The least squares collocation method for the biharmonic equation in irregular and multiply-connected domains

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Abstract. This paper reports new h- and p-versions of the least squares collocation method of high-order accuracy proposed and implemented for solving boundary value problems for the biharmonic equation in irregular and multiply-connected domains. This paper shows that approximate solutions obtained by the least squares collocation method converge with high order and agree with analytical solutions of test problems with high degree of accuracy. There has been a comparison made for the results achieved in this study and results of other authors who used finite difference and spectral methods.

1. Introduction

Biharmonic equations are of great importance for many areas of science and technology. Classical examples can be found in hydrodynamics, solid mechanics, theory of thin plates and many other areas. For example, in fluid dynamics (zero Reynolds number), the stream function satisfies the biharmonic equation. In solid mechanics, the solution of the biharmonic equation can be used to represent the Airy stress function. In the theory of thin plates, the solution of the biharmonic equation can be used for analysing the stress-strain state of an isotropic plate under the action of a transverse load \cite{1}. However, biharmonic equations are difficult to numerically solve due to the fourth order derivatives in the differential equation.

Numerical solutions of boundary value problems for the biharmonic equation are usually obtained through the use of the finite difference method (FDM), because of the ease of grid generation and fast solving of different problems. However, there are limited publications on FDMs for biharmonic equations in irregular domains \cite{2,3}. The finite element method (FEM) is another popular method for solving such problems using unstructured grids \cite{4–8}. As referred to in \cite{9}, although the solution techniques for fourth order equations by FDMs and FEMs are well developed, there are not many results available for dealing with arbitrary shapes and complex boundary conditions. Spectral methods and methods of spectral elements have become increasingly popular in the computation of continuum mechanics problems. The main advantage of these methods are the exponential rate of convergence if the unknown solution is sufficiently smooth \cite{9–12}. Spectral methods are less effective or flexible than FEMs if the smoothness of
solutions is weak. There are other numerical methods to the biharmonic equation. These include
the fast multipole method, the spline collocation, the mimetic method, the mixed methods. The
readers are referred to [13] for more references on these methods.

This paper is a continuation of our early studies [14,15] and extends into multiply-connected
domains. The irregular domain is embedded in a fictitious rectangle covered by the regular
grid with rectangular cells. An approximate problem is assigned to a problem for PDE by
projecting into a finite-dimensional linear functional space in the least squares collocation (LSC)
method. The solution of the approximate problem is reduced to solving a system of linear
algebraic equations (SLAE). The approximate solution is obtained as a linear combination
with indeterminate coefficients of the basis elements defined in this space. The overdetermined
SLAE for unknown coefficients is constructed of collocation equations, boundary conditions and
matching conditions in each cell of the grid. Krylov subspaces [16], multi-grid complexes [17]
and the diagonal preconditioner [18] are used in the new h-version of the LSC method. On top
of that, the new p-version of the LSC method with the diagonal preconditioner is implemented
for solving the biharmonic problem. The approximate solution in this case is represented in the
form of direct products of Chebyshev polynomials series. The collocation points are selected in
the roots of Chebyshev polynomials [19]. We compare our results with those in [2,3,9].

2. Statement of the problem and the description of the LSC method

2.1. Statement of the problem

Let us consider the Dirichlet problem for the biharmonic equation for $u(x_1, x_2)$

$$\Delta^2 u = f(x_1, x_2), \quad (x_1, x_2) \in \Omega, \quad u|_{\delta \Omega} = g_1(x_1, x_2), \quad u_n|_{\delta \Omega} = g_2(x_1, x_2) \quad (2.1)$$

in the irregular domain $\Omega \subset \mathbb{R}^2$ with the boundary $\delta \Omega$, where $\Delta^2 = \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^4}$.

$u_n = \frac{\partial u}{\partial n}$, $\vec{n}$ is the unit normal pointing outward, $u(x_1, x_2)$ is the unknown function, $f(x_1, x_2)$,
g1(x1, x2) and g2(x1, x2) are given functions. The boundary conditions are also set on inner
boundaries of the computational domain if the domain is multiply-connected. Let us present the
description of the LSC method by the example of solving the biharmonic equation in irregular
simply connected domains. Other boundary value problems for the biharmonic equation is
carried out similarly to the algorithm described below.

In this paper $\Omega$ is the domain with a discrete boundary (figures 1–3). The program builds a
continuous double spline $(x_1(t), x_2(t))$ if boundary is smooth or a piece-wise spline if boundary
has breakpoints for describing $\delta \Omega$.

2.2. h-version of the LSC method

The irregular domain is embedded in the fictitious rectangle (figure 1) covered by the regular
grid $N_1 \times N_2$ with rectangular cells. An “onefold” layer of boundary irregular cells (i-cells)
appears near the domain boundary. The “onefold” layer is cut of by the boundary from the
rectangular cells of the initial regular grid. An i-cell that does not contain the center of the
initial rectangular cell that contains it is attached to the neighboring one (figure 1). The readers
are referred to [15,20] for more information about this technique.

Let $N_{cells}$ be the number of cells in each version of the LSC method. For convenience we
introduce the local coordinate system in each $j$-th cell

$$y_1 = \frac{(x_1 - x_{1j})}{h_1}, \quad y_2 = \frac{(x_2 - x_{2j})}{h_2}, \quad (2.2)$$
Figure 1. Grid with i-cells in the h-version. The symbol ◦ denotes the collocation points, × — the matching points, □ — the points for recording the boundary conditions (also for figure 2). The symbol • denotes points giving the domain boundary (also for figure 2 and 3), ◦ — the centers of i-cells. The i-cells 8 and 14 join with the cells 7 and 10.

Figure 2. The computational domain in the p-version. For example, here the collocation points (◦) are selected in the roots of the direct product of sixth degree Chebyshev polynomials. Dashed lines denote the boundary of the fictitious rectangle (also for figure 3).

where \((x_{1j}, x_{2j})\) is the cell center, \(h_1\) and \(h_2\) are the characteristic sizes of the \(j\)-th cell, \(j = 1, ..., N_{\text{cells}}\), \(v(y_1, y_2) = u(x_1(y_1), x_2(y_2))\). Here \(2h_1\) and \(2h_2\) are the sizes of the rectangular cells in the \(x_1\) and \(x_2\) directions, respectively.

According to [15], the approximate solution in the h-version in each \(j\)-th cell is sought as

\[
v_{hj}(y_1, y_2) = \sum_{i_1=0}^{K} \sum_{i_2=0}^{K-1} b_{i_1i_2} y_1^{i_1} y_2^{i_2},
\]

\(K = 4\) in all calculations in this paper.

2.3. p-version of the LSC method

The initial domain is also embedded in the fictitious rectangle (figure 1) in the p-version. A single piece of the polynomial solution is constructed in the computational domain. Local coordinates are similarly entered by formula (2.2) and \(N_{\text{cells}} = N_1 = N_2 = j = 1\). The approximate solution is sought in the form

\[
v_{hj}(y_1, y_2) = \sum_{i_1=0}^{K_1-1} \sum_{i_2=0}^{K_2-1} b_{i_1i_2} \phi_{i_1}(y_1) \phi_{i_2}(y_2), \quad (y_1, y_2) \in [-1, 1] \times [-1, 1],
\]

where \(\phi_{i_1}(y_1) = \cos(i_1 \arccos(y_1))\), \(\phi_{i_2}(y_2) = \cos(i_2 \arccos(y_2))\).
2.4. Equations of the approximate problem and their solution

The unknown coefficients \( b_{ij} \) in the LSC method are determined from an overdetermined “local” system of equations in each cell consisting of the collocation equations, the matching conditions (h-version) and boundary conditions.

The collocation equations multiplied by \( h_1^2 h_2^2 \) are written down at the collocation points \((y_{1c}, y_{2c})\) in each \( j \)-th cell as

\[
k_c \left( \frac{h_2^4 \partial^4 v_{hj}}{h_1^4} + 2 \frac{\partial^4 v_{hj}}{\partial y_1^2 \partial y_2^2} + \frac{h_1^2 \partial^4 v_{hj}}{h_2^2 \partial y_2^4} \right) = k_c h_1^2 h_2^2 f(x_1(y_{1c}), x_2(y_{1c})),
\]

(2.3)

where \( c = 1, \ldots, 16 \) when \( K = 4 \) (h-version, figure 1), \( c = 1, \ldots, 36 \) when \( K_1 = K_2 = 6 \) (p-version, figure 2) and \( k_c \) is positive weight parameter.

The matching conditions (multiplied by \( h_1^2 h_2 \) in (2.5)) are written down at the matching points (figure 1) in the h-version in each \( j \)-th cell as

\[
k_{m_0} v_{hj} + k_{m_1} \frac{\partial v_{hj}}{\partial n_j} = k_{m_0} \hat{v}_h + k_{m_1} \frac{\partial \hat{v}_h}{\partial n_j},
\]

(2.4)

\[
h_1 h_2 \left( k_{m_2} \frac{\partial^2 v_{hj}}{\partial n_j^2} + k_{m_3} \frac{\partial^3 v_{hj}}{\partial n_j^3} \right) = h_1 h_2 \left( k_{m_2} \frac{\partial^2 \hat{v}_h}{\partial n_j^2} + k_{m_3} \frac{\partial^3 \hat{v}_h}{\partial n_j^3} \right),
\]

(2.5)

where \( n_j \) denotes the unit outer normal to the boundary of the \( j \)-th cell, \( v_{hj} \) and \( \hat{v}_h \) are the limits of the function \( v_{hj} \) as its arguments tend to the cell side from within and outside the cell, \( k_{m_0}, k_{m_1}, k_{m_2}, k_{m_3} \) are positive weight parameters.

The boundary conditions for the Dirichlet problem (2.1) are written down at several points belonging to \( \delta \Omega \) (figure 1 and 2) as

\[
k_{b_0} v_{hj} = 0,
\]

(2.6)

\[
k_{b_1} \left( \frac{n_1}{h_1} \frac{\partial v}{\partial y_1} + \frac{n_2}{h_2} \frac{\partial v}{\partial y_2} \right) = 0,
\]

(2.7)

where \( k_{b_0}, k_{b_1} \) are positive weight parameters.

The overdetermined SLAE obtained by combining equations (2.3)–(2.7) in all cells of the computational domain (global system) is solved iteratively by the Gauss – Seidel method in the h-version. In this process one “global iteration” consists of sequential solution of the local SLAE in all cells of the domain. A system matrix of equations is reduced to the upper triangular form by the Householder method (also in the p-version) when constructing the solution in each cell. The iterative process continues while the condition is true

\[
\max_{i_1 i_2 j} \left| b_{i_1 i_2 j}^{n+1} - b_{i_1 i_2 j}^n \right| > \epsilon,
\]

(2.8)

where \( b_{i_1 i_2 j}^n \) — coefficient of the polynomial which approximates the solution in the cell with the number \( j \) on the \( n \)-th iteration. The value \( \epsilon \) is a given small constant [15, 20].

3. Numerical examples

3.1. Dirichlet problem in a simply connected irregular domain

We try to recover the values of the function \( u(x_1, x_2) = x_1^2 \ln(1 + x_2) + \frac{x_2}{1 + x_1} \) in the ellipse \( \frac{x_1^2}{0.5^2} + \frac{x_2^2}{0.15^2} \leq 1 \) from the knowledge of \( \Delta^2 u \) inside the domain and \( u, u_{nn} \) on the domain boundary. Table 1 shows the absolute errors \( K_{\infty} \) (maximal difference between the computed solution and the exact one) for various grids (h-version). We compare our results to numerical
examples given in [2, 3]. The FDM proposed in [3] has the bigger convergence rate than the h-version of the LSC method when $K = 4$ [15]. The boundary value problem for the biharmonic equation of the fourth order was splitted into two coupled Poisson equations in the other papers [2, 3]. It is known that the Dirichlet problem for the Poisson equation is better conditioned than the boundary value problem for the biharmonic equation (2.1). Therefore, this approach makes it relatively easy to build a highly accurate solution of the original problem. It is hereby shown that the high accuracy solution of the boundary value problem for the biharmonic equation is obtained by the LSC method without using above described technique. A more accurate solution than in [2, 3, 15] was obtained by the p-version of the LSC method for solving this problem. It can be seen that the $L_\infty$ error yields $1e-7$ with much less computational cost in the p-version compared with the h-version. In the p-version the following parameters were used in the calculations: $k_c = 0.01$, $k_b = k_{b1} = 1$.

### Table 1. Results of numerical experiments for the example 3.1.

|                  | Shapeev, Belyaev [15] | Ben-Artzi et al. [3] |
|------------------|-----------------------|----------------------|
| $N_1 \times N_2$| $L_\infty$            | Rate                 | $L_\infty$            | Rate                 |
| 8x4              | 1.97e-5               | —                    | —                    | —                    |
| 16x8             | 1.56e-6               | 3.65                 | 3.0e-6               | —                    |
| 32x16            | 1.52e-7               | 3.35                 | 1.1e-7               | 4.8                  |
| 64x32            | 1.24e-8               | 3.61                 | 1.5e-9               | 6.2                  |

|                  | Chen et al. [2]       | This paper, p-version |
|------------------|-----------------------|----------------------|
| $N_1 \times N_2$| $L_\infty$            | Rate                 | $K_1 = K_2$          | $L_\infty$           |
| 64x32            | 3.65e-4               | —                    | 10                   | 6.23e-7              |
| 128x64           | 9.54e-05              | 1.9                  | 15                   | 9.93e-10             |
| 256x128          | 2.08e-05              | 2.2                  | 20                   | 3.90e-12             |
| 512x256          | 4.98e-06              | 2.1                  | 25                   | 3.05e-14             |

#### 3.2. Dirichlet problem in a multiply-connected irregular domain

Let us consider the Dirichlet problem for the biharmonic equation (2.1) with analytical solution $u(x_1, x_2) = e^{x_1} + e^{x_2}$. The multiply-connected domain outer boundary is the cardioid defined by

$$
\begin{align*}
    x_1(t) &= 16\sin^3(t)/30 + 0.6, \\
    x_2(t) &= (13\cos(t) - 5\cos(2t) - 2\cos(3t) - \cos(4t))/30 + 0.6, \\
    t &\in [0, 2\pi].
\end{align*}
$$

Let this domain have three circular holes: $(x_1 - 0.35)^2 + (x_2 - 0.7)^2 \leq 0.1^2$, $(x_1 - 0.85)^2 + (x_2 - 0.78)^2 \leq 0.07^2$, $(x_1 - 0.7)^2 + (x_2 - 0.45)^2 \leq 0.04^2$ (figure 3).

Table 2 shows the numerical results arising from the h- and p-versions of the LSC method. Computations were done using Intel Core i7-4700MQ CPU 2.80 GHz, DIMM DDR3 800 MHz 6 Gb. The fifth column is the CPU time in seconds spent in solving the global system in table 2. This table also shows the number of iterations $N_{iter}$.

#### 3.3. Various types of boundary conditions

Let us consider an example as in [9] of the linear bending problem of a plate of irregular form (figure 2) under the action of the external load $q_0 \sin(\pi x_1) \sin(\pi x_2)$, where $q_0 = \text{const}$, $\nu = 0.3$ is
Figure 3. The computational domain for the example 3.2. The symbol × denotes the breakpoints of the domain outer boundary. Two double splines were built for describing the domain outer boundary.

Table 2. Results of numerical experiments for the example 3.2.

| $N_1 \times N_2$ | $L_\infty$ | Rate | $N_{iter}$ | CPU time (s) | $K_1 = K_2$ | $L_\infty$ |
|----------------|-----------|------|------------|-------------|-------------|-----------|
| 6×6            | 2.12e-5   | 42   | —          | 0.062       | 7           | 7.49e-7   |
| 12×12          | 3.24e-6   | 81   | 2.70       | 0.313       | 9           | 1.99e-9   |
| 24×24          | 1.67e-7   | 121  | 4.27       | 1.500       | 11          | 3.65e-12  |
| 48×48          | 8.75e-9   | 241  | 4.25       | 11.735      | 13          | 8.88e-15  |

Poisson’s ratio. The exact values of the $M_n w$ and $V_n w$ are set on the circular arc, $w$ and $M_n w$ are set on two straight edges adjacent to the arc, $w$ and $\frac{\partial w}{\partial n}$ are set on the remaining straight edge. In the present study $M_n$ is the second order differential operator, $V_n$ is the third order differential operator [1]. The exact value of the solution $u(x_1, x_2) = \frac{q_0 \sin(\pi x_1) \sin(\pi x_2)}{(4\pi^4 D)}$, where $D$ is the bending stiffness. The values of the weight parameters have been found for each $K_1$ and $K_2$. Table 3 compares the numerical results arising from the LSC method and the method proposed in [9].

4. Conclusions

The new h- and p-versions of the LSC method of high-order accuracy are proposed and implemented for the numerical solution of the nonhomogeneous biharmonic equation in the irregular and multiply-connected domains. The convergence of approximate solutions to the exact ones was verified on a sequence of grids in the h-version of the LSC method and by increasing the degree of polynomials in the p-version in numerical experiments with analytical solutions. It is shown that the accuracy in the LSC method is achieved not lower than the accuracy of solutions obtained by other method.
Table 3. Relative $L_2$ errors, denoted by $N_e$.

| $K_1$, $K_2$ | 4  | 6  | 8  | 10 | 12 |
|--------------|----|----|----|----|----|
| $N_e$  | 1.75e-02 | 1.47e-04 | 2.61e-06 | 1.29e-07 | 7.10e-10 |

This paper, p-version

| $K_1$, $K_2$ | 4  | 6  | 8  | 10 | 12 |
|--------------|----|----|----|----|----|
| $N_e$  | 2.82e-02 | 6.01e-04 | 5.93e-06 | 4.55e-08 | 3.59e-10 |

The solutions were sufficiently smooth in the tests considered. Spectral methods and the p-version of the LSC method implemented in this paper result in exponential order of convergence. The solution smoothness is generally weak in real and practically important problems. Therefore, spectral methods [9–12] and the p-version of the LSC method become less effective than h- and hp-versions of the LSC method.

It is expected that the new versions of the LSC method can be successfully applied to solve more complex problems, when the solutions are not smooth enough.

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