A New Approach To The Hierarchy Problem

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Abstract

We argue that identifying the electroweak Higgs particle with the extra components of the gauge field in $4 + d$ dimensions provides a solution to the hierarchy problem. The absence of ultraviolet quadratic divergences is due to the fact that the Higgs mass is protected by an exact gauge symmetry at energies beyond the cutoff. The idea is implemented within explicit models which also provide a link between fermion chirality and electroweak symmetry breaking in four dimensions.

1 Introduction and Outline

Despite the experimental success of the Standard Model as a theory describing the strong and electroweak interactions of elementary particles, this model is not theoretically satisfactory for various reasons. There are two main sources of theoretical dissatisfaction; the first has to do with the model itself (when it comes to explaining, flavor, and charge quantization, for example), and the second arises when the standard model is discussed within a more general context where the fourth fundamental force of nature, gravity, is present; leading subsequently to an instability of the weak scale at the quantum level. This latter obstacle, known as the hierarchy problem between the

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1An extended version of the talk presented during the activities of the European Meeting (CERN, 18-22 April 2001), and at University of Bonn (29 May 2001), based on hep-ph/0102307.

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3In the recent past there has also been dissatisfaction from the experimental point of view, since the Standard Model is in shortage of explaining, for instance, neutrino oscillations and the recent data of $g-2$ of the Muon.
electroweak and gravity scales, was the main motivation for particle physicists to start searching for new physics beyond the standard model. These searches lead to the birth of many hoping-to-be-physical theories like technicolor, grand unified theories, supersymmetry, and recently models with large extra dimensions.

The Hierarchy problem is present at both classical and quantum levels. At the tree level, there is a huge difference between the scales associated with the electroweak and the gravitational interactions, $M_W/M_P \sim 10^{-17}$. Although this hierarchy is softened when GUTs are considered, it persists to exist $M_W/M_{GUT} \sim 10^{-13}$.

If quantum corrections were not to significantly alter the value of the Higgs (mass)$^2$ computed classically, one could consider the above mentioned hierarchy at the tree level as one of the many extreme ratios existing in nature. However, there are huge quadratic corrections to the Higgs (mass)$^2$ at the quantum level due to the fact that the Higgs particle is described by a fundamental scalar field. These corrections change the classical value of the Higgs (mass)$^2$ by many orders of magnitude, and adjusting the value back to its classical one requires a fine-tuning of order $10^{-34}$ in the case of a gravitational cutoff, and another of order $10^{-26}$ in GUTs.

Supersymmetry is a very good example where the problem is nicely solved at the quantum level, in fact once the SUSY breaking scale is set classically, usually taken to be around few TeV, only logarithmic corrections will alter this value. The key principle for the absence of quadratic divergences in this theory is that the Higgs mass is protected by the symmetry above the cutoff. Being in a multiplet with fermions, the Higgs is deemed to be massless, due to chiral symmetry, as long as SUSY is unbroken.

In the work [1], we employ the above key principle, though without supersymmetry, to explicit models where the Higgs mass is protected by a gauge symmetry in $4+d$ dimensions, the Higgs being identified with the components of a gauge field in higher dimensions.

The talk is organized as the following; in the beginning a short reminder of the original idea behind Kaluza–Klein theories, the idea on which our constructions are based, is given. Then I will describe in words the principle ideas we are proposing in our models, after which I will move to discuss these models in more details. Firstly I will present a toy model for leptons in 6 dimensions, then a toy model for quarks and leptons in 10 dimensions where Higgs mass is produced at the tree-level. Then I will show how Higgs mass
can be induced at the loop level, within the same framework. I will conclude with a summary and outlook.

2 In words

2.1 A historical remark

Although having seven heavens is not a particularly new idea, the first to be published in a scientific journal, proposing our observed world to be an effective theory of a fundamental theory existing in more than four dimensions, was Nordström’s back in 1914. Having no general relativity at that time, Gunnar Nordström wrote down Maxwell’s equations in 5 dimensional space-time, and by wrapping the fifth dimension on a circle, he reduced the equations to Maxwell-Nordström electromagnetic-gravitational theory in 4 dimensions.

Five years later, in 1919, the mathematician Theodor Kaluza proposed obtaining a four dimensional Einstein-Maxwell theory starting from Einstein’s gravity equations in 5 dimensions. Assuming the five dimensional manifold, $W$, to be a product of a 4 dimensional space-time $M_4$ and a circle $S^1$, $W = M_4 \times S^1$, the fifteen components of the metric can be decomposed from a four dimensional point of view into 10 describing the gravity tensor, four forming the components of a $U(1)$ gauge field, and one degree of freedom representing a scalar field. By Fourier expanding those fields, retaining only the zero modes, and integrating along the circle $S^1$, one obtains a theory in four dimensions which is invariant under both four-dimensional general coordinate transformations and abelian gauge transformations. and a massless vector boson which can be checked to have $U(1)$ gauge transformations.

In his original work, Kaluza assumed the zero mode of the scalar field to be constant, $\phi^0 = 1$. In any case, the value of $\phi^0$ had to be positive in order to insure the proper relative sign of the Einstein and Maxwell terms so that the energy is positive. This in turn means that the fifth dimension must be space like. This can also be easily understood in terms of causality; clearly a compact time-like dimension would lead to closed time-like curves. The abelian gauge symmetry arising in four dimensions upon compactification originates from the isometry of the circle. Those last two point, the requirement that the extra dimensions must be space-like, and that the isometry
of the compact space results in a gauge symmetry of the effective action are general arguments.

Seven years later, Oskar Klein used Kaluza’s idea in an attempt to explain the underlying quantum mechanics of Schrödinger equation by deriving it from a five-dimensional space in which the Planck constant is introduced in connection with the periodicity along the closed fifth dimension. In this paper he also discusses the size of the compactified circle, getting closer to giving the extra dimensions a physical meaning than his predecessors. In a separate work, still in 1926, Klein proposed a relativistic generalization of Schrödinger’s equation by starting from a massless wave equation in five dimensions and arriving at four-dimensional Klein-Gordon equation for individual harmonics.

Afterwards, many people adopted Kaluza’s idea, starting from Einstein early last century and continuing by several physicists today. During this period, the idea of having new dimensions to propagate in inspired many to write the first complete models for Lagrangians unifying Yang-Mills and gravity theories, supergravity in 11 dimensions, and superstring theories which has to be considered in 10 dimensions for the theory to be anomaly free and hence consistent at the quantum level. In the original Kaluza–Klein framework, the particles are free to move in the compact space, as well as the to-be-observable one, and hence the volume of the new dimensions should be small in order not to undesirably interfere with the present observations and existing experimental data, since the new idea of 4 + d dimensions has a strong potential to naturally modify the expansion of the Universe and the cross sections of elementary particle interactions. Hence, the typical scale of compactification in string theory is taken to be of order of the four dimensional Planck length \( M_P \).

A recently activity of model building inspired by string theory, though not as mathematically rigorous, have been developed, and were launched by V. Fock and H. Mandel, though others worked for and achieved the same aim as well.
main aim of those models was to solve the hierarchy problem between the gravity and electroweak scales, by lowering the fundamental scale of gravity in $4 + d$ from $10^{19}\text{GeV}$ down to 1TeV. In Kaluza–Klein type models, this requires assuming a bigger compactification radii than the ones used in string theory, and therefore new physics is expected to be observed at energies approaching the typical mass scale of compactification. In the warp compactification models, the extra space is non-compact. The exponentially damping warp factor considered in those models enters into the formulae linking the fundamental gravity scale and the four dimensional one. Again, new dimensionful parameters have to be introduced and tuned in order to solve the hierarchy problem.

So far no full and consistent model has been constructed, neither in tensor compactification nor in the warp one, however the phenomenological and cosmological implications have been studied extensively, and attempts towards creating a theoretically appealing model persist.

2.2 What we are doing

The models we are interested in constructing are in the direction of answering the following question: “How close can one get to the Standard Model, in four dimensions, by Kaluza–Klein compactifying large extra dimensions of an Einstein Yang-Mills theory coupled to fermions in $4 + d$ dimensions?”.

Eventually, the effective action of a “good” theory should provide us in four dimensions with:

- Chiral fermions.
- The standard model gauge group, $SU(3) \times SU(2) \times U(1) \rightarrow U(1)$, or a group containing it.
- Fermions and Higgs particles in the correct representations of standard model gauge group.
- A good solution for the hierarchy problem.
- Correct quarks and lepton masses.
- Suppressed proton decay.
We will give examples where the first four points mentioned above can be realized, with some hints to achieving the last two.

In an attempt to approaching the answer of the above question, we propose two principal ideas:

- Identifying the electroweak Higgs with the extra components of the Yang-Mills field as a solution for the hierarchy problem.

- Linking fermion chirality to the spontaneous electroweak symmetry breaking.

2.3 Approaching the hierarchy problem

Our suggestion for the hierarchy problem is that the electroweak Higgs particle is identified with the extra components of the gauge field in $4 + d$ dimensions. Suppose that we start from an Einstein Yang-Mills theory coupled to fermions on a $4 + d$ dimensional manifold, $W$, $W = M_4 \times Y_d$. The field content to start with consists of a graviton, $g_{MN}(x,y)$, a gauge field $A_a^\alpha(x,y)$ in the algebra of a Lie group $G$, and a fermion $\psi(x,y)$. Where $M,N = 0,1,2,...,d+3$, $a = 1,...,\dim G$, and $x$ and $y$ are the coordinates on $M_4$ and $Y_d$ respectively. We take $Y$ to be a compact manifold with a typical volume of order $\text{TeV}^d$, and $\times$ to indicate a tensor product. Arguing that the Higgs particle is

\[ H(x) \equiv A_\alpha(x) \quad \alpha \in Y \]

implies a solution for a hierarchy problem at the quantum levels, as was first pointed out by [12].

Classically, the Planck scale is related to the fundamental $4 + d$ dimensional gravity scale by the relation

\[ M_P = a^d M^{\frac{d}{2}+1} \]

When $Y_d = Y_1 \times Y_2 \times \ldots \times Y_n$, the $a^d$ should be replaced by the product of the volumes of each manifold. As an example, we take

\[ W = M_4 \times S^2 \times \mathbb{C}P^2 \]

as in the model we discussed in [1] for quarks and leptons. In this example we have the gravity scale $M \sim 10^4 \text{TeV}$, and hence new physics is expected to
show off at around $1/a \sim 1$TeV. Whether one considers the cutoff to be $M$, $1/a$, or the scale at which the gauge coupling in 10 dimensions becomes strong (in our case this happens at around 1TeV as well), the hierarchy between the weak scale and the cutoff, $\Lambda$, is much milder than the one in the ordinary gravity or grand unified theories:

$$\frac{m_H}{\Lambda} \sim 10^{-5} - 10^{-1}$$

Quantum mechanically, the absence of quadratic, or large, divergences can be understood via the argument of symmetry, as in the case of SUSY, however with no fundamental scalar to start with. The gauge symmetry in this case is spontaneously broken due to the presence of a topologically non-trivial background (as I will explain later on), however, at energies larger than the compactification scale, it is recovered and the Higgs field is massless being a component of the massless gauge field. In other words the Higgs mass

$$m_H^2 \leq \frac{1}{a^2} f(Ea)$$

where $E$ is the common energy scale, and $a$ is the typical radius of compactification. We conjecture that $\lim_{E \to \infty} f(Ea) = 0$. In fact, it was shown in [12] that the function $f(Ea)$ is exponentially damping at energies higher than $1/a$. Finding the explicit form of $f$ in our case is technically more complicated due to the presence of a monopole background, however we believe that the $\lim_{E \to \infty} f(Ea)$ will always be finite.

2.4 Linking chirality and SSB

As it is well known, a topologically non-trivial background is needed in Kaluza–Klein field theories in order to get chiral fermions in four dimensions as proposed by [13] and discussed in [14]. In the following examples we will show that the same monopole background used to get chiral fermions is also responsible for the spontaneous symmetry breaking (SSB) of the standard model gauge group.
3 Monopole Background ⇒ Chirality + SSB

Let us start from the following general action

\[ S = \int d^Dx \sqrt{-G} \left( \frac{1}{k^2} \cal{R} + \lambda - \frac{1}{2g^2} \text{Tr} F^2 + \bar{\psi} i \gamma \psi + \ldots \right) \] (1)

where \( \cal{R} \) is the Ricci scalar, \( F \) is the field strength of the Yang–Mills potential, \( \lambda \) is a cosmological constant, and \( \psi \) is a fermion in some representation of the non-abelian gauge group \( G \). The dots imply all possible higher order non-renormalizable operators.

In order to illustrate the idea, let us begin by a simple example of leptons in six dimensions.

3.1 \( SU(3) \) in \( D = 6 \)

The bosonic Einstein and Yang–Mills equations of motion derived from the action (1) admit the following solutions; for the metric

\[ ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + a^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \]

(in other words \( W = M_4 \times S^2 \) is a solution), and

\[ \langle A_\varphi \rangle = \frac{1}{2} H (\cos \theta - 1) d\varphi \] (2)

for the background gauge field.\(^5\) Where \( H = \text{diag}(n_1, n_2, -n_1 - n_2) \), is in the Cartan subalgebra of \( SU(3) \), and \( n_1, n_2 \) have to be integers in order for the fermions to patch properly on the upper and lower hemispheres.

It is clear that the monopole background (2) breaks the \( SU(3) \) symmetry to \( U(1) \times \tilde{U}(1) \) for a generic \( n_1 \) and \( n_2 \). The metric is also invariant under the rotation of the \( S^2 \) covered by \( \theta \) and \( \varphi \). Thus for arbitrary values of \( n_1 \) and \( n_2 \) the effective d=4 theory will be invariant under \( SU(2) \times U(1) \times \tilde{U}(1) \). In case that \( n_1 = n_2 = n \), or its equivalent\(^6\), the maximal \( SU(2) \times U(1) \) will remain unbroken thereby enlarging the 4 dimensional symmetry to \( SU(2) \times U(1) \).

\(^5\)We show the calculations performed on the upper hemisphere only for simplicity. However, the same can be done, with consistent patching, on the lower hemisphere, as well, by starting from the solution \( \langle A_\varphi \rangle = \frac{1}{2} H (\cos \theta + 1) d\varphi \).

\(^6\) For example, the case \( (n_1, n_2) = (1, 1) \) is equivalent to \( (n_1, n_2) = (-1, 2) \).
$SU(2) \times \tilde{U}(1)$. We are going to adopt this second case, and shall show that
the two $SU(2)$’s (call them as $SU(2)_L$ and $SU(2)_R$, indicating the isometry
group and the subgroup of $SU(3)$ respectively) act chirally.

Let us start by computing the fermion spectrum.

### 3.1.1 Spectrum of chiral fermions

The Dirac equation on a generic manifold is

$$\mathcal{D}_W \psi(x, y) = 0$$

Suppose $W = M_4 \times S^2$, the above equation can be written, with the appro-
priate choice of $\Gamma$ matrices, as

$$\mathcal{D}_4 \psi(x, y) + \mathcal{D}_{S^2} \psi(x, y) = 0$$

Note that in the absence of a background gauge field

- $\mathcal{D}_{S^2} \psi = 0$ has no regular solutions.
- Index $\mathcal{D}_{S^2} = 0$.

The first point can be understood easily, with and without going into ex-
plicit computations, since, according to Lichnerowicz’s theorem, all positively
curved smooth compact manifolds, including $S^2$, do not admit harmonic
spinors. In order for the resulting fermion in 4 dimensions to be chiral, it
should be massless to start with. Hence the first requirement for the Dirac
operator on the compact internal manifold is to have at least one zero mode.
Also, the index of the Dirac operator is often, not always, zero.

The only way to get a solution for $\mathcal{D}_{S^2} \psi = 0$ is to couple it to a topolog-
ically non-trivial background $[13, 14]$. This also changes Index $\mathcal{D} = 0$ and
hence allows for achieving a chiral theory, as the standard model, in four
dimensions.

Consider the Dirac operator on $S^2$ coupled to the monopole background
$[13]$:

$$\mathcal{D}_{S^2} = \Gamma^m E^\alpha_m (\partial_\alpha - \frac{1}{2} \omega_{\alpha[k,l]} \Sigma^{kl} - i \langle A_\alpha(y) \rangle)$$

The background solution is of the form:

$$\langle A_\alpha(y) \rangle = \frac{1}{2} H \omega_\alpha$$

(3)
where $\omega_\alpha (\alpha = 1, 2)$ is; $\omega_\theta = 0$, and $\omega_\phi = -(\cos \theta - 1)$.

Now let $\psi(x, y)$ be a Weyl spinor in 6 dimensions, in the $\mathbf{3}$ of $SU(3)$. Using the chirality matrix, $\gamma_5$, in four dimensions, $\psi$ can be written as:

$$\psi = \frac{1 + \gamma_5}{2} \psi + \frac{1 + \gamma_5}{2} \psi = \psi_R + \psi_L$$

The Dirac equation in a monopole background hence simplifies to

$$\partial_\theta \psi_R + \frac{i}{\sin \theta} \psi_R + \lambda_R \psi_R \ctg \theta = 0$$

$$-\partial_\theta \psi_L + \frac{i}{\sin \theta} \psi_L + \lambda_L \psi_L \ctg \theta = 0$$

where the “isohelicities” of the various components of the fermions can be tabulated in

$$\lambda(\psi_R) = \begin{pmatrix}
\frac{1}{2}(1 + n_1) \\
\frac{1}{2}(1 + n_2) \\
\frac{1}{2}(1 - n_1 - n_2)
\end{pmatrix}$$

and

$$\lambda(\psi_L) = \begin{pmatrix}
\frac{1}{2}(-1 + n_1) \\
\frac{1}{2}(-1 + n_2) \\
-\frac{1}{2}(1 + n_1 + n_2)
\end{pmatrix}$$

It can be checked that regular solutions exist only for $\lambda_L \geq 0$ and $\lambda_R \leq 0$.

For instance, adopting the choice $(n_1, n_2) = (-1, 2)$, the spectrum consists of two right-handed singlets of the Kaluza–Klein $SU(2)_L$ and one left-handed doublet of the same $SU(2)_L$. Including the transformation under $SU(2)_R$ and $\tilde{U}(1)$, we can summarize the chiral fermion spectrum as $(2_L, 1_L)_1 + (1_R, 2_R)_{1/2}$ where the subscripts indicate the $\tilde{U}(1)$ charges.

### 3.1.2 Higgs spectrum

Having explained above how coupling to a monopole background can result in chiral fermions upon compactification, the aim now is to show how coupling to the same background derives spontaneous symmetry breaking.

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7See [13].
To start with, there are two sources for the d=4 boson fields, those originating from the fluctuations of the metrical backgrounds and those stemming from the d=6 gauge field fluctuations. In general the effective four dimensional fields are linear combination of these two types of fluctuations. The modes we are interested in, however, do not mix with gravity. Therefore in this paper we shall not include the gravitational fluctuations. The bosonic part of the effective 4-dimensional theory is then obtained by expanding the 6-dimensional Yang-Mills action on the background of our classical solution.

We shall write
\[ A_C(x, y) = \langle A_C(y) \rangle + V_C(x, y) \]
where \( \langle A_\mu(y) \rangle = 0, \mu = 0, 1, 2, 3 \), and \( \langle A_\alpha(y) \rangle \neq 0 \) is the background background monopole configuration given in (4). For the gauge field fluctuations \( V_\mu \) we shall retain only the ones in the algebra of the unbroken gauge group, while for the fields \( V_\alpha \) tangent to \( S^2 \), we shall retain the components in the complementary subspace in the Lie algebra. It is only these ones which become tachyonic in the monopole background and hence lead to the spontaneous symmetry breaking.

Substitute the above in the Yang-Mills field strength,
\[ F_{AB} = \partial_A A_B - \partial_B A_A - i[A_A, A_B] = D_A V_B - D_B V_A - i[V_A, V_B] . \]
where \( D_A V_B \) is defined by
\[ D_A V_B = \nabla_A V_B - i[A_A, V_B] . \]
where \( \nabla_A V_B \) represents the ordinary Riemannian derivative.

As has been explained in detail in [13] and [15] in performing the harmonic expansion on \( S^2 \) it is convenient to use the \( U(1) \) basis on the tangent space to \( S^2 \). The mode structure of each field is then completely fixed by its "isohelicity" \( \lambda \). The isohelicitics can be tabulated as [15]

\[
\lambda(V_\alpha) = \begin{pmatrix}
0 & \frac{1}{2}(n_1 - n_2) & \frac{1}{2}(2n_1 + n_2) \\
-\frac{1}{2}(n_1 - n_2) & 0 & \frac{1}{2}(n_1 + 2n_2) \\
-\frac{1}{2}(2n_1 + n_2) & -\frac{1}{2}(n_1 + 2n_2) & 0
\end{pmatrix}
\]

\[
\lambda(V_\pm) = \begin{pmatrix}
\pm 1 & \frac{1}{2}(n_1 - n_2) \pm 1 & \frac{1}{2}(2n_1 + n_2) \pm 1 \\
-\frac{1}{2}(n_1 - n_2) \pm 1 & \pm 1 & \frac{1}{2}(n_1 + 2n_2) \pm 1 \\
-\frac{1}{2}(2n_1 + n_2) \pm 1 & -\frac{1}{2}(n_1 + 2n_2) \pm 1 & \pm 1
\end{pmatrix}
\]
The harmonic expansion on $S^2$ then proceeds as in [13] and [15], viz,

\[ V_a(x, \theta, \phi) = \sum_{l \geq |\lambda|} \sum_m (2l+1)^{1/2} D_{\lambda,m}^l(\phi, \theta) V^l_{am}(x), \]
\[ V_{\pm}(x, \theta, \phi) = 2^{-1/2}(V_4 \mp iV_5) \]
\[ = \sum_{l \geq |\lambda|} \sum_m (2l+1)^{1/2} D_{\lambda,\pm,m}^l(\phi, \theta) V^{l}_{\pm m}(x), \]

where $D_{\lambda m}^l$ are the $SU(2)$ -representation matrices with Euler angles $\phi, \theta$, and $\pm \phi$.

The linearized Yang-Mills equations, in the covariant gauge $D_A V_A = 0$ satisfied by the component fields $V^l_{bm}(x), V^l_{\pm m}(x)$ are

\[ \{ \partial^2 - a^{-2}[l(l+1) - \lambda^2] \} V_\phi = 0, \quad l \geq |\lambda| \]
\[ \{ \partial^2 - a^{-2}[l(l+1) - (\lambda_\pm \mp 1)^2] \} V_{\pm} = 0, \quad l \geq |\lambda_{\pm}| \]

where $a$ is the radius of $S^2$.

The components $V_{\pm}$ clearly exhibit tachyons. For example, in $V_+$ the component $l = |\lambda_+|$ for $\lambda_+ \leq 0$ carries negative (mass)\(^2\) [15] 

\[ -(|\lambda_+| + 1)/a^2. \]

As we saw while deriving the fermionic spectrum, it is necessary to choose $(n_1, n_2) = (-1, 2)$, or its equivalent, in order to obtain left handed chiral fermions in the doublet representation of $SU(2)_L$. It is an interesting accident that for the same choice of the magnetic charges the leading term in the expansion of $V^6_{\phi} + iV^7_{\tau}$ is tachyonic and has $l = 1/2$. We thus obtain a d=4 Higgs doublet of $SU(2)_L$.

Retaining only the massless gauge fields and this tachyonic field and integrating over the coordinates of $S^2$ we obtain the following 4 dimensional effective action:

\[ \mathcal{L}_B = -\frac{1}{2g^2} \int_0^{2\pi} \frac{d\phi}{\sin \theta} \int_0^{\pi} d\theta \text{ Tr } F_{MN} F^{MN} \]
\[ = -\frac{1}{4g_1^2} F_{\mu \nu}^8 \ - \frac{1}{4g_2^2} F_{\mu \nu}^r - \frac{1}{4e^2} W_{\mu \nu}^2 \]
\[ - \text{ Tr } \left\{ (\partial_\mu - \frac{3}{2} iV_{\mu}^8 - iV_{\mu}^r \sigma_{\mu}^r - iW_{\mu}^r \tau_{\mu}^r) \phi |^2 - \frac{3}{2a^2} \phi^\dagger \phi + 2g_1^2 (\phi^\dagger \phi)^2 \right\} \]
\[ \mathcal{L}_F = \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \, \bar{\psi} i \nabla \psi \]
\[ = \bar{\lambda}_L i \gamma^\mu \left( \partial_\mu - ig_1 V_\mu^8 - ieW^i_\mu \tau^i \right) \lambda_L \]
\[ + \bar{\lambda}_R i \gamma^\mu \left( \partial_\mu - ig_2 V_\mu^8 - ig_2 V_\mu^i \sigma^i \right) \lambda_R \]
\[ - 2g_1 \{ \bar{\lambda}_L \phi (i\sigma_2) \lambda_R - \bar{\lambda}_R (i\sigma_2) \phi^\dagger \lambda_L \} \]

where \( W^i_\mu \) indicate the Kaluza–Klein \( SU(2)_L \) gauge potential and \( W^i_\mu \) is its corresponding field strength. \( e \) denotes the Kaluza–Klein gauge coupling constant. The Higgs doublet \( \phi \), with some appropriate constant rescalings, is the \( x \)-dependent coefficient of the \( l = 1/2 \) term in the expansion of \( V^6_+ + iV^7_+ \).

The \( SU(2)_R \times \tilde{U}(1) \) couplings \( g_2 \) and \( g_1 \) are related to the six dimensional gauge coupling and the radius of \( S^2 \) via Einstein equations
\[ g_2 = \frac{1}{2\sqrt{\pi}} a = \sqrt{3} g_1. \]

With above choice of magnetic charges, the components \( V^4_+ + iV^5_+ \) and \( V^6_+ + iV^7_+ \) form an \( SU(2)_R \) doublet which is charged under the \( \tilde{U}(1) \). Again its \( l = 1/2 \) harmonics on \( S^2 \) is tachyonic. As can be seen, the Higgs field transforms as \( (2,2)_{1/2} \). The chiral fermions are \( \lambda_L = (2_L, 1) \), and \( \lambda_R = (1, 2_R)_{1/2} \).

### 3.2 \( U(6) \) in \( D = 10 \)

Solution for the bosonic equations of motions are:
\[ ds^2 = a_1^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) + \frac{4a_2^2}{1 + \bar{\zeta}^T \zeta} d\bar{\zeta}^a \left( \delta^{ab} - \frac{\zeta^a \bar{\zeta}^b}{1 + \bar{\zeta}^T \zeta} \right) d\zeta^b \]
for the metric, or in other words
\[ W = M_4 \times S^2 \times \mathbb{C}P^2, \]
and
\[ \langle A(y) \rangle = \frac{n}{2} (\cos \theta - 1) d\varphi + qi\omega \]  

(5)
where \((\cos \theta - 1)d\varphi\) and \(\omega\) are the connections on \(S^2\) and \(\mathbb{C}P^2\) respectively.

\[
\omega(\zeta, \bar{\zeta}) = \frac{1}{2(1 + \zeta^\dagger \zeta)} (\zeta^\dagger d\zeta - d\zeta^\dagger \zeta)
\]

\(d\omega\) is the self dual Kähler form on \(\mathbb{C}P^2\). It is thus an instanton type solution of the Yang-Mills equation in \(\mathbb{C}P^2\) (remember that the first term is nothing but the monopole solution \(3\) on \(S^2\)). The coupling of the \(\mathbb{C}P^2\) instanton to the fermions is crucial in defining spinors globally on \(\mathbb{C}P^2\).

We chose the matrices \(n\) and \(q\) to be \(n = \text{diag}(-2, +1, +1, -2, +1, +1)\), and \(q = \text{diag}(+\frac{5}{2}, +\frac{5}{2}, +\frac{5}{2}, +\frac{3}{2}, +\frac{3}{2}, +\frac{3}{2})\) (again \(n\) has to be an integer while \(q\) a half of an odd integer for topological reasons).

The procedure to adopt is very similar to the previous model for leptons explained before, however the computations here are more subtle. In order to cut the long story short, I will give here only the results.

The vacuum in 4 dimensions is invariant under the isometry groups of both \(S^2\) and \(\mathbb{C}P^2\) which are \(SU(2)\) and \(SU(3)\) respectively. We start from a Weyl spinor in the 6 of \(U(6)\) in 10 dimensions. The fermion spectrum we obtain in 4 dimensions can be summarized as

\[
(2L, 3) + (1R, 3) + (1R, \bar{3}) + (2L, 1) + (1R, 1) + (1R, 1)
\]

\[
\uparrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow
\]

\[
\begin{pmatrix}
  u \\
  d
\end{pmatrix}_L, \quad u_R, \quad d_R, \quad \begin{pmatrix}
  \nu_e \\
  e
\end{pmatrix}_L, \quad e_R, \quad \psi_R
\]

where the arrow are pointing towards the corresponding degrees of freedoms in the standard model. The additional right handed singlet can be removed by an appropriate choice of \(q\). The Higgs spectrum consists of two doublets of \(SU(2)\), \(\phi_1 = (2, 1)\), and \(\phi_2 = (2, 1)\), which are singlets of \(SU(3)\).

The vacuum is also invariant under the left unbroken subgroup of \(U(6)\) which is \(SU(2) \times SU(2) \times U(1)^3\). We assume in [1] that this group is weakly coupled while the standard model group originates from the isometry group. Nevertheless, we include the transformation properties of the fermions under this group:

\[
(2L, 3) \sim (1, 1)_{(-2,0,1)}
\]

\[
(1R, 3) + (1R, \bar{3}) \sim (2, 1)_{(1,0,1)}
\]
The Higgs doublets, on the other hand, have the transformations \( \phi_1 = (2, 1)_{(-3,0,0)} \) and \( \phi_2 = (1, 2)_{(0,-3,0)} \).

This model is free of global anomalies, and can be made free of local anomalies as well \([1]\).

These two models can not be considered realistic the way they stand. One of the reasons is that the mass of fermions in four dimensions are of the order of the compactification scale. This would not have been a problem by itself since those fermions could in principle be identified with the third generation. The problem is that, even in this case, there is no justification for ignoring the higher Kaluza–Klein modes which also have masses of the same order. Another reason is that the resulting group in four dimensions is bigger than the standard model group, and the assignments of hypercharges are not correct.

4 Loop induced Higgs mass

As a way to overcome the issue of incompatible scales, the first thing which comes to mind is to produce the Higgs mass by quantum corrections. The idea is to construct a model where Higgs mass is zero at the tree level. The sign of the one loop induced effective mass will depend on the imbalance between the contribution of fermions and bosons. By a judicious choice of the fermionic degrees of freedom this sign can be made tachyonic. We proceed by showing that a zero tree level Higgs mass is possible in a monopole background by considering \( U(N) \) gauge theory in \( D = 10 \) compactified on three spheres \( W = M_4 \times S^2 \times S^2 \times S^2 \) (with radii \( a, a', \) and \( a'' \) respectively), with three monopoles on each sphere. The ansatz for the background gauge field is

\[
A = \frac{n}{2}(\cos \theta - 1)d\phi + \frac{n'}{2}(\cos \theta' - 1)d\phi' + \frac{n''}{2}(\cos \theta'' - 1)d\phi''
\]

where \( n, n', n'' \subset U(N) \), and they are \( N \times N \) diagonal real matrices. The scalars of interest for us are those components of the fluctuations of the vector potential which are tangent to \( S^2 \times S^2 \times S^2 \) and are in the directions of perpendicular to the Cartan subalgebra of \( U(N) \). Consider the field \( V^j_i \) tangent to \( S^2 \). The masses of these fields are related to the eigenvalues of
the Laplacians on the three spheres which are, in turn, determined from the
isohelicities of $V^j_{-1}$

$$\lambda = -1 + \frac{1}{2}(n_i - n_j), \quad \lambda' = \frac{1}{2}(n'_i - n'_j), \quad \lambda'' = \frac{1}{2}(n''_i - n''_j).$$

For illustration, we consider an $n$ matrix which has only the elements $n_1$ and $n_2$ different from zero and such that $n_1 - n_2 \geq 2$. Then $\lambda(V^2_{-1}) \geq 0$ and according to our general rule the leading mode in this field can be
tachyonic. The question we would like to answer is if by an appropriate
choice of magnetic charges we can make the mass of this field to vanish. The
answer, as shown in [1], is yes. The masses of the infinite tower of modes of
$V^2_{-1}$ are given by

$$a^2M^2 = l(l+1) - \lambda^2 + \frac{a^2}{a'^2}(l'(l'+1) - \lambda'^2) + \frac{a^2}{a''^2}(l''(l''+1) - \lambda''^2) + 1 - (n_1 - n_2)$$

and by employing the bosonic background equations, and making a specific
choice for the magnetic charges, the leading mode is indeed massless [1].
For the same choice there will of course be a similar massless mode in the
fluctuations $V^2_{-1}$ tangent to $S^2$. One can make all other modes to have
positive masses by appropriate choices of the remaining magnetic charges.

5 Summary and Outlook

We argued that the electroweak Higgs scalar field originates from the extra
components of a Yang-Mills potential in $4 + d$ dimensions. No $\Lambda^2$ divergences
are expected to be present at high energies, simply because the Higgs will
be a component of a massless gauge boson at energies beyond $1/a$. The
computation of the Higgs mass in low energies is basically the one of vacuum
polarization in $4 + d$ dimensions, after integrating over the compactified space
and restricting the on-shell momenta to low energies.

This scalar induces spontaneous symmetry breaking in 4 dimensions due
to the presence of a magnetic monopole like background. The same back-
ground produces chiral fermions in 4 dimensions, hence linking both the
electroweak symmetry breaking and fermion chirality in the standard model
to have one source: the monopole background, and/or the instanton, in the
internal space.
Chiral fermions obtained are in the correct representations of $SU(3) \times SU(2)$. Two models were worked out, where the standard model gauge group is assumed to be stemming from the isometry groups of the compactified space which are $SU(2)$ and $SU(3)$ for $S^2$ and $CP^2$ respectively (see section 3.2). The first was a toy model for leptons, using an $SU(3)$ gauge theory in 6 dimensions compactified on an $S^2$, in the presence of a monopole. The second was a toy model for quarks and leptons, with a $U(6)$ gauge theory in 10 dimensions on $S^2 \times CP^2$, in the background of a monopole and an instanton. In the latter model, the tachyonic mode, to be identified with the Higgs, is a singlet of $SU(3)$, and hence only $SU(2)$ gets spontaneously broken.

These models can not be considered realistic as they stand for several reasons. Mainly the hypercharge assignments are incorrect, and the resulting group in four dimensions is larger than the standard model one. Besides, the masses of quarks and leptons are too big $\sim O(1/a)$ (no justification for ignoring KK modes which have masses of the same order).

The problem of fermion mass scales can be overcome by producing the Higgs mass at the quantum level. An existence example was given; a $U(N)$ Yang-Mills in 10 dimensions compactified on three spheres in the background of three monopoles, where the Higgs mass is zero at the tree level and is expected to develop a tachyonic mass at one loop due to the imbalance between the bosonic and fermionic degrees of freedom. A work in progress may reveal some further phenomenological aspects of the idea.

Acknowledgment
I am grateful to Seif Randjbar-Daemi for many instructive and enlightening discussions. This work is partially supported by the European TMR project “Supersymmetry And The Early Universe” under the contract HPRN-CT-2000-0152.

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