AUTOMORPHIC FORMS AND LORENTZIAN KAC–MOODY ALGEBRAS. PART I.

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ABSTRACT. Using the general method which was applied to prove finiteness of the set of hyperbolic generalized Cartan matrices of elliptic and parabolic type, we classify all symmetric (and twisted to symmetric) hyperbolic generalized Cartan matrices of elliptic type and of rank 3 with a lattice Weyl vector.

We develop the general theory of reflective lattices $T$ with 2 negative squares and reflective automorphic forms on homogeneous domains of type IV defined by $T$. We consider this theory as mirror symmetric to the theory of elliptic and parabolic hyperbolic reflection groups and corresponding hyperbolic root systems. We formulate Arithmetic Mirror Symmetry Conjecture relating both these theories and prove some statements to support this Conjecture. This subject is connected with automorphic correction of Lorentzian Kac–Moody algebras. We define Lie reflective automorphic forms which are the most beautiful automorphic forms defining automorphic Lorentzian Kac–Moody algebras and formulate finiteness Conjecture for these forms.

Detailed study of automorphic correction and Lie reflective automorphic forms for generalized Cartan matrices mentioned above will be given in Part II.

0. Introduction

This paper continues series of our papers [GN1]—[GN4], [N9]—[N11] devoted to automorphic forms related with Lorentzian Kac–Moody algebras and Mirror Symmetry.

In this paper we try to develop regularly the theory of Lorentzian Kac–Moody algebras and the related theory of automorphic forms.

Like the classical theory of semi-simple Lie algebras and affine Kac–Moody algebras, the theory of Lorentzian Kac–Moody algebras and the related theory of automorphic forms start with some root systems and reflection groups. For Lorentzian Kac–Moody algebras, they are hyperbolic reflection groups and corresponding root systems which are given by reflection groups of hyperbolic lattices (over $\mathbb{Z}$). It was shown in [N9], [N10] that to have a Lorentzian Kac–Moody algebra with good properties one is forced to consider hyperbolic reflection groups having some condition of finiteness of volume for a fundamental polyhedron of the group. Hyperbolic reflection groups of this type (and corresponding hyperbolic root systems) are divided in elliptic, when volume is finite, and parabolic, when volume is finite after factorization of a fundamental polyhedron by its Abelian cusp group of symmetries. Hyperbolic lattices having these groups as reflection groups are respectively called reflective (elliptic reflective or parabolic reflective). It was shown in [N9] and [N10]
(in fact, it was done in [N4], [N5] long time ago) that sets of elliptic and parabolic hyperbolic reflection groups and corresponding hyperbolic generalized Cartan matrices of elliptic and parabolic type (which define the corresponding Lorentzian Kac–Moody algebras) are essentially finite. More exactly, the set of reflective hyperbolic lattices is finite for \( \text{rk} \geq 3 \). It shows that “right” Lorentzian Kac–Moody algebras are all exceptional, there are no infinite series (at least, for elliptic type).

The main idea of proving finiteness of elliptic and parabolic hyperbolic reflection groups is not only theoretical. This gives an effective method for classification. It has been applied in [N7] to classify all hyperbolic lattices having elliptic groups of reflections in vectors with equal squares (so called elliptic 2-reflective hyperbolic lattices). (For rank \( \geq 5 \) this classification had been obtained in [N3]; for rank = 4 it had been done by É.B. Vinberg, see [N8].) We mention that this gives a description of all symmetric hyperbolic generalized Cartan matrices of elliptic type and of rank 3. They are divided in a finite set of series (possibly infinite) corresponding to their 2-reflective hyperbolic lattices. These hyperbolic lattices are important for the theory of K3 surfaces (e.g., for automorphism groups and for discriminants of moduli). For automorphic correction of Lorentzian Kac–Moody algebras, which we discuss below, one needs to find among these matrices all matrices with so called lattice Weyl vector (possibly for a twisted generalized Cartan matrix). Only for these matrices one may hope to find an automorphic form on IV type domains which gives an automorphic correction of the corresponding Kac–Moody algebra. Here we don’t discuss possible non-holomorphic automorphic forms, forms on different type domains or possible vector-valued holomorphic forms. From our point of view, the case of lattice Weyl vector is the most symmetric and beautiful to be considered.

In Sect. 1, we apply this general method to classify all symmetric (and twisted to symmetric) hyperbolic generalized Cartan matrices of elliptic type and of rank 3 with a lattice Weyl vector. One needs existence of a lattice Weyl vector to have a formal denominator function of the corresponding Kac–Moody algebra as a function on IV type domain. We prove that there are exactly 16 symmetric hyperbolic generalized Cartan matrices of elliptic type with a lattice Weyl vector and of the rank 3. Between them 12 are non-compact (i.e. a fundamental polyhedron of the corresponding hyperbolic reflection group has a vertex at infinity). The list of these 12 non-compact cases was announced in [GN4]. We also find conjecturally all hyperbolic generalized Cartan matrices of elliptic type and of rank 3 with a lattice Weyl vector which are twisted to symmetric generalized Cartan matrices. There are 60 of them (including 16 above). All these cases are important because of their relation with discriminants of K3 surfaces moduli which we will discuss in Part 2 of the paper. We classify corresponding parabolic cases in Part 2.

We are sure that similar method permits to find all hyperbolic generalized Cartan matrices of elliptic and parabolic type and of rank 3 with a lattice Weyl vector, and there should not be too many of them. One can adapt this method for higher-dimensional case (with combination of other methods). It is expected that number of possibilities is decreasing when dimension is increasing. Thus, our calculations suggest that it is reasonable to find all hyperbolic generalized Cartan matrices of elliptic and parabolic type with a lattice Weyl vector. There should not be too many cases.

In Sect. 2 we discuss the generalization of the theory of reflective hyperbolic lattices and corresponding elliptic and parabolic hyperbolic reflection groups and
their elliptic and parabolic hyperbolic root systems on complex Hermitian symmetric domains of type IV. Using Mirror Symmetry ideology (related, at least, with K3 surfaces), we are trying to find the corresponding mirror symmetric theory.

From our point of view, this mirror symmetric theory is related with so called reflective automorphic forms $\Phi$ on IV type domains $\Omega(T)$ defined by lattices $T$ with 2 negative squares, i.e. of signature $(n, 2)$.

We call an automorphic form $\Phi$ on $\Omega(T)$ with respect to a subgroup $G \subset O^+(T)$ of finite index reflective if it is a holomorphic automorphic form and the set of zeros of $\Phi$ is union of quadratic divisors orthogonal to primitive roots of $T$. Here a primitive element $\delta \in T$ is called a root if $(\delta, \delta) > 0$ and the reflection $s_\delta$ in $\delta$ belongs to the automorphism group of the lattice $T$. A lattice $T$ with 2 negative squares having a reflective automorphic form $\Phi$ is called reflective also. We consider the set $\Delta(\Phi)$ of all primitive roots of $T$ orthogonal to components of the zero divisor of $\Phi$ as a generalization of hyperbolic primitive root systems on lattices with 2 negative squares. The geometry of the automorphic form $\Phi$ should define a more detailed root system with the primitive root system $\Delta(\Phi)$ as the base.

We propose Arithmetic Mirror Symmetry Conjecture which relates reflective hyperbolic lattices and elliptic and parabolic hyperbolic root systems with reflective lattices $T$ with 2 negative squares and primitive root systems $\Delta(\Phi)$ of their reflective automorphic forms $\Phi$. (This Conjecture is a generalization of our Mirror Symmetry Conjectures and results for K3 in [GN1]—[GN4] and [N10], [N11].) Roughly speaking, Arithmetic Mirror Symmetry Conjecture says that a reflective lattice $T$ with 2 negative squares and a root system $\Delta(\Phi)$ of a reflective automorphic form $\Phi$ of $T$ should give hyperbolic reflective lattices and elliptic and parabolic hyperbolic root systems at cusps (here and in what follows we consider only 0-dimensional cusps). All elliptic and parabolic reflective lattices and hyperbolic root systems appear at cusps of reflective lattices $T$ and systems of primitive roots $\Delta(\Phi)$ of their reflective automorphic forms $\Phi$. Number of reflective lattices $T$ of $rk T \geq 5$ is finite.

The main reason why Arithmetic Mirror Symmetry Conjecture is valid is the finiteness results above about reflective hyperbolic lattices and elliptic and parabolic hyperbolic root systems, and proper formulated Necessary condition 2.2.2 (see Necessary condition 2.2.2 in Sect. 2) imposed by the Koecher principle. It was shown in [N11] that Necessary condition 2.2.2 gives extremely strong restrictions on a lattice $T$ to have a reflective automorphic form.

We prove two statements which support Arithmetic Mirror Symmetry Conjecture. First we show that lattices $T$ and root systems satisfying Arithmetic Mirror Symmetry Conjecture almost satisfy Necessary condition 2.2.2 (see Theorem 2.2.5). We have mentioned that this condition is extremely strong. The second argument is related with Fourier expansions of reflective automorphic forms $\Phi$ at cusps. We show that if an automorphic form $\Phi$ has a Fourier expansion at a cusp with a generalized lattice Weyl vector (see Sect. 2.4), then Arithmetic Mirror Symmetry Conjecture is valid for $\Phi$ at the cusp. This is exactly the same argument which was applied in [N10], [N9] to show that to have a good Lorentzian Kac–Moody algebra (e.g., having an automorphic correction) one is forced to consider hyperbolic generalized Cartan matrices of elliptic or parabolic type with a lattice Weyl vector. Infinite products, recently introduced and studied by R. Borcherds [B5] and which are connected with automorphic forms having rational quadratic divisors, give examples of Fourier expansions with generalized lattice Weyl vectors. (See also the important paper by J. Harvey and G. Moore [HM1] closely related with
Thus, if $\Phi$ has this infinite product expansion, it satisfies Arithmetic Mirror Symmetry Conjecture. Since sets of zeros of reflective automorphic forms $\Phi$ are unions of quadratic divisors, it is possible that reflective automorphic forms $\Phi$ always have infinite product expansions of this type at cusps.

We mention that Arithmetic Mirror Symmetry Conjecture together with finiteness results about elliptic and parabolic hyperbolic reflection groups and reflective hyperbolic lattices are extremely important for finding all reflective lattices $T$ with 2 negative squares and their reflective automorphic forms $\Phi$.

In Sect. 2.5 we consider automorphic forms $\Phi$ having Fourier expansion at a cusp with integral Fourier coefficients and connected with a hyperbolic generalized Cartan matrix $A$ with a lattice Weyl vector (by [N10] the matrix $A$ should have elliptic or parabolic type; otherwise, the automorphic form $\Phi$ does not exist). We call this Fourier expansion Fourier expansion of Lie type. Fourier expansions of this type were considered in [N10] and [GN1]—[GN4] where, in particular, it was shown that some very classical automorphic forms have Fourier expansions of this type. In connection with generalized Kac–Moody algebras theory, first examples of automorphic forms with Fourier expansions of Lie type were found by R. Borcherds ([B4]—[B6]). A Fourier expansion, at a cusp, of Lie type of an automorphic form defines generalized Lorentzian Kac–Moody superalgebras which have the denominator function defined by this expansion. One can call these algebras as automorphic Lorentzian Kac–Moody superalgebras. Using terminology introduced in [N10], [GN1]—[GN4], this automorphic form, considered at the cusp, and its generalized Lorentzian Kac–Moody superalgebras give automorphic corrections of the Kac–Moody algebra with the generalized Cartan matrix $A$.

In Sect. 2.5 we define the most interesting (from our point of view) class of Lie reflective automorphic forms. They are Lie reflective automorphic forms $\Phi$ having Fourier expansions of Lie type at all cusps. All automorphic forms with Fourier expansions of Lie type found in our papers [GN1]—[GN4] are automorphic forms of this type (Lie reflective). Main Conjecture is that there exists only finite number of Lie reflective automorphic forms. This Conjecture should not be very difficult because it is very closely related with finiteness results for hyperbolic generalized Cartan matrices of elliptic and parabolic type and with a lattice Weyl vector. Some complete list of these matrices for rank 3 we have in Sect. 1.

We suppose that all finiteness results and conjectures above are related with conjectured finiteness of families of Calabi–Yau 3-folds and $n$-folds for $n \geq 3$. One can find some arguments which support this thesis in [N12], [HM1], [HM2], [Ka1], [Ka2], [DVV]. We hope to consider this point in further publications.

In Part II of our paper (it will appear in a short time) we find automorphic corrections of Lorentzian Kac–Moody algebras with the most part of 60 hyperbolic generalized Cartan matrices of elliptic type classified in Sect. 1 and for some hyperbolic generalized Cartan matrices of parabolic type. For all their correcting automorphic forms $\Phi$ we prove that they are Lie reflective finding their Fourier expansions and infinite product expansions. Using general theory described in Sect. 2, we can see that these automorphic forms are not just accidental. They are very universal and natural from the point of view of the general theory of reflective automorphic forms and elliptic and parabolic hyperbolic root systems described above. From this point of view, their Fourier expansions at cusps give very natural automorphic Lorentzian Kac–Moody superalgebras.
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1. Classification of symmetric hyperbolic generalized Cartan matrices of rank 3 of elliptic arithmetic type and with a lattice Weyl vector

1.1. Reminding of hyperbolic generalized Cartan matrices.

In this Section, we recall elementary facts about connection between hyperbolic generalized Cartan matrices and reflection groups in hyperbolic spaces. See [K1] about general definitions of generalized Cartan matrices and their properties, [V3] about elementary properties of hyperbolic reflection groups, which we need here, and [N9], [N10].

For a countable set of indices $I$, a finite rank matrix $A = (a_{ij})$, $i, j \in I$, is called a generalized Cartan matrix if

- $a_{ii} = 2$ for any $i \in I$;
- $a_{ij}$ are non-positive integers for $i \neq j$;
- $a_{ij} = 0$ implies $a_{ji} = 0$.

We always suppose that $A$ is indecomposable which means that there does not exist a decomposition $I = I_1 \cup I_2$ such that both $I_1$ and $I_2$ are non-empty and $a_{ij} = 0$ for any $i \in I_1$ and any $j \in I_2$.

A generalized Cartan matrix $A$ is called symmetrizable if there exists an invertible diagonal matrix $D = \text{diag}(..., \epsilon_i, ...)$ and a symmetric matrix $B = (b_{ij})$, such that

$$A = DB; \quad \text{or} \quad (a_{ij}) = (\epsilon_i b_{ij})$$

One always can suppose that

$$\epsilon_i \in \mathbb{Q}, \quad \epsilon_i > 0, \quad b_{ij} \in \mathbb{Z}, \quad b_{ii} \in 2\mathbb{Z},$$

equivalently,

$$b_{ij} \in \mathbb{Z}, \quad b_{ii} \in 2\mathbb{Z}, \quad b_{ij} \leq 0, \quad b_{ii} > 0,$$

for any $i, j \in I$.

Since the matrix $A$ is indecomposable, the matrices $D$ and $B$ are defined uniquely up to a multiplicative constant. We call the matrix $B$ a symmetrized generalized Cartan matrix. It obviously defines the matrix $A$:

$$(a_{ij}) = \left( \frac{2b_{ij}}{b_{ii}} \right)$$

where

$$b_{ii} | 2b_{ij} \quad \text{for any} \quad i, j \in I.$$
A has exactly one negative eigenvalue). For this case, there exists a geometric realization of \( A \) (and \( B \)) in a hyperbolic space. We describe it below.

Let us consider an integral hyperbolic symmetric bilinear form (lattice, to be short) which is defined by the symmetric hyperbolic matrix \( B \) modulo kernel. We denote the form by \( (\ , \ ) \) and its free \( \mathbb{Z} \)-module by \( M \). Then the matrix \( B \) is the Gram matrix of the corresponding elements \( \alpha_i \in M, \ i \in I \), which generate the module \( M \):

\[
B = (b_{ij}) = ((\alpha_i, \alpha_j)).
\]

Then

\[
A = (a_{ij}) = \left( \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \right). \tag{1.1.8}
\]

The lattice \( M \) is hyperbolic (i.e., it has exactly one negative square), even (i.e. \((x,x)\) is even for any \( x \in M \)). To be shorter, we will denote a lattice \((M, (\ , \ ))\) by one letter \( M \). The lattice \( M \) defines the cone

\[
V(M) = \{ x \in M \otimes \mathbb{R} \mid (x,x) < 0 \}. \tag{1.1.9}
\]

There exists unique choice of its half \( V^+(M) \) such that the cone

\[
\mathbb{R}_+M = \{ x \in V^+(M) \mid (x, \alpha_i) \leq 0 \ \text{for any} \ i \in I \} \tag{1.1.10}
\]

is non-empty. Here each inequality \((x, \alpha_i) \leq 0, \ i \in I \), is essential (i.e. the cone will be changed if one removes this inequality).

Using hyperbolic geometry language, the projectivization \( \mathcal{L}^+(M) = V^+(M)/\mathbb{R}_+^+ \) is a hyperbolic space, and the projectivisation \( \mathcal{M} \) of the cone \( \mathbb{R}_+M \) is a locally finite polyhedron in the hyperbolic space \( \mathcal{L}^+(M) \) with the set of orthogonal vectors (to faces of \( \mathcal{M} \) and directed outwards) \( P(\mathcal{M}) = \{ \alpha_i \mid i \in I \} \). Since the lattice \( M \) is integral hyperbolic and because of conditions

\[
(\alpha_i, \alpha_i)|2(\alpha_i, \alpha_j) \ \text{for any} \ i \in I, \tag{1.1.11}
\]

\[
(\alpha_i, \alpha_j) \leq 0 \ \text{if} \ i \neq j \in I \tag{1.1.12}
\]

(see (1.1.4) and (1.1.6)), the polyhedron \( \mathcal{M} \) is a fundamental polyhedron for a discrete reflection group \( W \) generated by reflections \( s_{\alpha_i} \) in hyperplanes

\[
\mathcal{H}_{\alpha_i} = \{ \mathbb{R}_+^+x \in \mathcal{L}^+(M) \mid (x, \alpha_i) = 0 \}. \tag{1.1.13}
\]

It changes places half spaces \( \mathcal{H}_{\alpha_i}^+ \) and \( \mathcal{H}_{\alpha_i}^- \) bounded by the hyperplane \( \mathcal{H}_{\alpha_i} \), where

\[
\mathcal{H}_{\alpha_i}^+ = \{ \mathbb{R}_+^+x \in \mathcal{L}^+(M) \mid (x, \alpha_i) \leq 0 \}. \tag{1.1.14}
\]

On the hyperbolic lattice \( M \) the reflection \( s_{\alpha_i} \) is defined by the formula

\[
x \mapsto x - (2(\alpha_i, x)/(\alpha_i, \alpha_i))\alpha_i, \ x \in M. \tag{1.1.15}
\]

Thus, an indecomposable hyperbolic generalized Cartan matrix \( A \) is equivalent to a triplet

\[
(M, W, P(\mathcal{M})). \tag{1.1.16}
\]
Here $M$ is a primitive even hyperbolic lattice and $W \subset W(M) \subset O^+(M)$ is a reflection subgroup generated by reflections in a set of elements of the lattice $M$ with positive squares. $W(M)$ denote the group generated by reflections in all elements of the lattice $M$ with positive squares, $O(M)$ is the orthogonal group of the lattice $M$ and $O^+(M)$ is its subgroup which fixes the half-cone $V^+(M)$. The third object $P(M) \subset M$ in (1.1.16) is an acceptable set of elements orthogonal to the faces of a fundamental polyhedron $\mathcal{M} \subset \mathcal{L}^+(M)$ of the group $W$ (the set $P(M)$ contains exactly one element orthogonal to each face of $M$). The set $P(M)$ is called acceptable if it generates the lattice $M$ and has the property

\[(\alpha, \alpha)|2(\alpha, M) \text{ for any } \alpha \in P(M).\] (1.1.17)

Moreover, we suppose the set $P(M)$ has an indecomposable Gram matrix. Any triplet (1.1.16) defines a hyperbolic generalized Cartan matrix

\[A = \left(\frac{2(\alpha, \alpha')}{(\alpha, \alpha)}\right), \quad \alpha, \alpha' \in P(M).\] (1.1.18)

See [N9], [N10] for details. We call the triplet (1.1.16) a geometric realization of the hyperbolic generalized Cartan matrix (1.1.18).

Suppose that there are natural $\lambda(\alpha), \alpha \in P(M)$, such that all $\lambda(\alpha)$ are coprime:

\[\text{g.c.d.}(\{\lambda(\alpha) \mid \alpha \in P(M)\}) = 1,\] (1.1.19)

and elements $\tilde{\alpha} = \lambda(\alpha)\alpha$ satisfy the condition (1.1.11), i.e.

\[(\tilde{\alpha}, \tilde{\alpha})|2(\tilde{\alpha}, \tilde{\alpha}') \text{ for any } \alpha, \alpha' \in P(M).\] (1.1.20)

Equivalently,

\[\lambda(\alpha)(\alpha, \alpha)|2\lambda(\alpha')(\alpha, \alpha') \text{ for any } \alpha, \alpha' \in P(M).\] (1.1.21)

Then the matrix

\[\tilde{A} = \left(\frac{2(\tilde{\alpha}, \tilde{\alpha}')}{(\alpha, \alpha)}\right) = \left(\frac{2\lambda(\alpha')(\alpha, \alpha')}{\lambda(\alpha)(\alpha, \alpha)}\right), \quad \alpha, \alpha' \in P(M),\] (1.1.22)

is also a hyperbolic generalized Cartan matrix with the same reflection group $W$. This matrix is called twisted to $A$. Coefficients $\lambda(\alpha)$ satisfying (1.1.19) and (1.1.20) (equivalently, (1.1.19) and (1.1.21)) are called twisted coefficients. The corresponding generalized Cartan matrix $\tilde{A}$ is called twisted to $A$. Thus, the geometric realization of the generalized Cartan matrix $\tilde{A}$ is equal to

\[(\tilde{M}, \tilde{W}, \tilde{P}(\tilde{M})) = (\{\lambda(\alpha)\alpha \mid \alpha \in P(M)\} \subset M, W, \{\lambda(\alpha)\alpha \mid \alpha \in P(M)\}).\] (1.1.23)

where we identify hyperbolic spaces of $M$ and $\tilde{M}$. (Here $[X]$ denotes a submodule, sublattice generated by the set $X$.) Thus, the basic relation between $A$ and its twisted $\tilde{A}$ is that they have the same reflection (or Weyl) group $W$ and the same fundamental polyhedron $M$. 
A generalized Cartan matrix is called *untwisted* if it is not twisted to any generalized Cartan matrix different from itself. To find all possible generalized Cartan matrices, it is sufficient to find all untwisted generalized Cartan matrices and find for them all possible sets of twisted coefficients. For a finite generalized Cartan matrix $A$ the number of possible twisted coefficients and twisted to $A$ generalized Cartan matrices $\tilde{A}$ is finite.

Let $A$ be a hyperbolic generalized Cartan matrix and (1.1.16) its geometric realization. We introduce the group

$$\text{Sym } (A) = \text{Sym } (P(M)) = \{ g \in O^+(M) \mid g(P(M)) = P(M) \} \quad (1.1.24)$$

which is called *group of symmetries* of the generalized Cartan matrix $A$ (or $P(M)$). One can consider this group as a subgroup of $O(M)$ or a subgroup of symmetries of the fundamental polyhedron $\mathcal{M}$.

We use the following definitions from [N10] which are important for automorphic correction of Lorentzian Kac–Moody algebras corresponding to hyperbolic generalized Cartan matrices. One has a chance to find an automorphic correction only for cases described below. See Sects. 2.4, 2.5.

**Definition 1.1.1.** A hyperbolic generalized Cartan matrix $A$ has *restricted arithmetic type* if it is not empty and the semi-direct product

$$W \rtimes \text{Sym } (A) = W \rtimes \text{Sym } (P(M)) \quad (1.1.25)$$

has a finite index in $O^+(M)$.

**Definition 1.1.2.** A hyperbolic generalized Cartan matrix $A$ has a *lattice Weyl vector* if there exists an element $\rho \in M \otimes \mathbb{Q}$ such that for a constant $N > 0$

$$(\rho, \alpha) = -(\alpha, \alpha)/2 \quad \text{for any } \alpha \in P(M). \quad (1.1.26)$$

Then $\rho$ is called a *lattice Weyl vector*.

More generally, $A$ has a *generalized lattice Weyl vector* if there exists a non-zero element $\rho \in M \otimes \mathbb{Q}$ such that for a constant $N > 0$ one has

$$0 \leq -(\rho, \alpha) \leq N \quad \text{for any } \alpha \in P(M). \quad (1.1.27)$$

The element $\rho$ is called a *generalized lattice Weyl vector*.

One can prove (see [N10]) that if $\text{rk } A = \text{rk } M \geq 3$ (equivalently, $\text{dim } \mathcal{L}^+(M) \geq 2$) and $A$ has a restricted arithmetic type, then $\mathbb{R}_{++} \rho \in \mathcal{M}$ and $(\rho, \rho) \leq 0$. Therefore, one should consider two cases:

**Definition 1.1.3.** A hyperbolic generalized Cartan matrix has *elliptic type* if it has a restricted arithmetic type and has a generalized lattice Weyl vector $\rho$ with negative square: $(\rho, \rho) < 0$. This is equivalent to finiteness of index $[O(M) : W] < \infty$ or finiteness of volume of $\mathcal{M}$ (in particular, the set $P(M)$ is finite).

A hyperbolic generalized Cartan matrix $A$ has *parabolic type* if it has restricted arithmetic type and has a generalized lattice Weyl vector $\rho$ with zero square: $(\rho, \rho) = 0$, and it does not have a generalized lattice Weyl vector with negative square (i.e., it is not elliptic). For this case the group of symmetries $\text{Sym } (A) = W \rtimes \text{Sym } (P(M))$ is trivial.
Sym \((P(\mathcal{M})) \subset O(\rho_{\mathcal{M}})\) is a crystallographic group and is Abelian up to a finite index.

It is proved in \([N10]\) (in fact, it was done in \([N3]\), \([N4]\), \([N5]\)) that number of these cases is essentially finite.

The case when \(A\) has elliptic or parabolic type and at the same time has a lattice Weyl vector is especially important from the point of view of the theory of Lorentzian Kac–Moody algebras. See Sects 2.4, 2.5. It is proved in \([N10]\) that number of hyperbolic generalized Cartan matrices of elliptic type and with a lattice Weyl vector is finite for rank \(\geq 3\).

In the next Section we want to classify all symmetric generalized Cartan matrices of elliptic type and of rank \(3\) with a lattice Weyl vector.

**1.2. The classification of generalized Cartan matrices of elliptic type and of rank 3 with a lattice Weyl vector which are twisted to symmetric generalized Cartan matrices.**

Let \(A\) be a generalized Cartan matrix of elliptic type. Suppose that \(A\) is twisted to a symmetric generalized Cartan matrix \(\overline{A}\). Obviously, then \(\overline{A}\) is also elliptic.

Let us take a geometric realization \(G(A) = (\mathcal{M}, W, P(\mathcal{M}))\) of \(A\). Then for any \(\alpha \in P(\mathcal{M})\) one has \(\alpha = \lambda(\alpha)\delta(\alpha)\) where \((\delta(\alpha), \delta(\alpha)) = 2\), \(\lambda(\alpha) \in \mathbb{N}\) and the matrix \(\overline{A} = (\delta(\alpha), \delta(\alpha'))\), \(\alpha, \alpha' \in P(\mathcal{M})\), is a symmetric generalized Cartan matrix. Suppose additionally that \(\text{rk } A = 3\). Then \(\mathcal{M}\) is a polygon \(A_1A_2A_3\ldots A_n\) on a hyperbolic plane, and we can numerate elements of \(P(\mathcal{M})\) and \(\overline{P}(\mathcal{M}) = \{\delta(\alpha) = \alpha/\lambda(\alpha) \mid \alpha \in P(\mathcal{M})\}\) by \(1, \ldots, n\) where \(\alpha_i\) is orthogonal to the side \(A_iA_{i+1}\) of the polygon. We naturally denote \(\delta_i = \delta(\alpha_i)\) and \(\lambda_i = \lambda(\alpha_i)\). Then \(A\) and its geometric realization are equivalent to \((1 + \lfloor n/2 \rfloor) \times n\) matrix \(G(A)\). It has the first line

\[
\lambda_1, \ldots, \lambda_n
\]

and it has \(i + 1\)-th line

\[
-(\delta_1, \delta_{1+i}), -(\delta_2, \delta_{2+i}), \ldots, -(\delta_n, \delta_{n+i}),
\]

\(1 \leq i \leq \lfloor n/2 \rfloor\), where we consider the second index \(j\), \(1 \leq j \leq n\), modulo \(n\). Thus, a \(j\)-th column of \(G(A)\) is

\[
(\lambda_j; (\delta_j, \delta_{j+1}), (\delta_j, \delta_{j+2}), \ldots, (\delta_j, \delta_{j+\lfloor n/2 \rfloor}))^t.
\]

We want to find all matrices \(G(A)\) of this type having a lattice Weyl vector \(\rho\) which is an element \(\rho \in M \otimes \mathbb{Q}\) satisfying (1.1.26). Equivalently,

\[
(\rho, \delta_i) = -\lambda_i, \quad i = 1, \ldots, n.
\]  

**1.2.1**

The geometric realization \(G(A)\) is convenient because it easily permits to find the group of symmetries \(\text{Sym } (A)\), it has smaller coefficients and smaller size than the generalized Cartan matrix \(A\) and the corresponding symmetric generalized Cartan matrix \(B\).

We want to prove the following classification,
Theorem 1.2.1. All geometric realizations $G(A)$ of hyperbolic generalized Cartan matrices $A$ of elliptic type with a lattice Weyl vector and with properties: $A$ has rank 3, the matrix $A$ is twisted to a symmetric generalized Cartan matrix, all twisting coefficients $\lambda_i$ satisfy the inequality $\lambda_i \leq 12$, are given in Table 1 below.

Conjecture 1.2.2. Table 1 gives the complete list of hyperbolic generalized Cartan matrices $A$ of elliptic type with a lattice Weyl vector and with the properties: $A$ has rank 3 and the matrix $A$ is twisted to a symmetric generalized Cartan matrix (i.e. one can drop the inequalities $\lambda_i \leq 12$ from the Theorem 1.2.1).

We have the following arguments to support Conjecture 1.2.2. First, it is known [N10] that the number of all hyperbolic generalized Cartan matrices of elliptic type with a lattice Weyl vector is finite for rank $\geq 3$. Thus, there exists an absolute constant $m$ such that all $\lambda_i \leq m$. Second, we did our calculations for all $\lambda_i \leq 12$, but the result of these calculations has only matrices with all $\lambda_i \leq 6$. Thus, in between 6 and 12 there are no new solutions. In further publications we hope to present additional arguments which prove Conjecture 1.2.2.

Table 1.

Geometric realizations of hyperbolic generalized Cartan matrices of elliptic type with a lattice Weyl vector which are twisted to symmetric generalized Cartan matrices and have rank 3.

| $r$       | $G(A)$                                  |
|-----------|-----------------------------------------|
| $r = -59/2$ | $\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 1 \end{pmatrix}$ |
| $r = -16$  | $\begin{pmatrix} 1 & 4 & 2 \\ 0 & 2 & 2 \\ 1 \end{pmatrix}$ |
| $r = -10$  | $\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 1 \end{pmatrix}$ |
| $r = -7$   | $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 3 & 3 & 3 \\ 3 \end{pmatrix}$ |
| $r = -6$   | $\begin{pmatrix} 1 & 6 & 3 & 2 \\ 0 & 2 & 0 & 2 \\ 3 & 6 & 3 & 6 \\ 3 \end{pmatrix}$ |
| $r = -11/2$ | $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 2 \\ 1 \end{pmatrix}$ |
| $r = -4$   | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 4 & 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 4 & 4 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 4 & 4 & 2 \\ 2 & 2 & 1 \\ 2 & 2 & 2 \\ 3 \end{pmatrix}$ |
\[
\begin{align*}
\text{LORENTZIAN KAC–MOODY ALGEBRAS} & \quad 11 \\
\end{align*}
\]

\[
\begin{align*}
 r = -7/2 : & \quad G(A) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix}. & r = -5/2 : & \quad G(A) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix}. \\
 r = -13/6 : & \quad G(A) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}. & r = -17/8 : & \quad G(A) = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 4 & 4 & 4 & 4 \end{pmatrix}. \\
 r = -2 : & \quad G(A) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{pmatrix}, & r = -17/8 : & \quad G(A) = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 4 & 4 & 4 & 4 \end{pmatrix}, \\
 r = -3/2 : & \quad G(A) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 6 & 6 & 6 \\ 7 & 6 \end{pmatrix}. & r = -1 : & \quad G(A) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 4 & 4 & 4 \\ 6 & 6 & 6 & 6 \end{pmatrix}, \quad r = -3/2 : & \quad G(A) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 4 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 \end{pmatrix}. \\
 r = -2/3 : & \quad G(A) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 5 & 5 & 5 \end{pmatrix}. & r = -5/8 : & \quad G(A) = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 & 0 \\ 4 & 1 & 4 & 8 & 8 \end{pmatrix}. \\
 r = -1/2 : & \quad G(A) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 6 & 6 & 6 \\ 6 & 10 & 6 & 10 & 6 \end{pmatrix}, \quad r = -1/2 : & \quad G(A) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \\ 6 & 6 & 6 \\ 6 & 10 & 6 & 10 & 6 \end{pmatrix}. \\
 r = -7/18 : & \quad G(A) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 6 & 6 & 6 & 7 & 0 \end{pmatrix}. \\
\end{align*}
\]
\[ r = -1/4 : \quad G(A) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 7 & 7 & 7 & 7 & 7 \\ 10 & 10 & 10 & 10 & 10 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 6 & 14 & 14 & 6 & 14 & 14 \\ 10 & 34 & 10 & 10 & 34 & 10 \end{pmatrix}. \]

\[ r = -2/9 : \quad G(A) = \begin{pmatrix} 0 & 2 & 0 & 2 & 0 & 2 \\ 6 & 6 & 12 & 12 & 6 & 12 & 12 \\ 7 & 18 & 34 & 18 & 7 & 18 & 34 & 18 \\ 11 & 38 & 38 & 11 & 38 & 38 & 11 \end{pmatrix}. \]

\[ r = -1/6 : \quad G(A) = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 10 & 10 & 10 & 10 & 10 \\ 14 & 14 & 14 & 14 & 14 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 10 & 10 & 10 & 10 & 10 \\ 16 & 16 & 10 & 10 & 16 & 10 \end{pmatrix}. \]
Proof of Theorem 1.2.1. We fix $m \in \mathbb{N}$. We give a finite algorithm which permits to find all matrices $G(A)$ of our type with $\lambda_i \leq m$ for all $1 \leq \lambda_i \leq n$.

Step 1. It is known [N4], that there exist three consecutive sides of the polygon $A_1\ldots A_n$ which we denote by $A_1A_2$, $A_2A_3$, $A_3A_4$ such that for the orthogonal vectors $\delta_1, \delta_2, \delta_3 \in \mathcal{P}(\mathcal{M})$ to these sides one has

$$2 \leq \langle \delta_1, \delta_2 \rangle \leq 0; \quad 14 \leq \langle \delta_1, \delta_3 \rangle \leq 0; \quad 2 \leq \langle \delta_2, \delta_3 \rangle \leq 0. \quad (1.2.2)$$
We remark that since

\[(\delta_i, \delta_i) = 2\]  

(1.2.3)

for any \(i\), one should have

\[-2 \leq (\delta_i, \delta_{i+1}) \leq 0\]  

(1.2.4)

since the lines \(A_iA_{i+1}\) and \(A_{i+1}A_{i+2}\) should have a common point and the polygon \(A_1...A_n\) has acute \((\leq \pi/2)\) angles. The last is equivalent to

\[(\delta_i, \delta_j) \leq 0\]  

if \(i \neq j\).  

(1.2.5)

Obviously, one can find all (in a finite set) possibilities for the symmetric hyperbolic integral matrices \(g = (\delta_i, \delta_j), 1 \leq i, j \leq 3\), satisfying (1.2.2).

We fix one of these \(g\). It defines the rational hyperbolic bilinear form \(M \otimes \mathbb{Q}\). For the fixed \(g\) we find all natural \((\lambda_1, \lambda_2, \lambda_3)\) such that

\[\lambda_i \leq m\]  

(1.2.6)

and \(\lambda_i^2(\delta_i, \delta_i) \mid 2\lambda_i\lambda_j(\delta_i, \delta_j),\) equivalently

\[\lambda_i(\delta_i, \delta_i) \mid 2\lambda_j(\delta_i, \delta_j)\]  

(1.2.7)

for all \(1 \leq i, j \leq 3\). Further we consider only these \((g, (\lambda_1, \lambda_2, \lambda_3))\). For the fixed \((g, (\lambda_1, \lambda_2, \lambda_3))\), there exists a unique \(\rho \in M \otimes \mathbb{Q}\) (a lattice Weyl vector) such that

\[(\rho, \delta_i) = -\lambda_i,\] and we calculate

\[r = (\rho, \rho)\]  

(1.2.8)

Considering all \((g, (\lambda_1, \lambda_2, \lambda_3))\) above and calculating \(r\) for them, we find a finite set \(R\) of all such \(r\) with \(r < 0\).

**Step 2.** We fix \(r \in R\). Like above, we find all possible hyperbolic Gram matrices \(g_i, \delta_i, \delta_{i+1}, \delta_{i+2}\) and twisting coefficients \(\Lambda_i = (\lambda_i, \lambda_{i+1}, \lambda_{i+2})\) which satisfy the conditions (1.2.3)—(1.2.7) and calculations as above of the lattice Weyl vector \(\rho\) give the fixed square \((\rho, \rho) = r\) (i.e. (1.2.8) is valid for the fixed \(r\) where we calculate \(\rho\) using \(\delta_i, \delta_{i+1}, \delta_{i+2}\)). For fixed \(\lambda_i, \lambda_{i+1}, \lambda_{i+2}\) the set of possible such Gram matrices \(g_i\) is finite. Really, geometrically it is clear that if \(\lambda_i, \lambda_{i+1},\) \(\lambda_{i+2}\) and \((\delta_i, \delta_{i+1}), (\delta_{i+1}, \delta_{i+2})\) are fixed, then the function \((\rho, \rho)\) will be increasing for decreasing (negative) \((\delta_i, \delta_{i+2})\). Thus, we get a finite set \(K_r^3\) of possibilities for \((g_i, \Lambda_i)\). Actually, one can prove that the set \(K_r^3\) will be also finite if we drop inequalities \(\lambda_i \leq m\).

Consider datum \((g, \Lambda)\) from \(K_r^3\) such that \(\text{ and } -2 \leq g_{i(i+2)}\). The ones with coprime \(\Lambda\) give all \(G(A)\) with \(n = 3\) we are looking for (with the fixed invariant \(r\)). We remove all datum with \(-2 \leq g_{i(i+2)}\) from \(K_r^3\) and denote the rest by \((K_r^3)'\).

**Step 3.** Using the described set \((K_r^3)'\), we find all Gram matrices

\[(g_{ij}) = (\delta_i, \delta_j), \quad 1 \leq i, j \leq 4,\]  

(1.2.9)

and \(\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) such that \((g_{ij})\) is hyperbolic of rank 3 and the properties (1.2.4) for \(1 \leq i \leq 3\), (1.2.3), (1.2.5), (1.2.6), (1.2.7) for \(1 \leq i, j \leq 4\), and (1.2.8) for main submatrices

\[(g_{ij}) = (\delta_i, \delta_j), \quad 1 \leq i, j \leq 3,\]  

(1.2.10)
λ are valid. Equivalently, (1.2.10) with (λ₁, λ₂, λ₃) and (1.2.11) with (λ₂, λ₃, λ₄) both belong to (Kₚ)′.

The last condition defines finite set of possibilities for all coefficients of the matrix (1.2.9) except the coefficients g₁₄ = g₄₁ = (δ₁, δ₄). Using (1.2.8) for the matrix (gᵢⱼ), 1 ≤ i, j ≤ 3, we can calculate the coefficient g₁₄ using other coefficients. Really, elements δ₁, δ₂, μ generate over Q the same 3-dimensional lattice as δ₁, δ₂, δ₃. Thus, if we know g₂₄ = (δ₂, δ₄), g₄₂ = (δ₃, δ₄), μ = −λ₄, we can find g₁₄ = (δ₁, δ₄).

Further, we should check that g₁₄ is integral, g₁₄ ≤ 0, the condition (1.2.7) is satisfied for g₁₄ = (δ₁, δ₄), and the determinant of the matrix (1.2.9) is equal to 0. Thus, we can find the finite set of matrices (1.2.9) and Λ. We denote this set by (Kₚ)′.

Consider datum (g, Λ) from Kₚ⁴ such that −2 ≤ g₁₄. Those with coprime Λ give all G(A) with n = 4 we are looking for. We remove all datum with −2 ≤ g₁₄ from Kₚ⁴ and denote the rest by (Kₚ⁴)′.

**Step n − 1.** Now we define the sets Kₚⁿ and (Kₚⁿ)′, n ≥ 5. The set Kₚⁿ is the set of all pairs (g, Λ) where g is n × n hyperbolic integral Gram matrix

\[ g = (gᵢⱼ) = (δᵢ, δⱼ), \quad 1 ≤ i, j ≤ n, \]  

(1.2.12)

or rank 3,

\[ Λ = (λ₁, ..., λₙ) \]  

(1.2.13)

where λᵢ ∈ ℤ, 1 ≤ i ≤ n, such that the condition (1.2.4) is valid for all 1 ≤ i ≤ n−1 and conditions (1.2.3), (1.2.5) — (1.2.8) are valid for all 1 ≤ i, j ≤ n. The set (Kₚⁿ)′ is a subset of Kₚⁿ which is characterized by the inequality

\[ g₁ₙ < −2. \]  

(1.2.14)

To find Kₚⁿ, we remark that (gᵢⱼ), 1 ≤ i, j ≤ n−1, and (λ₁, ..., λₙ₋₁) should belong to (Kₚⁿ−1)′ and (gᵢⱼ), 2 ≤ i, j ≤ n, and (λ₂, ..., λₙ) as well. Like above, one should calculate g₁ₙ = gₙ₁ using other coefficients, and check that g₁ₙ is integral non-positive and it satisfies the condition (1.2.7). Then the pair (g, Λ) belongs to Kₚⁿ.

The set Kₚⁿ = (Kₚⁿ)′ with coprime Λ gives all G(A) of the size n we are looking for.

The calculations will stop when the set Kₚⁿ+₁ is empty.

**Remark 1.2.9.** For r = 0 the same algorithm gives all possible hyperbolic generalized Cartan matrices of parabolic type and of rank 3 with a lattice Weyl vector which are twisted to symmetric generalized Cartan matrices. Here one does not need Step 1 and should start with Step 2.

In Table 2 below we give a realization of the algorithm given above using GP/PARI calculator, Version 1.38, by C. Batut, D. Bernardi, H. Cohen and M. Olivier. Table 1 gives matrices G(A) which we obtain using this program. To check (at least) that a matrix G(A) of Table 1 gives a generalized Cartan matrix of Theorem 1.2.1, one should construct a matrix C of the size (n + 1) × n as follows. The first line of C is equal to the first line (λ₁, ..., λₙ) of the matrix G(A). The next lines of C are equal to the second and the third lines of Table 1.
of $C$ are equal to the lines of the symmetric Gram matrix $(\delta_i, \delta_j)$, $1 \leq i, j \leq n$. For $i = j$ one has $(\delta_i, \delta_j) = 2$. For $i < j$ the number $-(\delta_i, \delta_j)$ is equal to the coefficient of the $i$-th column and $j - i + 1$-th line of the matrix $G(A)$. One should check that the matrix $C$ has rank 3; all coefficients $-(\delta_i, \delta_{i+1})$ are not less than $-2$ and $\lambda_i(\delta_i, \delta_i)|2\lambda_j(\delta_i, \delta_j)$ for all $1 \leq i, j \leq n$.

### Table 2.

PARi program for calculation of the hyperbolic generalized Cartan matrices of elliptic and parabolic type with a lattice Weyl vector which are twisted to symmetric generalized Cartan matrices and have rank 3

```pari
\% Unified program for calculation of twisted cases.
\% To calculate with multiplicities up to k, one should
\% write main(k). The result will be in the file pari.log

gram(a,b,c,g)=
g=2*idmat(3);g[1,2]=–a;g[2,1]=–a;g[1,3]=–b;g[3,1]=–b;g[2,3]=–c;g[3,2]=–c; g;
rho(a,b,c,ta,tc,g)=
g=2*idmat(3);g[1,2]=–a;g[2,1]=–a;g[1,3]=–b;g[3,1]=–b;g[2,3]=–c;g[3,2]=–c; \ gauss(g,[-ta,-tb,-tc]);
rhosq(a,b,c,ta,tb,tc,g,r)=
g=2*idmat(3);g[1,2]=–a;g[2,1]=–a;g[1,3]=–b;g[3,1]=–b;g[2,3]=–c;g[3,2]=–c; \ r=gauss(g,[-ta,-tb,-tc]);r~*g*r;
rad(p,rhoo,rho1,a,b,c,ta,tb,tc,g,r,r1,n,k,alpha)=
n=1;rhoo=vector(n,j,0);rho1=rhoo;
for(a=0,2,\)
for(b=0,14,\)
for(c=0,2,\)
for(ta=1,p,\)
for(tb=1,p,\)
for(tc=1,p,\)
if((type(tb*a/ta)!=1)||(type(ta*a/tb)!=1)||(type(tb*c/tc)!=1)||(type(tc*c/tb)!=1)||
type(tc*b/ta)!=1,\)
g=2*idmat(3);g[1,2]=–a;g[2,1]=–a;g[1,3]=–b;g[3,1]=–b;g[2,3]=–c;g[3,2]=–c; \ if(det(g)>0,\)
r1=gauss(g,[-ta,-tb,-tc]);r=r1~*g*r1;\)
if(r>0,\)
alpha=1;k=1;\)
while(alpha,\)
if(rhoo[k]<r,k=k+1,\)
if(rhoo[k]==r, alpha=0,n=n+1;rho1=vector(n,j,0);\)
for(l=1,k-1,rhoo[l]=rhoo[l]); rho1[k]=r;for(l=k+1,n,rhoo[l]=rho1[l-1]; \ rhoo=rho1;alpha=0)))))))))\);rhoo;

rot(v,s,v1)=
v1=v;for(j=1,s-1,for(k=j+1,s,\)
if(b<c,v1[(i-1)*(2*s-1)+2*k-j]–v[i*(2*s-1)+2*k-j],\)
if(b<c,v1[2*s*(j-1)+k]–v1[2*s*(j-1)+k]));
```
\begin{verbatim}
    v1[(j-1)*(2*s-j)/2+k-j]=v[j]);for(j=1,s,
if(j<s,v1[(s-1)*s/2+j]=v[(s-1)*s/2+j+1], v1[s*(s+1)/2]=v[(s-1)*s/2+1]));v1;

    sym(v,s,v1)=,
    v1=v;for(j=1,s-1,for(k=j+1,s,
    v1[(j-1)*(2*s-j)/2+k-j]=v[(s-k)*(s+k-1)/2+k-j]);
    for(j=1,s,
    v1[(s-1)*s/2+j]=v[(s-1)*s/2+s-j+1]);v1;

    sorting(a,s,n,m1,m2,a1,a2,a3,t,alpha,beta,v,v1)=
    if(type(a)==1
    ||type(a)==17,a,
    n=matsize(a);m1=n[1];
    a1=a;t=1;
    v=a1[1,];v1=sym(v,s);if(t>=m1,alpha=0,alpha=1);
    while(alpha,
    a2=matrix(t,n[2],j,k,a1[j,k]);m2=t;
    for(j=t+1,m1,
    beta=1;
    for(k=1,s,v=rot(v,s);v1=rot(v1,s);
    if(a1[j,]!=v&&a1[j,]!=v1,,beta=0));
    if(beta==0,,
    m2=m2+1;a3=matrix(m2,n[2],j,k,0);
    for(l=1,m2–1,a3[l,]=a2[l,]);a3[m2,]=a1[j,];a2=a3;));
    a1=a2;m1=m2;
    if(t>=m1,alpha=0,
    t=t+1;v=vector(n[2],j,a1[t,j]);v1=sym(v,s)));a1);

    gr(d,s,b,g)=
    g=[2,0,-1;0,2,-2;–1,–2,2];
    b=vector((s–1)*s/2,j,0);
    for(j=1,s–1,
    for(k=j+1,s,
    b[(j–1)*(2*s–j)/2+k–j]=–d[j,]*g*d[lift(mod(j+t,s))+1,]);b;

    \ Calculation of the polygonal matrix from the matrix d of vectors
    \ delta_i in bases a,b,c
    polygon(d,s,g,c)=
    g=[2,0,–1;0,2,–2;–1,–2,2];
    c=matrix(floor(s/2),s,j,k,0);
    for(t=1,floor(s/2),for(j=0,s–1,
    c[t,j+1]=–d[j+1,]*g*d[lift(mod(j+t,s))+1,]);c;

    \ Calculation of the polygonal matrix from the gram vector (with multiplicities for delta_i)
    polygon1(b,s,a,c)
    a=2*idmat(s);
    for(j=1,s–1,for(k=j+1,s,
    c[j,k]=b[(i–1)*(2*s–i)/2+k–i];a[k,j]=c[j,k]));
\end{verbatim}
\begin{verbatim}
c=matrix(floor(s/2)+2,s,j,k,0);
for(t=1,floor(s/2),for(j=0,s–1,
  c[t+2,j+1]=–a[j+1,lift(mod(j+t,s))+1]);
for(j=1,s,c[1,j]=b[s*(s–1)/2+j]);c;

\end{verbatim}

\begin{verbatim}
\textbackslash calculation of the symmetric generalized Cartan
\textbackslash matrix (Gram matrix of m_i \delta_j) from the Gram vector

symcartan(b,s,a,sa)=
  a=2*idmat(s);
  for(j=1,s–1,for(k=j+1,s,
    a[j,k]=–b[(j–1)*(2*s–j)/2+k–j];a[k,j]=a[j,k]);
  sa=matrix(s,s,j,k,b*[s*(s–1)/2+j]*b*[s*(s–1)/2+k]*a[j,k]);sa;

\end{verbatim}

\begin{verbatim}
\textbackslash function eq3(p,r)
eq3(p,r,s,m,u,r1,alpha)=s=matrix(60,6,j,k,0);m=0;s[1,]=[0,0,0,0,0,0];
  for(a=0,2,for(c=0,2,for(ta=1,p,for(tc=1,p,for(tb=1,p,
    b=1;alpha=1;
    while(alpha,\n      if(det(gram(a,b,c))>=0,b=b+1,\n      r1=rhosq(a,b,c,ta,tb,tc);
      if(r1<r,b=b+1,);\n      if(r1>r,alpha=0,);\n      if(r1==r,\n        if(type(tb*a/ta)==1,\n        if(type(ta*a/tb)==1,\n        if(type(tb*c/tc)==1,\n        if(type(tc*c/tb)==1,\n        if(type(ta*b/tc)==1,\n        if(type(tc*b/ta)==1,\n        m=m+1;s[m,]=[a,b,c,ta,tb,tc],);),),),),),);
    alpha=0,))));
  if(m==0,0,u=matrix(m,6,j,k,0);for(j=1,m,u[j,]=s[j,]);u);

\end{verbatim}

\begin{verbatim}
\textbackslash function eq4(p,r,s=4
 eq4(p,r,a,aa,aaa,n,m,u,v,w,x,y,z, talpha,tbeta,tgamma,tdelta,ra)=
  u=eq3(p,r);
  if(type(u)==1,0,
    n=matsize(u);n=n[1];v1=matrix(n,6,j,k,0);m=0;
    for(x=1,n,if(u[x,2]>2,m=m+1;v1[m,]=u[x,]);
    if(m==0,a=0,\n      a=matrix(m,6,j,k,0);for(x=1,m,a[x,]=v1[x,]);
      if(type(a)==1,0,\n      n=matsize(a);n=n[1];aa=matrix(100,10,j,k,0);m=0;
      for(j=1,n,u=a[j,1];v=a[j,2];w=a[j,3];talpa=a[j,4];\n      tbeta=a[j,5];tgamma=a[j,6];\n      for(k=1,n,\n        if([w,tbeta,tgamma]!=[a[k,1],a[k,4],a[k,5]],\n          y=a[k,2];z=a[k,3]; tdelta=a[k,6];ra=rhom(u,v,w,talpha,tbeta,tgamma);
          x=(tdelta, ra[2]*ra[2]^(-1), ra[2]*ra[2]^(-1)[1]);
        );\n      ));\n    ));\n  ));
\end{verbatim}
if(x<0, \
if(type(x)!=1||type(tdelta*x/talpha)!=1|| type(talpha*x/tdelta)!=1,, \
if(det([[2,–u,–v,–x;–u,2,–w,–y;–v,–w,2,–z;–x,–y,–z,2]])!=0,, \
m=m+1;aa[m]=[[u,v,x,w,y,z,talpha,tbeta,tdelta]])); 
if(m==0,aaa=0,aaa=matrix(m,10,j,k,0); for(j=1,m,aaa[j,j]=aa[j,j]);\aaa);

eq(p,r,a,aa,n,s,m,m,vv,ww,u,v,w,u1,v1,w1,g,g1, 
 delta4,delta4du,deltalast,deltalastdu,deltalast1)=\n s=3;a=eq3(p,r);\n if(type(a)==1,,n=matsize(a); 
 aa1=matrix(n[1],n[2],j,k,0);m1=0; 
 for(j=1,n[1], 
 if(a[j,s–1]>2,, 
 aaa=vector(s,k,a[j,n[2]–s+k]); 
 if(content(aaa)!=1,,m1=m1+1;aa1[m1,j]=a[j,j])); 
 if(m1==0,a1=0, 
 a1=matrix(m1,n[2],j,k,aa1[j,k])); 
 a1=sorting(a1,s); 
 if(type(a1)==1,,n1=matsize(a1);for(j=1,n1[1], 
 q=aa1[j,j]; 
 pprint(q);texprint(q); 
 pprint(symcartan(q,s));texprint(symcartan(q,s)); 
 pprint(polygon1(q,s));texprint(polygon1(q,s))); 
 s=4;a=eq4(p,r);\n alpha=1; \n while(alpha,, 
 if(type(a)==1,alpha=0,n=matsize(a); 
 aa=matrix(n[1],n[2],j,k,0);m=0; 
 aa1=matrix(n[1],n[2],j,k,0);m1=0; 
 for(j=1,n[1], 
 if(a[j,s–1]>2,m=m+1;aa[m]=a[j,j], 
 aaa=vector(s,k,a[j,n[2]–s+k]); 
 if(content(aaa)!=1,,m1=m1+1;aa1[m1,j]=a[j,j])); 
 if(m1==0,a1=0, 
 a1=matrix(m1,n[2],j,k,aa1[j,k])); 
 a1=sorting(a1,s); 
 if(type(a1)==1,,n1=matsize(a1);for(j=1,n1[1], 
 q=a1[j,j]; 
 pprint(q);texprint(q); 
 pprint(symcartan(q,s));texprint(symcartan(q,s)); 
 pprint(polygon1(q,s));texprint(polygon1(q,s))); 
 if(m==0,a=0;alpha=0, 
 a=matrix(m,n[2],j,k,aa[j,k]); 
 aa=matrix(100,n[2]+s+1,j,k,0);mm=0; 
 for(j=1,m,for(k=1,m, 
 vv=vector(n[2]–s,l,0); 
 for(l=1,n[2]–2*s+1,vv[l]=a[j,s–1+l]); 
 for(l=n[2]–2*s+2,n[2]–s,vv[l]=a[j,s+l]); 
 vv=vector(n[2]–2,s,0)); 
}
Remark 1.2.4. Table 1 contains 60 matrices \( G(A) \). Seven of them correspond to compact case (i.e., a fundamental polygon \( \mathcal{M} \) for \( W \) is compact in the hyperbolic plane, it has only finite vertices). These cases correspond to matrices \( G(A) \) with second line without 2 (for 2 the corresponding angle of \( \mathcal{M} \) is equal to 0, for 1 it is equal to \( \pi/3 \), for 0 it is equal to \( \pi/2 \)).

Between 7 compact cases there are 4 non-twisted ones (i.e. they give symmetric generalized Cartan matrices). Non-twisted cases correspond to matrices \( G(A) \) with first line containing 1 only. Between 53 non-compact cases there are 12 non-twisted ones.

We consider these last 12 cases below.

1.3. Symmetric hyperbolic generalized Cartan matrices of rank 3 having elliptic type and a lattice Weyl vector. Non-compact case.

From Theorem 1.2.1, we get 12 symmetric generalized Cartan matrices \( A \) corresponding to 12 non-compact and non-twisted matrices \( G(A) \) of Table 1 (see Remark 1.2.4). We announced this list in [GN4].

Theorem 1.3.1. There are exactly 12 symmetric hyperbolic generalized Cartan matrices of elliptic type and of rank 3 which have Weyl group with a non-compact (i.e., with an infinite vertex) fundamental polygon and have a lattice Weyl vector.
They are matrices $A_{1,0} - A_{3,III}$ below:

$$A_{1,0} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}, \quad A_{1,I} = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

$$A_{1,II} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}, \quad A_{1,III} = \begin{pmatrix} 2 & -2 & -6 & -6 & -2 \\ -2 & 2 & 0 & -6 & -7 \\ -6 & 0 & 2 & -2 & -6 \\ -6 & -6 & -2 & 2 & 0 \\ -2 & -7 & -6 & 0 & 2 \end{pmatrix},$$

$$A_{2,0} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix},$$

$$A_{2,I} = \begin{pmatrix} 2 & -2 & -4 & 0 \\ -2 & 2 & 0 & -4 \\ -4 & 0 & 2 & -2 \\ 0 & -4 & -2 & 2 \end{pmatrix}, \quad A_{2,II} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -2 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix},$$

$$A_{2,III} = \begin{pmatrix} 2 & -2 & -8 & -16 & -18 & -14 & -8 & 0 \\ -2 & 2 & 0 & -8 & -14 & -18 & -16 & -8 \\ -8 & 0 & 2 & -2 & -8 & -16 & -14 & -14 \\ -16 & -8 & -2 & 2 & 0 & -8 & -14 & -18 \\ -18 & -14 & -8 & 0 & 2 & -2 & -8 & -16 \\ -14 & -18 & -16 & -8 & 2 & 0 & -8 & -14 \\ -8 & -16 & -18 & -14 & -8 & 0 & 2 & -2 \\ 0 & -8 & -14 & -18 & -16 & -8 & -2 & 2 \end{pmatrix},$$

$$A_{3,0} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}, \quad A_{3,I} = \begin{pmatrix} 2 & -2 & -5 & -1 \\ -2 & 2 & -1 & -5 \\ -5 & -1 & 2 & -2 \\ -1 & -5 & -2 & 2 \end{pmatrix},$$

$$A_{3,II} = \begin{pmatrix} 2 & -2 & -10 & -14 & -10 & -2 \\ -2 & 2 & -2 & -10 & -14 & -10 \\ -10 & -2 & 2 & -2 & -10 & -14 \\ -14 & -10 & -2 & 2 & -2 & -10 \\ -10 & -14 & -10 & -2 & 2 & -2 \\ -2 & -10 & -14 & -10 & -2 & 2 \end{pmatrix},$$

$$A_{3,III} = \begin{pmatrix} 2 & -2 & -11 & -25 & -37 & -47 & -50 & -46 & -37 & -23 & -11 & -1 \\ -2 & 2 & -1 & -11 & -23 & -37 & -46 & -50 & -47 & -37 & -25 & -11 \\ -11 & -1 & 2 & -2 & -11 & -25 & -37 & -47 & -50 & -46 & -37 & -23 \\ -25 & -11 & -2 & 2 & -11 & -23 & -37 & -46 & -50 & -47 & -37 & -23 \\ -37 & -23 & -11 & -1 & 2 & -2 & -11 & -25 & -37 & -47 & -50 & -46 \\ -47 & -37 & -25 & -11 & -2 & 2 & -1 & -11 & -23 & -37 & -46 & -50 \\ -50 & -46 & -37 & -23 & -11 & -1 & 2 & -2 & -11 & -25 & -37 & -47 \\ -46 & -50 & -47 & -37 & -25 & -11 & -2 & 2 & -1 & -11 & -23 & -37 \\ -37 & -47 & -50 & -46 & -37 & -23 & -11 & -1 & 2 & -2 & -11 & -25 \\ -23 & -37 & -46 & -50 & -47 & -37 & -25 & -11 & -2 & 2 & -1 & -11 \\ -11 & -25 & -37 & -50 & -46 & -37 & -23 & -11 & -1 & 2 & -2 & -11 \\ -11 & -25 & -37 & -50 & -46 & -37 & -23 & -11 & -1 & 2 & -2 & -11 \end{pmatrix}. $$
Proof. We leave the trivial calculation of matrices $A$ from matrices $G(A)$ to a reader. In Table 1, non-compact non-twisted cases give respectively:

$r = -23/2$ gives $A_{1,0}$;
$r = -4$ gives $A_{1,1}$;
$r = -7/2$ gives $A_{2,0}$;
$r = -13/6$ gives $A_{3,0}$;
$r = -3/2$ gives $A_{1,11}$;
$r = -1$ gives $A_{2,1}$;
$r = -2/3$ gives $A_{3,1}$;
$r = -1/2$ gives $A_{2,11}$;
$r = -7/18$ gives $A_{1,111}$;
$r = -1/6$ gives $A_{3,11}$;
$r = -1/8$ gives $A_{2,111}$;
$r = -1/24$ gives $A_{3,111}$.

We numerate 12 matrices of Theorem 1.3.1 by two indices $i, j = 0, I, II, III$. For non-twisted cases the matrices $A_{i,j}$ are the Gram matrices of sets $P(M_{i,j})$ of the orthogonal vectors (normalized by square 2) to fundamental polygons $M_{i,j}$ of reflection groups $W_{i,j}$ on a hyperbolic plane. The polygons $M_{i,*}$ corresponding to $A_{i,*}$, $i = 1, 2, 3$, are naturally composed from a triangle with two angles 0 and $\pi/2$ and the third angle $\pi/3$, $\pi/4$ and $\pi/6$ respectively. The second index $j$ shows how we get the polygon $M_{i,j}$ from this triangle. Below we describe the polygons $M_{i,j}$, the sets $P(M_{i,j})$, the lattices $M_{i,j}$ and the lattice Weyl vectors $\rho_{i,j}$ corresponding to $A_{i,j}$.

We use the following notations for lattices and their discriminant forms:

$\langle A \rangle$ is a lattice with a matrix $A$;
$\oplus$ denotes an orthogonal sum of lattices, $K^n$ is orthogonal sum of $n$ copies of a lattice $K$ (the same for forms);
$K(n)$ denotes a lattice which is obtained multiplying by $n \in \mathbb{Q}$ the form of a lattice $K$;
$nK, n \in \mathbb{Q}$, denotes a lattice $(nM, \phi|_{nM})$ if $K = (M, \phi)$, evidently, $nK \cong K(n^2)$;
$b_K$ and $q_K$ are the discriminant bilinear and quadratic forms respectively on the discriminant group $A_K = K^*/K$ of a lattice $K$; for discriminant forms we use notations from [N2];
b$_{\epsilon}(p^k), q_{\epsilon}(p^k)$ are the discriminant bilinear and quadratic form respectively of a one-dimensional $p$-adic lattice (i.e. over the ring $\mathbb{Z}_p$ of $p$-adic integers) $K_{\epsilon}(p^k) = \langle \epsilon p^k \rangle, \epsilon \in \mathbb{Z}_p^*$;
u$_{-}(2^k), v_{-}(2^k)$ and u$_-^{(2)}(2^k), v_{+}^{(2)}(2^k)$ are discriminant bilinear and quadratic forms of 2-adic lattices $U^{(2)}(2^k) = U(2^k) \otimes \mathbb{Z}_2$ and $V^{(2)}(2^k) = A_2 \otimes \mathbb{Z}_2$ respectively where

\[
U = \begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix};
\]

$W^{(m_1,\ldots,m_k)}(K)$ denotes a subgroup of $O^+(K)$ of a lattice $K$ generated by reflections $s_{\delta}$ in all primitive elements $\delta \in K$ with square $(\delta, \delta) \in \{m_1, \ldots, m_k\}$; here $O^+(K)$ is a subgroup of the full orthogonal group $O(K)$ which fixes the cone $V^+(K)$.

1.3.1. Generalized Cartan matrices $A$
In Figure 1 we consider the following construction on a hyperbolic plane. Let $ABC$ be a triangle with the angles $0$, $\pi/3$ and $\pi/2$ respectively. Considering the image of $ABC$ by the reflection in $AC$, we get a triangle $BB_1A$ with the angles $\pi/3$, $\pi/3$ and $0$ respectively. Considering the image of the triangle $ABC$ by the group (of type $D_3$) generated by reflections in $BC$ and $BA$, we get a right triangle $AA_2A_3$ (here $AA_2$ contains $C$) with zero angles. The right triangle $AA_2A_3$ is divided by medians $AC_2$, $A_2C_3$ and $A_3C$ (all containing the center $B$). Considering a reflection in $A_2A_3$ of the triangle $A_2A_3C_3$ and a reflection in $AA_3$ of the triangle $AC_2A_3$, we get a pentagon $AA_2C_3'A_3'C_2$ with the angles $0$, $0$, $\pi/2$, $0$, $\pi/2$ respectively. Here $C_3'$ is the image of $C_3$ and $C_2'$ is the image of $C_2$.

It is easy to see that the matrix $A_{1,0}$ is the Gram matrix (of orthogonal vectors with square 2 to the sides) of the triangle $M_{1,0} = ABC$. The matrix $A_{1,1}$ is the Gram matrix of the triangle $M_{1,1} = BB_1A$, the matrix $A_{1,II}$ is the Gram matrix of the triangle $M_{1,II} = AA_2A_3$, and $A_{1,III}$ is the Gram matrix of the pentagon $M_{1,III} = AA_2C_3'A_3'C_2$.

![Figure 1. Fundamental polygons of $A_{1,j}$](image-url)
We introduce a hyperbolic lattice $M_{1,0} \cong U \oplus \langle 2 \rangle$ with the corresponding standard basis $\xi_1, \xi_2, \xi_3$ having the Gram matrix
\[
\begin{pmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

Thus, $\xi_1, \xi_2$ and $\xi_3$ are the standard basis of orthogonal lattices $U$ and $\langle 2 \rangle$ respectively. We identify the hyperbolic planes of the lattice $M_{1,0}$ and Figure 1 as follows.

Point $A_2 = \mathbb{R}^+ \xi_1$, $A = \mathbb{R}^+ \xi_2$, elements $a = -\xi_1 + \xi_2$, $b = \xi_3$ and $c = \xi_1 - \xi_3$ are orthogonal to sides $CB$, $AC$ and $AB$ of the triangle $ABC$ and are directed outward.

The triangle $M_{1,0} = BCA$ has
\[
P(M_{1,0}) = \{ \delta_1 = a = -\xi_1 + \xi_2, \delta_2 = b = \xi_3, \delta_3 = c = \xi_1 - \xi_3 \} \quad (1.3.1)
\]

with the Gram matrix $A_{1,0}$ and the lattice Weyl vector
\[
\rho_{1,0} = 3\xi_1 + 2\xi_2 - (1/2)\xi_3. \quad (1.3.2)
\]
The lattice $M_{1,0}$ is generated by $P(M_{1,0})$ and is then the lattice of $A_{1,0}$. Thus,
\[
M_{1,0} \cong U \oplus \langle 2 \rangle. \quad (1.3.3)
\]

We have
\[
W_{1,0} = W^{(2)}(M_{1,0}), \quad \text{Sym } (P(M_{1,0})) \text{ is trivial.} \quad (1.3.4)
\]
The (1.3.4) are well-known. For example, they follow from Vinberg’s algorithm [V1] for calculation of a fundamental polyhedron of a hyperbolic reflection group.

The triangle $M_{1,I} = B_1AB$ has
\[
P(M_{1,I}) = \{ \delta_1 = \xi_1 + \xi_3, \delta_2 = \xi_1 - \xi_3, \delta_3 = -\xi_1 + \xi_2 \} \quad (1.3.5)
\]

with the Gram matrix $A_{1,I}$ and the lattice Weyl vector
\[
\rho_{1,I} = 2\xi_1 + \xi_2. \quad (1.3.6)
\]
The lattice $M_{1,I} = [a, 2b, c]$ is a sublattice of the lattice $M_{1,I}$. It has the discriminant form
\[
q_{M_{1,I}} = q_5^{(2)}(8). \quad (1.3.7)
\]
We have
\[
W_{1,I} = W^{(2)}(M_{1,I}), \quad \text{Sym } (M_{1,I}) = D_1 \quad (1.3.8)
\]
where the second group is generated by the reflection in $2\xi_3 = \delta_1 - \delta_2$. Here and in similar calculations below we use results of [N7] where all 2-reflective integral hyperbolic lattices $M$ of rank 3 were classified and their reflection groups $W^{(2)}(M)$ were calculated. A lattice $M$ is called 2-reflective if $W^{(2)}(M)$ has finite index in $O(M)$.

The triangle $M_{1,II} = AA_3A_2$ has
\[
P(M_{1,II}) = \{ \delta_1 = 2\xi_1 + \xi_3, \delta_2 = 2\xi_2 + \xi_3, \delta_3 = \xi_1 - \xi_2 \} \quad (1.3.9)
\]
with the Gram matrix $A_{1,II}$ and the lattice Weyl vector

$$\rho_{1,II} = \xi_1 + \xi_2 - (1/2)\xi_3.$$  
(1.3.10)

The lattice $M_{1,II} = [2\xi_1, 2\xi_2, \xi_3] = [2a, b, 2c]$ is

$$M_{1,II} \cong U(4) \oplus \langle 2 \rangle.$$  
(1.3.11)

We have

$$W_{1,II} = W^{(2)}(M_{1,II}), \quad \text{Sym}(P(M_{1,II})) = D_3$$  
(1.3.12)

where the second group is generated by the reflections in $c = \xi_1 - \xi_3$ and $a = -\xi_1 + \xi_2$.

The pentagon $M_{1,III} = AA_2C_3'A_3C_2'$ has

$$P(M_{1,III}) = \{\delta_1 = \xi_3, \delta_2 = 3\xi_2 - \xi_3, \delta_3 = 2\xi_1 + 4\xi_2 - 3\xi_3, \delta_4 = 4\xi_1 + 2\xi_2 - 3\xi_3, \delta_5 = 3\xi_1 - \xi_3\}$$  
(1.3.13)

with the Gram matrix $A_{1,III}$ and the lattice Weyl vector

$$\rho_{1,III} = (2/3)\xi_1 + (2/3)\xi_2 - (1/2)\xi_3.$$  
(1.3.14)

The lattice $M_{1,III} = [a, b, 3c]$ has the discriminant form

$$q_{M_{1,III}} = q^{(2)}_1(2) \oplus q^{(3)}_1(9).$$  
(1.3.15)

We have

$$W_{1,III} \prec W^{(2)}(M_{1,III}) \quad \text{and} \quad W^{(2)}(M_{1,III})/W_{1,III} = D_1$$  
(1.3.16)

where the second group is generated by the reflection $s_a$. The reflection $s_a$ also generates the group Sym$(P(M_{1,III}))$.

We remark that all lattices $M_{1,0}, M_{1,I}, M_{1,II}, M_{1,III}$ are characterized by their discriminant quadratic forms: i.e. a lattice of signature $(2,1)$ with one of these discriminant quadratic forms is unique.

1.3.2. Generalized Cartan matrices $A_{2,j}$.

On Figure 2 $ABC$ is a triangle with the angles 0, $\pi/4$ and $\pi/2$ respectively. Considering the orbit of $ABC$ by the group generated by the reflection in $BC$, we get a triangle $ABA_2$ with the angles 0, $\pi/2$ and 0 respectively. Considering the orbit of the triangle $ABC$ by the group (of type $D_2$) generated by the reflections in $BC$ and $AC$, we get a quadrangle $ABA_2B_1$ with the angles 0, $\pi/2$, 0 and $\pi/2$ respectively. Considering the orbit of the triangle $ABC$ by the group (of type $D_4$) generated by the reflections in $BC$ and $BA$, we get a right quadrangle $AA_2A_3A_4$ (here $AA_2$ contains $C$) with zero angles and the center $B$. Considering images of the center $B$ by reflections in the sides $AA_2$, $A_2A_3$, $A_3A_4$ and $A_4A$, we get points $B_1$, $B_2$, $B_3$ and $B_4$ respectively which together with $A$, $A_2$, $A_3$ and $A_4$ define an 8-gon $AB_1A_2B_2A_3B_3A_4B_4$ with the angles 0, $\pi/2$, 0, $\pi/2$, 0, $\pi/2$, 0, $\pi/2$ respectively. This 8-gon also is the orbit of the triangle $ABB_1$ with the angles 0, $\pi/4$, $\pi/4$ respectively by the group (of type $D_4$) generated by reflections in his sides $BA$, $BB_1$.

It is easy to see that the matrix $A_{2,0}$ is the Gram matrix (of orthogonal vectors with squares 2 to the sides) of the triangle $M_{1,0} = ABA_2$. The matrix $A_{2,0}$ is the
We introduce a hyperbolic lattice $M_{2,0} \cong U(2) \oplus (2)$ with the corresponding standard bases $\xi_1, \xi_2, \xi_3$ having the Gram matrix

$$
\begin{pmatrix}
0 & -2 & 0 \\
-2 & 0 & 0 \\
0 & 0 & 2
\end{pmatrix}.
$$

We identify the hyperbolic plane of the lattice $M_{2,0}$ and of Figure 2: Point $A_2 = \mathbb{R}^+ \xi_1$, $A = \mathbb{R}^+ \xi_2$, elements $a = -\xi_1 + \xi_2$, $b = \xi_3$ and $c = \xi_1 - \xi_3$ are orthogonal to sides $CB$, $AC$ and $AB$ of the triangle $ABC$ and directed outward. Here $(a, a) = 4$ and $(b, b) = (c, c) = 2$.

We have for $M_{2,0} = AA_2B$

$$
P(M_{2,0}) = \{ \xi_1, \xi_2, \xi_3, \xi_1 - \xi_2, \xi_1 - \xi_3, \xi_2 - \xi_3 \} \quad (1.3.17)
$$

Figure 2. Fundamental polygons of $A_{2,j}$.
with the Gram matrix $A_{2,0}$ and the lattice Weyl vector

$$\rho_{2,0} = \xi_1 + \xi_2 - (1/2)\xi_3.$$  \hfill (1.3.18)

The lattice $M_{2,0}$ generated by $P(M_{2,0})$ is then $M_{2,0} = [a, b, c]$, and

$$M_{2,0} \cong U(2) \oplus \langle 2 \rangle.$$  \hfill (1.3.19)

We have

$$W_{2,0} = W^{(2)}(M_{2,0}), \quad \text{Sym} (P(M_{2,0})) = D_1$$

where the second group is generated by the reflection $s_a$.

The quadrangle $M_{2,I} = B_1ABA_2$ has

$$P(M_{2,I}) = \{\delta_1 = \xi_1 + \xi_3, \delta_2 = \xi_1 - \xi_3, \delta_3 = \xi_2 - \xi_3, \delta_4 = \xi_2 + \xi_3\}$$  \hfill (1.3.21)

with the Gram matrix $A_{2,I}$ and the lattice Weyl vector

$$\rho_{2,I} = (1/2)\xi_1 + (1/2)\xi_2.$$  \hfill (1.3.22)

The lattice $M_{2,I} = [a, 2b, c]$ is

$$M_{2,I} \cong \langle -8 \rangle \oplus \langle 2 \rangle \oplus \langle 2 \rangle.$$  \hfill (1.3.23)

We have

$$W_{2,I} = W^{(2)}(M_{2,I}), \quad \text{Sym} (M_{2,I}) = D_2$$

where the second group is generated by the reflections $s_a$ and $s_{2b}$.

The right quadrangle $M_{2,II} = AA_2A_3A_4$ has

$$P(M_{2,II}) = \{\delta_1 = \xi_3, \delta_2 = 2\xi_1 - 3\xi_3, \delta_3 = 2\xi_1 + 2\xi_2 - 3\xi_3, \delta_4 = 2\xi_2 - 3\xi_3\}$$  \hfill (1.3.25)

with the Gram matrix $A_{2,II}$ and the lattice Weyl vector

$$\rho_{2,II} = (1/2)\xi_1 + (1/2)\xi_2 - (1/2)\xi_3.$$  \hfill (1.3.26)

The lattice $M_{2,II} = [2\xi_1, 2\xi_2, \xi_3] = [2a, b, 2c]$ is

$$M_{2,II} \cong U(8) \oplus \langle 2 \rangle.$$  \hfill (1.3.27)

We have

$$W_{2,II} = W^{(2)}(M_{2,II}), \quad \text{Sym} (P(M_{2,II})) = D_4$$

where the second group is generated by the reflections $s_{2a}$ and $s_{2a}$.

The 8-gon $M_{2,III} = B_1A_2B_2A_3B_3A_4B_4A$ has

$$P(M_{2,III}) = \{\delta_1 = \xi_2 + \xi_3, \delta_2 = 3\xi_2 - \xi_3, \delta_3 = \xi_1 + 4\xi_2 - 3\xi_3, \delta_4 = 3\xi_1 + 4\xi_2 - 5\xi_3,$$

$$\delta_5 = 4\xi_1 + 3\xi_2 - 5\xi_3, \delta_6 = 4\xi_1 + \xi_2 - 3\xi_3, \delta_7 = 3\xi_1 - 3\xi_3, \delta_8 = \xi_1 + \xi_3\}$$  \hfill (1.3.29)

with the Gram matrix $A_{2,III}$ and the lattice Weyl vector

$$\rho_{2,III} = (1/4)\xi_1 + (1/4)\xi_2 - (1/4)\xi_3.$$  \hfill (1.3.29)
The lattice $M_{2,III} = [a, 2b + c, 2c]$ is

$$M_{2,III} \cong \langle -32 \rangle \oplus \langle 2 \rangle \oplus \langle 2 \rangle. \quad (1.3.31)$$

We have

$$W_{2,III} = W^{(2)}(M_{2,III}), \quad \text{Sym} \ (P(M_{2,III})) = D_4 \quad (1.3.32)$$

where the second group is generated by the reflections $s_a$ and $s_{2c}$.

All lattices $M_{2,0}$, $M_{2,1}$, $M_{2,II}$, $M_{2,III}$ are characterized by their discriminant quadratic forms.

1.3.3. Generalized Cartan matrices $A_{3,j}$.

On Figure 3 $ABC$ is a triangle with angles 0, $\pi/6$ and $\pi/2$ respectively. Considering the orbit of $ABC$ by the group generated by the reflection in $BC$, we get a triangle $ABA_2$ with the angles 0, $\pi/3$ and 0 respectively. Considering the orbit of the triangle $ABC$ by the group (of type $D_2$) generated by reflections in $BC$ and $AC$, we get a quadrangle $ABA_2B_1$ with the angles 0, $\pi/3$, 0 and $\pi/3$ respectively. Considering the orbit of the triangle $ABC$ by the group (of type $D_6$) generated by reflections in $BC$ and $BA$, we get a right 6-gon $AA_2A_3A_4A_5A_6$ (here $AA_2$ contains $C$) with zero angles and the center $B$. Considering images of the center $B$ by reflections in sides of the 6-gon, we get points $B_1, B_2, B_3, B_4, B_5, B_6$ which together with vertices of the 6-gon define an 12-gon $AB_1A_2B_2A_3B_3A_4B_4A_5B_5A_6B_6$ with the angles 0, $\pi/3$, 0, $\pi/3$, 0, $\pi/3$, 0, $\pi/3$, 0, $\pi/3$ respectively. This 12-gon also is the orbit of the triangle $ABB_1$ with the angles 0, $\pi/6$, $\pi/6$ respectively by the group (of type $D_6$) generated by reflections in his sides $BA$ and $BB_1$.

The matrix $A_{3,0}$ is the Gram matrix of the triangle $\mathcal{M}_{3,0} = ABA$. The matrix $A_{3,1}$ is the Gram matrix of the quadrangle $\mathcal{M}_{3,1} = ABA_2B_1$. The matrix $A_{3,II}$ is the Gram matrix of the right 6-gon $\mathcal{M}_{3,II} = AA_2A_3A_4A_5A_6$, and $A_{3,III}$ is the Gram matrix of the 12-gon $\mathcal{M}_{3,III} = AB_1A_2B_2A_3B_3A_4B_4A_5B_5A_6B_6$.

We consider a hyperbolic lattice $M_{3,0} \cong U(3) \oplus \langle 2 \rangle$ with the standard bases $\xi_1$, $\xi_2$, $\xi_3$ having the Gram matrix

$$\begin{pmatrix}
0 & -3 & 0 \\
-3 & 0 & 0 \\
0 & 0 & 2
\end{pmatrix}.$$

We identify the hyperbolic plane of the lattice $M_{3,0}$ and of Figure 3: Points $A_2 = \mathbb{R}_{++} \xi_1$, $A = \mathbb{R}_{++} \xi_2$, elements $a = -\xi_1 + \xi_2$, $b = \xi_3$ and $c = \xi_1 - \xi_3$ are orthogonal to sides $BC$, $CA$ and $AB$ respectively of the triangle $ABC$ and directed outward. Here $(a, a) = 6$ and $(b, b) = (c, c) = 2$.

For $\mathcal{M}_{3,0} = AA_2B$, we have

$$P(\mathcal{M}_{3,0}) = \{\delta_1 = \xi_3, \delta_2 = \xi_2 - \xi_3, \delta_3 = \xi_1 - \xi_3\} \quad (1.3.33)$$

with the Gram matrix $A_{3,0}$ and the lattice Weyl vector

$$\rho_{3,0} = (2/3)\xi_1 + (2/3)\xi_2 - (1/2)\xi_3. \quad (1.3.34)$$

The lattice $M_{3,0}$, generated by $P(\mathcal{M}_{3,0})$, is then $M_{3,0} = [a, b, c]$, and

$$M_{3,0} \cong U(2) \oplus \langle 2 \rangle. \quad (1.3.35)$$
We have
\[ W_{3,0} = W^{(2)}(M_{3,0}), \quad \text{Sym} (P(M_{3,0})) = D_1 \] (1.3.36)

where the second group is generated by the reflection \( s_a \).

The quadrangle \( M_{3,l} = B_1 ABA_2 \) has
\[ P(M_{3,l}) = \{ \delta_1 = \xi_1 + \xi_3, \ \delta_2 = \xi_1 - \xi_3, \ \delta_3 = \xi_2 - \xi_3, \ \delta_4 = \xi_2 + \xi_3 \} \] (1.3.37)
with the Gram matrix \( A_{3,l} \) and the lattice Weyl vector
\[ \rho_{3,l} = (1/3)\xi_1 + (1/3)\xi_2. \] (1.3.38)

The lattice \( M_{3,l} = [a, 2b, c] \), and it has the discriminant quadratic form
\[ q_{M_{3,l}} \sim \sigma^{(3)}(2) \oplus \sigma^{(3)}(2) \oplus \sigma^{(2)}(8). \] (1.3.39)
We have

\[ W_{3,I} = W^{(2)}(M_{3,I}), \quad \text{Sym} \left( M_{3,I} \right) = D_2 \]  

(1.3.40)

where the second group is generated by the reflections \( s_a \) and \( s_{2b} \).

The right 6-gon \( M_{3,II} = A A_2 A_3 A_4 A_5 A_6 \) has

\[
P(M_{3,II}) = \{ \delta_1 = \xi_2, \delta_2 = 2\xi_2 - \xi_3, \delta_3 = 2\xi_1 + 4\xi_2 - 5\xi_3, \\
\delta_4 = 4\xi_1 + 4\xi_2 - 7\xi_3, \delta_5 = 4\xi_1 + 2\xi_2 - 5\xi_3, \delta_6 = 2\xi_1 - \xi_3 \}
\]

(1.3.41)

with the Gram matrix \( A_{3,II} \) and the lattice Weyl vector

\[
\rho_{3,II} = (1/3)\xi_1 + (1/3)\xi_2 - (1/2)\xi_3.
\]

(1.3.42)

The lattice \( M_{3,II} = [2\xi_1, 2\xi_2, \xi_3] = [2a, b, 2c] \) is

\[ M_{3,II} \cong U(12) \oplus \langle 2 \rangle. \]  

(1.3.43)

We have

\[ W_{3,II} = W^{(2)}(M_{3,II}), \quad \text{Sym} \left( P(M_{3,II}) \right) = D_6 \]  

(1.3.44)

where the second group is generated by the reflections \( s_{2s} \) and \( s_{2a} \).

The 12-gon \( M_{3,III} = B_1 A_2 B_2 A_3 B_3 A_4 B_4 A_5 B_5 A_6 B_6 A \) has

\[
P(M_{3,III}) = \{ \delta_1 = \xi_2 + \xi_3, \delta_2 = 3\xi_2 - \xi_3, \delta_3 = \xi_1 + 5\xi_2 - 4\xi_3, \\
\delta_4 = 3\xi_1 + 7\xi_2 - 8\xi_3, \delta_5 = 5\xi_1 + 8\xi_2 - 11\xi_3, \delta_6 = 7\xi_1 + 8\xi_2 - 13\xi_3, \\
\delta_7 = 8\xi_1 + 7\xi_2 - 13\xi_3, \delta_8 = 8\xi_1 + 5\xi_2 - 11\xi_3, \delta_9 = 7\xi_1 + 3\xi_3 - 8\xi_3, \\
\delta_{10} = 5\xi_1 + \xi_2 - 4\xi_3, \delta_{11} = 3\xi_1 - \xi_3, \delta_{12} = \xi_1 + \xi_3 \}
\]

(1.3.45)

with the Gram matrix \( A_{3,III} \) and the lattice Weyl vector

\[
\rho_{3,III} = (1/6)\xi_1 + (1/6)\xi_2 - (1/4)\xi_3.
\]

(1.3.46)

The lattice \( M_{3,III} = M_{3,I} = [a, 2b, c] \) has the discriminant quadratic form

\[ q_{M_{3,III}} = q_1^{(3)}(3) \oplus q_-^{(3)}(3) \oplus q_5^{(2)}(8). \]  

(1.3.47)

We have

\[ W_{3,III} \triangleleft W^{(2)}(M_{3,III}) \text{ and } W^{(2)}(M_{3,III})/W_{3,III} = D_3 \]  

(1.3.48)

where the second group is generated by the reflections \( s_{\xi_2-\xi_3} \) (in \( BA_2 \)) and \( s_c \) (in \( BA \)). Moreover, \( W_{3,III} \) is normal in a bigger subgroup

\[ W_{3,III} \triangleleft W^{(2,6)}(M_{3,III}) \text{ and } W^{(2,6)}(M_{3,III})/W_{3,III} = D_6 \]  

(1.3.49)

where the group \( W^{(2,6)}(M_{3,III}) \) is generated by the reflections \( s_a, s_c \) and \( s_{\xi_1+\xi_3} \) in all sides of the triangle \( B_1 BA \). The group

\[ A(P(M_{3,III})) = D_3. \]  

(1.3.50)
is generated by the reflections \( s_\alpha \) and \( s_\gamma \).

All lattices \( M_{3,0}, M_{3,1} = M_{3,II} \) and \( M_{3,II} \) are characterized by their discriminant quadratic forms.

2. Reflective automorphic Forms and reflective Lattices, and Arithmetic Mirror Symmetry Conjecture. Lie reflective Automorphic Forms. General Theory

In this Section we want to generalize some results of our papers [N9], [N10], [GN1] — [GN4] and [N11]. The results of this section were subjects of the lectures given by us at the meeting of Moscow Mathematical Society and Steklov Mathematical Institute (April 1996), European Algebraic Geometry meeting at Warwick (July 1996), RIMS at Kyoto and Nagoya University (September 1996).

The main subject of this section is to define proper analogs of elliptic and parabolic reflection groups and corresponding root systems (related with lattices of signature \((n,1)\)) for lattices with 2 negative squares. These two notions are related by arithmetic mirror symmetry conjecture. This section also contains our general point of view to the theory of automorphic forms related with Mirror Symmetry and the theory of automorphic Lorentzian Kac–Moody algebras.

2.1. Reflective hyperbolic lattices and reflective lattices with 2 negative squares. Reflective automorphic forms on IV type domains.

Let \( K \) be a lattice with form \((\cdot,\cdot)\). An element \( \delta \in K \) is called a \textit{primitive root} if \( \delta \) is primitive in \( K \), \((\delta,\delta) > 0 \) and the reflection \( s_\delta : x \mapsto x - (2(x,\delta)/(\delta,\delta))\delta \) in \( \delta \) belongs to the automorphism group \( O(K) \) of the lattice \( K \). Equivalently, \((\delta,\delta)|2(K,\delta)\). Remark that any element \( \delta \in K \) with square \((\delta,\delta) = 1 \) or 2 is a primitive root. It is called 1 or 2 root respectively. The squares \((\delta,\delta)\) of primitive roots are bounded: \((\delta,\delta) \leq N\) where \( N \) is some constant depending on the lattice \( K \). We denote by \( \Delta(K)_\text{pr} \) the set of all primitive roots of a lattice \( K \). A subset \( \Delta \subset \Delta(K)_\text{pr} \) is called a \textit{primitive root system} if \( \Delta \) is invariant with respect to the reflection group \( W(\Delta) \) generated by all reflections in \( \Delta \). If \( \Delta \subset K \) is a root system and \( K' \subset K \) is a sublattice, then \( \Delta \cap K' \) is a root system in \( K' \).

First, we recall some definitions and results about reflective hyperbolic lattices and corresponding reflection groups and root systems (see [N10], [N4], [N5], [V2], [V3]). If \( M \) is a hyperbolic lattice (i.e. \( M \) has signature \((n,1)\)), a primitive root system \( \Delta \subset \Delta(M)_\text{pr} \) of \( M \) is equivalent to a reflection group \( W(\Delta) \subset O^+(M) \) generated by reflections in \( \Delta \). An element \( w \in W(\Delta) \) is a reflection with respect to some primitive element \( \delta \in M \) if and only if \( \delta \in \Delta \). A reflection group \( W \) of a hyperbolic lattice \( M \) is called \textit{elliptic} if index \([O(M) : W]\) is finite. This is equivalent to finiteness of volume of the fundamental polyhedron \( \mathcal{M} \) of \( W \) or to the restricted arithmetic type of \( W \) together with existence of a generalized lattice Weyl vector \( \rho \) with square \((\rho,\rho) < 0 \) for the set \( P(M) \) of orthogonal vectors to \( \mathcal{M} \) (see Definition 1.1.3). A reflection group \( W \) of a hyperbolic lattice \( M \) is called \textit{parabolic} if \( W \) has restricted arithmetic type and the set \( P(M)_\text{pr} \) of primitive orthogonal vectors to \( \mathcal{M} \) has a generalized lattice Weyl vector \( \rho \) with zero square \((\rho,\rho) = 0 \) and does not have a generalized lattice Weyl vector with negative square. Using this, we say that a primitive root system \( \Delta \subset \Delta(M)_\text{pr} \) is \textit{elliptic} or \textit{parabolic} if the corresponding reflection group \( W(\Delta) \) is elliptic or parabolic respectively. A lattice \( M \) is called \textit{reflective} if \( O(M) \) has at least one elliptic or parabolic reflection group \( W \subset O^+(M) \) or the corresponding elliptic or parabolic primitive root system. If the
group $W$ is elliptic or parabolic, one can also call $M$ elliptic or parabolic reflective respectively. The main result we need is (see [N10], [N4], [N5] and [V2])

**Theorem 2.1.1.** For $rk M \geq 3$ the number of reflective hyperbolic lattices (elliptic and parabolic) is finite up to multiplication of the form of a lattice on $n \in \mathbb{Q}$. In particular, the number of maximal elliptic and parabolic primitive root systems $\Delta$ and corresponding elliptic and parabolic reflection groups $W$ is finite.

We say that a lattice $T$ has $2$ negative squares if $T$ has signature $(n, 2)$. We want to suggest a definition of reflective lattices with $2$ negative squares and formulate a conjectured analog of Theorem 2.1.1.

Let $T$ be a lattice with $2$ negative squares. We denote by $\Omega^+(T)$ a Hermitian symmetric domain of type IV which is one of the two connected components of

$$\Omega(T) = \{Z \in \mathbb{P}(T \otimes \mathbb{C}) \mid (Z, Z) = 0, (Z, \overline{Z}) < 0\}.$$  

We consider the corresponding homogeneous cones (without zeros) $\tilde{\Omega}(T) \subset T \otimes \mathbb{C}$ and $\tilde{\Omega}^+(T) \subset T \otimes \mathbb{C}$ such that $\Omega(T) = \tilde{\Omega}(T)/\mathbb{C}^*$ and $\Omega^+(T) = \tilde{\Omega}^+(T)/\mathbb{C}^*$. Let $O^+(T)$ be the subgroup of index two of $O(T)$ which keeps the component $\Omega^+(T)$. It is well known that $O^+(T)$ is discrete in $\Omega^+(T)$ and has a fundamental domain of finite volume.

A function $\Phi$ on $\tilde{\Omega}^+(T)$ is called an automorphic form of weight $k$ ($k \in \mathbb{Z}/2$) if $\Phi$ is holomorphic on $\tilde{\Omega}^+(T)$, $\Phi(c\omega) = c^{-k}\Phi(\omega)$ for any $c \in \mathbb{C}^*$, $\omega \in \tilde{\Omega}^+(T)$, and $\Phi(\gamma(\omega)) = \chi(\gamma)\Phi(\omega)$ for any $\gamma \in G, \omega \in \tilde{\Omega}^+(T)$. Here $G \subset O(T)^+$ is a subgroup of finite index and $\chi : G \rightarrow \mathbb{C}^*$ is some character or a multiplier system with the kernel of finite index in $G$. Then $\Phi$ is called automorphic with respect to $G$ with $\chi$. The function $\Phi$ additionally should be holomorphic at infinity $\Omega(T)_\infty$ of $\Omega(T)$. If $\text{codim}_{\Omega(T)}\Omega(T)_\infty \geq 2$, this condition is automatically valid according to the Koecher principle. Since $\dim \Omega(T)_\infty \leq 1$, this is in particular true if $rk T \geq 5$.

For $e \in T$ with $(e, e) > 0$ we denote $\mathcal{H}_e = \{Z \in \Omega^+(T) \mid (Z, e) = 0\}$. The $\mathcal{H}_e$ is called a quadratic divisor orthogonal to $e$. The quadratic divisor $\mathcal{H}_e$ does not change if one changes $e$ to $te$, $t \in \mathbb{Q}$. The union $D(T)$ of all quadratic divisors orthogonal to the primitive roots $\Delta(T)_{pr}$ is called discriminant of $T$.

**Definition 2.1.2.** Let $T$ be a lattice with $2$ negative squares. An automorphic form $\Phi$ on $\Omega(T)$ is called reflective for $T$ if the set of zeros $(\Phi)_0$ of $\Phi$ is the union of quadratic divisors orthogonal to primitive roots $\delta \in \Delta(T)$ (with some multiplicities). The lattice $T$ is called reflective if there exists at least one reflective automorphic form for $T$.

We denote by $\Delta(\Phi)$ the set of all primitive roots $\delta \in \Delta(T)$ such that $\Phi$ is equal to zero on $\mathcal{H}_\delta$. Evidently, if $\delta \in \Delta(\Phi)$, then $-\delta \in \Delta(\Phi)$. An automorphic form $\Phi$ is called strongly reflective if $\Delta(\Phi)$ is a root system and $\Phi$ is automorphic with respect to the group $W(\Delta(\Phi))$ generated by reflections in all elements of $\Delta(\Phi)$.

The union $D(\Phi)$ of all quadratic divisors orthogonal to $\delta \in \Delta(\Phi)$ is called discriminant of $\Phi$. It is why reflective automorphic forms are also called discriminant automorphic forms. They are related with discriminants of K3 surfaces moduli (see [GN3], [N11]).

Below we will also use notation $D(\Delta)$ for the union of all quadratic divisors orthogonal to elements of some subset $\Delta \subset \Delta(T)_\text{pr}$ of primitive roots.
Suppose that \( \Phi \) is a reflective automorphic form for \( T \). Suppose that \( \Phi \) is invariant with respect to a subgroup \( G \subset O^+(T) \) of a finite index. The product
\[
\Phi' = \prod_{g \in G \setminus O^+(T)} g^* \Phi,
\]
is also a reflective automorphic form, but \( \Delta(\Phi') \) is invariant with respect to \( O^+(T) \). In particular, \( \Delta(\Phi') \) is a root system. Thus, \( \Phi' \) is a strongly reflective automorphic form. The advantage of strongly reflective automorphic forms is that they are automorphic with some quadratic character with respect to the reflection group \( W(\Delta(\Phi)) \) generated by the reflections in \( \Delta(\Phi) \). We remark that \( W(\Delta(\Phi)) \) always has finite index in \( O(T) \) because of the general results by G.A. Margulis about normal subgroups of arithmetic groups of rank \( \geq 2 \).

It is a very interesting and difficult problem to find all reflective lattices \( T \) with \( 2 \) negative squares and all their reflective automorphic forms. Some reflective lattices \( T \) and their reflective automorphic forms were found in [B1]—[B6] and [GN1]–[GN4]. Many other will be found in Part II of this paper.

2.2. Arithmetic Mirror Symmetry Conjectures.

Conjecture below was first formulated in [N11] for 2-roots and corresponding 2-reflective automorphic forms. We consider this conjecture as a mirror symmetric statement to Theorem 2.1.1.

**Conjecture 2.2.1.** The number of reflective lattices \( T \) with 2 negative squares and \( \text{rk} \ T \geq 5 \) is finite up to multiplication of the form of the lattice \( T \). In particular, for \( \text{rk} \ T \geq 5 \) the number of maximal systems of primitive roots \( \Delta(\Phi) \) or reflective automorphic forms \( \Phi \) is finite up to multiplication of the form of \( T \).

The main reason to have so strong statement is the Koecher principle for automorphic forms (see [Ba], for example).

Let \( T \) be a reflective lattice with 2 negative squares and \( \Phi \) be a reflective automorphic form for \( T \). Let \( \Delta(\Phi) \) be the set of primitive roots of \( \Phi \). Let \( T_1 \subset T \) be a primitive sublattice of \( T \) with 2 negative squares. Then \( \Omega(T_1) \subset \Omega(T) \) is a symmetric subdomain of type IV of \( \Omega(T) \) and restriction \( \Phi|_{\Omega(T_1)} \) is an automorphic form for the lattice \( T_1 \). By Koecher principle, \( \Phi|_{\Omega(T_1)} \) should have a non-empty set of zeros if \( \text{codim}_{\Omega(T_1)} \Omega(T_1)_{\infty} \geq 2 \). Thus
\[
D(\Delta(\Phi)) \cap \Omega(T_1) \neq \emptyset \text{ if } \text{codim}_{\Omega(T_1)} \Omega(T_1)_{\infty} \geq 2.
\]
We get the following condition:

**Necessary Condition 2.2.2.** Let \( T \) be a lattice with 2 negative squares and \( \Delta \subset \Delta(T)_{pr} \) a set of primitive roots. If there exists a reflective automorphic form \( \Phi \) with the set of roots \( \Delta(\Phi) \subset \Delta \), then for any primitive sublattice \( T_1 \subset T \) with 2 negative squares and \( \text{codim}_{\Omega(T_1)} \Omega(T_1)_{\infty} \geq 2 \), one has
\[
D(\Delta) \cap \Omega(T_1) \neq \emptyset.
\]
In particular, for \( \Delta = \Delta(T)_{pr} \) we get: If \( T \) is a reflective lattice, then for any primitive sublattice \( T_1 \subset T \) with 2 negative squares and \( \text{codim}_{\Omega(T_1)} \Omega(T_1)_{\infty} \geq 2 \), the set \( D(\Delta(T)_{pr}) \cap \Omega(T_1) \) is not empty.

It was shown in [N11] that Necessary Condition 2.2.2 for a lattice \( T \) to be reflective is extremely strong. Actually, we know only one way to prove that a lattice \( T \) is reflective: this way is the so-called lattice method.
of \( \text{rk} \ T \geq 5 \) satisfies this condition: to construct a reflective automorphic form for \( T \). It is why we suggest more strong than Conjecture 2.2.1

**Conjecture 2.2.3.** The set of lattices \( T \) with 2 negative squares and with \( \text{rk} \ T \geq 5 \) which satisfy Necessary Condition 2.2.2 is finite up to multiplication of the form of \( T \) on \( n \in \mathbb{Q} \). In particular, the set of maximal systems of primitive roots \( \Delta \) which satisfy Necessary Condition 2.2.2 is finite up to multiplication of the form on \( n \in \mathbb{Q} \).

Now we have two types of reflective lattices: hyperbolic reflective lattices and reflective lattices with 2 negative squares. We think that these two types of lattices are related by Arithmetic Mirror Symmetry Conjecture which we formulate below. It is inspired by Mirror Symmetry (at least, for K3 surfaces). We consider this conjecture as much more strong variant of our Mirror Symmetry Conjectures in [GN3] and [N11]. But the main ingredient of our Conjecture in [GN3] is preserved. We believe that the right variant of mirror symmetry for K3 surfaces is connected with the very global objects like Automorphic Forms on a IV type domains. Finiteness Conjectures 2.2.1 and 2.2.3. together with Theorem 2.1.1 show that there actually exist very few cases when this kind of theory takes place. We believe that these finiteness results and Conjectures are related with conjectured finiteness of families of Calabi–Yau 3-folds and \( n \)-folds (one can try to find some connection in [N12], [HM1], [HM2], [Ka1], [Ka2], [DVV]).

**Arithmetic Mirror Symmetry Conjecture 2.2.4.**

(i) Let \( T \) be a reflective lattice with 2 negative squares and \( \Phi \) be a reflective automorphic form for \( T \) with a primitive root system \( \Delta(\Phi) \). Let \( c \in T \) be a primitive isotropic element of \( T \) and \( S = c_T^\perp / [c] \) be the corresponding hyperbolic lattice with the root system

\[
\Delta(\Phi)|_S = \Delta(\Phi) \cap c_T^\perp \quad \text{mod } [c].
\]

Then the root system \( \Delta(\Phi)|_S \subset S \) is elliptic or parabolic (in particular, the lattice \( S \) is reflective) if the set of roots \( \Delta(\Phi)|_S \) is non-empty.

(ii) Any hyperbolic reflective lattice \( S \) with an elliptic or parabolic primitive root system \( \Delta \) may be obtained from some lattice \( T \) with 2 negative squares and a reflective automorphic form \( \Phi \) of \( T \) by the construction (i) above.

We have two arguments to support Conjecture 2.2.4. First we will show in Theorem 2.2.5 below that if \( T \) together with a primitive root system \( \Delta \) satisfies the condition (i) of Conjecture 2.2.4, then \( T \) with \( \Delta \) satisfies Necessary Condition 2.2.2 for “the most part” of primitive sublattices \( T_1 \subset T \). We have mentioned that this condition is extremely strong which had been shown in [N11].

**Theorem 2.2.5.** Let \( T \) be a lattice with 2 negative squares and \( \Delta \) be a primitive root system of \( T \). Suppose that for any primitive isotropic \( c \in T \) and \( S = c_T^\perp / [c] \) the root system \( \Delta|_S \subset S \) is elliptic or parabolic if it is not empty (i.e. the condition (i) of Conjecture 2.2.4 is valid for \( \Delta \).) Then \( T \) satisfies Necessary Condition 2.2.2 for all primitive sublattices \( T_1 \subset T \) such that \( T_1 \) has a primitive isotropic element \( e \) with property \( e_T^\perp \cap \Delta \neq \emptyset \).

**Proof.** Let \( S = e_T^\perp / [e] \) and let \( S_1 = e_T^\perp / [e] \) be its hyperbolic sublattice. It is sufficient to show that the root system \( \Delta|_{S_1} \) has the following property which is a...
hyperbolic analog of Necessary Condition 2.2.2:

\[
( \bigcup_{\delta \in \Delta_s} \mathcal{H}_\delta \bigcap \mathcal{L}(S_1) \neq \emptyset. \tag{2.2.1}\]

Here \( \mathcal{L}(S_1) \subset \mathcal{L}(S) \) are the hyperbolic spaces corresponding to \( S_1 \subset S \), and \( \mathcal{H}_\delta \) is the hyperplane in \( \mathcal{L}(S) \) orthogonal to \( \delta \). It is true that \( \text{rk } S_1 \geq 2 \) and \( S_1 \) does not have isotropic elements if \( \text{rk } S_1 = 2 \) (it follows from the condition \( \text{codim}_{\Omega(T_1)}\Omega(T_1) \geq 2 \) where we consider an empty set as having dimension \((-1))

It follows that the subspace \( \mathcal{L}(S_1) \) has a non-empty finite (not at infinity) intersection with a face of one of fundamental polyhedron \( \mathcal{M} \) for \( W(\Delta_s) \) acting in \( \mathcal{L}(S) \). This is obvious if \( \mathcal{M} \) has finite volume (elliptic case). One should draw a picture to see that the same is true for a parabolic case when the polyhedron \( \mathcal{M} \) is parabolic with respect to a cusp in \( S \). This finishes the proof.

Another support of Conjecture 2.2.4 is that if a reflective automorphic form \( \Phi \) has a Fourier expansion with a generalized lattice Weyl vector (one can consider this as a special type of Fourier expansion), then the statement (i) of Conjecture 2.2.4 is valid for this cusp (see Sect. 2.4). Infinite products “a la Borcherds” (see R. Borcherds, [B5]) give a particular case of Fourier expansions with a generalized lattice Weyl vector and are related with automorphic forms with rational quadratic divisors (see [B5]). It is possible that reflective automorphic forms \( \Phi \) always have these infinite product expansions at cusps because of their special sets of zeros. One can also believe that this is true because of finiteness Conjectures 2.2.1 and 2.2.3. It seems, all known reflective automorphic forms \( \Phi \) have these infinite product expansions at cusps.

2.3. Fourier expansion of an automorphic form \( \Phi \) on the domain \( \Omega(T) \).

Let \( T \) be a lattice with 2 negative squares. Here we analyze Fourier expansion of an automorphic form \( \Phi \) on \( \Omega^+(T) \) at a 0-dimensional cusp in terms of the lattice \( T \). We describe canonical linear complex coordinates at the cusp and the group of translations which determines the Fourier expansion.

A cusp (of dimension 0) of \( \Omega^+(T) \) is defined by an isotropic sublattice of \( T \) of rank one, i.e. by a primitive isotropic element \( c \in T \) which is defined by the cusp canonically up to replacing \( c \) to \( -c \). For the given isotropic primitive \( c \in T \) we put

\[
S_c = c^\perp / c := c_T^\perp / \mathbb{Z}c
\]

which is a hyperbolic lattice canonically defined by the cusp. Let \( O^+(T)_c = \{ g \in O^+(T) \mid g(c) = c \} \) be the corresponding to \( c \) parabolic subgroup and let

\[
O^+(T)_{c^+ / c} = \{ \phi \in O^+(T)_c \mid \phi \text{ is identical on } c^+ / c \}
\]

be its unipotent subgroup which is a subgroup of finite index of the additive group \( S^*_c \subset S \otimes \mathbb{Q} \). The corresponding embedding \( f \) is defined as follows: for \( \phi \in O^+(T)_{c^+ / c} \) the function \( f(\phi) \in S^*_c \) is determined by the rule:

\[
\phi(x) = x + f(\phi)(x)c, \quad x \in c^+.
\]

One can prove that \( f(\phi) \) can be extended to a function on \( T \) from the module

\[
T^*_c = \{ x \in T^*_c \mid \langle x, c \rangle = 0 \}
\]
It means that in fact
\[ O^+(T)_{c^\perp/c} \subset \tilde{T}^*_{c,0} \subset S^*_c \]  
(2.3.1)
where \( T^*_{c,0} \rightarrow \tilde{T}^*_{c,0} \subset S^*_c \) is the natural homomorphism.

Let us define canonical coordinates at the cusp \( c \). For an arbitrary element \( C \omega \in \Omega(T) \) one can choose canonically its representative \( \omega \in C \omega \) by the condition \( (\omega, c) = -1 \). This choice identifies \( \Omega(T) \) with a domain of the affine quadric
\[ \Omega(T)_c = \{ \omega \in T \otimes \mathbb{C} \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) < 0, (\omega, c) = -1 \} \cong \Omega(T) \]
and \( \Omega^+(T) \) with a connected component \( \Omega^+(T)_c \) of \( \Omega(T)_c \). We consider a cone
\[ V(S_c) = \{ x \in S_c \otimes \mathbb{R} \mid (x, x) < 0 \} \]
of the hyperbolic lattice \( S_c \), its half-cone \( V^+(S_c) \) and the corresponding complexified cone \( \Omega(V^+(S_c)) = S_c \otimes \mathbb{R} + iV^+(S_c) \) which is a complex tube domain of type IV.

An element
\[ h \in T^*_{c,-1} = \{ x \in T^* \mid (x, c) = -1 \} \]
defines an orthogonal decomposition \( T \otimes \mathbb{Q} = S_c \otimes \mathbb{Q} \oplus (\mathbb{Q}c + \mathbb{Q}h) \) of \( T \) over \( \mathbb{Q} \) and an isomorphism (after replacing the cone \( V^+(S_c) \) by the opposite half-cone \( -V^+(S_c) \) if necessary)
\[ \Omega(V^+(S_c)) \rightarrow \Omega^+(T)_c \cong \Omega^+(T) \text{ by } z \mapsto z + \frac{(z, z) + (h, h)}{2} c + h \in \Omega^+(T)_c. \]
The coordinate \( z \in \Omega(V^+(S_c)) \) depends on the choice of \( h \in T^*_{c,-1} \). If one replaces \( h \) with another \( h' \in T^*_{c,-1} \), then \( h' - h = r \in T^*_{c,0} \). The new coordinate \( z' = h' \in \Omega(V^+(S_c)) \) is then \( z' = z - \tilde{r} \) where \( \tilde{r} \) is the natural image of \( r \) in \( S^*_c \) (see (2.3.1)).

Using the coordinate \( z \in \Omega(V^+(S_c)) \) defined by the choice of \( h \in T^*_{c,-1} \), we can identify an arbitrary automorphic form \( \Phi \) of weight \( k \) on the domain \( \Omega^+(T) \) with an automorphic form \( \Phi_{c,h} \) on the domain \( \Omega(V^+(S_c)) \). They are related by the formula:
\[ \Phi(\lambda(z + \frac{(z, z) + (h, h)}{2} c + h)) = \lambda^{-k} \Phi_{c,h}(z), \quad z \in \Omega(V^+(S_c)), \ \lambda \in \mathbb{C}^*. \]
In other words, after the \( h \)-identification of the domains \( \Omega(V^+(S_c)) = \Omega^+(T)_c \), we get \( \Phi_{c,h} = \Phi_{|\Omega^+(T)_c} \). For the coordinates defined by \( h, h' \in T^*_{c,-1} \) with \( r = h' - h \in T^*_{c,0} \), one has \( \Phi_{c,h'}(z') = \Phi_{c,h}(z) \) if \( z' = z - \tilde{r} \). Thus,
\[ \Phi_{c,h'}(z) = \Phi_{c,h}(z + \tilde{r}) \]
(2.3.2)
for any \( z \in \Omega(V^+(S_c)), \) any \( h, h' \in T^*_{c,-1} \) with \( r = h' - h \) and \( \tilde{r} \in T^*_{c,0} \subset S^*_c \).

Let us assume that \( \Phi \) is an automorphic form with respect to a group \( G \subset O^+(T) \) of finite index with a multiplier system \( \chi \). The unipotent subgroup
\[ G_{c^\perp/c} = \{ g \in O^+(T)_{c^\perp/c} \cap G \mid g^* \Phi = \Phi \} \]
has a finite index in the lattice \( S^*_c \). The invariance of \( \Phi \) with respect to the subgroup \( G_{c^\perp/c} \) is equivalent to the periodicity of \( \Phi_{c,h} \)
\[ \Phi_{c,h}(z + c) = \Phi_{c,h}(z), \quad z \in \Omega(V^+(S_c)), c \in G_{c^\perp/c} \subset \tilde{T}^*_{c,0} \subset S^*_c. \]
Thus, one can consider the Fourier expansion of $\Phi_{c,h}$ by characters of $(G_{c^\perp/c})^*$. We may consider $(G_{c^\perp/c})^*$ as a subset of $S_c^* \otimes \mathbb{Q} = S_c \otimes \mathbb{Q}$. Then $S_c \subset (G_{c^\perp/c})^*$. As a result we get

$$\Phi_{c,h}(z) = \sum_{a \in (G_{c^\perp/c})^*} m(a) \exp(-2\pi i (a, z)). \quad (2.3.3)$$

This is called the Fourier expansion of the automorphic form $\Phi$ at the cusp $c$. In general this Fourier expansion depends on the choice of $h \in T_{c,-1}^*$ and according to (2.3.2) there is in fact a homogeneous space over the finite Abelian group

$$\tilde{T}_{c,0}^*/(G_{c^\perp/c})$$

of Fourier expansions of $\Phi$ for different canonical coordinates. All such Fourier expansions are related according to (2.3.2). For another $h' \in T_{c,-1}^*$, we get

$$\Phi_{c,h'}(z) = \sum_{a \in (G_{c^\perp/c})^*} \exp(-2\pi i (a, \tilde{r})) m(a) \exp(-2\pi i (a, z)). \quad (2.3.5)$$

where $r = h' - h$, $\tilde{r} \in \tilde{T}_{c,0}^*$ and $\exp(-2\pi i (a, \tilde{r}))$ is an element from the group of unities of the order equals to the exponent of the finite Abelian group $\tilde{T}_{c,0}^*/(G_{c^\perp/c})$. Below we will be especially interesting in Fourier expansions with integral Fourier coefficients $m(a)$. Using (2.3.5), one can find all Fourier expansions (in the different $h$-coordinates) of the automorphic form $\Phi$ with integral Fourier coefficients from one of them.

Further, when we speak about some properties of Fourier expansion of an automorphic form $\Phi$ at a cusp $c$, we mean existence of one of the Fourier expansions of $\Phi$ at the cusp $c$ with this property. Below we omit the index $h \in T_{c,-1}$ in Fourier expansion of an automorphic form $\Phi$.

2.4. Fourier expansions of reflective automorphic forms and Arithmetic Mirror Symmetry Conjecture.

Let $T$ be a lattice with 2 negative squares and $\Phi$ be an automorphic form on $\Omega^+(T)$ possibly with some character with respect to a subgroup $G \subset O^+(T)$ of finite index. Let $c \in T$ be a primitive isotropic element and $S = c^\perp/[c]$ the corresponding hyperbolic lattice. Then the automorphic form $\Phi(z)$, $z \in \Omega(V^+(S))$, is invariant with some character $\epsilon$ with respect to a subgroup $H \subset O^+(S)$ of finite index. Let us suppose that $H$ is a semi-direct product

$$H = W \rtimes A$$

where $W$ is a reflection group with a fundamental polyhedron $\mathcal{M}$ in $L^+(S)$ and $A$ is a group of symmetries of $\mathcal{M}$. Then $\Phi(z)$ has a Fourier expansion

$$\Phi(z) = \sum_{w \in W} \epsilon(w) \sum_{a \in \mathbb{R}^+_+ \mathcal{M}} m(a) \exp(-2\pi i (w(a), z)). \quad (2.4.1)$$

Here $a$ runs through some lattice $M$ in $S \otimes \mathbb{Q}$, and we call elements $a$ of this lattice Fourier exponents. The corresponding $m(a) \in \mathbb{C}$ is called Fourier coefficient.
Definition 2.4.1. We say that a Fourier exponent $\rho \in \mathbb{R}^+\mathcal{M}$ is a generalized lattice Weyl vector for $\Phi$ (with respect to $W$ and $\mathcal{M}$) if there exists a partial ordering $\leq$ of Fourier exponents $a \in \mathbb{R}^+\mathcal{M}$ such that this ordering is $A$-invariant and $\rho$ is the unique minimal element for this ordering.

We remark that $(\rho, \rho) \leq 0$ since $\mathbb{R}^+\mathcal{M} \subset \mathcal{V}^+(S)$.

Proposition 2.4.2. If there exists a generalized lattice Weyl vector $\rho$ for $\Phi$, then the reflection group $W$ is elliptic if $(\rho, \rho) < 0$, and $W$ is elliptic or parabolic if $(\rho, \rho) = 0$. In particular, the lattice $S$ is reflective.

Proof. Obviously, $\rho$ is fixed by $A$. If $(\rho, \rho) < 0$, then $A$ is finite since $A$ is discrete in $L^+(S)$. It follows that $W$ has finite index in $O^+(S)$ because $W \cdot A$ does. Thus, $W$ is elliptic. If $(\rho, \rho) = 0$, but $A$ is finite, we get the previous case. If $A$ is infinite, then $W$ is parabolic with respect to the cusp $\rho$, by definition. It follows the statement.

As an example of a possible ordering for exponents in $\mathbb{R}^+\mathcal{M}$, we can consider a standard ordering defined by the cone $\mathcal{V}^+(S)$. Thus, $x \geq y$ if there exists $z \in \mathcal{V}^+(S)$ such that $x = y + z$. For the standard ordering, the Fourier expansion of $\Phi(z)$ with a generalized lattice Weyl vector $\rho \in \mathbb{R}^+\mathcal{M}$ has the form

$$
\Phi(z) = \sum_{w \in W} \epsilon(w) \sum_{\rho + a \in \mathbb{R}^+\mathcal{M}, a \in \mathcal{V}^+(S)} m(a) \exp(-2\pi i (w(\rho + a), z)).
$$

(2.4.2)

Let $\Delta \subset S$ be the set of all primitive roots of $W$. A root $\delta \in \Delta$ is called positive if $(\delta, \mathcal{M}) \leq 0$. Otherwise, $\delta$ is called negative. We denote by $\Delta_+$ the set of all positive roots of $\Delta$. The most general ordering related with $W$, we can introduce, is the ordering defined by the closed convex cone $\mathcal{I}$ below:

$$
x \geq y \text{ if } x - y \in \mathcal{I} = \sum_{\delta \in \Delta_+} \mathbb{R}_+ \delta + \mathcal{V}^+(S).
$$

(2.4.3)

Fourier expansion of $\Phi(z)$ with a generalized lattice Weyl vector $\rho \in \mathbb{R}^+\mathcal{M}$ corresponding to the ordering (2.4.3) is

$$
\Phi(z) = \sum_{w \in W} \epsilon(w) \sum_{\rho + a \in \mathbb{R}^+\mathcal{M}, a \in \mathcal{I}} m(a) \exp(-2\pi i (w(\rho + a), z)).
$$

(2.4.4)

A particular case of Fourier expansion of $\Phi$ of type (2.4.4) with a generalized lattice Weyl vector is given by the infinite product “a la Borcherds” (see [B5]). This is a formal infinite product expansion of the form

$$
\Phi(z) = C \exp(-2\pi i (\rho, z)) \prod_{\alpha > 0} (1 - u(\alpha) \exp(-2\pi i (\alpha, z)))^{\text{mult } \alpha}.
$$

(2.4.5)

Here $C$ is a constant, $\rho \in \mathbb{R}^+\mathcal{M}$, $\alpha > 0$ means that $\alpha \in \mathbb{N} \Delta_+ \cup (\mathcal{M} \cap \mathcal{V}^+(S))$, coefficients $u(\alpha) \in \mathbb{C}$ and multiplicities $\text{mult } \alpha \in \mathbb{Q}$. Obviously, then $\Phi(z)$ has Fourier expansion of type (2.4.4) with the generalized lattice Weyl vector $\rho \in \mathbb{R}^+\mathcal{M}$.

Thus, from considerations above and Proposition 2.4.2, we get
Proposition 2.4.3. Let an automorphic form $\Phi$ has a Fourier expansion of type (2.4.4) with a generalized lattice Weyl vector $\rho \in \mathbb{R}_{++}M$. Then the reflection group $W$ is elliptic or parabolic and the lattice $S$ is reflective. In particular, this is valid if $\Phi(z)$ has a formal infinite product expansion of type (2.4.5).

From Proposition 2.4.3, we get the statement which supports Arithmetic Mirror Symmetry Conjecture 2.2.4.

Corollary 2.4.4. Let $T$ be a lattice with 2 negative squares and $\Phi$ a reflective automorphic form with primitive roots system $\Delta(\Phi)$. Suppose that at a cusp $c$ the form $\Phi$ has a Fourier expansion of type (2.4.4) with a generalized lattice Weyl vector $\rho \in \mathbb{R}_{++}M$ for $W = W((\Delta(\Phi) \cap c^\perp) \mod [c])$ (for example, assume that $\Phi$ has a formal infinite product expansion (2.4.5)). Then Arithmetic Mirror Symmetry Conjecture 2.2.4 is valid for $\Phi$ at the cusp $c$.

It is possible that all reflective automorphic forms have infinite product expansions of type (2.4.5) at cusps because of their very special sets of zeros.

2.5. Lie reflective automorphic forms.

In notation above, let $P(M) \subset S$ be an acceptable set of orthogonal vectors to $M$. It means that $(\alpha, \alpha) \mid 2(\alpha, \alpha')$ for all $\alpha, \alpha' \in P(M)$. We choose a subset $P(M)_\mathbb{T} \subset P(M)$ which is called odd subset. Its complement $P(M)_{\mathbb{P}} = P(M) - P(M)_\mathbb{T}$ is called even subset. We additionally suppose that $(\alpha, \alpha) \mid (\alpha, \alpha')$, for all $\alpha \in P(M)_\mathbb{T}$, $\alpha' \in P(M)$.

The set $P(M)$ together with its subdivision $P(M) = P(M)_{\mathbb{P}} \bigsqcup P(M)_\mathbb{T}$ on even and odd subsets satisfying (2.5.1) and (2.5.2) is called acceptable. We remark that the main invariant of $W$, $M$ and $P(M)$ is the symmetrized generalized Cartan matrix

$$B = ((\alpha, \alpha')), \quad \alpha, \alpha' \in P(M),$$

or (if one permits to multiply the matrix $B$ by constants) a generalized Cartan matrix

$$A = \left( \frac{2(\alpha, \alpha')}{(\alpha, \alpha)} \right),$$

and the subset of odd indices $P(M)_\mathbb{T} \subset P(M)$.

Definition 2.5.1. We say that $\Phi$ has Fourier expansion at the cusp $c$ of Lie type (with the symmetrized generalized Cartan matrix $B$ or with the generalized Cartan matrix $A$) if its Fourier expansion has the form (it is a very special type of the expansion (2.4.2) which corresponds to the subcone $\mathbb{R}_{++}M \subset V^+(S)$)

$$\Phi(z) = \sum_{w \in W} \epsilon(w) \left( \exp(-2\pi i(w(\rho), z)) - \sum_{a \in M \cap \mathbb{R}_{++}M} m(a) \exp(-2\pi i(w(\rho + a), z)) \right)$$

(2.5.5)

where

(i) $M \subset S \otimes \mathbb{Q}$ is a lattice in $S \otimes \mathbb{Q}$ such that $P(M) \subset M$, $(\alpha, \alpha) \mid 2(M, \alpha)$ for any $\alpha \in P(M)$, and $(\alpha, \alpha) \mid (M, \alpha)$ for any $\alpha \in P(M)_\mathbb{T}$; the lattice $M$ is called a root lattice.
(ii) \( \epsilon(s_\alpha) = (-1)^{1+\bar{\gamma}} \) for any \( \alpha \in P(M)_\bar{\gamma} \) and any \( \bar{\gamma} = \overline{0}, \overline{1} \).

(iii) The element \( \rho \in \mathbb{R}_+ M \cap (S \otimes \mathbb{Q}) \) satisfies the equality: \( (\rho, \alpha) = -(\alpha, \alpha)/2 \) for any \( \alpha \in P(M) \); it is called a lattice Weyl vector.

(iv) All Fourier coefficients \( m(a), a \in M \cap \mathbb{R}_+ M \), are integral.

Using construction [K1], [K3], [B1], [GN1], [R], one can correspond to Lie type Fourier expansion (2.5.5) a generalized Kac–Moody superalgebra. It is defined by a set of simple roots \( s\Delta \) which is divided on the set of simple real roots \( s\Delta^\text{re} \) and the set of simple imaginary roots \( s\Delta^\text{im} \). Both these sets are divided in subsets of even and odd roots which are marked by indices \( \overline{0} \) and \( \overline{1} \) respectively. We have

\[
s\Delta^\text{re}_\overline{0} = P(M)_\overline{0}, \quad s\Delta^\text{re}_\overline{1} = P(M)_\overline{1}
\]

and

\[
s\Delta^\text{im} = s\Delta^\text{im}_\overline{0} \cup s\Delta^\text{im}_\overline{1}, \quad s\Delta^\text{im}_\overline{0} = \{m(a)_\overline{0} a \mid a \in M \cap \mathbb{R}_+ M \}
\]

where non-negative integers \( m(a)_\overline{0} \) and \( m(a)_\overline{1} \) are related with Fourier coefficients \( m(a) \) in (2.5.5) as follows:

\[
m(a)_\overline{0} - m(a)_\overline{1} = m(a), \text{ for any } a \in \mathbb{R}_+ M \text{ with } (a, a) < 0,
\]

and for any primitive \( a_0 \in M \cap \mathbb{R}_+ M \) with \( (a_0, a_0) = 0 \) one has the identity of formal power series with one variable \( q \):

\[
\prod_{n \in \mathbb{N}} (1 - q^n)^{m(na_0)_\overline{0} - m(na_0)_\overline{1}} = 1 - \sum_{t \in \mathbb{N}} m(ta_0) q^t.
\]

In (2.5.7) \( m(a)_\overline{1} a \) means that we repeat the element \( a \) in the sequence \( s\Delta^\text{im}_\overline{1} \) exactly \( m(a)_\overline{1} \) times. The set \( s\Delta = s\Delta^\text{re}_\overline{1} \cap s\Delta^\text{im}_\overline{1} \) is called the set of simple roots.

The generalized Kac–Moody superalgebra \( \mathfrak{g} = \mathfrak{g}''(s\Delta) \) corresponding to \( s\Delta \) is a Lie superalgebra over \( \mathbb{R} \) generated by \( h_r, e_r, f_r, r \in s\Delta \). All \( h_r \) are even; \( e_r, f_r \) are even (respectively odd) if \( r \) is even (respectively odd). The algebra \( \mathfrak{g} \) has the defining relations (where \( r, r' \) are arbitrary elements of \( s\Delta \)):

1. The map \( r \mapsto h_r \) gives an embedding of \( M \otimes \mathbb{R} \) into \( \mathfrak{g}''(s\Delta) \) as an Abelian subalgebra (it is even since all \( h_r \) are even). In particular, all elements \( h_r \) commute.
2. \([h_r, e_{r'}] = (r, r')e_{r'}, \text{ and } [h_r, f_{r'}] = -(r, r')f_{r'}\).
3. \([e_r, f_{r'}] = h_r \text{ if } r = r', \text{ and is } 0 \text{ if } r \neq r'\).
4. \((\text{ad } e_r)^{1-2(r, r')/(r, r)} e_{r'} = (\text{ad } f_r)^{1-2(r, r')/(r, r)} f_{r'} = 0 \text{ if } r \neq r' \text{ and } (r, r) > 0 \) (equivalently, \( r \in s\Delta^\text{re} \)).
5. If \( (r, r') = 0, \text{ then } [e_r, e_{r'}] = [f_r, f_{r'}] = 0\).

The superalgebra \( \mathfrak{g} = \mathfrak{g}''(s\Delta) \) is graded by the root lattice \( M \) as follows. Let

\[
Q_+ = \sum_{\alpha \in s\Delta} \mathbb{Z}_+ \alpha \subset M
\]

be the integral cone (semi-group) generated by all simple roots \( \Delta \). We have

\[
\mathfrak{g} = \left( \bigoplus_{\alpha} \mathfrak{g}_\alpha \right) \bigoplus \mathfrak{g}_0 \bigoplus \left( \bigoplus_{\alpha} \mathfrak{g}_{-\alpha} \right)
\]
where \(e_r\) and \(f_r\) have degree \(r \in Q_+\) and \(-r \in Q_+\) respectively, \(r \in \Delta\); and \(\mathfrak{g}_0 = M \otimes \mathbb{R}\). An element \(\alpha \in \pm Q_+\) is called a root if \(\alpha \neq 0\) and \(\mathfrak{g}_\alpha\) is non-zero. Let \(\Lambda\) be the set of all roots and \(\Lambda = \Delta \cap \pm Q_+\). For a root \(\alpha \in \Delta\) we denote \(\text{mult}_{\sigma \alpha} = \dim \mathfrak{g}_{\alpha, \overline{\sigma}}, \ \text{mult}_{\varpi \alpha} = -\dim \mathfrak{g}_{\alpha, \overline{\varpi}}\) and

\[
\text{mult} \alpha = \text{mult}_{\sigma \alpha} + \text{mult}_{\varpi \alpha} = \dim \mathfrak{g}_{\alpha, \overline{\sigma}} - \dim \mathfrak{g}_{\alpha, \overline{\varpi}}. \tag{2.5.12}
\]

The set of roots and multiplicities of roots are \(W\)-invariant. We have the formal denominator identity for \(\mathfrak{g}\) (which is similar to (2.4.5)):

\[
\Phi(z) = \sum_{w \in W} \epsilon(w) \left( \exp(-2\pi i(w(\rho), z)) - \sum_{a \in M \cap \mathbb{R}_{++} M} m(a) \exp(-2\pi i(w(\rho + a), z)) \right) = \exp(-2\pi i(\rho, z)) \prod_{\alpha \in \Delta_+} (1 - \exp(-2\pi i(\alpha, z)))^{\text{mult} \alpha}. \tag{2.5.13}
\]

We remark that \(\mathfrak{g}_{m\alpha} = 0\) for \(m > 1\) and \(\mathfrak{g}_\alpha\) is even of dimensions one if \(\alpha\) is a real even root (i.e. it belongs to \(W(P(M)_{\overline{\sigma}})\)). Then \(\text{mult} m\alpha = 0\) for \(m > 1\), and \(\text{mult} \alpha = 1\). If \(\alpha\) is a real odd root, (i.e. it belongs to \(W(P(M)_{\overline{\varpi}})\), the space \(\mathfrak{g}_{m\alpha} = 0\) for \(m > 2\), \(\mathfrak{g}_\alpha\) is odd of dimension one, and \(\mathfrak{g}_{2\alpha}\) is even of dimension one. Thus, \(\text{mult} \alpha = -1\), \(\text{mult} 2\alpha = 1\) and \(\text{mult} m\alpha = 0\) for \(m > 2\). Suppose that the infinite product (2.5.13) converges at a neighborhood of the cusp \(c\) (i.e. if \((\text{Im} \ z, \text{Im} \ z) \ll 0\)). It then follows that all zeros of \(\Phi\) at a neighborhood of the cusp \(c\) are quadratic divisors of multiplicity one orthogonal to some elements of the form \(\alpha + mc, m \in \mathbb{Q}\), where \(\alpha\) is a real root.

**Definition 2.5.2.** A generalized Kac–Moody superalgebra \(\mathfrak{g}''(\Delta)\) defined above using the automorphic form \(\Phi\) on IV type domain with Fourier expansion of Lie type, is called an automorphic Lorentzian Kac–Moody superalgebra with the symmetrized generalized Cartan matrix \(B\) (or with a generalized Cartan matrix \(A\)) and with the subset of odd indices \(P(M)_{\overline{\sigma}} \subset P(M)\).

The algebra \(\mathfrak{g}''(\Delta)\) is also called an automorphic correction of a Lorentzian Kac–Moody superalgebra \(\mathfrak{g}''(\Delta_{\text{re}}), \ \Delta_{\text{re}} = P(M)\), defined by the symmetrized generalized Cartan matrix \(B\) (or by a generalized Cartan matrix \(A\)) and with the subset of odd indices \(P(M)_{\overline{\sigma}} \subset P(M)\).

To find all possible automorphic generalized Lorentzian Kac–Moody superalgebras, one should find all automorphic forms \(\Phi\) on IV type domains with Fourier expansion of Lie type at a cusp. There are two cases:

Case (A): the Weyl group \(W\) is non-trivial, equivalently \(P(M) \neq \emptyset\).

Case (B): the Weyl group \(W\) is trivial, equivalently \(P(M) = \emptyset\).

From Proposition 2.4.3, we get for the case (A):

**Theorem 2.5.3.** Assume there exists an automorphic form \(\Phi\) with Fourier expansion of Lie type related with a non-trivial reflection group \(W\) (or \(P(M) \neq \emptyset\)). Equivalently, suppose that there exists an automorphic correction of the Lorentzian Kac–Moody superalgebra \(\mathfrak{g}''(P(M))\) with \(P(M) \neq \emptyset\). Then \(P(M)\) has a lattice Weyl vector \(\rho \in \mathbb{R}_{++} M\) and \(W\) is elliptic if \((\rho, \rho) < 0\), and \(W\) is parabolic if \((\rho, \rho) = 0\).

One can consider Theorem 2.5.3 as a generalization to superalgebras of some results from [N10] (see also [N9]) where it was shown that to have a “good” Lorentzian superalgebra, it is necessary to have a lattice Weyl vector which is real and parabolic.
Kac–Moody algebra and to have its automorphic correction by an automorphic form on IV type domain, one is forced to restrict with Weyl groups $W$ of elliptic or parabolic type and sets $P(M)$ with a lattice Weyl vector. One has no choice if he wants to restrict with holomorphic automorphic forms on IV type domains as correcting automorphic forms.

Type IV domain may have several cusps, and the automorphic form $\Phi$ may have different Fourier expansions at different cusps. Due to R. Borcherds [B5] and J. Harvey and G. Moore [HM1], under some quite general conditions, the infinite product (2.5.13) converges at a neighborhood of the cusp and has zeros which are quadratic divisors with multiplicity one orthogonal to real roots. Globalizing these local data, we suggest a definition of “the most beautiful” automorphic forms related with Kac–Moody algebras:

**Definition 2.5.4.** An automorphic form $\Phi$ on the domain $\Omega^+(T)$ where $T$ is a lattice with 2 negative squares is called a **Lie reflective automorphic form** if
(i) $\Phi$ is reflective for $T$ and all zeros of $\Phi$ have multiplicity one;
(ii) at any cusp $c$ in $\Omega(T)$, the form $\Phi$ has a Lie type Fourier expansion (2.5.5) with respect to a reflection subgroup $W$ of the reflection group of the lattice $T$.

The lattice $T$ having a Lie reflective automorphic form is called **Lie reflective**.

It is an interesting but very difficult problem to find all Lie reflective lattices $T$ and all Lie reflective automorphic forms and the corresponding automorphic Kac–Moody superalgebras. We believe that because of Theorems 2.2.1, 2.5.3 and Necessary Condition 2.2.2 we have the following statement:

**Conjecture 2.5.5.** The numbers of Lie reflective lattices $T$ with 2 negative squares and their Lie reflective automorphic forms are finite up to multiplication of the form of the lattice $T$.

This statement is mirror symmetric to finiteness of generalized Cartan matrices of elliptic and parabolic type with a lattice Weyl vector proved in [N10].

We suggest the following physical speculation about Definition 2.5.4. The symmetric domain $\Omega^+(T)$ might be related with moduli space of some physical theory, a Lie reflective automorphic form $\Phi$ might be related with some global function (like Lagrangian) which defines this physical theory. When we approach to a cusp of $\Omega^+(T)$ (that corresponds to a degeneration of the theory), an automorphic Lorentzian Kac–Moody algebra defined by Fourier expansion of $\Phi$ at the cusp appears. It is related with a quantization of the physical theory.

In Part II, using the methods of [G1]—[G4], we find many Lie reflective automorphic forms and Lie reflective lattices which have Lie type Fourier expansions with hyperbolic generalized Cartan matrices of elliptic and parabolic type classified in Sect. 1. Thus we will restrict considering 3-dimensional case, but all these results may be transferred to higher-dimensional case.

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