INSTANTANEOUS CONVEXITY BREAKING FOR THE QUASI-STATIC DROPLET MODEL

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Abstract. We consider a well-known quasi-static model for the shape of a liquid droplet. The solution can be described in terms of time-evolving domains in $\mathbb{R}^n$. We give an example to show that convexity of the domain can be instantaneously broken.

1. Introduction

We consider the following system of equations for a function $u(x,t)$ and domains $\Omega_t \subset \mathbb{R}^n$, for $t \geq 0$. This system is used to model the quasi-static shape evolution of a liquid droplet of height $u(x,t)$ occupying the region $\Omega_t$:

$$
\begin{align*}
-\Delta u &= \lambda_t, \quad \text{on } \Omega_t \\
 u &= 0, \quad \text{on } \partial \Omega_t \\
 V &= F(|Du|), \quad \text{on } \partial \Omega_t \\
 \int_{\Omega_t} u \, dx &= 1
\end{align*}
$$

In the above, $V$ is the velocity of the free boundary $\partial \Omega_t$ in the direction of the outward unit normal and $F : (0, \infty) \to \mathbb{R}$ is an analytic function with $F'(r) > 0$ for $r > 0$. The constant $\lambda_t > 0$ is determined by the integral condition on $u$.

The initial data is given by a domain $\Omega_0$ which we assume is bounded with smooth boundary $\partial \Omega_0$. Note that the domains $\Omega_t$ (assuming they are bounded with sufficiently regular boundary $\partial \Omega_t$) determine uniquely the solution $x \mapsto u(x,t)$. Thus we may denote a solution of (1.1) by a family of evolving domains $\Omega_t$. In Section 2 we will explain what is meant by a classical solution to this problem.

The system of equations (1.1) has long been accepted as a model for droplet evolution in the physical literature [1, 5, 7, 10, 11]. There have been results on weak formulations of this equation by Glasner-Kim [6] and Grunewald-Kim [8]. Feldman-Kim [3] gave some conditions for global existence and convergence to an equilibrium. Escher-Guidotti [2] proved a short time existence result for classical solutions, which we describe in Section 2 below.

In this note we address the following natural question:

**Question 1.1.** Is the convexity of $\Omega_t$ preserved by the system (1.1)?

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This question is implicit in the work of Glasner-Kim [6]. It was raised explicitly by Feldman-Kim [3, p.822], “Let us point out that, in particular, it is unknown whether the convexity of the drop is preserved in the system (1.1).”

In this note, we answer Question 1.1 by showing that convexity is not generally preserved. We make an assumption on \( F \), namely that

\[
\lim_{r \to 0^+} \frac{F''(r)}{F'(r)} \geq \gamma,
\]

for some \( \gamma > 0 \).

This includes the important cases \( F(r) = r^3 - 1 \) and \( F(r) = r^2 - 1 \) considered in [6] and [3, 8] respectively.

We construct an example where \( \Omega_t \) is convex for \( t = 0 \), but not convex for \( t \in (0, \delta) \) for some \( \delta > 0 \).

**Theorem 1.1.** Assume \( F \) satisfies assumption (1.2). There exists \( \delta > 0 \) and a bounded convex domain \( \Omega_0 \subset \mathbb{R}^2 \) with smooth boundary such that the solution \( \Omega_t \) to (1.1) with this initial data is not convex for any \( t \in (0, \delta) \).

Escher-Guidotti [2] showed that as long as \( \Omega_0 \) is a bounded domain with sufficiently smooth boundary, there always exists a unique classical solution for a short time, and this is what is meant by “the solution \( \Omega_t \)” in the statement of Theorem 1.1. In Section 2, we describe more precisely the results of [2].

In Section 3 we give the proof of Theorem 1.1 The starting point is an explicit solution of the equation \( -\Delta u = \lambda_0 \) on an equilateral triangle [9]. We smooth out the corners to obtain our convex domain \( \Omega_0 \), and show that it immediately breaks convexity.

2. Short time existence

In this section, we recall the short time existence result of Escher-Guidotti [2].

We first give a definition of a solution of (1.1), following [2]. Note that the domains \( \Omega_t \) determine uniquely the functions \( u \), so we will describe the solution of (1.1) in terms of varying domains - given as graphs over the original boundary.

Fix \( \alpha \in (0, 1) \). Assume \( \Omega_0 \) is a bounded domain in \( \mathbb{R}^n \) whose boundary \( \Gamma_0 := \partial \Omega_0 \) is a smooth hypersurface. Let \( \nu(x) \) denote the unit outward normal to \( \Gamma_0 \) at \( x \). Then there exists a maximal constant \( \sigma(\Omega_0) > 0 \) such that for any given function \( \rho \in C^{2+\alpha}(\Gamma_0) \) with \( \|\rho\|_{C^1(\Gamma_0)} \leq \sigma \), the set

\[
\Gamma_\rho = \{ x + \rho(x)\nu(x) \mid x \in \Gamma_0 \},
\]

is a \( C^{2+\alpha} \) hypersurface in \( \mathbb{R}^n \) which is the boundary of a bounded domain \( \Omega = \Omega(\rho) \).

We can now describe a solution of (1.1) in terms of a time-varying family \( \rho(x, t) \). Namely, given \( \rho \in C([0, T], C^{2+\alpha}(\Gamma_0)) \cap C^1([0, T], C^{1+\alpha}(\Gamma_0)) \),
with \( \sup_{t \in [0,T]} \| \rho(\cdot,t) \|_{C^1(\Gamma_0)} < \sigma(\Omega_0) \), write \( \Omega_t \), for \( t \in [0,T] \) for the corresponding family of domains, with boundaries \( \Gamma_t := \Gamma_{\rho(t)} \). The velocity \( V \) of the boundary in the direction of the outward normal, at a point \( y = x + \rho(x,t)\nu(x) \in \Gamma_t \) is given by

\[
V = \frac{\partial \rho}{\partial t}(x,t)\nu(x) \cdot n(y,t),
\]

where \( n(y,t) \) is the outward unit normal to \( \Gamma_t \) at the point \( y \).

Since the domains \( \Omega_t \) have \( C^{2+\alpha} \) boundaries, there exists for each \( t \) a unique solution \( u(\cdot,t) \in C^{2+\alpha}(\Omega_t) \) and \( \lambda_t \in \mathbb{R} \) of

\[
-\Delta u = \lambda_t, \quad \text{on } \Omega_t, \quad u|_{\Gamma_t} = 0, \quad \int_{\Omega_t} u \, dx = 1,
\]

(see for example [4, Theorem 6.14]).

Then we say that such a \( \rho \) is a classical solution of (1.1) with initial domain \( \Omega_0 \) if the velocity \( V(y) \) at each \( y \in \Gamma_t \), for \( t \in [0,T] \) satisfies

\[
V = F(|Du|).
\]

The main theorem of Escher-Guidotti [2] implies in particular the following:

**Theorem 2.1.** There exists a \( T > 0 \) and a unique classical solution

\[
\rho \in C([0,T], C^{2+\alpha}(\Gamma_0)) \cap C^1([0,T], C^{1+\alpha}(\Gamma_0))
\]

of the quasi-static droplet model (1.1) with initial domain \( \Omega_0 \) whose boundary \( \Gamma_0 \) is smooth.

In fact they prove more: they also allow their initial domain to have boundary in \( C^{2+\alpha} \). Note that this result does not require the assumption (1.2).

3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. We work in \( \mathbb{R}^2 \), using \( x \) and \( y \) as coordinates. The heart of the proof is the following lemma, which makes use of the assumption (1.2).

**Lemma 3.1.** There exists a bounded convex domain \( \Omega_0 \) with smooth boundary \( \Gamma_0 \), and real numbers \( 0 < x_0 < x_1 \) with the following properties:

(i) \( \Omega_0 \) is contained in \( \{ y \geq 0 \} \).
(ii) \( (x,0) \in \partial \Omega_0 \) for \( x_0 \leq x \leq x_1 \).
(iii) Let \( u(x,y) \) solve

\[
-\Delta u = \lambda_0, \quad \text{on } \Omega_0, \quad u|_{\Gamma_0} = 0, \quad \int_{\Omega_0} u \, dx \, dy = 1,
\]

for a constant \( \lambda_0 \). Then \( V(x) := F(|Du(x,0)|) \) satisfies

\[
\frac{V(x_0) + V(x_1)}{2} > V \left( \frac{x_0 + x_1}{2} \right).
\]
Proof. We begin with the following explicit solution of the “torsion problem,” \(-\Delta v = \text{const}\), on the equilateral triangle \([9]\). Let \(D\) be the equilateral triangle of side length \(2a\) given by
\[ y > 0, \quad \sqrt{3}|x| > y - a\sqrt{3}. \]
The function
\[ v = cy((y - a\sqrt{3})^2 - 3x^2), \quad \text{for } c := \frac{5}{3a^5}, \]
satisfies
\[ -\Delta v = 4ac\sqrt{3}, \]
vanishes on the boundary of \(D\) and satisfies
\[ \int_D v \, dx \, dy = 1. \]
On the bottom edge of the triangle
\[ E_1 = \{(x,0) \in \mathbb{R}^2 \mid -a \leq x \leq a\}, \]
we have
\[ v_y(x,0) = 3c(a^2 - x^2). \]
Hence
\[ V(x) := F(3c(a^2 - x^2)), \]
and
\[ V''(x) = 36c^2x^2F''(3c(a^2 - x^2)) - 6cF'(3c(a^2 - x^2)). \]
Recalling that \(c = 5/(3a^5)\), then we may choose \(a > 0\) sufficiently small so that
\[ 36c^2x^2 \geq 2\frac{6c}{\gamma}, \quad \text{for } |x| \geq a/2, \]
where \(\gamma > 0\) is given by our assumption \((1.2)\). From now on we fix this \(a\) (and hence \(c\)).

It follows from \((3.1)\), \((3.2)\) and \((1.2)\) that \(V''(x) > 0\) for \(|x|\) sufficiently close to \(a\). In particular there exists \(0 < x_0 < x_1 < a\) with
\[ \frac{V(x_0) + V(x_1)}{2} > V\left(\frac{x_0 + x_1}{2}\right), \]
The above example above readily implies the existence of a smooth domain \(\Omega_0\) satisfying the conditions in the Lemma. Indeed, we only have to “smooth the corners” of the triangle domain \(D\).

Denote the vertices of \(D\) by \(p_1, p_2, p_3\). Let \(\{D_k\}_{k=1}^\infty\) be a sequence of bounded convex domains with smooth boundaries such that for each \(k \geq 1:\)

(1) \(D_k \subset D_{k+1} \subset D\) (the sequence is nested and increasing).
(2) \(D \setminus D_k \subset \bigcup_{i=1}^3 B_{k-1}(p_i)\), where \(B_r(p)\) denotes the ball of radius \(r\) centered at \(p\).
Such a sequence \( \{D_k\} \) can be constructed by “rounding out the corners” of the triangle \( D \) in a ball of radius \( k^{-1} \) centered at each corner.

For each \( k \geq 1 \) let \( u_k \) on \( D_k \) be the solutions of
\[
-\Delta u_k = 4ac\sqrt{3}, \quad \text{on } D_k, \quad u|_{\partial D_k} = 0,
\]
where we recall that \( a \) and \( c \) are fixed constants.

It follows from property (1) above and the maximum principle that for each \( k \geq 1 \)
\begin{equation}
0 < u_k \leq u_{k+1} \leq v, \quad \text{on } D_k,
\end{equation}
from which we conclude a pointwise limit on the triangle \( D \)
\begin{equation}
0 \leq u_{\infty}(x) := \lim_{k \to \infty} u_k(x) \leq v(x), \quad \text{for } x \in D,
\end{equation}
and define \( u_{\infty}(x) \) to be zero on \( \partial D \).

By standard elliptic estimates (see for example [4, Theorem 6.19] and the remark after it), the convergence above will hold in \( C^{\ell}(K) \) for any compact set \( K \subset \subset (D \setminus \{p_1, p_2, p_3\}) \) and any \( \ell \geq 0 \). Hence \( u_{\infty} \in C^{\infty}(D \setminus \{p_1, p_2, p_3\}) \) and 
\[-\Delta u_{\infty} = 4ac\sqrt{3} \quad \text{on } D.\]
Moreover, by (3.4) and the continuity of \( v \) it is easily verified that \( u_{\infty} \) is also continuous at the corners \( p_1, p_2, p_3 \) and thus on all of \( D \). By the maximum principle, \( u_{\infty} = v \). Note also that
\[
\int_{D_k} u_k \, dx \, dy \to 1, \quad \text{as } k \to \infty.
\]

Then for sufficiently large \( k \) the domain \( \Omega_0 := D_k \) will satisfy conditions (i), (ii), (iii), with
\[
u := \frac{u_k}{\int_{D_k} u_k \, dx \, dy}, \quad \lambda_0 := \frac{4ac\sqrt{3}}{\int_{D_k} u_k \, dx \, dy}.
\]
Here we are using (3.3) and the fact that \( x \mapsto F(|Du_k(x, 0)|) \) will converge uniformly to \( x \mapsto F(|Dv(x, 0)|) \) on \( [x_0, x_1] \) as \( k \to \infty \). This completes the proof of the lemma. \( \square \)

**Proof of Theorem 1.1.** Let \( \Omega_0 \) and \( u \) be given as in Lemma 3.1. By Theorem 2.1, there exists a unique classical solution of (1.1) for a short time interval \([0, T]\) with \( T > 0 \).

The boundaries \( \Gamma_t \) of \( \Omega_t \) can be written as graphs over \( \Gamma_0 := \partial \Omega_0 \). In particular, using \( x \) as a coordinate, part of \( \Gamma_t \) is given by a graph \( y = g(x, t) \) for \( x_0 \leq x \leq x_1 \), with \( g(x, 0) = 0 \) for \( x_0 \leq x \leq x_1 \), with the unit normal to \( \Omega_0 \) being in the negative \( y \) direction.

We may assume that \( g \in C([0, T], C^{2+\alpha}([x_0, x_1])) \cap C^1([0, T], C^{1+\alpha}([x_0, x_1])). \)

Moreover, \( (\partial g/\partial t)(x, 0) \) represents the negative of the velocity in the normal direction at time \( t = 0 \). Hence by (iii) of Lemma 3.1,
\[
\frac{1}{2} \left( \frac{\partial g}{\partial t}(x_0, 0) + \frac{\partial g}{\partial t}(x_1, 0) \right) < \frac{\partial g}{\partial t} \left( \frac{x_0 + x_1}{2}, 0 \right).
\]
Then for $t \in (0, \delta]$ for $\delta > 0$ sufficiently small, we have
\[
\frac{1}{2} \left( g(x_0, t) + g(x_1, t) \right) < g\left( \frac{x_0 + x_1}{2}, t \right).
\]
In particular, $x \mapsto g(x, t)$ is not convex for $(x, t) \in [x_0, x_1] \times (0, \delta]$. Hence $\Omega_t$ is not a convex domain for $t \in (0, \delta]$.

\[\Box\]

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