Flat affine subvarieties in Oeljeklaus-Toma manifolds

Liviu Ornea\(^1\), Misha Verbitsky\(^2\) and Victor Vuletescu\(^1\)

Abstract:
The Oeljeklaus-Toma (OT-) manifolds are compact, complex, non-Kähler manifolds constructed by Oeljeklaus and Toma, and generalizing the Inoue surfaces. Their construction uses the number-theoretic data: a number field \(K\) and a torsion-free subgroup \(U\) in the group of units of the ring of integers of \(K\), with rank of \(U\) equal to the number of real embeddings of \(K\). OT-manifolds are equipped with a torsion-free flat affine connection preserving the complex structure (this structure is known as “flat affine structure”). We prove that any complex subvariety of smallest possible positive dimension in an OT-manifold is also flat affine. This is used to show that if all elements in \(U \setminus \{1\}\) are primitive in \(K\), then \(X\) contains no proper analytic subvarieties.

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1 Introduction

The OT (Oeljeklaus-Toma) manifolds were discovered by K. Oeljeklaus and M. Toma in 2005 (OT). These manifolds are complex solvmanifolds generalizing the Inoue surfaces of class \(S^0\) (I).

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The construction of Oeljeklaus and Toma is based on number-theoretic date. However, the geometry of OT-manifolds is best understood using the Lie group theory.

Let $G$ be a Lie group equipped with a right-invariant integrable complex structure. Recall that a **group manifold** is the quotient $G/\Gamma$ of $G$ by the right action of a discrete, cocompact subgroup $\Gamma \subset G$. A **complex solvmanifold** is a group manifold with a solvable group $G$. The notion of a solvmanifold is due to G. D. Mostow, who proved a structure theorem for (real) solvmanifolds in his first paper [Mo]. The corresponding notion of a **complex solvmanifold** is probably due to K. Hasegawa [H] who also classified 2-dimensional complex solvmanifolds.

In the case of Oeljeklaus-Toma manifolds, the solvable Lie group is obtained as follows. Recall that a **metabelian** group is a semidirect product of two abelian groups. Consider two abelian Lie groups $A_\mathbb{R}$ and $H_\mathbb{R}$ associated with a number field $K$. We define $A_\mathbb{R} := \mathcal{O}_K \otimes \mathbb{Z}_\mathbb{R}$ and $H_\mathbb{R} := U \otimes \mathbb{Z}_\mathbb{R}$, where $U$ is a free abelian subgroup in the group $\mathcal{O}_K^*$ of units in the integers ring $\mathcal{O}_K$ of $K$. There is a natural action of $U$ on $\mathcal{O}_K$, allowing one to define the semidirect product $G := A_\mathbb{R} \rtimes H_\mathbb{R}$. The corresponding cocompact discrete group is $\Gamma := \mathcal{O}_K^+ \ltimes U$, where $\mathcal{O}_K^+$ is the additive group of $\mathcal{O}_K$. The OT-manifold is $G/\Gamma$, with the right-invariant complex structure defined explicitly in Section 5.

OT-manifolds provide a counterexample to Vaisman’s conjecture [Va] which was open for 25 years. They are non-Kähler flat affine complex manifolds$^{\dag}$ of algebraic dimension 0 ([OT]). Since their discovery in 2005, OT-manifolds were the subject of much research of complex geometric and number theoretic nature ([BO], [OV], [MT], [Ve1], [PV], [Bra]).

It is known that OT-manifolds have no complex curves, [Ve1], and for $t = 1$ they have no complex subvarieties (see [OV], where the proof makes explicit use of the LCK structure). Moreover, all surfaces contained in OT-manifolds are blow-ups of Inoue surfaces $S^0$, [Ve2]. However, in general, there is no characterization of the possible subvarieties of OT manifolds. The aim of this paper is to give a sufficient condition for an OT-manifold to not have submanifolds. In Section 5 we prove:

**Theorem 1.1.** Let $X = X(K, U)$ be an OT-manifold. Assume that any element $u \in U \setminus \{1\}$ is a primitive element for the number field $K$. Then $X$ contains no proper complex analytic subvarieties.

We also prove the following theorem. Recall that a **flat affine manifold** is

$^{\dag}$A complex manifold is called **flat affine** if it is equipped with a flat torsion-free connection preserving the complex structure.
a manifold equipped with a torsion-free flat connection. By construction, OT-manifolds come equipped with a flat affine structure. A submanifold of $Z \subset M$ of a flat affine manifold is called flat affine if locally around any smooth point $z \in Z$, the sub-bundle $TZ \subset TM |_z$ is preserved by the flat affine connection. Notice that all flat affine manifolds are equipped with local coordinates such that the transition functions are affine, and in these coordinates $Z$ is an affine subspace.

**Theorem 1.2.** Let $M$ be an OT-manifold, and $Z \subset M$ an irreducible complex subvariety of smallest possible positive dimension. Then $Z$ is a smooth flat affine submanifold of $M$.

**Proof:** See Remark 5.2.

The paper is organized as follows. Section 2 will describe the construction and main properties of OT-manifolds, Section 3 provides examples of OT submanifolds, in Section 4 we prove that all holomorphic maps from tori to OT manifolds are constant, while in Section 5 we give the proof of Theorem 1.1.

## 2 OT-manifolds

We briefly describe the construction of Oeljeklaus-Toma manifolds, following [OT].

Let $K$ be a number field which has $2t$ complex embeddings denoted $\tau_i, \bar{\tau}_i$ and $s$ real ones denoted $\sigma_j$, $s > 0$, $t > 0$ (for what needed in this paper about number theory, see e.g. [MI]).

Denote $\mathcal{O}_K^{*,+} := \mathcal{O}_K^* \cap \bigcap_i \sigma_i^{-1}(\mathbb{R}^*)$. Clearly, $\mathcal{O}_K^{*,+}$ is a finite index subgroup of the group of units of $\mathcal{O}_K$.

Let $\mathbb{H} = \{y \in \mathbb{C}; \text{Im } y > 0\}$ be the upper half-plane. For any $\zeta \in \mathcal{O}_K$ define the automorphism $T_\zeta$ of $\mathbb{H}^s \times \mathbb{C}^t$ by

$$T_\zeta(x_1, \ldots, x_t, y_1, \ldots, y_s) = \left(x_1 + \tau_1(\zeta), \ldots, x_t + \tau_t(\zeta), y_1 + \sigma_1(\zeta), \ldots, y_s + \sigma_s(\zeta)\right).$$

Similarly, for any totally positive unit $\xi$, let $R_\xi$ be the automorphism of $\mathbb{C}^t \times \mathbb{H}^s$ defined by

$$R_\xi(x_1, \ldots, x_t, y_1, \ldots, y_s) = \left(\tau_1(\xi)x_1, \ldots, \tau_t(\xi)x_t, \sigma_1(\xi)y_1, \ldots, \sigma_t(\xi)y_s\right).$$

Note that the totally positivity of $\xi$ is needed for $R_\xi$ to act on $\mathbb{H}^s \times \mathbb{C}^t$.

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1 An element of a number field is called “totally positive” if it is mapped to a positive number under all real embeddings.
For any subgroup $U \subset \mathcal{O}_K^{*+}$, the above maps define a free action of the semidirect product $U \ltimes \mathcal{O}_K$ on $\mathbb{H}^0 \times \mathbb{C}^t$.

It is proven in [OT] that one can always find admissible subgroups $U$ such that the above action is discrete and cocompact. Note that if $U$ is an admissible subgroup then necessarily one has

$$\text{rank}_\mathbb{Z}(U) + \text{rank}_\mathbb{Z}(\mathcal{O}_K) = 2(s + t),$$

and hence $\text{rank}_\mathbb{Z}(U) = s$. This explains why the condition $t > 0$ is needed: otherwise we would have $\text{rank} \mathcal{O}_K^* = s - 1$, strictly less than $s$, and thus admissible subgroups could not exist.

**Definition 2.1.** ([OT]) For an admissible subgroup $U$, the quotient $X(K, U) := (\mathbb{H}^0 \times \mathbb{C}^t)/(U \ltimes \mathcal{O}_K)$ is called an Oeljeklaus-Toma manifold (OT-manifold for short).

**Remark 2.2.** It was observed in [MT] that in the previous construction one may take instead of the ring of integers $\mathcal{O}(K)$ any (additive) subgroup $H \subset (K, +)$ which equals $\mathcal{O}(K)$ up to finite index, i.e. either $H \subset \mathcal{O}(K)$ or $\mathcal{O}(K) \subset H$ with finite index. We let $X(K, H, U)$ the resulting manifold. Note that the OT-manifolds in [Definition 2.1] correspond to the case $H = \mathcal{O}_K$. Clearly, any such $X(K, H, U)$ is isogeneous to $X(K, U)$, that is, $X(K, H, U)$ is a finite cover of a finite quotient of $X(K, U)$.

**Remark 2.3.** For $s = t = 1$, one recovers a version of the classical construction used by Inoue to define the Inoue surfaces of class $S^0$ ([I]). In [I], no number theory was employed. However, the matrix $M \in \text{SL}(3, \mathbb{Z})$ used in [I, SS2] to construct the Inoue surface of class $S^0$ gives a cubic number field, generated by its root, and this field can be used to recover $M$ in a usual way. If one applies the Oeljeklaus-Toma construction to this cubic field, one would obtain the Inoue surface associated with $M$.

**Remark 2.4.** All OT-manifolds ($s > 0$) are non-Kähler, but for $t = 1$ they admit locally conformally Kähler (LCK) metrics (see [DO] for this notion).

**Definition 2.5.** ([OT]) An OT-manifold is called of simple type, if $U \not\subset \mathcal{Z}$ and it does not satisfy any of the following equivalent conditions:

1. The action of $U$ on $\mathcal{O}_K$ admits a proper, non-trivial, invariant submodule of lower rank.

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$^2$Equivalence follows from [OT] Lemma 1.4.
2. There exists some proper, intermediate field extension \( Q \subset K' \subset K \), with \( U \subset O_{K'}^* \).

**Remark 2.6.** a) A simple type OT-manifold has no proper OT submanifolds with the same group of units \( U \). Also, note the difference towards the notion of simplicity in [CDV].

b) If \( K \) is a number field, then, by Dirichlet’s theorem, under the logarithmic embedding its group of units identifies (up to its subgroup of roots of unity) with a full lattice in a real vector space. Denote this space by \( V_K \). Similarly, if \( K' \subset K \) is some subfield of \( K \), then the group of units of \( K' \) identifies with a lattice in a proper vector subspace \( V_{K'} \subset V_K \). As \( K \) has finitely many subfields, and as the admissible group of units \( U \) of an OT-manifold can be chosen generically in the group of units of \( K \), we see that the OT-manifolds of simple type are generic.

3 **Examples of submanifolds in OT-manifolds**

A simple example of an OT-manifold embedded in a larger OT-manifold which is not of simple type in the sense of Oeljeklaus-Toma is constructed in [OT, Remark 1.7].

We now provide an example of an OT submanifold embedded in an OT-manifold which is of simple type in the sense of Oeljeklaus-Toma.

**Example 3.1:** Take \( L = \mathbb{Q}[X]/(X^3 - 2) \); then \( L \) has one real embedding \( \tau_1 \) and \( 2 \) complex ones \( \tau_2, \tau_3 = \overline{\tau_2} \). Note that \( U_L = O_L^* \) is a free group of rank one, and denote \( u_1 \) be a generator for \( U_L \). Then \( U_L \) is an admissible group, and let \( S = \text{X}(L, U_L) \) is the corresponding OT-manifold (an Inoue surface \( S^0 \)).

Now take \( K = \mathbb{Q}[X]/(X^6 - 2) \). The field \( K \) is an extension of degree \( 2 \) of \( L \) which has two real embeddings \( \sigma_1, \sigma_2 \) (which both extend the embedding \( \tau_1 \) of \( L \)) and four complex embeddings: \( \sigma_3, \sigma_4 \) (which extend \( \tau_2 \)) and \( \sigma_5 = \overline{\sigma_3}, \sigma_6 = \overline{\sigma_4} \) (which extend \( \tau_3 = \overline{\tau}_2 \)). Consider the unit \( u_2 \in O_K^* \) such that \( \sigma_1(u_2) = (\sqrt{2} - 1)^2 \). Then \( \sigma_2(u_2) = (\sqrt{2} + 1)^2 \), and hence the subgroup \( U_K \subset O_K^* \) generated by \( u_1 \) and \( u_2 \) is admissible, since the projection on the first two factors of their logarithmic embedding is

\[
\begin{pmatrix}
\log(u_1) & \log(u_1) \\
2\log(\sqrt{2} - 1) & 2\log(\sqrt{2} + 1)
\end{pmatrix}
\]

which is of maximal rank.
Let $X = X(K, U_K)$ be the corresponding OT-manifold, $X = \mathbb{H}^2 \times \mathbb{C}^2 / (U_K \ltimes \mathcal{O}_K)$. Define the map $i : S \to X$ by

$$i([w, z]) = [w, w, z, z],$$

where we denoted by $[x]$ the equivalence class of $x$. Clearly, $i$ is well-defined.

**Claim 3.2.** The map $i$ is injective. Indeed, if $i([w, z]) = i([w', z'])$ then there exists $u \in U_K, a \in \mathcal{O}_K$ such that

$$w = \sigma_1(u)w' + \sigma_1(a),$$
$$w = \sigma_2(u)w' + \sigma_2(a).$$

This implies

$$(\sigma_1(u) - \sigma_2(u))w' = \sigma_2(a) - \sigma_1(a).$$

If $\sigma_1(u) \neq \sigma_2(u)$, then $w' \in \mathbb{R}$, which is not possible, and hence $\sigma_1(u) = \sigma_2(u)$. This yields $u \in L$, thus $u \in U_L$. Moreover, $\sigma_1(a) = \sigma_2(a)$, and hence also $a \in \mathcal{O}_L$. But then $[w, z] = [w', z']$.

**Remark 3.3.** The above constructed $X$ is of simple type. Indeed, the unit $u_2$ is a primitive element for $K$, hence there is no proper subfield $K' \subset K$ containing $u_2$.

### 4 Holomorphic maps from and to tori

In the proof of the main result we shall need the following result interesting in itself:

**Proposition 4.1.** Let $X$ be an OT-manifold and $T$ a complex torus. Then any holomorphic map $f : T \to X$ must be constant.

**Proof:** Let $T = \mathbb{C}^d / \Lambda$, and $X = \mathbb{H}^s \times \mathbb{C}^t / (U \ltimes A)$. Note that $\pi_1(T) = \Lambda$, and $\pi_1(X) = U \ltimes A$. Let $I$ be the image of the natural morphism $f_* : \pi_1(T) \to \pi_1(X)$. With the above identifications, we let:

$$f_*(\lambda) = \gamma_\lambda = (u_\lambda, a_\lambda), \quad \lambda \in \Lambda.$$

Let now $\tilde{f} : \tilde{T} \to \tilde{X}$ be a lift of $f$ at the universal covers. We then have:

$$\tilde{f}(t + \lambda) = \gamma_\lambda(\tilde{f}(t)), \forall t \in \tilde{T}. \quad (4.1)$$
Let \( f_1 = \text{pr}_1 \circ \tilde{f} \), where \( \text{pr}_1 : \mathbb{H}^d \times \mathbb{C}^t \to \mathbb{H} \) is the projection onto the first factor. Then \( f_1 \) is a map from \( \mathbb{C}^d \) into \( \mathbb{H} \), and hence by Liouville’s theorem it must be constant. It follows that the first component of the map \( \tilde{f} \) is a constant, say \( w_1 \). Then (4.1) implies

\[
w_1 = \sigma_1(u_\lambda) w_1 + \sigma_1(a_\lambda).
\]

Now if \( \sigma_1(u_\lambda) \neq 1 \) for some \( \lambda \in \Lambda \), then \( w_1 \) would be real, which is impossible. It follows that \( u_\lambda = 1 \) for all \( \lambda \in \Lambda \) and thus \( I \) is actually a subgroup of \( A \).

But then we have \( w_1 = w_1 + \sigma_1(a_\lambda) \), and hence \( \sigma_1(a) = 0 \) for all \( \lambda \in \Lambda \). This implies \( I = \{0\} \), and thus \( \tilde{f} \) factors through a map from \( T \) to the universal cover \( \tilde{X} \) of \( X \), which is constant as \( T \) is compact.

**Remark 4.2.** It is easy to see that conversely, every holomorphic map from an OT-manifold to a torus must be constant. This is because the Albanese torus of an OT-manifold is trivial, since there are no non-zero closed holomorphic 1-forms on an OT-manifold ([OT, Proposition 2.5]).

## 5 The proof of the main result

**Theorem 5.1.** Let \( X = X(K, U) \) be an OT-manifold. Assume that any element \( u \in U \setminus \{1\} \) is a primitive element for \( K \). Then \( X \) contains no proper analytic subspaces.

**Proof.** We argue by contradiction. Let \( Z \subset X \) be an analytic connected proper subspace of minimum positive dimension. By a result of S.M. Verbitskaya, an OT-manifold cannot contain curves, and hence \( \dim(Z) \geq 2 \) (note that the proof in [Ve1] can be easily extended to cover the singular case, too). Moreover, as \( Z \) is of minimum positive dimension, we deduce that \( Z \) contains no Weil divisors, and it has at most finitely many isolated singularities.

Let \( Z_{\text{reg}} = Z \setminus \text{Sing}(Z) \) be the regular part of \( Z \). Then the Remmert-Stein’s theorem (see e.g. [FG, Theorem 6.9, p. 150]) implies that \( Z \setminus \text{reg} \) has no divisors.

For any \( i = 1, \ldots, \dim(X) \) let \( L_i \) be the flat line bundle on \( X \) associated with the representation \( \rho_i : \pi_1(X) \to \mathbb{C}^* \), \( \rho_i(u, a) = \sigma_i(u) \) (here we identified \( \pi_1(X) \) with \( U \ltimes \mathcal{O}_X \)). Then \( L_i \) is locally generated by \( \frac{\partial}{\partial z_i} \) and the tangent bundle \( T_X \) is naturally identified with the direct sum

\[
T_X = \bigoplus_{i=1}^n L_i.
\]

We want to understand the restriction of \( T_X \) to \( Z \). It will be enough to look at
the regular part $Z_{\text{reg}}$ on which we have the exact sequence:

$$0 \longrightarrow T_{Z_{\text{reg}}} \overset{i}{\longrightarrow} \bigoplus_{i=1}^{n} L_{i|_{Z_{\text{reg}}}} \overset{\mathcal{N}_{Z_{\text{reg}}|X}}{\longrightarrow} 0 \quad (5.1)$$

Let $I \subset \{1, \ldots, \dim(X)\}$ and let

$$\text{pr}_{\mathcal{I}} \colon \bigoplus_{i=1}^{\dim(X)} L_{i} \longrightarrow \bigoplus_{i \in \mathcal{I}} L_{i}$$

be the canonical projection.

We claim that there exists $J \subset \{1, \ldots, \dim(X)\}$ with $\#J = \dim(Z_{\text{reg}})$ such that $i_{J} = \text{pr}_{\mathcal{J}} \circ i$ is an isomorphism, and hence $T_{Z_{\text{reg}}} \cong \bigoplus_{i \in \mathcal{J}} L_{i}$.

Indeed, there must be a subset $J \subset \{1, \ldots, \dim(X)\}$ with $\#J = \dim(Z_{\text{reg}})$ such that the map $i_{J}$ is injective, otherwise $i$ in (5.1) would not be injective. As $Z_{\text{reg}}$ has no divisors, the degeneracy locus of $i_{J}$ is empty, and hence $i_{J}$ is an isomorphism, as claimed.

Thus we can see the map $i$ as a matrix

$$A = (a_{ij}), \ i = 1, \ldots, \dim(Z_{\text{reg}}), \ j = 1, \ldots, \dim(X)$$

with each $a_{ij} \in \text{Hom}(L_{i|_{Z_{\text{reg}}}}, L_{j|_{Z_{\text{reg}}}})$.

Let $\pi : \tilde{X} \longrightarrow X$ be the universal cover of $X$. As $Z_{\text{reg}}$ has no divisors, any morphism of line bundles of $Z_{\text{reg}}$ is either zero or multiplication by a non-zero constant. As a consequence, the entries of the matrix $A$ are all constant, and hence the image of the bundle morphism

$$T_{\pi^{-1}(Z_{\text{reg}})} \longrightarrow T_{\tilde{X}|_{\pi^{-1}(Z_{\text{reg}})}}$$

is the vector subspace generated by the vectors $\{f_{i}\}, \ i \in J$, given by

$$f_{i} = \sum_{j=1}^{\dim(X)} a_{ij} \frac{\partial}{\partial z_{j}}.$$

In particular, the preimage $\pi^{-1}(Z_{\text{reg}})$ of $Z_{\text{reg}}$ on the universal cover $\tilde{X} = \mathbb{H}^{s} \times \mathbb{C}^{t}$ of $X$ is locally an open subset of an affine subspace $\mathcal{A}$ of $\mathbb{C}^{s+t}$ where the direction of $\mathcal{A}$ is spanned by

$$f_{i} = \sum_{j=1}^{\dim(X)} a_{ij} e_{j}$$

where $\{e_{j}\}, \ j = 1, \ldots, \dim(X)$ is the canonical basis in $\mathbb{C}^{s+t}$. 

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It follows that any connected component \( \tilde{Z}_{\text{reg}} \subset \pi^{-1}(Z_{\text{reg}}) \) is contained as an open subset of an affine subspace \( \mathcal{A}_{\tilde{Z}_{\text{reg}}} \) of the same dimension, hence any connected component of \( \tilde{Z} \subset \pi^{-1}(Z) \) is also contained into such an affine subspace. We derive that \( \pi^{-1}(Z) \) is in fact smooth, and hence \( Z \) is smooth, too.

Moreover, since \( \tilde{Z} \) is closed in \( \mathbb{H}^s \times \mathbb{C}^t \), the following equality holds:

\[
\tilde{Z} = \mathcal{A}_{\tilde{Z}} \cap (\mathbb{H}^s \times \mathbb{C}^t).
\]

**Remark 5.2.** Notice that this observation also proves Theorem 1.2.

Now fix a connected component \( \tilde{Z} \subset \pi^{-1}(Z) \). Then

\[
Z = \tilde{Z} / \text{Stab}(\tilde{Z})
\]

where

\[
\text{Stab}(\tilde{Z}) = \{ \gamma \in \pi_1(X) | \gamma(\tilde{Z}) = \tilde{Z} \}.
\]

Analysing the structure of the group \( \text{Stab}(\tilde{Z}) \) will eventually lead to a contradiction. In the first place, observe that \( \text{Stab}(\tilde{Z}) \) cannot consist of translations only, since otherwise \( Z \) would be a torus\(^1\), contradicting with Proposition 4.1.

Fix then \( \gamma \in \text{Stab}(\tilde{Z}) \) which is not a translation and let \( R_u \) be the linear map induced by \( \gamma \). The direction \( \tilde{Z} \) of the affine subspace \( \tilde{Z} \) is then left invariant by \( R_u \). Since \( R_u \) is diagonal, either \( \tilde{Z} \) has a basis among \( \{e_1, \ldots, e_n\} \) or (at least) two of the eigenvalues of \( R_u \) are equal. The second case is excluded by the assumption on \( M \), so there exists a subset \( I \subset \{1, \ldots, n\} \) such that \( \tilde{Z} = \mathcal{A}_{\{e_i\}} \). It follows that for any \( j \in I' := \{1, \ldots, n\} \setminus I \) there exists constants \( c_j \in \mathbb{C} \) such that \( \tilde{Z} \) is given by the equations \( z_j = c_j, \forall j \in I' \).

Let now \( \gamma \in \text{Stab}(\tilde{Z}) \) be arbitrary. Note that \( \gamma \) cannot be a translation by some \( (\sigma_1(a), \ldots, \sigma_n(a)) \), since then for any \( j \in I' \) we would have \( c_j = c_j + \sigma_j(a) \), yielding \( \sigma_j(a) = 0 \), and hence \( a = 0 \). This implies that any nontrivial \( \gamma \in \text{Stab}(\tilde{Z}) \) is of the form

\[
\gamma(z_1, \ldots, z_n) = (\sigma_1(u)z_1 + \sigma_1(a), \ldots, \sigma_n(u)z_n + \sigma_n(a)),
\]

for some \( u \in U, u \neq 1 \) and for some \( a \in A_M \).

But since \( \gamma \in \text{Stab}(\tilde{Z}) \) we see that for any \( j \in J \) we have

\[
c_j = \sigma_j(u)c_j + \sigma_j(a),
\]

\(^1\) A bit more details are needed here..
and thus

\[ c_j = \frac{\sigma_j(a)}{1 - \sigma_j(u)}, \quad \text{for all } j \in J. \]

The point \( P_0 \in \mathbb{C}^s \times \mathbb{C}^t \),

\[ P_0 = \left( \frac{\sigma_1(a)}{1 - \sigma_1(u)}, \ldots, \frac{\sigma_n(a)}{1 - \sigma_n(u)} \right), \]

is thus fixed by all \( \gamma \in \text{Stab}(\tilde{Z}) \). But then, after changing the coordinates in \( \tilde{Z} \) via

\[ z_i \mapsto z_i - \frac{\sigma_i(a)}{1 - \sigma_i(u)}, \quad i = 1, \ldots, n, \]

we see that \( \text{Stab}(\tilde{Z}) \) acts on \( \tilde{Z} \) by linear diagonal transformations. This means that \( \tilde{Z} \) has a compact quotient under the action of a group of diagonal transformations. But this is easily seen to be impossible since on one hand, if a free abelian group \( G \) of linear diagonal transformations acts discretely then its rank is 1 (look at the orbit through \( G \) of any point), while since \( \tilde{Z} \) is a contractible manifold of real dimension at least 2, the rank of any free abelian group acting cocompactly on it must equal its real dimension (cf [CE], Application 3, pp 357 and [Bro] Example 5, pp 185).

This contradiction completes the proof of Theorem 5.1.\( \blacksquare \)

**Remark 5.3.** The condition that “any element \( u \in U, u \neq 1 \) is a primitive element for \( K \)” is satisfied by a wide class of choices for \( K \) and \( U \). For instance, if \( K \) is a number field of prime degree over \( \mathbb{Q} \) then any choice of the admissible group of units will satisfy this condition.

**Remark 5.4.** Although the Example 3.1 may suggest that the condition that any \( u \in U, u \neq 1 \) is a primitive element may be equivalent to the fact that the OT-manifold has no proper subvarieties, this is not entirely correct. There are cases when this condition is not satisfied, but the OT-manifold still has no proper complex subvarieties. For instance, if \( K \) is a number field with a single complex place \( (t = 1) \), the OT-manifold \( X \) has no proper complex subvarieties by [OV]. But for such a number field \( K \) we see that the rank of the group of units of \( K \) is \( s + 1 - 1 = s \) so if the number field \( K \) contains a proper subfield \( \mathbb{Q} \subset L \subset K \) then for any choice of the admissible group of units \( U \) there are elements in \( U \) which belong to \( L \), hence not all elements in \( U \) are primitive elements for \( K \).

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**Liviu Ornea**

University of Bucharest, Faculty of Mathematics,
14 Academiei str., 70109 Bucharest, Romania, and:
Institute of Mathematics "Simion Stoilow" of the Romanian Academy,
21, Calea Grivitei Str. 010702-Bucharest, Romania
lornea@fmi.unibuc.ro, Liviu.Ornea@imar.ro

**Misha Verbitsky**

Instituto Nacional de Matemática Pura e Aplicada (IMPA)
Estrada Dona Castorina, 110
Jardim Botânico, CEP 22460-320
Rio de Janeiro, RJ - Brasil
also:
Laboratory of Algebraic Geometry,
National Research University Higher School of Economics,
Department of Mathematics, 6 Usacheva street, Moscow, Russia.

**Victor Vuletescu**

University of Bucharest, Faculty of Mathematics,
14 Academiei str., 70109 Bucharest, Romania.
vuli@fmi.unibuc.ro