Contractibility of deformation spaces of $G$–trees

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Abstract  Forester has defined spaces of simplicial tree actions for a finitely generated group, called deformation spaces. Culler and Vogtmann’s Outer space is an example of a deformation space. Using ideas from Skora’s proof of the contractibility of Outer space, we show that under some mild hypotheses deformation spaces are contractible.

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Culler and Vogtmann’s Outer space is a good geometric model for $\text{Out}(F_n)$, the outer automorphism group of a finitely generated free group of rank $n \geq 2$, for three reasons:

(1) Outer space is contractible;
(2) point stabilizers are finite; and
(3) there is an equivariant deformation retract on which the action is cocompact [1].

Outer space is the analog of Teichmüller space for the mapping class group of a closed negatively curved surface or of the symmetric space for an arithmetic group. See [1] and [12] for a survey of some results about $\text{Out}(F_n)$ obtained from using this connection between the three classes of groups. Also see [2] for some open questions about the similarities and differences.

Recall that Outer space is the moduli space of free actions of a free group on a simplicial tree. Forester has defined a generalization of Outer space for an arbitrary finitely generated group $G$ [5]. The generalization allows actions which are not free but requires the subgroups with fixed points to be the same among all actions in the moduli space. Unfortunately these spaces are not $\text{Out}(G)$–invariant in general. Nevertheless, in the cases when the space is invariant under $\text{Out}(G)$ these spaces have the potential to provide information about the structure of $\text{Out}(G)$. The purpose of this paper is to show that these spaces share the first of the above mentioned properties with Outer space, i.e. they are contractible.
For a finitely generated group \( G \), a \( G \)-tree is a metric simplicial tree on which \( G \) acts by isometries. Two \( G \)-trees \( T \) and \( T' \) are equivalent if there is a \( G \)-equivariant isometry between them. When we speak of a \( G \)-tree we will always mean the equivalence class of the \( G \)-tree. A subgroup is called an elliptic subgroup for \( T \) if it has a fixed point in \( T \). Given a \( G \)-tree there are two moves one can perform to the tree that do not change whether or not subgroups of \( G \) are elliptic. These moves correspond to the isomorphism \( A \cong A \ast_C C \) and are called collapse and expansion. For a detailed description of the moves see \cite{5}. In \cite{5} Forester proves the converse, namely if two cocompact \( G \)-trees have the same elliptic subgroups, then there is a finite sequence of collapses and expansions (called an elementary deformation) transforming one \( G \)-tree to the other. A \( G \)-tree \( T \) is cocompact if the quotient \( T/G \) is a finite graph.

We let \( \mathcal{X} \) denote a maximal set of cocompact \( G \)-trees which are related by an elementary deformation. By the theorem of Forester mentioned above, an equivalent definition is as the set of all cocompact \( G \)-trees that have the same elliptic subgroups as some fixed \( G \)-tree. Both of these interpretations are utilized in the following. This set \( \mathcal{X} \) is called a unnormalized deformation space. We will always assume that the \( G \)-trees are minimal, irreducible and that \( G \) acts without inversions. See section \[ \] for these definitions.

As is common practice in spaces of this nature, we projectivize by taking the quotient of \( \mathcal{X} \) under the action of \( \mathbb{R}^+ \) by homothety. The quotient \( \mathcal{X}/\mathbb{R}^+ \) is called a deformation space and is denoted \( \mathcal{D} \). Outer space is an example of a deformation space for a finitely generated free group where the only elliptic subgroup is the trivial group. Culler and Vogtmann described a contraction of the spine of Outer space using combinatorial methods and a “Morse-like” function \cite{4}. Skora showed in a different manner that Outer space is contractible \cite{11}. The method of Skora is to homotope the unnormalized deformation space projecting to Outer space to a set homeomorphic to a simplex\( \times \mathbb{R}^+ \) by continuously unfolding \( G \)-trees in the unnormalized deformation space. This homotopy descends to Outer space, proving its contractibility. It is this idea which we extend to show:

\textbf{Theorem 6.7} For a finitely generated group \( G \), any irreducible deformation space which contains a \( G \)-tree with finitely generated vertex groups is contractible.

The outline of the proof is as follows: starting with an unnormalized deformation space \( \mathcal{X} \), we look at the space \( \mathcal{M}(\mathcal{X}) \) of morphisms between elements of \( \mathcal{X} \). A morphism is a \( G \)-equivariant map between \( G \)-trees which on each
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segment either folds or is an isometry. Given a morphism $\phi: T \to Y$ we show that we can continuously interpolate between the two $G$–trees. We then fix some reduced $G$–tree $T \in \mathcal{X}$. For another $G$–tree $Y \in \mathcal{X}$ we define a map $B(Y): T \to Y$, which is not a morphism but is nice in certain respects. The assignment $Y \mapsto B(Y)$ is a continuous function between the appropriate spaces. We redefine the metric on $T$ to obtain another $G$–tree $T_Y$ (equivariantly homeomorphic to $T$) such that $B(T): T_Y \to Y$ is a morphism. Thus we can homotope $\mathcal{X}$ to the space of trees equivariantly homeomorphic to $T$. We show this space is homeomorphic to a simplex $\times \mathbb{R}^+$, thus $\mathcal{X}$ is contractible. This homotopy descends to a contraction of the deformation space $\mathcal{D}$.

Originally, the following proof was only for finitely generated generalized Baumslag–Solitar groups, for which there is a natural $\text{Out}(G)$–invariant deformation space. A generalized Baumslag–Solitar group is a group which admits an action on a simplicial tree where the stabilizer of any point is isomorphic to $\mathbb{Z}$. However after a research announcement by Guirardel and Levitt [7], which contains Theorem 6.7, we noticed that our proof for generalized Baumslag–Solitar groups went through in the general case after modifying case (ii) in Lemma 6.4. We are grateful for their announcement. They have proven Theorem 6.7 in the case of a free product and have given several consequences [8].

The majority of material presented within is in Skora’s preprint [11]. As this preprint was never published, we present the full details here. The main difference from [11] is section 6.

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1 Preliminaries

For a $G$–tree $T$, the length function $l_T: G \to [0, \infty)$ is defined by $l_T(g) = \min_{x \in T} d(x, gx)$. The characteristic set $T_g$, of a element $g \in G$ is where this minimum is realized, i.e. $T_g = \{x \in T \mid d(x, gx) = l_T(g)\}$. An element is elliptic if $l_T(g) = 0$ and hyperbolic otherwise. For $g \in G$ hyperbolic, the characteristic set is isometric to $\mathbb{R}$ and $g$ acts on $T_g$ by translation by $l_T(g)$. In this case the characteristic set of $g$ is often called the axis of $g$. Note that $d(x, gx) = 2d(x, T_g) + l_T(g)$ for both $g$ elliptic or $g$ hyperbolic. If a subgroup
$H \subseteq G$ is elliptic, we define the characteristic set of $H$ as $T_H = \{ x \in T \mid hx = x \ \forall h \in H \}$. For a closed set $A \subseteq T$ we let $p_A : T \to A$ denote the nearest point projection. A map between metric simplicial trees $\phi : T \to T'$ is morphism if for any segment $[x, y] \subseteq T$ there is a subsegment $[x, x'] \subseteq [x, y]$ on which $\phi$ is an isometry. If $T$ and $T'$ are $G$–trees, we also require that $\phi$ is $G$–equivariant.

We have following dictionary of group actions on trees [3]. A $G$–tree is trivial if there is a fixed point and minimal if there is no proper invariant subtree. A $G$–tree $T$ is reducible if:

1. every element fixes a point (equivalent to being trivial for finitely generated groups); or
2. $G$ fixes exactly one end of $T$; or
3. $G$ leaves a set of two ends of $T$ invariant.

If $T$ is not reducible, it is irreducible. A $G$–tree is irreducible if and only if there are two hyperbolic elements whose axes are either disjoint or intersect in a compact set [3]. This feature is preserved by elementary deformations [3], hence any $G$–tree obtained via an elementary deformation from an irreducible $G$–tree is also irreducible.

Unless otherwise stated, we will always assume $G$–trees are minimal and irreducible. We say a deformation space is irreducible if every $G$–tree in the space is irreducible. By the above statement, a deformation space is irreducible if any $G$–tree in the space is irreducible.

# Topology on deformation spaces

We endow an unnormalized deformation space $X$ with the Hausdorff–Gromov topology. Gromov introduced this topology as a way to compare two distinct metric spaces [6]. This topology generalizes the Hausdorff distance between two closed sets in a metric space. The deformation space $D$ is then topologized as the quotient $X/\mathbb{R}^+$. 

The Hausdorff–Gromov topology is defined as follows. Let $X, Y$ be metric $G$–spaces, i.e. metric spaces equipped with isometric $G$–actions. For any $\epsilon > 0$, an $\epsilon$–approximation is a set $R \subseteq X \times Y$ that surjects onto each factor such that if $x, x' \in X$ and $y, y' \in Y$ with $xRy$ (i.e. $(x, y) \in R$) and $x'Ry'$ then $|d(x, x') - d(y, y')| < \epsilon$. We say that $R$ is a closed $\epsilon$–approximation if $R$ is closed in $X \times Y$. For a finite subset $P \subseteq G$ and subspaces $K \subseteq X, L \subseteq Y$ the
\(\epsilon\)-approximation in \(K \times L\) is \(P\)-equivariant if whenever \(g \in P, x, gx \in K\) and \(y \in L\) with \(xRy\) then \(gy \in L\) and \(gxRgy\).

Given an \(\epsilon\)-approximation \(R \subseteq X \times Y\), we let \(R_\delta\) denote the closed \(\delta\)-neighborhood of \(R\) using the \(L^1\) metric. In other words \(R_\delta = \{(x, y) \in X \times Y \mid \exists(x', y') \in R\ \text{with} \ d(x, x') + d(y, y') \leq \delta\}\). One can show that \(R_\delta\) is a \((\epsilon + 2\delta)\)-approximation. If \(R\) is \(P\)-equivariant, then \(R_\delta\) is \(P\)-equivariant.

These \(\epsilon\)-approximations can topologize any set of metric \(G\)-spaces. In particular, they can topologize any unnormalized deformation space \(X\). Let \(S\) be such a set of metric \(G\)-spaces. Then for \(X \in S, K \subseteq X\) compact, \(P \subseteq G\) finite and \(\epsilon > 0\) define a basic open set \(U(X, K, P, \epsilon)\) to be the set of all \(Y \in S\) such that there is a compact set \(L \subseteq Y\) and a \(P\)-equivariant closed \(\epsilon\)-approximation \(R \subseteq K \times L\). If \(K \subseteq K'\) and \(P \subseteq P'\) then \(U(X, K', P', \epsilon) \subseteq U(X, K, P, \epsilon)\). This will allow us to assume that certain subsets of \(X\) and \(G\) are contained in \(K\) and \(P\) respectively by shrinking our basic open set.

Given an \(\epsilon\)-approximation \(R \subseteq X \times Y\), we will assume it is full: i.e. if \(xRy\) and \(x'Ry'\) then every point in \([x, x']\) is related by \(R\) to some point in \([y, y']\) and vice versa. This is not necessary but it cleans up some of the proofs in sections and \(\Box\) When the set \(S\) contains only trees the two topologies generated are the same. For \(X, P, \epsilon\) as above let \(U_f(X, K, P, \epsilon)\) be the set of all \(Y \in S\) such that there is a finite subtree \(L \subseteq Y\) and a \(P\)-equivariant closed full \(\epsilon\)-approximation \(R \subseteq K \times L\). Clearly we have \(U_f(X, K, P, \epsilon) \subseteq U(X, K, P, \epsilon)\). We now show the opposite inclusion of bases.

For two trees \(X, Y\), subsets \(K \subseteq X, L \subseteq Y\) related by an \(\epsilon\)-approximation \(R \subseteq K \times L\) and a finite segment \([x_1, x_2] \subseteq K\), let \(R([x_1, x_2]) = \{z \in L \mid \exists x \in [x_1, x_2] \text{ with } xRz\}\). For \(z \in R([x_1, x_2])\) with \(x_iRy_i\) for some \(y_i \in Y, i = 1, 2\) we have \(d(z, [y_1, y_2]) < 2\epsilon\). We have the following statement about the density of \(R([x_1, x_2])\).

**Lemma 2.1** If \(z_0 \in [y_1, y_2] \subseteq Y\) where \(x_iRy_i\) for \(i = 1, 2\), then there is a \(z \in R([x_1, x_2])\) such that \(d(z_0, z) < 2\epsilon\).

**Proof** We assume this is not the case. Let \(d(y_1, z_0) = d_1, d(y_2, z_0) = d_2\). As \(y_1, y_2 \in R([x_1, x_2])\) we can assume both \(d_1\) and \(d_2\) are larger than \(\epsilon\). Take \(x \in [x_1, x_2]\) such that \(d(x_1, x) = d_1\). Therefore \(d(x_2, x) < d_2 + \epsilon\). There is a \(z \in R([x_1, x_2])\) such that \(xRz\). For this \(z\), \(d(y_1, z) < d_1 + 2\epsilon\) and \(d(y_2, z) < d_2 + 2\epsilon\). Now we let \(z' = p_{y_1, y_2}(z)\). Hence by our initial assumption \(d(z_0, z') + d(z, z') \geq d(z_0, z) \geq 2\epsilon\). Assume without loss of generality that \(z'\) is closer to \(y_1\) than \(z_0\) is. Then \(d(y_2, z) = d(y_2, z_0) + d(z_0, z') + d(z', z) \geq d_2 + 2\epsilon\), a contradiction. \(\Box\)
To finish up the claim that the two above mentioned topologies are the same we let \( \delta = \frac{\epsilon}{5} \). Then for \( Y \in U(X, K, P, \delta) \) we have a \( P \)-equivariant \( \delta \)-approximation between \( K \) and some finite subtree \( L \subseteq Y \). By the above \( R_{2\delta} \) is a full \( P \)-equivariant \( \epsilon \)-approximation between \( K \) and \( L \). Therefore \( U(X, K, P, \delta) \subseteq U_f(X, K, P, \epsilon) \) and the two topologies are indeed the same.

We will also topologize the space of morphisms between elements in a deformation space. Let \( \phi: X \to X' \) and \( \psi: Y \to Y' \) be \( G \)-equivariant maps, a closed \( \epsilon \)-approximation between these two maps is a pair \((R, R')\) such that:

1. \( R \subseteq X \times Y \) and \( R' \subseteq X' \times Y' \) are closed \( \epsilon \)-approximations; and
2. for \( x \in X, y \in Y \) if \( xRy \) then \( \phi(x)R'\psi(y) \).

Let \( P \subseteq G \) be finite and \( K \subseteq X, K' \subseteq X', L \subseteq Y, L' \subseteq Y' \) be subspaces. The \( \epsilon \)-approximation \((R, R')\) is \( P \)-equivariant if \( R \) and \( R' \) are \( P \)-equivariant on the appropriate subspaces. Note that if \( z \in \text{graph}(\phi) \) then by definition there is a \( w \in \text{graph}(\psi) \) with \( z(R, R')w \).

As above this allows us to topologize a set of \( G \)-equivariant maps between \( G \)-spaces. In particular we can topologize \( \mathcal{M}(\mathcal{X}) \), the set of morphisms between elements of \( \mathcal{X} \). Let \( \mathcal{S}' \) a set of \( G \)-equivariant maps between \( G \)-spaces. For \( \phi: X \to X' \) in \( \mathcal{S}' \), and \( K \subseteq X, K' \subseteq X' \) both compact with \( \phi(K) \subseteq K' \), \( P \subseteq G \) finite and \( \epsilon > 0 \) define the basic open set \( U(\phi, K \times K', P, \epsilon) \) to be the set of all maps \( \psi: Y \to Y' \) in \( \mathcal{S}' \) such that there are compact sets \( L \subseteq Y \), \( L' \subseteq Y' \) with \( \psi(L) \subseteq L' \) and a \( P \)-equivariant closed \( \epsilon \)-approximation \((R, R')\) bewtween \( \phi: K \to K' \) and \( \psi: L \to L' \).

For a space \( \mathcal{S} \) of metric \( G \)-spaces and a space \( \mathcal{S}' \) of \( G \)-equivariant maps between the elements of \( \mathcal{S} \) we have the two continuous maps \( \mathcal{D}o \) and \( \mathcal{R}a \) defined from \( \mathcal{S}' \) to \( \mathcal{S} \) which send a map to its domain and range respectively. In other words, for \( \phi: X \to X' \) an element of \( \mathcal{S}' \) we have \( \mathcal{D}o(\phi) = X \) and \( \mathcal{R}a(\phi) = X' \).

There are two other topologies one might use to topologize a deformation space. Let \( C \) be the set of conjugacy classes for \( G \). Then we have a function \( l: \mathcal{X} \to \mathbb{R}^C \) where the coordinates are given by the length functions \( l_T(c) \) where \( c \in C \). Culler and Morgan showed that for minimal irreducible actions on \( \mathbb{R} \)-trees this function is injective. This defines a topology on \( \mathcal{X} \) (and hence on \( \mathcal{D} \)) called the \textit{axes} topology. Paulin proved that for spaces of minimal irreducible actions on \( \mathbb{R} \)-trees, the Hausdorff–Gromov topology is the same as the axes topology.

We can define the \textit{weak} topology directly on \( \mathcal{D} \). The \textit{volume} of a \( G \)-tree \( T \), denoted \( \text{vol}(T) \), is the sum of the lengths of the unoriented edges of \( T/G \). We
identify \( \mathcal{D} \) with the \( G \)-trees in \( \mathcal{X} \) that have volume one. By reassigning the lengths of the edges of \( T/G \) in a manner to hold the volume constant we can define a simplex in \( \mathcal{D} \). The weak topology is defined by considering \( \mathcal{D} \) as the union of such simplicies. In general, the weak topology is different from the axes and Hausdorff–Gromov topology, see [9] for an example.

3 Deforming trees

A morphism \( \phi: T \to T' \) between trees in an unnormalized deformation space \( \mathcal{X} \) can be decomposed into elementary deformations [5]. We will define trees \( T_t \) which continuously interpolate between \( T \) and \( T' \).

For the morphism \( \phi: T \to T' \), a nontrivial segment \([x, x'] \subseteq T\) is folded if \( \phi(x) = \phi(x') \). A folded segment is maximally folded if it cannot be locally extended to a segment which is folded. On a maximally folded segment \([x, x']\) the function \( d(\phi(z), \phi(x)) \) attains a local maximum at possibly several points. Such points are called fold points of the morphism \( \phi \). The points at which the global maxima are obtained are called maximal fold points. We remark that every fold point is a maximal fold point for some maximally folded segment. A fold point \( z \) is \( d \)-deep if \( d(\phi(x), \phi(z)) > d \) for some maximally folded segment \([x, x']\) of which \( z \) is a maximal fold point.

We let \( m(\phi) = \sup \{ d(\phi(z), \phi(x)) \mid z \in [x, x'] \text{ where } \phi(x) = \phi(x') \} \). Then \( m(\phi) \) is finite as elementary deformations are quasi-isometries [5]. Notice that \( m(\phi) = 0 \) if and only if \( \phi \) is an isometry and hence \( T = T' \) as \( G \)-trees. For \( 0 \leq t \leq 1 \) we define \( V_t = \{ (x, y) \in T \times T' \mid d(\phi(x), y) \leq m(\phi)t \} \). For \((x, y) \in V_t \) let \( C_t(x, y) \) denote the path component of \( V_t \cap (T \times \{ y \}) \) which contains \((x, y)\). Finally, we define:

\[
W_t = \{(x, y) \in V_t \mid C_t(x, y) \cap \text{graph}(\phi) \neq \emptyset \}.
\]

Thus \( W_t \) is a thickening of \( \text{graph}(\phi) \subseteq T \times T' \). We will write \( W_t(\phi) \) when we need to specify the morphism. Let \( F_t \) be a partition of \( W_t \) into sets which are the path components of \( W_t \cap (T \times \{ y \}) \) for \( y \in T \) and \( T_t = W_t/F_t \). We denote points in \( T_t \) by \([z]_t \) for \( z \in W_t \). As \( F_t \) is \( G \)-equivariant, \( T_t \) is a \( G \)-tree. For \( 0 \leq s \leq t \leq 1 \) the inclusions \( W_s \to W_t \) induce \( G \)-equivariant maps \( \phi_{st}: T_s \to T_t \).

A path \( \gamma: [0, 1] \to W_t \) is taut if for components \( A \) in \( F_t \), \( \gamma^{-1}(A) \) is connected. For \( t > 0 \), a non-backtracking path \( \gamma \) in \( T_t \) lifts to a path \( \tilde{\gamma} \) in \( W_t \) with endpoints in \( \text{graph}(\phi) \). This lift \( \tilde{\gamma} \) is homotopic relative to these endpoints to
a taut product of paths $\gamma_1 \cdots \gamma_k$ where each $\gamma_i$ lies either in a component of $F_t$ or is a non-backtracking path in $\text{graph}(\phi)$. For $t = 0$ a non-backtracking path $\gamma$ in $T_0$ lifts to a path $\tilde{\gamma}$ which is homotopic relative to its endpoints to a taut product of paths $\gamma_1 \cdots \gamma_k$ where $\phi$ is an isometry on each $\gamma_i$. We call these decompositions taut corner paths, the pieces lying in $\text{graph}(\phi)$ are called essential, the pieces lying in some component of $F_t$ are called nonessential, see Figure 2. Metrize $T_t$ by setting $\text{length}(\gamma)$ equal to the sum of the lengths of the essential pieces measured in $T'$ (or equivalently measured in $T$). With this metric the maps $\phi_{st}$ are morphisms.

**Lemma 3.1** For the above definitions: $T_0 = T$, $T_1 = T'$ as $G$–trees, $\phi_{00} = \text{Id}_T$ and $\phi_{01} = \phi$.

**Proof** The only nonobvious claim here is $T_1 = T'$. This is equivalent to saying that the sets $W_1 \cap (T \times \{y\})$ are connected. Let $(x_1, y), (x_2, y) \in W_1 \cap (T \times
\{y\}). We will show that these two points lie in the same component. Choose 
\((z_1, y), (z_2, y) \in W_1 \cap (T \times \{y\})\) such that \(\phi(z_i) = y\) and 
\((x_i, y), (z_i, y)\) are in the same component of \(W_1 \cap (T \times \{y\})\) for \(i = 1, 2\). For \(z \in [z_1, z_2]\) we have 
d(\phi(z), y) \leq m(\phi). Thus the pairs of points \((z_1, y), (z_2, y)\) are in the same 
component of \(W_1 \cap (T \times \{y\})\). Then as \((x_1, y)\) is in the same component as 
\((z_1, y)\) and \((x_2, y)\) is in the same component as \((z_2, y)\), the points \((x_1, y)\) and 
\((x_2, y)\) are in the same component. Thus \(W_1 \cap (T \times \{y\})\) is connected. □

**Lemma 3.2** If \(T'\) is irreducible, then so is \(T_t\) for \(0 \leq t \leq 1\).

**Proof** As the \(G\)-tree \(T'\) is irreducible, there are \(g, h \in G\) which act hyperbolically on \(T'\) such that \(T_g \cap T_h\) is empty or compact \([3]\). As equivariant maps cannot make elliptic elements act hyperbolically, \(g, h\) act hyperbolically in \(T_t\). The maps \(\phi_{st}\) are quasi-isometries, hence the axes of \(g\) and \(h\) have empty or compact intersection. This implies that the \(G\)-tree \(T_t\) is irreducible. □

The following lemma is obvious.

**Lemma 3.3** \(T_t\) is in the same unnormalized deformation space as \(T\) and \(T'\) for \(0 \leq t \leq 1\).

**Remark 3.4** For future reference we remark that the above construction is invariant under the \(\mathbb{R}^+\)-action. In other words if we scale both \(T\) and \(T'\) by a nonzero positive number \(k\), then the trees \(T_t\) are scaled by \(k\).

## 4 Continuity of deformation

Fix an unnormalized deformation space \(\mathcal{X}\). Recall that \(\mathcal{M}(\mathcal{X})\) is the space of all morphisms between \(G\)-trees in \(\mathcal{X}\). Define \(\Phi:\ \mathcal{M}(\mathcal{X}) \times \{(s, t) \mid 0 \leq s \leq t \leq 1\} \to \mathcal{M}(\mathcal{X})\) by \(\Phi(\phi, (s, t)) = \phi_{st}\). The goal of this section is the following theorem:

**Theorem 4.1** \(\Phi\) is continuous.

We have some work before we can prove this. The approach is the same as in
Skora’s preprint \([11]\), with the addition of Lemmas \([12]\) and \([13]\). We will consider
a fixed morphism \(\phi: X \to X'\) between finite simplicial trees and prove some
results about morphisms \(\psi: Y \to Y'\) which are close to \(\phi\). The main step is
to show Lemma 4.7 if the two morphisms $\phi: X \to X'$ and $\psi: Y \to Y'$ are close and we fold both $X$ and $Y$ for a similar amount of time, then the two folded trees have comparable lengths. To prove this, we show that for a taut corner path in $W_t(\phi)$, the individual pieces are related to taut corner paths of comparable length in $W_s(\psi)$ when both $s$ and $t$ are close and $\phi$ and $\psi$ are close.

Our first step is to show that maps close to $\phi$ have similar folding data.

**Lemma 4.2** Let $\phi: X \to X'$ be a morphism of finite simplicial trees. Then for all $\epsilon > 0$ there is a $\delta > 0$ such if $(R, R')$ is a $\delta$-approximation between $\phi: X \to X'$ and $\psi: Y \to Y'$ then $|m(\phi) - m(\psi)| < \epsilon$.

**Proof** Let $\epsilon > 0$ be arbitrary. There are two cases to deal with. Notice that the lemma is symmetric with respect to $\phi$ and $\psi$.

**Case 1** $m(\phi), m(\psi) > 0$

Set $\delta = \min\{\frac{\epsilon}{2}, \frac{m(\phi)}{3}, \frac{m(\psi)}{3}\}$. Let $z \in [x, x']$ be such that $d(\phi(z), \phi(x)) = m(\phi)$ and $\phi(x) = \phi(x')$. There are corresponding points $y, y', w \in Y$ such that $x R y, x' R y'$ and $z R w$. As we can assume that $R$ is full, we may assume that $w \in [y, y']$. Then $d(\psi(y), \psi(y')) < \delta$ and $d(\psi(w), \psi(y)), d(\psi(w), \psi(y')) > m(\phi) - \delta > \delta$. Hence there is a subsegment contained in $[y, y']$ and containing $w$ which is folded. Thus $m(\psi) > m(\phi) - 2\delta$. Repeating the argument for $Y$ we see that $m(\phi) > m(\psi) - 2\delta$. Hence we see that $|m(\phi) - m(\psi)| < \epsilon$.

**Case 2** $m(\phi) = 0$ and $m(\psi) > 0$

Set $\delta = \frac{\epsilon}{2}$. Let $[y, y'] \subseteq Y$ be a folded segment where $w \in [y, y']$ attains $d(\psi(w), \psi(y)) = m(\psi)$. For corresponding points $x, x', z \in X$, we have that $[x, x']$ is embedded and $d(\phi(x), \phi(x')) < \delta$. Hence for all $z \in [x, x']$, we have $d(\phi(z), \phi(x)) < \delta$. Thus $m(\psi) < \epsilon$.

Let $F(\phi)$ denote the number of fold points for the morphism $\phi: X \to X'$. Thus for $N(\phi) = 3(F(\phi) + 1)$ we have that any taut corner path $\gamma$ in $W_t(\phi)$ can be written as $\gamma = \gamma_1 \cdots \gamma_n$ with $n \leq N(\phi)$ where each $\gamma_i$ is either essential or nonessential.

We need a similar statement about morphisms close to $\phi$. It is easy to see that we cannot expect a universal bound, but we can bound the number of large folds, which is sufficient. For $d > 0$, we introduce an equivalence relation on the set of fold points defined by $z \sim_d z'$ if there is a sequence of fold points: $z = z_0, \ldots, z_n = z'$ such that $d(z_i, z_{i+1}) < 2d$. Let
\[ F_d(\psi) = \{ \{ \text{fold points for } \psi \}/ \sim_d \} \setminus \{ \text{classes without a } d \text{-deep point} \}. \]

Notice that \( |F_{m(\psi)}s(\psi)| \) is the number of fold points for the map \( \psi_{s1} : Y_s \rightarrow Y' \). Suppose \((R, R')\) is a \( \delta \)-approximation between \( \phi : X \rightarrow X' \) and \( \psi : Y \rightarrow Y' \) where \( \delta \leq d \). Then \( |F_d(\psi)| \) is bounded independent of \( \psi \) as for each class in \( F_d(\psi) \) we have a \( \frac{d}{2} \) neighborhood in \( X \), and the neighborhoods for different classes are disjoint. Set \( F_d(\phi) \) to be the maximum of \( |F_d(\psi)| \) over all morphisms \( \psi : Y \rightarrow Y' \) for which there is a \( d \)-approximation between \( \phi : X \rightarrow X' \) and \( \psi : Y \rightarrow Y' \). As above we let \( N_d(\phi) = 3(F_d(\phi) + 1) \). Thus if \( \zeta \) is a taut corner path in \( W_s(\psi) \) then we can write \( \zeta = \zeta_1 \cdots \zeta_n \) with \( n \leq N_d(\phi) \) where each \( \zeta_i \) is either nonessential or has length equal to the length of its image in \( Y' \).

We now show that for a taut corner path in \( W_t(\phi) \), the individual pieces are related to a taut corner path in \( W_s(\psi) \) of comparable length. This is proven for the essential pieces first. As a convention when taking several points in \( X \) and points related to them in \( Y \), if some of the points in \( X \) are the same we require that the related points in \( Y \) are the same.

**Lemma 4.3** Let \( \phi : X \rightarrow X' \) be a morphism of finite simplicial trees. Let \( z_1, z_2 \) be points in \( \text{graph}(\phi) \subseteq W_t(\phi) \) such that the taut corner path \( \gamma \) between them lies entirely in \( \text{graph}(\phi) \). Then for all \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \((R, R')\) is a \( \delta \)-approximation between \( \phi : X \rightarrow X' \) and \( \psi : Y \rightarrow Y' \) and \( w_i \in \text{graph}(\psi) \) where \( z_i(R, R')w_i \) for \( i = 1, 2 \), then \( |\text{length}(\gamma) - \text{length}(\zeta)| < \epsilon \) where \( \zeta \) is the taut corner path \( \zeta \) in \( W_s(\psi) \) from \( w_1 \) to \( w_2 \).

**Proof** Let \( \epsilon > 0 \) be arbitrary. Let \( \delta = \epsilon \) and assume the data in the hypothesis. Let \( z_i = (x_i, x'_i), w_i = (y_i, y'_i) \) for \( i = 1, 2 \) and let \( \zeta \) be the taut corner path in \( W_s(\psi) \) connecting \( w_1 \) to \( w_2 \). By hypothesis \( \text{length}(\gamma) = d(x_1, x_2) = d(x'_1, x'_2) \). As \( \text{length}(\zeta) \leq d(y_1, y_2) < d(x_1, x_2) + \delta \) and \( \text{length}(\zeta) \geq d(y'_1, y'_2) > d(x'_1, x'_2) - \delta \), we have the conclusion of the lemma.

Next we have a similar statement for the nonessential pieces:

**Lemma 4.4** Let \( \phi : X \rightarrow X' \) be a morphism of finite simplicial trees. Let \( z_1, z_2 \) be points in \( \text{graph}(\phi) \subseteq W_t(\phi) \) such that the taut corner path \( \gamma \) between them lies entirely in a component of \( F_1 \). Then for all \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \((R, R')\) is a \( \delta \)-approximation between \( \phi : X \rightarrow X' \) and \( \psi : Y \rightarrow Y' \), \( |m(\psi)s - m(\phi)t| < \delta \), and \( w_i \in \text{graph}(\psi) \) where \( z_i(R, R')w_i \) \( i = 1, 2 \), then \( \text{length}(\zeta) < \epsilon \) where \( \zeta \) is the taut corner path in \( W_s(\psi) \) from \( w_1 \) to \( w_2 \).
Thus we have length($\zeta$) approximations to see: length($\max$ of $\zeta$) nonessential. This might not be the case, but the length of an essential piece $\max$ of $N$ Let $t = 0$ or $m(\phi) = 0$

Let $\delta = 1$. Then as $z_1 = z_2$, we have $w_1 = w_2$ by the above convention. Hence length($\zeta$) = 0.

Case 2 $t > 0$ and $m(\phi) > 0$

Let $N = N_d(\phi)$ as above where $d = \frac{2m(\phi)t}{3}$ and set $\delta = \min\{\frac{\epsilon}{2N}, \frac{m(\phi)t}{3}\}$. As $m(\psi)s > m(\phi)t - \delta \geq d$, the number of fold points for $\psi s_1$ is less than $F_d(\phi)$.

Therefore we can write $\zeta = \zeta_1 \cdots \zeta_n$ where $n \leq N$ and each $\zeta_i$ is nonessential or has length equal to the length of its image in $Y'$.

If $\psi([y_1,y_2])$ is contained within a $m(\psi)s$ neighborhood about $y_1$, then $\zeta$ is nonessential. This might not be the case, but the length of an essential piece of $\zeta$ is bounded by how far $\psi([y_1,y_2])$ travels away from $y_1$: length($\zeta_i$) $\leq \max\{d(\psi(y),y_1') - m(\psi)s \mid y \in [y_1,y_2]\},0\}$. Now we use fullness of the approximations to see: length($\zeta_i$) $\leq \max\{d(\phi(x),x_1') - m(\psi)s + \delta \mid x \in [x_1,x_2]\},0\} \leq \delta$.

Thus we have length($\zeta$) $\leq \sum$ length($\zeta_i$) $\leq 2\delta n < \epsilon$. \hfill \Box

Putting together the previous two lemmas we have:

**Lemma 4.5** Let $\phi: X \to X'$ be a morphism of finite simplicial trees. Let $z_1 , \ldots , z_n$ be points in $\text{graph}(\phi \subseteq W_{t}(\phi)$ and $\gamma = \gamma_1 \cdots \gamma_n$ the taut corner path between them. Then for all $\epsilon > 0$ there is a $\delta > 0$ such if $(R,R')$ is a $\delta$–approximation between $\phi: X \to X'$ and $\psi: Y \to Y'$, $|\text{length}(\gamma_i) - \text{length}(\zeta_i)| < \epsilon$ for $i = 1,\ldots , n$.

The next lemma is a converse to Lemma 4.4 and the proof is simpler as we know how many fold points $\phi$ has. Recall that the image of $z \in W_i$ in the quotient tree is denoted $[z]_i$.

**Lemma 4.6** Let $\phi: X \to X'$ be a morphism of finite simplicial trees. Then for all $\epsilon > 0$ there is a $\delta > 0$ such that if $(R,R')$ is a $\delta$–approximation between...
\(\phi: X \to X'\) and \(\psi: Y \to Y'\) and \(|m(\psi)s - m(\phi)t| < \delta\) then \(d([z_1]_t, [z_2]_t) < \epsilon\) where \(z_i \in \text{graph}(\phi), w_i \in \text{graph}(\psi)\) with \(z_i(R,R')w_i\) for \(i = 1, 2\) and \([w_1]_s = [w_2]_s\).

**Proof** Let \(\epsilon > 0\) be arbitrary and \(\delta = \frac{\epsilon}{2N}\), where \(N = N(\phi)\). Let \(\gamma = \gamma_1 \cdots \gamma_n\) be the taut corner path from \(z_1\) to \(z_2\) where each piece is either essential or nonessential and \(n \leq N\). Using the same argument as in case 2 for Lemma 4.3 we can bound the lengths of the \(\gamma_i\) by \(2\delta\). Thus \(\text{length}(\gamma) \leq 2\delta n < \epsilon\). 

Now using the previous two lemmas, we are able to show that close morphisms which are folded for a similar amount of time have comparable lengths. We will also remove the dependence on the folding data using Lemma 4.2.

**Lemma 4.7** Let \(\phi: X \to X'\) be a morphism of finite simplicial trees. Then for all \(\epsilon > 0\) there is a \(\delta > 0\) such that if \((R,R')\) is a \(\delta\)--approximation between \(\phi: X \to X'\) and \(\psi: Y \to Y'\) and \(|s - t| < \delta\) then \(|d([z_1]_t, [z_2]_t) - d([w_1]_s, [w_2]_s)| < \epsilon\) where \(z_i \in \text{graph}(\phi), w_i \in \text{graph}(\psi)\) with \(z_i(R,R')w_i\) for \(i = 1, 2\).

**Proof** Let \(\epsilon\) be arbitrary. Set \(\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2N}\), where \(N = N(\phi)\). Use these to find \(\delta_1, \delta_2\) from Lemma 4.5 and Lemma 4.6 respectively. Let \(\epsilon_3 = \frac{1}{2} \min\{\delta_1, \delta_2\}\) and take \(\delta_3\) from Lemma 4.2 using \(\epsilon_3\). Finally set \(\delta = \min\{\epsilon_2, \frac{\epsilon_3}{m(\phi)}, \delta_3\}\).

The choice of these parameters implies that if \((R,R')\) is a \(\delta\)--approximation between \(\phi: X \to X'\) and \(\psi: Y \to Y'\), and \(|s - t| < \delta\), then \(|m(\psi)s - m(\phi)t| < \epsilon_3\). Thus \(m(\psi)s - m(\phi)t < (m(\phi) + \epsilon_3)s - m(\phi)t < m(\phi)(s - t) + \epsilon_3 < \delta_1, \delta_2\) and similarly \(m(\phi)t - m(\psi)s < \delta_1, \delta_2\). Therefore we can use Lemma 4.5 and Lemma 4.6.

We can write the taut corner path connecting \(z_1\) and \(z_2\) as \(\gamma = \gamma_1 \cdots \gamma_n\) where \(n \leq N\) and each \(\gamma_i\) is either essential or nonessential. Hence by Lemma 4.5 we have a path \(\zeta = \zeta_1 \cdots \zeta_n\) connecting \(w_1\) to \(w_2\) where each piece is a taut corner path and \(|\text{length}(\gamma_i) - \text{length}(\zeta_i)| < \epsilon_1\). Hence \(d([w_1]_s, [w_2]_s) \leq \sum\text{length}(\gamma_i) < \sum\text{length}(\zeta_i) + \epsilon_1 < d([z_1]_t, [z_2]_t) + \epsilon\).

If \(d([w_1]_s, [w_2]_s) \leq d([z_1]_t, [z_2]_t) - \epsilon\), then as \(d([z_1]_t, [z_2]_t) = \sum\text{length}(\gamma_i) < (\sum\text{length}(\zeta_i) + \frac{\epsilon_1}{2})\) we get that \(d([w_1]_s, [w_2]_s) < \sum\text{length}(\zeta_i) - \frac{\epsilon_1}{2}\). Since the only folds of \([\zeta]\) in \(Y_s\) are at the intersection points of \([\zeta_i]\) with \([\zeta_{i+1}]\), there are two points \(q_1\) and \(q_2\) on \(\zeta\) such that the length along \(\zeta\) between these two points is greater than \(\frac{\epsilon_1}{2N}\) but these are the same point in \(Y_s\). Thus for points \(p_1, p_2 \in W_s(\phi)\) with \(p_i(R,R')q_i\), for \(i = 1, 2\) we have \(d([p_1]_t, [p_2]_t) > \frac{\epsilon_1}{2N} = \delta \geq \epsilon_2\). However the choice of \(\delta_3\) implies that \(d([p_1]_t, [p_2]_t) < \epsilon_2\) by Lemma 4.6. Hence we have a contradiction. Therefore \(|d([z_1]_t, [z_2]_t) - d([w_1]_s, [w_2]_s)| < \epsilon\). 

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Thus the folded trees have comparable lengths. We can use this to build an \( \epsilon \)-approximation between these trees. For morphisms \( \phi: X \to X' \), \( \psi: Y \to Y' \) which are related by an \( \epsilon \)-approximation \((R, R')\) we define a new relation \([R, R]'_{ts}\) from \(X_t\) to \(Y_s\) by \([z]_{t}[R, R]'_{ts}[w]_{s}\) whenever \(z(R, R')w\) for \(z \in \text{graph}(\phi)\) and \(w \in \text{graph}(\psi)\). We now prove a lemma about this relation when \(s\) and \(t\) are close.

**Lemma 4.8** Let \( \phi: X \to X' \) be a morphism of finite simplicial trees. For all \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if \((R, R')\) is a \( \delta \)-approximation between \( \phi: X \to X' \) and \( \psi: Y \to Y' \) and \( |s - t| < \delta \) then \([R, R]'_{ts}\) is an \( \epsilon \)-approximation from \(X_t\) to \(Y_s\). If \( R \) and \( R' \) are \( P \)-equivariant, then so is \([R, R]'_{ts}\).

**Proof** Let \( \epsilon > 0 \) be arbitrary and choose \( \delta \) from Lemma 4.7. Given data as in the hypothesis, \([R, R]'_{ts}\) is an \( \epsilon \)-approximation. It also follows that if \( R \) and \( R' \) are \( P \)-equivariant, then so is \([R, R]'_{ts}\). \( \square \)

Given an arbitrary morphism \( \phi: T \to T' \) between \( G \)-trees in the unnormalized deformation space \( X \), for subtrees \( X \subseteq T, X' \subseteq T' \) such that \( \phi(X) \subseteq X' \) we can define \( X_t \) as \( W_t(\phi|_X)/(F_t \cap (X \times X')) \). We can now prove that \( \Phi \) is continuous.

**Proof** Let \( \epsilon > 0 \) be arbitrary. Let \( \phi: T \to T' \) and \( 0 \leq s \leq t \leq 1 \) be given. Assume \( U \) is the basic open set around \( \phi_{st} \) given by \( U = U(\phi_{st}, X \times X', P, \epsilon) \) where \( X \subseteq T, X' \subseteq T' \) are finite subtrees and \( P \) is a finite subset of \( G \). Let \( \delta \) be given by Lemma 4.8 and \( V = U(\phi, X \times X', P, \delta) \).

Suppose \( \psi: \tilde{T} \to \tilde{T}' \) with \( \psi \in V \) and \( |p - s| < \delta, |q - t| < \delta \). We will show that \( \psi_{pq} \in U \).

For some finite subtrees \( Y \subseteq \tilde{T}, Y' \subseteq \tilde{T}' \) there is a \( \delta \)-approximation \((R, R')\) from \( \phi: X \to X' \) to \( \psi: Y \to Y' \). The claim is that \(([R, R]'_{sp}, [R, R]'_{tq})\) is a closed \( \epsilon \)-approximation from \( \phi_{st}: X_s \to X_t \) to \( \psi_{pq}: Y_p \to Y_q \). The choice of \( \delta \) implies that both \([R, R]'_{sp}\) and \([R, R]'_{tq}\) are \( \epsilon \)-approximations by Lemma 4.8. If \([z]_s[R, R]'_{sp}[w]_p\) then we have that \([z]_t[R, R]'_{tq}[w]_q\). Therefore \( \psi_{pq} \in U \). \( \square \)

### 5 Continuity of base point

For a \( G \)-tree \( T \in \mathcal{X} \) define \( l_T(S) = \min_{x \in T} \max_{g \in S} d(x, gx) \), where \( S \) is some finite subset of \( G \). The characteristic set of \( S \) is \( T_S = \{ x \in T \mid l_T(S) \leq \} \)
max\(_g\in S\) \(d(x, gx)\}). This agrees with the earlier notion for characteristic set when the subgroup generated by \(S\) is elliptic. Clearly for \(g \in S\) we have \(l_T(g) \leq l_T(S)\). We let \(S'\) be the subset of \(S\) where this is an equality, i.e. \(S' = \{g \in S \mid l_T(g) = l_T(S)\}\). Finally we define \(Z_S = \bigcap_{g \in S'} T_g\).

**Lemma 5.1** Let \(T\) be a \(G\)–tree and let \(S\) be a finite subset of \(G\). Then \(T_S\) is contained in the union of a finite simplicial tree and \(Z_S\). In particular, if \(Z_S\) is a finite simplicial tree, then \(T_S\) is a finite simplicial tree.

**Proof** Let \(x \in T\) and \(X\) be the union of all arcs from \(x\) to \(T_g\) for \(g \in S\), then \(X\) is a finite simplicial tree. If \(y \in T_S\) is not in \(X\), let \(z\) be the closest point in \(X\) to \(y\). Then \(d(y, gy) \geq d(z, gz)\) for all \(g \in S\) as \(d(y, T_g) \geq d(z, T_g)\) with equality only if \(y \in T_g\). If \(g \in S'\) then \(d(y, gy) \geq d(z, gz) \geq l_T(S)\). As \(y \in T_S\) we have \(l_T(S) \geq d(y, gy)\). Hence we have equality \(d(y, gy) = d(z, gz)\) for \(g \in S'\). Thus \(y \in T_g\) for all \(g \in S'\) and hence \(y \in Z_S\).

Let \(S\) generate \(G\). Then for irreducible \(G\)–trees \(T\), \(Z_S\) is finite, hence so is \(T_S\). We have some simple lemmas on the shape and position of \(T_S\) based on \(l_T(S)\) and \(l_T(g)\) that will be used in Proposition 5.3.

**Lemma 5.2** Suppose that \(T_S\) is finite. Then \(T_S\) is either a point or a segment. Moreover, the latter only occurs when there is a \(g \in S\) such that \(l_T(g) = l_T(S)\). In both cases, there are distinct \(g_1, g_2 \in S\) such that \(d(x, g_1 x) = d(x, g_2 x) = l_T(S)\) for all \(x \in T_S\).

**Proof** Suppose \(l_T(S) > \max_{g \in S} l_T(g)\) and there are distinct points \(x_1, x_2 \in T_S\). Let \(g_1, g_2 \in S\) be such that \(\max_{g \in S} d(x_i, gx_i) = d(x_i, g_i x_i) = l_T(S)\) for \(i = 1, 2\). Thus \(x_i \notin T_{g_i}\). Consider the segment \([x_1, x_2]\). Let \(y \in [x_1, x_2]\) and \(y \neq x_1, x_2\). Then for any \(g \in S\), \(d(y, T_g) < d(x_i, T_g)\) for either \(i = 1\) or \(2\), hence \(\max_{g \in S} d(y, gy) < l_T(S)\). This is a contradiction, therefore \(T_S = \{x\}\).

Now notice that there are \(g_1, g_2 \in S\) such that \(d(x, g_1 x) = l_T(S)\). For if there was only one such \(g\), then for some point \(y\) near \(x\) on the arc from \(x\) to \(T_g\), \(\max_{g \in S} d(y, gy) < d(x, gx) = l_T(S)\), which is a contradiction.

If \(l_T(S) = l_T(g)\) for \(g \in S\) then \(T_S \subset T_g\). Therefore \(T_S\) is either a point or a segment. If there were only one such \(g \in S\) such that \(l_T(S) = l_T(g)\), then \(T_S\) is open by a similar argument as above. This is a contradiction.

Recall that for \(A \subseteq T\) closed, we let \(p_A\): \(T \to A\) denote the nearest point projection.
Lemma 5.3 Let $z \in T \setminus T_S$ and $x = p_{T_S}(z)$. Then for some $g \in S$ such that $d(x,gx) = l_T(S)$, we have that $x$ is on the arc from $z$ to $T_g$.

Proof Suppose not. Then for points $x' \in [x,z]$ near $x$, $d(x',T_g) \leq d(x,T_g)$ for all $g \in S$ such that $d(x,gx) = l_T(S)$. This is a contradiction. □

For an irreducible $G$–tree $T$, let $x_*$ denote the midpoint of $T_S$. This is called the basepoint of the action. Define a map $b(T) : G \to T$ by $g \mapsto gx_*$. This defines a map $b : \mathcal{X} \to \mathcal{E}(G, \mathcal{X})$ where $\mathcal{E}(G, \mathcal{X})$ is the space of equivariant maps from $G$ to $G$–trees in $\mathcal{X}$. The topology for $\mathcal{E}(G, \mathcal{X})$ is the Gromov-Hausdorff topology defined in section 2 where we consider $G$ as a metric $G$–space. The actual metric we place on $G$ does not matter as the domain is fixed in $\mathcal{E}(G, \mathcal{X})$. The remainder of this section is used to prove that $b : \mathcal{X} \to \mathcal{E}(G, \mathcal{X})$ is a continuous function.

Proposition 5.4 $b$ is continuous.

Proof This amounts to showing that close $G$–trees in $\mathcal{X}$ have close basepoints. Let $T \in \mathcal{X}$, there are two cases depending on $l_T(S)$.

Case 1 $l_T(S) > \max_{g \in S} l_T(g)$

By Lemma 5.2 we have that $T_S = \{x_*\}$. Within the set of $g \in S$ such that $d(x_*, gx_*) = l_T(S)$, there are two elements $g_1, g_2$ such that $x_*$ is on the spanning arc from $T_{g_1}$ to $T_{g_2}$. Let $x_i$ be the point on $T_{g_i}$ nearest to $x_*$. Thus $x_* \in [x_1, x_2]$ and $d(x_1, x_2) = d(x_1, x_*) + d(x_*, x_2)$.

Let $U$ be the basic open set $U = U(b(T), P \times K, P, \epsilon)$, where $S \subseteq P$ and $P(\{x_*, x_1, x_2\}) \subseteq K$. By the remark in section 2 we can assume that $P$ and $K$ contain these subsets by shrinking $U$. Also let $V = U(T, K, P, \delta)$, where $\delta = \frac{1}{4} \min\{\epsilon, d(x_*, x_1), d(x_*, x_2)\}$. Suppose that $Y \in V$, we will show that $b(Y) \subseteq U$. By definition, there is a $P$–equivariant closed $\delta$–approximation $R \subseteq K \times L$ for some finite subtree $L \subseteq Y$.

Fix related points in $L$: $x_* Ry_i, x_1 Ry_i$ for $i = 1, 2$. By fullness of $R$, we may assume that $y_* \in [y_1, y_2]$. Our object now is to show that $y_*$ is close to every point in $Y_S$, in particular, the midpoint of $Y_S$. This involves some inequalities.

As $|d(x_*, gx_*) - d(y_*, gy_*)| \leq \delta$ for all $g \in S$ we have $\max_{g \in S} d(y_*, gy_*) < \max_{g \in S} d(x_*, gx_*) + \delta = d(x_*, g_1x_*) + \delta < d(y_* gy_*) + 2\delta$ for $i = 1, 2$. Therefore, if $y \in Y_S$, then $d(y, gy) \leq \max_{g \in S} d(y, gy) \leq \max_{g \in S} d(y_*, gy_*) < d(y_*, g_1y_*) + \delta$.
Let $y$ be related to $x$ in $T$. Let $\delta > 0$ be such that $d(y, x) > \delta$. Let $\epsilon > 0$ be such that $d(y, x) > \epsilon$. Let $\delta' > 0$ be such that $d(y, x) > \delta'$. Let $\delta'' > 0$ be such that $d(y, x) > \delta''$. Let $\delta''' > 0$ be such that $d(y, x) > \delta'''$. Let $\delta'''' > 0$ be such that $d(y, x) > \delta''''$. Let $\delta''''' > 0$ be such that $d(y, x) > \delta'''''$. Let $\delta'''''' > 0$ be such that $d(y, x) > \delta''''''$. Let $\delta''''''' > 0$ be such that $d(y, x) > \delta'''''''$. Let $\delta'''''''' > 0$ be such that $d(y, x) > \delta''''''''$. Let $\delta''''''''' > 0$ be such that $d(y, x) > \delta'''''''''$. Let $\delta'''''''''' > 0$ be such that $d(y, x) > \delta''''''''''$. Let $\delta''''''''' > 0$ be such that $d(y, x) > \delta'''''''''$. Let $\delta'''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''$. Let $\delta''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''$. Let $\delta'''''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''''$. Let $\delta''''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''''$. Let $\delta'''''''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''''''$. Let $\delta''''''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''''''$. Let $\delta'''''''''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''''''''$. Let $\delta''''''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''''''$. Let $\delta'''''''''''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''''''''''$. Let $\delta''''''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''''''$. Let $\delta'''''''''''''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''''''''''''$. Let $\delta''''''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''''''$. Let $\delta'''''''''''''''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''''''''''''''$. Let $\delta''''''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''''''$. Let $\delta'''''''''''''''''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''''''''''''''''$. Let $\delta''''''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''''''$. Let $\delta'''''''''''''''''''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''''''''''''''''''$. Let $\delta''''''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''''''$. Let $\delta'''''''''''''''''''''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''''''''''''''''''''$. Let $\delta''''''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''''''$. Let $\delta'''''''''''''''''''''''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''''''''''''''''''''''$. Let $\delta''''''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''''''$. Let $\delta'''''''''''''''''''''''''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''''''''''''''''''''''''$. Let $\delta''''''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''''''$. Let $\delta'''''''''''''''''''''''''''''''''''' > 0$ be such that $d(y, x) > \delta''''''''''''''''''''''''''''''''''''$. Let $\delta''''''''''''''' > 0$ be such that $d(y, x) > \delta'''''''''''''''.
Hausdorff distance between \([y_1, y_2]\) and \(Y_S\) is small. As before, this involves some inequalities. Our first step is to show that points in \([y_1, y_2]\) are close to some point in \(Y_S\).

Let \(z \in [y_1, y_2]\) and \(x \in [x_1, x_2] = T_S\) where \(xRz\). Then \(\max_{\gamma \in S} d(z, gz) < \max_{\gamma \in S} d(x, gx) + \delta = d(x, hx) + \delta < d(z, hx) + 2\delta\). Since \(R\) is full, this is true for any \(z \in [y_1, y_2]\). Note that the above inequality implies \(l_Y(S) < l_T(S) + \delta\).

We now show that the segment \([h^{-1}y_1, hy_2]\) is close to the axis \(Y_h\). Let \(z, z' \in [h^{-1}y_1, hy_2]\) and \(x, x' \in [h^{-1}x_1, h_x]\) where \(xRz, x'Rz'\).

\[
|d(z, Y_h) - d(z', Y_h)| = \left|\frac{1}{2}(d(z, hx) - l_Y(h)) - \frac{1}{2}(d(z', hx') - l_Y(h))\right|
\]

\[
= \frac{1}{2}|d(z, hx) - d(x, hx)| + (d(x', hx') - d(z, hx'))| < \delta.
\]

In particular \(|d(h^{-1}y_1, Y_h) - d(hy_2, Y_h)| < \delta\), as \(d(h^{-1}y_1, hy_2) > 2l_T(h) - \delta > 2\delta\) this implies that there is a \(z_0 \in [h^{-1}y_1, hy_2] \cap Y_h\). Hence for any \(z \in [h^{-1}y_1, hy_2]\) we have \(d(z, Y_h) < \delta\). Likewise the same is true for \(z \in [y_1, y_2]\).

Now for \(z \in [y_1, y_2]\), \(l_Y(S) - 2\delta \leq \max_{\gamma \in S} d(z, gz) - 2\delta < d(z, hx) < l_Y(h) + 2\delta < l_Y(S) + 2\delta\). For \(z \in [y_1, y_2]\) that are not in \(Y_S\), let \(y = p_{Y_S}(z)\) and let \(g' \in S\) be given by Lemma \ref{lem}. Then \(\frac{1}{2}|d(z, g'z) - d(y, g'y)| \leq \frac{1}{2}\max_{\gamma \in S} d(z, gz) - l_Y(S) < \delta\). Hence for \(z \in [y_1, y_2]\), we have \(d(z, y) < \delta\).

For the opposite inequality we show that points in \(Y_S\) are close to some point in \([y_1, y_2]\). We do so by showing that points far enough away from \([y_1, y_2]\) cannot lie in \(Y_S\). First note that the above inequality implies: \(l_Y(S) - l_Y(h) < 4\delta\). Hence if \(y' \in Y_S\), then \(2d(y', Y_S) = d(y', hy') - l_Y(h) < l_Y(S) - (l_Y(S) - 4\delta)\). Thus \(d(y', Y_h) < 2\delta\). Recall that we have shown \(l_Y(S) < l_T(S) + \delta\).

The idea now is to use Lemma \ref{lem} on points far from \([y_1, y_2]\). Assume that \(y' \in Y_S\) and \(d(y', [y_1, y_2]) \geq 4\delta\). Then there is some point \(y \in Y_h \cap L\) with \(d(y, [y_1, y_2]) \geq 2\delta\). Without loss of generality, we assume that \(y\) is closer to \(y_1\) than to \(y_2\). Let \(x \in T_h \cap K\) be such that \(xRy\). Then \(d(x, x_1) \geq \delta\). Hence by Lemma \ref{lem} there is a \(g \in S\) such that \(d(x, gx) \geq l_T(S) + 2\delta\). Therefore \(l_Y(S) \geq d(y, gy) \geq l_T(S) + \delta > l_Y(S)\), which is a contradiction. Therefore
the Hausdorff distance between $Y_S$ and $[y_1, y_2]$ is less than $4\delta$. Let $y_0 \in Y_S$ be the basepoint, then $d(y_0, y_0) < 4\delta$. Now proceed as in case 1 using the $P$–equivariant closed $\epsilon$–approximation $(Id_G, R_{4\delta})$.

This completes the proof.

Remark 5.5 The technical statement proved in the above which is used later on in Lemma 6.4 is that if two trees $Y$ and $Z$ have subtrees, $L \subseteq Y$, $M \subseteq Z$ with $P\{b(Y)(1)\} \subseteq L$, $S \subseteq P$ and a $P$–equivariant $\epsilon$–approximation $R \subseteq L \times M$, then if $z \in Z$ with $b(Y)(1)Rz$, we have $d(z, b(Z)(1)) < 4\epsilon$. In other words, any point related to the basepoint of $Y$ is within $4\epsilon$ of the basepoint of $Z$.

6 Contractibility of deformation space

To prove the contractibility of the unnormalized deformation space $\mathcal{X}$, we construct a homotopy onto a contractible subset. To define the homotopy, for any $G$–tree $T' \in \mathcal{X}$ we need to build a nice map from some fixed $G$–tree $T \in \mathcal{X}$ to $T'$. To ensure that the map $T \to T'$ is nice, we will need $T$ to be reduced.

Definition 6.1 A $G$–tree $T$ is reduced if for all edges $e = [u, v]$, $u$ is $G$–equivalent to $v$ if $G_e = G_u$.

This is equivalent to Forester’s definition in [5] where a tree is said to be reduced if it admits no collapse moves. We will use this notion via the next lemma.

Lemma 6.2 Let $T$ be a reduced $G$–tree and $u, v \in T$ vertices such that there is an edge $e = [u, v]$ and $x$ a vertex with $G_u, G_v \subseteq G_x$. Then $u$ is $G$–equivalent to $v$.

Proof Without loss of generality, assume that $v$ is closer to $x$ than $u$ is. Then $[u, x] = e \cup [v, x]$ and as $G_u$ stabilizes $[u, x]$ this implies that $G_u = G_e$. Hence as $T$ is reduced, the two endpoints of $e$ are $G$–equivalent.

We now require that our unnormalized deformation space $\mathcal{X}$ contains a $G$–tree with finitely generated vertex groups. In particular as all $G$–trees in $\mathcal{X}$ are cocompact, there is a reduced tree $T \in \mathcal{X}$ with finitely generated vertex groups. Define $T(T, \mathcal{X})$ as the space of all continuous maps from $T$ to $G$–trees in $\mathcal{X}$ that take vertices to vertices and are injective on the edges of $T$. We call such maps...
transverse. This has a different meaning than in [11], where transverse only implies cellular. We topologize \( T(T,X) \) using the Gromov-Hausdorff topology from section 2.

Our aim now is to build a section \( B: \mathcal{X} \to T(T,X) \). Let \( G \) be finitely generated by \( S \) and fix \( X \subseteq T \) a subtree whose edges map bijectively to \( T/G \). We follow Forester’s construction from Proposition 4.16 in [5]. Order the vertices of \( X \) as \( \{v_1, \ldots, v_k\} \) where vertices in the same orbit are consecutive. For the \( i \)th orbit \( v_{i0}, \ldots, v_{i0+d} \) let \( g_{i0} = 1 \) and fix \( g_{i0+q} \in G \) such that \( g_{i0+q}v_{i0} = v_{i0+q} \) for \( 1 \leq q \leq d \). As the path \([v_{i0}, g_{i0+q}g_{i0+p}v_{i0}]\) for \( 1 \leq p, q \leq d \) is contained in \( X \), it maps bijectively to \( T/G \). Therefore the products \( g_{i0+q}g_{i0+p} \) are hyperbolic for \( p \neq q \) (Lemma 2.7(b) [5]).

Given \( Y \in \mathcal{X} \), we define the map \( B(Y): T \to Y \) first on the vertices of \( X \). Let \( y_s = b(Y)(1) \), where \( b \) is the basepoint map of Proposition 5.4. Recall that \( p_A \) is projection onto the closed subset \( A \). Consider an orbit \( \{v_{i0}, \ldots, v_{i0+d}\} \). If \( G_{v_{i0}} \neq \{1\} \) then let \( Y_{i0} \subseteq Y \) be the characteristic set of \( G_{v_{i0}} \). Otherwise, let \( Y_{i0} = y_s \). As \( T \) is reduced, \( G_{v_{i0}} = \{1\} \) can only happen if \( G \) is a finitely generated free group of rank at least 2. In which case \( T/G \) is a rose and there is only one orbit of vertices in \( X \). Define \( B(Y) \) on the orbit by: \( v_{i0+d} \mapsto g_{i0+d}pY_{v_{i0}}(y_s) \).

We now show that \( B(Y) \) can be extended to a transverse map. If there is an edge \( e \subseteq X \) where \( e = [u,v] \) with \( B(Y)(u) = B(Y)(v) = x' \in Y \), then \( G_u, G_v \subseteq G_{x'} \). This subgroup must fix a vertex \( x \in T \), hence \( G_u, G_v \subseteq G_x \) and by Lemma 6.2 \( u \) and \( v \) must be in the same orbit. But if \( v_i \) and \( v_j \) are in the same orbit then as \( g_{i}^{-1}g_{j} \) is hyperbolic for \( i \neq j \) necessarily \( B(Y)(v_i) \neq B(Y)(v_j) \). Thus we can linearly map each edge of \( X \) injectively into \( Y \). Now extend \( B(Y) \) to all of \( T \) equivariantly. As \( B(Y) \) is injective on each edge this defines \( B: \mathcal{X} \to T(T,X) \).

For the \( i \)th orbit, let \( G_i \) be the vertex stabilizer of the first vertex in this orbit and denote the characteristic set for \( G_i \) by the subscript \( i \), i.e. \( Y_{G_i} = Y_i \). If \( G_i = 1 \) then as before, set \( Y_i = y_s \). Let \( G_i \) be finitely generated by \( S_i \), then for any \( G \)–tree \( Y \in \mathcal{X} \) we have \( Y_i = \cap_{s \in S_i} Y_s \). Let \( Q \subseteq G \) be the union of the \( S_i \)’s and \( S' \) a finite generating set for \( G \).

Lemma 6.3 \( B \) is continuous and \( Ra(B(Y)) = Y \) for all \( Y \in \mathcal{X} \).

If \( G \) is finitely generated free group of rank at least 2, then this follows from Proposition 5.4. Thus we assume that \( G \) is not free. Before we prove this lemma in general, we prove a statement about the position of the basepoint relative the fixed point sets.
Lemma 6.4 Let \( Y, Z \in \mathcal{X} \), and \( Y_i, Z_i \) be the characteristic sets as described above. Let \( L \subseteq Y, M \subseteq Z \) be subtrees with \( R \subseteq L \times M \) a \( P \)-equivariant \( \epsilon \)-approximation where \( Q \subseteq P \) and \( P\{y_*, p_{Y_i}(y_*)\} \subseteq L \). Then for \( z \in Z \) such that \( p_{Y_i}(y_*)Rz \) we have \( d(z, p_{Z_i}(z_\ast)) < 10\epsilon \).

**Proof** Fix \( \hat{z} \in Z \) where \( y_*, R\hat{z} \), then by Remark 5.5, \( d(\hat{z}, z_\ast) < 4\epsilon \). The lemma follows from 3 observations:

(i) \(|d(z_\ast, z) - d(y_*, p_{Y_i}(y_*))| < |d(\hat{z}, z) - d(y_*, p_{Y_i}(y_*))| + 4\epsilon < 5\epsilon .\)

(ii) Let \( g \in S_i \) be such that \( p_{Z_i}(z_\ast) = p_{Z_0}(z_\ast) \). Then:

\[
2d(z_\ast, p_{Z_i}(z_\ast)) = d(z_\ast, g_{z_\ast}) < d(\hat{z}, g\hat{z}) + 8\epsilon = d(g_{y_*}, y_*) + 9\epsilon = 2d(y_*, p_{Y_i}(y_*)) + 9\epsilon.
\]

Likewise, running this argument with \( h \in S_i \) such that \( p_{Y_i}(y_*) = p_{Y_h}(y_*) \), we see that \(|d(z_\ast, p_{Z_i}(z_\ast)) - d(y_*, p_{Y_i}(y_*))| < 5\epsilon .\)

(iii) Let \( g \in S_i \) be such that \( p_{Z_i}(z) = p_{Z_0}(z) \). Then: \( 2d(z, p_{Z_i}(z)) = d(z, g_{z}) < d(p_{Y_i}(y_*), gp_{Y_i}(y_*)) + \epsilon = \epsilon .\)

Putting (i) and (ii) together: \(|d(z_\ast, z) - d(z_\ast, p_{Z_i}(z_\ast))| < 10\epsilon .\) If \([z_\ast, z]\) passes through \( Z_i \) then \( d(z, p_{Z_i}(z_\ast)) < 10\epsilon .\) If \([z_\ast, z]\) doesn’t pass through \( Z_i \), then \( p_{Z_i}(z) = p_{Z_0}(z_\ast) \) and hence \( d(z, p_{Z_i}(z)) = d(z, p_{Z_0}(z)) < \epsilon \) by (iii).

Now we can prove Lemma 6.3

**Proof** Let \( Y \in \mathcal{X} \) and let \( U \) be a basic open set of \( B(Y), U = U(B(Y), K \times L, P, \epsilon) \) where \( Q \subseteq P \) and \( P\{y_*, p_{Y_i}(y_*)\} \subseteq L \). Enlarge \( P \) such that \( K \subseteq PX \). Also let \( V \) be a basic open set for \( Y \in \mathcal{X} \), \( V = U(Y, L, P, \delta) \) where \( \delta = \frac{\epsilon}{10} \). If \( Z \in V \), we have for definition a \( \delta \)-approximation \( R \subseteq L \times M \) for some \( M \subseteq Z \). This is the set-up in Lemma 6.4. As before, we will show that \( B(Z) \in U \).

We claim that \((Id, R_{10\delta})\) is an \( \epsilon \)-approximation from \( B(Y) \) to \( B(Z) \), hence \( B(Z) \in U \). As in the proof of Proposition 5.4 the only nontrivial check is that \( B(Y)(x)R_{10\delta}B(Z)(x) \) for \( x \in K \). Without loss of generality, we can assume that \( x \) is a vertex as the maps are linear on the edges. Now \( x = gv \) for some \( v \in X \), ordered first in its orbit and some \( g \in P \). Let \( A = Y_G, B = Z_G \), then \( B(Y)(x) = gp_A(y_*) \) and \( B(Z)(x) = gp_B(z_\ast) \). Let \( z \in Z \) be such that \( p_{A}(y_*)Rz \). Thus by Lemma 5.4

\[
d(gp_{A}(y_*), B(Y)(x)) + d(gz, B(Z)(x)) = d(gz, gp_B(z_\ast)) = d(z, p_{B}(z_\ast)) < 10\delta.
\]

This completes the proof. 

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The map $B(Y)$ is not a morphism in the sense used within this paper. However we can redefine the metric on $T$ to get a new $G$–tree $T_Y$ such that the map $B(Y)$ when regarded as a map $B(Y) : T_Y \to Y$ is a morphism. As each edge of $T$ is mapped injectively via $B(Y)$ to $Y$ we can remetrize each edge by pulling back the metric on $Y$. Thus we can remetrize $T$ by setting the distance between two points to be the length of the geodesic path between them. Call this new $G$–tree $T_Y$. Then $T_Y$ is equivariantly homeomorphic to the $G$–tree $T$. Let $G(T)$ denote the set of $G$–trees in $X$ which are equivariantly homeomorphic to the $G$–tree $T$. Recall that the volume of $T$ is defined as $\text{vol}(T) = \sum \text{length}(e)$ where the sum is over the unoriented edges of $T/G$.

**Proposition 6.5** $G(T)$ is homeomorphic to $\sigma \times \mathbb{R}^+$ where $\sigma$ is an open simplex of dimension one less than the number of edges of $T/G$.

**Proof** Fix an ordering $e_1, \ldots, e_n$ of the edges of $T$. This in turn gives an ordering of the edges of $T' \in G(T)$. Let $h : G(T) \to \sigma \times \mathbb{R}^+$ be defined by:

$$h(T') = \left( \frac{1}{\text{vol}(T')} \text{length}(e_1), \ldots, \text{length}(e_n), \text{vol}(T') \right).$$

(1)

It is clear that this map gives a bijection between the sets. As we are working with irreducible $G$–trees, as mentioned in section 2 the Gromov–Hausdorff topology is the same as the axes topology. A small change in $G(T)$ of the length functions results in a small change in the lengths of the edges. And conversely, a small change in the length of the edges of $T/G$ results in a small change of the length functions only for the hyperbolic conjugacy classes whose axis project down to paths which cross the rescaled edges. Therefore, $h$ is a homeomorphism.

Denote by $\beta(Y) : T_Y \to Y$ the morphism induced by the transverse map $B(Y) : T \to Y$. Hence $\beta$ defines a map $\beta : \mathcal{X} \to \mathcal{M}(G(T), \mathcal{X})$. As $B : \mathcal{X} \to \mathcal{T}(T, \mathcal{X})$ is continuous and the newly defined metric on $T_Y$ depends continuously on the metric on $Y$, $\beta$ is continuous. We can now define a homotopy equivalence from $\mathcal{X}$ to $G(T)$.

**Theorem 6.6** For a finitely generated group $G$, any irreducible unnormalized deformation space which contains a $G$–tree with finitely generated vertex groups is contractible.

**Proof** Let $\beta : \mathcal{X} \to \mathcal{M}(G(T), \mathcal{X}), \Phi : \mathcal{M}(\mathcal{X}) \times \{(s, t) \mid 0 \leq s \leq t \leq 1\} \to \mathcal{M}(\mathcal{X})$ and $Ra : \mathcal{M}(\mathcal{X}) \to \mathcal{X}$ be the continuous functions defined above.
Define a homotopy $H: \mathcal{X} \times [0,1] \rightarrow \mathcal{X}$ by $H_{(1-t)}(Y) = Ra(\Phi(\beta(Y)), 0, t))$. Then $H_0(Y) = Ra(\Phi(\beta(Y)), 0, 1)) = Ra(\beta(Y)) = Y$ and $H_1(\mathcal{X}) = \mathcal{G}(T)$, which is contractible by 6.5.

Recall that $\mathcal{D} = \mathcal{X}/\mathbb{R}^+$. As $Ra \circ \Phi$ is $\mathbb{R}^+$–invariant (Remark 3.3) and $\beta$ clearly is also, $H$ descends to a homotopy of $\mathcal{D}$. Therefore we have the following theorem as stated in the introduction:

**Theorem 6.7** For a finitely generated group $G$, any irreducible deformation space which contains a $G$–tree with finitely generated vertex groups is contractible.

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