot \hat{Q}_p)^{\times}$ such that $y = \prod_{\tau \in \Sigma} t_\tau^{n_\tau} y_0$.

**Proof.** If $y \cdot B_{e,E}$ is stable under $G_E$, then $g(y)/y \in B_{e,E}$ for all $g \in G_E$. Note that if $z \in B_{e,E}^\times$, then $g(z)/z \in B_{e,E}^\times$. This implies that $g(y)/y \in B_{e,E} \cap (E \otimes \mathbb{Q}_p B_{e,E}^\times)$. By proposition 2.1 $g(y)/y \in E^\times$. The map $\delta : G_E \to E^\times$ given by $\delta(g) = g(y)/y$ is a crystalline character of $G_E$, and hence of the form $\prod_{\tau \in \Sigma} \tau(\chi_\tau)^{n_\tau} \eta_0$ where $n_\tau \in \mathbb{Z}$ and $\eta_0 : G_E \to E^\times$ is unramified. This implies that there exists $y_0 \in (E \cdot \hat{Q}_p)^{\times}$ such that $y = \prod_{\tau \in \Sigma} t_\tau^{n_\tau} y_0$. If $y \in B_{e,E}$, then $\phi_E(y) = y$ so that $\sum_{\tau \in \Sigma} n_\tau = 0$ by (1) of proposition 2.4 and hence $y \in B_{e,E}^\times$. □

**Corollary 2.7.** If $\delta : G_E \to E^\times$ is a $B_{e,E}$-admissible character, then $\delta$ is de Rham and the sum of its weights at all $\tau \in \Sigma$ is 0. Conversely, any character $\delta : G_E \to E^\times$ that is de Rham with the sum of its weights at all $\tau \in \Sigma$ equal to 0 is the product of a $B_{e,E}$-admissible character by a potentially unramified character.

**Proof.** The first assertion follows immediately from proposition 2.6. We now prove the second assertion. If $\delta : G_E \to E^\times$ is de Rham, it is of the form $\prod_{\tau \in \Sigma} \tau(\chi_\tau)^{n_\tau} \eta_0$ where $n_\tau \in \mathbb{Z}$ and $\eta_0 : G_E \to E^\times$ is potentially unramified. Let $n = \{n_\tau\}_{\tau \in \Sigma}$ and $u$ be the corresponding unit (proposition 2.5). If $g \in G_E$, then $g(u)/u = \prod_{\tau \in \Sigma} \tau(\chi_\tau(\pi))^n \eta_\pi(g)$ where $\eta_\pi : G_E \to E^\times$ is unramified. The second assertion then follows from this. □

A $B_{e,E}$-representation of $G_K$ is a free $B_{e,E}$-module of finite rank with a semi-linear and continuous action of $G_K$ (recall that $G_K$ acts linearly on $E$). If $\delta : G_K \to E^\times$ is a character (or, more generally, an element of $H^1(G_K, B_{e,E})$), we denote by $B_{e,E}(\delta)$ the resulting rank 1 $B_{e,E}$-representation of $G_K$.

**Proposition 2.8.** If $W_e$ is a $B_{e,E}$-representation of $G_K$, and if $X_e$ is a sub $B_{e,E}$-module of $W_e$ stable under $G_K$, then $X_e$ is a free $B_{e,E}$-module, and it is saturated in $W_e$.

**Proof.** See lemma 1.10 of [Nak09]. □

**Proposition 2.9.** If $W$ is a rank 1 $B_{e,E}$-representation of $G_E$, then there exists $\delta : G_E \to E^\times$ such that $W = B_{e,E}(\delta)$.

**Proof.** If we choose a basis $w$ of $W$, then $g(w) = \delta(g)w$ with $\delta(g) \in B_{e,E}$, so that $\delta(g)$ is of the form $\prod_{\tau \in \Sigma} t_\tau^{n_\tau(u)} u(n_\pi)$ by proposition 2.5. Since $\delta(gh) = \delta(g)g(\delta(h))$, (1) of proposition 2.4 implies that the maps $n_\tau : G_E \to \mathbb{Z}$ are continuous homomorphisms. They are therefore trivial, and this implies that $\delta(g) \in E^\times$. □
Proposition 2.13. We have

Corollary 3.2. If

Proof. Take $E = \mathbb{Q}_p(\sqrt{p})$ and $K = \mathbb{Q}_p$ and $W = (E \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{e = n} = t_{id} \cdot B_{e,E}$. The $E$-linear action of $G_{\mathbb{Q}_p}$ on $W$ is given by the map $\delta : g \mapsto g(t_{id})/t_{id}$. If $g \in G_E$, then $\delta(g) = \chi_{\pi}(g)$. If $u = t_{id}^{n} t_{r}^{-n} u_{n}, \ldots \in B_{e,E}$ as in proposition 2.5 and $g \notin G_E$, then $g(u t_{id}) / ut_{id} = t_{id}^{-2n-1} t_{r}^{2n+1} v$ with $v \in (E \cdot \hat{Q}_p^{nr})^\times$. Therefore, there is no character $\eta : G_{\mathbb{Q}_p} \to E^\times$ such that $W = B_{e,E}(\eta)$.

Note that $W$ is the $B_{e,E}$-component of the $B_{K}^\text{\text{E}}$-pair $W_0^{-1}$ of §1.4 of [Nak09]. □

Remark 2.12. The results of this section provide a new proof of proposition [7].

Proof. By theorem 6.5.2 of [FF18], the ring $(E \otimes_{\mathbb{Q}_p} B_{\text{cris}}[1/t_{id}])^{e = 1}$ is a PID. Since $B_{e,E}$ is a localization of $(E \otimes_{\mathbb{Q}_p} B_{\text{cris}}[1/t_{id}])^{e = 1}$, it is itself a PID. □

Proposition 2.13. We have $\text{Frac}(B_{e,E})^{G_K} = E$.

Proof. Take $x/y \in \text{Frac}(B_{e,E})^{G_K}$ with $x, y \in B_{e,E}$ coprime. If $g \in G_K$, then $g(x) y = x g(y)$ so that $x$ divides $g(x)$ and $y$ divides $g(y)$ in $B_{e,E}$ (recall that $B_{e,E}$ is a PID). By proposition 2.6 $x$ and $y$ belong to $B_{e,E}^\times$. This implies that $x/y \in B_{e,E}^{G_K} = E$. □

Corollary 2.14. If $W_e$ is a $B_{e,E}$-representation of $G_K$, then $\dim_E W_e^{G_K} \leq \text{rk} W_e$.

Proof. By a standard argument, proposition 2.13 implies that the map $B_{e,E} \otimes_{E} W_e^{G_K} \to W_e$ is injective. This implies the corollary. □

3. Triangulable representations

In this section, we study triangulable $B_{[K]}^{\alpha E}$-pairs and $B_{e,E}$-representations of $G_K$. We say that a $B_{[K]}^{\alpha E}$-pair is irreducible if it has no non-trivial saturated sub $B_{[K]}^{\alpha E}$-pair (see §2.1 of [Ber08]).

Proposition 3.1. If $W = (W_e, W_{dR}^\times)$ is an irreducible $B_{[K]}^{\alpha E}$-pair, then $W_e$ is an irreducible $B_{e,E}$-representation of $G_K$.

Proof. Let $X_e$ be a sub-object of $W_e$. By proposition 2.8, it is a saturated and free sub-module of $W_e$. The space $X_{dR}^+ = X_{dR} \cap W_{dR}^+$ is an $E \otimes_{\mathbb{Q}_p} B_{dR}^+$ lattice of $X_{dR}$ stable under $G_K$. Hence $X = (X_e, X_{dR}^+)$ is a saturated sub $B_{[K]}^{\alpha E}$-pair of $W$. □

Corollary 3.2. If $W$ is a $B_{[K]}^{\alpha E}$-pair, then $W$ is split triangulable as a $B_{[K]}^{\alpha E}$-pair if and only if $W_e$ is split triangulable as a $B_{e,E}$-representation of $G_K$. 
Proof. It is clear that if $W$ is split triangulable, then so is $W_e$. Conversely, the proof of proposition 3.1 shows how to construct a triangulation of $W$ from a triangulation of $W_e$. □

Let $Δ$ be a set of rank 1 $E ⊗ _Q B_e$-representations of $G_K$. Recall that a $B_{iK}$-pair is split $Δ$-triangulable if it is split triangulable, and the rank 1 $E ⊗ _Q B_e$-representations of $G_K$ that come from the triangulation are all in $Δ$.

**Proposition 3.3.** If $0 → W' → W → W'' → 0$ is an exact sequence of $B_{iK}$-pairs, then $W$ is split $Δ$-triangulable if and only if $W'$ and $W''$ are split $Δ$-triangulable.

Proof. If $W'$ and $W''$ are split $Δ$-triangulable, then $W$ is obviously split $Δ$-triangulable. We now prove the converse. If $W_e$ admits a triangulation, then so do $W'_e$ and $W''_e$. By corollary 3.2, $W'$ and $W''$ are therefore split triangulable. Proposition 2.8 implies that two different triangulations of $W_e$ give rise to two composition series of $W_e$ (seen as an $E ⊗ _Q B_e$-representation of $G_K$). The set of rank 1 objects attached to any triangulation of $W_e$ is therefore well-defined up to permutation by the Jordan-Hölder theorem. Hence if $W$ is split $Δ$-triangulable, then so are $W'$ and $W''$. □

**Proposition 3.4.** If $W_e$ is an irreducible $B_{e,E}$-representation of $G_K$, then every surjective map $π : End(W_e) → B_{e,E}(δ)$ of $B_{e,E}$-representations of $G_K$ is split.

Proof. Write $B_{e,E}(δ) = B_{e,E} · e_δ$, where $g(ε_δ) = δ(g) ε_δ$ with $δ(g) ∈ B_{e,E}$. Recall that if $A$ is a ring and $M$ is a free $A$-module, then $End_A(M)$ is its own dual, for the pairing $(f, g) → Tr(fg)$. The map $π$ is therefore of the form $f → Tr(fh) · ε_δ$ for some $h ∈ End(W_e)$. The map $h$ satisfies $g(h) = δ(g)^{-1}h$, and therefore gives rise to a $G_K$-equivariant map $h : W_e → W_e(δ)$. Since $W_e$ is irreducible, $h$ is invertible. We can then write $End(W_e) = ker(π) ⊕ B_{e,E} · h^{-1}$, which shows that $π$ is split. □

**Theorem 3.5.** If $W_e$ is an irreducible $B_{e,E}$-representation of $G_K$ such that $End(W_e)$ is split triangulable, then the triangulation of $End(W_e)$ splits.

Proof. Write $\{0\} = X_0 ⊂ X_1 ⊂ · · · ⊂ X_d = End(W_e)$, with $X_i/X_{i-1} = B_{e,E}(δ_i)$ for all $i$. By proposition 3.4, the exact sequence $0 → X_{d-1} → End(W_e) → B_{e,E}(δ_d) → 0$ is split, and therefore $End(W_e) = X_{d-1} ⊕ B_{e,E}(δ_d)$.

Suppose that we have an isomorphism $End(W_e) = X_j ⊕ B_{e,E}(δ_{j+1}) ⊕ · · · ⊕ B_{e,E}(δ_d)$. Let $π_j$ denote the composition $End(W_e) → X_j → B_{e,E}(δ_j)$. By proposition 3.1, $End(W_e) = ker(π_j) ⊕ B_{e,E}(δ_j)$. We have $ker(π_j) = X_{j-1} ⊕ B_{e,E}(δ_{j+1}) ⊕ · · · ⊕ B_{e,E}(δ_d)$, so that $End(W_e) = X_{j-1} ⊕ B_{e,E}(δ_j) ⊕ · · · ⊕ B_{e,E}(δ_d)$. The claim follows by induction. □

**Remark 3.6.** Theorem 3.5 is reminiscent of the following result of Chevalley: if $G$ is any group and if $X$ and $Y$ are finite dimensional semi-simple characteristic 0 representations
of $G$, then $X \otimes Y$ is also semi-simple. The same holds for semi-linear representations and, more generally, in any Tannakian category over a field of characteristic 0 [Del16].

4. Cohomology of $B$-pairs

The cohomology of $B_{|K}^\otimes E_i$-pairs is defined and studied in §2.1 of [Nak09]. We recall what we need. Let $W$ be a $B_{|K}^\otimes E_i$-pair. Nakamura constructs an $E$-vector space $H^1(G_K, W)$ that has the following properties

1. $H^1(G_K, W) = \text{Ext}^1(B, W)$ (i.e. it classifies the extensions of $B_{|K}^\otimes E_i$-pairs);
2. there is an exact sequence of $E$-vector spaces

$$W_{\text{dR}}^G \rightarrow H^1(G_K, W) \rightarrow H^1(G_K, W_e) \oplus H^1(G_K, W_{\text{dR}}^+)$$

If $W$ is a rank 1 $B_{|K}^\otimes E_i$-pair with $W_e \in \Delta(Q_p)$, then $W_{\text{dR}}^G$ is an $E$-vector space of dimension 1 or 0, depending on whether $W_e$ (extended to $G_{Q_p}$) is de Rham or not. Since $W_{\text{dR}}^G = K \otimes_{Q_p} W_{\text{dR}}^{G_{Q_p}}$, this implies that $W_{\text{dR}}^G = \{0\}$ if $W$ is not de Rham. Note that if $W$ is a rank 1 $B_{|K}^\otimes E_i$-pair with $K \neq Q_p$, then $W$ may be “partially de Rham” in the sense of [Din17], so that in general $W_{\text{dR}}^G$ can be non-zero even if $W$ is not de Rham.

**Proposition 4.1.** If $W$ is a rank 1 $B_{|K}^\otimes E_i$-pair with $W_e \in \Delta(Q_p)$, then the map $H^1(G_K, W_{\text{dR}}^+) \rightarrow H^1(G_K, W_{\text{dR}}^+)$ is injective.

**Proof.** If $W$ is de Rham, this follows from lemma 2.6 of [Nak09]. Assume now that $W$ is not de Rham. The lattice $W_{\text{dR}}^+$ is not necessarily $G_{Q_p}$-stable in $W_{\text{dR}}$. Let $Y_{\text{dR}}^+ = \sum_{g \in G_{Q_p}/G_K} g(W_{\text{dR}}^+)$, which is a $G_{Q_p}$-stable lattice, and let $Y = (W_e, Y_{\text{dR}}^+)$. This is a $B_{|Q_p}^\otimes E_i$-pair, which is not de Rham and hence not Hodge-Tate by lemma 4.1 of [Nak09]. It has a single Hodge-Tate weight $y$, with $y \notin Z$. All the Hodge-Tate weights of $Y$ seen as a $B_{|K}^\otimes E_i$-pair are therefore equal to $y$. By proposition 2.4 of [BC10], the Hodge-Tate weights of $W$ are all of the form $y + a$ with $a \in Z$, and hence none of them is in $Z$. Therefore for every $i \in Z$, we have $(t^i W_{\text{dR}}^+ / t^{i+1} W_{\text{dR}}^+)^{G_K} = 0$.

This implies that $(W_{\text{dR}}^+ / W_{\text{dR}}^+)^{G_K} = 0$. Since we have an exact sequence

$$(W_{\text{dR}}^+ / W_{\text{dR}}^+)^{G_K} \rightarrow H^1(G_K, W_{\text{dR}}^+) \rightarrow H^1(G_K, W_{\text{dR}}^+)$$

the second arrow is injective. \qed

**Corollary 4.2.** If $X$ is a direct sum of rank 1 $B_{|K}^\otimes E_i$-pairs whose $E \otimes_{Q_p} B_{|Q_p}$-components are in $\Delta(Q_p)$, then the map $H^1(G_K, X_{\text{dR}}^+) \rightarrow H^1(G_K, X_{\text{dR}}^+)$ is injective.

**Proposition 4.3.** If a $B_{|K}^\otimes E_i$-pair $W$ is split $\Delta(Q_p)$-triangulable, and if the triangulation of $W_e$ splits as a direct sum of 1-dimensional $B_{|Q_p}^\otimes E_i$-representations $B(\delta_i)_e$ such that $\delta_i \delta_j^{-1}$ is not de Rham for any $i \neq j$, then the triangulation of $W$ splits.
Proof. Let $0 = W_0 \subset W_1 \subset \cdots \subset W_d = W$ be the given triangulation of $W$. We prove by induction on $j$ that $W_j = B(\delta_1) \oplus \cdots \oplus B(\delta_j)$. This is true for $j = 1$, assume it holds for $j - 1$. Write $0 \rightarrow W_{j-1} \rightarrow W_j \rightarrow B(\delta_j) \rightarrow 0$ and $W_{j-1} = B(\delta_1) \oplus \cdots \oplus B(\delta_{j-1})$. Let $X = W_{j-1}(\delta_j^{-1})$ and $Y = W_j(\delta_j^{-1})$. The $B^{\otimes E}_{[L]}$-pair $Y$ corresponds to a class in $H^1(G_K, X)$. The $B_{e,E}$-representation $Y_e$ is split, and therefore so is $Y_{\text{dir}}$. By corollary 4.2 so is $Y_{\text{dir}}^+$. The class of $Y$ in $H^1(G_K, X)$ is therefore in the kernel of $H^1(G_K, X) \rightarrow H^1(G_K, X_e) \oplus H^1(G_K, X_{\text{dir}}^+)$. Since $X_{\text{dir}}^{G_K} = 0$ by hypothesis, the class of $Y$ is trivial and hence $W_j = W_{j-1} \oplus B(\delta_j)$. The proposition follows by induction. \hfill \Box

5. Proof of the main theorem

In this section, we prove theorem B. Let $F$ be a finite extension of $E$ of degree $\geq 2$, and write $F \otimes_E F = \oplus_i F_i$. There are at least two summands since $F$ itself is one of them.

Proposition 5.1. Let $F/E$ be as above, and let $W$ be an $F$-linear representation of $G_K$. We have $F \otimes_E W = \oplus_i (F_i \otimes_F W)$ as $F$-linear representations of $G_K$.

Proof. We have $F \otimes_E W = (F \otimes_F W) \otimes_F W = \oplus_i (F_i \otimes_F W)$. \hfill \Box

Corollary 5.2. If $W$ is a $B_{e,E}$-representation of $G_K$ that has an $F$-linear structure, then $W$ becomes reducible after extending scalars from $E$ to $F$.

Let us say that a $B^{\otimes E}_{[K]}$-pair $W$ is completely irreducible if $(F \otimes_E W)|_{G_L}$ is an irreducible $B^{\otimes F}_{[L]}$-pair for every finite extensions $F$ of $E$ and $L$ of $K$.

Proposition 5.3. If $X$ and $Y$ are two completely irreducible $B^{\otimes E}_{[K]}$-pairs such that $\text{Hom}(X, Y)$ is split $\Delta(Q_p)$-triangulable, then $X$ and $Y$ are of rank 1.

Proof. There exists a rank 1 $B^{\otimes E}_{[K]}$-pair $B(\delta)$ and an inclusion $B(\delta) \subset \text{Hom}(X, Y)$. This gives rise to a non-zero map $X \rightarrow Y(\delta^{-1})$ of $B^{\otimes E}_{[K]}$-pairs. Write $B(\delta)_e = B_{e,E}(\mu)$. Since $X$ and $Y$ are irreducible, $X_e$ and $Y_e$ are irreducible $B_{e,E}$-representations of $G_K$ (proposition 4.4), and the map $X_e \rightarrow Y_e(\mu^{-1})$ is therefore an isomorphism. This implies that $\text{Hom}(X_e, Y_e) = \text{End}(X_e)(\mu)$, so that $\text{End}(X_e)$ is split triangulable. By theorem 3.13 the triangulation of $\text{End}(X_e)$ splits. The triangulation of $\text{Hom}(X_e, Y_e) = \text{End}(X_e)(\mu)$ therefore also splits. Write $\text{Hom}(X_e, Y_e) = \oplus_i B(\delta_i)_e$.

Suppose that none of the $\delta_i \delta_j^{-1}$ are de Rham for any $i \neq j$. By proposition 4.4 applied to $W = \text{Hom}(X, Y)$, the triangulation of $\text{Hom}(X, Y)$ splits. We can therefore write $\text{Hom}(X, Y) = \oplus_i B(\delta_i)$. Since $X$ and $Y$ are both irreducible, they are pure of some slopes $s$ and $t$ by theorem 3.13. The $B^{\otimes E}_{[K]}$-pair $\text{Hom}(X, Y)$ is then pure of slope $t - s$ by (1) of proposition 4.4. By (3) of ibid, each of the $B(\delta_i)$ is also pure of slope $t - s$. Each $B(\delta_i)$ gives rise to a map $X \rightarrow Y(\delta_i^{-1})$, which is an isomorphism of $B^{\otimes E}_{[K]}$-pairs by (2)

\hfill \Box
of ibid, since $X$ and $Y$ are both pure of slope $s$. By taking determinants, we get 
$$\delta^i = \det(Y) \det(X)^{-1}$$ for every $i$. This implies that $(\delta_1, \delta_2, \ldots, \delta_n)^n = 1$ so that $\delta \delta_i$ is of finite order, and hence de Rham (lemma 4.1 of [Nak09]), contradicting our assumption.

Therefore, one of the $\delta_i \delta_j$ is de Rham for some $i \neq j$. Write $B(\delta_i) = B_{e, E}(\mu)$ and $B(\delta_j) = B_{e, E}(\mu)$ where $\mu$ and the $\mu_j$ are characters $G_E \to E^\times$ (see proposition 2.9), so that $\text{End}(X_\delta)(\mu) = \bigoplus_k B_{e, E}(\mu)$ as $B_{e, E}$-representations of $G_E$. The fact that $\delta_i \delta_j$ is de Rham implies that $\mu_i \mu_j^{-1}$ is de Rham. We then have $X_\delta = X_\delta(\mu_i \mu_j^{-1}) = X_\delta(\mu_i \mu_j^{-1})$, so that $X_\delta = X_\delta(\mu_i \mu_j^{-1})$. By taking determinants, we find that $B_{e, E}(\mu_i \mu_j^{-1}) = B_{e, E}$ and therefore by corollary 2.7, $(\mu_i \mu_j^{-1})^n : G_E \to E^\times$ is de Rham and the sum of its weights is 0. This implies that the sum of the weights of $\mu_i \mu_j^{-1} : G_E \to E^\times$ is 0. By corollary 2.7, $\mu_i \mu_j^{-1} = \chi \eta$ with $\chi : G_E \to E^\times$ a $B_{e, E}$-admissible character and $\eta : G_E \to E^\times$ potentially unramified. Since $X_\delta(\chi \eta) = X_\delta$ and $X_\delta(\chi) = X_\delta$, we get $X_\delta(\eta) = X_\delta$. By taking determinants, we get that $\eta^n$ is $B_{e, E}$-admissible. Since $\eta^n$ is potentially unramified, it is trivial. Hence $\eta$ is a character of finite order of $G_E$, and so there exists a finite extension $L$ of $K$ such that $\mu_i = \chi \mu_j$ on $G_L$.

The space $\text{End}(X_\delta)(\mu)$ contains $B_{e, E}(\mu_j) \oplus B_{e, E}(\mu_j)$, which is isomorphic to $B_{e, E}(\mu_j) \oplus B_{e, E}(\mu_j)$ after restricting to $G_L$. Let $f$ and $g$ be the two resulting isomorphisms $X_\delta \to X_\delta(\mu_i \mu_j^{-1})$. The map $h = f^{-1} \circ g : X_\delta \to X_\delta$ is $G_L$-equivariant and is not in $E^\times \cdot \text{Id}$ since $f$ and $g$ are $B_{e, E}$-linearly independent. Therefore, $\text{End}(X_\delta)^{G_L}$ is strictly larger than $E$.

Since $X_\delta|_{G_L}$ is irreducible, Schur’s lemma and corollary 2.14 imply that $\text{End}(X_\delta)^{G_L}$ contains a field $F$ such that $[F : E] \geq 2$ (for example, $F = E(h)$). Hence $X_\delta|_{G_L}$ has an $F$-linear structure. Corollary 3.1 implies that $(F \otimes_E X_\delta)|_{G_L}$ is reducible. By proposition 3.3, $X$ is not completely irreducible. This is a contradiction, so $X$ had to be of rank 1. By symmetry, the same holds for $Y$.

We now recall and prove theorem B. A strict sub-quotient of a $B^{\otimes E}$-pair is a quotient of a saturated sub $B^{\otimes E}$-pair.

**Theorem 5.4.** If $X$ and $Y$ are two $B^{\otimes E}$-pairs whose tensor product is $\Delta(Q_p)$-triangulable, then $X$ and $Y$ are both potentially triangulable.

**Proof.** We can replace $E$ and $K$ by finite extensions $F$ and $L$ if necessary, and write $X$ and $Y$ as successive extensions of completely reducible $B^{\otimes F}$-pairs. If $X'$ and $Y'$ are two strict sub-quotients of $X$ and $Y$, then $X' \otimes Y'$ is a strict sub-quotient of $X \otimes Y$, and it is $\Delta(Q_p)$-triangulable by proposition 3.3. Proposition 5.3 applied to $(X')^*$ and $Y'$ so that $X' \otimes Y' = \text{Hom}((X')^*, Y')$, tells us that $X'$ and $Y'$ are of rank 1.

Hence the $B^{\otimes F}$-pairs $(F \otimes_E X)|_{G_L}$ and $(F \otimes_E Y)|_{G_L}$ are split triangulable. 

\qed
Corollary 5.5. If $X_e$ and $Y_e$ are two $B_{e,E}$-representations of $G_K$ whose tensor product is triangulable, with the rank 1 sub-quotients extending to $B_{e,E}$-representations of $G_{Q_p}$, then $X_e$ and $Y_e$ are both potentially triangulable.

Proof. By proposition 1.2, $X_e$ and $Y_e$ extend to $B_{K}^\otimes E$-pairs. The result follows from corollary 3.2 and theorem 5.4. □

We finish with an example of a representation $V$ such that $V \otimes E V$ is trianguline, but $V$ itself is not trianguline. This shows that the “potentially” in the statement of theorem A cannot be avoided. Let $Q_8$ denote the quaternion group. If $p \equiv 3 \mod 4$, there is a Galois extension $K/Q_p$ such that $\text{Gal}(K/Q_p) = Q_8$ (see II.3.6 of [JY88]). Choose such a $p$ and $K$, and let $E$ be a finite extension of $Q_p$ containing $\sqrt{-1}$. The group $Q_8$ has a (unique) irreducible 2-dimensional $E$-linear representation, which we inflate to a representation $V$ of $G_{Q_p}$. One can check that $V \otimes E V$ is a direct sum of characters, hence trianguline, and that the semi-linear representation $\text{Frac}(B_{e,E}) \otimes E V$ is irreducible. This holds for all $E$ as above, so that $V$ is not trianguline.

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