RELATIVELY PROJECTIVITY AND THE GREEN CORRESPONDENCE FOR COMPLEXES

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Abstract. We investigate a version of the Green correspondence for categories of complexes, including homotopy categories and derived categories. The correspondence is an equivalence between a category defined over a finite group $G$ and the same for a subgroup $H$, often the normalizer of a $p$-subgroup of $G$. We present a basic formula for deciding when categories of modules or complexes have a Green correspondence and apply it to many examples. In several cases the equivalence is an equivalence of triangulated categories, and in special cases it is an equivalence of tensor triangulated categories.

1. Introduction

The classical Green correspondence was defined by J. A. Green [11] more than half a century ago. It was one of several papers by Green that changed the face of modular representation theory, with an emphasis more on modules and maps rather than characters. The correspondence expressed a relationship given by induction and restriction, between relative categories of modules of a finite group $G$ and a subgroup $H$, where $H$ is usually taken as the normalizer of the vertex of some module of interest. Green originally stated it in terms of the representation rings of the groups, i.e., as a correspondence of objects, with little regards for the maps. In a later paper [12], he described it in terms of an equivalence of categories, with induction and restriction being functors. Still, he assumes that the modules are finitely generated and the coefficient ring $k$ is either a field of characteristic $p > 0$ or a complete DVR whose residue ring is a field of characteristic $p$. Basically, the assumption assures that the categories have a Krull-Schmidt Theorem. Many other extensions such as [3] [15], to name just a couple, also rely largely on the Krull-Schmidt property. Later, a sweeping generalization by Benson and Wheeler [6], proved equivalences of categories not only for infinitely generated modules, but they...
also allowed the coefficients to be from any commutative ring $k$ in which the order of the group is not invertible.

In this paper we carry the study a step further looking at a variation on the Green correspondence for categories of complexes, including homotopy categories and derived categories. We build somewhat on the work [21] of the second and third authors. The main issue is that we generalize the Benson-Wheeler proof [6] thereby providing axioms insuring that induction and restriction give categorical equivalences. The main theorem is presented in Section 6. The primary difficulty in applying the axioms is to show that the categories under consideration have the idempotent completion property, i.e. they are Karoubian. This property is the substitute for the Krull-Schmidt property, which is lacking in many of the categories that we consider. Generally, the property holds whenever a triangulated category has infinite direct sums.

The earlier sections of the paper are concerned with some explanation of the numerous categories that we consider. In Section 2, we recall the notion of an exact category and state some preliminaries. A main interest is the quotient categories of complexes defined by relative projectivity, relative to a $kG$-module. Included are categories of complexes of $kG$-modules, and those complexes bounded or bounded above or below, homotopy categories defined by term split sequences, or relative term split sequences or relatively split sequences, sequences that split on tensoring with a specific module. A primary goal in Section 3 is to determine the projective objects associated to the exact category and to show that these are Frobenius categories.

In Section 4, we address the issue of idempotent completions. The categories of complexes and their homotopy categories from the previous sections are shown to be idempotent complete by usual methods provided the coefficients are either Artinian or infinitely generated modules are allowed. For complexes of modules of finite composition length, it is proved that the associated quotient categories defined by relative projective objects are idempotent complete.

Section 5 introduces the subcategories of acyclic complexes, and the associated derived categories. Among other things, it is shown that certain subcategories of acyclic objects defined by relative projectivity are thick subcategories.

The main theorem on equivalences is Theorem 6.1. In Section 7, we recall some of the standard constructions for group representations, such as Frobenius Reciprocity and the Mackey Theorem and show that these also hold for the categories of complexes that we consider. This demonstrates that the categories satisfy many of the conditions of the hypothesis of Theorem 6.1. The actual application of Theorem 6.1 takes place in Section 8. The classical Green correspondence is extended to the categories of complexes and their associated homotopy categories. For derived categories, it is necessary to add an additional assumption.
In some cases, the equivalences associated to the Green correspondence are equivalences of triangulated categories. For example, suppose that $B$ is a block of $kG$ with defect group $P$ and $b$ is its Brauer correspondent, a block of $kH$ where $H$ is some subgroup that contains $N_G(P)$. Then there is a triangulated equivalence between the quotient category of homotopy classes of bounded complexes of $B$-modules modulo $\mathfrak{X}$-projective complexes and the same for $b$-modules modulo $\mathfrak{X}$-projective complexes in $b$. Here $\mathfrak{X}$ is the collection of proper intersections of $P$ with its conjugates by elements not in $H$. See Section 9 for precise details. While the results are mostly for the homotopy categories and derived categories of blocks, they apply also to the stable category of modules. If $\mathfrak{Q}$ is the set of all proper intersections of a Sylow $p$-subgroup of $G$, then the Green correspondence as above for $kG$-modules is an equivalence of tensor triangulated categories.

For notation in this paper, let $G$ be a finite group and let $k$ be a commutative ring. Let $\text{Mod}(kG)$ denote the category of all $kG$-modules and let $\text{mod}(kG)$ be the category of finitely generated $kG$-modules. These are tensor categories in that, given two objects $M$ and $N$, there is tensor product which is also an object in the category. The $G$-action on $M \otimes_k N$ is given by the diagonal $g \mapsto g \otimes g$. Throughout the paper, the symbol $\otimes$ means $\otimes_k$ unless otherwise indicated.

In the first five sections of the paper, it seems helpful to make a clear distinction between modules and complexes. So a complex $X^*$ is marked with the superscript "*", standing in place of a specific degree. This convention is relaxed in later sections where the notation presents other difficulties.

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2. Basics on categories

In this section we review a few basic categorical constructions that are needed for the results of this paper. Most of this material is aimed at stating and proving facts concerned with modules over group algebras, and for this reason we make little attempt at great generality. For background information see the papers [16, 17] or the books [18, 14].

A category is $k$-linear if all of its hom sets are $k$-modules and composition of morphisms is $k$-linear. It implies that there is a forgetful functor to the category of $k$-modules. A $k$-linear category $\mathcal{C}$ is hom-finite provided for any two objects $M$ and $N$, $\text{Hom}_\mathcal{C}(M, N)$ has finite composition length.
An additive category $C$ is a Krull-Schmidt category if the objects satisfy a Krull-Schmidt theorem, namely every object has a decomposition as a direct sum of a finite number of indecomposable objects and this decomposition is unique up to ordering of the summands and isomorphisms of the summands. This is equivalent to the condition that the endomorphism ring of an indecomposable object in $C$ be a local ring.

Let $C_{\text{plx}}(kG)$ be the category of complexes of $kG$-modules and chain maps. Thus an object $X^*$ in $C_{\text{plx}}(kG)$ is a complex

$$
\cdots \xrightarrow{\partial_n} X^n \xrightarrow{\partial_{n+1}} X^{n+1} \xrightarrow{\partial_{n+2}} X^{n+2} \xrightarrow{\partial_{n+3}} \cdots
$$

of $kG$-modules and $kG$-module homomorphisms. For complexes $X^*$ and $Y^*$, a chain map $\mu : X^* \to Y^*$ is a sequence of maps $\{\mu_n : X^n \to Y^n\}$ such that $\partial^Y_n \mu_n = \mu_{n+1} \partial^X_n$.

Let $C_{\text{plx}}^+(kG)$, $C_{\text{plx}}^-(kG)$ and $C_{\text{plx}}^b(kG)$ denote the full subcategories of $C_{\text{plx}}(kG)$ consisting of complexes that are bounded (respectively) above, below or both above and below. All of these categories are $k$-linear. Let $\text{cpx}(kG) = C_{\text{plx}}(\text{mod}(kG))$ be the category of complexes of finitely generated $kG$-modules, and let $\text{cpx}^+(kG)$, $\text{cpx}^-(kG)$ and $\text{cpx}^b(kG)$ be the bounded versions. Again these are all $k$-linear. Note that if $k$ is a field, then also these are tensor categories, except for $\text{cpx}(kG)$. The latter suffers from the fact that the tensor product of two complexes, which are unbounded in both directions may not have finitely generated terms even when the two complexes have finitely generated terms.

**Proposition 2.1.** Suppose that $k$ is a field. The category $\text{cpx}^b(kG)$ of complexes of finitely generated $kG$-modules is a $k$-linear, hom-finite, Krull-Schmidt category.

**Proof.** That $\text{cpx}^b(kG)$ is $k$-linear and hom-finite is clear from the definition. The fact that it is a Krull-Schmidt category follows from a result of Atiyah [1] which states that any abelian category satisfying a certain "bichain condition" has the Krull-Schmidt property. However, it is easy to see that any hom-finite category satisfies the condition. \qed

The notion of an exact category was first introduced by Quillen in [17], and has been extensively developed since then. Start with an additive category $C$. Let $\mathcal{E}$ be a collection of exact sequences of objects and maps in $C$. We assume that $\mathcal{E}$ satisfies certain axioms. Among these are statements such as that any exact sequence isomorphic to an element of $\mathcal{E}$ is in $\mathcal{E}$. The first maps in the exact sequences are called admissible monomorphisms, while the second maps are admissible epimorphisms. The composition of two admissible monomorphisms is an admissible monomorphism, and similarly for admissible epimorphisms. Also, admissible monomorphisms are preserved by arbitrary pushouts while admissible epimorphisms are preserved by arbitrary pullbacks. See [17] for precise details.
If $\mathcal{E}$ is a collection of exact sequences in $\mathcal{C}$ as above, then the pair $(\mathcal{C}, \mathcal{E})$ is called an exact category. An object $X \in \mathcal{C}$ is called $\mathcal{E}$-projective if for each exact sequence

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

in $\mathcal{E}$, the sequence of groups

$$0 \rightarrow \text{Hom}(X, A') \rightarrow \text{Hom}(X, A) \rightarrow \text{Hom}(X, A'') \rightarrow 0$$

is exact. The notion of $\mathcal{E}$-injective is defined dually.

An exact category $(\mathcal{C}, \mathcal{E})$ has enough injectives, if for each $A \in \mathcal{C}$, there is an admissible monomorphism $A \rightarrow I$ where $I$ is injective. Also if for each $A \in \mathcal{C}$, there is an admissible epimorphism $P \rightarrow A$, where $P$ is $\mathcal{E}$-projective, then we say that $(\mathcal{C}, \mathcal{E})$ has enough projectives. In such a category, we denote the subcategory of projectives by $\mathcal{E}\text{-Proj}$.

An exact category $(\mathcal{C}, \mathcal{E})$ is a Frobenius category if it has enough projectives and injectives and if the collections of projective and injective objects coincide.

As an example, we recall that, if $k$ is a field, then the projective objects and injective objects in $\text{Mod}(kG)$ and in $\text{mod}(kG)$ coincide. Then the category $\text{Cpx}(kG)$, where we assume that the collection of sequence $\mathcal{E}$ to be all exact sequences, is a Frobenius category. The projectives in this category are the complexes of projective modules that are both split and exact. These are complexes $X^*$ which are direct sums of two term exact complexes having the form $0 \rightarrow P \rightarrow P \rightarrow 0$ where $P$ is a projective $kG$-module and nonzero terms occur in degree $n$ and $n+1$ for some $n$. That is, the $\mathcal{E}$-projective objects are the null-homotopic complexes of projective modules. It is easy to see that these complexes are also injective.

Suppose that $(\mathcal{C}, \mathcal{E})$ is a Frobenius category. Associated to $(\mathcal{C}, \mathcal{E})$, there is a stable category or quotient category which we denote $\mathcal{D} = \mathcal{C}/(\mathcal{E}\text{-Proj})$. The objects in the category coincide with the objects in $\mathcal{C}$. For $X$ and $Y$ objects in $\mathcal{C}$ the morphisms are given by

$$\text{Hom}_\mathcal{D}(X, Y) = \frac{\text{Hom}_\mathcal{C}(X, Y)}{\text{PHom}_\mathcal{C}(X, Y)}$$

where $\text{PHom}_\mathcal{C}$ are the homomorphisms that factor through an $\mathcal{E}$-projective module. This is a triangulated category, the triangles corresponding to sequences in $\mathcal{E}$. The method is briefly described as follows. See [14] for more details.

If $M$ is an object in $\mathcal{C}$, there is a sequence in $\mathcal{E}$

$$0 \rightarrow M \rightarrow I_\mathcal{E} \rightarrow \Omega^{-1}_\mathcal{E}(M) \rightarrow 0$$

where $I_\mathcal{E} = I_\mathcal{E}(M)$ is an $\mathcal{E}$-injective object that serves the purpose of a relative injective hull. It defines (up to isomorphism in $\mathcal{D}$) the object $\Omega^{-1}_\mathcal{E}(M)$. The operator
\(\Omega_{\xi}^{-1}\) is a functor on the stable category \(\mathcal{D}\). Then a given morphism, \(\varphi : M \to N\), is fit into a triangle by means of the pushout diagram

\[
\begin{array}{c}
0 \longrightarrow M \longrightarrow I \longrightarrow \Omega_{\xi}^{-1}(M) \longrightarrow 0 \\
\varphi \\
\downarrow \\
0 \longrightarrow N \longrightarrow U \longrightarrow \Omega_{\xi}^{-1}(M) \longrightarrow 0
\end{array}
\]

Here \(I\) is the \(\mathcal{E}\)-injective hull of \(M\) as above, and \(U\) is the pushout in the left square. Then a triangle containing the class of \(\varphi\) is given as

\[
M \xrightarrow{\varphi} N \xrightarrow{\beta} U \xrightarrow{\gamma} \Omega_{\xi}^{-1}(M)
\]

An important example is the collection \(\mathcal{T} = \mathcal{T}\text{-seq}(\text{Cpx}(kG))\) of term split sequences of complexes in \(\text{Cpx}(kG)\) \cite{13}. A sequence \(0 \to A^* \to B^* \to C^* \to 0\) of objects in \(\text{Cpx}^*(kG)\) is term split if for every degree \(d\) the sequence of terms \(0 \to A^d \to B^d \to C^d \to 0\) is a split sequence of \(kG\)-modules. The \(\mathcal{T}\text{-seq}\)-projective objects in the exact category \((\text{Cpx}(kG), \mathcal{T})\) are the split exact complexes. A complex \(X^*\) is split exact if for every \(d\) there is a map \(s_d : X^{d+1} \to X^d\) such that \(\partial_{d-1}s_{d-1} + s_d\partial_d : X^d \to X^d\) is the identity homomorphism. Such a complex is a direct sum of two-term complexes \(0 \to X \to X \to 0\) where the nontrivial map is the identity. In other words, \(\mathcal{T}\text{-Proj}\) is the collection of complexes that are homotopic to the zero complex.

Let \(\mathcal{K}(kG) = \mathcal{K}(\text{Cpx}(kG))\), denote the homotopy category of complexes of \(kG\)-modules and homotopy classes of chain maps. It is a triangulated category and its translation functor \(\Omega_{\mathcal{T}\text{-seq}}^{-1}\) is the shift functor that takes \(X^*\) to \(X[1]^*\) where \(X[1]^n = X^{n+1}\) for all \(n\) and the boundary maps are all multiplied by \(-1\). If \(f : Y^* \to X^*\) is a chain map then the third object in the triangle of \(f\) is isomorphic to the usual mapping cone \(M(f)\). Here \(M(f)^n = X^n \oplus Y^{n+1}\) and the boundary map \(M(f)^n \to M(f)^{n+1}\) is given by \(\partial(x, y) = (\partial(x) + f(y), -\partial(y))\). We let \(\mathcal{K}^*(kG) = \mathcal{K}(\text{Cpx}^*(kG))\) for \(\ast = +, -\) or \(b\) be the full subcategory of \(\mathcal{K}(kG)\) with objects in \(\text{Cpx}^*(kG)\).

The following is well known.

**Proposition 2.2.** The quotient category \(\text{Cpx}(kG)/\mathcal{T}\text{-Proj} = \mathcal{K}(kG)\), by the projectives of the set of term split sequences, \(\mathcal{T} = \mathcal{T}\text{-seq}(\text{Cpx}(kG))\), is the homotopy category \(\mathcal{K}(kG)\) of complexes of \(kG\)-modules and homotopy classes of maps. Likewise, for \(\ast = +, -, b\), \(\mathcal{K}(\text{Cpx}^*(kG)) = \text{Cpx}^*(kG)/\mathcal{T}\text{-seq}(\text{Cpx}^*(kG))\cdot\mathcal{Proj}\). We similarly denote the homotopy categories of complexes of finitely generated modules \(\mathcal{K}(\text{cpx}(kG))\) and \(\mathcal{K}(\text{Cpx}^*(kG))\). These are tensor triangulated category (except for \(\mathcal{K}(\text{cpx}(kG))\) which has no tensor). The exact categories \((\text{Cpx}^*(kG), \mathcal{T})\) and \((\text{cpx}^*(kG), \mathcal{T}\text{-seq}(\text{Cpx}^*(kG)))\) are Frobenius categories.
Proof. The fact that $\mathcal{K}(kG)$ is triangulated follows from standard arguments and is well known. It has a tensor structure because the class $\mathcal{T}\mathcal{S}$-seq is closed under arbitrary tensors. That is, the sequence of objects in a term split sequence splits as a sequence of $k$-modules. So if $0 \to A \to B \to C \to 0$ is a term split sequence of $kG$-complexes then so is $0 \to A^* \otimes X^* \to B^* \otimes X^* \to C^* \otimes X^* \to 0$ for any complex $X^*$ in $\text{Cpx}(kG)$. It does not matter that tensoring with any $X^*$ might not be an exact functor. The same holds for the considered subcategories. That is, for example, the tensor of a term split sequence in $\text{cpx}^+(kG)$ with any object in $\text{cpx}^+(kG)$ is again a term split sequence in $\text{cpx}^+(kG)$. \qed

Proposition 2.3. Suppose that $k$ is a field. The category $\mathcal{K}(\text{cpx}^b(kG))$ of bounded complexes of finitely generated $kG$-modules and homotopy classes of maps is a Krull-Schmidt category.

Proof. As above it can be seen that $\mathcal{K}(\text{cpx}^b(kG))$ is the stable or quotient category of the exact category $(\text{cpx}^b(kG), \mathcal{T})$, where $\mathcal{T}$ is the collection of term split sequences of bounded complexes of finitely generated $kG$-modules. Now, $\mathcal{K}^b(\text{Cpx}(kG))$ is a hom-finite, Krull-Schmidt category. Thus, the Krull-Schmidt property is a consequence of the observation that the endomorphism ring of an indecomposable object is a quotient of a finite dimensional local ring, and hence remains a local ring. That $\mathcal{K}^b(\text{Cpx}(kG))$ is a Frobenius category follows from the arguments given below. \qed

Let $\mathcal{S}(\text{cpx}^b(kG))$ denote the full subcategory of $\text{cpx}^b(kG)$ consisting of all complexes of $kG$-modules that are free and split on restriction to $k$. That is, the restriction to $k$ of such a complex is a finite direct sum of complexes having the form either

\[
\ldots \to 0 \to k \overset{\text{id}}{\to} k \to 0 \to \ldots \quad \text{or} \quad \ldots \to 0 \to \\frac{\text{id}}{0} \to 0 \to \ldots
\]

In particular, it is direct sum of one- and two-term sequences, and the terms are $k$-isomorphic to $k$.

For $X^*$ in $\mathcal{S}(\text{cpx}^b(kG))$, let $(X^*)^* = \text{Hom}_k(X^*, k)$ be its $k$-dual. It is the complex $(X^*)^d = \text{Hom}_k(X^{-d}, k)$ and with boundary map being the dual of the boundary map for $X^*$, adjusted by a sign. That is, the boundary map $\delta_d : (X^*)^d \to (X^*)^{d+1}$ is given by $\delta_d = (-1)^{d+1}\partial_{d-1}^\#$, where $\partial_{d-1}^\# : (X^{d+1})^\# \to (X^d)^\#$ is the ordinary dual: $(\partial_{d-1}^\#(\lambda))(x) = \lambda(\partial_{d-1}(x))$ for $\lambda \in (X^{d+1})^\#$, and $x \in X^d$. For $Y^*$ any element of $\text{Cpx}(kG)$, there is an isomorphism $\text{hom}_k^*(X, Y) \cong (X^*)^* \otimes Y^*$. Here $\text{hom}_k^*(X, Y)$ is the collection $\sum \text{Hom}_k(X^i, Y^{i+j})$. The aggregate $\text{hom}_k^*(X, Y)$ is a complex. An element in $\text{hom}_k^*(X, Y)$ is an indexed sequence of maps $\{f_i\}$ where $f_i : X^i \to Y^{i+j}$ is $k$-homomorphism. The boundary map on the complex takes $f_i$ to $\partial(f_i) = (-1)^i(\partial_Y \circ f_i - f_i \circ \partial_X)$. 

There is the usual adjointness

$$\text{Hom}_{\text{cpx}(kG)}(V^* \otimes X^*, U^*) \cong \text{Hom}_{\text{cpx}(kG)}(V^*, (X^*)^* \otimes U^*),$$

for any complexes $U^*$ and $V^*$.

Suppose that $U$ is a finitely generated $kG$-module. There is a trace map $\text{Tr} = Tr_U : U^# \otimes U \to k$ given by $\text{Tr}(\lambda \otimes u) = \lambda(u)$ for $\lambda$ in $U^#$ and $u$ in $U$. Viewed from the isomorphism $U^# \otimes U \cong \text{Hom}_k(U, U)$, it becomes the usual trace map on matrices. With a sign convention, it extends to a chain map on complexes. Let $k^*$ be the complex having only one nonzero term which is in degree zero and is equal to $k$. Then for any $U^*$ in $\mathcal{S}(\text{cpx}^b(kG))$ there is a trace map $\text{Tr} : (U^*)^# \otimes U^* \to k^*$, which in degree zero $\text{Tr} : ((U^*)^# \otimes U^*)_0 \to k$ is the super trace map. That is, on $(U_d^d)^# \otimes U^d$, the super trace is $(-1)^d \text{Tr}_{U_d^d}$.

Likewise for $U$ a finitely generated $kG$-module, there is a unit homomorphism $\iota = \iota_U : k \to U^# \otimes U$, which sends 1 to the identity homomorphism $\text{Id}_U \in \text{Hom}_k(U, U)$. Here $\text{Id}_U = \sum \lambda_i \otimes v_i$ where $\{\lambda_i\}$ and $\{v_i\}$ are dual bases of $U^#$ and $U$, respectively. This map is dual to the trace map. For $U^*$ a complex in $\mathcal{S}(\text{cpx}^b(kG))$ there is also a unit homomorphism $\iota : k^* \to (U^*)^# \otimes U^*$ which is dual to the trace map.

It can be calculated that the composition

$$U^* \xrightarrow{\iota \otimes 1} (U^#)^* \otimes U^* \otimes U^* \cong U^* \otimes (U^#)^* \otimes U^* \xrightarrow{1 \otimes \text{Tr}} U^*$$

is the identity map.

**Lemma 2.4.** Assume that $U^* \in \mathcal{S}(\text{cpx}^b(kG))$. The maps $\text{Tr}_U$ and $\iota_U$ are chain maps. Moreover, $U^*$ is a direct summand of $U^* \otimes (U^*)^# \otimes U^*$.

**Proof.** By hypothesis, $U^*$ is a bounded complex of finitely generated $kG$-modules whose restriction to $k$ is both free and split.

$$\ldots \to U^{-1} \xrightarrow{\partial} U^0 \xrightarrow{\partial} U^1 \to \ldots$$

Its dual has boundary map $\delta : (U^i)^# \to (U^{i-1})^#$ is given by $\delta(\lambda) = (-1)^{i-1} \lambda \circ \partial$. Then the tensor product has the form

$$(U^# \otimes U)^*: \ldots \to (U^# \otimes U)^{-1} \to (U^# \otimes U)^0 \xrightarrow{\partial} (U^* \otimes U)^1 \to \ldots$$

Let $k^*$ be the complex with $k^0 = k$ and all other terms zero. To check that the super trace $\text{Tr} : (U^# \otimes U)^* \to k^*$ is a chain map, we need only check that $\text{Tr} \partial : (U^# \otimes U)^{-1} \to k^0 \cong k$ is the zero map. For this, choose $\lambda_i \in (U^i)^#$ and $x_{i-1} \in U^{i-1}$. 
Then
\[
\text{Tr} \partial(\lambda_i \otimes u_{i-1}) = \text{Tr}(\delta(\lambda_i) \otimes u_{i-1} + (-1)^i \lambda_i \otimes \partial(u_{i-1})) \\
= (-1)^{i-1}\delta(\lambda_i)(u_{i-1}) + (-1)^i(-1)^{i-1}\lambda_i\partial(u_{i-1}) \\
= (1 + (-1)^i)\lambda_i\partial(u_{i-1}) = 0.
\]
The unit map is the dual of the (super) trace map and hence is also a chain map.

For the second statement, we note that the composition of $I \otimes 1$ with $1 \otimes \text{Tr}$ is the identity of $U^* \cong k^* \otimes U^* \cong U^* \otimes k^*$ (see [2] for the module version). Because $\text{cpx}^b(kG)$ is an abelian category, this gives us a direct sum splitting. \hfill \square

We say that collection $\mathcal{E}$ of exact sequences in a tensor subcategory $\mathcal{C}$ of $\text{Cpx}(kG)$ is closed under arbitrary tensor products, if whenever a sequence $0 \to A^* \to B^* \to C^* \to 0$ is in $\mathcal{E}$, then so also is $0 \to A^* \otimes X^* \to B^* \otimes X^* \to C^* \otimes X^* \to 0$ for any complex $X^*$ in $\mathcal{C}$.

With these notions in mind we can prove the following.

**Theorem 2.5.** Suppose that $\mathcal{E}$ is a collection of sequences in $\text{Cpx}(kG)$ such that $(\text{Cpx}(kG), \mathcal{E})$ is an exact category. We assume the following.

a. $\mathcal{E}$ is closed under arbitrary tensor products.

b. The trivial complex $k^*$ has a projective cover $\psi : P(k)^* \to k^*$ relative to $\mathcal{E}$ in $\mathcal{S}(\text{cpx}^b(kG))$. In particular, we have an exact sequence in $\mathcal{E}$
\[
E_k : \quad 0 \to \Omega_\mathcal{E}(k)^* \to P(k)^* \xrightarrow{\psi} k^* \to 0
\]
where $P(k)^*$ is in $\mathcal{E}$-$\text{Proj}$ and in $\mathcal{S}(\text{cpx}^b(kG))$. That is, each $P(k)^n$ is a free $k$-module of finite rank.

c. For any object $X^*$, $X^* \otimes P(k)^*$ is $\mathcal{E}$-projective.

d. The dual sequence to $E_k$ is in $\mathcal{E}$ and $(P(k)^*)^#$ is $\mathcal{E}$-injective.

e. For any object $X^*$, $X^* \otimes (P(k)^*)^#$ is $\mathcal{E}$-injective.

Then we have the following.

(1) $(P(k)^*)^#$ is $\mathcal{E}$-projective.

(2) An object is $\mathcal{E}$-projective if and only if it is a direct summand of an object having the form $X^* \otimes P(k)^*$ for some complex $X^*$,

(3) Every $\mathcal{E}$-injective complex is a direct summand of an object having the form $X^* \otimes (P(k)^*)^#$ for some complex $X^*$,

(4) For any complex $X^*$ and any $n$, $\Omega^n_\mathcal{E}(k)^* \otimes X^* \cong \Omega^n_\mathcal{E}(X^*) \oplus Y^*$ for some $\mathcal{E}$-projective complex $Y^*$.

(5) The exact category $(\text{Cpx}^b(kG), \mathcal{E})$ has enough projectives and injectives, and is a Frobenius category.
Proof. The first statement is a direct consequence of assumption (c) and Lemma 2.4, since \((P(k)^*)^\#\) is a direct summand of \((P(k)^*)^\# \otimes P(k)^* \otimes (P(k)^*)^\#\). For the second statement, suppose that \(X^*\) is \(\mathcal{E}\)-projective. Then the sequence \(\text{Hom}(X^*, E_k \otimes X^*)\) is exact, since \(E_k \otimes X^*\) is in \(\mathcal{E}\). However, this implies that \(X^*\) is a direct summand of \(P(k)^* \otimes X^*\). The converse is statement (c).

The dual argument concludes that if \(X^*\) is \(\mathcal{E}\)-injective then it is a direct summand of \((P(k)^*)^\# \otimes X^*\) and hence it is projective. Likewise, \(\mathcal{E}\)-projective objects are also relatively injective. The last two statements use the facts that the sequence \(E_k \otimes X^*\) is a relative projective cover of \(X^*\) and \(E_k^\# \otimes X^*\) is a relative injective hull of \(X^*\). \(\square\)

Note that if \(\mathcal{E}\) is the collection of term split sequences, then \(P(k)^*\) is the two term complex with the nonzero terms in degrees 0 and 1, as in the diagram

\[
\begin{array}{c}
P(k)^*: & \ldots & \rightarrow 0 & \rightarrow k & \rightarrow k & \rightarrow 0 & \rightarrow \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
k^*: & \ldots & \rightarrow 0 & \rightarrow k & \rightarrow 0 & \rightarrow \ldots \\
\end{array}
\]

In this case \(P(k)^*\) satisfies all of the conditions of the above theorem.

Theorem 2.6. Let \(C\) denote one of the categories \(\text{Cpx}^*(kG)\), \(\text{cpx}(kG)\) or \(\text{cpx}^*(kG)\) for \(* = +, -\) or \(b\). Suppose that \(\mathcal{E}\) is a collection of sequences of objects in \(\mathcal{E}\) such that \((C, \mathcal{E})\) is an exact category, and that \(\mathcal{E}\) satisfies the hypotheses of Theorem 2.5. Then the conclusions of that theorem also hold for \((C, \mathcal{E})\).

Proof. The proof is the same as for Theorem 2.5. It is only necessary to notice such thing as the fact that \(P(k)^* \otimes X^*\) is in \(C\) whenever \(X^*\) is in \(C\). \(\square\)

3. Relative projectivity

In this section we present the basics of relative projectivity, relative to a module and to a collection of subgroups. The ideas take place in the context of \(kG\)-modules where \(G\) is a finite group and \(k\) is a commutative ring such that the order of \(G\) is not invertible in \(k\). For some background see the paper [9]. Throughout the section, the symbol \(V\) denotes a \(kG\)-module that, as a module over \(k\), is free and has finite rank.

Definition 3.1. [19] Suppose that \(V\) is a finitely generated \(kG\)-module. A \(kG\)-module \(M\) is said to be relatively \(V\)-projective provided \(M\) is a direct summand of \(V \otimes X\) for some \(kG\)-module \(X\). A complex \(X^*\) in \(\text{Cpx}(kG)\) is said to be \(V\)-projective if it is a direct summand of \(V \otimes Y^*\) for some complex \(Y^*\) in \(\text{Cpx}(kG)\).

The full subcategory of \(V\)-projective complexes is denoted \(\text{V-Proj}(\text{Cpx}(kG))\) or just \(V-Proj\) if there is no confusion. It is closed under direct sums and summands,
but not under extensions. It is also closed under tensor products with arbitrary modules and complexes, by associativity.

Suppose that $\mathcal{H}$ is a collection of subgroups of $G$. We say that $X^* \in \text{Cpx}(kG)$ is relatively $\mathcal{H}$-projective if it is $V$-projective for $V = \sum_{H \in \mathcal{H}} k_H$. In other words, by Frobenius reciprocity (see Theorem [7.1]), $X^*$ is $\mathcal{H}$-projective if it is a summand of a direct sum of complexes of modules induced from $\text{Cpx}(kH)$ for $H \in \mathcal{H}$.

**Definition 3.2.** An exact sequence of objects $E : 0 \to A^* \to B^* \to C^* \to 0$ in $\text{Cpx}(kG)$ is $V$-split if the tensor product $V \otimes E$ is split. It is $V$-term split if $V \otimes E$ is term split. It is term+$V$-split if it is both term split and also $V$-split.

We note one fact that will be of some use in later sections. Its proof is simply that if $E$ is a sequence of $k$-modules, the $k \otimes_k E$ is split if and only if $E$ is split.

**Lemma 3.3.** Suppose that $U$ is free as a module over $k$. Then any $U$-split sequence is split as a sequence of $k$-modules.

The collections of $V$-split sequences, $V$-term split sequences and term+$V$-split sequences are denoted $V$-$\text{Split-seq}$, $V$-$\text{TSS-seq}$ and $T\text{S}+V$-$\text{Split-seq}$, respectively. All of these collections are closed under arbitrary tensors since $V$ is free as a $k$-module. The next task is to identify the projective objects relative to the collections and to show that the exact categories are Frobenius categories. For this, it is helpful to have some additional information on the relative projectivity.

Note here that, if $k$ is a field of characteristic $p$ dividing the order of $G$ and if $V$ is an absolutely indecomposable, finitely generated $kG$-module, then the trace map $V^# \otimes V \to k$ is split if and only if $p$ does not divide the dimension of $V$ (see [5]). That is, the trace map is split if and only if $\text{Tr}(\text{Id}_V) \neq 0$. In the case that $p$ does not divide the dimension of $V$ or of any direct summand of $V$, then we have that $k$ is $V$-projective and hence every $kG$-module is $V$-projective. For the rest of this paper we avoid this situation.

We introduce the following variation on the homotopy category. As we use the notation several times we here give it a label. For the remainder of the section assume the following notation.

**Notation 3.4.** Let $\mathcal{C}$ denote any one of the categories $\text{Cpx}(kG)$, $\text{cpx}(kG)$, $\text{Cpx}^+(kG)$, or $\text{cpx}^+(kG)$ for $\ast = +, -$ or $b$. If the group is in question, denote the category $\mathcal{C}_G$.

**Proposition 3.5.** Assume that $V$ is free of finite rank as a $k$-module. Let $\mathcal{E} = V$-$\text{Split-seq}$ the collection of $V$-split sequences in $\mathcal{C}$. Then the quotient category $\mathcal{K}_{\mathcal{E}-\text{Proj}}(\mathcal{C}) = \mathcal{C} / \mathcal{E}$-$\text{Proj}$ is a Frobenius category. The projective objects form the set $\mathcal{E}$-$\text{Proj}$ which consists of all direct summands of objects having the form $X^* \otimes V^# \otimes V$ for $X^*$ in $\mathcal{C}(kG)$. In addition, if $k$ is a field then, the category $\mathcal{K}_{V-\text{Split}}(\text{cpx}^b(kG))$ is a Krull-Schmidt category.
Proof. If $k$ is a field, then the category $\mathcal{K}_{V\text{-}\text{Split}}(\text{cpx}^b(kG))$, is a quotient of a Krull-Schmidt category. That is, the endomorphism ring of any indecomposable object is the quotient of a finite dimensional local $k$-algebra and hence is a local ring.

The rest of the proof follows from Theorem 2.5 once we determine the projective objects. Let $V^*$ be the complex with only one nonzero term, which is $V^* \otimes V$ in degree zero. Then there is an exact sequence

$$
0 \longrightarrow W^* \longrightarrow V^* \overset{\mu}{\longrightarrow} k^* \longrightarrow 0
$$

where $\mu_0 = \text{Tr} : V^* \otimes V \rightarrow k$ is the trace map in degree zero. This sequence is $V$-split by Lemma 2.4, and the middle term is $V$-projective, implying that the middle term is in $\mathcal{E}\text{-Proj}$. Consequently, it is the sequence of a relative projective cover of $k^*$ plus, perhaps, a sequence having the form $0 \rightarrow U^* \cong U^* \rightarrow 0$. Here, $U^*$ is a complex with only one nonzero term that is a direct summand of $V^* \otimes V$, which is $V$-projective. In a similar fashion we see that the dual sequence is an injective hull of $k^*$. The fact that $\mathcal{E}$ is closed under arbitrary tensor products is obvious. Observe that $V^* \otimes X^* \cong V^* \otimes V \otimes X^*$. Thus $V^* \otimes X^*$ is $\mathcal{E}$-projective since for any sequence $E$ in $\mathcal{E}$, we have that $\text{Hom}_{kG}(V^* \otimes X^*, E) \cong \text{Hom}_{kG}(X^*, V^* \otimes V \otimes E)$ which is exact. The proofs of statements about duals in Theorem 2.5 are similar. Hence the hypotheses of that theorem are all satisfied and the proof is complete. $\square$

**Proposition 3.6.** Assume that $V$ is free of finite rank as a $k$-module. Let $\mathcal{E} = V\text{-}\mathcal{T}\mathcal{S}\text{-}\text{seq}$ in $\mathcal{C}$. The quotient category $\mathcal{K}_{V\text{-}\mathcal{T}\mathcal{S}}(\mathcal{C}) = \mathcal{C} / \mathcal{E}\text{-}\text{proj}$ is a Frobenius category. The projective objects form the collection $\mathcal{E}\text{-Proj}$ consisting of all objects that are both split and contractible complexes of $V$-projective modules. That is $\mathcal{E}\text{-Proj}$ contains all complexes having the form $\ldots 0 \rightarrow W \rightarrow W \rightarrow 0 \ldots$ where $W$ is $V$-projective, the nonzero terms occur in consecutive degrees and the boundary map is the identity. It consists of all direct sums of such sequences when such a direct sum is in $\mathcal{C}$. If $k$ is a field, then the category $\mathcal{K}_{V\text{-}\mathcal{T}\mathcal{S}}(\text{cpx}^b(kG))$ is a Krull-Schmidt category.

**Proof.** The proof is similar to the previous proof except for the issue of the projective cover of the trivial complex. Again, $\mathcal{E}$ is closed under arbitrary tensor products. Let $V^*$ be the complex with all zero terms except for $V^* \otimes V$ in degrees zero and one. Then we have a map of complexes

$$
\begin{array}{ccccccc}
\ldots & 0 & \longrightarrow & V^* \otimes V & \overset{\text{Id}_V}{\longrightarrow} & V^* \otimes V & \longrightarrow & 0 & \longrightarrow & \ldots \\
\downarrow{\mu} & & & \downarrow{\text{Tr}} & & & \downarrow{\text{Tr}} & & \\
\ldots & 0 & \longrightarrow & k & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots \\
\end{array}
$$

If $U^*$ is the kernel of $\mu$, then the sequence

$$
0 \longrightarrow U^* \longrightarrow V^* \overset{\mu}{\longrightarrow} k^* \longrightarrow 0
$$
is in $\mathcal{E}$. Hence, all $\mathcal{E}$-projective complexes are direct summands of complexes having the form $V^\ast \otimes X^\ast$ for some complex $X^\ast$. It is now an easy exercise to show that they have the stated form. The rest follows from Theorem 2.5.

**Proposition 3.7.** Assume that $V$ is free of finite rank as a $k$-module. Let $\mathcal{E} = TS + V$-Split-Seq. The quotient category $\mathcal{K}_{TS + V}$-Split $(C) = C / \mathcal{E}$-Proj is Frobenius category. The projective objects is the set $\mathcal{E}$-Proj consisting of all complexes that can be written as a direct sum of an object in $TS$-Proj and $V$-Split-Proj.

*Proof.* Suppose that $\hat{k}^\ast$ is the complex with $k$ in degrees 0 and 1, the map between them being the identity and all other terms equal to zero. Let $V$ be the complex with $V^\# \otimes V$ in degree zero and all other terms equal to zero. Then we have diagram

$$
\begin{array}{ccccccccc}
V^\ast \oplus \hat{k}^\ast : & \cdots & 0 & \longrightarrow & (V^\# \otimes V) \oplus k^{(0, 1d_k)} & 0 & \longrightarrow & \cdots \\
\mu \downarrow & & & & (\text{Tr}, Id_k) & \downarrow & & \\
\hat{k}^\ast : & \cdots & 0 & \longrightarrow & k & 0 & \longrightarrow & \cdots
\end{array}
$$

The chain map $\mu$ is both $V$-split and term split, and the complex $V^\ast \oplus \hat{k}^\ast$ is in $TS + V$-Split-Proj. Thus, the projective objects are as stated. The rest of the proof proceeds as before.  

\[\square\]

4. Finite length complexes and idempotent completions

In this section we consider which of the categories that we have discussed have idempotent completions, as well as investigate idempotent completion property for homotopy categories of complexes of modules of finite length. This discussion is crucial to the application of the Green correspondence as formulated in Section 6. The idempotent complete property is a weak substitute for the Krull-Schmidt property in categories. Let $X$ be an object in an additive category $C$. An idempotent $e \in \text{Hom}_C(X, X)$ is said to be split provided $X$ is a direct sum $X = X' \oplus X''$ in such a way that the restriction of $e$ to $X'$ is the identity map and to $X''$ is zero. The category $C$ is idempotent complete provided every idempotent splits in $C$. An abelian category is idempotent complete simply because it has kernels in cokernels. A triangulated category that has countable direct sums is idempotent complete [7]. Of course, any Krull-Schmidt category has idempotent completions. With this information we can prove the following.

**Proposition 4.1.** Assume that $k$ is a commutative ring and $G$ is a finite group. The categories $C = C(kG)$ where $C$ is one Cpx, Cpx+, Cpx or Cpx are idempotent complete. So also are their homotopy categories $\mathcal{K}(C)$, $\mathcal{K}_{V-Split}(C)$, $\mathcal{K}_{V-TS}(C)$ and $\mathcal{K}_{TS + V-Split}(C)$ where $V$ is a $kG$-module that is free of finite rank as a $k$-module.
Proof. The categories denoted $\mathcal{C}$ are abelian and therefore idempotent complete. For the homotopy categories, the only problem is that $\text{Cpx}^+(kG)$, $\text{Cpx}^-(kG)$ and $\text{Cpx}^b(kG)$ do not have arbitrary direct sums and hence neither do their homotopy categories. That is, the direct sum of an infinite number of bounded complexes may no longer be bounded. However, the proof [7] of the existence of a splitting for an idempotent requires a homotopy limit construction that, in turn, requires only a direct sum of a countable number of copies of the complex on which the idempotent is defined. Such a direct sum exists in these categories, and hence the splitting of any idempotent exists. \[\square\]

For the homotopy categories of complexes of finitely generated modules the problem is more difficult. The categories $\text{cpx}(kG)$, $\text{cpx}^+(kG)$, $\text{cpx}^-(kG)$ and $\text{cpx}^b(kG)$ have idempotent completions because they are abelian. If $k$ is a field, then the same conclusion holds for the homotopy categories $K(\text{cpx}^b(kG))$, $K_{V-Splt}(\text{cpx}^b(kG))$, $K_{T S+V-Splt}(\text{cpx}^b(kG))$ and $K_{V-T S}(\text{cpx}^b(kG))$ because they are Krull-Schmidt categories. These are part of a more general collection of categories that are idempotent complete.

Let $\text{cpxFL}(kG)$ denote the subcategory of $\text{cpx}(kG)$, consisting of all complexes with the property that every term in the complex has finite composition length. That is, every term has a composition series of finite length in which the quotients of successive terms in the series are simple $kG$-modules. Similarly let $\text{cpxFL}^+(kG)$, $\text{cpxFL}^-(kG)$ and $\text{cpxFL}^b(kG)$ be the categories of complexes of finite length modules that are bounded above or below or just bounded.

Notice that $\text{cpxFL}(kG)$ is a full subcategory of $\text{cpx}(kG)$ and that a complex $X^*$ in $\text{cpx}(kG)$ is in $\text{cpxFL}(kG)$ if and only if every term $X^i$ has finite length as a module over $k$. If $V$ is a $kG$-module that is free of finite rank over $k$ then $V \otimes X^*$ is in $\text{cpxFL}(kG)$ if and only if $X^*$ is in $\text{cpxFL}(kG)$. Moreover, $X^*$ is projective relative to $V$ if and only if it is a direct summand of $X^* \otimes V \otimes V^\#$. 

We are indebted to Jeremy Rickard for most of the proof of the following.

**Theorem 4.2.** Let $\mathcal{C}$ denote any of the FL categories of complexes: $\text{cpxFL}(kG)$, $\text{cpxFL}^+(kG)$, $\text{cpxFL}^-(kG)$ or $\text{cpxFL}^b(kG)$. Then $\mathcal{C}$ has idempotent completions and so does any of the homotopy categories $\mathcal{K}(\mathcal{C})$, $\mathcal{K}_{V-Splt}(\mathcal{C})$, $\mathcal{K}_{T S+V-Splt}(\mathcal{C})$ and $\mathcal{K}_{V-T S}(\mathcal{C})$. We are assuming here that $V$ is a $kG$-module that is free of finite rank over $k$.

**Proof.** Note that the categories of complexes are abelian and hence also idempotent complete. So we may assume that we are in one of the homotopy categories which we call $\mathcal{K}$. Suppose that $X^*$ is an object in $\mathcal{K}$ and that $e : X^* \to X^*$ is a chain map such that $e^2 = e$ in $\mathcal{K}$. That is, $e^2$ is homotopic to $e$ in the sense that $e^2 - e$ factors through the appropriate relatively projective object. For each $i$, we have
nested sequences of submodules

\[ e_i X^i \supset e_i^2 X^i \supset e_i^3 X^i \supset \ldots \]

and

\[ \text{Ker}(e_i) \subseteq \text{Ker}(e_i^2) \subseteq \text{Ker}(e_i^3) \subseteq \ldots \]

Because \( X^i \) has finite composition length, both sequences stabilize. Let \( Y^i \) and \( Z^i \) be the limit modules. Then we have that for \( n \) sufficiently large, depending on \( i \), \( e_i^n Y^i = Y^i \) and \( e_i^n Z^i = \{0\} \). The boundary map commutes with \( e \), and hence, \( X^* = Y^* \oplus Z^* \). On the complex \( Y^* \), \( e \) acts as \( e_Y \), an isomorphism, while on \( Z^* \) it acts as \( e_Z \) which is nilpotent in every degree.

Next we should note that both \( e_Y \) and \( e_Z \) are idempotent. That is, the homotopy can be made to respect the decomposition of \( X^* \) into a direct sum. For example, suppose we are in the category \( \mathcal{K}_{TS + V \cdot Sph}(\mathbf{C}) \). Then we have that \( e - e^2 = \partial d + d\partial + f \) where \( d \) is a homotopy in the usual sense, \( d_j : X^j \to X^{j-1} \), and \( f \) factors through a \( V \)-projective complex, which we can assume to be \( X^* \otimes V \otimes V^* \). Put in matrix form we have that, on \( X^i \),

\[
\begin{bmatrix}
  e_Y - e_Y^2 & 0 \\
  0 & e_Z - e_Z^2
\end{bmatrix} = \begin{bmatrix}
  d_{11}^{i+1} & d_{12}^{i+1} \\
  d_{21}^{i+1} & d_{22}^{i+1}
\end{bmatrix} \begin{bmatrix}
  \partial_Y & 0 \\
  0 & \partial_Z
\end{bmatrix} + \begin{bmatrix}
  \partial_Y & 0 \\
  0 & \partial_Z
\end{bmatrix} \begin{bmatrix}
  d_{11} & d_{12} \\
  d_{21} & d_{22}
\end{bmatrix} + \begin{bmatrix}
  f_{11} & f_{12} \\
  f_{21} & f_{22}
\end{bmatrix}
\]

Thus we see that \( e_Y - e_Y^2 = d_{11} \partial_Y + \partial_Y d_{11} + f_{11} \), and similarly for \( e_Z \). Consequently, in the category, \( e_Y \) is the identity on \( Y^* \), and \( e_Z \) is both idempotent and nilpotent on every term of \( Z^* \). Now let \( w = 1 + e_Z + e_Z^2 + e_Z^3 + \ldots \). This is a chain map from \( Z^* \) to itself. It is well defined because on every term of \( Z^* \) the sum is finite. On the other hand,

\[ e_Z = (e_Z - e_Z^2)w = (d_{22} \partial_Z + \partial_Z d_{22} + f_{22})w = d_{22}w \partial_Z + \partial_Z d_{22} w + f_{22}w. \]

Hence, \( e_Z \) is the zero map in \( \mathcal{K}_{TS + V \cdot Sph}(\mathbf{C}) \). This proves that the idempotent \( e \) is split. The proof in the other categories is similar. \( \square \)

5. ACYCLIC COMPLEXES AND LOCALIZATIONS

In this section we introduce the derived categories that are the Verdier localizations of the homotopy categories at thick subcategories of acyclic complexes. Several variations on the standard theme are discussed. It turns out that some different constructions yield the same end object. The primary results concern the existence of the derived categories and the idempotent completions. Let \( \mathbf{C} \) be as in 3.4.

We recall that a subcategory \( \mathcal{L} \) of a triangulated category \( \mathcal{C} \) is thick if it a full triangulated subcategory of \( \mathcal{C} \) and if it is closed under taking direct summands. In this context, triangulated means close under the shift functor and if two of three objects in any triangle in \( \mathcal{C} \) are in \( \mathcal{L} \) then so is the third. The definition that we give
has been called Rickard’s Criterion (see [18]). It expresses precisely the conditions needed to construct a Verdier localization of \( \mathcal{C} \) by inverting any morphism such that third object in the triangle of that morphism is also in subcategory \( \mathcal{L} \).

Given an exact category \((\mathcal{C}, \mathcal{E})\), the subcategory of acyclic objects \( \mathfrak{A}(\mathcal{C}, \mathcal{E}) \) is the full subcategory in \( \mathbf{Cpx}(\mathcal{C}) \) consisting of all exact complexes of the form

\[
X^* : \ldots \to X^{n-1} \to X^n \to X^{n+1} \to \ldots
\]

such that for every \( n \), the map \( X^n \to X^{n+1} \) decomposes as a composition of an admissible epimorphism \( X^n \to A^n \) with an admissible monomorphism \( A^n \to X^{n+1} \) where \( A^n \to X^{n+1} \to A^{n+1} \) is an exact sequence in \( \mathcal{E} \).

In a category \( \mathcal{C} \) of complexes over \( kG \), we are interested in some subcategories of acyclic complexes. The first is the subcategory of all acyclic complexes, where here acyclic means being exact or simply having zero homology. We denote it \( \mathcal{A}(\mathcal{C}) \). If \( \mathcal{C} = \mathbf{Cpx}(kG) \), it is an easy check to see that this is the subcategory \( \mathfrak{A}(\mathbf{Mod}(kG), \mathbf{seq}(\mathbf{Mod}(kG))) \) of all \( kG \)-modules where \( \mathbf{seq}(\mathbf{Mod}(kG)) \) denotes all exact sequences. If \( \mathcal{C} \) is \( \mathbf{cpx}(kG) \), then the subcategory of acyclic complexes is the collection \( \mathfrak{A}(\mathbf{mod}((kG), \mathbf{seq}(\mathbf{mod}(kG)))) \) of the indicated exact category.

Conditions can be put on the types of acyclic complexes that are acceptable. For example, let \( V \) be a fixed \( kG \)-module, which is free and finitely generated as a \( k \)-module, and let

\[
\mathcal{A}_{V\text{-split}}(\mathbf{Cpx}(kG)) = \mathcal{A}(\mathbf{Mod}((kG), V\text{-split seq}(\mathbf{Mod}(kG))))
\]

where \( V\text{-split seq}(\mathbf{Mod}(kG)) \) is the collection of \( V \)-split sequences of \( kG \)-modules. In this case an acyclic object is a complex of \( kG \)-modules that is both \( V \)-split and exact. Hence its tensor product with \( V \) is contractible.

Similarly, it is possible to define acyclic complexes in the homotopy categories \( \mathcal{K}(\mathcal{C}), \mathcal{K}_{V\text{-split}}(\mathcal{C}) \) and \( \mathcal{K}_{V\text{-Tors}}(\mathcal{C}) \). These are the classes of the corresponding acyclic complexes in \( \mathcal{A}(\mathcal{C}) \). That is, for example, the objects in \( \mathcal{A}(\mathcal{K}_{V\text{-split}}(\mathcal{C})) \) are \( V \)-split-homotopy classes of complexes in \( \mathcal{C} \) that are exact.

**Proposition 5.1.** Assume that \( V \) is free of finite rank as a module over \( k \). The subcategories \( \mathcal{A}(\mathcal{K}(\mathcal{C})) \) and \( \mathcal{A}_{V\text{-split}}(\mathcal{K}(\mathcal{C})) \), are thick subcategories of \( \mathcal{K}(\mathcal{C}) \), for \( \mathcal{C} \) as in [3.4].

**Proof.** See Lemma 1.2 of [18]. \( \square \)

There is warrant for some care here. It seems that the subcategory \( \mathcal{A}(\mathcal{K}_{V\text{-split}}(\mathcal{C})) \) and \( \mathcal{A}_{V\text{-split}}(\mathcal{K}_{V\text{-split}}(\mathcal{C})) \) are not thick subcategories of \( \mathcal{K}_{V\text{-split}}(\mathcal{C}) \), in general. The problem here is that the notion of acyclic is muddled. One can consider the image of the subcategory of acyclic complexes of \( \mathcal{C} \) in the homotopy category. But the relative projective \( \mathbf{Proj}(V\text{-split seq}) \) are not acyclic in the usual sense of being
exact. Hence, there exist zero objects in the homotopy category that are not acyclic in the usual sense. Still we can prove the following.

**Proposition 5.2.** Assume that $V$ is free of finite rank as a $k$-module. The subcategories $\mathcal{A}(K_{V-\mathcal{TS}(C)})$ and $\mathcal{A}_{V\text{-}\text{split}}(K_{V-\mathcal{TS}(C)})$ are thick subcategories $K_{V-\mathcal{TS}(C)}$.

**Proof.** Because $\text{Proj}(V-\mathcal{TS}\text{-seq})$ consists of acyclic complexes (see Proposition 3.6), the subcategory $\mathcal{A}(K_{V-\mathcal{TS}(C)})$ is triangulated. It is clear that it is closed under taking direct summands, and hence, it is a thick subcategory.

The problem left is to show that the category $\mathcal{A}_{V\text{-}\text{split}}(K_{V-\mathcal{TS}(C)})$ is triangulated. Suppose that $X^*$ and $Y^*$ are $V$-split acyclic complexes and that $f: X^* \to Y^*$ is a chain map. We need to show that the third object in the triangle of $f$ is $V$-split and acyclic. The third object is the object $B^*$, given by the diagram:

$$
\begin{array}{ccc}
\varepsilon' & X^* & X^* \otimes V^* \\
\downarrow f & \downarrow \Omega_{\varepsilon'}^{-1}(X^*) & \\
0 & Y^* & B^* \\
\end{array}
$$

where $\varepsilon'$ is in $V-\mathcal{TS}\text{-seq}$ and $B^*$ is the pushout in the left square. Here $V^*$ is given in the proof of Proposition 3.6. To see that $B$ is a $V$-split acyclic complex, begin by tensoring the entire diagram with $V$ and consider the upper row which is the exact sequence

$$
\begin{array}{ccc}
0 & X^* \otimes V & \alpha \ X^* \otimes V^* \otimes V \\
\downarrow \Omega_{\varepsilon'}^{-1}(X^*) & \\
0 & \Omega_{\varepsilon'}^{-1}(X^*) & 0
\end{array}
$$

Now recall that the map $k \otimes V \rightarrow V \otimes V^# \otimes V$ is split. So the complex $V^* \otimes V$ is a direct sum of two complexes one of which has the form

$$
\begin{array}{ccc}
\ldots & 0 & V \\
\ldots & V & 0 \\
\ldots & 0 & \ldots
\end{array}
$$

where the nonzero terms occur in degree -1 and 0. Moreover, this summand contains the image of the chain map $\alpha$. Thus the first part of the upper row of the previous diagram looks like

$$
\begin{array}{ccc}
\ldots & 0 & X^* \otimes V \\
\downarrow \alpha & \\
\ldots & 0 & \ldots
\end{array}
$$

Because $X^* \otimes V$ is a split sequence, a straightforward exercise shows that the chain map $\alpha$ is also split. Consequently, the top row of the original diagram when tensored with $V$ is a split sequence of complexes, and the bottom row, being the pushout of the top row, when tensored with $V$ is also split. Hence $B$ is $V$-split as asserted.

Thus we have that $\mathcal{A}_{V\text{-}\text{split}}(K_{V-\mathcal{TS}(C)})$ is triangulated. It is obviously closed under taking direct summands. Hence, it is a thick subcategory of $K_{V-\mathcal{TS}(C)}$. □
If the category $A$ is the thick subcategory of acyclic complexes in a homotopy category $C$ of complexes, then the derived category $D_A(C)$ is the Verdier quotient or localization of $C$ at $A$. The objects in the derived category are the same as those in $C$. But the morphisms between two objects $X^*$ and $Y^*$ are obtained by inverting any morphism such that the third object in the triangle of that morphism is in $A$. Such a morphism is called a quasi-isomorphism. Thus a morphism is a composition $g^{-1}f$ as in the diagram

$$X^* \xrightarrow{f} Z^* \xleftarrow{g} Y^*$$

where the third object in the triangle containing $g$ is in $A$.

We use the notation $D_a(K_{\mathcal{X}}(C))$ to mean the derived category of $K_{\mathcal{X}}(C)$ for $\mathcal{X}$ one of $TS$ or $VTS$, with respect to the subcategory of acyclic complexes $A_a$. Here $a$ is either $-$ (blank) or $V$-split, meaning that either all acyclic complexes or the $V$-split acyclic complexes. Thus $K_{VTS}(Cpx^b(kG))$ means the quotient category $Cpx^b(kG)/(VTS\text{-Proj})$ of bounded complexes of $kG$-modules by the projectives of the exact category $(Cpx^b(kG),VTS\text{-seq})$, and $D_{V\text{-split}}(K_{VTS}(Cpx^b(kG)))$ is its localization by inverting any map such that the third object of the triangle of that map is an acyclic complex that splits on tensoring with $V$. As we see below, the notation can be simplified even more. Indeed, there is some contraction in the list of derived categories.

**Proposition 5.3.** Assume that $V$ is free of finite rank as a $k$-module. Let $C$ be as in 3.4. The natural functor $K_{VTS}(C) \to K(C)$ induces equivalences on the derived categories $D(K_{VTS}(C)) \to D(K(C))$ and $D_{V\text{-split}}(K_{VTS}(C)) \to D_{V\text{-split}}(K(C))$. Moreover, these are triangular equivalences.

**Proof.** Let $TS_k\text{-seq}$ be the collection of sequence that are term split on restriction to $k$. By Frobenius Reciprocity (see Theorem 7.1(b)), $TS_k\text{-seq} = kG\text{-TS}\text{-seq}$. We have a sequence of subcategories

$$TS_k\text{-seq} \text{-Proj} \subseteq VTS\text{-seq} \text{-Proj} \subseteq TS\text{-seq} \text{-Proj},$$

leading to a sequence of functors

$$K_{TS_k}(C) \xrightarrow{\sim} K_{VTS}(C) \xrightarrow{\sim} K_{TS}(C).$$

It is well known that the composition of the two functors induces an equivalence between $K_{kG\text{-TS}}(C)$ and $K(C)$. The point is that the projectives of each of the exact categories in question are acyclic complexes by 3.6. Any acyclic complex is quasi-isomorphic to the zero complex and hence becomes the zero object in the derived category. So for example, if a map between complexes factors through an element of $\text{Proj}(VTS\text{-seq})$, then it becomes the zero map in the derived category. The same argument applies to prove that the first statement of the theorem.
The equivalence $\mathcal{D}_{V-Split}(\mathcal{K}_{V-TS}(C)) \to \mathcal{D}_{V-Split}(\mathcal{K}(C))$ is proved similarly. The fact that these functors induce triangle equivalences is an exercise that we leave to the reader.

**Theorem 5.4.** All of the derived categories $\mathcal{D}_a(\mathcal{K}_b(C))$, that we have considered and have countable direct sums or allow the direct sum of a countable number of copies of any object, have idempotent completions.

*Proof.* This follows from [7], because the infinite categories have countable coproducts (direct sums).

**Theorem 5.5.** Suppose that $k$ is a field. The category $\mathcal{D}^b(kG) = \mathcal{D}(\mathcal{K}(cpx^b(kG)))$ is a Krull-Schmidt category.

*Proof.* It is straightforward to show that $\mathcal{D}^b(kG)$ is a hom-finite category, and it has idempotent completions. Hence, if $X^*$ is a complex in $\mathcal{D}^b(kG)$, then its endomorphism ring is a finite dimensional $k$-algebra that has a complete collection of primitive idempotents. Thus it has a complete collection of primitive idempotents that sum to the identity. This provides a decomposition of $X^*$ into indecomposable subcomplexes. The uniqueness of the decomposition can be proved from the structure of the endomorphism ring.

Finally, we should note that all of the categories that have been discussed respect the block structure of $kG$. The group algebra can be written as a direct sum of indecomposable two-sided ideals $kG = B_1 \oplus \cdots \oplus B_n$. Each $B_i$ contains an idempotent $e_i$ which acts as the identity of $B_i$, so that $B_i = e_i kG$ and $e_i = e_j$ for $j \neq i$. If $X$ is a $kG$-module or complex of $kG$-modules then $X = \oplus_i e_iX$, and we say that $e_iX$ is in the block $B_i$. There are no nonzero homomorphisms between modules or complexes that are in different blocks.

For the record, we state the following.

**Theorem 5.6.** Suppose that $B$ is a block of $kG$. Let $C(B)$ be any of the categories of complexes in [3.4] or any of the categories of complexes of finite length modules as in Section 4 restricted to modules in the block $B$. The homotopy categories $\mathcal{K}(C(B))$, $\mathcal{K}_{V-Split}(C(B))$ and $\mathcal{K}_{V-TS}(C(B))$ as well as the derived categories $\mathcal{D}(\mathcal{K}(C))$ and $\mathcal{D}_{V-Split}(\mathcal{K}(C(B)))$ are triangulated categories. These categories have idempotent completions provided they have countable direct sums or permit the countable direct sum of an object with itself, or have objects that are complexes of finite length modules. If $k$ is a field, then the categories $cpx^b(B)$, $\mathcal{K}(cpx^b(B))$, $\mathcal{K}_{V-Split}(cpx^b(B))$, $\mathcal{K}_{V-TS}(cpx^b(B))$, $\mathcal{D}(\mathcal{K}(cpx^b(B))) = \mathcal{D}^b(B)$, and $\mathcal{D}_{V-Split}(\mathcal{K}(cpx^b(B)))$ are Krull-Schmidt categories.

As a cautionary note, it should be added that seldom do any of the above categories, that are associated to blocks, have a tensor structure.
6. A functorial version of the Green correspondence

The purpose of this section is to lay a functorial framework for the Green correspondence. The aim is to isolate, in an abstract way, the condition necessary to define the correspondence. By this process, we see that the correspondence can be defined in many contexts. Our specific applications are to categories of complexes and their homotopy categories and derived categories. The reader might notice that, even though the setting is far more general, the development here follows closely the same steps as in the paper of Benson and Wheeler [6]. Indeed that paper was a big inspiration.

We wish to consider the following diagram of categories and functors. In the diagram, all vertical arrows are inclusions of full subcategories. For a category $\mathcal{D}$ the notation $\text{Ad}(\mathcal{D})$ means the closure of $\mathcal{D}$ under taking direct summands. If $\mathcal{C}$ and $\mathcal{D}$ are subcategories of $\mathcal{G}$, then $\mathcal{C} + \mathcal{D}$ means the full subcategory of all objects that can be written as the direct sum of an object in $\mathcal{C}$ and an object in $\mathcal{D}$.

\[
\begin{array}{ccc}
\mathcal{L}' = \mathcal{L} + \text{Ad}(F(\mathcal{L})) & \leftrightarrow & \mathcal{M} = \text{Ad}(I(\mathcal{L})) \\
\mathcal{L} & \leftrightarrow & \mathcal{M} \\
\mathcal{X} & \leftrightarrow & I(\mathcal{X})
\end{array}
\]

Here $\mathcal{Y} = \mathcal{X} + \text{Ad}(F(\mathcal{L}))$. The arrow from $\mathcal{M}$ to $\mathcal{L}'$ is dashed because it is not a functor, though there is a functor to $\mathcal{L}'/\mathcal{Y}$, as is explained below.

Our main theorem is the following.

**Theorem 6.1.** Suppose we have categories given as in the above diagram. We assume the following.

1. All of the categories are additive categories.
2. The subcategories $\mathcal{L}$ and $\mathcal{X}$ are closed under direct sums and summands in $\mathcal{H}$.
The quotient categories $\mathcal{H}/\mathcal{X}$ and $\mathcal{G}/I(\mathcal{X})$ are defined.

(4) $I$ and $R$ are an adjoint pair of functors with natural transformations $\varepsilon : 1_\mathcal{H} \to RI$ and $\eta : IR \to 1_\mathcal{G}$.

(5) There is a functor $F : \mathcal{H} \to \mathcal{H}$ and a natural transformation $\varphi : F \to RI$ such that the sum of the transformations $(\varepsilon, \varphi) : 1_\mathcal{H} \oplus F \to RI$ is an isomorphism of functors. In particular, for every object $M$ in $\mathcal{H}$ we have that $RI(M) \cong M \oplus F(M)$.

(6) For objects $L$ in $\mathcal{L}$ and $M$ in $\mathcal{M}$, every map $\gamma : L \to F(M)$ and every map $\delta : F(M) \to L$ that factors through an object in $\mathcal{Y}$, factors also through an object in $\mathcal{X}$.

(7) The quotient category $\mathcal{H}/\mathcal{X}$ has idempotent completions.

Then we have an adjoint pair of $\Pi$ and $\Pi$ functors

$$
\begin{array}{ccc}
\mathcal{L}/\mathcal{X} & \xrightarrow{1} & \mathcal{M}/I(\mathcal{X}) \\
\xleftarrow{R} & & \xleftarrow{\Pi} \\
\end{array}
$$

that give categorical equivalences.

To define the functors, we require some preliminary information. Throughout, we use the notation of the theorem.

**Lemma 6.2.** There exist functors $U$ and $V$,

$$
\begin{array}{ccc}
\mathcal{L}/\mathcal{X} & \xrightarrow{U} & \mathcal{L}'/\mathcal{Y} \\
\xleftarrow{V} & & \xleftarrow{1} \\
\end{array}
$$

giving equivalences of categories.

**Proof.** First note that $\mathcal{L}' = \mathcal{L} + \text{Ad} F(\mathcal{M}) = \mathcal{L} + \mathcal{Y}$. Define $U$ by $U(L) = L \oplus 0$ for $L \in \mathcal{L}$, that is, the functor induced by the inclusion of $\mathcal{L}$ into $\mathcal{L}'$. It is clear that any map that factors through an object in $\mathcal{X}$ also factors through one in $\mathcal{Y}$.

For $V$, suppose that $L \oplus Y$ is an object in $\mathcal{L} + \mathcal{Y}$, i.e. $L$ in $\mathcal{L}$ and $Y$ in $\mathcal{Y}$. Then, let $V(L \oplus Y) = L$. We first check that this is well defined. For suppose that $L \oplus Y \cong L' \oplus Y'$. The isomorphism between the two is given by a matrix

$$
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix}
$$

By condition (6) of the theorem, $\beta$ and $\gamma$ factor through an element of $\mathcal{X}$. It follows, with some computation, that $L$ and $L'$ are isomorphic modulo $\mathcal{X}$. The other details are likewise straightforward. Thus, $UV$ and $VU$ are the identity functors. \qed

**Lemma 6.3.** The category $\mathcal{L}'/\mathcal{Y}$ has idempotent completions.
Proof. Note that the category \( \mathcal{L}' \) can not be assumed to have idempotent completions. However we have seen that its quotient by \( \mathcal{Y} \) is equivalent to \( \mathcal{L}/\mathcal{X} \). Hence, it is sufficient to show that \( \mathcal{L}/\mathcal{X} \) has idempotent completions and we know this by Condition (2) and (7) of the theorem. That is, \( \mathcal{L}/\mathcal{X} \subseteq \mathcal{H}/\mathcal{X} \), and the latter is idempotent complete. \( \square \)

Lemma 6.4. Suppose that \( L \) is in \( \mathcal{L}' \), and that \( M \) is a direct summand of \( L \). Then there exists \( Y \) in \( \mathcal{Y} \) such that \( M \oplus Y \) is in \( \mathcal{L}' \).

Proof. Suppose that \( e : L \to L \) is the idempotent corresponding to \( M \), the projection of \( L \) to \( M \). By the previous lemma, we know that \( e \) splits. So that \( L \cong L_e \oplus L_{1-e} \), where \( L_e \cong M \) in \( \mathcal{L}'/\mathcal{Y} \). So there exist \( Y \) and \( Y' \) in \( \mathcal{Y} \) such that \( L_e \oplus Y' \cong M \oplus Y \) in \( \mathcal{H} \). Consequently, \( M \oplus Y \) is in \( \mathcal{L}' \). \( \square \)

Corollary 6.5. The functor \( R \) induces a functor \( \hat{R} : \mathcal{M} \to \mathcal{L}'/\mathcal{Y} \).

Proof. Suppose that \( M \) is an object in \( \mathcal{M} \). Then \( M \) is a direct summand of \( I(L) \) for some \( L \) in \( \mathcal{L} \). Now, \( RI(L) = L \oplus F(L) \), and by the previous lemma, there exists \( Y \) in \( \mathcal{Y} \) such that \( R(M) \oplus Y \) is in \( \mathcal{L}' = \mathcal{L} + \mathcal{A}(F(M)) \). Thus, we define \( \hat{R}(M) = R(M) \oplus Y \). Note that this does not depend on the choice of \( Y \). \( \square \)

We are now ready to prove the main theorem of the section.

Proof of Theorem 6.1. We define \( I : \mathcal{L}/\mathcal{X} \to \mathcal{M}/I(\mathcal{X}) \) to be the functor induced by the restriction of \( I \) to \( \mathcal{L} \). The functor \( \hat{R} \) is the composition \( \hat{R} = V \hat{R} \).

Suppose that \( M \) is an object in \( \mathcal{M} \). We know that there exists \( Y \) in \( \mathcal{Y} \) such that \( \hat{R}(M) = R(M) \oplus Y \) is in \( \mathcal{L}' \). That is \( R(M) \oplus Y \cong L' \oplus Y' \) for \( L' \) in \( \mathcal{L} \) and \( Y' \) in \( \mathcal{Y} \). Then \( V \hat{R}(M) = L' \). The map \( \hat{R}(M) = R(M) \oplus Y \to L' \oplus Y' \) has the form of a matrix

\[
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix}
\]

where \( \beta \) and \( \gamma \) factor through objects in \( \mathcal{X} \). Consequently, the relevant portion is the map \( \alpha = \alpha_M : R(M) \to L' \).

For \( L \) in \( \mathcal{L} \) and \( M \) in \( \mathcal{M} \), we define

\[
\gamma : \text{Hom}_{\mathcal{L}/I(\mathcal{X})}(I(L), M) \to \text{Hom}_{\mathcal{M}/\mathcal{X}}(L, R(M))
\]

by \( \gamma(f) = \alpha_M R(f) \varepsilon_L \). That is, this is the composition modulo \( \mathcal{X} \)

\[
L \xrightarrow{\varepsilon_L} RI(L) \xrightarrow{R(f)} R(M) \oplus Y \xrightarrow{\gamma} L' \oplus Y' \xrightarrow{\delta} L'
\]
Note here that if $f$ factors through $I(X)$ for $X$ in $\mathcal{X}$, then $\gamma(f)$ factors through an object in $\mathcal{X}$. This happens because any map from $\mathbb{L}$ to $F(X)$ factors through an object in $\mathcal{X}$ by Condition (6) of the Theorem.

Now we define $\beta : \text{Hom}_{\mathcal{M}/\mathcal{X}}(L, \mathbb{R}(M)) \to \text{Hom}_{\mathcal{L}/I(\mathcal{X})}(\mathbb{I}(L), M)$ by letting $\beta(g) = \eta_M I(\alpha_M^{-1} g)$. That is, it is the composition

$$I(L) \xrightarrow{I(g)} IV \hat{R}(M) \cong I(L' \oplus Y') \xrightarrow{I(\alpha_M^{-1})} IR(M) \xrightarrow{\eta_M} M$$

Of course, the map which we are calling $\alpha^{-1}$ is only an inverse for $\alpha$ modulo maps that factor through elements of $\mathcal{X}$. The isomorphism $R(M) \oplus Y \cong L' \oplus Y'$ guarantees that there is such a map. Note that if $g$ factors through an element of $\mathcal{X}$, then the composition $\eta_M I(\alpha^{-1} g)$ factors through an element of $I(\mathcal{X})$.

Suppose that $g \in \text{Hom}_{\mathcal{M}/\mathcal{X}}(L, \mathbb{R}(M))$. Let $f = \beta(g)$. Then,

$$\gamma(f) = \alpha_M R(f) \varepsilon_L = \alpha_M R(\eta_M I(\alpha_M^{-1} g)) \varepsilon_L$$

$$= \alpha_M R(\eta_M) RI(\alpha_M^{-1} g) \varepsilon_L = \alpha_M \alpha_M^{-1} g = g$$

modulo maps that factor through objects in $\mathcal{X}$. The next to last step in the above sequence of equations is a consequence of the adjunction between $R$ and $I$ which implies that $R(\eta_M) RI(\mu) \varepsilon_L = \mu$ for any map $\mu : L \to R(M)$.

On the other hand if $f \in \text{Hom}_{\mathcal{L}/I(\mathcal{X})}(\mathbb{I}(L), M)$, then let $g = \gamma(f)$. So

$$\beta(g) = \eta_M I(\alpha_M^{-1} g) = \eta_M I(\alpha_M^{-1} \alpha_M R(f) \varepsilon_L)$$

$$= \eta_M IR(f) I(\varepsilon_L) = f$$

Thus we have shown that $\gamma \beta$ and $\beta \gamma$ are the identities and that $\mathbb{I}$ and $\mathbb{R}$ are an adjoint pair as asserted. \qed

**Remark 6.6.** The primary reason for the assumption of $\mathcal{H} / \mathcal{X}$ having idempotent completions is to make possible the proof of Lemma 6.4. That is, we require that $\mathcal{L}' / \mathcal{Y}$ have idempotent completions as in Lemma 6.3. The same thing would be accomplished if we assumed that $\mathcal{G}$ and $\mathcal{H}$ are Krull-Schmidt categories, or that $\mathcal{G}$ and $\mathcal{H}$ have countable direct sums.

### 7. Relative projective theory.

Let $k$ be a commutative ring, and suppose that $H$ is a subgroup of a finite group $G$. In this section, we consider the induction and restriction functors and remind the reader that many of the standard results associated with these functors, by virtue of their combinatorial nature, hold for complexes and homotopy classes of
complexes as well as for modules. In particular, some of the conditions of Theorem 6.1 are classical results in representation theory. For notation, let $\mathcal{C}_G$ denote one of the categories of complexes such as $\text{Cpx}$, $\text{cpx}$, $\text{Cpx}^+$, $\text{cpx}FL$ of $kG$-modules or its homotopy categories $K(\mathcal{C}_G)$. There are induction and restriction functors, which we denote $\text{Ind}_H^G$ and $\text{Res}_H^G$. When $X$ is a complex or module for $kH$ and when there is little chance of confusion, we often use the standard notation $X^{\uparrow G} = \text{Ind}_H^G(X) = kG \otimes_{kH} X$ to denote its induction to $G$. We also use $X^{\downarrow H} = \text{Res}_H^G(X)$ to denote the restriction to $kH$ of a $kG$-complex or module $X$. For any object $X$ in $\mathcal{C}$, there is a natural decomposition

$$\text{Ind}_H^G(X) = X^{\uparrow G} = kG \otimes_{kH} X = \oplus_{g \in G/H} g \otimes X$$

as complexes of vector spaces, where the sum is over a set of representatives of the left cosets of $H$ in $G$. Likewise for a map $f : X \to Y$, $\text{Ind}_H^G(f) = \sum_{g \in G/H} g \otimes f$.

The following results are standard in representation theory. The statement (a) is usually called the Mackey Decomposition Theorem, while (b) is Frobenius Reciprocity. The statement (c) which is a form of Frobenius reciprocity, is often called the Eckmann-Shapiro Lemma. While the proofs are classical, we give a quick review here in order to make it clear that the theorems are valid for complexes regardless of the coefficients. As the constructions are all well known and straightforward, we leave it to the reader to check a great many details.

In the notation of the theorem below, we note that if $t \in G$, then $(t \otimes \cdot)$ is a functor from $\mathcal{C}_{K \cap tH}$ to $\mathcal{C}_{Kt \cap H}$, taking an object $X$ to $t \otimes X$. For $K$ a subgroup of $G$ and $t \in G$, $tK = tKt^{-1}$ and $K^t = t^{-1}Kt$.

**Theorem 7.1.** Suppose that $\mathcal{C}$ is as in 3.4 or a category of complexes of finite length modules as in Section 4. Let $H$ and $K$ be subgroups of $G$. Then the following hold in $\mathcal{C}$.

(a) There is a natural transformation of functors

$$\alpha : \text{Res}_K^G \text{Ind}_H^G(\cdot) \longrightarrow \sum_{t \in K\backslash G/H} \text{Ind}_{Kt \cap H}^K(t \otimes \text{Res}_H^K(\cdot))$$

where the sum is over a set of representatives of the $K-H$-double coset in $G$. The transformation is an isomorphism on objects. Thus for an object $X$ in $\mathcal{C}_H$, there is an isomorphism

$$\alpha_X : (X^{\uparrow G})^{\downarrow K} \cong \oplus_{t \in K \backslash G/H} (t \otimes X^{\downarrow Kt \cap H})^{\uparrow K}.$$

(b) Assume that the tensor products of objects in $\mathcal{C}_G$ and $\mathcal{C}_H$ are defined. There is a natural transformation of functors

$$\beta : \cdot_G \otimes \text{Ind}_H^G(\cdot_H) \longrightarrow \text{Ind}_H^G(\text{Res}_H^K(\cdot_G) \otimes \cdot_H)$$
from $\mathbf{C}_G \times \mathbf{C}_H$ to $\mathbf{C}_G$, that is an isomorphism on objects. Thus for objects $X$ in $\mathbf{C}_G$ and $Y$ in $\mathbf{C}_H$, $\beta_{X,Y} : X \otimes Y^G \cong (X_{\downarrow H} \otimes Y)^G$.

(c) The functors $\text{Res}_H^G$ and $\text{Ind}_H^G$ are adjoints of each other.

(d) There is a functor $F : \mathbf{C}_H \to \mathbf{C}_H$ such that $\text{Res}_H^G \text{Ind}_H^G = 1_{\mathbf{C}_H} \oplus F$.

Proof. The point of the Mackey Theorem (a) is that for any object $X$ in $\mathbf{C}$

$$\text{Ind}_H^G(X) = X^G = kG \otimes_{kH} X = \bigoplus_{g \in G/H} g \otimes X$$

where the sum is over a complete set of representatives of the left cosets of $H$ in $G$. If one restricts to $K$, then the sum over all of the left cosets in a single $K$-$H$-double coset is a $kK$-subcomplex. It remains to show that as $kK$-objects $\sum_{g \in K\cap tH t^{-1}} g \otimes X \cong \text{Ind}_{K\cap tH t^{-1}}^K \text{Res}_{K\cap tH t^{-1}}^H (t \otimes X)$ where the sum is over a set of representatives of the left cosets of $H$ that are contained in $KtH$ for $t \in G$. This proof is fairly straightforward. It should be checked that the decomposition given by the Mackey Theorem commutes with the differentials of a complex, and that the isomorphism, which is defined internally on an object $X$, is actually a natural transformation of the functors.

For the Frobenius reciprocity (b), there is a map on objects that sends

$$X \otimes (kG \otimes_{kH} Y) \to kG \otimes_{kH} (X_{\downarrow H} \otimes Y)$$

defined by the formula

$$x \otimes (g \otimes y) \mapsto g \otimes (g^{-1}x \otimes y),$$

for $x \in X$, $y \in Y$ and $g \in G$. The inverse isomorphism send $g \otimes (x \otimes y)$ to $gx \otimes (g \otimes y)$. It can be seen that the maps commute with the differentials on the complexes and with maps between complexes. Thus they are natural transformations of the functors.

To prove (c), we define natural transformation $\eta : \text{Ind}_H^G \text{Res}_H^G \to 1_{\mathbf{C}_G}$ and $\varepsilon : 1_{\mathbf{C}_H} \to \text{Res}_H^G \text{Ind}_H^G$, by $\eta_X(g \otimes x) = gx$ for $g \in G$ and $x \in X$ in $\mathbf{C}_G$, and $\varepsilon_Y(y) = 1 \otimes y$ for $y \in Y$ in $\mathbf{C}_H$. Then the isomorphism

(adj1) \quad $\text{Hom}_{\mathbf{C}_H}(Y, \text{Res}_H^G(X)) \to \text{Hom}_{\mathbf{C}_G}(\text{Ind}_H^G(Y), X)$

sends a map $f$ to $\eta \text{Ind}_H^G(f)$, while the inverse isomorphism sends $f$ to $\text{Res}_H^G(\varepsilon)(f)$. Likewise we have natural transformation $\eta' : \text{Res}_H^G \text{Ind}_H^G \to 1_{\mathbf{C}_H}$ and $\varepsilon' : 1_{\mathbf{C}_G} \to \text{Ind}_H^G \text{Res}_H^G$ by $\eta'_Y(f)(\sum g_{\downarrow H} g \otimes y_g) = y_1$, for $y_g \in Y$ and the sum over a set of representatives of the left cosets of $H$ in $G$, and $\varepsilon'(x) = \sum g_{\downarrow H} g \otimes g^{-1}x$ for $x \in X$ in $\mathbf{C}_G$. Then, the isomorphism

(adj2) \quad $\text{Hom}_{\mathbf{C}_H}(\text{Res}_H^G(X), Y) \to \text{Hom}_{\mathbf{C}_G}(X, \text{Ind}_H^G(Y))$

takes $f$ to $\eta' \text{Ind}_H^G(f)$, while its inverse takes a map $f$ to $R(f)\varepsilon'$. 
Statement (d) is a direct consequence of the Mackey Theorem, letting
\[ F(\cdot) = \sum_{H \setminus G/H, x \neq 1} \text{Ind}^H_{H \cap H}(t \otimes \text{Res}^H_{H \cap H}(\cdot)) \]
where the sum is over a set or representatives of the \( H \)-\( H \)-double cosets in \( G \) that are not the identity. Note that \( \varepsilon : 1_{C_H} \to \text{Res}^G_H \text{Ind}^G_H = 1_{C_H} \oplus F \) is the injection and it is split by \( \eta' : \text{Res}^G_H \text{Ind}^G_H \to 1_{C_H} \).
\[ \square \]

For the homotopy categories we have the following.

**Theorem 7.2.** Let \( H \) be a subgroup of \( G \). Let \( C \) be a category of complexes as in Theorem 7.1. Suppose that \( V \) is a \( kG \)-module that is free of finite rank as a module over \( k \). Let \((\mathcal{K}_*(C_G), \mathcal{K}_*(C_H)) \) be one of the pairs of homotopy categories \((\mathcal{K}(C_G), \mathcal{K}(C_H)), (\mathcal{K}_{V, \text{split}}(C_G), \mathcal{K}_{V, \text{split}}(C_H)), (\mathcal{K}_{V, \tau S}(C_G), \mathcal{K}_{V, \tau S}(C_H)), \) or \((\mathcal{K}_{\tau S + V, \text{split}}(C_G), \mathcal{K}_{\tau S + V, \text{split}}(C_H)) \). Then induction and restriction define functors \( \text{Ind}^G_H : \mathcal{K}_*(C_H) \to \mathcal{K}_*(C_G) \) and \( \text{Res}^G_H : \mathcal{K}_*(C_G) \to \mathcal{K}_*(C_H) \). Moreover, these functors satisfy the conclusions of Theorem 7.1.

**Proof.** It suffices to show that the relative projective objects for the homotopy are preserved by the functors. For the ordinary homotopy, a relative projective object is a direct sum of two-term complexes of the form \( \cdots \to 0 \to W \to W \to 0 \to \cdots \). It is obvious that the induction or restriction of such a complex has the same form. Consequently, the induction or restriction of a map that factors through such a complex also factors through a relative projective.

Notice that the restriction of a \( V \)-projective complex of \( kG \)-modules to \( H \) is a \( V_{\downarrow H} \)-projective complex. On the other hand, if \( X \) is a \( V_{\downarrow H} \)-projective module or complex, then \( X \) is a direct summand of \( Y \otimes V_{\uparrow H}^\# \otimes V_{\downarrow H} \) for some object \( Y \). By Frobenius Reciprocity (Theorem 7.1(b)), \( X^{\uparrow G} \) is a direct summand of \( Y^{\uparrow G} \otimes V^\# \otimes V \). Hence the induction of a relative \( V_{\downarrow H} \)-projective object is a relative \( V \)-projective object. This proves that the induced functors are defined. Furthermore, we have the following commutative diagram:

\[
\begin{array}{ccc}
C_H & \xrightarrow{\text{Ind}^G_H} & C_G \\
\text{Res}^G_H \downarrow & & \downarrow \text{Res}^G_H \\
\mathcal{K}_*(C_H) & \xrightarrow{\text{Ind}^G_H} & \mathcal{K}_*(C_G).
\end{array}
\]

Since the functors \( q_{H} \) and \( q_{G} \) preserve direct sums, the statements (a) follows.

If \( X \) and \( Y \) are isomorphic in \( C_G \), then \( X \) and \( Y \) are isomorphic in \( \mathcal{K}(C_G) \). Then the statement (b) follows from the definition of the induction functors above. With the above commutative diagram, the statement (c) and (d) follows from the above.
discussion and from the explicit description of the isomorphisms given in the proof of statement (c) and (d) of Theorem 7.1. □

We remind the reader that if $H$ is a subgroup of $G$, then an object in $C_G$ is (relatively) $H$-projective if it factors through a direct summand of $Y^H$ for some object $Y$ in $C_H$. By Frobenius reciprocity (see part (b) above) this is the same as being $V$-projective for $V = (k_H)^G$. If $D$ is a collection of subgroups of $G$, we say that an object is $D$-projective if it factors through a direct sum of $D$-projective objects for $D \in D$. This is the same as being $V$-projective for $V = \sum_{D \in D} k^G_D$. A map is $D$-projective if it factors through a $D$-projective object. For a subgroup $H$ of $G$ and complexes $X$ and $Y$, there is a relative trace map $\text{Tr}^G_H : \text{Hom}_{C_H}(X, Y) \to \text{Hom}_{C_G}(X, Y)$ given by $\text{Tr}^G_H(f) = \sum_{G/H} gf g^{-1}$. Observer that, if $\alpha \in \text{Hom}_{kG}(W, X)$ and $\beta \in \text{Hom}_{kG}(Y, Z)$ for objects $W$ and $Z$, then $\beta \text{Tr}^G_H(f) \alpha = \text{Tr}^G_H(\beta f \alpha)$.

It is clear that the map $\text{Tr}^G_H$ is well defined in any of the categories of complexes, i.e. whenever a map of objects is a map or sets. Some more care is needed for the homotopy categories.

**Lemma 7.3.** Let $K_G$ be one of the homotopy categories $K_* (C_G)$ as in Theorem 7.2. Then the map $\text{Tr}^G_H$ induces a map $\text{Tr}^G_H : \text{Hom}_{K_H}(X, Y) \to \text{Hom}_{K_G}(X, Y)$ for any objects $X$ and $Y$.

**Proof.** Let $P_H$ be the collection of projective objects in $C_H$ relative to the homotopy. It suffices to show is that if $f : X \to Y$ is a $kH$-map of objects in $C_G$ that is zero on $K_H$ (that is, factors through an object in $P_H$) then $\text{Tr}^G_H(f)$ factors through an object in $P_G$. If $f$ factors through an object in $P_H$, then it factors through a relative projective cover for $Y_{\downarrow H}$ which is in $P_H$. It can be seen from the description of the projectives as in Propositions 3.5, 3.6, and 3.7 that the restriction to $H$ of a relative projective cover for an object $Y$ can serve as a relative projective cover for $Y_{\downarrow H}$. Let $\beta : P \to Y$ be such a cover for $Y$, with $P$ in $P_G$. Then we have that $f = \beta \alpha$ for $\alpha : X_{\downarrow H} \to P_{\downarrow H}$. But then $\text{Tr}^G_H(f) = \text{Tr}^G_H(\alpha) \beta$ factors through an object in $P$. □

**Theorem 7.4.** Suppose that $C$ is as in 7.1 or a homotopy category of a category of such complexes as in Theorem 7.2. Let $H$ be a subgroup of $G$ and let $D$ be a collection of subgroups of $G$. Then the following hold in $C$.

(a) If an object $X$ in $C$ is $H$-projective, then $\text{Id}_X = \text{Tr}^G_H(\mu)$ for some $kH$-map $\mu : X \to X$.

(b) A map $f : X \to Y$ factors through an $H$-projective object if and only if $f = \text{Tr}^G_H(\mu)$ for some $\mu : X_{\downarrow H} \to Y_{\downarrow H}$ in $C_H$.

(c) Assuming the $C_G$ has idempotent completions, an object $X$ is $H$-projective if and only if $X$ is a direct summand of $\text{Ind}_H^G \text{Res}_H^G(X)$.

(d) A map $f : X \to Y$ is $D$-projective if and only if $f = \sum_{D \in D} \text{Tr}^G_D(\gamma_D)$ where for $D \in D$, $\gamma_D : X_{\downarrow D} \to Y_{\downarrow D}$ is a $kD$-map.
(e) If \( f : X \to Y \) is \( \mathcal{D} \)-projective and \( g : Y \to Z \) is \( \mathcal{E} \)-projective, for \( \mathcal{E} \) another collection of subgroups of \( G \), then \( gf \) is \( \mathcal{F} \)-projective where \( \mathcal{F} = \{ D \cap E | D \in \mathcal{D}, E \in \mathcal{E} \} \).

(f) Suppose that \( k \) is a field and \( B \) is a block of \( kG \) having defect group \( Q \) and if \( X \) is an object in \( \mathcal{C} \) and in the block \( B \), then \( X \) is \( Q \)-projective.

Proof. Because we have the commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(X_{\downarrow H}, Y_{\downarrow H}) \overset{\text{Tr}_{G}^{H}}{\longrightarrow} \text{Hom}_{\mathcal{C}}(X, Y) \\
\downarrow \hspace{2cm} \downarrow \hspace{2cm} \\
\text{Hom}_{\mathcal{K}}(X_{\downarrow H}, Y_{\downarrow H}) \overset{\text{Tr}_{G}^{H}}{\longrightarrow} \text{Hom}_{\mathcal{K}}(X, Y),
\end{array}
\]

it suffices to prove the Theorem in category of complexes.

If \( X = \text{Ind}_{H}^{G}(Y) \) for \( Y \in \mathcal{C}_{H} \), then, writing \( X = \sum_{g \in G/H} (g \otimes y) \), define \( f : X \to X \) by \( f(1 \otimes y) = 1 \otimes y \) and \( f(g \otimes y) = 0 \) if \( g \notin H \). Then \( \text{Id}_{X} = \text{Tr}_{H}^{G}(f) \).

To prove (b) we first note that if \( f : X \to Y \) factors through an \( H \)-projective object, then it factors through \( Z^{\uparrow G} \) for some object \( Z \) in \( \mathcal{C}_{H} \), and it is a composition with the identity of \( Z^{\uparrow G} \) which is a relative trace. For the converse, suppose that \( f = \text{Tr}_{H}^{G}(\mu) \) for some \( \mu : X_{\downarrow H} \to Y_{\downarrow H} \). then define \( \sigma : X \to (Y_{\downarrow H})^{\uparrow G} \) by \( \sigma(x) = \sum_{g \in G/H} g \otimes \mu(g^{-1}x) \) for \( x \in X \), and \( \tau : (Y_{\downarrow H})^{\uparrow G} \to Y \) by \( \tau(g \otimes y) = gy \) for \( y \in Y \).

Then note that \( \tau \sigma = \text{Tr}_{H}^{G}(\mu) \) in \( \mathcal{C}_{G} \).

For (c) we notice in the above proof that the condition that \( \text{Id}_{X} = \text{Tr}_{H}^{G}(\mu) \) for some \( \mu \) implies the existence of \( \sigma : X \to (X_{\downarrow H})^{\uparrow G} \) and \( \tau : (X_{\downarrow H})^{\uparrow G} \to X \) with \( \tau \sigma = \text{Id}_{X} \). Thus \( \sigma \tau \) is an idempotent in the endomorphism ring of \( (X_{\downarrow H})^{\uparrow G} \). Assuming that it splits, \( X \) is a direct summand of \( (X_{\downarrow H})^{\uparrow G} \).

Statement (d) follows from (b). For (e), notice that if \( H \) and \( J \) are subgroups then a composition \( \text{Tr}_{H}^{G}(\alpha \text{Tr}_{J}^{G}(\beta)) = \text{Tr}_{H}^{G}(\alpha \text{Res}_{H}^{J}(\text{Tr}_{J}^{G}(\beta))) \). The remainder of the proof is a consequence of the Mackey Theorem and the transitivity of induction.

Similarly, the last statement is a consequence of the fact that the central idempotent that is the identity for the block \( B \) is a relative trace from the defect group \( D \). See a standard text such as [10].

8. The Green correspondence

The purpose of this section is present a version of the Green correspondence for derived categories and categories of complexes associated to group algebras. The
classical Green correspondence assumes that $k$ is a field of characteristic $p$ and that $H$ is a subgroup containing the normalizer of a $p$-subgroup and the correspondences is between relatively $\mathfrak{X}$-projective $kG$-modules and relatively $\mathfrak{Y}$-projective $kH$-modules for certain collections of subgroups $\mathfrak{X}$ and $\mathfrak{Y}$. The approach here uses Theorem 6.1 and is somewhat more general as far as the choices of the collections of subgroups.

Suppose that $H$ is a subgroup of the finite group $G$. Let $\mathfrak{P}$ be a nonempty collection of subgroups of $H$: 

$$ \mathfrak{X} = \{ gP_1g^{-1} \cap P_2 \mid P_1, P_2 \in \mathfrak{P} \text{ and } g \in G \setminus H \} $$

and

$$ \mathfrak{Y} = \{ gPg^{-1} \cap H \mid P \in \mathfrak{P} \text{ and } g \in G \setminus H \}. $$

Let

$$ V_\mathfrak{X} = \sum_{\mathfrak{P} \in \mathfrak{X}} H_p^\mathfrak{X} \quad \text{and} \quad V_\mathfrak{Y} = \sum_{\mathfrak{P} \in \mathfrak{Y}} H_p^\mathfrak{X}. $$

Note here that $V_\mathfrak{X}$ and $V_\mathfrak{Y}$ are both free and finitely generated as modules over the coefficient ring $k$.

Let $C_G$ denote any one of the categories $\text{Cpx}(kG)$, $\text{cpxFL}(kG)$, $\text{Cpx}^*(kG)$, or $\text{cpxFL}^*(kG)$ for $* = +, -, b$. Likewise, we let $C_H$ be the same with $G$ replaced by $H$. Let $K_G = K_*(C_G)$ be one of the homotopy categories $K(C_G)$, $K_{V-S\text{plt}}(C_G)$, $K_{V-TS}(C_G)$ or $K_{V-+TS}(C_G)$ (where $V$ is a finitely generated $kG$-module that is free as a $k$-module). Then let $K_H$ be $K(C_H)$, $K_{V(H)-S\text{plt}}(C_H)$, $K_{V+H-TS}(C_G)$ or $K_{V+H+S\text{plt}}(C_G)$, to correspond to $K_G$. Let $\mathcal{C}_G$ be $C_G$ or $C_H$ for some choice.

In $C_G$, let $\mathfrak{X}-\text{Proj}(C_G)$ be the collection of $V_\mathfrak{X}$-projective objects. Such an object is a direct summand of a direct sum of objects induced from objects in $\mathcal{C}_P$ for $P \in \mathfrak{X}$. Similarly, an exact sequence of objects in $\mathcal{C}_G$ is $\mathfrak{X}$-split, if it is $V_\mathfrak{X}$-split, thus implying that the sequence splits on restriction to every subgroup $P \in \mathfrak{X}$.

The idea expressed in the following lemma is used to establish idempotent completions in the proofs of some theorems.

**Lemma 8.1.** Let $\mathcal{C}$ be a category of complexes as above. Let $V$ be a finitely generated $kG$-module that is free as a module over $k$. For a collection of subgroups $\mathfrak{U}$, let $V_\mathfrak{U} = \sum_{\mathfrak{P} \in \mathfrak{U}} H_p^\mathfrak{U}$. Let $K = K_{TS+V-S\text{plt}}(C)$. Then

$$ \frac{K}{\mathfrak{U}-\text{Proj}(K)} = K_{TS+(V \oplus V_\mathfrak{U})-S\text{plt}}(C). $$

**Proof.** Suppose that $X$ and $Y$ are objects in $\mathcal{C}$ and $\theta : X \to Y$ is a morphism in $K$. Then $\theta$ if $\mathfrak{U}$-projective if and only if $\theta = \beta \alpha$, $\alpha : X \to Z$, $\beta : Z \to Y$, where $Z$ is $\mathfrak{U}$-projective and $\alpha, \beta$ are morphisms in $K$. But then $\theta - \beta \alpha$ is zero in $K$ and hence factors through an object that is projective relative to $TS + V\text{-Splt}$ sequences. This means that $\theta$ is a projective relative to $TS + (V \oplus V_\mathfrak{U})\text{-Splt}$ sequences. \qed
The main theorem is the following.

**Theorem 8.2.** Let $C_G$, $C_H$, $\mathfrak{P}$, $\mathfrak{X}$, $\mathfrak{Y}$ be as above. Then there are equivalences of categories

\[
\begin{array}{ccc}
\mathfrak{P} \text{-Proj}(C_H) & \overset{I}{\leftarrow} & \mathfrak{P} \text{-Proj}(C_G) \\
\mathfrak{X} \text{-Proj}(C_H) & \overset{R}{\rightarrow} & \mathfrak{X} \text{-Proj}(C_G)
\end{array}
\]

that are induced by the induction and restriction operations.

**Proof.** The proof is by application of Theorem 6.1. The problem is to show that the hypotheses hold in every case. The setup is that $H = C_H$, $G = C_G$, $R = \text{Res}_{H}^G$, $I = \text{Ind}_{H}^G$, $\mathcal{L} = \mathfrak{P} \text{-Proj}(C_H)$, By transitivity of induction, $\mathcal{M} = \mathfrak{P} \text{-Proj}(C_G)$.

Notice that $\mathfrak{X} = \mathfrak{X} \text{-Proj}(C_H)$, while $\text{Ad}(I(\mathfrak{X})) = \mathfrak{X} \text{-Proj}(C_G)$.

Conditions (1), (2) and (3) of Theorem 6.1 are clearly satisfied. Conditions (4) and (5), follow from Theorem 7.1 parts (c) and (d), respectively, and Theorem 7.2.

For condition (6), we need a lemma, which says that a subgroup of some element of $\mathfrak{P}$ that is also a subgroup of some element in $\mathfrak{Y}$ is contained in a subgroup in $\mathfrak{X}$. This is a standard result. That is, if $Q \subseteq P_1$ for $P_1 \in \mathfrak{P}$ and $Q \subseteq H \cap gP_2g^{-1}$ for $P_2 \in \mathfrak{P}$ and $g \notin H$, then $Q \subseteq P_1 \cap gP_2g^{-1} \in \mathfrak{X}$. If $L \in \mathcal{L}$, $M \in \mathcal{M}$ and $\gamma : L \rightarrow F(M)$ factors through an $\mathfrak{P}$-projective object, then $\gamma = \gamma \text{Id}_L$ factors through an $\mathfrak{X}$-projective object by statements (d) and (e) of Theorem 7.4.

To prove (7), it is only necessary to show that any of the quotient categories $U = \mathcal{C}_H / \mathfrak{X} \text{-Proj}(C_H)$ has idempotent completions. Note that $U$ is a triangulated category. In every case that we consider, by Lemma 8.1, $U$ is a category that has been discussed in Section 4 with regard to the question of idempotent completions. Thus, $U$ has idempotent completions by Proposition 4.1 and Theorem 4.2.

As is pointed out in [6], the functors are not precisely the restriction and induction functors. The problem is that the restriction of a $\mathfrak{P}$-projective $kG$-module is not $\mathfrak{P}$-projective as a $kH$-module. So the inverse of the induction functor is actually the composition of the restriction with another categorical equivalence (called “\( V \)” in Lemma 6.2) as in the proof of Theorem 6.1.

**Theorem 8.3.** Let $C_G$, $C_H$, $\mathfrak{P}$, $\mathfrak{X}$, $\mathfrak{Y}$ be as above. Let either $A_G$ be $A(C_G)$ and $A_H$ be $A(C_H)$ or $A_G = A_{V \text{-Sph}}(C_G)$ and $A_H = A_{V_H \text{-Sph}}(C_H)$. Assume that $A_G$ is a thick subcategory of $C_G$ and $A_H$ is a thick subcategory of $C_H$ (see the remark following Proposition 7.7). Then there are equivalences of categories

\[
\begin{array}{ccc}
\mathfrak{P} \text{-Proj}(A_H) & \overset{I}{\leftarrow} & \mathfrak{P} \text{-Proj}(A_G) \\
\mathfrak{X} \text{-Proj}(A_H) & \overset{R}{\rightarrow} & \mathfrak{X} \text{-Proj}(A_G)
\end{array}
\]
that are induced by the induction and restriction operations.

**Proof.** The categories involved are additive, thick subcategories of $C_G$ or $C_H$. For $C_G$, the restriction and induction of an acyclic complex is again an acyclic complex. For $C_H$, the restriction and induction of an acyclic complex is again an acyclic complex. Regarding the hypothesis of Theorem 6.1, conditions (1), (2) and (4) are automatic. For condition (3), it should be noted that if $f : X \to Y$ is a map of objects in $A_H$ that factors through an $X$-projective object, then it factors through the relative $X$-projective cover of the object $Y$, which is an acyclic object. Hence, there are no $X$-projective maps in $C_H$ of objects in $A_H$, that are not also $X$-projective in $A_H$. The same hold of $I(X)$-projective maps between objects in $A_G$. These facts require some checking on a case by case basis depending on the category $C$. We leave the check to the reader.

Condition (5) of Theorem 6.1 is essentially the Mackey theorem which holds for acyclic complexes. It should be noted that the functor $F$ takes acyclic objects to acyclic objects. For Condition (6), we observe that if $F : L \to F(M)$ factors through an $\mathfrak{Y}$-projective object where $L$ is an acyclic complex in $\mathfrak{YProj}(A_H)$ and $M$ is in $\mathfrak{YProj}(A_G)$, then it factors through the $\mathfrak{Y}$-projective cover of $F(M)$ which is acyclic. Because $X$ is $\mathfrak{Y}$-projective, the map $f$ factors through an $X$-projective complex, which we can take to be acyclic as before.

Finally, there is the question of Condition (7). However, as the subcategory $A_H$ is thick in $C_H$, the property of the quotient category of $C_H$ by the $X$-projective objects having idempotent completions, extends to the category of acyclic objects. That is, the splitting of an idempotent on an acyclic object in the quotient of $C_H$ gives the direct sum of two objects that must be acyclic. □

Another approach to a proof for the above theorem is that category $A_G$ and $A_H$ are subcategories of $C_G$ and $C_H$, and the induction and restriction functors for $A$ are the restrictions of those for $C$. Moreover, an object in $A_H$ is $X$-projective in $A_H$ if and only if it is $X$-projective in $C_H$. So the question might be whether the induced equivalences in Theorem 8.2 extends to those of Theorem 8.3. The latter theorem asserts an affirmative answer and the real reason is embedded in the proof. Essentially, it is that a map between acyclic objects in $A_H$, that factors through an $X$-projective objective in $C_H$, factors through an $X$-projective object in $A_H$. It is a consequence of the fact that it factors through a projective cover.

For derived categories we come down to the following.

**Theorem 8.4.** Let $D_G = D(C_G)$ with $D_H = D(C_H)$ or $D_G = D_{V}\text{-}S\text{plt}(C_G)$ with $D_H = D_{V_H}\text{-}S\text{plt}(C_H)$ for $C_G$, $C_H$ as in the previous theorem. Assume, as in that theorem, that the subcategories of acyclic objects are thick. Then there are equivalences of
categories
\[
\begin{array}{ccc}
\mathfrak{P} \cdot \text{Proj}(\mathcal{D}_H) & \xrightarrow{\mathcal{I}} & \mathfrak{P} \cdot \text{Proj}(\mathcal{D}_G) \\
\mathfrak{X} \cdot \text{Proj}(\mathcal{D}_H) & \leftarrow & \mathfrak{X} \cdot \text{Proj}(\mathcal{D}_G)
\end{array}
\]
that are induced by the induction and restriction operations.

Proof. The derived categories \( \mathcal{D}_G \) and \( \mathcal{D}_H \) have the same objects as \( \mathcal{C}_G \) and \( \mathcal{C}_H \), respectively, and we know that we have well defined equivalences on objects. It is easy to see that the induction functor takes exact sequences of complexes to exact sequences of complexes and in the homotopy categories, and takes triangles to triangles. This from \( \mathcal{C}_H \) to \( \mathcal{C}_G \). By the equivalences, the same happens for the inverse. In the derived category \( \mathcal{D}_G \) or \( \mathcal{D}_H \), a morphism between objects \( X \) and \( Y \) is an equivalence class of diagrams
\[
X \xrightarrow{\phi} Z \xrightarrow{\theta} Y
\]
where the third object in the triangle (in \( \mathcal{C}_G \) or \( \mathcal{C}_H \) as appropriate) of \( \phi \) is acyclic. Because, the functors take triangles to triangles and acyclic objects to acyclic objects, they are equivalences also on morphisms. So we have equivalences of the derived categories as additive categories.

\[ \square \]

Remark 8.5. The proof of the above theorem avoids the question of idempotent completeness of any of the derived categories. We know that idempotent completions do exist in several cases. Balmer and Schlichting \cite{4} verify idempotent completions in the bounded derived category of an exact category that has idempotent completions, and also for the category of bounded below complexes. However, in general, the localization of an idempotent complete triangulated category, by an idempotent complete thick subcategory, may not be idempotent complete.

9. Blocks and triangulations

Suppose that \( k \) is an algebraically closed field of characteristic \( p > 0 \), or a complete discrete valuation ring whose residue field is an algebraically closed field of characteristic \( p > 0 \). Let \( \mathcal{C} \) be one of the categories of complexes as before. We remind the reader that all of the categories that we have discussed respect the block structure. That is, if \( B \) and \( B' \) are two different blocks of \( kG \) then there are no nonzero morphism from any complex of modules in \( B \) to any complex of modules in \( B' \). This is simply because the idempotents for the blocks, which act as identity on modules in the block, annihilate each other. A block \( B \) has a defect group \( Q \) with the property that every module or complex in \( B \) is \( Q \)-projective, i.e. is a direct summand of an object induced from \( Q \). The same holds for complexes of \( B \)-modules, and in fact, every morphism between two modules or complexes in \( B \) is in the image of the relative trace map \( \text{Tr}^G_Q \).
The Brauer correspondent of $B$ is a block of $kN_G(Q)$, with the property that the product of the central idempotent of $B$ and $b$ is not zero. But more importantly we have the following. We set the notation, as this will be used again.

**Notation 9.1.** For a block $B$ of $kG$, let $Q$ be its defect group and let $b$ be the Brauer correspondent of $B$. Let $\mathcal{P} = \{Q\}$, $H = N_G(Q)$, $\mathcal{X} = \{Q \cap Q^\sigma | \sigma \in G \setminus H\}$, and $\mathcal{Y} = \{H \cap Q^\sigma | \sigma \in G \setminus H\}$. For $C$ a category of complexes such as $\text{Cpx}$ or $\text{Cpx}^+$, let $C(B)$ be the full subcategory of $C_G$ consisting of those complexes that lie in $B$.

Note that in the above notation, $\mathcal{P} \text{-Proj} = C(B)$.

**Proposition 9.2.** Use Notation 9.1. Suppose that $B$ is a block of $kG$. Let $C$ be a category of complexes as in Theorem 8.2. For the functors

$$f : \mathcal{P} \text{-Proj}(C_G) \longrightarrow \mathcal{P} \text{-Proj}(C_H) \quad \text{and} \quad g : \mathcal{P} \text{-Proj}(C_H) \longrightarrow \mathcal{P} \text{-Proj}(C_G)$$

we have that

$$f : \frac{C(B)}{\mathcal{X} \text{-Proj}(C(b))} \longrightarrow \frac{C(b)}{\mathcal{Y} \text{-Proj}(C(b))} \quad \text{and} \quad g : \frac{C(b)}{\mathcal{Y} \text{-Proj}(C(b))} \longrightarrow \frac{C(B)}{\mathcal{X} \text{-Proj}(C(B))}$$

**Proof.** It follows from Theorem 2.7 in [15]. The proof in [15] is only for finitely generated modules, but it holds equally well for complexes and homotopy classes of complexes. \(\square\)

It is worth noting the following. Its proof follows from Theorem 7.4(e), after recalling that any object in $C(b)$ is $\mathcal{P}$-projective.

**Lemma 9.3.** With the above notation, we have that

$$\frac{C(b)}{\mathcal{Y} \text{-Proj}(C(b))} = \frac{C(B)}{\mathcal{X} \text{-Proj}(C(B))}.$$ 

That is, a map between objects in $C(b)$ factors through an $\mathcal{Y}$-projective object if and only it factors through an $\mathcal{X}$-projective object.

The main theorem of the section is the following.

**Theorem 9.4.** Use Notation 9.1. Suppose that $B$ is a block of $kG$. Suppose that $C$ is a category of complexes or a homotopy category as in Theorem 8.2. Then the equivalences

$$\frac{C(b)}{\mathcal{X} \text{-Proj}(C(b))} \xrightarrow{\mathcal{Y}} \frac{C(B)}{\mathcal{X} \text{-Proj}(C(B))}$$

are equivalences of triangulated categories.
Proof. Let \( V = \sum_{P \in \mathcal{X}} k_P^G \). The object of the proof is to show that given a triangle in the domain category, its image is a triangle in the target category. To this suppose that \( \sigma : X^* \to Y^* \) is a map of complexes in \( \mathcal{C}(B) \). That is, it is a complex in \( \mathcal{C}_G \) whose terms are all in the block \( B \). We view the quotient of the homotopy category as in Theorem 8.1. We construct the triangle of \( \sigma \) by constructing the diagram:

\[
\begin{array}{ccccccccc}
0 & \to & X^* & \to & I^* & \to & \Omega_S^{-1}(X^*) & \to & 0 \\
\downarrow{\sigma} & & \downarrow{\sigma} & & \downarrow{\sigma} & & \downarrow{\sigma} & & \\
0 & \to & Y^* & \to & E^* & \to & \Omega_S^{-1}(X^*) & \to & 0 \\
\end{array}
\]

Here, the upper row is the first step in relative injective resolution of \( X^* \), and the complex \( E^* \) is the pushout of the upper left corner. In this particular case, it means that \( I^* \) is the direct sum of a split complex and a complex of \( V \)-projective modules, and the upper row is term split and split on restriction to every subgroup in \( \mathcal{X} \).

Now consider the effect of the restriction functor \( f \) on the diagram. The restriction of an \( \mathcal{X} \)-projective complex is \( \mathcal{Y} \)-projective. The restriction of a split complex remains split, the restriction of the upper row remains term-split. Finally, the upper row splits on restriction to any subgroup of \( \mathcal{Y} \). Thus, the restriction to \( H \) of the triangle

\[
X^* \to Y^* \to B^* \to \Omega_S^{-1}(X^*)
\]

is again a triangle. \( \square \)

To extend the theorem to the derived category, we need the next lemma.

**Theorem 9.5.** Let \( \mathcal{D}_G, \mathcal{D}_H \) be as in Theorem 8.4 Suppose that \( \mathcal{D}(B) \) is the full subcategory of \( \mathcal{D}_G \) consisting of those classes of complexes in the block \( B \). Let \( \mathcal{D}(b) \) be the same for the Brauer correspondent of \( B \). Then we have equivalences of triangulated categories

\[
\begin{array}{ccc}
\mathcal{D}(B) & \xrightarrow{\mathcal{I}} & \mathcal{D}(b) \\
\mathfrak{X} \text{-Proj}(\mathcal{D}(B)) & \xleftarrow{\mathcal{R}} & \mathfrak{X} \text{-Proj}(\mathcal{D}(b))
\end{array}
\]

**Proof.** The triangles in the derived category are the same as those in the homotopy category. Consequently the theorem is proved by application of Theorems 8.4 and 9.4. \( \square \)

Suppose that \( S \) is a Sylow \( p \)-subgroup of \( G \). Let \( \mathfrak{X} = \{ S \cap S^x | x \in G \setminus N_G(S) \} \). This is the collection of nontrivial Sylow intersections. A theorem of J. A. Green says that the defect group of any block is a Sylow intersection. Consequently, the defect group of any block is either equal to \( S \) or is in \( \mathfrak{X} \). This information allows the following observation.
Theorem 9.6. Let $C$ be one of the categories of complexes or homotopy classes of complexes that as in Theorem 8.2 or a derived category as in Theorem 8.4. Assume that $C_G$ is a tensor category. Let $H = N_G(S)$. Then the equivalences

$$
\begin{array}{c}
C_H \\
\xrightarrow{\mathcal{I}} \\
\mathfrak{R} \\
\xrightarrow{=} \\
C_G
\end{array}
$$

are equivalences of tensor triangulated triangles.

Proof. Let $B_1, \ldots, B_t$ be the blocks of $kG$ that have $S$ as defect group. Then notice that

$$
\frac{C_G}{\mathfrak{P} \text{-Proj}(C_G)} = \sum_{i=1}^{t} \frac{C(B_i)}{\mathfrak{P} \text{-Proj}(C(B_i))}
$$

since any module or complex in any block with smaller defect group is $\mathfrak{P}$-projective. Thus by Theorem 9.4, these are triangle equivalences. Hence, the only question here is the tensor structure. These categories have tensor structures because the subcategories being factored out are closed under arbitrary tensor product. That is, for example, if $X^*$ is in $\mathcal{E} - \text{Proj}$ for $\mathcal{E} = T S + V - \text{seq}$, then so is $X^* \otimes Y^*$ for any appropriate $Y^*$. So finally, the proof is the observation that the restriction map commutes with tensor products. □

Remark 9.7. As we previously noticed, there is no tensor product of arbitrary objects in $\text{cpx}(kG)$. However, one should still be able to use the tensor structure for the category $\text{cpx}(kG)$ in some constructive way.

Remark 9.8. It might be tempting to use the above result to accomplish something such as classifying thick subcategories or localizing subcategories. However, such structures may be very complicated and it is likely that the Balmer spectrum of thick subcategories is not Noetherian. See [8].

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