CLUSTER EXPANSION FORMULAS AND PERFECT MATCHINGS

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Abstract. We study cluster algebras with principal coefficient systems that are associated to unpunctured surfaces. We give a direct formula for the Laurent polynomial expansion of cluster variables in these cluster algebras in terms of perfect matchings of a certain graph $G_{X,\gamma}$, that is constructed from the surface by recursive glueing of elementary pieces that we call tiles. We also give a second formula for these Laurent polynomial expansions in terms of subgraphs of the graph $G_{X,\gamma}$.

1. Introduction

Cluster algebras, introduced in [FZ1], are commutative algebras equipped with a distinguished set of generators, the cluster variables. The cluster variables are grouped into sets of constant cardinality $n$, the clusters, and the integer $n$ is called the rank of the cluster algebra. Starting with an initial cluster $x$ (together with a skew symmetrizable integer $n \times n$ matrix $B = (b_{ij})$ and a coefficient vector $y = (y_i)$ whose entries are elements of a torsion-free abelian group $\mathbb{P}$) the set of cluster variables is obtained by repeated application of so called mutations. To be more precise, let $\mathcal{F}$ be the field of rational functions in the indeterminates $x_1, x_2, \ldots, x_n$ over the quotient field of the integer group ring $\mathbb{ZP}$. Thus $x = \{x_1, x_2, \ldots, x_n\}$ is a transcendence basis for $\mathcal{F}$. For every $k = 1, 2, \ldots, n$, the mutation $\mu_k(x)$ of the cluster $x = \{x_1, x_2, \ldots, x_n\}$ is a new cluster $\mu_k(x) = x \setminus \{x_k\} \cup \{x'_k\}$ obtained from $x$ by replacing the cluster variable $x_k$ by the new cluster variable

\begin{equation}
 x'_k = \frac{1}{x_k} \left( y_k^+ \prod_{b_{ki} > 0} x_i^{b_{ki}} + y_k^- \prod_{b_{ki} < 0} x_i^{-b_{ki}} \right)
\end{equation}

in $\mathcal{F}$, where $y_k^+, y_k^-$ are certain monomials in $y_1, y_2, \ldots, y_n$. Mutations also change the attached matrix $B$ as well as the coefficient vector $y$, see [FZ1].

The set of all cluster variables is the union of all clusters obtained from an initial cluster $x$ by repeated mutations. Note that this set may be infinite.

It is clear from the construction that every cluster variable is a rational function in the initial cluster variables $x_1, x_2, \ldots, x_n$. In [FZ1] it is shown that every cluster variable $u$ is actually a Laurent polynomial in the $x_i$, that is, $u$ can be written as a reduced fraction

\begin{equation}
 u = \frac{f(x_1, x_2, \ldots, x_n)}{\prod_{i=1}^n x_i^{d_i}},
\end{equation}

where $f \in \mathbb{ZP}[x_1, x_2, \ldots, x_n]$ and $d_i \geq 0$. The right hand side of equation (2) is called the cluster expansion of $u$ in $x$. 

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The cluster algebra is determined by the initial matrix $B$ and the choice of the coefficient system. A canonical choice of coefficients is the principal coefficient system, introduced in [FZ2], which means that the coefficient group $P$ is the free abelian group on $n$ generators $y_1, y_2, \ldots, y_n$, and the initial coefficient tuple $y = \{y_1, y_2, \ldots, y_n\}$ consists of these $n$ generators. In [FZ2], the authors show that knowing the expansion formulas in the case where the cluster algebra has principal coefficients allows one to compute the expansion formulas for arbitrary coefficient systems.

Inspired by the work of Fock and Goncharov [FG1, FG2, FG3] and Gekhtman, Shapiro and Vainshtein [GSV1, GSV2] which discovered cluster structures in the context of Teichmüller theory, Fomin, Shapiro and Thurston [FST, FT] initiated a systematic study of the cluster algebras arising from triangulations of a surface with boundary and marked points. In this approach, cluster variables in the cluster algebra correspond to arcs in the surface, and clusters correspond to triangulations. In [S2], building on earlier results in [S1, ST], this model was used to give a direct expansion formula for cluster variables in cluster algebras associated to unpunctured surfaces, with arbitrary coefficients, in terms of certain paths on the triangulation.

Our first main result in this paper is a new parametrization of this formula in terms of perfect matchings of a certain weighted graph that is constructed from the surface by recursive gluing of elementary pieces that we call tiles. To be more precise, let $x_\gamma$ be a cluster variable corresponding to an arc $\gamma$ in the unpunctured surface and let $d$ be the number of crossings between $\gamma$ and the triangulation $T$ of the surface. Then $\gamma$ runs through $d + 1$ triangles of $T$ and each pair of consecutive triangles forms a quadrilateral which we call a tile. So we obtain $d$ tiles, each of which is a weighted graph, whose weights are given by the cluster variables $x_\tau$ associated to the arcs $\tau$ of the triangulation $T$.

We obtain a weighted graph $G_{T,\gamma}$ by gluing the $d$ tiles in a specific way and then deleting the diagonal in each tile. To any perfect matching $M$ of this graph we associate its weight $w(M)$, which is the product of the weights of its edges, hence a product of cluster variables. We prove the following cluster expansion formula:

**Theorem 3.1**

$$x_\gamma = \sum_M \frac{w(M) y(M)}{x_{i_1} x_{i_2} \cdots x_{i_d}},$$

where the sum is over all perfect matchings $M$ of $G_{T,\gamma}$, $w(M)$ is the weight of $M$, and $y(M)$ is a monomial in $y$.

We also give a formula for the coefficients $y(M)$ in terms of perfect matchings as follows. The $F$-polynomial $F_{\gamma}$, introduced in [FZ2] is obtained from the Laurent polynomial $x_\gamma$ (with principal coefficients) by substituting 1 for each of the cluster variables $x_1, x_2, \ldots, x_n$. By [S2] Theorem 6.2, Corollary 6.4, the $F$-polynomial has constant term 1 and a unique term of maximal degree that is divisible by all the other occurring monomials. The two corresponding matchings are the unique two matchings that have all their edges on the boundary of the graph $G_{T,\gamma}$. We denote by $M_-$ the one with $y(M_-) = 1$ and the other by $M_+$. Now, for an arbitrary perfect matching $M$, the coefficient $y(M)$ is determined by the set of edges of the symmetric difference $M_- \ominus M = (M_- \cup M) \setminus (M_- \cap M)$ as follows.
The set $M_\ominus \odot M$ is the set of boundary edges of a (possibly disconnected) subgraph $G_M$ of $G_{T,\gamma}$ which is a union of tiles $G_M = \cup_{j \in J} S_j$. Moreover,

$$y(M) = \prod_{j \in J} y_j$$

As an immediate corollary, we see that the corresponding $g$-vector, introduced in [FZ2], is

$$g_\gamma = \text{deg} \left( \frac{w(M_\ominus)}{x_{i_1} \cdots x_{i_d}} \right).$$

Our third main result is yet another description of the formula of Theorem 3.1 in terms of the graph $G_{T,\gamma}$ only. In order to state this result, we need some notation. If $H$ is a graph, let $c(H)$ be the number of connected components of $H$, let $E(H)$ be the set of edges of $H$, and denote by $\partial H$ the set of boundary edges of $H$. Define $H_k$ to be the set of all subgraphs $H$ of $G_{T,\gamma}$ such that $H$ is a union of $k$ tiles $H = S_{j_1} \cup \cdots \cup S_{j_k}$ and such that the number of edges of $M_\ominus$ that are contained in $H$ is equal to $k + c(H)$. For $H \in H_k$, let

$$y(H) = \prod_{S_{j_i} \text{ tile in } H} y_{i_j}.$$

**Theorem 6.1** The cluster expansion of the cluster variable $x_\gamma$ is given by

$$x_\gamma = \sum_{k=0}^{d} \sum_{H \in H_k} \frac{w(\partial H \ominus M_\ominus) y(H)}{x_{i_1} x_{i_2} \cdots x_{i_d}}.$$

Theorem 6.1 has interesting intersections with work of other people. In [CCS2], the authors obtained a formula for the denominators of the cluster expansion in types $A, D$ and $E$, see also [BMR]. In [CC, CK, CK2] an expansion formula was given in the case where the cluster algebra is acyclic and the cluster lies in an acyclic seed. Palu generalized this formula to arbitrary clusters in an acyclic cluster algebra [Pa]. These formulas use the cluster category introduced in [BMRRT], and in [CCS] for type $A$, and do not give information about the coefficients.

Recently, Fu and Keller generalized this formula further to cluster algebras with principal coefficients that admit a categorification by a 2-Calabi-Yau category [FK], and, combining results of [A] and [ABCP, LF], such a categorification exists in the case of cluster algebras associated to unpunctured surfaces.

In [SZ, CZ, Z, MP] cluster expansions for cluster algebras of rank 2 are given, in [Pr1, CP, FZ3] the case $A$ is considered. In section 4 of [Pr1], Propp describes two constructions of snake graphs, the latter of which are unweighted analogues for the case $A$ of the graphs $G_{T,\gamma}$ that we present in this paper. Propp assigns a snake graph to each arc in the triangulation of an $n$-gon and shows that the numbers of matchings in these graphs satisfy the Conway-Coxeter frieze pattern induced by the Ptolemy relations on the $n$-gon. In [M] a cluster expansion for cluster algebras of classical type is given for clusters that lie in a bipartite seed, and the forthcoming work of [MW] will concern cluster expansions for cluster algebras of classical type with principal coefficients, for an arbitrary seed.

The formula for $y(M)$ given in Theorem 5.1 also can be formulated in terms of height functions, as found in literature such as [EKLP] or [Pr2]. We discuss this connection in Remark 5.3 of section 5.
The paper is organized as follows. In section 2, we recall the construction of cluster algebras from surfaces of \cite{FST}. Section 3 contains the construction of the graph \( G_{T, \gamma} \) and the statement of the cluster expansion formula. Section 4 is devoted to the proof of the expansion formula. The formula for \( y(M) \) and the formula for the \( g \)-vectors is given in section 5. In section 6, we present the expansion formula in terms of subgraphs and deduce a formula for the \( F \)-polynomials. We give an example in section 7.

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2. Cluster algebras from surfaces

In this section, we recall the construction of \cite{FST} in the case of surfaces without punctures.

Let \( S \) be a connected oriented 2-dimensional Riemann surface with boundary and \( M \) a non-empty set of marked points in the closure of \( S \) with at least one marked point on each boundary component. The pair \((S, M)\) is called bordered surface with marked points. Marked points in the interior of \( S \) are called punctures.

In this paper we will only consider surfaces \((S, M)\) such that all marked points lie on the boundary of \( S \), and we will refer to \((S, M)\) simply by unpunctured surface.

We say that two curves in \( S \) do not cross if they do not intersect each other except that endpoints may coincide.

**Definition 1.** An arc \( \gamma \) in \((S, M)\) is a curve in \( S \) such that

(a) the endpoints are in \( M \),
(b) \( \gamma \) does not cross itself,
(c) the relative interior of \( \gamma \) is disjoint from \( M \) and from the boundary of \( S \),
(d) \( \gamma \) does not cut out a monogon or a digon.

Curves that connect two marked points and lie entirely on the boundary of \( S \) without passing through a third marked point are called boundary arcs. Hence an arc is a curve between two marked points, which does not intersect itself nor the boundary except possibly at its endpoints and which is not homotopic to a point or a boundary arc.

Each arc is considered up to isotopy inside the class of such curves. Moreover, each arc is considered up to orientation, so if an arc has endpoints \( a, b \in M \) then it can be represented by a curve that runs from \( a \) to \( b \), as well as by a curve that runs from \( b \) to \( a \).

For any two arcs \( \gamma, \gamma' \) in \( S \), let \( e(\gamma, \gamma') \) be the minimal number of crossings of \( \gamma \) and \( \gamma' \), that is, \( e(\gamma, \gamma') \) is the minimum of the numbers of crossings of arcs \( \alpha \) and \( \alpha' \), where \( \alpha \) is isotopic to \( \gamma \) and \( \alpha' \) is isotopic to \( \gamma' \). Two arcs \( \gamma, \gamma' \) are called compatible if \( e(\gamma, \gamma') = 0 \). A triangulation is a maximal collection of compatible arcs together with all boundary arcs. The arcs of a triangulation cut the surface into triangles. Since \((S, M)\) is an unpunctured surface, the three sides of each triangle are distinct (in contrast to the case of surfaces with punctures). Any triangulation has \( n + m \) elements, \( n \) of which are arcs in \( S \), and the remaining \( m \) elements are boundary arcs. Note that the number of boundary arcs is equal to the number of marked points.
Proposition 2.1. The number $n$ of arcs in any triangulation is given by the formula $n = 6g + 3b + m - 6$, where $g$ is the genus of $S$, $b$ is the number of boundary components and $m = |M|$ is the number of marked points. The number $n$ is called the rank of $(S, M)$.

Proof. [FST 2.10] □

Note that $b > 0$ since the set $M$ is not empty. Table 1 gives some examples of unpunctured surfaces.

Following [FST], we associate a cluster algebra to the unpunctured surface $(S, M)$ as follows. Choose any triangulation $T$, let $\tau_1, \tau_2, \ldots, \tau_n$ be the $n$ interior arcs of $T$ and denote the $m$ boundary arcs of the surface by $\tau_{n+1}, \tau_{n+2}, \ldots, \tau_{n+m}$. For any triangle $\Delta$ in $T$ define a matrix $B^\Delta = (b^\Delta_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ by

\[ b^\Delta_{ij} = \begin{cases} 
1 & \text{if } \tau_i \text{ and } \tau_j \text{ are sides of } \Delta \text{ with } \tau_j \text{ following } \tau_i \text{ in the counter-clockwise order;} \\
-1 & \text{if } \tau_i \text{ and } \tau_j \text{ are sides of } \Delta \text{ with } \tau_j \text{ following } \tau_i \text{ in the clockwise order;} \\
0 & \text{otherwise.}
\end{cases} \]

Then define the matrix $B_T = (b_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ by $b_{ij} = \sum_\Delta b^\Delta_{ij}$, where the sum is taken over all triangles in $T$. Note that the boundary arcs of the triangulation are ignored in the definition of $B_T$. Let $\tilde{B}_T = (b_{ij})_{1 \leq i \leq 2n, 1 \leq j \leq n}$ be the $2n \times n$ matrix whose upper $n \times n$ part is $B_T$ and whose lower $n \times n$ part is the identity matrix. The matrix $B_T$ is skew-symmetric and each of its entries $b_{ij}$ is either $0, 1, -1, 2$, or $-2$, since every arc $\tau$ can be in at most two triangles. An example where $b_{ij} = 2$ is given in Figure 4.

Let $A(x_T, y_T, B_T)$ be the cluster algebra with principal coefficients in the triangulation $T$, that is, $A(x_T, y_T, B_T)$ is given by the seed $(x_T, y_T, B_T)$ where $x_T = \{x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_n}\}$ is the cluster associated to the triangulation $T$, and the initial coefficient vector $y_T = (y_1, y_2, \ldots, y_n)$ is the vector of generators of $\mathbb{P} = \text{Trop}(y_1, y_2, \ldots, y_n)$.

For the boundary arcs we define $x_{\tau_k} = 1$, $k = n + 1, n + 2, \ldots, n + m$.

For each $k = 1, 2, \ldots, n$, there is a unique quadrilateral in $T \setminus \{\tau_k\}$ in which $\tau_k$ is one of the diagonals. Let $\tau'_{k}$ denote the other diagonal in that quadrilateral. Define the flip $\mu_k T$ to be the triangulation $T \setminus \{\tau_k\} \cup \{\tau'_k\}$. The mutation $\mu_k$ of the seed $\Sigma_T$ in the cluster algebra $A$ corresponds to the flip $\mu_k$ of the triangulation $T$ in the following sense. The matrix $\mu_k(B_T)$ is the matrix corresponding to the triangulation $\mu_k T$, the cluster $\mu_k(x_T)$ is $x_T \setminus \{x_{\tau_k}\} \cup \{x_{\tau'_k}\}$, and the corresponding

| $b$ | $g$ | $m$ | surface               |
|-----|-----|-----|-----------------------|
| 1   | 0   | $n+3$ | polygon              |
| 1   | 1   | $n-3$ | torus with disk removed |
| 1   | 2   | $n-9$ | genus 2 surface with disk removed |
| 2   | 0   | $n$  | annulus              |
| 2   | 1   | $n-6$ | torus with 2 disks removed |
| 2   | 2   | $n-12$ | genus 2 surface with 2 disks removed |

Table 1. Examples of unpunctured surfaces
Figure 1. A triangulation with $b_{23} = 2$

Figure 2. The tile $\overline{S}_k$

The exchange relation is given by

$$x_{\tau_k}x_{\tau'_k} = x_{\rho_1}x_{\rho_2}y^+ + x_{\sigma_1}x_{\sigma_2}y^-,$$

where $y^+, y^-$ are some coefficients, and $\rho_1, \sigma_1, \rho_2, \sigma_2$ are the sides of the quadrilateral in which $\tau_k$ and $\tau'_k$ are the diagonals, such that $\rho_1, \rho_2$ are opposite sides and $\sigma_1, \sigma_2$ are opposite sides too.

3. Expansion Formula

In this section, we will present an expansion formula for the cluster variables in terms of perfect matchings of a graph that is constructed recursively using so-called tiles.

3.1. Tiles. For the purpose of this paper, a tile $\overline{S}_k$ is a planar four vertex graph with five weighted edges having the shape of two equilateral triangles that share one edge, see Figure 2. The weight on each edge of the tile $\overline{S}_k$ is a single variable. The unique interior edge is called diagonal and the four exterior edges are called sides of $\overline{S}_k$. We shall use $S_k$ to denote the graph obtained from $\overline{S}_k$ by removing the diagonal.

Now let $T$ be a triangulation of the unpunctured surface $(S, M)$. If $\tau_k \in T$ is an interior arc, then $\tau_k$ lies in precisely two triangles in $T$, hence $\tau_k$ is the diagonal of a unique quadrilateral $Q_{\tau_k}$ in $T$. We associate to this quadrilateral a tile $\overline{S}_k$ by assigning the weight $x_k$ to the diagonal and the weights $x_a, x_b, x_c, x_d$ to the sides of
3.2. The graph $G_{T,\gamma}$. Let $T$ be a triangulation of an unpunctured surface $(S, M)$ and let $\gamma$ be an arc in $(S, M)$ which is not in $T$. Choose an orientation on $\gamma$ and let $s \in M$ be its starting point, and let $t \in M$ be its endpoint. We denote by $p_0 = s, p_1, p_2, \ldots, p_{d+1} = t$ the points of intersection of $\gamma$ and $T$ in order. Let $i_1, i_2, \ldots, i_d$ be such that $p_k$ lies on the arc $\tau_{ik} \in T$. Note that $i_k$ may be equal to $i_j$ even if $k \neq j$. Let $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_d$ be a sequence of tiles so that $\tilde{S}_k$ is isomorphic to the tile $\bar{S}_{i_k}$, for $k = 1, 2, \ldots, d$.

For $k$ from 0 to $d$, let $\gamma_{k}$ denote the segment of the path $\gamma$ from the point $p_k$ to the point $p_{k+1}$. Each $\gamma_k$ lies in exactly one triangle $\Delta_k$ in $T$, and if $1 \leq k \leq d - 1$ then $\Delta_k$ is formed by the arcs $\tau_{ik}, \tau_{ik+1}$, and a third arc that we denote by $\tau_{[\gamma_k]}$.

We will define a graph $\overline{G}_{T,\gamma}$ by recursive glueing of tiles. Start with $\overline{G}_{T,\gamma,1} \cong \bar{S}_1$, where we orient the tile $\tilde{S}_1$ so that the diagonal goes from northwest to southeast, and the starting point $p_0$ of $\gamma$ is in the southwest corner of $\tilde{S}_1$. For all $k = 1, 2, \ldots, d - 1$ let $\overline{G}_{T,\gamma,k+1}$ be the graph obtained by adjoining the tile $\tilde{S}_{k+1}$ to the tile $\tilde{S}_k$ of the graph $\overline{G}_{T,\gamma,k}$ along the edge weighted $x_{[\gamma_k]}$, see Figure 3. We always orient the tiles so that the diagonals go from northwest to southeast. Note that the edge weighted $x_{[\gamma_k]}$ is either the northern or the eastern edge of the tile $\tilde{S}_k$. Finally, we define $\overline{G}_{T,\gamma}$ to be $\overline{G}_{T,\gamma,d}$.

Let $G_{T,\gamma}$ be the graph obtained from $\overline{G}_{T,\gamma}$ by removing the diagonal in each tile, that is, $G_{T,\gamma}$ is constructed in the same way as $\overline{G}_{T,\gamma}$ but using tiles $S_{ik}$ instead of $\bar{S}_{i_k}$.

A perfect matching of a graph is a subset of the edges so that each vertex is covered exactly once. We define the weight $w(M)$ of a perfect matching $M$ to be the product of the weights of all edges in $M$.

3.3. Cluster expansion formula. Let $(S, M)$ be an unpunctured surface with triangulation $T$, and let $A = A(x_T, y_T, B)$ be the cluster algebra with principal coefficients in the initial seed $(x_T, y_T, B)$ defined in section 2. Each cluster variable in $A$ corresponds to an arc in $(S, M)$. Let $x_{\gamma}$ be an arbitrary cluster variable corresponding to an arc $\gamma$. Choose an orientation of $\gamma$, and let $\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_d}$ be the arcs of the triangulation that are crossed by $\gamma$ in this order, with multiplicities possible. Let $G_{T,\gamma}$ be the graph constructed in section 3.2.
Theorem 3.1. With the above notation
\[ x_\gamma = \sum_M \frac{w(M) y(M)}{x_{i_1} x_{i_2} \cdots x_{i_d}}, \]
where the sum is over all perfect matchings \( M \) of \( G_{T,\gamma} \), \( w(M) \) is the weight of \( M \), and \( y(M) \) is a monomial in \( y_T \).

The proof of Theorem 3.1 will be given in Section 4.

4. Proof of Theorem 3.1

Throughout this section, \( T \) is a triangulation of an unpunctured surface \( (S, M) \), \( \gamma \) is an arc in \( S \) with a fixed orientation, and \( s \in M \) is its starting point and \( t \in M \) is its endpoint. Moreover, \( p_0 = s, p_1, p_2, \ldots, p_{d+1} = t \) are the points of intersection of \( \gamma \) and \( T \) in order, and \( i_1, i_2, \ldots, i_d \) are such that \( p_k \) lies on the arc \( \tau_{i_k} \in T \).

4.1. Complete \((T, \gamma)\)-paths. Following [ST], we will consider paths \( \alpha \in S \) that are concatenations of arcs in the triangulation \( T \), more precisely, \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)}) \) with \( \alpha_i \in T \), for \( i = 1, 2, \ldots, \ell(\alpha) \) and the starting point of \( \alpha_i \) is the endpoint of \( \alpha_{i-1} \). Such a path is called a \( T \)-path.

We call a \( T \)-path \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)}) \) a complete \((T, \gamma)\)-path if the following axioms hold:

\section*{(T1) The even arcs are precisely the arcs crossed by \( \gamma \) in order, that is, \( \alpha_{2k} = \tau_{i_k} \).

(T2) For all \( k = 0, 1, 2, \ldots, d \), the segment \( \gamma_k \) is homotopic to the segment of the path \( \alpha \) starting at the point \( p_k \) following \( \alpha_{2k}, \alpha_{2k+1} \) and \( \alpha_{2k+2} \) until the point \( p_{k+1} \).

We define the Laurent monomial \( x(\alpha) \) of the complete \((T, \gamma)\)-path \( \alpha \) by
\[ x(\alpha) = \prod_{i \text{ odd}} x_{\alpha_i} \prod_{i \text{ even}} x_{\alpha_i}^{-1}. \]

Remark 4.1.\begin{itemize}
- Every complete \((T, \gamma)\)-path starts and ends at the same point as \( \gamma \), because of (T2).
- Every complete \((T, \gamma)\)-path has length \( 2d + 1 \).
- For all arcs \( \tau \) in the triangulation \( T \), the number of times that \( \tau \) occurs as \( \alpha_{2k} \) is exactly the number of crossings between \( \gamma \) and \( \tau \).
- In contrast to the ordinary \((T, \gamma)\)-paths defined in [ST], complete \((T, \gamma)\)-paths allow backtracking.
- The denominator of the Laurent monomial \( x(\alpha) \) is equal to \( x_{i_1} x_{i_2} \cdots x_{i_d} \).
\end{itemize}

4.2. Universal cover. Let \( \pi : \tilde{S} \to S \) be a universal cover of the surface \( S \), and let \( \tilde{M} = \pi^{-1}(M) \) and \( \tilde{T} = \pi^{-1}(T) \).

Choose \( \tilde{s} \in \pi^{-1}(s) \). There exists a unique lift \( \tilde{\gamma} \) of \( \gamma \) starting at \( \tilde{s} \). Then \( \tilde{\gamma} \) is the concatenation of subpaths \( \tilde{\gamma}_0, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{d+1} \) where \( \tilde{\gamma}_k \) is a path from a point \( \tilde{p}_k \) to a point \( \tilde{p}_{k+1} \) such that \( \tilde{\gamma}_k \) is a lift of \( \gamma_k \) and \( \tilde{p}_k \in \pi^{-1}(p_k) \), for \( k = 0, 1, \ldots, d+1 \). Let \( \ell = \tilde{p}_{d+1} \in \pi^{-1}(t) \).

For \( k \) from 1 to \( d \), let \( \tilde{\tau}_{i_k} \) be the unique lift of \( \tau_{i_k} \) running through \( \tilde{p}_k \) and let \( \tilde{\tau}_{[\gamma_k]} \) be the unique lift of \( \tau_{[\gamma_k]} \) that is bounding a triangle in \( \tilde{S} \) with \( \tilde{\tau}_{i_k} \) and \( \tilde{\tau}_{i_{k+1}} \). Each \( \tilde{\gamma}_k \) lies in exactly one triangle \( \tilde{\Delta}_k \) in \( \tilde{T} \). Let \( \tilde{S(\gamma)} \subset \tilde{S} \) be the union of the triangles \( \tilde{\Delta}_0, \tilde{\Delta}_1, \ldots, \tilde{\Delta}_{d+1} \) and let \( \tilde{M(\gamma)} = \tilde{M} \cap \tilde{S(\gamma)} \) and \( \tilde{T(\gamma)} = \tilde{T} \cap \tilde{S(\gamma)} \).

Then \( (\tilde{S(\gamma)}, \tilde{M(\gamma)}) \) is a simply connected unpunctured surface of which \( \tilde{T(\gamma)} \) is a
triangulation. This triangulation \( \tilde{T}(\gamma) \) consists of arcs \( \tilde{\tau}_k, \tilde{\tau}_{\gamma} \) with \( k = 1, 2, \ldots, d \), and two arcs incident to \( \tilde{s} \) and two arcs incident to \( \tilde{t} \).

The underlying graph of \( \tilde{T}(\gamma) \) is the graph with vertex set \( \tilde{M}(\gamma) \) and whose set of edges consists of the (unoriented) arcs in \( \tilde{T}(\gamma) \).

By \cite[Section 5.5]{S2}, we can compute the Laurent expansion of \( x_\gamma \) using complete \((\tilde{T}(\gamma), \tilde{\gamma})\)-paths in \((\tilde{S}(\gamma), \tilde{M}(\gamma))\).

4.3. Folding. The graph \( \tilde{G}_{T,\gamma} \) was constructed by gluing tiles \( \tilde{S}_{k+1} \) to tiles \( \tilde{S}_k \) along edges with weight \( x_{\gamma} \), see figure 8. Now we will fold the graph along the edges weighted \( x_{\gamma} \), thereby identifying the two triangles incident to \( x_{\gamma} \), \( k = 1, 2, \ldots, d - 1 \).

To be more precise, the edge with weight \( x_{\gamma} \), that lies in the two tiles \( \tilde{S}_{k+1} \) and \( \tilde{S}_k \), is contained in precisely two triangles \( \Delta_k \) and \( \Delta'_k \) in \( \tilde{G}_{T,\gamma} \): \( \Delta_k \) lying inside the tile \( \tilde{S}_k \) and \( \Delta'_k \) lying inside the tile \( \tilde{S}_{k+1} \). Both \( \Delta_k \) and \( \Delta'_k \) have weights \( x_{\gamma} \), \( x_k, x_{k+1} \), but opposite orientations. Cutting \( \tilde{G}_{T,\gamma} \) along the edge with weight \( x_{\gamma} \), one obtains two connected components. Let \( R_k \) be the component that contains the tile \( \tilde{S}_k \) and \( R_{k+1} \) the component that contains \( \tilde{S}_{k+1} \).

The folding of the graph \( \tilde{G}_{T,\gamma} \) along \( x_{\gamma} \) is the graph obtained by flipping \( R_{k+1} \) and then gluing it to \( R_k \) by identifying the two triangles \( \Delta_k \) and \( \Delta'_k \).

The graph obtained by consecutive folding of \( \tilde{G}_{T,\gamma} \) along all edges with weight \( x_{\gamma} \) for \( k = 1, 2, \ldots, d - 1 \), is isomorphic to the underlying graph of the triangulation \( \tilde{T}(\gamma) \) of the unpunctured surface \((\tilde{S}(\gamma), \tilde{M}(\gamma))\). Indeed, there clearly is a bijection between the triangles in both graphs, and, in both graphs the way the triangles are glued together is uniquely determined by \( \gamma \). Note that the two graphs may have opposite orientations.

We obtain a map that we call the folding map

\[
\phi : \left\{ \text{perfect matchings in } \tilde{G}_{T,\gamma} \right\} \rightarrow \left\{ \text{complete } (\tilde{T}(\gamma), \tilde{\gamma})\text{-paths in } (\tilde{S}(\gamma), \tilde{M}(\gamma)) \right\}
\]

\[
M \quad \mapsto \quad \tilde{\alpha}_M
\]
as follows. First we associate a path \( \alpha_M \) in \( \tilde{G}_{T,\gamma} \) to the matching \( M \), by inserting a diagonal between any two consecutive edges of the perfect matching. More precisely, \( \alpha_M \) is the path starting at \( s \) going along the unique edge of \( M \) that is incident to \( s \), then going along the diagonal of the first tile \( \tilde{S}_1 \), then along the unique edge of \( M \) that is incident to the endpoint of that diagonal, and so forth.

Since \( M \) has cardinality \( d + 1 \), the path \( \alpha_M \) consists of \( 2d + 1 \) edges, thus \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2d+1}) \). Now we define \( \tilde{\alpha}_M = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{2d+1}) \) by folding the path \( \alpha_M \). Thus, if \( M = \{\beta_1, \beta_3, \ldots, \beta_{2d-1}, \beta_{2d+1}\} \), where the edges are ordered according to \( \gamma \), then \( \phi(M) = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{2d+1}) \), where \( \tilde{\alpha}_{2k+1} \) is the image of \( \beta_{2k+1} \) under the folding and \( \tilde{\alpha}_{2k} = \tilde{\tau}_k \) is the arc crossing \( \gamma \) at \( p_k \). Then \( \phi(M) \) satisfies the axiom (T1) by construction. Moreover, \( \phi(M) \) satisfies the axiom (T2), because, for each \( k = 0, 1, \ldots, d \), the segment of the path \( \phi(M) \), which starts at the point \( p_k \) following \( \tilde{\alpha}_{2k}, \tilde{\alpha}_{2k+1} \) and \( \tilde{\alpha}_{2k+2} \) until the point \( p_{k+1} \), is homotopic to the segment \( \gamma_k \), since both segments lie in the simply connected triangle \( \Delta_k \) formed by \( \tau_k, \tilde{\tau}_{k+1} \) and \( \tilde{\tau}_{\gamma_k} \). Therefore, the folding map \( \phi \) is well defined.

Note that it is possible that \( \tilde{\alpha}_k, \tilde{\alpha}_{k+1} \) is backtracking, that is, \( \tilde{\alpha}_k \) and \( \tilde{\alpha}_{k+1} \) run along the same arc \( \tilde{\tau} \in \tilde{T}(\gamma) \).
4.4. Unfolding the surface. Let $\alpha$ be a boundary arc in $(\tilde{S}(\gamma), \tilde{M}(\gamma))$ that is not adjacent to $\tilde{s}$ and not adjacent to $\tilde{t}$. Then there is a unique triangle $\Delta$ in $\tilde{T}(\gamma)$ in which $\alpha$ is a side. The other two sides of $\Delta$ are two consecutive diagonals, which we denote by $\tilde{\tau}_j$ and $\tilde{\tau}_{j+1}$, see Figure 3.

By cutting the underlying graph of $\tilde{T}(\gamma)$ along $\tilde{\tau}_j$, we obtain two pieces. Let $R_{j+1}$ denote the piece that contains $\alpha$, $\tilde{\tau}_{j+1}$ and $t$. Similarly, cutting $(\tilde{S}(\gamma),\tilde{M}(\gamma))$ along $\tilde{\tau}_{j+1}$, we obtain two pieces, and we denote by $R_j$ the piece that contains $s$, $\tilde{\tau}_j$ and $\alpha$.

The graph obtained by unfolding along $\alpha$ is the graph obtained by flipping $R_j$ and then gluing it to $R_{j+1}$ along $\alpha$. In this new graph, we label the edge of $R_j$ that had the label $\tilde{\tau}_{j+1}$ by $\tilde{\tau}^b_j+1$ and the edge of $R_{j+1}$ that had the label $\tilde{\tau}_j$ by $\tilde{\tau}^b_{j+1}$, indicating that these edges are on the boundary of the new graph, see Figure 4.

Lemma 4.2. The graph obtained by repeated unfolding of the underlying graph of $\tilde{T}(\gamma)$ along all boundary edges not adjacent to $s$ or $t$ is isomorphic to the graph $\mathcal{G}_{T,\gamma}$. Moreover, for each unfolding along an edge $\alpha$, the edges labeled $\tilde{\tau}^b_j$, $\tilde{\tau}^b_{j+1}$ are on the boundary of $\mathcal{G}_{T,\gamma}$ and carry the weights $x_j$, $x_{j+1}$ respectively, the edges labeled $\tilde{\tau}_j$, $\tilde{\tau}_{j+1}$ are diagonals in $\mathcal{G}_{T,\gamma}$ and carry the weights $x_j$, $x_{j+1}$ respectively, and $\alpha$ is an interior edge of $\mathcal{G}_{T,\gamma}$ that is not a diagonal and carries the weight $x_1$.

Proof. This follows from the construction. □

4.5. Unfolding map. We define a map

\[
\begin{align*}
\{\text{complete } (\tilde{T}(\gamma), \tilde{\gamma}) - \text{paths}\} & \rightarrow \{\text{perfect matchings of } G_{T,\gamma}\} \\
\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{2d+1}) & \mapsto M_\tilde{\alpha} = \{\beta_1, \beta_2, \ldots, \beta_{2d+1}\}
\end{align*}
\]

where $\beta_1 = \tilde{\alpha}_1$, $\beta_{2d+1} = \tilde{\alpha}_{2d+1}$ and

\[
\beta_{2k+1} = \begin{cases} 
\tilde{\alpha}_{2k+1} & \text{if } \tilde{\alpha}_{2k+1} \text{ is a boundary arc in } \tilde{T}(\gamma), \\
\tilde{\tau}_j^b & \text{if } \tilde{\alpha}_{2k+1} = \tilde{\tau}_j \text{ is a diagonal in } \tilde{T}(\gamma).
\end{cases}
\]

We will show that this map is well-defined. Suppose $\beta_{2k+1}$ and $\beta_{2\ell+1}$ have a common endpoint $x$. Then $\alpha_{2k+1}$ and $\alpha_{2\ell+1}$ have a common endpoint $y$ in $(\tilde{S}(\gamma), \tilde{M}(\gamma))$ and the two edges are not separated in the unfolding described in Lemma 4.2. Consequently, there is no triangle in $\tilde{T}(\gamma)$ that is contained in the subpolygon spanned by $\alpha_{2k+1}$ and $\alpha_{2\ell+1}$, hence $\alpha_{2k+1}$ is equal to $\alpha_{2\ell+1}$. This implies that every arc in the subpath $(\alpha_{2k+1}, \alpha_{2k+2} \ldots \alpha_{2\ell+1})$ is equal to the same diagonal $\tilde{\tau}_j$, and the only way this can happen is when $\ell = k + 1$ and $(\alpha_{2k+1}, \alpha_{2k+2} \ldots \alpha_{2\ell+1}) = (\tilde{\tau}_j, \tilde{\tau}_j, \tilde{\tau}_j)$ and both endpoints of $\tilde{\tau}_j$ are incident to an interior arc other than $\tilde{\tau}_j$. In this case, $\tilde{\tau}_j$ bounds the two triangles $\tilde{\tau}_{j-1}, \tilde{\tau}_j, \tilde{\tau}_{j+1}$ and $\tilde{\tau}_j, \tilde{\tau}_{j+1}, \tilde{\tau}_{j+2}$ in $\tilde{T}(\gamma)$. Unfolding along $\tilde{\tau}_{j-1}$ and $\tilde{\tau}_{j+1}$ will produce edges $\beta_{2k+1}$ and $\beta_{2\ell+1}$ that are not adjacent, see Figure 3. This shows that no vertex of $G_{T,\gamma}$ is covered twice in $M_\alpha$. 

![Figure 4. Completion of paths](image-url)
Indeed, the number of vertices of \( \alpha \tau \) follows since every \( \alpha \) is complete, and thus \( M_\alpha \) has \( d + 1 \) edges. The statement follows since every \( \beta_j \in M_\alpha \) has two distinct endpoints. This shows that \( M_\alpha \) is a perfect matching and our map is well-defined.

Lemma 4.3. The unfolding map \( \tilde{\alpha} \mapsto M_\tilde{\alpha} \) is the inverse of the folding map \( M \mapsto \tilde{\alpha}_M \). In particular, both maps are bijections.

Proof. Let \( \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{2d+1}) \) be a complete \((\tilde{T}(\gamma), \tilde{\gamma})\)-path. Then \( \tilde{\alpha}_M = (\alpha_1, \alpha_2, \ldots, \alpha_{2d+1}) \) where \( \alpha_{2k+1} \) is the image under folding of the arc \( \tilde{\alpha}_j \) if \( \tilde{\alpha}_{2k+1} = \tilde{\tau}_j \) is a diagonal in \( \tilde{T}(\gamma) \) or, otherwise, the image under the folding of the arc \( \alpha_{2k+1} \). Thus \( \alpha_{2k+1} = \tilde{\alpha}_{2k+1} \). Moreover, \( \alpha_{2k} = \tau_i = \tilde{\alpha}_{2k} \), and thus \( \tilde{\alpha}_M = \tilde{\alpha} \).

Conversely, let \( M = \{\beta_1, \beta_2, \ldots, \beta_{2d-1}, \beta_{2d+1}\} \) be a perfect matching of \( G_{T,\gamma} \). Then \( M_{\tilde{\alpha}_M} = \{\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_{2d-1}, \tilde{\beta}_{2d+1}\} \) where

\[
\tilde{\beta}_{2k+1} = \begin{cases} 
\tilde{\alpha}_{2k+1} & \text{if } \alpha_{2k+1} \text{ is a boundary arc}, \\
\tilde{\tau}_j^b & \text{if } \alpha_{2k+1} = \tilde{\tau}_j \text{ is a diagonal} \\
\tilde{\tau}_j & \text{if } \beta_{2k+1} = \tilde{\tau}_j, \\
\tilde{\tau}_j^b & \text{if } \beta_{2k+1} = \tilde{\tau}_j^b.
\end{cases}
\]

Hence \( M_{\tilde{\alpha}_M} = M \).

Combining Lemma 4.3 with the results of \([S2]\), we obtain the following Theorem.

Theorem 4.4. There is a bijection between the set of perfect matchings of the graph \( G_{T,\gamma} \) and the set of complete \((T, \gamma)\)-paths in \((S, M)\) given by \( M \mapsto \pi(\tilde{\alpha}_M) \), where \( \tilde{\alpha}_M \) is the image of \( M \) under the folding map and \( \pi \) is induced by the universal cover \( \pi : \tilde{S} \to S \). Moreover, the numerator of the Laurent monomial \( x(\pi(\tilde{\alpha}_M)) \) of the complete \((T, \gamma)\)-path \( \pi(\tilde{\alpha}_M) \) is equal to the weight \( w(M) \) of the matching \( M \).

Proof. The map in the Theorem is a bijection, because it is the composition of the folding map, which is a bijection, by Lemma 4.3 and the map \( \pi \), which is a bijection, by \([S2] \) Lemma 5.8. The last statement of the Theorem follows from the construction of the graph \( G_{T,\gamma} \). \( \square \)
4.6. Proof of Theorem 3.1 It has been shown in [S2, Theorem 3.2] that

\[ x_\gamma = \sum_\alpha x(\alpha) y(\alpha), \]

where the sum is over all complete \((T, \gamma)\)-paths \(\alpha\) in \((S, M)\), \(y(\alpha)\) is a monomial in \(y_T\), and

\[ x(\alpha) = \prod_{k \text{ odd}} x_{\alpha_k} \prod_{k \text{ even}} x_{\alpha_k}^{-1}. \]

Applying Theorem 4.4 to equation (3) yields

\[ x_\gamma = \sum_M w(M) y(M)(x_{i_1} x_{i_2} \cdots x_{i_d})^{-1}, \]

where the sum is over all perfect matchings \(M\) of \(G_{T,\gamma}\), \(w(M)\) is the weight of the matching and \(y(M) = y(\pi(\tilde{\alpha}_M))\). This completes the proof of Theorem 3.1.

5. A formula for \(y(M)\)

In this section, we give a description of the coefficients \(y(M)\) in terms of the matching \(M\). First, we need to recall some results from [S2].

Recall that \(T\) is a triangulation of the unpunctured surface \((S, M)\), \(\gamma\) is an arc in \((S, M)\) that crosses \(T\) exactly \(d\) times, we have fixed an orientation for \(\gamma\) and denote by \(s = p_0, p_1, \ldots, p_d, p_{d+1} = t\) the intersection points of \(\gamma\) and \(T\) in order of occurrence on \(\gamma\). Let \(i_1, i_2, \ldots, i_d\) be such that \(p_k\) lies on the arc \(\tau_{i_k} \in T\), for \(k = 1, 2, \ldots, d\).

For \(k = 0, 1, \ldots, d\), let \(\gamma_k\) denote the segment of the path \(\gamma\) from the point \(p_k\) to the point \(p_{k+1}\). Each \(\gamma_k\) lies in exactly one triangle \(\Delta_k\) in \(T\). If \(1 \leq k \leq d - 1\), the triangle \(\Delta_k\) is formed by the arcs \(\tau_{i_k}, \tau_{i_{k+1}}\), and a third arc that we denote by \(\tau_{i_{\gamma_k}}\).

The orientation of the surface \(S\) induces an orientation on each of these triangles in such a way that, whenever two triangles \(\Delta, \Delta'\) share an edge \(\tau\), then the orientation of \(\tau\) in \(\Delta\) is opposite to the orientation of \(\tau\) in \(\Delta'\). There are precisely two such orientations, we assume without loss of generality that we have the “clockwise orientation”, that is, in each triangle \(\Delta\), going around the boundary of \(\Delta\) according to the orientation of \(\Delta\) is clockwise when looking at it from outside the surface.

Let \(\alpha\) be a complete \((T, \gamma)\)-path. Then \(\alpha_{2k} = \tau_{i_k}\) is a common edge of the two triangles \(\Delta_{k-1}\) and \(\Delta_k\). We say that \(\alpha_{2k}\) is \(\gamma\)-oriented if the orientation of \(\alpha_{2k}\) in the path \(\alpha\) is the same as the orientation of \(\tau_{i_k}\) in the triangle \(\Delta_k\), see Figure 6.

It is shown in [S2, Theorem 3.2] that

\[ y(\alpha) = \prod_{k : \alpha_{2k} \text{ is } \gamma\text{-oriented}} y_{i_k}. \]

Each perfect matching \(M\) of \(G_{T,\gamma}\) induces a path \(\alpha_M\) in \(\overline{G}_{T,\gamma}\) as in the construction of the folding map in section 4.3. The even arcs of \(\alpha_M\) are the diagonals of the graph \(\overline{G}_{T,\gamma}\). We say that an even arc of \(\alpha_M\) has upward orientation if \(\alpha_M\) is directed from southeast to northwest on that even arc, otherwise we say that the arc has downward orientation. If going upward on the first even arc of \(\alpha_M\) is \(\gamma\)-oriented then we have that the \((2k)\)-th arc of \(\pi(\tilde{\alpha}_M)\) is \(\gamma\)-oriented if and only if the \(2k\)-th arc of \(\alpha_M\) is upward if \(k\) is odd, and downward if \(k\) is even. If, on the
other hand, going downward on the first even arc of \( \alpha_M \) is \( \gamma \)-oriented then we have that the \( (2k) \)-th arc of \( \pi(\tilde{\alpha}_M) \) is \( \gamma \)-oriented if and only if the \( 2k \)-th arc of \( \alpha_M \) is downward if \( k \) is odd, and upward if \( k \) is even.

There are precisely two perfect matchings \( M_+ \) and \( M_- \) of \( G_{T,\gamma} \) that contain only boundary edges of \( G_{T,\gamma} \). The orientations of the even arcs in both of the induced \( (T, \gamma) \)-paths \( \alpha_{M_+} \) and \( \alpha_{M_-} \) are alternatingly upward and downward, thus for one of the two paths, say \( M_+ \), each even arc of \( \pi(\tilde{\alpha}_{M_+}) \) is \( \gamma \)-oriented, whereas for \( M_- \) none of the even arcs of \( \pi(\tilde{\alpha}_{M_-}) \) is \( \gamma \)-oriented. That is, \( y(M_-) = 1 \) and \( y(M_+) = y_1 y_2 \cdots y_d \).

For an arbitrary perfect matching \( M \), the coefficient \( y(M) \) is determined by the set of edges of the symmetric difference \( M \ominus M = (M \cup M) \setminus (M \cap M) \) as follows.

**Theorem 5.1.** The set \( M \ominus M \) is the set of boundary edges of a (possibly disconnected) subgraph \( G_M \) of \( G_{T,\gamma} \) which is a union of tiles

\[
G_M = \bigcup_{j \in J} S_j.
\]

Moreover,

\[
y(M) = \prod_{j \in J} y_j
\]

**Proof.** Choose any edge \( e_1 \) and either endpoint in \( M_- \setminus (M_- \cap M) \), and walk along that edge until its other endpoint. Since \( M \) is a perfect matching, this endpoint is incident to an edge \( e_2 \) in \( M \), which is different from \( e_1 \) and, hence, not in \( M_- \). Thus \( e_2 \in M \setminus (M_- \cap M) \). Now walk along \( e_2 \) until its other endpoint. This endpoint is incident to an edge \( e_3 \) in \( M_- \) which is different from \( e_2 \), and, hence, not in \( M \). Thus \( e_3 \in M_- \setminus (M_- \cap M) \). Continuing this way, we construct a sequence of edges in \( M_- \ominus M \). Since \( G_{T,\gamma} \) has only finitely many edges, this sequence must become periodic after a certain number of steps; thus there exist \( p, N \) such that \( e_k = e_{k+p} \) for all \( k \geq N \).

We will show that one can take \( N = 1 \). Suppose to the contrary that \( N \geq 2 \) is the smallest integer such that \( e_k = e_{k+p} \) for all \( k \geq N \). Then \( e_{N+1}, e_N \) and \( e_{N+p-1} \)
share a common endpoint. But \( e_{N-1}, e_N \) and \( e_{N+p-1} \) are elements of the union of two perfect matchings, hence \( e_{N-1} = e_{N+p-1} \), contradicting the minimality of \( N \).

Therefore the sequence \( e_1, e_2, \ldots, e_p \) in \( M \cap M_- \) is the set of boundary edges of a connected subgraph of \( G_{T, \gamma} \) which is a union of tiles.

The graph \( G_M \) is the union of these connected subgraphs and, hence, it is a union of tiles. Let \( H \) be a connected component of \( G_M \). There are precisely two perfect matchings \( M_-(H) \) and \( M_+(H) \) of \( H \) that consist only of boundary edges of \( H \). Clearly, these two matchings are \( M_- \cap E(H) \) and \( M_+ \cap E(H) \), where \( E(H) \) is the set of edges of the graph \( H \). Therefore, in each tile of \( H \), the orientation of the diagonal in \( \alpha_{M_-} \) and \( \alpha_M \) are opposite. The restrictions of \( M_- \) and \( M \) to \( E(G_{T, \gamma}) \setminus E(G_M) \) are identical, hence in each tile of \( G_{T, \gamma} \setminus G_M \), the orientations of the diagonal in \( \alpha_{M_-} \) and \( \alpha_M \) are equal. It follows from equation \( 4 \) that \( y(M) = \prod_{i \in J} y_{i} \).

It has been shown in \([EKP]\) that, for any cluster variable \( x_\gamma \) in \( A \), its Laurent expansion in the initial seed \( (x_T, y_T, B_T) \) is homogeneous with respect to the grading given by \( \deg(x_i) = e_i \) and \( \deg(y_i) = B_T e_i \), where \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{Z}^n \) with 1 at position \( i \). By definition, the \( g \)-vector \( g_\gamma \) of a cluster variable \( x_\gamma \) is the degree of its Laurent expansion with respect to this grading.

**Corollary 5.2.** The \( g \)-vector \( g_\gamma \) of \( x_\gamma \) is given by

\[
g_\gamma = \deg \frac{w(M_-)}{x_{i_1} x_{i_2} \cdots x_{i_d}}.
\]

**Proof.** This follows from the fact that \( y(M_-) = 1 \). \( \square \)

**Remark 5.3.** The formula for \( y(M) \) can also be phrased in terms of height functions. As described in section 3 of \([EKP]\), one way to define the height function on the faces of a bipartite planar graph \( G \), covered by a perfect matching \( M \), is to superimpose each matching with the fixed matching \( M_0 \) (the unique matching of minimal height). In the case where \( G \) is a snake graph, we take \( M_0 \) to be \( M_- \), one of the two matchings of \( G \) only involving edges on the boundary. Color the vertices of \( G \) black and white so that no two adjacent vertices have the same color. In this superposition, we orient edges of \( M \) from black to white, and edges of \( M_- \) from white to black. We thereby obtain a spanning set of cycles, and removing the cycles of length two exactly corresponds to taking the symmetric difference \( M \cap M_- \). We can read the resulting graph as a relief-map, in which the altitude changes by \( +1 \) or \( -1 \) as one crosses over a contour line, according to whether the counter-line is directed clockwise or counter-clockwise. By this procedure, we obtain a height function \( h_M : F(G) \to \mathbb{Z} \) which assigns integers to the faces of graph \( G \). When \( G \) is a snake graph, the set of faces of \( F(G) \) is simply the set of tiles \( \{S_j\} \) of \( G \). Comparing with the definition of \( y(M) \) in Theorem 4.3, we see that

\[
y(M) = \prod_{S_j \in F(G)} y_j^{h_M(j)}.
\]

An alternative definition of height functions comes from \([EKP]\) by translating the matching problem into a domino tiling problem on a region colored as a checkerboard. We imagine an ant starting at an arbitrary vertex at height 0, walking along the boundary of each domino, and changing its height by \( +1 \) or \( -1 \) as it traverses the boundary of a black or white square, respectively. The values of the height function under these two formulations agree up to scaling by four.
6. Cluster expansion without matchings

In this section, we give a formula for the cluster expansion of $x_\gamma$ in terms of the graph $G_{T,\gamma}$ only.

For any graph $H$, let $c(H)$ be the number of connected components of $H$. Let $E(H)$ be the set of edges of $H$, and denote by $\partial H$ the set of boundary edges of $H$. Define $\mathcal{H}_k$ to be the set of all subgraphs $H$ of $G_{T,\gamma}$ such that $H$ is a union of $k$ tiles $H = S_{j_1} \cup \cdots \cup S_{j_k}$, and the number of edges of $M_-$ that are contained in $H$ is equal to $k + c(H)$. For $H \in \mathcal{H}_k$, let

$$y(H) = \prod_{S_i \text{ tile in } H} y_i,$$

**Theorem 6.1.** The cluster expansion of the cluster variable $x_\gamma$ is given by

$$x_\gamma = \sum_{k=0}^{d} \sum_{H \in \mathcal{H}_k} \frac{w(\partial H \ominus M_-)}{x_1 x_2 \cdots x_d} y(H),$$

Proof. It follows from the theorems [5.1] and [5.2] that

$$x_\gamma = \sum_{k=1}^{d} \sum_{M: |y(M)| = k} \frac{w(M) y(G_M)}{x_1 x_2 \cdots x_{i_k}},$$

where $|y(M)|$ is the number of tiles in $G_M$. We will show that for all $k$, the map $M \mapsto G_M$ is a bijection between the set of perfect matchings $M$ of $G_{T,\gamma}$ such that $|y(M)| = k$ and the set $\mathcal{H}_k$.

- The map is well-defined. Clearly, $G_M$ is the union of $k$ tiles. Moreover, $E(G_M) \cap M_-$ is a perfect matching of $G_M$, since $M_-$ consists of every other boundary edge of $G_{T,\gamma}$. Thus the cardinality of $(E(G_M) \cap M_-)$ is half the number of vertices of $G_M$, which is equal to $2k + 2c(G_M)$. Therefore, the cardinality of $(E(G_M) \cap M_-)$ is $k + c(G_M)$ and $G_M \in \mathcal{H}_k$.

- The map is injective, since two graphs $G_M, G_M'$ are equal if and only if their boundaries are.

- The map is surjective. Let $H = S_{j_1} \cup \cdots \cup S_{j_k}$ such that the cardinality of $E(H) \cap M_-$ equals $k + c(H)$. The boundary of $H$ consists of $2k + 2c(H)$ edges, half of which lie in $M_-$. As in the proof of Theorem [5.2], let $M_-(H) = E(H) \cap M_-$ and $M_+(H)$ be the two perfect matchings of $H$ that consist of boundary edges only. Let $M = M_+(H) \cup (M_- \setminus M_-(H))$. Then $M$ is a perfect matching of $G_{T,\gamma}$ such that $G_M = H$, and moreover, $|y(M)|$ is equal to the number of tiles in $H$, which is $k$. Thus the map is surjective.

Now the boundary edges of $G_M$ are precisely the elements of $M \ominus M_-$, which implies that $\partial(G_M) \ominus M_- = (M \ominus M_-) \ominus M_- = M \ominus (M_\ominus M_-)$, and this completes the proof.

**Corollary 6.2.** The $F$-polynomial of $\gamma$ is given by

$$F_\gamma = \sum_{k=0}^{d} \sum_{H \in \mathcal{H}_k} y(H).$$
7. Example

We illustrate Theorem 3.1, Theorem 5.1 and Theorem 6.1 in an example. Let \((S, M)\) be the annulus with two marked points on each of the two boundary components, and let \(T = \{\tau_1, \ldots, \tau_8\}\) be the triangulation shown in Figure 7. Let \(\gamma\) be the dotted arc in that figure. It has \(d = 6\) crossings with the triangulation. The sequence of crossed arcs \(\tau_{i_1}, \ldots, \tau_{i_6}\) is \(\tau_1, \tau_2, \tau_3, \tau_4, \tau_1, \tau_2\), and the corresponding segments \(\gamma_0, \ldots, \gamma_6\) of the arc \(\gamma\) are labeled in the figure. Moreover, \(\tau_{[\gamma_1]} = \tau_6\), \(\tau_{[\gamma_2]} = \tau_8\), \(\tau_{[\gamma_3]} = \tau_7\), \(\tau_{[\gamma_4]} = \tau_5\) and \(\tau_{[\gamma_5]} = \tau_6\).

The graph \(G_{T, \gamma}\) is obtained by gluing the corresponding six tiles \(\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{S}_4, \tilde{S}_1, \) and \(\tilde{S}_2\). The result is shown in Figure 8.
Theorems 3.1 and 5.1 imply that \( x_\gamma(x_{i_1}x_{i_2} \cdots x_{i_d}) \) is equal to

\[
\begin{align*}
& x_4x_6x_8x_4x_6x_8y_1y_3y_4y_1 + x_4x_6x_8x_4x_6x_8y_1y_3y_4y_2 \\
+ & x_4x_6x_8x_4x_6x_8y_1y_3y_4y_1 + x_4x_6x_2x_2x_1x_2x_8y_1 \\
+ & x_4x_6x_2x_2x_1x_2x_8y_1y_4 + x_4x_6x_2x_2x_1x_2x_8y_1y_1 \\
+ & x_4x_1x_3x_4x_6x_8y_1y_2y_3y_4y_1 + x_4x_1x_3x_4x_1x_3y_1y_2y_3y_4y_1y_2 \\
+ & x_4x_1x_3x_4x_8x_8y_1y_2y_3y_4y_1 + x_4x_2x_2x_4x_4x_6x_8y_1y_4y_1 \\
+ & x_5x_2x_8x_4x_1x_3y_3y_4y_1y_2 + x_5x_2x_8x_4x_5x_2x_8y_3y_4 \\
+ & x_5x_2x_8x_4x_3x_1x_2x_8 + x_5x_2x_8x_4x_6x_8y_1y_1 \\
+ & x_5x_2x_2x_7x_4x_1x_3y_4y_1y_2 + x_5x_2x_2x_7x_5x_2x_8y_4, \\
\end{align*}
\]

which is equal to
The matching horizontal edges of the first three tiles and the horizontal edges of the last two tiles. Thus $y$ and fifth tile, whence $y(M) = y_1 y_3 y_4 y_5 = y_1 y_3 y_4 y_5$.

To illustrate Theorem 6.1 let $k = 2$. Then $\mathcal{H}_k$ consists of the subgraphs $H$ of $G_{T, 2}$ which are unions of two tiles and such that $E(H) \cap M_-$ has three elements if $H$ is connected, respectively four elements if $H$ has two connected components. Thus $\mathcal{H}_2$ has three elements

$$\mathcal{H}_2 = \{S_{i_3} \cup S_{i_4}, S_{i_4} \cup S_{i_5}, S_{i_1} \cup S_{i_4}\}$$

corresponding to the three terms

$$x_2^2 x_4 y_3 y_4, x_2^2 x_4 y_1 y_4$$

and $x_2^2 x_4 y_1 y_4$.

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\[ x_2^2 x_4 y_3 y_4 + x_2^2 x_4 y_1 y_4 \]

For example, the first term corresponds to the matching $M$ consisting of the horizontal edges of the first three tiles and the horizontal edges of the last two tiles. The matching $M_- \cap M$ consists in the boundary edges weighted $x_5$ and $x_2$ in the first tile, $x_2$ in the third tile, $x_1$ and $x_3$ in the forth, $x_2$ in the fifth and $x_8$ in the sixth tile. Thus $M_- \cap M = (M_- \cup M) \setminus (M_- \cap M)$ is the union of the first, third, forth and fifth tile, whence $y(M) = y_1 y_3 y_4 y_5 = y_1 y_3 y_4 y_5$.

\[ x_2^2 x_4 y_3 y_4, x_2^2 x_4 y_1 y_4 \]
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