Many cliques in $H$-free subgraphs of random graphs

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Abstract

For two fixed graphs $T$ and $H$ let $ex(G(n, p), T, H)$ be the random variable counting the maximum number of copies of $T$ in an $H$-free subgraph of the random graph $G(n, p)$. We show that for the case $T = K_m$ and $\chi(H) > m$ the behavior of $ex(G(n, p), K_m, H)$ depends strongly on the relation between $p$ and $m^2(H) = \max_{H' \subseteq H, |V(H')| \geq 3} \frac{e(H')-1}{\chi(H')-2}$.

When $m^2(H) > m^2(K_m)$ we prove that with high probability, depending on the value of $p$, either one can maintain almost all copies of $K_m$, or it is asymptotically best to take a $\chi(H)-1$ partite subgraph of $G(n, p)$. Thus, the transition between the two behaviors in this case occurs at $p = n^{-1/m^2(H)}$.

To show that the second case is not redundant we present a construction which may be of independent interest. For each $k \geq 4$ we construct a family of $k$-chromatic graphs $G(k, \epsilon_i)$ where $m_2(G(k, \epsilon_i))$ tends to $\frac{(k+1)(k-2)}{2(k-1)} < m_2(K_{k-1})$ as $i$ tends to infinity. This is tight for all values of $k$ as for any $k$-chromatic graph $G$, $m_2(G) > \frac{(k+1)(k-2)}{2(k-1)}$.

Key words: Turán type problems, random graphs, chromatic number.

1 Introduction

The well known Turán function, denoted $ex(n, H)$, counts the maximum number of edges in an $H$-free subgraph of the complete graph on $n$ vertices (see for example [22] for a survey). A natural generalization of this question is to change the base graph and instead of taking a subgraph of the complete graph consider a subgraph of a random graph. More precisely let $G(n, p)$ be the random graph on $n$ vertices where each edge is chosen randomly and independently with probability $p$. Let

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ex(G(n, p), H) denote the random variable counting the maximum number of edges in an H-free subgraph of G(n, p).

The behavior of ex(G(n, p), H) is studied in [8], and additional results appear in [18], [13], [11], [12] and more. Taking an extremal graph G which is H-free on n vertices with ex(n, H) edges and then keeping each edge of G randomly and independently with probability p shows that w.h.p., that is, with probability tending to 1 as n tends to infinity,

\[ ex(G(n, p), H) \geq (1 + o(1))ex(n, H)p. \]

In [13] Kohayakawa, Łuczak and Rödl and in [11] Haxell, Kohayakawa and Łuczak conjectured that the opposite inequality is asymptotically valid for values of p for which each edge in G(n, p) takes part in a copy of H.

This conjecture was proved by Conlon and Gowers in [6], for the balanced case, and by Schacht in [20] for general graphs (see also [5] and [19]). Motivated by the condition that each edge is in a copy of H with the maximum number of copies of H, then keeping each edge of G randomly and independently with probability p shows that w.h.p.,

\[ ex(G(n, p), H) \geq (1 + o(1))ex(n, H)p. \]

The Erdős-Simonovits-Stone theorem states that \( ex(n, H) = \binom{n}{2} \left( 1 - \frac{1}{\chi(H)-1} + o(1) \right) \), and so the theorem proved in the papers above, restated in simpler terms is the following

**Theorem 1.1 ([6], [20]).** For any fixed graph H the following holds w.h.p.

\[ ex(G(n, p), H) = \begin{cases} 
(1 - \frac{1}{\chi(H)-1} + o(1))\binom{n}{2}p & \text{for } p \gg n^{-1/m_2(H)} \\
(1 + o(1))\binom{n}{2}p & \text{for } p \ll n^{-1/m_2(H)}
\end{cases} \]

where here and in what follows we write \( f(n) \gg g(n) \) when \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \).

Another generalization of the classical Turán question is to ask for the maximum number of copies of a graph T in an H-free subgraph of the complete graph on n vertices. This function, denoted ex(n, T, H), is studied in [3] and in some special cases in the references therein. Combining both generalizations we define the following. For two graphs T and H, let \( ex(G(n, p), T, H) \) be the random variable whose value is the maximum number of copies of T in an H-free subgraph of G(n, p). Note that as before the expected value of \( ex(G(n, p), T, H) \) is at least \( ex(n, T, H)p^{e(T)} \) for any T and H.

In [3] it is shown that for any H with \( \chi(H) = k > m \), \( ex(n, K_m, H) = (1 + o(1))\binom{k-1}{m}(\frac{n}{k-1})^m \). This motivates the following question analogous to the one answered in Theorem 1.1. For which values of p is it true that \( ex(G(n, p), K_m, H) = (1 + o(1))\binom{k-1}{m}(\frac{n}{k-1})^m p^{\binom{m}{2}} \) w.h.p.?

We show that the behavior of \( ex(G(n, p), K_m, H) \) depends strongly on the relation between \( m_2(K_m) \) and \( m_2(H) \). When \( m_2(H) > m_2(K_m) \) there are two regions in which the random variable behaves differently. If p is much smaller than \( n^{-1/m_2(H)} \) then the H-free subgraph of G \( \sim G(n, p) \) with the maximum number of copies of K_m has w.h.p. most of the copies of K_m in G as only a negligible number of edges take part in a copy of H. When p is much bigger than \( n^{-1/m_2(H)} \) we can no longer keep most of the copies of K_m in an H-free subgraph and it is asymptotically best to take a \((k-1)\)-partite subgraph of G(n, p). The last part also holds when \( m_2(H) = m_2(K_m) \). Our first theorem is the following:
Theorem 1.2. Let $H$ be a fixed graph with $\chi(H) = k > m$. If $p$ is such that $\binom{n}{m}p^{m(\frac{m}{2})}$ tends to infinity as $n$ tends to infinity then w.h.p.

$$ex(G(n, p), K_m, H) = \begin{cases} (1 + o(1))\binom{\frac{n-1}{m}}{\frac{m}{2}}p^{m(\frac{m}{2})} & \text{for } p \gg n^{-1/m_2(H)} \text{ provided } m_2(H) \geq m_2(K_m) \\ (1 + o(1))\binom{m}{m}p^{m(\frac{m}{2})} & \text{for } p \ll n^{-1/m_2(H)} \text{ provided } m_2(H) > m_2(K_m) \end{cases}$$

Theorem 1.2 is valid when $m_2(H) > m_2(K_m)$. What about graphs $H$ with $\chi(H) = k > m$ as before but $m_2(H) < m_2(K_m)$? Do such graphs $H$ exist at all?

A graph $H$ is $k$-critical if $\chi(H) = k$ and for any subgraph $H' \subset H$, $\chi(H') < k$. In [15] Kostochka and Yancey show that if $k \geq 4$ and $H$ is $k$-critical, then

$$e(H) \geq \frac{(k + 1)(k - 2)v(H) - k(k - 3)}{2(k - 1)}.$$  

This implies that for every $k$-critical $n$-vertex graph $H$,

$$\frac{e(H) - 1}{v(H) - 2} \geq \frac{(k + 1)(k - 2)n - k(k - 3) - 2(k - 1)}{2(k - 1)(n - 2)} > \frac{(k + 1)(k - 2)}{2(k - 1)}. \quad (1)$$

Therefore for any $H$ with $\chi(H) = k$ one has

$$m_2(H) > \frac{(k + 1)(k - 2)}{2(k - 1)}.$$ 

This implies that Theorem 1.2 covers any graph $H$ for which $\chi(H) \geq m + 2$, since $m_2(K_m) = \frac{m+1}{2}$.

When $\chi(H) = m + 1$ the situation is more complicated. Before investigating the function $ex(G(n, p), K_m, H)$ for these graphs we show that the case $m_2(H) < m_2(K_m)$ and $\chi(H) = m + 1$ is not redundant. To do so we prove the following theorem, which may be of independent interest. The theorem strengthens the result in [1] for $m = 3$, expands it to any $m$, and by [15] it is tight.

Theorem 1.3. For every fixed $k \geq 4$ and $\epsilon > 0$ there exist infinitely many $k$-chromatic graphs $G(k, \epsilon)$ with

$$m_2(G(k, \epsilon)) \leq (1 + \epsilon)\frac{(k + 1)(k - 2)}{2(k - 1)}.$$  

This theorem shows that there are infinitely many $m + 1$ chromatic graphs $H$ with $m_2(H) < m_2(K_m)$. For these graphs there are three regions of interest for the value of $p$: $p$ much bigger than $n^{-1/m_2(K_m)}$, $p$ much smaller than $n^{-1/m_2(H)}$, and $p$ in the middle range.

One might suspect that as before the function $ex(G(n, p), K_m, H)$ will change its behavior at $p = n^{-1/m_2(H)}$ but this is no longer the case. We prove that for some graphs $H$ when $p$ is slightly bigger than $n^{-1/m_2(H)}$ we can still take w.h.p. an $H$-free subgraph of $G(n, p)$ that contains most of the copies of $K_m$:

Theorem 1.4. Let $H$ be a graph such that $\chi(H) = m + 1 \geq 4$, $m_2(H) < c$ for some $c < m_2(K_m)$ and there exists $H_0 \subset H$ for which $\frac{v(H_0) - 1}{v(H_0) - 2} = m_2(H)$ and $v(H_0) > M(m, c)$ where $M(m, c)$ is large enough. If $p \leq n^{-\frac{1}{m_2(H_0)} + \delta}$ for $\delta := \delta(m, c) > 0$ small enough and $\binom{n}{m}p^{m(\frac{m}{2})}$ tends to infinity as $n$ tends to infinity, then w.h.p.

$$ex(G(n, p), K_m, H) = (1 + o(1))\binom{n}{m}p^{m(\frac{m}{2})}.$$
On the other hand, we prove that for big enough values of $p$ one cannot find an $H$-free subgraph of $G(n,p)$ with $(1 + o(1))(n)p(n)$ copies of $K_m$ and it is asymptotically best to take a $k - 1$-partite subgraph of $G(n,p)$.

As an example we show that the theorem above can be applied to the graphs constructed in Theorem 1.3.

**Lemma 1.5.** For every two integers $k$ and $N$ there is $\epsilon > 0$ small enough such that $v(G_0(k, \epsilon)) > N$, where $G_0(k, \epsilon)$ is a subgraph of $G(k, \epsilon)$ for which $\frac{v(G_0(k, \epsilon)) - 1}{v(G_0(k, \epsilon)) - 2} = m_2(G(k, \epsilon))$.

The rest of the paper is organized as follows. In Section 2 we establish some general results for $G(n,p)$. In Section 3 we prove Theorem 1.2. In Section 4 we describe the construction of sparse graphs with a given chromatic number and prove Theorem 1.3. In Section 5 we prove Theorem 1.4 and Lemma 1.5. We finish with some concluding remarks and open problems in Section 6.

## 2 Auxiliary Results

We need the following well known Chernoff bounds on the upper and lower tails of the binomial distribution (see e.g. [4], [17]).

**Lemma 2.1.** Let $X \sim \text{Bin}(n, p)$ then

1. $\mathbb{P}(X < (1 - a)\mathbb{E}X) < e^{-\frac{a^2\mathbb{E}X}{2}}$ for $0 < a < 1$
2. $\mathbb{P}(X > (1 + a)\mathbb{E}X) < e^{-\frac{a^2\mathbb{E}X}{3}}$ for $0 < a < 1$
3. $\mathbb{P}(X > (1 + a)\mathbb{E}X) < e^{-\frac{a\mathbb{E}X}{3}}$ for $a > 1$

The following known result is used a few times

**Theorem 2.2** (see, e.g., Theorem 4.4.5 in [4]). Let $H$ be a fixed graph. For every subgraph $H'$ of $H$ (including $H$ itself) let $X_{H'}$ denote the number of copies of $H'$ in $G(n,p)$. Assume $p$ is such that $\mathbb{E}[X_{H'}] \to \infty$ for every $H' \subseteq H$. Then w.h.p.

$$X_H = (1 + o(1))\mathbb{E}[X_H].$$

In addition we prove technical lemmas to be used in Sections 3 and 5. From here on for two graphs $G$ and $H$ we denote by $N(G, H)$ the number of copies of $H$ in $G$.

**Lemma 2.3.** Let $G \sim G(n,p)$ with $p \gg n^{-1/m_2(K_m)}$ then w.h.p.

1. Every set of $o(pm^2)$ edges takes part in $o(N(G, K_m))$ copies of $K_m$,
2. For every $\epsilon > 0$ small enough every set of $n^{-\epsilon}pm^2$ edges takes part in at most $n^{-\epsilon/3}N(G, K_m)$ copies of $K_m$.

**Proof.** Let $G \sim G(n,p)$ and let $X$ be the random variable counting the number of copies of $K_m$ on a randomly chosen edge of $G(n,p)$. First we show that $\mathbb{E}[X^2] \leq O(\mathbb{E}[X])$. Given an edge let $\{A_1, \ldots A_l\}$ be all the possible copies of $K_m$ using this edge in $K_n$ and let $|A_i \cap A_j|$ be the number
of vertices the copies share. Let $X_{A_i}$ be the indicator of the event $A_i \subset G$. Then $X = \sum X_{A_i}$ and we get that

$$\mathbb{E}^2[X] = (\sum \mathbb{E}[X_{A_i}])^2 = \Theta([n^{m-2}p^{(m)}(2)-1]^2)$$

$$\mathbb{E}[X^2] = \mathbb{E}[\sum_{k=2}^{m} \sum_{|A_i \cap A_j| = k} X_{A_i} X_{A_j}]$$

$$\leq \sum_{k=2}^{m} n^{2m-k-2}p^{(m)}(2)^{(m-k)+(m-k)-1}$$

Put $S_k = n^{2m-k-2}p^{(m)}(2)^{(m-k)+(m-k)-1}$ and note that $S_2 = \Theta(\mathbb{E}^2[X])$. Furthermore, for any $2 < k \leq m$ the following holds $S_2/S_k = n^{k-2}p^{(k)}(2)-1 n^{-\epsilon} \propto \infty$ as $p \gg n^{-1/m_2(K_m)} \geq n^{-1/m_2(K_k)}$ and from this

$$\mathbb{E}[X^2] \leq O(\mathbb{E}^2[X]).$$

(Note that in fact $\mathbb{E}[X^2] = (1 + o(1))\mathbb{E}^2[X]$ but the above estimate suffices for our purpose here)

Let $M = N(G, K_m)$. To prove the first part assume towards a contradiction that there is a set of edges, $E_0 \subseteq E(G)$, which is of size $o(n^2p)$ and that there exists $c > 0$ such that there are $cM$ copies of $K_m$ containing at least one edge from it.

On one hand, $\mathbb{E}^2[X] = [M \left(\frac{m}{2}\right) e(G)]^2$. On the other hand by Jensen’s inequality

$$\mathbb{E}[X^2] \geq \mathbb{E}[X^2 \mid e \in E_0]\mathbb{P}[e \in E_0] \geq \left(\frac{cM}{|E_0|} \right)^2 \cdot \frac{|E_0|}{e(G)} = \left(\frac{M \left(\frac{m}{2}\right)}{e(G)} \right)^2 \frac{e(G)}{\left(\frac{m}{2}\right)^2} \frac{e(G)}{|E_0|} = \omega(\mathbb{E}^2[X])$$

where the last equality holds as $|E_0| = o(e(G))$. This is a contradiction to (2) and so the first part of the Lemma holds.

For the second part assume there is a set $E_0$ such that $|E_0| = n^{-\epsilon}mn^2$ and the set of copies of $K_m$ using edges of $E_0$ is of size at least $n^{-\epsilon/3}M$. Note that w.h.p. $e(G) \geq \frac{1}{4}n^2p$. Repeating the calculation above we get that

$$\mathbb{E}[X^2] \geq \mathbb{E}[X^2 \mid e \in E_0]\mathbb{P}[e \in E_0] = \left(\frac{n^{-\epsilon/3}M}{n^{-\epsilon}e(G)} \right)^2 \cdot \frac{n^{-\epsilon}}{4} = \left(\frac{M^2}{e(G)} \right)^2 \frac{n^{\epsilon/3}}{4} = \omega(\mathbb{E}^2[X])$$

which is again a contradiction, and thus the second part of the lemma holds.

**Lemma 2.4.** Let $G \sim G(n, p)$ for $p = n^{-a}$ with $-a < -1/m_2(K_m)$. Then w.h.p. the number of copies of $K_m$ sharing an edge with other copies of $K_m$ is $o(n^m p^{(m)}(2))$.

**Proof.** First note that $n^{m-2}p^{(m)}(2)-1 = (np^{(m+1)/2})^{m-2} = n^{-\alpha(m-2)}$ for some $\alpha > 0$. The expected number of pairs of copies of $K_m$ sharing $a$ vertices, where $m-1 \geq a \geq 2$ is at most

$$n^{2m-a}p^{(m)}(2)^{(m-a)+(m-a)} = n^{m}p^{(m)}(2) \cdot (np^{(m+1)/2})^{(m-a)}$$

$$< n^{m}p^{(m)}(2)n^{(m-a)}$$

$$= n^{m}p^{(m)}(2)n^{-\alpha}.$$
Here we used the fact that \( np^{\frac{m+1}{2}} < 1 \) and \( p < 1 \).

Using Markov’s inequality we get that the probability that \( G \) has more than \( 2n^mp^{\binom{m}{2}}n^{-\alpha/2} \) copies of \( K_m \) sharing an edge is no more than \( n^{-\alpha/2} \).

### 3 Proof of Theorem 1.2

To prove Theorem 1.2, we prove three lemmas for three ranges of values of \( p \) using different approaches. Lemmas 3.1 and 3.2 are stated in a more general form as they are also used in Section 5. An explanation on how the lemmas prove Theorem 1.2 follows after the statements.

**Lemma 3.1.** Let \( H \) be a fixed graph with \( \chi(H) = k > m \) and let \( p \gg \max\{n^{-\frac{1}{2m^2(H)}}, n^{-\frac{1}{2m^2(K_m)}}\} \). Then

\[
ex(G(n,p), K_m, H) = (1 + o(1)) \left( \frac{k - 1}{m} \right) \left( \frac{n}{k - 1} \right)^m p^{\binom{m}{2}}.
\]

**Lemma 3.2.** Let \( H \) be a fixed graph with \( \chi(H) = k > m \), let \( p < \min\{n^{-\frac{1}{2m^2(H)}}, n^{-\frac{1}{2m^2(K_m)}}\} \) for some fixed \( \delta > 0 \), and assume \( n^m p^{\binom{m}{2}} \) tends to infinity as \( n \) tends to infinity. Then

\[
ex(G(n,p), K_m, H) = (1 + o(1)) \left( \frac{n}{m} \right)^{\binom{m}{2}}.
\]

**Lemma 3.3.** Let \( H \) be a fixed graph with \( \chi(H) = k > m \) and let \( n^{-1/m^2(K_m)} - \epsilon < p \ll n^{-1/m^2(H)} \) where \( \epsilon > 0 \) is sufficiently small. Then

\[
ex(G(n,p), K_m, H) = (1 + o(1)) \left( \frac{n}{m} \right)^{\binom{m}{2}}.
\]

Lemma 3.1 takes care of the first part of Theorem 1.2. If \( m_2(H) \geq m_2(K_m) \) then \( n^{-1/m^2(H)} \geq n^{-1/m^2(K_m)} \) and this lemma covers values of \( p \) for which \( p \gg n^{-1/m^2(H)} \).

For the second part of Theorem 1.2 we have Lemmas 3.2 and 3.3. If \( m_2(H) > m_2(K_m) \) Lemma 3.2 covers values of \( p \) for which \( p < n^{-1/m^2(K_m)} - \epsilon \) and Lemma 3.3 covers the range \( n^{-1/m^2(K_m)} - \epsilon < p \ll n^{-1/m^2(H)} \). Choosing \( \epsilon > \delta \) makes sure we do not miss values of \( p \).

We mostly focus on the proof of Lemma 3.1 as the other two are simpler. Lemmas 3.1 and 3.2 are also relevant for the case \( m_2(H) < m_2(K_m) \), and are used again in Section 5. For the proof of Lemma 3.1 we need several tools.

**Lemma 3.4.** Let \( G \) be a \( k \)-partite complete graph with each side of size \( n \), let \( p \in [0,1] \) and let \( G' \) be a random subgraph of \( G \) where each edge is chosen randomly and independently with probability \( p \). If \( n^m p^{\binom{m}{2}} \) goes to infinity together with \( n \) then the number of copies of \( K_m \) for \( m < k \) with each vertex in a different \( V_i \) is w.h.p.

\[
(1 + o(1)) \left( \frac{k}{m} \right)n^m p^{\binom{m}{2}}.
\]

To prove the lemma, we use the following concentration result:
Lemma 3.5 (see, e.g., Corollary 4.3.5 in [4]). Let $X_1, X_2, ..., X_r$ be indicator random variables for events $A_i$, and let $X = \sum_{i=1}^r X_i$. Furthermore assume $X_1, ..., X_r$ are symmetric (i.e. for every $i \neq j$ there is a measure preserving mapping of the probability space that sends event $A_i$ to $A_j$). Write $i \sim j$ if the events $A_i$ and $A_j$ are not independent. Set $\Delta^* = \sum_{i \sim j} \mathbb{P}(A_j \mid A_i)$ for some fixed $i$. If $\mathbb{E}[X] \to \infty$ and $\Delta^* = o(\mathbb{E}[X])$ then $X = (1 + o(1))\mathbb{E}(X)$.

Proof of lemma 3.4 The expected number of copies of $K_m$ in $G'$ is $(1 + o(1))\binom{k}{m} n^m p_{\frac{m}{2}}(\Delta)$. So we only need to show that it is indeed concentrated around its expectation. To do so we use Lemma 3.5.

Let $A_i$ be the event that a specific copy of $K_m$ appears in $G'$, and $X_i$ be its indicator function. Clearly the number of copies of $K_m$ in $G'$ is $X = \sum X_i$. In this case $i \sim j$ if the corresponding copies of $K_m$ share edges. We write $i \cap j = a$ if the two copies share exactly $a$ vertices. It is clear that the variables $X_i$ are symmetric. By the definition in the lemma,

$$\Delta^* = \sum_{i \sim j} \mathbb{P}(A_j \mid A_i)$$

$$= \sum_{2 \leq a \leq m-1} \sum_{i \cap j = a} \mathbb{P}(A_j \mid A_i)$$

$$\leq \sum_{2 \leq a \leq m-1} \binom{m}{a} \binom{k-a}{m-a} n^{m-a} p^{(m-a)+(m-a)a}$$

$$= o\left( \left(\frac{k}{m}\right) n^m p_{\frac{m}{2}} \right).$$

The last inequality holds as $n^m p_{\frac{m}{2}} = n^{m-a} p^{(m-a)+(m-a)a} \cdot n^a p_{\frac{a}{2}}$ and $n^a p_{\frac{a}{2}} = (np^{\frac{a}{2}})^a$ tends to infinity as $n$ tends to infinity for $a < m$. \hfill \square

To prove the upper bound in Lemma 3.1 we use a standard technique for estimating the number of copies of a certain graph inside another. This is done by applying Szemerédi’s regularity lemma and then a relevant counting lemma. The regularity lemma allows us to find an equipartition of any graph into a constant number of sets $\{V_i\}$, such that most of the pairs of sets $\{V_i, V_j\}$ are regular (i.e. the densities between large subsets of sets $V_i$ and $V_j$ do not deviate by more than $\epsilon$ from the density between $V_i$ and $V_j$).

In a sparse graph (such as a dense subgraph of a sparse random graph) we need a stronger definition of regularity than the one used in dense graphs. Let $U$ and $V$ be two disjoint subsets of $V(G)$. We say that they form an $(\epsilon, p)$-regular pair if for any $U' \subseteq U, V' \subseteq V$ such that $|U'| \geq \epsilon |U|$ and $|V'| \geq \epsilon |V|$: $|d(U', V') - d(U, V)| \leq \epsilon p$, where $d(X, Y) = \frac{|E(X, Y)|}{|X||Y|}$ is the edge density between two disjoint sets $X, Y \subseteq V(G)$.

Furthermore, an $(\epsilon, p)$-partition of the vertex set of a graph $G$ is an equipartition of $V(G)$ into $t$ pairwise disjoint sets $V(G) = V_1 \cup ... \cup V_t$ in which all but at most $t^2$ pairs of sets are $(\epsilon, p)$-regular. For a dense graph, Szemerédi’s regularity lemma assures us that we can always find a regular partition of the graph into at most $t(\epsilon)$ parts, but this is not enough for sparse graphs. For
the case of subgraphs of random graphs, one can use a variation by Kohayakawa and Rödl [14] (see also [21], [2] and [16] for some related results).

In this regularity lemma we add an extra condition. We say that a graph \( G \) on \( n \) vertices is \((\eta, p, D)\)-upper-uniform if for all disjoint sets \( U_1, U_2 \subset V(G) \) such that \( |U_i| > \eta n \) one has \( d(U_1, U_2) \leq Dp \). Given this definition we can now state the needed lemma:

**Theorem 3.6** ([14]). For every \( \epsilon > 0, t_0 > 0 \) and \( D > 0 \), there are \( \eta, T \) and \( N_0 \) such that for any \( p \in [0, 1] \), each \((\eta, p, D)\)-upper-uniform graph on \( n > N_0 \) vertices has an \((\epsilon, p)\)-regular partition into \( t \in [t_0, T] \) parts.

In order to estimate the number of copies of a certain graph after finding a regular partition one needs counting lemmas. We use a proposition from [7] to show that a certain cluster graph is \( H \)-free, and to give a direct estimate on the number of copies of \( K_m \). To state the proposition we need to introduce some notation. For a graph \( H \) on \( k \) vertices, \( \{1, ..., k\} \), and for a sequence of integers \( m = (m_{ij})_{ij \in E(H)} \), we denote by \( G(H, n', m, \epsilon, p) \) the following family of graphs. The vertex set of each graph in the family is a disjoint union of sets \( V_1, ..., V_k \) such that \( |V_i| = n' \) for all \( i \). As for the edges, for each \( ij \in E(H) \) there is an \((\epsilon, p)\)-regular bipartite graph with \( m_{ij} \) edges between the sets \( V_i \) and \( V_j \), and these are all the edges in the graph. For any \( G \in G(H, n', m, \epsilon, p) \) denote by \( G(H) \) the number of copies of \( H \) in \( G \) in which every vertex \( i \) is in the set \( V_i \).

**Proposition 3.7** ([7]). For every graph \( H \) and every \( \delta, d > 0 \), there exists \( \xi > 0 \) with the following property. For every \( \eta > 0 \), there is a \( C > 0 \) such that if \( p \geq C n^{-1/m_2(H)} \) then w.h.p. the following holds in \( G(n, p) \).

1. For every \( n' \geq \eta n \), \( m \) with \( m_{ij} \geq dp(n')^2 \) for all \( ij \in E(H) \) and every subgraph \( G \) of \( G(n, p) \) in \( G(H, n', m, \epsilon, p) \),

\[
G(H) \geq \xi \left( \prod_{ij \in E(H)} \frac{m_{ij}}{(n')^2} \right) (n')^{v(H)}.
\]  

(3)

2. Moreover, if \( H \) is strictly balanced, i.e. for every proper subgraph \( H' \) of \( H \) one has \( m_2(H) > m_2(H') \), then

\[
G(H) = (1 \pm \delta) \left( \prod_{ij \in E(H)} \frac{m_{ij}}{(n')^2} \right) (n')^{v(H)}.
\]  

(4)

Note that the first part tells us that if \( G \) is a subgraph of \( G(n, p) \) in \( G(H, n', m, \epsilon, p) \), then it contains at least one copy of \( H \) with vertex \( i \) in \( V_i \).

We can now proceed to the proof of Lemma 3.1 starting with a sketch of the argument. Note that the same steps can be applied to determine \( \epsilon x(G(n, p), T, H) \) for graphs \( T \) and \( H \) for which \( \epsilon x(n, T, H) = \Theta(n^{v(T)}) \) and \( p \gtrsim \max\{n^{-1/m_2(H)}, n^{-1/m_2(T)}\} \).

Let \( G \) be an \( H \)-free subgraph of \( G(n, p) \) maximizing the number of copies of \( K_m \). First apply the sparse regularity lemma (Theorem 3.6) to \( G \) and observe using Chernoff and properties of the regular partition that there are only a few edges inside clusters and between sparse or irregular pairs. By lemma 2.3 these edges do not contribute significantly to the count of \( K_m \). We can thus consider only graphs \( G \) which do not have such edges.

By Proposition 3.7 the cluster graph must be \( H \)-free and taking \( G \) to be maximal we can assume all pairs in the cluster graph have the maximal possible density. Applying Proposition 3.7 again to count the number of copies of \( K_m \) reduces the problem to the dense case solved in [3].
We continue with the full details of the proof.

**Proof of Lemma 3.7.** A \((k-1)\)-partite graph with sides of size \(\frac{n}{k-1}\) each is an \(n\)-vertex \(H\)-free graph containing \((1 + o(1))\left(\frac{k-1}{m}\right)^m\) copies of \(K_m\). We can get a random subgraph of it by keeping each edge with probability \(p\), independently of the other edges. Then by Lemma 3.4 the number of copies of \(K_m\) in it is \((1 + o(1))\left(\frac{n}{k-1}\right)^m p^{(m/2)}\) w.h.p., proving the required lower bound on \(\text{ex}(G(n, p), K_m, H)\).

For the upper bound we need to show that no \(H\)-free subgraph of \(G(n, p)\) has more than \((1 + o(1))\left(\frac{k-1}{m}\right)^m p^{(m/2)}\) copies of \(K_m\). Let \(G\) be an \(H\)-free subgraph of \(G(n, p)\) with the maximum number copies of \(K_m\). To use Theorem 3.6 we need to show that \(G\) is \((\eta, p, D)\)-upper-uniform for some constant \(D\), say \(D = 2\), and \(\eta > 0\). Indeed, taking any two disjoint subsets \(V_1, V_2\) of size \(\geq \eta n\), we get that the number of edges between them is bounded by the number of edges between them in \(G(n, p)\), which is distributed like \(\text{Bin}(|V_1| \cdot |V_2|, p)\). Applying Part 3 of Lemma 2.1 and the union bound gives us that w.h.p. the number of edges between any two such sets is \(\leq 2|V_1| \cdot |V_2| p\) and so \(d(V_1, V_2) < 2p\) as needed. Thus by Theorem 3.6 \(G\) admits an \((\epsilon, p)\)-regular pair.

Define the \textit{cluster graph} of \(G\) to be the graph whose vertices are the sets \(V_i\) of the partition and there is an edge between two sets if the density of the bipartite graph induced by them is at least \(\delta p\) for some fixed small \(\delta > 0\), and they form an \((\epsilon, p)\)-regular pair.

First we show that w.h.p. the cluster graph is \(H\)-free. Assume that there is a copy of \(H\) in the cluster graph, induced by the sets \(V_1, \ldots, V_{\nu(G)}\). Consider these sets in the original graph \(G\). To apply Part 1 of Proposition 3.7 first note that indeed \(p \geq C n^{-1/m_2(H)}\). Furthermore if \(ij \in E(H)\) then by the definition of the cluster graph \(V_i\) and \(V_j\) form an \((\epsilon, p)\)-regular pair and there are at least \(\delta p(\frac{n}{k-1})^2\) edges between them. Thus the graph spanned by the edges between \(V_1, \ldots, V_{\nu(G)}\) in \(G\) is in \(G(H, \frac{n}{k-1}, m, \epsilon, p)\) where \(m_{ij} \geq \delta p(\frac{n}{k-1})^2\), and so w.h.p. it contains a copy of \(H\) with vertex \(i\) in the set \(V_i\). This contradicts the fact that \(G\) was \(H\)-free to start with.

If the cluster graph is indeed \(H\)-free, as proven in \(3\), Proposition 2.2, since \(\chi(H) > m\) then \(\text{ex}(t, K_m, H) = (1 + o(1))\left(\frac{k-1}{m}\right)^m\). This gives a bound on the number of copies of \(K_m\) in the cluster graph. For sets \(V_1, \ldots, V_m\) that span a copy of \(K_m\) in the cluster graph we would like to bound the number of copies of \(K_m\) with a vertex in each set in the original graph \(G\).

To do this, we use Part 2 of Proposition 3.7. Note that we cannot use Lemma 3.4 as we need it for every subgraph of \(G(n, p)\) and not only for a specific one. Part 2 can be applied only to balanced graphs, and indeed any subgraph of \(K_m\) is \(K_{m'}\) for some \(m' < m\) and \(m_2(K_{m'}) = \frac{m'^2}{2} < \frac{m^2}{2} = m_2(K_m)\). As we would like to have a upper bound on the number of copies of \(K_m\) with a vertex in each set, we can assume that the bipartite graph between \(V_i\) and \(V_j\) has all of the edges from \(G(n, p)\).

By Parts 1 and 2 of Lemma 2.1 w.h.p. for any \(V_i\) and \(V_j\) of size \(\frac{n}{k-1}\), \(|E(V_i, V_j)| = (1 + o(1))p(\frac{n}{k-1})^2\). Thus the graph induced by the sets \(V_1, \ldots, V_m\) in \(G(n, p)\) is in \(G(K_m, \frac{n}{k-1}, m, \epsilon, p)\) where \(m_{ij} = (1 + o(1))p(\frac{n}{k-1})^2\) for any pair \(ij\). From this the number of copies of \(K_m\) in \(G\) with a vertex in every \(V_i\) is at most \((1 + o(1))p(\frac{n}{k-1})^m\). Plugging this into the bound on the number of copies of \(K_m\) in the cluster graph implies that the number of copies of \(K_m\) coming from copies of \(K_m\) in the cluster is w.h.p. at most

\[
(1 + o(1))\left(\frac{k-1}{m}\right)^m \cdot p^{(m/2)}(\frac{n}{k-1})^m = (1 + o(1))\left(\frac{k-1}{m}\right)^m \cdot p^{(m/2)}.
\]

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It is left to show that the number of copies of $K_m$ coming from other parts of the graph is negligible.

To do this we show that the number of edges inside clusters and between non-dense or irregular pairs is negligible. By Chernoff (Part 3 of Lemma 2.4) the number of edges inside a cluster is at most $2p(n^2/2)t \leq 2p^2/n^2$. The number of irregular pairs is at most $\epsilon t^2$, and again by Chernoff there are no more than $2p(n^2/2) \cdot \epsilon t^2 = 2\epsilon pn^2$ edges between these pairs. Finally, the number of edges between non-dense pairs is at most $\delta p(n^2/2)t^2 = \delta pn^2$.

As $\epsilon$, $\delta$ and $1/\delta$ can be chosen as small as needed we get that the number of such edges is $o(n^2p)$

Thus we may apply Lemma 2.3 and conclude that the number of copies of $H$ is negligible.

Thus we may apply Lemma 2.3 and conclude that the number of copies of $K_m$ containing at least one of these edges is $o(n^m p(m^2))$.

Therefore, for any $H$-free $G \subset G(n, p)$ the number of copies of $K_m$ in $G$ is at most $(1 + o(1))(n^{-k-1}(np)^{m-2}m^m)$ as needed.

The proofs of the other two lemmas are a bit simpler.

Proof of Lemma 3.2. As $p < n^{-1/m_2(K_m) - \delta}$ we can first delete all copies of $K_m$ sharing an edge with other copies and by Lemma 2.4 we deleted w.h.p. only $o(n^m p(m^2))$ copies of $K_m$. Let $H'$ be a subgraph of $H$ for which $e(H') = m_2(H)$. Let $e$ be an edge of $H'$ and define $\{H_i\}$ to be the family of all graphs obtained by gluing a copy of $K_m$ to the edge $e$ in $H'$ and allowing any further intersection. Note that the number of graphs in $\{H_i\}$ depends only on $H'$ and $m$. One can make $G$ into an $H$-free graph by deleting the edge $e$ from every copy of a graph from $\{H_i\}$ and every edge that does not take part in a copy of $K_m$. As we may assume every edge takes part in at most one copy of $K_m$ it is enough to show that the number of copies of graphs from $\{H_i\}$ is $o(n^m p(m^2))$.

For a fixed graph $J$, let $X_J$ be the random variable counting the number of copies of $J$ in $G \sim G(n, p)$. With this notation,

$$\mathbb{E}(X_{K_m}) = \Theta(n^m p(m^2)) = \Theta(n^2 p(np^{m_2(K_m)})^{m-2}),$$

$$\mathbb{E}(X_{H_i}) = \Theta(n^2 p(np^{m_2(H_i)})v(H_i)^{-2}).$$

As $m_2(H_i) \geq m_2(K_m)$ and $p < n^{-1/m_2(K_m)}$, we get $np^{m_2(H_i)} \leq np^{m_2(K_m)} \ll 1$. Furthermore as $v(H_i) > m$ (otherwise $H'$ would be a subgraph of $K_m$) we get that $(np^{m_2(H_i)})v(H_i)^{-2} = o((np^{m_2(K_m)})^{m-2})$ and thus $\mathbb{E}(X_{H_i}) = o(\mathbb{E}(X_{K_m}))$.

If $p$ is such that the expected number of copies of $K_m$, the graphs $\{H_i\}$ and any of their subgraphs goes to infinity as $n$ goes to infinity we can apply Theorem 2.2 and get that $X_{K_m} = (1 + o(1))\binom{n}{m}p(m^2)$ and the number of copies of $H_i$ is w.h.p. $(1 + o(1))\mathbb{E}(X_{H_i}) = o(\mathbb{E}(X_{K_m}))$. Thus if we remove all edges playing the part of $e$ in any $H_i$ the number of copies of $K_m$ will still be $(1 + o(1))\binom{n}{m}p(m^2)$.

Finally, if the number of copies of some subgraph of $H_i$ does not tend to infinity as $n$ tends to infinity we can remove all of the edges taking part in it, and the number of edges removed is $o(\binom{n}{m}p(m^2))$. As each edge takes part in a single copy of $K_m$, we still get that the number of copies of $K_m$ in this graph is $(1 + o(1))\binom{n}{m}p(m^2)$, as needed.
Proof of Lemma 3.3. Let \( n^{-1/m_2(K_m)} - \epsilon < p \ll n^{-1/m_2(H)} \) and \( G \sim G(n, p) \). Let \( H' \) be a subgraph of \( H \) for which \( \frac{e(H') - 1}{v(H') - 2} = m_2(H) \). We show that if \( G \) is made \( H \)-free by removing a single edge from every copy of \( H' \) then the number of copies of \( K_m \) deleted is \( o\left(n^{m}p^{(m)}\right)\). Theorem 2.2 assures us that the number of copies of \( K_m \) in \( G \) is \( (1 + o(1))\left(n^{m}p^{(m)}\right) \) and so it stays essentially the same after removing all copies of \( H' \).

The expected number of copies of \( H' \) in \( G \) is

\[
E[N(G, H')] = \Theta(n^2p(np^{m_2(H')})^{v(H')-2}) = o(n^2p).
\]

Thus by Markov’s inequality w.h.p. \( N(G, H') = o(n^2p) \). If \( p \gg n^{-1/m_2(K_m)} \) then by Lemma 2.3 deleting all these edges removes only \( o(n^m p^{(m)}) \) copies of \( K_m \).

As for smaller values of \( p \), namely \( p \leq O(n^{-1/m_2(K_m)}) \), it follows that

\[
E[N(G, H')] = \Theta(n^2p(np^{m_2(H')})^{v(H')-2}) \leq n^{-\beta}n^2p
\]

for some \( \beta > 0 \). By Markov’s inequality w.h.p. the number of edges taking part in a copy of \( H' \) in \( G \) is at most \( n^{-\alpha}n^2p \) for, say, \( \alpha = \frac{\beta}{2} \).

Since \( p \leq O(n^{-1/m_2(K_m)}) \) Lemma 2.3 cannot be applied directly. To take care of this, define \( q = n^2p \gg n^{-1/m_2(K_m)} \). Lemma 2.3 applied to \( G(n, q) \) implies that a set of at most \( n^{-\alpha}qn^2 \) edges takes part in no more than \( n^{-\beta}n^m q^{(m)} \) copies of \( K_m \), where \( \delta = \delta(\alpha) > 0 \).

The number of copies of \( K_m \) containing a member of a set of edges in \( G(n, p) \) is monotone in \( p \) and in the size of the set. Thus when deleting a single edge from each copy of \( H' \) in \( G(n, p) \) the number of copies of \( K_m \) removed is w.h.p. at most \( n^{-\delta} n^m q^{(m)} = n^{-\delta+2}n^m p^{(m)} \). Choosing \( \epsilon \) small enough implies that the number of copies of \( K_m \) removed is \( o(n^m p^{(m)}) \) as needed. \( \square \)

4 Construction of graphs with small 2-density

In the proof of Theorem 1.3 we construct a family of graphs \( \{G(k, \epsilon)\} \) that are \( k \)-critical and \( m_2(G(k, \epsilon)) = (1 + \epsilon)M_k \) where \( M_k \) is the smallest possible value of \( m_2 \) for a \( k \)-chromatic graph. The following notation will be useful. For a graph \( G \) and \( A \subseteq V(G) \) such that \( |A| \geq 3 \), let \( d^2_G(A) = \frac{e(G[A])-1}{|A|-2} \). By definition, \( m_2(G) = \min_{A \subseteq V(G)}:|A|=3 d^2_G(A) \).

Proof of Theorem 1.3. We construct the graphs \( G(k, \epsilon) \) in three steps. In Step 1 we construct so called \( (k, t) \)-towers and derive some useful properties of them. In Step 2 we make from \( (k, t) \)-towers more complicated \( (k, t) \)-complexes and supercomplexes, and in Step 3 we replace each edge in a copy of \( K_k \) with a supercomplex and prove the needed.

Step 1: Towers

Let \( t = t(\epsilon) = \lceil k^3/\epsilon \rceil \). The \( (k, t) \)-tower with base \( \{v_0, 0v_1\} \) is the graph \( T_{k, t} \) defined as follows. The vertex set of \( T_{k, t} \) is \( V_0 \cup V_1 \cup \ldots \cup V_t \), where \( V_0 = \{v_0, 0, v_1\} \) and for \( 1 \leq i \leq t \), \( V_i = \{v_i, 0, v_1, \ldots, v_{i+k-1}\} \). For \( i = 1, \ldots, t \), \( T_{k, t}[V_i] \) induces \( K_{k-1} - e \) with the missing edge \( v_{i+k-2}v_1 \). Also for \( i = 1, \ldots, t \), vertex \( v_{i-1, 0} \) is adjacent to \( v_{i,j} \) for all \( 0 \leq j \leq (k-2)/2 \) and vertex \( v_{i-1, 1} \) is adjacent to \( v_{i,j} \) for all \( (k-1)/2 \leq j \leq k-2 \). There are no other edges.
By construction, $|E(T_{k,t})| = t \left( \binom{k-1}{2} - 1 + (k-1) \right) = t \left( \frac{(k+1)(k-2)}{2} - \frac{(k+1)(k-2)}{2(k-1)} \right)$, that is,

$$d^{(2)}_{T_{k,t}}(V(T_{k,t})) = \frac{(k+1)(k-2)}{2(k-1)} - \frac{1}{|V(T_{k,t})| - 2}.$$

Also, since for each $i = 1, \ldots, t$, $|N(v_{i-1,1}) \cap V_i| \leq (k-1)/2$ and among the $[(k-1)/2]$ neighbors of $v_{i-1,0}$ in $V_i$, $v_{i,0}$ and $v_{i,1}$ are not adjacent to each other,

$$\omega(T_{k,t}) = k - 2. \quad (6)$$

Our first goal is to show that $T_{k,t}$ has no dense subgraphs. We will use the language of potentials to prove this. For a graph $H$ and $A \subseteq V(H)$, let

$$\rho_{k,H}(A) = (k+1)(k-2)|A| - 2(k-1)|E(H[A])|$$

be the potential of $A$ in $H$. A convenient property of potentials is that if $|A| \geq 3$, then

$$\rho_{k,H}(A) \geq 2(k+1)(k-2) - 2(k-1)$$

if and only if $d_H^{(2)}(A) \leq \frac{(k+1)(k-2)}{2(k-1)}, \quad (7)$

but potentials are also well defined for sets with cardinality two or less.

**Lemma 4.1.** Let $T = T_{k,t}$. For every $A \subseteq V(T)$,

if $|A| \geq 2$, then $\rho_{k,T}(A) \geq 2(k+1)(k-2) - 2(k-1). \quad (8)$

Moreover,

if $V_0 \subseteq A$, then $\rho_{k,T}(A) \geq 2(k+1)(k-2). \quad (9)$

**Proof.** Suppose the lemma is not true. Among $A \subseteq V(T)$ with $|A| \geq 2$ for which (8) or (9) does not hold, choose $A_0$ with the smallest size. Let $a = |A_0|$. If $a = 2$, then $\rho_{k,T}(A_0) = 2(k+1)(k-2) - 2(k-1)|E(T[A_0])| \geq 2(k+1)(k-2) - 2(k-1).$ Moreover, if $a = 2$ and $V_0 \subseteq A$, then $V_0 = A_0$ and so $E(T[A_0]) = \emptyset$. This contradicts the choice of $A_0$. So

$$a \geq 3. \quad (10)$$
Let \( i_0 \) be the maximum \( i \) such that \( A_0 \cap V_i \neq \emptyset \). By (11), \( i_0 \geq 1 \). Let \( A' = A_0 \cap V_{i_0} \) and \( a' = |A'| \).

**Case 1:** \( a' \leq k - 2 \) and \( a - a' \geq 2 \). Since \(|(A_0 - A') \cap V_0| = |A_0 \cap V_0|\), by the minimality of \( a \), (8) and (9) hold for \( A_0 - A' \). Thus,

\[
\rho_{k,T}(A_0) \geq \rho_{k,T}(A_0 - A') + a'(k + 1)(k - 2) - 2(k - 1) \left( a' + \frac{a'}{2} \right)
\]

\[
= \rho_{k,T}(A_0 - A') + a' \left( (k^2 - k) - 2k + 2 - (k - 1)(a' - 1) \right).
\]

Since \( k \geq 4 \) and \( a' \leq k - 2 \), the expression in the brackets is at least \( k^2 - 3k - (k - 1)(k - 3) = k - 3 > 0 \), contradicting the choice of \( A_0 \).

**Case 2:** \( A' = V_{i_0} \) and \( a - a' \geq 2 \). Then \( a' = k - 1 \). As in Case 1, (8) and (9) hold for \( A_0 - A' \). Thus,

\[
\rho_{k,T}(A_0) \geq \rho_{k,T}(A_0 - A') + a'(k + 1)(k - 2) - 2(k - 1) \left( a' + \frac{a'}{2} - 1 \right)
\]

\[
= \rho_{k,T}(A_0 - A') + (k - 1) \left[ (k^2 - k - 2) - 2(k - 1) - (k - 1)((k - 1) - 1) + 2 \right]
\]

\[
\geq \rho_{k,T}(A_0 - A') + (k - 1)^2 \left[ (k - 2) - (k - 2) \right] = \rho_{k,T}(A_0 - A'),
\]

contradicting the minimality of \( A_0 \).

**Case 3:** \( a = a' \), i.e., \( A_0 = A' \). Then \( V_0 \not\subseteq A_0 \) and \( a' \geq 3 \). If \( a \leq k - 2 \), then

\[
\rho_{k,T}(A_0) \geq a(k + 1)(k - 2) - 2(k - 1) \left( \frac{a}{2} \right) = a(\lfloor (k + 1)(k - 2) - (k - 1)(a - 1) \rfloor).
\]

(11)

Since the RHS of (11) is quadratic in \( a \) with the negative leading coefficient, it is enough to evaluate the RHS of (11) for \( a = 2 \) and \( a = k - 2 \). For \( a = 2 \), it is \( 2(k + 1)(k - 2) - 2(k - 1) \), exactly as in (8). For \( a = k - 2 \), it is

\[
(k - 2)[(k + 1)(k - 2) - (k - 1)(k - 2)] = (3k - 5)(k - 2),
\]

and \( (3k - 5)(k - 2) \geq 2(k + 1)(k - 2) - 2(k - 1) \) for \( k \geq 4 \). If \( a = k - 1 \), then \( A_0 = V_i \) and

\[
\rho_{k,T}(A_0) = a(k + 1)(k - 2) - 2(k - 1) \left( \frac{a}{2} - 1 \right) = (k - 1)((k + 1)(k - 2) - (k - 1)(k - 2) + 2)
\]

\[
= 2(k - 1)^2 > 2(k + 1)(k - 2) - 2(k - 1).
\]

**Case 4:** \( a - a' = 1 \). As in Case 3, \( V_0 \not\subseteq A_0 \) and \( a' \geq 2 \). Let \( \{z\} = A_0 - A' \). Repeating the argument of Case 3, we obtain that \( \rho_{k,T}(A') \geq 2(k + 1)(k - 2) - 2(k - 1) \). So, if \( d_{T[A_0]}(z) \leq \frac{k - 1}{2} \), then

\[
\rho_{k,T}(A_0) \geq \rho_{k,T}(A') + (k + 1)(k - 2) - 2(k - 1) \left( \frac{k - 1}{2} \right) = \rho_{k,T}(A') + k - 3 > \rho_{k,T}(A'),
\]

a contradiction to the choice of \( A_0 \). And the only way that \( d_{T[A_0]}(z) > \frac{k - 1}{2} \), is that \( z = v_{i-1,0} \), \( k \) is even, and \( A' \supseteq \{v_{i,0}, \ldots, v_{i,(k-2)/2}\} \). Then edge \( v_{i,0}v_{i,1} \) is missing in \( T[A'] \), and hence

\[
\rho_{k,T}(A_0) = (a' + 1)(k + 1)(k - 2) - 2(k - 1) \left( \frac{a'}{2} - 1 + k/2 \right).
\]

(12)
Since the RHS of (12) is quadratic in $a'$ with the negative leading coefficient and $a' \geq k/2$, it is enough to evaluate the RHS of (12) for $a' = k/2$ and $a' = k - 1$. For $a' = k/2$, it is

$$\frac{k + 2}{2} (k + 1) (k - 2) - (k - 1) \left( \frac{k(k - 2)}{4} - 2 + k \right) = \frac{k - 2}{4} (k^2 + 3k + 8).$$

Since $\frac{k^2 + 3k + 8}{4} > 2k$ for $k \geq 4$ and $2k(k - 2) > 2(k + 1)(k - 2) - 2(k - 1)$, we satisfy (8). If $a' = k - 1$, then the RHS of (12) is

$$k(k + 1)(k - 2) - (k - 1) ((k - 1)(k - 2) - 2 + k) = (k - 2) [k(k + 1) - (k - 1)^2 - k + 1]$$

$$= 2(k - 2) > 2(k + 1)(k - 2) - 2(k - 1).$$

Graph $T_{k, t}$ also has good coloring properties.

**Lemma 4.2.** Suppose $T_{k, t}$ has a $(k - 1)$-coloring $f$ such that

$$f(v_{0,1}) = f(v_{0,0}).$$

Then for every $1 \leq i \leq t$,

$$f(v_{i,1}) = f(v_{i,0}).$$

**Proof.** We prove (14) by induction on $i$. For $i = 0$, this is (13). Suppose (14) holds for $i = j < t$. Since $V_{j+1} \subseteq N(v_{i,0}) \cup N(v_{i,1})$, the color $f(v_{j,1}) = f(v_{j,2})$ is not used on $V_{j+1}$ and thus $f(v_{j+1,1}) = f(v_{j+1,0})$, as claimed.

**Step 2: Tower complexes**

A tower complex $C_{k, t}$ is the union of $k$ copies $T^1_{k, t}, \ldots, T^k_{k, t}$ of the tower $T_{k, t}$ such that every two of them have the common base $V^0_0 = \ldots = V^k$, are vertex-disjoint apart from that, and have no edges between $T^i_{k, t} - V^i_0$ and $T^j_{k, t} - V^j_0$ for $j \neq i$. This common base $V^0 = \{v_{0,0}, v_{0,1}\}$ will be called the base of $C_{k, t}$.

Lemma 3.1 naturally extends to complexes as follows.

**Lemma 4.3.** Let $C = C_{k, t}$. For every $A \subseteq V(C)$,

$$\text{if } |A| \geq 2, \text{ then } \rho_{k,C}(A) \geq 2(k + 1)(k - 2) - 2(k - 1).$$

Moreover,

$$\text{if } A \supseteq V_0, \text{ then } \rho_{k,C}(A) \geq 2(k + 1)(k - 2).$$

**Proof.** Let $A \subseteq V(C)$ with $|A| \geq 2$, and $A_0 = A \cap V_0$. Let $A_i = A \cap V(T^i_{k, t})$ if $A \cap V(T^i_{k, t}) - V_0 \neq \emptyset$, and $A_i = \emptyset$ otherwise. Let $I = \{i \in [t] : A_i \neq \emptyset\}$. If $|I| \leq 1$, then $A$ is a subset of one of the towers, and we are done by Lemma 3.1. So let $|I| \geq 2$.

**Case 1:** $V_0 \subseteq A$. Then for each nonempty $A_i$, $|A_i| \geq 3$ and by Lemma 3.1 $\rho_{k,C}(A_i) \geq 2(k + 1)(k - 2)$. So, by the definition of the potential,

$$\rho_{k,C}(A) = \sum_{i \in I} \rho_{k,C}(A_i) - (|I| - 1)2(k + 1)(k - 2) \geq |I|2(k + 1)(k - 2)-(|I|-1)2(k + 1)(k - 2) = 2(k + 1)(k - 2).$$
The main properties of $S \in \mathcal{I}$ are stated in the next three lemmas.

**Lemma 4.1.** For each pair $(i, j)$ with $1 \leq i < j \leq k$, an edge in $B_{k,t}$ connects $\{v_{i,0}^i, v_{i,1}^i\}$ with $\{v_{j,0}^j, v_{j,1}^j\}$. There are no other edges.

![Figure 2: $B_{4,t}$](image)

By construction, $B_{k,t}$ has exactly $\binom{k}{2}$ edges, and the maximum degree of $B_{k,t}$ is $\lfloor k/2 \rfloor$. It is important that

$$\text{for each } 1 \leq i < j \leq k, \text{ an edge in } B_{k,t} \text{ connects } \{v_{i,0}^i, v_{i,1}^i\} \text{ with } \{v_{j,0}^j, v_{j,1}^j\}. \quad (17)$$

**Lemma 4.4.** For each $(k-1)$-coloring $f$ of $S_{k,t}$,

$$f(v_{0,1}) \neq f(v_{0,0}). \quad (18)$$

**Proof.** Suppose $S_{k,t}$ has a $(k-1)$-coloring $f$ with $f(v_{0,1}) = f(v_{0,0})$. Then by Lemma 4.2, $f(v_{i,1}^i) = f(v_{i,0}^i)$ for every $1 \leq i \leq k$. Thus by (17), the $k$ colors $f(v_{1,0}^1), f(v_{2,0}^2), \ldots, f(v_{k,0}^k)$ are all distinct, a contradiction. \hfill \Box

**Lemma 4.5.** Let $S = S_{k,t}$ with base $V_0$. For every $A \subseteq V(S) - V_0$,

$$\text{if } |A| \geq 2, \text{ then } \rho_{k,S}(A) \geq 2(k+1)(k-2) - 2(k-1). \quad (19)$$
Proof. Suppose the lemma is not true. Let $C$ be the copy of $C_{k,t}$ from which we obtained $S$ by adding the edges of $B = B_{k,t}$. Among $A \subseteq V(S) - V_0$ with $|A| \geq 2$ and $\rho_{k,S}(A) < 2(k+1)(k-2) - 2(k-1)$, choose $A_0$ with the smallest size. Let $a = |A_0|$. Let $I = \{ i \in [t] : A_0 \cap V(T^i_{k,t}) \neq \emptyset \}$. If $|I| \leq 1$, then $A$ is a subset of one of the towers, and we are done by Lemma 4.1. So let $|I| \geq 2$.

If $a = 2$, then
\[
\rho_{k,S}(A_0) = a(k+1)(k-2) - 2(k-1)|E(S[A_0])| \geq 2(k+1)(k-2) - 2(k-1),
\]
contradicting the choice of $A_0$. So $a \geq 3$. Furthermore, if $a = 3$, then since $|I| \geq 2$, $B_{k,t}$ is bipartite, and $v^i_{k,0}v^i_{k,1} \notin E(S)$ for any $i$, the graph $S[A_0]$ has at most two edges and so $\rho_{k,S}(A_0) \geq 3(k+1)(k-2) - 2(2(k-1)) > 2(k+1)(k-2) - 2(k-1)$. Thus
\[
a \geq 4. \tag{20}
\]

If $d_{S[A_0]}(w) \leq \frac{k-1}{2}$ for some $w \in A_0$, then
\[
\rho_{k,S}(A_0 - w) \leq \rho_{k,S}(A_0) - (k+1)(k-2) + \frac{k-1}{2}2(k-1) = \rho_{k,S}(A_0) + 3 - k < \rho_{k,S}(A_0).
\]

By (20), this contradicts the minimality of $a$. So,
\[
\delta(S[A_0]) \geq \frac{k}{2}. \tag{21}
\]

Let $E(A_0, B)$ denote the set of edges of $B$ both ends of which are in $A_0$. Then since $A_0 \cap V_0 = \emptyset$,
\[
\rho_{k,S}(A_0) = \rho_{k,C}(A_0) - 2(k-1)|E(A_0, B)| = \sum_{i \in I} \rho_{k,C}(A_i) - 2(k-1)|E(A_0, B)|. \tag{22}
\]

Let $I_1 = \{ i \in I : |A_0 \cap V(T^i_{k,t})| = 1 \}$ and $I_2 = I - I_1$. By Lemma 4.1, for each $i \in I_2$, $\rho_{k,S}(A_i) \geq 2(k+1)(k-2) - 2(k-1)$. Thus if $I_1 = \emptyset$, then by (22) and the fact that $|E(A_0, B)| \leq \binom{|I|}{2}$, we have
\[
\rho_{k,S}(A_0) \geq |I|(2(k+1)(k-2) - 2(k-1)) - \binom{|I|}{2}2(k-1) = |I|(2k^2 - 3k - 3) - |I|(k-1)).
\]
The minimum of the last expression is achieved either for $|I| = 2$ or for $|I| = k$. If $|I| = 2$, this is $2(2k^2 - 5k - 1) > 2(k+1)(k-2) - 2(k-1)$. If $|I| = k$, this is $k(k^2 - 2k - 3)$, which is again greater than $2(k+1)(k-2) - 2(k-1)$. Thus $|I| \neq \emptyset$.

Suppose $i, i' \in I_1$, $w \in A_i$, $w' \in A_{i'}$, and $ww' \in E(S)$. Let $A' = A_0 - w - w'$. By the definition of $I_1$, all edges of $S[A_0]$ incident with $w$ or $w'$ are in $E(B)$. Since $\Delta(B) \leq \frac{k}{2}$, $|E(S[A_0])| - |E(S[A'])| \leq k - 1$. Thus
\[
\rho_{k,S}(A') \leq \rho_{k,S}(A_0) - 2(k+1)(k-2) + (k-1)2(k-1) = \rho_{k,S}(A_0) - 2k + 6.
\]

But by (20), $|A'| \geq 2$, a contradiction to the minimality of $a$. It follows that for every $i \in I_1$, each neighbor in $A_0$ of the vertex $w \in A_i$ is in some $A_j$ for $j \in I_2$. This implies $|E(A_0, B)| \leq \binom{|I|}{2} - \binom{|I_2|}{2}$. Together with (21) and $\Delta(B) = \lfloor k/2 \rfloor$, this yields that for each $i \in I_1$, the vertex $w \in A_i$ has exactly $k/2$ neighbors in $B$, and all these neighbors are in $A$. In particular, $|I_2| \geq \frac{k}{2}$ and $k$ is even.
Moreover, if \( i, i' \in I_1, w \in A_i \) and \( w' \in A_{i'} \), then their neighborhoods in \( B \) are distinct, and thus in this case \( |I_2| > \frac{k}{2} \). Since \( k \) is even, this implies

\[
|I_2| \geq \frac{k + 2}{2}.
\]  

(23)

Since the potential of a single vertex is \((k+1)(k-2)\),

\[
\rho_{k,S}(A_0) \geq |I|(2(k+1)(k-2) - 2(k-1)) - |I_1|((k+1)(k-2) - 2(k-1)) - \left(\left(\frac{|I|}{2}\right) - \left(\frac{|I_1|}{2}\right)\right)2(k-1).
\]

(24)

The expression \(-|I_1|((k+1)(k-2) - 2(k-1) + \frac{|I|}{2})2(k-1)\) in (24) decreases when \(|I_1|\) grows but is at most \(\frac{k-2}{2}\). Thus by (23), it is enough to let \(|I_1| = |I| - \frac{k+2}{4}\) in (24). So,

\[
\rho_{k,S}(A_0) \geq |I|(k+1)(k-2) + \frac{k+2}{2}((k+1)(k-2) - 2(k-1)) - (k-1)(k+2)(\frac{|I|}{4} - \frac{k+4}{4})
\]

\[
= -2k|I| + \frac{k+2}{2}\left[k^2 - 3k + \frac{k^2 + 3k - 4}{2}\right] \geq -2k^2 + \frac{(k+2)(3k^2 - 3k - 4)}{4} > 2(k+1)(k-2) - 2(k-1)
\]

for \(k \geq 4\).

\(\square\)

**Lemma 4.6.** Let \( S = S_{k,t} \) with base \( V_0 \). Let \( A \subseteq V(S) \) and \(|A| \leq t + 1\).

If \(|A| \geq 2\), then \(\rho_{k,S}(A) \geq 2(k+1)(k-2) - 2(k-1)\).

Moreover,

\[\text{if } A \supseteq V_0, \text{ then } \rho_{k,S}(A) \geq 2(k+1)(k-2).\]

(25)

(26)

**Proof.** Suppose the lemma is not true. Among \( A \subseteq V(S) \) with \(|A| \geq 2\) for which (25) or (26) does not hold, choose \( A_0 \) with the smallest size. Let \( a = |A_0| \). By Lemma 4.3, \( S[A_0] \) contains an edge \( uv \) in \( B \). By Lemma 4.5, \( A_0 \) contains a vertex \( v \in V_0 \). In particular, \( a \geq 3 \).

If \( S[A_0] \) is disconnected, then \( A_0 \) is the disjoint union of nonempty \( A' \) and \( A'' \) such that \( S \) has no edges connecting \( A' \) with \( A'' \). Since \( a \geq 3 \), we may assume that \(|A'| \geq 2\). By the minimality of \( A_0 \), \( \rho_{k,S}(A') \geq 2(k+1)(k-2) - 2(k-1) \). Also, \( \rho_{k,S}(A'') \geq (k+1)(k-2) \). Thus

\[
\rho_{k,S}(A_0) = \rho_{k,S}(A') + \rho_{k,S}(A'') \geq 2(k+1)(k-2) - 2(k-1) + (k+1)(k-2) > 2(k+1)(k-2),
\]

contradicting the choice of \( A_0 \). Therefore, \( S[A_0] \) is connected.

Since the distance in \( S \) between \( v \in V_0 \) and \( \{w, w'\} \subseteq V(B) \) is at least \( t \), \( a \geq t + 2 \), a contradiction.

\(\square\)

**Step 3: Completing the construction**

Let \( G = G(k, \varepsilon) \) be obtained from a copy \( H \) of \( K_k \) by replacing every edge \( uv \) in \( H \) by a copy \( S(uv) \) of \( S_{k,t} \) with base \( \{u,v\} \) so that all other vertices in these graphs are distinct. Suppose \( G \) has a \((k-1)\)-coloring \( f \). Since \(|V(H)| = k\), for some distinct \( u, v \in V(H) \), \( f(u) = f(v) \). This contradicts Lemma 4.4. Thus \( \chi(G) \geq k \).

Suppose there exists \( A \subseteq V(G) \) with

\[
|A| \geq 2 \text{ and } |E(G[A])| > 1 + (1 + \varepsilon)\frac{(k+1)(k-2)}{2(k-1)}(|A| - 2).
\]

(27)
Choose a smallest $A_0 \subseteq V(G)$ satisfying (27) and let $a = |A_0|$. Since a 2-vertex (simple) graph has at most one edge, $a \geq 3$. We claim that

$$G[A_0] \text{ is 2-connected.} \quad (28)$$

Indeed, if not, then since $a \geq 3$, there are $x \in A_0$ and subsets $A_1, A_2$ of $A_0$ such that $A_1 \cap A_2 = \{x\}$, $A_1 \cup A_2 = A_0$, $|A_1| \geq 2$, $|A_2| \geq 2$, and there are no edges between $A_1 - x$ and $A_2 - x$ (this includes the case that $G[A_0]$ is disconnected). By the minimality of $a$, $|E(G[A_j])| \leq 1 + (1 + \epsilon)\frac{(k+1)(k-2)}{2(k-1)}(|A_j| - 2)$ for $j = 1, 2$. So,

$$|E(G[A_0])| = |E(G[A_1])| + |E(G[A_2])| \leq 2 + (1 + \epsilon)\frac{(k+1)(k-2)}{2(k-1)}(|A_1| + |A_2| - 4)$$

$$= 2 + (1 + \epsilon)\frac{(k+1)(k-2)}{2(k-1)}(a - 3) \leq 1 + (1 + \epsilon)\frac{(k+1)(k-2)}{2(k-1)}(a - 2),$$

contradicting (27). This proves (28).

Let $J = \{uv \in E(H) : A_0 \cap (V(S(uv) - u - v) \neq \emptyset)\}$. For $uv \in J$, let $A_{uv} = A_0 \cap (V(S(uv)))$. Since $G[A_0]$ is 2-connected, for each $uv \in J$,

$$\{u, v\} \subset A_{uv} \text{ and } G[A_{uv}] \text{ is connected. In particular, } |A_{uv}| \geq 4. \quad (29)$$

Our next claim is that for each $uv \in J$,

$$|E(G[A_{uv}])| \leq (1 + \epsilon)\frac{(k+1)(k-2)}{2(k-1)}(|A_{uv}| - 2). \quad (30)$$

Indeed, if $|A_{uv}| \leq t + 1$, this follows from Lemma 4.6. If $|A_{uv}| \geq t + 2$, then by the part of Lemma 4.3 dealing with $A \supseteq V_0$,

$$|E(G[A_{uv}])| \leq |E(B_{k,t})| + \frac{(k+1)(k-2)}{2(k-1)}(|A_{uv}| - 2) = \left(\frac{k}{2}\right) + \frac{(k+1)(k-2)}{2(k-1)}(|A_{uv}| - 2).$$

But since $t \geq k^3/\epsilon$, $(k/2) < \epsilon t\frac{(k+1)(k-2)}{2(k-1)}$. This proves (30).

By (30),

$$|E(G[A_0])| = \sum_{uv \in J} |E(G[A_{uv}])| \leq (1 + \epsilon)\frac{(k+1)(k-2)}{2(k-1)} \sum_{uv \in J} (|A_{uv}| - 2) \quad (31)$$

Since each $A_{uv}$ has at most two vertices in common with the union of all other $A_{u'v'}$, $\sum_{uv \in J} (|A_{uv}| - 2) \leq a - 2$. Thus (31) contradicts the choice of $A_0$. It follows that no $A \subseteq V(G)$ satisfies (27), which exactly means that $m_2(G) \leq (1 + \epsilon)\frac{(k+1)(k-2)}{2(k-1)}$. \[\square\]

5 The case $m_2(H) < m_2(K_m)$

When $m_2(H) < m_2(K_m)$ we show that as in the previous case there are two typical behaviors of the function $ex_t(G(n, p), K_m, H)$. For small values of $p$ Lemma 3.2 shows that there exists w.h.p. an $H$-free subgraph of $G(n, p)$ which contains all but a negligible part of the copies of $K_m$. For large values of $p$ Lemma 3.1 shows that w.h.p. every $H$-free graph will have to contain a much smaller proportion of the copies of $K_m$. 18
However, unlike in the case \( m_2(H) > m_2(K_m) \) discussed in Section 3, the change between the behaviors for \( p = n^{-a} \) does not happen at \(-a = -1/m_2(H)\). Theorem 1.4 shows that if \( p = n^{-a} \) and \(-a\) is slightly bigger than \(-1/m_2(H)\) we can still take all but a negligible number of copies of \( K_m \) into an \( H\)-free subgraph. As for a conjecture about where the change happens (and if there are indeed two regions of different behavior and not more) see the discussion in the last section.

**Proof of Theorem 1.4** Let \( G \sim G(n, p) \) with \( p = n^{-a} \) where \(-a = -c + \delta \) for some small \( \delta > 0 \) to be chosen later. Let \( G' \) be the graph obtained from \( G \) by first removing all pairs of copies of \( K_m \) sharing an edge and then removing all edges that do not take part in a copy of \( K_m \). As \( \delta \) is small, we may assume that \(-a < -1/m_2(K_m)\), apply Lemma 2.4 and deduce that w.h.p. the number of copies of \( K_m \) removed in the first step is \( o\left(\binom{n}{m}p^{\frac{a}{2}}\right)\). In the second step there are no copies of \( K_m \) removed, and thus w.h.p. \( N(G, K_m) = (1 + o(1))N(G', K_m) \). Furthermore, if there is a copy of \( H_0 \) in \( G' \) then each edge of it must be contained in a copy of \( K_m \) and not in two or more such copies.

Let \( \mathcal{H}_m \) be the family of the following graphs. Every graph in \( \mathcal{H}_m \) is an edge disjoint union of copies of \( K_m \), it contains a copy of \( H_0 \) and removing any copy of \( K_m \) makes \( H_0 \)-free. Note that if \( G \) is \( \mathcal{H}_m \)-free then \( G' \) is \( H_0 \)-free.

To show that \( G \) is indeed \( \mathcal{H}_m \)-free w.h.p. we prove that for any \( H' \in \mathcal{H}_m \) the expected number of copies of it in \( G \) is \( o(1) \). We will show this for \( p = n^{-\frac{m+1}{2} + \delta} \), and it will thus clearly hold for smaller values of \( p \) as well. For every \( H' \) the expected number of copies of it in \( G(n, p) \) is \( \Theta(p^{e(H')}n^{v(H')}) = \Theta(n^{-\frac{m+1}{2} + \delta} \cdot n^{e(H')}m^{v(H')}) \) and we want to show that it is equal \( o(1) \) for any \( H' \). For this it is enough to show that \(-\frac{e(H')}{v(H')} + m_2(H) - \delta \cdot \frac{e(H')}{v(H')} m_2(H) < 0 \). We first prove that

\[
d(H') := \frac{e(H')}{v(H')} > m_2(H) + \delta'
\]

for some \( \delta' := \delta'(m, c) \) and then to finish show that \( \frac{e(H')}{v(H')} m_2(H) \leq g(m) \) for some function \( g \).

Note that every \( H' \in \mathcal{H}_m \) contains a copy of \( H_0 \) and that \( H_0 \) itself does not contain a copy of \( K_m \) as \( m_2(H_0) < m_2(K_m) \). The vertices of copies of \( K_m \) in \( H' \) can be either all from \( H_0 \) or use some external vertices. Let \( E_1 \) be the edges between two vertices of \( H_0 \) that are not part of the original \( H_0 \) and let \( |E_1| = e_1 \). Furthermore, let \( V_1 \cup \ldots \cup V_k = V(H') \setminus V(H_0) \) be the external vertices, where each \( V_i \) creates a copy of \( K_m \) with the other vertices from \( H_0 \) and let \( |V_i| = v_i \).

Each edge in \( H_0 \) must be a part of a copy of \( K_m \). An edge in \( E_1 \) takes care of at most \( \binom{m}{2} - 1 \) edges from \( H_0 \), and each \( V_i \) takes care of at most \( \binom{m - v_i}{2} \) edges. From this we get that

\[
e(H_0) \leq \sum_{i=1}^{k} \binom{m - v_i}{2} + e_1(\binom{m}{2} - 1)
\leq k(\binom{m - 1}{2}) + e_1(\binom{m}{2} - 1)
\leq \frac{m^2}{2}(k + e_1).
\]

We will take care of two cases, either \( e_1 \geq \frac{e(H_0)}{m^2} \) or \( k \geq \frac{e(H_0)}{m^2} \). In the first case let \( H_1 \) be the graph \( H_0 \) together with the edges in \( E_1 \). Then

\[
\frac{e(H_1)}{v(H_1)} = \frac{e(H_0) + e_1}{v(H_0)} \geq (1 + \frac{1}{m^2})\frac{e(H_0)}{v(H_0)}.
\]

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We can assume $v(H_0)$ is large enough so that $\frac{e(H_0)}{v(H_0)} \geq (1 - \frac{1}{2m^2})$ and as $m_2(H_0)$ is bounded from below by a function of $m$, we get that for some $\delta' := \delta'(m)$ small enough we get

$$\frac{e(H_1)}{v(H_1)} \geq m_2(H_0) + \delta'.$$

Hence w.h.p. there is no copy of $H_1$ in $G$, and thus no copy of $H'$.

Now let us assume that $k \geq \frac{e(H_0)}{m^2}$ and let $\gamma = m_2(K_m) - m_2(H) \geq m_2(K_m) - c$. The expression $\frac{\binom{v_i}{2} + v_i(m - v_i)}{v_i}$ decreases with $v_i$, and as $V_i$ creates a copy of $K_m$ with an edge of $H_0$, we get that $v_i \leq m - 2$ and so $\frac{\binom{v_i}{2} + v_i(m - v_i)}{v_i} \geq \frac{\binom{m - 1}{2}}{m - 2}$. It follows that

$$\sum_{i=0}^{k} \frac{\binom{v_i}{2}}{2} + v_i(m - v_i) \geq \sum_{i=0}^{k} v_i \frac{\binom{m - 1}{2}}{m - 2} = \sum_{i=0}^{k} v_i(m_2(H_0) + \gamma). \quad (32)$$

Every set of vertices $V_i$ uses at least one edge in $H_0$ for a copy of $K_m$, and as there are no two copies of $K_m$ sharing an edge, it follows that:

$$v(H') = v(H_0) + \sum_{i=0}^{k} v_i \leq e(H_0) + (m - 1)e(H_0) = m \cdot e(H_0).$$

Combining this with the assumption on $k$ we conclude

$$\sum_{i=0}^{k} v_i \geq \frac{e(H_0)}{m^2} \geq \frac{v(H')}{m^3}. \quad (33)$$

Finally a direct calculation yields

$$e(H_0) + e_1 > e(H_0) - 1 = \frac{e(H_0) - 1}{v(H_0) - 2}(v(H_0) - 2) = m_2(H_0)(v(H_0) - 2). \quad (34)$$

Applying the above inequalities we get

$$e(H') = e(H_0) + e_1 + \sum_{i=0}^{k} \frac{\binom{v_i}{2}}{2} + v_i(m - v_i) \geq m_2(H)(\sum_{i=0}^{k} v_i + v(H_0) - 2) + \sum_{i=0}^{k} v_i \gamma$$

$$= m_2(H)(v(H') - 2) + \sum_{i=0}^{k} v_i \gamma \geq m_2(H)(v(H') - 2) + \frac{v(H')}{m^3} \gamma$$

$$\geq v(H')(m_2(H_0) + \frac{1}{2m^3} \gamma).$$
The last inequality holds if $2m_2(H) \leq v(H') \frac{2}{2m^3}$, but this is true as $v(H_0)$ is large enough. Thus, for $\delta' := \delta'(m, c)$ small enough,

$$\frac{e(H')}{v(H')} \geq m_2(H_0) + \frac{1}{2m^3} \gamma \geq m_2(H_0) + \frac{1}{2m^3}(m_2(K_m) - c) \geq m_2(H_0) + \delta'$$

and again, w.h.p. $G$ will not have a copy of $H'$.

It is left to show that indeed $\frac{e(H')}{v(H')}m_2(H) \leq g(m)$. By the definition of $H'$ we get that $\frac{e(H')}{v(H')} < \frac{e(H_0) + (m - 2)e(H_0)}{v(H')} = (m - 1)\frac{e(H_0)}{v(H_0)}$. As we may assume that $v(H_0)$ is large, it follows that $\frac{e(H_0)}{v(H_0)} \leq m_2(H_0)(1 + \frac{1}{m})$, and as $m_2(H) < m_2(K_m)$, we conclude that for some $g(m)$ the needed inequality holds.

To finish this section, we show that indeed the theorem can be applied to $G(m + 1, \epsilon)$.

Proof of Lemma 1.2: To prove this we will use the following fact. If $\frac{p}{q}$ and $\frac{b}{c}$ are rational numbers such that $0 < |\frac{p}{q} - \frac{b}{c}| \leq \frac{1}{6M}$ then $p \geq M$. Indeed, assume towards a contradiction that $q < M$, but then $|\frac{p}{q} - \frac{b}{c}| = |\frac{pq - bc}{q^2}| \geq \frac{1}{q^2} > \frac{1}{6M}$.

Let $G_0 := G_0(m + 1, \epsilon)$, and take $\frac{p}{q} = \frac{(m+2)(m-1)}{2m}$ and $\frac{b}{c} = \frac{e(G_0) - 1}{v(G_0) - 2}$. By Theorem 1.3 it follows that $|\frac{p}{q} - \frac{b}{c}| = \epsilon \frac{(m+2)(m-1)}{2m}$. Choosing $\epsilon$ small enough will make $v(G_0)$ as large as needed.

6. Concluding remarks and open problems

- It is interesting to note that there are two main behaviors of the function $ex(G(n,p), K_m, H)$ that we know of. For $K_m$ and $H$ with $\chi(H) = k > m$ for small $p$ one gets that an $H$-free subgraph of $G \sim G(n,p)$ can contain w.h.p. most of the copies of $K_m$ in the original $G$. On the other hand, when $p > \max\{n^{-1/m_2(H)}, n^{-1/m_2(K_m)}\}$ then an $H$-free graph with the maximal number of $K_m$'s is essentially w.h.p. $k - 1$ partite, thus has a constant proportion less copies of $K_m$ than $G$.

If $m_2(H) > m_2(K_m)$ then Theorem 1.2 shows that the behavior changes at $p = n^{-1/m_2(H)}$, but if $m_2(H) < m_2(K_m)$ the critical value of $p$ is bounded away from $n^{-1/m_2(H)}$ and it is not clear where exactly it is.

Looking at the graph $G \sim G(n,p)$ and taking only edges that take part in a copy of $K_m$ yields another random graph $G|_{K_m}$. The probability of an edge to take part in $G|_{K_m}$ is $\Theta(p \cdot n^{m-2}p^{(m-1)}$. A natural conjecture is that if $n^{m-2}p^{(m-1)}$ is much bigger than $n^{-1/m_2(H)}$ then when maximizing the number of $K_m$ in an $H$-free subgraph we cannot avoid a copy of $H$ by deleting a negligible number of copies of $K_m$ and when $n^{m-2}p^{(m-1)}$ is much smaller than $n^{-1/m_2(H)}$ we can keep most of the copies of $K_m$ in an $H$-free subgraph of $G \sim G(n,p)$. It would be interesting to decide if this is indeed the case.
• Another possible model of a random graph, tailored specifically to ensure that each edge lies in a copy of $K_m$, is the following. Each $m$-subset of a set of $n$ labeled vertices, randomly and independently, is taken as an $m$-clique with probability $p(n)$. In this model the resulting random graph $G$ is equal to its subgraph $G|_{K_m}$ defined in the previous paragraph, and one can study the behavior of the maximum possible number of copies of $K_m$ in an $H$-free subgraph of it for all admissible values of $p(n)$.

• There are other graphs $T$ and $H$ for which $ex(n, T, H)$ is known, and one can study the behavior of $ex(G(n, p), T, H)$ in these cases. For example in [10] and independently in [9] it is shown that $ex(n, C_5, K_3) = (n/5)^5$ when $n$ is divisible by 5.

Using some of the techniques in this paper we can prove that for $p \gg n^{-1/2} = n^{-1/m_2(K_3)}$, $ex(n, C_5, K_3) = (1 + o(1))(np/5)^5$ w.h.p. whereas if $p \ll n^{-1/2}$ then w.h.p. $ex(n, C_5, K_3) = \left(\frac{1}{5} + o(1)\right)(np)^5$. Similar results can be proved in additional cases for which $ex(n, T, H) = \Omega(n^t)$ where $t$ is the number of vertices of $T$. As observed in [3], these are exactly all pairs of graphs $T, H$ where $H$ is not a subgraph of any blowup of $T$.

• When investigating $ex(G(n, p), T, H)$ here we focused on the case that $T$ is a complete graph. It is possible that a variation of Theorem 1.2 can be proved for any $T$ and $H$ satisfying $m_2(T) > m_2(H)$, even without knowing the exact value of $ex(n, T, H)$.

• In the cases studied here for non-critical values of $p$, $ex(G(n, p), T, H)$ is always either almost all copies of $T$ in $G(n, p)$ or $(1 + o(1))ex(n, T, H)p^e(T)$. It would be interesting to decide if such a phenomenon holds for all $T, H$.

• As with the classical Turán problem, the question studied here can be investigated for a general graph $T$ and finite or infinite families $\mathcal{H}$.

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