Photon Dispersion in a Supernova Core

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While the photon forward-scattering amplitude on free magnetic dipoles (e.g. free neutrons) vanishes, the nucleon magnetic moments still contribute significantly to the photon dispersion relation in a supernova (SN) core where the nucleon spins are not free due to their interaction. We study the frequency dependence of the relevant spin susceptibility in a toy model with only neutrons which interact by one-pion exchange. Our approach amounts to calculating the photon absorption rate from the inverse bremsstrahlung process $\gamma n \to nm$, and then deriving the refractive index $n_{\text{refr}}$ with the help of the Kramers-Kronig relation. In the static limit ($\omega \to 0$) the dispersion relation is governed by the Pauli susceptibility $\chi_{\text{Pauli}}$ so that $n_{\text{refr}}^2 - 1 \approx \chi_{\text{Pauli}} > 0$. For $\omega$ somewhat above the neutron spin-relaxation rate $\Gamma_n$ we find $n_{\text{refr}}^2 - 1 < 0$, and for $\omega > \Gamma_n$ the photon dispersion relation acquires the form $\omega^2 - k^2 = m_{\text{trans}}^2$. An exact expression for the “transverse photon mass” $m_{\text{trans}}$ is given in terms of the $f$-sum of the neutron spin autocorrelation function; an estimate is $m_{\text{trans}}^2 \approx \chi_{\text{Pauli}} T \Gamma_n$. The dominant contribution to $n_{\text{refr}}$ in a SN core remains the electron plasma frequency so that the Cherenkov processes $\gamma \nu \leftrightarrow \nu$ remain forbidden for all photon frequencies.

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I. INTRODUCTION

The photon dispersion relation in astrophysical plasmas is usually dominated by the electromagnetic interaction with the electrons of the medium. It was recently claimed [1], however, that in a supernova (SN) core the photon interaction with the magnetic moments of the nucleons yields the dominant contribution to the refractive index $n_{\text{refr}}$. Because the new contribution has the opposite sign of the usual plasma term, the photon four-momentum $K$ would actually become space-like, allowing for the Cherenkov processes $\gamma \nu \to \nu$ and $\nu \to \nu \gamma$ which could be of great importance for the neutrino opacities. Due to a numerical error in Ref. [1] the overall magnitude of the nucleon magnetic-moment effect is in fact much smaller [2], but even after this correction it is not very much smaller than the electron plasma effect and thus deserves a closer look.

It is surprising, at first, that the neutron magnetic moments contribute at all to the refractive index because the photon forward-scattering amplitude on fermions with a magnetic moment is identically zero. Most recently the photon polarization tensor for an ensemble of noninteracting spin-$\frac{1}{2}$ particles was calculated in great detail [3] and it was found that indeed the magnetic moments alone do not produce any contribution to $n_{\text{refr}}$. However, the underlying assumption of a collisionless system is far from justified in a SN core where the nucleon spin-spin interaction plays a dominant role. For photon frequencies below the nucleon spin relaxation rate $\Gamma_n$, the hydrodynamic limit is the physically appropriate description (not the collisionless limit) justifying the use of the Pauli susceptibility in Ref. [1]. In a SN core the spin relaxation rate is likely to be of the same order as the temperature $T$, typical photon frequencies are also of that order, so that for the entire spectrum of relevant photon frequencies neither the hydrodynamic nor the collisionless limits are truly justified. Therefore, an understanding of the photon refractive index and its frequency dependence requires a more general analysis than has been offered in either Ref. [1] or [3].

Perhaps the easiest way to appreciate this point is to consider photon absorption. In a collisionless system of neutral fermions (“neutrons”) with a magnetic moment $\mu_n$ the refractive index is $n_{\text{refr}} = 1$ up to order $\mu_n^2$ which implies that there is no Landau damping, i.e. no Cherenkov effect $\gamma n \leftrightarrow n$. The only photon damping occurs at order $\mu_n^4$ from magnetic Compton scattering $\gamma n \to n \gamma$. On the other hand, if our “neutrons” interact by a spin-dependent force (which for real neutrons is caused by pion exchange) we have the inverse-bremsstrahlung absorption process $\gamma nn \to nn$ so that we do have photon absorption to order $\mu_n^2$. Its rate far exceeds that of magnetic Compton scattering because in a SN core the $nn$ interaction rate is large. By virtue of the
Kramers-Kronig relation one can then derive the associated refractive index \(n_{\text{refr}}\) which does not vanish to order \(\mu_0^2\). (Note that we always take \(n_{\text{refr}}\) to be a real quantity even though one sometimes describes absorption by an “imaginary part of the refractive index.”)

We proceed in Sec. [II] with the general photon dispersion relation in a pure neutron medium in terms of the dynamical spin-density structure function by virtue of the fluctuation-dissipation theorem and the Kramers-Kronig relation. In Sec. [III] we use a semi-heuristic expression for the dispersion relation of propagating modes is determined by \(\varepsilon_{\text{L}}\) and \(\varepsilon_{\text{T}}\) with \(\varepsilon_{\text{L}} \equiv \varepsilon + (1 - \mu^{-1}) k^2/\omega^2\), the transverse dielectric permittivity. They are related to the polarization functions by \(\varepsilon_{\text{L}} = 1 - \pi_{\text{L}}/(\omega^2 - k^2)\) and \(\varepsilon_{\text{T}} = 1 - \pi_{\text{T}}/\omega^2\). In this language the dispersion relations take on their standard form \(\varepsilon_{\text{L}}(\omega, k) = 0\) and \(\omega \varepsilon_{\text{T}}(\omega, k) = k^2\).

We are presently only concerned with the dispersion relation of transverse modes (“photons”) because a medium consisting of magnetic dipoles is not expected to support longitudinal modes (longitudinal plasmons). From the above it follows immediately that the photon dispersion relation can be written in the form

\[
\frac{k^2}{\omega^2} = \epsilon(\omega, k) \mu(\omega, k).
\]

With the usual definition of the photon refractive index

\[
\begin{align*}
n_{\text{refr}} &\equiv \frac{k}{\omega} \quad (2) \\
n_{\text{refr}}^2(\omega, k) &\equiv \epsilon(\omega, k) \mu(\omega, k) \quad (3)
\end{align*}
\]

we arrive at the classical result \(n_{\text{refr}}^2 = \epsilon \mu\). It follows that the refractive index must be determined self-consistently as a solution of

\[
n_{\text{refr}}^2(\omega, k) = \epsilon(\omega, k) \mu(\omega, k)
\]

with \(k = n_{\text{refr}} \omega\) for any frequency \(\omega\) of a propagating mode. Depending on the properties of the medium the long-wavelength approximation \(\epsilon(\omega, k) \approx \epsilon(\omega, 0)\) and \(\mu(\omega, k) \approx \mu(\omega, 0)\) may be justified, leading to the much simpler dispersion relation \(n_{\text{refr}}^2(\omega) = \epsilon(\omega, 0) \mu(\omega, 0)\).

Sometimes it will be more useful to write the photon dispersion relation in the form \(\omega^2 - k^2 = m_{\text{eff}}^2\) in terms of a frequency-dependent “effective mass”

\[
m_{\text{eff}}^2(\omega) = (1 - n_{\text{refr}}^2(\omega))^2, 
\]

where in fact \(m_{\text{eff}}^2 < 0\) is possible. For electric interactions and frequencies well above all resonances we obtain the well-known plasma effect dispersion relation which implies that \(m_{\text{eff}}^2 > 0\) and independent of frequency.

We will show that the same holds true for our magnetic case. Therefore, it is useful to define

\[
m_\gamma \equiv \lim_{\omega \to \infty} m_{\text{eff}}(\omega)
\]

as the (transverse) “photon mass” in the medium.

We will mostly be concerned with a medium of neutrons which interact with the electromagnetic field by their magnetic dipole moment. In the nonrelativistic limit they do not respond at all to an applied electric field so that we may use the approximation \(\gamma = 1\). The magnetic permeability can be written as \(\mu = 1 + \chi\) in
terms of the magnetic susceptibility $\chi$. (We use rationalized units where $\alpha = e^2/4\pi \approx 1/137$ or else we would have to write $\mu = 1 + 4\pi \chi$.)

In general the magnetic susceptibility is a complex function of the real variables $\omega$ and $k$. Following common practice we write it in the form

$$\chi(\omega, k) = \chi'(\omega, k) + i \chi''(\omega, k)$$

in terms of its real and imaginary parts. It is found that that $\chi'$ is an odd function of $\omega$ while $\chi'$ is even. Because we have defined the refractive index to be a real quantity the dispersion relation is

$$n_{\text{refr}}^2(\omega, k) - 1 = \chi'(\omega, k)$$

with $k = n_{\text{refr}}\omega$.

The imaginary part of the susceptibility describes dissipation: Usually one pictures a stationary beam of frequency $\omega$ along the $z$-direction which is characterized by a (real) wavenumber $k = n_{\text{refr}}\omega$ and a damping wavenumber $\kappa = \frac{1}{2}\lambda^{-1}$ with $\lambda$ the photon mean free path. The amplitude of this beam varies as $e^{-ikz}e^{-\kappa z}$, its intensity as $e^{-2kz} = e^{-\lambda z}$. The relativistic limit $n_{\text{refr}} - 1 \ll 1$ implies $n_{\text{refr}}^2 - 1 = (n_{\text{refr}} + 1)(n_{\text{refr}} - 1) \approx 2(n_{\text{refr}} - 1)$ or $n_{\text{refr}} - 1 \approx \frac{\chi}{2}$. Therefore, one can picture $\frac{1}{2}\chi'''$ to be an “imaginary part of the refractive index”, leading to the identification $\kappa = \frac{1}{2}\chi'\omega$ or $\chi'' = (\lambda \omega)^{-1}$.

We stress that at finite temperature this simple interpretation is not complete because the medium can both absorb and spontaneously emit photons. The two processes are related by the usual detailed-balance factor $e^{-\omega/T}$. What is actually damped is not a mode $k$ of the electromagnetic field, but rather the deviation of its occupation number from a thermal distribution. It is easy to show that this damping occurs at a rate $1 - e^{-\omega/T}$ times the “naive” absorption rate $\Gamma_{\text{abs}}$. Therefore, the appropriate interpretation of the imaginary part of the susceptibility is

$$\chi''(\omega, k) = \frac{1}{\omega} \left( 1 - e^{-\omega/T} \right) \Gamma_{\text{abs}}(\omega)$$

with $k = n_{\text{refr}}\omega$. In the limit $|n_{\text{refr}} - 1| \ll 1$ the “naive” absorption rate is $\Gamma_{\text{abs}} = \lambda^{-1}$; it is given by the standard formula “absorption cross section times target density.”

### B. Fluctuation-Dissipation Theorem

To lowest order the neutrons can absorb photons only because they interact by a spin-dependent force which enables the inverse-bremsstrahlung process $\gamma nn \rightarrow nn$. At the same time this spin-dependent force causes the neutron spins to fluctuate. The relationship between spin fluctuations and the absorbptive part of the spin susceptibility is encapsulated in the fluctuation-dissipation theorem which will help us to understand some general properties of the photon refractive index.

In our case the most useful quantity to describe the neutron spin fluctuations is the dynamical spin-density structure function. Following the normalization convention of Ref. [1] it is defined as

$$S_\sigma(\omega, k) = \frac{4}{3n_n} \int_{-\infty}^{+\infty} dt e^{i\omega t} \langle \sigma(t, k) \cdot \sigma(0, -k) \rangle,$$ (9)

where $\sigma(t, k)$ is the spatial Fourier transform of the neutron spin-density operator $\sigma(t, r) = \frac{1}{2}[\psi^\dagger(t, r) \tau \psi(t, r)]$. Here $\psi(t, r)$ is a Pauli two-spinor describing the nucleon field and $\tau$ is a vector of Pauli matrices. Further, $n_n$ is the neutron number density and $\langle \cdots \rangle$ denotes a thermal ensemble average. Of course, in an isotropic system the structure function depends only on $k = |k|$. The normalization was chosen such that

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_\sigma(\omega, 0) = 1$$ (10)

for a case where there are no static spin-spin correlations between different neutrons which are taken to be nondegenerate. In the limit of vanishing spin-spin interactions we have

$$S_\sigma(\omega, 0) \rightarrow 2\pi\delta(\omega).$$ (11)

Moreover, it satisfies

$$S_\sigma(\omega, -k) = e^{-\omega/T} S_\sigma(\omega, k)$$ (12)

as required by the principle of detailed balance.

We next observe that the operator for the magnetization density for neutrons is $M = 2\mu_n \sigma$ where the factor 2 is the gyromagnetic ratio for a spin-$\frac{1}{2}$ particle and $\mu_n$ is the neutron magnetic moment, not to be confused with the magnetic permeability $\mu$ of the previous section. A relationship between the dissipative part of $\chi$ and spontaneous spin fluctuations can then be written in the form [3]

$$\chi''(\omega, k) = \frac{1}{2} \int_{-\infty}^{+\infty} dt e^{i\omega t} \left\langle \frac{1}{3} \sum_{i=1}^{3} [M_i(t, k), M_i(0, -k)] \right\rangle,$$ (13)

where $[\cdot, \cdot]$ is the usual commutator. Comparing this with Eqs. [6] and [12] reveals that this relationship is equivalent to

$$\chi''(\omega, k) = \frac{1}{2} \mu_n^2 n_n \left( 1 - e^{-\omega/T} \right) S_\sigma(\omega, k).$$ (14)

In this form it is known as the fluctuation-dissipation theorem [3][4].
C. Kramers-Kronig Relation

Once the imaginary part of the magnetic susceptibility is known the real part can be found by virtue of the well-known Kramers-Kronig relation

\[
\chi''(\omega, k) = P \int_{-\infty}^{+\infty} \frac{d\hat{\omega}}{\pi} \frac{\chi''(\hat{\omega}, k)}{\hat{\omega} - \omega}
\]  

(15)

where \(P\) denotes a Cauchy principal value integral. With the help of the fluctuation-dissipation theorem Eq. (14) we find a direct relationship between the dispersive part of the magnetic susceptibility and the spin-density structure function

\[
\chi''(\omega, k) = 2\mu_n^2 n_n \int_{-\infty}^{+\infty} \frac{d\hat{\omega}}{2\pi} \frac{\hat{\omega} S_\sigma(\hat{\omega}, k)}{\hat{\omega}^2 - \omega^2}.
\]

(16)

One may use detailed balance to write this in the form

\[
\chi''(\omega, k) = \chi_{\text{Pauli}} P \int_{0}^{\infty} \frac{d\hat{\omega}}{\pi} \frac{\hat{\omega}^2 S_\sigma(\hat{\omega}, k)}{\hat{\omega}^2 - \omega^2},
\]

(17)

where

\[
\chi_{\text{Pauli}} \equiv \frac{\mu_n^2 n_n}{T}
\]

(18)

is the usual Pauli susceptibility for a system of collisionless spin-\(\frac{1}{2}\) particles with a magnetic moment \(\mu_n\).

D. Limiting Cases

In order to understand the general behavior of the refractive index we begin with the static limit \(\omega \to 0\). The static susceptibility \(\chi_0(k) \equiv \chi'(0, k)\) has no imaginary part because \(\chi''\) is an odd function of \(\omega\). From Eq. (17) we find for the real part

\[
\chi_0(k) = \chi_{\text{Pauli}} \int_{0}^{\infty} \frac{d\hat{\omega}}{\pi} \frac{1 - e^{-\hat{\omega}/T}}{\hat{\omega}/T} S_\sigma(\hat{\omega}, k).
\]

(19)

In the collisionless limit the structure function becomes narrowly peaked around \(\omega = 0\). With the help of Eq. (11) we thus recover the usual long-wavelength result \(\chi_0(0) = \chi_{\text{Pauli}}\). When \(S_\sigma(\omega, k)\) is not narrowly peaked on scales of the temperature, the static susceptibility decreases relative to the Pauli value—we show this effect explicitly in Fig. 4 below in the framework of a heuristic toy model.

How large may the frequencies be that the static result is still approximately justified? The structure function in the long-wavelength limit \(S_\sigma(\omega, 0)\) has the interpretation of the autocorrelation function of a single neutron spin. Therefore, it is a broad, decreasing function of \(\omega\) with a width representing something like the spin-relaxation or spin-fluctuation rate \(\Gamma_{\sigma}\). If the external electromagnetic perturbation has a frequency much less than this, \(\omega \ll \Gamma_{\sigma}\), we are in the hydrodynamic limit where the neutron spins may fully relax to a new thermodynamic equilibrium state on the time scale of a period of the perturbation. In this case we may use the static susceptibility to estimate the photon refractive index. Moreover, even though we just saw that the static susceptibility is not independent of the width of \(S_\sigma(\omega)\), this dependence is weak so that in the hydrodynamic limit the Pauli susceptibility is a good estimate, justifying the approach of Ref. [1] to photon dispersion in the limit \(\omega \ll \Gamma_{\sigma}\).

The opposite limiting case is that of very large \(\omega\). If \(S_\sigma(\omega)\) falls off sufficiently fast beyond some frequency \(\omega_0\) which is determined by the nature of the \(nn\) interaction potential, then for \(\omega \gg \omega_0\) the integral in Eq. (16) is dominated by \(|\hat{\omega}| \lesssim \omega_0\), leading to

\[
\chi''(\omega, k) = -2\mu_n^2 n_n \int_{-\infty}^{+\infty} \frac{d\hat{\omega}}{2\pi} \frac{\hat{\omega}}{\hat{\omega}^2 - \omega^2} S_\sigma(\hat{\omega}, k).
\]

(20)

The integral in this equation is the so-called \(f\)-sum of the structure function. Independently of the nature of the assumed \(nn\) interaction the \(f\)-sum always exists and is given as a thermal expectation value of the tensor part of the \(nn\) interaction potential [12]. Moreover, the \(f\)-sum is always positive because of the detailed-balance property Eq. (12).

For photon dispersion, this result corresponds to a positive value for the squared effective mass defined in Eq. (4). With Eq. (7) we find

\[
m_{\text{eff}}^2 = 2\mu_n^2 n_n \int_{-\infty}^{+\infty} \frac{d\hat{\omega}}{2\pi} \frac{\hat{\omega}}{\hat{\omega}^2 - \omega^2} S_\sigma(\hat{\omega}, k).
\]

(21)

If the momentum dependence of this expression is weak so that we may use the long-wavelength limit then the photon dispersion relation is that of a massive particle \(\hat{\omega}^2 - k^2 = m_{\text{eff}}^2\) with the transverse photon mass given by Eq. (21) with \(k = 0\) on the right-hand side. The appearance of this form has the same cause as in the usual plasma case, i.e. \(n_{\text{eff}}\) is given by the \(f\)-sum of the relevant dynamical structure functions.

The Pauli susceptibility is a positive number (the neutrons are a paramagnetic medium) so that in the hydrodynamic limit the photon dispersion relation is approximately characterized by \(n_{\text{eff}}^2 - 1 = \chi_{\text{Pauli}}\) or \(m_{\text{eff}}^2 = -\chi_{\text{Pauli}}\omega^2 < 0\). On the other hand in the large-frequency limit we have \(m_{\text{eff}}^2 > 0\) as given in Eq. (21). Moreover, on dimensional grounds the \(f\)-sum must take on the approximate value \(\Gamma_{\sigma}\). Therefore,

\[
m_{\text{eff}}^2 \approx \chi_{\text{Pauli}} \times \begin{cases} -\omega^2 & \text{for } \omega \ll \Gamma_{\sigma} \\ + \Gamma_{\sigma} & \text{for } \omega \gg \Gamma_{\sigma} \end{cases}
\]

(22)

gives us a rough picture of the behavior of the photon dispersion relation in a medium of neutron spins.
III. SEMI-HEURISTIC MODEL

In a SN core neither the collisionless nor the hydrodynamic limits are appropriate so that we need to come up with a concrete expression for the dynamical spin-density structure function in order to estimate the photon refractive index. In a dilute medium one may use the usual perturbative methods to compute the processes $\gamma n \leftrightarrow nn$. Because the relevant photon energies are small compared with the neutron mass the momentum transfer of the radiation to the neutron system may be neglected, an approximation which amounts to the long-wavelength limit which we shall henceforth adopt with the notation $S_\sigma(\omega) \equiv \lim_{k \to 0} S_\sigma(\omega,k)$. Next, one may extract $S^{(1)}_\sigma(\omega)$, where the superscript indicates this is a lowest-order perturbative result. Independently of the details of the assumed $nn$ interaction potential one finds the generic representation

$$S^{(1)}_\sigma(\omega) = \frac{\Gamma}{\omega^2 + \Gamma^2/4} s(\omega/T),$$

where $s(x)$ with $x = \omega/T$ is a slowly varying function of order unity. This factorization is somewhat arbitrary; we define what we call the neutron “spin-fluctuation rate” $\Gamma_\sigma$ such that for nondegenerate neutrons $s(0) = 1$. Moreover, we have

$$s(-x) = s(x) e^{-x}$$

so that the detailed-balance relation for $S_\sigma(\omega)$ Eq. (12) is satisfied.

The lowest-order perturbative representation $S^{(1)}_\sigma(\omega)$ diverges at $\omega = 0$ and thus violates the normalization rule Eq. (10). However, including multiple-scattering effects suggests the “summed” representation

$$S_\sigma(\omega) = \frac{\Gamma_\sigma}{\omega^2 + \Gamma^2/4} s(\omega/T).$$

In a very dilute medium this function is strongly peaked around $\omega = 0$ so that it approaches $2\pi \delta(\omega)$. In this limit we have $\Gamma = \Gamma_\sigma$, i.e. we approach the classical limit of a Lorentzian correlation function $S_\sigma(\omega) \equiv \Gamma_\sigma/\omega^2 + \Gamma^2/4)$. We stress that the representation Eq. (23) is completely general if we interpret $\Gamma$ as a function of $\omega$ which in linear-response theory is related to the neutron spin’s “memory function” Eq. (12). In our heuristic description, however, we will use a constant value for $\Gamma$ which is fixed by the normalization requirement Eq. (17).

In order to calculate $\Gamma_\sigma$ and $s(x)$ in a dilute neutron medium we model the $nn$ interaction by one-pion exchange in Born approximation, an approach which has been common practice for SN and neutron-star physics since Friman and Maxwell’s seminal paper [13] and which is further justified in Ref. [14]. Further, we take the neutrons to be nondegenerate which is not a bad approximation during the early phases of SN core cooling. Finally, we neglect the mass in the pion propagator which is also a reasonable approximation for the large momentum transfers in typical $nn$ collisions in a SN core. All of these approximations go in the same direction of somewhat overestimating the $nn$ spin interaction rate. We also ignore static spin-spin correlations which could, in principle, both enhance or diminish our results.

Within this framework the spin-fluctuation rate is explicitly found to be [14]

$$\Gamma_\sigma = 4\sqrt{\pi} \alpha_{\pi}^2 n_n T^{1/2} m_N^{-5/2},$$

where $\alpha_{\pi} \equiv (f_2 m_N/m_\pi)^2/4\pi \approx 15$ with $f \approx 1$ is the pion-nucleon “fine-structure constant,” $n_n$ is the neutron density, and $m_N$ the nucleon mass. Numerically we find

$$\gamma_\sigma = \Gamma_\sigma/T = 8.6 \rho_{14} T_{10}^{-1/2},$$

where $\rho_{14} \equiv \rho/10^{14}$ g cm$^{-3}$ and $T_{10} \equiv T/10$ MeV. Moreover, one finds [13]

$$s(x) = \int_{\max(0,-x)}^{\infty} dv \ e^{-v}[\sqrt{v(v+x)}$$

$$- \frac{x^2}{2(2v+x)} \log \left( \frac{\sqrt{v+x} + \sqrt{v}}{\sqrt{v+x} - \sqrt{v}} \right) \right],$$

an expression which indeed fulfills the detailed balance requirement Eq. (22) and which is smooth at $x = 0$ with the derivative $s'(0) = 1/2$ (Fig. 1). We will use a simple analytic approximation to this integral [14]

$$s(x) \mid_{x \geq 0} \approx \left( \frac{x}{4\pi} + \left[ 1 + \left( \frac{12 + \frac{3}{\pi}}{3} \right) x \right]^{-1/2} \right)^{-1/2}$$

which reproduces the correct limiting behavior for $x \gg 1$ and for $x = 0$ where it also has the correct derivative. It deviates from the true value by no more than 2.5% anywhere. For $x < 0$ we use $s(x) = s(-x) e^x$ in accordance with detailed balance.

FIG. 1. The function $s(x)$ as defined in Eq. (28). The analytic approximation Eq. (29) is identical to within plotting accuracy.
Our semiheuristic toy model is thus completely defined. In Fig. 2 we show the “downstairs Γ” of Eq. (25) as a function of $Γ_σ$ such that the normalization requirement Eq. (10) is obeyed. By construction we have $Γ = Γ_σ$ for $Γ_σ → 0$ with smaller values for a larger $Γ_σ$. This reduction is mostly due to the detailed-balance behavior which suppresses the classical structure function for negative $ω$.

Further we consider the $f$-sum which is for our present model

$$\int_{-∞}^{+∞} \frac{dω}{2π} ω S_σ(ω) = Γ_σ \int_{-∞}^{+∞} \frac{dx}{2π} \frac{x}{x^2 + γ^2/4} s(x), \quad (30)$$

where $γ ≡ Γ/T$. As claimed before it is equal to the spin fluctuation rate times a factor of order unity which is shown in Fig. 3 as a function of $Γ_σ$.

Finally we show in Fig. 4 the static long-wavelength susceptibility in units of the Pauli susceptibility for our toy model according to Eq. (19). It is a slowly decreasing function of the spin fluctuation rate.

The overall spin-density structure function $S_σ(ω)$ in our toy model is shown in Fig. 5 for several values of $γ_σ = Γ_σ/T$.

Next we study the photon dispersion relation implied by our model. In Fig. 6 we show the long-wavelength limit $χ'(ω) = n^2_{ref} - 1$ in units of the Pauli susceptibility as a function of $x = ω/T$ for several values of $γ_σ$. The overall behavior is exactly as expected from our general discussion in Sec. II D. To see the large-$ω$ behavior more clearly we show in Fig. 7 the equivalent quantity $m^2_{eff}$ in units of $χ_{Pauli}Γ_σ T$. As predicted, $m^2_{eff}$ approaches an asymptotic value which is independent of frequency and which is of order $χ_{Pauli}Γ_σ T$. Of course, for small $ω$ the squared “effective mass” $m^2_{eff}$ begins at negative values. However, for all frequencies and all values of $γ_σ$ we find that $|m^2_{eff}| < m^2_γ$ where the latter is the asymptotic value for $ω → ∞$. 

![Figure 2](image2.png)

FIG. 2. $Γ$ appearing in Eq. (25) to normalize $S_σ(ω)$ as a function of $Γ_σ$.

![Figure 3](image3.png)

FIG. 3. $f$-sum of $S_σ(ω)$ as defined in Eq. (30) as a function of $Γ_σ$.

![Figure 4](image4.png)

FIG. 4. Static long-wavelength limit of the magnetic susceptibility according to Eq. (19) as a function of $Γ_σ$.

![Figure 5](image5.png)

FIG. 5. Dynamical spin-density structure function $S_σ(ω)$ in our heuristic model Eq. (25) for $γ_σ = 3, 10, and 30$.

![Figure 6](image6.png)

FIG. 6. The overall spin-density structure function $S_σ(ω)$ in our toy model is shown in Fig. 5 for several values of $γ_σ$. The overall behavior is exactly as expected from our general discussion in Sec. II D.

![Figure 7](image7.png)

FIG. 7. The equivalent quantity $m^2_{eff}$ in units of $χ_{Pauli}Γ_σ T$. As predicted, $m^2_{eff}$ approaches an asymptotic value which is independent of frequency and which is of order $χ_{Pauli}Γ_σ T$. Of course, for small $ω$ the squared “effective mass” $m^2_{eff}$ begins at negative values. However, for all frequencies and all values of $γ_σ$ we find that $|m^2_{eff}| < m^2_γ$ where the latter is the asymptotic value for $ω → ∞$. 

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We next consider the corresponding quantity caused by the interaction with electrons. One finds \[ m_{\gamma}^{2} \big|_{\text{electrons}} = \frac{2\alpha}{\pi} p_{F,e}^{2}. \] Numerically, this corresponds to

\[ m_{\gamma}^{2} \big|_{\text{electrons}} = 16.3 \text{ MeV} \ Y_{e}^{1/3} \rho_{14}, \] (34)

where \( Y_{e} \) is the number of electrons per baryon. Evidently the center of a SN core with \( \rho_{14} \approx 8, T_{10} \approx 4 \) and initially \( Y_{e} \approx 0.3 \) the magnetic moment contribution could be almost as large as the electronic term. However, because we have probably overestimated the magnetic term by a factor of a few the electrons still dominate.

IV. DISCUSSION AND SUMMARY

We have calculated the photon refractive index due to the interaction with the magnetic moments of the nucleons. For simplicity we have limited our discussion to nondegenerate neutrons. In the collisionless limit the forward-scattering amplitude vanishes identically so that the neutron magnetic moments alone do not cause any deviation of the photon dispersion relation from the vacuum behavior. However, because of strong neutron spin interactions the collisionless limit is far from justified in a SN core. On the basis of the fluctuation-dissipation theorem and the Kramers-Kronig relation we have derived a general expression for the photon refractive index in terms of the dynamical neutron spin-density structure function \( S_{\sigma}(\omega, k) \). In an interacting medium it is a broad function of \( \omega \), in contrast to the collisionless limit where it is proportional to \( \delta(\omega) \).

We have found that for \( \omega \ll \Gamma_{\sigma} \) (the neutron spin fluctuation rate) the “effective photon mass” \( m_{\gamma}^{2} \big|_{\text{eff}} \) begins with negative values \( -\chi_{\text{Pauli}} \omega^{2} \) in terms of the Pauli susceptibility of the neutron ensemble. However, as shown in Fig. 7 this function quickly turns around and then grows asymptotically to a positive value \( m_{\gamma}^{2} \approx \chi_{\text{Pauli}} T \Gamma_{\sigma} \). In absolute terms this “transverse photon mass” is much larger than the maximum excursion of \( m_{\gamma}^{2} \big|_{\text{eff}} \) to negative values.

A numerical comparison for conditions relevant for a SN core reveals that the transverse photon mass caused by the neutron magnetic moment tends to be much smaller than that caused by the electron plasma effect, except for extreme densities and low electron fractions where the magnetic term may actually compete with the electronic one. A numerically accurate comparison is not possible because the neutron dynamical spin-density structure function is not known in any detail. We have only performed a relatively schematic estimate which involved many simplifying assumptions. However, it still
appears safe to conclude that the negative magnetic $m_{\text{eff}}^2$ at small frequencies cannot compete with the electronic plasma effect. This indicates that the total $m_{\text{eff}}^2$ is always positive, i.e. the photon refractive index is always less than 1 and it is reasonably well estimated by the electronic plasma effect. This implies that the Cherenkov processes $\nu \leftrightarrow \nu \gamma$ remain forbidden.

A more quantitative analysis than has been presented here requires a better understanding of the dynamical nucleon spin-density structure function, or more precisely, of the dynamical spin and isospin susceptibilities of a hot and dense nuclear medium. We stress that the $\omega$ dependence is crucial for the photon dispersion relation as well as the neutrino opacities [4], the static susceptibilities alone which have sometimes been studied in the literature are not enough.

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