DECAY RATES AND INITIAL VALUES FOR TIME-FRACTIONAL
DIFFUSION-WAVE EQUATIONS

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Abstract. We consider a solution $u(\cdot,t)$ to an initial boundary value problem for time-
fractional diffusion-wave equation with the order $\alpha \in (0,2) \setminus \{1\}$ where $t$ is a time variable.
We first prove that a suitable norm of $u(\cdot,t)$ is bounded by $\frac{1}{t^\alpha}$ for $0 < \alpha < 1$ and $\frac{1}{t^{\alpha-1}}$ for
$1 < \alpha < 2$ for all large $t > 0$. Moreover we characterize initial values in the cases where
the decay rates are faster than the above critical exponents. Differently from the classical
diffusion equation $\alpha = 1$, the decay rate can give some local characterization of initial
values. The proof is based on the eigenfunction expansions of solutions and the asymptotic
expansions of the Mittag-Leffler functions for large time.

Key words. fractional diffusion-wave equation, decay rate, initial value

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial \Omega$ and let $\nu(x) := (\nu_1(x), \ldots, \nu_d(x))$
be the unit outward normal vector to $\partial \Omega$ at $x$. We assume that

$$0 < \alpha < 2, \quad \alpha \neq 1.$$ 

By $\partial^\alpha_t$ we denote the Caputo derivative:

$$\partial^\alpha_t g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} g(s) \, ds$$

for $\alpha \notin \mathbb{N}$ satisfying $n-1 < \alpha < n$ with $n \in \mathbb{N}$ (e.g., Podlubny [13]). For $\alpha = 1$, we write
$\partial_t g(t) = \frac{dg}{dt}$ and $\partial_t g(x,t) = \frac{\partial g}{\partial t}(x,t)$.

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We consider an initial boundary value problem for a time-fractional diffusion-wave equation:

\[
\begin{align*}
\partial_t^\alpha u(x, t) &= -Au(x, t), \quad x \in \Omega, \ 0 < t < T, \\
\left. u \right|_{\partial \Omega \times (0,T)} &= 0, \\
u(x, 0) &= a(x), \quad x \in \Omega \quad \text{if } 0 < \alpha \leq 1, \\
u(x, 0) &= a(x), \quad \partial_t u(x, 0) = b(x), \quad x \in \Omega \quad \text{if } 1 < \alpha < 2.
\end{align*}
\]

(1.1)

Throughout this article, we set

\[
(-Av)(x) = \sum_{i,j=1}^{d} \partial_i (a_{ij}(x) \partial_j v(x)) + c(x)v(x), \quad x \in \Omega,
\]

where \(a_{ij} = a_{ji}, \ 1 \leq i, j \leq n\) and \(c\) are sufficiently smooth on \(\overline{\Omega}\), and \(c(x) \leq 0\) for \(x \in \overline{\Omega}\), and we assume that there exists a constant \(\sigma > 0\) such that

\[
\sum_{i,j=1}^{d} a_{ij}(x) \zeta_i \zeta_j \geq \sigma \sum_{i=1}^{d} \zeta_i^2 \quad \text{for all } x \in \overline{\Omega} \quad \text{and } \zeta_1, ..., \zeta_d \in \mathbb{R}.
\]

For \(\alpha \in (0, 2) \setminus \{1\}\), the first equation in (1.1) is called a fractional diffusion-wave equation, which models anomalous diffusion in heterogeneous media. As for physical backgrounds, we are restricted to a few references: Metzler and Klafter [11], Roman and Alemany [14], and one can consult Chapter 10 in [13].

The properties such as asymptotic behavior as \(t \to \infty\) of solution \(u\) to (1.1) are proved to depend on the fractional order \(\alpha\) of the derivative. Moreover decay rates can characterize the initial values which is very different from the case \(\alpha = 1\). The main purpose of this article is to study these topics.

Throughout this article, \(L^2(\Omega), H^\mu(\Omega)\) denote the usual Lebesgue space and Sobolev spaces (e.g., Adams [11]), and by \(\| \cdot \|\) and \((\cdot, \cdot)\) we denote the norm and the scalar product in \(L^2(\Omega)\) respectively. When we specify the norm in a Hilbert space \(Y\), we write \(\| \cdot \|_Y\). All the functions under consideration are assumed to be real-valued.

We define the domain \(D(A)\) of \(A\) by \(H^2(\Omega) \cap H^1_0(\Omega)\). Then the operator \(A\) in \(L^2(\Omega)\) has positive eigenvalues with finite multiplicities. We denote the set of all the eigenvalues by

\[0 < \lambda_1 < \lambda_2 \cdots \to \infty.\]

We set \(\text{Ker} (A - \lambda_n) := \{v \in D(A); Av = \lambda_n v\}\) and \(d_n := \text{dim Ker} (A - \lambda_n)\). We denote an orthonormal basis of \(\text{Ker} (A - \lambda_n)\) by \(\{\varphi_{nk}\}_{1 \leq k \leq d_n}\).
Then we define a fractional power $A^\gamma$ with $\gamma \in \mathbb{R}$ (e.g., Paazy [12]), and we see
\[
D(A^\gamma) = \left\{ a \in L^2(\Omega); \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} \lambda_n^2 (a, \varphi_{nk})^2 < \infty \right\} \quad \text{if } \gamma > 0
\]
and $D(A^\gamma) \supset L^2(\Omega)$ if $\gamma \leq 0$,
\[
A^\gamma a = \sum_{n=1}^{\infty} \lambda_n^\gamma \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk}, \quad \| A^\gamma a \| = \left( \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} \lambda_n^{2\gamma} (a, \varphi_{nk})^2 \right)^{\frac{1}{2}}, \quad a \in D(A^\gamma). \quad (1.2)
\]
In particular,
\[
A^{-1} a = \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} \frac{1}{\lambda_n} (a, \varphi_{nk}) \varphi_{nk}, \quad \| A^{-1} a \| = \left( \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} \frac{1}{\lambda_n^2} (a, \varphi_{nk})^2 \right)^{\frac{1}{2}}. \quad (1.3)
\]
Moreover it is known that
\[
D(A^\gamma) \subset H^{2\gamma}(\Omega) \quad \text{for } \gamma \geq 0.
\]
The well-posedness for (1.1) is studied for example in Gorenflo, Luchko and Yamamoto [8], Kubica, Ryszewska and Yamamoto [10], Sakamoto and Yamamoto [15]. As for the asymptotic behavior, we know
\[
\| u(\cdot, t) \| \leq \frac{C}{t^\alpha} \| a \|, \quad t > 0 \quad (1.4)
\]
(e.g., [10], [15], Vergara and Zacher [16]). The article [16] first established (1.4) for $t$-dependent operator $A$. Moreover by the eigenfunction expansion of $u(x, t)$ (e.g., [15]), one can prove
\[
\| u(\cdot, t) \| \leq \frac{C}{t^\alpha} \| a \| + \frac{C}{t^{\alpha-1}} \| b \|, \quad t > 0 \quad (1.5)
\]
for $1 < \alpha < 2$.

First we improve (1.4) and (1.5) with stronger norm of $u$.

**Theorem 1.**

Let $t_0 > 0$ be arbitrarily fixed. There exists a constant $C > 0$ depending on $t_0$ such that
\[
\| u(\cdot, t) \|_{H^2(\Omega)} \leq \begin{cases} \frac{C}{t^\alpha} \| a \| & \text{if } 0 < \alpha < 1, \\ \frac{C}{t^\alpha} \| a \| + \frac{C}{t^{\alpha-1}} \| b \| & \text{if } 1 < \alpha < 2 \end{cases}
\]
for $t \geq t_0$.

This theorem means that the Sobolev regularity of initial values is improved by 2 after any time $t > 0$ passes.

The fractional diffusion-wave equation (1.1) models slow diffusion, which the decay estimates (1.4) and (1.5) describe. For $\alpha = 1$, by the eigenfunction expansion of $u$, we can readily
prove that $\|u(\cdot, t)\| \leq e^{-\lambda_1 t}\|a\|$. Needless to say, Theorem 1 does not reject the exponential decay $e^{-\lambda_1 t}$, but as this article shows, the decay rates in the theorem are the best possible in a sense.

For further statements, we introduce a bounded linear operator $F : \mathcal{D}(A^\gamma) \rightarrow Y$, where $\gamma > 0$ and $Y$ is a Hilbert space with the norm $\| \cdot \|_Y$. We interpret that $F$ is an observation mapping, and we consider the following four kinds of $F$.

**Case 1.**
Let $\omega \subset \Omega$ be a subdomain. Let

$$F_1(v) = v|_\omega, \quad \mathcal{D}(F_1) = L^2(\Omega), \quad Y = L^2(\omega).$$

Then $F_1 : L^2(\Omega) \rightarrow L^2(\omega)$ is bounded.

**Case 2.**
Let $\Gamma \subset \partial \Omega$ be a subboundary. Let

$$F_2(v) = \partial_{\nu_A}v|_\Gamma, \quad \mathcal{D}(F_2) = H^2(\Omega), \quad Y = L^2(\Gamma).$$

Here we set

$$\partial_{\nu_A}v := \sum_{i,j=1}^d a_{ij}(x)(\partial_i v)(x)\nu_j(x).$$

The trace theorem (e.g., Adams [1]) implies that $F_2 : H^2(\Omega) \rightarrow L^2(\Gamma)$ is bounded.

**Case 3.**
Let $x^1, ..., x^M \in \Omega$ be fixed and let $\gamma > \frac{d}{4}$, where $d$ is the spatial dimensions. We consider

$$F_3(v) = (v(x^1), ..., v(x^M)), \quad \mathcal{D}(F_3) = \mathcal{D}(A^\gamma), \quad Y = \mathbb{R}^M.$$

Then the Sobolev embedding implies that $\mathcal{D}(F_3) \subset C(\overline{\Omega})$, and so $F_3 : \mathcal{D}(A^\gamma) \rightarrow \mathbb{R}^M$ is bounded. We interpret that $F_3$ are pointwise data.

**Case 4.**
Let $\rho_1, ..., \rho_M \in L^2(\Omega)$ be given and let $Y = \mathbb{R}^M$. Let

$$F_4(v) = \left(\int_{\Omega} \rho_k(x)v(x)dx\right)_{1 \leq k \leq M}, \quad \mathcal{D}(F_4) = L^2(\Omega), \quad Y = \mathbb{R}^M.$$

Then $F_4 : L^2(\Omega) \rightarrow \mathbb{R}^M$ is bounded and corresponds to distributed data with weight functions $\rho_k$ whose supports concentrate around some points in $\Omega$.

Now we state

**Theorem 2.**

In (1.1) we assume that $a, b \in L^2(\Omega)$ for $F_1, F_2, F_4$ and $a, b \in \mathcal{D}(A^{\gamma_0})$ with $\gamma_0 = 0$ if $\frac{d}{4} < 1$
and \( \gamma_0 > \frac{4}{3} - 1 \) if \( \frac{4}{3} \geq 1 \) for \( F_3 \). Let \( u = u(x,t) \) satisfy (1.1). For \( j = 3, 4 \), let \( F_j \) satisfy \( F_j|_{\text{Ker}(\lambda_n - A)} \) is injective for all \( n \in \mathbb{N} \).

Furthermore we assume that for \( j = 1, 2, 3, 4 \), there exist sequences \( \tau_n, n \in \mathbb{N} \) and \( C_n > 0, n \in \mathbb{N} \) which may depend on \( u \), such that

\[
\tau_n > 0, \quad \lim_{n \to \infty} \tau_n = \infty \tag{1.10}
\]

and

\[
\|F_j(u(\cdot, t))\|_Y \leq \frac{C_n}{t^{\tau_n}} \quad \text{as} \quad t \to \infty \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.11}
\]

Then \( u = 0 \) in \( \Omega \times (0, \infty) \).

For \( 0 < \alpha < 1 \), a similar result is proved as Theorem 4.3 in \cite{15}, and Theorem 2 is an improvement.

**Example of \( F_3 \) such that \( F_3|_{\text{Ker}(\lambda_n - A)} \) is injective.**

Let

\[
A = -\Delta, \quad d = 2, \quad \Omega = \{(x_1, x_2); 0 < x_1 < L_1, 0 < x_2 < L_2\}.
\]

Then \( \dim \text{Ker} \ (A - \lambda_n) = 1 \) for each \( n \in \mathbb{N} \) if \( \frac{L_1}{L_2} \not\in \mathbb{Q} \). Indeed, the eigenvalues are given by \( \lambda_{mn} := \left( \frac{m^2}{L_1^2} + \frac{n^2}{L_2^2} \right) \pi^2 \), \( m, n \in \mathbb{N} \) and the corresponding eigenfunction \( \varphi_{mn}(x) \) is given by \( \sin \left( \frac{m\pi}{L_1} x_1 \right) \sin \left( \frac{n\pi}{L_2} x_2 \right) \). Therefore, by \( \frac{L_1}{L_2} \not\in \mathbb{Q} \) we see that if \( \lambda_{mn} = \lambda_{m'n'} \) with \( m, n, m', n' \in \mathbb{N} \), then \( m = m' \) and \( n = n' \).

Let \( x^1 = (x_1^1, x_2^1) \in \Omega \) satisfy \( \frac{x_1^1}{L_1}, \frac{x_2^1}{L_2} \not\in \mathbb{Q} \). We set \( F_3(v) := v(x^1) \) and \( M = 1 \). Then we can readily verify that \( F_3|_{\text{Ker}(\lambda_n - A)} \) is injective for all \( n \in \mathbb{N} \).

The corresponding result to Theorem 2 can be proved for the classical diffusion equation \( \alpha = 1 \): if there exist sequences \( \tau_n, n \in \mathbb{N} \) and \( C_n > 0, n \in \mathbb{N} \) which can depend on \( u \) such that \( \tau_n > 0, \lim_{n \to \infty} \tau_n = \infty \) and

\[
\|u(\cdot, t)\|_{L^2(\omega)} \leq C_n e^{-\tau_n t} \quad \text{as} \quad t \to \infty,
\]

then \( u = 0 \) in \( \Omega \times (0, \infty) \).

Next we consider characterizations of initial values yielding faster decay than \( \frac{1}{t^\alpha} \) and/or \( \frac{1}{t^{\infty / n = \infty}} \).

**Theorem 3.**

(i) Let \( F_1 \) be defined by (1.6).

**Case I:** \( 0 < \alpha < 1 \).
If
\[ \|u(\cdot, t)\|_{L^2(\omega)} = o\left(\frac{1}{t^\alpha}\right) \quad \text{as } t \to \infty, \] (1.12)
then
\[ A^{-1}a = a = 0 \quad \text{in } \omega. \] (1.13)
Moreover, assuming further that either \( a \geq 0 \) in \( \Omega \) or \( a \leq 0 \) in \( \Omega \), then (1.12) yields \( a = 0 \) in \( \Omega \).

**Case II: \( 1 < \alpha < 2 \).**

If
\[ \|u(\cdot, t)\|_{L^2(\omega)} = o\left(\frac{1}{t^{\alpha-1}}\right) \quad \text{as } t \to \infty, \] (1.14)
then
\[ A^{-1}b = b = 0 \quad \text{in } \omega. \] (1.15)
If (1.12) holds, then we have \( u(x_0, 0) = \partial_t u(x_0, 0) = 0 \). Moreover, assuming further that either \( b \geq 0 \) in \( \Omega \) or \( b \leq 0 \) in \( \Omega \), then (1.14) yields \( b = 0 \) in \( \Omega \), and the same conclusion holds for \( a \).

**(ii)** Let \( F_3 \) be defined by (1.8) with \( M = 1 \) and \( a, b \in \mathcal{D}(A^\gamma) \) with \( \gamma > \frac{d}{4} \).

**Case 1: \( 0 < \alpha < 1 \).**

\[ |Au(x_0, t)| = |\partial_t^\alpha u(x_0, t)| = o\left(\frac{1}{t^\alpha}\right) \quad \text{as } t \to \infty \] (1.16)
if and only if
\[ u(x_0, 0) = 0. \]

**Case II: \( 1 < \alpha < 2 \).**

\[ |Au(x_0, t)| = |\partial_t^\alpha u(x_0, t)| = o\left(\frac{1}{t^{\alpha-1}}\right) \quad \text{as } t \to \infty \] (1.17)
if and only if
\[ \partial_t u(x_0, 0) = 0. \]
Moreover (1.16) holds if and only if
\[ u(x_0, 0) = \partial_t u(x_0, 0) = 0. \]

Theorem 3 asserts that the faster decay than \( \frac{1}{t^\alpha} \) or \( \frac{1}{t^{\alpha-1}} \) provides information that initial values vanishes at some point or in a subdomain.
In a special case, we prove

**Proposition 1.**

*Let* $a, b \in \mathcal{D}(A^\gamma)$ *with* $\gamma > \frac{d}{4}$, *and*

\[
\begin{cases}
    a \geq 0 \text{ in } \Omega \quad \text{or} \quad a \leq 0 \text{ in } \Omega, \\
    b \geq 0 \text{ in } \Omega \quad \text{or} \quad b \leq 0 \text{ in } \Omega.
\end{cases}
\]  

**(1.18)**

**Case I:** $0 < \alpha < 1$.

\[|u(x_0, t)| = o\left(\frac{1}{t^{\alpha}}\right) \quad \text{as } t \to \infty,\]

if and only if

\[u(x, 0) = 0, \quad a \in \Omega.\]

**Case II:** $1 < \alpha < 2$.

\[|u(\cdot, t)| = o\left(\frac{1}{t^{\alpha-1}}\right) \quad \text{as } t \to \infty\]

if and only if

\[\partial_t u(x, 0) = 0, \quad x \in \Omega.\]

We cannot expect similar results to Theorem 2 for the classical diffusion equation, i.e., $\alpha = 1$.

**Example of the classical diffusion equation.**

\[
\begin{cases}
    \partial_t u(x, t) = \partial_x^2 u(x, t), \quad 0 < x < 1, \ t > 0, \\
    u(0, t) = u(1, t) = 0, \quad t > 0, \\
    u(x, 0) = a(x), \quad 0 < x < 1.
\end{cases}
\]

Then it is well-known that for arbitrary $t_0 > 0$ and $a \in L^2(0, 1)$, we can choose a constant $C > 0$ such that

\[|u(x_0, t)| \leq Ce^{-\pi^2t} \quad t > 0,\]

and

\[|u(x_0, t)| = o(e^{-\pi^2t}) \quad \text{as } t \to \infty\]

if and only if

\[\sin \pi x_0 \int_0^1 a(x) \sin \pi x dx = 0. \]  

**(1.21)**

In other words, Theorem 2 means that for $\alpha \in (0, 2) \setminus \{1\}$, the faster decay at a point $x_0$ or in a subdomain $\omega$ still keeps some information of the initial value $a(x)$ at $x_0$ or in $\omega$. On the
other hand, in the case of $\alpha = 1$, the decay rate is influenced only by averaged information (1.21) of the initial value. However under extra assumption that the initial value $a$ does not change the signs, by (1.21) we can conclude that $a = 0$ in $\Omega$ by $\sin \pi x \geq 0$ for $0 < x < 1$ if $\sin \pi x_0 \neq 0$. This is true for general dimensions, because one can prove that the eigenfunction for $\lambda_1$ does not change the signs.

This article is composed of five sections. In Section 2, we show lemmata which we use for the proofs of Theorems 1 - 3 and Proposition 1. Sections 3 and 4 are devoted to the proofs of Theorems 1-2 and Theorem 3 and Proposition 1, respectively. In Section 5, we give concluding remarks.

2. Preliminaries

For $\alpha > 0$, we define the Mittag-Leffler functions by

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 2)}, \quad z \in \mathbb{C}$$

and it is know that $E_{\alpha,1}(z)$ and $E_{\alpha,2}(z)$ are entire functions in $z \in \mathbb{C}$ (e.g. Gorenflo, Kilbas, Mainardi and Rogosin [7], Podlubny [13]).

First we show

Lemma 1.

Let $\beta = 1, 2$ and $\alpha \in (0, 2) \setminus \{1\}$.

(i) For $p \in \mathbb{N}$ we have

$$E_{\alpha,\beta}(-\eta) = \sum_{\ell=1}^{p} \frac{(-1)^{\ell+1}}{\Gamma(\beta - \alpha \ell)} \frac{1}{\eta^\ell} + O \left( \frac{1}{\eta^{p+1}} \right) \quad \text{as } \eta > 0, \rightarrow \infty. \quad (2.1)$$

(ii)

$$|E_{\alpha,\beta}(\eta)| \leq \frac{C}{1 + \eta} \quad \text{for all } \eta > 0. \quad (2.2)$$

Proof of Lemma 1.

As for (2.1), see Proposition 3.6 (pp.25-26) in [7] or Theorem 1.4 (pp.33-34) in [13]. The estimate (2.2) is seen by Theorem 1.6 (p.35) in [13] for example, $\blacksquare$

Moreover, by the eigenfunction expansion of the solution $u$ to (1.1) (e.g., Theorems 2.1 and 2.3 in [15]), we have

Lemma 2.
\[ u(x, t) = \sum_{n=1}^{\infty} E_{\alpha, 1}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk}(x) \quad \text{if } 0 < \alpha < 1, \]
\[ u(x, t) = \sum_{n=1}^{\infty} \left[ E_{\alpha, 1}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk}(x) \right. \]
\[ \left. + t E_{\alpha, 2}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (b, \varphi_{nk}) \varphi_{nk}(x) \right] \quad \text{if } 1 < \alpha < 2 \quad (2.4) \]
in \( C([0, T]; L^2(\Omega)) \cap C((0, T]; H^2(\Omega) \cap H_0^1(\Omega)) \).

By Lemma 1, we can prove

**Lemma 3.**

(i) Let \( a, b \in D(A^{\gamma_0}) \) where \( \gamma_0 = 0 \) if \( \frac{d}{4} < 1 \) and \( \gamma_0 > \frac{d}{4} - 1 \) if \( \frac{d}{4} \geq 1 \). Then the series in (2.4) are convergents in \( C(\Omega \times [t_0, T]) \).

(ii) Let \( a, b \in L^2(\Omega) \). Then
\[ \partial_{\nu_A} u(x, t) = \sum_{n=1}^{\infty} E_{\alpha, 1}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (a, \varphi_{nk}) \partial_{\nu_A} \varphi_{nk}(x) \quad \text{if } 0 < \alpha < 1, \]
\[ \partial_{\nu_A} u(x, t) = \sum_{n=1}^{\infty} \left[ E_{\alpha, 1}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (a, \varphi_{nk}) \partial_{\nu_A} \varphi_{nk}(x) \right. \]
\[ \left. + t E_{\alpha, 2}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (b, \varphi_{nk}) \partial_{\nu_A} \varphi_{nk}(x) \right] \quad \text{if } 1 < \alpha < 2 \]
in \( C([t_0, T]; L^2(\partial \Omega)) \).

For the proof of Lemma 3, we show

**Lemma 4.**

Let \( \gamma \in \mathbb{R} \), and let \( t_0 \in (0, T) \) be given arbitrarily. We assume that \( a, b \in D(A^{\gamma}) \). Then there exists a constant \( C = C(t_0, \gamma) > 0 \) such that
\[ \| A^{\gamma+1} u(\cdot, t) \| \leq \begin{cases} C t^{-\alpha} \| A^{\gamma} a \| & \text{if } 0 < \alpha < 1, \\ C(t^{-\alpha} \| A^{\gamma} a \| + t^{-\alpha+1} \| A^{\gamma} b \|) & \text{if } 1 < \alpha < 2 \end{cases} \]
for all \( t \geq t_0 \).

**Proof of Lemma 4.**

For \( \gamma \in \mathbb{R} \), by each \( u_0 \in D(A^{\gamma}) \), applying (1.2) we see
\[ A^{\gamma+1}(u_0, \varphi_{nk}) \varphi_{nk} = (u_0, \varphi_{nk}) \lambda^{\gamma+1}_n \varphi_{nk} = \lambda_n (u_0, \lambda^{\gamma}_n \varphi_{nk}) \varphi_{nk} = \lambda_n (u_0, A^{\gamma} \varphi_{nk}) \varphi_{nk} \]
$$= \lambda_n (A^\gamma u_0, \varphi_{nk}) \varphi_{nk}.$$  

Here we used \((u_0, A^\gamma \varphi_{nk}) = (A^\gamma u_0, \varphi_{nk})\) by (1.2). Therefore, in view of (2.4), we have

\[
A^{\gamma+1} u(x, t) = \sum_{n=1}^{\infty} \lambda_n E_{\alpha,1} (-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (A^\gamma a, \varphi_{nk}) \varphi_{nk}(x) + t \sum_{n=1}^{\infty} \lambda_n E_{\alpha,2} (-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (A^\gamma b, \varphi_{nk}) \varphi_{nk}(x)
\]

in \(C([0, T]; L^2(\Omega))\). We fix \(t_0 > 0\) arbitrarily. Let \(1 < \alpha < 2\). By (2.2) we see

\[
\|A^{\gamma+1} u(\cdot, t)\|^2 \leq \sum_{n=1}^{\infty} \lambda_n^2 |E_{\alpha,1} (-\lambda_n t^\alpha)|^2 \sum_{k=1}^{d_n} \| (A^\gamma a, \varphi_{nk}) \|^2 + t^2 \sum_{n=1}^{\infty} \lambda_n^2 |E_{\alpha,2} (-\lambda_n t^\alpha)|^2 \sum_{k=1}^{d_n} \| (A^\gamma b, \varphi_{nk}) \|^2
\]

\[
\leq C(t_0) \left( \frac{1}{t^2} \sum_{n=1}^{\infty} \lambda_n^2 \sum_{k=1}^{d_n} \| (A^\gamma a, \varphi_{nk}) \|^2 \frac{1}{\lambda_n^2} + \frac{1}{t^2} \sum_{n=1}^{\infty} \lambda_n^2 \sum_{k=1}^{d_n} \| (A^\gamma b, \varphi_{nk}) \|^2 \frac{1}{\lambda_n^2} \right)
\]

for \(t \geq t_0\). The proof for \(0 < \alpha < 1\) is similar. Thus we complete the proof of Lemma 4. ■

Now we proceed to

**Proof of Lemma 3 (i).**

By the condition on \(\gamma\), we apply the Sobolev embedding to have

\[
\|u(\cdot, t)\|_{C(\Omega)} \leq C \|A^{\gamma+1} u(\cdot, t)\|_{L^2(\Omega)}.
\]

Therefore, Lemma 4 yields that the series in (2.4) converge in \(C(\Omega \times [t_0, T])\). Part (ii) is seen by the trace theorem:

\[
\|\partial_\nu A u(\cdot, t)\|_{L^2(\partial \Omega)} \leq C \|Au(\cdot, t)\|_{L^2(\Omega)}.
\]

■

We conclude this section with

**Lemma 5.**

We assume that \(p_n \in \mathbb{R}, \{\ell_m\}_{m \in \mathbb{N}} \subseteq \mathbb{N}\) satisfying \(\lim_{m \to \infty} \ell_m = \infty\), and there exist constants \(C > 0\) and \(\theta_0 \geq 0\) such that

\[
\sup_{n \in \mathbb{N}} |p_n| \leq C \lambda_n^{\theta_0}.
\]  \hspace{1cm} (2.5)

If

\[
\sum_{n=1}^{\infty} \frac{p_n}{\lambda_n^{\ell_m}} = 0 \quad \text{for all } m \in \mathbb{N},
\]
then \( p_n = 0 \) for all \( n \in \mathbb{N} \).

**Proof.**

By \( \mu_n, n \in \mathbb{N} \), we renumber the eigenvalues \( \lambda_n \) of \( A \) according to the multiplicities:

\[
\mu_k = \lambda_1 \text{ for } 1 \leq k \leq d_1, \quad \mu_k = \lambda_2 \text{ for } d_1 + 1 \leq k \leq d_1 + d_2, \ldots.
\]

Then \( \mu_n \leq \lambda_n \) for \( n \in \mathbb{N} \).

On the other hand, there exists a constant \( c_1 > 0 \) such that

\[
\mu_n = c_1 n^{\frac{d}{2}} + o(1) \quad \text{as } n \to \infty
\]

(e.g., Agmon [2], Theorem 15.1). Here we recall that \( d \) is the spatial dimensions. Therefore, we can find a large constant \( \theta_1 > 0 \), for example \( \theta_1 > \frac{d}{2} \), such that

\[
\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\theta_1}} < \infty.
\]

We set

\[
r_n := \frac{p_n}{\lambda_n^{\theta_0 + \theta_1}}, \quad n \in \mathbb{N}.
\]

Then (2.5) implies

\[
\sum_{n=1}^{\infty} |r_n| \leq \sum_{n=1}^{\infty} \left| \frac{p_n}{\lambda_n^{\theta_0}} \right| \frac{1}{\lambda_n^{\theta_1}} \leq C \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\theta_1}} < \infty.
\]

Since \( \sum_{n=1}^{\infty} \frac{r_n}{\lambda_n^{\kappa_m}} = 0 \), we obtain \( \sum_{n=1}^{\infty} \frac{r_n}{\lambda_n^{\kappa_m}} = 0 \) for all \( m \in \mathbb{N} \), where \( \kappa_m = \ell_m - \theta_0 - \theta_1 \), so that

\[
\frac{r_1}{\lambda_1^{\kappa_m}} + \sum_{n=2}^{\infty} \frac{r_n}{\lambda_n^{\kappa_m}} = 0, \quad \text{that is,} \quad r_1 + \sum_{n=2}^{\infty} r_n \left( \frac{\lambda_1}{\lambda_n} \right)^{\kappa_m} = 0.
\]

Hence

\[
|r_1| = \left| - \sum_{n=2}^{\infty} r_n \left( \frac{\lambda_1}{\lambda_n} \right)^{\kappa_m} \right| \leq \left( \sum_{n=2}^{\infty} |r_n| \right) \left( \frac{\lambda_1}{\lambda_2} \right)^{\kappa_m}.
\]

By \( 0 < \lambda_1 < \lambda_2 < \ldots. \), we see that

\[
\left| \frac{\lambda_1}{\lambda_2} \right| < 1.
\]

Letting \( m \to \infty \), we see that \( \kappa_m \to \infty \), and so \( r_1 = 0 \), that is, \( p_1 = 0 \). Therefore,

\[
\sum_{n=2}^{\infty} \frac{r_n}{\lambda_n^{\kappa_m}} = 0.
\]

Repeating the above argument, we have \( p_2 = p_3 = \ldots = 0 \). Thus the proof of Lemma 5 is complete. ■
3. PROOFS OF THEOREMS 1 AND 2

3.1. Proof of Theorem 1.
Now, by noting that \( \| u(\cdot, t) \|_{H^2(\Omega)} \leq C \| Au(\cdot, t) \| \) by \( u(\cdot, t) \in D(A) \), Theorem 1 follows directly from Lemma 4 with \( \gamma = 0 \) in Section 2.

3.2. Proof of Theorem 2.
First Step.
It suffices to prove in the case \( 1 < \alpha < 2 \), because the case \( 0 < \alpha < 1 \) is similar and even simpler. In view of Lemma 3, for \( a \) and \( b \) satisfying the conditions in the theorem, we have

\[
F_j(u(\cdot, t)) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha) F_j \left( \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk} \right) + t \sum_{n=1}^{\infty} \lambda_n E_{\alpha,2}(-\lambda_n t^\alpha) F_j \left( \sum_{k=1}^{d_n} (b, \varphi_{nk}) \varphi_{nk} \right), \quad j = 1, 2, 3, 4
\]

in \( C([t_0, T]; Y) \), where

\[
Y = \begin{cases} 
L^2(\omega) & \text{for } F_1, \\
L^2(\partial\Omega) & \text{for } F_2, \\
\mathbb{R}^M & \text{for } F_3 \text{ and } F_4.
\end{cases}
\]

Applying (2.1) in Lemma 1, we obtain

\[
F_j(u(\cdot, t)) = \sum_{\ell=1}^{p} \frac{(-1)^{\ell+1}}{\Gamma(1-\alpha\ell)} t^{\alpha\ell} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^\ell} F_j \left( \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk} \right) + O \left( \frac{1}{t^{\alpha+\alpha-1}} \right) \sum_{n=1}^{\infty} F_j \left( \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk} \right) + O \left( \frac{1}{t^{\alpha+\alpha-1}} \right) \sum_{n=1}^{\infty} F_j \left( \sum_{k=1}^{d_n} (b, \varphi_{nk}) \varphi_{nk} \right) + O \left( \frac{1}{t^{\alpha+\alpha-1}} \right)
\]

(3.1)

Therefore, (3.1) yields

\[
F_j(u(\cdot, t)) = \sum_{\ell=1}^{p} \frac{(-1)^{\ell+1}}{\Gamma(1-\alpha\ell)} t^{\alpha\ell} \sum_{n=1}^{p_n} p_n^\ell + \sum_{\ell=1}^{p} \frac{(-1)^{\ell+1}}{\Gamma(2-\alpha\ell)} t^{\alpha\ell-1} \sum_{n=1}^{q_n} q_n^\ell + O \left( \frac{1}{t^{\alpha+\alpha-1}} \right) \quad (3.2)
\]

as \( t \to \infty \). Here we set

\[
p_n^j = p_n = F_j \left( \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk} \right), \quad q_n^j = q_n = F_j \left( \sum_{k=1}^{d_n} (b, \varphi_{nk}) \varphi_{nk} \right)
\]

for \( j = 1, 2, 3, 4 \).

In the above series, we exclude \( \ell \in \mathbb{N} \) such that \( 1 - \alpha\ell, 2 - \alpha\ell \in \{0, -1, -2, \ldots\} \), that is, the terms do not appear if \( \alpha\ell \in \mathbb{N} \).
Second Step.

We see that

$$\{ \ell \in \mathbb{N}; \alpha \ell \not\in \mathbb{N} \} \text{ is an infinite set if } \alpha \not\in \mathbb{N}. \quad (3.3)$$

Indeed if not, then \( \{ \frac{n}{\alpha} \}_{n \in \mathbb{N}} \cap \mathbb{N} \) is an infinite set. Therefore there exists \( N_0 \in \mathbb{N} \) such that \( \{ \frac{n}{\alpha} \}_{n \in \mathbb{N}} \supset \{ N_0, N_0 + 1, ... \} \). Hence we can choose \( n', n'' \in \mathbb{N} \) such that \( N_0 + 1 = \frac{n''}{\alpha} \) and \( N_0 = \frac{n'}{\alpha} \), and so \( \frac{n'' - n'}{\alpha} = 1 \). By \( \alpha \not\in \mathbb{N} \), this is impossible. Therefore (3.3) holds.

We number the infinite set \( \{ \ell \in \mathbb{N}; \alpha \ell \not\in \mathbb{N} \} \) by \( \ell_1, \ell_2, \ell_3, ... \) and for each \( N \in \mathbb{N} \), we can rewrite (3.2) as

$$F_j(u(\cdot, t)) = \sum_{m=1}^{N} \frac{(-1)^{\ell_m + 1}}{\Gamma(1 - \alpha \ell_m)} \sum_{n=1}^{\infty} \frac{p_n}{\lambda_n^{\ell_m}} + \sum_{m=1}^{N} \frac{(-1)^{\ell_m + 1}}{\Gamma(2 - \alpha \ell_m)} \sum_{n=1}^{\infty} \frac{q_n}{\lambda_n^{\ell_m}} + O \left( \frac{1}{t^{\alpha \ell_{N+1}-1}} \right) \quad \text{as } t \to \infty. \quad (3.4)$$

Moreover

$$\{ \alpha n \}_{n \in \mathbb{N}} \cap \{ \alpha n - 1 \}_{n \in \mathbb{N}} = \emptyset \quad \text{for } 1 < \alpha < 2. \quad (3.5)$$

Indeed let \( \alpha n' = \alpha n'' - 1 \) with some \( n', n'' \in \mathbb{N} \). Then \( \alpha \ell_0 = 1 \) with \( \ell_0 := n'' - n' \), which means \( \alpha \leq 1 \) and this is a contradiction by \( 1 < \alpha < 2 \).

By (3.5), we number \( \{ \alpha \ell_m \}_{m \in \mathbb{N}} \cup \{ \alpha \ell_m - 1 \}_{m \in \mathbb{N}} \) by \( \alpha \ell_1 - 1 =: s_1 < s_2 < \cdots < s_{2N} := \alpha \ell_N \) and then

$$F_j(u(\cdot, t)) = \sum_{m=1}^{2N} \frac{Q_m}{t^{s_m}} + O \left( \frac{1}{t^{\alpha \ell_{N+1}-1}} \right) \quad \text{in } C([t_0, T]; Y) \text{ as } t \to \infty, \quad (3.6)$$

where

$$Q_m = \frac{(-1)^{\ell + 1}}{\Gamma(1 - \alpha \ell_m)} \sum_{n=1}^{\infty} \frac{p_n}{\lambda_n^{\ell_m}} \quad \text{or} \quad Q_m = \frac{(-1)^{\ell + 1}}{\Gamma(2 - \alpha \ell_m)} \sum_{n=1}^{\infty} \frac{q_n}{\lambda_n^{\ell_m}}.$$

Third Step.

We fix \( N \in \mathbb{N} \) arbitrarily. In terms of (1.11), by (3.6) we see that for each \( n \in \mathbb{N} \) there exists a constant \( C_n > 0 \) such that

$$\|Q_1\|_Y \frac{t^{s_1}}{t^{s_1}} - \sum_{m=2}^{2N} \frac{\|Q_m\|_Y}{t^{s_m}} - \frac{C}{t^{\alpha \ell_{N+1}-1}} \leq \frac{C_n}{t^{s_n}}. \quad (3.7)$$

Then

$$\|Q_1\|_Y \leq \sum_{m=2}^{2N} \frac{\|Q_m\|_Y}{t^{s_m-s_1}} + \frac{C}{t^{\alpha \ell_{N+1}-1-s_1}} + \frac{C_n}{t^{s_n-s_1}}. \quad (3.8)$$

We note that \( \alpha \ell_N < \alpha \ell_{N+1} - 1 \) by \( \alpha > 1 \) and \( \ell_n, \ell_{N+1} \in \mathbb{N} \), so that \( s_{2N} < \alpha \ell_{N+1} - 1 \).
Since \( \lim_{n \to \infty} \tau_n = \infty \), we can choose \( n \in \mathbb{N} \) such that \( \tau_n > s_1 \). Hence, letting \( t \to \infty \), we have \( Q_1 = 0 \) in \( Y \). Continuing this argument, we reach \( Q_m = 0 \) for \( 1 \leq m \leq 2N \). Since \( N \in \mathbb{N} \) is arbitrary, we obtain \( Q_m = 0 \) for all \( m \in \mathbb{N} \), that is,

\[
\sum_{n=1}^{\infty} \frac{p_n}{\lambda_n^{m_n}} = \sum_{n=1}^{\infty} \frac{q_n}{\lambda_n^{m_n}} = 0 \quad \text{for all } m \in \mathbb{N}.
\]

In order to apply Lemma 5, we have to verify (2.5). It suffices to consider for \( p_n \), because the verification for \( q_n \) is the same.

**Case:** \( F_1(u(\cdot, t)) \).

By the Sobolev embedding (e.g., [1]), fixing \( \mu_0 > 0 \) with \( 2\mu_0 > d \), we have

\[
\|p_n\|_{C(\overline{\Omega})} \leq C\|p_n\|_{H^{\mu_0}(\Omega)} \leq C\left\|A^{\frac{\mu_0}{2}} \left( \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk} \right) \right\|_{L^{2}(\Omega)} \leq C\lambda_n^{\frac{\mu_0}{n}} \left\|a\right\|_{L^{2}(\Omega)}.
\]

For the second inequality, we need sufficient smoothness of the coefficients \( a_{ij} \) and \( c \) of the elliptic operator \( A \) (e.g., Gilbarg and Trudinger [9]). Therefore

\[
\|p_n\|_{C(\overline{\Omega})} \leq C\lambda_n^{\frac{\mu_0}{n}}, \quad n \in \mathbb{N}.
\]

Therefore, we see (2.5) for \( F_1, F_3 \) and \( F_4 \) with \( \theta_0 = \frac{\mu_0}{2} \).

**Case:** \( F_2(u(\cdot, t)) \).

We fix \( \mu_0 > 0 \) such that \( 2\mu_0 > d \). Then by the Sobolev embedding, we obtain

\[
\left\| \partial_{\nu A} \left( \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk} \right) \right\|_{C(\partial \Omega)} \leq C \left\| \left( \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk} \right) \right\|_{C^1(\overline{\Omega})} \leq C \lambda_n^{\frac{\mu_0}{n} + \frac{1}{2}} \left\| \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk} \right\|_{L^2(\Omega)} \leq C \lambda_n^{\frac{\mu_0}{n} + \frac{1}{2}} \|a\|_{L^2(\Omega)}.
\]

Hence (2.5) is satisfied with \( \theta_0 = \frac{\mu_0}{2} + \frac{1}{2} \).

Therefore, Lemma 5 yields \( p_n = q_n = 0 \) for all \( n \in \mathbb{N} \), that is,

\[
F_j \left( \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk} \right) = F_j \left( \sum_{k=1}^{d_n} (b, \varphi_{nk}) \varphi_{nk} \right) = 0, \quad j = 1, 2, 3, 4, \quad n \in \mathbb{N}. \quad (3.7)
\]

**Fourth Step.**
It suffices to verify that \( p_n = 0 \) for \( n \in \mathbb{N} \) imply \( a = 0 \) in \( \Omega \). For \( F_3 \) and \( F_4 \), the assumption in Theorem 2 yields
\[
\sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk} = \sum_{k=1}^{d_n} (b, \varphi_{nk}) \varphi_{nk} = 0 \quad \text{in } \Omega
\]
for all \( n \in \mathbb{N} \). Therefore, \( a = b = 0 \) in \( \Omega \), that is, \( u = 0 \) in \( \Omega \times (0, \infty) \). Thus the proof of Theorem 2 is complete for \( F_3 \) and \( F_4 \).

**Case: \( F_1 \).** By (3.7), we have
\[
p_n(x) = \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk}(x) = 0, \quad n \in \mathbb{N}, \, x \in \omega.
\]
Since \((A - \lambda_n)p_n = 0\) in \( \Omega \), we apply the unique continuation for the elliptic operator \( A - \lambda_n \) (e.g., Choulli [3], Hörmander [9]) to see that \( p_n = 0 \) in \( \Omega \) for \( n \in \mathbb{N} \). Since \( a = \sum_{n=1}^{\infty} p_n \) in \( L^2(\Omega) \), we reach \( a = 0 \) in \( \Omega \).

**Case: \( F_2 \).** We set \( u_n(x) = \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk}(x) \) for \( x \in \Omega \). By \( u_n \in \mathcal{D}(A) \), we have \( u_n = 0 \) on \( \Gamma \) and so
\[
\partial_{\nu_A} u_n(x) = u_n(x) = 0, \quad n \in \mathbb{N}, \, x \in \Gamma.
\]
Therefore, since \((A - \lambda_n)u_n = 0\) in \( \Omega \), the unique continuation (e.g., [3], [9]) yields \( \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk}(x) = 0 \) for all \( n \in \mathbb{N} \) and \( x \in \Omega \). Hence, we can see \( a = 0 \) in \( \Omega \). Thus the proof of Theorem 2 is complete.

### 4. Proofs of Theorem 3 and Proposition 1

#### 4.1. Proof of Theorem 3

**Case: \( F_1 \).**

It is sufficient to prove the case \( 1 < \alpha < 2 \). Let (1.14) hold. By (3.2) with \( p = 1 \), noting that \( \Gamma(1 - \alpha) \) and \( \Gamma(2 - \alpha) \) are finite, we see
\[
\left\| \frac{1}{\Gamma(1 - \alpha)} \frac{1}{t^{\alpha}} \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} \frac{(a, \varphi_{nk}) \varphi_{nk}}{\lambda_n} + \frac{1}{\Gamma(2 - \alpha)} \frac{1}{t^{\alpha-1}} \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} \frac{(b, \varphi_{nk}) \varphi_{nk}}{\lambda_n} \right\|_{L^2(\omega)} = o \left( \frac{1}{t^{1-\alpha}} \right).
\]
(4.1)

Therefore, in terms of (1.3), we obtain
\[
\frac{1}{\Gamma(2 - \alpha)} \frac{1}{t^{\alpha-1}} \| A^{-1} b \|_{L^2(\omega)} - \frac{1}{\Gamma(1 - \alpha)} \frac{1}{t^{\alpha}} \| A^{-1} a \|_{L^2(\omega)} = o \left( \frac{1}{t^{1-\alpha}} \right)
\]
as \( t \to \infty \). Multiplying with \( t^{\alpha-1} \) and letting \( t \to \infty \), we obtain \( A^{-1} b = 0 \) in \( \omega \).
Next let (1.12) hold. Then, by \( o\left(\frac{1}{t}\right) \leq o\left(\frac{1}{t^{\alpha}}\right) \), we have also (1.14), so that we have already proved \( A^{-1}b = 0 \) in \( \Omega \). Therefore, since

\[
\frac{(-1)^2}{\Gamma(2 - \alpha)} \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} \frac{(b, \varphi_{nk}) \varphi_{nk}}{\lambda_n} = \frac{1}{\Gamma(2 - \alpha)} \frac{1}{t^{\alpha - 1}} A^{-1}b = 0 \quad \text{in } \Omega,
\]
equality (3.2) with \( p = 1 \) and (1.12) yield

\[
\frac{1}{\Gamma(1 - \alpha)} \frac{1}{t^{\alpha}} \|A^{-1}a\|_{L^2(\Omega)} + o\left(\frac{1}{t^{2\alpha}}\right) + o\left(\frac{1}{t^{2\alpha - 1}}\right) = o\left(\frac{1}{t^{\alpha}}\right).
\]

Multiplying with \( t^\alpha \) and letting \( t \to \infty \), by \( \alpha - 1 > 0 \), we see that \( A^{-1}a = 0 \) in \( \Omega \).

Moreover \( A^{-1}a = 0 \) in \( \Omega \) implies \( a = 0 \) in \( \Omega \). Indeed, setting \( g := A^{-1}a \) in \( \Omega \), we have \( g = 0 \) in \( \Omega \) and \( Ag = a \) in \( \Omega \). Therefore, \( a = A0 = 0 \) in \( \Omega \). Similarly \( A^{-1}b = 0 \) in \( \Omega \) yields \( b = 0 \) in \( \Omega \).

Finally we have to prove that the extra condition

\[
a \geq 0 \quad \text{in } \Omega \quad \text{or} \quad a \leq 0 \quad \text{in } \Omega, \tag{4.2}
\]

implies \( a = 0 \) in \( \Omega \).

Let \( a \geq 0 \) in \( \Omega \). Then \( g := A^{-1}a \) satisfies

\[
\sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j g(x)) + c(x)g(x) \geq 0 \quad \text{in } \Omega.
\]

By \( c \leq 0 \) in \( \Omega \) and \( g = 0 \) on \( \partial \Omega \), the weak maximum principle (e.g., Theorem 3.1 (p.32) in Gilbarg and Trudinger \[6\]) implies that \( g \leq 0 \) on \( \overline{\Omega} \). Since \( g(x) = 0 \) for \( x \in \omega \), we see that \( g \) achieves the maximum 0 at an interior point \( x_0 \in \Omega \). Again by \( c \leq 0 \) in \( \Omega \), the strong maximum principle (e.g., Theorem 3.5 (p.35) in \[6\]) yields that \( g \) is a constant function, that is, \( g(x) = 0 \) for all \( x \in \Omega \). Hence, \( a = Ag = 0 \) in \( \Omega \). Thus the proof in the case \( F_1 \) is complete.

**Case: \( F_3 \).**

It suffices to prove only in the case \( 1 < \alpha < 2 \). By Lemma 2, for arbitrarily chosen \( t_0 \in (0, T) \), we see

\[
Au(x, t) = \sum_{n=1}^{\infty} E_{\alpha, 1}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (a, \varphi_{nk}) \lambda_n \varphi_{nk} \\
+ \sum_{n=1}^{\infty} t E_{\alpha, 2}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (b, \varphi_{nk}) \lambda_n \varphi_{nk} \quad \text{in } C([t_0, T]; L^2(\Omega)).
\]

Using \( a, b \in \mathcal{D}(A^\gamma) \) with \( \gamma > \frac{d}{4} \) and

\[
A^\gamma(a, \varphi_{nk}) \lambda_n \varphi_{nk} = \lambda_n^{1+\gamma}(a, \varphi_{nk}) \varphi_{nk} = \lambda_n(a, A^\gamma \varphi_{nk}) \varphi_{nk} = \lambda_n(A^\gamma a, \varphi_{nk}) \varphi_{nk}, \quad \text{etc.,}
\]
we obtain

\[ A^{1+\gamma}u(x, t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (A^{\gamma}a, \varphi_{nk}) \lambda_n \varphi_{nk} \]

\[ + \sum_{n=1}^{\infty} t E_{\alpha,2}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (A^{\gamma}b, \varphi_{nk}) \lambda_n \varphi_{nk}. \]

Consequently, by Lemma 1, we can prove

\[ \|A^{1+\gamma}u\|_{L^\infty(t_0, T; L^2(\Omega))} < \infty, \]

and so the above series is convergent in \( L^\infty(t_0, T; L^2(\Omega)) \). Since the Sobolev embedding implies \( D(A^{\gamma}) \subset C(\Omega) \) with \( \gamma > \frac{d}{4} \), we obtain

\[ Au(x_0, t) = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (a, \varphi_{nk}) \lambda_n \varphi_{nk}(x_0) \]

\[ + \sum_{n=1}^{\infty} t E_{\alpha,2}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (b, \varphi_{nk}) \lambda_n \varphi_{nk}(x_0), \quad t_0 < t < T \quad \text{in} \quad C[t_0, T]. \]

Substituting (2.1) with \( p = 1 \) and \( \beta = 1, 2 \), we have

\[ Au(x_0, t) = \binom{1}{\Gamma(1-\alpha)} \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk}(x_0) \frac{1}{t^\alpha} \]

\[ + \binom{1}{\Gamma(2-\alpha)} \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} (b, \varphi_{nk}) \varphi_{nk}(x_0) \frac{1}{t^{\alpha-1}} + O \left( \frac{1}{t^{2\alpha-1}} \right) \quad \text{as} \quad t \to \infty. \]

By \( a, b \in D(A^{\gamma}) \subset C(\Omega) \), we find

\[ Au(x_0, t) = \frac{1}{\Gamma(1-\alpha)} t^\alpha a(x_0) + \frac{1}{\Gamma(2-\alpha)} t^{\alpha-1} b(x_0) + O \left( \frac{1}{t^{2\alpha-1}} \right) \quad \text{as} \quad t \to \infty. \quad \text{(4.3)} \]

By an argument similar to Case \( F_1 \) in Theorem 3, we see that (1.16) and (1.17) imply \( a(x_0) = 0 \) and \( b(x_0) = 0 \) respectively. The converse assertion in the theorem directly follows from (4.3).

4.2. Proof of Proposition 1

It is sufficient to prove in the case \( 1 < \alpha < 2 \). By \( a, b \in D(A^{\gamma}) \subset C(\Omega) \) with \( \gamma > \frac{d}{4} \), similarly to (4.1), we obtain

\[ u(x_0, t) = \frac{1}{\Gamma(1-\alpha)} t^\alpha \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} (a, \varphi_{nk}) \frac{1}{\lambda_n} \varphi_{nk}(x_0) + \frac{1}{\Gamma(2-\alpha)} t^{\alpha-1} \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} (b, \varphi_{nk}) \frac{1}{\lambda_n} \varphi_{nk}(x_0) + O \left( \frac{1}{t^{2\alpha-1}} \right) \]
\[
\frac{1}{\Gamma(1-\alpha)} \frac{1}{t^\alpha} (A^{-1}a)(x_0) + \frac{1}{\Gamma(2-\alpha)} \frac{1}{t^{\alpha-1}} (A^{-1}b)(x_0) + O \left( \frac{1}{t^{2\alpha-1}} \right) \quad \text{as } t \to \infty.
\]

Similarly to the case \( F_1 \) in the proof of Theorem 3, we can prove that (1.19) and (1.20) imply \((A^{-1}a)(x_0) = 0\) and \((A^{-1}b)(x_0) = 0\) respectively. Under the assumption that \( a \) and \( b \) do not change the signs in \( \Omega \), in view of the weak and the strong maximum principles, we can argue similarly to the final part of the proof of Theorem 3 in the case of \( F_1 \), so that we can reach \( a = 0 \) in \( \Omega \) and/or \( b = 0 \) in \( \Omega \). Therefore, we prove that (1.19) and (1.20) imply \( a(x) = b(x) = 0 \) and \( b(x) = 0 \) for \( x \in \Omega \), respectively. The converse statement of the proposition is readily seen. Thus the proof of Proposition 1 is complete.

5. Concluding remarks

5.1. Time-fractional diffusion-wave equations with order \( \alpha \in (0, 2) \setminus \{1\} \) describe slow diffusion and is known not to have strong smoothing property as the classical diffusion equation. Such a weak smoothing property is characterized by the norm equivalence between \( \|u(\cdot, t)\|_{H^2(\Omega)} \) and \( \|u(\cdot, 0)\|_{L^2(\Omega)} \) for any \( t > 0 \) in the case of \( 0 < \alpha < 1 \). The weak smoothing property allows that the backward problem in time is well-posed for \( \alpha \in (0, 2) \setminus \{1\} \) (Floridia, Li and Yamamoto [4], Floridia and Yamamoto [5], Sakamoto and Yamamoto [15]), which is a remarkable difference from the case \( \alpha = 1 \).

The current article establishes that local properties of initial values affect the decay rate of solution as \( t \to \infty \), which indicates that a time-fractional equation can keep some profile of the initial value even for very large \( t > 0 \), which can be understood related to the backward well-posedness in time and is essentially different from the case \( \alpha = 1 \).

The essence of the argument relies on that the behavior of a solution \( u \) for large \( t > 0 \) admits an asymptotic expansion with respect to \( (\frac{1}{t})^{\alpha \ell} \) and \( (\frac{1}{t})^{\alpha \ell - 1} \) with \( \ell \in \mathbb{N} \).

5.2. We can generalize Theorem 3 (ii). For simplicity, we consider only the case \( 0 < \alpha < 1 \).

**Proposition 2.**

Let \( a \in \mathcal{D}(A^\gamma) \) with \( \gamma > \frac{d}{4} \) and \( 0 < \alpha, \beta < 1 \). Then

(i) \( |\partial_t^\beta u(x_0, t)| \leq \frac{C}{t^\beta} \|a\| \).

(ii) If \( |\partial_t^\beta u(x_0, t)| = o \left( \frac{1}{t^\beta} \right) \) as \( t \to \infty \),
then \( u(x_0, 0) = 0 \).

The proof relies on

\[
\partial_t^\beta u(x, t) = -t^{\alpha-\beta} \sum_{n=1}^{\infty} \lambda_n E_{\alpha, \alpha+1-\beta}(-\lambda_n t^\alpha) \sum_{k=1}^{d_n} (a, \varphi_{nk}) \varphi_{nk}(x) \quad \text{in } C((0, T]; L^2(\Omega))
\]  \hspace{1cm} (5.1)

and then we can argue similarly to Theorem 3 (ii) by (2.1). The equation (5.1) can be verified as follows:

\[
\partial_t^\beta (t^{\alpha_k}) = \frac{\Gamma(\alpha_k + 1)}{\Gamma(\alpha_k + 1 - \beta)} t^{\alpha_k - \beta}, \quad k \in \mathbb{N},
\]

and so the termwise differentiation yields

\[
\partial_t^\beta E_{\alpha,1}(-\lambda_n t^\alpha) = -\lambda_n t^{\alpha-\beta} E_{\alpha, \alpha+1-\beta}(-\lambda_n t^\alpha), \quad t > 0.
\]

Then (2.4) yields (5.1).

We omit the details of the proof of Proposition 2.

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