Emmy Noether looks at the deconfined quantum critical point

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Noether’s theorem is one of the fundamental laws of physics, relating continuous symmetries and conserved currents. Here we explore the role of Noether’s theorem at the deconfined quantum critical point (DQCP), which is the quantum phase transition beyond the Landau-Ginzburg-Wilson paradigm. It was expected that a larger continuous symmetry could emerge at the DQCP, which, if true, should lead to emerged conserved current at low energy. By identifying the emergent current fluctuation in the spin excitation spectra, we can quantitatively study the current-current correlation in large-scale quantum Monte Carlo simulations. Our results reveal the conservation of the emergent current, as signified by the vanishing anomalous dimension of the current operator, and hence provide supporting evidence for the emergent symmetry at the DQCP. Our study demonstrates an elegant yet practical approach to detect emergent symmetry by probing the spin excitation, which could potentially guide the ongoing experimental search for DQCP in quantum magnets.

Introduction.- Noether’s theorem is a profound theorem in physics that states every continuous (differentiable) symmetry of a physical system is associated with a corresponding conservation law [1]. Well-known examples include the angular momentum, linear momentum and energy conservations, when the system respects rotation, space and time translation symmetries. The conservation law usually manifests itself in the form of a conserved current $J_\mu$, which satisfies the equation of current conservation $\partial_\mu J_\mu = 0$. Likewise, the observation of a conserved current in a physical system usually serves as a direct evidence of the associated continuous symmetry.

In this paper, we introduce an explicit application of the Noether’s theorem in identifying the emergent continuous symmetry in a case study of exotic quantum phase transition – the deconfined quantum critical point (DQCP), which is the quantum phase transition beyond the Landau-Ginzburg-Wilson paradigm. The DQCP describes a direct continuous transition between two symmetry breaking phases that spontaneously breaks very different symmetries. In particular, we focus on a type of DQCP which is only recently identified in quantum Monte Carlo (QMC) simulations [6, 7], dubbed the easy-plane DQCP. It is a direct quantum phase transition in a (2+1)D quantum spin model between the antiferromagnetic XY (AFXY) ordered phase and the columnar valence bond solid (VBS) phase. The AFXY phase spontaneously breaks the in-plane $U(1)$ spin rotation symmetry, while the VBS phase spontaneously breaks the $Z_4$ lattice rotation symmetry. In the conventional Landau-Ginzburg-Wilson (LGW) paradigm, without fine-tuning, one would expect an intermediate phase in which the AFXY and VBS order parameters either coexist or both vanish. The DQCP goes beyond the LGW paradigm [2–4] where both $U(1)$ and $Z_4$ symmetries are restored at and only at the transition point. Remarkably the $U(1) \times Z_4$ microscopic symmetry could further be enlarged into an emergent $O(4)$ symmetry that governs the low-energy critical fluctuations. The emergent $O(4)$ symmetry, if exists, is the hallmark of the easy-plane DQCP [8–10].

FIG. 1. (a) The easy-plane J-Q (EPJQ) model and its phase diagram. The $Q$ term describes the three-dimmer interaction in both horizontal and vertical directions, with the arrangement of site indices shown on the right. (b) Obtained spin excitation spectra in $S^x$ channel of the EPJQ model at the DQCP in our previous work [7]. Darker color indicates higher intensity. The high symmetry points in the Brillouin zone (BZ) are defined on the right.

Several methods have been developed to test the emer-
gent symmetry at the DQCP, including comparing the critical exponents in spin and VBS channels [6, 11–14], plotting the order parameter histograms [5, 11, 15–20], and probing the degenerated correlation spectra of spin-0 and spin-1 excitations [21, 22]. But the conserved Noether current that is directly associated with the emergent continuous symmetry [23] has not been measured. In this work, we directly probe the SO(4) current fluctuation at the easy-plane DQCP and justify the current conservation by measuring the scaling dimensions of the current-current correlations.

Based on the field theoretical analysis, which captures the emergent O(4) symmetry of the DQCP, we are able to identify different components of the SO(4) current operator in terms of the original spin operator at different moment points. We then measure imaginary-time correlation functions of these spin fluctuations at the designated momenta in large-scale QMC simulations to study the scaling behavior of the current operator. The conservation of the Noether current \( \partial \mu J_\mu = 0 \) implies that the flux of the current \( \int_{\partial \Omega} e^{\mu \lambda} J_\mu dx_\lambda \) through a close manifold \( \partial \Omega \) in the \((2+1)\)D spacetime must remain constant. As a result, the conserved current should follow a precise scaling law \( J_\mu \sim x^{-\gamma_1} \) with the scaling dimension pinned at 2, an integer instead of a usually fractional critical exponent [24]. By a systematic finite size scaling of the numerical data, we are able to make quantitative comparison of the measured scaling dimensions with field theoretical expectation.

Our results consistently reveal the presence of the emergent O(4) symmetry at the easy-plane DQCP, also suggests that at the present value of easy-plane anisotropy the transition is actually of exotic weakly first order type [25]. More importantly, our attempt of bridging the conserved current correlation from basic law of physics with large-scale numerical calculation of quantum many-body systems, provides a complementary and elegant way of identifying the emergent continuous symmetry at quantum phase transitions (not necessarily to be continuous), which are becoming ubiquitously present in the new paradigms of quantum matter such as various forms of DQCP [2–6, 14–18, 26, 27], frustrated magnets [19, 28–30], interacting topological phases [14, 31–33] and quantum electrodynamics systems [10, 30]. Our numerical study of the conserved current at an emergent O(4) symmetric DQCP will also guide future spectroscopy experiments, neutron scattering for example, in search of DQCP in candidate materials, such as the Shnasy-Sutherland lattice compound SrCu_2(BO_3)_2 [22].

Model and Symmetry.- The easy-plane DQCP was reported in our previous QMC simulation of the easy-plane J-Q model [6],

\[
H_{\text{EPJQ}} = -J \sum_{\langle ij \rangle} (P_{ij} + \Delta S_i^z S_j^z) - Q \sum_{\langle jklmn \rangle} P_{ij} P_{kl} P_{mn}, \tag{1}
\]

where \( P_{ij} = \frac{1}{4} - S_i \cdot S_j \) is the singlet-projection operator between nearest-neighbor sites and \( S_i \) denotes the spin-1/2 operator on each site \( i \). The two- and six-spin terms are both illustrated in Fig. 1 (a). At \( \Delta = 0 \), the model goes back to the SU(2)_spin J-Q_3 model [11, 34, 35]. The three singlet projectors strengthen the VBS state while keeping \( J > 0 \) in sign-free QMC simulations. The term \( \Delta S_i^x S_j^x \) with \( \Delta \in \{0, 1\} \) introduces the easy-plane anisotropy that breaks the SU(2)_spin symmetry down to U(1)_spin. In Ref. [6] it is shown that at \( \Delta = 1/2 \), the EPJQ model exhibits a direct quantum phase transition between the AFXY and VBS phases at \( q_c = \frac{Q}{2J} = 0.62(1) \), realizing the easy-plane DQCP. As argued based on dualities [8, 10], the critical point is expected to exhibit an O(4) symmetry at low-energy, which rotates the two-component XY order parameter and the two-component VBS order parameter together as an O(4) vector.

The easy-plane DQCP can be described by the \( N_f = 4 \) quantum electrodynamics (QED) theory with an easy-plane anisotropy, as proposed in Refs. [3, 4, 6, 7, 10, 23, 31, 36]. In this description, the spin operator \( S_i \) is first fractionalized into two fermionic spinons \( f_i = (f_i^\uparrow, f_i^\downarrow)^\tau \) following \( S_i = \frac{1}{2} f_i^\dagger f_i \), under the constraint \( f_i^\dagger f_i = 1 \) on each site \( i \). An emergent U(1) gauge structure (generated by the gauge transformation \( f_i \rightarrow e^{i \theta_i} f_i \)) arises from the fractionalization [37]. At the DQCP, the system can be viewed as a U(1) Dirac spin liquid, where the spinons are placed in a Dirac dispersion modeled by the square-lattice \( \pi \)-flux state [7, 23, 30, 38]. Specific spinon interactions will also be introduced to implement the easy-plane anisotropy. At the mean-field level (ignoring the gauge fluctuation and the spinon interaction for a moment), the spinon is described by \( \pi \)-flux model Hamiltonian

\[
H_{\text{MF}} = \sum_i f_i^{\dagger} f_i^{\dagger} f_i + (-)^x f_i^{\dagger} f_{i+\hat{x}} f_i + \text{h.c.} + \cdots, \tag{2}
\]

where each site \( i \) is equivalently labeled by its coordinate \( r_i = (x_i, y_i) \) on the square lattice, and \( \hat{x} = (1, 0) \) and \( \hat{y} = (0, 1) \) are the lattice vectors. We take a \( 2 \times 2 \) unit cell with the sublattices labeled by \( A(0, 0), B(0, 1), C(1, 0), D(1, 1) \) and define the spinon operator in the momentum space as \( f_k = (f_{kA}, f_{kB}, f_{kC}, f_{kD})^\tau \), where \( f_{kA} = \sum_{i \in A} f_i e^{-ik \cdot r_i} \) (and similar for other sublattices) and \( k \in [0, \pi] \times [0, \pi] \). In the momentum space, the mean-field Hamiltonian \( H_{\text{MF}} \) in Eq. (2) becomes

\[
H_{\text{MF}} = 2 \sum_k f_k^{\dagger} (\cos k_x \sigma^{100} + \cos k_y \sigma^{310}) f_k, \tag{3}
\]

where \( \sigma^{\mu\nu\lambda} = \sigma^\mu \otimes \sigma^\nu \otimes \sigma^\lambda \) (with \( \mu, \nu, \lambda = 0, 1, 2, 3 \)) denotes the direct product of Pauli matrices. The Hamiltonian in Eq. (3) describes a spinon band structure with four degenerated Dirac cones at the momentum \( k = (\pi/2, \pi/2) \), matching the matter content of the \( N_f = 4 \) QED theory [30]. In Ref. [7], it is explicitly shown that
the spectral properties at the easy-plane DQCP can be qualitatively captured by the spinon mean-field theory in Eq. (3). Ref. [7] also shows that gauge fluctuations and spinon interactions must be included to obtain quantitatively correct spin excitation spectra. Fig. 1 (b) depicts the there-obtained QMC spin spectra along the high-symmetry path in BZ.

| Q       | lattice model | field theory |
|---------|---------------|--------------|
| (0, 0)  | (σ^{300}_x, σ^{300}_y, σ^{300}_z) | (J_{15}, J_{27}, J_{35}) |
| (π, 0)  | (σ^{301}_x, σ^{302}_y, σ^{303}_z) | (J_{14}, J_{23}, J_{34}) |
| (0, π)  | (σ^{311}_x, σ^{332}_y, σ^{333}_z) | (n^1, n^2, n^3) |
| (π, π)  | (σ^{311}_x, σ^{332}_y, σ^{333}_z) | (n^1, n^2, n^3) |

TABLE I. Identification of the low-energy spin operators to the field theory operators.

Conserved currents.- It was found in Ref. [7] that gapless continua appear in the spin excitation spectra at four distinct momenta \( Q = (0, 0), (π, 0), (0, π) \) and \((π, π)\). Spin operators at these momenta can be written in terms of spinon bilinear operators,

\[
S_Q = \sum_i S_i e^{-i Q \cdot r_i} = \frac{1}{2} \sum_k f^\dagger_k s_Q f_k,
\]

where the matrix forms of \( s_Q \) are listed in Tab. I for various \( Q \). In particular, \( S_{(π, π)} \) corresponds to the Néel order parameter. The VBS order parameter \( D = \sum_i D_i = \sum_i (D^x_i, D^y_i) \) can also be written in terms of spinon bilinear operators,

\[
D^x_i = \frac{1}{2} (-\gamma^3_i f^\dagger_{i+\hat{x}} f_i + h.c.,
\]

\[
D^y_i = \frac{1}{2} (-\gamma^3_i + \gamma^y_i f^\dagger_{i+y} f_i + h.c.,
\]

which, in the momentum space, reads

\[
D = \sum_k f^\dagger_k (\sin k_x \sigma^{200}, \sin k_y \sigma^{320}) f_k.
\]

We focus on the low-energy spinon \( f \) and expand the spinon band structure in Eq. (3) around the Dirac point at \( k = (π/2, π/2) \). The low-energy effective theory turns out to be a \( N_f = 4 \) QED theory described by the following Lagrangian,

\[
\mathcal{L} = \bar{f} \gamma^\mu (\partial_\mu - i a_\mu) f + \mathcal{L}_{\text{int}} + \cdots,
\]

where \( (\gamma^0, \gamma^1, \gamma^2) = (σ^{210}, σ^{310}, σ^{100}) \) and \( f = f^\dagger γ^0 \). \( \mathcal{L}_{\text{int}} \) contains the spinon interaction to be specified later. One can find the following five matrices \( (Γ^1, Γ^2, Γ^3, Γ^4, Γ^5) = (σ^{121}, σ^{122}, σ^{123}, σ^{010}, σ^{130}) \) in the spinon single-particle space, which anticommute with each other and all commute with the \( γ^\mu \) matrices. They can be used to construct an \( O(5) \) vector \( n = \bar{f} \Gamma f \) at the field theory level. The physical meaning of the \( O(5) \) vector is a combined order parameter of the three-component Néel and two-component VBS order parameters,

\[
(n^1, n^2, n^3) = f (Γ^1, Γ^2, Γ^3) f = f^\dagger (σ^{311}, σ^{332}, σ^{333}) f \sim S_{(π, π)},
\]

\[
(n^4, n^5) = f (Γ^4, Γ^5) f = f^\dagger (σ^{200}, σ^{320}) f \sim D.
\]

The SO(5) group that rotates the \( O(5) \) vector \( n \) is generated by ten generators \( Σ^{ab} = \frac{1}{\pi}[Γ^a, Γ^b] \) \((a, b = 1, \cdots, 5)\). This SO(5) group is expected to be an emergent symmetry for the spin SU(2) symmetric DQCP [3–5, 8–10, 14, 15]. According to Noether’s theorem, there exist ten emergent conserved currents associated to SO(5) symmetry generators as

\[
J^{ab}_\mu = f γ^\mu Σ^{ab} f.
\]

However, in our case, the emergent SO(5) symmetry is broken down to O(4) explicitly by the easy-plane anisotropy, which singles out \( n^5 \) (Néel z-component) as a special component. This amounts to adding the interaction \( \mathcal{L}_{\text{int}} = \frac{2}{\pi} f \langle f^\dagger Σ^5 f \rangle^3 \) to the Lagrangian in Eq. (7) explicitly (where a momentum cutoff \( Λ \) is introduced to render the coupling \( g \) dimensionless). The remaining six SO(4) conserved currents are just a subset of \( J^{ab}_\mu \) for \( a, b = 1, 2, 4, 5 \). With this setup, we can identify the current operators \( J^{ab}_\mu \) in the field theory to the spin operator \( S_Q \) by matching their matrix representation \( s_Q \) of spinon bilinear forms, as summarized in Tab. I.

In particular, we focus on the emergent conserved current \( J^{15}_\mu \), which is associated to an emergent rotation symmetry between AFXY and VBS order parameters. Using the spinon representation, we found \( J^{15}_\mu = f γ^2 Σ^{15}_\mu f = \frac{1}{2} f σ^{100} [σ^{210}, σ^{130}] f = \frac{1}{2} f^\dagger σ^{301} f \) corresponds to the \( S^{x}_{(π, 0)} \) spin fluctuation (as \( S^{x}_{(π, 0)} = σ^{301} \)), which are measured in QMC simulation. For comparison, we also study another conserved current \( J^{12}_\mu \) in association with the microscopic U(1) spin rotation symmetry, which appears as the \( S^{x}_{(0, 0)} \) spin fluctuation following the similar derivation.

Numerical Results.- Suppose the easy-plane DQCP has the proposed SO(4) emergent symmetry, this will put a strong constraint on the correlation function of current operators, and the consequence can be tested in our QMC simulation. For an emergent conserved current \( J^{15}_\mu \), its correlation function will be universally given by

\[
\langle J^{15}_{\mu} J^{15}_{\nu} \rangle ≈ \frac{|q|}{q^2} \left( δ_{\mu ν} - \frac{q_\mu q_\nu}{|q|^2} \right), (a, b = 1, 2, 4, 5),
\]

which will not receive corrections from gauge fluctuations and spinon interactions. Here \( q = (q_0, q) = (q_0, q_1, q_2) \) stands for the Euclidian frequency-momentum vector and \( |q|^2 = q_0^2 + q_1^2 + q_2^2 \). Using the operator correspondence in Tab. I, the current-current correlation in the field theory can be translated to the spin-spin correlation in the
lattice model. For example,

\[ \langle S^x(\pi, 0) S^x(\pi, 0) \rangle \sim \langle J_{15}^z J_{15}^z \rangle \sim (q^2 + q^4)/|q|^{1-\eta_{(\pi, 0)}}, \]
\[ \langle S^z(0, 0) S^z(0, 0) \rangle \sim \langle J_{0}^{12} J_{0}^{12} \rangle \sim (q^2 + q^4)/|q|^{1-\eta_{(0, 0)}}. \] (11)

We have introduced two anomalous exponents \( \eta_{(\pi, 0)} \) and \( \eta_{(0, 0)} \) for general considerations. They will vanish separately if their corresponding currents are indeed conserved. In particular, the vanishing \( \eta_{(\pi, 0)} \) will be non-trivial, as it corresponds to the conservation of the emergent current \( J_{15}^z \) of AFXY-VBS rotation, which is not expected on the microscopic level. Thus measuring the anomalous exponents of Noether currents provide a clear answer: only a single exponent is needed to give an confirmation about whether certain enlarged symmetry indeed emerges at low-energy. This is different from measuring the non-vanishing anomalous exponents of the order parameters (Eq. (8)) in previous works for SO(4) [6, 16] and SO(5) [5, 11, 13, 35, 39] cases, where one needs to compare exponents of different order parameters to determine the emergent symmetry. The conserved current correlation therefore offers another independent probe of emergent continuous symmetry at the DQCP [6, 7, 30], which is complementary to previous approaches such as the order parameter histogram [5, 15, 18].

In QMC simulations, the spin-spin correlation \( G_{\alpha}^Q(\tau, q) = \sum_i \langle S_i^\alpha(\tau) S_i^\alpha(0) \rangle e^{i(Q+q)(r_i-r_j)} \) (for \( a = x, y, z \)) can be directly measured in the imaginary time domain and the momentum space. In order to make comparison with the numerics, we need to Fourier transform the previous field theory predictions in Eq. (11) from Matsubara frequency to imaginary time, following \( G_{\alpha}^Q(\tau, q) = \int dq_0 e^{-i q_0 \tau} \langle S_i^\alpha(\tau) S_i^\alpha(0) \rangle \). The results are

\[ G_{(\pi, 0)}(\tau, q) \sim q^2 F_2(\tau, q) + q^{\eta_{(\pi, 0)}} F_{\eta_{(\pi, 0)}+1}(\tau, q), \]
\[ G_{(0, 0)}(\tau, q) \sim q^2 F_2(\tau, q) + q^{\eta_{(0, 0)}} F_{\eta_{(0, 0)}+1}(\tau, q), \] (12)

where \( F_\alpha(\tau, q) = \frac{2\pi}{\alpha} K_\alpha(|q|\tau) \) and \( K_\alpha(|q|\tau) \) is the \( \alpha \)-th order Bessel K-function (detailed derivations of Eq. (12) are given in Sec.1 of Supplemental Material (SM) [40]). The anomalous dimension \( \eta_{(\pi, 0)} \) and \( \eta_{(0, 0)} \) are fitting parameters to be determined from the data. The numerical determination of these exponents from finite size QMC results will be the focus of narrative below.

Fig. 2 (a) and (b) depict the imaginary time correlations \( G_{(\pi, 0)}(\tau, q = 0) \) and \( G_{(0, 0)}(\tau, q) \), respectively. We note that around \( Q = (\pi, 0) \) the spin-spin correlation remains finite, so we take the QMC measurements at \( (\pi, 0) \); whereas around \( Q = (0, 0) \) the spin-spin correlation vanishes with \( q \), so we take QMC measurements at a small momentum deviation \( \frac{2\pi}{L} \) away from \( (0, 0) \). Nevertheless, the momentum deviation \( q \) in the fitting formula Eq. (12) is still treated as a fitting parameter (of the order \( \sim \frac{2\pi}{L} \)) to partially take care of the finite-size effect. One can see that for the system sizes considered, \( L = 32, 48, 64 \)
and 96 (the others are not shown), the Bessel functions in Eq. (12) fit the data well. In Fig. 2 (a) and (b), we fit the imaginary time data with \( \eta_{(\pi,0)} \) and \( \eta_{(0,0)} \) as free fitting parameters. Because the short (imaginary-)time data includes significant contributions from high energy excitations, for which the fitting function is no longer valid, we dynamically choose the fitting range starting from an appropriate short-time cutoff such that \( \chi^2 \) of the fitting is close to one. After fitting all system sizes from \( L = 16 \) to \( L = 96 \), the scaling dimensions \( \eta_{(\pi,0)} \) and \( \eta_{(0,0)} \) are obtained, and their finite size scaling are given in Fig. 3.

The extrapolated values of the fitted scaling dimensions, as shown in Fig. 3, converges to zero for infinite size within numerical errors. With the system size up to \( L = 96 \), we obtain \( \eta_{(\pi,0)} = 0.002(9) \) and \( \eta_{(0,0)} = 0.004(6) \), see the inset of Fig. 3, indicating that the current \( J_2^{15} \) and \( J_0^{12} \) are conserved. The conservation of \( J_0^{12} \) is just a consequence of the spin \( U(1) \) symmetry, but the conservation of \( J_2^{15} \) is a remarkable observation in favor of the emergent \( O(4) \) symmetry at the easy-plane DQCPC.

The absence or presence of emergent continuous symmetry at DQCPC has been the central topic of intensive investigations [5–7, 11, 12, 15, 39, 41, 42], yet direct measurements of the emergent continuous symmetry is hard to verify. The order parameter histogram, serves as qualitative probe, can distinguish strong first transition from the all other possibilities. But the other possibilities still include first order transition (weakly) [43, 44], continuous transition (the original scenario of DQCPC), or, simply one is mislead by the finite size effect of the Monte Carlo simulation itself. The measurements of the critical exponents associated with the order parameters (in Eq. (8)), on the other hand, suffers from finite size scaling corrections [35, 41], where non-trivial finite size analysis methods are still under development [13]. In this sense, the probe of scaling dimensions \( \eta_{(\pi,0)} \) and \( \eta_{(0,0)} \) of the conserved currents, provides a numerically complementary way of demonstrating the emergent continuous symmetry at the DQCPC [10, 30].

Discussions.- Lastly, let’s discuss the extrapolation of the order parameters at the DQCPC. As shown in Fig. 4, the AFXY and VBS order parameters \( \langle m_{xy}^2 \rangle \) and \( \langle D^2 \rangle = \frac{1}{2} \langle (D_x^2 + D_y^2) \rangle \) are measured for various system sizes at their corresponding Binder ratio crossings, and the largest system size is \( L = 96 \). Inset shows the histogram of extrapolated \( m_{xy}^2 = 0.0009(2) \) and \( D^2 = 0.0010(3) \).

To conclude, we have successfully demonstrated Noether’s theorem in action in the frontier research of quantum matter, helping us to identify the emergent \( SO(4) \) continuous symmetry at easy-plane DQCPC. Our attempt stands out as a new numerical tool in identifying emergent continuous symmetry, ubiquitously present at novel quantum phase transitions in DQCPC, frustrated magnets, interacting topological phases and quantum electrodynamic systems. Comparing with the analyses of order parameter histogram and critical exponents, our approach provides a complementary view point both in numerical accessibility and theoretical elegance.

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SUPPLEMENTAL MATERIAL

Emmy Noether looks at the deconfined quantum critical point

In this supplemental material, we provide the technical details concerning the QMC measurements of conserved currents (Sec. SI) and the finite size scaling of the order parameters at the easy-plane DQCP (Sec. SII).

SI. Conserved and non-conserved currents

In this section, we will give a detailed derivation of the fitting function we proposed in Eq. (12) in the main text. Let us start with the more general form of the current-current correlation with an anomalous exponent $\eta$,

$$\langle J_{\mu}^{ab}(-q)J_{\nu}^{ab}(q) \rangle \sim |q|^{1+\eta} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{|q|^2} \right) = \frac{|q|^2\delta_{\mu\nu} - q_\mu q_\nu}{|q|^{1-\eta}}$$  \hspace{1cm} (S1)

without the assumption that the current is conserved. The result can later be applied to the case of conserved current by setting $\eta = 0$.

To make connection to the imaginary time data measured in the QMC simulation, we need to Fourier transform $\langle J_{\mu}^{ab}(-q)J_{\nu}^{ab}(q) \rangle$ from the imaginary frequency $q_0$ to the imaginary time $\tau$ domain (where $q = (q_0,q)$ contains both the imaginary frequency and the momentum components),

$$\Pi_{\mu\nu}(\tau,q) = \int dq_0 e^{-i q_0 \tau} \langle J_{\mu}^{ab}(-q_0)J_{\nu}^{ab}(q) \rangle. \hspace{1cm} (S2)$$

To carry out the Fourier transform, we use the mathematical fact that

$$\int dq_0 e^{-i q_0 \tau} \frac{1}{|q|^{1-\eta}} = \int dq_0 e^{-i q_0 \tau} \frac{1}{(q_0^2 + q^2)^{(1-\eta)/2}} = \frac{2\sqrt{\pi}}{\Gamma\left(\frac{1-\eta}{2}\right)} \frac{|q|^{\eta/2}}{2} K_{\frac{\eta}{2}}(|q|\tau) \hspace{1cm} (S3)$$

where the function $F_\alpha(\tau,q) = \frac{2\pi}{\sqrt{\pi}} K_{\alpha}(|q|\tau)$ was introduced to simplify the notation. With this we can evaluate the following

$$\Pi_{00}(\tau,q) = \int dq_0 e^{-i q_0 \tau} \frac{q^2}{|q|^{1-\eta}} = \frac{2\sqrt{\pi}}{\Gamma\left(\frac{1-\eta}{2}\right)} q^2 F_{\frac{\eta}{2}}(\tau,q), \hspace{1cm} (S4)$$

$$\Pi_{22}(\tau,q) = \int dq_0 e^{-i q_0 \tau} \frac{|q|^2 - q_0^2}{|q|^{1-\eta}} = \int dq_0 e^{-i q_0 \tau} \left( \frac{1}{|q|^{1-(\eta+2)}} - \frac{q_0^2}{|q|^{1-\eta}} \right)$$

$$= -\frac{2\sqrt{\pi}}{\Gamma\left(\frac{1-\eta}{2}\right)} \left( \pi \frac{\eta+1}{2} F_{\frac{\eta+1}{2}}(\tau,q) + q_0^2 F_{\frac{\eta}{2}}(\tau,q) \right). \hspace{1cm} (S5)$$

We will use these results later.

The current-current correlations are measured as spin-spin correlations around different momenta in different spin channels. We will focus on the following two correlations

$$G_0^a(\tau,q) = \int dq_0 e^{-i q_0 \tau} \langle S_{(0,0)}^a(-q)S_{(0,0)}^a(q) \rangle, \hspace{1cm} (S6)$$

$$G_{(\pi,0)}^a(\tau,q) = \int dq_0 e^{-i q_0 \tau} \langle S_{(\pi,0)}^a(-q)S_{(\pi,0)}^a(q) \rangle.$$
According to Tab. I in the main text, the operator $S_{a(0,0)}^0$ corresponds to the current $J_{0c}^b$ and the operator $S_{a(\pi,0)}^0$ corresponds to the current $J_{2c}^b$. So we can make the following identifications

$$G_{(0,0)}^a \sim \Pi_{00} \propto q^2 F_2(\tau, q),$$
$$G_{(\pi,0)}^a \sim \Pi_{22} \propto q_2^2 F_2(\tau, q) + \frac{n+1}{2} F_{2n+1}(\tau, q).$$

which lead to the fitting functions in Eq. (12) in the main text.

For conserved currents, we expect $\eta = 0$, thus

$$G_{(0,0)}^a(\tau, q) \propto q^2 K_0(|\tau|),$$
$$G_{(\pi,0)}^a(\tau, q) \propto q^2 K_0(|\tau|) + |q/\tau| K_1(|q\tau|).$$

Around $Q = (0, 0)$ the spin-spin correlation vanishes with $q$, so the measurement can only be done at a small momentum deviation $q$ away from $(0, 0)$. Around $Q = (\pi, 0)$, the spin-spin correlation remains finite as $q \to 0$, which takes the simple power-law form

$$G_{(\pi,0)}^a(\tau, q = 0) \propto |\tau|^{-2}.$$

### SII. Extrapolation of the order parameters

In this section, we provide details of the determination of the position $q_c(L)$ of the DQCP of EPJQ model, and explain how the extrapolation of the order parameters are performed in Fig. 4 in the main text.

![FIG. S1. Determination of the transition point between AFXY and VBS phases in the EPJQ model in Eq. (1) in main text. From the Binder ratios of $R^m$ and $R^D$ in Eq. (S10), the crossings of two consecutive $L$-s give $q_c(L)$.](image)

First we measured the Binder ratio for both AFXY order parameter and VBS order parameters, given as
Thermodynamic limit values of $q_c(\infty)$ can be obtained as the extrapolated value at $1/L = 0$.

$$R_m = \frac{\langle m^4_{xy} \rangle}{\langle m^2_{xy} \rangle^2}$$
$$R_D = \frac{\langle D^4 \rangle}{\langle D^2 \rangle^2}$$  \hspace{1cm} (S10)$$

where $m^2_{xy} = \frac{1}{2}(m_x^2 + m_y^2)$ with $m_x^2 = \frac{1}{L^2} \sum_i (-1)^{i_x+i_y} S_i^x)^2$ the square of magnetic moment along the $S^x$ with $(\pi, \pi)$ order on the lattice. And $D^2 = \frac{1}{2}(D_x^2 + D_y^2)$ with $D_x^2 = \frac{1}{L^2} \sum_i (-1)^{i_x} S_i \cdot S_{i+x}$ the square of the dimer singlet along the $x$-axis with the $(\pi, 0)$ order on the lattice.

Fig. S1 show the crossing of the two Binder ratios at some representative system sizes. As $L$ increases, the drift of the Binder ratio can be seen, both in $R_m$ and $R_D$. From the crossing of two consecutive $L$-s, for example $L$ and $2L$, the finite size transition point $q_c(L)$ can be determined. And the size dependence of $q_c(L)$ is given by

$$q_c(L) = q_c(\infty) + aL^{-1/\nu-\omega_1} + ...$$  \hspace{1cm} (S11)$$

where $q_c(\infty)$ is the transition point at the thermodynamic limit and $\nu$ is the correlation length exponent and $\omega_1$ is the correction exponent. We then fit the crossing points $q_c(L)$ from both $R_m$ and $R_D$ using Eq. (S11) and extrapolate to $q_c(\infty)$. In Fig. S2 we find the anisotropic behaviour of $q_c(L)$ from $R_m$, so one more term $a_2 L^{-1/\nu-\omega_2}$ in Eq. (S11) is included in the fitting [46] to take care of such higher order corrections. At the thermodynamic limit the two order parameters actually extrapolate to the same value within errorbar as $q_c = 0.61832(8)$ from $R_m$ and $q_c = 0.6187(5)$ from $R_D$. In Fig. 4 of the main text, we plot $m^2_{xy}$ and $D^2$ for each system size $L$ at their corresponding $q_c(L)$, obtained in Fig. S1 here, and then as $L \to \infty$, the value of the order parameters at the thermodynamic limit, with $L = 12, 24, 36, 48, \cdots, 84, 96$, are obtained.