Quantum probing topological phase transitions by non-Markovianity

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Understanding the physical significance and probing the global invariants characterizing quantum topological phases in extended systems is a main challenge in modern physics with major impact in different areas of science. Here we propose a quantum-information-inspired probing method where topological phase transitions are revealed by a non-Markovianity quantifier. We illustrate our idea by considering the decoherence dynamics of an external read-out qubit that probes a Su-Schrieffer-Heeger (SSH) chain with either pure dephasing or dissipative coupling. Qubit decoherence features and non-Markovianity measure clearly signal the topological phase transition of the SSH chain.

1 Introduction

The discovery of the quantum Hall effect [1, 2] has revolutionized our understanding of quantum phases of matter, with major impact in different areas of physics. Symmetry-protected topological phases are characterized by topological invariants and symmetries such that quantum phases with different topological invariants are not connected to each others by perturbations, unless a spectral gap is closed. Remarkable examples are quantum Hall systems [1,2], topological insulators, and topological superconductors [3–5]. While in conventional Ginzburg-Landau theory phase transitions are identified by symmetry-breaking of a local order parameter [6], topological phases are described by global invariants such as the Zak phase [7,8], the Chern number [9,10], or the Thouless-Kohmoto-Nightingale-denNijs invariant [11].

Since non-local topological invariants can escape a direct measure, identifying ways to relate topological invariants with measurable quantities has engaged physicists for long time. The bulk-boundary correspondence [5], relating the bulk topological invariants with the number of edge states in finite systems with open boundaries, provides the simplest route to measure topological numbers [12–14]. However, edge states are not always accessible and the bulk-boundary correspondence can even fail in certain models, such as in non-Hermitian systems [15,16]. Direct measurements of topological invariants in the bulk have been proposed and demonstrated in a series of experiments with synthetic matter, such as those based on Bloch oscillations [17,18], unitary and non-unitary quantum walks [19–25], and out-of-equilibrium (quench) dynamics [26–33]. In one dimension, the simplest model exhibiting a topological phase transition is the celebrated Su-Schrieffer-Heeger (SSH) model, describing a tight binding one-dimensional lattice with staggered hopping amplitudes [34,35]. The SSH model exhibits two distinct topological phases characterized by different values of Zak phase [17]. Such topological phases have been experimentally observed in different physical settings, including cold-atom platforms [17], photonic systems [20,21,23] and topological circuits [36].

The identification of topological phases by means of quantum-information-oriented indicators is a rather new and exciting area of research, which remains largely unexplored. Previous studies highlighted the interplay between topology and entanglement entropy, entanglement spectrum and multipartite entanglement [37–40]. In particular, entanglement was used as a probe in the spin- 1/2 Heisenberg model on the kagome lattice [41] and in the Kitaev model [42] while, for the SSH model, bipartite ground-state entanglement was shown to exhibit a non-analytical behavior around the topological phase transition [43].

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One of the most important topics recently developed in the field of open quantum systems is certainly the search for memory effects in large environments. Then, a variety of approaches has been proposed to quantify the amount of non-Markovianity focusing on Markovian approximations in open quantum systems, the characterization of memory effects and their quantitative estimate by non-Markovianity measures [44–47]. This is because a precise knowledge of the environmental properties can be exploited to mitigate dissipation or to exploit it by engineering the bath [51, 52].

In this paper, we propose a quantum probing scheme based on an external read-out qubit where topological phase transitions are revealed by non-Markovianity measure. While previous bulk probing methods require rather generally to access the dynamical evolution in the lattice [17, 20, 21, 23], in our setting the probing is fully external and topological phases are retrieved by the decoherence dynamics of the qubit. In open quantum systems, non-Markovian quantum probes have been recently explored to extract information from environments [48–50, 52–54]. In particular, local and global external probes of many-body systems have been considered looking at their non-Markovianity in connection with quantum phase transition in Ising models [54], superfluid-to-Mott-insulator transition [50] and Anderson localization in disordered environments [48].

Our main idea is illustrated by considering decoherence dynamics of a qubit locally probing a SSH lattice by either pure dephasing or dissipative interaction. A sharp transition between non-Markovian and Markovian dynamics is observed when the SSH gap closes and reopens into a different topological phase. The local interaction between the chain and the probe gives rise to new states, localized around the probe [bound states (BSs)], the number and the symmetry properties of which are strongly dependent on the topological phase of the SSH chain. This enables the sharp transition between non-Markovian and Markovian dynamics of the (out-of-equilibrium) probe. For a dephasing qubit the relation between Loschmidt echo and the information flow between the probe and the system was reported in Ref. [54]. This read-out of the topological phase is robust, as different measures of non-Markovianity coincide for dephasing probe and, furthermore, it does not need fine tuning of probing strengths.

On the other hand the dissipative qubit scheme allows to read-out the topological phase with an even sharper transition but its operation is limited to a restricted probing strengths regime.

The paper is organized as follows. Section 2 provides the description of the SSH model and of its dephasing interaction with the external qubit used as a probe. The decoherence dynamics of the probe and the exact energy spectrum of the coupled qubit-bath system are presented in Section 3. The non-markovian features of the probe dynamics are discussed in Section 4, where we show how the topological phase transition of the SSH chain can be revealed by a non-Markovian quantifier. The main conclusions are outlined in Section 5. Finally, some technical details and a different coupling scheme are presented in two Appendices.

2 Model

We consider a one-dimensional SSH lattice probed by a qubit interacting with the unitary cell of the lattice (Fig. 1). Indeed, if the interaction were limited to a single site, symmetry reasons would rule out any difference between the two topological phases. The Hamiltonian of the full system is given by

$$H = H_{SSH} + H_I + H_P$$  \hspace{1cm} (1)

where $H_P$ is the Hamiltonian of the probe qubit,

$$H_{SSH} = t_1 \sum_{n=1}^{N} (a_n^\dagger b_n + \text{hc}) + t_2 \sum_{n=1}^{N} (a_n^\dagger b_{n-1} + \text{hc}) $$  \hspace{1cm} (2)

is the Hamiltonian of the SSH chain, with $a, b$ either fermionic or bosonic annihilation operators and $H_I$ describes the qubit-chain interaction. The SSH chain comprises $N$ unit cells, and periodic boundary conditions are assumed. As for interaction term, it can be either of dissipative or dephasing nature for the probe. Here we will primarily focus on the dephasing scenario, as it allows for a direct link between topology detection and non-Markovianity. Signatures of the topological phases can be also found looking at the non-Markovianity of the probe dynamics in the dissipative case, but the results there are less direct and will be described in detail in Appendix A.

The interaction Hamiltonian in the pure dephasing case, (where the probe Hamiltonian can be written as $H_P = (\sigma_0/2)\sigma_z$, with $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ the pseudospin operator associated to the probe qubit), analogous to what normally used in quantum nondemolition protocols, can be obtained working in the dispersive coupling limit: if we denote with $t$ and we label with $0$ the chain cell at which the probe is coupled,

$$H_I = \gamma \left( \frac{1}{2} + \sigma_z \right) \otimes (a_0^\dagger a_0 + b_0^\dagger b_0) $$  \hspace{1cm} (3)

where $\gamma$ measures the coupling strength. This $\sigma_z \otimes n$ interaction has been reported in the 90’s for atom-light interaction in the dispersive limit [55, 56]. In these experiments, Rydberg atoms interact with microwave photons.
Taking as initial state at $t_0$ a coherent state for the probe $C_e |e\rangle + C_g |g\rangle$ and a generic chain state $|\Psi_0\rangle$ (that will be chosen in the one-excitation sector), the evolution of the probe reduced density matrix reads

$$\rho_p(t) = \begin{pmatrix} |C_e|^2 & C_e C_g^* q(t) \\ C_g^* C_e q^*(t) & |C_g|^2 \end{pmatrix}$$

where the only time-dependent coefficient is given by

$$q(t) = \langle \Psi_0 | e^{iH_{SSH} t} e^{-i\hat{H} t} | \Psi_0 \rangle,$$

with $\hat{H} = H_{SSH} + \gamma (a_0^\dagger a_0 + b_0^\dagger b_0)$. Its modulus square $L(t) = |q(t)|^2$ is known as Loschmidt echo (LE) and is a direct measure of decoherence. This is typical of pure dephasing dynamics, where the probe decoherence rate is fully determined by a quantity that only depends on environmental degrees of freedom. In order to determine the dynamics of the LE, it is then necessary to study the properties of the renormalized Hamiltonian $\hat{H}$, which corresponds to an effective SSH chain displaying a potential-site defect at the $n = 0$ unit cell. For a sufficiently large number of sites, $\hat{H}$ keeps the two-band structure of $H_{SSH}$. However, up to four BSs around the impurity can emerge that can deeply affect the probe dynamics. A way to determine the presence and the shape of BSs is based on a simple Ansatz that exploits the mirror symmetry of $\hat{H}$. If such states exist, their wavefunction must be either even or odd around the impurity cell 0, that is, they can be written as

$$|\psi\rangle_{\pm} = |c_0(a_0^\dagger \pm b_0^\dagger) + c_1(b_{-1}^\dagger \pm a_1^\dagger) + c_2(a_{-2}^\dagger \pm b_1^\dagger) + \ldots\rangle |0\rangle$$

From the time-independent Schrödinger equation, these two families of states lead to two sets of coupled equations that can be solved analytically in the thermodynamic limit through a scaling Ansatz. Taking into account the structure of the model, we can introduce the dimer localization parameter $X$ and the dimer imbalance parameter $Y$ such that

$$c_{2n} = c_0 X^{-n} \quad n \geq 1$$
$$c_{2n+1} = c_0 Y^n \quad n \geq 0$$

with $|X| > 1$ for localization. The eigenenergy and localization/imbalance parameters are found as solutions of coupled algebraic equations; details are given in Appendix B. Depending on the ratio $t_1/t_2$, we face two different scenarios: (i) For $t_1 < t_2$, there are always two symmetric BSs, irrespective of the strength of the coupling constant $\gamma$. As for the antisymmetric equations, we have two solutions for $t_1 < \gamma/2$, while no solutions are found whenever $t_1 > \gamma/2$. (ii) For $t_1 > t_2$, there exist one symmetric and one antisymmetric solutions, without any dependence on $\gamma$. The band structure of $\hat{H}$ and diagram of BSs is presented in Figs. 2(a) and (b) for the two topological phases as a function of the coupling $\gamma$; the behavior of the localization parameter $X$ as a function of $t_1/t_2$, which determines the boundary of existence of BSs, is given in Appendix B. The energy spectrum is composed by the two minbands of the SSH lattice, with gap separation $|t_1 - t_2|$, and a number of BSs (point spectrum) with energies either in the gap or above the upper minband (Fig. 2). As commented above, the symmetry properties of such BSs change drastically across the critical point $t_1/t_2 = 1$: irrespective of the value of $\gamma$, in the phase $t_1 < t_2$, there is an even number of symmetric BSs and an even number of antisymmetric BSs, while both symmetric and antisymmetric BSs appear in an odd number in the phase $t_1 > t_2$.

In the case $\gamma < 2t_1$, which is the most typical operational regime, the qubit-SSH system sustains two BSs. The two BSs are both symmetric in the topological phase $t_2 > t_1$, whereas in the topological phase $t_2 < t_1$ they have opposite parity. This is because, at the gap closing point $t_2 = t_1$, the parity of the in-gap bound state changes from symmetric to anti-symmetric [see Fig. 2(c)]. The flip of parity of the in-gap bound state at the band closing point is a topological property, related to the topological phase transition of the SSH chain, and it turns out to be robust.
against off-diagonal disorder in the chain, as shown in the next section. Indeed, the two bound states present for \( \gamma < 2t_1 \) (assuming \( \gamma > 0 \), which makes the defect Hamiltonian nonnegative) directly derive from the two upper band-edge states of the SSH chain, which are expelled from their respective bands in the presence of the renormalization induced by the probe, as their energies are lifted up. These band-edge states have energies respectively \( |t_1 + t_2| \) and \( -|t_1 - t_2| \), as it can be obtained from equations (B-3) and (B-4) taking \( \gamma \rightarrow 0 \). As the cell defect does not break the spatial mirror symmetry of the model, the new BSs must belong to the same symmetry sector of their zero-defect limit states. Now, while the band-edge state with energy \( |t_1 + t_2| \) is completely uniform along the chain and symmetric independently on the topological phase of the SSH model, the state with energy \( -|t_1 - t_2| \) is a symmetric one for \( t_1 > t_2 \) while is antisymmetric for \( t_1 < t_2 \). This change of symmetry directly originates in the SSH winding number, as it is critically related to the eigenvalues in the momentum space. Thus, the flip of parity of the in-gap bound state is due to the change of topological phase of the SSH chain. In the context of different topological models, a dependence of defect-induced in-band BSs on the symmetries of the model was also reported in Refs. [60,61].

4 Non-Markovianity measure and topological phase transition

The presence or the absence of BSs and their symmetry properties deeply change the decoherence dynamics of the qubit. In particular, the parity flip of the in-gap BS at the band closing point can be exploited to reveal the topological phase transition of the SSH bath. In fact, depending on the system parameters and on the initial state of the chain, the coherence (5) can experience either a pure exponential decay or oscillatory behavior. In particular, we are interested in studying the behavior of the coherence and LE around the topological critical point \( t_1 = t_2 \) and to relate the phase transition to the non-Markovianity measure. Let us assume that the initial state of the lattice is an anti-symmetry state \( |\Psi_0^A\rangle \) around the impurity cell. The choice of such a state is motivated by the fact that its symmetry properties makes it especially sensitive to the topological phase, however we stress that our results hold even though the initial state is not perfectly antisymmetric, i.e. our probing protocol is robust against unavoidable imperfections in the initial state preparation of the system. The dynamics of \( |\Psi_0^A(t)\rangle \) will be greatly affected by the presence of BSs and by their symmetries, which are in turn established by the topological phase of the SSH bath. In fact, for \( t_1 < t_2 \), \( |\Psi_0^A\rangle \) is orthogonal to both BSs, that are indeed symmetric, while for \( t_1 > t_2 \) one of the two BSs will contribute to the dynamics (see Fig. 2), which will eventually give rise to an oscillatory behavior of \( L(t) \). Since the symmetry flipping of the in-gap BS requires the gap to close and reopen in the other topological phase of the SSH chain (fig. 2(c)), the qualitative change of the LE, from exponential-like to oscillatory-like, is thus of topological origin. Examples of the decay dynamics of the coherence \( |q(t)\rangle \) in the two different topological phases are given in the inset of Fig. 3. In the simulations, the initial state belongs to the one-excitation sector

\[
|\Psi_0^A\rangle = \frac{1}{2}(a_0^+ - b_0^+ + a_1^+ - b_1^+)|0\rangle
\]

corresponding to an anti-symmetric state homogeneously distributed over two unit cells, and the time evolution is computed in the two topological phases for a fixed value of the coupling \( \gamma / t_1 \). For \( t_2 > t_1 \), the dynamics of \( |\Psi_0^A\rangle \) is blind with respect to the two BSs and only depends on
The initial state is \(|\Psi_0^A\rangle = (1/2)(a_0^+ - b_0^+ + a_1^+ - b_1^+)|0\rangle\). The behavior of \(\mathcal{N}_T\) is shown for three different values of the ratio \(\gamma/t_1\) (\(\gamma/t_1 = 0.8\) (red), \(\gamma/t_1 = 1.2\) (blue), and \(\gamma/t_1 = 1.6\) (black)). In all cases a clear phase transition is observed at \(t_1 = t_2\), with \(\mathcal{N}_T\) almost vanishing in the \(t_2 > t_1\) topological phase. Longer times would slightly modify the value of \(\mathcal{N}_T\), as the initial oscillations would weight less. The inset shows the detailed dynamics of the coherence \(|q(t)|\) in the two distinct topological phases for the same coupling strength \(\gamma = 1.2 t_1\). Curve 1: \(t_2 = 0.6 t_1\) (red); curve 2: \(t_2 = 1.4 t_1\) (blue). A chain of \(N = 200\) cells has been considered in the numerical simulations.

As already pointed out in Refs. [48, 65], \(\mathcal{N}\) diverges if \(|q(t)|\) exhibits oscillations, as it happens in our case. This drawback can be simply circumvented by time average \(\mathcal{N}\) over several periods of oscillations: we can in fact define

\[
\mathcal{N}_T = \frac{1}{T} \int_0^T dt \frac{d\sqrt{L(t)}}{dt},
\]

From an experimental point of view, using \(\mathcal{N}_T\) instead of \(\mathcal{N}\) has no physical implications, as the probe dynamics has to be measured in finite time intervals in both cases. In Fig. 3, \(\mathcal{N}_T\) is displayed for three different chain-probe coupling constants as a function of \(t_2/t_1\). The transition observed is continuous around the critical point, as it is determined by the spectral weight of the antisymmetric BS with respect to the initial state. This quantity is exactly zero for \(t_1 = t_2\) and then starts building up in a continuous fashion as the BS gets more localized. The discontinuity in the derivative (as in ordinary first-order phase transitions) of \(\mathcal{N}_T\) showed in Fig. 3 tells us that non-Markovianity provides a sharp indicator that can be used as an effective order parameter to identify the topological phase.

The oscillations of the LE correspond to a periodic reflux of information from the bath back to the probe, which can be studied in terms of non-Markovianity through, for instance, the quantifier \(\mathcal{N}\) of Ref. [62]. Such a quantifier of memory flow-back is based on the observation that in Markovian processes the distinguishability between pairs of quantum states decreases in time. Then, the existence of time windows where this contractive property is violated witnesses the presence of non-Markovianity in the dynamical map. In the case of pure dephasing, as shown in Ref. [63], \(\mathcal{N}\) can be directly calculated as the integral of \(d\sqrt{L(t)}/dt\) extended over the time intervals where the derivative \(\dot{L}\) is positive. It is particularly convenient that, actually, in the presence of pure dephasing, different measures of non-Markovianity coincide [44, 47, 62], as they only depend on a single time-dependent function (the LE in our case). This also implies that \(\mathcal{N}\) is easily accessible in experiments, as shown for instance in Ref. [64] in the case of photons.
is broken, so that BSs cannot be classified anymore as symmetric or antisymmetric states. Nevertheless, we can introduce an indicator of the symmetry of an eigenstate about the defect unit cell via a parity index $\mathcal{P}$ defined as follows:

$$\mathcal{P} = \frac{1}{2} \sum_n |a_n + b_{-n}|^2 - \frac{1}{2} \sum_n (|a_n|^2 + |b_n|^2)$$

(11)

with $0 \leq \mathcal{P} \leq 1$, $\mathcal{P} = 1$ for an exact symmetric state ($a_n = b_{-n}$) and $\mathcal{P} = 0$ for an exact anti-symmetric state ($a_n = -b_{-n}$). Hence an eigenstate with parity index $\mathcal{P}$ close to one means that the eigenstate is dominantly a symmetric state, whereas an eigenstate with $\mathcal{P}$ close to zero indicates that it is a dominant anti-symmetric function.

Figure 4 shows the numerically-computed behavior of the energy spectra and non-Markovianity quantifier $\mathcal{N}_T$ versus $t_2/\langle t_1 \rangle$ in a disordered SSH chain, comprising 301 unit cells, for two different values of disorder strength and for $\gamma = 0.8\langle t_1 \rangle$. The initial state is $|\Psi_0^{(\delta)}\rangle = (1/2)(|a_0\rangle - |b_0\rangle + |a_1\rangle - |b_1\rangle)|0\rangle$. An inspection of the energy spectra indicates that, even for a moderate disorder in the chain, the BSs induced by coupling to the qubit keep a high degree of parity, as measured by the parity index $\mathcal{P}$ [see the insets in Figs.4(b) and (e)], and that the parity of the in-gap BS is flipped as the gap closes and reopens at $t_2 = \langle t_1 \rangle$. The main difference as compared to the ordered lattice is that the value of $\mathcal{N}_T$ in the $\langle t_1 \rangle < t_2$ topological phase does not vanish but settles down to a steady value that increases as the disorder strength increases. This behavior is likely to be ascribed to Anderson localization of bulk states in the lattice, which enhances non-Markovianity [48]. Such an enhancement is featureless as $t_2/\langle t_1 \rangle$ is varied, so that the signature of the topological phase transition is still clearly visible and the effect of disorder is basically to provide a bias to $\mathcal{N}_T$, as shown in Figs.4(c) and (f).

Before concluding, let us remark that the connection between non-Markovianity and probing of topological phases is strengthened by the fact that it is not strictly related to the model discussed so far but can also be found in the presence of a very dissimilar probe-chain interaction (dissipative coupling). In fact, as detailed in Appendix A, also in this different probing protocol, characterized by the interaction Hamiltonian $H_I = \gamma(\sigma_+ (a_0 + b_0) + \sigma_- (a_0^\dagger + b_0^\dagger))$ [66], it is possible to detect the topological phase transition by looking at the behavior of $\mathcal{N}_T$. The main difference with respect to the dephasing case is that an abrupt transition in the non-Markovianity measure only takes place if we set $\gamma = t_1$. Otherwise, $\mathcal{N}_T$ cannot be taken as an order parameter and a direct inspection of the shape of the output would be necessary to infer the topological phase (see Appendix A). Indeed, the number of BSs would in any case determine the number of frequencies observed in the output signal, which could be detected by a spectral analysis. This feature can also be found in the dephasing scenario in the case where initial state is not an antisymmetric one. Indeed, the dependence of the parity of the number of BSs on the topological phase of the SSH chain would in any case warrant qualitatively different probe dynamics and then different spectral contents.

### 5 Conclusions

Non-Markovianity and revivals of quantum coherence have attracted a great attention in the past recent years as potential resources in different contexts [44–47]. Here we have shown how non-Markovianity can be harnessed to externally probe the topological phases of quantum matter, without the need to detect the dynamics of the entire system. We have proposed a quantum-information-inspired strategy to measure global topological invariants of extended quantum systems by monitoring the decoherence dynamics of an external probe locally coupled to the system. The main idea has been illustrated by considering the non-Markovian dynamics of an external read-out qubit probing a SSH chain both for dephasing and dissipative reduced dynamics. Exploiting the bound-state phase diagram, one can choose a convenient set of possible initial states and tailor a very efficient probing protocol. Indeed, it turns out that non-Markovianity does not vanish solely in one of the two topological phases of the chain, thus revealing its topological order and serving as a sharp indicator for the transition between different phases. The system we have described is suitable for experimental implementation both in photonic and cold-atom platform, where the SSH chain can be simulated, as well as its coupling to the external probing qubit [66]. As commented above, the dephasing dynamics can be achieved working in the dispersive limit (strong detuning). Our results, besides of shedding new light onto the topological significance of memory effects in the dynamics of open quantum systems, suggest that topological phases can provide a powerful means to control (either enhance or suppress) quantum decoherence.

### Appendix

#### A Dissipative Model

We consider a two-level system (qubit) as a probe of the topological phase, dissipatively coupled to the SSH lattice...
Figure 4 (Color online) Effect of hopping disorder on energy spectra and non-Markovianity quantifier $\mathcal{N}_T$. (a) Behavior of hopping rate $t_1$ versus number of unit cell in the lattice for a realization of disorder (disorder strength $\delta = 0.15$). The qubit-bath coupling is $\gamma = 0.8 \langle t_1 \rangle$. (b) Numerically-computed energy spectrum of $\tilde{H}$ versus $t_2/\langle t_1 \rangle$. The insets show the behavior of the parity index $\mathcal{P}$ of the two BSs, indicating that parity flipping at the gap closing point is still observed in the presence of disorder. (c) Numerically-computed non-Markovianity quantifier $\mathcal{N}_T (T = 150 \langle t_1 \rangle)$ as a function of $t_2/\langle t_1 \rangle$. The inset shows the detailed dynamics of the coherence $|q(t)|$ in the two distinct topological phases for $t_2 = 0.6 \langle t_1 \rangle$ (curve 1, red) and $t_2 = 1.4 \langle t_1 \rangle$ (curve 2, blue). (d-f) Same as (a-c), but for a disorder strength $\delta = 0.25$.

as schematically shown in Fig. 1. The model of the qubit dissipatively-coupled to a SSH lattice is defined by the Hamiltonian Eq. (1), which is here rewritten for the sake of clearness

$$H = H_{SSH} + H_P + H_I,$$  \hfill (A-1)

where

$$H_P = \frac{\omega_0}{2} \sigma_z$$  \hfill (A-2)

is the Hamiltonian of the qubit with Rabi frequency $\omega_0$,  
$$H_{SSH} = \omega_0 \sum_n (a_n^\dagger a_n + b_n^\dagger b_n)$$  \hfill (A-3)

$$+ t_1 \sum_n (a_n^\dagger b_n + h.c.) + t_2 \sum_n (a_n^\dagger b_{n-1} + h.c.)$$
is the SSH Hamiltonian, with the band gap center at energy \( \omega_0 \), and
\[
H_I = \gamma [\sigma_+(a_0 + b_0) + \sigma_-(a_0^\dagger + b_0^\dagger)]
\] (A-4)
is the interaction term in the rotating-wave approximation. In the above equations, \( \sigma_z = |e\rangle\langle e| - |g\rangle\langle g| \), \( \sigma_+ = |e\rangle\langle g| \), \( \sigma_- = |g\rangle\langle e| \) are the Pauli pseudo spin operators associated to the qubit with ground and excited states \( |g\rangle \) and \( |e\rangle \), respectively, and \( \gamma \) is the strength of the qubit-bath dissipative coupling. Here we consider the case \( \omega_0 = \omega_\alpha \) corresponding to absence of decay in the weak-coupling (markovian) approximation, however, we do not make any assumption on the coupling \( \gamma \), which could be of the same order of magnitude or even larger than the hopping rates \( t_{1,2} \) of the SSH chain, i.e. of the width of energy bands/gap (strong coupling regime).

**Bound states and decay dynamics.** Let us assume that at initial time \( t = 0 \) the SSH chain is in the vacuum state, while the probing two-level system is prepared in the excited state, i.e. \( |\psi(0)\rangle = |e\rangle \otimes |0\rangle \). Apart from a phase term rotating at the frequency \( \omega_0 \), the time-evolved state can be written as
\[
|\psi(t)\rangle = q(t)|e\rangle \otimes |0\rangle + |g\rangle \sum_n \left( \alpha_n(t) a_n^\dagger + \beta_n(t) b_n^\dagger \right) |0\rangle
\] (A-5)
where the amplitudes \( q(t) \), \( \alpha_n(t) \) and \( \beta_n(t) \) satisfy the coupled evolution equations
\[
i \frac{d\alpha_n}{dt} = t_2 \beta_n + t_1 \beta_{n-1} + \delta_{n,0} \gamma q
\] (A-6)
\[
i \frac{d\beta_n}{dt} = t_2 \alpha_n + t_1 \alpha_{n+1} + \delta_{n,0} \gamma q
\] (A-7)
\[
i \frac{dq}{dt} = \gamma (a_0 + b_0)
\] (A-8)
with the initial conditions \( \alpha_n(0) = \beta_n(0) = 0 \), \( q(0) = 1 \). Equations (A-6–A-8) can be written in compact form as
\[
i \frac{d}{dt} \begin{pmatrix} \alpha_n \\ \beta_n \\ q \end{pmatrix} = H \begin{pmatrix} \alpha_n \\ \beta_n \\ q \end{pmatrix}
\] (A-9)
where \( H \) is the Hamiltonian of the system in the single-excitation sector. Clearly, \( |q(t)|^2 \) is the probability that the two-level system remains in the excited state at time \( t \). More generally, if at initial time the qubit is in a mixed state described by the density matrix \( \rho_{i,k} = \langle i|\rho(0)|k\rangle \) \( (i, k = g, e) \) and the SSH lattice is in the vacuum state, the reduced density matrix \( \rho(t) \) of the qubit at time \( t \), obtained after tracing out the lattice degrees of freedom, reads
\[
\rho(t) = \begin{pmatrix} |q(t)|^2 \rho_{ee} & q(t) \rho_{eg} \\ q^*(t) \rho_{ge} & (1 - |q(t)|^2) \rho_{ee} + \rho_{gg} \end{pmatrix}
\] (A-10)
Therefore, the coherence of the qubit decays as \( \sim |q(t)| \), while the population as \( \sim |q(t)|^2 \). The temporal decay law of \( q(t) \) can be formally written as contour integral in complex plane after solving Eqs. (A-6–A-8) using Laplace transform (or Green-function) method. After an initial transient, the behavior of \( q(t) \) is determined by the poles of the Green function, which correspond to BSs of the Hamiltonian \( \mathcal{H} \). Therefore, the decay behavior of the coherence \( \rho_{eg}(t) = q(t)\rho_{eg} \) is ultimately determined by the number and frequencies of the BSs of the Hamiltonian \( \mathcal{H} \).

Figures A-1(a) and (b) show the energy spectrum of the Hamiltonian \( \mathcal{H} \) versus the coupling \( \gamma \) for the two topological phases \( t_1 < t_2 \) and \( t_1 > t_2 \) of the SSH lattice. The energy spectrum clearly comprises a continuous spectrum which reproduces the two bands of the SSH lattice, separated by the gap \( |t_1 - t_2| \) and independent of the topological phase of the lattice, and a number of BSs, localized near \( n = 0 \), which depend on the topological phase of the lattice. Clearly, for symmetry reasons the BSs can be classified as even (symmetric) and odd (anti-symmetric) modes, corresponding to \( \beta_{-n} = \pm \alpha_n \). It can be readily shown that BSs with odd symmetry \( \beta_{-n} = -\alpha_n \) do not exist, while even-symmetric BSs \( (\beta_{-n} = \alpha_n) \) can be searched by making the Ansatz
\[
\alpha_n = c_1 X^{-n} \exp(-iEt), \quad n \geq 1
\] (A-11)
\[
\beta_n = c_2 X^{-n} \exp(-iEt), \quad n \geq 0
\] (A-12)
\[
q = q_0 \exp(-iEt)
\] (A-13)
In the above equations, \( E \) is the eigenenergy of the BS, \( X \) is related to its localization length (with \( |X| > 1 \) for localization), \( c_{1,2} \) are constant amplitudes that determine the relative occupation of the the excitation in the two sublattices of the SSH chain, and \( q_0 \) is the occupation amplitude of the qubit. The above Ansatz provides an eigenstate of the Hamiltonian \( \mathcal{H} \) provided that the following conditions are met
\[
Ec_1 = (t_1 + t_2 X)c_2
\] (A-14)
\[
Ec_2 = (t_1 + t_2 X)c_1
\] (A-15)
\[
(E - t_1)c_2 = c_1 t_2 X + \gamma q_0
\] (A-16)
\[
E q_0 = 2\gamma c_2.
\] (A-17)
From Eqs.(A-14) and (A-15), it readily turns out that the energy \( E \) of the BS is related to the localization parameter
Figure A-1 (Color online) (a,b) Energy spectrum (band diagram) of the Hamiltonian $\mathcal{H}$, for (a) the topological phase $t_1 < t_2$, and (b) $t_1 > t_2$. In (a), there are three bound states (BSs), denoted by Ia, Ila, and IIa: a zero-energy topologically-protected state (Ia), and two BSs without topological protection that emanate from above and from below of the upper and lower bands. Mode IIa is thresholdless but disappears for $\gamma > \gamma_1 \equiv \sqrt{t_1(t_1 + t_2)}$. In (b) all three modes do not have topological protection. Mode Ib is thresholdless but disappears for $\gamma > \gamma_2 \equiv \sqrt{t_1(t_1 - t_2)}$, mode IIb is thresholdless and does exist for any value of coupling $\gamma$, whereas mode IIlb exists for $\gamma > \gamma_1 \equiv \sqrt{t_1(t_1 + t_2)}$. Parameter values used in the plots are $t_1 = 1$, $t_2 = 2$ in (a), and $t_1 = 2$, $t_2 = 1$ in (b). The insets show the coupling of the qubit with the SSH chain in the flat band limit [$t_1 = 0$ in (a) and $t_2 = 0$ in (b)], where only few sites of the lattice can be populated and the number of bound states [three in (a) and two in (b)] can be readily derived from symmetry considerations. (c) Domain of existence of BSs of the Hamiltonian $\mathcal{H}$ in the $(t_2/t_1, \gamma/t_1)$ plane. The vertical dashed line separates the two topological phases $t_1 < t_2$ and $t_1 > t_2$ of the SSH lattice.

$X$ by the dispersion relation

$$E^2 = t_1^2 + t_2^2 + t_1 t_2 (X + 1/X)$$

(A-18)

while the amplitudes $c_2$ and $q_0$ are the solutions to the homogeneous system of equations

$$(t_1^2 + t_1 t_2 X - E t_1) c_2 - \gamma E q_0 = 0$$

(A-19)

$$2 \gamma c_2 - E q_0 = 0.$$  

(A-20)

The solvability condition of the homogeneous system yields either $E = 0$ or

$$E = t_1 + t_2 X - 2 \gamma^2 / t_1.$$  

(A-21)

For $E = 0$ one obtains $c_2 = 0$, $X = -t_2/t_1$, $q_0/c_1 = t_1/\gamma$. The localization condition $|X| > 1$ indicates that the zero-energy BS exists in the topological phase $t_2 > t_1$ solely. Note that this state shows topological protection, i.e. its energy does not change as the ratio $t_1/t_2$ is varied until the gap is closed. The other possible values of energies are obtained by solving the nonlinear system of equations (A-18) and (A-21). After elimination of the energy $E$, the following cubic equation for the localization parameter $X$ is obtained

$$X^3 + s_1 X^2 + s_2 X + s_3 = 0$$

(A-22)

with coefficients

$$s_1 = \frac{t_1}{t_2} - 4 \frac{\gamma^2}{t_1 t_2}$$

(A-23)

$$s_2 = -1 - 4 \frac{\gamma^2}{t_2^2} + 4 \frac{\gamma^4}{t_1^2 t_2^2}$$

(A-24)

$$s_3 = -\frac{t_1}{t_2}.$$  

(A-25)

The acceptable roots of Eq.(A-22) are those with $|X| > 1$ (for localization of the BS) and $X \neq 0$ (because the energy $E$ should be real). Once an acceptable root $X$ of the cubic equation has been found, the corresponding energy eigenvalue is obtained from Eq.(A-21).

The boundaries for existence of BSs in the $(t_2/t_1, \gamma/t_1)$ can be readily found by setting $X = \pm 1$ in Eq.(A-22). The states with $X = 1$ correspond to emergent BSs that emanate from the top (or the bottom) of the upper (or lower) band, whereas the states with $X = -1$ emanates from the bottom (or the top) of the upper (or lower) band. The domain of existence of BSs and typical band diagrams for the two different topological phase $t_1 < t_2$ and $t_2 > t_1$ are illustrated in Fig.A-1. As shown above, in the topological phase $t_1 < t_2$ there is always one BS at zero energy (denoted by Ia in the figure), which shows topological
There are also other two edge states without topological protection, one being thresholdless (mode IIA) while the other one (mode IIIa) emerges from the bottom of the lower band at $\gamma > \gamma_1 \equiv \sqrt{t_1(t_1 + t_2)}$; see Fig.A-1(a). In the other topological phase $t_1 > t_2$ one can find most two BSs without topological protection. A first BS (mode Ib) is thresholdless and falls into the continuum of lower band at $\gamma > \gamma_2 \equiv \sqrt{t_1(t_1 - t_2)}$. The second BS (mode IIb) is thresholdless and its energy lies above the upper band, while the third BS (mode IIIb) emerges from the bottom of the lower band at $\gamma > \gamma_1$; see Fig.A-1(b). Note that in the limit of flat bands $t_1 \rightarrow 0$ or $t_2 \rightarrow 0$ in the two topological phases, there are three and two BSs, respectively. The different number of BSs stem from the fact that, in the flat band limit, the qubit probes a finite but different number of sites of the SSH chain, as schematically shown in the insets of Fig.A-1(a) and (b). In Fig.A-1(a), there are five sites overall that can be occupied, corresponding to five states, two with odd symmetry and the other three with even symmetry. Since only the even modes have excitation in the qubit, one concludes that there are three BSs relevant for the qubit decay dynamics. On the other hand, in Fig.A-1(b) there are only three sites overall that can be occupied, corresponding to three states with defined symmetry. Only the two states with even symmetry have excitation in the qubit, and therefore in this case we have two BSs relevant for the decay dynamics of the qubit.

**Recoherence and non-Markovianity measure.** The decay dynamics of the coherence $|q(t)|$ comprises an initial transient decay followed by a non-decay dynamics which can be oscillatory or non-oscillatory depending on the number of BSs of the Hamiltonian $\mathcal{H}$, as in the dephasing coupling scheme discussed in the main text. The most interesting case is attained when the coupling $\gamma$ of the qubit with the SSH bath is set equal to $t_1$. As $t_2$ is varied from below to above $t_1$, so as the gap closes and the SSH lattice undergoes a topological phase transition, the decay of coherence shows a qualitative change, as depicted in Fig.A-2. In the topological phase $t_1 < t_2$, there are two BSs so as the asymptotic behavior of $|q(t)|$ is oscillatory at a frequency given by the energy separation of the two BSs [see Fig.A-2(b)]. On the other hand, in the topological phase $t_1 > t_2$ there is only one BS and, after an initial transient, the coherence settles down to a steady-state (non-oscillatory) value [Fig.A-2(a)]. As discussed in the main text and according to [62], non-Markovianity can be quantified through the lack of contractiveness of the dynamical map. Given the map of Eq. (A-10), $\mathcal{N}$ is fully determined by the recoherence time windows of $|q(t)|$ [67]. As discussed in the pure dephasing scenario, the oscillatory behavior of $\rho(t)$ would lead to a divergence of
non-Markovianity, whose behavior can be regularized introducing the temporal average

$$\mathcal{N}_T = \frac{1}{T} \int_{0}^{T} dt \frac{d|q(t)|}{dt},$$  \hspace{1cm} (A-26)

where the integral is extended over the time intervals where the derivative \(d|q|/dt\) is positive. For long times \(T\) it turns out that \(\mathcal{N}_T\) vanishes in the topological phase \(t_1 > t_2\), while it reaches a steady and non-vanishing value in the \(t_1 < t_2\) topological phase. The vanishing of non-Markovianity for a unique value of the system parameters was also reported in a different context in Ref. [68]. Therefore, the non-Markovianity measure \(\mathcal{N}_T\) provides an order parameter of the topological quantum phase transition. This is shown in Fig.A-2(c), where we numerically computed the non-Markovianity measure \(\mathcal{N}_T\), after an initial transient decay dynamics, versus the ratio \(t_2/t_1\) with \(\gamma = t_1\).

Clearly, for a coupling \(\gamma\) different than \(t_1\), the non-Markovianity measure does not provide an order parameter of the topological quantum phase transition of the SSH lattice. Nevertheless, in the strong coupling regime \(\gamma > \gamma_1\), the signature of the quantum topological phase transition can be detected by the qualitative distinct behavior of the coherence \(|q(t)|\) after the initial fractional decay transient, as shown in Fig.A-3. In the topological phase \(t_1 > t_2\), there are two BSs and therefore the coherence \(|q(t)|\) shows a single-frequency oscillatory dynamics [Fig.A-3(a)], which is clearly visible in the three-peaks structure of the frequency spectrum of \(L(t)\) [see the inset of Fig.A-3(a)]. On the other hand, in the topological phase \(t_1 < t_2\), there are three BSs and therefore the coherence \(|q(t)|\) shows a multi-frequency and rather generally aperiodic oscillatory dynamics [Fig.A-3(b)], which is clearly visible in the seven-peaks structure of the frequency spectrum of \(L(t)\) [see the inset of Fig.A-3(b)].

### B Dephasing coupling: bound states

In this section, we will mainly focus on the analytical calculation of the BSs for the dephasing model of Eqs. (1-3). Starting from the states defined by Eq. (6) in the main text, we look for possible solutions to the Schrödinger equations \(\hat{H}\ket{\psi_\pm} = E_\pm \ket{\psi_\pm}\), with \(\hat{H} = H_{SSH} + \gamma (a_d^\dagger a_0 + b_0^\dagger b_0)\). One readily obtains

\[
\hat{H}\ket{\psi}_\pm = (\bar{\gamma}_\pm c_0 (a_0^\dagger \pm b_0^\dagger) + c_0 t_2 (b_{-1}^\dagger \pm a_1^\dagger)) + c_1 [t_2 (a_0^\dagger \pm b_0^\dagger) + t_1 (a_{-1}^\dagger \pm b_1^\dagger)] + \ldots |0\rangle = E_{\pm} \ket{\psi}_\pm,
\]

where \(\bar{\gamma}_\pm = \gamma \pm t_1\). Notice that solving these equations is equivalent to taking as an effective Hamiltonian a semi-infinite chain obtained by cutting the original one in the middle of the impurity cell and replacing the impurity energy by \(\gamma\). This renormalization is due to a "bounce" around the impurity. In the limit of a chain much longer than the localization of the BSs, the dimerically tailored Ansatz \(c_0 = X^nc_2n\) and \(c_{2n+1} = Y c_{2n}\) allows one to write two sets of closed equations

\[
\begin{align*}
\bar{\gamma}_\pm t_1 + Y t_2 &= E_\pm \\
t_2 + t_1/X &= E_\pm Y \\
t_2 + t_1 X &= E_\pm/Y.
\end{align*}
\]

(B-2)

The quantity \(X\) determines the localization length of the BSs expressed in cell units, and the condition \(|X| > 1\) should be satisfied for localization. In other words, solutions of (B-3) corresponding to \(|X| < 1\) are not BSs of (B-1) and must be discarded. The explicit solutions (two for any of the systems) for the energies are the following:

\[
E^+_\pm = \frac{\gamma^2 + 2t_1^2 + 2\gamma t_1 \pm \sqrt{(\gamma^2 + 2\gamma t_1)^2 + 4t_2^2(\gamma + t_1)^2}}{2(\gamma + t_1)}
\]

(B-3)

and

\[
E^-_\pm = \frac{\gamma^2 + 2t_1^2 - 2\gamma t_1 \pm \sqrt{(\gamma^2 - 2\gamma t_1)^2 + 4t_2^2(\gamma - t_1)^2}}{2(\gamma - t_1)}
\]

(B-4)

while the respective localization length parameters are

\[
X^+_\pm = \frac{\gamma^2 + 2\gamma t_1 \pm \sqrt{(\gamma^2 + 2\gamma t_1)^2 + 4t_2^2(\gamma + t_1)^2}}{2t_1 t_2}
\]

(B-5)

\[
X^-_\pm = \frac{\gamma^2 - 2\gamma t_1 \pm \sqrt{(\gamma^2 - 2\gamma t_1)^2 + 4t_2^2(\gamma - t_1)^2}}{2t_1 t_2}
\]

and the dimer imbalances are

\[
Y^+_\pm = \frac{\gamma + t_1 \pm \frac{1}{X^+_\pm}}{t_1}
\]

\[
Y^-_\pm = \frac{\gamma - t_1 \mp \frac{1}{X^-_\pm}}{t_1}
\]

(B-6)

Finally, the normalization condition of states \(\ket{\psi}_\pm\) determines the value of \(c_0\), which reads

\[
c_0 = \frac{1}{2(1 + Y^2)}.
\]

(B-7)

for any of the four solutions. In Fig. B-1, we plot the inverse of the localization length for a fixed value of \(\gamma\) as a function
The reason why in one case there are always two BSs and in the other one there can be four of them can be understood looking at the flat-band limit (Fig.B-2). In the phase \( t_1 > t_2 \), this limit corresponds to taking \( t_2 = 0 \), which means that the impurity cell is completely decoupled from the rest of the dimers and can be described by the cell Hamiltonian \( H_0^{t,t_2} = t_1 (a_0^t b_0 + b_0^t a_0) + \gamma (a_1^t a_0 + b_1^t b_0) \), which admits the pair of eigenstates of opposite parity \((a_0^t \pm b_0^t)\ket{0}\) with eigenvalues \( \gamma \pm t_1 \) [Fig.B-2(a)]. On the other hand, in the other phase we would have \( t_1 = 0 \). Then, the Hamiltonian around the impurity would be \( H_0^{t,0} = t_2 (a_{-1}^t b_{-1} + b_{-1}^t a_{-1} + h.c.) + \gamma (a_1^t a_0 + b_1^t b_0) \), which correspond to two disconnected dimers, each of them admitting two eigenstates of different parity [Fig.B-2(b)]. The transition from four to two BSs cannot be analyzed using the flat-band argument, as it only happens for finite values of \( t_1 \).

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