Simulations of Overstable Inertial-acoustic Modes in Black-Hole Accretion Discs

Wen Fu\textsuperscript{1,2,3*} and Dong Lai\textsuperscript{1*}
\textsuperscript{1} Department of Astronomy, Cornell University, Ithaca, NY 14853, USA
\textsuperscript{2} Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA
\textsuperscript{3} Department of Physics and Astronomy, Rice University, Houston, TX 77005, USA

ABSTRACT
We present two-dimensional inviscid hydrodynamic simulations of overstable inertial-acoustic oscillation modes (p-modes) in black-hole accretion discs. These global spiral waves are trapped in the inner-most region of the disc, and are driven overstable by wave absorption at the corotation resonance ($r_c$) when the gradient of the background disc vortensity (vorticity divided by surface density) at $r_c$ is positive and the disc inner boundary is sufficiently reflective. Previous linear calculations have shown that the growth rates of these modes can be as high as 10% of the rotation frequency at the disc inner edge. We confirm these linear growth rates and the primary disc oscillation frequencies in our simulations when the mode amplitude undergoes exponential growth. We show that the mode growth saturates when the radial velocity perturbation becomes comparable to the disc sound speed. During the saturation stage, the primary disc oscillation frequency differs only slightly (by less than a few percent) from the linear mode frequency. Sharp features in the fluid velocity profiles at this stage suggest that the saturation results from nonlinear wave steepening and mode-mode interactions.

Key words: accretion, accretion discs – hydrodynamics – waves – instabilities – X-ray: binaries.

1 INTRODUCTION

In several recent papers (Lai & Tsang 2009; Tsang & Lai 2009c; Fu & Lai 2011; Horak & Lai 2013), we have presented detailed study of the linear instability of non-axisymmetric inertial-acoustic modes [also called p-modes; see Kato (2001) and Wagoner (2008) for review] trapped in the inner-most region of black-hole (BH) accretion discs. This global instability arises because of wave absorption at the corotation resonance (where the wave pattern rotation frequency matches the background disc rotation rate) and requires that the disc vortensity has a positive gradient at the corotation radius (see Narayan et al. 1987, Tsang & Lai 2008 and references therein). The disc vortensity (vorticity divided by surface density) is given by

$$\zeta = \frac{\kappa^2}{2\Omega\Sigma},$$

where $\Omega(r)$ is the disc rotation frequency, $\kappa(r)$ is the radial epicyclic frequency and $\Sigma(r)$ is the surface density. General relativistic (GR) effect plays an important role in the instability: For a Newtonian disc, with $\Omega = \kappa r^{-3/2}$ and relatively flat $\Sigma(r)$ profile, we have $d\zeta/dr < 0$, so the corotational wave absorption leads to mode damping. By contrast, $\kappa$ is non-monotonic near a BH (e.g., for a Schwarzschild BH, $\kappa$ reaches a maximum at $r = 8GM/c^2$ and goes to zero at $r_{\text{ISCO}} = 6GM/c^2$), the vortensity is also non-monotonic. Thus, p-modes with frequencies such that $d\zeta/dr > 0$ at the corotation resonance are overstable. Our calculations based on several disc models and Paczynski-Witta pseudo-Newtonian potential (Lai & Tsang 2009, Tsang & Lai 2009c) and full GR (Horak & Lai 2013) showed that the lowest-order p-modes with $m = 2, 3, 4, \cdots$ have the largest growth rates, with the mode frequencies $\omega \simeq \beta m\Omega_{\text{ISCO}}$ (thus giving commensurate frequency ratio $2 : 3 : 4, \cdots$), where the dimensionless constant $\beta \lesssim 1$ depends weakly the disc prop-

\* Email: wenfu@astro.cornell.edu (WF); dong@astro.cornell.edu (DL)

Equation (1) applies to barotropic discs in Newtonian (or pseudo-Newtonina) theory. See Tsang & Lai (2009c) for non-barotropic discs and Horak & Lai (2013) for full general relativistic expression.
properties. These overstable p-modes could potentially explain the High-frequency Quasi-Periodic Oscillations (HFQPOs) observed in BH X-ray binaries (e.g., Remillard & McClintock 2006; Belloni et al. 2012).

The effects of magnetic fields on the oscillation modes of BH accretion discs have been investigated by Fu & Lai (2009, 2011, 2012) and Yu & Lai (2013). Fu & Lai (2009) showed that the the basic wave properties (e.g., propagation diagram) of p-modes are not strongly affected by disc magnetic fields, and it is likely that these p-modes are robust in the presence of disc turbulence (see Arras, Blaes & Turner 2006; Reynolds & Miller 2009). By contrast, other diskoseismic modes with vertical structure (such as g-modes and c-modes) may be easily “destroyed” by the magnetic field (Fu & Lai 2009) or suffer damping due to corotation resonance (Kato 2003; Li et al. 2003; Tsang & Lai 2009a). Although a modest toroidal disc magnetic field tends to reduce the growth rate of the p-mode (Fu & Lai 2011), a large-scale poloidal field can enhance the instability (Yu & Lai 2013; see Tagger & Pallet 1999; Tagger & Varniere 2000). The p-modes are also influenced by the magnetosphere that may exist inside the disc inner edge (Fu & Lai 2012).

So far our published works are based on linear analysis. While these are useful for identifying the key physics and issues, the nonlinear evolution and saturation of the mode growth can only be studied by numerical simulations. It is known that fluid perturbations near the corotation resonance are particularly prone to become nonlinear (e.g., Balmforth & Korycansky 2001; Ogilvie & Lubow 2003). Moreover, real accretion discs are more complex than any semi-analytic models considered in our previous works. Numerical MHD simulations (including GRMHD) are playing an increasingly important role in unraveling the nature of BH accretion flows (e.g., De Villiers & Hawley 2003; Machida & Matsumoto 2003; Fragile et al. 2007; Noble et al. 2009, 2011; Reynolds & Miller 2009; Beckwith et al. 2008, 2009; Mosibrodzka et al. 2009; Penna et al. 2010; Kulkarni et al. 2011; Hawley et al. 2011; O’Neill et al. 2011; Dolence et al. 2012; McKinney et al. 2012; Henisey et al. 2012). Despite much progress, global GRMHD simulations still lag far behind observations, and so far they have not revealed clear signs of HFQPOs that are directly comparable with the observations of BH X-ray binaries. If the corotation instability and its magnetic counterparts studied in our recent papers play a role in HFQPOs, the length-scale involved would be small and a proper treatment of flow boundary conditions is important. It is necessary to carry out “controlled” numerical experiments to capture and evaluate these subtle effects.

In this paper, we use two-dimensional hydrodynamic simulations to investigate the nonlinear evolution of corotational instability of p-modes. Our 2D model has obvious limitations. For example it does not include disc magnetic field and turbulence. However, we emphasize that since the p-modes we are studying are 2D magnetic waves with no vertical structure, their basic radial “shapes” and real frequencies may be qualitatively unaffected by the turbulence (see Arras et al. 2006; Reynolds & Miller 2009; Fu & Lai 2009). Indeed, several local simulations have indicated that density waves can propagate in the presence of MRI turbulence (Gardiner & Stone 2005; Fromang et al. 2007; Heinemann & Papaloizou 2009). Our goal here is to investigate the saturation of overstable p-modes and the their nonlinear behaviours.

2 NUMERICAL SETUP

Our accretion disc is assumed to be inviscid and geometrically thin so that the hydynamical equations can be reduced to two-dimension with vertically integrated quantities. We adopt an isothermal state of equation throughout this study, i.e. \( P = c_s^2 \Sigma \) where \( P \) is the vertically integrated pressure, \( \Sigma \) is the surface density and \( c_s \) is the constant sound speed. Self-gravity and magnetic fields are neglected.

We use the Paczynski-Witt Pseudo-Newtonian potential (Paczynski & Witt 1980) to mimic the GR effect:

\[
\Phi = -\frac{GM}{r - r_S},
\]

where \( r_S = 2GM/c^2 \) is the Schwarzschild radius. The corresponding Keplerian rotation frequency and radial epicyclic frequency are

\[
\Omega_K = \sqrt{\frac{GM}{r(r - r_S)}},
\]

\[
\kappa = \Omega_K \sqrt{\frac{r - 3r_S}{r - r_S}}.
\]

In our computation, we will adopt the units such that the inner disc radius (at the Inner-most Stable Circular Orbit or ISCO) is at \( r = 1.0 \) and the Keperian frequency at the ISCO is \( \Omega_{\text{ISCO}} = 1 \). In these units, \( r_S = 1/3 \), and

\[
\Omega_K = \frac{2}{3} \frac{1}{r - r_S} \frac{1}{\sqrt{r}}.
\]

Our computation domain extends from \( r = 1.0 \) to \( r = 4.0 \) in the radial direction and from \( \phi = 0 \) to \( \phi = 2\pi \) in the azimuthal direction. We also use the Keplerian orbital period \( T = 2\pi/\Omega_K = 2\pi r \) at \( r = 1 \) as the unit for time. The equilibrium state of the disc is axisymmetric. The surface density profile has a simple power-law form

\[
\Sigma_0 = r^{-1},
\]

which leads to a positive vortensity gradient in the inner disc region. The equilibrium rotation frequency of the disc is given by

\[
\Omega_0(r) = \sqrt{\frac{4/9}{r(r - 1/3)^2} - \frac{c_s^2}{r^2}}.
\]

Throughout our simulation, we will adopt \( c_s = 0.1 \) so that \( \Omega_0 \approx \Omega_K \).
Inertial-acoustic modes in BH Accretion Discs

Figure 1. Evolution of the radial velocity amplitude $|u_r|_{\text{max}}$ (evaluated at $r = 1.1$) for three runs with initial azimuthal mode number $m = 2$ (left panel), $m = 3$ (middle panel) and with random perturbations (right panel). The dashed lines are the fits for the exponential growth stage (between $\sim 10$ and $\sim 30$ orbits) of the mode amplitude.

Figure 2. Comparison of the radial profiles of velocity perturbations from non-linear simulation and linear mode calculation. The top and bottom panels show the radial and azimuthal velocity perturbations, respectively. The left and right panels are for cases with azimuthal mode number $m = 2$ and $m = 3$, respectively. In each panel, the dashed line is taken from the real part of the complex wavefunction obtained in linear mode calculation, while the solid line is from the non-linear simulation during the exponential growth stage (at $T = 20$ orbits), with the quantities evaluated at $\phi = 0.4\pi$. Note that the normalization factor is given in the $y$-axis label for the nonlinear simulation results.

We solve the Euler equations that govern the dynamics of the disc flow with the PLUTO code\footnote{publicly available at http://plutocode.ph.unito.it/} (Mignone et al. 2007), which is a Godunov-type code with multiphysics and multialgorithm modules. For this study, we choose a Runge-Kutta scheme (for time integration) and piecewise linear reconstruction (for space integration) to achieve second order accuracy, and Roe solver as the solution of Rie-
manner. In the last stage (beyond approximately the same level. A fit to the exponential growth stage occupies roughly the first 10 orbits, during which the stage (from approximately 30 orbits), the fastest growing mode becomes dominant and undergoes exponential growth. After about 10 orbits of evolution, one of them (the fastest growing mode) has its oscillation amplitude grown by a large amount such that it dominates over other modes. This corresponds to the primary spike in the top-left panel. The other spikes in the same panel are harmonics of this primary spike (see the labels of the dashed vertical lines). Table 1 shows that the frequency of this fastest growing mode (\(\omega_{r=1}\)) differs from the frequency obtained in linear mode calculation by only 0.3\%, which again demonstrates the consistency of these two studies. After the perturbation saturates (bottom-left panel), we see that the basic structure of the power density spectrum does not change much except that these spikes are not as “clean” as in the linear regime; this is probably due to the interaction of different modes. Compared with the upper panel, the location of the primary spike (\(\omega_{r=2}\) in Table 1) is increased by 2\%. In Table 1 we also include the results from both linear mode calculation and numerical simulation for modes with other mode number \(m\). The comparison illustrates two main points: First, the frequencies of the fastest growing modes during the exponentially growth stage of numerical simulations are exceptionally close to the linear calculation results (differ by less than 1%); second, the frequencies of the fastest growing modes during the saturation stage are only slightly higher (except for the \(m = 8\) mode which shows lower frequency) than ones during the exponential growing stage. These indicate that the mode frequencies obtained in linear mode calculation are fairly robust and can be reliably applied in the interpretation of HFQPOs.

In the right columns of Fig. 3 the simulation starts with a random initial perturbation which excites modes with various \(m\)’s. During the exponential growth stage, six modes stand out. By examining the location of the corresponding spikes, we know that these are the \(m = 3, 4, 5, 6, 7, 8\) modes. Although the \(m = 3\) mode seems to be the most prominent one in this figure, we note that this is because for this particular run the initial random perturbation happens to contain more \(m = 3\) wave components. If we were to start the run with a different initial random perturbation, then the relative strengths of those peaks would also be different (not necessarily with \(m = 3\) being dominant).

Fig. 4 shows the comparison of the velocity perturbations during the linear stage (\(T = 20\) orbits), at the end of the linear stage (\(T = 30\) orbits) and during the sat-

---

4 All the frequencies in this paper are angular frequencies unless otherwise noted.
Table 1. Comparison of results from linear and nonlinear studies of overstable disc p-modes

| $m$ | $\omega_1^b$ | $\omega_i^b$ | $\omega_r^b/m\Omega^b$ | $\omega_r^b$ | $|\omega_r - \omega_1|/\omega_r^b$ | $\omega_r^b/\omega_r^2$ | $|\omega_r^2 - \omega_r|/\omega_r^2$ |
|-----|--------------|--------------|----------------------|-------------|---------------------|----------------------|---------------------|
| 2   | 1.4066       | 0.0632       | 0.7033               | 1.3998      | 0.5%                | 1.4296               | 2.1%                |
| 3   | 2.1942       | 0.0733       | 0.7314               | 2.1997      | 0.3%                | 2.2445               | 2.0%                |
| 4   | 3.0051       | 0.0763       | 0.7512               | 2.9996      | 0.2%                | 3.0594               | 2.0%                |
| 5   | 3.8294       | 0.0751       | 0.7659               | 3.7995      | 0.8%                | 3.8886               | 2.3%                |
| 6   | 4.6621       | 0.0714       | 0.7770               | 4.6494      | 0.3%                | 4.7749               | 2.6%                |
| 7   | 5.5007       | 0.0664       | 0.7858               | 5.4992      | 0.03%               | 5.6756               | 3.1%                |
| 8   | 6.3436       | 0.0607       | 0.7930               | 6.3492      | 0.09%               | 6.3189               | 0.5%                |

a Azimuthal mode number  
b Mode frequency from the linear calculation (in units of Keplerian orbital frequency at the inner disc boundary; same for $\omega_i$, $\omega_r^1$ and $\omega_r^2$)  
c Mode growth rate from the linear calculation  
d Ratio of wave pattern speed to the Keplerian orbital frequency at the inner disc boundary  
e Mode frequency during the exponential growth stage of nonlinear simulation (peak frequency of the power density spectrum between $\sim 10$ orbits and $\sim 30$ orbits)  
f Difference between $\omega_r$ (linear result) and $\omega_r^1$ (nonlinear result)  
g Mode frequency during the saturation stage of nonlinear simulation (peak frequency of the power density spectrum between $\sim 30$ orbits and $\sim 100$ orbits)  
h Difference between $\omega_r^1$ and $\omega_r^2$

Figure 3. Power density spectra of the radial velocity perturbations near the disk inner boundary ($r = 1.1$). Each panel shows the normalized FFT magnitude as a function of frequency. The left and right columns are for runs with initial $m = 3$ and random perturbation, respectively. In the top and bottom panels, the Fourier transforms are sampled for time periods of [10, 30] orbits and [30, 100] orbits, respectively.

uration stage ($T = 50$ orbits) for simulations with different initial perturbations. At $T = 20$ orbits, the oscillation mainly comes from the single fastest growing mode and has a smooth radial profile. At $T = 30$ orbits, the perturbation starts to saturate, and the oscillation now consists of many different modes, and its radial profile exhibits sharp variations at several locations. These sharp features remain after the saturation ($T = 50$ orbits).

To see this evolution from a different perspective, in Fig. 5 we show the color contours of radial velocity for runs with different initial perturbations (different columns) at different times (different rows). We can clearly see that spiral

© 2012 RAS, MNRAS 000, 1–
waves gradually develop due to the instability. As the system evolves, sharp features in velocities emerge. At the end (the bottom row) the spiral arms becomes more irregular, which is related to the emergence and interaction of multiple modes after saturation.

The sharp velocity variations shown in Figs. 4-5 suggest shock-like features. Note that since our simulations adopt isothermal equation of state (with $P/\Sigma =\text{constant}$), there is no entropy generation and shock formation in the strict sense. The radial velocity jump (see Fig. 4) is comparable but always smaller than the sound speed. Nevertheless, these sharp features imply that wave steepening plays an important role in the mode saturation.

4 CONCLUSIONS

We have carried out high-resolution, two-dimensional hydrodynamical simulations of overtstable inertial-acoustic modes (p-modes) in BH accretion discs with various initial conditions. The evolution of disc p-modes exhibits two stages. During the first (linear) stage, the oscillation amplitude grows exponentially. In the cases with a specific azimuthal mode number ($m$), the mode frequency, growth rate and wavefunctions agree well with those obtained in our previous linear mode analysis. In the cases with random initial perturbation, the disc power-density spectrum exhibits several prominent frequencies, consistent with those of the fastest growing linear modes (with various $m$’s). These comparisons with the linear theory confirm the physics of corotational instability that drives disc p-modes presented in our previous studies (Lai & Tsang 2009; Tsang & Lai 2009c; Fu & Lai 2011, 2012). In the second stage, the mode growth saturates and the disc oscillation amplitude remains at roughly a constant level. In general, we find that the primary disc oscillation frequency (in the cases with specific initial $m$) is larger than the linear mode frequency by less than 4%, indicating the robustness of disc oscillation frequency in the non-linear regime. Based on the sharp, shock-like features of fluid velocity profiles, we suggest that the nonlinear saturation of disc oscillations is caused by wave steepening and mode-mode interactions.

As noted in Section 1, our 2D hydrodynamical simulations presented in this paper do not capture various complexities (e.g., magnetic field, turbulence, radiation) associated with real BH accretion discs. Nevertheless, they demonstrate that under appropriate conditions, disc p-modes can grow to nonlinear amplitudes with well-defined frequencies that are similar to the linear mode frequencies. A number of issues must be addressed before we can apply our theory to the interpretation of HFQPOs. First, magnetic fields may play an important role in the disc oscillations. Indeed, we have shown in previous linear calculations (Fu & Lai 2011,2012) that the growth of p-modes can be significantly affected by disc toroidal magnetic fields. A strong, large-
Inertial-acoustic modes in BH Accretion Discs

Figure 5. Evolution of the radial velocity for runs with initial $m = 2$ (left), $m = 3$ (middle) and random (right) perturbations, respectively. From top to bottom, the times are $T = 20, 30$ and $50$ orbits, respectively. Note that the color scale varies from panel to panel.

scale poloidal field can also change the linear mode frequency (Yu & Lai 2013). Whether or not these remain true in the non-linear regime is currently unclear. Second, understanding the nature of the inner disc boundary is crucial. Our calculations rely on the assumption that the inner disc edge is reflective to incoming spiral waves. In the standard disc model with zero-torque inner boundary condition, the radial inflow velocity is not negligible near the ISCO, and the flow goes through a transonic point. While the steep density and velocity gradients at the ISCO give rise to partial wave reflection (Lai & Tsang 2009), such radial inflow can lead to significant mode damping such that the net growth rates of $p$-modes become negative. This may explain the absence of HFQPOs in the thermal state of BH x-ray binaries. However, it is possible that the inner-most region of BH accretion discs accumulates significant magnetic flux and forms a magnetosphere. The disc-the magnetosphere boundary will be highly reflective, leading to the growth of disc oscillations (Fu & Lai 2012; see also Tsang & Lai 2009b). Finally, the effect of MRI-driven disc turbulence on the $p$-modes requires further understanding. In particular, turbulent viscosity may lead to mode growth or damping, depending on the magnitude and the density-dependence of the viscosity (R. Miranda & D. Lai 2013, in prep).

ACKNOWLEDGEMENTS

This work has been supported in part by the NSF grants AST-1008245, AST-1211061 and the NASA grant NNX12AF85G. WF also acknowledges the support from the Laboratory Directed Research and Development Program at LANL.

REFERENCES

Arras P., Blaes O. M., Turner N. J., 2006, ApJ, 645, L65
Balmforth N. J., Korycansky D. G. 2001, MNRAS, 326, 833
Beckwith K., Hawley J. F., Krolik J. H., 2008, MNRAS, 390, 21
Beckwith K., Hawley J. F., Krolik J. H., 2009, ApJ, 707, 428
Belloni T. M., Sanna A., Mendez M., 2012, MNRAS, 426, 1701
Chan C.-K., 2009, ApJ, 704, 68
de Val-Borro M., et al., 2006, MNRAS, 370, 529
De Villiers J.-P., Hawley J. F., 2003, ApJ, 592, 1060
Dolence, J.C., Gammie, C.F., Shiokawa, H., Noble, S.C.
2012, ApJ, 746, L10
Fragile P. C., Blaes O., Anninos P., Salmonson J. D., 2007, ApJ, 668, 417
Fromang S., Papaloizou, J., Lesur G., Heinemann T., 2007, A&A, 476, 1123
Fu W., Lai D., 2009, ApJ, 690, 1386
Fu W., Lai D., 2011, MNRAS, 410, 399
Fu W., Lai D., 2012, MNRAS, 423, 831
Gardiner T. A., Stone J. M., 2005, AIP Conference Proceedings, 784, 475
Hawley J. F., Guan X., Krolik J. H., 2011, ApJ, 738, 84
Heinemann T., Papaloizou J. C. B., 2009, MNRAS, 397, 52
Henisey K. B., Blaes O. M., Fragile P. C., Ferreira B. T.,
2009, ApJ, 706, 705
Henisey K. B., Blaes O. M., Fragile P. C., 2012, ApJ, 761, 18
Horak J., Lai D. 2013, MNRAS, in press
Kato S., 2001, PASJ, 53, 1
Kato S., 2003, PASJ, 55, 257
Kulkarni A. K., et al., 2011, MNRAS, 414, 1183
Lai D., Tsang D., 2009, MNRAS, 393, 979
Li L., Goodman J., Narayan R., 2003, ApJ, 593, 980
Machida M., Matsumoto R., 2003, ApJ, 585, 429
McKinney J. C., Tchekhovskoy A., Blandford R. D., 2012, MNRAS, 423, 3083
Mignone A., Bodo G., Massaglia S., Matsakos T., Tesileanu
O., Zanni C., Ferrari A., 2007, ApJS, 170, 228
Miranda R., Lai D., 2013, in preparation
Moscibrodzka M., Gammie C. F., Dolence J. C., Shiokawa
H., Leung Po Kin, 2009, ApJ, 706, 497
Narayan R., Goldreich P., Goodman J., 1987, MNRAS, 228, 1
Noble S. C., Krolik J. H., Hawley J. F., 2009, ApJ, 692, 411
Noble S. C., Krolik J. H., Schnittman J. D., Hawley J. F.,
2011, ApJ, 743, 115
Ogilvie G. I., Lubow S. H., 2003, ApJ, 587, 398
O’Neill S. M., Reynolds C. S., Miller C. M., 2009, ApJ, 693, 1100
O’Neill S. M., Reynolds C. S., Miller C. M., Sorathia K.,
2011, ApJ, 736, 107
Paczynski B., Witt P. J., 1980, A&A, 88, 23
Penna R. F., McKinney J. C., Narayan R., Tchekhovskoy
A., Shafee R., McClintock J. E., 2010, MNRAS, 408, 752
Remillard R. A., McClintock J. E., 2006, ARA&A, 44, 49
Reynolds C. S., Miller M. C., 2009, ApJ, 692, 869
Tagger M., Pellat R., 1999, A&A, 349, 1003
Tagger M., Varniere P., 2006, ApJ, 652, 1457
Tsang D., Lai D., 2008, MNRAS, 387, 446
Tsang D., Lai D., 2009a, MNRAS, 393, 992
Tsang D., Lai D., 2009b, MNRAS, 396, 589
Tsang D., Lai D., 2009c, MNRAS, 400, 470
Wagoner R. V., 2008, J. Phys: Conf. Series, Vol.118, 012006