A New Metatheorem and Subdirect Product Theorem for $L$-Subgroups

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ABSTRACT

This paper is a continuation of the work of Tom Head 'Metatheorem for deriving fuzzy theorems from crisp versions'. The concept of natural extension is introduced which is then applied in the development of a new metatheorem in $L$-setting, where $L$ is a complete chain. The application of this theorem is demonstrated through the notions of generated $L$-subgroups and commutator $L$-subgroups. Moreover, a new subdirect product theorem is developed wherein it is demonstrated that for a group $G$, the $L$-subgroup lattice can be represented as a subdirect product of copies of its associated lattice of crisp subgroups. The significance of this theorem is exhibited by applying it to a characterization of generalized cyclic groups in terms of the distributivity of the lattice of $L$-subgroups of a group.

1. Introduction

Lotfi Zadeh in his seminal paper introduced the concept of a fuzzy set along with his well known extension principle [1]. The extension principle was utilized by researchers and mathematicians in all the disciplines of fuzzy mathematics and fuzzy logic in order to obtain images and pre-images of fuzzy sets under the action of various types of mappings. In the year 1995, Prof. Tom Head in his pioneering paper [2] formulated the concept of convolutional extension which is similar in nature and importance to that of Zadeh's extension principle. Using this notion and the notion of Rep function which is, in fact, based on the notion of strong level subsets, Tom Head formulated his well known metatheorem and subdirect product theorem. These theorems were in turn used to obtain fuzzy versions of numerous results of classical algebra. This made number of published research papers redundant in the area of fuzzy algebraic structures.

For any algebra $X$, Tom Head by using convolutional method very conveniently and elegantly mirrored any $n$-ary operation of $X$ to its fuzzy power set $F(X)$, power set $P(X)$, crisp set $C(X)$ and finally to the cartesian product $C(X)^J$ where $J = [0, 1)$. The intricate relationship and interplay of these subsets produced some amazing results which are formulated in the form of metatheorem and subdirect product theorems. The purpose of the formulation of these results was to obtain the fuzzy versions of the corresponding crisp...
results of algebra. Tom Head successfully accomplished this task and demonstrated how to extend the results of semigroup to fuzzy setting by an application of metatheorem. These results were earlier established by several researchers. It was observed by Tom Head that in the process of extending the results, the arguments used are having a lot of reasoning in common. In the formulation of the metatheorem this sort of situation is taken care of and this common reasoning is incorporated in the structure of the proof of the theorem.

Unfortunately after the emergence of metatheorem, not much attention has been paid on its further development or its application in other areas of algebra such as group theory and ring theory. There are only few papers related to this topic [3–6]. In [3], a new subdirect product theorem for fuzzy congruences is established. This result along with the subdirect product theorem established by Tom Head is applied to prove that the lattice of fuzzy congruences is isomorphic to the lattice of fuzzy normal subgroups. In [6], a special case of metatheorem for semi groups is stated and some results of semigroups from classical algebra are transferred to fuzzy setting. In the course of this development, in his erratum [7], Tom Head also defined a very important concept of tip extended pair of fuzzy subgroups in order to formulate the join of two fuzzy normal subgroups of a group. Using this technique for constructing the join, A. Jain [5] has provided a much simpler and direct proof of modularity of the lattice of fuzzy normal subgroups. In the similar manner, I. Jahan [4] devised the join of two \( L \)-ideals by using the notion of tip extended pair of \( L \)-ideals and established the modularity of the lattice of \( L \)-ideals of a ring which was an open problem. Here in this case the metatheorem or the subdirect product theorem were not applicable.

The metatheorem of Tom Head is successfully applied in obtaining the fuzzy versions of the results of a semigroup. However, in extending the results from group theory to fuzzy setting by an application of metatheorem, certain expressions cannot be formulated. It is due to the fact that the operations involved in these expressions are not convolutional extensions, they are rather binary or unary operations on the power set of the given group. In order to overcome this limitation, we have introduced the concept of a natural extension. A natural extension extends an operation (unary or binary) from the power set \( P(G) \) of a group \( G \) to the set \( L^{G} \) of \( L \)-subsets in \( G \). This natural extension also extends a concept from a class \( C \) of crisp subsets of \( G \) to the class \( \mathcal{C} \) of \( L \)-subsets in \( G \) generically. Further, this notion is used in the development of a new metatheorem in \( L \)-setting when the lattice \( L \) is a complete chain. Subsequently, this metatheorem is applied to the notions of generated \( L \)-subgroups and commutator \( L \)-subgroups. Tom Head has established a subdirect theorem for a restricted class of fuzzy subgroups namely fuzzy normal subgroups. In this work, a more general representation theorem for the lattice \( \mathcal{L}(G) \) of \( L \)-subgroups of a group \( G \) is established when the lattice \( L \) is a complete chain. Tom Head’s representation function is being used to demonstrate that the lattice \( \mathcal{L}(G) \) can be represented as a subdirect product of the copies of lattice of crisp subgroups \( \mathcal{L}_{c}(G) \). In order to do this, firstly the sup operation for an arbitrary family of \( L \)-subgroups is defined. Then it is shown that the function \( \text{Rep} \) commutes with arbitrary sups in the lattice \( \mathcal{L}(G) \). Our subdirect product theorem follows as a consequence of this commutation. In a forthcoming paper, we demonstrate an application of metatheorem for transferring the results of nilpotent groups, solvable groups and related concepts to \( L \)-setting.
2. Preliminaries

We briefly review here some of the basic concepts and definitions which will be used in the sequel [2,8–14].

The system \((L, \leq, \lor, \land)\) denotes a complete chain, where \(\leq\) denotes the partial ordering of \(L\), the join (sup) and the meet (inf) of the elements of \(L\) are denoted by \(\lor\) and \(\land\), respectively. Also, we write 1 and 0 for the maximal and the minimal elements of \(L\), respectively.

An \(L\)-set in a non empty set \(X\) is a function from \(X\) into \(L\). The set of \(L\)-sets in \(X\) is called the \(L\)-power set of \(X\) and is denoted by \(L^X\). For \(\mu \in L^X\), the set \(\{\mu(x) : x \in X\}\), denoted by \(\text{Im} \mu\), is called the image of \(\mu\) and the tip of \(\mu\), denoted by \(\text{sup} \mu\), is defined as \(\bigvee_{x \in X} \mu(x)\). The containment of an \(L\)-set \(\mu\) in an \(L\)-set \(\eta\) is defined as usual and is denoted by \(\subseteq\). We use the same notation \(\subseteq\) for the ordinary set inclusion. The union and the intersection of a family of \(L\)-sets \(\{\mu_i\}_{i \in \lambda}\), where \(\lambda\) is a non empty indexing set, are defined as usual and are denoted by \(\bigcup_{i \in \lambda} \mu_i\) and \(\bigcap_{i \in \lambda} \mu_i\), respectively. For an \(L\)-set \(\mu\) in a set \(X\) and \(a \in L\) the notions of a level subset and a strong level subset \(\mu > a\) are, respectively, defined by:

\[
\mu_a = \{x \in X \mid \mu(x) \geq a\} \quad \text{and} \quad \mu_a^> = \{x \in X \mid \mu(x) > a\}.
\]

Let \(\{\mu_i\}_{i \in \lambda}\), where \(\lambda\) is a non empty indexing set, be a family of \(L\)-sets in \(X\). Then

\[
\left(\bigcup_{i \in \lambda} \mu_i\right)_a^> = \bigcup_{i \in \lambda} (\mu_i)_a^>, \quad \forall a \in L \sim \{1\}.
\]

Hence onwards \(G\) denotes a group with the identity element \(e\) and \(\lambda\) denotes a non empty indexing set. Also, for a non empty set \(X\), by \(1_A\) we shall denote the characteristic function of any subset \(A \subseteq X\).

**Definition 2.1:** Let \(\mu \in L^G\). Then \(\mu\) is said to be an \(L\)-subgroup of \(G\), if the following conditions are satisfied:

(i) \(\mu(xy) \geq \min(\mu(x), \mu(y))\) \forall x, y \in G.

(ii) \(\mu(x^{-1}) = \mu(x)\) \forall x \in G.

If \(\mu\) is an \(L\)-subgroup, then it attains supremum at the identity \(e\) of \(G\). We shall call \(\mu(e)\) to be the tip of \(\mu\). The set of all \(L\)-subgroups of \(G\) is denoted by \(\mathcal{L}(G)\).

**Definition 2.2:** Let \(\mu \in \mathcal{L}(G)\). Then \(\mu\) is said to be an \(L\)-normal subgroup of \(G\) if \(\mu(xy) = \mu(yx), \forall x, y \in G\).

The set of all \(L\)-normal subgroups of \(G\) is denoted by \(\mathcal{L}_n(G)\).

**Definition 2.3:** Let \(\mu \in L^G\). Then the least \(L\)-subgroup of \(G\) containing \(\mu\) is called the \(L\)-subgroup of \(G\) generated by \(\mu\) and is denoted by \(\langle \mu \rangle\).

We use the same notation \(\langle A \rangle\) for the subgroup generated by an ordinary subset \(A\) of \(G\). By \(L(G)\), we shall denote the set of all subgroups of \(G\) and by \(L_n(G)\), the set of all normal subgroups of \(G\). By \(\mathcal{L}_c(G)\), let us denote the set of all characteristic functions of the members
of $L(G)$. On the other hand, by $L_{nc}(G)$, we shall denote the set of all characteristic functions of the members of $L_n(G)$.

**Proposition 2.4:** Let $A \subseteq G$ be any subset. Then $A \in L(G)$ if and only if $1_A \in L(G)$. Thus $L_c(G) \subseteq L(G)$.

**Proposition 2.5:** Let $A \subseteq G$ be any subgroup. Then $A \in L_n(G)$ if and only if $1_A \in L_n(G)$. Thus $L_{nc}(G) \subseteq L_n(G)$.

It is well known that for any group $G$, $L(G)$ forms a complete lattice under the ordering of set inclusion $\subseteq$, whereas $L_n(G)$ constitute a complete sublattice of $L(G)$. On the other hand, $L(G)$ and $L_n(G)$ form complete lattices under the ordering of $L$-set inclusion. The meet and the join operations for a family $\{\mu_i\}_{i \in \lambda}$ of $L$-subgroup of $G$ are, respectively, defined by:

\[
\bigwedge_{i \in \lambda} \mu_i = \bigcap_{i \in \lambda} \mu_i \quad \text{and} \quad \bigvee_{i \in \lambda} \mu_i = \left\langle \bigcup_{i \in \lambda} \mu_i \right\rangle,
\]

where $\left\langle \bigcup_{i \in \lambda} \mu_i \right\rangle$ is the $L$-subgroup of $G$ generated by the union of the family $\{\mu_i\}_{i \in \lambda}$. Further recall the following result from [15,16]:

**Theorem 2.6:** Let $\mu \in L^G$. Define an $L$-set $\mu^*$ in $G$ by

\[
\mu^*(x) = \bigvee_{a \leq \sup \mu} \{a : x \in \left\langle \mu_a \right\rangle\}.
\]

Then $\mu^* = \left\langle \mu \right\rangle$.

In view of the above theorem, the following is easy to verify:

**Proposition 2.7:** Let $\mu \in L^G$ and $\mu(xy) = \mu(yx) \forall x, y \in G$. Then $\left\langle \mu \right\rangle \in L_n(G)$.

### 3. Natural Extensions

For a non empty set $X$, let $C(X)$ denote the set of all crisp subsets of $X$, that is, the set of all functions from $X$ to \{0, 1\} where ‘0’ is the minimal element and ‘1’ is the maximal element of $L$. Let $P(X)$ be the ordinary power set of $X$. It is well known that $P(X)$ is a complete lattice under the ordering of set inclusion where arbitrary join and meet are, respectively, the union and the intersection of an arbitrary family of subsets of $X$. Also, $L^X$ is a complete lattice under the usual ordering of $L$-set inclusion $\subseteq$ and $C(X)$ is a complete sublattice of $L^X$. Recall that the function $\chi : P(X) \to C(X)$ defined by $\chi(A) = 1_A$, is a bijection from $P(X)$ onto $C(X)$. Moreover, the following facts can be verified:

1. The function $\chi$ commutes with arbitrary infs of $P(X)$ and $C(X)$.
2. The function $\chi$ commutes with arbitrary sups of $P(X)$ and $C(X)$.

As a consequence of the above facts, the Boolean lattice $P(X)$ is isomorphic with $C(X)$. The Rep function plays a key role in the development of this exposition. We recall the following from [2]:
Definition 3.1: For a non empty set \( X \), let \( \text{Rep} : L^X \to C(X)^J \) where \( J = L \sim \{1\} \), be defined for each \( \mu \in L^X \), for each \( a \in J \) and \( x \in X \), by

\[
\text{Rep} \mu(a)(x) = \begin{cases} 
0 & \text{if } \mu(x) \leq a, \\
1 & \text{otherwise}.
\end{cases}
\]

It can be established that \( \text{Rep} \mu \) is an injective function. Also, \( \text{Rep} \mu \) is an order reversing function from \( J \) into \( C(X) \). The ordering of \( C(X) \) induces a pointwise partial order on \( C(X)^J \). That is for \( P \) and \( Q \) in \( C(X)^J \)

\[
P \leq Q \text{ if and only if } P(a) \leq Q(a) \quad \forall \ a \in J.
\]

Further as \( C(X) \) is a lattice, \( C(X)^J \) is also a lattice. Tom Head enlarged the domain of a binary operation on \( G \) to \( LG \) by convolutional extension method. Also, the binary operation \( * \) on \( G \) induces a binary operation \( * \) on \( P(G) \) and \( C(G) \).

Definition 3.2: Let \( \langle G,^{-1}, * \rangle \) be a group. Then the convolutional extension of \( * \) from \( G \) to \( LG \) is defined by

\[
\eta * \theta(x) = \bigvee_{x = * (y, z)} \{ \eta(y) \land \theta(z) \} \quad \forall \ x \in G \text{ and } \forall \ \eta, \theta \in LG,
\]

where \( *(y, z) \) is an expression in \( G \) obtained by using the elements \( y, z \) or their inverses.

The above extension of the binary operation on \( G \) provides us the well known product of fuzzy sets introduced by Liu [11].

As pointed out earlier that there are certain expressions formed on a power set of a group \( G \) involving unary or binary operations of \( P(G) \) which cannot be extended by convolutional extension method. Thus the metatheorem cannot be applied to establish the results which involve such type of expressions. In order to overcome this limitation, we introduce the concept of natural extension. The process is the same as that of defining a fuzzy \( \mathcal{C} \)-class from the corresponding \( \mathcal{C} \)-class of crisp subsets by generic method. Let us first exhibit that how a unary operation \( \tilde{\cdot} \) on \( P(G) \) can be extended to provide a natural extension or generic extension on \( L^G \).

Let \( \langle G,^{-1}, * \rangle \) be a group and \( \tilde{\cdot} : P(G) \to P(G) \) be any unary operation. Then the domain of the unary operation can be extended to \( L^G \) by defining \( \tilde{\cdot} : L^G \to L^G \) generically as follows:

Let \( \mu \in L^G \) and \( x \in G \). Then \( \hat{\mu} \in L^G \) is given by

\[
\hat{\mu}(x) = \bigvee_{a < \sup \mu} \{ a : x \in (\mu_{a}) \}.
\]

Further if \( \tilde{\cdot} : A \to \hat{A} \) where \( \hat{A} = \langle A \rangle \), then

\[
\hat{\mu}(x) = \bigvee_{a < \sup \mu} \{ a : x \in (\mu_{a}) \}.
\]
It is interesting to see that the \( L \)-subset \( \hat{\mu} \) coincide with the \( L \)-subset \( \mu^{**} \) discussed by Ajmal [3,15] who also established that if

\[
\mu^*(x) = \bigvee_{a \leq \sup \mu} \{a : x \in \langle \mu_a \rangle\} \quad \text{and} \quad \mu^{**}(x) = \bigvee_{a < \sup \mu} \{a : x \in \langle \mu^>_a \rangle\},
\]
then \( \mu^* = \mu^{**} = \langle \mu \rangle \). That is \( \mu^* = \hat{\mu} = \langle \mu \rangle \).

Next we formulate the concept of natural extension for a binary operation on \( P(G) \).

**Definition 3.3:** Let \( \langle G,^{-1},* \rangle \) be a group and \( \circ : P(G) \times P(G) \rightarrow P(G) \) be any binary operation. Then the domain of the binary operation \( \circ \) can be extended to \( L^G \) by defining \( \circ : L^G \times L^G \rightarrow L^G \) as follows:

Let \( \eta, \theta \in L^G \) and \( x \in G \). Then \( \eta \circ \theta \in L^G \) is given by

\[
\eta \circ \theta(x) = \bigvee_{a < \sup \eta \land \sup \theta} \{a : x \in \eta^>_a \circ \theta^>_a \}.
\]

The binary operation \( \circ \) on \( L^G \) so obtained is called the natural extension or generic extension of the binary operation \( \circ \) on \( P(G) \). A natural extension is said to be commutative if the Rep function commutes with it.

A natural extension of a unary \( \sim \) or a binary operation \( \circ \) from \( P(G) \) to \( L^G \) induces an operation on \( C(G) \) as follows:

\[
\hat{1}_A = 1_\hat{A} \quad \text{and} \quad 1_A \circ 1_B = 1_{A\cap B}.
\]

This in turn provides a pointwise operation on \( C(G)^J \).

**Proposition 3.4:** For each unary operation \( \sim \) or a binary operation \( \circ \) on \( P(G) \), the crisp set \( C(G) \) is closed with respect to that natural extensions of \( \sim \) or \( \circ \) to \( L^G \). The bijection \( \chi_i : P(G) \rightarrow C(G) \) commutes with operations \( \sim \) or \( \circ \) on \( P(G) \) and \( C(G) \).

As demonstrated, any finitely many unary or binary operations say \( \circ_1, \circ_2, \ldots, \circ_k \) where \( k \geq 1 \) on \( P(G) \) extend to operations on \( C(G), L^G \) and \( C(G)^J \). Also, the binary operation \( \ast \) on \( G \) extends to these sets by convolutional extension method. Moreover, \( P(G) \) has two well known binary operations \( \cap = \inf \) and \( \cup = \sup \). Thus \( C(G), L^G \) and \( C(G)^J \) become algebras having the corresponding operations \( \cap = \inf, \cup = \sup, \ast, \circ_1, \circ_2, \ldots, \circ_k \).

It can be verified easily that the function \( \text{Rep} : L^G \rightarrow C(G)^J \) where \( J = L \sim \{1\} \) commutes with the natural extensions of the binary operations union and intersection. As the Rep function is injective, it follows that \( \text{Rep} \) is an order isomorphism of \( L^G \) onto \( I(G) \) where \( I(G) \) is the image of Rep function. Further, for commutative natural extensions, Rep becomes an algebraic isomorphism of \( L^G \) onto \( I(G) \).

If \( L \) is a complete dense chain, then the convolutional extension introduced by Tom Head provides a special instance of commutative natural extension. Moreover, it is easy to verify that the function \( \text{Rep} \) commutes with the natural extension of the unary operation \( \sim : P(G) \rightarrow P(G) \) defined by \( \hat{A} = \langle A \rangle \). That is if \( \mu \in L^G \), then

\[
\text{Rep}(\langle \mu \rangle)(a) = \langle \text{Rep} \mu(a) \rangle \quad \forall \ a \in J. \tag{1}
\]

Below we provide some more examples of commutative natural extensions.
(a) Let a binary operation $\circ : P(G) \times P(G) \rightarrow P(G)$ be defined by $A \circ B = (A \cup B)$. Then the natural extension of $\circ$ to $L^G$ is given by

$$\eta \circ \theta(x) = \bigvee_{a < \sup \eta \land \sup \theta} \{ a : x \in \eta_a^\sup \circ \theta_a^\sup \},$$

and we write $\eta \circ \theta = (\eta \cup \theta)$. This natural extension is commutative as

$$\text{Rep}(\eta \cup \theta)(a) = \text{Rep}(\eta \cup \theta)(a) \quad \text{(by (1))}$$

$$= \left( \text{Rep} \eta(a) \cup \text{Rep} \theta(a) \right).$$

(as the Rep function commutes with the union of L-subsets)

(b) Let a binary operation $\circ : P(G) \times P(G) \rightarrow P(G)$ be defined by $A \circ B = (A, B)$, where $(A, B)$ is the commutator of $A$ and $B$. Then the natural extension of this binary operation $\circ$ to $L^G$ is given by

$$\eta \circ \theta(x) = \bigvee_{a < \sup \eta \land \sup \theta} \{ a : x \in \eta_a^\sup \circ \theta_a^\sup \}$$

and we write $\eta \circ \theta = (\eta, \theta)$. Moreover, it is a commutative natural extension provided $L$ is a dense chain (see Proposition 3.5).

(c) Let a binary operation $\circ : P(G) \times P(G) \rightarrow P(G)$ be defined by $A \circ B = [A, B]$, where $[A, B]$ is the commutator subgroup of $A$ and $B$. Then the natural extension of this binary operation $\circ$ from $P(G)$ $(L(G))$ to $L^G$ $(L(G))$ is given by

$$\eta \circ \theta(x) = \bigvee_{a < \sup \eta \land \sup \theta} \{ a : x \in \eta_a^\sup \circ \theta_a^\sup \}$$

and we write $\eta \circ \theta = [\eta, \theta]$. Note that this natural extension is a composition of the natural extension of the of the unary operation $\hat{\cdot}$ on $P(G)$ given by $\hat{\cdot} = (\hat{\cdot})$ and the binary operation $\circ$ on $P(G)$ given by $A \circ B = (A, B)$. As in a dense chain the operations $\hat{\cdot}$ and $\circ$ are commutative, it follows that the given binary operation is commutative provided $L$ is a dense chain.

**Proposition 3.5:** Let $L$ be a dense chain. Then the function $\text{Rep}$ commutes with the natural extension of the binary operation $\circ : P(G) \times P(G) \rightarrow P(G)$ defined by $A \circ B = (A, B)$ where $(A, B)$ is the commutator of $A$ and $B$.

**Proof:** We have to prove $\text{Rep}(\eta, \theta)(a) = (\text{Rep} \eta(a), \text{Rep} \theta(a)) \forall a \in J$. In view of Proposition 3.4, it is sufficient to prove that

$$\text{Rep}(\eta, \theta)(a)(x) = 1 \text{ if and only if } x \in (\eta_a^\sup, \theta_a^\sup) \forall a \in J \text{ and } \forall x \in G. \quad \blacksquare$$
Let \( a \in J \) and \( x \in G \). Suppose \( \text{Rep}(\eta, \theta)(a)(x) = 1 \). Then by the definition of \( \text{Rep} \) function \( (\eta, \theta)(x) > a \). Hence in view of the definition of natural extension \( \exists a_0 > a \) such that \( x \in (\eta_{a_0}, \theta_{a_0}) \). As \( a_0 > a, \eta_{a_0} \sqsubseteq \eta_{a} \) and \( \theta_{a_0} \sqsubseteq \theta_{a} \). This implies \( x \in (\eta_{a}, \theta_{a}) \). Conversely, let \( x \in (\eta_{a}, \theta_{a}) \). Then \( x = yz^{-1}x^{-1} \) where \( y \in \eta_{a} \) and \( z \in \theta_{a} \) i.e. \( \eta(y) > a \) and \( \theta(z) > a \) so that \( \eta(y) \land \theta(z) > a \). As \( L \) is a dense chain, choose \( r \in L \) such that \( \eta(y) \land \theta(z) > r > a \). Hence

\[
(\eta, \theta)(x) = \bigvee_{b \leq \sup \eta \land \sup \theta} \{ b : x \in (\eta_b, \theta_b) \} > r > a.
\]

Consequently by the definition of \( \text{Rep} \) function \( \text{Rep}(\eta, \theta)(x) = 1 \).

Below we provide a simple proof of the following result from [10]:

**Lemma 3.6:** The function \( \chi : P(G) \to C(G) \) commutes with natural extension of the binary operation \( \circ : P(G) \times P(G) \to P(G) \) defined by \( A \circ B = [A, B] \). That is

\[
1_{[A, B]} = [1_A, 1_B].
\]

**Proof:** Let \( A, B \in P(G) \). By using the Proposition 3.4, we have

\[
1_{[A, B]} = \chi(A, B) = \chi(A, B) = (\chi A, \chi B) = [1_A, 1_B].
\]

\[\square\]

### 4. Metatheorem for \( L \)-Subgroups

For the meaning of an expression over a set of variables ‘Var’ and a set of operations ‘Op’ in any algebra the readers are referred to [2].

Let \( \langle G, ^{-1}, \ast \rangle \) be a group and \( P(G) \) has finitely many binary or unary operation say \( \circ_1, \circ_2, \ldots, \circ_k \) where \( k \geq 1 \). Then \( P(G) \) is an algebra with the operations \( \cap = \inf, \cup = \sup, \ast, \circ_1, \circ_2, \ldots, \circ_k \) where \( \ast \) is a binary operation on \( P(G) \) induced by the binary operation \( \ast \) on \( G \). Then \( C(G) \) and \( L^G \) also become algebras with operations \( \cap = \inf, \cup = \sup, \ast, \circ_1, \circ_2, \ldots, \circ_k \). Let \( E = E(v_1, v_2, \ldots, v_m) \) be an expression over the set of variables \( \text{Var} = \{v_1, v_2, \ldots, v_m\} \) and the set of operations \( \text{Op} = \{\cap = \inf, \cup = \sup, \ast, \circ_1, \circ_2, \ldots, \circ_k\} \). Then for any \( m \) elements \( \theta_1, \theta_2, \ldots, \theta_m \in L^G, E = E(\theta_1, \theta_2, \ldots, \theta_m) \) is the element of \( L^G \) which results when each occurrence of \( v_i \) is replaced by \( \theta_i \) and the result is evaluated in \( L^G \).

**Definition 4.1:** A class \( C \) of \( L \)-sets in a non empty set \( X \) is closed under projections if, for each \( \theta \in C \) and for each \( a \in J \), the crisp set \( \text{Rep} \theta(a) \in C \).

Here we provide a metatheorem for groups in \( L \)-setting and its proof can be obtained on the lines of the proof of Tom Head’s metatheorem.

**Theorem 4.2:** Let \( \langle G, ^{-1}, \ast \rangle \) be a group. Let \( L^G \) be provided with the set of operations \( \text{Op} = \{\cap = \inf, \cup = \sup, \ast, \circ_1, \circ_2, \ldots, \circ_k\} \) where \( \ast \) is a convolutional extension and each \( \circ_i \) is a commutative natural extension. Let \( D(v_1, \ldots, v_m) \) and \( E(v_1, \ldots, v_m) \) be expressions over the set of variables \( \text{Var} = \{v_1, v_2, \ldots, v_m\} \) and the operation set \( \text{Op} \). Let \( C_1, \ldots, C_m \) be classes of \( L \)-sets in \( G \) that are closed under projections. Then the inequality

\[
D(\theta_1, \ldots, \theta_m) \text{ REL } E(\theta_1, \ldots, \theta_m)
\]

holds for all \( L \)-sets \( \theta_1 \in C_1, \ldots, \theta_m \in C_m \) if and only if it holds for all crisp sets \( \theta_1 \in C_1, \ldots, \theta_m \in C_m \) where \( \text{REL} \) is any of the relations \( \leq, = \) or \( \geq \).
**Proposition 4.3:** Let $C$ and $D$ be the classes of crisp subsets of a group $G$ and $\mathcal{C}$ and $\mathcal{D}$ be their corresponding $L$-classes that are projection closed. Then

1. $C \subseteq D$ if and only if $\mathcal{C} \subseteq \mathcal{D}$.
2. $C = D$ if and only if $\mathcal{C} = \mathcal{D}$.

**Proof:** Let $C \subseteq D$ and $\mu \in C$. Since the class $\mathcal{C}$ is projection closed, therefore $\text{Rep} \mu(a) \in C$ for each $a \in J$. This implies $\text{Rep} \mu(a) \in D$ for each $a \in J$. Hence $\mu \in \mathcal{D}$. Converse is obvious.

**Proposition 4.4:** Let $G$ be a group and $L$ be a chain. Then

1. $L(G)$ is closed under projections.
2. $L_n(G)$ is closed under projections.

**Proof:** (i) Let $x, y \in G$ and $a \in G$. Then for $\mu \in L(G)$ we show that

$$\text{Rep} \mu(a)(xy^{-1}) \geq \text{Rep} \mu(a)(x) \land \text{Rep} \mu(a)(y).$$

In view of the definition of $\text{Rep}$ function, $\text{Rep} \mu(a)(x) \land \text{Rep} \mu(a)(y) = 0$ or $1$. If $\text{Rep} \mu(a)(x) \land \text{Rep} \mu(a)(y) = 0$, then there is nothing to prove. So we let $\text{Rep} \mu(a)(x) \land \text{Rep} \mu(a)(y) = 1$. This implies $\text{Rep} \mu(a)(x) = 1$ and $\text{Rep} \mu(a)(y) = 1$. Thus $\mu(x) > a$ and $\mu(y) > a$. As $\mu \in L(G)$, we have

$$\mu(xy^{-1}) \geq \mu(x) \land \mu(y) > a.$$ 

Hence $\text{Rep} \mu(a)(xy^{-1}) = 1$. Consequently $\text{Rep} \mu(a) \in L(G)$ $\forall$ $a \in G$. Therefore $L(G)$ is closed under projections.

(ii) The result can be verified easily by using the fact that $\mu(xy) = \mu(yx)$ $\forall$ $\mu \in L_n(G)$.

We prove the following results by an application of metatheorem:

**Theorem 4.5:** Let $L$ be a dense chain and $\eta, \theta \in L_n(G)$. Then $[\eta, \theta] \in L_n(G)$ and $[\eta, \theta] \subseteq \eta \cap \theta$.

**Proof:** Let us consider the class of crisp normal subgroups $L_{nc}(G)$. Now define the following classes:

$$C_{cn} = \{1_H \circ 1_K : H, K \in L_n(G)\} \quad \text{and} \quad C_{cn} = \{\eta \circ \theta : \eta, \theta \in L_n(G)\},$$

where the operation $\circ$ is the natural extension of the binary operation $\circ : P(G) \times P(G) \to P(G)$ given by $H \circ K = [H, K]$. That is $1_H \circ 1_K = [1_H, 1_K]$, $\eta \circ \theta = [\eta, \theta]$. Here the function chi provides an order theoretic and algebraic isomorphism from $P(G)$ to $C(G)$. Next, in view of Lemma 3.6, we have

$$[1_H, 1_K] = 1_{[H,K]} \in L_{nc}(G).$$

Thus $C_{cn} \subseteq L_{nc}(G)$. Also, as $L$ is a complete dense chain, by example (c) of commutative natural extension on page 7, the class $C_{cn}$ is closed under projections. By Proposition 4.4,
the class $\mathcal{L}_n(G)$ is closed under projection. Thus by Proposition 4.3 (i), $\mathcal{C}_{cn} \subseteq \mathcal{L}_n(G)$. That is

$$[\eta, \theta] \in \mathcal{L}_n(G), \text{ whenever } \eta, \theta \in \mathcal{L}_n(G).$$

To prove the second part, we use metatheorem for $L$-subgroups. We construct the following two expressions in $C(G)$:

$$D(H, K) = 1_H \circ 1_K \quad \text{and} \quad E(H, K) = 1_H \cap 1_K,$$

where $1_H \circ 1_K = [1_H, 1_K]$ and $H$ and $K \in L_n(G)$. In view of Lemma 3.6 and by the isomorphism chi from $P(G)$ to $C(G)$ and we have

$$[1_H, 1_K] = 1_{[H,K]} \subseteq 1_{H \cap K} = 1_H \cap 1_K.$$

Thus $D(H, K) \leq E(H, K)$ holds in $\mathcal{L}_{nc}(G)$. Again as $L$ is a complete dense chain, the natural extension $\circ$ is commutative. By Proposition 4.4, the class $\mathcal{L}_n(G)$ of $L$-normal subgroups of $G$ is closed under projection. So the metatheorem for $L$-subgroups applies and hence the above containment holds for corresponding $L$-class $\mathcal{L}_n(G)$. That is

$$D(\eta, \theta) \leq E(\eta, \theta) \quad \forall \eta, \theta \in \mathcal{L}_n(G).$$

Therefore $[\eta, \theta] \subseteq \eta \cap \theta, \forall \eta, \theta \in \mathcal{L}_n(G).$ ■

**Proposition 4.6:** Let $L$ be a dense chain and $\eta, \theta \in L^G$ and $\eta \subseteq \theta$. Then $[\mu, \eta] \subseteq [\mu, \theta] \forall \mu \in L^G$.

**Proof:** We prove the result by an application of metatheorem for $L$-subgroups. Let $D_1(H) = 1_H, D_2(H, M) = 1_H \circ 1_M, E_1(K) = 1_K$ and $E_2(K, M) = 1_K \circ 1_M$ be two expressions in $C(G)$ where $H, K$ and $M$ are subsets of $G$ such that $H \subseteq K$ and the operation $\circ$ is the natural extension of the binary operation $\circ : P(G) \times P(G) \to P(G)$ given by $A \circ B = [A, B]$. We show that

$$\text{if } D_1(H) \leq E_1(K), \text{ then } D_2(H, G) \leq E_2(K, G).$$

(2)

In view of Lemma 3.6 and by the isomorphism chi from $P(G)$ to $C(G)$, we have

$$[1_H, 1_G] = 1_{[H,G]} \subseteq 1_{[K,G]} = [1_K, 1_G].$$

Hence (2) holds. Again, as $L$ is a complete dense chain, the natural extension $\circ$ is commutative. So the metatheorem for $L$-subgroups applies and hence the above containment holds for corresponding $L$-class. That is

$$\text{if } D_1(\eta) \leq E_1(\theta), \text{ then } D_2(\eta, \mu) \leq E_2(\theta, \mu) \forall \eta, \theta, \mu \in L^G.$$

Therefore

$$\text{if } \eta \subseteq \theta, \text{ then } [\mu, \eta] \subseteq [\mu, \theta] \quad \forall \eta, \theta, \mu \in L^G.$$

The proof of the following result is another exhibition of an application of metatheorem.
Proposition 4.7: Let \( \eta, \theta \in L^G \). Then \([\eta, \theta] = [\theta, \eta]\).

Proof: Let \( D(H, K) = 1_H \circ 1_K \), and \( E(H, K) = 1_K \circ 1_H \) be two expressions in \( C(G) \) where \( H \) and \( K \) are subsets of \( G \) and the operation \( \circ \) is the natural extension of the binary operation \( \circ : P(G) \times P(G) \rightarrow P(G) \) given by \( A \circ B = [A, B] \). In view of Lemma 3.6 and by the isomorphism \( \chi \) from \( P(G) \) to \( C(G) \), we have

\[
[1_H, 1_K] = 1_{[H,K]} = 1_{[K,H]} = [1_K, 1_H].
\]

Thus \( D(H, K) = E(H, K) \). As \( L \) is a complete dense chain, the natural extension \( \circ \) is commutative. So the metatheorem applies and this equality holds for corresponding \( L \)-class also. That is

\[
D(\eta, \theta) = E(\eta, \theta) \quad \forall \eta, \theta \in L^G.
\]

Therefore \([\eta, \theta] = [\theta, \eta] \forall \eta, \theta \in L^G\). \(\blacksquare\)

In the following result, the authors [10] prove only one sided containment and the equality is established for the class of fuzzy subgroups having identical tips. Here by an application of metatheorem we establish the equality without taking any restriction on the tips of \( L \)-subgroups. Also, in the following theorem \( \ast \) denotes the set product of \( L \)-subsets which is a convolutional extension.

Theorem 4.8: Let \( L \) be a dense chain and \( \eta, \theta \in \mathcal{L}_n(G) \) and \( \sigma \in \mathcal{L}(G) \). Then

\[
[\eta \ast \sigma, \theta \ast \sigma] = [\eta, \sigma] \ast [\theta, \sigma].
\]

Proof: Consider the following two expressions in \( C(G) \):

\[
D(H, K, M) = (1_H \ast 1_M) \circ (1_K \ast 1_M) \quad \text{and} \quad E(H, K, M) = (1_H \circ 1_M) \ast (1_K \circ 1_M),
\]

where \( H, K \in \mathcal{L}_n(G), M \in L(G) \) and the operation \( \circ \) is the natural extension of the binary operation \( \circ : P(G) \times P(G) \rightarrow P(G) \) given by \( A \circ B = [A, B] \) and \( \ast \) is the convolution extension of binary operation on \( G \). We show that

\[
D(H, K, M) = E(H, K, M). \tag{3}
\]

In view of Lemma 3.6 and by the isomorphism \( \chi \) from \( P(G) \) to \( C(G) \), it follows that

\[
[1_H \ast 1_M, 1_K \ast 1_M] = [1_{HM}, 1_{KM}] = 1_{[HM,KM]} = 1_{[H,M] \ast [K,M]}.
\]

Thus (3) holds. Now as \( L \) is a complete dense chain, the natural extension \( \circ \) and the convolutional extension \( \ast \) are commutative. By Proposition 4.4, the classes \( \mathcal{L}(G) \) and \( \mathcal{L}_n(G) \) are closed under projection. So the metatheorem applies and the above equality holds for corresponding expressions for projection closed \( L \)-classes \( \mathcal{L}_n(G) \) and \( \mathcal{L}(G) \). Thus we have

\[
D(\eta, \theta, \sigma) = E(\eta, \theta, \sigma) \quad \forall \eta, \theta \in \mathcal{L}_n(G) \quad \text{and} \quad \forall \sigma \in \mathcal{L}(G).
\]

That is \( [\eta \ast \sigma] \circ [\theta \ast \sigma] = (\eta \circ \sigma) \ast (\theta \circ \sigma) \). Therefore

\[
[\eta \ast \sigma, \theta \ast \sigma] = [\eta, \sigma] \ast [\theta, \sigma] \quad \forall \eta, \theta \in \mathcal{L}_n(G) \quad \text{and} \quad \forall \sigma \in \mathcal{L}(G).
\]

\(\blacksquare\)
5. Subdirect Product Theorem for \( L \)-Subgroups

Tom Head in his important papers [2,7] has presented the fuzzy power algebra as a subdirect product of copies of its associated crisp power algebra. Here, we are using the same technique to represent the \( L \)-subgroup lattice \( \mathcal{L}(G) \) as a subdirect product of the copies \( \mathcal{L}_c(G) \) in the lattice \( \mathcal{L}_c(G)^J \). Recall that an algebra \( A \) is said to be a subdirect product of a family of algebras \( \{ A_b : b \in B \} \), where \( B \) is an arbitrary index set, if \( A \) is isomorphic to a subalgebra of the product algebra \( \{ A_b : b \in B \} \) with the property that its projection into each co-ordinate space \( A_b \) is a surjection. Recall that if \( X \) is an algebra having the \( n \)-ary operations \( *_1, \ldots, *_k, n \geq 1 \), then these operations extends to operations \( *_1, \ldots, *_k, n \geq 1 \) on \( P(X) \), \( C(X) \) and \( L^X \) by convolutional extension method. Moreover, as \( P(X) \), \( C(X) \) and \( L^X \) have two additional operations sup and inf, these sets become \( (*, \ldots, *, \inf, \sup) \)-algebras. Since the function \( \text{Rep} : L^X \rightarrow C(X)^J \) commutes with \( *_1, \ldots, *_k \), finite inf and arbitrary sup ; the function \( \text{Rep} \) turns out to be an injective homomorphism of the algebra \( L^X \) into the product algebra \( C(X)^J \). Therefore \( \text{Rep} \) is an algebraic as well as order theoretic isomorphism of \( L^X \) with its image \( I(X) \). Here we state subdirect product theorem for the algebra of \( L^X \) by Tom Head [2].

**Theorem 5.1:** Let \( X \) be an algebra having \( n \)-ary operations \( *_1, \ldots, *_k \), for various values of \( n \geq 1 \). Then \( \text{Rep} : L^X \rightarrow C(X)^J \) is a representation of the \( (\land, \lor, *_1, \ldots, *_k) \)-algebra \( L^X \) as subdirect product of copies of the \( (\land, \lor, *_1, \ldots, *_k) \)-algebra \( C(X) \).

\[
\begin{array}{ccc}
C(X) \subseteq L^X & \xrightarrow{\text{Rep}} & I(X) \subseteq C(X)^J \\
\downarrow & & \downarrow \\
C(X) & &
\end{array}
\]

Diagram 1.

Next we come to the main theorem of this paper. Let \( \langle G, ^{-1}, * \rangle \) be a group and let \( \mathcal{L} \text{Rep} \) be the restriction of the function \( \text{Rep} : L^G \rightarrow C(G)^J \) to \( \mathcal{L}(G) \). Also, let \( \mathcal{L}I(G) \) be the image of \( \mathcal{L} \text{Rep} \). The inf in \( \mathcal{L}(G) \) coincides with inf in the lattice \( L^G \) whereas the sup is defined as

\[
\text{sup}\{\eta, \theta\} = <\eta \cup \theta> \quad \forall \eta, \theta \in \mathcal{L}(G).
\]

**Theorem 5.2:** The representation function \( \mathcal{L} \text{Rep} : \mathcal{L}(G) \rightarrow C(G)^J \) commutes with arbitrary sups in the lattice \( \mathcal{L}(G) \).

**Proof:** Let \( \{\mu_i\}_{i \in \lambda} \) be a family of \( L \)-subgroups in \( \mathcal{L}(G) \). Then the join of this family is given by \( \bigvee_{i \in \lambda} \mu_i = \langle \bigcup_{i \in \lambda} \mu_i \rangle \). Now,

\[
\mathcal{L} \text{Rep} \left( \bigvee_{i \in \lambda} \mu_i \right)(a) = \mathcal{L} \text{Rep} \left( \bigcup_{i \in \lambda} \mu_i \right)(a)
\]

\[
= \chi \left( \left( \bigcup_{i \in \lambda} \mu_i \right)_a \right) \quad \text{(by the definition of Rep function)}
\]
\[\chi\left(\bigcup_{i \in \lambda} \mu_i\right) \left(a^\ast\right) = \chi\left(\bigcup_{i \in \lambda} \mu_i\right) \left(a^\ast\right) = \chi\left(\bigcup_{i \in \lambda} \mu_i\right) \left(a^\ast\right) = \chi\left(\bigcup_{i \in \lambda} \mu_i\right) \left(a^\ast\right) = \chi\left(\bigcup_{i \in \lambda} \mu_i\right) \left(a^\ast\right) \] (by Proposition 3.4)

**Theorem 5.3:** Let \(\langle G,^{-1},\ast\rangle\) be a group. Then \(\mathcal{L}\text{Rep} : \mathcal{L}(G) \rightarrow \mathcal{L}_c(G)^J\) is an isomorphism of the lattice \(\mathcal{L}(G)\) onto \(\mathcal{L}_I(G)\) that represents \(\mathcal{L}(G)\) as a subdirect product of the copies of \(\mathcal{L}_c(G)\) in the lattice \(\mathcal{L}_c(G)^J\).

**Proof:** Since \(\text{Rep}\) is a bijection from \(\mathcal{L}^G\) onto \(\mathcal{I}(G)\), the function \(\mathcal{L}\text{Rep}\) is also a bijection from \(\mathcal{L}(G)\) onto \(\mathcal{L}_I(G)\). Moreover, the projection in each co-ordinate space \(\mathcal{L}_c(G)\) is a sur-jection, since for each \(a \in J\) and \(\mu \in \mathcal{L}_c(G)\), \(\text{Rep} \mu(a) = \mu\). Also \(\mathcal{L}(G)\) is closed under inf and sup, since for \(\eta, \theta \in \mathcal{L}(G)\), \(\sup\{\eta, \theta\} = \langle \eta \cup \theta \rangle\), is an \(\mathcal{L}\)-subgroup. To show that \(\mathcal{L}\text{Rep}\) is an isomorphism, it remains to prove that \(\mathcal{L}\text{Rep}\) commutes with inf and sup. Since \(\text{Rep}\) commutes with inf, \(\mathcal{L}\text{Rep}\) commutes with inf in \(\mathcal{L}(G)\). Also, \(\mathcal{L}\text{Rep}\) commutes with sup in \(\mathcal{L}(G)\) by Theorem 5.2. This completes the proof of the theorem.

The above theorem can be illustrated by the following diagram:

```
\[\mathcal{L}_c(G) \subset \mathcal{L}(G) \xrightarrow{\mathcal{L}\text{Rep}} \mathcal{L}_I(G) \subset \mathcal{L}_c(G)^J\]
```

**Diagram 2.**
Corollary 5.4: For each group \( \langle G,^{-1}, * \rangle \), the lattice \( \mathcal{L}(G) \) of \( L \)-subgroups of \( G \) and the lattice of crisp subgroups \( \mathcal{L}_{c}(G) \) of \( G \) satisfy precisely the same (inf = \( \cap \), sup = \( \langle \rangle \))-identities.

Tom Head established a subdirect product theorem for the lattice \( \mathcal{F}\mathcal{L}_{n}(G) \) of fuzzy normal subgroups of a group \( G \). For this purpose, he had to demonstrate that the function \( \text{Rep} \) commutes with the inf of the lattice \( \mathcal{F}\mathcal{L}_{n}(G) \) and also with its sup. The first requirement is satisfied since the inf of \( \mathcal{F}\mathcal{L}_{n}(G) \) coincides with inf of lattice \( \mathcal{F}(G) \) of fuzzy susets in \( G \). To show the other commutation, for a pair of fuzzy subgroups \( \eta \) and \( \theta \), Tom Head has defined what he calls a tip extended pair of fuzzy subgroups \( \eta^{1} \) and \( \theta^{1} \) (see Erratum [7]). Then he asserts that \( \text{sup}\{\eta, \theta\} = \eta^{1} * \theta^{1} \), where the operation ‘\(*\)’ in \( \mathcal{F}\mathcal{L}_{n}(G) \) is the convolutional extension of the binary operation ‘\(*\)’ of the group \( G \). The function \( \text{Rep} \) commutes with the sup of \( \mathcal{F}\mathcal{L}_{n}(G) \) follows in view of the fact that the \( \text{Rep} \) function commutes with the convolutional extension ‘\(*\)’. Here we provide a new subdirect product theorem for the lattice of \( L \)-normal subgroups of \( G \) which is a sublattice of the lattice \( \mathcal{L}(G) \). Firstly, note that the inf in \( \mathcal{L}_{n}(G) \) coincides with inf in the lattice \( \mathcal{L}(G) \). Since \( \langle \eta \cup \theta \rangle \in \mathcal{L}_{n}(G) \forall \eta, \theta \in \mathcal{L}_{n}(G) \), the sup in \( \mathcal{L}_{n}(G) \) coincides with the sup in the lattice \( \mathcal{L}(G) \). Let \( \mathcal{L}_{n}\text{Rep} \) be the restriction of the function \( \text{Rep} : \mathcal{L}_{n}(G) \rightarrow \mathcal{C}(G)^{j} \) to \( \mathcal{L}_{n}(G) \) and let \( \mathcal{L}_{n}\text{I}(G) \) be the image of \( \mathcal{L}_{n}\text{Rep} \). Then, we have the following:

**Theorem 5.5:** Let \( \langle G,^{-1}, * \rangle \) be a group. Then \( \mathcal{L}_{n}\text{Rep} : \mathcal{L}_{n}(G) \rightarrow \mathcal{L}_{nc}(G)^{j} \) is an isomorphism of the lattice \( \mathcal{L}_{n}(G) \) onto \( \mathcal{L}_{n}\text{I}(G) \) that represents \( \mathcal{L}_{n}(G) \) as a subdirect product of the copies of \( \mathcal{L}_{nc}(G) \) in the lattice \( \mathcal{L}_{nc}(G)^{j} \).

\[
\begin{align*}
\mathcal{L}_{nc}(G) \subseteq \mathcal{L}_{n}(G) & \xrightarrow{\mathcal{L}_{n}\text{Rep}} \mathcal{L}_{n}\text{I}(G) \subseteq \mathcal{L}_{nc}(G)^{j} \\
\mathcal{L}_{nc}(G) & \xrightarrow{\mathcal{L}_{n}\text{I}(G)} \mathcal{L}_{nc}(G)
\end{align*}
\]

Diagram 3.

Note that the subdirect product theorem by Tom Head [2] is obtained as a corollary. Moreover, we obtain the following:

**Corollary 5.6:** For each group \( \langle G,^{-1}, * \rangle \), the lattice \( \mathcal{L}_{n}(G) \) of \( L \)-normal subgroups of \( G \) is a modular sublattice of \( \mathcal{L}(G) \).

In classical algebra, a generalized cyclic group can be characterized by the distributivity of its lattice of subgroups. Here we demonstrate that such groups can also be characterized in terms of the distributivity of its lattice of \( L \)-subgroups.

**Definition 5.7:** A group \( G \) is said to be generalized cyclic if for any two elements \( a, b \in G \), there exists an element \( c \in G \) such that

\[
a = c^{m} \quad \text{and} \quad b = c^{n}
\]

for some integers \( m \) and \( n \).
The group of rational numbers under addition is an example of a generalized cyclic group which is not cyclic. It is worthwhile to mention here that a generalized cyclic group $G$ is an Abelian group. Moreover for its subgroup lattice $L(G)$, the join and the meet of $H, K \in L(G)$ are defined by

\[ H \lor K = HK \quad \text{and} \quad H \land K = H \cap K. \]

The following result is well known and can be found in Birkhoff [8]:

**Theorem 5.8:** A group $G$ is generalized cyclic if and only if its subgroup lattice $L(G)$ is distributive.

In [17] the authors obtain a characterization of generalized cyclic groups in terms of distributivity of its $L$-subgroup lattice. Here we apply the new subdirect product theorem and the main result of [17] is obtained as a corollary to Theorem 5.5.

**Corollary 5.9:** Let $\langle G, -^1, * \rangle$ be a group. Then $G$ is a generalized cyclic if and only if the lattice $L(G)$ of $L$-subgroups of $G$ is distributive.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

**Funding**

The first author was supported by University Grants Commission, EMERITUS FELLOWSHIP, India [grant number F.6-6/2016-17/EMERITUS-2015-17-GEN-7877] during the course of development of this paper.

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