Gravity From Topological Field Theory

by

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Abstract
We construct a topological field theory which, on the one hand, generalizes BF theories in that there is non-trivial coupling to 'topological matter fields'; and, on the other, generalizes the three-dimensional model of Carlip and Gegenberg to arbitrary dimensional manifolds. Like the three-dimensional model, the theory can be considered to describe a gravitational field interacting with topological matter. In particular, in two dimensions, the model is that of gravity on a torus. In four dimensions, the model is shown to admit constant curvature black hole solutions.
1 Introduction

The considerable amount of progress made in recent years in quantum gravity has in large part resulted from advances in (2+1)-dimensional general relativity. In the absence of matter, the field equations imply that spacetime is locally flat. Consequently the theory has no propagating modes, and the reduced phase space is finite dimensional, allowing one to fully quantize the theory [1]. Unfortunately, the addition of matter has the general effect of destroying this exact solvability, introducing propagating modes which render the phase space infinite dimensional.

However there is a theory in which a class of matter fields in (2 + 1) dimensions does not destroy solvability. Referred to as BCEA theory, it minimally couples a pair of topological matter fields (one-forms $B^a$ and $C^a$) to gravity in such a way that the connection $A^a$ remains locally flat [2], while allowing the triad $E^a$ to have non-trivial structure. Hence, gravitational and matter degrees of freedom are inextricably mixed. In addition, the field equations admit constant curvature black hole solutions that are analogous (but not identical) to the (2 + 1)-dimensional BTZ black hole [3]. In the context of BCEA theory, the role of inner and outer event horizons for the (spinning) BTZ black hole are interchanged [4].

Furthermore BCEA theory has a supersymmetric generalization for which one can show that this black hole solution is supersymmetric [5]. The supersymmetrized theory can be quantized, and it turns out that the partition function is a topological invariant which in a sense contains both the sum of the squares of the Ray-Singer analytic torsions and the Casson invariant of flat SO(3) bundles over the three dimensional spacetime manifold [2, 6].

In this paper we construct higher-dimensional versions of BCEA theory, which we call BQPA theories. The class of BQPA theories minimally couple topological matter (i.e. matter which does not couple directly to a metric) to gravity. The metric is a derived quantity which can have non-trivial structure due to the presence of the matter fields. In particular, black holes of constant negative curvature are solutions to the field equations, with the (negative) cosmological constant being a constant of integration. These black holes are analogous (but not identical) to the class of topological black holes recently considered in the literature [7, 8, 9].

The outline of the paper is as follows. In Section 2, we briefly review the original 3D BCEA theory. In Sections 3 and 4 we develop the higher dimensional generalization- BQPA theory. In Section 5 we show how BQPA theory may be considered as a theory of gravity interacting with (topolog-
ical) matter fields. In Section 6 we explicate the slightly distinct case of the two dimensional theory, showing, in particular, that it can be considered as a theory of gravity on a torus, with the various topological matter fields related to the complex geometry of the torus. In Section 7 the four dimensional theory is shown to admit cosmological solutions and in Section 8 constant curvature black hole solutions are obtained. We close with some some speculations and suggestions for further work.

2 BCEA Gravity

The BQPA theory is the generalization to n dimensions of a 3-D topological field theory [3], which we have called “BCEA theory” [3]. In this section, we review BCEA theory, highlighting those features useful in the generalization to spacetimes with arbitrary dimension.

The action for BCEA theory is

$$S_{[B, C, E, A]} = -\frac{1}{2} \int_{M^3} \left( E^i \wedge F_i[A] + B^i \wedge D_{(A)} C_i \right), \quad (2.1)$$

where $D_{(A)}$ is the covariant derivative with respect to the SO(3) (or SO(2,1)) connection $A^i$ with curvature $R_{a[A]}$ given by

$$F_i[A] := dA_i + \frac{1}{2} [A, A], \quad (2.2)$$

and $E^i, B^i, C^i$ are 1-form fields over $M_3$ taking their values in the Lie algebra so(3), respectively so(2,1). In particular, $A = A^i G_i$, with $G_i$ the generators of the Lie group:

$$[G_i, G_j] = \frac{1}{2} \epsilon_{ijk} G_k, \quad (2.3)$$

where the indices $i, j, ... = 0, 1, 2$ are raised and lowered with respect to the Kronecker delta $\delta_{ij}$ in the case of SO(3), and the Minkowski metric $\eta_{ij} = \text{diagonal}(-1, +1, +1)$ in the case of SO(2,1).

The stationary points of $S_{[B, C, E, A]}$ are determined by the field equations

$$F_i[A] = 0,$$

$$D_{(A)} B^i = 0,$$

$$D_{(A)} C_i = 0,$$

$$D_{(A)} E^i + \frac{1}{2} \epsilon^{ijk} B_j \wedge C_k = 0. \quad (2.4)$$
It is because of the term in $B_j \wedge C_k$ in the last equation of motion, that the triad $E^i$ is not, in general, compatible with the locally flat spin-connection $A_i$. Nevertheless, the equations of motion above determine a non-trivial spacetime geometry on $M^3$: if we define a one-form field $Q_i$ by the requirement

$$\varepsilon^{ijk} (Q_j \wedge E_k - B_j \wedge C_k) = 0,$$

then the equation of motion for the $E^i$ can be written as

$$dE^i + \frac{1}{2} \varepsilon^{ijk} \omega_j \wedge E_k = 0,$$

where

$$\omega_i := A_i + Q_i.$$  \hspace{1cm} (2.7)

Eq. (2.6) may be recognized as the condition that the frame-field $E^i$ be compatible with the (nonflat) spin connection $\omega_i$.

We may thus interpret BCEA theory as a model of (2+1)-dimensional gravity with a triad $E^i$ and a connection $\omega_i$ coupled to matter fields $B^i$ and $C^i$. The geometry is determined by the metric $g_{\mu\nu} = \eta_{ij} E^i_{\mu} E^j_{\nu}$ in the case where the gauge group is SO(2,1), and with $\eta_{ij}$ replaced by $\delta_{ij}$ for SO(3).

The action functional $S[B, C, E, A]$ is invariant under a twelve-parameter group whose infinitesimal generators are [2]

$$\delta B^i = D_{(A)} \rho^i + \frac{1}{2} \varepsilon^{ijk} B_j \tau_k,$$

$$\delta C^i = D_{(A)} \lambda^i + \frac{1}{2} \varepsilon^{ijk} C_j \tau_k,$$

$$\delta E^i = D_{(A)} \xi^i + \frac{1}{2} \varepsilon^{ijk} (E_j \tau_k + B_j \lambda_k + C_j \rho_k),$$

$$\delta A^i = D_{(A)} \tau^i.$$  \hspace{1cm} (2.8)

This group may be recognized as I(ISO(2,1)), where the notation IG denotes the semi-direct product of the Lie group G with its own Lie algebra $\mathcal{L}_G$. Like the action for ordinary Einstein gravity in three dimensions [10], the BCEA action can be obtained from a Chern-Simons functional, now for the gauge group I(ISO(2,1)).

Although BCEA theory has not been widely explored, a few of its properties are known. Quantization of the theory has been discussed: one can show [3] that the partition function is given by the sum of the squares of the Ray-Singer torsions for each flat connection $A$. Black hole solutions of various types were later obtained as solutions of the equations of motion.
Eq. (2.4) [3], including an analogue of the BTZ black hole [11]. Although the metric formally is the same as that of the BTZ metric, in BCEA theory the constants of integration have a different interpretation, and the black hole thermodynamics is not of the Bekenstein-Hawking form, except in the extremal case [3].

3 BQPA Theory

We now generalize the BCEA theory to \( n \) dimensions. The group involved is the Poincare group \( I'G \) built from the \( n \)-dimensional Lorentz group, \( G = SO(n-1,1) \) (or the rotation group \( G = SO(n) \)). If the generators of \( G \) are \( G_{ij} \), with \( i, j, \ldots = 0, 1, \ldots, (n-1) \) and \( T_j \) generate translations, then the set \( \{G_{ij}, T_k\} \) generates \( I'G \):

\[
\begin{align*}
[G_{ij}, T_k] &= \frac{1}{2} \left( g_{jk} \delta_i^l - g_{ik} \delta_j^l \right) T_l; \\
[G_{ij}, G_{kl}] &= \frac{1}{2} \left( g_{kj} G_{il} + g_{ji} G_{ki} + g_{ik} G_{lj} + g_{il} G_{jk} \right); \\
[T_i, T_j] &= 0,
\end{align*}
\] (3.1)

where \( g_{ij} := \eta_{ij} \) for \( SO(n-1,1) \) and \( g_{ij} = \delta_{ij} \) for \( SO(n) \). The simplest way to do this is via the action functional

\[
S[B, Q, P, A] = -\frac{1}{2} \int_{M_n} P^{ij} \wedge F_{ij}[A] + D_{(A)} B^i \wedge Q_i. \quad (3.2)
\]

Here, \( A := A^{ij} G_{ij} \) is an \( SO(n-1,1) \) (or \( SO(n) \)) connection. The curvature \( F^{ij}[A] = \frac{1}{2} F_{\mu\nu}[A] dx^\mu dx^\nu \) is given by

\[
F^{ij}[A] = dA^{ij} + A^i_k \wedge A^{kj}. \quad (3.3)
\]

The 1-form fields \( B = B^iT_i \) and \((n-2)\)-form fields \( Q = Q^iT_i \) take their values in the translation subgroup of \( I'SO(n-1,1) \) while the \((n-2)\)-form fields \( P = P_{ij} G^{ij} \) take their values in the dual to the Lie algebra generated by the \( G_{ij} \).

The exterior covariant derivatives with respect to the \( SO(n-1,1) \) connection \( A \) of the ‘translation’ and ‘rotation’ type fields are, respectively,

\[
\begin{align*}
D_{(A)} B^i &= dB^i + A^i_j \wedge B^j, \\
D_{(A)} P_{ij} &= dP_{ij} + A^k_i \wedge P_{kj} + A^k_j \wedge P_{ik}.
\end{align*}
\] (3.4)
The stationary configurations are given by the classical equations of motion

\[ F_{ij}[A] = 0, \quad (3.5) \]
\[ D_{(A)}P_{ij} + B_{ij} \wedge Q_i = 0, \quad (3.6) \]
\[ D_{(A)}B^i = 0, \quad (3.7) \]
\[ D_{(A)}Q_i = 0. \quad (3.8) \]

From the analysis \cite{12} it is straightforward to show that the symmetry group of the theory is \( I'(I'SO(n - 1, 1)) \), i.e., the semi-direct product of \( I'SO(n - 1, 1) \) with its Lie algebra. Indeed, the infinitesimal gauge transformations which preserve the action Eq. \( (3.2) \) are given by \[ \delta A_{ij} = D_{(A)}\tau_{ij}, \quad (3.9) \]
\[ \delta Q_i = D_{(A)}\xi_i - \tau_i^j Q_j, \quad (3.10) \]
\[ \delta B^i = D_{(A)}\lambda^i - \tau^i_j B^j, \quad (3.11) \]
\[ \delta P_{ij} = D_{(A)}\nu_{ij} + \lambda_i [Q_j] - \xi_i [B_j] + 2\tau_{ij}^k P_{jk}. \quad (3.12) \]

The gauge parameters are form fields, with \( \tau_{ij} = -\tau_{ji} \) and \( \lambda^i \) both 0-form fields, while \( \xi^i \) and \( \nu_{ij} = -\nu_{ji} \) are \((n - 3)\)-form fields. For \( n > 3 \) the gauge symmetry is reducible.

We emphasize here that the action Eq. \( (3.2) \) is a topological field theory – like Chern-Simons or BF theories there are no propagating modes – in other words the (reduced) phase space is a finite dimensional manifold. Also, as in other topological field theories, the diffeomorphisms in the ‘base manifold’ \( M_n \) are gauge transformations if the equations of motion are satisfied \cite{12}.

4 Hamiltonian Analysis

In the 3-D theory, the 1-form fields \( E^i \) are the momenta canonically conjugate to the connection components \( A_i \) \cite{2}. In the Ashtekar formulation of canonical general relativity theory, the spatial triads are conjugate to the complex connections \cite{3}. This motivated extracting gravitational physics from the 3-D theory by identifying the \( E^i \) with the spacetime triad. In the following we generalize the Hamiltonian analysis of the 3-D theory to \( n \)-dimensions as a prelude to extracting gravitational physics. We generalize the results obtained by Bi \cite{12} for \( n = 4 \).
Locally the splitting of the manifold $M_n$ into space $\Sigma_{n-1}$ and time $\mathbb{R}$ is accomplished by defining a volume element $\epsilon^{a_1\ldots a_{n-1}}$ on $\Sigma_{n-1}$ from the permutation symbol $\epsilon^{\mu_1\ldots\mu_n}$ on $M_n$ by

$$\epsilon^{a_1\ldots a_{n-1}} = \epsilon^{0a_1\ldots a_{n-1}}. \quad (4.1)$$

Let $t$ denote the ‘time coordinate’ on $\mathbb{R}$ and $x^a$ the ‘spatial coordinates’ on a patch on $\Sigma_{n-1}$. Then after integrating terms in $\partial_0 A_{ij}$ and $\partial_0 B^i_a$ by parts and using the notation $f := \partial_0 f, D_a f^i := \partial_a f^i + A^i_j f^j$ we get for the action Eq. (3.2)

$$S = \frac{1}{2(n-2)!} \int dt \int_{\Sigma_{n-1}} d^{n-1}x \left\{ \epsilon^{a_1\ldots a_{n-2}a} \left[ \dot{A}_{ija} P^{ij}_{a_1\ldots a_{n-2}} + \dot{B}^i_a Q_{ia_1\ldots a_{n-2}} \right] + 
A_{ij0} \left( D_a P^{ij}_{a_1\ldots a_{n-2}} + B^i_a Q^{ij}_{a_1\ldots a_{n-2}} \right) + B^i_0 D_a Q_{ia_1\ldots a_{n-2}} \right\} + 
\frac{1}{2(n-2)!} \int dt \int_{\partial\Sigma_{n-1}} dS_a \epsilon^{a_1\ldots a_{n-2}a} \left( A_{ij0} P^{ij}_{a_1\ldots a_{n-2}} + B^i_0 Q_{ia_1\ldots a_{n-2}} \right). \quad (4.2)$$

The first two terms give the momenta canonically conjugate to $A_{ija}$ and $B^i_a$ respectively

$$\Pi^{ija} := \frac{\delta S}{\delta A_{ija}} = \frac{-1}{(n-2)!} \epsilon^{a_1\ldots a_{n-2}a} P^{ij}_{a_1\ldots a_{n-2}}, \quad (4.3)$$

$$\Pi^a_i := \frac{1}{2(n-2)!} \epsilon^{a_1\ldots a_{n-2}a} Q_{ia_1\ldots a_{n-2}}. \quad (4.4)$$

On the other hand, the coefficients of the Lagrange multipliers $A_{ij0}, B^i_0, P^{ij}_{0a_1\ldots a_{n-3}}, Q_{iaa_1\ldots a_{n-3}}$ are the constraints

$$C^{ij} := D_a \Pi^{ija} + B^i_a \Pi^{ija} \approx 0, \quad (4.5)$$

$$C^a_i := D_a \Pi^a_i \approx 0, \quad (4.6)$$

$$C_{ijab} := F_{ijab} \approx 0, \quad (4.7)$$

$$C^a_{ab} := D_{[a} B^i_{b]} \approx 0. \quad (4.8)$$

It can be shown that the Poisson bracket algebra satisfied by the constraints is the Lie algebra of the group $I(I^r SO(n-1,1))$. See [12] for the case $n = 4$.

Now the $\Pi^{ija}$ are Lie algebra valued components of a spatial vector field density. The most natural way to extract a spatial frame-field is

$$\hat{E}^{fa} = \frac{(-1)^n}{[(n-2)!]^2} \epsilon^{a_1\ldots a_{n-2}a} P^{ij}_{a_1\ldots a_{n-2}}. \quad (4.9)$$
The indices $I, J, ..., = 1, 2, ..., (n - 1)$ denote components in the Lie algebra $so(n - 1)$ of the little group of $SO(n - 1, 1)$. The $\tilde{E}^I_a$ are the densitized spatial components of the contravariant frame-fields. Note that $\tilde{E}^I_a$ is the spatial Hodge dual of the spatial components of the $(n - 2)$–form $P^I_0$. We will use this in the next section to construct Lorentzian (or Riemannian) spacetime geometries. We note here that, only in the case $n = 4$, is it equally natural to define the densitized contravariant spatial triad by

$$\ast \tilde{E}^I_a := \frac{1}{2} \epsilon^I_{JK} \Pi^{JKa}$$

$$= -\frac{1}{2} \epsilon^I_{JK} \epsilon^{abc} P^{JK}_{bc}. \tag{4.10}$$

5 The Gravitational Sector of BQPA Theory

We saw in Section 1 that the 3-D theory could be interpreted as a gravity plus gauge field theory by identifying the 1-form fields $E^i, \omega^i$ with the spacetime geometry for any configuration in which the $\omega^i$ are solutions of the algebraic equations Eq. (2.5). We shall generalize this to the $n$–dimensional case in this section.

In the case $n > 3$, the the field $P^{ij}$, corresponding to $E^i$, is an $(n - 2)$–form field and, in general, we saw that a densitized spatial contravariant frame-field $\tilde{E}^I_a$ can be defined via Eq. (4.9). Now we define in the case that $n \geq 3$ a contravariant vector frame-field $E^I_a$ by

$$E^I_a := \tilde{E}^{-1/n-2} \tilde{E}^I_a, \tag{5.1}$$

in a region where $\tilde{E} \neq 0$, with $\tilde{E} := \det(\tilde{E}^I_a)$. In this region, $E^I_a$ has an inverse which we denote by $E^a_I$. It follows that $E := \det(E^a_I) = 1/\det(E^I_a)$, so that $E = \tilde{E}^{1/n-2}$.

A Riemannian metric $h_{ab}$ on $\Sigma_{n-1}$ is then given by the the inverse of

$$h^{ab} = \delta^{IJ} E^a_I E^b_J. \tag{5.2}$$

The spacetime geometry is specified, in the ADM formulation, by the spatial frame-field $E^I = E^I_a dx^a$ together with the lapse function $N$ and shift vector $N^a$ on $\Sigma_{n-1}$.

$$ds^2 = -N^2 dt^2 + (\delta_{IJ} E^I_a E^J_b) (dx^a + N^a dt)(dx^b + N^b dt) \tag{5.3}$$

1For the case of $n = 2$, this construction fails. See the following section for the case $n = 2$. 7
We define the lapse and shift in terms of the $P^{ij}$ via
\begin{align}
N &= \frac{1}{(n-1)!(n-2)!} \epsilon_{aba_1...a_{n-3}} E I_a E_{jb} P^{IJ}_{0a_1...a_{n-3}}, \\
N^a &= \frac{(-1)}{[(n-2)!]^2} \epsilon_{aba_1...a_{n-3}} E I_b P^{0I}_{0a_1...a_{n-3}}.
\end{align}

We have defined the ADM components- the spatial metric, lapse function and shift vector- from some of the components of the form $P^{ij}$. We can construct a covariant frame-field $e^I_\mu$ from the ADM components via
\begin{equation}
e^I_a = E^I_a, \quad e^I_0 = E^I_a N^a, \quad e^0_0 = N.
\end{equation}

Note that the components $e^0_a$ are not determined from the ADM variables. One may use the gauge freedom to choose, say, $e^0_a = 0$. This is consistent with
\begin{equation}
P^{ij} = \epsilon^{ij}_{k_1...k_{n-2}} e^{k_1} \wedge ... \wedge e^{k_{n-2}},
\end{equation}
as long as the frame-field satisfies $e^0_a = 0$, and then it follows that $P^{IJ}_{a_1...a_{n-2}} = 0$. Note that the $N, N^a$ are Lagrange multipliers enforcing constraints in both cases.

In the case $n = 4$, the components $P^{IJ}_{ab}$ are subject to an intriguing interpretation. As we saw at the end of Section 3- Eq. (4.10)- we could define the densitized spatial frame field with respect to the components $P^{ij}_{ab}$ rather than the $P^{0I}_{ab}$. Now on-shell the $A^{ij}_a$- the spatial components of the $SO(3,1)$ connection- are flat. One can choose a gauge in which the $A^{ij}_a$ are self-(or anti-self-)dual, i.e.
\begin{equation}
* A^{ij}_a := \frac{1}{2} \epsilon^{ijkl} A^{kl}_a = \pm A^{ij}_a.
\end{equation}

The corresponding components of the canonical momentum $\Pi^{ija}$ are self-(anti- self-)dual. Hence the densitized spatial frame fields satisfy
\begin{equation}
* \tilde{E}^{ja} = \pm \tilde{E}^{ja}.
\end{equation}

Hence, up to a gauge transformation, the two geometries defined by $*\tilde{E}$ and $\tilde{E}$ are equivalent. This construction is reminiscent of the formulation of Einstein gravity in terms of the Ashtekar variables \cite{13}.

\footnote{We use the convention that if $w$ is a p-form, then $w = \frac{1}{p!} w_{a_1...a_p} dx^{a_1} \wedge ... \wedge dx^{a_p}$.}
6 The Two Dimensional Case

The two dimensional case is completely and explicitly solvable, though, as we will see, somewhat degenerate. It is informative to display the solution for this case, since it illustrates many of salient features of the BQPA theories.

We will show below that, in the simple case where the spacetime is two dimensional, that the gauge group of the BQPA model is \( I'\text{SO}(2) \). We note here that this case is rather singular compared with that of higher spacetime dimension, where the gauge group is \( I'\text{SO}(N) \).

In this case, \( P_{ij} = \epsilon_{ij} P \) is a 0-form. The indices \( i, j, ... = 0, 1 \). The \( \text{SO}(2) \)-connection is \( A_{ij} = \epsilon_{ij} A \). The \( Q_i \) and \( B^i \) are respectively 0-and 1-form fields taking values in \( \text{so}(2) \).

Since the group \( \text{SO}(2) \) is abelian, the vanishing curvature condition gives, in a coordinate patch \( U \),

\[
A = d\alpha, \tag{6.1}
\]

where \( \alpha \) is a 0-form field. Now define

\[
Q := Q_0 + iQ_1, \quad B := B_0 + iB_1. \tag{6.2}
\]

Then the equations of motion for \( Q_i \) and \( B^i \) can be written respectively in forms

\[
dQ = i\alpha Q, \\
 dB = i\alpha B, \tag{6.3}
\]

and have respective general solutions

\[
Q = q e^{i\alpha}, \\
B = (\bar{B} + dB)e^{i\alpha}, \tag{6.4}
\]

where \( q := q_0 + iq_1 \) is an arbitrary complex constant, \( \bar{B}^i \) is a harmonic 1-form field and \( \beta^i \) is an arbitrary 0-form field, with \( \bar{B} = \bar{B}^0 + i\bar{B}^1 \) and similarly \( \beta = \beta^0 + i\beta^1 \).

The equations of motion for \( P^{ij} \) reduce to

\[
dP + \frac{1}{2}(Q_0 B^1 - Q_1 B^0) = 0. \tag{6.5}
\]
It is straightforward to show that in the Hamiltonian analysis there are two first class constraints:

\[
\begin{align*}
   C &= 2 \partial_x P + \epsilon^i_j B^j Q_i, \\
   C_i &= \partial_x Q_i + \epsilon^j_i A_x Q_j.
\end{align*}
\]

(6.6) (6.7)

It follows that the Poisson bracket algebra is precisely that of the Lie algebra of the two dimensional Poincare group \( \mathcal{I}'SO(2) \).

We now proceed to construct a spacetime geometry from the \( \mathcal{I}'SO(2) \) structure. We are of course dealing with the Riemannian case. We restrict to the case that the 2-manifold is closed and compact. Indeed, it is not surprising that our \( \mathcal{I}'SO(2) \) structure forces the torus topology onto the 2-manifold, since the isometry group of the torus is precisely \( \mathcal{I}'SO(2) \). The form of Eq. (6.5) suggests that \( dP \) is the imaginary part of the product \( \frac{1}{2} \bar{Q}B \), where \( Q \) is the complex conjugate of \( Q \). Hence we define

\[
\begin{align*}
   d\hat{P} &= -\frac{1}{2} \bar{Q}B \\
   &= -\frac{1}{2} (Q_i B^i + \epsilon^j_i Q_i B^j).
\end{align*}
\]

(6.8) (6.9)

We see that \( d\hat{P} \) is closed if \( B \) and \( Q \) are closed under the exterior covariant derivative \( D_A \) with respect to the flat Abelian connection \( A \). Furthermore, the explicit dependence on the flat connection \( A \) in \( B \) and \( Q \) cancel in \( \bar{Q}B \). We construct \( B \) locally from the Beltrami form \( dz + \mu d\bar{z} \) and the flat connection \( A = d\alpha \) as follows:

\[
B = 2e^{i\alpha} (dz + \mu d\bar{z}).
\]

(6.10)

Note that \( B \) above is a non-trivial harmonic 1-form field if \( \partial_\mu \mu = 0 \). Then

\[
d\hat{P} = \bar{q} (dz + \mu d\bar{z}).
\]

(6.11)

This gives the usual form for the metric on 2-torus:

\[
ds^2 = |d\hat{P}|^2 = |q|^2 |dz + \mu d\bar{z}|^2.
\]

(6.12)

With \( \mu = \)constant, the metric is that of a flat torus with Teichmüller parameter \( \tau \) determined by \( \mu = (1 + i\tau)/(1 - i\tau) \). Hence, \( \hat{P} \) is the complex coordinate in which the metric on the torus is isothermal.
7 Cosmological Solutions

We consider in this section spacetime solutions to the BQPA theory. These are straightforward to obtain. For a spacetime of constant positive curvature the solution corresponding to the metric
\[ ds^2 = -dt^2 + e^{2t/\ell} \left( dx^2 + dy^2 + dz^2 \right), \] (7.1)
is given by
\[ P_{ij} = \epsilon^{ijkl} e_k \wedge e_l, \] (7.2)
\[ B_I = 0 \quad B_0 = -4e^{2t/\ell} dt, \] (7.3)
\[ Q_I = \frac{1}{\ell} \epsilon_{ijk} dx^j \wedge dx^K \quad Q_0 = 0, \] (7.4)
\[ A_i^j = 0, \] (7.5)
where
\[ e^0 = dt \quad e^I = e^{t/\ell} \delta^I_a dx^a, \] (7.6)
and I, J, ... = 1, 2, 3.
Likewise the solution with constant negative curvature
\[ ds^2 = dz^2 + e^{2z/\ell} \left( -dt^2 + dx^2 + dx^2 \right), \] (7.7)
is given by Eq. (7.2) and Eq. (7.5), but with
\[ B_I = 0 \quad B_3 = 4e^{2z/\ell} dz, \] (7.8)
\[ Q_I = \frac{1}{\ell} \epsilon_{ijk} dx^j \wedge dx^K \quad Q_3 = 0, \] (7.9)
(7.10)
where
\[ e^I = e^{z/\ell} \delta^I_a dx^a \quad e_3 = dz, \] (7.11)
with indices I, J, ... = 0, 1, 2 raised and lowered by the 2 + 1-dimensional Minkowski metric \( \eta^{IJ} = \text{diagonal}(-1,+1,+1). \)
The solutions derived above illustrate an application of our approach toward deriving gravity from topological field theory in the simplest non-trivial case, namely that of a spacetime with constant curvature.
8 Black Holes

The method for obtaining black hole solutions from spacetimes of constant negative curvature are by now well-established \[11, 7, 8\]. Since spacetimes of constant negative curvature are solutions to the field equations, we can expect that black hole spacetimes are solutions to the field equations as well. We find that this is indeed the case. We obtain the solution

\[
[P_{ij}] =
\begin{bmatrix}
0 & -2R^2 \sinh \theta \, d\theta \wedge d\phi & 2R \sqrt{f} \sinh \theta \, dR \wedge d\phi & -2R \sqrt{f} \, dR \wedge d\theta \\
2R^2 \sinh \theta \, d\theta \wedge d\phi & 0 & 2R \sqrt{f} \sinh \theta \, dT \wedge d\phi & -2R \sqrt{f} \, dT \wedge d\theta \\
-2R \sqrt{f} \, dR \wedge d\phi & -2R \sqrt{f} \sinh \theta \, dT \wedge d\phi & 0 & 2dT \wedge dR \\
2R \sqrt{f} \, dR \wedge d\theta & 2R \sqrt{f} \, dT \wedge d\theta & -2dT \wedge dR & 0
\end{bmatrix}
\]  

(8.1)

\[
[B_i] = -4 e^{2T/\ell} \sqrt{f} \left( dT + \frac{R}{\ell f} \, dR \right) \left[ 1, \frac{R}{\ell}, 0, 0 \right],
\]  

(8.2)

and

\[
[Q_i] = 2 \frac{R}{\ell f} e^{-2T/\ell} \left[ \frac{R^2}{\ell \sqrt{f}} \sinh \theta \, d\theta \wedge d\phi, -\frac{R}{\sqrt{f}} \sinh \theta \, d\theta \wedge d\phi, \sinh \theta \lambda \wedge d\phi, -\lambda \wedge d\theta \right].
\]  

(8.3)

In the above, the quantity \( f := -1 + R^2/\ell^2 \) and

\[
\lambda := \left( \frac{R}{\ell} \, dT + \frac{1}{f} \, dR \right).
\]

The connection is

\[
[A^i_j] :=
\begin{bmatrix}
0 & \frac{dR}{\ell f} & -\frac{Rd\theta}{\ell \sqrt{f}} & -\frac{R \sinh \theta d\phi}{\ell \sqrt{f}} \\
-\frac{dR}{\ell f} & 0 & \frac{d\theta}{\sqrt{f}} & -\frac{\sinh \theta}{\sqrt{f}} \, d\phi \\
-\frac{R}{\ell \sqrt{f}} \, d\theta & -\frac{d\theta}{\sqrt{f}} & 0 & -\cosh \theta d\phi \\
-\frac{R}{\ell \sqrt{f}} \sinh \theta \, d\phi & -\frac{l}{\sqrt{f}} \sinh \theta \, d\phi & \cosh \theta \, d\phi & 0
\end{bmatrix}
\]  

(8.4)
and it is straightforward to show that it is flat, i.e. $F^{ij}[A] = dA^{ij} + A^i_k \wedge A^{kj} = 0$.

Using the prescription given in section (3) we find that $P^{ij} = \epsilon^{ijkl} e_k \wedge e_l$, consistent with eq. (5.7), and so the metric is

$$ds^2 = -f \ dT^2 + f^{-1} \ dR^2 + R^2 \left( d\theta^2 + \sinh^2(\theta) \ d\phi^2 \right),$$

(8.5)

where $l = \sqrt{3/|\Lambda|}$, with $\Lambda$ the (negative) cosmological constant. The $(\theta, \phi)$ section of the metric (8.5) describes a compact space of genus $g > 2$ provided $\theta$ and $\phi$ are appropriately identified. The section described by these coordinates is that of a hyperbolic space or pseudosphere. This space becomes compact upon identifying the opposite edges of a $4g$-sided polygon (whose sides are geodesics) centered at the origin $\theta = \phi = 0$ of the pseudosphere. An octagon is the simplest such polygon, yielding a compact space of genus $g = 2$ under these identifications. Further details of this construction can be found in refs. [7, 8].

Once these identifications are made, the metric (8.5) describes the spacetime of a topological black hole of genus $g > 2$, with event horizon at $r = l$. Indeed, the above solution (8.1–8.4) is a $(3 + 1)$ dimensional generalization of the black hole solution obtained in BCEA theory [8]. Note that this is a local solution which yields the metric (8.5).

9 Summary

We have in this paper demonstrated how one can extend to higher dimensions the understanding of gravity derived from a topological field theory that has been previously given in $(2 + 1)$ dimensions [2, 3]. In these BQPA theories the connection is flat and the metric is a derived quantity. However the field equations give rise to spacetime structures that are (at least) as interesting as in $(2 + 1)$ dimensions, as shown in the previous two sections. We suggest that this latter feature – in conjunction with the finite number of degrees of freedom these models have – make them attractive candidates to study, particularly in terms of their quantum properties.

A number of interesting questions remain. What is the physical interpretation of the remaining degrees of freedom $P_{a_1\ldots a_{n-2}}^{IJ}$? In particular, are there topological obstructions to the ‘gauge’ $P_{a_1\ldots a_{n-2}}^{IJ} = 0$? Are there other interesting solutions to the theory which have a clear physical interpretation? How does the coupling of other forms of matter affect the basic physical properties and structure of BQPA theories? What resemblance do
these theories have to their \((2 + 1)\) dimensional counterparts? These and other topics remain the subjects of future investigation.

**ACKNOWLEDGEMENTS**

We would like to acknowledge the partial support of the Natural Sciences and Engineering Research Council of Canada and of the National Science Foundation (Grant No. PHY94-07194). R.B. Mann would like to thank the hospitality of the ITP in Santa Barbara, where some of this work was carried out.

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