JUCYS-MURPHY ELEMENTS
AND A SYMMETRIC FUNCTION IDENTITY

JENNIFER R. GALOVICH
St. John’s University
Collegeville, MN 56321

Abstract: Consider the elements of the group algebra \( C S_n \) given by
\( R_j = \sum_{i=1}^{j-1} (ij) \), for \( 2 \leq j \leq n \). Jucys [3-5] and Murphy[7] showed that these elements act diagonally on elements of \( S_n \) and gave explicit formulas for the diagonal entries. We give a new, combinatorial proof of this work in case \( j = n \) and present several similar results which arise from these combinatorial methods.

In a series of papers published early in the twentieth century, Alfred Young described three forms for the irreducible representations of the symmetric group \( S_n \) [12]. Among these, the seminormal form enjoys several nice properties:
(i) Matrices corresponding to adjacent transpositions can be computed explicitly and easily.
(ii) The representation restricts from \( S_n \) to \( S_{n-1} \) in block diagonal form with no change of basis required.
A.A. Jucys [3-5] and G.E. Murphy [7] gave a different construction by introducing elements of the group algebra \( C S_n \) which act diagonally, and from which Young’s seminormal form can be recovered. Moreover, the diagonal entries of these Jucys-Murphy elements are easy to describe.

In 1994, the late S. Kerov asked for a combinatorial proof of a certain symmetric function identity. [1]. As it happens, that identity is equivalent to the action of a particular Jucys-Murphy element. In this note, therefore, we first provide the requested combinatorial proof. Then we present several variations on the Jucys-Murphy theme which are suggested by these methods.

We begin with some definitions and notation. A partition of a positive integer \( n \) is a sequence \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \) where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \) and \( \sum \lambda_i = n \). We write \( \lambda \vdash n \). To each \( \lambda \vdash n \) we associate its Ferrers diagram \( F_\lambda \), a left- and top-justified array of squares or cells. More precisely, if \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \) then \( F_\lambda \) consists of \( k \) left-justified rows of lengths \( \lambda_1, \lambda_2, ..., \lambda_k \), reading from top to bottom. If \( x \) is a cell in \( F_\lambda \) located in row \( i \) and column \( j \) we define \( c(x) \), the content of \( x \), by \( c(x) = j - i \). (See Figure 1a.)

Note that by deleting any corner cell in \( F_\lambda \) we automatically obtain the Ferrers diagram of a partition of \( n-1 \). Partitions obtained in this way will be of particular interest; such a partition will be denoted \( \lambda - x \), where \( x \) names the deleted corner cell. Similarly, by adding a cell \( x \) to an unoccupied corner of
we obtain a partition of \( n + 1 \). A partition which arises in this way will be denoted \( \lambda + x \).

If \( \lambda \) and \( \mu \) are partitions we define the skew diagram \( \lambda - \mu \) to be the set theoretic difference \( F(\lambda) \setminus F(\mu) \). A rimhook is a skew diagram which contains no \( 2 \times 2 \) square as a subset. The length of a rimhook is the number of cells it contains; the height, \( ht \), is one less than the number of rows it occupies. (See Figure 1b.)

\[
\begin{align*}
\text{Figure 1a: } & F(\lambda) \text{ for } \lambda = (4, 3, 2) \\
\text{Figure 1b: } & \text{The rimhook } \lambda - \mu \text{ for } \lambda = (4, 3, 2) \text{ and } \mu = (2, 1, 1). \\
&\text{Length of } \lambda - \mu \text{ is 5; } ht(\lambda - \mu) = 2.
\end{align*}
\]

Let \( S_n \) denote the symmetric group on \( n \) letters. Throughout, we will write elements of \( S_n \) using cycle notation, identifying the cycle type of a permutation as the partition determined by the cycle lengths.

To each \( \lambda \vdash n \) we also associate \( \chi^\lambda \), the irreducible character of \( S_n \) corresponding to \( \lambda \). As functions on \( S_n \) the \( \chi^\lambda \)'s are constant on the conjugacy classes of \( S_n \); indeed, the collection \( \{ \chi^\lambda : \lambda \vdash n \} \) forms an orthonormal basis for \( CF_n \), the space of all class functions on \( S_n \).

We use \( \Lambda_n \) to denote the space of homogeneous symmetric functions of degree \( n \). Of the six standard bases for \( \Lambda_n \) two are important for the present work. They are \( \{ p_\lambda : \lambda \vdash n \} \) (the power sum symmetric functions) and \( \{ s_\lambda : \lambda \vdash n \} \) (the Schur functions), an orthonormal basis for \( \Lambda_n \). (We refer the reader to [6, 9, 10] for more detailed information about symmetric functions.)

Our main theoretical tool is the (Frobenius) characteristic map which relates the spaces \( CF_n \) and \( \Lambda_n \):

**Definition:** Let \( f \in CF_n \). The characteristic map \( \text{ch}^n : CF_n \to \Lambda_n \) is defined by

\[
\text{ch}^n(f) = \sum_{\mu \vdash n} f(\mu) \cdot \frac{p_\mu}{z_\mu}
\]

where \( p_\mu \) is the power sum symmetric function corresponding to \( \mu \) and \( z_\mu = m_1!^{m_1} m_2!^{m_2} \cdots m_k!^{m_k} \).

The map \( \text{ch}^n \) is an isometry; moreover, \( \text{ch}^n(\chi^\lambda) = s_\lambda \). [9, p. 163].
Given any $\sigma \in S_n$ and $2 \leq j \leq n$ we define the Jucys- Murphy element $R_j(\sigma) \in \mathbb{C}S_n$ by

$$R_j(\sigma) = \sum_{i=1}^{j-1} \sigma \cdot (i, j).$$

The Jucys-Murphy elements $R_j$ have many interesting properties and applications [2]. Chief among these is the fact that for any $\lambda \vdash n$ the character

$$\chi^\lambda(R_j(\sigma)) \equiv \sum_{i=1}^{j-1} \chi^\lambda(\sigma \cdot (i, j))$$

can be computed easily in terms of the contents of the cells of $F_\lambda$ [7]. In case $j = n$, that theorem can be formulated as follows:

**Theorem 1 [Murphy, 7].** Let $\sigma \in S_n$ with $\sigma(n) = n$. Let $\sigma'$ denote the restriction of $\sigma$ to $\{1, 2, \ldots, n-1\}$. For any $\lambda \vdash n$ we have

$$\chi^\lambda(R_n(\sigma)) = \sum_{i=1}^{n-1} \chi^\lambda(\sigma \cdot (i, n)) = \sum_x (1 + c(x)) \cdot \chi^\lambda - x(\sigma)$$

where the sum on the right is taken over all corner cells $x$ of $F_\lambda$.

Remark: A slightly different formulation of (1) follows immediately from the Branching Rule [9, p. 77]:

$$\chi^\lambda(R_n(\sigma) + \chi^\lambda(\sigma) = \sum_{i=1}^{n} \chi^\lambda(\sigma \cdot (i, n)) = \sum_x (1 + c(x)) \cdot \chi^\lambda - x(\sigma)$$

**Proof of Theorem 1.** Note that $\chi^\lambda(R_n(\sigma))$ depends only on the cycle type $\mu$ of $\sigma$, for $(\sigma \cdot (i, n)$ has the same cycle structure as $\sigma$ except that the cycle in $\sigma$ containing $i$ is augmented by inserting $n$. For example, let $\sigma = (253)(1)(4) \in S_6$. Then $\sigma' = (253)(1)(4)$ and has cycle type $\mu = 1^23^1$. In this case

$$R_6 = (2653)(1)(4) + (2563)(1)(4) + (2536)(1)(4) + (253)(16)(4) + (253)(1)(46).$$

Of these five summands, two have type $1^12^13^1$ and three have type $1^23^04^1$. In general, if the cycle type of $\sigma$ is $\mu = 1^{m_1}2^{m_2} \cdots k^{m_k}$ then among the $n-1 = \sum j \cdot m_j$ summands of $R_n(\sigma)$ there are $j \cdot m_j$ with cycle type $\tilde{\mu} = 1^{m_1}2^{m_2} \cdots j^{m_j-1}(j+1)^{m_{j+1}} \cdots k^{m_k}$.

When $\tilde{\mu}$ is obtained from $\mu$ in this way, by replacing an existing part of size $j$ with a part of size $j + 1$, we write $\tilde{\mu} > j \mu$. Thus

$$\sum_{i=1}^{n-1} \chi^\lambda(\sigma \cdot (i, n)) = \sum_{j \geq 1} \sum_{\tilde{\mu} > j \mu} \chi^\lambda(\tilde{\mu}) \cdot (j \cdot m_j).$$
Since the right side of (1) can also be viewed as a function of $\mu$, the key is to apply the Frobenius characteristic function to both sides.

Working first with the left side of (1), let $F(\mu) = \sum_{\mu} \sum_{\hat{\mu} \succ j \mu} \chi^\lambda(\hat{\mu}) \cdot (j \cdot m_j)$. Then

$$\text{ch}^n(F) = \sum_{\mu} F(\mu) \cdot \frac{p_{\mu}}{z_{\mu}}$$

$$= \sum_{\mu} \left[ \sum_{\hat{\mu} \succ j \mu} \chi^\lambda(\hat{\mu}) \cdot (j \cdot m_j) \right] \cdot \frac{p_{\mu}}{z_{\mu}}$$

Note that $\hat{\mu} > j \mu$ implies

$$\frac{j \cdot m_j}{z_{\mu}} = \frac{(j+1)(1+m_{j+1})}{z_{\hat{\mu}}}$$

and similarly, $p_{\mu} = p_{\hat{\mu}} \frac{p_j}{p_{j+1}}$. Substituting and reversing the order of summation, we have

$$\text{ch}^n(F) = \sum_{\mu} \sum_{\hat{\mu} \succ j \mu} \chi^\lambda(\hat{\mu}) \cdot \frac{(j+1)(1+m_{j+1})}{z_{\hat{\mu}}} \cdot \frac{p_{\hat{\mu}}}{p_{j+1}} \cdot \frac{p_j}{p_{j+1}}$$

$$= \sum_{\hat{\mu}} \chi^\lambda(\hat{\mu}) \cdot \frac{1}{z_{\hat{\mu}}} \cdot \left[ \sum_{j \geq 1} p_j (j+1) \frac{(1+m_{j+1})}{p_{j+1}} \right]$$

$$= \sum_{\hat{\mu}} \chi^\lambda(\hat{\mu}) \cdot \frac{1}{z_{\hat{\mu}}} \cdot \left[ \sum_{j \geq 1} p_j (j+1) \frac{\partial}{\partial p_{j+1}} (p_{\hat{\mu}}) \right]$$

since $1 + m_{j+1}$ is the multiplicity of $j + 1$ in $\hat{\mu}$. Continuing,

$$\text{ch}^n(F) = \sum_{j \geq 1} p_j (j+1) \frac{\partial}{\partial p_{j+1}} \left( \sum_{\hat{\mu}} \chi^\lambda(\hat{\mu}) \cdot \frac{p_{\hat{\mu}}}{z_{\hat{\mu}}} \right)$$

$$= \sum_{j \geq 1} p_j (j+1) \frac{\partial}{\partial p_{j+1}} (s_{\lambda}).$$

Returning now to the right side of (1), we have

$$\text{ch}^n \sum_{x} \chi^{\lambda-x} \cdot c(x) = \sum_{x} s_{\lambda-x} \cdot c(x);$$

thus it suffices to show that

$$\sum_{j \geq 1} p_j (j+1) \frac{\partial}{\partial p_{j+1}} (s_{\lambda}) = \sum_{x} s_{\lambda-x} \cdot c(x). \quad (3)$$

As operators on Schur functions, both $p_j$ and $Dp_j = j \frac{\partial}{\partial p_j}$ can be interpreted in terms of rimhooks:
Lemma. Let $\lambda$ be a partition, $j$ a positive integer. Then

(i) $p_j s_\lambda = \sum_\nu (-1)^{ht(\nu - \lambda)} s_\nu$

(ii) $(Dp_j) \cdot s_\lambda = \sum_\nu (-1)^{ht(\lambda - \nu)} s_\nu$

where the sums are taken over all partitions $\nu$ such that $\nu - \lambda$ (resp. $\lambda - \nu$) is a rimhook of length $j$.

Proof. (i) [6, p. 31]

(ii) Since $\{s_\lambda\}$ is an orthonormal basis for $\Lambda_n$ it is enough to compute $\langle D(p_j) s_\lambda, s_\nu \rangle$. Using part (i) and the fact that, for any symmetric function $f$, the operator $D(f)$ is the adjoint of multiplication by $f$ [6, p. 43] we have

\[
\langle D(p_j) s_\lambda, s_\nu \rangle = \langle s_\lambda, p_j s_\nu \rangle = \langle s_\lambda, \sum_\zeta (-1)^{ht(\zeta - \nu)} s_\zeta \rangle = \sum_\zeta (-1)^{ht(\zeta - \nu)} \langle s_\lambda, s_\zeta \rangle = (-1)^{ht(\lambda - \nu)}
\]

where the sums are taken over all partitions $\zeta$ such that $\zeta - \nu$ is a rimhook of length $j$.

We use the Lemma to recast the left side of (2) as follows:

\[
\sum_{j \geq 1} p_j (j + 1) \frac{\partial}{\partial p_{j+1}} (s_\lambda) = \sum_{j \geq 1} p_j D(p_{j+1})(s_\lambda)
\]

\[
= \sum_{j \geq 1} p_j \sum_\nu (-1)^{ht(\lambda - \nu)} s_\nu
\]

\[
= \sum_{j \geq 1} \sum_\nu (-1)^{ht(\lambda - \nu)} p_j s_\nu
\]

\[
= \sum_{j \geq 1} \sum_\nu (-1)^{ht(\lambda - \nu)} \sum_\zeta (-1)^{ht(\zeta - \nu)} s_\zeta
\]

where the first sum is taken over partitions $\nu$ such that $\lambda - \nu$ is a rimhook of length $j + 1$ and the second is over partitions $\zeta$ such that $\zeta - \nu$ is a rimhook of length $j$.

To complete the proof of Theorem 1, we need to establish the identity

\[
\sum_{j \geq 1} \sum_\nu (-1)^{ht(\lambda - \nu)} \sum_\zeta (-1)^{ht(\zeta - \nu)} s_\zeta = \sum_\nu s_\lambda - x \cdot c(x).
\]

(4)

Note that the coefficient of each $s_\zeta$ arises by considering all possible ways in which one may obtain the shape $F_\zeta$ ($\zeta \vdash n - 1$) by removing a rimhook of length $j + 1$ from $F_\lambda$ to obtain a shape $F_\nu$ then adding to $F_\nu$ a rimhook of length $j$. 5
For example, if $\lambda = 332$ and $\zeta = 322$ then the coefficient of $s_{\zeta}$ arises from the following cases:

| $j$ | $\nu$ | $ht(\lambda - \nu)$ | $ht(\zeta - \nu)$ | net contribution |
|-----|-------|----------------------|---------------------|-------------------|
| 1   | (2, 2, 2) | 1                    | 0                   | -1               |
| 2   | (3, 1, 1) | 1                    | 1                   | +1               |
| 3   | (3, 1)    | 1                    | 1                   | +1               |

Therefore the coefficient of $s_{322}$ on the left side of (4) is +1; note that $322 = 332 - x$ where $x = (2, 3)$, so $c(x) = +1$ is also the coefficient of $s_{322}$ on the right, as predicted. However, if $\zeta = 43$ then the coefficient of $s_{\zeta}$ is zero, since the cases $j = 1 (\nu = 33)$ and $j = 4 (\nu = 21)$ give signs +1 and -1, respectively. This result is consistent with the right side, since $\zeta = 43$ is not of the form $\lambda - x$ for any $x$. In the same way, the general argument divides into two cases:

Case (i). $\zeta = \lambda - x$ for some $x$: Suppose that $x = (p, q)$. Then $\zeta$ arises from $\lambda$ by removing a rimhook which either begins or ends with the cell $(p, q)$. There are $q - 1$ possibilities beginning with $(p, q)$ and any rimhook added on must have the same sign, since $(p, q)$ is a corner in $F_{\lambda}$. There are $p - 1$ ways to remove a rimhook ending at $(p, q)$ and any rimhook added on must have the opposite sign. Therefore the net contribution is $(q - 1) - (p - 1) = q - p = c(x)$.

Case (ii). $\zeta \neq \lambda - x$ for any $x$: I claim that in all such instances $\zeta$ arises in exactly two ways of opposite sign. Note first that in these cases, both of the skew shapes $\lambda - \zeta$ and $\zeta - \lambda$ must be (non-empty) rimhooks since they are contained in the set of deleted or added cells. For example, if $\lambda = (4, 3, 2, 2)$ and $\zeta = (6, 3, 1)$ then $\lambda - \zeta = \bullet \bullet$ and $\zeta - \lambda = \bullet \bullet$. Moreover, there are exactly two ways in which $\zeta$ arises from removing and then adding a rimhook: For considering the cells which connect $\lambda - \zeta$ and $\zeta - \lambda$, either all are removed and then replaced, or none of them, as illustrated in Figure 2.

![Figure 2: $\lambda = (4, 3, 2, 2); \zeta = (6, 3, 1)$](image-url)
To show that these two ways have opposite signs, let $d(\lambda, \zeta)$ be the number of rows in which some cells are deleted but not replaced; let $a(\lambda, \zeta)$ be the number of rows in which some cells are added without having been deleted; let $r(\lambda, \zeta)$ be the number of rows in which cells are both deleted and replaced. If $r(\lambda, \zeta) = 0$ then the sign associated with $s_\mu$ is $(-1)^{a(\lambda, \zeta) + d(\lambda, \zeta)}$. However, the connectedness of a rimhook guarantees that if $r(\lambda, \zeta) \neq 0$ then some row counted by $r(\lambda, \zeta)$ is also counted by either $a(\lambda, \zeta)$ or $d(\lambda, \zeta)$. Therefore the sign in that case is $(-1)^{a(\lambda, \zeta) + 2r(\lambda, \zeta) + d(\lambda, \zeta) - 1} = (-1)^{a(\lambda, \zeta) + d(\lambda, \zeta) - 1}$.

and the coefficient of $s_\lambda$ must be 0.

This completes the proof of Theorem 1.

The same ideas used in the proof of Theorem 1 can also be used to produce several interesting variations. In the first variation we replace the Jucys-Murphy element $R_n$ by the analogous sum of 3-cycles. The summands on the right side of equation (1) become values of characters corresponding to partitions of $n - 2$ obtained from $F_\lambda$ by removing two cells. Such a partition will be denoted $\lambda - (x, y)$. The content $c(x, y)$ of a pair of deleted cells is defined as follows:

(i) $c(x, y) = c(x)$ if $(x, y)$ forms a horizontal domino $x \ y$

(ii) $c(x, y) = -c(x)$ if $(x, y)$ forms a vertical domino $x \ y$

(iii) $c(x, y) = -1$ if $x$ and $y$ are not contiguous.

**Theorem 2:** Let $\sigma \in S_n$ with $\sigma(n) = n$ and $\sigma(n - 1) = n - 1$, and define $T_n(\sigma) = \sum_{i=1}^{n-1} \sigma(i \ n - 1 \ n)$. Let $\overline{\sigma}$ denote the restriction of $\sigma$ to $\{1, 2, ... n - 2\}$. Then if $\lambda$ is any partition of $n$ we have

$$\chi^\lambda(T_n(\sigma)) = \sum_{i=1}^{n-2} \chi^\lambda(\sigma \cdot (i \ n - 1 \ n)) = \sum_{(x, y)} \chi^{\lambda - (x, y)}(\overline{\sigma}) \cdot c(x, y)$$

(5)

**Proof:** The Frobenius characteristic function can be applied to both sides yielding

$$\sum_{j \geq 1} p_j D_{p_j + 2}(s_\lambda) = \sum_{(x, y)} s_{\lambda - (x, y)} \cdot c(x, y)$$

(6)

The rest of the argument is essentially identical to that of Theorem 1 except for the case in which $x$ and $y$ are not contiguous. That situation arises exactly when $x$ and $y$ are the head and tail of a rimhook $\nu$ of length $j + 2$ replaced by the length $j$ rimhook $\nu - (x, y)$. The latter occupies one fewer row than $\nu$ so contributes the resulting shape with multiplicity $-1$. 

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A different variation on Theorem 1 is obtained by reversing the roles of \( j \) and \( j + 1 \) in equation (3). We have the following theorem; again the proof is essentially the same as that of Theorem 1.

**Theorem 3.** Let \( \lambda \vdash n - 1 \). Let \( \sigma \in S_n \) and set \( V_n(\sigma) = \sum_{i \neq \sigma(i)} (i \sigma(i)). \) When \( i \neq \sigma(i) \) the permutation \( \sigma \cdot (i \sigma(i)) \) has a fixed point and so may be considered as an element of \( S_{n-1} \). With this in mind, we have

\[
\chi^\lambda(V_n(\sigma)) = \sum_{i \neq \sigma(i)} \chi^\lambda(\sigma \cdot (i \sigma(i))) = \sum_x \chi^\lambda+x(\sigma) \cdot c(x).
\] (7)

As in the case of Theorem 1, the Branching Rule immediately gives:

\[
\sum_{i=1}^{n} \chi^\lambda(\sigma \cdot (i \sigma(i))) = \sum_x \chi^\lambda+x(\sigma) \cdot (1 + c(x)).
\] (8)

Remark: The Jucys-Murphy element \( R_n \) acts on a permutation \( \sigma \) which has a fixed point and, as noted earlier, the summands of \( R_n(\sigma) \) are obtained by removing the fixed point and inserting it into each of the cycles of \( \sigma \) in all possible ways. On the other hand, for an arbitrary \( \sigma \in S_n \), the element \( V_n(\sigma) \) “inverses” this action by creating fixed points in all possible ways.

Specializations of Theorem 3 lead to various corollaries. For example, if we choose \( \sigma \) to be the identity permutation, then the sum on the left in equation (7) is empty and we have:

\[
\sum_x f^{\lambda+x} \cdot c(x) = 0
\] (9)

where \( f^{\lambda+x} \) is the dimension of the representation corresponding to \( \lambda + x \).

Finally, the symmetric function approach we have described extends to the hyperoctahedral group \( B_n \) as well. One can use an analog of the characteristic map described by John Stembridge [11], referring also to the work of Arun Ram [8] who has in fact extended the entire Jucys-Murphy construction to types \( B_n \), \( D_n \), and \( G_2 \).

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