BEL–ROBINSON ENERGY AND CONSTANT MEAN CURVATURE FOLIATIONS

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ABSTRACT. An energy estimate is proved for the Bel–Robinson energy along a constant mean curvature foliation in a spatially compact vacuum spacetime, assuming an $L^\infty$ bound on the second fundamental form, and a bound on a spacetime version of Bel–Robinson energy.

1. Introduction

Let $(\bar{M}, \bar{g})$ be a 3+1 dimensional $C^\infty$ maximal globally hyperbolic vacuum (MGHV) space–time, which is spatially compact, i.e. $\bar{M}$ has compact Cauchy surfaces. One of the main conjectures (the CMC conjecture, see [5] for background) concerning spatially compact MGHV spacetimes states that if there is a constant mean curvature (CMC) Cauchy surface $M_0$, in such a spacetime $\bar{M}$, then there is a foliation in $\bar{M}$ of CMC Cauchy surfaces with mean curvatures taking on all geometrically allowed values. Specifically, in case the Cauchy surface $M_0$ is of Yamabe type $-1$ or $0$, then the mean curvatures take all values in $(-\infty, 0)$, or $(0, \infty)$, depending on the sign of the mean curvature of $M_0$, while in case $M_0$ is of Yamabe type $+1$, the mean curvatures take on all values in $(-\infty, \infty)$.

The only progress towards proving the CMC conjecture so far has been made under conditions of symmetry, cf. [11], or curvature bounds [3], [6].

One approach to the CMC conjecture is to view it as a statement about the global existence problem for the Einstein vacuum field equations

$$\bar{R}_{\alpha\beta} = 0, \quad \text{(EFE)}$$

in the CMC time gauge. It is known that in the CMC gauge with zero shift, the (EFE) are non–strictly hyperbolic [8] while in other gauges such as wave coordinates, the (EFE) form a system of quasi–linear wave equations for the metric $\bar{g}$. In this context, it has been conjectured that the Cauchy problem for the (EFE) is well–posed for data in $H^2 \times H^1$ (the $H^2$ conjecture, see [10]).

From this point of view it is interesting to consider continuation principles for the (EFE), in CMC gauge. In this note we will use a scaling argument to prove an energy estimate for CMC foliations. The energy we consider is
a version of the Bel–Robinson energy. For a spatial hypersurface \( M \) in \( \bar{M} \), the energy expression we consider is defined by
\[
Q(M) = \int_M (|E|^2 + |B|^2) \mu_g,
\]
where \( E, B \) are the electric and magnetic parts of the Weyl tensor (defined w.r.t. the timelike normal \( T \) of \( M \). Roughly speaking, \( Q \) bounds Cauchy data \((g, K)\) on \( M \) in \( H^2 \times H^1 \). Here, \( g \) is the induced metric on \( M \) and \( K \) is the second fundamental form of \( K \). Therefore, if the \( H^2 \) conjecture is true, apriori bounds for the Bel–Robinson energy can be expected to be relevant to the global existence problem for the (EFE).

Let \( H = \text{tr} K \) denote the mean curvature, and assume the CMC gauge condition \( H = t \). Define the spacetime Bel–Robinson energy of a CMC foliation \( F_I = \{ M_t, t \in I \} \) by
\[
Q(F_I) = \int_I dt \int_{M_t} N(|E|^2 + |B|^2) \mu_g, \tag{1.1}
\]
where \( N \) is the lapse function.

We are now ready to state our main result

**Theorem 1.1.** Let \((\bar{M}, \bar{g})\) be a MGHV space–time, and let \( I = (t_-, t_+) \) with \(-\infty < t_- < t_+ < 0\), be such that there is a CMC foliation \( F_I \) in \((\bar{M}, \bar{g})\). Let \( t_0 \in I \).

Suppose that \( \limsup_{t \to t_+} Q(t) = \infty \). Then at least one of the following holds:

1. \( \limsup_{t \to t_+} Q(F_{[t_0, t)}) = \infty \),
2. \( \limsup_{t \to t_+} \frac{||K(t)||_{L^\infty}}{|H(t)|} = \infty \).

The time reversed statement with \( t_+ \) replaced by \( t_- \) also holds.

**Remark 1.1.** Let \((M, \gamma)\) be a compact hyperbolic 3–manifold with sectional curvature \(-1\). Then the metric \( \tilde{\gamma} = -d\rho^2 + \rho^2 \gamma \) on \( \bar{M} = (0, \infty) \times M \) is flat. It follows from the work of Andersson and Moncrief \[7\] that for small perturbations of \((M, \tilde{\gamma})\), there is a global CMC foliation \( F_{[t_0, t)} \) in the expanding direction, and for this foliation, the Bel–Robinson energy decays as \( Q(t) = O(H^2(t)) \), which implies that the space–time Bel–Robinson energy \( Q(F) \) is bounded in this case. It is interesting to consider the behavior of \( Q(F_{[t_0, t)}) \) when \( t_0 \) decreases.

The proof of Theorem 1.1 is based on a scaling argument, which we now sketch. The statement of the theorem is symmetric in time, but here we consider only the future time direction, the argument in the reverse direction is similar. Suppose for a contradiction there is a constant \( \Lambda < \infty \) so that \( Q(F_{[t_0, t_+)}) \leq \Lambda, \frac{||K||_{L^\infty}}{|H|^2} \leq \Lambda \) for all \( t \in [t_0, t_+) \), and that
\[
\limsup_{t \to t_+} Q(t) = \infty.
\]
An energy estimate shows that \( r_h(t)Q(t) \leq C \), where \( r_h \) is an \( L^{1,p} \) harmonic radius, for some fixed \( p, 3 < p < 6 \), and hence if \( r_h \) is bounded from below there is nothing to prove. Suppose for a contradiction that \( r_h \to 0 \) as \( t \to t_\ast \). The combination \( r_hQ \) is scale invariant, and hence by rescaling \( \bar{g} \) to \( \bar{g}' = r_h^{-2}\bar{g} \), we get a sequence of metrics \( g' \) with \( Q \) bounded. \( Q \) bounds \( g' \) in \( L^{2,2} \) and hence we may pick out a subsequence of \( (g', N') \), which converges weakly to a solution \( (g_\infty, N_\infty) \) of the static vacuum Einstein equations,

\[
\Delta N = 0, \\
\nabla^2 N = NRic.
\]

It follows from our assumptions that the limit \( g_\infty \) is complete, and the limiting \( N_\infty \) is bounded from above and below. Then by [2], \( g_\infty \) must be flat, with infinite harmonic radius, which contradicts \( r'_h = 1 \), by the weak continuity of \( r_h \) on \( L^{2,2} \). We conclude that in fact \( r_h \) is bounded away from zero, and hence that \( Q \) does not blow up, which proves the theorem.

2. Preliminaries

For a space–like hypersurface \( M \) in \( \bar{M} \) we denote its timelike normal \( T \) and induced metric and second fundamental form \((g, K)\). We assume all fields are \( C^\infty \) unless otherwise stated. Let lower case greek indices run over \( 0, \ldots, 3 \) while lower case latin indices run over \( 1, \ldots, 3 \). We work in an adapted frame \( e_\alpha \), with \( e_0 = \partial_t \). Our convention for \( K \) is \( K_{ab} = -\frac{1}{2}L_T \bar{g}_{ab} \), so that if the mean curvature \( H = \tr K \) is negative, \( T \) points in the expanding direction. We will sometimes use an index \( T \) to denote contraction with \( T \), for example \( u_T = u_\alpha T^\alpha \).

In a nonflat spatially compact, globally hyperbolic, vacuum spacetime, the maximum principle implies uniqueness of constant mean curvature (CMC) Cauchy surfaces. In particular, each \( x \in \bar{M} \) is contained in at most one CMC Cauchy surface, and for each \( t \in \mathbb{R} \), there is at most one \( M_t \) with mean curvature \( t \).

Let \( I \subset \mathbb{R} \) be an interval. A CMC foliation \( \mathcal{F}_I \) in \( \bar{M} \) is a foliation \( \mathcal{F}_I = \{ M_t, t \in I \} \) such that for each \( t \in I \), \( M_t \) is a \( C^\infty \) CMC Cauchy surface with mean curvature \( t \). When convenient we will write \( g(t), K(t) \) for the data induced on \( M_t \). Introducing coordinates \( x^\alpha \) with \( x^0 = t \), the lapse and shift \( N, X \) of the foliation are defined by \( \partial_t = NT + X \). We may without loss of generality assume \( X = 0 \).

We call \( \mathcal{F}_I \) a maximal CMC foliation in \( \bar{M} \) if there is no interval \( I' \) containing \( I \) as a strict subset with a CMC foliation \( \mathcal{F}_{I'} \). Given a foliation in \( \bar{M} \), we write \( \bar{M}_\mathcal{F} \) for the support of \( \mathcal{F} \).

Assume that \( \bar{M} \) contains a compact, constant mean curvature (CMC) Cauchy surface \( M_0 \) with mean curvature \( H^0 < 0 \). By standard results there is then an interval \( I = (t_-, t_+) \subset \mathbb{R} \), \( H^0 \in I \), such that there is a CMC foliation \( \mathcal{F}_I \), and by uniqueness, \( M_{H^0} = M_0 \). Hence if \( \mathcal{F}_I \) is a maximal CMC foliation, then \( I \) is open.
2.1. **The Bel–Robinson energy.** Let $W$ be the Weyl tensor of $(\tilde{M}, \tilde{g})$ and let $^*W$ denote its (left) dual (in vacuum $^*W = W^*$). The Bel–Robinson tensor $Q$ of $(\tilde{M}, \tilde{g})$ is defined by

$$Q_{\alpha \beta \gamma \delta} = W_{\alpha \mu \gamma \nu} W^\mu_{\beta \delta} + ^*W_{\alpha \mu \gamma \nu} ^*W^\mu_{\beta \delta}.$$  \hspace{1cm} (2.1)

Then $Q$ is totally symmetric and trace–less, and in vacuum, $Q$ has vanishing divergence.

Let $E_{\alpha \beta} = W_{\alpha T \beta T}$, $B_{\alpha \beta} = ^*W_{\alpha T \beta T}$ be the electric and magnetic parts of the Weyl tensor. Then $E, B$ are symmetric, $t$–tangent (i.e. $E_{\alpha T} = B_{\alpha T} = 0$) and trace invariant, $g^{ab}E_{ab} = g^{ab}B_{ab} = 0$.

In vacuum, we have

$$E_{ab} = \text{Ric}_{ab} + HK_{ab} - K_{ac}K^c_b,$$

$$B_{ab} = -\text{curl}K_{ab},$$

where for a symmetric tensor in dimension 3,

$$\text{curl}A_{ab} = \frac{1}{2}(\epsilon_a^{st} \nabla_t A_{sb} + \epsilon_b^{st} \nabla_t A_{sa}).$$

Recall that for symmetric traceless tensors in dimension 3, the Hodge system $A \mapsto (\text{div}A, \text{curl}A)$ is elliptic.

The following identities, see [7], relate $Q$ to $E$ and $B$,

$$Q_{TTTT} = E_{ab}E^{ab} + B_{ab}B^{ab} = |E|^2 + |B|^2,$$

$$Q_{aTTT} = 2(E \wedge B)_a,$$

$$Q_{abTT} = -(E \times E)_{ab} - (B \times B)_{ab} + \frac{1}{3}(|E|^2 + |B|^2)g_{ij},$$  \hspace{1cm} (2.2a,b,c)

where by definition, for symmetric tensors $A, B$ in dimension 3,

$$(A \wedge B)_a = \epsilon_a^{bc} A_b^d B_{dc},$$

$$(A \times B)_{ab} = \epsilon_a^{cd} \epsilon_b^{ef} A_{ce}B_{df} + \frac{1}{3}(A \cdot B)g_{ab} - \frac{1}{3} (\text{tr}A)(\text{tr}B)g_{ab}.$$  \hspace{1cm} (2.2d)

From equation (2.2a), it follows that $Q_{TTTT} \geq 0$ with equality if and only if $W = 0$. Let $\mathcal{F}$ be a foliation in $M$. The Bel–Robinson energy $Q(t)$ of $M_t \in \mathcal{F}$ w.r.t. the time–like normal $T$, is defined by

$$Q(t) = Q(M_t) = \int_{M_t} Q_{TTTT} \mu_g = \int_{M_t} (|E|^2 + |B|^2) \mu_g.$$  \hspace{1cm} (2.3)

An application of the Gauss law gives in vacuum,

$$\partial_t Q(t) = -3 \int_{M_t} NQ_{\alpha\beta TTT} \pi^{\alpha\beta} \mu_g,$$

where $\pi_{\alpha\beta} = \tilde{\nabla}_a T_{\beta}$. A computation shows that the only nonzero components of $\pi_{\alpha\beta}$ are $\pi_{ab} = -K_{ab}$, $\pi_{Ta} = N^{-1} \nabla_a N$. Thus

$$NQ_{\alpha\beta TTT} \pi^{\alpha\beta} = -NQ_{abTT} K^{ab} + Q_{aTTT} \nabla^a N.$$  \hspace{1cm} (2.3)
3. Proof of Theorem 1.1

We will assume that the complement of points 1, 2 of Theorem 1.1 holds, and prove from this that \( Q(t) \) does not blow up. Assume for a contradiction there is a constant \( \Lambda > 1 \) so that for \( t \in [t_0, t_*) \),

\[
Q(F_{[t_0, t]}) \leq \Lambda, \quad \frac{||K(t)||^2_{L^\infty}}{H^2(t)} \leq \Lambda, \tag{3.1}
\]

and that \( \limsup_{t \to t_*} Q(t) = \infty \).

We let \( L^s_p \) denote the \( L^p \) Sobolev spaces and write \( H^s \) for \( L^s_2 \). We will sometimes use subindices \( x \) or \( t, x \) to distinguish function spaces defined w.r.t. space or space–time.

For a foliation \( F_I \), we may without loss of generality assume that \( T = N^{-1}\partial_t \), where \( N > 0 \) is the lapse function of the foliation. Then \( \bar{g} \) is of the form

\[
\bar{g} = -N^2 dt^2 + g_{ab} dx^a dx^b.
\]

The lapse function satisfies

\[
-\Delta N + |K|^2 N = 1, \tag{3.2}
\]

which using the maximum principle implies the estimate

\[
1/||K||^2_{L^\infty} \leq N \leq 3/H^2. \tag{3.3}
\]

Let \( F \) be a foliation in \( \bar{M} \). From (2.3) we get

\[
|\partial_t Q| \leq C_1 (||\nabla N||_{L^\infty} + ||N||_{L^\infty} ||K||_{L^\infty}) Q,
\]

with \( C_1 \) a universal constant. By assumption, \( |K|^2/H^2 \leq \Lambda \). Let \( \hat{K} = K - (H/3)g \) be the traceless part of \( K \). Using \( |K|^2 = H^2/3 + |\hat{K}|^2 \geq H^2/3 \), we get

\[
|\partial_t Q| \leq C (||\nabla N||_{L^\infty} + \frac{\Lambda}{|H|}) Q. \tag{3.4}
\]

We may assume without loss of generality that \( ||\nabla N||_{L^\infty} \geq \Lambda/|H| \), since otherwise there would be nothing to prove. Therefore we may absorb \( \Lambda/|H| \) in the constant in (3.4) to get

\[
|\partial_t Q| \leq C ||\nabla N||_{L^\infty} Q. \tag{3.5}
\]

3.1. The blowup. Choose once and for all a fixed \( p \) satisfying

\[
3 < p < 6. \tag{3.6}
\]

On the Riemannian manifold \( (M_t, g_t) \), let \( r_h(x) \) denote the \( L^{1,p} \) harmonic radius of \( (M_t, g_t) \) at \( x \in M_t \); thus \( r_h(x) \) is the radius of the largest geodesic ball about \( x \) on which there is a harmonic coordinate chart in which the metric coefficients \( g_{ab} \) satisfy

\[
r_h(x)^{-3/p} ||g_{ab} - \delta_{ab}||_{L^p(B_x(r_h(x)))} + r_h(x)^{(p-3)/p} ||\partial g_{ab}||_{L^p(B_x(r_h(x)))} \leq C, \tag{3.7}
\]

where \( C \) is a fixed constant (say \( C = 1 \)), cf. [1, 4]. By Sobolev embedding, in \( B_x(r_h(x)) \), the \( C^\beta \) norm of \( g_{ab} \) is controlled, for \( \beta = 1 - \frac{3}{p} \). The presence...
of the factors of \( r_h(x) \) in (3.7) means the estimate (3.7) is scale invariant. It follows from this that \( r_h(x) \) scales as a distance.

It is well known that the Laplacian in such a local harmonic coordinate chart on \( B_x(r_h(x)) \) has the form

\[
\Delta u = g^{ab} \partial_a \partial_b u.
\]

Thus, within \( B_x(r_h(x)) \), \( \Delta \) is given in these local coordinates as a non–divergence form elliptic operator, with uniform \( C^\beta \) control on the coefficients \( g^{ab} \), and uniform bounds on the ellipticity constants.

We have the following standard (interior) \( L^p \) elliptic estimate for this Laplace operator, c.f. [9, Thm. 9.11]. Let \( B = B_x(r_h(x)) \) and \( B' = B_x(\frac{1}{2} r_h(x)) \). Then

\[
||N||_{L^2, p(B')} \leq C ||\Delta N||_{L^p(B)} + ||N||_{L^p(B)}. \tag{3.8}
\]

We drop the dependence on \( p \), since \( p \) is fixed. We need to make explicit the dependence of the constant \( C \) on \( r_h(x) \). This is done by a standard scaling argument. Thus, assume (by rescaling if necessary), that \( r_h(x) = 1 \). Then (3.8) becomes

\[
||N||_{L^2, p(B')} \leq C ||\Delta N||_{L^p(B)} + ||N||_{L^p(B)}. \tag{3.9}
\]

By Sobolev embedding, since \( p > 3 \) is fixed, and \( B' = B(\frac{1}{2}) \), we have

\[
||\nabla N||_{L^\infty(B')} \leq c \cdot ||N||_{L^2, p(B')},
\]

so that

\[
||\nabla N||_{L^\infty(B')} \leq C_o ||\Delta N||_{L^p(B)} + ||N||_{L^p(B)},
\]

and in particular,

\[
||\nabla N||_{L^\infty(B')} \leq C_o ||\Delta N||_{L^\infty(B)} + ||N||_{L^\infty(B)}, \tag{3.9}
\]

where \( C_o \) is an absolute constant, (i.e. independent of \( N \), given control on \( \Delta \) from definition of \( r_h = 1 \)). Now we put in scale factors to make (3.9) scale invariant and write (3.9) as

\[
r_h(x)||\nabla N||_{L^\infty(B')} \leq C_o [r_h(x)]^2 ||\Delta N||_{L^\infty(B)} + ||N||_{L^\infty(B)}. \tag{3.10}
\]

Note that the function \( N \) is itself scale invariant. Each term in (3.10) is invariant under scaling, and thus (3.10) holds in any scale. Therefore, it holds in the metric \( g(t) \).

Let

\[
r_h = r_h(t) = \inf_{s \leq t} \inf_{x \in M_s} r_h(x).
\]

From the lapse equation,

\[
\Delta N = N|K|^2 - 1.
\]

Using (3.3) and \( |K|^2 = H^2/3 + |\hat{K}|^2 \) we find

\[
0 \leq \Delta N \leq 3 \frac{|\hat{K}|^2}{H^2} \leq 3 \frac{|K|^2}{H^2} \leq 3 \Lambda.
\]
Thus, we have
\[ r_h \| \nabla N \|_{L^\infty(B')} \leq 3C_0 \left( r_h^2 \Lambda + \frac{1}{H^2} \right). \]  
(3.11)

In particular, this gives the estimate
\[ \| \nabla N(t) \|_{L^\infty} \leq C(\Lambda, t_*)/r_h(t). \]  
(3.12)

Integrating (3.11), (recall we absorbed the term \( \Lambda/|H| \) in (3.4) into the constant), gives
\[ Q(t_1) \leq Q(t_0) + C \int_{t_0}^{t_1} ds \| \nabla N(s) \|_{L^\infty} Q(s). \]  
We may without loss of generality assume the last term is bigger than 1, so we may absorb \( Q(t_0) \) into \( C \).

Multiplying both sides by \( r\), and using (3.11) we have
\[ r_h Q(t_1) \leq C \int_{t_0}^{t_1} ds \left( r_h^2 \Lambda + \frac{1}{H^2(s)} \right) Q(s). \]  
We may without loss of generality assume \( r_h \leq 1/|H| \), since otherwise there would be nothing to prove, and therefore we can absorb the term \( r_h^2 \Lambda \) into the constant. Then we have
\[ r_h Q(t_1) \leq C \int_{t_0}^{t_1} ds \frac{1}{H^2(s)} Q(s). \]  
The inequality (3.13) implies
\[ N \geq 3 \Lambda^{-1} H^{-2}, \]
which by the definition of the spacetime Bel–Robinson energy \( Q(F_{[t_0, t_1]}) \), see (1.1), gives
\[ r_h(t_1)Q(t_1) \leq C Q(F_{[t_0, t_1]}). \]  
(3.13)

By assumption, \( \limsup_{t \nearrow t_*} Q(t) = \infty \), which by the assumed bound on \( Q(F_{[t_0, t_1]}) \) implies
\[ \lim_{t \nearrow t_*} r_h(t) = 0. \]
We will show that this contradicts (3.1).

Suppose then that there is an increasing sequence of times \( t_i, t_i \to t_* \) as \( i \to \infty \), so that \( r_i = r_h(t_i) \) satisfy \( \lim_{i \to \infty} r_i = 0 \) (recall that by construction \( r_h(t) \) is decreasing).

Now we have from (3.13) and our assumptions,
\[ r_i Q(t_i) \leq C. \]  
(3.14)

Now we introduce the blowup scale. Let \( \tilde{g}' = r_i^{-2} \tilde{g} \). We will denote the scaled versions of \( g, K \) by \( g'_i, K'_i \). We scale the coordinates as \( t'_i = r_i^{-1} t, x'_i = r_i^{-1} x' \), so that the coordinate components of \( \tilde{g}'_i \) are scale invariant. Then \( |K'_i| \leq \Lambda r_i \), while the lapse \( N \) does not scale, \( N'_i(x'_i) = N(x) \). After translating the time coordinate as \( t'_i = r_i^{-1}(t - t_i) \), we focus our attention
on the time interval $t_i' \in [-1, 0]$. We further translate the space coordinate so that the center of the coordinate system $(0,0)$ is the point where the harmonic radius achieves its minimum value.

Since $r_h \mathcal{Q}$ is scale invariant, we have $r_h \mathcal{Q} = \mathcal{Q}'$, and hence the inequality

$$\mathcal{Q}'(t_i') \leq C$$

(3.15)

holds. This means in view of the definition of the Bel–Robinson energy that at the blowup scale, $\text{Ric}'_i$ is bounded in $L^2$. By construction $r_h' \geq 1$ and by [4], it follows from the Ricci bound that $g'_i$ is bounded in $L^2_{\text{loc}}$. Similarly the Hodge system relating $K$ to $B$ leads to $K'_i$ bounded in $L^1_{\text{loc}}$.

The Einstein vacuum equation is scale invariant, and therefore holds at the blowup scale. We will argue in the next section, that the above bounds on $g'_i, K'_i$ allow us to pick out a weakly convergent subsequence of $(g'_i, N'_i)$ with limit $g_{\infty}, N_{\infty}$ solving the static vacuum Einstein equation, cf. equation (1.2) below, with $g_{\infty}$ complete.

### 3.2. Weak convergence.

Let $\tilde{g}'_i$ be the sequence of rescaled spacetime metrics. We consider rescaled time $t_i'$ in the interval $[-1,0]$. By construction, the $L^{1,\infty}$ harmonic radius satisfies $r'_i(0) = 1$, and $r'_i(t) \geq 1$ for $t \in [-1,0]$. Equation (3.16) implies that the rescaled lapse is bounded from above and below,

$$\frac{1}{t^2_i \Lambda} \leq N'_i \leq \frac{3}{t_i^2}.$$  

(3.16)

By (3.15), we have $\mathcal{Q}'(t) \leq C$, for $t \in [-1,0]$, and hence we have $(g'_i, K'_i)$ bounded in $L^\infty([-1,0]; L^2_{\text{loc}} \times L^1_{\text{loc}})$. It follows that there is a subsequence which converges weak-\(\ast\) to a limit $(g_{\infty}, K_{\infty}) \in L^\infty([-1,0]; L^2_{\text{loc}} \times L^1_{\text{loc}})$, with corresponding spacetime metric $\tilde{g}_{\infty}$. By passing to a further subsequence if necessary, which we still denote using the index $i$, we may assume that $g'_i(0) \rightarrow g_{\infty}(0)$ weakly in $L^2_{\text{loc}}$.

Let us consider the properties of this limit. First note that $|K'_i| \leq \Lambda r_i \rightarrow 0$ as $i \rightarrow \infty$, and hence $K_{\infty} \equiv 0$. The relation $\partial_t g'_i = -2N'_i K'_i$ holds in the limit and since $K_{\infty} \equiv 0$, we conclude that $g_{\infty}$ is time independent, so that the limiting spacetime metric $\tilde{g}_{\infty}$ is static. The lapse equation now implies that the limiting lapse function satisfies

$$\Delta_\infty N_{\infty} = 0,$$  

(3.17a)

where $\Delta_\infty$ is the Laplace operator defined w.r.t. $g_{\infty}$.

The rescaled spacetime metrics $\tilde{g}'_i$ are solutions of the Einstein vacuum equation and the evolution equation for $K$,

$$\partial_t K = -\nabla^2 N + N(\text{Ric} + H K - 2K : K),$$

where $K : K_{ab} = K_{ac}K^c_b$, holds weakly in the limit. In view of the fact that $g \mapsto \text{Ric}$ is weakly continuous on $L^2_{\text{loc}}$ and $K_{\infty} \equiv 0$, we get the equation

$$0 = -\nabla^2_{\infty} N_{\infty} + N_{\infty} \text{Ric}_{\infty}.$$  

(3.17b)
By construction, \( g_\infty \) is complete, and hence in view of (3.17) we have a complete solution of the static Einstein equations with \( N_\infty > 0 \). It follows by [2, Theorem 3.2] that \( g_\infty \) is flat and \( N_\infty \) is constant. In particular, \( r_h[g_\infty](0) = \infty \). Now, since \( r_h \) is by definition the \( L^{1,p} \) harmonic radius, \( 3 < p < 6 \), the map \( g \to r_h \) is weakly continuous on \( L^{2,2}_\text{loc} \) and hence by construction \( r_h[g_\infty](0) = 1 \). This is a contradiction, and it follows that in fact \( \lim \inf_{i \to \infty} r_i > 0 \), which by the BR energy estimate (3.14) implies that \( Q(t) \) does not blow up. This completes the proof of Theorem 1.1.

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