EXISTENCE AND ASYMPTOTIC BEHAVIOR OF HELICOIDAL TRANSLATING SOLITONS OF THE MEAN CURVATURE FLOW

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ABSTRACT. Translating soliton is a special solution for the mean curvature flow (MCF) and the parabolic rescaling model of type II singularities for the MCF. By introducing an appropriate coordinate transformation, we first show that there exist complete helicoidal translating solitons for the MCF in $\mathbb{R}^3$ and we classify the profile curves and analyze their asymptotic behavior. We rediscover the helicoidal translating solitons for the MCF which are founded by Halldorsson [10]. Second, for the pinch zero we rediscover rotationally symmetric translating solitons in $\mathbb{R}^{n+1}$ and analyze the asymptotic behavior of the profile curves using a dynamical system. Clearly rotational hypersurfaces are foliated by spheres. We finally show that translating solitons foliated by spheres become rotationally symmetric translating solitons with the axis of revolution parallel to the translating direction. Hence, we obtain that any translating soliton foliated by spheres becomes either an $n$-dimensional translating paraboloid or a winglike translator.

1. Introduction. A smooth family of immersions $F : \Sigma \times [0, T) \to \mathbb{R}^{n+1}$ is a solution of the mean curvature flow (MCF) if $F$ satisfies the following parabolic equation

$$\frac{\partial}{\partial t} F(p, t) = \overrightarrow{H}(p, t),$$

for all $(p, t) \in \Sigma \times [0, T)$, where $\overrightarrow{H}$ is the mean curvature vector. MCF is the negative gradient flow of the area functional. It is well known that any closed (hyper)surface flows in the direction of steepest descent for area and occurs singularities in the finite time under MCF. Thus it is important to study singularities of MCF. There

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are two types of singularities type I and type II which are represented by the self-similar solution and translating soliton, respectively. Huisken [13] introduced the rescaling technique and the monotonicity formula for MCF and proved that if MCF has type I singularity, then there exists a smoothly convergent subsequence of the rescaling such that its limit tends to the self-similar solution. Huisken and Sinestrari [14] showed that the translating solitons arise as parabolic rescalings of type II singularities. More precisely, a surface Σ is a translating soliton if it satisfies \( \vec{H} = \frac{1}{\tau} \) where \( \tau \) is a constant vector. The translating soliton is an eternal solution \( F(p,t) = F(p) + \tau t \) of (1), that is, it translates with constant velocity \( \tau \) without deforming its shape under MCF. In addition, we can assume the velocity vector \( \tau \) as a unit vector via homothety in \( \mathbb{R}^3 \). From another point of view, translating solitons in \( \mathbb{R}^{n+1} \) are weighted minimal surfaces in the smooth metric measure space \( (\mathbb{R}^{n+1}, ds_0, f) \), where \( ds_0 \) is the Euclidean metric and \( f \) is a linear function on \( \mathbb{R}^{n+1} \). A weighted minimal surface is a critical point of the weighted area functional \( e^{-f} dv \) and minimal in \( (\mathbb{R}^{n+1}, g) \), where \( g \) is the conformal metric \( g = e^{-2f} ds_0 \) (see [3, 6, 15, 28]).

There are some interesting examples of translating solitons for MCF. Altschuler and Wu [2] showed that there exists a complete, noncompact, 2-dimensional rotationally symmetric translating soliton called a translating paraboloid. Clutterbuck, Schnürer and Schulze [7] found the non-convex \( n \)-dimensional rotationally symmetric translating solitons in \( \mathbb{R}^{n+1} \) which are called winglike translators. Nguyen [23] constructed the new examples of self-translating surfaces by the gluing technique with the intersection of a grim reaper and a plane. Wang [33] found other convex translating solitons without rotational symmetry. Nguyen [24] also constructed various complete embedded translating solitons in \( \mathbb{R}^3 \). Dávila, del Pino and Nguyen [8] constructed embedded, complete translating solitons homeomorphic to the Costa-Hoffman-Meeks minimal surfaces which are desingularization of intersection with a winglike translator and a translating paraboloid.

From the examples of translating solitons, we could obtain some interesting results and a direction to classify them. For a geometric surface in a space, a helicoidal surface is very natural object. Do Carmo and Dajczer [9] described that every helicoidal surface with constant mean curvature in \( \mathbb{R}^3 \), which is a family of isometric surfaces, continuously changes from unduloid to a nodoid. Ripoll [30] proved that there exists a one-parameter family of complete simply connected minimal helicoidal surfaces in \( \mathbb{H}^3 \) with angular pitch \( \alpha \) which foliates \( \mathbb{H}^3 \). Perdomo [27] proved that curves, which intersect with the plane perpendicular to the helicoidal axis and the minimal helicoidal surfaces in \( \mathbb{R}^3 \), is characterized by TreadmillSled. Earp and Toubiana [31] dealt with a minimal or constant mean curvature surface invariant under screw motions in \( \mathbb{H}^2 \times \mathbb{R} \) and \( \mathbb{S}^2 \times \mathbb{R} \). There have been several interesting results for helicoidal surface [4, 19, 21, 22, 26].

Halldorsson [10] showed that the existence of the helicoidal rotating soliton under MCF which is equivalent to the helicoidal translating soliton with translating direction parallels to the axis of rotation under MCF. We consider the helicoidal translating soliton under MCF. In Section 2, we prove that the translating direction of the helicoidal translating solitons must be parallel to the helicoidal axis and rediscover their existence which are translating solitons invariant under the helicoidal motions. Previous studies [1, 5, 32] used a coordinate transformation and analyzed the dynamical system to construct homothetic invariant surface like minimal and
zero scalar curvature surface. Since translating soliton is not invariant by homothety, we introduce an appropriate coordinate transformation by modifying coordinate transformation in [1, 5, 32]. From the appropriate coordinate transformation, we can change the second order ordinary differential equation to the system of the first order ordinary differential equations and analyze a phase plane of an associated vector field from the system to prove the existence of helicoidal translating solitons of MCF. As a result, we classify the kinds of profile curves and their asymptotic behavior. Furthermore, we proved that the profile curve of a helicoidal translating soliton has the same asymptotic end of either a translating paraboloid or a wing-like translator. This is determined by analyzing the eigenspace on the phase plane associated with the helicoidal translating soliton.

In Section 3, we rediscover rotationally symmetric translating solitons as a special case, helicoidal surface with pitch \( h = 0 \). In [2, 7], they dealt with the existence of rotationally symmetry translating solitons. We give a different point of view by analyzing a dynamical system using the coordinate transformation used in Section 2, which is more intuitive.

Except for planes parallel to translating direction, grim reapers cylinder which are cylindrical surfaces and rotational translating solitons are fundamental examples of translating solitons. Rigidity results have already been reported in [11, 20, 29]. As characterizations of rotationally symmetric translating solitons, the authors [18] proved that the 2-dimensional translating soliton foliated by circles becomes a surface of revolution with the axis of revolution parallel to the translating direction and restricted results of \( n \)-dimensional rotationally symmetric translating solitons.

In Section 4, we extend this result to the \( n \)-dimensional translating soliton foliated by spheres in \( \mathbb{R}^{n+1} \). We basically use the construction used in [16, 25].

2. Helicoidal translating soliton for MCF in \( \mathbb{R}^3 \). We call a surface that is invariant under helicoidal motions a helicoidal surface. The helicoidal axis is the axis of rotation and translation of the helicoidal motion. We deal with a translating soliton for MCF that is invariant under helicoidal motions, which is called a helicoidal translating soliton. In this section, we prove the existence of helicoidal translating solitons in \( \mathbb{R}^3 \) and find its profile curves by analyzing an associated dynamical system.

Let \( X : \Sigma \rightarrow \mathbb{R}^3 \) be an immersion of surface \( \Sigma \) with the following partial derivatives

\[
X_s = \frac{\partial X}{\partial s}, \quad X_t = \frac{\partial X}{\partial t},
\]

where \((s, t)\) is a local coordinate of \( \Sigma \). The first fundamental form and its coefficients are expressed as follows:

\[
I = \langle dX, dX \rangle = E ds^2 + 2F ds dt + G dt^2,
\]

\[
E = \|X_s\|^2, \quad F = \langle X_s, X_t \rangle, \quad G = \|X_t\|^2.
\]

Let \( \nu \) be a unit normal vector field on \( \Sigma \)

\[
\nu = \frac{X_s \times X_t}{\|X_s \times X_t\|} = \frac{X_s \times X_t}{\sqrt{EG - F^2}}.
\]

The second fundamental form and its coefficients are

\[
II = -(d\nu, dX) = L ds^2 + 2M ds dt + N dt^2, \quad (2)
\]

\[
L = \langle \nu, X_{ss} \rangle, \quad M = \langle \nu, X_{st} \rangle, \quad N = \langle \nu, X_{tt} \rangle. \quad (3)
\]
The mean curvature $H$ of $\Sigma$ is

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)}.$$  

Consider a helicoidal surface $\Sigma$ with pitch $h$ and the $z$-axis as helicoidal axis. We can locally parametrize $\Sigma$ as $X : \Sigma \to \mathbb{R}^3$ by

$$X(s, t) = (f(s) \cos(t), f(s) \sin(t), g(s) + ht),$$

where $f$ and $g$ are functions of $s$. When $h = 0$, this surface is a surface of revolution and we deal with the case in Section 3. We may assume that $h \neq 0$ and the profile curve $(f(s), g(s))$ is a unit speed curve on $\mathcal{Q} = \{(x, z) \in \mathbb{R}^2 : x \geq 0\}$. The unit normal vector field $\nu$ and the mean curvature $H$ in this parametrization are

$$\nu = \frac{1}{\sqrt{f'^2 + h^2 f'^2}} (hf' \sin(t) - fg' \cos(t), -hf' \cos(t) - fg' \sin(t), ff'),$$

$$H = \frac{f^2 + 2h^2 f'^2)g' + f(h^2 + f^2)(f'g'' - g'f'')}{2(f^2 + h^2 f'^2)^{3/2}},$$

where a prime mark $'$ denotes a derivative of a function with respect to $s$.

**Proposition 1.** Let $\Sigma$ be a helicoidal surface with pitch $h$. If $\Sigma$ is a translating soliton for MCF, then the translating direction must be parallel to the helicoidal axis.

**Proof of Proposition 1.** Up to rotating and rescaling in $\mathbb{R}^3$, we may assume that the helicoidal axis is parallel to the $z$-axis and the translating direction is a unit vector $v = (a, b, c)$. We have the following equation for a helicoidal translating soliton with pitch $h$ and translating direction $v$ for MCF

$$0 = H - \langle v, v \rangle = \frac{1}{2(f^2 + h^2 f'^2)^{3/2}} (A_0 + A_1 \cos(t) + B_1 \sin(t)),$$

where $A_0$, $A_1$ and $B_1$ are functions of $s$ given by

$$A_0 = (f^2 + 2h^2 f'^2)g' + f(h^2 + f^2)(f'g'' - g'f'') - 2cf'f' + h^2 f'^2),$$

$$A_1 = 2(f^2 + h^2 f'^2)(hhf' + afg'),$$

$$B_1 = 2(f^2 + h^2 f'^2)(-hbf' + bgf').$$

Since it is a trigonometric polynomial with linearly independent trigonometric terms $\cos(t)$ and $\sin(t)$, the coefficient of each trigonometrical function must vanish identically. Assume that $a^2 + b^2$ is not zero and we consider $aA_1 + bB_1 = 0$, which implies that $2(fg'(a^2 + b^2)(f^2 + h^2 f'^2) = 0$, which is a contradiction. Thus, we have $a = b = 0$ and $c = \pm 1$. 

A helicoidal translating soliton for MCF has a helicoidal axis parallel to the translating direction $v$ according to Proposition 1. We may assume that the helicoidal axis is parallel to $e_3$ and the translating direction is equal to $e_3$ up to rotating, rescaling and reflection in $\mathbb{R}^3$. From Proposition 1, we obtain an equation

$$H - \langle v, e_3 \rangle = \frac{(f^2 + 2h^2 f'^2)g' + f(h^2 + f^2)(f'g'' - g'f'' - 2f'f'(f^2 + h^2 f'^2))}{2(f^2 + h^2 f'^2)^{3/2}},$$

which implies a condition for the helicoidal translating soliton with pitch $h$ for MCF as follows:

$$(f^2 + 2h^2 f'^2)g' + f(h^2 + f^2)(f'g'' - g'f'') - 2f'f'(f^2 + h^2 f'^2) = 0. \quad (6)$$
We analyze the phase space \((a, b)\) corresponding to ordinary differential equation (6). We define a coordinate transformation taking suitable parameter \(\tilde{s} (-\infty < \tilde{s} < \infty)\) and use the same notation \(s\) for convenience, which is an appropriate coordinate transformation as follows:

\[
\begin{align*}
\sin(a(s)) &= g', \\
\cos(a(s)) &= f', \\
\tan(b(s)) &= f.
\end{align*}
\]

Differentiating (9) with respect to \(s\) leads to the derivative of \(b(s)\)

\[b'(s) = \cos(a(s))\cos(b(s))^2.\]

Next, changing the coordinate and multiplying by \(b'\cos(b(s))^3\), we obtain

\[a' A + b' B = 0, \quad (10)\]

where \(A\) and \(B\) are as follows:

\[
\begin{align*}
A &= \sin(b)\cos(a)\cos^2(b)(h^2\cos^2(b) + \sin^2(b)), \\
B &= 2h^2\cos^2(a)\cos^2(b)\sin(a - b) + \sin^2(b)(\sin(a)\cos(b) - 2\cos(a)\sin(b)).
\end{align*}
\]

This gives us a system of first order ordinary differential equations for \(a(s)\) and \(b(s)\)

\[
\begin{align*}
a' &= -B, \\
b' &= A.
\end{align*}
\]

Thus, we define an associated vector field \(V(a, b) = (V_1(a, b), V_2(a, b))\) such that \(V_1(a, b) = -B\) and \(V_2(a, b) = A\) on \((-\infty, \infty) \times (0, \pi/2)\). We can find the singular points of the vector field \(V\) from the following lemma.

**Lemma 2.1.** The vector field \(V = (V_1, V_2)\) has period \(2\pi\) along the \(a\)-axis and satisfies the following properties for each \(k \in \mathbb{Z}\):

1. \(V_1\) vanishes at \((k\pi + \pi/2, 0)\).
2. There exists a function \(b_k(a)\) for \(a \in [k\pi, k\pi + \pi/2]\) satisfying \(b_k(k\pi) = 0\), \(b_k(\pi/2 + k\pi) = \pi/2\), \(b'_k(k\pi) = 1\) and \(b'_k(\pi/2 + k\pi) = 2\) such that \(V_1(a, b_k(a)) = 0\).
3. \(V_2\) vanishes along the lines \(a = k\pi + \pi/2\) and \(b = k\pi/2\).

**Proof of Lemma 2.1.** Dividing \(V_1\) by \(\cos(b)^3\), we obtain the following cubic equation for \(\tan(b)\) as follows:

\[\tan^2(b)(2\cos(a)\tan(b) - \sin(a)) - 2h^2\cos^2(a)(\cos(a)\tan(b) + \sin(a)) = 0.\]

There is a unique solution \(\tan(b) = x(a)\) if \(a \in [k\pi, k\pi + \pi/2]\), but there is a multiple root. All roots are real if \(a = k\pi + \pi/2\), otherwise, there is no real root according to the discriminant of the cubic equation

\[-4h^2\cos^2(a)(16h^4\cos^8(a) + 71h^2\cos^4(a)\sin^2(a) + 2\sin^4(a)) \leq 0.\]

By direct computation, one root is \(b_k(a) = \arctan(x(a)) + k\pi\) such that \(b_k(k\pi) = 0\), \(b_k(\pi/2 + k\pi) = \pi/2\), \(b'_k(k\pi) = 1\) and \(b'_k(\pi/2 + k\pi) = 2\), and the other is \((k\pi + \pi/2, 0)\) for each \(k \in \mathbb{Z}\).

A singular point is a vanishing point of a given vector field and we obtain singular points of the associated vector \(V\) from Lemma 2.1.
Corollary 1. Singular points of the associated vector $V$ are $p_{6k} = (2k\pi, 0)$, $p_{6k+1} = (\pi/2 + 2k\pi, \pi/2)$, $p_{6k+2} = (\pi/2 + 2k\pi, 0)$, $p_{6k+3} = (\pi + 2k\pi, 0)$, $p_{6k+4} = (3\pi/2 + 2k\pi, \pi/2)$ and $p_{6k+5} = (3\pi/2 + 2k\pi, 0)$ for any $k \in \mathbb{Z}$.

Proposition 2. Singular points $p_{6k}$ and $p_{6k+3}$ are saddle points, $p_{6k+1}$ and $p_{6k+4}$ are (single) stable and unstable degenerate equilibriums, respectively, and $p_{6k+2}$ and $p_{6k+5}$ are doubly degenerate equilibriums for any $k \in \mathbb{Z}$.

Proof of Proposition 2. The Jacobian matrix of the vector field $V$ is that

\[ DV(a, b) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \]

where

\[
\begin{align*}
 a_{11} &= -h^2 \cos(a) \cos^2(b) (3 \cos(2a - b) - \cos(b)) \\
 &\quad - \sin^2(b) (\cos(a - b) + \sin(a) \sin(b)), \\
 a_{12} &= \cos(a) \cos(b) \left( h^2 \cos(a)(3 \cos(a - 2b) - \cos(a)) + 6 \sin^2(b) \right) \\
 &\quad - \frac{1}{2} \sin(a) \sin(b)(1 + 3 \cos(2b)), \\
 a_{21} &= - \cos^2(b) \sin(a) \sin(b)(h^2 \cos^2(b) + \sin^2(b)), \\
 a_{22} &= \cos(a) \cos(b) \left( \sin^2(b)(3 \cos^2(b) - 2 \sin^2(b)) + h^2(\cos(b)^4 - \sin^2(2b)) \right). 
\end{align*}
\]

For any $k \in \mathbb{Z}$, we have

\[
DV(p_{6k}) = \begin{pmatrix} -2h^2 & 2h^2 \\ 0 & h^2 \end{pmatrix}, \quad DV(p_{6k+3}) = \begin{pmatrix} 2h^2 & -2h^2 \\ 0 & -h^2 \end{pmatrix},
\]

which means the points $p_{6k}$ and $p_{6k+3}$ are saddle points with eigenvalues $\lambda_1(p_{6k}) = -2h^2$, $\lambda_2(p_{6k}) = h^2$, $\lambda_1(p_{6k+3}) = 2h^2$ and $\lambda_2(p_{6k+3}) = -h^2$, and corresponding eigenvectors $v_1(p_{6k}) = (1, 0)$, $v_2(p_{6k}) = (2, 3)$, $v_1(p_{6k+3}) = (1, 0)$ and $v_2(p_{6k+3}) = (2, 3)$, respectively.

For any $k \in \mathbb{Z}$, we have

\[
DV(p_{6k+1}) = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}, \quad DV(p_{6k+4}) = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}.
\]

The eigenvalues at points $p_{6k+1}$ and $p_{6k+4}$ are $\lambda_1(p_{6k+1}) = -2$, $\lambda_2(p_{6k+1}) = 0$, $\lambda_1(p_{6k+4}) = 2$ and $\lambda_2(p_{6k+4}) = 0$, and corresponding eigenvectors are $v_1(p_{6k+1}) = (1, 0)$, $v_2(p_{6k+1}) = (1, 0)$ and $v_1(p_{6k+4}) = (-1, -2)$, $v_2(p_{6k+4}) = (1, 0)$ and $v_2(p_{6k+4}) = (-1, -2)$. These points are single degenerate equilibriums. In particular, $p_{6k+1}$ and $p_{6k+4}$ are stable and unstable degenerate equilibriums, respectively.

For any $k \in \mathbb{Z}$, we have

\[
DV(p_{6k+2}) = DV(p_{6k+5}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

The eigenvalues at points $p_{6k+2}$ and $p_{6k+5}$ are $\lambda_1(p_{6k+2}) = 0$, $\lambda_2(p_{6k+2}) = 0$, $\lambda_1(p_{6k+5}) = 0$ and $\lambda_2(p_{6k+5}) = 0$, and the corresponding eigenvectors are $v_1(p_{6k+2}) = (1, 0)$, $v_2(p_{6k+2}) = (0, 1)$, $v_1(p_{6k+5}) = (1, 0)$ and $v_2(p_{6k+5}) = (0, 1)$, respectively. Thus, these points are doubly degenerate equilibriums.

The associated vector field $V$ is periodic with period $2\pi$ along the $a$-axis. It suffices to investigate the vector field $V$ on $D = [0, 5\pi/2] \times \{0, \pi/2\}$ to observe the solution of (6). Thus, we consider nine singular points $p_i$ for $i = 0, \cdots, 8$ on $D$. We
define five separated parts $D_k = [(k-1)\pi/2, k\pi/2] \times (0, \pi/2)$ for $k = 1, \ldots, 5$ to observe the behavior of the associated vector field.

**Lemma 2.2.** The associated vector field $V$ does not have a periodic trajectory on $D$.

**Proof of Lemma 2.2.** We can easily check that (see Fig. 1)

\[
\begin{aligned}
V_2 &\geq 0 \text{ on } D_1, \\
V_1 &< 0 \text{ on } D_2, \\
V_2 &\leq 0 \text{ on } D_3, \\
V_1 &> 0 \text{ on } D_4, \\
V_2 &\geq 0 \text{ on } D_5.
\end{aligned}
\]

In particular, $V_1$ does not vanish on the lines $a = \pi/2$, $a = 3\pi/2$ and $a = 5\pi/2$. This implies that every trajectory leaves each $D_k$ and does not come back to $D_k$. Thus, there is no periodic orbit on $D$. 

There are some notations defined for the convenience of the stable and unstable manifolds at points. For a given flow $\psi_t : \mathbb{R}^2 \to \mathbb{R}^2$, the stable manifold and unstable manifold of $p$ are denoted by $W^s(p) = \{ x : \psi_t(x) \to p \text{ as } t \to \infty \}$ and $W^u(p) = \{ x : \psi_t(x) \to p \text{ as } t \to -\infty \}$, respectively. We use these notation to observe the behavior of integral curves $\phi(s)$.

**Lemma 2.3.** Every trajectory $\{ \phi(s) \}$ of $V$ contained on $D$ defined for all $s \in \mathbb{R}$ is one of the following types (see Fig. 1):

1. There exists a unique integral curve $\phi(s)$ such that for each $k = 0, 1, 2$,

\[
\lim_{s \to -\infty} \phi(s) = p_{3k}, \quad \lim_{s \to \infty} \phi(s) = p_{3k+1}.
\]

2. There are a family $\{ \phi(s) \}$ of integral curves such that for each $k = 1, 7$,

\[
\lim_{s \to -\infty} \phi(s) = p_1, \quad \lim_{s \to \infty} \phi(s) = p_k.
\]

**Proof of Lemma 2.3.** There is no periodic trajectory on $D$ by Lemma 2.2. Let $\Omega_1$ and $\Omega_2$ be positive and negative parts of $V_1$ on $D$, respectively. $\Omega_1$ and $\Omega_2$ each consist of two connected components. We consider the Poincaré-Bendixon theorem with Lemma 2.2, which implies that all trajectory has two ends that each tends to one of singular points.
The point $p_0$ is a saddle point from Proposition 2 and it has a one-dimensional stable manifold and a one-dimensional unstable manifold by the stable manifold theorem, so there are two integral curves starting from $p_0$. Since one of the integral curves is a straight line on the $a$-axis, we ignore the integral curve whose tangent vector is $v_1(p_0)$. By Proposition 2 and Lemma 2.2, we have $\lambda_2(p_0) = h^2$, $v_2(p_0) = (2, 3)$ and $b'_0(0) = 1$. This implies that eigenvector $v_2(p_0)$ always points upward for the graph of $b_0$. Thus, an integral curve starting from $p_0$ goes inside $\Omega_1 \cap D_1$ and tends to $p_1$, which is contained in both $W^u(p_0)$ and $W^s(p_1)$. Similarly, we have two integral curves, one is starting from $p_4$ and tending to $p_3$ and the other is from $p_6$ to $p_7$.

The points $p_1$ and $p_7$ are stable degenerate equilibriums and $p_4$ is an unstable degenerate equilibrium from Proposition 2. For each $k = 0, 1, 2$, eigenvector $v_2(p_{3k+1})$ is contained in the tangent space of $b_{3k+1}$ at $p_{3k+1}$. Thus, we have two kinds of integral curves along the line of equilibria at $p_4$, one intersects $(D_2 \cup D_3) \cap \Omega_2$, goes upward in $D_1$ and finally tends to $p_1$, while the other intersects $D_3 \cap \Omega_1$, goes upward in $D_4 \cup D_5$ and tends to $p_7$. That is, the integral curves are contained in $W^u(p_4)$ and $W^s(p_{3k+1})$ for each $k = 0, 2$, respectively.

**Proposition 3.** Let $\phi(s)$ be a trajectory of $V$ defined for all $s \in \mathbb{R}$ and $\gamma(s) = (f(s), g(s)) \in Q$ be the corresponding curve. Then, $\gamma$ is one of the following types (see Fig. 2)

1. $\gamma$ is a convex curve with two ends, one approaches the $z$-axis orthogonally and the other tends to $(\infty, \infty)$ (corresponding to (1) in Lemma 2.3).

2. $\gamma$ is a non-convex curve without self-intersection and with two ends tending to $(\infty, \infty)$ (corresponding to (2) in Lemma 2.3).

**Proof of Proposition 3.** We analyze (1) and (2) in Lemma 2.3. We know that the integral curves tend to go somewhere. Firstly, we observe the case of (1) in Lemma 2.3. The curve $\gamma$ satisfies

$$\lim_{s \to -\infty} \tan(a(s)) = \lim_{s \to -\infty} \frac{g'(s)}{f'(s)} = 0,$$
\[
\lim_{s \to -\infty} \tan(b(s)) = \lim_{s \to -\infty} f(s) = 0,
\]
\[
\lim_{s \to -\infty} \tan(a(s)) = \lim_{s \to -\infty} \frac{g'(s)}{f'(s)} = \infty,
\]
\[
\lim_{s \to -\infty} \tan(b(s)) = \lim_{s \to -\infty} f(s) = \infty.
\]
This deduce that \( \gamma \) approaches to the \( z \)-axis perpendicularly as \( s \to -\infty \) and the gradient and distance from the \( z \)-axis tend to infinity as \( s \to \infty \). Furthermore, this integral curve is not included in \( D_1 \cap \Omega_2 \), which means that the profile curve is increasing and convex. Similarly, we have the same profile curves form the integral curves starting from \( p_4 \) and \( p_6 \) to \( p_3 \) and \( p_7 \), respectively.

Secondly, observing the case of (2) in Lemma 2.3, it suffices to consider an integral curve from \( p_4 \) to \( p_7 \), which implies that the curve \( \gamma \) satisfies
\[
\lim_{s \to -\infty} \tan(a(s)) = \lim_{s \to -\infty} \frac{g'(s)}{f'(s)} = \infty,
\]
\[
\lim_{s \to -\infty} \tan(b(s)) = \lim_{s \to -\infty} f(s) = \infty,
\]
\[
\lim_{s \to \infty} \tan(a(s)) = \lim_{s \to \infty} \frac{g'(s)}{f'(s)} = \infty,
\]
\[
\lim_{s \to \infty} \tan(b(s)) = \lim_{s \to \infty} f(s) = \infty.
\]

This integral curve goes through the lines \( a = \frac{3\pi}{2} \) and \( a = 2\pi \), which implies the profile curve has two points corresponding to passing each line, one is the point whose tangent vector is vertical and another is the point whose tangent vector is horizontal. Similarly, we have the same observation for an integral curve from \( p_4 \) to \( p_1 \).}

\begin{thm}
Let \( \Sigma \) be a helicoidal surface with pitch \( h \) in \( \mathbb{R}^3 \). If \( \Sigma \) is a translating soliton for MCF, then its profile curve is one of the two curves described in Proposition 3.
\end{thm}

\begin{rem}
Any connected set consisting of trajectories and fixed points in the phase plane is a solution of a given dynamical system by the Poincaré-Bendixon theorem. Since there is a point reflection of \( D \) for the origin from the coordinate transformation used at (7), (8) and (9), there is an integral curve starting from \( p_0 \) to \((-\pi/2, -\pi/2)\). A connected set consisting of two integral curves from \( p_0 \) to \((-\pi/2, -\pi/2)\) and from \( p_0 \) to \( p_1 \), and a fixed point \( p_0 \) is a solution of (10). Thus, we have a complete helicoidal translating soliton with a profile curve corresponding to the connected set.
\end{rem}

\begin{thm}
The profile curves of the helicoidal translating solitons with pitch \( h \) in \( \mathbb{R}^3 \) have asymptotic behavior of \( y = x^2 \).
\end{thm}

\begin{proof}[Proof of Theorem 2.5]
We know that the profile curves of the helicoidal translating solitons corresponding integral curves have at least one point tending to \((\infty, \infty)\). We observe the asymptotic behavior of the profile curves from (7), (8), (9) and \( v_1(p_1) = (-1, -2) \). We first consider a line \( a = \frac{1}{2}(b - \frac{3\pi}{2}) + \frac{\pi}{2} \) and the equation \( \arctan(f) + \arctan \left( \frac{1}{f} \right) = \frac{\pi}{2} \) for all \( f > 0 \). Then, we have
\[
\frac{dg}{df} = \frac{g'}{f'} = \tan(a) = \tan\left(\frac{1}{2} \left( b - \frac{\pi}{2} \right) + \frac{\pi}{2} \right) = \tan\left( -\frac{1}{2} \arctan\left( \frac{1}{f} \right) + \frac{\pi}{2} \right).
\]

Form Taylor series for \(\tan\left( -\frac{1}{2} \arctan\left( \frac{1}{t} \right) + \frac{\pi}{2} \right)\) at infinity, we have the following series:

\[
2t + \frac{1}{2t} + O\left( \frac{1}{t^3} \right).
\]

Thus, we obtain the ordinary differential equation \(\frac{dg}{df} = 2f\) and the limit behavior of the profile curves of helicoidal translating solitons is the same with \(y = x^2\). □

**Remark 2.** Theorem 2.5 indicates that the profile curves of the helicoidal translating solitons have the same asymptotic behavior of the translating paraboloid and winglike.

**Remark 3.** Halldorsson [10] constructed the helicoidal surface rotating with unit speed under MCF (shortly, helicoidal rotating soliton of MCF) satisfying

\[
H = \langle RX, \nu \rangle,
\]

where the matrix \(R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) generates the rotation and in [10], we can straightforward check that the helicoidal rotating solitons is equivalent to the helicoidal translating solitons with the translating direction parallels to \(e_3\), which is the axis of rotation, with speed equals to \(-h\). Halldorsson showed that for each \(h > 0\) there exists a one-parameter family of helicoidal surfaces with pitch \(h\), that rotate with unit speed around their helicoidal axis under the mean curvature flow.

3. **Rotationally symmetric translating soliton for MCF in** \(\mathbb{R}^{n+1}\). The existence of an \(n\)-dimensional rotationally symmetric translating soliton in \(\mathbb{R}^{n+1}\) was proved in [2, 7]. However, we propose a more intuitive point of view in this section. We follow the process in Section 2, in which we use a coordinate transformation defined by (9) and analyze the phase space \((a, b)\) corresponding the ordinary differential equation.

Let \(M\) be an \(n\)-dimensional rotationally symmetric hypersurface for MCF in \(\mathbb{R}^{n+1}\). \(M\) is represented by the level set of a smooth function \(f\) on \(\mathbb{R}^{n+1}\), that is \(M = f^{-1}(\{0\})\),

\[
f(x_1, \cdots, x_n, t) = \sum_{i=1}^{n} x_i^2 - r(t)^2,
\]

where \(r(t)\) is the radius of sphere. We have

\[
\nu = \frac{\nabla f}{\|\nabla f\|},
\]

\[
H = \frac{1}{n} \text{div} \left( -\frac{\nabla f}{\|\nabla f\|} \right).
\]

We consider an \(n\)-dimensional rotationally symmetric translating soliton with translating direction parallel to the axis of revolution in \(\mathbb{R}^{n+1}\). We may assume
that, up to rotating and rescaling, the translating direction is $e_{n+1}$. Since $M$ satisfies $H - \langle \nu, e_{n+1} \rangle = 0$, we have the following equation:

$$\left(1 - n + nrr'\right)(1 + r'^2) + rr'' = 0. \quad (11)$$

We can change parameterization by $t = \beta(s)$ to obtain a parametric curve, not a graph. We define a new function $f(x_1, \cdots, x_n, s)$ with the same zero level set of $f(x_1, \cdots, x_n, t)$

$$\tilde{f}(x_1, \cdots, x_n, s) = f(x_1, \cdots, x_n, \beta(s)) = \sum_{i=1}^{n} x_i^2 - \alpha(s)^2,$$

where $\alpha = r \circ \beta$ and we have the derivative of $f$ with respect to $s$. Then, we rewrite (11) as follows:

$$\frac{1}{\alpha'^2} \left((\alpha'(1 - n) + n\beta\beta')(\alpha'^2 + \beta'^2) + \beta(\alpha'\beta'' - \alpha''\beta')\right) = 0. \quad (12)$$

A curve $\gamma : I \to Q$ represented by $\gamma(s) = (\alpha(s), \beta(s))$ is the profile curve of $M$ where $Q = \{(r, x_{n+1}) \in \mathbb{R}^2 : r = \sqrt{x_1^2 + \cdots + x_n^2} > 0\}$. We can assume that $\gamma$ is a unit speed curve on $Q$ and rewrite (12) as an equivalent equation

$$\alpha'(1 - n) + n\beta\beta' + \beta(\alpha'\beta'' - \alpha''\beta') = 0. \quad (13)$$

We use the coordinate transformation used in (7), (8) and (9)

$$\cos(a(s)) = \beta', \quad (14)$$
$$\sin(a(s)) = \alpha', \quad (15)$$
$$\tan(b(s)) = \beta. \quad (16)$$

Differentiating (16) with respect to $s$ leads to the derivative of $b(s)$,

$$b'(s) = \cos(a(s)) \cos(b(s))^2.$$ 

We analyze the phase plane $(a, b)$ corresponding to this ordinary differential equation. Changing the coordinate in (13) and multiplying by $-b' \cos(b(s))$, we get

$$a' \cos(a) \cos^2(b) \sin(b) - b' ((1 - n) \sin(a) \cos(b) + n \cos(a) \sin(b)) = 0. \quad (17)$$

This equation gives us the following system of first order ordinary differential equations for $a$ and $b$:

$$a' = (1 - n) \sin(a) \cos(b) + n \cos(a) \sin(b),$$
$$b' = \cos(a) \cos^2(b) \sin(b).$$

We define an associated vector field $V(a, b) = (V_1(a, b), V_2(a, b))$ on $(-\infty, \infty) \times (0, \pi/2)$ such that

$$V_1(a, b) = a' = (1 - n) \sin(a) \cos(b) + n \cos(a) \sin(b),$$
$$V_2(a, b) = b' = \cos(a) \cos^2(b) \sin(b).$$

Singular points of the vector field $V$ can be found by the following lemma.

**Lemma 3.1.** The vector field $V = (V_1, V_2)$ has period $2\pi$ along the $a$-axis and satisfies the following properties for any $k \in \mathbb{Z}$:

1. $V_1$ vanishes along a graph of

$$b_k(a) = \arctan \left( \frac{n - 1}{n} \tan(a + k\pi) \right).$$
Moreover, $b_k$ is smooth, increasing and convex on $(k\pi, k\pi + \pi/2)$, and satisfies $b_k(k\pi) = 0$, $b_k(k\pi + \pi/2) = \pi/2$, $b'_k(k\pi) = (n-1)/n$ and $b'_k(k\pi + \pi/2) = n/(n-1)$.

(2) $V_2$ vanishes along the lines $a = \pi/2 + k\pi$, $b = 0$ and $b = \pi/2$.

**Proof of Lemma 3.1.** By direct computation, we obtain $V_1(a, b) = V_1(a + 2\pi, b)$ and by solving $V_1 = 0$, we obtain $b_k(a)$. Differentiating $b_k(a)$ with respect to $a$ yields

$$\frac{db_k}{da} = \frac{2n(n-1)}{n^2 + (n-1)^2 + (2n-1)\cos(2(a + k\pi))},$$

which is always positive for any $k \in \mathbb{Z}$. The second derivative of $b_k(a)$ is

$$\frac{d^2b_k}{da^2} = \frac{4n(n-1)(2n-1)\sin(2(a + k\pi))}{(n^2 + (n-1)^2 + (2n-1)\cos(2(a + k\pi)))^2},$$

and its sign depends only on $\sin(2(a + k\pi))$. Thus, we have the graph of $b_k$ is smooth, increasing and convex on $(k\pi, k\pi + \pi/2)$. 

**Corollary 2.** Singular points of the associated vector field $V$ are $p_{4k} = (2k\pi, 0)$, $p_{4k+1} = (\pi/2 + 2k\pi, \pi/2)$, $p_{4k+2} = (\pi + 2k\pi, 0)$ and $p_{4k+3} = (3\pi/2 + 2k\pi, \pi/2)$ for any $k \in \mathbb{Z}$.

**Proposition 4.** Singular points $p_{4k}$ and $p_{4k+2}$ are saddle points and singular points $p_{4k+1}$ and $p_{4k+3}$ are (single) stable and unstable degenerate equilibriums for any $k \in \mathbb{Z}$.

**Proof of Proposition 4.** The Jacobian matrix of the vector field $V$ is

$$DV(a, b) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$a_{11} = (1 - n) \cos(a) \cos(b) - n \sin(a) \sin(b),$$

$$a_{12} = n \cos(a) \cos(b) - (1 - n) \sin(a) \sin(b),$$

$$a_{21} = \cos^2(b) \sin(a) \sin(b),$$

$$a_{22} = \frac{1}{2} \cos(a) \cos(b)(3 \cos(2b) - 1).$$

For any $k \in \mathbb{Z}$, we have

$$DV(p_{4k}) = \begin{pmatrix} 1 - n & n \\ 0 & 1 \end{pmatrix}, \quad DV(p_{4k+2}) = \begin{pmatrix} n - 1 & -n \\ 0 & -1 \end{pmatrix},$$

which imply that point $p_{4k}$ and $p_{4k+2}$ are saddle points with eigenvalues $\lambda_1(p_{4k}) = 1$, $\lambda_2(p_{4k}) = 1 - n$, $\lambda_1(p_{4k+2}) = -1$ and $\lambda_2(p_{4k+2}) = n - 1$, and corresponding eigenvectors $v_1(p_{4k}) = (1, 1)$, $v_2(p_{4k}) = (1, 0)$, $v_1(p_{4k+2}) = (1, 1)$ and $v_2(p_{4k+2}) = (1, 0)$, respectively.

For any $k \in \mathbb{Z}$,

$$DV(p_{4k+1}) = \begin{pmatrix} -n & n - 1 \\ 0 & 0 \end{pmatrix}, \quad DV(p_{4k+3}) = \begin{pmatrix} n & 1 - n \\ 0 & 0 \end{pmatrix}.$$ 

The corresponding eigenvalues are $\lambda_1(p_{4k+1}) = 0$, $\lambda_2(p_{4k+1}) = -n$, $\lambda_1(p_{4k+3}) = 0$ and $\lambda_2(p_{4k+3}) = n$, and corresponding eigenvectors are $v_1(p_{4k+1}) = (1 - n, -n)$, $v_2(p_{4k+1}) = (1, 0)$, $v_1(p_{4k+3}) = (1 - n, -n)$ and $v_2(p_{4k+3}) = (1, 0)$, respectively.
Thus, these points are singly degenerate equilibriums. In particular, $p_{4k+1}$ and $p_{4k+3}$ are stable and unstable degenerate equilibriums, respectively.

Since the associated vector field $V$ is periodic with period $2\pi$ along the $a$-axis, it suffices to investigate the vector field $V$ on $D = [0, 5\pi/2] \times (0, \pi/2)$ to observe the solution of (17). Thus, we consider six singular points $p_i$ for $i = 0, \cdots, 5$ on $D$. We define separated parts $D_k = [(k-1)\pi/2, k\pi/2] \times (0, \pi/2)$ for $k = 1, \cdots, 5$ to observe the behavior of the associated vector field.

**Lemma 3.2.** An associated vector field $V$ does not have a periodic trajectory on $D$.

**Proof of Lemma 3.2.** We can easily check that (see Fig. 4)
\[
\begin{align*}
V_2 \geq 0 & \text{ on } D_1, \\
V_1 < 0 & \text{ on } D_2, \\
V_2 \leq 0 & \text{ on } D_3, \\
V_1 > 0 & \text{ on } D_4, \\
V_2 \geq 0 & \text{ on } D_5.
\end{align*}
\]
In particular, $V_2$ vanishes with $V_1 \neq 0$ on the lines $a = \pi/2, a = 3\pi/2$ and $a = 5\pi/2$. Thus, there are no periodic orbits on $D$.

**Lemma 3.3.** Every trajectory $\{\phi(s)\}$ of $V$ contained on $D$ defined for all $t \in \mathbb{R}$ is one of the following types (see Fig 3):

1. There exists a unique integral curve $\phi(s)$ such that for each $k = 0, 1, 2$,
   \[\lim_{s \to -\infty} \phi(s) = p_{2k}, \quad \lim_{s \to +\infty} \phi(s) = p_{2k+1} .\]

2. There is a family $\{\phi(s)\}$ of integral curves for each $k = 1, 5$ satisfying
   \[\lim_{s \to -\infty} \phi(s) = p_3, \quad \lim_{s \to +\infty} \phi(s) = p_k .\]

**Proof of Lemma 3.3.** There is no periodic trajectory on $D$ by Lemma 3.2. Let $\Omega_1$ and $\Omega_2$ be positive and negative parts of $V_1$ on $D$. We consider the Poincaré-Bendixon theorem from Lemma 3.2, which implies that all trajectories tend to one of singular points. This is almost the same phase plane as in the case of the helicoidal translating soliton. Thus, we can easily check this lemma from the proof of Lemma 2.3.

We translate from the behavior of the trajectories $\{\phi(s)\}$ of $V$ described in Lemma 3.3 to corresponding the curves.

**Proposition 5.** Let $\{\phi(s)\}$ be a trajectory of $V$ defined for all $s \in \mathbb{R}$ and $\gamma(s) = (\alpha(s), \beta(s)) \in Q$ be the corresponding curve. Then, $\gamma$ is one of the following types (see Fig. 4)

1. $\gamma$ has two ends, one approaches orthogonally the $x_{n+1}$-axis as $s \to -\infty$ and the other tends to $(\infty, \infty)$ as $s \to \infty$ (corresponding to (1) in Lemma 3.3). Specially, it is called a translating paraboloid if $n = 2$.
2. $\gamma$ is a non-convex curve without self-intersection and with two ends tending to $(\infty, \infty)$ as $s \to \pm \infty$ (corresponding to (2) in Lemma 3.3). Specially, it is called a winglike translator if $n = 2$. 
Proof of Proposition 5. Analyzing (1) and (2) in Lemma 3.3, we observe that the integral curves tend to the same points in Proposition 3 and we can easily check this from the proof of Proposition 3.

We call type (1) and type (2) curves in Proposition 5 the \( n \)-dimensional translating paraboloid and winglike translator.

**Theorem 3.4.** Let \( M \) be an \( n \)-dimensional hypersurface of revolution in \( \mathbb{R}^{n+1} \). If \( M \) is a translating soliton for MCF, then it is either an \( n \)-dimensional translating paraboloid or an \( n \)-dimensional winglike (see Fig. 4).

We consider the asymptotic behavior of the profile curves of the \( n \)-dimensional rotational symmetric translating solitons in \( \mathbb{R}^{n+1} \).

**Theorem 3.5.** The profile curves of the \( n \)-dimensional rotational symmetric translating solitons in \( \mathbb{R}^{n+1} \) have the asymptotic behavior of

\[
y = \frac{n}{2(n-1)} x^2.
\]

Proof of Theorem 3.5. The profile curve of the \( n \)-dimensional translating paraboloid has one end tending to \((\infty, \infty)\) and the profile curve of the \( n \)-dimensional winglike translator has two ends tending to \((\infty, \infty)\). The integral curves tending to \( p_1 \) and \( p_5 \) are along the lines of equilibria at \( p_1 \) and \( p_5 \), respectively. The integral curves from \( p_3 \) are along the line of equilibria at \( p_3 \). We observe the asymptotic behavior of the profile curves from (7), (8), (9) and \( v_1(p_1) = (1-n, -n) \). We follow the same process with Theorem 2.5 in Section 2. We consider the line \( a = \frac{n-1}{n} (b - \frac{\pi}{2}) + \frac{\pi}{2} \). Then, we have

\[
\frac{dg}{df} = \frac{g'}{f'} = \tan(a) = \tan\left(\frac{n-1}{n} (b - \frac{\pi}{2}) + \frac{\pi}{2}\right) = \tan\left(-\frac{n-1}{n} \arctan\left(\frac{1}{f}\right) + \frac{\pi}{2}\right).
\]

Form Taylor series for \( \tan\left(-\frac{n-1}{n} \arctan\left(\frac{1}{f}\right) + \frac{\pi}{2}\right) \) at infinity, we have the following series:

\[
\frac{n}{n-1} f + \frac{2n-1}{3(n-1)n} f^3 + O\left(\frac{1}{f^5}\right).
\]

Thus, we obtain the ordinary differential equation \( \frac{dg}{df} = \frac{n}{n-1} f \) and the limit behavior of the profile curves of the \( n \)-dimensional rotationally symmetric translating soliton.
Figure 4. Profile curves \((n = 2, 5)\).

is the same with \(y = \frac{n}{2(n-1)}x^2\) (the coefficient is depending on the definition of the mean curvature \(H\) of the hypersurface).

This indicates that the profile curves of an \(n\)-dimensional translating paraboloid and winglike translator have the same asymptotic behavior in [7].

4. **Translating soliton foliated by spheres for MCF in \(\mathbb{R}^{n+1}\).** The authors in [18] proved that a 2-dimensional translating soliton foliated by circles becomes a surface of revolution with the axis parallel to the translating direction. In this section, we consider the dimension of the hypersurface is \(n\) more than 2. Firstly, we consider an \(n\)-dimensional translating soliton foliated by spheres lying on one-parameter family of hyperplanes and show that hyperplanes of the family are parallel. Secondly, we prove that if a hypersurface foliated by spheres lying on parallel hyperplanes is a translating soliton for MCF, then it is a hypersurface of revolution whose axis of revolution is parallel to the translating direction.

We follow the construction of submanifold \(M\) in \(\mathbb{R}^{n+1}\) which is regarded as an \(n\)-dimensional translating soliton foliated by spheres lying on one-parameter family of hyperplanes for MCF [16, 17, 25]. It gives us a coordinate system on an open set around \(M\) in \(\mathbb{R}^{n+1}\). We will use the same notation in [16, 17, 25].

Let \(N_0\) be a unit vector field to the hyperplanes of the family and \(\gamma(t)\) be an integral curve of \(N_0\). We assume that the hyperplanes of the family are not parallel and it follows that the first curvature \(\kappa_0\) of \(\gamma\) does not vanish except in a discrete set of points. Away from those points, we can take the Frenet frame \(\{N_i\}_{i=0,\ldots,n}\) of \(\gamma\) which satisfies \(\dot{N}_0 = \kappa_0 N_1\), \(\dot{N}_i = -\kappa_{i-1} N_{i-1} + \kappa_i N_{i+1}\) for \(1 \leq i \leq n-1\) and \(\dot{N}_n = -\kappa_{n-1} N_{n-1}\) where \(\cdot\) denotes \(\partial/\partial t\). We define \(\Pi_t\) as the hyperplane containing \(\gamma(t)\) whose intersection with \(M\) is a sphere centered at \(c(t)\) with radius \(r(t)\). Since the vector \(c(t) - \gamma(t)\) is in \(\Pi_t\), there are smooth functions \(a_k(t)\) and \(\alpha_k(t)\) for \(k = 1 \cdots n\) such that

\[
\begin{align*}
c(t) &= \gamma(t) + \sum_{i=1}^{n} a_i(t) N_i(t), \\
\dot{c}(t) &= \sum_{k=0}^{n} \alpha_k(t) N_k(t).
\end{align*}
\]

We define a map \(X : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}\) to construct a coordinate by

\[
X(t, v_1, \ldots, v_n) = c(t) + r(t) \sum_{i=1}^{n} v_i N_i(t).
\]
By direct computation, we obtain the induced metric $G = [g_{ij}]$ for $0 \leq i, j \leq n$ such that

$$g_{00} = (\alpha_0 - rv_0 v_1)^2 + \frac{1}{r^2} \sum_{k=1}^{n} g_{0k}^2,$$

$$g_{01} = r\alpha_1 + rv_1 - r^2 \kappa_1 v_2,$$

$$g_{0i} = r\alpha_i + rv_i + r^2(\kappa_{i-1} v_{i-1} - \kappa_i v_{i+1}), \quad \text{for } 2 \leq i \leq n - 1,$$

$$g_{0n} = r\alpha_n + rv_n + r^2 \kappa_{n-1} v_{n-1},$$

$$g_{ij} = \delta_{ij} r^2, \quad \text{for } 1 \leq i, j \leq n.$$

This leads to the following two equations:

$$\sum_{k=1}^{n} g_{0k} v_k = r(\langle \alpha, v \rangle + \dot{r} |v|^2),$$

$$g = \det(G) = r^{2n}(\alpha_0 - rv_1 \kappa_0)^2,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $v = (v_1, \cdots, v_n)$ and $\langle \cdot, \cdot \rangle$ and $| \cdot |$ are the standard Euclidean metric and norm in $\mathbb{R}^n$. If $g$ is identically zero, then $X$ is not an immersion.

We assume that $g$ does not vanish, which implies an inverse matrix $G^{-1} = [g^{ij}]$ for $0 \leq i, j \leq n$ such that

$$g^{00} = \frac{1}{g} r^{2n},$$

$$g^{0i} = -\frac{1}{g} r^{2n-2} g_{0i}, \quad \text{for } 1 \leq i \leq n,$$

$$g^{ii} = \frac{1}{g} r^{2n-4} (r^2(\alpha_0 - rv_1 \kappa_0)^2 + g_{00}^2), \quad \text{for } 1 \leq i \leq n,$$

$$g^{ij} = \frac{1}{g} r^{2n-4} g_{0i} g_{0j}, \quad \text{for } i \neq j.$$

Thus, $M$ is the level set of a smooth function $f$, that is $M = f^{-1}(\{0\})$, defined by

$$f(t, v_1, \cdots, v_n) = \sum_{i=1}^{n} v_i^2 - 1.$$

We calculate a unit normal vector field $\nu$ and the mean curvature $H$ as follows:

$$\nu = \frac{\nabla f}{\|\nabla f\|},$$

$$H = \frac{1}{n} \text{div} \left( -\frac{\nabla f}{\|\nabla f\|} \right),$$

where $\| \cdot \|$, $\nabla$ and div are the norm, gradient and divergence with respect to the metric $G$.

Firstly, we have

$$\nabla f = 2 \sum_{i=1}^{n} g^{0i} v_i \frac{\partial}{\partial t} + 2 \sum_{i,j=1}^{n} g^{ij} v_i \frac{\partial}{\partial v_j}$$

$$= \frac{2}{g} \left\{ \sum_{i=1}^{n} g g^{0i} v_i \frac{\partial}{\partial t} + \sum_{i \neq j}^{n} g g^{ij} v_i \frac{\partial}{\partial v_j} + \sum_{i=1}^{n} g g^{ii} v_i \frac{\partial}{\partial v_i} \right\}.$$
\[
= \frac{2r^{2n-4}}{g} \left\{ -r^2 \sum_{i=1}^{n} g_{0i} v_i \frac{\partial}{\partial t} + \sum_{i \neq j} g_{0j} g_{0i} v_i \frac{\partial}{\partial v_j} \\
+ \sum_{i=1}^{n} \left( r^2 (\alpha_0 - rv_1 \kappa_0)^2 + g_{0i}^2 \right) v_i \frac{\partial}{\partial v_i} \right\}
\]

\[
= \frac{2r^{2n-4}}{g} \left\{ -r^3 (\langle \alpha, v \rangle + \dot{r} |v|^2) \frac{\partial}{\partial t} \\
+ \sum_{i=1}^{n} \left( r (\langle \alpha, v \rangle + \dot{r} |v|^2) g_{0i}^2 + r^2 (\alpha_0 - rv_1 \kappa_0)^2 \right) v_i \frac{\partial}{\partial v_i} \right\}
\]

\[
= \frac{2r^{n-1}}{g} \left( T_0 \frac{\partial}{\partial t} + \sum_{i=1}^{n} T_i \frac{\partial}{\partial v_i} \right),
\]

where

\[
T_0 = -r^n (\langle \alpha, v \rangle + \dot{r} |v|^2),
\]

\[
T_i = r^{n-2} (r (\alpha_0 - rv_1 \kappa_0)^2 v_i + (\langle \alpha, v \rangle + \dot{r} |v|^2) g_{0i}).
\]

By a direct computation, we also have

\[
\| \nabla f \|^2 = 4 \sum_{i,j=1}^{n} g^{ij} v_i v_j
\]

\[
= \frac{4}{g} \left\{ \sum_{i \neq j} g^{ij} v_i v_j + \sum_{i=1}^{n} g^{ii} v_i^2 \right\}
\]

\[
= \frac{4r^{2n-4}}{g} \left\{ \sum_{i,j=1}^{n} g_{0i} g_{0j} v_i v_j + r^2 (\alpha_0 - rv_1 \kappa_0)^2 |v|^2 \right\}
\]

\[
= \frac{4r^{2n-2}}{g} \left\{ \sum_{i,j=1}^{n} (\langle \alpha, v \rangle + \dot{r} |v|^2)^2 + (\alpha_0 - rv_1 \kappa_0)^2 |v|^2 \right\}
\]

\[
= \frac{4r^{2n-2}}{g} D,
\]

where

\[
D = (\alpha_0 - rv_1 \kappa_0)^2 |v|^2 + (\langle \alpha, v \rangle + \dot{r} |v|^2)^2.
\]

Combining the above two equations leads to a unit normal vector field as follows:

\[
\nu = \frac{1}{\sqrt{D}} \left( \frac{T_0}{\sqrt{D}} \frac{\partial}{\partial t} + \sum_{i=1}^{n} T_i \frac{\partial}{\partial v_i} \right)
\]

Secondly, we recall the result about the mean curvature \( H \) of the submanifold \( M \) in [16] to consider a translating soliton foliated by spheres lying on one-parameter family of hyperplanes in \( \mathbb{R}^{n+1} \)

\[
H = -\frac{1}{n \sqrt{g}} \left\{ \frac{\partial}{\partial t} \left( \frac{T_0}{\sqrt{D}} \right) + \sum_{i=1}^{n} \frac{\partial}{\partial v_i} \left( \frac{T_i}{\sqrt{D}} \right) \right\}.
\]
Theorem 4.1. Let \( M \) be an \( n \)-dimensional hypersurface foliated by spheres lying on one-parameter hyperplanes with \( n > 2 \). If \( M \) is a translating soliton for MCF, then the hyperplanes of the family are parallel.

Proof of Theorem 4.1. Let \( M \) be an \( n \)-dimensional hypersurface foliated by sphere lying on one-parameter family of hyperplanes in \( \mathbb{R}^{n+1} \). Suppose that the hyperplanes of the family are not parallel, that is, the first curvature \( \kappa_0 \) of \( \gamma \) does not vanish except in a discrete set of points. Away from those points, we can construct \( M \) by the level set of a smooth function \( f(t, v_1, \cdots, v_n) = \sum_{i=1}^{n} v_i^2 - 1 \) as a submanifold in \( \mathbb{R}^{n+1} \) from a map

\[
X(t, v_1, \cdots, v_n) = c(t) + r(t) \sum_{i=1}^{n} v_i N_i(t).
\]

We have a unit normal vector field \( \nu \) and the mean curvature \( H \) of \( M \) as follows:

\[
\nu = \frac{1}{\sqrt{g}} \left( T_0 \frac{\partial}{\partial t} + \sum_{i=1}^{n} T_i \frac{\partial}{\partial v_i} \right),
\]

\[
H = -\frac{1}{n \sqrt{g}} \left\{ \frac{\partial}{\partial t} \left( \frac{T_0}{\sqrt{D}} \right) + \sum_{i=1}^{n} \frac{\partial}{\partial v_i} \left( \frac{T_i}{\sqrt{D}} \right) \right\}.
\]

We define an equation for vanishing from \( M \) being a translating soliton with translating direction \( v \)

\[
P(t, v_1, \cdots, v_n) = 2ngD^{3/2} (H - \langle \nu, v \rangle),
\]

where \( \langle \cdot, \cdot \rangle \) is the Euclidean inner product in \( \mathbb{R}^{n+1} \).

If \( \kappa_0 \) vanishes, then the hyperplanes of the family are parallel each other (see Lemma 1 in [17]). If \( \alpha_0 \) vanishes, then the spheres of the hypersurface lie on a hyperplane. We assume that both \( \alpha_0 \) and \( \kappa_0 \) do not vanish and the vector \( v \) is a constant vector. We consider three cases: the first (\( j = 1 \)) is \( v_1 = \cos(\theta), v_2 = \sin(\theta) \) and \( v_i = 0 \) for \( i = 3, \cdots, n \), the second (\( j = 2 \)) is \( v_1 = \cos(\theta), v_k = \sin(\theta) \) and \( v_i = 0 \) for \( i = 2, \cdots, k-1, n \) and the third (\( j = 3 \)) is \( v_1 = \cos(\theta), v_n = \sin(\theta) \) and \( v_i = 0 \) for \( i = 2, \cdots, n-1 \), where the symbols with \( \hat{\cdot} \) are omitted. In each case, we obtain trigonometric polynomials of \( \cos(k\theta) \) and \( \sin(k\theta) \) for \( k = 0, \cdots, 5 \) and \( j = 1, 2, 3, 5 \)

\[
\bar{P}_j(t, \theta) = \sum_{k=0}^{5} (A_{j,k}(t) \cos(k\theta) + B_{j,k}(t) \sin(k\theta)),
\]

where \( A_{j,k}(t) \) and \( B_{j,k}(t) \) are functions of \( t \). Since the polynomials are presented by independent trigonometric terms, all coefficients must be vanished. We denote \( V_0 \) and \( V_i \) for \( \langle \partial/\partial t, v \rangle \) and \( \langle \partial/\partial v_i, v \rangle \) for \( i = 1, \cdots, n \).

Case 1. \( j = 1 \) (\( v_1 = \cos(\theta), v_2 = \sin(\theta) \) and \( v_i = 0 \) for \( i = 3, \cdots, n \)).

We assume that \( V_1^2 + V_2^2 \neq 0 \). Considering \( A_{1,5}V_2 - B_{1,5}V_1 = 0 \), we have

\[
\frac{1}{4}nr^{n+1} (V_1^2 + V_2^2) \kappa_0 \alpha_1 \alpha_2 = 0.
\]

We have two possibilities according to the values of \( \alpha_1 \) and \( \alpha_2 \).
1. \( \alpha_1 = 0. \)
   We have \( \alpha_2 = \pm r\kappa_0 \) from
   
   \[
   A_{1,5} = -\frac{1}{8}nr^{n+1}V_1\kappa_0^2(r^2\kappa_0^2 - \alpha_2^2),
   \]
   
   \[
   B_{1,5} = -\frac{1}{8}nr^{n+1}V_2\kappa_0^2(r^2\kappa_0^2 - \alpha_2^2).
   \]

   This implies a contradiction from
   
   \[
   A_{1,4}V_1 + B_{1,4}V_2 = \frac{1}{2}nr^{n+2}(V_1^2 + V_2^2)\kappa_0^3\alpha_0.
   \]

2. \( \alpha_2 = 0. \)
   We have a contradiction from
   
   \[
   A_{1,5} = -\frac{1}{8}nr^{n+1}V_1\kappa_0^2(r^2\kappa_0^2 + \alpha_1^2),
   \]
   
   \[
   B_{1,5} = -\frac{1}{8}nr^{n+1}V_2\kappa_0^2(r^2\kappa_0^2 + \alpha_1^2).
   \]

   Thus, we have \( V_1 = V_2 = 0. \)

**Case 2.** \( j = 2 \) (\( v_1 = \cos(\theta), v_k = \sin(\theta) \) and \( v_i = 0 \) for \( i = 2, \ldots, k \cdots, n \)).

Assume that \( V_k \neq 0 \). The highest coefficients are

\[
A_{2,5} = \frac{1}{4}nr^{n+1}V_k\kappa_0^2\alpha_1\alpha_k,
\]

\[
B_{2,5} = -\frac{1}{8}nr^{n+1}V_k\kappa_0^2(r^2\kappa_0^2 + \alpha_1^2 - \alpha_k^2).
\]

If \( \alpha_k = 0 \), then it is a contradiction by \( B_{2,5} \neq 0 \). Thus, we consider \( \alpha_1 = 0 \) from \( A_{2,5} = 0 \) and obtain

\[
B_{2,4} = \frac{1}{2}nr^nV_k\kappa_0\alpha_0(2r^2\kappa_0^2 - \alpha_k^2)
\]

\[
B_{2,5} = -\frac{1}{8}nr^{n+1}V_k\kappa_0^2(r^2\kappa_0^2 - \alpha_k^2).
\]

It implies \( r^2\kappa_0^2 = 0 \), which is a contradiction. Thus, we have \( V_k = 0. \)

**Case 3.** \( j = 3 \) (\( v_1 = \cos(\theta), v_n = \sin(\theta) \) and \( v_i = 0 \) for \( i = 2, \ldots, n - 1 \)).

Assuming that \( V_n \neq 0 \), the coefficients are

\[
A_{3,5} = \frac{1}{4}nr^{n+1}V_n\kappa_0^2\alpha_1\alpha_n,
\]

\[
B_{3,5} = -\frac{1}{8}nr^{n+1}V_n\kappa_0^2(r^2\kappa_0^2 + \alpha_1^2 - \alpha_n^2).
\]

If \( \alpha_n = 0 \), then it is a contradiction by \( B_{3,5} \neq 0 \). Thus, we consider \( \alpha_1 = 0 \) from \( A_{3,5} = 0 \). We have \( \alpha_n^2 = r^2\kappa_0^2 \) and \( \alpha_n^2 = 2r^2\kappa_0^2 \) from

\[
B_{3,5} = -\frac{1}{8}nr^{n+1}V_n\kappa_0^2(r^2\kappa_0^2 - \alpha_n^2),
\]

\[
B_{3,4} = \frac{1}{2}nr^nV_n\kappa_0\alpha_0(2r^2\kappa_0^2 - \alpha_n^2).
\]

This is a contradiction. Thus, we have \( V_n = 0. \)

Therefore, we have \( V_k = 0 \) for \( k = 1, \cdots, n \) and \( V_0 \) is a constant, which means the integral curve of \( N_0 \) is a straight line parallel to \( v \) and \( \kappa_0 = 0. \) This is a contradiction. The proof is complete. \( \square \)
We want to use the maximum principle for comparing surfaces. We consider an $n$-dimensional hypersurface $M$ in $\mathbb{R}^{n+1}$ locally, in a neighborhood of $p$, as a graph on a domain $D$ of a tangent plane at $p$, that is,

$$M = \text{Graph}(\phi) = \{\exp_q(\phi(q)\nu_p(q)) : q \in D\},$$

where $\phi \in C^2(D)$, $\nu_p$ is a unit normal vector field along $T_pM$ and $\exp$ is the exponential map. Let $M_i$ be two hypersurfaces with the same mean curvature $H = \langle \nu, v \rangle$ in $\mathbb{R}^{n+1}$. It follows a differential equation $L(\phi_i) = f(p)$ from $M_i = \text{Graph}(\phi_i)$. The difference $w = \phi_1 - \phi_2$ verifies a differential equation $L(w) = 0$ which is an elliptic partial differential equation on $D$. From the Hopf maximum principle for elliptic partial differential equations [12], the tangency principle is as follows:

**Proposition 6** (Maximum principle). Let $M_1$ and $M_2$ be two connected $n$-dimensional hypersurfaces with the mean curvature $H_1(p) = H_2(p)$ at a common point $p$ in $\mathbb{R}^{n+1}$ where $H_1$ and $H_2$ are the mean curvatures of $M_1$ and $M_2$, respectively. Then, we have the following:

1. (Interior maximum principle) Suppose that there is a common point $p \in \text{int}(M_1) \cap \text{int}(M_2)$ and the unit normal vectors of $M_1$ and $M_2$ coincide at $p$. If $M_1$ lies on one side of $M_2$ in a neighborhood of $p$, then $M_1 = M_2$.

2. (Boundary maximum principle) Suppose $M_1$ and $M_2$ have $C^2$-boundaries and there is a common point $p \in \partial M_1 \cap \partial M_2$ such that $\partial M_1$ and $\partial M_2$ are tangent spaces and the interior conormal vectors of $\partial M_1$ and $\partial M_2$ coincide at $p$. If $M_1$ lies on one side of $M_2$ in a neighborhood of $p$, then $M_1 = M_2$.

**Theorem 4.2.** Let $M$ be an $n$-dimensional hypersurface foliated by spheres lying on parallel hyperplanes with $n > 2$. If $M$ is a translating soliton with translating direction $v$ for MCF, it is a hypersurface of revolution whose revolution axis is parallel to $v$.

**Proof of Theorem 4.2.** Let $M$ be a translating soliton foliated by spheres in parallel hyperplanes with translating direction $v$ for MCF. Let $e_0$ be a unit normal vector field to the hyperplanes containing each foliated sphere and $\gamma_0$ be an integral curve of $e_0$. Since $\gamma_0$ is assumed to be a straight line, there is a constant orthonormal vector frame $\{E_i\}_{i=1, \cdots, n+1}$ such that $E_{n+1} = e_0$. Each sphere is in the parallel hyperplanes spanned by $\{E_i\}_{i=1, \cdots, n}$. Without loss of generality, we may assume that for each $i = 1, \cdots, n+1$, $E_i$ is equal to $e_i$ up to rotations in $\mathbb{R}^{n+1}$.

Let $\pi_i = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = t_i\}$ for $i = 1, 2$ be two parallel planes with $t_1 < t_2$. Each sphere lies in a horizontal hyperplane $\pi_t = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = t\}$. We consider $M^*$ a piece of $M$ between $\pi_1$ and $\pi_2$, whose boundary is $M^* \cap (\pi_1 \cup \pi_2)$. Since any rotation around $v$ does not affect the mean curvature and inner product with a unit normal vector $\nu$ and $v$, we may assume that the centers of two sphere $M^* \cap \pi_1$ and $M^* \cap \pi_2$ lie in the $x_1x_{n+1}$-plane by a rotation around $v$. The symmetries of $\partial M^*$ inherit $M^*$ by applying Proposition 6 and Alexandrov reflection method to the hyperplanes orthogonal to the $x_1x_{n+1}$-plane. This implies that the center of level sets $M^* \cap \pi_t$ lies on the $x_1x_{n+1}$-plane for $t \in [t_1, t_2]$. Thus, the center $C(t)$ of spheres is

$$C(t) = (c(t), 0, \cdots, 0, t),$$
where $c(t)$ is a function of $t$. $M$ is represented by the level set of a smooth function $f$, that is $M = f^{-1}(\{0\})$

$$f(x_1, \cdots, x_n, t) = (x_1 - c(t))^2 + \sum_{i=2}^{n} x_i^2 - r(t)^2,$$

where $r(t)$ is the radius of the sphere. By direct computations, we obtain the following:

$$\nabla f = 2(x_1 - c, x_2, \cdots, x_n, -rr' - c'(x_1 - c)),$$

$$\|\nabla f\|^2 = 4(x_1 - c)^2 + 4 \sum_{i=2}^{n} x_i^2 + 4(rr' + c'(x_1 - c))^2,$$

$$\triangle f = 2(n - rr'' - r'^2 - c''(x_1 - c) + c'^2).$$

We calculate the mean curvature $H$ and a unit normal vector field $\nu$ as follows:

$$\nu = \frac{\nabla f}{\|\nabla f\|},$$

$$H = \frac{1}{n} \text{div} \left( - \frac{\nabla f}{\|\nabla f\|} \right)$$

$$= \frac{1}{n\|\nabla f\|^2} \langle \nabla f, \nabla f \rangle.$$  

We have the following equation to obtain a translating soliton with translating direction $\nu$ for MCF

$$H - \langle \nu, \nu \rangle = \frac{1}{n\|\nabla f\|^3} \left( \frac{1}{2} \|\nabla f\|^2 - \|\nabla f\|^2 \triangle f - n\|\nabla f\|^2 \langle \nabla f, \nu \rangle \right). \quad (18)$$

If $M$ is a translating soliton for MCF, then equation (18) is identically zero. Thus, we define an equation $P(x_1, \cdots, x_n, t)$ to vanish as follows:

$$P(x_1, \cdots, x_n, t) = \frac{1}{2} \|\nabla f\|^2 - \|\nabla f\|^2 \triangle f - n\|\nabla f\|^2 \langle \nabla f, \nu \rangle. \quad (19)$$

We substitute $x_1 = \cos(\theta)$, $x_k = \sin(\theta)$ (possibly $k = 2, \cdots, n$) and $x_i = 0$ for $i \neq 1, k$. We have a trigonometric polynomial equation with respect to the variable $\theta$, which must vanish by assumption,

$$\tilde{P}(\theta, t) = \sum_{j=0}^{3} (A_j(t) \cos(j\theta) + B_j(t) \sin(j\theta)),$$

where $A_j(t)$ and $B_j(t)$ are functions that depend only on $t$. Since this is an expression of independent trigonometric terms, all coefficients must vanish.

The coefficients for the highest order of $j$ are

$$A_3 = 2nc^2 \langle e_{n+1}, \nu \rangle c'(e_1, \nu),$$

$$B_3 = -nc^2 \langle e_k, \nu \rangle,$$

where $\{e_i\}$ is the standard basis of $\mathbb{R}^{n+1}$.

To prove by contradiction, we assume that $c'$ does not vanish. We have $\langle e_k, \nu \rangle$ is identically zero for $k = 2, \cdots, n$ from $B_3 = 0$. If either $\langle e_1, \nu \rangle = 0$ or $\langle e_{n+1}, \nu \rangle = 0$, then both cases are contradictions of $A_3 = 0$. Thus, we get $\langle e_{n+1}, \nu \rangle c'(e_1, \nu) = 0$.
and \( \langle e_k, v \rangle = 0 \) and by substituting these into (19), we have the following polynomial equation with respect to \( x_1 \) only:

\[
P(x_1, \ldots, x_n, t) = \sum_{l=0}^{2} C_l(t)x_1^l.
\]  

(20)

Since this is expressed as a polynomial of degree two of the variable \( x_1 \), all coefficients \( C_l(t) \) are identically zero to vanish. By direct computation and inserting \( c' = \langle e_1, v \rangle/\langle e_{n+1}, v \rangle \) into equation (20), we have two cases from

\[
C_2 = \frac{8\langle e_1, v \rangle}{\langle e_{n+1}, v \rangle} (2 - n + n\langle e_{n+1}, v \rangle rr').
\]

From \( 2 - n + n\langle e_{n+1}, v \rangle rr' = 0 \), rearranging for \( r' \) and substituting it into \( C_1 \), we have

\[
C_1 = -\frac{16(n-2)\langle e_1, v \rangle}{n\langle e_{n+1}, v \rangle^2},
\]

which is a contradiction. Thus, \( c' \) identically vanishes.

Finally, with \( c' = 0 \) and \( \langle e_k, v \rangle = 0 \) for \( k = 2, \ldots, n \), we have

\[
P(x_1, \ldots, x_n, t) = C_1(t)x_1 + C_0(t),
\]

where \( C_1(t) \) and \( C_2(t) \) are as follows:

\[
C_1 = -8mr^2(1 + r'^2)\langle e_1, v \rangle,
\]

\[
C_0 = 8r^2 \left( (1 - n + n\langle e_{n+1}, v \rangle rr')(1 + r'^2) + r''r \right).
\]

We get \( \langle e_1, v \rangle = 0 \) and \( \langle e_{n+1}, v \rangle = 1 \) from \( C_1 = 0 \). Therefore, \( v \) is the same as \( e_{n+1} \). \( M \) is a hypersurface of revolution with the revolution axis parallel to \( v \). Moreover, every coefficient vanishes except

\[
C_0 = 8r^2 \left( (1 - n + nrr')(1 + r'^2) + rr'' \right).
\]

(21)

This ordinary differential equation yields the profile curve of this hypersurface of revolution. \( \square \)

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