Chaos and sub-diffusion in wave packets with periodically kicked interactions

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We study the quantum dynamics of a peculiar driven system, a Bose gas subjected to periodically-kicked interactions. In the limit of infinitely short kicks, this system was recently shown to exhibit a fast exponential spreading of the wave function. Here we examine this problem for kicks of arbitrary duration and show that, in this case, the spreading is not exponential but rather sub-diffusive at long time. This phenomenon stems from the competition between the kinetic and interaction energies within the kicks, which is absent in the limit of delta kicks. Our analysis further shows that the breakdown of exponential spreading occurs at relatively short times even for extremely short kicks, suggesting that, in practice, sub-diffusion should be more the rule than the exception in this system.

I. INTRODUCTION

Advances in non-equilibrium quantum physics have recently revealed the richness of periodically-driven systems [1–3], which in general “heat” to an infinite-temperature state due to the interplay between external forcing and interactions [4–6]. Among driven systems, quantum kicked rotors have played a central role, as paradigmatic models for quantum chaos [7, 8] exhibiting the phenomenon of dynamical localization in momentum space [9–12]. In the quantum kicked rotor, the role of interactions has also been explored on the basis of the Gross-Pitaevskii equation [13–17]. It has been found, in particular, that even a weak nonlinearity may have a dramatic impact on the dynamics, by breaking the localization of wave packets and leading to a sub-diffusive spreading. A similar phenomenon was also pointed out in nonlinear, spatially disordered chains [15, 18–23].

In the recent years, there has also been a growing interest in driven quantum systems involving temporally-modulated interactions [24–27], used for instance to design synthetic gauge fields or to modify many-body quantum transport. In cold-atom experiments, modulations of the interaction are typically implemented using the technique of Feshbach resonances [28]. In the present work, we explore the quantum dynamics of a Bose-Einstein condensate subjected to periodically-kicked interactions. In the case of infinitely short (delta) kicks, it has been recently shown that this system exhibits an ultrafast, exponential spreading of the wave function in momentum space [29, 30]. Such an exponential spreading was confirmed by methods of classical chaos based on the calculation of Lyapunov exponents [29] and on a mapping with a generalized kicked rotor [30]. Here, we revisit this problem by considering interaction kicks or arbitrary duration. While we recover the exponential spreading in the limit of delta kicks, we find that, as soon as the kicks are finite, the spreading is no longer exponential but rather sub-diffusive at long time. This phenomenon stems from the competition between the kinetic and interaction energies within the kicks, which is absent in the limit of delta kicks. At the microscopic level, we interpret this sub-diffusive spreading in terms of a mechanism of incoherent coupling of the momentum sites. As regards the momentum distribution of the Bose gas, we find that the periodically-kicked interactions first give rise to an early-time exponential depletion of the condensate mode, quickly followed by the emergence of a “thermal” background of particles spreading sub-diffusively. Our analysis finally shows that the time scale where exponential spreading breaks down scales logarithmically with the kick duration. This indicates that, as soon as the kicks are finite, the sub-diffusive motion tends to take over the exponential spreading at relatively short times, even if extremely short kicks are considered.

The manuscript is organized as follows. In Sec. II, we present our model of a Bose gas subjected to a sequence of interaction kicks. The time evolution of a wave packet in the limit of delta kicks is then presented in Sec. III, and the results of previous works are recalled. In Sec. IV, we consider kicks of finite duration and show that wave packets spread sub-diffusively in that case. A simple model for sub-diffusion is introduced. In Secs. V and VI, we then discuss how the sub-diffusive motion shows up in the condensate fraction and the momentum distribution. We finally summarize and discuss our results in Sec. VII. Technical details are collected in the appendix.

II. THE MODEL

We study the mean-field, dynamical evolution of a one-dimensional Bose gas with a time-dependent interaction potential. This dynamics is described by the Gross-Pitaevskii equation

$$i\hbar \partial_t \Psi(x, t) = \frac{\hat{p}^2}{2m} \Psi(x, t) + g(t) |\Psi(x, t)|^2 \Psi(x, t),$$

with normalization \(\int dx |\Psi(x, t)|^2 = 1\) for the wave function \(\Psi(x, t)\). The momentum operator is \(\hat{p} = -i\hbar \partial_x\). We consider a periodic, temporal modulation of the interaction term taking the form of a sequence of square pulses

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(or “kicks”) of period $T$, width $\delta t$ and amplitude $gN$, with $g$ the interaction parameter and $N$ the total number of atoms:

$$g(t) = \begin{cases} 0 & \text{if } t \in [nT, (n+1)T - \delta t], \ n \in \mathbb{Z}, \\ gN & \text{otherwise}. \end{cases}$$ (2)

In practice, such a sequence can be realized by applying a periodic magnetic field modulation to the atomic cloud, exploiting a Feshbach resonance. From now on, we denote by $L$ the system size and assume periodic boundary conditions, thus describing a ring geometry. This implies that the eigenstates $p$ of the momentum operator are quantized in units of $2\pi\hbar/q$, where $q \in \mathbb{Z}$ is an integer.

In order to numerically study the time evolution entailed by the sequence (2), it is convenient to work with a dimensionless version of Eq. (1). To this aim, we first rescale position, time and wave function according to

$$t = \frac{t}{T}, \ x = 2\pi x/L, \ |\psi|^2 = |\Psi|^2 L/2\pi,$$ (3)

and introduce the effective Planck constant

$$\hbar_{\text{eff}} = \frac{\hbar}{m} \left(\frac{2\pi}{L}\right)^2.$$ (4)

This leads to the dimensionless Gross-Pitaevskii equation

$$i\hbar_{\text{eff}} \partial_t \psi(x,t) = \frac{\hbar_{\text{eff}}q^2}{2} \psi(x,t) + \gamma(t)|\psi(x,t)|^2 \psi(x,t),$$ (5)

where the reduced momentum operator is $\hat{q} = -i\partial_x$. The dimensionless position $x$ lies in the interval $[0,2\pi)$, and the new wave function still obeys $\int dx |\psi(x,t)|^2 = 1$. The dimensionless, self-interaction modulation is now given by

$$\gamma(t) = \begin{cases} 0 & \text{if } t \in [n, n+1 - \delta t/T], \ n \in \mathbb{Z}, \\ \gamma & \text{otherwise}, \end{cases}$$ (6)

where $\gamma = 2\pi g N \hbar_{\text{eff}} T/L\hbar$.

In [29, 30], Eqs. (5) and (6) were investigated in the limit of Dirac delta kicks, i.e., for $\delta t/T \rightarrow 0$, $\gamma \rightarrow \infty$ with the product $\gamma \delta t/T$ constant: $\gamma(t) = \gamma \delta t/T \sum_n \delta(t-n)$. With this model, which is known as the Gross-Pitaevskii map [29], the authors of [29, 30] observed a strongly chaotic dynamics characterized by an exponential spreading of the wave function in momentum space. In [31], this model was also shown to support stroboscopic solitonic solutions. A particular consequence of taking the limit of pure delta kicks is that the kinetic energy is irrelevant at the specific times where the kicks are nonzero. This is no longer the case as soon as the kick duration is finite: during the kicks, the kinetic energy cannot be neglected and competes with the interaction term. This is precisely the situation we explore in the following, where we will show that this competition dramatically modifies the spreading of wave packets.

The time evolution of the state vector during one period (free evolution and kick) is governed by the evolution operator

$$\hat{U}(n) = \mathcal{T} \exp \left[-i \int_{n \delta t/T}^{n+1 \delta t/T} dt' \left(\frac{\hbar_{\text{eff}}q^2}{2} + \frac{\gamma|\psi|^2}{\hbar_{\text{eff}}} \right)\right] \times \exp \left[-i \int_{n}^{n+1} dt' \left(\frac{\hbar_{\text{eff}}q^2}{2}\right)\right],$$ (7)

where $\mathcal{T}$ is the time-ordering operator. In this expression, the first exponential refers to the evolution during kick $n$, while the second one describes the free evolution stage before it. To study the system’s dynamics, from now on we focus for simplicity on the specific limit $\hbar_{\text{eff}} \gg 1$, where a random phase approximation can be used. Indeed, in this case the phase $\sim \hbar_{\text{eff}}$ accumulated during the free evolution stage is very large, such that it can be accurately replaced by a random variable $\phi_q$ uniformly distributed over $[0,2\pi]$. Note that we cannot apply the same approximation for the kinetic phase in the first exponential of Eq. (7), which is of the order of the product $\hbar_{\text{eff}} \delta t/T$, not necessarily large. To deal with the latter, it is convenient to introduce the change of variables $s = (T/\delta t) t' + n(1 - T/\delta t)$, so that

$$\hat{U}(n) = \mathcal{T} \exp \left[-i \int_{n}^{n+1} ds \left(\frac{\hat{q}^2}{2f^2} + \gamma^*|\psi|^2\right)\right] \exp(-i\phi_q),$$ (8)

where

$$\gamma^* = \frac{\gamma \delta t}{T\hbar_{\text{eff}}} = \frac{2\pi g N \delta t}{L\hbar}$$ (9)

is the effective interaction strength, and

$$f = \sqrt{\frac{T}{\delta t \hbar_{\text{eff}}}} = \frac{L}{2\pi} \sqrt{\frac{m}{\hbar}}$$ (10)

controls the amplitude of the kinetic energy during the kicks. Information about the finite duration of the kicks is entirely contained in this parameter. In particular, when $f = \infty$ the kinetic term in Eq. (8) vanishes and one effectively recovers the delta-kick limit of [29, 30].

The evolution operator (8) can be readily implemented numerically to describe the chaotic dynamics entailed by Eq. (5) for arbitrary kick durations. To this aim, we consider the evolution of the wave function in momentum space:

$$\psi_q(t) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} dx e^{iqx} \psi(x,t).$$ (11)

Recalling that permissible reduced momenta $q \in \mathbb{Z}$, this relation can be inverted as

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \sum_q e^{-iqx} \psi_q(t),$$ (12)

with the normalization $\sum_q |\psi_q(t)|^2 = 1$. In the following, we take as an initial state the wave function

$$\psi_q(t = 0) \propto \exp(-\lambda^2 q^2),$$ (13)
of momentum width $\lambda^{-1}$ typically smaller than 1. In practice, this state is a good model for the narrow momentum distribution of a Bose-Einstein condensate. Note that the corresponding spatial distribution is broad, nearly uniform at the scale of the system size, and it remains uniform on average during the time evolution. In contrast, we will see in the next section that its momentum distribution exhibits a non-trivial behavior as a result of the combined effect of the nonlinear and kinetic terms in Eq. (8).

In order to numerically describe the evolution of the wave packet $\psi(q,t)$, we successively apply the evolution operator (8) to the initial state (13), using a second-order split-step method to evaluate the wave function at each time step $\Delta s$ [32]. The latter is always chosen much smaller than the unit time scale, typically $\Delta s \sim 10^{-4}$.

In the simulations we discretize the interval $[0, 2\pi]$ into $N_s$ spatial steps, where $N_s \gg 1$. All our results, finally, are averaged over typically $N_q \sim 10^4$ realizations of the random phase $\phi_q$. Some observables of interest, like $|\psi_q|^2$, are very sensitive to the numerical instability inherent to the non-linear Schrödinger equation [33, 34]. This is especially true for initial conditions that are very narrow in momentum space. In order to bypass this problem, we have worked with a high-precision arithmetic whenever exponential sensitivity to initial conditions was the limiting factor. Typically our algorithm ensured $N_d = 10^4$ significant decimal digits. We have always checked that increasing $1/\Delta s$, $N_s$ or $N_d$ does not alter our numerical calculations.

III. WAVE-PACKET SPREADING FOR DELTA KICKS

To characterize temporal spreading of the wave packet (13) subjected to the interaction kicks, we examine the temporal evolution of its mean-square width in momentum space,

$$\sigma^2(t) = \sum_q q^2 |\psi_q(t)|^2,$$

(14)

where the overbar refers to averaging over the random phase $\phi_q$ accumulated between the kicks. In this section, we focus on the limit $f = \infty$ of delta kicks. The corresponding time evolution of $\sigma^2(t)$ is shown in Fig. 1, and emphasizes two distinct dynamical regimes. $\sigma^2(t)$ first grows exponentially up to a certain characteristic time $t_E$ ($t_E \simeq 80$ in Fig. 1). Then, for $t > t_E$, the growth slows down albeit it remains exponential.

To clarify the origin of this result, we first discuss the short times $t < t_E$. The growth of $\sigma^2(t)$ in this regime corresponds to a fast initial depletion of the condensate from $q = 0$ to the neighboring momentum sites $q \neq 0$. To describe it quantitatively, we start from the evolution equation of the Fourier modes during a given kick $n$:

$$i\partial_t \psi_j = \frac{\gamma^*}{\pi} \sum_{q_1, q_2} \psi_{q_1}^* \psi_{q_2} \psi_{q+q_1-q_2}.$$

(15)

This equation corresponds to the first exponential term in the evolution operator (8). At short time, mostly modes $q = -1, 0, 1$ are populated. Neglecting the other modes and assuming $|\psi_j|^2 \ll |\psi_0|^2$ ($j = \pm 1$), we can linearize Eq. (15), which leads to $\psi_0(t) \simeq \exp(-i\gamma^* t/2\pi)$ and

$$i\partial_t \psi_j \simeq \frac{\gamma^*}{\pi} \psi_j + \frac{\gamma^*}{2\pi} \psi_0^2 \psi_{-j}^*.$$

(16)

These equations contain oscillatory factors that are conveniently removed with the gauge transformation $\tilde{\psi}_j = \psi_j \exp(i\gamma^* t/2\pi t)$. Then we introduce the circular state vector for the first Fourier mode after the kick $n$, $\Gamma(n) = (\Re \tilde{\psi}_1(n), \Im \tilde{\psi}_1(n))^\Gamma$, and assume for simplicity $\tilde{\psi}_1 = \tilde{\psi}_{-1}$. The propagation of this state vector over one period obeys $\Gamma(n) = U \Gamma(n-1)$, where the transfer matrix $U$ is given by

$$U \simeq \begin{pmatrix} 1 & 0 \\ -\gamma^* / \pi & 1 \end{pmatrix} \begin{pmatrix} \cos \phi_1 - \sin \phi_1 \\ \sin \phi_1 \cos \phi_1 \end{pmatrix}.$$

(17)

The second matrix in the right-hand side describes the free-space propagation between the kicks $n - 1$ and $n$, which involves the uniformly distributed kinetic phase $\phi_1$, see Eq. (8). The first matrix, on the other hand, describes the propagation during the kick $n$, and is inferred from Eq. (16) and its complex-conjugated version. The
population of the first Fourier mode after \( n \) kicks, finally, follows from:

\[
|\psi_1(t = n)|^2 = \frac{1}{2\pi} \int_0^{2\pi} d\phi_1 \|U^n \Gamma(0)\|^2.
\]  

(18)

The calculation of the integral over \( \phi_1 \) is detailed in the appendix. At weak interaction strength \( \gamma^*/\pi \ll 1 \), we find

\[
|\psi_1(t)|^2 \approx e^{-2\lambda^2} \left[ 1 + \frac{1}{2\pi} \sqrt{\frac{\gamma^*}{2t}} e^{\gamma^*/\pi} \right].
\]  

(19)

This indeed describes an exponential growth of the first Fourier mode, at the rate \( \gamma^*/\pi \). In the short-time regime where Eq. (19) is valid, the mean-square width (14) is dominated by the contribution of the first mode, i.e., \( \sigma^2(t) \approx 2|\psi_1(t)|^2 \). This result is shown in Fig. 1 (dashed curve), and matches very well the numerical simulations.

The linearization approach above can also be used to access \( t_E \). Physically, \( t_E \) is the typical time needed for the wave function to spread over the accessible phase space, the so-called Ehrenfest time [37]. In practice, it is also the time scale where the linearization used to obtain Eq. (16) breaks down. It can be simply estimated by the condition \( \max_{\phi_1} \|U^t \Gamma(0)\|^2 = 1/2 \), i.e. the specific configuration where \( |\psi_1|^2 \) is maximal and becomes of the order of \( |\psi_0|^2 \) (recall that, as long as only two modes are considered, \( |\psi_0|^2 + 2|\psi_1|^2 = 1 \) due to the normalization). This criterion leads to (see appendix)

\[
t_E \approx \frac{2\pi\lambda^2}{\gamma^*}.
\]  

(20)

Notice that the more sites the initial wave packet covers, the earlier one deviates from the perturbative approach. The Ehrenfest time (20) is reported in Fig. 1, and it well describes the crossover to the long-time expansion regime.

When \( t > t_E \), many \( q \)-sites start to be populated and the linearization approach no longer applies. Study of this regime was the main goal of the previous works [29, 30], where it was found that

\[
\sigma^2(t) \propto \exp[t \ln(1 + (\gamma^*/\pi)^2)].
\]  

(21)

In [29], in particular, the authors derived this exponential growth by rewriting Eq. (5) in the form of a generalized kicked-rotor model and by studying the evolution of \( \sigma^2(t) \) in the corresponding classical map. A similar exponential growth was also found in [35], in a slightly different model involving a linear kicking potential on top of the nonlinear sequence of kicks. The exponential law (21) is shown in Fig. 1, and well captures the numerical results at long time.

IV. FINITE KICKS: LONG-TIME SUB-DIFFUSION

Let us now consider the case of finite kicks, which is the main object of the paper. The evolution of the wave-packet mean-square width for finite values of \( f \) is displayed in Fig. 2. The limit \( f = \infty \) is also shown for comparison. The figure shows that when \( f \) is finite, the behavior of \( \sigma^2(t) \) at times \( t < t_E \) remains well captured by Eq. (19), except, perhaps, for small values of \( f \). On the other hand, a dramatically different evolution emerges beyond \( t_E \): the growth of \( \sigma^2(t) \) is no longer exponential but rather algebraic, with a prefactor increasing with \( f \).

An analysis up to \( t \simeq 5 \times 10^3 \) is shown in the inset, and suggests the following algebraic scaling at long time:

\[
\sigma^2(t) \propto t^{1/2}.
\]  

(22)

Such a sub-diffusive behavior can be understood in terms of a mechanism of incoherent heating of the neighboring sites of the spreading wave packet. A similar mechanism was shown to also take place in the context of the nonlinear Schrödinger equation in the presence of disorder [13, 14]. To make this idea more quantitative, we start from the evolution equation during a kick at finite \( f \):

\[
\mathbf{i} \partial_t \psi_q = \frac{\sigma^2}{2f^2} \psi_q + \frac{\gamma^*}{2\pi} \sum_{q_1, q_2} \psi^*_q \psi_{q_1} \psi_{q_2} \psi_{q + q_1 - q_2}.
\]  

(23)

Let us now consider a certain momentum-site \( q \) located at the border of the spreading wave packet. At the contact with the wave packet, the amplitude of this site evolves according to Eq. (23). To simplify this equation, we first suppose that the complex amplitudes \( \psi_q \) become completely random at long enough time. This assumption is corroborated by the numerical simulations in Fig. 3, which show that the distribution of the \( |\psi_q(t)|^2 \) becomes exponential at long time. We then make the hypothesis that the coupling between the site \( q \) and the spreading

FIG. 2. Main panel: mean-square width of the wave packet as a function of the number of kicks \( t \) up to \( t = 600 \), for finite, increasing values of \( f \). Solid curves from bottom to top correspond to \( f = 1, 2, 4, 8, 16, 32, 64 \). Here \( \gamma^* = 4, \lambda = 3.03 \). The black curve corresponds to numerical results in the limit \( f = \infty \) (delta kicks). Inset: mean-square width up to \( t = 5000 \) at \( f = 16 \), emphasizing the sub-diffusive behavior.
wave packet consists in an incoherent heating mechanism, i.e., an incoherent increase of the amplitude \( \psi_q(t) \). This leads us to approximate Eq. (23) by a Langevin equation
\[
 i\partial_t \psi_q \simeq \frac{q^2}{2f^2} \psi_q + \frac{\gamma^*}{2\pi} f^2 \rho(t)^{3/2} \eta(t),
\]
where \( \rho(t) \) is the momentum density of the spreading wave packet, which for the simplicity of the argument we here take uniform, and \( \eta(t) \) is an uncorrelated random noise, satisfying \( \eta(t)\eta(t') = \delta(t-t') \). The prefactor \( f^2 = N^2 \) stems from the number \( N \) of terms effectively involved in each sum in the right-hand side of Eq. (23). In the sub-diffusive regime, this number is such that the kinetic term in Eq. (23) is appreciable, i.e., \( q \sim N \sim f \).

At long time, the solution of the Langevin equation for the average squared amplitude at the site \( q \) is \( |\psi_q(t)|^2 \sim f^4 \gamma^*^2 \rho^3 t \). From this, we infer that the typical time \( \tau \) it takes for the wave packet to “heat” the site \( q \) is such that \( \rho \sim f^4 \gamma^*^2 \rho^3 \tau \), giving \( \tau^{-1} \sim f^4 \gamma^*^2 \rho^3 \). Finally, we assume that the wave-packet spreading can be described by a nonlinear diffusion equation of the type \( d\sigma^2(t)/dt = D(t) \), with a diffusion coefficient \( D(t) \) proportional to the heating rate \( \tau^{-1} \). This leads to
\[
 \sigma^2(t) \sim f^4 \gamma^*^2 \rho^2 t.
\]
For a uniform wave packet, \( \rho(t) = 1/\sigma(t) \) due to norm conservation. The solution of Eq. (25) for \( \sigma^2(t) \) then immediately yields
\[
 \sigma^2(t) \sim f^2 \gamma^* t^{1/2},
\]
which well describes the behavior observed in the numerical simulations.

\section{Condensate Fraction and Crossover to the Delta-Kick Limit}

Another relevant quantity for probing the difference between finite and delta kicks is the average Bose condensate fraction, defined as \( \langle |\psi_0(t)|^2 \rangle \). This quantity is shown in the main panel of Fig. 4. Like for \( \sigma^2(t) \), we first discuss the case of delta kicks, \( f = \infty \), which is represented by the solid black line. Numerically, we find that in this limit the condensate fraction decays exponentially at all times beyond \( t_d \) \( \langle |\psi_0(t)|^2 \rangle \sim \exp[-(t - t_d)/\lambda] \). To understand this exponential decay and the associated time scale \( \tau \), we first note that \( \langle |\psi_0(t)|^2 \rangle \) is essentially the probability for the condensate mode to remain populated at time \( t \). Then, we use the qualitative argument that, at a time \( t = n \lambda \), this probability is also the probability that the first \( n - 1 \) momentum excitations have not grown exponentially (see Sec. III). As seen in the appendix, at small interaction strength the probability for a given excitation to grow exponentially is \( \mathcal{P} \sim \gamma^*/\pi^2 \ll 1 \). We infer
\[
 \langle |\psi_0(t)|^2 \rangle \sim (1 - \mathcal{P})^{n-1} \sim \exp[-(n - 1)\mathcal{P}]
\]
\[
 = \exp[-\gamma^*(t - t_d)/(\pi^2 t_d)].
\]
We conclude that the characteristic time \( \tau \) to deplete the condensate is given by
\[
 \tau \simeq \frac{\pi^2 t_d}{\gamma^*} = \frac{2\pi^3 \lambda^2}{\gamma^* \tau^2}.
\]
We have also studied this time scale numerically as a function of \( \gamma^* \) and \( \lambda \), as shown by the two plots in Fig. 5, and the results agree with the prediction (28).
Let us now consider kicks of finite duration. The time evolution of the condensate fraction in that case is illustrated by the colored curves in Fig. 4. As for $\sigma^2$, as soon as $f$ is finite we observe a clear deviation from the exponential scaling, $|\langle \psi_0(t) \rangle|^2$ decreasing much more slowly. An analysis of the condensate fraction over longer times, shown in the inset of Fig. 4, again points toward a sub-diffusive behavior at finite $f$. We find $|\langle \psi_0(t) \rangle|^2 \sim 1/t^{1/4}$, which is fully consistent with the sub-diffusive law (22) for the mean-square width (see also Sec. VI).

The time evolutions of the condensate fraction at finite and infinite $f$ discussed above can be used to estimate the characteristic time $t_f$ beyond which the model $f = \infty$ of delta kicks can no longer be reliably utilized to describe the chaotic dynamics. This question is crucial from a practical point of view, since in a real experiment the duration of the kicks cannot be made arbitrarily small, especially if the bosonic interactions are modulated using Feshbach resonances. To find $t_f$, we interpolate the temporal scalings of the condensate fraction at finite and infinite $f$, where $|\langle \psi_0(t) \rangle|^2 \sim 1/ft^{1/4}$ and $|\langle \psi_0(t) \rangle|^2 \sim \exp[-(t - t_E)/\tau]$, respectively. This method, illustrated in the inset of Fig. 4, yields

$$t_f \sim \tau \ln f \sim \frac{2\pi^3 \lambda^2}{\gamma^2} \ln f.$$  

(29)

The logarithmic dependence of $t_f$ on $f$ has a remarkable consequence. Even for extremely large values of $f$, i.e., for extremely short kick durations, the breakdown of the exponential decay of the condensate fraction [or the exponential growth of $\sigma^2(t)$] occurs at relatively short times (this phenomenon is, in fact, visible by eye in Fig. 2).

VI. MOMENTUM DISTRIBUTION

All the above findings can be summarized by looking at the average momentum distribution of the Bose gas at different times. Such distributions are displayed in the upper panel of Fig. 6. The distributions first exhibit an exponential decay of the condensate mode, $|\langle \psi_0(t) \rangle|^2$, at the scale of $\tau$, quickly accompanied by a slow growth of the “thermal” modes $q \neq 0$. The latter control the sub-diffusive evolution of the wave-packet mean-square width according to Eq. (22). The lower panel is a zoom on the central part of the distribution at $t = 200$ and $t = 500$. These profiles are very well described by the Gaussian distribution (30), without any fit parameter.

This implies that, in a real experiment that unavoidably involves finite kicks, the sub-diffusive behavior described in the present work should be more the rule than the exception.
approximated by the (normalized) Gaussian profile
\[ |\psi(t)|^2 \sim \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp \left( -\frac{q^2}{2\sigma^2(t)} \right), \]
where \( \sigma^2(t) \) is the mean-square width of the whole distribution [satisfying Eq. (22)]. Note that this law in particular implies \( |\psi_0(t)|^2 \sim 1/t^{1/4} \), in accordance with the results of the previous section. This Gaussian shape is another marked difference with the behavior observed in the \( f = \infty \) limit, for which the profile is known to be exponential at all momenta [29, 30, 35].

When \( f \) is finite, nevertheless, our numerics suggests that the far \( q \)-wings of the momentum distribution also decay exponentially, see Fig. 6. At variance with the \( f = \infty \) limit, however, here the exponential decay length \( \xi(t) \) does not grow exponentially in time, but rather sub-diffusively. The numerical results of Fig. 7, indeed, suggest \( |\psi_q(t)|^2 \sim \exp[-|q|/\xi(t)] \) with \( \xi(t) \sim t^\alpha \) and \( \alpha \) close to 1/3. Although the degree of universality of this value is not yet clear to us, this sub-diffusive law is apparently different from the one governing the variance \( \sigma^2(t) \), see Fig. 7. There is, of course, no contradiction at this stage, since the exponential wings provide a negligible contribution to the variance of the whole distribution. While we have not been able to find an analytical basis for the scaling of the far wings, it could stem from a mechanism different from the incoherent heating discussed in Sec. IV, involving, for example, resonant coherent coupling between the spreading wave packet. Clarifying this question would constitute an interesting challenge for future work.

FIG. 7. Time evolution of the exponential decay length \( \xi(t) \) governing the far wings of the momentum distribution. A linear fit (dashed line) suggests an algebraic scaling close to \( \xi(t) \sim t^{1/3} \). The time evolution of the variance \( \sigma^2(t) \) is also shown for comparison. Here \( f = 16, \gamma^* = 4 \) and \( \lambda = 3.03 \).

VII. DISCUSSION AND SUMMARY

By considering a Bose gas subjected to a periodic modulation of the interactions taking the form of finite kicks, we have found evidence for the emergence of a mechanism of sub-diffusive spreading of the wave function beyond a characteristic Ehrenfest time. This result has to be contrasted with the case of delta kicks, where the spreading is always exponential. We have interpreted the sub-diffusive motion in terms of an incoherent heating process for the nonlinear coupling of momentum sites. Beyond this analysis, however, one may ask what fundamental mechanism could explain the different dynamics observed for finite and delta kicks. A possible explanation could be the different nature of the quantities conserved within a given kick in the two scenarios. Indeed, when \( f = \infty \), the evolution equation during one kick can be immediately integrated to yield
\[ \psi(x, n+1) = e^{-i\gamma^* |\psi(x,n)|^2} \psi(x,n). \]

The norm of the wave function is thus conserved for all point \( x \) in that case. This local constraint in position space suggests that, conversely, the coupling between modes is weakly constrained in momentum space, resulting in a very fast spreading of the wave packet. In contrast, when \( f \) is finite, such a local solution no longer exists and, instead, the nonlinear Schödinger equation involves only global integrals of motions of the form \( \int dx \psi^* (x,t)Q_j(x,t) \psi (x,t) \) with the \( Q_j(x,t) \) defined via recursion relations [36]. We expect this global character to translate into a much weaker coupling between the modes in reciprocal space.

Our analysis has also revealed that, for finite kicks, the sub-diffusive motion takes over the exponential spreading at a characteristic time that scales logarithmically with the kick duration. This characteristic time is thus always relatively short, even in the limit of the extremely short kicks. This suggests that, in practice, sub-diffusion rather than exponential spreading of wave packets should be more naturally observed in this system.

APPENDIX: EXPONENTIAL GROWTH OF THE FIRST FOURIER MODE

In this appendix, we calculate the population of the first Fourier mode at short time,
\[ |\psi_1(t = n)|^2 = \frac{1}{2\pi} \int_0^{2\pi} d\phi_1 \|U^n \Gamma(0)\|^2, \]
where the transfer matrix \( U \) is given by Eq. (17), and the initial state vector is \( \Gamma(0) = (e^{-\lambda^2}, 0)^T \). By symmetry, the contribution of negative \( \phi_1 \) equals the one of positive \( \phi_1 \), so that the integral average in Eq. (32) can be replaced by \( 1/\pi \int_0^{\pi} d\phi_1 \). Explicitly, the matrix \( U \) reads
\[ U = \begin{pmatrix} \cos \phi_1 & -\sin \phi_1 \\ -\frac{\gamma^*}{\pi} \cos \phi_1 + \sin \phi_1 & \cos \phi_1 + \frac{\gamma^*}{\pi} \sin \phi_1 \end{pmatrix}. \]
whose eigenvalues are of the form $\mu \pm \sqrt{\mu^2 - 1}$, with $\mu = \cos \phi_1 + (\gamma^*/2\pi) \sin \phi_1$. For values of $\phi_1$ such that $\mu^2 - 1 < 0$, the spectrum of $U$ is unimodular. The exponential growth of the first Fourier mode observed in the numerical simulations, on the other hand, stems from the contributions of $\phi_1$ such that $\mu^2 - 1 > 0$. Indeed, in this case one of the two (distinct) eigenvalues is of modulus strictly larger than one, eventually leading to

$$|\gamma| < 1,$$ 

in this case one of the two (distinct) eigenvalues is of modulus strictly larger than one, eventually leading to an exponential growth of $|\psi_1(t)|^2$. For weak interactions $\gamma^*/2\pi \ll 1$, $\mu^2 - 1 \simeq \phi_1(\gamma^*/\pi - \phi_1)$ such that the values of $\phi_1$ leading to $\mu^2 - 1 > 0$ lie in the interval $[0, \gamma^*/\pi]$. Note that this upper bound also defines the probability $P = \gamma^*/\pi^2$ that the first excitation grows exponentially when one "draws" a value of $\phi_1$.

The diagonalization of $U$ provides

\[
U^n \Gamma(0) = -\exp(-\lambda^2) \left( \frac{2}{\pi} \sinh x_n - \cosh x_n \right),
\]

for $\gamma^* \ll 1$, where $\eta = \phi_1 \gamma^*/2\pi$, $x = \sqrt{\phi_1(\gamma^*/\pi - \phi_1)}$ and $\nu = \gamma^*/\pi - \phi_1$. At leading order in $\gamma^*$, this leads to

$$|\psi_1(t = n)|^2 \simeq \left. e^{-2\lambda^2 + \gamma^* e^{-2\lambda^2} \int_0^{\pi/2} \frac{d\phi_1}{\phi_1} \sin^2[t \sqrt{\phi_1(\gamma^*/\pi - \phi_1)}]\right|,$$ 

The integral over $\phi_1$ can be calculated by a saddle point approximation, the saddle point being $\phi_1 = \gamma^*/2\pi$. This gives

$$|\psi_1(t)|^2 \simeq e^{-2\lambda^2 \left[ 1 + \frac{1}{2\pi} \sqrt{\gamma^*/2\pi} e^{-\gamma^*/\pi} \right]},$$

which is Eq. (19) of the main text. To find the Ehrenfest time (20), finally, we simply apply the criterion given in the main text, $\max_{\phi_1} ||U^{e\Gamma(0)}|| = 1/2$, together with Eq. (A.35), noting that the maximum of the integrand is attained at the saddle point $\phi_1 = \gamma^*/2\pi$.

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