Finsler black holes induced by noncommutative anholonomic distributions in Einstein gravity

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Abstract
We study Finsler black holes induced from Einstein gravity as possible effects of quantum spacetime noncommutativity. Such Finsler models are defined by nonholonomic frames not on tangent bundles but on (pseudo)Riemannian manifolds being compatible with standard theories of physics. We focus on noncommutative deformations of Schwarzschild metrics into locally anisotropic stationary ones with spherical/rotoid symmetry. The conditions are derived when black hole configurations can be extracted from two classes of exact solutions depending on noncommutative parameters. The first class of metrics is defined by nonholonomic deformations of the gravitational vacuum by noncommutative geometry. The second class of such solutions is induced by noncommutative matter fields and/or effective polarizations of cosmological constants.

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1. Introduction

The study of noncommutative black holes is an active topic in both gravity physics and modern geometry; see [1] for a recent review of results. Noncommutative geometry, quantum gravity and string/brane theory appear to be connected strongly in low energy limits. We can model physical effects in such theories using deformations on noncommutative parameters of some classes of exact and physically important solutions in general relativity.

Different approaches to quantum field theory (including gauge and gravity models) on noncommutative spaces were elaborated using, for instance, the simplest example of a Moyal–Weyl spacetime, with and without Seiberg–Witten maps and various applications in cosmology and black hole physics (see [2–5] and references therein). Our constructions are based on the nonlinear connection formalism and Finsler geometry methods in commutative and noncommutative geometry [6, 7]. They were applied to generalized Seiberg–Witten
theories derived for the Einstein gravity equivalently reformulated (at classical level, using nonholonomic constraints) and/or generalized as certain models of Poincaré de Sitter gauge gravity [8]; see also extensions to noncommutative (super) gravity/string gravity theories and [9]. Here we note that different models of noncommutative gauge gravity theories [10–16] were also elaborated based on generalizations of some commutative/complex/nonsymmetric geometries. Our approach was oriented to unify the constructions on commutative and noncommutative gravity theories in the language of geometry of nonholonomic manifolds/bundle spaces1.

In [17], following the so-called anholonomic deformation method (see recent reviews [18, 19]), we provided the first examples of black hole/ellipsoid/toroidal solutions in noncommutative and/or nonholonomic variables in Einstein gravity and gauge and string gravity generalizations2. The bulk of metrics for noncommutative black holes reviewed in [1] can be included as certain holonomic (non)commutative configurations of nonholonomic solutions. This provides additional arguments that a series of important physical effects for noncommutative black holes can be derived by using nonholonomic and/or noncommutative deformations of well-known solutions in general relativity.

In this paper, we study two classes of Finsler-type black hole solutions, with zero and non-zero matter field sources/cosmological constant, induced by noncommutative anholonomic variables3 in Einstein gravity. Especially, we wish to point out that such nonholonomic configurations may ‘survive’ even in the classical (commutative) limits and that Finsler-type variables can be considered both in noncommutative gravity (defining complex nonholonomic distributions) and in Einstein gravity (stating some classes of real nonholonomic distributions).

Noncommutative relations on coordinates positively result in generic off-diagonal metrics4. The possibility of formulating a geometric method of systematical derivation of exact solutions for noncommutative deformations of general relativity brings a number of new physical insights. For instance, we can generate new classes of noncommutative Finsler-like black hole solutions which are more general than the well-known Kerr solutions (when the off-diagonal terms of metric can be modeled by rotation frames/coordinates) and depend on noncommutative parameters. Such black hole objects can be considered for models of Finsler gravity on tangent bundles (with metrics and connections depending on ‘velocities’), as nonholonomic configurations in Einstein gravity and generalizations and/or extended to complex distributions in noncommutative gravity. This allows us to examine important features concerning deformations of horizons and topologies of locally anisotropic and/or noncommutative black holes, their stability, phase structure and transitions, singularities and symmetries, quantum corrections, self-consistent imbedding into nontrivial solitonic backgrounds, etc.

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1 In modern geometry and applications to physics and mechanics, equivalent terms like anholonomic and nonintegrable manifolds are also used; for our purposes, it is convenient to use all such terms. A pair \((V, N)\), where \(V\) is a manifold and \(N\) is a nonintegrable distribution on \(V\), is called a nonholonomic manifold.

2 The anholonomic deformation method should not be confused with the Cartan’s moving frame method; even in the first case ‘moving frames’ can also be included. In our approach, we consider arbitrary real/complex, in general, noncommutative/supersymmetric nonholonomic distributions on certain manifolds and adapt the geometric constructions with respect to such distributions. This results in (nonlinear) deformations of connection and metric structures, which is not the case for moving frames, when geometric objects are re-defined with respect to moving/different systems of reference. Selecting some convenient nonholonomic distributions, we obtain separations of equations and reparametrizations of variables in some physically important nonlinear systems of partial differential equations which allows us to integrate such systems in general forms. Then constraining correspondingly certain general solutions, we select some subclasses of exact solutions, for instance, in general relativity.

3 Let us say to be defined by certain quantum corrections in quasi-classical limits of quantum gravity models.

4 Such metrics cannot be diagonalized by coordinate transforms, see below formulas (1) and (10).
In this context, a systematical search of possible solutions in noncommutative generalizations of Einstein gravity is of great significance. Using the fact that any type of noncommutative coordinate relations, and other structures like the star product, noncommutative symmetries, etc, can be considered as certain complex distributions on a (pseudo)Riemannian manifold, or vector/tangent bundle, we can apply the formalism of nonlinear connections and adapted geometric constructions originally developed in Finsler and Lagrange geometry. The resulting anholonomic deformation method provides us not only a new technique for finding solutions in certain gravity theories but also a promising unified geometric scheme to (in general, nonholonomic) Ricci flow theory [7, 20–22], deformation and A-brane quantization of gravity [23–26] and possible applications in modern particle physics.

The content of this work is as follows. In section 2 we outline the geometry of complex nonholonomic distributions defining noncommutative gravity models. Section 3 is devoted to a generalization of the anholonomic frame method for constructing exact solutions with a noncommutative parameter. The conditions are formulated when such solutions define effective off-diagonal metrics in Einstein gravity. We analyze noncommutative nonholonomic deformations of Schwarzschild spacetimes in section 4 (being considered vacuum configurations, with nontrivial matter sources and with noncommutative ellipsoidal symmetries). In section 5 we provide a procedure of extracting black hole and rotoid configurations for small noncommutative parameters. We show how (non)commutative gravity models can be described using Finsler variables. Finally, in section 6 the conclusions of this work are formulated.

2. Complex nonholonomic distributions and noncommutative gravity models

There exist many formulations of noncommutative geometry/gravity based on nonlocal deformation of spacetime and field theories starting from noncommutative relations of type

\[ u^\alpha u^\beta - u^\beta u^\alpha = i\theta^{\alpha\beta}, \]

where \( u^\alpha \) are local spacetime coordinates, \( i \) is the imaginary unity, \( i^2 = -1 \) and \( \theta^{\alpha\beta} \) is an antisymmetric second-rank tensor (which, for simplicity, for certain models, is taken to be with constant coefficients). Following our unified approach to (pseudo)Riemannian and Finsler–Lagrange spaces [6, 17, 19] (using the geometry of nonholonomic manifolds), we consider that for \( \theta^{\alpha\beta} \to 0 \) the local coordinates \( u^\alpha \) are on a four-dimensional (4D) nonholonomic manifold \( \mathcal{V} \) of a necessary smooth class. Such spacetimes can be enabled with a conventional \( 2 + 2 \) splitting (defined by a nonholonomic, equivalently, anholonomic/non-integrable real distribution), when local coordinates \( u = (x, y) \) on an open region \( U \subset \mathcal{V} \) are labeled in the form \( u^\alpha = (x^i, y^a) \), with indices of type \( i, j, k, \ldots = 1, 2 \) and \( a, b, c, \ldots = 3, 4 \). The coefficients of tensor-like objects on \( \mathcal{V} \) can be computed with respect to a general (non-coordinate) local basis \( e_a = (e_i, e_a) \).

For our purposes, we consider a subclass of nonholonomic manifolds \( \mathcal{V} \), called \( N \)-anholonomic spaces (spacetimes, for corresponding signatures), enabled with a nonintegrable distribution stating a conventional horizontal (\( h \)) space, (\( h\mathcal{V} \)), and vertical (\( v \)) space, (\( v\mathcal{V} \)):

\[ T\mathcal{V} = h\mathcal{V} \oplus v\mathcal{V}, \]

if \( \mathcal{V} = TM \) is the total space of a tangent bundle \( (TM, \pi, M) \) on a two-dimensional (2D) base manifold \( M \), the values \( x^i \) and \( y^a \) are respectively the base coordinates (on a low-dimensional space/spacetime) and fiber coordinates (velocity like). Alternatively, we can consider that \( \mathcal{V} = V \) is a 4D nonholonomic manifold (in particular, a pseudo-Riemannian one) with a local fibered structure.

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\[ 5 \text{ If } \mathcal{V} = TM \text{ is the total space of a tangent bundle } (TM, \pi, M) \text{ on a two-dimensional (2D) base manifold } M, \text{ the values } x^i \text{ and } y^a \text{ are respectively the base coordinates (on a low-dimensional space/spacetime) and fiber coordinates (velocity like). Alternatively, we can consider that } \mathcal{V} = V \text{ is a 4D nonholonomic manifold (in particular, a pseudo-Riemannian one) with a local fibered structure.} \]
which by definition determines a nonlinear connection (N-connection) structure \( N = N^\mu_i(x) dx^i \otimes dy^\mu \), see details in \([6, 17, 19, 29, 30]\). On a commutative \( V \), any (prime) metric \( g = \sum_{\alpha} \epsilon_{\alpha}^{\alpha} \otimes \epsilon_{\alpha}^{\alpha} \) (a Schwarzschild, ellipsoid, ring or other type solution, their conformal transforms and nonholonomic deformations which, in general, are not solutions of the Einstein equations) can be parametrized in the form

\[
g = g_i(u)dx^i \otimes dx^i + h_i(u)dx^i \otimes e^i, \tag{3}
\]

\[
e^i = e^\alpha_i(u)du^\alpha = \left( e^1 = dx^1, e^\alpha = dy^\alpha + N^\alpha_i dx^i \right). \tag{4}
\]

The nonholonomic frame structure is characterized by relations

\[
[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = w_\alpha^\beta e_\beta, \tag{5}
\]

where

\[
e_\alpha = e^\alpha_i(u)\partial/\partial u^\alpha = (e^1 = \partial/\partial x^1 - N^\alpha_i \partial/\partial y^\alpha, e^\alpha = \partial/\partial y^\alpha) \tag{6}
\]

are dual to (4). The nontrivial anholonomy coefficients are determined by the N-connection coefficients \( N = \{ N^\mu_i \} \) following formulas \( w^\mu_i = \partial_i N^\mu_i \) and \( w^\mu_i = \Omega^\mu_i \), where

\[
\Omega^\mu_i = e^i \left( N^\alpha_i \right) - e^i \left( N^\alpha_i \right) \tag{7}
\]

define the coefficients of N-connection curvature\(^6\).

On an \( N \)-anholonomic manifold, it is convenient to work with the so-called canonical distinguished connection (in brief, canonical d-connection \( \tilde{\Gamma} = \{ \tilde{\Gamma}^\gamma_{\alpha \beta} \} \)) which is metric compatible, \( \tilde{\nabla}g = 0 \), and completely defined by the coefficients of a metric \( g \) (3) and an \( N \)-connection \( N \), subjected to the condition that the so-called \( h \) and \( v \)-components of torsion are zero\(^7\). Using the deformation of linear connections’ formula \( \Gamma^\gamma_{\alpha \beta} = \tilde{\Gamma}^\gamma_{\alpha \beta} + Z^\gamma_{\alpha \beta} \), where \( V = \{ V^\gamma_{\alpha \beta} \} \) is the Levi-Civita connection (this connection is metric compatible, torsionless and completely defined by the coefficients of the same metric structure \( g \)), we can perform all geometric constructions in two equivalent forms: applying the covariant derivative \( \tilde{\nabla} \) and/or \( V \). This is possible because all values \( \Gamma, \tilde{\Gamma} \) and \( Z \) are completely determined in unique forms by \( g \) for a prescribed nonholonomic splitting (see details and coefficient formulas in \([6, 17, 19, 27]\)).

Any class of noncommutative relations (1) on an \( N \)-anholonomic spacetime \( V \) defines additionally a complex distribution and transforms this space into a complex nonholonomic manifold \( 4^V \).\(^8\) We shall follow the approach to noncommutative geometry based on the Groenewold-Moyal product (star product or \( * \)-product) \([31, 32]\) inspired by the foundations of quantum mechanics \([33, 34]\). For the Einstein gravity and its equivalent lifts on de Sitter/affine bundles and various types of noncommutative Lagrange–Finsler geometries, we defined star products adapted to N-connection structures \([8, 9, 17]\); see also \([7]\) and part III in \([6]\) on alternative approaches with nonholonomic Dirac operators and Ricci flows of noncommutative geometries. In general, such constructions are related to deformations of noncommutative structures and/or \( h \)- and \( v \)-components of torsion.

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\(^{6}\) We use boldface symbols for spaces (and geometric objects on such spaces) enabled with the N-connection structure. Here we note that the particular holonomic/integrable case is selected by the integrability conditions \( w^\mu_i = 0 \).

\(^{7}\) By definition, a d-connection is a linear connection preserving under parallelism a given N-connection splitting (2); in general, a d-connection has a nontrivial torsion tensor but for the canonical d-connection the torsion is induced by the anholonomy coefficients which in their turn are defined by certain off-diagonal N-coefficients in the corresponding metric.

\(^{8}\) Here we note that a noncommutative distribution of type (1) mixes the \( h \)- and \( v \)-components, for instance, of coordinates \( x^i \) and \( y^\mu \). Nevertheless, it is possible to redefine the constructions in a language of projective modules with certain conventional irreversible splitting of type \( T^V = h^V \oplus v^V \), see details in \([7]\) and part III in \([6]\). Here we also note that we shall use the label \( \theta \) both for tensor-like values \( \theta_{\alpha \beta} \) and a set of parameters, for instance, \( \theta_{\delta \alpha} \).
the commutative algebra of bounded (complex valued) continuous functions $C(V)$ on $V$ into a noncommutative algebra $\mathcal{A}(V)$. We considered different constructions of $\mathcal{A}$ corresponding to different choices of the so-called symbols of operators, see details and references in [2, 3, 33, 34], and the extended Weyl ordered symbol $\mathcal{W}$, to get an algebra isomorphism with properties

$$\mathcal{W}[1 f \star 2 f] = \mathcal{W}[1 f] \mathcal{W}[2 f] = \hat{f} \star \hat{f},$$

for $f, 2 f \in \mathcal{C}(V)$ and $\hat{f}, \hat{2 f} \in \mathcal{A}(V)$, when the induced $\star$-product is associative and noncommutative. Such a product can be introduced on nonholonomic manifolds [8, 9, 17] using the $N$-elongated partial derivatives (6),

$$\hat{f} \star \hat{2 f} = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{\imath}{2} \right)^k \theta^{\alpha_1 \beta_1} \cdots \theta^{\alpha_k \beta_k} e_{\alpha_1} \cdots e_{\alpha_k} \hat{1} f(u) e_{\beta_1} \cdots e_{\beta_k} \hat{2 f}(u).$$

For nonholonomic configurations, we have two types of ‘noncommutativity’ given by relations (1) and (5).

For a noncommutative nonholonomic spacetime model $\mathcal{V}$ of a spacetime $V$, we can derive an $N$-adapted local frame structure $\theta e_a = (\theta e_a, \theta e_{\alpha a})$ which can be constructed by noncommutative deformations of $e_a$,

$$\theta e_\alpha^a = e_\alpha^a + i \theta^{\alpha_1 \beta_1} e_\alpha^{\alpha_1} \beta_1 + \theta^{\alpha_1 \beta_1} \theta^{\alpha_2 \beta_2} e_\alpha^{\alpha_1 \beta_1 \alpha_2 \beta_2} + O(\theta^3),$$

$$\theta e_\alpha^a = e_\alpha^a + i \theta^{\alpha_1 \beta_1} e_\alpha^{\alpha_1} \beta_1 + \theta^{\alpha_1 \beta_1} \theta^{\alpha_2 \beta_2} e_\alpha^{\alpha_1 \beta_1 \alpha_2 \beta_2} + O(\theta^3),$$

subjected to the condition $\theta e_\alpha^a \star \theta e_\beta^a = \delta_\beta^a$ for $\delta_\beta^a$ being the Kronecker tensor, where $e_\alpha^{\alpha_1 \beta_1}$ and $e_\alpha^{\alpha_1 \beta_1 \alpha_2 \beta_2}$ can be written in terms of $e_\beta^a$, $\theta^{\alpha_1 \beta_1}$ and the spin-distinguished connection corresponding to $D$. Such formulas were introduced for noncommutative deformations of the Einstein and de Sitter/Poincaré-like gauge gravity [8, 9] and complex gauge gravity [11] and then generalized for noncommutative nonholonomic configurations in string/brane and generalized Finsler theories in part III in [6, 7, 17] (we note that we can also consider alternative expansions in ‘non’ $N$-adapted form working with the spin connection corresponding to the Levi-Civita connection).

The noncommutative deformation of a metric (3), $g \rightarrow \theta g$, can be defined in the form

$$\theta g_{\alpha\beta} = \frac{1}{2} \eta_{\alpha\beta} \left[ e_\alpha^a \star \left( \theta e_\beta^a \right)^* + \theta e_\alpha^a \star \left( \theta e_\beta^a \right)^* \right].$$

where $(\cdots)^*$ denotes Hermitian conjugation and $\eta_{\alpha\beta}$ is the flat Minkowski space metric. In the $N$-adapted form, as nonholonomic deformations, such metrics were used for constructing exact solutions in string/gauge/Einstein and Lagrange–Finsler metric-affine and noncommutative gravity theories in [6, 17]. In explicit form, formula (10) was introduced in [5] for decompositions of type (9) performed for the spin connection corresponding to the Levi-Civita connection. In our approach, the ‘boldface’ formulas allow us to extend the formalism to various types of commutative and noncommutative nonholonomic and generalized Finsler spaces and to also compute noncommutative deforms of $N$-connection coefficients.

The target metrics resulting after noncommutative nonholonomic transforms (to be investigated in this work) can be parametrized in the general form:

$$\theta g = \theta g_i(u, \theta) dx^i \otimes dx^i + \theta h_a(u, \theta) \theta e^a \otimes \theta e^a,$$

$$\theta e^a = \theta e_\alpha^a(u, \theta) du^a = (dx^i, \theta e^a = dx^i + \theta N^a(u, \theta) dx^i),$$

where it is convenient to consider conventional polarizations $\eta_{\alpha\beta}$ when

$$\theta g_i = \check{h}_i(u, \theta) g_i,$$

$$\theta h_a = \check{h}_a(u, \theta) h_a,$$

$$\theta N^a(u, \theta) = \check{h}_a(u, \theta) N^a$$

(12)
for $g_i, h_{ij}, N^a_i$ given by a prime metric (3). How to construct exact solutions of gravitational and matter field equations defined by a very general ansatz of type (11), with coefficients depending on arbitrary parameters $\theta$ and various types of integration functions, in Einstein gravity and (non)commutative string/gauge/Finsler, etc, like generalizations, is considered in [6, 17–19, 27, 28].

In this work, we shall analyze noncommutative deformations induced by (1) for a class of four-dimensional (4D (pseudo)Riemannian) metrics (or 2D (pseudo) Finsler metrics) defining (non)commutative Finsler–Einstein spaces as exact solutions of the Einstein equations:

$$\theta^\alpha\theta^\beta = \delta^\alpha_\beta, \quad \theta^\alpha_\beta, \quad \theta^\alpha_\beta = \delta^\alpha_\beta = 0,$$

where $\delta^\alpha_\beta$ are components of the unit tensor computed for the canonical distinguished connection (d-connection) $\theta\theta$, see details in [6, 18, 19, 27] and, on Finsler models on tangent bundles, [29, 30]. Functions $\theta^\alpha_\beta$ and $\theta^\alpha_\beta$ are considered to be defined by certain matter fields in a corresponding model of (non)commutative gravity. The geometric objects in (13) must be computed using the $\theta\theta$-product (8) and the coefficients contain in general the complex unity $i$. Nevertheless, it is possible to prescribe such nonholonomic distributions on the ‘prime’ $\theta\theta$ when, for instance,

$$\theta^\alpha\theta^\beta = \delta^\alpha_\beta, \quad \theta^\alpha_\beta, \quad \theta^\alpha_\beta = \delta^\alpha_\beta = 0,$$

and we get generalized Lagrange–Finsler and/or (pseudo) Riemannian geometries, and corresponding gravitational models, with parametric dependences of geometric objects on $\theta$.

Solutions of nonholonomic equations (13) are typical ones for the Finsler gravity with metric compatible d-connections$^9$ or in the so-called Einsteing/stringbrane/gauge gravity with nonholonomic/Finsler-like variables. In the standard approach to the Einstein gravity, when $D \rightarrow \theta\theta$, the Einstein spaces are defined by metrics $g$ as solutions of the equations

$$\theta^\alpha_\beta \theta\theta = \theta^\alpha_\beta,$$

where $\hat{E}^a_i$ is the Einstein tensor for $\theta\theta$ and $\theta^\alpha_\beta$ is proportional to the energy–momentum tensor of matter in general relativity. Of course, for noncommutative gravity models in (14), we must consider values of type $\theta^\alpha_\beta, \theta^\alpha_\beta, \theta^\alpha_\beta, \theta^\alpha_\beta, \theta^\alpha_\beta$ that are solutions of (13) transformed (u, $\theta$) are solutions of (13) transformed into a system of partial differential equations (with parametric dependence of coefficients on $\theta$) which after certain further restrictions on coefficients determining the nonholonomic distribution can result in generic off–diagonal solutions for general relativity$^{10}$.

$^9$ We emphasize that Finsler-like coordinates can be considered on any (pseudo), or complex Riemannian manifold and various types of complex Finsler/Riemannian geometries/gravity models.

$^{10}$ The metrics for such spacetimes cannot be diagonalized by coordinate transforms.
3. General solutions with noncommutative parameters

A noncommutative deformation of coordinates of type (1) defined by \( \theta \) together with correspondingly stated nonholonomic distributions on \( \tilde{\mathbf{V}} \) transform prime metrics \( g \) (for instance, a Schwarzschild solution on \( \mathbf{V} \)) into respective classes of target metrics \( \tilde{g} = \tilde{\mathbf{g}} \) as solutions of Finsler-type gravitational field equations (13) and/or standard Einstein equations (14) in general gravity. The goal of this section is to show how such solutions and their noncommutative/nonholonomic transforms can be constructed in the general form for vacuum and non-vacuum locally anisotropic configurations.

We parametrize the noncommutative and nonholonomic transform of a metric \( g \) into \( \tilde{g} \) resulting from formulas (9) and (10) and expressing of polarizations in (12), as

\[
\begin{align*}
\tilde{g}_{i} &= g_{i}(u, \theta) + \tilde{g}_{i}(u, \theta, \theta^{2}) + \mathcal{O}(\theta^{4}), \\
\tilde{h}_{a} &= h_{a}(u, \theta) + \tilde{h}_{a}(u, \theta, \theta^{2}) + \mathcal{O}(\theta^{4}), \\
\tilde{N}_{i}^{a} &= N_{i}^{a}(u, \theta), \\
\end{align*}
\]

where \( \tilde{g}_{i} = g_{i} \) and \( \tilde{h}_{a} = h_{a} \) for \( \tilde{h}_{a} = 1 \), but for general \( \tilde{h}_{a}(u) \) we get nonholonomic deformations which do not depend on \( \theta \).

3.1. Nonholonomic Einstein equations depending on a noncommutative parameter

The gravitational field equations (13) for a metric (11) with coefficients (15) and sources of type \( \Theta_{\beta}^{i} = [\Theta_{1} = \Theta_{2}(x^i, v, \theta), \Theta_{2}^{2} = \Theta_{2}(x^i, v, \theta), \Theta_{3}^{3} = \Theta_{4}(x^i, \theta), \Theta_{2}^{4} = \Theta_{4}(x^i, \theta) ] \) (16)

transform into this system of partial differential equations:

\[
\begin{align*}
\tilde{R}_{i}^{1} &= \frac{1}{2\tilde{g}_{1}^{1}\tilde{g}_{2}^{2}} \times \left[ \tilde{g}_{1}^{1}\tilde{g}_{2}^{2} - \tilde{g}_{2}^{1}\tilde{g}_{1}^{2} \right] = -\Theta_{4}(x^i, \theta), \\
\tilde{R}_{i}^{3} &= -\tilde{w}_{i}^{*} - \frac{\alpha_{i}}{\tilde{h}_{4}^{*}} = 0, \\
\tilde{R}_{i}^{4} &= -\frac{\tilde{h}_{3}^{*}}{2\tilde{h}_{4}^{*}} \left[ \tilde{a}_{i}^{*} + \gamma \tilde{a}_{i}^{*} \right] = 0,
\end{align*}
\]

where, for \( \tilde{a}_{i}^{*}, \tilde{a}_{j}^{*} \neq 0 \),

\[
\begin{align*}
\alpha_{i} &= \frac{\tilde{g}_{i}^{*}}{\tilde{h}_{3}^{*}} \partial \phi, \\
\beta &= \frac{\tilde{g}_{i}^{*}}{\til{g}_{4}^{*}} \phi^{*}, \\
\gamma &= \frac{3\til{h}_{3}^{*}}{2\til{h}_{4}^{*}} - \frac{\til{h}_{4}^{*}}{\til{h}_{3}^{*}}, \\
\phi &= \ln |\til{h}_{3}^{*}/\til{h}_{4}^{*}|, \\
\end{align*}
\]

when the necessary partial derivatives are written in the form \( \til{a}^{*} = \partial a/\partial x^{i}, \til{a}' = \partial a/\partial x^{2}, \)

\( \til{a}^{*} = \partial a/\partial v \). In the vacuum case, we must consider \( \tilde{\gamma}_{2,4} = 0 \). Various classes of

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11 See similar details on computing the Ricci tensor coefficients \( \tilde{R}_{ij}^{m} \) for the canonical d-connection \( \tilde{\mathbf{D}} \) in parts II and III of [6] and reviews [18, 19], revising those formulas for the case when the geometric objects depend on noncommutative parameter \( \theta \).
(non)holonomic Einstein, Finsler–Einstein and generalized spaces can be generated if the sources (16) are taken \( \Upsilon_2,4 = \lambda \), where \( \lambda \) is a nonzero cosmological constant; see examples of such solutions in [6, 17–19, 27, 28].

3.2. Exact solutions for the canonical d-connection

Let us express the coefficients of a target metric (11), and respective polarizations (12), in the form

\[
\begin{align*}
\theta g_k &= \epsilon_k \epsilon \psi(x^i, \theta), \\
\theta h_3 &= \epsilon_3 h_0^2(x^i, \theta) f^*(x^i, v, \theta) \overline{\varphi}^2 |\psi(x^i, v, \theta)|, \\
\theta h_4 &= \epsilon_4 [f(x^i, v, \theta) - f_0(x^i, \theta)]^2, \\
\theta N^k_{\alpha} &= w_k(x^i, v, \theta), \quad \theta N^k_{\beta} = n_k(x^i, v, \theta),
\end{align*}
\]

(22)

with arbitrary constants \( \epsilon_\alpha = \pm 1 \), and \( h^*_{3} \neq 0 \) and \( h^*_4 \neq 0 \), when \( f^* = 0 \). By straightforward verification, or following methods outlined in [6, 17–19], we can prove that any off-diagonal metric

\[
\theta g = \epsilon^\delta \left[ \epsilon_1 \overline{\varphi} dx^1 \otimes dx^1 + \epsilon_2 dx^2 \otimes dx^2 \right] + \epsilon_3 h_0^2 f^*(x^i, v, \theta) \delta v \otimes \delta v + \epsilon_4 (f - f_0) \delta y^4 \otimes \delta y^4,
\]

\[
\delta v = dv + w_3(x^i, v, \theta) dx^3, \quad \delta y^4 = dy^4 + n_k(x^i, v, \theta) dx^k
\]

(23)

defines an exact solution of the system of partial differential equations (17)–(20), i.e. of the Einstein equation for the canonical d-connection (13) for a metric of type (11) with the coefficients of form (22), if the following conditions are satisfied:

1. function \( \psi \) is a solution of equation \( \epsilon_1 \psi^{**} + \epsilon_2 \psi' = \Upsilon_2 \);
2. the value \( \varsigma \) is computed following formula

\[
\varsigma(x^i, v, \theta) = \varsigma_0(x^i, \theta) - \frac{\epsilon_3}{8} h_0^2(x^i, \theta) \int \Upsilon_2 f^* (f - f_0) dv
\]

and taken \( \varsigma = 1 \) for \( \Upsilon_2 = 0 \);
3. for a given source \( \Upsilon_2 \), the N-connection coefficients are computed following the formulas

\[
\begin{align*}
w_i(x^k, v, \theta) &= - \partial_i \varsigma / \varsigma^*, \\
n_k(x^i, v, \theta) &= 1 n_k(x^i, \theta) + 2 n_k(x^i, \theta) \int \frac{[f^*]^2 \varsigma dv}{(f - f_0)^3}.
\end{align*}
\]

(24)

(25)

and \( w_i(x^k, v, \theta) \) are arbitrary functions if \( \varsigma = 1 \) for \( \Upsilon_2 = 0 \).

It should be emphasized that such solutions depend on arbitrary nontrivial functions \( f \) (with \( f^* \neq 0 \), \( f_0, h_0, \varsigma_0 \), \( 1 n_k \) and \( 2 n_k \), and sources \( \Upsilon_2 \) and \( \Upsilon_4 \). Such values for the corresponding quasi-classical limits of solutions to metrics of signatures \( \epsilon_\alpha = \pm 1 \) have to be defined by certain boundary conditions and physical considerations.

The ansatz of type (11) for coefficients (22) with \( h_3^* = 0 \) but \( h_4^* \neq 0 \) (or, inversely, \( h_3^* \neq 0 \) but \( h_4^* = 0 \)) consists more special cases and request a bit different method of constructing exact solutions, see details in [6].

\[12\] The reason to choose such forms of parametrizations is that they generate very general classes of exact solutions in general relativity and Finsler gravity theories with one Killing vector symmetry, see details in [6, 19]. We proved in [18] that such solutions can be generalized to possess dependences on certain families of real parameters. In this work, we modify the constructions for metrics when such \( \theta \)-parameters are for noncommutative deformations.

\[13\] We put the left symbol \( ' \circledast ' \) in order to emphasize that such a metric is a solution of gravitational field equations.
3.3. Off-diagonal solutions for the Levi-Civita connection

The solutions for the gravitational field equations for the canonical d-connection (which can be used for various models of noncommutative Finsler gravity and generalizations) presented in the previous subsection can be constrained additionally and transformed into solutions of the Einstein equations for the Levi-Civita connection (14), all depending, in general, on the parameter $\theta$. Such classes of metrics are of type

$$
\theta \circ g = e^{\psi(x^i, \theta)} \left[ \epsilon_1 \, dx^1 \otimes dx^1 + \epsilon_2 \, dx^2 \otimes dx^2 \right] + h_3(x^i, v, \theta) \, \delta v \otimes \delta v + h_4(x^i, v, \theta) \, \delta y^4 \otimes \delta y^4,
$$

(26)

with the coefficients restricted to satisfy the conditions

$$
\epsilon_1 \psi_{\bullet \bullet} + \epsilon_2 \psi_{\prime \prime} = \Upsilon_4, \quad h_3^* / h_4 = \Upsilon_2,

w_1' - w_2^* + w_2 w_1^* - w_1 w_2^* = 0, \quad n_1' - n_2^* = 0
$$

(27)

for $w_i = \partial_i \phi / \phi^*$, see (21), for given sources $\Upsilon_4(x^k, \theta)$ and $\Upsilon_2(x^k, v, \theta)$. We note that the second equation in (27) relates two functions $h_3$ and $h_4$ and the third and fourth equations from the mentioned conditions select such nonholonomic configurations when the coefficients of the canonical d-connection and the Levi-Civita connection are the same with respect to $N$-adapted frames (4) and (6), even such connections (and corresponding derived Ricci and Riemannian curvature tensors) are different by definition.

Even the ansatz (26) depends on three coordinates $(x^k, v)$ and noncommutative parameter $\theta$; it allows us to construct more general classes of solutions with dependence on four coordinates if such metrics can be related by chains of nonholonomic transforms.

4. Noncommutative nonholonomic deformations of Schwarzschild metrics

Solutions of type (23) and/or (26) are very general ones induced by noncommutative nonholonomic distributions and it is not clear what type of physical interpretation can be associated with such metrics. In this section, we analyze certain classes of nonholonomic constraints which allow us to construct black hole solutions and noncommutative corrections to such solutions.

The goal of this subsection is to formulate the conditions when spherical symmetric noncommutative (Schwarzschild type) configurations can be extracted.

4.1. Vacuum noncommutative nonholonomic configurations

In the simplest case, we analyze a class of holonomic noncommutative deformations, with $\Theta_i^j = 0$, of the Schwarzschild metric

$$
\text{Sch}_1 = |g_1| dr \otimes dr + |g_2| d\theta \otimes d\theta + |h_3| d\varphi \otimes d\varphi + |h_4| dt \otimes dt,
$$

$$
|g_1| = -\left(1 - \frac{\alpha}{r}\right)^{-1}, \quad |g_2| = -r^2, \quad |h_3| = -r^2 \sin^2 \theta, \quad |h_4| = 1 - \frac{\alpha}{r},
$$

written in spherical coordinates $u^\alpha = (x^1 = \xi, x^2 = \vartheta, x^3 = \varphi, x^4 = t)$ for $\alpha = 2G\mu_0/c^2$, correspondingly defined by the Newton constant $G$, a point mass $\mu_0$ and the light speed $c$.

---

14 Computed in [35].
Taking
\[ \dot{g}_1 = |g_1|, \dot{h}_a = |h_a|, \]
\[ \dot{g}_1 = -\frac{\alpha(4\gamma^2 - 3\gamma)}{16\gamma^2(r - \gamma)^2}, \quad \dot{g}_2 = -\frac{2\gamma^2 - 17\alpha(r - \gamma)}{32\gamma^2(r - \gamma)^2}, \]
\[ \dot{h}_3 = -\frac{(r^2 + \alpha\gamma^2 - \gamma^3) \sin \vartheta - \gamma(2\gamma - \gamma^2)\sin \vartheta}{16\gamma^2(r - \gamma)^2}, \quad \dot{h}_4 = -\frac{\alpha(8\gamma - 11\gamma)}{16\gamma^4} \] (28)
for
\[ \dot{g}_1 = |\dot{g}_1| + |\dot{g}_2|\gamma^2 + \mathcal{O}(\gamma^4), \quad \dot{h}_a = |\dot{h}_a| + |\dot{h}_3|\gamma^2 + \mathcal{O}(\gamma^4), \]
we get a ‘degenerated’ case of solutions (23), see details in [6, 17–19], because \( \partial h_a/\partial \varphi = 0 \) which is related to the case of holonomic/integrable off-diagonal metrics. For such metrics, the deformations (28) are just those presented in [1, 5, 35].

A more general class of noncommutative deformations of the Schwarzschild metric can be generated by the nonholonomic transform of type (12) when the metric coefficients polarizations, \( \bar{h}_a \), and N-connection coefficients, \( \bar{N}_a \), for
\[ \parallel g_1 = \bar{h}_1(r, \vartheta, \gamma) |g_1|, \quad \parallel h_a = \bar{h}_a(r, \vartheta, \gamma) |h_a|, \]
\[ \parallel N_3 = w_3(r, \vartheta, \gamma), \quad \quad \parallel N_4 = n_4(r, \vartheta, \gamma), \]
are constrained to define a metric (23) for \( \gamma_4 = \gamma_2 = 0 \). The coefficients of such metrics, computed with respect to N-adapted frames (4) defined by \( \parallel N_a \), can be re-parametrized in the form
\[ \parallel g_k = \epsilon_k e^{\psi(r, \vartheta, \gamma)} |g_k| + \delta |g_k| + \varepsilon |g_k| + \Omega_\gamma^2(r, \vartheta, \gamma), \]
\[ \parallel h_3 = \epsilon_3 h_3^0[f^*(r, \vartheta, \gamma) + \Omega_\gamma^2(r, \vartheta, \gamma) - f_0(r, \vartheta, \gamma)]^2 + \Omega_\gamma^2(r, \vartheta, \gamma), \quad \gamma_0 = \text{const} \neq 0; \]
for the nonholonomic deformations \( \delta |g_k|, \delta |h_a|, \delta |h_a| \) are for correspondingly given generating functions \( \psi(r, \vartheta, \gamma) \) and \( f(r, \vartheta, \gamma) \) expressed as series on \( \gamma^k \) for \( k = 1, 2, \ldots \).

Such coefficients define noncommutative Finsler-type spacetimes being solutions of the Einstein equations for the canonical d-connection. They are determined by the (prime) Schwarzschild data \( |g_1| \) and \( |h_a| \) and certain classes on noncommutative nonholonomic distributions defining off-diagonal gravitational interactions. In order to get solutions for the Levi-Civita connection, we have to constrain (29) additionally in a form to generate metrics of type (26) with coefficients subjected to conditions (27) for zero sources \( \gamma_a \).

### 4.2. Noncommutative deformations with nontrivial sources

In the holonomic case, such noncommutative generalizations of the Schwarzschild metric are known (see, for instance, [36–38] and review [1]) when
\[ \gamma g = \gamma g_1 dr \otimes dr + \gamma g_2 d\vartheta \otimes d\vartheta + \gamma g_3 d\vartheta \otimes d\varphi + \gamma g_4 dr \otimes d\varphi, \]
\[ \gamma g_1 = -\left(1 - \frac{4M_0\gamma}{\sqrt{\pi}r}\right)^{-1}, \quad \gamma g_2 = -r^2, \]
\[ \gamma h_3 = -r^2 \sin^2 \vartheta, \quad \gamma h_4 = 1 - \frac{4M_0\gamma}{\sqrt{\pi}r} \] (30)
for \( \gamma \) being the so-called lower incomplete Gamma function
\[ \gamma \left(3 - \frac{r^2}{4\gamma}\right) = \int_0^{\gamma_0} p^{1/2} e^{-p} dp. \]
is the solution of a noncommutative version of the Einstein equation

\[
\theta E_{\alpha\beta} = \frac{8\pi G}{c^2} \theta T_{\alpha\beta},
\]

where \( \theta E_{\alpha\beta} \) is formally left unchanged (i.e. is for the commutative Levi-Civita connection in commutative coordinates) but

\[
\theta T_{\alpha\beta} = \begin{pmatrix}
-p_1 & -p_\perp & -p_\perp \\
-p_\perp & -p_\perp & \rho_0
\end{pmatrix}
\]

(31)

with \( p_1 = -\rho_0 \) and \( p_\perp = -\rho_0 - \xi \partial_\nu \rho_0(r) \) is taken for a self-gravitating, anisotropic fluid-type matter modeling noncommutativity.

Via nonholonomic deforms, we can generalize the solution (30) to off-diagonal metrics of type

\[
\eta S^\alpha = \begin{pmatrix}
\eta_1 (\rho_0, \theta, \phi, \theta) & \eta_2 (\rho_0, \theta, \phi, \theta) & \eta_3 (\rho_0, \theta, \phi, \theta) & \eta_4 (\rho_0, \theta, \phi, \theta)
\end{pmatrix}
\]

being exact solutions of the Einstein equation for the canonical d-connection (13) with locally anisotropically self-gravitating source

\[
\Upsilon^\alpha = \begin{pmatrix}
\Upsilon_1 (\rho_0, \theta, \phi, \theta) & \Upsilon_2 (\rho_0, \theta, \phi, \theta) & \Upsilon_3 (\rho_0, \theta, \phi, \theta) & \Upsilon_4 (\rho_0, \theta, \phi, \theta)
\end{pmatrix}
\]

Such sources should be taken with certain polarization coefficients when \( \Upsilon \sim \eta T \) is constructed using the matter energy–momentum tensor (31).

The coefficients of metric (32) are computed to satisfy correspondingly the conditions:

1. function \( \psi (r, \theta, \phi) \) is a solution of equation \( \psi^{**} + \psi'' = -\Upsilon_4 \);
2. for a nonzero constant \( h_0^2 \), and given \( \Upsilon_2 \),

\[
\zeta (r, \theta, \phi, \theta) = \zeta_{00} (r, \theta, \phi) + h_0 \int \Upsilon_2 \psi^* [f - f_0] d\psi;
\]

3. the N-connection coefficients are

\[
w_i (r, \theta, \phi, \theta) = -\partial_\nu \zeta / \zeta^*,
\]

\[
n_k (r, \theta, \phi, \theta) = n_k (r, \theta, \phi, \theta) + n_k (r, \theta, \phi, \theta) \int \left[ \psi^* \right]^2 \psi / \left[ f - f_0 \right] d\psi.
\]

The above-presented class of metrics describes nonholonomic deformations of the Schwarzschild metric into (pseudo) Finsler configurations induced by the noncommutative parameter. Subjecting the coefficients of (32) to additional constraints of type (27) with nonzero sources \( \Upsilon_\alpha \), we extract a subclass of solutions for noncommutative gravity with an effective Levi-Civita connection.

4.3. Noncommutative ellipsoidal deformations

In this section, we provide a method of extracting ellipsoidal configurations from a general metric (32) with coefficients constrained to generate solutions on the Einstein equations for the canonical d-connection or Levi-Civita connection.
We consider a diagonal metric depending on the noncommutative parameter $\theta$ (in general, such a metric is not a solution of any gravitational field equations):
\[
\text{\textcolor{red}{\textgreek{g}}} = -\text{\textgreek{d}}\xi \otimes \text{\textgreek{d}}\xi - r^2(\xi) \text{\textgreek{d}}\vartheta \otimes \text{\textgreek{d}}\vartheta - r^2(\xi) \sin^2 \vartheta \text{\textgreek{d}}\varphi \otimes \text{\textgreek{d}}\varphi + \sigma^2(\xi) \text{\textgreek{d}}t \otimes \text{\textgreek{d}}t, \tag{33}
\]
where the local coordinates and nontrivial metric coefficients are parametrized in the form
\[
x^1 = \xi, \quad x^2 = \vartheta, \quad x^3 = \varphi, \quad x^4 = t, \quad \hat{g}_1 = -1, \quad \hat{g}_2 = -r^2(\xi), \quad \hat{h}_3 = -r^2(\xi) \sin^2 \vartheta, \quad \hat{h}_4 = \sigma^2(\xi) \tag{34}
\]
for
\[
\xi = \int \text{\textgreek{d}}t \left| 1 - \frac{2\mu_0}{r} + \frac{\theta}{r^2} \right|^{1/2} \quad \text{and} \quad \sigma^2(r) = 1 - \frac{2\mu_0}{r} + \frac{\theta}{r^2}.
\]
For $\theta = 0$ and variable $\xi(r)$, this metric is just the Schwarzschild solution written in spacetime spherical coordinates $(r, \vartheta, \varphi, t)$.

Target metrics are generated by nonholonomic deformations with $\hat{g}_i = \eta_i \hat{g}_i$ and $\hat{h}_a = \eta_a \hat{h}_a$ and some nontrivial $w_i, n_i$, where $(\hat{g}_i, \hat{h}_a)$ are given by data $(34)$ and parametrized by an ansatz of type $(32)$,
\[
\hat{g} = -\eta_1(\xi, \vartheta, \varphi, \theta) \text{\textgreek{d}}\xi \otimes \text{\textgreek{d}}\xi - \eta_2(\xi, \vartheta, \varphi, \theta) \text{\textgreek{d}}\vartheta \otimes \text{\textgreek{d}}\vartheta - \eta_3(\xi, \vartheta, \varphi, \theta) r^2(\xi) \sin^2 \vartheta \text{\textgreek{d}}\varphi \otimes \text{\textgreek{d}}\varphi + \eta_4(\xi, \vartheta, \varphi, \theta) \sigma^2(\xi) \text{\textgreek{d}}t \otimes \text{\textgreek{d}}t,
\]
\[
\hat{h}_a = b^a = \eta_a(\xi, \vartheta, \varphi, \theta) \sigma^2(\xi) \tag{36}
\]
for $[\eta_3] = (h_0)^2 \left[ \frac{\text{\textgreek{h}}_a}{\text{\textgreek{h}}_1} \left( \sqrt{\eta_{4}} \right) \right]^2$. In these formulas, we have to chose $h_0 = \text{\textgreek{c}}$ in order to satisfy condition $(27)$, where $\eta_4$ can be any function satisfying the condition $\eta_4 \neq 0$. We generate a class of solutions for any function $b(\xi, \vartheta, \varphi, \theta)$ with $b^* \neq 0$.

It is possible to compute the polarizations $\eta_1$ and $\eta_2$, when $\eta_1 = \eta_2 r^2 = \epsilon^{\text{\textgreek{h}}(\xi, \vartheta, \varphi)}$, from $(17)$ with $\bar{\Upsilon}_4 = 0$, i.e. from $\psi^{**} + \psi^{\prime\prime} = 0$.

Putting the above-defined values of coefficients in the ansatz $(35)$, we find a class of exact vacuum solutions of the Einstein equations defining stationary nonholonomic deformations of the Schwarzschild metric
\[
\epsilon^\text{\textgreek{g}} = -\epsilon^{\text{\textgreek{h}}(\xi, \vartheta, \varphi)}(\text{\textgreek{d}}\xi \otimes \text{\textgreek{d}}\xi + \text{\textgreek{d}}\vartheta \otimes \text{\textgreek{d}}\vartheta)
- 4(\sqrt{\eta_4(\xi, \vartheta, \varphi, \theta)})^2 \sigma^2(\xi) \delta \varphi \otimes \delta \varphi
+ \eta_4(\xi, \vartheta, \varphi, \theta) \sigma^2(\xi) \delta t \otimes \delta t,
\]
\[
\delta \varphi = \epsilon^{\text{\textgreek{h}}(\xi, \vartheta, \varphi)} \text{\textgreek{d}}\varphi + \epsilon^{\text{\textgreek{h}}(\xi, \vartheta, \varphi)} \text{\textgreek{d}}\varphi + \epsilon^{\text{\textgreek{h}}(\xi, \vartheta, \varphi)} \text{\textgreek{d}}\varphi,
\]
\[
\delta t = \epsilon^{\text{\textgreek{h}}(\xi, \vartheta, \varphi)} \epsilon^{\text{\textgreek{h}}(\xi, \vartheta, \varphi)} \text{\textgreek{d}}t + \epsilon^{\text{\textgreek{h}}(\xi, \vartheta, \varphi)} \text{\textgreek{d}}t. \tag{37}
\]
The $N$-connection coefficients $w_i$ and $n_i$ in $(37)$ must satisfy the last two conditions from $(27)$ in order to get vacuum metrics in Einstein gravity. Such vacuum solutions are for
nonholonomic deformations of a static black hole metric into (non)holonomic noncommutative Einstein spaces with locally anisotropic backgrounds (on coordinate $\psi$) defined by an arbitrary function $\eta_4(\xi, \vartheta, \varphi, \theta)$ with $\partial_{\varphi} \eta_4 \neq 0$, an arbitrary $\psi(\xi, \vartheta, \theta)$ solving the 2D Laplace equation and certain integration functions $1^i w_i(\xi, \vartheta, \varphi, \theta)$ and $1^i n_i(\xi, \vartheta, \theta)$. The nonholonomic structure of such spaces depends parametrically on noncommutative parameter(s) $\psi$.

In general, the solutions from the target set of metrics (35) or (37), do not define black holes and do not describe obvious physical situations. Nevertheless, they preserve the singular character of the coefficient $\sigma^2(\xi)$ vanishing on the horizon of a Schwarzschild black hole if we take only smooth integration functions for some small noncommutative parameters $\theta$. We can also consider a prescribed physical situation when, for instance, $\eta_4$ mimics 3D, or 2D, solitonic polarizations on the coordinates $\xi, \vartheta, \varphi$ or on $\xi, \varphi$.

5. Extracting black hole and rotoid configurations

From a class of metrics (37) defining nonholonomic noncommutative deformations of the Schwarzschild solution depending on parameter $\theta$, it is possible to select locally anisotropic configurations with possible physical interpretation of gravitational vacuum configurations with spherical and/or rotoid (ellipsoid) symmetry.

5.1. Linear parametric noncommutative polarizations

Let us consider generating functions of type

$$b^2 = q(\xi, \vartheta, \varphi) + \theta s(\xi, \vartheta, \varphi)$$

and, for simplicity, restrict our analysis only with linear decompositions on a small dimensionless parameter $\bar{\theta} \sim \theta$ with $0 < \bar{\theta} \ll 1$. This way, we shall construct off-diagonal exact solutions of the Einstein equations depending on $\bar{\theta}$ which for rotoid configurations can be considered as a small eccentricity. For a value (38), we obtain

$$(b^*)^2 = [\sqrt{|q|}]^2 \left[ 1 + \bar{\theta} \left( \frac{1}{\sqrt{|q|}} \right) \left( \frac{s}{\sqrt{|q|}} \right)^* \right],$$

which allows us to compute the vertical coefficients of d-metric (37) (i.e $h_{3}$ and $h_{4}$ and corresponding polarizations $\eta_{3}$ and $\eta_{4}$) using formulas (36).

One should emphasize that nonholonomic deformations are not obligatory related to noncommutative ones. For instance, in a particular case, we can generate nonholonomic deformations of the Schwarzschild solution not depending on $\bar{\theta}$: we have to put $\bar{\theta} = 0$ in the above formulas and consider $b^2 = q$ and $(b^*)^2 = [\sqrt{|q|}]^2$. Such classes of black hole solutions are analyzed in [27].

Nonholonomic deformations to rotoid configurations can be generated for

$$q = 1 - \frac{2\mu(\xi, \vartheta, \varphi)}{r} \quad \text{and} \quad s = \frac{q_0(r)}{4\mu^2} \sin(\omega_0 \varphi + \varphi_0),$$

with $\mu(\xi, \vartheta, \varphi) = \mu_{0} + \bar{\theta} \mu_{1}(\xi, \vartheta, \varphi)$ (locally anisotropically polarized mass) with certain constants $\mu, \omega_{0}$ and $\varphi_{0}$ and arbitrary functions/polarizations $\mu_{1}(\xi, \vartheta, \varphi)$ and $q_0(r)$ to be determined from some boundary conditions, with $\bar{\theta}$ treated as the eccentricity of an ellipsoid.

---

15 We can summarize on all orders $\tilde{(\bar{\theta})}^{(\bar{\theta})} \ldots$ stating such recurrent formulas for coefficients when obtaining convergent series to some functions depending both on spacetime coordinates and a parameter $\theta$, see a detailed analysis in [18].

16 We can relate $\bar{\theta}$ to an eccentricity because the coefficient $h_{4} = b^2 = \eta_{4}(\xi, \vartheta, \varphi, \theta)$ becomes zero for data (39) if $r_{s} \simeq 2\mu_{0}/[1 + \bar{\theta} \frac{\mu_{0}}{4\mu^2} \sin(\omega_0 \varphi + \varphi_0)]$, which is the ‘parametric’ equation for an ellipse $r_x(\varphi)$ for any fixed values $\frac{\mu_{0}}{4\mu^2}, \omega_{0}, \varphi_{0}$ and $\mu_{0}$.
Such a noncommutative nonholonomic configuration determines a small deformation of the Schwarzschild spherical horizon into an ellipsoidal one (rotoid configuration with eccentricity $\bar{\theta}$).

We provide the general solution for noncommutative ellipsoidal black holes determined by nonholonomic $h$-components of metric and $N$-connection coefficients which ‘survive’ in the limit $\bar{\theta} \to 0$, i.e. such values do not depend on the noncommutative parameter. The dependence on noncommutativity is contained in $v$-components of metric. This class of stationary rotoid-type solutions is parametrized in the form

$$
\text{rot}_{\bar{\theta}} g = -e^{\psi} (d\xi \otimes d\xi + d\vartheta \otimes d\vartheta)
- 4[(\sqrt{|q|})^*]^2 \left[ 1 + \bar{\theta} \frac{1}{(\sqrt{|q|})^*} \left( \frac{s}{\sqrt{|q|}} \right)^* \right] \delta \varphi \otimes \delta \varphi
+ (q + \bar{\theta}s) \delta t \otimes \delta t,
$$

where the $N$-connection coefficients are taken the same as for (40).}

$$
\delta \varphi = d\varphi + w_1 d\xi + w_2 d\vartheta, \quad \delta t = dt + \frac{1}{n_1} d\xi + \frac{1}{n_2} d\vartheta,
$$

with functions $q(\xi, \vartheta, \varphi)$ and $s(\xi, \vartheta, \varphi)$ given by formulas (39) and $N$-connection coefficients $w_i(\xi, \vartheta)$ subjected to conditions

$$
\begin{align*}
& w_1 w_2 \left( \ln \left| \frac{w_1}{w_2} \right| \right)^* = w_2^* - w_1^*, \quad w_i^* \neq 0 \\
\text{or } \quad & w_2^* - w_1^* = 0, \quad w_1^* = 0; \quad 1_{n_1}(\xi, \vartheta) - 1_{n_2}(\xi, \vartheta) = 0
\end{align*}
$$

and $\psi(\xi, \vartheta)$ being any function for which $\psi^{**} + \psi'' = 0$.

For small eccentricities, a metric (40) defines stationary configurations for the so-called black ellipsoid solutions (their stability and properties can be analyzed following the methods elaborated in [17, 39, 40], see also a summary of results and generalizations for various types of locally anisotropic gravity models in [6]). There is a substantial difference between solutions provided in this section and similar black ellipsoid ones constructed in [27]. In this work, such metrics transform into the usual Schwarzschild one if the values $e^\psi, w_i, 1_{n_i}$ have the corresponding limits for $\bar{\theta} \to 0$, i.e. for commutative configurations. For ellipsoidal configurations with generic off-diagonal terms, an eccentricity $\varepsilon$ may be non-trivial because of generic nonholonomic constraints.

### 5.2. Rotoids and noncommutative solitonic distributions

There are static three-dimensional solitonic distributions $\eta(\xi, \vartheta, \varphi, \theta)$, defined as solutions of a solitonic equation

$$
\eta^{**} + \varepsilon (\eta^* + 6\eta \eta^* + \eta^{***})^* = 0, \quad \varepsilon = \pm 1,
$$

resulting in stationary black ellipsoid–solitonic noncommutative spacetimes $^\theta \mathbb{V}$ generated as further deformations of a metric $^\text{rot}_{\bar{\theta}} g$ (40). Such metrics are of type

$$
\text{rot}_{\text{solit}} g = -e^{\psi} (d\xi \otimes d\xi + d\vartheta \otimes d\vartheta)
- 4[(\sqrt{|q|})^*]^2 \left[ 1 + \bar{\theta} \frac{1}{(\sqrt{|q|})^*} \left( \frac{s}{\sqrt{|q|}} \right)^* \right] \delta \varphi \otimes \delta \varphi
+ \eta(q + \bar{\theta}s) \delta t \otimes \delta t,
$$

where the $N$-connection coefficients are taken the same as for (40).

\[17\] A function $\eta$ can be a solution of any three-dimensional solitonic and/or other nonlinear wave equations.
For small values of $\bar{\theta}$, a possible spacetime noncommutativity determines nonholonomic embedding of the Schwarzschild solution into a solitonic vacuum. In the limit of small polarizations, when $|\eta| \sim 1$, the black hole character of metrics is preserved and the solitonic distribution can be considered as on a Schwarzschild background. It is also possible to take such parameters of $\eta$ when a black hole is nonholonomically placed on a ‘gravitational hill’ defined by a soliton induced by spacetime noncommutativity.

A vacuum metric (41) can be generalized for (pseudo) Finsler spaces with canonical $d$-connection as a solution of equations

$$\hat{R}_{\alpha\beta} = 0$$

if the metric is generalized to a subclass of (35) with stationary coefficients subjected to conditions

$$\psi^{\bullet\bullet}(\xi, \vartheta, \bar{\theta}) + \psi^\prime(\xi, \vartheta, \bar{\theta}) = 0;$$

$$h_3 = \pm e^{-2\phi} \left( h_4 \right)^2$$

for given $h_4(\xi, \vartheta, \varphi, \bar{\theta})$, $\phi = \phi = \text{const}$;

$$u_i = u_i(\xi, \vartheta, \varphi, \bar{\theta})$$

are any functions;

$$n_i = n_i(\xi, \vartheta, \varphi, \bar{\theta})$$

are any functions;

$$h_4 = h_4 = \pm e^{-2\phi} \left( h_4 \right)^2$$

for given $h_4(\xi, \vartheta, \varphi, \bar{\theta})$. In the limit $\bar{\theta} \to 0$, we get a Schwarzschild configuration mapped nonholonomically on an $N$-anholonomic (pseudo)Riemannian spacetime with a prescribed nontrivial $N$-connection structure.

The above constructed classes of noncommutative and/or nonholonomic black hole-type solutions (40) and (41) are stationary. It is also possible to generalize such constructions for nonholonomic propagation of black holes in extra dimension and/or as Ricci flows, in our case induced by spacetime noncommutativity is also possible. We have to apply the geometric methods elaborated in [20–22, 41], see also reviews of results, with solutions for the metric-affine gravity, noncommutative generalizations etc, in [6, 19].

5.3. Noncommutative gravity and (pseudo) Finsler variables

In [27], we formulated a procedure of nonholonomic transforms of (pseudo)Finsler metrics into (pseudo)Riemannian ones, and inversely, and further deformations of both types of metrics to exact solutions of the Einstein equations. In this section, we show that such constructions can be performed for nontrivial noncommutative parameters $\theta$ which emphasize that (in general, complex) Finsler geometries can be induced by spacetime noncommutativity. For certain types of nonholonomic distributions, the constructions provide certain models of stationary black hole solutions. Of course, such geometric/physical models are equivalent if they are performed for the same canonical $d$–connection and/or Levi-Civita connection.

We summarize the main steps of such noncommutative complex Finsler–(pseudo) Riemannian transform.

1. Let us consider a solution for (non)holonomic noncommutative generalized Einstein gravity with a metric

$$\theta^\alpha_k = \hat{g}_k dx^i \otimes dx^i + \hat{h}_a(dy^a + \hat{N}_a^j dx^j) \otimes (dy^a + \hat{N}_b^j dx^j)$$

$$= \hat{g}_a^\epsilon \otimes \epsilon^i + \hat{h}_a^\epsilon \otimes \hat{\epsilon}^\epsilon = \hat{g}_a^\epsilon \otimes \epsilon^i + \hat{h}_a^\epsilon \otimes \hat{\epsilon}^\epsilon$$

related to an arbitrary (pseudo)Riemannian metric with transforms of type

$$\theta^\alpha_{a'}^\beta' = \hat{\epsilon}^a_{a'} \hat{\epsilon}^\beta_{\beta'} \theta^\alpha_{\beta'}$$

18 We shall omit the left label $\theta$ in this section if this will not result in ambiguities.
parametrized in the form
\[ \hat{g}_{ij'} = g_{ij'} \hat{e}^i_{j'} \hat{e}^j_{j'} + h_{a'b'} \hat{e}^i_{j'} \hat{e}^j_{j'}, \quad \hat{h}_{aa'} = g_{ij'} \hat{e}^i_{j'} \hat{e}^j_{j'} + h_{a'b'} \hat{e}^i_{j'} \hat{e}^j_{j'}. \]

For \( \hat{e}^i_{j'} = \delta^i_{j'} \), we write (42) as
\[ \hat{g}_{ij'} = g_{ij'} + h_{a'b'} (\hat{e}^i_{j'})^2, \quad \hat{h}_{aa'} = g_{ij'} (\hat{e}^i_{j'})^2 + h_{a'b'}, \]
i.e. in a form of four equations for eight unknown variables \( \hat{e}^i_{j'} \) and \( \hat{e}^i_{a'a'} \) and
\[ \hat{N}^a_{aa'} = \hat{e}^i_{a'a'} N^a_{aa'} = N^a_{aa'}. \]

2. We choose on a fundamental Finsler function
\[ F = \hat{F}(x^i, v, \theta, \tilde{F}(x^i, v, \theta)) \]
inducing canonically a d-metric of type
\[ \hat{f} = f_i dx^i \otimes dx^i + f_2 (dy^a + \gamma N^a_i dx^i) \otimes (dy^a + \gamma N^a_i dx^i), \]
\[ = f_i e^i \otimes e^i + f_2 e^a \otimes e^a \]
determined by data \( \hat{f}_{ab} = [f_i, f_a, \gamma N^a_i] \) in a canonical N-elongated base \( e^a = (dx^i, \gamma e^a) = dy^a + \gamma N^a_i dx^i \).

3. We define
\[ g_{i'} = f_{i'} \left( \frac{\hat{u}_{i'}}{w_{i'}} \right)^2 \frac{h_{i'}}{f_{i'}} \quad \text{and} \quad \tilde{g}_{i'} = f_{i'} \left( \frac{\hat{h}_{i'}}{n_{i'}} \right)^2 \frac{h_{i'}}{f_{i'}}. \]
Both formulas are compatible if \( \hat{u}_{i'} \) and \( \hat{h}_{i'} \) are constrained to satisfy the conditions\(^{19}\)
\[ \Theta_{i'} = \Theta_{i''} = \emptyset, \]
where \( \Theta_{i'} = \left( \frac{\hat{u}_{i'}}{w_{i'}} \right)^2 \left( \frac{h_{i'}}{f_{i'}} \right)^2 \) and \( \Theta = \left( \frac{\hat{u}_{i'}}{w_{i'}} \right)^2 \left( \frac{\hat{h}_{i'}}{n_{i'}} \right)^2 = \left( \frac{\hat{u}_{i'}}{w_{i'}} \right)^2 \left( \frac{\hat{h}_{i'}}{n_{i'}} \right)^2 \). Using \( \Theta \), we compute
\[ \hat{g}_{i'} = \left( \frac{\hat{u}_{i'}}{w_{i'}} \right)^2 \frac{f_{i'}}{f_{i'}} \quad \text{and} \quad \hat{h}_{i'} = h_{i'} \Theta, \]
where (in this case) there is no summing on indices. So we constructed the data \( g_{i'}, h_{a'} \)
and \( u_{i'}, n_{i'} \).

4. The values \( \hat{e}^i_{j'} \) and \( \hat{e}^i_{a'a'} \) are determined as any nontrivial solutions of
\[ \hat{g}_{i'} = g_{i'} + h_{a'b'} (\hat{e}^i_{j'})^2, \quad \hat{h}_{a'a'} = g_{i'} (\hat{e}^i_{a'a'})^2 + h_{a'b'}, \quad \hat{N}^a_{aa'} = N^a_{aa'}. \]

For instance, we can choose
\[ \hat{e}^i_{1'} = \pm \sqrt{\frac{f_1}{\hat{g}_{1'}}}, \quad \hat{e}^i_{2'} = \pm \sqrt{\frac{f_2}{\hat{g}_{2'}}}, \quad \hat{e}^i_{3'} = \pm \sqrt{\frac{f_3}{\hat{g}_{3'}}}, \quad \hat{e}^i_{4'} = \pm \sqrt{\frac{f_4}{\hat{g}_{4'}}}. \]

Finally, in this section, we conclude that any model of noncommutative nonholonomic gravity with distributions of type (1) and/or (2) can be equivalently re-formulated as a Finsler gravity induced by a generating function of type \( F = \hat{F} + \gamma F \) in the limit \( \theta \to 0 \), for any
\(^{19}\) See details in [27].
solution $\dot{g}$, there is a scheme of two nonholonomic transforms which allows us to rewrite the Schwarzschild solution and its noncommutative/nonholonomic deformations as a Finsler metric $\dot{f}$.

6. Concluding remarks

In this paper we have constructed new classes of exact solutions with generic off-diagonal metrics depending on a noncommutative parameter $\theta$. In particular we have studied nonholonomic noncommutative deformations of Schwarzschild metrics which can be induced by effective energy–momentum tensors/effective cosmological constants and/or nonholonomic vacuum gravitational distributions. Such classes of solutions define complex Finsler spacetimes, induced parametrically from Einstein gravity, which can be equivalently modeled as complex Riemannian manifolds enabled with nonholonomic distributions. We provided a procedure of extracting stationary black hole configurations with ellipsoidal symmetry and possible solitonic deformations.

In the presence of noncommutativity, the nonholonomic frame structure and matter energy–momentum tensor have contributions from the noncommutative parameter. The anholonomic deformation method of constructing exact solutions in gravity allows us to define real (pseudo) Finsler configurations if we choose to work with the canonical distinguished connection. Further restrictions on the metric and nonlinear connection coefficients can be chosen in such a way that we can generate generic off-diagonal solutions on general relativity.

Our geometric method allows us to consider immersing of different types of (pseudo) Riemannian metrics, and/or exact solutions in Einstein gravity, (‘prime’ metrics) in noncommutative backgrounds which effectively polarize the interaction constants, deforms nonholonomically the frame structure, metrics and connections. The resulting ‘target’ metrics are positively constructed to solve gravitational field equations but, in general, it is difficult to understand what kind of physical importance they may have in modern gravity. We have chosen small rotoid and solitonic noncommutative deformations because there are explicit proofs that they are stable under perturbations and have much similarity with stationary black hole solutions in general relativity [17, 39, 40].

In this work, we emphasized constructions when black hole configurations are imbedded self-consistently into nonholonomic backgrounds induced by noncommutativity. The main difference from similar ellipsoidal configurations and rotoid black holes considered in [27] is that, in our case, the eccentricity is just a dimensionless variant of noncommutative parameter (in general, we can construct solutions with an infinite number of parameters of different origins, see details in [18]). So, such types of stationary black hole solutions are induced by noncommutative deformations with additional nonholonomic constraints. They are different from all those outlined in the review [1] and [2, 5, 35–38] (those classes of noncommutative solutions can be extracted from more general nonholonomic ones, constructed in our works, as certain holonomic configurations).

Finally, we emphasize that the provided noncommutative generalization of the anholonomic frame method can be applied to various types of commutative and noncommutative (in general, nonsymmetric) models of gauge [8, 17] and string/brane gravity [26], Ricci flows [7, 20–22] and nonholonomic quantum deformations of Einstein gravity [23–25] as we emphasized in [6, 9, 19]. All parameters of classical and quantum deformations and/or of flow evolution, physical constants and coefficients of metrics and connections, considered in those works, can be redefined to contain effective noncommutative constants and polarizations.
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References

[1] Nicolini P 2009 Noncommutative black holes, the final appeal to quantum gravity: a review Int. J. Mod. Phys. A 24 1229–308
[2] Aschieri P, Blomhann C, Dimitrijevic M, Meyer F, Schupp P and Wess J 2005 A gravity theory on noncommutative spaces Class. Quantum Grav. 22 3511–32
[3] Szabo R J 2006 Symmetry, gravity and noncommutativity Class. Quantum Grav. 23 R199–242
[4] Muller-Hoissen F 2008 Noncommutative geometries and gravity AIP Conf. Proc. 977 12
[5] Chaichian M, Setare M R, Tureanu A and Zet G 2008 On black holes and cosmological constant in noncommutative gauge theory of gravity J. High Energy Phys. JHEP04(2008)064
[6] Vacaru S, Stavrinos P, Gabarov E and Goanta D 2006 Clifford and Riemann–Finsler Structures in Geometric Mechanics and Gravity (Selected Works, Differential Geometry—Dynamical Systems. Monograph vol 7) (Bucharest: Geometry Balkan Press) (www.mathem.pub.ro/dgds/mono/va-t.pdf and arXiv:gr-qc/0508023)
[7] Vacaru S 2009 Spectral functionals, nonholonomic Dirac operators, and noncommutative Ricci flows J. Math. Phys. 50 073503
[8] Vacaru S 2001 Gauge and Einstein gravity from non-Abelian gauge models on noncommutative spaces Phys. Lett. B 498 74–82
[9] Vacaru S I, Chiosa I A and Vicol N A 2001 Locally anisotropic supergravity and gauge gravity on noncommutative spaces NATO Advanced Research Workshop Proceedings “Noncommutative Structures in Mathematics and Physics” (23–27 September, Kyiv, Ukraine) ed Duplij S and Wess J (Dordrecht: Kluwer) pp 229–43 (arXiv:hep-th/0011221)
[10] Moffat J 2000 Perturbative noncommutative quantum gravity Phys. Lett. B 493 142–8
[11] Chamseddine A H 2001 Deforming Einstein’s gravity Phys. Lett. B 504 33–7
[12] Sahakian V 2001 Transcribing spacetime data into matrices J. High Energy Phys. JHEP06(2001)037
[13] Cardella M A and Zanon D 2003 Noncommutative deformation of four dimensional Einstein gravity Class. Quantum Grav. 20 L95–104
[14] Nishino H and Rajpoot S 2002 Noncommutative nonlinear supersymmetry arXiv:hep-th/0212329
[15] Ardalan F, Arfaei H, Garousi M R and Ghodosi A 2003 Gravity on noncommutative D-branes Int. J. Mod. Phys. A 18 1051–66
[16] Yang H S 2006 Emergent gravity from noncommutative spacetime arXiv:hep-th/0611174
[17] Vacaru S 2005 Exact solutions with noncommutative symmetries in Einstein and gauge gravity J. Math. Phys. 46 042503
[18] Vacaru S 2007 Parametric nonholonomic frame transforms and exact solutions in gravity Int. J. Geom. Methods Mod. Phys. 4 1285–334
[19] Vacaru S 2008 Finsler and Lagrange geometries in Einstein and string gravity Int. J. Geom. Methods Mod. Phys. 5 473–511
[20] Vacaru S 2009 Nonholonomic Ricci flows: exact solutions and gravity Electron. J. Theor. Phys. 6 27–58 (http://www.ejtp.com, arXiv:0705.0728 [math-ph])
[21] Vacaru S 2009 Nonholonomic Ricci flows and parametric deformations of the solitonic pp-waves and Schwarzschild solutions Electron. J. Theor. Phys. 6 63–93 (http://www.ejtp.com, arXiv:0705.0729 [math-ph])
[22] Vacaru S 2009 Nonholonomic Ricci flows, exact solutions in gravity, and symmetric and nonsymmetric metrics Int. J. Theor. Phys. 48 579–606
[23] Vacaru S 2007 Deformation quantization of almost Kähler models and Lagrange–Finsler Spaces J. Math. Phys. 48 123509
[24] Vacaru S 2008 Deformation quantization of nonholonomic almost Kähler models and Einstein gravity Phys. Lett. A 372 2949–55
[25] Anastasiei M and Vacaru S 2009 Fedosov quantization of Lagrange–Finsler and Hamilton–Cartan spaces and Einstein gravity lifts on (co)tangent bundles J. Math. Phys. 50 013510 (23 pp)
[26] Vacaru S 2009 Branes and quantization of an A-model complexification for Einstein gravity in almost Kähler variables Int. J. Geom. Methods Mod. Phys. 6 873–909
[27] Vacaru S 2009 Black holes, ellipsoids, and nonlinear waves in Pseudo Finsler spaces and Einstein gravity arXiv:0905.4401 [gr-qc]

[28] Anastasiei M and Vacaru S 2009 Nonholonomic black ring and solitonic solutions in Finsler and extra dimension gravity theories arXiv:0906.3811 [hep-th]

[29] Miron R and Anastasiei M 1997 Vector Bundles and Lagrange Spaces with Applications to Relativity (Bucharest: Geometry Balkan Press) (translation from Romanian (Editura Academiei Romane, 1987))

[30] Miron R and Anastasiei M 1994 The Geometry of Lagrange Spaces: Theory and Applications (FTPH vol 59) (Dordrecht: Kluwer)

[31] Groenewold H H 1946 On the principles of elementary quantum mechanics Physica 12 405–60

[32] Moyal J E 1949 Proc. Quantum mechanics as a statistical theory Proc. Camb. Phil. Soc. 45 99–124

[33] Weyl H 1931 The Theory of Groups and Quantum Mechanics (New York: Dover)

[34] Wigner E P 1932 On the quantum correction for thermodynamic equilibrium Phys. Rev. 40 749–759

[35] Chaichian M, Tureanu A and Zet G 2008 Corrections to Schwarzschild solution in noncommutative gauge theory of gravity Phys. Lett. B 660 573–8

[36] Nicolini P, Smailagic A and Spallucci E 2006 Noncommutative geometry inspired Schwarzschild black hole Phys. Lett. B 632 547–51

[37] Kobakhidze A 2007 Noncommutative corrections to classical black holes arXiv:0712.0642 [gr-qc]

[38] Rizzo T G 2006 Noncommutative inspired black holes in extradimensions J. High Energy Phys. JHEP09/2006/021

[39] Vacaru S 2003 Horizons and geodesics of black ellipsoids Int. J. Mod. Phys. D 12 479–94

[40] Vacaru S 2003 Perturbations and stability of black ellipsoids Int. J. Mod. Phys. D 12 461–78

[41] Vacaru S and Singleton D 2002 Warped solitonic deformations and propagation of black holes in 5D vacuum gravity Class. Quantum Grav. 19 3583–602