Abstract

We consider hypothesis testing problems for a single covariate in the context of a linear model with Gaussian design when $p > n$. Under minimal sparsity conditions of their type and without any compatibility condition, we construct an asymptotically Gaussian estimator with variance equal to the oracle least-squares. The estimator is based on a weighted average of all models of a given sparsity level in the spirit of exponential weighting. We adapt this procedure to estimate the signal strength and provide a few applications. We support our results using numerical simulations based on algorithm which approximates the theoretical estimator and provide a comparison with the de-biased lasso.

1 Introduction

In the past decade, there has been much interest in high-dimensional linear models, particularly following the work of Tibshirani (1996). However, it was not until the past few years that there have been methods to construct confidence intervals and p-values for particular covariates in the model. Consider the standard high-dimensional linear model

$$Y = X\beta + Z\gamma + \epsilon. \quad (1)$$
To simplify notation, we have separated the covariates into $X$ and $Z$ and are interested in analyzing $\beta$. In recent years, there have been mainly two approaches to constructing confidence intervals in high-dimensional linear models. There have been approaches such as Lee, Sun, Sun & Taylor (2016), which construct conditional confidence intervals for $\beta$ given that $\beta$ was selected by a procedure, such as the lasso. Simultaneously, there has been work to construct unconditional confidence intervals for $\beta$, where $X$ is the a priori selected covariate of interest, such as Javanmard & Montanari (2014), van de Geer, Bühlmann, Ritov & Dezeure (2014), and Zhang & Zhang (2013), which is also our focus. To avoid digressions, we will not elaborate on the former. A review of many of the current methods is available in Dezeure, Bühlmann, Meier & Meinshausen (2015). Much of the existing literature relies on using a version of the de-sparsified lasso introduced simultaneously by Javanmard & Montanari (2014), van de Geer et al. (2014), and Zhang & Zhang (2013). The aforementioned methods are applicable for sub-Gaussian designs under certain technical assumptions, but we will focus on the case where $(X, Z)$ are jointly Gaussian with covariance matrix $\Sigma$. The idea behind the existing approaches is to invert the KKT conditions of the lasso and perform node-wise lasso to approximate the inverse covariance matrix of the design, which attempts to correct the bias introduced by the lasso.

Since the lasso forms the basis for the procedure, certain assumptions must be made in order to ensure that the lasso enjoys the nice theoretical properties that have been developed over the past two decades. The paper by van de Geer & Bühlmann (2009) provides an overview of various assumptions that have been used to prove oracle inequalities for the lasso. These assumptions are a consequence of the fact the lasso is used rather than being grounded in applicability. In particular, for confidence intervals, van de Geer et al. (2014) assume that the compatibility condition holds for the Gram matrix, which is the weakest assumption from van de Geer & Bühlmann (2009), and is essentially a necessary assumption for the lasso to enjoy the fast rate (cf Bellec (2018)). However, this raises an important question on necessity: Is the compatibility condition necessary for constructing confidence intervals in high-dimensions?

To this end, we provide an estimator which does not require the compatibility condition but still enjoys an asymptotic normal distribution with the parametric length $1/\sqrt{n}$ and the same asymptotic variance as the oracle least-squares. Letting $s$ denote the sparsity of $\gamma$, our estimator requires the ultra-sparse regime, in which $s = o(\sqrt{n}/\log(p))$. Letting $s_\Omega$ the number of
covariates $Z$ on which $X$ depends (see assumption (B)), this estimator is not adaptive in the sense of Javanmard & Montanari (2018), which appears to be a trade-off with the compatibility condition.

As a consequence of our estimation procedure for $\beta$, we are able to construct a $\sqrt{n}$-consistent estimator of the signal strength, which we denote by $\sigma_2^2$, also without the compatibility condition. Cai & Guo (2018) consider a more general problem in the semi-supervised setting but their results for the supervised framework require minimal non-zero eigenvalues on the covariance matrix. Moreover, we show that the estimator attains an asymptotic variance equal to that of the efficient estimator in low-dimensions. In addition to the applications considered by Cai & Guo (2018), we consider the implications in assessing model fit and trying to determining the sparsity of $\gamma$.

1.1 Organization of the paper

In Section 2, we formally introduce the problem setups and estimators considered as well as the notation and assumptions used throughout the paper. Section 3 includes the main results, which can roughly be split into three parts, corresponding to the risk for prediction, inference on $\beta$, and inference on $\sigma_2^2$. We start with preliminary results on the risk for prediction under our proposed procedure, which is used in asymptotic analyses later. We then proceed to inference on $\beta$ in Section 3.2. Section 3.3 considers the extension for the estimation of $\sigma_2^2$ as well as three applications. Finally, an overview on computation as well as numerical simulations are provided in Section 4. All of the proofs of the relevant results are presented in Section 6.1.

2 Model

Let $(X_1, Z_1, Y_1), ..., (X_n, Z_n, Y_n)$ be independent and identically distributed where $(X_i, Z_i, Y_i) \in (\mathbb{R} \times \mathbb{R}^p \times \mathbb{R})$ with $p > n$, which we think of as forming a triangular array. We will restrict our attention to the case where the design is fully Gaussian. That is, we assume that $(X_i, Z_i)^T \overset{i.i.d.}\sim \mathcal{N}_{p+1}(0_{p+1}, \Sigma)$ for some covariance matrix $\Sigma$. We will write $X$ to denote the matrix $(X_1, ..., X_n)^T$ and define $Z$ and $Y$ analogously. Then, the model we consider is given in equation (1). This is the focus of the first part of the paper, regarding inference on $\beta$. In the second half, we are primarily interested in estimating the signal
strength. In this scenario, we do not distinguish between $X$ and $Z$, and will therefore consider the simplified model

$$Y = Z\gamma + \epsilon. \quad (2)$$

Then, our parameter of interest will be $\sigma^2 = n^{-1}\mathbb{E}\|Z\gamma\|^2 = n^{-1}\text{Var}(Z\gamma)$.

### 2.1 Notation

Throughout, all of our variables have a dependence on $n$, but, when it should not cause confusion, we will suppress this dependence. Consistent with other works on high-dimensional linear models, sparsity will play an important role. There are different notions of sparsity and we only consider the case of strong sparsity. We will let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$ and $\|\cdot\|_0$ the $L_0$-norm on $\mathbb{R}^p$. Then, we have the following definition.

**Definition 2.1.** A vector $\gamma \in \mathbb{R}^p$ is said to be strongly sparse with sparsity $s$ and strongly sparse set $S$ if there exists a vector $\gamma_S \in \mathbb{R}^p$ such that $\gamma = \gamma_S$ and $\|\gamma_S\|_0 = s$.

We will let $S$ denote the strongly sparse set of covariates for $\gamma$ with sparsity $s$. Similarly, we will assume that $X$ depends on $Z$ only through a small subset $Z_T$ with $s_\Omega \triangleq |T|$. For $u \in \mathbb{N}$, we will let $\mathcal{M}_u$ denote the collection of all models of size $u$. When $u = s$, we will write $\mathcal{M} = \mathcal{M}_s$.

For each model $m \in \mathcal{M}_u$, $Z_m$ will denote the $n \times u$ sub-matrix of $Z$ corresponding to the columns indexed by $m$ and $Z_i$ to denote the $i$th row of $Z$, with the interpretation being clear from context. For simplicity, we will write $\xi \triangleq Z_S\gamma_S$ with $n^{-1}\text{Var}(\xi) = \sigma^2_\xi$. We will use $Z_S\gamma_S$ and $\xi$ interchangeably. We note that $\sigma^2_\xi$ is the signal strength in the second part of the paper, on which we will perform inference.

Then, to each model $m$, we will let $Q_m$ denote the projection onto the column space of $Z_m$, $P_m$ denote the projection onto the column space of $(X, Z_m)$, and $R_m$ denote the projection onto the column space of $(\xi, Z_m)$. Accordingly, we will use $Q_m^\perp$, $P_m^\perp$, and $R_m^\perp$ to denote the projections onto their respective ortho-complements. Let $\Sigma$ denote the covariance matrix of $(X, Z)$.

Later, we will need to utilize a data-split. We will write $\tilde{X}$, $\tilde{Z}$, and $\tilde{Y}$ to denote an independent copy of $X$, $Z$, and $Y$. Then, $Q_m^\perp$ will denote the analogue of $Q_m^\perp$ but computed on the second half of the data, and analogously...
for other items. Instead of dividing everything by two, we will assume that
our sample size is not $n$, but rather $2n$.

When looking at asymptotic variances, we frequently refer to the “low-
dimensional problem”, by which we mean the setting where $s = p < n$ and
is fixed.

2.2 Motivation and Exponential Weighting

2.2.1 Estimating $\beta$

Suppose temporarily that $S$ is known. Then, we are in the low-dimensional
linear model and $\beta$ would be estimated by using the classical least-squares
approach. The least-squares solution can be framed as a two-stage proce-
dure. In the first stage, $Y$ is regressed on $Z_S$ and $X$ on $Z_S$. In the second
stage, we regress the residuals against each other. Therefore, we can view
the estimation of $\beta$ as a prediction problem; after predicting $Y$ and $X$ as
well as possible using $Z_S$, we consider the linear relationship between the
residuals. This is essentially the idea of $\hat{\beta}_{EW,u}$. It turns out this estimator
has a bias which converges at a slower order than $\sqrt{n}$ due to imperfect or-
thogonalization; so we will correct this estimator later to obtain $\hat{\beta}_{DEW,u}$, a
de-biased version of $\hat{\beta}_{EW,u}$.

Our motivation was to construct confidence intervals for $\beta$ under minimal
assumptions on the design. Since we have now transformed our view of the
problem into one of prediction, we now need an estimate of $Z_S\gamma_S$ that is
consistent under minimal assumptions. For our analyses, since we do not
have access to the true level of sparsity $s$, we will consider models of size
$u$ but will defer discussion on how to choose $u$ to Section 4. We propose
using a modified version of exponential weighting introduced by Leung &
Barron (2006). The idea is to obtain many least-squares estimates and to
average them in a data-driven way to obtain an aggregate estimate of the
mean. Leung & Barron (2006) prove that exponential weighting has a sharp
oracle bound on the risk regardless of the design and how ill-conditioned it
may be. The reason why the compatibility condition is not required is that
exponential weighting does not take into account the actual or estimated
values of the coefficients, rather only the projection of the data using each of
the parametric sub-models over which we are averaging.

Our approach differs from the original exponential weighting since the
least-squares estimates and the weights as proposed by Leung & Barron
(2006) are estimated on the same sample whereas we utilize a data-split, which simplifies our analyses in the second stage. The idea of data-splitting for the aggregation of estimators has been around for almost two decades, with the first proposal by Nemirovski (2000), however under a slightly different problem setup. To the best of our knowledge, this is the first use of data-splitting for the exponential weights of Leung & Barron (2006). We will need to use exponential weighting twice, an extra set for the additional de-biasing step. Define the first set of exponential weights over $\mathcal{M}_u$ as

$$
\tilde{w}_{m,Y} \triangleq \frac{\exp \left( -\frac{1}{\alpha_Y} \left\| \widetilde{P}_m Y \right\|^2 \right)}{\sum_{k \in \mathcal{M}_u} \exp \left( -\frac{1}{\alpha_Y} \left\| \widetilde{P}_k Y \right\|^2 \right)},
$$

where $\alpha_Y > 4(\sigma^2 \lor \sigma^2)$. Then, for each $m \in \mathcal{M}_u$, let $\hat{\beta}_m$ denote the least-squares estimator of $\beta$ using the covariates $(X, Z_m)$. The estimator $\hat{\beta}_{EW,u}$ is given by

$$
\hat{\beta}_{EW,u} \triangleq \sum_{m \in \mathcal{M}_u} \tilde{w}_{m,Y} \hat{\beta}_m.
$$

Thus, $\hat{\beta}_{EW,u}$ is simply the exponentially weighted average of the $\hat{\beta}_m$ over $\mathcal{M}_u$. Similarly, we will define $\hat{\gamma}_{EW,u}$ as the exponentially weighted average of the $\hat{\gamma}_m$. Later, it is shown that $\hat{\beta}_{EW,u}$ is an asymptotically consistent estimator. However, it is not necessarily an asymptotically normal estimator due to a large bias term. To correct the bias in the second stage, we need another set of exponential weights.

$$
\tilde{w}_{m,X} \triangleq \frac{\exp \left( -\frac{1}{\alpha_X} \left\| \widetilde{Q}_m X \right\|^2 \right)}{\sum_{k \in \mathcal{M}_u} \exp \left( -\frac{1}{\alpha_X} \left\| \widetilde{Q}_k X \right\|^2 \right)},
$$

where $\alpha_X > 4(\sigma^2 \lor n^{-1}\text{Var}(Z))$. Then, let $\hat{\delta}_m$ denote the least-squares estimate of $\delta$ using covariates $Z_m$. This gives rise to the estimator

$$
\hat{\delta}_{EW,u} \triangleq \sum_{m \in \mathcal{M}_u} \tilde{w}_{m,X} \hat{\delta}_m.
$$
Then, the bias corrected estimator for $\beta$ is

$$
\hat{\beta}_{DEW,u} \triangleq \frac{(X - Z\hat{\delta}_{EW,u})(Y - Z\hat{\delta}_{EW,u}\hat{\beta}_{EW,u} - Z\hat{\gamma}_{EW,u})}{\|X - Z\hat{\delta}_{EW,u}\|^2}.
$$

Intuitively, $X - Z\hat{\delta}_{EW,u}$ is the orthogonalization of $X$ in the presence of all of the $Z$. Similarly, $Y - Z\hat{\delta}_{EW,u}\hat{\beta}_{EW,u} - Z\hat{\gamma}_{EW,u}$ is the orthogonalization of $Y$ in the presence of all the $Z$. The remaining correlation between the residuals will be due to the effect of $X$ on $Y$.

Since only a lower bound is required on both $\alpha_X$, $\alpha_Y$, for simplicity, we write $\alpha \triangleq \alpha_X \lor \alpha_Y$ and use the same constant for both set of weights. This will enter as a constant factor in the risk bound, but, as our analysis is asymptotic, this will not play any role. Our value of $\alpha$ is larger compared to Leung & Barron (2006), who only require $\alpha \geq 4\sigma^2$, since we incorporate the randomness of the signal. Since large values of $\alpha$ equate to perfect averaging of models while small values of $\alpha$ act as model selection, we require more averaging than Leung & Barron (2006). We remark that the needed value of $\alpha$ defies a Bayesian interpretation of the procedure since the Bayes procedure requires a leading constant of 2, as shown by Leung & Barron (2006).

### 2.2.2 Estimating $\sigma^2_\xi$

Temporarily, we will assume that $p < n$ and is fixed. Then, letting $Q$ denote the projection onto the column space of $Z$, in the low-dimensional linear model we would estimate $\sigma^2_\xi$ with

$$
\frac{1}{n} \|QY\|^2 \overset{a.s.}{\to} \sigma^2_\xi
$$

as $n \to \infty$ (see Lemma 3.8). This seems to suggest, like before, we may use exponential weights to average various estimates from parametric sub-models in the high-dimensional case. Unlike before, since we are not interested in consistently estimating any particular component of the vector $Z\gamma$, but rather $n^{-1}E\|Z\gamma\|^2$, we will not need an extra de-biasing step.
For a model \( m \in \mathcal{M}_u \), define the exponential weights as

\[
\tilde{w}_{m,\xi} \triangleq \frac{\exp \left( -\frac{1}{\alpha_Y} \| \tilde{Q}_m \tilde{Y} \|_2^2 \right)}{\sum_{k \in \mathcal{M}_u} \exp \left( -\frac{1}{\alpha_Y} \| \tilde{Q}_k \tilde{Y} \|_2^2 \right)},
\]

with \( \alpha_Y \) as defined previously. Then, our estimate of \( \sigma^2_\xi \) will be

\[
\hat{\sigma}^2_{\xi, EW, u} \triangleq \sum_{m \in \mathcal{M}} \tilde{w}_{m,\xi} \hat{\sigma}^2_{\xi,m},
\]

where

\[
\hat{\sigma}^2_{\xi,m} \triangleq \frac{1}{n} \| Q_m Y \|_2^2.
\]

### 2.3 Assumptions

Before presenting the results, we will state all of the assumptions that we will make in the paper. We will assume without the loss of generality that \( X \) and each column of \( Z \) have squared norm that is \( \mathcal{O}_p(n) \). Since our analysis in this paper is mainly asymptotic, we will need to make some assumptions regarding sparsity and the structure of the design.

The assumptions that will be used are:

1. **(A)** The rows of \((X, Z)\) satisfy

   \[(X_1, Z_1), \ldots, (X_n, Z_n) \overset{i.i.d.}{\sim} \mathcal{N}_{p+1}(0_{p+1}, \Sigma).\]

2. **(B)** There exists a Gaussian vector \( \eta \sim \mathcal{N}_n(0_n, \sigma^2_\eta I_n) \) independent of \( Z \) with \( \sigma^2_\eta > 0 \) such that

   \[X = Z\delta + \eta \quad (3)\]

   Moreover, \( \delta \) is a sparse vector with strongly sparse set \( T \) such that \( \delta = \delta_T \) and \( s_\Omega = \| \delta_T \|_0 \).

3. **(C)** The sequence of constants \( u \) satisfies:

   
   (i) \[\liminf(u - s \vee s_\Omega) \geq 0,\]
(ii) \[ \frac{u \log(p)}{\sqrt{n}} \to 0. \]

(D) The sequence of constants $u$ satisfies:

(i) \[ \lim \inf (u - s) \geq 0, \]

(ii) \[ \frac{u \log(p)}{\sqrt{n}} \to 0. \]

**Remark 2.2.** The definition of $s_\Omega$ from assumption (B) is consistent with that of Javanmard & Montanari (2018). Suppose that $\Sigma$ is invertible with inverse $\Omega$. Let \[ s_\Omega = \max_j |\{k \neq j : \Omega_{j,k} \neq 0\}|. \] Then, since the design is Gaussian, this implies the existence of a set $T \in \mathcal{M}_{s_\Omega}$ satisfying assumption (B).

**Remark 2.3.** There are two assumptions above on sparsity. The first assumption, (C), is for asymptotic normality in estimating $\beta$. The second assumption, (D), will be used for asymptotic normality in estimating $\sigma^2_\xi$.

**Remark 2.4.** The assumption that $\lim \inf (u - s) \geq 0$ is to ensure that we consider models that are sufficiently large so that it will be possible to remove the full effect of $Z_S \gamma_S$ from $Y$.

Since our procedure is non-adaptive to the row sparsity, we will assume without the loss of generality that $s_\Omega \leq s$ by enriching $S$ if necessary. We note that these assumptions are a proper subset of the assumptions used in van de Geer et al. (2014), most notably omitting the compatibility condition.

## 3 Theoretical Results

In this section, we present all of the main results regarding $\hat{\beta}_{\text{EW},u}$, $\hat{\beta}_{\text{DEW},u}$, and $\hat{\sigma}^2_{\xi,\text{EW},u}$ and defer all the proofs to Section 6.1. We start by stating an oracle inequality when using the data-splitted exponential weights for prediction. Then, we consider the consistency of $\hat{\beta}_{\text{EW},u}$, which is then used for the asymptotic normality of $\hat{\beta}_{\text{DEW},u}$. Finally, we analyze $\hat{\sigma}^2_{\xi,\text{EW},u}$ and consider its applications.
3.1 Risk for Prediction

One salient feature of the exponential weighting procedure given by Leung & Barron (2006) and Rigollet & Tsybakov (2012) is the risk bound on prediction. They show that, under no assumptions on the eigenvalues of the design, the risk of the exponentially weighted estimate of the mean tends to zero. Specifically,

\[
\frac{1}{n} \mathbb{E} \left\| \left( \sum_{m \in M_u} \tilde{w}_{m,Y} \tilde{P}_m Y \right) - (\tilde{X} \beta + \tilde{Z} \gamma) \right\|^2 \\
\leq \inf_{m \in M_u} \left\{ \frac{1}{n} \mathbb{E} \left\| \tilde{P}_m \tilde{Z} \gamma \right\|^2 + \frac{9\sigma^2 s}{n} \log \left( \frac{enp}{s} \right) + \frac{2\sigma^2}{n} \right\}.
\]

By assumption (C), the above risk tends to zero as \( n \) tends to infinity even with multiplication of \( \sqrt{n} \). However, the estimates of the mean and the weights are built using the same sample whereas we perform a data split to obtain independence between the two estimates, something that is used in the second stage to reduce the bias. Under the additional assumption of (A), the following proposition shows that there is sufficient concentration of the weights to ensure that the risk tends to zero.

**Proposition 3.1.** Consider the linear model given in (1). Assume that (A) and (C) are both satisfied. Then,

\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \mathbb{E} \left\| (X \hat{\beta}_{EW,u} + Z \hat{\gamma}_{EW,u}) - (X \beta + Z \gamma) \right\|^2 = 0.
\]

By the exact same line of reasoning, we have the following proposition, the proof of which is omitted.

**Proposition 3.2.** Consider the linear model given in (3). Assume that (A) and (C) are both satisfied. Then,

\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \mathbb{E} \left\| Z \hat{\delta}_{EW,u} - Z \delta \right\|^2 = 0.
\]

This provides corresponding risk bounds under data-splitting for Gaussian designs for exponential weighting. We emphasize that the proofs given here only require the Gaussian assumption and the sparsity rate; there is no
assumption regarding the design matrix or the population covariance matrix. Analogous results regarding the risk in the lasso require additional assumptions, such as a compatibility condition or a restricted eigenvalue condition (for example, cf Bickel, Ritov & Tsybakov (2009)).

Hence, by using exponential weighting, it implies that \( X\hat{\beta}_{\text{EW},u} + Z\hat{\gamma}_{\text{EW},u} \) and \( Z\hat{\delta}_{\text{EW},u} \) are good estimates of \( X\beta + ZS\gamma \) and \( ZT\delta_T \) respectively. However, we actually do not need an estimate of \( X\beta + ZS\gamma \), rather an estimate of \( ZS\gamma \). In the following section, we will show that \( X\hat{\beta}_{\text{EW},u} \) is a good estimate of \( X\beta \), which will imply that \( Z\hat{\gamma}_{\text{EW},u} \) is a good estimate of \( ZS\gamma \).

3.2 Inference on \( \beta \)
In this section, we analyze the asymptotic behavior of the estimator \( \hat{\beta}_{\text{EW},u} \) and \( \hat{\beta}_{\text{DEW},u} \). By showing that \( \hat{\beta}_{\text{EW},u} \) is a good estimate of \( \beta \), we can hope to show that \( X\hat{\beta}_{\text{EW},u} \) is a good estimate of \( X\beta \). Then, by the triangle inequality in conjunction with Proposition 3.1, it will follow that \( Z\hat{\gamma}_{\text{EW},u} \) is a consistent estimator for \( ZS\gamma \). For any fixed \( m \in M_u \), the least-squares estimator of \( \beta \) using \( (X,Z_m) \) is equal to the following

\[
\hat{\beta}_m = \frac{X^TQ_m^\perp Y}{XQ_m^\perp X} = \beta + \frac{X^TQ_m^\perp ZS\gamma}{X^TQ_m^\perp X} + \frac{X^TQ_m^\perp \epsilon}{X^TQ_m^\perp X}.
\]

Therefore, \( \hat{\beta}_{\text{EW},u} \) can be written as

\[
\hat{\beta}_{\text{EW},u} = \beta + \sum_{m \in M_u} \tilde{w}_{m,Y} \frac{X^TQ_m^\perp ZS\gamma}{X^TQ_m^\perp X} + \sum_{m \in M_u} \tilde{w}_{m,Y} \frac{X^TQ_m^\perp \epsilon}{X^TQ_m^\perp X}. \tag{4}
\]

The first term is the bias term and the last term is the noise term. Using this decomposition, we can prove the following lemma.

Lemma 3.3. Consider the linear model given in (1). Assuming (A), (B), and (C), then

\[
n^{1/4} \left( \hat{\beta}_{\text{EW},u} - \beta \right)  \xrightarrow{L^2} 0.
\]

As an immediate corollary, we have the following result about the risk using \( Z\hat{\gamma}_{\text{EW},u} \) for estimating \( ZS\gamma \).
Corollary 3.3.1. Consider the linear model given in (1). Assuming (A), (B), and (C), then

\[
\frac{1}{\sqrt{n}} \|Z_{\hat{\gamma}_{EW,u}} - Z_{S \gamma_S}\|^2 \overset{P}{\to} 0.
\]

Remark 3.4. We want to note that \(\hat{\beta}_{EW,u}\) might not necessarily attain the \(\sqrt{n}\) rate since the bias will be too large. In equation (4), the first term is the bias term. Since the design is fully Gaussian, for a fixed \(m \in M_u\), there exists a vector of regression coefficients \(\theta_m\) and an independent Gaussian vector \(\zeta_m \sim \mathcal{N}_n(0, \tau^2_m I_n)\) such that

\[
Z_{S \gamma_S} = (X, Z_m)\theta_m + \zeta_m.
\]

Let \(\theta_{m,1}\) denote the first coordinate of \(\theta_m\). Thus, the bias can be expressed as

\[
\sum_{m \in M_u} \bar{w}_{m,y} \frac{X^T Q^\perp_m Z_S \gamma_S}{X^T Q^\perp_m X} = \sum_{m \in M_u} \bar{w}_{m,y} \left( \theta_{m,1} + \frac{X^T Q^\perp_m \zeta_m}{X^T Q^\perp_m X} \right).
\]

It is shown in the proof of Lemma 3.3 that, for \(n\) sufficiently large, \(\theta^2_{m,1} \leq 2\tau^2_m / \sigma^2_n\). For a corresponding lower bound, note that

\[
\sigma^2_X \theta^2_{m,1} n = \theta^2_{m,1} E \|X\|^2 \geq E \|R^\perp_{m,X} \theta_{m,1}\|^2 = \tau^2_m (n - u - 1).
\]

Now, solving for \(\theta^2_{m,1}\) and taking limits, it follows that \(\theta^2_{m,1} \sim \tau^2_m\). From the proof of Proposition 3.1, for any value of \(t > 0\), we defined the set

\[
A_t \overset{\Delta}{=} \left\{ m \in M_u : \tau^2_m \leq \frac{t}{\sqrt{n}} \right\}.
\]

\(A_t\) represents the collection of all the models that do well at explaining \(X \beta + Z_S \gamma_S\). Models that differ structurally at a rate smaller than \(\sqrt{n}\) are indistinguishable since the dominating term is no longer the bias, rather it is the effect of the noise from \(\epsilon\). In the ultra-sparse regime, as in the low-dimensional case, the normalization factor is \(\sqrt{n}\), which is the noise level. But, from the above equation, the bias in \(\hat{\beta}_{EW,u}\) is on the \(n^{1/4}\) rate if \(\tau^2_m \sim n^{-1/2}\). In such a scenario, the bias could dominate, implying that we do not attain the \(\sqrt{n}\) rate.
This suggests that we need to further de-bias $\hat{\beta}_{EW,u}$ to obtain an asymptotically normal estimator, leading to $\hat{\beta}_{DEW,u}$.

**Theorem 3.5.** Consider the linear model given in (1). Assuming (A), (B), and (C), then

$$\sqrt{n} \left( \hat{\beta}_{DEW,u} - \beta \right) \big| (X,Z) \overset{D}{\rightarrow} N \left( 0, \frac{\sigma_\xi^2}{\sigma^2} \right).$$

Recall that, due to the data-split, the effective sample size that we have been working with is $2n$. However, the asymptotic distribution of $\hat{\beta}_{DEW,u}$ does not depend on the other split of the data. Therefore, by reversing the roles of $(X,Z,Y)$ with $(\tilde{X}, \tilde{Z}, \tilde{Y})$, we can construct another estimator via exponential weighting by the exact same procedure. These two estimators are asymptotically independent; hence, we may average the two independent estimates $\beta$ to obtain an estimator that makes full use of the data. Let $\tilde{\beta}_{DEW,u}$ denote $\hat{\beta}_{DEW,u}$ but with the roles of $(X,Z,Y)$ and $(\tilde{X}, \tilde{Z}, \tilde{Y})$ reversed. We have the following corollary about the averaged estimator.

**Corollary 3.5.1.** Consider the linear model given in (1). Assuming (A), (B), and (C), then

$$\sqrt{2n} \left( \frac{1}{2} \left( \hat{\beta}_{DEW,u} + \tilde{\beta}_{DEW,u} \right) - \beta \right) \big| (X,Z), (\tilde{X}, \tilde{Z}) \overset{D}{\rightarrow} N \left( 0, \frac{\sigma_\xi^2}{\sigma^2} \right).$$

Note that this is the same variance as the de-biased lasso estimator, which is known to be asymptotically efficient in a semi-parametric sense (cf van de Geer et al. (2014) Theorem 2.3).

### 3.3 Estimation of $\sigma_\xi^2$ and its Applications

In this section, we start by stating the main result regarding $\hat{\sigma}_\xi^2_{EW,u}$ and then show a few applications.

**Theorem 3.6.** Consider the linear model given in (2). Assume (A). If

(1) $\sigma_\xi^2 = 0$, then

$$\hat{\sigma}_{\xi,EW,u}^2 = \frac{1}{n} \sum_{m \in M_u} \tilde{w}_{m,\xi} \|Q_m \xi\|^2. \quad (5)$$
\( \sigma^2_\xi > 0 \) and assuming (D), then
\[
\sqrt{n} \left( \hat{\sigma}^2_\xi,_{EW,u} - \sigma^2_\xi \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, 2\sigma^4_\xi + 4\sigma^2_\xi \sigma^2_\epsilon \right).
\] (6)

As an immediate corollary, we also have the following.

**Corollary 3.6.1.** Consider the linear model given in (2). Assuming (A) and (D), then
\[
\hat{R}^2_{EW,u} \triangleq \frac{n \hat{\sigma}^2_\xi,_{EW,u}}{\|Y\|^2} \xrightarrow{P} \frac{\sigma^2_\xi}{\sigma^2_\xi + \sigma^2_\epsilon}.
\]

**Remark 3.7.** If only a consistent estimator of \( R^2 \) is required, assumption (D) from the above corollary can be replaced with the weaker condition that \( s = o(n/\log(p)) \).

From Theorem 3.6, a natural question that arises is regarding the efficiency in the estimation of \( \sigma^2_\xi \) in high-dimensions. In the low-dimensional problem, by Central Limit Theorem considerations, if \( \sigma^2_\xi > 0 \), then
\[
\sqrt{n} \left( \frac{1}{n} \|QY\|^2 - \sigma^2_\xi \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, 2\sigma^4_\xi + 4\sigma^2_\xi \sigma^2_\epsilon \right).
\]

Hence, \( \hat{\sigma}^2_\xi,_{EW,u} \) has the same asymptotic variance as the above estimator. The following lemma shows that \( n^{-1} \|QY\|^2 \) is the efficient estimator of \( \sigma^2_\xi \) in the low-dimensional case.

**Lemma 3.8.** Consider the linear model given by equation (2). Assume that \( Z \) is a Gaussian design with full column rank and \( p < n \) is fixed. Then, the estimator \( n^{-1} \|QY\|^2 \) is efficient for estimating \( \sigma^2_\xi \).

**Remark 3.9.** Cai & Guo (2018), in Theorem 4, obtain the same asymptotic distribution for a Gaussian linear model using the CHIVE estimator under the additional assumption that \( \Sigma \) has bounded eigenvalues.

We will primarily elaborate on three applications of \( \hat{\sigma}^2_\xi,_{EW,u} \). Theorem 3.6 implies that we may conduct hypothesis tests on the entire vector \( \gamma \). In particular, we may be interested in testing:

\[
H_0 : \|\gamma\|_0 = 0 \quad \quad \quad H_1 : \|\gamma\|_0 > 0
\]
The null hypothesis corresponds to testing $\sigma^2_{\xi} = 0$. In this setting, equation (5) provides us with a test statistic since the distribution of $\hat{\sigma}^2_{\xi,EW,u}$ can be easily simulated given the covariates $Z$ and equation (6) provides an approximation for the power of the test assuming (D). The test is still valid under mis-specification of $u < s$ but the power will vary accordingly.

The second application involves the coefficient of determination, which is commonly used in the sciences. Olkin & Finn (1995) provides a procedure to construct confidence intervals for $R^2$ in a low-dimensional linear regression. Again, by Central Limit Theorem considerations,

$$\sqrt{n} \left( \frac{1}{n} \| Y \|^2 - (\sigma^2_{\xi} + \sigma^2_{\epsilon}) \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, 2 \big( \sigma^2_{\xi} + \sigma^2_{\epsilon} \big)^2 \right).$$

By the Delta Method applied to equations (6) and (7), it follows immediately that $\hat{R}^2_{EW,u}$ has an asymptotic Gaussian distribution, which may be used to construct confidence intervals.

The last application we consider deals with determining the sparsity of $\gamma$. There has been much work over the years to try to consistently estimate or obtain a lower bound on the sparsity of $\gamma$. For example, the lasso is known to be model selection consistent under an Irrepresentable Condition “$\beta$-min” assumption, which is essentially a necessary and sufficient (cf Zhao & Yu (2006)). Similarly, Kim, Kwon & Choi (2012) develop a consistent model selection criteria based on the generalized information criterion, but require strong conditions like the “$\beta$-min” assumption.

Our approach is motivated by low-dimensional procedures and computing high-dimensional analogues. Consider the case when $p < n$ and fixed. Common suggestions for performing model selection include: adjusted-$R^2$, Mallows’s $C_P$, AIC, and BIC. However, all of the aforementioned procedures can be rewritten as functions of the statistics ($R^2, \| Y \|^2$). In the high-dimensional regime, Corollary 3.6.1 asserts that we have a consistent estimator of $R^2$. Therefore, this yields various procedures for estimating the sparsity of the vector $\gamma$.

There have been other extensions of the above criteria to high-dimensions, such as Chen & Chen (2008), who propose the extended BIC criterion. This approach consistently selects the correct model under an asymptotic identifiability condition when $p > n$ but does not allow $s$ to grow with $n$.

However, we note that our approach is only a heuristic for determining the sparsity as some theoretical properties of the low-dimensional procedures do
not carry over. For example, Rao & Wu (1989) proved that BIC is consistent when \( p \) is fixed, which no longer applies to our procedure.

4 Implementation and Empirical Results

In this section, we propose a procedure to estimate \( \hat{\beta}_{EW,u} \) and \( \hat{\beta}_{DEW,u} \) and provide a simulation study to analyze the coverage and length of the resultant confidence intervals.

4.1 Estimation

Up to this point, to simplify analyses, we have assumed that we have access to a sequence of sparsities \( u \) and the values of the tuning parameters \( \alpha_X \) and \( \alpha_Y \). However, in practical applications, neither is viable. Moreover, we need to compute \( |M| = \binom{p}{u} = O(p^u) \) least square models, which is not tenable for any high-dimensional problem. Furthermore, Zhang, Wainwright & Jordan (2014) show that dependence on the restricted eigenvalue is unavoidable for polynomial-time algorithms. Therefore, in this section, we will provide procedures to overcome these problems.

Temporarily, consider the exponential weighting for \( \hat{\beta}_{EW,u} \). We assume the linear model given in equation (1) and need to choose a value of \( \alpha_Y > 4(\sigma^2 \lor \sigma^2) \). However, due to independence of \( X\beta + Z\gamma + \epsilon \), it suffices to use \( \text{Var}(Y) \), which is equal to than \( \sigma^2 + \sigma^2 \). We can do a similar decomposition for \( \alpha_X \). Therefore, we may use the plug-in estimators \( \hat{\alpha}_X = 8\text{Var}(X) \) and \( \hat{\alpha}_Y = 8\text{Var}(Y) \).

Before returning to the question on how to decide a value of \( u \), we will first discuss the computation of \( \hat{\beta}_{EW,u} \). As suggested by Rigollet & Tsybakov (2011) Section 7, we will also use a Metropolis Hastings approach. Our approach differs slightly since they consider models of differing sizes while we restrict our attention to \( u \)-sparse models. By viewing \( M \) as a hypercube with each vertex having weight \( \tilde{w}_m \), we may view \( \hat{\beta}_{EW,u} \) as the expectation of the \( \binom{p}{u} \) least-squares models over \( M_u \). Before describing the procedure, we need to introduce a bit of notation. For any model \( m \in M_u \), we will let \( K_m \) denote the neighbors of \( M_u \), defined as

\[
K_m \triangleq \{ k \in M_u : \| k - m \|_0 = 2 \}.
\]
That is, \( \mathcal{K}_m \) is the set of all models that swap out exactly one covariate. Moreover, let \( \text{RSS}_m \triangleq \| \overline{P}_m \hat{Y} \|_2^2 \). This implies the following algorithm, which closely parallels Rigollet & Tsybakov (2011):

Initialize a random point \( m_0 \in \mathcal{M}_u \) and compute \( \text{RSS}_{m_0} \). For \( t \geq 0 \),

1. Uniformly select \( k \in \mathcal{K}_{m_t} \) and compute \( \text{RSS}_k \).

2. Then, generate a random variable \( m_{t+1} \) by

\[
m_{t+1} = \begin{cases} 
m_t & \text{with probability } \exp \left( -\frac{1}{\alpha} (\text{RSS}_k - \text{RSS}_m) \right) \\
k & \text{with probability } 1 - \exp \left( -\frac{1}{\alpha} (\text{RSS}_k - \text{RSS}_m) \right)
\end{cases}
\]

3. Compute \( \hat{\beta}_{t+1} \) as the least-squares estimator under model \( m_{t+1} \)

Then, for some fixed burnin time \( T_0 \) and number of trials \( T \), the approximation of \( \hat{\beta}_{\text{EW},u} \) will be given by

\[
\hat{\beta}_{\text{EW},u} = \frac{1}{T} \sum_{t=0}^{T} \hat{\beta}_t.
\]

Analogous to Theorem 7.1 of Rigollet & Tsybakov (2011), it will follow that

\[
\lim_{T \to \infty} \hat{\beta}_{\text{EW},u} = \hat{\beta}_{\text{EW},u}.
\]

A similar procedure can be used to estimate \( \hat{\gamma}_{\text{EW},u} \) and \( \hat{\delta}_{\text{EW},u} \).

Estimating \( u \) is not as straightforward unfortunately. A simple solution would be to use cross-validation to tune the value of \( u \), but such a procedure would be extremely computationally extensive, especially over many splits of the data. Instead, there are several heuristics which may be applied to try to upper bound the sparsity level. One possibility is to apply the exponential screening procedure of Rigollet & Tsybakov (2011), and threshold the coefficients. Let \( \hat{\gamma}_{\text{ES}} \) denote the exponential screening estimate of \( \gamma \). Following Rigollet & Tsybakov (2011), a covariate \( Z_j \) is said to be selected if
\( \hat{\gamma}_{ES,j} > 1/n \). Then, since we only need an upper bound, we may set \( u \) to simply be a multiple of the number of selected covariates or refine a grid for cross-validation based on how many covariates are selected.

We propose another approach to estimating sparsity, utilizing the results from Section 3.3. We may construct a coarse grid of sparsities and compute \( \hat{\sigma}^2_{\xi,EW,u} \) as \( u \) ranges over the grid. Then, we apply a low-dimensional model selection criteria to select the sparsity (like AIC). Since we only need a lower bound on the sparsity, we may take 1.5 times the selected value to overcome problems of distinguishing model sizes that are close to the true level \( s \).

### 4.2 Simulations

In this section, we implement the procedure presented in the previous section and compare the performance of \( \hat{\beta}_{DEW,u} \) to the de-biased lasso and the oracle least-squares that knows the set \( S \).

#### 4.2.1 Methods and Models

For ease of comparison, our simulations will closely parallel those given in van de Geer et al. (2014). For the linear model

\[
Y = X\beta + Z\gamma + \epsilon,
\]

we will consider the case when \( n = 100, p = 501 \) and \( s \in \{3, 15\} \). Rather than sampling the values of \( \beta \) and \( \gamma \) from a uniform distribution, we fix the values of \( \gamma \) to be equal to 1 on \( S \) and 0 on \( S^c \). For \( \beta \), we will consider the cases when \( \beta \in \{0, 1\} \) to compare the coverage probabilities and power under two different setups. For the correlation matrix \( \Sigma \), we will use the following:

- **Equi-correlation:** \( \Sigma_{i,j} = \begin{cases} 
1 & \text{if } i = j \\
\rho & \text{if } i \neq j 
\end{cases} \)

- **Toeplitz:** \( \Sigma_{i,j} = 0.9^{|i-j|} \)

For the equi-correlation case, we consider when \( \rho \in \{0, 0.8, 0.999\} \). When \( \rho = 0 \), this corresponds to an identity covariance matrix and will serve as a useful benchmark. The second case, \( \rho = 0.8 \), corresponds to the analysis given in van de Geer et al. (2014) and the last case highlights a very salient feature of exponential weighting. Unlike the lasso, which is dependent on the
compatibility constant in the oracle bound, exponential weighting is more robust to the ill-conditioning of the design. The higher the value of \( \rho \), the smaller the compatibility constant, which will in turn affect the performance of the lasso. Since exponential weighting depends primarily on the linear span of the covariates, it should be more robust under such a scenario.

All of our simulations are conducted in R to take advantage of the \texttt{hdi} package from Dezeure et al. (2015). For each of the covariance matrices, we generate \( n = 100 \) observations from

\[
\begin{pmatrix} X \\ Z \end{pmatrix} \sim \mathcal{N}_{p+1} \left( \begin{pmatrix} 0_1 \\ 0_p \end{pmatrix}, \Sigma \right).
\]

In the equi-correlation case, we take the first \( s \) coordinates of \( \gamma \) to be equal to one and the remaining \( p-s \) coordinates to be zero. In the Toeplitz case, we randomly sample \( s \) coordinates to be one and the remaining \( p-s \) coordinates to be zero. Then, independently, we generate 100 trials of \( \epsilon \) and compute \( Y \).

For each trial, we compute the least-squares estimator, the de-biased lasso, and the de-biased exponential weighting.

We compute the least-squares estimator assuming we know the true sparse set \( S \). That is, we solve the regression problem

\[
Y = X\beta + ZS\gamma_S + \epsilon.
\]

We included the least-squares in our simulations to act as an oracle to assess the other two procedures, in particular when assessing the minimax length of the confidence intervals.

For the de-biased lasso, we use the default settings of the \texttt{lasso.proj} from the \texttt{hdi} package.

For the de-biased exponential weighting, as suggested by Rigollet & Tsybakov (2011), we take \( T_0 = 3000 \) and \( T = 7000 \) for the Metropolis-Hastings algorithm. Moreover, we set \( \alpha_X = 8\text{Var}(X) \) and \( \text{Var}(Y) = 8\text{Var}(Y) \) where both are computed on the whole data. To choose the value of \( u \), we compute \( \hat{\sigma}_{\xi,\text{EW},u}^2 \) over \{4, 8, 12, 16, 20\} and use 1.5 times the sparsity level chosen by AIC.

To compare the three procedures, we consider three different measures:

1. Average Coverage: The percentage of time the true value of \( \beta \) falls inside the confidence interval.

2. Average Power: The percentage of time zero does not fall inside the confidence interval (only in the simulations when \( \beta = 1 \)).
Table 1: Simulations under equi-correlation and $\beta = 0$

| $s$ | 3  | 3  | 3  | 15 | 15 | 15 |
|-----|----|----|----|----|----|----|
| $\rho$ | 0.00 | 0.80 | 0.999 | 0.00 | 0.80 | 0.999 |
| AvgCov | LS | 0.94 | 0.94 | 0.98 | 0.94 | 0.92 | 0.94 |
|       | DL | 0.96 | 0.98 | 0.59 | 1.00 | 0.99 | 0.72 |
|       | DEW | 0.98 | 0.87 | 0.95 | 0.86 | 1.00 | 0.91 |
| AvgLen | LS | 0.46 | 0.72 | 11.26 | 0.47 | 0.94 | 12.83 |
|       | DL | 0.53 | 0.86 | 0.72 | 0.70 | 1.05 | 1.37 |
|       | DEW | 0.78 | 1.05 | 13.57 | 1.48 | 1.73 | 11.98 |

3. Average Length: The average length of the confidence interval, taken as the upper endpoint minus the lower endpoint.

4.2.2 Results

Table 1 compares the three procedure when $\beta = 0$ under the three different values of $\rho$ in the equi-correlation case. In the case when $\rho \in \{0, 0.8\}$, all three procedures seem comparable. In the high correlation case, $\rho = 0.999$, the de-biased lasso performs poorly due to a large compatibility constant. The length of the confidence intervals are overly optimistic compared to both least-squares and de-biased exponential weighting. The length of the de-biased exponential weighting intervals seems to track the length of the least-squares intervals well regardless of the amount of correlation, while still maintaining comparable coverage. When the correlation is low, the de-biased exponential weighting is more conservative compared to the de-biased lasso.

Figure 1 shows the confidence intervals of the three procedures over the first ten trials of the simulation of the equi-correlation design when $\beta = 0$. The horizontal line indicates the true value of $\beta$.

Table 2 shows the results when $\beta = 1$ under the equi-correlation case. The results are comparable to the case when $\beta = 0$, with the three having comparable power as well. The notable exception is when $s = 15$ and $\rho = 0$, where both procedures do poorly. The power of the de-biased lasso in this setting is inflated due to the poor coverage of the procedure. Again, the de-biased lasso performs poorly when the correlation amongst the covariates is large. Figure 2 provides the confidence intervals for the first ten trials when $\beta = 1$.

Finally, Table 3 and Figure 3 consider the case of the Toeplitz correlation
Figure 1: Plots of the estimated confidence intervals in the first 10 simulations under equi-correlation and $\beta = 0$

(a) $\rho = 0$ and $s = 3$

(b) $\rho = 0$ and $s = 15$

(c) $\rho = 0.800$ and $s = 3$

(d) $\rho = 0.800$ and $s = 15$

(e) $\rho = 0.999$ and $s = 3$

(f) $\rho = 0.999$ and $s = 15$
Figure 2: Plots of the estimated confidence intervals in the first 10 simulations under equi-correlation and $\beta = 1$

(a) $\rho = 0$ and $s = 3$

(b) $\rho = 0$ and $s = 15$

(c) $\rho = 0.800$ and $s = 3$

(d) $\rho = 0.800$ and $s = 15$

(e) $\rho = 0.999$ and $s = 3$

(f) $\rho = 0.999$ and $s = 15$
Table 2: Simulations under equi-correlation and $\beta = 1$

| $s$ | 3   | 3   | 3   | 15  | 15  | 15  |
|-----|-----|-----|-----|-----|-----|-----|
| $\rho$ | 0   | 0.8 | 0.999 | 0   | 0.8 | 0.999 |
| AvgCov | LS  | 0.95 | 0.95 | 0.96 | 0.94 | 0.97 | 0.95 |
|       | DL  | 0.93 | 0.90 | 0.11 | 0.05 | 0.73 | 0.38 |
|       | DEW | 0.97 | 0.97 | 0.89 | 0.33 | 0.95 | 0.91 |
| AvgPow | LS  | 1.00 | 1.00 | 0.04 | 1.00 | 0.98 | 0.03 |
|       | DL  | 1.00 | 0.95 | 0.40 | 0.80 | 0.73 | 0.29 |
|       | DEW | 1.00 | 0.91 | 0.11 | 0.04 | 0.91 | 0.07 |
| AvgLen | LS  | 0.42 | 0.78 | 11.79 | 0.37 | 0.99 | 12.19 |
|       | DL  | 0.50 | 0.94 | 0.82 | 0.58 | 0.96 | 1.26 |
|       | DEW | 0.86 | 1.29 | 12.51 | 1.24 | 1.85 | 10.99 |

and the results are similar to those under equi-correlation.

5 Discussions

In this paper, we considered two seemingly distinct but related problems: inference on both $\beta$ and $\sigma_x^2$. It is shown that, under a linear model with a Gaussian design when $p > n$, inference can be conducted without the compatibility condition, both with asymptotic variance equal to the oracle procedure using least-squares. By removing the compatibility condition, we demonstrate via simulations that, in finite samples with large correlation, confidence intervals constructed from the de-biased lasso can perform poorly relative to the asymptotic coverage, unlike the de-biased exponential weighting.

However, there are two related limitations to our approach. We are unable to extend our results beyond the Gaussian design since the proofs currently rely heavily on the conditional distribution of Gaussian vectors. Since the proof of the risk for prediction utilizes Laplace transforms, this seems to suggest that de-biased exponential weighting should be applicable to sub-Gaussian designs. Similarly, we are unable to extend our results to partially linear models for the same reason.

Another question is regarding the sparsity rate. We currently require the balanced assumption that $(s \lor s_{11}) \log(p)/\sqrt{n} \to 0$ while Javanmard & Montanari (2018) only demand $(s \land s_{11}) \log(p)/\sqrt{n} \to 0$ and $s \log^2(p)/n \to 0$. 
Figure 3: Plots of the estimated confidence intervals in the first 10 simulations under Toeplitz correlation

(a) $\beta = 0$ and $s = 3$

(b) $\beta = 0$ and $s = 15$

(c) $\beta = 1$ and $s = 3$

(d) $\beta = 1$ and $s = 15$
Table 3: Simulations under Toeplitz correlation

| $s$ | 3   | 3  | 15 | 15 |
|-----|-----|----|----|----|
| $\beta$ | 0   | 1  | 0  | 1  |
| AvgCov | LS  | 0.97 | 0.93 | 0.94 | 0.95 |
|       | DL  | 0.97 | 0.89 | 0.99 | 0.82 |
|       | DEW | 0.71 | 0.96 | 0.75 | 0.97 |
| AvgPow | LS  | 1.00 | 1.00 | 1.00 | 1.00 |
|       | DL  | 1.00 | 0.86 | 0.86 | 0.86 |
|       | DEW | 1.00 | 0.98 | 0.98 | 0.98 |
| AvgLen | LS  | 0.42 | 0.39 | 0.37 | 0.46 |
|       | DL  | 0.78 | 0.74 | 0.69 | 0.91 |
|       | DEW | 0.63 | 0.72 | 1.02 | 1.31 |

Our sparsity assumption, though, is not directly comparable as we allow a larger class of designs in terms of correlation. We conjecture that, without further assumptions on the design, up to logarithmic factors, $s \Omega s < n$ is required for the $\sqrt{n}$ rate, which we satisfy through the balanced assumption.

6 Appendix

6.1 Proof of Results

6.1.1 Proof of Risk for Prediction

Before proving Proposition 3.1, we will state and prove a fact about Gaussian random variable for the sake of completeness.

**Lemma 6.1.** Let $X$ and $Y$ be independent standard Gaussian random variables. Then, for any $|\lambda| < 1$, the Laplace transform of the product, $XY$, is given by

$$\mathcal{L}\{XY\}(\lambda) = \frac{1}{\sqrt{1 - \lambda^2}}.$$

**Proof.** Indeed, noting that

$$4XY = (X + Y)^2 - (X - Y)^2.$$
Then, $X + Y$ and $X - Y$ are independent Gaussians. Recall that for $\lambda < 1/2$, the Laplace Transform of a $\chi^2_k$ random variable is

$$\mathcal{L}\{\chi^2_k\}(\lambda) = (1 - 2\lambda)^{-k/2}. \quad (8)$$

Thus, for $\lambda < 1$,

$$\mathcal{L}\{XY\}(\lambda) = \left(\mathcal{L}\left\{\frac{(X + Y)^2}{4}\right\}(\lambda)\right) \left(\mathcal{L}\left\{\frac{(X - Y)^2}{4}\right\}(-\lambda)\right) = \frac{1}{\sqrt{1 - \lambda}} \frac{1}{\sqrt{1 + \lambda}} = \frac{1}{\sqrt{1 - \lambda^2}},$$

which finishes the proof. \hfill \square

**Proof of Proposition 3.1.** Consider the randomized estimator given by

$$\hat{\beta}_{\text{rand}} \triangleq \hat{\beta}_m \text{ w.p. } \tilde{w}_m,Y, \quad \hat{\gamma}_{\text{rand}} \triangleq \hat{\gamma}_m \text{ w.p. } \tilde{w}_m,Y.$$

Recall $\xi \triangleq ZS\gamma_S$. Note that

$$\left\| \left( X\hat{\beta}_m + Z\hat{\gamma}_m \right) - (X\beta + \xi) \right\|^2 = \left\| P_m\xi - P_m\epsilon \right\|^2.$$

Since the design is fully Gaussian, there exists a vector of regression coefficients $\theta_m$ and a Gaussian vector $\zeta_m \sim \mathcal{N}_n(0_n, \tau^2_m I_n)$ independent of $(X, Z_m)$ such that

$$\xi = (X, Z_m)\theta_m + \zeta_m.$$

Now, fix a value of $t > 0$ and define the set

$$\mathcal{A}_t \triangleq \left\{ m \in \mathcal{M}_u : \tau^2_m \leq \frac{t}{\sqrt{n}} \right\}.$$

Letting $r_{\text{rand}}$ denote the risk of the randomized estimator scaled by $\sqrt{n}$, it
follows that
\[
 r_{\text{rand}} \leq \frac{1}{\sqrt{n}} \mathbb{E} \sum_{m \in M_u} \tilde{w}_{m,Y} \left( \| P_m^\top \xi \|^2 + \| P_m \epsilon \|^2 \right)
\]
\[
 = \frac{1}{\sqrt{n}} \sum_{m \in M_u} \mathbb{E} \tilde{w}_{m,Y} \left( \tau_m^2 (n - u - 1) + \sigma_\epsilon^2 (u + 1) \right)
\]
\[
 \leq \sigma_\epsilon^2 u + \frac{1}{\sqrt{n}} + \sum_{m \in A_t} \mathbb{E} \tilde{w}_{m,Y} \frac{n - u - 1}{n} \sum_{\tilde{m} \in A_t} \mathbb{E} \tilde{w}_{\tilde{m},Y} \sigma_\xi^2 \frac{n - u - 1}{\sqrt{n}}
\]
\[
 \leq \sigma_\epsilon^2 u + \frac{1}{\sqrt{n}} + \frac{n - u - 1}{n} \sum_{m \in A_t} \mathbb{E} \tilde{w}_{m,Y} \sigma_\xi^2 \frac{n - u - 1}{\sqrt{n}}
\]
Then,
\[
 \tilde{w}_{m,Y} \leq \exp \left( -\frac{1}{\alpha} \left( \left\| \tilde{P}_m^\top Y \right\|^2 - \left\| \tilde{P}_s^\top Y \right\|^2 \right) \right)
\]
\[
 \leq \exp \left( -\frac{1}{\alpha} \left( \left\| \tilde{P}_m^\top \xi \right\|^2 + 2 \xi^\top \tilde{P}_m^\top \epsilon + \left\| \tilde{P}_s^\top \epsilon \right\|^2 - \left\| \tilde{P}_m^\top \epsilon \right\|^2 \right) \right)
\]
\[
 \leq \exp \left( -\frac{1}{\alpha} \left( \left\| \tilde{P}_m^\top \xi \right\|^2 + \left\| \tilde{P}_s^\top \epsilon \right\|^2 + 2 \xi^\top \tilde{P}_m^\top \epsilon \right) \right)
\]
From Lemma 6.1 and equation (8), it follows by applying Cauchy-Schwarz and conditioning on \((\tilde{X}, \tilde{Z})\) that
\[
 \mathbb{E} \tilde{w}_{m,Y} \leq \left( \mathbb{E} \exp \left( -\frac{2}{\alpha} \left( \left\| \tilde{P}_m^\top \xi \right\|^2 + \left\| \tilde{P}_s^\top \epsilon \right\|^2 \right) \right) \right)^{1/2} \left( \mathbb{E} \exp \left( -\frac{4}{\alpha} \xi^\top \tilde{P}_m^\top \epsilon \right) \right)^{1/2}
\]
\[
 \leq \left( 1 + \frac{4 \tau_m^2}{\alpha} \right)^{-(n-u-1)/4} \left( 1 + \frac{4 \sigma_\epsilon^2}{\alpha} \right)^{-(u+1)/4} \left( 1 - \frac{16 \tau_m^2 \sigma_\epsilon^2}{\alpha^2} \right)^{-(n-u-1)/4}
\]
\[
 \sim \exp \left( -\frac{1}{\alpha} \left( 1 - 4 \frac{\sigma_\epsilon^2}{\alpha} \right) \tau_m^2 (n - u - 1) \right).
\]
Since \( \alpha > 4 (\sigma_\epsilon^2 \vee \sigma_\xi^2) \), assumption \((C)\) implies
\[
 \sum_{m \in A_t} \mathbb{E} \tilde{w}_{m,Y} \sigma_\xi^2 \frac{n - u - 1}{\sqrt{n}}
\]
\[
 \leq \exp \left( -\frac{1}{\alpha} \left( 1 - 4 \frac{\sigma_\epsilon^2}{\alpha} \right) \frac{n - u - 1}{\sqrt{n}} + \log \left( \sigma_\xi^2 \frac{n - u - 1}{\sqrt{n}} \right) + \log (|A_t|) \right)
\]
\[
 \to 0.
\]
Hence,

$$\limsup_{n \to \infty} r_{\text{rand}} \leq t.$$ 

Since $t > 0$ was arbitrary, $r_{\text{rand}} \to 0$. Applying the Rao-Blackwell device finishes the proof. \hfill \square

6.1.2 Proof of Consistency of $\hat{\beta}_{\text{EW},u}$

We start by proving a useful consequence of assumption (B).

**Lemma 6.2.** Assume (A), (B), and (C). Then,

$$\liminf_{n \to \infty} \inf_{m \in \mathcal{M}_u} \frac{1}{n} \mathbb{E}\|R_m^\perp X\|^2 = \sigma^2_\eta > 0.$$ 

Moreover,

$$\limsup_{n \to \infty} \sup_{m \in \mathcal{M}_u} n\mathbb{E}\|Q_m^\perp X\|^2 \leq \limsup_{n \to \infty} \sup_{m \in \mathcal{M}_u} n\mathbb{E}\|R_m^\perp X\|^2 = \frac{1}{\sigma^2_\eta}.$$ 

**Proof.** Temporarily fix $m \in \mathcal{M}_u$ arbitrarily. By assumption (B),

$$X = Z^T \delta_T + \eta.$$ 

Thus,

$$\mathbb{E}\|R_m^\perp X\|^2 = \mathbb{E}\|R_m^\perp (Z^T \delta_T + \eta)\|^2 \geq \mathbb{E}\|R_m^\perp \eta\|^2 = \sigma^2_\eta (n - u - 1).$$ 

Rearranging,

$$\frac{1}{n - u - 1} \mathbb{E}\|R_m^\perp X\|^2 \geq \sigma^2_\eta.$$ 

Since $m \in \mathcal{M}$ was arbitrary, it follows that

$$\liminf_{n \to \infty} \inf_{m \in \mathcal{M}_u} \frac{1}{n} \mathbb{E}\|R_m^\perp X\|^2 \geq \sigma^2_\eta.$$ 

For equality, letting $m = T \in \mathcal{M}_u$ proves the first claim. For the second claim, again fix $m \in \mathcal{M}_u$ arbitrarily. The first inequality is immediate since
$R_m$ projects onto a subspace of $Q_m$. Next, since the design is fully Gaussian, we may write

$$Z_T\delta_T = Z_m\kappa_m + \zeta_m,$$

where $\kappa_m$ is a vector of regression coefficients and a Gaussian vector $\zeta_m \sim \mathcal{N}_n(0_n, \nu_m^2 I_n)$ independent of $(\eta_m, Z_m)$. Then,

$$n\mathbb{E}\left\| R_m^\perp X \right\|^2 = n\mathbb{E}\left\| R_m^\perp (\zeta_m + \eta_m) \right\|^2 = \frac{n}{(\nu_m^2 + \sigma^2)(n - u - 3)} \leq \frac{n}{\sigma^2(n - u - 3)}.$$

Since $m \in \mathcal{M}$ was arbitrary, it follows that

$$\limsup_{n \to \infty} \sup_{m \in \mathcal{M}_u} n\mathbb{E}\left\| R_m^\perp X \right\|^2 \leq \frac{1}{\sigma^2}.$$

For equality, letting $m = T \in \mathcal{M}_u$ finishes the proof. □

Instead of directly proving Lemma 3.3, by the triangle inequality, it suffices to show each individual term converges in $L_2$, which is the content of the following two lemmata.

**Lemma 6.3.** Consider the linear model given in (1). Assuming (A), (B), and (C), then

$$\left( n^{1/4} \sum_{m \in \mathcal{M}} \bar{w}_{m,Y} \frac{X^T Q_m^\perp Z_m \gamma_s}{X^T Q_m^\perp X} \right)^2 \xrightarrow{L_2} 0.$$

**Proof.** Recall $\xi \triangleq Z_m \gamma_s$. Since the design is fully Gaussian, there exists a vector of regression coefficients $\theta_m$ and a Gaussian vector $\zeta_m \sim \mathcal{N}_n(0_n, \tau_m^2 I_n)$ independent of $(X, Z_m)$ such that

$$\xi = (X, Z_m)\theta_m + \zeta_m.$$

Letting $\theta_{m,1}$ denote the first entry of $\theta_m$, it follows that

$$n^{1/4} \sum_{m \in \mathcal{M}_u} \bar{w}_{m,Y} \frac{X^T Q_m^\perp Z_s \gamma_s}{X^T Q_m^\perp X} = n^{1/4} \sum_{m \in \mathcal{M}_u} \bar{w}_{m,Y} \left( \theta_{m,1} + \frac{X^T Q_m^\perp \zeta_m}{X^T Q_m^\perp X} \right).$$

Temporarily fix $m \in \mathcal{M}$. Now, writing

$$\mathbb{E} \left\| R_m^\perp X \theta_{m,1} \right\|^2 = \mathbb{E} \left\| R_m^\perp \zeta_m \right\|^2 = \tau_m^2(n - u - 1),$$

...
it follows by Lemma 6.2 that for all \( n \) sufficiently large,
\[
\theta^2_{m,1} \leq \frac{2\tau^2_m}{\sigma^2_n} \tag{9}
\]
uniformly in \( m \). Furthermore, by a direct variance calculation and Lemma 6.2 again, for all \( n \) sufficiently large,
\[
\mathbb{E} \left( n^{1/4} \frac{X^T Q_m^\perp \epsilon}{X^T Q_m^\perp X} \right)^2 \leq \tau^2_m \sqrt{n} \mathbb{E} \| Q_m X \|^2 = \frac{2\tau^2_m}{\sqrt{n}}
\]
uniformly in \( m \). Thus, for \( n \) sufficiently large,
\[
\mathbb{E} \left( n^{1/4} \sum_{m \in \mathcal{M}_u} \bar{w}_{m,Y} \frac{X^T Q_m^\perp Z S^\gamma S}{X^T Q_m^\perp X} \right)^2 \leq 4 \sum_{m \in \mathcal{M}_u} \mathbb{E} \bar{w}_{m,Y} \tau^2_m \left( \frac{\sqrt{n}}{\sigma^2_n} + \frac{1}{\sqrt{n}} \right).
\]
Following as in the proof of Proposition 3.1 finishes the proof. \( \square \)

To finish the proof of consistency, we need one more lemma.

**Lemma 6.4.** Consider the linear model given in (1). Assuming (A), (B), and (C), then
\[
n^{1/4} \sum_{m \in \mathcal{M}_u} \bar{w}_{m,Y} \frac{X^T Q_m^\perp \epsilon}{X^T Q_m^\perp X} \overset{L_2}{\rightarrow} 0.
\]

**Proof.** Indeed,
\[
\mathbb{E} \left( n^{1/4} \sum_{m \in \mathcal{M}_u} \bar{w}_{m,Y} \frac{X^T Q_m^\perp \epsilon}{X^T Q_m^\perp X} \right)^2 \leq \sqrt{n} \sum_{m \in \mathcal{M}_u} \mathbb{E} \bar{w}_{m,Y} \mathbb{E} \left( \frac{X^T Q_m^\perp \epsilon}{X^T Q_m^\perp X} \right)^2 \leq \sigma^2_\epsilon \sqrt{n} \sum_{m \in \mathcal{M}_u} \mathbb{E} \bar{w}_{m,Y} \mathbb{E} \| Q_m X \|^{-2}.
\]

From Lemma 6.2, for all \( n \) sufficiently large,
\[
\mathbb{E} \| Q_m X \|^{-2} \leq \frac{2}{\sigma^2_n}
\]
uniformly in \( m \). Therefore,
\[
\mathbb{E} \left( n^{1/4} \sum_{m \in \mathcal{M}_u} \bar{w}_{m,Y} \frac{X^T Q_m^\perp \epsilon}{X^T Q_m^\perp X} \right)^2 \leq \frac{2\sigma^2_\epsilon}{\sigma^2_n \sqrt{n}} \sum_{m \in \mathcal{M}_u} \mathbb{E} \bar{w}_{m,Y} \leq \frac{2\sigma^2_\epsilon}{\sigma^2_n \sqrt{n}} \rightarrow 0,
\]
which finishes the proof. \( \square \)
Combining the previous two results proves Lemma 3.3.

Proof of Corollary 3.3.1. By the triangle inequality,
\[ \| Z\hat{\gamma}_{EW,u} - Z_S\delta_S \|^2 \]
\[ \leq 2 \left\| (X\hat{\beta}_{EW,u} + Z\hat{\gamma}_{EW,u}) - (X\beta + Z_S\gamma_S) \right\|^2 + 2 \| X \|^2 \left( \hat{\beta}_{EW,u} - \beta \right)^2. \]
Then, note that \( \| X \|^2 = \mathcal{O}_P(n) \). Applying Proposition 3.1 and Lemma 3.3 finishes the proof.

6.1.3 Proof of Asymptotic Normality of \( \hat{\beta}_{DEW,u} \)
We start by decomposing the estimator.
\[ \hat{\beta}_{DEW,u} = \frac{(X - Z\hat{\delta}_{EW,u})^\top (Y - Z\hat{\delta}_{EW,u}\beta - Z\hat{\gamma}_{EW,u})}{\| X - Z\hat{\delta}_{EW,u} \|^2} \]
\[ = \beta + \frac{\left( ZT\delta_T - Z\hat{\delta}_{EW,u} \right)^\top \left( Z\hat{\delta}_{EW,u} (\beta - \hat{\beta}_{EW,u}) + Z (\gamma_S - \hat{\gamma}_{EW,u}) \right)}{\| X - Z\hat{\delta}_{EW,u} \|^2} \]
\[ + \frac{\eta^\top \left( Z\hat{\delta}_{EW,u} (\beta - \hat{\beta}_{EW,u}) + Z (\gamma_S - \hat{\gamma}_{EW,u}) \right) + (X - Z\hat{\delta}_{EW,u})^\top \epsilon}{\| X - Z\hat{\delta}_{EW,u} \|^2}. \]
Again, we will proceed in steps and start with the denominator.

Lemma 6.5. Consider the linear model given in (3). Assuming (A), (B), and (C), then
\[ \frac{1}{n} \| X - Z\hat{\delta}_{EW,u} \|^2 \overset{p}{\to} \sigma_\eta^2. \]
Proof. Indeed,
\[ \frac{1}{\sqrt{n}} \| \eta \| - \frac{1}{\sqrt{n}} \| ZT\delta_T - Z\hat{\delta}_{EW,u} \| \leq \frac{1}{\sqrt{n}} \| X - Z\hat{\delta}_{EW,u} \| \]
\[ \leq \frac{1}{\sqrt{n}} \| \eta \| + \frac{1}{\sqrt{n}} \| ZT\delta_T - Z\hat{\delta}_{EW,u} \|. \]
By Proposition 3.2,
\begin{equation*}
\frac{1}{\sqrt{n}} \left\| Z_T \delta_T - Z \hat{\delta}_{EW,u} \right\| \xrightarrow{P} 0.
\end{equation*}

Now, by the Strong Law of Large Numbers,
\begin{equation*}
\frac{1}{n} \| \eta \|^2 \xrightarrow{P} \sigma^2_{\eta},
\end{equation*}
which finishes the proof. \hfill \square

**Lemma 6.6.** Consider the linear model given in (1). Assuming (A), (B), and (C), then
\begin{equation*}
\sqrt{n} \left( Z_T \delta_T - Z \hat{\delta}_{EW,u} \right)^T \left( Z \hat{\delta}_{EW,u} \left( \beta - \hat{\beta}_{EW,u} \right) + Z (\gamma_T - \hat{\gamma}_{EW,u}) \right) \xrightarrow{P} 0.
\end{equation*}

**Proof.** By Cauchy-Schwarz,
\begin{equation*}
\frac{\left( Z_T \delta_T - Z \hat{\delta}_{EW,u} \right)^T \left( Z \hat{\delta}_{EW,u} \left( \beta - \hat{\beta}_{EW,u} \right) \right)}{\left\| X - Z \hat{\delta}_{EW,u} \right\|^2} \leq \frac{\left\| Z_T \delta_T - Z \hat{\delta}_{EW,u} \right\| \left\| Z \hat{\delta}_{EW,u} \right\| \beta - \hat{\beta}_{EW,u}}{\left\| X - Z \hat{\delta}_{EW,u} \right\|^2}.
\end{equation*}

Applying Proposition 3.2 and Lemmata 3.3 and 6.5, it follows that
\begin{align*}
\left\| Z_T \delta_T - Z \hat{\delta}_{EW,u} \right\| &= o_P(n^{1/4}), \\
\beta - \hat{\beta}_{EW,u} &= o_P(n^{-1/4}), \\
\left\| X - Z \hat{\delta}_{EW,u} \right\|^2 &= O_P(n).
\end{align*}

Finally, by Proposition 3.2 again,
\begin{equation*}
\left\| Z \hat{\delta}_{EW,u} \right\| \leq \left\| Z \hat{\delta}_{EW,u} - Z T \delta_T \right\| + \left\| Z T \delta_T \right\| = o_P(n^{1/4}) + O_P(\sqrt{n}) = O_P(\sqrt{n}).
\end{equation*}

Thus,
\begin{equation*}
\sqrt{n} \frac{\left( Z_T \delta_T - Z \hat{\delta}_{EW,u} \right)^T \left( Z \hat{\delta}_{EW,u} \left( \beta - \hat{\beta}_{EW,u} \right) \right)}{\left\| X - Z \hat{\delta}_{EW,u} \right\|^2} \leq \sqrt{n} \frac{o_P(n^{1/4}) O_P(\sqrt{n}) o_P(n^{-1/4})}{O_P(n)} = o_P(1).
\end{equation*}
Similarly, by Cauchy-Schwarz,
\[
\left( Z_T \delta_T - Z \hat{\delta}_{EW,u} \right)^T \left( Z (\gamma - \hat{\gamma}_{EW,u}) \right) \leq \left\| Z_T \delta_T - Z \hat{\delta}_{EW,u} \right\| \left\| Z (\gamma - \hat{\gamma}_{EW,u}) \right\|
\]

By Corollary 3.3.1,
\[
\left\| Z (\gamma - \hat{\gamma}_{EW,u}) \right\| = \mathcal{O}_P(n^{1/4}).
\]
Therefore,
\[
\sqrt{n} \left( Z_T \delta_T - Z \hat{\delta}_{EW,u} \right)^T \left( Z (\gamma - \hat{\gamma}_{EW,u}) \right) \leq \sqrt{n} \mathcal{O}_P(n^{1/4}) = \mathcal{O}_P(1).
\]

Combining these two results finishes the proof.

Lemma 6.7. Consider the linear model given in (1). Assuming (A), (B), and (C), then
\[
\sqrt{n} \left( Z_T \delta_T - Z \hat{\delta}_{EW,u} \right)^T \left( Z (\gamma - \hat{\gamma}_{EW,u}) \right) \left\| X - Z \hat{\delta}_{EW,u} \right\| \xrightarrow{P} 0.
\]

Proof. Indeed,
\[
\eta^T Z \hat{\delta}_{EW,u} = \sum_{m \in M_u} \tilde{w}_{m,Y} \eta^T Q_m X = \sum_{m \in M_u} \tilde{w}_{m,Y} Q_m Z_T \delta_T + \sum_{m \in M_u} \tilde{w}_{m,Y} \eta^T Q_m \eta.
\]

By the triangle inequality and Cauchy-Schwarz,
\[
\mathbb{E} \left| \sum_{m \in M_u} \tilde{w}_{m,Y} \eta^T Q_m Z_T \delta_T \right| \leq \sum_{m \in M_u} \mathbb{E} \tilde{w}_{m,Y} \mathbb{E} \left| \eta^T Q_m Z_T \delta_T \right| 
\]
\[
\leq \sum_{m \in M_u} \mathbb{E} \tilde{w}_{m,Y} \sqrt{\mathbb{E} \left| Q_m \eta \right|^2 \mathbb{E} \left| Q_m Z_T \delta_T \right|^2} 
\]
\[
\leq \sum_{m \in M_u} \mathbb{E} \tilde{w}_{m,Y} \sqrt{un \sigma_X^2 \sigma_y^2} = \sigma_X \sigma_y \sqrt{un},
\]

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where \( n^{-1} \text{Var}(X) \triangleq \sigma_X^2 \). By a direct computation,

\[
\mathbb{E} \sum_{m \in M_u} \tilde{w}_{m,Y} \eta^T Q_m \eta = \sigma_u^2 u.
\]

Hence,

\[
\eta^T Z \hat{\beta}_{EW,u} = O_P(\sqrt{un}).
\]

Combining this with Lemma 3.3 and 6.5, it follows that

\[
\sqrt{n} \left\| \eta^T Z \hat{\beta}_{EW,u} \left( \beta - \hat{\beta}_{EW,u} \right) \right\|^2 = \frac{O_p(\sqrt{un})o_P(n^{-1/4})}{O_P(n)} = o_P(1).
\]

For the other term,

\[
\left| \eta^T Z (\gamma - \hat{\gamma}_{EW,u}) \right| \leq \sum_{m \in M_u} \tilde{w}_{m,Y} \left| \eta^T Q_m (Z \gamma - Z \hat{\gamma}_m) \right|
\]

\[
+ \left| \sum_{m \in M_u} \tilde{w}_{m,Y} \eta^T Q_m^\perp Z \gamma \right|
\]

\[
\leq \sum_{m \in M_u} \tilde{w}_{m,Y} \|Q_m\| \|Q_m (Z \gamma - Z \hat{\gamma}_m)\|
\]

\[
+ \left| \sum_{m \in M_u} \tilde{w}_{m,Y} \eta^T Q_m^\perp Z \gamma \right|
\]

Then, for a fixed \( m \in M \), writing

\[
\xi = (X, Z_m) \theta_m + \zeta_m
\]

with \( \zeta_m \sim \mathcal{N}_n(0_n, \tau^2_m) \) and substituting this into equation (1),

\[
Y = X(\beta + \theta_{m,1}) + Z_m \theta_{m,-1} + \zeta_m + \epsilon.
\]

Therefore, \( \hat{\gamma}_m \) is the least-squares estimator for \( \theta_{m,-1} \). By classical least-squares theory,

\[
\mathbb{E} \|Z_m(\theta_{m,-1} - \hat{\gamma}_m)\|^2 = O(u).
\]
Hence,
\[ \| Q_m (Z_S \gamma - \hat{Z}_m \gamma) \| \leq \| Q_m X_{\theta,1} \| + \| Z_m (\theta_{m,-1} - \hat{\gamma}_m) \| + \| Q_m \zeta_m \|. \]

By Cauchy-Schwarz,
\[
\mathbb{E} (\| Q_m \eta \| \| Q_m (Z_S \gamma - Z_m \hat{\gamma}_m) \|)
\leq 2\sqrt{\mathbb{E} \| Q_m \eta \|^2 \mathbb{E} \| Q_m X_{\theta,1} \|^2 + \mathbb{E} \| Z_m (\theta_{m,-1} - \hat{\gamma}_m) \|^2 + \mathbb{E} \| Q_m \zeta_m \|^2}
\leq 2\sigma \sqrt{u} n \sigma^2 X_{\theta,1} \tau_m + O(u) + u \tau_m^2.
\]

But, as was shown in equation (9), for all \( n \) sufficiently large,
\[ \theta_{m,1}^2 \leq \frac{2 \tau_m^2}{\sigma^2}. \]

Hence, for \( n \) sufficiently large,
\[ \mathbb{E} (\| Q_m \eta \| \| Q_m (Z_S \gamma - Z_m \hat{\gamma}_m) \|) \leq 4\sigma X \sqrt{u} n \tau_m. \]

Following in the proof of Proposition 3.1,
\[ \sum_{m \in \mathcal{M}_u} \mathbb{E} \bar{w}_{m,Y} 4\sigma X \sqrt{u} n \tau_m = o(\sqrt{n}). \]

Now, for \( n \) sufficiently large,
\[
\mathbb{E} \left( \sum_{m \in \mathcal{M}_u} \bar{w}_{m,Y} \eta^T Q_m^\perp Z_S \gamma S \right)^2 \leq \sum_{m \in \mathcal{M}_u} \mathbb{E} \bar{w}_{m,Y} \mathbb{E} (\eta^T Q_m^\perp \xi)^2 \leq \sum_{m \in \mathcal{M}_u} \mathbb{E} \bar{w}_{m,Y} (2\sigma^2 X n + (n - u)) \tau_m^2.
\]

Again, following in the proof of Proposition 3.1,
\[ \mathbb{E} \left( \sum_{m \in \mathcal{M}_u} \bar{w}_{m,Y} \eta^T Q_m^\perp Z_S \gamma S \right)^2 = o(\sqrt{n}). \]

Thus,
\[ \eta^T Z (\gamma - \hat{\gamma}_{EW,u}) = o_F(\sqrt{n}). \]
which implies that
\[
\sqrt{n}\eta^T Z (\gamma_S - \hat{\gamma}_{EW,u}) \overline{\|X - Z \hat{\delta}_{EW,u}\|^2} = \sqrt{n} \frac{o_P(\sqrt{n})}{\mathcal{O}_P(n)} = o_P(1).
\]
Combining the results finishes the proof.

And finally, the asymptotic normality.

**Lemma 6.8.** Consider the linear model given in (1). Assuming (A), (B), and (C), then
\[
\sqrt{n} \left( X - Z \hat{\delta}_{EW,u} \right)^T \epsilon \overline{\|X - Z \hat{\delta}_{EW,u}\|^2} (X,Z) \overset{\mathcal{L}}{\to} \mathcal{N} \left( 0, \frac{\sigma^2 \epsilon}{\sigma^2} \right).
\]

**Proof.** Indeed, normality follows from the fact that \( \epsilon \sim \mathcal{N}_n(0_n, \sigma^2 \epsilon I_n) \). For the variance, note that
\[
\text{Var} \left( \sqrt{n} \left( X - Z \hat{\delta}_{EW,u} \right)^T \epsilon \overline{\|X - Z \hat{\delta}_{EW,u}\|^2} (X,Z) \right) = \frac{\sigma^2 n}{\|X - Z \hat{\delta}_{EW,u}\|^2}.
\]
Applying Lemma 6.5 finishes the proof.

**6.1.4 Proof of \( \hat{\sigma}^2_{\xi,EW,u} \)**

**Proof of Theorem 3.6.** We begin by noting the following decomposition.
\[
\hat{\sigma}^2_{\xi,EW,u} = \frac{1}{n} \sum_{m \in M_u} \tilde{w}_{m,\xi} \|Q_m Y - \xi + \xi\|^2
\]
\[
= \frac{1}{n} \sum_{m \in M_u} \tilde{w}_{m,\xi} \left( \|Q_m Y - \xi\|^2 + 2 \xi^T (Q_m Y - \xi) + \|\xi\|^2 \right)
\]
\[
= \frac{1}{n} \sum_{m \in M_u} \tilde{w}_{m,\xi} \left( \|Q_m Y - \xi\|^2 + 2 \|Q_m Y\|^2 - 2 \epsilon^T Q_m \xi - 2 \|Q_m \epsilon\|^2 - \|\xi\|^2 \right)
\]
\[
= 2 \hat{\sigma}^2_{\xi,EW,u} + \frac{1}{n} \sum_{m \in M_u} \tilde{w}_{m,\xi} \left( \|Q_m Y - \xi\|^2 - 2 \epsilon^T Q_m \xi - 2 \|Q_m \epsilon\|^2 - \|\xi\|^2 \right).
\]
Rearranging, it follows that
\[
\hat{\sigma}^2_{\xi,EW,u} = \frac{1}{n} \|\xi\|^2 + \frac{2}{n} \epsilon^T \xi - \frac{2}{n} \sum_{m \in M_u} \tilde{w}_{m,\xi} \epsilon^T Q_m \xi
\]
\[-\frac{1}{n} \sum_{m \in M_u} \tilde{w}_{m,\xi} \|Q_m Y - \xi\|^2 + \frac{2}{n} \sum_{m \in M_u} \tilde{w}_{m,\xi} \|Q_m \epsilon\|^2.
\]
The first result follows immediately by setting $\xi \equiv 0_n$. Now, applying Proposition 3.2 with $Y$ and $\xi$ playing the role of $X$ and $Z_\delta$ respectively, it follows that
\[
\frac{1}{\sqrt{n}} \sum_{m \in M_u} \tilde{w}_{m,\xi} \|Q_m Y - \xi\|^2 \overset{p}{\to} 0.
\]
Moreover, by the Strong Law of Large Numbers
\[
\frac{2}{\sqrt{n}} \sum_{m \in M_u} \tilde{w}_{m,\xi} \|Q_m \epsilon\|^2 \overset{p}{\to} 0.
\]
Furthermore, we have that
\[
\mathbb{E} \left( \frac{1}{\sqrt{n}} \sum_{m \in M_u} \tilde{w}_{m,\xi} \epsilon^T Q_m^\perp \xi \right)^2 \leq \frac{1}{n} \mathbb{E} \sum_{m \in M_u} \tilde{w}_{m,\xi} \left( \epsilon^T Q_m^\perp \xi \right)^2
\]
\leq \frac{\sigma_x^2}{n} \mathbb{E} \sum_{m \in M_u} \tilde{w}_{m,\xi} \|Q_m^\perp \xi\|^2
\leq \frac{\sigma_x^2}{n} \mathbb{E} \sum_{m \in M_u} \tilde{w}_{m,\xi} \|Q_m^\perp \xi - Q_m \epsilon + Q_m \epsilon\|^2
\leq \frac{4\sigma_x^2}{n} \mathbb{E} \sum_{m \in M_u} \tilde{w}_{m,\xi} \left( \|Q_m^\perp \xi - Q_m \epsilon\|^2 + \|Q_m \epsilon\|^2 \right)
\to 0.
\]
The last line from the above equation follows again from Proposition 3.2. Therefore,
\[
\sqrt{n} \left( \hat{\sigma}^2_{\xi,EW,u} - \sigma_x^2 \right) = \sqrt{n} \left( \frac{1}{n} \|\xi\|^2 - \sigma_x^2 \right) + \frac{2}{\sqrt{n}} \epsilon^T \xi + o_\mathbb{P}(1).
\]
The asymptotic distribution follows from two applications of the Central Limit Theorem. \qed

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Proof of Lemma 3.8. It is known that the least-squares estimator, $\hat{\gamma}_{LS}$, is efficient for estimating $\gamma$ in the low-dimensional linear model. Since $Z$ is assumed to be of full rank, there exists a smooth re-parameterization of the problem given by $(\gamma, \sigma^2) \mapsto (\sigma^2_\xi, \vartheta, \sigma^2_\epsilon)$, where $(\sigma^2_\xi, \vartheta)$ is the $p$-dimensional polar representation of $\|Z\gamma\|^2$. Taking the bowl-shaped loss to be quadratic in the first component, the result follows from the arguments of Section 2.3 of Bickel, Klaassen, Ritov & Wellner (1993) since $\|QY\|^2 = \|Z\hat{\gamma}_{LS}\|^2$. □

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