Experimental verification of the minimal dynamical-system modeling of autonomous heat engine

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(Dated: November 18, 2019)
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The characterization of autonomous heat engines is one of the goals of the recent challenges in constructing finite-time thermodynamics. However, the high complexity of the engine involving an intimate coupling among heat, gas flow, and mechanics has prevented a simple modeling. We experimentally demonstrate that the nonequilibrium dynamics of a low-temperature-differential Stirling engine, which is a model autonomous engine, is reproduced quantitatively by a recently-derived minimal dynamical model with only two variables. The model characterizes the engine cycle as a limit cycle and reveals its bifurcation dynamics. Thus, our experiments will support a novel approach to explore the finite-time thermodynamics of autonomous heat engines based on a simple dynamical system.

PACS numbers: 07.20.Pe, 05.45.-a, 05.70.Ln

The finite-time thermodynamics has made efforts to incorporate the finite-time dynamics into conventional thermodynamics which is originally formulated based on infinitely slow quasistatic processes\cite{1}. It was successful in characterizing the efficiency at maximum power\cite{2,3}, optimal control with the minimal energy cost\cite{4,5}, the trade-off relations between the power and efficiency\cite{6,7,8}, and stochastic heat engines\cite{9,10}. One of the goals of the finite-time thermodynamics is to elucidate the dynamics and energetics of the autonomous heat engines. As the heat engines have played essential roles in constructing thermodynamics since Carnot\cite{21}, the autonomous heat engines working far from equilibrium would contribute fundamentally to construct finite-time thermodynamics. However, the high complexity of the engine dynamics involving a mechanical motion, heat flow, and gas flow has prevented a simple modeling.

Recently, a simple dynamical model of an autonomous heat engine has been proposed \cite{22}. This model characterizes the engine dynamics as a limit cycle of dynamical equations and predicts the bifurcation dynamics of the engine cycle. Its derivation is based on a low-temperature-differential Stirling engine (LTD-SE), which is a model autocatalytic heat engine with a minimal structure \cite{23}. However, once verified by experiments, the model is expected to be applicable to a wide range of autonomous heat engines with small modifications because of its simple and intuitive formulation. In this letter, we demonstrate experimentally that this simple model reproduced the essential characteristics of an LTD-SE, such as a bifurcation diagram and pressure-volume curve.

The Stirling engine is an autonomous and closed heat engine \cite{24,25}. Given a temperature difference, the engine cycles the volume, temperature, and pressure inside a cylinder autonomously without external timing control and rotates a flywheel unidirectionally. Theoretically, an ideal Stirling engine, i.e., a Stirling cycle, achieves the Carnot efficiency. The LTD-SE \cite{22,23} consists of a power piston, displacer, flywheel, two cranks, and two rods connecting the piston and displacer to the flywheel (Fig. 1). The flywheel rotates when a temperature difference exceeding a threshold value is given between the top and bottom plates of the cylinder. When \(\Delta T \equiv T_{btm} - T_{top} > \Delta T_c > 0\), the heating from the bottom increases the internal pressure and pushes the power piston upward. This drives the flywheel rotation via the rod connecting the power piston and the crank attached to the flywheel, which then pushes the displacer downward via the rod connecting the displacer and another crank. The displacer serves to switch the heat baths.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{(Color online) Thermodynamic diagram of an autonomous heat engine with the rotation cycle of the low-temperature-differential Stirling engine as a model system. The theory \cite{22} assumes that the internal gas is effectively in contact with a single heat bath with a temperature oscillating between \(T_{top}\) and \(T_{btm}\) synchronized with the displacer motion.}
\end{figure}
When the displacer moves downward, most air in the cylinder moves to the upper side and makes contact with the top plate at a lower temperature of $T_{\text{top}}$. The cooling of gas results in the pressure decrease and pushes the power piston downward, and the cycle restores to the initial state. The $\pi/2$ out of phase of the two cranks makes a cycle. The flywheel provides inertia necessary for a smooth rotation. When $\Delta T < \Delta T_c^- < 0$, the flywheel rotates in the opposite direction with an inverted mechanism. However, the dynamics are not symmetric between the positive and negative $\Delta T$, as seen below.

**Theory.** One of the authors proposed the equation of motion of the flywheel [22]:

$$\dot{\theta} = \omega,$$

$$I \dot{\omega} = s [p(\theta, \omega) - p_0] r \sin \theta - \Gamma \omega. \quad (1)$$

Here, $\theta$ and $\omega$ are the angular position and velocity of the flywheel, respectively (Fig. 1). $I$ and $\Gamma$ are the moment of inertia and frictional coefficient, respectively, of the engine’s rotational degree of freedom. $s$ is the sectional area of the power piston. $s[p(\theta, \omega) - p_0] \equiv s \Delta p$ corresponds to the force on the crank applied by the power piston via rod, and $s \Delta p \cdot \pi r \sin \theta$ is the torque on the flywheel (piston-crank mechanism). $p_0$ is the external pressure. The theory [22] approximates that the gas is in contact with a single heat bath at an effective temperature $T_{\text{eff}}(\theta) \equiv T_{\text{top}} + \frac{1 + \sin \theta}{2} \Delta T$, which oscillates between $T_{\text{top}}$ and $T_{\text{btm}}$, synchronized with the displacer motion. The gas temperature $T(\theta, \omega)$ is calculated by considering the heat transfer between the gas and the top and bottom plates of the cylinder under an adiabatic assumption that the temperature equilibration is sufficiently fast compared to the flywheel dynamics. $p(\theta, \omega)$ is calculated using the equation of the state.

The above model contains $\Delta T$ explicitly through $p(\theta, \omega)$ [22], proposing a novel concept that the LTD-SE is a thermodynamic pendulum driven by a thermodynamic force. As noted, the steady cycling of the LTD-SE is only possible for $\Delta T$ exceeding threshold values $\Delta T_c^\pm$ [22]: $\Delta T > \Delta T_c^+ > 0$ or $\Delta T < \Delta T_c^- < 0$. The theory predicts that the cycle observed is a stable limit cycle of $\mathcal{U}$, which disappears at $\Delta T_c^\pm$ by colliding with a saddle point of $\mathcal{U}$ as $|\Delta T|$ is decreased. This bifurcation is the homoclinic bifurcation, a kind of a global bifurcation, and seen in, for example, a driven pendulum and a Josephson junction [29]. At $\Delta T_c^- < \Delta T < \Delta T_c^+$, the stable fixed point of $\mathcal{U}$ is the unique stable attractor.

For analyzing experimental data in the below, we modeled the system simply as

$$T(\theta, \omega) = T_{\text{top}} + \frac{1 + \alpha \sin(\theta - \omega \tau)}{2} \Delta T,$$  

$$p(\theta, \omega) = \beta \frac{nRT(\theta, \omega)}{V(\theta)}. \quad (3)$$

Here, the effect of the heat transfer on the gas temperature is simply implemented by two parameters, the proportional coefficient $\alpha$ and the time delay $\tau$. The temperature $T(\theta, \omega)$ oscillates with the amplitude of $\alpha \Delta T/2$ around $T_{\text{top}} + \Delta T/2$. The gas pressure $p(\theta, \omega)$ is calculated based on an effective equation of the state for the ideal gas. $n$ is the mol number of the internal gas, and $R$ is the gas constant. $V(\theta) = V_0 + rs(1 - \cos \theta)$ is the volume of the cylinder, where $V_0$ is the cylinder volume excluding the displacer volume. The temperature and pressure can be nonuniform inside the cylinder, and therefore the equation of the state for the ideal gas may not hold as it is. The coefficient $\beta$ is introduced to compensate such an effect. We will determine $\alpha$, $\beta$, and $\tau$ by fitting experimental data.

**Experiment.** An LTD-SE (N-92 type) was bought from Kontax (UK). We controlled $T_{\text{top}}$ and $T_{\text{btm}}$ by Peltier modules equipped with water flowing block (Fig. 2b). The temperatures were monitored at 2.5 Hz by Platinum resistance temperature detectors attached to the

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**FIG. 2.** (Color online) Rotation. a, Experimental set up. See the main text for details. b, Steady rotational trajectories at different $\Delta T$. c, Rotational trajectories at $\Delta T = 8 \text{ K} > \Delta T_c^+$ initiated with small (dashed) or large (solid) angular velocity. d, Pressure-volume curves for $\Delta T = 5 \text{ K} < \Delta T_c^-$. e, Pressure-volume curves for $\Delta T = 24 \text{ K} \text{ (red)}$, $-24 \text{ K} \text{ (navy)}$, $9 \text{ K} \text{ (yellow)}$, and $-9 \text{ K} \text{ (cyan)}$ obtained by experiments (solid) and theories (dashed). The curves are colored in the same way as in b. The cycling direction was clockwise under all the conditions as indicated by black arrows. The average of $\Delta p$ for the theoretical curves was forced to zero. f, Relaxation dynamics at $\Delta T = 0 \text{ K}$. 


surface of the plates. The image of the target pattern (three circles aligned in an isosceles triangle) attached to the crank screw was recorded by a high-speed camera (Basler, Germany) at 100 Hz and analyzed in real time to obtain $\theta$ and $\omega$. A pressure sensor (Copal electronics, Japan) was fixed at the side of the cylinder to monitor $\Delta p$ at 100 Hz.

Without perturbation, the engine was settled at a stationary position $\theta_0 \simeq -35^\circ$, where the pressure difference across the power piston and the gravity force on the power piston, displacer, crank screws, and rods are presumably balanced. The theory (1) neglects the gravity force for the simplification and locates the stationary position at $0^\circ$.

At a sufficiently large $|\Delta T|$, whereas the stationary state at $\theta_0$ is still stable, the flywheel rotated steadily when an initial angular momentum with a sufficiently large magnitude was given by hand. The rotation direction depended on the sign of $\Delta T$ (Fig. 2b). As expected, the angular velocity became smaller when $|\Delta T|$ decreased and stalled at a finite value of $\Delta T$, which was slightly different for the sign of $\Delta T$; $\Delta T_+ < 7.4$ K and $\Delta T_- = -7.2$ K (Fig. 3a).

Figure 2 shows two typical trajectories started with different initial angular velocities at $\Delta T > \Delta T_c^+$. With a large initial angular velocity, we observed that the trajectory eventually converged to the periodic trajectory determined by $\Delta T$. When the engine in this steady state was perturbed externally by hand, the trajectory soon returned to the original periodic one, implying a stable limit cycle. With a small initial angular velocity, the trajectory converged to the stable fixed point at $\theta \simeq -35^\circ$, failing in converging to the stable limit cycle. When $\Delta T_- < \Delta T < \Delta T_c^+$, the stable fixed point at $\theta_0 \simeq -35^\circ$ was the unique stable attractor (Fig. 2c).

The pressure-volume curve exhibited a circular diagram (Fig. 2d), demonstrating a heat engine. The cycling direction was the same independent of the sign of $\Delta T$, and the PV curves were nearly symmetric for the sign of $\Delta T$. The area increased with $|\Delta T|$.

For evaluating the bifurcation diagram, we manually provided an initial angular momentum at $\Delta T = 36$ K or $-30$ K with keeping $T_{\text{top}} = 24^\circ$C and waited for about one hour for the sufficient relaxation of the temperatures and flywheel rotation. Then, we varied $\Delta T$ from 36 K to 0 K or from $-30$ K to 0 K in a stepwise manner at a rate of $\pm 1$ K every 180 s for $|\Delta T| > 8.5$ K and $\pm 0.02$ K every 60 s otherwise with keeping $T_{\text{top}} = 24^\circ$C.

The bifurcation diagram is shown in Fig. 3a superposed by a theoretical curve (1) with the following parameters. $V_0 = 44900$ mm$^3$, $s = 71$ mm$^2$, $r = 3.5$ mm, $I = 5.7 \times 10^{-5}$ kg m$^2$, and $p_0 = 101.3$ kPa. $n = 0.00185$ mol, $R = 8.3144 /$ K mol. We estimated $\Gamma$ roughly to be $2.1 \times 10^{-6}$ kg m$^2$/J/s from the relaxation curve of $\omega(t)$ at $\Delta T = 0$ (Fig. 2f) although the relaxation curve was more linear and much less oscillatory than the theoretical curve expected from (1). The linear relaxation curve implies large friction at small $\omega$ region, which is typical for the dry friction of the bearing. The nonlinear friction was not explicitly included in the model (1) for simplicity. We determined $\alpha$, $\beta$, and $\gamma$ as 0.12, 0.92, and 10 ms, respectively, by fitting. The theory reproduced the bifurcation diagram quantitatively (Fig. 3a) including the steep change in the vicinity of $\Delta T_c$, supporting the two-variable model. The pressure-volume curve was also quantitatively reproduced (Fig. 2f).

For further tests, we induced additional frictional load by pressing a polishing buff to a crank disc. The magnitude of the friction was manually changed by shifting the pressure of the buff on the crank disc using a mechanical translational stage. The additional friction decreased the angular velocity and increased $\Delta T_c^+$ (Fig. 3b)). The theory with the same parameters except for the frictional coefficient, which was fitted by eye, reproduced the experimental curves quantitatively. These results demonstrate that the autonomous engine cycles of the LTD-SE can be described by a simple two-variable model of (1).

We also discovered an oscillation branch at $\Delta T \leq -28$ K (Fig. 3b). Here, for exploring a small $\Delta T$ region, $T_{\text{top}}$ was set to a relatively large value, 60$^\circ$C. When we shifted the flywheel angle a little bit from the stable fixed angle.

FIG. 3. (Color online) Bifurcation diagrams. a, The time-averaged angular velocity ($\omega$) of the limit cycle was plotted against the temperature differences $\Delta T$. At $|\Delta T| \geq 9$ K, three experimental traces were averaged (circle). The error bar corresponds to S.D. At $|\Delta T| < 9$ K, six ($\Delta T > 0$) or four ($\Delta T < 0$) traces were superposed (solid lines). b, Bifurcation diagrams with/without additional frictional load. Stable limit cycles (open symbols) and stable fixed points (closed symbols). The values of $\Gamma$ were obtained by fitting.

\[ |\Delta T| = 314 \text{ J} / \text{K} \text{mol.} \]
the complex autonomous heat engine are reproduced by a simple and intuitive two-variable model quantitatively. Thus, our experiments support a novel approach to explore the finite-time thermodynamics of autonomous heat engines based on a simple dynamical-system description. The results also validate the concept that the engine can be considered as a thermodynamic pendulum driven by $\Delta T$.

The two-variable model is a minimal model of the autonomous heat engines in the sense that at least two variables are required to describe a limit cycle. We expect that the model is applicable to a wide range of autonomous heat engines by modifying, for example, the cycle shape $T(\theta, \omega)$ in (2) and the piston-crank mechanism $r \sin \theta$ in (1). The model did not reproduce the oscillation branch. At $\Delta T \leq -33$ K, the trajectory $\theta(\omega)$ possessed an intersection (Fig. 4b), meaning that the description by only $\theta$ and $\omega$ does no longer describe the oscillation dynamics correctly at some points. At the intersection points, the pressure was a multiple-valued function of $(\theta, \omega)$, suggesting that (3) is not valid at these points. Although the oscillation is not an essential mode of the engine, it would be intriguing to explore what modification to the theory could describe the detailed dynamics of the LTD-SE including the oscillation mode.

The formulation of the thermodynamic efficiency of the autonomous heat engine would be of crucial importance, which is complementary to the formulation in non-autonomous heat engines (2, 3). The evaluation of the efficiency requires the measurement of the heat flowing through the engine and remains for future studies.

This work was supported by JSPS KAKENHI (18H05427 and 19K03651).

$\theta, \omega$, $\Delta T$}

FIG. 4. (Color online) Oscillation mode observed at $T_{\text{top}} = 60^\circ$ C. a, $\theta$ as the functions of time (left), $\omega$ (center), and $\Delta p$ (right). b, Peak angles of the oscillation (circle), stable fixed points (closed square), unstable fixed points (open square), and expected unstable limit cycles (dashed line).

We demonstrated that the essential characteristics of the flywheel started a periodic oscillation with a finite amplitude and a period of about 10 seconds, which can be considered as an oscillatory stable limit cycle. This limit cycle showed complicated behaviors; the amplitude increases accompanied by seemingly period-doubling bifurcations ($-28$ K $\geq \Delta T > -37$ K), aperiodic oscillation similar to chaos ($-37$ K $\geq \Delta T > -40$ K), and again periodic oscillations accompanied by small additional oscillations ($-40$ K $\geq \Delta T > -55$ K). The oscillation branch disappeared at $\Delta T \leq -55$ K, and a small perturbation got drawn into a rotation branch. The rotation mode was observed for all $\Delta T < 0$ with a sufficiently large initial angular velocity. The oscillation was not observed for $\Delta T > 0$. The stable fixed point (spiral) at $\theta_0 \simeq -35^\circ$ became unstable at $\Delta T = -40$ K (Fig. 4b), which may be considered to be a subcritical Hopf bifurcation. An unstable limit-cycle branch associated with this Hopf bifurcation, which was difficult to be identified by experiments (dashed line in Fig. 4b) and the stable branch may be created and annihilated in a pair at $\Delta T = -28$ K. We need further studies to determine the bifurcation characteristics.

We demonstrated that the essential characteristics of

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