ON HOM-GERSTENHABER ALGEBRAS AND HOM-LIE ALGEBROIDS

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Abstract. We define the notion of hom-Batalin-Vilkovisky algebras and strong differential hom-Gerstenhaber algebras as a special class of hom-Gerstenhaber algebras and provide canonical examples associated to some well-known hom-structures. Representations of a hom-Lie algebroid on a hom-bundle are defined and a cohomology of a regular hom-Lie algebroid with coefficients in a representation is studied. We discuss about relationship between these classes of hom-Gerstenhaber algebras and geometric structures on a vector bundle. As an application, we associate a homology to a regular hom-Lie algebroid and then define a hom-Poisson homology associated to a hom-Poisson manifold.

1. Introduction

Hom-Lie algebras were introduced in the context of $q$-deformation of Witt and Virasoro algebras. In a sequel, various concepts and properties have been derived to the framework of other hom-algebras as well. The study of hom-algebras appeared extensively in the work of J. Hartwig, D. Larsson, A. Makhlouf, S. Silvestrov, D. Yau and other authors ([1], [6], [9], [10], [15]). More recently, the notion of hom-Lie algebroid is introduced in [8] by going through a formulation of hom-Gerstenhaber algebra and following the classical bijective correspondence between Lie algebroids and Gerstenhaber algebras. On the other hand, there are canonically defined adjoint functors between category of Lie-Rinehart algebras and category of Gerstenhaber algebras. This leads us to proceed further and define the category of hom-Lie-Rinehart algebras in [11], and discuss adjoint functors between the category of hom-Gerstenhaber algebras and the category of hom-Lie-Rinehart algebras. Furthermore, the notion of a hom-Lie bialgebroid and a hom-Courant algebroid is defined in [3].

There are some well known algebraic structures such as Batalin-Vilkovisky algebras and differential Gerstenhaber algebras satisfying nice relationship with Lie-Rinehart algebras and different geometric structures on a vector bundle (in [14], [12], [7] and the references therein). In this paper, our first goal is to define hom-Batalin-Vilkovisky algebras and strong differential hom-Gerstenhaber algebras as a special class of hom-Gerstenhaber algebras and present several examples which are obtained canonically.

In [3], a hom-bundle of rank $n$ is defined, as a triplet $(A, \psi, \alpha)$ where $A$ is a rank $n$ vector bundle over a smooth manifold $M$, the map $\psi : M \rightarrow M$ is a smooth map and $\alpha : \Gamma A \rightarrow \Gamma A$ is a $\psi^*$-function linear map, i.e. $\alpha(f, x) = \psi^*(f) \cdot \alpha(x)$ for $f \in C^\infty(M)$, $x \in \Gamma A$. It is proved in [8] that hom-Lie algebroid structures on the hom-bundle $(A, \psi, \alpha)$ are in bijective correspondence with hom-Gerstenhaber algebra structures on $(A, \tilde{\alpha})$, where $A = \oplus_{0 \leq k \leq n} \Gamma(\wedge^k A)$ and the map $\tilde{\alpha}$ is an extension of $\alpha$ to higher degree elements. Let $(A, \psi, \alpha)$ be an invertible hom-bundle. Then, the second goal of this paper is to analyse the following.

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The relationship between hom-Batalin-Vilkovisky algebra structures on \((\mathfrak{A}, \tilde{\alpha})\) and hom-Lie algebroid structures on the hom-bundle \((A, \psi, \alpha)\) with a representation of this hom-Lie algebroid on the hom-bundle \((\wedge^n A, \psi, \tilde{\alpha})\).

The relationship between strong differential hom-Gerstenhaber algebra structures on \((\mathfrak{A}, \tilde{\alpha})\) and hom-Lie bialgebroid structures on the hom-bundle \((A, \psi, \alpha)\).

For the first relationship, we prove a more general result (Corollary 3.16) for hom-Lie-Rinehart algebras (see Definition 2.14). As an application, we get a homology associated to a hom-Poisson manifold \((M, \psi, \pi)\) where the map \(\psi : M \rightarrow M\) is a diffeomorphism. We call this homology as “hom-Poisson homology” since for the case when \(\psi\) is the identity map on \(M\), this gives the Poisson homology. The paper is organized as follows:

In Section 2, we recall preliminaries on hom-structures. This will help us to fix several notations to be used in the later part of our discussion.

In Section 3, we define a hom-Batalin-Vilkovisky algebra as an exact hom-Gerstenhaber algebra and give some examples. Right modules and subsequently a homology is studied for a hom-Lie-Rinehart algebra with coefficients in a right module. We also express homology of a hom-Lie-Rinehart algebra with trivial coefficients in terms of the associated hom-Gerstenhaber algebra. We study a cohomology of regular hom-Lie-Rinehart algebra \((\mathcal{L}, \alpha)\) with coefficients in a left module \((M, \beta)\). If the underlying \(A\)-module \(L\) is projective of rank \(n\), then we prove a one-one correspondence between right \((\mathcal{L}, \alpha)\)-module structures on \((A, \phi)\) and left \((\mathcal{L}, \alpha)\)-module structures on \((\wedge^n L, \alpha)\).

In Section 4, we define representations of a hom-Lie algebroid on a hom-bundle. If \(\mathcal{A} := (A, \phi, [-,-], \rho, \alpha)\) is a regular hom-Lie algebroid, where \(A\) is a rank \(n\) vector bundle over \(M\), then it is proved that there exists a bijective correspondence between exact generators of the associated hom-Gerstenhaber algebra and representations of \(\mathcal{A}\) on the hom-bundle \((\wedge^n A, \psi, \tilde{\alpha})\). We associate a homology to a regular hom-Lie algebroid when the underlying vector bundle is a real vector bundle. For a hom-Poisson manifold \((M, \psi, \pi)\) (where \(\psi : M \rightarrow M\) is a diffeomorphism), we define hom-Poisson homology as the homology of its associated cotangent hom-Lie algebroid. Moreover, we define a cochain complex for a regular hom-Lie algebroid with coefficients in a representation.

In the last Section, we define strong differential hom-Gerstenhaber algebras and give several examples. Finally, We discuss the relationship between strong differential hom-Gerstenhaber algebras and hom-Lie bialgebroids.

### 2. Preliminaries on hom-structures

In this section, we recall some basic definitions from [8], [11], and [13]. Let \(R\) be a commutative ring with unity and \(\mathbb{Z}_+\) be the set of all non-negative integers.

**Definition 2.1.** A hom-Lie algebra is a triplet \((\mathfrak{g}, [-,-], \alpha)\) where \(\mathfrak{g}\) is a \(R\)-module equipped with a skew-symmetric \(R\)-bilinear map \([-,-] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}\) and a Lie algebra homomorphism \(\alpha : \mathfrak{g} \rightarrow \mathfrak{g}\) such that the hom-Jacobi identity holds, i.e., \([\alpha(x), [y,z]] + [\alpha(y), [z,x]] + [\alpha(z), [x,y]] = 0\) for all \(x, y, z \in \mathfrak{g}\).

If \(\alpha\) is an automorphism of \(\mathfrak{g}\), then \((\mathfrak{g}, [-,-], \alpha)\) is called a regular hom-Lie algebra.

**Definition 2.2.** A representation of a hom-Lie algebra \((\mathfrak{g}, [-,-], \alpha)\) on a \(R\)-module \(V\) is a pair \((\rho, \alpha_V)\) of \(R\)-linear maps \(\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V), \alpha_V : V \rightarrow V\) such that

\[\rho(\alpha(x)) \circ \alpha_V = \alpha_V \circ \rho(x),\]
Given an associative commutative algebra

\[ B \]

Batalin-Vilkovisky algebra if it has a generator

A Gerstenhaber algebra is a triple

\[ (A, \alpha, \delta) \]

Definition 2.7. - is said to be a generator of the bracket \( \phi \) for all \( x \in g \).

Example 2.3. For any integer \( s \), we can define the \( \alpha^s \)-adjoint representation of the regular hom-Lie algebra \( (g, [-,-], \alpha) \) on \( g \) by \( (ad_s, \alpha) \), where \( ad_s(g)(h) = [\alpha^s(g), h] \) for all \( g, h \in g \).

Definition 2.4. A graded hom-Lie algebra is a triplet \( (g, [-,-], \alpha) \), where \( g = \oplus_{i \in \mathbb{Z}} g_i \) is a graded module, \( [-,-] : g \otimes g \rightarrow g \) is a graded skew-symmetric bilinear map of degree \(-1\), and \( \alpha : g \rightarrow g \) is a homomorphism of \( (g, [-,-]) \) of degree \( 0 \), satisfying:

\[
(-1)^{i(j-1)k} [a(x), [y, z]] + (-1)^{j(i-1)k}[a(y), [x, z]] + (-1)^{i(j-1)k} [a(z), [x, y]] = 0,
\]

for all \( x \in g_i, y \in g_j, \) and \( z \in g_k \).

Definition 2.5. Let \( \mathfrak{A} = \oplus_{k \in \mathbb{Z}} \mathfrak{A}_k \) be a graded commutative algebra, \( \sigma \) and \( \tau \) be 0-degree endomorphism of \( \mathfrak{A} \), then a \( \{\sigma, \tau\}\)-differential graded commutative algebra is quadruple \( (\mathfrak{A}, \sigma, \tau, d) \), where \( d \) is a degree 1 square zero operator on \( \mathfrak{A} \) satisfying the following:

1. \( d \circ \sigma = \sigma \circ d, \) \( d \circ \tau = \tau \circ d; \)
2. \( d(ab) = d(a)\tau(b) + (-1)^{|a|}\sigma(a)d(b) \) for \( a, b \in \mathfrak{A} \).

Definition 2.6. A Gerstenhaber algebra is a triple \( (\mathcal{A} = \oplus_{i \in \mathbb{Z}} \mathcal{A}_i, \wedge, [-,-]) \) where \( \mathcal{A} \) is a graded commutative associative \( R \)-algebra, and \( [-,-] : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \) is a bilinear map of degree \(-1\) such that:

1. \( (\mathcal{A}, [-,-]) \) is a graded Lie algebra.
2. The following Leibniz rule holds:

\[
[X,Y \wedge Z] = [X,Y] \wedge Z + (-1)^{|X|} Z \wedge [X,Z],
\]

for all \( X \in \mathcal{A}_i, Y \in \mathcal{A}_j, Z \in \mathcal{A}_k \).

For a Gerstenhaber algebra \( (\mathcal{A} = \oplus_{i \in \mathbb{Z}} \mathcal{A}_i, \wedge, [-,-]) \), a linear operator \( D : \mathcal{A} \rightarrow \mathcal{A} \) of degree \(-1\) is said to be a generator of the bracket \([-,-]\) if for every homogeneous \( a, b \in \mathcal{A} \):

\[
[a, b] = (-1)^{|a|} (D(ab) - (Da)b - (-1)^{|a|} a(Db));
\]

Definition 2.7. A Gerstenhaber algebra \( \mathcal{A} \) is called an exact Gerstenhaber algebra or a Batalin-Vilkovisky algebra if it has a generator \( D \) of square zero.

Definition 2.8. Given an associative commutative algebra \( \mathcal{A} \), an \( \mathcal{A} \)-module \( M \) and an algebra endomorphism \( \phi : \mathcal{A} \rightarrow \mathcal{A} \), we call a \( \mathcal{A} \)-linear map \( \delta : \mathcal{A} \rightarrow M \) a \( \phi \)-derivation of \( \mathcal{A} \) into \( M \) if it satisfies

\[
\delta(ab) = \phi(a) \delta(b) + \phi(b) \delta(a)
\]

for all \( a, b \in \mathcal{A} \). Let us denote the set of \( \phi \)-derivations by \( \text{Der}_\phi(\mathcal{A}) \).

Remark 2.9. Given a manifold \( M \) and a smooth map \( \psi : M \rightarrow M \), then the space of \( \psi^* \)-derivations of \( C^\infty(M) \) into \( C^\infty(M) \) can be identified with the space of sections of the pull back bundle of \( TM \), i.e. \( \Gamma(\psi^*TM) \).

Definition 2.10. A hom-Lie algebroid is a quintuple \( (A, \phi, [-,-], \rho, \alpha) \), where \( A \) is a vector bundle over a smooth manifold \( M \), \( \phi : M \rightarrow M \) is a smooth map, \( [-,-] : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A) \) is a bilinear map, \( \rho : \phi A \rightarrow \phi^*TM \) is a bundle map, called anchor, and \( \alpha : \Gamma(A) \rightarrow \Gamma(A) \) is a linear map, such that:
(1) \( \alpha(f.X) = \phi^*(f).\alpha(X) \) for all \( X \in \Gamma(A), f \in C^\infty(M) \);
(2) The triplet \( (\Gamma(A), [-, -], \alpha) \) is a hom-Lie algebra;
(3) The following hom-Leibniz identity holds: \( [X, fY] = \phi^*(f)[X,Y] + \rho(X)[f\alpha(Y)] \); for all \( X,Y \in \Gamma(A), f \in C^\infty(M) \);
(4) The pair \( (\rho, \phi^*) \) is a representation of the hom-Lie algebra \( (\Gamma(A), [-, -], \alpha) \) on the space \( C^\infty(M) \).

Here, \( \rho(X)[f] \) stands for the function on \( M \), such that

\[
\rho(X)[f](m) = \langle d_{\phi(m)}f, \rho_m(X_{\phi(m)}) \rangle
\]

for \( m \in M \). Here, \( \rho_m : (\phi^! A)_m \cong A_{\phi(m)} \rightarrow (\phi^! TM)_m \cong T_{\phi(m)}M \) is the anchor map evaluated at \( m \in M \) and \( X_{\phi(m)} \) is the value of the section \( X \in \Gamma(A) \) at \( \phi(m) \in M \).

Moreover, a hom-Lie algebroid is said to be regular (or invertible) if the map \( \alpha : \Gamma(A) \rightarrow \Gamma(A) \) is an invertible linear map and the map \( \phi : M \rightarrow M \) is a diffeomorphism.

**Definition 2.11.** A hom-Gerstenhaber algebra is a quadruple \( (A, \oplus_{i \in \mathbb{Z}_+} A_i, \wedge, [-, -], \alpha) \) where \( A \) is a graded commutative associative \( R \)-algebra, \( \alpha \) is an endomorphism of \( (A, \wedge) \) of degree 0 and \([-,-]: A \otimes A \rightarrow A \) is a bilinear map of degree \(-1\) such that:

1. the triple \( (A, [-, -], \alpha) \) is a graded hom-Lie algebra.
2. the hom-Leibniz rule holds:

\[
[X, Y \wedge Z] = [X, Y] \wedge \alpha(Z) + (-1)^{(i-1)j}\alpha(Y) \wedge [X, Z],
\]

for all \( X \in A_i, Y \in A_j, Z \in A_k \).

**Example 2.12.** Let \( (A, [-, -], \wedge) \) be a Gerstenhaber algebra and \( \alpha : (A, [-, -], \wedge) \rightarrow (A, [-, -], \wedge) \) be an endomorphism, then the quadruple \( (A, \wedge, \alpha \circ [-, -], \alpha) \) is a hom-Gerstenhaber algebra.

**Example 2.13.** Given a hom-Lie algebra \( (g, [-, -], \alpha) \), one can define a hom-Gerstenhaber algebra \( (\mathfrak{g} = \wedge^* g, \wedge, [-, -], \alpha, \alpha) \), where

\[
[x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_m] = \sum_{i=1}^{n} \sum_{j=1}^{m} (-1)^{i+j} x_i \wedge \alpha_G(x_1 \wedge \cdots \hat{x_i} \wedge \cdots \wedge x_n \wedge y_1 \wedge \cdots \hat{y_j} \wedge \cdots \wedge y_m)
\]

for all \( x_1, \cdots, x_n, y_1, \cdots, y_m \in g \) and

\[
\alpha_{\mathfrak{g}}(x_1 \wedge \cdots \wedge x_n) = \alpha(x_1) \wedge \cdots \wedge \alpha(x_n).
\]

See [3] for further details.

**Definition 2.14.** A hom-Lie Rinehart algebra over \( (A, \phi) \) is a tuple \( (A, L, [-, -], \phi, \alpha, \rho) \), where \( A \) is an associative commutative algebra, \( L \) is an \( A \)-module, \([-,-]: L \times L \rightarrow L \) is a skew symmetric bilinear map, \( \phi : A \rightarrow A \) is an algebra homomorphism, \( \alpha : L \rightarrow L \) is a linear map satisfying \( \alpha([x,y]) = [\alpha(x), \alpha(y)] \), and \( \rho : L \rightarrow \text{Der}_\phi A \) such that

1. The triple \( (L, [-, -], \alpha) \) is a hom-Lie algebra.
2. \( \alpha(a.x) = \phi(a).\alpha(x) \) for all \( a \in A, x \in L \).
3. \( (\rho, \phi) \) is a representation of \( (L, [ , ], \alpha) \) on \( A \).
4. \( \rho(a.x) = \phi(a).\rho(x) \) for all \( a \in A, x \in L \).
5. \( [x, a.y] = \phi(a)[x, y] + \rho(x)(a)\alpha(y) \) for all \( a \in A, x, y \in L \).

A hom-Lie-Rinehart algebra \( (A, L, [-, -], \phi, \alpha, \rho) \) is called regular if the map \( \phi : A \rightarrow A \) is an algebra automorphism and the map \( \alpha : L \rightarrow L \) is a bijection.
Let us denote a hom-Lie-Rinehart algebra \((A, L, [-, -], \phi, \alpha, \rho)\) simply by \((\mathcal{L}, \alpha)\). In [11], we discussed several examples of hom-Lie-Rinehart algebras.

3. Hom-Batalin-Vilkovisky algebras and hom-Lie-Rinehart algebras

3.1. Hom-Batalin-Vilkovisky algebras.

**Definition 3.1.** A hom-Gerstenhaber algebra \((A = \oplus_{i \in \mathbb{Z}_+} A_i, \wedge, [-, -], \alpha)\) is said to be generated by an operator \(D : A \to A\) of degree \(-1\) if \(D \circ \alpha = \alpha \circ D\), and

\[
[X, Y] = (-1)^{|X|}(DXY - DX\alpha(Y) - (-1)^{|X|}\alpha(X)(DY));
\]

for homogeneous elements \(X, Y \in A\). Furthermore, if \(D^2 = 0\), then \(D\) is called an exact generator and hom-Gerstenhaber algebra \(A\) is called an exact hom-Gerstenhaber algebra or hom-Batalin-Vilkovisky algebra.

**Example 3.2.** Recall from Example 2.13 that there is a canonical hom-Gerstenhaber algebra structure on the exterior algebra \(\mathcal{E} = \wedge^* \mathfrak{g}, \wedge, [-, -], \alpha_G\) associated to a hom-Lie algebra \((\mathfrak{g}, [-, -], \alpha)\). If we consider the boundary operator \(d : \wedge^\mathfrak{g} \to \wedge^{\mathfrak{g}-1} \mathfrak{g}\) of a hom-Lie algebra with coefficients in the trivial module \(R\) (defined in [15]), given by

\[
d(x_1 \wedge \cdots \wedge x_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j}[x_i, x_j] \wedge \alpha_G(x_1 \wedge \cdots \hat{x}_i \cdots \wedge \hat{x}_j \cdots \wedge x_n)
\]

for all \(x_1, \cdots, x_n \in \mathfrak{g}\). Then, \(d : \mathcal{E} \to \mathcal{E}\) is a map of degree \(-1\) such that \(d^2 = 0\), and \(d \circ \alpha = \alpha \circ d\). More importantly, this operator \(d\) generates the graded hom-Lie bracket \([-,-]_\mathcal{E}\) in the following way:

\[
[X, Y]_\mathcal{E} = (-1)^{|X|}(dXY - DX\alpha(Y) - (-1)^{|X|}\alpha(X)(DY))
\]

for \(X, Y \in \mathcal{E}\). Hence the operator \(d\) is an exact generator of the hom-Gerstenhaber algebra \(\mathcal{E}\). This yields a hom-BV algebra associated to a hom-Lie algebra.

**Example 3.3.** Given a Batalin-Vilkovisky algebra \((\mathcal{A}, [-, -], \wedge, \partial)\) and an endomorphism \(\alpha : (\mathcal{A}, [-, -], \wedge) \to (\mathcal{A}, [-, -], \wedge)\) of the underlying Gerstenhaber algebra satisfying \(\partial \circ \alpha = \alpha \circ \partial\), then the quadruple \((\mathcal{A}, \wedge, [-, -], \alpha = \alpha \circ [-, -], \alpha)\) is a hom-Gerstenhaber algebra. Since for any homogeneous elements \(a, b \in \mathcal{A}\), we have

\[
[a, b] = (-1)^{|a|}(\partial(ab) - \partial(a)b - (-1)^{|a|}a\partial(b)),
\]

so by applying \(\alpha\) on both sides we obtain the following relation:

\[
[a, b]_\alpha = (-1)^{|a|}(\partial_\alpha(ab) - \partial_\alpha(a)b - (-1)^{|a|}\alpha(a)\partial_\alpha(b)),
\]

i.e. \(\partial_\alpha = \alpha \circ \partial\) generates the hom-Gerstenhaber bracket \([-,-]_\alpha\).

3.2. Homology of a hom-Lie-Rinehart algebra. Let \(A\) be an associative and commutative \(R\)-algebra, and \(\phi\) be an algebra homomorphism of \(A\). Suppose \((\mathcal{L}, \alpha)\) is a hom-Lie-Rinehart algebra over the pair \((A, \phi)\).

**Definition 3.4.** Let \(M\) be an \(A\)-module and \(\beta \in \text{End}_R(M)\). Then the pair \((M, \beta)\) is a right module over a hom-Lie Rinehart algebra \((\mathcal{L}, \alpha)\) if the following conditions hold.

1. There is a map \(\theta : M \otimes L \to M\) such that the pair \((\theta, \beta)\) is a representation of the hom-Lie algebra \((L, [-, -], \alpha)\) on \(M\), where \(\theta(m, x)\) is usually denoted by \(\{m, x\}\) for \(x \in L, m \in M\).
(2) $\beta(a.m) = \phi(a).\beta(m)$ for $a \in A$ and $m \in M$.
(3) $\{a.m, x\} = \{m, a.x\} = \phi(a).\{m, x\} - \rho(x)(a).\beta(m)$ for $a \in A$, $x \in L$, $m \in M$.

If $\alpha = \text{Id}_L$ and $\beta = \text{Id}_M$, then $M$ is a right Lie-Rinehart algebra module. Note that there is no canonical right module structure on $(A, \phi)$ as one would expect from the case of Lie-Rinehart algebras.

Let $(M, \beta)$ be a right module over a hom-Lie-Rinehart algebra $(L, \alpha)$. Define $C_n(L, M) := M \otimes_A \wedge^n_A L$, for $n \geq 0$. Also define the boundary map $d : C_n(L, M) \rightarrow C_{n-1}(L, M)$ by

\[
d(m \otimes (x_1 \otimes \cdots \otimes x_n)) = \sum_{i=1}^{n} (-1)^{i+1} \{m, x_i\} \otimes (\alpha(x_1) \otimes \cdots \otimes \alpha(x_i) \otimes \cdots \otimes \alpha(x_n)) + \\
\sum_{i<j} (-1)^{i+j} \beta(m) \otimes ([x_i, x_j], \alpha(x_1) \otimes \cdots \otimes \alpha(x_i) \otimes \cdots \otimes \alpha(x_j) \otimes \cdots \otimes \alpha(x_n))
\]

for $m \in M$ and $x_1, \ldots, x_n \in L$. By the definition of the right hom-Lie-Rinehart algebra module structure on $(M, \beta)$, it follows that $d^2 = 0$. Thus, $(C_\ast(L, M), d)$ forms a chain complex. The homology of the hom-Lie-Rinehart algebra $(L, \alpha)$ with coefficient in the right module $(M, \beta)$ is given by

\[H^{hLR}_\ast(L, M) := H_\ast(C_\ast(L, M)).\]

**Remark 3.5.** If $\alpha = \text{Id}_L$ and $\beta = \text{Id}_M$, then $M$ is a right Lie-Rinehart algebra module and $H^{hLR}_\ast(L, M)$ is the Lie-Rinehart algebra homology with coefficients in $M$.

**Remark 3.6.** If $A = R$ then $(\theta, \beta)$ is a representation of hom-Lie algebra $(L, [-, -], \alpha)$ on $M$ and $H^{hLR}_\ast(L, M)$ is the homology of a hom-Lie algebra with coefficients in $M$, which is defined in [15].

Let $(L, \alpha)$ be a hom-Lie-Rinehart algebra, then there is no canonical right module structure on $(A, \phi)$. In the following theorem we give a bijective correspondence between the exact generators of hom-Gerstenhaber bracket on $\wedge^\ast_A L$ and right module structures on $(A, \phi)$.

**Theorem 3.7.** There is a bijective correspondence between right $(L, \alpha)$-module structures on $(A, \phi)$ and exact generators of the hom-Gerstenhaber algebra bracket on $\wedge^\ast_A L$.

**Proof.** Let $(A, \phi)$ be a right module structure over hom-Lie-Rinehart algebra $(L, \alpha)$, then we can define an operator $D : \wedge^\ast_A L \rightarrow \wedge^\ast_A L$ as follows:

\[
D(x_1 \wedge \cdots \wedge x_n) = \sum_{i=1}^{n} (-1)^{i+1} \{1, x_i\} \cdot (\alpha(x_1) \wedge \cdots \wedge \alpha(x_i) \wedge \cdots \wedge \alpha(x_n)) + \\
\sum_{i<j} (-1)^{i+j} ([x_i, x_j] \wedge \alpha(x_1) \wedge \cdots \wedge \alpha(x_i) \wedge \cdots \wedge \alpha(x_j) \wedge \cdots \wedge \alpha(x_n))
\]

(4)

where $\{-,-\}$ denotes the right action of $L$ on $A$. It follows from straightforward calculations that $D$ commutes with $\alpha$ and generates the bracket on $\wedge^\ast_A L$, i.e.

\[\{X, Y\} = (-1)^{|X|}(D(XY) - (DX)\alpha(Y) - (-1)^{|X|}\alpha(X)(DY));\]

for any homogeneous elements $X, Y \in \wedge^\ast_A L$. 

Conversely, let $D : \wedge^*_A L \to \wedge^*_A L$ generates the hom-Gerstenhaber bracket on $\wedge^*_A L$ and $D^2 = 0$. Then define the right action of $L$ on $A$ as follows

$$\{ab, x\} = \phi(ab)(D(x)) - \rho(x)(ab) = \phi(a)(\phi(b)(D(x)) - \rho(x)(b)) - \phi(b)\rho(x)(a) = \phi(a)\{b, x\} - \phi(b)\rho(x)(a).$$

Here, $\rho$ is the anchor map of the hom-Lie-Rinehart algebra $(\mathcal{L}, \alpha)$. Furthermore, using equation (1) we have the following:

$$\{b, ax\} = \phi(b)(D(ax)) - \rho(ax)(b) = \phi(b)(\phi(a)(D(x)) - \rho(x)(a)) - \phi(a)\rho(x)(b) = \phi(a)\{b, x\} - \phi(b)\rho(x)(a).$$

Let us define $\theta : A \otimes L \to A$ as $\theta(a, x) = \{a, x\}$, then the conditions: $D^2 = 0$, and $D \circ \alpha = \alpha \circ D$ implies that $(\theta, \phi)$ is a representation of the hom-Lie algebra $(L, \{-,-\}, \alpha)$ on $A$. Hence, an exact generator gives a right $(\mathcal{L}, \alpha)$-module structure on the pair $(A, \phi)$.

Suppose the canonical hom-Gerstenhaber algebra structure on $\wedge^*_A L$ corresponding to the hom-Lie-Rinehart algebra $(\mathcal{L}, \alpha)$, has an exact generator $D$. Then by the above Theorem 3.7, we get the corresponding right $(\mathcal{L}, \alpha)$-module structure on $(A, \phi)$. Moreover, by the canonical isomorphism of $A$-modules: $A \otimes_A \wedge^*_A L \cong \wedge^*_A L$, and the definition of the boundary from equation (3), we have the following result:

**Proposition 3.8.** The homology $H^{hLR}_*(\mathcal{L}, A)$ of hom-Lie-Rinehart algebra $(\mathcal{L}, \alpha)$ with coefficients in right $(\mathcal{L}, \alpha)$-module $(A, \phi)$ is isomorphic to the homology of the chain complex $(\wedge^*_A L, D)$.

**Definition 3.9.** Let $M$ be an $A$-module, and $\beta \in \text{End}_R(M)$. Then the pair $(M, \beta)$ is a left-module over the hom-Lie Rinehart algebra $(\mathcal{L}, \alpha)$ if following conditions hold.

1. There is a map $\theta : L \otimes M \to M$, such that the pair $(\theta, \beta)$ is a representation of the hom-Lie algebra $(L, \{-,-\}, \alpha)$ on $M$. We denote $\theta(X)(m)$ by $\{X, m\}$ for $X \in L$ and $m \in M$.
2. $\beta(a.m) = \phi(a)\beta(m)$ for $a \in A$ and $m \in M$. 
3. $\{a.X, m\} = \phi(a)\{X, m\}$ for $a \in A$, $X \in L$, $m \in M$. 
4. $\{X, a.m\} = \phi(a)\{X, m\} + \rho(X)(a)\beta(m)$ for $X \in L$, $a \in A$, $m \in M$.

If we consider $\alpha = Id_L$, and $\beta = Id_M$, then the hom-Lie-Rinehart algebra $(\mathcal{L}, \alpha)$ turns out to be simply a Lie-Rinehart algebra and any left $(\mathcal{L}, \alpha)$-module $(M, \beta)$ is just a Lie-Rinehart algebra module.

**Example 3.10.** Suppose $(\mathcal{L}, \alpha)$ is a hom-Lie-Rinehart algebra over $(A, \phi)$. Then $(A, \phi)$ is a left $(\mathcal{L}, \alpha)$-module, where left action is given by the anchor map.

**3.3. Regular Hom-Lie-Rinehart algebras.** Let $(\mathcal{L}, \alpha)$ be a regular hom-Lie-Rinehart algebra over $(A, \phi)$ and $(M, \beta)$ be a left $(\mathcal{L}, \alpha)$-module. Let $\theta : L \times M \to M$ be the left action of $L$ on $M$, then define a cochain complex

$$(\text{Alt}_A(\mathcal{L}, M) = \oplus_{n \geq 0}\text{Alt}_A^n(\mathcal{L}, M), \delta),$$

where $\delta$ is the coboundary operator.


where $\text{Alt}_A^n(L, M) := \text{Hom}_A(\wedge^n_A L, M)$, and the coboundary map $\delta : \text{Alt}_A^n(L, M) \to \text{Alt}_A^{n+1}(L, M)$ is defined as follows:

$$
\delta f(x_1, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \theta(\alpha^{-1}(x_i))(f(\alpha^{-1}(x_1), \cdots, \hat{x}_i, \cdots, \alpha^{-1}(x_{n+1}))
+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \beta(f(\alpha^{-2}([x_i, x_j]), \alpha^{-1}(x_1), \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, \alpha^{-1}(x_{n+1}))
$$
for $f \in \text{Alt}_A^n(L, M)$, and $x_i \in L$, for $1 \leq i \leq n+1$.

**Proposition 3.11.** If $f \in \text{Alt}_A^n(L, M)$, then $\delta f \in \text{Alt}_A^{n+1}(L, M)$ and $\delta^2 = 0$.

**Proof.** Here, we need to check that for any $f \in \text{Hom}_A(\wedge^n_A L, M)$, $\delta f \in \text{Hom}_A(\wedge^{n+1}_A L, M)$ and $\delta^2 = 0$. This follows using the fact that $(\rho, \phi)$ is a representation of the hom-Lie algebra $(L, [-, -], \alpha)$ on $A$ and $(M, \beta)$ is a left $(L, \alpha)$-module. \qed

In view of this proposition we obtain that $(\text{Alt}_A(L, M), \delta)$ is a cochain complex. We denote the cohomology space of the cochain complex $(\text{Alt}_A(L, M), \delta)$ by $H^*_hR(L, M)$.

In particular, if we consider a regular hom-Lie algebra simply as a hom-Lie-Rinehart algebra, then the above cochain complex with coefficients in a hom-Lie algebra module is same as the cochain complex mentioned in Section 3 of [4].

Moreover, a hom-Lie algebroid is a particular case of hom-Lie-Rinehart algebras. Thus by following the above discussion, we define representations of a hom-Lie algebroid, and subsequently define a cohomology of a hom-Lie algebroid with coefficients in the representation in the next section.

**Example 3.12.** If $\alpha = \text{Id}_L$, then the hom-Lie-Rinehart algebra $(L, \alpha)$ is simply a Lie-Rinehart algebra $L$ over $A$ and the algebra $A$ is a Lie-Rinehart algebra module over $L$. Also, the cohomology $H^*_hR(L, M)$ is same as the Lie-Rinehart algebra cohomology with coefficients in the module $A$.

**Example 3.13.** Let $M$ be a smooth manifold, $A = C^\infty(M)$ and $L = \chi(M)$- the space of smooth vector fields on $M$. Let $\alpha = \text{Id}_L$, then the cohomology $H^*_hR(L, A)$ is the de-Rham cohomology of $M$.

Let $(L, \alpha)$ be a regular hom-Lie-Rinehart algebra over $(A, \phi)$. Then the pair $(A, \phi)$ is a left $(L, \alpha)$-module. Consider the cochain complex $(\text{Alt}_A(L, A), \delta)$ of the hom-Lie-Rinehart algebra $(L, \alpha)$ with coefficients in $(A, \phi)$. Now, define a multiplication $\wedge : \text{Alt}_A(L, A) \otimes_R \text{Alt}_A(L, A) \to \text{Alt}_A(L, A)$

$$
(\xi \wedge \eta)(x_1, \cdots, x_{p+q}) = \sum_{\sigma \in S(p,q)} \text{sgn}(\sigma)(\xi(x_{\sigma(1)}, \cdots, x_{\sigma(p)}), \eta(x_{\sigma(p+1)}, \cdots, x_{\sigma(p+q)})).
$$
for $x_1, \cdots, x_{p+q} \in L$, $\xi \in \text{Alt}_A^p(L, A)$, and $\eta \in \text{Alt}_A^q(L, A)$. Here, $S(p, q)$ denotes the set of all $(p, q)$-shuffles in the symmetric group $S_{p+q}$. Also, define $\Phi : \text{Alt}_A(L, A) \to \text{Alt}_A(L, A)$ as follows

$$
\Phi(\xi)(x_1, \cdots, x_p) = \phi(\xi(\alpha^{-1}(x_1), \cdots, \alpha^{-1}(x_p)))
$$
for $x_1, \cdots, x_p \in L$, and $\xi \in \text{Alt}_A^p(L, A)$. The multiplication $\wedge$ makes the chain complex $\text{Alt}_A(L, A)$, a graded commutative algebra since $A$ is a commutative algebra. Next, we have the following result:

**Lemma 3.14.** Let $(L, A)$ be a regular hom-Lie-Rinehart algebra, where the underlying $A$-module $L$ is projective of rank $n$. Then the pair $(\wedge_A^n L^*, \Phi)$ is a right $(L, \alpha)$-module.
Proof. For any \( x \in L \), define the contraction \( i_x : Alt_A(\mathcal{L}, A) \rightarrow Alt_A(\mathcal{L}, A) \) by
\[
i_x(\eta)(x_1, \cdots, x_{m-1}) = \Phi(\eta)(\alpha(x), x_1, \cdots, x_{m-1})
\]
for \( x_1, \cdots, x_{m-1} \in L \) and \( \eta \in Alt_A^m(\mathcal{L}, A) \). By a straightforward calculation, for any \( \eta_1 \in Alt_A^m(\mathcal{L}, A) \), and \( \eta_2 \in Alt_A^k(\mathcal{L}, A) \), we get the following equation:
\[
i_x(\eta_1 \wedge \eta_2) = i_x(\eta_1) \wedge \Phi(\eta_2) + (-1)^m \Phi(\eta_1) \wedge i_x(\eta_2)
\]
Define the right action \( \Theta : \wedge^n_A L^\ast \times L \rightarrow \wedge^n_A L^\ast \) as follows:
\[
\Theta(\xi, x) = -d(i_{\alpha^{-1}(x)}(\Phi^{-1}(\xi))).
\]
for all \( X \in \wedge^n_A L \) and \( \xi \in \wedge^n_A L^\ast \). Then it follows that the pair \( (\wedge^n_A L^\ast, \Phi) \) is a right \((L, \alpha)\)-module. \( \square \)

Theorem 3.15. Let \((\mathcal{L}, \alpha)\) be a regular hom-Lie-Rinehart algebra over \((A, \phi)\). If \( L \) is a projective \( A \)-module of rank \( n \), then there is a bijective correspondence between right \((\mathcal{L}, \alpha)\)-module structures on \((A, \phi)\) and left \((\mathcal{L}, \alpha)\)-module structures on \((\wedge^n_A L, \alpha)\).

Proof. First assume that \((A, \phi)\) is a right \((\mathcal{L}, \alpha)\)-module and the right action is given by \((a, x) \mapsto a.x\).
Here, \( \wedge^n_A L \) is projective \( A \)-module of rank \( 1 \), so we have an isomorphism of \( A \)-modules \( \Psi : \wedge^n_A L \rightarrow Hom_A(\wedge^n_A L^\ast, A) \) given by
\[
\Psi(X)(\xi) = \xi(X)
\]
for \( X \in \wedge^n_A L \) and \( \xi \in \wedge^n_A L^\ast \). For each \( X \in \wedge^n_A L \), denote \( \Psi(X) \) by \( \Psi_X \) and define an invertible map \( \gamma : Hom_A(\wedge^n_A L^\ast, A) \rightarrow Hom_A(\wedge^n_A L^\ast, A) \) as follows
\[
\gamma(\alpha.X) = \phi,a, \Psi_X(\alpha(X))
\]
for each \( X \in \wedge^n_A L \).

Now, for any \( x \in L \), \( X \in \wedge^n_A L \), and \( \xi \in \wedge^n_A L^\ast \), denote \( \Phi^{-1}(\xi) \) by \( \tilde{\xi} \) and define a left action \( \nabla : L \times Hom_A(\wedge^n_A L^\ast, A) \rightarrow Hom_A(\wedge^n_A L^\ast, A) \) as follows
\[
\nabla(x, \Psi_X)(\xi) = \gamma(\Psi_X(\Theta(\tilde{\xi}, x))) - (\Psi_X(\tilde{\xi})).x
\]
Here the map \( \Theta : \wedge^n_A L^\ast \times L \rightarrow \wedge^n_A L^\ast \) is the right action of the \( A \)-module \( L \) on \( \wedge^n_A L^\ast \), as defined in Lemma 3.14. By direct calculation, the left action \( \nabla \) makes the pair \((Hom_A(\wedge^n_A L^\ast, A), \gamma)\) a left \((\mathcal{L}, \alpha)\)-module. Subsequently, it gives a left \((\mathcal{L}, \alpha)\)-module structure on \((\wedge^n_A L, \alpha)\).

Conversely, let us consider \((\wedge^n_A L, \alpha)\) is a left \((\mathcal{L}, \alpha)\)-module, where the left action is given by \((x, X) \mapsto \nabla x(X)\). Since \( \wedge^n_A L \) is a projective \( A \)-module of rank \( 1 \), we have an isomorphism \( \theta : A \rightarrow Hom_A(\wedge^n_A L, \wedge^n_A L) \) which maps \( a \mapsto \theta a \), where \( \theta a(X) = \theta(a)(X) = a.X \) for \( a \in A \), \( X \in \wedge^n_A L \). The isomorphism \( \phi : A \rightarrow A \) induces an invertible map \( \tilde{\phi} : Hom_A(\wedge^n_A L, \wedge^n_A L) \rightarrow Hom_A(\wedge^n_A L, \wedge^n_A L) \), defined as: \( \phi(\theta a) = \theta(\phi(a)) \).

Define a right action \( \mu : Hom_A(\wedge^n_A L, \wedge^n_A L) \times L \rightarrow Hom_A(\wedge^n_A L, \wedge^n_A L) \) by:
\[
\mu(\theta a, x)(X) = \tilde{\phi}(\theta a)((x, \alpha^{-1}(X))_\theta) - \nabla x(\theta a(\alpha^{-1}(X)))
\]
here, \( X \in \wedge^n_A L, x \in L, a \in A \) and \([\cdot, \cdot]_\theta\) is the hom-Gerstenhaber bracket obtained from the hom-Lie bracket on \( L \). A straightforward but long calculation shows that the pair \((Hom_A(\wedge^n_A L, \wedge^n_A L), \tilde{\phi})\) is a right \((\mathcal{L}, \alpha)\)-module. Hence, there is a right \((\mathcal{L}, \alpha)\)-module structure on \((A, \phi)\). \( \square \)

Now, by Theorem 3.7 and Theorem 3.15, we immediately get the following result:

Corollary 3.16. Let \((\mathcal{L}, \alpha)\) be a regular hom-Lie-Rinehart algebra over \((A, \phi)\). If \( L \) is a projective \( A \)-module of rank \( n \), then there is a bijective correspondence between exact generators of the hom-Gerstenhaber algebra bracket on \( \wedge^n_A L \) and left \((\mathcal{L}, \alpha)\)-module structures on \((\wedge^n_A L, \alpha)\).
4. Applications to hom-Lie algebroids

Here we consider hom-Lie algebroids as defined by Laurent-Gengoux and Teles in [8]. There is also a modified version of hom-Lie algebroids presented by L. Cai, et al. in [3] and an equivalence is shown for the case of regular (or invertible) hom-Lie algebroids. First, we define Representations of a hom-Lie algebroid.

4.1. Representations of hom-Lie algebroids. Let \( \mathcal{A} := (A, \phi, [-,-], \rho, \alpha) \) be a hom-Lie algebroid and \((E, \phi, \beta)\) be a hom-bundle over a smooth manifold \(M\). A bilinear map \( \nabla : \Gamma A \otimes \Gamma E \to \Gamma E \), denoted by \( \nabla(x, s) := \nabla_x(s) \), is a representation of \( \mathcal{A} \) on a hom-bundle \((E, \phi, \beta)\) if it satisfies the following properties:

1. \( \nabla_{f.x}(s) = \phi^*(f).\nabla_x(s) \);
2. \( \nabla_x(f.s) = \phi^*(f).\nabla_x(s) + \rho(x)[f].\beta(s) \);
3. \( (\nabla, \beta) \) is a representation of the underlying hom-Lie algebra \((\Gamma A, [-,-], \alpha)\) on \(\Gamma E\);

for all \(x \in \Gamma A\), \(s \in \Gamma E\) and \(f \in C^\infty(M)\).

**Example 4.1.** Let \( \mathcal{A} = (A, \phi, [-,-], \rho, \alpha) \) be a hom-Lie algebroid over \(M\). Suppose \( \phi^* : C^\infty(M) \to C^\infty(M) \) denotes the algebra homomorphism induced by the smooth map \( \phi : M \to M \). Define a map \( \nabla^{\phi^*} : \Gamma A \otimes C^\infty(M) \to C^\infty(M) \) given by \( \nabla^{\phi^*}(x, f) = \rho(x)[f] \) for \(x \in \Gamma A\) and \(f \in C^\infty(M)\). Then \( \nabla^{\phi^*} \) is a canonical representation of \( \mathcal{A} \) on the hom-bundle \((M \times \mathbb{R}, \phi, \phi^*)\).

**Example 4.2.** Let \( \mathcal{A} = (A, \phi, [-,-], \rho, \alpha) \) be a hom-Lie algebroid over \(M\) and \((E, \phi, \beta)\) be a hom-bundle over \(M\), where \(E\) is a trivial line bundle over \(M\). Assume \(s \in \Gamma E\) is a nowhere vanishing section of the trivial line bundle \(E\) over \(M\). Define a map \( \nabla : \Gamma A \otimes \Gamma E \to \Gamma E \) by

\[
\nabla(x, f) = \rho(x)[f].\beta(s)
\]

for all \(x \in \Gamma A\) and \(f \in C^\infty(M)\). Then the map \( \nabla \) is a representation of \( \mathcal{A} \) on \((E, \phi, \beta)\).

Let \( \mathcal{A} = (A, \phi, [-,-], \rho, \alpha) \) be a hom-Lie algebroid, where \(A\) is a vector bundle of rank \(n\) over \(M\), then \(\wedge^n A\) is a line bundle over \(M\). Extend the map \(\alpha : \Gamma A \to \Gamma M\) to a map \(\tilde{\alpha} : \Gamma(\wedge^n A) \to (\Gamma \wedge^n A)\) defined by

\[
\tilde{\alpha}(x_1 \wedge \cdots \wedge x_n) = \alpha(x_1) \wedge \cdots \wedge \alpha(x_n)
\]

for any \(x_1, x_2, \ldots, x_n \in \Gamma A\).

**Proposition 4.3.** Let \( \mathcal{A} = (A, \phi, [-,-], \rho, \alpha) \) be a regular hom-Lie algebroid. Then there is a one-one correspondence between representations of \( \mathcal{A} \) on the hom-bundle \((\wedge^n A, \phi, \tilde{\alpha})\) and exact generators of the associated hom-Gerstenhaber algebra \(\mathfrak{g} := (\oplus_{k \geq 0} \Gamma \wedge^k A^*, \wedge, [-,-]_A, \tilde{\alpha})\) (here, \(\tilde{\alpha}\) is extension of the map \(\alpha\) to higher degree elements).

**Proof.** This result follows from the Corollary 3.16. More precisely, given an exact generator \(D\) of the associated hom-Gerstenhaber algebra \(\mathfrak{g}\), define a map \(\nabla : \Gamma A \otimes \Gamma(\wedge^n A) \to \Gamma(\wedge^n A)\) by

\[
\nabla(a, X) = [a, X]_A - (Da)\tilde{\alpha}(X),
\]

where \(a \in A\), and \(X \in \Gamma(\wedge^n A)\). It is immediate to check that \(\nabla\) is a representation of \(\mathcal{A}\) on the hom-bundle \((\wedge^n A, \phi, \tilde{\alpha})\).
Conversely, let $\nabla$ is a representation of $\mathcal{A}$ on the hom-bundle $(\wedge^n A, \phi, \alpha)$, then there exists a unique generator $D$ of the hom-Gerstenhaber bracket such that for any $X \in \Gamma(\wedge^n A)$ the following condition is satisfied:

$$D(a)\tilde{\alpha}(X) = [a, X]_A - \nabla(a, X).$$

Define $D$ on higher degree elements by the following relation:

$$D(a \wedge b) = -[a, b]_A + D(a)\tilde{\alpha}(b) - \tilde{\alpha}(a) \wedge D(b),$$

for $a \in A$, and $b \in \Gamma(\wedge^k A)$. By using this relation and the fact that $a \wedge X = 0$ for any $X \in \Gamma(\wedge^n A)$, the following condition is equivalent to equation (8):

$$\nabla(a, X) = -\tilde{\alpha}(a) \wedge D(X).$$

\[ \square \]

Let $\mathcal{A} = (A, \phi, [-, -], \rho, \alpha)$ be a hom-Lie algebroid and $(E, \phi, \beta)$ be a hom-bundle over a smooth manifold $M$, where $E$ is a line bundle. Then the following proposition extracts a representation of $\mathcal{A}$ on the hom-bundle $(E, \phi, \beta)$ from a given representation of $\mathcal{A}$ on the square hom-bundle given by the triplet $(E^2 := E \otimes E, \phi, \beta := \beta \otimes \beta)$ over $M$.

**Proposition 4.4.** Let $\mathcal{A} := (A, \phi, [-, -], \rho, \alpha)$ be a hom-Lie algebroid, and the triplet $(E, \phi, \beta)$ be a hom-bundle over a smooth manifold $M$, where $E$ is a line bundle. If the map $\nabla$ is a representation of $\mathcal{A}$ on the hom-bundle $(E, \phi, \beta)$, then the map $\nabla : \Gamma A \otimes \Gamma E \to \Gamma E$ is a representation of $\mathcal{A}$ on the hom-bundle $(E^2, \phi, \beta)$, which is defined as follows:

Let $s$ be a section of $E$ and $U$ be an open subset of $M$, then $s = f.t$ for some $f \in C^\infty(U)$ and some section $t \in \Gamma M$, which vanishes nowhere over $U$. Then define

$$\nabla(x, s) \big|_{\phi^{-1}(U)} = (\rho(x)(f)(\beta(t)) \big|_{\phi^{-1}(U)} + \frac{1}{2}(\nabla(x, t^2)(\beta(t^2)). \beta(s) \big|_{\phi^{-1}(U)}).

Furthermore, the map $\nabla : \Gamma A \otimes \Gamma E \to \Gamma E$ gives back the initial map $\tilde{\nabla} : \Gamma A \otimes \Gamma(E \otimes E) \to \Gamma(E \otimes E)$ by the following equation:

$$\tilde{\nabla}(x, s_1 \otimes s_2) = \nabla(x, s_1) \otimes \beta(s_2) + \beta(s_1) \otimes \nabla(x, s_2)$$

for $x \in \Gamma A$, $s_1, s_2 \in \Gamma E$.

**Proof.** First we prove that $(\nabla, \beta)$ is a representation of $(\Gamma A, [-, -], \alpha)$ on $\Gamma E$. We need to show the following:

- $\nabla([x, y], \beta(s)) = \nabla(\alpha(x), \nabla(y, s)) - \nabla(\alpha(y), \nabla(x, s))$, and

- $\nabla(\alpha(x), \beta(s)) = \beta(\nabla(x, s))$.

Let $s$ be a section of $E$ and $U \subset M$ be an open subset of $M$, then $s = f.t$ for some $f \in C^\infty(U)$ and some section $t \in \Gamma M$, which vanishes nowhere over $U$. Then

$$\nabla([x, y], \beta(s)) \big|_{\phi^{-2}(U)} = (\rho([x, y])(\phi(f)), \beta^2(t)) \big|_{\phi^{-2}(U)} + \frac{1}{2}(\nabla([x, y], \beta(t^2)) / \beta^2(t^2), \beta(s) \big|_{\phi^{-2}(U)}.$$

Here,

$$(\rho([x, y])(\phi(f)), \beta^2(t)) \big|_{\phi^{-2}(U)} = \left(\rho(\alpha(x))\rho(y)(f) - \rho(\alpha(y))\rho(x)(f)\right) \beta^2(t) \big|_{\phi^{-2}(U)}.$$

Since $\beta$ is a bijective map, the section $\beta(t^2)$ vanishes nowhere over the open subset $\phi^{-1}(U)$ of $M$, so we can write

$$\nabla(x, t^2) \big|_{\phi^{-1}(U)} = f_x \beta(t^2) \big|_{\phi^{-1}(U)}$$

(10)
for some $f_x \in C^\infty(M)$, and $x \in \Gamma A$. The map $\nabla$ is a representation of $\mathcal{A}$ on the hom-bundle $(E^2, \phi, \tilde{\beta})$, i.e.

$$\nabla([x, y], \tilde{\beta}(t^2)) = \nabla(\alpha(x), \nabla(y, t^2)) - \nabla(\alpha(y), \nabla(x, t^2))$$

and

$$\nabla(\alpha(x), \tilde{\beta}(t^2)) = \tilde{\beta}(\nabla(x, t^2)).$$

Thus,

$$\phi^*(f_{\alpha^{-1}([x, y])}) = \rho(\alpha(x))(f_y) - \rho(\alpha(y))(f_x).$$

Also, by the definition of $\nabla$, we have the following:

$$\nabla(\alpha(x), \nabla(y, s))\big|_{\phi^{-2}(U)}$$

$$= \nabla\left(\alpha(x), (\rho(y)(f)\cdot\beta(t) + \frac{1}{2}(\nabla(y, t^2)/\tilde{\beta}(t^2)), \beta(s)\right)\big|_{\phi^{-2}(U)}$$

$$= \left(\rho(\alpha(x))\rho(y)(f)\cdot\beta^2(t) + \frac{1}{2}\nabla(\alpha(x), \tilde{\beta}(t^2)), \beta(\rho(y)(f)\cdot\beta(t)) \right.$$  

$$+ \frac{1}{2}\rho(\alpha(x))\left(\frac{\nabla(y, t^2)}{\beta(t^2)}\cdot\phi^*(f)\right), \beta^2(t) + \frac{1}{4}\nabla(\alpha(x), \tilde{\beta}(t^2)), \beta\left(\frac{\nabla(y, t^2)}{\beta(t^2)}\cdot\phi^*(f)\right)\bigg)\bigg|_{\phi^{-2}(U)}$$

(12)  

$$= \left(\rho(\alpha(x))\rho(y)(f)\cdot\beta^2(t) + \frac{1}{2}\nabla(\alpha(x), \tilde{\beta}(t^2)), \beta(\rho(y)(f)\cdot\beta(t)) \right.$$  

$$+ \frac{1}{2}\rho(\alpha(x))\left(\frac{\nabla(y, t^2)}{\beta(t^2)}\cdot\phi^*(f)\right), \beta^2(t) + \frac{1}{4}\nabla(\alpha(x), \tilde{\beta}(t^2)), \beta\left(\frac{\nabla(y, t^2)}{\beta(t^2)}\cdot\phi^*(f)\right)\bigg)\bigg|_{\phi^{-2}(U)}.$$  

By using equation (10), the equation (12) can also be expressed as:

(13)  

$$\nabla(\alpha(x), \nabla(y, s))\big|_{\phi^{-2}(U)} = \left(\rho(\alpha(x))\rho(y)(f)\cdot\beta^2(t) + \frac{1}{2}\phi^*(f_x\cdot\rho(\alpha(y))(\phi^*(f)), \beta^2(t) \right.$$  

$$+ \frac{1}{2}\rho(\alpha(x))(f_y\cdot\phi^*(f)), \beta^2(t) + \frac{1}{4}\phi^*(f_x\cdot f_y), \beta^2(s)\bigg)\bigg|_{\phi^{-2}(U)}.$$  

Similarly,

(14)  

$$\nabla(\alpha(y), \nabla(x, s))\big|_{\phi^{-2}(U)} = \left(\rho(\alpha(y))\rho(x)(f)\cdot\beta^2(t) + \frac{1}{2}\phi^*(f_y\cdot\rho(\alpha(x))(\phi^*(f)), \beta^2(t) \right.$$  

$$+ \frac{1}{2}\rho(\alpha(y))(f_x\cdot\phi^*(f)), \beta^2(t) + \frac{1}{4}\phi^*(f_x\cdot f_y), \beta^2(s)\bigg)\bigg|_{\phi^{-2}(U)}.$$  

By equations (9), (11), (13), and (14), we get the following equation:

(15)  

$$\nabla(\alpha(x), \nabla(y, s)) - \nabla(\alpha(y), \nabla(x, s)) = \nabla([x, y], \beta(s)).$$
Moreover,
\[
\nabla(x(s), \beta(s)) |_{\phi^{-2}(U)} = \left( \frac{\rho(x(s)) \phi^*(f.s)}{\beta^*(t)} \cdot \beta^*(s) \right) \mid_{\phi^{-2}(U)}
\]
\[
= \left( \beta(\rho(x(f), \beta(t)) \cdot \frac{1}{2} \phi^*(f.x.s) \cdot \beta^*(s)) \right) \mid_{\phi^{-2}(U)}
\]
\[
= \left( \beta(\rho(x(f), \beta(t)) \cdot \frac{1}{2} f.x.s, \beta(s)) \right) \mid_{\phi^{-2}(U)}
\]
\[
= \beta(\nabla(x(s))) \mid_{\phi^{-2}(U)},
\]
i.e.,
\[
(16) \quad \nabla(x(s), \beta(s)) = \beta(\nabla(x, s)).
\]
Thus by equations (15), and (16), the pair \((\nabla, \beta)\) is a representation of \((\Gamma A, [-, -], \alpha)\) on \(\Gamma E\). It is immediate to observe that:

- \(\nabla(f.x, s) = \phi^*(f) \cdot \nabla(x, s),\)
- \(\nabla(x, f.s) = \phi^*(f) \cdot \nabla(x, s) + \rho(x(f), \beta(s)),\)

for all \(x \in \Gamma A, f \in C^\infty(M), \) and \(s \in \Gamma E\). Therefore, \(\nabla\) is a representation of \(\mathcal{A}\) on the hom-bundle \((E, \phi, \beta)\). Let \(U \subset M\) be an open subset of \(M\), then

\[
\bar{\nabla}(x, s_1 \otimes s_2) |_{\phi^{-1}(U)} = \nabla(x, s_1) \otimes \beta(s_2) |_{\phi^{-1}(U)} + \beta(s_1) \otimes \nabla(x, s_2) |_{\phi^{-1}(U)}
\]

for \(x \in \Gamma A, s_1, s_2 \in \Gamma E\). Hence, squaring the map \(\nabla: \Gamma A \otimes \Gamma E \to \Gamma E\) gives back the original map \(\bar{\nabla}: \Gamma A \otimes \Gamma(E^2) \to \Gamma(E^2)\).

The Proposition 4.4 generalises Proposition 4.2 of [5], which states: If a representation of a Lie algebroid on the square of a line bundle is given then there exists a representation of the Lie algebroid on the line bundle.

**Remark 4.5.** Let \(A\) be a real vector bundle of rank \(n\) over a manifold \(M\). If \(\mathcal{A} := (A, \phi, [-, -], \rho, \alpha)\) is a hom-Lie algebroid over \(M\) then \(\wedge^n A\) is a real line bundle over \(M\). Note that \(\wedge^n A \otimes \wedge^n A\) is a trivial line bundle over \(M\). Now, define a map \(\tilde{\alpha}: \wedge^n A \otimes \wedge^n A \to \wedge^n A \otimes \wedge^n A\) as follows:

\[
\tilde{\alpha}(X \otimes Y) = \tilde{\alpha}(X) \otimes \tilde{\alpha}(Y)
\]

for \(X, Y \in \wedge^n A\). By Example 4.2, there exists a representation of \(\mathcal{A}\) on the hom-bundle \((\wedge^n A \otimes \wedge^n A, \phi, \tilde{\alpha})\). Consequently, by the Proposition 4.4 we get a representation of \(\mathcal{A}\) on the hom-bundle \((\wedge^n A, \phi, \tilde{\alpha})\).

### 4.2. Cohomology of regular hom-Lie algebroids.

Let \(\mathcal{A} := (A, \phi, [-, -], \rho, \alpha)\) be a regular hom-Lie algebroid over \(M\) and the map \(\nabla\) be a representation of \(\mathcal{A}\) on the hom-bundle \((E, \phi, \beta)\). We define a cochain complex \((C^\ast(A; E), d_{A,E})\) for \(\mathcal{A}\) with coefficients in this representation as follows:
Theorem 4.6. Let \( \mathcal{A} \) be a regular hom-bundle over \( M \), i.e. the map \( \phi : M \to M \) is a diffeomorphism and \( \alpha : \Gamma A \to \Gamma A \) is an invertible map. Then a hom-Lie algebroid structure \( \mathcal{A} := (\mathcal{A}, \phi, [-,-], \rho, \alpha) \) on the hom-bundle \((\mathcal{A}, \phi, \alpha)\) is equivalent to a \((\hat{\alpha}, \hat{\alpha})\)-differential graded commutative algebra on \( \oplus_{n \geq 0} \Gamma(\wedge^n \mathcal{A}^*) \).

Proof. Let \( \mathcal{A} := (\mathcal{A}, \phi, [-,-], \rho, \alpha) \) be a hom-Lie algebroid over \( M \). Consider the coboundary operator \( d_A \) defined by equation (15). Then we need to prove that

\[
d_A(\zeta \wedge \eta) = d_A(\zeta) \wedge \Psi(\eta) + (-1)^{|\zeta|} \Psi(\zeta) \wedge d_A(\eta)
\]

for all \( \zeta \in \Gamma(\wedge^p \mathcal{A}^*), \eta \in \Gamma(\wedge^q \mathcal{A}^*), \) and \( p, q \geq 0 \). It simply follows by induction on the degree \( p \).

Then the tuple \( (\oplus_{n \geq 0} \Gamma(\wedge^n \mathcal{A}^*), \hat{\alpha}, d_A) \) is a \((\hat{\alpha}, \hat{\alpha})\)-differential graded commutative algebra.

Conversely, let \( (\oplus_{n \geq 0} \Gamma(\wedge^n \mathcal{A}^*), \hat{\alpha}, d) \) be a \((\hat{\alpha}, \hat{\alpha})\)-differential graded commutative algebra. We define the anchor map \( \rho \) given by \( \rho(x)[f] = d_A f, \alpha(x) > \), and the hom-Lie bracket \([-,-]\) is given by

\[
< [x, y], \xi > = \rho(\alpha^2(x)) < \hat{\alpha}^{-1}(\xi), y > - \rho(\alpha^2(y)) < \hat{\alpha}^{-1}(\xi), x > - (d(\hat{\alpha}^{-1}(\xi))(\alpha(x), \alpha(y))
\]

for all \( x, y \in \Gamma A, f \in C^\infty(M) \) and \( \xi \in \Gamma A^* \). Then it follows that \((\mathcal{A}, \phi, [-,-], \rho, \alpha)\) is a hom-Lie algebroid. \( \square \)
A version of the above theorem is proved in [3], by considering a modified definition of hom-Lie algebroid and the associated cochain complex. Let us recall the following definitions of interior multiplication and Lie derivative from [3]:

- For any $X \in \Gamma(\wedge^k A)$, define the interior multiplication $i_X : \Gamma(\wedge^n A^*) \to \Gamma(\wedge^{n-k} A^*)$ by
  \[
  (i_X \Xi)(x_1, x_2, \cdots, x_{n-k}) = (\hat{\alpha}(\Xi))(\hat{\alpha}(X), x_1, \cdots, x_{n-k})
  \]
  for any $x_1, x_2, \cdots, x_{n-k} \in \Gamma A$.
- Let $X \in \Gamma(\wedge^k A)$, then define the Lie derivative $L_X : \Gamma(\wedge^n A^*) \to \Gamma(\wedge^{n-k+1} A^*)$ by
  \[
  L_X \circ \hat{\alpha} = i_X \circ d_A - (-1)^k d_A \circ i_{\hat{\alpha}^{-1}}(X).
  \]

Now for all $X \in \Gamma(\wedge^k A)$, $Y \in \Gamma(\wedge^l A)$, $f \in C^\infty(M)$, and $\Xi \in \Gamma(\wedge^n A^*)$ the interior multiplication satisfies the following properties:

1. $i_f \Xi = i_X(f \Xi) = \phi^*(f) i_X$,
2. $\hat{\alpha}(i_X(\Xi)) = i_{\hat{\alpha}(X)}(\hat{\alpha}(\Xi))$,
3. $\hat{\alpha}(\Xi(x_1, x_2)) = i_{\hat{\alpha}(x_1)}(i_{\hat{\alpha}(x_2)}(\Xi)) - (-1)^{kl} i_{\hat{\alpha}(x_1)}(i_{\hat{\alpha}(x_2)}(\Xi))$.

The Lie derivative $L_X : \Gamma(\wedge^n A^*) \to \Gamma(\wedge^{n-k+1} A^*)$, for any $X \in \Gamma(\wedge^k A)$ satisfies the following equation:

\[
L_{f,X}\Xi = \phi^*(f) L_X\Xi - (-1)^k d_A f \wedge i_X(\Xi)
\]

for all $f \in C^\infty(M)$, and $\Xi \in \Gamma(\wedge^n A^*)$. If $k = 1$, i.e. $X \in \Gamma A$, then for all $f \in C^\infty(M)$, and $\Xi \in \Gamma(\wedge^n A^*)$, we have

\[
L_X(f,\Xi) = \phi^*(f) L_X\Xi + \rho(x)(f)\hat{\alpha}(\Xi)
\]

Moreover, for $x \in \Gamma A$, the interior multiplication $i_x$ and Lie derivative $L_x$ satisfy the following identities:

\[
i_x(\Xi_1 \wedge \Xi_2) = i_x \Xi_1 \wedge \hat{\alpha}(\Xi_2) + (-1)^m \hat{\alpha}(\Xi_1) \wedge i_x \Xi_2,
\]
\[
L_x(\Xi_1 \wedge \Xi_2) = L_x \Xi_1 \wedge \hat{\alpha}(\Xi_2) + \hat{\alpha}(\Xi_1) \wedge L_x \Xi_2,
\]

for any $\Xi_1 \in \Gamma(\wedge^n A^*)$, $\Xi_2 \in \Gamma(\wedge^n A^*)$. The above properties and identities follows from [3], by replacing the differential $d : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$, defined in Section 3 of [3], by the differential $d_A$ given by equation (18).

**Remark 4.7.** For a manifold $M$ with a diffeomorphism $\psi : M \to M$, the tangent hom-Lie algebroid $T$ is given by the tuple $(\psi^! TM, \psi, [-, -]_{\psi^*}, \text{Ad}_{\psi^*}, \text{Ad}_{\psi^*})$, where the bracket $[-, -]_{\psi^*}$ and anchor map $\text{Ad}_{\psi^*}$ are defined as follows:

- $[X, Y]_{\psi^*} = \psi^* \circ X \circ (\psi^* )^{-1} \circ Y \circ (\psi^* )^{-1} - \psi^* \circ Y \circ (\psi^* )^{-1} \circ X \circ (\psi^* )^{-1}$ for $X, Y \in \psi^! TM$;
- $\text{Ad}_{\psi^*}(X) = \psi^* \circ X \circ (\psi^* )^{-1}$ for $X \in \psi^! TM$.

If $(M, \psi, \pi)$ is a hom-Poisson manifold, where $\psi : M \to M$ is a diffeomorphism and $\pi$ is a hom-Poisson bivector, then $T^* := (\psi^! T^* M, \psi, [-, -]_{\pi^#}, \text{Ad}_{\psi^*}^!, \pi^# \circ \text{Ad}_{\psi^*}^!)$ is the cotangent hom-Lie algebroid as defined in [3], where

- $[\xi, \eta]_{\pi^#} = L_{\pi^#}(\xi) \eta - L_{\pi^#}(\eta) \xi - d\pi(\xi, \eta)$ for $\xi, \eta \in \psi^! T^* M$,
- $\text{Ad}_{\psi^*}^!(\xi)(X) = \psi^*(\xi(\text{Ad}_{\psi^*}^{-1}(X)))$ for $\xi \in \psi^! T^* M$, $X \in \psi^! TM$, and
- the anchor map is $\pi^# \circ \text{Ad}_{\psi^*}^!$ instead of $\pi^#$ (Note that $\pi^#$ is the anchor map for the cotangent hom-Lie algebroid in [3]), since we are using the Definition 2.10 of hom-Lie algebroid.
Let us consider \((\Gamma(\wedge^\text{top}\psi^1T^*M))^2 := \Gamma(\wedge^\text{top}\psi^1T^*M) \otimes \Gamma(\wedge^\text{top}\psi^1T^*M)\), where \(\wedge^\text{top}\) denotes the highest exterior power. Define a map \(D: \Gamma(\psi^1T^*M) \otimes (\Gamma(\wedge^\text{top}\psi^1T^*M))^2 \rightarrow (\Gamma(\wedge^\text{top}\psi^1T^*M))^2\) as follows:

\[
D_\xi(\eta_1 \otimes \eta_2) := D(\xi, (\eta_1 \otimes \eta_2)) = [[\xi, \eta_1]]_{\pi^\#} \otimes A\psi_{\ast}^1(\eta_2) + A\phi_{\ast}^1(\eta_1) \otimes L_{\pi^\#(\xi)}(\eta_2)
\]

for \(\xi \in \Gamma(\psi^1T^*M)\) and \(\eta_1, \eta_2 \in \Gamma(\wedge^\text{top}\psi^1T^*M)\). Then it follows by a long but straightforward calculation that the map \(D\) is a representation of \(T^*\) on the hom-bundle \((\Gamma(\wedge^\text{top}\psi^1T^*M))^2, \psi, A\psi_{\ast}^1\), where the map \(A\psi_{\ast}^1\) is the extension of the map \(A\psi_{\ast}\) to \((\Gamma(\wedge^\text{top}\psi^1T^*M))^2\).

4.3. Homology of regular hom-Lie algebroids. Let \(\mathcal{A} := (A, \phi, [-,-], \rho, \alpha)\) be a regular hom-Lie algebroid over a manifold \(M\), where \(A\) is a real vector bundle of rank \(n\) over \(M\), then by Remark 4.5, we get a representation of \(\mathcal{A}\) on the hom-bundle \((\wedge^k A, \phi, \alpha)\). Let us denote this homology by \(H_\mathcal{A}^k\). If \(\nabla^1\) and \(\nabla^2\) are two representations of \(\mathcal{A}\) on the hom-bundle \((\wedge^n A, \phi, \alpha)\), then it is natural to ask about the relation between the induced homologies.

Firstly, let \(D_{\nabla^1}\) and \(D_{\nabla^2}\) be exact generators of the associated hom-Gerstenhaber algebra \(\mathfrak{G}\) to the hom-Lie algebroid \(\mathcal{A}\), obtained by Proposition 4.3 respectively. Then first observe that

\[
D_{\nabla^1} - D_{\nabla^2}(f, x) = \phi(f). (D_{\nabla^1} - D_{\nabla^2})(x)
\]

for \(f \in C^\infty(M)\) and \(x \in \Gamma A\). Therefore, there exists \(\xi \in \Gamma A^\ast\) such that

\[
D_{\nabla^1} - D_{\nabla^2}(x) = \phi(\xi(x)) = i_\xi(x),
\]

for \(x \in \Gamma A\). Since \(D_{\nabla^1}\) and \(D_{\nabla^2}\) commute with the map \(\alpha\), we have \(\alpha^\ast(i_\xi(x)) = i_\xi(\alpha(x))\). By equation (11), \(D_{\nabla^1} - D_{\nabla^2}\) satisfies the following condition:

\[
(D_{\nabla^1} - D_{\nabla^2})(X \wedge Y) = (D_{\nabla^1} - D_{\nabla^2})(X) \wedge \alpha(Y) + (\alpha(Y) \wedge (D_{\nabla^1} - D_{\nabla^2})(Y))
\]

for \(X, Y \in \mathfrak{G}\), \(|X|\) denotes degree of \(X\). Then for any \(X \in \mathfrak{G}\), it follows that \(D_{\nabla^1} - D_{\nabla^2}(X) = i_\xi(X)\) and \(\alpha \circ i_\xi = i_\xi \circ \alpha\). The equation (12) is equivalent to the following equation:

\[
(\nabla^1_x - \nabla^2_x)(X) = i_\xi(x). \alpha(X)
\]

for \(x \in \Gamma A\) and \(X \in \Gamma(\wedge^n A)\). Now, let us make the following observations:

- \((D_{\nabla^2} \circ i_\xi + i_\xi \circ D_{\nabla^2})(x, y) = -i_{df}(\alpha(x), \alpha(y))\) for any \(x, y \in \Gamma A\), and
- by using the fact that \(D_{\nabla^1} \circ D_{\nabla^1} = 0\), and \(D_{\nabla^2} \circ D_{\nabla^2} = 0\), we get \(df = 0\), i.e. \(\xi\) is a 1-cocycle.

We say that the maps \(\nabla^1\) and \(\nabla^2\) are homotopic if \(\xi\) is a 1-coboundary, i.e. \(\xi = df\) for some \(f \in C^\infty(M)\). Similarly, in this case the corresponding generators \(D_{\nabla^1}\) and \(D_{\nabla^2}\) are said to be homotopic.

**Theorem 4.8.** If representations \(\nabla^1\) and \(\nabla^2\) of \(A\) on the hom-bundle \((\wedge^n A, \phi, \alpha)\) are homotopic, then \(H_\mathcal{A}^1\cong H_\mathcal{A}^2\).

**Proof.** It is given that maps \(\nabla^1\) and \(\nabla^2\) are homotopic, i.e. for \(X \in \Gamma(\wedge^\text{top} A)\) and \(x \in \Gamma A\) we have the following equation

\[
(\nabla^1_x - \nabla^2_x)(X) = i_{df}(x). \alpha(X),
\]

or equivalently

\[
D_{\nabla^1} - D_{\nabla^2} = i_{df}
\]
for some $f \in C^\infty(M)$ such that $\tilde{\alpha} \circ i_{df} = i_{df} \circ \tilde{\alpha}$. Now, let us define a map $F : \oplus_{k \geq 0} \Gamma \wedge^k A \to \oplus_{k \geq 0} \Gamma \wedge^k A$ by

$$F(\lambda) = e^f \lambda$$

for $\lambda \in \Gamma \wedge^k A$, $k \geq 0$. By using the fact that $[e^f, \lambda]_A = -\rho(\lambda)(e^f) = e^{\phi(f)}[f, \lambda]$, and equation \([1]\) for generators $D_\nabla$ and $D_\nabla^2$, we get the following relation:

$$D_\nabla(f.e^f, \lambda) = e^{\phi(f)}D_\nabla^2(\lambda)$$

for any $\lambda \in \Gamma \wedge^k A$, $k \geq 0$. Now, it is clear that $F(Ker(D_\nabla^2)) \subset Ker(D_\nabla^1)$ and thus $F$ induces a map $\tilde{F} : Ker(D_\nabla^2) \to Ker(D_\nabla^1)$. Let $\lambda = D_\nabla^1(R)$ for $\lambda \in \Gamma(\wedge^k A)$, $k > 0$ and for some $R \in \Gamma(\wedge^{k-1} A)$, $k > 0$. Then

$$F(\lambda) = e^f \lambda = e^f.D_\nabla^2(R) = D_\nabla^1(e^{\phi(f)}(R)). \lambda$$

Hence $\tilde{F}$ induces an isomorphism $F : H^*_\nabla(A) \to H^*_\nabla(A)$. □

Let $(M, \psi, \pi)$ be a hom-Poisson manifold and recall the associated cotangent hom-Lie algebroid from remark \([17]\) which is given by $T^* := (\psi^! TM, \psi, [-,-]_\pi^#, Ad_{\psi^*} \#_0, \#_0 \circ Ad_{\psi^*})$. Also recall that the map $D$ is a representation of $T^*$ on the hom-bundle $((\wedge^{top} \psi^! TM)^2, \psi, (Ad_{\psi^*})^2)$. Then by Proposition \([4,3]\) we have a representation of $T^*$ on the hom-bundle $(\wedge^{top} \psi^! TM, \psi, Ad_{\psi^*})$. In particular, we have the following result:

**Proposition 4.9.** Let $(M, \psi, \pi)$ be a hom-Poisson manifold. Define a map $D : \Gamma(\psi^! TM) \otimes \Gamma(\wedge^{top} \psi^! TM) \to \Gamma(\wedge^{top} \psi^! TM)$ as follows:

$$\bar{D}(\xi, \mu) = [\xi, \mu]_\pi^# - \pi(d\xi).Ad_{\psi^*}(\mu)$$

for any $\xi \in \Gamma(\psi^! TM)$, $\mu \in \Gamma(\wedge^{top} \psi^! TM)$. Then the map $\bar{D}$ is a representation of $T^*$ on the hom-bundle $(\wedge^{top} \psi^! TM, \psi, Ad_{\psi^*}^\dagger)$. 

**Proof.** Let us denote $\tilde{\gamma} := Ad_{\psi^*}^\dagger(\gamma)$, and $\bar{\gamma} := (Ad_{\psi^*})^{-1}(\gamma)$ for $\gamma \in \Gamma(\wedge^k \psi^! TM)$. Then

$$D([\xi, \eta]_\pi^#, \tilde{\mu}) = [[\xi, \eta]_\pi^#, \tilde{\mu}]_\pi^# - \pi(d[\xi, \eta]_\pi^#) \tilde{\mu}$$

Also, by using equation \([22]\), we have the following:

$$\bar{D}(\xi, \bar{D}(\eta, \mu)) = \bar{D}([\xi, \eta]_\pi^#, \pi(d\eta) \tilde{\mu})$$

$$\bar{D}([\xi, \eta]_\pi^#, \pi(d\eta) \tilde{\mu}) = [[\xi, [\eta, \mu]]_\pi^#]_\pi^# - \pi(d\xi).[\tilde{\eta}, \tilde{\mu}]_\pi^# - [[\xi, \pi(d\eta) \tilde{\mu}]_\pi^#$$

$$\bar{D}([\xi, [\eta, \mu]]_\pi^#, \pi(d\eta) \tilde{\mu}) = \bar{D}([\xi, [\eta, \mu]]_\pi^#, \pi(d\eta) \tilde{\mu})$$

Similarly,

$$\bar{D}(\bar{\eta}, \bar{D}(\xi, \mu)) = [[\tilde{\eta}, [\xi, \mu]]_\pi^#]_\pi^# - \pi(d\tilde{\eta}).[[\tilde{\xi}, \tilde{\mu}]_\pi^# - \psi^*(\pi(d\xi))[[\tilde{\xi}, \tilde{\mu}]_\pi^#$$

Now let us observe the following:

- $L_X(\gamma(\alpha)) = L_X(\gamma)(Ad_{\psi^*} \alpha) + \bar{\gamma}(L_X(\alpha))$ for all $X \in \Gamma \psi^! TM$, $\alpha \in \Gamma(\wedge^k \psi^! TM)$, and $\gamma \in \Gamma(\wedge^k \psi^! TM)$. 


Using above observations and the fact that \( Ad_{\psi^*} \circ \pi^\# = \pi^\# \circ Ad_{\psi^*} \), we have the following:

\[
- \pi^\#(\tilde{\xi})(\pi(d\eta)) + \pi^\#(\tilde{\eta})(\pi(d\xi)) = -[\pi^\#(\tilde{\xi}), \pi]|_{\psi^*}(d\eta) - Ad_{\psi^*} \circ \pi(L_{\pi^\#(\tilde{\xi})}d\eta) + [\pi^\#(\tilde{\eta}), \pi]|_{\psi^*}(d\xi) + Ad_{\psi^*} \circ \pi(L_{\pi^\#(\tilde{\eta})}d\xi)
\]

\[
= [[\pi^\#(\tilde{\eta}), \pi]|_{\psi^*}(d\xi)] - [[\pi^\#(\tilde{\xi}), \pi]|_{\psi^*}(d\eta) + \pi(L_{\pi^\#(\tilde{\eta})}d\xi - L_{\pi^\#(\tilde{\xi})}d\eta)
\]

\[
= [[\pi^\#(\tilde{\eta}), \pi]|_{\psi^*}(d\xi)] - [[\pi^\#(\tilde{\xi}), \pi]|_{\psi^*}(d\eta) + \pi(d(L_{\pi^\#(\tilde{\eta})}d\xi - L_{\pi^\#(\tilde{\xi})}d\eta + d\pi(\xi, \eta)))]
\]

Thus, by using equations (22), (23), (24), and (25), we immediately get the following:

- \( \bar{D}(\tilde{\xi}, \tilde{\eta}) = D(\tilde{\xi}, \tilde{\eta}) \)
- \( D(\tilde{\xi}, \tilde{\mu}) = Ad_{\psi^*} D(\tilde{\xi}, \tilde{\mu}) \)

i.e. the pair \((\bar{D}, Ad_{\psi^*})\) is a representation of hom-Lie algebra \((\Gamma^*(\Lambda^1(TM))^*, [-, -]_{\pi^\#}, Ad_{\psi^*})\) on the line bundle \(\Gamma(\Lambda^1(T^*M))\) with respect to the map \(Ad_{\psi^*}\). Furthermore, \( \bar{D}(f \tilde{\xi}, \tilde{\mu}) = \psi^*(f) \bar{D}(\tilde{\xi}, \tilde{\mu}) \) and \( \bar{D}(\tilde{\xi}, f \tilde{\mu}) = \psi^*(f) \bar{D}(\tilde{\xi}, \tilde{\mu}) + \pi^\#(\tilde{\xi})(f) \tilde{\mu} \) for any \( f \in C^\infty(M) \), \( \tilde{\xi} \in \Gamma(\Lambda^1(T^*M)) \), and \( \tilde{\mu} \in \Gamma(\Lambda^1(T^*M)) \). Hence, the map \( \bar{D} \) is a representation of \( T^* \) on the hom-bundle \((\Lambda^1(T^*M), \psi, Ad_{\psi^*})\).

**Remark 4.10.** Note that Equation (22) can be rewritten as

\[
\bar{D}(\tilde{\xi}, \tilde{\mu}) = L_{\pi^\#(\tilde{\xi})}(\tilde{\mu}) + \pi(d\xi)(\tilde{\mu})
\]

By Proposition 4.3, the representation \( \bar{D} \) of \( T^* \) on hom-bundle \((\Lambda^1(T^*M), \psi, Ad_{\psi^*})\) corresponds to an exact generator of the hom-Gerstenhaber algebra associated to the cotangent hom-Lie algebroid \( T^* \). In particular, we get the operator \([i_{\pi}, d]_{Ad_{\psi^*}} : \Gamma(\Lambda^k(T^*M)) \rightarrow \Gamma(\Lambda^{k-1}(T^*M))\), defined as

\[
[i_{\pi}, d]_{Ad_{\psi^*}} = Ad_{\psi^*} \circ i_{\pi} \circ (Ad_{\psi^*})^{-1} \circ d \circ (Ad_{\psi^*})^{-1} - Ad_{\psi^*} \circ d \circ (Ad_{\psi^*})^{-1} \circ i_{\pi} \circ (Ad_{\psi^*})^{-1}
\]

We call the homology of the chain complex \((\oplus_{k \geq 0} \Gamma(\Lambda^k(T^*M)), [i_{\pi}, d]_{Ad_{\psi^*}})\), the **hom-Poisson homology** associated to hom-Poisson manifold \((M, \psi, \pi)\). Note that in the case \( \psi = id_M \), it gives the Poisson homology.

### 5. Strong differential hom-Gerstenhaber algebras

Let \((A, \psi, \alpha)\) be invertible hom-bundle over \(M\), \((A, \psi, [-, -]_A, \rho_A, \alpha)\) and \((A^*, \psi, [-, -]_{A^*}, \rho_{A^*}, \alpha^\dagger)\) be two Hom-Lie algebroids in duality (here, \( \alpha^\dagger(\xi)(X) = \psi^*(\xi(\alpha^{-1}(X))) \) for \( \xi \in \Gamma A^* \) and \( X \in \Gamma A \)). Then recall from [3] that the pair \((A, A^*)\) is called a hom-Lie bialgebroid if

\[
d_{s}(s, y) = [d_{s}x, \alpha(y)] + [\alpha(x), d_{s}y], \text{ for all } x, y \in \Gamma A.
\]

Here, the map \( d_{s} \) is the coboundary map given by (17) for the hom-Lie algebroid \((A^*, \psi, [-, -]_{A^*}, \rho_{A^*}, \alpha^\dagger)\) and the bracket on the right hand side is the hom-Gerstenhaber bracket induced on exterior bundle over \(A\) by the hom-Lie algebroid structure \((A, \psi, [-, -]_A, \rho_A, \alpha)\).

**Definition 5.1.** A differential hom-Gerstenhaber algebra is a hom-Gerstenhaber algebra \( \mathcal{A} := (\oplus_{i \in \mathbb{Z}_+} A_i, \wedge, [-, -], \alpha) \) equipped with a degree 1 map \( d : \mathcal{A} \rightarrow \mathcal{A} \) such that
the map \( d \) is a \((\alpha, \alpha)\)-derivation of degree 1 with respect to the graded commutative and associative product \( \wedge \), i.e. \( d(X \wedge Y) = [dX, \alpha(Y)] + [\alpha(X), dY] \) for \( X, Y \in \mathfrak{A} \).

- \( d^2 = 0 \) and the map \( d \) commutes with \( \alpha \), i.e. \( d \circ \alpha = \alpha \circ d \).

The hom-Gerstenhaber algebra \( \mathfrak{A} \) is said to be a strong differential hom-Gerstenhaber algebra if \( d \) also satisfies the equation: \( d[X, Y] = [dX, \alpha(Y)] + [\alpha(X), dY] \) for \( X, Y \in \mathfrak{A} \). Let us denote this strong differential hom-Gerstenhaber algebra by the tuple \((\oplus_{i \in \mathbb{Z}}, \mathfrak{A}, \wedge, [-,-], \alpha, d)\). If the map \( \alpha : \mathfrak{A} \to \mathfrak{A} \) is an invertible map, then we say \( \mathfrak{A} \) is a strong differential regular hom-Gerstenhaber algebra.

Example 5.2. Let \((\mathfrak{g}, [\cdot, \cdot], \alpha)\) and \((\mathfrak{g}^*, [\cdot, \cdot]^*, \alpha^1)\) be two regular hom-Lie algebras (where \( \alpha^1(\xi)(X) = \xi(\alpha^{-1}(X)) \) for \( \xi \in \mathfrak{g}^* \) and \( X \in \mathfrak{g} \)). Recall from \([2]\), \((L, L^*)\) is a purely hom-Lie bialgebra if the following compatibility condition holds:

\[
\Delta([x, y]_L) = [\alpha^{-1}(x), \Delta(y)]_\mathfrak{g} + [\Delta(x), \alpha^{-1}(y)]_\mathfrak{g}.
\]

Here, \( x, y \in L \), the bracket \([\cdot, \cdot]_\mathfrak{g}\) is the hom-Gerstenhaber bracket obtained by extending \([\cdot, \cdot]_\mathfrak{g}\), and \( \Delta : L \to \wedge^2 L \) is the dual map of the hom-Lie algebra bracket \([\cdot, \cdot]_\mathfrak{g} : \wedge^2 \mathfrak{g} \to \mathfrak{g}^* \).

Let us first recall from Section 3 of \([2]\) that the adjoint representation of \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \alpha)\) on \( \wedge^k \mathfrak{g} \) is given by \( \text{ad}_x(Y) = [x, Y]_\mathfrak{g} \) for \( x, Y \in \mathfrak{g} \) and \([\cdot, \cdot]_\mathfrak{g}\) is the associated hom-Gerstenhaber bracket on the exterior algebra \( \wedge \mathfrak{g} \). Then the above equation (27) can also be expressed as

\[
\Delta([x, y]_L) = \text{ad}_{\alpha^{-1}(x)}(\Delta(y)) - \text{ad}_{\alpha^{-1}(y)}(\Delta(x)).
\]

Also, recall that the co-adjoint representation of \((\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \alpha)\) on \( \mathfrak{g}^* \) with respect to \( \alpha^1 = (\alpha^{-1})^* \) is given by the map \( \text{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*) \) such that \( < \text{ad}^*_x(\xi), y > = - < \xi, \text{ad}_x(y) > \), and \( \text{ad}^*_x(\xi) = \alpha_x^*((\alpha^2)\xi) \) for \( x, y \in \mathfrak{g} \) and \( \xi \in \mathfrak{g}^* \). The map \( \text{ad}^*_x : \mathfrak{g}^* \to \mathfrak{g}^* \) for any \( x \in \mathfrak{g} \) is extended to higher degree elements by the following equation:

\[
\text{ad}^*_x(\xi_1 \wedge \cdots \wedge \xi_n) = \sum_{1 \leq i \leq n} \alpha^*_x(\xi_1) \wedge \text{ad}^*_x(\xi_i) \wedge \cdots \wedge \alpha^*_x(\xi_n)
\]

where \( \xi_i \in \mathfrak{g}^* \) for all \( 1 \leq i \leq n \). Let \( d_* \) be the coboundary operator of the hom-Lie algebra \((\mathfrak{g}^*, [\cdot, \cdot]^*, \alpha^1)\) given by equation (15) (Consider the hom-Lie algebra \((\mathfrak{g}^*, [-,-]^*, \alpha^1)\) as a hom-Lie algbroid over point manifold \( M \)). Note that for any \( x, y \in \mathfrak{g} \), and \( \eta \in \mathfrak{g}^* \) we have

\[
d_*[x, y](\xi \wedge \eta) = < [x, y], ((\alpha^2)^*([\xi, \eta])) > = - < \Delta(\alpha^2[x, y]), \xi \wedge \eta >.
\]

Then by using equations (28), and (29), we get the following derivation condition:

\[
d_*[X, Y]_\mathfrak{g} = [d_*X, \alpha_\mathfrak{g}(Y)]_\mathfrak{g} + [\alpha_\mathfrak{g}(X), d_*Y]_\mathfrak{g}.
\]

for any \( X, Y \in \wedge \mathfrak{g}^* \). Thus the hom-Gerstenhaber algebra \((\mathfrak{g} = \wedge^\bullet \mathfrak{g}, \wedge, [-,-]_\mathfrak{g}, \alpha)\) associated to the hom-Lie algebra \((\mathfrak{g}, [-,-]_\mathfrak{g}, \alpha)\) gives a strong differential hom-Gerstenhaber algebra with differential \( d_* \).

Moreover, considering the coboundary operator \( d \) of the hom-Lie algebra \((\mathfrak{g}, [-,-]_\mathfrak{g}, \alpha)\) given by (18), the hom-Gerstenhaber algebra \((\mathfrak{g}^* = \wedge^\bullet \mathfrak{g}^*, \wedge, [-,-]^*_\mathfrak{g}, \alpha^*_\mathfrak{g})\) associated to the hom-Lie algebra \((\mathfrak{g}^*, [-,-]^*_\mathfrak{g}, \alpha^1)\) gives a strong differential hom-Gerstenhaber algebra with the strong differential \( d : \mathfrak{g}^* \to \mathfrak{g}^* \).

Example 5.3. Suppose \((M, \psi, \pi)\) is a hom-Poisson manifold and \( \psi : M \to M \) is a diffeomorphism. Then we have the tangent hom-Lie algebroid \((\psi^1TM, \psi, [-,-]_\psi, \text{Ad}_\psi^*, \text{Ad}_\psi^*)\) and the cotangent
hom-Lie algebroid \((\psi^T M, \psi, [-,-]_\pi, Ad_\psi^\dagger, \pi^\# \circ Ad_\psi^\dagger)\) respectively. Let us denote the corresponding differentials of tangent and cotangent hom-Lie algebroids with trivial representation by \(d\) and \(d_a\), respectively. Then we have the following observations:

- For any \(X \in \Gamma \wedge^k (\psi^T M)\), \(d_a X = [(\pi^\# X)]^\psi\), where \([[-,-]]^\psi\) is the hom-Gerstenhaber bracket obtained by extending the bracket \([-,-]\) with respect to graded commutative product \([\alpha]\). By using the graded hom-Jacobi identity for \([[-,-]]^\psi\), we obtain the following equation:

\[
d_a[x, y]_\psi = [[d_a x, Ad_\psi^\dagger (y)]_\psi + [Ad_\psi^\dagger (x), d_a y]_\psi
\]

for any \(x, y \in \Gamma \psi^1 TM\). Finally, by using the hom-Leibniz rule for the bracket \([[-,-]]^\psi\) with respect to the graded commutative product, one obtains the following derivation condition:

\[
d_a[[X, Y]]_\psi = [[d_a X, Ad_\psi^\dagger (Y)]_\psi + [Ad_\psi^\dagger (X), d_a Y]_\psi
\]

for any \(X, Y \in \bigoplus_{i \in \mathbb{Z}} \Gamma \wedge^i (\psi^T M)\). Thus the hom-Gerstenhaber algebra associated to tangent hom-Lie algebroid, is a strong differential hom-Gerstenhaber algebra with the differential \(d_a\).

- Hom-Gerstenhaber algebra \(\mathfrak{A} = (\bigoplus_{i \in \mathbb{Z}} \Gamma \wedge^i (\psi^T M), \wedge, \alpha, d)\) is the extension of the map \(Ad_\psi^\dagger\) to the exterior bundle) associated to the cotangent hom-Lie algebroid, is a strong differential hom-Gerstenhaber algebra with the differential: \(d_a\).

Example 5.4. Given a Gerstenhaber algebra \((A, [-,-], \wedge)\) with a strong differential \(d\) and an endomorphism \(\alpha : (A, [-,-], \wedge) \to (A, [-,-], \wedge)\) satisfying \(d \circ \alpha = \alpha \circ d\), then first note that the quadruple \((A, [-,-], \wedge, \alpha)\) is a hom-Gerstenhaber algebra. Let us define \(d_\alpha := \alpha \circ d\), then \(d_\alpha : A \to A\) is a map of degree 1. Since the differential \(d\) is a derivation of degree 1 with respect to graded commutative product \(\wedge\), the map \(d_\alpha\) satisfies the following derivation condition with respect to \(\wedge\):

\[
d_\alpha (X \wedge Y) = d_\alpha (X) \wedge \alpha (Y) + (-1)^{|X|} \alpha (X) \wedge d_\alpha (Y)
\]

for all \(X, Y \in A\). Furthermore, the map \(d_\alpha\) satisfies square zero condition and for all \(X, Y \in A\) it satisfies the following derivation condition with respect to \([-,-]\):

\[
d_\alpha [X, Y]_\alpha = [d_\alpha (X), \alpha (Y)]_\alpha + [\alpha (X), d_\alpha (Y)]_\alpha.
\]

Thus the above hom-Gerstenhaber algebra \((A, \wedge, [-,-], \alpha)\) is a strong differential hom-Gerstenhaber algebra with strong differential \(d_\alpha\).

Let \(A\) be a vector bundle over \(M\), \(\psi : M \to M\) be a smooth map, \(\beta : \Gamma A \to \Gamma A\) be a linear map then the space \(\bigoplus_{i \in \mathbb{Z}} \Gamma (\wedge^i A)\) has a hom-Gerstenhaber algebra structure if and only if we have a hom-Lie algebroid structure on hom-bundle \((A, \psi, \beta)\), see Theorem 4.4, [8].

Now, let \((A, \psi, \beta)\) be a hom-bundle, where \(\psi : M \to M\) is a diffeomorphism, \(\beta : \Gamma A \to \Gamma A\) is a bijective linear map. Also let \(\mathfrak{A} = (\bigoplus_{i \in \mathbb{Z}} \Gamma (\wedge^i A), \wedge, [-,-], \alpha, d)\) be a strong differential regular hom-Gerstenhaber algebra structure on the space of multisections \(\bigoplus_{i \in \mathbb{Z}} \Gamma (\wedge^i A)\), where \(\alpha : \mathfrak{A} \to \mathfrak{A}\) is a degree 0 map such that \(\alpha_0 = \psi^* : C^\infty (M) \to C^\infty (M)\) is an algebra automorphism induced by \(\psi\), \(\alpha_1 = \beta\) and the map \(\alpha\) is an extension of \(\alpha_0\) and \(\alpha_1\) to the higher sections.

Theorem 5.5. The tuple \((\bigoplus_{i \in \mathbb{Z}} \Gamma (\wedge^i A), \wedge, [-,-], \alpha, d)\) is a strong differential regular hom-Gerstenhaber algebra if and only if \((A, A^\dagger)\) is a hom-Lie bialgebroid.

Proof. Let us assume that \(\mathfrak{A} = (\bigoplus_{i \in \mathbb{Z}} \Gamma (\wedge^i A), \wedge, [-,-], \alpha, d)\) is a strong differential regular hom-Gerstenhaber algebra. It is clear that \((A, \psi, [-,-], A, \rho_A, \alpha_1)\) is a hom-Lie algebroid, where \([-,-]_A\) and \(\rho_A\) are obtained by restricting \([-,-]\) on \(\Gamma A \times \Gamma A\) and \(\Gamma A \times C^\infty (M)\), respectively (Theorem 4.4, [8]). Since \((\bigoplus_{i \in \mathbb{Z}} \Gamma (\wedge^i A), \alpha, d)\) is an \((\alpha, \alpha)\)-differential graded algebra, by using Theorem 4.4.
we obtain the bracket $[-, -]_{A^*}$ on $A^*$ and the anchor map $\rho_{A^*}$ such that $(A^*, \phi, [-, -]_{A^*}, \rho_{A^*}, \beta^\dagger)$ is a hom-Lie algebroid (here, $\beta^\dagger(\xi)(X) = \psi(\xi(\beta^{-1}(X)))$ for $\xi \in \Gamma A^*$ and $X \in \Gamma A$). Note that the differential corresponding to this hom-Lie algebroid structure, given by equation (18), coincides with the differential $d$. Moreover, the derivation property of the differential $d$ with respect to the Lie bracket $[-, -]_A$ on $A$, i.e. the equation:

$$d[X, Y]_A = [dX, \alpha_1(Y)] + [\alpha_1(X), dY].$$

for $X, Y \in \Gamma A$ implies that $(A, A^*)$ is a hom-Lie bialgebroid.

Conversely, for a given hom-Lie bialgebroid $(A, A^*)$ we obtain a hom-Gerstenhaber algebra structure given by $\mathfrak{g} = (\oplus_{i \in \mathbb{Z}} \Gamma(\wedge^i A), \wedge, [-, -], \alpha)$ since there is a hom-Lie algebroid structure on the hom-bundle $(A, \psi, \beta)$ over $M$. Let $d$ be the differential given by equation (18), for the hom-Lie algebroid structure on $(A^*, \psi, \beta^\dagger)$. Then $\mathfrak{g}$ is an $(\alpha, \alpha)$-differential graded commutative algebra with the differential $d$. Now, by equation (26), it is clear that $\mathfrak{g} = (\oplus_{i \in \mathbb{Z}} \Gamma(\wedge^i A), \wedge, [-, -], \alpha, d)$ is a strong differential regular hom-Gerstenhaber algebra.

**Example 5.6.** Let $(M, \psi, \pi)$ be a given hom-Poisson manifold where the smooth map $\psi : M \to M$ is a diffeomorphism, then there are hom-Lie algebroid structures on both the pull back bundles $\psi^*TM$ and $\psi^*TM^*$. Let $\mathfrak{g}$ be the hom-Gerstenhaber algebra corresponding to the hom-Lie algebroid structure on $\psi^*TM$. Here, the map $\pi : \mathfrak{g} \to \mathfrak{g}$ defined by $\pi = [\pi, -]_{\mathfrak{g}}$ makes $\mathfrak{g}$ a strong differential regular hom-Gerstenhaber algebra (since $\pi$ is a hom-Poisson bivector). Furthermore, the map $d = d_\pi$, differential obtained by equation (18), for the hom-Lie algebroid structure on $\psi^*TM^*$. Therefore, the hom-Lie bialgebroid structure on $(\psi^*TM, \psi^*TM^*)$ corresponding to this strong differential regular hom-Gerstenhaber algebra is the canonical hom-Lie bialgebroid structure on $(\psi^*TM, \psi^*TM^*)$ for a given hom-Poisson manifold as defined in [3].

**Example 5.7.** Let $(\mathfrak{g}, [-, -]_\mathfrak{g}, \alpha)$ and $(\mathfrak{g}^*, [-, -]_{\mathfrak{g}^*}, \alpha^\dagger)$ be two regular hom-Lie algebras (where $\alpha^\dagger(\xi)(X) = \xi(\alpha^{-1}(X))$ for $\xi \in L^*$ and $X \in L$). If $(L, L^*)$ is a purely hom-Lie bialgebra, then $(L, L^*)$ is a hom-Lie bialgebroid over a point manifold. In this case the associated strong differential hom-Gerstenhaber algebra is $(\mathfrak{g} = \Lambda^\mathfrak{g}, \wedge, [-, -]_{\mathfrak{g}, \alpha, \phi, d_\phi})$ given in Example 5.2.

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