UNIFORM SZEGÖ COCYCLES OVER STRICTLY ERGODIC SUBSHIFTS

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Dedicated to Barry Simon on the occasion of his 60th birthday.

Abstract. We consider ergodic families of Verblunsky coefficients generated by minimal aperiodic subshifts. Simon conjectured that the associated probability measures on the unit circle have essential support of zero Lebesgue measure. We prove this for a large class of subshifts, namely those satisfying Boshernitzan’s condition. This is accomplished by relating the essential support to uniform convergence properties of the corresponding Szegö cocycles.

1. Introduction

(Ω, T) is called a subshift over A if A is finite with discrete topology and Ω is a closed T-invariant subset of A^Z, where A^Z carries the product topology and T : A^Z → A^Z is given by (Ts)(n) = s(n + 1). A function F on Ω is called locally constant if there exists a non-negative integer N with

F(ω) = F(ω') whenever (ω(−N), . . . , ω(N)) = (ω'(−N), . . . , ω'(N)).

The subshift (Ω, T) is called minimal if every orbit {T^nω : n ∈ Z} is dense in Ω. It is called aperiodic if T^nω ̸= ω for all ω ∈ Ω and n ̸= 0.

Now, let a minimal subshift and f : Ω → D continuous be given. For ω ∈ Ω, define dµω to be the probability measure on the unit circle associated with the Verblunsky coefficients α_n(ω) = f(T^nω), n ≥ 0. Then, by minimality, the essential support of the dµω does not depend on ω. It will be denoted by Σ.

In fact, dµω is the spectral measure of the CMV matrix Cω given by

\[
\begin{pmatrix}
\tilde{\alpha}_0(\omega) & \tilde{\alpha}_1(\omega)\rho_0(\omega) & \rho_1(\omega)\rho_0(\omega) & 0 & 0 & \ldots \\
\rho_0(\omega) & -\tilde{\alpha}_1(\omega)\alpha_0(\omega) & -\rho_1(\omega)\alpha_0(\omega) & 0 & 0 & \ldots \\
0 & \tilde{\alpha}_2(\omega)\rho_1(\omega) & -\tilde{\alpha}_2(\omega)\alpha_1(\omega) & \tilde{\alpha}_3(\omega)\rho_2(\omega) & \rho_3(\omega)\rho_2(\omega) & \ldots \\
0 & \rho_2(\omega)\rho_1(\omega) & -\rho_2(\omega)\alpha_1(\omega) & -\tilde{\alpha}_3(\omega)\alpha_2(\omega) & -\rho_3(\omega)\alpha_2(\omega) & \ldots \\
0 & 0 & 0 & \tilde{\alpha}_4(\omega)\rho_3(\omega) & -\tilde{\alpha}_4(\omega)\alpha_3(\omega) & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

Here, ρ_n(ω) = (1 − |α_n(ω)|)^{-1/2}. Therefore, Σ is the essential spectrum of Cω. In a completely analogous way, it is also possible to define an extended CMV matrix, \E_ω, acting on ℓ^2(Z) using the Verblunsky coefficients α_n(ω) = f(T^nω), n ∈ Z. Then, Σ is the spectrum of \E_ω for each ω ∈ Ω.

See Simon [9, 10] for background on polynomials orthogonal with respect to a probability measure on the unit circle (OPUC) and the associated Verblunsky coefficients and CMV matrix. In [10] Conjecture 12.8.2, Simon conjectures the following:

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Simon’s Subshift Conjecture. Suppose $\mathcal{A}$ is a subset of $\mathbb{D}$, the subshift $(\Omega, T)$ is minimal and aperiodic and let $f : \Omega \to \mathbb{D}$, $f(\omega) = \omega(0)$. Then, $\Sigma$ has zero Lebesgue measure.

Our goal in this paper is to prove this conjecture (actually, in a stronger form) under a mild assumption on the subshift that is satisfied in many cases of interest. We will give some examples at the end of the paper and refer the reader to [4] for a more detailed discussion.

Before we formulate this assumption, let us recall that each subshift $(\Omega, T)$ over $\mathcal{A}$ gives rise to the associated set of words

$$W(\Omega) = \{\omega(k) \cdots \omega(k + n - 1) : k \in \mathbb{Z}, n \in \mathbb{N}, \omega \in \Omega\}.$$  

For $w \in W(\Omega)$, we define $V_w = \{\omega \in \Omega : \omega(1) \cdots \omega(|w|) = w\}$.

$(\Omega, T)$ is said to satisfy the Boshernitzan condition if it is minimal and there exists an ergodic probability measure $\nu$ on $\Omega$, a constant $C > 0$ and a sequence $(l_n)$ with $l_n \to \infty$ for $n \to \infty$ such that $|w|\nu(V_w) \geq C$ whenever $w \in W(\Omega)$ with $|w| = l_n$ for some $n$; compare [2].

**Theorem 1.** Suppose the subshift $(\Omega, T)$ satisfies the Boshernitzan condition. Let $f : \Omega \to \mathbb{D}$ be locally constant and $\alpha_n(\omega) = f(T^n \omega)$ for $\omega \in \Omega$ and $n \geq 0$. If for some $\omega \in \Omega$, $\alpha_n(\omega)$ is not periodic, then for every $\omega \in \Omega$, the essential support of the measure $d\mu_{\omega}$ associated with Verblunsky coefficients $(\alpha_n(\omega))_{n \geq 0}$ has zero Lebesgue measure.

Note that by minimality of $\Omega$ and local constancy of $f$, $\alpha_n(\omega)$ is periodic for some $\omega$ if and only if it is periodic for every $\omega$.

Theorem 1 is the unit circle analogue of a result obtained in [3] for discrete Schrödinger operators. Apart from addressing Simon’s Subshift Conjecture, it is our intention to introduce certain dynamical systems methods in the study of ergodic Verblunsky coefficients and CMV matrices. These methods have been very successful in recent studies of ergodic discrete Schrödinger operators, but have not yet found their way into the OPUC literature to the degree they deserve.

**2. A Dynamical Characterization of the Essential Spectrum**

A continuous map $A : \Omega \to \mathbb{GL}(2, \mathbb{C})$ gives rise to a so-called cocycle, which is a map from $\Omega \times \mathbb{C}^2$ to itself given by $(\omega, v) \mapsto (T\omega, A(\omega)v)$. This map is usually denoted by the same symbol. When studying the iterates of the cocycle, the following matrices describe the dynamics of the second component:

$$A(n, \omega) = \begin{cases} 
A(T^{n-1} \omega) \cdots A(\omega) & : n > 0 \\
Id & : n = 0 \\
A(T^n \omega)^{-1} \cdots A(T^{-1} \omega)^{-1} & : n < 0.
\end{cases}$$

By the multiplicative ergodic theorem, there exists a $\gamma(A) \in \mathbb{R}$ with

$$\gamma(A) = \lim_{n \to \infty} \frac{1}{n} \log \|A(n, \omega)\|$$

for $\mu$-almost every $\omega \in \Omega$. We say that $A$ is uniform if (2) holds for all $\omega \in \Omega$ and the convergence is uniform on $\Omega$.

**Lemma 2.1.** Suppose $(\Omega, T)$ is a uniquely ergodic dynamical system. Let $A : \Omega \to \mathbb{SL}(2, \mathbb{R})$ be continuous. Then the following are equivalent:

(i) $A$ is uniform and $\gamma(A) > 0$. 

(ii) There exists a continuous map \( P \) from \( \Omega \) to the (not necessarily orthogonal) projections on \( \mathbb{R}^2 \) and \( C, \beta > 0 \) such that
\[
\|A(n, \omega)P(\omega)A(m, \omega)^{-1}\| \leq C e^{-\beta(n-m)} \quad n \geq m
\]
\[
\|A(n, \omega)(1 - P(\omega))A(m, \omega)^{-1}\| \leq C e^{-\beta(m-n)} \quad n \leq m.
\]

(iii) The exist continuous maps \( U, V \) from \( \Omega \) to the projective space \( \mathbb{P}\mathbb{R}^2 \) over \( \mathbb{R}^2 \) and \( \tilde{C}, \tilde{\beta} > 0 \) such that
\[
\|A(n, \omega)u\| \leq \tilde{C} e^{-\tilde{\beta}n}\|u\| \quad \text{for} \quad n \geq 0, \ u \in U(\omega)
\]
\[
\|A(-n, \omega)v\| \leq \tilde{C} e^{-\tilde{\beta}n}\|v\| \quad \text{for} \quad n \geq 0, \ v \in V(\omega).
\]

Proof. The equivalence of (i) and (iii) is discussed in [8, Theorem 4].

The implication \((ii) \implies (iii)\) follows from the case \( m = 0 \) in (ii) after one defines \( U(\omega) = \text{Range } P(\omega) \) and \( V(\omega) = \text{Range } (1 - P(\omega)) \).

It remains to show \((iii) \implies (ii)\). It is not hard to see that \( U(\omega) \neq V(\omega) \) for every \( \omega \in \Omega \) (see [8] as well). Thus, for \( \omega \in \Omega \) fixed, any \( x \in \mathbb{R}^2 \) can be written uniquely as
\[
x = u + v
\]
with \( u \in U(\omega) \) and \( v \in V(\omega) \). We then define
\[
P(\omega)x := u.
\]

Thus,
\[
P(\omega)x \in U(\omega), \quad (1 - P(\omega))x \in V(\omega)
\]
for any \( \omega \in \Omega \) and \( x \in \mathbb{R}^2 \). As \( U \) and \( V \) are continuous, so is \( P \). Now, the case \( m = 0 \) of (ii) follows directly. Moreover, as \( U(T^m \omega) = A(m, \omega)U(\omega) \) and \( V(T^m \omega) = A(m, \omega)V(\omega) \), we have
\[
P(T^m \omega)A(m, \omega) = A(m, \omega)P(\omega)
\]
and therefore
\[
A(m, \omega)^{-1}P(T^m \omega) = P(\omega)A(m, \omega)^{-1}
\]
and similar with \( P \) replaced by \( 1 - P \). Given this, the case of general \( m \) in (ii) follows from the case \( m = 0 \) with \( \omega \) replaced by \( T^m \omega \). \( \square \)

Remark. Condition (ii) is known as exponential dichotomy. For a study of this condition in the context of Schrödinger operators on the real line we refer the reader to [10].

The cocycles we deal with do not take values in SL(2, \( \mathbb{R} \)). They take values in SU(1, 1), the set of 2 \times 2-matrices \( A \) with determinant equal to one and \( A^*JA = J \) for \( J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

Lemma 2.2. With the unitary matrix
\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}
\]
we have that \( U^{-1}SU(1, 1)U = \text{SL}(2, \mathbb{R}) \).

Proof. This follows from [10] Proposition 10.4.1] and the discussion preceding it. \( \square \)
Theorem 2. Suppose \((Ω, T)\) is strictly ergodic and \(f : Ω → D\) is non-constant and continuous. Define, for \(z ∈ ∂D\), the cocycle \(A_z : Ω → U(1, 1)\) by

\[
A_z(ω) = (1 − |f(ω)|^2)^{-1/2} \begin{pmatrix} z & -f(ω) \\ -f(ω)z & 1 \end{pmatrix}.
\]

Then, \(∂D \setminus Σ = \{z ∈ ∂D : A_z is uniform and \(γ(A_z) > 0\}\}\).

Proof. By Theorem 2.6 of [5] (see also Remark 1 on page 143),

\[
∂D \setminus Σ = \{z ∈ ∂D : U^{-1}A_zU satisfies (ii) of Lemma 2.1\}.
\]

By Lemma 2.1 we can replace condition (ii) by condition (i) in the preceding equality. As \(U\) is unitary, the assertion follows.

Remark. Since the matrices \(A_z(n, ω)\) are the transfer matrices associated with the Szegő recursion of the orthonormal polynomials with respect to the measures \(dμ_ω\), it is natural to call cocycles of the form (3) Szegő cocycles.

3. Boshernitzan’s Condition and Uniform Convergence

Proposition 3.1. Suppose the subshift \((Ω, T)\) satisfies the Boshernitzan condition. Then, every locally constant cocycle \(A : Ω → U(1, 1)\) is uniform.

Proof. Since \(|det A(ω)| = 1\), it suffices to prove uniformity for the cocycle \(\tilde{A} : Ω → SU(1, 1)\) given by \(\tilde{A}(ω) = (det A(ω))^{-1/2}A(ω)\). Here we choose an arbitrary but fixed branch of the square root. Note that \(\tilde{A}\) is locally constant since \(A\) is locally constant. Obviously, \(\tilde{A}\) is uniform if and only if the locally constant cocycle \(\bar{A} = U^{-1}\tilde{A}U\) with \(U\) as in Lemma 2.2 is uniform. Now, by Lemma 2.2 \(\bar{A}\) takes values in \(SL(2, R)\).

It was shown in [3, Theorem 1] that every locally constant \(SL(2, R)\) cocycle over \((Ω, T)\) is uniform if \((Ω, T)\) satisfies the Boshernitzan condition.

4. Proof of the Main Result

The OPUC version of Kotani theory yields the following consequence: If the Verblunsky coefficients are ergodic, aperiodic, and take finitely many values, then

\[
\text{Leb}\left(\{z ∈ ∂D : γ(A_z) = 0\}\right) = 0.
\]

See, for example, [10, Theorem 10.11.4]. This result applies in the setting of Theorem 1.

Proof of Theorem 1. By Theorem 2 the common essential support of the measures \(dμ_ω\) is given by

\[
\{z ∈ ∂D : γ(A_z) = 0\} \cup \{z ∈ ∂D : γ(A_z) > 0 \text{ and } A_z \text{ is non-uniform}\}.
\]

The first set has zero Lebesgue measure by (4) and the second set is empty by Proposition 3.1.
5. Examples of Subshifts Satisfying Boshernitzan’s Condition

In this section we give a number of examples of aperiodic subshifts that satisfy the Boshernitzan condition. As was shown in [4], most of the commonly studied minimal subshifts satisfy the Boshernitzan condition. For example, subshifts obtained by codings of rotations and interval exchange transformations.

Here we focus on subshifts associated with codings of rotations, that is, models displaying a certain kind of quasi-periodicity. This class has attracted a large amount of attention in other settings (e.g., discrete Schrödinger operators and Jacobi matrices) but their study in the context of OPUC is still in its early stages.

Let \( \alpha \in (0,1) \) be irrational, \( 0 = \beta_0 < \beta_1 < \cdots < \beta_{p-1} < \beta_p = 1 \), and associate the intervals of the induced partition with \( p \) symbols \( v_1, \ldots, v_p \):

\[
v_n(\theta) = v_k \iff \theta + n\alpha \mod 1 \in [\beta_{k-1}, \beta_k).
\]

We obtain a subshift over the alphabet \( \{v_1, \ldots, v_p\} \),

\[
\Omega_{\alpha,\beta_1,\ldots,\beta_{p-1}} = \{v(\theta) : \theta \in [0,1)\}.
\]

The case \( p = 2 \) is often of special interest. The following theorem summarizes the results obtained in [4] for this case.

**Theorem 3.** Let \( \alpha \) be irrational.

(a) If \( \beta = m\alpha + n \mod 1 \) with integers \( m, n \), then \( \Omega_{\alpha,\beta} \) satisfies (B).

(b) If \( \alpha \) has bounded partial quotients, then \( \Omega_{\alpha,\beta} \) satisfies (B) for every \( \beta \in (0,1) \).

(c) If \( \alpha \) has unbounded partial quotients, then \( \Omega_{\alpha,\beta} \) satisfies (B) for Lebesgue almost every \( \beta \in (0,1) \).

Recall that \( \alpha \) has bounded partial quotients if and only if the sequence \( \{a_n\} \) in the continued fraction expansion of \( \alpha \),

\[
\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},
\]

is bounded.

It was also shown in [4] that in the case where \( \alpha \) has unbounded partial quotients, there always exists a \( \beta \in (0,1) \) such that \( \Omega_{\alpha,\beta} \) does not satisfy (B). These results show that the validity of (B) is well understood for codings of rotations with respect to a partition of the circle into two intervals.

In the general case, we have the following theorem, which was derived in [4] from a result of Boshernitzan [1].

**Theorem 4.** Let \( \alpha \in (0,1) \) be irrational and suppose that \( \beta_1, \ldots, \beta_{p-1} \in \mathbb{Q} \). Then the subshift \( \Omega_{\alpha,\beta_1,\ldots,\beta_{p-1}} \) satisfies the Boshernitzan condition (B).

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