Scaling of the dynamics of homogeneous states of one-dimensional long-range interacting systems

T. M. Rocha Filho, A. E. Santana, M. A. Amato, and A. Figueiredo

Instituto de Física and International Center for Condensed Matter Physics
Universidade de Brasília, CP: 04455, 70919-970 - Brasília, Brazil

Abstract

Quasi-Stationary States of long-range interacting systems have been studied at length over the last fifteen years. It is known that the collisional terms of the Balescu-Lenard and Landau equations vanish for one-dimensional systems in homogeneous states, thus requiring a new kinetic equation with a proper dependence on the number of particles. Here we show that previous scalings described in the literature are due either to small size effects or the use of improper variables to describe the dynamics. The correct scaling is proportional to the square of the number of particles and deduce the kinetic equation valid for the homogeneous regime and numerical evidence is given for the Hamiltonian Mean Field and ring models.

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Classical systems with long range interaction can present unusual properties such as non-ergodicity, anomalous diffusion, aging, non-Gaussian Quasi-Stationary States, negative microcanonical heat capacity and, owing to the known result of the heat capacity being always positive in the canonical ensemble, also ensemble inequivalence [1–4, 9–11]. A pair interaction potential is said to be long ranged if it decays at long distances as \( r^{-\alpha} \) such that \( \alpha \leq d \), with \( d \) the spatial dimension [5]. The dynamics of these systems has essentially three stages. The first one corresponds to a violent relaxation into a Quasi Stationary State (QSS) in a short time roughly independent on the number of particles \( N \). The second stage corresponds to the QSS with a very long relaxation time to thermodynamic equilibrium which diverges with \( N \). Finally the third step corresponds to the final thermodynamic equilibrium. In the thermodynamic limit this final stage is never attained. After the violent relaxation the system may also oscillate around a QSS with an amplitude decreasing with time due to a non-linear Landau damping [12]. The slowly varying state remains Vlasov stable (see Ref. [30] and references therein) and in some cases may loose its stability and rapidly evolves into another QSS and thence resuming the slow evolution towards equilibrium. This slow dynamics of the QSS have been extensively studied in the literature for different systems such as the Hamiltonian Mean Field (HMF) model [13–17] define by the Hamiltonian

\[
H = \frac{1}{2} \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{N} \sum_{i<j=1}^{N} \left[ 1 - \cos(\theta_i - \theta_j) \right],
\]

(1)

where \( \theta_i \) is the position angle of particle \( i \) on a circle and \( p_i \) its conjugate momentum, the Ring model with Hamiltonian [18]:

\[
H = \frac{1}{2} \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{N} \sum_{i<j=1}^{N} \frac{1}{\sqrt{2} \sqrt{1 - \cos(\theta_i - \theta_j) + \epsilon}},
\]

(2)

and also one [19, 23], two [24, 27] and three-dimensional self-gravitating systems [25, 26, 28, 29].

The first stage of the dynamics is described by the Vlasov equation which is exact equation satisfied by the one particle distribution function in the \( N \to \infty \) limit [30, 31]. For finite \( N \) this equation is valid only for short times encompassing the initial violent relaxation, a term coined by Lynden-Bell who first developed a statistical theory for predicting its outcome [32], even though a satisfactory theory is still lacking [10, 33–35]. After this initial stage collisional effects (graininess) accumulate and the Vlasov equation must be corrected
by considering higher order terms in $1/N$ leading to kinetic equations such as the Landau or Balescu-Lenard equations \[36\]. A sensible revision of the known kinetic equations for long-range interacting systems and their deductions, with all proper references, is presented by Chavanis in Refs. \[37–40\]. These equations usually can be obtained from the BBGKY hierarchy by taking into account the contribution from the two-body correlation functions, which are of order $1/N$ \[36, 37\] and therefore lead to a time scale of collisional relaxation proportional to $N$ \[37\]. For three-dimensional gravity the dynamics scales as $N \log N$ (the so-called Chandrasekhar scaling) \[41\].

The Balescu-Lenard equation for a one-dimensional homogeneous system is written as \[36\]:

$$\frac{\partial}{\partial t} f_1(p_1; t) = 2\pi^2 n \frac{\partial}{\partial p_1} \int dp_2 \int dk \frac{k^2 \tilde{V}(k)^2}{|\varepsilon(k, kp_1)|^2} \delta(k(p_1 - p_2)) \partial_{12} f_1(p_1; t) f_1(p_2; t), \quad (3)$$

where $f_1(p_1; t)$ is the one-particle reduced distribution function, $n$ the particle density, $p_i$ the momentum of particle $i$, $\tilde{V}(k)$ the Fourier transform of the pair interaction potential and $\varepsilon(k, kp_1)$ the dielectric function. Setting $\varepsilon(k, kp_1) = 1$ amounts to not considering collective effects and yields the Landau equation. The right-hand side of Eq. (3) vanishes identically due to the Dirac delta function \[42, 43\]. Therefore higher order terms must be kept when truncating the BBGKY hierarchy, leading to a different scaling of the time evolution in a homogeneous state. More recently Sato proposed a derivation of a kinetic equation for one-dimensional homogeneous systems by summing contributions of all orders in the hierarchy \[44\]. Unfortunately his approach is limited to dilute gases and is not relevant for the problems at hand here. It would be natural to expect that in this case the predominant collisional corrections to the Kinetic equation come from higher order proportional to $1/N^2$ implying a relaxation scaling proportional to $N^2$. On the other hand different scalings proportional to $N^{1.7}$ and exp$(N)$ were reported in Refs. \[13, 14\] for different initial conditions while the a $N^2$ scaling was estimated numerically for one-dimensional plasmas \[45, 46\]. The authors obtained in Ref. \[30\] for the HMF model and a different type of initial condition a scaling proportional to $N^2$. These facts seems to indicate a dominant contribution from the next term in the $1/N$ expansion. A possible explanation for these discrepancies comes from the fact that the number of particles in the simulations described in \[30\] are much greater than in \[13, 14\], and a possible finite size effect should be carefully investigated. In the present letter we show that indeed the observed scaling $N^{1.7}$ is due to finite size effect (small
and the correct scaling for higher values of $N$ is the more obvious value $N^2$. We illustrate this point from numerical simulations for the HMF model for the type of initial conditions considered in Refs. [13, 14] and varying the number of particles up to larger values than in previous studies. The $\exp(N)$ scaling was obtained in Ref. [14] by considering the dynamics of the phase of the magnetization vector. We show that the conclusions in Ref. [14] are misleading and that looking at the time evolution of higher order momenta of the velocity distribution leads to a correct estimation of the scaling in the dynamics. We also show that the $N^2$ scaling also occurs in homogeneous states for the Ring-model. The scaling of the dynamics of the QSS of the latter model has not been studied before, up to the authors knowledge, due to the difficulty to pinpoint homogeneous QSS with a finite lifetime and the numerical effort necessary for numerical simulations.

To determine the correct scaling for the dynamics we look at the fourth momentum $M_4 = \langle p^4 \rangle$ of the momentum distribution starting from a homogeneous waterbag initial condition defined by

$$f(p; 0) = \begin{cases} 
  1/2p_0 & \text{if } |p| < p_0, \\
  0, & \text{otherwise},
\end{cases}$$

for $p_0$ constant. Even in a QSS $M_4$ varies slowly with time due to the collisional corrections to the Vlasov equation and the dependence of the dynamics on $N$ being the inverse of the collisional term in the Kinetic equation. Figure 1 shows the time evolution of $M_4$ with two different scalings: $N^{1.7}$ and $N^2$ for the same simulation parameters as in Ref. [13]. Clearly the data collapse is better for the former scaling. Nevertheless the number of particles used in Ref. [13] is not so expressive, so we show in Fig. 2 simulations with much larger values of $N$. This time the $N^2$ scaling clearly gives a better data collapse. Figure 3 shows the observed lifetimes of the QSS for the initial conditions in Figs. 1 and 2. In the left pane a scaling close to $N^{1.7}$ is obtained for $N < 10,000$ while the $N^2$ scaling is obtained for $N > 10,000$ in the right hand panel. Note also that the error bars for smaller $N$ are still non-negligible. To address the scaling $\exp(N)$ as presented in Ref. [14] we redo the same simulations with the same type of semi-elliptic initial conditions (see [14] for a description). Figures 4 and 5 show the time evolution of $M_4$ beyond the lifetime of the QSS (in Ref [14] the total simulation time was smaller than the lifetime of the QSS). Again the correct scaling is $N^2$.

We also studied the scaling with time of the lifetime of homogeneous QSS for the ring-model. The kinetic energy for different values of $N$ are shown in Fig. 6. Although some
FIG. 1. Moments $M_4$ as a function of time for the HMF model, with a homogeneous waterbag initial condition with energy per particle $e = 0.69$ and $N = 4,000; 6,000; 8,000; 10,000$. Left Panel: the time was rescaled as $t \rightarrow t/(N \times 10^{-3})^{1.7}$. Right Panel: time rescaled as $t \rightarrow t/(N \times 10^{-3})^2$.

important fluctuations are still present the $N^2$ scaling is also observed in this case. All the numeric data here presented points out to a kinetic equation obtained from the next order correction in $N^{-1}$, which implies the observed $N^2$ scaling for homogeneous states with $N$ sufficiently large. Other values for the scaling exponent are due either to small size effects or a misinterpretation of numerical data.

The kinetic equation for a homogeneous one-dimensional system can be obtained starting from the BBGKY hierarchy [48]:

$$
\left( \frac{\partial}{\partial t} - \hat{L}_s \right) = (N - s) \sum_{i=1}^{s} \frac{\partial}{\partial p_i} \int G_{i,s+1} f_{s+1} d(s + 1), \quad (5)
$$

where $f_s(x_1, \ldots, x_s; t) \equiv f_s(x_1, p_1, \ldots, x_s, p_s; t)$ is the normalized $s$-particle reduced distribution, $x_i$ and $p_i$ the position and momentum variables of particle $i$, respectively, $di \equiv dx_i dp_i$ and $G_{i,j} = -\partial V(x_i - x_j)/\partial x_i$ the force on particle $i$ due to particle $j$, with $V(x_i - x_j)$ the pair interaction potential. The Liouvillian operator $\hat{L}_s$ in Eq. (5) is defined by

$$
\hat{L}_s = -\sum_{i=1}^{s} \hat{k}_i + \sum_{i<j=1}^{2} \hat{O}_{i,j}, \quad (6)
$$

$$
\hat{k}_i = \frac{p_i}{m} \frac{\partial}{\partial x_i}, \quad \hat{O}_{i,j} = -G_{i,j} \partial_{12}, \quad \partial_{12} \equiv \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_j}. \quad (7)
$$

By using the Kac prescription [49] the pair interaction potential can be written as $V(x_i - x_j) = N^{-1}v(x_i - x_j)$ with $v$ independent of $N$. The hierarchy for large $N$ (and small $s$) can
FIG. 2. Moments $M_4$ as a function of time for the HMF model, with a homogeneous waterbag initial condition with energy per particle $e = 0.69$ and $N = 20,000; 40,000; 60,000; 80,000; 100,000$. a) Time is rescaled as $t \rightarrow t/(N \times 10^{-3})^{1.7}$. b) Time window corresponding to the duration of the QSS with the same time scaling as (a). c) Time rescaled as $t \rightarrow t/(N \times 10^{-3})^{2}$. d) Time window corresponding to the duration of the QSS with the same time scaling as (a). The graphics shows clearly that for larger number of particles the correct scaling is $N^2$.

then be written as:

$$\left( \frac{\partial}{\partial t} - \hat{K}_s + \frac{1}{N} \hat{\Theta}_s \right) f_s = \hat{I}_s f_{s+1},$$

(8)

with

$$\hat{K}_s = \sum_{i=1}^{s} \hat{k}_i, \quad \hat{\Theta}_s = - \sum_{i<j=1}^{s} g_{ij} \partial_{ij},$$

(9)

$$g_{ij} = - \frac{\partial}{\partial x_i} v(x_i - x_j), \quad \hat{I}_s = - \sum_{i=1}^{s} \frac{\partial}{\partial p_i} \int d(s+1) g_{i,s+1}.$$

(10)

In order to obtain a closed equation for $f_1$ we must determine the two-particle correlation.
FIG. 3. Left Panel: Life-times for the homogeneous waterbag state of the HMF model with energy per particle $e = 0.69$. The best fit yields a power law scaling $N^{1.76}$. Right Panel: same as the left panel but with greater number of particles and a best fit scaling of $N^{1.99}$.

FIG. 4. Moments $M_4$ as a function of time for the HMF model, with a homogeneous semi-elliptic initial condition with energy per particle $e = 0.69$ and $N = 20,000; 30,000; 50,000; 100,000$. Left Panel: the time was rescaled as $t \rightarrow t/(N \times 10^{-3})^{1.7}$. Right Panel: time rescaled as $t \rightarrow t/(N \times 10^{-3})^2$.

function. Two-particle correlations require the interaction of the two particles, which is of order $N^{-1}$. For three-particle correlations a two particle interaction between say particles 1 and 2, and then between particles 2 and 3 is required, and therefore this correlation function if or order $N^{-2}$, and so on. The irreducible cluster representation (correlation expansion)
FIG. 5. Fourth moment $M_4$ of the velocity distribution as a function of time for the HMF model, with a homogeneous semi-elliptic initial condition with energy per particle $e = 0.8$ and $N = 20,000; 30,000; 50,000; 100,000$. The time variable is rescaled as $t \rightarrow t/(N/2 \times 10^4)^2$ revealing the data collapse.

for the reduced distribution functions $f_s$ up to $s = 3$ is given by [36, 48]:

\begin{align}
    f_2(1, 2) &= f_1(1)f_1(2) + C_2(1, 2), \\
    f_3(1, 2, 3) &= f_1(1)f_1(2)f_1(3) + \sum_{P(1,2,3)} f_1(1)C_2(2, 3) + C_3(1, 2, 3),
\end{align}

where $\sum_{P(1,2,3)}$ stands for the sum of the different permutations between particles 1, 2 and 3. Note that since the reduced distributions are taken as fully symmetric by the permutation of any two particles, the same is valid for the pure correlation functions $C_2$ and $C_3$. We thus write:

\begin{align}
    C_2(i, j) &= \frac{1}{N} C_2^{(1)}(i, j) + \frac{1}{N^2} C_2^{(2)}(i, j) + O(\frac{1}{N^3}), \\
    C_3(i, j, l) &= \frac{1}{N^2} C_3^{(3)}(i, j, l).
\end{align}

The Vlasov and Landau equations are obtained at order $N^0$ and $N^{-1}$, respectively. Plugging Eqs. (12), (13) and (14) in Eq. (8) for $s = 1, 2$, and equating equal powers of $N$ on each side yields in a cumbersome but quite straightforward way the equations for the coefficients in Eq. (13) and (14). We first note that some preliminary simplifications are possible. Since the states are always homogeneous here we have that

\begin{align}
    \hat{K}_s f_1(i) &= 0.
\end{align}
On the other hand, the vanishing of the collisional integral in the Landau equation is equivalent to \( \hat{I}_2 C_2^{(1)}(1, 2) = 0 \). The mean field force always vanishes in a homogeneous state, and therefore the action of \( \hat{I}_s \) on any product of any number of functions \( f_1^{(n)}(i) \) and \( C_2^{(1)}(j, l) \) also vanishes. In this way, the dynamics of the one-particle distribution function is given by the second order correction in \( N^{-1} \):

\[
\frac{\partial}{\partial t} f_1(1) = \frac{1}{N^2} \hat{I}_1 C_2^{(2)}(1, 2) .
\]  

(16)

At order \( N^{-1} \) we obtain:

\[
\frac{\partial}{\partial t} C_2^{(1)}(1, 2) + \hat{K}_2 C_2^{(1)}(1, 2) - \hat{\Theta}_2 f_1(1) f_1(2) = 0 ,
\]  

(17)

\[
\sum_{P(1,2,3)} \left[ \frac{\partial}{\partial t} C_2^{(1)}(1, 2) \right]_{f_1(3)} + \sum_{P(1,2,3)} \left[ \hat{K}_2 C_2^{(1)}(1, 2) \right]_{f_1(3)}
\]

\[
- \sum_{P(1,2,3)} \left[ \hat{\Theta}_3 f_1(1) \right]_{f_1(2) f_1(3)} = 0 .
\]  

(18)

It is easily checked out that Eq. (18) ensues from Eq. (17). At order \( N^{-2} \) we have:

\[
\frac{\partial}{\partial t} C_2^{(2)}(1, 2) + \hat{K}_2 C_2^{(2)}(1, 2) - \sum_{P(1,2,3)} \hat{I}_2 \left[ f_1(1) C_2^{(2)}(2, 3) \right] - \hat{\Theta}_2 C_2^{(1)}(1, 2)
\]

\[
+ \sum_{P(1,2)} \left[ \frac{\partial}{\partial t} f_1(1) \right]_{f_1(2)} - \hat{\Theta}_2 \left[ f_1(1) f_1(2) \right] = 0 .
\]  

(19)

Another equation is obtained at this order, but involves \( C_3^{(2)}(1, 2, 3) \) and as we are going to show the determination of the three-particle correlation is not required here. From Eq. (16) and the definition of \( \hat{I}_2 \) in Eq. (10) it also straightforward to show that the third and fifth terms in Eq. (19) cancel mutually (the error is of order \( N^3 \)) and we are left with:

\[
\frac{\partial}{\partial t} C_2^{(2)}(1, 2) + \hat{K}_2 C_2^{(2)}(1, 2) - \hat{\Theta}_2 C_2^{(1)}(1, 2) - \hat{\Theta}_2 \left[ f_1(1) f_1(2) \right] = 0 .
\]  

(20)

The strategy to obtain the kinetic equation up to order \( N^{-2} \) is thus to solve equation (17) for \( C_2^{(1)}(1, 2) \), and then solve Eq. (20) for \( C_2^{(2)}(1, 2) \). Thence the kinetic equation is obtained from Eq. (16).

At the lowest order the dynamics of the system corresponds to a vanishing mean force field and thus \( \dot{\mathbf{p}}_i = \mathcal{O}(N^{-1}) \) and \( \mathbf{x}_i(t) = \mathbf{x}_i(t_0) + (\mathbf{p}_i/m)(t - t_0) + \mathcal{O}(N^{-1}) \) (ballistic motion) and thus:

\[
\frac{d}{dt} C_2^{(1)}(1, 2; t) = \left[ \frac{\partial}{\partial t} + \frac{p_1}{m} \frac{\partial}{\partial x_1} + \frac{p_2}{m} \frac{\partial}{\partial x_2} \right] C_2^{(1)}(1, 2; t) = \left[ \frac{\partial}{\partial t} + \hat{K}_2 \right] C_2^{(1)}(1, 2; t) .
\]  

(21)
Equation (17) is then rewritten as:

\[
\frac{d}{dt} C^{(1)}_2 (1, 2) = \dot{\Theta}_2 f_1 (1) f_1 (2),
\]  

with solution

\[
C^{(1)}_2 (1, 2; t) = \int_{t_0}^{t} dt' \dot{\Theta}_2 f_1 (1; t) f_1 (2; t) + C^{(1)}_2 (1, 2; t_0)
\]

\[
= - \int_{t_0}^{t} dt' g_{12} (x_1 (t') - x_2 (t')) \partial_{12} f_1 (p_1; t') f_1 (p_2; t') + C^{(1)}_2 (1, 2; t_0). \tag{23}
\]

We now assume that particles are uncorrelated at \( t = t_0 \) and put \( C_2 (1, 2; t_0) = 0 \). Similarly Eq. (20) can be written as:

\[
\frac{d}{dt} C^{(2)}_2 (1, 2; t) = \dot{\Theta}_2 C^{(1)}_2 (1, 2; t) + \dot{\Theta}_2 [f_1 (1) f_1 (2)] = 0,
\]  

with solution

\[
C^{(2)}_2 (1, 2; t) = - \int_{t_0}^{t} dt'' g_{12} (x_1 (t'') - x_2 (t'')) \partial_{12} (f_1 (p_1, t'') f_1 (p_2, t''))
\]

\[
- \int_{t_0}^{t} dt'' g_{12} (x_1 (t'') - x_2 (t'')) \partial_{12} \int_{t_0}^{t'} dt' g_{12} (x_1 (t') - x_2 (t')) \partial_{12} f_1 (p_1; t') f_1 (p_2; t'), \tag{25}
\]

where \( g_{12} (x_1 (t) - x_2 (t)) = g_{12} (x_1 - v_{12} (t - t_0)), \) \( x_{12} = x_1 (t_0) - x_2 (t_0) \) and \( v_{12} = (p_1 - p_2)/m \).

We note that the action of \( \dot{I}_1 \) on the first term on the right-hand side of Eq. (25) vanishes and therefore only the second term contributes to the kinetic equation as obtained from Eq. (16):

\[
\frac{\partial}{\partial t} f_1 (p_1; t) = - \frac{1}{N^2} \frac{\partial}{\partial p_1} \int dx_2 dp_2 g (x_1 (t) - x_2 (t)) C^{(2)}_2 (1, 2; t). \tag{26}
\]

To show that the right hand side of the last equation does indeed yield a non-vanishing contribution, we first use Eq. (25) in Eq. (26), take the limit \( t_0 \to \infty \) and then change variables as \( \tau'' = t - t'' \) and \( \tau' = t - \tau' - t' \):

\[
\frac{\partial}{\partial t} f_1 (p_1; t) = - \frac{1}{N^2} \frac{\partial}{\partial p_1} \int dp_2 [G_1 \partial_{12}^2 + G_2 \partial_{12}] f_1 (p_1; t) f_1 (p_2; t), \tag{27}
\]

with

\[
G_1 = \int dx_2 \int_0^{\infty} d\tau'' \int_0^{\infty} d\tau' g_{12} (t) g_{12} (t - \tau' - \tau''),
\]

\[
G_2 = \int dx_2 \int_0^{\infty} d\tau'' \int_0^{\infty} d\tau' g_{12} (t) g_{12} (t - \tau' \tau''),
\]

where \( g_{12} (t) \equiv g_{12} (x_1 (t) - x_2 (t)) \). Now taking the Fourier transforms:

\[
g_{12} (x) = - \frac{1}{2\pi} \int dk \, ik \, e^{ikx} \tilde{\nu} (k), \quad \tilde{\nu} (k) = \int dk \, e^{-ikx} \nu (x), \tag{30}
\]
with \( p_0 = 0.59 \) \((f(p) = 1/2p_0\) if \(-p_0 < p < p_0\), for a few different numbers of particles. The time is rescaled as \( t \rightarrow t/(N/10240)^2 \).

and using the ballistic approximation in expressions such as \( g_{12}(t - \tau'') = g_{12}(x_1(t) - x_2(t) - v_{12}\tau'') \), \( v_{12} \equiv v_1 - v_2 \), which yields an error or order \( N^{-3} \), we obtain after some manipulations:

\[
    G_1 = -\frac{i}{4} \int dk\,dk' \, kk' \, \delta_+((k + k')v_{12}) \delta_+((k + k')v_{12}) \tilde{v}(k)\tilde{v}(k')\tilde{v}(k + k'),
\]

\[
    G_2 = \frac{i}{2v_{12}} \int dk\,dk' \frac{\partial}{\partial k} \left[ kk' \, \delta_+((k + k')v_{12}) \delta_+((k + k')v_{12}) \right] \tilde{v}(k)\tilde{v}(k')\tilde{v}(k + k'),
\]

In Eqs. (31) and (32) we used the Cauchy integral \( \int_0^\infty dk \exp(ikx) = \pi\delta_+(x) = \pi\delta(x) + i\text{P}(1/x) \) where the last term in the principal part of \( 1/x \) [36]. Since only the real parts of \( G_1 \) and \( G_2 \) contribute to Eq. (27), we finally obtain:

\[
    \frac{\partial}{\partial t} f_1(p_1; t) = -\frac{\pi}{4N^2} \frac{\partial}{\partial p_1} \int dp_2 \int dk \mathcal{P} \left( \frac{1}{k v_{12}} \right) \left[ k^4\tilde{v}(k)\tilde{v}(k')\tilde{v}(k + k')\partial_{12}^2 \right. \\
    \left. + \frac{\partial}{\partial k} \left( k^4\tilde{v}(k)\tilde{v}(k')\tilde{v}(k + k') \right) \partial_{12} \right] f_1(p_1; t) f_1(p_2; t),
\]

which is our final form of a kinetic equation obtained in a \( 1/N \) expansion for one-dimensional homogeneous systems. It explains the \( N^2 \) time scaling observed for the dynamics of the HMF and ring models as presented here and closes a long-standing debate in the community studying long-range interacting systems.

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