Einstein’s equations and the chiral model

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Abstract

The vacuum Einstein equations for spacetimes with two commuting spacelike Killing field symmetries are studied using the Ashtekar variables. The case of compact spacelike hypersurfaces which are three-tori is considered, and the determinant of the Killing two-torus metric is chosen as the time gauge. The Hamiltonian evolution equations in this gauge may be rewritten as those of a modified SL(2) principal chiral model with a time dependent ‘coupling constant’, or equivalently, with time dependent SL(2) structure constants. The evolution equations have a generalized zero-curvature formulation. Using this form, the explicit time dependence of an infinite number of spatial-diffeomorphism invariant phase space functionals is extracted, and it is shown that these are observables in the sense that they Poisson commute.

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with the reduced Hamiltonian. An infinite set of observables that have SL(2) indices are also found. This determination of the explicit time dependence of an infinite set of spatial-diffeomorphism invariant observables amounts to the solutions of the Hamiltonian Einstein equations for these observables.

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I. INTRODUCTION

In classical field theory one would like to find physically interesting solutions and to study the question of integrability. This is a difficult problem for four-dimensional non-linear theories such as Einstein gravity, and most of the integrable theories are two dimensional. The classic examples are the Kortweg-de Vries and Sine-Gordon models, whose study has led to systematic methods for addressing the integrability question for two-dimensional field theories [1,2]. Self-dual Yang-Mills and gravity theories are the only ones in four dimensions which are considered to be integrable. This is because of twistor constructions of the general solutions [3,4]. However, self-dual theories are unusual in that they have formulations as two dimensional theories, and so the standard two dimensional methods are likely to be applicable. For example, self-dual gravity has a formulation as the two-dimensional principal chiral model [5–7].

Vacuum Einstein gravity with two commuting spacelike Killing vector fields is a two-dimensional field theory which has two local degrees of freedom. Perhaps the main result in this reduction of Einstein gravity is the discovery by Geroch of an infinite dimensional ‘hidden symmetry’ of the field equations [8]. This symmetry leads to a solution generating method for this sector of the Einstein equations, which has since been studied from various points of view [9–13].

As for other two-dimensional theories, hidden symmetries suggest that this reduction of the Einstein equations is integrable. However there is so far no proof of integrability in the Liouville sense - that is, it has not been shown that there exists an infinite number of Poisson commuting constants of the motion. In order to address this question, one would first have to identify the hidden symmetry generators on the phase space, or otherwise give a solution to the Hamiltonian equations of motion.

Apart from the purely classical reasons, classical integrability is very useful for constructing quantum theories. This is because in canonical quantization, one would like to convert gauge invariant phase space functionals into Hermitian operators on a suitable representa-
tion space. In generally covariant theories it is difficult to find a sufficiently large number of such functionals. If the classical theory is known to have hidden symmetries, then it should be possible to find them explicitly on the phase space. The phase space generators of the symmetries would then serve as observables.

In this paper we study the two commuting spacelike Killing field reduced equations from a Hamiltonian point of view. This will be with a view to addressing the integrability question and identifying explicitly the hidden symmetries on the phase space. We will use the Ashtekar canonical formalism \cite{14,15} for complex relativity, where the conjugate phase space variables are complex, but are defined on a real manifold.

There has been previous work by the author on this reduction using the Ashtekar variables \cite{16}, where it was claimed that the Hamiltonian evolution equations were equivalent to those of the \( SL(2) \) principal chiral model. It was then pointed out \cite{17} that this identification was based on a further reduction of the theory, because the gauge fixing used in it could not be achieved for generic two Killing field reduced spacetimes. In particular it could not be achieved for the Gowdy models \cite{18}. In this paper we will show, among other things, how to rectify this situation by using a gauge fixing that was suggested in Ref. \cite{16}.

The outline of this paper is as follows: In the next section we review the reduction of the Einstein theory to the sector where the metric has two commuting spacelike Killing fields. Details of this may be found in a previous work by the author and Smolin \cite{19}. In Section III we focus on the three-torus Gowdy cosmology, and fix the determinant of the Killing two-torus metric as the time gauge. In this gauge the Einstein evolution equations become those of a modified (complexified) \( SL(2) \) principal chiral model with a time dependent ‘coupling constant’. Section IV shows how the evolution equations may be written as a generalized zero-curvature equation. The generalization is in the fact that our equation has explicit time dependence whereas the standard zero curvature equations for integrable two-dimensional models do not. Section V describes how to extract the time dependence of a specific infinite set of diffeomorphism invariant observables on the phase space, and identify explicitly the generators of the hidden symmetries of the reduced Hamiltonian. The last section gives the
main conclusions, and contains a discussion of the relevance of the results for constructing a quantum theory for this sector of Einstein gravity.

II. TWO KILLING VECTOR FIELD REDUCTION

The Ashtekar Hamiltonian variables [14,15] for complexified general relativity are the (complex) canonically conjugate pair \((A^i_a, \tilde{E}^{ai})\) where \(A^i_a\) is an SO(3) connection and \(\tilde{E}^{ai}\) is a densitized dreibein. \(a, b, ..\) are three dimensional spatial indices, \(i, j, .. = 1, 2, 3\) are internal SO(3) indices, and the tilde denotes a density of weight one. The constraints of general relativity are

\[
\mathcal{G}^i := D_a \tilde{E}^{ai} = 0, \quad (2.1)
\]

\[
\mathcal{C}_a := F_{ab} \tilde{E}^{ai} = 0, \quad (2.2)
\]

\[
\mathcal{H} := \epsilon^{ijk} F_{ab} \tilde{E}^{aj} \tilde{E}^{bk} = 0, \quad (2.3)
\]

where

\[
D_a \lambda^i = \partial_a \lambda^i + \epsilon^{ijk} A^j_a \lambda^k \quad (2.4)
\]
is the covariant derivative, and

\[
F_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^{ijk} A^j_a A^k_b \quad (2.5)
\]
is its curvature.

Since the phase space variables are complex, reality conditions need to be imposed to obtain the Euclidean or Lorentzian sectors. These are

\[
A^i_a = \bar{A}^i_a \quad E^{ai} = \bar{E}^{ai} \quad (2.6)
\]

for the former, and

\[
A^i_a + \bar{A}^i_a = 2\Gamma^i_a(E), \quad E^{ai} = \bar{E}^{ai} \quad (2.7)
\]
for the latter. The $\Gamma^{i}_{\alpha}(E)$ is the connection for spatial indices and the bar denotes complex conjugation.

We now review the two commuting spacelike Killing field reduction of these constraints which was first presented in \[19\]. Working in spatial coordinates $x, y$, such that the Killing vector fields are $(\partial/\partial x)^{a}$ and $(\partial/\partial y)^{a}$ implies that the phase space variables will depend on only one of the three spatial coordinates. Specifically, we assume that the spatial topology is that of a three torus so that the phase space variables depend on the time coordinate $t$ and one angular coordinate $\theta$. This situation corresponds to one of the Gowdy cosmological models [18]. (The other permitted spatial topologies for the Gowdy cosmologies are $S^{1} \times S^{2}$ and $S^{3}$.)

In addition to these Killing field conditions, we set to zero some of the phase space variables as a part of the symmetry reduction:

$$\tilde{E}^{x3} = \tilde{E}^{y3} = \tilde{E}^{\theta1} = \tilde{E}^{\theta2} = 0,$$

$$A^{3}_{x} = A^{3}_{y} = A^{1}_{\theta} = A^{2}_{\theta} = 0.$$ \hfill (2.8)

These conditions may be viewed as implementing a partial gauge fixing and solution of some of the resulting second class constraints. Details of these steps are given in Ref. \[19\]. The end result, Eqns. \((2.9, 2.11)\) below, is a simplified set of first class constraints that describe a two dimensional field theory with two local degrees of freedom on $S^{1} \times R$. Renaming the remaining variables $A := A^{3}_{\theta}$, $E := \tilde{E}^{\alpha3}$ and $A^{I}_{\alpha}$, $\tilde{E}^{\alpha I}$, where $\alpha, \beta, \ldots = x, y$ and $I, J, \ldots = 1, 2$, the reduced constraints are

$$G := \partial E + J = 0,$$ \hfill (2.9)

$$C := A \partial E - \tilde{E}^{\alpha I} \partial A^{I}_{\alpha} = 0,$$ \hfill (2.10)

$$H := -2\epsilon^{IJ} F^{I}_{\alpha} \tilde{E}^{\alpha J} E + F_{\alpha \beta} \tilde{E}^{\alpha I} \tilde{E}^{\beta J} \epsilon_{IJ}$$

$$= -2(\tilde{E}^{\alpha I} \epsilon^{IJ} \partial A^{I}_{\alpha} + 2AEK - K^{\alpha}_{\beta} K^{\alpha}_{\beta} + K^{2} = 0,$$ \hfill (2.11)

where $\partial = (\partial/\partial \theta) = \iota$. 

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\[ K^\beta_\alpha := A^I_\alpha \bar{E}^{\beta I}, \quad K := K^\alpha_\alpha, \]  
(2.12)

\[ J^\beta_\alpha := \epsilon^{IJ} A^I_\alpha \bar{E}^{\beta J}, \quad J := J^\alpha_\alpha, \]  
(2.13)

and \( \epsilon^{12} = 1 = -\epsilon^{21} \).

The SO(3) Gauss law has been reduced to U(1), and the spatial diffeomorphism constraint to Diff(\(S^1\)). This may be seen by calculating the Poisson algebra of the constraints smeared by functions \( \Lambda(t, \theta) \), the shift \( V(t, \theta) \), and the lapse \( N(t, \theta) \) (which is a density of weight \(-1\)). With

\[ G(\Lambda) = \int_0^{2\pi} d\theta \, \Lambda G, \]  
(2.14)

\[ C(V) = \int_0^{2\pi} d\theta \, VC, \]  
(2.15)

\[ H(N) = \int_0^{2\pi} d\theta \, NH, \]  
(2.16)

the constraint algebra is

\[ \{G(\Lambda), G(\Lambda')\} = \{G(\Lambda), H(N)\} = 0, \]  
(2.17)

\[ \{C(V), C(V')\} = C(\mathcal{L}_V V'), \]  
(2.18)

\[ \{H(N), H(N')\} = C(W) - G(AW), \]  
(2.19)

where

\[ W \equiv E^2(N \partial N' - N' \partial N). \]  
(2.20)

This shows that \( C \) generates Diff(\(S^1\)). Also we note that this reduced first class system still describes a sector of general relativity due to the Poisson bracket \( \{H(N), H(N')\} \), which is the reduced version of that for full general relativity in the Ashtekar variables.

The variables \( K^\beta_\alpha \) and \( J^\beta_\alpha \) defined above will be used below in the discussion of observables. Here we note their properties. They are invariant under the reduced Gauss law (2.13), transform as densities of weight \(+1\) under the Diff(\(S^1\)) generated by \( C \), and form the Poisson algebra.
\[ \{ K_\alpha^\beta, K_\gamma^\sigma \} = \delta_\alpha^\gamma K_\beta^\beta - \delta_\beta^\gamma K_\alpha^\sigma, \]  
\[ \{ J_\alpha^\beta, J_\gamma^\sigma \} = -\delta_\alpha^\gamma K_\beta^\beta + \delta_\beta^\gamma K_\alpha^\sigma, \]  
\[ \{ K_\alpha^\beta, J_\gamma^\sigma \} = \delta_\alpha^\gamma J_\beta^\beta - \delta_\beta^\gamma J_\alpha^\sigma. \]  
(2.21)

This shows that \( K_\alpha^\beta \) form the \( \text{gl}(2) \) Lie algebra, and hence generate \( \text{gl}(2) \) rotations on variables with indices \( \alpha, \beta, .. = x, y. \)

The following linear combinations of \( K_\alpha^\beta \) form the \( \text{sl}(2) \) subalgebra of \( \text{gl}(2) \):

\[ L_1 = \frac{1}{2}(K_y^x + K_x^y), \quad L_2 = \frac{1}{2}(K_x^x - K_y^y), \quad L_3 = \frac{1}{2}(K_y^x - K_x^y). \]  
(2.24)

The Poisson bracket algebra of these is

\[ \{ L_i, L_j \} = C_{ij}^k L_k, \]  
(2.25)

where \( C_{12}^3 = -1, C_{23}^1 = 1, C_{31}^2 = 1 \) are the \( \text{sl}(2) \) structure constants. (From here on the indices \( i, j, k... \) will denote \( \text{sl}(2) \) indices, and not the \( \text{so}(3) \) internal indices of the Ashtekar formulation). The corresponding linear combinations of \( J_\alpha^\beta \) are denoted by \( J_i, i = 1, 2, 3. \) Their Poisson brackets are

\[ \{ L_i, J_j \} = C_{ij}^k J_k, \quad \{ J_i, J_j \} = -C_{ij}^k L_k. \]  
(2.26)

We also have

\[ \{ J, J_i \} = \{ J, L_i \} = \{ K, J_i \} = \{ K, L_i \} = 0. \]  
(2.27)

For discussing observables, it will turn out to be very convenient to replace the eight canonical phase space variables \( A_\alpha^I \) and \( \tilde{E}^{\alpha I} \) by the eight Gauss law invariant variables \( K_\alpha^\beta \) and \( J_\alpha^\beta. \) We will refer to the latter as the \( \text{gl}(2) \) variables.

**III. GAUGE FIXING**

In this section we pick a preferred foliation of the two-Killing field reduced spacetimes by fixing the time coordinate. We will show, by giving the spacetime metric, that the time
gauge choice is the one made by Gowdy \cite{18}. It will turn out that the evolution equations in this gauge become those of a modified SL(2) chiral model which has a time dependent ‘coupling constant’.

Since the theory we are considering is complex general relativity on a real manifold, we need to fix, as the (real) time coordinate, a (complex) phase space variable that transforms as a scalar under the spatial diffeomorphisms \cite{2,10}. Since $E$ transforms like a scalar in the reduction we are considering, we set

\[
\text{Im}(E) = 0, \quad \text{Re}(E) = t
\]  

as the time gauge choice. The (complex) reduced Hamiltonian density is by definition the negative of the variable conjugate to time ($= E$):

\[
H_R := -A = -\frac{1}{K} \tilde{E}^{\alpha J} \epsilon^{IJ} \partial_{A_i^J} + \frac{1}{2Kt} (K^2 - K^\alpha_{\beta} K_{\beta}^\alpha).
\]  

(3.2)

The lapse density $N$ is determined by requiring that the gauge fixing condition be preserved in time:

\[
\dot{E} = \dot{t} = 1 = \{ E, H(N) \},
\]  

(3.3)

gives

\[
N = -\frac{1}{2tK}.
\]  

(3.4)

With this gauge choice, we also find that the Gauss law constraint (2.9) reduces further to

\[
J = 0,
\]  

(3.5)

and remains first class.

The evolution equations of the gl(2) variables in this gauge are derived from Hamilton’s equation

\[
\dot{X} = \{ X, \int_0^{2\pi} d\theta \ [ H_R + \Lambda J \] + C(V) \},
\]  

(3.6)
where $C(V)$ generates spatial diffeomorphisms and $J$ generates the Gauss law rotations.

In the following we focus only on the contributions from $H_R$, because below we will make a $\theta$ coordinate fixing that sets the shift $V$ to zero. Also, the Gauss law term in the full Hamiltonian (3.3) does not contribute to the evolution of the $gl(2)$ variables, because these variables are already Gauss law invariant.

The equations for $J$ and $K$ are

\[
\dot{J} = 0,
\]

\[
\dot{K} = \partial (\frac{J}{K}) = 0,
\]  

(3.7)  

(3.8)

where the last equality follows because of the reduced Gauss law in this gauge. These imply that the Gauss law is preserved under evolution, and that $K = f(\theta)$, where $f$ is an arbitrary density on the circle.

The equations for $L_i$ and $J_i$ are

\[
\dot{L}_i = \partial (\frac{J_i}{K}),
\]

\[
\dot{J}_i = -\partial (\frac{L_i}{K}) - \frac{2}{iK} C_{ij}^k J_j L_k.
\]  

(3.9)  

(3.10)

The explicit time dependence of the reduced Hamiltonian appears only in the $\dot{J}_i$ equation.

These evolution equations may be further simplified by setting a $\theta$ coordinate condition which makes the spatial diffeomorphism constraint (2.10) second class. As for the time gauge fixing above, a scalar phase space function must be fixed as the $\theta$ coordinate. Since $K$ transforms as a density of weight one, we can set

\[
\theta = -\frac{1}{\alpha} \text{Re}(\int^\theta d\theta' K(\theta', t)), \quad \text{Im}(\int^\theta d\theta' K(\theta', t)) = 0.
\]  

(3.11)

This gives $K = -\alpha \neq 0$, a constant density. We note that a scalar density transforms non-trivially as $K'(\theta', t) = \partial \theta / \partial \theta' K(\theta, t)$ under coordinate transformations. Therefore, setting a density on the spatial surface to be a constant indeed fixes a spatial coordinate. We also note that the constant of motion $\int_0^{2\pi} d\theta K(\theta, t)$ now takes the value $-2\pi \alpha$.

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1 This type of coordinate fixing condition - setting a phase space density on the circle to be a
The shift vector $V$ is determined by requiring that the gauge condition be preserved under Hamiltonian evolution. We have

$$\dot{\theta} = 0 = \{-\frac{1}{\alpha} \int_{0}^{\theta} d\theta' K(\theta', t), \int_{0}^{2\pi} d\theta [ H_R + \Lambda J ] + C(V)\} = -\frac{1}{\alpha} \int_{0}^{\theta} d\theta' L_{V}K(\theta', t),$$

(3.12)

where $L_{V}$ denotes the Lie derivative, and the last equality follows because $\{ \int_{0}^{2\pi} d\theta H_R, J \} = \{ \int_{0}^{2\pi} d\theta H_R, K \} = 0$. Therefore $V = 0$.

The evolution equations (3.9-3.10) for $L_i$ and $J_i$, which are the only non-trivial ones, then become

$$\dot{L}_i + \frac{1}{\alpha} J'_i = 0 \quad (3.13)$$
$$\dot{J}_i - \frac{1}{\alpha} L'_i + \frac{2}{\alpha t} C_{i}^{jk} L_j J_k = 0. \quad (3.14)$$

We now make a further change of variable, the time rescaling $\tau = t/\alpha$, and then replace $\tau$ by $t$. This changes the above equations to

$$\dot{L}_i + J'_i = 0 \quad (3.15)$$
$$\dot{J}_i - L'_i + \frac{2}{\alpha t} C_{i}^{jk} L_j J_k = 0. \quad (3.16)$$

These resemble the first order form of the evolution equations of the principal chiral model, which for the two-dimensional Lie algebra valued gauge field $A_{\mu} (\mu = x, t)$ are

$$\partial_{\mu} A_{\mu} = 0,$$
$$\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}] = 0. \quad (3.17)$$

constant, has been used in the quantization of the one polarization Gowdy cosmology by Berger \cite{20}.

\footnote{We note that $\int_{0}^{2\pi} d\theta H_R$ is a first class Hamiltonian because it commutes with the second class pair consisting of the diffeomorphism constraint and the $\theta$ coordinate fixing condition. Therefore the Dirac brackets give the same evolution equations as the ordinary Poisson brackets.}
The difference from the latter is the $1/t$ factor in (3.10), which is like a time dependent coupling constant. This factor may be absorbed in the structure constants to make them time dependent. Thus, this reduction of the Einstein equations is equivalent to an SL(2) chiral model in which the structure constants scale like $1/t$.

Of the original eight $\text{gl}(2)$ variables we have the six $\text{sl}(2)$ variables $J_i$ and $L_i$ left. The two conditions still to be imposed to gauge fix completely are the spatial diffeomorphism constraint

$$\tilde{E}^\alpha_A \partial A_\alpha = 0,$$

which is now second class, and a gauge condition for fixing the remaining Gauss law, $J = 0$. These will reduce the six $\text{sl}(2)$ variables down to the four phase space degrees of freedom per point for gravity. We will not reduce the system completely by solving these two conditions, but rather focus on studying the symmetries associated with the evolution equations (3.15-3.16).

We now give the spacetime metric that results from the time gauge fixing described above. The doubly densitized inverse of the spatial metric in the Ashtekar formulation is $\tilde{q}^{ab} = \tilde{E}^a_i \tilde{E}^b_i$. Therefore, with $q \equiv \det(q_{ab})$, the spatial metric is given by

$$\tilde{q}^{ab} = q q^{ab} = \begin{pmatrix} E^2 & 0 \\ 0 & e^{\alpha \beta} \end{pmatrix},$$

where $e^{\alpha \beta} := \tilde{E}^{\alpha l} \tilde{E}^{\beta l}$. The determinant of the spatial metric is $q = t \sqrt{\det(e^{\alpha \beta})} =: te$. Therefore

$$q_{ab} = \begin{pmatrix} e/t & 0 \\ 0 & t ee^{\alpha \beta} \end{pmatrix}.$$

From (3.4), the lapse function $\mathcal{N}$ is

$$\mathcal{N} = \sqrt{q} N = \sqrt{\frac{e}{4t}}.$$

The line element (after the above rescaling $t \rightarrow t\alpha$) in terms of phase space variables is then
\[ ds^2 = \frac{e(t, \theta)}{\alpha t} \left( -\frac{1}{4} dt^2 + d\theta^2 \right) + \alpha t e(t, \theta) e_{\alpha\beta}(t, \theta) \, dx^\alpha \, dx^\beta. \]  \tag{3.22}

From this we see that the determinant of the Killing two-torus metric is \( \text{det}(\alpha t e e_{\alpha\beta}) = (\alpha t)^2 \). Since \( e_{\alpha\beta} \) is complex the metric is also. The real Lorentzian section of this is determined by the reality condition \( \text{Im}(e_{\alpha\beta}) = 0 \). This gives the Gowdy \( T^3 \) metric. We note that \( t = 0 \) is the initial spacelike cosmological singularity, and that the time gauge choice \( E = t \) turns out to be the same as Gowdy’s gauge \[18\], namely the determinant of the Killing two-torus metric is time.

**IV. EVOLUTION EQUATIONS AS A ‘ZERO-CURVATURE’ CONDITION**

The zero curvature formulation for a non-linear field theory, which is basically the same as the equation for the Lax pair for the theory, arises from a linear system of equations which are also known as the inverse scattering equations. The latter is a pair of equations whose integrability condition gives the non-linear field theory in question. This formulation is important for determining integrability \[1,2\] because the conservation laws associated with zero curvature equations are relatively easy to obtain. All known integrable models have such formulations. Our form below \[17\] for the two Killing field reduced Einstein equations is different from all the other known models in that it contains explicit time dependence. This is a direct consequence of the fact that we have the time dependent reduced Hamiltonian \[3.2\]. Nevertheless, as we will see in the following section, an infinite set of symmetries of the reduced Hamiltonian can still be obtained as a consequence of \[17\].

The evolution equations \[3.15-3.16\] derived in the last section may be rewritten in a compact form using the \( \text{sl}(2) \) matrix generators

\[ g_1 = \frac{1}{2\alpha} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_2 = \frac{1}{2\alpha} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g_3 = \frac{1}{2\alpha} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]  \tag{4.1}

which satisfy the relations

\[ [g_i, g_j] = \frac{1}{\alpha} C_{ij}^k g_k, \quad g_i \, g_j = \frac{1}{2\alpha} C_{ij}^k g_k. \]  \tag{4.2}
Defining the matrices

\[ A_0 := 2L_i g_i , \quad A_1 := 2J_i g_i , \]

the evolution equations (3.15-3.16) become

\[ \partial_0 A_0 + \partial_1 A_1 = 0 \quad (4.4) \]

\[ \partial_0 A_1 - \partial_1 A_0 + \frac{1}{t} [ A_0 , A_1 ] = 0. \quad (4.5) \]

Equations (4.4-4.5) resemble the first order form of the SL(2) chiral model field equations, but with the ‘coupling constant’ factor $1/t$ multiplying the commutator.

The two evolution equations (4.4-4.5) may be rewritten as a single equation in the following way. Define for a parameter $\lambda$

\[ a_0 := \frac{1}{1 + \lambda^2} (A_0 - \lambda A_1) \quad a_1 := \frac{1}{1 + \lambda^2} (\lambda A_0 + A_1) \quad (4.6) \]

Then equations (4.4-4.5) follow from the single time dependent ‘zero-curvature’ equation

\[ \dot{a}_1 - a_0' + \frac{1}{t} [ a_0 , a_1 ] = 0. \quad (4.7) \]

This equation is the main result of this section, and represents a compact way of writing the evolution equations for this reduction of general relativity.

**V. OBSERVABLES**

In this section, the dynamical equation (4.7) is used to obtain the observables, which are the phase space functionals that Poisson commute with the reduced Hamiltonian (3.2). We

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3We note that in the derivation of these equations we used the fundamental Poisson bracket relation for Euclidean complexified general relativity \( \{ A_i^a(x), E^b_j(y) \} = \delta^b_a \delta^i_j \delta(x, y) \), without the factor $i$ of the Lorentzian Ashtekar variables on the right hand side. When this factor is put in one gets the Lorentzian chiral model equation, where a relative minus sign appears in (3.15) above. All the results of this paper go through with appropriate sign changes in the definitions (4.6).
note that the standard procedure that applies to two-dimensional models \[1,2\] now has to be modified since our zero curvature equation (4.7) has explicit time dependence. We also note that the standard (time independent) zero-curvature equations allow one to establish two results: (1) The extraction of an infinite number of phase space functionals that commute with the Hamiltonian, and (2) a simple proof that these functionals are in involution. Below we establish the first result for our system using the generalized ‘zero-curvature’ equation.

The transfer matrix used in the study of two-dimensional models is like the Wilson loop in non-Abelian Yang-Mills theory. The zero curvature formulation of the evolution equations of a theory implies that the trace of the transfer matrix is a conserved quantity, or ‘observable’. This is true for essentially the same reason as that which makes the trace of the Wilson loop an observable in 2+1 gravity \[21,22\].

We consider the following time dependent analog of the transfer matrix

\[
U[A_0, A_1](0, \theta) := \text{Pexp} \left[ \frac{1}{t} \int_0^\theta d\theta' a_1(t, \theta', \lambda) \right]
\equiv I + \frac{1}{t} \int_0^\theta d\theta' a_1(\theta, t, \lambda)
+ \frac{1}{t^2} \int_0^\theta d\theta' \int_0^{\theta'} d\theta'' a_1(\theta'', t, \lambda) a_1(\theta', t, \lambda) + ..., \tag{5.1}
\]

where \(I\) is the \(2 \times 2\) identity matrix. \(U(0, \theta)\) depends on time explicitly, and also implicitly through the gravitational variables. Defining

\[
M = \text{Tr} \ U(0, 2\pi), \tag{5.2}
\]

we have

\[
\frac{dM}{dt} = \frac{\partial M}{\partial t} + \int_0^{2\pi} d\theta \ \frac{\delta M}{\delta a_{1j}(\theta, t)} \frac{\partial a_{1j}(\theta, t)}{\partial t}, \tag{5.3}
\]

where \(a_1 := a_{1i}g_i\). The second term on the right hand side is zero because

\[
\int_0^{2\pi} d\theta \ \frac{\delta M}{\delta a_{1j}(\theta, t)} \frac{\partial a_{1j}(\theta, t)}{\partial t} = \int_0^{2\pi} d\theta \ \text{Tr} \left[ U(0, \theta) \frac{1}{t} \frac{\partial a_1}{\partial t} U(\theta, 2\pi) \right]
= \int_0^{2\pi} d\theta \ \text{Tr} \left[ U(0, \theta) \frac{1}{t} (\frac{\partial a_0}{\partial \theta} - \frac{1}{t} [a_0, a_1]) U(\theta, 2\pi) \right]
= \frac{1}{t^2} \int_0^{2\pi} d\theta \ \text{Tr} \left[ U(0, \theta)(-a_1 a_0 + a_0 a_1 - [a_0, a_1]) U(\theta, 2\pi) \right].
\]
\[ + \frac{1}{t} \text{Tr} \left[ U(0, 2\pi)a_0(2\pi) - a_0(0)U(0, 2\pi) \right] \]
\[ = 0, \quad (5.4) \]

where we have used (4.7), integrated by parts, and used

\[ U'(0, \theta) = \frac{1}{t} U(0, \theta)a_1(\theta), \quad U'(\theta, 2\pi) = -\frac{1}{t} a_1(\theta)U(\theta, 2\pi). \quad (5.5) \]

The second term in the third equality in (5.4) is the surface term arising from the integration by parts, which without the trace on \( M \) gives a non-zero contribution.

The time dependence of \( M \) is given by Hamilton’s equation

\[ \frac{dM}{dt} = \{ M, \int_0^{2\pi} d\theta H_R \} + \frac{\partial M}{\partial t}, \quad (5.6) \]

where the first term on the right hand side gives the implicit time dependence. Therefore, the calculation (5.4) above gives the crucial result

\[ \{ M, \int_0^{2\pi} d\theta H_R \} = 0. \quad (5.7) \]

Therefore each coefficient of \( \lambda \) in \( M \) is a phase space functional that generates a symmetry of the reduced Hamiltonian. This is one of the main results of this paper. We note two points about the observables generated using \( M \): (1) The observables, while being symmetries of the reduced Hamiltonian, are not constants of the motion because of their explicit time dependence, (indeed the reduced Hamiltonian itself is explicitly time dependent), and (2) if the Poisson bracket in (5.7) is replaced by the Dirac bracket the result is the same, because (as also noted above) the reduced Hamiltonian is first class and \( M \) is spatial-diffeomorphism invariant. Furthermore, \( M \) also has vanishing Poisson and Dirac brackets with the remaining first class constraint \( J \), again because \( M \) is spatial-diffeomorphism invariant, and because \( J \) commutes with \( K \) (which is used in the \( \theta \) fixing condition), and also with the \( J_i \) and \( L_i \) (out of which \( M \) is made). Therefore the functionals generated via \( M \) are indeed time dependent observables of the theory.

There are a set of three constants of the motion for this system that have been given before [19]. These are
\[ l_i := \int_0^{2\pi} d\theta \ L_i(t, \theta). \quad (5.8) \]

It is obvious from (3.15) that these are conserved.

These \( l_i \) may be used to obtain an infinite number of phase space functionals with \( \text{sl}(2) \) indices that commute with the reduced Hamiltonian \( H_R \). These arise from the ‘generating functional’

\[ \alpha_i := \{ l_i, M \}. \quad (5.9) \]

Expanding \( \alpha_i \) in a power series in \( \lambda \) gives the infinite set of functionals \( \alpha_i^n \) as coefficients of \( \lambda^n \). Explicitly

\[ \alpha_i^n := \frac{\partial^n}{\partial \lambda^n} \{ l_i, M \}|_{\lambda=0}. \quad (5.10) \]

The first three of these are

\[ \alpha_i^0 = \{ l_i, \text{TrPexp} \int_0^{2\pi} d\theta \ A_1(\theta) \}, \quad (5.11) \]

\[ \alpha_i^1 = \{ l_i, \frac{1}{t} \int_0^{2\pi} d\theta \ \text{Tr} \left[ V(0, \theta)A_0(\theta)V(\theta, 2\pi) \right] \}, \quad (5.12) \]

\[ \alpha_i^2 = \{ l_i, \frac{1}{t} \int_0^{2\pi} d\theta \ \text{Tr} \left[ V(0, \theta)A_1V(\theta, 2\pi) \right] \]
\[ + \frac{2}{t^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \ \text{Tr} \left[ V(0, \theta')A_0(\theta')V(\theta', \theta)A_0(\theta)V(\theta, 2\pi) \right] \}, \quad (5.13) \]

where

\[ V(0, \theta) := \text{TrPexp}[ \frac{1}{t} \int_0^\theta d\theta' \ A_1(\theta', t) ]. \quad (5.14) \]

The functionals on the right hand sides in the Poisson brackets resemble the loop variables in 3+1 gravity in the Ashtekar formulation [23]. In the two Killing field reduction here, there is effectively only the loop that wraps around the \( \theta \) circle. However, unlike the 3+1 gravity loop variables, these Poisson commute with the reduced Hamiltonian. We note that there is a factor of \( 1/t \) associated with each insertion of \( A_0 \) or \( A_1 \) on the loop, and that \( n \) counts the number of such insertions. This suggests an affine algebra structure for the \( \alpha_i^n \) Poisson algebra.
What we have given are the observables in the Ashtekar variables for the spacetime metric (3.22), which is the standard form for metrics with two commuting spacelike Killing vector fields. To calculate the observables explicitly from a given spacetime metric of the class we are considering is a straightforward procedure. The steps are: (i) Calculate the extrinsic curvature \( k^i_a \) and Christoffel connection \( \Gamma^i_a(E) \), which gives the Ashtekar connection \( A^i_a = \Gamma^a_i + i k^i_a \), (ii) calculate the \( \text{sl}(2) \) variables \( L_i \) and \( J_i \) (2.24), which gives the variable \( a_1 \) (4.6), and finally, (iii) calculate the generating functional \( M[a_i] \) (5.2), whose expansion in powers of \( \lambda \) gives all the observables. This will give the observables explicitly as functions of the spacetime metric variables, rather than as functions of Hamiltonian variables.

**VI. REALITY CONDITIONS**

So far the theory we have been discussing is a reduction of complex general relativity. Therefore the phase space observables given in the last section are also complex. In order to obtain the observables for the real Euclidean and Lorentzian theories we must impose the reality conditions (2.6) and (2.7).

The Euclidean conditions simply imply that the phase space variables must be real from the start. Hence the specialization of the observables to this case is easy - we set \( J_i \) and \( L_i \) in the generating functional \( M \) (5.2) to be real, which leads directly to real observables.

For the Lorentzian theory, we note first that a real observable can always be defined for the complex theory. To see this we first set \( E^{ai} \) to be real, which is one of the reality conditions. The reality condition on \( A^i_a \) implies that \( \delta / \delta A = -\delta / \delta A^\dagger \). This implies that the complex conjugate of an observable is also an observable because the complex conjugates of the constraints are also constraints. Then if \( O[E, A] \) is an observable for the complex theory, a real observable for the complex theory is \( O[A, E] + O[\bar{A}, E] \) (24). Therefore the observables in the Lorentzian theory are given by

\[
O_L[A, E] := (O[A, E] + O[\bar{A}, E])|_{A = 2\Gamma^i - A} .
\]  

(6.1)

For our case, we can define the Lorentzian generating functional \( M_L \) for the Lorentzian
observables from $M$ (5.2) in exactly this way. Equivalently, this can be done separately for each observable derived from $M$. Therefore the symmetries described above go through for the Lorentzian theory as well.

**VII. CONCLUSIONS AND DISCUSSION**

There are three main results presented in this paper. The first is a rewriting of the vacuum Einstein equations for metrics with two commuting spacelike Killing vector fields, such that they resemble the field equations of the SL(2) principal chiral model. The only difference from the latter is that the 'coupling constant' is explicitly time dependent. The second result is a further rewriting of the reduced equations which leads to a generalized zero curvature formulation. The third is the explicit identification of an infinite set of phase space functionals which generate the hidden symmetries of the reduced Hamiltonian via Poisson brackets. These phase space functionals are spatial-diffeomorphism invariant and their time dependence is explicit. *This amounts to a solution of the equations of motion for this infinite set of variables.*

We have not addressed the question of Liouville integrability for this system, though the above results may provide a first step in this direction. The infinite set of phase space functionals given above that commute with the Hamiltonian do not have non-vanishing Poisson brackets with one another. It has been shown in Ref. [11] that the Lie algebra of the Geroch group is in fact the sl(2) affine algebra. It is possible that the algebra of our observables is exactly this, since they appear to form an sl(2) loop algebra.

For integrability one would like to show whether there are sums of products of observables which do commute with one another. For the two-dimensional models with standard (time independent) zero-curvature equations, it is possible to show that there are two distinct symplectic structures on their phase spaces. This fact leads to a relatively easy proof of integrability [1]. In the present case, with the understanding achieved so far, we do not know how to do this.
Finding observables in the classical theory is a prerequisite for certain quantization schemes. There is some debate concerning how observables should be defined [25] in a generally covariant theory. One view is that observables should be fully gauge invariant, which means that they are constants of the motion or ‘perennials’. The quantum theory would then be constructed by finding suitable representations of the Poisson algebra of these observables. This raises the question of how one would see time evolution in the quantum theory, since constants of the motion do not evolve. Another view is that only kinematically gauge invariant functionals should be used for quantization, and that the Hamiltonian constraint should be converted into a functional Schrödinger equation. In this approach, if it can be carried through, time evolution would be seen in the same way as for non-generally covariant theories.

The observables we have given fall into neither category because we have fixed a specific time gauge. The observables commute with the reduced Hamiltonian, but also have explicit time dependence. Thus, this situation appears to have the virtues of both the above viewpoints. In particular, as stated above, these observables are solutions to the equations of motion. A drawback may be that, although two-volume may be a physically reasonable definition of time in cosmology, the quantum theory would be dependent on this preferred choice.

To proceed with quantization one would first need to know what the algebra of the $\alpha_i^n$ is. Since each $n$ counts the number of insertations of $A_0$ or $A_1$ in the $\theta$ circle, the algebra structure already resembles that of an affine (Kac-Moody) algebra. This is because the Poisson bracket of elements with $m$ and $n$ insertions would lead to one with $m + n$ insertions. We conjecture, from the above similarity with the chiral model and the result of Ref. [11], that the algebra of certain sums of these observables is the sl(2) affine algebra. The quantum theory would then arise as a representation of this algebra. Since the time evolution of each $\alpha_i^n$ is already known, the result would be an evolving quantized algebra of observables. To see that this indeed comes about is a topic for further work.

This paper has been restricted to the case of spacetimes that have compact spatial
surfaces. For the non-compact case one would have to keep track of the boundary terms that arise in the constraints. In particular there will be a surface contribution to the Hamiltonian. Here also it would be of interest to find a gauge fixing that leads to some generalized zero-curvature form of the evolution equations that allows the extraction of physical observables.

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