DECIDING THE EXISTENCE OF MINORITY TERMS

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Abstract. This paper investigates the computational complexity of deciding if a given finite idempotent algebra has a ternary term operation \( m \) that satisfies the minority equations

\[
m(y, x, x) \approx m(x, y, x) \approx m(x, x, y) \approx y.
\]

We show that a common polynomial-time approach to testing for this type of condition will not work in this case and that this decision problem lies in the class \( \text{NP} \).

1. Introduction

It is not difficult to see that for the 2-element group \( \mathbb{Z}_2 = \langle \{0, 1\}, + \rangle \), the term operation \( m(x, y, z) = x + y + z \) satisfies the equations

\[
(1) \quad m(y, x, x) \approx m(x, y, x) \approx m(x, x, y) \approx y.
\]

A slightly more challenging exercise is to show that a finite Abelian group will have such a term operation if and only if it is isomorphic to a Cartesian power of \( \mathbb{Z}_2 \).

A ternary operation \( m(x, y, z) \) on a set \( A \) is called a minority operation on \( A \) if it satisfies the identities (1). A ternary term \( t(x, y, z) \) of an algebra \( A \) is a minority term of \( A \) if its interpretation as an operation on \( A \), \( t^A(x, y, z) \), is a minority operation on \( A \). Given a finite algebra \( A \), one can decide if it has a minority term by constructing all of its ternary term operations and checking to see if any of them satisfy the equations (1). Since the set of ternary term operations of \( A \) can be as big as \( |A|^3 \), this procedure will have a runtime that in the worst case will be exponential in the size of \( A \).

In this paper we consider the computational complexity of testing for the existence of a minority term for finite algebras that are idempotent. An \( n \)-ary operation \( f \) on a set \( A \) is idempotent if it satisfies the equation \( f(x, x, \ldots, x) \approx x \) and an algebra is idempotent if all of its basic operations are. We observe that every minority operation is idempotent. While idempotent algebras are rather special, one can always form one by taking the idempotent reduct of a given algebra \( A \). This is the algebra with universe \( A \) whose basic operations are all of the idempotent term operations of \( A \). It

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turns out that many important properties of an algebra and the variety that it generates are governed by its idempotent reduct \[9\].

The condition of an algebra having a minority term is an example of a more general existential condition on the set of term operations of an algebra called a strong Maltsev condition. Such a condition consists of a finite set of operation symbols along with a finite set of equations involving them. An algebra is said to satisfy the condition if for each \(k\)-ary operation symbol from the condition, there is a corresponding \(k\)-ary term operation of the algebra so that under this correspondence, the equations of the condition hold. For a more careful and complete presentation of this notion and related ones, we refer the reader to \[6\].

Given a strong Maltsev condition \(\Sigma\), the problem of determining if a finite algebra satisfies \(\Sigma\) is decidable and lies in the complexity class \(\text{EXPTIME}\). As in the minority term case, one can construct all term operations of an algebra up to the largest arity of an operation symbol in \(\Sigma\) and then check to see if any of them can be used to witness the satisfaction of the equations of \(\Sigma\). In general, we cannot do any better than this, since for some strong Maltsev conditions, it is known that the corresponding decision problem is \(\text{EXPTIME}\)-complete \[5\].

The situation for finite idempotent algebras appears to be better than in the general case since there are a number of strong Maltsev conditions for which there are polynomial-time procedures to decide if a finite idempotent algebra satisfies them \[5, 7, 8\]. At present there is no known characterization of these strong Maltsev conditions and we hope that the results of this paper may help to lead to a better understanding of them. We refer the reader to \[3\] or to \[1\] for background on the basic algebraic notions and results used in this work.

2. MINORITY IS A JOIN OF TWO WEAKER CONDITIONS

One approach to understanding the minority term condition is to see if maybe there exist two weaker Maltsev conditions \(\Sigma_1\) and \(\Sigma_2\) such that a finite algebra \(A\) has a minority term if and only if \(A\) satisfies both \(\Sigma_1\) and \(\Sigma_2\). In this situation, we would say that the minority term condition is the join of \(\Sigma_1\) and \(\Sigma_2\). Were this the case, we could decide if \(A\) has a minority term by deciding \(\Sigma_1\) and \(\Sigma_2\).

On the surface, the minority term condition is already quite concise and natural; it is not clear if having a minority term can be expressed as a join of weaker conditions. In this section, we show that it is a join of having a Maltsev term with a condition which we call having a minority-majority term (not to be confused with the ‘generalized minority-majority’ terms from \[4\]). Maltsev terms are a classical object of study in universal algebra – deciding if an algebra has them is in \(P\) for finite idempotent algebras. The minority-majority terms are much less understood.
Definition 1. A ternary term \( p(x, y, z) \) of an algebra \( A \) is a Maltsev term for \( A \) if it satisfies the equations
\[
p(x, x, y) \approx p(y, x, x) \approx y
\]
and a 6-ary term \( t(x_1, \ldots, x_6) \) is a minority-majority term of \( A \) if it satisfies the equations
\[
t(y, x, x, z, y, y) \approx y \quad t(x, y, x, y, z, y) \approx y \quad t(x, x, y, y, y, z) \approx y.
\]

We point out that if an algebra has a minority term then it also, trivially, has a Maltsev term, but that the converse does not hold (as witnessed by the cyclic group \( \mathbb{Z}_4 \)). Our definition of a minority-majority term is a strengthening of the term condition found by M. Olšák in [12]. Olšák has shown that his terms are a weakest non-trivial strong Maltsev condition whose terms are all idempotent.

We observe that by padding variables, any algebra that has a minority term or a majority term (just replace the final occurrence of the variable \( y \) in the equations (1) by the variable \( x \) to define such a term) also has a minority-majority term. Since the 2-element lattice has a majority term but no minority term, it follows that having a minority-majority term is strictly weaker than having a minority term.

Theorem 2. An algebra has a minority term if and only if it has a Maltsev term and a minority-majority term.

Proof. The discussion preceding this theorem establishes one direction of this theorem. For the other we need to show that if an algebra \( A \) has a Maltsev term \( p(x, y, z) \), and a minority-majority term \( t(x_1, \ldots, x_6) \) then \( A \) has a minority term. Given such an algebra \( A \), define
\[
m(x, y, z) = t(x, y, z, p(z, x, y), p(x, y, z), p(y, z, x)).
\]
Verifying that \( m(x, y, z) \) is a minority term for \( A \) is straightforward; we show one of the three required equalities here as an example:
\[
m(x, x, y) \approx t(x, x, y, p(y, x, x), p(x, x, y), p(x, y, x)) \approx t(x, x, y, y, p(x, y, x)) \approx y.
\]

Corollary 3. The problem of deciding if a finite algebra has a minority term can be reduced to the problems of deciding if it has a Maltsev term and if it has a minority-majority term.

As was demonstrated in [5, 7], there is a polynomial-time algorithm to decide if a finite idempotent algebra has a Maltsev term. Therefore, should testing for a minority-majority term for finite idempotent algebras prove to be tractable, then this would lead to a fast algorithm for testing for a minority
term, at least for finite idempotent algebras. From the hardness results found in [5] it follows that in general, the problem of deciding if a finite algebra has a minority-majority term is \textsc{Exptime}-complete.

3. **Local Maltsev terms**

In [5, 7, 8, 13] polynomial-time algorithms are presented for deciding if certain Maltsev conditions hold in the variety generated by a given finite idempotent algebra. One particular Maltsev condition that is addressed by all of these papers is that of having a Maltsev term. In all but [5], the polynomial-time algorithm produced is based on testing for the presence of enough ‘local’ Maltsev terms in the given algebra.

**Definition 4.** Let $A$ be an algebra and $S \subseteq A^2 \times \{0, 1\}$. A term operation $t(x, y, z)$ of $A$ is a **local Maltsev term operation** for $S$ if:

- whenever $((a, b), 0) \in S$, $t(a, b, b) = a$, and
- whenever $((a, b), 1) \in S$, $t(a, a, b) = b$.

Clearly, if $A$ has a Maltsev term then it has a local Maltsev term operation for every subset $S$ of $A^2 \times \{0, 1\}$ and conversely, if $A$ has a local Maltsev term operation for $S = A^2 \times \{0, 1\}$ then it has a Maltsev term. In [7, 8, 13] it is shown that if a finite idempotent algebra $A$ has local Maltsev term operations for all two element subsets of $A^2 \times \{0, 1\}$ then $A$ will have a Maltsev term. This fact is then used as the basis for a polynomial-time test to decide if a given finite idempotent algebra has a Maltsev term.

In this section we extract an additional piece of information from this approach to testing for a Maltsev term, namely that if a finite idempotent algebra has a Maltsev term, then we can produce an operation table or a circuit for a Maltsev term operation in time polynomial in the size of the algebra.

To measure the size of an input algebra, we use the following formula. For $A$ a finite algebra, let

$$\|A\| = \sum_{i=0}^{r} k_i |A|^i,$$

where $k_i$ is the number of $i$-ary basic operations of $A$ and $r$ is the maximum arity of the basic operations of $A$. So $\|A\|$ is roughly equal to the sum of the sizes of the operation tables of the basic operations of $A$.

**Theorem 5.** Let $A$ be a finite idempotent algebra. There is an algorithm whose run-time can be bounded by a polynomial in the size of $A$ that will produce the table of some Maltsev term operation of $A$, should one exist.

**Proof.** Let $A$ be a finite idempotent algebra and let $n = |A|$. Assume that $|A| > 1$ and that $A$ has at least one basic operation of positive arity since otherwise $A$ is either trivial or can’t have a Maltsev term operation, respectively. Thus $\|A\| \geq n$. 
We will represent a ternary term operation $t$ of $A$ by a table of values – such a table has $n^3$ entries, a size polynomial in $n$.

In the first stage of the algorithm, we will check to see if $A$ has a local Maltsev term operation $t_{a,b,c,d}(x,y,z)$ for each two element subset
\[
\{((a,b),0),((c,d),1)\}
\]
of $A^2 \times \{0,1\}$. If for some such subset no local Maltsev term operation exists, then the algorithm halts, since in this case, $A$ will not have a Maltsev term. Otherwise, the algorithm will store the operation tables for each term operation $t_{a,b,c,d}$ found. Note that the algorithm will need to store $n^4$ of these tables.

Let us fix $a,b,c,d \in A$ and explain in more detail how to find the local Maltsev term $t_{a,b,c,d}$: To this end, we will examine the subalgebra $R$ of $A^2$ generated by $\{(a,c),(b,c),(b,d)\}$. This subalgebra can be generated in time $O(||A||^3)$ by an algorithm that, in essence, keeps adding elements to $R$ as required by operations of $A$ (see, for example, Theorem 2.2 of [13], or just follow the rest of this proof). During the subalgebra generation process, we will remember for each $(u,v) \in R$ the operation table of a ternary term operation $q_{(u,v)}$ such that
\[
q_{(u,v)}\left(\begin{array}{ccc} a & b & b \\ c & c & d \end{array}\right) = \left(\begin{array}{c} u \\ v \end{array}\right).
\]
It is easy to verify that if $(a,d) \not\in R$, then there is no local Maltsev operation $t_{a,b,c,d}$, while if $(a,d) \in R$, we can let $t_{a,b,c,d} = q_{(a,d)}$.

If the new element $(u,v)$ arises by applying the $p$-ary basic operation $f$ to elements $(u_i,v_i)$ for $1 \leq i \leq p$ that have already been produced, then we let
\[
q_{(u,v)} = f(q_{(u_1,v_1)},q_{(u_2,v_2)},\ldots,q_{(u_p,v_p)}).
\]
From this formula and stored operation tables for the $q_{(u_i,v_i)}$’s, it is easy to compute the table for $q_{(u,v)}$.

Once the first stage has been completed, $A$ is guaranteed to have a Maltsev term (see, for example, Theorem 2.2 of [13], or just follow the rest of this proof). For the rest of this proof fix an enumeration $(a_1,b_1),(a_2,b_2),\ldots,(a_{n^2},b_{n^2})$ of $A^2$.

The second stage of our algorithm will produce and store, for each $a,b \in A$, the table of a term operation $t_{a,b}(x,y,z)$ of $A$ such that $t_{a,b}(a,b,b) = a$ and $t_{a,b}(x,x,y) = y$ for all $x,y \in A$. The construction of this operation will proceed by induction.

For $1 \leq j \leq n^2$ we define the operation $t_{a,b}^j(x,y,z)$ on $A$ inductively as follows:

- $t_{a,b}^1(x,y,z) = t_{a,b,a_1,b_1}(x,y,z)$, and
- for $1 \leq j < n^2$, $t_{a,b}^{j+1}(x,y,z) = t_{a,b,u,v}(t_{a,b}^j(x,y,z),t_{a,b}^j(y,y,z),z)$, where $u = t_{a,b}^j(a_{j+1},a_{j+1},b_{j+1})$ and $v = b_{j+1}$.

An easy inductive argument shows that $t_{a,b}^j(a,b,b) = a$ and $t_{a,b}^j(a_i,a_i,b_i) = b_i$ for all $i \leq j \leq n^2$, and so setting $t_{a,b}(x,y,z) = t_{a,b}^{n^2}(x,y,z)$ works.
In the final stage of the algorithm, we construct and store, for each \( j \leq n^2 \), the operation table of a term operation \( t_j(x, y, z) \) such that \( t_j(a, a, b) = b \) for all \( a, b \in A \) and \( t_j(a_i, b_i, b_i) = a_i \) for all \( i \leq j \). We define this sequence of operations inductively again:

- \( t_1(x, y, z) = t_{a_1, b_1}(x, y, z) \), and
- for \( 1 \leq j < n^2 \), \( t_{j+1}(x, y, z) = t_{u,v}(x, t_j(x, y, y), t_j(x, y, z)) \), where \( u = a_{j+1} \) and \( v = t_j(a_{j+1}, b_{j+1}, b_{j+1}) \).

Inductively, it can be shown that for \( 1 \leq j \leq n^2 \), the operation \( t_j(x, y, z) \) satisfies the claimed properties and so \( t_{n^2}(x, y, z) \) will be a Maltsev term operation for \( A \). Without going into detail, we claim that the runtime of each stage of this algorithm can be bounded by a polynomial in \( \|A\| \). □

In the scope of the previous theorem, it is natural to ask whether one can obtain a term that would represent the minority term operation in the algebra \( A \). There is a problem with constructing a term by composing terms for the functions \( t_{a,b,c,d} \) as indicated in the above proof: the term is extended by one layer in each step of the algorithm which results in a term of exponential size. Therefore, the bookkeeping of this term would increase the running time of the algorithm beyond polynomial. Nevertheless, this can be circumvented by constructing a succinct representation of the term operation, namely by considering circuits instead of terms.

Informally, a circuit over an algebraic language (as a generalization of logical circuits) is a collection of gates labeled by function symbols, where the number of inputs of each gate corresponds to the arity of the operation symbol, so that the inputs are either connected to outputs of some gate, or designated as inputs of the circuit, an output of one of the gates is designated as an output of the circuit. Furthermore, these connections allow for straightforward evaluation, i.e., there are no oriented cycles.

Formally, we define an \( n \)-ary circuit in the language of an algebra \( A \) as a directed acyclic graph with possibly multiple edges that has two kinds of vertices: inputs and gates. There are exactly \( n \) inputs, labeled by variables \( x_1, \ldots, x_n \), and each of them is a source, and a finite number of gates. Each gate is labeled by an operation symbol of \( A \), the in-degree corresponds to the arity of the operation, and the in-edges are ordered. One of the vertices is designated as the output of the circuit. We define the size of the circuit to be the number of its vertices.

A value of a circuit at a tuple \( a_1, \ldots, a_n \) is defined by the following recursive computation: A value on an input vertex labeled by \( x_i \) is \( a_i \), a value on a gate labeled by \( g \) is the value of the operation \( g^A \) applied to the values of its in-neighbours in the specified order. Finally, the output value of the circuit is then the value of the output vertex. It is easy to see that the value of a circuit on a given tuple can be computed in linear time (in the size of the circuit) in a straightforward way. For a fixed circuit the function that maps the input tuple to the output is a term function of \( A \). Indeed, to find such a term it is enough to evaluate the circuit in the free (term) algebra on the
tuple \(x_1, \ldots, x_n\). The converse is also true since any term can be represented as a ‘tree’ circuit (it is an oriented tree if we omit all input vertices). Many terms can be expressed by considerably smaller circuits. We give one such example in Figure 1.

In the proof of the theorem below, we will also use circuits with multiple outputs. The only difference in the definition is that several vertices are designated as outputs. Any such circuit then computes a tuple of term functions.

**Theorem 6.** Let \(A\) be a finite idempotent algebra. There is an algorithm whose run-time can be bounded by a polynomial in the size of \(A\) that will either prove that \(A\) has no Maltsev term operation, or output a circuit for some Maltsev term operation of \(A\).

**Proof.** The proof follows that of Theorem 5. As before, let \(n = |A|\). To rule out trivial cases, we assume that \(A\) has at least one basic operation of positive arity and that \(n \geq 2\). Then \(|A| \geq n\).

We construct a circuit representing a Maltsev operation in three steps: The first step produces circuits that compute \(t_{a,b,c,d}\), the second step produces circuits that compute \(t_{a,b}\), and the final step produces a circuit for a Maltsev operation \(t\). We note that the algorithm can fail only in the first step.

Step 1. Given \(a, b, c, d\), we aim to produce a circuit that computes the term function \(t_{a,b,c,d}\). To do that, we consider the sequence \(r_1, r_2, \ldots\) of elements of \(R\) found by the algorithm that generates \(R\) from the proof of Theorem 5; we also assume that \(r_1 = (a, c)\), \(r_2 = (b, c)\), and \(r_3 = (b, d)\).

We produce a sequence of ternary circuits \(C_{a,b,c,d}^3 \subseteq C_{a,b,c,d}^4 \subseteq \ldots\) such that each \(C_{a,b,c,d}^k\) has \(k\) outputs, and the values of \(C_{a,b,c,d}^k\) on \(r_1, r_2, r_3\) give \(r_1, \ldots, r_k\). We define \(C_{a,b,c,d}^0\) to be the circuit with no gates, and outputs \(x_1, x_2, x_3\). The circuit \(C_{a,b,c,d}^{k+1}\) is defined inductively from \(C_{a,b,c,d}^k\): Consider an operation \(f\) and \(r_{i_1}, \ldots, r_{i_p}\) with \(i_j \leq k\) such that \(r_{k+1} = f(r_{i_1}, \ldots, r_{i_p})\); add a gate labeled \(f\) to \(C_{a,b,c,d}^k\) connecting its inputs with the outputs of \(C_{a,b,c,d}^k\) numbered by \(i_j\) for \(j = 1, \ldots, p\). We designate the output of this gate as the \((k + 1)\)-st output of \(C_{a,b,c,d}^{k+1}\). It is straightforward to check that the circuits \(C_{a,b,c,d}^k\) satisfy the requirements. We also note that the size of \(C_{a,b,c,d}^k\) is exactly \(k\). We stop this inductive construction if either \(r_k = (a, d)\), in which case we define the circuit \(C_{a,b,c,d}\) from \(C_{a,b,c,d}^k\) by indicating a single
output to be the $k$-th output of $C_{a,b,c,d}^k$, or if no new elements $r_{k+1}$ can be generated, in that case the algorithm stops with failure.

Step 2. At this point we assume that the functions $t_{a,b,c,d}$ are part of the signature. It is clear that the full circuit can be obtained by substituting the circuits $C_{a,b,c,d}$ for gates labeled by $t_{a,b,c,d}$, and this can be still done in polynomial time. Our task is to obtain a circuit for $t_{a,b}$.

We do this by inductively constructing circuits $C_{a,b}^j$ that compute two values of the terms $t_{a,b}^j$, namely $t_{a,b}^j(x, y, z)$ and $t_{a,b}^j(y, y, z)$. Starting with $j = 0$ and $t^0(x, y, z) = x$, we define $C_{a,b}^0$ to be the circuit with no gates and outputs $x, y$. Further, we define circuit $C_{a,b}^{j+1}$ inductively from $C_{a,b}^j$ by adding two gates labeled by $t_{a,b,u,v}$, where $u = t_{a,b}^j(a_{j+1}, a_{j+1}, b_{j+1})$ and $v = b_{j+1}$: the first gate has as inputs the two outputs of $C_{a,b}^j$ and $z$, the second gate has as inputs two copies of the second output of $C_{a,b}^j$ and $z$. See Figure 2 for a graphical representation. Again, it is straightforward to check that these circuits have the required properties. Also note that the size of $C_{a,b}^j$ is bounded by $2j + 3$ which is a polynomial. The final circuit $C_{a,b}$ computing $t_{a,b}$ is obtained from $C_{a,b}^{n^2}$ by designating the first output of $C_{a,b}^{n^2}$ to be the only output of $C_{a,b}$.

Step 3. Again, we assume that $t_{a,b}$ are basic operations, and construct circuits $C_j$ computing two values $t_j(x, y, y)$ and $t_j(x, y, z)$ of $t_j$ inductively. The proof is analogous to Step 2, with the only difference that we use Figure 3 for the inductive definition.

Each step can be run separately in quadratic time, outputting a polynomial size circuit. This also implies that expanding the gates according to their definitions in previous steps can be also done in polynomial time, namely the time as well as the size of the output is bounded by $O(n^6)$. □
4. LOCAL MINORITY TERMS

In contrast to the situation for Maltsev terms highlighted in the previous section, we will show that having plenty of ‘local’ minority terms does not guarantee that a finite idempotent algebra will have a minority term. One consequence of this is that an approach along the lines in [7, 8, 13] to finding an efficient algorithm to decide if a finite idempotent algebra has a minority term will not work.

In this section, we will construct for each natural number \( n > 2 \) a finite idempotent algebra \( A_n \) with the following properties: The universe of \( A_n \) has size \( 4n \) and \( A_n \) does not have a minority term, but for every subset \( E \) of \( A_n \) of size \( n - 1 \) there is a term of \( A_n \) that acts as a minority term on the elements of \( E \).

We start our construction by fixing some \( n > 2 \) and some minority operation \( m \) on the set \( [n] = \{1, 2, \ldots, n\} \). To make things concrete we set

\[
m(x, y, z) = \begin{cases} 
  x & y = z \\
  y & x = z \\
  z & \text{else,}
\end{cases}
\]

but note that any minority operation on \([n]\) will do.

Since there are two nonisomorphic groups of order 4, we have two different natural group operations on \( \{0,1,2,3\} \): addition modulo 4, which we will denote by ‘\(+_4\)’ (its inverse is ‘\(-_4\)’), and bitwise XOR, which we denote by ‘\(\oplus\)’ (this operation takes bitwise XOR of the binary representations of input numbers, so for example \(1 \oplus 3 = 2\)).

In any of the arithmetic expressions involving \(\oplus\) and \(+_4\) in this section, the operation \(\oplus\) will take precedence. Observe that the operations \(+_4\) and \(\oplus\) agree on inputs from \(\{0,2\}^2\) (e.g., \(2 +_4 2 = 2 \oplus 2 = 0\)).
Let $A_n = [n] \times [4]$. For $i \in [n]$, we define $t_i(x, y, z)$ to be the following operation on $A_n$: For most inputs $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in A_n$, we set

$$t_i((a_1, b_1), (a_2, b_2), (a_3, b_3)) = (m(a_1, a_2, a_3), b_1 \oplus b_2 \oplus b_3).$$

The only exception occurs when $a_1 = a_2 = a_3 = i$. In that case we have

$$t_i((i, b_1), (i, b_2), (i, b_3)) = (i, b_1 - 4b_2 + 4b_3).$$

Let $A_n$ be the algebra with the universe $A_n$ and basic operations $t_1, \ldots, t_n$. By construction, the following claim is true:

**Claim 7.** For every $(n-1)$-element subset $E$ of $A_n$, there is a term operation of $A_n$ that satisfies the minority term equations when restricted to elements from $E$.

**Proof.** Pick $i \in [n]$ such that no element of $E$ has its first coordinate equal to $i$; the operation $t_i$ is a local minority for this $E$. □

**Proposition 8.** For $n > 1$, the algebra $A_n$ does not have a minority term.

**Proof.** Given some $(i, a) \in A_n$, we will refer to $a$ as the interesting part of $(i, a)$. This is to avoid talking about ‘second coordinates’ in the confusing situation when $(i, a)$ itself is a part of a tuple of elements of $A_n$.

To prove the proposition, we will define a certain subuniverse $R$ of $(A_n)^{3n}$ and then show that $R$ is not closed under any minority operation on $A_n$ (applied coordinate-wise).

Let $R \subseteq (A_n)^{3n}$ be the set of all $3n$-tuples of the form

$$\begin{pmatrix}
(1, 2x_1 + 4a) \\
(2, 2x_2 + 4a) \\
\vdots \\
(n, 2x_n + 4a) \\
(1, 2x_{n+1} + 4b) \\
(2, 2x_{n+2} + 4b) \\
\vdots \\
(n, 2x_{2n} + 4b) \\
(1, 2x_{2n+1} + 4c) \\
(2, 2x_{2n+2} + 4c) \\
\vdots \\
(n, 2x_{3n} + 4c)
\end{pmatrix}$$

where $x_1, \ldots, x_{3n}, a, b, c \in \{0, 1\}$ and we have

$$a + b + c + 2 \sum_{i=1}^{3n} x_i = 2 \pmod{4}. \quad (2)$$

Note that we can view the left hand side of equation (2) as the sum of the interesting parts of, say, the 1st, $(n+1)$-th and $(2n+1)$-th entry of the $3n$-tuple, plus twice the sum of the $x_i$’s where $i$ ranges over $[3n] \setminus \{1, n+1, 2n+1\}$. This will be useful later.
Claim 9. The relation $R$ is a subuniverse of $(A_n)^{3n}$.

Proof. By the symmetry of the $t_i$'s and $R$, it is enough to show that $t_1$ preserves $R$. Let us take three arbitrary members of $R$:

\[
\begin{pmatrix}
1, 2x_{1,1} + 4 a_1 \\
2, 2x_{1,2} + 4 a_1 \\
\vdots \\
n, 2x_{1,n} + 4 a_1 \\
1, 2x_{1,n+1} + 4 b_1 \\
2, 2x_{1,n+2} + 4 b_1 \\
\vdots \\
n, 2x_{1,2n} + 4 b_1 \\
1, 2x_{1,2n+1} + 4 c_1 \\
2, 2x_{1,2n+2} + 4 c_1 \\
\vdots \\
n, 2x_{1,3n} + 4 c_1 \\
\end{pmatrix}
\]

and apply $t_1$ to them to get (after using that $\oplus$ and $+4$ agree on inputs of the form $2x$):

\[
\bar{r} = \begin{pmatrix}
1, a_1 - 4 a_2 + 4 a_3 + 4 2x_{1,1} + 4 2x_{2,1} + 4 2x_{3,1} \\
2, a_1 \oplus a_2 \oplus a_3 + 4 2x_{1,2} + 4 2x_{2,2} + 4 2x_{3,2} \\
\vdots \\
n, a_1 \oplus a_2 \oplus a_3 + 4 2x_{1,n} + 4 2x_{2,n} + 4 2x_{3,n} \\
1, b_1 - 4 b_2 + 4 b_3 + 4 2x_{1,n+1} + 4 2x_{2,n+1} + 4 2x_{3,n+1} \\
2, b_1 \oplus b_2 \oplus b_3 + 4 2x_{1,n+2} + 4 2x_{2,n+2} + 4 2x_{3,n+2} \\
\vdots \\
n, b_1 \oplus b_2 \oplus b_3 + 4 2x_{1,2n} + 4 2x_{2,2n} + 4 2x_{3,2n} \\
1, c_1 - 4 c_2 + 4 c_3 + 4 2x_{1,2n+1} + 4 2x_{2,2n+1} + 4 2x_{3,2n+1} \\
2, c_1 \oplus c_2 \oplus c_3 + 4 2x_{1,2n+2} + 4 2x_{2,2n+2} + 4 2x_{3,2n+2} \\
\vdots \\
n, c_1 \oplus c_2 \oplus c_3 + 4 2x_{1,3n} + 4 2x_{2,3n} + 4 2x_{3,3n}
\end{pmatrix}
\]

We want to verify that $\bar{r} \in R$. Looking at the interesting parts of entries of $\bar{r}$, we see that we have three length $n$ blocks of entries of the same parity modulo 2, so the structure of the tuple matches the definition of $R$. All that remains is to verify (2). Recalling the note below equation (2), we see that we need to prove that when we sum the interesting parts of entries 1, $n+1$ and $2n+1$ and add to them the sum of $2x_{i,j}$'s for $i = 1, 2, 3$ and $j \in [3n] \setminus \{1, n+1, 2n+1\}$, we get 2 modulo 4.
However, we know that

\[
\begin{align*}
    a_1 + b_1 + c_1 + 2 \sum_{j=1}^{3n} x_{1,j} & \equiv 2 \pmod{4} \\
-ag 2 - b_2 - c_2 + 2 \sum_{j=1}^{3n} x_{2,j} & \equiv 2 \pmod{4} \\
ag 3 + b_3 + c_3 + 2 \sum_{j=1}^{3n} x_{3,j} & \equiv 2 \pmod{4},
\end{align*}
\]

where in the middle line we used \(-2 = 2 \pmod{4}\). Taking the sum of these three equalities, we get

\[
\begin{align*}
    a_1 - a_2 + a_3 + b_1 - b_2 + b_3 + c_1 - c_2 + c_3 + 2 \sum_{i=1}^{3} \sum_{j=1}^{3n} x_{i,j} & \equiv 2 \pmod{4},
\end{align*}
\]

which shows that indeed \(\vec{r} \in R\). \(\square\)

It is easy to see that \(R\) contains the three tuples:

\[
\begin{align*}
    (1, 1, 0) & \quad (1, 1, 0) & \quad (1, 0, 0) \\
    (2, 1, 0) & \quad (2, 1, 0) & \quad (2, 0, 0) \\
    \vdots & \quad \vdots & \quad \vdots \\
    (n, 1, 0) & \quad (n, 1, 0) & \quad (n, 0, 0) \\
    (1, 1, 0) & \quad (1, 0, 0) & \quad (1, 1, 0) \\
    (2, 1, 0) & \quad (2, 0, 0) & \quad (2, 1, 0) \\
    \vdots & \quad \vdots & \quad \vdots \\
    (n, 1, 0) & \quad (n, 0, 0) & \quad (n, 1, 0) \\
    (1, 0, 0) & \quad (1, 1, 0) & \quad (1, 1, 0) \\
    (2, 0, 0) & \quad (2, 1, 0) & \quad (2, 1, 0) \\
    \vdots & \quad \vdots & \quad \vdots \\
    (n, 0, 0) & \quad (n, 1, 0) & \quad (n, 1, 0)
\end{align*}
\]
but does not contain the tuple:

$$\begin{pmatrix}
(1, 0, 0) \\
(2, 0, 0) \\
\vdots \\
(n, 0, 0) \\
(1, 0, 0) \\
(2, 0, 0) \\
\vdots \\
(n, 0, 0)
\end{pmatrix}.$$ 

However, this last tuple can be obtained from the first three by applying any minority operation on the set $A_n$ coordinate-wise. From this we conclude that $A_n$ does not have a minority term. □

The algebras $A_n$ can also be used to witness that having a lot of local minority-majority terms does not guarantee the presence of an actual minority-majority term. By padding with dummy variables, any local minority term of an algebra $A_n$ is also a term that locally satisfies the minority-majority term equations. But since each $A_n$ has a Maltsev term but not a minority term, then by Theorem [2] it follows that $A_n$ cannot have a minority-majority term.

5. Deciding minority in idempotent algebras is in NP

The results from the previous section imply that one cannot base an efficient test for the presence of a minority term in a finite idempotent algebra on checking if it has enough local minority terms. This does not rule out that the problem is in the class $P$, but to date no other approach to showing this has worked. As an intermediate result, we show, at least, that this decision problem is in $NP$ and so is not $EXPTIME$-complete (unless $NP = EXPTIME$).

Define $\text{MINORITY}^{ld}$ to be the decision problem:

- INPUT: A finite idempotent algebra $A$.
- QUESTION: Does $A$ have a minority term?

We first show that an instance $A$ of the decision problem $\text{MINORITY}^{ld}$ can be expressed as a particular instance of the subpower membership problem for $A$.

**Definition 10.** Given a finite algebra $A$, the *subpower membership problem* for $A$, denoted by $\text{SMP}(A)$, is the following decision problem:

- INPUT: $\bar{a}_1, \ldots, \bar{a}_k, \bar{b} \in A^n$
- QUESTION: Is $\bar{b}$ in the subalgebra of $A^n$ generated by $\{\bar{a}_1, \ldots, \bar{a}_k\}$?
In general, it is known that for some finite algebras this problem can be \textsc{EXPTIME}-complete \cite{al2} and that for some others, e.g., for any algebra that has only trivial or constant basic operations, it lies in the class \textsc{P}. In \cite{al3}, it is shown that when \textit{A} has a Mal'tsev term, then \textit{SMP(\textit{A})} is in \textsc{NP}. We base the main result of this section on this fact and its proof.

To build an instance of \textit{SMP(\textit{A})} that expresses that \textit{A} has a minority term, let \( I = \{(a, b, c) \mid a, b, c \in A \text{ and } |\{a, b, c\}| \leq 2\} \). So \( |I| = 3|A|^2 - 2|A| \).

For \((a, b, c) \in I\), let \( \min(a, b, c) \) be the minority element of this triple. So
\[
\min(a, b, b) = \min(b, a, b) = \min(b, b, a) = \min(a, a, a) = a.
\]

For \( 1 \leq i \leq 3 \), let \( \bar{\pi}_i \in A^I \) be defined by \( \bar{\pi}_i(a_1, a_2, a_3) = a_i \) and define \( \bar{\mu}_A \in A^I \) by \( \bar{\mu}_A(a_1, a_2, a_3) = \min(a_1, a_2, a_3) \), for all \((a_1, a_2, a_3) \in I\). Denote the instance \( \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3, \) and \( \bar{\mu}_A \) of \textit{SMP(\textit{A})} by \( \min(\textit{A}) \).

**Proposition 11.** An algebra \textit{A} has a minority term if and only if \( \bar{\mu}_A \) is a member of the subpower of \( A^I \) generated by \( \{\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3\} \), i.e., if and only if \( \min(\textit{A}) \) is a ‘yes’ instance of \textit{SMP(\textit{A})} when \textit{A} is finite.

**Proof.** If \( m(x, y, z) \) is a minority term for \textit{A}, then applying \( m \) coordinatewise to the generators \( \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3 \) will produce the element \( \bar{\mu}_A \). Conversely, any term that produces \( \bar{\mu}_A \) from these generators will be a minority term for \textit{A}.

Examining the definition of \( \min(\textit{A}) \), we see that the parameters from Definition \cite{al2} are \( k = 3 \) and \( n = 3|A|^2 - 2|A| \), which is (for algebras with at least one at least unary basic operation) polynomial in \( |\textit{A}| \). For \textit{A} idempotent, we can in fact improve \( n \) to \( 3|A|^2 - 3|A| \), since then we do not need to include in \( I \) entries of the form \((a, a, a)\).

As noted earlier, in \cite{al3} P. Mayr shows that when \textit{A} is a finite algebra that has a Mal'tsev term, then \textit{SMP(\textit{A})} is in the class \textsc{NP}. We claim that the sizes of the certificates that P. Mayr produces in his proof can be bounded by a polynomial in the size of the instance and in \( |\textit{A}| \) and that the runtime of the algorithm to process a certificate can also be bounded by such a polynomial, as long as the operation table for a Mal'tsev term for \textit{A} is available. As such, we have the following.

**Theorem 12.** The decision problem \textit{MINORITY}\textsuperscript{id} is in the class \textsc{NP}.

**Proof.** We show that the class of finite idempotent algebras that have a minority term is polynomially verifiable using Theorem \cite{al4} and ideas found in the proof of Theorem 1.1 of \cite{al3}. Let \textit{A} be a finite idempotent algebra. If \( |A| = 1 \), then, trivially, \textit{A} has a minority term, and so we may assume that \( |A| > 1 \).

We first check, using the algorithm from Theorem \cite{al4} to see if \textit{A} has a Mal'tsev term. If it does not, then \textit{A} will not have a minority term, and so we answer ‘no’. Otherwise, we augment the list of basic operations of \textit{A} by adding the Mal'tsev operation on \textit{A} that the algorithm produced. Denote this ternary operation by \( m \). Note that this modification of \textit{A} will increase
its size by (the negligible amount) $|A|^3$ and that the amount of time needed to compute the operation table of $m$ can be bounded by a polynomial in $\|A\|$. Also, the resulting algebra is term equivalent to the original one and so the subpower membership problem is the same for both algebras.

Certifying that $A$ has a minority term is equivalent, according to Proposition 11 to certifying that min$(A)$ is a ‘yes’ instance of SMP$(A)$. Let us denote the subpower of $A$ generated by $\vec{r}_1, \vec{r}_2, \vec{r}_3$ by $S$. Our goal in the rest of the proof is to show that there is a polynomial size (measured by $\|A\|$) certificate for $\vec{\mu}_A \in S$.

We now describe in some detail the certificate that is produced in the proof of Theorem 1.1 of [11] to verify a ‘yes’ instance of SMP$(A)$ for some finite algebra $A$; we will consider this for the algebra $A$ and the instance min$(A)$.

A key notion that is used in the proof is that of a canonical representation of a subpower $B$ of $A^n$ with respect to the Mal'tsev operation $m$. It is a presentation $R$ of a special type of generating set for $B$ that consists of pairs of elements from $A^n$ called forks, along with a subset $R_1$ of $A^n$. A fork at index $i$, for $1 < i \leq n$, is a pair of $n$-tuples over $A$, $(\vec{r}, \vec{s})$, with $r_j = s_j$ for all $j < i$ and with $r_i \neq s_i$. The set of forks at index $i$ in $R$ is denoted $R_i$.

We record two facts about canonical representations that we will use.

**Claim 13.** Given a subalgebra $B$ of $A^n$, there is a canonical representation for $B$ with respect to $m$. Every such canonical representation $R$ satisfies:

- $|R_i| \leq |A|$ and $|R_i| \leq |A|^2$ for $1 < i \leq n$, and
- if $\vec{b} \in B$ then

$$\vec{b} = m(m(\ldots m(\vec{r}_1, \vec{s}_2, \vec{r}_2), \ldots, \vec{s}_{n-1}, \vec{r}_{n-1}), \vec{s}_n, \vec{r}_n)$$

for some $\vec{r}_1 \in R_1$ and $(\vec{r}_i, \vec{s}_i) \in R_i$ for $i > 1$.

Consequently, every canonical representation of $A$ will consist of at most $|A| + 2(n - 1)|A|^2$ tuples from $A^n$ together with a record of which tuple – or pair of tuples – belongs in which set $R_i$ (this record is easily polynomial in $|A|, n$). Crucially, every element $\vec{b}$ of $B$ can be obtained by applying a term in the operation $m$ of length at most $3n$ to some elements found in the pairs of the canonical representation.

Thus a canonical representation for $S$ together with a formula for $\vec{\mu}_A$ of the form (3) will certify that $\vec{\mu}_A \in S$. The only missing piece is how to efficiently verify that a given family $R_1, R_2, \ldots$ is a canonical representation for $S$. Thankfully, if we follow the algorithm from [11], we will get a short certificate of membership for every $\vec{r} \in R_1$ and every $(\vec{r}, \vec{s}) \in R_i$ ($i = 2, \ldots, n$).

In the proof of Theorem 1.1 from [11] an algorithm called Canonical-Representation is defined that takes as input a finite algebra $A$ and a set of generators $\{\vec{a}_1, \ldots, \vec{a}_k\} \subseteq A^n$ and outputs a canonical representation $R$ of the subpower of $A$ generated by $\vec{a}_1, \ldots, \vec{a}_k$. At the start of the algorithm, the proto-canonical representation is initialized to be empty, i.e., $R_i = \emptyset$ for $1 \leq i \leq n$ and as the algorithm proceeds, elements are added to $R$ until it becomes a canonical representation. Let $R[j]$ denote the contents of $R$ just
after the $j$th element has been added to it. Since the size (total number of
tuples or pairs of tuples) of $R$ is bounded above by $|A| + (n - 1)|A|^2$
then for some $j$ no bigger than this number we will have that $R = R[j]$.

As explained in the proof of Theorem 1.1 of [11], every addition to $R$
by CanonicalRepresentation is either one of the generators or can be
witnessed by a short term that demonstrates that the element is in the
subalgebra generated by some elements that have already been added to $R$.
More precisely, the $j$-th addition to $R$ by CanonicalRepresentation can
be of two kinds:

- if $\vec{r}$ is added to $R_1$, then either $\vec{r}$ is one of the generators $\vec{a}_1, \ldots, \vec{a}_k$,
or is of the form $f(\vec{v}_1, \ldots, \vec{v}_t)$ for some basic operation $f$ of $A$ and
some elements $\vec{v}_p$ such that for $1 \leq p \leq t$, $\vec{v}_p$ results from applying
some term of length at most $3n$ in the operation $m$ to some elements
from $R[j - 1]$.

- if $(\vec{r}, \vec{s})$ is added to some $R_i$ for $i > 1$, then $\vec{s}$ results from applying
some term of length at most $3n$ in the operation $m$ to some elements
from $R[j - 1]$ and $\vec{r}$ is as in the previous case.

In both cases, the new tuple or pair of tuples can be specified by a string
whose length can be bounded above by a polynomial in $n$, $k$, and in $\|A\|$.
This string will specify the generators, basic operations, terms in the operation
$m$, and pointers to the specifications of the elements from $R[j - 1]$ that
can be used to produce $\vec{r}$ or $(\vec{r}, \vec{s})$.

Going back to our problem, the sequence of strings certifying $\vec{r} \in S$ or
$\vec{r}, \vec{s} \in S$ (polynomially many, each of polynomial length) can be viewed as
a certificate that $R$ is a canonical representation of $S$. This was what we
needed to get a certificate of $\vec{\mu}_A \in S$ that one can verify in polynomial time:
If $\vec{\mu}_A \in S$, the certificate will be a canonical representation of $S$ (certified by
the strings for the $\vec{r}$’s and $\vec{r}, \vec{s}$’s as per the previous paragraph) and a formula
for $\vec{\mu}_A$ in the form (3). \hfill $\Box$

6. Conclusion

While Theorem [12] establishes that testing for a minority term for finite
idempotent algebras is not as hard as it could be, the true complexity of
this decision problem is still open. Our proof of this theorem closely ties the
complexity of MINORITY$^{id}$ to the complexity of the subpower membership
problem for finite Maltsev algebras. Thus any progress on determining the
complexity of SMP($A$) for finite Maltsev algebras may have a bearing on the
complexity of MINORITY$^{id}$. From the previous section, it follows that if there
is a polynomial-time algorithm to settle the following uniform version of the
subpower membership problem, then MINORITY$^{id}$ will be polynomial-time
decidable.

- INPUT: A finite algebra $A$ that has a Maltsev term as a basic oper-
ation, and $\vec{a}_1, \ldots, \vec{a}_k, \vec{b} \in A^n$.
- QUESTION: Is $\vec{b}$ in the subalgebra of $A^n$ generated by $\{\vec{a}_1, \ldots, \vec{a}_k\}$?
While not explicitly stated in [11], it can be seen there that this uniform version of the problem lies in the complexity class \( \mathbf{NP} \) but its actual complexity is currently unknown.

It might be that a new algorithm for this problem will place \( \text{MINORITY}^{Id} \) into \( \mathbf{P} \). There has certainly been progress on the algorithmic side of \( \text{SMP} \); a major recent paper has shown in particular that \( \text{SMP}(A) \) is tractable for \( A \) with cube term operations (of which a Maltsev term operation is a special case) as long as \( A \) generates a residually small variety [2] (the statement from the paper is actually stronger than this, allowing multiple algebras in place of \( A \)).

In Section 2 we introduced the notion of a minority-majority term and showed that if testing for such a term for finite idempotent algebras could be done by a polynomial-time algorithm, then \( \text{MINORITY}^{Id} \) would lie in the complexity class \( \mathbf{P} \). This is why we conclude our paper with a question about deciding minority-majority terms.

**Open problem.** What is the complexity of deciding if a finite idempotent algebra has a minority-majority term?

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