A generalized closure concept based on neighborhood-equivalence and preserving graph Hamiltonicity

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Abstract
A graph is Hamiltonian if it contains a cycle which goes through all vertices exactly once. Determining if a graph is Hamiltonian is known as a NP-complete problem and no satisfactory characterization for these graphs has been found.

In 1976 Bondy and Chvátal introduced a way to get round the Hamiltonicity problem complexity by using a closure of the graph. This closure is a supergraph of G which preserves Hamiltonicity, that is, which is Hamiltonian if and only if G is. Since this seminal work, several closure concepts preserving Hamiltonicity were introduced. In particular Ryjáček defined in 1997 a closure concept for claw-free graphs based on local completion. The completion is performed for every eligible vertex of the graph.

Extending these works, Bretto and Vallée recently introduced a new closure concept preserving Hamiltonicity and based on local completion. The local completion is performed for each neighborhood-equivalence eligible vertex of the graph.

In this article, we generalize the main results of Bretto and Vallée by introducing a broader notion of neighborhood-equivalence eligibility, allowing the definition of a denser graph closure which still preserves Hamiltonicity.

Keywords: graph closure, Hamilton cycles, Hamiltonicity problem, local completion.

1 Introduction
A graph is Hamiltonian if it contains a cycle which goes through all vertices exactly once. Determining if a graph is Hamiltonian is known as a NP-complete problem and no satisfactory characterization for these graphs has been found. A huge body of literature exists on the subject surveyed for instance in [8, 9].
In [1], Bondy and Chvátal introduced a way to get round the Hamiltonicity problem complexity by using a closure of the graph. The closure is then proved to be Hamiltonian if and only if the graph is. In particular, if the closure is a complete graph then the graph is Hamiltonian. Since this seminal article, several closure concepts preserving Hamiltonicity were introduced (for a survey on the topic, see for instance [6]). In particular Z. Ryjáček defined in [10] a closure concept for claw-free graphs based on local completion. The local completion is repeatedly performed on every eligible vertex, as long as such a vertex exists.

Following a different approach, Goodman and Hedetniemi gave in [7] a sufficient condition for Hamiltonicity based on the existence of a clique-covering of the graph. This condition was recently generalized in [2, 3] using the notion of Eulerian clique-covering. It was also shown in [11], that there exists an Eulerian clique-covering of a graph if and only if there exists a normal one, where a clique-covering is normal if it contains the closed neighborhood of every simplicial vertex of the graph. In this context, closure concepts based on local completion are interesting since, then, the closure of a graph contains more simplicial vertices than the graph itself, making the search for a normal clique-covering easier. For instance, a closure in the sense of [10] has at most one normal Eulerian clique-covering.

In [4] a new closure concept based on local completion and preserving Hamiltonicity for all graphs is studied. The closure is defined using the notion of neighborhood-equivalence as first introduced in [2], and is obtained by performing a local completion at all neighborhood-equivalence eligible (N-eligible) vertices of the graph.

In the sequel, we generalize N-eligibility using the notion 2-weighted N-eligibility (N2-eligibility) (cf. Definition 11). We show how to obtain the N2-closure of a graph by performing recursively a local completion at a chosen N2-eligible vertex. The N2-closure is then proved to be Hamiltonian if and only if the graph is. In particular, the N2-closure is a supergraph of the N-closure whenever the N-eligible vertices are chosen first during the completion process.

In a first section, we introduce some notations and remind the reader of the definitions of local completion and neighborhood-equivalence. In a second section, we introduce alternating paths and present some of their properties. Finally, in Section 4, these paths are used to show that, for all graph G and choice function ρ on the set of vertices of G, the N2-closure of G is a N2-eligible free graph which circumference is equal to the circumference of G. We conclude by giving an example (Figure 2) which shows that there are in general more than one N2-closure for a given graph.

## 2 Preliminaries

### 2.1 General Notations

In the sequel, |X| denotes the cardinal of the set X, and X \ Y = \{x ∈ X : x \notin Y\}. We also define ℙ(X) = \{\{x, y\} ⊆ X : x ≠ y\} and [X, Y] = \{\{x, y\} ∈
\( P(X \cup Y) : x \in X, y \in Y \). Notice that \([X, Y] = [Y, X]\) and that \( X \subseteq X' \) implies \([X, Y'] \subseteq [X', Y']\) and \([Y, X] \subseteq [Y', X']\).

We always suppose a graph to be undirected, simple and finite. Thus a graph \( G \) is a pair \((V, E)\) where \( V \) is the vertex set of \( G \), and \( E \) is a subset of \( P(V) \).

To simplify notations a pair \( \{x, y\} \in P(V) \) is simply written \( xy \). If \( X \subseteq V \), \( E(X) = \{xy \in E : x, y \in X\} \).

The \((open)\) neighborhood of \( x \in V \) is the set \( N(x) = \{y : xy \in E\} \). Its \((closed)\) neighborhood is the set \( N[x] = N(x) \cup \{x\} \). The \((closed)\) neighborhood of \( X \subseteq V \) is the set \( N[X] = \bigcup_{x \in X} N[x] \). Its \((open)\) neighborhood is the set \( N(X) = N[X] \setminus X \).

A walk in \( G \) is a sequence of vertices \( w = x_0 \ldots x_k \) such that \( x_i, x_{i+1} \in E \), for every \( i \in \{0, \ldots, k-1\} \). The integer \( k \) is the length of \( w \), \( x_0, x_k \) are its endpoints, \( x_0 \) its starting point and \( x_k \) its ending point. In particular \( x \) is a walk of length 0 with starting and ending point \( x \). A walk is \((closed)\) if \( k \geq 3 \) and \( x_0 = x_k \). A closed walk is a cycle if it contains no repetition of vertex except for the endpoint. We denote by \( c(G) \) the circumference of \( G \), that is, the length of the longest cycle in \( G \). A cycle is Hamilton if it contains every vertex of the graph. A graph is Hamiltonian if it contains a Hamilton cycle.

If \( w = x_0 \ldots x_k \) is a walk then \( V(w) = \{x_0, \ldots, x_k\} \) and \( E(w) = \{x_i, x_{i+1} : i \in \{0, \ldots, k-1\}\} \). Notice that there are infinitely many graphs in which a given sequence \( w \) is a walk and that \( V(w) \subseteq V \) and \( E(w) \subseteq E \), for every such graph.

If \( w = x_0 \ldots x_k \) is a walk then \( x_i \overset{w}{\rightarrow} x_j \), where \( 0 \leq i \leq j \leq k \), denotes the subwalk of \( w \) with endpoints \( x_i, x_j \), and \( x_j \overset{w}{\leftarrow} x_i \) the reverse walk \( x_jx_{j-1}\ldots x_i \). In particular, if \( i = j \) then \( x_i \overset{w}{\rightarrow} x_j = x_i \) and we let \( \overset{w}{\rightarrow} = x_k \ldots x_0 \). Notice that \( w = x_0 \overset{w}{\rightarrow} x_k \). For every \( i \in \{0, \ldots, k-1\} \), \( x_i^+ \) is the successor of \( x_i \) in \( w \), that is, \( x_i^+ = x_{i+1} \). For every \( i \in \{1, \ldots, k\} \), \( x_i^- \) is the predecessor of \( x_i \) in \( w \), that is, \( x_i^- = x_{i-1} \). Finally, if \( w' = y_0 \ldots y_n \) is a walk then \( \overset{w'}{w} \) denotes the sequence \( x_0 \ldots x_ky_0 \ldots y_n \). Clearly \( \overset{w'}{w} \) is a walk if and only if \( x_ky_0 \in E \).

If \( C = x_0 \ldots x_kx_0 \) is a cycle then, for every \( i \in \{1, \ldots, k\} \), the walk \( x_i \overset{C}{\rightarrow} x_kx_0 \) is also a cycle. Such a cycle is said to be a rotation of \( C \). Clearly each rotation of \( C \) has the same vertices than \( C \) and so is of maximal length if and only if \( C \) is. The same remark applies to \( \overline{C} \).

A path is a walk containing no repetition of vertex. For every path \( P \), we define the neighborhood of \( x \) in \( P \) as the set \( P(x) \) which contains, when defined, the predecessor and the successor of \( x \). That is, if \( P = x \) then \( P(x) = \emptyset \); and if \( k \neq 0 \) and \( P = x_0 \ldots x_k \) then \( P(x_0) = \{x_0^+\} \), \( P(x_k) = \{x_k^-\} \) and \( P(x_i) = \{x_i^+, x_i^-\} \), for every \( i \in \{1, \ldots, k-1\} \). Notice that clearly \( P(x) \subseteq N(x) \cap \bigcup_{y \in P} N(y) \), but, since \( P(x) \) contains only the immediate predecessor and immediate successor of \( x \) in \( P \), we may have \( P(x) \neq N(x) \cap \bigcup_{y \in P} N(y) \). For every \( X \subseteq \bigcup_{y \in P} N(y) \), we let \( P(X) = \bigcup_{x \in X} P(x) \). Notice that, contrarily to \( N(X) \cap \bigcup_{y \in P} N(y) \), \( P(X) \cap \bigcup_{y \in P} N(y) \) may not be empty.

A set \( X \subseteq V \) is a clique of \( G \) if \( xy \in E \), for all distinct \( x, y \in X \). A vertex is simplicial if \( N[x] \) is a clique of \( G \). We denote by \( S \) the set of simplicial vertices of \( G \) and by \( NS \) the set of non-simplicial vertices of \( G \).
For every $X \subseteq V$, the size $\sigma(X)$ of $X$ in $G$ is $|E(X)|$. Its size $\sigma_P(X)$ in $P$, where $P$ is a path, is $|E(P) \cap [X, X]|$.

A graph $G$ is connected if all distinct vertices $x, y$ are connected by a walk, and complete if $E = P(V)$. From now on, $G$ is always supposed to be connected.

### 2.2 Local completion, neighborhood equivalence

We define local completion below using the notion of neighborhood-equivalence as defined in [2]. We also give some easy results.

**Definition 1** Two vertices $x, y$ of $G$ are neighborhood-equivalent if $N[x] = N[y]$. We write $x \equiv y$ to express that $x, y$ are neighborhood equivalent and $\bar{x}$ is the class of $x$ modulo $\equiv$.

**Fact 2** For all $x \in V$:
1. $N[\bar{x}] = N[x]$.
2. $\bar{x}$ is a clique and $[N(\bar{x}), \bar{x}] \subseteq E$.

**Definition 3** The local completion of $G$ at $x$ is the graph $G_x = (V, E_x)$, where $E_x = E \cup B_x$ and $B_x = \{yz : yz \notin E, y, z \in N(\bar{x})\}$.

Obviously $E \cap B_x = \emptyset$. Moreover, if $y, z \in N[x]$ and $yz \notin E$ then $y, z \in N(\bar{x})$ by Fact 2. Hence, it is easy to see that $B_x = \{yz : yz \notin E, y, z \in N[x]\}$ and so that $N[x]$ is complete in $G_x$. It is also clear that $G_x$ is connected if $G$ is.

In the sequel, we denote respectively by $N_x[z]$ (resp. $\bar{z}$) the neighborhood (resp. the neighborhood-equivalence class of $z$) in $G_x$ and by $S_x$ (resp. $NS_x$) the set of simplicial (resp. non-simplicial) vertices of $G_x$.

**Fact 4** For every $x \in V$, $G$ is a spanning subgraph of $G_x$ and:
1. For every $x' \in \bar{x}$, $N[x'] = N_x[x']$.
2. $\bar{x} \subseteq S_x \supseteq S$.

**Proof.** Clearly $G$ is a spanning subgraph of $G_x$ and so $N[y] \subseteq N_x[y]$ for every $y \in V$. Moreover, clearly, by definition of $B_x$, if $N[y] \neq N_x[y]$ then $y \in N(\bar{x})$ and so $y \notin \bar{x}$. That proves the first point which in turn implies easily $\bar{x} \subseteq S_x$, since $N_x[x] = N[x]$ is a clique in $G_x$. Notice now that $N(\bar{x}) = N[x] \setminus \bar{x}$ by Fact 2.1 and so $N(\bar{x}) \subseteq N(x)$.

It remains to show that $S \subseteq S_x$. So let $y \in S$. If $N[y] = N_x[y]$ then $y \in S_x$, since $N[y]$ is a clique in $G$ and so in $G_x$. If now $N[y] \neq N_x[y]$, we have $y \in N(\bar{x}) \subseteq N(x)$ and so $x \in N(y)$. Let now $u, v \in N_x[y]$, we must show $uv \in E_x$. If $uv \in E$ then the result is immediate, since $E \subseteq E_x$, so we can suppose $uv \notin E$. Hence, by simpliciality of $y$, at least one vertex among $u, v$ is not in $N[y]$. Without loss of generality, we can suppose $u$ this vertex. We have $y \in N_x[u] \setminus N[u]$, and so $N_x[u] \neq N[u]$ and $u \in N(\bar{x}) \subseteq N(x)$. If now $v \in N(\bar{x})$
then \( uv \in B_x \), by definition of \( B_x \). Finally, if \( v \notin N(x) \) then \( N_x[v] = N[v] \) and, since \( v \in N_x[y] \), we get \( y \in N[v] \) and so \( v \in N[y] \). Hence, since \( x \in N(y) \), we have \( xv \in E \) by simpliciality of \( Y \). So \( v \in N[x] \) and, since \( u \in N[x] \), we conclude \( uv \in E_x \) from the fact that \( N[x] \) is clique in \( G_x \). Thus \( N_x[y] \) is a clique in \( G_x \). Hence, since \( x \in N(y) \), we have \( xv \in E \) by simpliciality of \( Y \). So \( v \in N[x] \) and, since \( u \in N[x] \), we conclude \( uv \in E_x \) from the fact that \( N[x] \) is clique in \( G_x \). Thus \( N_x[y] \) is a clique in \( G_x \) and so \( y \in S_x \). That proves \( S \subseteq S_x \) and so the second point.

**Lemma 5** Let \( X \subseteq V \) and \( y, z \in V \) such that \([\{y, z\}, X] \subseteq E \). If \( C \) is a cycle of maximal length such that \( yz \in E(C) \) then \( V(C) \cap X = X \).

**Proof.** Obviously \( V(C) \cap X \subseteq X \). Up to a rotation we can suppose that \( y \) is the starting point of \( C \), and so either \( C = yz \rightarrow C y \) or \( C = y \rightarrow C zy \). We can suppose the second case (otherwise the proof is done for \( C \)). Moreover, if we suppose \( x \in X \setminus V(C) \) then, since \( yx, zx \in [\{y, z\}, X] \subseteq E \), \( y \rightarrow C zxy \) is a cycle in \( G \) strictly longer than \( C \), contradicting the maximality of \( C \). ■

### 3 Alternating paths

In this section, we assume a graph \( G \) with set of vertices \( V \) and set of edges \( E \).

**Definition 6** Let \( P \) be a path in \( G \) and \( X, Y \) be disjoint subsets of \( V(P) \). If the endpoints of \( P \) are in \( Y \) then:

1. \( P \) is \( YX \)-pseudo-alternating if moreover \( P(X) \subseteq X \cup Y \).
2. \( P \) is \( YX \)-semi-alternating if moreover \( P(X) \subseteq Y \).
3. \( P \) is \( YX \)-alternating if moreover \( P(X) \subseteq Y \) and \( P(Y) \subseteq X \).

A \( YX \)-pseudo-alternating path is proper if it is not \( YX \)-semi-alternating. A \( YX \)-semi-alternating path is proper if it is not \( YX \)-alternating.

Notice that a \( YX \)-alternating path is a particular case of \( YX \)-semi-alternating path which in turn is a particular case of \( YX \)-pseudo-alternating path. Moreover, the \( YX \)-pseudo-alternating path \( P \) is proper if and only if \( P(X) \cap X \neq \emptyset \). Finally, notice that \( P = y \) is the unique \( \{y\} \emptyset \)-alternating path.

**Fact 7** If \( P \) is a path and \( X, Y \) are disjoint subsets of \( V(P) \) then \( P \) is \( YX \)-alternating if and only if \( P \) satisfies the following conditions:

1. \( |X| = |Y| - 1 \)
2. There exist two enumerations \( y_0, \ldots, y_n \) and \( x_0, \ldots, x_{n-1} \) of \( Y \) and \( X \) such that \( P = y_0x_0 \ldots x_{n-1}y_n \).
3. \( V(P) = X \cup Y \), \( P(X) = Y \) and \( P(Y) = X \).
Proof. If is easy to check that if \( P \) satisfies the conditions above then \( P \) is \( YX \)-alternating. Now if \( P \) is \( YX \)-alternating then, using the fact that the endpoints of \( P \) are in \( Y \), that \( P(Y) \subseteq X \) and that \( P(X) \subseteq Y \), it is easy to build the enumerations \( y_0 \ldots y_n \) and \( x_0 \ldots x_{n-1} \), where \( y_0 \) is the starting point of \( P \), \( x_0 \) its successor, and so on, until we reach the ending point \( y_n \) of \( P \). It is then easy to check that \( P \) satisfies the other conditions. ■

We remind the reader that \( \sigma_P(Y) \) is the size of \( Y \) in \( P \), that is, \( \sigma_P(Y) = |E(P) \cap |Y, Y||. \)

Lemma 8 If \( P \) is \( YX \)-semi-alternating then:

1. \( |X| < |Y| - \sigma_P(Y) \).
2. If \( \forall P \setminus (X \cup Y) \neq \emptyset \) then \( |X| < |Y| - 1 \).

Proof. Let \( P = z_0 \ldots z_k \) be a path verifying the conditions of the lemma. For every \( i \in \{0, \ldots, k\} \), let \( P_i = z_0 \ldots z_i \), \( X_i = \forall(P_i) \cap X_i \), \( Y_i = \forall(P_i) \cap Y_i \). Let also \( Z_i = \forall(P_i) \setminus X_i \cup Y_i \) and define \( b_i = 0 \) if \( Z_i = \emptyset \), and \( b_i = 1 \) otherwise. Notice that \( X_i, Y_i, Z_i \) are pairwise disjoint, since \( X \) and \( Y \) are disjoint by Definition 6, and that \( \forall(P) = X_i \cup Y_i \cup Z_i \). We let \( Z = \forall(P) \setminus (X \cup Y) \).

We show first by induction on \( i \in \{0, \ldots, k\} \) that:

1. If \( z_i \in X \) then \( |X_i| \leq |Y_i| - \sigma_P(Y_i) \) and \( |X_i| \leq |Y_i| - b_i \).
2. If \( z_i \in Y \) then \( |X_i| < |Y_i| - \sigma_P(Y_i) \) and \( |X_i| < |Y_i| - b_i \).
3. If \( z_i \in Z \) then \( |X_i| < |Y_i| - \sigma_P(Y_i) \) and \( |X_i| \leq |Y_i| - b_i \).

If \( i = 0 \) then the result comes easily from the fact that \( z_0 \in Y \) (Definition 6). As induction hypothesis suppose now the result true for \( i \in \{0, \ldots, k-1\} \). Notice that the induction hypothesis implies that if \( z_i \in Y \cup Z \) then \( |X_i| < |Y_i| - \sigma_P(Y_i) \) and if \( z_i \in X \cup Z \) then \( |X_i| \leq |Y_i| - b_i \). It also implies, in any case, \( |X_i| \leq |Y_i| - \sigma_P(Y_i) \) and \( |X_i| \leq |Y_i| - b_i \). Let now \( z = z_{i+1} \), we have three possibilities:

1. If \( z \in X \) then \( X_{i+1} = X_i \cup \{z\} \), \( Y_{i+1} = Y_i \), \( \sigma_P(Y_{i+1}) = \sigma_P(Y_i) \) and \( b_{i+1} = b_i \). Hence, in particular, \( |X_{i+1}| = |X_i| + 1 \), since \( P \) does not contain repetition of vertex. Moreover, since \( P(X) \subseteq Y \) (Definition 6), we have \( z_i \in Y_i \) and so \( |X_i| < |Y_i| - \sigma_P(Y_i) \) and \( |X_i| < |Y_i| - b_i \) by induction hypothesis. Hence, obviously \( |X_{i+1}| \leq |Y_{i+1}| - \sigma_P(Y_{i+1}) \) and \( |X_{i+1}| \leq |Y_{i+1}| - b_{i+1} \).

2. If \( z \in Y \) then \( X_{i+1} = X_i \), \( Y_{i+1} = Y_i \cup \{z\} \) and \( b_{i+1} = b_i \). Hence, \( |Y_{i+1}| = |Y_i| + 1 \) and, since \( |X_i| \leq |Y_i| - b_i \) by induction hypothesis, it comes easily \( |X_{i+1}| < |Y_{i+1}| - b_{i+1} \). Now, if \( z \in X \) then \( z \notin E(Y) \), and so \( \sigma_P(Y_{i+1}) = \sigma_P(Y_i) \). Hence, since \( |X_i| \leq |Y_i| - \sigma_P(Y_i) \) by induction hypothesis, it comes \( |X_{i+1}| < |Y_{i+1}| - \sigma_P(Y_{i+1}) \). If \( z \in Y \cup Z \), we have \( |X_i| < |Y_i| - \sigma_P(Y_i) \) by induction hypothesis. Moreover, we have either \( \sigma_P(Y_{i+1}) = \sigma_P(Y_i) + 1 \) if \( z_i \in Y \), or \( \sigma_P(Y_{i+1}) = \sigma_P(Y_i) \) if \( z_i \in Z \). Hence
\[ \sigma_P(Y_{i+1}) \leq \sigma_P(Y_i) + 1 \] and so \( |Y_i| - \sigma_P(Y_i) \leq |Y_{i+1}| - \sigma_P(Y_{i+1}) \). It comes \( |X_{i+1}| < |Y_{i+1}| - \sigma_P(Y_{i+1}) \).

3. If \( z \in Z \) then we have \( X_{i+1} = X_i, Y_{i+1} = Y_i, \sigma_P(Y_{i+1}) = \sigma_P(Y_i) \) and either \( b_{i+1} = b_i + 1 \), if \( Z_i = \emptyset \), or \( b_{i+1} = b_i \) otherwise. Moreover, \( z_i \notin X_i \), since \( P(X) \subseteq Y \), and so \( z_i \notin Y \cup Z \). Hence we have \( |X_i| < |Y_i| - \sigma_P(Y_i) \) by induction hypothesis and so it comes immediately \( |X_{i+1}| < |Y_{i+1}| - \sigma_P(Y_{i+1}) \). If now \( z_i \in Y \), we have \( |X_i| < |Y_i| - b_i \) by induction hypothesis and, since \( b_{i+1} \leq b_i + 1 \), it comes \( |X_{i+1}| \leq |Y_{i+1}| - (b_i + 1) \leq |Y_{i+1}| - b_{i+1} \).

Finally, if \( z_i \in Z \), we have \( Z_i \neq \emptyset \) and so \( b_{i+1} = b_i \). We have also \( |X_i| \leq |Y_i| - b_i \) by induction hypothesis and so \( |X_{i+1}| \leq |Y_{i+1}| - b_{i+1} \).

Notice now that \( P = P_k, X = X_k, Y = Y_k \) and \( Z = Z_k \). Hence, since \( z_k \in Y \) by hypothesis on \( P \), we have \( |X| < |Y| - \sigma_P(Y) \). Moreover, if \( Z \neq \emptyset \) then \( b_k = 1 \) and so \( |X| < |Y| - 1 \). ■

**Lemma 9** A YX-semi-alternating path \( P \) is YX-alternating if and only if \( |X| \geq |Y| - 1 \).

**Proof.** From Fact 7 it is obvious that if \( P \) is YX-alternating then \( P \) is YX-semi-alternating and such that \( |X| \geq |Y| - 1 \). Suppose now that \( P \) is YX-semi-alternating and \( |X| \geq |Y| - 1 \). By Definition 8, it remains to show that \( P(Y) \subseteq X \). It is done by contradiction. Indeed, suppose that there exists \( y \in Y \) and \( z \notin X \) such \( yz \in E(P) \). We have either \( z \notin Y \) and so \( \sigma_P(Y) \geq 1 \), or \( z \notin V(P) \setminus X \cup Y \). In the first case, we get \( |X| \geq |Y| - 1 \geq |Y| - \sigma_P(Y) \), contradicting Lemma 8. In the second case we get \( |X| < |Y| - 1 \) by Lemma 8.2, contradicting \( |X| \geq |Y| - 1 \). ■

## 4 N2-closures and Hamiltonicity

The notions of N-eligible vertex and N-closure of a graph were defined in [4]. A vertex \( x \) is N-eligible if it is non-simplicial and \( |x| \geq |N(x)| \). The N-closure \( cl_N(G) \) of \( G \) is obtained by performing a local completion for every N-eligible vertex of \( G \). The main result of [4] states that, for every graph \( G \), \( G \) is a spanning subgraph of \( cl_N(G) \). \( cl_N(G) \) does not contain any N-eligible vertex and \( c(G) = c(cl_N(G)) \).

In this section, we first introduce a generalization of the notion of N-eligibility called the 2-weighted N-eligibility (N2-eligibility). Then, after having described some techniques for transforming semi-alternating paths into cycles, we show in Theorem 17 that the main result of [4] can be extended to N2-closures of graphs.

### 4.1 N2-eligibility: definition

The N2-eligibility is defined using a kind of weight-function \( \chi_2 \), the weight depending on the number of edges in \( E(N(x)) \). More precisely, \( \chi_2 \) counts the
number of edges in $E(N(\bar{x}))$ up to 2. It is easy to see that every N-eligible vertex is also N2-eligible.

**Definition 10** Let $G$ be a graph. We define the function $\chi_2 : V \mapsto \{0, 1, 2\}$ by, for every $x \in V$:

$$
\chi_2(x) = \begin{cases} 
0 & \text{if } \sigma(N(\bar{x})) = 0 \\
1 & \text{if } \sigma(N(\bar{x})) = 1 \\
2 & \text{otherwise}
\end{cases}
$$

**Definition 11** A vertex $x$ of $G$ is 2-weighted N-eligible (N2-eligible) if $x \in NS$ and $|\bar{x}| \geq |N(\bar{x})| - \chi_2(x)$.

### 4.2 Building cycles from semi-alternating paths

In this section, we assume a graph $G = (V, E)$, a N2-eligible vertex $x \in V$ and a $Y\bar{x}$-semi-alternating path $P$ in $G_x$ such that $Y \subseteq N(\bar{x})$. We remind the reader that $G$ is a spanning subgraph of $G_x$. Notice that the graph $G$ of the previous section corresponds here to $G_x$.

**Fact 12** Since $P$ is $Y\bar{x}$-semi-alternating and $Y \subseteq N(\bar{x})$, it comes:

1. $|Y, \bar{x}| \subseteq E$.
2. $|Y| - \sigma_p(Y) > |\bar{x}| \geq |Y| - \chi_2(x)$.
3. $\chi_2(x) > \sigma_p(Y)$ and $\chi_2(x) \in \{1, 2\}$.
4. $\chi_2(x) - \sigma_p(Y) \in \{1, 2\}$.

**Proof.** Since $Y \subseteq N(\bar{x})$, the first point comes immediately from Fact [22] and we have $|\bar{x}| \geq |N(\bar{x})| - \chi_2(x) \geq |Y| - \chi_2(x)$ by N2-eligibility of $x$. We have also $|\bar{x}| < |Y| - \sigma_p(Y)$ by Lemma [81]. That proves the second point which in turn implies $|Y| - \sigma_p(Y) > |Y| - \chi_2(x)$ and so $\chi_2(x) > \sigma_p(Y)$. We get $\chi_2(x) \neq 0$ and so the third point, which implies easily the fourth one.

**Lemma 13** If $P$ is $Y\bar{x}$-alternating then, for every path $Q$ in $G_x$ with distinct endpoints in $Y$ and without any other common vertex with $P$, there exists a cycle $C$ in $G_x$ such that $\mathbb{V}(C) = \mathbb{V}(P) \cup \mathbb{V}(Q)$ and $\mathbb{E}(C) \subseteq \mathbb{E}(P) \cup \mathbb{E}(Q)$.

**Proof.** Let $P = y_0x_0 \ldots x_{n-1}y_n$, where $y_0, \ldots , y_n$ is an enumeration of $Y$ and $x_0, \ldots , x_{n-1}$ an enumeration of $\bar{x}$. Let $Q$ be a path satisfying the conditions of the lemma and let $y_i, y_j \in \mathbb{V}(P) \cap Y$, $i, j \in \{0, \ldots , n\}$, be the endpoints of $Q$, that is $Q = \overrightarrow{y_i y_j}$. We have $i < j$, since these endpoints are distinct, and we can suppose $i < j$. Notice that $\mathbb{Y}(\{x_i, x_j\}) \subseteq \mathbb{E}$ (Fact [22]) and that $\mathbb{V}(Q) \cap \mathbb{V}(P) = \{y_i, y_j\}$ by hypothesis. Hence, if $j = n$ then $C = \overrightarrow{y_0 y_1 \ldots y_{n-1} y_n} \overrightarrow{x_j y_0}$ is a cycle in $G_x$. If $j < n$ then $C = \overrightarrow{y_0 y_1 \ldots y_{j-1} y_j} \overrightarrow{x_{j+1} y_0} \overrightarrow{y_j x_{j+1} \ldots x_n y_n}$ is a cycle in $G_x$. Clearly, in both cases, $\mathbb{V}(C) = \mathbb{V}(P) \cup \mathbb{V}(Q)$ and every edge of $C$ not already in $P$ is either in $Q$ or in $\mathbb{Y}(\{x_i, x_j\}) \subseteq \mathbb{Y}(\bar{x})$. Hence, $\mathbb{E}(C) \subseteq \mathbb{E}(P) \cup \mathbb{E}(Q) \cup \mathbb{E}(Q)$.

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Lemma 14 If $P$ is $Y\bar{x}$-alternating and there exists a cycle $C$ in $G_x$ of maximal length such that $\forall(C) = \forall(P)$ then there exists a cycle $C'$ in $G_x$ such that $\forall(C') = \forall(P)$ and $E(C') \subseteq E(P) \cup E$.

Proof. Suppose that $P$ is $Y\bar{x}$-alternating and let $C$ be a cycle in $G_x$ of maximal length such that $\forall(C) = \forall(P)$. Let $d = \chi_2(x) - \sigma_P(Y)$. We have $\sigma_P(Y) = \chi_2(x) - d$ and so, from Fact 12.2, we have $|Y| - \chi_2(x) + d > |\bar{x}| \geq |N(\bar{x})| - \chi_2(x) \geq |Y| - \chi_2(x)$. Hence, in particular: (1) $|Y| + d > |N(\bar{x})| \geq |Y|$.

We show now that there is an edge $uv \in E(Y)$ such that $uv \notin E(P)$. If $d = 1$ then there exists a unique edge $uv \in E(N(\bar{x}))$ such that $uv \notin E(P) \cap [Y, Y]$. From (1), we get $|Y| + 1 > |N(\bar{x})| \geq |Y|$ and so, since $Y \subseteq N(\bar{x}), Y = N(\bar{x})$ and so $uv \in E(Y)$. If now $d = 2$ then there exist two edges $uv, u'v' \in E(N(\bar{x}))$ such that $uv, u'v' \notin E(P) \cap [Y, Y]$. If at least three distinct vertices among $u, v, u', v'$ belong to $Y$ then at least two belong to the same edge $uv$ or $u'v'$, and we get the result we are looking for. Suppose now that at most two vertices among $u, v, u', v'$ belong to $Y$. From (1), we have $|Y| + 2 > |N(\bar{x})| \geq |Y|$. Hence there exists at most one vertex in $N(\bar{x}) \setminus Y$. So $uv$ and $u'v'$ must have a common vertex, otherwise $u, v, u', v'$ would be distinct vertices of $N(\bar{x})$ and so at least three of them would be in $Y$. Without loss of generality, we can suppose $v = v'$. Notice that, since $|N(\bar{x}) \setminus Y| \leq 1$, at least two vertices among $u, v, v'$ are in $Y$ and so exactly two of them are. If they both belong to the same edge, we have the result we are looking for. We show now that the second case is impossible. Indeed, if we suppose $u, v' \in Y$, and since $v \notin \bar{x} \cup Y = \forall(P)$ (Fact 7.3), the path $Q = uvv'$ satisfies the conditions of Lemma 13. Hence, there exists a cycle $C'$ in $G_x$ such that $\forall(C') = \forall(P) \cup \forall(Q)$. But $\forall(P) \cup \forall(Q) = \forall(P) \cup \{v\}$ and since $\forall(P) = \forall(C)$ by hypothesis, $C'$ is a cycle containing one more vertex than $C$, contradicting the maximality of $C$.

Hence, we have shown that there exists an edge $uv \in E(Y)$ such that $uv \notin E(P)$. It remains to notice that $Q = uv$ is a path satisfying the condition of Lemma 13. Hence there exists a cycle $C'$ such that $\forall(C') = \forall(P) \cup \forall(Q)$ and $E(C') \subseteq E(P) \cup [Y, \bar{x}] \cup \{uv\}$. Since $\forall(P) \cup \forall(Q) = \forall(P)$ and $|Y, \bar{x}] \cup \{uv\} \subseteq E$ (Fact 12.1 and $uv \in E(Y)$), $C'$ is the cycle we are looking for.

Lemma 15 If $P$ is a proper $Y\bar{x}$-semi-alternating path then there exists a cycle $C$ such that $\forall(C) = \forall(P)$ and $E(C) \subseteq E(P) \cup E$.

Proof. Suppose that $x$ and $P$ satisfy the conditions of the lemma. Since $P$ is proper, we have $|\bar{x}| < |Y| - 1$ by Lemma 9. Hence $|Y| - 2 \geq |\bar{x}|$ and since $\chi_2(x) \leq 2$, it comes also $|\bar{x}| \geq |Y| - \chi_2(x) \geq |Y| - 2$ by Fact 12.2. Hence, we have $|\bar{x}| = |Y| - 2$ and so $\chi_2(x) = 2$, otherwise we would have $\chi_2(x) = 1$ by Fact 12.3 and so $|Y| - 2 = |\bar{x}| \geq |Y| - 1$ by Fact 12.2. From $Y \subseteq N(\bar{x})$, we have also $Y = N(\bar{x})$, otherwise we would have $|Y| - 2 = |\bar{x}| \geq |N(\bar{x})| - 2 > |Y| - 2$. Let now $n = |Y|, m = |\bar{x}|, y_1, \ldots, y_n$ be an enumeration of the vertices of $Y$ ordered as they appear in $P$ and $x_1, \ldots, x_m$ be a similar enumeration for $\bar{x}$. Since $P$ is not $Y\bar{x}$-alternating and $P(\bar{x}) \subseteq Y$ (Definition 8), there is a smaller $i \in \{1, \ldots, n - 1\}$ such that $y_i^+ \notin \bar{x}$. Let $P_0 = y_1 \rightarrow P y_i$ and $P_1 = y_{i+1} \rightarrow P y_n$. 

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Let also \( X_j = \nabla(V_j) \cap \bar{x} \) and \( Y_j = \nabla(V_j) \cap Y \), \( n_j = |Y_j| \) and \( m_j = |X_j| \), where \( j \in \{0, 1\} \). Clearly, since \( \bar{P}(\bar{x}) \subseteq Y \), we have \( P_j(X_j) \subseteq Y_j \) and so \( P_j \) is \( Y_j \times X_j \)-semi-alternating, \( j \in \{0, 1\} \). By minimality of \( i \), \( P_0 = y_1 x_1 \ldots x_{i-1} y_i \) and so \( P_0 \) is \( Y_0 \times X_0 \)-alternating. Moreover, since \( y_{i+1} \) is the next element of \( Y \) appearing in \( P \) after \( y_i \), there is no vertex of \( Y \) in between \( y_i \) and \( y_{i+1} \). In addition, since \( P(\bar{x}) \subseteq Y \) and \( y_i^+ \notin \bar{x} \), it is easy to see there is no vertex of \( \bar{x} \) between \( y_i \) and \( y_{i+1} \). Hence, \( \bar{x} = x_0 \cup X_1 \) and \( Y = Y_0 \cup Y_1 \) and, since \( P_0 \) contains \( i \) vertices of \( Y \) and \( i-1 \) vertices of \( \bar{x} \), we have \( n_1 = n - i \) and \( m_1 = n - 2 - (i - 1) \). It comes \( m_1 = n_1 - 1 \) and so by Lemma \( [9] \) \( P_1 \) is the alternative sequence \( y_{i+1} x_i \ldots x_n - 2 y_n \).

We show now that there exists an edge \( uv \in E(Y) \) such that \( uv \notin E(P) \) and \( uv \neq y_i y_{i+1} \). Let \( d = \chi_2(x) - \sigma_P(Y) \). Since \( d \geq 1 \) (Fact 122), there exists an edge \( uv \in E(N(\bar{x})) \) such that \( uv \notin E(P) \cap [Y, Y] \). Since \( Y = N(\bar{x}) \), we have \( uv \in E(Y) \). If now \( y_{i+1} = y_i^+ \), that is, \( y_i y_{i+1} \in E(P) \) and \( P = P_0 P_1 \), we have \( uv \neq y_i y_{i+1} \), since \( uv \notin E(P) \), and so \( uv \) is the edge we are looking for. Now if \( y_{i+1} \neq y_i^+ \), since \( uv \) is the next vertex of \( Y \) after \( y_i \) on \( P \) and since neither \( P_0 \) nor \( P_1 \) contains an edge in \( [Y, Y] \), it is clear that \( P \) contains no such edge. Hence \( d = 2 - 0 = 2 \) and there exists another edge \( u'v' \in E(N(\bar{x})) \) such that \( u'v' \notin E(P) \cap [Y, Y] \). Since \( Y = N(\bar{x}) \), we have \( u'v' \in E(Y) \) and so at least one edge among \( uv \) and \( u'v' \) must be different from \( y_i y_{i+1} \). Without loss of generality, we can suppose \( uv \) this edge.

Finally, we show that there exists a cycle \( C \) in \( G_x \) such that \( \forall(C) = \forall(P) \) and \( E(C) \subseteq E(P) \cup E \). Since \( [Y, \bar{x}] \cup \{uv\} \subseteq E \) (Fact 121 and \( uv \in E(Y) \)), it is sufficient to show that there is a cycle \( C \) in \( G_x \) such that \( \forall(C) = \forall(P) \) and \( E(C) \subseteq E(P) \cup [Y, \bar{x}] \cup \{uv\} \). We define now \( Q = \emptyset \) if \( y_{i+1} = y_i^+ \), and \( Q = y_i^+ \to P \) \( y_{i+1}^- \) otherwise. We have \( P = P_0 Q P_1 \) where \( P_0 Q P_1 \) is defined as \( P_0 P_1 \) if \( Q = \emptyset \). Since \( uv \in E(Y) \), there are \( k, l \in \{1, \ldots, n\} \) such that \( uv = y_i y_j \). Without loss of generality we can suppose \( k < l \), and so \( k \neq n \) and \( l \neq 1 \). Moreover, we have \( y_k y_l \neq y_i y_{i+1} \) since \( y_k y_l \neq y_i y_{i+1} \). Finally, since \( P = y_1 x_1 \ldots x_{i-1} y_i y_{i+1} x_{i+1} \ldots x_n - 2 y_k y_l \) and \( V(Q) \cap \bar{Y} = \emptyset = V(Q) \cap X \), it is clear that the successor \( y^+ \) in \( P \) of every vertex \( y \in Y \) is in \( \bar{x} \), except for \( y_k \) and \( y_l \). In particular \( y_n^+ \) is not defined. Similarly, the predecessor \( y^- \) of every \( y \in Y \) is in \( \bar{x} \) except for \( y_{i+1} \) and \( y_i \), where \( y_i^- \) is not defined. We make now two cases:

- Suppose first \( k = i \) and so, since \( y_k y_l \neq y_i y_{i+1} \), \( i + 1 < l \leq n \). We have \( y_i \in Y_1 \), \( y_l^- \in X_1 \) and so \( [Y, \{y_i^-\}] \subseteq [Y, \bar{x}] \subseteq E \). Now if \( y_i \neq \bar{x} \), we have either \( k = 1 \) or \( k = i + 1 \). The second case is impossible, since \( k = i \), and so \( k = 1 \). Hence, \( P_0 = y_i \) and \( C = y_i y_l \to P \) \( y_i y_{l-1} \to P \) \( y_1 \) is the cycle we are looking for. If now \( y_i \in \bar{x} \) then \( k \neq 1 \), \( [Y, \{y_i^-\}] \subseteq [Y, \bar{x}] \subseteq E \) and \( C = y_i y_l \to P \) \( y_k y_l \to P \) \( y_n \) \( y_k \) \( P \) \( y_1 \) is the cycle we are looking for.

- Suppose now \( k \neq i \). We have \( y_k^+ \in \bar{x} \) and so \( [Y, \{y_k^+\}] \subseteq [Y, \bar{x}] \subseteq E \). If \( y_i^- \in \bar{x} \) then \( [Y, \{y_i^-\}] \subseteq [Y, \bar{x}] \subseteq E \) and so \( C = y_i \to P \) \( y_k y_l \to P \) \( y_n \) \( y_k \) \( P \) \( y_1 \) is the cycle we are looking for. If \( y_i^+ \in \bar{x} \) then \( [Y, \{y_i^+\}] \subseteq [Y, \bar{x}] \subseteq E \) and
of

\[ V_P = \text{maximality of } c \]

for every \( c \). We can suppose the second case (otherwise the proof is done using \( \mathbb{Z} \)).

The case \( l = i \) being impossible, we have \( l = i + 1 = n \), \( P_1 = y_n \) and \( C = y_1 \xrightarrow{P} y_ky_n \xrightarrow{P} y_i^+y_1 \) is the cycle we are looking for.

\[ \square \]

### 4.3 The N2-closure operation preserves Hamiltonicity

Theorem 17 is essentially a corollary of the proposition below, which states that the circumference is preserved by local completion at a N2-eligible vertex.

**Proposition 16** If \( x \) is a N2-eligible vertex of \( G \) then \( c(G_x) = c(G) \).

**Proof.** Since \( G \) is a spanning subgraph of \( G_x \), clearly \( c(G) \leq c(G_x) \). Now to prove \( c(G_x) \leq c(G) \) it is sufficient to prove that every cycle \( C \) of \( G_x \) can be transformed into a cycle \( \bar{C} \) of \( G \) such that \( \forall(C) = \forall(C) \). The proof is by induction on the number of edges of \( C \) which are in \( B_x \). Indeed, if \( C \) contains no edge of \( B_x \) then \( C \) is a cycle of \( G \) and the result is immediate. Now suppose that \( C \) contains \( k + 1 \) edges of \( B_x \) and let \( yz \in B_x \cap E(C) \). By Definition 3 \( y, z \in N(\bar{x}) \) and \( yz \notin E \). Moreover, since \( \{y, z\} \subseteq E \subseteq E_x \) by Fact 4.2 the maximality of \( C \) implies \( \forall(C) \cap \bar{x} = \bar{x} \) by Lemma 5.

Up to a rotation, we can suppose that \( y \) is the starting and ending point of \( C \), and so either \( C = yz \xrightarrow{C} y \), or \( C = y \xrightarrow{C} yz \). Without loss of generality, we can suppose the second case (otherwise the proof is done using \( \bar{C} \)). Let now \( P = y \xrightarrow{C} z \). Clearly \( \forall(P) = \forall(C) \) and \( E(P) = E(C) \setminus \{yz\} \). Hence in particular, \( |B_x \cap E(P)| = k \) and \( \bar{x} \subseteq \forall(P) \).

We show now that there exists a cycle \( C' \) in \( G_x \) such that \( \forall(P) = \forall(C') \) and \( E(C') \subseteq E(P) \cup E \). Let \( Y = \forall(P) \cap N(\bar{x}) \). Clearly \( Y \) and \( \bar{x} \) are disjoint subsets of \( \forall(P) \) and \( y, z \in Y \). Moreover, by Facts 4.1 and 4.1 we have \( N[\bar{x}] = N_{\bar{x}}[x'] \), for every \( x' \in \bar{x} \), and so it is straightforward to check that \( P(\bar{x}) \subseteq \bar{x} \cup Y \). Hence \( P \) is a \( Y \bar{x}\)-pseudo-alternating path (cf. Definition 5). We have also \( |Y, \bar{x}| \subseteq E \) by Fact 4.2. Now, if \( P \) is proper, that is, if \( P(\bar{x}) \cap \bar{x} \neq \emptyset \), then there exists an edge \( uv \in E(P) \cap [\bar{x}, \bar{x}] \). Since \( uz, vz \in [\bar{x}, Y] \subseteq E \), \( C' = y \xrightarrow{P} uz \xrightarrow{P} vz \) is the cycle we are looking for. Now, if \( P \) is \( Y \bar{x}\)-semi-alternating then there exists a cycle \( C' \) such that \( \forall(C') = \forall(P) \) and \( E(C') \subseteq E(P) \cup E \) by Lemma 4.4 and 4.5.

Hence, in any case, there exists a cycle \( C' \) such that \( \forall(P) = \forall(C') \) and \( E(C') \subseteq E(P) \cup E \). So, in particular, we have \( B_x \cap E(C') \subseteq B_x \cap E(P) \) and the induction hypothesis applies to \( C' \). Hence there is a cycle \( C \) of \( G \) such that \( \forall(C') = \forall(C) \). So, since \( \forall(C') = \forall(P) = \forall(C) \), \( C \) is the cycle of \( G \) we are looking for. \( \square \)

Notice that the notion of N2-eligibility can be generalized to every positive integer \( k \), in the obvious way, by using a weight-function \( \chi_k \) which counts the edges of \( E(N(\bar{x})) \) up to \( k \). Nevertheless, the N2-eligibility is optimal in the sense that Proposition 5.6 is not always true as soon as \( x \) is \( Nk \)-eligible for some
A non-Hamiltonian graph $G$ where $x$ is $N3$-eligible.

The graph $G_x$ with Hamilton cycle $zuwvexy$.

Figure 1: A non-Hamiltonian graph with a Hamiltonian $N3$-completion.

$k \geq 3$. A counter-example is given in Figure 1 for $k = 3$, and so for every $k \geq 3$, since every $N3$-eligible vertex is also $Nk$-eligible for every $k > 3$.

We remind the reader that a choice function on $V$ is a function $\rho : \mathcal{P}(V) \mapsto V$ such that, for every non-empty $X \in \mathcal{P}(V)$, $\rho(X) \in X$.

**Theorem 17** For all graph $G$ and choice function $\rho$ on $V$ there exists a graph $cl_{\rho}(G)$ containing no $N2$-eligible vertex and such that $G$ is a spanning subgraph of $cl_{\rho}(G)$ and $c(cl_{\rho}(G)) = c(G)$.

Proof. The proof is by induction on the number of non-simplicial vertices of $G$. If $NS = \emptyset$ or if there is no $N2$-eligible vertex in $G$ then we define $cl_{\rho}(G) = G$. Otherwise, let $x = \rho(\nu)$, where $\nu$ is the set of $N2$-eligible vertices of $G$. Notice that $x \in NS$ and, since $\bar{x} \subseteq S_x \supseteq S$ by Fact 12 it comes $x \in S_x \setminus S$, $S \subseteq S_x$ and so $NS_x \subseteq NS$. Hence, by induction hypothesis, there exists a graph $cl_{\rho}(G_x)$ which contains no $N2$-eligible vertex, such that $G_x$ is a spanning subgraph of $cl_{\rho}(G_x)$ and such that $c(G_x) = c(cl_{\rho}(G_x))$. Since $G$ is a spanning subgraph of $G_x$ and $c(G) = c(G_x)$ by Proposition 16 the result comes by letting $cl_{\rho}(G) = cl_{\rho}(G_x)$. $lacksquare$

**Corollary 18** $cl_{\rho}(G)$ is Hamiltonian if and only if $G$ is, if and only if $cl_{\rho'}(G)$ is, for every choice function $\rho'$ on $V$.

5 Conclusion

In this article, we introduced another closure concept preserving Hamiltonicity which is essentially a generalization of $N$-closure defined in [4]. Nevertheless, due to its greater generality, the $N2$-closure obtained in Theorem 17 by recursively choosing a $N2$-eligible vertex $x$ may depend of the choice of $x$. Hence there are often more than one $N2$-closure for a given graph. As shown in Figure 2 below, this can be due to the fact that a non-simplicial vertex may be $N2$-eligible in $G$ but not in $G_x$, although it is still non-simplicial in $G_x$. Hence, contrary to
A graph $G$ where $x, x'$ and $z, z'$ form two classes of $N_2$-eligible vertices

The $N_2$-closure of $G$ obtained by choosing $x$ and where $z, z'$ are non-simplicial and not $N_2$-eligible

The $N_2$-closure of $G$ obtained by choosing $z$ and where $x, x'$ are non-simplicial and not $N_2$-eligible

Figure 2: A graph with two distinct non-optimal $N_2$-closures.

the $N$-closure, the $N_2$-closure is not optimal in the sense that every $N_2$-eligible vertex of $G$ would be simplicial in $cl_{ρ}(G)$. Nevertheless, it seems that a strategy to build an optimal $N_2$-closure is possible.

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