COMMUTATIVE MATCHING ROTA-BAXTER OPERATORS, DECORATED SHUFFLE PRODUCTS AND MATCHING ZINBIEL ALGEBRAS

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ABSTRACT. The Rota-Baxter algebra and the shuffle product are both algebraic structures arising from integral operators and integral equations. Free commutative Rota-Baxter algebras provide an algebraic framework for integral equations with the simple Riemann integral operator. The Zinbiel algebra provides a category to characterize the shuffle product algebra as the free object. Motivated by algebraic structures underlying integral equations involving multiple integral operators and kernels, we study commutative matching Rota-Baxter algebras and construct the free objects making use of the decorated shuffle product. We also construct free commutative matching Rota-Baxter algebras in a relative context, to emulate the action of the integral operators on the coefficient functions in an integral equation. We finally show that the decorated shuffle product has the universal property as the free matching Zinbiel algebra.

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1. Introduction

In view of applications to Volterra integral equations with multiple kernels, this paper gives an explicit construction of free commutative matching Rota-Baxter algebras in both the absolute and relative contexts. The decorated shuffle product needed in the construction is shown to give the free object in the category of matching Zinbiel algebras.
1.1. **Integrations, Rota-Baxter algebras and shuffle product.** Algebraic approaches of integrals led to the notions of the Rota-Baxter algebra and the shuffle product. The notion of a Rota-Baxter algebra has its origin in the probability study of G. Baxter on fluctuation theory [5]. It is a pair \((R, P)\) where \(R\) is an associative algebra and \(P\) is a linear operator on \(R\) satisfying the **Rota-Baxter identity**

\[
P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy) \quad \text{for all } x, y \in R.
\]

Here \(\lambda\) is a fixed scalar in the base ring. Even though the primary interest of Baxter was when \(\lambda = -1\), he observed that the equation when \(\lambda = 0\) is the integration by parts formula for the **simple Riemann integral** (here simple means having a trivial kernel)

\[
I(f)(x) := \int_0^x f(t) \, dt
\]

on continuous functions. Thus when \(\lambda = 0\), the pair \((R, P)\) can be called an integral algebra as an integral analog of the differential algebra [31, 36, 39] originated from the algebraic study of differential equations. The free differential algebra, realized as the differential polynomial algebra, provides a uniform setting to consider all differential equations, just like what the polynomial algebra provides for algebraic equations. Indeed a large part of differential algebra is centered on developing an algebraic theory of differential equations in parallel to the theories of algebraic equations, including the corresponding Galois theory, algebraic groups and algorithms.

The recent decades witnessed a tremendous development of the Rota-Baxter algebra through its applications and connections in diverse areas in mathematics and physics [1, 3, 4, 14, 15, 24, 26, 41]. However, much still needs to be done about Rota-Baxter algebras of weight zero as an algebraic framework for integral equations. In this regard, free commutative Rota-Baxter algebras of weight zero should serve as the universal space for integral equations with the simple integral operator in Eq. (1), similar to the role served by free differential algebras for differential equations. For some related literature, see [6, 21, 40].

Incidently, shortly before Baxter introduced the algebraic formulation of the integration by parts formula that later bore his name, another notion also merged in the algebraic study of integration, that is the one of the shuffle product. A fundamental notion in many areas from combinatorics to Hopf algebras and free Lie algebras [17, 37, 38], the shuffle product has been the key in the study of integrals from various viewpoints. One is the theory of iterated path integrals invented by K. T. Chen who, as a prominent differential geometer, pursued this subject throughout his career [10, 11, 12, 13]. The subject has since become an important tool in various branches of algebraic geometry, topology, number theory and mathematical physics [32].

In number theory, through the integral representations of multiple zeta values by Kontsevich, the shuffle product, together with the quasi-shuffle product, provides the algebraic study of multiple zeta values with the (extended) double shuffle framework that conjecturally dictates all the algebraic relations among multiple zeta values [7, 27, 28, 29, 30].

On the categorical level, the ubiquitous role of the shuffle product algebra is manifested algebraically by the fact that the shuffle product algebra is the free Zinbiel algebra which is an equivalent form of the free commutative dendriform algebra [34].

Relating these two algebraic interpretations of the simple integral operator, the free commutative Rota-Baxter algebras of weight zero naturally builds on the shuffle product algebra, while commutative Rota-Baxter algebras naturally give rise to Zinbiel algebras.
1.2. *Multiple Rota-Baxter algebras and decorated shuffle product*. With the simple integral operator in Eq. (1) as the inceptive case, there are many other integral operators including the various Fredholm operators and Volterra operators [44]. Further there can be different integral operators appearing in the same integral equation. This presents the need to consider integral type or Rota-Baxter type algebras with multiple operators, leading to the subject of study of this paper and its predecessors [22, 42] which were also motivated by multiple pre-Lie algebras arising from the recent work [8, 18] on algebraic renormalization of regularity structures and polarized associative Yang-Baxter equations [1, 9]. Application to integral equations can be found in [25].

Thus the purpose of this paper is two fold. On the one hand, we construct free commutative matching Rota-Baxter algebras as an algebraic framework to consider integral equations with multiple integral operators of Rota-Baxter nature. There we need to consider two cases. To begin with, the free commutative matching Rota-Baxter algebra is generated on an algebra. Since in an integral equation, the integral operators also act on the functions in the base algebra, there is the need to construct free matching Rota-Baxter algebras for which the generating algebra has matching Rota-Baxter operators of its own that is compatible with the matching Rota-Baxter operators on the whole free object. In this case, the constructions are based a decorated or colored version of the shuffle product algebra.

On the other hand, in analogous to Zinbiel algebras as the category in which to characterize the shuffle product algebra by a universal property [34], we introduce matching Zinbiel algebras in order to provide a suitable category for the universal property for the decorated shuffle product.

With these goals in mind, here is the layout of the paper. In Section 2, we first recall notions and basic examples of matching Rota-Baxter algebras and the construction of free commutative Rota-Baxter algebra by the shuffle product. We then construct free commutative matching Rota-Baxter algebras by a decorated shuffle product (Theorem 2.10).

In Section 3, we introduce the notion of a relative Rota-Baxter algebra for the need of integral equations where the coefficient functions already carry an integral operator, as noted above. As in the case of commutative algebras, here relative means we only consider matching Rota-Baxter algebras on a given base matching Rota-Baxter algebra. We then construct free matching Rota-Baxter algebras in the relative context (Theorem 3.3). The free object is a subalgebra of the free object in the non-relative case in Section 2, but the linear operators have to be dealt with carefully in verifying that they satisfy the required conditions.

In Section 4, we consider commutative dendriform algebras and its equivalent notion of Zinbiel algebras in the matching or multiple context. We then characterize the decorated shuffle product algebras in the construction of free commutative matching Rota-Baxter algebras in Section 2 as the free objects in the category of matching Zinbiel algebras (Theorem 4.10), generalizing the shuffle product algebra construction of free Zinbiel algebras of Loday [34].

It is worth noting that quite much progress has been made in another algebraic structure with multiple Rota-Baxter operators, called the Rota-Baxter family algebra with origin in a Lie theoretic approach to renormalization [16, 23]. This includes the constructions of free objects, related families of dendriform algebras and pre-Lie algebras, and the generalizations from the perspectives of monoidal categories and algebraic operad [2, 18, 19, 20, 35, 43].

**Notation.** Throughout this paper, let $k$ be a unitary commutative ring which will be the base ring of all modules, algebras, tensor products, as well as linear maps, unless otherwise stated. By an algebra we mean an associative unitary ($k$)-algebra.
2. Free commutative matching Rota-Baxter algebras

In this section, we first recall the concept of a matching Rota-Baxter algebra and the construction of free commutative Rota-Baxter algebras utilizing the shuffle product. We then generalize the shuffle product to a decorated version in order to construct free commutative matching Rota-Baxter algebras.

2.1. Matching Rota-Baxter algebras. Let us recall the concept of matching Rota-Baxter algebras [42] as a generalization of Rota-Baxter algebras.

Definition 2.1. Let $\Omega$ be a nonempty set and $\lambda_\Omega := (\lambda_\omega)_{\omega \in \Omega}$ a family of elements of $k$ parameterized by $\Omega$. Equivalently, $\lambda_\Omega$ is a map $\lambda_\Omega : \Omega \to k$. A matching Rota-Baxter algebra (or simply MRBA) of weight $\lambda_\Omega$ is a pair $(R, P_\Omega)$ consisting of an algebra $R$ and a family $P_\Omega := (P_\omega)_{\omega \in \Omega}$ of linear operators $P_\omega : R \to R, \omega \in \Omega$ that satisfy the matching Rota-Baxter equation

$$P_\omega(x)P_\beta(y) = P_\alpha(xP_\beta(y)) + P_\beta(P_\alpha(x)y) + \lambda_\beta P_\alpha(xy) \quad \text{for } x, y \in R, \alpha, \beta \in \Omega.$$ 

For each $\omega \in \Omega$, $(R, P_\omega)$ is a Rota-Baxter algebra of weight $\lambda_\omega$. When $\lambda_\Omega = \{\lambda\}$, that is, when the map $\lambda_\Omega : \Omega \to k$ is a constant function, we also call the MRBA $(R, \lambda_\Omega)$ to have weight $\lambda$.

Definition 2.2. Let $(R, P_\Omega)$ and $(R', P'_\Omega)$ be MRBAs of the same weight $\lambda_\Omega$. A linear map $\phi : R \to R'$ is called an MRBA homomorphism if $\phi$ is an algebra homomorphism such that $\phi P_\omega = P'_\omega \phi$ for all $\omega \in \Omega$.

A natural example of Rota-Baxter operator of weight 0 is the operator of Riemann integral in Eq. (1). Consider the $\mathbb{R}$-algebra $R := \text{Cont}(\mathbb{R})$ of continuous functions on $\mathbb{R}$. Then $(R, I)$ is a Rota-Baxter algebra of weight 0 [5]. We generalize this to the multiple case.

Example 2.3. Fix a family $k_\omega(x)$ of functions (called kernels [44]) in $R$ parameterized by $\omega \in \Omega$. Define the Volterra integral operators

$$I_\omega : R \to R, \quad f(x) \mapsto \int_0^x k_\omega(t)f(t)\, dt, \quad \omega \in \Omega. \quad (2)$$

Note that $I_\omega(f) = I(k_\omega f)$ for the integral operator $I$ in Eq. (1). Then from the Rota-Baxter property of $I$, we obtain, for $\alpha, \beta \in \Omega$ and $f, g \in R$,

$$I_\alpha(f)I_\beta(g) = I(k_\alpha f)I(k_\beta g) = I(k_\alpha fI(k_\beta g)) + I(I(k_\alpha f)k_\beta g) = I_\alpha(fI_\beta(g)) + I_\beta(I_\alpha(f)g).$$

Thus $(R, (I_\omega)_{\omega \in \Omega})$ is a matching Rota-Baxter algebra of weight zero. Understanding integral equations with such a family of Volterra operators is our main motivation in the construction of free commutative MRBAs, especially in the relative context (Section 3). See [25] for further study of Volterra operators and integral equations from an algebraic point of view.

Here are some further examples and properties of MRBAs [42].

Remark 2.4. (a) Let $(R, P_\Omega)$ be an MRBA of weight $\lambda_\Omega$. For a linear combination

$$P := \sum_{\omega \in \Omega} a_\omega P_\omega, \quad a_\omega \in k,$$

with finite support, the $(R, P)$ is a Rota-Baxter algebra of weight $\sum_{\omega \in \Omega} a_\omega \lambda_\omega$.

(b) Matching Rota-Baxter algebras have a close connection with matching pre-Lie algebras introduced by Foissy [18]. Let $(R, P_\Omega)$ be a MRBA of weight $\lambda_\Omega$. Define

$$x *_\omega y := P_\omega(x)y - yP_\omega(x) - \lambda_\omega xy \quad \text{for } x, y, z \in R, \omega \in \Omega.$$

Then the pair $(R, (*_\omega)_{\omega \in \Omega})$ is a matching pre-Lie algebra.
(c) For \( r, s \in R \otimes R \), let
\[
   r_{13}s_{12} - r_{12}s_{23} + r_{23}s_{13} = -as_{13}
\]
be the **polarized associative Yang-Baxter equation** of weight \( \lambda \). Then a solution of this equation gives a matching Rota-Baxter operator of weight \( \lambda \).

2.2. **Decorated shuffle product.** We first recall the notion of the shuffle product and its application to the construction of free commutative Rota-Baxter algebras \([24, 26]\). Then this process is extended to the construction of free commutative MRBAs.

2.2.1. **Shuffle product and free commutative Rota-Baxter algebras.** For \( k \geq 1 \), let \( S_k \) denote the symmetric group on the set \([k]\) of first \( k \) integers. For \( m, n \geq 1 \), define the set of \((m, n)\)-shuffles by
\[
   S(m, n) := \left\{ \sigma \in S_{m+n} \bigg| \sigma^{-1}(1) < \cdots < \sigma^{-1}(m), \sigma^{-1}(m+1) < \cdots < \sigma^{-1}(m+n) \right\}.
\]

Let \( V \) be a \( k \)-module. Define the \( k \)-module
\[
   T(V) := \bigoplus_{k \geq 0} V^{\otimes k} = k \oplus V \oplus V^{\otimes 2} \oplus \cdots
\]
with the convention that \( V^{\otimes 0} = k \). For \( m, n \geq 1 \), let
\[
a = a_1 \otimes \cdots \otimes a_m \in V^{\otimes m} \quad \text{and} \quad b = b_1 \otimes \cdots \otimes b_n \in V^{\otimes n}
\]
be two pure tensors. For \( \sigma \in S(m, n) \), the **shuffle** of \( a \) and \( b \) via \( \sigma \) is given by
\[
   \sigma(a \otimes b) := u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(m+n)} \in V^{\otimes (m+n)},
\]
where
\[
u_k = \begin{cases} 
   a_k & \text{if } 1 \leq k \leq m, \\
   b_{k-m} & \text{if } m + 1 \leq k \leq m + n.
\end{cases}
\]
The sum
\[
   a \shuffle b := \sum_{\sigma \in S(m, n)} \sigma(a \otimes b)
\]
is called the **shuffle product** of \( a \) and \( b \). By the convention, \( a \shuffle b \) is the scale product when \( m = 0 \) or \( n = 0 \). Then the pair \((T(V), \shuffle)\) is a commutative algebra.

The shuffle product \( \shuffle \) on \( T(V) \) can also be defined recursively, so that for \( a = a_1 \otimes a' \in V^{\otimes m} \) and \( b = b_1 \otimes b' \in V^{\otimes n} \) with \( a' \in V^{\otimes (m-1)} \) and \( b' \in V^{\otimes (n-1)} \), we have
\[
   a \shuffle b = a_1 \otimes (a' \shuffle b') + b_1 \otimes (a \shuffle b').
\]

Now let \( A \) be a commutative algebra. Define the \( k \)-module
\[
   \Pi^+(A) := T(A)
\]
with the shuffle product. Further define
\[
   \Pi(A) := A \otimes \Pi^+(A) = \bigoplus_{k \geq 1} A^{\otimes k}
\]
to be the tensor product algebra whose product \( \odot \) is called the **augmented shuffle product**. More precisely, \( \odot \) is defined by
\[
   (a_0 \otimes a) \odot (b_0 \otimes b) := (a_0b_0) \otimes (a \shuffle b) \quad \text{for } a_0, b_0 \in A, a, b \in \Pi^+(A).
\]

Define
\[
   P : \Pi(A) \to \Pi(A), \quad a \mapsto P(a) = 1 \otimes a.
\]
2.2.2. Decorated shuffle product. Now we are going to construct a free commutative MRBA on $A$. We note that $\Pi^+(A)$ is naturally identified with the subalgebra $1 \otimes \Pi^+(A)$ of $\Pi(A)$ and, under this identification, we have $\Pi(A) = A(1 \otimes \Pi^+(A))$ as a product of subalgebras. In fact, $\Pi(A)$ is the internal tensor product of the subalgebras $A$ and $1 \otimes \Pi^+(A)$ in the sense that $\Pi(A) = A(1 \otimes \Pi^+(A))$ and the two subalgebras $A$ and $1 \otimes \Pi^+(A)$ are linearly disjoint.

We next apply this idea to the construction of free commutative MRBAs.

From the above construction of the free commutative Rota-Baxter algebra, we note that the Rota-Baxter operator $P$ can be replaced by the notation $\otimes$ to obtain $\Pi(A)$. Strongly motivated by this, we now generalized tensor product to a colored version.

Let $\Omega$ be a nonempty set. For $k \geq 0$, denote

$$A \otimes_{\Omega} := A \otimes_{\Omega} \cdots \otimes_{\Omega} A := \bigoplus_{\omega_1, \ldots, \omega_k \in \Omega} A \otimes_{\omega_1} \cdots \otimes_{\omega_k} A,$$

with the convention that $A \otimes_{\Omega} = k$ when $k = 0$. Here for each $\omega \in \Omega$, $\otimes_{\omega}$ is the usual tensor product $\otimes$ over $k$. But we use the subscript $\omega$ to distinguish the tensor products as $\omega$ varies in $\Omega$.

We denote

$$\Pi_{\Omega}(A) := \bigoplus_{k \geq 1} A \otimes_{\Omega}$$

as a multiple operator variation of $\Pi(A)$.

Note that $A$ and

$$k \otimes_{\Omega} A \otimes_{\Omega} := \bigoplus_{\omega_1, \ldots, \omega_k \in \Omega} k \otimes_{\omega_1} A \otimes_{\omega_2} \cdots \otimes_{\omega_k} A$$

are submodules of $A \otimes_{\Omega}$ and the the natural map

$$A \otimes (k \otimes_{\Omega} A \otimes_{\Omega}) \to A \otimes_{\Omega}$$

is a linear isomorphism. So $A \otimes_{\Omega}$ is the linear disjoint product of $A$ and $(k \otimes_{\Omega} A \otimes_{\Omega})$. To emphasize this property, we write

$$A \otimes_{\Omega} = A \otimes (k \otimes_{\Omega} A \otimes_{\Omega}).$$

Then denoting

$$\Pi^+_{\Omega}(A) := \bigoplus_{n \geq 0} (k \otimes_{\Omega} A \otimes_{\Omega} \cdots) \otimes_{\Omega} = k \otimes (A \otimes (k \otimes_{\Omega} A \otimes_{\Omega} \cdots)), \quad (k \otimes_{\Omega} A \otimes_{\Omega} \cdots),$$

we have the linear disjoint of subalgebras

$$\Pi^+_{\Omega}(A) = A \otimes \left(\bigoplus_{k \geq 1} \otimes_{\Omega} A \otimes_{\Omega} \right) = A \otimes (\Pi^+_{\Omega}(A)).$$

Now we equip a commutative product $\otimes_{\Omega}$ on $\Pi_{\Omega}(A)$ by first defining a product $\Pi^+_{\Omega}$ on $\Pi^+_{\Omega}(A)$, called the decorated shuffle product.

For $m, n \geq 1$, let

$$1 \otimes_{\alpha_1} a := 1 \otimes_{\alpha_1} a_1 \otimes_{\alpha_2} \cdots \otimes_{\alpha_m} a_m \in k \otimes_{\Omega} A \otimes_{\Omega}$$

and

$$1 \otimes_{\beta_1} b := 1 \otimes_{\beta_1} b_1 \otimes_{\beta_2} \cdots \otimes_{\beta_n} b_n \in k \otimes_{\Omega} A \otimes_{\Omega}$$

be pure tensors. For $\sigma \in S(m, n)$, the decorated shuffle of $1 \otimes_{\alpha_1} a$ and $1 \otimes_{\beta_1} b$ via $\sigma$ is given by

$$\sigma(1 \otimes_{\alpha_1} a, 1 \otimes_{\beta_1} b) := 1 \otimes_{\alpha_{\sigma(1)}} u_{\sigma(1)} \otimes_{\alpha_{\sigma(2)}} u_{\sigma(2)} \otimes_{\alpha_{\sigma(3)}} \cdots \otimes_{\alpha_{\sigma(m+n)}} u_{\sigma(m+n)} \in k \otimes_{\Omega} A \otimes_{\Omega}.$$
where
\[
\gamma_k := \begin{cases} \alpha_k, & \text{if } 1 \leq k \leq m, \\ \beta_{k-m}, & \text{if } m + 1 \leq k \leq m + n, \end{cases} \quad u_k = \begin{cases} a_k, & \text{if } 1 \leq k \leq m, \\ b_{k-m}, & \text{if } m + 1 \leq k \leq m + n. \end{cases}
\]

We then define the **decorated shuffle product** of \(1 \otimes_{\alpha_1} a \) and \(1 \otimes_{\beta_1} b\) to be the sum
\[
(1 \otimes_{\alpha_1} a) \omega (1 \otimes_{\beta_1} b) := \sum_{\sigma \in S(m, n)} \sigma(1 \otimes_{\alpha_1} a, 1 \otimes_{\beta_1} b).
\]

By convention, the product is the scale product when \(m = 0\) or \(n = 0\).

To see that the pair \((III^+_{\Omega}(A), \omega_{\Omega})\) is a commutative algebra, we use a compact notation for pure tensors in \(III^+_{\Omega}(A)\):
\[
\begin{bmatrix} \omega_1 \\ a_1 \\ \alpha_1 \\ 1 \\ \cdots \\ \omega_k \\ a_k \end{bmatrix} := 1 \otimes_{\omega_1} a_1 \otimes_{\omega_2} a_2 \otimes_{\omega_3} \cdots \otimes_{\omega_k} a_k.
\]

With this notation, we see that the decorated shuffle in Eq. (8) is simply the usual shuffle in Eq. (3): for integers \(m, n \geq 1\), \((m, n)\)-shuffle \(\sigma \in S_{m, n}\) and pure tensors
\[
\begin{align*}
1 \otimes_{\alpha_1} a &= 1 \otimes_{\alpha_1} a_1 \otimes_{\alpha_2} a_2 \cdots \otimes_{\alpha_m} a_m = \begin{bmatrix} \alpha_1 \\ a_1 \\ \alpha_1 \\ 1 \\ \cdots \\ \alpha_m \\ a_m \end{bmatrix} \in k \otimes_{\Omega} A^{\otimes m} \\
1 \otimes_{\beta_1} b &= 1 \otimes_{\beta_1} b_1 \otimes_{\beta_2} b_2 \cdots \otimes_{\beta_n} b_n = \begin{bmatrix} \beta_1 \\ b_1 \\ \beta_1 \\ 1 \\ \cdots \\ \beta_n \\ b_n \end{bmatrix} \in k \otimes_{\Omega} A^{\otimes n},
\end{align*}
\]
we have
\[
\sigma(1 \otimes_{\alpha_1} a, 1 \otimes_{\beta_1} b) = \sigma \left( \begin{bmatrix} \alpha_1 \\ a_1 \\ \alpha_1 \\ 1 \\ \cdots \\ \alpha_m \\ a_m \end{bmatrix} \begin{bmatrix} \beta_1 \\ b_1 \\ \beta_1 \\ 1 \\ \cdots \\ \beta_n \\ b_n \end{bmatrix} \right),
\]
where the right hand side is the shuffle in Eq. (3). Consequently,
\[
(1 \otimes_{\alpha_1} a) \omega (1 \otimes_{\beta_1} b) := \sum_{\sigma \in S(m, n)} \sigma(1 \otimes_{\alpha_1} a_1 \otimes_{\alpha_2} a_2 \otimes_{\alpha_3} a_3 \cdots \otimes_{\alpha_m} a_m, 1 \otimes_{\beta_1} b_1 \otimes_{\beta_2} b_2 \otimes_{\beta_3} b_3 \cdots \otimes_{\beta_n} b_n).
\]

It then follows immediately that the product \(\omega_{\Omega}\) on \(III^+_{\Omega}(A)\) is commutative associative. Furthermore, the recursive formula of \(\omega_{\Omega}\) in Eq. (4) gives the recursive formula of \(\omega_{\Omega}\): for \(m, n \geq 1\) we have
\[
(1 \otimes_{\alpha_1} a) \omega (1 \otimes_{\beta_1} b) = \begin{bmatrix} \alpha_1 \\ a_1 \\ \alpha_1 \\ 1 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ a_2 \\ \alpha_2 \\ 1 \\ \cdots \\ \alpha_m \\ a_m \\ \alpha_m \\ 1 \\ \cdots \end{bmatrix} \omega_{\Omega} \begin{bmatrix} \beta_1 \\ b_1 \\ \beta_2 \\ b_2 \\ \cdots \\ \beta_n \\ b_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ b_1 \end{bmatrix} \left( (1 \otimes_{\alpha_1} a_1 \otimes_{\alpha_2} a_2 \otimes_{\alpha_3} a_3 \cdots \otimes_{\alpha_m} a_m) \omega (1 \otimes_{\beta_1} b) \right) + \begin{bmatrix} \alpha_1 \\ a_1 \end{bmatrix} \left( (1 \otimes_{\beta_1} b_1 \otimes_{\beta_2} b_2 \otimes_{\beta_3} b_3 \cdots \otimes_{\beta_n} b_n) \right).
\]

Here are some examples to compare the two notations.

**Example 2.5.** Let \(m = n = 1\). Then
\[
(1 \otimes_{\alpha_1} a_1) \omega (1 \otimes_{\beta_1} b_1) = \begin{bmatrix} \alpha_1 \\ a_1 \end{bmatrix} \omega_{\Omega} \begin{bmatrix} \beta_1 \\ b_1 \end{bmatrix}
\]
Thus the pair \((X, \oplus, \otimes)\) is a commutative algebra, called the \emph{decorated shuffle algebra}.

\subsection{Decorated augmented shuffle algebra}

By Eq. (9), \(\omega_\Omega\) is still defined by a shuffle product. Thus the pair \((\Pi^+_\Omega(A), \omega_\Omega)\) is a commutative algebra, called the \emph{decorated shuffle algebra}.

Now we recall the linear disjoint factorization from Eqs. (6) and (7):

\[
A^{\otimes n} = A \boxtimes \left( k \otimes_\Omega A^{\otimes (k-1)} \right) = A \boxtimes \left( \bigoplus_{x_1, \ldots, x_{k-1} \in \Omega} k \otimes_\Omega A \otimes_{\omega_1} \cdots \otimes_{\omega_{k-1}} A \right)
\]

and

\[
\Pi_\Omega(A) = \otimes_{k \geq 1} A^{\otimes n} = A \boxtimes \left( \bigoplus_{k \geq 1} k \otimes_\Omega A^{\otimes (k-1)} \right) = A \boxtimes \Pi^+_\Omega(A),
\]

with

\[
\Pi^+_\Omega(A) := \bigoplus_{n \geq 0} (k \otimes_\Omega A^{\otimes n}) = k \oplus (k \otimes_\Omega A) \oplus (k \otimes_\Omega A^{\otimes 2}) \oplus \cdots.
\]

Then we define a product \(\odot_\Omega\) on \(\Pi_\Omega(A)\) to be the tensor product of the product on \(A\) and the decorated shuffle product \(\omega_\Omega\) on \(\Pi^+_\Omega(A)\). To be precise, for pure tensors

\[
a := a_0 \otimes_{\omega_\Omega} a' := a_0 \otimes_{\omega_\Omega} a_1 \otimes_{\omega_\Omega} \cdots \otimes_{\omega_\Omega} a_m \in A^{\otimes n+1} \subseteq \Pi_\Omega(A),
\]

\[
b := b_0 \otimes_{\omega_\Omega} b' := b_0 \otimes_{\omega_\Omega} b_1 \otimes_{\omega_\Omega} \cdots \otimes_{\omega_\Omega} b_n \in A^{\otimes n+1} \subseteq \Pi_\Omega(A)
\]

with \(m, n \geq 0\), define

\[
a \odot_\Omega b := a_0 b_0 \left( (1 \otimes_{\omega_\Omega} a') \odot_\Omega (1 \otimes_{\omega_\Omega} b') \right),
\]
called the \emph{decorated augmented shuffle product}. The product \(\odot_\Omega\) also has a recursive expression:

\[\text{Lemma 2.7. For pure tensors } a \text{ and } b, \text{ we have}
\]

\[
a \odot_\Omega b := \begin{cases} a_0 b_0, & \text{if } a = a_0, b' = b_0, \\ a_0 b_0 \otimes_{\omega_\Omega} a', & \text{if } a \neq a_0, b = b_0, \\ a_0 b_0 \otimes_{\omega_\Omega} b', & \text{if } a = a_0, b \neq b_0, \\ a_0 b_0 \otimes_{\omega_\Omega} (a' \odot_\Omega (1 \otimes_{\omega_\Omega} b')) + a_0 b_0 \otimes_{\omega_\Omega} ((1 \otimes_{\omega_\Omega} a') \odot_\Omega b'), & \text{if } a \neq a_0, b \neq b_0. \end{cases}
\]
Proof. The cases when \( a = a_0 \) or \( b = b_0 \) follows from observation. The case of \( a \neq a_0 \) and \( b \neq b_0 \) follows from
\[
a \cdot \omega \ b = a_0 b_0 ((1 \otimes a_1, a') \circ \omega (1 \otimes \beta_1, b')) \\
= a_0 b_0 \otimes a_1 \left( (1 \otimes a_2, a_3 \cdots \otimes a_m \omega (1 \otimes \beta_1, \beta_2') \right) \\
\quad \quad \quad + a_0 b_0 \otimes \beta_1 \ b_1 \left( (1 \otimes a_1, a') \cdot \omega (1 \otimes \beta_1, \beta_2, \cdots \otimes \beta_n, \beta_n) \right) \quad \text{(by Eq. (10))}
\]
\[
= a_0 b_0 \otimes \omega \left( (1 \otimes a_2, a_3 \cdots \otimes a_m \omega (1 \otimes \beta_1, \beta_2') \right) \\
\quad \quad \quad + a_0 b_0 \otimes \omega \left( (1 \otimes a_1, a') \omega (1 \otimes \omega_1, \omega_2, \cdots \otimes \omega_n, b_n) \right) \quad \text{(by Eq. (13))}
\]
\[
= a_0 b_0 \otimes a_1 \left( a' \cdot \omega (1 \otimes \beta_1, b') \right) + a_0 b_0 \otimes \beta_1 \left( (1 \otimes a_1, a') \cdot \omega b' \right). \quad \Box
\]

To summarize, by Eq. (13) we have

**Proposition 2.8.** The pair \((\Omega \omega, \cdot \omega)\) is a commutative algebra.

2.3.2. The universal property. We now show that the decorated augmented shuffle product algebra \(\Omega \omega \cdot \omega\) is the free commutative matching Rota-Baxter algebra of weight zero on the algebra \(A\).

**Definition 2.9.** Let \(\omega\) be a nonempty set and \(A\) a commutative algebra. A free commutative MRBA of weight zero on \(A\) is a commutative MRBA \(F_{\text{MRB}}(A)\) of weight zero together with an algebra homomorphism \(j_A : A \to F_{\text{MRB}}(A)\) such that for any commutative MRBA \((R, (P_{\omega, R})_{\omega \in \Omega})\) of weight zero and any algebra homomorphism \(f : A \to R\), there is a unique MRBA homomorphism \(\tilde{f} : F_{\text{MRB}}(A) \to R\) such that \(f = \tilde{f} j_A\).

Now we arrive at our main result in this section. For each \(\omega \in \Omega\), define
\[
P_{\omega} := P_{\omega, A} : \Omega \omega \cdot \omega \to \Omega \omega \cdot \omega, \quad a \mapsto 1 \otimes a, \text{ for } a \in \Omega \omega \cdot \omega.
\]

**Theorem 2.10.** Let \(\omega\) be a nonempty set and \(A\) a commutative algebra. Then the triple \((\Omega \omega \cdot \omega, (P_{\omega})_{\omega \in \Omega}),\) together with the natural embedding \(j_A : A \to \Omega \omega \cdot \omega\), is the free commutative MRBA of weight zero on \(A\).

**Proof.** We first show that \((\Omega \omega \cdot \omega, (P_{\omega})_{\omega \in \Omega})\) is a commutative MRBA of weight zero on \(A\). Since \((\Omega \omega \cdot \omega, \cdot \omega)\) is a commutative algebra by Proposition 2.8, we only need to verify that \((P_{\omega})_{\omega \in \Omega}\) satisfy the matching Rota-Baxter equation in the case of \(\lambda = 0\). For \(\alpha_1, \beta_1 \in \Omega\) and \(a', b' \in \Omega \omega \cdot \omega\), we have
\[
P_{\alpha_1}(a') \cdot \omega P_{\beta_1}(b') \\
= (1 \otimes a_1, a') \cdot \omega (1 \otimes \beta_1, b') \quad \text{(by Eq. (15))}
\]
\[
= 1 \otimes a_1 (a' \cdot \omega (1 \otimes \beta_1, b')) + 1 \otimes \beta_1 ((1 \otimes a_1, a') \cdot \omega b') \quad \text{(by Eq. (14))}
\]
\[
= 1 \otimes a_1 (a' \cdot \omega P_{\beta_1}(b')) + 1 \otimes \beta_1 (P_{\alpha_1}(a') \cdot \omega b') \quad \text{(by Eq. (15))}
\]
\[
= P_{\alpha_1}(a' \cdot \omega P_{\beta_1}(b')) + P_{\beta_1}(P_{\alpha_1}(a') \cdot \omega b').
\]

It remains to verify the universal property of \((\Omega \omega \cdot \omega, (P_{\omega})_{\omega \in \Omega})\). Let \((R, (P_{\omega, R})_{\omega \in \Omega})\) be a commutative MRBA of weight zero and let \(f : A \to R\) be an algebra homomorphism.

**Existence.** To construct a linear map \(\tilde{f} : \Omega \omega \cdot \omega \to R\), it suffices to define \(\tilde{f}(a)\) for pure tensor \(a = a_0 \otimes \omega_1, a' \in A^\otimes (m+1)\) with \(m \geq 0\), \(a_0 \in A\) and \(a' \in A^\otimes m\). For this we employ in induction on \(m \geq 0\). For the initial step of \(m = 0\), we have \(a = a_0\) and define
\[
\tilde{f}(a) = \tilde{f}(a_0) := f(a_0).
\]
For the induction step of \( m \geq 1 \), we define
\[
\tilde{f}(a) = \tilde{f}(a_0 \otimes_{a_1} a') := f(a_0)P_{a_1, R}(\tilde{f}(a')).
\]
For \( a \in III_\Omega(A) \) and \( \omega \in \Omega \),
\[
(\tilde{f}P_\omega)(a) = \tilde{f}(P_\omega(a)) = \tilde{f}(1 \otimes_\omega a) = f(1)P_{\omega, R}(\tilde{f}(a)) = P_{\omega, R}(\tilde{f}(a)) = (P_{\omega, R}\tilde{f})(a).
\]
Hence
\[
(18) \quad \tilde{f}P_\omega = P_{\omega, R}\tilde{f} \quad \text{for} \quad \omega \in \Omega.
\]
We next prove the compatibility of \( \tilde{f} \) with the multiplication \( \circ_\Omega \):
\[
(19) \quad \tilde{f}(a \circ_\Omega b) = \tilde{f}(a)\tilde{f}(b) \quad \text{for} \quad a = a_0 \otimes_{a_1} a' \in A^{\otimes_\Omega(m+1)} \quad \text{and} \quad b = b_0 \otimes_{\beta_1} b' \in A^{\otimes_\Omega(n+1)},
\]
by induction on \( m + n \geq 0 \). When \( m = n = 0 \), we have
\[
a = a_0, \quad b = b_0 \in A \quad \text{and} \quad a \circ_\Omega b = a_0b_0,
\]
and so
\[
\tilde{f}(a \circ_\Omega b) = \tilde{f}(a_0b_0) = f(a_0)b_0 = f(a_0)f(b_0) = \tilde{f}(a)\tilde{f}(b),
\]
by \( f \) being an algebra homomorphism. Suppose that Eq. (19) has been validated for \( m + n \leq k \) with a \( k \geq 0 \), and consider the case of \( m + n = k + 1 \). Then
\[
a = a_0 \otimes_{a_1} a' = a_0 \circ_\Omega P_{a_1}(a') \in A^{\otimes_\Omega(m+1)}, \quad b = b_0 \otimes_{\beta_1} b' = b_0 \circ_\Omega P_{\beta_1}(b') \in A^{\otimes_\Omega(n+1)},
\]
and so
\[
\tilde{f}(a \circ_\Omega b)
= \tilde{f}((a_0 \circ_\Omega P_{a_1}(a')) \circ_\Omega (b_0 \circ_\Omega P_{\beta_1}(b')))
= \tilde{f}((a_0b_0) \circ_\Omega P_{a_1}(a') \circ_\Omega P_{\beta_1}(b')) \quad \text{(by Proposition 2.8 and Eq. (14))}
= \tilde{f}((a_0b_0) \circ_\Omega (P_{a_1}(a') \circ_\Omega P_{\beta_1}(b')) + P_{\beta_1}(P_{a_1}(a') \circ_\Omega b')) \quad \text{(by Eq. (16))}
= \tilde{f}(a_0b_0)(\tilde{f}P_{a_1})(a' \circ_\Omega (1 \otimes_{\beta_1} b')) + \tilde{f}(a_0b_0)(\tilde{f}P_{\beta_1})(1 \otimes_{a_1} a') \circ_\Omega b')
= \tilde{f}(a_0b_0)(\tilde{f}P_{a_1})(\tilde{f}(1 \otimes_{\beta_1} b')) + \tilde{f}(a_0b_0)P_{\beta_1,R}(\tilde{f}(1 \otimes_{a_1} a') \tilde{f}(b'))
= \tilde{f}(a_0b_0)(\tilde{f}P_{a_1,R})(\tilde{f}(a') \tilde{f}(P_{\beta_1}(b'))) + \tilde{f}(a_0b_0)P_{\beta_1,R}(\tilde{f}(P_{a_1}(a')) \tilde{f}(b'))
= \tilde{f}(a_0b_0)(\tilde{f}P_{a_1,R})(\tilde{f}(a') \tilde{f}(P_{\beta_1}(b')) + \tilde{f}(a_0b_0)P_{\beta_1,R}(\tilde{f}(P_{a_1}(a')) \tilde{f}(b'))
= \tilde{f}(a_0)f(b_0)P_{\beta_1,R}(\tilde{f}(b'))
= \tilde{f}(a_0)\tilde{f}(b) \quad \text{(by Eq. (17)).}
\]
(\textbf{Uniqueness}). In fact, for \( a = a_0 \otimes_{a_1} a_1 \otimes_{a_2} a_2 \otimes_{a_3} \cdots \otimes_{a_m} a_m \in A^{\otimes(m+1)} \), since
\[
a_0 \otimes_{a_1} a_1 \otimes_{a_2} a_2 \otimes_{a_3} \cdots \otimes_{a_m} a_m \in A^{\otimes(m+1)} = a_0P_{a_1}(a_1P_{a_2}(a_2P_{a_3}(a_3\cdots P_{a_m}(a_m))))
\]
in order for \( \tilde{f} \) to be a matching Rota-Baxter algebra homomorphism, we must have
\[
\tilde{f}(a) = f(a_0)P_{a_1,R}(f(a_1))P_{a_2,R}(f(a_2)) \cdots P_{a_m,R}(f(a_m)) \cdots)
\]
So the uniqueness of \( \tilde{f} \) is proved. This completes the proof. \( \square \)

Let \( X \) be a set and \( k[X] \) the polynomial algebra on \( X \) with the natural embedding \( X \hookrightarrow k[X] \). As a special case of Theorem 2.10, we obtain a free commutative matching Rota-Baxter algebra of weight zero on a set.

**Corollary 2.11.** Let \( X \) be a set. The matching Rota-Baxter algebra \( \llbracket X \rrbracket(k[X]), (P_{\omega,k[X]},\omega \in \Omega) \) of weight zero, together with the natural embedding
\[
j_X : X \hookrightarrow k[X] \hookrightarrow \llbracket X \rrbracket(k[X]),
\]
is the free commutative matching Rota-Baxter algebra of weight zero on \( X \), described by the following universal property: for any commutative MRBA \( (R, (P_{\omega,R},\omega \in \Omega)) \) of weight zero and any set map \( \phi : X \to R \), there exists a unique matching MRBA homomorphism \( \tilde{\phi} : \llbracket X \rrbracket(k[X]) \to R \) such that \( \phi = \tilde{\phi}j_X \).

3. Free commutative relative matching Rota-Baxter algebras

With the application to integral equations [25] in mind, we consider MRBAs in a relative context, in the sense that the base ring is already a MRBA.

3.1. Relative MRBAs and the construction of their free objects. We begin with the following analog of algebras over a base ring.

**Definition 3.1.** Let \( (F, \kappa_\Omega) \) be a fixed MRBA of weight \( \lambda_\Omega \).

(a) An \( (F, \kappa_\Omega) \)-MRBA or simply an \( (F, \kappa_\Omega) \)-algebra is an MRBA \( (R, P_{\Omega,R}) \) of weight \( \lambda_\Omega \) together with an MRBA homomorphism \( I = i_R : (F, \kappa_\Omega) \to (R, P_{\Omega,R}) \).

(b) An homomorphism \( \phi : (R, P_{\Omega,R}) \to (R', P'_{\Omega,R'}) \) of \( (F, \kappa_\Omega) \)-algebras is an algebra homomorphism \( \phi : R \to R' \) such that
\[
\phi P_{\omega,R} = P'_{\omega,R'} \phi \quad \text{for} \quad \omega \in \Omega.
\]

As in the case of an \( F \)-algebra, the structure map \( i_R \) is usually suppressed. For \( (F, \kappa_\Omega) \)-algebra \( (R, P_\Omega) \), \( R \) is an algebra over \( F \) and the following identity holds:
\[
\kappa_\omega(k)P_{\beta,R}(u) = P_{\alpha,R}(kP_{\beta,R}(u)) + P_{\beta,R}(\kappa_\omega(k)u) + \lambda_\beta P_{\alpha,R}(ku) \quad \text{for} \quad k \in F, u \in R, \alpha, \beta \in \Omega.
\]

We note the presence to two base algebras \( k \) and \( F \), where \( k \) is the ring of constants for linear maps and tensor products, while the Rota-Baxter operators \( \kappa_\omega \) and \( P_{\omega}, \omega \in \Omega, \) are not \( F \)-linear.

The concept of free \( (F, \kappa_\Omega) \)-MRBA is explicitly defined as follows.

**Definition 3.2.** Let \( (F, \kappa_\Omega) \) be a fixed commutative MRBA of weight \( \lambda_\Omega \) and \( A \) a commutative algebra. A free \( (F, \kappa_\Omega) \)-MRBA on \( A \) is a \( (F, \kappa_\Omega) \)-MRBA \( (F_{\text{MRB}}(A), P_{\text{MRB}}(A), \kappa_\Omega) \), together with an algebra homomorphism \( j_A : A \to F_{\text{MRB}}(A) \) satisfying the following universal property: for any \( (F, \kappa_\Omega) \)-MRBA \( (R, P_{\Omega,R}) \) and any algebra homomorphism \( f : A \to R \), there is a unique \( (F, \kappa_\Omega) \)-MRBA homomorphism \( \tilde{f} : F_{\text{MRB}}(F, \kappa_\Omega) \to R \) such that \( f = \tilde{f}j_A \).

When all the algebras involved are assumed to be commutative, we have the notion of a free commutative \( (F, \kappa_\Omega) \)-algebra on \( A \).
Let $A$ be an augmented algebra. So $A$ can be taken as the unitization $A = k \oplus A^+$ of an algebra $A^+$. We will construct the free commutative relative MRBA of weight zero on $A$. Let $(F, \kappa_\Omega)$ be a fixed commutative MRBA of weight zero. Denote by

$$\mathfrak{A} := F \otimes A \text{ and } \mathfrak{A}^+ := F \otimes A^+. $$

So $\mathfrak{A} = F \oplus \mathfrak{A}^+$.

Recall the free commutative MRBA $\mathbb{III}_\Omega(\mathfrak{A})$ of weight zero on $\mathfrak{A}$ introduced in Eq. (11):

$$\mathbb{III}_\Omega(\mathfrak{A}) = \bigoplus_{k \geq 1} \mathfrak{A} \otimes \mathfrak{A}^{\otimes k} = \mathfrak{A} \oplus \mathfrak{A}^{\oplus 2} \oplus \cdots .$$

Consider the $k$-submodule $\mathbb{III}_\Omega^{\text{rel}}(F, A)$ of $\mathbb{III}_\Omega(\mathfrak{A})$ spanned by pure tensors of the form

$$u_0 \otimes_{\omega_1} u_1 \otimes_{\omega_2} \cdots \otimes_{\omega_k} u_k, \quad u_0 \in \mathfrak{A}, u_1, \cdots, u_k \in \mathfrak{A}^+. $$

Thus

$$\mathbb{III}_\Omega^{\text{rel}}(F, A) = \mathfrak{A} \oplus \bigoplus_{k \geq 1} \mathfrak{A} \otimes A^{\otimes k}.$$

Using the notation in Eq. (12), denote

$$\mathbb{III}_\Omega^{\text{rel}}(\mathfrak{A}^+) := \bigoplus_{n \geq 0} k \otimes (\mathfrak{A}^+)^{\otimes n}$$

with the convention that $(\mathfrak{A}^+)^{\otimes 0} := k$. Then

$$\mathbb{III}_\Omega^{\text{rel}}(F, A) = \mathfrak{A} \otimes \mathbb{III}_\Omega^{\text{rel}}(\mathfrak{A}^+) := \bigoplus_{n \geq 0} \mathfrak{A} \otimes (\mathfrak{A}^+)^{\otimes n} = \mathfrak{A} \oplus (\mathfrak{A} \otimes \mathfrak{A}^+) \oplus (\mathfrak{A} \otimes (\mathfrak{A}^+)^{\otimes 2}) \oplus \cdots .$$

Hence $\mathbb{III}_\Omega^{\text{rel}}(F, A)$ is closed under the multiplication $\diamond_\Omega$ by Eq. (13), and so $(\mathbb{III}_\Omega^{\text{rel}}(F, A), \diamond_\Omega)$ is a subalgebra of the free commutative MRBA $\mathbb{III}_\Omega(\mathfrak{A})$ of weight zero on $\mathfrak{A}$.

Next we define linear operators

$$P_\omega : \mathbb{III}_\Omega^{\text{rel}}(F, A) \to \mathbb{III}_\Omega^{\text{rel}}(F, A) \quad \text{for } \omega \in \Omega.$$

Let $u = u_0 \otimes_{\omega_1} \cdots \otimes_{\omega_n} u_n$ be a pure tensor in $\mathbb{III}_\Omega^{\text{rel}}(F, A)$. Since $\mathfrak{A} = F \oplus \mathfrak{A}^+$, the first tensor factor $u_0$ is either in $F$ or in $\mathfrak{A}^+$. We accordingly define

$$P_\omega(u) := P_{\omega, F, A}(u) := \begin{cases} 
\kappa_\omega(u_0), & \text{if } u_0 \in F, n = 0, \\
\kappa_\omega(u_0) \otimes_{\omega_1} u_1 \otimes_{\omega_2} \cdots \otimes_{\omega_n} u_n & \text{if } u_0 \in F, n \geq 1, \\
-1 \otimes_{\omega_1} \kappa_\omega(u_0) u_1 \otimes_{\omega_2} u_2 \otimes_{\omega_3} \cdots \otimes_{\omega_n} u_n, & \text{if } u_0 \in F, n \geq 1, \\
1 \otimes_{\omega_0} u_0 \otimes_{\omega_1} u_1 \otimes_{\omega_2} \cdots \otimes_{\omega_n} u_n, & \text{if } u_0 \in \mathfrak{A}^+. 
\end{cases}$$

Then $\mathbb{III}_\Omega^{\text{rel}}(F, A)$ is closed under the operators $P_\omega$ for $\omega \in \Omega$. Denote by $P_{\Omega, F, A} := (P_\omega)_{\omega \in \Omega}$. Let

$$i : F \hookrightarrow \mathfrak{A} \subseteq \mathbb{III}_\Omega^{\text{rel}}(F, A)$$

and

$$j_A : A \hookrightarrow \mathfrak{A} \hookrightarrow \mathbb{III}_\Omega^{\text{rel}}(F, A)$$

be the nature embeddings.

**Theorem 3.3.** Let $(F, \kappa_\Omega)$ be a commutative MRBA of weight zero and $A$ a commutative augmented algebra. Then the triple $(\mathbb{III}_\Omega^{\text{rel}}(F, A), \diamond_\Omega, P_{\Omega, F, A})$, together with the maps $i$ and $j_A$, is the free commutative $(F, \kappa_\Omega)$-MRBA on $A$. 
3.2. The proof Theorem 3.3. We will divide the proof of Theorem 3.3 into two steps:

Step 1. Together with \( i : F \rightarrow \mathcal{B}_{\Omega}^{\text{rel}}(F, A) \), the triple \( (\mathcal{B}_{\Omega}^{\text{rel}}(F, A), \circ_{\Omega}, \mathcal{C}_{\Omega,F,A}) \) is a \((F, \kappa_{\Omega})\)-MRBA;

Step 2. Together with \( J_{\mathcal{C}} : A \rightarrow \mathcal{B}_{\Omega}^{\text{rel}}(F, A) \), the \((F, \kappa_{\Omega})\)-algebra \( (\mathcal{B}_{\Omega}^{\text{rel}}(F, A), \circ_{\Omega}, \mathcal{C}_{\Omega,F,A}) \) satisfies the universal property of a free commutative \((F, \kappa_{\Omega})\)-algebra on \( A \).

3.2.1. Step 1. The triple \( (\mathcal{B}_{\Omega}^{\text{rel}}(F, A), \circ_{\Omega}, \mathcal{C}_{\Omega,F,A}) \) is a \((F, \kappa_{\Omega})\)-MRBA. Since \( \circ_{\Omega} \) is commutative associative and, by Eq. (21), the map \( i : F \rightarrow \mathcal{B}_{\Omega}^{\text{rel}}(F, A) \) is compatible with the operators \( \kappa_{\omega} \) on \( F \) and \( P_{\omega}, \omega, \tau \in \Omega \) on \( \mathcal{B}_{\Omega}^{\text{rel}}(F, A) \), it is sufficient to verify

\[
P_{\omega}(u) \circ_{\Omega} P_{\tau}(v) = P_{\omega}(P_{\omega}(u) \circ_{\Omega} \omega) \circ_{\Omega} P_{\tau}(v)
\]

for \( u, v \in \mathcal{B}_{\Omega}^{\text{rel}}(F, A), \omega, \tau \in \Omega \).

By additivity, we only need to consider pure tensors \( u = u_{0} \otimes_{\omega_{1}} u_{1} \otimes_{\omega_{2}} \cdots \otimes_{\omega_{m}} u_{m} =: u_{0} \otimes_{\omega_{1}} u^{' \prime} \) and \( v = v_{0} \otimes_{\tau_{1}} v_{1} \otimes_{\tau_{2}} \cdots \otimes_{\tau_{n}} v_{n} =: v_{0} \otimes_{\tau_{1}} v^{' \prime} \).

We have four cases to consider.

Case 1. \( u_{0} \in F \) and \( v_{0} \in F \). In this case, by Eq. (21), we have

\[
P_{\omega}(u) = \kappa_{\omega}(u_{0}) \otimes_{\omega_{1}} u^{' \prime} - 1 \otimes_{\omega_{1}} \kappa_{\omega}(u_{0}) u^{' \prime} \quad \text{and} \quad P_{\tau}(v) = \kappa_{\tau}(v_{0}) \otimes_{\tau_{1}} v^{' \prime} - 1 \otimes_{\tau_{1}} \kappa_{\tau}(v_{0}) v^{' \prime}.
\]

On the one hand,

\[
P_{\omega}(u) \circ_{\Omega} P_{\tau}(v)
\]

\[
= P_{\omega}(u) \circ_{\Omega} \left( \kappa_{\tau}(v_{0}) \otimes_{\tau_{1}} v^{' \prime} - 1 \otimes_{\tau_{1}} \kappa_{\tau}(v_{0}) v^{' \prime} \right)
\]

\[
= P_{\omega}(u_{0}\kappa_{\tau}(v_{0})) \otimes_{\omega_{1}} \left( u^{' \prime} \circ_{\Omega} \left( 1 \otimes_{\tau_{1}} v^{' \prime} \right) + u_{0}\kappa_{\tau}(v_{0}) \otimes_{\tau_{1}} \left( 1 \otimes_{\omega_{1}} \left( 1 \otimes_{\omega_{2}} \cdots \otimes_{\omega_{m}} u_{m} \right) \otimes_{\omega_{1}} u^{' \prime} \right) \right)
\]

\[
- P_{\omega}(u_{0} \otimes_{\omega_{1}} \left( u^{' \prime} \otimes_{\Omega} \left( 1 \otimes_{\tau_{1}} \kappa_{\tau}(v_{0}) v^{' \prime} \right) + u_{0} \otimes_{\tau_{1}} \left( 1 \otimes_{\omega_{1}} \left( 1 \otimes_{\omega_{2}} \cdots \otimes_{\omega_{m}} u_{m} \right) \otimes_{\omega_{1}} \left( 1 \otimes_{\omega_{1}} \left( 1 \otimes_{\omega_{2}} \cdots \otimes_{\omega_{m}} u_{m} \right) \otimes_{\omega_{1}} \left( 1 \otimes_{\omega_{1}} \left( 1 \otimes_{\omega_{2}} \cdots \otimes_{\omega_{m}} u_{m} \right) \otimes_{\omega_{1}} \left( 1 \otimes_{\omega_{1}} \left( 1 \otimes_{\omega_{2}} \cdots \otimes_{\omega_{m}} u_{m} \right) \otimes_{\omega_{1}} u^{' \prime} \right) \right) \right) \right) \right)
\]

\[
= \kappa_{\omega}(u_{0}\kappa_{\tau}(v_{0})) \otimes_{\omega_{1}} \left( u^{' \prime} \otimes_{\Omega} \left( 1 \otimes_{\tau_{1}} v^{' \prime} \right) - 1 \otimes_{\omega_{1}} \kappa_{\omega}(u_{0}\kappa_{\tau}(v_{0})) \right) + \kappa_{\omega}(u_{0}\kappa_{\tau}(v_{0})) \otimes_{\tau_{1}} \left( 1 \otimes_{\omega_{1}} \left( 1 \otimes_{\omega_{2}} \cdots \otimes_{\omega_{m}} u_{m} \right) \otimes_{\omega_{1}} u^{' \prime} \right)
\]

\[
- \kappa_{\omega}(u_{0}) \otimes_{\omega_{1}} \left( u^{' \prime} \otimes_{\Omega} \left( 1 \otimes_{\tau_{1}} \kappa_{\tau}(v_{0}) v^{' \prime} \right) + 1 \otimes_{\omega_{1}} \kappa_{\omega}(u_{0}) \right) \left( 1 \otimes_{\omega_{1}} u^{' \prime} \right)
\]

\[
= \kappa_{\omega}(u_{0}) \otimes_{\omega_{1}} u^{' \prime} - 1 \otimes_{\omega_{1}} \kappa_{\omega}(u_{0}) u^{' \prime}
\]

By the same argument, we have

\[
P_{\tau}(P_{\omega}(u) \circ_{\Omega} v)
\]

\[
= P_{\tau}(P_{\omega}(u) \circ_{\Omega} v)
\]

\[
= \kappa_{\tau}(\kappa_{\omega}(u_{0}) v_{0}) \otimes_{\omega_{1}} \left( u^{' \prime} \otimes_{\Omega} \left( 1 \otimes_{\tau_{1}} v^{' \prime} \right) - 1 \otimes_{\omega_{1}} \kappa_{\tau}(\kappa_{\omega}(u_{0}) v_{0}) \right) + \kappa_{\tau}(\kappa_{\omega}(u_{0}) v_{0}) \otimes_{\tau_{1}} \left( 1 \otimes_{\omega_{1}} \left( 1 \otimes_{\omega_{2}} \cdots \otimes_{\omega_{m}} u_{m} \right) \otimes_{\omega_{1}} u^{' \prime} \right)
\]

\[
- \kappa_{\tau}(v_{0}) \otimes_{\omega_{1}} \left( \kappa_{\omega}(u_{0}) u^{' \prime} \otimes_{\Omega} \left( 1 \otimes_{\tau_{1}} v^{' \prime} \right) + 1 \otimes_{\omega_{1}} \kappa_{\tau}(v_{0}) \right) \left( \kappa_{\omega}(u_{0}) u^{' \prime} \otimes_{\Omega} \left( 1 \otimes_{\tau_{1}} v^{' \prime} \right) \right)
\]

\[
= \kappa_{\tau}(v_{0}) \otimes_{\omega_{1}} \left( \kappa_{\omega}(u_{0}) u^{' \prime} \otimes_{\Omega} \left( 1 \otimes_{\tau_{1}} v^{' \prime} \right) + 1 \otimes_{\omega_{1}} \kappa_{\tau}(v_{0}) \right) \left( \kappa_{\omega}(u_{0}) u^{' \prime} \otimes_{\Omega} \left( 1 \otimes_{\tau_{1}} v^{' \prime} \right) \right)
\]

\[
- \kappa_{\tau}(v_{0}) \otimes_{\omega_{1}} \left( \kappa_{\omega}(u_{0}) u^{' \prime} \otimes_{\Omega} \left( 1 \otimes_{\tau_{1}} v^{' \prime} \right) + 1 \otimes_{\omega_{1}} \kappa_{\tau}(v_{0}) \right) \left( \kappa_{\omega}(u_{0}) u^{' \prime} \otimes_{\Omega} \left( 1 \otimes_{\tau_{1}} v^{' \prime} \right) \right).
\]
This allows us to rewrite Eq. (23) as

\[
P_\omega(u \otimes P_\tau(v)) + P_\tau(P_\omega(u) \otimes v)
= \kappa_\omega(u_0 \kappa_\tau(v_0)) \otimes \omega_1 \left( \frac{u'}{\omega_1} \otimes_{\tau_1} (1 \otimes_{\tau_1} v') \right) - 1 \otimes_{\omega_1} \kappa_\omega(u_0 \kappa_\tau(v_0)) \left( \frac{u'}{\omega_1} \otimes_{\tau_1} (1 \otimes_{\tau_1} v') \right)
- \kappa_\omega(u_0) \otimes \omega_1 \left( \frac{u'}{\omega_1} \otimes_{\tau_1} (1 \otimes_{\tau_1} v') \right) + 1 \otimes_{\omega_1} \kappa_\omega(u_0)(1 \otimes_{\tau_1} \kappa_\tau(v_0)v')
\]

by gathering the second, tenth, fourteenth terms, and the fourth, eighth, twelfth terms of Eq. (23) and the definition of \(\kappa_\tau\).

On the other hand,

\[
P_\omega(u) \otimes P_\tau(v)
= (\kappa_\omega(u_0) \otimes \omega_1 \frac{u'}{\omega_1} - 1 \otimes_{\omega_1} \kappa_\omega(u_0)v') \otimes_{\tau_1} (1 \otimes_{\tau_1} \kappa_\tau(v_0)v')
= (\kappa_\omega(u_0) \otimes \omega_1 \frac{u'}{\omega_1} \otimes_{\tau_1} \kappa_\tau(v_0)v') - (\kappa_\omega(u_0) \otimes \omega_1 \frac{u'}{\omega_1} \otimes_{\tau_1} \kappa_\tau(v_0)v')
- (1 \otimes_{\omega_1} \kappa_\omega(u_0)v') \otimes_{\tau_1} (1 \otimes_{\tau_1} \kappa_\tau(v_0)v') + (1 \otimes_{\omega_1} \kappa_\omega(u_0)v') \otimes_{\tau_1} (1 \otimes_{\tau_1} \kappa_\tau(v_0)v')
- \kappa_\omega(u_0)(1 \otimes_{\omega_1} \kappa_\omega(u_0)v') + \kappa_\omega(u_0)(1 \otimes_{\omega_1} \kappa_\omega(u_0)v')
\]

by gathering the second, tenth, fourteenth terms, and the fourth, eighth, twelfth terms of Eq. (23) and the definition of \(\kappa_\tau\).
Then on the one hand,
\[
\begin{align*}
&= \kappa(u_0 \kappa(v_0)) \otimes \omega_1 (u' \otimes \Omega (1 \otimes \tau, v')) + \kappa \big( \kappa(u_0 \kappa(v_0)) \otimes \omega_1 (u' \otimes \Omega (1 \otimes \tau, v')) \\
&+ \kappa(u_0 \kappa(v_0)) \otimes \tau \big( (1 \otimes \omega_1, u') \otimes \Omega v' \big) + \kappa \big( \kappa(u_0 \kappa(v_0)) \otimes \tau \big( (1 \otimes \omega_1, u') \otimes \Omega v' \big) \\
&- \kappa(u_0) \otimes \omega_1 (u' \otimes \Omega (1 \otimes \tau, \kappa(v_0)v')) - \kappa(u_0) \otimes \tau \big( (1 \otimes \omega_1, u') \otimes \Omega \kappa(v_0)v' \big) \\
&- \kappa(v_0) \otimes \omega_1 (\kappa(u_0)u' \otimes \Omega (1 \otimes \tau, v')) - \kappa(v_0) \otimes \tau \big( (1 \otimes \omega_1, \kappa(u_0)u') \otimes \Omega v' \big) \\
&+ 1 \otimes \omega_1 (\kappa(u_0)u' \otimes \Omega (1 \otimes \tau, \kappa(v_0)v')) + 1 \otimes \tau \big( (1 \otimes \omega_1, \kappa(u_0)u') \otimes \Omega \kappa(v_0)v' \big)
\end{align*}
\]
(by Eq. (24)).

Then the \( i \)-th term in the expansion of \( P_\omega(u) \circ \Omega P_\tau(v) \) is equal to the \( \sigma(i) \)-th term in the expansion of \( P_\omega(u \circ \Omega P_\tau(v)) + P_\tau(P_\omega(u) \circ \Omega v) \), where \( \sigma \) is the permutation of order 10:
\[
\begin{pmatrix}
i \\
\sigma(i)
\end{pmatrix} =
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 6 & 2 & 7 & 3 & 5 & 8 & 9 & 4 & 10
\end{pmatrix}
\]

**Case 2.** \( u_0 \in F \) and \( v_0 \in \mathfrak{A}^+ \). In this case, by Eq. (21), we have
\[
P_\omega(u) = \kappa(u_0) \otimes \omega_1 u' - 1 \otimes \omega_1 \kappa(u_0)u' \quad \text{and} \quad P_\tau(v) = 1 \otimes \tau v.
\]

Then on the one hand,
\[
P_\omega(u) \circ \Omega P_\tau(v)
\]
\[
= \left( \kappa(u_0) \otimes \omega_1 u' - 1 \otimes \omega_1 \kappa(u_0)u' \right) \circ \Omega (1 \otimes \tau v)
\]
\[
= \left( \kappa(u_0) \otimes \omega_1 u' \right) \circ \Omega (1 \otimes \tau v) - \left( 1 \otimes \omega_1 \kappa(u_0)u' \right) \circ \Omega (1 \otimes \tau v)
\]
\[
= \kappa(u_0) \otimes \omega_1 (u' \circ \Omega (1 \otimes \tau v) + \kappa(v_0) \otimes \tau \big( (1 \otimes \omega_1, u') \otimes \Omega v \big) - 1 \otimes \omega_1 \big( (1 \otimes \omega_1, \kappa(u_0)u') \otimes \Omega v \big)
\]
(by Eq. (14))
\[
= \kappa(u_0) \otimes \omega_1 (u' \circ \Omega (1 \otimes \tau v) + \kappa(v_0) \otimes \tau \big( (1 \otimes \omega_1, u') \otimes \Omega v \big) - 1 \otimes \omega_1 \big( (1 \otimes \omega_1, \kappa(u_0)u') \otimes \Omega v \big)
\]
(by Eq. (22))
\[
= \kappa(u_0) \otimes \omega_1 (u' \circ \Omega (1 \otimes \tau v) + \kappa(v_0) \otimes \tau \big( (1 \otimes \omega_1, u') \otimes \Omega v \big) - 1 \otimes \omega_1 \big( (1 \otimes \omega_1, \kappa(u_0)u') \otimes \Omega v \big)
\]
\[
= \kappa(u_0) \otimes \omega_1 (u' \circ \Omega (1 \otimes \tau v) + \kappa(v_0) \otimes \tau \big( (1 \otimes \omega_1, u') \otimes \Omega v \big) - 1 \otimes \omega_1 \big( (1 \otimes \omega_1, \kappa(u_0)u') \otimes \Omega v \big)
\]
(by Eq. (14)).

On the other hand,
\[
P_\omega(u \circ \Omega P_\tau(v)) + P_\tau(P_\omega(u) \circ \Omega v)
\]
\[
= P_\omega \bigg( (u_0 \otimes \omega_1 u') \circ \Omega (1 \otimes \tau v) \bigg) + P_\tau \left( \left( \kappa(u_0) \otimes \omega_1 u' - 1 \otimes \omega_1 \kappa(u_0)u' \right) \circ \Omega (v_0 \otimes \tau, v') \right)
\]
\[
= P_\omega \left( u_0 \otimes \omega_1 (u' \circ \Omega (1 \otimes \tau v)) + u_0 \otimes \tau \big( (1 \otimes \omega_1, u') \otimes \Omega v \big) \right)
\]
\[
+ P_\tau \left( \kappa(u_0) \otimes \omega_1 (u' \circ \Omega (1 \otimes \tau v)) + \kappa(u_0) \otimes \tau \big( (1 \otimes \omega_1, u') \otimes \Omega v \big) \right)
\]
that there is unique (∇\varOmega \eta̅_\omega) \circ \Omega (1 \otimes \tau \nu') + v_0 \otimes \tau_1 \left( (1 \otimes \omega_1 \kappa_\omega(u_0)u') \circ \Omega \nu' \right) \right) \quad \text{(by Eq. (14))}

\begin{align*}
= & \kappa_\omega(u_0) \otimes \omega_1 \left( u' \circ \Omega (1 \otimes \tau \nu) \right) - 1 \otimes \omega_1 \kappa_\omega(u_0) \left( u' \circ \Omega (1 \otimes \tau \nu) \right) \\
& + \kappa_\omega(u_0) \otimes \tau_1 \left( (1 \otimes \omega_1 u') \circ \Omega \nu \right) - 1 \otimes \tau_1 \kappa_\omega(u_0) \left( (1 \otimes \omega_1 u') \circ \Omega \nu' \right) \\
& + 1 \otimes \tau \kappa_\omega(u_0)v_0 \otimes \omega_1 \left( u' \circ \Omega (1 \otimes \tau_1 \nu') \right) + 1 \otimes \tau \kappa_\omega(u_0)v_0 \otimes \tau_1 \left( (1 \otimes \omega_1 u') \circ \Omega \nu' \right) \\
& - 1 \otimes \tau v_0 \otimes \omega_1 \left( \kappa_\omega(u_0)u' \circ \Omega (1 \otimes \tau \nu') \right) - 1 \otimes \tau v_0 \otimes \tau_1 \left( (1 \otimes \omega_1 \kappa_\omega(u_0)u') \circ \Omega \nu' \right) \quad \text{(by Eq. (21))}
\end{align*}

\begin{align*}
= & \kappa_\omega(u_0) \otimes \omega_1 \left( u' \circ \Omega (1 \otimes \tau \nu) \right) - 1 \otimes \omega_1 \kappa_\omega(u_0) \left( u' \circ \Omega (1 \otimes \tau \nu) \right) \\
& + \kappa_\omega(u_0) \otimes \tau_1 \left( (1 \otimes \omega_1 u') \circ \Omega \nu \right) - 1 \otimes \tau_1 \kappa_\omega(u_0) \left( (1 \otimes \omega_1 u') \circ \Omega (v_0 \otimes \tau_1 \nu') \right) \\
& + 1 \otimes \tau \kappa_\omega(u_0)v_0 \otimes \omega_1 \left( u' \circ \Omega (1 \otimes \tau_1 \nu') \right) + 1 \otimes \tau \kappa_\omega(u_0)v_0 \otimes \tau_1 \left( (1 \otimes \omega_1 u') \circ \Omega \nu' \right) \\
& - 1 \otimes \tau v_0 \otimes \omega_1 \left( \kappa_\omega(u_0)u' \circ \Omega (1 \otimes \tau_1 \nu') \right) - 1 \otimes \tau v_0 \otimes \tau_1 \left( (1 \otimes \omega_1 \kappa_\omega(u_0)u') \circ \Omega \nu' \right) \quad \text{(by Eq. (22))}
\end{align*}

\begin{align*}
= & \kappa_\omega(u_0) \otimes \omega_1 \left( u' \circ \Omega (1 \otimes \tau \nu) \right) - 1 \otimes \omega_1 \kappa_\omega(u_0) \left( u' \circ \Omega (1 \otimes \tau \nu) \right) \\
& + \kappa_\omega(u_0) \otimes \tau_1 \left( (1 \otimes \omega_1 u') \circ \Omega \nu \right) - 1 \otimes \tau_1 \kappa_\omega(u_0) \left( (1 \otimes \omega_1 u') \circ \Omega \nu \right) \\
& - 1 \otimes \tau v_0 \otimes \omega_1 \left( \kappa_\omega(u_0)u' \circ \Omega (1 \otimes \tau \nu') \right) - 1 \otimes \tau v_0 \otimes \tau_1 \left( (1 \otimes \omega_1 \kappa_\omega(u_0)u') \circ \Omega \nu' \right),
\end{align*}

where the last equality is by gathering the fourth, fifth, sixth terms by Eq. (14).

Thus by the F-linearity, we obtain

\begin{equation*}
P_\omega(u) \circ \Omega P_\tau(v) = P_\omega(u \circ \Omega P_\tau(v)) + P_\tau(P_\omega(u) \circ \Omega v).
\end{equation*}

**Case 3.** $u_0 \in \mathfrak{H}^+$ and $v_0 \in F$. It is similar to the proof of Case 2.

**Case 4.** $u_0 \in \mathfrak{H}^+$ and $v_0 \in \mathfrak{H}^+$. In this case, we can write

\begin{equation*}
P_\omega(u) = 1 \otimes \omega \; u \quad \text{and} \quad P_\tau(v) = 1 \otimes \tau \; v.
\end{equation*}

Then the proof of this case is similar to the proof of Eq. (16) in Theorem 2.10. This completes the proof.

3.2.2. **Step 2. The universal property of $(\mathbb{R}_{\Omega^1}^\tau(F, A), \circ \Omega, P_{\Omega,F,A})$.** Let $(R, P_{\Omega,R})$ be a commutative relative MRBA of weight zero and let $f : A \to R$ be an algebra homomorphism. We show that there is unique $(F, \kappa_\Omega)-$MRBAs homomorphism $\tilde{f} : (\mathbb{R}_{\Omega^1}^\tau(F, A), P_\Omega) \to (R, P_{\Omega,R})$ such that $\tilde{f} \circ R = f$.

(\textbf{The existence}). For any pure tensor $u = u_0 \otimes \omega_1 u' \in \mathfrak{H} \otimes \Omega (\mathfrak{H}^+)^{\otimes m}$ with $m \geq 0$, we use induction on $m \geq 0$ to define $\tilde{f}(u)$. For the initial step of $m = 0$, we have $u = u_0$. If $u_0 \in F$, then define

\begin{equation}
\tilde{f}(u_0) := u_0 \tilde{f}(1) := u_0 1.
\end{equation}

If $u_0 \in \mathfrak{H}^+$, then we may write $u_0 = u_{0,a} \otimes u_{0,v} = u_{0,a} u_{0,v}$ for some $u_{0,a} \in F$ and $u_{0,v} \in A^+$, and we define

\begin{equation}
\tilde{f}(u_0) := \tilde{f}(u_{0,a} u_{0,v}) := u_{0,a} f(u_{0,v}).
\end{equation}

Consider the induction step of $m \geq 1$. If $u_0 \in F$, then define

\begin{equation}
\tilde{f}(u) := \tilde{f}(u_0 \otimes \omega_1 u') := \tilde{f}(u_0 P_{\omega_1}(u')) := u_0 \tilde{f}(P_{\omega_1}(u')) := u_0 P_{\omega_1,R}(\tilde{f}(u')).
\end{equation}
If \( u_0 \in \mathcal{A}^+ \), then define
\[
\tilde{f}(u_0 \otimes_{\omega_1} u') := \tilde{f}(u_0 P_{\omega_1}(u')) := \tilde{f}(u_0)P_{\omega_1,R} \tilde{f}(u').
\]

By Eqs. (25) and (27), \( \tilde{f} \) is \( F \)-linear. Now we prove that \( \tilde{f} \) is compatible with the operators \( P_\omega \) and \( P_{\omega,R}, \omega \in \Omega \).

Let \( u = u_0 \otimes_{\omega_1} u' \in \mathcal{A} \otimes \Omega (\mathcal{A}^+)_{\otimes m}^0 \) and \( P_\omega \in P_\Omega \). If \( u_0 \in F \), then
\[
\tilde{f} P_\omega(u) = \tilde{f} (u_0 \otimes_{\omega_1} u') = \tilde{f} (1 \otimes_{\omega_1} \kappa_\omega(u_0) u') \\
= \tilde{f} (\kappa_\omega(u_0) \otimes_{\omega_1} u' - 1 \otimes_{\omega_1} \kappa_\omega(u_0) u') \quad \text{(by Eq. (21))}
\]
\[
= \tilde{f} (\kappa_\omega(u_0) \otimes_{\omega_1} u') - \tilde{f} (1 \otimes_{\omega_1} \kappa_\omega(u_0) u') \\
= \kappa_\omega(u_0) P_{\omega,R} (\tilde{f}(u')) - P_{\omega,R} (\tilde{f}(\kappa_\omega(u_0) u')) \quad \text{(by Eq. (27))}
\]
\[
= \kappa_\omega(u_0) P_{\omega,R} (\tilde{f}(u')) - P_{\omega,R} (\tilde{f}(u') - F\text{-linearity})
\]
\[
= P_{\omega,R} (\tilde{f}(u_0 P_{\omega,R}(\tilde{f}(u'))) \quad \text{(by Eq. (20))}
\]
\[
= P_{\omega,R} \tilde{f}(u_0 \otimes_{\omega_1} u') \quad \text{(by Eq. (27))}
\]
\[
= P_{\omega,R} \tilde{f}(u) \quad \text{(by Eq. (27)).}
\]

If \( u_0 \in \mathcal{A}^+ \), then
\[
\tilde{f} P_\omega(u) = \tilde{f} (1 \otimes_{\omega_1} u') = \tilde{f} (1) P_{\omega,R} \tilde{f}(u) = P_{\omega,R} \tilde{f}(u).
\]

Thus
\[
\tilde{f} P_\omega = P_{\omega,R} \tilde{f} \quad \text{for} \quad \omega \in \Omega.
\]

Next we check the compatibility of \( f \) with the multiplication \( \odot_\Omega \): for \( u = u_0 \otimes_{\omega_1} u' \in \mathcal{A} \otimes \Omega (\mathcal{A}^+)_{\otimes m}^0 \) and \( v = v_0 \otimes_{\tau_1} v' \in \mathcal{A} \otimes \Omega (\mathcal{A}^+)_{\otimes m}^0 \),
\[
\tilde{f}(u \odot_\Omega v) = \tilde{f}(u) \tilde{f}(v).
\]

We will check this utilizing the induction on \( m + n \geq 0 \). When \( m = n = 0 \), we have
\[
u = u_0 \quad \text{and} \quad v = v_0.
\]

If \( u_0 \in F \) and \( v_0 \in F \), then it follows from Eq. (25) that
\[
\tilde{f}(u \odot_\Omega v) = \tilde{f}(u_0 \odot_\Omega v_0) = \tilde{f}(u_0) \tilde{f}(v_0) = \tilde{f}(u) \tilde{f}(v).
\]

If \( u_0 \in F \) and \( v_0 \in \mathcal{A}^+ \) or \( u_0 \in \mathcal{A}^+ \) and \( v_0 \in F \), without loss of generality, letting \( u_0 \in F \) and \( v_0 \in \mathcal{A}^+ \), then we may write \( v_0 = v_{0,a} \otimes v_{0,v} = v_{0,a} v_{0,v} \) for some \( v_{0,a} \in F \) and \( v_{0,v} \in A^+ \), and we have
\[
\tilde{f}(u \odot_\Omega v) = \tilde{f}(u_0 \odot_\Omega v_0) = \tilde{f}(u_0) \tilde{f}(v_0) = u_0 v_{0,a} f(v_{0,v}) = u_0 1 v_{0,a} f(v_{0,v}) = \tilde{f}(u_0) \tilde{f}(v_0) = \tilde{f}(u) \tilde{f}(v).
\]

If \( u_0, v_0 \in \mathcal{A}^+ \), then
\[
u_0 = u_{0,a} \otimes v_{0,v} = u_{0,a} u_{0,v} \quad \text{and} \quad v_0 = v_{0,a} \otimes v_{0,v} = v_{0,a} v_{0,v}
\]

for some \( u_{0,a}, v_{0,a} \in F \) and \( u_{0,v}, v_{0,v} \in A^+ \). We get
\[
\tilde{f}(u \odot_\Omega v) = \tilde{f}(u_0 \odot_\Omega v_0) = \tilde{f}(u_0) \tilde{f}(v_0) = \tilde{f}(u_0, v_0) = \tilde{f}(u_{0,a} v_{0,a} u_{0,v} v_{0,v})
\]
\[
= u_{0,a} v_{0,a} f(u_{0,v} v_{0,v}) = u_{0,a} v_{0,a} f(u_{0,v}) f(v_{0,v}) = u_{0,a} f(u_{0,v}) v_{0,a} f(v_{0,v})
\]
\[
= \tilde{f}(u_0) \tilde{f}(v_0) = \tilde{f}(u) \tilde{f}(v).
\]

Suppose that Eq. (30) has been validated for \( m + n \leq k \) with \( k \geq 0 \), and consider the case of \( m + n = k + 1 \geq 1 \). There are four cases to consider.
Case 1. \( u_0 \in F \) and \( v_0 \in F \). In this case, we have

\[
\tilde{f}(u \circ_{\Omega} v) = \tilde{f}(u_0 \otimes_{\omega_1} u' \circ_{\Omega} (v_0 \otimes_{\tau_1} v'))
\]

\[
= \tilde{f}\left(u_0 v_0 \otimes_{\omega_1} \left(u' \circ_{\Omega} (1 \otimes_{\tau_1} v')\right)\right) + u_0 v_0 \otimes_{\tau_1} \left((1 \otimes_{\omega_1} u') \circ_{\Omega} v'\right) \quad \text{(by Eq. (14))}
\]

\[
= u_0 v_0 P_{\omega_1,R}(\tilde{f}(u' \circ_{\Omega} (1 \otimes_{\tau_1} v'))) + u_0 v_0 P_{\tau_1,R}(\tilde{f}(1 \otimes_{\omega_1} u') \circ_{\Omega} v') \quad \text{(by Eq. (27))}
\]

\[
= u_0 v_0 P_{\omega_1,R}(\tilde{f}(u') \tilde{f}(1 \otimes_{\omega_1} v')) + u_0 v_0 P_{\tau_1,R}(\tilde{f}(1 \otimes_{\omega_1} u') \tilde{f}(v'))
\]

(by the induction hypothesis)

\[
= u_0 v_0 P_{\omega_1,R}(\tilde{f}(u') P_{1,R}(\tilde{f}(v'))) + u_0 v_0 P_{\tau_1,R}(P_{\omega_1,R}(\tilde{f}(u')) \tilde{f}(v')) \quad \text{(by Eq. (27))}
\]

\[
= u_0 v_0 P_{\omega_1,R}(\tilde{f}(u')) P_{1,R}(\tilde{f}(v')) \quad \text{(by \( (R, P_{\omega_1,R}) \) being a matching Rota-Baxter algebra)}
\]

\[
= \left(u_0 P_{\omega_1,R}(\tilde{f}(u'))\right)\left(v_0 P_{1,R}(\tilde{f}(v'))\right) \quad \text{(by the commutativity)}
\]

\[
= \tilde{f}(u_0 \otimes_{\omega_1} u') \tilde{f}(v_0 \otimes_{\tau_1} v')
\]

\[
= \tilde{f}(u) \tilde{f}(v).
\]

Case 2. \( u_0 \in F \) and \( v_0 \in \mathcal{A}^+ \). In this case, we have

\[
\tilde{f}(u \circ_{\Omega} v) = \tilde{f}\left((u_0 \otimes_{\omega_1} u') \circ_{\Omega} (v_0 \otimes_{\tau_1} v')\right)
\]

\[
= \tilde{f}\left(u_0 v_0 \otimes_{\omega_1} \left(u' \circ_{\Omega} (1 \otimes_{\tau_1} v')\right)\right) + u_0 v_0 \otimes_{\tau_1} \left((1 \otimes_{\omega_1} u') \circ_{\Omega} v'\right) \quad \text{(by Eq. (14))}
\]

\[
= \tilde{f}(u_0 v_0) P_{\omega_1,R}(\tilde{f}(u' \circ_{\Omega} (1 \otimes_{\tau_1} v'))) + \tilde{f}(u_0 v_0) P_{\tau_1,R}(\tilde{f}(1 \otimes_{\omega_1} u') \circ_{\Omega} v') \quad \text{(by Eq. (27))}
\]

\[
= \tilde{f}(u_0 v_0) P_{\omega_1,R}(\tilde{f}(u') \tilde{f}(1 \otimes_{\omega_1} v')) + \tilde{f}(u_0 v_0) P_{\tau_1,R}(\tilde{f}(1 \otimes_{\omega_1} u') \tilde{f}(v'))
\]

(by the induction hypothesis)

\[
= \tilde{f}(u_0 v_0) P_{\omega_1,R}(\tilde{f}(u') P_{1,R}(\tilde{f}(v'))) + \tilde{f}(u_0 v_0) P_{\tau_1,R}(P_{\omega_1,R}(\tilde{f}(u')) \tilde{f}(v')) \quad \text{(by Eq. (27))}
\]

\[
= \tilde{f}(u_0 v_0) P_{\omega_1,R}(\tilde{f}(u')) P_{1,R}(\tilde{f}(v')) \quad \text{(by \( (R, P_{\omega_1,R}) \) being a matching Rota-Baxter algebra)}
\]

\[
= u_0 \tilde{f}(v_0) P_{\omega_1,R}(\tilde{f}(u')) P_{1,R}(\tilde{f}(v')) \quad \text{(by \( \tilde{f} \) being \( (F, \kappa_{\Omega}) \)-linearity)}
\]

\[
= \left(u_0 P_{\omega_1,R}(\tilde{f}(u'))\right)\left(\tilde{f}(v_0) P_{1,R}(\tilde{f}(v'))\right) \quad \text{(by Eqs. (27) and (28))}
\]

\[
= \tilde{f}(u_0 \otimes_{\omega_1} u') \tilde{f}(v_0 \otimes_{\tau_1} v')
\]

\[
= \tilde{f}(u) \tilde{f}(v).
\]

Case 3. \( u_0 \in \mathcal{A}^+ \) and \( v_0 \in F \). This case is similar to Case 2.

Case 4. \( u_0 \in \mathcal{A}^+ \) and \( v_0 \in \mathcal{A}^+ \). In this case, we write \( u_0 \) and \( v_0 \) in the form of Eq. (31) and obtain

\[
u = u_0 \otimes_{\omega_1} u' = u_0 P_{\omega_1}(u') \in \mathcal{A} \otimes_{\Omega} (\mathcal{A}^+) \otimes_{\Omega}^m,
\]

\[
v = v_0 \otimes_{\tau_1} v' = v_0 P_{\tau_1}(v') \in \mathcal{A} \otimes_{\Omega} (\mathcal{A}^+) \otimes_{\Omega}^n,
\]

which implies

\[
\tilde{f}(u \circ_{\Omega} v)
\]
= \tilde{f}(u_0 \otimes_{\omega_1} u') \circ_{\Omega} (v_0 \otimes_{\tau_1} v')
= \tilde{f}(u_0 \odot_{\Omega} P_{\omega_1}(u') \odot_{\Omega} (v_0 \odot_{\tau_1} P_{\tau_1}(v'))
= \tilde{f}(u_0 v_0) \odot_{\Omega} P_{\omega_1}(u') \odot_{\Omega} P_{\tau_1}(v') \quad \text{(by } \odot_{\Omega} \text{ being commutative and Eq. (14)})
= \tilde{f}(u_0 v_0) \odot_{\Omega} (P_{\omega_1}(u') \odot_{\Omega} P_{\tau_1}(v')) + P_{\tau_1}(P_{\omega_1}(u') \odot_{\Omega} v') \quad \text{(by Step 1)}
= \tilde{f}(u_0 v_0) P_{\omega_1}(u') + u_0 v_0 P_{\tau_1}(P_{\omega_1}(u') \odot_{\Omega} v') \quad \text{(by Eq. (14))}
= \tilde{f}(u_0 v_0)(P_{\omega_1,R} \tilde{f})(u') \odot_{\Omega} (1 \otimes_{\tau_1} v') + \tilde{f}(u_0 v_0)(P_{\tau_1,R} \tilde{f})(1 \otimes_{\omega_1} u') \odot_{\Omega} v' \quad \text{(by Eq. (28))}
= \tilde{f}(u_0 v_0) P_{\omega_1,R} \tilde{f}(u') \odot_{\Omega} (1 \otimes_{\tau_1} v') + \tilde{f}(u_0 v_0) P_{\tau_1,R} \tilde{f}(1 \otimes_{\omega_1} u') \odot_{\Omega} v' \quad \text{(by the induction hypothesis)}
= \tilde{f}(u_0 v_0) P_{\omega_1,R} \tilde{f}(u') \odot_{\Omega} (P_{\tau_1}(v')) + P_{\tau_1,R} (P_{\omega_1,R} \tilde{f}(u')) \odot_{\Omega} v' \quad \text{(by Eq. (29))}
= \tilde{f}(u_0 v_0) P_{\omega_1,R} \tilde{f}(u') \odot_{\Omega} P_{\tau_1,R} \tilde{f}(v') \quad \text{(by } (R, P_{\Omega,R}) \text{ being a matching Rota-Baxter algebra)}
= \tilde{f}(u_0) \tilde{f}(v_0) P_{\omega_1,R} \tilde{f}(u') \odot_{\Omega} P_{\tau_1,R} \tilde{f}(v') \quad \text{(by the initial step of this subcase )}
= \tilde{f}(u_0) P_{\omega_1,R} \tilde{f}(u') \odot_{\Omega} \tilde{f}(v_0) P_{\tau_1,R} \tilde{f}(v') \quad \text{(by Step 1)}
= \tilde{f}(u_0) \tilde{f}(v) \quad \text{(by Eq. (28)).}

(The uniqueness). Since \( \tilde{f} \) is a matching \((F, \kappa_{\Omega})\)-Rota-Baxter algebra homomorphism with \( f = \tilde{f} J_{\Lambda} \), it must be determinate uniquely by Eq. (25)—Eq. (28).

4. Matching dendriform algebras and matching Zinbiel algebras

In this section, we introduce the concept of commutative matching dendriform algebras and the equivalent matching Zinbiel algebras. We then establish their relationship with MRBAs, generalizing the connection of Zinbiel algebras with commutative Rota-Baxter algebras. Finally, free matching Zinbiel algebras are constructed.

4.1. Commutative matching dendriform algebras. Motivated by the natural connection of Rota-Baxter algebras (of weight zero) with dendriform algebras [11] on the one hand, and the connection with pre-Lie algebras on the other, dendriform algebras have been generalized to matching dendriform algebras in [42]. We recall this notion and basic properties.

Definition 4.1. Let \( \Omega \) be a nonempty set. A matching dendriform algebra is a \( k \)-module \( D \) together with a family of binary operations \((<_{\omega}, >_{\omega})_{\omega \in \Omega}\), such that, for \( x, y, z \in T \) and \( \alpha, \beta \in \Omega \),

\begin{align*}
(32) & \quad (x <_{\alpha} y) <_{\beta} z = x <_{\alpha} (y <_{\beta} z) + x <_{\beta} (y >_{\alpha} z), \\
(33) & \quad (x >_{\alpha} y) <_{\beta} z = x >_{\alpha} (y <_{\beta} z), \\
(34) & \quad (x <_{\beta} y) >_{\alpha} z = (x >_{\alpha} y) >_{\beta} z = x >_{\alpha} (y >_{\beta} z).
\end{align*}

Remark 4.2. (a) Let \((D, (<_{\omega}, >_{\omega})_{\omega \in \Omega})\) be a matching dendriform algebra. Consider linear combinations

\begin{align*}
<_{A} := \sum_{\omega \in \Omega} a_{\omega} <_{\omega} \quad \text{and} \quad >_{A} := \sum_{\omega \in \Omega} a_{\omega} >_{\omega}, \quad a_{\omega} \in k,
\end{align*}
with a finite support. Then \((D, \langle A, > A)\) is a dendriform algebra.

(b) An MRBA \((R, (P_\omega)_{\omega \in \Omega})\) of weight \(\lambda_\Omega = (\lambda_\omega)_{\omega \in \Omega}\) induces a matching dendriform algebra \((R, \langle \lambda_\omega, \lambda_\omega \rangle)_{\omega \in \Omega}\), where

\[
x <_\omega y := xP_\omega(y) + \lambda_\omega xy \quad \text{and} \quad x >_\omega y := P_\omega(x)y \quad \text{for} \ x, y \in R, \omega \in \Omega.
\]

Generalizing commutative dendriform algebras, we give

**Definition 4.3.** Let \(\Omega\) be a nonempty set. A matching dendriform algebra \((D, \langle_; >; \rangle)_{\omega \in \Omega}\) is called **commutative** if

\[
x >_\omega y = y <_\omega x \quad \text{for} \ x, y \in D, \omega \in \Omega.
\]

We give an example of matching dendriform algebras from Volterra integral operators.

**Example 4.4.** As in Example 2.3, consider the \(\mathbb{R}\)-algebra \(R := \text{Cont}(\mathbb{R})\) and a family \((k_\omega(x))_{\omega \in \Omega}\) of continuous functions on \(R\). Define a **Volterra integral operator** \(I_\omega : R \to R\) by taking

\[
I_\omega(f(x)) := \int_0^x k_\omega(t)f(t)\, dt \quad \text{for} \ f \in R, \omega \in \Omega.
\]

By Example 2.3, \((R, (I_\omega)_{\omega \in \Omega})\) is a commutative MRBA of weight 0. Applying Remark 4.2 (b), the operations

\[
f(x) <_\omega g(x) := f(x) \int_0^x k_\omega(t)g(t)\, dt, \quad f(x) >_\omega g(x) := g(x) \int_0^x k_\omega(t)f(t)\, dt
\]

equip \(R\) with a commutative matching dendriform algebra structure.

The commutative dendriform algebra is equivalent to the Zinbiel algebra which arose as the Koszul dual to a Leibniz algebra introduced by Loday [33]. Free Zinbiel algebras were shown to be precisely the shuffle product algebra [34].

**Definition 4.5.** A **(left) matching Zinbiel algebra** is a \(k\)-module \(Z\) together with a family of binary operations \((\circ_\omega)_{\omega \in \Omega}\) such that, for \(x, y, z \in Z\) and \(\alpha, \beta \in \Omega\),

\[
(x \circ_\alpha y) \circ_\beta z = x \circ_\alpha (y \circ_\beta z) + x \circ_\beta (z \circ_\alpha y).
\]

**Remark 4.6.** (a) Any Zinbiel algebra can be viewed as a matching Zinbiel algebra by taking \(\Omega\) to be a singleton.

(b) In a matching Zinbiel algebra \((Z, (\circ_\omega)_{\omega \in \Omega})\), \((Z, \circ_\omega)\) is a Zinbiel algebra for any \(\omega \in \Omega\).

(c) For a matching Zinbiel algebra, it follows from Eq. (35) that

\[
(x \circ_\beta z) \circ_\alpha y = x \circ_\beta (z \circ_\alpha y) + x \circ_\alpha (y \circ_\beta z).
\]

Note that the right hand side is invariant when \((y, \alpha)\) is replaced by \((z, \beta)\). Thus

\[
(x \circ_\alpha y) \circ_\beta z = (x \circ_\beta z) \circ_\alpha y,
\]

which is precisely the second axiom of **multiple permutative algebras** introduced by Foissy [18, Proposition 12].

It is well-known that a Zinbiel algebra is equivalent to a commutative dendriform algebra [1], which we now generalize to the matching context.

**Proposition 4.7.** Let \(\Omega\) be a nonempty set.

(a) If \((D, \langle; _, >_; \rangle)_{\omega \in \Omega}\) is a commutative matching dendriform algebra, then \((D, \langle_; \rangle)_{\omega \in \Omega}\) is a match Zinbiel algebra.
Further, for $x >_\alpha y >_\beta z$,

$$x >_\alpha (y >_\beta z) = (z <_\beta y) <_\alpha x = z <_\beta (y <_\alpha x) + z <_\alpha (x <_\beta y)$$

(by Eq. (35))

$$= (y <_\alpha x) >_\beta z + (x <_\beta y) >_\alpha z$$

(by Eq. (36))

$$= (x >_\alpha y) >_\beta z + (x <_\beta y) >_\alpha z$$

(by Eq. (36))

$$= (x <_\beta y) >_\alpha z + (x >_\alpha y) >_\beta z,$$

as required. \hfill \Box

The following result shows that a matching Zinbiel algebra gives rise to other matching Zinbiel algebras by arbitrary finite linear combinations.

**Proposition 4.8.** Let $\Omega$ and $I$ be nonempty sets and let $A_i : \Omega \to k$, $i \in I$, be a family of maps with finite supports, identified with $A_i = (a_{i,\omega})_{\omega \in \Omega}$. Let $(Z, (\circ_{\omega})_{\omega \in \Omega})$ be a matching Zinbiel algebra. Consider the linear combinations

$$\circ_i := \circ_{A_i} := \sum_{\omega \in \Omega} a_{i,\omega} \circ_{\omega} \text{ for } i \in I. \tag{38}$$

Then $(Z, (\circ_i)_{i \in I})$ is a matching Zinbiel algebra. In particular, $(Z, \circ_i)$ is a Zinbiel algebra for each $\omega \in \Omega$.

**Proof.** For $x, y, z \in Z$ and $\alpha, \beta \in \Omega$, we have

$$\left(x \circ_i y\right) \circ_j z = \sum_{\beta \in \Omega} a_{j,\beta} \left(\sum_{\omega \in \Omega} a_{i,\omega} x \circ_{\omega} y\right) \circ_{\beta} z \quad \text{(by Eq. (38))}$$
4.2. The free matching Zinbiel algebra. In this subsection, we construct the free commutative matching Zinbiel algebra on a module.

**Definition 4.9.** Let $\Omega$ be a nonempty set and $M$ a module. A free matching Zinbiel algebra on $M$ is a matching Zinbiel algebra $(D, (\omega)_{\omega \in \Omega})$ together with a module homomorphism $j_M : M \to D$ satisfying the universal property: for any matching Zinbiel algebra $(D', (\omega)_{\omega \in \Omega})$ and any module homomorphism $f : M \to D'$, there is a unique matching Zinbiel algebra homomorphism $\tilde{f} : D \to D'$ such that $f = \tilde{f} j_M$.

For a $k$-module $M$, we can regard $M$ as a commutative algebra equipped with the zero multiplication. By Theorem 2.10, the free commutative of weight zero on $M$ is

$$\Pi_\Omega(M) = \bigoplus_{n \geq 1} M^{\otimes n},$$

equipped with the product $\diamond$ defined in Eq. (13) and operators defined in Eq. (15). Let

$$a := a_0 \otimes_{a_1} a_1 \otimes_{a_2} \cdots \otimes_{a_m} a_m = a_0 \otimes_{a_1} a' \in \Pi_\Omega(M),$$

$$b := b_0 \otimes_{b_1} b_1 \otimes_{b_2} \cdots \otimes_{b_n} b_n = b_0 \otimes_{b_1} b' \in \Pi_\Omega(M)$$

with $m, n \geq 0$. For $\omega \in \Omega$, define

$$\prec_{\omega} : \Pi_\Omega(M) \otimes \Pi_\Omega(M) \to \Pi_\Omega(M),$$

$$a \otimes b \mapsto a_0 \left( (1 \otimes_{a_1} a') \Pi_{\omega} (1 \otimes_{b} b) \right) = a \diamond_{\omega} (1 \otimes_{\omega} b) = a \diamond_{\omega} P_{\omega}(b).$$

Now we are ready for free matching Zinbiel algebra.

**Theorem 4.10.** Let $\Omega$ be a nonempty set and $M$ a module. Then $(\Pi_\Omega(M), (\prec_{\omega})_{\omega \in \Omega})$, together with the natural algebra homomorphism $j_M : M \to \Pi_\Omega(M)$, is the free matching Zinbiel algebra on $M$.

**Remark 4.11.** By the definition of matching Zinbiel products $(\prec_{\omega})_{\omega \in \Omega}$ and Eq. (39), the product $\prec_{\omega, \omega}$, can be given by the decorated shuffle product as follows.

$$a \prec_{\omega} b = a_0 \left( (1 \otimes_{a_1} a') \Pi_{\omega} (1 \otimes_{b} b) \right)$$
\[ = a_0 \left( \sum_{\sigma \in S(m+n+1)} \sigma \left( \left[ \begin{array}{c|c|c|c|c} a_1 & a_2 & \cdots & a_m & b_0 \\ \hline b_1 & b_2 & \cdots & b_n \end{array} \right] \right) \right). \]

When \( \Omega \) is a singleton, this is the product for the free Zinbiel algebra in terms of shuffle product given by Loday [33].

**Proof.** By Theorem 2.10, the triple \((\Pi_\Omega(M), \circ_\Omega, (P_\omega.M)_{\omega \in \Omega})\) is the free commutative matching Rota-Baxter algebra of weight zero on \( M \). So \((\Pi_\Omega(M), (\prec_\omega, \succ_\omega)_{\omega \in \Omega})\) is a matching dendriform algebra by Remark 4.2 (b), where

\[ a \prec_\omega b := P_\omega(a) \circ_\Omega b = (1 \otimes_\omega a) \circ_\Omega b = b_0 \left( (1 \otimes_\omega a) \omega_\Omega (1 \otimes_{\beta_1} b') \right). \]

(40)

\[ a \succ_\omega b := a \circ_\Omega P_\omega(b) = a \circ_\Omega (1 \otimes_\omega b) = a_0 \left( (1 \otimes_{\alpha_1} a') \omega_\Omega (1 \otimes_\omega b) \right). \]

Further, it is commutative, as \((\Pi_\Omega(M), \circ_\Omega)\) is commutative. Thus it follows from Proposition 4.7 (a) that \((\Pi_\Omega(M), (\prec_\omega, \succ_\omega)_{\omega \in \Omega})\) is a matching Zinbiel algebra.

We next prove that \((\Pi_\Omega(M), (\prec_\omega, \succ_\omega)_{\omega \in \Omega})\) is the free matching Zinbiel algebra on \( M \) by verifying its universal property. For this, let \((D, (\prec_\omega, \succ_\omega)_{\omega \in \Omega})\) be a matching Zinbiel algebra and let \( f : M \to D \) be a module homomorphism. We prove that there is unique homomorphism \( \tilde{f} : \Pi_\Omega(M) \to D \) of matching Zinbiel algebras such that \( \tilde{f}_f = f \).

(The existence). Define a linear map \( \tilde{f} : \Pi_\Omega(M) \to D \) as follows. For \( a = a_0 \otimes_{\alpha_1} a' \in M^{\otimes_\Omega m+1} \) with \( a_0 \in M \) and \( a' \in M^{\otimes_\Omega m} \), we define \( \tilde{f}(a) \) by induction on \( m \geq 0 \). For the initial step of \( m = 0 \), we have \( a = a_0 \) and define

\[ \tilde{f}(a) := \tilde{f}(a_0) := f(a_0). \]

For the induction step of \( m \geq 1 \), we define

\[ \tilde{f}(a) := \tilde{f}(a_0 \otimes_{\alpha_1} a') := f(a_0) \prec_{\alpha_1, D} \tilde{f}(a'). \]

We now prove that \( \tilde{f} \) is a homomorphism of matching dendriform algebras:

\[ \tilde{f}(a \prec_\omega b) = \tilde{f}(a) \prec_{\omega, D} \tilde{f}(b) \]

(43)

for \( a = a_0 \otimes_{\alpha_1} a' \in M^{\otimes_\Omega (m+1)} \) and \( b = b_0 \otimes_\omega b' \in M^{\otimes_\Omega (n+1)} \) with \( m, n \geq 0 \).

We proceed to prove Eq. (43) by induction on \( m + n \geq 0 \). When \( m + n = 0 \), we have \( a = a_0 \) and \( b = b_0 \) are in \( M \). So by Eqs. (39), (41) and (42),

\[ \tilde{f}(a \prec_\omega b) = \tilde{f}(a_0 \otimes_\omega b_0) = \tilde{f}(a_0 (1 \otimes_\omega b_0)) = \tilde{f}(a_0 \circ_\Omega b_0) = f(a_0) \prec_{\omega, D} f(b_0) = \tilde{f}(a) \prec_{\omega, D} \tilde{f}(b). \]

Assume that Eq. (43) has been proved when \( m + n = k \) for a \( k \geq 0 \), and consider the case of \( m + n = k + 1 \). Then

\[ \tilde{f}(a \prec_\omega b) = \tilde{f}(a_0 (1 \otimes_{\alpha_1} a') \omega_\Omega (1 \otimes_\omega b)) \]

\[ = \tilde{f}(a_0 \otimes_{\alpha_1} a_1 (1 \otimes_{\alpha_2} a_2 \otimes_{\alpha_3} \cdots \otimes_{\alpha_m} a_m) \omega_\Omega (1 \otimes_\omega b)) + a_0 \otimes_\omega b_0 \left( (1 \otimes_{\alpha_1} a') \omega_\Omega (1 \otimes_{\beta_1} b_1 \otimes_{\beta_2} b_2 \otimes_{\beta_n} b_n) \right) \]

(by Eq. (10))

\[ = f(a_0) \prec_{\alpha_1, D} \tilde{f}(a') \prec_\omega b + \tilde{f}(a_0) \prec_{\omega, D} \tilde{f}(b) \]

(by Eqs. (40) and (42))

\[ = \tilde{f}(a_0) \prec_{\alpha_1, D} \left( \tilde{f}(a') \prec_{\omega, D} \tilde{f}(b) \right) + \tilde{f}(a_0) \prec_{\omega, D} \left( \tilde{f}(b) \prec_{\alpha_1, D} \tilde{f}(a') \right) \]

(by the induction hypothesis)
\[
= (\tilde{f}(a_0) \prec_{\alpha_1, D} \tilde{f}(a')) \prec_{\omega, D} \tilde{f}(b) \quad \text{(by Eq. (35))}
\]
\[
= (f(a_0) \prec_{\alpha_1, D} \tilde{f}(a')) \prec_{\omega, D} \tilde{f}(b) \quad \text{(by Eq. (41))}
\]
\[
= \tilde{f}(a) \prec_{\omega, D} \tilde{f}(b) \quad \text{(by Eq. (42))},
\]

(The uniqueness). Suppose that \( \tilde{f}' : \Pi_{\Omega}(M) \to D \) is a homomorphism of matching Zinbiel algebras with \( \tilde{f}'j_M = f \). We verify \( \tilde{f}'(a) = \tilde{f}(a) \) for \( a = a_0 \otimes_{\alpha_1} a' \in M^{\otimes_0(m+1)} \) by induction on \( m \geq 0 \). For the initial step of \( m = 0 \), we have \( a = a_0 \). By Eq. (41), we have
\[
\tilde{f}'(a) = \tilde{f}'(a_0) = f(a_0) = \tilde{f}(a).
\]

For the induction step of \( m \geq 1 \), we have
\[
\tilde{f}'(a) = \tilde{f}'(a_0 \otimes_{\alpha_1} a') = \tilde{f}'(a_0) \prec_{\alpha_1} a' = \tilde{f}'(a_0) \prec_{\alpha_1, D} \tilde{f}'(a') = f(a_0) \prec_{\alpha_1, D} \tilde{f}'(a').
\]

By the induction hypothesis, \( \tilde{f}'(a') = \tilde{f}(a') \). Then \( \tilde{f}'(a) = \tilde{f}(a) \) by the construction of \( \tilde{f}(a) \). This completes the induction. \( \square \)

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