Nilpotent Cone and Bivariant Theory

Vincenzo Di Gennaro, Davide Franco, and Carmine Sessa

Abstract. We exhibit a new proof, relying on bivariant theory, that the nilpotent cone is rationally smooth. Our approach enables us to prove a slightly more general statement.

Mathematics Subject Classification. Primary 14L30; Secondary 14B05, 14E15, 20C30, 58K15.

Keywords. Nilpotent cone, decomposition theorem, Springer theory, Weyl group, bivariant theory, Gysin homomorphism, homology manifold, resolution of singularities, Grothendieck simultaneous resolution.

1. Introduction

In [2] Borho and MacPherson proved that the nilpotent cone is a rational homology manifold. The proof relies on the celebrated Decomposition Theorem by Beilinson, Bernstein, Deligne and Gabber [1] and on the Springer’s theory of Weyl group representations (see [2] and the references therein).

The aim of this paper is to present a new proof, in our opinion conceptually very simple, based on the bivariant theory founded by Fulton and MacPherson in [4]. Actually, our approach enables us to prove a slightly more general statement (see Remark 2.4 below). By bivariant theory we intend the topological bivariant homology theory with coefficients in a Noetherian commutative ring with identity $\mathbb{A}$ [4, pp. 32, 83 and p. 86, Corollary 7.3.4].

That the nilpotent cone is a rational homology manifold can be seen as an easy consequence of a characterization of homology manifolds we recently proved in [3, Theorem 6.1]: given a resolution of singularities $\pi : \tilde{N} \to N$ of a quasi-projective variety $N$, then $N$ is a homology manifold if and only if there exists a bivariant class of degree one for $\pi$. A bivariant class of degree one for
π is an element \( \eta \in H^0(\tilde{N} \to N) \) such that the induced Gysin homomorphism \( \eta_0 : H^0(\tilde{N}) \to H^0(N) \) sends \( 1_{\tilde{N}} \) to \( 1_N \).

2. The Main Result

Theorem 2.1. Let \( \pi' : \tilde{g} \to g \) be a projective morphism between complex quasi-projective nonsingular varieties of the same dimension. Assume that \( \pi' \) is generically finite, of degree \( \delta \). Let \( N \subset g \) be a closed irreducible subvariety. Consider the induced fibre square diagram:

\[
\begin{array}{ccc}
\tilde{N} & \to & \tilde{g} \\
\downarrow \pi & & \downarrow \pi' \\
N & \to & g \\
\end{array}
\]

where \( \tilde{N} := N \times_g \tilde{g} \). If \( \tilde{N} \) is irreducible and nonsingular and \( \pi \) is birational, then \( N \) is an \( \mathbb{A} \)-homology manifold for every Noetherian commutative ring with identity \( \mathbb{A} \) for which \( \delta \) is a unit.

Proof. Since \( \pi' : \tilde{g} \to g \) is a projective morphism between complex quasi-projective nonsingular varieties of the same dimension, it is a local complete intersection morphism of relative codimension 0 [4, p. 130]. Let \( \theta' \in H^0(\tilde{g} \to g) \cong \text{Hom}_{D^b_c(g)}(R\pi'_* A_{\tilde{g}}, A_g) \) be the orientation class of \( \pi' \) [4, p. 131]. Let \( \theta'_0 : H^0(\tilde{g}) \to H^0(g) \) be the induced Gysin map. It is clear that \( \theta'_0(1_{\tilde{g}}) = \delta \cdot 1_g \in H^0(g) \), where \( \delta \) is the degree of \( \pi' \). Therefore, if we denote by \( \theta := i^* \theta' \in H^0(\tilde{N} \to N) \cong \text{Hom}_{D^b_c(N)}(R\pi_* A_{\tilde{N}}, A_N) \) the pull-back of \( \theta' \), then \( \delta^{-1} \cdot \theta \) is a bivariant class of degree one for \( \pi \) [3, 2. Notations, (ii)]. At this point, our claim follows by [3, Theorem 6.1]. For the Reader’s convenience, let us briefly summarize the argument.

Since \( \delta^{-1} \cdot \theta \) is a bivariant class of degree one for \( \pi \), it follows that \( (\delta^{-1} \cdot \theta) \circ \pi^* = \text{id}_{A_N} \) in \( D^b_c(N) \), i.e. that \( \delta^{-1} \cdot \theta \) is a section of the pull-back \( \pi^* : A_N \to R\pi_* A_{\tilde{N}} \) [3, Remark 2.1, (i)]. Hence, \( A_N \) is a direct summand of \( Rf_* A_{\tilde{N}} \) in \( D^b_c(N) \) [3, Lemma 3.2] and so we have a decomposition

\[
Rf_* A_{\tilde{N}} \cong A_N \oplus \mathcal{K}. \tag{1}
\]

Now, set \( \nu = \dim \tilde{N} = \dim N \) and let \( [\tilde{N}] \in H_{2\nu}(\tilde{N}) \) be the fundamental class of \( \tilde{N} \). We have:

\[
[\tilde{N}] \in H_{2\nu}(\tilde{N}) \cong H^{-2\nu}(\tilde{N} \to \text{pt.}) \cong \text{Hom}_{D^b_c(\tilde{N})}(A_{\tilde{N}}[\nu], D(A_{\tilde{N}}[\nu])),
\]

where \( D \) denotes Verdier dual. Therefore, \( [\tilde{N}] \) corresponds to a morphism

\[
A_{\tilde{N}}[\nu] \to D(A_{\tilde{N}}[\nu]), \tag{2}
\]
whose induced map in hypercohomology is nothing but the duality morphism
\[ \mathcal{D}_{\tilde{N}} : x \in H^\bullet(\tilde{N}) \to x \cap [\tilde{N}] \in H_{2\nu-\bullet}(\tilde{N}). \] (3)
If we assume that \( \tilde{N} \) is nonsingular (actually it suffices that \( \tilde{N} \) is an \( \mathbb{A} \)-homology manifold), the morphisms (2) and (3) are isomorphisms. The first one induces an isomorphism
\[ R\pi_\ast \mathbb{A}_{\tilde{N}}[\nu] \to D\big( R\pi_\ast \mathbb{A}_{\tilde{N}}[\nu] \big), \]
which in turn, via decomposition (1), induces two projections
\[ \mathbb{A}_N[\nu] \to D(\mathbb{A}_N[\nu]), \quad \mathbb{K}[\nu] \to D(D(\mathbb{K}[\nu])). \] (4)
Making explicit the isomorphism induced in cohomology and homology by (1), one may prove [3, Corollary 5.1] that \( \mathcal{D}_{\tilde{N}} \) is the direct sum of \( P_1 \) and \( P_2 \), where
\[ P_1 : H^\bullet(\mathcal{N}) \to H_{2\nu-\bullet}(\mathcal{N}) \quad \text{and} \quad P_2 : \mathbb{H}(\mathbb{K}[\nu]) \to \mathbb{H}(D(\mathbb{K}[\nu])) \]
are the maps induced in hypercohomology by the projections (4). It follows that \( P_1 \) is an isomorphism, because so is \( \mathcal{D}_{\tilde{N}} \), and this holds true when restricting to every open subset \( U \) of \( \mathcal{N} \). For instance (see also [3, Corollary 5.1]), if \( \mathcal{U} = \pi^{-1}(U) \), the vanishing of the morphism \( \mathbb{H}^\bullet(\mathbb{K}_U[\nu]) \to H_{2\nu-\bullet}(U) \) derives from projection formula [4, p. 26, G4, (ii)]:
\[ \pi_\ast([\mathcal{U}] \cap \lambda_*w) = \pi_\ast(\delta^{-1}\theta^*[U] \cap \lambda_*w) = \delta^{-1}(\theta_*\lambda_*w) \cap [U] = 0, \quad \forall w \in \mathbb{H}^\bullet(\mathbb{K}_U[\nu]), \]
where \( \lambda_* \) is the morphism induced in hypercohomology by \( \mathbb{K}_U[\nu] \to R\pi_\ast \mathbb{K}_U[\nu] \).

Therefore, we have \( \mathbb{A}_N[\nu] \cong D(\mathbb{A}_N[\nu]) \), which is equivalent to say that \( \mathcal{N} \) is an \( \mathbb{A} \)-homology manifold. \( \square \)

**Remark 2.2.** Observe that, as a scheme, \( \tilde{N} \) could also be nonreduced, but what matters is that, for the usual topology, it is a nonsingular variety [4, p. 32, 3.1.1].

**Corollary 2.3.** The nilpotent cone is a rational homology manifold.

**Proof.** Let \( \pi : \tilde{N} \to \mathcal{N} \) be the Springer resolution of the nilpotent cone \( \mathcal{N} \). It extends to a generically finite projective morphism \( \pi' : \tilde{g} \to g \), known as the Grothendieck simultaneous resolution, between complex quasi-projective nonsingular varieties of the same dimension [2, p. 49]. Therefore, Theorem 2.1 applies. \( \square \)

**Remark 2.4.** If the Grothendieck simultaneous resolution \( \pi' : \tilde{g} \to g \) has degree \( \delta \), by Theorem 2.1 we deduce that the nilpotent cone \( \mathcal{N} \) is an \( \mathbb{A} \)-homology manifold for every Noetherian commutative ring with identity \( \mathbb{A} \) for which \( \delta \) is a unit. For instance, for the variety \( \mathcal{N} \) of nilpotent matrices in \( \text{GL}(n, \mathbb{C}) \), we have \( \delta = n! \). Therefore, in this case, \( \mathcal{N} \) is also a \( \mathbb{Z}_h \)-homology manifold for every integer \( h \) relatively prime with \( n! \) in \( \mathbb{Z} \).
**Funding**  Open access funding provided by Università degli Studi di Napoli Federico II within the CRUI-CARE Agreement. The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

**Data Availability Statement**  Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Declarations**  
**Conflict of interest** The authors have no relevant financial or non-financial interests to disclose.

**Open Access.**  This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit [http://creativecommons.org/licenses/by/4.0/](http://creativecommons.org/licenses/by/4.0/).

**References**

[1] Beilinson, A., Bernstein, J., Deligne, P.: Faisceaux pervers, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, 100, Soc. Math. France, Paris, 5–171 (1982)

[2] Borho, W., MacPherson, R.: Partial Resolutions of Nilpotent Varieties. Astérisque 101–102, 23–74 (1983)

[3] Di Gennaro, V., Franco, D., Sessa, C.: Bivariant class of degree one. Results Math. 77(6),(2022). Paper No. 227

[4] Fulton, W., MacPherson, R.: Categorical framework for the study of singular spaces. Mem. Am. Math. Soc. 31(243) (1981). pp. vi+165

Vincenzo Di Gennaro  
Dipartimento di Matematica  
Università di Roma “Tor Vergata”  
Via della Ricerca Scientifica  
00133 Rome  
Italy  
e-mail: digennar@exp.mat.uniroma2.it
Davide Franco and Carmine Sessa  
Dipartimento di Matematica e Applicazioni “R. Caccioppoli”  
Università di Napoli “Federico II”  
Via Cintia  
80126 Napoli  
Italy  
e-mail: davide.franco@unina.it; carmine.sessa2@unina.it  

Received: April 20, 2023.  
Accepted: July 29, 2023.  

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.